Multitime controlled linear PDE systems

Cristian Ghiu and Constantin Udrişte

University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics II, Splaiul Independenţei 313, 060042 Bucharest, Romania, e-mail: crisghiu@yahoo.com

University Politehnica of Bucharest, Faculty of Applied Sciences, Department of Mathematics-Informatics I, Splaiul Independenţei 313, 060042 Bucharest, Romania, e-mail: udriste@mathe.m.pub.ro, anet.udri@yahoo.com

Abstract

We derive new results regarding the controllability and the reachability of multitime controlled linear PDE systems of first order. These systems describe some important multitime evolution in engineering, economics and biology. Some of them come from evolution PDEs of superior order. The original results include a refinement and a supplement of multitime optimal control theory, developed in some recent papers by the second author. They refer to the complete integrability conditions, conditions for the existence of solutions, path independent curvilinear integrals, the multitime fundamental matrix, multitime adjoint Cauchy problems, control space, controllability and reachability of phases, controllability gramian, reachability gramian, controllability matrix, counter-examples and commentaries.

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1 Introduction

Here a controlled system is a dynamic multitime linear PDE system on which one can act by using appropriate controls. Among the most common problems that appear when studying such systems are multitime controllability problem and multitime reachability problem.
The multitime controllability refers to pairs of states that can be moved from the first one to the second one and the multitime reachability operates on the reverse order of states. Of course, the order of states is given by the product order (partial order) on multitime source space. The study of controllability of dynamical systems represented by normal PDEs starts in the papers [12] – [23], [3] (multitime maximum principle), [7], [8], [24] (maximum principle in the context of weak derivatives), [4] (numerical methods for robust control), [10], [11] (reachability of hybrid systems). Though many of situations are rather well understood, there are still quite challenging open problems due to the fact that the product order relation on multitime space is not total.

This paper deals with control theory for systems governed by multitime linear PDE systems (m-flows). Section 2 presents a new and complete framework for the multitime nonautonomous linear PDE systems of first order. Section 3 contains original results about controllability and reachability of the controlled multitime nonautonomous linear PDE systems of first order. The controllability and the reachability of multitime autonomous linear PDE systems of first order is analyzed in Section 4. The comments (Section 5) show that in some other situations can occur strange mathematical phenomena due to the discontinuity of controls in multitime evolutions.

2 Nonautonomous linear PDE system of first order

We start with some mathematical ingredients related to evolution PDEs (m-flows). Let $t = (t^1, \ldots, t^m) \in \mathbb{R}^m$, called multitime, $x = (x^1, \ldots, x^n) \top \in \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$, and $G \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open subset. We consider the evolution PDE system

$$\frac{\partial x}{\partial t}(t) = X_\alpha(t, x(t)), \quad \forall \alpha = 1, m, \quad (2.1)$$

where $X_\alpha : G \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$, $X_\alpha = (X^1_\alpha, \ldots, X^m_\alpha)$.

**Definition 2.1.** The PDE system (2.1) is called completely integrable if $\forall (t_0, x_0) \in G$, $\exists D_0 \subseteq \mathbb{R}^m$, $D_0$ open with $t_0 \in D_0$ and $\exists x : D_0 \to \mathbb{R}^n$, $x$ differentiable, such that $(t, x(t)) \in G$, $\forall t \in D_0$, $x$ verifies (2.1) and $x(t_0) = x_0$.

The following Theorems, 2.1 to 2.4, represent new versions of some well-known results [3], [12] – [23].
Theorem 2.1. Suppose the components $X_\alpha$ are of class $C^1$, $\forall \alpha = 1, m$.

i) Any solution of the PDE system (2.1) is of class $C^2$.

ii) If the PDE system (2.1) is completely integrable, then

\[
\frac{\partial X_\alpha}{\partial t^\beta}(t, x) + X_j^\beta(t, x) \frac{\partial X_\alpha}{\partial x^j}(t, x) = \frac{\partial X_\alpha}{\partial t^\alpha}(t, x) + X_j^\alpha(t, x) \frac{\partial X_\beta}{\partial x^j}(t, x),
\]

\[\forall (t, x) \in G, \forall \alpha, \beta = 1, m.\]  (2.2)

or in matrix notations

\[
\frac{\partial X_\alpha}{\partial t^\beta} + \left( \frac{\partial X_\alpha}{\partial x^1} \cdots \frac{\partial X_\alpha}{\partial x^n} \right) X_\beta = \frac{\partial X_\alpha}{\partial t^\alpha}(t, x) + \left( \frac{\partial X_\beta}{\partial x^1} \cdots \frac{\partial X_\beta}{\partial x^n} \right) X_\alpha,
\]

\[\forall (t, x) \in G, \forall \alpha, \beta = 1, m.\]  (2.3)

The relations (2.2) or (2.3) are called the complete integrability conditions.

Theorem 2.2 (Frobenius). Let $G \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open subset and

\[X_\alpha : G \rightarrow \mathbb{R}^n = M_{n,1}(\mathbb{R}), \ X_\alpha \text{ of class } C^1, \forall \alpha = 1, m.\]

a) If the conditions (2.2) are satisfied, then the PDE system (2.1) is completely integrable.

b) Let $D \subseteq \mathbb{R}^m$ be an open and convex subset and $G = D \times \mathbb{R}^n$. Suppose that the following condition is fulfilled: $\exists R \geq 0$ and there exist the continuous functions $\varphi, \psi : D \rightarrow [0, \infty)$ such that

\[||X_\alpha(t, x)|| \leq \varphi(t)||x|| + \psi(t), \ \forall t \in D, \ \forall x \in \mathbb{R}^n, \ ||x|| \geq R, \ \forall \alpha = 1, m.\]  (2.4)

(For example, if the PDE (2.1) is linear, then the conditions (2.4) are satisfied).

If the complete integrability conditions (2.2) are satisfied, then: $\forall (t_0, x_0) \in D \times \mathbb{R}^n$, $\exists x : D \rightarrow \mathbb{R}^n, x$ of class $C^2$, solution of the PDE system (2.1) and $x(t_0) = x_0$.

Theorem 2.3. Let $G \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be an open subset and

\[X_\alpha : G \rightarrow \mathbb{R}^n = M_{n,1}(\mathbb{R}), \ X_\alpha \text{ of class } C^1, \forall \alpha = 1, m.\]

Let $D_1, D_2 \subseteq \mathbb{R}^m$ be open subsets and $y : D_1 \rightarrow \mathbb{R}^n$, $z : D_2 \rightarrow \mathbb{R}^n$ be solutions of the PDE system (2.1). If $D_1 \cap D_2$ is connected and there exists $t_0 \in D_1 \cap D_2$ such that $y(t_0) = z(t_0)$, then $y(t) = z(t)$, $\forall t \in D_1 \cap D_2$. 

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Definition 2.2. Let $D \subseteq \mathbb{R}^m$ be an open subset and $P_\alpha : D \to \mathcal{M}_{n,k}(\mathbb{R})$ be functions of class $C^1$. We say that the curvilinear integral $\int_\gamma P_\alpha(t)dt^\alpha$ is path independent (on $D$), if for any two points $t_0, t_1 \in D$ and any two piecewise $C^1$ curves $\eta : [a, b] \to D$, $\lambda : [c, d] \to D$, with $\eta(a) = \lambda(c) = t_0$ and $\eta(b) = \lambda(d) = t_1$, we have

$$\int_\eta P_\alpha(t)dt^\alpha = \int_\lambda P_\alpha(t)dt^\alpha.$$

Theorem 2.4. Let $D \subseteq \mathbb{R}^m$ be an open subset and $P_\alpha : D \to \mathcal{M}_{n,k}(\mathbb{R})$ be $C^1$ functions, $\forall \alpha = 1, m$. If $D$ is a convex set (sufficiently, connected and simply connected), then the following statements are equivalent:

i) $\frac{\partial P_\alpha}{\partial t^\beta}(t) = \frac{\partial P_\beta}{\partial t^\alpha}(t), \forall t \in D, \forall \alpha = 1, m.$

ii) $\exists \xi : D \to \mathcal{M}_{n,k}(\mathbb{R})$ solution of the PDE system

$$\frac{\partial \xi}{\partial t^\alpha}(t) = P_\alpha(t), \forall t \in D, \forall \alpha = 1, m.$$

iii) The curvilinear integral

$$\int_\gamma P_\alpha(t)dt^\alpha$$

is path independent on the set $D$.

In the conditions i) - iii), we have:

a) If $\xi$ is a solution of the PDE system of ii), and $\gamma : [a, b] \to D$ is a piecewise $C^1$ curve, then

$$\int_\gamma P_\alpha(t)dt^\alpha = \xi(\gamma(b)) - \xi(\gamma(a)).$$

b) Let $t_0 \in D$ be a fixed point. For $t \in D$, let $\gamma_{t_0,t} : [a, b] \to D$ be a piecewise $C^1$ curve, from $t_0$ to $t$. The primitive

$$\xi : D \to \mathcal{M}_{n,k}(\mathbb{R}), \xi(t) = \int_{\gamma_{t_0,t}} P_\alpha(s)ds^\alpha$$

is a solution of the PDE system of ii).
For example, if $D$ is a star-shaped set with respect to $t_0$, then the primitive $\xi$ can be written alternatively

$$\xi(t) = \int_0^1 (t^\alpha - t_0^\alpha) P_\alpha((1 - \tau)t_0 + \tau t) \, d\tau.$$ 

In case that the PDE system $(2.1)$ is linear, i.e.,

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + F_\alpha(t), \quad \forall \alpha = 1, m,$$

with $M_\alpha : D \to \mathcal{M}_n(\mathbb{R})$ and $F_\alpha : D \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$ of class $C^1$, the complete integrability conditions become

$$\frac{\partial M_\alpha}{\partial t^\beta}(t)x + \frac{\partial F_\alpha}{\partial t^\beta}(t) = \frac{\partial M_\beta}{\partial t^\alpha}(t)x + \frac{\partial F_\beta}{\partial t^\alpha}(t),$$

$$\forall t \in D, \forall x \in \mathbb{R}^n, \forall \alpha, \beta = 1, m,$$

which is equivalent to

$$\frac{\partial M_\alpha}{\partial t^\beta}(t) + M_\alpha(t)M_\beta(t) = \frac{\partial M_\beta}{\partial t^\alpha}(t) + M_\beta(t)M_\alpha(t) \quad (2.6)$$

$$M_\alpha(t)F_\beta(t) + \frac{\partial F_\alpha}{\partial t^\beta}(t) = M_\beta(t)F_\alpha(t) + \frac{\partial F_\beta}{\partial t^\alpha}(t). \quad (2.7)$$

We have obtained the following result:

**Theorem 2.5.** *Let $D \subseteq \mathbb{R}^m$ be an open and convex subset, let $M_\alpha : D \to \mathcal{M}_n(\mathbb{R})$ be $C^1$ matrix functions, $\forall \alpha = 1, m$ and let $F_\alpha : D \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$ be $C^1$ vector functions, $\forall \alpha = 1, m$. Suppose the relations $\eqref{integrability1}, \eqref{integrability2}$ are true, $\forall t \in D, \forall \alpha, \beta = 1, m$. Then the problem

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + F_\alpha(t), \quad \forall \alpha = 1, m,$$

$$x(t_0) = x_0$$

has a unique solution $x : D \to \mathbb{R}^n$. This solution is of class $C^2$.***
Further, everywhere, $D$ will be an open and convex subset of $\mathbb{R}^m$, and $M_\alpha : D \to \mathcal{M}_n(\mathbb{R})$, $\forall \alpha = 1, m$, are matrix functions of class $\mathcal{C}^1$, which verifies the relations (2.6), $\forall t \in D$, $\forall \alpha, \beta = 1, m$.

There exists a unique matrix solution
\[
\chi(\cdot, t_0) : D \to \mathcal{M}_n(\mathbb{R})
\]
of the problem
\[
\begin{align*}
\frac{\partial X}{\partial t^\alpha} &= M_\alpha(t)X, \quad \forall \alpha = 1, m \\
X(t_0) &= I_n.
\end{align*}
\] (2.8)

(For those $n$ problems equivalent to the matrix problem, we apply the Theorem 2.5).

**Definition 2.3.** The matrix function
\[
\chi(\cdot, \cdot) : D \times D \to \mathcal{M}_n(\mathbb{R})
\]
is called the fundamental matrix.

**Proposition 2.1.** The fundamental matrix has the following properties:

a) $\chi(t_0, t_1)\chi(t_0, t_1) = \chi(t, t_1)$, $\forall t, t_0, t_1 \in D$,

b) $\chi(t_0, t_0) = I_n$, $\forall t_0 \in D$,

c) $\chi(t, t_0)^{-1} = \chi(t_0, t)$, $\forall t_0, t \in D$.

d) $\frac{\partial}{\partial t^\alpha}(\chi(t, t)) = -\chi(t, t)M_\alpha(t)$, $\forall t \in D$, $\forall \alpha$.

**Proof.**
a) If $Y(t) = \chi(t, t_0)\chi(t_0, t_1)$, then
\[
\begin{align*}
\frac{\partial Y}{\partial t^\alpha} &= \frac{\partial}{\partial t^\alpha}(\chi(t, t_0)\chi(t_0, t_1)) = M_\alpha(t)\chi(t_0, t_1) + \chi(t, t_0)\frac{\partial}{\partial t^\alpha}(\chi(t_0, t_1)) = M_\alpha(t)Y; \\
Y(t_0) &= \chi(t_0, t_0)\chi(t_0, t_1) = I_n\chi(t_0, t_1) = \chi(t_0, t_1).
\end{align*}
\]

Hence $Y(t)$ and $\chi(t, t_1)$ are both solutions of the matrix PDE system
\[
\frac{\partial X}{\partial t^\alpha} = M_\alpha(t)X, \quad \forall \alpha = 1, m,
\]
which coincide for $t = t_0$. From uniqueness it follows that $Y(t) = \chi(t, t_1)$, $\forall t$.

b) Direct consequence of the definition of the function $\chi(t, t_0)$.

c) It follows readily from a) and b). For a), we take $t_1 = t$, etc.

d) Differentiating the identity
\[
\chi(t_0, t_0)\chi(t_0, t) = I_n
\]
with respect to \( t^\alpha \), we find
\[
\frac{\partial}{\partial t^\alpha} (\chi(t, t_0))\chi(t_0, t) + \chi(t, t_0) \frac{\partial}{\partial t^\alpha} (\chi(t_0, t)) = 0
\]
or
\[
M_\alpha(t)\chi(t, t_0)\chi(t_0, t) + \chi(t, t_0) \frac{\partial}{\partial t^\alpha} (\chi(t_0, t)) = 0,
\]
i.e.,
\[
\chi(t, t_0) \frac{\partial}{\partial t^\alpha} (\chi(t_0, t)) = -M_\alpha(t).
\]
Multiplying at the left-hand side by \( \chi(t, t_0)^{-1} = \chi(t_0, t) \), we get
\[
\frac{\partial}{\partial t^\alpha} (\chi(t_0, t)) = -\chi(t_0, t)M_\alpha(t).
\]

\[ \square \]

Proposition 2.2. The Cauchy problem
\[
\frac{\partial x}{\partial t^\alpha} = M_\alpha(t) x, \quad \forall \alpha = 1, m, \tag{2.9}
\]
\[
x(t_0) = x_0
\]
has the solution \( x : D \to \mathbb{R}^n, x(t) = \chi(t, t_0)x_0 \).

Definition 2.4. Let us consider the PDE system \textcolor{red}{(2.9)}. The homogeneous PDE system
\[
\frac{\partial y}{\partial t^\alpha}(t) = -M_\alpha^T(t)y(t), \quad \forall \alpha = 1, m \tag{2.10}
\]
is called the adjoint system.

The complete integrability conditions of the adjoint system are
\[
-\frac{\partial M_\alpha^T}{\partial t^\beta} + M_\alpha^T M_\beta^T = -\frac{\partial M_\beta^T}{\partial t^\alpha} + M_\beta^T M_\alpha^T
\]
or
\[
\frac{\partial M_\alpha}{\partial t^\beta} + M_\beta M_\alpha = -\frac{\partial M_\beta}{\partial t^\alpha} + M_\alpha M_\beta
\]
\[
\frac{\partial M_\alpha}{\partial t^\beta} + M_\beta M_\alpha = -\frac{\partial M_\beta}{\partial t^\alpha} + M_\alpha M_\beta
\]
i.e., identical to the relations \textcolor{red}{(2.6)} of complete integrability of the system \textcolor{red}{(2.9)}.

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Proposition 2.3. a) The matrix solution of the Cauchy problem

\[ \frac{\partial X}{\partial t^\alpha} = -M_\alpha^T X, \quad \forall \alpha = 1, m \]

is \( \Phi(t, t_0) = \chi(t_0, t)^T \).

b) The solution of the adjoint Cauchy problem

\[ \frac{\partial \varphi}{\partial t^\alpha} = -M_\alpha^T \varphi, \quad \forall \alpha = 1, m \]

is

\[ \varphi(t_0) = \varphi_0 \]

The solution of the Cauchy problem

\[ \frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + F_\alpha(t), \quad \forall \alpha = 1, m, \]

is

\[ x(t_0) = x_0 \]

Proof. a) We use the Proposition 2.1, d), i.e.,

\[ \frac{\partial}{\partial t^\alpha}(\chi(t_0, t)) = -\chi(t_0, t)M_\alpha, \]

which is equivalent to

\[ \frac{\partial}{\partial t^\alpha}(\chi(t_0, t)^T) = -M_\alpha^T \chi(t_0, t)^T. \]

b) follows immediately from a). \( \square \)

Theorem 2.6. In the conditions of Theorem 2.5 the solution of the Cauchy problem

\[ \frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + F_\alpha(t), \quad \forall \alpha = 1, m, \]

is

\[ x : D \to \mathbb{R}^n, \quad x(t) = \chi(t, t_0)x_0 + \int_{\gamma_{t_0,t}} \chi(t, s)F_\alpha(s)ds^\alpha, \]

where \( \gamma_{t_0,t} \) is a piecewise \( C^1 \) curve, included in \( D \), covered from \( t_0 \) to \( t \).

The curvilinear integral \( \int_{\gamma} \chi(t, s)F_\alpha(s)ds^\alpha \) is path independent.
Proof. We show that the curvilinear integral is path independent. According to the Theorem 2.4, we must show that

$$\frac{\partial}{\partial s} (\chi(t, s) F_\alpha(s)) = \frac{\partial}{\partial s} (\chi(t, s) F_\beta(s))$$

or

$$-\chi(t, s) M_\beta(s) F_\alpha(s) + \chi(t, s) \frac{\partial F_\alpha}{\partial s}(s) = -\chi(t, s) M_\alpha(s) F_\beta(s) + \chi(t, s) \frac{\partial F_\beta}{\partial s}(s),$$

and these are equivalent to the relations (2.7).

Now we write the sheet $x(t)$ as

$$x(t) = \chi(t, t_0)x_0 + \chi(t, t_0) \int_{\gamma_{t_0, t}} \chi(t_0, s) F_\alpha(s) ds.$$  

According to the Theorem 2.4 we get

$$\frac{\partial}{\partial t} \left( \int_{\gamma_{t_0, t}} \chi(t_0, s) F_\alpha(s) ds \right) = \chi(t_0, t) F_\beta(t).$$

It follows that

$$\frac{\partial x}{\partial t}(t) = M_\beta(t) \chi(t, t_0)x_0 +$$

$$+ M_\beta(t) \chi(t, t_0) \int_{\gamma_{t_0, t}} \chi(t_0, s) F_\alpha(s) ds + \chi(t, t_0) \chi(t_0, t) F_\beta(t)$$

$$= M_\beta(t) \left( \chi(t, t_0)x_0 + \int_{\gamma_{t_0, t}} \chi(t, s) F_\alpha(s) ds \right) + F_\beta(t) = M_\beta(t) x(t) + F_\beta(t).$$

One verifies easily the initial condition $x(t_0) = x_0$. 

3 Controlled nonautonomous linear PDE system of first order

Our main results include generalizations to multitime case of the single-time control (see, for example, [2], [9]) in the vision of Lawrence C. Evans, Lev S. Pontryagin. They are complementary to the results in [3], [4], [7], [8], [10] – [23]. Related topics can be found in the papers [1], [6].
Let $D \subseteq \mathbb{R}^m$ be an open and convex subset, let $M_\alpha : D \to \mathcal{M}_n(\mathbb{R})$ be $C^1$ quadratic matrix functions, let $N_\alpha : D \to \mathcal{M}_{n,k}(\mathbb{R})$ be $C^1$ rectangular matrix functions, and let $u_\alpha : D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R})$ be $C^1$ vector functions, all indexed after $\alpha = 1, \ldots, m$.

We consider the evolution PDE system

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + N_\alpha(t)u_\alpha(t), \quad \forall \alpha = 1, m. \tag{3.1}$$

Its complete integrability conditions are equivalent to

$$\frac{\partial M_\alpha}{\partial t^\beta}(t) + M_\alpha(t)M_\beta(t) = \frac{\partial M_\beta}{\partial t^\alpha}(t) + M_\beta(t)M_\alpha(t),$$

$$M_\alpha(t)N_\beta(t)u_\beta(t) + \frac{\partial N_\alpha}{\partial t^\beta}(t)u_\alpha(t) + N_\alpha(t)\frac{\partial u_\alpha}{\partial t^\beta}(t)$$

$$= M_\beta(t)N_\alpha(t)u_\alpha(t) + \frac{\partial N_\beta}{\partial t^\alpha}(t)u_\beta(t) + N_\beta(t)\frac{\partial u_\beta}{\partial t^\alpha}(t),$$

$\forall t \in D, \forall \alpha, \beta = 1, m. \tag{3.2}$

**Definition 3.1.** Suppose that the matrix functions $M_\alpha(\cdot)$ verify the relations (2.6), $\forall t \in D, \forall \alpha, \beta = 1, m$. The vector space

$$\mathcal{U} = \left\{ u = (u_\alpha)_{\alpha=1,m} \mid u_\alpha : D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R}), \text{ of class } C^1, \forall \alpha = 1, m \right\}$$

and which verify the relations (3.2) for all $\alpha, \beta$ is called the control space.

From the Theorem 2.6 we obtain immediately

**Theorem 3.1.** If the matrix functions $M_\alpha(\cdot)$ verify the relations (2.6), $\forall t \in D, \forall \alpha, \beta = 1, m$ and $u = (u_\alpha)_{\alpha=1,m}$ is a control, then the Cauchy problem

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + N_\alpha(t)u_\alpha(t), \quad \forall \alpha = 1, m.$$  

$$x(t_0) = x_0 \quad (t_0 \in D, \ x_0 \in \mathbb{R}^n)$$

has a unique solution

$$x : D \to \mathbb{R}^n, \ x(t) = \chi(t, t_0)x_0 + \int_{\gamma_{t_0, t}} \chi(t, s)N_\alpha(s)u_\alpha(s)ds^\alpha,$$
where $\gamma_{t_0,t}$ is a piecewise $C^1$ curve, included in $D$, covered from $t_0$ to $t$.

The curvilinear integral $\int_\gamma \chi(t,s)N_\alpha(s)u_\alpha(s))ds$ is path independent and the solution $x(\cdot)$ is of class $C^2$.

Further, in this paper, $D$ will be an open and convex subset of $\mathbb{R}^m$, the $C^1$ quadratic matrix functions $M_\alpha : D \to \mathcal{M}_n(\mathbb{R}), \forall \alpha = \overline{1,m}$ will verify the relations $(2.6)$, $\forall t \in D$, $\forall \alpha, \beta = \overline{1,m}$ and the rectangular matrix functions $N_\alpha : D \to \mathcal{M}_{n,k}(\mathbb{R}),$ will be of class $C^1, \forall \alpha = \overline{1,m}$.

**Definition 3.2.** The pair $(s,y)$, $s \in D$, $y \in \mathbb{R}^n$ is called phase of the PDE system $(3.1)$.

a) Let $(t_0, x_0), (s,y) \in D \times \mathbb{R}^n$. We say that the phase $(t_0, x_0)$ transfers to the phase $(s,y)$ if the Cauchy problems $\{(3.1), x(t_0) = x_0\}$ and $\{(3.1), x(s) = y\}$ have the same solution (for the same control $u(\cdot)$); or, equivalently, the solution $x(t)$ of the Cauchy problem $\{(3.1), x(t_0) = x_0\}$ verifies also the condition $x(s) = y$. We will say that the control $u(\cdot)$ transfers the phase $(t_0, x_0)$ into the phase $(s,y)$.

b) The phase $(t,x)$ is called reachable (respectively pseudo-reachable) if there exists a point $t_0 \in D$, with $t_0^\alpha < t_\alpha$, $\forall \alpha$, (respectively, if there exists a point $t_0 \in D$, $t_0 \neq t$), and there exists a control $u(\cdot)$ which transfers the phase $(t_0,0)$ into the phase $(t,x)$.

c) The phase $(t,x)$ is called controllable (respectively, pseudo-controllable) if there exists a point $s \in D$, with $s^\alpha > t_\alpha$, $\forall \alpha$, (respectively, if there exists a point $s \in D$, $s \neq t$), and a control $u(\cdot)$ which transfers the phase $(t,x)$ into the phase $(s,0)$.

d) Let $t_0,t \in D$, with $t_0^\alpha < t_\alpha$, $\forall \alpha$, (respectively, let $t_0,t \in D$, $t_0 \neq t$).

The PDE system $(3.1)$ is called completely reachable (respectively completely pseudo-reachable) from $t_0$ to $t$ if for any point $x \in \mathbb{R}^n$, the phase $(t_0,0)$ transfers to the phase $(t,x)$, i.e., for any $x$, the phase $(t,x)$ is reachable (respectively, pseudo-reachable) with the same $t_0$.

e) Let $t \in D$. The PDE system $(3.1)$ is called completely reachable (respectively, completely pseudo-reachable) at the moment $t$, if for any $t_0 \in D$, with $t_0^\alpha < t_\alpha$, $\forall \alpha$, (respectively $\forall t_0 \in D$, $t_0 \neq t$), and for any $x \in \mathbb{R}^n$, the phase $(t_0,0)$ transfers into the phase $(t,x)$.

f) Let $t_0,t \in D$, with $t_0^\alpha < t_\alpha$, $\forall \alpha$ (respectively, let $t_0,t \in D$, $t_0 \neq t$).

The PDE system $(3.1)$ is called completely controllable (respectively, completely pseudo-controllable) from $t_0$ to $t$ if for any point $x \in \mathbb{R}^n$, the phase $(t_0,x)$ transfers into the phase $(t,0)$, i.e., for any point $x$ the phase $(t_0,x)$ is controllable (respectively, pseudo-controllable) with the same $t$. 

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g) Let \( t_0 \in D \). The PDE system (3.1) is called completely controllable (respectively, completely pseudo-controllable) at the moment \( t_0 \), if \( \forall t \in D \), with \( t^* > t_0 \), \( \forall \alpha \) (respectively, \( \forall t \in D \), \( t_0 \neq t \)), and for any point \( x \in \mathbb{R}^n \), the phase \((t_0, x)\) transfers into the phase \((t, 0)\).

h) The PDE system (3.1) is called completely reachable (respectively, completely pseudo-reachable) if it is completely reachable (respectively, completely pseudo-reachable) at any moment of \( D \).

The PDE system (3.1) is called completely controllable (respectively, completely pseudo-controllable) if it is completely controllable (respectively, completely pseudo-controllable) at any moment of \( D \).

The multitime control property does not only depend on the dimensions \( m \) and \( n \) but on how matrices \( M_\alpha \) and \( N_\alpha \) interact.

The phase \((t_0, x_0)\) transfers into the phase \((t_1, y)\) \iff \exists u(\cdot) = (u_\alpha(\cdot)) a control such that the solution \( x(\cdot) \) of the problem \( \{3.1\}, x(t_0) = x_0 \) verifies also \( x(t_1) = y \), equivalent to

\[
\exists u(\cdot) = (u_\alpha(\cdot)) \text{ a control such that }
\]

\[
x(t) = \chi(t, t_0)x_0 + \int_{\gamma_{t_0,t}} \chi(t, t_0)\chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds^\alpha \text{ and } x(t_1) = y
\]

\[
\iff \exists u(\cdot) = (u_\alpha(\cdot)) \text{ a control such that }
\]

\[
y = \chi(t_1, t_0)(x_0 + \int_{\gamma_{t_0,t_1}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds^\alpha)
\]

\[
\iff \exists u(\cdot) = (u_\alpha(\cdot)) \text{ a control such that }
\]

\[
\chi(t_0, t_1)y - x_0 = \int_{\gamma_{t_0,t_1}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds^\alpha.
\]

We introduce the set

\[
\mathcal{V}(t_0, t) := \left\{ \int_{\gamma_{t_0,t}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds^\alpha \right\} | (u_\alpha)_{\alpha=1}^{m} \text{ is a control}.
\]

The set \( \mathcal{V}(t_0, t) \) is a vector subspace of \( \mathbb{R}^n \). It is called the controllability space. Since the curvilinear integral is path independent, we remark that \( \mathcal{V}(t_0, t) \) does not depend on the curve \( \gamma_{t_0,t} \), which joins \( t_0 \) to \( t \), but depends on the multetimes \( t_0 \) and \( t \). Also \( \chi(t, t_0)\mathcal{V}(t_0, t) = \mathcal{V}(t, t_0) \).

From the foregoing arguments, it follows immediately
Theorem 3.2. Let us consider the system (3.1), with the matrix functions $M_\alpha(\cdot)$ verifying the relations (2.6).

i) The control $(u_\alpha)_{\alpha=1,\ldots,m}$ transfers the phase $(t_0, x_0)$ to the phase $(t, y)$ if and only if

\[ \chi(t_0, t)y - x_0 = \int_{\gamma_{t_0,t}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds. \]

ii) The control $(u_\alpha)_{\alpha=1,\ldots,m}$ transfers the phase $(t_0, x_0)$ to the phase $(t, 0)$ if and only if

\[ x_0 + \int_{\gamma_{t_0,t}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds = 0. \]

iii) The phase $(t_0, x_0)$ transfers into the phase $(t, y)$ if and only if

\[ x_0 - \chi(t_0, t)y \in \mathcal{V}(t_0, t) \]

equivalent to

\[ y - \chi(t, t_0)x_0 \in \mathcal{V}(t, t_0). \]

iv) The phase $(t_0, x_0)$ is controllable (respectively, pseudo-controllable) if and only if \( \exists t \in D, \text{ with } t^\alpha > t_0^\alpha, \forall \alpha \) (respectively, \( \exists t \in D, \text{ with } t \neq t_0 \)) such that

\[ x_0 \in \mathcal{V}(t_0, t). \]

v) The phase $(t, y)$ is reachable (respectively, pseudo-reachable) if and only if \( \exists t_0 \in D, \text{ with } t_0 < t^\alpha, \forall \alpha \) (respectively, \( \exists t_0 \in D, \text{ with } t_0 \neq t \)) such that

\[ y \in \mathcal{V}(t, t_0). \]

vi) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha, \forall \alpha$ (respectively, let $t_0, t \in D$, $t_0 \neq t$). The PDE system is completely controllable (respectively, completely pseudo-controllable) from the multitime $t_0$ into the multitime $t$ if and only if

\[ \mathcal{V}(t_0, t) = \mathbb{R}^n, \quad \text{equality equivalent to} \quad \mathcal{V}(t_0, t) = \mathbb{R}^n. \]

vii) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha, \forall \alpha$ (respectively, let $t_0, t \in D$, $t_0 \neq t$). The PDE system is completely reachable (respectively, completely pseudo-reachable) from the multitime $t_0$ into the multitime $t$ if and only if

\[ \mathcal{V}(t, t_0) = \mathbb{R}^n, \quad \text{equality equivalent to} \quad \mathcal{V}(t, t_0) = \mathbb{R}^n. \]

viii) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha, \forall \alpha$ (respectively, let $t_0, t \in D$, $t_0 \neq t$). The PDE system is completely controllable (respectively, completely pseudo-controllable) from the multitime $t_0$ into the multitime $t$ if and only if it is completely reachable (respectively, completely pseudo-reachable) from $t_0$ to $t$.  

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According to the Theorem [2.4] the curvilinear integral

\[ \int_{\gamma} \chi(t_0, s)N_\alpha(s)N_\alpha^T(s)\chi(t_0, s)^T ds^\alpha \]

is path independent if and only if, for any \( \alpha, \beta = \overline{1, m} \), the following conditions are satisfied:

\[
\frac{\partial}{\partial s^\beta}(\chi(t_0, s))N_\alpha N_\alpha^T \chi(t_0, s)^T + \chi(t_0, s) \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T \chi(t_0, s)^T + \\
\chi(t_0, s) N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} \chi(t_0, s)^T + \chi(t_0, s) N_\alpha \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T \chi(t_0, s)^T + \\
\chi(t_0, s) N_\beta \frac{\partial N_\beta}{\partial s^\alpha} \chi(t_0, s)^T + \chi(t_0, s) N_\beta \frac{\partial N_\alpha}{\partial s^\beta} \chi(t_0, s)^T \]

or

\[
-\chi(t_0, s) M_\beta N_\alpha N_\alpha^T \chi(t_0, s)^T + \chi(t_0, s) \frac{\partial N_\beta}{\partial s^\alpha} N_\alpha^T \chi(t_0, s)^T + \\
\chi(t_0, s) M_\alpha N_\beta N_\beta^T \chi(t_0, s)^T + \chi(t_0, s) N_\beta \frac{\partial N_\beta}{\partial s^\alpha} N_\alpha^T \chi(t_0, s)^T + \\
\chi(t_0, s) N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} \chi(t_0, s)^T - \chi(t_0, s) N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T \chi(t_0, s)^T + \\
\chi(t_0, s) N_\beta \frac{\partial N_\beta}{\partial s^\alpha} \chi(t_0, s)^T - \chi(t_0, s) N_\beta \frac{\partial N_\beta}{\partial s^\alpha} N_\alpha^T \chi(t_0, s)^T.
\]

Since the fundamental matrix \( \chi(t_0, s) \) is invertible, the foregoing equality is equivalent to

\[
-M_\beta N_\alpha^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} - N_\alpha N_\alpha^T M_\beta^T \\
= -M_\beta N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} - N_\beta N_\beta^T M_\alpha^T.
\]

In this way, we have proved

**Proposition 3.1.** Let \( t_0 \in D \), fixed. The curvilinear integral

\[ \int_{\gamma} \chi(t_0, s)N_\alpha(s)N_\alpha^T(s)\chi(t_0, s)^T ds^\alpha \]
is path independent on $D$ (in the sense of definition 2.1) if and only if, for any $\alpha, \beta = \overline{1, m}$, the relations

$$
M_\alpha N_\beta N_\beta^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} + N_\beta N_\beta^T M_\alpha^T \\
= M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} + N_\alpha N_\alpha^T M_\beta^T
$$

(3.3)

are verified on $D$. This is equivalent to,

$$
M_\alpha N_\beta N_\beta^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + \left( M_\alpha N_\beta N_\beta^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T \right)^T \\
= M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + \left( M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T \right)^T
$$

(3.4)

or

$$
M_\alpha N_\beta N_\beta^T + N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + \left( M_\alpha N_\beta N_\beta^T + N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} \right)^T \\
= M_\beta N_\alpha N_\alpha^T + N_\beta \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + \left( M_\beta N_\alpha N_\alpha^T + N_\beta \frac{\partial N_\beta}{\partial s^\alpha} \right)^T.
$$

(3.5)

It is sufficient, for example, that for any $\alpha, \beta = \overline{1, m}$, to have

$$
M_\alpha N_\beta N_\beta^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T = M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T
$$

(3.6)

or

$$
M_\alpha N_\beta N_\beta^T + N_\alpha \frac{\partial N_\alpha}{\partial s^\beta} = M_\beta N_\alpha N_\alpha^T + N_\beta \frac{\partial N_\beta}{\partial s^\alpha}.
$$

(3.7)

**Proposition 3.2.** Let us suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2.6), $\forall t \in D$, $\forall \alpha, \beta = \overline{1, m}$. We fix $t_0 \in D$. For each $v \in \mathbb{R}^n$ and $\alpha = \overline{1, m}$, we consider the functions

$$
u_{\alpha, v} : D \to \mathbb{R}^k, \quad u_{\alpha, v}(s) = N_\alpha^T(s) \chi(t_0, s)^T v, \quad \forall s \in D.$$

The following statements are equivalent

i) For any $v \in \mathbb{R}^n$, the family $(u_{\alpha, v})_{\alpha = \overline{1, m}}$ is a control for the PDE system (3.3).

ii) For any $\alpha, \beta = \overline{1, m}$, the relations (3.3) are satisfied on the set $D$, i.e.,

$$
M_\alpha N_\beta N_\beta^T + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} + N_\beta N_\beta^T M_\alpha^T \\
= M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} + N_\alpha N_\alpha^T M_\beta^T.
$$
iii) The curvilinear integral

\[ \int_\gamma \chi(t_0, s) N_\alpha(s) N_\alpha^T(s) \chi(t_0, s) \, ds^\alpha \]

is path independent on the set \( D \).

**Proof.** The family \((u_\alpha, v_\alpha)_{\alpha=1}^m\) is a control if and only if it verifies, \( \forall \alpha, \beta = 1, m \), the relations (3.2) on the set \( D \), i.e.,

\[
M_\alpha N_\beta N_\beta^\top \chi(t_0, s)^\top v + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T \chi(t_0, s)^\top v +
\]

\[
+ N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} \chi(t_0, s)^\top v =
\]

\[
M_\beta N_\alpha N_\alpha^T \chi(t_0, s)^\top v + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T \chi(t_0, s)^\top v +
\]

\[
+ N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} \chi(t_0, s)^\top v =
\]

equivalent to

\[
M_\alpha N_\beta N_\beta^\top \chi(t_0, s)^\top v + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T \chi(t_0, s)^\top v +
\]

\[
+ N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} \chi(t_0, s)^\top v - N_\alpha N_\alpha^T M_\beta^\top \chi(t_0, s)^\top v =
\]

\[
M_\beta N_\alpha N_\alpha^T \chi(t_0, s)^\top v + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T \chi(t_0, s)^\top v +
\]

\[
+ N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} \chi(t_0, s)^\top v - N_\beta N_\beta^T M_\alpha^\top \chi(t_0, s)^\top v
\]

or

\[
\left( M_\alpha N_\beta N_\beta^\top + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} - N_\alpha N_\alpha^T M_\beta^\top \right) \chi(t_0, s)^\top v =
\]

\[
\left( M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} - N_\beta N_\beta^T M_\alpha^\top \right) \chi(t_0, s)^\top v,
\]

equivalent to

\[
\left( M_\alpha N_\beta N_\beta^\top + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^T + N_\alpha \frac{\partial N_\alpha^T}{\partial s^\beta} + N_\beta N_\beta^T M_\alpha^\top \right) \chi(t_0, s)^\top v =
\]

\[
\left( M_\beta N_\alpha N_\alpha^T + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^T + N_\beta \frac{\partial N_\beta^T}{\partial s^\alpha} + N_\alpha N_\alpha^T M_\beta^\top \right) \chi(t_0, s)^\top v.
\]
The implication $ii) \implies i)$ follows immediately.

Let us prove $i) \implies ii)$. Since for each $v \in \mathbb{R}^n$, the family $(u_{\alpha,v})_{\alpha=1}^{m}$ is a control, it follows that the relations $(\ast)$ hold for any $v \in \mathbb{R}^n$, whence we deduce that for any matrix $A \in \mathcal{M}_{n,\mathbb{R}}$, $\forall p$, we have

$$
\begin{align*}
(M_{\alpha}N_{\beta}N_{\alpha}^\top + \frac{\partial N_{\alpha}}{\partial s^\beta}N_{\alpha}^\top + N_{\alpha}\frac{\partial N_{\beta}^\top}{\partial s^\alpha} + N_{\beta}N_{\beta}^\top M_{\alpha}^\top)\chi(t_0,s)^\top A
&= (M_{\beta}N_{\alpha}N_{\alpha}^\top + \frac{\partial N_{\beta}}{\partial s^\alpha}N_{\beta}^\top + N_{\beta}\frac{\partial N_{\alpha}^\top}{\partial s^\beta} + N_{\alpha}N_{\alpha}^\top M_{\beta}^\top)\chi(t_0,s)^\top A.
\end{align*}
$$

Taking $A = I_n$, we find

$$
\begin{align*}
(M_{\alpha}N_{\beta}N_{\alpha}^\top + \frac{\partial N_{\alpha}}{\partial s^\beta}N_{\alpha}^\top + N_{\alpha}\frac{\partial N_{\beta}^\top}{\partial s^\alpha} + N_{\beta}N_{\beta}^\top M_{\alpha}^\top)\chi(t_0,s)^\top
&= (M_{\beta}N_{\alpha}N_{\alpha}^\top + \frac{\partial N_{\beta}}{\partial s^\alpha}N_{\beta}^\top + N_{\beta}\frac{\partial N_{\alpha}^\top}{\partial s^\beta} + N_{\alpha}N_{\alpha}^\top M_{\beta}^\top)\chi(t_0,s)^\top.
\end{align*}
$$

The matrix $\chi(t_0,s)^\top$ is invertible. The last equality is multiplied in the right-hand side by $(\chi(t_0,s)^\top)^{-1}$, obtaining the relation (3.3).

The equivalence of the statements $ii)$ and $iii)$ is just the Proposition 3.1. \(\Box\)

**Definition 3.3.** Suppose that, for any $\alpha, \beta = 1, m$, the relations (3.3) are true. The matrix function

$$
\mathcal{C} : D \times D \to \mathcal{M}_n(\mathbb{R}), \mathcal{C}(t_0, t) := \int_{\gamma_{t_0,t}} \chi(t_0,s)N_{\alpha}(s)N_{\alpha}^\top(s)\chi(t_0,s)^\top ds^\alpha
$$

is called the controllability gramian.

The matrix function

$$
\mathcal{R}(t_0, t) := \int_{\gamma_{t_0,t}} \chi(t,s)N_{\alpha}(s)N_{\alpha}^\top(s)\chi(t,s)^\top ds^\alpha
$$

is called the reachability gramian.

The controllability gramian is used to determine whether or not a linear PDE system is controllable. The reachability gramian is used to determine whether or not a linear PDE system is reachable. One observes immediately that

$$
\mathcal{R}(t_0, t) = -\mathcal{C}(t, t_0), \ \forall t_0, t \in D
$$

and

$$
\chi(t,t_0)\mathcal{C}(t_0,t)\chi(t,t_0)^\top = -\mathcal{C}(t, t_0), \ \forall t_0, t \in D.
$$

Hence the matrices $\mathcal{C}(t_0, t), \mathcal{C}(t, t_0), \mathcal{R}(t_0, t), \mathcal{R}(t, t_0)$ have all the same rank.
Definition 3.4. Let $A \in \mathcal{M}_{p,q}(\mathbb{R})$ be a real matrix. Denote $\text{Im}(A)$ and $\text{Ker}(A)$, the image, respectively the kernel of the linear map

$$f : \mathbb{R}^q = \mathcal{M}_{q,1}(\mathbb{R}) \to \mathbb{R}^p = \mathcal{M}_{p,1}(\mathbb{R}), \quad f(x) = Ax.$$  

Of course, the subset $\text{Im}(A)$ is a vector subspace of $\mathcal{M}_{p,1}(\mathbb{R})$ generated by the columns of the matrix $A$.

Theorem 3.3. In the conditions of Theorem 3.1, if $\forall \alpha, \beta = \overline{1,m}$, the conditions (3.3) are true, then, for any $t$ and $t_0$ with $t^\alpha \geq t_0^\alpha$, $\forall \alpha = \overline{1,m}$ (or $t^\alpha \leq t_0^\alpha$, $\forall \alpha = \overline{1,m}$), we have

$$\mathcal{V}(t_0, t) = \text{Im}(C(t_0, t)).$$

Proof. The inclusion $\mathcal{V}(t_0, t) \subseteq \text{Im}(C(t_0, t))$ is equivalent to

$$(\mathcal{V}(t_0, t))^\perp \supseteq (\text{Im}(C(t_0, t)))^\perp = \text{Ker}((C(t_0, t))^\top).$$

We have $b \in \text{Ker}((C(t_0, t))^\top) \iff ((C(t_0, t))^\top)b = 0 \iff b^\top C(t_0, t) = 0.$  

Hence $b^\top C(t_0, t)b = 0.$

The controllability gramian is independent on the curve $\gamma$ covered from the multitime $t_0$ to the multitime $t$. Particularly, we fix $\gamma$ as being the straight line segment which joins the points $t_0, t$, i.e., $\gamma(\tau) = \tau(t - t_0) + t_0, \tau \in [0, 1]$. It follows

$$C(t_0, t) = \int_0^1 \sum_{\alpha=1}^m (t^\alpha - t_0^\alpha)\chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))N_\alpha^\top(\gamma(\tau))\chi(t_0, \gamma(\tau))^\top d\tau$$

$$= \int_0^1 \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} (t^\alpha - t_0^\alpha)\chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))N_\alpha^\top(\gamma(\tau))\chi(t_0, \gamma(\tau))^\top d\tau$$

On the other hand we get

$$\int_0^1 \sum_{\alpha \text{ with } t^\alpha > t_0^\alpha} (t^\alpha - t_0^\alpha)b^\top \chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))N_\alpha^\top(\gamma(\tau))\chi(t_0, \gamma(\tau))^\top b d\tau = 0,$$

or

$$\int_0^1 \sum_{\alpha \text{ with } t^\alpha > t_0^\alpha} (t^\alpha - t_0^\alpha)\left\|b^\top \chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))\right\|^2 d\tau = 0.$$
It follows that for any \( \alpha \) with \( t^\alpha \neq t^\alpha_0 \) and, \( \forall \tau \in [0, 1] \), we have
\[
  b^\top \chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau)) = 0.
\]
Hence
\[
b^\top \int_0^1 \sum_{\alpha=1}^m \chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))u_\alpha(\gamma(\tau))\gamma^\alpha(\tau) \, d\tau
\]
\[
= \int_0^1 \sum_{\alpha \text{ with } t^\alpha \neq t^\alpha_0} b^\top \chi(t_0, \gamma(\tau))N_\alpha(\gamma(\tau))u_\alpha(\gamma(\tau))(t^\alpha - t^\alpha_0) \, d\tau = 0.
\]
Consequently \( b \in \left( V(t_0, t) \right)^\perp \).

Let us prove the inclusion \( \text{Im}(\mathcal{C}(t_0, t)) \subseteq V(t_0, t) \). For that, we select
\[
w = \mathcal{C}(t_0, t)v \in \text{Im}(\mathcal{C}(t_0, t)), \quad w = \int_{\gamma_{t_0, t}} \chi(t_0, s)N_\alpha(s)N_\alpha^\top(s)\chi(t_0, s)^\top v \, ds^\alpha.
\]
We choose
\[
u_\alpha(s) = N_\alpha^\top(s)\chi(t_0, s)^\top v.
\]
According to the Proposition 3.2, the family \((u_\alpha)_{\alpha=1}^m\) is a control, hence
\[
w = \int_{\gamma_{t_0, t}} \chi(t_0, s)N_\alpha(s)u_\alpha(s) \, ds^\alpha \in V(t_0, t).
\]

If, for any \( \alpha, \beta = 1, m \), the relations (3.3) are true (hence we have also \( V(t_0, t) = \text{Im}(\mathcal{C}(t_0, t)) \) for \( t^\alpha > (or <) t^\alpha_0, \forall \alpha \)), then from the Theorem 3.2 it follows

**Theorem 3.4.** Suppose that we are in the conditions of the Theorem 3.1 and furthermore, for any \( \alpha, \beta = 1, m \), the conditions (3.3) are true.

i) Let \( t_0, t \in D, t_0 \neq t, t^\alpha_0 \leq t^\alpha, \forall \alpha \) or \( t^\alpha \leq t^\alpha_0, \forall \alpha \). Then the phase \((t_0, x_0)\) transfers to the phase \((t, y)\) if and only if
\[
x_0 - \chi(t_0, t)y \in \text{Im}(\mathcal{C}(t_0, t)),
\]
which is equivalent to
\[
y - \chi(t, t_0)x_0 \in \text{Im}(\mathcal{C}(t, t_0)).
\]
ii) The phase \((t_0, x_0)\) is controllable if and only if \(\exists t \in D\), with \(t^\alpha > t^0_0\), \(\forall \alpha\), such that
\[x_0 \in \text{Im}(C(t_0, t)).\]

iii) The phase \((t, y)\) is reachable if and only if \(\exists t_0 \in D\), with \(t^0_0 < t^\alpha\), \(\forall \alpha\), such that
\[y \in \text{Im}(C(t, t_0)).\]

iv) Let \(t_0, t \in D\), with \(t^0_0 < t^\alpha\), \(\forall \alpha\). The PDE system is completely controllable from \(t_0\) to \(t\) if and only if
\[\text{rank } C(t_0, t) = n \quad \iff \quad \mathbb{R}^n = \text{Im}(C(t_0, t)).\]

v) Let \(t_0, t \in D\), with \(t^0_0 < t^\alpha\), \(\forall \alpha\). The PDE system is completely reachable from \(t_0\) into \(t\) if and only if
\[\text{rank } C(t_0, t) = n \quad \iff \quad \mathbb{R}^n = \text{Im}(C(t_0, t)).\]

4 Controlled autonomous linear PDE system of first order

A very special case is those of controlled autonomous linear PDE system of first order, when the matrix functions \(M_\alpha, N_\alpha\) are constants. Then the relation (2.6) becomes
\[M_\alpha M_\beta = M_\beta M_\alpha,\]
the relation (3.2) reduces to
\[M_\alpha N_\beta u_\beta + N_\alpha \frac{\partial u_\alpha}{\partial s^\beta} = M_\beta N_\alpha u_\alpha + N_\beta \frac{\partial u_\beta}{\partial s^\alpha}\]
and the relation (3.3) can be written as
\[M_\alpha N_\beta N_\beta^\top + N_\beta N_\beta^\top M_\alpha^\top = M_\beta N_\alpha N_\alpha^\top + N_\alpha N_\alpha^\top M_\beta^\top.\]

On the other hand, we have
\[\chi(t, t_0) = e^{M_\alpha(t^\alpha - t^0_0)}.\]

In this case, the fundamental matrix function \(\chi(t, t_0)\) is defined for any \((t, t_0) \in \mathbb{R}^m \times \mathbb{R}^m\), and not only for \((t, t_0) \in D \times D\). Moreover, \(\chi(t, t_0) = \chi(t - t_0, 0)\).
The controllability gramian matrix becomes
\[ C(t_0, t) = \int_{t_0}^{t} e^{-M_\beta(s^\beta-t_0^\beta)}N_\alpha N_\alpha^\top e^{-M_\beta(s^\beta-t_0^\beta)}\, ds^\alpha. \]

Since the relations (3.3) are verified on \( \mathbb{R}^m \), from the Proposition 3.1 it follows that the curvilinear integral is path independent on \( \mathbb{R}^m \). Hence the controllability gramian \( C(t_0, t) \) is defined on \( \mathbb{R}^m \times \mathbb{R}^m \) and also \( C(t_0, t) = C(0, t - t_0) \).

In fact, if the matrix functions \( M_\alpha \) and \( N_\alpha \) are constant \( \forall \alpha \), one can take \( D = \mathbb{R}^m \).

**Definition 4.1.** For \( \alpha = 1, m \), let us consider the constant matrices \( M_\alpha \in \mathcal{M}_n(\mathbb{R}) \), \( N_\alpha \in \mathcal{M}_{n,k}(\mathbb{R}) \), such that
\[ M_\alpha M_\beta = M_\beta M_\alpha, \quad \forall \alpha, \beta = 1, m. \]

For each \( \alpha = 1, m \), we define the matrix
\[ G_\alpha = \left( N_\alpha \quad M_1 N_\alpha \quad M_2 N_\alpha \quad \ldots \quad M_m N_\alpha \quad \ldots \quad M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} \cdot N_\alpha \quad \ldots \right) \]
made from all block matrices of the form
\[ M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} \cdot N_\alpha \]
with \( 0 \leq k_1; k_2; \ldots; k_m \leq n - 1 \). Further, we need to specify the order in which one ranges the block matrices \( M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} \cdot N_\alpha \) in the matrix \( G_\alpha \). In this way, the matrix \( G_\alpha \) will be well defined.

On the set
\[ \left\{(k_1; k_2; \ldots; k_m) \in \mathbb{N}^m \mid 0 \leq k_1; k_2; \ldots; k_m \leq n - 1 \right\}, \]
we define the following order relation, denoted by \( \preceq \):
\[ (k_1; k_2; \ldots; k_m) \preceq (q_1; q_2; \ldots; q_m) \quad \text{if} \]
\[ k_1 + k_2 + \cdots + k_m < q_1 + q_2 + \cdots + q_m \]
\[ \text{or} \quad k_1 + k_2 + \cdots + k_m = q_1 + q_2 + \cdots + q_m \quad \text{and} \]
\[ k_1 > q_1 \text{ or } \exists j = 2, \ldots, m \text{ such that } k_1 = q_1, k_2 = q_2, \ldots, k_{j-1} = q_{j-1}, k_j > q_j \]
\[ \text{or} \]
\[(k_1; k_2; \ldots; k_m) = (q_1; q_2; \ldots; q_m).\]

One verifies quickly that \(\preceq\) is an order relation. The block matrices

\[
M_{k_1}^1 \cdot M_{k_2}^2 \cdots M_{k_m}^m \cdot N_\alpha
\]

are ranged in \(G_\alpha\) in increasing order of \((k_1; k_2; \ldots; k_m)\), relative to the order relation \(\preceq\). This means in fact that the block matrices are written in the increasing order of the sum \(k_1 + k_2 + \cdots + k_m\); in case that two such sums are equal, the block matrices are written in lexicographic decreasing order of \((k_1; k_2; \ldots; k_m)\).

The matrix

\[
G := \left( \begin{array}{cccc}
G_1 & G_2 & \ldots & G_m
\end{array} \right).
\]

is called the controllability matrix of the PDE system (3.1).

**Theorem 4.1.** Let \(t_0, t \in \mathbb{R}^m\) such that \(t_0^\alpha < t^\alpha\), \(\forall \alpha = 1, m\). Then

\[
\text{rank } C(t, t_0) = \text{rank } C(t_0, t) = \text{rank } G,
\]

relation equivalent to

\[
\text{Im } C(t_0, t) = \text{Im } G,
\]

or

\[
\text{Ker}(C(t_0, t)^\top) = \text{Ker}(G^\top).
\]

**Proof.** We already have seen that rank \(C(t, t_0) = \text{rank } C(t_0, t)\). Hence, it is enough to show the equality \(\text{Ker}(C(t_0, t)^\top) = \text{Ker}(G^\top)\).

The inclusion \(\text{Ker}(C(t_0, t)^\top) \subseteq \text{Ker}(G^\top)\): let \(b\) such that \((C(t_0, t))^\top b = 0\), or \(b^\top C(t_0, t) b = 0\). Consequently we have and \(b^\top C(t_0, t) b = 0\).

Let \(s\) such as \(t_0^\alpha < s^\alpha < t^\alpha\), \(\forall \alpha = 1, m\), \(s\) fixed but arbitrarily chosen. Instead of the curve \(\gamma_{t_0, t}\) we select the union of the segments \([t_0, s]\) and \([s, t]\), covered from \(t_0\) to \(s\), respectively from \(s\) to \(t\). Let us parameterize these segments by

\[
\begin{align*}
\lambda_1(\tau) &= (1 - \tau)t_0 + \tau s, \quad \forall \tau \in [0, 1], \\
\lambda_2(\tau) &= (1 - \tau)s + \tau t, \quad \forall \tau \in [0, 1].
\end{align*}
\]

Obviously \(\gamma_{t_0, t}\) is a piecewise \(C^1\) curve. From \(b^\top C(t_0, t) b = 0\), we find

\[
\begin{align*}
\int_0^1 b^\top e^{-M_\beta(\lambda_1^\beta(\tau) - t_0^\beta)} N_\alpha N_\alpha^\top \left( e^{-M_\beta(\lambda_1^\beta(\tau) - t_0^\beta)} \right)^\top b(s^\alpha - t_0^\alpha) d\tau \\
+ \int_0^1 b^\top e^{-M_\beta(\lambda_2^\beta(\tau) - t_0^\beta)} N_\alpha N_\alpha^\top \left( e^{-M_\beta(\lambda_2^\beta(\tau) - t_0^\beta)} \right)^\top b(t^\alpha - s^\alpha) d\tau &= 0;
\end{align*}
\]

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\[
\int_{0}^{1} \left( \left\| b^\top e^{-M_\beta(\lambda_1^\beta(\tau) - t^\beta_0)} N_\alpha \right\|^2 (s^\alpha - t^\alpha_0) + \left\| b^\top e^{-M_\beta(\lambda_2^\beta(\tau) - t^\beta_0)} N_\alpha \right\|^2 (t^\alpha - s^\alpha) \right) d\tau = 0.
\]

Since the integrand is positive, it follows
\[
\sum_{\alpha=1}^{m} \left\| b^\top e^{-M_\beta(\lambda_1^\beta(\tau) - t^\beta_0)} N_\alpha \right\|^2 (s^\alpha - t^\alpha_0) + \sum_{\alpha=1}^{m} \left\| b^\top e^{-M_\beta(\lambda_2^\beta(\tau) - t^\beta_0)} N_\alpha \right\|^2 (t^\alpha - s^\alpha) = 0, \ \forall \tau \in [0; 1].
\]

But \(s^\alpha - t^\alpha_0 > 0\) and \(t^\alpha - s^\alpha > 0\), hence
\[
b^\top e^{-M_\beta(\lambda_1^\beta(\tau) - t^\beta_0)} N_\alpha = 0, \ \forall \tau \in [0; 1], \ \forall \alpha = \overline{1, m}
\]
and
\[
b^\top e^{-M_\beta(\lambda_2^\beta(\tau) - t^\beta_0)} N_\alpha = 0, \ \forall \tau \in [0; 1], \ \forall \alpha = \overline{1, m}.
\]

We set \(\tau = 1\), hence \(\lambda_3^\beta(1) = s^\beta\) and one obtains
\[
b^\top e^{-M_\beta(s^\beta - t^\beta_0)} N_\alpha = 0, \ \forall \alpha = \overline{1, m};
\]
valid equalities \(\forall s^\beta \in (t^\beta_0; t^\beta), \ \beta = \overline{1, m}\). But, taking into account the continuity, it follows
\[
b^\top e^{-M_\beta(s^\beta - t^\beta_0)} N_\alpha = 0, \ \forall \alpha = \overline{1, m}, \quad (4.1)
\]
and \(\forall s\) such that \(t^\beta_0 \leq s^\beta < t^\beta, \ \forall \beta\).

Let \(0 \leq k_1, k_2, \ldots, k_m \leq n - 1\). In (4.1) we differentiate with respect to \(s^1\) of \(k_1\) times, with respect to \(s^2\) of \(k_2\) times, \ldots, with respect to \(s^m\) of \(k_m\) times (we take also into account the relation (2.6)):
\[
(-1)^{k_1+k_2+\ldots+k_m} b^\top e^{-M_\beta(s^\beta - t^\beta_0)} M_1^{k_1} M_2^{k_2} \ldots M_m^{k_m} N_\alpha = 0,
\]
while for \(s = t_0\) we have
\[
b^\top M_1^{k_1} M_2^{k_2} \ldots M_m^{k_m} N_\alpha = 0,
\]

hence also \(b^\top G = 0\) or \(G^\top b = 0\), i.e., \(b \in \text{Ker}(G^\top)\).
The inclusion $\ker(G^\top) \subseteq \ker(C(t_0, t)^\top)$: let $b$ such that $G^\top b = 0$, or $b^\top G = 0$. It follows that
\[ b^\top M_{k_1}^{k_1} M_{k_2}^{k_2} \cdots M_{k_m}^{k_m} N_\alpha = 0, \quad \forall 0 \leq k_1, k_2, \ldots, k_m \leq n - 1. \]
From the Hamilton-Cayley Theorem, and taking into account the relations $M_\alpha M_\beta = M_\beta M_\alpha$, we deduce
\[ b^\top M_{k_1}^{k_1} M_{k_2}^{k_2} \cdots M_{k_m}^{k_m} N_\alpha = 0, \quad \forall k_1, k_2, \ldots, k_m \geq 0, \]
valid equalities $\forall k_1, k_2, \ldots, k_m \geq 0$. Hence we have $\forall p \geq 0, \forall \alpha = 1, m,$
\[ b^\top \left( \sum_{\beta=1}^m b^\top \left( M_\beta (t_0^\beta - s^\beta) \right)^p \right) N_\alpha = 0. \]
It follows that $b^\top C(t_0, t) = 0$ or $(C(t_0, t))^\top b = 0$, hence $b \in \ker(C(t_0, t)^\top)$. □

**Remark 4.1.** From the proof of the Theorem 4.1, we see that the inclusion
\[ \ker(G^\top) \subseteq \ker(C(t_0, t)^\top), \quad \text{i.e., } \im C(t_0, t) \subseteq \im G, \]
is true for any $t_0, t \in \mathbb{R}^m$, hence we have also
\[ \rank C(t_0, t) \leq \rank G, \quad \forall t_0, t \in \mathbb{R}^m. \]

**Example 4.1.** There exist linear autonomous PDE systems for which we have $t_0, t \in \mathbb{R}^m$, $t_0 \neq t$, $t_0^\alpha \leq t^\alpha$, $\forall \alpha$, such that $\rank C(t_0; t) < \rank G$. Indeed, let us take
\[ m = 2, \quad n = 2, \quad k = 1, \quad D = \mathbb{R}^2 \]
\[ N_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = 0. \]
These verify $M_1 M_2 = M_2 M_1 = 0$ and $M_1 N_1 = N_1$, $M_1 N_2 = 0$. Control space: $u = (u_1, u_2)$ is a control if and only if
\[ M_1 N_2 u_2 + N_1 \frac{\partial u_1}{\partial t_2} = M_2 N_1 u_1 + N_2 \frac{\partial u_2}{\partial t_1}; \]
the relations $M_1 N_2 = 0$, $M_2 = 0$ transforms the foregoing PDE in

$$N_1 \frac{\partial u_1}{\partial t_2} = N_2 \frac{\partial u_2}{\partial t_1}$$

equivalent to

$$\frac{\partial u_1}{\partial t_2} = 0; \quad \frac{\partial u_2}{\partial t_1} = 0.$$

Consequently $u = (u_1, u_2)$ is a control if and only if there exist $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$, of class $C^1$ such that

$$u_1(t_1, t_2) = f_1(t_1), \quad u_2(t_1, t_2) = f_2(t_2), \quad \forall (t_1, t_2) \in \mathbb{R}^2.$$

The relation

$$M_1 N_2 N_2^\top + N_2 N_2^\top M_1^\top = M_2 N_1 N_1^\top + N_1 N_1^\top M_2^\top$$

(see the condition (3.3)) is obvious since $M_1 N_2 = 0$ and $M_2 = 0$.

The rank of the matrix

$$G = \begin{pmatrix} N_1 & M_1 N_1 & M_2 N_1 & M_1 M_2 N_1 & N_2 & M_1 N_2 & M_2 N_2 & M_1 M_2 N_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is 2.

We compute the matrix $C(t_0, t)$, with $t_0 = 0 = (0, 0)$, $t = (t_1, 0)$, $(t_1 \neq 0)$, i.e., $t_0^2 = t^2 = 0$. For that, we select $\gamma(\tau) = (\tau, 0)$, $\tau \in [0, t_1]$; $\gamma$ is a curve joining the two-time $(0, 0)$ with $t = (t_1, 0)$. Then

$$C((0, 0); (t^1, 0)) = \int_0^{t^1} e^{-\tau} M_1 \cdot N_1 N_1^\top \cdot (e^{-\tau} M_1)^\top d\tau.$$

But

$$e^{-\tau} M_1 = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & 0 \end{pmatrix}, \quad e^{-\tau} M_1 \cdot N_1 = \begin{pmatrix} e^{-\tau} \\ 0 \end{pmatrix};$$

$$C((0, 0); (t^1, 0)) = \int_0^{t^1} \begin{pmatrix} e^{-\tau} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\tau} & 0 \\ 0 & 0 \end{pmatrix} d\tau = \int_0^{t^1} \begin{pmatrix} e^{-2\tau} & 0 \\ 0 & 0 \end{pmatrix} d\tau;$$

$$C((0, 0); (t^1, 0)) = \begin{pmatrix} 1 - e^{-2t_1} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Consequently the rank of the matrix $C((0, 0); (t^1, 0))$ is 1, strictly smaller than the rank of $G$.

The Theorem 3.4 can be rewritten as
Theorem 4.2. For \( \alpha = 1, m \), let us consider the constant matrices \( M_\alpha \in M_n(\mathbb{R}) \), \( N_\alpha \in M_{n,k}(\mathbb{R}) \), such that

\[
M_\alpha M_\beta = M_\beta M_\alpha, \quad \forall \alpha, \beta = 1, m,
\]

\[
M_\alpha N_\beta N_\beta^\top + N_\beta N_\beta^\top M_\alpha^\top = M_\beta N_\alpha N_\alpha^\top + N_\alpha N_\alpha^\top M_\beta^\top, \quad \forall \alpha, \beta = 1, m.
\]

We consider the autonomous PDE system

\[
\frac{\partial x}{\partial t^\alpha} = M_\alpha x + N_\alpha u_\alpha(t), \quad \forall \alpha = 1, m,
\]

where \( u = (u_\alpha)_{\alpha=1,m} \) is a control, i.e., \( u_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^k = M_{k,1}(\mathbb{R}) \) is of class \( C^1 \), \( \forall \alpha \), and

\[
M_\alpha N_\beta u_\beta(t) + N_\alpha \frac{\partial u_\alpha}{\partial s^\beta}(t) = M_\beta N_\alpha u_\alpha(t) + N_\beta \frac{\partial u_\beta}{\partial s^\alpha}(t), \quad \forall t \in \mathbb{R}^m, \quad \forall \alpha, \beta = 1, m.
\]

Let \( G \) be the controllability matrix of this PDE system.

i) If \( t^\alpha > (\text{or} <) t^\alpha_0 \), \( \forall \alpha \), then the phase \((t_0, x_0)\) transfers to the phase \((t, y)\) if and only if

\[
x_0 - e^{M_\alpha(t^\alpha - t^\alpha_0)} y \in \text{Im}(G),
\]

equivalent to

\[
y - e^{M_\alpha(t^\alpha - t^\alpha_0)} x_0 \in \text{Im}(G).
\]

ii) The phase \((t_0, x_0)\) is controllable if and only if \( x_0 \in \text{Im}(G) \).

One observes that if exists a multitime \( t_0 \) such that the phase \((t_0, x_0)\) is controllable, then for any multitime \( t \), the phases \((t, x_0)\) are controllable.

iii) The phase \((t, y)\) is reachable if and only if \( y \in \text{Im}(G) \).

One observes that if there exists a multitime \( t \) such that the phase \((t, y)\) is reachable, then for any multitime \( s \), the phases \((s, y)\) are reachable.

iv) If the phase \((t_0, x_0)\) is controllable (or reachable), then for any multitime \( t \), the phases \((t, x_0)\) are controllable and reachable.

v) Let \( t_0, t \in D \), with \( t_0^\alpha < t^\alpha \), \( \forall \alpha \). The PDE system is completely controllable from \( t_0 \) to \( t \) if and only if \( \text{rank } G = n \).

vi) Let \( t_0, t \in D \), with \( t_0^\alpha < t^\alpha \), \( \forall \alpha \). The PDE system is completely reachable from \( t_0 \) to \( t \) if and only if \( \text{rank } G = n \).

vii) If there exist \( t_0, t \in D \), with \( t_0^\alpha < t^\alpha \), \( \forall \alpha \) and if the PDE system is completely controllable (or completely reachable) from \( t_0 \) to \( t \), then the PDE system is completely controllable and completely reachable (equivalent to \( \text{rank } G = n \)).
Example 4.2. Let us give an example of multitime linear PDE system, with constant matrices $M_\alpha, N_\alpha$, for which we have rank $G = n$, but no state $(t_0, x_0)$, with $x_0 \neq 0$, is controllable. Let us consider

$$m = 3; \quad n = 3; \quad k = 1; \quad D = \mathbb{R}^3;$$

$$M_1 = M_2 = M_3 \overset{\text{not}}{=} M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The foregoing matrices verify the relations

$$MN_1 = N_2, \quad MN_2 = N_3, \quad MN_3 = N_1$$

and the relations (2.6) are obvious.

Let us determine the control space: $u = (u_1, u_2, u_3)$ is a control if and only if the relations (3.2) are true. If in (3.2) we put $\alpha = 2, \beta = 3$, then

$$M_2N_3u_3(t) + N_2\frac{\partial u_2}{\partial t^3}(t) = M_3N_2u_2(t) + N_3\frac{\partial u_3}{\partial t^2}(t), \quad \forall t \in \mathbb{R}^3,$$

or

$$u_3(t)N_1 + \frac{\partial u_2}{\partial t^2}(t)N_2 = u_2(t)N_3 + \frac{\partial u_3}{\partial t^2}(t)N_3, \quad \forall t \in \mathbb{R}^3.$$

Since $N_1, N_2, N_3$ are linearly independent, it follows that

$$u_3(t) = 0, \quad u_2(t) + \frac{\partial u_3}{\partial t^2}(t) = 0, \quad \forall t \in \mathbb{R}^3,$$

hence $u_3(t) = u_2(t) = 0, \forall t \in \mathbb{R}^3$.

If in (3.2) we take $\alpha = 1, \beta = 2$, then we get

$$M_1N_2u_2(t) + N_1\frac{\partial u_1}{\partial t^2}(t) = M_2N_1u_1(t) + N_2\frac{\partial u_2}{\partial t^2}(t), \quad \forall t \in \mathbb{R}^3,$$

or

$$N_1\frac{\partial u_1}{\partial t^2}(t) = N_2u_1(t), \quad \forall t \in \mathbb{R}^3,$$

hence $u_1(t) = 0, \forall t \in \mathbb{R}^3$. We have proved that the control space is $\{0\}$, and consequently

$$\mathcal{V}(t_0, t) = \{0\}, \quad \forall (t_0, t) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

These results and the Theorem 3.2 show that no phase $(t_0, x_0)$, with $x_0 \neq 0$, is controllable or reachable.

We remark that the controllability matrix $G$ has the rank $3 = n$. The conclusion of the Theorem 4.2 is no longer valid. The reason is that the conditions (3.3) are not true.
5 Comments

In this paper, the functions which define the PDE systems (for example (2.1), (3.1), etc.) are of class $C^1$ and satisfy the complete integrability conditions (of type (2.2), (2.6)+(2.7), (2.6)+(3.2)). Also, the general relations (2.4) are verified - we have linear PDE systems. Hence, throughout, the Cauchy problem \{(3.1), x(t_0) = x_0\} has a unique solution, global defined, and it is of class $C^2$ (Theorems 2.1, 2.2, 2.3).

Sometimes, in the papers [12] – [23], the functions which define the PDE systems (for example, the controls) are piecewise $C^1$ functions; in this case, the complete integrability conditions are piecewise satisfied. Identically, the solutions will verify the PDEs in the piecewise sense. Generally, the solutions are not continuous functions.

In the paper [13] it is indicated a construction of the solution of a Cauchy problem associated to a linear PDE system. It is similar (in a certain sense) with those obtained in the Theorem 3.1. But, in this context, the Cauchy problem has not a unique solution. To maintain this idea, we give the following example: let $m = 2$, $n = 1$ (hence $x(\cdot) = x_1(\cdot)$), $k = 1$, $D = \mathbb{R}^2$,

$$M_1(t) = M_2(t) = 0, \quad N_1(t) = N_2(t) = 1, \quad \forall t \in \mathbb{R}^2;$$

$$u_1(t^1, t^2) = \begin{cases} 1, & \text{if } t^1 + t^2 \geq 1 \\ 0, & \text{if } t^1 + t^2 < 1 \end{cases}; \quad u_2(t^1, t^2) = 0, \forall (t^1, t^2) \in \mathbb{R}^2.$$

Set $t_0 = (0, 0)$, $x_0 = 0$, i.e., $x(0, 0) = 0$ and we formulate the Cauchy problem

$$\frac{\partial x}{\partial t^1} = u_1(t^1, t^2), \quad \frac{\partial x}{\partial t^2} = 0, \quad x(0, 0) = 0.$$

This PDE system satisfies the piecewise complete integrability conditions (2.6) and (3.2) (this can be easily checked); the conditions (3.2) are true there where the control $u_1(\cdot)$ is of class $C^1$, i.e., on the non-connected set

$$\mathbb{R}^2 \setminus \{(t^1, t^2)|t^1 + t^2 = 1\}.$$

Here, we have $\chi(t, s) = 1$, $\forall (t, s) \in \mathbb{R}^2 \times \mathbb{R}^2$. If $t = (t^1, t^2) \in \mathbb{R}^2$ is an arbitrary fixed two-time, then

$$x(t^1, t^2) = \chi(t, t_0)x_0 + \int_{\gamma_{(0, t)}} \chi(t, s)N_\alpha(s)u_\alpha(s)ds^\alpha = \int_{\gamma_{(0, t)}} u_1(s^1, s^2)ds^1.$$

For $t^1 + t^2 < 1$, we obtain obviously $x(t^1, t^2) = 0$. It remains to study the case $t^1 + t^2 > 1.$
Let \( a \in \mathbb{R} \) be an arbitrary point. We consider the curve \( \gamma_{t_0,t} \) consisting in two segments: the first is the segment which joins the point \( t_0 = (0,0) \) to the point \((a,1-a)\), on the straight line \( s^1 + s^2 = 1 \), where \( u_1(\cdot) \) is discontinuous; the second segment joins the point \((a,1-a)\) to the point \((t^1,t^2)\). The first segment, without the point \((a,1-a)\), is situated in the semiplane \( \{(s^1,s^2)|s^1 + s^2 < 1\} \), and the second, without the point \((a,1-a)\), is included in the semiplane \( \{(s^1,s^2)|s^1 + s^2 > 1\} \). A parametrization of the second segment is

\[
s^1(\tau) = (1-\tau)a + \tau t^1, \quad s^2(\tau) = (1-\tau)(1-a) + \tau t^2, \quad \tau \in [0,1].
\]

Taking into account that \( u_1(\cdot) \) vanishes on the first segment, we find

\[
x(t^1,t^2) = \int_0^1 1 \cdot (t^1 - a) d\tau = t^1 - a.
\]

Consequently, the solution is given by the formula

\[
x(t^1,t^2) = \begin{cases} 
t^1 - a, & \text{if } t^1 + t^2 > 1 \\
0, & \text{if } t^1 + t^2 < 1,
\end{cases}
\]

on the set

\[\mathbb{R}^2 \setminus \{(t^1,t^2)|t^1 + t^2 = 1\}\]

and \( x(0,0) = 0 \). Here we recognize an infinity of solutions since \( a \) is an arbitrary point.

The foregoing solution \( x(\cdot) \) can be extended to a continuous function at \((a,1-a)\), but in rest the function \( x(\cdot) \) is discontinuous on the straight line \( \{(t^1,t^2)|t^1 + t^2 = 1\} \) (for any given values on this straight line).

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