Dynamical Breakdown of Chirality and Parity in (2+1)-dimensional QED

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CHIBA-EP-77-REV
July 1994
hep-ph/9404361

Abstract

In the (2+1)-dimensional QED with and without the Chern-Simons term, we find the non-local gauge in which there is no wavefunction renormalization for the fermion in the framework of the Schwinger-Dyson equation. By solving the Schwinger-Dyson equation in the non-local gauge, we find a finite critical value $N_f^c$ for the number of flavors $N_f$ of four-component Dirac fermions, above which the chiral symmetry restores irrespective of presence or absence of the Chern-Simons term. In the same framework, we study the possibility of dynamical breakdown of the parity. It is shown that the parity is not dynamically broken. We discuss this reason from the viewpoint of the Schwinger-Dyson equation.
1 Introduction

The study of the dynamical symmetry breaking in the (2+1)-dimensional QED (QED3) has attracted a great deal of attention recently. In the framework of the Schwinger-Dyson (SD) equation in the relativistic quantum field theory, Appelquist et al. \cite{1,2} investigated the dynamical symmetry breaking of $U(2N_f)$ to $U(N_f) \times U(N_f)$ in QED3 with $N_f$ flavors of four-component Dirac fermions. The study has given rise to much controversy as follows. In the bare vertex approximation: $\Gamma_\mu(p,q) = \gamma_\mu$, Appelquist, Nash and Wijewardhana (ANW) \cite{2} have shown that there exists the finite critical number of flavors
\[ N_f^c = \frac{32}{\pi^2} \approx 3.2, \] above which ($N_f > N_f^c$) the chiral symmetry restores \footnote{In the quenched limit $N_f \to 0$, the theory has only one phase, chiral-symmetry-breaking phase, as shown by the SD equation \cite{3} and the Monte Carlo simulation \cite{6}.} and for $N_f < N_f^c$ the fermion mass is dynamically generated and obeys the scaling of the essential singularity type:
\[ m \sim \Lambda f(N_f), \quad f(N_f) = \exp \left[ -\frac{2n\pi}{\sqrt{N_f^c/N_f - 1}} \right]. \] This result was obtained under the assumption of no wavefunction renormalization $A(p) \equiv 1$ for the fermion in the Landau gauge where the fermion propagator is written as $S(p) = [A(p)\not{p} - B(p)]^{-1}$. They justified their approximations based on the argument of $1/N_f$ expansion in which $A(p) = 1 + O(1/N_f)$ and $\Gamma_\mu(p,q) = \gamma_\mu + O(1/N_f)$.

However, this was criticised by Pennington and Webb (PW) \cite{4} and Atkinson, Johnson and Pennington (AJP) \cite{5}. They claimed that the one loop correction to the wavefunction renormalization and the corresponding vertex correction of the type $\Gamma_\mu(p,q) = \gamma_\mu A(q)$ \footnote{Similar ansatz for the vertex, e.g., $\Gamma_\mu(p,q) = \frac{1}{2}\gamma_\mu[A(p) + A(q)]$ or $\Gamma_\mu(p,q) = \gamma_\mu[A(p)\theta(p^2 - q^2) + A(q)\theta(q^2 - p^2)]$, leads to the same result.} leads to the scaling law of the exponential type:
\[ f(N_f) = \exp \left[ -\frac{3\pi^2}{8N_f} \right] \] and that there does not exist a finite critical flavour, $N_f^c = \infty$. Similar claim was stated \footnote{In the quenched limit $N_f \to 0$, the theory has only one phase, chiral-symmetry-breaking phase, as shown by the SD equation \cite{3} and the Monte Carlo simulation \cite{6}.} in the approach of the effective potential in contrast to the original result by Matsuki, Miao and Viswanathan \cite{7}.

On the other hand, the Monte Carlo simulation of non-compact QED on a lattice by Daggoto, Kogut and Kocic \cite{8} suggests the existence of two phases which are separated by a finite $N_f^c \sim 3.5 \pm 0.5$. However Azcoiti et al. \cite{9} recently report different and delicate results. Hence this issue has not yet been confirmed by the simulation of lattice non-compact QED.

In the framework of the SD equation, it was also pointed out that the inclusion of the infrared (IR) cutoff which plays the role of the finite size of the lattice simulation may drastically change the critical phenomenon associated with the dynamical symmetry breaking. Especially the ansatz of PW leads to the mean-field type scaling law and the
cutoff-dependent finite $N_f$ in the large IR cutoff, while the exponential type scaling and infinite $N_f$ are obtained in the sufficiently small IR cutoff [10].

The subtlety of the problem in the framework of the SD equation comes from the fact that, in order to take into account the wavefunction renormalization, it is indispensable to include the vertex correction as a consequence of the Ward-Takahashi (WT) identity. However incorporating the vertex correction properly is quite difficult in the non-perturbative study of gauge theory, although there are many works of going along such a line under the appropriate ansatz for the vertex in QED3 [11, 12] as well as the (3+1)-dimensional case, QED4 [13, 15, 14].

In this paper we investigate this problem from a different point of view. Instead of considering the vertex correction, we look for the situation in which we do not have to take into account the vertex correction. This is achieved only when there is no wavefunction renormalization. In the quenched approximation which neglects the vacuum polarization in the photon propagator, this is simply realized by taking the Landau gauge [16]. In the presence of the vacuum polarization and/or the Chern-Simons (CS) term [17], however, we cannot choose the (covariant) gauge fixing parameter such that $A(p) \equiv 1$ follows. Recently it has been recognized that this can be done by taking the non-local gauge [18, 19, 20]. This standpoint leads to the complicated non-local gauge instead of the vertex correction. However this method has some advantages that we do not have to treat the coupled SD equation for two functions $A(p)$ and $B(p)$, and somewhat arbitrary (non-perturbative) ansatz for the vertex so as to satisfy the WT identity.

In the same scheme, we study the dynamical breakdown of the parity. The bare CS term breaks the parity explicitly. Even if there exists no bare CS term, the CS term may be induced by the radiative correction through the non-perturbative effect. Therefore there is a possibility that the induced CS term breaks the parity. However previous studies so far support the claim that the dynamical breakdown of the parity does not occur [21, 22, 23, 24] in agreement with the general argument [25]. We clarify this reason from the viewpoint of SD equation.

## 2 SD equation and the non-local gauge

We consider the (2+1)-dimensional QED (QED3) with the following (euclidean) lagrangian

$$\mathcal{L}_{\text{QED3}} = \bar{\psi}^{i}(i\gamma_{\mu}\partial_{\mu} - m_{e} - m_{\sigma}\tau)\psi^{i} + \frac{1}{4}F_{\mu\nu}^{2} + e\bar{\psi}^{i}\gamma_{\mu}\psi^{i}A_{\mu} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{GF}},$$

with the Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{i}{2}\theta\epsilon_{\mu\nu\rho}A_{\mu}\partial_{\nu}A_{\rho},$$

and the gauge fixing term $\mathcal{L}_{\text{GF}}$ whose form will be specified in what follows. According to Appelquist et al. [1], $\psi^{i}$ denotes the 4-component Dirac fermion with a flavour index $i(1, ..., N_f)$. Here $\gamma_{a}(a = 0, 1, 2)$ are $4 \times 4$ matrices which satisfy the clifford algebra
\(\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}\) and are explicitly defined by

\[
\gamma_0 := \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma_1 := \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix},
\]

(6)

with \(\sigma_a (a = 1, 2, 3)\) being the Pauli matrices. We further introduce the \(4 \times 4\) matrices \(\gamma_3, \gamma_5\) and \(\tau\) as

\[
\gamma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau := \gamma_3\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(7)

First we define the chiral transformation by \(\psi(x) \to \gamma_3\psi(x), A_\mu(x) \to A_\mu(x)\). Then the mass term \(m_e\psi\bar{\psi}\) and \(m_o\bar{\psi}\tau\psi\) are respectively odd and even under the chiral transformation. Next the parity transformation is defined by \(\psi(x_0, x_1, x_2) \to P\psi(x_0, -x_1, x_2), A_\mu(x_0, x_1, x_2) \to (-1)^{\delta_{\mu 1}} A_\mu(x_0, -x_1, x_2)\), with \(P = -i\gamma_5\gamma_1\). Then \(m_e\psi\bar{\psi}\) and \(m_o\bar{\psi}\tau\psi\) are respectively parity even and odd mass term. The bare CS term \(L_{CS}\) is parity odd and hence the inclusion of the bare CS term breaks the parity explicitly even if \(m_o = 0\).

Corresponding to the diagram of Fig.1, the SD equation for the full (exact) fermion propagator \(S(p) = [A(p)\dot{\psi} - B(p)]^{-1}\) is written as follows.

\[
A(p)\dot{\psi} - B(p) = \dot{\psi} - m_e - m_o\tau
\]

\[
+ e^2 \int \frac{d^3q}{(2\pi)^3} D_{\mu\nu}(q - p) \gamma_\mu \frac{A(q)\dot{\psi} - B(q)}{A^2(q)q^2 + B^2(q)} \Gamma_\nu(p, q),
\]

(8)

where \(D_{\mu\nu}(k)\) is the full photon propagator and \(\Gamma_\nu(p, q)\) the full vertex function. In this paper we take the bare vertex \(\gamma_\mu\) instead of the full vertex \(\Gamma_\mu(p, q)\). In order for the bare vertex approximation \(\Gamma_\mu(p, q) = \gamma_\mu\) to be justified in this SD equation, we must guarantee no wavefunction renormalization \(A(p) \equiv 1\) in light of the Ward-Takahashi (WT) identity: \((p_\mu - q_\mu)\Gamma_\mu(p, q) = S^{-1}(p) - S^{-1}(q)\). Actually, in the case of the quenched approximation (without CS term):

\[
D_{\mu\nu}(k) = D_{\mu\nu}^0(k) = \frac{1}{k^2} \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} (1 - \xi) \right]
\]

(9)

for the covariant gauge-fixing \(L_{GF} = \frac{1}{2}(\partial_\mu A_\mu)^2\), this is simply achieved by taking the Landau gauge \(\xi = 0\), see e.g. [3].

In the presence of the vacuum polarization and/or the CS term, this cannot be achieved by choosing an appropriate value for the usual gauge fixing parameter \(\xi\) and instead we must adopt the non-local gauge in the sense that \(\xi\) becomes a function of the photon momentum: \(\xi = \xi(k^2)\) [18, 19, 20]. In QED3, therefore, we consider the photon propagator of the form:

\[
D_{\mu\nu}(k) = \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] D_T(k) + \frac{k_\mu k_\nu}{k^2} D_L(k) + \epsilon_{\mu\nu\rho} \frac{k_\rho}{\sqrt{k^2}} D_O(k),
\]

(10)

3In what follows, \(f := q\) implies that \(f\) is defined by \(g\).
4Note that \(\tau\) anticommutes with \(\gamma_3, \gamma_5\) and commutes with \(\gamma_0, \gamma_1, \gamma_2\), and that \(\gamma_5^3 = -\gamma_5, \gamma_5^2 = -1\).
with the non-local gauge \[\xi(k^2) = \frac{D_L(k)}{D_T(k)}\] (11)

The vacuum polarization tensor in QED3 is written as

\[
\Pi_{\mu\nu}(k) = \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \Pi_T(k) + \frac{k_\mu k_\nu}{k^2} \Pi_L(k) + \epsilon_{\mu\nu\rho} \frac{k_\rho}{\sqrt{k^2}} \Pi_O(k).
\]

From the SD equation for the photon propagator,

\[D_{\mu\nu}(k) = [D_{\mu\nu}(k)]^{-1} - \Pi_{\mu\nu}(k),\]

the general form for \(D_T(k)\) and \(D_O(k)\) is written as

\[
D_T(k) = \frac{k^2 - \Pi_T(k)}{(k^2 - \Pi_T(k))^2 + (\Pi_O(k) - \theta\sqrt{k^2})^2},
\]

\[
D_O(k) = \frac{\Pi_O(k) - \theta\sqrt{k^2}}{(k^2 - \Pi_T(k))^2 + (\Pi_O(k) - \theta\sqrt{k^2})^2}.
\]

Now we define the parity eigenmatrix decomposition by \(\chi_\pm = \frac{1}{2}(1 \pm \tau)\). Then we have \(A\hat{p} - B = A_+\hat{p} + A_\tau \hat{p} - B_e - B_0\tau = (A_+\hat{p} - B_+)\chi_+ + (A_-\hat{p} - B_-)\chi_-\) and \([A\hat{p} - B]^{-1} = [A_+\hat{p} - B_+]^{-1}\chi_+ + [A_-\hat{p} - B_-]^{-1}\chi_-\). Then the parity even and odd parts are obtained as \(A_e = (A_+ + A_-)/2\) and \(A_o = (A_+ - A_-)/2\) and similarly for \(B_e\) and \(B_o\): \(B_e = (B_+ + B_-)/2\) and \(B_o = (B_+ - B_-)/2\). Thus the SD equation for the fermion propagator is decomposed into the simultaneous integral equation for \(A\) and \(B\):

\[
A_\pm(p) = 1 + e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2A_\pm^2(q) + B_\pm^2(q)} \left\{ \pm 2B_\pm(q) \frac{p \cdot (q - p)}{|q - p|} D_O(k) + A_\pm(q) \left[ (p \cdot q)\xi(k^2) + 2\frac{(k \cdot q)(k \cdot p)}{k^2} (1 - \xi(k^2)) \right] D_T(k) \right\},
\]

\[
B_\pm(p) = m_\pm + e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2A_\pm^2(q) + B_\pm^2(q)} \left\{ B_\pm(q)[2 + \xi(k^2)] D_T(k) + 2A_\pm(q) \frac{q \cdot (q - p)}{|q - p|} D_O(k) \right\},
\]

where \(m_\pm := m_e \pm m_o\) and \(k := q - p\).

As shown in Appendix A, if we take the following non-local gauge function \(\xi(s := k^2)\):

\[
\xi(s) = 2 - \frac{2}{s^2 D_T(s)} \int_0^s dt D_T(\sqrt{t}) t,
\]

\[\quad \text{This non-local gauge function} \ \xi \text{ reduces to the usual gauge fixing parameter} \ a \text{ of the covariant gauge} \ \frac{1}{2\pi}(\partial_\mu A_\mu)^2 \text{ in the limit} \ \frac{1}{T^2} \to \frac{1}{T^2} \text{ and} \ \frac{D_L(k)}{T^2} \to \frac{a}{T^2}.\]
then we have

\[ A_\pm(p) = 1 \pm \frac{e^2}{p^2} \int \frac{d^3q}{(2\pi)^3} \frac{2B_\pm(q)}{q^2 A_\pm^2(q) + B_\pm^2(q)} D_O(k) \frac{p \cdot (q-p)}{|q-p|}. \]  

(19)

If we do not take into account the CS term \( D_O(k) \equiv 0 \), the above expression for the non-local gauge gives exactly \( A_\pm \equiv 1 \) for any \( B \). In the case of the induced CS term, i.e., \( \theta = 0 \) (see section 4), the term \( B_\pm(p)D_O(k) \) is of the order \( B^2 \), which is negligible in the linearized SD equation which we are going to study in what follows. Even in the presence of bare CS term (see section 3), the contribution from the second term in the right hand side of eq. (19) is very small compared with 1 in the neighborhood of the critical point of the chiral phase transition which we pay special attention to in this paper.  

If we adopt this non-local gauge, therefore, we obtain \( A_e(p) \cong 1 \) and \( A_o(p) \cong 0 \), and the even and odd mass functions obey

\[ B_e(p) = m_e + e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{|q^2 + B_e^2(q) + B_o^2(q)|^2 - 4B_e^2(q)B_o^2(q)} \times \left\{ [2 + \xi(k^2)][|q^2 + B_e^2(q) - B_o^2(q)]B_e(q)D_T(k) - 4B_e(q)B_o(q) \frac{q \cdot (q-p)}{|q-p|} D_O(k) \right\}, \]  

(20)

\[ B_o(p) = m_o + e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{|q^2 + B_e^2(q) + B_o^2(q)|^2 - 4B_e^2(q)B_o^2(q)} \times \left\{ [2 + \xi(k^2)][|q^2 - B_e^2(q) + B_o^2(q)]B_o(q)D_T(k) + 2|q^2 + B_e^2(q) + B_o^2(q)| \frac{q \cdot (q-p)}{|q-p|} D_O(k) \right\}, \]  

(21)

In the non-local gauge given above, \( B_\pm(p) \) obey

\[ B_\pm(p) = m_\pm + \frac{e^2}{2\pi^2} \int_0^\Lambda dq \frac{q^2}{q^2 + B_\pm^2(q)} [B_\pm(q)K(p,q) \mp L(p,q)], \]  

(22)

where we have introduced the ultraviolet (UV) cutoff \( \Lambda \) and defined

\[ K(p,q) := \int_0^\pi d\vartheta \sin \vartheta \left[ 1 + \frac{\xi(q-p)}{2} \right] D_T(q-p), \]  

(23)

and

\[ L(p,q) := \int_0^\pi d\vartheta \sin \vartheta \frac{q \cdot (q-p)}{|q-p|} D_O(q-p), \]  

(24)

\[ ^6 \text{This has been confirmed numerically, see [35].} \]

\[ ^7 \text{It is shown that the solution damps very quickly in the region } p > \alpha \text{ and hence the UV cutoff can be substantially identified with } \alpha. \]
for the angle $\vartheta$ defined by $(q-p)^2 = q^2 + p^2 - 2pq \cos \vartheta$.

This integral equation is non-linear in $B_\pm$ and difficult to be solved analytically. Therefore we adopt the approximation to get the appropriate linear equation. Here it should be remarked that, in the linearized equation, if $B(p)$ is a solution, the rescaled function $\kappa B(p)$ by a constant $\kappa$ is also a solution when $L(p,q) \equiv 0$ and $m_\pm = 0$. Hence the magnitude of the linearized integral equation can not be determined in this case. Therefore, in the linearized equation, we must introduce an additional condition, the normalization condition, which specifies the magnitude of the solution.

According to [28, 29], we linearize eq. (22) by replacing the denominator $q^2 + B^2(p)$ with $q^2$ and introduce the infrared (IR) cutoff $M$ in the region of integration: $M \leq q \leq \Lambda$. Then we adopt the normalization condition $M = B(p = M)$. This procedure enforces us to consider the solution only in the region: $M \leq p \leq \Lambda$. In this approximation, the SD equation reduces to

$$B_\pm(p) = m_\pm + \frac{e^2}{2\pi^2} \int_{M_\pm}^\Lambda dq \left[ B_\pm(q)K(p,q) + L(p,q) \right],$$

supplemented with the normalization condition $M_\pm = B_\pm(p = M_\pm)$.

### 3 Chiral symmetry breaking

First of all, we consider the dynamical breaking of the chiral symmetry. In this section, we neglect the induced CS term, $\Pi_O(k) \equiv 0$ and take into account the one-loop vacuum polarization:

$$\Pi_T(k) = -\alpha \sqrt{k^2} + O(B^2), \quad \alpha := \frac{e^2 N_f}{8}.$$ (26)

Then $D_T(k)$ and $D_O(k)$ read

$$D_T(k) = \frac{k^2 + \alpha \sqrt{k^2}}{(k^2 + \alpha \sqrt{k^2})^2 + \theta^2 k^2}, \quad D_O(k) = -\frac{\theta \sqrt{k^2}}{(k^2 + \alpha \sqrt{k^2})^2 + \theta^2 k^2},$$

where $k^2 = (p-q)^2 := s$ denotes the squared photon momentum. In this case, eq. (18) leads to the non-local gauge function:

$$\xi(s) = 2 - 2 \left\{ \frac{(s + \alpha \sqrt{s})^2 + \theta^2 s}{s^2 (s + \alpha \sqrt{s})} \right\} \left\{ -2\alpha \sqrt{s} + s - 4\alpha \theta \left[ \arctan \frac{\alpha}{\theta} - \arctan \frac{\alpha + \sqrt{s}}{\theta} \right] \right\}$$

$$+ (\alpha^2 - \theta^2) \ln \left\{ \frac{(\alpha + \sqrt{s})^2 + \theta^2}{\alpha^2 + \theta^2} \right\}.$$ (28)

The function $\xi = \xi(k^2)$ of $k^2$ is not singular even at $k^2 = 0$ and decreases to zero as $k^2 \rightarrow \infty$, see Fig.2. The non-local gauge just obtained eq. (28) has quite complicated

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8In the linearization of eq. (20) we should keep the coupling between $B_+(q)$ and $B_+(q)$, which leads to an infrared cutoff depending on both $B_+(0)$ and $B_+(0)$. In the linearization of eq. (22), leading to eq. (25), the equations for $B_+$ and $B_-$ decouple completely, but it should be noted that $M_+$ and $M_-$ are different from each other due to the presence of $\theta$. [24].
form. In order to solve the SD equation analytically, therefore, we expand the non-local
gauge eq. (28) around \( k^2 = 0 \) and take into account the first and the second terms, 
since it changes rapidly in the infrared (IR) region \( k/\Lambda \ll 1 \). The effect of higher order
terms will be discussed in the final section. Of course, it is necessary to perform the
numerical calculation to include the non-local gauge completely, which is a subject of
the subsequent paper [35]. It is easy to show that \( \xi(k^2) \) of eq. (28) has the following
expansion in the IR region \( k/\alpha \ll 1 \):

\[
\xi(k^2) = \frac{2}{3} - \frac{1}{3} \frac{\alpha^2 - \theta^2 \sqrt{k^2}}{\alpha^2 + \theta^2} + O\left(\frac{k^2}{\alpha}\right).
\]

(29)

This expansion can be matched with the \( 1/N_f \) expansion (when \( \theta = 0 \)). If we expand
the non-local gauge into such a series and truncate the series up to \( O\left(\frac{k}{\alpha N_f}\right) \),
the truncated non-local gauge \( \xi \) guarantees that \( A(p) = 1 + O\left(\frac{1}{N_f^{N+1}}\right) \). This
implies that the WT identity is satisfied at least up to \( O\left(\frac{1}{N_f^{N+1}}\right) \) by the vertex
function \( \Gamma_\mu(p,q) = \gamma_\mu + O\left(\frac{1}{N_f^{N+1}}\right) \) and the wavefunction renormalization function
\( A(p) = 1 + O\left(\frac{1}{N_f^{N+1}}\right) \). The first term \( 2/3 \) corresponds to the gauge choice of
Carena, Clark and Wagner [31], which is obtained as a consequence of our systematic
treatment of non-local gauge.

We substitute the above form eq. (28) of the non-local gauge into the SD equation
eq. (17) for \( B_\pm \). In order to solve the equation analytically, we linearize the equation
based on the second approximation explained in the previous section:

\[
B_\pm(p) = m_\pm \pm f(p) + \frac{4\alpha}{\pi^2 N_f} \int_{M_\pm}^\Lambda dq B_\pm(q) K(p,q),
\]

(30)

with

\[
f(p) := -\frac{4\alpha}{\pi^2 N_f} \int_{M_\pm}^\Lambda dq L(p,q).
\]

(31)

Here \( K(p,q) \) is obtained by using eq. (29) as

\[
K(p,q) = \frac{8}{3} \frac{\alpha}{\alpha^2 + \theta^2} \left[ \frac{p+q-|p-q|}{2pq} - \frac{9\alpha^2 - \theta^2}{8} \frac{1}{\alpha^2 + \theta^2} \right].
\]

(32)

where, corresponding to eq. (29) of the non-local gauge-function, we have used the
following expansion:

\[
D_T(k) = \frac{\alpha}{\alpha^2 + \theta^2} \frac{1}{\sqrt{k^2}} - \frac{\alpha^2 - \theta^2}{(\alpha^2 + \theta^2)^2} + O(\sqrt{k^2}).
\]

(33)

Similarly \( L(p,q) \) is obtained as

\[
L(p,q) = -\frac{\alpha}{\alpha^2 + \theta^2} \left[ 1 - \frac{p^2 - q^2}{2pq} \ln \frac{p+q}{|p-q|} \right] + \frac{2\alpha \theta}{(\alpha^2 + \theta^2)^2} \left[ \frac{q}{p} (p+q-|p-q|) \right. \\
- \left. \frac{(p+q)(p^2-q^2) - |p-q|(p^2+pq+q^2)}{3pq} \right],
\]

(34)
where we have used
\[ D_O(k) = -\frac{\theta}{\alpha^2 + \theta^2} \frac{1}{\sqrt{k^2}} + \frac{2\alpha \theta}{(\alpha^2 + \theta^2)^2} + O(\sqrt{k^2}). \quad (35) \]

Thus we obtain the SD equation
\[ B_\pm(p) = m_\pm \pm f(p) + \lambda \int_{M_\pm}^{\Lambda} dq B_\pm(q) \left[ \frac{p + q - |p - q|}{2pq} - \frac{9\alpha^2 - \theta^2}{8\alpha^2 + \theta^2} \right], \quad (36) \]
with
\[ \lambda := \frac{32}{3\pi^2 N_f \alpha^2 + \theta^2}. \quad (37) \]

Note that, if \( \theta = 0 \), \( f(p) \equiv 0 \) and \( B_\pm(p) \) obeys the same equation.

The integral equation eq. (36) can be solved as the boundary value problem of the second order differential equation as follows. It can be converted into the differential equation
\[ \frac{d}{dp} \left( p^2 \frac{dB_\pm(p)}{dp} \right) + \lambda B_\pm(p) = \pm r(p), \quad (38) \]
where
\[ r(p) := \frac{d}{dp} \left( p^2 \frac{df(p)}{dp} \right). \quad (39) \]

Then the solution of the SD integral equation can be obtained from the general solution of the differential equation eq. (38) which satisfies both the IR boundary condition (BC):
\[ B_\pm'(M_\pm) = \pm f'(M_\pm), \quad (40) \]
and the UV BC:
\[ m_\pm = B_\pm(\Lambda) + [1 - J(\alpha, \theta)] \Lambda [B_\pm'(\Lambda) \mp f'(\Lambda)] \mp f(\Lambda), \quad (41) \]
where
\[ J(\alpha, \theta) := \frac{9\alpha^2 - \theta^2 \Lambda}{8\alpha^2 + \theta^2}. \quad (42) \]

Here it should be remarked that the second term in the non-local gauge function eq. (29) does not change the differential equation eq. (38) and simply changes the UV BC. Therefore the second term takes no effect on the general solution of the differential equation, but may change the scaling law through the UV BC. Without it, UV BC reads \( m_\pm = B_\pm(\Lambda) + \Lambda [B_\pm'(\Lambda) \mp f'(\Lambda)] \mp f(\Lambda). \)

If \( r(p) \equiv 0 \), the general solution of eq. (38) is given by the linear combination of two special solutions: \( p^{(-1+\tau)/2} \) and \( p^{(-1-\tau)/2} \) where \( \tau := \sqrt{1 - 4\Lambda} \). In the presence of the inhomogeneous term \( r(p) \), the general solution of eq. (38) is given by
\[ B_\pm(p) = M_\pm \left[ c_1 \left( \frac{p}{M_\pm} \right)^{\frac{-1+\tau}{2}} + c_2 \left( \frac{p}{M_\pm} \right)^{\frac{-1-\tau}{2}} \right] \pm b(p), \quad (43) \]
where
\[ b_\theta(p) := p^{-\frac{1+\tau}{2}} \int^q dq \ q^{-1-\tau} \int^q dk r(k) k^{-\frac{1+\tau}{2}}. \] (44)

Two constants \( c_1 \) and \( c_2 \) are determined from the IR BC and the normalization condition in what follows.

In order to write \( b_\theta(p) \) explicitly, we must calculate the integration as
\[ f(p) = \frac{3\lambda\theta}{8\alpha} \int^\Lambda_0 dq \left( 1 - \frac{p^2 - q^2}{2pq} \ln \frac{p + q}{|p - q|} \right). \] (45)

Numerical integration shows that the function \( f(p) \) is well approximated in the interval \([0, \Lambda]\) by the function:
\[ f(p) = \frac{3\lambda\theta}{8\alpha} \Lambda \left[ 2 - a \frac{p}{\Lambda} + b \left( \frac{p}{\Lambda} \right)^2 \right], \] (46)
with \( a = 2.5329 \) and \( b = 0.7913 \), see Fig. 3. Especially, in the region \( p \ll \Lambda \), \( f(p) \) is well approximated by the linear function with \( a = 2, b = 0 \). This can be derived by expanding the integrand of \( f(p) \) in powers of \( q/p \) or \( p/q \) after dividing the integral region into \( p > q \) and \( p < q \), respectively.

Hence the function \( r(p) \) reads
\[ r(p) = \frac{3\lambda\theta}{4\alpha} \Lambda \left[ -a \frac{p}{\Lambda} + 3b \left( \frac{p}{\Lambda} \right)^2 \right]. \] (47)

This leads to
\[ b_\theta(p) = \frac{3\lambda\theta}{4\alpha} \Lambda \left[ -a \frac{p}{\Lambda} + 3b \left( \frac{p}{\Lambda} \right)^2 \right]. \] (48)

Note that \( b_\theta(p) \) is monotonically decreasing in \( p \) and \( b_\theta(p) \downarrow 0 \) as \( \theta \downarrow 0 \) uniformly in \( p \).

By using \( f'(M) = \frac{3\lambda\theta}{4\alpha} \left( -\frac{\theta}{2} + b \frac{M}{\Lambda} \right) \), IR BC eq. (II) reads
\[ c_1 -\frac{1+\tau}{2} + c_2 -\frac{1-\tau}{2} = \pm \frac{3\lambda\theta}{4\alpha} \left[ -\frac{\lambda}{2(2+\lambda)} + b \frac{\lambda}{6+\lambda} M_\pm \right]. \] (49)

From the normalization condition, \( B_\pm (M_\pm) = M_\pm \), we obtain
\[ c_1 + c_2 = 1 \pm \frac{3\lambda\theta}{4\alpha} \left[ a \frac{1}{2+\lambda} - b \frac{3}{6+\lambda} \frac{M_\pm}{\Lambda} \right]. \] (50)

Therefore two coefficients \( c_1, c_2 \) are determined as
\[ c_1 = \frac{1+\tau}{2\tau} \pm \frac{3\lambda\theta}{4\alpha} \left[ a \frac{1-\lambda+\tau}{2(2+\lambda)} + b \frac{M_\pm -3+2\lambda-3\tau}{\Lambda \ 2(6+\lambda)} \right], \]
\[ c_2 = -\frac{1+\tau}{2\tau} \pm \frac{3\lambda\theta}{4\alpha} \left[ a \frac{-1-\lambda+\tau}{2(2+\lambda)} + b \frac{M_\pm 3-2\lambda-3\tau}{\Lambda \ 2(6+\lambda)} \right]. \] (51)

Finally the UV BC eq. (III) is written as
\[ \frac{m_\pm}{\Lambda} = \left[ 1 + J + \tau(1 - J) \right]^{\frac{c_1}{2}} \left( \frac{M_\pm}{\Lambda} \right)^{\frac{3\lambda\theta}{4\alpha}} + \left[ 1 + J - \tau(1 - J) \right]^{\frac{c_1}{2}} \left( \frac{M_\pm}{\Lambda} \right)^{\frac{3\lambda\theta}{4\alpha}} \]
\[ \mp \frac{3\lambda\theta}{4\alpha} C(\lambda, \theta), \] (52)

where
\[ C(\lambda, \theta) := 1 - \left( \frac{\lambda(2 - J)}{2(2+\lambda)} + b \frac{\lambda(3 - 2J)}{2(6+\lambda)} \right). \] (53)
3.1 \( \theta = 0 \)

In the absence of \( \theta \), \( B_+ \) and \( B_- \) obey the same SD equation when \( m_+ = 0 \). Therefore \( B_+(p) \equiv B_+(p) \) and \( B_-(p) \equiv 0 \). In this case \( B_+(p) \equiv 0 \) is a trivial solution of eq. (17). Here we define \( M_e := M_\pm \). Then the UV BC eq. (52) yields

\[
0 = [1 + J + \tau(1 - J)] \frac{c_1}{2} \left( \frac{M_e}{\Lambda} \right) \frac{1 + \tau}{1 - \tau} + [1 + J - \tau(1 - J)] \frac{c_2}{2} \left( \frac{M_e}{\Lambda} \right) \frac{1 + \tau}{1 - \tau},
\]

when \( m_\pm = 0 \). By using the coefficient obtained in eq. (51):

\[
c_1 = \frac{1 + \tau}{2\tau}, \quad c_2 = -\frac{1 + \tau}{2\tau},
\]

the ratio \( M_e/\Lambda \) is obtained as

\[
\left( \frac{M_e}{\Lambda} \right)^\tau = \frac{(1 + \tau) [1 + J + (1 - J)\tau]}{(1 - \tau) [1 + J - (1 - J)\tau]}.
\]

The right hand side (RHS) is greater than 1, since \( 0 < \tau := \sqrt{1 - 4\lambda} < 1 \) and \( J > 0 \). Therefore this equation can not have the solution \( M_e/\Lambda < 1 \) for real \( \tau \), i.e., \( \lambda < 1/4 \).

In the region \( \lambda > \lambda_c := 1/4 \), we define

\[
\omega := \sqrt{4\lambda - 1}, \quad \tau = i\omega, \quad \lambda = \frac{32}{3\pi^2 N_f}.
\]

Then we look for the solution of the equation

\[
\left( \frac{M_e}{\Lambda} \right)^{i\omega} = \frac{1 + J - (1 - J)\omega^2 + 2i\omega}{1 + J - (1 - J)\omega^2 - 2i\omega}.
\]

This equation has the solution \( 0 < M_e/\Lambda < 1 \) of

\[
\frac{M_e}{\Lambda} = \exp \left[ -\frac{2n\pi - 2\phi}{\omega} \right], \quad \phi = \arctan \frac{2\omega}{1 + J - (1 - J)\omega^2}.
\]

In this region the non-trivial solution is obtained from eq. (53) and eq. (51).

Thus it is shown that, in the chiral limit \( m_e = 0 \), the non-trivial chiral-symmetry-breaking solution exists only for \( N_f < N_f^c \) as

\[
B_e(p) = \frac{M_e^{3/2} \sin \left( \frac{1}{2} \sqrt{N_f^c/N_f} - 1 \ln \frac{p}{M_e} + \delta \right)}{\sin (\delta)},
\]

with \( \delta = \arctan \omega = \arctan \sqrt{N_f^c/N_f} - 1 \). The scaling law for the fermion dynamical mass is given by

\[
\frac{M_e}{\Lambda} = \exp \left[ -\frac{2n\pi - 2\phi}{\sqrt{N_f^c/N_f} - 1} \right], \quad \phi = \arctan \frac{2 \sqrt{N_f^c/N_f} - 1}{2 - N_f^c/N_f \left( 1 - \frac{3\Lambda}{8\alpha} \right)}.
\]
with a critical value for the flavour:

\[ N_f^c = \frac{128}{3\pi^2}. \] (62)

This shows that there exists the finite critical number of flavors \( N_f^c \) above which \( N_f > N_f^c \) the chiral symmetry restores.

In the absence of CS term, the value \( N_f^c = \frac{128}{3\pi^2} \sim 4.3 \) that we have just found is 4/3 times as large as the value \( 32/\pi^2 \) of ANW \(^2\) and coincides with that of the 1/\( N_f \) analysis by Nash \(^3\). This result clearly comes from the first term \( \xi = 2/3 \) of the non-local gauge eq. (29). The second term of the non-local gauge does change neither the critical value nor the type of the scaling law for the fermion mass, the essential singularity at \( N_f^c \), and merely decreases the critical coefficient, i.e., the magnitude of the fermion mass, since \( M_e/\Lambda \sim K \exp \left[ -\frac{2n\pi}{\sqrt{N_f^c/N_f - 1}} \right], \ K = \exp \left[ \frac{4}{1 + \frac{2\Lambda}{\alpha}} \right], \) in the neighborhood of \( N_f^c \).

### 3.2 \( \theta \neq 0 \)

In this subsection we consider the region \( N_f < N_f^c = \frac{128}{3\pi^2} \). By substituting eq. (51) into the UV BC eq. (52), the equation of state is obtained as

\[
m_{\pm} = \frac{(M_{\pm}^{2} \lambda^{2} \omega)}{4i\omega} \left[ R_{\pm}(\lambda, \theta) + iI_{\pm}(\lambda, \theta) \right] - \frac{(M_{\pm}^{2} \lambda^{2} \omega)}{4i\omega} \left[ R_{\pm}(\lambda, \theta) - iI_{\pm}(\lambda, \theta) \right]
\]

\[
\mp \frac{3\lambda^{2} \theta}{4\alpha} C(\lambda, \theta),
\]

(63)

where we have defined

\[
R_{\pm}(\lambda, \theta) := 1 + J - \omega^{2}(1 - J) \pm 2\Theta[(1 - \lambda)(1 + J) - \omega^{2}(1 - J)],
\]

\[
I_{\pm}(\lambda, \theta) := 2\omega\{1 \pm \Theta[2 - \lambda(1 - J)]\},
\]

(64)

with \( J = J(\alpha, \theta) \) defined by eq. (42) and

\[
\Theta := \frac{3a\lambda \theta}{8(2 + \lambda) \alpha}.
\]

(65)

In deriving this equation eq. (63), we have neglected the term with the factor \( b \) in eq. (51), since \( b(M/\Lambda)^2 \ll a(M/\Lambda) \).

When \( m_{\pm} = 0 \), we look for the solution for

\[
\zeta_{\pm} := \frac{M_{\pm}}{\Lambda}.
\]

(66)

From the equation of state, \( \zeta_{\pm} \) obeys

\[
\sin(\phi_{\pm} - \frac{\omega}{2} \ln \zeta_{\pm}) = \pm \frac{3}{2} C(\lambda, \theta) \omega K_{\pm}^{1/2} \frac{\lambda^{2} \theta}{\alpha} \zeta_{\pm}^{-3/2},
\]

(67)

\(^9\)Nash’s non-local gauge corresponds to a special case of ours: \( D_L(k)/D_T(k) = \text{constant, independently of } k.\)
where we have defined
\[ \phi_{\pm} := \arctan \frac{I_{\pm}(\lambda, \theta)}{R_{\pm}(\lambda, \theta)}, \]  
(68)
and
\[ K_{\pm} = R_{\pm}(\lambda, \theta)^2 + I_{\pm}(\lambda, \theta)^2. \]  
(69)
For the non-trivial solution to exist, the above equation must have the solution for
\[ 0 < \zeta_{\pm} < 1. \]
We start from the limit \( \theta \to 0 \). In this limit, RHS of eq. (67) vanishes. Hence the solution of eq. (67) is given by
\[ t_{\pm} := \phi_{\pm} - \frac{\omega}{2} \ln \zeta_{\pm} = n\pi \ (n = 1, 2, \ldots). \]  
(70)
Then, in the region \( N_f < N_f^c \) and \( \theta = 0 \), \( M_{\pm} \) obeys the scaling law of the essential singularity type:
\[ M_{\pm} = \Lambda \exp \left[ -\frac{2n\pi - 2\phi_{\pm}}{\omega} \right] > 0, \]  
(71)
as shown in the previous subsection. From the argument \[ (2, 27) \] using the effective potential of Cornwall-Jackiw-Tomboulis \[ (34) \], it is shown that the solution with \( n = 1 \) corresponds to the nodeless solution and gives the ground state solution and that the solution with \( n \geq 2 \) corresponds to the excited solution with nodes \( n - 1 \). The magnitude of the solution with nodes \( n \) is related to the ground state solution as
\[ M_{\pm} = \Lambda e^{\frac{2\phi_{\pm}}{\omega}} \left( e^{-\frac{2\phi_{\pm}}{\omega}} M_{\pm}^{n=1} \right)^n. \]  
(72)
This implies the hierarchy of the mass scale:
\[ M_{\pm}^{n=1} > M_{\pm}^{n=2} > M_{\pm}^{n=3} > \ldots. \]
Next we keep the value of \( N_f \) below the critical point \( N_f^c \) and increases \( \theta > 0. \) To find the solution, we rewrite eq. (67) as
\[ \sin t_{\pm} = H_{\pm}(t_{\pm}), \quad H_{\pm}(t) := \pm \frac{3}{2} C(\lambda, \theta) \omega K_{\pm}^{-1/2} \frac{\lambda \theta}{\alpha} e^{\frac{-3\phi_{\pm}}{\omega}} \exp \left[ \frac{3t}{\omega} \right]. \]  
(73)
The solution of eq. (73) is given as the intersection point of the \( \sin t \) curve with the curve \( H_{\pm}(t) \), see Fig.4. In the absence of the CS term, \( \theta = 0 \), eq. (67) has (countably) infinite number of solutions as shown above. However, in the presence of the CS term, \( \theta \neq 0 \), the strongly oscillating solutions with many nodes \( (n \gg 1) \) disappear and eq. (73) has finite number of solutions, no matter how \( \theta \) is small, since \( |H_{\pm}(t)| \) is a rapidly and monotonically increasing function in \( t \) and \( H_{\pm}(t) \to \pm \infty \) as \( t \to \infty \).
For small \( \theta > 0 \), \( H_+(t) > 0 \) and \( H_-(t) < 0 \) for our choice: \( a = 2.5 \) and \( J \) given by eq. (42). Paying attention to the region \( 0 < t < 2\pi \), we see in this case that \( t_+ \) (resp. \( t_- \)) has a solution at \( t = \pi - \delta_+(\theta) \) (resp. \( t = \pi + \delta_-(\theta) \)) slightly smaller (resp. larger) than \( \pi \) where \( \delta_+(\theta) \geq 0 \) and \( \delta_+(\theta = 0) = 0 \), see Fig. 4. Hence we obtain the solution
\[ M_{\pm} = \Lambda e^{\pm \frac{2\delta_{\pm}(\theta)}{\omega}} \exp \left[ -\frac{2n\pi - 2\phi_{\pm}}{\omega} \right]. \]  
(74)
Therefore the solution \( M_+^0 \) gets larger value than \( M_-^0 = 0 \), while \( M_-^0 \) gets smaller one than \( M_+^0 = 0 \). By substituting the coefficients eq. (71) into eq. (13), the non-trivial solution is given by

\[
B_{\pm}(p) = \frac{M_{\pm}^{3/2}}{\sqrt{p}} \left[ \sin \left( \frac{\pi}{2} \ln \frac{p}{M_{\pm}} + \delta \right) \pm 2\Theta \sin \left( \frac{\pi}{2} \ln \frac{p}{M_{\pm}} + \epsilon \right) \right] \pm b_\Theta(p),
\]

where \( b_\Theta(p) \) is given by eq. (15) and \( \delta = \arctan \omega \) and \( \epsilon = \arctan \frac{\omega}{\chi} \). This solution satisfies the normalization condition \( M_\pm = B_\pm(M_\pm) \) as should does.

As discovered in [13], in the presence of the Chern-Simons term, a first order phase transition does occur. This phenomenon can be explained in our scheme as follows. Consider the situation that \( \theta \) is increased gradually with \( N_f \) being kept. We find that there is a certain value of \( \theta \), say \( \theta_c(N_f) \), such that at \( \theta = \theta_c(N_f) \) the nodeless solution \( B_-(p) \) (corresponding to \( n = 1 \)) does disappear suddenly, since \( H_-(t) \) does not intersect with \( \sin t \) any more in the interval \( 0 < t < \pi \) for sufficiently large \( \theta \). At \( \theta = \theta_c \), the solution \( B_-(p) \) cease to exist. For \( \theta > \theta_c(N_f) \), only the nodeless solution \( B_+(p) \) exists. This discontinuity at \( \theta = \theta_c(N_f) \) persists for \( B_e = (B_+ - B_-)/2 \) and \( B_o = (B_+ + B_-)/2 \). This discontinuity can be observed as the first order phase transition in the direction of \( \theta \). By using \( S(p) = [A_p - B]^{-1} = [A_+ p - B_+ - B_-]^{-1} \chi_+ + [A_ - p - B_-]^{-1} \chi_- \), the chiral order parameter \( \langle \bar{\psi} \psi \rangle \) is written as

\[
\langle \bar{\psi} \psi \rangle/N_f = \int_0^\Lambda \frac{p^2 dp}{\pi^2} \left[ \frac{B_+(p)}{p^2 + B_+(p)^2} + \frac{B_-(p)}{p^2 + B_-(p)^2} \right].
\]

Therefore such a first order transition can be observed as the discontinuity of the chiral order parameter.

The critical line is obtained as follows. Substituting eq. (74) back into eq. (77), we obtain

\[
\frac{\lambda \theta}{\alpha} = (-1)^{n+1} \frac{2}{3} \sin(\delta_\pm(\theta)) \frac{K_\pm^{1/2}}{\omega} C(\lambda, \theta)^{-1} e^{\pm \frac{\delta_\pm(\theta)}{\omega}} \exp \left[ -\frac{3n\pi - 3\phi_\pm}{\omega} \right].
\]

Up to \( O(\theta) \), the critical line in the phase diagram \((N_f, \theta)\) is given by

\[
\frac{\lambda \theta}{\alpha} \sim \exp \left[ -\frac{3n\pi}{\omega} \right].
\]

Above this critical line in the phase diagram, the chiral symmetry restores. This shows that the CS term has the effect to decrease the critical number of flavors, \( N_f^c(\theta = 0) \). However the chiral-symmetry restoring transition is the first order phase transition.

At typical points in the phase diagram, \((\theta, N_f)\), which is drawn schematically in Fig.5, we find \((M_e = (M_+ + M_-)/2, M_o = (M_+ - M_-)/2)\)

\[
A: \quad N_f > N_f^c, \theta = 0; M_e = 0, M_o = 0, M_+ = 0, M_- = 0,
B: \quad N_f < N_f^c, \theta = 0; M_e > 0, M_o = 0, M_+ = M_- > 0,
C: \quad N_f < N_f^c, \theta \neq 0; M_e > 0, M_o > 0, M_+ > M_- > 0,
D: \quad N_f < N_f^c, \theta = \theta_c(N_f); (first order transition)
E: \quad N_f > N_f^c, \theta \neq 0; M_e = 0, M_o = 2M_+ > 0, M_+ = -M_- > 0,
\]
where the points A, E are in the chiral symmetric phase and B, C, D are in the chiral-symmetry-breaking phase. In the above we have considered the route starting at $\theta = 0$ ($A \to B \to C \to D$) to justify the linearization procedure adopted in this paper, since $B_\pm(p) \equiv 0$ is a trivial solution at $\theta = 0$. It should be remarked that we can adopt another linearization \[27\] by replacing the denominator $q^2 + B^2(q)$ with $q^2 + M^2$ where $M$ determines the normalization of the solution of the linear SD equation, e.g., $M = B(0)$. In \[35\] and the subsequent paper we give the thorough analysis on the choice of linearization procedure which is compatible with the route of approaching the critical point, especially, $A \to E \to D$ starting at $\theta \neq 0$ from the beginning where $B_\pm(p) \equiv 0$ is not a trivial solution.

4 Parity breaking

We consider a problem whether or not the parity is dynamically broken in (2+1)-dimensional QED, in other words, the non-perturbatively induced CS term may or may not break the parity. When $\theta = 0$, we obtain

$$D_T(k) = \frac{k^2 + \alpha \sqrt{k^2}}{(k^2 + \alpha \sqrt{k^2})^2 + \Pi_O(k)^2},$$  \hspace{1cm} (80)

$$D_O(k) = \frac{\Pi_O(k)}{(k^2 + \alpha \sqrt{k^2})^2 + \Pi_O(k)^2}. \hspace{1cm} (81)$$

Here it should be remarked that the parity odd vacuum polarization $\Pi_O(k)$ is at least linearly proportional to $B_o(p)$, since

$$\Pi_O(k) = 64 \frac{\alpha}{\sqrt{k^2}} \int \frac{d^3p}{(2\pi)^3} (k^2 + p \cdot k) \frac{B_o(p)}{(p + k)^2 p^2} + O(B_o^2). \hspace{1cm} (82)$$

Therefore the quantity $\Pi_O^2(k)$ in the denominator of $D_T$ and $D_O$ should be neglected in the linearized SD equation. This is sufficient to study the critical phenomena in the neighborhood of the critical point.

From eq. (21), the linearized SD equation for the odd part $B_o$ reads

$$B_o(p) = m_o + \frac{8\alpha}{N_f} \int \frac{d^3q}{(2\pi)^3} \frac{B_o(q)}{q^2} \left[ 2 + \xi((p - q)^2) \right] D_T(q - p)$$

$$- \frac{16\alpha}{N_f} \int \frac{d^3q}{(2\pi)^3} \frac{q \cdot (q - p)}{q^2 \sqrt{(q - p)^2}} D_O(q - p), \hspace{1cm} (83)$$

where the non-local gauge is given by $\xi(k^2) = \frac{2}{k^2} - \frac{\sqrt{k^2}}{3\alpha} + O(k^2)$. By mimicking the procedure of CCW \[31\], the linearized SD equation can be rewritten as

$$B_o(p) = m_o + \frac{32}{3\pi^2 N_f} \int \Lambda dq K(p, q) B_o(q), \hspace{1cm} (84)$$

15
with the kernel
\[
K(p, q) = \frac{p + q - |p - q|}{2pq} - \frac{9}{8\alpha} - 6\pi \int_{\Lambda}^\Lambda \frac{dk}{k^2} \frac{1}{(k + \alpha)^2} \left[ 1 + \frac{k^2 - p^2}{2pk} \ln \left( \frac{p + k}{|p - k|} \right) \right] \left[ 1 + \frac{k^2 - q^2}{2qk} \ln \left( \frac{q + k}{|q - k|} \right) \right].
\] (85)

This integral equation is too difficult to be solved analytically. However it is quite easy to see that the SD equation can not have the nodeless (ground state) solution with a definite sign, since the kernel \(K(p, q)\) is always negative for any \(p, q \in [0, \Lambda]\) as shown in Fig. 6. This implies that the SD equation has only a trivial solution \(B_0(p) \equiv 0\). Therefore the dynamical breakdown of parity does not occur, which is in agreement with the previous analyses [21, 22, 23, 24].

5 Conclusion and Discussion

In this paper we have reexamined the dynamical breakdown of chirality and parity in \((2+1)\)-dimensional QED. In order to get rid of the ambiguity coming from the vertex correction, we have derived the non-local gauge fixing function which guarantees the absence of the wavefunction renormalization for the fermion. Using the non-local gauge, we have written down explicitly the SD equation for the fermion mass function.

In the analyses done in this paper, we have taken into account the leading and next-to-leading terms of the non-local gauge function \(\xi(k^2)\) in the IR region to solve the SD equation analytically. Within this approximation we have found that there exists the finite critical number of flavors \(N_c^f = 128/3\pi^2\) and that the scaling form of the essential singularity type is obtained in the absence of the CS term. The critical value \(N_c^f\) obtained in this paper reproduces Nash’s result [30] which is slightly larger than the previous one \(32/\pi^2\) in the absence of the CS term \(\theta = 0\).

In the presence of the CS term, it is shown [35] that the phase transition in the direction of \(\theta\) shows the discontinuity at some critical value, \(\theta_c(N_f)\), which depends on \(N_f\). The bare CS term decreases the critical flavour number. The order of chiral transition as well as the critical line obtained in [35] is completely different from that of Hong and Park [33]. In Hong and Park [33] the first order term of \(\theta\) was neglected in the SD equation for the fermion mass function which corresponds to eq. (20) and hence this approximation inevitably forces the parity-odd mass part to decouple completely from the SD equation for the parity-even mass. This careless treatment leads to the totally different conclusion: 1) the critical number of flavors as a function of \(\theta\): \(N_c^f(\theta) = N_c^f(\theta = 0) \frac{1}{1 + (\theta/\alpha)^2}\), which implies the critical line \(\frac{\theta}{\alpha} = \sqrt{N_c^f/N_f - 1}\) where \(N_c^f = N_c^f(\theta = 0)\). 2) the continuous (infinite order) phase transition characterized by the essential singularity on the whole critical line: \(\frac{M}{\Lambda} \sim \exp \left[ \frac{-2n\pi}{\sqrt{N_c^f(\theta)/N_f - 1}} \right]\), as in the absence of CS term. However this treatment is not correct as was noticed in the footnote of the linear approximation in section 2, since the CS term dominates at large momentum and gives the essential contribution to the UVBC which determines the character of the phase transition. This fact was confirmed by the direct numerical calculation of the non-linear SD equation [35] without relying on the specific linearization procedure.
In the same scheme we have shown that the dynamical breakdown of parity does not occur, which agrees with the previous analyses [21, 22, 23, 24] and the general argument [25]. An interpretation of absence of dynamical parity breaking is as follows. The gauge interaction acts as an attractive force to cause the (chiral) condensation between fermion and antifermion, \( \langle \bar{\psi}\psi \rangle \neq 0 \). As can be seen from the analysis in the previous section, however, the induced CS term generates the same effect as the repulsive force and this gets superior to the original attractive force to result in the net repulsive force (minus sign of the kernel) which prevents the condensation, \( \langle \bar{\psi}\tau\psi \rangle = 0 \). Therefore it is necessary to incorporate other attractive interactions so as to induce the parity breakdown \( \langle \bar{\psi}\tau\psi \rangle \neq 0 \), e.g., the four-fermion interaction as done in CCW [32].

We must comment on the limitation of our approach. The explicit form of the non-local gauge we have found in this paper can not be applied to the vertex which is not of the form \( \Gamma_\mu(p,q) = \gamma_\mu F(p,q) \) where \( F(p,q) \) is the functional of \( A \) and has the limit: \( F(p,q) \to 1 \) as \( A(p) \to 1 \). For the given vertex which has the different tensor structure [11], therefore, we must rederive the corresponding non-local gauge, if this is possible. In such a case, we can not say anything about the result at this stage.

In this paper we have truncated the \( 1/N_f \) series up to some finite order and have taken into account the first two terms. To improve our analyses, we should mention the effect of the third term in the non-local gauge:

\[
\xi(k^2) = \frac{2}{3} - \frac{1}{3} \frac{\alpha^2 - \theta^2}{\alpha^2 + \theta^2} \frac{\sqrt{k^2}}{\alpha} + \frac{(\alpha^2 - 5\theta^2)(3\alpha^2 + \theta^2)}{15(\alpha^2 + \theta^2)^2} \frac{k^2}{\alpha^2} + O \left( \left( \frac{k}{\alpha} \right)^3 \right) .
\]

The third term changes the equivalent differential equation and hence the solution of the SD equation may change. It may be expected that the qualitative feature of the dynamical symmetry breaking will be unchanged. If the \( 1/N_f \) series can be summed as pointed out in [11], however, the wavefunction renormalization effect can alter the behavior of the dynamically important low momentum region, which is recognized also in [11]. Moreover we have neglected all the terms of the order \( O(B^2) \) to obtain the closed SD equation for \( B_e \) and \( B_o \). The resulting equation is a linear integral equation where the IR cutoff must be introduced to define the equation in the IR region. This procedure may change the low momentum behavior of the solution, in contrast to the UV region. Anyway it is quite necessary to perform the numerical calculation based on the non-local gauge without the expansion in order to confirm our results.

Finally we mention another problem. Though the inclusion of the bare CS term breaks the parity explicitly, it is an interesting question whether the induced CS term recovers the parity invariance which is explicitly broken at the tree level, \( \theta \neq 0 \). In order to answer this question, we must include both the bare and the induced CS terms, in which case it is difficult to solve the corresponding SD equation analytically. This problem will be discussed in a forthcoming paper.

One of the authors (K.-I. K.) is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No. 05640339), and the International Collaboration Program of the Japan Society for the Promotion of Science. We would like to thank Pieter Maris for having pointed out the incomplete point in the first version of this paper. K.-I. K is grateful to Yoonbai Kim for valuable comments on the effect of the Chern-Simons term.
A Derivation of the non-local gauge

In QED, we introduce the non-local gauge fixing function \( \xi(k^2) \) in the photon propagator:

\[
D_{\mu\nu}(k) = \frac{d(k^2)}{k^2} \left( g_{\mu\nu} - \eta(k^2) \frac{k_\mu k_\nu}{k^2} \right),
\]

where

\[
d(k^2) := k^2 D_T(k), \quad \eta(k^2) := 1 - \xi(k^2).
\]

In the euclidean QED in D-dimensional space, the SD equation for the fermion wave function renormalization \( A \) is given by

\[
p^2 A(p^2) - p^2 = e^2 \int \frac{d^D q}{(2\pi)^D} \frac{A(q^2)}{q^2 A^2(q^2) + B^2(q^2)}
\times d(k^2) \left[ (D - 2) \frac{p \cdot q}{k^2} + \left( \frac{p \cdot q}{k^2} - 2 \frac{p^2 q^2 - (p \cdot q)^2}{k^4} \right) \eta(k^2) \right],
\]

and the SD equation for the fermion mass function \( B \) reads

\[
B(p^2) = e^2 \int \frac{d^D q}{(2\pi)^D} \frac{B(q^2)}{q^2 A^2(q^2) + B^2(q^2)}
\times d(k^2) \left[ (D - \eta(k^2)) k^2 \right].
\]

Separating the angle \( \vartheta \) defined by

\[
k^2 := (q - p)^2 = x + y - 2\sqrt{xy} \cos \vartheta, \quad x := p^2, \quad y := q^2,
\]

we find

\[
p^2 A(p^2) - p^2 = C_D e^2 \int_0^{\Lambda^2} dy \frac{y^{(D-2)/2} A(y)}{y A^2(y) + B^2(y)} \int_0^\pi d\vartheta \sin^{D-2} \vartheta \cos \vartheta \left[ \sqrt{x y} \cos \vartheta (D - 2 + \eta(k^2)) - 2 xy - \frac{(\sqrt{x y} \cos \vartheta)^2}{k^4} \right],
\]

and

\[
B(p^2) = C_D e^2 \int_0^{\Lambda^2} dy \frac{y^{(D-2)/2} B(y)}{y A^2(y) + B^2(y)} \int_0^\pi d\vartheta \sin^{D-1} \vartheta \left[ D - \eta(k^2) \right],
\]

where

\[
C_D := \frac{1}{2D\pi(D+1)/2\Gamma(D-1/2)}.
\]

We perform the angular integration by parts as

\[
\int_0^\pi d\vartheta \sin^{D-2} \vartheta \cos \vartheta \sqrt{x y} f(z) = \frac{\sqrt{x y}}{D - 1} \left[ \sin^{D-1} \vartheta f(z) \right]_0^\pi - \frac{2xy}{D - 1} \int_0^\pi d\vartheta \sin^D \vartheta f'(z),
\]

(95)
Define the angle

\[ z := k^2 = (q - p)^2. \]  

Then we find

\[
p^2 A(p^2) - p^2 = - e^2 C_D \int_0^\Lambda d\nu \frac{y \sqrt{(D-2)/2} A(y)(2xy)}{yA^2(y) + B^2(y)} \int_0^\pi d\theta \sin^D \theta \times \left[ \frac{1}{D-1} \frac{D}{D-2} \frac{A(y)}{A^2(y) + B^2(y)} \int_0^\pi d\theta \sin^D \theta \right].
\]

This is further rewritten as

\[
p^2 A(p^2) - p^2 = - \frac{C_D e^2}{D-1} \int_0^\Lambda d\nu \frac{y \sqrt{(D-2)/2} A(y)(2xy)}{yA^2(y) + B^2(y)} \int_0^\pi d\theta \sin^D \theta \times \frac{1}{D-1} \left\{ \left( z^{D-2} d(z) \eta(z) \right)' - (D - 2) z^{D-3} [d(z) - z d'(z)] \right\},
\]

where the prime denotes the differentiation with respect to \( z \).

The requirement \( A(p^2) \equiv 1 \) is achieved by taking \( \eta(z) \) such that \( (z^{D-2} d(z) \eta(z))' - (D - 2) z^{D-3} [d(z) - z d'(z)] \equiv 0 \). This is simply solved as follows.

\[
\eta(z) = \frac{D - 2}{z^{D-2} d(z)} \int_0^z dt [d(t) - t d'(t)] t^{D-3},
\]

where we have assumed that \( [z^{D-2} d(z) \eta(z)]_{z=0} = 0 \) so as to eliminate the \( 1/z^{D-2} \) singularity in \( \eta(z) \) and \( \xi(z) = 1 - \eta(z) \). This should be checked after having obtained the function \( \eta(z) \).

The gauge-fixing function \( \xi(z) = 1 - \eta(z) \) is simplified as

\[
\xi(z) = D - 1 - \frac{(D - 1)(D - 2)}{D^{D-2} d(z)} \int_0^z dt d(t) t^{D-3},
\]

where we have assumed \( [z^{D-2} d(z)]_{z=0} = 0 \).

For \( D = 3 \), eq. (100) reduces to eq. (103), since \( k^4 D_T(k^2), k^4 D_T(k^2) \xi(k^2) \to 0 \) as \( k^2 \to 0 \).

\[ \text{B Integration formulae} \]

Define the angle \( \vartheta \) from \( k^2 := (q - p)^2 = p^2 + q^2 - 2pq \cos \vartheta \), where \( k = \sqrt{k^2}, p = \sqrt{p^2}, q = \sqrt{q^2} \). Then we obtain the following results.

\[
\int_0^\pi d\theta \sin \frac{1}{k} = \frac{p + q - |p - q|}{pq} = \frac{2}{\max(p, q)},
\]

\[
\int_0^\pi d\theta \sin \frac{\vartheta}{k} = \frac{(p + q)(p^2 - pq + q^2) - |p - q|(p^2 + pq + q^2)}{3p^2 q^2} = \frac{2 \min(p, q)}{3 \max(p^2, q^2)},
\]

\[
\int_0^\pi d\theta \sin \frac{1}{k^2} = \frac{1}{pq} \ln \frac{p + q}{|p - q|},
\]

\[
\int_0^\pi d\theta \sin \frac{\vartheta}{k^2} = -\frac{1}{pq} + \frac{p^2 + q^2}{2p^2 q^2} \ln \frac{p + q}{|p - q|}.
\]
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Figure Captions

Fig.1: Schwinger-Dyson equation for the fermion propagator.

Fig.2: Non-local gauge function $\xi(s)$ for $s := k^2$ (when $\theta = 0$). The lines from above to below correspond to $\alpha/\Lambda = 1, 10^{-1}, 10^{-2}, 10^{-4}$.

Fig.3: Plot of the function $f(p)$ in eq. (45), apart from the factor of the integral.

Fig.4: Schematic plot of the function $H_\pm(t)$ eq. (73) and $\sin t$. The solution $M_\pm$ is obtained from the intersection point of the curve $H_\pm(t)$ with $\sin t$.

Fig.5: Phase diagram (schematic).

Fig.6: Integral kernel $K(p, q)$ as a function of $p$ and $q$ where $\Lambda = \alpha$. 


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