Exploring the Rindler vacuum and the Euclidean Plane

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Abstract

In flat spacetime, two inequivalent vacuum states which arise rather naturally are the Rindler vacuum $|R\rangle$ and the Minkowski vacuum $|M\rangle$. We discuss several aspects of the Rindler vacuum, concentrating on the propagator and Schwinger (heat) kernel defined using $|R\rangle$, both in the Lorentzian and Euclidean sectors. We start by exploring an intriguing result due to Candelas and Raine [1], viz., that $G_R$, the Feynman propagator corresponding to $|R\rangle$, can be expressed as a curious integral transform of $G_M$, the Feynman propagator in $|M\rangle$. We show that, this relation actually follows from the well known result that, $G_M$ can be written as a periodic sum of $G_R$, in the Rindler time $\tau$, with the period (in proper units) $2\pi i$. We further show that, the integral transform result holds for a wide class of pairs of bi-scalars $\{F_M, F_R\}$, provided $F_M$ can be represented as a periodic sum of $F_R$ with period $2\pi i$. We provide an explicit procedure to retrieve $F_R$ from its periodic sum $F_M$, for a wide class of functions. An example of particular interest is the pair of Schwinger kernels $\{K_M, K_R\}$, corresponding to the Minkowski and the Rindler vacua. We obtain explicit expression for $K_R$ and clarify several conceptual and technical issues related to these biscalars both in the Euclidean and Lorentzian sector. In particular we address the issue of retrieving the information contained in all the four wedges of the Rindler frame in the Lorentzian sector, starting from the Euclidean Rindler (polar) coordinates. This is possible but require four different types of analytic continuations, based on one unifying principle. Our procedure allows generalization of these results to any (bifurcate Killing) horizon in curved spacetime.

1 Introduction and Motivations

In standard quantum field theory, the Fock basis is introduced by identifying the creation ($a_j$) and annihilation ($a_j^\dagger$) operators, defining a vacuum state through $a_j|0\rangle = 0$, and building multi particle states by the repeated action of $a_j^\dagger$ on the vacuum state. The creation and annihilation operators are identified, in turn, by using a suitable definition for positive frequency mode functions. As is well known, such procedures are not unique and one can easily construct inequivalent class of creation/annihilation operators thereby leading to different vacuum states [2].

One such situation — extensively studied in literature — corresponds to the notions of Rindler vacuum and Minkowski vacuum [3–7], which arise along the following lines. In a $D$-dimensional flat spacetime, introduce the standard Lorentzian coordinates $x^a = (t, x, x_\perp)$. The $x - t$ plane is divided into four wedges $(R, L, F, P)$ by the $x = \pm t$ planes in the standard manner. Introduce the Rindler coordinates $(\tau, \rho, x_\perp)$ in the right $(R)$ and the left $(L)$ wedges with, for e.g., $x = \rho \cosh \tau, t = \rho \sinh \tau$ in $R$ and $x = -\rho \cosh \tau, t = -\rho \sinh \tau$ in $L$, with similar transformations in other wedges [8]. (We assume that $\rho > 0$)

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and \(-\infty < \tau < \infty\). The metric is static with respect to both \(t\) and \(\tau\) coordinates and hence one can find mode functions which are positive frequency with respect to \(t\) or with respect to \(\tau\). The corresponding creation/annihilation operators can be used to define the Minkowski vacuum \(|M\rangle\) and the Rindler vacuum \(|R\rangle\). One can then build standard QFT based on these two vacua and study their inter-relationship. In particular, one can define the Minkowski and Rindler propagators by the standard procedure:

\[
G_M(x_2, x_1) \equiv \langle M|T_t[\phi(x_2)\phi(x_1)]|M\rangle; \quad G_R(x_2, x_1) \equiv \langle R|T_\tau[\phi(x_2)\phi(x_1)]|R\rangle
\]  

(1)

where the subscripts on \(T\) indicate the time-coordinate used for time-ordering.

It is also well known that \(|M\rangle\) behaves like a thermal state filled with Rindler particles at the temperature \(T = (1/2\pi)\). This thermality implies that the Minkowski propagator \(G_M\) can be thought of as a ‘thermalised’ version of the Rindler propagator \(G_R\) in the following sense [9]:

\[
G_M(i\tau) = \sum_{n=-\infty}^{\infty} G_R(i\tau + 2\pi n)
\]

(2)

This equation is to be understood as summarizing a three-step process\(^1\): (i) Take the function \(G_R(\tau)\) and analytically continue it to the Euclidean time \(\tau_E \equiv i\tau\). (ii) Construct the sum in the right hand side by changing \(\tau_E \rightarrow \tau_E + 2\pi n\) and summing over all \(n\). (iii) Analytically continue back to \(\tau\). The result is formally expressed by Eq. (2).

1.1 Motivation 1: Probing an intriguing relation

There is, however, another intriguing relationship between \(G_M\) and \(G_R\), which has received very little attention in the literature. It turns out that, for events \((x_1, x_2)\) in \(R\), there is a curious relation between \(G_R\) and \(G_M\) given by:

\[
G_R(x_1, x_2) = G_M[\sigma(x_1, x_2)] - \int_{-\infty}^{\infty} d\lambda \frac{G_M[\sigma(x_1, x_{re}(\lambda))]}{\pi^2 + (\lambda - \tau_1)^2}
\]

(3)

where \(\sigma^2(x, y)\) is the square of the invariant distance between the two events and \(x_{re}\) is the event obtained by reflecting the event \(x\) from right wedge \(R\) to left wedge \(L\). That is,

\[
\sigma^2(x_1, x_{re}(\lambda)) = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cosh(\lambda - \tau_2) + (\Delta x_\perp)^2 = \sigma^2(\tau_1, \tau_2 + i\pi)
\]

(4)

In the last expression, we have only displayed the dependence on the Rindler time coordinate. This relation was first obtained in 1976 by Candelas and Raine [1] by a fairly lengthy, detailed, computation. In the last four decades this result has received very little attention or elaboration in the literature — e.g., the reviews [10, 11] do not even mention it — though the structure of \(G_R\) has been investigated by several authors (for example, [9, 12–14] etc.) in the intervening years.\(^2\) This relation, however, is rather intriguing because of the following features:

(a) The second term in the right hand side of Eq. (3) uses \(G_M\) between the events \(x_1\) in \(R\) and \(x_{re}\) in \(L\). The origin of this reflected point has never been clarified in the literature, including the original paper.

\(^1\)Hereafter, we will say that a function \(f_M(\tau)\) is a thermalised version of a function \(f_R(\tau)\) when the two functions are related by this procedure.

\(^2\)In fact, the only two other papers in which we could find this result quoted briefly, without any elaboration, were Refs. [15, 16].
(b) As we said before, it is well known — and has been extensively discussed in literature — that the
Minkowski propagator \( G_M \) can be thought of as a ‘thermalised’ version of the Rindler propagator \( G_R \) in
the following sense:

\[
G_M(i\tau) = \sum_{n=-\infty}^{\infty} G_R(i\tau + 2\pi n)
\]  

(5)

This periodicity in the Euclidean time is, of course, the root cause of the thermal behaviour which arises
in the standard approach. It is not clear whether the relation in Eq. (3) is connected with this basic fact,
and if so, how. (We will show that they are intimately related.)

c) The original derivation makes use of the fact that \( G_R \) and \( G_M \) are the Feynman propagators in the
two vacua. It is not clear whether the same relation holds for a much wider class of functions (and we will
see that it does) and if so what are the essential ingredients which go into this relation. (We will discover
these ingredients.)

d) If you analytically continue \( \tau \to \tau_E = i\tau \), the coordinate transformation in the right wedge changes
from \((x = \rho \cosh \tau, t = \rho \sinh \tau)\) to \((x = \rho \cos \tau_E, t_E = \rho \sin \tau_E)\). Therefore, this analytic continuation
of the right wedge alone fills the entire Euclidean plane in Cartesian coordinates \((t_E, x)\). In particular,
the relation \( x = \rho \cosh \tau \) (with \( \rho > 0 \)) implies that \( x \) is always positive; but on analytic continuation
\( x = \rho \cos \tau_E \) (with \( \rho > 0 \)) covers the entire real line \((-\infty < x < \infty)\) including negative values of \( x \). This,
in turn, implies that there is no distinct “reflected event” \( x_{re} \) in the Euclidean sector (or, more precisely,
both \( L \) and \( R \) wedges maps to the entire Euclidean plane on their own). So the question arises as to how
Eq. (3) transforms into the Euclidean sector and how it is to be interpreted.

e) The second term in the right hand side of Eq. (3) convolves \( G_M \) with a kernel of the type \((x^2 + \pi^2)^{-1}\).

Though the authors of \([1]\) make a passing comment of some ‘line charge density’, this interpretation doesn’t
make any clear physical sense. On the other hand, this particular kernel is well known in complex analysis
and is a special case of a Poisson kernel \([17]\) with the factor \((x^2 + y^2)^{-1}\) with \( y = \pi \). The Poisson kernel can
be used to extend functions defined on the real line to the upper half plane in such a way that they remain
bounded (unlike the usual analytic continuation) in the upper half plane.\(^3\) Roughly speaking the role
of the Poisson kernel is to allow \( G_M \) to be extended into the upper complex plane and, in particular, along
the line \( z = \tau + i\pi \). This is important because the transformation \( \tau \to \tau + i\pi \) actually reflects the event
\((x = \rho \cosh \tau, t = \rho \sinh \tau)\) to \((x = -\rho \cosh \tau, t = -\rho \sinh \tau)\) thereby taking \( x \) to \( x_{re} \). Since \( G_M \) in
the second term of Eq. (3) is already evaluated at \( x_{re} \) this convolution with a Poisson kernel actually brings it
back to \( R \). It is not a priori obvious what exactly is going on in the reflection followed by the convolution
by the Poisson kernel.

A closely related aspect is the introduction of the transformation \( \tau \to \tau + i\pi \) of the Rindler time, which is
different from the usual Euclideanisation obtained by \( \tau \to i\tau \). In Euclideanisation we rotate the
time axis by \( \pi/2 \) in going from \( \text{Re} \ \tau \) to \( \text{Im} \ \tau \). Here we are shifting \( \text{Re} \ \tau \) parallel to itself by an amount
\( \pi \) in the complex plane. The properties of such an analytic extension for various functions plays a crucial
role in the structure of Eq. (3).

This paper arose, partially, out of our attempt to probe these issues and to distill the essential mathe-
matical structure behind Eq. (3). We will show that a large class of functions satisfy Eq. (3) — of which
the Feynman propagator is just a special case — and identify the key structural factor common to all these
functions. We will provide a very simple, straight forward proof of Eq. (3) for a broad class of functions in
Section 2 and will follow it up with a more elegant and formal proof which highlights the role of complex
plane, Poisson kernel and the analytic extension \( \tau \to \tau + i\pi \).

\(^3\)For example, the standard analytical continuation of \( \cos x \) will lead to \( \cos z \equiv \cos(x + iy) \) while Poisson kernel will extend
\( \cos x \) to \( e^{-y} \cos x \).
1.2 Motivation 2: Schwinger (heat) kernel and inequivalent vacua

There is a second motivation for this work which is related to the Schwinger kernel approach to the inequivalent vacua. We will now describe this motivation.

The canonical quantization (based on identifying creation/annihilation operators, vacuum state, Fock basis etc) is the familiar procedure one usually uses. It is, however, possible to approach QFT in curvilinear coordinates/curved spacetime from a different perspective based on the Schwinger (heat) kernel approach. In this approach the central quantity is the Schwinger kernel which, in Lorentzian spacetime, satisfies the covariant differential equation and the boundary condition given by

\[
\frac{i}{s} \frac{\partial K}{\partial s} + \Box K = \delta(s) \delta(x, x_0); \quad \lim_{s \to 0} K(x, x_0; s) = \delta(x, x_0)
\] (6)

where \(\delta(x, x_0)\) is the properly densitised Dirac delta function in curved spacetime or curvilinear coordinates. It is possible to obtain all other relevant constructs — e.g., the Feynman propagator, effective Lagrangian etc. — from the kernel by:

\[
iG(x, x_0) = \int_0^\infty ds K(x, x_0; s); \quad iL_{\text{eff}}(x) = \int_0^\infty \frac{ds}{s} K(x, x_0; s)
\] (7)

The Eq. (6) shows that \(K\) just encodes the properties of the Laplacian in a given curved spacetime and is independent of the physics of the scalar field.\(^4\) There are two aspects related to the Schwinger kernel which we will focus on in this work.

First, the relation between the kernel and the propagator in Eq. (7), as well as the fact that the propagators for two vacuua are related by the thermalisation procedure in Eq. (2) tells us that the two kernels will also satisfy a similar relation:

\[
K_M(i\tau) = \sum_{n=-\infty}^{\infty} K_R(i\tau + 2\pi n)
\] (8)

We will show in Section 3 that any two functions which obey this thermalisation condition will also satisfy a result like Eq. (3) which can be obtained as the “inverse” of the thermalisation condition. Using this fact, we can immediately obtain \(K_R\) from the well-known expression for \(K_M\). We could not find explicit expressions for the Schwinger kernel, corresponding to Rindler vacuum, in the published literature. Its simple derivation shows the power of our general technique, involved in “inverting” Eq. (8).

The above procedure will lead to an expression for \(K_R\) which is distinct from \(K_M\). This is, of course, as it should be because \(K_R\) and \(K_M\) have to lead to very different propagators \(G_R\) and \(G_M\). This, fact, however, raises a second question: We can incorporate both the differential equation and the boundary condition in Eq. (6) and write down the formal solution to this equation in the form

\[
K = e^{i\frac{s}{s}} \delta(x, x_0)
\] (9)

Once we determine \(K\) we can obtain the Feynman propagator by using Eq. (7). In this approach it appears that we are obtaining a Feynman propagator directly from the solution for the Schwinger kernel given in Eq. (9). It is not a priori obvious how this approach will lead to different vacua, different kernels and

\(^4\)If we consider a massive scalar field then the corresponding kernel will involve the mass \(m\) and is related to massless heat kernel by the simple relation: \(K_m \neq 0(x, x_0; s) = e^{-im^2s} K_{m=0}(x, x_0; s)\) so that it is enough to concentrate on the massless case, most of the time.
different propagators; at first sight, the solution defining the kernel in Eq. (9) appears to be unique and generally covariant, suggesting that there is a unique kernel for the quantum field, independent of the coordinates used to describe it.\footnote{Defining the Schwinger kernel through an explicit solution in Eq. (9) is somewhat different — conceptually and technically — from defining the kernel simply as a solution to the partial differential equation Eq. (6). In the latter approach one has certain level of ambiguity in the solution since one could add any solution to the homogeneous part of the equation. This ambiguity is removed when we define the kernel directly through Eq. (9). Therefore one cannot claim that the difference between Rindler vacuum and the Minkowski vacuum can be taken care of by adding a suitable solution to the homogeneous part of the differential equation in Eq. (6).} This, of course, cannot be true because we know from the canonical approach that the definition of propagators also depend on the vacuum state and they are (non-trivially) different when computed using $\ket{M}$ or $\ket{R}$. If, instead, we use Eq. (9) to determine $K$ and Eq. (7) to determine $G$ the information about vacuum states has to slip in through boundary conditions. It is important to identify how exactly this comes about.

Interestingly enough, the difference between Rindler and Minkowski vacuum is encoded in the way we choose to represent the Dirac delta function $\delta(x, x_0)$ in Eq. (9). Notice that in Eq. (9) both the operator $\Box$ as well as $\delta(x, x_0)$ are generally covariant constructs. So if we change the coordinate system, the right hand side of Eq. (9) will transform in a generally covariant manner and so will the kernel $K(x, x_0; s)$. To get a distinctly different kernel, we need to actually tinker with the Dirac delta function. Given the rather curious nature of this result, we will discuss it at some detail. (As we said before, we have not seen the expression for $K_R$ in the published literature and the expressions we derive in this paper might be also of intrinsic interest.)

### 1.3 Motivation 3: From the Euclidean plane to all the four Rindler wedges

The standard coordinates in right Rindler wedge $(\tau, \rho)$ are related to the inertial $(t, x)$ coordinates by: $x = \rho \cosh \tau, t = \rho \sinh \tau$ with $\rho > 0, -\infty < \tau < \infty$. Clearly the $(\tau, \rho)$ coordinates only cover the right wedge. The analytic continuation $it \rightarrow t_E$ and $i\tau \rightarrow \tau_E$ leads to $x = \rho \cos \tau_E, t_E = \rho \sin \tau_E$ with $\rho > 0, 0 < \tau < 2\pi$. We see that the Euclidean coordinates $(\tau_E, \rho)$ covers the entire Euclidean plane $(t_E, x)$. In other words the analytic continuation of the Rindler right wedge alone fills the entire Euclidean plane, covered by the inertial (Euclidean) coordinates. While the transformation $x = \rho \cosh \tau$ with real $\tau$ only covers $x > 0$, the shift $\tau \rightarrow \tau + i\pi$ takes us from $x > 0$ to $x < 0$; the imaginary values of $\tau$ knows about the region beyond the horizon. The horizon $(x^2 - t^2 = 0)$ in the Lorentzian sector collapses to the origin $(x_E^2 + t_E^2 = 0)\footnote{Defining the Schwinger kernel through an explicit solution in Eq. (9) is somewhat different — conceptually and technically — from defining the kernel simply as a solution to the partial differential equation Eq. (6). In the latter approach one has certain level of ambiguity in the solution since one could add any solution to the homogeneous part of the equation. This ambiguity is removed when we define the kernel directly through Eq. (9). Therefore one cannot claim that the difference between Rindler vacuum and the Minkowski vacuum can be taken care of by adding a suitable solution to the homogeneous part of the differential equation in Eq. (6).}$ in the Euclidean plane and $F, L, P$ wedges disappear. It can be easily checked that the same phenomenon occurs when we analytically continue from, say, the left wedge as well.

This raises an interesting question, which provides the third motivation for this work. It is generally believed that, at least flat spacetime, one can define the quantum field theory — in particular, the Feynman propagator — properly in the Euclidean sector and then analytically continue it to the Lorentzian sector. This certainly works when we use inertial coordinates and go from $(t_E, x)$ to $(t, x)$. But it appears that if we start with Rindler (‘polar’) coordinates $(\tau_E, \rho)$ in the Euclidean sector, write down the Euclidean propagator, and analytically continue to the Rindler coordinates in Lorentzian sector, we only seem to recover the Lorentzian propagator in the right wedge! How do we obtain the form of the propagator when one or both of the coordinates are outside the right wedge? Since the Euclidean Rindler coordinates cover the entire Euclidean inertial manifold, there must exist a way of obtaining the results for all the four wedges in the Lorentzian sector, starting from the Euclidean Rindler coordinates.

We will see that this is indeed possible but the analytic continuation is nontrivial especially when the two events in $G(x_1, x_2)$ are in different sectors. (This was briefly mentioned by one of the authors, in the
Appendix of Ref. [18] and we will elaborate on this construction, especially for the Schwinger kernel.) This construction will also provide a relatively simple route to the propagator in different sectors obtained, for example, by a more complicated procedure in Ref. [19]. We have not seen the corresponding results for the Schwinger kernel in the literature and we will provide their explicit forms.

2 A general result for a class of functions

We will start with the task of understanding Eq. (3). It is well known that the time dependence of the form \( \exp(-i\omega|\tau|) \) with \( \omega > 0 \) plays a crucial role in the structure of Feynman propagator. It is this factor which propagates positive frequency modes forward in time and negative frequency modes backward in time. Consider a function built by an arbitrary superposition of \( \exp(-i\omega|\tau|) \) in the form

\[
F_R(\tau) = \int_0^\infty d\omega \ A(\omega)e^{-i\omega|\tau|} = \int_0^\infty d\omega \ A(\omega)(\sinh \pi\omega)e^{-i\omega|\tau|} \tag{10}
\]

The first equality defines the superposition in terms of the weightage \( A(\omega) \); in the second equality we have set \( A(\omega) \equiv A(\omega)\sinh \pi\omega \) for future convenience. We have only displayed the dependence on the Rindler time coordinate \( \tau \) but both \( F_R \) as well \( A(\omega) \) will depend on all the other coordinates. Most of the time, we will be interested in biscalars which depend on two events \( x_1 \) and \( x_2 \) with (i) a dependence in Rindler time coordinates through \( \tau \equiv \tau_2 - \tau_1 \) and (ii) a dependence in the transverse coordinates through \( x_\perp \equiv x_2^2 - x_1^2 \). That is, the biscalars respect the translation invariance of the Rindler metric in the Rindler time coordinate and the transverse coordinates. While dealing with such biscalars, we actually have \( F_R = F_R(\rho_1, \rho_2, x_\perp, \tau) \) and \( A = A(\rho_1, \rho_2, x_\perp, \omega) \) but for clarity, we suppress the display of the dependence on \( \rho_1, \rho_2, x_\perp \) in both \( F_R \) and \( A(\omega) \). In all these discussions, the transverse coordinates go for ride; so it is often convenient to Fourier transform all the relevant functions with respect to \( x_\perp \) and work with \( F_R(\rho_1, \rho_2, k_\perp, \tau) \) and \( A(\rho_1, \rho_2, k_\perp, \omega) \); this is what we will do most of the time.

Let us next construct another function obtained by thermalising \( F_R(\tau) \) along the lines described just after Eq. (5). This is easily done using the result:

\[
(\sinh \pi\omega) \sum_{n=-\infty}^{\infty} e^{-\omega|\tau_E + 2\pi n|} = \cosh \omega (|\tau_E| - \pi) ; \quad (0 < |\tau_E| < 2\pi) \tag{11}
\]

and analytically continuing back from \( \tau_E \) to \( \tau \). (For completeness, we have provided a proof of this relation in Appendix A). In Eq. (11), the left hand side is clearly periodic in \( \tau_E \) with period \( 2\pi \) by construction. It is obvious that if, say, \( \tau_E \) lies between \( 2\pi N \) and \( 2\pi(N + 1) \) the sum can only depend on \( \tau_E - 2\pi N \). In the right hand side, therefore, the range of \( \tau_E \) is restricted to the interval \((0, 2\pi)\). This procedure leads us to the second function \( F_M(\tau) \) which is the thermalised version of \( F_R(\tau) \):

\[
F_M(\tau) = \int_0^\infty d\omega \ A(\omega) \cosh \omega [i|\tau| - \pi] \tag{12}
\]

It is now possible to prove that \( F_R \) and \( F_M \) — defined by Eq. (10) and Eq. (12) — satisfy the relation in Eq. (3) we are trying to understand, for any choice of \( A(\omega) \). In other words, only two ingredients have gone into proving the result in Eq. (3) for this class of functions.\(^{6}\) First is that \( F_R \) is built from

\(^{6}\text{In fact, the result in Eq. (3) holds for a wider class of functions, viz., all even functions } F_M(\tau). \text{ Such a function can be expressed as a superposition of } \exp(-i\omega|\tau|) \text{ with positive and negative } \omega \text{ with an weightage factor that is even in } \omega. \text{ We will comment on this generalization towards the end.}\)
an arbitrary superpositions of \( \exp(-i\omega|\tau|) \) with \( \omega > 0 \). Second, \( F_M \) is constructed by thermalising \( F_R \). We know that these two conditions are indeed satisfied by the Rindler and Minkowski propagators. The Feynman propagator for the Rindler vacuum is built from a superposition of \( \exp(-i\omega|\tau|) \) with \( \omega > 0 \); and we know from Eq. (5) that \( G_M \) is a thermalised version of \( G_R \). So clearly, these two functions satisfy Eq. (3).

Proving that the functions defined by Eq. (10) and Eq. (12) satisfies the relation like Eq. (3) is fairly straightforward. We first note that the factor \( A(\omega) \) and the integration over \( \omega \) goes for a while establishing Eq. (3) between \( F_R \) and \( F_M \). So all we need to prove is an identity satisfied by hyperbolic functions in the form:

\[
\sinh(\pi\omega)e^{-i\omega|\tau|} = \cosh(\omega(|r| - \pi)) = \cosh(\omega(|\mu| - \pi) + \pi) - \int_{-\infty}^{\infty} d\mu \frac{\cosh(\omega[||\mu| - \pi| + \pi])}{(\mu - \tau)^2 + \pi^2}
\]  
(13)

(We stress that this is merely an identity involving hyperbolic functions, devoid of physics content.) In the integral in the second term on the right hand side, we actually have just \( \cosh(i\omega|\mu|) = \cos(\omega\mu) \) in the numerator, which we have written in such a manner that it can be compared with the first term on the right hand side, evaluated with a shift of \( \pi \). When \( \tau > 0 \), this shift of \( \pi \) in \( i\tau \) leads to a shift of \(-i\pi \) in \( \tau \) and thus “reflects” the event from the right wedge to the left wedge. (When \( \tau < 0 \), the shift is by \( i\pi \), which again leads to the same ‘reflection’.) Once we have the result in Eq. (13) we can multiply the whole equation by \( A(\omega) \) and integrate over \( \omega \) to establish that \( F_M \) and \( F_R \) satisfy the relation in Eq. (3).

The proof of Eq. (13) is straightforward, almost trivial. Writing the numerator of the integral as \( \cos(\omega\mu) \) and performing the integral, we get this term to be:

\[
-\int_{-\infty}^{\infty} d\mu \frac{\cos(\omega\mu)}{(\mu - \tau)^2 + \pi^2} = -e^{-\pi\omega} \cos(\omega\tau)
\]  
(14)

The result in Eq. (13) now reduces to a simple, easily verified, hyperbolic identity:

\[
(\sinh(\pi\omega)) e^{-\omega z} = \cosh(\omega|z - \pi|) - e^{-\pi\omega} \cosh(\omega z)
\]  
(15)

evaluated for \( z = i|\tau| \). This completes the proof of Eq. (13). As explained earlier, the only two ingredients which went into establishing this result are: (i) The \( F_R \) is built by a superposition of \( \exp(-i\omega|\tau|) \) and (ii) \( F_M \) is obtained from \( F_R \) by thermalisation.

It is easy to verify that the Rindler and Minkowski propagators indeed have the form of \( F_M \) and \( F_R \) with a particular choice for \( A(\omega) \) given by

\[
A(\omega) = \frac{1}{\pi^2} K_{i\omega}(\mu\rho_1) K_{i\omega}(\mu\rho_2); \quad \mu^2 = m^2 + k_\perp^2
\]  
(16)

With this choice of \( A(\omega) \) in Eq. (10) and Eq. (12) we will correctly reproduce the two propagators \( G_R(\tau, k_\perp, \rho_1, \rho_2) \) and \( G_M(\tau, k_\perp, \rho_1, \rho_2) \) which are the Fourier transform of the propagators in transverse coordinates. The cognoscenti will immediately see that Eq. (10), with the \( A(\omega) \) in Eq. (16), gives the Rindler propagator because it is built from the normalized mode functions of the form \( u_{\omega, k_\perp} \propto (\sinh(\pi\omega))^{1/2} K_{i\omega}(\mu\rho) e^{i(k_\perp \cdot x_\perp - \omega \tau)} \). Proving that Eq. (16), substituted into Eq. (12) gives the Minkowski propagator is also not difficult. To do this most directly, start from the Schwinger representation for the Minkowski propagator given by

\[
G_M(x_1, x_2) = i \left( \frac{1}{4\pi i} \right)^{D/2} \int_0^\infty \frac{d\lambda}{\lambda^{D/2}} e^{-i\lambda m^2 - (i/4\lambda)|\sigma|^2}
\]  
(17)
where $\sigma^2(x_1, x_2)$ is the interval between the two events in $R$ expressed in Rindler coordinates. It contains the square of transverse separation $(\Delta x_\perp)^2$. Fourier transforming with respect to this separation leads to the result

$$G_M(\tau, k_\perp, \rho_1, \rho_2) = \frac{1}{2\pi} K_0(\mu \ell)$$

(18)

where $\ell^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cosh \tau$ and $\mu^2 = m^2 + k_\perp$. We now recall the convenient identity (see Eq(24), p382 of [20])

$$\frac{\pi}{2} K_0(\mu \ell) = \int_0^\infty d\omega K_{i\omega}(\mu \rho_1) K_{i\omega}(\mu \rho_2) \cosh[\omega(\pi - \tau_E)]$$

(19)

analytically continue it from $\tau_E$ to $\tau$, and express $K_0(\mu \ell)$ as a product of two $K_{i\omega}$ functions. This will prove that the choice of $A(\omega)$ in Eq. (16), when substituted into Eq. (12), does reproduce the Minkowski propagator.

We will now comment on a wider class of functions for which this result holds. Any even function, $F_R(\tau)$, can be expressed in the form:

$$F_R(\tau) = \int_{-\infty}^{+\infty} d\omega B(|\omega|) \exp(-i\omega|\tau|)$$

(20)

which differs from our original set in Eq. (10) by the fact that we are now integrating over the whole range of $\omega$ without just restricting to the positive frequencies. Once we obtain the result in Eq. (13) we can multiply the whole equation by $B(|\omega|)$ and integrate over all $\omega$ and again obtain the result in Eq. (3) for all $F_R(\tau)$ which is even in $\tau$. A simple example of such a function is the Schwinger (heat) kernel defined with respect to Rindler and Minkowski vacua. The Rindler heat kernel $K_R(\tau)$ is an even function of $\tau$ and can indeed be expressed in the form an integral in Eq. (20). So the result in Eq. (3) holds not only for the propagators but also for the heat kernels and we will discuss these features in detail in Sections 4 and 5.

We will now provide a more elegant proof of Eq. (3) for all even functions $F_R(\tau)$ which clearly proves that it is an “inversion” of the thermalisation condition.

### 3 Inversion of the thermalisation condition: An elegant approach

We will next provide a more elegant, general, but somewhat formal proof of the results obtained in the last section. This proof has the advantage that — besides being fairly elegant — it can be generalized to a wide class of spacetimes and a variety of functions related to Rindler-like and Minkowski-like vacua. The basic procedure is as follows. We start with two functions $F_M(\tau)$ and $F_R(\tau)$ related by the thermalisation condition in Eq. (5). The only assumption we make is that $F_R(\tau) = F_R(-\tau)$, viz. $F_R$ is an even function of its relevant argument. We then express this in the form of an integral relation given by

$$F_M(z) = \int_{C'} \frac{du}{(i\pi)} H(z; u) F_R(u).$$

(21)

where $H$ is a suitable integral kernel and $C'$ is a specific contour in the complex plane given in Figure 1a. We have implicitly assumed that $F_R$ has the required limits for the right hand side of Eq. (21) to exist. Once $F_M$ is related to $F_R$ by this integral equation, we can invert the kernel and find $F_R$ in terms of $F_M$ thereby again getting a relation of the form

$$F_R(z) = \int_{C} \frac{du}{(i\pi)} G(z; u) F_M(u)$$

(22)
where $G$ is the ‘inverse’ of $H$ which appears in Eq. (21). (Again, we require that $F_M$ vanishes sufficiently faster on the asymptotic regions of $C$ for the last integral to exist.) This result, in turn, will lead to an expression of the form in Eq. (3) for the two functions $F_M$ and $F_R$. We will now provide the details of this approach.

Since $F_M$ is an infinite periodic sum of $F_R$, Eq. (21) will lead to a result like Eq. (5) if $H$ has poles at
\[ u = \pm (z + 2\pi n); \quad n \in \mathbb{Z} \]  
with a constant value of residues, independent of $n$. There are several functions which will satisfy this criterion and we will choose for our purpose the function
\[ H(z; u) = \frac{\sinh u}{4(cosh u - \cosh z)} \]  
with the integration contour $C'$ shown in Figure 1a and $p_i$ (see the figure) represents a generic pole of the function $F_M$. A straightforward calculation using the residue theorem now leads to the result
\[ F_M(z) = \int_{C'} \frac{du}{(i\pi)} H(z; u) F_R(u) \]  
\[ = \frac{1}{2} \sum_{n=-\infty}^{\infty} F_R(z + 2\pi n) + \frac{1}{2} \sum_{n=-\infty}^{\infty} F_R(-z + 2\pi n) \]  
\[ = \sum_{n=-\infty}^{\infty} F_R(z + 2\pi n) \]  
where the last step uses our assumption that $F_R(z)$ is an even function. This completes the first part of our task, viz., expressing $F_M$ as a contour integral involving $F_R$. The next step is to invert the relation

![Figure 1: (a) Contour C' and (b) contour C.](image)

in Eq. (21) and express $F_R$ as an integral over $F_M$ in the form of Eq. (22). It turns out that the relevant
The inverse function to be used in Eq. (22) is given by

\[
G(z; u) = \frac{u}{(u^2 - z^2)}
\] (28)

where the contour is \(C\) shown in Figure 1b. We will first demonstrate that, with this choice of \(G\) and the contour \(C\), we do reproduce a relation of the form in Eq. (3). Having done that, we will provide a direct proof that \(G\) and \(H\) are indeed inverses of each other in Appendix B.

With the choice of \(G\) in Eq. (28) and the contour shown in Fig. 1, the relevant integral in the complex plane becomes the sum of three parts.

\[
\int_C \frac{du}{(i\pi)} G(z; u) F_M(u) = \text{Residue term} + \text{vertical integral} + \text{horizontal integrals} \tag{29}
\]

The residue term is just \(F_M(z)\). Since \(F_M(z)\) is an even function, \(G(z; iy) F_M(iy)\) will be an odd function of \(y\), for real \(y\). Hence, the vertical integral vanishes. Let us now look at the horizontal integral.

\[
\text{Horizontal integrals} = \frac{i}{\pi} \int_0^\infty \frac{(x - i\pi)}{(x - i\pi)^2 + z^2} F_M(x - i\pi) dx + \frac{i}{\pi} \int_0^\infty \frac{(x + i\pi)}{(x + i\pi)^2 + z^2} F_M(x + i\pi) dx \tag{30}
\]

The evenness and pseudo-periodicity condition of \(F_M\) implies that \(F_M(x - i\pi) = F_M(x + i\pi)\). Hence,

\[
\text{Horizontal integrals} = -\int_0^\infty dx \left[ \frac{1}{\pi^2 + (x - z)^2} + \frac{1}{\pi^2 + (x + z)^2} \right] F_M(x + i\pi) \tag{31}
\]

\[
= \frac{1}{2} \int_{-\infty}^\infty dx \left[ \frac{1}{\pi^2 + (x - z)^2} + \frac{1}{\pi^2 + (x + z)^2} \right] F_M(x + i\pi) \tag{32}
\]

\[
= -\int_{-\infty}^\infty dx \frac{F_M(x + i\pi)}{\pi^2 + (x - z)^2} \tag{33}
\]

This gives us the result

\[
F_R(z) = F_M(z) - \int_{-\infty}^\infty dx \frac{F_M(x + i\pi)}{\pi^2 + (x - z)^2} \tag{34}
\]

which has the same form as in Eq. (3) in the context when \(x + i\pi\) leads to the reflected trajectory. Thus the result in Eq. (3) is extremely general and holds under the following generic conditions: (a) \(F_R(z)\) is an even function of \(z\) and \(F_M(z)\) is obtained by the thermalisation of the function \(F_M\). We do not use any other specific property of the spacetime or the nature of the two vacua etc.

It is clear that the very general nature of our proof allows it to be extended to several other spacetimes, like for e.g., Schwarzschild, deSitter etc. Whenever the spacetime has a bifurcate Killing horizon, the orbits of the Killing vector allow the partitioning of the relevant plane into four wedges just as in the case of Rindler coordinatisation of Minkowski spacetime. In a general context, we will not have explicit/closed form expressions for the mode functions of the scalar field and hence one may not be able to carry out explicit computation of, say, the propagators \(G_R\) and \(G_M\). However, in all these contexts (for e.g., Schwarzschild, deSitter, etc.) we can prove that the propagator in the global, Minkowski-like vacuum, \(G_M\) is a thermalised version of the propagator \(G_R\) defined in the Rindler-like vacuum. Given this thermalisation condition and the evenness of \(G_R(\tau)\) one can immediately obtain Eq. (34) in all these spacetimes. To complete the generalization, we should be able to arrange matters such that the shift \(\tau + i\pi\) leads to the reflected trajectory. This holds, again, in any spacetime with a bifurcate Killing horizon. In such
spacetimes, one can introduce coordinate systems in such a way that the thermalisation condition — viz., the periodicity of $\tau_E$ in $2\pi$ — arises as a two step process: the shift $\tau_E \rightarrow \tau_E + i\pi$ reflects the coordinates from the right wedge to the left wedge and a further shift by $-i\pi$ brings it back to the right wedge thereby ensuring periodicity. So our result holds in all these contexts which is a significant generalization of the original result of Ref. [1].

All that remains to be shown is that the integral kernels $G$ and $H$ (appearing in Eq. (21) and Eq. (22)) given by Eq. (24) and Eq. (28) are indeed inverses of each other. To do this, we only need to show that Eq. (22) implies Eq. (21) and vice-versa with these choices. This is completely straightforward and hence we delegate the technical details to Appendix B.

The result in Eq. (34) implies an simple relationship between the Fourier transform of $F_M$ and $F_R$. To derive this relation, we can restrict ourselves to real values of $z$. For simplifying the notation we will write, $F_M(x + i\pi) \equiv \tilde{F}_M^{(r)}(x)$ with the superscript “$r$” reminding us of the reflection in the coordinates. Then, for real values of $z$, our result in Eq. (34) reduces to:

$$F_R(\tau) = F_M(\tau) - \int_{-\infty}^{\infty} dx \frac{F_M(x + i\pi)}{\pi^2 + (x - \tau)^2}; \quad (\tau \in \mathbb{R}) \quad (35)$$

We will now take the Fourier transform of both sides of this equation with respect to $\tau$. Note that, the second term in Eq. (35) is just a convolution of $F_M^{(r)}$ with a normalized Lorentzian function, which has the Fourier transform $e^{-\pi|\omega|}$. So the convolution theorem leads to the final result:

$$\tilde{F}_R(\omega) = \tilde{F}_M(\omega) - e^{-\pi|\omega|} \tilde{F}_M^{(r)}(\omega) \quad (36)$$

where the tide over a function indicates the Fourier transform. We will see later that, this remarkably simple relation can be used to derive the Fourier transform with respect to the Rindler time $\tau$ of $G_R$, the Feynman propagator in the Rindler vacuum. Obtaining the same by direct calculation involves a tricky deformation of a contour integral. Further, by choosing $F_R$ and $F_M$ appropriately, the relation Eq. (36) can be applied to Feynman propagators in appropriate vacuum states in any (bifurcate Killing) horizon in curved spacetime. [We mention the following point to avoid possible misunderstanding. Since $F_M^{(r)}(x) \equiv F_M(x + i\pi)$, these two functions are related by a shift through a purely imaginary quantity, viz., $i\pi$. For shifts by real $\ell$, the Fourier transforms of $F_M(x)$ and $f_M(x) \equiv F_M(x + \ell)$, with respect to $x$, are related by $\hat{f}_M(\nu) = e^{-i\ell\nu} \tilde{F}_M(\nu)$, where the tide over a function indicates the Fourier transform. In general one cannot generalize this result to the case of a purely imaginary value of $\ell$, say $\ell = iy$, where $y \in \mathbb{R}$, and write $\hat{f}_M(\nu) = e^{iy\nu} \tilde{F}_M(\nu)$. This is because shifting the contour in the complex plane may not in general be legitimate. In that case, $\tilde{F}_M^{(r)}(\omega)$ has to be computed by a careful, independent, calculation. This happens to be the situation here and we will discuss later an explicit example, in which this fact will become clearer.]

Before concluding this section, we will briefly consider the Euclidean version of our results. For this purpose, let us consider an even function $f_M(z)$ which satisfies the condition $f_M(z + 2\pi n) = f_M(z)$. Hence, along real line, $f_M$ is a periodic function with a period $2\pi$. Let us now suppose that we can write $f_M(z)$ as a periodic sum of another even function $f_R(z)$. That is,

$$f_M(z) = \sum_{n=\infty}^{\infty} f_R(z + 2\pi n) \quad (37)$$

A natural question to ask is: can we retrieve the function $f_R(z)$ from $f_M(z)$ by an integral transformation analogous to Eq. (22). By an argument similar to the previous one in the Lorentzian sector, we can see
that this transformation is given by

$$f_R(z) = \int_{\pi + i\infty}^{-\pi + i\infty} f_M(u) \left( \frac{2u}{u^2 - z^2} \right) \frac{du}{(2\pi i)}$$

(38)

It is convenient to restrict the variable $z$ in the range $(-\pi, \pi)$ (which can be done without loss of generality), and call the variable $z$ with this restriction as $\theta$. We can now break down the integral in Eq. (38) into four parts: (i) the integral along the vertical line joining $\pi + i\infty$ and $\pi$ (ii) the integral along the vertical line joining $-\pi$ and $-\pi + i\infty$ (ii) integral from $\pi$ to $-\pi$ along the real line and (iv) the integral along infinitesimal semicircles around the poles of the integrand at $z = \pm \theta$. This is explicitly given below:

$$f_R(\theta) = \int_{-\infty}^{0} \frac{2(\pi + i\lambda)f_M(\pi + i\lambda)}{(\pi + i\lambda)^2 - \theta^2} \frac{d\lambda}{(2\pi)} + \int_{0}^{\infty} \frac{2(-\pi + i\lambda)f_M(-\pi + i\lambda)}{(-\pi + i\lambda)^2 - \theta^2} \frac{d\lambda}{(2\pi)}$$

$$+ \int_{\pi}^{-\pi} \frac{2\pi f_M(x)}{x^2 - \theta^2} dx + \frac{1}{2} \text{Res} \left[ f_M(u) \left( \frac{2u}{u^2 - \theta^2} \right) \right]_{u=\theta} + \frac{1}{2} \text{Res} \left[ f_M(u) \left( \frac{2u}{u^2 - \theta^2} \right) \right]_{u=-\theta}$$

The last two terms combine to give $f_R(\theta)$. The third term vanishes because the integrand is an odd functions. The surviving terms can be combined to get the following expression for $f_R(\theta)$:

$$f_R(\theta) = f_M(\theta) - \int_{-\infty}^{\infty} \frac{f_M(\pi + i\lambda)}{\pi^2 + (\lambda - i\theta)^2} d\lambda; \quad |\theta| < \pi$$

(40)

Once again, we see that the a Poisson-like kernel emerges. Note that this is exactly the form that one expects, given our result in Eq. (34).

4 The Schwinger kernel for the Rindler vacuum

We shall now turn our attention to the derivation and discussion of the Schwinger kernel corresponding to the Rindler vacuum. In particular, we will concentrate on two aspects. First, we will use the relations obtained in the previous sections which allows us to invert the thermalisation property and obtain the Rindler kernel in the Lorentzian sector directly. This shows the power of the result obtained in Section 3 by its application in a new non-trivial context. Second, we will address the question of how the Rindler kernel can be obtained in the Euclidean sector. This requires careful consideration of the boundary conditions and we will provide a derivation from first principles working entirely in the Euclidean sector. We will start by inverting the thermalisation condition in the Lorentzian sector.

4.1 Inversion of the thermalisation condition

As we said before, the Schwinger kernel also satisfies the thermalisation condition in the form of Eq. (8). Therefore, our result in section 3 tells us that it can be obtained from the Minkowski kernel through the relation

$$K_R(\tilde{\tau}) = \int_{C} \frac{du}{(i\pi)} \mathcal{G}(\tilde{\tau}; u) K_M(u)$$

(41)

where the “inverter” is given by

$$\mathcal{G}(\tilde{\tau}; u) = \frac{u}{u^2 - \tilde{\tau}^2}$$

(42)
and the Minkowski kernel, expressed in Rindler coordinates, is given by:

\[ K_M = \frac{1}{(4\pi is)} \exp \left( -\frac{\rho^2 + \rho'^2}{4i} \right) \exp (z \cosh \tilde{\tau}) \equiv N(\rho, \rho'; s) \exp (z \cosh \tilde{\tau}) \] (43)

with

\[ z = \frac{\rho \rho'}{2is}; \quad \tilde{\tau} = |\tau - \tau'| \] (44)

Note that, we have not bothered to include the transverse part and the mass term, since they go for a ride during the “inversion” process. All the non trivial aspects of the kernels that we need in this section are in their \( \tau, \rho \) dependent parts. Therefore, for the purpose of deriving the main result of this section, it suffices to work in the 1+1 dimensions. However, for completeness, by suitable generalization of 1+1 dimensional results, we shall explicitly present the d+1 dimensional case, towards the end.

In order to evaluate the integral in Eq. (41) we shall first assume that \( \text{Im} \bar{\tau} < -\pi \) and then analytically continue our result to \( \text{Im} \bar{\tau} = 0 \). We can then use

\[ \frac{u}{u^2 - \tilde{\tau}^2} = \int_0^\infty d\omega \sin(\omega u) e^{-i\omega \tilde{\tau}}; \quad (\text{Im}[\tilde{\tau}] < -|\text{Im}[u]|) \] (45)

to rewrite \( K_R \) as,

\[ K_R(\tilde{\tau}) = N \int_C \frac{du}{(i\pi^2)} \left( \int_0^\infty d\omega \sin(\omega u) e^{-i\omega \tilde{\tau}} \right) e^{z \cosh u}; \quad (\text{Im}[\tilde{\tau}] < -|\text{Im}[u]|) \] (46)

The order of integration can now be interchanged to get,

\[ K_R(\tilde{\tau}) = N \int_0^\infty d\omega e^{-i\omega \tilde{\tau}} \left( \int_C \frac{du}{(i\pi^2)} \sin(\omega u) e^{z \cosh u} \right) ; \quad (\text{Im}[\tilde{\tau}] < -|\text{Im}[u]|) \] (47)

\[ = N \int_0^\infty d\omega e^{-i\omega \tilde{\tau}} \left( \int_C \frac{du}{(i\pi^2)} \left[ \frac{e^{i\omega u} - e^{-i\omega u}}{2i} \right] e^{z \cosh u} \right) \] (48)

Recall that the Modified Bessel function \( I_\nu \) has the following integral representation [21].

\[ I_\nu(z) = \int_C \frac{du}{2\pi i} e^{z \cosh u} e^{-\nu u} \] (49)

where, the contour \( C \) is as shown in Figure 2. Hence, the expression for \( K_R \) reduces to:

\[ K_R(\tilde{\tau}) = N \int_0^\infty \frac{d\omega}{i} \frac{1}{i} e^{-i\omega \tilde{\tau}} \left( I_{-i\omega}(z) - I_{i\omega}(z) \right) \] (50)

\[ = N \int_0^\infty d\omega e^{-i\omega \tilde{\tau}} \left( \frac{2}{\pi} \sinh(\pi \omega) K_{i\omega}(z) \right) \] (51)

\[ = \frac{1}{(2\pi is)} \exp \left( -\frac{\rho^2 + \rho'^2}{4is} \right) \int_0^\infty \frac{d\nu}{\pi} e^{-i\nu \tilde{\tau}} \sinh(\pi \nu) K_{\nu} \left( \frac{\rho \rho'}{2is} \right) \] (52)

This gives the explicit form of the Rindler kernel in terms of a single integral. Unfortunately, the final

\[ ^{7} \text{It is assumed that } s \text{ actually stands for } s e^{-i\epsilon} \text{ to ensure convergence in the expressions; we shall not explicitly display it except when required.} \]
We now provide the result for the, massive, $d+1$ dimensional case. Let $K_R^{(m)}(k_\perp, \tilde{\tau})$ be the Fourier transform of the massive kernel $K_R^{(m)}$ with respect to the transverse Cartesian coordinate differences $\Delta x_\perp$. We also recall that the massive kernel is just $e^{-im^2s}$ times the massless kernel. Hence, the generalization of Eq. (52) can be conveniently written in terms of $K_R^{(m)}(k_\perp, \tilde{\tau})$ as,

$$K_R^{(m)}(k_\perp, \tilde{\tau}) = e^{-is\mu^2/(2\pi is)} \exp\left(-\frac{\rho^2 + \rho'^2}{4is}\right) \int_0^\infty d\nu \frac{\pi}{\nu} e^{-i\nu \tilde{\tau}} \sinh(\pi \nu) K_m \left(\frac{\rho\rho'}{2is}\right).$$

(53)

where, \(\mu^2 = m^2 + |k_\perp|^2\). (The $k_\perp$-dependent exponential factor coming from the Fourier transform of $e^{-\Delta x_\perp^2/(4is)}$ with respect to $\Delta x_\perp$ and $e^{-im^2s}$ are combined to give the $\mu^2$ dependent part of the kernel. This is why working with the Fourier transform with respect to the transverse coordinates is useful.) The rest of the terms in Eq. (53) can be seen to match exactly with the 1+1 dimensional massless kernel given in Eq. (52). To check the correctness of Eq. (52), let us calculate the Feynman propagator $G_R$ for a massive scalar field from $K_R^{(m)}(k_\perp, \tilde{\tau})$ and compare it with the known results in literature. The Fourier transform $G_R(k_\perp, \tilde{\tau})$ with respect to $x_\perp$ of the Feynman propagator $G_R$ for a massive scalar field can be evaluated using Eq. (7) to give:

$$G_R(k_\perp, \tilde{\tau}) = -i \int_0^\infty ds e^{-is\mu^2/(2\pi is)} \exp\left(-\frac{\rho^2 + \rho'^2}{4is}\right) \int_0^\infty d\nu \frac{\pi}{\nu} e^{-i\nu \tilde{\tau}} \sinh(\pi \nu) K_m \left(\frac{\rho\rho'}{2is}\right).$$

(54)

where, $\mu^2 = m^2 + |k_\perp|^2$. To simplify this expression further, we need the following identity:

$$\int_0^\infty \frac{du}{2s} \exp\left(-u - \frac{\rho_1^2 + \rho_2^2}{4s}\right) K_\nu \left(\frac{\rho_1\rho_2}{2u}\right) = K_\nu(\rho_1) K_\nu(\rho_2)$$

(55)
By a straightforward application of this result, we can simplify Eq.(54) to arrive at

\[ G_R(k_\perp, \tilde{\tau}) = \int_0^\infty \frac{d\nu}{\pi^2} e^{-i\nu|\tau - \tau'|} \sinh(\pi\nu) K_{\nu}(\mu\nu) K_{\nu}(\mu\nu'). \]  

(56)

which matches with the known expressions in the literature (see for example, [1, 12]). A useful mnemonic is worth mentioning here; the \( G_R(k_\perp, \tilde{\tau}) \) for a scalar field of mass \( m \) in \( d+1 \) dimensions can obtained from the propagator for a scalar field of mass \( m \) in \( 1+1 \) dimension by the replacement: \( m \rightarrow \mu \).

5 Solving the Euclidean heat kernel equation

The line interval in the 2-dimensional plane can be expressed in polar and Cartesian coordinates in the form:

\[ ds^2 = dx^2 + dy^2 = r^2 d\theta^2 + dr^2 \]  

(57)

As is well known, these are the Euclidean versions of the corresponding line element which we encounter in Minkowski and Rindler coordinates:

\[ ds^2 = -dt^2 + dx^2 = -\rho^2 d\tau^2 + d\rho^2 \]  

(58)

The Euclidean quantum field theory can be reformulated in terms of an appropriate 'heat kernel', the Euclidean analogue of Schwinger kernel, and is defined by:

\[ \frac{\partial K}{\partial s} - \Box_E K = \delta(s)\delta(x,x_0); \lim_{s \rightarrow 0} K = \delta(x,x_0) \]  

(59)

Naively one would have expected that the heat kernel obtained in the Euclidean sector of Rindler and Minkowski coordinates to match with the that in Cartesian and polar coordinates, respectively.

The question then arises as to why the Rindler heat kernel \( K^{Eu}_R \) – the Euclidean continuation of Schwinger kernel corresponding to the Rindler vacuum (and the corresponding Rindler propagator \( G^{Eu}_R \)) – is distinct from the Minkowski heat kernel \( K^{Eu}_M \) – the Euclidean continuation of the Schwinger kernel corresponding to the Minkowski vacuum (and the Minkowski propagator \( G^{Eu}_M \)). Kernels for diffusion and Schrödinger equations, in flat Euclidean space, with non trivial boundary conditions, have been studied extensively in literature (for example, [22–24]). However, we review the solutions of heat kernel equation with the intention of tracing how the distinction between different vacua are encoded in the boundary conditions. The formal solution to Eq.(59), incorporating the correct boundary condition, can be given as

\[ K = \exp(s\Box_E)[\delta(x,x_0)] \]  

(60)

where \( \delta(x,x_0) \) is a properly densitised Dirac delta function. Note that both the operator \( \Box_E \) and \( \delta(x,x_0) \) are generally covariant, making \( K \) a covariant biscalar. If you solve this equation in one coordinate system (say, in the Cartesian coordinates) and transform everything covariantly to a new coordinate system (say, polar coordinates), you will only get back the same heat kernel (and Feynman propagator) expressed in

\(^8\text{If one approaches the problem via standard canonical quantization, this question has a seemingly natural answer. You first obtain the relevant positive frequency modes in Rindler and Minkowski coordinates and define the corresponding Rindler and Minkowski vacua. The propagators are then defined as expectation values of products of field operators in these two vacua. However, we are approaching it from the point of view of solutions to a particular differential equation. It is then necessary to understand how the distinction between the two vacua is encoded in these solutions.}\)
terms of the new coordinates. To get something like the (Euclidean) Rindler kernel, in this approach, we have to use two distinct forms of Dirac delta functions in Cartesian and polar coordinate systems. We will now see in some detail what this implies.

The basic difference between the approaches that lead to $K_M$ and $K_R$ can be understood through different representations of the Dirac delta functions. To see what is involved, recall that, using the properties of Dirac delta function, we can write

$$
\delta(x-x')\delta(y-y') = \delta(r-r')\delta(r\sin\theta-r'\sin\theta') = \frac{\delta(r-r')}{r}\delta\left(2\sin\left[\frac{(\theta-\theta')}{2}\right]\right) \tag{61}
$$

We now want to re-express this result as a Dirac delta function involving the $\theta$ coordinates. Using the basic property of Dirac delta function

$$
\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \tag{62}
$$

and the periodicity of sine function, it immediately follows that

$$
\delta(x-x')\delta(y-y') = \sum_{n=-\infty}^{\infty} \frac{\delta(r-r')}{r}\delta(\theta-\theta'-2\pi n) \tag{63}
$$

In other words, the Dirac delta function in polar coordinates is an infinite sum. In fact, it is similar to what appears in our thermalisation of functions encountered in earlier sections. We could even say that when we proceed from the Cartesian coordinates to the polar coordinates the new Dirac delta function is obtained by thermalising $\delta(\theta-\theta')$.

Since Eq. (9) gives the kernel as a result of a linear operation on the delta function, it immediately follows that each of the terms in Eq. (63) will lead to a kernel $K_n$ parameterized by the integer $n$. The full kernel is provided by the infinite sum over these kernels $K_n$. To see how this comes about from first principles, let us represent the delta function in polar coordinates in Eq. (63) in terms of an appropriately chosen mode functions $\phi_m(\lambda; r, \theta)$ such that

$$
\sum_{m=-\infty}^{\infty} \int_0^\infty \phi_m(\lambda; r, \theta)\phi_m^*(\lambda; r', \theta')d\lambda = \sum_{n=-\infty}^{\infty} \frac{\delta(r-r')}{r}\delta(\theta-\theta'-2\pi n)\tag{64}
$$

where the functions

$$
\phi_m(\lambda; r, \theta) = \frac{e^{im\theta}}{\sqrt{2\pi}} \sqrt{\lambda} J_m(\lambda r) \tag{65}
$$

are eigenfunctions of the $\Box_E$ operator with eigen value $-\lambda^2$. If we now use Eq. (9) with this eigenfunction expansion, we will obtain

$$
\exp(s\Box_E) \left[ \sum_{n=-\infty}^{\infty} \frac{\delta(r-r')}{r}\delta(\theta-\theta'-2\pi n) \right] = \sum_{m=-\infty}^{\infty} \int_0^\infty e^{-\lambda^2 s} \phi_m(\lambda; r, \theta)\phi_m^*(\lambda; r', \theta')d\lambda \tag{66}
$$

Using the identity [25, Eq.(10.22.67)]

$$
\int_0^\infty e^{-\lambda^2 s} J_m(\lambda r)J_m(\lambda r')\lambda d\lambda = \frac{1}{2s} \exp\left(-\frac{r^2 + r'^2}{4s}\right) I_m\left(\frac{rr'}{2s}\right) \tag{67}
$$

16
\[ e^a \cos z = \sum_{m=-\infty}^{\infty} I_m(a)e^{imz} \]  

(68)

with \( a = \frac{rr'}{2s} \) and \( z = \theta - \theta' \), we get the final result

\[ K_{\text{Eu}}^E = \exp(sE) \left[ \sum_{n=-\infty}^{\infty} \frac{\delta(r-r')}{r} \delta(\theta - \theta' - 2\pi n) \right] = \frac{1}{(4\pi s)^2} e^{-\frac{(r^2 + r'^2)}{4s}} e^{i\omega \cos(\theta - \theta')} \]  

(69)

which, of course, matches with the result obtained by transforming the standard Minkowski heat kernel \( K_{\text{Eu}}^E \) from the Cartesian coordinates to polar coordinates, treating it as biscalar. So, by this procedure, we have just verified — the rather trivial — property of the kernel viz., it is generally covariant.

There is another way of working with Eq. (63). We can use the rule that we will confine to angular differences which satisfies \( |\theta - \theta'| < 2\pi \). In that case only the \( n = 0 \) term in Eq. (63) contributes and the expression for Dirac delta function becomes

\[ \delta(x - x')\delta(y - y') = \frac{\delta(r-r')}{r} \delta(\theta - \theta') \]  

(70)

With this interpretation we will get back the same result as long as we keep ‘\( m \)’ in the eigenfunction in Eq. (65) to be an integer. In that case, the eigenfunctions are clearly periodic under \( \theta \to \theta + 2\pi \) and this is the key reason for us getting back \( K_{\text{Eu}}^E \). In this procedure, the mode functions will satisfy Eq. (64) with only the \( n = 0 \) term being present in the right hand side.

To obtain the analog of Rindler kernel we have to get out of the implicit periodicity in \( \theta \). This can be achieved as follows. Suppose we insist that the relevant Dirac delta function should not be taken as the one in Eq. (63) but with just the \( n = 0 \) term. Further, we assume that the eigenvalue \( m \) now takes all real values rather than integer values. That is, we will assume that we can replace the eigenfunction \( \phi_m \) by \( \phi_\omega \) with \( \omega \) taking all real values. (This is, of course, essential for breaking the periodicity; if we take the eigenfunctions with \( e^{im\theta} \) with integer values for \( m \), they are formally periodic, even if we assume that \( 0 \leq \theta < 2\pi \).) It would be interesting to ask how the previous analysis will change and what kind of kernel one will get. The entire analysis goes through with the new eigenfunction \( \phi_\omega \). The completeness relation Eq. (64) now gets replaced by

\[ \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\lambda \phi_\omega(\lambda; r, \theta)\phi_\omega^*(\lambda; r', \theta') = \frac{\delta(r-r')}{r} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(\theta - \theta')} \]  

(71)

\[ = \frac{\delta(r-r')}{r} \delta(\theta - \theta') \]  

(72)

This will lead to the Rindler kernel, \( K_{\text{Eu}}^R \):

\[ K_{\text{Eu}}^R = \exp(sE) \frac{\delta(r-r')}{r} \delta(\theta - \theta') \]  

(73)

\[ = \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\lambda e^{-\lambda s^2} \phi_\omega(\lambda; r, \theta)\phi_\omega^*(\lambda; r, \theta) \]  

(74)

which, using the definition of \( \phi_\omega \) from Eq. (65) and the identity in Eq. (67) can be expressed as

\[ K_{\text{Eu}}^R = \frac{1}{(4\pi s)^2} \exp \left( \frac{-\frac{r^2 + r'^2}{4s}}{4s} \right) \int_{-\infty}^{\infty} d\omega I_{|\omega|} \left( \frac{rr'}{2s} \right) e^{i\omega(\theta - \theta')} \]  

(75)
From the known asymptotic behaviour of \( I_n(z) \) one can directly verify that this kernel indeed satisfies the boundary condition we started with, namely:

\[
\lim_{s \to 0} K_{E_R}^{Eu} = \frac{\delta (r-r')}{r} \delta (\theta - \theta')
\]  

(76)

In other words, the expression for the kernel in Eq. (9) correctly reproduces both \( K_{E_R}^{Eu} \) and \( K_{M}^{Eu} \) depending on how we interpret the Dirac delta function on the right hand side. If you take the “natural” option and insist that \( \theta \) is an angle with periodicity \( 2\pi \) then you will only get \( K_{M}^{Eu} \), which, of course, has this periodicity arising from the thermalising condition Eq. (8). In order to get \( K_{E_R}^{Eu} \) one has to explicitly break this periodicity condition in the Euclidean sector.\(^9\) It therefore appears in the Euclidean sector \( K_{E_R}^{Eu} \) arises very naturally while we need to do something artificial to get \( K_{M}^{Eu} \), namely allowing the kernel to be multivalued in the Euclidean plane. However, \( K_{E_R}^{Eu} \) does in fact arise naturally as the heat kernel in an infinitely sheeted, locally flat, 2-d Riemann surface.

Incidentally, the situation is somewhat different in the Lorentzian sector. If you do the coordinate transformation of the Dirac delta function in the Lorentzian sector, then instead of Eq. (61) we will get an expression with \( \delta [2 \sinh ((\tau - \tau')/2)] \). This leads to just \( \delta [\tau - \tau'] \) as long as we stick to real values of \( \tau \) and \( \tau' \), as we should. (The periodicity now is in the imaginary values of \( \tau - \tau' \) which do not play a role as long as we decide to live in Lorentzian spacetime.) This would suggest that an analysis similar to what we did above will not recognize any kind of periodicity in the Lorentzian sector. As a result this will lead to the Rindler kernel \( K_R \) in a natural fashion. To obtain the Minkowski kernel \( K_M \) in the Lorentzian sector, working entirely in Rindler coordinates, we have to explicitly thermalise \( K_R \). We find it rather intriguing that the natural solution to Eq. (9) leads to the Rindler kernel \( K_R \) in the Lorentzian sector while it leads to the Minkowski kernel \( K_M \) in the Euclidean sector. We will come back to this feature in the last section.

It can be directly verified that the Euclidean kernels \( K_{E_R}^{Eu} \) and \( K_{E_M}^{Eu} \), on integration over \( s \) leads to the correct propagators \( G_{E_R}^{Eu} \) and \( G_{E_M}^{Eu} \), respectively. This can be obtained most easily by first rewriting the relevant expressions in terms of \( K_{\nu\nu} \) in the following forms:

\[
K_{M}^{Eu} = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty \frac{d\nu}{\pi} \cosh[\nu(\pi - \phi)] K_{i\nu} \left( \frac{r'r''}{2s} \right)
\]  

(77)

and

\[
K_{R}^{Eu} = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty \frac{d\nu}{\pi} e^{-\nu \phi} \sinh(\pi \nu) K_{i\nu} \left( \frac{r'r''}{2s} \right)
\]  

(78)

These expressions are derived in Appendix C. The integration leads to the following expressions in the Euclidean sector which are well known in the literature

\[
G_{R}^{Eu} = \int_0^\infty \frac{d\nu}{\pi^2} e^{-\nu |\theta - \theta'|} \sinh(\pi \nu) K_{i\nu} (mr) K_{i\nu} (mr')
\]  

(79)

and

\[
G_{M}^{Eu} = \int_0^\infty \frac{d\nu}{\pi^2} \cosh[\nu(\pi - |\theta - \theta'|)] K_{i\nu} (mr) K_{i\nu} (mr').
\]  

(80)

\(^9\)In the context of QM in 2-dimensional plane, one would not dream of writing the Dirac delta function with just \( \delta (\theta - \theta_0) \). In QM, the kernel evolves an initial wave function to a final wave function; in fact \( K(x, x_0; s) \) is indeed the wave function for a particle at time \( s \) if it started out as a delta function, \( \delta (x, x_0) \) at time \( s = 0 \). So to justify the choice of delta function with a factor \( \delta (\theta - \theta_0) \) one should be able to interpret it as the initial wave function of a particle in the plane. This is tricky because one would expect the wave functions to be single valued. Since \( \theta \) and \( \theta + 2\pi \) represent the same physical point, the function \( \delta (\theta - \theta_0) \) cannot be an acceptable choice for the initial wave function. Therefore the kernel \( K_{u=0} \) never arises in the context of, say, normal QM.
This matches exactly with the previous results, for example, that in [12].

6 From the Euclidean Plane to the four Rindler wedges

Finally we will take up the third motivation for this work, mentioned in Sec. 1.3. We will show that the Lorentzian biscalars (like $K_M(x_1, x_2; s)$ and $G_M(x_1, x_2)$) can indeed be obtained from their corresponding Euclidean avatars ($K_{M}^{E}(x_1, x_2; s)$ and $G_{M}^{E}(x_1, x_2)$) by analytic continuation. However the standard analytic continuation, discussed in the literature — which involves replacing $t_{E} \rightarrow -it$ and $\tau_{E} \rightarrow -i\tau$ in $x = \rho \cos \tau_{E}, t_{E} = \rho \sin \tau_{E}$ — will only lead us to the events in the right Rindler wedge. To obtain the form of the biscalars in other wedges, we actually need different sets of analytic continuations. This is particularly important when the two events in the biscalars are located in different wedges.

The key (unifying) principle which helps us to discover the correct analytic continuation is the following: The invariant Euclidean distance squared ($\sigma_{E}^{2}$) should be analytically continued to the invariant Lorentzian distance squared ($\sigma^{2}$) such that the latter has an infinitesimal imaginary part. Depending on the location of the two events that define the Lorentzian kernel, different cases lead to different mappings which we will list below. We first discuss the analytic continuation of the kernel and then describe the corresponding results for the propagator.

6.1 The recipe for analytic continuation

The analytic continuation from Euclidean to Minkowski space for simple scalar functions was discussed in [18]. We generalize this prescription and propose a recipe for analytically continuing kernels and Feynman propagators, which are biscalars. To accomplish this it suffices to analytically continue two basic biscalars in Euclidean space: (i) $\sigma_{E}^{2}$, the square of Euclidean distance and $\Theta_{E} = |\theta - \theta'|$, the angle between vectors, say $\mathbf{x}$ and $\mathbf{x}'$, representing the two points. To reproduce the analytical structure of Feynman propagator and kernel, we should ensure that $\sigma_{E}^{2}$ be analytically continued to $\sigma^{2} + i0^{+}$ in the Lorentzian sector. To analytically continue $\Theta_{E}$, let us start by writing its explicit form:

$$\Theta_{E}(\mathbf{x}, \mathbf{x}') = \cos^{-1}\left[\frac{\mathbf{x} \cdot \mathbf{x}'}{||\mathbf{x}|| ||\mathbf{x}'||}\right]$$

(81)

where, $\mathbf{x}$ and $\mathbf{x}'$ are the Cartesian coordinates of two points in the Euclidean plane and dot denotes the standard inner product in the Euclidean flat space. It is convenient at this stage to define the quantity in the square bracket as $Z(\mathbf{x}, \mathbf{x}')$. In terms of the polar coordinates, the biscalar $Z$ can be written as

$$Z(\mathbf{x}, \mathbf{x}') = \frac{r^{2} - r'^{2} - \sigma_{E}^{2}}{2rr'}.$$  

(82)

Note that $\Theta_{E}$ is a function of $Z$ alone; more specifically we have, $\Theta_{E} = \cos^{-1}(Z)$. However, since we are interested in analytic continuation of $\Theta_{E}$, we need an expression for $\Theta_{E}(Z)$ that is valid in the whole of complex $Z$–plane. For this purpose, recall that the expression for $\cos^{-1}(y)$, for real $y$ in the range $[-1, 1]$ is given by

$$\cos^{-1}(y) = \frac{\pi}{2} - \sin^{-1}(y) = \frac{\pi}{2} + i \log(iy + \sqrt{1 - y^{2}})$$

(83)
which is easy to verify.\(^\text{10}\) This expression enables us to analytically continue the \(\cos^{-1}\) function into the complex plane. However, one cannot naively extend it into an analytic function in the entire complex place, since it inherits branch cuts from the ‘log’ and the ‘square root function’ that appear in its definition. Hence, a complex extension of \(\Theta_Z\) needs to be defined using appropriate \(i\epsilon\) prescription as:

\[
\Theta_E(Z) = \frac{\pi}{2} + i \log \left( \sqrt{1 - Z^2} + i0^+ + iZ \right)
\]  

(84)

With these two inputs we get a consistent analytic continuation of the Euclidean biscalars \(K_{EM}^{Eu}\) and \(G_{EM}^{Eu}\) to the whole of Minkowski space. We give the explicit recipe in Table 1 for three cases: (i) \(RR\), when both points are in \(R\) wedge (ii) \(FF\), when both points are in \(F\) wedge and (iii) \(RF\), when one point is in \(F\) and the other in \(R\). (More algebraic details are given in Appendix D). The other cases (involving \(L\) and \(P\)) can be found in a similar manner.

Table 1: Recipe for analytic continuation

| Case | Euclidean \(\rightarrow\) Lorentzian | \(\sigma_F^\theta \rightarrow \sigma^z\) | \(\Theta_E \rightarrow \Theta\) |
|------|-----------------------------------|----------------------------------|-------------------------------|
| \(RR\) | \((r, \theta) \rightarrow (\rho, i\tau e^{-i\theta})\) \((r', \theta') \rightarrow (\rho', i\tau'e^{-i\theta'})\) | \(\rho^2 + \rho'^2 - 2\rho \rho' \cosh(\tau - \tau') + i0^+\) | \(i|\tau - \tau'| + 0^+\) |
| \(RF\) | \((r, \theta) \rightarrow (\rho_R, i\tau R)\) \((r', \theta') \rightarrow (i\rho_F, i\tau_F + \frac{\pi}{2} + \epsilon)\) | \(-\rho_F^2 + \rho_R^2 - 2\rho_F \rho_R \sinh(\tau_F - \tau_R) + i0^+\) | \(i(\tau_F - \tau_R) + \frac{\pi}{2} + \epsilon\) |
| \(FF\) | \((r_<, \theta) \rightarrow (-e^{i\theta}i\rho_<, i\tau + \frac{\pi}{2})\) \((r_>, \theta') \rightarrow (i\rho_>, i\tau' - \frac{\pi}{2})\) | \(-\rho_<^2 - \rho_>^2 + 2\rho_< \rho_> \cosh(\tau - \tau') + i0^+\) | \(-i|\tau - \tau'| + \pi\) |

6.2 Analytic continuation of \(K_{EM}^{Eu}\) and \(G_{EM}^{Eu}\)

With the recipe in Table 1 and the analytic continuation of the Schwinger propertime \(s \rightarrow ise^{-i\theta}\), we can obtain the expressions for \(K_M\) and \(G_M\), by analytic continuation of \(K_{EM}^{Eu}\) and \(G_{EM}^{Eu}\). We will start with \(K_M\) in the \(RR\), \(RF\) and \(FF\) sectors. (We use the notation \(AB\) to describe the situation in which the two events are in the wedges \(A\) and \(B\).) The Feynman propagator can then be found using the standard relation

\[
iG_M(x, x') = \int_0^\infty ds\ e^{-im^2s}K_M(x, x'; s)\]  

(85)

It is clear that we will encounter the following type of integral, while evaluating \(G_M\) using the above relation (after rotating the contour of integration via \(s \rightarrow -is\)):

\[
I(z_1, z_2) = \int_0^\infty \frac{ds}{2s}e^{-m^2s} \exp \left( -\frac{z_1^2 + z_2^2}{4s} \right) K_\nu \left( \frac{z_1 z_2}{2s} \right)
\]  

(86)

When \(z_1\) and \(z_2\) satisfies the following conditions,

\[
|\arg[z_1]| < \pi; \quad |\arg[z_2]| < \pi; \quad |\arg[z_1 + z_2]| < \frac{\pi}{4}.
\]  

(87)

\(^{10}\text{Put } y = \cos \theta, \text{ where } 0 \leq \theta \leq \pi, \text{ so that } iy + \sqrt{1 - y^2} = ie^{-i\theta}. \text{ Therefore, } \cos^{-1}(y) = \frac{\pi}{2} - i\log(ie^{-i\theta}) = \theta.\)
one can evaluate the integral explicitly to get,
\[
I(z_1, z_2) = K_{i\nu}(m z_1) K_{i\nu}(m z_2).
\]
(88)

Our strategy will be to analytically continue \(I(z_1, z_2)\) for the desired values of \(z_1\) and \(z_2\) by using the appropriate analytic continuation of \(K_{i\nu}\). We will now work out the calculations explicitly for the RR, FF and RF cases. (These results have been obtained earlier in Ref.[19] by a more complicated procedure.)

### 6.2.1 RR: both events in the \(R\) wedge

Application of our prescription to Eq.(77) yields the following expression for Minkowski kernel \(K^{(m)}_M\) for a massive scalar field, when both events are in the \(R\) wedge:
\[
K^{(m)}_M = \frac{e^{-i\nu^2 z_1}}{(4\pi is)} \exp \left(-\frac{\rho^2 + \rho^\prime^2}{4is}\right) \int_{-\infty}^{\infty} \frac{d\nu}{\pi} e^{-i\nu[|\tau-\tau'|+i(\pi-\epsilon)]} K_{i\nu}(\rho \rho^\prime) \]  
(90)

This is for a \((1+d)\) dimensional, massive case and — as discussed earlier in connection with Eq. (53) — we have taken a Fourier transform with respect to the transverse coordinate differences and set \(\mu^2 = m^2 + k^2_\perp\). (A similar result can be obtained when both points are in the \(L\)-wedge, which we will not discuss explicitly.) The Feynman propagator can now be found by using Eq. (85) and (88) to get:
\[
G_M = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} e^{-i\nu[|\tau-\tau'|+i(\pi-\epsilon)]} K_{i\nu}(\mu \rho) K_{i\nu}(\mu \rho^\prime) d\nu
\]  
(91)

To arrive at a more familiar form of \(K_M\), we can use the following result:
\[
\int_{-\infty}^{\infty} e^{-i(\xi_R+i\xi_I)\omega} K_{i\omega}(\alpha) K_{i\omega}(\beta) d\omega = \pi K_0 \left[\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cosh(\xi_R + i\xi_I)}\right];
\]
(91)

which is valid for
\[
|\arg[\alpha]| + |\arg[\beta]| + |\xi_I| < \pi
\]
(92)

With \(\alpha = \mu \rho, \beta = \mu \rho^\prime, \xi_R = |\tau-\tau'|\) and \(\xi_I = (\pi - \epsilon)\), we arrive at:
\[
G_M = \frac{1}{2\pi} K_0 \left(\mu \sqrt{\rho^2 + \rho^\prime^2 - 2\rho \rho^\prime \cosh(\tau - \tau') + i0^+}\right)
\]  
(93)

which gives the correct propagator for a massive scalar field (see e.g.[12]). The following point needs to be emphasised in the above calculation: We need a negative sign for the cosh term in the argument of \(K_0\) to give the correct propagator while Eq. (91) has a positive sign for the cosh term in the argument of \(K_0\). We can flip this sign if we could take \(\xi_I = \pi\) but this is forbidden due to the the condition of validity, given by Eq. (92). We can, of course, set \(\xi_I = (\pi - \epsilon)\) to flip the sign of the cosh term in the argument of \(K_0\), which is what we have done. But this procedure adds an infinitesimal imaginary part — coming from \(e^{-i\epsilon}\) to the argument, which is indicated by \(i0^+\) in Eq. (93). But this is exactly what we need in the Feynman propagator and thus everything works out consistently, though somewhat subtly.

There is another representation of this propagator which is useful. This representations is motivated by the fact that Eq. (90) contains \(|\tau - \tau'|\) while it will be useful to have just the Fourier transform with \((\tau - \tau')\).

(The Fourier transform of the propagator with respect to \((\tau - \tau)\) can be interpreted as the amplitude for propagation in energy space and is closely related to the thermality of Rindler horizon [18]). This can be
achieved as follows: Using the result Eq. (91) again, but this time with \( \xi_R = (\tau - \tau') \), \( \xi_I = 0 \), \( \alpha = e^{i(\pi - \epsilon)}\mu \rho_< \) and \( \beta = \mu \rho_> \) we arrive at a different representation for \( G_M \), given by

\[
G_M = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} e^{-i\nu(\tau - \tau')} K_{\nu}(e^{i(\pi - \epsilon)}\mu \rho_<)K_{i\nu}(\mu \rho_>)d\nu
\]

This expression reveals that, the Fourier transform \( \tilde{G}_M(\nu) \) (with the dependence on other variables being suppressed) of \( G_M \) with respect to \((\tau - \tau')\) is given by

\[
\tilde{G}_M(\nu) = \frac{1}{\pi} K_{i\nu}(e^{i(\pi - \epsilon)}\mu \rho_<)K_{i\nu}(\mu \rho_>) = \frac{1}{\pi} K_{i\nu}(-\mu \rho_<)K_{i\nu}(\mu \rho_>)
\]

where the \( i\epsilon \) is suppressed in the last equality. This result was derived in [18] by a different procedure.

It is also possible to obtain a similar Fourier transform for the ‘reflected’ propagator, that will be useful later on. This uses another choice of parameters in Eq. (91) corresponding to \( \alpha = \mu \rho, \beta = \nu \rho' \), \( \xi_R = (\tau - \tau') \) and \( \xi_I = 0 \). A direct substitution of this choice into the RHS of Eq. (91) gives \( \pi K_0(\sqrt{\rho^2 + \rho'^2 + 2\rho \rho' \cosh(\tau - \tau'))} \). A comparison of this expression (divides by \( 2\pi^2 \)) with Eq. (93) shows that it corresponds to \( G_M^{(r)} \), the Feynman propagator between \((\rho, \tau)\) and another event in \( L \) obtained by reflecting \( \rho', \tau' \) about the origin of \( x - t \) plane. Hence, again using Eq. (91), with out latest choice of parameters, gives us

\[
\tilde{G}_M^{(r)}(\nu) = \frac{1}{\pi} K_{i\nu}(\mu \rho)K_{i\nu}(\mu \rho')
\]

This result, along with Eq. (94), will be made use later for a simple derivation of the Rindler propagator \( G_R \).

Another important technical remark concerning the structure of \( \tilde{G}_M(\nu) \) and \( \tilde{G}_M^{(r)}(\nu) \) in Eq. (95) and Eq. (96) is in order. We know that, the propagator \( G_M \) and the ‘reflected’ propagator \( G_M^{(r)} \) are related by

\[
G_M^{(r)}(\tau - \tau') = G_M(\tau - \tau' + i\pi)
\]

This can be obtained by \( \epsilon \rightarrow -\epsilon \) application of the shift theorem for the Fourier transform (valid for shifts by real quantities) will now give, \( \tilde{G}_M^{(r)}(\nu) = e^{\pi \nu} \tilde{G}_M(\nu) \) which, as we can see from Eq. (95) and Eq. (96), is incorrect. As we have mentioned earlier — see the discussion after Eq. (36) — the Fourier transforms of two functions \( f(x) \) and \( f_y(x) \equiv f(x + iy) \), with respect to \( x \), are in general not related by \( \tilde{f}_y(\nu) = e^{iy\nu} \tilde{f}(\nu) \), for real \( y \). The Fourier transform of \( f(x) \) exists and is given by \( \tilde{f}(\nu) \), if and only if the following integral converges:

\[
\int_{-\infty}^{\infty} \tilde{f}(\nu)e^{-i\nu y} \frac{d\nu}{2\pi} = f(x)
\]

However, the convergence of this integral in Eq. (97) does not guarantee that the following integral also converges, for an arbitrary real \( y \):

\[
I = \int_{-\infty}^{\infty} \left[ e^{iy\nu} \tilde{f}(\nu) \right] e^{-i\nu y} \frac{d\nu}{2\pi}
\]

For the integral in Eq. (98) to converge, \( \tilde{f}(\nu) \) should satisfy the additional condition that \( e^{iy\nu} \tilde{f}(\nu) \) vanishes sufficiently fast as \( \nu \rightarrow \pm\infty \), where the \( +/\) sign is for positive/negative values of \( y \), respectively. Hence, in general, \( e^{iy\nu} \tilde{f}(\nu) \) will not be the Fourier transform of \( f_y(x) \) and one cannot obtain the result by shifting.
Let us illustrate this aspect in the case of the Minkowski propagators $G_M$ and $G_M^{(r)}$. Recall that we derived the Fourier transform of $G_M$ using the relation Eq. (91), with the parameters being chosen as $\xi_R = (\tau - \tau')$, $\xi_I = 0$, $\alpha = e^{i(\pi - \epsilon)}\mu_\rho$ and $\beta = \mu_\rho$. Since, $G_M(\tau - \tau + i\pi) = G_M^{(r)}$, one may naïvely choose $\xi_R = (\tau - \tau')$, $\xi_I = i\pi$, $\alpha = e^{i(\pi - \epsilon)}\mu_\rho$ and $\beta = \mu_\rho$ to ‘derive’ the Fourier transform $\tilde{G}_M^{(r)}$. Using this choice in Eq. (91) would then lead to the wrong result, given by shifting: $\tilde{G}_M^{(r)}(\nu) = e^{i\pi}G_M(\nu)$. This result is wrong because, for this choice of parameters, the left hand side Eq. (92) is given by:

$$|\arg[\alpha]| + |\arg[\beta]| + |\xi_I| = 2\pi - \epsilon > \pi$$

thereby violating the condition in Eq. (92). Since Eq. (92) is a necessary condition for using the integral in Eq. (91), violating it leads to a wrong result. More precisely, our choice of parameters does not satisfy the necessary condition for the integral in Eq. (91) to converge, thereby invalidating the procedure of shifting the contour in the complex plane; this is the reason why $\tilde{G}_M^{(r)}(\nu) \neq e^{i\pi}G_M(\nu)$. To derive the correct Fourier transform of $G_M^{(r)}$, one has to proceed exactly as we did earlier; by choosing $\alpha = \mu_\rho, \beta = \nu^\prime, \xi_R = (\tau - \tau')$ and $\xi_I = 0$. With this choice of parameters, the right hand side of Eq. (91) takes the required form that is relevant for $G_M^{(r)}$ and the condition $|\arg[\alpha]| + |\arg[\beta]| + |\xi_I| < \pi$ is also satisfied\footnote{The consideration of this condition also appears in another important context. This also concerns the interrelationship between $\tilde{G}_M^{(r)}(\nu)$ and $\tilde{G}_M^{(r)}(\nu)$. Note that, since the right hand side of Eq. (96) is symmetric under $\rho \leftrightarrow \rho'$, we could as well make the replacement $\rho \rightarrow \rho_<$ and $\rho' \rightarrow \rho_>$ in this equation. Now, we could have arrived at the expression hence obtained, by replacing $\rho_\rho \rightarrow e^{-i\pi}\rho_\rho$ in Eq. (95). Since, $\rho_\rho \rightarrow e^{-i\pi}\rho_\rho$ is also essentially ‘reflection’, this provides another independent route of arriving at Eq. (96). However, even though $\rho_\rho \rightarrow e^{-i\pi}\rho_\rho$ also implies a ‘reflection’, we cannot make this replacement in Eq. (95) to derive $\tilde{G}_M^{(r)}$. This is, once again, because such a replacement would imply the choice of parameters $\xi_R = (\tau - \tau'), \xi_I = 0, \alpha = e^{i(\pi - \epsilon)}\mu_\rho$ and $\beta = \mu_\rho$, which violates the necessary condition for the convergence of Eq. (91).}. This leads to a convergent integral and the correct Fourier transform in Eq. (96).

### 6.2.2 FF: both events in the F wedge

Once again, using Table 1 in Eq. (77) we get

$$K_M^{(m)} = \frac{e^{-i\mu\pi s}}{4\pi is} \exp \left( \frac{\rho_\rho + \rho_\rho^2}{4is} \right) \int_{-\infty}^{\infty} d\nu e^{-i\nu(\tau - \tau')} K_{iv} \left[ \frac{e^{i\tau} \rho_\rho}{2is} \right]$$

In this case, we have replaced $|\tau - \tau'|$ by $(\tau - \tau')$ to arrive at this expression in the form of a Fourier transform. We could do this in this case — but not in the case of Eq. (89) — because the rest of the integrand is symmetric under $\nu \rightarrow -\nu$. The $\nu$ integration can be easily done to reduce this expression to the familiar form:

$$K_M^{(m)} = \frac{e^{-i\nu^2 s}}{4\pi is} \exp \left[ -\frac{1}{4is}(-\rho^2 - \rho'^2 + 2\rho\rho' \cosh(\tau - \tau') + i0^+) \right]$$

To find the expression for $G_M$, we need to apply the following substitution in Eq. (88),

$$z_1 = e^{i(-\frac{\pi}{2} + \tau')} \rho_\rho; \quad z_2 = e^{i\frac{\pi}{2}} \rho_\rho.$$  

Further, using the standard connection formulas between the MacDonald and Hankel functions, we arrive at

$$G_M = \frac{1}{4} \int_{-\infty}^{\infty} e^{-i\nu(\tau - \tau')} H_{iv}^{(1)}(\mu e^{i\tau} \rho_\rho) H_{iv}^{(2)}(\mu \rho_\rho) d\nu$$

(103)
This expression has a simple interpretation. It is well known that the solutions to the massive Klein-Gordon equation in \( F \), which is a positive frequency mode with respect to the inertial time is given by (see for example [26]):

\[
\psi_{\nu,k_\perp}(\rho,\tau) \propto e^{\pi|\nu|/2}H_{i|\nu|}^{(2)}(\mu\rho)e^{-i\nu\tau}
\]  

(104)

In terms of these modes, with appropriate normalization, Eq.(103) reduces to the suggestive form

\[
G_M = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \psi_{\nu,k_\perp}(\rho,\tau)\psi_{\nu,k_\perp}^\ast(\rho,\tau').
\]

(105)

In the \( F \) wedge, \( \rho \) acts as the time coordinate and \( \tau \) acts as the space coordinate. So the subscripts < and > in the above expression actually give us a time-ordered expression. So Eq. (105) has the correct form of the time ordered correlator in the Minkowski vacuum, expressed in terms of positive frequency mode functions, as is appropriate for the Feynman propagator.

6.2.3 RF: one event in \( R \) and the other in \( F \)

In this case, \( K_M \) takes the form:

\[
K_M^{(m)} = \frac{e^{-i\mu^2s}}{(4\pi is)} \exp \left( -\frac{\rho_R^2 - \rho_F^2}{4is} \right) \int_{-\infty}^{\infty} d\nu e^{-i\nu(\tau_F - \tau_R)} \left\{ \frac{e^{\pi|\nu|/2}}{\pi} K_{i\nu} \left[ \frac{\rho_F \rho_R}{2s} \right] \right\}
\]

(106)

The term in curly brackets is just the Fourier transform of \( e^{iz\sinh(\tau_F - \tau_R)} \) with respect to \( \tau_F \), where \( z = (\rho_F \rho_R)/(2s) \). Therefore this expression may be simplified to get;

\[
K_M^{(m)} = \frac{e^{-i\mu^2s}}{(4\pi is)} \exp \left[ -\frac{1}{4is}(\rho_R^2 - \rho_F^2 + 2\rho_F \rho_R \sinh(\tau_F - \tau_R) + i0^+) \right]
\]

(107)

Once again, we obtain the familiar expression. To find the Feynman propagator, in Eq.(88) choose \( z_1 \) and \( z_2 \) to be the following:

\[
z_1 = \rho_R; \quad z_2 = e^{i\pi/2} \rho_F.
\]

(108)

The form of \( G_R \) in this case can then be found to be

\[
G_M = \frac{-i}{4\pi} \int_{-\infty}^{\infty} e^{-i\nu(\tau_F - \tau_R)}H_{i\nu}^{(2)}(\mu\rho_F)K_{i\nu}(\mu\rho_R) d\nu.
\]

(109)

In a recent work [18], this form of the Minkowski propagator was a key ingredient in a simple derivation of thermality of the Rindler horizon.

6.3 Analytic continuation of \( K_{Eu}^R \) and \( G_{Eu}^R \)

In exactly the same manner as we analytically continued the Euclidean-Minkowski kernel and propagator, we can also analytically continue the Euclidean-Rindler kernel and propagator. Purely algebraically, we can do this in all the four wedges. However, the Rindler vacuum state — used implicitly in the construction of the Rindler kernel and Rindler propagator — has a natural definition only in \( R \) and \( L \) wedges. So we
shall confine our analysis in this paper to the two cases in which the analytic continuation of $K_{R}^{E\nu}$ and $G_{R}^{E\nu}$ has a natural interpretation viz., in RR and LL. We will only present the RR case here, since the LL case can be dealt in an identical manner. In RR our procedure leads to the kernel:

$$K_{R}^{(m)} = \frac{e^{-i\mu^{2}s}}{(2\pi^{2}i\hbar)} \exp\left(\frac{-\mu^{2} + \mu'^{2}}{4i\hbar}\right) \int_{0}^{\infty} d\nu \, e^{-i\nu|\tau-\tau'|-\epsilon\nu} \left(\sinh \pi\nu\right) K_{i\nu} \left[\frac{\rho\rho'}{2i\hbar}\right]$$  \hspace{1cm} (110)

The Feynman propagator $G_{R}$ can now be computed to give expressions previously known in literature (see e.g.[1,12]):

$$G_{R} = \frac{1}{\pi^{2}} \int_{0}^{\infty} e^{-i\nu|\tau-\tau'|} \sinh(\pi\nu) K_{i\nu}(\mu\rho) K_{i\nu}(\mu\rho') \, d\nu$$  \hspace{1cm} (111)

The Rindler modes $u_{\nu,k}(\rho,\tau)$, which are positive frequency solutions with respect to the Rindler time coordinate, are given by [27]

$$u_{\nu,k}(\rho,\tau) = \frac{\sqrt{\sinh(\pi\nu)}}{\pi} K_{i\nu}(m\rho) e^{-i\nu\tau}; \quad \nu > 0$$  \hspace{1cm} (112)

The Feynman propagator can then be rewritten in the expected form:

$$G_{R} = \int_{0}^{\infty} d\nu \, u_{\nu,k}(\rho,\tau>_{\nu}) u_{\nu,k}(\rho',\tau<_{\nu})$$  \hspace{1cm} (113)

where, $\tau>_{\nu} > \tau<_{\nu}$. This representation confirms that the propagator $G_{R}$ that we derived is indeed a time ordered correlation function.

Another important quantity that we are interested in is the Fourier transform of $G_{R}$ with respect to $(\tau-\tau')$. We will first derive it from Eq. (111) and then show that our result Eq.(36) reproduces it in an alternative, simple, manner. For the former derivation, let us first rewrite the integral in Eq.(111) as:

$$G_{R} = \frac{1}{2\pi} \int_{0}^{\infty} d\nu \, e^{-i\nu|\tau-\tau'|} I_{-i\nu}(\mu\rho<_{\nu}) K_{i\nu}(\mu\rho>_{\nu}) - \frac{1}{2\pi} \int_{0}^{\infty} d\nu \, e^{-i\nu|\tau-\tau'|} I_{i\nu}(\mu\rho<_{\nu}) K_{i\nu}(\mu\rho>_{\nu})$$  \hspace{1cm} (114)

where, we have used the identity $(2/\sinh(\pi\nu)) K_{i\nu}(x) = I_{-i\nu}(x) - I_{i\nu}(x)$. Since, there are no poles for $I_{i\nu}(m\rho<_{\nu}) K_{i\nu}(m\rho>_{\nu})$ in the lower half complex $\nu-$plane and the condition $\rho>_{\nu} > \rho<_{\nu}$ ensures that the this terms vanishes sufficiently fast as $\text{Im}[\nu] \rightarrow -\infty$, we can rotate the $\nu$ integration in the second term of Eq. (114) to the straight line contour along $\tau e^{i(\pi-\epsilon)}$ connecting $r = 0$ and $r = \infty$. Therefore, the expression for $G_{R}$ reduces to the form

$$G_{R} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \, e^{-i\nu|\tau-\tau'|} I_{-i\nu}(\mu\rho<_{\nu}) K_{i\nu}(\mu\rho>_{\nu})$$  \hspace{1cm} (115)

Since the integrand in the last expression is symmetric under $\nu \rightarrow -\nu$, we can as well replace $|\tau-\tau'|$ with $(\tau-\tau')$ to reveal the Fourier transform of $G_{R}$ with respect to $(\tau-\tau')$. Hence, we obtain

$$\tilde{G}_{R} = -i I_{-i\nu}(\mu\rho<_{\nu}) K_{i\nu}(\mu\rho>_{\nu})$$  \hspace{1cm} (116)

which is the Fourier transform of the Rindler propagator. As far as we know this simple expression has not been obtained before in the literature.
We shall now present an alternative, simpler, derivation of this result by making use of Eq.(36). Note that, since $G_M$ and $G_R$ are related by the thermalization condition in Eq. (5), if follows that their Fourier transforms with respect to $(\tau - \tau')$ satisfies Eq.(36). Hence, the Fourier transform of $G_R$ is expected to satisfy:

$$\tilde{G}_R(\nu) = \tilde{G}_M(\nu) - e^{-\pi|\nu|} \tilde{G}_M^{(\tau)}(\nu) \quad (117)$$

Now, let us use Eq.(95) and Eq.(96) in this equation to get

$$\tilde{G}_R(\nu) = \frac{1}{\pi} \left[ K_{\nu}(e^{i\pi} \mu \rho_<) - e^{-\pi|\nu|} K_{\nu}(\mu \rho_<) \right] K_{\nu}(\mu \rho_>) \quad (118)$$

From the well known identity,

$$K_{\nu}(e^{i\pi} x) = e^{-\pi|\nu|} K_{\nu}(x) - i\pi I_{-i|\nu|}(x); \quad x > 0$$

we can simplify Eq.(118) to finally obtain:

$$\tilde{G}_R(\nu) = -i I_{-i|\nu|}(\mu \rho_>) K_{\nu}(\mu \rho_>) \quad (120)$$

This matches exactly with the expression for $\tilde{G}_R$ in Eq.(117) that we derived using more direct method.

### 7 Discussion

The physics in the Rindler frame, vis-a-vis the Minkowski frame has been investigated extensively in the literature. In view of this, it is useful to highlight new results and insights that this work offers.

- It is well known that Minkowski vacuum appears as a thermal state to a Rindler observer. A corollary of this result is that, given the Feynman propagator $G_R$, one can find $G_M$ as a periodic sum of $G_R$ (in the time variable $\tau$) with the period being $2\pi i$ (in appropriate units). It is natural to ask, whether there exists an inversion of this procedure. That is, given the expression for $G_M$, can we find $G_R$? In fact, one could ask whether such a result, if it exists, holds for a wider class of functions $\{F_M, F_R\}$, wherein the former is a periodic sum of the latter with a period $2\pi i$? We have shown explicitly that there exist an inverse transformation that retrieve $F_R$ from its periodic sum $F_M$, for a general class of functions $F_R$. The Schwinger kernels $\{K_M, K_R\}$ also form such a pair. Using the inverse procedure, we derived an integral representation for $K_R$ in 1+1 dimensional flat spacetime, which is consistent with known results in literature. (However, as far as we know, the explicit expression for $K_R$ does not seem to exist in the previous literature.)

- In reference [1], the authors derived an expression for $G_R$ in terms if a curious integral transformation of $G_M$. However, this result has not been explored further in the literature, in spite of the fact that $G_R$ itself is a well studied object. We have shown that this result is a direct consequence of the fact that $G_M$ is a periodic sum of $G_R$ with period, $2\pi i$. Specifically, it follows directly from the inverse transformation mentioned in the previous item. Hence, the result also holds true for the wider class of functions $\{F_M, F_R\}$. In particular, the Schwinger kernel $K_R$ can also be expressed as the integral transform of $K_M$ in exactly the same manner.
• The Schwinger kernel for a scalar field, in the Euclidean sector, is just the heat kernel. For the
two inequivalent vacua $|M\rangle$ and $|R\rangle$, the corresponding heat kernels are clearly different. However,
it is well-known that: (a) the right(left) Rindler wedge, under Euclideanization with respect to $\tau$,
maps to the whole of Euclidean flat space and (b) the heat kernel, in the Euclidean plane, maps to
$K_M$ in the Lorentzian sector. The non-trivial task is to find the heat kernel, in the same Euclidean
plane, that maps to $K_R$. We explicitly demonstrate how this can accomplished by solving the
differential equation satisfied by the heat kernel, using appropriate mode functions. We show that,
the existence of two inequivalent heat kernels, is related to the fact that Dirac delta function, can be
represented using two different sets of orthonormal mode functions; we obtain (i) (Euclidean) $K_M$,
when the modes used to represent the Dirac delta function are invariant under rotation by $2\pi$ and
(ii) (Euclidean) $K_R$, when the mode functions are not invariant under rotation by any common finite
angle.

• The analytic continuation of the Euclidean polar coordinates — which involves replacing $t_E \rightarrow -it$
and $\tau_E \rightarrow -i\tau$ in $x = \rho \cos \tau_E, t_E = \rho \sin \tau_E$ — will lead us only to the events in the right Rindler
wedge. The question arises as to how one can extract the information contained in the other four
wedges of the Lorentzian sector from the an expression valid in the Euclidean sector. We have
provided the four different analytic continuation of the Euclidean polar coordinates such that we
can reach all the four wedges in the Lorentzian sector. The procedure is based on a simple unified
principle, viz that the analytic continuation should map Euclidean squared distance $\sigma_E^2$ to $(\sigma_M^2 + i\epsilon)$,
with a positive, infinitesimal, imaginary part in the Lorentzian sector. We explicitly demonstrate
that this procedure leads to the correct expressions for the propagators in the Lorentzian sector,
even when the two events are in two different wedges. Again, as far as we know, this problem has
not been explicitly addressed in the previous literature.

This approach also allows us to discover relatively simple expressions for the temporal Fourier trans-
form of the Minkowski and Rindler propagators. These two are connected by the relation in Eq. (117),
which we hope to study in detail in a future publication.

Our procedure can be generalized to any bifurcate Killing horizons in curved spacetime like, e.g., the de
Sitter horizon or black hole horizon. This implies that one can expect results similar to Eq. (3) in much
more general situations, even when the explicit forms of $G_M$ and $G_R$ are not available. We hope to study
this in detail in a future publication.

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A Derivation of Eq.(11)

Let us denote the desired sum by $S$.

$$S = \sum_{n=-\infty}^{\infty} e^{-|\tau_E + 2\pi n|}$$ (121)
It is convenient to assume that:

\[-2\pi N < \tau_E < -2\pi(N - 1)\]  \hfill (122)

where, \(N\) is an integer (which can either be positive, negative or zero). Then the sum \(S\) can be split into two parts as follows:

\[S = \sum_{n=-\infty}^{N-1} e^{i\omega(\tau_E + 2\pi n)} + \sum_{n=N}^{\infty} e^{-i\omega(\tau_E + 2\pi n)}\]  \hfill (123)

Each of the two summations on the right hand side of the above equation is a geometric series. Hence, using standard results, we get

\[S = \frac{e^{i\omega[\tau_E + 2\pi(N-1)]}}{1 - e^{-2\pi\omega}} + \frac{e^{-i\omega(\tau_E + 2\pi N)}}{1 - e^{-2\pi\omega}}\]  \hfill (124)

This can be further simplified into,

\[S = \frac{\cosh[\omega (\tau_E + \pi(2N - 1))]}{\sinh(\pi\omega)} = \frac{\cosh[\omega (\tau_E - \pi)]}{\sinh(\pi\omega)}\]  \hfill (125)

where, \(\Lambda_E = \tau_E \text{ mod } 2\pi\). Now, when \(0 < \tau_E < 2\pi\), the above equation simplifies to

\[S = \frac{\cosh[\omega (\tau_E - \pi)]}{\sinh(\pi\omega)}\]  \hfill (126)

Similarity, when \(-2\pi < \tau_E < 0\), we get

\[S = \frac{\cosh[\omega (\tau_E + \pi)]}{\sinh(\pi\omega)}\]  \hfill (127)

Combining the last two equations, the expression for \(S\), when \(|\tau_E| < 2\pi\) is given by

\[S = \frac{\cosh[\omega (|\tau_E| - \pi)]}{\sinh(\pi\omega)}\]  \hfill (128)

\section*{B Inverting the periodic summation}

We will begin with the proof that Eq. (22) implies Eq. (21). Note that by definition, \(F_M(z+2\pi n) = F_M(z)\). Hence, for convenience, we assume that \(-\pi < \text{Im}[z] < \pi\). Now, consider the RHS of Eq. (21):

\[
\int_{C'} \frac{du}{i\pi} \mathcal{H}(z; u) F_R(u) = \int_{C'} \frac{du}{i\pi} \mathcal{H}(z; u) \left[ \int_{C} \frac{dv}{i\pi} G(u; v) F_M(v) \right] (129)
\]

\[
\phantom{= \int_{C'} \frac{du}{i\pi} \mathcal{H}(z; u) G(u; v)} = \int_{C'} \frac{dv}{i\pi} \left[ \int_{C} \frac{du}{i\pi} \mathcal{H}(z; u) G(u; v) \right] F_M(v) (130)
\]

Consider the terms in the square bracket; using residue theorem, we get:

\[
\int_{C} \frac{du}{i\pi} \mathcal{H}(z; u) G(u; v) = \sum_{n=-\infty}^{\infty} \left( \frac{v}{v^2 - (z + 2\pi in)^2} \right) (131)
\]

\[
= \frac{\sinh v}{2(cosh v - cosh z)} \]  \hfill (132)
Using this, Eq. (130) can be simplified to

\[
\int_{C'} \frac{dv}{i\pi} \left[ \int_{C'} \frac{du}{i\pi} H(z; u) G(u; v) \right] F_M(v) = \int_{C'} \frac{dv}{2i\pi} \frac{\sinh v}{(\cosh v - \cosh z)} F_M(v)
\]

(133)

It is convenient to deform $C$ into the $\tilde{C}$ given in Figure 3. There are three contributions to this integral: (1) the residue at $z$, (2) integral over the vertical line from $-i\pi$ to $i\pi$ and (3) integrals over the horizontal lines.

\[
\int_{C'} \frac{dv}{(2i\pi)} \frac{\sinh v}{(\cosh v - \cosh z)} F_M(v) = F_M(z) + \int_{-\pi}^{\pi} \frac{dy}{2i\pi} \frac{i \sin y}{(\cos y - \cosh z)} F_M(iy)
\]

(134)

\[
+ \int_{0}^{\infty} \frac{dx}{(2i\pi)} \frac{\sinh x}{(\cosh x + \cosh z)} F_M(x - i\pi)
\]

\[
+ \int_{0}^{\infty} \frac{dx}{(2i\pi)} \frac{\sinh x}{(\cosh x + \cosh z)} F_M(x + i\pi)
\]

The second term in the first line vanishes, because $F_M(iy)$ is even. From the pseudo periodicity and evenness of $F_M$, we also have $F_M(x - i\pi) = F_M(x + i\pi)$. Hence, the sum of third and forth terms vanishes. Finally, we have

\[
\int_{C'} \frac{du}{i\pi} H(z; u) F_R(u) = F_M(z).
\]

(135)

Let us next show that Eq. (21) implies Eq. (22). We start with the RHS of Eq. (22).

\[
\int_{C} \frac{du}{(i\pi)} G(z; u) F_M(u) = \int_{C} \frac{du}{(i\pi)} G(z; u) \left[ \int_{C'} \frac{dv}{i\pi} H(u; v) F_R(v) \right]
\]

(136)

\[
= \int_{C'} \frac{dv}{(i\pi)} \left[ \int_{C} \frac{du}{(i\pi)} G(z; u) H(u; v) \right] F_R(v)
\]

(137)
Again, we will deform the contour $C$ to $\tilde{C}$, so that the term in the square bracket becomes,

$$\int \frac{du}{(i\pi)} G(z; u) H(u; v) = \text{Residue term + vertical integral + horizontal integrals} \quad (138)$$

Note that the residue term is present only when $z$ lies in the shaded region of Figure 1. In this case, we have

$$\int \frac{du}{(i\pi)} G(z; u) H(u; v) = \frac{\sinh v}{4(\cosh v - \sinh z)} + \int_{-\pi}^{\pi} \frac{dy}{\pi} \left[ \frac{-iy}{(y^2 + z^2)} \right] \left[ \frac{\sinh v}{4(\cosh v - \cos y)} \right]$$

$$\int_{-\infty}^{0} \frac{dx}{(i\pi)} \left[ \frac{(x - i\pi)}{(x - i\pi)^2 - z^2} \right] \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] + \int_{0}^{\infty} \frac{dx}{(i\pi)} \left[ \frac{(x + i\pi)}{(x + i\pi)^2 - z^2} \right] \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right]$$

The second term in the last equation vanishes, since the integrand is odd. The third and forth term can be combined to get:

$$- \int_{0}^{\infty} \frac{dx}{(i\pi)} \left\{ \frac{2i\pi(\pi^2 + \pi^2 + z^2)}{\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] \right\}$$

$$= - \int_{-\infty}^{\infty} dx \left\{ \frac{i\pi(\pi^2 + \pi^2 + z^2)}{\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] \right\}$$

$$= - \int_{-\infty}^{\infty} dx \frac{1}{2\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] - \int_{-\infty}^{\infty} dx \frac{1}{2\pi^2 + (z + x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] \quad (141)$$

$$= - \int_{-\infty}^{\infty} \frac{dx}{\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] \quad (142)$$

Therefore, we have

$$\int \frac{du}{(i\pi)} G(z; u) H(u; v) = \frac{\sinh v}{4(\cosh v - \sinh z)} - \int_{-\infty}^{\infty} \frac{dx}{\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] \quad (143)$$

So that,

$$\int \frac{dv}{(i\pi)} \left[ \int \frac{du}{(i\pi)} G(z; u) H(u; v) \right] F_R(v) \quad (144)$$

$$= \int \frac{dv}{(i\pi)} \frac{\sinh v}{4(\cosh v - \sinh z)} F_R(v) - \int \frac{dv}{(i\pi)} \int_{-\infty}^{\infty} \frac{dx}{\pi^2 + (z - x)^2} \left[ \frac{\sinh v}{4(\cosh v + \cosh x)} \right] F_R(v) \quad (145)$$

$$= F_M(z) - \int_{-\infty}^{\infty} \frac{dx}{\pi^2 + (z - x)^2} F_M(x + i\pi) \quad (146)$$

$$= F_R(z) \quad (147)$$
C The two Kernels: Technical details

C.1 Derivation of $K_{Eu}^R$

We start with the expression for $K_{Eu}^R$ given in Eq. (75).

$$K_{Eu}^R = \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \int_0^\infty d\omega I_{|\omega|} \left(\frac{r'r}{2s}\right) \cos \omega(\theta - \theta')$$  \hspace{1cm} (148)

Recall that the modified Bessel function has the following contour integral representation.

$$I_\nu(z) = \frac{1}{2\pi i} \int_C du e^{z \cosh u - \nu u}; \quad |\arg[z]| < \frac{\pi}{2}$$  \hspace{1cm} (149)

where, the contour $C$ is as given in Figure 1. Using this, the kernel $K_R$ can be rewritten as

$$K_{Eu}^R = \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \int_0^\infty d\omega \left[\frac{1}{2\pi i} \int_C du e^{z \cosh u - \omega u}\right] \cos \omega\phi$$  \hspace{1cm} (150)

where, $z = (rr')/(2s)$ and $\phi = |\theta - \theta'|$. Using the following identity,

$$\int_0^\infty \cos \omega \phi e^{-\omega u} = \frac{u}{u^2 + \phi^2}; \quad |\Im(\phi)| < |\Re(u)|$$  \hspace{1cm} (151)

we can further simplify $K_{Eu}^R$ to be,

$$K_{Eu}^R = \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \left[\frac{1}{2\pi i} \int_C du e^{z \cosh u} \left(\frac{u}{u^2 + \phi^2}\right)\right]$$  \hspace{1cm} (152)

Let us now assume that $\phi > \pi$, and simplify this expression, which can later be analytically continued for $\phi < \pi$. The following identity may be used,

$$\int_0^\infty d\nu \sin(\nu u) e^{-\nu\phi} = \frac{u^2}{u^2 + \phi^2}; \quad |\Im(u)| < |\Re(\phi)|$$  \hspace{1cm} (153)

When $\phi > \pi$, the condition for the above integral is satisfied, so that $K_{Eu}^R$ may be written as

$$K_{Eu}^R = \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \left[\frac{1}{2\pi i} \int_C du e^{z \cosh u} \left(\int_0^\infty d\nu \sin(\nu u) e^{-\nu\phi}\right)\right]$$  \hspace{1cm} (154)

$$= \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \int_0^\infty d\nu e^{-\nu\phi} \left[\frac{1}{2\pi i} \int_C du e^{z \cosh u} \left(\frac{e^{i\nu u} - e^{-i\nu u}}{2i}\right)\right]$$  \hspace{1cm} (155)

Again, using the integral representation of $I_\nu$, we obtain

$$K_{Eu}^R = \frac{1}{(2\pi s)} \exp \left(-\frac{r^2 + r'^2}{4s}\right) \int_0^\infty d\nu e^{-\nu\phi} \left(\frac{I_{-\nu}(z) - I_{\nu}(z)}{2i}\right)$$  \hspace{1cm} (156)

Recall that the MacDonald $K_{\mu}(z)$ is given by

$$K_{\mu}(z) = \frac{\pi}{2} \left(\frac{I_{-\mu}(z) - I_{\mu}(z)}{\sin(\pi\mu)}\right)$$  \hspace{1cm} (157)

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Hence,

\[ K_{Eu}^E = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty \frac{d\nu}{\pi} e^{-\nu\phi} \sinh(\pi \nu) K_{iv} (z) \]  

(158)

\[ = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty \frac{d\nu}{\pi} e^{-\nu\phi} \sinh(\pi \nu) K_{iv} \left( \frac{rr'}{2s} \right) \]  

(159)

\section*{C.2 \textit{K}_{Eu}^E in terms of MacDonald function}

Let us start with the following representation of \( K_{Eu}^E \) that follows from Eq. (67).

\[ K_{Eu}^E = \frac{1}{(4\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \sum_{m=-\infty}^{\infty} I_m (z) e^{im\phi} \]  

(160)

where, \( z = (rr'/2s) \) and \( \phi = |\theta - \theta'| \). Using the contour representation of \( I_{\nu} \), we have

\[ K_{Eu}^E = \frac{1}{(4\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \sum_{m=-\infty}^{\infty} \left\{ \frac{1}{2\pi i} \int_C du e^{z \cosh u - mu} \right\} e^{im\phi} \]  

(161)

The following identity is useful,

\[-1 + \sum_{m=0}^{\infty} 2 \cos(m\phi) e^{-mu} = \frac{\sinh(u)}{\cosh(u) - \cos(\phi)}; \quad \Re[u] \geq 0 \]  

(162)

\[ = \sum_{n=-\infty}^{\infty} \frac{2u}{u^2 + (\phi + 2\pi n)} \]  

(163)

\[ = 2 \sum_{n=-\infty}^{\infty} \left[ \int_0^\infty \sin(\nu u) e^{-\nu|\phi + 2\pi n|} \right] \]  

(164)

\[ = 2 \int_0^\infty d\nu \sin(\nu u) \left[ \frac{\cosh \{ \nu(\pi - \phi) \}}{\sinh(\pi \nu)} \right]; \quad 0 < \phi < \pi. \]  

(165)

Using this in Eq. (161) we get,

\[ K_{Eu}^E = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty d\nu \frac{\cosh \{ \nu(\pi - \phi) \}}{\sinh(\pi \nu)} \left[ \frac{1}{2\pi i} \int_C du e^{z \cosh u \sin(\nu \mu)} \right] \]  

(166)

Again using the integral representation of \( I_{\nu} \) and the relation between \( K_{\nu} \) and \( I_{\nu} \), we get

\[ K_{Eu}^E = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty d\nu \frac{\cosh \{ \nu(\pi - \phi) \}}{\sinh(\pi \nu)} K_{iv} (z) \]  

(167)

\[ = \frac{1}{(2\pi s)} \exp \left( -\frac{r^2 + r'^2}{4s} \right) \int_0^\infty d\nu \frac{\cosh \{ \nu(\pi - \phi) \}}{\sinh(\pi \nu)} K_{iv} \left( \frac{rr'}{2s} \right) \]  

(168)

\section*{D Details of analytic continuation}

In this section we will discuss some technical details of our recipe for analytic continuation in Table 1. Here, we explicitly show that under these transformations (i) \( \sigma^2 \rightarrow \sigma^2 + i\epsilon \) and (ii) the analytic continuation of \( \Theta_E \) that follows from Eq.(84) matches exactly with that in given Table 1.
D.1 RR: both events on the R wedge

The transformations in this case are given by
\[ (r, \theta) \rightarrow (\rho, i\tau e^{-i\epsilon}); \quad (r', \theta') \rightarrow (\rho', i\tau' e^{i\epsilon}). \] (169)

Therefore, to leading order in \( \epsilon \), the invariant distance squared is given by
\[ \sigma^2_E \rightarrow \left[ \rho^2 - 2\rho\rho' \cosh (\tau - \tau') + (\rho')^2 \right] + 2i\epsilon \rho\rho' (\tau - \tau') \sinh (\tau - \tau') + O (\epsilon^2). \] (170)

Clearly the imaginary part of the right hand side is positive. Now, to find the analytic continuation of \( \Theta_E \), let us first look at the expression for \( Z \).
\[ Z = \cosh(e^{-i\epsilon|\tau - \tau'|}); \] (171)
\[ \Rightarrow 1 - Z^2 = -\sinh^2 (\tau - \tau') + i\epsilon (\tau - \tau') \sinh [2(\tau - \tau')] + O (\epsilon^2) \] (172)

Since \( 1 - Z^2 \) has a small positive imaginary component, the square root in the expression for \( \Theta_E \) will evaluate to \( \sqrt{1 + Z^2} \). Hence, the final expression for \( \Theta_E \) simplifies to:
\[ \Theta_E \rightarrow \frac{\pi}{2} + i \log(ie^{i|\tau - \tau'|1-\epsilon}) = i|\tau - \tau'| + \pi. \] (173)

D.2 RF: one event on R and the other on F wedge

According to our recipe, the transformation of points in this case is given by
\[ (r, \theta) \rightarrow (\rho_R, i\tau_R); \quad (r', \theta') \rightarrow (i\rho_F, i\tau_F + \frac{\pi}{2} + \epsilon). \] (174)

Hence, \( \sigma^2_E \) transforms to
\[ \sigma^2_E \rightarrow \left[ -\rho_F^2 - 2\rho_F \rho_R \sinh (\tau_F - \tau_R) + \rho_R^2 \right] + 2i\epsilon \rho_F \rho_R \cosh (\tau_F - \tau_R) + O (\epsilon^2). \] (175)

Once again, we see that the imaginary part of the right hand side is positive. The biscalar \( Z \) in this case turns out to be:
\[ Z = -i \sinh [(\tau_F - \tau_R) - i\epsilon]; \] (176)
\[ \Rightarrow 1 - Z^2 = \cosh [(\tau_F - \tau_R) - i\epsilon] \] (177)

Therefore, the analytic continuation of \( \Theta_E \) becomes;
\[ \Theta_E \rightarrow \frac{\pi}{2} + i \log(e^{i(\tau_F - \tau_R) - i\epsilon}) = i(\tau_F - \tau_R) + \frac{\pi}{2} + \epsilon. \] (178)

Notice that there is no ‘modulus’ on \( \tau_F - \tau_R \).
D.3 FF: both events on the F wedge

Following our prescription in Table 1, the analytic continuation of coordinates is given by:

\[(r_\prec, \theta) \rightarrow (-e^{i\rho_\prec}i\tau + \frac{\pi}{2}); \quad (r_\succ, \theta') \rightarrow (i\rho_\succ, i\tau' - \frac{\pi}{2}).\] (179)

Under this transformation, the square of distance \(\sigma^2_E\) continues to:

\[\sigma^2_E \rightarrow (2\rho_\succ\rho_\prec \cosh(\tau - \tau') - \rho_\succ^2 - \rho_\prec^2 - 2i\epsilon\rho_\prec (\rho_\succ \cosh(\tau - \tau') - \rho_\prec) + O(\epsilon^2)\] (180)

Since, by definition \(\rho_\succ \geq \rho_\prec\), we find that the imaginary part of the right hand is positive. The biscalar \(Z\) in this case becomes:

\[Z = -\cosh(\tau - \tau');\] (181)

\[\Rightarrow 1 - Z^2 = -\sinh^2(\tau - \tau').\] (182)

Therefore, \(\sqrt{1 - Z^2 + i0^+} = i \sinh |\tau - \tau'|\). Finally, the expression for \(\Theta_E\) reduces to:

\[\Theta_E \rightarrow \frac{\pi}{2} + i \log(-ie^{-i|\tau - \tau'|}) = -i|\tau - \tau'| + \pi\] (183)

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