Convergence of Optimistic Gradient Descent Ascent in Bilinear Games

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Abstract

We study the convergence of Optimistic Gradient Descent Ascent in unconstrained bilinear games. In a first part, we consider the general zero-sum case and extend previous results by Daskalakis et al. [4], Liang and Stokes [14] and others: we prove for the first time the exponential convergence of OGDA to a saddle point, for any payoff matrix. We also provide a new, optimal, geometric ratio for the convergence, and in case of multiple Nash equilibria we characterize the limit equilibrium as a function of the initialization. In a second part, we introduce OGDA for general-sum bilinear games: we provide sufficient conditions for the convergence of OGDA to a Nash equilibrium, and show that in an interesting class of games, including common payoffs games and zero-sum games, either OGDA converges exponentially fast to a Nash equilibrium, or the payoffs of both players converge exponentially fast to $+\infty$ (which might be interpreted as the endogenous emergence of coordination, or cooperation, among players). We also present a very simple trick, the Double OGDA, ensuring convergence to a Nash equilibrium in any general-sum bilinear game. We finally illustrate our results on an example of Generative Adversarial Networks.

1 Introduction

Min-max optimization is receiving a lot of attention, due in particular to the popularity of generative adversarial networks (GANs, introduced in [10]) and adversarial training (see for instance [15]). In the original version of GANs, two neural networks are in competition: a generator that aims at generating data as close as possible to potential true data, and a discriminator whose goal is to tell between generated and true data. Because the discriminator wants to maximize a loss that the generator wants to minimize, GANs can be seen as a min-max, or zero-sum game, problem.

Formally, consider a “payoff” function $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and assume we want to find $x^*$ achieving $\min_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} g(x, y)$. Assuming the zero-sum game associated to $g$ has a Nash equilibrium (i.e., a saddle-point), the
game-theoretic approach is to find an equilibrium, i.e. \((x^*, y^*)\) in \(\mathcal{X} \times \mathcal{Y}\) such that:
\[
\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, g(x, y^*) \leq g(x^*, y) \leq g(x^*, y^*). \tag{1}
\]

A natural algorithm to search for such an equilibrium is the Gradient Descent-Ascent algorithm (GDA): given a small parameter \(\eta > 0\) and a starting point \((x_0, y_0)\), define inductively the sequence:
\[
\forall t \geq 0, \begin{cases}
x_{t+1} = x_t + \eta \frac{\partial g}{\partial x}(x_t, y_t), \\
y_{t+1} = y_t - \eta \frac{\partial g}{\partial y}(x_t, y_t).
\end{cases}
\]

Most GANs currently use Gradient Descent Ascent (GDA) under the hood. In some cases, GDA was acknowledged to converge in average to the solution of the min-max problem in several zero-sum games [17]. However, it is well known that even for bilinear functions \(g(x, y) = x^T Ay\) with \(\mathcal{X} = \mathcal{Y} = \mathbb{R}^n\), GDA may exhibit a cyclic behavior and the last iterate may not converge.

A nice variant of the GDA is the Optimistic Gradient Descent-Ascent algorithm (OGDA):
\[
\forall t \geq 0, \begin{cases}
x_{t+1} = x_t + 2\eta \frac{\partial g}{\partial x}(x_t, y_t) - \eta \frac{\partial g}{\partial x}(x_{t-1}, y_{t-1}), \\
y_{t+1} = y_t - 2\eta \frac{\partial g}{\partial y}(x_t, y_t) + \eta \frac{\partial g}{\partial y}(x_{t-1}, y_{t-1}).
\end{cases} \tag{2}
\]

This optimistic variant has been introduced by Popov in [19] and rediscovered in the GANs literature ([4] for example). A challenge is then to determine when OGDA converges to an equilibrium of the game.

In this paper, we focus on the bilinear unconstrained case, where \(\mathcal{X} = \mathbb{R}^n\), \(\mathcal{Y} = \mathbb{R}^p\) and payoffs are bilinear. We study here the zero-sum case as in (1), as well as the general sum-case: given matrices \(A\) and \(B\) in \(\mathbb{R}^{n \times p}\), does OGDA lead to a Nash equilibrium of the corresponding game, i.e. to a point \((x^*, y^*)\) such that
\[
\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, x^T Ay^* \leq x^* T Ay^* \text{ and } x^* T By^* \leq x^T By^*. \tag{3}
\]

We first consider in section 2 the zero-sum case where \(g(x, y) = x^T Ay\) for some matrix \(A \in \mathbb{R}^{n \times p}\). The difficulty is not to find a saddle-point of \(g\) (this is fairly easy: take \(x^* = y^* = 0\)) but to prove or disprove the convergence of OGDA to a saddle-point in this simple setup. In this context, Daskalakis et al. [4] proved the last iterate convergence of OGDA, in the sense that for each \(\varepsilon > 0\) one can find \(\eta > 0\) and \(T\) large enough such that \((x_T, y_T)\) (as in (2)) is \(\varepsilon\)-close to a Nash equilibrium. Liang and Stokes [14], as well as Mokthari et al. [16] and Peng et al. [18] proved that in the case of a square invertible matrix \(A\), OGDA converges to \((0, 0)\) at exponential speed.

We prove here in Theorem 2.5 a stronger result: for any matrix \(A\) and any
initial conditions, OGDA does converge, and the limit is a Nash equilibrium. Moreover, the speed of convergence is exponential, and we exhibit the optimal ratio for the convergence bound, strictly improving the previous bounds obtained for square invertible matrices. We also characterize the limit Nash equilibrium as a function of the initialization, the matrix $A$ and the stepsize $\eta$. We believe these results matter because they show a strong stability of the OGDA algorithm for any matrix: not only OGDA can be used to find $\varepsilon$-equilibria, but they will not exhibit cyclic behaviors within the (possibly large) set of Nash equilibria. Moreover as soon as the stepsize $\eta$ is below some threshold, a fixed $\eta$ is enough to guarantee convergence, and the larger $\eta$ below this threshold, the better the geometric ratio of convergence. The proof is based on a precise spectral analysis of the linear system with variable $(x_{t+1}, y_{t+1}, x_t, y_t)$, one difficulty being to control the angles between eigenspaces in order to obtain proper rates of convergence. In section 3, Theorem 3.1, we extend the results of section 2 to any bilinear payoff function $g$, i.e. we consider the cases where $g(x,y) = x^T Ay + b^T x + c^T y + d$.

We then consider OGDA for general-sum bilinear games. We stick here to the case where $X$ and $Y$ are Euclidean spaces, and in section 4, we introduce the natural extension of OGDA for bilinear general-sum games. It is easy to see that OGDA need not always converge to a Nash equilibrium (3) of the game here. We give in Theorem 4.4 sufficient conditions for convergence of OGDA to a Nash equilibrium, including the case where all eigenvalues of $B^T A$ are real negative. Next in Theorem 4.8 we introduce a very simple variant of OGDA, called DOGDA (Double Optimistic Gradient Descent Ascent), which always converges to a Nash equilibrium in a bilinear general-sum game $(x^T Ay, x^T By)$. The "DOGDA trick" is simply based on a strong specificity of the games considered, namely that $(x^*, y^*)$ is a Nash equilibrium of the game if and only if $(x^*$ is an optimal strategy for player 1 in the zero-sum game with matrix $-B$ and $y^*$ is an optimal strategy for player 2 in the zero-sum game with matrix $A$).

The last part of section 4 considers OGDA for a particular class of payoff matrices, including in particular the case of common payoff games where $A = B$ (implying that there is a potential for coordination or cooperation among players), or more generally the cases where $B = \alpha A$ for some $\alpha \neq 0$. It is proved that for this class of games, either OGDA converges to a Nash equilibrium, or under OGDA the payoff of both players tends to $+\infty$ exponentially fast as $t \to +\infty$. We believe this result is a meaningful extension of the convergence of OGDA, since both players having $+\infty$ payoffs can be seen as a generalized Nash equilibrium of the game and a very desirable outcome of the interaction.

In section 5 we recall the theoretical GANs and WGANs setups, and present an application to Wasserstein GANs borrowed from Daskalakis’ pa-
per [4]. We also discuss limitations of the bilinear model with respect to real GANs applications. We finally conclude and present a few possible directions for future research.

Other Related Works:
Besides [4], [14], [16] and [18], Zhang and Yu [22] also consider convergence of gradient methods in bilinear zero-sum games. Gidel et al. [9] study GANs through variational inequalities, with a convergence ratio not better than in [14].

In [20], Rakhlin and Sridharan introduced an algorithm called Optimistic Mirror Descent (OMD), based on the Mirror Descent, and showed that it is a no-regret algorithm. Optimistic Gradient Descent Ascent (OGDA) is a particular case of OMD, for which the regularizer used is the 2-norm. Another algorithm close to OGDA is the Extra-Gradient method (EG) introduced by Korpelevich in [13]. Both OGDA and EG can be seen as approximations of the Proximal Point (PP) method. In [16], Mokhtari et al. showed again the exponential convergence of PP, OGDA, and of EG, for square invertible bilinear games, for a single particular value of $\eta$, and with a convergence bound of smaller than ours. Another article studying Extra-Gradient, including a stochastic setting, is the one of Hsieh et al. [11].

Beyond bilinear games, some papers consider OGDA for concave/convex games. In particular, Daskalakis et al. [6] study the stability of fixed points of the dynamics. Non-concave/non-convex setting exhibit problematic properties: [12] shows there may exist attractors of OGDA containing no stationary points, and [7] shows that finding an approximate local minmax equilibrium is PPAD-hard. However, in [8], Diakonikolas et al. showed positive results in a new class of non concave/non convex minmax problems.

Many studies where also done in a compact settings, that is when $\mathcal{X}$ and $\mathcal{Y}$ are compact sets. In [5], Daskalakis et al. proved the convergence of an Optimistic version of Multiplicative Weight Updates on the simplex, for an appropriate value of the learning rate. Later, Wei et al. improved their results in [21], by showing the convergence for any learning rate smaller than some constant.

In a recent paper [1], Anagnostides et al. considers the convergence of a projected version of OGDA for general-sum games, where after each iteration, a projection on the simplex is applied.

Notations 1.1. We use the following standards notations. $A^T$ denotes the transpose of a matrix $A$ in $\mathbb{C}^{n \times p}$. If $A$ is square, $\text{Sp}(A)$ is the set of complex eigenvalues of $A$, and $\rho(A) = \max\{ |\lambda|, \lambda \in \text{Sp}(A) \}$ is the spectral radius of $A$. Given column vectors $x$, $x'$ in $\mathbb{R}^n$ or more generally in $\mathbb{C}^n$, we use the Hermitian scalar product $\langle x, x' \rangle = x^T x'$ where $\overline{x'}$ is the complex conjugate.
of $x'$, and the Euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$. $I_n$ denotes the identity matrix in $\mathbb{R}^{n \times n}$. To simplify, we use $I$ when the dimension is $2(n + p)$.

2 Zero-Sum Games $x^T Ay$

2.1 Setup

Let $A$ be a payoff matrix in $\mathbb{R}^{n \times p}$. We consider the zero-sum game where simultaneously player 1 chooses $x$ in $\mathbb{R}^n$, player 2 chooses $y$ in $\mathbb{R}^p$, and finally player 2 pays $x^T Ay$ to player 1. Here $x$ and $y$ are seen as column vectors. This game has a value which is 0, and both players have optimal strategies:

$$\max_{x \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^p} x^T Ay = \min_{y \in \mathbb{R}^p} \sup_{x \in \mathbb{R}^n} x^T Ay = 0.$$ 

The set of optimal strategies of player 1 is

$$\{ x \in \mathbb{R}^n, x^T A = 0 \} = \text{Ker}(A^T),$$

and the set of optimal strategies of player 2 is

$$\{ y \in \mathbb{R}^p, Ay = 0 \} = \text{Ker}(A).$$

So $(x, y)$ is a Nash equilibrium (or saddle-point) of the game if and only if $x \in \text{Ker}(A^T)$ and $y \in \text{Ker}(A)$.

Definition 2.1. The Gradient Descent Ascent is defined by:

$$\forall t \geq 0, \begin{cases} x_{t+1} = x_t + \eta Ay_t, \\ y_{t+1} = y_t - \eta A^T x_t, \end{cases}$$

where $\eta > 0$ is a fixed parameter (the gradient step), and $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^p$ is the initialization.

Notice that the fixed points of the GDA are exactly the Nash equilibria of the game. It is well known that the GDA does not converge in general, as one can see in the simple “Matching Pennies” case where $n = p = 1$ and $A = (1)$, so that $x^T Ay$ is simply $xy$. In this case, we obtain:

$$\begin{cases} x_{t+1} = x_t + \eta y_t, \\ y_{t+1} = y_t - \eta x_t, \end{cases}$$

which implies $x_{t+1}^2 + y_{t+1}^2 = (x_t^2 + y_t^2)(1 + \eta^2)$, and $||x_t, y_t||_{t \to \infty} \to +\infty$.

To overcome this problem, [20] (see also [6]) introduced “optimism” to GDA.

Definition 2.2. The Optimistic Gradient Descent Ascent (OGDA) is defined by:

$$\forall t \geq 0, \begin{cases} x_{t+1} = x_t + 2\eta Ay_t - \eta Ay_{t-1}, \\ y_{t+1} = y_t - 2\eta A^T x_t + \eta A^T x_{t-1}, \end{cases}$$

where $\eta > 0$ is a fixed parameter, and $(x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$ is the initialization.
The term of time $t-1$ is here to thwart the divergence of GDA, by bringing back toward the center the iterates at time $t+1$. We can see empirically the importance of this “optimistic” addition in figure 1 for the matching pennies problem where $A = (1)$.

![Gradient Descent Ascent on Matching Pennies](image1)

![Optimistic Gradient Descent Ascent on Matching Pennies](image2)

**Figure 1:** Comparison of GDA and OGDA for Matching Pennies

We will see our optimistic GDA as a dynamical system in $E := \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$.

**Definition 2.3.** Define the matrix

$$
\Lambda = \begin{pmatrix}
I_n & 2\eta A & 0 & -\eta A \\
-2\eta A^T & I_p & \eta A^T & 0 \\
I_n & 0 & 0 & 0 \\
0 & I_p & 0 & 0
\end{pmatrix} \in \mathbb{R}^{(n+p+n+p) \times (n+p+n+p)},
$$

and for $t \geq 0$, let $Z_t$ be the column vector $Z_t = \begin{pmatrix} x_t \\ y_t \\ x_{t-1} \\ y_{t-1} \end{pmatrix} \in E$.

The matrix $\Lambda$ has already been introduced in [18]. We now have:

$$Z_0 \in E \text{ and } \forall t \geq 0, Z_{t+1} = \Lambda Z_t. \quad (4)$$

### 2.2 Convergence of OGDA for a zero-sum game $x^T A y$

The convergence of OGDA when $A = (1)$ is well known and can be shown as follows. We have

$$
\Lambda = \begin{pmatrix}
1 & 2\eta & 0 & -\eta \\
-2\eta & 1 & \eta & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
$$

6
with characteristic polynomial \( \det(\Lambda - \lambda I) = \lambda^2(1 - \lambda)^2 + \eta^2(1 - 2\lambda)^2 \).
Assuming \( \eta < 1/2 \), easy computations show that the spectral radius of \( \Lambda \) is strictly smaller than 1. Then \( \Lambda^t \xrightarrow{t \to \infty} 0 \) by Gelfand’s theorem, so \( (x_t, y_t) \xrightarrow{t \to \infty} (0, 0) \).

We now generalize this linear algebra approach to any matrix \( A \), focusing not only on the convergence property but also on the characterization of the limit and on the speed of convergence.

For this, we first need the following definition:

**Definition 2.4.** For two matrices \( A \) and \( B \), we denote by \( S(A, B) \) the set \( \text{Sp}(B^T A) \cup \text{Sp}(AB^T) \). To simplify, we use the notation \( S(A) \) to denote \( S(A, A) \).

**Theorem 2.5.** Let \( A \in \mathbb{R}^{n \times p} \) and \( \eta \in (0, \frac{1}{\sqrt{\mu_{\max}}} \), where \( \mu_{\max} = \rho(A^T A) = \rho(A A^T) \) is the largest eigenvalue of \( A^T A \). Given \( (x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p \), we consider the Optimistic Gradient Descent:

\[
\begin{align*}
  x_{t+1} &= x_t + 2\eta A y_t - \eta A y_{t-1}, \\
  y_{t+1} &= y_t - 2\eta A^T x_t + \eta A^T x_{t-1}.
\end{align*}
\]

Then we have:
1) \((x_t, y_t)_t\) converges to a Nash equilibrium.
2) The limit vector \((x_\infty, y_\infty)\) is such that \(x_\infty\) is the orthogonal projection of \(x_0\) onto \(\text{Ker}(A^T)\) and \(y_\infty\) is the orthogonal projection of \(y_0\) onto \(\text{Ker}(A)\).
3) And the convergence is exponential: for all \(t \geq 0\),

\[
\| (x_t, y_t) - (x_\infty, y_\infty) \| \leq C D \lambda_{\max}^t,
\]

with \( C = \sqrt{\frac{2}{1 - \sqrt{1 + 5\eta^2 \mu_{\max}^2}}} \), \( D \) is the distance from the initialization \(Z_0\) to the set \(\text{Ker}(\Lambda - I)\), and

\[
\lambda_{\max} = \max \left\{ \sqrt{\frac{1}{2} (1 + \sqrt{1 - 4\eta^2 \mu})}, \mu > 0, \mu \in S(A) \right\} = \max\{ |\lambda|, \lambda \in \text{Sp}(\Lambda), \lambda \neq 1 \} < 1, \text{ with the convention } \lambda_{\max} = 0 \text{ if } A = 0.
\]

The proof is in the Appendix. The most technical part is the control of the angles between eigenvectors of the matrix \( \Lambda \), to obtain a fast convergence speed.
2.3 Comments on Theorem 2.5

Several remarks should be made on this theorem:

a) Ker($\Lambda - I$) is the set \[
\begin{pmatrix}
x \\
y \\
x \\
y
\end{pmatrix}, (x, y) \text{ Nash equilibrium}
\] Since it contains 0, $D \leq \| (x_0, y_0, x_{-1}, y_{-1}) \| = \| Z_0 \|.$

b) Notice that the speed of convergence is independent of the dimension: $C, D$ and $\lambda_{\text{max}}$ do not depend on $n$ nor $p$.

c) The step-size $\eta$ is fixed here, and does not evolve with time. Given that it is strictly below $\frac{1}{2\sqrt{\mu_{\text{max}}}}$, the ratio of convergence $\lambda_{\text{max}}$ is better when $\eta$ is large, so that having $\eta$ too small is not a good option for large $t$.

d) In the particular case where Ker($A$) = \{0\}, one can show that convergence to 0 holds as soon as $0 < \eta < \frac{1}{\sqrt{3\sqrt{\mu_{\text{max}}}}}$. Here all eigenvalues of $\Lambda$ have modulus strictly less than 1, so $\Lambda^t \rightarrow 0$ by Gelfand’s formula.

e) When looking at the expression of the limit $(x_\infty, y_\infty)$ with regard to the initialization $(x_0, y_0, x_{-1}, y_{-1})$, one should notice that $x_\infty$ depends only on $x_0$, and that $y_\infty$ depends only on $y_0$. Thus, $y_0$, $y_{-1}$ and $x_{-1}$ will only have an influence of the curve $(x_t)_t$, but not on its limit.

f) It is possible to show that our ratio $\lambda_{\text{max}}$ is optimal. Take any matrix $A \neq 0$, consider $\mu_{\text{min}}$ the smallest positive eigenvalue of $A^T A$ and $\lambda = \frac{1}{2}(1 + \sqrt{1 - 4\eta^2 \mu_{\text{min}} + 2i\eta \sqrt{\mu_{\text{min}}}})$. Lemmas A.2 and A.3 will imply that $\lambda$ is an eigenvalue of $\Lambda$ with modulus $\lambda_{\text{max}} = \sqrt{\frac{1}{2}(1 + \sqrt{1 - 4\eta^2 \mu_{\text{min}}})} < 1$. Consider $Z_0 \neq 0$ such that $\Lambda Z_0 = \lambda Z_0$, we have for all $t$, $Z_t = \lambda^t Z_0 \rightarrow 0$ as $t \rightarrow \infty$. So for all $t \geq 0$, 
\[
\| (x_t, y_t, x_{t-1}, y_{t-1}) - 0 \| = \lambda_{\text{max}}^t \| Z_0 \|.
\] This proves that our result is better than the previous ones. We will still do a tour of the best paper so far on the subject, to see how we compare with them, and what we improved with Theorem 2.5. This tour is summarized in Table 1.

g) [4] was the first paper to study last iterate convergence of OGDA in saddle point problems.
They consider the case of a matrix $A$ with $\mu_{\text{max}} \leq 1$ and $0 < \eta < \frac{1}{4} \mu_{\text{min}}$, where $\mu_{\text{min}} > 0$ is the smallest positive eigenvalue of $A^T A$. Assuming $x_{-1} = x_0$ and $y_{-1} = y_0$, and introducing $\Delta_t = \|A^T x_t\|^2 + \|Ay_t\|^2$, they show that for $t \geq 2$,

$$
\Delta_t \leq (1 - \eta^2 \mu_{\text{min}}^2) \Delta_{t-1} + 16\eta^3 \Delta_0,
$$

This does not prove convergence of $(\Delta_t)_t$ to 0 but shows that $\Delta_t$ is small for large $t$ and small $\eta$.

Let us fix an $\epsilon > 0$, and let $\eta$ be at the utmost of order $\epsilon^2$. Denote by $T$ the number of epochs needed for $\|(x_t, y_t)\|$ to be smaller than $\epsilon$. Then:

$$
\frac{16 \epsilon^2 \Delta_0}{\mu_{\text{min}}^2} + (1 - \epsilon^4 \mu_{\text{min}}^{-2})^{-2} (\Delta_2 - \frac{16 \epsilon^2 \Delta_0}{\mu_{\text{min}}^2}) \leq \epsilon \iff T \geq \alpha \epsilon^{-4} \log \left( \frac{1}{\epsilon} \right)
$$

for some constant $\alpha$. Due to the term in $\epsilon^{-4}$, they have a polynomial convergence, which is worth than the exponential convergence that we (in 2.5), and others, proved.

h) Let us now compare our result to the more recent work [14] of Liang and Stokes. Theorem 2.5 holds for any initialization $(x_0, y_0, x_{-1}, y_{-1})$, for any matrix, and for any small enough value of $\eta$, and not only for a particular value of $\eta$ and only for squared invertible matrices, as in the paper of Liang and Stokes. Consider for instance the particular case where $\eta^2 \mu_{\text{max}} = 1/8$, studied by Liang and Stokes [14] for the particular case of $A$ square invertible. We still denote by $\mu_{\text{min}} > 0$ the smallest eigenvalue of $A^T A$. Liang and Stokes obtained:

$$
\|(x_t, y_t) - (0, 0)\| \leq 4 \sqrt{2} \lambda_L^t \max\{\|(x_0, y_0)\|, \|(x_1, y_1)\|\},
$$

with

$$
\lambda_L = \exp\left(-\frac{\mu_{\text{min}}}{16 \mu_{\text{max}}} \right).
$$

| Reference | Convergence Speed | Exponential Parameter | Constant | Learning Rate |
|-----------|-------------------|-----------------------|----------|---------------|
| [4]       | Polynomial        | $\times$              | $\times$ | $\eta < \mu_{\text{min}}^2$ |
| [14]      | Exponential       | $\exp(-\eta^2 \mu_{\text{min}})$ | 4        | $\eta = \frac{1}{\sqrt{8} \mu_{\text{max}}}$ |
| [18]      | Exponential       | $\sqrt{\frac{1}{2} \left(1 + \sqrt{1 - \eta^2 \mu_{\text{min}}}\right)}$ | Unbounded | $\eta \leq \frac{1}{2 \sqrt{\mu_{\text{max}}}}$ |
| [16]      | Exponential       | $\sqrt{\frac{1}{2} \left(1 - \mu_{\text{min}} \eta^2\right)}$ | 1        | $\eta = \frac{1}{\sqrt{4 \mu_{\text{max}}}}$ |
| This paper | Exponential       | $\sqrt{\frac{1}{2} \left(1 + \sqrt{1 - \frac{4 \eta^2 \mu_{\text{min}}}{\mu_{\text{max}}}\right)}$ | $C = \sqrt{\frac{2}{1 - \frac{1 + 5 \eta^2 \mu_{\text{max}}}{2 + \eta^2 \mu_{\text{max}}}}$ | $\eta \leq \frac{1}{2 \sqrt{\mu_{\text{max}}}}$ |

Table 1: Comparison table of the convergence rate
Since \( \|(x_0, y_0, x_{-1}, y_{-1})\| \leq \sqrt{2} \max\{\|(x_0, y_0)\|, \|(x_{-1}, y_{-1})\|\} \), the constants in the above inequality (8) is just slightly bigger than the constant \( C_\eta \) we find for \( \eta = \frac{1}{\sqrt{8} \mu_{\text{max}}} \).

Moreover, by comparing the derivative of the functions \( f(x) = \sqrt{\frac{1}{2}(1 + \sqrt{1 - x/2})} \) and \( g(x) = \exp\left(-\frac{x}{16}\right) \), we can show that \( \lambda_{\text{max}} = f\left(\frac{\mu_{\text{min}} \mu_{\text{max}}}{\mu_{\text{max}}}\right) \leq \lambda_L = g\left(\frac{\mu_{\text{min}} \mu_{\text{max}}}{\mu_{\text{max}}}\right) \) for all \( \frac{\mu_{\text{min}}}{\mu_{\text{max}}} \in [0, 1] \), showing that moreover than generalizing the convergence results of Liang and Stokes to a larger class of matrices, we also improved their bound.

i) Another exponential bound can be seen in [18]. They showed that for any learning rate \( \eta \),

\[
\|(x_t, y_t) - (x_\infty, y_\infty)\| \leq C_P \lambda_P^T
\]

where \( \lambda_P = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \mu_{\text{min}}^2 \eta^2}} \). Their value of \( \lambda_P \) is still higher than \( \lambda_{\text{max}} \) for any value of \( \eta \) and of \( A \), and they have no control on their constant \( C_P \).

j) In their paper [16], Mokhtari et al. showed once again the convergence of OGDA for square invertible matrices and for a particular value of \( \eta \). They showed their bound for \( \eta \) such that \( \eta^2 \mu_{\text{max}} = 40 \). The bound they finally found is

\[
\|(x_t, y_t) - (x_\infty, y_\infty)\| \leq \lambda_M^T \hat{r}_0
\]

where \( \lambda_M = \sqrt{1 - \frac{\mu_{\text{min}}}{6400 \mu_{\text{max}}}} \). Once again, it can be shown that for any value of \( \frac{\mu_{\text{min}}}{\mu_{\text{max}}} \), \( \lambda_{\text{max}} \leq \lambda_M \).

k) Here, we finally illustrate a part of Theorem 2.5: the expression of the limit value of OGDA with respect to the initialization. To show this, we plotted several run of OGDA on the game with matrix \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \).

They can be seen on Figure 2 below. On the left image, we plotted the first coordinate of \( x_t \) with the first coordinate of \( y_t \), and on the right image, we plotted the second coordinate of \( x_t \) with the second coordinate of \( y_t \). As was proved in Theorem 2.5, we can see that in each case, OGDA converges to the orthogonal projection of the initialization vector \( (x_0, y_0) \) onto \( \text{Ker}(A^T) \times \text{Ker}(A) \).

Each of the target point \( (x_\infty, y_\infty) \) verifies \( x_{\infty,1} = x_{\infty,2} \) and \( y_{\infty,1} = y_{\infty,2} \), because \( x_\infty \) and \( y_\infty \) are in the kernel of \( A \). This can be found back in the plots: the targets, designated by the cross, are at the same place on the left and on the right figure.
Let us now consider a more general case, where we add to the bilinear term $x^T Ay$, a linear function in $(x,y)$. Fix $A \in \mathbb{R}^{n \times p}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^p$ and $d \in \mathbb{R}$. We consider the zero-sum game where simultaneously player 1 chooses $x$ in $\mathbb{R}^n$, player 2 chooses $y$ in $\mathbb{R}^p$, and finally player 2 pays the quantity $g(x, y) = x^T Ay + b^T x + c^T y + d$ to player 1. This allows us to model any general bilinear functions. The Nash equilibria are the vectors $(x,y)$ such that $g(x, y) = \sup_{y' \in \mathbb{R}^p} g(x, y') = \inf_{x' \in \mathbb{R}^n} g(x', y)$, i.e. such that $A^T x + c = 0$ and $Ay + b = 0$. Here, there exists a Nash equilibrium if and only if $-b$ is in the range of $A$ and $-c$ is in the range of $A^T$.

The next result is the extension of Theorem 2.5 to this setup. The proof is in the Appendix.

**Theorem 3.1.** Let $A \in \mathbb{R}^{n \times p}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}$ and $\eta$ be such that $0 < \eta < \frac{1}{2 \sqrt{\mu_{\text{max}}}}$, where $\mu_{\text{max}} = \rho(A^T A)$ is the largest eigenvalue of $A^T A$. Given $(x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$, we consider the Optimistic Gradient Descent Ascent:

$$
\forall t \geq 0, \quad \left\{ \begin{array}{l}
x_{t+1} = x_t + 2\eta(Ay_t + b) - \eta(Ay_{t-1} + b), \\
y_{t+1} = y_t - 2\eta(A^T x_t + c) + \eta(A^T x_{t-1} + c).
\end{array} \right.
$$

If the set $\{(x, y)| A^T x + c = 0, Ay + b = 0\}$ of Nash equilibria is empty, then $(x_t, y_t)_t$ diverges.

If the set of Nash equilibria is not empty, $(x_t, y_t)$ converges to a Nash equilibrium $(x_\infty, y_\infty)$. The limit vector $(x_\infty, y_\infty)$ verifies that $x_\infty$ is the...
orthogonal projection of $x_0$ onto the affine subspace \( \{ x \in \mathbb{R}^n | A^T x + c = 0 \} \) and that $y_\infty$ is the projection of $y_0$ onto \( \{ y \in \mathbb{R}^p | Ay + b = 0 \} \).

Moreover, the convergence is exponential: for all $t \geq 0$,
\[
\| (x_t, y_t) - (x_\infty, y_\infty) \| \leq C D \lambda_\text{max}^t,
\]
with $C = \sqrt{\frac{2}{1 - \sqrt{1 + \eta^2 \mu_{\text{max}}^2}}}$, $D$ is the distance from \((x_0, y_0, x_{-1}, y_{-1})\) to the set \( \{ (x, y, x, y) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p, A^T x + c = 0, Ay + b = 0 \} \), and
\[
\lambda_\text{max} = \max \left\{ \sqrt{\frac{1}{2} (1 + \sqrt{1 - 4 \eta^2 \mu})}, \mu > 0, \mu \in S(A) \right\}
\]
with the convention $\lambda_\text{max} = 0$ if $A = 0$.

## 4 General-sum games \((x^T Ay, x^T By)\)

We consider here the general case of a non zero-sum game given by two matrices $A$ and $B$ in $\mathbb{R}^{n \times p}$. Simultaneously player 1 chooses $x$ in $\mathbb{R}^n$ and player 2 chooses $y$ in $\mathbb{R}^p$, then the payoff to player 1 is $x^T Ay$ and the payoff to player 2 is $x^T By$. By definition, \((x, y) \in \mathbb{R}^n \times \mathbb{R}^p\) is a Nash equilibrium of the game if:
\[
x^T Ay = \max_{x' \in \mathbb{R}^n} x'^T Ay \quad \text{and} \quad x^T By = \max_{y' \in \mathbb{R}^p} x^T By'.
\]
The set of Nash equilibria is $\text{Ker}(B^T) \times \text{Ker}(A)$, which always contains $(0, 0)$.

We introduce the OGDA for general-sum games:

**Definition 4.1.** The Optimistic Gradient Descent Ascent algorithm for the general-sum game \((x^T Ay, x^T By)\) is defined by:
\[
\forall t \geq 0, \quad \left\{ \begin{array}{l}
x_{t+1} = x_t + 2\eta Ay_t - \eta Ay_{t-1}, \\
y_{t+1} = y_t + 2\eta B^T x_t - \eta B^T x_{t-1},
\end{array} \right. \tag{9}
\]
where $\eta > 0$ is a fixed parameter, and $(x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$ is the initialization.

Notice that if $(x_t, y_t)_t$ converges, the limit is a Nash equilibrium.

### 4.1 Non convergence of OGDA for general-sum games

We will see in this section that there exist some general-sum games for which OGDA does not converge to a Nash equilibrium.
A simple case is when $A = B = (1)$. Assume for simplicity that $x_0 = y_0$ and $x_{-1} = y_{-1}$. Then for all $t$, $x_t = y_t$ and

$$x_{t+1} = x_t(1 + 2\eta) - \eta x_{t-1}.$$ 

This can be written $x_{t+1} - x_t = \eta x_t + \eta(x_t - x_{t-1})$, so that if $x_1 \geq \max\{x_0, 0\}$ then by induction $x_{t+1} \geq \max\{x_t, 0\}$ for all $t$. It implies $x_{t+1} \geq x_t(1 + \eta)$ for all $t$, so $x_t \xrightarrow{t \to \infty} +\infty$. We immediately obtain:

**Proposition 4.2.** Convergence of OGDA may fail for general-sum games.

In spite of this proposition, one can slightly extend the analysis of section 2 to the non zero-sum case. Define here the matrix:

$$\Lambda_{A,B} = \begin{pmatrix}
I_n & 2\eta A & 0 & -\eta A \\
2\eta B^T & I_p & -\eta B^T & 0 \\
I_n & 0 & 0 & 0 \\
0 & I_p & 0 & 0
\end{pmatrix} \in \mathbb{R}^{(n+p+n+p) \times (n+p+n+p)}.
$$

Lemma A.3 easily extends as follows. Let $S^*(\mu)$ stands for the set $\{\lambda \in \mathbb{C}, \lambda^2(1 - \lambda)^2 = \mu \eta^2(1 - 2\lambda)^2\}$.

**Proposition 4.3.**

$$\text{Sp}(\Lambda_{A,B}) = \{\lambda|\lambda \in S^*(\mu), \mu \in S(A,B)\}.$$

“Unfortunately”, for each $\mu \notin \mathbb{R}_-$ there exists $\eta_0 > 0$ such that for each $0 < \eta < \eta_0$ there exists $\lambda$ in $S^*(\mu)$ with $|\lambda| > 1$. The next proposition gives sufficient conditions for convergence of OGDA in general-sum games, including the cases where $B = \lambda A$ for some $\lambda \leq 0$.

**Theorem 4.4.** Let $A, B$ be in $\mathbb{R}^{n \times p}$. Assume that $S(A, B) \subset \mathbb{R}_-$, and that $(B^T A$ is invertible or $\Lambda_{A,B}$ is diagonalizable). Then for every initial condition, the Optimistic Gradient Descent Ascent for the general-sum game $(x^T Ay, x^T By)$ converges to a Nash equilibrium of the game.

Let $A, B$ be in $\mathbb{R}^{n \times p}$ with $S(A, B) \subset \mathbb{R}_-$ and $(B^T A$ invertible or $\Lambda_{A,B}$ diagonalizable). Furthermore, assume $\eta \in (0, \frac{1}{\sqrt{\mu_{\text{max}}}})$, where $\mu_{\text{max}} = \rho(B^T A) = \rho(AB^T)$ is the largest eigenvalue of $B^T A$. Given $(x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$ an initialization, we consider the Optimistic Gradient Descent algorithm defined in 9.

Then we have:

1) $(x_t, y_t)$ converges to a Nash equilibrium.

2) The limit vector $(x_\infty, y_\infty)$ is such that \(\begin{pmatrix} x_\infty \\ y_\infty \\ x_\infty \\ y_\infty \end{pmatrix}\) is the linear projection of the initialization \(\begin{pmatrix} x_0 \\ y_0 \\ x_{-1} \\ y_{-1} \end{pmatrix}\) onto $\text{Ker}(\Lambda_{A,B} - I) \oplus \lambda \neq 1 \text{Ker}(\Lambda_{A,B} - \lambda I)$. 

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Proof. Assume that $S(A,B) \subset \mathbb{R}^\ldots$. Then as in Corollary A.4, for $\eta > 0$ small enough we have $\text{Sp}(\Lambda_{A,B}) \subset \{1\} \cup \{\lambda \in \mathbb{C}, |\lambda| < 1\}$.

1) Assume $B^TA$ is invertible. Then $\text{Sp}(AB^T) = \text{Sp}(B^T A) \subset \{\mu \in \mathbb{R}, \mu < 0\}$, and for $\eta > 0$ small enough $\text{Sp}(\Lambda_{A,B}) \subset \{\lambda \in \mathbb{C}, |\lambda| < 1\}$. Then $\Lambda_{tA,B} \overset{t \to \infty}{\longrightarrow} 0$ by Gelfand’s theorem, so $(x_t, y_t) \overset{t \to \infty}{\longrightarrow} (0, 0)$.

Ker($\Lambda_{A,B} - I$) being restricted to the vector $(0, 0, 0, 0)$, $(x_\infty, y_\infty, x_\infty, y_\infty)$ is the projection onto Ker($\Lambda_{A,B} - I$) along the direct sum of the other eigenspaces of $\Lambda_{A,B}$.

2) Let $Z_0 = (x_0, y_0, x_{-1}, y_{-1})$ and $Z_t = (x_t, y_t, x_{t-1}, y_{t-1}) = \Lambda_{tA,B}^t Z_0$.

Assume $\Lambda_{A,B}$ is diagonalizable. Since, in addition its spectrum is included in $\{1\} \cup \{\lambda \in \mathbb{C}, |\lambda| < 1\}$, the sequence $(\Lambda_{tA,B}^t Z_0)_t$ converges to the projection of $Z_0$ onto the eigenspace associated to the eigenvalue 1, along the direct sum of all the other eigenspaces. Moreover, $(x_\infty, y_\infty, x_\infty, y_\infty) \in \text{Ker}(\Lambda_{A,B} - I)$ implies that $(x_\infty, y_\infty)$ is in Ker($B^T \times \text{Kernel}(A)$). Finally, this proves that $(x_t, y_t)_t$ converges to a Nash equilibrium, and gives its expression with regard to the initialization. \qed

Remark 4.5. Consider the case where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $B^TA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonalizable with spectrum reduced to $\{0\}$. However one can show that $\Lambda_{A,B}$ is not diagonalizable, and $(\Lambda_{tA,B})_t$ does not converge, so that for many initial conditions OGDA does not converge here.

Remark 4.6. Let $A$ be the payoff matrix for a zero-sum game. There is a link between changing the learning rate of OGDA from $\eta$ to $\eta' = \frac{\eta}{m}$ for this game, and looking to OGDA for the general-sum game $(A, -m^2 A)$ with parameter $\eta$. Both algorithm will converge to the same points with the same convergence speed.

Thus, if the learning rate is fixed for any reason, studying a well-modified general-sum game will improve the convergence rate.

4.2 Double Optimistic Gradient Descent Ascent

Recall the set of Nash equilibria of the game $(x^TAy, x^TB) = \text{Ker}(B^T) \times \text{Ker}(A)$. The set $\text{Ker}(B^T)$ is the set of optimal strategies of player 1 in the zero-sum game $x^T(-B)y$ (where player 1 wants to minimize the payoff $x^TB$) and $\text{Ker}(A)$ is the set of optimal strategies of player 2 in the zero-sum game $x^TAy$ (where player 2 wants to minimize $x^TAy$). As a consequence, convergence of OGDA for zero-sum games gives here a simple algorithm converging to a Nash equilibrium of a general-sum game.

Definition 4.7. The Double Optimistic Gradient Descent Ascent (DOGDA)
for the general-sum game $(x^T Ay, x^T By)$ is defined by:

\[
\begin{align*}
    x_{t+1} &= x_t - 2\eta By_{t} + \eta By_{t-1}, \\
    y_{t+1}' &= y_t' + 2\eta B^T x_t - \eta B^T x_{t-1}, \\
    x_{t+1}' &= x_t' + 2\eta Ay_t - \eta Ay_{t-1}, \\
    y_{t+1} &= y_t - 2\eta A^T x_{t}', + \eta A^T x_{t-1}'.
\end{align*}
\]

where $\eta > 0$ is a fixed parameter, and $z_0 := (x_0, y_0', x_0', y_0, x_{-1}, y_{-1}', x_{-1}', y_{-1}) \in (\mathbb{R}^n \times \mathbb{R}^p)^4$ is the initialization.

**Theorem 4.8.** Convergence of DOGDA to a Nash Equilibrium of the game $(x^T Ay, x^T By)$.

Let $A, B \in \mathbb{R}^{n \times p}$ and $0 < \eta < \frac{1}{2\sqrt{\mu_{\text{max}}}}$, where $\mu_{\text{max}} = \max\{\rho(A^T A), \rho(B^T B)\}$. Consider $z_0 \in (\mathbb{R}^n \times \mathbb{R}^p)^4$ and the sequence $(x_t, y_t', x_t', y_t)$ induced by the DOGDA algorithm (10) with initialization $z_0$. Then $(x_t, y_t)$ converges to a Nash equilibrium $(x_\infty, y_\infty) \in \text{Ker}(B^T) \times \text{Ker}(A)$, and the convergence is exponential: for all $t \geq 0$,

\[
\| (x_t, y_t) - (x_\infty, y_\infty) \| \leq C \lambda_{\text{max}}^t \| z_0 \|,
\]

where $C = \sqrt{2} \left( 1 - \frac{1 + 5\eta^2 \mu_{\text{max}}}{2 + \eta^2 \mu_{\text{max}}} \right)^{-1/2}$ and $\lambda_{\text{max}} = \max\left\{ \sqrt{\frac{1}{2}(1 + \sqrt{1 - 4\eta^2 \mu}), \mu > 0, \mu \in \mathcal{S}(A) \cup \mathcal{S}(B) } \right\}$, with the convention $\lambda_{\text{max}} = 0$ if $A = B = 0$.

**Proof.** Directly comes from Theorem 2.5, since DOGDA is the juxtaposition of 2 uncoupled OGDA dynamics. \qed

**Remark 4.9.** One can replace $B$ with $-B$ and/or $A$ with $-A$ in the OGDA or DOGDA algorithms. This is a strong specificity of our bilinear games. We have chosen (10) in definition 4.7 because of the following remark 4.10.

**Remark 4.10.** The DOGDA trick is due to the following property $(P)$ of our bilinear games: if $g_1$ denotes the payoff function $x^T Ay$ for player 1 and $g_2$ is the payoff function $x^T By$ for player 2, the Nash equilibria of the game $(g_1, g_2)$ are the couples $(x, y)$ where $x$ is optimal for player 1 in the zero-sum game $(-g_2, g_2)$ and $y$ is optimal for player 2 in the zero-sum game $(g_1, -g_1)$. One can ask how often this property holds for general games $(g_1, g_2)$? In the case of general-sum games with 2 players and 2 actions for each player played with mixed strategies, assuming that the 8 real coefficients are a priori independently selected according to the uniform law on a given compact interval with positive length, one can show that the probability to obtain a game where this property holds is $2/9 \approx 22\%$.  

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4.3 “Cooperation” induced by OGDA

Assume the game \((x^TAy, x^TBy)\) has a “potential for cooperation”, in the sense that there exist actions \((x, y) \in \mathbb{R}^n \times \mathbb{R}^p\) such that both \(x^TAy\) and \(x^TBy\) are positive. Define the sequence \((x_t, y_t)\) by \((x_t, y_t) = t(x, y)\) for each \(t\), then both \(x_t^TAy_t \xrightarrow{t \to +\infty} +\infty\) and \(x_t^TBy_t \xrightarrow{t \to +\infty} +\infty\). Even though we are not reaching a Nash equilibrium, this may be seen as a suitable outcome of the interaction between the players, who aim at maximizing their payoff.

A particular case of such games are common-payoff games, for which \(A = B\). This includes the case where \(A = B = (1)\), for which there is no hope to obtain convergence of OGDA to a Nash equilibrium of the game except if the vectors at initialization are already null. However, we reach an infinite payoff with OGDA, as was seen in section 4.1. We now generalize this property to a larger group of matrices.

We use the usual version of OGDA for general-sum games, that is algorithm (9). And the matrix describing the dynamics is \(\Lambda_{A,B}\), as described in section 4.1. In the proof of Theorem 2.5, we showed that a matrix \(\Lambda_{A,-A}\) is always diagonalizable. In the general case of matrices \(A, B\), assuming that the spectrum of \(B^TA\) is included in \(\mathbb{R}\) is not sufficient to conclude that \(\Lambda_{A,B}\) is diagonalizable. However if \(A = \alpha B\) for some real parameter \(\alpha\), we will show that \(\Lambda_{A,B}\) is diagonalizable for \(\eta\) small enough.

We know from Proposition 4.3 that: \(\text{Sp}(\Lambda_{A,B}) = \bigcup_{\mu \in S(A,B)} S^*(\mu)\), with \(S^*(\mu) = S(-\mu) = \{\lambda \in \mathbb{C}, \lambda^2(1 - \lambda)^2 = \mu\eta^2(1 - 2\lambda)^2\}\). We now count the eigenvalues of the matrix \(\Lambda_{A,B}\).

**Lemma 4.11.** Let \(A, B \in \mathbb{R}^{n \times p}\) be two matrices such that \(S(A, B) \subset \mathbb{R}\), and assume \(\eta < \frac{1}{2\sqrt{\mu_{\text{max}}}}\), where \(\mu_{\text{max}} = \rho(B^TA) = \rho(AB^T)\). Then, for \(\mu \in S(A, B)\), there are three cases:
- If \(\mu < 0\), define \(\nu = i\sqrt{|\mu|}\) and \(\delta = \sqrt{1 - 4\eta^2|\mu|} \geq 0\). Then \(S^*(\mu)\) has exactly 4 elements, which are \(\lambda_1^*(\mu) = \frac{1}{2}(1 + \delta + 2\eta\nu), \lambda_2^*(\mu) = \frac{1}{2}(1 - \delta + 2\eta\nu), \lambda_3^*(\mu) = \overline{\lambda_1^*(\mu)}, \lambda_4^*(\mu) = \overline{\lambda_2^*(\mu)}\). Their modulus are all strictly smaller than 1.
- If \(\mu = 0\), then \(\lambda_1^*(\mu) = \lambda_3^*(\mu) = 1\) and \(\lambda_2^*(\mu) = \lambda_4^*(\mu) = 0\).
- If \(\mu > 0\), define \(\nu = \sqrt{\mu}\) and \(\delta = \sqrt{1 + 4\eta^2\nu^2}\). Then \(S^*(\mu)\) has exactly 4 elements which are real, denoted: \(\lambda_1^*(\mu) = \frac{1}{2}(1 + 2\eta\nu + \delta) > 1, \lambda_2^*(\mu) = \frac{1}{2}(1 + 2\eta\nu - \delta) \in (0, 1), \lambda_3^*(\mu) = \frac{1}{2}(1 - 2\eta\nu + \delta) \in (0, 1)\) and \(\lambda_4^*(\mu) = \frac{1}{2}(1 - 2\eta\nu - \delta) \in (-1, 0)\).

We can now prove that \(\Lambda_{A,B}\) is diagonalizable whenever \(A\) and \(B\) are colinear. Define \(E^*_\lambda\) as the set of real eigenvectors of \(\Lambda_{A,B}\) associated to an eigenvalue \(\lambda\).
Proposition 4.12. Assume that $A = \alpha B$ for some real $\alpha \neq 0$, and $\eta < \frac{2}{3\sqrt{\mu_{\max}}}$. Then $\mathbb{R}^{(n+p+n+p)} = \oplus_{\lambda \in \text{Sp}(\Lambda_{A,B})} E^*_\lambda$.

Similarly, one can show that if $A$ and $B$ are square matrices in $\mathbb{R}^{n \times n}$ such that $B^T A$ is diagonalizable with $n$ distinct nonzero real eigenvalues.

We can now present our last theorem, related to the convergence to a Nash equilibrium or to infinite payoffs.

Theorem 4.13. Let $A, B \in \mathbb{R}^{n \times p}$ and assume that $S(A, B) \subset \mathbb{R}$, $\Lambda_{A,B}$ is diagonalizable and let $0 < \eta < \frac{1}{2\sqrt{\mu_{\max}}}$. Given $(x_0, y_0, x_{-1}, y_{-1}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p$, we consider the Optimistic Gradient Descent Ascent:

$$\forall t \geq 0, \begin{cases} x_{t+1} = x_t + 2\eta A y_t - \eta A y_{t-1}, \\ y_{t+1} = y_t + 2\eta B^T x_t - \eta B^T x_{t-1}. \end{cases}$$

Then when $t \to \infty$, either $(x_t, y_t)$ converges to a Nash equilibrium of the game $(x^T Ay, x^T By)$, or the current payoffs $x^T_t Ay_t$ and $x^T_t By_t$ converge exponentially fast to $+\infty$.

The assumptions of the theorem are in particular true in each of the two following cases:

1) $B = \alpha A$ for some real number $\alpha \neq 0$,

2) $A$ and $B$ are square matrices in $\mathbb{R}^{n \times n}$ such that $B^T A$ is diagonalizable with $n$ distinct nonzero real eigenvalues.

Notice that case 1) includes the case of common payoffs games $A = B$, as well as the case of zero-sum games $B = -A$ (where $(x_t, y_t)$, converges to a Nash equilibrium).

5 Illustration: Generative Adversarial Networks

Algorithms to approximate probability distributions and generate new data from some unknown laws are more and more needed. Despite this necessity, the theory behind this problem is lacking, resolving in inconsistent performances. The main type of generative algorithm are Generative Adversarial Networks (GANs) introduced in [10] by Goodfellow et al.

5.1 GANs and WGANs

In GANs, two neural networks (a discriminator and a generator) are set in competition against each other. The goal of the generator is to generate new data as close as possible to the true data. And in the original version of GANs, the discriminator aims at recognizing true data from the created ones. We will here mostly consider Wasserstein GANs, a very popular sort
of GANs that was introduced by Arjovsky in [3] and uses the Wasserstein distance between probability distributions $\mu$ and $\nu$ defined by

$$W_1(\mu, \nu) = \sup_{f \in \text{Lip}_1} \mathbb{E}_\mu[f] - \mathbb{E}_\nu[f].$$

There is a true distribution $\mu^*$, and we (the generator) want to generate a distribution as close as possible to $\mu^*$. Unfortunately, we do not have access to the true distribution, but only to an empirical approximation $\hat{\mu}$, via some samples generated according to $\mu^*$. We also cannot choose any distribution but are restricted to choose a parameter $\theta$, and we want to find the parameter minimizing the Wasserstein distance between the generated distribution and the empirical distribution $\hat{\mu}$.

More precisely, the generator $G_\theta$ is a usual neural network from a latent space $\mathbb{R}^n$ to the feature space $\mathbb{R}^d$. From a Gaussian distribution $Z \sim \mathcal{N}(0, I_n)$ it defines a probability measure $\mu_\theta$, which is the law of the random variable $G_\theta(Z)$. In order to find the generated measure which is the closest to $\hat{\mu}$, we want to find the parameter $\theta$ minimizing the Wasserstein distance between $\hat{\mu}$ and $\mu_\theta$. This is given by the problem

$$\inf_{\theta} W_1(\hat{\mu}, \mu_\theta) = \inf_{\theta} \sup_{f \in \text{Lip}_1} \mathbb{E}_{\hat{\mu}}[f] - \mathbb{E}_{\mu_\theta}[f]$$

where $f$ plays the role of the discriminator.

However, in practice, we cannot compute the supremum over 1-Lipschitz function, thus we need to approximate this set using neural networks. We parameterize the set of discriminators, and denote by $D_\alpha$ the 1-Lipschitz neural network from $\mathbb{R}^d$ to $\mathbb{R}$ with parameter $\alpha$. In practice, we want the set $\{D_\alpha|\alpha\}$ to be dense in the set of 1-Lipschitz function (as shown possible, for instance, in Anil et al. in [2]).

The WGAN problem we finally want to solve can thus be written as

$$\inf_{\theta} \sup_{\alpha} \mathcal{L}(\theta, \alpha) \text{ where } \mathcal{L}(\theta, \alpha) = \mathbb{E}_{\hat{\mu}}[D_\alpha] - \mathbb{E}_{\mu_\theta}[D_\alpha]$$

(11)

5.2 Example

We now look at a simple illustrative example introduced by Daskalakis et al. in [4], and see how tight is our theoretical bound for convergence. We assume that the data follows a multivariate normal distribution of mean $v \in \mathbb{R}^n$: $\mu^* = \mathcal{N}(v, I_n)$, where $v$ is unknown. We begin from a multivariate normal distribution $Z$ with mean 0: $Z \sim \mathcal{N}(0, I_n)$, and study linear generators of the
form $G_\theta(z) = z + \theta$. For the set of discriminators, we use the very small set of all $D_\alpha : x \mapsto \langle \alpha, x \rangle$, where $\|\alpha\|_2 \leq 1$ so that $D_\alpha$ is 1-Lipschitz. Fortunately, for this problem, if $\alpha_0$ respects $\|\alpha_0\|_2 \leq 1$, then, for all $t$, $\|\alpha_t\|_2 \leq 1$, which means that we can drop this constraint as long as we choose well $\alpha_0$.

The WGAN problem becomes:

$$\inf_{\theta} \sup_{\alpha} \mathbb{E}_{x \sim \mathcal{N}(v, I_n)}[\langle \alpha, x \rangle] - \mathbb{E}_{z \sim \mathcal{N}(0, I_n)}[\langle \alpha, z + \theta \rangle]$$

If we consider the case with true expectation, we get $\inf_{\theta} \sup_{\alpha} \langle \alpha, v - \theta \rangle$.

We are in the case $x^T A y + b^T x + c^T y + d$ as in section 3 with $A = -I_n \in \mathbb{R}^{n \times n}$, $b = 0$, $c = v$ and $d = 0$. We showed that in this case, for $\eta < \frac{1}{2\sqrt{\lambda_{\max}}}$, OGDA will converge exponentially fast. Running OGDA on this example with $\eta = 0.3 < \frac{1}{\sqrt{2}}$ gives us $\lambda_{\max} = \frac{3}{\sqrt{5}} \approx 1.34$ and $C \approx 3.45$.

Figure 3: Log plot of the distance to the Nash equilibrium and predicted bounds

We plotted the graph of the distance of OGDA to the set of Nash equilibria with regard to time, on figure 3, where we took $n = 2$ and $v = \left(\frac{3}{4}\right)$. In the pictures, we see the exponential convergence of OGDA for different values of $\eta$. We also plotted the different bounds that were found previously.

In the first picture, $\eta$ is close to the maximum possible level, so the convergence is really fast. We have seen in Remark 2.3 that our ratio $\lambda_{\max}$ is the best possible, and these figures mark well this fact: the plots of the log of the distance to the limit Nash equilibrium and of the log of the bound we found are parallel. However, if $\eta = 0.03$ our constant $C$ is not optimal, as can be seen by the space between the two parallel lines.
5.3 Discussion and limitations

Our results only concern bilinear games, which can be seen as toy-models compared to the complexity of real WGANs. To go towards more realism, several issues could be considered.

In practice, neural networks have at least one million parameters, if not billions, and they suffer from the curse of dimensionality. However, OGDA will not suffer more than GDA in this account, because we only need to do matrix/vector operations, and still do not need to multiply matrices together, or to inverse a matrix. OGDA, as GDA is a first order method with a single oracle call per iteration. Moreover we have seen that the speed of convergence to equilibria is independent of the dimensions $n$ and $p$.

Another issue is to take into account the stochastic character of the answer that we receive. In practice, we won’t exactly have access to $E_{x \sim N(\nu, I_d)} [\langle \alpha, x \rangle]$, nor to $E_{z \sim N(0, I_d)} [\langle \alpha, z + \theta \rangle]$, but only to empirical average of data following the according distribution. In [4], Daskalakis et al. already considered this problem. A deeper study on stochasticity for OGDA and Extra-Gradient was done by Hsieh et al. in [12]. They showed a geometric convergence of the expected distance to the saddle points for strongly concave/convex games, with a bound depending on the variance of the noise.

A further step is not to assume the unboundedness of the problem. In this article like in many others, we assume that the set of parameters is an Euclidean space: we are doing minimization and maximization on $\mathbb{R}^p$ and $\mathbb{R}^n$. But in reality, neural networks are computer-based, thus the set of possible neural-networks is far more complex. However, the goal is to model the largest possible set of functions, and the maxima and minima reached by the weights are very large, and will get larger if float maximum bit-size gets higher.

A last important factor, and not the least of them, is the fact that most of the GANs problem used in practice are non-concave/non-convex, because neural networks do not have a convex dependence with their parameters. Convergence to equilibria for non-concave/non-convex games is currently far from being well-understood.

6 Conclusion

We have proved for the first time the exponential convergence of OGDA to a Nash equilibrium for any bilinear game, and also provided the best ratio for the convergence. This implies an important stability property for the OGDA system as the number of steps increases. In case of multiple Nash equilibria, we also characterize the limit equilibrium as a function of the initialization.
We also extended the study to general-sum games. We gave a sufficient condition for OGDA to converge to a Nash equilibrium there, and proved that in an important class of games (including the common payoffs case as well as the zero-sum case) either OGDA converges to a Nash equilibrium, or the payoffs of both players converge to $+\infty$. We also presented a simple trick to guarantee, via a variant of OGDA, convergence to Nash equilibria in our general-sum games.

The bilinear games we have studied are widely used as a toy model for many applications, and we illustrated our results on a GAN example. These might be used as an additional argument in favor of using the Optimistic version of GDA in GANs. Nevertheless, if one wants to go further in the application to GANs, there are several factors to take into account to be closer to real applications. One of them would be the stochasticity of the payoffs. Some experiments were made in [4], but no theoretical convergence speed is known, nor any bound on the proximity to a Nash equilibrium depending on the noise of the stochastic payoff. A second and harder factor is the non-concave/non-convex properties of real instances of GANs.

All in all, the general comprehension of OGDA is improving, and so are the applications to the stability of first-order methods for GANs. This paper aims at continuing this dynamics, in giving more insights into the min-max problem.
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A Appendix: Proofs

A.1 Proof of Theorem 2.5

We fix \( A \in \mathbb{R}^{n \times p} \) and \( \eta > 0 \).

A.1.1 Convergence to Nash equilibria

Recalling the characteristic polynomial of \( \Lambda \) in the particular case where \( A = (1) \), we introduce the following sets:

**Definition A.1.** Given \( \mu \in \mathbb{C} \), we define

\[
S(\mu) = \{ \lambda \in \mathbb{C}, \lambda^2(1 - \lambda)^2 + \mu \eta^2(1 - 2\lambda)^2 = 0 \}.
\]

Notice that if \( \eta = 0 \) or \( \mu = 0 \), then \( S(\mu) = \{0, 1\} \). Simple computations imply the following lemma.

**Lemma A.2.** Assume \( \mu \) is real positive, with \( 1 - 4\eta^2\mu > 0 \). Define \( \nu = i\sqrt{\mu} \) and \( \delta = \sqrt{1 - 4\eta^2\mu} \geq 0 \). Then \( S(\mu) \) has exactly four elements, denoted:

\[
\lambda_1(\mu) = \frac{1}{2}(1 + \delta + 2\eta\nu), \quad \lambda_2(\mu) = \frac{1}{2}(1 - \delta + 2\eta\nu), \quad \lambda_3(\mu) = \overline{\lambda_1(\mu)}, \quad \lambda_4(\mu) = \overline{\lambda_2(\mu)}.
\]

We have \( |\lambda_1(\mu)|^2 = |\lambda_3(\mu)|^2 = \frac{1}{4}(1 + \sqrt{1 - 4\eta^2\mu}) < 1 \), and \( |\lambda_2(\mu)|^2 = |\lambda_4(\mu)|^2 = \frac{1}{4}(1 - \sqrt{1 - 4\eta^2\mu}) < \frac{1}{2} \).

The link between the eigenvalues of the two matrices \( A^T A \) and \( AA^T \), and the eigenvalues of \( \Lambda \) is the following.

**Lemma A.3.**

\( \text{Sp}(\Lambda) = \{ \lambda | \lambda \in S(\mu), \mu \in S(A) \} \)

*Proof.* For \( \lambda \in \mathbb{C} \) and \( Z = (x^T, y^T, x'^T, y'^T)^T \in E \), we have:

\[
\Lambda Z = \lambda Z \iff \begin{cases}
  x + 2\eta A y - \eta A y' = \lambda x \\
  y - 2\eta A^T x + \eta A^T x' = \lambda y \\
  \quad x = \lambda x' \\
  \quad y = \lambda y'
\end{cases} \iff \begin{cases}
  \lambda(1 - \lambda)x' + (2\lambda - 1)\eta A y' = 0 \\
  \lambda(1 - \lambda)y' - (2\lambda - 1)\eta A^T x' = 0 \\
  \quad x = \lambda x' \\
  \quad y = \lambda y'
\end{cases}
\]

Assume \( \Lambda Z = \lambda Z \) and \( Z \neq 0 \). Multiplying the first line by \((2\lambda - 1)\eta A^T\) and the second line by \((2\lambda - 1)\eta A\), we get:

\[
\begin{cases}
  \lambda^2(1 - \lambda)^2 y' + (2\lambda - 1)^2 \eta^2 A^T A y' = 0 \\
  \lambda^2(1 - \lambda)^2 x' + (2\lambda - 1)^2 \eta^2 A A^T x' = 0
\end{cases}
\]
Proposition A.5. Assume $\lambda \neq 1/2$. This implies that $-\frac{\lambda^2(1-\lambda)^2}{(2\lambda-1)^2}$ is an eigenvalue of $A^TA$ or of $AA^T$. If we denote by $\mu$ such eigenvalue, then $\lambda \in S(\mu)$.

Conversely, let $\mu$ be an eigenvalue of $A^TA$, and consider $\lambda \in S(\mu)$. Let $x' \neq 0$ be such that $AA^Tx' = \mu x'$ and consider $y'$ satisfying:

$$\lambda(1-\lambda)y' = \eta(2\lambda-1)A^Tx'.$$

If $\mu \neq 0$, then $\lambda \neq 0, 1$ and $y$ is uniquely defined; if $\mu = 0$ then $AA^Tx' = 0$ so $A^Tx' = 0$ and any $y'$ will do. In each case, one can check that $\Lambda(\lambda x', \lambda y', x', y')^T = \lambda Z$, so that $\lambda$ is an eigenvalue of $\Lambda$. This also stands for $\mu$ an eigenvalue of $AA^T$.

Thus, the set of eigenvalues of $\Lambda$ is $\{\lambda | \lambda \in S(\mu), \mu \in S(A)\}$. \qed

Corollary A.4. Assume $1 - 4\eta^2\mu \geq 0$ for all $\mu \in S(A)$, or equivalently, assume $\eta \leq \frac{1}{\sqrt{\mu_{\max}}}$. Then

$$\text{Sp}(A) \subset \{1\} \cup \{\lambda \in \mathbb{C}, |\lambda| < 1\}.$$  

Proof. Consider $\mu \in S(A)$. Since $A^TA$ and $AA^T$ are positive semi-definite, $\mu$ is a non negative real number. If $\mu \neq 0$, $S(\mu) \subset \{\lambda \in \mathbb{C}, |\lambda| < 1\}$ by Lemma A.2. If $\mu = 0$, then $S(\mu) = \{0, 1\}$. \qed

We now show that $\Lambda$ is diagonalizable in $\mathbb{C}$. For each eigenvalue $\lambda$ of $\Lambda$, we denote by $E_\lambda$ the associated eigenspace, i.e. $E_\lambda = \text{Ker}(\Lambda - \lambda I)$.

Proposition A.5. Assume $\eta < \frac{1}{\sqrt{\mu_{\max}}}$. Then

$$\mathbb{C}^{(n+p+n+p)} = \bigoplus_{\lambda \in \text{Sp}(A)} E_\lambda.$$  

Proof. Let $\lambda \in \text{Sp}(A)$, with $\lambda \notin \{0, 1\}$. Then, according to the proof of Lemma A.3,

$$E_\lambda = \left\{ (x, y, x', y') \in \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^p, x = \lambda x', y = \lambda y', A^T y' = \mu y', x' = \frac{(1-2\lambda)\eta}{\lambda(1-\lambda)} A^T y' \right\},$$

where $\mu \in \text{Sp}(A^TA)$ is uniquely defined by $\lambda^2(1-\lambda)^2 = -\eta^2\mu(2\lambda - 1)^2$. Thus $\dim(E_\lambda) = \dim(\text{Ker}(A^TA - \mu I))$. Notice that thanks to Lemma A.2, for each $\mu$ in $\text{Sp}(A^TA) \setminus \{0\}$ we have exactly 4 elements in $S(\mu)$.

Regarding $\lambda = 0$, we have $E_0 = \left\{ (x, y, x', y'), x = 0, y = 0, A^T y' = 0, A^T x' = 0 \right\}$, so that $\dim(E_0) = \dim(\text{Ker}(A)) + \dim(\text{Ker}(A^T))$. And if $\lambda = 1$, then $E_1 = \left\{ (x, y, x', y'), x = x', y = y', A^T y' = 0, A^T x' = 0 \right\}$ and again, $\dim(E_1) =$
dim(Ker(A)) + dim(Ker(ATA)).

Recall that rank(ATA) = rank(A) = rank(AT) and, AT being positive semi-definite, rank(ATA) = \sum_{\mu>0} \dim(Ker(ATA - \mu I)). We can now sum the dimension of the eigenspaces:

\[
\sum_{\lambda \in \text{Sp}(A)} \dim(E_{\lambda}) = \dim(E_0) + \dim(E_1) + \sum_{\lambda \in \text{Sp}(A), \lambda \neq 0,1} \dim(E_{\lambda})
\]

\[
= 2 \dim(Ker(A)) + 2 \dim(Ker(AT)) + \sum_{\mu \in \text{Sp}(ATA), \mu \neq 0} 4 \dim(Ker(ATA - \mu I_d))
\]

\[
= 2 \dim(Ker(A)) + 2 \dim(Ker(AT)) + 4 \text{rank}(AT) = 2(\text{rank}(A) + \text{dim}(\text{Ker}(A))) + 2(\text{rank}(AT) + \text{dim}(\text{Ker}(AT)))
\]

\[
= 2(n + p).
\]

As a consequence \(\Lambda\) is diagonalizable in \(\mathbb{C}\).

Write now \(Z = Z_0\) for simplicity. \(Z\) can now be uniquely written:

\[
Z = \sum_{\lambda \in \text{Sp}(A)} z_{\lambda}, \text{ with } z_{\lambda} \in E_{\lambda} \text{ for all } \lambda.
\]

For all \(t \geq 0\), \(Z_t = \sum_{\lambda \in \text{Sp}(A)} \lambda^t z_{\lambda}\) and \(Z_t \xrightarrow{t \to \infty} Z_\infty := z_1\). We have obtained the convergence of \(Z_t\) to the projection of \(Z_0\) onto \(\text{Ker}(\Lambda - I)\) along \(\bigoplus_{\lambda \neq 1} \text{Ker}(\Lambda - \lambda I)\).

**Lemma A.6.** For any nonzero eigenvalue \(\mu\) of \(AT\) and any eigenvector \(y\) associated with \(\mu\), \(y\) is orthogonal to \(\text{Ker}(A)\).

**Proof.** Let \(\hat{y}\) be a vector in the kernel of \(A\). Then:

\[
\mu \langle y, \hat{y} \rangle = \langle \mu y, \hat{y} \rangle = \langle ATy, \hat{y} \rangle = \langle Ay, A\hat{y} \rangle = \langle Ay, 0 \rangle = 0
\]

Hence, \(\langle y, \hat{y} \rangle = 0\) for any \(\hat{y}\) in \(\text{Ker}(A)\). This means that \(y\) is orthogonal to \(\text{Ker}(A)\).

Now, let us write \(z_{\lambda}\) as \((\hat{x}_\lambda, \hat{y}_\lambda, \hat{x}'_\lambda, \hat{y}'_\lambda)\) for all \(\lambda\) in \(\mathcal{S}(A)\), and we recall that \(Z_t = (x_t, y_t, x'_t, y'_t)\). Thanks to A.6, we know that for all \(\lambda \neq \{0,1\}, \hat{y}_\lambda\) is orthogonal to \(\text{Ker}(A)\). Moreover, \(\hat{y}_0 = 0\), thus \(\hat{y}_1 = y_\infty\) is the orthogonal projection of \(y_0\) onto \(\text{Ker}(A)\). In the same way, we can prove that \(x_\infty\) is the orthogonal projection of \(x_0\) onto \(\text{Ker}(AT)\).
A.1.2 Speed of convergence

For all \( t \geq 1 \),

\[
Z_t - Z_\infty = \sum_{\lambda \in \text{Sp}(A), \lambda \neq 0, 1} \lambda^t z_\lambda = \sum_{\mu \in \text{Sp}(A^T A), \mu > 0} \sum_{l=1}^{4} \lambda_l^t z_{\lambda_l(\mu)}.
\]

The matrix \( A \) is not diagonalizable in an orthogonal basis, and the vectors \( (z_\lambda) \) are not orthogonal in general. We will show however that the situation is close to it.

One can first check that for all \( \mu > 0 \) in \( \text{Sp}(A^T A) \), if \( l = 1, 2 \),

\[
E_{\lambda_l(\mu)} = \left\{ (x, y, x', y')^T \in \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^p, x = \lambda_l(\mu)x', y = \lambda_l(\mu)y', A^T A y' = \mu y', x' = \frac{1}{\mu} A y' \right\},
\]

and if \( l = 3, 4 \),

\[
E_{\lambda_l(\mu)} = \left\{ (x, y, x', y')^T \in \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^p, x = \lambda_l(\mu)x', y = \lambda_l(\mu)y', A^T A y' = \mu y', x' = -\frac{1}{\mu} A y' \right\}.
\]

For the case where \( \mu = 0 \), we have:

\[
E_0 = \left\{ (0, 0, x', y')^T \in \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^p, Ay' = 0, A^T x' = 0 \right\}
\]

and

\[
E_1 = \left\{ (x', y', x', y')^T \in \mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^p, Ay' = 0, A^T x' = 0 \right\}
\]

Lemma A.7.

a) For \( \mu_1 \neq \mu_2 \) in \( \text{S}(A) \), the vector subspaces \( \oplus_{l=1,2,3,4} E_{\lambda_l(\mu_1)} \) and \( \oplus_{l=1,2,3,4} E_{\lambda_l(\mu_2)} \) are orthogonal.

b) For \( \mu \in \text{S}(A) \setminus \{0\} \), \( l \in \{1, 2\} \) and \( l' \in \{3, 4\} \), the vector subspaces \( E_{\lambda_l(\mu)} \) and \( E_{\lambda_{l'}(\mu)} \) are orthogonal.

Proof. a) Consider \( z_1 = (x_1, y_1, x_1', y_1') \) in \( E_{\lambda_l(\mu_1)} \) and \( z_2 = (x_2, y_2, x_2', y_2') \) in \( E_{\lambda_{l'}(\mu_2)} \). We have both:

\[
\langle Ay_1', Ay_2' \rangle = \langle y_1', A^T A y_2' \rangle = \langle y_1', \mu_2 y_2' \rangle = \mu_2 \langle y_1', y_2' \rangle,
\]

\[
\langle Ay_1', Ay_2' \rangle = \langle A^T A y_1', y_2' \rangle = \langle \mu_1 y_1', y_2' \rangle = \mu_1 \langle y_1', y_2' \rangle.
\]

Since \( \mu_1 \neq \mu_2 \), \( \langle y_1', y_2' \rangle = 0 \). Then \( \langle y_1, y_2 \rangle = \langle x_1, x_2 \rangle = 0 \), and finally \( \langle z_1, z_2 \rangle = 0 \).

b) Consider \( z_1 = (x_1, y_1, x_1', y_1') \) in \( E_{\lambda_l(\mu)} \) and \( z_2 = (x_2, y_2, x_2', y_2') \) in \( E_{\lambda_{l'}(\mu)} \).

\[
\langle x_1', x_2' \rangle = \langle \frac{1}{\mu_1} A y_1', -\frac{1}{\mu_2} A y_2' \rangle = \frac{1}{\mu_1^2} \langle A y_1', A y_2' \rangle = \frac{1}{\mu_1^2} \langle y_1', A^T A y_2' \rangle = \frac{1}{\mu_2} \langle y_1', \mu y_2' \rangle = -\langle y_1', y_2' \rangle.
\]

Then \( \langle x_1, x_2 \rangle = -\langle y_1, y_2 \rangle \), and \( \langle z_1, z_2 \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \langle x_1', x_2' \rangle + \langle y_1', y_2' \rangle = 0 \). □
Remark A.8. A consequence of the previous lemma is that for $t \geq 1$:

$$\|Z_t-Z_\infty\|^2 = \sum_{\mu \in S(A), \mu > 0} (\|\lambda_1(\mu)^t z_{\lambda_1(\mu)} + \lambda_2(\mu)^t z_{\lambda_2(\mu)}\|^2 + \|\lambda_3(\mu)^t z_{\lambda_3(\mu)} + \lambda_4(\mu)^t z_{\lambda_4(\mu)}\|^2).$$

We now fix $\mu > 0$ in $S(A)$ and simply write $\lambda_l$ for $\lambda_l(\mu)$ and $z_l$ for $z_{\lambda_l}$.

We need to handle the terms $\|\lambda_1^t z_1 + \lambda_2^t z_2\|^2$ and $\|\lambda_3^t z_3 + \lambda_4^t z_4\|^2$. Consider the first term (the second one will be treated similarly), we want to write the dimension. This would be easy if $\lambda_1$ and $\lambda_2$ were orthogonal, and $C = 1$ would do. This would be impossible if $z_1 + z_2 = 0$ or $z_1 \neq 0$, but this can not happen since $z_1$ and $z_2$ belong to different eigenspaces. The key idea is that the “angle” between the subspaces $E_{\lambda_1}$ and $E_{\lambda_2}$ can not be too low and is, unformally speaking, close to $\pi/4$: think of $\lambda_1$ close to 1, and of $\lambda_2$ close to 0 so that for some vector subspace $F$, $E_{\lambda_1}$ can be seen as the set of vectors $\{(\alpha, \alpha), \alpha \in F\}$ whereas $E_{\lambda_2}$ can be seen as the set of vectors $\{(0, \alpha), \alpha \in F\}$. If as a thought experiment we imagine $F$ being the real line, we would have the horizontal axis and the 45-degree line in the Euclidean plane.

Lemma A.9. $z_l$ being in $E_{\lambda_l(\mu)}$ for each $l = 1, 2, 3, 4$, we have:

a) $$\langle z_1, z_2 \rangle \leq \sqrt{\frac{1 + 5n^2\mu}{2 + \eta^2\mu}} \|z_1\| \|z_2\| \text{ and } \langle z_3, z_4 \rangle \leq \sqrt{\frac{1 + 5n^2\mu}{2 + \eta^2\mu}} \|z_3\| \|z_4\|.$$  

b) Introducing $C = \sqrt{\frac{2}{1 - \sqrt{\frac{1 + 5n^2\mu_{\text{max}}}{2 + \eta^2\mu_{\text{max}}}}}}$, we have:

$$\|\lambda_1^t z_1 + \lambda_2^t z_2\|^2 \leq C^2 |\lambda_1|^{2t} \|z_1 + z_2\|^2 \text{ and } |\lambda_3^t z_1 + \lambda_4^t z_2\|^2 \leq C^2 |\lambda_1|^{2t} \|z_3 + z_4\|^2.$$  

Proof. a) Write $z_1 = (x_1, y_1, x'_1, y'_1)$ and $z_2 = (x_2, y_2, x'_2, y'_2)$.

$$\langle x'_1, x'_2 \rangle = \langle \frac{1}{\mu} Ay'_1, \frac{1}{\mu} Ay'_2 \rangle = \frac{1}{\mu} \langle Ay'_1, Ay'_2 \rangle = \frac{1}{\mu} \langle y'_1, A^T Ay'_2 \rangle = \frac{1}{\mu} \langle y'_1, \mu y'_2 \rangle = \langle y'_1, y'_2 \rangle,$$

and $\langle x_1, x_2 \rangle = \lambda_1 \langle y'_1, y'_2 \rangle = \langle y_1, y_2 \rangle$. So

$$\langle z_1, z_2 \rangle = 2(1 + \lambda_1 \lambda_2) \langle y'_1, y'_2 \rangle,$$

$$\|z_1\|^2 = 2(1 + |\lambda_1|^2) \|y'_1\|^2 \text{ and } \|z_2\|^2 = 2(1 + |\lambda_2|^2) \|y'_2\|^2.$$  

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We obtain:
\[
\langle z_1, z_2 \rangle = \frac{(1 + \lambda_1 \lambda_2)}{\sqrt{(1 + |\lambda_1|^2)(1 + |\lambda_2|^2)}} \langle \gamma', \gamma \rangle \frac{||\gamma'||||\gamma||}{||z_1||||z_2||}.
\]

Computations show that \(\sqrt{(1 + |\lambda_1|^2)(1 + |\lambda_2|^2)} = \frac{1}{2}\sqrt{9 - \delta^2} = \sqrt{2 + \eta^2\mu},\) and \(|1 + \lambda_1 \lambda_2| = \sqrt{1 + 5\eta^2\mu}.\) Since \(\|\gamma'||\|\gamma\| \leq 1\) by Cauchy-Schwartz inequality, we obtain the upper bound for \(|\langle z_1, z_2 \rangle|\). The proof is similar for \(|\langle z_3, z_4 \rangle|\).

b) On the one-hand,
\[
\|\lambda_1' z_1 + \lambda_2' z_2\|^2 \leq 2 \left( \|\lambda_1' z_1\|^2 + \|\lambda_2' z_2\|^2 \right) \leq 2|\lambda_1|^{2t}\left(\|z_1\|^2 + \|z_2\|^2\right).
\]

On the other hand,
\[
\|z_1 + z_2\|^2 = \|z_1\|^2 + \|z_2\|^2 + 2 \text{Re}(\langle z_1, z_2 \rangle),
\]
\[
\geq \|z_1\|^2 + \|z_2\|^2 - 2\sqrt{1 + 5\eta^2\mu} \|z_1\| \|z_2\|,
\]
\[
\geq \|z_1\|^2 + \|z_2\|^2 - \frac{1 + 5\eta^2\mu}{2 + \eta^2\mu} (\|z_1\|^2 + \|z_2\|^2),
\]
\[
\geq (\|z_1\|^2 + \|z_2\|^2)(1 - \frac{1 + 5\eta^2\mu}{2 + \eta^2\mu}).
\]

Notice that \(\eta^2\mu < 1/4\) implies \(1 - \frac{1 + 5\eta^2\mu}{2 + \eta^2\mu} > 0\). We obtain:
\[
\|\lambda_1' z_1 + \lambda_2' z_2\|^2 \leq \frac{2}{1 - \frac{1 + 5\eta^2\mu}{2 + \eta^2\mu}}|\lambda_1|^{2t}\|z_1 + z_2\|^2.
\]

The last inequality is proved similarly. \(\square\)

We can now bound \(\|Z_t - Z_\infty\|^2\).
\[
\|Z_t - Z_\infty\|^2 = \sum_{\mu \in \mathcal{S}(A), \mu > 0} \left( \|\lambda_1(\mu)^t z_{\lambda_1(\mu)} + \lambda_2(\mu)^t z_{\lambda_2(\mu)}\|^2 + \|\lambda_3(\mu)^t z_{\lambda_3(\mu)} + \lambda_4(\mu)^t z_{\lambda_4(\mu)}\|^2 \right),
\]
\[
\leq \sum_{\mu \in \mathcal{S}(A), \mu > 0} C^2 \lambda_{\text{max}}^{2t} \left( \|z_{\lambda_1(\mu)} + z_{\lambda_2(\mu)}\|^2 + \|z_{\lambda_3(\mu)} + z_{\lambda_4(\mu)}\|^2 \right),
\]
\[
\leq C^2 \lambda_{\text{max}}^{2t} \|Z_0\|^2.
\]

And we get:
\[
\|Z_t - Z_\infty\| \leq C \lambda_{\text{max}}^{t} \|Z_0\|. \quad (13)
\]
We can now conclude. Take any Nash equilibrium \((x^*, y^*)\) in \(\text{Ker}(A^T) \times \text{Ker}(A)\), and define for each \(t \geq -1\):

\[
x'_t = x_t - x^* \quad \text{and} \quad y'_t = y_t - y^*.
\]

We have proved (part 2) of Theorem 2.5 that \((x'_t, y'_t)\) converges to a limit \((x'_\infty, y'_\infty)\). By linearity of the projection, and since \((x^*, y^*)\) is invariant for the OGDA, this limit is \((x^\infty, y^\infty) = (x^*, y^*)\). So

\[
\| (x'_\infty, y'_\infty) - (x'_t, y'_t) \| = \| (x^\infty, y^\infty) - (x_t, y_t) \|.
\]

Using inequality (13), we obtain:

\[
\| (x^\infty, y^\infty) - (x_t, y_t) \| \leq C\lambda_{\max} t \| Z'_0 \|
\]

with \(\| Z'_0 \| = \| (x_0 - x^*, y_0 - y^*, x'_{-1} - x^*, y'_{-1} - y^*) \|\). Letting \((x^*, y^*)\) vary in \(\text{Ker}(A^T) \times \text{Ker}(A)\) concludes the proof of Theorem 2.5.

### A.2 Proof of Theorem 3.1

Notice that the fixed points of the OGDA are exactly the Nash equilibria of the game:

\[
\begin{align*}
x^* &= x^* + 2\eta(Ay^* + b) - \eta(Ay^* + b) \\
y^* &= y^* - 2\eta(A^T x^* + c) + \eta(A^T x^* + c)
\end{align*}
\]

Thus, saying that the set of Nash equilibria is empty is equivalent to saying that OGDA has no fixed point: if there is no Nash equilibrium, OGDA diverge.

In the sequel we assume that the set of Nash equilibria is not empty, and fix a Nash equilibrium \((x^*, y^*)\). Define for each \(t \geq -1\):

\[
x'_t = x_t - x^* \quad \text{and} \quad y'_t = y_t - y^*.
\]

Easy calculations show that:

\[
\forall t \geq 0, \quad \begin{cases} x'_{t+1} = x'_t + 2\eta Ay'_t - \eta Ay'_{t-1} \\
y'_{t+1} = y'_t - 2\eta A^T x'_t + \eta A^T x'_{t-1}
\end{cases}
\]

So the sequence \((x'_t, y'_t)\) follows the OGDA algorithm of section 2 (see 5). Thus, according to Theorem 2.5, the sequence \((x'_t, y'_t)\) converges to a Nash equilibrium \((x'_\infty, y'_\infty) \in \text{Ker}(A^T) \times \text{Ker}(A)\). As a consequence \((x_t, y_t)_t\) converges to the limit \((x^\infty, y^\infty) = (x'_\infty + x^*, y'_\infty + y^*)\) which is a fixed point of the OGDA, hence a Nash equilibrium of the game.

Denote by \(\Pi\) the orthogonal projection onto \(\text{Ker}(A^T) \times \text{Ker}(A)\). According to Theorem 2.5, we have \((x'_\infty, y'_\infty) = \Pi(x_0, y_0)_t\), so:

\[
(x^\infty, y^\infty) = (x^*, y^*) + \Pi(x_0, y_0) - \Pi(x^*, y^*) = (x^*, y^*) + \Pi((x_0, y_0) - (x^*, y^*)).
\]
Thus $(x_{\infty}, y_{\infty})$ is the orthogonal projection of $(x_0, y_0)$ onto the affine space $\text{Ker}(A^T) \times \text{Ker}(A) + (x^*, y^*)$, which is equal to $\{(x, y)|A^T x + c = 0, Ay + b = 0\}$.

It remains to study the convergence speed. For all $t$, we have $\|(x_t, y_t) - (x_{\infty}, y_{\infty})\| = \|(x_t', y_t') - (x_{\infty}', y_{\infty}')\|$ and we obtain from Theorem 2.5 an exponential speed of convergence: for all $t \geq 0$,

$$\|(x_t, y_t) - (x_{\infty}, y_{\infty})\| \leq C' \lambda_{\max}^t,$$

with $C = \sqrt{\frac{2}{1 - \sqrt{1 + 4\eta_2^2\mu_{\max}}}}$, $D'$ is the distance from $(x_0', y_0', x_{-1}', y_{-1}')$ to the set $\{(x, y, x, y) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p, A^T x = 0, Ay = 0\}$, and

$$\lambda_{\max} = \max \left\{ \sqrt{\frac{1}{2} (1 + \sqrt{1 - 4\eta_2^2\mu})}, \mu > 0, \mu \in \mathcal{S}(A) \right\}.$$

Let us finally notice that:

$$D' \leq \|(x_0', y_0', x_{-1}', y_{-1}')\| = \|(x_0, y_0, x_{-1}, y_{-1}) - (x^*, y^*, x^*, y^*)\|.$$ Since this is true for all Nash equilibria $(x^*, y^*)$, we obtain that $D'$ is not greater than the distance from $(x_0, y_0, x_{-1}, y_{-1})$ to the set $\{(x, y, x, y), A^T x + c = 0, Ay + b = 0\}$.

### A.3 Proof of Proposition 4.12

Let $\mu \neq 0$ be in $\text{Sp}(B^T A)$, and $\lambda \in S^*(\mu)$. Then

$$E_{\lambda}^* = \left\{ (x, y, x', y')^T \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p, x = \lambda x', y = \lambda y', A^T Ay' = \frac{\mu}{\alpha} y', x' = (1 - 2\lambda)\eta \lambda (1 - \lambda) A^T y' \right\},$$

with $\lambda^2 (1 - \lambda)^2 = \eta_2^2 \mu (2\lambda - 1)^2$. Thus $\dim(E_{\lambda}^*) = \dim(\text{Ker}(A^T A - \mu I))$.

Thanks to Lemma 4.11, for each $\mu$ in $\text{Sp}(A^T A) \setminus \{0\}$ we have exactly 4 elements in $S^*(\mu)$.

Regarding $\lambda = 0$, we have $E_{0}^* = \left\{ (x, y, x', y'), x = 0, y = 0, Ay' = 0, B^T x' = 0 \right\}$, so that $\dim(E_{0}^*) = \dim(\text{Ker}(A)) + \dim(\text{Ker}(B^T)) = \dim(\text{Ker}(A)) + \dim(\text{Ker}(A^T))$.

And if $\lambda = 1$, then $E_{1}^* = \left\{ (x, y, x', y'), x = x', y = y', Ay' = 0, B^T x' = 0 \right\}$, and again, $\dim(E_{1}^*) = \dim(\text{Ker}(A)) + \dim(\text{Ker}(A^T))$.

We proceed exactly as in the proof of proposition 4.5 to obtain

$$\sum_{\lambda \in \text{Sp}(A_{A,B})} \dim(E_{\lambda}) = 2(n + p).$$

As a consequence $\Lambda_{A,B}$ is diagonalizable.

### A.4 Proof of Theorem 4.13

We have for each $t$, $Z_t = \Lambda_{A,B}^t Z_0$, with $Z_t$ the column vector $(x_t^T, y_t^T, x_{t-1}^T, y_{t-1}^T)^T$.

Because $\Lambda_{A,B}$ is assumed to be diagonalizable, $Z_0$ can be uniquely written
$Z_0 = \sum_{\mu \in S(A,B)} \sum_{\lambda \in S^*(\mu)} z_{0,\lambda},$ with $z_{0,\lambda} \in E^*_\lambda$ for each $\lambda$.

Then for each $t$,

$$Z_t = \sum_{\mu \in S(A,B)} \sum_{\lambda \in S^*(\mu)} \lambda^t z_{0,\lambda}.$$  

Define $\rho_* = \max\{ |\lambda| \in \text{Sp}(\Lambda_{A,B}), z_{0,\lambda} \neq 0 \}$, and $\lambda_*$ an eigenvalue verifying $|\lambda_*| = \rho_*$. 

If $\rho_* < 1$, then $Z_t \xrightarrow{t \to \infty} 0$ and $(x_t, y_t) \xrightarrow{t \to \infty} (0, 0)$, which is a Nash equilibrium of the game. 

If $\rho_* = 1$, looking at Lemma 4.11, we can see that $\lambda_* = 1$ is the only possible option such that $|\lambda_*| = \rho_*$. Then $Z_t \xrightarrow{t \to \infty} z_{0,1} \in E^*_1$. So $(x_t, y_t)$ converges to a vector $(x, y)$ with $Ay = 0$ and $B^T x = 0$, that is a Nash equilibrium of the game.

Assume finally that $\rho_* > 1$, we will prove that $x_t^T Ay_t \xrightarrow{t \to \infty} +\infty$ and $x_t^T Ay_t \xrightarrow{t \to \infty} +\infty$. Let $\lambda_*$ be such that $|\lambda_*| = \rho_*$. Then, because $|\lambda_*| > 1$, according to Lemma 4.11, $\lambda_*$ is a positive real: $\lambda_* \geq 1$.

Consider $z_{0,\lambda_*} = (x, y, x', y')^T \in E^*_{\lambda_*}$, we have $z_{0,\lambda_*} \neq 0$, $x = \lambda_* x'$, $y = \lambda_* y'$, $B^T Ay' = \mu^* y'$ and $x' = \frac{1}{\lambda_*}Ay'$, where $\mu^*$ is such that $\lambda_* \in S^*(\mu^*)$.

Since $\lambda_* > 1$, we not only have $\lambda_* \in S^*(\mu^*)$, but we also have $\lambda_*(1 - \lambda_*) = \sqrt{\mu^2 \eta(1 - 2\lambda_*)}$.

We obtain:

$$\langle x', Ay' \rangle = \frac{1}{\sqrt{\mu^*}} \|Ay'\|^2 > 0,$$

and for the matrix $B$:

$$\langle B^T x', y' \rangle = \|B^T y'\|^2 > 0,$$

Then, because $x_t \simeq \lambda_*^t x$ and $y_t \simeq \lambda_*^t y$, we finally have for the payoffs of stage $t$:

$$\langle x_t, Ay_t \rangle \simeq \lambda_*^{2t} \langle x, Ay \rangle \xrightarrow{t \to \infty} +\infty.$$

and

$$\langle B^T x_t, y_t \rangle \simeq \lambda_*^{2t} \langle B^T x, y \rangle \xrightarrow{t \to \infty} +\infty.$$