SUP × INF INEQUALITIES FOR THE SCALAR CURVATURE EQUATION IN DIMENSIONS 4 AND 5.

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ABSTRACT. We consider the following problem on bounded open set $\Omega$ of $\mathbb{R}^n$:

$$\begin{cases}
-\Delta u = Vu^{\frac{n+2}{n-2}} \quad &\text{in } \Omega \subset \mathbb{R}^n, \quad n = 4, 5, \\
u > 0 \quad &\text{in } \Omega.
\end{cases}$$

We assume that:

- $V \in C^{1,\beta}(\Omega), \quad 0 < \beta \leq 1$
- $0 < a \leq V \leq b < +\infty$,
- $|\nabla V| \leq A$ in $\Omega$,
- $|\nabla^{1+\beta} V| \leq B$ in $\Omega$.

then, we have a sup × inf inequality for the solutions of the previous equation, namely:

$$(\sup_K u)^{\frac{n}{n-2}} \times \inf_{\Omega} u \leq c = c(a, b, A, B, \beta, K, \Omega), \quad \text{for } n = 4,$$

and,

$$(\sup_K u)^{1/3} \times \inf_{\Omega} u \leq c = c(a, b, A, B, K, \Omega), \quad \text{for } n = 5, \quad \text{and } \beta = 1.$$

1. INTRODUCTION AND MAIN RESULT

We work on $\Omega \subset \subset \mathbb{R}^4$ and we consider the following equation:

$$\begin{cases}
-\Delta u = Vu^{\frac{n+2}{n-2}} \quad &\text{in } \Omega \subset \mathbb{R}^n, \quad n = 4, 5, \\
u > 0 \quad &\text{in } \Omega.
\end{cases} \quad (E)$$

with,

$$
\begin{cases}
V \in C^{1,\beta}(\Omega), \\
0 < a \leq V \leq b < +\infty \text{ in } \Omega, \\
|\nabla V| \leq A \text{ in } \Omega, \\
|\nabla^{1+\beta} V| \leq B \text{ in } \Omega.
\end{cases} \quad (C_\beta)
$$

Without loss of generality, we suppose $\Omega = B_1(0)$ the unit ball of $\mathbb{R}^n$.

The corresponding equation in two dimensions on open set $\Omega$ of $\mathbb{R}^2$ is:

$$-\Delta u = V(x)e^u, \quad (E')$$

The equation $(E')$ was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_{\Omega} V, ||V||_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [19], we have:

$$C \sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, ||V||_{L^\infty(\Omega)}, K, \Omega),$$

and,
where $K$ is a compact subset of $\Omega$, $C$ is a positive constant which depends on $\inf_{\Omega} V$, and, $\alpha \in (0, 1]$. For $n \geq 3$ we have the following general equation on a riemannian manifold:

$$-\Delta u + hu = V(x)u^{n+2}, \quad u > 0. \quad (E_n)$$

Where $h, V$ are two continuous functions. In the case $c_n h = R_g$ the scalar curvature, we call $V$ the prescribed scalar curvature. Here $c_n$ is a universal constant.

The equation $(E_n)$ was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = S^n$ see for example, [2-4], [11], [15]. In this case we have a sup $\times$ inf inequality.

In the case $V \equiv 1$ and $M$ compact, the equation $(E_n)$ is Yamabe equation. T.Aubin and R.Schoen proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary.

When $M$ is a compact Riemannian manifold, there exist some compactness result for equation $(E_n)$ see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose $M$ not diiffeormorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose $M$ Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product sup $\times$ inf is bounded. On other handm see [3], [5] and [6] for other Harnack type inequalities, and, see [3] and [7] about some caracterisation of the solutions of this equation $(E_n)$ in this case ($V \equiv 1$).

Here we extend a result of [11] on an open set of $\mathbb{R}^n, n = 4, 5$. In fact we consider the prescribed scalar curvature equation on an open set of $\mathbb{R}^n, n = 4, 5$, and, we prove a sup $\times$ inf inequality on compact set of the domain when the derivative of the prescribed scalar curvature is $\beta$-holderian, $\beta > 0$.

Our proof is an extension of Chen-Lin result in dimension 4 and 5, see [11], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method.

We have the following result in dimension 4, which is the consequence of the work of Chen-Lin.

**Theorem A.** For all $a, b, m, A, B > 0$, and for all compact $K$ of $\Omega$, there exists a positive constant $c = c(a, b, A, B, K, \Omega)$ such that:

$$\sup_K u \times \inf_{\Omega} u \leq c,$$

where $u$ is solution of $(E)$ with $V, C^2$ satisfying $(C_\beta)$ for $\beta = 1$.

Here, we give an inequality of type sup $\times$ inf for the equation $(E)$ in dimension 4 and with general conditions on the prescribed scalar curvature, exactly we take a $C^{1, \beta}$ condition. In fact we extend the result of Chen-Lin in dimension 4.

Here we prove:

**Theorem 1.1.** For all $a, b, A, B > 0, 1 \geq \beta > 0$, and for all compact $K$ of $\Omega$, there exists a positive constant $c = c(a, b, A, B, \beta, K, \Omega)$ such that:

$$(\sup_K u)^\beta \times \inf_{\Omega} u \leq c,$$

where $u$ is solution of $(E)$ with $V$ satisfying $(C_\beta)$. 

We have the following result in dimension 5, which is the consequence of the work of Chen-Lin.

**Theorem B.** For all \(a, b, m, A, B > 0\), and for all compact \(K\) of \(\Omega\), there exists a positive constant \(c = c(a, b, m, A, B, K, \Omega)\) such that:

\[
\sup_K u \leq c, \quad \text{if} \quad \inf_{\Omega} u \geq m,
\]

where \(u\) is solution of \((E)\) with \(V\) satisfying \((C_\beta) = (C_1)\) for \(\beta = 1\).

Here, we give an inequality of type \(\sup \times \inf\) for the equation \((E)\) in dimension 5 and with general conditions on the prescribed scalar curvature, exactly we take a \(C^2\) condition \((\beta = 1\) in \((C_\beta))\). In fact we extend the result of Chen-Lin in dimension 5.

Here we prove:

**Theorem 1.2.** For all \(a, b, A, B > 0\), and for all compact \(K\) of \(\Omega\), there exists a positive constant \(c = c(a, b, A, B, K, \Omega)\) such that:

\[
(\sup_K u)^{1/3} \times \inf_{\Omega} u \leq c,
\]

where \(u\) is solution of \((E)\) with \(V\) satisfying \((C_\beta)\) for \(\beta = 1\).

2. The method of moving-plane.

In this section we will formulate a modified version of the method of moving-plane for use later. Let \(\Omega\) an open set and \(\Omega^c\) the complement of \(\Omega\). We consider a solution \(u\) of the following equation:

\[
\begin{cases}
\Delta u + f(x, u) = 0, \\
\quad u > 0,
\end{cases}
\]

where \(f(x, u)\) is nonegative, Holder continuous in \(x\), \(C^1\) in \(u\), and defined on \(\bar{\Omega} \times (0, +\infty)\).

Let \(e\) be a unit vector in \(\mathbb{R}^n\). For \(\lambda < 0\), we let

\(\lambda_1 \equiv \sup \{\lambda \leq 0, \Omega^c \subset \Sigma_\lambda\}\),

\(
\Sigma'_\lambda = \Sigma_\lambda - \Omega^c\)

for \(\lambda \leq \lambda_1\), and \(\Sigma'_\lambda\) the closure of \(\Sigma'_\lambda\). Let \(u_\lambda(x) = u(x^\lambda)\) and \(w_\lambda(x) = u(x) - u_\lambda(x)\) for \(x \in \Sigma'_\lambda\). Then we have, for any arbitrary function \(b_\lambda(x)\),

\[
\Delta w_\lambda(x) + b_\lambda(x)w_\lambda(x) = Q(x, b_\lambda(x)),
\]

where,

\[
Q(x, b_\lambda(x)) = f(x^\lambda, u^\lambda) - f(x, u) + b_\lambda(x)w_\lambda(x).
\]

The hypothesis \((*)\) is said to be satisfied if there are two families of functions \(b_\lambda(x)\) and \(h^\lambda(x)\) defined in \(\Sigma'_\lambda\), for \(\lambda \in (-\infty, \lambda_1]\) such that, the following assertions holds:

\[
0 \leq b_\lambda(x) \leq c(x)|x|^{-2},
\]

where \(c(x)\) is independant of \(\lambda\) and tends to zero as \(|x|\) tends to \(+\infty\),

\[
h^\lambda(x) \in C^1(\Sigma_\lambda \cap \Omega),
\]

and satisfies:

\[
\begin{cases}
\Delta h^\lambda(x) \geq Q(x, b_\lambda(x)) \text{ in } \Sigma_\lambda \cap \Omega \\
h^\lambda(x) > 0 \text{ in } \Sigma_\lambda \cap \Omega
\end{cases}
\]

in the distributional sense and,
\[ h^\lambda(x) = 0 \text{ on } T_{\lambda} \text{ and } h^\lambda(x) = O(|x|^{-t_1}) , \]
as \(|x| \to +\infty\) for some constant \(t_1 > 0\),

\[ h^\lambda(x) + \epsilon < w^\lambda(x), \]
in a neighborhood of \(\partial\Omega\), where \(\epsilon\) is a constant positive independent of \(x\).

\[ \begin{aligned}
&h^\lambda(x) \text{ and } \nabla x h^\lambda \text{ are continuous with respect to both variables } \\
&x \text{ and } \lambda, \text{ and for any compact set of } \Omega, w^\lambda(x) > h^\lambda(x)
\end{aligned} \]
holds when \(-\lambda\) is sufficiently large.

We have the following lemma:

**Lemma 2.1.** Let \(u\) be a solution of \((E'')\). Suppose that \(u(x) \geq C > 0\) in a neighborhood of \(\partial\Omega\) and \(u(x) = O(|x|^{-t_2})\) at \(+\infty\) for some positive \(t_2\). Assume there exist \(b^\lambda(x)\) and \(h^\lambda(x)\) such that the hypothesis \((\ast)\) is satisfied for \(\lambda \leq \lambda_1\). Then \(w^\lambda(x) > 0\) in \(\Sigma_{\lambda}'\), and \(\langle \nabla u, e \rangle > 0\) on \(T_{\lambda}\) for \(\lambda \in (-\infty, \lambda_1)\).

For the proof see Chen and Lin, [10].

**Remark 2.2.** If we know that \(w^\lambda - h^\lambda > 0\) for some \(\lambda = \lambda_0 < \lambda_1\) and \(b^\lambda\) and \(h^\lambda\) satisfy the hypothesis \((\ast)\) for \(\lambda_0 \leq \lambda \leq \lambda_1\), then the conclusion of the lemma 2.1 holds.

### 3. PROOF OF THE RESULT:

**Proof of the theorem 1, \(n = 4\):**

To prove the theorem, we argue by contradiction and we assume that the \((\sup)^{\beta} \times \inf\) tends to infinity.

**Step 1: blow-up analysis**

We want to prove that:

\[ \tilde{R}_i^2(\sup_{B_{\tilde{R}_i}(0)} u)^{\beta} \times \inf_{B_{\tilde{R}_i}(0)} u \leq c = c(a, b, A, B, \beta), \]

If it is not the case, we have:

\[ \tilde{R}_i^2(\sup_{B_{a_i}(0)} u_i)^{\beta} \times \inf_{B_{\tilde{R}_i}(0)} u_i = i^6 \to +\infty, \]

For positive solutions \(u_i > 0\) of the equation \((E)\) and \(\tilde{R}_i \to 0\). Thus,

\[ \frac{1}{i} \tilde{R}_i(\sup_{B_{a_i}(0)} u_i)^{1+\beta/2} \to +\infty, \]

and,

\[ \frac{1}{i} \tilde{R}_i(\sup_{B_{a_i}(0)} u_i)^{1+\beta/2} \to +\infty, \]

Let \(a_i\) such that:

\[ u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i, \]

We set,

\[ s_t(x) = (\tilde{R}_i - |x - a_i|)^{2/(1+\beta)} u_i(x), \]

we have,

\[ s_t(\tilde{a}_i) = \max_{B_{\tilde{R}_i}(a_i)} s_i \geq s_t(a_i) = \tilde{R}_i^{2/(1+\beta)} \sup_{B_{\tilde{R}_i}(0)} u_i \to +\infty, \]
we set, 
\[ R_i = \frac{1}{2}(\tilde{R}_i - |\bar{x}_i - a_i|) \]
We have, for \(|x - \bar{x}_i| \leq \frac{R_i}{2}\),
\[ \tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| - |x - a_i| \geq 2R_i - R_i = R_i \]
Thus,
\[ \frac{u_i(x)}{u_i(\bar{x}_i)} \leq \beta_i \leq 2^{2/(1+\beta)} \]
with \(\beta_i \to 1\).
We set,
\[ M_i = u_i(\bar{x}_i), \quad v^*_i(y) = \frac{u_i(\bar{x}_i + M_i^{-1}y)}{u_i(\bar{x}_i)} \]
with \(|y| \leq 1\).
And,
\[ \frac{1}{i^2} R_i^2 M_i^\beta \times \inf_{B_{\tilde{R}_i}(0)} u_i \to +\infty. \]
By the elliptic estimates, \(v^*_i\) converge on each compact set of \(\mathbb{R}^4\) to a function \(U^*_0 > 0\) solution of:
\[ \begin{cases} -\Delta U^*_0 = V(0)U^*_0^3 \quad \text{in} \quad \mathbb{R}^4, \\ U^*_0(0) = 1 = \max_{\mathbb{R}^4} U^*_0. \end{cases} \]
For simplicity, we assume that \(0 < V(0) = n(n-2) = 8\). By a result of Caffarelli-Gidas-Spruck, see [10], we have:
\[ U^*_0(y) = (1 + |y|^2)^{-1}. \]
We set,
\[ v_i(y) = v^*_i(y + \epsilon), \]
where \(v^*_i\) is the blow-up function. Then, \(v_i\) has a local maximum near \(-\epsilon\).\[ U_0(y) = U^*_0(y + \epsilon). \]
We want to prove that:
\[ \min_{\{0 \leq |y| \leq r\}} v^*_i \leq (1 + \epsilon)U^*_0(r). \]
for \(0 \leq r \leq L_i\), with \(L_i = \frac{1}{2i} R_i M_i^{(1+\beta)/2} = 2L_i\).
We assume that it is not true, then, there is a sequence of number \(r_i \in (0, L_i)\) and \(\epsilon > 0\), such that:
\[ \min_{\{0 \leq |y| \leq r_i\}} v^*_i \geq (1 + \epsilon)U^*_0(r_i). \]
We have:
\[ r_i \to +\infty. \]
Thus, we have for \(r_i \in (0, L_i)\):
\[ \min_{\{0 \leq |y| \leq r_i\}} v_i \geq (1 + \epsilon)U_0(r_i). \]
Also, we can find a sequence of number \(l_i \to +\infty\) such that:
Thus, 
\[ \min_{\{0 \leq |y| \leq l_i\}} v_i \geq (1 - \epsilon/2)U_0(l_i). \]

**Step 2:** The Kelvin transform and the Moving-plane method

**Step 2.1:** a linear equation perturbed by a term, and, the auxiliary function

**Step 2.1.1:**

\[ D_i = |\nabla V_i(x_i)| \rightarrow 0. \]

We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider \( r_i \in (0, L_i) \) where \( L_i \) is the number of the blow-up analysis.

\[ L_i = \frac{1}{2i} R_i M_i^{(1+\beta)/2}. \]

We use the assumption that the sup times inf is not bounded to prove \( w_\lambda > h_\lambda \) in \( \Sigma_\lambda = \{ y, y_1 > \lambda \} \) and on the boundary.

The function \( v_i \) has a local maximum near \(-e\) and converge to \( U_0(y) = U_0^*(y + e) \) on each compact set of \( \mathbb{R}^5 \). \( U_0 \) has a maximum at \(-e\).

We argue by contradiction and we suppose that:

\[ D_i = |\nabla V_i(x_i)| \not\rightarrow 0. \]

Then, without loss of generality we can assume that:

\[ \nabla V_i(x_i) \rightarrow e = (1, 0, \ldots, 0). \]

Where \( x_i \) is:

\[ x_i = \bar{x}_i + M_i^{-1}e, \]

with \( \bar{x}_i \) is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

\[ I_\delta(y) = \frac{|y|}{|y| - \delta e} - \frac{\delta e}{|y| - \delta e}, \]

\[ v_i^\delta(y) = \frac{v_i(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}}, \]

and,

\[ V_\delta(y) = V_i(x_i + M_i^{-1}I_\delta(y)). \]

\[ U_\delta(y) = \frac{U_0(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}}. \]

Then, \( U_\delta \) has a local maximum near \( e_\delta \rightarrow -e \) when \( \delta \rightarrow 0 \). The function \( v_i^\delta \) has a local maximum near \(-e\).

We want to prove by the application of the maximum principle and the Hopf lemma that near \( e_\delta \) we have not a local maximum, which is a contradiction.

We set on \( \Sigma_\lambda^\prime = \Sigma_\lambda - \{ y, |y - \delta e| \leq \frac{\delta}{R_i} \} \simeq \Sigma_\lambda - \{ y, |I_\delta(y)| \geq r_i \}: \)

\[ h_\lambda(y) = -\int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta. \]

with,
\[ Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\lambda(\eta^\lambda))^3. \]

And, by the same estimates, we have for \( \eta \in A_1 = \{ \eta, |\eta| \leq R = \epsilon_0/\delta \}, \)
\[ V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1}(\eta_1 - \lambda) + o(1)M_i^{-1}|\eta^\lambda|, \]
and, we have for \( \eta \in A_2 = \Sigma_\lambda - A_1: \)
\[ |V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq CM_i^{-1}(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|), \]
And, we have for some \( \lambda_0 \leq -2 \) and \( C_0 > 0: \)
\[ w_\lambda(y) = v_i^\lambda(y) - v_i^\lambda(y^\lambda) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n}, \]
for \( y_1 > \lambda_0. \)

Because, by the maximum principle:
\[
\min_{\{i, \leq I_\delta(y) \leq r_i \}} v_i = \min \{ \min_{\{I_\delta(y) = l_i \}} v_i, \min_{\{I_\delta(y) = r_i \}} v_i \} \geq (1 - \epsilon)U_\delta(\frac{\epsilon}{\delta}) \\
\geq (1 + c_1 \delta - \epsilon)U_\delta(\frac{\epsilon}{\delta}) \geq (1 + c_1 \delta - 2\epsilon)v_i^\lambda(y^\lambda),
\]
and for \( |I_\delta(y)| \leq l_i \) we use the \( C^2 \) convergence of \( v_i^\lambda \) to \( U_\delta. \)

Thus,
\[ w_\lambda(y) > 2\epsilon > 0, \]
By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section),
and by our hypothesis on \( v_i, \) we have:
\[ 0 < h_\lambda(y) = O(1)M_i^{-1}(y_1 - \lambda)(1 + |y|)^{-n} < 2\epsilon < w_\lambda(y). \]
also, we have the same estimate on the boundary, \( |I_\delta(\eta)| = r_i \) or \( |y - \epsilon/\delta| = c_2r_i^{-1}: \)

Step 2.1.1: \( |\nabla V_i(x_i)|^{1/\beta}[u_i(x_i)] \leq C \)

Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we choose the same \( h_\lambda, \) except the fact that here we use the computation with \( M_i^{-1(1+\beta)} \) in front the regular part of \( h_\lambda. \)

Here also, we consider \( r_i \in (0, L_i) \) where \( L_i \) is the number of the blow-up analysis.
\[ L_i = \frac{1}{2i}R_iM_i^{(1+\beta)/2}. \]

We argue by contradiction and we suppose that:
\[ M_i^\beta D_i \to +\infty. \]

Then, without loss of generality we can assume that:
\[ \frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \to e = (1, 0, ...0). \]

We use the Kelvin transform twice and around this point and around 0.
\[ h_\lambda(y) = \epsilon r_i^{-2}G_\lambda(y, \frac{\epsilon}{\delta}) - \int_{\Sigma_\lambda} G_\lambda(y, \eta)Q_\lambda(\eta)d\eta, \]
with,
\[ Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\lambda(\eta^\lambda))^3. \]
And, by the same estimates, we have for \( \eta \in A_1 \)
\[ V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1}D_i(\eta_1 - \lambda) + o(1)M_i^{-1}|\eta^\lambda|, \]
and, we have for \( \eta \in A_2, |I_0(\eta)| \leq c_2 M_i D_i^{1/\beta} \),

\[
|V_0(\eta) - V_0(\eta^\lambda)| \leq C M_i^{-1} D_i (|I_0(\eta)| + |I_0(\eta^\lambda)|),
\]

and for \( M_i D_i^{1/\beta} \leq |I_0(\eta)| \leq r_i \),

\[
|V_0(\eta) - V_0(\eta^\lambda)| \leq M_i^{-1} D_i |I_0(\eta)| + M_i^{-(1+\beta)} |I_0(\eta)|^{(1+\beta)},
\]

By the same estimates, we have for \( |I_0(\eta)| \leq r_i \) or \( |y - e/\delta| \geq c_3 r_i^{-1} \):

\[
h_\lambda(y) \simeq c r_i^{-2} G_\lambda(y, \frac{e}{\delta}) + c_4 M_i^{-1} D_i \left( \frac{y_1 - \lambda}{|y|^{n}} + o(1) M_i^{-1} D_i \frac{y_1 - \lambda}{|y|^{n}} + o(1) M_i^{-(1+\beta)} G_\lambda(y, \frac{e}{\delta}) \right),
\]

with \( c_4 > 0 \).

And, we have for some \( \lambda_0 \leq -2 \) and \( C_0 > 0 \):

\[
v^\ast_i(y) - v^\ast_i(y^\lambda) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n},
\]

for \( y_1 > \lambda_0 \).

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on \( v_\ast \), we have:

\[
0 < h_\lambda(y) < 2e < w_\lambda(y).
\]

also, we have the same estimate on the boundary, \( |I_0(\eta)| = r_i \) or \( |y - e/\delta| = c_3 r_i^{-1} \)

**Step 2.2 conclusion : a linear equation perturbed by a term, and, the auxiliary function**

Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function \( h_\lambda \) (which correspond to this step), except the fact that here in front the regular part of this function we have \( M_i^{-(1+\beta)} \).

Here also, we consider \( r_i \in (0, L_i) \) where \( L_i \) is the number of the blow-up analysis.

\[
L_i = \frac{1}{2t} R_i M_i^{(1+\beta)/2}.
\]

We set,

\[
v_i(z) = v^\ast_i(z + e),
\]

where \( v^\ast_i \) is the blow-up function. Then, \( v_i \) has a local maximum near \( -e \).

\[
U_0(z) = U^\ast_0(z + e).
\]

We have, for \( |y| \geq L_i^{-1}, L_i = \frac{1}{2} R_i M_i \),

\[
\tilde{v}_i(y) = \frac{1}{|y|^{n-2}} v_i \left( \frac{y}{|y|^2} \right).
\]

\[
|V_i(\tilde{x}_i + M_i^{-1} y / |y|^2) - V_i(\tilde{x}_i)| \leq M_i^{-1} (1 + |y|^{-1}).
\]

\[
x_i = \tilde{x}_i + M_i^{-1} e,
\]

Then, for simplicity, we can assume that, \( \tilde{v}_i \) has a local maximum near \( e^* = (-1/2, 0, \ldots 0) \).

Also, we have:

\[
|V_i(x_i + M_i^{-1} y / |y|^2) - V_i(x_i + M_i^{-1} y^\lambda / |y|^2)| \leq M_i^{-(1+\beta)} (1 + |y|^{-1}).
\]

\[
h_\lambda(y) \simeq c r_i^{-2} G_\lambda(y, 0) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.
\]

where, \( \Sigma_\lambda = \{ \eta, |\eta| \leq r_i^{-1} \} \), and,
we have by the same computations that:
\[
\int_{\Sigma'} G_{\lambda}(y, \eta) Q_\lambda(\eta) d\eta \leq CM_i^{-(1+\beta)} G_\lambda(y, 0) << \epsilon r_i^{-2} G_\lambda(y, 0).
\]
By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on \( v_i \), we have:
\[
0 < h_\lambda(y) < 2 \varepsilon < w_\lambda(y).
\]
also, we have the same estimate on the boundary, \(|y| = \frac{1}{r_i}\).

**Proof of the theorem 2, \( n = 5 \):**

To prove the theorem, we argue by contradiction and we assume that the \( (\sup)^{1/3} \times \inf \) tends to infinity.

**Step 1: blow-up analysis**

We want to prove that:
\[
\tilde{R}_i^3 (\sup_{B_{	ilde{R}_i}(0)} u_i)^{1/3} \times \inf_{B_{3\tilde{R}_i}(0)} u_i \leq c = c(a, b, A, B),
\]
If it is not the case, we have:
\[
\tilde{R}_i^3 (\sup_{B_{\tilde{R}_i}(0)} u_i)^{1/3} \times \inf_{B_{3\tilde{R}_i}(0)} u_i = \epsilon^6 \to +\infty,
\]
For positive solutions \( u_i > 0 \) of the equation \((E)\) and \( \tilde{R}_i \to 0 \).
Thus,
\[
\frac{1}{\epsilon} \tilde{R}_i (\sup_{B_\tilde{R}_i} u_i)^{2/3} \to +\infty,
\]
and,
\[
\frac{1}{\epsilon} \tilde{R}_i \sup_{B_\tilde{R}_i} u_i^{4/9} \to +\infty,
\]
Let \( a_i \) such that:
\[
u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i,
\]
We set,
\[
s_i(x) = (\tilde{R}_i - |x - a_i|)^{9/4} u_i(x),
\]
we have,
\[
s_i(\bar{x}_i) = \max_{B_{\tilde{R}_i}(0)} s_i \geq s_i(a_i) = \tilde{R}_i^{9/4} \sup_{B_{\tilde{R}_i}(0)} u_i \to +\infty,
\]
we set,
\[
R_i = \frac{1}{2} (\tilde{R}_i - |\bar{x}_i - a_i|),
\]
We have, for \( |x - \bar{x}_i| \leq \frac{R_i}{\epsilon} \),
\[
\tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| \geq 2R_i - R_i = R_i
\]
Thus,
\[
\frac{u_i(x)}{u_i(x_i)} \leq \beta_i \leq 2^{9/4}.
\]

with \( \beta_i \to 1 \).

We set,
\[
M_i = u_i(x_i), \quad v_i^*(y) = \frac{u_i(x_i + M_i^{-2/3}y)}{u_i(x_i)}, \quad |y| \leq \frac{1}{i} R_i M_i^{4/9} = 2L_i.
\]

And,
\[
\frac{1}{i^3} R_i^3 M_i^{1/3} \times \inf \limits_{B_{3R_i}(0)} u_i \to +\infty,
\]

By the elliptic estimates, \( v_i^* \) converge on each compact set of \( \mathbb{R}^5 \) to a function \( U_0^* > 0 \) solution of:

\[
\begin{cases}
-\Delta U_0^* = V(0) U_0^{*7/3} & \text{in } \mathbb{R}^5, \\
U_0^*(0) = 1 = \max_{\mathbb{R}^5} U_0^*.
\end{cases}
\]

For simplicity, we assume that \( 0 < V(0) = n(n-2) = 15 \). By a result of Caffarelli-Gidas-Spruck, see [10], we have:

\[
U_0^*(y) = (1 + |y|^2)^{-3/2}.
\]

We set,
\[
v_i(y) = v_i^*(y + \epsilon),
\]

where \( v_i^* \) is the blow-up function. Then, \( v_i \) has a local maximum near \( -\epsilon \).

\[
U_0(y) = U_0^*(y + \epsilon).
\]

We want to prove that:

\[
\min_{\{0 \leq |y| \leq r\}} v_i^* \leq (1 + \epsilon) U_0^*(r).
\]

for \( 0 \leq r \leq L_i \), with \( L_i = \frac{1}{2i} R_i M_i^{4/9} \).

We assume that it is not true, then, there is a sequence of number \( r_i \in (0, L_i) \) and \( \epsilon > 0 \), such that:

\[
\min_{\{0 \leq |y| \leq r_i\}} v_i^* \geq (1 + \epsilon) U_0^*(r_i).
\]

We have:

\[
r_i \to +\infty.
\]

Thus, we have for \( r_i \in (0, L_i) \):

\[
\min_{\{0 \leq |y| \leq r_i\}} v_i \geq (1 + \epsilon) U_0(r_i).
\]

Also, we can find a sequence of number \( l_i \to +\infty \) such that:

\[
||v_i^* - U_0||_{C^2(B_{l_i}(0))} \to 0.
\]

Thus,

\[
\min_{\{0 \leq |y| \leq l_i\}} v_i \geq (1 - \epsilon/2) U_0(l_i).
\]

**Step 2**: The Kelvin transform and the Moving-plane method

**Step 2.1**: a linear equation perturbed by a term, and, the auxiliary function
Step 2.1.1: \( D_i = |\nabla V_i(x_i)| \to 0. \)

We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider \( r_i \in (0, L_i) \) where \( L_i \) is the number of the blow-up analysis.

\[
L_i = \frac{1}{2}\frac{R_i}{M_i^{4/9}}.
\]

We use the assumption that the sup times inf is not bounded to prove \( w_{\lambda} > h_{\lambda} \) in \( \Sigma_{\lambda} = \{ y, y_1 > \lambda \} \), and on the boundary.

The function \( v_i \) has a local maximum near \(-e\) and converge to \( U_0(y) = U_0^*(y + e) \) on each compact set of \( \mathbb{R}^5 \). \( U_0 \) has a maximum at \(-e\).

We argue by contradiction and we suppose that:

\[
D_i = |\nabla V_i(x_i)| \not\to 0.
\]

Then, without loss of generality we can assume that:

\[
\nabla V_i(x_i) \to e = (1, 0, ... 0).
\]

Where \( x_i \) is:

\[
x_i = \bar{x}_i + M_i^{-2/3}e,
\]

with \( \bar{x}_i \) is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

\[
I_\delta(y) = \frac{|y|}{|y|^2} - \frac{\delta e}{|y| - \delta |e|};
\]

\[
v_i^\delta(y) = \frac{v_i(I_\delta(y))}{|y|^{-2}|y - e/\delta|^{n-2}},
\]

and,

\[
V_\delta(y) = V_i(x_i + M_i^{-2/3}I_\delta(y)).
\]

\[
U_\delta(y) = \frac{U_0(I_\delta(y))}{|y|^{-2}|y - e/\delta|^{n-2}}.
\]

Then, \( U_\delta \) has a local maximum near \( e_\delta \to -e \) when \( \delta \to 0 \). The function \( v_i^\delta \) has a local maximum near \(-e\).

We want to prove by the application of the maximum principle and the Hopf lemma that near \( e_\delta \) we have not a local maximum, which is a contradiction.

We set on \( \Sigma_{\lambda}^\prime = \Sigma_{\lambda} - \{ y_1 \leq \frac{\delta}{R_i} \} \simeq \Sigma_{\lambda} - \{ y, |I_\delta(y)| \geq r_i \} \):

\[
h_{\lambda}(y) = -\int_{\Sigma_{\lambda}} G_{\lambda}(y, \eta)Q_{\lambda}(\eta)d\eta.
\]

with,

\[
Q_{\lambda}(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}.
\]

And, by the same estimates, we have for \( \eta \in A_1 = \{ \eta, |\eta| \leq R = e_0/\delta \} \),

\[
V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-2/3}(\eta_1 - \lambda) + o(1)M_i^{-2/3}|\eta^\lambda|,
\]

and, we have for \( \eta \in A_2 = \Sigma_{\lambda} - A_1 \):

\[
|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq CM_i^{-2/3}(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),
\]

And, we have for some \( \lambda_0 \leq -2 \) and \( C_0 > 0 \):
\[ v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n}, \]

for \( y_1 > \lambda_0 \).

By the same estimates, and by our hypothesis on \( v_i \), we have, for \( c_1 > 0 \):

\[ 0 < h_\lambda(y) < 2\epsilon < w_\lambda(y). \]

also, we have the same estimate on the boundary, \( |I_\delta(\eta)| = r_i \) or \( |y - e/\delta| = c_2 r_i^{-1} \).

Step 2.1: \(|\nabla V_i(x_i)||u_i(x_i)|^{2/3} \leq C^\lambda\)

Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we take \( \alpha = 2 \) and we choose the same \( h_\lambda \), except the fact that here we use the computation with \( M_i^{-4/3} \) in front the regular part of \( h_\lambda \).

Here also, we consider \( r_i \in (0, L_i) \) where \( L_i \) is the number of the blow-up analysis.

\[ L_i = \frac{1}{2\epsilon} R_0 M_i^{4/9}. \]

We argue by contradiction and we suppose that:

\[ M_i^{2/3} D_i \to +\infty. \]

Then, without loss of generality we can assume that:

\[ \frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \to e = (1, 0, \ldots, 0). \]

We use the Kelvin transform twice and around this point and around 0.

\[ h_\lambda(y) = c_1 r_i^{-3} G_\lambda(y, \frac{e}{\delta}) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta, \]

with,

\[ Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}. \]

And, by the same estimates, we have for \( \eta \in A_1 \)

\[ V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-2/3} D_i(\eta - \lambda) + o(1) M_i^{-2/3} |\eta^\lambda|, \]

and, we have for \( \eta \in A_2, |I_\delta(\eta)| \leq c_2 M_i^{2/3} D_i, \)

\[ |V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{2/3} D_i [I_\delta(\eta)] + |I_\delta(\eta^\lambda)|, \]

and for \( M_i^{2/3} D_i \leq |I_\delta(\eta)| \leq r_i, \)

\[ |V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq M_i^{-2/3} D_i |I_\delta(\eta)| + M_i^{-4/3} |I_\delta(\eta)|^{2}, \]

By the same estimates, we have for \( |I_\delta(\eta)| \leq r_i \) or \( |y - e/\delta| \geq c_3 r_i^{-1} \):

\[ h_\lambda(y) \simeq c_1 r_i^{-3} G_\lambda(y, \frac{e}{\delta}) + c_4 M_i^{-2/3} D_i (y_1 - \lambda) |y|^n + o(1) M_i^{-2/3} D_i (y_1 - \lambda) |y|^n + o(1) M_i^{-4/3} G_\lambda(y, \frac{e}{\delta}), \]

with \( c_4 > 0. \)

And, we have for some \( \lambda_0 \leq -2 \) and \( C_0 > 0 \):

\[ v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n}, \]

for \( y_1 > \lambda_0. \)

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on \( v_i \), we have:

\[ 0 < h_\lambda(y) < 2\epsilon < w_\lambda(y). \]
also, we have the same estimate on the boundary, \(|I_\delta(\eta)| = r_i \) or \(|y - e/\delta| = c_5 r_i^{-1}\):

**Step 2.2 conclusion:** a linear equation perturbed by a term, and, the auxiliary function

Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function \(h_\lambda\) (which correspond to this step), except the fact that here in front the regular part of this function we have \(M_i^{-4/3}\).

Here also, we consider \(r_i \in (0, L_i)\) where \(L_i\) is the number of the blow-up analysis.

\[ L_i = \frac{1}{2} R_i M_i^{4/9}. \]

We set,

\[ v_i(z) = v_i^*(z + e), \]

where \(v_i^*\) is the blow-up function. Then, \(v_i\) has a local maximum near \(-e\).

\[ u_0(z) = u_0^*(z + e). \]

We have, for \(|y| \geq L_i^{-1}, L_i = \frac{1}{2} R_i M_i^{2/3},\)

\[ \bar{v}_i(y) = \frac{1}{|y|^{n-2}} v_i \left( \frac{y}{|y|^2} \right). \]

\[ |V_i(x_i + M_i^{-2/3} \frac{y}{|y|^2}) - V_i(x_i)| \leq M_i^{-4/3}(1 + |y|^{-2}). \]

\[ x_i = \bar{x}_i + M_i^{-2/3} e, \]

Then, for simplicity, we can assume that, \(\bar{v}_i\) has a local maximum near \(e^* = (-1/2, 0, ... 0)\).

Also, we have:

\[ |V_i(x_i + M_i^{-2/3} \frac{y}{|y|^2}) - V_i(x_i + M_i^{-2/3} \frac{y^\lambda}{|y|^2})| \leq M_i^{-4/3}(1 + |y|^{-2}). \]

\[ h_\lambda(y) = \epsilon r_i^{-3} G_\lambda(y, 0) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta. \]

where, \(\Sigma_\lambda = \Sigma - \{\eta, |\eta| \leq r_i^{-1}\}\), and,

\[ Q_\lambda(\eta) = \left( V_i(x_i + M_i^{-2/3} \frac{y^\lambda}{|y|^2}) - V_i(x_i + M_i^{-2/3} \frac{y^{\lambda}}{|y|^2}) \right) (v_i(y^\lambda))^{n+2}. \]

we have by the same computations that:

\[ \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta \leq CM_i^{-4/3} G_\lambda(y, 0) < < \epsilon r_i^{-3} G_\lambda(y, 0). \]

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on \(v_i\), we have:

\[ 0 < h_\lambda(y) < 2 \epsilon < w_\lambda(y). \]

also, we have the same estimate on the boundary, \(|\eta| = \frac{1}{r_i}\).
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