FIXED POINTS OF THE EQUIVARIANT ALGEBRAIC $K$-THEORY OF SPACES

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Abstract. In a recent work Malkiewich and Merling proposed a definition of the equivariant $K$-theory of spaces for spaces equipped with an action of a finite group. We show that the fixed points of this spectrum admit a tom Dieck-type splitting. We also show that this splitting is compatible with the splitting of the equivariant suspension spectrum. The first of these results has been obtained independently by John Rognes.

1. Introduction

In [3] Malkiewich and Merling proposed a definition of the equivariant $K$-theory of spaces $A_G(X)$ in the case when $G$ is a finite group and $X$ is a $G$-space. They also showed that the fixed spectrum of $A_G(X)$ can be described as follows. We will say that a space $Y$ is $G$-retractive over $X$ if $Y$ is a $G$-space equipped with $G$-equivariant maps $r: Y \rightarrow X: s$ such that $s$ is a $G$-cofibration and $rs = id_X$. Let $R_G(X)$ denote the category of $G$-retractive spaces dominated by finite relative $G$-CW complexes with $G$-equivariant maps over and under $X$ as morphisms. This is a Waldhausen category with weak equivalences and cofibrations given by $G$-homotopy equivalences and $G$-cofibrations. Applying Waldhausen’s $S_*$-construction to $R_G(X)$ we obtain a spectrum $A^G(X)$. This spectrum can be identified with the fixed point spectrum of $A_G(X)$.

The goal of this note is to investigate some properties of the spectrum $A^G(X)$. First, we will show that $A^G(X)$ admits a tom Dieck-type splitting. For a group $G$ and a subgroup $H \subseteq G$ let $NH$ denote the normalizer of $H$ in $G$ and let $WH = NH/H$. Denote also by $C_G$ the set of all conjugacy classes $(H)$ of subgroups of $G$.

1.1. Theorem. Let $G$ be a finite finite group. For any $G$-space $X$ there is a weak equivalence of spectra

$$\tau^A_X: A^G(X) \xrightarrow{\cong} \prod_{(H) \in C_G} A(EWH \times_{WH} X^H)$$

Moreover, this weak equivalence is natural in $X$. 

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This result has been obtained independently by John Rognes.

Next, recall that in a non-equivariant setting for a space $X$ we have the assembly map $a_X : Q(X_+) \to A(X)$ where $Q(X_+)$ denotes the suspension spectrum of $X$. We will show that similarly, for a finite group $G$ and a $G$-space $X$ there is a natural map $a^G : Q^G(X_+) \to A^G(X)$ where $Q^G(X_+)$ is the fixed point spectrum of the equivariant suspension spectrum of $X$. Since $Q^G(X_+)$ admits tom Dieck-type splitting a natural question is if this splitting is compatible with the splitting of $A^G(X)$ given by Theorem 1.1. We will show that this is the case:

1.2. Theorem. Let $G$ be a finite group. For any $G$-space $X$ the following diagram is commutes up to homotopy:

$$
\begin{align*}
Q^G(X_+) & \xrightarrow{\sim} \prod_{H \in C_G} Q(EWH \times_{WH} X^H) \\
\downarrow a^G & \\
A^G(X) & \xrightarrow{\sim} \prod_{H \in C_G} A(EWH \times_{WH} X^H)
\end{align*}
$$

where $a^G_H : Q(EWH \times_{WH} X^H) \to A(EWH \times_{WH} X^H)$ is the assembly map.

1.3. Note. Throughout this paper we will freely use the machinery of [7]. In particular by a Waldhausen category we mean a category with cofibrations and weak equivalences defined in [7, §1.2]. Given a Waldhausen category $\mathcal{C}$ by $K(\mathcal{C})$ we will denote the $K$-theory spectrum of $\mathcal{C}$ obtained by applying Waldhausen’s $S_\bullet$-construction.

2. Proof of Theorem 1.1

Let $G$ be a finite group. Recall that $C_G$ denotes the set of conjugacy classes $(H)$ of subgroups of $G$. Given a subgroup $H \subseteq G$ denote by $\mathcal{R}_H^G(X)$ the full subcategory of $\mathcal{R}^G(X)$ whose object are all retractive $G$-spaces $Y$ over $X$ such that all orbits of the action of $G$ on $Y \setminus X$ are of the type $G/H$. This is a Waldhausen category with weak equivalences and cofibrations inherited from $\mathcal{R}^G(X)$.

2.1. Proposition. Let $G$ be a finite group, and let $X$ be a $G$-space. There is a natural equivalence, natural in $X$:

$$
\tilde{\tau}_X^A : A^G(X) \xrightarrow{\sim} \prod_{(H) \in C_G} K(\mathcal{R}_H^G(X))
$$

Proof. Consider $C_G$ with an ordering:

$$
C_G = \{(H_1), (H_2), \ldots, (H_n)\}
$$

such that if $H_j$ has a subgroup conjugate to $H_i$ then $i \geq j$. Denote by $\mathcal{R}^G_{ij}(X)$ the subcategory of $\mathcal{R}^G(X)$ consisting of all retractive spaces $Y$ such that each orbit of
the action of $G$ on $Y \setminus X$ is isomorphic to $G/H_j$ for some $j \leq i$. We claim that for any $i \leq n$ there exists a weak equivalence

$$K(R^G_{\leq i}(X)) \xrightarrow{\simeq} K(R^G_{\leq i-1}(X)) \times K(R^G_{H_i}(X))$$

To see this denote by $E$ the Waldhausen category of cofibration sequences in $R^G(X)$, and let $E^i$ denote the subcategory of $E$ consisting of cofibration sequences

$$Y' \to Y \to Y/ Y'$$

such that $Y \in R^G_{\leq i}(X)$, $Y' \in R^G_{H_i}(X)$ and $Y/ Y' \in R^G_{\leq i-1}(X)$. By [7, Proposition 1.3.2] the functor $(Y' \to Y \to Y/ Y') \to (Y', Y/ Y')$ induces a weak equivalence

$$K(E^i) \xrightarrow{\simeq} K(R^G_{\leq i-1}(X)) \times K(R^G_{H_i}(X))$$

Next, notice that we have an exact functor $\Phi: E^i \to R^G_{\leq i}(X)$ given by $\Phi(Y' \to Y \to Y/ Y') = Y$, and an exact functor $\Psi: R^G_{\leq i}(X) \to E^i$ which assigns to a space $Y$ the cofibration sequence $\Psi(Y) = (Y' \to Y \to Y/ Y')$ where $Y' = X \cup Y^H$. These functors are inverse equivalences of categories and so they yield a weak equivalence $K(R^G_{\leq i}(X)) \simeq K(E^i)$.

Since $R^G(X) = R^G_{\leq n}(X)$ arguing by induction we obtain

$$A^G(X) = K(R^G(X)) \xrightarrow{\simeq} \prod_{(H_i) \in C_G} K(R^G_{H_i}(X))$$

Our next goal will be to identify the spectrum $K(R^G_{H}(X))$ for $H \subseteq G$. Notice that if $X$ is a $G$-space then the group $WH$ acts the space $X^H$. Consider the space $EWH \times X^H$ with the action of $WH$ given by $g(y, x) = (gy, yx)$. We will show that the following holds:

**2.2. Proposition.** Let $G$ be a finite group, $H \subseteq G$ be a subgroup, and let $X$ be a $G$-space. There exists an exact weak equivalence of categories

$$R^G_H(X) \to R^{WH}(EWH \times X^H)$$

which induces a weak equivalence of spectra $K(R^G_H(X)) \xrightarrow{\simeq} A^{WH}(EWH \times X^H)$.

We will first consider a special case where $H = \{e\}$ is the trivial subgroup of $G$:

**2.3. Lemma.** Let $G$ be a finite group, and let $X$ be a $G$-space. There exists an exact weak equivalence of categories

$$R^G_{\{e\}}(X) \to R^G(EG \times X)$$

**Proof.** Consider the exact functor

$$\Phi: R^G_{\{e\}}(X) \to R^G(EG \times X)$$
FIXED POINTS OF THE EQUIVARIANT ALGEBRAIC $\kappa$-THEORY OF SPACES

given by $\Phi(Y) = EG \times Y$ for $Y \in \mathcal{R}_e^G(X)$ where $EG \times Y$ is given the structure of a $G$-retractive space over $EG \times X$ in the obvious way. We also have an exact functor

$$\Psi: \mathcal{R}^G(EG \times X) \to \mathcal{R}_e^G(X)$$

defined as follows. Given a retractive space $(r: Y \hookrightarrow EG \times X; s) \in \mathcal{R}^G(EG \times X)$ define

$$\pi_*Z := \text{colim} (Z \leftarrow EG \times X \xrightarrow{\pi} X)$$

where $\pi$ is the projection map. The space $\pi_*Z$ has the structure of a $G$-retractive space over $X$. Moreover, since $G$ acts freely on $\pi_*Z \setminus X$ thus $\pi_*Z \in \mathcal{R}_e^G(X)$. We set:

$$\Psi(Z) = \pi_*Z.$$ 

For a retractive space $(r: Y \hookrightarrow X; s) \in \mathcal{R}_e^G(X)$ consider the morphism $\Psi\Phi(Y) \to Y$ in $\mathcal{R}_e^G(X)$ induced by the map of pushouts

$$
\begin{array}{ccc}
EG \times Y & \xleftarrow{id \times s} & EG \times X \\
\downarrow & & \downarrow \pi \\
Y & \xleftarrow{s} & X
\end{array}
\quad
\begin{array}{ccc}
\xrightarrow{\pi} & & \xrightarrow{=} X \\
\xrightarrow{} & & \xrightarrow{=}
\end{array}
$$

By [6, II.2.11, Exercise 5] we obtain that this morphism is a weak equivalence in $\mathcal{R}_e^G(X)$. In this way we get a natural weak equivalence between $\Psi\Phi$ and the identity functor on $\mathcal{R}_e^G(X)$.

Finally, for $(r: Z \hookrightarrow EG \times X; s) \in \mathcal{R}_e^G(X)$ let $\varphi_Z: Z \to \pi_*Z$ induced by the map of pushouts:

$$
\begin{array}{ccc}
Z & \xleftarrow{s} & EG \times X \\
\downarrow & & \downarrow \pi \\
Z & \xleftarrow{s} & EG \times X
\end{array}
\quad
\begin{array}{ccc}
= & & = \\
\xrightarrow{=} & & \xrightarrow{=}
\end{array}
$$

and let $\pi_{EG}: EG \times X \to EG$ be the projection map. The maps $(\pi_{EG}, \varphi_Z): Z \to \Phi\Psi(Z) = EG \times \pi_*Z$ define a natural weak equivalence between the identity functor on $\mathcal{R}^G(EG \times X)$ and the functor $\Phi\Psi$. \hfill \Box

**Proof of Proposition 2.2.** By Lemma 2.3 it will suffice to show that for any subgroup $H \subseteq G$ there exists an exact equivalence of categories

$$\mathcal{R}_e^G(X) \to \mathcal{R}_e^{WH}(X^H)$$

Let $\Gamma: \mathcal{R}_e^G(X) \to \mathcal{R}_e^{WH}(X^H)$ be the exact functor which associates to a retractive space $Y \in \mathcal{R}_e^G(X)$ the space $Y^H \in \mathcal{R}_e^{WH}(X^H)$. We also have an exact functor

$$\Lambda: \mathcal{R}_e^{WH}(X^H) \to \mathcal{R}_e^G(X)$$

defined as follows. Consider \(G/H\) as a left \(G\)-space and a right \(WH\)-space. Given a retractive space \((r': Z \xrightarrow{r} X^H, s) \in \mathcal{R}^{WH}_{[e]}(X^H)\) the twisted product \(G/H \times_{WH} Z\) is a \(G\)-retractive space over \(G/H \times_{WH} X^H\). We have a \(G\)-map
\[
\lambda_X: G/H \times_{WH} X^H \to X
\]
given by \(\lambda_X(gH, x) = gx\). We set:
\[
\Lambda(Z) := \text{colim} (G/H \times_{WH} Z \xleftarrow{id \times s} G/H \times_{WH} X^H \xrightarrow{\lambda_X} X)
\]
We claim that \(\Gamma\) and \(\Lambda\) are inverse equivalences of categories. To see this consider first a retractive space \((r: Y \xrightarrow{r} X: s) \in \mathcal{R}^{G}_{[e]}(X)\). The map of pushouts diagrams
\[
\begin{array}{ccc}
G/H \times_{WH} Y^H & \xleftarrow{id \times s} & G/H \times_{WH} X^H \\
\downarrow{\lambda_Y} & & \downarrow{\lambda_X} \\
Y & = & X
\end{array}
\]
induces a map \(\eta_Y: \Lambda\Gamma(Y) \to Y\). This map is natural in \(Y\), and so defines a natural transformation of functors \(\eta: \Lambda\Gamma \Rightarrow \text{Id}_{\mathcal{R}^{G}_{[e]}(X)}\). We will show that \(\eta_Y\) is an isomorphism for any space \(Y\) by constructing its inverse. Take the \(G\)-equivariant embedding
\[
\mu_1: s(X) \to \Lambda\Gamma(Y)
\]
We also have an \(NH\)-equivariant map \(\mu_2: Y^H \to \Lambda\Gamma(Y)\) which is the composition of the map \(Y^H \to G/H \times_{WH} Y^H\) that sends \(y \in Y^H\) to the point \([eH, y]\) and the inclusion \(G/H \times_{WH} Y^H \to \Lambda\Gamma(Y)\). Since the sets \(s(X)\) and \(Y^H\) are closed in \(Y\), and the maps \(\mu_1\) and \(\mu_2\) coincide on \(s(X)\) and \(Y^H\) we obtain a continuous map \(\mu': s(X) \cup Y^H \to \Lambda\Gamma(Y)\). Moreover, notice that for every \(y \in s(X) \cup Y^H\) and for every \(g \in G\) such that \(gy \in s(X) \cup Y^H\) we have \(\mu'(gy) = gy\mu'(y)\) (we use here the assumption that all orbits of \(Y\) \(\setminus s(X)\) are isomorphic to \(G/H\)). By [1, Ch. 1, Theorem 3.3] the map \(\mu'\) extends uniquely to a \(G\)-equivariant map \(\mu: Y \to \Lambda\Gamma(Y)\). It is straightforward to check that \(\mu\) is the inverse of \(\eta_Y\).

Construction of a natural isomorphism between the functor \(\Gamma\Lambda\) and the identity functor on \(\mathcal{R}^{WH}_{[e]}(X^H)\) is straightforward. 

Combining Proposition 2.1 with Proposition 2.2 for any \(G\)-space \(X\) we obtain a weak equivalence
\[
A^G(X) \xrightarrow{\sim} \prod_{(H) \in \text{CG}} A^{WH}(EWH \times X^H)
\]
In order to complete the proof of Theorem 1.1 it suffices to show for any subgroup \(H \subseteq G\) there exists a weak equivalence \(A^{WH}(EWH \times X^H) \approx A(EWH \times_{WH} X^H)\). Since the action of \(WH\) on \(EWH \times X^H\) is free this follows from the following fact:

**2.4. Lemma.** If \(X\) is a space with a free action of a finite group \(G\) then \(A^G(X) \approx A(X/G)\).
Proof: Recall that the spectrum $A(X/G)$ is obtained by applying Waldhausen’s $S^1$-construction to the category $\mathcal{R}(X/G)$ of homotopy finitely dominated retractive spaces over $X/G$. For $(r: Y \leftrightarrow X: s) \in \mathcal{R}^G(X)$ define $\Gamma(Y) = Y/G$. This space is in a natural way a retractive space over $X/G$ so it is an object in $\mathcal{R}(X/G)$. We also have a functor

$$\Lambda: \mathcal{R}(X/G) \to \mathcal{R}^G(X)$$

defined as follows. Let $q: X \to X/G$ be the quotient map. For $(r: Y \leftrightarrow X/G: s) \in \mathcal{R}(X/G)$ let

$$q^*Y = \{(x, y) \in X \times Y \mid q(x) = r(y)\}$$

This is a $G$-space with the action of $G$ given by $g(x, y) = (gx, y)$. Moreover, $q^*Y$ is in a natural way a $G$-retractive space over $X$, and it is an object of the category $\mathcal{R}^G(X)$. We set: $\Lambda(Y) := q^*Y$. It is straightforward to check that $\Gamma$ and $\Lambda$ are inverse equivalences of categories.

\[\square\]

3. Compatibility with the splitting of $Q^G(X_+)$

Our next goal is to prove Theorem 1.2 which says that the splitting of $A^G(X)$ is compatible with the splitting of $Q^G(X_+)$, the fixed point spectrum of the equivariant suspension spectrum of $X$.

Let $G$ be a finite group and let $X$ be a $G$-space. Denote by $\mathcal{F}^G(X)$ the subcategory of $\mathcal{R}^G(X)$ whose objects are $G$-retractive spaces $r: Y \leftrightarrow X: s$ satisfying the following conditions:

(i) $Y = Y_X \sqcup Y_e$ such that $s(X) \subseteq Y_X$, and the maps $r: Y_X \leftrightarrow X: s$ are inverse $G$-homotopy equivalences.

(ii) $Y_e$ isomorphic to a finite disjoint union of $G$-spaces of the form $D \times G/H$ where $H$ a subgroup of $G$, $D$ is a contractible space, and the $G$-action on $D \times G/H$ is given by $g(x, kH) = (x, gkH)$.

A morphism between objects $r: Y_X \sqcup Y_e \leftrightarrow X: s$ and $r': Y'_X \sqcup Y'_e \leftrightarrow X: s'$ is a $G$-map $f: Y_X \sqcup Y_e \to Y'_X \sqcup Y'_e$ that satisfies two conditions:

1) if $y \in Y_e$ and $f(y) \in Y'_e$ then $f(y) = s' r(y)$;

2) the induced map $f_*: \pi_0(f^{-1}(Y'_e)) \to \pi_0(Y'_e)$ is a monomorphism.

The category $\mathcal{F}^G(X)$ is a Waldhausen category with weak equivalences and cofibrations inherited from $\mathcal{R}^G(X)$. The inclusion functor $\mathcal{F}^G(X) \to \mathcal{R}^G(X)$ is exact, so it induces a map of spectra $a^G: K(\mathcal{F}^G(X)) \to A^G(X)$.
3.1. Proposition. There exists a weak equivalence of spectra

$$\tau^Q_X : K(\mathcal{F}^G(X)) \xrightarrow{\sim} \prod_{(H) \in C_G} K(\mathcal{F}^{WH}(EWH \times X^H))$$

natural in $X$ such that the following diagram commutes:

$$\begin{array}{ccc}
K(\mathcal{F}^G(X)) & \xrightarrow{\tau^Q_X} & \prod_{(H) \in C_G} K(\mathcal{F}^{WH}(EWH \times X^H)) \\
\downarrow \phi^G & & \downarrow \prod \phi^{WH} \\
A^G(X) & \xrightarrow{\tau^A_X} & \prod_{(H) \in C_G} A^{WH}(EWH \times X^H)
\end{array}$$

Proof: The weak equivalence $\tau^Q_X$ can be constructed using the same arguments we used in Section 2 in proofs of Propositions 2.1 and 2.2, taking the category $\mathcal{F}^G(X)$ in place of $\mathcal{R}^G(X)$ and the category $\mathcal{F}^{WH}_H(X) = \mathcal{F}^G(X) \cap \mathcal{R}^{WH}_H(X)$ in place of $\mathcal{R}^G(X)_H$. Commutativity of the diagram follows directly from this construction. □

In order to complete the proof of Theorem 1.2 it will suffice to show the following holds:

(i) $K(\mathcal{F}^G(X)) = Q^G(X_+)$
(ii) $K(\mathcal{F}^{WH}(EWH \times X^H)) = Q(EWH \times WH X^H)$ for each $H \in G$.

Notice that it is enough to prove the second of these statements. Indeed, it will give $K(\mathcal{F}^G(X)) \cong \prod_{(H) \in C_G} Q(EWH \times WH X^H)$, and by [5, 2, Theorem 11.1], the product of on the right hand side is equivalent to $Q^G(X_+)$.

The proof of (ii) will be split in two steps. First, for a space $X$ let $\mathcal{F}(X)$ denote the category whose objects are (non-equivariant) retractive space $r : Y \rightrightarrows X : s$ satisfying the following conditions:

1) $Y = Y_X \sqcup \bigcup_{i \in I} B_i$ where $I$ is a finite set and $B_i$ is a contractible space for each $i \in I$;
2) the map $s : X \to Y$ is a cofibration and $s(X) \subseteq Y_X$;
3) the maps $s : X \to Y_X$ and $r : Y_X \to X$ are inverse homotopy equivalences.

A morphism between objects $r : Y_X \sqcup \bigcup_{i \in I} B_i \rightrightarrows X : s$ and $r' : Y_X \sqcup \bigcup_{i \in I} B_i \rightrightarrows X : s'$ is a continuous map $f : Y_X \sqcup \bigcup_{i \in I} B_i \to Y'_X \sqcup \bigcup_{j \in J} B'_j$ over and under $X$ that satisfies two conditions:

1) if $f(B_i) \subseteq Y'_X$ then $f|_{B_i} = s' r$.
2) the induced map $f_* : \pi_0(f^{-1}(\bigcup_{j \in J} B'_j)) \to \pi_0(\bigcup_{j \in J} B'_j)$ is a monomorphism.
The category \(F(X)\) is a Waldhausen category where a morphism \(f: Y \to Y'\) is a weak equivalence if it is a weak homotopy equivalences and \(f\) is a cofibration it has the homotopy extension property.

3.2. Lemma. Let \(G\) be a finite group, and let \(X\) be a \(G\)-space. If \(G\) acts freely on \(X\) then there exists a weak equivalence of spectra \(K(F^G(X)) \simeq K(F(X/G))\)

The proof is essentially the same as the proof of Lemma 2.4.

Going back to claim (ii), since the action of \(WH\) on \(EWH \times X^H\) is free by Lemma 3.2 we obtain a weak equivalence \(K(F^{WH}(EWH \times X^H)) \simeq K(F(EWH \times_{WH} X^H))\).

In order to complete the proof of existence of the weak equivalence (ii), and thus complete the proof of Theorem 1.2 in remains to show that the following holds:

3.3. Lemma. For any space \(X\) there is a weak equivalence of spectra \(K(F(X)) \simeq Q(X_*)\)

Proof. We will assume that \(X\) is connected space. The general case follows from essentially the same argument. Notice that for any cofibration sequence \(Y' \to Y \to Y/Y'\) in \(F(X)\) there is a functorial weak equivalence \(Y \simeq Y' \sqcup Y/Y'\). This implies that we have a weak equivalence of spectra \(K(F(X)) \simeq \Omega|wN_*F(X)|\) where \(wN_*F(X)\) is a simplicial category defined as in [7, §1.8]. Let \(L_X\) denote the loop group of \(X\). Let \(R^0_k(\ast, L_X)\) be the category of 0-spherical objects of rank \(k\) defined as in [7, p. 386] and let \(R^0(\ast, L_X) = \bigcup_k R^0_k(\ast, L_X)\). We have:

\[
\Omega|wN_*F(X)| \simeq \Omega|wR^0(\ast, L_X)| \simeq \mathbb{Z} \times \lim_k |wR^0_k(\ast, L_X)|^+ \simeq Q(X_*)
\]

The first of these weak equivalences can be obtained by essentially the same argument as one used in the proof of [7, Proposition 2.1.4], while the other two comes from a theorem of Segal [4] (see also [7, p.386]).

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