Fermion masses and mixings in a $U(1)_X$ model based on the $\Sigma(18)$ discrete symmetry

V. V. Vien$^{a,b}$, A. E. Cárcamo Hernández$^{c,d,e}$, and H. N. Long$^f$

$^a$Theoretical Particle Physics and Cosmology Research Group, Advanced Institute of Materials Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam
$^b$Faculty of Applied Sciences, Ton Duc Thang University, Ho Chi Minh City, Vietnam
$^c$Departamento de Física, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile
$^d$Centro Científico-Tecnológico de Valparaíso, Casilla 110-V, Valparaíso, Chile
$^e$Millennium Institute for Subatomic Physics at High-Energy Frontier (SAPHIR), Fernández Concha 700, Santiago, Chile
$^f$Institute of Physics, Vietnam Academy of Science and Technology, 10 Dao Tan, Ba Đình, Hanoi, Vietnam.

(Dated: September 10, 2021)

We have built a renormalizable $U(1)_X$ model with a $\Sigma(18) \times Z_4$ symmetry, whose spontaneous breaking yields the observed SM fermion masses and fermionic mixing parameters. The tiny masses of the light active neutrinos are produced by the type I seesaw mechanism mediated by very heavy right handed Majorana neutrinos. To the best of our knowledge, this model is the first implementation of the $\Sigma(18)$ flavor symmetry in a renormalizable $U(1)_X$ model. Our model allows a successful fit for the SM fermion masses, fermionic mixing angles and CP phases for both quark and lepton sectors. The obtained values for the physical observables of both quark and lepton sectors are in accordance with the experimental data. We obtain an effective neutrino mass parameter of $\langle m_{ee} \rangle = 1.51 \times 10^{-3} \text{eV}$ for normal ordering and $\langle m_{ee} \rangle = 4.88 \times 10^{-2} \text{eV}$ for inverted ordering which are well consistent with the recent experimental limits on neutrinoless double beta decay.

PACS numbers: 12.15.Ff; 12.60.Cn; 12.60.Fr; 14.60.Pq; 14.60.St.
I. INTRODUCTION

In recent years neutrino oscillation experiments have confirmed that the leptonic mixing angles and neutrino mass squared differences are measured with high precision which require us to extend the standard model (SM) to successfully explain the current pattern of lepton masses and mixing angles. Among the possible extensions of the SM, the versions with an extra $U(1)_X$ gauge symmetry are promising scenarios since the simplest possibility is to introduce three right-handed neutrinos that we need to incorporate the neutrino masses in the SM. In this type of model, many phenomena including neutrino masses, dark matter, the muon anomalous magnetic moment, inflation, leptogenesis, gravitational wave radiation are explained, however, the most minimal versions of $U(1)_X$ models do not include a description of SM fermion masses and mixings.

In order to explain the pattern on fermion masses and mixings, many extensions of the SM have been proposed with the inclusion of non-Abelian discrete groups, which have brought many outstanding advantages, see for instance, $S_3$, $T'$, $D_4$, $Q_6$, $A_4$, $Q_8$, etc. However, there are substantial differences between our present work and others since in most of previous works the lepton and/or quark masses and mixings are generated by the texture zero mass matrices, via non-renormalizable terms; at loop levels; and by combining with other gauge symmetries and/or supplementing other discrete symmetries. The $U(1)_{B-L}$ extension of the SM based on $S_3$, $D_4$ and $Q_6$ has been studied in Refs. in which the fermion masses and mixings are obtained at the first order of perturbation theory.

In this work, we propose a $U(1)_X$ renormalizable theory based on the $\Sigma(18)$ flavor symmetry, supplemented by the $Z_4$ discrete group capable of reproducing the SM fermion masses and mixings at tree-level. We use the $\Sigma(18)$ discrete group, since it is the simplest non-trivial group of the type $\Sigma(2N^2)$ with $N = 3$ which is isomorphic to $(Z_3 \times Z_3') \rtimes Z_2$. The $\Sigma(18)$ discrete group has 18 elements which are divided into nine conjugacy classes and has nine irreducible representations: the six singlets $1_{+0}$, $1_{+1}$, $1_{+2}$, $1_{-0}$, $1_{-1}$, $1_{-2}$ and the three doublets $2_{10}$, $2_{20}$ and $2_{21}$. Mathematical properties of the $\Sigma(18)$ discrete group are discussed in detail in Ref. 63. However, for convention,
we present briefly the tensor products of $\Sigma(18)$ in Appendix A. The reason for adding the auxiliary symmetry $U(1)_X$ was introduced in Ref. [94] in another different multiHiggs model based on the $A_4$ discrete symmetry where the global $U(1)_X$ symmetry is softly broken in the scalar potential in order to prevent the appearance of a Goldstone boson; thus, we do not further discuss on this issue here. Let us note that the $\Sigma(18)$ symmetry has not been considered before in this type of models and to the best of our knowledge the model proposed in this work is the first implementation of the $\Sigma(18)$ flavor symmetry in a renormalizable $U(1)_X$ model.

The layout of the remainder of the paper is as follows. In Section II we describe our proposed SM extension by adding the $U(1)_X, \Sigma(18)$ and $Z_4$ symmetries and considering an extended scalar sector and right handed Majorana neutrinos. In Section III we describe the implications of our model in lepton masses and mixings. Section IV deals with quark masses and mixings. The implications of our model in $K - \bar{K}$ and $B - \bar{B}$ mixings are discussed in Section V. The consequences of our model in charged lepton flavor violation are analyzed in section VI. We conclude in Section VII. A brief description of the Clebsch Gordan coefficients for the $\Sigma(18)$ group is presented in Appendix A.

II. THE MODEL

The electroweak gauge group of the SM is supplemented by a $\Sigma(18) \times Z_4$ discrete symmetry and a global symmetry $U(1)_X$ where $\psi_{iL}, l_{iR}$ ($i = 1, 2, 3$) and $\varphi, \varphi'$ carry $X = 1$ while all other fields have $X = 0$. In addition to the SM model particle content, three right-handed neutrinos ($\nu_{1R}, \nu_{2R}$), one $SU(2)_L$ doublet $\phi$ with $X = 0$ are assigned as $2_{10}$, two $SU(2)_L$ doublets $\varphi, \varphi'$ with $X = 1$, respectively, put in $1_{-0}$ and $2_{20}$ under $\Sigma(18)$ and two $SU(2)_L$ singlets $\chi, \rho$ with $X = 0$ respectively put in $2_{10}$ and $1_{+1}$ under $\Sigma(18)$ are introduced. The particle content of the model are summarized in Tables I and II.

---

1 In this model, fermion masses and mixing angles are generated from renormalizable Yukawa interactions. Non-Abelian discrete groups $S_3, T', Q_4, D_4, Q_6$ contain one-and two-dimensional representations, however, their singlet/doublet components are combined in different ways. Furthermore, $\Sigma(18)$ contains three two dimensional representations $2_{10}, 2_{20}, 2_{21}$ where $2^*_{10} = 2_{20}$ and $2^*_{20} = 2_{10}$ while $2_{21}$ is a real representation together with its tensor products presented in Appendix A make $\Sigma(18)$ group has some advantages compared to the other discrete groups and our proposed model is completely different from previous works.
Table I: Fermion assignments under the symmetry $SU(2)_L \times U(1)_Y \times U(1)_X \times \Sigma(18) \times Z_4 \equiv G$. Here $\alpha = 2, 3$ and $\beta = 1, 2$.

| Fields | $\psi_{1L}$ | $\psi_{\alpha L}$ | $l_{1R}$ | $l_{\alpha R}$ | $\nu_{1R}$ | $\nu_{\alpha R}$ | $Q_{\beta L}$ | $Q_{3L}$ | $u_{\beta R}$ | $u_{3R}$ | $d_{\beta R}$ | $d_{3R}$ |
|--------|-------------|------------------|----------|----------------|-----------|----------------|-------------|---------|-------------|---------|-------------|---------|
| $SU(2)_L$ | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 |
| $U(1)_Y$ | $-1$ | $-1$ | $-2$ | $-2$ | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{4}{3}$ | $\frac{4}{3}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| $U(1)_X$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Sigma(18)$ | $1_{+0}$ | $2_{10}$ | $1_{+1}$ | $2_{21}$ | $1_{+1}$ | $2_{10}$ | $2_{10}$ | $1_{+2}$ | $2_{20}$ | $1_{+1}$ | $2_{21}$ | $1_{+0}$ |
| $Z_4$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $-i$ | $i$ | $-i$ | $i$ | $-i$ | $i$ |

Table II: Scalar assignments under $G$ symmetry.

| Fields | $H$ | $\phi$ | $\phi'$ | $\varphi$ | $\varphi'$ | $\chi$ | $\rho$ |
|--------|-----|--------|---------|----------|----------|-------|-------|
| $SU(2)_L$ | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| $U(1)_Y$ | 1 | 1 | 1 | $-1$ | $-1$ | 0 | 0 |
| $U(1)_X$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $\Sigma(18)$ | $1_{+2}$ | $2_{10}$ | $2_{10}$ | $1_{-0}$ | $2_{20}$ | $2_{10}$ | $1_{+1}$ |
| $Z_4$ | 1 | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ |

The charged lepton masses can arise from the couplings of $\bar{\psi}_{(1,\alpha)L}l_{(1,\alpha)R}$ to scalars and the neutrino masses are generated by the couplings of $\bar{\psi}_{(1,\alpha)L}\nu_{(1,\alpha)R}$ and $\bar{\nu}_{(1,\alpha)R}\nu_{(1,\alpha)R}$ to scalars while quarks masses can arise from couplings of $\bar{Q}_{(3,\beta)L}u_{(3,\beta)R}$ and $\bar{Q}_{(3,\beta)L}d_{(3,\beta)R}$ to scalars. Under $G$ symmetry these couplings are summarized in Table III. In order to generate all SM fermion masses, we introduce seven scalars as shown in Table II where $H$, $\phi$ and $\phi'$ give the charged-lepton and quarks masses, whereas $\varphi$, $\varphi'$ are responsible for generating the Dirac mass terms and $\chi$, $\rho$ yield the Majorana mass terms. The Yukawa interactions for leptons and quarks invariant under all the
Table III: List of couplings which can give masses to the fermions

| Couplings | $[SU(2)_L, U(1)_Y, U(1)_X, \Sigma(18), Z_4]$ |
|-----------|-----------------------------------------------|
| $\overline{\psi}_{1L}l_{1R}$ | $(2, -1, 0, \frac{1}{1+1}, 1)$ |
| $\overline{\psi}_{1L}l_{\alpha R}$ | $(2, -1, 0, \frac{2}{21}, 1)$ |
| $\overline{\psi}_{\alpha L}l_{1R}$ | $(2, -1, 0, \frac{2}{10}, 1)$ |
| $\overline{\psi}_{\alpha L}l_{\alpha R}$ | $(2, -1, 0, 1_{+1} \oplus \frac{1}{1-1} + \frac{1}{2}, 1)$ |
| $\overline{\psi}_{1L}\nu_{1R}$ | $(2, 1, -1, 1_{+1}, 1)$ |
| $\overline{\psi}_{1L}\nu_{\alpha R}$ | $(2, 1, -1, \frac{2}{10}, 1)$ |
| $\overline{\psi}_{\alpha L}\nu_{1R}$ | $(2, 1, -1, \frac{2}{10}, 1)$ |
| $\overline{\psi}_{\alpha L}\nu_{\alpha R}$ | $(2, 1, -1, 1_{+1} \oplus \frac{1}{1-0} + \frac{1}{2}, 1)$ |
| $\overline{\nu}_{1R}\nu_{1R}$ | $(1, 0, 0, 1_{+2}, -1)$ |
| $\overline{\nu}_{1R}\nu_{\alpha R}$ | $(1, 0, 0, \frac{2}{21}, -1)$ |
| $\overline{\nu}_{\alpha R}\nu_{1R}$ | $(1, 0, 0, \frac{2}{21}, -1)$ |
| $\overline{\nu}_{\alpha R}\nu_{\alpha R}$ | $(1, 0, 0, 1_{+1} \oplus \frac{1}{1-1} + \frac{1}{2}, -1)$ |
| $\overline{Q}_{\beta L}u_{\beta R}$ | $(2, 1, 0, 1_{+2} + 1_{-2} + \frac{2}{10}, 1)$ |
| $\overline{Q}_{\beta L}u_{3R}$ | $(2, 1, 0, \frac{2}{10}, -1)$ |
| $\overline{Q}_{3L}u_{\beta R}$ | $(2, 1, 0, \frac{2}{10}, -1)$ |
| $\overline{Q}_{3L}u_{3R}$ | $(2, 1, 0, 1_{+2}, -1)$ |
| $\overline{Q}_{\beta L}d_{\beta R}$ | $(2, -1, 0, 1_{+1} + 1_{-1} + \frac{2}{20}, 1)$ |
| $\overline{Q}_{\beta L}d_{3R}$ | $(2, -1, 0, \frac{2}{20}, -1)$ |
| $\overline{Q}_{3L}d_{\beta R}$ | $(2, -1, 0, \frac{2}{20}, -1)$ |
| $\overline{Q}_{3L}d_{3R}$ | $(2, -1, 0, 1_{+1}, 1)$ |

Symmetries of the model are\(^2\):

\[
-\mathcal{L}_Y = h_1 \overline{\psi}_{1L}Hl_{1R} + h_2 (\overline{\psi}_{\alpha L}l_{\alpha R})_{1+1} H + h_3 (\overline{\psi}_{\alpha L}l_{\alpha R})_{\frac{2}{20}} \phi \\
+ \frac{x_1}{2} (\overline{\psi}_{\alpha L}l_{\alpha R})_{1-0} \nu_{\alpha R} + \frac{x_2}{2} (\overline{\psi}_{\alpha L}l_{\alpha R})_{1+0} \nu_{\alpha R} + \frac{x_3}{2} (\overline{\psi}_{\alpha L}l_{\alpha R}) \frac{1}{1+2} \nu_{1R} \\
+ \frac{y_1}{2} (\overline{\nu}_{\alpha R}l_{\alpha R})_{1+2} \nu_{R} + \frac{y_2}{2} (\overline{\nu}_{\alpha R}l_{\alpha R})_{1+1} \nu_{R} + \frac{y_3}{2} (\overline{\nu}_{\alpha R}l_{\alpha R})_{\frac{2}{20}} \chi + \text{H.c.}
\]

\[
-\mathcal{L}_q = h_1u (Q_{\beta L}u_{\beta R})_{1+2} \overline{H} + h_2u(Q_{3L}H)_{1+3}u_{3R} + h_3u(Q_{\beta L}u_{\beta R})_{\frac{2}{10}} \overline{\phi} \\
+ h_4u(Q_{\beta L}u_{3R})_{\frac{2}{10}} \overline{\phi} + h_5u(Q_{3L}u_{3R})_{\frac{2}{10}} \overline{\phi} \\
+ h_6d(Q_{\beta L}d_{\beta R})_{1+2} \overline{H} + h_7d(Q_{3L}H)_{1+3}d_{3R} + h_8d(Q_{\beta L}d_{\beta R})_{\frac{2}{20}} \overline{\phi} \\
+ h_9d(Q_{\beta L}d_{3R})_{\frac{2}{20}} \overline{\phi} + h_{10}d(Q_{3L}d_{3R})_{\frac{2}{20}} \overline{\phi} + \text{H.c.}
\]

\(^2\) Here, $\phi$, $\phi'$ and $\overline{H}$ are respectively the complex conjugate fields of $\phi$, $\phi'$ and $H$, i.e., $\phi = i \sigma_2 \phi^* = (\phi_\alpha^0 - \phi_\gamma) \overline{\phi} \sim [2, -1, 0, \frac{2}{20}, 1]$, $\overline{\phi'} \sim [2, -1, 0, \frac{2}{20}, -1]$, $\overline{H} \sim [2, -1, 0, 1_{-1+1}, 1]$. 
It is important to note that the $U(1)_X$ and $\Sigma(18)$ symmetries forbid some Yukawa interactions thus giving rise to the desired textures for the lepton and quark sectors as shown in Eqs. (32), (34) and (75) and this is an interesting feature of these symmetries. For instance, for the known scalars in Table 1 in the charged lepton sector, the following interactions $(\overline{\psi}_1 L^1 R)\phi_i$, $(\overline{\psi}_1 L\alpha R)\phi_i$, $(\overline{\psi}_\alpha L^1 R)\phi_i$, $(\overline{\psi}_1 L\alpha R)H$, $(\overline{\psi}_\alpha L^1 R)H$ are forbidden by the $\Sigma(18)$ symmetry; in the neutrino sector, the following interactions $(\overline{\psi}_1 L^1 R)\varphi_i$, $(\overline{\psi}_1 L^1 R)\varphi_i$, $(\overline{\psi}_1 L\alpha R)\varphi_i$ and $(\overline{\psi}_\alpha L\alpha R)\varphi_i$, $(\nu^c_{1R} \nu^C_{1R})\chi_i$, $(\nu^c_{1R} \nu^C_{1R})\chi_i$, $(\nu^c_{1R} \nu^C_{1R})\rho$ and $(\nu^c_{1R} \nu^C_{1R})\rho$ are forbidden by the $\Sigma(18)$ symmetry; in quark sector, $(Q_{3L} u_{3R})\overline{H}$, $(Q_{3L} u_{3R})\overline{H}$, $(Q_{3L} u_{3R})\overline{\phi}$, $(Q_{3L} u_{3R})\overline{\phi}$, $(Q_{3L} u_{3R})\overline{H}$, $(Q_{3L} d_{3R})H$, $(Q_{3L} d_{3R})\overline{H}$, $(Q_{3L} d_{3R})\overline{\phi}$, $(Q_{3L} d_{3R})\overline{\phi}$ and $(Q_{3L} d_{3R})\overline{\phi}$ are prevented by the $\Sigma(18)$ symmetry, whereas the following interactions $(\overline{\psi}_\alpha L\alpha R)\varphi_i$, $(\overline{\psi}_\alpha L\alpha R)\varphi_i$, $(\overline{\psi}_\alpha L\alpha R)\varphi_i$, $(\overline{\psi}_\alpha L\alpha R)\varphi_i$ and $(\overline{\psi}_\alpha L\alpha R)\varphi_i$ are prevented by the $U(1)_X$ symmetry.

In order to generate the observed pattern of SM fermion masses and mixing angles, from the potential minimization condition, we consider the following VEV configuration for the scalar fields:

$$
\langle H \rangle = \begin{pmatrix} 0 \\ v_H \end{pmatrix}, \quad \langle \phi \rangle = (\langle \phi_1 \rangle \quad \langle \phi_2 \rangle), \quad \langle \phi_i \rangle = \begin{pmatrix} 0 \\ v_i \end{pmatrix} \quad (i = 1, 2),
$$

$$
\langle \phi' \rangle = (\langle \phi'_1 \rangle \quad \langle \phi'_1 \rangle), \quad \langle \phi'_1 \rangle = \begin{pmatrix} 0 \\ v' \end{pmatrix}, \quad \langle \varphi \rangle = \begin{pmatrix} v_\varphi \\ 0 \end{pmatrix},
$$

$$
\langle \varphi' \rangle = (\langle \varphi'_1 \rangle \quad \langle \varphi'_1 \rangle), \quad \langle \varphi'_1 \rangle = \begin{pmatrix} v_{\varphi'} \\ 0 \end{pmatrix}, \quad \langle \chi \rangle = (0 \quad \langle \chi_2 \rangle), \quad \langle \chi_2 \rangle = v_\chi, \quad \langle \rho \rangle = v_\rho.
$$

(3)

In order to prove that the scalar fields with the VEV alignments as chosen in Eq. (3) is obtained from the minimization condition of $\mathcal{V}_{\text{total}}$ in Appendix B, let us put

$$
v_{\phi'_2} = v_{\phi'_1} = v', \quad v_{\varphi'_2} = v_{\varphi'_1} = v_{\varphi'}, \quad v_{\chi_1} = 0, \quad v_{\chi_2} = v_\chi,
$$

$$
v_H = v_H, \quad v_1 = v_1, \quad v_2 = v_2, \quad v' = v', \quad v_{\varphi} = v_{\varphi'}, \quad v_{\chi} = v_\chi, \quad v_\rho = v_\rho,
$$

(4)

which leads to

$$
\frac{\partial \mathcal{V}_{\text{total}}}{\partial v_j^2} = \frac{\partial^2 \mathcal{V}_{\text{total}}}{\partial v_j^2}, \quad \frac{\partial^2 \mathcal{V}_{\text{total}}}{\partial v_j^2} = \frac{\partial^2 \mathcal{V}_{\text{total}}}{\partial v_j^2} (v_j = v_H, v_1, v_2, v', v_{\varphi}, v_{\varphi'}, v_\chi, v_\rho),
$$

(6)

and the minimization condition of $\mathcal{V}_{\text{total}}$ become

$$
\frac{\partial \mathcal{V}_{\text{total}}}{\partial v_j} = 0, \quad \frac{\partial^2 \mathcal{V}_{\text{total}}}{\partial v_j^2} > 0 \quad (v_j = v_H, v_1, v_2, v', v_{\varphi}, v_{\varphi'}, v_\chi, v_\rho).
$$

(7)

Furthermore, for simplicity and without loss of generality, we consider the following benchmark
point of the Yukawa couplings:

\[
\begin{align*}
\lambda^\phi &= \lambda_2^\phi = \lambda_3^\phi = \lambda^\phi, \\
\lambda^H &= \lambda_2^H = \lambda_3^H = \lambda^H, \\
\lambda^{\phi'} &= \lambda_2^{\phi'} = \lambda_3^{\phi'} = \lambda^{\phi'}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\
\lambda^{H\phi} &= \lambda_2^{H\phi} = \lambda_3^{H\phi} = \lambda^{H\phi}, \\
\lambda^{H\phi'} &= \lambda_2^{H\phi'} = \lambda_3^{H\phi'} = \lambda^{H\phi'}, \\end{align*}
\]

The expressions of the scalar potential minimum equations in Eq. (7) thus reduce to the expressions in Appendix C in which the system of Eqs. (C1)–(C8) always have the solution

\[
\begin{align*}
\lambda^H &= \frac{\beta_H}{2(v_1 - v_2)(2v_1^2 + v_1v_2 + 2v_2^2)v_H}, \\
\lambda^\phi &= \frac{\beta_\phi}{2(v_1 - v_2)(2v_1^2 + v_1v_2 + 2v_2^2)}, \\
\lambda^{\phi'} &= -\frac{\beta_{\phi'}}{2v_2^2}, \\
\lambda^{H\phi'} &= -\frac{\beta_{H\phi'}}{24v_2^3}, \\
\lambda^{H\phi} &= -\frac{\beta_{H\phi}}{4v_2^3}, \\
\lambda^\rho &= -\frac{\beta_\rho}{v_2^3},
\end{align*}
\]

where \(\beta_\Phi (\Phi = H, \phi, \phi', \varphi, \varphi', \chi, \rho)\) and \(\beta_{H\phi}\) are defined in Appendix D.

We will show that, with \(\lambda_\Phi\) and \(\lambda_{H\phi}\) in Eqs. (21)–(23), there exist Yukawa couplings such that expressions in (7) are always give the solution as chosen in Eq. (3). For instance, for the following
the expressions in (7) are always satisfied in the case of the natural solution of the potential minimum condition. In models with more than one SU(2) L Higgs doublet, as in the present model, the Flavor Changing Neutral Current (FCNC) processes exist however they can be suppressed by adding discrete symmetries which have been presented in Refs. [90, 95–101]. In addition, the large amount of parametric freedom allows to find a suitable region of parameter space where these FCNC can be suppressed. A numerical analysis of the FCNC, along with other phenomenological aspects in a multiHiggs doublet model with the $D_4$ discrete symmetry is presented in [59]. The implications of our model in the FCNC interactions are discussed in section 4.

III. LEPTON MASSES AND MIXINGS

From the lepton Yukawa terms given by Eq. (1) and the tensor product of $\Sigma(18)$ in Appendix A we can rewrite the Yukawa interactions in the lepton sector:

$$-\mathcal{L}_Y = h_1 \bar{\psi}_1 L H_1 R + h_2 (\bar{\psi}_2 L H_1 R + \bar{\psi}_3 L H_3 R) + h_3 (\bar{\psi}_2 L \phi_2^0 R + \bar{\psi}_3 L \phi_1^0 R)$$

$$+ \frac{x_1}{2} (\bar{\psi}_2 L \nu_2 R - \bar{\psi}_3 L \nu_3 R) \varphi + \frac{x_2}{2} (\bar{\psi}_1 L \nu_2 R + \bar{\psi}_1 L \nu_1 R) + \frac{x_3}{2} (\bar{\psi}_2 L \nu_1 R + \bar{\psi}_3 L \nu_1 R)$$

$$+ \frac{y_1}{2} (\nu_1^c R \nu_1 R) \varphi + \frac{y_2}{2} (\bar{\nu}_2^c R \nu_3 R + \bar{\nu}_3^c R \nu_2 R) \rho + \frac{y_3}{2} (\nu_2^c R \chi_1 \nu_2 R + \nu_3^c R \chi_2 R \nu_3 R) + H.c. \quad (29)$$

Here, we have used the notations: $\delta_{\nu}^2 = \frac{\partial^2 Y_{\nu \nu \nu}}{\partial \nu^2}$, $\delta_{\nu}^2 = \frac{\partial^2 Y_{\nu \nu \nu}}{\partial \nu^2}$, $\delta_{\nu}^2 = \frac{\partial^2 Y_{\nu \nu \nu}}{\partial \nu^2}$, $\delta_{\nu}^2 = \frac{\partial^2 Y_{\nu \nu \nu}}{\partial \nu^2}$.
With the help of Eq. (3), we get the mass terms for leptons as follows:

\[-L_{lep}^{mass} = h_1 v_H \tilde{l}_1 l_1 R + h_2 v_H (\tilde{l}_2 l_2 R + \tilde{l}_3 l_3 R) + h_3 (v_2 \tilde{l}_2 l_3 R + v_3 \tilde{l}_3 l_2 R) + \frac{x_1 v_\phi}{2} \bar{\nu}_2 l_2 R - \frac{x_1 v_\phi}{2} \bar{\nu}_3 l_3 R + \frac{x_2 v_\phi'}{2} \bar{\nu}_1 l_2 R + \frac{x_2 v_\phi'}{2} \bar{\nu}_1 l_3 R + \frac{x_3 v_\phi'}{2} \bar{\nu}_2 l_3 R + \frac{x_3 v_\phi'}{2} \bar{\nu}_3 l_1 R + \frac{y_1 v_\rho}{2} \bar{\nu}_1 l_1 R + \frac{y_1 v_\rho}{2} \bar{\nu}_1 l_3 R + \frac{y_2 v_\rho^*}{2} (\bar{\nu}_2^c l_3 R + \bar{\nu}_3^c l_2 R) + \frac{y_3 v_\chi}{2} \bar{\nu}_3 R \bar{\nu}_3 R + H.c. \quad (30)\]

which can be written in the matrix form:

\[-L_{lep}^{mass} = \bar{l}_L M_d l_R + \frac{1}{2} \bar{n}_L M_\nu n_L + H.c. \quad (31)\]

where

\[l_l = (l_1, l_2, l_3)^T, \quad l_R = (l_1, l_2, l_3)^T, \quad M_d = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & b_1 & a_2 \end{pmatrix}, \quad (32)\]

\[n_L = (\nu^c_L, \nu^c_R)^T, \quad M_\nu = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (33)\]

with \(\nu^c = (\nu^c_L, \nu^c_R, \nu^c_R)^T\), \(\nu_R = (\nu_1, \nu_2, \nu_3)^T\) and \(M_D, M_R\) are respectively Dirac and Majorana neutrino mass matrices,

\[M_D = \begin{pmatrix} 0 & b_D & b_D \\ c_D & a_D & 0 \\ c_D & 0 & -a_D \end{pmatrix}, \quad M_R = \begin{pmatrix} a_R & 0 & 0 \\ 0 & 0 & b_R \\ 0 & b_R & c_R \end{pmatrix}, \quad (34)\]

and

\[a_i = h_i v_H, \quad b_i = h_3 v_i \quad (i = 1, 2), \quad (35)\]

\[a_D = x_1 v_\phi, \quad b_D = x_2 v_\phi', \quad c_D = x_3 v_\phi', \quad (36)\]

Let us define a Hermitian matrix \(M_l\) as follows

\[M_l = M_{cl} M_{cl}^T = \begin{pmatrix} |a_1|^2 & 0 & 0 \\ 0 & |a_2|^2 + |b_2|^2 & a_2 b_1^* + a_2^* b_2 \\ 0 & (a_2 b_1^* + a_2^* b_2)^* & |a_2|^2 + |b_1|^2 \end{pmatrix}, \quad (37)\]
which can be diagonalised by $\mathcal{U}_{L,R}$ satisfying $\mathcal{U}_L^+ \mathcal{M}_\nu \mathcal{U}_R = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$, where

$$\mathcal{U}_L = \mathcal{U}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_l & -\sin \theta_l e^{-i \alpha} \\ 0 & \sin \theta_l e^{i \alpha} & \cos \theta_l \end{pmatrix},$$

(38)

$$m_e^2 = |a_1|^2, \hspace{1cm} m_{\mu,\tau}^2 = \frac{1}{2} (\gamma_1 \mp \gamma_2),$$

(39)

with

$$\gamma_2 = \sqrt{(|b_1|^2 - |b_2|^2)^2 + 4|a_2|^2 (|b_1|^2 + |b_2|^2) + 8|a_2|^2 |b_1||b_2| \cos(2\alpha_2 - \beta_1 - \beta_2)},$$

$$\gamma_1 = 2|a_2|^2 + |b_1|^2 + |b_2|^2, \hspace{0.5cm} \alpha_2 = \text{arg}(a_2), \hspace{0.5cm} \beta_i = \text{arg}(b_i) \hspace{0.5cm} (i = 1, 2),$$

(40)

$$\alpha = \frac{i}{2} \log \left[ \frac{a_2 b_2^* + a_1 b_1^*}{(a_2 b_1^* + a_1 b_2^*)^2} \right], \hspace{0.5cm} \theta_l = \arctan \left[ \frac{(a_2 b_1 + a_1 b_2)e^{-i \alpha}}{|a_2|^2 + |b_2|^2 - m_\tau^2} \right].$$

(41)

Comparing the result in Eq. (39) with the experimental values of the charged lepton masses given in Ref. [102], $m_e \simeq 0.51999$ MeV, $m_\mu \simeq 105.65837$ MeV, $m_\tau = 1776.86$ MeV, we obtain:

$$|a_1| = 0.510999 \times 10^6 \text{ eV}, \hspace{0.5cm} \gamma_1 = 3.1684 \times 10^{18} \text{ eV}^2, \hspace{0.5cm} \gamma_2 = 3.14607 \times 10^{18} \text{ eV}^2.$$  

(42)

In the case $\alpha_2 = \beta_1 = \beta_2$ and $|v_1| \sim |v_2|$, we get:

$$|h_1| \sim 5 \times 10^5 \left| \frac{v_H}{|v_H|} \right|, \hspace{0.5cm} |h_2| \sim 8 \times 10^8 \left| \frac{v_H}{|v_H|} \right|, \hspace{0.5cm} |h_3| \sim 9 \times 10^8 \left| \frac{v_H}{|v_H|} \right|.$$  

(43)

As we will see below, since the charged lepton mixing matrix $\mathcal{U}_L$ is non trivial in our model, it can contribute to the final leptonic mixing matrix, defined by $U = \mathcal{U}_L^+ \mathcal{U}_\nu$ where $\mathcal{U}_L$ refers to the left-handed charged-lepton mixing matrix and $\mathcal{U}_\nu$ is the neutrino mixing matrix.

Regarding the neutrino sector, from Eq. (34), the light active neutrino mass matrix arises from the type-I seesaw mechanism as follows:

$$\mathcal{M}_\nu = -\mathcal{M}_D M_R^{-1} \mathcal{M}_D^T = \begin{pmatrix} \frac{b_1^2 (c_R - 2 b_R)}{b_R^2} & \frac{a_D b_D (c_R - b_R)}{b_R^2} & \frac{a_D b_R}{b_R} \\ \frac{a_D b_D (c_R - b_R)}{b_R^2} & \frac{a_D^2 c_R}{b_R} - \frac{c_D^2}{a_R} & \frac{a_D^2}{a_R} \\ \frac{a_D b_R}{b_R} & \frac{a_D^2}{a_R} - \frac{c_D^2}{a_R} & -\frac{c_D^2}{a_R} \end{pmatrix},$$

(44)

which has three exact eigenvalues

$$m_1 = 0, \hspace{0.5cm} m_{2,3} = \kappa_1 \mp \kappa_2,$$

(45)

where

$$\kappa_1 = \frac{(a_D^2 + b_D^2) c_R}{2 b_R^2} - \frac{b_D^2}{a_R} - \frac{c_D^2}{a_R}, \hspace{0.5cm} \kappa_2 = \frac{\sqrt{k}}{2 a_R b_R^2},$$

(46)

$$k = b_R^4 \left\{ 4 b_R^2 \left[ a_R (a_D^2 + b_D^2) - b_R c_D^2 \right]^2 - 4 a_R b_D^2 b_R \left[ a_R (a_D^2 + b_D^2) - b_R c_D^2 \right] c_R 
+ a_R^2 (a_D^2 + b_D^2)^2 c_R^2 \right\},$$
and the corresponding mixing matrix is:

$$\mathcal{U} = \begin{pmatrix}
\frac{1}{\sqrt{\nu^2 + 2}} & \frac{N_+}{\sqrt{\nu^2 + N_+^2 + 1}} & \frac{N_+}{\sqrt{\nu^2 + N_+^2 + 1}} \\
\frac{K_-}{\sqrt{\nu^2 + 2}} & \frac{1}{\sqrt{\nu^2 + N_+^2 + 1}} & \frac{1}{\sqrt{\nu^2 + N_+^2 + 1}} \\
\frac{1}{\sqrt{\nu^2 + 2}} & \frac{1}{\sqrt{\nu^2 + N_+^2 + 1}} & \frac{1}{\sqrt{\nu^2 + N_+^2 + 1}}
\end{pmatrix} \cdot \mathbb{P},$$

(47)

where $\mathbb{P} = \text{diag}(1, 1, i)$ and $\mathbb{K}, \mathbb{K}_+, \mathbb{N}_+$ are defined

$$\mathbb{K} = \frac{a_D}{b_D}, \quad \mathbb{K}_+ = \kappa_{11} \mp \kappa_{12}, \quad \mathbb{N}_+ = \epsilon_{11} \mp \epsilon_{12},$$

(48)

where

$$\kappa_{11} = \frac{b_D}{2a_D} \left\{ \frac{(a_D^2 + b_D^2)a_{RCR}}{b_R[(a_D^2 + b_D^2)a_R - b_Rc_D^2]} - 2 \right\}, \quad \kappa_{12} = \frac{b_D\sqrt{\kappa}}{2a_Db_R^3[(a_D^2 + b_D^2)a_R - c_D^2b_R]},$$

$$\epsilon_{11} = \frac{(a_D^2 + b_D^2)a_{RCR}}{2b_R[a_R(a_D^2 + b_D^2) - c_D^2b_R]}, \quad \epsilon_{12} = \frac{\sqrt{\kappa}}{2a_Db_R^3[(a_D^2 + b_D^2)a_R - c_D^2b_R]},$$

(49)

and $a_D, b_D, c_D, a_R, b_R, c_R$ are given in Eq. (50).

From the explicit expressions of $m_{2,3}, \mathbb{K}, \mathbb{K}_+$ and $\mathbb{N}_+$ in Eqs. (45), (46), (48) and (49), the following relations hold:

$$1 + \mathbb{K}_- - \mathbb{N}_- = 0, \quad 1 + \mathbb{K}_+ - \mathbb{N}_+ = 0, \quad 1 + \mathbb{K}_- \mathbb{K}_+ + \mathbb{N}_- \mathbb{N}_+ = 0,$$

(50)

$$a_D = \frac{(\mathbb{N}_- + \mathbb{N}_+ - 2)b_D}{\mathbb{K}_- + \mathbb{K}_+},$$

$$a_R = \frac{(\Lambda_N + 2)\Lambda_K^2 b_R c_D^2}{[\Lambda_K^2 + \Lambda_N(\Lambda_N - 2)](\Lambda_N - 2)b_D^2 - (m_2 + m_3)\Lambda_K^2 b_R},$$

$$c_R = \frac{2[2\Lambda_K^2 + (\Lambda_N - 2)^2]b_D^2 + \Lambda_K^2 (m_2 + m_3)b_R}{[\Lambda_K^2 + (\Lambda_N - 2)^2](\Lambda_N + 2)b_D^2},$$

(51)

with

$$\Lambda_K = \mathbb{K}_- + \mathbb{K}_+, \quad \Lambda_N = \mathbb{N}_- + \mathbb{N}_+. $$

(52)

The effective neutrino mass matrix $\mathcal{M}_\nu$ in Eq. (41) is diagonalized as

$$\mathcal{U}_\nu^\dagger \mathcal{M}_\nu \mathcal{U}_\nu = \begin{pmatrix}
0 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix}, \quad \mathcal{U}_\nu = \begin{pmatrix}
\frac{K_-}{\sqrt{\nu^2 + 2}} & \frac{K_-}{\sqrt{\nu^2 + N_+^2 + 1}} & \frac{\sqrt{i}K_+}{\sqrt{\nu^2 + i}} \\
\frac{N_+}{\nu^2 + 2} & \frac{N_+}{\nu^2 + N_+^2 + 1} & \frac{N_+}{\nu^2 + N_+^2 + 1} \\
\frac{1}{\nu^2 + 2} & \frac{1}{\nu^2 + N_+^2 + 1} & \frac{1}{\nu^2 + N_+^2 + 1}
\end{pmatrix} \quad \text{for NO},$$

(53)

$$\mathcal{U}_\nu^\dagger \mathcal{M}_\nu \mathcal{U}_\nu = \begin{pmatrix}
m_2 & 0 & 0 \\
0 & m_3 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \mathcal{U}_\nu = \begin{pmatrix}
\frac{K_-}{\sqrt{\nu^2 + 2}} & \frac{K_-}{\sqrt{\nu^2 + N_+^2 + 1}} & \frac{\sqrt{i}K_+}{\sqrt{\nu^2 + i}} \\
\frac{N_+}{\nu^2 + 2} & \frac{N_+}{\nu^2 + N_+^2 + 1} & \frac{N_+}{\nu^2 + N_+^2 + 1} \\
\frac{1}{\nu^2 + 2} & \frac{1}{\nu^2 + N_+^2 + 1} & \frac{1}{\nu^2 + N_+^2 + 1}
\end{pmatrix} \quad \text{for IO},$$

(53)
where \( m_{2,3} \) and \( \mathcal{K}, \mathcal{K}_+, N_+ \) are respectively given in Eqs. (45) and (48).

The final leptonic mixing matrix then reads:

\[
U = U_L^* U_R = \left( \begin{array}{ccc}
\frac{\mathcal{K}}{\sqrt{K^2+1}} & \frac{-iK_+}{\sqrt{K_+^2+N_+^2+1}} & \frac{iK_-}{\sqrt{K_-^2+N_-^2+1}} \\
\frac{-i\alpha \sin \theta_1 - \cos \theta_1}{\sqrt{K^2+1}} & \frac{-i\sin \theta_2 N_- + e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+N_-^2+1}} & \frac{-i\sin \theta_2 N_+}{\sqrt{K^2+N_+^2+1}} \\
\frac{\cos \theta_1 N_- + e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+N_-^2+1}} & \frac{-i\cos \theta_1 e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+N_-^2+1}} & \frac{\sin \theta_1 N_+}{\sqrt{K^2+N_+^2+1}} \\
\end{array} \right)
\]

for NO,

\[
U = U_L^* U_R = \left( \begin{array}{ccc}
\frac{\mathcal{K}}{\sqrt{K^2+1}} & \frac{\mathcal{K}_-}{\sqrt{K_-^2+N_-^2+1}} & \frac{i\mathcal{K}_+}{\sqrt{K_+^2+N_+^2+1}} \\
\frac{\cos \theta_1 + e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+1}} & \frac{-\sin \theta_2 N_- + e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+N_-^2+1}} & \frac{-\sin \theta_2 N_+}{\sqrt{K^2+N_+^2+1}} \\
\cos \theta_1 e^{-i\alpha} \sin \theta_1 & \frac{-\cos \theta_1 e^{-i\alpha} \sin \theta_1}{\sqrt{K^2+N_-^2+1}} & \frac{\sin \theta_1 N_+}{\sqrt{K^2+N_+^2+1}} \\
\end{array} \right)
\]

for IO.

In the three neutrino oscillation picture, the lepton mixing matrix can be parametrized as

\[
U_{PMNS} = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix}. \quad (55)
\]

whereby, \( \theta_{12}, \theta_{23}, \theta_{13} \) can be defined via the elements of the leptonic mixing matrix:

\[
s_{13}^2 = |U_{13}|^2, \quad t_{12}^2 = \left| \frac{U_{12}}{U_{11}} \right|^2, \quad t_{23}^2 = \left| \frac{U_{23}}{U_{33}} \right|^2. \quad (56)
\]

As it is well known, the neutrino mass spectrum is currently unknown and it can be NO or IO depending on the sign of \( \Delta m_{32}^2 \) which will be presented in the next section.

### A. Normal spectrum

In NO, the Jarlskog invariant \( J_{CP} \) which determines the magnitude of CP violation in neutrino oscillations \[102\], determined from Eqs. (56) and (54), takes the form

\[
J_{CP}^N = \text{Im}(U_{23}^* U_{13}^* U_{12}^* U_{22}^*) = \frac{\mathcal{K}_+^2 (1 - N_+^2) \cos \theta_1 \sin \theta_2 \sin \alpha}{2\mathcal{K}_+^2 + (N_- - 1)^2} \left( 1 + \mathcal{K}_+^2 + N_+^2 \right). \quad (57)
\]

Comparing Eq. (57) with its corresponding expression in the standard parametrization of the neutrino mixing matrix given in Ref. [102], \( J_{CP} = s_{13} c_{13}^2 s_{12} c_{12} s_{23} c_{23} \sin \delta \), we get:

\[
\sin \delta^N = \frac{\mathcal{K}_+^2 (1 - N_+^2)}{2\mathcal{K}_+^2 + (N_- - 1)^2} \frac{\cos \theta_1 \sin \theta_2 \sin \alpha}{s_{13} c_{13}^2 s_{12} c_{12} s_{23} c_{23}}. \quad (58)
\]

\[4\] Here, \( \delta \) is the Dirac CP violating phase and \( c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij} \) with \( \theta_{12}, \ \theta_{23} \) and \( \theta_{13} \) being the solar, atmospheric and reactor angle, respectively. \( P \) contains two Majorana phases \( (\alpha_{21}, \alpha_{31}) \) which play no role in neutrino oscillations, \( P = \text{diag} \left( 1, e^{i\alpha_{21}}, e^{i\alpha_{31}} \right) \), and thus will be ignored.
Furthermore, from Eqs. (54) and (56), for NO, we get:

\[ s_{13}^2 = \frac{K_+^2}{1 + K_+^2 + N_+^2}, \quad t_{12}^2 = \frac{K_+^2 (N_+ + 1)^2}{(N_+ - 1)^2 (1 + K_+^2 + N_+^2)}, \]

\[ t_{23}^2 = \frac{\sin^2 \theta_t + \cos^2 \theta_t N_+^2 + \sin(2\theta_t) \cos \alpha N_+}{\cos^2 \theta_t + \sin^2 \theta_t N_+^2 - \sin(2\theta_t) \cos \alpha N_+}, \]

(59)

Combining Eqs. (50) and (59) yields:

\[ K_+ = \sqrt{2s_{13}^2 t_{12}^2}, \quad N_+ = 1 + \frac{2s_{13}}{t_{12}^2 - s_{13}} \]

\[ \cos \alpha = \frac{(1 - t_{23}^2) (s_{13}^2 + t_{12}^2) + 2(1 - 2\sin^2 \theta_t) (1 + t_{23}^2) s_{13} t_{12}}{(s_{13}^2 - t_{12}^2) (1 + t_{23}^2) \sin(2\theta_t)}, \]

(60)

Next, substituting Eq. (60) into Eq. (58) yields:

\[ \sin \delta = -\frac{\sin(2\theta_t) \sin \alpha}{\sin(2\theta_{23})}. \]

(62)

We note that the elements \( U_{1i} (i = 1, 2, 3) \) depend only on \( \theta_{12} \) and \( \theta_{13} \) while \( U_{2i} \) and \( U_{3i} (i = 1, 2, 3) \) depend on all lepton mixing angles \( \theta_{ij} (ij = 12, 23, 13) \) and \( \theta_t \).

For NO, by taking the best-fit values of leptonic mixing angles \( \theta_{ij} (i, j = 1, 2, 3) \) \cite{102}, \( \sin^2 \theta_{12} = 0.307, \sin^2 \theta_{13} = 2.18 \times 10^{-2}, \sin^2 \theta_{23} = 0.545 \), we obtain \( U_{11}^N = 0.823, U_{12}^N = -0.548, U_{13}^N = 0.148i \) and \( U_{ij}^N (i = 2, 3; j = 1, 2, 3) \) depend only on \( \sin \theta_t \) that is plotted in Fig. 2 with \( \sin \theta_t \in (0.8, 0.9) \).

In the case \( \delta = 1.36\pi \) \cite{102}, from Eq. (62) we obtain the model parameters as shown in Tab. IV

| Parameters | The derived values |
|------------|-------------------|
| \( K \)    | 2.05              |
| \( K_- \)  | -0.734            |
| \( K_+ \)  | 0.278             |
| \( N_- \)  | -0.507            |
| \( N_+ \)  | 1.57              |
| \( \theta_t \) | 55.1°            |
| \( \alpha \) | 73.9°            |

The lepton mixing matrix in Eq.(54) then takes the form:

\[ U^N = \begin{pmatrix}
0.823 & -0.548 & 0.148i \\
-0.138 - 0.316i & -0.046 - 0.588i & 0.419 + 0.598i \\
0.321 + 0.316i & 0.513 + 0.298i & 0.657 + 0.113i
\end{pmatrix}, \]

(63)
which is unitary and consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. [117].

Now, by using the recent best-fit values for the squared-neutrino mass differences [102], $\Delta m_{21}^2 = 7.53 \times 10^{-5} \text{eV}^2$, $\Delta m_{32}^2 = 2.453 \times 10^{-3} \text{eV}^2$ for the NO, we get a solution

$$\kappa_1 = 2.95 \times 10^{-2}, \quad \kappa_2 = 2.08 \times 10^{-2},$$

$$m_1 = 0 \text{eV}, \quad m_2 = 8.68 \times 10^{-3} \text{eV}, \quad m_3 = 5.03 \times 10^{-2} \text{eV}. \quad (64)$$

The absolute neutrino mass, defined as the sum of the mass of the three neutrino mass eigenstates, is found to be $\sum_{i=1}^{3} m_{\nu_i}^N = 5.90 \times 10^{-2} \text{eV}$. At present, the sum of the three neutrino masses has not been precisely determined, however, the result obtained from our model is well consistent with the strongest bound from cosmology, $\sum m_{\nu} < 0.078 \text{eV}$ [103].

Next, substituting explicit expressions of $b_{PD}, b_{CD}, a_R, b_R, c_R$ from Eq. (36) and the obtained values of $K_+, N_+$ in Tab. IV and $m_{2,3}$ in Eq. (64) into Eq. (51), we get the following relations:

$$x_1 = 2.05 \left( \frac{v_{\nu_1}'}{v_{\nu_2}'} \right) x_2, \quad y_1 = \frac{1}{1.16 \left( \frac{v_{\nu_1}'}{v_{\nu_2}'} \right) \frac{x_2^2}{y_2^2} - 0.0192 \left( \frac{v_{\nu_1}'}{v_{\nu_2}'} \right) \frac{1}{x_3^2}},$$

$$y_3 = \frac{0.827 v_{\nu_1} y_2}{v_0} + \frac{0.00393 (v_{\nu_1}')^2 y_2^2}{v_0 v_{\nu_2}' x_2^2}. \quad (65)$$

### B. Inverted spectrum

Similar to the NO, from Eqs. (54) and (56) for IO, we get a solution:

$$K_+ = -\sqrt{2} c_{13} t_{12}, \quad N_+ = \frac{2}{1 + s_{13} t_{12}} - 1, \quad \cos \alpha = \frac{1}{\sin(2\theta)} \frac{1 - t_{23}^2}{1 + t_{23}^2}, \quad (66)$$

and the Jarlskog invariant $J_{CP}$ determined from (54) and $\sin \delta$ take the form:

$$J_{CP}^I = -\frac{\mathbb{K}K_+(1 + N_+) \cos \theta \sin \theta \sin \alpha}{(K^2 + 2)(K_+^2 + N_+^2 + 1)}, \quad (67)$$

$$\sin \delta^I = \frac{\mathbb{K}K_+(1 + N_+) \cos \theta \sin \theta \sin \alpha}{(K_+^2 + 2)(1 + K_+^2 + N_+^2) s_{13}c_{13}s_{12}c_{12}s_{23}c_{23}}. \quad (68)$$

With the help of Eq. (50), it is easy to show that $J_{CP}^N = J_{CP}^I$ and $\sin \delta^N = \sin \delta^I$ thus the relations in Eqs. (57) and (58) are satisfied for both normal and inverted orderings and the differences start from Eqs. (60) and (66).

Next, by taking the best-fit values of leptonic mixing angles $\theta_{ij} \ (i, j = 1, 2, 3)$ for IO [102], $s_{12}^2 = 0.307$, $s_{23}^2 = 0.547$ and $s_{13}^2 = 2.18 \times 10^{-2}$, we get $U_{11}^I = 0.823$, $U_{12}^I = -0.548$, $U_{13}^I = 0.148i$ and the other elements of $U$ depend only on $\sin \theta_i$ that is plotted in Fig. 3.
In the case the CP violating phase takes the best-fit values \( \delta = 1.36\pi \), we find \( \sin \theta_l = -0.537 \) \( (\theta_l = 327.5^\circ) \) and the other model parameters are explicitly given in Tab. \( \Box \)

Table V: The model parameters in the case \( \delta = 1.36\pi \) in IO

| Parameters | The derived values |
|------------|-------------------|
| \( K \)    | 0.211             |
| \( K^- \)  | 2.7               |
| \( K^+ \)  | -0.848            |
| \( N^- \)  | 1.57              |
| \( N^+ \)  | 0.821             |
| \( \alpha \)| 84.0°             |

The PMNS leptonic mixing matrix of Eq.\( (54) \) takes the form:

\[
U^I = \begin{pmatrix}
0.823 & -0.548 & 0.148i \\
0.387 + 0.163i & 0.412 + 0.345i & -0.373 - 0.629i \\
0.284 + 0.256i & 0.575 + 0.283i & 0.373 + 0.551i \\
\end{pmatrix},
\]

which is unitary and consistent with the constraint on the absolute values of the entries of the lepton mixing matrix given in Ref. \( \Box \).

Now, by using the recent best-fit values for the squared-neutrino mass differences \( \Delta m_{21}^2 = 7.53 \times 10^{-5} \text{eV}^2, \Delta m_{32}^2 = -2.546 \times 10^{-3} \text{eV}^2 \) for IO, we get a solution

\[
\begin{align*}
\kappa_1 &= 5.01 \times 10^{-2}, \quad \kappa_2 = 3.76 \times 10^{-4}, \\
m_1 &= 4.97 \times 10^{-2} \text{eV}, \quad m_2 = 5.05 \times 10^{-2} \text{eV}, \quad m_3 = 0 \text{eV}.
\end{align*}
\]

Using our best fit results given above, we find that the sum of three light neutrino masses is given by \( \sum_{i=1}^{3} m_{\nu_i} = 0.1 \text{eV} \). Currently, the cosmological data set limits on the sum of light active neutrino masses \( \Box \), \( \sum m_\nu < 0.152 \) in the minimal \( \Lambda \text{CDM} + \sum m_\nu \) model, \( \sum m_\nu < 0.118 \text{ eV} \) in the minimal \( \Lambda \text{CDM} + \sum m_\nu \) model with the high-l polarization data, \( \sum m_\nu < 0.305 \text{ eV} \) in the DDE model with \( \text{TT} + \text{BAO} + \text{PAN} + \tau_0p055 \), \( \sum m_\nu < 0.305 \text{ eV} \) in the DDE model with \( \text{TT} + \text{BAO} + \text{PAN} + \tau_0p055 \), \( \sum m_\nu < 0.247 \text{ eV} \) in the DDE model with \( \text{TTTEEE} + \text{BAO} + \text{PAN} + \tau_0p055 \) and \( \sum m_\nu < 0.101 \text{ eV} \) in NPDDE model with \( \text{TTTEEE} + \text{BAO} + \text{PAN} + \tau_0p055 \).

Thus, our obtained value for the sum of the light active neutrino masses is well consistent with the aforementioned bounds arising from cosmology.
By substituting explicit expressions of \( b_D, b_D, c_D, a_R, b_R, c_R \) from Eq. (36) and the obtained values of \( K, N \) in Tab. V and \( m_{1,2} \) in Eq. (70) into Eq. (51), we get the relations:

\[
x_1 = \frac{0.211 v'_\phi x_2}{v_\phi}, \quad y_1 = \frac{1}{v_\phi x_2^2 y_2} - \frac{0.0228 v'_\phi}{v'_\phi x_2^2},
\]

\[
y_3 = \frac{2.13 v'_\rho y_2}{v_\chi} + \frac{0.052 (v'_\rho)^2 y_2^2}{v_\rho^2 x_2^2},
\]

(71)

C. Effective neutrino mass parameters

The effective neutrino mass parameters governing the beta decay and neutrinoless double beta decay are defined as \( m_\beta = \sqrt{\sum_{k=1}^{3} |U_{ek}|^2 m_k^2} \), \( \langle m_{ee} \rangle = \sum_{k=1}^{3} U_{ek}^2 m_k \), where \( U_{ek} \) \((k = 1, 2, 3)\) are the leptonic mixing matrix elements and \( m_k \) correspond to the masses of three light neutrinos. Using the model parameters obtained in subsections IIIA and IIIB, we find the following numerical values for the above mentioned mass parameters:

\[
\langle m_{ee} \rangle = \begin{cases} 
1.51 \times 10^{-3} \text{eV} & \text{for NO,} \\
4.88 \times 10^{-2} \text{eV} & \text{for IO,}
\end{cases}
\]

(72)

and

\[
m_\beta = \begin{cases} 
8.82 \times 10^{-3} \text{eV} & \text{for NO,} \\
4.94 \times 10^{-2} \text{eV} & \text{for IO.}
\end{cases}
\]

(73)

The resulting effective neutrino mass parameters in Eqs. (72) and (73), for both normal and inverted orderings, are below all the upper bounds arising from present 0\(\nu\beta\beta\) decay experiments, such as, KamLAND-Zen[104] \(\langle m_{ee} \rangle < 0.05 \div 0.16 \text{eV}\), GERDA[105] \(\langle m_{ee} \rangle < 0.12 \div 0.26 \text{eV}\), MAJORANA[106] \(\langle m_{ee} \rangle < 0.24 \div 0.53 \text{eV}\), EXO[107,109] \(\langle m_{ee} \rangle < 0.17 \div 0.49 \text{eV}\), CUORE[110,111] \(\langle m_{ee} \rangle < 0.11 \div 0.5 \text{eV}\). Hence, our obtained effective neutrino mass parameter is beyond the reach of the present and forthcoming 0\(\nu\beta\beta\)-decay experiments.

IV. QUARK MASSES AND MIXINGS

In this section, we show that our model is able to successfully reproduce the observed pattern of SM quark masses and mixing parameters. Indeed, from the quark Yukawa terms given by Eq. (2) and the tensor product of \(\Sigma(18)\) in Appendix A we can rewrite the Yukawa interactions in the
We look for the eigenvalue problem solutions reproducing the experimental values of the quark parameters \[112, 113\]:

quark sector in the form:

\[
-\mathcal{L}_q = h_{1u}(\bar{Q}_{1L}H u_{2R} + \bar{Q}_{2L}H u_{1R}) + h_{2u}\bar{Q}_{3L}H u_{3R} + h_{3u}(\bar{Q}_{1L}\tilde{\phi}_1 u_{1R} + \bar{Q}_{2L}\tilde{\phi}_2 u_{2R}) \\
+ h_{4u}(\bar{Q}_{1L}\tilde{\phi}'_2 u_{3R} + \bar{Q}_{2L}\tilde{\phi}'_1 u_{3R}) + h_{5u}(\bar{Q}_{3L}\tilde{\phi}_2 u_{1R} + \bar{Q}_{3L}\tilde{\phi}'_1 u_{2R}) \\
+ h_{1d}(\bar{Q}_{1L}H d_{1R} + \bar{Q}_{2L}H d_{2R}) + h_{2d}\bar{Q}_{3L}H d_{3R} + h_{3d}(\bar{Q}_{1L}\phi_2 d_{1R} + \bar{Q}_{2L}\phi_1 d_{1R}) \\
+ h_{4d}(\bar{Q}_{1L}\phi'_1 d_{3R} + \bar{Q}_{2L}\phi'_2 d_{3R}) + h_{5d}(\bar{Q}_{3L}\phi'_2 d_{1R} + \bar{Q}_{3L}\phi'_1 d_{2R}) + \text{H.c.} \\
\]

With the VEV alignments of $H$ and $\phi$ as chosen in Eq. \[3\], the mass Lagrangian of quarks reads

\[
-\mathcal{L}_q^\text{mass} = h_{1u}v_H(\bar{u}_{1L}u_{2R} + \bar{u}_{2L}u_{1R}) + h_{2u}v_H\bar{u}_{3L}u_{3R} + h_{3u}(v'_1\bar{u}_{1L}u_{1R} + v'_2\bar{u}_{2L}u_{2R}) \\
+ h_{4u}(v'_2\bar{u}_{1L}u_{3R} + v'_2\bar{u}_{2L}u_{3R}) + h_{5u}(v'_2\bar{u}_{3L}u_{1R} + v'_2\bar{u}_{3L}u_{2R}) \\
+ h_{1d}v_H(\bar{d}_{1L}d_{1R} + \bar{d}_{2L}d_{2R}) + h_{2d}v_H\bar{d}_{3L}d_{3R} + h_{3d}(v'_2\bar{d}_{1L}d_{1R} + v'_2\bar{d}_{2L}d_{1R}) \\
+ h_{4d}(v'_2\bar{d}_{1L}d_{3R} + v'_2\bar{d}_{2L}d_{3R}) + h_{5d}(v'_2\bar{d}_{3L}d_{1R} + v'_2\bar{d}_{3L}d_{2R}) + \text{H.c.} \\
\equiv (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L})M_u(u_{1R}, u_{2R}, u_{3R})^T + (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L})M_d(d_{1R}, d_{2R}, d_{3R})^T + \text{H.c.},
\]

where the up- and down-quark mass matrices are

\[
M_u = \begin{pmatrix} b_{1u} & a_{1u} & c_{2u} \\ a_{1u} & b_{2u} & c_{1u} \\ g_{2u} & g_{1u} & a_{2u} \end{pmatrix}, \quad M_d = \begin{pmatrix} a_{1d} & b_{1d} & c_{1d} \\ b_{2d} & a_{1d} & c_{2d} \\ g_{2d} & g_{1d} & a_{2d} \end{pmatrix},
\]

with

\[
a_{(1,2)u} = h_{(1,2)u}v_H, \quad b_{(1,2)u} = h_{3u}v'^*_2, \quad c_{(1,2)u} = h_{4u}v'^*_1, \quad g_{(1,2)u} = h_{5u}v'^*_1, \\
a_{(1,2)d} = h_{(1,2)d}v_H, \quad b_{(1,2)d} = h_{3d}v(2,1), \quad c_{(1,2)d} = h_{4d}v(1,2), \quad g_{(1,2)d} = h_{5d}v(1,2).
\]

Now we turn our attention to the experimental values of the SM quark masses and CKM parameters \[112, 113\]:

\[
m_u(\text{MeV}) = 1.24 \pm 0.22, \quad m_d(\text{MeV}) = 2.69 \pm 0.19, \quad m_s(\text{MeV}) = 53.5 \pm 4.6, \\
m_c(\text{GeV}) = 0.63 \pm 0.02, \quad m_t(\text{GeV}) = 172.9 \pm 0.4, \quad m_b(\text{GeV}) = 2.86 \pm 0.03, \\
\sin \theta_{12} = 0.2245 \pm 0.00044, \quad \sin \theta_{23} = 0.0421 \pm 0.00076, \quad \sin \theta_{13} = 0.00365 \pm 0.00012, \\
J = (3.18 \pm 0.15) \times 10^{-5}.
\]

We look for the eigenvalue problem solutions reproducing the experimental values of the quark
masses and the CKM parameters given by Eq. (77), finding the following solution:

\[
M_u = \begin{pmatrix}
-27.6375 - 62.7392i & -31.3158 - 66.0758i & 24.5526 - 34.1468i \\
-31.3158 - 66.0758i & -36.0052 - 69.6769i & 25.5349 - 37.6027i \\
24.5526 - 34.1468i & 25.5349 - 37.6027i & 25.9351 + 2.79953i
\end{pmatrix} \text{GeV},
\]

\[
M_d = \begin{pmatrix}
1.24842 & 1.22041 - 0.00504449i & 0.37787 - 0.563654i \\
1.22041 + 0.00504449i & 1.24842 & 0.368746 - 0.599054i \\
0.37787 + 0.563654i & 0.368746 + 0.599054i & 0.419347
\end{pmatrix} \text{GeV}.
\] (78)

This shows that our model is consistent with and successfully accommodate the experimental values of the physical observables of the quark sector: the six quark masses, the quark mixing angles and the CP violating phase in the quark sector.

V. \( K - \bar{K} \) AND \( B - \bar{B} \) MIXINGS.

In this section we discuss the implications of our model in the FCNC interactions in the down type quark sector. The FCNC Yukawa interactions in the down type quark sector give rise to meson oscillations. Here we focus on the \( K - \bar{K} \) mixing, whose corresponding \( \Delta M_K \) parameter arises from the following effective Hamiltonian:

\[
\mathcal{H}^{(\Delta S=2)}_{\text{eff}} = \frac{G_F m_W^2}{16\pi^2} \sum_i C_i (\mu) O_i (\mu).
\] (79)

In our analysis of the \( K - \bar{K} \) mixing we follow the approach of \cite{114, 115}. As in Ref. \cite{114, 115}, the \( K - \bar{K} \) mixing in our model mainly arise from the tree level exchange of neutral CP even and CP odd scalars, thus giving rise to the following operators:

\[
O_1^{LL} = (\bar{s} P_L d) (\bar{s} P_L d), \quad O_1^{RR} = (\bar{s} P_R d) (\bar{s} P_R d), \quad O_1^{LR} = (\bar{s} P_L d) (\bar{s} P_R d),
\] (80)

where the corresponding Wilson coefficient are given by:

\[
C_1^{LL} = \frac{16\pi^2}{G_F m_W^2} \left( \sum_{i=1}^N \frac{y_{H^0_{\pi R L} d_L}^2}{m_{H_i^0}^2} - \sum_{i=1}^{N-4} \frac{y_{A^0_{\pi R L} d_L}^2}{m_{H_i^0}^2} \right) = \frac{16\pi^2}{G_F m_W^2} \tilde{C}_1^{LL},
\] (81)

\[
C_1^{RR} = \frac{16\pi^2}{G_F m_W^2} \left( \sum_{i=1}^N \frac{y_{H^0_{\pi L d R} d_R}^2}{m_{H_i^0}^2} - \sum_{i=1}^{N-4} \frac{y_{A^0_{\pi L d R} d_R}^2}{m_{H_i^0}^2} \right) = \frac{16\pi^2}{G_F m_W^2} \tilde{C}_1^{RR},
\] (82)

\[
C_2^{LR} = \frac{16\pi^2}{G_F m_W^2} \left( \sum_{i=1}^N \frac{y_{H^0_{\pi R L} d_L} y_{H^0_{\pi L d R} d_R}}{m_{H_i^0}^2} - \sum_{i=1}^{N-4} \frac{y_{A^0_{\pi R L} d_L} y_{A^0_{\pi L d R} d_R}}{m_{H_i^0}^2} \right) = \frac{16\pi^2}{G_F m_W^2} \tilde{C}_2^{LR},
\] (83)
with

$$\tilde{C}_{1}^{LL} = \sum_{i=1}^{N} \frac{y_{H_{i}^{0}\pi_{R}d_{L}}^{2}}{m_{H_{i}^{0}}^{2}} - \sum_{i=1}^{N-4} \frac{y_{A_{i}^{0}\pi_{R}d_{L}}^{2}}{m_{A_{i}^{0}}^{2}}, \quad (84)$$

$$\tilde{C}_{1}^{RR} = \sum_{i=1}^{N} \frac{y_{H_{i}^{0}\pi_{L}d_{R}}^{2}}{m_{H_{i}^{0}}^{2}} - \sum_{i=1}^{N-4} \frac{y_{A_{i}^{0}\pi_{L}d_{R}}^{2}}{m_{A_{i}^{0}}^{2}}, \quad (85)$$

$$\tilde{C}_{2}^{LR} = \sum_{i=1}^{N} \frac{y_{H_{i}^{0}\pi_{R}d_{L}} y_{H_{i}^{0}\pi_{L}d_{R}}}{m_{H_{i}^{0}}^{2}} - \sum_{i=1}^{N-4} \frac{y_{A_{i}^{0}\pi_{R}d_{L}} y_{A_{i}^{0}\pi_{L}d_{R}}}{m_{A_{i}^{0}}^{2}}. \quad (86)$$

Here $N = 11$ is the number of CP even scalars of our model, whereas $N - 4 = 7$ is the number of CP odd scalars. Let us note that our model is an extended 8HDM where the scalar sector is enlarged by the inclusion of 3 real gauge singlet scalars.

On the other hand, the $K - \bar{K}$ mass splitting has the form:

$$\Delta m_{K} = 2 \text{Re} \left< K^{0} | \frac{(\Delta S = 2)}{H_{\text{eff}}} | K^{0} \right> = \frac{G_{F} m_{W}^{2}}{12 \pi^{2}} m_{K}^{2} f_{K} \eta_{K} B_{K} \left[ P_{2}^{LR} \tilde{C}_{2}^{LR} + P_{1}^{LL} \left( \tilde{C}_{1}^{LL} + \tilde{C}_{1}^{RR} \right) \right] = \frac{4}{3} m_{K} f_{K}^{2} \eta_{K} B_{K} \left[ P_{2}^{LR} \tilde{C}_{2}^{LR} + P_{1}^{LL} \left( \tilde{C}_{1}^{LL} + \tilde{C}_{1}^{RR} \right) \right]. \quad (87)$$

Then, it follows that:

$$M_{12} = \frac{\Delta m_{K}}{m_{K}} = \frac{4}{3} f_{K}^{2} \eta_{K} B_{K} \left[ P_{2}^{LR} \tilde{C}_{2}^{LR} + P_{1}^{LL} \left( \tilde{C}_{1}^{LL} + \tilde{C}_{1}^{RR} \right) \right]. \quad (88)$$

Using the following parameters [114, 115]:

$$\Delta m_{K} = 3.483 \times 10^{-12} \text{MeV}, \quad m_{K} = 497.614 \text{MeV}, \quad f_{K} = 160 \text{MeV},$$

$$B_{K} = 0.85 \pm 0.15, \quad \sqrt{B_{K} f_{K}} = 135 \text{MeV}, \quad \eta_{K} = 0.57, \quad P_{1}^{LL} = -9.3,$$

$$P_{2}^{LR} = 30.6, \quad M_{12} = \frac{\Delta m_{K}}{m_{K}} = 7.2948 \times 10^{-15}. \quad (89)$$

We get the following constraint arising from $K - \bar{K}$ mixing:

$$P_{2}^{LR} \tilde{C}_{2}^{LR} + P_{1}^{LL} \left( \tilde{C}_{1}^{LL} + \tilde{C}_{1}^{RR} \right) \leq 4.41 \times 10^{-19} \text{MeV}^{-2}. \quad (90)$$

Given the large amount parametric freedom in both fermion and scalar sectors of our model, such constraint can be fulfilled. To show explicitly that the above given constraint resulting from $K - \bar{K}$ mixing is successfully fullfilled and given the large amount of parameters in our model, we consider a simplified benchmark scenario where:

$$y_{H_{i}^{0}\pi_{R}d_{L}} = y_{H_{i}^{0}\pi_{R}d_{L}} = y_{H_{i}^{0}\pi_{L}d_{R}} = y_{A_{i}^{0}\pi_{R}d_{L}} = y_{A_{i}^{0}\pi_{L}d_{R}} = y, \quad (91)$$

$$m_{H_{i}^{0}} = m_{h} = 126 \text{GeV}, \quad m_{A_{i}^{0}} = m_{H}, \quad (92)$$

$$m_{A_{i}^{0}} = m_{A}, \quad j = 2, 3, \cdots, 11, \quad i = 1, 2, \cdots, 7. \quad (93)$$
Here we identified $H_0^1$ with 126 GeV SM like Higgs boson. We plot in Figure 4 the allowed parameter space in the $m_H - m_A$ plane consistent with the constraint arising from $K - \bar{K}$ mixing in the aforementioned simplified benchmark scenario of our model. Here, for the sake of simplicity we have set $y = 2 \times 10^{-5}$. Figure shows that our model can successfully accommodate the constraint arising from $K - \bar{K}$ mixing, in a large region of parameter space. It is worth mentioning that we are considering a scenario where the down type quark Yukawa couplings have been taken to be real, which implies that the CP violation in the quark sector only arises from the up type quark sector. Consequently, in that scenario the stringent constraints that are usually imposed on any possible new contribution to the $K - \bar{K}$ mixing from CP violating processes, are not relevant in our case. Furthermore, in what regards the $B_d - \bar{B}_d$ and $B_s - \bar{B}_s$ mixings, we have numerically checked that in the aforementioned simplified benchmark scenario and above described region of parameter space with flavour violating Yukawa coupling of the order of $10^{-5}$, the obtained values for the $\Delta m_{B_d}$ and $\Delta m_{B_s}$ parameters are about two and four orders of magnitude, respectively, below their corresponding experimental values $\Delta m_{B_d} = 3.337 \times 10^{-10}\text{MeV}$ and $\Delta m_{B_s} = 1.042 \times 10^{-8}\text{MeV}$. On the other hand, in the simplified benchmark scenario, when the couplings of the flavor changing neutral Yukawa interactions responsible for the $B_d - \bar{B}_d$ and $B_s - \bar{B}_s$ mixings take values of about $2 \times 10^{-4}$ and $10^{-3}$, respectively, the $B_d - \bar{B}_d$ and $B_s - \bar{B}_s$ mixings arising from these interactions reach values close to their experimental upper bounds, thus giving rise to the allowed regions in the $m_H - m_A$ plane consistent with these constraints and displayed in Figure 5.

VI. CHARGED LEPTON FLAVOR VIOLATION

In this section we analyze the implications of our model in charged lepton flavor violation. From the charged lepton Yukawa interactions it follows that the $\mu \rightarrow e\gamma$ and $\tau \rightarrow e\gamma$ decays are absent in our model since there are no flavor changing neutral (FCN) scalar interactions involving the first family of SM charged leptons with the remaining ones. However, the are flavor changing neutral scalar interactions involving the tau and the muon that give rise to the $\tau \rightarrow \mu\gamma$ decay. The $\tau \rightarrow \mu\gamma$ decay appears at one loop level and involve the exchange of electrically neutral CP even and CP odd scalars and the tau and muon leptons running in the internal lines of the loop. Its branching...
ratio is given by \[116\]:

\[
\text{Br}(\tau \to \mu \gamma) \simeq \frac{3(4\pi)^3 \alpha_{EM}}{4G_F^2} \left( \frac{1}{16\pi^2} \right)^2 \sum_{l=\mu,\tau} \sum_{i=1}^{N} \frac{y_{H_i^0 \tau} y_{H_i^0 \mu}}{m_{H_i^0}^2} \left\{ \frac{1}{6} - \frac{m_l}{m_\mu} \left[ \frac{3}{2} + \ln \left( \frac{m_l}{m_{H_i^0}} \right) \right] \right\}
\]

\[
+ \sum_{l=e,\mu} \sum_{i=1}^{N-4} \frac{y_{A_i^0 \tau} y_{A_i^0 \mu}}{m_{A_i^0}^2} \left\{ \frac{1}{6} + \frac{m_l}{m_\mu} \left[ \frac{3}{2} + \ln \left( \frac{m_l}{m_{A_i^0}} \right) \right] \right\}^2 .
\] (94)

To simplify our analysis we choose the benchmark scenario described in section 4. We display in Figure the allowed parameter space in the \(m_H - m_A\) plane consistent with the existing \(\tau \to \mu \gamma\) experimental constraints. Here we set the Yukawa couplings of the FCN leptonic Yukawa interactions equal to \(10^{-2}\) and \(5 \times 10^{-3}\) for the left and right plots, respectively. Consequently, our model is highly consistent with the constraints arising from lepton flavor violating decays for a large region of parameter space. Furthermore, it allows for charged lepton flavor violating (CLFV) processes within the reach of future experimental sensitivity.

VII. CONCLUSIONS

We have built a renormalizable theory where the SM gauge symmetry is extended by the inclusion of the global \(U(1)_X\) symmetry and the \(\Sigma(18) \times Z_4\) discrete group which leads to a successful fit of SM fermion masses and mixings. The right-handed neutrinos are responsible for the generation of the tiny active neutrino masses through a type I seesaw mechanism mediated by heavy right handed Majorana neutrinos. The resulting physical parameters are in accordance with the recent experimental data. We find values for the effective neutrino mass parameters equal to \(\langle m_{ee} \rangle = 1.51 \times 10^{-3}\) eV for normal ordering and \(\langle m_{ee} \rangle = 4.88 \times 10^{-2}\) eV for inverted ordering, which are well in accordance with the recent experimental limits on neutrinoless double beta decay. The proposed model also successfully accommodates the recent experimental values of the physical observables of the quark sector, including the six quark masses, the quark mixing angles and the CP violating phase in the quark sector. Furthermore, our model can also accommodate the constraints arising from \(K - \bar{K}, B_d - \bar{B}_d\) and \(B_s - \bar{B}_s\) mixings as well as the constraints arising from charged lepton flavor violation.

Acknowledgments

This research is funded by ANID-Chile FONDECYT 1210378. H. N. Long acknowledges the financial support of the International Centre of Physics at the Institute of Physics, Vietnam Academy of Science and Technology with Grant number CIP.2021.02.
Appendix A: The Clebsch-Gordan coefficients of $\Sigma(18)$ group

$\Sigma(18)$ is the simplest non-trivial group of $\Sigma(2N^2)$ with $N = 3$ which is isomorphic to $(Z_3 \times Z_3') \times Z_2$. It has 18 elements, $b^k a^m a^n$ for $k = 0, 1$ and $m, n = 0, 1, 2$, where $a, a'$ and $b$ satisfy $a^3 = a'^3 = e$, $b^2 = e$, $aa' = a'a$ and $bab = a'$. All elements of $\Sigma(18)$ are divided into nine conjugacy classes with $1_{+0}$, $1_{+1}$, $1_{+2}$, $1_{-0}$, $1_{-1}$, $1_{-2}$, $2_{10}$, $2_{20}$ and $2_{21}$ as its nine irreducible representations.

The tensor products between doublets of $\Sigma(18)$ are given by [93]:

\[
\begin{align*}
2_{10} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+1}(x_1y_2 + x_2y_1) \oplus 1_{-1}(-x_1y_2 + x_2y_1) \oplus 2_{20} \left( \begin{array}{c} x_1y_1 \\ x_2y_2 \end{array} \right), \\
2_{20} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{20} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+2}(x_1y_2 + x_2y_1) \oplus 1_{-2}(-x_1y_2 + x_2y_1) \oplus 2_{10} \left( \begin{array}{c} x_1y_1 \\ x_2y_2 \end{array} \right), \\
2_{21} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{21} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+0}(x_1y_2 + x_2y_1) \oplus 1_{-0}(-x_1y_2 + x_2y_1) \oplus 2_{21} \left( \begin{array}{c} x_2y_2 \\ x_1y_1 \end{array} \right), \\
2_{20} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+0}(x_1y_1 + x_2y_2) \oplus 1_{-0}(x_1y_1 - x_2y_2) \oplus 2_{21} \left( \begin{array}{c} x_1y_2 \\ x_2y_1 \end{array} \right), \\
2_{21} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+2}(x_1y_2 + x_2y_1) \oplus 1_{-2}(-x_1y_2 + x_2y_1) \oplus 2_{10} \left( \begin{array}{c} x_2y_2 \\ x_1y_1 \end{array} \right), \\
2_{21} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \otimes 2_{20} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 1_{+1}(x_1y_1 + x_2y_2) \oplus 1_{-1}(x_1y_1 - x_2y_2) \oplus 2_{20} \left( \begin{array}{c} x_1y_2 \\ x_2y_1 \end{array} \right),
\end{align*}
\]

where $x_i, y_i (i = 1, 2)$ are the components of two different representations.

The tensor products between singlets and doublets of $\Sigma(18)$ are obtained as [93]:

\[
\begin{align*}
1_{\pm 0}(x) \otimes 2_{21} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{21} \left( \begin{array}{c} xy_1 \\ xy_2 \end{array} \right), & 1_{\pm 1}(x) \otimes 2_{21} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{20} \left( \begin{array}{c} xy_2 \\ xy_1 \end{array} \right), \\
1_{\pm 2}(x) \otimes 2_{21} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{10} \left( \begin{array}{c} xy_1 \\ xy_2 \end{array} \right), & 1_{\pm 0}(x) \otimes 2_{20} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{20} \left( \begin{array}{c} xy_1 \\ xy_2 \end{array} \right), \\
1_{\pm 1}(x) \otimes 2_{20} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{10} \left( \begin{array}{c} xy_2 \\ xy_1 \end{array} \right), & 1_{\pm 2}(x) \otimes 2_{20} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{21} \left( \begin{array}{c} xy_2 \\ xy_1 \end{array} \right), \\
1_{\pm 0}(x) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{10} \left( \begin{array}{c} xy_1 \\ xy_2 \end{array} \right), & 1_{\pm 1}(x) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{21} \left( \begin{array}{c} xy_1 \\ xy_2 \end{array} \right), \\
1_{\pm 2}(x) \otimes 2_{10} \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) &= 2_{20} \left( \begin{array}{c} xy_2 \\ xy_1 \end{array} \right).
\end{align*}
\]
The tensor products between singlets of $\Sigma(18)$ are obtained as \[93:]

\begin{align*}
1_{\pm 0}(x) \otimes 1_{\pm 0}(y) &= 1_{\pm 0}(xy), \\
1_{\pm 2}(x) \otimes 1_{\pm 2}(y) &= 1_{\pm 1}(xy), \\
1_{\pm 1}(x) \otimes 1_{\pm 0}(y) &= 1_{\pm 1}(xy), \\
1_{\pm 2}(x) \otimes 1_{\pm 0}(y) &= 1_{\pm 2}(xy), \\
1_{\pm 0}(x) \otimes 1_{\mp 0}(y) &= 1_{\pm 0}(xy), \\
1_{\pm 1}(x) \otimes 1_{\mp 1}(y) &= 1_{\pm 1}(xy), \\
1_{\pm 2}(x) \otimes 1_{\mp 2}(y) &= 1_{\pm 2}(xy), \\
1_{\pm 0}(x) \otimes 1_{\mp 0}(y) &= 1_{\mp 0}(xy), \\
1_{\pm 1}(x) \otimes 1_{\mp 1}(y) &= 1_{\mp 1}(xy), \\
1_{\pm 2}(x) \otimes 1_{\mp 2}(y) &= 1_{\mp 2}(xy), \\
1_{\pm 0}(x) \otimes 1_{\mp 0}(y) &= 1_{\mp 0}(xy).
\end{align*}

(A3)

The rules to conjugate of all the representations of $\Sigma(18)$ are given by:

\begin{align*}
1^*_0(x^*) &= 1_{\pm 0}(x^*), \\
1^*_1(x^*) &= 1_{\pm 1}(x^*), \\
1^*_2(x^*) &= 1_{\pm 2}(x^*),
\end{align*}

(A4)

\begin{align*}
2^*_{10} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} &= 2_{20} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}, \\
2^*_{20} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} &= 2_{10} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}.
\end{align*}

(A5)

Appendix B: Renormalizable Higgs potential invariant under $G$ symmetry

The general renormalizable potential invariant under $G$ symmetry is a sum of the following components:\footnote{Here, we have used the notation: $\mathcal{V}(a_1 \rightarrow a_2, b_1 \rightarrow b_2, \cdots) \equiv \mathcal{V}(a_1, b_1, \cdots)_{(a_1=a_2, b_1=b_2, \cdots)}$.}

\begin{align*}
\mathcal{V}_{\text{total}} &= \mathcal{V}(H) + \mathcal{V}(\phi) + \mathcal{V}(\phi') + \mathcal{V}(\varphi) + \mathcal{V}(\varphi') + \mathcal{V}(\chi) + \mathcal{V}(\rho) + \mathcal{V}(H, \phi) + \mathcal{V}(H, \varphi) \\
&+ \mathcal{V}(H, \varphi') + \mathcal{V}(H, \chi) + \mathcal{V}(H, \rho) + \mathcal{V}(\phi, \varphi') + \mathcal{V}(\phi, \varphi) + \mathcal{V}(\phi, \rho) + \mathcal{V}(\varphi, \varphi') \\
&+ \mathcal{V}(\varphi, \chi) + \mathcal{V}(\varphi, \rho) + \mathcal{V}(\varphi', \varphi) + \mathcal{V}(\varphi', \varphi') + \mathcal{V}(\varphi', \chi) + \mathcal{V}(\varphi', \rho) + \mathcal{V}(\varphi', \rho) + \mathcal{V}(\varphi, \rho) + \mathcal{V}(\chi, \rho) + \mathcal{V}_{\text{three}} + \mathcal{V}_{\text{four}},
\end{align*}

(B1)
where
\[
\mathcal{V}(H) = \mu_H^2 H\dagger H + \lambda_H^2 (H\dagger H)_{1+0} (H\dagger H)_{1+0},
\]
\[
\mathcal{V}(\phi) = \mu_\phi^2 \phi\dagger \phi + \lambda_1^2 \phi\dagger \phi)_{1+0} (\phi\dagger \phi)_{1-0} + \lambda_2^2 (\phi\dagger \phi)_{1-0} (\phi\dagger \phi)_{1-0} + \lambda_3^2 (\phi\dagger \phi)_{2+1} (\phi\dagger \phi)_{2+1},
\]
\[
\mathcal{V}(\phi') = \mathcal{V}(\phi \rightarrow \phi'), \quad \mathcal{V}(\varphi) = \mathcal{V}(H \rightarrow \varphi), \quad \mathcal{V}(\varphi') = \mathcal{V}(\phi \rightarrow \varphi'), \quad \mathcal{V}(\chi) = \mathcal{V}(\phi \rightarrow \chi),
\]
\[
\mathcal{V}(\rho) = \mathcal{V}(H \rightarrow \rho),
\]
\[
\mathcal{V}(H, \phi) = \lambda_1^2 \phi (H\dagger H)_{1+0} (\phi\dagger \phi)_{1+0} + \lambda_2^2 \phi (H\dagger H)_{2+1} (\phi\dagger \phi)_{2+1}
\]  
\[+ \lambda_3^2 \phi (H\dagger H)_{2+1} (\phi\dagger \phi)_{2+1}, \quad \mathcal{V}(H, \phi) = \mathcal{V}(H, \phi \rightarrow \phi'),
\]
\[
\mathcal{V}(H, \varphi) = \lambda_1^2 \varphi (H\dagger H)_{1+0} (\varphi\dagger \varphi)_{1+0} + \lambda_2^2 \varphi (H\dagger H)_{1+0} (\varphi\dagger \varphi)_{1+0},
\]
\[
\mathcal{V}(H, \varphi') = \lambda_1^2 \varphi' (H\dagger H)_{1+0} (\varphi'\dagger \varphi')_{1+0} + \lambda_2^2 \varphi' (H\dagger H)_{1+0} (\varphi'\dagger \varphi')_{1+0} + \lambda_3^2 \varphi' (H\dagger H)_{2+1} (\varphi'\dagger \varphi')_{2+1},
\]
\[
\mathcal{V}(H, \chi) = \lambda_1^2 \chi (H\dagger H)_{1+0} (\chi\dagger \chi)_{1+0} + \lambda_2^2 \chi (H\dagger H)_{1+0} (\chi\dagger \chi)_{1+0} + \lambda_3^2 \chi (H\dagger H)_{2+1} (\chi\dagger \chi)_{2+1},
\]
\[
\mathcal{V}(H, \rho) = \lambda_1^2 \rho (H\dagger H)_{1+0} (\rho\dagger \rho)_{1+0} + \lambda_2^2 \rho (H\dagger H)_{1+0} (\rho\dagger \rho)_{1+0},
\]
\[
\mathcal{V}(\phi, \varphi) = \lambda_1^2 \phi \varphi (\phi\dagger \varphi)_{1+0} (\varphi\dagger \varphi)_{1+0} + \lambda_2^2 \phi \varphi (\phi\dagger \varphi)_{1+0} (\varphi\dagger \varphi)_{1+0} + \lambda_3^2 \phi \varphi (\phi\dagger \varphi)_{2+1} (\varphi\dagger \varphi)_{2+1},
\]
\[
\mathcal{V}(\phi, \varphi') = \lambda_1^2 \phi \varphi' (\phi\dagger \varphi')_{1+0} (\varphi'\dagger \varphi')_{1+0} + \lambda_2^2 \phi \varphi' (\phi\dagger \varphi')_{1+0} (\varphi'\dagger \varphi')_{1+0} + \lambda_3^2 \phi \varphi' (\phi\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1},
\]
\[
\mathcal{V}(\phi, \chi) = \mathcal{V}(\phi, \phi' \rightarrow \chi), \quad \mathcal{V}(\phi, \rho) = \mathcal{V}(H \rightarrow \phi, \varphi' \rightarrow \rho), \quad \mathcal{V}(\phi, \varphi) = \mathcal{V}(\phi \rightarrow \phi', \varphi),
\]
\[
\mathcal{V}(\phi', \varphi') = \mathcal{V}(\phi \rightarrow \phi', \varphi'), \quad \mathcal{V}(\phi', \chi) = \mathcal{V}(\phi \rightarrow \phi', \chi), \quad \mathcal{V}(\phi', \rho) = \mathcal{V}(H \rightarrow \phi', \varphi' \rightarrow \rho),
\]
\[
\mathcal{V}(\varphi, \varphi') = \mathcal{V}(\phi \rightarrow \varphi, \varphi' \rightarrow \varphi'), \quad \mathcal{V}(\varphi, \chi) = \mathcal{V}(H \rightarrow \varphi, \varphi' \rightarrow \varphi'), \quad \mathcal{V}(\varphi, \rho) = \mathcal{V}(H \rightarrow \varphi, \varphi' \rightarrow \rho),
\]
\[
\mathcal{V}(\varphi', \chi) = \mathcal{V}(\phi \rightarrow \chi, \varphi' \rightarrow \chi), \quad \mathcal{V}(\varphi', \rho) = \mathcal{V}(H \rightarrow \chi, \varphi' \rightarrow \rho), \quad \mathcal{V}(\chi, \rho) = \mathcal{V}(H \rightarrow \chi, \varphi' \rightarrow \rho),
\]
\[
\mathcal{V}_{\text{three}} = \lambda_1^2 \phi \varphi' (H\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1} + \lambda_2^2 \phi \varphi' (H\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1} + \lambda_3^2 \phi \varphi' (H\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1},
\]
\[
\mathcal{V}_{\text{four}} = \lambda_1^2 \varphi' (\varphi'\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1} + \lambda_2^2 \varphi' (\varphi'\dagger \varphi')_{2+1} (\varphi'\dagger \varphi')_{2+1}.
\]

It is noted that all the other renormalizable three-and four-scalar interactions are forbidden by one/some of the model symmetries.
Appendix C: The potential minimum condition

\[
\begin{align*}
\mu_H^2v_H + \lambda H\phi\phi'(v_1 + v_2)v_\phi^2 &+ 2\lambda H\phi(v_1 + v_2)^2v_H + \lambda H\phi\phi'(v_1 + v_2)v_H^2 \\
+ 2v_H(\lambda H\phi\phi'(v_1 + v_2)v_\phi^2 + 2\lambda H\phi'v_\phi^2 + \lambda H\chi v_\chi^2 + \lambda H^2v_H^2 + 4\lambda H\phi'v_H^2 + \lambda H^2v_H^2) &= 0, \tag{C1}
\end{align*}
\]

\[
\begin{align*}
\mu_\phi^2v_1 + 2\lambda \phi v_1(2v_1^2 + v_2^2) + 2\lambda \phi v_1v_2^2 &+ 3\lambda \phi v_1v_\phi^2 + 3\lambda \phi v_2v_\phi^2 \\
+ \lambda \phi v_1v_\chi^2 &+ \lambda H\phi\phi'v_\phi^2v_H + 2\lambda \phi v_1v_H^2 + 2\lambda \phi v_2v_H^2 + 5\lambda \phi v_1v_H^2 \\
+ \lambda \phi v_2v_H^2 &+ \lambda H\phi\phi'v_Hv_H^2 + 2\lambda \phi v_1v_H^2 = 0, \tag{C2}
\end{align*}
\]

\[
\begin{align*}
\mu_\phi^2v_2 + 2\lambda \phi v_2(2v_2^2 + v_1^2) + 2\lambda \phi v_2v_\phi^2 &+ 3\lambda \phi v_1v_\phi^2 + 3\lambda \phi v_2v_\phi^2 \\
+ \lambda \phi v_2v_\chi^2 &+ \lambda H\phi\phi'v_\phi^2v_H + 2\lambda \phi v_1v_H^2 + 2\lambda \phi v_2v_H^2 + 5\lambda \phi v_2v_H^2 \\
+ \lambda \phi v_1v_H^2 &+ \lambda H\phi\phi'v_Hv_H^2 + 2\lambda \phi v_1v_H^2 = 0, \tag{C3}
\end{align*}
\]

\[
\begin{align*}
2\mu_\phi^2 + \lambda \phi\phi'(5v_1^2 + 2v_1v_2 + 5v_2^2) + 4\lambda \phi\phi'v_\phi^2 &+ 12\lambda \phi\phi'v_\phi^2 + 5\lambda \phi\chi v_\chi^2 \\
+ 2v_H \left[\lambda H\phi\phi'(v_1 + v_2) + 4\lambda H\phi'v_H^2\right] &+ 12\lambda \phi\phi'v_H^2 + 4\lambda \phi\phi'v_\phi^2 = 0, \tag{C4}
\end{align*}
\]

\[
\begin{align*}
2\mu_\phi^2v_\phi &+ 4v_\phi \left[\lambda \phi\phi'(v_1^2 + v_2^2) + \lambda \phi\phi'v_\phi^2 + 2\lambda \phi\phi'v_\phi^2 + \lambda \phi v_\chi^2 + \lambda H\phi\phi'v_H^2 + 2\lambda \phi\phi'v_H^2 \right] \\
+ 2\lambda \phi\phi'v_\phi &+ 4v_\phi \left[\lambda \phi\phi'(v_1^2 + v_2^2) + \lambda \phi\phi'v_\phi^2 + 2\lambda \phi\phi'v_\phi^2 + \lambda \phi v_\chi^2 + \lambda H\phi\phi'v_H^2 + 2\lambda \phi\phi'v_H^2 \right] = 0, \tag{C5}
\end{align*}
\]

\[
\begin{align*}
2\mu_\phi^2v_\phi &+ 4v_\phi \left[3\lambda \phi\phi'(v_1^2 + v_2^2) + 4\lambda \phi\phi'v_\phi^2 + 12\lambda \phi\phi'v_\phi^2 + 3\lambda \phi\chi v_\chi^2 + 12\lambda \phi\phi'v_H^2 \\
+ 2v_H \left[\lambda H\phi\phi'(v_1 + v_2) + 2\lambda H\phi'v_H^2\right] \right] &+ 12\lambda \phi\phi'v_H^2 + 4\lambda \phi\phi'v_\phi^2 = 0, \tag{C6}
\end{align*}
\]

\[
\begin{align*}
2v_\chi \left[\lambda \phi\chi(v_1^2 + 4v_2^2) + 2\lambda \phi\chi v_\chi^2 + 3\lambda \phi\chi v_\phi^2 + 4\lambda \phi\chi v_\chi^2 + 2\lambda H\chi v_H^2 + 5\lambda \phi\chi v_H^2 \right] \\
+ 2v_\chi \left[\lambda \phi\chi(v_1^2 + 4v_2^2) + 2\lambda \phi\chi v_\chi^2 + 3\lambda \phi\chi v_\phi^2 + 4\lambda \phi\chi v_\chi^2 + 2\lambda H \phi v_H^2 + 5\lambda \phi\chi v_H^2 \right] = 0, \tag{C7}
\end{align*}
\]

\[
\begin{align*}
2v_\rho \left[\lambda \phi\phi(v_1^2 + v_2^2) + \lambda \phi v_\phi^2 + 2\lambda \phi v_\phi^2 + \lambda \phi v_\phi^2 + \lambda H\rho v_H^2 + 2\lambda \phi\phi v_\phi^2 + \rho v_\rho^2 \right] \\
+ v_\rho H v_\rho + 2\lambda \phi\phi v_\phi v_\rho v_\phi v_\rho &= 0, \tag{C8}
\end{align*}
\]

\[
\begin{align*}
\mu_H^2 + 2 \left[\lambda H\phi(v_1 + v_2)^2 + \lambda H\phi v_\phi^2 + 2\lambda H\phi'v_\phi^2 + \lambda H\chi v_\chi^2 + 3\lambda H v_H^2 + 4\lambda H\phi'v_H^2 + \lambda H^2v_H^2 \right] &> 0, \tag{C9}
\end{align*}
\]

\[
\begin{align*}
\mu_\phi^2 + 2\lambda \phi(6v_1^2 + v_2^2) + 2\lambda \phi v_\phi^2 + 3\lambda \phi v_\phi^2 + \lambda \phi x v_\chi^2 + 2\lambda H\phi v_H^2 + 5\lambda \phi\phi v_H^2 + 2\lambda \phi\phi v_\rho^2 > 0, \tag{C10}
\end{align*}
\]

\[
\begin{align*}
\mu_\phi^2 + 2\lambda \phi(v_1^2 + 6v_2^2) + 2\lambda \phi v_\phi^2 + 3\lambda \phi v_\phi^2 + \lambda \phi x v_\chi^2 + 2\lambda H\phi v_H^2 + 5\lambda \phi\phi v_H^2 + 2\lambda \phi\phi v_\rho^2 > 0, \tag{C11}
\end{align*}
\]

\[
\begin{align*}
2\mu_\phi^2 + \lambda \phi\phi'(5v_1^2 + 2v_1v_2 + 5v_2^2) + 4\lambda \phi\phi'v_\phi^2 + 12\lambda \phi\phi'v_\phi^2 + 5\lambda \phi\chi v_\chi^2 + 36\lambda \phi\phi'v_H^2 + 4\lambda \phi\phi'v_H^2 \\
+ 2v_H \left[\lambda H\phi\phi'(v_1 + v_2) + 4\lambda H\phi'v_H^2\right] > 0, \tag{C12}
\end{align*}
\]

\[
\begin{align*}
\mu_\phi^2 + 2 \left[\lambda \phi\phi(v_1^2 + v_2^2) + 3\lambda \phi v_\phi^2 + 2\lambda \phi v_\phi^2 + \lambda \phi x v_\chi^2 + \lambda H\phi v_H^2 + 2\lambda \phi\phi v_H^2 + 2\lambda \phi\phi v_\rho^2 \right] &> 0, \tag{C13}
\end{align*}
\]

\[
\begin{align*}
2\mu_\phi^2 + 3\lambda \phi\phi'(v_1 + v_2)^2 + 4\lambda \phi\phi'v_\phi^2 + 36\lambda \phi\phi'v_\phi^2 + 3\lambda \phi x v_\chi^2 + 12\lambda \phi\phi'v_H^2 + 4\lambda \phi\phi'v_H^2 \\
+ 2v_H \left[\lambda H\phi\phi'(v_1 + v_2) + 2\lambda H\phi'v_H^2\right] > 0, \tag{C14}
\end{align*}
\]

\[
\begin{align*}
\mu_\chi^2 + \lambda \phi\chi(v_2^2 + 4v_2^2) + 2\lambda \phi\chi v_\rho^2 + 3\lambda \phi\chi v_\phi^2 + 12\lambda \phi\chi v_H^2 + 2\lambda H \phi v_H^2 + 5\lambda \phi\chi v_H^2 + 2\lambda \phi\phi v_\rho^2 > 0, \tag{C15}
\end{align*}
\]

\[
\begin{align*}
\mu_\rho^2 + 2 \left[\lambda \phi\rho v_1^2 + v_2^2 + 2\lambda \phi\rho v_\phi^2 + \lambda \phi v_\phi^2 + \lambda H \rho v_H^2 + 2\lambda \phi\rho v_H^2 + 3\lambda \phi v_\rho^2 \right] > 0. \tag{C16}
\end{align*}
\]
Appendix D: The explicit expressions of $\beta_H, \beta_\phi, \beta_{\phi'}, \beta_{\varphi'}, \beta_\chi, \beta_\rho$ and $\beta_{H\phi}$

\[
\beta_H = \left( \mu_0^2 + 2\lambda\phi\varphi v_\varphi^2 \right) v_1 v_2 (v_1 - v_2)(v_1 + v_2)^2
+ \lambda\phi\varphi' \left( 6v_1^5 + 9v_1^4 v_2 + 3v_1^3 v_2^2 - 3v_1^2 v_2^3 - 9v_1 v_2^4 - 6v_2^5 \right) v_\varphi^2
+ \left( \mu_H^2 + 2\lambda H\phi + 4\lambda H\phi' + 2\lambda H\chi \right) \left( 2v_1^3 - v_1^2 v_2 + v_1 v_2^2 - 2v_2^3 \right) v_H^2
+ \lambda\phi\phi' \left( 2v_1^5 + 7v_1^4 v_2 + 5v_1^3 v_2^2 - 5v_1^2 v_2^3 - 7v_1 v_2^4 - 2v_2^5 \right) v'^2
+ \lambda\phi\chi \left( 7v_1^3 + 7v_1^2 v_2 + 2v_1 v_2^2 + 2v_2^3 \right) v_1 v_2 v_2^2 - 8\lambda H\phi' \left( 2v_1^3 - v_1^2 v_2 + v_1 v_2^2 - 2v_2^3 \right) v_H^2
+ 2(v_1 - v_2) \left[ \lambda\phi\chi v_1 v_2 (v_1 + v_2)^2 - \lambda H\rho (2v_1^3 + v_1 v_2^2 + 2v_2^3) v_H^2 \right] v_2^2.
\]

\[
\beta_\phi = \left( \mu_0^2 + 2\lambda\phi\varphi v_\varphi^2 + 4\lambda\phi\phi v_\phi^2 + 2\lambda\phi\phi v_\rho^2 \right) (v_2 - v_1) v_\varphi^2,
\]

\[
\beta_{\phi'} = 2\mu_0^2 + \lambda\phi\phi' (5v_1^2 + 2v_1 v_2 + 5v_2^2) + 4\lambda\phi\phi' v_\phi^2 + 12\lambda\phi\phi' v_\rho^2
+ 4\lambda\phi\varphi' v_\varphi^2 + 5\lambda\phi\chi v_\chi^2 + 2v_H \left[ \lambda H\phi' (v_1 + v_2) + 4\lambda H\phi' v_H \right],
\]

\[
\beta_{\varphi'} = v_\varphi \left\{ \mu_0^2 + 2 \left[ \lambda\phi\varphi (v_1^2 + v_2^2) + 2\lambda\phi\varphi' v_\varphi^2 + \lambda\phi\chi v_\chi^2 + \lambda H\phi v_H^2 + 2\lambda\phi\rho v_\rho^2 \right] \right\}
+ \lambda\phi\rho v_\varphi v_\chi v_\rho + 2\lambda\phi\rho v_\rho^2.
\]

\[
\beta_\chi = \mu_0^2 + \lambda\phi\chi (v_1^2 + 4v_2^2) + 2\lambda\phi\chi v_\chi^2 + 3\lambda\phi\chi v_\varphi^2 + 2\lambda H\chi v_H^2 + 5\lambda\phi\chi v_\rho^2
+ \lambda\phi\rho v_\varphi v_\chi v_\rho + 2\lambda\phi\chi v_\rho^2,
\]

\[
\beta_\rho = \left\{ \mu_0^2 + 2 \left[ \lambda\phi\rho (v_1^2 + v_2^2) + \lambda\phi\rho v_\varphi^2 + 2\lambda\phi\rho v_\rho^2 + \lambda\phi\rho v_\chi + \lambda H\rho v_H^2 + 2\lambda\phi\rho v_\rho^2 \right] \right\} v_\rho
+ \lambda\phi\rho v_\varphi v_\chi v_\rho,
\]

\[
\beta_{H\phi} = (\mu_0^2 + 2\lambda\phi\varphi v_\varphi^2) v_1 v_2 (v_2^3 - v_1^3) - 3\lambda\phi\varphi' (2v_1^4 + v_1^3 v_2 - v_1 v_2^3 - 2v_2^4) v_\varphi^2
+ \lambda\phi\chi (7v_1^2 + 2v_1 v_2 + 2v_2^2) v_1 v_2 v_2^2 - \lambda H\phi' \left( 2v_1^3 - v_1^2 v_2 + v_1 v_2^2 - 2v_2^3 \right) v_H v_2^2
+ 2\lambda\phi\rho v_1 v_2 (v_2^2 - v_1^2) v_\rho^2.
\]

[1] T. Appelquist, B. A. Dobrescu, A. R. Hopper, Phys. Rev. D 68, 035012 (2003) [arXiv: hep-ph/0212073].
[2] N. Okada and S. Okada, Phys. Rev. D 95, 035025 (2017) [arXiv: 1611.02672 [hep-ph]].
[3] A Das, P.S. B. Dev and N. Okada, Phys. Lett. B 799, 135052 (2019) [arXiv: 1906.04132 [hep-ph]].
[4] A. Davidson, Phys. Rev. D 20, 776 (1979).
[5] R. N. Mohapatra and R. E. Marshak, Phys. Rev. Lett. 44, 1316 (1980).
1401.0937 [hep-ph]].

[37] A. E. Cárcamo Hernández, I. de Medeiros Varzielas and N. A. Neill, Phys. Rev. D 94, 033011 (2016) arXiv:1511.07420 [hep-ph]].

[38] A. E. Cárcamo Hernández, Eur. Phys. J. C 76, 503 (2016) arXiv:1512.09092 [hep-ph]].

[39] A. E. Cárcamo Hernández, S. Kovalenko and I. Schmidt, J. High Energy Phys. 02, 125 (2017) arXiv:1611.09797 [hep-ph]].

[40] A. E. Cárcamo Hernández, I. de Medeiros Varzielas and E. Schumacher, Phys. Rev. D 93, 016003 (2016) arXiv:1509.02083 [hep-ph]].

[41] C. Arbeláez, A. E. Cárcamo Hernández, S. Kovalenko and I. Schmidt, Eur. Phys. J. C 76, 422 (2017) arXiv:1612.03607 [hep-ph]].

[42] A. E. Cárcamo Hernández, J. Vignatti and A. Zerwekh, J. Phys. G 46, 115007 (2019) [arXiv:1807.05321 [hep-ph]].

[43] E. A. Garcés, J. C. Gómez-Izquierdo and F. Gonzalez-Canales, Eur. Phys. J. C 78, 812 (2018) arXiv:1807.02727 [hep-ph]].

[44] S. Pramanick, Phys. Rev. D 100, 035009 (2019) arXiv:1904.07558 [hep-ph]].

[45] A. E. Cárcamo Hernández, Y. Hidalgo Velásquez, S. Kovalenko, H. N. Long, N. A. Pérez-Julve and V. V. Vien, Eur. Phys. J. C 81, No. 2 (2021) 191, arXiv:2002.07347 [hep-ph]].

[46] J. D. García-Aguilar and J. C. Gómez-Izquierdo, arXiv: 2010.15370 [hep-ph]].

[47] M.-C. Chen and K. T. Mahanthappa, Phys. Lett. B 652, 34 (2007) [arXiv: 0705.0714 [hep-ph]].

[48] G.-J. Ding, Phys. Rev. D 78, 036011 (2008) [arXiv: 0803.2278 [hep-ph]].

[49] D. A. Eby, P. H. Frampton, X. -G. He and T. W. Kephart, Phys. Rev. D 84, 037302 (2011) [arXiv: 1103.5737 [hep-ph]].

[50] P. H. Frampton, C. M. Ho, T. W. Kephart, Phys. Rev. D 89 (2014) 027701 [arXiv: 1305.4402 [hep-ph]].

[51] A. E. Cárcamo Hernández, Y. Hidalgo Velásquez and N. A. Pérez-Julve, Eur. Phys. J. C 79, 828 (2019) arXiv:1905.02323 [hep-ph]].

[52] V. V. Vien, H. N. Long, A. E. Cárcamo Hernández, Mod. Phys. Lett. A 34, 1950005 (2019) [arXiv: 1812.07263 [hep-ph]].

[53] W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura, M. Tanimoto, J. High Energy Phys. 0407, 078 (2004) [arXiv: hep-ph/0407112].

[54] H. Ishimori et al., Phys. Lett. B 662, 178 (2008) [arXiv: 0802.2310 [hep-ph]].

[55] A. Adulpravitchai, A. Blum, C. Hagedorn, J. High Energy Phys. 0903, 046 (2009) [arXiv: 0812.3799 [hep-ph]].

[56] V. V. Vien and H. N. Long, Int. J. Mod. Phys. A, 28, 1350159 (2013) [arXiv: 1312.5034 [hep-ph]].

[57] V. V. Vien, Mod. Phys. Lett. A 29, 1450122 (2014).

[58] V. V. Vien and H. N. Long, J. Korean Phys. Soc. 66, 1809 (2015) [arXiv: 1408.4333 [hep-ph]].

[59] A. E. Cárcamo Hernández, C. O. Dib and U. J. Saldaña-Salazar, Phys. Lett. B 809, 135750 (2020) [arXiv: 2001.07140 [hep-ph]].
[60] K. S. Babu and J. Kubo, Phys.Rev. D 71, 056006 (2005) [arXiv: hep-ph/0411226 [hep-ph]].
[61] Y. Kajiyama, E. Itou and J. Kubo, decay, Nucl. Phys. B 743, 74 (2006) [arXiv: hep-ph/0511268].
[62] Y. Kajiyama, J. High Energy Phys. 04, 007 (2007) [arXiv: hep-ph/0702056].
[63] N. Kifune, J. Kubo and A. Lenz, Symmetry, Phys.Rev. D 77, 076010 (2008) [arXiv: 0712.0503 [hep-ph]].
[64] K. Babu and Y. Meng, Phys.Rev. D 80, 075003 (2009) [arXiv: 0907.4231 [hep-ph]].
[65] K. Kawashima, J. Kubo and A. Lenz, Phys.Lett. B 681, 60 (2009) [arXiv: 0907.2302 [hep-ph]].
[66] K. Babu, K. Kawashima and J. Kubo, Phys.Rev. D 83, 095008 (2011) [arXiv:1103.1664 [hep-ph]].
[67] J. C. Gmez-Izquierdo, F. Gonzalez-Canales and M. Mondragn, Int. J. Mod. Phys. A 32, 1750171 (2017) [arXiv: 1705.06324 [hep-ph]].
[68] T. Araki and Y. F. Li, Phys. Rev. D 85, 065016 (2012) [arXiv: 1112.5819 [hep-ph]].
[69] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775, 120 (2007) [arXiv: hep-ph/0610165].
[70] K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552, 207 (2003) [arXiv: hep-ph/0206292].
[71] G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005) [arXiv: hep-ph/0504165].
[72] E. Ma, Phys. Rev. D 73, 057304 (2006) [arXiv: hep-ph/0601225].
[73] X. G. He, Y. Y. Keum and R. R. Volkas, J. High Energy Phys. 0604, 039 (2006) [arXiv: hep-ph/0601001].
[74] S. Morisi, M. Picariello, and E. Torrente-Lujan, Phys. Rev. D 75, 075015 (2007) [arXiv: hep-ph/0702034].
[75] F. Bazzocchi, S. Kaneko and S. Morisi, J. High Energy Phys. 0803, 063 (2008) [arXiv: 0707.3032 [hep-ph]].
[76] F. Bazzocchi, M. Frigerio, and S. Morisi, Phys. Rev. D 78, 116018 (2008) [arXiv: 0809.3573 [hep-ph]].
[77] G. Altarelli, F. Feruglio and C. Hagedorn, J. High Energy Phys.0803, 052 (2008) [arXiv: 0802.0090 [hep-ph]].
[78] M. Hirsch, S. Morisi and J. W. F. Valle, Phys. Rev. D 78, 093007 (2008) [arXiv: 0804.1521 [hep-ph]].
[79] E. Ma, Phys. Lett. B 671, 366 (2009) [arXiv: 0808.1729 [hep-ph]].
[80] G. Altarelli and D. Meloni, J. Phys. G 36, 085005 (2009) [arXiv: 0905.0620 [hep-ph]].
[81] Y. Lin, Nucl. Phys. B 813, 91 (2009) [arXiv: 0804.2867 [hep-ph]].
[82] Y. H. Ahn and C. S. Chen, Phys. Rev. D 81, 105013 (2010) [arXiv: 1001.2869 [hep-ph]].
[83] J. Barry and W. Rodejohann, Phys. Rev. D 81, 093002 (2010); Erratum: Phys.Rev. D 81 119901 (2010) [arXiv: 1003.2385 [hep-ph]].
[84] P. V. Dong, L. T. Hue, H. N. Long and D. V. Soa, Phys. Rev. D 81, 053004 (2010) [arXiv: 1001.4625 [hep-ph]].
[85] G. J. Ding and D. Meloni, Nucl. Phys. B 855, 21 (2012) [arXiv: 1108.2733 [hep-ph]].
[86] H. Ishimori et al., Prog. Theor. Phys. Suppl. 183, 1 (2010) [arXiv:1003.3552 [hep-th]].
[87] V. V. Vien, H. N. Long, Int. J. Mod. Phys. A 30, 1550117 (2015) [arXiv: 1405.4665 [hep-ph]].
[88] T. Phong Nguyen, L. T. Hue, D. T. Si and T. T. Thuc, Prog. Theor. Exp. Phys. 2020, 033B04 (2020)
[89] S. Dev and S. Verma, Mod. Phys. Lett. A 25, 2837 (2010) [arXiv: 1005.4521 [hep-ph]].
[90] J. C. Gómez-Izquierdo and M. Mondragón, Eur. Phys. J. C 79, 285 (2019).
[91] V. V. Vien, J. Phys. G 47, 055007 (2020).
[92] V. V. Vien, Nucl. Phys. B 956, 115015 (2020).
[93] H. Ishimori et. al., Prog. Theor. Phys. Suppl. 183, 1 (2010) [arXiv: 1003.3552 [hep-th]].
[94] V. V. Vien, J. Phys. G 47, 055007 (2020).
[95] V. V. Vien, Nucl. Phys. B 956, 115015 (2020).
[96] P. M. Ferreira, Joao P. Silva, Phys.Rev.D 78, 116007 (2008) [arXiv: 0809.2788 [hep-ph]].
[97] J. Kubo, Fortsch.Phys. 61, 597 (2013) [arXiv: 1210.7046 [hep-ph]].
[98] A. Pennelas, A. Pich, J. High Energy Phys. 12, 084 (2017) [arXiv: 1710.02040 [hep-ph]].
[99] Miguel P. Bento, Howard E. Haber, J. C. Romao, Joao P. Silva, J. High Energy Phys. 10, 143 (2018) [arXiv: 1808.07123 [hep-ph]].
[100] M. Arroyo-Urena, J. Lorenzo Díaz-Cruz, Bryan O. Larios-López and M. A. Pérez-de León, Chin. Phys. C 45, 023118 (2021) [arXiv: 1901.01304 [hep-ph]].
[101] W. Rodejohann, U. Saldana-Salazar, J. High Energy Phys. 07, 036 (2019) [arXiv: 1903.00983 [hep-ph]].
[102] P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020).
[103] S. Roy Choudhury and S. Choubey, JCAP 1809, 017 (2018) [arXiv: 1806.10832 [astro-ph.CO]].
[104] A. Gando et al. (KamLAND-Zen Collaboration), Phys. Rev. Lett. 117, 082503 (2016) [arXiv: 1605.02889 [hep-ph]].
[105] M. Agostini et al. (GERDA Collaboration), Phys. Rev. Lett. 120, 132503 (2018) [arXiv:1803.11100 [nucl-ex]].
[106] C. E. Aalseth et al. (Majorana Collaboration), Phys. Rev. Lett. 120, 132502 (2018) [arXiv: 1710.11608 [nucl-ex]].
[107] M. Auger et al. (EXO-200 collaboration), JINST 7, P05010 (2012).
[108] J. B. Albert et al. (EXO-200 collaboration), Nature 510, 229 (2014).
[109] J. B. Albert et al. (EXO-200 Collaboration), Phys. Rev. Lett. 120, 072701 (2018) [arXiv: 1707.08707 [hep-ex]].
[110] C. Alduino et al.(CUORE collaboration), JINST 11, P07009 (2016).
[111] C. Alduino et al.(CUORE collaboration), Phys. Rev. Lett. 120, 132501 (2018) [arXiv: 1710.07988 [nucl-ex]].
[112] Z. z. Xing, Phys. Rept. 854, 1 (2020) [arXiv: 1909.09610 [hep-ph]].
[113] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D 98, 030001 (2018).
[114] A. Dedes and A. Pilaftsis, Phys. Rev. D 67, 015012 (2003) doi:10.1103/PhysRevD.67.015012
[115] A. Aranda, C. Bonilla and J. L. Diaz-Cruz, Phys. Lett. B 717, 248-251 (2012) doi:10.1016/j.physletb.2012.09.011 [arXiv:1204.5558 [hep-ph]].

[116] M. Lindner, M. Platscher and F. S. Queiroz, Phys. Rept. 731, 1-82 (2018) doi:10.1016/j.physrep.2017.12.001 [arXiv:1610.06587 [hep-ph]].

[117] I. Esteban et al., J. High Energy Phys. 01, 106 (2019) [arXiv: 1811.05487 [hep-ph]].
Figure 1: $\delta_2 v_h, \delta_2 v_1, \delta_2 v_2, \delta_2 v_4, \delta_2 v_6, \delta_2 v_8$ and $\delta_2 v_{v_4}$ versus $\lambda^x, \lambda^y$ and $\lambda^z$ with $\lambda^x \in (-10^{-3}, -10^{-5}), \lambda^y \in (-10^{-3}, -10^{-5})$ and $\lambda^z \in (-10^{-3}, -10^{-5})$. 
Figure 2: $|U_{ij}^N|$ ($i = 2, 3; j = 1, 2, 3$) as functions of $\sin \theta_l$ with $\sin \theta_l \in (0.8, 0.9)$ for NO.

Figure 3: $|U_{ij}^I|$ ($i = 2, 3; j = 1, 2, 3$) as functions of $\sin \theta_l$ with $\sin \theta_l \in (-0.9, -0.8)$ for IO.

Figure 4: Allowed region in the $m_H - m_A$ plane consistent with the constraint arising from $K - \overline{K}$ mixing.
Figure 5: Allowed region in the $m_H - m_A$ plane consistent with the constraint arising from $B_d - \bar{B}_d$ (left-plot) and $B_s - \bar{B}_s$ (right-plot) mixings. The couplings of the flavor changing neutral Yukawa interactions have been set to be equal to $2 \times 10^{-4}$ and $10^{-3}$ for the left and right plots, respectively.

Figure 6: Allowed region in the $m_H - m_A$ plane consistent with the charged lepton flavor-violating constraints. The Yukawa couplings of the FCN leptonic Yukawa interactions have been set to be equal to $10^{-2}$ and $5 \times 10^{-3}$ for the left and right plots, respectively.