Periodic analogues of the Kerr solutions:
a numerical study

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Abstract

In recent years black hole configurations with non standard topology or with
non-standard asymptotic behavior have gained considerable attention. In this
article we carry out numerical investigations aimed to find periodic coaxial
configurations of co-rotating $3 + 1$ vacuum black holes, for which existence
and uniqueness has not yet been theoretically proven. The aimed configura-
tions would extend Myers/Korotkin-Nicolai’s family of non-rotating (static)
coaxial arrays of black holes. We find that numerical solutions with a given
value for the area $A$ and for the angular momentum $J$ of the horizons appear
to exist only when the separation between consecutive horizons is larger than
a certain critical value that depends only on $A$ and $|J|$. We also establish that
the solutions have the same Lewis’s cylindrical asymptotic behavior as van
Stockum’s infinite rotating cylinders. Below the mentioned critical value the
rotational energy appears to be too big to sustain a global equilibrium and a
singularity shows up at a finite distance from the bulk. This phenomenon is
a relative of van Stockum’s asymptotic collapse, manifest when the angular
momentum (per unit of axial length) reaches a critical value compared to the
mass (per unit of axial length), and that results from a transition in the Lewis’s
class of the cylindrical exterior solution. This remarkable phenomenon seems
to be unexplored in the context of coaxial arrays of black holes. Ergospheres
and other global properties are also presented in detail.

Keywords: vacuum black holes, Stockum’s cylindric solutions,
Myers/Korotkin-Nicolai solutions

(Some figures may appear in colour only in the online journal)
1. Introduction

In recent years vacuum black hole configurations with non standard topology or with non-standard asymptotic behavior have gained considerable attention. In five dimensions, Emparan and Reall [1] have found asymptotically flat stationary solutions with ring-like $S^1 \times S^2$ horizon and Elvang and Figueras [2] have found asymptotically flat black Saturns, where a black ring $S^1 \times S^2$ rotates around a black sphere $S^3$. More recently Khuri, Weinstein and Yamada [3, 4], have found periodic static coaxial arrays of horizons either with spherical $S^3$ or ring-like $S^3 \times S^2$ topology. Rather than being asymptotically flat, these latter configurations are asymptotically Levi-Civita/Kasner solutions and generalize to five dimensions the important Myers/Korotkin-Nicolai (MKN) family of vacuum static $3+1$ solutions [5–7], referred sometimes as ‘periodic Schwarzschild’, as they are obtained as ‘linear’ superpositions of Schwarzschild’s solutions via Weyl’s method. The search for new solutions has branched rapidly to higher dimensions and to different fields, giving by now a rich landscape of topologies, [8]. However, the question whether ‘periodic Kerr’ configurations generalizing the ‘periodic Schwarzschild’ exist, either in four or higher dimensions, remains still open. In this article we aim to investigate this latter problem in four dimensions numerically. Specifically, we carry out numerical investigations pointing at constructing periodic coaxial configurations of an infinite number of co-rotating $3+1$ black holes. The equally spaced horizons are all intended to have the same area $A$, angular momentum $J$ and angular velocity $\Omega_H$, (see figure 1). This problem has been discussed by Korotkin and Nicolai in [7] (section 4) and also by Mavrin in [9], however not conclusively. Numerical investigations and an analysis of the solution space and of the global properties of the solutions appear to be carried out here for the first time.

This problem poses a number of difficulties on the numerical side. The central, relevant problem to solve is a harmonic map on a rectangle in two dimensions, whose solutions are found as the stationary regime of a harmonic map heat flow. The main difficulty is rooted in finding and implementing the appropriate boundary conditions, specially those fixing the area $A$ and the angular momentum $J$, and those required to drive the flow to the right asymptotic behavior for the given $A$ and $J$.

The results of this paper can be summarized as follows. Periodic configurations having a given value of $J$ and $A(> 8\pi |J|)$, appear to exist only when the separation between the horizons is larger than a certain critical value that depends only on $A$ and $|J|$. The smaller the separation the bigger the angular velocity $|\Omega_H|$ and the bigger the rotational energy and the total energy. The solution at asymptotically large radial values matches always a vacuum Lewis’s solution [10] as in the van Stockum’s rotating cylinders [11]. Furthermore, as the separation between horizons gets smaller than the critical value, no asymptotic model can hold the given amount of energy and we evidence a singularity formation at a finite distance from the bulk. Roughly speaking, the solution ‘closes up’ for a large enough radial value due to too much rotational energy. This phenomenon is similar to van Stockum’s asymptotic collapse, studied in [11], and manifesting when the angular momentum (per unit of axial length) increases past a critical value compared to the mass (per unit of axial length). When the rotation increases, the exterior Lewis solution transits from one extending to infinity to one blowing up at a finite distance from the material cylinder. There is a change in the Lewis’s class. This change of class, that prevents black holes from getting too close, appears to be entirely novel in our context and is in sharp contrast to what occurs for the periodic Schwarzschild configurations, where horizons
can get arbitrarily near. The ergo-regions are always bounded and disjoint. We observe though that below the critical separation the ergospheres can indeed merge, but such solutions do not extend to infinity. The Komar mass $M$ per black hole satisfies the relevant inequality,

$$\sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}} \leq M,$$

and equality is approached as the separation between the black holes grows unboundedly and the geometry near the horizons approaches that of Kerr.

The paper is organized as follows. In section 2 we overview the theoretical and numerical problem commenting on the difficulties and strategies. We present the main equations to be solved and recall the Lewis’s classes. We also discuss the boundary condition for the harmonic map heat flow and give the precise set up for the numerical study. In section 3 we discuss the numerical techniques used to solve the equations. In section 4 we present our results and discuss several features of the numerical solutions, and in section 5 we contextualize our work and give some possible future directions of research.
2. The setup

2.1. The reduced stationary equations

The black hole configurations that we are looking for will be in Weyl–Papapetrou form (see [12] and references therein),

\[
ds^2 = -V dt^2 + 2W dt d\phi + \eta d\phi^2 + \frac{e^{2\gamma}}{\eta} (d\rho^2 + dz^2),
\]

(2)

where \((t, \rho, z, \phi) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times S^1\) and where the components \(V, W, \eta\) and \(\gamma\) depend only on \((\rho, z)\). We require \(V, W, \eta\) and \(\gamma\) to be \(z\)-periodic with period \(L\), and, to prevent struts on the axis, we demand in addition \(V, W, \eta\) and \(\gamma\) to be symmetric with respect to the reflection \(z \rightarrow -z\).

If we denote by \(\omega\) the twist potential associated to the Killing vector \(\partial_\phi\), then \(\eta(\rho, z)\) and \(\omega(\rho, z)\) satisfy the harmonic map equations (Ernst Equations),

\[
\Delta \eta = \frac{\left| \nabla \eta \right|^2 - \left| \nabla \omega \right|^2}{\eta},
\]

(3)

\[
\Delta \omega = 2 \langle \nabla \omega, \nabla \eta \rangle, \tag{4}
\]

where \(\nabla f := (\partial_\rho f, \partial_z f)\), the two-gradient, \(\langle , \rangle\) is the flat space inner product in the \((\rho, z)\) embedded space, and \(\Delta := \partial_{\rho \rho} + \frac{1}{\rho} \partial_\rho + \partial_z^2\) is the 3D axisymmetric flat-space Laplacian operator in cylindrical coordinates \((\rho, z, \phi)\). Once these equations have been solved, the angular velocity function \(\Omega\) is obtained by line integration of the quadratures,

\[
\partial_{z} \Omega = \rho \frac{\partial_\rho \omega}{\eta^2}, \quad \partial_\rho \Omega = -\rho \frac{\partial_z \omega}{\eta^2}, \tag{5}
\]

and the exponent \(\gamma\) is found after line integration of,

\[
\partial_{\rho} \gamma = \frac{\rho}{2\eta^2} (\partial_\rho \eta \partial_z \eta + \partial_\rho \omega \partial_z \omega),
\]

\[
\partial_z \gamma = \frac{\rho}{4\eta^2} \left( (\partial_\rho \eta)^2 - (\partial_z \eta)^2 + (\partial_\rho \omega)^2 - (\partial_z \omega)^2 \right). \tag{6}
\]

Finally, the metric components \(W\) and \(V\) follow from the identities,

\[
W = \eta \Omega, \quad V = \frac{\rho^2 - W^2}{\eta}. \tag{7}
\]

Thus, the stationary axisymmetric Einstein equations are solved in a ladder-like fashion: first solve the harmonic map equations (3) and (4), then solve the quadratures (5) and (6), and finally get from them the other coefficients \(V\) and \(W\) of the metric (2).

2.2. Lewis solutions and the asymptotic models

The Lewis solutions [10] (see also [11]) are cylindrically symmetric (i.e. independent on \(\phi\) and \(z\)) rotating stationary vacuum solutions. Some of the solutions extend to infinity \((\rho = +\infty)\) and some do not. The possible forms for \(\eta\) are the following,

\[(\pm) : \eta = \rho \frac{|w|}{a} \sin(\pm a \ln(\rho) + b), \quad a > 0, \ b \in \mathbb{R}, \tag{8}\]

\[(\pm) : \eta = \rho |w|(\pm \ln(\rho) + b), \quad b \in \mathbb{R}, \tag{9}\]
where $a$ and $b$ are free parameters and $w$ is related to the twist potential by,

$$\omega = wz, \quad w \neq 0.$$  

The solutions extending to infinity are (II+) and (III+) (note that they are positive only after some $\rho$). For the case (III+), the metric components $V$, $W$, $\eta$ and $e^{2\gamma}/\eta$ get the form,

$$V = \frac{2a}{|w|} e^{-b\rho^{1-a}}, \quad W = s(w) e^{-b\rho^{1-a}},$$

$$\eta = \frac{|w|}{2a} (e^{b\rho^{1+a}} - e^{-b\rho^{1-a}}), \quad \frac{e^{2\gamma}}{\eta} = c \rho^{(a^2-1)/2},$$

where $s(w) = w/|w|$ is the sign of $w$ and $c > 0$ is another free parameter. The angular velocity function $\Omega$ is,

$$\Omega = \frac{2a}{w} \frac{e^{-b\rho^{-a}}}{(e^{b\rho^a} - e^{-b\rho^{-a}})},$$

and note that $\Omega$ is set to be zero at infinity. It turns out that the solutions (III+) with $a \geq 1$ cannot model the asymptotic behavior ($\rho \to \infty$) of a periodic array of black holes. Indeed, if one such array of black holes was asymptotically a solution (III+) with $a \geq 1$, then a simple computation shows that the lapse function (i.e. the component of $\partial_t$ normal to $\{t = 0\}$) would tend to zero at infinity and as it is zero on the horizons, it would have an absolute maximum thus contradicting the maximum principle on the lapse equation (the Laplacian of the lapse is always less than or equal to zero). As $a$ tends to zero, $\eta$ in (III+) degenerates into (II+). The asymptotic models are therefore (III+) with $0 < a < 1$ and (II+).

The metrics of the models (III+) are Kasner to leading order (except for the cross term $-2s(w)e^{-b\rho^{1-a}}dtd\phi$). Recalling that the Kasner metrics, given by their line elements, have the form,

$$ds^2_K = -e^{-b\rho^a} dt^2 + e^b \rho^{2-\alpha} d\phi^2 + c \rho^{(a^2-1)/2} (d\rho^2 + dz^2),$$

we see that in case (III+) the Kasner exponent is $\alpha = 1 - a$, hence can vary in $(0,1)$.

### 2.3. Overview of the problem

The numerical implementation of the problem has a number of subtleties that we would like to discuss in what follows.

The configurations that we are looking for have three degrees of freedom that we will take to be the area $A$, the angular momentum $J$, and the period $L$, with $L$ linked to the physical separation between consecutive black holes. These parameters need to be incorporated into the boundary conditions of the main equations (3) and (4) for $\eta$ and $\omega$.

As we look for metrics periodic in $z$, we restrict the analysis to the region $\{\rho \geq 0, -L/2 \leq \rho \leq L/2\}$ keeping appropriate periodicity conditions on the top and bottom lines $\{\rho \geq 0, \rho = \pm L/2\}$. Here $L$ is the period mentioned earlier and is a free parameter. The equations for $\omega$ and $\eta$ need to be supplied with boundary data on the border $\{\rho = 0, -L/2 \leq \rho \leq L/2\}$. This boundary contains the horizon $H = \{\rho = 0, -m \leq z \leq m\}$ and the two components of the axis $A$ which is the complement of $H$. The boundary conditions arise from two sides: from the natural conditions that $\eta$ and $\omega$ must verify on horizons and axes, and from fixing the values of the area $A$ and of the angular momentum $J$. The angular momentum $J$ is set by specifying Dirichlet data for $\omega$ on the axis, whereas the area $A$ is set by specifying the limit values of $\eta/\rho^2$.
at the poles $z = \pm m$. The fact that one can incorporate $A$ and $J$ into the boundary conditions in this simple way is a great advantage of our formulation. The parameter $m$ that may seem to be a free parameter, is equal to the surface gravity $\kappa$ ($\kappa^2 := -\langle \nabla \partial_t, \nabla \partial_t \rangle / 2$), of the horizons times $A/4\pi$ and therefore can be fixed using the freedom in the definition of the stationary Killing field $\partial_t$, or the definition of time in (2). In this article we chose $\kappa = \kappa(A, J)$ equal to the surface gravity of the Kerr black holes for the given $A$ and $J$. The full set of boundary conditions on $\{\rho = 0, -L/2 \leq z \leq L/2\}$ is discussed in section 2.4.

To numerically solve the harmonic map equations we formulate the problem on a finite rectangle $[0, \rho_{\text{MAX}}] \times [-L/2, L/2]$. This adds the extra difficulty of finding natural boundary conditions also at $\rho = \rho_{\text{MAX}}$, for $\omega$ and for $\eta$ (see figure 2). On physical grounds one expects the solutions to become asymptotically independent of $z$ as $\rho \to \infty$ and to approach a Lewis model. The problem is that one does not know a priori which Lewis solution shows up for the given $A$, $J$ and $L$. If that information were known, then one could easily supply appropriate boundary conditions at $\rho_{\text{MAX}}$. Now, all Lewis solutions have $\omega = wz$, so we make $\omega(\rho_{\text{MAX}}, z) = wz$. For $\eta$ however no such single choice is possible. We set a Neumann type of boundary condition for $\eta$ and for that we use that, on actual solutions, the Komar mass expression $M(\rho)$, in (35), is constant, so that $M(\rho_{\text{MAX}}) = M(\rho)$ easily relates $\partial_\rho \eta(\rho_{\text{MAX}})$ to $\eta$ and $\omega$ at any $\rho < \rho_{\text{MAX}}$. Then, to define the condition for $\eta$ we make use of this relation, equating $\partial_\rho \eta(\rho_{\text{MAX}})$ to the average of the Komar mass expression in the bulk $0 < \rho < \rho_{\text{MAX}}$. We discuss this condition in section 2.4, and state it in (35). This peculiar Neumann condition for $\eta$ at $\rho_{\text{MAX}}$ is what gave us the best numerical results in terms of the speed and the stability of our code.

In order to find solutions to the equations (3) and (4), we look for stationary solutions of the harmonic map heat flow equations,

$$\partial_\tau \eta = -\Delta \eta + \frac{|
abla \eta|^2 - |
abla \omega|^2}{\eta}, \quad (16)$$

$$\partial_\tau \omega = -\Delta \omega + 2 \langle \nabla \omega, \nabla \eta \rangle / \eta, \quad (17)$$

where $\tau$ is a measure of the flow time, starting from certain initial data $(\omega_0, \eta_0)$. In the next subsections we make explicit the boundary conditions and the initial data for this flow.
2.4. Boundary conditions

From now on it will be more convenient to work with \( \sigma = \ln \eta - 2 \ln \rho \) instead of \( \eta \) (this choice has some advantages, see \([12–14]\)). In terms of \((\sigma, \omega)\) the harmonic map equations (3) and (4) become,

\[
\Delta \sigma = -\frac{e^{-2\sigma}|\nabla \omega|^2}{\rho^2},
\]

\[
\Delta \omega = 4\left(\frac{\langle \nabla \omega, \nabla \rho \rangle}{\rho} + 2\langle \nabla \omega, \nabla \sigma \rangle\right).
\]

The boundary conditions for this system will be the same as the boundary condition for the harmonic map heat flow, and are as follows.

**Boundary conditions on \( A \) and \( H \).** Recall that \( H = \{-m < z < m\} \) and that the axis \( A \) has two regions in \([-L/2, L/2] \), \( A_+ = \{m < z < L/2\} \) and \( A_- = \{-L/2 < z < -m\} \).

The boundary conditions for \( \omega \) are,

\[
\omega \big|_{A_+} = 4J, \quad \omega \big|_{A_-} = -4J,
\]

\[
\partial_\rho \omega \big|_H = 0.
\]

The first, Dirichlet condition, comes from fixing the angular momentum of the horizons to \( J \). The second, Neumann condition, can be obtained by demanding the spacetime smoothness of \( \Omega = W/\eta \) that is linked to \( \omega \) by (5).

The boundary conditions for \( \sigma \) are,

\[
\partial_\rho \sigma \big|_{A \setminus \partial H} = 0,
\]

\[
\partial_\rho (\sigma + 2 \ln \rho) \big|_H = 0,
\]

\[
\lim_{(\rho, z) \to (0, \pm m)} \sigma \sigma_0 = 1,
\]

where \( \sigma_0 \) is the reference Kerr solution given \( A, J \).

Now, recall that for Kerr,

\[
m = \sqrt{\frac{A}{16\pi} \frac{1 - (8\pi J/A)^2}{\sqrt{1 + (8\pi J/A)^2}}},
\]

and recall too, as explained in section 2.3, that this is the choice of \( m \) that we make. In particular we have the algebraic identity \( A = 4\pi m/\kappa \) where \( \kappa \) is the surface gravity of Kerr. On the other hand, the condition (24) enforces \( \gamma \) on \( H \) (which is constant) to be that of Kerr too. As \( e^{-\gamma} \big|_H = \kappa \), we deduce,

\[
\text{Area}(H) = 2\pi \int_{-m}^{m} e^\gamma dz = 2\pi \int_{-m}^{m} \frac{1}{\kappa} dz = \frac{4\pi m}{\kappa} = A.
\]

So the area of the horizon is indeed \( A \). This gives us a prescription for an indirect normalization of the Killing vector field \( \partial_\rho \) outside the ergoregion.

**Boundary conditions at \( \rho_{\text{MAX}} \).** Since we expect Lewis asymptotic model and the Komar angular momentum is \( J \), for the boundary condition of \( \omega \) we simply set,

\[
\omega(\rho_{\text{MAX}}, z) = (8J/L)z.
\]

The boundary condition for \( \sigma \) is delicate and we motivate it as follows. Given a solution to (3) and (4) one can get a Smarr type of expression for the Komar mass \( M \) (per black hole)
for the Killing field $\partial t$ on any 2-torus $t = 0$ and $\rho = \text{const}$ of the spacetime, of course after identifying $z = -L/2$ to $z = L/2$. The expression is,
\[ M(\rho) = \frac{1}{4} \int_{-L/2}^{L/2} (-\rho \partial_\rho \sigma + \Omega \partial_\z \omega) dz, \]
(28)
which we know indeed to be $\rho$–independent. For the Lewis solutions (III+) or (II+) we have $\Omega \to 0$ as $\rho \to \infty$, $\rho \partial_\rho \sigma \to a - 1$ as $\rho \to \infty$, and $\partial_\z \omega = w$, so we obtain,
\[ M = \frac{L}{4} (1 - a). \]
(29)
Thus, the Kasner exponent $\alpha = 1 - a$ is equal to $4M/L$. Hence, for any solution asymptotically (III+) or (II+) we have the decay,
\[ \partial_\rho \sigma \bigg|_\rho = \frac{4M}{L\rho} + o(1/\rho), \quad \rho \gg 1. \]
(30)
This suggests taking as boundary condition,
\[ \partial_\rho \sigma \bigg|_{\rho = \rho_{\text{max}}} = \frac{4M}{L\rho_{\text{max}}}. \]
(31)
But this condition faces a couple of problems. First, we do not know a priori the dependence of $M$ with $A$, $J$ and $L$. Another problem is that, since we are evolving a harmonic map heat flow, at any time $\tau \geq 0$ the integral in (28) associated to $(\sigma, \omega)$ at $\tau$ is not necessarily constant as a function of $\rho$, and even the function $\Omega$ is not well defined since the integrability conditions (5) do not necessarily hold. Therefore, we have to give a definition for $M(\rho, \tau)$ so that we can use (31) as boundary condition. By integrating by parts the last term in (28), and using (5), we have a candidate
\[ M(\rho, \tau) = 2\Omega(\rho, \tau) \bigg|_{z = -L/2} - \frac{1}{4} \int_{-L/2}^{L/2} (\rho \partial_\rho \sigma + \frac{\rho}{\eta^2} \partial_\rho \omega \omega) dz, \]
(32)
where $\Omega(\rho, \tau)$ is prescribed to be
\[ \Omega(\rho, \tau) = \int_{\rho_{\text{max}}}^{\rho} \frac{\rho'}{\eta^2} \partial_\z \omega d\rho', \]
(33)
with the functions evaluated at $z = -L/2$. If $(\sigma, \omega)$ converges to a solution, then $M(\rho, \tau)$ converges to a constant function. Taking this into account, a natural dynamical condition arises by taking the $\rho$-average of $M(\rho, \tau)$ over the numerical domain, at each time step, and impose
\[ M(\rho_{\text{max}}) = \overline{M}(\tau), \]
(34)
which in particular links the values of $M$ near the axis with those values far away. We tried several definitions of $\rho$-average in equation (34), though in the end we use a simple, uniform average in the interior of the interval $(0, \rho_{\text{max}})$. Then, our boundary condition for $\sigma$ at $\rho_{\text{max}}$ is given by
\[ \partial_\rho \sigma \bigg|_{\rho = \rho_{\text{max}}} = -\frac{4\overline{M}(\tau)}{L\rho_{\text{max}}}. \]
(35)
\[^3\] In our numerical codes this means taking uniform average of the values corresponding to the interior points of the discretized $\rho$ coordinate, excluding just the values at the boundary $\rho = 0$ and the two grid points next to it, and the values at the outer boundary $\rho = \rho_{\text{max}}$.8
Observe that in this condition there is no reference to any specific asymptotic model and can be used in other periodic configurations as well.

It remains to fix the boundary conditions for the integration of the quadratures (5) and (37). For $\Omega$, we impose

$$\Omega|_{\rho=\max} = 0,$$

(36)

which is the natural condition for the asymptotic models, and also consistent with the definition of $M$. The boundary condition on $q$ is just that it vanishes at one point in $A$. As it is explained below, the symmetries on $\sigma$ and $\omega$ and the integrability equation (37) imply this same condition holds at any other point of the regions $A_+$ and $A_-$ of the axis, thus preventing struts.

2.4.1. Absence of angle defects. After having $\eta$ and $\omega$, the exponent $\gamma$ is found by integration of (6) and is therefore determined up to a constant. One needs to show that this constant can be adjusted so that there are no angle defects on $A_+$ and $A_-$, where, recall, the angle defect in an axis point $z_0$ is $\delta(z_0) = 2\pi - \lim_{r \to 0} l(x)/r(x)$, and where here $r(x)$ is the radius of the disc $\{z = z_0, \rho \leq x\}$ and $l(x)$ its perimeter. This defect can be better computed with the help of the auxiliar function $q = \gamma - \sigma - \ln \rho$ as $\delta(z_0) = 2\pi - \pi e^{-q(0,0)}$. The function $q$ is found by line integration of,

$$\partial_\rho q = -\frac{\rho}{4}(\partial_\rho \sigma)^2 - (\partial_\rho \sigma)^2 + \frac{\rho}{4\eta^2}((\partial_\rho \omega)^2 - (\partial_\rho \omega)^2),$$

$$\partial_z q = \frac{\rho}{2}\left(\partial_z \sigma \partial_\rho \sigma + \frac{1}{\eta^2} \partial_z \omega \partial_\rho \omega\right),$$

(37)

and a simple inspection of these equations shows that $q$ is constant on each axis component. We will show below that if $q(0,L/2) = 0$ then $q(0,-L/2) = 0$. By periodicity $q$ would be 0 on $A_+$ and $A_-$. Adding $-\ln 2$ we obtain angle defect 0 everywhere. Now, the integral of the closed 1-form,

$$\left(\frac{\rho}{4}(\partial_\rho \sigma)^2 - (\partial_\rho \sigma)^2 + \frac{\rho}{4\eta^2}((\partial_\rho \omega)^2 - (\partial_\rho \omega)^2)\right) d\rho + \frac{\rho}{2}\left(\partial_z \sigma \partial_\rho \sigma + \frac{1}{\eta^2} \partial_z \omega \partial_\rho \omega\right) dz$$

(38)

on the segment from $(1,L/2)$ to $(1,-L/2)$ is zero by the symmetries of $\sigma$ and $\omega$. Also by these symmetries, the integral on the segments $[0,1] \times \{L/2\}$ and $[0,1] \times \{-L/2\}$, oriented in the same direction are equal. Therefore the integral on the three consecutive intervals is zero, so $0 = q(L/2) = q(-L/2)$.

2.5. Initial data for the harmonic map heat flow

We need some initial condition at $\tau = 0$ of the heat flow, which we will call seed. In particular, it is desirable that the seed contains the prescribed singular behavior of the solutions at the horizons, so that we can define a well-posed numerical problem without singularities. We decompose $\sigma$ and $\omega$ following (13); we split them as a sum of known solutions to the non-periodic problem plus a perturbation $\bar{\sigma}, \bar{\omega}$. In the case a single horizon per period, the sum of known solutions can be computed in the same fashion as the function $\sigma_{MKN}$ was defined in [5, 6]: as a series of superimposed solutions. Let $\sigma_0(\rho, z)$ and $\omega_0(\rho, z)$ be the solutions to the asymptotically flat Kerr black hole with momentum $J$ and horizon $H_0$, and define

$$\sigma = \sigma_0 + \sigma_r + \bar{\sigma},$$

$$\omega = \omega_0 + \omega_r + \bar{\omega},$$

(39)

(40)
where,
\[
\sigma_r(\rho, z) = C + \sum_{n=1}^{\infty} \left( \sigma_0(\rho, z - nL, J) + \sigma_0(\rho, z + nL, J) - \frac{4M}{nL} \right),
\]
(41)
\[
\omega_r(\rho, z) = \sum_{n=1}^{\infty} \left( \omega_0(\rho, z - nL, J) + \omega_0(\rho, z + nL, J) \right),
\]
(42)
and where \(C\) is a constant such that \(\sigma_r |_{\partial H} = 0\), that is, its value at the poles is zero. Here the constants \(4M/nL\) are needed for the series to be convergent (since asymptotically, each term goes as \(-2M/\sqrt{(z^2 - nL^2)^2 + \rho^2}\) and therefore we need to cancel out this divergent term, as in [6]). In our actual numerical calculations we use a cut-off value \(N_L\) for \(n\), which we take \(N_L \geq 1\), which can be thought of as the number of ‘domains’ we stack on both the top and below the central domain.

We expect \((\bar{\omega}, \bar{\sigma})\) to be regular throughout the evolution. By inserting the decomposition (39) into (18) and (19), and using the fact that \((\sigma_0, \omega_0)\) is a solution, we obtain the evolution equations for \(\bar{\sigma}\) and \(\bar{\omega}\),

\[
\partial_t \bar{\sigma} = \Delta \bar{\sigma} + \Delta \sigma_r + \frac{e^{-2n|\nabla \omega_0|^2}}{\rho^2} \left( e^{-2(\sigma + \sigma_r)} - 1 \right)
+ \frac{e^{-2(\sigma + \sigma_r)}}{\rho^4} \left( |\nabla \omega|^2 + |\nabla \bar{\omega}|^2 + 2 \left( \partial_i \omega_i \partial^i \omega_0 \right) \right)
\]
(43)
\[
\partial_t \bar{\omega} = \Delta \bar{\omega} + \Delta \omega_r - \frac{4}{\rho} \left( \partial_i \omega_i + \partial_r \bar{\omega} \right)
- 2 \left( \partial_i \omega_0 \partial^i \sigma_r \right)
+ \partial_i \omega_i \partial^i \sigma + \partial_i \omega_i \partial^i \sigma_0 + \partial_i \omega_i \partial^i \sigma + \partial_i \omega_i \partial^i \bar{\sigma}
+ \partial_i \omega_i \partial^i \sigma_0 + \partial_i \omega_i \partial^i \sigma + \partial_i \omega_i \partial^i \sigma \right).
\]
(44)
where Einstein summation notation was used for the index \(i\) that runs from 1 to 2 (i.e. representing \(\rho\) and \(z\)). Equations (43) and (44) are the equations that we solve numerically, with the following boundary conditions, which can be read off from the conditions for the total functions \((\omega, \sigma)\),

\[
\bar{\omega}(\rho_{MAX}) = 0, \quad \partial_r \bar{\omega} |_{\partial H} = 0, \quad \bar{\omega} |_{A} = 0,
\]
(45)
since \(\omega_0 + \omega_r\) already satisfies asymptotically the linear behavior for \(\omega\), and

\[
\partial_r \bar{\sigma} |_{(A \cup H) \setminus \partial H} = 0, \quad \bar{\sigma} |_{\partial H} = 0,
\]
(46)
with the asymptotic condition,

\[
\partial_r \sigma |_{\rho_{MAX}} = - \frac{4M(\rho)}{L_{\rho_{MAX}}} \left( \partial_\rho (\sigma_0 + \sigma_r) \right) |_{\rho_{MAX}}
\]
(47)
The last term is not \(z\)-independent, since the series are truncated, and therefore we take its average on \(z\), denoting by \(\bar{X}\) the average along \(x\) coordinate of the variable \(X\). We will call \(\beta\) the dynamical quantity given by the right hand side of equation (47).
3. Numerical Implementation

We use a grid adapted to a finite computational region where the Weyl–Papapetrou coordinates range as follows,

\[(\rho, z) \in [0, \rho_{\text{MAX}}] \times [-L/2, L/2].\]

We use \(N_\rho + 1\)-point Chebyshev grid to discretize \(\rho\) and a uniform grid of \(N_z\) points, which are semi-displaced with respect to the boundaries \(z = \pm L/2\), to discretize \(z\). Along this section we use sub indices to identify grid points and grid values:

\[
\rho_i = \frac{1}{2} \rho_{\text{MAX}} \left( 1 - \cos \left( \frac{\pi i}{N_\rho} \right) \right) \quad i = 0, \ldots, N_\rho,
\]

\[
z_j = -\frac{L}{2} + \frac{L}{N_z} \left( j + \frac{1}{2} \right) \quad j = 0, \ldots, N_z - 1.
\]  

(48)

Observe that the symmetry axis, \(\{\rho = 0\}\), is included in the grid while the axis \(\{z = 0\}\) is not. Also, the \(z\)-grid is defined in such a way that the poles \(\mathcal{H} \cap \mathcal{A}\) are at the middle of two consecutive grid points. Derivatives with respect to \(\rho\) are approximated by the derivatives of the polynomial interpolation on the Chebyshev grid, while derivatives with respect to \(z\) are approximated as the derivatives of the standard Fourier interpolation on the uniform grid. This is, we use pseudo spectral and spectral collocation methods in \(\rho\) and \(z\) respectively.

We wrote two independent versions of Python codes to carry out the numerical computations to cross check the results. The implementation of the spectral method is through the standard \texttt{rfft} routines provided by \texttt{NumPy}, while for the pseudo spectral derivatives and integrals we tried various matrix implementations \cite{15–17} that produce no significant differences between them. The values of the analytical Kerr solution and their derivatives were obtained as Python codification using \texttt{SymPy} and \texttt{Maple}.

As explained in section 2.4, every solutions to our problem is obtained as the final state of the parabolic flow (43) and (44). The singularities of \(\sigma\) at \(\mathcal{H}\), and the starting point for the evolution of the parabolic flow, are handled via the splitting of \(\sigma\) and \(\omega\) as in equation (39) with the introduction of the seed \(\sigma_0 + \sigma_r\) and \(\omega_0 + \omega_r\). This is, the initial value for \(\sigma\) and \(\omega\) is always taken as zero, and the boundary conditions are given by equations (45)–(47).

A particular numerical problem is defined once the values of physical parameters: \(J, A, L\), the value of \(N_d\) which amounts the number of periods we stack to build the seed, and the values of the numerical parameters: \(\rho_{\text{MAX}}, N_\rho\) and \(N_z\), are chosen. As explained before, we choose the values of \(L\) judiciously so that the poles fall at the middle of two consecutive gridpoints at \(\rho = 0\). In section 4 we show the precise values used in our runs.

The time evolution for the parabolic flow is implemented with Euler’s method. One could argue that Euler’s method is a low precision method, but it is explicit, simple to implement, and more important we are only seeking for the final stationary solutions of the equations where all time variations go to zero together with the associated truncation error. We are not interested in the precision along the time evolution. Near the stationary state the truncation error is dominated by that of the space discretization. Of course the time step is subordinated to the grid sizes so as to obtain a numerically stable scheme at all times. The closer to the symmetry axis, the stricter the Courant-Friedrichs-Lewy (CFL) condition becomes, since the Chebyshev mesh size is smaller and the derivatives of various functions involved are larger. In most of our runs a time step \(\delta \tau = 10^{-4}\) turned out to be suitable.
### 3.1. The boundary conditions

Let us denote, as usual, the grid functions as

\[ \bar{\sigma}_{i,j}(\tau) = \bar{\sigma}(\rho_i, z_j, \tau), \quad \text{and} \quad \bar{\omega}_{i,j}(\tau) = \bar{\omega}(\rho_i, z_j, \tau), \]

\[ i = 0, \ldots, N_\rho, \quad j = 0, \ldots, N_z - 1, \]

and the pseudo spectral derivative matrix associated to the Chebyshev \( \rho \)-grid as

\[ D_{i,k}, \quad i, k = 0, \ldots, N_\rho. \]

Given the grid functions at time \( \tau \), a single Euler step determines both grid functions at time \( \tau + \delta \tau \) in all gridpoints with \( 1 \leq i \leq N_\rho - 1 \). Periodicity in \( z \) is an intrinsic part of the implementation. The values at \( i = 0 \) (axis and horizon) and \( i = N_\rho \) (outer boundary) at time \( \tau + \delta \tau \) are determined by the boundary conditions as follows.

1. \( \bar{\omega}_{N_\rho,j}(\tau + \delta \tau) = 0 \) for all \( j \) (homogenous Dirichlet condition for \( \bar{\omega} \) at \( \rho_{\text{MAX}} \)).
2. \( \bar{\omega}_{0,j}(\tau + \delta \tau) = 0 \) for all \( j \) such that \( z_j \in A \) (homogenous Dirichlet condition for \( \bar{\omega} \) at the axis).
3. Solve \( \sum_{k=1}^{N_\rho} D_{0,k} \bar{\omega}_{k,j}(\tau + \delta \tau) = 0 \) for \( \bar{\omega}_{0,j}(\tau + \delta \tau) \), for those \( j \) such that \( z_j \in H \) (homogenous Neumann condition for \( \bar{\omega} \) on the horizon).
4. \( \bar{\sigma}_{0,j}(\tau + \delta \tau) \) and \( \bar{\sigma}_{N_\rho,j} \) are determined by solving a \( 2 \times 2 \) system that implements the homogenous Neumann condition for \( \bar{\sigma} \) at the axis and the dynamical inhomogeneous Neumann condition for \( \bar{\sigma} \) at the outer boundary \( \rho = \rho_{\text{MAX}} \). For each value of \( j \) the system is

\[
\begin{pmatrix}
D_{0,0} & D_{0,N_\rho} \\
D_{N_\rho,0} & D_{N_\rho,N_\rho}
\end{pmatrix}
\begin{pmatrix}
\bar{\sigma}_{0,j}(\tau + \delta \tau) \\
\bar{\sigma}_{N_\rho,j}(\tau + \delta \tau)
\end{pmatrix}
= \begin{pmatrix} c \\ d \end{pmatrix}
\]

where the inhomogeneity is

\[
c = - \sum_{i=1}^{N_\rho-1} D_{0,i} \bar{\sigma}_{i,j}(\tau + \delta \tau),
\]

\[
d = - \sum_{i=1}^{N_\rho-1} D_{N_\rho,i} \bar{\sigma}_{i,j}(\tau + \delta \tau) + \beta,
\]

where \( \beta \) is the dynamical value given by the right hand side of equation (47).

5. Finally, to keep the homogeneous Dirichlet condition for \( \bar{\sigma} \) at the poles we compute the (very small) violation \( \delta_{\text{pole}} \) as the average of the \( \bar{\sigma}(\tau + \delta \tau) \) values at the two nearest neighbor grid points on \( \rho = 0 \) to any of the poles, and subtract this value from \( \bar{\sigma}(\tau + \delta \tau) \) on the whole grid.

The evolution of the parabolic flow approaches the stationary state only in an asymptotic manner. To measure the distance to stationarity, we compute the \( L^2 \) norm of the right hand side equations (43) and (44). This is an absolute measure of the "error". Then, we compute the relative errors

\[
\varepsilon_\sigma = \frac{\| \text{rhs}(\bar{\sigma}) \|}{\| \bar{\sigma} \|}, \quad \varepsilon_\omega = \frac{\| \text{rhs}(\bar{\omega}) \|}{\| \bar{\omega} \|},
\]

\[
\varepsilon_\sigma = \frac{\| \text{rhs}(\bar{\sigma}) \|}{\| \sigma \|}, \quad \text{and} \quad \varepsilon_\omega = \frac{\| \text{rhs}(\bar{\omega}) \|}{\| \omega \|}.
\]
Finally, the quadratures for the various functions that need to be done are implemented by standard spectral \texttt{rfft} routines along \( z \) direction and Clenshaw–Curtis integration along \( \rho \) direction.

4. Results

In this section we present two series of simulations computed with our code for two different values of angular momentum: \( J = 1/4 \) and \( J = 1/2 \). For the series with \( J = 1/4 \) we present and analyze in detail several aspects of the solutions obtained. For the series with \( J = 1/2 \), not to be redundant, we simply show a Table with some relevant quantities computed. We choose to compute solutions whose horizon’s area is \( A = 16\pi \). Thus, recalling that \( \kappa \) is the Kerr temperature given \( A \) and \( J \), the horizon’s semi-length becomes

\[
m = \frac{4 - J^2}{2\sqrt{4 + J^2}}.
\]

In the cases we study, \( m = 0.9095 \) for \( J = 1/2 \) and \( m = 0.9768 \) for \( J = 1/4 \). The various solutions in each series correspond to different values of the parameter \( L \), that we choose as

\[
L = \frac{N_z}{N_h} m,
\]

where \( N_h \) is the number of \( z \)-gridpoints inside each horizon. In all cases the computational domain has \( \rho_{\text{MAX}} = 40 \), and the computing grid is defined with \( N_\rho = 79 \) and \( N_z = 100 \) (see (48)).

4.1. First series: \( J = 1/4 \)

4.1.1. Convergence of the parabolic flow and regularity of the solution. The convergence of the parabolic flow to stationary state turns out to be slow. For all the solutions in this series, we stopped the flow after computing \( 8 \times 10^6 \) steps with \( \delta \tau = 10^{-4} \), where we found that the relative errors \( \bar{\sigma} \) and \( \bar{\omega} \) are comparably small. Typical plots of \( \bar{\sigma} \) and \( \bar{\omega} \), in logarithmic scale, along the evolution of the flow are shown in figure 3.

We now check that \( \rho_{\text{MAX}} = 40 \) defines a computational domain which is large enough so as not to alter significantly the asymptotic behavior of the solutions (in the spirit explained at the end of section 2.3). To this end we compute some solutions using \( \rho_{\text{MAX}} = 60 \) and compare the results. The most sensitive function to compare is \( \bar{\sigma} \). In figure 4 the plots of \( \bar{\sigma}(\rho, z = 0) \) just at the middle of the horizon, for the case \( L = 6.9770 \) are shown. It is clear the difference between the solutions is not significant. Also, the regularity of the solution at this final time is checked by computing \( \Delta q = q(0, L/2) - q(0, -L/2) \), by path integrating around the horizon from just above the upper pole to just below the lower pole. As it can be seen, the violation of regularity turns out to be extremely small. The final values of the relative errors for all the runs in this series, together with the values of \( \Delta q \) are shown in table 1. We choose the run with \( L = 4.8839 \) (corresponding to \( N_h = 40 \)) as an example to show the plots of the solution and relevant functions, figure 5 shows the plots of \( \bar{\sigma}, \bar{\omega}, \sigma \) and \( \omega \).

We also want to check the convergence of the numerical solution with respect to the smallness of the discretization parameters (mesh sizes and time step). To remind one of the definitions, let \( h \) denote the smallness parameter of the numerical method representing consistently a differential problem. Formally, the limit \( h \to 0 \), applied to the discrete method gives the differential equation plus boundary conditions. Let \( v^h(x, \tau) \) denote a solution to the numerical
Figure 3. Convergence of the parabolic flow to stationary state as a function of time steps for the solution with $N_h = 40$ in table 1.

Figure 4. Plot of $\bar{\sigma}(z = 0)$ when computed using two values of $\rho_{\text{MAX}}$ for the case $L = 6.9770$.

Table 1. Relative error after $8 \times 10^6$ time steps and violation of regularity, $\Delta q$, for the solutions in the series.

| $N_h$ | $L$         | $\varepsilon_{\theta}$ | $\varepsilon_\omega$ | $\varepsilon_\pi$ | $\varepsilon_\sigma$ | $\Delta q$    |
|------|-------------|------------------------|-----------------------|-------------------|---------------------|----------------|
| 22   | 8.8798      | $2.19 \times 10^{-6}$  | $1.40 \times 10^{-6}$ | $7.94 \times 10^{-8}$ | $3.22 \times 10^{-9}$ | $-6.66 \times 10^{-15}$ |
| 28   | 6.9770      | $2.21 \times 10^{-5}$  | $4.83 \times 10^{-6}$ | $1.31 \times 10^{-7}$ | $1.29 \times 10^{-8}$ | $-4.17 \times 10^{-14}$ |
| 34   | 5.7457      | $2.46 \times 10^{-5}$  | $1.06 \times 10^{-5}$ | $2.34 \times 10^{-7}$ | $3.25 \times 10^{-8}$ | $-5.10 \times 10^{-14}$ |
| 40   | 4.8839      | $2.01 \times 10^{-5}$  | $1.78 \times 10^{-5}$ | $3.04 \times 10^{-7}$ | $6.19 \times 10^{-8}$ | $-1.50 \times 10^{-13}$ |
| 46   | 4.2468      | $1.59 \times 10^{-5}$  | $2.62 \times 10^{-5}$ | $3.84 \times 10^{-7}$ | $1.03 \times 10^{-7}$ | $2.49 \times 10^{-14}$ |
| 50   | 3.9071      | $1.47 \times 10^{-5}$  | $3.27 \times 10^{-5}$ | $4.89 \times 10^{-7}$ | $1.40 \times 10^{-7}$ | $4.39 \times 10^{-14}$ |
| 54   | 3.7568      | $1.47 \times 10^{-5}$  | $3.63 \times 10^{-5}$ | $5.77 \times 10^{-7}$ | $1.62 \times 10^{-7}$ | $4.59 \times 10^{-14}$ |
| 58   | 3.6177      | $1.52 \times 10^{-5}$  | $4.01 \times 10^{-5}$ | $7.07 \times 10^{-7}$ | $1.87 \times 10^{-7}$ | $2.07 \times 10^{-13}$ |
| 60   | 3.4885      | $1.63 \times 10^{-5}$  | $4.42 \times 10^{-5}$ | $9.04 \times 10^{-7}$ | $2.16 \times 10^{-7}$ | $8.26 \times 10^{-14}$ |
|      | 3.3682      | $1.83 \times 10^{-5}$  | $4.87 \times 10^{-5}$ | $1.22 \times 10^{-6}$ | $2.48 \times 10^{-7}$ | $1.32 \times 10^{-13}$ |
|      | 3.2559      | $2.17 \times 10^{-5}$  | $5.36 \times 10^{-5}$ | $1.75 \times 10^{-6}$ | $2.85 \times 10^{-7}$ | $2.35 \times 10^{-13}$ |
method and $u(x, \tau)$ the solution of the consistent differential equation plus boundary conditions. It is said that $v^h(x, \tau)$ converges to $u(x, \tau)$ when $h \to 0$, if $v^h(x, \tau) = u(x, \tau) + \mathcal{O}(h^k)$ for all $(x, \tau) \in D$ and $k > 0$. The constant $k$ is called the order of convergence of the method. In a nonlinear problem like ours, we can compute $k$ by evaluating the quotient

$$Q = \frac{||v^h - v^{h/2}||}{||v^{h/2} - v^{h/4}||}$$

where $v^h$, $v^{h/2}$ and $v^{h/4}$ are three numerical approximations to the same differential problem (same parameters, initial data and boundary data) computed on different grids or with different time steps. The norm $|| \cdot ||$ is, for example, the discrete version of the $L^1$ norm on $D$. This quotient should be, for $h$ small enough, close to $2^k$.

We first test the convergence of the time discretization (Euler method) by fixing the space discretization (i.e. by fixing $N_\rho$ and $N_z$). To do this we take solution already close to the stationary state with parameters: $J = 1/4$, $L = 6.9770$, $\rho_{\text{MAX}} = 40$, $N_\rho = 80$, $N_z = 100$ and $\tau = 500$, and compute from there three solution with three time steps $\delta \tau = 10^{-5}$, $\delta \tau = 2 \times 10^{-5}$, and

Figure 5. Plots of the solution corresponding to $L = 4.8839$ of table 1. From left to right, from top to bottom: $\bar{\sigma}$, $\bar{\omega}$, $\sigma$, $\omega$. 
\( \delta \tau = 4 \times 10^{-5} \) up to a time \( \tau_1 = 510 \). We then compute the quotient \( Q \) as in (50). We obtain linear convergence, as expected for Euler’s method. For example, for \( \bar{\sigma} \), we get

\[
\frac{\| \bar{\sigma}^{4 \delta \tau} - \bar{\sigma}^{2 \delta \tau} \|}{\| \bar{\sigma}^{2 \delta \tau} - \bar{\sigma}^{\delta \tau} \|} = 2.00068
\]

indicating linear convergence \((k = 1)\).

We now study the convergence of the full numerical scheme we use to compute the parabolic flow. There are three smallness parameters: The time step \( \delta \tau \) of Euler’s method; the mesh size of the uniform grid for the \( \rho \) coordinate which is proportional to \( 1/N_\rho \); and the mesh size of the non uniform grid for the \( \rho \) coordinate (see (48)) whose smaller values, close to the boundaries \( \rho = 0 \) and \( \rho = \rho_{\text{MAX}} \), are proportional to \( 1/(N_\rho)^2 \).

Numerical stability for Euler’s method applied to a parabolic problem like ours requires that

\[
\delta \tau \leq C \min \left\{ \left( \frac{1}{N_\rho} \right)^2, \left( \frac{1}{N_\rho} \right)^2 \right\},
\]

where \( C \) is some positive constant depending on the equation and the exact solution we are approximating. The limit \( \delta \tau \to 0, N_\rho \to \infty \) and \( N_\rho \to \infty \) that recovers the differential problem needs to be taken always satisfying (52).

To check the order of convergence of our method we compute the solution on three different grids. Starting with a solution, on a coarse grid, already close to the stationary state with parameters \( J = 1/4, L = 6.9770, \rho_{\text{MAX}} = 40, N_\rho = 40, N_\rho = 50 \) and \( \tau = 500 \), we refine this solution to obtain initial data on two finer grids, with \( N_\rho = 80, N_\rho = 100 \) and with \( N_\rho = 160, N_\rho = 200 \). To do this refinement we use quadratic interpolation. Then, we evolve these initial data with different time steps chosen to fulfill the relation (52) until time \( \tau_1 = 510 \) (see table 2).

We now compute the quotient (50). To do this we need to subtract the solutions on the same grid. We use quadratic interpolation to restrict the solution on grid 2, to the grid 1, and the solution on grid 3 to grid 2. Then we use the grid 1 and grid 2 to compute the numerator and denominator, respectively, in (50), obtaining

\[
\frac{\| \bar{\omega}^1 - \bar{\omega}^2 \|}{\| \bar{\omega}^2 - \bar{\omega}^3 \|} = 4.881215, \quad \frac{\| \bar{\sigma}^1 - \bar{\sigma}^2 \|}{\| \bar{\sigma}^2 - \bar{\sigma}^3 \|} = 1.742365.
\]

These values show convergence at different rates. For \( \bar{\omega} \) we obtain higher than quadratic convergence \( k = 2.29 \) while for \( \bar{\sigma} \) we obtain somewhat sub linear convergence \( (k = 0.80) \). We believe that the difference in the convergence rates is because of the boundary conditions. For \( \bar{\omega} \) the boundary conditions (homogeneous Dirichlet and homogeneous Neuman) are fixed along the evolution. For \( \bar{\sigma} \), on the other hand, the boundary condition at the outer boundary \( (\rho = \rho_{\text{MAX}}) \) is a dynamical one; the boundary data depends on the solution in the bulk. Thus, the evolution changes the boundary condition at every time step, making the convergence rate for \( \bar{\sigma} \) low.
Table 3. Relevant quantities computed for the solutions in the series with $J = 1/4$.

| $L$     | $M$ (mass) | Angular velocity | $\alpha$ (from $M$) | $\alpha$ (from $V$) |
|---------|------------|------------------|----------------------|----------------------|
| 8.8798  | 1.0095     | 6.5753$\times 10^{-2}$ | 4.5476$\times 10^{-1}$ | 4.5477$\times 10^{-1}$ |
| 6.9770  | 1.0119     | 7.0513$\times 10^{-2}$ | 5.8014$\times 10^{-1}$ | 5.8017$\times 10^{-1}$ |
| 5.7457  | 1.0164     | 7.9507$\times 10^{-2}$ | 7.0758$\times 10^{-1}$ | 7.0768$\times 10^{-1}$ |
| 4.8839  | 1.0250     | 9.6689$\times 10^{-2}$ | 8.3947$\times 10^{-1}$ | 8.3977$\times 10^{-1}$ |
| 4.2468  | 1.0422     | 1.3130$\times 10^{-1}$ | 9.8167$\times 10^{-1}$ | 9.8264$\times 10^{-1}$ |
| 3.9071  | 1.0640     | 1.7495$\times 10^{-1}$ | 1.0893               | 1.0916               |
| 3.7568  | 1.0807     | 2.0831$\times 10^{-1}$ | 1.1506               | 1.1541               |
| 3.6177  | 1.1036     | 2.5413$\times 10^{-1}$ | 1.2202               | 1.2257               |
| 3.4885  | 1.1358     | 3.1881$\times 10^{-1}$ | 1.3024               | 1.3113               |
| 3.3682  | 1.1831     | 4.1348$\times 10^{-1}$ | 1.4050               | 1.4201               |
| 3.2559  | 1.2557     | 5.5907$\times 10^{-1}$ | 1.5426               | 1.5698               |

4.1.2. **Mass, angular velocity and Kasner parameter.** At the final time we compute, for every solution, several relevant quantities. The mass $M$ is computed as the integral $M(\rho_{\text{MAX}})$. The horizon’s angular velocity is obtained as the averaged value of $\Omega(\rho = 0)$ in one of the horizons.$^4$ We also compute the Kasner exponent in two different ways: the first value is obtained from the mass, as $4M/L$, while the second value is obtained from the asymptotic behavior of the function $V$; more precisely, as the slope of a linear regression of $\ln(V)$ as function of $\ln(\rho)$ in the asymptotic region of the computational domain, see (12). We arbitrarily define the asymptotic region of the domain as the portion of the domain, adjacent to $\rho_{\text{MAX}}$, corresponding to 30% of $\rho$ grid points. The two values obtained for the Kasner exponent are in very good agreement. All these quantities, for the solutions in this series are shown in table 3.

Figure 6 shows the plots of relevant metric functions obtained for the solution corresponding to $N_h = 40$.

4.1.3. **The Smarr identity.** We found that a very sensitive test for checking the computations is the validity of the Smarr identity, this is, the constancy of $M(\rho)$ (see equation (28)) as a function of $\rho$. In this sense the plot of $M(\rho)$ became crucial to test to correctness of the outer boundary condition for $\sigma$. In figure 7 we show plots that compare $M(\rho)$ computed for the seed (initial data for the flow) and $M(\rho)$ at final time for the six numerically computed solutions with larger $L$ in the series.

4.1.4. **Best fitting asymptotic models.** We want to check which of the asymptotic candidate solutions fits better the numerically computed solution. To this end, we take the average on the $z$ axis for the function $\eta$ as to get a $z$ independent function $\bar{\eta}(\rho)$. We then compute the best fitting model $\eta$-function for the six possibilities given by models (I+), (I−), (II+), (II−), (III+) and (III−) (see equations (8)–(10)). To do this we minimize the deviation of the model $\eta(a, b, \rho)$ from $\bar{\eta}$ by varying the parameter $a$, or taking $a = 0$ for the models (II), and choosing $b$

---

$^4$ The computation of $\Omega$ is singular at the horizon; we compute $\Omega$ strictly in the interior and get the value of $\Omega(\rho = 0)$ by simple linear extrapolation from the first and second internal gridpoints.
in such a way that \( \eta \) functions are coincident at the outer boundary. We measure the mentioned deviation by computing the integrated square difference in the asymptotic region:

\[
\Delta \eta = \int_{\text{asympt. region}} \left( \hat{\eta}(\rho) - \eta(a, b, \rho) \right)^2 d\rho.
\]

The results of fitting the eleven solutions in table 1 are shown in table 4. It is very interesting to see how well the \( z \)-averaged \( \eta \) function of our numerically computed solution fits, in the whole \( \rho \) range, one of the model \( \eta \) functions. As expected, for large values of \( L \) the best fitting model is (III\( ^+ \)). Then, for decreasing values of \( L \) the best fitting model becomes (I\( ^+ \)), then to (I\( ^- \)) and finally (III\( ^- \)). Figure 8 show six examples taken from table 4.

It is interesting to observe here that, on theoretical grounds, there must indeed be a critical \( L \) below which no solution solutions extending to infinity indeed exist. Assume that we have an actual solution extending to infinity, with any of the two possible asymptotic models mentioned in section 2.2. Recall that the Smarr formula is,
Figure 7. Plots of $M(\rho)$ for the seed and for the numerically computed solution compared for six of the solutions in the series. From left to right from top to bottom the plots correspond to the cases with $N_h = 22, 28, 34, 40, 46, 50$ in table 1.

\[ M = \frac{1}{4\pi} \kappa A + 2\Omega H J, \]  

(53)

where $M$ and $J$ are the Komar mass and angular momentum, respectively, $\kappa$ is the horizon temperature, $A$ is the horizon area and $\Omega H$ is the horizon angular velocity. For the asymptotic models (III\(+\)) or (II\(+\)) with $0 \leq a < 1$, we obtain,

\[ (1 - a)L/4 = \frac{1}{4\pi} \kappa A + 2\Omega H J. \]  

(54)

Now, the positivity of $a$ allows to obtain the lower bound for $L$. Indeed, one can easily show that $\Omega H J > 0$ (integrate (18) between $\rho = 0$ and $\infty$), forcing,

\[ 0 < 2\Omega H J \leq \frac{L}{4} - \frac{1}{4\pi} \kappa A = \frac{L}{4} - m, \]  

(55)

and therefore, $L > 4m$. This relation is easily verified in the numerical Tables. This is a rough yet remarkable lower bound for the critical $L$. 

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Table 4. Fit of the solutions of table 1 with the six possible models given by equations (8)–(10).

| L    | a     | b     | Δη   | (III+) | a   | b   | Δη   |
|------|-------|-------|------|--------|-----|-----|------|
| 8.8798 | 0.5451 | 2.6149 | 2.4×10⁻⁷ | 89.9297 | 5.2×10¹ | 0.0001 | 0.0090 | 5.2×10¹ |
| 6.9770 | 0.4194 | 2.2513 | 1.4×10⁻⁷ | 49.4932 | 1.6×10¹ | 0.0001 | 0.0049 | 1.6×10¹ |
| 5.7457 | 0.2906 | 1.8077 | 9.0×10⁻⁹ | 26.8568 | 3.2   | 0.0001 | 0.0027 | 3.2   |
| 4.8839 | 0.1500 | 1.1372 | 1.5×10⁻⁸ | 13.7707 | 2.3×10⁻¹ | 0.0001 | 0.0014 | 2.3×10⁻¹ |
| 4.2468 | 0.0001 | 0.0006 | 6.5×10⁻² | 6.0951 | 6.5×10⁻² | 0.1005 | 1.0170 | 6.0×10⁻⁷ |
| 3.9071 | 0.0001 | 0.0003 | 2.9×10⁻¹ | 2.8198 | 2.9×10⁻¹ | 0.1536 | 0.9808 | 4.3×10⁻² |
| 3.7568 | 0.0001 | 0.0002 | 4.0×10⁻¹ | 1.5649 | 4.0×10⁻¹ | 0.1903 | 0.8483 | 8.5×10⁻² |
| 3.6177 | 0.0001 | 0.0001 | 5.0×10⁻¹ | 0.5104 | 5.0×10⁻¹ | 0.2381 | 0.6756 | 1.3×10⁻¹ |
| 3.4885 | 0.0001 | 0.0000 | 5.9×10⁻¹ | −0.3766 | 5.9×10⁻¹ | 0.3019 | 0.4511 | 1.6×10⁻¹ |
| 3.3682 | 0.0001 | 0.0001 | 6.5×10⁻¹ | −1.1241 | 6.5×10⁻¹ | 0.3898 | 0.1107 | 1.9×10⁻¹ |
| 3.2559 | 0.0001 | 0.0002 | 7.1×10⁻¹ | −1.7558 | 7.1×10⁻¹ | 0.5172 | 0.3566 | 2.1×10⁻¹ |

| L    | a     | b     | Δη   | (I−)   | a   | b   | Δη   |
|------|-------|-------|------|--------|-----|-----|------|
| 8.8798 | 0.0106 | 1.4862 | 5.4×10¹ | 97.3074 | 5.6×10¹ | 0.0001 | 0.0097 | 5.6×10¹ |
| 6.9770 | 0.0188 | 1.6213 | 1.7×10¹ | 56.8710 | 1.9×10¹ | 0.0001 | 0.0057 | 1.9×10¹ |
| 5.7457 | 0.0327 | 1.6434 | 4.2   | 34.2346 | 5.1   | 0.0001 | 0.0034 | 5.1   |
| 4.8839 | 0.0572 | 1.7306 | 5.9×10⁻¹ | 21.1485 | 1.1   | 0.0001 | 0.0021 | 1.1   |
| 4.2468 | 0.1022 | 1.9351 | 3.6×10⁻³ | 13.4728 | 1.4×10⁻¹ | 0.0001 | 0.0013 | 1.4×10⁻¹ |
| 3.9071 | 0.1250 | 1.4114 | 8.5×10⁻⁹ | 10.1975 | 2.1×10⁻² | 0.0001 | 0.0010 | 2.1×10⁻² |
| 3.7568 | 0.1164 | 1.0874 | 1.6×10⁻⁹ | 8.9426 | 5.8×10⁻³ | 0.0001 | 0.0009 | 5.8×10⁻³ |
| 3.6177 | 0.0909 | 0.7270 | 1.1×10⁻⁹ | 7.8882 | 8.4×10⁻⁴ | 0.0001 | 0.0008 | 8.4×10⁻⁴ |
| 3.4885 | 0.0001 | 0.0007 | 2.3×10⁻⁷ | 7.0012 | 2.3×10⁻¹ | 0.0147 | 0.1029 | 1.4×10⁻⁹ |
| 3.3682 | 0.0001 | 0.0006 | 2.6×10⁻⁴ | 6.2537 | 2.6×10⁻⁴ | 0.1089 | 0.6775 | 2.2×10⁻⁹ |
| 3.2559 | 0.0001 | 0.0006 | 4.7×10⁻⁴ | 5.6220 | 4.7×10⁻⁴ | 0.1629 | 0.9108 | 3.3×10⁻⁹ |

4.15. Ergo-region merging. For large values of L, the ergo-region associated to the horizon does not touch the z = ±L/2 boundaries of the domain. The boundary of the ergo-region is thus topologically S². When the value of L decreases the ergo-region gets closer to the boundaries and at some point touches them. At a critical value of L the outer boundary of the ergo-region changes topology, becoming a torus T². The change however takes place for L below the critical value, so that such solutions do not extend to infinity. Ergo-regions changes has been studied in binary systems, so is not a surprise to see it in the periodic set up. This process is shown in figure 9.

4.2. Second series: J = 1/2

We include here, for the sake of comparison, results corresponding to a series of four solutions with a higher value of J and the same horizon’s area as the previous series. The three solutions with larger value of L are better fitted by the asymptotic model (III+) of (10), while the solution with $L = 4.5475$ is better fitted with the model (I+) of equation (8).

In Table 5 the relevant physical quantities of the solutions in this series are shown.
Figure 8. Plots of $\bar{\eta}$ together with the fitting models (III$+$), (I$+$), (I$+$) and (III$+$) for the six solutions in table 4 corresponding, from left to right, from top to bottom, to $L = 8.8798, 4.8839, 4.2468, 3.7568, 3.4885, 3.2559$. The $\bar{\eta}$ curve overlaps the best fitting model in all cases, while some of the bad fitting model curves overlap among themselves, that is why not all curves are visible in all plots.

Figure 9. The shaded regions are the ergospheres of the last six solutions of table 1. From left to right, from top to bottom the six plots correspond $L = 8.8798, 4.8839, 4.2468, 3.7568, 3.4885, 3.2559$. 
Table 5. Relevant quantities computed for solutions with $J = 1/2$.

| $N_h$ | $L$  | $M$ (mass) | Angular velocity | $\alpha$ (from $M$) | $\alpha$ (from $V$) |
|-------|------|------------|------------------|----------------------|---------------------|
| 22    | 8.2683 | 1.0391     | $1.3024 \times 10^{-1}$ | $5.0272 \times 10^{-1}$ | $5.0278 \times 10^{-1}$ |
| 28    | 6.4965 | 1.0518     | $1.4287 \times 10^{-1}$ | $6.4761 \times 10^{-1}$ | $6.4781 \times 10^{-1}$ |
| 32    | 5.3501 | 1.0775     | $1.6860 \times 10^{-1}$ | $8.0560 \times 10^{-1}$ | $8.0632 \times 10^{-1}$ |
| 40    | 4.5475 | 1.1340     | $2.2516 \times 10^{-1}$ | $9.9745 \times 10^{-1}$ | $1.0003$ |

5. Conclusions

In this work we present numerical solutions of the Einstein equations in a periodic set up, for stationary, coaxial co-rotating black holes, thus improving the knowledge of stationary solutions outside the standard asymptotically flat scenario and trivial topology. The solutions add angular momentum to the important MKN configurations of static, coaxial and periodic black holes, thus realizing a periodic analogue of the Kerr solution. The solution space contains a rich structure, sharing many qualitative and quantitative properties with the family of van Stockum infinite rotating cylinders.

Periodic Kerr solutions and van Stockum solutions both have asymptotic Lewis solution behavior. Furthermore, both families display a critical phenomenon: in the case of periodic Kerr we find that numerical solutions with a given value for the area $A$ and for the angular momentum $J$ of the horizons appear to exist only when the separation between consecutive horizons is larger than a certain critical value that depends only on $A$ and $|J|$. Below the mentioned critical value the rotational energy appears to be too big to sustain a global equilibrium and a singularity shows up at a finite distance from the bulk. For van Stockum cylinders instead, the asymptotic collapse is manifest when the angular momentum (per unit of length) reaches a critical value compared to the mass (per unit of length). In both families the collapse is through a transition in the Lewis models.

The Smarr identity proved to be a powerful tool to define an appropriate Neumann-like asymptotic boundary condition for the harmonic map heat flow. This new dynamical Neumann condition can be used in other periodic configurations, for example in periodic arrangements of counter-rotating black holes, a situation that we are presently investigating. In this case, the asymptotic model is not that of a rotating rod, but Kasner as in the MKN family. The solution space seems to have quite different properties. This in particular poses interesting questions regarding the collapse at large radius and the existence of solutions when the horizons get closer.

This work explores several features which are important in black hole physics in other contexts as well, such as string theory [18–23], supergravity [24, 25] and binary solutions [26–29].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
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Appendix. Simple deduction of Lewis’s models

Assume then that \( \partial_z \) is a Killing field so that the metric coefficient \( V, W, \eta \) and \( \gamma \) are \( z \)-independent. This implies that \( \sigma \) is independent of \( z \) and it is simple to see that \( \omega \) must be independent of \( \rho \). The equations for \( \sigma \) and \( \omega \) thus become,

\[
\partial^2_{\sigma} \sigma + \frac{1}{\rho} \partial_\rho \sigma = -\frac{e^{-2\sigma}(\partial_\rho \omega)^2}{\rho^4}, \quad \partial^2_{\omega} \omega = 0. \tag{56}
\]

Hence \( \omega = wz + w_1 \) with \( w \neq 0 \) and \( w_1 \) constants. Therefore,

\[
\partial^2_{\sigma} \sigma + \frac{1}{\rho} \partial_\rho \sigma = -\frac{e^{-2\sigma}w^2}{\rho^4}. \tag{57}
\]

After making the change of variables \( \bar{u} = -\sigma - \ln \rho \) the previous equation implies,

\[
(\rho \bar{u})' = w^2 e^{2\bar{u}} + C, \tag{58}
\]

where \( ' = \frac{d}{d\rho} \). The sign of the constant \( C \) will be relevant. Set \( C = e a^2 \), with \( a \geq 0 \) and \( \epsilon = -1, 1 \).

Now, letting \( x = e^{-\bar{u}} (= \eta / \rho) \) and defining the new variable \( \zeta = \ln \rho / \rho_0 \) (for some \( \rho_0 > 0 \)), we arrive at (we use \( \partial_\zeta x = \dot{x} \)), \( \dot{x}^2 = w^2 + e a^2 \rho_0^2 \), to obtain after \( \zeta \)-derivation\(^5\),

\[
\ddot{x} = e a^2 x. \tag{59}
\]

In figure 10 we present the phase space \((\dot{x}, x)\) of the solutions. There are three types of solutions:

1. For \( a = 0 \), we have,

\[
x = |w| (\pm \ln \rho + b), \tag{60}
\]

2. For \( a > 0 \) and \( \epsilon = -1 \), we have,

\[
x = \frac{|w|}{a} \sin(\pm a \ln \rho + b), \tag{61}
\]

\(^5\) \( \dot{x} = 0 \) or \( x = \epsilon w/a \) are not solutions of the original problem and were introduced in the deduction of (58).
3. For $a > 0$ and $\varepsilon = 1$, we have,

$$x = \frac{|w|}{a} \sinh(\pm a \ln \rho + b).$$

(62)

In the three cases $b$ is an arbitrary constant.

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