HIGH-LOW TEMPERATURE DUALITIES FOR THE CLASSICAL \( \beta \)-ENSEMBLES

PETER J. FORRESTER

Abstract. The loop equations for the \( \beta \)-ensembles are conventionally solved in terms of a \( 1/N \) expansion. We observe that it is also possible to fix \( N \) and expand in inverse powers of \( \beta \). At leading order, for the one-point function \( W_1(x) \) corresponding to the average of the linear statistic \( A = \sum_{j=1}^{N} 1/(x - \lambda_j) \), and specialising the classical weights, this reclaims well known results of Stieltjes relating the zeros of the classical polynomials to the minimum energy configuration of certain log-gas potential energies. Moreover, it is observed that the differential equations satisfied by \( W_1(x) \) in the case of classical weights — which are particular Riccati equations — are simply related to the differential equations satisfied by \( W_1(x) \) in the high temperature scaled limit \( \beta = 2\alpha/N \) (\( \alpha \) fixed, \( N \to \infty \)), implying a certain high-low temperature duality. A generalisation of this duality, valid without any limiting procedure, is shown to hold for \( W_1(x) \) and all its higher point analogues in the classical \( \beta \)-ensembles.

1. Introduction

The eigenvalue probability density function (PDF), supported on \( \mathbb{R}^N \) and proportional to

\[
\prod_{l=1}^{N} e^{-\beta \lambda_l^2/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta
\]

is said to specify the Gaussian \( \beta \)-ensemble. Best known are the cases \( \beta = 1 \) and \( \beta = 2 \). Then with \( X \) a real (respectively complex) Gaussian random matrix, forming the real symmetric (respectively complex Hermitian) random matrix \( H = \frac{1}{2}(X + X^\dagger) \) specifies ensembles of random matrices with eigenvalue PDF given by (1.1). For general \( \beta > 0 \), an ensemble of tridiagonal random matrices, with entries on and above the diagonal each independent, can be specified which realise the eigenvalue PDF (1.1) [15].

The PDF (1.1) can be written in Boltzmann factor form

\[
e^{-\beta U^G}, \quad U^G := \sum_{l=1}^{N} x_l^2/2 - \sum_{1 \leq j < k \leq N} \log |x_k - x_j|.
\]

It is a classical result due to Stieltjes that the minimum of the potential energy \( U^G \) is unique and occurs at the ordered zeros, \( z_1 < z_2 < \cdots < z_N \) say, of the Hermite polynomials \( H_N(x) \) [35, 36]; for a recent work on this general theme see [27]. Thus, for \( N \) fixed, the eigenvalues specified by (1.1) crystallise to these zeros. Define the Hessian matrix \( \mathcal{H} \) at the critical point \( x = z \), where \( z = (z_1, z_2, \ldots, z_N) \) according to

\[
\mathcal{H} = \left[ \frac{\partial^2 U^G}{\partial x_j \partial x_k} \right]_{j,k=1}^{N} \bigg|_{x=z}.
\]
Set \( y_j = x_j - z_j \). Then to leading order as \( \beta \to \infty \), upon a multi-variable Taylor expansion, we see that (1.2) is proportional to the Boltzmann factor of an oscillator system specified by
\[
\exp \left( -\frac{\beta}{2} y H y^T \right). \tag{1.4}
\]

Generally, for a continuous classical statistical mechanical system with a (unique) ground state, expanding about this point in configuration space is referred to as the harmonic approximation. For the log-gas on a circle of radius \( L \), where the potential energy term in (1.2) is now given by
\[
U^C := - \sum_{1 \leq j < k \leq N} \log |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|, \quad x_l \in [0, L), \tag{1.5}
\]
and the ground state is equal spacing in units of \( 2\pi L/N \), a study of the harmonic approximation has been given in [19], [21, §5], [20, §4.8.2] and [34]. Note that for this system, the ground state only becomes unique upon fixing one of the particles (say setting \( x_1 = 0 \)) due to the rotation invariance. Specifically for (1.2), the harmonic approximation was first studied in [5], and subsequent in [17] where the analysis proceeded through the tridiagonal realisation of the Gaussian \( \beta \)-ensemble introduced in [15]. Much earlier, for the two-dimensional version of (1.2) — the so-called two-dimensional one-component plasma in a disk — the harmonic approximation was studied in [2].

The equilibrium statistical mechanical system (1.2) admits generalisations, which furthermore permit analysis in the \( \beta \to \infty \) limit. One generalisation is to consider the evolution under Brownian dynamics; see e.g. [20, Ch. 11]. Works considering the behaviour for \( \beta \to \infty \), of this precise model, and variants for other log-potential systems in one-dimension, include [6–9, 37, 40, 41]. Another generalisation is to consider the evolution under Brownian dynamics, with the associated \( \beta \)-ensemble, with the Gaussian case (1.2) being the simplest [24]. Such a process in the limit \( \beta \to \infty \) has been studied in the works [11, 25, 26]; see also [39, Remark 13]. In both these lines of work, a mathematical structure to have shown itself is that of the associated classical orthogonal polynomials. Here one recalls (see e.g. [10]) that for a family of orthogonal polynomials \( \{ p_n(x) \} \) with three term recurrence
\[
p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \tag{1.6}
\]
the associated orthogonal polynomials \( \{ p_n(x; \alpha) \} \) satisfy the modified three term recurrence
\[
p_{n+1}(x; c) = (A_{n+c} x + B_{n+c}) p_n(x; c) - C_{n+c} p_{n-1}(x; c). \tag{1.7}
\]

It is also the case that the weight function for the orthogonality relation of the associated classical orthogonal polynomials has appeared in studies of the respective \( \beta \)-ensembles in the scaled high temperature regime, \( \beta = 2\alpha/N \) and \( N \to \infty \) [3, 4, 29, 38]. Specifically, with the weight function \( e^{-\beta \lambda^2/2} \) modified to \( e^{-\lambda^2/2} \), and with the corresponding PDF then denoted by use of the label "\( G^* \)" and similarly for the density \( \rho_{(1)}(x) \) normalised to integrate to unity, we have the limit formula [3]
\[
\rho_{(1), \alpha}(x; \alpha) := \lim_{\beta \to 2\alpha/N} \rho_{(1)}^G(x; \alpha) = \frac{1}{\sqrt{2\pi} \Gamma(1 + \alpha)} \frac{1}{|D_{-\alpha}(ix)|^2}, \tag{1.8}
\]
where \( D_{-\alpha}(z) \) is the so-called parabolic cylinder function. This same expression is known to be the weight function for the orthogonality relation of the associated Hermite polynomials [10].
In relation to the scaled high temperature regime, it is also known that \[ W_1^{0,G^*}(x; \alpha) := \lim_{\beta \to 2N/\alpha} \left( \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right)^{G^*} = -\frac{1}{\alpha} \frac{d}{dx} \left( e^{-x^2/4} \log D_{-\alpha}(ix) \right); \] (1.9)
in fact it was after first deriving this expression that (1.8) was deduced in [3]. Thus
\[ W_1^{0,G^*}(x; \alpha) = \int_{-\infty}^{\infty} \frac{\rho^{G^*}_{(1),0}(y; \alpha)}{x - y} \, dy, \] (1.10)
and so \( \rho^{G^*}_{(1),0} \) can be deduced from knowledge of \( W_1^{0,G^*} \) by applying the formula for the inverse Stieltjes transform. Starting from the known differential equation for the parabolic cylinder function [33, §12.2], and with \( g(z) := e^{z^2/4}D_{-\alpha}(z) \), we have that \( g(z) \) satisfies the second order linear differential equation
\[ g''(z) - zg'(z) - ag(z) = 0. \] (1.11)
On the other hand, Stieltjes result gives that in relation to (1.1) itself (this will be labelled by the use of "G" as used in (1.2)), now taking the low temperature limit with \( N \) fixed, we have
\[ W_1^{0,G}(x; N) := \lim_{\beta \to \infty} \left( \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right)^{G} = \frac{d}{dx} \log H_N(x). \] (1.12)
The differential equation satisfied by \( H_N(x) \) is
\[ f''(x) - 2xf'(x) + 2Nf(x) = 0. \] (1.13)
Thus (Corollary 2.3)
\[ \frac{i}{\sqrt{2N}} W_1^{0,G}(ix/\sqrt{2}; N) = W_1^{0,G^*}(x; -N). \] (1.14)

It is shown in Remark 2.4, point 1, that the known functional equation for the moments of the density of (1.1), for \( N \) and \( \kappa := \beta/2 \) fixed [16]
\[ m_{2p}^{G}(N, \kappa) = (-1)^{p+1} \kappa^{-p-1} m_{2p}^{G}(-\kappa N, \kappa^{-1}), \] (1.15)
permits an independent derivation of (1.14). With regards to (1.15), note that as well as mapping \( \kappa \) to \( 1/\kappa \), and thus high temperature to low temperature, the variable \( N \) is mapped to \( -\kappa N \), which can thus no longer be interpreted as the number of eigenvalues in (1.1). Nevertheless, it is well defined since \( m_{2p}^{G} \) is a polynomial of degree \( p + 1 \) in \( N \); see e.g. [42].

In addition to providing the details of the derivation of (1.14) and its relationship to (1.15), we will extend the working to the Laguerre and Jacobi \( \beta \)-ensembles. Thus we will proceed by considering the differential equation satisfied by \( g(x) \) in the formula \( W_1^{0,C^*}(x) \propto \frac{d}{dx} \log g(x) \), where \( C^* \) denotes the scaled high temperature limit of the respective classical \( \beta \)-ensemble, and that satisfied by \( f(x) \) in the formula \( W_1^{0,C}(x) = \frac{d}{dx} \log f(x) \) corresponding to the low temperature limit \( \beta \to \infty \) as implied by the results of Stieltjes relating \( f(x) \) to a classical orthogonal polynomial. Moreover, we will show that the functional equation (1.14) can be generalised to higher point extensions of the averages in (1.9) and (1.12), specified for \( n = 2, 3 \) by appropriate limits of
\[ \overline{W}_n(x_1, \ldots, x_n; \kappa, N) := \left( \prod_{l=1}^{n} (A_l - \langle A_l \rangle) \right), \quad A_i = \sum_{j=1}^{N} \frac{1}{x_i - \lambda_j}. \] (1.16)
For general \( n \geq 2 \) define \( \bar{W}_n \) by the requirement that it be the generating function for the mixed cumulants \( \{ \mu_{p_1, \ldots, p_n} \} \) as indexed by \( p_1, \ldots, p_n \),

\[
\bar{W}_n(x_1, \ldots, x_n; N, \kappa) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{\mu_{p_1, \ldots, p_n}(N, \kappa)}{x_1^{p_1} \cdots x_n^{p_n}}, \tag{1.17}
\]

considered as a formal series. With \( n = n_0 \geq 4 \) this is can equivalently be given as a multinomial in the averages on the RHS of (1.16) with \( n = 2, \ldots, n_0 \), homogeneous of degree \( n \) in the total number of factors of the form \( (A_l - \langle A_l \rangle) \). The resummation holds as an analytic function when the eigenvalue support is compact (Jacobi case) and the \( x_i \) are large enough; otherwise (1.17) is the asymptotic expansion of the definition in terms of averages. The definition in terms of averages and moments in the case \( n = 1 \) is

\[
\bar{W}_1(x_1; N, \kappa) = \langle A_1 \rangle = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_p(N, \kappa)}{x^p}; \tag{1.18}
\]

in relation to the first equality note that (1.16) is not appropriate as it vanishes for \( n = 1 \), while in relation to the second note that for \( n = 1 \) the cumulants and the moments are the same quantity.

**Proposition 1.1.** For the Gaussian \( \beta \)-ensemble with weight \( e^{-\kappa x^2} \), denote the quantity \( \bar{W}_n \) specified in the above paragraph by \( \bar{W}_n^G(x_1, \ldots, x_n; N, \kappa) \). Specifically the expansion (1.17) is to be used to provide a meaning for continuous values of \( N \). We have

\[
\bar{W}_n^G(x_1, \ldots, x_n; N, \kappa) = (-i/\kappa)^n \bar{W}_n^G(i \sqrt{\kappa} x_1, \ldots, i \sqrt{\kappa} x_n; -\kappa N, \kappa^{-1}). \tag{1.19}
\]

To see the relevancy of (1.19) to (1.15), substitute the moment expansion (1.18) and equate coefficients of like powers of \( 1/x \) on both sides to reclaim the latter. The proof of Proposition 1.1 is given in Section 2, along with other high-low temperature dualities in the Gaussian case such as (1.14). Analogous workings are undertaken in Sections 3 and 4 for the Laguerre and Jacobi \( \beta \)-ensembles respectively.

2. **High-low temperature duality for \( W_1^G \) and generalisation to \( W_n^G \)**

2.1. **The zero temperature limit of \( W_1^G \)**. Generalising (1.1), consider the family of eigenvalue PDFs proportional to

\[
\prod_{l=1}^{N} w(\lambda_l) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \tag{2.1}
\]

where \( w(\lambda) \) is referred to as the weight function. We use the notation \( \text{ME}_{\beta, N}[w] \) in reference to (2.1), and the terminology \( \beta \)-ensemble, with the name associated with the weight specified as an adjective. Thus, for example, \( \text{ME}_{\beta, N}[e^{-\beta \lambda^2/2}] \) is referred to as the Gaussian \( \beta \)-ensemble, as used in association with (1.1).

Let \( A = \sum_{j=1}^{N} a(\lambda_j) \) be a general linear statistic, and consider its average \( \langle A \rangle_{\text{ME}_{\beta, N}[w]} \). Let \( \rho_{(1)}(\lambda) \) denote the corresponding eigenvalue density. This is specified by the requirement that \( \int_a^b \rho_{(1)}(\lambda) \, d\lambda \) be equal to the expected number of eigenvalues in a general interval \([a, b]\). It relates to the average of the linear statistic \( A \) by

\[
\langle A \rangle_{\text{ME}_{\beta, N}[w]} = \int_{-\infty}^{\infty} a(\lambda) \rho_{(1)}(\lambda) \, d\lambda. \tag{2.2}
\]
The two-point correlation \( \rho_{(2)}(\lambda_1, \lambda_2) \) can be specified by the requirement when divided by \( \rho_{(1)}(\lambda_2) \), it is equal to the eigenvalue density at \( \lambda_1 \), given there is an eigenvalue at \( \lambda_2 \). It relates to the average of the product of two linear statistics

\[
\langle AB \rangle_{ME_\beta,N[w]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\lambda_1) b(\lambda_2) \left( \rho_{(2)}(\lambda_1, \lambda_2) + \delta(\lambda_1 - \lambda_2) \rho_{(1)}(\lambda) \right) d\lambda_1 d\lambda_2.
\]

(2.3)

Equivalently, for the covariance

\[
\text{Cov} (A, B) := \langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle_{ME_\beta,N[w]}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\lambda_1) b(\lambda_2) \left( \rho_{(2)}^T(\lambda_1, \lambda_2) + \delta(\lambda_1 - \lambda_2) \rho_{(1)}(\lambda) \right) d\lambda_1 d\lambda_2,
\]

where \( \rho_{(2)}^T(\lambda_1, \lambda_2) := \rho_{(2)}(\lambda_1, \lambda_2) - \rho_{(1)}(\lambda_1) \rho_{(1)}(\lambda_2) \).

Introduce the linear statistics and covariances

\[
\overline{w}_1(x) := \left\langle \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right\rangle_{ME_\beta,N[w]}, \quad \overline{w}_2(x_1, x_2) := \text{Cov} \left( \sum_{j=1}^{N} \frac{1}{x_1 - \lambda_j}, \sum_{j=1}^{N} \frac{1}{x_2 - \lambda_j} \right)_{ME_\beta,N[w]}.
\]

(2.5)

Later, we will have use too for the \( n \)-point generating function (1.17) viewed in terms of the averages (1.16), which extends the above definition of \( \overline{w}_2(x_1, x_2) \). For the \( \beta \)-ensemble with Gaussian weight \( w(x) = e^{-\beta x^2/2} \), the quantities (2.5) are inter-related by the particular loop equation [12, 14, 32, 42]

\[
\left( (\kappa - 1) \frac{\partial}{\partial x_1} - 2\kappa x_1 \right) \overline{W}^G_1(x_1; N, \kappa) + 2N\kappa + \kappa \left( \overline{W}^G_2(x_1, x_1; N, \kappa) + (\overline{W}^G_1(x_1; N, \kappa))^2 \right) = 0,
\]

(2.6)

where \( \kappa = \beta/2 \) as in (1.15). The equation (2.6) is in fact the first in an infinite hierarchy involving \( \{ \overline{w}_n \}_{n=1}^\infty \); see subsection 2.3 below.

We see that the equation (2.6) contains two unknowns; generally the \( n \)-th loop equation involves \( \{ \overline{w}_n(x_1, \ldots, x_k; N, \kappa) \}_{k=1}^{n+1} \). It has been known for some time (see [31]) that a triangular system of equations results from an appropriate \( 1/N \) expansion of each of the \( \overline{w}_n \), with an essential point being that this quantity decays at leading order as \( N^{2-n} \). In the recent work [22] it was shown that a triangular system also results in the case \( \beta \) is proportional to \( 1/N \), even though each \( \overline{w}_n \) is proportional to \( N \). Here our interest is \( N \) fixed, but \( \beta \) large. For this we make use of the expansions

\[
\overline{W}^G_1(x; N, \kappa) = W^{0,G}_1(x; N) + \frac{1}{\kappa} W^{1,G}_1(x; N) + \cdots, \quad \overline{W}^G_2(x_1, x_2; N, \kappa) = \frac{1}{\kappa} W^{0,G}_2(x_1, x_2; N) + \cdots,
\]

(2.7)

where higher order terms are in higher powers of \( 1/\kappa \). Note that the meaning of \( W^{0,G}_1(x; N) \) here is consistent with (1.12).

In relation to justifying (2.7), in particular the first relation, as is consistent with (1.18) recall that the moments of the spectral density \( \{ m_{2p}^G \} \) relate to the linear statistic \( \overline{W}^G_1(x) \) by the large \( x \) expansion

\[
\overline{W}^G_1(x; N, \kappa) = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_{2p}^G(N, \kappa)}{x^{2p}}, \quad m_{2p}^G(N, \kappa) = \int_{-\infty}^{\infty} x^{2p} \rho_{(1)}(x) \, dx.
\]

(2.8)

where use has been made of the fact that since the Gaussian weight is symmetric about the origin, the odd moments vanish. Thus the expansion for \( \overline{W}^G_1(x; N, \kappa) \) in (2.7) implies the corresponding
large $\beta$ expansion of the moments

$$m_{2p}^G(N, \kappa) = m_{2p}^{0,G}(N) + \frac{1}{\kappa} m_{2p}^{1,G}(N) + \cdots ,$$  (2.9)

where the $m_{2p}^{k,G}(N)$ are independent of $\beta$. Rescaling $m_{2p}^G(N, \kappa) = 2^{-p} \tilde{m}_{2p}^G(N, \kappa)$ for notational convenience, for low orders we have the exact results [42, §4.3]

$$\tilde{m}_0^G(N, \kappa) = N$$
$$\tilde{m}_2^G(N, \kappa) = N^2 - N + \kappa^{-1} N$$
$$\tilde{m}_4^G(N, \kappa) = 2N^3 - 5N^2 + 3N + \kappa^{-1}(5N^2 - 5N) + \cdots$$
$$\tilde{m}_6^G(N, \kappa) = 5N^4 - 22N^3 + 32N^2 - 15N + \kappa^{-1}(22N^3 - 54N^2 + 32N) + \cdots$$
$$\tilde{m}_8^G(N, \kappa) = 14N^5 - 93N^4 + 234N^3 - 260N^2 + 105N + \kappa^{-1}(93N^4 - 398N^3 + 565N^2 - 260N) + \cdots,$$  (2.10)

where terms not shown are higher order in $\kappa^{-1}$ (each $\tilde{m}_{2p}$ is a polynomial in $\kappa^{-1}$ of degree $p$).

This is consistent with (2.9) and thus the first expansion in (2.7). Furthermore, with $N$ large the $\kappa$ dependence in (2.7) has previously been established [42, §3].

The formal justification for fixed $N$ is to use the Laplace method of asymptotic analysis, whereby the integrand corresponding to the averages in (2.8) is expanded about the critical point $\lambda = z$. Let $A_i := \sum_{j=1}^N 1/(x_i - \lambda_j)$. Recalling the working leading to (1.4), this shows

$$\langle A_1 \rangle = \sum_{j=1}^N \frac{1}{x - z_j} + \langle y \mathcal{H}_1 y^T \rangle_Q + \cdots , \quad \langle A_2 \rangle = \langle y \mathcal{H}_2 y^T \rangle_Q + \cdots .$$

Here higher order terms involve powers of $y_j^{2p}$ for $p \geq 2$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are some $N \times N$ symmetric matrices, and the average over $Q$ is with respect to the PDF corresponding to (1.4). Changing variables $\sqrt{\beta} y \rightarrow y$ implies (2.7).

Substituting (2.7) in (2.6) and equating leading terms for large $\beta$ gives a differential equation involving $W_{1}^{0,G}(x)$ only,

$$\left( \frac{1}{2} \frac{d}{dx} - x \right) W_{1}^{0,G}(x; N) + N + \frac{1}{2} (W_{1}^{0,G}(x; N))^2 = 0.$$  (2.11)

This Riccati equation can readily be solved in terms of Hermite polynomials.

**Proposition 2.1.** We have

$$W_{1}^{0,G}(x; N) = \frac{d}{dx} \log H_N(x) = \frac{N}{x} + \frac{d}{dx} \left( 1 + N! \sum_{m=1}^{\infty} \frac{(-1)^m (2x)^{-2m}}{m!(N - 2m)!} \right)$$
$$= \frac{N}{x} + \frac{N^2 - N}{2x^3} + \frac{2N^3 - 5N^2 + 3N}{2^2 x^5} + \frac{5N^4 - 22N^3 + 32N^2 - 15N}{2^3 x^7} + \cdots $$  (2.12)

Furthermore, writing

$$W_{1}^{0,G}(x; N) = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_{2p}^{G}(N)}{x^{2p}}$$  (2.13)

as is consistent with (2.8) and (2.9), we have that $\{m_{2p}^{G}\}$ satisfy the recurrence

$$2m_{k+2}^{0,G}(N) = \sum_{s=0}^{k/2} m_{2s}^{G}(N) m_{k-2s}^{0,G}(N) - (k + 1)m_{k}^{0,G}(N), \quad m_{0}^{0,G}(N) = N,$$  (2.14)

valid for $k = 0, 2, 4, \ldots$. 
DUALITIES FOR THE CLASSICAL $\beta$-ENSEMBLES

Proof. Setting

\[ W_1^{0,G}(x; N) = \frac{d}{dx} \log f(x), \quad f(x) \sim c x^N, \]

where $c$ is a nonzero constant, we see from (2.11) that $f$ satisfies the linear second order differential equation

\[ f''(x) - 2 x f'(x) + 2 N f(x) = 0. \]  

(2.15)

The unique solution of this equation satisfying the required boundary condition is the Hermite polynomial $H_N(x)$. The recurrence (2.14) follows by substituting (2.13) in (2.11) and equating like powers of $1/x$.

Remark 2.2. 1. We see that the explicit form of the moments $\{m_{2p}^{0,G}\}$ as read off from (2.12) and (2.13) agree with the leading (with respect to $1/\kappa$) terms in (2.10).

2. For $\beta \to \infty$, the result of Stieltjes discussed below (1.2) tells us that

\[ W_1^{0,G}(x; N) = \sum_{j=1}^{N} \frac{1}{x - z_j}, \]

where $\{z_j\}$ denote the zeros of $H_N(x)$. Thus

\[ W_1^{0,G}(x; N) = \frac{d}{dx} \log H_N(x) \]  

(2.16)

in agreement with (2.12).

3. It follows from (2.14) that each $m_{2p}^{0,G}$ is a polynomial of degree $p + 1$ in $N$ which vanishes when $N = 0$ and $N = 1$. The coefficient of $N^{p+1}$ is recognised as $C_p/2^p$, where $\{C_p\}$ denotes the Catalan numbers. Indeed, writing $m_{2p}^{0,G} \sim a_p N^{p+1}/2^p$ for $N \to \infty$ we see that (2.14) simplifies to read

\[ a_{p+1} = \sum_{s=0}^{p} a_s a_{p-s}, \quad a_0 = 1, \]  

(2.17)

which has the unique solution $a_p = C_p$. This is in keeping with the well known result (see e.g. [28]) that the density of the zeros of the Hermite polynomials, scaled by $\sqrt{2/N}$, is given by the Wigner semi-circle law

\[ \rho_{(1)}^{W}(x) = \frac{1}{\pi} \sqrt{1 - (x/2)^2}, \]  

(2.18)

supported on $|x| < 2$, as follows from the moment formula

\[ \int_{-2}^{2} x^{2p} \rho_{(1)}^{W}(x) dx = C_p. \]  

(2.19)

2.2. High-low temperature duality of $W_1^{G}$. Making the replacements

\[ x \mapsto ix/\sqrt{2}, \quad W_1^{0,G}(ix/\sqrt{2}; N) \mapsto (\sqrt{2N}/i) g(x; N) \]  

(2.20)

in (2.11) shows

\[ \left( - \frac{\partial}{\partial x} - x \right) g(x; N) + 1 - N (g(x; N))^2 = 0, \quad g(x; N) \sim \frac{1}{x}. \]  

(2.21)

This same differential equation (up to the meaning of $N$) is known in the study of the Gaussian $\beta$-ensemble $\text{ME}_{\beta,N}[e^{-x^2/2}]$, now scaled at high temperature by setting $\beta = 2\alpha/N$ then taking $N \to \infty$ [3]. Hence writing

\[ \frac{1}{N} \left( \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right)_{\text{ME}_{\beta,N}[e^{-x^2/2}]}_{\beta=2\alpha/N} = W_1^{0,G^*}(x; \alpha) + \frac{1}{N} W_1^{1,G^*}(x; \alpha) + \cdots \]  

(2.22)
(here we are using $G^*$ to indicate the Gaussian $\beta$-ensemble based on the weight $e^{-x^2/2}$ rather than on the weight $e^{-\beta x^2/2}$ as relates to Proposition 2.1) one has that $W_{1,0}^{0,G^*}(x;\alpha)$ satisfies (2.21) but with $-N$ replaced by $\alpha$. We can thus relate $W_{1,0}^{0,G^*}$ to $W_{1}^{0,G^*}$, and similarly relate the corresponding moments.

**Corollary 2.3.** We have the identity (1.14). Also, with the moments $\{m_{2p}^{0,G^*}(N)\}$ specified as in (2.13), and the moments $\{m_{2p}^{0,G^*}(\alpha)\}$ specified according to

$$W_{1,0}^{0,G^*}(x;\alpha) = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_{2p}^{0,G^*}(\alpha)}{x^{2p}}, \quad (2.23)$$

it follows from (1.14) that

$$\frac{(-2)^p}{N} m_{2p}^{0,G^*}(N) = m_{2p}^{0,G^*}(-N). \quad (2.24)$$

**Remark 2.4.** 1. For the Gaussian $\beta$-ensemble with weight $e^{-\beta x^2/2}$ it is known that the moments $m_{2p}^G(N,\kappa)$, for fixed $N$ and $\kappa$, satisfy the functional equation (1.15). From the definitions $m_{2p}^G(N,\kappa) = \beta^{-p} m_{2p}^{G^*}(N,\kappa)$. Setting $\kappa = \alpha/N$, dividing by $N$ and taking $N \to \infty$, it follows

$$m_{2p}^{G^*}(\alpha) = (-1)^{p+1} (2p/\alpha) \lim_{N \to \infty} m_{2p}^G(-\alpha, N/\alpha) = (-1)^{p+1} (2p/\alpha) m_{2p}^{0,G^*} \bigg|_{N \to -\alpha}, \quad (2.25)$$

in agreement with (2.24). We remark that in [18] this duality between high and low temperature for the Gaussian $\beta$-ensemble has been used to study the density of states in the scaled high temperature limit from knowledge of underlying structures at zero temperature.

2. The solution of the differential equation (2.21) with $-N = \alpha$ is [3] (see [1] for a generalisation)

$$W_{1,0}^{0,G^*}(x;\alpha) = \frac{x}{2\alpha} - \frac{1}{\alpha} \frac{d}{dx} \log D_{-\alpha}(ix), \quad (2.26)$$

where $D_{-\alpha}(z)$ is the so-called parabolic cylinder function. Now (see e.g. [44])

$$D_N(z) = e^{-z^2/4} e^{-N/2} H_N(z/\sqrt{2})$$

and so

$$W_{1,0}^{0,G^*}(ix;\alpha) \bigg|_{\alpha = -N} = \frac{1}{iN} \frac{d}{dx} \log H_N(x/\sqrt{2}).$$

With knowledge of (1.14), this reclaims (2.16).

### 2.3. High-low temperature duality of $W_n^G$. Our aims in this subsection are to prove Proposition 1.1, and to proceed to deduce from the proposition generalisations of (1.14) and (1.15).

**Proof of Proposition 1.1.** For $n \geq 2$ the $n$-th loop equation for the Gaussian $\beta$-ensemble with weight $e^{-\beta x^2/2}$, and thus the generalisation of (2.6) which corresponds to the case $n = 1$, is (see the references listed in relation to (2.6))

$$0 = \left[ (\kappa - 1) \frac{\partial}{\partial x_1} - 2\kappa x_1 \right] W_{n}(x_1, J_n; N, \kappa)$$

$$+ \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \left( W_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; N, \kappa) - W_{n-1}(J_n; N, \kappa) \right)$$

$$+ \kappa \left[ W_{n+1}(x_1, x_1, J_n; N, \kappa) + \sum_{J \subseteq J_n} W_{|J|+1}(x_1, J; N, \kappa) W_{n-|J|}(x_1, J \setminus J; N, \kappa) \right]. \quad (2.27)$$
Here the notation \( \hat{x}_k \) indicates that the variable \( x_k \) is not present in the argument, and thus \( \mathcal{W}_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; N, \kappa) = \mathcal{W}_{n-1}(\{x_j\}_{j=1}^n \setminus \{x_k\}; N, \kappa) \). Also for \( n \geq 2 \), \( J_n = (x_2, \ldots, x_n) \), while \( J_1 = \emptyset \). In (2.27) change variables \( x_j \mapsto i\sqrt{\kappa}x_j \), replace \( \kappa \) by \( \kappa^{-1} \) and \( N \) by \( \tilde{N} \) where \( \tilde{N} \) is arbitrary. Then with

\[
\tilde{W}_n^G(x_1, \ldots, x_j; N, \kappa) = \mathcal{W}_n^G(i\sqrt{\kappa}x_1, \ldots, i\sqrt{\kappa}x_j; \tilde{N}, \kappa^{-1})
\]

we see that \( \{\tilde{W}_n^G\} \) satisfy

\[
0 = i \left[ (\kappa - 1) \frac{\partial}{\partial x_1} - 2\kappa x_1 \right] \tilde{W}_n(x_1, J_n; \tilde{N}, \kappa) - \kappa^{1/2} \sum_{k=2}^n \frac{\partial}{\partial x_k} \left\{ \frac{\tilde{W}_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; \tilde{N}, \kappa) - \tilde{W}_{n-1}(J_n; \tilde{N}, \kappa)}{x_k - \hat{x}_k} \right\} + \kappa^{1/2} \tilde{W}_{n+1}(x_1, x_1, J_n; \tilde{N}, \kappa) + \sum_{J \subseteq J_n} \tilde{W}_{|J|+1}(x_1, J; \tilde{N}, \kappa)\tilde{W}_{n-|J|}(x_1, J_n \setminus J; \tilde{N}, \kappa). \tag{2.28}
\]

Now substituting

\[
\tilde{W}_n^G(x_1, \ldots, x_j; \tilde{N}, \kappa) = (i\sqrt{\kappa})^n \tilde{W}_n^G(x_1, \ldots, x_j; \tilde{N}, \kappa)
\]

shows that the equations satisfied by \( \tilde{W}_n^G \) are precisely (2.27) for each \( n = 2, 3, \ldots \).

We can check too that with the same change of variables and mappings of the previous paragraph, the loop equation for \( n = 1 \) (2.6) is similarly transformed to its original form, provided \( \tilde{N} = -\kappa N \). Thus all the loop equations are invariant under the mapping implied by the right hand side of (1.19). Since upon a \( 1/N \) expansion, the validity of which is rigorously established in [13], the loop equations uniquely determine \( \{W_n^G\} \), the functional equation (1.19) has been validated.

\[ \square \]

We can use (1.19) to deduce the analogue of (1.14), relating the low temperature \( n \)-point quantity \( W_n^{0,G} \) in the \( \beta \to \infty \) expansion

\[
\mathcal{W}_n^G(x_1, \ldots, x_n; N, \kappa) = \frac{1}{\kappa^{n-1}} W_n^{0,G}(x_1, \ldots, x_n; N, \kappa) + \cdots, \tag{2.29}
\]

(cf. (2.7); the derivation involving the Laplace method of asymptotic expansion is the same, making essential use of the structure of the \( \mathcal{W}_n^G \) when expressed in terms of the averages on the RHS of (1.16) as noted below (1.17)) to the high temperature \( n \)-point quantity \( W_n^{0,G^*} \) in the \( N \to \infty \) expansion

\[
\frac{1}{N} \mathcal{W}_n^{G^*}(x_1, \ldots, x_n; N, \kappa) |_{\kappa = \alpha/N} = W_n^{0,G^*}(x_1, \ldots, x_n; \alpha) + \cdots \tag{2.30}
\]

(cf. (2.22)). Noting the analogues of (1.17)

\[
W_n^{0,G}(x_1, \ldots, x_n; N) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{\mu_0^{0,G}(N)}{x_1^{p_1} \cdots x_n^{p_n}}, \tag{2.31}
\]

and

\[
W_n^{0,G^*}(x_1, \ldots, x_n; \alpha) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{\mu_0^{0,G^*}(\alpha)}{x_1^{p_1} \cdots x_n^{p_n}}, \tag{2.32}
\]

this can be equivalently be written as a generalisation of (2.24).
Corollary 2.5. We have
\[-\left(\frac{-i}{\sqrt{2}}\right)^n \frac{1}{N} W_n^{0,G}(ix_1/\sqrt{2}, \ldots, ix_n/\sqrt{2}; N) = W_n^{0,G^*}(x_1, \ldots, x_n; -N).\] (2.33)

Equivalently
\[-\frac{(-\sqrt{2})^{p_1+\cdots+p_n}}{N} \mu_n^{0,G}(p_1, \ldots, p_n; N) = \mu_n^{0,G^*}(p_1, \ldots, p_n; -N).\] (2.34)

Proof. Noting from the definitions that
\[W_n^{G^*}(x_1, \ldots, x_n; N, \kappa) = (\sqrt{2\kappa})^{-n} W_n^{G}(ix_1/\sqrt{2}, \ldots, ix_n/\sqrt{2}; -\kappa N, \kappa),\] (2.35)
it follows from (1.19) that
\[W_n^{G^*}(x_1, \ldots, x_n; N, \kappa) = (-i/\sqrt{2}\kappa)^n W_n^{G}(ix_1/\sqrt{2}, \ldots, ix_n/\sqrt{2}; -\kappa N, \kappa^{-1}).\]

Hence, substituting for \(W_n^{G^*}\) in (2.30) it follows
\[\frac{W_n^{0,G^*}(x_1, \ldots, x_n; \alpha)}{\alpha} = \frac{1}{\alpha} \left(\frac{-i}{\sqrt{2}}\right)^n \lim_{N \to \infty} (N/\alpha)^{n-1} W_n^{G}(ix_1/\sqrt{2}, \ldots, ix_n/\sqrt{2}; -\alpha N/\alpha)\]
\[= \frac{1}{\alpha} \left(\frac{-i}{\sqrt{2}}\right)^n W_n^{0,G}(ix_1/\sqrt{2}, \ldots, ix_n/\sqrt{2}; -\alpha),\]
where the second equality follows from (2.29) Setting \(-\alpha = N\) gives (2.33).

The equation (2.34) now follows by substituting the expansions (2.31) and (2.32) in (2.33) and equating coefficients of like powers. \(\square\)

Remark 2.6. 1. The functional equation (1.19) also has consequence for the expansion coefficients \(\mu_n^{G}(p_1, \ldots, p_n; N, \kappa)\) in (1.17), by providing a generalisation of (1.15). Thus we see
\[\mu_n^{G}(p_1, \ldots, p_n; N, \kappa) = (1/i\sqrt{\kappa})^{2n+\Sigma_{k=1}^n p_k} \mu_n^{G}(\kappa, \kappa^{-1}).\] (2.36)

2. A coupled recurrence scheme to compute \(\mu^{0,G^*}(p_1, p_2; \alpha)\) has been given in [22] (the coupling is with \(\mu^{0,G^*}(p; \alpha)\)). In light of (2.34) this implies an analogous computation of \(\mu_n^{G^*}(p_1, p_2; N)\).

3. We can check from (2.35) that the functional equation (1.19) is formally unchanged by use of the weight \(e^{-x^2/2}\) corresponding to the Gaussian \(\beta\)-ensemble denoted by \(W^*\) and thus
\[W_n^{G^*}(x_1, \ldots, x_n; N, \kappa) = (-i/\sqrt{\kappa})^n W_n^{G^*}(i\sqrt{\kappa}x_1, \ldots, i\sqrt{\kappa}x_n; -\kappa N, \kappa^{-1}).\] (2.37)

3. High-low temperature duality for the Laguerre \(\beta\)-ensemble

3.1. The zero temperature limit of \(W_1^{L}\). The considerations of the previous section can be extended to the other classical \(\beta\)-ensembles, namely those with the Laguerre and the Jacobi weights. In the Laguerre case, we will denote by "L" the \(\beta\)-ensemble with weight \(x^{\beta \alpha/2}e^{-\beta x^2/2}x_{x>0}\). We know from [23] that the first loop equation is
\[\left[(\kappa-1) \frac{d}{dx} + \left(\frac{\alpha x}{x} - \kappa\right)\right] W_1^{L}(x; N, a, \kappa) + \left(\frac{N}{x} \kappa \right) + x [W_2^{L}(x; N, a, \kappa) + W_1^{L}(x; N, a, \kappa)]^2 = 0, \] (3.1)
where as in the previous section \(\kappa = \beta/2\). Introducing low temperature expansions of the form (2.7) shows that in the low temperature limit (3.1) reduces to the Riccati type equation for the quantity \(W_1^{0,L}(x; N, a),\)
\[\left[\frac{d}{dx} + \left(\frac{a}{x} - 1\right)\right] W_1^{0,L}(x; N, a) + \frac{N}{x} + (W_1^{0,L}(x; N, a))^2 = 0, \] (3.2)
which can readily be solved in terms of Laguerre polynomials.
Proposition 3.1. We have

\[ W_{1}^{0,L}(x; N, a) = \frac{d}{dx} \log L_{N}^{a-1}(x) = \frac{N}{x} + \frac{d}{dx} \left( 1 + N!(N + a - 1)! \sum_{m=1}^{\infty} m!(N + a - 1 - m)!(N - m)! \left( \frac{-x}{m} \right)^{m} \right) \]

\[ = \frac{N}{x} + \frac{N(N + a - 1)}{x^{2}} \left( 1 + \frac{2(N + a - 2)}{x} + \frac{(5N^{2} + (-11 + 5a)N + a^{2} - 5a + a^{2})}{x^{2}} + \cdots \right) \quad \] (3.3)

Furthermore, writing

\[ W_{1}^{0,L}(x; N, a) = \frac{1}{x} \sum_{p=0}^{\infty} m_{p}^{0,L}(N, a) \frac{1}{x^{p}} \] (3.4)

as is consistent with (2.8) and (2.9), we have that \( \{m_{p}^{0,L}\} \) satisfy the recurrence

\[ m_{k+1}^{0,L}(N, a) = \sum_{s=0}^{k} m_{s}^{0,L}(N, a)m_{k-s}^{0,L}(N, a) - (k + 1 - a)m_{k}^{0,L}(N, a), \quad m_{0}^{0,L}(N, a) = N, \] (3.5)

valid for \( k = 0, 1, \ldots \).

Proof. Setting

\[ W_{1}^{0,L}(x; N, a) = \frac{d}{dx} \log f(x), \quad f(x) \sim cx^{N}, \]

where \( c \) is a nonzero constant, we see from (3.2) that \( f \) satisfies the linear second order differential equation

\[ xf''(x) + (a - x)f'(x) + Nf(x) = 0. \] (3.6)

Up to a proportionality constant, the unique solution of this equation satisfying the required boundary condition is the Laguerre polynomial \( L_{N}^{a-1}(x) \). The recurrence (3.5) follows by substituting (3.4) in (3.2) and equating like powers of \( 1/x \).

\[ \square \]

Remark 3.2. 1. It follows from [23, Prop. 3.11] (see also [30]) that with Laguerre weight \( x^{\beta a/2}e^{-x^{2}/2} \), the corresponding moments of the density \( m^{(L)}(N, a, \kappa) \) are, for low order, given by

\[ \frac{1}{N} m_{1}^{(L)}(N, a, \kappa) = N + \frac{1}{\kappa}(1 - \kappa + \kappa a) \]

\[ \frac{1}{N} m_{2}^{(L)}(N, a, \kappa) = 2N^{2} + \frac{N}{\kappa}(4 - 4\kappa + 3a\kappa) + \frac{1}{\kappa^{2}}(2 - 4\kappa + 2\kappa^{2} + 3a\kappa - 3a\kappa^{2} + \kappa^{2}a^{2}). \] (3.7)

Expanding with respect to \( 1/\kappa \), we see the leading terms are the coefficients of \( 1/x^{2} \) and \( 1/x^{3} \), as is consistent with (3.3).

2. Writing the PDF for the Laguerre \( \beta \)-ensemble in Boltzmann factor form

\[ e^{-\beta U^{L}}, \quad U^{L} := \sum_{l=1}^{N} (x_{l} - a \log x_{l}) - \sum_{1 \leq j < k \leq N} \log |x_{j} - x_{k}|, \]

from a limiting case of (4.1) below, as considered by Stieltjes in [36], we know that the minimum of \( U^{L} \) is unique and occurs at the zeros of the Laguerre polynomial \( L_{N}^{a-1}(x) \). Arguing as in the derivation of (2.16) gives an alternative derivation of the first equality in (3.3).
3.2. High-low temperature duality of $W_1^\kappa$. The recent work [22] considered the Laguerre $\beta$-ensemble with weight $x^\alpha e^{-x} \chi_{x>0}$ (to distinguish this from the Laguerre weight of the previous subsection, the notation "L." will be used) in the scaled high temperature limit obtained by setting $\kappa = \alpha/N$ and taking $N \to \infty$. Writing

$$
\frac{1}{N} \left( \sum_{j=1}^{N} \frac{1}{x - \lambda_j} \right)^{L^*} \bigg|_{\kappa = \alpha/N} = W_1^{0,L^*}(x; \alpha, a) + \frac{1}{N} W_1^{1,L^*}(x; \alpha, a) + \cdots \tag{3.8}
$$

it was shown that $W_1^{0,L^*}$ satisfies the differential equation

$$
\left[ -\frac{d}{dx} + \left( \frac{a}{x} - 1 \right) \right] W_1^{0,L^*}(x; \alpha, a) + \frac{1}{x} + \alpha (W_1^{0,L^*}(x; \alpha, a))^2 = 0. \tag{3.9}
$$

We see that upon the substitutions $x = -y$, $W_1^{0,L^*}(x; N, a) = -N g(y)$ in (3.9) that $g(y)$ so specified satisfies precisely the differential equation (3.2), except that $a \mapsto -a$ and $\alpha = -N$. Considering the requirement of the behaviour at infinity we conclude

$$
-\frac{1}{N} W_1^{0,L^*}(-x; N, a) = W_1^{0,L^*}(x; -N, -a). \tag{3.10}
$$

In fact we know from [4] (see also [22]) that

$$
W_1^{0,L^*}(x; \alpha, a) = -\frac{1}{\alpha} \frac{d}{dx} \log \left( x^{-a/2} e^{-x/2} W_{-a/2}(1+a/2)(-x) \right), \tag{3.11}
$$

where $W_{\mu,\kappa}(z)$ denotes the Whittaker function. Substituting this in the right hand side of (3.10), and substituting the first equality of (3.3) for the left hand side shows that we must have

$$
L_N^{\alpha-1}(x) = Ce^{x/2, a/2} W_{N+a/2, (1-a)/2}(x) \tag{3.12}
$$

for some constant $C$. This is a known identity [33, Eq. (13.18.17) and the general fact $W_{\mu,\kappa}(z) = W_{\mu,\kappa}(z)$]. Also, writing

$$
W_1^{0,L^*}(x; \alpha, a) = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_p^{0,L^*}(\alpha, a)}{x^p}, \tag{3.13}
$$

it follows from (3.4) and (3.10) that

$$
(-1)^p \frac{1}{N} m_p^{0,L^*}(N, a) = m_p^{0,L^*}(-N, -a). \tag{3.14}
$$

Substituting in (3.5), this implies

$$
m_{k+1}^{0,L^*}(-N, -a) = -N \sum_{s=0}^{k} m_s^{0,L^*}(-N, -a) m_{k-s}^{0,L^*}(-N, -a) + (k + 1 - a) m_k^{0,L^*}(-N, -a), \tag{3.15}
$$

subject to the initial condition $m_0^{0,L^*}(-N, -a) = 1$. With $(-N, -a)$ replaced by $(\alpha, a)$ this is known from [22, Eq. (4.13)].
3.3. High-low temperature duality of $W_n^L$. The equation (3.1) is the case $n = 1$ of a hierarchy of loop equations which for $n \geq 2$ read [23, Eq. (3.9)]

$$0 = \left[ (\kappa - 1) \frac{\partial}{\partial x_1} + \left( \frac{\kappa a}{x_1} - \kappa \right) \right] W_n^L(x_1, J_n)$$

$$+ \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \left( \frac{W_{n-1}^L(x_1, \ldots, \hat{x_k}, \ldots, x_n) - W_{n-1}^L(J_n)}{x_1 - x_k} + \frac{1}{x_1} W_{n-1}^L(J_n) \right)$$

$$+ \kappa \left[ W_{n+1}^L(x_1, x_1, J_n) + \sum_{J \subseteq J_n} W_{|J|+1}^L(x_1, J) W_{n-|J|}^L(x_1, J_n \setminus J) \right]. \quad (3.16)$$

Moreover, we know from [23] that these equations become triangular upon an appropriate $1/N$ expansion of each of the $W_n^L$, and so uniquely determine the latter. Analogous to (3.7) for $n = 1$. With this understood, we can make use of the loop equations to derive a functional equation for the $W_n^L$.

**Proposition 3.3.** For the Laguerre $\beta$-ensemble with weight $x^{\beta a}/e^{-\beta x^2}$, denote the quantity (1.16) by $W_n^L(x_1, \ldots, x_n; N, \kappa, a)$ for $n \geq 2$, and similarly the quantity (1.18) in the case $n = 1$. Extend their meaning to general values of $N, a, \kappa$; this latter point is illustrated by (3.7) for $n = 1$. With this understood, we can make use of the loop equations to derive a functional equation for the $W_n^L$.

We can use Proposition 3.3 to obtain a generalisation of (1.14) and (2.34). For this purpose, analogous to (2.29) expand for $\kappa \to \infty$

$$W_n^L(x_1, \ldots, x_n; N, \kappa, a) = \frac{1}{\kappa^{n-1}} W_n^{0, L}(x_1, \ldots, x_n; N, a) + \cdots \quad (3.20)$$

and analogous to (2.30) also expand for $N \to \infty$

$$\frac{1}{N} W_n^{L, \kappa}(x_1, \ldots, x_n; N, \kappa, a) \bigg|_{\kappa=\alpha/N} = W_n^{0, L, \kappa}(x_1, \ldots, x_n; \alpha, a) + \cdots \quad (3.21)$$

Further expand the terms on the right hand side of each of these,

$$W_n^{0, L}(x_1, \ldots, x_n; N, a) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{\mu_{p_1, \ldots, p_n}^L(N, a)}{x_1^{p_1} \cdots x_n^{p_n}} \quad (3.22)$$

and

$$W_n^{0, L, \kappa}(x_1, \ldots, x_n; \alpha, a) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{\mu_{p_1, \ldots, p_n}^{0, L, \kappa}(\alpha, a)}{x_1^{p_1} \cdots x_n^{p_n}} \quad (3.23)$$

(cf. (2.31) and (2.32)).
Corollary 3.4. We have
\[ -\frac{1}{N} W_{n}^{0,L}(x_{1}, \ldots, x_{n}; N, a) = W_{n}^{0,L}(-x_{1}, \ldots, -x_{n}; -N, -a) \]  \hspace{1cm} (3.24)
and
\[ \frac{1}{N} \mu_{np_{1}, \ldots, p_{n}}^{0,L}(N, a) = (-1)^{p_{1}+\cdots+p_{n}+n^{-1}} \mu_{np_{1}, \ldots, p_{n}}^{0,L}(-N, -a). \]  \hspace{1cm} (3.25)

Proof. Noting from the definitions that
\[ \prod_{n}^{L}(x_{1}, \ldots, x_{n}; N, \kappa, a) = \kappa^{-n} \prod_{n}^{L}(x_{1}/\kappa, \ldots, x_{n}/\kappa; N, \kappa, a/\kappa), \]
it follows from (3.18) that
\[ \prod_{n}^{L}(x_{1}, \ldots, x_{n}; N, \kappa, a) = \kappa^{-n} \prod_{n}^{L}(-x_{1}, \ldots, -x_{n}; -\kappa, \kappa^{-1}, -a). \]
Setting \( \kappa = a/N \), dividing both sides by \( N \) and taking the limit \( N \to \infty \) on both sides using (3.20) and (3.21) shows
\[ \prod_{n}^{L}(x_{1}, \ldots, x_{n}; a, a) = \frac{1}{a} \prod_{n}^{L}(-x_{1}, \ldots, -x_{n}; -a, -a). \]
Changing the sign of each of \( (x_{1}, \ldots, x_{n}; a, a) \) in this equation, interchanging the role of the LHS and RHS and setting \( a = N \) gives (3.24). Substituting (3.22) and (3.23) gives (3.25). \( \square \)

4. HIGH-LOW TEMPERATURE DUALITY FOR THE JACOBI \( \beta \)-ENSEMBLE

4.1. The zero temperature limit of \( W_{1}^{J} \). We will denote by "J" the \( \beta \)-ensemble implied by the weight \( x^{\beta a/2}(1 - x)^{\beta b/2} \chi_{0 < x < 1} \). Writing the eigenvalue PDF in Boltzmann factor form, as proportional to
\[ e^{-\beta U_{1}}, \quad U_{1} := -\sum_{i=1}^{N} (a \log x_{i} + b \log(1 - x_{i})) - \sum_{1 \leq j < k \leq N} \log |x_{j} - x_{k}|, \]  \hspace{1cm} (4.1)
we know from the work of Stieltjes [36] that the minimum of \( U_{1} \) for \( x_{1} \in (0, 1) \) is unique and occurs at the zeros of the Jacobi polynomials \( P_{N}^{(a-1, b-1)}(1 - 2x) \). As in the Gaussian and Laguerre cases above, this result can be reclaimed using a loop equation formalism.

Thus we know from [23] that for the Jacobi \( \beta \)-ensemble the first loop equation is
\[ 0 = \left( (\kappa - 1) \frac{d}{dx_{1}} + \left( \frac{\kappa a}{x_{1}} - \frac{\kappa b}{1 - x_{1}} \right) \right) \prod_{n}^{L}(x_{1}; N, a, b, \kappa) + \frac{1}{x_{1}(1 - x)} [(\kappa a + \kappa b + 1)N + \kappa N(N - 1)] \\
+ \kappa \left[ \prod_{n}^{L}(x_{1}; N, a, b, \kappa) + (\prod_{n}^{L}(x_{1}; N, a, b, \kappa))^{2} \right]. \]  \hspace{1cm} (4.2)
Introducing low temperature expansions of the form (2.7) we see that in the low temperature limit (4.2) reduces to a Riccati type equation for the quantity \( W_{1}^{0,3}(x; N, a, b) \). This equation reads
\[ 0 = \left( \frac{d}{dx} + \frac{a}{x} - \frac{b}{1 - x} \right) W_{1}^{0,3}(x; N, a, b) + \frac{(a + b - 1)N + N^{2}}{x(1 - x)} + (W_{1}^{0,3}(x; N, a, b))^{2}. \]  \hspace{1cm} (4.3)
Solving (4.3) with an appropriate boundary condition gives rise to Jacobi polynomials.
Proposition 4.1. We have
\[ W_1^{0, J}(x; N, a, b) = \frac{d}{dx} \log P_N^{(a-1, b-1)}(1-2x) \]
\[ = \frac{N}{x} + \frac{d}{dx} \left( \frac{N! \Gamma(a + N)}{\Gamma(a + b + 2N - 1)} \sum_{m=1}^{\infty} \frac{(-x)^{-m} \Gamma(a + b + 2N - m - 1)}{m!(N - m)! \Gamma(a + N - m)} \right) \]
\[ = \frac{N}{x} + \frac{N(N + a - 1)}{(2N + a + b - 2)x^2} \]
\[ + \frac{N(N + a - 1)(4 + a^2 + a(-4 + b) - 2b + (-7 + 3a + 2b)N + 3N^2)}{(-3 + a + b + 2N)(-2 + a + b + 2N)^2 x^3} + \ldots \] \hspace{1cm} (4.4)

Furthermore, writing
\[ W_1^{0, J}(x; N, a, b) = \frac{1}{x} \sum_{p=0}^{\infty} m_p^{0, J}(N, a, b) x^p \] \hspace{1cm} (4.5)
we have that \( \{m_p^{0, J}\} \) satisfy the recurrence
\[ m_p^{0, J} = \frac{1}{2N + a + b - 1 - p} \left( (N + a - 1)N + b \sum_{s=1}^{p-1} m_s^{0, J} + N \sum_{s=1}^{p-1} m_s^{0, J} m_{p-s}^{0, J} \right), \quad m_0^{0, J} = N, \] \hspace{1cm} (4.6)
valid for \( p = 1, 2, \ldots \).

Proof. Let \( c \) be a nonzero constant and set
\[ W_1^{0, J}(x; N, a, b) = \frac{d}{dx} \log f(x), \quad f(x) \sim cx^N. \]
We see from (4.3) that \( f \) satisfies the second order linear differential equation
\[ x(1-x)f''(x) + (a(1-x) - bx)f'(x) + ((a + b - 1)N + N^2)f(x) = 0. \] \hspace{1cm} (4.7)
Up to a proportionality constant, the unique solution of this equation satisfying the required boundary condition is the Jacobi polynomial \( P_N^{(a-1, b-1)}(1-2x) \). The recurrence (4.6) follows by substituting (4.5) in (4.3) and equating like powers of \( 1/x \). \qed

4.2. High-low temperature duality of \( W_1^J \). The recent work [22] considered the Jacobi \( \beta \)-ensemble with weight \( x^a(1-x)^b \chi_{0<x<1} \) (to distinguish this from the Jacobi weight of the previous subsection, the notation "J∗∗" will be used) in the scaled high temperature limit \( \kappa = \alpha/N \) and \( N \to \infty \). Writing the analogue of the expansion (3.8), it was shown that \( W_1^{0, J^∗}(x; \alpha, a, b) \) satisfies the differential equation
\[ \left( -\frac{d}{dx} + \frac{a}{x} - \frac{b}{1-x} \right) W_1^{0, J^∗}(x; \alpha, a, b) + \frac{1}{x(1-x)} (1 + a + b + \alpha) + \alpha \left( W_1^{0, J^∗}(x; \alpha, a, b) \right)^2 = 0. \] \hspace{1cm} (4.8)
Substituting \( W_1^{0, J^∗}(x) = -g(x)/N \) we see that \( g(x) \) satisfies the differential equation (4.7), except that \( a \mapsto -a, b \mapsto -b \) and \( \alpha = -N \). Considering the requirement of the behaviour at infinity we conclude
\[ -\frac{1}{N} W_1^{0, J^∗}(x; N, a, b) = W_1^{0, J^∗}(x; -N, -a, -b). \] \hspace{1cm} (4.9)
From [22], in terms of the Gauss hypergeometric function, we have
\[ W_1^{0, J^∗}(x; \alpha, a, b) = -\frac{1}{\alpha} \frac{d}{dx} \log \left( x^{-a} F_1(\alpha, a + 1, 2\alpha + a + b + 2; x^{-1}) \right). \] \hspace{1cm} (4.10)
Substituting this along with the first equality in (4.4) shows
\[ P_{N}^{(a-1,b-1)}(1 - 2x) = C x^{N} 2F_{1} \left( -N, -N - a + 1, -2N - a - b + 2; x^{-1} \right) \]  (4.11)
for some constant \( C \), which can readily be checked by equating like powers of \( x \).

4.3. **High-low temperature duality of \( W_{n}^{J} \).** The hierarchy of loop equations generalising (4.12) for \( n \geq 2 \) reads \([23, \text{Eq. (4.6)}]\)

\[
0 = \left( (\kappa - 1) \frac{\partial}{\partial x_{1}} + \left( \frac{\kappa a}{x_{1}} - \frac{\kappa b}{1 - x_{1}} \right) \right) \overline{W}_{n}^{0}(x_{1}, J_{n}) - \frac{1}{x_{1}(1-x_{1})} \sum_{k=2}^{n} x_{k} \frac{\partial}{\partial x_{k}} \overline{W}_{n-1}^{0}(J_{n}) + \sum_{k=2}^{n} \frac{\partial}{\partial x_{k}} \left\{ \frac{\overline{W}_{n-1}^{0}(x_{1}, \ldots, \hat{x}_{k}, \ldots, x_{n}) - \overline{W}_{n-1}^{0}(J_{n})}{x_{1} - x_{k}} + \frac{1}{x_{1}} \overline{W}_{n-1}^{0}(J_{n}) \right\} + \kappa \left[ \overline{W}_{n+1}^{0}(x_{1}, x_{1}, J_{n}) + \sum_{J_{n+1}} \overline{W}_{|J|+1}^{0}(x_{1}, J) \overline{W}_{n-|J|}^{0}(x_{1}, J_{n} \setminus J) \right]. \tag{4.12}
\]

These equations become triangular upon an appropriate \( 1/N \) expansion of each of the \( \overline{W}_{n}^{J} \), and so uniquely determine the latter [23]. As in the Gaussian and Laguerre cases, introducing the expansion about infinity

\[
\overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, \kappa, a, b) = \frac{1}{x_{1} \cdots x_{n}} \sum_{p_{1}, \ldots, p_{n} = 0}^{\infty} \frac{\mu_{p_{1}, \ldots, p_{n}}^{0}(N, \kappa, a, b)}{x_{1}^{p_{1}} \cdots x_{n}^{p_{n}}}, \tag{4.13}
\]

allows for a meaning to \( \overline{W}_{n}^{J} \) for general \( N, a, b, \kappa \), since it is known that the \( \mu_{p_{1}, \ldots, p_{n}}^{0} \) are rational functions in \( N, a, b, \kappa \); see [23].

**Proposition 4.2.** For the Jacobi \( \beta \)-ensemble with weight \( x^{\beta a/2}(1 - x)^{\beta b/2} \), denote the quantity (1.16) by \( \overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, \kappa, a, b) \) for \( n \geq 2 \), and similarly the quantity (1.18) in the case \( n = 1 \). Extend their meaning to general values of \( N \) using (4.13). We have

\[
\overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, \kappa, a, b) = (-\kappa)^{-n} \overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; -\kappa N, \kappa^{-1}, -\kappa a, -\kappa b). \tag{4.14}
\]

Equivalently

\[
\mu_{p_{1}, \ldots, p_{n}}^{0}(N, \kappa, a, b) = (-\kappa)^{-n} \mu_{p_{1}, \ldots, p_{n}}^{0}(-\kappa N, \kappa^{-1}, -a, -b). \tag{4.15}
\]

**Proof.** Proceeding as in the Gaussian and Laguerre cases, the functional equation (4.14) follows as an invariance of the full set of loop equations, and (4.15) is a consequence of this which follows upon substituting (4.13) in (4.14) and equating coefficients. \( \square \)

As in the Gaussian and Laguerre cases, we can use Proposition 4.2 to obtain a generalisation of (1.14) and (2.34). To begin we expand for \( \kappa \to \infty \)

\[
\overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, \kappa, a, b) = \frac{1}{\kappa^{n-1}} \overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, a, b) + \cdots \tag{4.16}
\]

and also expand for \( N \to \infty \)

\[
\left. \frac{1}{N} \overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; N, \kappa, a, b) \right|_{\kappa = \alpha/N} = \overline{W}_{n}^{0}(x_{1}, \ldots, x_{n}; \alpha, a, b) + \cdots \tag{4.17}
\]
Then we expand the terms on the right hand side of each of these,

\[ W_n^{0,J} (x_1, \ldots, x_n; N, a, b) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n = 0}^{\infty} \mu_{p_1, \ldots, p_n}^{0,J} (N, a, b), \tag{4.18} \]

and

\[ W_n^{0,J^*} (x_1, \ldots, x_n; \alpha, a) = \frac{1}{x_1 \cdots x_n} \sum_{p_1, \ldots, p_n = 0}^{\infty} \mu_{p_1, \ldots, p_n}^{0,J^*} (\alpha, a). \tag{4.19} \]

**Corollary 4.3.** We have

\[ W_n^{J^*} (x_1, \ldots, x_n; N, \kappa, a, b) = (-1)^n N W_n^{J} (x_1, \ldots, x_n; -N, -a, -b), \tag{4.20} \]

and

\[ \mu_{p_1, \ldots, p_n}^{0,J^*} (N, a, b) = (-1)^n N \mu_{p_1, \ldots, p_n}^{0,J} (-N, -a, -b). \tag{4.21} \]

**Proof.** It follows from (4.14) and the definition of \( J^* \) that

\[ \frac{1}{N} \mathcal{P}_n^{J^*} (x_1, \ldots, x_n; N, \kappa, a, b) = (-\kappa)^{-n} N W_n^{J} (x_1, \ldots, x_n; -\kappa N, \kappa^{-1}, -a, -b). \]

Taking the limit \( N \to \infty \) on both sides using (4.16) and (4.17) implies (4.20). Substituting (4.18) and (4.19) gives (4.21). \( \square \)

**Remark 4.4.** It is known that a limiting case of the Jacobi \( \beta \)-ensemble implies the \( \beta \)-ensemble on the circle corresponding to (1.5); see [20, §3.9]. In the case of the latter, playing the role of the statistics (1.16) and (1.18) is the modification of the \( A_i \) therein by

\[ A_i = \sum_{p=1}^{N} e^{i\theta_p} + z_i \]

From [43, Prop. 4.7] we know

\[ \mathbb{W}_n (z_1, \ldots, z_n; N, \kappa) = (-\kappa)^{-n} \mathbb{W}_n (z_1, \ldots, z_n; -\kappa N, \kappa^{-1}) \]

(cf. (4.14)).

**Acknowledgments.** The research of PJF is part of the program of study supported by the Australian Research Council Centre of Excellence ACEMS, and the Discovery Project grant DP210102887. An input to this work was the knowledge gained from the presentation of M. Voit as part of the Bielefeld-Melbourne on-line random matrix seminar in late October 2020. Thanks are due to G. Akemann for organising this. The feedback of the referee, by way of a thorough reading and helpful remarks, is much appreciated.

**References**

[1] G. Akemann and S.S. Byun, *The high temperature crossover for general 2D Coulomb gases*, J.Stat. Phys. 175 (2019), 1043–1065.

[2] A. Alastuey and B. Jancovici, *On the two-dimensional one-component Coulomb plasma*, J. Physique 42 (1981), 1–12.

[3] R. Allez, J.P. Bouchard and A. Guionnet, *Invariant beta ensembles and the Gauss-Wigner crossover*, Phys. Rev. Lett. 109 (2012), 09412.

[4] R. Allez, J.-P. Bouchaud, S. N. Majumdar, P. Vivo, *Invariant \( \beta \)-Wishart ensembles, crossover densities and asymptotic corrections to the Marchenko-Pastur law*, J. Phys. 46, 015001 (2013).
[5] A. Andersen, A.D. Jackson and H.J. Pedersen, *Rigidity and normal modes in random matrix spectra*, Nucl. Phys. A 650 (1999), 213–223.

[6] S. Andraus, K. Hermann and M. Voit, *Limit theorems and soft edge of freezing random matrix models via dual orthogonal polynomials*, arXiv:2009.01418.

[7] S. Andraus, M. Katori, and S. Miyashita, *Interacting particles on the line and Dunkl intertwining operator of type A: Application to the freezing regime*, J. Phys. A: Math. Theor., 45 (2012), 395201.

[8] S. Andraus, M. Voit, *Limit theorems for multivariate Bessel processes in the freezing regime*, Stoch. Proc. Appl. 129 (2019), 4771–4790.

[9] S. Andraus, M. Voit, *Central limit theorems for multivariate Bessel processes in the freezing regime. II. The covariance matrices*, J. Approx. Theory 246 (2019), 65–84.

[10] R. Askey and J. Wimp, *Associated Laguerre and Hermite polynomials*, Proc. Roy. Soc. Edinburgh A, 84, (1984) 15–37.

[11] A. Borodin and V. Gorin, *General β-Jacobi corners process and the Gaussian free field*, Comm. Pure Appl. Math., 68 (2015), 1774–1844.

[12] G. Borot, B. Eynard, S.N. Majumdar, and C. Nadal, *Large deviations of the maximal eigenvalue of random matrices*, J. Stat. Mech. 2011 (2011), P11024.

[13] G. Borot and A. Guionnet, *Asymptotic expansion of β matrix models in the one-cut regime*, Commun. Math. Phys. 317 (2013), 447–483.

[14] A. Brini, M. Mariño, and S. Stevan, *The uses of the refined matrix model recursion*, J. Math. Phys. 52 (2011), 35–51.

[15] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. 43 (2002), 5830–5847.

[16] I. Dumitriu and A. Edelman, *Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models*, J. Math. Phys. 47 (2006), 063302.

[17] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, Ann. I. H. Poincaré – PR 41 (2005), 1083–1099.

[18] T.K. Duy and T. Shirai, *The mean spectral measures of random Jacobi matrices related to Gaussian beta ensembles*, Electron. Commun. Probab. 20 (2015), paper no. 68.

[19] P.J. Forrester, *Recurrence equations for the computation of correlations in the 1/r² quantum many body system*, J. Stat. Phys. 72 (1993), 39–50.

[20] P.J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.

[21] P.J. Forrester, B. Jancovici, and D.S. McAnally, *Analytic properties of the structure function for the one-dimensional one-component log-gas*, J. Stat. Phys. 102 (2000), 737–780.

[22] P.J. Forrester and G. Mazzuca, *The classical β-ensembles with β proportional to 1/N: from loop equations to Dyson’s disordered chain*, arXiv:2102.09201.

[23] P.J. Forrester, A.A. Rahman, and N.S. Witte, *Large N expansions for the Laguerre and Jacobi β ensembles from the loop equations*, J. Math. Phys. 58 (2017), 113303.

[24] P.J. Forrester and E.M. Rains, *Interpretations of some parameter dependent generalizations of classical matrix ensembles*, Prob. Theory Related Fields 131 (2005), 1–61.

[25] V. Gorin and V. Kleptsyn, *Universal objects of the infinite beta random matrix theory*, arXiv:2009.02006.
[26] V. Gorin, A. Marcus, *Crystallization of random matrix orbits*, IMRN, **2020** (2020), 883–913.

[27] K. Johnson, and B. Simanek, *Electrostatic equilibria on the unit circle via Jacobi polynomials*, J. Math. Phys. **61** (2020), 122901.

[28] M. Kornik and G. Michaletzky, *Wigner matrices, the moments of roots of Hermite polynomials and the semicircle law*, J. Approx. Th. **211** (2016), 29–41.

[29] G. Mazzuca, *On the mean density of states of some matrices related to the beta ensembles and an application to the Toda lattice*, arXiv:2008.04604

[30] F. Mezzadri, A.K. Reynolds and B. Winn, *Moments of the eigenvalue densities and of the secular coefficients of $\beta$-ensembles*, Nonlinearity **30** (2017), 1034.

[31] A.A. Migdal, *Loop equations and $1/N$ expansions*, Phys. Rep. **102** (1983), 199–290.

[32] A.D. Mironov, A.Yu. Morozov, A.V. Popolitov, and Sh.R.Shakirov, *Resolvents and Seiberg-Witten representation for a Gaussian $\beta$-ensemble*, Theor. Math. Phys., **171** (2012), 505–522.

[33] NIST Digital Library of Mathematical Functions.

[34] A. Soshnikov and Y. Xu, *Gaussian approximation of the distribution of strongly repelling particles on the unit circle*, Theory Probab. Appl. **65** (2021), 588–615.

[35] T.J. Stieltjes, *Sur quelques théorèmes d’algèbre*, Comptes Rendus de l’Académie des Sciences **100** (1885), 439–440.

[36] T.J. Stieltjes, *Sur les polynômes de Jacobi*, Comptes Rendus de l’Académie des Sciences **100** (1885), 620–622.

[37] K.D. Trinh, *Limit theorems for moment processes of beta Dyson’s Brownian motions and beta Laguerre processes*, in preparation.

[38] H.D. Trinh and K.D. Trinh, *Beta Jacobi ensembles and associated Jacobi polynomials*, arXiv:2005.01100 (2020).

[39] R. Van Peski, *Limits and fluctuations of $p$-adic random matrix products*, arXiv:2011.09356.

[40] M. Voit, *Central limit theorems for multivariate Bessel processes in the freezing regime*, J. Approx. Theory **239** (2019), 210–231.

[41] M. Voit and J.H.C. Woerner, *Functional central limit theorems for multivariate Bessel processes in the freezing regime*, Stoch. Anal. Appl. **39** (2021), 136–156.

[42] N.S. Witte and P.J. Forrester, *Moments of the Gaussian $\beta$ ensembles and the large $N$ expansion of the densities*, J. Math. Phys. **55** (2014), 083302.

[43] N.S. Witte and P.J. Forrester, *Loop equation analysis of the circular ensembles*, JHEP **2015** (2015), 173.

[44] A. Wünsche, *Associated Hermite polynomials related to parabolic cylinder functions*, Advances in Pure Mathematics **9** (2019), 15–42.

School of Mathematical and Statistics, ARC Centre of Excellence for Mathematical and Statistical Frontiers, The University of Melbourne, Victoria 3010, Australia

*Email address: pjforr@unimelb.edu.au*