Primes with a missing digit: Distribution in arithmetic progressions and an application in sieve theory

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Abstract
We prove Bombieri–Vinogradov type theorems for primes with a missing digit in their $b$-adic expansion for some large positive integer $b$. The proof is based on the circle method, which relies on the Fourier structure of the integers with a missing digit and the exponential sums over primes in arithmetic progressions. Combining our results with the semi-linear sieve, we obtain an upper bound and a lower bound of the correct order of magnitude for the number of primes of the form $p = 1 + m^2 + n^2$ with a missing digit in a large odd base $b$.

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PART I. MAIN RESULTS AND OUTLINE OF THE PROOF

1 | INTRODUCTION

Let $b \geq 3$ be an integer and let $a_0 \in \{0, 1, \ldots, b - 1\}$. Consider

$$ \mathcal{A} := \left\{ \sum_{j \geq 0} n_j b^j : n_j \in \{0, \ldots, b - 1\} \setminus \{a_0\} \text{ for all } j \right\} $$

the set of nonnegative integers without the digit $a_0$ in their $b$-adic expansion. For any $k \in \mathbb{N}$, the cardinality of the set $\mathcal{A} \cap [1, b^k)$ is $\approx (b - 1)^k$.

For the rest of the paper, we set $X = b^k$ and note that there are $\approx X^{\zeta}$ elements in $\mathcal{A}$ less than $X$, where

$$ \zeta := \frac{\log(b - 1)}{\log b} < 1. $$

This reveals that $\mathcal{A}$ is a “sparse set”. It is often the case that sparseness is one of the obstacles in analytic number theory. However, the set $\mathcal{A}$ admits some interesting structure in the sense that its Fourier transform has an explicit description, which is often small in size. There has been a considerable amount of work (see Dartyge–Mauduit [1, 2], Erdős–Mauduit–Sárközy [4, 5], Konyagin [11], Maynard [15–17], and Pratt [19]) in this direction by exploiting the Fourier structure of the set $\mathcal{A}$.

Remark. Note that $\zeta \to 1$ as $b \to \infty$. We shall have many occasions to use this fact in the paper, and we do so without further comment.

It is a natural question to ask if the set $\mathcal{A}$ contains infinitely many primes. We expect the answer to be affirmative. In his celebrated paper [16], Maynard showed that the set $\mathcal{A}$ contains infinitely many primes for any base $b \geq 10$. Moreover, for a large base, say $b \geq 2 \times 10^6$, he established an asymptotic formula (see [15, Theorem 2.5] or [17, Theorem 1.1]).

Prior to Maynard’s work, Dartyge–Mauduit [1, 2] showed the existence of infinitely many almost-primes (integers with at most 2 prime factors) in $\mathcal{A}$ for any base $b \geq 3$. They used crucially the fact that $\mathcal{A}$ is well-distributed in arithmetic progressions (see, for example, [1, 4]). In that spirit, we are interested in understanding how the primes of $\mathcal{A}$ are distributed in arithmetic progressions. For $(c, d) = 1$ and $(d, b) = 1$, one expects that as $X \to \infty$,

$$ \#\{p < X : p \equiv c \pmod{d}, p \in \mathcal{A}\} \sim \frac{1}{\varphi(d)} \#\{p < X : p \in \mathcal{A}\} $$

where $\varphi(d)$ is Euler's totient function.
holds uniformly for \( d \leq X^{\varepsilon(1-\varepsilon)} \) with any fixed \( \varepsilon > 0 \). This seems to be a difficult question at present. Instead, we aim for a Bombieri–Vinogradov theorem of the following type:

\[
\sum_{d \leq D} \max_{(c,d) = 1} \left\{ \frac{1}{\varphi(d)} \# \{ p < X : p \equiv c \pmod{d}, p \in \mathcal{A} \} - \frac{1}{\varphi(d)} \# \{ p < X : p \in \mathcal{A} \} \right\} \ll_{A,b} \frac{X^\varepsilon}{(\log X)^A},
\]

where \( D \leq X^{1/2-\varepsilon} \), for any fixed \( \varepsilon > 0 \), provided that \( b \) is large enough in terms of \( \varepsilon \) (so that \( \zeta \) is close enough to 1). However, using the current techniques, we are not able to prove that the above estimate holds for \( D \leq X^{1/2-\varepsilon} \). Nevertheless, we can make some progress in this direction.

For technical convenience, we will work with the von Mangoldt function \( \Lambda \) (recall that \( \Lambda(n) = \log p \) if \( n = p^m \), and 0 otherwise). For \( X = b^k \) with \( k \in \mathbb{N} \) and for \( (c,d) = (r,b) = 1 \), we set

\[
\mathcal{E}(X; d, c; b, r) := \sum_{n < X} \Lambda(n) \chi_1(n) \frac{1}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A(n)}(n).
\]

Note that

\[
\sum_{n < X} 1_{A(n)}(n) = \frac{X^\varepsilon}{b-1}
\]

whenever \( r \not\equiv a_0 \pmod{b} \); otherwise, both sums in the definition of \( \mathcal{E}(X; d, c; b, r) \) is 0. Moreover, the condition \( n \equiv r \pmod{b} \) is equivalent to \( n \) having \( r \) as its last digit in its \( b \)-adic expansion. We add this condition in order to simplify some technical details later on.

**Theorem 1.** Let \( \delta > 0 \) and let \( b \) be an integer that is sufficiently large in terms of \( \delta \). Let \( D \in [1, X^{1/3-\delta}] \) and let \( r \in \mathcal{A} \cap [0, b) \) be an integer such that \( (r, b) = 1 \). Then for any \( A > 0 \), we have

\[
\sum_{d \leq D} \max_{(c,d) = 1} \left| \mathcal{E}(X; d, c; b, r) \right| \ll_{A,b,\delta} \frac{X^\varepsilon}{(\log X)^A}.
\]

We can do a little better if we allow the moduli to be the product of two integers. However, the parameter \( c \) is now fixed, so we must drop the expression \( \max_{(c,d) = 1} \) from (1.1).

**Theorem 2.** Let \( \delta > 0 \), let \( b \) be an integer that is sufficiently large in terms of \( \delta \), and let

\[
D_1 \in [1, X^{1/3-\delta} \] and \( D_2 \in [1, X^{1/9}] \).
\]

Let \( c \) be a nonzero integer and let \( r \in \mathcal{A} \cap [0, b) \) with \( (r, b) = 1 \). Then for any \( A > 0 \), we have

\[
\sum_{d_1 \leq D_1} \sum_{d_2 \leq D_2} \left| \mathcal{E}(X; d_1 d_2, c; b, r) \right| \ll_{A,b,\delta} \frac{X^\varepsilon}{(\log X)^A},
\]

where \( ^* \) in the sum denotes the conditions \( (c, d_1 d_2) = (b, d_1 d_2) = (d_1, d_2) = 1 \).
We can further have better results in this direction when we replace the absolute value inside the sum over \( d \) with a well-factorable function. Before proceeding to state our result, we formally define the “well-factorable” function.

**Definition 1 (Well-factorable).** Let \( D \geq 1 \) be a real number. We say an arithmetic function \( \xi : \mathbb{N} \to \mathbb{R} \) well-factorable of level \( D \) if, for any choice of factorization \( D = D_1D_2 \) with \( D_1, D_2 \geq 1 \), there exist two arithmetic functions \( \xi_1, \xi_2 : \mathbb{N} \to \mathbb{R} \) such that

(i) \( |\xi_1|, |\xi_2| \leq 1 \);  
(ii) \( \xi_1 \) is supported on \([1, D_1]\) and \( \xi_2 \) is supported on \([1, D_2]\);  
(iii) we have

\[
\xi(d) = \sum_{\substack{d_1d_2=d \ \ (d_1,d_2)=1}} \xi_1(d_1)\xi_2(d_2).
\]

With this definition, we are now ready to state the following result.

**Theorem 3.** Let \( \delta > 0 \) and let \( b \) be an integer that is sufficiently large in terms of \( \delta \). Let \( \xi : \mathbb{N} \to \mathbb{R} \) be a well-factorable arithmetic function of level \( D \in [1, X^{1/2-\delta}] \). Let \( c \) be a nonzero integer and let \( r \in \mathbb{A} \cap [0, b) \) be such that \((r, b) = 1\). Then, for any \( A > 0 \), we have

\[
\sum_{d \leq D, (d,bc)=1} \xi(d)E(X; d, c; b, r) \ll_{A,b,\delta} X^\xi (\log X)^A.
\]

**Remark.** We have not explicitly mentioned in the above three theorems the size of \( b \). In fact, it will be evident from the proof that \( \delta \) and \( b \) are inversely related to each other. A simple calculation suggests that the size of \( b \) is approximately of order \( 10^{632} \) if we take \( \delta = 1/100 \). Therefore, we will refrain from explicitly calculating \( \delta \) and \( b \) in the above three theorems.

The key point of Theorems 2 and 3 is the quantitative improvement over Theorem 1 allowing us to handle moduli as large as \( X^{4/9-\delta} \) and \( X^{1/2-\delta} \), respectively (instead of \( X^{1/3-\delta} \)). However, Theorem 2 has the disadvantage that it has a stronger requirement that the moduli need to be composite and Theorem 3 requires moduli weighted by a well-factorable function. But, in some of the applications of sieve theory, we do have well-factorable moduli. In fact, we give such an application in this paper: we prove the existence of infinitely many primes of the form \( p = 1 + m^2 + n^2 \) with a missing digit in a large odd base \( b \). The following theorem gives a precise statement.

**Theorem 4.** Let \( b \) be an odd integer that is sufficiently large, and let

\[
\mathbb{B} = \{ n \mid n = n_1^2 + n_2^2 \text{ for some } (n_1, n_2) = 1 \}.
\]

\(^{1}\) In general, we do not require the co-primality condition in the definition of \( \xi \). However, in order to avoid some technical issues, we impose this condition here.
Let \( r \in A \cap [0, b) \) with \( (r(r - 1), b) = 1 \). Then, we have

\[
\sum_{\substack{p < X \\ p \equiv r \pmod{b}}} 1_A(p) 1_B(p - 1) \approx_b \frac{X^5}{(\log X)^{3/2}}.
\]

**Remark.** The implicit upper bound in Theorem 4 follows from Theorem 1 and a standard upper bound sieve estimate (for example, see Lemma 3.1). However, for the lower bound, we need to be more careful and use an argument due to Iwaniec [8, 9] that allows sieving for primes of the form \( 1 + m^2 + n^2 \) using level of distribution slightly less than \( X^{1/2} \). Additionally, in order to use the sieve estimates efficiently, we need two technical results, namely, Theorems 5 and 6 (similar in nature to Theorems 2 and 3).

**Notations.** We employ some standard notation that will be used throughout the paper.

- Expressions of the form \( f(X) = O(g(X)) \), \( f(X) \ll g(X) \) and \( g(X) \gg f(X) \) signify that \( |f(X)| \leq C |g(X)| \) for all sufficiently large \( X \), where \( C > 0 \) is an absolute constant. A subscript of the form \( \ll_A \) means the implied constant may depend on the parameter \( A \). The notation \( f(X) \approx g(X) \) indicates that \( f(X) \ll g(X) \ll f(X) \). Here all the quantities should be thought of \( X = b^k \) with \( k \) an integer and \( k \to \infty \).
- All sums, products and maxima will be taken over \( \mathbb{N} = \{1, 2, \ldots\} \) unless specified otherwise.
- We reserve the letters \( p, p', p_1, p_2 \) to denote primes.
- The letter \( \gamma \) will always denote the Euler–Mascheroni constant.
- As usual, \( \mathbb{R} \) will denote the set of real numbers, \( \mathbb{P} \) the set of primes and \( \mathbb{Z} \) the set of integers. Furthermore, \( p^v \| m \) means that \( p^v | m \) and \( p^{v+1} \nmid m \).
- Throughout the paper, \( \varphi \) will denote the totient function, \( \mu \) the Möbius function, and \( \tau_h(n) \) the number of ways of writing \( n \) as a product of \( h \) natural numbers.
- As it is customary, we denote \( e(y) = e^{2\pi i y} \) for any real number \( y \). We write \( n \sim N \) to denote \( N < n \leq 2N \). We use \( \| y \| \) to denote \( \min_{n \in \mathbb{Z}} |y - n| \).
- Unless otherwise specified, \( \chi \) will always denote a Dirichlet character modulo some positive integer. The symbol \( \chi_0 \) will always denote a principal character.
- We will set \( (a, b) \) to be the greatest common divisor of integers \( a \) and \( b \) and by abuse of notation it will also denote the open interval on the real line. On the other hand, \( [a, b] \) will denote the closed interval on the real line, and sometimes it will denote the least common multiple of integers \( a \) and \( b \). Its exact meaning will always be clear from the context.
- For co-prime integers \( m \) and \( n \), we set \( \overline{n} \) to denote the inverse of \( n \) modulo \( m \), that is, \( n\overline{n} \equiv 1 \pmod{m} \).
- We let \( 1_E \) to be the characteristic function of the set \( E \) (so \( 1_E(x) = 1 \) if \( x \in E \), and 0, otherwise).
- For any set \( E \), \#\( E \) denotes the cardinality of the set \( E \).
- For any two arithmetic functions \( f \) and \( g \), we write \( (f * g)(n) := \sum_{ab=n} f(a)g(b) \) for their Dirichlet convolution.
- For any arithmetic function \( f : \mathbb{N} \to \mathbb{C} \), we set \( \|f\|_2 := (\sum_n |f(n)|^2)^{1/2} \).
- For any arithmetic function \( F \), we also set \( F_{\leq U}(n) := F(n) \cdot 1_{n \leq U} \) and \( F_{> U}(n) := F(n) \cdot 1_{n > U} \).
- We set \( B = \{n \in \mathbb{Z} : n = n_1^2 + n_2^2 \text{ for some } (n_1, n_2) = 1\} \) and \( B = \{n \geq 1 : p|n \Rightarrow p \equiv 1 \pmod{4}\} \).
We set $X = b^k$ with $k \in \mathbb{N}$ and $k \to \infty$ for the rest of the paper except Part III. Throughout, we fix a choice of an integer $b \geq 3$ and $a_0 \in \{0, 1, \ldots, b - 1\}$, and we set

$$\mathcal{A} := \left\{ \sum_{j \geq 0} n_j b^j : n_j \in \{0, \ldots, b - 1\} \setminus \{a_0\} \forall j \right\}.$$ 

In addition, given an integer $r \in \mathcal{A} \cap [0, b)$, we let

$$\mathcal{A}_r = \{n \in \mathcal{A} : n \equiv r \pmod{b}\}.$$ 

For $(c, d) = (r, b) = 1$, we set

$$\mathcal{E}(X; d, c; b, r) = \sum_{n < X} \Lambda(n) 1_{\mathcal{A}}(n) - \frac{1}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X \atop n \equiv r \pmod{b}} 1_{\mathcal{A}}(n).$$

Furthermore, we set $\zeta := \frac{\log(b-1)}{\log b}$ for the rest of the paper.

**Organization of the Paper.** We will give a proof outline in Section 2 following Maynard [17], which is based on the circle method.

We devote Part II to establishing Theorem 4.

The graphical structure for Sections 3 and 4 can be described below:

In Part III, we will establish exponential sum estimates over primes in arithmetic progressions, which is one of the key ingredients to prove our main results.

Finally, in Part IV we will employ the circle method to establish our main theorems. In particular, we will deduce Theorems 1, 2, 3, 5, and 6 from a more general theorem, Theorem 7 in Section 8.
The dependency graph for Part IV leading to the proofs of Theorems 1, 2, 3, 5, and 6 is given below:

**2 SET-UP AND OUTLINE OF THE PROOF**

The strategy to prove Theorems 1, 2, and 3 is to apply the circle method. For the sake of exposition, we will outline the proof of Theorem 1 following the set-up from Maynard [17].

Let $\hat{1}_\mathcal{A}$ be the Fourier transform of the set $\mathcal{A}$ restricted to $\{1, \ldots, X\}$ with $X = b^k$. Then, for any real number $\theta \in [0, 1)$, we have

$$\hat{1}_\mathcal{A}(\theta) := \sum_{n<X} 1_\mathcal{A}(n)e(n\theta) = \prod_{j=0}^{k-1} \left( \sum_{0 \leq n_j < b} 1_\mathcal{A}(n_j) e(n_j b^j \theta) \right),$$

(2.1)

where $n = \sum_{j=0}^{k-1} n_j b^j$. Next, for $r \in \mathcal{A}$, we set

$$\mathcal{A}_r = \{ n \in \mathcal{A} : n \equiv r (\text{mod } b) \}.$$

We then define

$$\hat{1}_{\mathcal{A}_r}(\theta) := \sum_{n<X} 1_{\mathcal{A}_r}(n)e(n\theta)$$

$$= e(r\theta) \prod_{j=1}^{k-1} \left( \sum_{0 \leq n_j < b} 1_{\mathcal{A}_r}(n_j) e(n_j b^j \theta) \right).$$

(2.2)

Note that for $r \in \mathcal{A} \cap [0, b)$ and for any real number $\theta \in [0, 1)$, we have the trivial bound:

$$|\hat{1}_{\mathcal{A}_r}(\theta)| \leq \frac{X^\epsilon}{b - 1} \leq X^\epsilon,$$

which we will often use in the paper.
Next, by Fourier inversion on $\mathbb{Z}/X\mathbb{Z}$, for $n < X$, we have

$$1_{A_r}(n) = \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) e \left( -\frac{nt}{X} \right).$$

(2.3)

To prove Theorem 1, we consider the following set-up. For $(c, d) = 1$ and for any real number $\theta \in [0, 1)$, we set

$$\hat{\Lambda}_{d, c}(\theta) = \sum_{n < X \atop n \equiv c (\mod d)} \Lambda(n) e(n\theta).$$

(2.4)

Then, by the relations (2.3) and (2.4), we have

$$\sum_{n < X \atop n \equiv c (\mod d)} \Lambda(n)1_{A_r}(n) = \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{\Lambda}_{d, c} \left( -\frac{t}{X} \right).$$

(2.5)

Therefore, our task in (1.1) reduces to showing that

$$\sum_{d \leq D \atop (d, b) = 1} \max_{1 \leq c < d} \left| \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{\Lambda}_{d, c} \left( -\frac{t}{X} \right) - \frac{1}{\varphi(d) \varphi(b)} \sum_{n < X} 1_{A_r}(n) \right| \ll A, b, \delta \frac{X^\varepsilon}{(\log X)^A}. \tag{2.6}$$

We then consider two cases according to whether $t/X$ is close to a rational number with a small denominator or not, namely, major arcs and minor arcs, respectively.

**Major arcs:** The major arcs $\mathcal{M}$ are those $t$’s in $[0, X) \cap \mathbb{Z}$ such that

$$\left| \frac{t}{X} - \frac{a}{q} \right| \leq (\log X)^C \frac{C}{X}$$

for some $(a, q) = 1, 0 \leq a < q, 1 \leq q \leq (\log X)^C$ with $C > 0$ to be chosen later in terms of $A$. It will be convenient to divide the major arcs $\mathcal{M}$ into three disjoint subsets:

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3,$$

(2.7)

where

$$\mathcal{M}_1 = \left\{ t \in [0, X) \cap \mathbb{Z} : \left| \frac{t}{X} - \frac{a}{q} \right| \leq \frac{(\log X)^C}{X} \text{ for some } (a, q) = 1, 1 \leq a < q \leq (\log X)^C, q \mid X \right\},$$

$$\mathcal{M}_2 = \left\{ t \in [0, X) \cap \mathbb{Z} : \frac{t}{X} = \frac{a}{q} + \frac{\eta}{X} \right\}$$

for some $(a, q) = 1, 0 \leq a < q \leq (\log X)^C, q \geq 1, q \mid X, 0 < |\eta| \leq (\log X)^C,$

$$\mathcal{M}_3 = \left\{ t \in [0, X) \cap \mathbb{Z} : \frac{t}{X} = \frac{a}{q} \text{ for some } (a, q) = 1, 0 \leq a < q \leq (\log X)^C, q \geq 1, q \mid X \right\}.$$
(a) We use the $L^\infty$ bound for the Fourier transform of the set $\mathcal{A}_r$ and the trivial bound for $\hat{\Lambda}_{d,c}(t/X)$ to estimate the sum (2.6) when $t \in \mathcal{M}_1$.

(b) It turns out that when $t \in \mathcal{M}_2$, we can use the Bombieri–Vinogradov theorem to handle the exponential sum $\hat{\Lambda}_{d,c}(t/X)$ and the trivial bound for $\hat{1}_{\mathcal{A}_r}(t/X)$ in (2.6).

(c) When $t \in \mathcal{M}_3$, we get the main term in (2.6) and the error term is again controlled by using the Bombieri–Vinogradov theorem. We note that $\hat{1}_{\mathcal{A}_r}(t/X)$ is large if $t$ is close to a number with few nonzero base-$b$ digits.

This will establish our estimate in (2.6) when $t/X$ is in major arcs.

Minor arcs: The “minor arcs” $m$ are those $t \in [0,X) \cap \mathbb{Z}$ such that $t \not\in \mathcal{M}$. We use a $L^\infty - L^1$ bound to handle minor arcs as follows:

$$\sum_{d \leq D} \max_{(c,d)=1} \left| \frac{1}{X} \sum_{t \in m} \hat{1}_{\mathcal{A}_r} \left( \frac{t}{X} \right) \hat{\Lambda}_{d,c} \left( \frac{-t}{X} \right) \right| \leq \left( \sup_{t \in m} \sum_{d \leq D} \max_{(c,d)=1} \left| \hat{\Lambda}_{d,c} \left( \frac{-t}{X} \right) \right| \right) \sum_{t \in m} \frac{1}{X} \left| \hat{1}_{\mathcal{A}_r} \left( \frac{t}{X} \right) \right|. \tag{2.8}$$

As in Maynard [17], we use a large-sieve type argument to control the $L^1$ sum of $\hat{1}_{\mathcal{A}_r}$, which is shown to be small in Lemma 9.3. Next, our goal is to save over the trivial bound on $\sum_{d \leq D} \max_{(c,d)=1} |\hat{\Lambda}_{d,c}(t/X)|$ when $t \in m$ and $D$ as large as possible. We use estimates from exponential sum over primes in arithmetic progressions from the works of Matomäki [14], Mikawa [18], and Teräväinen [20] to handle those sums over primes in Part III. Combining these $L^1$ and $L^\infty$ bounds, we will show that

$$\left( \sup_{t \in m} \sum_{d \leq D} \max_{(c,d)=1} \left| \hat{\Lambda}_{d,c} \left( \frac{-t}{X} \right) \right| \right) \sum_{t \in m} \frac{1}{X} \left| \hat{1}_{\mathcal{A}_r} \left( \frac{t}{X} \right) \right| \ll A,b,\delta \frac{X^\epsilon}{(\log X)^A}.$$ 

This completes the rough outline of the proof of Theorem 1.

The key difference in the proofs of Theorems 2 and 3 compared to Theorem 1 is better exponential sums estimate over primes in arithmetic progressions, which allows us to take a bigger range of the moduli $d \leq D$.

Remark. Note that we will establish a much more general theorem, Theorem 7, for an arithmetic function $\mathfrak{f}$ satisfying some appropriate conditions in Part IV. In particular, Theorem 7 will incorporate Theorems 1, 2, and 3 by choosing $\mathfrak{f}$ and other parameters appropriately.

PART II. SIEVE METHODS AND THEIR APPLICATIONS

3  |  PRELIMINARIES FROM SIEVE METHODS

In this section, we collect some technical results from sieve methods that will be needed to prove Theorem 4.
Given a sequence of weights \( C = (c(n))_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0} \) with \( \sum_{n=1}^{\infty} c(n) < \infty \) and a set of primes \( P \), we consider the sifting function,

\[
S(C, P, z) := \sum_{(n, P(z)) = 1} c(n),
\]

where for some real number \( z > 1 \),

\[
P(z) := \prod_{\substack{p \leq z \\ p \in P}} p.
\]

Here, \( z \) is often called the sifting parameter in the sieve setting.

To proceed further, for any \( x \geq 1 \) and for each integer \( d \geq 1 \), we set

\[
C_d(x) := \sum_{\substack{n \leq x \\ d \mid n}} c(n),
\]

and we impose the following axioms of sieve theory:

(A1) For some multiplicative function \( g \), we have

\[
C_d(x) = \frac{g(d)}{d} C(x) + E(d),
\]

where \( C(x) \) can be interpreted as an approximation to \( \sum_{n \leq x} c(n) \) and \( E(d) \) is a real number which we think of as an error term.

(A2) We assume that the multiplicative function \( g \) satisfies \( g(p) \leq \min\{2, p - 1\} \) for all primes \( p \in P \).

(A3) There is a constant \( A > 0 \), and a quantity \( D \geq 1 \) such that

\[
\sum_{d \leq D} \mu^2(d)|E(d)| \ll_A \frac{C(x)}{(\log x)^A}.
\]

If such an estimate holds, then we say \( C \) has level of distribution \( D \).

(A4) We have

\[
\sum_{\substack{p \leq x \\ p \in P}} \frac{g(p)\log p}{p} = \alpha \log x + O(1) \quad \text{for all } x.
\]

Here we say \( \alpha \) as the dimension of the sieve.

Next, we state the definition of what is an upper bound sieve and a lower bound sieve.

**Definition 2** (Upper bound sieve). An arithmetic function \( \lambda^+ : \mathbb{N} \to \mathbb{R} \) that is supported on the set \( \{d \mid P(z) : d \leq D\} \) and satisfies the relation \( (\lambda^+ * 1)(n) \geq 1_{(n, P(z)) = 1} \) is called an upper bound sieve of level \( D \) for the set of primes \( P \).
**Definition 3** (Lower bound sieve). An arithmetic function \( \lambda^- : \mathbb{N} \to \mathbb{R} \) that is supported on the set \( \{ d | P(z) : d \leq D \} \) and satisfies the relation \( 1_{n, P(z)} = 1 \geq (\lambda^- \ast 1)(n) \) is called a lower bound sieve of level \( D \) for the set of primes \( P \).

**Remark.** We will refer to \( \lambda^\pm \) as the sieve weights or sifting weights in this paper.

Now we are ready to state the Fundamental Lemma of Sieve Theory in the special case when the dimension \( \varkappa \) equals \( 1/2 \), often referred to as the semi-linear sieve or the half-dimensional sieve.

**Lemma 3.1** (Fundamental Lemma for the semi-linear sieve). Consider a sequence \( C = (c(n))_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0} \) and a set of primes \( P \) satisfying axioms (A1), (A2), and (A4) with \( \varkappa = 1/2 \). If \( u_1 > 0 \) and \( D = z_1^{u_1} \), then there exist two arithmetic functions \( \lambda^\pm_{\text{sem}} : \mathbb{N} \to [-1,1] \) supported on the set \( \{ d | P(z_1) : d \leq D \} \), and we have

\[
S(C, P, z_1) \geq C(x) \left\{ f_{\text{sem}}(u_1) + o(1) \right\} \prod_{p \leq z_1, p \in P} \left( 1 - \frac{g(p)}{p} \right) - \sum_{p \mid p \in P \atop d \leq D} \lambda^-_{\text{sem}}(d)E(d), \tag{3.1}
\]

and

\[
S(C, P, z_1) \leq C(x) \left\{ F_{\text{sem}}(u_1) + o(1) \right\} \prod_{p \leq z_1, p \in P} \left( 1 - \frac{g(p)}{p} \right) + \sum_{p \mid p \in P \atop d \leq D} \lambda^+_{\text{sem}}(d)E(d), \tag{3.2}
\]

where \( f_{\text{sem}}, F_{\text{sem}} \) are continuous functions in \( u_1 = \log D / \log z_1 \) such that

\[
\begin{cases}
\sqrt{u_1}F_{\text{sem}}(u_1) = 2\sqrt{e^\gamma / \pi} & \text{if } 0 < u_1 \leq 2, \\
f_{\text{sem}}(u_1) = 0 & \text{if } 0 < u_1 \leq 1,
\end{cases} \tag{3.3}
\]

where \( \gamma \) is the Euler–Mascheroni constant, and for \( 1 \leq u_1 \leq 3 \) we have

\[
\frac{\sqrt{u_1}f_{\text{sem}}(u_1)}{\sqrt{e^\gamma / \pi}} = \int_1^{u_1} \frac{dy}{\sqrt{y(y-1)}} = \log \left( 1 + 2(u_1 - 1) + 2\sqrt{u_1(u_1 - 1)} \right). \tag{3.4}
\]

**Proof.** The proof follows from [6, Theorems 11.12–11.13] with \( \beta = 1 \) and [6, chapter 14, pp. 275–276]. \( \square \)

We also state the partial well-factorability (see Definition 1) of the semi-linear sieve in the next lemma.

**Lemma 3.2** (Partial well-factorability of semi-linear sieve). Let \( \varepsilon > 0 \) be small. Let \( \delta \in (0, 10^{-3}] \) and let \( \rho_{\text{sem}} = \frac{2}{\gamma} (1 - 4\delta) - \varepsilon \). Then the lower bound semi-linear sieve weights \( \lambda^-_{\text{sem}} \) as given in Lemma 3.1 with level \( X^\rho_{\text{sem}} \) and sifting parameter \( z_1 \leq X^{1/3 - 2\delta - 2\varepsilon^2} \) is supported in the set

\[
\mathfrak{S}^{-, \text{sem}} = \{ p_1 \cdots p_r < X^{\rho_{\text{sem}}} : z_1 \geq p_1 > \ldots > p_r, p_1 \cdots p_{2m-1}p_{2m}^2 \leq X^{\rho_{\text{sem}}} \forall m \geq 1 \}, \tag{3.5}
\]
where \( p_1, \ldots, p_r \) denote primes. In addition, for any \( D_0 \in [X^{1/5} - 4\delta - 2\varepsilon^2, X^{\rho_{\text{lin}}}] \), every \( d \in \mathfrak{D}^{-,\text{sem}} \cap [X^{1/10}, X^{\rho_{\text{lin}}}] \) can be factorized as \( d = d_1 d_2 \) such that \( d_1 \in [X^{1/10}, D_0] \) and \( d_1 d_2^2 \leq X^{1-4\delta-2\varepsilon^2}/D_0 \).

Proof. This is [20, Lemma 9.2] with \( \theta = \delta \) and \( D = D_0 \).

Next, we state the Fundamental Lemma for the linear sieve, that is, for dimension \( \kappa = 1 \).

**Lemma 3.3** (Fundamental Lemma for the linear sieve). Consider a sequence \( C = (c(n))_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0} \) and a set of primes \( \mathcal{P} \) satisfying axioms (A1), (A2), and (A4) with \( \kappa = 1 \). If \( u_2 > 0 \) and \( D = z_{2u}^{(2)} \), then there exist two arithmetic functions \( \lambda_{\text{lin}}^\pm : \mathbb{N} \to [-1, 1] \) supported on the set \( \{d \mid \mathcal{P}(z_2) : d \leq D\} \), and we have

\[
S(C, \mathcal{P}, z_2) \geq C(x) \left\{ f_{\text{lin}}(u_2) + o(1) \right\} \prod_{p \in z_2, p \mid P} \left( 1 - \frac{\vartheta(p)}{p} \right) - \sum_{p \mid d \leq D} \lambda_{\text{lin}}^-(d)E(d), \tag{3.6}
\]

and

\[
S(C, \mathcal{P}, z_2) \leq C(x) \left\{ F_{\text{lin}}(u_2) + o(1) \right\} \prod_{p \in z_2, p \mid P} \left( 1 - \frac{\vartheta(p)}{p} \right) + \sum_{p \mid d \leq D} \lambda_{\text{lin}}^+(d)E(d), \tag{3.7}
\]

where \( f_{\text{lin}}, F_{\text{lin}} \) are continuous functions in \( u_2 = \log D / \log z_2 \) such that

\[
\begin{aligned}
  u_2 F_{\text{lin}}(u_2) &= 2e^\gamma & \text{if} & \ 1 \leq u_2 \leq 3, \\
  f_{\text{lin}}(u_2) &= 0 & \text{if} & \ 0 < u_2 < 2,
\end{aligned}
\tag{3.8}
\]

where \( \gamma \) is the Euler–Mascheroni constant.

Proof. The proof follows from [6, Theorems 11.12–11.13] with \( \beta = 2 \) and [6, chapter 12, pp. 235–236].

To deal with the linear sieve in Theorem 6, we need the following well-factorability lemma.

**Lemma 3.4** (Well-factorability of linear sieve). Let \( \varepsilon > 0 \) be small. Let \( \delta \in (0, 10^{-3}] \) and let \( \rho_{\text{lin}} = \frac{1}{2} - 2\delta - \varepsilon \). Then the upper bound linear sieve weights \( \lambda_{\text{lin}}^+ \) as given in Lemma 3.3 with level \( X^{\rho_{\text{lin}}} \) and sifting parameter \( z_2 \leq X^{1/2} \) is supported in the set

\[
\mathfrak{D}^{+, \text{lin}} = \{ p_1 \cdots p_r \leq X^{\rho_{\text{lin}}} : z_2 \geq p_1 > \cdots > p_r, p_1 \cdots p_{2m-2}P_{2m-1}^3 \leq X^{\rho_{\text{lin}}} \ \forall \ m \geq 1 \}, \tag{3.9}
\]

where \( p_1, \ldots, p_r \) denote primes. In addition, for any \( D_0 \in [X^{1/5}, X^{\rho_{\text{lin}}} \cap [X^{1/10}, X^{\rho_{\text{lin}}}] \) can be factorized as \( d = d_1 d_2 \) such that \( d_1 \in [X^{1/10}, D_0] \) and \( d_1 d_2^2 \leq X^{1-4\delta-2\varepsilon^2}/D_0 \).

Proof. See [20, Lemma 9.1] or [6, Lemma 12.16].
4 | PROOF OF THEOREM 4

4.1 | Upper bound in Theorem 4

We first establish the upper bound in Theorem 4 by using Lemma 3.1 and assuming Theorem 1.

**Proposition 4.1.** Let \( b \) be a sufficiently large odd integer and \( r \in \mathcal{A} \cap [0, b) \) be such that \((r, b) = (r - 1, b) = 1\). Then, we have
\[
\sum_{\substack{p < X \\ p \equiv r \pmod{b}}} 1_{\mathcal{A}}(p)1_{\equiv}(p - 1) \ll_b \frac{X^\zeta}{(\log X)^{3/2}}.
\]

**Proof.** Let \( z \in [2, X] \) be a parameter to be chosen later. We let
\[
P_3 = \{p \equiv 3 \pmod{4}, p \nmid b\} \text{ and } P_3(z) = \prod_{p \in P_3} p.
\]

Then, we have
\[
\sum_{\substack{p < X \\ p \equiv r \pmod{b}}} 1_{\mathcal{A}}(p)1_{\equiv}(p - 1) \leq \sum_{\substack{p < X \\ (p-1,P_3(z))=1}} 1_{\mathcal{A}_r}(p) \tag{4.1}
\]
\[
< \frac{10}{9\log X} \sum_{X^{9/10} < p < X} \Lambda(n)1_{\mathcal{A}_r}(n) + X^{9/10}\tag{4.2}
\]

Next, for \( d | P_3(z) \), we set
\[
E(d) = \sum_{n \equiv 1 \pmod{d}} \Lambda(n)1_{\mathcal{A}_r}(n) - \frac{1}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{\mathcal{A}_r}(n).
\]

Therefore, by Theorem 1 with \( D = X^{3/10} \), for any large real number \( A > 0 \), we find that
\[
\sum_{d \leq X^{3/10} \atop d | P_3(z)} |E(d)| \ll \frac{X^\zeta}{(\log X)^A}.
\]

Now, we choose \( c(n) = \Lambda(n)1_{\mathcal{A}_r}(n) \) for \( n < X \) and \( z_1 = D = z = X^{3/10} \) in Lemma 3.1. Clearly, the sequence \( c(n) \) satisfies the axioms of sieve theory with \( q(d) = d/\varphi(d) \) and \( x = 1/2 \). Therefore, by the upper bound semi-linear sieve \((3.2)\) with \( u_1 = 1 \), we have
\[
\sum_{n < X \atop (n-1,P_3(z))=1} \Lambda(n)1_{\mathcal{A}_r}(n) \leq \left(\frac{2e^{\gamma/2}}{\pi^{1/2}} + o(1)\right) \frac{b}{\varphi(b)} \sum_{n < X} 1_{\mathcal{A}_r}(n) \prod_{p \leq z} \left(1 - \frac{1}{p - 1}\right) + O\left(\frac{X^\zeta}{(\log X)^A}\right),
\]
where \( \gamma \) is the Euler–Mascheroni constant. The above estimate together with the estimates from (4.1) and (4.2) allows us to obtain
\[
\sum_{p < X \atop p \equiv r (\text{mod } b)} 1 A(p) 1_{\Xi}(p - 1) \leq \left( \frac{2e^\gamma / 2}{\pi^{1/2}} + o(1) \right) \frac{10b}{9\varphi(b) \log X} \sum_{n < X} 1 A_r(n) \prod_{p < z} \left( 1 - \frac{1}{p} \right) \\
+ O \left( \frac{X^5}{(\log X)^A} + X^{9/10} \right).
\]
We can now use Mertens’ estimate [12, Theorem 3.4(c)] to the product over the primes (for example, see [6, p. 278] for a detailed estimate) and the fact that \( \sum_{n < X} 1 A_r(n) = X^5 / (b - 1) \) to deduce that
\[
\sum_{p < X \atop p \equiv r (\text{mod } b)} 1 A(p) 1_{\Xi}(p - 1) \ll_b \frac{X^5}{(\log X)^{3/2}}
\]
as desired. \( \square \)

### 4.2 Lower bound in Theorem 4

The lower bound in Theorem 4 can also be obtained from [20, Theorem 6.5] by choosing \( \omega_n = 1 A(n) \cdot 1_{n \equiv r (\text{mod } b)} \), where Hypothesis 6.4 holds by considering variants of Theorems 5 and 6. For the sake of completeness, we will establish the lower bound from scratch in this paper. To do so, we consider the following sieve set-up.

#### 4.2.1 Sieve set-up for the lower bound

For \( r \in A \cap [0, b) \) with \((r(r - 1), b) = 1\), we set
\[
F = \{ p - 1 : p < X, p \in A_r, p \equiv 3 (\text{mod } 8) \},
\]
\[
P_3 = \{ p \equiv 3 (\text{mod } 4), p \nmid b \}, \quad \text{and} \quad P_3(z) = \prod_{p < z \atop p \in P_3} p.
\] (4.3)

Note that, as \( p \equiv r (\text{mod } b) \) for the primes we are considering here, and we have assumed that \((r - 1, b) = 1\), so there are no primes that divide both \( p - 1 \) and \( b \). So, we have that
\[
\sum_{p < X \atop p \equiv r (\text{mod } b)} 1 A(p) 1_{\Xi}(p - 1) \geq S(F, P_3, X^{1/2}) = \sum_{p < X \atop (p - 1, P_3(X^{1/2})) = 1 \atop p \equiv 3 (\text{mod } 8)} 1 A_r(p).
\] (4.4)

For notational convenience, we set \( z = X^{1/2} \) for some \( \alpha \in [2, 4) \). Later, we will choose \( \alpha \approx 3 \).

By the Buchstab identity (see [6, eq. (6.4)]), we have
\[
S(F, P_3, \sqrt{X}) = S(F, P_3, z) - \sum_{z < p_1 \leq \sqrt{X} \atop p_1 \equiv 3 (\text{mod } 4)} S(F_{p_1}, P_3, p_1) =: S - T.
\] (4.5)
We will give a lower bound for $S$ using the semi-linear sieve and Theorem 5. On the other hand, an upper bound for $T$ is given using the linear sieve and Theorem 6.

As $p - 1$ has an even number of prime factors in the class $3 \pmod{4}$ and by our choice of $z$, we can write the sum $T$ as

$$T = \sum_{p < X} \sum_{p - 1 = 2n_1p_1p_2} 1_{A_r}(p),$$

where $B = \{ n : p | n \Rightarrow p \equiv 1 \pmod{4} \}$. Following Matomäki [14], we define

$$L = \{ \ell = n_1p_1 : n_1 \leq X^{1 - 2/\alpha}, n_1 \in B, X^{1/\alpha} \leq p_1 < (X/n_1)^{1/2}, p_1 \in P_3 \}, \quad (4.6)$$

and for each $\ell \in L$,

$$M(\ell) = \{ m = 2\ell p_2 + 1 : m \in A_r, p_2 < X/2\ell, p_2 \in P_3, p_2 \geq X^{1/\alpha} \}. \quad (4.7)$$

Note that for each $m \in M(\ell)$, we have $m \equiv r \pmod{b}$. As, by our assumption $(r - 1, b) = 1$, we have that $(\ell', b) = 1$. This allows us to bound the sum $T$ as

$$T \leq \sum_{\ell \in L} \left( S(M(\ell), P(\ell), X^{1/\nu}) + O(X^{1/\nu}) \right), \quad (4.8)$$

where $P(\ell) = \{ p : p \nmid 2b\ell \}$ and we will choose $\nu$ appropriately later. In fact, we will choose $\nu \approx 5$.

**Remark.** Note that if $m \in M(\ell)$ in (4.7), we have $m \equiv r \pmod{b}$. As $(r - 1, b) = 1$ this implies that $(2\ell p_2, b) = 1$, which in turn restricts the base $b$ to be odd.

Now we are ready to bound the sums $S$ from below and $T$ from above separately in the following two propositions.

**Proposition 4.2.** Assume the above sieve set-up. Let $\varepsilon > 0$ be small. Let $\delta \in (0, 10^{-3}]$ and let $b$ be an odd integer that is sufficiently large in terms of $\delta$. Let $\alpha = (1/3 - 2\delta)^{-1} + \varepsilon$ be such that $\alpha \in [2, 4)$ and let $\rho_{\text{sem}} \leq \frac{3}{7}(1 - 4\delta) - \varepsilon$. Then we have

$$S \geq \mathcal{G} + o(1) \frac{b}{(\log X)^{3/2} \varphi(b)} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p - 1} \right)^{-1} I_{\text{sem}}(\rho_{\text{sem}}, \alpha) \sum_{n < X} 1_{A_r}(n), \quad (4.9)$$

where

$$\mathcal{G} = \frac{1}{4\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p^2} \right)^{1/2} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{(p - 1)^2} \right) \quad (4.10)$$

and

$$I_{\text{sem}}(\rho_{\text{sem}}, \alpha) = \frac{1}{\sqrt{\rho_{\text{sem}}}} \int_1^{\alpha \rho_{\text{sem}}} \frac{dy}{\sqrt{y(y - 1)}}. \quad (4.11)$$
Proposition 4.3. Assume the above sieve set-up. Let $\varepsilon > 0$ be small. Let $\delta \in (0, 10^{-3}]$ and let $b$ be an odd integer that is sufficiently large in terms of $\delta$. Let $\alpha = (1/3 - 2\delta)^{-1} + \varepsilon$ be such that $\alpha \in [2, 4)$ and let $\rho_{\text{lin}} \leq 1/2 - 2\delta - \varepsilon$. Then we have

$$T \leq \frac{10\mathcal{S} + o(1)}{9(\log X)^{3/2}} \frac{b}{\varphi(b)} \prod_{\substack{p\mid b \mod 4 \mid b}} \left(1 - \frac{1}{p - 1}\right)^{-1} I_{\text{lin}}(\rho_{\text{lin}}, \alpha) \sum_{n < X} 1_{\mathcal{A}_{\rho}}(n),$$

where $\mathcal{S}$ is given by the relation (4.10) and

$$I_{\text{lin}}(\rho_{\text{lin}}, \alpha) = \frac{1}{\rho_{\text{lin}}} \int_{2}^{\alpha} \frac{\log(y - 1)}{y(1 - y/\alpha)^{1/2}} \, dy.$$  

(4.13)

Now we can give the proof of Theorem 4 from Propositions 4.1, 4.2, and 4.3.

Proof of Theorem 4 assuming Propositions 4.2 and 4.3. From (4.4), (4.5), (4.9), and (4.12), we have

$$\sum_{\substack{p < X \atop p \equiv r \mod b}} 1_{\mathcal{A}}(p) 1_{\mathbb{B}}(p - 1) \geq S(F, \mathcal{P}_3, \sqrt{X})$$

$$= S(F, \mathcal{P}_3, X^{1/\alpha}) - T$$

$$\geq \frac{\mathcal{S} + o(1)}{(\log X)^{3/2}} \frac{b}{\varphi(b)} \prod_{\substack{p\mid b \mod 4 \mid b}} \left(1 - \frac{1}{p - 1}\right)^{-1}$$

$$\times \left( I_{\text{sem}}(\rho_{\text{sem}}, \alpha) - \frac{10}{9} \cdot I_{\text{lin}}(\rho_{\text{lin}}, \alpha) + o(1) \right) \sum_{n < X} 1_{\mathcal{A}_{\rho}}(n).$$

A simple numerical computation yields that

$$I_{\text{sem}}(\rho_{\text{sem}}, \alpha) - \frac{10}{9} \cdot I_{\text{lin}}(\rho_{\text{lin}}, \alpha) > 1.60492 - 1.4566 = 0.1482 > 0$$

for $\rho_{\text{sem}} = 3(1 - 4\delta)/7 - \varepsilon$, $\rho_{\text{lin}} = 1/2 - 2\delta - \varepsilon$, $\alpha = (1/3 - 2\delta)^{-1} + \varepsilon$, $\delta = 1/1000$ with $\varepsilon > 0$ small. Hence, we obtain

$$\sum_{\substack{p < X \atop p \equiv r \mod b}} 1_{\mathcal{A}}(p) 1_{\mathbb{B}}(p - 1) \gg \frac{X^{\varepsilon}}{(\log X)^{3/2}}.$$

This establishes the lower bound in Theorem 4. Along with Proposition 4.1, this completes the proof of Theorem 4.

\[ \square \]

4.3 Auxiliary results

We collect two key estimates essential for us while computing the lower bound.
Lemma 4.4. We have

\[
\prod_{\substack{p \leq y \\
p \equiv 3 \pmod{4}}} \left( 1 - \frac{1}{\varphi(p)} \right) = 2C_2C_3(1 + o(1)) \left( \frac{\pi e^{-\gamma}}{\log y} \right)^{1/2} \quad \text{as} \quad y \to \infty,
\]

where \( \gamma \) is the Euler–Mascheroni constant,

\[
C_2 = \frac{1}{2\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p^2} \right)^{1/2} \quad \text{and} \quad C_3 = \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{(p-1)^2} \right).
\]

Proof. The proof is standard and can be easily derived following [6, pp. 277–278].

Lemma 4.5. Let \( \mathcal{L} \) be as in (4.6) and let \( \alpha \in [2,4) \). For any positive integer \( n \geq 3 \), let

\[
t(n) = \prod_{\substack{p \mid n \\
p > 2}} \frac{p-1}{p-2}.
\]

Then, we have

\[
\sum_{\substack{\ell \in \mathcal{L} \\
(\ell, 2b) = 1}} \frac{t(\ell)}{\ell \log(X/\ell)} = \frac{1 + o(1)}{(\log X)^{1/2}} \frac{C_2}{2C_1} \prod_{p \equiv 1 \pmod{4}} \left( 1 + \frac{1}{p-2} \right)^{-1} \int_{2}^{\alpha} \frac{\log(y-1)}{\log(y-\alpha)^{1/2}} dy,
\]

where

\[
C_2 = \frac{1}{2\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p^2} \right)^{1/2} \quad \text{and} \quad C_1 = \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{1}{(p-1)^2} \right).
\]

Proof. The proof follows from the proof of [13, Lemma 5] in conjunction with [21, Satz 1] to incorporate the extra condition \((\ell, 2b) = 1\).

4.4 Proof of Proposition 4.2

We establish Proposition 4.2 assuming Theorem 5, given below.

Theorem 5 (Semi-linear sieve equidistribution estimate). Let \( \varepsilon > 0 \) be small. Let \( \delta \in (0, 10^{-3}] \) and let \( b \) be an odd integer that is sufficiently large in terms of \( \delta \). Let \( r \in \mathcal{A} \cap [0, b) \) with \( (r^2 - 1, b) = 1 \). Let \( \lambda_{\text{sem}} \) be as in Lemma 3.1 and Lemma 3.2 with \( z_1 \leq X^{1/3 - 2\delta - 2\varepsilon^2} \) and \( D = X^{\rho_{\text{sem}}} \), where \( \rho_{\text{sem}} = 3(1 - 4\delta)/7 - \varepsilon \). Then for any \( A > 0 \), we have

\[
\sum_{\substack{d \leq D \\
(d, 2b) = 1}} \lambda_{\text{sem}}(d) \left( \sum_{\substack{n < X \\
(n, d) = 1 \pmod{3} \pmod{8}}} \Lambda(n)1_{\mathcal{A}_{r}}(n) - \frac{1}{4\varphi(d)\varphi(b)} \sum_{n < X} 1_{\mathcal{A}_{r}}(n) \right) \ll_{A,b,\delta,\varepsilon} \frac{X^\varepsilon}{(\log X)^A}.
\]
Proof of Proposition 4.2 assuming Theorem 5. We have

\[ S \geq \frac{1}{\log X} \sum_{\substack{p < X \\ (p-1, P_3(X^{1/\alpha}))=1 \\ p \equiv 3 \pmod{8}}} 1_{A_r}(p) \log p. \quad (4.15) \]

Next, for \( d | P_3(X^{1/\alpha}) = \prod_{p < X^{1/\alpha}, p \in \mathcal{P}_3} p \), where \( \mathcal{P}_3 = \{ p \equiv 3 \pmod{4} : p \nmid b \} \), let

\[ E_1(d) = \sum_{\substack{p < X \\ p \equiv 1 \pmod{d} \\ p \equiv 3 \pmod{8}}} 1_{A_r}(p) \log p - \frac{1}{4\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n). \]

Now we choose \( c(n) = 1_{A_r,r \nmid p}(n) \log n \) for \( n < X \) and \( n \equiv 3 \pmod{8} \) in Lemma 3.1. Then, for \( 1 \leq u_1 \leq 3 \), the lower bound semi-linear sieve (3.1) yields

\[ \sum_{\substack{p < X \\ (p-1, P_3(X^{1/\alpha}))=1 \\ p \equiv 3 \pmod{8}}} 1_{A_r}(p) \log p \geq \frac{1}{4} (f_{\text{sem}}(u) + o(1)) V_{\text{sem}}(X^{1/\alpha}) \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n) \]

\[ + \sum_{\substack{d \leq X^{u_1/\alpha} \\ (d,2b)=1}} \lambda_{-\text{sem}}(d) E_1(d), \quad (4.16) \]

where \( \lambda_{-\text{sem}} \) are the lower bound semi-linear sieve weights with sifting parameter \( z_1 = X^{1/\alpha} \), \( f_{\text{sem}}(u_1) \) is given by (3.4), and

\[ V_{\text{sem}}(X^{1/\alpha}) = \prod_{\substack{p < X^{1/\alpha} \\ p \equiv 3 \pmod{4} \\ (p,b)=1}} \left( 1 - \frac{1}{\varphi(p)} \right). \quad (4.17) \]

We have \( z_1 = X^{1/\alpha} \leq X^{1/3-2\delta-2\epsilon^2} \), so that we can take \( u_1 = \rho_{\text{sem}} \alpha \), where \( \rho_{\text{sem}} = \frac{3}{7}(1-4\delta) - \epsilon \) in Theorem 5. We can then use Theorem 5 and the fact that the contribution of prime powers is negligible to bound the error term \( E_1(d) \). In fact, using Chebyshev’s estimate [12, Theorem 2.4], the contribution of prime powers can be bounded by

\[ \ll \sum_{d \leq X^{1/3-2\delta-2\epsilon}} \sum_{\substack{p^m < X \\ p \equiv 1 \pmod{d} \\ p \equiv 3 \pmod{8} \\ m \geq 2}} 1_{A_r}(p) \log p \ll (\log X) \sum_{d \leq X^{1/3-2\delta-2\epsilon}} \sum_{p \leq X^{1/2}} 1 \ll X^{13/14-12\delta/7}, \]

which is admissible. Hence, the error term in (4.16) can be bounded as

\[ \sum_{d \leq X^{z_{\text{sem}}}} \lambda_{-\text{sem}}(d) E_1(d) \ll_{A,b,\delta,\epsilon} \frac{X^z}{(\log X)^A}. \quad (4.18) \]
Next, we simplify the main term in (4.16) using Lemma 4.4, so that
\[ V_{\text{sem}}(X^{1/\alpha}) = (1 + o(1)) \prod_{\substack{p | b \pmod{4} \atop p \equiv 3}} \left(1 - \frac{1}{p-1}\right)^{-1} \cdot 2C_2C_3 \cdot \left(\frac{\alpha \pi e^{-\gamma}}{\log X}\right)^{1/2}, \tag{4.19} \]
where
\[ C_2 = \frac{1}{2\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p^2}\right)^{1/2} \quad \text{and} \quad C_3 = \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{(p-1)^2}\right). \tag{4.20} \]
Putting the estimates from (4.16), (4.18), and (4.19) in (4.15), and noting that \( u_1 = \alpha \rho_{\text{sem}} \), we have
\[ S \geq \frac{2C_2C_3(1 + o(1))}{4(\log X)^{3/2}} \prod_{\substack{p | b \pmod{4} \atop p \equiv 3}} \left(1 - \frac{1}{p-1}\right)^{-1} \left(\frac{\alpha}{u_1}\right)^{1/2} \int_1^{u_1} \frac{dy}{\sqrt{y(y-1)}} \times \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n) \]
\[ = \frac{C_2C_3(1 + o(1))}{2(\log X)^{3/2}} \prod_{\substack{p | b \pmod{4} \atop p \equiv 3}} \left(1 - \frac{1}{p-1}\right)^{-1} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n) I_{\text{sem}}(\rho_{\text{sem}}, \alpha), \]
where \( I_{\text{sem}}(\rho_{\text{sem}}, \alpha) \) is given by (4.11) Therefore,
\[ S \geq \mathcal{O} + o(1) \frac{b}{(\log X)^{3/2} \varphi(b)} \prod_{\substack{p | b \pmod{4} \atop p \equiv 3}} \left(1 - \frac{1}{p-1}\right)^{-1} I_{\text{sem}}(\rho_{\text{sem}}, \alpha) \sum_{n < X} 1_{A_r}(n), \tag{4.21} \]
where
\[ \mathcal{O} = \frac{C_2C_3}{2} = \frac{1}{4\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p^2}\right)^{1/2} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{(p-1)^2}\right). \tag{4.22} \]
This completes the proof of Proposition 4.2. \( \square \)

Thus, we are left to establish Theorem 5, which we do in Part IV.

### 4.5 Proof of Proposition 4.3

Finally, we give the proof of Proposition 4.3 assuming Theorem 6, given below.

**Theorem 6** (Linear sieve equidistribution estimate). Let \( \varepsilon > 0 \) be small. Let \( \delta \in (0, 10^{-3}] \) and let \( b \) be an odd integer that is sufficiently large in terms of \( \delta \). Let \( r \in A \cap [0, b) \) with \( (r, b) = (r-1, b) = 1 \). Let \( L \) be a real number such that \( L \in [X^{1/3 - 2\varepsilon}, X^{2/3 + 2\varepsilon}] \). Suppose \( \lambda_{\text{lin}}^+ \) is a bounded arithmetic real-valued function, and \( \lambda_{\text{lin}}^+ \) is as in Lemmas 3.3 and 3.4 with \( z_2 = X^{1/5} \), and \( D = X^{\rho_{\text{lin}}} \) for \( \rho_{\text{lin}} = \)
Then for any $A > 0$, we have

$$\sum_{d \in X^{1/2-2\delta}} \lambda^+(d) \left( \sum_{\ell' \in \mathcal{L}} \sum_{\ell' \mod d} \sum_{n < X/2\ell' \mod (d,2b)} 1_{\mathcal{A}_r}(2\ell'n + 1) \Lambda(n) \right) \ll_{A,b,\delta,\varepsilon} X^{1/2-2\delta - \varepsilon}. \quad (4.23)$$

**Proof of Proposition 4.3 assuming Theorem 6.** By the inequality (4.8), for some parameter $\nu$ (to be chosen later), we find that

$$T \leq \sum_{\ell' \in \mathcal{L}} \left( S(M(\ell'), P(\ell), X^{1/\nu}) + O(X^{1/\nu}) \right), \quad (4.24)$$

where $\mathcal{L}$ and $M(\ell')$ are given by (4.6) and (4.7), respectively. Furthermore, $P(\ell') = \{ p : p \nmid 2b \ell' \}$. Next, we set $P_\ell(X^{1/\nu}) := \prod_{p < X^{1/\nu}, p \in \mathcal{P}(\ell')} p$ and note that

$$\sum_{\ell' \in \mathcal{L}} \left( S(M(\ell'), P(\ell'), X^{1/\nu}) + O(X^{1/\nu}) \right) \leq \sum_{\ell' \in \mathcal{L}} \sum_{p_2 < X/2\ell'} 1_{\mathcal{A}_r}(2\ell'p_2 + 1) + O(X^{1/\nu} \# \mathcal{L}),$$

where $\sum^b_{\ell' \in \mathcal{L}}$ denotes a sum over values of $p_2$ satisfying

$$\ell'p_2 \equiv 1 \pmod{4} \quad \text{and} \quad \left( 2\ell'p_2 + 1, P_\ell(X^{1/\nu}) \right) = 1.$$

As in the proof of Proposition 4.1, we first split the range of $p_2$ to obtain

$$\sum_{p_2 < X/2\ell'} 1_{\mathcal{A}_r}(2\ell'p_2 + 1) \leq \frac{10}{9 \log(X/\ell')} \sum_{n < X/2\ell' \mod (d,2b)} \Lambda(n)1_{\mathcal{A}_r}(2\ell'n + 1)$$

$$+ \sum_{p_2 < X/\ell' \# \mathcal{L}} 1_{\mathcal{A}_r}(2\ell'p_2 + 1).$$

Next, we use Chebyshev's bound [12, Theorem 2.4] for the sum over primes $p_2$. Note that, as $\alpha = (1/3 - 2\delta)^{-1} + \varepsilon$, by (4.6), we have

$$\mathcal{L} \subset [X^{1/\alpha}, X^{1-1/\alpha}] \subset [X^{1/3 - 2\delta - \varepsilon}, X^{2/3 + 2\delta + \varepsilon}]. \quad (4.25)$$

This allows us to bound the second sum as

$$\sum_{\ell' \in \mathcal{L}} \sum_{p_2 < X/\ell' \# \mathcal{L}} 1_{\mathcal{A}_r}(2\ell'p_2 + 1) \ll \sum_{\ell' \in \mathcal{L}} \frac{(X/\ell')^{9/10}}{\log(X/\ell')} \ll X^{29/30 + \delta/5 + \varepsilon}.$$
The above estimates yield

\[
T \leq \frac{10}{9} \sum_{\ell \in \mathcal{L}} \frac{1}{\log(X/\ell)} \sum_{n<X/2^\ell}^{\mathcal{P}} \Lambda(n)1_{A_\ell}(2\ell n + 1) + O\left(X^{29/30+5/5+\varepsilon} + \# \mathcal{L}X^{1/\varepsilon}\right), \tag{4.26}
\]

where \(\sum^{\mathcal{P}}\) denotes a sum over values of \(n\) satisfying

\[\ell n \equiv 1 \pmod{4} \quad \text{and} \quad \left(2\ell n + 1, P_\ell(X^{1/\nu})\right) = 1.\]

Next, for \(d \mid \prod_{p < z, p \in \mathcal{P}(\ell)} p\), where \(\mathcal{P}(\ell) = \{p : p \nmid 2b\ell\}\), we let

\[
E_2(d) = \sum_{\ell n \equiv 1 \pmod{4} \atop \ell n + 1 \equiv 0 \pmod{d} \atop 2\ell n \equiv 1 \pmod{4} \atop \ell n < X} \Lambda(n)1_{A_\ell}(2\ell n + 1) - \frac{1}{4\varphi(d)} \frac{1}{\ell} \sum_{n<X} 1_{A_\ell}(n).
\]

We now apply Lemma 3.3 with the sequence \(c(n) = \Lambda(n)1_{A_\ell}(2\ell n + 1)\) for \(n < X/2^\ell\) and \(\ell n \equiv 1 \pmod{4}\). Then given a parameter \(u_2 \in [1, 3]\) to be chosen later, the upper bound linear sieve (3.7) yields

\[
\sum_{n<X/2^\ell}^{\mathcal{P}} \Lambda(n)1_{A_\ell}(2\ell n + 1) \leq \frac{b}{4\varphi(b)} (F_{\text{lin}}(u_2) + o(1))V_{\text{lin}}(X^{1/\nu}) \frac{1}{\ell} \sum_{n<X} 1_{A_\ell}(n)
\]

\[+ \sum_{d \leq X^{u_2/\nu} \atop (d, 2b\ell) = 1} \lambda^+_{\text{lin}}(d)E_2(d), \tag{4.27}
\]

where \(\lambda^+_{\text{lin}}\) are the upper bound linear sieve weights with sifting parameter \(z_2 = X^{1/\nu}, F_{\text{lin}}(u_2) = 2e^\gamma/u_2\) and

\[
V_{\text{lin}}(X^{1/\nu}) = \prod_{p < X^{1/\nu} \atop (p, 2b\ell) = 1} \left(1 - \frac{1}{\varphi(p)}\right).
\]

As \((\ell, b) = 1\) in (4.26), we may use Mertens’ theorem [12, Theorem 3.4(c)] to obtain

\[
V_{\text{lin}}(X^{1/\nu}) = \prod_{p < X^{1/\nu} \atop (p, 2b\ell) = 1} \left(1 - \frac{1}{\varphi(p)}\right) = (1 + o(1)) \frac{2\nu C_1 C_3 e^{-\gamma} t(\ell) t(b)}{\log X}, \tag{4.28}
\]

where \(t(n)\) is given by (4.14),

\[
C_1 = \prod_{p \equiv 1 \pmod{4}} \left(1 - \frac{1}{(p - 1)^2}\right), \quad \text{and} \quad C_3 = \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{(p - 1)^2}\right). \tag{4.29}
\]

Now we take \(u_2 = \rho_{\text{lin}} \nu\) in the linear sieve, where \(\rho_{\text{lin}}\) corresponds to the level of the upper bound sieve in Theorem 6. Next, using (4.25), we write

\[
\sum_{d \leq X^{u_2/\nu}} \lambda^+_{\text{lin}}(d) \sum_{\ell \in \mathcal{L} \atop (\ell, 2bd) = 1} \frac{1}{\log(X/\ell)} E_2(d) = \sum_{d \leq X^{u_2/\nu}} \lambda^+_{\text{lin}}(d) \sum_{\ell \in \mathcal{L} \atop (\ell, 2bd) = 1} \frac{1}{\log(X/\ell)} E_2(d).
\]
We do a dyadic decomposition on the range of \( \ell' \), say \( \ell' \sim L \) with \( L \in [X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon}] \). As the number of such dyadic intervals are at most \( \log X \), we use Theorem 6 with \( \mathfrak{h}(\ell') = 1_{L}(\ell')/ \log(X/\ell') \) for \( \ell' \sim L \) to bound the above expression as

\[
\sum_{d \leq X^{\rho}} \lambda^{+}_{\text{lin}}(d) \sum_{X^{1/3-2\delta-\varepsilon} < \ell' \leq X^{2/3+2\delta+\varepsilon}} \mathfrak{h}(\ell')E_{2}(d) \ll (\log X) \sum_{d \leq X^{\rho}} \lambda^{+}_{\text{lin}}(d) \sum_{\ell' \sim L} \mathfrak{h}(\ell')E_{2}(d) \ll_{A,b,\delta,\varepsilon} \frac{X^{\zeta}}{(\log X)^{A}},
\]

which is admissible.

From (4.26), (4.27), and using the above bound for the error term, we have that

\[
T \leq \frac{10}{9} \cdot \frac{b}{4\varphi(b)}(F_{\text{lin}}(u_{2}) + o(1)) \cdot \sum_{n < X} 1_{\mathcal{A}_{r}}(n) \sum_{\ell' \in \mathcal{L}} \frac{V_{\text{lin}}(X^{1/\nu})}{\ell' \log(X/\ell')}
\]

\[
+ O_{A,b,\delta,\varepsilon} \left( \frac{X^{\zeta}}{(\log X)^{A}} + X^{29/30+\delta/5+\varepsilon} + \# \mathcal{L}X^{1/\nu} \right)
\]

\[
\leq \frac{10}{9} \cdot \frac{\nu}{2} \cdot C_{1}C_{2}e^{-\gamma} \cdot \frac{b \cdot t(b)}{\varphi(b)}(F_{\text{lin}}(u_{2}) + o(1)) \sum_{n < X} 1_{\mathcal{A}_{r}}(n) \sum_{\ell' \in \mathcal{L}} \frac{t(\ell')}{\ell' \log(X/\ell')}
\]

\[
+ O_{A,b,\delta,\varepsilon} \left( \frac{X^{\zeta}}{(\log X)^{A}} + X^{29/30+\delta/5+\varepsilon} + \# \mathcal{L}X^{1/\nu} \right),
\]

(4.30)

where we have used the asymptotic formula for \( V_{\text{lin}}(X^{1/\nu}) \) from the relation (4.28) in the last line.

Next, by Lemma 4.5, we have

\[
\sum_{\ell' \in \mathcal{L}} \frac{t(\ell')}{\ell' \log(X/\ell')} = \frac{(1 + o(1))C_{2}}{2C_{1}(\log X)^{1/2}} \prod_{p \mid b} \left( 1 + \frac{1}{p - 2} \right)^{-1} \int_{2}^{\alpha} \log(y - 1) y^{(1 - y/\alpha)^{1/2}} dy,
\]

(4.31)

where \( C_{2} \) is as in the relation (4.20) and \( C_{1} \) is given by (4.29).

We choose \( u_{2} = 5/2 \), so that \( F_{\text{lin}}(u_{2}) = 4e^{\gamma}/5 \). As by our choice, \( \nu_{\text{lin}} = 1/2 - 2\delta - \varepsilon \), we can choose \( \nu = 5 \). Note that as \( \nu = 5 \), \( \# \mathcal{L} \leq X^{2/3+2\delta+\varepsilon}, \varepsilon > 0 \) is small enough, \( \delta \in (0, 10^{-3}] \), and \( \zeta \) tends to 1 as \( b \to \infty \), we have that

\[
\# \mathcal{L}X^{1/\nu}, X^{29/30+\delta/5+\varepsilon} \ll_{A,\delta,\varepsilon} \frac{X^{\zeta}}{(\log X)^{A}}.
\]

(4.32)

Therefore, we substitute (4.31) and (4.32) in (4.30) to obtain

\[
T \leq \frac{10\mathfrak{S} + o(1)}{9(\log X)^{3/2}} \frac{b}{\varphi(b)} \prod_{p \mid b} \left( 1 - \frac{1}{p - 1} \right)^{-1} \prod_{p \equiv 1 (\mod 4)} \left( 1 + \frac{1}{p - 2} \right)^{-1} I_{\text{lin}}(\nu_{\text{lin}}, \alpha)
\]

\[
\times \sum_{n < X} 1_{\mathcal{A}_{r}}(n),
\]

(4.33)
where $I_{\text{lin}}(\rho_{\text{lin}}, \alpha)$ is given by (4.13) and $\Xi = C_2 C_3/2$ is given by the relation (4.22). Hence, the estimate (4.33) along with the fact that

$$
\prod_{p | b \atop p > 2} \left(1 - \frac{1}{p-1}\right)^{-1} \prod_{p | b \atop p \equiv 1 \pmod{4}} \left(1 + \frac{1}{p-2}\right)^{-1} = \prod_{p | b \atop p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p-1}\right)^{-1}
$$

yields the required bound for the sum $T$. $\square$

We have therefore established Proposition 4.3 assuming Theorem 6. So, we are left to establish Theorems 5 and 6, which we do in Part IV.

PART III. EXPONENTIAL SUMS

In this part, we estimate the exponential sums over primes in arithmetic progressions using Vinogradov’s method (see [12, chapter 23] for an introduction to the method), which we will employ in Part IV to deduce our main results. Note that some of the estimates in this part are well-known. See, for example, [14, 18].

Recall that we set $X = b^k$ with $k \in \mathbb{Z}$ and $k \to \infty$ throughout this paper. We remark that the results in this part of the paper hold for any large real number $X$.

5 | PRELIMINARY ESTIMATES AND TYPE I ESTIMATE

We begin with the following estimate.

**Lemma 5.1.** Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $0 < |\beta| < 1/q^2$. Then for any $M, N \geq 2$, we have

$$
\sum_{m=1}^{M} \min \left( N, \frac{1}{||m\theta||} \right) \ll \left( M + MNq|\beta| + \frac{1}{q|\beta|} \right) (\log 2qM).
$$

**Proof.** The proof of the lemma is a standard one. However, we need a variant of it to take advantage of $\beta$ in the sum. For a detailed proof, see [17, Lemma 4.1]. $\square$

Let us now deduce the following corollary from the above lemma.

**Corollary 5.2.** Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Then for any $M \geq 1$, we have

$$
\sum_{m \leq M} \min \left( X/m + 1, \frac{1}{||m\theta||} \right) \ll X \left( M/X + \frac{qH}{X} + \frac{1}{qH} \right)(\log 2qM)^2,
$$

where $H = 1 + |\beta|X$.

**Proof.** If $\beta \neq 0$, we perform a dyadic decomposition and then apply Lemma 5.1 to obtain

$$
\sum_{m \leq M} \min \left( X/m + 1, \frac{1}{||m\theta||} \right) \ll X \left( M/X + q|\beta| + \frac{1}{Xq|\beta|} \right)(\log 2qM)^2.
$$

(5.2)
Next, for all $\beta$, we apply [10, Lemma 13.7] to obtain
\[
\sum_{m \leq M} \left( \frac{X}{m} + 1, \frac{1}{\|m\bar{\theta}\|} \right) \ll X \left( \frac{M}{X} + \frac{1}{q} + \frac{q}{X} \right) (\log 2qM).
\] (5.3)

Combining the estimates from inequalities (5.2) and (5.3), we have
\[
\sum_{m \leq M} \left( \frac{X}{m} + 1, \frac{1}{\|m\bar{\theta}\|} \right) \ll X \left\{ \frac{M}{X} + \min \left( \frac{q|\beta| + \frac{1}{Xq|\beta|} + \frac{1}{X} + \frac{q}{X}}{q(1 + |\beta|X)} \right) (\log 2qM)^2 \right\}. \] (5.4)

Next, we note that
\[
\min \left( \frac{1}{q}, \frac{1}{Xq|\beta|} \right) \leq \frac{2}{q(1 + |\beta|X)}. \] (5.5)

Therefore, using (5.5) and recalling that $|\beta| < 1/q^2$, we obtain
\[
\min \left( q|\beta| + \frac{1}{Xq|\beta|} + \frac{1}{X} + \frac{q}{X} \right) \ll q|\beta| + \frac{q}{X(1 + |\beta|X)} = \frac{qH}{X} + \frac{1}{qH},
\]
where $H = 1 + |\beta|X$. Substituting the above inequality in (5.4) completes the proof of the corollary.

We now state the following bilinear sum estimates for the exponential sum.

**Lemma 5.3** (Bilinear estimate). Let $M, N \geq 1$ be such that $MN \leq X$. Let $\theta = a/q + \beta$ for some $(a, q) = 1$ and $|\beta| < 1/q^2$. Suppose $\alpha_1$ and $\alpha_2$ are two arithmetic functions supported in $[1, M]$ and $[1, N]$, respectively. Then we have
\[
\sum_{n \leq X} (\alpha_1 * \alpha_2)(n)e(n\theta) \ll X^{1/2} \|\alpha_1\|_2 \|\alpha_2\|_2 \left( \frac{M}{X} + \frac{N}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log 2qX),
\]
where $H = 1 + |\beta|X$.

**Proof.** The proof follows by combining the argument of [12, Theorem 23.6] with Lemma 5.1 and Corollary 5.2.

Next, we will need an auxiliary lemma due to Matomäki [14, Lemma 8], who improved on the earlier work of Mikawa [18].

**Lemma 5.4** (Matomäki). Let $M, N \geq 1$ be such that $M, N \leq X$. Let $\theta = a/q + \beta$ with $(a, q) = 1$, $|\beta| < 1/q^2$ and $q < X$. Then, for any $\psi > 0$, we have
\[
M \sum_{m \sim M} \sum_{n \sim N} \tau_3(n) \min \left( \frac{X}{m^2n} + 1, \frac{1}{\|m^2n\bar{\theta}\|} \right) \ll M^2 N(\log X)^3
+ X \left( \frac{1}{M} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2-\psi} (\log X)^8,
\]
where $H = 1 + |\beta|X$. 

**Proof.** The proof follows from the argument of [14, Lemma 8] in conjunction with Lemma 5.1 and Corollary 5.2.

### 5.1 Type I estimate

We will estimate the so-called Type I sum in the following lemma.

**Lemma 5.5 (Type I estimate).** Let \( v > 0 \). Let \( D, M \geq 1 \) be such that \( DM < X \). Let \( \vartheta = a/q + \beta \) with \( (a, q) = 1 \), \( |\beta| < 1/q^2 \) and \( q < X \). Suppose \( \alpha \) is an arithmetic function supported in \([1, M]\) and satisfies \( |\alpha| \leq \tau_{h_1} \cdot \log^{h_2} \) for some fixed integers \( h_1 \geq 1, h_2 \geq 0 \). Furthermore, let \( h_3 \geq 1 \) be a fixed integer. Then, we have

\[
\sum_{d \leq D} \tau_{h_3}(d) \cdot \max_{(c,d)=1} \left| \sum_{\substack{mn < X \atop 1 \leq m \leq M} \alpha(m)(\log n)^v e(m\vartheta) \right| \ll X \left( \frac{DM}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log X)^{(h_1+h_3)^2/2+h_2+v+1},
\]

where \( H = 1 + |\beta|X \).

**Proof.** Let \( S_{\text{Type I}} \) be the sum that we wish to estimate. Applying partial summation and then using the fact that \( \sum_{n \leq y} e(nt) \ll \min(y, ||t||^{-1}) \) for any real numbers \( y > 1 \) and \( t \), we have

\[
\sum_{\substack{n < X/m \atop n \equiv c \pmod{d}}} (\log n)^v e(m\vartheta) \ll (\log X/m)^v \min \left( \frac{X}{dm} + 1, \frac{1}{||dm\vartheta||} \right).
\]

This implies that

\[
|S_{\text{Type I}}| \ll (\log X)^v \sum_{d \leq D} \tau_{h_3}(d) \sum_{m \leq M} |\alpha(m)| \min \left( \frac{X}{dm} + 1, \frac{1}{||dm\vartheta||} \right),
\]

Next, we write \( d' = dm \), so that \( d' \leq DM \). Then, by the Cauchy–Schwarz inequality and Corollary 5.2 along with the fact that \( |\alpha| \leq \tau_{h_1} \cdot \log^{h_2} \), we have

\[
|S_{\text{Type I}}| \ll (\log X)^{v+h_2} \left( X \sum_{d' \leq DM} \frac{\tau_{h_1+h_3}(d')^2}{d'} \right)^{1/2} \left( \sum_{d' \leq DM} \min \left( \frac{X}{d'} + 1, \frac{1}{||d'\vartheta||} \right) \right)^{1/2}
\ll (\log X)^{v+h_2} X(\log X)^{(h_1+h_3)^2/2} \left( \frac{DM}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log X),
\]

where we have used the fact that \( \sum_{n \leq y} \tau_{h}(n)^2/n \ll (\log y)^{h^2} \) for any real number \( y \geq 2 \) and for any integer \( h \geq 1 \). The above estimate on simplification yields the desired result. \( \square \)
6 | TYPE II ESTIMATES

We use Vinogradov’s method to estimate the Type II sums in the following lemma.

**Lemma 6.1** (Pointwise Type II estimate). Let $M, N \geq 1$ be such that $MN \leq X$. Let $\theta = a/q + \beta$ with $(a, q) = 1$, $|\beta| < 1/q^2$ and $q < X$. Suppose $\alpha_1$ and $\alpha_2$ are two arithmetic functions supported in $[M, 2M]$ and $[N, 2N]$, respectively, and satisfy $|\alpha_1|, |\alpha_2| \leq \tau_h \cdot \log$ for some fixed integer $h \geq 1$. Let $c$ and $d$ be nonzero positive integers such that $(c, d) = 1$. Then, we have

$$
\left| \sum_{\substack{mn < X \\ m \sim M, n \sim N \\ mn \equiv (\mod d)}} \alpha_1(m)\alpha_2(n)e(mn\theta) \right| \ll X \left( \frac{M}{X} + \frac{N}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log X)^{h^2+2},
$$

(6.1)

where $H = 1 + |\beta|X$.

**Proof.** Let $\chi$ be Dirichlet character modulo $d$. Then, by the orthogonality of Dirichlet characters, we bound the sum in the left-hand side of (6.1) as

$$
\ll \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \left| \sum_{\substack{mn < X \\ m \sim M, n \sim N}} \alpha_1(m)\chi(m)\alpha_2(n)\chi(n)e(mn\theta) \right|.
$$

Now we use Lemma 5.3 with $\alpha_1 \cdot \chi$ and $\alpha_2 \cdot \chi$ to estimate the sum over $mn < X$ and the trivial bound to sum over $\varphi(d)$ characters modulo $\chi$ to show that the above sum is

$$
\ll X^{1/2}\|\alpha_1\|_2\|\alpha_2\|_2 \left( \frac{M}{X} + \frac{N}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log qX).
$$

(6.2)

Next, we recall that $|\alpha_1|, |\alpha_2| \leq \tau_h \cdot \log$ to obtain

$$
\|\alpha_1\|_2\|\alpha_2\|_2 \ll \left( \sum_{m \sim M} \tau_h(m)^2 \right)^{1/2} \left( \sum_{n \sim N} \tau_h(n)^2 \right)^{1/2} (\log X)^2 \ll (MN)^{1/2} (\log X)^{h^2+1},
$$

(6.3)

using the fact that $\sum_{n < y} \tau_h(n)^2 \ll y(\log y)^{h^2-1}$ for any real number $y \geq 2$. Substituting the estimate from (6.3) in (6.2) and using the fact that $MN \leq X$ and $q < X$ completes the proof of the lemma.

In the next lemma, we improve the bounds of the previous lemma by taking advantage of averaging.

**Lemma 6.2.** Let $c$ be a fixed nonzero integer. Let $D_1, D_2, M, N \geq 1$ be such that

$$
MN < X, \quad D_1M < X \quad \text{and} \quad D_1D_2^2N < X.
$$

Let $\theta = a/q + \beta$ with $(a, q) = 1$, $|\beta| < 1/q^2$ and $q < X$. Suppose $\alpha_1$ and $\alpha_2$ are two arithmetic functions with support $[M, 2M]$ and $[N, 2N]$, respectively, and satisfy $|\alpha_1|, |\alpha_2| \leq \tau_h \cdot \log$ for some fixed
integer \( h \geq 1 \). Then, for any integer \( h_1 \geq 1 \), we have

\[
\sum_{d_1 \sim D_1} \sum_{d_2 \sim D_2} \tau_{h_1}(d_1) \max_{(c',d_1) = 1} \sum_{mn < X} \alpha_1(m) \alpha_2(n) e(mn\theta) \leq X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_1 D_2 \sqrt{N}}{X} + \frac{1}{D_1^{1/4}} + \frac{1}{(qH)^{1/4}} \right)^{1/2} (\log X)^{h_2 + \frac{1}{2} + \frac{5}{2}},
\]

where \( H = 1 + |\beta|X \).

**Proof.** The proof of the lemma is closely related to the proofs of [14, Proposition 9] and [18, Theorem, p. 352], but for the convenience of the reader we include the proof here. We will estimate the sum:

\[
S_{\text{Type II}} := \sum_{d_1 \sim D_1} \tau_{h_1}(d_1) \sum_{m \sim M} \alpha_1(m) \sum_{d_2 \sim D_2} \lambda(d_1, d_2) \sum_{n \sim N} \alpha_2(n) e(mn\theta).
\]

Let us assume that the maximum over \( c' \) is attained at \( c_{d_1} \). Let \( \lambda(d_1, d_2) \in \mathbb{C} \) be of absolute value 1 whenever \( c' = c_{d_1} \) and \( (d_1, c_{d_1}) = (d_1, d_2) = (d_2, c) = 1 \) for \( d_1 \sim D_1 \) and \( d_2 \sim D_2 \). Then, we have

\[
S_{\text{Type II}} = \sum_{d_1 \sim D_1} \tau_{h_1}(d_1) \sum_{m \sim M} \alpha_1(m) \sum_{d_2 \sim D_2} \lambda(d_1, d_2) \sum_{n \sim N} \alpha_2(n) e(mn\theta).
\]

We apply the Cauchy–Schwarz inequality to obtain

\[
|S_{\text{Type II}}|^2 \leq D_1 (\log X)^{h_2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{m \sim M} \sum_{d_2 \sim D_2} \lambda(d_1, d_2) \sum_{n \sim N \sim X/m} \alpha_2(n) e(mn\theta) \|
\]

\[
\leq D_1 (\log X)^{h_2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{n_1, n_2 \sim N \sim X/m} \alpha_2(n_1) \overline{\alpha_2(n_2)}
\]

\[
\times \left| \sum_{m \sim M \sim X/n_1 X/n_2} e(m(n_1 - n_2)\theta) \right|
\]

\[
\ll D_1 (\log X)^{h_2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{n_1, n_2 \sim N \sim X/m} \alpha_2(n_1) \overline{\alpha_2(n_2)}
\]

\[
\times \left| \sum_{m \sim M \sim X/n_1 X/n_2} e(m(n_1 - n_2)\theta) \right|
\]

\[
\ll D_1 (\log X)^{h_2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{n_1, n_2 \sim N \sim X/m} \alpha_2(n_1) \overline{\alpha_2(n_2)}
\]

\[
\times \left| \sum_{m \sim M \sim X/n_1 X/n_2} e(m(n_1 - n_2)\theta) \right|
\]
\[
\ll D_1 (\log X)^{h_1^2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{j \in \{1, 2\}} \sum_{n_1, n_2 \sim N} \left| \alpha_2 (n_j) \right|^2
\]

\[
\times \left| \sum_{m \sim M} e \left( m(n_1 - n_2) \theta \right) \right|
\]

using the fact \( |\alpha_2(n_1 \overline{\alpha_2(n_2)}| \leq |\alpha_2(n_1)|^2 + |\alpha_2(n_2)|^2 \).

The above congruences \( mn_1 \equiv mn_2 \equiv c \ (\text{mod } d_1) \), \( mn_1 \equiv c \ (\text{mod } d_2) \), and \( mn_1 \equiv c \ (\text{mod } d_2') \) have a solution in \( m \) if and only if \( (n_1, d_1 d_2) = (n_2, d_1 d_2') = 1 \) and \( n_1 \equiv n_2 \ (\text{mod } d_1 (d_2, d_2')) \). Then, we have a unique solution \( m \equiv h' \ (\text{mod } d_1 [d_2, d_2']) \) for some \( h' \in \{0, 1, \ldots, d_1 [d_2, d_2'] - 1\} \). Next, we write

\[
n_1 - n_2 = n'd_1 (d_2, d_2') \quad \text{and} \quad m = h' + m'd_1 [d_2, d_2']
\]

so that \( |n'| < 4N/d_1 (d_2, d_2') \) and \( m' \ll 1 + M/d_1 [d_2, d_2'] \). This implies that

\[
m(n_1 - n_2) = h'n'd_1 (d_2, d_2') + d_1^2 d_2 d_2' n'm'.
\]

Then, we have

\[
|S_{\text{Type II}}|^2 \ll D_1 (\log X)^{h_1^2 - 1} \| \alpha_1 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{n_1 - N} \left| \alpha_2 (n_1) \right|^2
\]

\[
\times \left| \sum_{|n'| < 4N/d_1 (d_2, d_2')} \sum_{m'} e \left( m'n'^2 d_1^2 d_2 d_2' \theta \right) \right|
\]

\[
\ll D_1 (\log X)^{h_1^2 - 1} \| \alpha_1 \|^2 \| \alpha_2 \|^2
\]

\[
\times \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \sum_{|n'| < 4N/d_1 (d_2, d_2')} \min \left( \frac{M d_1 [d_2, d_2']}{|n'|^2 d_2 d_2' \theta |} + 1, \frac{1}{|n'|^2 d_2 d_2' \theta |} \right).
\]

(6.4)

The terms with \( n' = 0 \) in (6.4) contribute

\[
\ll D_1 (\log X)^{h_1^2 - 1} \| \alpha_1 \|^2 \| \alpha_2 \|^2 \sum_{d_1 \sim D_1} \sum_{d_2, d_2' \sim D_2} \left( \frac{M d_1 [d_2, d_2']}{|d_1 [d_2, d_2']| + 1} \right)
\]

\[
\ll D_1 M \| \alpha_1 \|^2 \| \alpha_2 \|^2 (\log X)^{h_1^2 + 2} + (D_1 D_2)^2 \| \alpha_1 \|^2 \| \alpha_2 \|^2 (\log X)^{h_1^2 - 1},
\]
using the fact that $\sum_{h_1,h_2 \leq y} 1/[h_1, h_2] \ll (\log y)^3$ for any $y \geq 2$. Therefore,

$$\left| S_{\text{Type II}} \right|^2 \ll \|\alpha_1\|_2^2 \|\alpha_2\|_2^2 (\log X)^{h_1^2-1} \left\{ MD_1 (\log X)^3 + D_1^2 D_2^2 + D_1 \sum_{d_1 \sim D_1} \sum_{d_2,d_2^\prime \sim D_2} \min_{1 \leq |n'| < 4N/d_1(d_2,d_2^\prime)} \left( \frac{M}{d_1[d_2,d_2^\prime]} + 1, \frac{1}{\|n'|d_2^2d_2^\prime\theta\|} \right) \right\}. \quad (6.5)$$

Next, we write $n'd_2d_2' = d''$, so that

$$0 < |d''| = |n'|d_2d_2' = |n'|(d_2,d_2')d_2d_2' < \frac{4D_2^2N}{D_1},$$

as $0 < |n'| < 4N/d_1(d_2,d_2')$ and $d_1 \sim D_1$. Moreover,

$$\frac{M}{d_1[d_2,d_2']} = \frac{Md_1(d_2,d_2')|n'|}{d_2^2|n'|d_2d_2'} \ll \frac{MN}{d_1^2|d''|}.$$ 

The above reduction yields

$$\left| S_{\text{Type II}} \right|^2 \ll \|\alpha_1\|_2^2 \|\alpha_2\|_2^2 \left\{ (D_1 M + D_1^2 D_2^2)(\log X)^{h_1^2+2} + D_1 (\log X)^{h_1^2-1} \sum_{d_1 \sim D_1} \sum_{1 \leq |d''| \leq D_2^2N/D_1} \tau_3(d'') \min \left( \frac{MN}{d_1^2|d''|} + 1, \frac{1}{\|d_1^2d''\theta\|} \right) \right\}. \quad (6.6)$$

We observe that if $D_2^2N/D_1 \ll 1$, then we can bound the sum

$$\sum_{d_1 \sim D_1} \sum_{1 \leq |d''| \leq D_2^2N/D_1} \tau_3(d'') \min \left( \frac{MN}{d_1^2|d''|} + 1, \frac{1}{\|d_1^2d''\theta\|} \right) \ll \sum_{d_1 \sim D_1} \left( \frac{MN}{d_1^2} + 1 \right) \ll M + D_1.$$ 

Therefore, we can assume that $D_2^2N/D_1 \gg 1$, otherwise the sum over $d''$ in (6.6) can be bounded trivially as above. Without loss of generality, we can assume that $d'' > 0$ in the above sum. 

Next, we apply Lemma 5.4 with $\psi = 1/4$ and recalling that $MN < X$ to obtain

$$D_1 \sum_{d_1 \sim D_1} \sum_{0 < d'' \leq D_2^2N/D_1} \tau_3(d'') \min \left( \frac{MN}{d_1^2d''} + 1, \frac{1}{\|d_1^2d''\theta\|} \right) \ll (\log X) \max_{1 \leq J \leq D_2^2N/D_1} \left| D_1 \sum_{d_1 \sim D_1} \sum_{d'' \leq J} \tau_3(d'') \min \left( \frac{X}{d_1^2d''} + 1, \frac{1}{\|d_1^2d''\theta\|} \right) \right| \ll \left\{ D_1 D_2^2N + X \left( \frac{1}{D_1} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/4} \right\} (\log X)^9.$$
Hence, from the above estimate together with (6.6), and recalling from (6.3) that \(\|\alpha_1\|_2\|\alpha_2\|_2 \ll X^{1/2}(\log X)^{h^2+1}\), we obtain

\[
|S_{\text{Type II}}| \ll X^{1/2} \left( MD_1 + D_1^2D_2^2 + D_1D_2^2N + X \left( \frac{1}{D_1} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/4} \right)^{1/2}
\]

\[
\times (\log X)^{h^2+h_1^2/2+5}.
\]

The above estimate on simplification completes the proof of the lemma.

Let us combine Lemmas 6.1 and 6.2 to obtain the following special case of Type II sums. In particular, we will use an optimization idea due to Mikawa [18].

**Corollary 6.3.** Let \(D, M, N \geq 1\) be such that

\[
DM < X, \quad N \leq M \quad \text{and} \quad MN < X.
\]

Let \(\theta = a/q + \beta\) with \((a, q) = 1\) and \(|\beta| < 1/q^2\). Suppose \(\alpha_1\) and \(\alpha_2\) are two arithmetic functions supported in \([M, 2M]\) and \([N, 2N]\), respectively, and satisfy \(|\alpha_1|, |\alpha_2| \leq \tau_h \cdot \log\) for some fixed integer \(h \geq 1\). Furthermore, let \(H = 1 + |\beta|X\) and \(qH \in [1, X]\). Then for any integer \(h_1 \geq 1\), we have

\[
\sum_{d \sim D_1} \tau_{h_1}(d) \cdot \max_{(c, d) = 1} \left| \sum_{m \sim M, n \sim N} \alpha_1(m)\alpha_2(n)e(mn\theta) \right| \ll X \left( \frac{DM}{X} + \frac{D^2}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} (\log X)^{h^2+h_1^2/2+5}.
\]

**Proof.** Let \(\Sigma_1\) be the sum we wish to estimate in the corollary. Then, by Lemma 6.1 and the fact that \(N \leq M\), we have

\[
\Sigma_1 \ll DX \left( \frac{M}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/2} (\log X)^{h^2+h_1+1}. \tag{6.7}
\]

Next, we apply Lemma 6.2 with \(D_1 = D\) and \(D_2 = 1\) along with the fact that \(N \leq M\) to obtain

\[
\Sigma_1 \ll X \left( \frac{DM}{X} + \frac{D^2}{X} + \frac{1}{D_1^{1/4}} + \frac{(qH)^{1/4}}{X^{1/4}} + \frac{1}{(qH)^{1/4}} \right)^{1/2} (\log X)^{h^2+h_1^2/2+5}. \tag{6.8}
\]

From the inequalities (6.7) and (6.8), we obtain

\[
\Sigma_1^2 \ll X^2 \left\{ \frac{DM}{X} + \frac{D^2}{X} + \frac{(qH)^{1/4}}{X^{1/4}} + \frac{1}{(qH)^{1/4}} + \min \left( \frac{1}{D_1^{1/4}} + \frac{D^2M}{X} + \frac{D^2qH}{X} + \frac{D^2}{qH} \right) \right\} \times (\log X)^{2h^2+h_1^2+10}. \tag{6.9}
\]
Next, we have

$$
\min \left( \frac{1}{D^{1/4}} \cdot \frac{D^2 M}{X} + \frac{D^2 qH}{X} + \frac{D^2}{qH} \right) \leq \left( \frac{1}{D^{1/4}} \right)^{8/9} \cdot \left( \frac{D^2 M}{X} + \frac{D^2 qH}{X} + \frac{D^2}{qH} \right)^{1/9} = \left( \frac{M}{X} + \frac{qH}{X} + \frac{1}{qH} \right)^{1/9}.
$$

Finally, we substitute the above estimate in (6.9) along with the fact that $qH \in [1, X]$ to complete the proof of the corollary.

Now we combine Lemma 6.2 and Corollary 6.3 to deduce the following corollary.

**Corollary 6.4.** Let $D_1, D_2, M, N \geq 1$ be such that

$$D_1 M < X, \quad N \leq M, \quad \text{and} \quad MN < X.$$  

Suppose that $\alpha_1$ and $\alpha_2$ are two arithmetic functions supported in $[M, 2M]$ and $[N, 2N]$, respectively, and satisfy $|\alpha_1|, |\alpha_2| \leq \tau h \cdot \log$ for some fixed integer $h \geq 1$. Let $\vartheta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Furthermore, let $H = 1 + |\beta| X$ and $qH \in [1, X]$. Set

$$S := \sum_{d_1 \sim D_1} \sum_{d_2 \sim D_2} \left| \sum_{\substack{m \sim M, \ n \sim N \ \text{mod} \ (d_1, d_2)}} \alpha_1(m) \alpha_2(n) e(mn \vartheta) \right|$$

Then the following estimates hold.

(a) If $D_1 D_2^2 N < X$, we have

$$S \ll X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_1 D_2^2 N}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(D_2 M)^{1/5}}{X^{1/5}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} \times (\log X)^{h^2 + 7}.$$

(b) If $D_1 D_2^{3/2} < X^{1/2}$, we have

$$S \ll X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X^{1/2}} + \frac{D_1 D_2^{3/2}}{X^{1/2}} + \frac{M^{1/9}}{X^{1/9}} + \frac{(D_2 M)^{1/5}}{X^{1/5}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} \times (\log X)^{h^2 + 7}.$$

**Proof.** We apply Lemma 6.2 with $h_1 = 1$ and $c' = c$ to obtain

$$S \ll X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_1 D_2^2 N}{X} + \frac{1}{D_1^{1/4}} + \frac{(qH)^{1/4}}{X^{1/4}} + \frac{1}{(qH)^{1/4}} \right)^{1/2} (\log X)^{h^2 + 6}. \quad (6.10)$$
Next, we write \( d = d_1d_2 \), so that \( d \in [D_1D_2, 4D_1D_2] \). We then apply Corollary \( 6.3 \) with \( D = D_1D_2 \) and \( h_1 = 2 \) to obtain

\[
S \ll X \left( \frac{D_1D_2M}{X} + \frac{(D_1D_2)^2}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} (\log X)^{h_2+7} \quad (6.11)
\]

From the inequalities \( (6.10) \) and \( (6.11) \), we obtain

\[
|S|^2 \ll X^2 \left\{ \frac{D_1M}{X} + \frac{(D_1D_2)^2}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right. \\
\left. + \min \left( \frac{D_1D_2^2N}{X} + \frac{1}{D_1^{1/4}}, \frac{D_1D_2M}{X} \right) \right\} (\log X)^{2h_2+14}, \quad (6.12)
\]

where we have used the fact that \( qH \in [1, X] \). Now we optimize the right-hand side of the above expression to obtain

\[
\min \left( \frac{D_1D_2^2N}{X} + \frac{1}{D_1^{1/4}}, \frac{D_1D_2M}{X} \right) \ll \frac{D_1D_2^2N}{X} + \left( \frac{1}{D_1^{1/4}} \right)^{4/5} \left( \frac{D_1D_2M}{X} \right)^{1/5} \ll \frac{D_1D_2^2N}{X} + \frac{(D_2M)^{1/5}}{X^{1/5}}.
\]

Substituting the above estimate in \( (6.12) \) completes the proof of the part (a) of the corollary. Next, we note that

\[
\min \left( \frac{D_1D_2^2N}{X} + \frac{1}{D_1^{1/4}}, \frac{D_1D_2M}{X} \right) \ll \left( \frac{D_1D_2^2N}{X} \cdot \frac{D_1D_2M}{X} \right)^{1/2} + \left( \frac{1}{D_1^{1/4}} \right)^{4/5} \left( \frac{D_1D_2M}{X} \right)^{1/5} \ll \frac{D_1D_2^{3/2}}{X^{1/2}} + \frac{(D_2M)^{1/5}}{X^{1/5}}.
\]

The above estimate together with \( (6.12) \) completes the proof of part (b). \( \square \)

7 | EXPONENTIAL SUMS OVER PRIMES IN ARITHMETIC PROGRESSIONS

7.1 | A general exponential sum estimate over primes in arithmetic progressions

We consider a general exponential sum estimate. Our key aim is to reduce the exponential sum over primes in arithmetic progressions into estimating Type I and Type II sums via the Vaughan identity.

**Proposition 7.1** (General exponential sum over primes in arithmetic progressions). Let \( \delta > 0 \) be small and let \( b \) be a fixed positive integer. Suppose that \( \sigma \) is an arithmetic function such that \( \sigma \) is
supported in \([1, D]\) with \(D \leq X^{1/2 - \delta}\), \(|\sigma| \leq \tau\), and for each \(d\) in the support of \(\sigma\), \(c_d\) is some reduced residue class modulo \(d\).

Let \(\delta = a/q + \beta\) with \((a, q) = 1\) and \(|\beta| < 1/q^2\). Furthermore, let \(H = 1 + |\beta|X\) and \(qH \in [1, X]\).

For any arithmetic functions \(\alpha_1, \alpha_2, \alpha_3\) with \(|\alpha_1|, |\alpha_2|, |\alpha_3| \leq \tau_2 \cdot \log\), suppose that the following two conditions holds.

(I) For \(j \in \{0, 1\}\), and for some constant \(C_1 > 0\), we have

\[
\left| \sum_{d \leq D} \sigma(d) \sum_{m_1 < X} \sum_{n \equiv c_d \pmod{d}} \alpha_1(m)(\log n)^j \epsilon(m \theta) \right| \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_1}.
\]

(II) For \(N \leq M, MN < X\), and for some constant \(C_2 > 0\), we have

\[
\max_{D' \leq D, M, N \leq X^{2/3}} \left| \sum_{d' \sim D'} \sigma(d) \sum_{m' < X} \sum_{n' \equiv c_d \pmod{d}} \alpha_2(m)\alpha_3(n) \epsilon(mn \theta) \right| \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_2}.
\]

Then, we have

\[
\left| \sum_{d \leq D} \sigma(d) \sum_{n < X} \Lambda(n) \epsilon(n \theta) \right| \ll_{b, \delta} X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_3},
\]

where \(C_3 = \max\{C_1, C_2 + 3\}\).

Proof. We may drop the condition \((n, b) = 1\) in the sum. Indeed, the contribution of \((n, b) > 1\) is

\[
\ll \left| \sum_{d \leq D} \sigma(d) \sum_{n < X} \Lambda(n) \epsilon(n \theta) \right| \ll D(\log D)(\log X)\tau(b)
\]

\[
\ll_{b, \delta} X^{1/2 - \delta} (\log X)^2 \ll_{b, \delta} X \cdot \left( \frac{(qH)^{1/2 + \delta}}{X^{1/2 + \delta}} \right) (\log X)^2,
\]

which is negligible. Therefore, we can focus on bounding the following sum

\[
\Sigma := \sum_{d \leq D} \sigma(d) \sum_{n < X} \Lambda(n) \epsilon(n \theta).
\]  

Let \(U = X^{1/3}\). Then, by Vaughan’s identity (see [12, Lemma 23.1]), we have

\[
\Lambda(n) = \Lambda_{\leq U}(n) + (\mu_{\leq U} * \log)(n) - (\mathfrak{f}_{\leq U} * 1)(n) - (\mathfrak{f}_{> U} * 1)(n) + (\mu_{> U} * \Lambda_{> U} * 1)(n),
\]

where \(\mathfrak{f} = \mu_{\leq U} * \Lambda_{\leq U}\) and note that \(|\mathfrak{f}| \leq \log\). This allows us to write the sum in (7.1) as

\[
\Sigma = \sum_{d \leq D} \sigma(d) \sum_{n < X} \left( \Lambda_{\leq U}(n) + (\mu_{\leq U} * \log)(n) - (\mathfrak{f}_{\leq U} * 1)(n) \right).
\]
\[-(f_{>U} \ast 1)(n) + (\mu_{>U} \ast \Lambda_{>U} \ast 1)(n)]e(n\theta)

= \Sigma_1 + \Sigma_2 - \Sigma_3 - \Sigma_4 + \Sigma_5, \tag{7.2}\]

say.

As \(\Lambda \ll \log\), we can bound the sum \(\Sigma_1\) as

\[
\Sigma_1 \ll (\log X) \sum_{d \leq D} \tau(d) \left( \frac{X^{1/3}}{d} + 1 \right) \ll (X^{1/3} + D)(\log X)^3 \ll \frac{X(qH)^{\delta/2}}{X^{\delta/2}}. \tag{7.3}\]

Next, we estimate the sums \(\Sigma_2\) and \(\Sigma_3\) using condition (I) with \(\alpha_1 \in \{\mu \leq X^{1/3}, f \leq X^{1/3}\}\) to obtain

\[
\Sigma_2, \Sigma_3 \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^C_1. \tag{7.4}\]

Now we estimate the sum \(\Sigma_4\) given by

\[
\Sigma_4 = \sum_{d \leq D} \sigma(d) \sum_{m \leq X} \sum_{n \leq X} f(m)e(mn\theta). \]

By a dyadic decomposition of summation ranges, we find that

\[
\Sigma_4 \ll (\log X)^3 \max_{1 \leq D' \leq D} \max_{X^{1/3} \leq M \leq X^{2/3}} \max_{1 \leq N \leq X^{2/3}} \left| \sum_{d \sim D'} \sigma(d) \sum_{m \leq X} \sum_{n \leq X} f(m)e(mn\theta) \right|. \tag{7.5}\]

Now we can apply condition (II) with \(\{\alpha_2, \alpha_3\} = \{f > X^{1/3}, 1\}\) by considering whether \(M\) or \(N\) is longer or not. The key point is that both \(M, N \leq X^{2/3}\). Therefore, we obtain

\[
\Sigma_4 \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_2+3}. \tag{7.5}\]

Similarly, by a dyadic decomposition of summation ranges in \(\Sigma_5\), we have

\[
\Sigma_5 \ll (\log X)^3 \max_{1 \leq D' \leq D} \max_{X^{1/3} \leq M, N \leq X^{2/3}} \left| \sum_{d \sim D'} \sigma(d) \sum_{m \leq X} \sum_{n \leq X} \Lambda(m)(\mu_{>X^{1/3}} \ast 1)(n)e(mn\theta) \right|. \tag{7.6}\]

As both \(M, N \in [X^{1/3}, X^{2/3}]\), without the loss of generality we can assume \(N \leq M\) and apply condition (II) with \(\alpha_2 = \Lambda_{>X^{1/3}}\) and \(\alpha_3 = \mu_{>X^{1/3}} \ast 1\) to obtain

\[
\Sigma_5 \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_2+3}. \tag{7.6}\]

Hence, substituting the estimates from (7.3), (7.4), (7.5), (7.6) in (7.2) completes the proof of the proposition. \(\square\)
Remark 1. Note that if $\delta > 0$ small, $\delta_1 \in \{\delta, \delta/2\}$, and $\delta_2, \delta_3 \geq \delta_1$, then we have the following estimate
\[
\frac{1}{X^{\delta_1}} + \frac{1}{X^{\delta_2}} + \frac{(qH)^{\delta_3}}{X^{\delta_3}} + \frac{1}{(qH)^{\delta_3}} \ll \frac{(qH)^{\delta_1}}{X^{\delta_1}} + \frac{1}{(qH)^{\delta_1}},
\]
where $H = 1 + |\beta|X$ and $qH \in [1, X]$.
We will use the above estimate on several occasions in the paper.

### 7.2 Exponential sum estimates over primes in arithmetic progression

We now employ Proposition 7.1 to establish the following exponential sum estimate.

**Proposition 7.2** (Exponential sum over primes in arithmetic progressions). Let $\delta > 0$, let $b$ be a fixed positive integer, and let $D \leq X^{1/3-\delta}$. Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Furthermore, let $H = 1 + |\beta|X$ and $qH \in [1, X]$. Then for some constant $C_1$, we have
\[
\sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{n < X} \Lambda(n) e(n\theta) \right| \ll b, \delta X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^C_1.
\]

**Proof.** Without the loss of generality, we may assume that the maximum over $c$ is attained at $c_d$. Let $\lambda(d) \in \mathbb{C}$ be of absolute value 1 whenever $c = c_d$ and $(d, bc_d) = 1$ for $d \in [1, D]$, so that
\[
\sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{n < X} \Lambda(n) e(n\theta) \right| = \sum_{d \leq D} \lambda(d) \sum_{n < X} \Lambda(n) e(n\theta).
\]

We may now use Proposition 7.1 with $\sigma = \lambda$ to establish the required bound. Note that $|\lambda| \leq 1$ in this case. So, it is enough to estimate the Type I and Type II sums.

**Verifying condition (I):** Recall that Type I sum in this case is of the following form
\[
\Sigma_{\text{Type I}} := \sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{n < X} \alpha_1(m) (\log n)^j e(mn\theta) \right|,
\]
where $|\alpha_1| \leq \tau_2 \cdot \log$ and $j \in \{0, 1\}$. We apply Lemma 5.5 with $M = X^{1/3}$, $v = 1$, $h_1 = 2$, $h_2 = 1$ and $h_3 = 1$ and Remark 1 to obtain
\[
\Sigma_{\text{Type I}} \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^6,
\]
as desired.

**Verifying condition (II):** We wish to estimate the following Type II sum
\[
\Sigma_{\text{Type II}} := \max_{D \leq X} \sum_{M, N \leq X^{2/3}} \max_{D - D'} \sum_{d \leq D'} \max_{(c, d) = 1} \left| \sum_{m < M, n < N} \alpha_2(m) \alpha_3(n) e(mn\theta) \right|.
\]
where $|\alpha_2|, |\alpha_3| \leq \tau_2 \cdot \log h$. Recalling that $D \leq X^{1/3-\delta}$, we may apply Corollary 6.3 with $h = 2$ and $h_1 = 1$ and Remark 1 to obtain

$$\Sigma_{\text{Type II}} \ll X^{\left(\frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}}\right)(\log X)^{1.0}}.$$ 

This completes the verification of condition (II), and hence the proof of the proposition. \hfill \square

### 7.3 Exponential sum over primes in arithmetic progressions with composite moduli

We now establish the exponential sum over primes in arithmetic progressions with composite moduli, which is one of the key inputs to prove Theorem 2.

**Proposition 7.3** (Exponential sum over primes with composite moduli). Let $\delta > 0$ be small, and let $b$ be a fixed positive integer. Let $D_1 \in [1, X^{1/3-\delta}]$ and $D_2 \in [1, X^{1/9}]$. Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Furthermore, let $H = 1 + |\beta|X$ and $qH \in [1, X]$. Let $c$ be a fixed nonzero integer. Then for some constant $C_2 > 0$, we have

$$\sum_{d_1 \leq D_1} \sum_{d_2 \leq D_2} \left| \sum_{\substack{n < X \atop n \equiv c \pmod{d_1d_2} \atop (n, b) = 1}} \Lambda(n)e(n\theta) \right| \ll_{b, \delta} X^{\left(\frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}}\right)(\log X)^{C_2}}, \quad (7.9)$$

where $*$ in the sum denotes the conditions $(d_1, d_2) = (d_1d_2, bc) = 1$.

**Proof.** We write

$$\sum_{d_1 \leq D_1} \sum_{d_2 \leq D_2} \left| \sum_{\substack{n < X \atop n \equiv c \pmod{d_1d_2} \atop (n, b) = 1}} \Lambda(n)e(n\theta) \right| = \sum_{d_1 \leq D_1} \sum_{d_2 \leq D_2} \Lambda(d_1, d_2) \sum_{\substack{n < X \atop n \equiv c \pmod{d_1d_2} \atop (n, b) = 1}} \Lambda(n)e(n\theta),$$

where $\lambda(d_1, d_2)$ is a complex number of absolute value 1 whenever $(d_1d_2, bc) = (d_1, d_2) = 1$ with $d_1 \in [1, D_1]$ and $d_2 \in [1, D_2]$. We now apply Proposition 7.1 with

$$\sigma(d) = \sum_{d_1d_2=d \atop d_i \in D_i \forall j} \lambda(d_1, d_2),$$

to establish the proposition. Note that $|\sigma| \leq \tau$ in this case. So, it is enough to estimate the Type I and Type II sums.

We can use Lemma 5.5 to estimate the Type I sums, which is similar to the proof of Proposition 7.2, so that condition (I) holds in Proposition 7.1.

For Type II sums, we need to estimate the following sum

$$\Sigma_{\text{Type II}} := \max_{D_1', D_2', M, N} \left| \sum_{d_1 \sim D_1'} \sum_{d_2 \sim D_2'} \sum_{\substack{mn < X \atop m \equiv c \pmod{d_1d_2} \atop m - M, n - N}} \alpha_1(m)\alpha_2(n)e(mn\theta) \right|.$$
where $|\alpha_1|, |\alpha_2| \leq \tau_2 \cdot \log$ and the maximum is taken over those $D'_1, D'_2, M, N$ that satisfy

$$D'_1 \in [1, D_1], \quad D'_2 \in [1, D_2], \quad M, N \leq X^{2/3}, \quad MN < X, \quad \text{and} \quad N \leq M. \quad (7.10)$$

We divide our analysis of the above Type II sum into two cases:

**Case I:** Suppose that $M \leq X^{1/2}$. Then we write $d = d_1 d_2$ so that $d_1, d_2 \in [D'_1 D'_2, 4D'_1 D'_2]$. We can now apply Corollary 6.3 with $h = 2$ and $h_1 = 2$ to obtain

$$\Sigma_{\text{Type II}} \ll \max_{M, N \leq D'_1 D'_2} \sum_{d \sim D} \tau(d) \left| \sum_{m \sim M, n \sim N} \alpha_1(m) \alpha_2(n) e(mn\theta) \right| \ll \max_{M, N \leq D'_1 D'_2} X \left( \frac{DM}{X} + \frac{D^2}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} (\log X)^{11}.$$  

Note that by assumption $D_1 \leq X^{1/3 - \delta}$ and $D_2 \leq X^{1/9}$. This implies that $D_1 D_2 \leq X^{4/9 - \delta} \leq X^{1/2 - \delta}$. Therefore,

$$\Sigma_{\text{Type II}} \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{11}$$

by Remark 1.

**Case 2:** Now we consider the case $M \geq X^{1/2}$. In this case, we have $N \leq X^{1/2}$. So, applying Corollary 6.4 (b) with $h = 2$, we obtain

$$\Sigma_{\text{Type II}} \ll \max_{D'_1, D'_2, M, N} X \left( \frac{D'_1 M}{X} + \frac{(D'_1 D'_2)^2}{X} + \frac{D'_1 (D'_2)^{3/2}}{X^{1/2}} + \frac{M^{1/9}}{X^{1/9}} + \frac{(D'_2 M)^{1/5}}{X^{1/5}} + \frac{(qH)^{1/9}}{X^{1/9}} \right)^{1/2} (\log X)^{11}.$$  

Recall the relation $(7.10)$, and note by assumption that $D_1 \leq X^{1/3 - \delta}$, and $D_2 \leq X^{1/9}$, so that $D_1 D_2 \leq X^{4/9 - \delta}$ and $D_1 D_2^{3/2} \leq X^{1/2 - \delta}$. Therefore, by Remark 1, we have

$$\Sigma_{\text{Type II}} \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{11}.$$  

The above two cases complete our analysis of Type II sum estimates. Hence, this completes the proof of the proposition. \qed

### 7.4 Exponential sum over primes with a well-factorable function

We now establish exponential sum over primes in arithmetic progressions weighted by a well-factorable function (see Definition 1 for the notion of well-factorable).

**Proposition 7.4** (Well-factorable exponential sum estimate). Let $\delta > 0$ and let $b$ be a fixed positive integer. Let $c$ be a fixed nonzero integer and let $\xi : \mathbb{N} \to \mathbb{R}$ be a well-factorable function of level $D \in$...
\([1, X^{1/2-\delta}]\) with \(|\xi| \leq 1\). Let \(\theta = a/q + \beta\) for some \((a, q) = 1\) and \(|\beta| < 1/q^2\). Furthermore, let \(H = 1 + |\beta|X\) and \(qH \in [1, X]\). Then for some constant \(C_3 > 0\), we have

\[
\sum_{d \leq D} \xi(d) \sum_{\substack{n < X \\mod d \\equiv c \mod d \\equiv 1 \\pmod{b} \\equiv 1 \\pmod{b}}} \Lambda(n) e(n\theta) \ll \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} (\log X)^C_3. \tag{7.11}
\]

Proof. If \(D \leq X^{1/3-\delta}\), we can apply Proposition 7.2 and the fact that \(|\xi| \leq 1\) to establish the required bound in the proposition with \(C_3 = C_1\). Therefore, we can assume that \(D > X^{1/3-\delta}\) for the rest of the proof.

We will use Proposition 7.1 with \(\sigma(d) = \xi(d)\) for \(d \in (X^{1/3-\delta}, X^{1/2-\delta}]\). The calculations for the Type I sums are analogous to, as in the proof of Proposition 7.2. We can apply Lemma 5.5 to estimate the Type I sum, so that condition (I) holds in Proposition 7.1. The key difference is the estimate for the Type II sums. So, we will explain the Type II sum estimates in this case. To do that, we must estimate the following Type II sum:

\[
\Sigma_{\text{well-fac, Type II}} := \max_{D', M, N} \left| \sum_{d \sim D'} \xi(d) \sum_{\substack{mn < X \\mod d \\equiv c \mod d \\equiv 1 \\pmod{b} \\equiv 1 \\pmod{b}}} \alpha_1(m) \alpha_2(n) e(mn\theta) \right|
\]

where \(|\alpha_1|, |\alpha_2| \leq \tau_2 \cdot \log\) and the maximum is taken over those \(D', M\) and \(N\) that satisfy

\[
D' \in (X^{1/3-\delta}, X^{1/2-\delta}], \quad M, N \leq X^{2/3}, \quad N \leq M, \quad MN < X.
\]

As in the proof of Proposition 7.3, we divide the analysis of \(\Sigma_{\text{well-fac, Type II}}\) into two cases:

**Case 1:** Suppose that \(M \leq X^{1/2}\). We apply Corollary 6.3 with \(h = 2, h_1 = 1\), and the fact that \(|\xi| \leq 1\) to obtain

\[
\Sigma_{\text{well-fac, Type II}} \ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{10}.
\]

**Case 2:** Suppose that \(M \in [X^{1/2}, X^{2/3}]\). For any \(d \sim D'\) in the support of \(\xi\), we write

\[
d = d_1d_2 \quad \text{with} \quad (d_1, d_2) = 1 \quad \text{for} \quad d_1 \sim D_1 \quad \text{and} \quad d_2 \sim D_2,
\]

so that \(D_1 D_2 \sim D' \leq D \leq X^{1/2-\delta}\). We take

\[
D_1 = \frac{D'X^{1/2}}{M}.
\]

As \(D' \in (X^{1/3-\delta}, X^{1/2-\delta}]\) and \(M \in [X^{1/2}, X^{2/3}]\), we have \(D_1 \leq D'\) and \(D' X^{1/2} \geq M\). Therefore,

\[
\frac{D_1 M}{X} \leq X^{-\delta}, \quad \frac{D_1 D_2 N}{X} \leq X^{-\delta}, \quad \frac{D_2 M}{X} \leq \frac{1}{X^{1/6}}.
\]
Therefore, we can now apply Corollary 6.4 (a) with $h = 2$ to obtain

$$
\Sigma_{\text{well-fac, Type II}} \ll (\log X)^2 \max_{M,N} \sum_{d_1 \sim D_1} \sum_{d_2 \sim D_2} | \sum_{m \sim M, n \sim N} \alpha_1(m) \alpha_2(n) e(mn\theta) |
$$

$$
\ll \max_{M,N} X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_2 N}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(D_2 M)^{1/5}}{X^{1/5}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2}
$$

$$
\times (\log X)^{13}
$$

$$
\ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{13}.
$$

The above two cases cover the entire range for the Type II sums. Therefore, condition (II) holds in Proposition 7.1. Hence, this completes the proof of the proposition.

7.5 | Exponential sum over primes with semi-linear sieve

We will use Lemma 3.2 to estimate the exponential sum in the following proposition.

**Proposition 7.5** (Semi-linear sieve exponential sum estimate). Let $\varepsilon > 0$ be small and let $\delta \in (0, 10^{-3}]$. Let $b$ be a fixed positive integer. Let $\lambda_{\text{sem}}^{\delta}$ be a lower bound semi-linear sieve weights of level $D \in \left[ 2, X^{3(1-4\delta)-\varepsilon} \right]$, as given in Lemma 3.1 and Lemma 3.2. Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Furthermore, let $H = 1 + |\beta| X$ and $qH \in [1, X]$. Then for some constant $C_4 > 0$, we have

$$
\sum_{d \leq D} \lambda_{\text{sem}}^{\delta} (d) \sum_{n < X} \Lambda(n) e(n \theta) \ll_{b, \delta, \varepsilon} X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{C_4}. \quad (7.12)
$$

The above proposition is closely related to [20, Theorem 1.5]. In fact, we will borrow a few ideas from [20] to establish the above proposition.

**Proof.** If $D \leq X^{1/10}$, the estimate in (7.12) follows from Proposition 7.2. So, we may assume throughout the proof that $D \geq X^{1/10}$.

We now apply Proposition 7.1 with $\sigma = \lambda_{\text{sem}}^{\delta}$. To do that, we consider the following Type I and Type II sums:

$$
\Sigma_{\text{sem, Type I}} := \sum_{d \leq D} \lambda_{\text{sem}}^{\delta} (d) \sum_{mn < X} \sum_{m \equiv 1 \pmod{d}} \alpha(m) \log^n(m) e(mn\theta),
$$

where $m \equiv 1 \pmod{d}$ and $n \equiv 3 \pmod{8}$.

$$
\Sigma_{\text{sem, Type II}} \ll (\log X)^2 \max_{M,N} \sum_{d \sim D_1} \sum_{d_2 \sim D_2} | \sum_{m \sim M, n \sim N} \alpha_1(m) \alpha_2(n) e(mn\theta) |
$$

$$
\ll \max_{M,N} X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_2 N}{X} + \frac{M^{1/9}}{X^{1/9}} + \frac{(D_2 M)^{1/5}}{X^{1/5}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2}
$$

$$
\times (\log X)^{13}
$$

$$
\ll X \left( \frac{(qH)^{\delta/2}}{X^{\delta/2}} + \frac{1}{(qH)^{\delta/2}} \right) (\log X)^{13}.
$$

The above two cases cover the entire range for the Type II sums. Therefore, condition (II) holds in Proposition 7.1. Hence, this completes the proof of the proposition.
\[ \Sigma_{\text{sem, Type II}} := \max_{D', M, N} \left| \sum_{d \sim D'} \lambda^{-}_{\text{sem}}(d) \sum_{m,n \leq X} \alpha_1(m)\alpha_2(n)e(mn\theta) \right|, \]

where \( j \in \{0,1\}, |\alpha|, |\alpha_1|, |\alpha_2| \leq \tau_2 \cdot \log \) and the maximum is over those \( D', M, N \) that satisfy

\[ D' \in \left[ X^{1/10}, X^3(1-4\delta)^{-\varepsilon} \right], \quad M, N \leq X^{2/3}, \quad N \leq M, \quad MN < X. \quad (7.13) \]

First, we use Lemma 5.5 to estimate the type I sum with \( h_1 = 2, h_2 = 1, h_3 = 1, M \leq X^{1/3} \) and \( D \leq X^{3(1-4\delta)/7-\varepsilon} \) to obtain

\[ \Sigma_{\text{sem, Type I}} \ll X \left( (qH)^{\delta} X^{\delta} \frac{1}{(qH)^{\delta}} \right) (\log X)^6. \]

This implies that condition (I) holds in Proposition 7.1.

Next, by orthogonality of the Dirichlet characters \( \chi_8 \) modulo 8, we have

\[ |\Sigma_{\text{sem, Type II}}| \ll \max_{D', M, N} \left| \sum_{d \sim D'} \lambda^{-}_{\text{sem}}(d) \sum_{m,n \leq X} \alpha_1(m)\chi_8(m)\alpha_2(n)\chi_8(n)e(mn\theta) \right|. \]

Next, we divide our analysis of the sum \( \Sigma_{\text{sem, Type II}} \) into two cases.

Case 1: Suppose that \( M \leq X^{1/2} \). In this case, we use Corollary 6.3 with \( D = D' \), \( c = 1, h = 2, h_1 = 1 \), and the facts that \( |\lambda^{-}_{\text{sem}}| \leq 1 \) and \( D' \leq X^{3/7} \leq X^{1/2-2\delta} \), to obtain

\[ \Sigma_{\text{sem, Type II}} \ll X \left( (qH)^{\delta} X^{\delta} \frac{1}{(qH)^{\delta}} \right) (\log X)^{10}. \]

Case 2: Suppose that \( M \in [X^{1/2}, X^{2/3}] \). The assumption on \( M \) implies that \( N \leq X^{1/2} \). We now consider two subcases.

Case 2(a): Suppose that \( D' \in [X^{1/10}, X^{3(1-4\delta)/7-\varepsilon}] \) and \( D' \leq X^{1-2\delta-\varepsilon^2}/M \). Recalling that \( |\lambda^{-}_{\text{sem}}| \leq 1 \), and by Corollary 6.3 with \( D = D' \), \( c = 1, h = 2, h_1 = 1 \), we obtain

\[ \Sigma_{\text{sem, Type II}} \ll X \left( (qH)^{\delta} X^{\delta} \frac{1}{(qH)^{\delta}} \right) (\log X)^{10}. \]

By assumption, \( D'M/X \leq X^{-2\delta-\varepsilon^2} \leq X^{-2\delta}, D' \leq X^{3/7} \leq X^{1/2-2\delta} \), and \( M \leq X^{2/3} \), so by Remark 1 we have

\[ \Sigma_{\text{sem, Type II}} \ll X \left( (qH)^{\delta} X^{\delta} \frac{1}{(qH)^{\delta}} \right) (\log X)^{10}. \]

Case 2(b): Finally, we consider the case when \( D' \in [X^{1/10}, X^{3(1-4\delta)/7-\varepsilon}] \) and \( D' > X^{1-2\delta-\varepsilon^2}/M \).

Note that the sifting parameter associated with \( \lambda^{-}_{\text{sem}} \) is \( X^{1/3-2\delta-2\varepsilon^2} \). We fix a parameter \( D_0 \in [X^{1/3-2\delta-2\varepsilon^2}, X^{3(1-4\delta)/7-\varepsilon}] \) to be chosen shortly. Then any \( d \sim D' \) in the support of \( \lambda^{-}_{\text{sem}} \) can be written as \( d = d_1d_2 \) with \( d_1 \in [X^{1/10}, D_0] \) and \( d_1d_2^2 \leq X^{1-4\delta-2\varepsilon^2}/D_0 \).

We take \( D_0 = X^{1-2\delta-\varepsilon^2}/M \). Note that as \( M \in \left[ X^{1/2}, X^{2/3} \right] \), this implies that \( D_0 \geq X^{1/3-2\delta-\varepsilon^2} \) and by assumption, \( D_0 = X^{1-2\delta-\varepsilon^2}/M < D' \leq X^{3(1-4\delta)/7-\varepsilon} \), so Lemma 3.2 is applicable in this case.
Next, we perform a dyadic decomposition of the range of the variables $d_1$ and $d_2$, so that

$$d_1 \sim D_1, \ d_2 \sim D_2, \ \text{where} \ \frac{X^{1/10}}{4} \leq D_1 \leq D_0, \ \frac{X^{1-4\delta-2\varepsilon}}{D_0} \leq D_1 D_2 \leq D'.$$

Therefore, we have

$$\frac{X^{1/10}}{4} \leq D_1 \leq \frac{X^{1-2\delta}}{M} \quad \text{and} \quad \frac{X^{1-4\delta}}{D_0} \leq \frac{M}{X^{2\delta}}. \quad (7.14)$$

By Lemma 6.2 with $h = 2$ and $h_1 = 1$, we obtain

$$\Sigma_{\text{sem, Type II}} \ll (\log X)^2 \max_{D_1, D_2 = D'} \left| \sum_{d_1 \sim D_1} \sum_{d_2 \sim D_2} \sum_{(d_1, d_2, 2bc) = 1} \sum_{(d_1, d_2) = 1} \alpha_1(m) \chi_8(m) \alpha_2(n) \chi_8(n) e(mn\theta) \right| \ll \max_{D_1, D_2 = D'} X \left( \frac{D_1 M}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_1 D_2^2 N}{X} + \frac{1}{D_1^{1/4}} \frac{1}{X^{1/4}} + \frac{1}{(qH)^{1/4}} + \frac{(qH)^{1/4}}{X} \right)^{1/2} \times (\log X)^{12}.$$

Using (7.14), recalling from (7.13) that

$$D_1 D_2 \simeq D' \leq X^{3(1-4\delta)/7-\varepsilon}, \ M N < X, \ M \leq X^{2/3},$$

and by Remark 1, we have

$$\Sigma_{\text{sem, Type II}} \ll X \left( \frac{(qH)^{\delta}}{X^{\delta}} + \frac{1}{(qH)^{\delta}} \right)(\log X)^{12}.$$

The above cases cover the entire range for the Type II sums. Noting that $\delta > \delta/2$, we see that condition (II) holds in Proposition 7.1. Hence, this completes the proof of the proposition. \hfill \square

**Remark.** We note that our proof of Case 2(b) in Proposition 7.5 can be generalized to any well-factorable sieve weights of level $D$ as long as $D \leq X^{1/2-2\delta}$. The same idea will feature in the proof of Proposition 7.6.

### 7.6 Exponential sum with linear sieve

We will use Lemma 3.4 to establish Proposition 7.6 given below.

**Proposition 7.6** (Linear sieve exponential sum estimate). Let $\varepsilon > 0$ be small and let $\delta \in (0, 10^{-3}]$. Let $b$ be a fixed positive integer. Let $\lambda_\text{lin}^+$ be an upper bound linear sieve weights of level $D \in [2, X^{1/2-2\delta-\varepsilon}]$, as given in Lemmas 3.3 and 3.4. Let $L$ be a real number such that $L \in \left[ X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon} \right]$ and let $h$ be a bounded arithmetic real-valued function. Let $\theta = a/q + \beta$ with $(a, q) = 1$ and $|\beta| < 1/q^2$. Furthermore, let $H = 1 + |\beta| \{X \text{ and } qH \in [1, X]\}$. Then for some constant
\[ C_5 > 0, \text{ we have} \]
\[
\sum_{d \leq D} \lambda^+(d) \sum_{\ell \sim L} \ell \sum_{n < X/2\ell} \Lambda(n) e(2\ell n + 1) \theta 
\]
\[
\ll_{b, \delta, \varepsilon} X \left( \frac{(qH)\delta}{X^{\delta}} + \frac{1}{(qH)^\delta} \right) (\log X)^{C_5}. \quad (7.15)
\]

Proof. Let \( \Sigma_{\text{lin}} \) be the sum we wish to estimate. The proof is similar to the proof of Proposition 7.5.

We note that \( \mathfrak{h}(\ell) \) is supported on \([L, 2L]\) with \( L \in [X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon}] \). We can proceed in the same way as in the proof of Proposition 7.5.

We begin with a dyadic decomposition of the range of \( n \) variable, say \( n \sim N \) in the sum \( \Sigma_{\text{lin}} \). Note that as \( L \in [X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon}] \) and \( n < X/2\ell \), we have that \( N \leq X^{2/3+2\delta+\varepsilon} \). Moreover, we also have that \( L \leq X^{2/3+2\delta+\varepsilon} \).

We therefore define two new parameters \( M' \) and \( N' \), where

\[
M' = \max\{L, N\}, \quad N' = \min\{L, N\}, \quad \text{so that} \quad N'M' < X, \quad N', M' \leq X^{2/3+2\delta+\varepsilon}.
\]

We also perform a dyadic decomposition on the range of \( d \) variable, say \( d \sim D' \), with \( D' \ll D \). Similarly to the proof of Proposition 7.5, we introduce Dirichlet characters \( \chi_4 \) modulo 4 to detect the congruence condition \( \ell n \equiv 1 \pmod{4} \). Therefore, we have

\[
\Sigma_{\text{lin}} \ll (\log X)^2 \max_{D', M', N'} \left| \sum_{d \sim D'} \lambda^+(d) \sum_{m \sim M'} \alpha_1(m) \alpha_2(n) e(mn \theta) \right|,
\]

where for \( m \sim \{M', N'\} \),

\[
\{\alpha_1(m), \alpha_2(m)\} = \{\mathfrak{h}(m) \chi_4(m) \cdot 1_{(m, b) = 1}, \Lambda(m/2) \chi_4(m/2) \cdot 1_{(m/2, b) = 1}, 2|m\},
\]

and the maximum is over those \( D', M', N' \) that satisfy

\[
D' \in [2, X^{1/2-2\delta-\varepsilon}], \quad M', N' \leq X^{2/3+2\delta+\varepsilon}, \quad N' \leq M', \quad M'N' < X. \quad (7.16)
\]

Note that as \( \mathfrak{h} \) is bounded and \( \Lambda \ll \log \), we have \( |\alpha_1|, |\alpha_2| \ll \log \). We also recall from Lemma 3.3 that \( |\lambda^+| \ll 1 \).

Next, we divide our analysis of the above sum into three cases.

Case 1: Suppose that \( D' \leq X^{1/10} \). We can then apply Corollary 6.3 with \( D = D' \) and \( h_1 = h = 1 \) to obtain

\[
\Sigma_{\text{lin}} \ll X \left( \frac{(qH)\delta}{X^{\delta}} + \frac{1}{(qH)^\delta} \right) (\log X)^9.
\]

For the rest of the two cases, we can assume that \( D' \geq X^{1/10} \).
Case 2: Suppose that $M' \leq X^{1/2}$ and $D' \in [X^{1/10}, X^{1/2-2\delta-\varepsilon}]$. In this case, we may apply Corollary 6.3 with $D = D'$, $c = -1$, $h = h_1 = 1$ to obtain

$$\Sigma_{\text{lin}} \ll X \left( \frac{(qH)^\delta}{X^\delta} + \frac{1}{(qH)^\delta} \right) \log X^9.$$

Case 3: Suppose that $M' \in [X^{1/2}, X^{2/3+2\delta+\varepsilon}]$. The assumption on $M'$ implies that $N' \leq X^{1/2}$. We now consider two subcases.

Case 3(a): Suppose that $D' \in [X^{1/10}, X^{1/2-2\delta-\varepsilon}]$ and $D' \leq X^{1-2\delta-\varepsilon} / M'$. We apply Corollary 6.3 with $D = D'$, $c = -1$, $h = h_1 = 1$ to obtain

$$\Sigma_{\text{lin}} \ll \log X^2 \max_{D', M', N'} X \left( \frac{D'M'}{X} + \frac{(D')^2}{X} + \frac{(M')^{1/9}}{X^{1/9}} + \frac{(qH)^{1/9}}{X^{1/9}} + \frac{1}{(qH)^{1/9}} \right)^{1/2} \log X^7.$$

By assumption, $D'M' / X \leq X^{-2\delta-\varepsilon} \leq X^{-\delta}$, $M' \leq X^{2/3+2\delta+\varepsilon}$ and $D' \leq X^{1/2-2\delta-\varepsilon}$. Therefore, by Remark 1, we see that

$$\Sigma_{\text{lin}} \ll X \left( \frac{(qH)^\delta}{X^\delta} + \frac{1}{(qH)^\delta} \right) \log X^9.$$

Case 3(b): Finally, we consider the case when $D' \in [X^{1/10}, X^{1/2-2\delta-\varepsilon}]$ and $D' > X^{1-2\delta-\varepsilon} / M'$. If $d \sim D'$ we write $d = d_1 d_2$, so that $d_1, d_2$ satisfy for every $D_0 \in [X^{1/5}, X^{1/2-2\delta-\varepsilon}]$, the inequalities $d_1 \in [X^{1/10}, D_0]$ and $d_1 d_2 \leq X^{1-4\delta-2\varepsilon} / D_0$.

We take $D_0 = X^{1-2\delta-\varepsilon} / M'$, which is in the range $[X^{1/5}, X^{1/2-2\delta-\varepsilon}]$ by the assumption on $D'$ and $M'$. This allows us to apply Lemma 3.2. Next, we do a dyadic decomposition of the range of $d_1$ and $d_2$ variables so that

$$d_1 \sim D_1, d_2 \sim D_2, \quad \text{where} \quad X^{1/10} \ll D_1 \leq D_0, \quad D_1 D_2 \leq X^{1-4\delta-2\varepsilon} / D_0, \quad D_1 D_2 \asymp D'.$$

Therefore, we have

$$X^{1/10} \ll D_1 \leq \frac{X^{1-2\delta-\varepsilon}}{M'} \leq \frac{X^{1-2\delta}}{M'} \quad \text{and} \quad D_1 D_2 \leq \frac{X^{1-4\delta-2\varepsilon}}{D_0} \leq \frac{M'}{X^{2\delta}}.$$

Recalling from (7.16) that $M'N' < X$ and $D' \leq X^{1/2-2\delta-\varepsilon}$, we can now use Lemma 6.2 to obtain the desired estimate

$$\Sigma_{\text{lin}} \ll \max_{D_1, D_2 \geq D'} \max_{M', N'} X \left( \frac{D_1M'}{X} + \frac{(D_1 D_2)^2}{X} + \frac{D_1 D_2^2 N'}{X} + \frac{1}{D_1^{1/4}} + \frac{(qH)^{1/4}}{X^{1/4}} + \frac{1}{(qH)^{1/4}} \right)^{1/2} \log X^{10}$$

$$\ll X \left( \frac{(qH)^\delta}{X^\delta} + \frac{1}{(qH)^\delta} \right) \log X^{10}.$$

The above three cases cover the entire range for the sum $\Sigma_{\text{lin}}$; hence, the proposition is established. □
PART IV. CIRCLE METHOD

In this part of the paper, we establish Theorems 1–3, 5, and 6. We will use the circle method and employ the exponential sums estimates from Part III to establish them.

8 | PROOF OF THEOREMS 1–3, 5, AND 6

8.1 | General theorem

In this section, we consider a general theorem for an arithmetic function \( \mathfrak{f} \) satisfying some conditions (see Theorem 7) to prove our main results.

Theorem 7 (General theorem). Let \( \delta > 0 \) and let \( b \) be an integer that is sufficiently large in terms of \( \delta \). Let \( k \) be a positive integer and set \( X := b^k \). Let \( D \) be a real number such that \( D \in [1, X^{1/2}) \). Let \( r \in A \cap \{0, b\} \) with \( (r, b) = 1 \) and let \( s \) be a positive integer such that \( (r - s, b) = 1 \). Let \( \mathfrak{f} \) be an arithmetic function supported on integers co-prime to \( b \) and \( |\mathfrak{f}| \ll \log X \). Suppose there exists an arithmetic function \( \sigma \) such that \( \sigma \) is supported on \([1, D]\), \( |\sigma| \leq \tau \), and for each \( d \) in the support of \( \sigma \), \( c_d \) is some reduced residue class modulo \( d \). Furthermore, assume that the following three conditions hold.

(a) (Partial sum estimate) For any \( y \in [X^{3/4}, X] \), for any \( A > 0 \) and for any integer \( d \in [1, X] \), there exists a parameter \( \lambda_d \) such that \( |\lambda_d| \ll \log X \) and the relation

\[
\sum_{n \leq y \atop (n, d) = 1} \mathfrak{f}(n) = y\lambda_d + O_{A, b} \left( \frac{y}{(\log y)^A} \right)
\]

holds.

(b) (Equidistribution estimate in arithmetic progressions) For any \( A, C > 0 \), we have

\[
\sum_{d \leq D \atop (d, b) = 1} \sum_{q \leq (\log X)^C \atop q | X} \max_{(c, d) = 1} \max_{1 \leq m \leq X^{3/4} \leq y \leq X} \left| \sum_{n \leq y \atop n \equiv c \pmod{d}} \mathfrak{f}(n) - \frac{y\lambda_d}{\phi(dq)} \right| \ll_{A, C, b} X \left( \frac{(\log X)^C}{(\log X)^A} \right),
\]

where \( \lambda_d \) is as described in condition (a).

(c) (Exponential sum estimate) Consider \( \theta = a/q + \beta \) with \( (a, q) = 1 \) and \( |\beta| < 1/q^2 \). Furthermore, let \( H = 1 + |\beta|X \) and \( qH \in [1, X] \). Let \( \omega \) be such that \( \omega \in (0, 1) \) and \( \alpha_b < \omega/2 \) (where \( \alpha_b \) is given by the relation \((9.4)\)). Then there exists an absolute constant \( C' > 0 \) such that

\[
\sum_{d \leq D \atop (d, b) = 1} \sigma(d) \sum_{n \leq X \atop n \equiv c \pmod{d}} \mathfrak{f}(n)e(n\theta) \ll_{b, \delta, \omega} X \left( \frac{(qH)^\omega}{X^\omega} + \frac{1}{(qH)^\omega} \right)(\log X)^{C'}.
\]

Then for any \( A > 0 \), we have

\[
\sum_{d \leq D \atop (d, b) = 1} \sigma(d) \left( \sum_{n \leq X \atop n \equiv c \pmod{d}} \mathfrak{f}(n)1_{A_r}(n + s) - \frac{\lambda_d}{\phi(d)} \frac{b}{\phi(b)} \sum_{n \leq X \atop n \equiv c \pmod{d}} 1_{A_r}(n) \right) \ll \frac{X^\varepsilon}{(\log X)^A},
\]

(8.1)

where the implicit constant in Vinogradov’s notation \( \ll \) depends at most on \( A, b, \delta, \) and \( \omega \).
Remark. In Section 9, we will see that \( \alpha_b \) given by (9.4) tends to 0 as \( b \to \infty \). So, our assumption that \( \alpha_b < \omega/2 \) in condition (c) of Theorem 7 is justified.

Before embarking into the proof of Theorem 7, we explain how to use it to deduce Theorems 1–3, 5, and 6.

8.2 Proof of Theorems 1–3, 5, and 6

Proof of Theorems 1, 2, 3, 5, and 6 assuming Theorem 7. We begin with the proof of Theorem 1.

Proof of Theorem 1. We will show that for any \( A > 0 \),

\[
\sum_{d \leq X^{1/3-\delta}} \max_{(c,d)=1} \left| \sum_{n<X \atop n \equiv c \pmod d} \Lambda(n) 1_{A_r}(n) - \frac{1}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n<X \atop n \equiv c \pmod d} 1_{A_r}(n) \right| \ll_{A,b,\delta} X^{\tau} \frac{X^5}{(\log X)^A}. \tag{8.2}
\]

Without loss of generality, we can assume that \( \max (c,d) = 1 \) is attained at some reduced residue class \( c_d \) modulo \( d \). Then, in Theorem 7, we take \( f(n) = \Lambda(n)1_{(n,b)=1} \) for \( n < X \), \( D = X^{1/3-\delta} \), \( s = 0 \) and \( \sigma \) to be the corresponding sign of the expression inside the absolute value of the left-hand side of (8.2) whenever \( (d,bc_d) = 1 \). Clearly, \(|\sigma| = 1 \leq \tau\).

Now we check the three conditions in Theorem 7.

Verifying condition (a): The condition (a) with \( \lambda_d = 1 \) follows from the Prime Number Theorem [3, chapter 18] together with the fact that for any \( y \geq 2 \),

\[
\sum_{n \leq y \atop (n,bd)>1} \Lambda(n) \ll (\log bd)(\log y). \tag{8.3}
\]

Verifying condition (b): To verify condition (b), we will show that, for any \( A,C > 0 \), the relation

\[
\sum_{d \leq D \atop (d,b)=1} \sum_{q \leq (\log X)^C \atop q \mid X} \max_{1 \leq c < d \atop 1 \leq m < q} \max_{X^{3/4} \leq y \leq X} \left| \sum_{n \leq y \atop n \equiv c \pmod d} \Lambda(n) - \frac{y}{\varphi(dq)} \right| \ll_{A,C,b,\delta} X \frac{X}{(\log X)^A}
\]

holds. By (8.3), we can drop the condition \( (n,b) = 1 \) in the above sum with an admissible error of \( \ll b X^{1/3-\delta} (\log X)^{C+2} \). Therefore, it is enough to show that

\[
\sum_{d \leq D \atop (d,b)=1} \sum_{q \leq (\log X)^C \atop q \mid X} \max_{1 \leq c < d \atop 1 \leq m < q} \max_{X^{3/4} \leq y \leq X} \left| \sum_{n \leq y \atop n \equiv c \pmod d} \Lambda(n) - \frac{y}{\varphi(dq)} \right| \ll_{A,C,b,\delta} X \frac{X}{(\log X)^A}.
\]

As \( q \mid X = b^k \) and \( (d,b) = 1 \), we have that \( (d,q) = 1 \). Without loss of generality, we can assume that the maximum over \( (c,d) = 1 \) is attained at some reduced residue class modulo \( d \), say, \( c_d \) and the maximum over \( (m,q) = 1 \) is attained at \( m_q \), a reduced residue class modulo \( q \). Then, by the Chinese Remainder Theorem, the system of congruences \( n \equiv c_d \pmod d \) and \( n \equiv m_q \pmod q \) has
a unique solution modulo $dq$. Let us call this solution $u_{dq}$. Then, we have

$$
\sum_{d \leq D} \sum_{q \in \{\log X\}^C} \max_{1 \leq c \leq d} \max_{1 \leq m < q} \max_{X^{1/4} \leq y \leq X} \left| \sum_{n \leq y} \Lambda(n) - \frac{y}{\varphi(dq)} \right| \ll \sum_{d' \leq D \{\log X\}^C} \tau(d') \max_{X^{1/4} \leq y \leq X} \left| \sum_{n \leq y} \Lambda(n) - \frac{y}{\varphi(d')} \right|.
$$

(8.4)

Note that

$$
\sum_{n \leq y} \Lambda(n) - \frac{y}{\varphi(d')} \ll \frac{y \log y}{d'}
$$

So, by the Cauchy–Schwarz inequality and by the Bombieri–Vinogradov theorem [3, chapter 28], the sum in (8.4) is

$$
\ll \left( X \log X \sum_{d' \leq D \{\log X\}^C} \frac{\tau(d')^2}{d'} \right)^{1/2} \left( \max_{X^{1/4} \leq y \leq X} \left| \sum_{n \leq y} \Lambda(n) - \frac{y}{\varphi(d')} \right| \right)^{1/2} \ll \frac{X}{(\log X)^A}.
$$

This completes the verification of condition (b).

Verifying condition (c): Condition (c) holds with $\omega = \delta/2$ and $C' = C_1$ by Proposition 7.2. As $b$ is large in terms of $\delta$, we have $\alpha_b < \delta/4$.

Thus, the estimate (8.1) in Theorem 7 holds for $\Lambda(n)1_{(n, b) = 1}$ for $n < X$. We can finally replace $\Lambda(n)1_{(n, b) = 1}$ by $\Lambda(n)$ for $n \in [1, X)$ by noting that

$$
\sum_{d \leq D} \sum_{n \leq X \mod c_d} \Lambda(n) \ll D \log b) \log D) \ll b, \delta \ X^{1/3 - \delta} (\log X)
$$

to complete the proof of Theorem 1.

The proofs of Theorems 2, 3, and 5 are similar to the above proof of Theorem 1. We will only briefly explain the key changes in the set-up.

Proof of Theorem 2. We apply Theorem 7 with $f(n) = \Lambda(n)1_{(n, b) = 1}$ for $n < X$, $s = 0$, $c_d = c$ (a fixed reduced residue class),

$$
\sigma(d) = \sum_{d|d_1d_2} \lambda(d_1, d_2),
$$

where $\lambda(d_1, d_2)$ is a complex number of absolute value 1, and $D = D_1D_2$ with $D_1 \leq X^{1/3 - \delta}$ and $D_2 \leq X^{1/9}$. Note that $|\sigma| \leq \tau$ in this case. We may now check three conditions of Theorem 7.
(i) By the Prime Number Theorem [3, chapter 18], it is evident that condition (a) holds with 
\( \lambda_d = 1 \) for any \( d \in [1,X) \).
(ii) Condition (b) follows from the Bombieri–Vinogradov theorem [3, chapter 28] and the Cauchy–Schwarz inequality.
(iii) Proposition 7.3 implies condition (c) with \( \omega = \delta/2 \) and \( C' = C_2 \).

As noted above in the proof of Theorem 1, we can remove the co-primality condition \((n, b) = 1\) with an admissible error \( \ll b, \delta X^{4/9-\delta} (\log X)^2 \). This establishes Theorem 2. \( \square \)

Proof of Theorem 3. To prove Theorem 3, we take \( f(n) = \Lambda(n)1_{n \equiv 1 \mod 8}1_{(n, b) = 1} \) for \( n < X \), \( s = 0 \), \( c_d = c \) (a fixed reduced residue class), \( \sigma = \xi \) and \( D = X^{1/2-\delta} \) in Theorem 7. In particular,

(i) condition (a) follows from the Prime Number Theorem [3, chapter 18] with \( \lambda_d = 1 \) for any \( d \in [1,X) \),
(ii) condition (b) follows from the Bombieri–Vinogradov theorem [3, chapter 28],
(iii) Proposition 7.4 to check condition (c) with \( \omega = \delta/2 \) and \( C' = C_3 \).

In this case, we extend it to \( \Lambda(n) \) with an admissible error \( \ll b, \delta X^{1/2-\delta} (\log X) \) to deduce Theorem 3. \( \square \)

Proof of Theorem 5. Theorem 5 follows from Theorem 7 by taking

\[ f(n) = \Lambda(n)1_{n \equiv 3 \mod 8}1_{(n, b) = 1} \text{ for } n \in [1,X), \]

\( s = 0, c_d = 1, \sigma = \lambda_{\text{sem}} \) and \( D = X^{3(1-4\delta)/7-\varepsilon} \). Clearly,

(i) condition (a) follows from the Prime Number Theorem in arithmetic progressions [3, chapters 20, 22] with \( \lambda_d = 1/4 \),
(ii) condition (b) follows from the Bombieri–Vinogradov theorem [3, chapter 28],
(iii) Proposition 7.5 implies condition (c).

Finally, we can replace \( \Lambda(n)1_{n \equiv 3 \mod 8}1_{(n, b) = 1} \) by \( \Lambda(n) \) with an admissible error \( \ll b, \delta X^{3(1-4\delta)/7}(\log X) \) to complete the proof of Theorem 5. \( \square \)

Proof of Theorem 6. Finally, we apply Theorem 7 to deduce Theorem 6 by taking

\[ f(n) = (\mathfrak{h} * \Lambda)(n/2)1_{(n,b)=1}, n \equiv 2 \mod 8 \text{ for } n \in [1,X), \]

where \( \mathfrak{h} \) is supported on \([L,2L]\) with \( L \in [X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon}] \). Furthermore, we take \( s = 1, c_d = -1, \sigma = \lambda_{\text{lin}} \) and \( D = X^{1/2-2\delta-\varepsilon} \) in Theorem 7. Now we check three conditions of Theorem 7.

(i) By the Prime Number Theorem in arithmetic progressions [3, chapters 20, 22], condition (a) holds with \( \lambda_d = \sum_{\ell \sim L, (\ell, 2bd)=1} \mathfrak{h}(\ell)/\ell \) for any \( d \in [1,X) \). As \( \mathfrak{h} \) is bounded, we have \( |\lambda_d| \ll \log L \ll \log X \).
(ii) Arguing as in the proof of Theorem 1, condition (b) follows from the Bombieri–Vinogradov theorem for the Dirichlet convolution and the Cauchy–Schwarz inequality. In particular, we apply [6, Theorem 9.17] with \( \alpha = \mathfrak{h} \) and \( \beta = \Lambda \). Note that as \( \mathfrak{h} \) is supported on \([L,2L]\) with \( L \in [X^{1/3-2\delta-\varepsilon}, X^{2/3+2\delta+\varepsilon}] \), for \( \ell n < y \) we have that \( n < y/L \). Moreover, by Siegel–Walfisz theorem, \( \Lambda \) satisfies the Siegel–Walfisz condition, and if \( y \in [X^{3/4}, X] \), then we have \( \ell, n < y/(\log y)^B \) for some \( B \) large. Therefore, the Bombieri–Vinogradov type estimate holds for the
above function \((\mathfrak{F} \ast \Lambda)(n/2)1_{n \equiv 2 (\text{mod } 8)}\) for \(n \leq y\). The Bombieri–Vinogradov type estimate together with the Cauchy–Schwarz inequality implies condition (b).

(iii) We can apply Proposition 7.6 to check condition (c) with \(\omega = \delta\) and \(C' = C_5\) to complete the proof of Theorem 6.

This completes the proof of Theorem 6.

\[\square\]

8.3 \ Proof outline of Theorem 7

We give a brief outline of the proof of Theorem 7 following the set-up from Section 2.

By Fourier inversion (see relation (2.3)), we have

\[
\sum_{\frac{t}{X} \leq n < X} f(n)1_{A_r}(n + s) = \frac{1}{X} \sum_{0 \leq l < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{1}_{d,c,d} \left( \frac{-l}{X} \right) e\left( \frac{-st}{X} \right),
\]

where for any \((c, d) = (d, b) = 1\) and for any real number \(\theta \in [0, 1)\),

\[
\hat{1}_{d,c}(\theta) := \sum_{\frac{t}{X} \leq n < X} f(n)e(n\theta).
\]

Remark. As \(|f| \ll \log\), we have for any real number \(\theta \in [0, 1)\) and for \(d < X\),

\[
|\hat{1}_{d,c}(\theta)| \leq \sum_{\frac{t}{X} \leq n < X} |f(n)| \ll \frac{X(\log X)}{d}.
\]

The strategy to prove Theorem 7 roughly goes as follows.

(i) As outline in Section 2, we dissect \(t/X\) into so-called major arcs and minor arcs.

(ii) The major arcs contribution is estimated in Proposition 10.1 by employing conditions (a) and (b) of Theorem 7.

(iii) The minor arcs contribution is estimated in Proposition 11.1 by using Lemma 9.3 (hybrid bound) and condition (c) of Theorem 7.

(iv) Finally, in Section 12 we combine Proposition 10.1 (major arcs estimate) and Proposition 11.1 (minor arcs estimate) to deduce Theorem 7.

9 \ FOURIER ESTIMATES FOR THE DIGIT FUNCTION

In this section, we collect the key properties of \(\hat{1}_{A_r}\) from Maynard [17]. For the purpose of this section, we introduce the following notation for brevity. For any integer \(j \in [1, k]\) and for any real number \(\theta \in [0, 1)\), we set

\[
\hat{1}_{\mathcal{N}[0, b^j]}(\theta) := \sum_{n < b^j} 1_{\mathcal{N}}(n)e(n\theta),
\]

where \(\mathcal{N} = A \) or \(\mathcal{A}_r\). In particular, \(\hat{1}_{A_r} = \hat{1}_{\mathcal{A}_r \cap [0, b^j]}\).
We begin with the $L^1$ bound in the following lemma.

**Lemma 9.1** ($L^1$ bound). There exists a constant $C_b \in [1/ \log b, 1 + 3/ \log b]$ such that

$$\sup_{\delta \in \mathbb{R}} \sum_{0 \leq t < b^k} \left| \hat{1}_{A_r} \left( \frac{t}{b^k} + \delta \right) \right| \ll_b (C_b b \log b)^k.$$  

**Proof.** We write $n = \sum_{j=0}^{k-1} n_j b^j$ with $n_0 = r$, so that for any real number $\theta \in [0, 1)$,

$$\hat{1}_{A_r}(\theta) = e(r\theta) \hat{1}_{A_r \cap [0, b^{k-1})}(b\theta). \quad (9.2)$$

The above factorization allows us to express our sum as

$$\sup_{\delta \in \mathbb{R}} \sum_{0 \leq t < b^k} \left| \hat{1}_{A_r} \left( \frac{t}{b^k} + \delta \right) \right| \ll b \cdot \sup_{\delta \in \mathbb{R}} \sum_{0 \leq t < b^{k-1}} \left| \hat{1}_{A} \left( \frac{t}{b^{k-1}} + b\delta \right) \right| \ll (C_b b \log b)^{k-1} \ll_b (C_b b \log b)^k,$$

where we have used [17, Lemma 5.1] with $b^k$ replace by $b^{k-1}$ to the sum over $t$. □

Next, we have the following large-sieve type estimate for the Fourier transform of the set $A_r$.

**Lemma 9.2** (Large-sieve type estimate). Let $Q \geq 1$. Then, we have

$$\sup_{\delta \in \mathbb{R}} \sum_{q \sim Q} \sum_{0 < a < q} \left| \hat{1}_{A_r} \left( \frac{a}{q} + \varepsilon + \delta \right) \right| \ll (Q^2 + b^k)(C_b \log b)^k, \quad (9.3)$$

where $C_b$ is the constant as in Lemma 9.1.

**Proof.** Note that $a/q + \varepsilon$ with $(a, q) = 1, q \sim Q$ and $|\varepsilon| < 1/2Q^2$ are well-spaced by $\gg 1/Q^2$ in the interval $[0, 1]$. Therefore, by the Gallagher–Sobolev type inequality (see [7]), we have

$$\sup_{\delta \in \mathbb{R}} \sum_{q \sim Q} \sum_{0 < a < q} \left| \hat{1}_{A_r} \left( \frac{a}{q} + \varepsilon + \delta \right) \right| \ll Q^2 \int_0^1 \left| \hat{1}_{A_r}(u) \right| du + \int_0^1 \left| \frac{d\hat{1}_{A_r}(u)}{du} \right| du.$$

By the relation (9.2) and arguing similarly as in the proof of [17, Lemma 5.2], we may estimate the above sum as

$$\ll Q^2 \int_0^1 \left| \hat{1}_{A_r \cap [0, b^{k-1})}(bu) \right| du + \int_0^1 \left| \frac{d\hat{1}_{A_r \cap [0, b^{k-1})}(bu)}{du} \right| du.$$

$$\ll_b (Q^2 + b^k)(C_b \log b)^k,$$

as desired. □
We also have the following hybrid bound for the Fourier transform of the set $A_r$.

**Lemma 9.3** (Hybrid estimate). Let $Q, B \geq 1$. Then, we have

$$
\sum_{q \sim Q} \sum_{1 \leq a < q} \sum_{\eta \not=} \left| \hat{1}_{A_r} \left( \frac{a}{q} + \frac{\eta}{b^k} \right) \right| \ll_b (b - 1)^k (Q^2 B)^{\alpha_b} + Q^2 B (C_b \log b)^k,
$$

where $C_b$ is the constant described in Lemma 9.1, and

$$
\alpha_b = \frac{\log \left( \frac{C_b \log b}{b-1} \right)}{\log b}.
$$

**Remark.** We note that $\alpha_b$ tends to 0 as $b \to \infty$. Therefore, $\alpha_b$ will be small if we take $b$ large enough, which is a crucial point in our entire Fourier analytic set-up.

**Proof.** The proof follows from the relation (9.2) in combination with the arguments of [17, Lemma 5.3].

We end this section with the $L^\infty$ bound for $\hat{1}_{A_r}$.

**Lemma 9.4** ($L^\infty$ bound). Let $q < b^{k/3}$ be of the form $q = q_1 q_2$ with $(q_1, b) = 1$ and $q_1 \neq 1$, and let $|\epsilon| < 1/2 b^{2k/3}$. Then, for any integer $a$ with $(a, q) = 1$, we have

$$
\left| \hat{1}_{A_r} \left( \frac{a}{q} + \epsilon \right) \right| \ll_b (b - 1)^k \exp \left( -c_b \frac{k}{\log q} \right),
$$

for some constant $c_b > 0$ depending only on $b$.

**Proof.** The proof follows from the relation (9.2) in conjunction with the argument of [17, Lemma 5.4].

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### 10 MAJOR ARCS

We devote this section to establishing the major arcs estimate. Throughout, $\hat{1}_{A_r}$ denotes the Fourier transform of the set $A_r$ given by (2.2) and $\hat{f}_{d, c}$ is given by (8.6).

**Proposition 10.1** (Major arcs estimate for Theorem 7). Let $C \geq 1$ be a large real number. Assume the setting of Theorem 7 and recall that $s$ is a positive integer such that $(r - s, b) = 1$. Then we have

$$
\sum_{d \leq D} \max_{(c, d) = 1} \left| \frac{1}{X} \sum_{0 < i < X \atop i \in \mathcal{M}} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{f}_{d, c} \left( \frac{-st}{X} \right) e \left( \frac{-st}{X} \right) \right| \ll \frac{\lambda_d}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n) \ll \frac{X^s}{(\log X)^{3C+5}},
$$

where $\mathcal{M}$ is given by the relation (2.7), and the implicit constant in $\ll$ depends at most on $b, C$ and $s$. 

Recall from the relation (2.7) that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. To prove Proposition 10.1, we will estimate separately the contribution coming from $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ in Lemmas 10.2, 10.3, and 10.4, respectively. We begin with estimating the contribution of $\mathcal{M}_1$ in the following lemma.

**Lemma 10.2.** Let $C \geq 1$, $D \in [1, X)$, and recall the set $\mathcal{M}_1$ is given by
\[
\mathcal{M}_1 = \left\{ t \in [0, X) \cap \mathbb{Z} : \left| \frac{t}{X} - \frac{a}{q} \right| \leq \frac{(\log X)^C}{X} \text{ for some } (a, q) = 1, 1 \leq a < q \leq (\log X)^C, q \nmid X \right\}.
\]
Assume the setting of Theorem 7. Then we have
\[
\frac{1}{X} \sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{0 \leq t < X} \sum_{\substack{0 \leq \xi < X \cr \xi \in \mathcal{M}_1}} \hat{\Lambda}_r \left( \frac{t}{X} \right) \hat{\chi}_{d, c} \left( \frac{-t}{X} \right) e \left( \frac{-st}{X} \right) \right| \ll \frac{X^\delta}{(\log X)^{5C+5}},
\]
where the implicit constant in $\ll$ depends at most on $b, C$ and $\delta$.

**Proof.** If $t \in \mathcal{M}_1$, we use Lemma 9.4 to obtain
\[
\left| \hat{\Lambda}_r \left( \frac{t}{X} \right) \right| \ll_{b, C} \frac{X^\delta}{(\log X)^{8C+7}}.
\]
We note that the cardinality of the set $\mathcal{M}_1$ is at most $\ll (\log X)^{3C}$. Therefore, by relation (8.7),
\[
\frac{1}{X} \sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{0 \leq t < X} \sum_{\substack{0 \leq \xi < X \cr \xi \in \mathcal{M}_1}} \hat{\Lambda}_r \left( \frac{t}{X} \right) \hat{\chi}_{d, c} \left( \frac{-t}{X} \right) \right| \ll_{b, C} \frac{1}{X} \cdot (\log X)^{3C} \cdot \frac{X^\delta}{(\log X)^{5C+5}} \cdot X(\log X) \cdot \sum_{d \leq D} \frac{1}{d}
\ll_{b, C} \frac{X^\delta}{(\log X)^{5C+5}}.
\]
This completes the proof of the lemma.

Now we estimate the contribution coming from $\mathcal{M}_2$.

**Lemma 10.3.** Let $C \geq 1$. Recall that the set $\mathcal{M}_2$ is given by
\[
\mathcal{M}_2 = \left\{ t \in [0, X) \cap \mathbb{Z} : \left| \frac{t}{X} - \frac{a}{q} \right| \leq \frac{X^\delta}{(\log X)^{5C+5}} \text{ for some } (a, q) = 1, 1 \leq a < q \leq (\log X)^C, q \nmid X \right\}.
\]
Assume the setting of Theorem 7. Then we have
\[
\frac{1}{X} \sum_{d \leq D} \max_{(c, d) = 1} \left| \sum_{0 \leq t < X} \sum_{\substack{0 \leq \xi < X \cr \xi \in \mathcal{M}_2}} \hat{\Lambda}_r \left( \frac{t}{X} \right) \hat{\chi}_{d, c} \left( \frac{-t}{X} \right) e \left( \frac{-st}{X} \right) \right| \ll \frac{X^\delta}{(\log X)^{5C+5}},
\]
where the implicit constant in $\ll$ depends at most on $b, C$ and $\delta$. 
Proof. We call the left-hand side of (10.2) as \( \Sigma_{\text{Major}} \) and simplify the sum as

\[
\Sigma_{\text{Major}} \leq \frac{1}{X} \sum_{d \leq D} \max_{(c,d)=1} \sum_{q \leq (\log X)^C} \sum_{a=0}^{q} \sum_{0 < |\eta| \leq (\log X)^C} \left| \hat{A}_r \left( \frac{a}{q} + \frac{\eta}{X} \right) \hat{f}_{d,c} \left( \frac{-a}{q} - \frac{\eta}{X} \right) \right|.
\]

Note that as \( q | X \) in the above expression, \( \eta \) is an integer in this case. Next, we use the trivial bound \( |\hat{A}_r(a/q + \eta/X)| \leq X^\epsilon \) in the above estimate to obtain

\[
\Sigma_{\text{Major}} \leq \frac{X^\epsilon}{X} \sum_{d \leq D} \max_{(c,d)=1} \sum_{q \leq (\log X)^C} \sum_{a=0}^{q} \sum_{0 < |\eta| \leq (\log X)^C} \left| \hat{f}_{d,c} \left( \frac{-a}{q} - \frac{\eta}{X} \right) \right|. \tag{10.3}
\]

Therefore, in order to establish the lemma it is enough to show that

\[
\Sigma'_{\text{Major}} := \sum_{d \leq D} \max_{(c,d)=1} \sum_{q \leq (\log X)^C} \sum_{a=0}^{q} \sum_{0 < |\eta| \leq (\log X)^C} \left| \hat{f}_{d,c} \left( \frac{-a}{q} - \frac{\eta}{X} \right) \right| \ll_C \frac{X}{(\log X)^{5C+5}}.
\]

We have

\[
\hat{f}_{d,c} \left( \frac{-a}{q} - \frac{\eta}{X} \right) = \sum_{m=1}^{q} e \left( \frac{-ma}{q} \right) \sum_{n \leq X, n \equiv c \ (\text{mod } d), n \equiv m \ (\text{mod } q)} \hat{f}(n) e \left( \frac{-n\eta}{X} \right).
\]

We note that as \( q | X = b^k \) and \( (n,b) = 1 \) (as \( \hat{f}(n) \) is supported on integers \( n \) such that \( (n,b) = 1 \)), the congruence \( n \equiv m \ (\text{mod } q) \) implies that we either have \( (m,q) = 1 \) or the above sum is empty. Furthermore, as \( (d,b) = 1 \) and \( q | X = b^k \), we have \( (d,q) = 1 \). Therefore, by the Chinese Remainder Theorem, the system of congruences

\[
n \equiv c \ (\text{mod } d) \quad \text{and} \quad n \equiv m \ (\text{mod } q)
\]

has a unique solution modulo \( dq \). This allows us to write

\[
\hat{f}_{d,c} \left( \frac{-a}{q} - \frac{\eta}{X} \right) = \sum_{m=1}^{q} \sum_{(m,q)=1} e \left( \frac{-ma}{q} \right) \sum_{n \leq X, n \equiv c \ (\text{mod } d), n \equiv m \ (\text{mod } q)} \hat{f}(n) e \left( \frac{-n\eta}{X} \right). \tag{10.4}
\]

Next, if \( (c,d) = (m,q) = 1 \), then for any \( y \geq 2 \), we denote

\[
\Psi_f(y; d, c; q, m) := \sum_{n \leq y, n \equiv c \ (\text{mod } d), n \equiv m \ (\text{mod } q)} \hat{f}(n).
\]

For any \( y \in [X^{3/4}, X] \), we denote

\[
\Delta_f(y; d, c; q, m) := \Psi_f(y; d, c; q, m) - \frac{y\lambda_d}{\varphi(dq)}.
\]
Using partial summation and the inequality (8.7), we have

\[
\sum_{\substack{n \leq X \\ n \equiv c \pmod{d} \\ n \equiv m \pmod{q}}} \hat{f}(n) e\left(-\frac{\eta n}{X}\right) = \int_{X^{3/4}}^{X} e\left(-\frac{\eta y}{X}\right) d\Delta(y; d, c, q, m) + \frac{\lambda_d}{\phi(dq)} \int_{X^{3/4}}^{X} e\left(-\frac{\eta y}{X}\right) dy
\]

\[
+ O\left(\log X + \frac{X^{3/4} \log X}{dq}\right)
\]

\[
= \int_{X^{3/4}}^{X} e\left(-\frac{\eta y}{X}\right) d\Delta(y; d, c, q, m) + O\left(\frac{\lambda_d X^{3/4}}{\phi(dq)} + \log X + \frac{X^{3/4} \log X}{dq}\right),
\]

(10.5)

where we have used the fact that \( \eta \) is an integer, so that \( \int_{1}^{X} e(-y\eta/X) dy = O(1) \). Next, using integration by parts, we have

\[
\int_{X^{3/4}}^{X} e\left(-\frac{\eta y}{X}\right) d\Delta(y; d, c, q, m) \ll (1 + |\eta|) \max_{X^{3/4} < y \leq X} |\Delta(y; d, c, q, m)|.
\]

(10.6)

Using the estimate from (10.6) in (10.5) along with the facts that \(|\eta| \leq (\log X)^C\) and that \(\lambda_d \ll \log X\), we obtain

\[
\sum_{\substack{n \leq X \\ n \equiv c \pmod{d} \\ n \equiv m \pmod{q}}} \hat{f}(n) e\left(-\frac{\eta n}{X}\right) \ll (\log X)^C \max_{X^{3/4} < y \leq X} |\Delta(y; d, c, q, m)| + \frac{X^{3/4} (\log X)^2}{dq}.
\]

(10.7)

Therefore, using the inequalities (10.4) and (10.7), we have

\[
\Sigma_{\text{Major}}' := \sum_{d \leq D} \max_{(c, d) = 1} \sum_{q \leq (\log X)^C} \frac{1}{q|X|} \sum_{a=0}^{q} \max_{0 < |\eta| \leq (\log X)^C} \left| \hat{f}_{d, c} \left( -\frac{a}{q} - \frac{\eta}{X} \right) \right|
\]

\[
\ll (\log X)^{4C} \sum_{d \leq D} \max_{(c, d) = 1} \max_{q \leq (\log X)^C} \sum_{(m, a) = 1} \max_{0 < |\eta| \leq (\log X)^C} |\Delta(y; d, c, q, m)| + X^{3/4} (\log X)^{3C+3}.
\]

We can now apply (b) with \( A = 9C + 5 \) to obtain

\[
\Sigma_{\text{Major}}' \ll (\log X)^{4C} \cdot \frac{X}{(\log X)^{9C+5}} + X^{3/4} (\log X)^{3C+3} \ll \frac{X}{(\log X)^{5C+5}},
\]

as desired. \( \square \)

Finally, we end this section by analyzing the set \( \mathcal{M}_3 \) (given below), which yields the expected main term in Proposition 10.1.
Lemma 10.4. Let $C \geq 1$. Recall that the set $\mathcal{M}_3$ is given by

$$\mathcal{M}_3 = \left\{ t \in [0, X) \cap \mathbb{Z} : t = \frac{a}{q}, \text{ for some } (a, q) = 1, 0 \leq a < q \leq (\log X)^C, q \geq 1, q|X \right\}.$$ 

Assume the setting of Theorem 7 and recall that $s$ is a positive integer such that $(r - s, b) = 1$. Then, we have

$$\sum_{(d, b) = 1} \max_{(c, d) = 1} \left| \frac{1}{X} \sum_{0 \leq t < X} \hat{\chi}_{A_r} \left( \frac{t}{X} \right) \hat{1}_{d, c} \left( \frac{-st}{X} \right) e\left( \frac{-st}{X} \right) - \frac{\lambda_d}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} \frac{1}{\varphi(n)} \right| \ll \frac{X^\varepsilon}{(\log X)^{5C+5}}, \quad (10.8)$$

where the implicit constant in $\ll$ depends at most on $b, C$ and $\delta$.

Proof. We begin with the following observation that for $k$ large enough

“$q \leq (\log X)^C, q|X = b^k$” is equivalent to “$q \leq (\log X)^C$, for every $p|q$, we have $p|b$”.

Therefore, we have

$$\frac{1}{X} \sum_{0 \leq t < X} \hat{\chi}_{A_r} \left( \frac{t}{X} \right) \hat{1}_{d, c} \left( \frac{-t}{X} \right) e\left( \frac{-st}{X} \right) = \frac{1}{X} \sum_{q \leq (\log X)^C} \sum_{0 \leq a < q, p|q \Rightarrow p|b} \left( (a, q) = 1 \right) \hat{1}_{A_r} \left( \frac{a}{q} \right) \hat{1}_{d, c} \left( \frac{-a}{q} \right) e\left( \frac{-sa}{q} \right). \quad (10.9)$$

If $(a, q) = 1$, we have

$$\hat{1}_{d, c} \left( \frac{-a}{q} \right) = \sum_{m=1}^{q} e\left( -\frac{ma}{q} \right) \sum_{\substack{n < X \n \equiv c (mod d) \n \equiv m (mod q) \n \equiv m (mod dq)}} f(n). \quad (10.10)$$

Arguing similarly as in Lemma 10.3, we note that $(d, q) = (m, q) = 1$. For brevity, let

$$\Delta_f(X; dq) := \max_{(c, d) = 1} \max_{(m, q) = 1} \left| \sum_{n < X, \substack{n \equiv (mod d) \n \equiv m (mod q)}} f(n) - \frac{\lambda_d X}{\varphi(dq)} \right|. \quad (10.11)$$

Then, we have

$$\hat{1}_{d, c} \left( \frac{-a}{q} \right) = \frac{\lambda_d X}{\varphi(dq)} \sum_{m=1}^{q} e\left( -\frac{ma}{q} \right) + O(\varphi(q)\Delta_f(X; dq))$$

$$= \frac{\lambda_d \mu(q)X}{\varphi(dq)} + O(\varphi(q)\Delta_f(X; dq)), \quad (10.12)$$

where we have used we have the expression for the Ramanujan sum (see [3, p. 149]): if $(a, q) = 1$, then

$$\sum_{m=1}^{q} e\left( -\frac{ma}{q} \right) = \mu(q). \quad (10.13)$$
Note that \(|\widehat{\lambda}_{\mathcal{A}_r}(a/q)| \leq X^\varepsilon\), so that the contribution of the big-Oh term from relation (10.12) to the expression in (10.8) is

\[
\ll \frac{1}{X} \sum_{d \leq D} \sum_{q \leq (\log X)^C} \sum_{0 \leq a < q} \left| \widehat{\lambda}_{\mathcal{A}_r} \left( \frac{a}{q} \right) \right| \varphi(q) \Delta_f(X; dq) \ll \frac{X^\varepsilon (\log X)^{2C}}{X} \sum_{d \leq D} \sum_{q \leq (\log X)^C} \Delta_f(X; dq).
\]

We apply condition (b) with \(A = 7C + 5\), so that the above sum is

\[
\ll b, C \frac{X^\varepsilon (\log X)^{2C}}{X} \cdot \frac{X}{(\log X)^{7C + 5}} \ll b, C \frac{X^\varepsilon (\log X)^{2C + 5}}{X},
\]

which is admissible.

We are therefore left with showing that

\[
\frac{\lambda_d}{X} \sum_{q \leq (\log X)^C} \sum_{0 \leq a < q} \widehat{\lambda}_{\mathcal{A}_r} \left( \frac{a}{q} \right) \frac{X \mu(q)}{\varphi(dq)} e \left( \frac{-sa}{q} \right) = \frac{\lambda_d}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{\mathcal{A}_r}(n).
\]

It is evident from the relation (10.12) that \(q\) is supported on square-free integers. Therefore, we have \(q|b\) because for every prime \(p|q\) implies \(p|b\). Then, by the relation (2.2) and the fact that \(q|b\), we have

\[
\widehat{\lambda}_{\mathcal{A}_r} \left( \frac{a}{q} \right) = e \left( \frac{ar}{q} \right) \sum_{n < X} 1_{\mathcal{A}_r}(n).
\]

(10.14)

We also note that \((d, q) = 1\) as \((d, b) = 1\) and \(q|b\). Furthermore, as by our assumption \((r - s, b) = 1\), we have \((r - s, q) = 1\). Therefore, using (10.12), (10.13), and (10.14) allows us to estimate the main term as

\[
\frac{1}{X} \sum_{q \leq (\log X)^C} \sum_{0 \leq a < q} \widehat{\lambda}_{\mathcal{A}_r} \left( \frac{a}{q} \right) \frac{X \mu(q)}{\varphi(dq)} e \left( \frac{-sa}{q} \right)
\]

\[
= \frac{\lambda_d}{\varphi(d)} \sum_{q \leq (\log X)^C} \sum_{0 \leq a < q} \widehat{\lambda}_{\mathcal{A}_r} \left( \frac{a}{q} \right) \frac{\mu(q)}{\varphi(q)} e \left( \frac{-sa}{q} \right)
\]

\[
= \frac{\lambda_d}{\varphi(d)} \sum_{n < X} 1_{\mathcal{A}_r}(n) \sum_{q|b} \sum_{0 \leq a < q} e \left( \frac{a(r - s)}{q} \right) \frac{\mu(q)}{\varphi(q)}
\]

\[
= \frac{\lambda_d}{\varphi(d)} b \frac{\mu^2(q)}{\varphi(q)} \sum_{n < X} 1_{\mathcal{A}_r}(n) = \frac{\lambda_d}{\varphi(d)} b \frac{\mu^2(q)}{\varphi(q)} \sum_{n < X} 1_{\mathcal{A}_r}(n)
\]

as desired. □

We can now combine Lemmas 10.2, 10.3, and 10.4 along with the fact that \(\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3\) (see the relation (2.7)) to complete the proof of Proposition 10.1.
11 | MINOR ARCS

In this section, we will establish the minor arc estimate for Theorem 7 by combining condition (c) of Theorem 7 and Lemma 9.3.

**Proposition 11.1** (Minor arc estimate for Theorem 7). Let $Q, B \geq 1$ with $QB \ll X^{1/2}$. Assume the set-up of Theorem 7. Then we have

$$
\sum_{q \sim Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{B<|\eta|+1 \leq 2B} \sum_{\substack{d \leq D \\ (d,b)=1}} \sigma(d) \hat{f}_{d,c_d} \left( \left( \frac{a}{q} + \frac{\eta}{X} \right) \right) \right| \ll X^{1+\varepsilon} \left( \frac{1}{(Q^2 B)^{\omega/2-\alpha_b}} + \frac{X^{\alpha_b}}{X^{\omega/2}} \right) (\log X)^{C'},
$$

where $\hat{f}_{d,c_d}$ and $\alpha_b$ are given by (2.2), (8.6), and (9.4), respectively. Furthermore, $\omega$ and $C'$ are as in condition (c) of Theorem 7, and the implicit constant in $\ll$ depends at most on $b$, $\delta$, and $\omega$.

**Proof.** We use the ideas of Maynard from the proof of [17, Lemma 6.1]. For $X = b^k$, we have

$$
\hat{f}_{d,c_d} \left( \frac{-a}{q} - \frac{\eta}{X} \right) = \sum_{n < X \atop n \equiv c_d (\text{mod} \ d)} f(n) e \left( -n \left( \frac{a}{q} + \frac{\eta}{X} \right) \right). \tag{11.1}
$$

We use condition (c) with $\theta = a/q + \beta$ where $\beta = \eta/X$. Hence, for $q \sim Q$ and $(1 + |\eta|) \sim B$ with $B \geq 1$, we note that $q(1 + |\beta|X) \prec QB$ to obtain

$$
\sup_{q \sim Q \atop (a,q)=1 \atop (|\eta|+1)\sim B} \sum_{d \leq D \atop (d,b)=1} \sigma(d) \hat{f}_{d,c_d} \left( \frac{-a}{q} - \frac{\eta}{X} \right) \ll X \left( \frac{(QB)^{\omega}}{X^{\omega}} + \frac{1}{(QB)^{\omega}} \right) (\log X)^{C'}. \tag{11.2}
$$

By assumption $QB \ll X^{1/2}$, so that $Q^2 B \ll X$. Therefore, Lemma 9.3 implies that

$$
\sum_{q \sim Q} \sum_{1 \leq a < q} \sum_{\substack{(|\eta|+1)\sim B \\ (a,q)=1 \atop X a/q + \eta \in \mathbb{Z}}} \left| \hat{f}_{d,c_d} \left( \frac{a}{q} + \frac{\eta}{X} \right) \right| \ll X^{\varepsilon} (Q^2 B)^{\alpha_b}, \tag{11.3}
$$

assuming that $b$ is large enough, so that $\alpha_b < 1$ and $\alpha_b \to 0$ as $b \to \infty$. Putting the estimates from (11.2) and (11.3) together, we have

$$
\sum_{q \sim Q} \sum_{a=1}^{q} \sum_{\substack{(|\eta|+1)\sim B \\ (a,q)=1 \atop X a/q + \eta \in \mathbb{Z}}} \left| \hat{f}_{d,c_d} \left( \frac{a}{q} + \frac{\eta}{X} \right) \right| \ll X^{1+\varepsilon} \left( \frac{(QB)^{\omega}(Q^2 B)^{\alpha_b}}{X^{\omega}} + \frac{(Q^2 B)^{\alpha_b}}{(QB)^{\omega}} \right) (\log X)^{C'}.
$$

By assumption $QB \ll X^{1/2}$, and by the fact that $(QB)^\omega > (Q^2B)^{\omega/2}$ for $B > 1$, the above estimate is

$$\ll X^{1+\varepsilon} \left( \frac{X^{\alpha_b}}{X^{\omega/2}} + \frac{1}{(Q^2B)^{\omega/2-\alpha_b}} \right) (\log X)^{C'}.$$ 

This establishes the desired result. □

12 PROOF OF THEOREM 7

We are now ready to give the proof of Theorem 7 by combining Proposition 10.1 (major arcs estimate) and Proposition 11.1 (minor arcs estimate).

Proof of Theorem 7. By Fourier inversion (relation (2.3)), we have

$$\sum_{n < X} f(n) 1_{A_r}(n + s) = \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{f}_{d,c,d} \left( -\frac{t}{X} \right) e \left( -\frac{st}{X} \right), \quad (12.1)$$

where $\hat{1}_{A_r}$ and $\hat{f}_{d,c,d}$ are given by (2.2) and (8.6), respectively.

We consider the parameter $C > 0$ to be chosen later. Then we dissect the fractions $t/X$ with $t \in [0,X) \cap \mathbb{Z}$ into two sets: major arcs $\mathcal{M}$ and minor arcs $\mathcal{m}$ (see Section 2 for definition of these two sets).

We may now use Proposition 10.1 to estimate the major arcs $\mathcal{M}$ contribution. We will show that

$$\sum_{d \leq D \atop (d,b) = 1} \sigma(d) \left( \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{f}_{d,c,d} \left( -\frac{t}{X} \right) e \left( -\frac{st}{X} \right) - \frac{\lambda_d}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n) \right) \ll \frac{X^\varepsilon}{(\log X)^C}. \quad (12.2)$$

For brevity, let us denote

$$\mathcal{E}(d) := \frac{1}{X} \sum_{0 \leq t < X} \hat{1}_{A_r} \left( \frac{t}{X} \right) \hat{f}_{d,c,d} \left( -\frac{t}{X} \right) e \left( -\frac{st}{X} \right) - \frac{\lambda_d}{\varphi(d)} \frac{b}{\varphi(b)} \sum_{n < X} 1_{A_r}(n).$$

We note that

$$\# \mathcal{M} \ll (\log X)^3 C, \quad |\hat{1}_{A_r}(t/X)| \ll X^\varepsilon, \quad \text{and} \quad |\hat{f}_{d,c,d}(-t/X)| \ll X(\log X)/d$$

so that trivially, we have

$$|\mathcal{E}(d)| \ll \frac{X^\varepsilon (\log X)^{C+1}}{d}.$$
Next, we apply the Cauchy–Schwarz inequality and use the assumption that $|\sigma| \leq \tau$ to obtain

$$
\sum_{d \leq D \atop (d,b)=1} \sigma(d)\mathcal{E}(d) \ll \left( X^\zeta (\log X)^{3C+1} \sum_{d \leq D} \frac{\tau(d)^2}{d} \right)^{1/2} \left( \sum_{d \leq D \atop (d,b)=1} |\mathcal{E}(d)| \right)^{1/2} \ll \frac{X^\zeta}{(\log X)^C},
$$

using Proposition 10.1. This completes our analysis of the major arcs.

We now use Proposition 11.1 for the remaining cases, that is, the minor arcs. We apply Dirichlet’s approximation theorem to find reduced fractions $a/q$ with $1 \leq q \leq X^{1/2}$ such that

$$
\left| \frac{t}{X} - \frac{a}{q} \right| \leq \frac{1}{qX^{1/2}}.
$$

Hence, we have

$$
\frac{t}{X} = \frac{a}{q} + \frac{\eta}{X},
$$

where $\max\{q, |\eta|\} \geq (\log X)^C$ and $q|\eta| \leq X^{1/2}$. Next, we perform a dyadic decomposition over $q \sim Q$ and $|\eta| + 1 \sim B$, so that $QB \ll X^{1/2}$. Also note that we have $\max\{Q, B\} \gg (\log X)^C$ in this case. Therefore, the contribution of minor arcs is

$$
\ll \frac{(\log X)^2}{X} \sum_{q \sim Q} \sum_{a=1}^{q} \sum_{B < |\eta| + 1 \leq 2B} \left| \hat{\chi}_q \left( \frac{a}{q} + \frac{\eta}{X} \right) \right| \sum_{d \leq D \atop (d,b)=1} \sigma(d) \hat{\chi}_d \left( -\frac{a}{q} - \frac{\eta}{X} \right)
$$

$$
\ll X^\zeta \left( \frac{(\log X)^C}{(\log X)^{C(\omega/2 - \alpha_b)}} + \frac{X^{\alpha_b}(\log X)^{C'}}{X^{\omega/2}} \right) (\log X)^2 \ll \frac{X^\zeta}{(\log X)^A},
$$

where we have chosen $C = (A + C' + 2)/(\omega/2 - \alpha_b)$. Note that $\alpha_b$ goes to 0 as $b \to \infty$ (see the relation (9.4)). In particular, as the base $b$ is sufficiently large, we have $\omega/2 > \alpha_b$. Along with (12.2), this completes the proof of Theorem 7.

\[ \square \]

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**REFERENCES**

1. C. Dartyge and C. Mauduit, *Nombres presque premiers dont l’écriture en base r ne comporte pas certain chiffres*, J. Number Theory 81 (2000), no. 2, 270–291.
2. C. Dartyge and C. Mauduit, *Ensembles de densité nulle contenant des entiers possédant au plus deux facteurs premiers*, J. Number Theory 91 (2001), no. 2, 230–255.
3. H. Davenport, *Multiplicative number theory*, 3rd ed., Revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, vol. 74, Springer, New York, 2000.
4. P. Erdős, C. Mauduit, and A. Sárközy, *On arithmetic properties of integers with missing digits. I. Distribution in residue classes*, J. Number Theory 70 (1998), no. 2, 99–120.
5. P. Erdős, C. Mauduit, and A. Sárközy, *On arithmetic properties of integers with missing digits. II. Prime factors*, *Paul Erdős memorial collection*, Discrete Math. 200 (1999), no. 1–3, 149–164.
6. J. Friedlander and H. Iwaniec, *Opera de Cribro*, American Mathematical Society Colloquium Publications, vol. 57, American Mathematical Society, Providence, RI, 2010.
7. P. Gallagher, *The large sieve*, Mathematika 14 (1967), 14–20.
8. H. Iwaniec, *Primes of the type $\phi(x, y) + a$ where $\phi$ is a quadratic form*, Acta Arith. 21 (1972), 203–234.
9. H. Iwaniec, *The half dimensional sieve*, Acta Arith. 29 (1976), 69–95.
10. H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.
11. S. Konyagin, *Arithmetic properties of integers with missing digits: distribution in residue classes*, Period. Math. Hungar. 42 (2001), no. 1–2, 145–162.
12. D. Koukoulopoulos, *The distribution of prime numbers*, Graduate Studies in Mathematics, vol. 203, American Mathematical Society, Providence, RI, 2019.
13. K. Matomäki, *Prime numbers of the form $p = m^2 + n^2 + 1$ in short intervals*, Acta Arith. 128 (2007), no. 2, 193–200.
14. K. Matomäki, *A Bombieri–Vinogradov type exponential sum result with applications*, J. Number Theory 129 (2009), no. 9, 2214–2225.
15. J. Maynard, *Digits of primes*, European Congress of Mathematics, 641–661, Eur. Math. Soc., Zürich, 2018.
16. J. Maynard, *Primes with restricted digits*, Invent. Math. 217 (2019), no. 1, 127–218.
17. J. Maynard, *Primes and polynomials with restricted digits*, Int. Math. Res. Not. IMRN 2022, no. 14, 1–23.
18. H. Mikawa, *On exponential sums over arithmetic progressions*, Tsukuba J. Math 24 (2000), no. 2, 351–360.
19. K. Pratt, *Primes from sums of two squares and missing digits*, Proc. Lond. Math. Soc. (3) 120 (2020), no. 6, 770–830.
20. J. Teräväinen, *The Goldbach problem for primes that are sums of squares plus one*, Mathematika 64 (2018), no. 1, 20–70.
21. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. 143 (1961), 75–102.
22. J. Wu, *Primes of the form $p = 1 + m^2 + n^2$ in short intervals*, Proc. Amer. Math. Soc. 126 (1998), no. 1, 1–8.