Detectability of Bell-CHSH nonlocality by qubit detectors with optimal local filters

Akira Matsumura* and Yasusada Nambu†

Department of Physics, Graduate School of Science, Nagoya University, Chikusa, Nagoya 464-8602, Japan

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Abstract

We investigate the detection problem of quantum nonlocal correlation by two qubit detectors. The detectors with an initial product state interact with a massless scalar field in the vacuum state, and then the out-state of the detectors are correlated after the interaction. Under the perturbative treatment in the second order of the coupling, the detectors’ state can be entangled but satisfies the Bell-CHSH inequality. It is known that the violation of the Bell-CHSH inequality for such an entangled state is obtained after a local filtering operation. In this paper, we construct the optimal filtering operation for the qubit detectors and derive the success probability of the filtering operation, which characterizes the reliability of revealing the Bell-CHSH nonlocality by the filtering operations. By applying the optimal filtering, it is shown that the detected Bell-CHSH nonlocality depends on the coherence of the detectors’ state and the spontaneous emission of scalar particles from each detector. We also comment on a trade-off relation between the success probability and the size of the parameter region showing quantum correlation.

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*Electronic address: matsumura.akira@h.mbox.nagoya-u.ac.jp
†Electronic address: nambu@gravity.phys.nagoya-u.ac.jp
I. INTRODUCTION

In the quantum information theory, the entanglement is a crucial property which describes a nonlocal correlation in the quantum mechanics. Due to the nonlocal feature of quantum entanglement, we can perform various protocols, such as quantum teleportation, superdense coding, quantum error correction, and so on [1]. The Bell-CHSH inequality characterizes a nonlocal correlation of the entanglement [2]. Based on the mathematically rigorous argument, the Bell-CHSH inequality [3] is satisfied only for the local hidden variable theories and a quantum state which violates this inequality cannot be yielded in hidden variable models.
The two aspects of quantum correlations, quantum entanglement and the violation of Bell-CHSH inequality (the Bell-CHSH nonlocality), are not equivalent and they are non-trivially related to each other \[4\]. Also in the quantum field theory, the quantum correlations play important roles. The vacuum state in the quantum field theory shows entanglement between spatial regions, which induces the Unruh effect \[5\] and determines the structure of the wave function of the vacuum (in particular, it is described by the tensor network \[6\]). Reeh and Schlieder showed that an arbitrary state of quantum field can be approximated by acting some local operators on the vacuum state \[7\], and such a property implies that the vacuum state is entangled among separated spatial regions. Also, it was shown that the free vacuum state violates the Bell-CHSH inequality by considering correlations between two spatial separated regions \[8\]. The vacuum of quantum field displays quantum entanglement and the violation of the Bell-CHSH inequality, hence these correlations essentially characterize the (many-body) property of the quantum field itself.

In connection with the vacuum entanglement of the quantum field, the detection of such quantum correlations by local observers has been investigated. The local detection problem tells us how the quantum resources of the vacuum is available, and provides the model of a suitable experimental setting to detect the spacelike quantum correlation of the vacuum. The local observer is usually modeled by a harmonic oscillator \[9–11\] or a qubit system \[12–19\]. Reznik \textit{et. al} \[12\] considered two qubit detectors initially not correlated. These detectors are coupled to a massless scalar field and do not interact directly with each other. Then it was shown that the entanglement can be detected but the violation of the Bell-CHSH inequality was found only after applying a local filter \[20\]. The local filtering operation is a kind of measurement process acted on each qubit by local observers Alice and Bob, which is constructed by post-selected (probabilistic) local operations and classical communication (LOCC). When we choose the operation properly, the Bell-CHSH nonlocality of the detectors’ state can be enhanced. This method is also applied to the cosmological situation to reveal the quantum nonlocality in the early universe \[21\]. In the quantum information theory, the optimal construction of the local filter which gives the maximal violation of the Bell-CHSH inequality was provided by papers \[22, 23\]. However, the optimal construction is not used in Ref. \[12\] and it is unclear whether its given filtering is optimal or not. Also, the local filtering operation is a probabilistic process and hence we should consider its probability to discuss the feasibility of the detection of the violation of the Bell-CHSH inequality.
In this paper, we investigate the detection problem of quantum nonlocality by two qubit detectors. The initial state of the detectors is usually assumed to be the uncorrelated ground state, however we also treat the excited state of the detectors. By such a generalization of the detectors' state, we clarify what is playing a crucial role to detect the quantum entanglement and the Bell-CHSH nonlocality. As an entanglement measure, we compute the negativity of the qubit detectors, which completely characterizes the entanglement for two qubits system [24]. Also we yield the optimal filtering operation for the two detectors by the construction method given in Ref. [23] to reveal the violation of the Bell-CHSH inequality. We show that the local filtering constructed by the systematic method corresponds to that given previously in Ref. [12], and the explicit formula of the success probability of the filtering operation is derived. For the reliable detection of the Bell-CHSH nonlocality between a spacelike regions in the vacuum, we explore the better setting of the detectors with a high success probability of the optimal local filtering. Through the analysis of the entanglement and the Bell-CHSH nonlocality revealed by the optimal filter for three different initial states, we show that the detected quantum correlation is determined by the two effects; one is the coherence of the detectors’ state and another is the spontaneous emission given by the local dynamics of each detector. In addition, it is shown that as the transition probability of the spontaneous emission grows, the quantum correlation between the detectors decreases and the success probability of the optimal filtering increases. Thus, there is a trade-off relation between the size of the parameter region indicating the quantum correlation and the success probability.

This paper organized as follows. In Sec. II, we introduce the system composed of two qubit detectors and a massless scalar field. For the second order of the coupling, we solve the dynamics under initial product states of the detectors and the vacuum state of the massless scalar field. Then we obtain the reduced density matrix of the detectors represented by a X state. In Sec. III, we calculate the negativity and the expectation value of the Bell operator for a X state. In Sec. IV, we explicitly construct the optimal filtering for a X state and derive the success probability of the filtering. In Sec V, we discuss the quantum entanglement and the Bell-CHSH nonlocality of detectors system and show the quantum correlation is determined by the coherence and the spontaneous emission of scalar particles. Sec. VI is devoted to summary and conclusion.
II. PERTURBATIVE DYNAMICS OF TWO DETECTORS COUPLED TO SCALAR FIELD

The vacuum state of a many body system or a quantum field has the nonlocal quantum (long-range) correlation. To investigate the detectability of the quantum correlation by local observers, we consider qubit detectors coupled to a massless scalar field. The free Hamiltonians is $H_{\text{free}} = H_A + H_B + H_\phi$ with

$$
H_A = \frac{\Omega}{2} \sigma_z^A, \quad H_B = \frac{\Omega}{2} \sigma_z^B, \quad H_\phi = \frac{1}{2} \int d^3x (\pi^2(x) + (\nabla \phi(x))^2), \quad (1)
$$

where $\sigma_{A,B}^z$ is the Pauli matrix, $\Omega$ is the energy gap of the qubits, $H_\phi$ is the free Hamiltonian of the massless scalar field $\phi$ and $\pi := \partial_t \phi$ is the conjugate momentum of the scalar field.

The interaction Hamiltonian is

$$
V(t) = g(t) \left[ \sigma_A^x \phi(x_A, t) + \sigma_B^x \phi(x_B, t) \right], \quad (2)
$$

where $x_A$ and $x_B$ denote each spatial position of the two detectors, that is, the two detectors are at rest at each position and locally interact with the scalar field. We assume that the switching function $g(t)$ is the Gaussian function

$$
g(t) = g_0 \exp \left[ -\frac{(t - t_0)^2}{2\sigma^2} \right], \quad (3)
$$

where $g_0$ is a coupling constant and $\sigma$ is a time interval while the interaction turns on. Roughly speaking, the detectors interact with the scalar field for $|t - t_0| \leq \sigma$. The choice of the Gaussian switching is more appropriate to extract the quantum correlations than a sudden switching function [16]. We assume that the initial state of the total system is a product state

$$
|\Psi_{\text{in}}\rangle = |a, b\rangle |0_\phi\rangle, \quad (4)
$$

where $a, b = \pm 1$ denote eigenvalues of $\sigma_{A,B}^x$ and $|0_\phi\rangle$ is the vacuum state of the scalar field. We also use the notation $|\uparrow\rangle = |+1\rangle, |\downarrow\rangle = |-1\rangle$ to represent the state of qubits. In the interaction picture, the out-state under the second order of the coupling is given by

$$
|\Psi_{\text{out}}\rangle = \left[ I - i \int_{-\infty}^{\infty} dt_1 \tilde{V}(t_1) - \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 T[\tilde{V}(t_1)\tilde{V}(t_2)] \right] |\Psi_{\text{in}}\rangle
$$

$$
= |a, b\rangle |0^\phi\rangle - i |a, -b\rangle \Phi^A_a |0_\phi\rangle - i |a, -b\rangle \Phi^B_{-a} |0^\phi\rangle
$$

$$
- \frac{1}{2} |a, b\rangle T[\Phi^A_a \Phi^A_{-a}] |0_\phi\rangle - \frac{1}{2} |a, b\rangle T[\Phi^B_{-a} \Phi^B_{-b}] |0^\phi\rangle - |a, -b\rangle T[\Phi^A_a \Phi^B_{-b}] |0^\phi\rangle, \quad (5)
$$

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where $\tilde{V}$ is the interaction Hamiltonian in the interaction picture, $T$ denotes the time ordering, and the operators $\Phi^A_a$ and $\Phi^B_b$ acting on the state of the scalar field are defined by
\[
\Phi^A_a = \int_{-\infty}^{\infty} dt \ g(t) e^{i\Omega_at} \phi(x_A, t), \quad \Phi^B_b = \int_{-\infty}^{\infty} dt \ g(t) e^{i\Omega_bt} \phi(x_B, t).
\] (6)

Each term in the equation (5) can be interpreted using the diagrammatic representation described in Fig. 1. For example, the second term in the equation (5) denotes that the detector A interacts once with the scalar field, then the qubit A is flipped.

\[
\rho_{AB} = \begin{bmatrix}
\rho_{11}(a, b) & 0 & 0 & \rho_{14}(a, b) \\
0 & \rho_{22}(a) & \rho_{23}(a, b) & 0 \\
0 & \rho_{23}^*(a, b) & \rho_{33}(b) & 0 \\
\rho_{14}^*(a, b) & 0 & 0 & \rho_{44}(a, b)
\end{bmatrix},
\] (7)

By tracing out the state of the scalar field, the reduced density matrix of the two detectors after the interaction is derived as follows:

FIG. 1: The diagrammatic representation of each term appeared in the equation (5).

By tracing out the state of the scalar field, the reduced density matrix of the two detectors after the interaction is derived as follows:

\[
\rho_{ij} = \langle i | \rho_{AB} | j \rangle \quad (i, j = 1, 2, 3, 4) \quad \text{and} \quad \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\} = \{|a, b\rangle, |-a, b\rangle, |a, -b\rangle, |-a, -b\rangle\}. \quad \text{This density matrix with non-diagonal components } \rho_{23}(a, b) \text{ and } \rho_{14}(a, b) \text{ is called the X state. That is, an X state has only the quantum coherence of the superposition } \{|1\rangle, |4\rangle\} \text{ or } \{|2\rangle, |3\rangle\} \text{ and this property makes the analysis of the quantum correlations}
\]
easier. Concretely, the non-diagonal components of the density matrix are

\[ \rho_{23}(a, b) = \langle 0_\phi | \Phi_b^A \Phi_{-a}^B | 0_\phi \rangle, \quad \rho_{14}(a, b) = -\langle 0_\phi | T[\Phi_{-a}^A \Phi_b^B] | 0_\phi \rangle, \]

and the diagonal components are given as

\[ \rho_{11}(a, b) = 1 - \rho_{22}(a) - \rho_{33}(b) - \rho_{44}(a, b), \]
\[ \rho_{22}(a) = \rho_{23}(a, a)|_{r=0}, \]
\[ \rho_{33}(b) = \rho_{23}(b, b)|_{r=0}, \]
\[ \rho_{44}(a, b) = \rho_{22}(a)\rho_{33}(b) + |\rho_{23}(a, b)|^2 + |\rho_{14}(a, b)|^2, \]

where \( r = |\mathbf{x}_A - \mathbf{x}_B| \) and the formula of \( \rho_{44}(a, b) \) is derived by the Wick theorem. Note that the non-diagonal components \( \rho_{23}(a, b) \) and \( \rho_{14}(a, b) \) depends on the Wightman function for the massless scalar field

\[ \langle 0_\phi | \phi(\mathbf{x}_A, t) \phi(\mathbf{x}_B, t') | 0_\phi \rangle = -\frac{1}{4\pi^2} \frac{1}{(t - t' - i\epsilon)^2 - r^2}, \]

where \( \epsilon \) is the UV cutoff parameter. The detectors with an initial product state can be entangled by the local interaction with the scalar field in the equation (2). We can explicitly compute \( \rho_{22}(a), \rho_{33}(b), \rho_{23}(a, b) \) and \( \rho_{14}(a, b) \) as

\[ \rho_{22}(a) = \frac{g_0^2}{4\pi} \left( e^{-(\Omega \sigma)^2} + 2a\Omega \sigma \text{Erfc}[-a\Omega \sigma] \right), \]
\[ \rho_{33}(b) = \frac{g_0^2}{4\pi} \left( e^{-(\Omega \sigma)^2} + 2b\Omega \sigma \text{Erfc}[-b\Omega \sigma] \right), \]
\[ \rho_{23}(a, b) = \frac{g_0^2 \sigma}{4\pi ir} e^{\Omega(a-b)t_0 - (\Omega \sigma)^2} \left( \exp \left[ \left( -\frac{\Omega \sigma}{2} (a - b) + i \frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ -\frac{\Omega \sigma}{2} (a + b) - i \frac{r}{2\sigma} \right] \right. \]
\[ \left. - \exp \left[ \left( -\frac{\Omega \sigma}{2} (a + b) + i \frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ -\frac{\Omega \sigma}{2} (a + b) + i \frac{r}{2\sigma} \right] \right), \]
\[ \rho_{14}(a, b) = \frac{g_0^2 \sigma}{4\pi ir} e^{\Omega(a+b)t_0 - (\Omega \sigma)^2} \left( \exp \left[ \left( \frac{\Omega \sigma}{2} (a - b) - i \frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ -\frac{\Omega \sigma}{2} (a - b) - i \frac{r}{2\sigma} \right] \right. \]
\[ \left. + \exp \left[ \left( -\frac{\Omega \sigma}{2} (a - b) - i \frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ -\frac{\Omega \sigma}{2} (a - b) - i \frac{r}{2\sigma} \right] \right), \]

where \( \text{Erfc}[z] \) is the error function defined by

\[ \text{Erfc}[z] = \int_0^\infty dt e^{-(t+z)^2}. \]

The detailed derivation (16) and (17) is presented in the Appendix A. From the explicit formulas of the density matrix, the quantum correlation of the scalar field detected via the two detectors can be computed.
III. NEGATIVITY AND BELL-CHSH INEQUALITY FOR X STATE

As the state of the detectors depends on the two-point function for the scalar field, we expect that the initially product state of the detectors becomes correlated after the interaction. To evaluate the quantum correlation between the two detectors, we consider the negativity and the Bell-CHSH inequality. The negativity is defined by the eigenvalues of a partial transposed density matrix $\rho_{AB}^{T_A}$ as

$$N = \sum_{\lambda_i < 0} |\lambda_i|, \quad (19)$$

where $\lambda_i$ are the eigenvalues of the partial transposed density matrix $\rho_{AB}^{T_A}$. If the negativity does not vanish, then the state is entangled. Especially, the opposite of the statement is true when the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$. Thus, the negativity has a nonzero value if and only if the given state is entangled [26], and hence the negativity completely characterizes whether the state of the detectors is entangled or not. For an X state, the negativity is explicitly obtained as

$$N_1 = \max \{N_1, 0\}, \quad (20)$$

$$N_1 = \frac{1}{2} \left( \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{23}|^2} - (\rho_{11} + \rho_{44}) \right), \quad (21)$$

$$N_2 = \frac{1}{2} \left( \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{14}|^2} - (\rho_{22} + \rho_{33}) \right). \quad (22)$$

The conditions $N_1 > 0$ or $N_2 > 0$ are rewritten in the simple form as

$$\sqrt{\rho_{11}\rho_{44}} < |\rho_{23}| \quad \text{or} \quad \sqrt{\rho_{22}\rho_{33}} < |\rho_{14}|. \quad (23)$$

For the detailed understanding of the quantum non-local correlation, it is important to evaluate the Bell-CHSH inequality [3] given by the correlation function for the qubit A and B. To compute the Bell-CHSH inequality, we introduce the Bell operator

$$B_{AB} = \frac{1}{2} n \cdot \sigma_A \otimes (m + m') \cdot \sigma_B + \frac{1}{2} n' \cdot \sigma_A \otimes (m - m') \cdot \sigma_B, \quad (24)$$

where $n, n', m, m'$ are unit vectors. We consider the maximum expectation value $\beta$ of the Bell-CHSH operator

$$\beta(\rho_{AB}) = \max_{n,n',m,m'} \text{Tr}[B_{AB} \rho_{AB}], \quad (25)$$
For separable states, $\beta(\rho_{AB})$ satisfies the following Bell-CHSH inequality

$$\beta(\rho_{AB}) \leq 1.$$ \hfill (26)

The inequality (26) holds for the local hidden variable theory which includes any separable states. For any physical states, $\beta(\rho_{AB})$ has the upper bound called the Tsirelson bound [27]

$$\beta(\rho_{AB}) \leq \sqrt{2}.$$ \hfill (27)

For an X state, the maximum value $\beta(\rho_{AB})$ can be calculated explicitly as

$$\beta(\rho_{AB}) = \max[\beta_1, \beta_2],$$ \hfill (28)

$$\beta_1 = \sqrt{(\rho_{11} + \rho_{44} - \rho_{22} - \rho_{33})^2 + 4(|\rho_{14}| + |\rho_{23}|)^2},$$ \hfill (29)

$$\beta_2 = 2\sqrt{(|\rho_{14}| + |\rho_{23}|)^2 + (|\rho_{14}| - |\rho_{23}|)^2},$$ \hfill (30)

where we used the Horodecki theorem [25] which provides the method to obtain the explicit form of $\beta$ from the singular value of the matrix $R^{ij} = \text{Tr}[\sigma^i_A \sigma^j_B \rho_{AB}]$. Note that the Bell-CHSH inequality is satisfied for the state of two detectors system given by (14)-(17) because of its perturbative treatment: The order of the coupling $g_0$ for the non-diagonal components $\rho_{23}$ and $\rho_{14}$ is $O(g_0^2)$, and then $\beta_1$ and $\beta_2$ for a small $g_0$ are evaluated as

$$\beta_1 = 1 - 2(\rho_{22} + \rho_{33}) + O(g_0^4), \quad \beta_2 = O(g_0^2),$$ \hfill (31)

where $\rho_{22}$ and $\rho_{33}$ are $O(g_0^2)$. Hence the maximum expectation value of the Bell operator $\beta$ is smaller than unity and the Bell-CHSH inequality is always satisfied. On the other hand, it is possible for the detectors to have a nonzero negativity because the condition for the entangled state (23) does not depend on the strength of coupling (the both sides of the inequality (23) have the same order for the coupling). Figure 2 shows the contour plot of the negativity in $(\Omega_r, \Omega_{\sigma})$ space for the detectors’ initial state $|\downarrow_A \downarrow_B\rangle$. The dashed line denotes the “null” curve $r = \sigma$ and then we find that the negativity has a nonzero value for a spacelike region $r > \sigma$. 

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FIG. 2: The contour plot of the negativity in the parameter space \((\Omega r, \Omega \sigma)\) with the initial detectors’ state \(|\downarrow_A \downarrow_B\rangle\). The dashed line represents the null curve \(r = \sigma\).

As we have seen above, the state of the detectors is entangled and satisfies the Bell-CHSH inequality. Interestingly, it is known that the violation of the Bell-CHSH inequality (the Bell-CHSH nonlocality) for such a state can be revealed by a local filtering operation [20].

IV. LOCAL FILTERING OPERATION FOR X STATE

We introduce a local filtering operation for the two qubit detector system. The local operation is defined by

\[
\rho_{AB} \rightarrow \rho'_{AB} = \frac{1}{p} M_A N_B \rho_{AB} M_A^\dagger N_B^\dagger,
\]

where \(M_A, N_B\) are local operators \((2 \times 2\) matrices) for each subsystem and \(p = \text{Tr}[M_A^\dagger M_A N_B^\dagger N_B \rho_{AB}]\) represents the success probability to attain the filtered state. Those operators have inverse matrices and satisfy the conditions

\[
M_A^\dagger M_A \leq I_A, \quad N_B^\dagger N_B \leq I_B.
\]

The local filtering operation is regarded as the local measurement process of each qubit and selects one outcome after this operation (regarded as the probabilistic LOCC). Although the stochastic process with the probability \(p\) is a local process, but the Bell-CHSH nonlocality of the bipartite system can be enhanced.
A. Key theorems

There are two important theorems to reveal the Bell-CHSH nonlocality by the local filtering operation [22, 23]:

**Theorem 1** [22] By a local filtering operation, a two-qubit state $\rho_{AB}$ can be uniquely transformed into a Bell diagonal state.

**Theorem 2** [23] If the optimized $\beta(\rho'_{AB})$ over all local operations $M_A$ and $N_B$ is larger than unity, then the filtered state $\rho'_{AB}$ is a Bell diagonal state. The statement is represented as

$$\max_{M_A,N_B} \beta(\rho'_{AB}) > 1 \implies \rho'_{AB} = \sum_{\mu=0}^{3} \lambda_\mu |\text{Bell}_\mu^{AB}\rangle \langle \text{Bell}_\mu^{AB}|,$$ \hspace{1cm} (34)

where $|\text{Bell}_\mu^{AB}\rangle := \sigma^\mu (|\uparrow_A \uparrow_B\rangle + |\downarrow_A \downarrow_B\rangle)/\sqrt{2}$ and $\sigma^\mu = \{I, \sigma^x, \sigma^y, \sigma^z\}$.

According to above theorems, we need the local operation transforming a given state to a Bell diagonal form to reveal the Bell-CHSH nonlocality for the state because the Bell diagonal form of the state is necessary for $\max \beta > 1$. In general, it is complicated to construct a local operation which transforms a given state to a Bell diagonal state, however we easily get it for an X state. We note that a Bell diagonal state $\sum_{\mu=0}^{3} \lambda_\mu |\text{Bell}_\mu^{AB}\rangle \langle \text{Bell}_\mu^{AB}|$ has the form of X state with its components given by

$$\frac{1}{2} \begin{bmatrix}
\lambda_0 + \lambda_3 & 0 & 0 & \lambda_0 - \lambda_3 \\
0 & \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 & 0 \\
0 & \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & 0 \\
\lambda_0 - \lambda_3 & 0 & 0 & \lambda_0 + \lambda_3 \\
\end{bmatrix}, \hspace{1cm} \sum_{\mu=0}^{3} \lambda_\mu = 1, \hspace{0.5cm} \lambda_\mu \geq 0. \hspace{1cm} (35)$$

This state corresponds is the X state with

$$\rho_{11} = \rho_{44}, \hspace{0.5cm} \rho_{22} = \rho_{33}, \hspace{0.5cm} \rho_{14} = \rho_{14}^*, \hspace{0.5cm} \rho_{23} = \rho_{23}^*.$$ \hspace{1cm} (36)

All we have to do is to transform a given X state to the X state satisfying these conditions by an appropriate filtering operation. We apply the local $z$ rotation $\exp[-i\theta \sigma_A^z/2 - i\phi \sigma_B^z/2]$ to a given X state. The diagonal components are invariant and the non-diagonal components are transformed as

$$\rho_{14} \rightarrow e^{-i(\theta + \phi)} \rho_{14}, \hspace{0.5cm} \rho_{23} \rightarrow e^{-i(\theta - \phi)} \rho_{23}.$$ \hspace{1cm} (37)

We can choose the parameters $\theta, \phi$ so that $\rho_{14}, \rho_{23}$ are positive and satisfy $\rho_{23} = \rho_{23}^*, \rho_{14} = \rho_{14}^*$. Without loss of generality, we assume that the diagonal components satisfy $\rho_{11} \geq \rho_{22} \geq \rho_{33} \geq$
\( \rho_{44} \). From the theorem 1, we can uniquely transform the two qubit system to a Bell diagonal form by a local filtering operation. Hence it is sufficient to find one of the filtering operations converting a given X state to a Bell diagonal state. For this purpose, we consider the local operation defined by

\[
M_A = \begin{bmatrix} \eta_A & 0 \\ 0 & 1 \end{bmatrix}, \quad N_B = \begin{bmatrix} \eta_B & 0 \\ 0 & 1 \end{bmatrix},
\]

where \( 0 < \eta_A^2 \leq 1 \) and \( 0 < \eta_B^2 \leq 1 \). This operation corresponds to the amplitude damping channel with a post selection and was used in Ref. [12] to detect the Bell-CHSH nonlocality. Under the local operations (38), the X state is transformed to

\[
\rho'_{AB} = \frac{1}{p} \begin{bmatrix}
\eta_A^2 \rho_{11} & 0 & 0 & \eta_A \eta_B |\rho_{14}\rangle \\
0 & \eta_A \rho_{22} & \eta_A \eta_B |\rho_{23}\rangle & 0 \\
0 & \eta_A \eta_B |\rho_{23}\rangle & \eta_B^2 \rho_{33} & 0 \\
\eta_A \eta_B |\rho_{14}\rangle & 0 & 0 & \rho_{44}
\end{bmatrix},
\]

where \( p = \eta_A^2 \eta_B^2 \rho_{11} + \eta_A^2 \rho_{22} + \eta_B^2 \rho_{33} + \rho_{44} \). If the parameters \( \eta_A \) and \( \eta_B \) satisfy \( \eta_A^2 \eta_B^2 \rho_{11} = \rho_{44} \), \( \eta_A^2 \rho_{22} = \eta_B^2 \rho_{33} \), that is,

\[
\eta_A^2 = \left( \frac{\rho_{44} \rho_{33}}{\rho_{11} \rho_{22}} \right)^{1/2}, \quad \eta_B^2 = \left( \frac{\rho_{44} \rho_{22}}{\rho_{11} \rho_{33}} \right)^{1/2},
\]

then the X state becomes the Bell diagonal state with the spectrum \( \{ \lambda_\mu \} \) given by

\[
\lambda_0 = \frac{\sqrt{\rho_{11} \rho_{44}} + |\rho_{14}|}{2(\sqrt{\rho_{11} \rho_{44}} + \sqrt{\rho_{22} \rho_{33}})}, \quad \lambda_1 = \frac{\sqrt{\rho_{22} \rho_{33}} + |\rho_{23}|}{2(\sqrt{\rho_{11} \rho_{44}} + \sqrt{\rho_{22} \rho_{33}})}
\]

\[
\lambda_2 = \frac{\sqrt{\rho_{22} \rho_{33}} - |\rho_{23}|}{2(\sqrt{\rho_{11} \rho_{44}} + \sqrt{\rho_{22} \rho_{33}})}, \quad \lambda_3 = \frac{\sqrt{\rho_{11} \rho_{44}} - |\rho_{14}|}{2(\sqrt{\rho_{11} \rho_{44}} + \sqrt{\rho_{22} \rho_{33}})}.
\]

Eq. (40) provides the optimal values of the local filters for detection of the Bell-CHSH nonlocality and the success probability \( p \) of the optimal filtering is

\[
p = 2 \rho_{44} \left[ 1 + \left( \frac{\rho_{22} \rho_{33}}{\rho_{11} \rho_{44}} \right)^{1/2} \right].
\]

This is probability characterizes the reliability of detecting the Bell-CHSH nonlocality by the local filtering operation.

**B. Quantum correlation of Bell diagonal state and coherence of X state**

To get clear understanding of the nonlocality represented by the X state, we investigate the detailed properties of the Bell diagonal state and its relationship to the X state. The
entanglement of the Bell diagonal state is completely characterized by the negativity. The conditions of non-zero negativity (23) for the Bell diagonal state are equivalent to

\[(\lambda_0 - 1/2)(\lambda_3 - 1/2) < 0 \quad \text{or} \quad (\lambda_1 - 1/2)(\lambda_2 - 1/2) < 0, \quad (43)\]

where we used the equation (35). Hence, whenever the largest eigenvalue of \(\lambda_\mu\) exceeds 1/2 (the spectrum \(\{\lambda_\mu\}\) is biased towards any one of the four Bell states), then the Bell diagonal state is entangled. Further, we focus on the Bell-CHSH nonlocality for the Bell diagonal state. When the maximum value \(\beta\) is larger than 1 (that is, \(\beta_1 > 1\) or \(\beta_2 > 1\) in the equation (28)), the eigenvalues satisfy

\[(\lambda_0 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 > 1/2 \quad \text{or} \quad (\lambda_0 - \lambda_1)^2 + (\lambda_2 - \lambda_3)^2 > 1/2, \quad (44)\]

where \(\lambda_0 \geq \lambda_3\) and \(\lambda_1 \geq \lambda_2\) are imposed by the equation (41). If we assume \(\lambda_0 > 1/2\) then \((\lambda_1 - \lambda_3)^2 \leq 1/4\) and \((\lambda_2 - \lambda_3)^2 \leq 1/4\) because \(\sum_{\mu} \lambda_\mu = 1\). Hence,

\[\lambda_0 - \lambda_2 > \frac{1}{2} \quad \text{or} \quad \lambda_0 - \lambda_1 > \frac{1}{2}. \quad (45)\]

To summarize, the typical region of the spectra satisfying the entanglement condition (43) and the Bell-CHSH nonlocality (necessary) conditions (45) are presented in Fig. 3. As is shown, the Bell diagonal state has the Bell-CHSH nonlocal correlation when \(\{\lambda_\mu\}\) concentrates in one of the Bell basis.

![Diagram](image)

**FIG. 3:** The typical region of \(\{\lambda_\mu\}\) revealing the quantum entanglement and the Bell-CHSH nonlocality.

Let us interpret the nonlocality of the Bell diagonal state in terms of the X state. In the equation (41), the spectrum \(\{\lambda_\mu\}\) depend on the components of the X state and their
dominant terms are $|\rho_{14}|$ and $|\rho_{23}|$. To make a biased distribution of $\{\lambda_{\mu}\}$, only one of $|\rho_{14}|$ and $|\rho_{23}|$ needs to be large. Each of the non-diagonal terms $\rho_{14}$ and $\rho_{23}$ represents the coherence for the superposition $\{|\uparrow \downarrow A\downarrow B\rangle, |\downarrow \uparrow A\downarrow B\rangle\}$ and $\{|\uparrow \downarrow A\downarrow B\rangle, |\downarrow \uparrow A\downarrow B\rangle\}$, respectively. Therefore, the filtered X state (the corresponding Bell diagonal state) shows the quantum correlation when either of the X state’s coherence $\{|\uparrow \downarrow A\downarrow B\rangle, |\downarrow \uparrow A\downarrow B\rangle\}$ or $\{|\uparrow \downarrow A\downarrow B\rangle, |\downarrow \uparrow A\downarrow B\rangle\}$ is dominant.

V. DETECTION OF BELL-CHSH NONLOCALITY WITH LOCAL FILTER

In this section, we examine the quantum entanglement and the Bell-CHSH nonlocality detected by the two qubit detectors with the initial conditions $|\downarrow \downarrow A\downarrow B\rangle, |\uparrow \uparrow A\downarrow B\rangle$ and $|\downarrow \uparrow A\downarrow B\rangle$. For the detection of the Bell-CHSH nonlocality, we apply the local filter to the qubit detectors’ state given in Sec. IV and evaluate the success probability of the optimal filtering. Then we clarify what properties determine the detection of the quantum correlation of the scalar field and the success probability of the filtering.

A. The initial condition $|\downarrow \downarrow A\downarrow B\rangle$

We consider the initial condition of the detectors $(a, b) = (-1, -1)$ corresponding to the state $|\downarrow \downarrow A\downarrow B\rangle$. From the equation (14),(15),(16) and (17), we derive

$$\rho_{22}(-1) = \frac{g_0^2}{4\pi} e^{-(\Omega\sigma)^2} - 2\Omega\sigma \text{Erfc}[\Omega\sigma], \quad (46)$$

$$\rho_{23}(-1, -1) = \frac{g_0^2}{4\pi r} e^{-(r/2\sigma)^2} \left( e^{-i\Omega r} \text{Erfc}\left[\Omega\sigma - i \frac{r}{2\sigma}\right] - e^{i\Omega r} \text{Erfc}\left[\Omega\sigma + i \frac{r}{2\sigma}\right]\right), \quad (47)$$

$$\rho_{14}(-1, -1) = \frac{g_0^2}{2\pi r} e^{-2i\Omega\sigma - (\Omega\sigma)^2 - (r/2\sigma)^2} \text{Erfc}\left[-i \frac{r}{2\sigma}\right], \quad (48)$$

and $\rho_{33}(-1) = \rho_{22}(-1)$. The left panel of Fig. 4 shows the contour plot of the negativity for the filtered X state with the initial condition $|\downarrow \downarrow A\downarrow B\rangle$. The coupling $g_0$ is fixed to $10^{-2}$. 

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FIG. 4: Left panel: The contour plot of the negativity with the initial condition $|\downarrow_A \downarrow_B\rangle$. The green dashed line denotes $\beta = 1$, and the above this line $\beta > 1$. Right panel: the behavior $\beta$ and the success probability $p$ of the optimal filtering as a function of $\Omega r$ with fixed $\Omega \sigma = 2.5$, $g_0 = 10^{-2}$.

The green dashed line denotes $\beta = 1$, and the region above this line represents $\beta > 1$ where the Bell-CHSH inequality is violated. In addition, we observe existence of the region where the Bell-CHSH nonlocality is not found even if the optimal filtering is acted on each detector.

In the right panel of Fig. 4, the value $\beta$ for the optimal filtering and the success probability $p$ of the optimal filtering are presented. According to the equation (42), the probability $p$ is $O(g_0^4)$ and given by

$$p \approx 2(\rho_{22}\rho_{33} + |\rho_{23}|^2 + |\rho_{14}|^2)\left[1 + \left(\frac{\rho_{22}\rho_{33} + |\rho_{23}|^2 + |\rho_{14}|^2}{\rho_{22}\rho_{33}}\right)^{1/2}\right].$$  \hspace{1cm} (49)

The value $p$ is around $10^{-15}$ for $\Omega \sigma = 2.5$ and $\Omega r = 3$ in the right panel of Fig. 4 and also for these parameter, $\beta$ is larger than 1. Hence, the probability $p$ is much smaller than $g_0^4$, which means that the success probability of the Bell-CHSH nonlocality detection is very small.

Also we analyze how the quantum correlation of the scalar field is detected through the detectors. In Sec. IV, we give the simple form of the spectrum $\{\lambda_\mu\}$ obtained from the components of the X state (41). Figure 5 shows the behavior of those spectra with $\Omega \sigma = 2.5$ and we observe that $\lambda_0$ is dominant compared to the others. The condition $\lambda_0 > 1/2$ is equivalent to $|\rho_{14}| > \sqrt{\rho_{22}\rho_{33}}$ in the equation (23). Thus the coherence $|\rho_{14}|$
FIG. 5: The behavior of the spectrum \( \{\lambda_\mu\} \) of the Bell diagonal state with fixed \( \Omega \sigma = 2.5 \) and \( g_0 = 10^{-2} \). The initial condition of the detectors' state is \( |\downarrow_A\downarrow_B\rangle \). \( \lambda_0 \) is larger than the other eigenvalues, that is, the coherence \( |\rho_{14}| \) is dominant.

of the superposition \( \{|\uparrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle\} \) is larger than \( \sqrt{\rho_{22}\rho_{33}} \), which gives the maximum of the coherence \( |\rho_{23}| \). To understand why the coherence \( |\rho_{14}| \) dominates, we remind that the detectors' state depends on the two-point function of the scalar field. The superposition \( \{|\uparrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle\} \) is realized by the exchange of the real or virtual scalar field. Figure 6 represents the diagrammatic picture of the superposition \( \{|\uparrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle\} \) in the second order dynamics.

FIG. 6: The diagrammatic picture of the coherence generation by the exchange of the scalar field.
B. The initial condition $|↑_A↑_B⟩$

We consider the detection of the quantum correlation for the initial state $|↑_A↑_B⟩$. The components $ρ_{22}, ρ_{33}, ρ_{23}$ and $ρ_{14}$ of the reduced density matrix are

\[
ρ_{22}(1) = \frac{g_0^2}{4π} \left( e^{-(Ωσ)^2} + 2Ωσ \text{Erfc}[−Ωσ] \right), \\
ρ_{23}(1, 1) = \frac{g_0^2}{4π i r} e^{-(r/2σ)^2} \left( e^{iΩr} \text{Erfc}[−Ωσ − i r / 2σ] − e^{−iΩr} \text{Erfc}[−Ωσ + i r / 2σ] \right), \\
ρ_{14}(1, 1) = \frac{g_0^2}{2π i r} e^{2iΩσ − (Ωσ)^2 − (r/2σ)^2} \text{Erfc}[−i r / 2σ],
\]

and $ρ_{33}(1) = ρ_{22}(1)$. These components are also given by replacing the frequency $Ω$ with $−Ω$ in the reduced density matrix for $(a, b) = (−1, −1)$. This is because the total Hamiltonian is invariant under the unitary transformations $σ^x_Aσ^x_B$ and $Ω \rightarrow −Ω$. Further, we find that $|ρ_{14}(1, 1)| = |ρ_{14}(−1, −1)|$, that is, those coherences with the two different initial conditions are equivalent. We have considered that the qubit A and B interact with the scalar field in the same manner and the total system evolves under the second order dynamics. Hence $|ρ_{14}(−1, −1)|$ is equal to the transition probability of $|↓_A↓_B⟩|0_φ⟩ \rightarrow |↑_A↑_B⟩|0_φ⟩$, which represents the exchange of the scalar particle between the two detectors. Because of the time translation and time reversal invariance of the vacuum, this transition probability is the same as the transition probability of $|↑_A↑_B⟩|0_φ⟩ \rightarrow |↓_A↓_B⟩|0_φ⟩$. The detail calculation is presented in the Appendix B.

The left panel of Fig. 7 presents the contour plot of the negativity with the initial state $|↑_A↑_B⟩$. The region with the nonzero negativity is much smaller compared to the result obtained with the initial state $|↓_A↓_B⟩$, and there is no spacelike region showing the quantum nonlocal correlation. In the right panel of Fig. 7, we gives the behavior of the spectrum of the Bell diagonal state for the initial condition $(a, b) = (1, 1)$ with fixed $Ωσ = 0.5$. Similar to $(a, b) = (−1, −1)$, we can confirm that the quantum correlation of the X state is carried by the exchange of the scalar particle.
Although the equality of the coherence $|\rho_{14}(-1,-1)| = |\rho_{14}(1,1)|$ holds, the detected region of entanglement is much different from that with the initial condition $|\downarrow_A\downarrow_B\rangle$. To understand this feature, we focus on the diagonal components $\rho_{22}$ and $\rho_{33}$. Evaluating the difference $\rho_{22}(1) - \rho_{22}(-1)$, we obtain

$$\rho_{22}(1) - \rho_{22}(-1) = \frac{g_0^2 \Omega \sigma}{2\pi} \int_{-\Omega \sigma}^{\Omega \sigma} dt \ e^{-t^2} \geq 0. \quad (53)$$

For $\Omega \sigma \gg 1$ the difference is proportional to $\Omega \sigma$, which corresponds to the Fermi’s golden rule. This implies that the detector with the initially excited state emits the scalar particle spontaneously. We note that the components $\rho_{22}$ and $\rho_{33}$ are the transition probability of $|a,b\rangle|0\phi\rangle \rightarrow |−a,b\rangle|1\phi\rangle$ and $|a,b\rangle|0\phi\rangle \rightarrow |a,−b\rangle|1\phi\rangle$, respectively ($|1\phi\rangle$ is a one-particle state of the scalar field). As the spontaneous emission is determined by the local dynamics, the detectors’ entanglement is not generated by such an emission. Indeed, from the equations (41) we note that the eigenvalue $\lambda_0$ is rewritten as

$$\lambda_0 = \frac{1}{2} + \frac{|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}}{2(\sqrt{\rho_{14}\rho_{44}} + \sqrt{\rho_{22}\rho_{33}})}, \quad (54)$$

and hence the inequality $|\rho_{14}(-1,-1)| - \rho_{22}(-1) \geq |\rho_{14}(1,1)| - \rho_{22}(1)$ implies the smallness of the eigenvalue $\lambda_0$. Therefore, it is difficult to reveal the spatial entanglement and the spatial Bell-CHSH nonlocality with initial excited state $|\uparrow_A\uparrow_B\rangle$. 

FIG. 7: Left panel: The contour plot of the negativity with the initial condition $|\uparrow_A\uparrow_B\rangle$. The inset is the enlarged version of the contour plot. The green dotted line represents $\beta = 1$. $N \neq 0$ region does not extend to the “space-like” region $r > \sigma$. Right panel: The behavior of the spectrum $\{\lambda_\mu\}$ of the Bell diagonal state with fixed $\Omega\sigma = 0.5$ and $g_0 = 10^{-2}$. The eigenvalue $\lambda_0$ is dominant due to the exchange of the scalar particle.
C. The initial condition $|\downarrow_A \uparrow_B\rangle$

We consider the detectors’ initial condition $|\downarrow_A \uparrow_B\rangle$. The components $\rho_{22}, \rho_{33}, \rho_{23}$ and $\rho_{14}$ of the reduced density matrix are given as

$$\rho_{22}(-1) = \frac{g_0^2}{4\pi} \left(e^{-(\Omega\sigma)^2} - 2\Omega\sigma \text{Erfc}[\Omega\sigma]\right),$$  \hfill (55)
$$\rho_{33}(1) = \frac{g_0^2}{4\pi} \left(e^{-(\Omega\sigma)^2} + 2\Omega\sigma \text{Erfc}[-\Omega\sigma]\right),$$  \hfill (56)
$$\rho_{23}(-1,1) = \frac{g_0^2\sigma}{4\pi i r} e^{-2i\Omega_0 -(r/2\sigma)^2} \left(\text{Erfc}\left[-i\frac{r}{2\sigma}\right] - \text{Erfc}\left[i\frac{r}{2\sigma}\right]\right),$$  \hfill (57)
$$\rho_{14}(-1,1) = \frac{g_0^2\sigma}{4\pi i r} e^{-(r/2\sigma)^2} \left(\text{Erfc}\left[-\Omega\sigma - i\frac{r}{2\sigma}\right] + \text{Erfc}\left[\Omega\sigma - i\frac{r}{2\sigma}\right]\right).$$  \hfill (58)

The left panel of Fig. 8 shows the contour plot of the negativity with the green dotted line $\beta = 1$. Further, we add the orange line which denotes $N = 0$ for the initial state $|\downarrow_A \downarrow_B\rangle$.

In this case, the spacelike entanglement is detected, however the size of the spacelike region is smaller compared to the case $|\downarrow_A \downarrow_B\rangle$. Right upper panel: The behavior of the spectrum $\{\lambda_\mu\}$. The eigenvalue $\lambda_0$ is the largest and the coherence $|\rho_{14}|$ is dominant in $N > 0$ region. Right lower panel: The ratio $|\rho_{14}(-1,1)|/|\rho_{14}(-1,-1)|$ is presented. The coherence $|\rho_{14}(-1,1)|$ is larger than $|\rho_{14}(1,1)|$.

**FIG. 8:** Left panel: The contour plot of the negativity with the initial condition $|\downarrow_A \uparrow_B\rangle$. The orange dashed line denotes the boundary of the nonzero negativity with the initial state $|\downarrow_A \downarrow_B\rangle$. In this case, the spacelike entanglement is detected, however the size of the spacelike region is smaller compared to the case $|\downarrow_A \downarrow_B\rangle$. Right upper panel: The behavior of the spectrum $\{\lambda_\mu\}$. The eigenvalue $\lambda_0$ is the largest and the coherence $|\rho_{14}|$ is dominant in $N > 0$ region. Right lower panel: The ratio $|\rho_{14}(-1,1)|/|\rho_{14}(-1,-1)|$ is presented. The coherence $|\rho_{14}(-1,1)|$ is larger than $|\rho_{14}(1,1)|$. 

We observe that the region with the nonzero negativity is smaller compared to the result obtained with the initial state $\left| \downarrow_A \downarrow_B \right>$, however there is the spacelike region which shows the quantum nonlocality unlike the results with $\left| \uparrow_A \uparrow_B \right>$. In the right upper panel of Fig. 8, the behavior of the eigenvalues of the Bell diagonal state is presented. As expected, the nonlocality detection is possible by the exchange of the scalar particle.

Also we denote the ratio $|\rho_{14}(1,-1)|/|\rho_{14}(-1,-1)|$ in the right lower panel of Fig. 8 and observe the coherence $|\rho_{14}(1,-1)|$ is larger than $|\rho_{14}(-1,-1)|$. This is because the detector with the initial excited state generates more real or virtual particles compared to the detector with the initial ground state $\left| \downarrow_A \downarrow_B \right>$. We have clarified that the process of the spontaneous emission from each detector reduces the negativity of the detectors’ state. The reduced density matrix of the detector with the initial condition $\left| \downarrow_A \uparrow_B \right>$ has the component $\rho_{33}(1)$ which is larger than $\rho_{33}(-1)$. Thus, the coherence $\rho_{14}(-1,1)$ and the transition probability $\rho_{33}(1)$ of the spontaneous emission non-trivially determine the eigenvalue $\lambda_0$ for the initial condition $\left| \downarrow_A \uparrow_B \right>$.

The left panel of Fig. 9 presents the violation of the Bell-CHSH inequality and the success probability of the optimal filtering. The qualitative behavior is similar to that with the initial state $\left| \downarrow_A \downarrow_B \right>$.

![Figure 9](image-url)

**FIG. 9:** Left panel: The behavior of $\beta$ and the detection probability $p$ with fixed $\omega \sigma = 2.5$ and $g_0 = 10^{-2}$. Right panel: The ratio $p(-1,1)/p(-1,-1)$. The detection probability $p(-1,1)$ of the Bell-CHSH nonlocality for the spacelike region is much larger than the probability $p(-1,-1)$. 


However, it shows the success probability of the Bell-CHSH nonlocality between two spacelike points is much larger for this setting. In the right panel of Fig. 9, the behavior of the ratio \( p(-1, 1)/p(-1, -1) \) is presented with fixed \( \Omega \sigma = 2.5 \). We find that the probability \( p(-1, 1) \) is much larger than the probability \( p(-1, -1) \) for the initial condition \( |\downarrow_A \downarrow_B\rangle \). Hence the detectors’ initial condition \( |\downarrow_A \uparrow_B\rangle \) is more suitable to detect the spacelike Bell-CHSH nonlocality compared to the initial condition \( |\downarrow_A \downarrow_B\rangle \).

Finally let us comment on the trade-off relation between the dimensions of the parameter space showing the Bell-CHSH nonlocality and the success probability \( p \) of the optimal filtering operation. Roughly speaking, from the equations (49) and (54), the success probability is determined by the sum \( |\rho_{14}|^2 + \rho_{22}\rho_{33} + |\rho_{23}|^2 \), and the eigenvalue \( \lambda_0 \) is given by the difference \( |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}} \). Thus as the success probability \( p \) increases, the eigenvalue \( \lambda_0 \) decreases because the transition probability \( \rho_{22} \) or \( \rho_{33} \) of the spontaneous emission grows.

In Fig. 8 and 9, the detected region of the Bell-CHSH nonlocality is small but the success probability is large. Hence the trade-off relation between the size of detectable parameter region of the Bell-CHSH nonlocality and the success probability is demonstrated for the detection problem with the qubit detectors.

VI. SUMMARY AND CONCLUSION

We investigated the detection of the quantum correlation of a massless scalar field by two qubit detectors. As an initial state, we considered a product state of the detectors and the vacuum state of the scalar field. Under the second order perturbation of the total system dynamics, the two detectors’ state can be entangled by the two-point function of the scalar field. Also we focused on the violation of the Bell-CHSH inequality for the qubit detectors. It is known that the Bell-CHSH nonlocality can be revealed only after the local filtering operation, which is post-selected LOCC by each of two local observers. In general, although it is complicated to construct the optimal filtering operation for revealing the Bell-CHSH nonlocality, we can simply obtain the optimal filtering as the considering detectors’ out-state is the X-state. The constructed filtering is given by a post selection after passing through an amplitude damping channel, and the probability for the post selection corresponds to the success probability of the optimal filtering. By examining the negativity and the violation of the Bell-CHSH inequality under the optimal filter, we found that the detection of nonlocal
correlation strongly depends on the initial state of the detectors. When the detectors are initially in the ground state, the spacelike region in the parameter space showing the quantum nonlocality is larger compared to the region obtained with the initially excited states. This is because the excited detectors spontaneously emit the scalar particles, and such a local dynamics cannot generate the quantum correlation between the detectors.

Further we focused on the success probability of the optimal filtering for the Bell-CHSH nonlocality detection between spatial separated regions. When the transition probability $\rho_{22}$ or $\rho_{33}$ describing the spontaneous emission is large, the Bell-CHSH nonlocality is small but the success probability is large. Due to this trade-off relation, the reliable detection of the Bell-CHSH nonlocality becomes non-trivial and we found that the detection of the spacelike Bell-CHSH nonlocality with a high success probability of the optimal filtering is performed when the detectors’ state is initially $|\downarrow_A \uparrow_B\rangle$. This result gives the suitable model for the reliable detection of the spacelike Bell-CHSH nonlocality through the two qubit detectors.

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Appendix A: components of reduced density matrix

The diagonal components $\rho_{22}$ and $\rho_{33}$ are obtained from $\rho_{23}$. By inserting the complete set in the equation (8), the non-diagonal component $\rho_{23}$ are represented by

$$
\rho_{23} = \langle 0_\phi | \Phi^A_a \Phi^B_{-b} | 0_\phi \rangle = \int d^3k \langle 0_\phi | \Phi^A_a | k_\phi \rangle \langle k_\phi | \Phi^B_{-b} | 0_\phi \rangle,
$$

where note that the inner product of $\Phi^A_a | 0_\phi \rangle$ and $n$-particle state for $n \geq 2$ or $n = 0$ is zero.

We introduce the regularized mode function of the Minkowski vacuum

$$
u_\epsilon^a(x,t) = \frac{e^{-i\omega_k(t-i\epsilon/2)+ik \cdot x}}{(2\pi)^{3/2} \sqrt{2\omega_k}},
$$

where $\omega_k = |k|$. The inner product $\langle 0_\phi | \Phi^A_a | k_\phi \rangle$ are calculated as

$$
\langle 0_\phi | \Phi^A_a | k_\phi \rangle = \int_{-\infty}^{\infty} dt g(t) e^{-i\Omega t} u_\epsilon^a(x_A, t) = g_0 \sqrt{2\pi\sigma^2} \frac{e^{-\frac{\sigma^2}{2}(\omega_k-\Omega \tau)^2-i\Omega \tau \omega}}{\sqrt{2\omega_k}} \nu_\epsilon(x_A, t_0).
$$
The component $\rho_{23}$ is computed as

$$\rho_{23} = \int d^3k \langle 0_\phi | \Phi_A^a | k_\phi \rangle \langle k_\phi | \Phi_B^b | 0_\phi \rangle$$

$$= 2\pi g_0^2 \sigma^2 e^{i\Omega(a-b)t_0} \int d^3k \ e^{-\frac{q^2}{2}(\omega_k-\Omegaa)^2 - \frac{q^2}{2}(\omega_k-\Omb)^2} u^a_k(x_A, t_0) u^b_k(x_B, t_0)$$

$$= \frac{g_0^2 \sigma}{4i\pi} e^{i\Omega(a-b)t_0} \int_0^\infty du \ e^{-\frac{1}{2}(u-\Omegaa)^2 - \frac{1}{2}(u-\Omb)^2} (e^{iur/\sigma} - e^{-iur/\sigma}) e^{-ru/\sigma}$$

$$= -\frac{g_0^2 \sigma}{4i\pi} e^{i\Omega(a-b)t_0 - (\Omegaa)^2} \left[ \exp \left[ \left( -\frac{\Omegaa}{2} (a+b) - i\frac{r}{2\sigma} + \frac{\epsilon}{2\sigma} \right)^2 \right] \operatorname{Erfc} \left[ -\frac{\Omegaa}{2} (a+b) - i\frac{r}{2\sigma} + \frac{\epsilon}{2\sigma} \right] \right.$$

$$+ \exp \left[ \left( -\frac{\Omb}{2} (a+b) + i\frac{r}{2\sigma} + \frac{\epsilon}{2\sigma} \right)^2 \right] \operatorname{Erfc} \left[ -\frac{\Omb}{2} (a+b) + i\frac{r}{2\sigma} + \frac{\epsilon}{2\sigma} \right] \right]. \quad (A4)$$

We get the equation (16) by taking the limit $\epsilon \to 0$. Next we derive the formula of $\rho_{14}$. Using the Wightman function $D^+(x-x', t-t') = \langle 0_\phi | \phi(x, t) \phi(x', t') | 0_\phi \rangle$ given by the equation (13), the non-diagonal component $\rho_{14}^*$ is

$$\rho_{14}^* = -\langle 0^\phi | [\Phi_A^a, \Phi_B^b] | 0^\phi \rangle$$

$$= -\int_{-\infty}^\infty dt_2 \int_{-\infty}^\infty dt_1 g(t_2)g(t_1) e^{i\Omega(-at_2-bt_1)} \left[ \theta(t_2 - t_1) D^+(r, t_2 - t_1) + \theta(t_1 - t_2) D^+(r, t_1 - t_2) \right]$$

$$= -2\sqrt{\pi} g_0^2 \sigma^2 (\Omegaa) e^{-i\Omega(a+b)t_0 - (\Omegaa)^2}$$

$$\times \int_{-\infty}^\infty dy e^{-(\Omegaa)^2(y+i(a-b)/2)^2} (\theta(-y) D^+(r, -2\Omegaa^2 y) + \theta(y) D^+(r, 2\Omegaa^2 y)), \quad (A5)$$

where the integral variables $t_1$ and $t_2$ are changed as

$$\Omegaa^2 x = \frac{(t_1 - t_0) + (t_2 - t_0)}{2}, \quad \Omegaa^2 y = \frac{(t_1 - t_0) - (t_2 - t_0)}{2} \quad (A6)$$

and we carried out the $x$ integration. By using the identity

$$e^{-y^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty d\eta \ e^{-\eta^2 + 2i\eta y}, \quad (A7)$$

the above equation (A5) can be rewritten as

$$\rho_{14}^* = -2g_0^2 \sigma^2 (\Omegaa) e^{-i\Omega(a+b)t_0 - (\Omegaa)^2}$$

$$\times \int_{-\infty}^\infty d\eta e^{-\eta^2} \left( e^{-(a-b)(\Omegaa)\eta} + e^{(a-b)(\Omb)\eta} \right) \int_{0}^{\infty} dy e^{2i(\Omegaa)\eta y} D^+(r, 2\Omegaa^2 y)$$

$$= \frac{g_0^2}{4\pi^2} e^{-i\Omega(a+b)t_0 - (\Omegaa)^2} \int_{-\infty}^\infty d\eta e^{-\eta^2} \left( e^{-(a-b)(\Omegaa)\eta} + e^{(a-b)(\Omb)\eta} \right) \int_{0}^{\infty} dy \frac{e^{i\eta y}}{(y - i\epsilon/\sigma)^2 - (r/\sigma)^2}. \quad (A8)$$
The \( y \) integration is equivalent to the complex integration given in Fig. 10. Hence,

\[
\int_0^\infty dy \frac{e^{i\eta y}}{(y - i\epsilon/\sigma)^2 - (r/\sigma)^2} = \left[ \frac{i\pi \sigma}{r} e^{i\eta (r/\sigma + i\epsilon/\sigma)} - i \int_0^\infty \frac{e^{-\eta y}}{(y - \epsilon/\sigma)^2 + (r/\sigma)^2} \right] \theta(\eta)
\]

\[
+ i \int_{-\infty}^0 \frac{e^{-\eta y}}{(y - \epsilon/\sigma)^2 + (r/\sigma)^2} \theta(-\eta),
\]

(A9)

where the second and third terms are the integration along the imaginary axis. For \( \epsilon \to 0 \) the sum of those terms is an odd function, and then it does not contribute to the \( \eta \) integration (note that the function of \( \eta \) in front of the equation (A9) is an even function).

Thus we get the following formula

\[
\rho_{14}^* = \frac{g_0^2}{4\pi^2} e^{-i\Omega(a+b)t_0 -(\Omega\sigma)^2} \int_0^\infty d\eta e^{-\eta^2} \left( e^{-\Omega(a-b)\Omega\sigma\eta} + e^{\Omega(a-b)\Omega\sigma\eta} \right) \frac{i\pi \sigma}{r} e^{i\eta r/\sigma}
\]

\[
= \frac{ig_0^2\sigma}{4\pi r} e^{-i\Omega(a+b)t_0 -(\Omega\sigma)^2} \left( \exp\left[ \left( \frac{\Omega\sigma}{2}(a-b) + i\frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ \frac{-\Omega\sigma}{2}(a-b) - i\frac{r}{2\sigma} \right] 
\]

\[
+ \exp\left[ \left( -\frac{\Omega\sigma}{2}(a-b) + i\frac{r}{2\sigma} \right)^2 \right] \text{Erfc} \left[ \frac{-\Omega\sigma}{2}(a-b) + i\frac{r}{2\sigma} \right].
\]

(A10)
Appendix B: equality of $|\rho_{14}(-1,-1)|$ and $|\rho_{14}(1,1)|$

Let us show the equality of $|\rho_{14}(-1,-1)|$ and $|\rho_{14}(1,1)|$. Under the second order of the coupling, the non-diagonal components $\rho_{14}(-1,-1)$ and $\rho_{14}(1,1)$ is

$$\rho_{14}(-1,-1) = \langle \downarrow_A \downarrow_B | \rho_{\text{out}} | \uparrow_A \uparrow_B \rangle$$

$$\approx -\langle \downarrow_A \downarrow_B | \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(t_2)\tilde{V}(t_1)|0_\phi \rangle | \uparrow_A \uparrow_B \rangle$$

$$= -2\langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 g(t_1)g(t_2)e^{-i\Omega(t_1+t_2)}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle,$$

$$\rho_{14}(1,1) = \langle \uparrow_A \uparrow_B | \rho_{\text{out}} | \downarrow_A \downarrow_B \rangle$$

$$\approx -\langle \uparrow_A \uparrow_B | \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{V}(t_2)\tilde{V}(t_1)|0_\phi \rangle | \downarrow_A \downarrow_B \rangle$$

$$= -2\langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 e^{i\Omega(t_1+t_2)}g(t_1)g(t_2)\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle, \quad (B1)$$

where $\rho_{\text{out}} = \text{Tr}_\phi[|\tilde{\Psi}_{\text{out}}\rangle\langle \tilde{\Psi}_{\text{out}}|]$ and we used the translation invariant of the vacuum state. Due to the time reversal invariance of the Minkowski vacuum, $\rho_{14}(-1,-1)$ is rewritten as

$$\langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 g(t_1)g(t_2)e^{-i\Omega(t_1+t_2)}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle$$

$$= \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 g(t_1+t_0)g(t_2+t_0)e^{-i\Omega(t_1+t_2)-2i\Omega t_0}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle$$

$$= \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{t_1}^{\infty} dt_2 g(-t_1+t_0)g(-t_2+t_0)e^{i\Omega(t_1+t_2)-2i\Omega t_0}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle.$$

$$\quad (B3)$$

The switching function $g(t)$ is a Gaussian function, and $g(t+t_0) = g(-t+t_0)$ holds. Thus,

$$\langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_1} dt_2 g(-t_1+t_0)g(-t_2+t_0)e^{i\Omega(t_1+t_2)-2i\Omega t_0}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle$$

$$= \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{t_1}^{\infty} dt_2 g(t_1+t_0)g(t_2+t_0)e^{i\Omega(t_1+t_2)-2i\Omega t_0}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle$$

$$= \langle 0_\phi \rangle \int_{-\infty}^{t_1} dt_1 \int_{t_1}^{\infty} dt_2 g(t_1)g(t_2)e^{i\Omega(t_1+t_2)-4i\Omega t_0}\phi(x_B-x_A,t_2-t_1)\phi(0)|0_\phi \rangle.$$  

Therefore, we get the equation $\rho_{14}(-1,-1) = e^{-4i\Omega t_0}\rho_{14}(1,1)$, that is, $|\rho_{14}(-1,-1)| =
$|\rho_{14}(1,1)|$ by the time translation and time reversal invariance of the vacuum.

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and Quantum Information* (Cambridge University Press, 2000).

[2] J. Bell, “On the Einstein Podolsky Rosen paradox”, *Physics (College. Park. Md)*. 1, (1964) 195–200.

[3] J. Clauser, M. Horne, A. Shimony, and R. Holt, “Proposed Experiment to Test Local Hidden-Variable Theories”, *Phys. Rev. Lett.* 23, (1969) 880–884.

[4] R. Werner, “Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model”, *Phys. Rev. A* 40, (1989) 4277.

[5] W. G. Unruh, “Notes on black-hole evaporation”, *Phys. Rev. D* 14, (1976) 870.

[6] J. I. Cirac and F. Verstraete, “Renormalization and tensor product states in spin chains and lattices”, *J. Phys. A: Math. Theor.* 42, (2009) 504004

[7] H. Reeh and S. Schlieder, *Nuovo Cimento* 22,105 (1961).

[8] S. J. Summers and R. Werner, “The vacuum violates Bell’s inequalities”, *Phys.Lett.A* 110, 257 (1985).

[9] Shih-Yuin Lin and B. L. Hu, “Accelerated detector-quantum field correlations: From vacuum fluctuations to radiation flux”, *Phys. Rev. D* 73, (2006) 124018.

[10] E. G. Brown, E. Martin-Martinez, N. C. Menicucci, and R. B. Mann, “Detectors for probing relativistic quantum physics beyond perturbation theory”, *Phys. Rev. D* 87, (2013) 084062

[11] E. G. Brown, “Thermal amplification of field-correlation harvesting”, *Phys. Rev. A* 88, (2013) 062336

[12] B. Reznik, A. Retzker, and J. Silman, “Violating Bell’s inequalities in vacuum”, *Phys. Rev. A* 71, (2005) 042104.

[13] A. Retzker, J. I. Cirac, and B. Reznik, “Detecting Vacuum Entanglement in a Linear Ion Trap”, *Phys. Rev. Lett* 94, (2005) 050504

[14] J. Silman and B. Reznik, “Long-range entanglement in the Dirac vacuum”, *Phys. Rev. A* 75, (2007) 052307

[15] J. Leon and C. Sabin, “Entanglement swapping between spacelike-separated atoms”, *Phys. Rev. A* 78, (2008) 052314
[16] A. Pozas-Kerstjens and E. Martin-Martinez, “Harvesting correlations from the quantum vacuum”, *Phys. Rev. D* **92**, (2015) 064042

[17] G. Salton, R. B. Mann and N. C. Menicucci, “Acceleration-assisted entanglement harvesting and rangefinding”, *New J. Phys.* **17**, (2015) 035001

[18] A. Sachs, R. B. Mann and E. Martin-Martinez, “Entanglement harvesting and divergences in quadratic Unruh-DeWitt detector pairs”, *Phys. Rev. D* **96**, (2017) 085012

[19] A. Pozas-Kerstjens, J. Louko, and E. Martin-Martinez, “Degenerate detectors are unable to harvest spacelike entanglement”, *Phys. Rev. D* **95**, (2017) 105009

[20] N. Gisin, “Hidden quantum nonlocality revealed by local filters”, *Phys. Lett. A* **210**, (1996) 151

[21] Y. Nambu and Y. Ohsumi, “Classical and quantum correlations of scalar field in the inflationary universe”, *Phys. Rev. D* **84**, (2011) 044028

[22] F. Verstraete, J. Dehaene, and B. DeMoor, “Local filtering operations on two qubits”, *Phys. Rev. A* **64**, (2001) 010101

[23] F. Verstraete and M. M. wolf, “Entanglement versus Bell Violations and Their Behavior under Local Filtering Operations”, *Phys. Rev. Lett* **89**, (2002) 170401

[24] G. Vidal and R. Werner, “Computable measure of entanglement”, *Phys. Rev. A* **65**, (2002) 032314.

[25] R. Horodecki, P. Horodecki, and M. Horodecki, “Violating Bell inequality by mixed spin-1/2 states: necessary and sufficient condition”, *Phys. Lett. A* **200**, (1995) 340.

[26] M. Horodecki, R. Horodecki, and P. Horodecki, “Separability of mixed states: necessary and sufficient conditions”, *Phys. Lett. A* **223**, (1996) 1-8.

[27] B. Tsirelson, “Quantum generalizations of Bell’s inequality”, *Lett. Math. Phys.* **4**, (1980) 93.