Closed-Form Delay-Optimal Power Control for Energy Harvesting Wireless System with Finite Energy Storage

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Abstract—In this paper, we consider delay-optimal power control for an energy harvesting wireless system with finite energy storage. The wireless system is powered solely by a renewable energy source with bursty data arrivals, and is characterized by a data queue and an energy queue. We consider a delay-optimal power control problem and formulate an infinite horizon average cost Markov Decision Process (MDP). To deal with the curse of dimensionality, we introduce a virtual continuous time system and derive closed-form approximate priority functions for the discrete time MDP at various operating regimes. Based on the approximation, we obtain an online power control solution which is adaptive to the channel state information as well as the data and energy queue state information. The derived power control solution has a multi-level water-filling structure, where the water level is determined jointly by the data and energy queue lengths. We show through simulations that the proposed scheme has significant performance gain compared with various baselines.

I. INTRODUCTION

Recently, green communication has received considerable attention since it will play an important role in enhancing energy efficiency and reducing carbon emissions in future wireless networks [1], [2]. To support green communication, energy harvesting techniques such as solar panels, wind turbines and thermoelectric generators [3] have become popular for enabling the transmission nodes to harvest energy from the environment. While the renewable energy sources may appear to be virtually free and they are random in nature, energy storage is needed to buffer the unstable supply of the renewable energy [4]. In [5] and [6], the authors propose transmission policies that minimize the transmission time for a given amount of data in point-to-point and broadcast energy harvesting networks with an infinite capacity battery. However, the infinite capacity battery assumption is not realistic in practice. In [7] and [8], offline power allocation policies are proposed by solving short-term throughput maximization problems under finite energy storage capacity in a finite time horizon. However, the above works [5]–[8] assume that the realizations of the energy arrival processes are known in advance (i.e., non-causal knowledge of future arrivals). Furthermore, the above proposed policies [5]–[8] are based on the assumption that there are infinite data backlogs at the transmitters so that the applications are delay-insensitive. In practice, it is very important to consider bursty data arrivals, bursty energy arrivals and delay requirements in designing the power control policy for delay-sensitive applications.

In this paper, we are interested in the online power control solution in a wireless system powered by a renewable energy source to support real-time delay-sensitive applications. The wireless transmitter is powered solely by an energy harvesting storage with limited energy storage capacity. Unlike the previous proposed schemes [5]–[8], we consider an online control policy, in the sense that we only have causal knowledge of the system states. Specifically, to support real-time applications with bursty data arrivals and bursty renewable energy arrivals, it is very important to dynamically control the transmit power that is adaptive to the channel state information (CSI), the data queue length (DQSI) and the energy queue length (EQSI). The CSI reveals the transmission opportunities of the time-varying physical channels. The DQSI reveals the urgency of the data flows and the EQSI reveals the availability of the renewable energy. It is highly non-trivial to strike a good balance between these factors.

Online power control adaptive to the CSI, the DQSI and the EQSI is quite challenging because the associated optimization problem belongs to an infinite-dimensional stochastic optimization problem. There is intense research interest in exploiting renewable energy in communication network designs. In [9], the authors use large deviations theory to find the closed-form expression for the buffer overflow probability and design an energy-efficient scheme for maximizing the decay exponent of this probability. In [10], the authors propose throughput-optimal control policies (in the stability sense) that are adaptive to the CSI, the DQSI and the EQSI for a point-to-point energy harvesting network. In [11] and [12], the authors extend the Lyapunov optimization framework to derive energy management algorithms, which can stabilize the data queue for energy harvesting networks with finite energy storage capacity. Note that the buffer overflow probability and the queue stability are weak forms of delay performance, and it is of great importance to study the control policies that minimize the average delay of the queueing network. A systematic approach in dealing with the delay-optimal control is to formulate the problem into an Markov Decision Process (MDP) [13], [14]. In [15], the authors propose several heuristic event-based adaptive transmission policies on the basis of a finite horizon MDP formulation. These solutions are suboptimal and with no performance guarantee. In [4], the authors consider online power control for the interference network with a renewable energy supply by solving an infinite horizon average cost
MDP. The authors in [10] also propose an online delay-optimal power control policy by solving an infinite horizon MDP for energy harvesting networks. However, the MDP problems in [4] and [10] are solved using numerical iteration algorithms, such as value iteration or policy iteration algorithms (Chap. 4 in Vol. 2 of [14]), which suffer from slow convergence and a lack of insight. There are some existing works that adopt MDP/POMDP approaches to solve the stochastic resource allocation problems for energy harvesting wireless sensor networks [16]–[24]. In [16], the authors consider a simple birth-death model for the energy queue dynamics and obtain a threshold-like data transmission scheme by maximizing an average data rate reward using the MDP approach. However, the energy queue model considered in [16] is a simplified model and the approach therein cannot be applied in our scenario with general energy queue dynamics. In [17]–[19], the authors propose an efficient power control scheme to minimize the average power consumption and the packet error rate. In [20] and [21], the authors consider on-off control of the sensor to maximize the event detection efficiency or maximize the average data rate reward using the MDP approach. In this paper, we focus on deriving a closed-form delay-optimal power control solution for an energy harvesting sensor network with finite energy storage capacity. The authors in [10] also consider periodic energy arrivals for designing the power control policy. However, the power control actions in [17]–[21] are chosen from discrete and finite action spaces. Hence, the approaches in [17]–[21] cannot be applied to our scenario where the power control action is chosen from a continuous action space. In [22] and [23], the authors propose a power allocation scheme for an energy harvesting sensor network with finite energy buffer capacity by solving a POMDP problem. However, they consider non-causal control, which means that the realizations of the energy arrival processes are known in advance. In [24], the authors consider general energy queue model with online causal power control schemes. However, the stochastic MDP/POMDP problems in [16]–[24] are solved using numerical value iteration or policy iteration algorithms [14]. In this paper, we focus on deriving a closed-form delay-optimal online power control solution that is adaptive to the CSI, the DQSI and the EQSI. There are several first order technical challenges associated with the stochastic optimization.

- **Challenges due to the Queue-Dependent Control:** In order to maintain low average delay performance and efficiently use the renewable energy in a finite capacity storage, it is important to dynamically control the transmit power based on the CSI, the DQSI and the EQSI. As a result, the underlying problem embraces both information theory (to model the physical layer dynamics) and the queueing theory (to model the data and energy queue dynamics) and is an infinite horizon stochastic optimization [13], [14]. Such problems are well-known to be very challenging due to the infinite-dimensional optimization (w.r.t. control policy) and lack of closed-form characterization of the value function in the optimality equation (i.e., the Bellman equation).

- **Complex Coupling between the Data Queue and the Energy Queue:** The service rate of the transmitter in the energy harvesting network depends on the current available energy stored in the energy queue buffer. As such, the dynamics of the data queue and the energy queue are coupled together. The associated stochastic optimization problem is a multi-dimensional MDP [25]. To solve the associated Bellman equation, numerical brute-force approaches (e.g., value iteration and policy iteration [14]) can be adopted, but they are not practical and provide no design insights. Therefore, it is desirable to obtain a low complexity and insightful solution for the dynamic power control in the energy harvesting system.

- **Challenges due to the Finite Energy Storage and Non-i.i.d. Energy Arrivals:** In practice, the energy storage (or battery) at the transmitter has finite capacity only. The finite renewable energy storage limit induces a difficult energy availability constraint (the energy consumption per time slot cannot exceed the available energy in the storage) in the stochastic optimization problem. Furthermore, in the previous literature (e.g., [4]–[9]), the bursty energy arrivals are modeled as an i.i.d. process for analytical tractability. In [10], the authors also consider periodic stationary energy arrivals for designing the power control policies. In practice, most of the renewable energy arrivals are not i.i.d.. Such non-i.i.d. nature will have a huge impact on the dimensioning of the battery capacity.

In this paper, we model the delay-optimal power control problem as an infinite horizon average cost MDP. Specifically, the stochastic MDP problem is to minimize the average delay of the transmitter subject to the energy availability constraint. By exploiting the special structure in our problem, we derive an equivalent Bellman equation to solve the MDP. We then introduce a virtual continuous time system (VCTS) where the evolutions of the data and energy queues are characterized by two coupled differential equations with reflections. We show that the priority function of the associated total cost problem in the VCTS is asymptotically optimal to that of the discrete time MDP problem when the slot duration is sufficiently small. Using the priority function in the VCTS as an approximation to the optimal priority function, we derive online power control solutions and obtain design insights from the structural properties of the priority function under different asymptotic regimes. The power control solution has a multi-level water-filling structure, where the DQSI and the EQSI determine the water level via the priority function. Finally, we compare the proposed solution with various baselines and show that significant performance gain can be achieved.

**II. System Model**

We consider a point-to-point energy harvesting system with finite energy storage. Fig. 1 illustrates the top-level system model, where the transmitter is powered solely by the energy harvesting storage with limited energy storage capacity. The
transmitter acts as a cross-layer controller, which takes the CSI, the DQSI and the EQSI as input and generates power control action as output. In this paper, the time dimension is partitioned into decision slot indexed by \( n (n = 0, 1, 2, \ldots) \) with duration \( \tau \). In the following subsections, we elaborate on the physical layer model and the bursty data arrival model, as well as the renewable energy arrival model.

### A. Physical Layer Model

We consider a point-to-point system as shown in Fig. 1. The transmitter sends information to the receiver. Let \( s \) be the transmitted information symbol and the received signal is given by

\[
y = h \sqrt{p} s + z
\]

where \( h \in \mathbb{C} \) is the complex channel fading coefficient between the transmitter and the receiver, \( p \) is the transmit power, and \( z \sim \mathcal{CN}(0, 1) \) is the i.i.d. complex Gaussian additive channel noise. We have the following assumption on the channel model.

**Assumption 1 (Channel Model):** \( h(n) \) remains constant within each decision slot and is i.i.d. over the slots. Specifically, we assume that \( h(n) \) follows a complex Gaussian distribution with zero mean and unit variance, i.e., \( h(n) \sim \mathcal{CN}(0, 1) \).

For given CSI realization \( h \) and power control action \( p \), the achievable data rate (bit/s/Hz) for the transmitter-receiver pair is given by

\[
R(h, p) = \log \left( 1 + \frac{\zeta p |h|^2}{1 + \zeta p} \right)
\]

where \( \zeta \in (0, 1] \) is a constant that is determined by the modulation and coding scheme (MCS) used in the system. For example, \( \zeta = 0.5 \) for QAM constellation at BER= 1% and \( \zeta = 1 \) for capacity-achieving coding (in which corresponds to the instantaneous mutual information). In this paper, our derived results are based on \( \zeta = 1 \) for simplicity, which can be easily extended to other MCS cases.

### B. Bursty Data Source Model and Data Queue Dynamics

As illustrated in Fig. 1, the transmitter maintains a data queue for the bursty traffic flow towards the receiver. Let \( \lambda(n) \tau \) be the random new data arrival (bits) at the end of the \( n \)-th decision slot at the transmitter. We have the following assumption on the data arrival process.

**Assumption 2 (Bursty Data Source Model):** The data arrival process \( \lambda(n) \) is i.i.d. over the slots according to a general distribution \( \Pr[\lambda] \) with finite average arrival rate \( \mathbb{E}[\lambda] = \lambda \).

Let \( Q(n) \in \mathbb{Q} \) denote the DQSI (bits) at the data queue of the transmitter at the beginning of the \( n \)-th slot, where \( \mathbb{Q} = [0, \infty) \) is the DQSI state space. We assume that the transmitter is causal in the sense that new data arrivals are observed after the control actions are performed at each decision slot. Hence, the data queue dynamics is given by

\[
Q(n + 1) = (Q(n) - R(h(n), p(n)) \tau)^+ + \lambda(n) \tau
\]

where \( x^+ = \max \{ x, 0 \} \). Note that \( p(n) \) is transmit power of the transmitter at time slot \( n \) and the power solely comes from the renewable energy source. We shall define the renewable energy source model in the next subsection.

### C. Renewable Energy Source Model and Energy Queue Dynamics

The power of the transmitter solely comes from the renewable energy source. Specifically, the transmitter is capable of harvesting energy from the environment, e.g., using solar panels, wind turbines and thermoelectric generators. We assume that the energy arrival process is block i.i.d. with block size \( N \). The block i.i.d. energy arrival model is used to take into account that the energy arrival process evolves at a different timescale w.r.t. that of the data arrival process. Let \( \alpha(n) \tau \) be the random renewable energy arrival (Joules) at the end of the \( n \)-th decision slot at the transmitter. We have the following assumption on the energy arrival process.

**Assumption 3 (Block i.i.d. Renewable Energy Source Model):** The energy arrival process \( \alpha(n) \) is block i.i.d. in the sense that \( \alpha(n) \) is constant for a block of \( N \) slots and is i.i.d. between blocks according to a general distribution \( \Pr[\alpha] \) with finite average energy arrival rate \( \mathbb{E} [\alpha] = \alpha \).

Due to the random nature of the renewable energy, there is limited energy storage capacity at the transmitter to buffer the renewable energy arrivals. Let \( E(n) \in \mathbb{E} \) denote the EQSI (Joules) at the beginning of the \( n \)-th slot, where \( \mathbb{E} = [0, N_E] \) is the EQSI state space and \( N_E \) denotes the energy queue buffer size (i.e., energy storage capacity in Joules).

**Remark 1 (Discussions on the Finite Energy Queue Capacity):** High-capacity renewable energy storage is very expensive and energy storage is one key cost component in renewable energy systems. As such, it is very important to consider the impact on how the finite renewable energy buffer affects the system performance. The analysis also serves as the first order dimensioning on how large an energy buffer is needed.

Note that when the energy buffer is full, i.e., \( E(n) = N_E \), additional energy cannot be harvested. Similarly, we assume that the transmitter is causal so that the renewable energy arrival \( E(n) \) is observed only after the power actions. Hence, the energy queue dynamics at the transmitter is given by

\[
E(n + 1) = \min \{ E(n) - p(n) \tau + \alpha(n) \tau, N_E \}
\]

where the renewable power consumption \( p(n) \) must satisfy the following energy availability constraint:

\[
p(n) \tau \leq E(n)
\]

The energy availability constraint means that the energy consumption at each time slot cannot exceed the current available energy in the energy storage. Due to this constraint, the energy queue \( E(n) \) in (4) will not go below zero (i.e., \( E(n) \geq 0 \) for all \( n \)).

**Remark 2: (Coupling Property of Data Queue and Energy Queue)** The data queue dynamics in (3) and the energy queue dynamics in (4) are coupled together. Specifically, the service rate \( R(n) \) in the data queue depends on the power control action \( p(n) \), which solely comes from the energy queue buffer.

\[1\] Specifically, \( \alpha(n) \) is constant when \( kN \leq t < (k+1)N \) for any given \( t \), where \( k \) is a positive integer.
III. DELAY-OPTIMAL PROBLEM FORMULATION

In this section, we formally define the power control policy and formulate the delay-optimal control problem for the point-to-point energy harvesting system.

A. Power Control Policy

For notation convenience, denote $\chi(n) = (h(n), Q(n), E(n))$. Let $\mathcal{F}(n) = \sigma(\{\chi(i) : 0 \leq i \leq n\})$ be the minimal $\sigma$-algebra containing the set $\{\chi(i) : 0 \leq i \leq n\}$, and $\{\mathcal{F}(n)\}$ be the associated filtration. At the beginning of the $n$-th slot, the transmitter determines the power control action based on the following control policy:

Definition 1 (Power Control Policy): A power control policy for the transmitter $\chi$ is $\mathcal{F}(n)$-adapted at each time slot $n$, meaning that the power control action $\gamma(n)$ is adaptive to all the information $\chi(i)$ up to some time $n$ (i.e., $\{\chi(i) : 0 \leq i \leq n\}$). Furthermore, the power control policy $\chi$ satisfies the energy availability constraint in (5), i.e., $\chi_n - E_n \leq \lambda_n \tau$ for any $n$. ■

Given a control policy $\chi$, the random process $\{\chi(n)\}$ is a controlled Markov chain with the following transition probability:

$$
\Pr[\chi(n+1)|\chi(n), \Omega(\chi(n))] = \Pr[h(n+1)] \Pr[Q(n+1)|Q(n), h(n), \Omega(\chi(n))] \cdot \Pr[E(n+1)|E(n), \Omega(\chi(n))]
$$

(6)

where $\Pr[Q(n+1)|Q(n), h(n), \Omega(\chi(n))]$ is the data queue transition probability which is given by

$$
\Pr[Q(n+1) = Q'|Q(n) = Q, h(n) = h, \Omega(\chi(n)) = \gamma] = \begin{cases} \Pr[p(n)], & \text{if } Q' = [Q - R(h, p) \tau + \lambda_n \tau] + \gamma \leq Q \\ 0, & \text{otherwise} \end{cases}
$$

(7)

and $\Pr[E(n+1)|E(n), \Omega(\chi(n))]$ is the energy queue transition probability which is given by

$$
\Pr[E(n+1) = E'|E(n) = E, \Omega(\chi(n)) = \gamma] = \begin{cases} \Pr[p(n)], & \text{if } E' = \min\{E - p\tau + \alpha(n) \tau, N_E\} \\ 0, & \text{otherwise} \end{cases}
$$

(8)

Furthermore, we have the following definition on the admissible control policy:

Definition 2 (Admissible Control Policy): A policy $\chi$ is admissible if the following requirements are satisfied:

- $\Omega$ is a unichain policy, i.e., the controlled Markov chain $\{\chi(n)\}$ under $\Omega$ has a single recurrent class and possibly some transient states.

- The queueing system under $\Omega$ is stable in the sense that $\lim_{n \to \infty} E^n[Q^2(n)] < \infty$, where $E^n$ means taking expectation w.r.t. the probability measure induced by the control policy $\Omega$.

B. Problem Formulation

As a result, under an admissible control policy $\Omega$, the average delay cost of the energy harvesting system starting from a given initial state $\chi(0)$ is given by

$$
\overline{D}(\Omega) = \lim_{N \to \infty} \sup \frac{1}{N} \sum_{n=0}^{N-1} E^n \left[ \frac{Q(n)}{\lambda} \right]
$$

(9)

We consider the following delay-optimal power control optimization for the energy harvesting system:

Problem 1 (Delay-Optimal Power Control Optimization):

$$
\min_{\Omega} \overline{D}(\Omega)
$$

(10)

where $\Omega$ satisfies the energy availability constraint according to Definition 1. ■

C. Optimality Conditions

While the MDP in Problem 1 is difficult in general, we utilize the i.i.d. assumption of the CSI to derive an equivalent optimality equation as summarized below.

Theorem 1 (Sufficient Conditions for Optimality): Assume there exists a $(\theta^*, \{V^*(Q, E)\})$ that solves the following equivalent optimality equation:

$$
\theta^* + V^*(Q, E) \geq \forall Q, E
$$

(11)

$$
\varepsilon E \left[ \min_{p \leq E/\tau} \left[ \frac{Q}{\lambda} + \sum_{Q', E'} \Pr[Q', E'|\chi(p) \tau)V^*(Q', E')] \right] \right | Q, E
$$

Furthermore, align all admissible control policy $\Omega$ and initial queue state $(Q(0), E(0))$, $V^*$ satisfies the following transversality condition:

$$
\lim_{N \to \infty} \frac{1}{N} E^n \left[ V^*(Q(N), E(N)) | Q(0), E(0) \right] = 0
$$

(12)

Then, we have the following results:

- $\theta^* = \min_{\Omega} \overline{D}(\Omega)$ is the optimal average cost for any initial state $\chi(0)$ and $V^*(Q, E)$ is called the priority function.

- Suppose there exists an admissible stationary control policy $\Omega^*$ with $\Omega^*(\chi) = p^*$ for any $\chi$, where $p^*$ attains the minimum of the R.H.S. of (11) for given $\chi$. Then, the optimal control policy of Problem 1 is given by $\Omega^*$. ■

Proof: please refer to Appendix A.

Based on the unichain assumption of the control policy in Definition 2, there is a unique solution for the Bellman equation in (11) and the transversality condition in (12). The solution $V^*(Q, E)$ captures the dynamic priority of the data flow for different $(Q, E)$. However, obtaining the priority function $V^*(Q, E)$ is highly non-trivial as it involves solving a large system of nonlinear fixed point equations. Brute-force approaches (such as value iteration and policy iteration) have huge complexity.

Challenge 1: Huge complexity in obtaining the priority function $V^*(Q, E)$. 

IV. VIRTUAL CONTINUOUS TIME SYSTEM AND APPROXIMATE PRIORITY FUNCTION

In this section, we adopt a continuous time approach so that we can exploit calculus techniques and theories of differential equations to obtain a closed-form approximate priority function. Specifically, we first reverse-engineer a virtual continuous time system (VCTS) and an associated total cost problem in the VCTS. We show that the optimality conditions of the VCTS is equivalent to that of the original MDP (up to $o(\tau)$ order optimal). Based on that, we exploit calculus techniques and theories of differential equations to obtain a closed-form characterization of the priority function $V^*(Q, E)$.

A. Virtual Continuous Time System

We first define the VCTS, which is a fictitious system with a continuous virtual queue state $(q(t), e(t))$, where $q(t) \in [0, \infty)$ and $e(t) \in [0, N_E]$ are the virtual data queue state and virtual energy queue state at time $t \in [0, \infty)$.

Let $\Omega^v$ be the virtual power control policy of the VCTS. Similarly, $\Omega^v$ is $\mathcal{F}^v_t$-adapted, where $\mathcal{F}^v_t = \sigma\{\{h(s), q(s), e(s) : 0 < s < t\}\}$ and $\{\mathcal{F}^v_t\}$ is the filtration of the VCTS. Furthermore, the virtual power control policy $\Omega^v$ satisfies the virtual energy availability constraint, i.e., $p(t)\tau \leq e(t) \forall t$. Given an initial virtual system state $(q_0, e_0)$ and a virtual policy $\Omega^v$, the trajectory of the virtual queuing system is described by the following coupled differential equations with reflections:

$$dq(t) = -E[R(h(t), p(t))|q(t), e(t)] + \lambda \tau dt + dL(t)$$
$$de(t) = -E[p(t)|q(t), e(t)] + \pi \tau dt - dU(t)$$

where $L(t)$ and $U(t)$ are the reflection processes associated with the lower data queue boundary $q(t) = 0$ and upper energy queue boundary $e(t) = N_E$, which are uniquely determined by the following equations ( Chap. 2.4 of [30]):

$$L(t) = \max\left\{0, -\min_{\tau \leq t} q_0 + \int_0^t (-E[R(h(s), p(s))|q(s), e(s)] + \lambda \tau)ds\right\}$$
$$U(t) = \max\left\{0, \max_{\tau \leq t} e_0 + \int_0^t (-E[p(s)|q(s), e(s)] + \pi \tau)ds\right\} - N_E$$

with $L(0) = U(0) = 0$.

Note that the process $L(t)$ ensures that the virtual data queue length $q(t)$ will not go below zero. The process $U(t)$ together with the virtual energy availability constraint ensures that the virtual energy queue length lies in the domain $[0, N_E]$.

Fig. 2 illustrate an example of the trajectories of $(q(t), L(t))$ and $(e(t), U(t))$ for a virtual policy $\Omega^v$.

Furthermore, we have the following definition on the admissible virtual control policy for the VCTS.

Definition 3 (Admissible Virtual Control Policy for VCTS): A virtual policy $\Omega^v$ for the VCTS is admissible if the following requirements are satisfied:

- For any initial virtual queue state $(q_0, e_0)$, the virtual queue trajectory $(q(t), e(t))$ in (13) and (14) under $\Omega^v$ is unique.
- For any initial virtual queue state $(q_0, e_0)$, the total cost $\int_0^\infty q(t) dt$ under $\Omega^v$ is bounded.

B. Total Cost Problem under the VCTS

Given an admissible virtual control policy $\Omega^v$, we define the total cost of the VCTS starting from a given initial virtual queue state $(q_0, e_0)$ as

$$V(q_0, e_0; \Omega^v) = \int_0^\infty q(t) dt$$

We consider the following infinite horizon total cost problem for the VCTS:
Problem 2 (Infinite Horizon Total Cost Problem for VCTS): For any initial virtual queue state \((q_0, e_0)\), the infinite horizon total cost problem for the VCTS is formulated as

\[
\min_{\Omega^v} V(q_0, e_0; \Omega^v)
\]  

(18)

where \(V(q_0, e_0; \Omega^v)\) is given in (17).

Note that the two technical conditions in Definition 3 on the admissible virtual policy are for the existence of an optimal policy for the total cost problem in Problem 2. The above total cost problem has been well-studied in the continuous time optimal control theory (Chap. 2.6 of [11]). The solution can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) equation as below.

Theorem 2 (Sufficient Conditions for Optimality under VCTS): Assume there exists a function \(V(q, e)\) that is of class \(C^1(\mathbb{R}_+^2)\), and \(V(q, e)\) satisfies the following HJB equation:

\[
\begin{align*}
\min_{p \leq e/\tau} & \quad \mathbb{E} \left[ \frac{q}{\mathcal{N}_r} + \frac{\partial V(q, e)}{\partial q} \left(-R(h, p) + \lambda \right) + \frac{\partial V(q, e)}{\partial e} \left(-p + \alpha \right) \right] | q, e | = 0 \quad \forall q, e
\end{align*}
\]  

(19)

Furthermore, for all admissible virtual control policy \(\Omega^v\) and initial virtual queue state \((q_0, e_0)\), the following conditions are satisfied:

\[
\begin{align*}
\limsup_{T \to \infty} & \quad \int_0^T \mathbb{E} \left[ \frac{\partial V(q(T), e(T))}{\partial q} \right] N_E U(t) dt = 0 \\
\lim_{T \to \infty} & \quad \int_0^T \mathbb{E} \left[ \frac{\partial V(q(t), e(t))}{\partial e} \right] U(t) dt = 0 \\
\limsup_{T \to \infty} & \quad \sup_{q, e \in \mathcal{E}} \mathbb{E} \left[ \frac{\partial V(q(T), e(T))}{\partial q} \right] = 0
\end{align*}
\]  

(20)

Then, we have the following results:

- \(V(q, e) = \min_{\Omega^v} V(q_0, e_0; \Omega^v)\) is the optimal total cost when \((q_0, e_0) = (q, e)\) and \(V(q, e)\) is called the virtual priority function.

- There exists an admissible virtual stationary control policy \(\Omega^{v*}(h, q, e)\) that satisfies (19) and (20) in Theorem 2 and is given by \(\frac{\partial V}{\partial e}(0, e(t)) = 0\). Then, the optimal control policy of Problem 2 is given by \(\Omega^{v*}\).

Proof: Please refer to Appendix B.

In the following theorem, we establish the relationship between the virtual priority function \(V(Q, E)\) in Theorem 2 and the optimal priority function \(V^*(Q, E)\) in Theorem 1.

Theorem 3 (Relation between \(V(Q, E)\) and \(V^*(Q, E)\)): If \(V(Q, E) = \mathcal{O}(Q^2)\) and \(V^{v*}\) is admissible in the discrete time system, then \(V^*(Q, E) = V(Q, E) + \alpha(\tau)\).

Proof: please refer to Appendix C.

Theorem 3 means that \(V(Q, E)\) can serve as an approximate priority function to the optimal priority function \(V^*(Q, E)\) with approximation error \(\alpha(\tau)\). As a result, solving the optimality equation in (11) is transformed into a calculus problem of solving the HJB equation in (19). In the next subsection, we shall focus on solving the HJB equation in (19) by leveraging the well-established theories of calculus and differential equations.

\(f(x)\) (\(x\) is a \(K\)-dim vector) is of class \(C^1(\mathbb{R}_+^K)\), if the first order partial derivatives w.r.t. each element of \(x \in \mathbb{R}_+^K\) are continuous.

V. CLOSED-FORM DELAY-OPTIMAL POWER CONTROL

The HJB equation in Theorem 2 is a coupled two-dimensional partial differential equation (PDE) and hence, one key obstacle is to obtain the closed-form solution to the PDE.

Challenge 2: Solution of the coupled two-dimensional PDE in Theorem 2

In this section, using asymptotic analysis, we obtain closed-form solutions to the multi-dimensional PDE in different operating regimes. We also discuss the control insights from the structural properties of the closed-form priority functions for different asymptotic regimes.

A. General Solution

We first have the following corollary on the optimal power control based on the HJB equation in Theorem 2 for given \(V(q, e)\):

Corollary 1 (Optimal Power Control based on Theorem 2): For given priority function \(V(q, e)\), the optimal power control action from the HJB equation in Theorem 2 is given by

\[
p^* = \min \left\{ \left( -\frac{\partial V(q, e)}{\partial q} + \frac{\partial V(q, e)}{\partial e} \right) \frac{1}{|h|^2}, e_T \right\}
\]  

(21)

Remark 3 (Structure of The Optimal Power Control Policy): The optimal power control policy in (21) depends on the instantaneous CSI, DQSI and EQSI. Furthermore, the power control action has a multi-level water-filling structure as illustrated in Fig. 4–Fig. 5, where the water level is adaptive control action has a DQSI and the EQSI affect the overall priority of the data flow.

We then establish the following theorem on the sufficient conditions to ensure the existence of solution to the PDE in Theorem 2.

Theorem 4 (Sufficient Conditions for the Existence of Solution): There exists a \(V(q, e)\) that satisfies (19) and (20) in Theorem 2 if the following conditions are satisfied:

\[
X < \exp \left( \frac{1}{x} \right) E_1 \left( \frac{1}{x} \right)
\]  

(22)

\[
N_E \geq e^*
\]  

(23)

where \(x\) satisfies \(x \exp \left( -\frac{1}{x} \right) - E_1 \left( \frac{1}{x} \right) = \alpha\) and \(E_1(x) \triangleq \int_1^\infty \frac{e^{-t}}{t} dt\) is the exponential integral function. \(e^*\) is the solution of the fixed point equation in (46) in Appendix C if \(X > E_1 \left( \frac{1}{x} \right)\), and \(e^* = \tau\alpha\) if \(X < E_1 \left( \frac{1}{x} \right)\).

Proof: Please refer to Appendix D.

The challenge is to find a priority function \(V(q, e)\) that satisfies (19) and (20). Note that the PDE in (19) is a two-dimensional PDE, which has no closed-form solution for the priority function \(V(q, e)\). In the next subsection V-B we consider different asymptotic regimes and obtain closed-form solutions of \(V(q, e)\) for these operating regimes.
B. Asymptotic Closed-Form Priority Functions and Control Insights

In this subsection, we obtain the closed-form priority functions $V(q,e)$ in different asymptotic regimes as illustrated in Fig. 3 and discuss the control insights for each regime.

1) Large-Data-Arrival-Energy-Sufficient Regime: In this regime, we consider the operating region with large $\lambda$ and large $\bar{\sigma}$, and $E_1\left(\frac{\alpha}{\bar{\lambda}}\right) < \bar{X} < \exp\left(\frac{\alpha}{\bar{\lambda}}\right) E_1\left(\frac{\alpha}{\bar{\lambda}}\right)$ (where $x$ satisfies $x \exp\left(-\frac{\alpha}{\bar{\lambda}}\right) - E_1\left(\frac{\alpha}{\bar{\lambda}}\right) = \bar{\sigma}$). This regime corresponds to the scenario that we have a large data arrival rate for the data queue and sufficient renewable energy supply for the energy queue to maintain the data queue stable. The closed-form priority function $V(q,e)$ for this regime is given by the following theorem:

**Theorem 5:** (Closed-Form $V(q,e)$ for the Large-Data-Arrival-Energy-Sufficient Regime) Under the large-data-arrival-energy-sufficient regime, the closed-form $V(q,e)$ of the PDE in Theorem 2 is given by

$$V(q,e) = \frac{e^\gamma}{4\alpha^2X^2\tau^2}\left(1 + 2\gamma_{eu} + 2\frac{\gamma}{\tau} - 2\log\left(\frac{e}{\tau}\right)\right) + \frac{eq}{\lambda\gamma\tau}$$

where $\gamma_{eu}$ is the Euler’s constant and $C_1 = \frac{\tau}{2\alpha}\left(1 + 2\gamma_{eu} + 2\frac{\gamma}{\tau} - 2\log\bar{\sigma}\right)$.

- When $0 < e < e^{th}$ ($e^{th}$ is the solution of $E_1\left(\frac{\alpha}{\bar{\lambda}}\right) = \bar{X}$), we have $V(q,e)$ as a function of $q$ only.

**Proof:** Please refer to Appendix E.

Based on Theorem 5 when $e \geq e^{th}$, since $V(q,e)$ is a function of $q$ only, we have $\frac{\partial V(q,e)}{\partial q} = 0$ for given $q$ and $e$. Therefore, the water level in Fig. 3 is infinite and hence, we have $p^* = \frac{\bar{X}}{\bar{\sigma}}$. Furthermore, based on the closed-form priority function in (24), we can calculate the closed-form expression of the water level $\frac{\partial V(q,e)}{\partial q}/\frac{\partial V(q,e)}{\partial e}$ in (21). We summarize the optimal power control structure for this regime in the following corollary:

**Corollary 2:** (Optimal Power Control Structure for the Large-Data-Arrival-Energy-Sufficient Regime) The optimal power control for the large-data-arrival-energy-sufficient regime is given by

- When $0 < e < e^{th}$ and $q > \frac{\bar{\sigma}}{\tau}(\gamma_{eu} + \frac{\bar{X}}{\bar{\sigma}} - \log\left(\frac{e}{\tau}\right))$, $p^* = 0$.
- When $0 < e < \bar{\sigma}q$ and $q < \bar{\sigma}\left(\gamma_{eu} + \frac{\bar{X}}{\bar{\sigma}} - \log\left(\frac{e}{\tau}\right)\right)$, the water level $-\frac{\partial V(q,e)}{\partial q}/\frac{\partial V(q,e)}{\partial e}$ is an increasing function of $q$ for a given $e$, and is a decreasing function of $e$ for a given $q$.
- When $\bar{\sigma}q < e < e^{th}$ and $q < \bar{\sigma}\left(\gamma_{eu} + \frac{\bar{X}}{\bar{\sigma}} - \log\left(\frac{e}{\tau}\right)\right)$, the water level $-\frac{\partial V(q,e)}{\partial q}/\frac{\partial V(q,e)}{\partial e}$ is an increasing function of $q$ for a given $e$, and is an increasing function of $e$ for a given $q$.
- When $e \geq e^{th}$, $p^* = \frac{\bar{X}}{\bar{\sigma}}$.

**Proof:** Please refer to Appendix F.

Fig. 4 illustrates the water level versus the data queue length and the energy queue length for the large-data-arrival-energy-sufficient regime, where $\tau = 0.1$ s, $\bar{X} = 1.8$ pkcs/s, $\bar{\sigma} = 10$ W, bandwidth is 1 MHz, and average packet length is 1 Mbits.
we have sufficient renewable energy, and it is appropriate to use all the available energy and make room for the future energy arrivals.

2) Small-Data-Arrival-Energy-Limited Regime: In this regime, we consider the operating region with small \( \bar{\lambda} \) and small \( \bar{\tau} \), and \( E_1(\frac{\lambda}{E}) < \bar{\lambda} < \exp\left(\frac{1}{\bar{\lambda}}\right) E_1\left(\frac{\lambda}{E}\right) \). This regime corresponds to the scenario that we have a small data arrival rate for the data queue and insufficient energy supply for the energy queue. The closed-form priority function \( V(q,e) \) for this regime is given by the following theorem:

Theorem 6: (Closed-Form \( V(q,e) \) for the Small-Data-Arrival-Energy-Limited Regime) Under the small-data-arrival-energy-limited regime, the closed-form \( V(q,e) \) of the PDE in Theorem 2 is given by

- When \( 0 < e < e^{th} \) (\( e^{th} \) is the solution of \( E_1\left(\frac{\bar{\lambda}}{E}\right) = \bar{\lambda} \)), we have
  \[
  V(q,e) = -\frac{e^3}{\lambda E r^2} + \frac{e^2}{2 \lambda E r^2} - \frac{q}{\lambda E} + C_2
  \]

- When \( e \geq e^{th} \), \( V(q,e) \) is a function of \( q \) only and \( C_2 = \frac{\bar{\lambda}}{E} - \frac{\lambda}{E} r \).

Proof: Please refer to Appendix G.

Based on the closed-form \( V(q,e) \) in Theorem 6, we have the following corollary summarizing the optimal power control structure for this regime:

Corollary 3: (Optimal Power Control Structure for the Small-Data-Arrival-Energy-Limited Regime) The optimal power control for the small-data-arrival-energy-limited regime is given by

- When \( 0 < e < e^{th} \) and \( q > -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), \( p^* = 0 \).
- When \( 0 < e < \sqrt{\frac{\bar{\tau} q}{\lambda}} \) and \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), the water level \( -\frac{\partial V(q,e)}{\partial q} / \partial V(q,e) \) is an increasing function of \( q \) for a given \( e \), and is a decreasing function of \( e \) for a given \( q \).
- When \( \sqrt{\frac{\bar{\tau} q}{\lambda}} < e < e^{th} \) and \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), the water level \( -\frac{\partial V(q,e)}{\partial q} / \partial V(q,e) \) is an increasing function of \( q \) for a given \( e \), and is an increasing function of \( e \) for a given \( q \).
- When \( e \geq e^{th} \), we have \( p^* = \frac{e^{th}}{\tau} \).

Proof: Please refer to Appendix H.

Fig. 5 illustrates the water level versus the data queue length and the energy queue length when \( e < e^{th} \). Specifically, Corollary 3 means that when \( 0 < e < e^{th} \) and for a large data queue length \( q > -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), we do not use any renewable energy to transmit data. The reason is that even though we can use the limited energy for data transmission, the data queue length will not decrease significantly, which contributes very little to the delay performance. Instead, if we do not use the energy at the current slot, we can save it and wait for the future good transmissions opportunities. On the other hand, for a small queue length \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), we can use the available energy for transmission and the water level is increasing w.r.t. \( q \), which is in accordance with the high urgency of the data

3) Small-Data-Arrival-Energy-Sufficient Regime: In this regime, we consider the operating region with \( \bar{\lambda} \leq E_1\left(\frac{\lambda}{E}\right) \). This regime corresponds to the scenario that we have a small data arrival rate for the data queue and sufficient renewable energy supply in the energy queue to maintain the data queue stable. The closed-form priority function \( V(q,e) \) for this regime is given by the following theorem:

Theorem 7: (Closed-Form \( V(q,e) \) for the Small-Data-Arrival-Energy-Sufficient Regime) Under the small-data-arrival-energy-sufficient regime, the closed-form \( V(q,e) \) of the PDE in Theorem 2 is given by

- When \( 0 < e < e^{th} \) (\( e^{th} \) is the solution of \( E_1\left(\frac{\bar{\lambda}}{E}\right) = \bar{\lambda} \)), we have
  \[
  V(q,e) = \frac{1}{2 \lambda E r^2} e^2 - \frac{q}{\lambda E} - \frac{1}{2 \lambda^2 E r^2} \left( q - \frac{\bar{\lambda}}{E} e \right)^2
  \]

- When \( e \geq e^{th} \), \( V(q,e) \) is a function of \( q \) only.

Proof: Please refer to Appendix I.

Based on the closed-form \( V(q,e) \) in Theorem 7, we have the following corollary summarizing the optimal power control structure in this regime:

Corollary 4: (Optimal Power Control Structure for the Small-Data-Arrival-Energy-Sufficient Regime) The optimal power control for the small-data-arrival-energy-sufficient regime is given by

- When \( 0 < e < e^{th} \) and \( q > -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), \( p^* = 0 \).
- When \( 0 < e < \sqrt{\frac{\bar{\tau} q}{\lambda}} \) and \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), the water level \( -\frac{\partial V(q,e)}{\partial q} / \partial V(q,e) \) is an increasing function of \( q \) for a given \( e \), and is a decreasing function of \( e \) for a given \( q \).
- When \( \sqrt{\frac{\bar{\tau} q}{\lambda}} < e < e^{th} \) and \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \), the water level \( -\frac{\partial V(q,e)}{\partial q} / \partial V(q,e) \) is an increasing function of \( q \) for a given \( e \), and is an increasing function of \( e \) for a given \( q \).
- When \( e \geq e^{th} \), we have \( p^* = \frac{e^{th}}{\tau} \).

9In order for \( q < -\frac{e^{2} + 3\bar{\lambda}e}{\lambda E} \) to hold, we require \( \sqrt{\bar{\tau} q} \leq \frac{\lambda}{E} \). For small \( \bar{\lambda} \), \( \frac{\bar{\lambda}}{E} \approx \exp\left(-\frac{1}{\bar{\lambda}}\right) \approx \frac{1}{\bar{\lambda}} \Rightarrow e^{th} \approx \bar{\lambda} \). Therefore, we have \( \sqrt{\bar{\tau} q} \in [0,e^{th}] \).

10Based on (26), we have \( \partial V(q,e)/\partial q = 0 \) for all \( q,e \), which induces an infinite water level in (21). Hence, we have \( p^* = \frac{e^{th}}{\lambda} \) when \( 0 < e < e^{th} \).
Corollary 4: (Optimal Power Control Structure for the Small-Data-Arrival-Energy-Sufficient Regime) The optimal power control for the small-data-arrival-energy-sufficient regime is given by

\[ p^* = \frac{e}{\tau} \tag{27} \]

Corollary 4 means that the optimal control policy for the small-data-arrival-energy-sufficient regime is to use all the available energy in the energy buffer. This is reasonable because in this regime we have \( \lambda \leq E_1 (\frac{1}{\alpha}) \), which means that there is plenty of renewable energy and it is sufficient to use all the available energy to support the data traffic.

Based on the closed-form solutions for the asymptotic operating regions in Theorem 5-7, we propose the following solution for the PDE in Theorem 2 that covers all regimes w.r.t. \((\overline{\lambda}, \overline{\tau})\):

\[
V(q, e) \approx \begin{cases} 
\text{sol. in Thm 5,} & \overline{\tau} \geq \overline{\tau}^h, E_1 \left( \frac{1}{\overline{\tau}} \right) < \overline{\lambda} < \exp \left( \frac{1}{x} \right) E_1 \left( \frac{1}{x} \right) \\
\text{sol. in Thm 6,} & \overline{\tau} < \overline{\tau}^h, E_1 \left( \frac{1}{\overline{\tau}} \right) < \overline{\lambda} < \exp \left( \frac{1}{x} \right) E_1 \left( \frac{1}{x} \right) \\
\text{sol. in Thm 7,} & \overline{\lambda} \leq E_1 \left( \frac{1}{\overline{\tau}} \right) 
\end{cases} \tag{28}
\]

where \( \overline{\tau}^h > 0 \) is a solution parameter.

C. Stability Conditions of using the Closed-Form Solution in the Discrete-Time System

In the previous subsection, we obtain the closed-form optimal power control solutions for different asymptotic regimes as in Theorem 5-7. We then establish the following theorem on the stability conditions when using the control policy in Corollary 4 in the original discrete time system in 3 and 4:

Theorem 8: (Stability Conditions of using the Closed-Form Solutions in the Discrete-Time System): Using (28) and the closed-form control policy in Corollary 4 if the following conditions are satisfied:

\[
\overline{\lambda} < E \left[ \exp \left( \frac{1}{\alpha} \right) E_1 \left( \frac{1}{\alpha} \right) \right] \tag{29}
\]

\[
N_E \geq Ne^* \tag{30}
\]

where \( e^* \) is defined in (23), then the data queue in the original discrete time system in 3 is stable, in the sense that \( \lim_{n \to \infty} E[Q^2(n)] < \infty \).

Proof: Please refer to Appendix J.

Theorem 8 means that using (28), the closed-form control policy in Corollary 4 is admissible according to Definition 2.

Remark 4 (Interpretation of the Conditions in Theorem 4):

- **Interpretation of the Condition on \( \overline{\lambda} \) and \( \overline{\tau} \) in (29):**
  The condition in (29) implies[1] that \( \overline{\tau} \) grows at least at the order of \( \exp(\lambda) \). It indicates that for given \( \lambda \), if \( \overline{\tau} \) is too small, even if we use all the available energy in the energy buffer at each time slot, the average data arrival rate will be larger than the average data departure rate for the data queue buffer. Therefore, the data queue cannot be stabilized.

- **Interpretation of the Condition on \( N_E \) in (30):**
  The condition in (30) gives a first order design guideline on the dimensioning of the energy storage capacity required at the transmitter. For example, \( N_E \) should be at least at a similar order of \( N\overline{\tau}\overline{\lambda} \). This condition on \( N_E \) ensures that the energy storage at the transmitter has sufficient energy to support data transmission for \( N \) slots when \( \alpha(t) \) is small.

VI. SIMULATIONS

In this section, we compare the performance of the proposed closed-form delay-optimal power control scheme in (21) with the following three baselines using numerical simulations:

- **Baseline 1, Greedy Strategy (GS) [10]:** At each time slot, the transmitter sends data to the receiver using the power \( p(t) = \min \left\{ \overline{\tau} - \epsilon, \frac{E(t)}{\tau} \right\} \) for a given small positive constant \( \epsilon \). The GS is a throughput-optimal policy in the stability sense, i.e., it ensures the stability of the queuing network.

- **Baseline 2, CSI-Only Water-Filling Strategy (COWFS) [10]:** At each time slot, the transmitter sends data to the receiver using the power \( p(t) = \left\{ \frac{Q(t)}{\gamma} - \frac{1}{|h(t)|^2}, \frac{E(t)}{\tau} \right\} \). Specifically, the water-filling solution in the COWFS is obtained by maximizing the ergodic capacity \( E[\log(1 + p|h|^2)] \) with the average power constraint \( E[p] = \overline{\tau} - \epsilon \) for a given small positive constant \( \epsilon \).

- **Baseline 3, Queue-Weighted Water-Filling Strategy (QWWFS) [11]:** At each time slot, the transmitter sends data to the receiver using the power \( p(t) = \left\{ \frac{Q(t)}{\gamma} - \frac{1}{|h(t)|^2}, \frac{E(t)}{\tau} \right\} \). The QWWFS is also a throughput-optimal policy. \( \gamma \) is the Lagrangian multiplier associated with the average power constraint \( E[p] = \overline{\tau} - \epsilon \) for a given small positive constant \( \epsilon \).

In the simulation, we consider a point-to-point energy harvesting system, where a base station (BS) communicates with a mobile station. The BS is equipped with a 40cm x 50cm solar
The average data arrival rate is the average packet arrival rate $\lambda$ (pck/s) and an exponentially distributed random packet size with mean $0.1$ Mbits. The decision slot duration $\tau$ is $50$ ms, and the total bandwidth is $1$ MHz. Furthermore, we consider Poisson energy arrival \cite{10} with average energy arrival rate $\overline{\alpha} = 1$~$\sim$~$10$ W. We assume that the block length of the energy arrival process is $N = 6000$, i.e., the energy arrival rate $\alpha(t)$ at the BS changes every $5$ min and the renewable energy is stored in a $1.2V$ $2000$ mAh lithium-ion battery. We compare the delay performance of the proposed scheme with the above three baselines.

A. Choice of the Solution Parameter $\overline{\alpha}^{th}$ in \cite{28}

Fig. 6 illustrates the performance loss ratio\textsuperscript{16} versus the average energy arrival rate with the average data arrival rate $\bar{\lambda} = \frac{1}{2} \left[ E_1 \left( \frac{1}{\overline{\alpha}} \right) + \exp \left( \frac{1}{\overline{\alpha}} \right) E_1 \left( \frac{1}{\overline{\alpha}} \right) \right]$. It can be observed that using the solution in Theorem\textsuperscript{5} the performance loss is small for large $\overline{\alpha}$ and it increases as $\overline{\alpha}$ decreases. In addition, using the solution in Theorem\textsuperscript{6} the performance loss is small for small $\overline{\alpha}$ and it increases as $\overline{\alpha}$ increases. It can be observed that choosing $\overline{\alpha}^{th} \approx 3.6$ can keep the performance loss down to $6\%$ over the entire operating regime w.r.t. $(\bar{\lambda}, \overline{\alpha})$.

B. Delay Performance for the Large-Data-Arrival-Energy-Sufficient Regime

Fig. 7 illustrates the average delay versus the average data arrival rate for the large-data-arrival-energy-sufficient regime. The average energy arrival rate is $\overline{\alpha} = 10$ W.

C. Delay Performance for the Small-Data-Arrival-Energy-Limited Regime

Fig. 8 illustrates the average delay versus the average data arrival rate for the small-data-arrival-energy-limited regime. The average data arrival rate is $\overline{\lambda} = 0.34 \sim 0.38$ pck/s and the average energy arrival rate is $\overline{\alpha} = 1$ W.

D. Delay Performance for the Small-Data-Arrival-Energy-Sufficient Regime

Fig. 9 illustrates the average delay versus the average data arrival rate for the small-data-arrival-energy-sufficient regime. The average data arrival rate is $\overline{\lambda} = 0.34 \sim 0.38$ pck/s and the average energy arrival rate is $\overline{\alpha} = 6$ W. The delay performance of the proposed scheme is very close to that of Baseline 3 and also better than those of Baselines 1 and 2. However, our proposed scheme has lower complexity compared with Baseline 3, which involves the gradient update to obtain the Lagrangian multiplier. Therefore, it is better to adopt our proposed scheme for the small-data-arrival-energy-sufficient regime.
Fig. 9: Average delay versus average data arrival rate for the small-data-arrival-energy-limited regime. The average energy arrival is $\tau = 6 \text{ W}$.

regime. Furthermore, the performance of the proposed closed-form solution is very close to that of the VIA.

E. Comparison of Complexity in Computational Time

Table I illustrates the comparison of the MATLAB computational time of the proposed solution, the baselines and the brute-force VIA [14]. Note that the proposed scheme has similar complexity to Baseline 1 due to the closed-form priority function. Therefore, our proposed scheme achieves significant performance gain with negligible computational cost.

VII. SUMMARY

In this paper, we propose a closed-form delay-optimal power control solution for an energy harvesting wireless network with finite energy storage. We formulate the associated stochastic optimization problem as an infinite horizon average cost MDP. Using a continuous time approach, we derive closed-form approximate priority functions for different asymptotic regimes. Based on the closed-form approximations, we propose a closed-form optimal control policy, which has a multi-level water filling structure and the water level is adaptive to the DQSI and the EQSI. Numerical results show that the proposed power control scheme has much better performance than the baselines.

APPENDIX A: PROOF OF THEOREM [1]

Following Proposition 4.6.1 of [14], the sufficient conditions for the optimality of Problem [1] is that there exists a $(\theta^*, \{V^*(\chi)\})$ that satisfies the following Bellman equation and $V^*$ satisfies the transversality condition in [12] for all admissible control policy $\Omega$ and initial state $\chi(0)$:

$$\theta^* + V^*(\chi) = \min_{p \in E/\tau} \left[ \frac{Q}{\lambda} + \sum_{\chi'} \Pr[\chi'|\chi, p] V^*(\chi') \right]$$ (31)

Then, $\theta^* = \min_{\Omega} \mathcal{D}(\Omega)$ is the optimal average cost for any initial state $\chi(0)$. Furthermore, suppose there exists a stationary admissible $\Omega^*$ with $V^*(\chi) = p^*$ for any $\chi$, where $p^*$ attains the minimum of the R.H.S. in (31) for given $\chi$. Then, the optimal control policy of Problem 1 is given by $\Omega^*$.

Taking expectation w.r.t. $t$ on both sizes of (31) and denoting $V^*(Q, E) = E[V^*(\chi)|Q, E]$, we obtain the equivalent Bellman equation in (31) in Theorem I.

APPENDIX B: PROOF OF THEOREM 2

Suppose $V(q, e)$ is of class $C^1(\mathbb{R}_+^2)$, we have $dV(q, e) = \frac{\partial V(q, e)}{\partial q} dq + \frac{\partial V(q, e)}{\partial e} de$. Substituting the dynamics in (13) and (14), we obtain

$$dV(q(t), e(t)) = D(q^*, e^*) \frac{\partial V(q, e)}{\partial q} dq + \frac{\partial V(q, e)}{\partial e} de$$ (32)

where $D(q^*, e^*) = \frac{\partial V(q, e)}{\partial q} (-E[R(h, p)|q, e] + \bar{\lambda}) \tau + \frac{\partial V(q, e)}{\partial e} (-E[p|q, e] + \bar{\tau}) \tau$. Integrating both sides w.r.t. $t$ from 0 to $T$, we have

$$V(q(T), e(T)) - V(q_0, e_0) = \int_0^T D(q(t), e(t)) dt + \int_0^T \frac{\partial V(q(t), e(t))}{\partial q} dq + \int_0^T \frac{\partial V(q(t), e(t))}{\partial e} de$$ (33)

where (a) is because $L(t)$ and $U(t)$ increase only when $q = 0$ and $e = N_E$ according to Chapter 2.4 of [30]. If $V(q, e)$ satisfies (19), from (34), we have for any admissible virtual policy $\Omega^*$,

$$V(q_0, e_0) \leq V(q(T), e(T)) - \int_0^T \frac{\partial V(0, e(t))}{\partial q} dq + \int_0^T \frac{\partial V(q(t), N_E)}{\partial e} de U(t) - \int_0^T \frac{q(t)}{\bar{\lambda}} dt$$ (35)

From the boundary conditions in (20), we have

$$\lim_{T \to \infty} \frac{\partial V(q(0), e)}{\partial q} dq + \int_0^T \frac{\partial V(q(t), N_E)}{\partial e} de U(t) = 0$$ and $\lim_{T \to \infty} V(q(T), e(T)) = 0$.
where the above equality is achieved if the admissible virtual stationary policy \( \Omega^*(q,e,h) \) attains the minimum in the HJB equation in (19) for all \((q,e,h)\). Hence, such \( \Omega^* \) is the optimal control policy of the total cost problem in VCTS in Problem 2.

APPENDIX C: PROOF OF THEOREM 3

A. Relationship between the Discrete Time and VCTS Optimality Equations

We first prove the following corollary based on Theorem 1.

Corollary 5 (Approximate Optimality Equation): Suppose there exist \( J(Q,E) \) of class \( C^1(\mathbb{R}^+_{Q,E}) \) that solve the following approximate optimality equation:

\[
\min_{p \leq E/\tau} \mathbb{E}\left[ \frac{Q}{\lambda\tau} + \frac{\partial J(Q,E)}{\partial Q} (-R(h,p) + \lambda) \right] + \frac{\partial J(Q,E)}{\partial E} (-p + \pi) |Q,E| = 0, \quad \forall Q,E
\] (37)

Furthermore, for all admissible control policy \( \Omega \) and initial queue state \( Q(0), E(0) \), the transversality condition in (12) is satisfied for \( J \). Then, we have \( V^*(Q,E) = J(Q,E) + o(\tau) \).

Proof of Corollary 5: We will establish the following Lemmas 4 and 5 to prove Corollary 5. For convenience, denote

\[
T_X(\theta, J, p) = \frac{Q}{\lambda} + \sum_{Q',E'} \Pr \left[ Q', E' \big| X, P \right] J(Q',E') - J(Q,E) - \theta
\] (38)

\[
T_X^1(\theta, J, p) = \frac{Q}{\lambda} + \frac{\partial J(Q,E)}{\partial Q} (-R(h,p) + \lambda) \tau + \frac{\partial J(Q,E)}{\partial E} (-p + \pi) \tau - \theta
\] (39)

Step 1, Relationship between \( T_X(\theta, J, p) \) and \( T_X^1(\theta, J, p) \):

Lemma 1: For any \( \chi \), \( T_X(\theta, J, p) = T_X^1(\theta, J, p) + \nu G_X(J,p) \) for some smooth function \( G_X \) and \( \nu = o(\tau) \).

Proof of Lemma 1: Let \( Q(n+1), E(n+1) = (Q',E') \) and \( Q(n), E(n) = (Q,E) \). For sufficiently small \( \tau \), according to the dynamics in (3) and (4), we have the following Taylor expansion on \( J(Q',E') \) in (11):

\[
\mathbb{E} \left[ J(Q',E') \big| Q,E \right] = J(Q,E) + \mathbb{E} \left[ \frac{\partial J(Q,E)}{\partial Q} (-R(h,p) + \lambda) \right] + \frac{\partial J(Q,E)}{\partial E} (-p + \pi) \tau + o(\tau)
\] (40)

Substituting (40) into \( T_X(\theta, J, p) \), we obtain \( T_X(\theta, J, p) = T_X^1(\theta, J, p) + \nu G_X(J,p) \) for some smooth function \( G_X \) and \( \nu = o(\tau) \).

Step 2, Growth Rate of \( \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] \):

Denote

\[
T_X(\theta, J, p) = \min_p T_X^1(\theta, J, p), \quad T_X^1(\theta, J, p) = \min_p T_X^1(\theta, J, p)
\] (41)

Suppose \( (\theta^*, V^*) \) satisfies the Bellman equation in (11) and (0, J) satisfies (37), we have for any \( \chi \).

\[
\mathbb{E} \left[ T_X(\theta^*, V^*) \big| Q,E \right] = 0, \quad \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = 0
\] (42)

Then, we establish the following lemma.

Lemma 2: \( \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = o(\tau), \forall Q,E \).

Proof of Lemma 2: For any \( \chi \), we have \( T_X(0, J) = \min_p T_X^1(0, J, p) \) and \( \nu G_X(J,p) \). On the other hand, \( T_X(0, J) \) is the optimal solution and the HJB Equation is satisfied for any \( \chi \).

\[
\mathbb{E} \left[ T_X^1(0, J, p) \big| Q,E \right] = \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = 0.
\] (43)

Step 3, Difference between \( V^*(Q,E) \) and \( J(Q,E) \):

Lemma 3: Suppose \( \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = 0 \) for all \( Q,E \) together with the transversality condition in (12) has a unique solution \( (\theta^*, V^*) \). If \( J \) satisfies (37) and the transversality condition in (12), then \( V^*(Q,E) - J(Q,E) = o(\tau) \).

Proof of Lemma 3: Suppose for some \( (Q',E') \), we have \( J(Q',E') = V^*(Q',E') + \alpha \) for some \( \alpha \neq 0 \) as \( \tau \to 0 \). From Lemma 2, we have \( \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = 0 \) for all \( Q,E \) and also \( J \) satisfies the transversality condition in (12). However, \( J(Q',E') \neq V^*(Q',E') \) because of the assumption that \( J(Q',E') = V^*(Q',E') + \alpha \). This contradicts the condition that \( (\theta^*, V^*) \) is a unique solution of \( \mathbb{E} \left[ T_X(0, J) \big| Q,E \right] = 0 \) for all \( Q,E \) and the transversality condition in (12). Hence, we must have \( V^*(Q,E) - J(Q,E) = o(\tau) \) for all \( Q,E \).

B. Relationship between the Discrete Time Optimality Equation and the HJB Equation

First, if \( V(Q,E) \) that is of class \( C^1(\mathbb{R}^+_{Q,E}) \) satisfies the optimality conditions of the total cost problem in VCTS (as shown in Theorem 2), then it also satisfies (37) in Corollary 5. Second, since \( V(Q,E) = O(Q^2) \), we have \( \lim_{n \to \infty} \mathbb{E} \left[ V(Q(n), E(n)) \right] ] < \infty \) for any admissible policy \( \Omega \) of the discrete time system according to Definition 2. Hence, \( V(Q,E) \) satisfies the transversality condition in (12). Using Corollary 5, we have \( V^*(Q,E) = V(Q,E) + o(\tau) \).

APPENDIX D: PROOF OF THEOREM 4

First, we simplify the PDE in (19). The optimal control policy that minimizes the L.H.S. of (19) is \( p^* = \min \left\{ -\frac{\partial V(q,e)}{\partial q} \big/ \frac{\partial V(q,e)}{\partial e} \big/ \frac{1}{\nu^2} \right\} \). Substituting it to the PDE in (19), we have

\[
\mathbb{E} \left[ \frac{q}{\lambda\tau} + \frac{\partial V(q,e)}{\partial q} (-R(h,p^*) + \lambda) \right] + \frac{\partial V(q,e)}{\partial e} (-p^* + \pi) \bigg| q,e \bigg| = 0
\] (43)
For convenience, denote $V_q \triangleq \frac{\partial V(q,e)}{\partial q}$ and $V_e \triangleq \frac{\partial V(q,e)}{\partial e}$. We then calculate the expectations in (43):

$$E[p^*] = \left[ \int_{-V_e+V_q}^{\infty} \frac{\partial V}{\partial q} (V_q - V_e - \frac{1}{\tau}) \exp (-x) dx + \int_{-V_e+V_q}^{-\infty} \frac{\partial V}{\partial e} (V_q - V_e - \frac{1}{\tau}) \exp (-x) dx \right] (V_q - V_e > \xi) + \left[ \int_{-V_e+V_q}^{\infty} \frac{\partial V}{\partial q} (V_q - V_e - \frac{1}{\tau}) \exp (-x) dx + E_1 \left( \frac{V_q}{V_e} + \frac{1}{\tau V_e} \exp \left( \frac{V_q}{V_e} - \frac{1}{\tau V_e} \right) \right) \right] \int \left( \frac{V_q}{V_e} < \xi \right) \neq G \left( \frac{V_q}{V_e} \right).
$$

Similarly, using the integration by parts, we have

$$E[R(h,p)] = \left[ \exp (\zeta) E_1 (\frac{\tau V_q}{V_e}) + E_1 (\frac{V_q}{V_e}) - E_1 (\tau V_q) \right] \int \left( \frac{V_q}{V_e} > \xi \right) + E_1 (\frac{V_q}{V_e}) \int \left( \frac{V_q}{V_e} < \xi \right) \neq F \left( \frac{V_q}{V_e}, \frac{\tau}{\tau V_e} \right).$$

Therefore, the PDE in (43) becomes:

$$\frac{q}{\lambda \tau} + V_q \left[ \frac{\lambda}{\tau} - F \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] + V_e \left[ \alpha - G \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] = 0$$

(44)

We then discuss the properties of $F$ and $G$ in (44) as follows:

- If $\frac{V_q}{V_e} \leq \xi$, $F$ is increasing w.r.t. $\frac{V_q}{V_e}$ and $F \in (0, E_1(\frac{\tau}{\tau V_e})]$. If $\frac{V_q}{V_e} > \xi$, $F$ is a function of $\frac{V_q}{V_e}$ and $\xi$, and is increasing w.r.t. $\frac{V_q}{V_e}$ and $F \in (E_1(\frac{\tau}{\tau V_e}), \exp(\zeta) E_1(\frac{\tau}{\tau V_e})].$

- If $\frac{V_q}{V_e} \geq \xi$, $G$ is increasing w.r.t. $\frac{V_q}{V_e}$ and $G \in [0, \tau \exp(-\zeta) - E_1(\frac{\tau}{\tau V_e})].$ If $\frac{V_q}{V_e} > \xi$, $G$ is a function of $\frac{V_q}{V_e}$ and $\xi$, and is increasing w.r.t. $\frac{V_q}{V_e}$ and $G \in (\tau \exp(-\zeta) - E_1(\frac{\tau}{\tau V_e}), \xi).$

For the continuous time queueing system in (13) and (14), there exists a steady data queue states $q_0 = 0$ and $e_0 \in [0, N_E]$, i.e., $\lim_{t \to \infty} q(t) = q_0$ and $\lim_{t \to \infty} e(t) = e_0$. At steady state, we require

$$\lambda \leq F \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right), \quad \alpha \geq G \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right)$$

(45)

The existence of solution for the HJB equation in Theorem 2 is equivalent to the existence of solution of (45). We shall discuss the solution of (45) in the following two cases:

**Case 1:** if the equalities are achieved in (45), i.e.,

$$\lambda = F \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right), \quad \alpha = G \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right)$$

(46)

there exists a $\tilde{e} \in [0, N_E]$ such that

$$\tilde{e} \tau \exp \left( -\frac{\tau}{\zeta} \right) - E_1 \left( \frac{\tau}{\zeta} \right) < \mu < \frac{\tilde{e}}{\zeta}$$

(47)

From the first equation above, we have $\pi \tau < \tilde{e} < x \tau$ where $x$ satisfies $x \exp \left( -\frac{1}{\zeta} \right) - E_1 \left( \frac{1}{\zeta} \right) = \pi$. For given $\tilde{e}$, we can obtain the range for $\lambda$ according to the second equation above: $E_1 \left( \frac{1}{\zeta} \right) < \lambda \leq \exp \left( \frac{1}{\zeta} \right) E_1 \left( \frac{1}{\zeta} \right)$. Furthermore, we denote the solution of (46) w.r.t. $e$ to be $e^*$. Then, it is sufficient that $N_E \geq e^*$ so that the solution of (46) is meaningful.

**Case 2:** if $\tilde{X} \leq E_1 \left( \frac{1}{\zeta} \right)$, we will show in Appendix I that the optimal control for this case achieves

$$\lambda < F \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right), \quad \alpha = G \left( \frac{V_q}{V_e}, \frac{e_0}{\tau} \right)$$

(48)

where the steady states are $q_0 = 0$, $e_0 = \alpha \tau$. In this case, we require that $N_E \geq e_0 = \alpha \tau$. Combining both cases, we obtain the conditions in (22) and (23). This completes the proof.

**Appendix E: Proof of Theorem 3.**

The PDE in (44) has different structures when $\frac{V_q}{V_e} < \xi$ and $\frac{V_q}{V_e} > \xi$. Specifically, when $\frac{V_q}{V_e} > \xi$,

$$\frac{q}{\lambda \tau} + V_q \left[ \lambda - F \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] + V_e \left[ \alpha - G \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] = 0$$

(49)

when $\frac{V_q}{V_e} < \xi$,

$$\frac{q}{\lambda \tau} + V_q \left[ \lambda - E_1 \left( \frac{-V_q}{V_q} \right) \right] + V_e \left[ \alpha + V_e \exp \left( \frac{V_q}{V_q} \right) + E_1 \left( \frac{-V_q}{V_q} \right) \right] = 0$$

(50)

A. Relationship among $\frac{V_q}{V_e}, \xi, \lambda$ and $\pi$

Dividing $-V_e$ on both sizes of (44), we have

$$J \left( \frac{V_q}{-V_e} \right) \triangleq \frac{-V_q}{V_e} \left[ F \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] - \left[ \alpha - G \left( \frac{V_q}{V_e}, \frac{e}{\tau} \right) \right] = -\frac{q}{\lambda \tau}$$

We first have the following lemma:

**Lemma 4**: From (51), we have $-V_e = \Theta \left( \frac{1}{\lambda \exp(\zeta)} \right).$

**Proof of Lemma 4**: We assume that $V(q,e) = o(g(\lambda))$ and $V(q,e) = O(f(\lambda))$ for some functions $f$ and $g$. Therefore, $V_q = o(g(\lambda)) = O(f(\lambda))$, and $V_e = o(g(\lambda)) = O(f(\lambda))$. According to (47), we have $\pi = o(\exp(\lambda))$. Combining (44), we have

$$\alpha(g(\lambda)) \Theta(\lambda) + o(g(\lambda)) \Theta(\exp(\lambda)) = -\Theta \left( \frac{1}{\lambda} \right)$$

(52)

$$O(f(\lambda)) \Theta(\lambda) + O(f(\lambda)) \Theta(\exp(\lambda)) = -\Theta \left( \frac{1}{\lambda} \right)$$

(53)

where (52) implies $g(\lambda) = -o \left( \frac{1}{\lambda \exp(\zeta)} \right)$, and (53) implies $f(\lambda) = -O \left( \frac{1}{\lambda \exp(\zeta)} \right)$. Hence, $V(q,e) = -\Theta \left( \frac{1}{\lambda \exp(\zeta)} \right)$, which induces $-V_e = \Theta \left( \frac{1}{\lambda \exp(\zeta)} \right).$

Based on Lemma 4, (51) implies $J \left( \frac{V_q}{V_e} \right) = -\Theta(\exp(\lambda))$. Let $e^{\Theta}$ satisfy $E_1 \left( \frac{\zeta}{e^{\Theta}} \right) = \tilde{X}$. We have the following discussions on the property of $J \left( \frac{V_q}{V_e} \right)$.
$e < e^{th}$: if $\exp \left( \frac{C}{\tau e} \right) E_1 \left( \frac{\tau e}{\tau c} \right) < \bar{\lambda}$, $J \left( \frac{V_q}{V_e} \right)$ is an increasing function w.r.t. $\frac{V_q}{V_e}$. Specifically, when $0 < \frac{V_q}{V_c} < x_0(e)$, where $F(x_0(e), \tau) = 0$, $J \left( \frac{V_q}{V_e} \right)$ is negative. When $\frac{V_q}{V_e} > x_0(e)$, $J \left( \frac{V_q}{V_e} \right)$ is positive. On the other hand, if $\exp \left( \frac{C}{\tau e} \right) E_1 \left( \frac{\tau e}{\tau c} \right) > \bar{\lambda}$, let $x_1(e)$ satisfy $F(x_1(e), \tau) = \bar{\lambda}$. When $0 < \frac{V_q}{V_e} < x_1(e)$, $J \left( \frac{V_q}{V_e} \right)$ is increasing w.r.t. $\frac{V_q}{V_e}$, and when $\frac{V_q}{V_e} > x_1(e)$, $J \left( \frac{V_q}{V_e} \right)$ is decreasing w.r.t. $\frac{V_q}{V_e}$.

$e \geq e^{th}$: let $x_2(e)$ satisfy $F(x_2(e), \tau) = \bar{\lambda}$. When $0 < \frac{V_q}{V_e} < x_2(e)$, $J \left( \frac{V_q}{V_e} \right)$ is negative and increasing.

Therefore, we have the following results on the relationship among $\frac{V_q}{V_e}$, $\tau$, $\bar{\lambda}$ and $\bar{\pi}$:

**Classification I (Relationship among $\frac{V_q}{V_e}$, $\tau$, $\bar{\lambda}$ and $\bar{\pi}$):**

1. $e < e^{th}$:
   - small $\bar{\lambda}$, small $\bar{\pi}$ and $E_1 \left( \frac{\tau c}{\tau e} \right) < \bar{\lambda} < \exp \left( \frac{\tau c}{\tau e} \right) E_1 \left( \frac{\tau c}{\tau e} \right)$: in this case, we have $\frac{V_q}{V_c} = \Theta \left( \exp \frac{\tau c}{\tau e} \right)$ which is large for sufficiently small $\bar{\lambda}$. Furthermore, since $e < e^{th}$, we have $\frac{V_q}{V_e} > \frac{\tau c}{\tau e}$. Therefore, the PDE in (51) becomes (49) with large $\frac{V_q}{V_e}$ and small $e$.
   - large $\bar{\lambda}$, large $\bar{\pi}$ and $E_1 \left( \frac{\tau c}{\tau e} \right) < \bar{\lambda} < \exp \left( \frac{\tau c}{\tau e} \right) E_1 \left( \frac{\tau c}{\tau e} \right)$: similar to the previous case, we have $\frac{V_q}{V_e} > \frac{\tau c}{\tau e}$. Since $e < e^{th}$ and we consider large $\bar{\lambda}$, $e$ is relatively large compared with $\frac{V_q}{V_e}$. Therefore, the PDE in (51) becomes (49) with large $\frac{V_q}{V_e}$ and large $e$.

2. $e \geq e^{th}$: in this case, since $0 < \frac{V_q}{V_e} < x_0(e)$ and $x_0(e) < \frac{\tau c}{\tau e}$, we have $\frac{V_q}{V_e} < \frac{\tau c}{\tau e}$, which means that the PDE in (51) becomes (50).

**B. Solving the HJB Equation under the Large-DATA-Arrival-Energy-Sufficient Regime**

According to Classification I when $e < e^{th}$, we have the PDE in (49) with large $\frac{V_q}{V_e}$ and large $e$, and when $e \geq e^{th}$, we have the PDE in (50). We first solve the PDE in (49) with large $\frac{V_q}{V_e}$ and large $e$. We have the following approximations for $\frac{V_q}{V_e} E [R(h, p^*)]$ in (51):

$$\frac{V_q}{V_c} E_1 \left( \frac{x e}{V_c} \right) = \frac{V_q}{V_c} \left( \frac{\alpha t}{e} \right) + o(1), \quad E_1 \left( \frac{x e}{V_c} \right) = -\gamma c + \frac{V_q}{V_c} \log \left( \frac{V_q}{V_c} - \frac{\tau c}{\tau e} \right) + 1 + o(1)$$

Therefore, we have

$$\frac{V_q}{V_c} E [R(h, p^*)] = \frac{V_q}{V_c} \left( \frac{\tau c}{\tau e} \right) + \frac{\tau c}{\tau e} \cdot \frac{V_q}{V_c} \left( \frac{\tau c}{\tau e} \right) + 1 + o(1)$$

Similarly, for $E [p^*]$, we have

$$\frac{V_q}{V_c} \exp \left( \frac{V_q}{V_e} \right) = \frac{V_q}{V_c} \left( \frac{\tau c}{\tau e} \right) + 1 + o(1)$$

Substituting (54) and (55) into (44) and for large $\bar{\lambda}$ and $\bar{\pi}$, we obtain the following simplified PDE:

$$\frac{q}{\tau V} + V_q \left( \bar{\lambda} - \exp \left( \frac{\tau e}{\tau} \right) E_1 \left( \frac{\tau e}{\tau} \right) \right) + V \pi = 0$$

For large $e$, we have $\exp \left( \frac{\tau e}{\tau} \right) E_1 \left( \frac{\tau e}{\tau} \right) = -\gamma c + \log \frac{\tau c}{\tau e} + o(1)$.

**C. Verification of the Admissibility of $\Omega^{*}$ under the Large-DATA-Arrival-Energy-Sufficient Regime**

We first calculate the water level when $e < e^{th}$ under (24):

$$V(q, e) = c_1 e + \phi(q, c_1) + c_2$$

where $\phi = O(q^2)$. Then, $\frac{dV(q, N_k)}{dq} = 0$ indicates that $c_1 = 0$, which means that $V(q, e)$ is a function of $q$ only. Hence, $V(q, e)$ is infinite which means that $p^* = \frac{\tau c}{\tau e}$ according to Corollary I. Combining (24) ($e < e^{th}$) and (57) ($e \geq e^{th}$), we obtain the full solution in this regime.
\[ + \int_{t_0}^{t} L(t') dt' \]
\[ \leq q(0) + q(t_0) - \int_{t_0}^{t} \left[ \exp \left( \frac{1}{\alpha + \epsilon / \tau} \right) E_1 \left( \frac{1}{\alpha + \epsilon / \tau} \right) - \chi \right] \tau' + \int_{t_0}^{t} L(t') dt' \]
\[ (60) \]

where (a) is due to \( e(t) < \chi \tau + \epsilon \) when \( t \geq t_0 \) according to (59). Since \( E_1 \left( \frac{\epsilon}{\alpha + \epsilon / \tau} \right) < \chi \), there exists a \( \delta > 0 \) such that \( \chi < \exp \left( \frac{\epsilon}{\alpha + \delta} \right) E_1 \left( \frac{1}{\alpha + \delta} \right) \). Choosing \( \epsilon = \delta \tau \) in (60), we obtain
\[ q(t) - q(0) < 0, \quad \text{if} \quad t \geq \frac{q(t_0)}{\exp \left( \frac{1}{\alpha + \delta} \right) E_1 \left( \frac{1}{\alpha + \delta} \right) - \chi} \]
\[ (61) \]
Therefore, we obtain the negative queue drift, which means that the greedy policy is a stabilizing policy [33, 34].

**APPENDIX F: PROOF OF COROLLARY 2**

Since \( V(q, e) \) is a function of \( q \) only when \( e \geq e^{th} \) and hence, \( \frac{V_{q}}{V_{\epsilon}} \) is infinite, which means that \( p^* = \frac{\epsilon}{\tau} \) according to Corollary 1. For \( e < e^{th} \), the water level (WL) is given in (58). When \( q > \frac{\lambda}{\pi} (\chi e + \chi - \log(\frac{\epsilon}{\tau})) \), the WL is negative, which results in \( p^* = 0 \). On the other hand, when \( q < \frac{\lambda}{\pi} (\chi e + \chi - \log(\frac{\epsilon}{\tau})) \), the WL is positive and increasing w.r.t. \( q \). Moreover, derivative of \( e \).

When \( e < \chi q \), the WL is decreasing w.r.t. \( e \), and when \( e < \chi q \), the WL is increasing w.r.t. \( e \).

**APPENDIX G: PROOF OF THEOREM 6**

A. Solving the HJB Equation under the Small-Data-Arrival-Energy-Sufficient Regime

According to Classification 1, when \( e < e^{th} \), we have the PDE in (49) with large \( \frac{V_{q}}{V_{\epsilon}} \) and small \( e \) and when \( e \geq e^{th} \), we have the PDE in (50). Following part B in Appendix E, we can obtain the simplified PDE as in (56). For small \( e \), we approximate \( \exp \left( \frac{\epsilon}{\tau} \right) E_1 \left( \frac{\epsilon}{\tau} \right) \) as \( \exp \left( \frac{\epsilon}{\tau} \right) E_1 \left( \frac{\epsilon}{\tau} \right) = 0 \) and (61). Substituting it to (56) and using 3.8.2.3 of (52), we obtain the solution for this case as in (25). Furthermore, the solution for \( e \geq e^{th} \) is the same as (57). Following the same procedure in Appendix E, it can be verified that the three conditions in (20) are satisfied.

B. Verification of the Admissibility of \( \Omega^{ws} \) under the Small-Data-Arrival-Energy-Limited Regime

We first calculate the WL when \( e < e^{th} \) under (25):
\[ \frac{V_q}{-V_e} = \frac{\chi\pi e}{-e^2 + \chi\pi e - \chi q} \]
\[ (62) \]
Therefore, for sufficiently large \( q \), we have \( \frac{V_q}{V_e} < 0 \), which means that there is no data transmission, and the energy buffer will harvest energy until \( e \geq e^{th} \) when the data queue will adopt the policy \( p^* = \frac{\epsilon}{\tau} \). Following the same proof as in part C in Appendix E, we can prove the negative data queue drift.

**APPENDIX H: PROOF OF COROLLARY 3**

Since \( V(q, e) \) is a function of \( q \) only when \( e \geq e^{th} \), we have \( p^* = \frac{\epsilon}{\tau} \). For \( e < e^{th} \), the WL is given in (62). When \( q > \frac{-e^2 + \chi q}{\chi q} \), the WL is negative, which results in \( p^* = 0 \). On the other hand, when \( q < \frac{\chi q}{\chi q} \), the WL is positive, which is increasing w.r.t. \( q \). Moreover, derivative of (62) w.r.t. \( e \).

When \( e < \sqrt{\chi\tau q} \), the WL is decreasing w.r.t. \( e \), and when \( e > \sqrt{\chi\tau q} \), the WL is increasing w.r.t. \( e \).

**APPENDIX I: PROOF OF THEOREM 7**

A. Solving the HJB Equation under the Small-Data-Arrival-Energy-Sufficient Regime

Following the same analysis as in part A in Appendix E, when \( e < e^{th} \), we have the PDE in (49), and when \( e \geq e^{th} \), we have the PDE in (50). For the PDE in (49), we require
\[ \frac{\partial V(0, e)}{\partial q} = 0 \]
\[ (63) \]
because the equalities in (45) cannot be achieved and \( L(t) \neq 0 \) after the virtual queueing system enters the steady state. Under this regime, the system operates at the region with small \( V_q \). We have the following approximations for \( E[R(h, p^*)] \) in (49):
\[ \frac{\partial V}{\partial q} = \frac{\epsilon}{\tau} E_1 \left( \frac{\epsilon}{\tau} \right) = 0 \]
\[ (64) \]
Similarly, for \( E[p^*] \), we have
\[ E[p^*] = 0 \]
\[ (65) \]
Substituting (64) and (65) into (44), we obtain the simplified PDE:
\[ \frac{\partial^2 V}{\partial q^2} + V_q \lambda + V, \pi = 0 \]
Using 3.8.2.3 of (52), we obtain the following solution for this case:
\[ V(q, e) = \frac{1}{2\gamma \tau} e^2 - \frac{eq}{\lambda \pi \tau} + \phi \left( \frac{q}{\lambda} + \frac{\chi}{\alpha} \right) \]
\[ (66) \]
From (63), we require that \( \phi' \left( \frac{\epsilon}{\alpha} \right) = \left( \frac{\epsilon}{\alpha} \right) \frac{1}{\alpha^2 \tau} \), \( \forall e \). We choose \( \phi(x) = x^2 \left( -\frac{\epsilon}{\alpha^2 \tau} \right) \). Therefore, the final solution is given in (26). Furthermore, the solution for \( e \geq e^{th} \) is the same as (57). Following the same procedure in Appendix E, it can be verified that the three conditions in (20) are satisfied.

B. Verification of the Admissibility of \( \Omega^{ws} \) under the Small-Data-Arrival-Energy-Sufficient Regime

Note that when \( e < e^{th} \), under the solution in (26), we have that \( \frac{\partial V(q, e)}{\partial q} = 0 \) for all \( q, e \), which results in \( p^* = \frac{\epsilon}{\tau} \). Following the same proof as in part C in Appendix E, we can prove the negative data queue drift.
APPENDIX J: PROOF OF THEOREM

We prove that for sufficiently large queue $Q(0)$, for the following case 1 $(E(0) > e^{th})$ and case 2 $(E(0) < e^{th})$, we have negative data queue drift.

Case 1, $E(0) > e^{th}$: In this case, the greedy policy $p^*(n) = \frac{E(n)}{E(0)}$ is adopted for all different asymptotic scenarios. Based on the energy queue dynamics in (65), we have $p^*(n) = \min\{\alpha(n-1) \cdot \frac{N \lambda}{2}, 1\}$ for $n \geq 1$. We then calculate the one-step queue drift as follows: for sufficiently large $Q$,

$$\begin{align*}
\mathbb{E} \left[ Q(n+1) - Q(n) \mid Q(n) = Q, E(n) = E \right] = & \mathbb{E} \left[ (Q - \log (1 + |\alpha|^2 E/\tau) \tau + \lambda \tau - Q) \right] \\
= & \left[ - \log (1 + |\alpha|^2 E/\tau) \tau + \lambda \tau \right] (Q(n) - E(n)) \ 	ext{(67)}
\end{align*}$$

where (a) is due to the fact that for given $E$ and sufficiently large $Q$, we have $\mathbb{Pr} \left[ Q > \log (1 + |\alpha|^2 E/\tau) \tau = \mathbb{Pr} \left[ |\alpha|^2 < \frac{\exp(Q-1)}{e^\tau} \right] > 1 - \delta \ (\forall \delta > 0) \right]$. In (67), if $\alpha(n-1) < \frac{N \lambda}{2}$, we have $\mathbb{E} \left[ - \log (1 + |\alpha|^2 E/\tau) \tau + \lambda \tau \right] = \mathbb{E} \left[ - \log (1 + |\alpha|^2 N \lambda/\tau) \tau + \lambda \tau \right] < 0$, where (b) is due to (29). If $\alpha(n-1) > \frac{N \lambda}{2}$, we have $\mathbb{E} \left[ - \log (1 + |\alpha|^2 E/\tau) \tau + \lambda \tau \right] = \mathbb{E} \left[ - \log (1 + |\alpha|^2 N \lambda/\tau) \tau + \lambda \tau \right] = 0$, where (c) due to $N \lambda/\tau > |\alpha| \tau$ and (d) is due to $\lambda \tau < 0$. Hence, we have negative drift.

Case 2, $E(0) < e^{th}$: In this case, we show that there exists some positive integer $n < N$ such that the $n$-step queue drift in the discrete time queueing system is negative. Since $E(0) < e^{th}$ and $Q(0)$ is sufficiently large, the data queue will not transmit in the beginning. For given $\alpha$, after $N - \frac{E(0)}{\alpha}$ number of time slots where $[x]$ is the ceiling function, the data queue will adopt the greedy policy to transmit. To prove the existence of $n$, it is sufficient to prove that

$$\begin{align*}
\mathbb{E} \left[ \left( N - \frac{E(0)}{\alpha} \right) \left( \exp \left( \frac{1}{\alpha} \right) E \left( \frac{1}{\alpha} \right) \right) \right] = & \mathbb{E} \left[ N \right] \\
\text{(68)}
\end{align*}$$

where the L.H.S. (R.H.S.) means the departure bits (arrival bits) before the end of the next change event of the energy arrival rate. From (68), we have

$$\mathbb{E} \left[ \left( 1 - \frac{1}{N} \left( \frac{e^{th} - E(0)}{\alpha} \right) \right) \left( \exp \left( \frac{1}{\alpha} \right) E \left( \frac{1}{\alpha} \right) \right) \right] > \mathbb{E} \left[ Q \right] \ 	ext{(69)}$$

and $\mathbb{E} \left[ \left( 1 - \frac{1}{N} \left( \frac{e^{th} - E(0)}{\alpha} \right) \right) \left( \exp \left( \frac{1}{\alpha} \right) E \left( \frac{1}{\alpha} \right) \right) \right] < \mathbb{E} \left[ Q \right] \ 	ext{(70)}$.

Therefore, we have negative drift for this case. Based on the Lyapunov theory [33, 34], negative state drift for both cases leads to the stability of $Q(n)$, i.e.,

$$\lim_{n \to \infty} \mathbb{E} \left[ Q^2(n) \right] < \infty.$$
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