SMALL VIOLATIONS OF STATISTICS

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Abstract

There are two motivations to consider statistics that are neither Bose nor Fermi: (1) to extend the framework of quantum theory and of quantum field theory, and (2) to provide a quantitative measure of possible violations of statistics. After reviewing tests of statistics for various particles, and types of statistics that are neither Bose nor Fermi, I discuss quons, particles characterized by the parameter $q$, which permit a smooth interpolation between Bose and Fermi statistics; $q = 1$ gives bosons, $q = -1$ gives fermions. The new result of this talk is work by Robert C. Hilborn and myself that gives a heuristic argument for an extension of conservation of statistics to quons with trilinear couplings of the form $\bar{f}fb$, where $f$ is fermion-like and $b$ is boson-like. We showed that $q_f^2 = q_b$. In particular, we related the bound on $q_\gamma$ for photons to the bound on $q_e$ for electrons, allowing the very precise bound for electrons to be carried over to photons. An extension of our argument suggests that all particles are fermions or bosons to high precision.

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1 INTRODUCTION

Michael Berry [1] reported on a very interesting new idea to derive the connection of spin and statistics without using relativity in this session. After hearing about this work it is going from the sublime to the ridiculous to consider theories in which particles can have statistics that are neither Bose nor Fermi. Nonetheless, I will do so for two reasons: to stretch the framework of quantum mechanics and of quantum field theory and to provide a formalism that allows a quantitative measure of the accuracy with which a given particle obeys either Bose or Fermi statistics. For an earlier general discussion of violations of statistics see [2].

I first review experiments that test statistics, and then survey the theoretical ways in which violations of statistics can be introduced for identical particles. I discuss quons, a type of particle that can have statistics that interpolate continuously between bosons and fermions, in some detail [3]. At present, the quon theory is the only theory that allows parametrization of small violations of statistics. The new result that I report in this talk is conservation of statistics for quons that relates the $q$-parameters for particles that couple to each other [1]. For electrons and photons the result is $q_{\text{photon}} = q_{\text{electron}}^2$, which allows the high-precision bound on possible violations of Fermi statistics for electrons to be carried over to a comparably high-precision bound on violations of Bose statistics for photons. In conclusion, I mention the need for a refined derivation of the above result. I also state a result for the statistics of composite systems of quons that Robert C. Hilborn and I found after the Orbis [5].

2 EXPERIMENTS

Until recently there were no high-precision tests of the Pauli exclusion principle for fermions nor were there such tests for violations of Bose statistics for bosons. The exclusion principle is deeply engrained in our understanding of quantum mechanics and there was no stimulus from either experiment or theory to question it. In the last few years, in part because of the great success of the standard model, long-accepted features of the standard model, such as Lorentz invariance [4] and CPT symmetry [7] have been questioned and, despite the absence of experimental signals
of violations, theories have been advanced that allow violations or, if no violations are seen, provide high-precision bounds on such violations for each type of particle. I am going to do the same for violations of statistics.

There are several types of experiments to detect violations of Fermi or Bose statistics if they occur. Here are three types: (i) search for transitions among anomalous states—in either solids or in gases, (ii) search for accumulation of particles in anomalous states, and (iii) search for deviations from the usual statistical properties of bulk matter. R. Amado and H. Primakoff [8] pointed out that there is a superselection rule separating states of identical particles in inequivalent irreducible representations of the symmetric group, and because of this there are no transitions between normal and anomalous states. One has to look for transitions among anomalous states rather than for transitions between normal and anomalous states. If transitions occur between states of the same symmetry type, they occur with the normal rate. Thus, for example, if the electrons in an atom are not in a totally antisymmetric representation so that the $K$-shell of the atom could have three electrons, then an electron in a higher shell would make the transition to the $K$-shell at the usual electromagnetic rate.

Atomic spectroscopy is the first place to search for violations of the exclusion principle since that is where Pauli discovered it [9]. One looks for funny lines which do not correspond to lines in the normal theory of atomic spectra. There are such lines, for example in the solar spectrum; however they probably can be accounted for in terms of highly ionized atoms in an environment of high pressure, high density and large magnetic fields. Laboratory spectra are well accounted for by theory and can bound the violation of the exclusion principle for electrons by something like $10^{-6}$ to $10^{-8}$ using the parametrization I describe in the next paragraph.

A useful quantitative measure of the violation, $v$, is that $v$ is the coefficient of the anomalous component of the two-particle density matrix; for fermions, the two-electron density matrix, $\rho_2$, is

$$\rho_2 = (1 - v_F)\rho_a + v_F\rho_s,$$

where $\rho_{a(s)}$ is the antisymmetric (symmetric) two-fermion density matrix. Mohapatra and I surveyed a variety of searches for violations of particle statistics in [10].
Next I discuss an insightful experiment by Maurice and Trudy Goldhaber that was designed to answer the question, “Are the electrons emitted in nuclear $\beta$-decay quantum mechanically identical to the electrons in atoms?” We know that the $\beta$-decay electrons have the same spin, charge and mass as electrons in atoms; however the Goldhabers realized that if the $\beta$-decay electrons were not quantum mechanically identical to those in atoms, then the $\beta$-decay electrons would not see the $K$-shell of a heavy atom as filled and would fall into the $K$-shell and emit an $x$-ray. They looked for such $x$-rays by letting $\beta$-decay electrons from a natural source fall on a block of lead. No such $x$-rays were found. The Goldhabers were able to confirm that electrons from the two sources are indeed quantum mechanically identical. At the same time, they found that any violation of the exclusion principle for electrons must be less than 5%.

E. Ramberg and G. Snow developed this experiment into one which yields a high-precision bound on violations of the exclusion principle. Their idea was to replace the natural $\beta$ source, which provides relatively few electrons, by an electric current, in which case Avogadro’s number is on our side. The possible violation of the exclusion principle is that a given collection of electrons can, with different probabilities, be in different permutation symmetry states. The probability to be in the “normal” totally antisymmetric state presumably would be close to one, the next largest probability would occur for the state with its Young tableau having one row with two boxes, etc. The idea of the experiment is that each collection of electrons has a possibility of being in an anomalous permutation state. If the density matrix for a conduction electron together with the electrons in an atom has a projection onto such an anomalous state, then the conduction electron will not see the $K$-shell of that atom as filled. Then a transition into the $K$-shell with $x$-ray emission is allowed. Each conduction electron which comes sufficiently close to a given atom has an independent chance to make such an $x$-ray-emitting transition, and thus the probability of seeing such an $x$-ray is proportional to the number of conduction electrons which traverse the sample and the number of atoms which the electrons visit, as well as the probability that a collection of electrons can be in the anomalous state. Ramberg and Snow chose to run 30 amperes through a thin copper strip for about a month. They estimated the energy of the $x$-rays which would be emitted due to the transition to the $K$-shell. No excess of $x$-rays above background
was found in this energy region. Ramberg and Snow set the limit

\[ v_F \leq 1.7 \times 10^{-26}. \]  

(2)

This is high precision, indeed! K. Deilamian, J.D. Gillaspy and D.E. Kelleher [13] searched for transitions atoms of helium in which the two electrons are in a symmetric state under permutations. They used precision calculations of the levels of such atoms made by G.W.F. Drake [14]. They found the limit \( v_F \leq 2 \times 10^{-7} \). M. De Angelis, et al [15] and, independently, R.C. Hilborn and C.L. Yuca [16] searched for forbidden bands in the \( O_2 \) spectrum and found the bounds \( v_B \leq 5 \times 10^{-7} \) and \( v_B \leq 5 \times 10^{-7} \), respectively, on violations of Bose statistics for the oxygen nuclei. Modugno, Ingusicio, and Tino [17] found that the probability of finding the two \( ^{16}O \) nuclei (spin 0) in carbon dioxide in a permutation antisymmetric state is less than \( 5 \times 10^{-9} \). Preliminary results on an experiment to bound violations of Bose statistics for photons give \( v_B \leq 10 \times 10^{-7} \) [18].

3. WAYS TO VIOLATE STATISTICS

It is difficult to violate the statistics of identical particles. The Hamiltonian must be totally symmetric in the dynamical variables of the identical particles; H cannot change the permutation symmetry type of the wave function. In particular, one cannot dial in a small violating term using \( H = H_S + \epsilon H_V \), since then the Hamiltonian would not be totally symmetric. Also one cannot, for example, have red electrons and blue electrons even if there were only red electrons in our neighborhood. This would lead to a doubling of the cross section \( \sigma(\gamma X \rightarrow e^+e^-X) \), since photons couple universally.

3.1 Gentile’s Intermediate Statistics

The first attempt to go beyond Bose and Fermi statistics seems to have been made by G. Gentile [19] who suggested an “intermediate statistics” in which at most \( n \) identical particles could occupy a given quantum state. In intermediate statistics, Fermi statistics is recovered for \( n = 1 \) and Bose statistics is recovered for \( n \rightarrow \infty \); thus intermediate statistics interpolates between Fermi and Bose statistics. However Gentile’s statistics is not a proper quantum statistics because the condition of having
at most \( n \) particles in a given quantum state is not invariant under change of basis. For example for intermediate statistics with \( n = 2 \) the state \(|\psi\rangle = |k, k, k\rangle\) does not exist; however the state \(|\chi\rangle = \sum_{l_1, l_2, l_3} U_{k, l_1} U_{k, l_2} U_{k, l_3} |l_1, l_2, l_3\rangle\) obtained from \(|\psi\rangle\) by the unitary change of single-particle basis \( |k\rangle' = \sum_l U_{k, l} |l\rangle\) does exist.

By contrast, parafermi statistics of order \( n \) (to be discussed just below) is invariant under change of basis. Parafermi statistics of order \( n \) not only allows at most \( n \) identical particles in the same state, but also allows at most \( n \) identical particles in a symmetric state. In the example just described, neither \(|\psi\rangle\) nor \(|\chi\rangle\) exist for parafermi statistics of order two.

### 3.2 Green’s Parastatistics

H.S. Green \[20\] proposed the first proper quantum mechanical generalization of Bose and Fermi statistics. Green noticed that the commutator of the number operator with the annihilation and creation operators is the same for both bosons and fermions

\[
[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger. \tag{3}
\]

The number operator can be written

\[
n_k = (1/2)[a_k^\dagger, a_k]_\pm + \text{const}, \tag{4}
\]

where the anticommutator (commutator) is for the Bose (Fermi) case. If these expressions are inserted in the number operator-creation operator commutation relation, the resulting relation is \textit{trilinear} in the annihilation and creation operators. Polarizing the number operator to get the transition operator \( n_{kl} \) that annihilates a free particle in state \( k \) and creates one in state \( l \) leads to Green’s trilinear commutation relation for his parabose and parafermi statistics,

\[
[[a_k^\dagger, a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm} a_k^\dagger. \tag{5}
\]

Since these rules are trilinear, the usual vacuum condition,

\[
a_k |0\rangle = 0, \tag{6}
\]

does not suffice to allow calculation of matrix elements of the \( a \)’s and \( a^\dagger \)’s; a condition on one-particle states must be added,

\[
a_k a_l^\dagger |0\rangle = \delta_{kl} |0\rangle. \tag{7}
\]
Green found an infinite set of solutions of his commutation rules, one for each integer, using an ansatz in terms of Bose and Fermi operators. Let

\[
    a_k^\dagger = \sum_{p=1}^{n} b_k^{(\alpha)}_p, \quad a_k = \sum_{p=1}^{n} b_k^{(\alpha)},
\]

and let the \( b_k^{(\alpha)} \) and \( b_k^{(\beta)} \) be Bose (Fermi) operators for \( \alpha = \beta \) but anticommute (commute) for \( \alpha \neq \beta \) for the “parabose” (“parafermi”) cases. This ansatz clearly satisfies Green’s relation. The integer \( p \) is the order of the parastatistics. The physical interpretation of \( p \) is that for parabosons \( p \) is the maximum number of particles that can occupy an antisymmetric state, while for parafermions \( p \) is the maximum number of particles that can occupy a symmetric state (in particular, the maximum number that can occupy the same state). The case \( p = 1 \) corresponds to the usual Bose or Fermi statistics. Later Messiah and I \[21\] proved that Green’s ansatz gives all Fock-like solutions of Green’s commutation rules. Local observables have a form analogous to the usual ones; for example, the local current for a spin-1/2 theory is \( j_\mu = (1/2)[\bar{\psi}(x), \psi(x)]_\_ \). From Green’s ansatz, it is clear that the squares of all norms of states are positive, since sums of Bose or Fermi operators give positive norms. Thus parastatistics gives a set of orthodox theories. Parastatistics is one of the possibilities found by Doplicher, Haag and Roberts \[22\] in a general study of particle statistics using algebraic field theory methods. Haag’s recent book \[23\] gives a good review of this work.

This is all well and good; however the violations of statistics provided by parastatistics are gross. Parafermi statistics of order two has up to two particles in each quantum state. High-precision experiments are not necessary to rule this out for the all particles we think are fermions.

3.3 The Ignatiev-Kuzmin Model and “Parons”

Interest in possible small violations of the exclusion principle was revived by a paper of Ignatiev and Kuzmin \[24\] in 1987. They constructed a model of one oscillator with three possible states: a vacuum state, a one-particle state and, with small probability, a two-particle state. They gave trilinear commutation relations for their
oscillator. Mohapatra and I showed that the Ignatiev-Kuzmin oscillator could be represented by a modified form of the order-two Green ansatz \cite{25}. We suspected that a field theory generalization of this model having an infinite number of oscillators would not have local observables and set about trying to prove this. To our surprise, we found that we could construct local observables and gave trilinear relations that guarantee the locality of the current \cite{23}. We also checked the positivity of the norms with states of three or fewer particles. At this stage, we were carried away with enthusiasm, named these particles “parons” since their algebra is a deformation of the parastatistics algebra, and thought we had found a local theory with small violation of the exclusion principle. We did not know that Govorkov \cite{26} had shown in generality that any deformation of the Green commutation relations necessarily has states with negative squared norms in the Fock-like representation. For our model the first such negative-probability state occurs for four particles in the representation of \(S_4\) with three boxes in the first row and one in the second. We were able to understand Govorkov’s result qualitatively as follows \cite{27}: Since parastatistics of order \(p\) is related by a Klein transformation to a model with exact \(SO(2)\) or \(SU(2)\) internal symmetry, a deformation of parastatistics that interpolates between Fermi and parafermi statistics of order two would be equivalent to interpolating between the trivial group whose only element is the identity and a theory with \(SO(2)\) or \(SU(2)\) internal symmetry. This is impossible, since there is no such interpolating group.

### 3.4 Apparent Violations of Statistics Due to Compositeness

Before getting to “quons,” the final type of statistics I will discuss, I want to interpolate some comments about apparent violations of statistics due to compositeness. Consider two \(^3He\) nuclei, each of which is a fermion. If these two nuclei are brought in close proximity, the exclusion principle will force each of them into excited states, plausibly with small amplitudes for the excited states. Let the creation operator for the nucleus at location \(A\) be

\[
b_A^\dagger = \sqrt{1 - \lambda_A^2} b_0^\dagger + \lambda_A b_1^\dagger + \cdots, |\lambda_A| << 1, \tag{9}\]

and the creation operator for the nucleus at location \(B\) be

\[
b_B^\dagger = \sqrt{1 - \lambda_B^2} b_0^\dagger + \lambda_B b_1^\dagger + \cdots, |\lambda_B| << 1. \tag{10}\]
Since these nuclei are fermions, the creation operators obey fermi statistics,

\[ [b_i^\dagger, b_j^\dagger]_+ = 0 \quad (11) \]

Then,

\[ b_i^\dagger b_j^\dagger |0\rangle = [\sqrt{1 - \lambda_i^2 \lambda_j^2} - \lambda_i \sqrt{1 - \lambda_j^2}] b_i^\dagger b_j^\dagger |0\rangle, \quad (12) \]

\[ \|b_i^\dagger b_j^\dagger |0\rangle\|^2 \approx (\lambda_i - \lambda_j)^2 << 1, \quad (13) \]

so with small probability, the two could even occupy the same location, because each could be excited into higher states with different amplitudes. This is not an intrinsic violation of the exclusion principle but rather only an apparent violation due to compositeness.

4 QUONS

4.1 Quon Algebra and Fock Representation

Now I come to my last topic, quons \[3\]. The quon algebra is

\[ a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij}. \quad (14) \]

For the Fock-like representation I impose the vacuum condition

\[ a_i |0\rangle = 0. \quad (15) \]

These two conditions determine all vacuum matrix elements of polynomials in the creation and annihilation operators. In the case of free quons all non-vanishing vacuum matrix elements must have the same number of annihilators and creators. For such a matrix element with all annihilators to the left and creators to the right, the matrix element is a sum of products of “contractions” of the form \[ \langle 0 | a a^\dagger | 0 \rangle \] just as in the case of bosons and fermions. The only difference is that the terms are multiplied by integer powers of \( q \). The power can be given as a graphical rule: Put \( \circ \)'s for each annihilator and \( \times \)'s for each creator in the order in which they occur in the matrix element on the \( x \)-axis. Draw lines above the \( x \)-axis connecting the pairs that are contracted. The minimum number of times these lines cross is the power of \( q \) for that term in the matrix element. Thus a modified Wick’s theorem holds for quon operators.
The physical significance of $q$ for small violations of Fermi statistics is that $q = 2v_F - 1$, where the parameter $v_F$ appears in Eq. (I). For small violations of Bose statistics, the two-particle density matrix is

$$\rho_2 = (1 - v_B)\rho_s + v_B\rho_a,$$

where $\rho_{s(a)}$ is the symmetric (antisymmetric) two-boson density matrix. Then $q = 1 - 2v_B$.

For $q$ in the open interval $(-1, 1)$ all representations of the symmetric group occur. As $q \to 1$ the symmetric representations are more heavily weighted and at $q = 1$ only the totally symmetric representation remains; correspondingly, as $q \to -1$ the antisymmetric representations are more heavily weighted and at $q = -1$ only the totally antisymmetric representation remains. Thus for a general $n$-quon state there are $n!$ linearly independent states for $-1 < q < 1$, but there is only one state for $q = \pm 1$. I emphasize something that many people find very strange: there is no commutation relation between two creation or between two annihilation operators, except for $q = \pm 1$, which, of course, correspond to Bose and Fermi statistics. Indeed, the fact that the general $n$-particle state with different quantum numbers for all the particles has $n!$ linearly independent states proves that there is no such commutation relation between any number of creation (or annihilation) operators. An even stronger statement holds: There is no two-sided ideal containing a term with only creation operators. Note that here quons differ from the “quantum plane” in which

$$xy = qyx$$

holds.

Quons are an operator realization of the “infinite statistics” that were found as a possible statistics by Doplicher, Haag and Roberts [22] in their general classification of particle statistics. The simplest case, $q = 0$ [28], suggested to me by Hegstrom [29], was discussed earlier in the context of operator algebras by Cuntz [30]. It seems likely that the Fock-like representations of quons for $|q| < 1$ are homotopic to each other and, in particular, to the $q = 0$ case, which is particularly simple. Thus it is convenient, as I will now do, to illustrate qualitative properties of quons for this simple case. All bilinear observables can be constructed from the
number operator, \( n_k \equiv n_{kk} \), or the transition operator, \( n_{kl} \), that obey
\[
[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger, \quad [n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger.
\] (18)

Although the formulas for \( n_k \) and \( n_{kl} \) in the general case are complicated, the corresponding formulas for \( q = 0 \) are simple [28]. Once Eq.(18) holds, the Hamiltonian and other observables can be constructed in the usual way; for example for free particles
\[
H = \sum_k \epsilon_k n_k, \quad \text{etc.}
\] (19)

The obvious thing is to try
\[
n_k = a_k^\dagger a_k.
\] (20)

Then
\[
[n_k, a_l^\dagger]_- = \delta_{kl} a_k^\dagger - a_l^\dagger a_k^\dagger a_k.
\] (21)

The first term in Eq.(21) is \( \delta_{kl} a_k^\dagger \) as desired; however the second term is extra and must be canceled. This can be done by adding the term \( \sum_l a_l^\dagger a_k^\dagger a_k a_l \) to the term in Eq.(20). This cancels the extra term, but adds a new extra term, that must be canceled by another term. This procedure yields an infinite series for the number operator and for the transition operator,
\[
n_{kl} = a_k^\dagger a_l + \sum_l a_l^\dagger a_k^\dagger a_l a_l + \sum_{l_1, l_2} a_{l_2}^\dagger a_{l_1}^\dagger a_k^\dagger a_l a_{l_1} a_{l_2} + \ldots
\] (22)

As in the Bose case, this infinite series for the transition or number operator defines an unbounded operator whose domain includes states made by polynomials in the creation operators acting on the vacuum. (As far as I know, this is the first case in which the number operator, Hamiltonian, etc. for a free field are of infinite degree. Presumably this is due to the fact that quons are a deformation of an algebra and are related to quantum groups.) For nonrelativistic theories, the \( x \)-space form of the transition operator is [32]
\[
\rho_1(x; y) = \psi^\dagger(x)\psi(y) + \int d^3 z \psi^\dagger(z)\psi^\dagger(x)\psi(y)\psi(z)
\]
\[
+ \int d^3 z_1 d^3 z_2 \psi(z_2)\psi^\dagger(z_1)\psi^\dagger(x)\psi(y)\psi(z_1)\psi(z_2) + \cdots,
\] (23)
which obeys the nonrelativistic locality requirement

\[ \left\{ \rho_1(x; y), \psi^\dagger(w) \right\}_- = \delta(y - w)\psi^\dagger(x), \quad \text{and} \quad \rho(x; y)|0\rangle = 0. \]  

(24)

The apparent nonlocality of this formula associated with the space integrals has no physical significance. To support this last statement, consider

\[ [Q j_\mu(x), Q j_\nu(y)]_- = 0, \quad x \sim y, \]  

(25)

where \( Q = \int d^3 x j^0(x) \). Equation (27) seems to have nonlocality because of the space integral in the \( Q \) factors; however, if

\[ [j_\mu(x), j_\nu(y)]_- = 0, \quad x \sim y, \]  

(26)

then Eq.(24) holds, despite the apparent nonlocality. What is relevant is the commutation relation, not the representation in terms of a space integral. (The apparent nonlocality of quantum electrodynamics in the Coulomb gauge is another such example.)

In a similar way,

\[ \left[ \rho_2(x, y; y', x'), \psi^\dagger(z) \right]_- = \delta(x' - z)\psi^\dagger(x)\rho_1(y, y') + \delta(y' - z)\psi^\dagger(y)\rho_1(x, x'). \]  

(27)

Then the Hamiltonian of a nonrelativistic theory with two-body interactions has the form

\[ H = (2m)^{-1} \int d^3 x \nabla_x \cdot \nabla_{x'} \rho_1(x, x')|_{x = x'} + \frac{1}{2} \int d^3 x d^3 y V(|x - y|)\rho_2(x, y; y, x). \]  

(28)

\[ \left[ H, \psi^\dagger(z_1) \cdots \psi^\dagger(z_n) \right]_- = \left[ -(2m)^{-1} \sum_{j=1}^n \nabla_{z_j}^2 + \sum_{i<j} V(|z_i - z_j|) \right] \psi^\dagger(z_1) \cdots \psi^\dagger(z_n) \]

\[ + \sum_{j=1}^n \int d^3 x V(|x - z_j|)\psi^\dagger(z_1) \cdots \psi^\dagger(z_n)\rho_1(x, x'). \]  

(29)

Since the last term on the right-hand-side of Eq.(29) vanishes when the equation is applied to the vacuum, this equation shows that the usual Schrödinger equation holds for the \( n \)-particle system. Thus the usual quantum mechanics is valid, with the sole exception that any permutation symmetry is allowed for the many-particle
system. This construction justifies calculating the energy levels of (anomalous) atoms with electrons in states that violate the exclusion principle using the normal Hamiltonian, but allowing anomalous permutation symmetry for the electrons [14]

4.2 Positivity of Squares of Norms

I have not yet addressed the question of positivity of the squares of norms that caused grief in the paron model. Several authors have given proofs of positivity [33, 34, 35, 36]. The proof of Zagier provides an explicit formula for the determinant of the $n! \times n!$ matrix of scalar products among the states of $n$ particles in different quantum states. Since this determinant is one for $q = 0$, the norms will be positive unless the determinant has zeros on the real axis. Zagier’s formula

\[ det \ M_n(q) = \prod_{k=1}^{n-1} (1 - q^{k(k+1)})^{(n-k)n/k(k+1)}, \]

has zeros only on the unit circle, so the desired positivity follows. Although quons satisfy the requirements of nonrelativistic locality, the quon field does not obey the relativistic requirement, namely spacelike commutativity of observables. Since quons interpolate smoothly between fermions, which must have odd half-integer spin, and bosons, which must have integer spin, the spin-statistics theorem, which can be proved, at least for free fields, from locality would be violated if locality were to hold for quon fields. It is amusing that, nonetheless, the free quon field obeys the TCP theorem and Wick’s theorem holds for quon fields [3].

4.3 Speicher’s ansatz

Speicher [35] has given an ansatz for the Fock-like representation of quons analogous to Green’s ansatz for parastatistics. Speicher represents the quon annihilation operator as

\[ a_k = \lim_{N \to \infty} N^{-1/2} \sum_{\alpha=1}^{N} b_{k}^{(\alpha)}, \]

where the $b_{k}^{(\alpha)}$ are Bose oscillators for each $\alpha$, but with relative commutation relations given by

\[ b_{k}^{(\alpha)} b_{l}^{(\beta)\dagger} = s^{(\alpha,\beta)} b_{l}^{(\beta)\dagger} b_{k}^{(\alpha)}, \alpha \neq \beta, \] where $s^{(\alpha,\beta)} = \pm 1$. \[ (32) \]
Equation (31) is taken as the weak limit, $N \to \infty$, in the vacuum expectation state of the Fock space representation of the $b_k^{(\alpha)}$. In this respect, Speicher’s ansatz differs from Green’s, which is an operator identity. Further to get the Fock-like representation of the quon algebra, Speicher chooses a probabilistic condition for the signs $s^{(\alpha,\beta)}$,

$$\text{prob}(s^{(\alpha,\beta)} = 1) = (1 + q)/2, \quad (33)$$
$$\text{prob}(s^{(\alpha,\beta)} = -1) = (1 - q)/2. \quad (34)$$

Since a sum of Bose operators acting on a Fock vacuum always gives a positive-definite norm, the positivity property is obvious with Speicher’s construction.

Speicher’s ansatz leads to the conjecture that there is an infinite-valued hidden degree of freedom underlying $q$-deformations analogous to the hidden degree of freedom underlying parastatistics.

If one asks “How well do we know that a given particle obeys Bose or Fermi statistics?,” we need a quantitative way to answer the question. That requires a formulation in which either Bose or Fermi statistics is violated by a small amount. As stated earlier, we cannot just add to the Hamiltonian a small term that violates Bose or Fermi statistics; such a term would not be invariant under permutations of the identical particles and thus would clash with the particles being identical. As mentioned above parastatistics, which does violate Bose or Fermi statistics, gives gross violations. The only way presently available to allow small violations of statistics is the quon theory just described.

Unfortunately, the quon theory is not completely satisfactory. The observables in quon theory do not commute at spacelike separation. If they did, particle statistics could change continuously from Bose to Fermi without changing the spin. Since spacelike commutativity of observables leads to the spin-statistics theorem, this would be a direct contradiction. Kinematic Lorentz invariance can be maintained, but without spacelike commutativity or anticommutativity of the fields the theory may not be consistent.

For nonrelativistic theories, however, quons are consistent. The nonrelativistic version of locality is

$$[\rho(x), \psi(y)] = -\delta(x - y)\psi(y) \quad (35)$$
for an observable $\rho(x)$ and a field $\psi(y)$ and this does hold for quon theories. It is the antiparticles that prevent locality in relativistic quon theories.

5. CONSERVATION OF STATISTICS

5.1 Conservation of Statistics for Bosons and Fermions

The first conservation of statistics theorem states that terms in the Hamiltonian density must have an even number of Fermi fields and that composites of fermions and bosons are bosons, unless they contain an odd number of fermions, in which case they are fermions \[37, 38\].

5.2 Conservation of Statistics for Parabosons and Parafermions

The extension to parabosons and parafermions is more complicated \[21\]; however, the main constraint is that for each order $p$ at least two para particles must enter into every reaction.

Reference \[39\] argues that the condition that the energy of widely separated subsystems be additive requires that all terms in the Hamiltonian be “effective Bose operators” in that sense that

$$[H(x), \phi(y)]_- \to 0, |x - y| \to \infty. \tag{36}$$

For example, $H$ should not have a term such as $\phi(x)\psi(x)$, where $\phi$ is Bose and $\psi$ is Fermi, because then the contributions to the energy of widely separated subsystems would alternate in sign. Such terms are also prohibited by rotational symmetry. This discussion was given in the context of external sources.

It is well known that external fermionic sources must be multiplied by a Grassmann number in order to be a valid term in a Hamiltonian. This is necessary, because additivity of the energy of widely separated systems requires that all terms in the Hamiltonian must be effective Bose operators. I constructed the quon analog of Grassmann numbers \[39\] in order to allow external quon sources. Because this issue was overlooked, the bound on violations of Bose statistics for photons claimed in \[40\] is invalid.
For a fully quantized field theory, one can replace Eq. (36) by the asymptotic causality condition, asymptotic local commutativity,

$$\left[ H(x), H(y) \right]_- = 0, \ |x - y| \to \infty$$  \hspace{1cm} (37)

or by the stronger causality condition, local commutativity,

$$\left[ H(x), H(y) \right]_- = 0, \ x \neq y.$$  \hspace{1cm} (38)

Studying this condition for quons in electrodynamics is complicated, since the terms in the interaction density will be cubic. It is simpler to use the description of the electron current or transition operator as an external source represented by a quonic Grassmann number.

5.3 Conservation of Statistics for Quons

Here we give a heuristic argument for conservation of statistics for quons based on a simpler requirement in the context of quonic Grassmann external sources [4]. The commutation relation of the quonic photon operator is

$$a(k)a^\dagger(l) - q_\gamma a^\dagger(l)a(k) = \delta(k - l),$$  \hspace{1cm} (39)

where $q_\gamma$ is the $q$-parameter for the photon quon field. We call the quonic Grassmann numbers for the electron transitions to which the photon quon operators couple $c(k)$. The Grassmann numbers that serve as the external source for coupling to the quon field for the photon must obey

$$c(k)c(l)^* - q_\gamma c(l)^*c(k) = 0,$$  \hspace{1cm} (40)

and the relative commutation relations must be

$$a(k)c(l)^* - q_\gamma c(l)^*a(k) = 0,$$  \hspace{1cm} (41)

etc. Since the electron current for emission or absorption of a photon with transition of the electron from one atomic state to another is bilinear in the creation and annihilation operators for the electron, a more detailed description of the photon emission would treat the photon as coupled to the electron current, rather than to an external source. We impose the requirement that the leading terms in the
commutation relation for the quonic Grassmann numbers of the source that couples to the photon should be mimicked by terms bilinear in the electron operators. The electron operators obey the relation

\[ b(k)b^\dagger(l) - q_e b^\dagger(l)b(k) = \delta(k - l), \quad (42) \]

where \( q_e \) is the \( q \)-parameter for the electron quon field.

To find the connection between \( q_e \) and \( q_\gamma \) we make the following associations,

\[ c(k) \Rightarrow b^\dagger(p)b(k + p), \quad c^*(l) \Rightarrow b^\dagger(l + r)b(r) \quad (43) \]

We now replace the \( c \)'s in Eq.(40) with the products of operators given in Eq.(43) and obtain

\[ \left[ b^\dagger(p)b(k + p)\right]\left[ b^\dagger(l + r)b(r)\right] - q_\gamma \left[ b^\dagger(l + r)b(r)\right]\left[ b^\dagger(p)b(k + p)\right] = 0. \quad (44) \]

This means that the source \( c(k) \) is replaced by a product of \( b \)'s that destroys net momentum \( k \); the source \( c^*(l) \) is replaced by a product of \( b \)'s that creates net momentum \( l \). We want to rearrange the operators in the first term of Eq.(44) to match the second term, because this corresponds to the standard normal ordering for the transition operators. For the products \( bb^\dagger \) we use Eq.(42). For the products \( bb \), as mentioned above, there is no operator relation; however on states in the Fock-like representation there is an approximate relation,

\[ b^\dagger(p)b(k + p) = q_e b(r)b(k + p) + \text{terms of order } 1 - q_e^2. \quad (45) \]

In other words, in the limit \( q_e \to -1 \), we retrieve the usual anticommutators for the electron operators. (The analogous relation for an operator that is approximately bosonic would be that the operators commute in the limit \( q_{\text{bosonic}} \to 1 \).) We also use the adjoint relation

\[ b^\dagger(p)b^\dagger(l + r) = q_e b^\dagger(l + r)b^\dagger(p) + \text{terms of order } 1 - q_e^2 \quad (46) \]

and, finally,

\[ q_e b^\dagger(p)b(r) = b(r)b^\dagger(p) - \delta(r - p). \quad (47) \]

We require only that the quartic terms that correspond to the quonic Grassmann relation Eq.(41) cancel, so we drop terms in which either \( k + p = l + r \) or \( r = p \). We also drop terms of order \( 1 - q_e^2 \). In this approximation, we find that Eq.(44) becomes

\[ (q_e^2 - q_\gamma)[b^\dagger(l + r)b(r)][b^\dagger(p)b(k + p)] \approx 0, \quad (48) \]
and conclude that
\[ q_e^2 \approx q_\gamma. \]  
(49)

This relates the bound on violations of Fermi statistics for electrons to the bound on violations of Bose statistics for photons and allows the extremely precise bound on possible violations of Fermi statistics for electrons to be carried over to photons. Eq. (49) is the quon analog of the conservation of statistics relation that the square of the phase for transposition of a pair of fermions equals the phase for transposition of a pair of bosons.

Arguments analogous to those just given, based on the source-quonic photon relation, Eq. (41), lead to
\[ q_e^2 \approx q_\gamma, \]  
(50)

where \( q_e \gamma \) occurs in the relative commutation relation
\[ a(k)b^\dagger(l) = q_e \gamma b^\dagger(l)a(k). \]  
(51)

Since the normal commutation relation between Bose and Fermi fields is for them to commute \([11]\), this shows that \( q_e \gamma \) is close to one.

6. HIGH-PRECISION BOUNDS

Since the Ramberg-Snow bound on Fermi statistics for electrons is
\[ v_e \leq 1.7 \times 10^{-26} \iff q_e \leq -1 + 3.4 \times 10^{-26}, \]  
(52)

the bound on Bose statistics for photons is
\[ q_\gamma \geq 1 - 6.8 \times 10^{-26} \iff v_\gamma \leq 3.4 \times 10^{-26}. \]  
(53)

This bound for photons is much stronger than could be gotten by a direct experiment. Nonetheless D. DeMille and N. Derr are performing an experiment that promises to give the best direct bound on Bose statistics for photons [18]. It is essential to test every basic property in as direct a way as possible. Thus experiments that yield direct bounds on photon statistics, such as the one being carried out by DeMille and Derr, are important.
Teplitz, Mohapatra and Baron have suggested a method to set a very low limit on violation of the Pauli exclusion principle for neutrons \[12\].

The argument just given that the $q_e$ value for electrons implies $q_\gamma \approx q_e^2$ for photons can be run in the opposite direction to find $q_\phi^2 \approx q_\gamma$ for each charged field $\phi$ that couples bilinearly to photons. Isospin and other symmetry arguments then imply that almost all particles obey Bose or Fermi statistics to a precision comparable to the precision with which electrons obey Fermi statistics.

7 CONCLUSION

In concluding, we note that further work should be carried out to justify the approximations made in deriving Eq.(49) and also to derive the relations among the $q$-parameters that follow from couplings that do not have the form $\bar{f}fb$. We plan to return to this topic in a later paper. After the Orbs, Hilborn and I derived a generalization of the Wigner–Ehrenfest-Oppenheimer rule of the statistics of bound states in terms of the quon statistics of their constituents, $q_{\text{composite}} = q_{\text{constituent}}^n$, where $n$ is the number of constituents in the bound state \[5\].

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