IMPLEMENTING MARKOV’S LIMITING CURVATURE HYPOTHESIS∗

ROBERT H. BRANDENBERGER

Department of Physics

Brown University, Providence, RI 02912, USA

ABSTRACT

The effective action for gravity at high curvatures is likely to contain higher derivative terms. These corrections may have profound consequences for the singularity structure of space-time and for early Universe cosmology. In this contribution, recent work is reviewed which demonstrates that it is possible to construct a class of effective gravitational actions for which all solutions with sufficient symmetries have limited curvature and are nonsingular. Near the limiting curvature, the coupling between matter and gravity goes to zero and in this sense the theory is asymptotically free.

∗ Contribution to the proceedings of the 6’th Quantum Gravity Seminar, Moscow, June 1995, ed. V. Berezin et al. (World Scientific, Singapore, 1996).
1. Introduction

In 1982, Academician M. A. Markov, to whose memory this meeting is dedicated, postulated\textsuperscript{1} the existence of a limiting density for all forms of matter, a density at which the microphysical differences between various types of matter were assumed to disappear. For homogeneous and isotropic cosmologies, this assumption would lead to a de Sitter phase at densities approaching the limiting density. A slightly modified hypothesis, the limiting curvature hypothesis\textsuperscript{2}, was put forwards a few years later. Here it is postulated that for any solution of the gravitational equations of motion, in particular for a black hole, the curvature remains bounded by a fundamental “maximal curvature”, whose value we should expect to be given by the Planck scale. In this case, the inside of a black hole would have to differ from the usual Schwarschild solution. One possibility\textsuperscript{2,3} is that it becomes a de Sitter space-time throat leading to a different universe. This possibility has recently also been suggested\textsuperscript{4} in the context of string theory.

In this contribution, we will summarize some recent work\textsuperscript{5–8} attempting to implement the limiting curvature hypothesis. We have constructed an effective action for gravity in which all solutions with sufficient symmetry are nonsingular. The theory is a higher derivative modification of the Einstein action, and is obtained by a constructive procedure well motivated in analogy with the analysis of point particle motion in special relativity. The resulting theory is asymptotically free in a sense which will be specified below.

The inclusion of higher derivative gravity terms in the fundamental Lagrangian when studying the evolution of the space-time metric $g_{\mu\nu}$ at high curvatures is well motivated, since it expected that Planck scale physics will generate such types of correction terms to the Einstein action. This can be seen by considering the effective action obtained by integrating out quantum matter fields in the presence of a dynamical metric, by calculating first order perturbative quantum gravity effects, or by studying the low energy effective action of a Planck scale unified theory such as string theory.
The question we wish to address in this work is whether it is possible to construct a class of effective actions for gravity which have improved singularity properties with the constraint that they give the correct low curvature limit. It is also interesting to explore the implications of such models for early Universe cosmology, in particular in connection with the possible occurrence of a period of inflation.

A possible objection to our approach is that near a singularity quantum effects will be important and therefore a classical analysis is doomed to fail. This argument is correct in the usual picture in which at high curvatures there are large fluctuations and space-time becomes more like a “quantum foam.” However, in our theory, at high curvature space-time becomes highly regular and thus a classical analysis is self-consistent. The property of asymptotic freedom is essential in order to reach this conclusion.

Our aim is to construct a theory with the property that the metric $g_{\mu\nu}$ approaches the de Sitter metric $g_{\mu\nu}^{DS}$, a metric with maximal symmetry which admits a geodesically complete and nonsingular extension, as the curvature $R$ approaches the Planck value $R_{pl}$. Here, $R$ stands for any curvature invariant. Naturally, from our classical considerations, $R_{pl}$ is a free parameter. However, if our theory is connected with Planck scale physics, we expect $R_{pl}$ to be set by the Planck scale.
Figure 1: Penrose diagrams for collapsing Universe (left) and black hole (right) in Einstein’s theory (top) and in the nonsingular Universe (bottom). C, E, DS and H stand for contracting phase, expanding phase, de Sitter phase and horizon, respectively, and wavy lines indicate singularities.

If successful, the above construction will have some very appealing consequences. Consider, for example, a collapsing spatially homogeneous Universe. According to Einstein’s theory, this Universe will collapse in finite proper time to a final “big crunch” singularity (top left Penrose diagram of Figure 1). In our theory, however, the Universe will approach a de Sitter model as the curvature increases. If the Universe is closed, there will be a de Sitter bounce followed by re-expansion (bottom left Penrose diagram in Figure 1). Similarly, in our theory spherically symmetric vacuum solutions would be nonsingular, i.e., black holes would have no singularities in their centers. The structure of a large black hole would be unchanged compared to what is predicted by Einstein’s theory (top right, Figure 1) outside and even slightly inside the horizon, since all curvature invariants are small in those regions. However, for \( r \to 0 \) (where \( r \) is the radial Schwarzschild coordinate), the solution changes and approaches a de Sitter solution (bottom right,
Figure 1). This would have interesting consequences for the black hole information loss problem.

To motivate our effective action construction, we turn to a well known analogy, point particle motion in the theory of special relativity.

2. An Analogy

The transition from the Newtonian theory of point particle motion to the special relativistic theory transforms a theory with no bound on the velocity into one in which there is a limiting velocity, the speed of light $c$ (in the following we use units in which $\hbar = c = 1$). This transition can be obtained$^5$ by starting with the action of a point particle with world line $x(t)$:

\[
S_{\text{old}} = \int dt \frac{1}{2} \dot{x}^2, \tag{2.1}
\]

and adding$^9$ a Lagrange multiplier which couples to $\dot{x}^2$, the quantity to be made finite, and which has a potential $V(\varphi)$:

\[
S_{\text{new}} = \int dt \left[ \frac{1}{2} \dot{x}^2 + \varphi \dot{x}^2 - V(\varphi) \right]. \tag{2.2}
\]

From the constraint equation

\[
\dot{x}^2 = \frac{\partial V}{\partial \varphi}, \tag{2.3}
\]

it follows that $\dot{x}^2$ is limited provided $V(\varphi)$ increases no faster than linearly in $\varphi$ for large $|\varphi|$. The small $\varphi$ asymptotics of $V(\varphi)$ is determined by demanding that at low velocities the correct Newtonian limit results:

\[
V(\varphi) \sim \varphi^2 \text{ as } |\varphi| \to 0, \tag{2.4}
\]

\[
V(\varphi) \sim \varphi \text{ as } |\varphi| \to \infty.
\]
Choosing the simple interpolating potential
\[ V(\varphi) = \frac{2\varphi^2}{1 + 2\varphi}, \]  
the Lagrange multiplier can be integrated out, resulting in the well-known action
\[ S_{\text{new}} = \frac{1}{2} \int dt \sqrt{1 - \dot{x}^2} \] for point particle motion in special relativity.

3. Construction

Our procedure for obtaining a nonsingular Universe theory\(^5\) is based on generalizing the above Lagrange multiplier construction to gravity. Starting from the Einstein action, we can introduce a Lagrange multiplier \(\varphi_1\) coupled to the Ricci scalar \(R\) to obtain a theory with limited \(R\):
\[ S = \int d^4x \sqrt{-g}(R + \varphi_1 R + V_1(\varphi_1)), \]
where the potential \(V_1(\varphi_1)\) satisfies the asymptotic conditions (2.4).

However, this action is insufficient to obtain a nonsingular gravity theory. For example, singular solutions of the Einstein equations with \(R = 0\) are not effected at all. The minimal requirements for a nonsingular theory is that all curvature invariants remain bounded and the space-time manifold is geodesically complete. Implementing Markov’s Limiting Curvature Hypothesis\(^1\)\(^2\), these conditions can be reduced to more manageable ones. First, we choose one curvature invariant \(I_1(g_{\mu\nu})\) and demand that it be explicitly bounded, i.e., \(|I_1| < I_{1}^{pl}\), where \(I_1^{pl}\) is the Planck scale value of \(I_1\). In a second step, we demand that as \(I_1(g_{\mu\nu})\) approaches \(I_1^{pl}\), the metric \(g_{\mu\nu}\) approach the de Sitter metric \(g_{\mu\nu}^{DS}\), a definite nonsingular metric with maximal symmetry. In this case, all curvature invariants are automatically bounded (they approach their de Sitter values), and the space-time can be extended to be geodesically complete.
Our approach is to implement the second step of the above procedure by another Lagrange multiplier construction\(^5\). We look for a curvature invariant \(I_2(g_{\mu\nu})\) with the property that

\[
I_2(g_{\mu\nu}) = 0 \iff g_{\mu\nu} = g_{\mu\nu}^{DS},
\]

(3.2)

introduce a second Lagrange multiplier field \(\varphi_2\) which couples to \(I_2\) and choose a potential \(V_2(\varphi_2)\) which forces \(I_2\) to zero at large \(|\varphi_2|\):

\[
S = \int d^4x \sqrt{-g}[R + \varphi_1 I_1 + V_1(\varphi_1) + \varphi_2 I_2 + V_2(\varphi_2)],
\]

(3.3)

with asymptotic conditions (2.4) for \(V_1(\varphi_1)\) and conditions

\[
\begin{align*}
V_2(\varphi_2) &\sim \text{const as } |\varphi_2| \to \infty \\
V_2(\varphi_2) &\sim \varphi_2^2 \text{ as } |\varphi_2| \to 0,
\end{align*}
\]

(3.4)

for \(V_2(\varphi_2)\). The first constraint forces \(I_2\) to zero, the second is required in order to obtain the correct low curvature limit.

These general conditions are reasonable, but not sufficient in order to obtain a nonsingular theory. It must still be shown that all solutions are well behaved, i.e., that they asymptotically reach the regions \(|\varphi_2| \to \infty\) of phase space (or that they can be controlled in some other way). This must be done for a specific realization of the above general construction.
4. Specific Model

At the moment we are only able to find an invariant \( I_2 \) which singles out de Sitter space by demanding \( I_2 = 0 \) provided we assume that the metric has special symmetries. The choice

\[
I_2 = (4R_{\mu\nu}R^{\mu\nu} - R^2 + C^2)^{1/2}, \quad (4.1)
\]
singles out the de Sitter metric among all homogeneous and isotropic metrics (in which case adding \( C^2 \), the Weyl tensor square, is superfluous), all homogeneous and anisotropic metrics, and all radially symmetric metrics.

We choose the action\(^5,6\)

\[
S = \int d^4x \sqrt{-g} \left[ R + \varphi_1 R - (\varphi_2 + \frac{3}{\sqrt{2}} \varphi_1) I_2^{1/2} + V_1(\varphi_1) + V_2(\varphi_2) \right] \quad (4.2)
\]

with

\[
V_1(\varphi_1) = 12 H_0^2 \frac{\varphi_1^2}{1 + \varphi_1} \left( 1 - \frac{\ln(1 + \varphi_1)}{1 + \varphi_1} \right) \quad (4.3)
\]

\[
V_2(\varphi_2) = -2\sqrt{3} H_0^2 \frac{\varphi_2^2}{1 + \varphi_2^2}. \quad (4.4)
\]

The general equations of motion resulting from this action are quite messy. However, when restricted to homogeneous and isotropic metrics of the form

\[
ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2), \quad (4.5)
\]

the equations are fairly simple. With \( H = \dot{a}/a \), the two \( \varphi_1 \) and \( \varphi_2 \) constraint
equations are

\[ H^2 = \frac{1}{12} V_1' \]  \hspace{1cm} (4.6)

\[ \dot{H} = -\frac{1}{2\sqrt{3}} V_2' \]  \hspace{1cm} (4.7)

and the dynamical \( g_{00} \) equation becomes

\[ 3(1 - 2\varphi_1)H^2 + \frac{1}{2}(V_1 + V_2) = \sqrt{3}H(\dot{\varphi}_2 + 3H\varphi_2) . \]  \hspace{1cm} (4.8)

The phase space of all vacuum configurations is the half plane \( \{(\varphi_1 \geq 0, \varphi_2)\} \).

Equations (4.6) and (4.7) can be used to express \( H \) and \( \dot{H} \) in terms of \( \varphi_1 \) and \( \varphi_2 \).

The remaining dynamical equation (4.8) can then be recast as

\[ \frac{d\varphi_2}{d\varphi_1} = -\frac{V_1''}{4V_2} \left[ -\sqrt{3}\varphi_2 + (1 - 2\varphi_1) - \frac{2}{V_1'}(V_1 + V_2) \right] . \]  \hspace{1cm} (4.9)

The solutions can be studied analytically in the asymptotic regions and numerically throughout the entire phase space.

The resulting phase diagram of vacuum solutions is sketched in Fig. 2 (for numerical results, see Ref. 6). The point \((\varphi_1, \varphi_2) = (0, 0)\) corresponds to Minkowski space-time \( M^4 \), the regions \( |\varphi_2| \to \infty \) to de Sitter space. As shown, all solutions either are periodic about \( M^4 \) or else they asymptotically approach de Sitter space. Hence, all solutions are nonsingular. This conclusion remains unchanged if we add spatial curvature to the model.
One of the most interesting properties of our theory is asymptotic freedom\(^6\), i.e., the coupling between matter and gravity goes to zero at high curvatures. It is easy to add matter (e.g., dust or radiation) to our model by taking the combined action

\[ S = S_g + S_m, \tag{4.10} \]

where \(S_g\) is the gravity action previously discussed, and \(S_m\) is the usual matter action in an external background space-time metric.

We find\(^6\) that in the asymptotic de Sitter regions, the trajectories of the solutions in the \((\varphi_1, \varphi_2)\) plane are unchanged by adding matter. This applies, for example, in a phase of de Sitter contraction when the matter energy density is increasing exponentially but does not affect the metric. The physical reason for asymptotic freedom is obvious: in the asymptotic regions of phase space, the space-time curvature approaches its maximal value and thus cannot be changed even by adding an arbitrary high matter energy density.
Naturally, the phase space trajectories near \((\varphi_1, \varphi_2) = (0, 0)\) are strongly affected by adding matter. In particular, \(M^4\) ceases to be a stable fixed point of the evolution equations.

5. Two Dimensional Results

The low energy effective actions for the space-time metric in 4 dimensions which come from string theory are only known perturbatively. They contain higher derivative terms, but not if the exact same form as the ones used in our construction. The connection between our limiting curvature construction and string theory-motivated effective actions is more apparent in two space-time dimensions\(^7,8\).

The most general renormalizable Lagrangian for string-induced dilaton gravity is

\[
\mathcal{L} = \sqrt{-g}[D(\varphi)R + G(\varphi)(\nabla \varphi)^2 + H(\varphi)],
\]

(5.1)

where \(\varphi(x,t)\) is the dilaton. In two space-time dimensions, the kinetic term for \(\varphi\) can be eliminated, resulting in a Lagrangian (in terms of rescaled fields) of the form

\[
\mathcal{L} = \sqrt{-g}[D(\varphi)R + V(\varphi)].
\]

(5.2)

We can now apply the limiting curvature construction to find classes of potentials for which the theory has nonsingular black hole\(^7\) and cosmological\(^8\) solutions. In the following, we discuss the nonsingular two-dimensional black hole. For other discussions of nonsingular two-dimensional black holes and cosmological solutions see Refs. 10 and 11, respectively.

To simplify the algebra, the dilaton is redefined such that

\[
D(\varphi) = \frac{1}{\varphi}.
\]

(5.3)
The most general static metric can be written as

$$ds^2 = f(r)dt^2 - g(r)dr^2$$  \hspace{1cm} (5.4)

and the gauge choice

$$g(r) = f(r)^{-1}$$  \hspace{1cm} (5.5)

is always possible. The variational equations are

$$f' = -V(\phi)\frac{\phi^2}{\phi'},$$  \hspace{1cm} (5.6)

$$\left(\frac{\phi'}{\phi^2}\right)' = 0$$  \hspace{1cm} (5.7)

and

$$\phi^{-2}R = \frac{\partial V}{\partial \phi},$$  \hspace{1cm} (5.8)

where a prime denotes the derivative with respect to $r$.

Equation (5.7) can be integrated to find (after rescaling $r$)

$$\phi = \frac{1}{Ar}.$$  \hspace{1cm} (5.9)

To give the correct large $r$ behavior for the metric, we need to impose that

$$f(r) \to 1 - \frac{2m}{r} \text{ as } r \to \infty.$$  \hspace{1cm} (5.10)

From (5.6) this leads to the asymptotic condition

$$V(\phi) \to 2mA^3\phi^2 \text{ as } \phi \to 0.$$  \hspace{1cm} (5.11)

The limiting curvature hypothesis requires that $R$ be bounded as $\phi \to \infty$. From
This implies

\[ V(\varphi) \to \frac{2}{\ell^2 \varphi} \quad \text{as} \quad \varphi \to \infty, \quad (5.12) \]

where \( \ell \) is a constant which determines the limiting curvature. As an interpolating potential we can choose

\[ V(\varphi) = \frac{2mA^3 \varphi^2}{1 + mA^3 \ell^2 \varphi^3}, \quad (5.13) \]

which allows (5.6) to be integrated explicitly\(^7\) to obtain \( f(r) \).

The resulting metric coefficient \( f(r) \) describes a nonsingular black hole with a single horizon at \( r \simeq 2m \). The metric is indistinguishable from the usual Schwarzschild metric until far inside of the horizon, where our \( f(r) \) remains regular and obtains vanishing derivative at \( r = 0 \), which allows for a geodesically complete extension of the manifold.

### 6. Discussion

We have shown that a class of higher derivative extensions of the Einstein theory exist for which many interesting solutions are nonsingular. Our class of models is very special. Most higher derivative theories of gravity have, in fact, much worse singularity properties than the Einstein theory. What is special about our class of theories is that they are obtained using a well motivated Lagrange multiplier construction which implements the limiting curvature hypothesis. We have shown that

i) all homogeneous and isotropic solutions are nonsingular\(^{5,6}\)

ii) the two-dimensional black holes are nonsingular\(^7\)

iii) nonsingular two-dimensional cosmologies exist\(^8\).

We also have evidence that four-dimensional black holes and anisotropic homogeneous cosmologies are nonsingular\(^{12}\).
By construction, all solutions are de Sitter at high curvature. Thus, the theories automatically have a period of inflation (driven by the gravity sector in analogy to Starobinsky inflation\textsuperscript{13}) in the early Universe.

A very important property of our theories is asymptotic freedom. This means that the coupling between matter and gravity goes to zero at high curvature, and might lead to an automatic suppression mechanism for scalar fluctuations.

In two space-time dimensions, there is a close connection between dilaton gravity and our construction. In four dimensions, the connection between fundamental physics and our class of effective actions remains to be explored. A promising direction for future research appears to be an exploration of the connection between the nonsingular cosmology described here and the “pre-big-bang” scenario\textsuperscript{14} which is based on string-inspired dilaton gravity. Using our implementation of the Limiting Curvature Hypothesis it might be possible to resolve the “graceful exit problem”\textsuperscript{15} of dilaton gravity.

More immediately, however, there are many important problems concerning the construction proposed here which remain to be resolved. In particular, does the theory remain well behaved when allowing for space-times without the special symmetries which we have assumed? What is the behavior of inhomogeneities at the linearized level? Does asymptotic freedom of the de Sitter phase have an effect on the magnitude of the density fluctuations produced during inflation? At first sight, no fundamental obstacles have appeared. However, the actual computations appear extremely tedious as a consequence of the higher derivative terms which appear in the action. Nevertheless, the potential benefits of our scenario make these computations well worth while.

Acknowledgements:

I am grateful to Professors V. Berezin and V. Rubakov for inviting me to write this contribution. I also wish to thank my collaborators Richhild Moessner, Masoud Mohazzab, Andrew Sornborger, Mark Trodden and in particular Slava
Mukhanov for the joy of collaboration. This work is supported in part by the US Department of Energy under Grant DE-FG0291ER40688, Task A.

REFERENCES

1. M. Markov, *Pis’ma Zh. Eksp. Theor. Fiz.* 36, 214 (1982); M. Markov, *Pis’ma Zh. Eksp. Theor. Fiz.* 46, 342 (1987).

2. V. Frolov, M. Markov and V. Mukhanov, *Phys. Lett.* B216, 272 (1989); V. Frolov, M. Markov and V. Mukhanov, *Phys. Rev.* D41, 383 (1990).

3. D. Morgan, *Phys. Rev.* D43, 3144 (1991); I. Dymnikova, *Gen. Rel. Grav.* 24, 235 (1992).

4. E. Martinec, *Class. Quant. Grav.* 12, 941 (1995).

5. V. Mukhanov and R. Brandenberger, *Phys. Rev. Lett.* 68, 1969 (1992).

6. R. Brandenberger, V. Mukhanov and A. Sornborger, *Phys. Rev.* D48, 1629 (1993).

7. M. Trodden, V. Mukhanov and R. Brandenberger, *Phys. Lett.* B316, 483 (1993).

8. R. Moessner and M. Trodden, *Phys. Rev.* D51, 2801 (1995).

9. B. Altshuler, *Class. Quant. Grav.* 7, 189 (1990).

10. R. Mann, S. Morsink, A. Sikkema and T. Steele, *Phys. Rev.* D43, 3948 (1991); R. Mann, *Gen. Rel. Grav.* 24, 433 (1992); D. Christensen and R. Mann, *Class. Quant. Grav.* 9, 1769 (1992).

11. T. Mishima and A. Nakamichi, *Prog. Theor. Phys. Suppl.* 114, 207 (1993); M. Yoshimura, *Phys. Rev.* D47, 5389 (1993); K. Chan and R. Mann, *Class. Quant. Grav.* 10 913 (1993); M. Osorio and M. Vazquez-Mozo, *Mod. Phys. Lett.* A8, 3111 (1993); M. Osorio and M. Vazquez-Mozo, *Mod. Phys. Lett.* A8, 3215 (1993).
12. R. Brandenberger, M. Mohazzab, V. Mukhanov, A. Sornborger and M. Trodden, in preparation (1995).

13. A. Starobinsky, *Phys. Lett.* B91, 99 (1980).

14. M. Gasperini and G. Veneziano, *Phys. Lett.* B277, 256 (1992);
   M. Gasperini and G. Veneziano, *Astropart. Phys.* 1, 317 (1993);
   J. Levin, *Phys. Rev.*, D51, 1536 (1995).

15. R. Brustein and G. Veneziano, *Phys. Lett.* B329, 429 (1994);
   N. Kaloper, R. Madden and K. Olive, ‘Towards a Singularity-Free Inflationary Universe’, [hep-th/9506027](http://arxiv.org/abs/hep-th/9506027) (1995).