CONLEY’S FUNDAMENTAL THEOREM FOR A CLASS OF HYBRID SYSTEMS

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ABSTRACT. We establish versions of Conley’s (i) fundamental theorem and (ii) decomposition theorem for a broad class of hybrid dynamical systems. The hybrid version of (i) asserts that a globally-defined hybrid complete Lyapunov function exists for every hybrid system in this class. Motivated by mechanics and control settings where physical or engineered events cause abrupt changes in a system’s governing dynamics, our results apply to a large class of Lagrangian hybrid systems (with impacts) studied extensively in the robotics literature. Viewed formally, these results generalize those of Conley and Franks for continuous-time and discrete-time dynamical systems, respectively, on metric spaces. However, we furnish specific examples illustrating how our statement of sufficient conditions represents merely an early step in the longer project of establishing what formal assumptions can and cannot endow hybrid systems models with the topologically well characterized partitions of limit behavior that make Conley’s theory so valuable in those classical settings.

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1. Introduction

In [Nor95], Norton argued that the following two theorems deserve to be called the “Fundamental Theorem of Dynamical Systems.”

**Theorem** ([Con78]). Any continuous flow on a compact metric space decomposes into a chain recurrent part and a gradient-like part. There exists a continuous Lyapunov function which strictly decreases along the flow on the gradient-like part.

**Theorem** ([Fra88]). The iteration of a continuous map on a compact metric space decomposes the space into a chain recurrent part and a gradient-like part. There exists a continuous Lyapunov function which strictly decreases under iteration of the map on the gradient-like part.

From the view of applications, these results endow models that achieve them with two important guarantees. First, the decomposition establishes a clear, deterministic notion of steady state behavior that, no matter how complicated its temporal manifestation [Lor64, May76, Hol90], imposes a computationally effective [KMV05] spatial partition into attractor basins [Mil06] whose topology persists under small perturbations. The passage from signal to symbol afforded by such partitions has great value for analyzing natural systems [AKK+09, GVdBV03], and has encouraged slowly growing use in the synthesis of engineered systems as well [ACR+02, CML+07, HCK11, HRK12]. Second, interpreted as a universal converse Lyapunov theorem, global analogue to the classical counterpart addressing a specific basin [Kel15], the long established value for classical [Son89], multistable [FA18] and hybrid control systems theory [GST09] is leveraged by a steadily advancing literature on constructive methods for their eventual feedback closed loops [BK06, GH15]. In our view, one of the most important applications for Lyapunov-expression of basin partitions is their long-proven role in sequential composition [BRK95] and their promise for parallel composition [Cow07, TVDK19], increasing the expressive richness of topologically grounded type theories [AH15] emerging from hybrid dynamical categories that admit them.

1.1. Contributions and organization of the paper. Motivated by problems of robotics and biomechanics, where the making and breaking of contacts intrinsic to most tasks necessitates the introduction of hybrid systems models, this paper addresses the question of what hybrid systems models admit a version of Conley’s fundamental theorem. We focus on a partial extension of a particularly simple but empirically useful class [JBK16], relative to which a closely related extension can be shown to generate a formal category equipped with the desired compositional operators [CGKS19]. Specifically, we introduce the class of topological hybrid systems (THS) and the subclass of metric hybrid systems (MHS) (Definition 1) that roughly generalizes the model of [JBK16] (see §A.1). After imposing the trapping guard condition (Definition 11) we prove an appropriately generalized version of Conley’s decomposition theorem (Theorem 1) as well as the existence (Theorem 2) of a globally-defined hybrid complete Lyapunov function (Definition 12). We illustrate the applicability of these results by presenting two broad MHS subclasses to which they apply: the smooth exit-boundary guarded MHS (Proposition 1) arising, for example in problems of legged locomotion [BRS15, DBK18]; and an extension (Proposition 2) of the Lagrangian hybrid systems (Corollary 1), a class of models (or near variations thereof) studied in the robotics literature [GAP01, WGK03, AZGS06, PG09, OA10, BCC17, RBCG17]. In contrast, a simple counterexample (Example 5, depicted in Figure 5) demonstrates that the conclusions of our version of Conley’s theorems for MHS can fail without the trapping guard condition. Finally, we use two variants of the Hamiltonian bouncing ball model to illustrate how these results apply to mechanical systems which undergo impacts (and to mechanical systems which have only Zeno maximal executions, in the case of the first variant). Bouncing against gravity (Example 6) generates an MHS satisfying the trapping guard condition, yielding the Conley decomposition and complete Lyapunov function (Figure 6) guaranteed by Theorems 1 and 2. In contrast, because linear time invariant vector fields are homogenous, the MHS generated by bouncing losslessly against a Hooke’s law spring (Example 7) fails the trapping guard condition, so this system does not satisfy the hypotheses of our main theorems; interestingly, however, this example does still satisfy our main theorems’ conclusions. We end with some more philosophically motivated remarks concerning the virtue of parsimony arising from these results that generalize both the discrete (Example 1) and the
continuous (Example 2) classical frameworks to unify the common but heretofore distinct assertions of [Con78, Fra88].

The remainder of this paper is organized as follows. After discussing related work below, we introduce the basic definitions and concepts relevant to our main results in §2. In §3 we state our main results. In §4 and §5, we present the applications and examples (some very specific and some quite general) as just discussed. The proofs of our main results rely on the reduction of suitably guarded MHS to classical dynamical systems on spaces obtained via the hybrid suspension technique introduced in §6.2; the latter technique may be of independent interest, generalizes the classical suspension of a discrete-time dynamical system [Sma67, BS02, p. 797, pp. 21–22], and is distinct from the hybrifold technique of [SJSL00, SJLS05] (see §A.4 and Appendix B for more details). We conclude with brief remarks of a more speculative nature in §7. Appendix A compares some of our constructions with selected prior work. Appendix B gives a primer on the classical suspension of a discrete-time dynamical system. Appendix C makes precise and proves the statement that, for a class of deterministic THS satisfying mild assumptions, the trapping guard condition holds if and only if a continuous hybrid suspension semiflow exists.

1.2. Related work. As reviewed above, for flows on compact metric spaces, Conley proved the existence of a complete Lyapunov function and that the chain recurrent set is the intersection of all attracting-repelling pairs [Con78]. Franks proved the corresponding results for maps on compact metric spaces [Fra88]. Hurley extended the decomposition theorem to maps and semiflows on arbitrary metric spaces [Hur95] and proved the existence of complete Lyapunov functions for maps on separable metric spaces [Hur98]. Using Hurley’s result, Patrão proved the existence of a complete Lyapunov function for any semiflow on a separable metric space [Pat11]. In the nondeterministic setting, McGehee and Wiandt generalized Franks’ results to the setting of iterations of closed relations [MW06, Wia08]; Bronstein and Kopanski generalized Conley’s results to a class of set-valued dynamical systems such as those arising from certain differential inclusions [BK88]. In the stochastic setting, Liu generalized the decomposition and fundamental theorems to random (semi-)dynamical systems such as those arising from stochastic (partial) differential equations [Liu05, Liu07a, Liu07b].

Motivated largely by mathematical models occurring in science and engineering, many investigators have worked to generalize results and tools from classical dynamical systems theory to the hybrid setting. Examples include extensions of local [SJLS01] and global [BPS01] structural stability results, contraction analysis [BC18, BLC18], and many theoretical tools concerning periodic orbits [BSKR16] including Floquet theory [BRS15] and the Poincaré-Bendixson theorem [SSJL02, CBC19, CB20]. In particular, to prove our results, we introduce the hybrid suspension of a THS, which is related to constructions appearing in [AS05, BGV +15] (see §A.4 for more details). Similarly, our THS and MHS definitions build on a long history of formal approaches to hybrid automata [SJLS05, HTP05, JBK16, Ler16, CGKS19]. Most specifically, our definition of hybrid $(\epsilon,T)$-chains is almost identical to the definition in [CGKS19] for smooth hybrid systems (with one important difference; see §A.3).

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2. Preliminaries

2.1. Two classes of hybrid systems. Following [HS06, Sec. 1.3], a local semiflow $\varphi$ on a topological space $F$ is a map $\varphi: \text{dom}(\varphi) \to F$ defined on an open neighborhood $\text{dom}(\varphi) \subseteq [0,\infty) \times F$ of $\{0\} \times F$ satisfying the following conditions, with $\varphi^t := \varphi(t, \cdot)$ and $t, s \in [0,\infty)$: (i) $\varphi^0 = \text{id}_F$, (ii) $(t+s,x) \in \text{dom}(\varphi) \iff (s,x) \in \text{dom}(\varphi)$ and $(t,\varphi^s(x)) \in \text{dom}(\varphi)$, and (iii) for all $(t+s,x) \in \text{dom}(\varphi)$,
Remark 1. Of references containing this style of definition include \[SJSL00, SJLS01, SSJL02, LJS05, BRS15, Ler16, JBK16, BLC18, BC18, CGKS19, CB20, LS20\].

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Discrete-time dynamics:

Continuous-time dynamics:

States: A topological state space \( I \) whose points are the possible states of the system.

Continuous-time dynamics: a continuous local semiflow \( \varphi \) defined on an open flow set \( F \subseteq I \).

Discrete-time dynamics: a closed guard set \( Z \subseteq I \) equipped with a continuous reset map \( r : Z \to I \).

If the topology of \( I \) arises from an extended metric \( \text{dist} : I \times I \to [0, +\infty] \), we say that \((H, \text{dist})\) is a metric hybrid system (MHS). (We will usually suppress the extended metric \( \text{dist} \) from the notation.)

Remark 1. Typical definitions of hybrid systems specify several distinct state spaces (usually smooth manifolds with corners), often called modes, each equipped with its own local semiflow (usually generated by a locally Lipschitz vector field). Each mode may contain a guard set, and discrete transitions from the guard sets to other modes are specified by reset maps (sometimes allowed to be more general relations).

At first glance one might incorrectly assume that Definition 1 can encapsulate only hybrid systems consisting of a single mode. However, this is not the case: given a hybrid system as defined in one of the aforementioned references (and having continuous reset maps), by defining \( I \) to be the disjoint union of the modes, \( F \) to be the disjoint union of the flow sets, \( Z \) to be the disjoint union of the guard sets, \( r : Z \to I \) to be the disjoint union of the reset maps, and \( \varphi \) to be the disjoint union of the local semiflows, one obtains a THS as in Definition 1. If additionally each of the modes of the given hybrid system is equipped with a compatible extended metric (by the Urysohn metrization theorem \[Mun00, Thm 34.1\], such a metric always exists if each mode is a smooth, paracompact manifold with corners), then one further obtains an

\[\varphi^{t+s}(x) = \varphi^t(\varphi^s(x))\]. Given \( x \in F \), the map \( t \mapsto \varphi^t(x) \) defined on some interval is called a trajectory of \( \varphi \). The local semiflow \( \varphi \) is a semiflow if \( \text{dom}(\varphi) = [0, \infty) \times F \).

The following definition follows much of the terminology of \[CGKS19\], but uses a simpler model for the discrete-time dynamics.\(^1\) Our definition of topological hybrid systems (THS) uses a more general framework for the continuous-time dynamics, i.e., local semiflows on topological spaces instead of vector fields on manifolds. For this reason, our definition of metric hybrid systems (MHS) differs from that of \[CGKS19, Def. 2.17\].\(^2\)

### Definition 1 (Topological and metric hybrid systems).

A topological hybrid system (THS) \( H = (I, F, Z, \varphi, r) \) consists of:

- **States:** a topological state space \( I \) whose points are the possible states of the system.
- **Continuous-time dynamics:** a continuous local semiflow \( \varphi \) defined on an open flow set \( F \subseteq I \).
- **Discrete-time dynamics:** a closed guard set \( Z \subseteq I \) equipped with a continuous reset map \( r : Z \to I \).

If the topology of \( I \) arises from an extended metric \( \text{dist} : I \times I \to [0, +\infty] \), we say that \((H, \text{dist})\) is a metric hybrid system (MHS). (We will usually suppress the extended metric \( \text{dist} \) from the notation.)

**Applications-oriented readers might be interested to consult footnote 28 for a brief discussion motivating the (essentially imperative) benefits of adopting this more general framework.**

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\(^1\) More specifically, we ignore any underlying graph structure and the fact that there may be various distinct “modes”; see Remark 1 for more details.

\(^2\) Applications-oriented readers might be interested to consult footnote 28 for a brief discussion motivating the (essentially imperative) benefits of adopting this more general framework.
MHS as in Definition 1 by leaving the distance between points in the same mode unchanged and defining the distance between points in distinct modes to be infinite. Thus, our results (including Theorems 1 and 2) can be applied to such hybrid systems, as long as they satisfy the relevant additional hypotheses.

**Definition 2.** Given a THS $H = (I, F, Z, \varphi, r)$, an execution in $H$ is a tuple $\chi = (N, \tau, \gamma)$ of

1. **Jump times:** a nondecreasing sequence $\tau = (\tau_j)_{j=0}^{N+1} \subseteq \mathbb{R} \cup \{+\infty\}$ where $N \in \mathbb{N}_0 \cup \{+\infty\}$, $\tau_0 = 0$, and $(\tau_j)_{j=0}^N \subseteq \mathbb{R}$.
2. **Flow arcs:** a sequence of continuous maps $\gamma = (\gamma_j : T_j \to I)_{j=0}^N$ with $[\tau_j, \tau_{j+1}) \subseteq T_j \subseteq [\tau_j, \tau_{j+1}) \cap \mathbb{R}$, $\gamma_j([-\tau_j, \tau_{j+1})) \subseteq F$, and such that the restriction $\gamma_j|_{[\tau_j, \tau_{j+1}]} : [\tau_j, \tau_{j+1}) \to F$ is a trajectory segment for the local semiflow $\varphi$. For all $0 \leq j < N$, we additionally require that $T_j = [\tau_j, \tau_{j+1})$, $\gamma_j(\tau_{j+1}) \in Z$, and $\gamma_{j+1}(\tau_{j+1}) = r(\gamma_j(\tau_{j+1}))$.

We call $\gamma_0(0)$ the initial state of $\chi$. If $N < \infty$ and $\tau_{N+1} \in T_N$, we call $\gamma_N(\tau_{N+1})$ the final state of $\chi$. We define the stop time of $\chi$ to be

$$\tau_{\text{stop}} := \begin{cases} \tau_{N+1} & N < \infty \\ \lim_{j \to \infty} \tau_j & N = \infty \end{cases}$$

If the stop time of $\chi$ is infinite, we say that $\chi$ is an infinite execution. If the stop time of $\chi$ is finite but $\chi$ has infinitely many jumps ($N = \infty$), we say that $\chi$ is a Zeno execution. We say that $\chi = (N, \tau, \gamma)$ is a maximal execution if, for any execution $\chi' = (N', \tau', \gamma')$ with $\gamma_0'(0) = \gamma_0(0)$ and each $\gamma_j$ equal to $\gamma_j'|_{T_j}$ for some $j' \in \{0, \ldots, N'\}$, $\chi = \chi'$.

We denote the set of executions in $H$ and executions with initial state $x \in I$ by $\mathcal{E}_H$ and $\mathcal{E}_H(x)$, respectively. For $\chi = (N, \tau, \gamma) \in \mathcal{E}_H$ and any $t \in \bigcup_{j=0}^N T_j$, we write $\chi(t) = \{\gamma_j(t) \mid 0 \leq j < N+1, t \in T_j\}$. We emphasize that $\chi(t)$ is generally a set with multiple elements if $t \in \{\tau_1, \ldots, \tau_N\}$, but otherwise $\chi(t)$ is a singleton and can thus be treated as a single point in $I$.

**Remark 2.** An important point concerning Definition 2 is that, if $\gamma_j(\tau_j) \in Z$ for some $0 \leq j \leq N$, then it is possible that $\tau_{j+1} = \tau_j$, i.e., an instantaneous reset might occur (and must occur if $H$ is deterministic; see Definition 3 below). In this case, the condition $\gamma_j([-\tau_j, \tau_{j+1})) = \emptyset \subseteq F$ is satisfied vacuously. A similar remark applies to Definition 4 below.

We also note that, since the domain of the final arc of an execution is not required to be closed, Definition 2 allows the final arc to be a trajectory of $\varphi$ which “blows up” or “escapes” in finite time (i.e., it cannot be extended to a $\varphi$ trajectory defined for all nonnegative time; c.f. [HS06, Sec. 1.3] for the latter terminology). However, our main results concerning THS $H = (I, F, Z, \varphi, r)$ include the assumption that every maximal execution of $H$ is infinite or Zeno, and this assumption implies that $\varphi$ trajectories may only “artificially” escape $F$ in finite time by converging to a limit in $Z$.

As in [CGKS19, Rem. 2.16], our definition of execution allows for infinitely many jumps in finite time (Zeno executions), but does not allow for subsequent execution after the stop time. In particular, while we do allow Zeno executions, in this paper we do not explicitly consider continuations of Zeno executions past the stop time. (Zeno continuations are considered, e.g., in [JELS99, AZGS06, OA10, JBK16, GS16].)

**Remark 3.** If $x \in I \setminus (F \cup Z)$, then Definition 2 implies that the only execution $\chi = (N, \tau, \gamma) \in \mathcal{E}_H(x)$ is the trivial execution: $N = 0$, $\tau = (0, 0)$, $\gamma_0 : \{0\} \to \{x\}$. It follows that, if every $x \in I$ has an execution $\chi \in \mathcal{E}_H(x)$ which is not trivial, then $I = F \cup Z$. In particular, if every $x \in I$ has an infinite or Zeno execution $\chi \in \mathcal{E}_H(x)$, it follows that $I = F \cup Z$.

**Definition 3.** A THS $H = (I, F, Z, \varphi, r)$ is deterministic if $Z \cap F = \emptyset$. A THS $H$ is nonblocking if for every $x \in I$ there is an infinite execution starting at $x$.

3Our definition of “infinite execution” follows that of [CGKS19, Def. 2.13]. However, this definition differs from that of [LJS"03, p. 4], which refers to both executions having infinite stop time and Zeno executions as “infinite.” Since our Definition 3 and [LJS"03, Def. III.1] both define “nonblocking” hybrid systems to be those for which all maximal executions are “infinite,” our definitions of “nonblocking” thus also differ in meaning. Our definition of “nonblocking” differs from that of [JBK16, Def. 4] in precisely the same way, although the latter reference does not introduce the “infinite execution” terminology.
For a deterministic THS, maximal executions are unique; c.f. [CGKS19, Prop. 2.21]. This justifies the terminology. For a deterministic THS, we use the notation $\chi_x$ to denote the unique maximal execution in $\mathcal{E}_H(x)$.

Most of our results do not assume the nonblocking condition; rather, most of our results (including Theorems 1 and 2) assume the weaker condition that all maximal executions of a given THS are either infinite or Zeno.

2.2. Hybrid chain equivalence, recurrence, and attracting-repelling pairs. The following definition of $(\epsilon, T)$-chains is essentially the same as [CGKS19, Def. 2.18] (see §A.3 for a comparison). As we show below, the corresponding Conley relation generalizes the classical notions for discrete-time and continuous-time systems on compact metric spaces. We recommend the reader glance at Figure 2 before reading the formal definition. We remark that $(\epsilon, T)$-chains can be viewed informally as “executions with errors,” with the values of $\epsilon, T$ determining the admissible errors.

Definition 4. Given an MHS $H = (I, F, Z, \varphi, r)$ and $\epsilon, T \geq 0$, an $(\epsilon, T)$-chain in $H$ is a tuple $\chi = (N, \tau, \eta, \gamma)$ of

4.1. Jump times: a nondecreasing sequence $\tau = (\tau_j)_{j=0}^{N+1} \subseteq \mathbb{R}$ where $N \in \mathbb{N}_{\geq 1}$ and $\tau_0 = 0$.

4.2. Continuous jump times: a subsequence $(\tau_{\eta_k})_{k=0}^M$ of $\tau$ such that $\eta_0 = 0, \eta_M \leq N$, and $\tau_{\eta_k} - \tau_{\eta_{k-1}} \geq T$ for all $k \geq 1$.

4.3. Flow arcs: a sequence $(\gamma_j)_{j=0}^N$ of continuous maps $\gamma_j : [\tau_j, \tau_{j+1}) \rightarrow I$ such that $\gamma_j([\tau_j, \tau_{j+1})) \subseteq F$ and the restriction $\gamma_j|_{[\tau_j, \tau_{j+1})} : [\tau_j, \tau_{j+1}) \rightarrow F$ is a trajectory segment for the local semiflow $\varphi$. In addition, we require the following:

4.3.1. Continuous-time jump condition: If $j = \eta_k$ for some $k \geq 1$, then $\gamma_j-1(\tau_j) \in F$ and $\text{dist}(\gamma_j(\tau_j), \gamma_j-1(\tau_j)) \leq \epsilon$.

4.3.2. Reset jump condition: If $1 \leq j \leq N$ and $j \neq \eta_k$ for any $k$, then $\gamma_j-1(\tau_j) \in Z$ and $\text{dist}(\gamma_j(\tau_j), r(\gamma_j-1(\tau_j))) \leq \epsilon$.

As in Definition 2, we call $\gamma_0(0)$ the initial state of $\chi$ and $\gamma_N(\tau_{N+1})$ the final state of $\chi$. We denote the set of $(\epsilon, T)$-chains in $H$, $(\epsilon, T)$-chains with initial state $x$, and $(\epsilon, T)$-chains with initial state $x$ and final state $y$ by $Ch^\epsilon_T, Ch^\epsilon_T(x)$ and $Ch^\epsilon_T(x, y)$, respectively. For $\chi = (N, \tau, \eta, \gamma) \in Ch^\epsilon_T$ and $0 \leq t \leq \tau_{N+1}$, we write $\chi(t) = \{\gamma_j(t) \mid 0 \leq j \leq N, \tau_j \leq t \leq \tau_{j+1}\}$.

Remark 4. Note that, unlike Definition 2 (and [CGKS19, Def. 2.18]), Definition 4 requires that $N \geq 1$. I.e., we require that an $(\epsilon, T)$-chain consist of at least two arcs.

Definition 5 (Hybrid Conley relation). Let $H = (I, F, Z, \varphi, r)$ be an MHS. The (hybrid) Conley relation $Ch_H \subseteq I \times I$ is defined by

$$(x, y) \in Ch_H \iff \forall \epsilon, T > 0 : Ch^\epsilon_T(x, y) \neq \emptyset.$$ 

As is standard with relations, we sometimes use the more intuitive notation $Ch_H(x, y)$ in place of $(x, y) \in Ch_H$.

Remark 5. Let $H = (I, F, Z, \varphi, r)$ be a THS such that $I$ is metrizable and compact. Then—since all compatible extended metrics on a compact metrizable space are uniformly equivalent—the Conley relation is independent of the choice of compatible metric on $I$ making $H$ into an MHS. Since the hypotheses for our main results (Theorems 1 and 2) include the assumption that $I$ is compact, the specific choice of extended metric is immaterial for the majority of our purposes in this paper.

The following two examples show that our hybrid Conley relation generalizes the classical notions in the discrete-time and continuous-time settings.

Example 1. Discrete-time (semi-)dynamical systems are instances of THS $H = (I, F, Z, \varphi, r)$ where $I = Z$ and $F = \emptyset$. If $H$ is also an MHS, then for any chain $\chi = (N, \tau, \eta, \gamma) \in Ch^\epsilon_T(x, y)$, we always have $(\eta_k) = (0)$ and $\tau_j = 0$ for all $j$ (because time never elapses). Each arc $\gamma_j$ is degenerate and corresponds to a single
An $(\epsilon, T)$-chain from $x$ to $y$

\[ x = \gamma_0(0) \]

\[ x = \gamma_0(0) \quad \rightarrow \quad \gamma_1(\tau_1) \quad \rightarrow \quad \gamma_2(\tau_2) \quad \rightarrow \quad \gamma_3(\tau_3) \quad \rightarrow \quad \gamma_4(\tau_4) \quad \rightarrow \quad \gamma_5(\tau_5) \quad \rightarrow \gamma_6(\tau_6) = y \]

Figure 2. An $(\epsilon, T)$-chain from $x \in I$ to $y \in I$ with $N = 5$. In this example, both $x, y \in F \subseteq I$. All open balls are of radius $\epsilon$, and we have $\gamma_1, (\tau_1 - \tau_3), (\tau_5 - \tau_4) \geq T$, but $(\tau_4 - \tau_3) < T$. In this example, the subsequence $(\eta_k)_{k=0}^M$ is given by $\eta_0 = 0, \eta_1 = 1, \eta_2 = 4,$ and $\eta_3 = 5$ so that there are $M = 3$ continuous-time jumps. Note that the final arc is not required to be degenerate (a single point), in contrast with the classical definitions for continuous-time (semi-)dynamical systems [Con78, Hur95]; however, Example 2 shows that this difference is immaterial in the continuous-time setting as far as the Conley relation (Definition 5) is concerned. Note also that the “double jump” $r(\gamma_2(\tau_2)) \sim (\gamma_3(\tau_3), \gamma_4(\tau_4))$ is permitted, even though $(\tau_4 - \tau_3) < T$, because $(\tau_4 - \tau_3) = (\tau_4 - \tau_3, \tau_4) \leq T$. Later in this paper we will also consider nice $(\epsilon, T)$-chains in which “double jumps” are not allowed; see §6.1 and Figure 7.

point. Thus $T$ is irrelevant, and we recover the usual notion of an $\epsilon$-chain for a discrete-time system which can also be specified by the more standard notation

\[ \chi = (x_0 = x, x_1, \ldots, x_N = y), \]

where $x_i = \gamma_i(\tau_i) = \gamma_i(0)$ for all $0 \leq i \leq N$. Thus, our notion of $(\epsilon, T)$-chain restricts to the classical notion when $H$ is a discrete-time system.

Example 2. Suppose $H = (I, F, Z, \varphi, r)$ is a continuous-time (semi-)dynamical system. That is, $I = F$ and $Z = \emptyset$. Then if $H$ is also an MHS, the classical notion of an $(\epsilon, T)$-chain for a semiflow [Con78, Hur95] is usually expressed (modulo indexing conventions) as a tuple

\[ \chi = (x_0 = x, x_1, \ldots, x_N = y; t_1, \ldots, t_N), \]

where each $t_i \geq T$ and $\text{dist}(\varphi^{t_i}(x_{i-1}), x_i) < \epsilon$ for all $1 \leq i \leq N$. It is easy to see that a classical $(\epsilon, T)$-chain always corresponds to an $(\epsilon, T)$-chain as in Definition 4; however, the converse does not hold. This is because a classical $(\epsilon, T)$-chain always ends with a jump [Con78, Hur95]—or equivalently, using the terminology of Definition 4, ends with a degenerate arc—but an $(\epsilon, T)$-chain in the sense of Definition 4 can end with a nondegenerate arc (see Figure 2).

However, if $I = F$ is compact then the corresponding classical Conley relation is equal to the Conley relation $Ch_H$. Indeed, if $\epsilon, T > 0$, we can use the uniform continuity of $\varphi$ on $[0, T] \times I$ to pick $\delta \in (0, \epsilon)$ such that $\text{dist}(p, q) < \delta$ implies $\text{dist}(\varphi^t(p), \varphi^t(q)) < \epsilon$ for all $t \in [0, T]$. Let $\chi = (N, \tau, \eta, \gamma) \in Ch_H^T(x, y)$; if $\tau_{N+1} - \tau_N \geq T$, then we can produce a classical $(\epsilon, T)$-chain by simply adding the degenerate arc at $\gamma_N(\tau_{N+1})$ to $\chi$. If instead $\tau_{N+1} - \tau_N < T$, then we can produce a classical $(\epsilon, T)$-chain from $x$ to $y$ by replacing the final arc with the degenerate arc at its terminal point, and extending the penultimate arc by $\tau_{N+1} - \tau_N$ (the penultimate arc necessarily exists since $N \geq 1$ by Definition 4).
Definition 6 (Hybrid chain recurrent set). Let \( H = (I, F, Z, \varphi, r) \) be an MHS. The \textit{(hybrid) chain recurrent set} \( R(H) \subseteq I \) is defined by

\[
R(H) = \{ x \in I \mid C_{hH}(x, x) \}
\]

Definition 7 (Hybrid chain equivalence). Let \( H = (I, F, Z, \varphi, r) \) be an MHS. Two points \( x, y \in I \) are \textit{chain equivalent} if \( C_{hH}(x, y) \) and \( C_{hH}(y, x) \). The \textit{chain equivalence class} of \( x \in I \) is the set \( \{ y \in I \mid C_{hH}(x, y) \text{ and } C_{hH}(y, x) \} \). (Note that every chain equivalence class is a subset of \( R(H) \).)

The following definition generalizes the definition of omega-limit set in [LJS+03, Def. II.7] (which considered omega-limit sets of singletons only), in addition to generalizing two of the standard definitions for discrete-time and continuous-time dynamical systems [BS02, Con78, p. 29, II.4.1.B].

Definition 8 (Hybrid omega-limit set). Let \( H = (I, F, Z, \varphi, r) \) be a THS. If \( U \subseteq I \), we define the omega-limit set \( \omega(U) \) of \( U \) via

\[
\omega(U) := \bigcap_{T > 0} \overline{\bigcup_{x \in U} \bigcup_{\chi \in (N, \tau, \gamma) \in E_H(x)} \{ \gamma_{j}(t) \mid j + t \geq T \}}.
\]

If \( x \in I \), we define \( \omega(x) := \omega(\{x\}) \).

Remark 6. If \( I \) is compact and \( U \subseteq I \) is such that every maximal execution \( \chi \in E_H(x) \) with \( x \in U \) is either infinite or Zeno, then \( \omega(U) \) is a decreasing intersection of nonempty compact sets, and is therefore compact and nonempty.

The following lemma shows that, as in the classical setting, an infinite or Zeno execution with initial state \( x \) converges to \( \omega(x) \) if \( I \) is compact.

Lemma 1 (Convergence to omega-limit set). Let \( H = (I, F, Z, \varphi, r) \) be a THS with \( I \) compact. Then for any \( x \in I \) and infinite or Zeno execution \( \chi = (N, \tau, \gamma) \in E_H(x) \), we have

\[
\gamma_j(t) \to \omega(x) \quad \text{as } j + t \to \infty.
\]

That is, for every neighborhood \( U \supseteq \omega(x) \), there exists \( M > 0 \) such that \( \gamma_j(t) \in U \) for all \( j + t > M \).

Proof. Suppose not. Then there exists an open neighborhood \( U \supseteq \omega(x) \) and subsequences \( (j_k)_{k \in \mathbb{N}}, (t_k)_{k \in \mathbb{N}} \) with \( j_k + t_k \to \infty \) such that \( \gamma_{j_k}(t_k) \notin I \setminus U \) for all \( k \). Compactness of \( I \setminus U \) implies that the sequence \( (\gamma_{j_k}(t_k))_{k \in \mathbb{N}} \) has a limit (accumulation) point \( y \in I \setminus U \). But Definition 8 implies that \( y \in \omega(x) \), and \( \omega(x) \) is disjoint from \( I \setminus U \) by the definition of \( U \), so we have a contradiction.

Definition 9 (Forward invariance). Let \( H = (I, F, Z, \varphi, r) \) be a THS and \( S \subseteq I \) a subset. We say that \( S \) is \textit{forward invariant} if for every \( x \in S \), \( \chi \in E_H(x) \), and \( t \geq 0 \), \( \chi(t) \subseteq S \).

The following example shows that, in spite of Lemma 1, hybrid omega-limit sets and chain recurrent sets can display behavior very different from that of a classical (semi-)dynamical system. In particular, the example shows that the hybrid chain recurrent set need not generally contain the “steady state behavior.” Such wild deviance motivates the introduction, in §3, of the \textit{trapping guard condition}; a THS satisfying the trapping guard condition does not suffer from such pathologies.

Example 3. Figure 3 and its caption specify an example which shows that, for a general MHS \( H = (I, F, Z, \varphi, r) \), the following hold: (i) omega-limit sets need not be forward invariant, (ii) omega-limit sets need not be contained in \( R(H) \), (iii) \( R(H) \) need not be forward invariant, and (iv) \( R(H) \) need not be closed. Furthermore, this example shows that these pathologies can occur even if \( I \) is compact and \( H \) is deterministic and nonblocking.

Remark 7. We will later give sufficient conditions (see Corollaries 2 and 4)—involving the soon-to-be-introduced \textit{trapping guard condition}—which ensure that none of the pathologies of Example 3 can occur.

Definition 10 (Hybrid trapping sets and attracting-repelling pairs). Let \( H = (I, F, Z, \varphi, r) \) be a THS. We say a precompact set \( U \subseteq I \) is a \textit{trapping set} if the following hold.
unrelated to grazing, such as the possibility that a wide variety of physically-relevant examples, as shown in the next sections. Third, Example 5 demonstrates that trapping guards, which we define below using the maximum flow time \( \mu: I \to [0, +\infty] \) given by

\[
\mu(x) := \begin{cases} 
\sup\{t \in [0, \infty): (t, x) \in \text{dom}(\phi)\}, & x \in F \\
0, & x \not\in F.
\end{cases}
\]

**Definition 11** (Trapping guards). Let \( H = (I, F, Z, \varphi, r) \) be a deterministic THS. Let \( \text{cl}(\text{dom}(\varphi)) \) be the closure of \( \text{dom}(\varphi) \) in \([0, \infty) \times I\). We say that \( Z \) is a **flow-induced retract** if there exists a neighborhood \( U \subseteq I \) of \( Z \) and a continuous retraction \( \rho: U \to Z \) (\( \rho|_Z = \text{id}_Z \)) such that (i) the \( U \)-restricted maximum flow time \( \mu|_U: U \to \mathbb{R} \) is continuous, and (ii) such that \( \varphi|_{\text{dom}(\varphi) \cap ([0, \infty) \times U)} \) admits a unique continuous extension \( \hat{\varphi} \) to \( \text{cl}(\text{dom}(\varphi)) \cap ([0, \infty) \times U) \) given by

\[
\hat{\varphi}(t, x) = \begin{cases} 
\varphi^t(x), & (t, x) \in \text{dom}(\varphi) \cap ([0, \infty) \times U) \\
\rho(x), & x \in U, t = \mu(x).
\end{cases}
\]

We say that \( \rho: U \to Z \) is a **flow-induced retraction**. If \( Z \) is a flow-induced retract, we also say that \( H \) has a **trapping guard** \( Z \) or that \( H \) satisfies the **trapping guard condition**.

**Remark 8.** Note that, in particular, the trapping guard condition rules out the possibility that trajectories of a continuous extension of \( \varphi \) “graze” \( Z \). However, the trapping guard condition also rules out behavior unrelated to grazing, such as the possibility that \( Z \) repel trajectories of \( \varphi \) initialized near \( Z \).

We justify Definition 11 in several ways. First, Example 3 shows that, in the absence of the trapping guard condition, omega-limit and chain recurrent sets need not satisfy many of the properties which are standard in the setting of classical dynamical systems. Second, THS satisfying this condition encompass a wide variety of physically-relevant examples, as shown in the next sections. Third, Example 5 demonstrates that our main theorems can fail without the trapping guard condition even for very simple MHS, hence
some such condition is necessary. Finally, in Remark 15 and Appendix C we point out that Definition 11 arises naturally from mathematical considerations.

Our second main result involves the notion of a complete Lyapunov function introduced by Conley [Con78, p. 39]; here we generalize the definition to MHS.

**Definition 12** (Hybrid complete Lyapunov function). Let $H = (I, F, Z, \varphi, r)$ be an MHS. A (hybrid) complete Lyapunov function for $H$ is a continuous function $L: I \to \mathbb{R}$ satisfying the following conditions.

12.1. For every $x \in F \setminus R(H)$, $\chi \in E_H(x)$, $t > 0$, and $y \in \chi(t)$, $L(y) < L(x)$.
12.2. If $x \in Z \setminus R(H)$, then $L(r(x)) < L(x)$.
12.3. For all $c \in L(R(H))$, $L^{-1}(c)$ is a chain equivalence class.
12.4. $L(R(H))$ is nowhere dense in $\mathbb{R}$.

We now state our main results, but postpone the proofs to §6.4.

**Theorem 1** (Conley’s decomposition theorem for MHS). Let $H = (I, F, Z, \varphi, r)$ be a deterministic metric hybrid system. Assume that $I$ is compact and that $Z$ is a trapping guard. Further suppose that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Then the hybrid chain recurrent set $R(H)$ admits a Conley decomposition:

$$R(H) = \bigcap \{A \cup A^* | A \text{ is an attracting set for } H\}.$$  

Furthermore, $x, y \in I$ are chain equivalent if and only if either $x, y \in A$ or $x, y \in A^*$ for every attracting-repelling pair $(A, A^*)$.

**Theorem 2** (Conley’s fundamental theorem for MHS). Let $H = (I, F, Z, \varphi, r)$ be a deterministic metric hybrid system. Assume that $I$ is compact and that $Z$ is a trapping guard. Further suppose that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Then there exists a complete Lyapunov function for $H$.

Simple examples illustrating these theorems—including an example which shows that the conclusions can fail without the trapping guard condition—are given in §5. §4 contains propositions which guarantee that Theorems 1 and 2 apply to various general classes of hybrid systems appearing in the literature.

**Remark 9.** As discussed in Examples 1 and 2, a discrete-time dynamical system is a deterministic hybrid system with $I = Z$, and a continuous-time dynamical system is a deterministic hybrid system with $I = F$. Viewed as hybrid systems, a continuous-time system has only infinite maximal executions, a discrete-time system has only Zeno maximal executions, and in both cases the trapping guard condition of Theorems 1 and 2 is satisfied vacuously. Additionally, Example 2 shows that, although Definition 4 does not specialize to the classical definition of $(\epsilon, T)$-chains in the continuous-time setting [Con78, Hur95], the Conley relations defined using both definitions coincide. Hence Theorems 1 and 2 strictly generalize the corresponding theorems of [Con78, Fra88].

**Remark 10.** Let $H = (I, F, Z, \varphi, r)$ be an MHS not satisfying the hypotheses of Theorems 1 and 2; e.g., $I$ could be noncompact. If there is a compact, forward invariant subset $K \subseteq I$, then the theorems can be applied to the hybrid system $H_K = (I \cap K, F \cap K, Z \cap K, \varphi_K, r|_{Z \cap K})$ (where $\varphi_K$ is the restriction of $\varphi$ to $\text{dom}(\varphi) \cap ([0, \infty) \times K)$) as long as $H_K$ satisfies the hypotheses of the theorems.

4. Applications

This section assumes some basic knowledge of smooth manifold theory [Lee13] (and, for Corollary 1, geometric mechanics [MR94]), and can safely be skipped by the reader whose background and/or motivation are lacking.
4.1. Preliminaries: smooth hybrid manifolds and related objects. Before proceeding to the results of this section, we first discuss some formalities for the purpose of considering state spaces which are disjoint unions of manifolds with different dimensions.\(^5\) We refer the reader to [Lee13] for all of the definitions in the standard setting of smooth manifolds with constant dimension.

Let \( J \) be an arbitrary index set. Similarly to [BRS15, JBK16, Sec. III.A, Sec. A.4],\(^6\) we define a smooth hybrid manifold (with boundary) \( M \) to be a disjoint union \( M = \bigsqcup_{j \in J} M_j \) of smooth, paracompact manifolds (with boundary) having possibly different dimensions. Given any other smooth hybrid manifold \( N = \bigsqcup_{j' \in J'} N_{j'} \) and \( k \in \mathbb{N}_{\geq 1} \cup \{+\infty\} \), we will say that a map \( f : M \to N \) is \( C^k \) if, for each \( j \in J \), there exists \( j' \in J' \) such that \( f(M_j) \subseteq N_{j'} \) and \( f|_{M_j} : M_j \to N_{j'} \) is \( C^k \).

Let \( \pi_j : T M_j \to M_j \) be the tangent bundles, we define the hybrid tangent bundle \( \pi : T M \to M \) by \( T M := \bigsqcup_{j \in J} T M_j \) and \( \pi := \bigsqcup_{j \in J} \pi_j \). We define a continuous vector field on \( M \) to be a continuous section \( X : M \to T M \) of the hybrid tangent bundle, so that each \( X|_{M_j} \) is a continuous vector field on \( M_j \). If \( M = \bigsqcup_{j \in J} M_j \) is a smooth hybrid manifold with boundary, we define the hybrid boundary \( \partial M \) of \( M \) to be \( \partial M := \bigsqcup_{j \in J} \partial M_j \). If \( X \) is a vector field on \( M \) and \( S \subseteq \partial M \), we say that \( X \) is strictly (resp. non-strictly) outward-pointing on \( S \) if each \( X|_{M_j} \) is strictly (resp. non-strictly) outward-pointing on \( S \); similarly, \( X \) is strictly (resp. non-strictly) inward-pointing on \( S \) if each \( X|_{M_j} \) is strictly (resp. non-strictly) inward-pointing on \( S \).

We say that the vector field \( X \) on \( M \) is locally Lipschitz if each \( X|_{M_j} \) is locally Lipschitz.\(^7\) An integral curve of \( X \) is an integral curve of some \( X|_{M_j} \). We say that \( X \) is a complete, locally Lipschitz (resp. \( C^k \)) vector field if each \( X|_{M_j} \) is complete; in this case, \( X \) generates a continuous (resp. \( C^k \)) flow \( \Phi : \mathbb{R} \times M \to M \) such that, for \( x \in M \), \( t \mapsto \Phi^t(x) \) is the integral curve of \( X \) with \( \Phi^0(x) = x \).

Given a \( C^k \) function \( \psi : M \to \mathbb{R} \) and a complete \( C^k \) vector field \( X \) on \( M \) with flow \( \Phi : \mathbb{R} \times M \to M \), we let \( L_X \psi := \bigsqcup_{j \in J} L_{X_j}(\psi|_{M_j}) = \frac{d}{dt}(\psi \circ \Phi^t)|_{t=0} \) denote the Lie derivative of \( \psi \) along \( X \); for all \( m \in \{2, \ldots, k\} \) we inductively define \( L_X^m \psi := L_X(L_X^{m-1} \psi) \).

Most other standard operations on manifolds, functions, and sections of fiber bundles can be straightforwardly transferred to this hybrid setting by taking disjoint unions where applicable.

4.2. General classes of MHS to which Theorems 1 and 2 apply. The following proposition provides a general class of hybrid systems to which Theorems 1 and 2 apply. We refer to hybrid systems satisfying the hypothesis of this proposition as smooth exit-boundary guarded THS, and by smooth exit-boundary guarded MHS if a compatible extended metric on \( I \) is also specified.

Proposition 1. Let \( H = (I, F, Z, \varphi, r) \) be a THS for which \( I = \bigsqcup_{j \in J} M_j \) is a smooth hybrid manifold with boundary, \( Z \) is a clopen subset of \( \partial I \), \( F := I \setminus Z \), and the local semiflow \( \varphi \) is generated by a locally Lipschitz vector field \( X \) on \( I \).

Assume that \( X \) is non-strictly inward pointing at every point of \( \partial I \setminus Z \), and is non-strictly outward pointing at every point of \( Z \). Further assume that, for each \( z \in Z \), the maximal integral curve \( t \mapsto \sigma(t) \) of \( X \) satisfying \( \sigma(0) = z \) is not defined for any positive values of \( t \). Then \( H \) satisfies the trapping guard condition. In particular, if \( I \) is compact then \( H \) admits a complete Lyapunov function and \( R(H) \) admits a Conley decomposition.

Remark 11. The conditions that (i) \( X \) point non-strictly outward on \( Z \) and (ii) “the maximal integral curve \( t \mapsto \sigma(t) \) of \( X \) satisfying \( \sigma(0) = z \) is not defined for any positive values of \( t \)” (for all \( z \in Z \)) are implied by the stronger assumption, appearing in the hybrid systems literature [BRS15, DBK18, CBC19, CB20], that \( X \) be strictly outward pointing at every point of \( Z \).

Proof. For each \( j \in J \) we view \( M_j \subseteq \mathbb{R}^{n_j} \) as a properly embedded submanifold (by Whitney’s theorem), extend \( X_j := X|_{M_j} \) arbitrarily to a locally Lipschitz vector field \( \tilde{X}_j \) on \( \mathbb{R}^{n_j} \) with local flow \( \tilde{\Phi}_j : \text{dom}(\tilde{\Phi}_j) \subseteq \mathbb{R}^{n_j} \) by Whitney’s theorem.\(^8\) Let \( \tilde{X}_j \) be a vector field on \( \mathbb{R}^{n_j} \) such that \( \tilde{X}_j|_{M_j} = X_j \). Then \( \tilde{X}_j \) is a vector field on \( \mathbb{R}^{n_j} \) such that \( \tilde{X}_j|_{M_j} = X_j \).

\(^5\)It seems that the majority of authors define a “smooth manifold” in such a way that all of its connected components are required to have the same dimension. One exception is [Tu10, p. 48] (and see [Tu17] for clarification).

\(^6\)Unlike the definitions in [BRS15, JBK16, Sec. III.A, Sec. A.4], here we do not require the index set \( J \) to be finite or even countable. Of course, \( J \) will necessarily be finite if \( M \) is compact.

\(^7\)This is a well-defined, metric-independent notion which depends only on the smooth structures of the \( M_j \) [KR19, Rem. 1].
$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and let $\pi^j_2: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto the second factor. Then

$$U_j := M_j \cap \pi^j_2 \left( (\hat{\Phi}_j)^{-1}(\mathbb{R}^n \setminus M_j) \right) \subseteq M_j$$

is an open neighborhood of $Z \cap M_j$ in $M_j$ such that (i) for all $x \in U_j$ and $T \geq 0$,

$$\left[ \hat{\Phi}_j^{[0,T]}(x) \subseteq \text{cl}(U_j) \subseteq M_j \right] \implies \left[ \hat{\Phi}_j^{[0,T]}(x) \subseteq U_j \right],$$

and (ii) for every $x \in U_j$, there exists $t > 0$ such that $\hat{\Phi}_j^t(x) \in Z \cap M_j$. Since also $Z \cap M_j$ is closed in $U_j$ and every point of $Z \cap M_j$ immediately flows into $\mathbb{R}^n \setminus M_j$, (i) and (ii) imply (using the terminology of [Con78, p. 24]) that $U_j$ is a Wazewski set for $\hat{\Phi}_j$ with eventual exit set $U_j$ and immediate exit set $Z \cap M_j$. In this case, the proof of Wazewski’s theorem [Con78, p. 25] and the definition of the disjoint union topology show that $Z$ is a trapping guard with $\bigcup_{j \in J} U_j$ the domain of a flow-induced retraction. If $I$ is also compact, then Theorems 1 and 2 imply the final statement of the proposition.

Proposition 1 gives one broad class of MHS which satisfy the hypotheses of Theorems 1 and 2. In the following Proposition 2, we present another broad class of MHS to which these theorems also apply. We then specialize Proposition 2 to a class of Lagrangian hybrid systems in Corollary 1. Instances of the latter class of systems (or slight variations thereof) have been studied, for example, in [GAP01, WGK03, AZGS06, PG09, OA10, BCC17, RBCG17]. Looking ahead to §5, this class of systems substantially generalizes the “gravitational-force bouncing ball” of Example 6 (but does not encompass systems like the “spring-force bouncing ball” of Example 7).

**Proposition 2.** Let $M$ be a smooth hybrid manifold, $k \in \mathbb{N} \cup \{+\infty\}$, $X$ be a complete $C^k$ vector field on $M$ with flow $\Phi: \mathbb{R} \times M \to M$, and $\psi: M \to \mathbb{R}$ be a $C^k$ function. We define $I := \psi^{-1}([0, \infty))$, $Z := \psi^{-1}(0) \cap (L_X \psi)^{-1}((\infty, 0])$, and $F := I \setminus Z$. We assume given a continuous map $r: Z \to I$. Defining the local semiflow $\varphi$ to be the restriction of the flow $\Phi$ to $\Phi^{-1}(F) \cap (0, \infty) \times F$ and equipping $M$ with any compatible extended metric yields an MHS $H = (I, F, Z, \varphi, r)$. $H$ is deterministic since $F \cap Z = \emptyset$, and every maximal execution in $H$ is either infinite or Zeno since $I = F \cup Z$ and $X$ is complete.

Further assume the following:

- There exists a compact set $K \subseteq I$ such that $r(Z \cap K) \subseteq K$ and, for all $x \in K$ and $T \geq 0$,

$$\left[ \Phi^{[0,T]}(x) \subseteq I \right] \implies \left[ \Phi^{[0,T]}(x) \subseteq K \right].$$

- For all $x \in Z \cap K \cap (L_X \psi)^{-1}(0)$, there exists an integer $m \in [2, k]$ such that

$$L_X^1 \psi(x) = \ldots = L_X^{m-1} \psi(x) = 0 \quad \text{and} \quad L_X^m \psi(x) < 0. \tag{4}$$

Then the restricted system

$$H_K = (I_K, F_K, Z_K, \varphi_K, r_K) = (I \cap K, F \cap K, Z \cap K, \varphi_K, r|_{Z \cap K})$$

(\text{where } \varphi_K \text{ is the restriction of } \varphi \text{ to } \text{dom}(\varphi) \cap (0, \infty) \times K) \text{ is well-defined, } H_K \text{ admits a complete Lyapunov function, and } R(H_K) \text{ admits a Conley decomposition.}

**Proof.** The first bulleted condition above implies that $H_K$ is a well-defined MHS, and the other hypotheses imply that $H_K$ is deterministic and that every $H_K$ execution is either infinite or Zeno.

We now prove that $Z \cap K$ is a trapping guard for $H_K$ by constructing a neighborhood $U$ of $Z \cap K$ in $K$ satisfying the conditions of Definition 11. We first define the impact time $\mu: I \to [0, +\infty)$ via

$$\mu(x) := \sup\{t \geq 0 \mid \Phi^{[0,t]}(x) \subseteq I\} = \inf\{t \geq 0 \mid \Phi^t(x) \in M \setminus I\}, \tag{5}$$

where the second equality holds because the mean value theorem, the definition of $Z$, and continuity of $\Phi$ imply that (i) $\Phi^{[0,t]}(z) \not\subseteq I$ for every $z \in Z$ and $\epsilon > 0$ and (ii) $\Phi^{\mu(x)}(x) \in Z$ for every $x \in I$.

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8This follows since $\mathbb{R}^n \setminus M_j$ is open, $\hat{\Phi}_j$ is continuous, $\text{dom}(\hat{\varphi}_j)$ is open, $\pi^j_2$ is an open map, and every $z \in Z \cap M_j$ immediately flows into $\mathbb{R}^n \setminus M_j$; this last property follows from the assumption that the $X$-integral curve through every $z \in Z$ is not defined for any positive times.

9Conley called this condition positive invariance of $K$ relative to $I$ [Con78, p. 46].
We now show that $Z \cap K$ has a neighborhood $U$ in $K$ such that $\mu(x) < \infty$ for every $x \in U$. Suppose not. Then, using (5) and the definition of $I$, for some $z \in Z \cap K$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq K$ with $x_n \to z$ and $\psi(\Phi^t(x_n)) \geq 0$ for all $t \geq 0$. This and the fact that $Z \subseteq \psi^{-1}(0)$ imply that $\psi(\Phi^t(z)) = 0$ for all $t \geq 0$. Thus, all Lie derivatives of $\psi$ at $z$ vanish, contradicting (4).

Next, we show that $\mu|_U$ is continuous. For any $x \in U$ and $\epsilon > 0$, there exists $T$ in $(\mu(x), \mu(x) + \epsilon)$ with $\Phi^T(x) \in M \setminus I$ by (5). Since $M \setminus I$ is open in $M$, the continuity of $\Phi^T$ yields a neighborhood $V \ni x$ with $\Phi^T(V) \subseteq M \setminus I$. Hence $\mu(V) \subseteq [0, T] \subseteq [0, \mu(x) + \epsilon]$, so $\mu|_U$ is upper semicontinuous. To show that $\mu|_U$ is lower semicontinuous, since $(\mu|_U)^{-1}(0) = Z \cap K$ it suffices to show that, for every $x \in U \setminus Z$ and every $\epsilon \in (0, \mu(x))$, $x$ has a neighborhood $V$ with $\mu(V) \subseteq (\mu(x) - \epsilon, \infty)$. Fix $x \in U \setminus Z$. By taking $\epsilon$ smaller if necessary, we may assume that $0 < \epsilon < \mu(x)$; pick $T \in (\mu(x) - \epsilon, \mu(x))$. By (5), we have $\Phi^{0, T}(x) \subseteq I \setminus Z$. By the continuity of $\Phi$, compactness of $[0, T]$, and openness of $I \setminus Z$ in $I$, there exists a neighborhood $V \ni x$ with $\Phi^{0, T}(V) \subseteq I \setminus Z$, so $\mu(V) \subseteq [T, \infty) \subseteq (\mu(x) - \epsilon, \infty)$ as desired.

It is easy to see that the definition (5) of $\mu$ coincides with that of (1). Since $\mu|_U$ is continuous and $\mu|_{Z \cap K} \equiv 0$, the flow induced-retraction $\rho: U \to Z \cap K$ defined by $\rho(x) := \Phi^{\mu(x)}(x)$ is continuous. Hence $\hat{\varphi}_K := \Phi|_{\text{cl(dom}(\varphi_K)) \cap ([0, \infty) \times U)}$ is a continuous extension of $\varphi_K$ to $\text{cl(dom}(\varphi_K)) \cap ([0, \infty) \times U)$ as required in Definition 11, and this extension is unique since $M$ is Hausdorff. Thus, $Z \cap K$ is a trapping guard for $H_K$, so $H_K$ satisfies all hypotheses of Theorems 1 and 2 (c.f. Remark 10). This completes the proof.  

We now specialize Proposition 2 to show that Theorems 1 and 2 can also be applied to a broad class of mechanical systems with unilateral constraints which undergo impacts, and which generalize the bouncing ball system that we will study in Example 6. Slightly generalizing [AZGS06, Def. 1], we define a hybrid Lagrangian to be a tuple

$$L = (Q, L, h),$$

where:

- $Q$ is a smooth hybrid manifold (the configuration space) with hybrid tangent bundle $\pi: TQ \to Q$;
- $L: TQ \to \mathbb{R}$ is a smooth, hyperregular Lagrangian [MR94, Sec. 7.3–7.4] (so that the associated Lagrangian vector field $X_L: TQ \to T(TQ)$ is well-defined and smooth (and second order: $T\pi \circ X_L = \text{id}_{TQ}$), and
- $h: Q \to \mathbb{R}$ is a smooth function.

We assume that the vector field $X_L$ is complete, so that it generates a smooth flow $\Phi: \mathbb{R} \times TQ \to TQ$.

**Corollary 1.** Let $E: TQ \to \mathbb{R}$ be the (total) energy associated to a hybrid Lagrangian $L = (Q, L, H)$; i.e., $E$ is the pullback of the Hamiltonian associated to $L$ via the Legendre transform [MR94, p. 183, p. 186]. Define $M := TQ, X := X_L, \psi := h \circ \pi; I, F, \varphi$ and $\varphi$ are then specified as in Proposition 2. Assuming $r: Z \to I$ is a given continuous map and endowing $TQ$ with any compatible extended metric, we obtain a deterministic MHS $H = (I, F, Z, \varphi, r)$ associated to $L$, such that every maximal execution is either infinite or Zeno.

Further assume the following:

- There exists $E_0 \in \mathbb{R}$ such that some connected component $K := S_{E_0}$ of $I \cap E^{-1}((-\infty, E_0])$ is compact and satisfies $r(Z \cap S_{E_0}) \subseteq S_{E_0}$.
- The set $Z \cap S_{E_0} \cap (L_{X_L}(h \circ \pi))^{-1}(0)$ satisfies the condition containing Equation (4).

Then the restricted system

$$H_{E_0} = (I_{E_0}, F_{E_0}, Z_{E_0}, \varphi_{E_0}, r_{E_0}) := (I \cap S_{E_0}, F \cap S_{E_0}, Z \cap S_{E_0}, \varphi_{E_0}, r|_{Z \cap S_{E_0}})$$

(where $\varphi_{E_0}$ is the restriction of $\varphi$ to $\text{dom}(\varphi) \cap ([0, \infty) \times S_{E_0})$) is well-defined, $H_{E_0}$ admits a complete Lyapunov function, and $R(H_{E_0})$ admits a Conley decomposition.

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10As opposed to [AZGS06, Def. 1], we do not require $Q$ to be a manifold of fixed dimension, and we do not require that $h^{-1}(0)$ be a smooth manifold. We could have also required less smoothness of $L$ and $h$, but we do not bother with this here; the interested reader can refer to Proposition 2 for more refined smoothness assumptions.

11In [AZGS06], the reset map $r$ is given a rather specific definition, which generalizes the reset map of Example 6, but we do not require the use of this specific definition here.
Figure 4. Depicted here are objects associated with the MHS $H = (I, F, Z, \varphi, r)$ of Example 4. $I$ is the pink shaded compact region contained in the $x$-$y$ plane. $F$ is the interior of $I$, and $Z$ is the circle shown in blue. A portion of the unique execution in the hybrid system of Example 4 with initial condition having polar coordinates $(\rho_0, \theta_0) = (\frac{1}{1000}, \frac{\pi}{2})$ is shown in black. Shown above is the graph of the complete Lyapunov function $L$ given by Equation (6) with choice of constants $a = \frac{1}{5}, b = -\frac{1}{10}, c = \frac{3}{10}, d = \frac{3}{10}$.

Proof. Conservation of energy [MR94, Prop. 7.3.1] implies that $E^{-1}((-\infty, E_0])$ is forward invariant under the flow of $X_{L}$, and so the first bulleted condition implies that $S_{E_0}$ is forward invariant for $H$; hence we may apply Proposition 2 to the restricted system $H_{E_0}$. \hfill \Box

5. Examples

We begin with a toy example (Example 4) which illustrates an easy application of Theorems 1 and 2. We then show in Example 5 that, even if an MHS satisfies all hypotheses of Theorems 1 and 2 except the trapping guard condition, the conclusions of both theorems can fail. We then proceed to give two examples motivated by toy models in classical mechanics. The first (Example 6, a special case of Corollary 1) illustrates a system to which Theorems 1 and 2 apply. The second (Example 7) illustrates a system to which they do not.

Example 4. Using polar coordinates $x = \rho \cos \theta, y = \rho \sin \theta$ on the plane, let $X$ be the complete $C^1$ vector field on $\mathbb{R}^2$ given by

$$X(\rho, \theta) = \rho (3 - \rho) \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \theta},$$

and let $\Phi: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the $C^1$ flow generated by $X$. From this data we now construct a (relatively tame) MHS satisfying the hypotheses of Theorems 1 and 2.

Let $I$ be the region $I = \{0 \leq \rho \leq 1\} \cup \{2 \leq \rho \leq 4\}$ consisting of the closed unit disk together with the annulus bounded by the circles of radius 2 and 4 centered at the origin; we endow $I$ with the metric induced by the Euclidean distance. Let $F$ be the interior of $I$, and let $Z$ be the unit circle; see Figure 4. Let $\varphi$ be the continuous semiflow given by the restriction of $\Phi$ to $(\Phi)^{-1}(F) \cap (F \times [0, \infty))$, and let the reset
of Definition 12, we have \( \implies \) that Theorem 2 is also violated. Indeed, suppose such an attracting-repelling pairs. It follows that the conclusions of Theorem 1 are violated.

**Example 6** that every subset \( r \) are satisfied. Define the MHS Theorems 1 and 2 can fail if the trapping guard hypothesis is violated, even if all of the other hypotheses are satisfied. Define the MHS (Failure in the absence of trapping guards) of Example 5. The hybrid system \( H = (I,F,Z,\varphi,r) \) as follows. Let

\[
I := [-1,0] \cup [1,3] \subseteq \mathbb{R}, \quad Z := \{0,2,3\}, \quad F := I \setminus Z.
\]

We let \( \varphi \) be generated by the vector field \( -t \frac{\partial}{\partial t} \) on \([-1,0]\) and by the constant vector field \( \frac{\partial}{\partial t} \) on \([1,3]\). We define \( r: Z \to I \) by \( r(0) = 1 \) and \( r(2) = r(3) = -1 \). Clearly \( H \) is deterministic and nonblocking.

It is easy to check that \( R(H) = [-1,0] \cup [1,2] \), that \( R(H) \) consists of a single chain equivalence class, and that every subset \( J \subseteq I \) satisfies \( \omega(J) = \{0\} \). But \( \{0\} \) is not forward invariant, so there are no nontrivial attracting-repelling pairs. It follows that the conclusions of Theorem 1 are violated.

We further claim that no complete Lyapunov function \( L: I \to \mathbb{R} \) for \( H \) exists, so that the conclusion of Theorem 2 is also violated. Indeed, suppose such an \( L \) exists. By the continuity of \( L \) and Condition 12.1 of Definition 12, we have \( L(2) > L(3) \). By Condition 12.2, we have \( L(3) > L(-1) \). Finally, Condition 12.3 implies that \( L(-1) = L(2) \). This implies that \( L(2) > L(3) > L(-1) = L(2) \), a contradiction.

We emphasize that \( H \) satisfies all hypotheses of Theorems 1 and 2 except for the trapping guard condition, and \( H \) violates the conclusions of both of these theorems.

**Example 5**. Let \( g > 0 \) and \( X \) be the complete analytic vector field on \( \mathbb{R}^2 \) given by

\[
X(x,y) = y \frac{\partial}{\partial x} - g \frac{\partial}{\partial y}.
\]

If we think of \( x \) as the vertical position of a point particle with unit mass moving under the influence of gravity, and think of \( y \) as its velocity, then the dynamics determined by \( X \) are equivalent to those determined by Newton’s second law of motion, \( \ddot{x} = -g \). Letting \( \Phi: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \) be the analytic flow.
generated by $X$, we now construct an MHS which represents a “bouncing ball,” and we show that this MHS satisfies all hypotheses of Theorems 1 and 2 after restriction to a compact forward invariant set.

Let $I$ be the closed half plane $I = \{x \geq 0\}$; we endow $I$ with the metric induced by the Euclidean distance. Let $F$ be the interior of $I$, and define $Z := \{(0, y) \mid y \leq 0\}$. Let $\varphi$ be the continuous semiflow given by the restriction of $\Phi$ to $(\Phi)^{-1}(F) \cap (F \times [0, \infty))$, and let the reset $r: Z \to I$ be given by

$$r(0, y) := (0, -dy),$$

where $d \in [0, 1]$ is the coefficient of restitution (which is related to the energy lost by the ball at impact). Clearly the MHS $H = (I, F, Z, \varphi, r)$ is deterministic.

We now verify that $H$ satisfies the trapping guard condition of Definition 11. We define the neighborhood $U$ of Definition 11 to be $U := I$. Integrating the vector field $X$ analytically yields

$$\Phi^t(x, y) = \left(x + yt - \frac{g}{2} t^2, y - gt\right).$$

Setting the first component on the right hand side equal to 0 and solving the resulting quadratic equation, we find that the maximum flow time $\mu: I \to [0, +\infty]$ (Equation 1) is given by

$$\mu(x, y) = \frac{y + \sqrt{y^2 + 2xg}}{g},$$

which is clearly continuous. We define the continuous flow-induced retraction $\rho: I \to Z$ by

$$\rho(x, y) := \Phi^\mu(x, y)(x, y) = (0, -\sqrt{y^2 + 2xg}).$$

Hence $\hat{\varphi} := |_{\text{cl(dom}(\varphi)) \cap [0, \infty) \times I}$ is a continuous extension of $\varphi$ to the closure of dom($\varphi$) in $[0, \infty) \times I$, as required in Definition 11, and this extension is unique since $\mathbb{R}^2$ is Hausdorff.
The above shows that $H$ satisfies the trapping guard condition. To investigate Zeno executions, we compute the total time elapsed during the execution initialized at $(x, y)$:

$$
\sum_{n=0}^{\infty} \mu \circ (r \circ \rho)^{on}(x, y) = \sum_{n=0}^{\infty} \mu \left( 0, 0, \sqrt{y^2 + 2xy} \right) = \frac{2\sqrt{y^2 + 2xy}}{g} \sum_{n=0}^{\infty} d^n.
$$

Since the last sum is finite if and only if $d \in [0, 1)$, it follows that (i) every maximal execution in $H$ initialized in $I \setminus \{0\}$ is Zeno if $0 \leq d < 1$, and (ii) every maximal execution in $H$ initialized in $I \setminus \{0\}$ is infinite if $d = 1$. (For any value of $d \in [0, 1)$, the execution initialized at $0$ is Zeno.)

The preceding shows that $H$ satisfies all hypotheses of Theorems 1 and 2 except for the hypothesis that $I$ is compact. However, for any finite $E_0 > 0$, the “energy” sublevel set $S_{E_0} := \{(x, y) \in \mathbb{I} \mid \frac{1}{2}y^2 + gx \leq E_0\}$ is compact and forward invariant for $H$. Therefore, the restricted system

$$
H_{E_0} = (I_{E_0}, F_{E_0}, Z_{E_0}, \varphi_{E_0}, r_{E_0}) = (I \cap S_{E_0}, F \cap S_{E_0}, Z \cap S_{E_0}, \varphi_{E_0}, r \mid Z \cap S_{E_0})
$$

(where $\varphi_{E_0}$ is the restriction of $\varphi$ to $\text{dom}(\varphi) \cap ([0, \infty) \times S_{E_0})$) satisfies all hypotheses of Theorems 1 and 2 (c.f. Remark 10).

When the coefficient of restitution $d \in [0, 1)$, it is easy to see that the only attracting-repelling pairs for $H_{E_0}$ are the trivial pairs $(A, A^*) = (I_{E_0}, \varnothing)$ and $(A, A^*) = (\varnothing, I_{E_0})$. According to Theorem 1, the chain recurrent set $R(H_{E_0})$ is therefore given by $R(H_{E_0}) = I_{E_0}$. According to Theorem 2, $H_{E_0}$ has a complete Lyapunov function; since $R(H_{E_0}) = I_{E_0}$ and since $I_{E_0}$ is connected, the complete Lyapunov functions are precisely the constant real-valued functions on $I_{E_0}$.

When the coefficient of restitution $d \in [0, 1)$, it is easy to see that $H_{E_0}$ has a unique nontrivial attracting-repelling pair $(A, A^*)$: the attracting set $A$ is the origin $0$, and the dual repelling set $A^*$ is the empty set. According to Theorem 1, the chain recurrent set is therefore given by $R(H_{E_0}) = \{0\}$. According to Theorem 2, $H_{E_0}$ has a complete Lyapunov function. One family of such complete Lyapunov functions are given by a linear combination of the maximum flow time and the square root of the total energy,

$$
L(x, y) = a\mu(x, y) + b\sqrt{\frac{1}{2}y^2 + gx},
$$

where $a, b > 0$ are constants satisfying $\frac{1-d}{\sqrt{2}}b > \frac{2d}{g}a$. The first term on the right strictly decreases along the continuous-time dynamics, while the second term is constant. On $Z \setminus \{0\}$, the first term on the right strictly decreases upon applying the reset map, while the second term strictly increases; however, the inequality $\frac{1-d}{\sqrt{2}}b > \frac{2d}{g}a$ ensures that

$$
L(r(0, y)) - L(0, y) = a \cdot \left( \frac{2d|y|}{g} - 0 \right) + b \cdot \left( \sqrt{\frac{1}{2}d^2y^2} - \sqrt{\frac{1}{2}y^2} \right)
$$

$$
= \left( a \frac{2d}{g} - b \frac{1-d}{\sqrt{2}} \right) |y| < 0,
$$

as desired. Because $L$ does not depend on $E_0$, $L$ is also a complete Lyapunov function for the full system $H$.

In closing, we remark that this example is a special case of the hybrid Lagrangian systems considered in Corollary 1 with (in the notation of the corollary) $Q := \mathbb{R}$, $L(x, y) := \frac{1}{2}y^2 - gx$, $h(x) := x$, and $r(0, y) := (0, -dy)$. Since the restriction of $E(x, y) = \frac{1}{2}y^2 + gx$ to $I$ is a proper function, $S_{E_0}$ is compact for every $E_0 > 0$. In this example, $Z \cap S_{E_0} \cap (\mathcal{L}_{X} \psi)^{-1}(0) = \{0\}$ is just the origin, and (4) is satisfied since $\mathcal{L}_{X}^2(h \circ \pi) = \dot{\pi} = -g < 0$ on all of $I$. Thus, Corollary 1 directly implies that $H_{E_0}$ satisfies all hypotheses of Theorems 1 and 2, although the above “hands-on” verification seems instructive.

Example 7. In this example we consider a system which, while superficially similar to the bouncing ball of Example 6, does not satisfy the trapping guard condition, so the hypotheses of Theorems 1 and 2 are not satisfied. Nevertheless, as we show below, this system does satisfy the conclusions of these two theorems. Thus, Theorems 1 and 2 could potentially be sharpened in future work.
We define the deterministic MHS $H = (I, F, Z, \varphi, r)$ to be given exactly as in Example 6, except we replace the vector field $X$ of Example 6 with

$$X(x, y) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$ 

Intuitively, $X$ is the vector field obtained from Newton’s second law of motion where the gravitational force of Example 6 is replaced by a Hookean spring with unit stiffness and rest position $x = 0$ (so $\dot{x} = -x$).

A computation as in Example 6 shows that, in this example, every maximal execution initialized in $I \setminus \{0\}$ is infinite, while the execution initialized at $0$ is Zeno. On $\mathbb{R}^2 \setminus \{0\}$, $X$ is given in polar coordinates $x = \rho \cos \theta$, $y = \rho \sin \theta$ as

$$X(\rho, \theta) = -\frac{\partial}{\partial \theta}.$$ 

Hence the maximum flow time $\mu: I \to [0, +\infty]$ (Equation 1) is given by

$$\mu(\rho, \theta) = \begin{cases} \theta + \frac{\pi}{2}, & \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \rho \neq 0 \\ 0, & \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \rho = 0 \end{cases},$$

which is discontinuous at the origin $0 \in Z \subseteq I$, so $H$ does not satisfy the trapping guard condition. Since also $0$ belongs to every energy sublevel $S_{E_0} := \{(x, y) \mid \frac{1}{2} \rho^2 \leq E_0\}$ for which $S_{E_0} \cap I \neq \emptyset$, the restricted system

$$H_{E_0} = (I_{E_0}, F_{E_0}, Z_{E_0}, \varphi_{E_0}, r_{E_0}) = (I \cap S_{E_0}, F \cap S_{E_0}, Z \cap S_{E_0}, \varphi_{E_0}, r \mid Z \cap S_{E_0})$$

defined as in Example 6 does not satisfy the hypotheses of Theorems 1 and 2 for any value of $E_0$ such that $S_{E_0} \cap I \neq \emptyset$.

However, in this example it is nevertheless easy to check that the conclusions of Theorem 1 and 2 still hold (for $H$, not merely $H_{E_0}$). When the coefficient of restitution $d = 1$, $H$ has only the trivial attracting-repelling pairs, $R(H) = I$, and any constant function on $I$ is a complete Lyapunov function.

When $d \in [0, 1)$, the only nontrivial attracting-repelling pair for $H$ is $(\{0\}, \emptyset)$, $R(H) = \{0\}$, and one family of complete Lyapunov functions on $I$ is given in polar coordinates by

$$L(\rho, \theta) = a \rho \mu(\rho, \theta) + b \rho,$$

where $a, b > 0$ satisfy $a < \frac{(1-d) b}{d \pi}$. To see that $L$ is indeed a complete Lyapunov function, note that $\rho$ is constant along $\varphi$ trajectories while $\mu$ strictly decreases, and $L$ decreases along resets from $Z \setminus \{0\}$ since

$$L(d\rho, \pi/2) - L(\rho, -\pi/2) = (ad\pi + b(d-1)) \rho < 0.$$

### 6. Proofs of the main results

This section culminates (in §6.4) with the proofs of Theorems 1 and 2. The earlier subsections develop the necessary tools.

#### 6.1. Removing “double jumps” from $(\epsilon, T)$-chains

In order to prove Theorems 1 and 2, we will show that the Conley relation for an MHS descends to the Conley relation for its hybrid suspension, which we introduce in §6.2. To facilitate this, in this section we show that the set of hybrid $(\epsilon, T)$-chains under consideration may be restricted to a “nice” subset for which there must be time-separation between any reset jump and a subsequent continuous-time jump. In other words, nice chains are not allowed to have “double jumps” like the one shown in Figure 2. A nice chain is shown in Figure 7.

**Definition 13.** Let $H$ be an MHS. We say that a chain $\chi = (N, \tau, \eta, \gamma) \in CH^T_H$ is nice if $\tau_{0k} - \tau_{(k-1)} \geq T$ for all $k \geq 1$. We denote the set of nice $(\epsilon, T)$-chains in $H$ by $CH^T_{H}$. We let $\hat{CH}_H$ denote the Conley relation with respect to the set of nice chains, i.e.,

$$(x, y) \in \hat{CH}_H \iff \text{for all } \epsilon, T > 0: \hat{CH}^T_{H}(x, y) \neq \emptyset.$$ 

As in Definition 5, we sometimes use the more intuitive notation $\hat{CH}_H(x, y)$ in place of $(x, y) \in \hat{CH}_H$. 

Lemma 2. Let $H = (I, F, Z, \varphi, r)$ be a deterministic THS satisfying the trapping guard condition. Further suppose that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$.

Then $\mu: I \to [0, +\infty]$ is continuous, the closure $\text{cl}(\text{dom}(\varphi))$ of $\text{dom}(\varphi)$ in $[0, \infty) \times I$ satisfies

$$
\text{cl}(\text{dom}(\varphi)) = \{(t, x) \in [0, \infty) \times \text{cl}(F) \mid t \leq \mu(x)\},
$$

and $\varphi$ has a unique continuous extension $\tilde{\varphi}$ defined on $\text{cl}(\text{dom}(\varphi))$ satisfying $\tilde{\varphi}^{\mu(x)}(x) \in Z$ for all $x \in \text{cl}(F) \cap \mu^{-1}([0, \infty))$.

Furthermore, $\tilde{\varphi}$ satisfies the following conditions, with $t, s \in [0, \infty)$: (i) $\varphi^0 = \text{id}_{\text{cl}(F)}$, (ii) $(t + s, x) \in \text{cl}(\text{dom}(\varphi))$, and (iii) for all $t \in [0, \infty)$ and $x \in \text{cl}(\text{dom}(\varphi))$, $\tilde{\varphi}^{t+s}(x) = \tilde{\varphi}^t(\tilde{\varphi}^s(x))$.

Proof. We first show that $\mu$ is continuous. Letting $U \supseteq Z$ be the domain of a flow-induced retraction, Definition 11 implies that $\mu_U$ is continuous. Since there is an infinite or Zeno execution starting at every $x \in I$, $\mu^{-1}([0, \infty)) = U \cup \bigcup_{t \geq 0}(\varphi^t)^{-1}(U)$ is a union of open subsets of $I$.

Since the restrictions $\mu|_U$ and $\mu|_{(\varphi^t)^{-1}(U)} = t + \mu_U \circ \varphi^t|_{(\varphi^t)^{-1}(U)}$ are all continuous, $\mu$ is continuous on $\mu^{-1}([0, \infty))$. Since $\text{dom}(\varphi)$ is open in $[0, \infty) \times I$ it follows that, for every $x \in \mu^{-1}(+\infty)$ and $T > 0$, there exists a neighborhood $V \ni x$ with $\mu(V) \subseteq [T, +\infty]$. Hence $\mu$ is also continuous at every point of $\mu^{-1}(+\infty)$, so $\mu: I \to [0, +\infty]$ is continuous.

We now show that $\text{cl}(\text{dom}(\varphi))$ is given by (8). Clearly $\text{dom}(\varphi)$ is contained in the sets on both sides of (8). If $(t, x) \not\in \text{dom}(\varphi)$ belongs to the set on the right of (8), then $t = \mu(x)$ since $\mu|_{I \setminus F} \equiv 0$ and the properties of a local semiflow imply that $\{t \mid (t, x) \in \text{dom}(\varphi)\} = [0, \mu(x))$ for all $x \in F$ [HS06, Sec. 1.3]. If $\mu(x) = 0$, then $(t, x) \in \{0\} \times \text{cl}(F) \subseteq \text{cl}(\text{dom}(\varphi))$ since $\{0\} \times F \subseteq \text{dom}(\varphi)$. If $\mu(x) > 0$, then $x \in F$ and $(t, x) \in \text{cl}([0, \mu(x)]) \times \{x\} \subseteq \text{cl}(\text{dom}(\varphi))$ since $[0, \mu(x)] \times \{x\} \subseteq \text{dom}(\varphi)$. Hence the set on the right of (8) is contained in the set on the left. On the other hand, if $(t, x)$ does not belong to the set on the right of (8), then either (i) $t \geq \mu(x)$ or (ii) $x \not\in \text{cl}(F)$. In case (i), continuity of $\mu$ implies that there are neighborhoods $V \ni x$ and $J \ni t$ such that $s \geq \mu(y)$ for all $(s, y) \in J \times V$, so $(J \times V) \cap \text{dom}(\varphi) = \emptyset$, and therefore $(t, x) \not\in \text{cl}(\text{dom}(\varphi))$. In case (ii), $[0, \infty) \times (I \setminus \text{cl}(F))$ is a neighborhood of $(t, x)$ disjoint from $\text{dom}(\varphi)$, so again $(t, x) \not\in \text{cl}(\text{dom}(\varphi))$. Hence the set on the left of (8) is also contained in the set on the right.

---

Figure 7. A nice $(\epsilon, T)$-chain from $x \in I$ to $y \in I$. Notice that, unlike in the $(\epsilon, T)$-chain of Figure 2, no “double jumps” are allowed.
From Remark 3 we have \( I = F \cup Z \), and this implies that \( \mu^{-1}(0) = Z \). Let \( \tilde{\varphi} \) be the unique continuous extension of \( \varphi|_{\text{dom}(\varphi) \cap ([0, \infty) \times U)} \) to \( \text{cl}(\text{dom}(\varphi)) \cap ([0, \infty) \times U) \) ensured by Definition 11. We now define \( \tilde{\varphi} : \text{cl}(\text{dom}(\varphi)) \to I \) via

\[
\tilde{\varphi}^t(x) = \begin{cases} 
\varphi'(x), & (t, x) \in \text{dom}(\varphi), \\
\tilde{\varphi}^s(\varphi^{t-s}(x)), & (t - s, x) \in \varphi^{-1}(U), \\
\tilde{\varphi}^t(x), & x \in U, (t, x) \in \text{cl}(\text{dom}(\varphi))
\end{cases}
\]

where \( s \in [0, t] \) ranges over all admissible values. It is clear that \( \tilde{\varphi} \) is well-defined by the definition of \( \tilde{\varphi} \) and the fact that \( \varphi \) satisfies the properties of a local semiflow. Since \( \tilde{\varphi} \) is defined by a family of continuous functions defined on open subsets of \( \text{cl}(\text{dom}(\varphi)) \), it follows that \( \tilde{\varphi} \) is continuous, so \( \tilde{\varphi} \) is indeed a continuous extension of \( \varphi \) to \( \text{cl}(\text{dom}(\varphi)) \). Uniqueness of \( \tilde{\varphi} \) follows from uniqueness of \( \varphi \) and the local semiflow properties of \( \varphi \).

If \( x \in \text{cl}(F) \cap Z \), then \( \tilde{\varphi}^\mu(x) = x \in Z \) since \( \mu^{-1}(0) = Z \). If instead \( x \in F \cap \mu^{-1}([0, \infty)) \), then \( y := \varphi^\mu(x) - s(x) \in U \) for some \( s \in [0, \mu(x)) \), so (9) and Definition 11 imply that \( \tilde{\varphi}^\mu(x) = \tilde{\varphi}^s(\varphi^{\mu(x)} - s(x)) = \tilde{\varphi}^\mu(y)(y) \in Z \) since \( s = \mu(y) \).

It remains only to verify the claimed properties (i–iii). (i) is immediate from (9), the definition of \( \tilde{\varphi} \), and the fact that \( \varphi^0 = \text{id}_F \). To prove (ii) first notice, since \( \tilde{\varphi}|_{\text{dom}(\varphi)} = \varphi \), the analogous property satisfied by \( \text{dom}(\varphi) \) is equivalent to

\[
\mu(x) = s + \mu(\tilde{\varphi}^t(x)) \quad \text{for all } x \in F \text{ and } s \in [0, \mu(x)].
\]

Taking the limit as \( s \to \mu(x) \) and using continuity of \( \mu \) implies that (10) also holds for \( x \in F \) and \( s \in [0, \mu(x)] \). On the other hand, (10) trivially holds for all \( x \in \text{cl}(F) \setminus F \) and \( s \in [0, \mu(x)] \) since then \( \mu(x) = 0 \) and \( \tilde{\varphi}^0(x) = x \). Hence (10) holds for all \( x \in \text{cl}(F) \) and \( s \in [0, \mu(x)] \), and this is equivalent to the claimed property (ii). Finally, the property (iii) is trivially verified for \( x \in \text{cl}(F) \setminus F \), and is verified for \( x \in F \) by taking sequences \( t_n \nearrow t, s_n \nearrow s \), using continuity of \( \tilde{\varphi} \), and using the analogous property satisfied by \( \varphi \).

**Lemma 3.** Let \( H = (I, F, Z, \varphi, \tau) \) be a deterministic MHS such that \( I \) is compact and \( Z \) is a trapping guard. Further suppose that, for every \( x \in I \), there is an infinite or Zeno execution starting at \( x \). Then \( \text{Ch}_H = \text{Ch}_H \).

In particular, two points of \( I \) are chain equivalent if and only if they are chain equivalent with respect to nice chains only.

**Proof.** It follows immediately from the definitions that \( \text{Ch}_H \subseteq \text{Ch}_H \).

For the reverse inclusion, suppose that \( (x, y) \in \text{Ch}_H \). Fix \( \epsilon, T > 0 \), and let \( U \) be a retraction domain (as in Definition 11) for \( Z \) with flow-induced retraction \( \rho : U \to Z \). Shrinking \( U \) if necessary, we may assume that \( U \) is compact and that \( \mu|_U \) is strictly bounded above by \( T \), where the maximum flow time \( \mu : I \to [0, +\infty] \) is defined in (1). By the uniform continuity of \( \rho \circ \rho \), there exists \( 0 < \delta < \epsilon / 2 \) such that \( \text{dist}(r \circ \rho(p), r \circ \rho(p')) < \epsilon / 2 \) whenever \( \text{dist}(p, p') < \delta \). If \( I \setminus U \) is nonempty, we may further assume that \( \delta < \text{dist}(Z, I \setminus U) \). By Lemma 2, \( \varphi \) has a continuous extension \( \tilde{\varphi} \) defined on the closure \( \text{cl}(\text{dom}(\varphi)) \) of \( \text{dom}(\varphi) \) in \( [0, \infty) \times I \). By compactness of \( I \), it follows that the restriction of \( \tilde{\varphi} \) to \( \text{cl}(\text{dom}(\varphi)) \cap ([0, 2T] \times I) \) is uniformly continuous. Pick \( 0 < \beta < \epsilon / 2 \) such that \( \text{dist}(\tilde{\varphi}^t(p), \tilde{\varphi}^t(p')) < \delta \) whenever \( \text{dist}(p, p') < \beta \), \( |t - t'| < \beta \), and \( t, t' \in [0, 2T] \). Let \( \chi = (N, \tau, \eta, \gamma) \in \text{Ch}_H^{2T}(x, y) \) be a \( (\beta, 2T) \)-chain from \( x \) to \( y \). Without loss of generality, we may assume that \( \tau_{N+1} - \tau_N \leq 2T \) by adding a trivial continuous-time jump at \( \max(\tau_N + 2T, \tau_{N+1} - 2T) \) if necessary.

We would like to modify \( \chi \) to get a nice \( (\epsilon, T) \)-chain. For each \( k \) such that \( 0 < \eta_k < N \) and \( \tau_{\eta_k} - \tau_{(\eta_k - 1)} < T \), we will remove the continuous-time jump at \( \tau_{\eta_k} \), continue on the execution prior to that jump, and return to a later point on the image of \( \chi \) via a new jump. Let \( u = \gamma_{(\eta_k - 1)}(\tau_{\eta_k}) \) and \( v = \gamma_{\eta_k}(\tau_{\eta_k}) \). Let \( t = \tau_{(\eta_k + 1)} - \tau_{\eta_k} \). One of two cases occurs: either \( \eta_k < 2T \) and \( \tau_{(\eta_k + 1)} \) is a reset jump, or \( \eta_k \geq 2T \). In the former case, we have two subcases based on whether \( \tilde{\varphi}^t(u) \) is either (a) defined or (b) undefined.

**Case (i)(a):** By the uniform continuity of the restriction of \( \tilde{\varphi} \) discussed above, we have \( \text{dist}(\tilde{\varphi}^t(u), \tilde{\varphi}^t(v)) < \delta \) since \( \text{dist}(u, v) < \beta \). Since \( \tau_{(\eta_k + 1)} \) is a reset jump, it follows that \( \tilde{\varphi}^t(v) \in Z \). Since \( \tilde{\varphi}^t(u) \) is defined, it
follows that $\tilde{\varphi}^s(u) \in U$ since $\delta < \text{dist}(Z, I \setminus U)$. Thus, $\text{dist}(r \circ \rho(\tilde{\varphi}^s(u)), r(\varphi^s(v))) < \epsilon/2$. By the definition of $\chi$, we have $\text{dist}(r(\tilde{\varphi}^s(v)), \gamma_{(n+1)}(\tau_{n+1})) < \beta < \epsilon/2$. Thus, by the triangle inequality we have
\[
\text{dist}(r \circ \rho(\tilde{\varphi}^s(u)), \gamma_{(n+1)}(\tau_{n+1})) < \epsilon, 
\]
so we can replace the jump at $\tau_{nk}$ by a modified jump at $\tau_{(n+1)}$ by extending the domain of $\gamma_{(n+1)}$ to $[\tau_{(n+1)}, \tau_{(n+1)}]$, i.e., by replacing $\gamma_{(n+1)}$ with $s \in [\tau_{(n+1)}, \tau_{(n+1)}]$ is $r(\tilde{\varphi}^{s-\tau_{(n+1)}}(\gamma_{(n+1)}(\tau_{(n+1)})))$. We obtain a modified chain after deleting $\tau_{nk}$ from the sequence $(\tau_j)_{j=0}^N$, deleting $\eta_k$ from the sequence $(\eta_j)_{j=M}^M$, and reindexing the sequences accordingly.

**Case (i)(b):** Since there exists a Zeno or infinite execution starting at $u$ and since $Z$ is closed, there exists a unique “first impact time” $t_0 < t < 2T$ such that $\varphi^0(U) \in Z$. By our uniform continuity considerations, we have $\text{dist}(\varphi^0(U), \varphi^0(v)) < \delta$. Since $\delta < \text{dist}(Z, I \setminus U)$, we have $\varphi^0(v) \in U$. Thus,
\[
\text{dist}(r(\varphi^0(u)), r(\gamma_{(n+1)}(\tau_{(n+1)}))) = \text{dist}(r(\varphi^0(u)), r \circ \rho(\varphi^0(v))) < \epsilon/2. 
\]
Moreover, $\text{dist}(r(\gamma_{(n+1)}(\tau_{(n+1)})), \gamma_{(n+1)}(\tau_{n+1})) < \beta < \epsilon/2$. Thus, by the triangle inequality we have
\[
\text{dist}(r(\varphi^0(u)), \gamma_{(n+1)}(\tau_{n+1})) < \epsilon, 
\]
so we can replace the jump at $\tau_{nk}$ by a jump at $t_0 + \tau_{nk}$ by extending the domain of $\gamma_{(n+1)}$ to $[\tau_{(n+1)}, t_0 + \tau_{nk}]$, i.e., by replacing $\gamma_{(n+1)}$ with $s \in [\tau_{(n+1)}, t_0 + \tau_{nk}]$ is $r(\tilde{\varphi}^{s-\tau_{(n+1)}}(\gamma_{(n+1)}(\tau_{(n+1)})))$. We obtain a modified chain after replacing $\tau_{nk}$ with $(t_0 + \tau_{nk})$, deleting $\tau_{nk}$ from the sequence $(\tau_j)_{j=0}^M$, deleting $\eta_k$ from the sequence $(\eta_j)_{j=M}^M$, and reindexing the sequences accordingly.

**Case (ii):** We want to replace the jump at $\tau_{nk}$ with a jump at $\tau_{nk} + T$. Since $\text{dist}(u, v) < \beta$, our uniform continuity considerations imply that $\text{dist}(\tilde{\varphi}^s(u), \tilde{\varphi}^s(v)) < \delta$ for all $s \in [0, 2T]$ such that the expression on the left is defined. Since $\tilde{\varphi}^s(v)$ is defined for all $s \in [0, 2T]$ and since $\mu_U < T$ by our choice of $U$, it follows that $\tilde{\varphi}^s(v) \in I \setminus U$ for all $s \in [0, T]$. Since $\delta < \text{dist}(Z, I \setminus U)$, it follows that $\tilde{\varphi}^s(u)$ is also defined for all $s \in [0, T]$. Hence $\text{dist}(\tilde{\varphi}^T(u), \tilde{\varphi}^T(v)) < \delta < \epsilon$, so we can replace the jump at $\tau_{nk}$ with a jump at $T + \tau_{nk}$ from $\tilde{\varphi}^T(u)$ to $\gamma_{(n+1)}(T + \tau_{nk})$. We obtain a modified chain after extending the domain of $\gamma_{(n+1)}$ to $[\tau_{(n+1)}, T + \tau_{nk}]$, replacing $\tau_{nk}$ with $T + \tau_{nk}$, and replacing $\gamma_{(n+1)}$ with its restriction $\gamma_{(n+1)}|_{T+\tau_{nk}} \gamma_{(n+1)}$. After applying the procedure described above in Cases (i)(a-b) and (ii), we obtain an $(\epsilon, T)$-chain for which $\tau_{nk} - \tau_{(n+1)} \geq T$ for all $k$ such that $0 < \eta_k < N$. The resulting chain will be nice unless $\eta_M = N$ and $\tau_N - \tau_{N-1} < T$; if this is the case, we end up with an $(\epsilon, T)$-chain $\chi = (N, \tau, \eta, \gamma) \in \mathcal{C}^2$ satisfying (I) $\tau_{nk} - \tau_{(n+1)} \geq T$ for all $0 < \eta_k < N$, (II) $\eta_M = N$, (III) $\tau_N - \tau_{N-1} > T$, and (IV) $\tau_{N+1} - \tau_N < 2T$ (from the second paragraph of the proof). We call such chains **almost-nice**: note that $N \geq 2$ for an almost-nice chain. By the above argument, for any $\epsilon, T > 0$ we can construct an $(\epsilon, T)$-chain from $x$ to $y$ which is either nice or almost-nice.

We now claim that, from the above, it follows that a nice $(\epsilon, T)$-chain between $x$ and $y$ exists. Suppose (to obtain a contradiction) that this is not the case. Then for each $n \in \mathbb{N}$, there exists an almost-nice $(1/n, T)$-chain $\chi^{(n)} = (N^{(n)}, \tau^{(n)}, \eta^{(n)}, \gamma^{(n)})$ from $x$ to $y$. For each $n$, define $t_n = \tau^{(n)}_{N^{(n)}+1} - \tau^{(n)}_{N^{(n)-1}} < T$ and $t'_n = \tau^{(n)}_{N^{(n)}+1} - \tau^{(n)}_{N^{(n)-1}} \leq 2T$. Similarly, let $u_n = r(\gamma^{(n)}_{N^{(n)}-2}(\tau^{(n)}_{N^{(n)}-1})), v_n = \gamma^{(n)}_{N^{(n)}-1}(\tau^{(n)}_{N^{(n)}-1})$, and $w_n = \gamma^{(n)}_{N^{(n)}-1}(\tau^{(n)}_{N^{(n)}-1})$. Since $I$ is compact, after passing to a subsequence we may assume that $t_n, t'_n \to t, t' \in [0, 2T]$ and $u_n, v_n, w_n \to u, v, w \in I$. Furthermore, each $(t_n, v_n)$ and $(t'_n, w_n)$ belong to dom($\tilde{\varphi}$) = cl(dom($\varphi$)), which is closed in $[0, \infty) \times I$; hence $(t, v), (t', w) \in \text{dom}(\tilde{\varphi})$. Since also $\tilde{\varphi}^{t_n}(v_n) \to w$ and $\tilde{\varphi}^{t'_n}(w_n) \to y$, we have $\tilde{\varphi}^t(v) = w$ and $\tilde{\varphi}^{t'}(w) = y$ by continuity of $\tilde{\varphi}$. Hence Lemma 2 implies that $(t + t', v) \in \text{dom}(\tilde{\varphi})$ and $\tilde{\varphi}^{t + t'}(v) = y$. Pick $n$ large enough so that $1/n < \epsilon/2$ and $\text{dist}(v_n, v) < \epsilon/2$. Since $\text{dist}(u_n, v_n) < 1/n < \epsilon/2$, the triangle inequality implies that $\text{dist}(u_n, v) < \epsilon$. Thus we can modify the chain $\chi^{(n)}$ so that the jump occurring at $\tau^{(n)}_{N^{(n)}-1}$ is from $u_n$ to $v$, so that $\gamma^{(n)}_{N^{(n)}-1} \gamma^{(n)}_{N^{(n)}-1}$ is replaced with the arc
\[
[\tau^{(n)}_{N^{(n)}-1}, t + t' + \tau^{(n)}_{N^{(n)}-1}] \to I, \quad s \mapsto \tilde{\varphi}^{s-\tau^{(n)}_{N^{(n)}-1}}(v) 
\]
terminating at \( y \) and \( \tau_{N(n)}^{(n)} \) is replaced with \( (t + t' + \tau_{N(n)+1}(n)) \), and so that the final arc \( \gamma_{N(n)+1} \) and time \( \tau_{N(n)+1} \) are deleted from the sequences \( \gamma(n), \tau(n) \). Since \( N \geq 2 \) for an almost-nice chain, the resulting chain is a nice \((\epsilon, T)\)-chain from \( x \) to \( y \) (consisting of \( N - 1 \) arcs), which contradicts our assumption that a nice \((\epsilon, T)\)-chain from \( x \) to \( y \) does not exist. This shows that \( \mathcal{Ch}_H \geq H' \) and completes the proof. \( \Box \)

### 6.2. Suspension of a hybrid system

Given a THS \( H = (I, F, Z, \varphi, r) \) satisfying the trapping guard condition (Definition 11), in this section we will construct two new systems: (i) a hybrid system \( H' = (I', F', Z', \varphi', r') \) with \( I' \) containing \( I \) as a proper subset which we term the relaxed hybrid system, and (ii) a continuous semiflow on a quotient \( \Sigma_H \) of \( H' \) which we term the hybrid suspension of \( H \). The basic ideas are contained in Figure 8. For the interested reader, §A.4 contains a comparison of \( H' \) and \( \Sigma_H \) to related constructions appearing in the literature.

The relaxed system \( H' \) formalizes the idea of requiring that \( H \) executions “wait” one time unit after impacting the guard before resetting. For us, the relaxed system is primarily a means to an end; it is an intermediate step in constructing the hybrid suspension, and it is also useful in analyzing some of the latter system’s properties. We state the definition of the relaxed system after the following lemma, which yields conditions under which the state space \( I' \) in the definition of \( H' \) is metrizable.

**Lemma 4.** Let \( X \) be a metrizable space and \( A \subseteq X \) a closed subset. Let \( f: A \to X \) be a continuous and closed map. Define an equivalence relation \( \sim \) on \( X \) by identifying each \( a \in A \) with \( f(a) \in X \). Then the quotient map \( \pi: X \to X/\sim \) is closed. If additionally \( f \) has compact fibers (i.e., \( f^{-1}(x) \) is compact for all \( x \in X \)), then the quotient space \( X/\sim \) is metrizable.

**Remark 12.** If \( A \) is compact and \( f \) is continuous, then \( f \) is automatically closed and proper; hence also \( f^{-1}(x) \) is compact for all \( x \in X \).

**Proof.** We first show that \( \pi \) is closed. Letting \( C \subseteq X \) be any closed subset, we compute

\[
\pi^{-1}(\pi(C)) = C \cup f^{-1}(C) \cup f(C \cap A).
\]

The first set on the right is closed by definition, and the second term is closed since \( A \) is closed and \( f \) is continuous. The third term is closed since \( f \) is a closed map. Hence \( \pi^{-1}(\pi(C)) \) is closed, and this in turn implies that \( \pi(C) \) is closed by the definition of the quotient topology. Hence \( \pi \) is closed.

We now prove that \( X/\sim \) is metrizable under the additional assumption that \( f \) has compact fibers. A theorem of Stone [Sto56, Thm 1] implies that, if \( \pi \) is closed, then \( X/\sim \) is metrizable if and only if \( \pi^{-1}(\pi(x)) \) has compact boundary for all \( x \in X \). Therefore, it suffices to show that \( \pi^{-1}(\pi(x)) \) has compact boundary for all \( x \).
We now show that (ii) holds. Fix $x \in X$ and substitute $C = \{x\}$ in (11) to obtain
\[
\pi^{-1}(\pi(x)) = \{x\} \cup f^{-1}(x) \cup f(\{x\} \cap A).
\]
The first and third terms on the right are compact because they are either singletons or empty. The second
term on the right is compact by our assumption that $f$ has compact fibers. Hence $\pi^{-1}(\pi(x))$ is compact,
and this in turn implies that $\pi^{-1}(\pi(x))$ has compact boundary. This completes the proof. \(\square\)

Although the following definition appears somewhat technical, the intuition is straightforward: delay
every reset by one unit of time. This is done by gluing the bottom of a cylinder to the original guard set,
and defining the new guard set to be the top of the cylinder (see Figure 8).

**Definition 14** (Relaxed hybrid system). Let $H = (I,F,Z,\varphi,r)$ be a deterministic THS with trapping
guard $Z$ such that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$.

We define another deterministic THS, which we call the **relaxed hybrid system** $H' = (I',F',Z',\varphi',r')$ associated to $H$, as follows. Let $\sim$ be the equivalence relation on $Y := I \sqcup (Z \times [0,1])$ which identifies $Z$ with $Z \times \{0\}$, $\pi_0 : Y \to Y/\sim$ be the corresponding quotient map, and $\pi_1 : [0,\infty) \times Y \to [0,\infty) \times (Y/\sim)$ be the quotient map $\pi_1 := \text{id}_{[0,\infty)} \times \pi_0$.\(^{14}\) Let $U \supseteq Z$ be the domain of a flow-induced retraction as in Definition 11, and let $\tilde{\varphi}$ be the unique continuous extension of $\varphi$ to the closure $\text{cl}(\text{dom}(\varphi))$ of $\text{dom}(\varphi)$ in $I$ ensured by Lemma 2. We define the spaces
\[
I' := \pi_0(I \sqcup (Z \times [0,1])) \quad Z' := \pi_0(Z \times [0,1]) \quad F' := \pi_0(I \sqcup (Z \times [0,1]))
\]
\[
\text{dom}(\varphi') := \pi_1 \left( \left\{(t,x) \mid 0 \leq t < \mu(x) + 1 \right\} \cup \bigcup_{t \in [0,1)} \{t\} \times \left(Z \times [0,1-t]\right) \right)
\]
where $\mu : I \to [0,\infty]$ is defined in (1), as well as the reset map $r' : Z' \to I'$ and flow $\varphi' : \text{dom}(\varphi') \to F'$ via
\[
r'(\pi_0(z,1)) := \pi_0(r(z)), \quad z \in Z
\]
\[
\varphi'(t,\pi_0(x)) = \begin{cases} 
\pi_0(\tilde{\varphi}'(x)), & (t,x) \in \text{cl}(\text{dom}(\varphi)) \\
\pi_0(\tilde{\varphi}(x)(t), t - \mu(x)), & x \in \text{cl}(F), t \in [\mu(x), \mu(x) + 1) \\
\pi_0(z, s + t), & x = (z, s) \in Z \times [0,1-t] 
\end{cases}
\]
Here $I'$ is equipped with the quotient topology. Lemma 2 implies that the three functions defining $\varphi'$ agree
on the intersections of their domains; since each such domain is closed in $\text{dom}(\varphi')$, the pasting lemma of point-set topology [Mun00, Thm 18.3] implies that $\varphi'$ is continuous. From the properties of $\tilde{\varphi}$ stated in Lemma 2, it is clear that $\varphi'$ is also a local semiflow. By the definition of the disjoint union and quotient
topologies, $Z'$ is closed in $I'$ and $F'$ is open in $I'$. Hence $H'$ is a THS, and $H'$ is deterministic since $Z' \cap F' = \emptyset$. Furthermore, $Z'$ is clearly a trapping guard.

If additionally $I$ is metrizable and $r$ is a closed map with compact fibers, then the same is true of $r'$, and therefore Lemma 4 implies that $I'$ is metrizable. (By Remark 12, these conditions on $r$ are automatically satisfied if $Z$ is compact, which in turn is automatically true if $I$ is compact.) In this case, it follows that we can make $H'$ an MHS by selecting an extended metric for $I'$ which is compatible with its topology.

We will later need the following two results. The first implies [Lee10, Ex. 2.29] that the “obvious embedding” $\iota : I \to I'$ defined by $\iota := \pi_0|_I$ is indeed a topological embedding (a homeomorphism onto its image).

**Lemma 5.** The quotient map $\pi_0 : I \sqcup (Z \times [0,1]) \to I'$ is closed.

**Proof.** Let $j : Z \to Z \times \{0\}$ be the obvious identification and $C \subseteq I \sqcup (Z \times [0,1])$ be closed. Then, since $Z$ is closed in $I$ and $j$ is a homeomorphism,
\[
\pi_0^{-1}(\pi_0(C)) = C \cup j^{-1}(C) \cup j(C \cap Z)
\]
\(^{14}\)That $\pi_1$ is a quotient map (where $[0,\infty) \times Y$ has the product topology) follows since (i) $\pi_0$ is a quotient map, (ii) $[0,\infty)$ is a locally compact Hausdorff space, and (iii) the product of a quotient map and the identity map of a locally compact Hausdorff space is always a quotient map [Lee10, Lem 4.72].
is closed by the definition of the disjoint union topology. By the definition of the quotient topology, \( \pi_0(C) \) is therefore closed.

**Lemma 6.** Let \( H = (I, F, Z, \varphi, r) \) be a deterministic MHS such that \( I \) is compact and \( Z \) is a trapping guard. Further suppose that, for every arc \( \gamma \) of \( H \).

Further suppose that, for every arc \( \gamma \) of \( H \).

For the converse, let \( \epsilon, T > 0 \) and \( (\iota(x), \iota(y)) \in Ch_H \). Let \( \rho : U \subseteq I \to Z \) be a flow-induced retraction. Shrinking \( U \) if necessary, we may assume that \( U \) is compact and that \( \mu_U \) is strictly bounded above by \( T \in (0, \infty) \), where the maximum flow time \( \mu : I \to [0, +\infty) \) is defined in Equation (1). Define \( U' := \iota(U) \cup \pi_0(Z \times [0, 1]) \subseteq I' \), define the flow-induced retraction \( \rho' : U' \to Z' \) by \( \rho'(\iota(x)) := \pi_0(\rho(x), 1) \) and \( \rho'(\pi_0(z, t)) := \pi_0(z, 1) \), and define \( \mu' : U' \to [0, 1+T] \) by \( \mu'(\iota(u)) := 1 + \mu(u) \) and \( \mu'(\pi_0(z, t)) := 1 - t \). Let \( \varphi \) be the continuous extension of \( \rho' \) to the closure \( \text{cl}(\text{dom}(\rho')) \) of \( \text{dom}(\rho') \) in \([0, \infty) \times I' \) ensured by Lemma 2. Since \( I, Z, \) and \( U' \) are compact and \( \iota : I \to \iota(I) \subseteq I' \) is a homeomorphism, there exists \( \delta > \zeta > 0 \) such that

\[
(12) \quad d_I'(\pi_0(Z \times [0, 1]), I \setminus \text{int}(U')) > \zeta
\]

and

\[
(13) \quad u, v \in I, d_I'(\iota(u), \iota(v)) < \delta \implies d_I'(u, v) < \epsilon
\]

\[
(14) \quad p, q \in U', d_I'(p, q) < \zeta \implies d_I'(r' \circ \rho'(p), r' \circ \rho'(q)) < \delta.
\]

By assumption there exists a chain \( \chi(0) = (N, \tau(0), \eta(0), \gamma(0)) \in \text{Ch}_H^{\delta,T+1}(\iota(x), \iota(y)) \); recall from Definition 4 that \( N \geq 1 \). We will now modify the chain \( \chi(0) \) inductively. Fix \( i \in \{0, \ldots, N-1\} \) and assume that, if \( i \geq 1 \), we have modified the first \( (i+1) \) arcs \((\gamma_{i+1}^{(i)}, \ldots, \gamma_i^{(i)})\) to obtain a chain \( \chi(i) = (N, \tau(i), \eta(i), \gamma(i)) \in \text{Ch}_H^{\delta,T+1}(\iota(x), \iota(y)) \) such that (a) the sub-chain obtained by throwing away the first \( (i+1) \) arcs of \( \chi(i) \) is either a single arc (if \( i = N - 1 \)) or a \((\zeta, T+1)\)-chain, and (b) for all \( j \in \{1, \ldots, i\} \): \( \gamma_j^{(i)} \nsubseteq \pi_0(Z \times [0, 1]) \) and \( \gamma_j^{(i)} \subseteq \pi_0(Z \times [0, 1]) \). If both \( \gamma_j^{(i)}(\tau_{i,j}^{(i)}) \in U' \), then we replace \( \gamma_j^{(i)} \) with the curve

\[
[t_{i,j}^{(i)}, t_{i,j}^{(i)} + \mu'(\gamma_j^{(i)}(\tau_{i,j}^{(i)}))] \to I',
\]

and \( \gamma_{i+1}^{(i)} \) with the degenerater curve \( t \to \gamma_{i+1}^{(i)} \). The upper bound

\[
\mu'(\cdot) < T \text{ and Equations (12), (14) can be used to show that, after redefining the sequences } \mu(t) \text{ and } \tau(t) \text{ accordingly, the result is a chain } \chi^{(i+1)} = (N, \tau^{(i+1)}, \eta^{(i+1)}, \gamma^{(i+1)}) \in \text{Ch}_H^{\delta,T+1}(\iota(x), \iota(y)) \text{ such that (a) the sub-chain obtained by throwing away the first } (i+2) \text{ arcs of } \chi^{(i+1)} \text{ is either empty (if } i = N - 1 \text{), a single arc (if } i = N - 2 \text{), or a } (\zeta, T+1) \text{-chain and (b) for all } j \in \{1, \ldots, i+1\} : \gamma_j^{(i+1)} \subseteq \pi_0(Z \times [0, 1]) \text{ and } \gamma_j^{(i+1)} \subseteq \pi_0(Z \times [0, 1]) \).

Hence by induction we obtain a chain \( \chi \in \text{Ch}_H^{\delta,T+1}(\iota(x), \iota(y)) \) satisfying \( \gamma_i(\tau_{i+1}) \subseteq \pi_0(Z \times [0, 1]) \) and \( \gamma_i(\tau_i) \subseteq \pi_0(Z \times [0, 1]) \) for every arc of \( \chi \). This implies that, if any arc of \( \chi \) meets \( \pi_0(Z \times [0, 1]) \), it must

\[\text{If } i = 0 \text{ we assume nothing, so that the base case of the induction argument is included in this one.}\]

\[\text{Note that, if } \gamma_i(\tau_{i+1}) \subseteq \pi_0(Z \times [0, 1]) \text{ is in } U', \text{ then (12) implies that it cannot belong to } \pi_0(Z \times [0, 1]).\]
pass through \( \pi_0(Z \times \{0\}) \) and terminate at \( \pi_0(Z \times \{1\}) \). Deleting the portions of the arcs that pass through \( \pi_0(Z \times [0,1]) \), composing the resulting arcs with \( \iota^{-1}: \iota(I) \rightarrow I \), and using (13), we finally obtain an \((e, T)\)-chain from \( x \) to \( y \) for \( H \). This completes the proof. \( \square \)

We note that the original motivation for relaxation—elimination of Zeno behavior [JELS99]—still holds in our setting.

**Remark 13** (Relaxation converts Zeno executions to infinite executions). Let \( H = (I, F, Z, \varphi, r) \) be a deterministic THS with trapping guard. Suppose that, for every \( x \in I \), there is an infinite or Zeno execution starting at \( x \). (By Remark 3, it follows that \( I = F \cup Z \).) Then the relaxed system \( H' \) is nonblocking, because every Zeno execution for \( H \) becomes an infinite execution for \( H' \). In particular, the nonblocking property implies that \( I' = F' \cup Z' \) (by Remark 3).

Using the relaxed hybrid system associated to \( H = (I, F, Z, \varphi, r) \), we now define the hybrid suspension \((\Sigma_H, \Phi_H)\) of \( H \), denoted by \( \Sigma_H \), together with an appropriate semiflow \( \Phi_H \) on \( \Sigma_H \). We choose to call this space a “suspension” because it generalizes the classical suspension of a discrete-time dynamical system [Sma67, BS02, p. 797, pp. 21–22]; indeed, if \( I = Z \) our construction reduces to the classical one (see Appendix B for details). We will later see that \((\Sigma_H, \Phi_H)\) is particularly compatible with \( H \) as far as omega-limit sets (Proposition 3), attracting-repelling pairs (Proposition 4), and chain recurrence (Proposition 5) are concerned. (This compatibility would not generally hold for the (generalized) hybrifold semiflow appearing in the literature and discussed in detail in §A.4.1; c.f. Remarks 17 and 19.)

**Definition 15** (Hybrid suspension). Let \( H = (I, F, Z, \varphi, r) \) be a deterministic THS with trapping guard \( Z \). Assume that, for every \( x \in I \), there is an infinite or Zeno execution starting at \( x \). Let \( H' = (I', F', Z', \varphi', r') \) be the relaxed hybrid system of Definition 14. Let \( \sim \) be the equivalence relation on \( I' \) which identifies each \( z' \in Z' \subseteq I' \) with \( r'(z') \in I' \). We define the topological space \( \Sigma_H := I'/\sim \) and let \( \pi: I' \rightarrow \Sigma_H \) be the quotient map. Since \( H' \) is deterministic and nonblocking with a trapping guard \( Z' \), since \( I' = F' \cup Z' \) (c.f. Remark 3), since \( \pi(\chi_{x'}(t)) \) is a singleton for all \( x' \in I' \) and \( t \geq 0 \) (by construction of \( \Sigma_H \)), and since \( \varphi' \) admits a unique continuous extension defined on the closure of \( \text{dom}(\varphi') \) in \([0, \infty) \times I' \) and satisfying the properties stated in Lemma 2, we obtain a well-defined continuous semiflow \( \Phi_H: \{0, \infty\} \times \Sigma_H \rightarrow \Sigma_H \) by defining \( \Phi'_H(\pi(x')) \) to be the unique element of the singleton \( \pi(\chi_{x'}(t)) \) for \( x' \in I' \) and \( t \geq 0 \). We define \((\Sigma_H, \Phi_H)\) to be the hybrid suspension and suspension semiflow of \( H \). For brevity, we sometimes simply refer to the pair \((\Sigma_H, \Phi_H)\) as the hybrid suspension.

If additionally \( I \) is metrizable and the reset map \( r \) is closed with compact fibers, then the same is true of \( r' \), and therefore \( \Sigma_H \) is metrizable (by Lemma 4 and Definition 14). By Remark 12, these conditions on \( r \) are automatically satisfied if the guard \( Z \) is compact, which in turn is automatically true if \( I \) is compact.

**Remark 14**. \( \Sigma_H \) is compact if \( I \) and \( Z \) are compact, since then \( I' \) is compact and \( \Sigma_H = \pi(I') \) is the continuous image of a compact set.

**Remark 15** (Further motivation for the trapping guard condition.). Let \( H = (I, F, Z, \varphi, r) \) be a deterministic THS having only infinite or Zeno maximal executions. Under the assumption that \( H \) satisfies the trapping guard condition (Definition 11), in §6.2 (Definitions 14 and 15) we defined the spaces \( I', \Sigma_H \) and maps \( \pi_0, \iota, \) and \( \pi \). We also constructed the suspension semiflow \( \Phi_H: \{0, \infty\} \times \Sigma_H \rightarrow \Sigma_H \) and showed that \( \Phi_H \) is continuous. It is immediate from the definitions that \( \Phi_H \) satisfies the following two properties.

1. \( \Phi_H^t(\pi \circ \pi_0(z, s)) = \pi \circ \pi_0(z, t + s) \) for all \( z \in Z, s \in [0,1], \) and \( t \in [0,1-s] \).
2. For all \((t, x) \in \text{dom}(\varphi), \pi \circ \iota(\varphi^t(x)) = \Phi_H^t(\pi \circ \iota(x)) \).

While for convenience of exposition we only defined the quantities \( I', \Sigma_H, \pi_0, \iota, \) and \( \pi \) under all of the above assumptions (in particular, assuming the trapping guard condition), their definitions make sense (verbatim) for any THS. Thus, for an arbitrary THS \( H \), it makes sense to ask the following question: under what circumstances does there exist a well-defined “suspension semiflow” \( \Phi \) on \( \Sigma_H \) for \( H \) in the sense that \( \Phi \) satisfies Conditions 1 and 2 (stated above for \( \Phi_H \))?
then there exists a continuous suspension semiflow $\Phi: [0, \infty) \times \Sigma_H \to \Sigma_H$ in the above sense if and only if $H$ satisfies the trapping guard condition. This further motivates the trapping guard condition.

Proposition 3 below is a fundamental result which relates hybrid omega-limit sets (Definition 8) to those of the hybrid suspension semiflow, and can be viewed as motivation for the definition of hybrid omega-limit sets.

**Proposition 3.** Let $H = (I, F, Z, \varphi, r)$ be a deterministic THS with $Z$ a trapping guard and $r: Z \to I$ a closed map. Assume that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$.

Let $H' = (I', F', Z', \varphi', r')$ be the relaxed hybrid system of Definition 14, $\pi_0: \Sigma_0 \subset (Z \times [0,1]) \to I'$ be the quotient of Definition 14, $\iota: I \to I'$ be the embedding $\pi_0 I$, and $(\Sigma_H, \Phi_H)$ be the hybrid suspension of Definition 15 with quotient $\pi: I' \to \Sigma_H$. Let $\rho_0: \pi_0(Z \times [0,1]) \to Z$ be the composition of the straight-line retraction $\pi_0(Z \times [0,1]) \to \iota(Z)$ with the identification $\iota(Z) \approx Z$. For all $B \subseteq \Sigma_H$, define

$$Z_B := \rho_0 \left( \pi^{-1}(B) \cap \pi_0(Z \times [0,1]) \right).$$

Then for all $B \subseteq \Sigma_H$:

$$\omega(B) = (\pi \circ \iota)^{-1}(\omega(B)) = \omega((\pi \circ \iota)^{-1}(B)) \cup \omega(Z_B).$$

In particular, for all $A \subseteq I$:

$$\omega(A) = (\pi \circ \iota)^{-1}(\omega(\pi \circ \iota(A))).$$

**Remark 16.** If $Z$ is compact and $I$ is Hausdorff, then $r$ is automatically a closed map. In particular, the proposition holds if $H$ satisfies the hypotheses of Theorems 1 and 2 (i.e., if $H$ is a deterministic MHS with $I$ compact, $Z$ a trapping guard, and such that all maximal executions are infinite or Zeno.)

**Remark 17.** The statement analogous to Proposition 3 obtained by replacing the hybrid suspension with the semiflow on the (generalized) hybrifold $M_H$ (defined in §A.4.1) is false. For example, consider a THS $H$ with $I = Z = \{0,1\}$ and $r(Z) = \{0\}$ (c.f. Example 1). Then $M_H$ is a singleton $\{\ast\}$. Letting $\pi_M: I \to M_H$ be the quotient defined in §A.4.1 and $j \in \{0,1\}$, we compute

$$\omega(j) = \{0\} \neq \{0,1\} = \pi^{-1}_M(\{\ast\}) = \pi^{-1}_M(\omega(\pi_M(j))),$$

so the analogue of (16) is false for the (generalized) hybrifold semiflow. The same example shows that the statement analogous to Proposition 4 below for the (generalized) hybrifold semiflow is also false.

**Proof of Proposition 3.** For purposes of readability, for this proof we define $f := \pi \circ \iota$ and $g := (\pi \circ \pi_0) | Z \times [0,1]$, and we introduce the following notation for $T > 0$ and subsets $A \subseteq I, B \subseteq \Sigma_H$:

$$R_{A,T} := \bigcup_{x \in A} \bigcup_{(N,\tau,\gamma) \in \mathcal{E}_H(x)} \{ \gamma_j(t) \mid j + t \geq T \}, \quad S_{B,T} := \Phi_H^{[T,\infty)}(B).$$

In the definition of $R_{A,T}$, note that each $\mathcal{E}_H(x)$ contains only a single maximal execution since we assume that $H$ is deterministic. For later use we note that $f$ is a closed map since it is the composition of (i) $\pi$, which is a closed map by Lemma 4, and (ii) $\iota$, which is a closed map since, for any closed $C \subseteq I$, $\pi^{-1}_0(\iota(C)) = C \cup ((C \cap Z) \times \{0\})$ is closed in $(I \cup Z \times [0,1])$. Furthermore, $g$ is a closed map by composition and restriction since (i) $\pi$ is closed, (ii) $\pi_0$ is closed by Lemma 5, and (iii) $Z \times [0,1]$ is closed in $I \cup (Z \times [0,1])$.

In order to prove the proposition, we first need to establish two facts: for any $B \subseteq \Sigma_H$,

$$\omega(B) = \omega(B \cap f(I)) \cup \omega(f(Z_B)).$$

and

$$f(I) \cap \bigcap_{T > 0} \text{cl} (S_{B,T}) = \bigcap_{T > 0} \text{cl} (f(I) \cap S_{B,T}).$$

We begin by establishing (17). First note that, for all $T \geq 0$, the definitions of $Z_B$ and $\Phi_H$ immediately imply that

$$\Phi_H^{[1+T,\infty)}(f(Z_B)) \subseteq \Phi_H^{[T,\infty)}(B \setminus f(I)) \subseteq \Phi_H^{[T,\infty)}(f(Z_B)).$$
Equation (19) and the definition of omega-limit sets imply that $\omega(f(Z_B)) = \omega(B \setminus f(I))$. Since the omega-limit set of a finite union is equal to the union of the omega limit sets\(^{17}\), $\omega(B) = (B \cap f(I)) \cup \omega(B \setminus f(I))$, from which (17) now follows.

We now establish (18) for fixed $B \subseteq \Sigma_H$; for readability, we define $C_B$ to be the set on the left side of (18) and $D_B$ to be the set on the right. Since $f$ is a closed map, $f(I)$ is closed in $\Sigma_H$, so it is immediate from general topology that $D_B \subseteq C_B$. Hence we need only prove that $C_B \subseteq D_B$. To obtain a contradiction, suppose that this is not the case, so that there exists $x \in C_B \setminus D_B$. Clearly we must have $x \notin \text{int}(f(I))$, so $x$ belongs to the boundary $g(Z \times \{0, 1\})$ of $f(I)$. Since $x \notin D_B$, there exists $T_0 > 0$ and a neighborhood $V_x$ of $x$ such that $V_x \cap f(I) \cap S_{B,T_0} = \emptyset$. Since the family $(S_{B,T})_{T > 0}$ decreases in $T$, this implies that

$$V_x \cap f(I) \cap S_{B,T} = \emptyset \text{ for all } T \geq T_0. \quad (20)$$

Let $Y := Z \times ((0, 1/4) \cup (3/4, 1])$ and $\tilde{\nu} : Y \rightarrow Z \times \{0, 1\}$ be the straight-line retraction. Since $g$ is a closed map, $g$ is a quotient map onto its image. Furthermore, since $Y$ is an open $g$-saturated subset of $Z \times [0, 1]$, it follows that (i) $g(Y)$ is open relative to $g(Z \times [0, 1])$ and (ii) $g|_{Y} : Y \rightarrow g(Y)$ is also a quotient map [Lee10, Prop. 3.62.d]. We also note that, if $g(z, t) = g(z', t')$ for some distinct points $(z, t), (z', t') \in Y$, then necessarily $(z, t), (z', t') \in Z \times \{0, 1\}$ and hence $(g \circ \tilde{\nu})(z, t) = g(z, t) = g(z', t') = (g \circ \tilde{\nu})(z', t')$ since $\tilde{\nu}|_{Z \times \{0, 1\}} = \text{id}|_{Z \times \{0, 1\}}$. Thus, by the universal property of the quotient topology [Lee10, Thm 3.70], the map $g \circ \tilde{\nu}$ descends to a continuous trajectory-preserving retraction $\nu : g(U) \rightarrow g(Z \times \{0, 1\})$. We now define a new set $U_x := V_x \cup \text{int}(V_x \cap g(Z \times \{0, 1\}))$. Since $U_x$ is open relative to $g(Y)$ which is in turn open relative to $g(Z \times [0, 1])$, $U_x$ is also open relative to $g(Z \times [0, 1])$ and hence the set $U_x \cup f(I)$ is a neighborhood of $x$. Thus, since $x \in C_B$, for every $T \geq T_0$ there exists $y_T \in S_{B,T} \cap (U_x \cup f(I)) \cap V_x = S_{B,T} \cap U_x$. Thus, by the definitions of the suspension semiflow, $\nu$, and $U_x$, we have $\nu(y_T) \in V_x \cap f(I) \cap S_{B,T} = \emptyset$ for any $T \geq T_0 + \frac{1}{4}$, contradicting (20). This establishes (18).

Armed with (17) and (18), we now proceed to prove the proposition. Since $B \cap f(I)$ and $f(Z_B)$ are subsets of $f(I)$, it is immediate from the definitions that, for all $T \geq 0$,

$$f(I) \cap S_{B \cap f(I),T} = f(R_{f^{-1}(B),T}), \quad f(I) \cap S_{f(Z_B),T} = f(R_{Z_B,0}). \quad (21)$$

The following computation, to be justified after, proves (15).

$$f^{-1}(\omega(B)) = f^{-1}(\omega(B \cap f(I)) \cup \omega(f(Z_B)))$$

$$= f^{-1} \left( \bigcap_{T > 0} \text{cl} \left( S_{B \cap f(I),T} \right) \cup \bigcap_{T > 0} \text{cl} \left( S_{f(Z_B),T} \right) \right)$$

$$= f^{-1} \left( \bigcap_{T > 0} f(I) \setminus \text{cl} \left( S_{B \cap f(I),T} \right) \cup \bigcap_{T > 0} f(I) \setminus \text{cl} \left( S_{f(Z_B),T} \right) \right)$$

$$= f^{-1} \left( \bigcap_{T > 0} \text{cl} \left( f(I) \cap S_{B \cap f(I),T} \right) \cup \bigcap_{T > 0} \text{cl} \left( f(I) \cap S_{f(Z_B),T} \right) \right)$$

$$= \bigcap_{T > 0} f^{-1} \left( \text{cl} \left( f(R_{f^{-1}(B),T}) \right) \right) \cup \bigcap_{T > 0} f^{-1} \left( \text{cl} \left( f(R_{Z_B,0}) \right) \right)$$

$$= \bigcap_{T > 0} \text{cl} \left( R_{f^{-1}(B),T} \right) \cup \bigcap_{T > 0} \text{cl} \left( R_{Z_B,0} \right)$$

$$= \omega(f^{-1}(B)) \cup \omega(Z_B),$$

The first equality follows from taking the preimage of both sides of (17). The second equality follows from the definition of omega-limit set. The third equality follows since intersecting a set with $f(I)$ does not change its $f$-preimage. The fourth equality follows from (18). The fifth equality follows from (21) and the

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\(^{17}\)Proof: since the closure of a finite union is the union of the closures, we compute $\omega(U \cup V) = \bigcap_{T > 0} \text{cl}(\Phi^{T,\infty}_H(U \cup V)) = \bigcap_{T > 0} \text{cl}(\Phi^{T,\infty}_H(U)) \cup \text{cl}(\Phi^{T,\infty}_H(V)) = \omega(U) \cup \omega(V)$ for any sets $U, V$. (This result is stated in [Con78, II.4.1.C] for the special case of a flow.) Repeating the same proof mutatis mutandis shows that hybrid omega-limit sets also possess this finite-union property.

\(^{18}\)Proof: $(r(Z) \cap Z \times \{0\}) \cup (r^{-1}(Z) \times \{1\}) \subseteq Z \times \{0, 1\} \subseteq Y$, so $g^{-1}(g(Y)) = Y \cup ((r(Z) \cap Z \times \{0\}) \cup (r^{-1}(Z) \times \{1\})) = Y.$
distributivity of preimages over intersections and unions. The sixth equality is justified by the facts that (i) $f$ is a continuous and closed map, so taking $f$-images commutes with taking closures [Lee10, Prop. 2.30], and (ii) $f$ is injective, so $f^{-1}(f(X)) = X$ for any $X \subseteq \Sigma_H$.

To complete the proof, it remains only to verify (16). Fix $A \subseteq I$. We have $Z_{f(A)} \subseteq A \cup r^{-1}(A)$ by the definition of $Z_{f(A)}$. Since $\omega\left(r^{-1}(A)\right) \subseteq \omega(A)$ by the definition of omega-limit sets, the finite-union property of omega-limit sets (footnote 17) implies that $\omega(Z_{f(A)}) \subseteq \omega(A) \cup \omega(r^{-1}(A)) \subseteq \omega(A)$. Taking $B = f(A)$ in (15), we thus obtain

$$f^{-1}(\omega(f(A))) = \omega(f^{-1}(f(A))) \cup \omega(Z_{f(A)}).$$

Since $f$ is injective, $A = f^{-1}(f(A))$; substituting this into the first term on the right above and using $\omega(Z_{f(A)}) \subseteq \omega(A)$ yields (16). This completes the proof. \hfill \Box

**Corollary 2.** Let $H = (I,F,Z,\varphi,r)$ be a deterministic THS with $Z$ a trapping guard and $r: Z \to I$ a closed map. Assume that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Then for any $U \subseteq I$, $\omega(U)$ is forward invariant.

**Proof.** Let all notation be as in Proposition 3 above, let $U \subseteq I$ be an arbitrary subset, and define $X := \pi \circ \iota(U)$. Since $\pi \circ \iota$ is injective, it follows that $U = (\pi \circ \iota)^{-1}(X)$. Hence Equation 16 implies that

$$\omega(U) = (\pi \circ \iota)^{-1}(\omega(X)).$$

It is well-known from classical dynamical systems theory that omega-limit sets of continuous semiflows are forward invariant (this is also easy to prove directly), so $\omega(X)$ is forward invariant for $\Phi_H$. Since the collection of maximal $H$ executions is precisely the collection of $(\pi \circ \iota)$-preimages of $\Phi_H$ trajectories, Equation (22) implies that $\omega(U)$ is forward invariant. \hfill \Box

**Proposition 4.** Let $H = (I,F,Z,\varphi,r)$ be a deterministic THS with $I$ compact, $Z$ a trapping guard, and $r: Z \to I$ a closed map. Further suppose that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Let $H' = (I',F',Z',\varphi',r')$ be the relaxed hybrid system of Definition 14, $\iota: I \to I'$ be the obvious embedding, and let $(\Sigma_H,\Phi_H)$ be the hybrid suspension of Definition 15 with quotient $\pi: I' \to \Sigma_H$.

Then $(A,A^*)$ is an attracting-repelling pair for $H$ if and only if $(A,A^*) = ((\pi \circ \iota)^{-1}(B),(\pi \circ \iota)^{-1}(B^*))$ for some attracting-repelling pair $(B,B^*)$ for $\Phi_H$.

**Proof.** For purposes of readability, for this proof we define $f := \pi \circ \iota$. We begin by noting that, if $W \subseteq \Sigma_H$ is any forward invariant set, then it follows from the definition of $\Phi_H$ that

$$\Phi^{1+T,\infty}(W \cap f(I)) \subseteq \Phi^{1+T,\infty}(W) \subseteq \Phi^{T,\infty}(W \cap f(I))$$

for any $T \geq 0$. By the definition of omega-limit set, this implies that

$$\omega(W) = \omega(W \cap f(I)).$$

Now let $U := f^{-1}(W)$. Since $r$ is a closed map, (23) together with Equation (16) of Proposition 3 yield

$$f^{-1}(\omega(W)) = f^{-1}(\omega(W \cap f(I))) = f^{-1}(\omega(f(U))) = \omega(U).$$

Now let $(B,B^*)$ be an attracting-repelling pair for $\Phi_H$ and let $W$ be an open trapping neighborhood for $B$. Then $U := f^{-1}(W)$ is open by continuity of $f$, and continuity of $f$ also implies that $\text{cl}(U) \subseteq f^{-1}(\text{cl}(W))$. Hence clearly $U$ satisfies the conditions of Definition 10 defining a trapping neighborhood, so $U$ determines an attracting set $A := \omega(U)$. Since $W$ is forward invariant, (24) implies that $A = f^{-1}(\omega(W)) = f^{-1}(B)$.

We now show that $A^* = f^{-1}(B^*)$. Equation (16) of Proposition 3 implies that $\omega(x) = f^{-1}(\omega(f(x)))$ for all $x \in I$. Since also $A = f^{-1}(B)$, it follows that, for all $x \in I$,

$$[\omega(x) \cap A = \emptyset] \iff [f^{-1}(\omega(f(x))) \cap f^{-1}(B) = \emptyset] \iff [\omega(f(x)) \cap B \cap f(I) = \emptyset].$$

The latter in turn holds if and only if $\omega(f(x)) \cap B = \emptyset$, since $\omega(f(x)) \cap B$ is forward invariant and $\Sigma_H \setminus f(I)$ contains no nonempty forward invariant subset. Thus, the repelling set $A^*$ dual to $A$ is given by $A^* = f^{-1}(B^*)$. We have now shown that, for every attracting-repelling pair $(B,B^*)$ for $\Phi_H$, $(f^{-1}(B), f^{-1}(B^*))$ is an attracting-repelling pair for $H$. 


To prove the converse, let \((A, A^*)\) be an attracting-repelling pair for \(H\) and let \(U \supseteq A\) be an open trapping neighborhood as in Definition 10. Then by the definition of the disjoint union and quotient topologies, \(U'_0 := \pi_0(U \sqcup ((U \cap Z) \times [0, 1]))\) is an open subset of \(I'\), where \(\pi_0 : I \sqcup (Z \times [0, 1]) \to I'\) is the quotient of Definition 14. Now define the set \(U' := U'_0 \cup \pi_0(r^{-1}(U) \times (0, 1]) \subseteq I'\), which is also open (for similar reasons). \(U'\) is saturated with respect to \(\pi\) since
\[
\pi^{-1}(\pi(U')) = r'(U' \cap Z') \cup U' \cup (r')^{-1}(U') = U' \cup (r')^{-1}(U') = U'.
\]
The first equality follows from the definition of \(\pi\). The second equality follows using the definition of \(U'\) and the fact that \(U\) is forward invariant: \(r'(U' \cap Z') = \pi(r(U \cap Z) \cup r(r^{-1}(U))] \subseteq \pi(U) \subseteq U', \) so \(r'(U' \cap Z') \cup U' = U'.\) The third equality follows since \(r'(Z') \subseteq \pi(I)\) and since \(U' \cap \pi(I) = \pi(U),\) so \((r')^{-1}(U') = (r')^{-1}(\pi(U)) = \pi_0(r^{-1}(U) \times \{1\}) \subseteq U'.\)

Since \(U'\) is open and saturated, \(W := \pi(U') \subseteq \Sigma_H\) is open, and \(\text{cl}(W) = \pi(\text{cl}(U'))\) since, by Lemma 4, \(\pi\) is a closed map [Lee10, Prop. 2.30]. These facts, together with the definitions of \(\Phi_H\) and \(U'\) and the fact that \(U'\) is a trapping neighborhood, imply that \(W\) is a trapping neighborhood for some attracting set \(B := \omega(W)\). Using the definition of \(U'\) and the fact that \(U'\) is saturated, it follows that \(f^{-1}(W) = U\). Since \(W\) is forward invariant, (24) therefore implies that \(A = \omega(U) = f^{-1}(\omega(W)) = f^{-1}(B).\) Finally, repeating the argument from the first part of the proof verbatim shows that \(A^* = f^{-1}(B^*).\) Hence every attracting repelling-pair \((A, A^*)\) for \(H\) is of the form \((f^{-1}(B), f^{-1}(B^*))\) for some attracting-repelling pair \((B, B^*)\) for \(\Phi_H\). This completes the proof. \(\square\)

6.3. Chain equivalence in the hybrid suspension. To prove our main results, we will relate the chain recurrent set of an MHS with the chain recurrent set of its hybrid suspension semiflow. Proposition 5 below establishes this relationship. In order to prove Proposition 5, we first prove the following lemma, a minor adaptation of [CK00, Prop. B.2.19].

Lemma 7. Let \(X\) be a compact metric space and \(\Phi : [0, \infty) \times X \to X\) be a continuous semiflow. Consider \(x, y \in X\) and fix \(T > 0\). If for every \(\epsilon > 0\) there exist \((\epsilon, T)\)-chains from (i) \(x\) to \(y\) and (ii) \(y\) to \(x\), then \(x\) and \(y\) are chain equivalent.

Remark 18. Formally speaking, Lemma 7 is discussing \((\epsilon, T)\)-chains and chain equivalence as defined in Definitions 4 and 7, as opposed to the standard definitions for semiflows (c.f. Example 2) to which [CK00, Prop. B.2.19] applies. However, the proofs are similar for either definition. Furthermore, our proof actually shows that \(x\) and \(y\) are also chain equivalent according to the standard definition, but this also follows from the stated conclusion and Example 2.

Proof. It suffices to show that, for every \(\epsilon > 0\), there are \((\epsilon, 2T)\)-chains from (i) \(x\) to \(y\) and (ii) \(y\) to \(x\). By the compactness of \(X\), the map \(\Phi\) is uniformly continuous on \(X \times [0, 4T]\). Hence there exists \(\delta \in (0, \epsilon/2)\) such that for all \(a, b \in X\) and \(t \in [0, 4T]\) we have \(\text{dist}(\Phi^t(a), \Phi^t(b)) < \epsilon/2\) whenever \(\text{dist}(a, b) < \delta\).

Let \(\chi = (N, \eta, \gamma) \in \mathcal{CH}_{\delta,T}(x, y)\). Without loss of generality, we may assume that \(\tau_{i} \leq \tau_{i+1} \leq [T, 2T]\) for all \(0 \leq i \leq N\). By concatenating \(\chi\) with a \((\delta, T)\)-chain from \(y\) to \(x\) for \(\tau_{i} \leq \tau_{i+1}\), which exists by Case 1 of Example 2, we may assume that \(N \geq 3\) (so that \(\chi\) contains at least four arcs). Thus, there exist integers \(N' \geq 1\) and \(r \in \{1, 2\}\) with \(N = 2N' + r\).

Let \(\tau' = (\tau'_i)_{i=0}^{N'+1}\) be the sequence given by \(\tau'_i := \tau_{2i}\) for \(0 \leq i \leq N'\) and \(\tau'_{N'+1} := \tau_{N+1}\). We (uniquely) define the corresponding arcs \(\gamma'_0, \ldots, \gamma'_N\) by their starting points \(\gamma'_i(\tau'_i) = (\gamma_i(\tau_i))\) for \(0 \leq i \leq N'.\) Then by the triangle inequality the chain \((N', \tau', \eta', \gamma')\) (where \(\eta' = (0, 1, \ldots, N')\)) defines an \((\epsilon, 2T)\)-chain from \(y\) to \(x\).

Repeating the above argument with the roles of \(x\) and \(y\) reversed also yields an \((\epsilon, 2T)\)-chain from \(x\) to \(y\). This completes the proof. \(\square\)

Proposition 5 (The chain equivalence classes of a compact hybrid system and its suspension). Let \(H = (I, F, Z, \varphi, r)\) be a deterministic MHS. Assume that \(I\) is compact and that \(Z\) is a trapping guard. Further suppose that, for every \(x \in I\), there is an infinite or Zeno execution starting at \(x\). Let \(H' = (I', F', Z', \varphi', r')\) be the relaxed hybrid system of Definition 14, let \(\iota : I \to I'\) be the obvious embedding, and let \((\Sigma_H, \Phi_H)\) be the hybrid suspension of Definition 15 with quotient \(\pi : I' \to \Sigma_H\).
Then for any choice of compatible extended metric on $\Sigma_H$, $x, y \in I$ are chain equivalent for $H$ if and only if $\pi \circ \iota(x), \pi \circ \iota(y) \in \Sigma_H$ are chain equivalent for $\Phi_H$. In particular, it follows that $R(H) = (\pi \circ \iota)^{-1}(R(\Phi_H))$, where $R(\Phi_H)$ is the chain recurrent set for $\Phi_H$.

**Remark 19.** Proposition 5 would become false if the hybrid suspension $\Sigma_H$ was replaced by the (generalized) hybrifold $M_H$ of $H$ discussed in §A.4.1. For example, consider a discrete-time dynamical system, i.e., a hybrid system $H$ with $I = Z$ (c.f. Example 1). Then every point of $M_H$ is a stationary point for the (generalized) hybrifold semiflow, and therefore every point of $M_H$ is chain recurrent. On the other hand, the chain recurrent set of $H$ is arbitrary.

The following immediate corollary of Proposition 5 concerns the classical suspension of a discrete-time dynamical system and is probably well-known, although we could not find a reference in the literature. See Appendix B or [Sma67, BS02, p. 797, pp. 21–22] for a primer on the classical suspension semiflow.

**Corollary 3.** Consider the discrete-time dynamical system defined by a continuous map $f: X \to X$ of a compact metric space. Let $\Sigma_f := \frac{X\times[0,1]}{(x,1)\sim(0,f(x))}$ be the classical suspension (mapping torus) of $f$, $\Phi: [0, \infty) \times \Sigma_f \to \Sigma_f$ be the suspension semiflow, $\iota: X \to X \times [0, 1]$ be the embedding $X \hookrightarrow X \times \{0\}$, and $\pi: X \times [0, 1] \to \Sigma_f$ be the quotient map.

Then for any choice of compatible extended metric on $\Sigma_f$, $x, y \in X$ are chain equivalent for $f$ if and only if $\pi \circ \iota(x), \pi \circ \iota(y)$ are chain equivalent for $\Phi$. In particular, $R(f) = (\pi \circ \iota)^{-1}(R(\Phi))$.

The classical analogue of the following corollary of Proposition 5 is well-known.

**Corollary 4.** Let $H = (I, F, Z, \varphi, r)$ be an MHS satisfying all hypotheses of Proposition 5. Then the chain equivalence classes of $H$ are closed and forward invariant. Similarly, $R(H)$ is closed and forward invariant. Additionally, $\omega(x) \subseteq R(H)$ for any $x \in I$.

**Proof of Corollary 4.** For the standard, classical definition of the Conley relation for the semiflow $\Phi_H$ (and for any choice of compatible metric on $\Sigma_H$), it is well known that all chain equivalence classes for $\Phi_H$, as well as the chain recurrent set for $\Phi_H$, are closed and forward invariant (this is also easy to prove directly).

By Example 2, the same is true if our definition of the Conley relation (Definition 5) for $\Phi_H$ is used instead. It follows that the same is true for $H$ since (i) Proposition 5 implies that the chain equivalence classes for $H$ (and hence also $R(H)$) are $(\pi \circ \iota)$-preimages of the chain equivalence classes for $\Phi_H$, (ii) $\pi \circ \iota$ is continuous, and (iii) the collection of maximal $H$ executions is precisely the collection of $(\pi \circ \iota)$-preimages of $\Phi_H$ trajectories.

Fix $x \in I$ and define $[x] := \pi \circ \iota(x)$. The final assertion follows from Proposition 5 and Equation (16) of Proposition 3 (which applies by Remark 19):

$$\omega(x) = (\pi \circ \iota)^{-1}(\omega([x])) \subseteq (\pi \circ \iota)^{-1}(R(\Phi_H)) = R(H).$$

The set inclusion follows from the well-known fact that, for a semiflow, the chain recurrent set contains the omega-limit set of any point (c.f. [Con78, II.6.3.C]).

We now prove Proposition 5.

**Proof of Proposition 5.** In the following, we let $d_I$ be the given extended metric on $I$ and $d_{I'}, d_{\Sigma_H}$ be any metrics on $I', \Sigma_H$ which are compatible with their respective topologies, and we use the notation $[x] := \pi(x)$ for $x \in I'$. Through a mild abuse of notation, we also use the notation $[x] := \pi \circ \iota(x)$ for $x \in I$ and $[z, t] := \pi \circ \pi_0(z, t)$, where $\pi_0: I \cup (Z \times [0, 1]) \to I'$ is the quotient of Definition 14.

We first show that, if $x, y \in I$ are chain equivalent, then $[x]$ and $[y]$ are chain equivalent for $\Phi_H$.\footnote{In this proof we are using Definitions 4 and 5 for the definition of the Conley relation for $\Phi_H$, although Example 2 shows that this is equivalent to the classical definition.} Since $I$ is compact it follows that the map $\pi \circ \iota: I \to \Sigma_H$ is uniformly continuous with respect to $d_I, d_{\Sigma_H}$. This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that every nice $(\delta, T + 1)$-chain for $H$ maps to an $(\epsilon, 1)$-chain for $\Phi_H$ under $\pi \circ \iota$. Lemma 3 implies that, for every $\delta, T > 0$, there are nice $(\delta, T + 1)$-chains from (i) $x$ to $y$ and (ii) $y$ to $x$, so it follows that there are $(\epsilon, 1)$-chains from (i) $[x]$ to $[y]$ and (ii) $[y]$ to $[x]$ for every $\epsilon > 0$. Hence Lemma 7 implies that $[x]$ and $[y]$ are chain equivalent for $\Phi_H$.\footnote{In this proof we are using Definitions 4 and 5 for the definition of the Conley relation for $\Phi_H$, although Example 2 shows that this is equivalent to the classical definition.}
To complete the proof we now show that, if \( x, y \in R(H) \) are such that \([x]\) and \([y]\) are chain equivalent for \(\tilde{H}_H\), then \([x]\) and \([y]\) are chain equivalent. It suffices to prove the stronger claim that \([x]\) being Conley related to \([y]\) for \(\tilde{H}_H\) implies \(Ch_H(x,y)\) for arbitrary \(x,y \in I\). And in order to prove this, by Lemma 6 it suffices to prove that \([x]\) being Conley related to \([y]\) for \(\tilde{H}_H\) implies \(Ch_H(\iota(x),\iota(y))\). We prove the latter claim below in two steps which we first briefly describe in the following paragraph. To improve readability in the remainder of the proof, we henceforth use the notation \([S] := \pi \circ \iota(S) \subseteq \Sigma_H\) if \(S \subseteq I\), \([S] := \pi(S) \subseteq \Sigma_H\) if \(S \subseteq I'\), and \([S] := \pi \circ \tilde{p}_0(S) \subseteq \Sigma_H\) if \(S \subseteq I' \cup (Z \times [0,1])\).

In Step 1 below, we will show that \((\epsilon,T)\)-chains for arbitrary \(\epsilon,T > 0\) can be constructed between \([x],[y] \in \pi \circ \iota(I)\) with the property that no jump points in the chain belong to \([Z \times (1,\frac{1}{2})]\). We refer to such chains as \((\epsilon,T)\)-special chains. In Step 2 we will use Step 1 as a tool to prove that, for any \(\epsilon,T > 0\), there exists an \((\epsilon,T)\)-chain from \(x\) to \(y\) for the relaxed system \(H'\) if \([x],[y]\) are Conley related for \(\tilde{H}_H\). This will show that \(Ch_H(x,y)\), and hence \(Ch_H(x,y)\) by Lemma 6, as desired.

**Step 1:** Define \(U' := \pi_0(Z \times [\frac{1}{4},1]) \subseteq I'\), \(V' := \pi_0(Z \times (\frac{1}{2},1)) \subseteq U'\), and the continuous maps \(\rho' : U' \to Z'\) via \(\rho'(\pi_0(z,t)) := \pi_0(z,1)\) and \(\mu' : U' \to [0,\frac{3}{4}] \) via \(\mu'(\pi_0(z,t)) := 1 - t\). Since \(U'\) and \(U'\) are compact metrizable spaces, \(\pi|_{U'} : U' \to [U']\) is a quotient map since it is a continuous, closed, and surjective map. The maps \(\pi U'\circ \rho'\) and \(\mu'\) are both constant on fibers of \(\pi|_{U'}\) since the restriction of \(\pi\) to \(\pi_0(Z \times [\frac{1}{4},1])\) is injective, \(\mu'|_{Z'} \equiv 0\), and \((\pi|_{U'}\circ \rho')|_{Z'} = \pi|_{Z'}\) since \(\rho'|_{Z'} = \text{id}_{Z'}\). Thus, by the universal property of the quotient topology [Lee10, Thm 3.70], \(\pi|_{U'}\circ \rho'\) and \(\mu'\) descend to a continuous retraction \(\nu : [U'] \to [Z']\) and a continuous map \(\alpha : [U'] \to [0,\frac{3}{4}]\), respectively. Note that \(\nu\) preserves trajectories, and that \(\alpha\) is the “time-to-impact-[\(Z']\) map” for points in \([U']\).

Now fix \(\epsilon,T > 0\) and let \([x],[y] \in [I]\) be Conley related for \(\tilde{H}_H\). Since \([U'],\Sigma_H,\) and \(cl([U'])\) are compact, there exists \(\delta \in (0,\epsilon)\) such that

\[
[x],[y] \in [U'], \quad d_{\Sigma_H}([x],[y]) < \delta \implies d_{\Sigma_H}(\nu([x]),\nu([y])) < \epsilon
\]

Indeed, suppose there did not exist \(\delta > 0\) such that (26) held. Then there exist sequences \((v_n)_{n \in \mathbb{N}} \subseteq [V']\) and \((w_n)_{n \in \mathbb{N}} \subseteq \Sigma_H \setminus \text{int}([U'])\) with \(d_{\Sigma_H}(v_n,w_n) \to 0\) and \(d_{\Sigma_H}(\nu(v_n),w_n) \geq \epsilon\) for all \(n\). By passing to subsequences, we may assume \(v_n \to v \in cl([V'])\) and \(w_n \to w \in \Sigma_H \setminus \text{int}([U'])\) with \(d_{\Sigma_H}(\nu(v),w) \geq \epsilon\).

Indeed, suppose there did not exist \(\delta > 0\) such that (26) held. Then there exist sequences \((v_n)_{n \in \mathbb{N}} \subseteq [V']\) and \((w_n)_{n \in \mathbb{N}} \subseteq \Sigma_H \setminus \text{int}([U'])\) with \(d_{\Sigma_H}(v_n,w_n) \to 0\) and \(d_{\Sigma_H}(\nu(v_n),w_n) \geq \epsilon\) for all \(n\). By passing to subsequences, we may assume \(v_n \to v \in cl([V'])\) and \(w_n \to w \in \Sigma_H \setminus \text{int}([U'])\) with \(d_{\Sigma_H}(\nu(v),w) \geq \epsilon\). Since \(d_{\Sigma_H}(v_n,w_n) \to 0\), we have \(v = w\). Thus, \(v \in cl([V']) \cap (\Sigma_H \setminus \text{int}([U'])) = [Z']\) and \(d_{\Sigma_H}(\nu(v),v) \geq \epsilon\), a contradiction since \(\nu|[Z'] = \text{id}|[Z']\).

Let \(\chi = (N,\tau^{(0)},\eta^{(0)},\gamma^{(0)}) \in Ch^{H,T+3/4}_{\tilde{H}_H}([x],[y])\); recall from Definition 4 that \(N \geq 1\). We will now modify the chain \(\chi\) inductively. Fix \(i \in \{0,\ldots,N-1\}\) and assume that, if \(i \geq 1\), we have modified the first \((i+1)\) arcs \((\gamma_0^{(i)},\ldots,\gamma_i^{(i)})\) to obtain a chain \(\chi = (N,\tau^{(0)},\eta^{(0)},\gamma^{(0)}) \in Ch^{H,T+3/4}_{\tilde{H}_H}([x],[y])\) such that (a) the sub-chain obtained by throwing away the first \((i+1)\) arcs of \(\chi\) is either a single arc (if \(i = N-1\)) or a \((\delta,T+3/4)\)-chain; and (b) for all \(j \in \{1,\ldots,i\}\): \(\gamma_j^{(i)}(\tau_j^{(i)}) \in [V']\).

- If both \(\gamma_i^{(i)}(\tau_i^{(i)}) = 1\) and \(\gamma_i^{(i)}(\tau_i^{(i)}) = 1\) are in \([U']\), then we replace \(\gamma_i^{(i)}\) with the curve

\[
[t \to \Phi_H^{t+\gamma_i^{(i)}(\tau_i^{(i)})}(\gamma_i^{(i)}(\tau_i^{(i)}))]
\]

and \(\tau_{i+1}^{(i)}\) with the curve

\[
[t \to \Phi_H^{t+\alpha(\gamma_i^{(i)}(\tau_i^{(i)}))}(\gamma_i^{(i)}(\tau_i^{(i)}))]
\]

- If \(\gamma_i^{(i)}(\tau_i^{(i)}) = [V']\) and \(\gamma_i^{(i)}(\tau_i^{(i)}) \in \Sigma_H \setminus \text{int}([U'])\), then we replace \(\gamma_i^{(i)}\) with the curve defined by (27), but we do not modify \(\gamma_i^{(i)}\).

\footnote{The semiflow \(\Phi_H\) defines an MHS \(\tilde{H}\) with guard \(\varnothing\) (c.f. Example 2). To avoid introducing extra notation, here we confine \(\Phi_H\) with \(\tilde{H}\) by writing, e.g., \(Ch^{H,T+3/4}_{\tilde{H}_H}([x],[y])\) instead of \(Ch^{H,T+3/4}_{\tilde{H}_H}([x],[y])\). We additionally remark that, since the guard for \(\tilde{H}\) is \(\varnothing\), chains can only have continuous-time jumps, so every chain \(\chi = (N,\tau,\eta,\gamma)\) for \(\Phi_H\) satisfies \(\eta \in (0,1,2,\ldots,N)\).}
If \( \gamma^{(i)}_{j+1}(\tau^{(i)}_{j+1}) \in [V'] \) and \( \gamma^{(i)}_{j+1}(\tau^{(i)}_{j+1}) \in \Sigma_H \setminus \text{int}([U']) \), then we replace \( \gamma^{(i)}_{j+1} \) with the curve defined by (28), but we do not modify \( \gamma^{(i)}_j \).

The upper bound \( \alpha(\cdot) \leq \frac{3}{4} \) and Equations (25), (26) can be used to show that, after redefining the sequences \( \eta^{(i)} \) and \( \tau^{(i)} \) accordingly, the result is a chain \( \chi^{(i+1)} = (N, \tau^{(i+1)}, \eta^{(i+1)}, \gamma^{(i+1)}) \in Ch^{\epsilon,T}_{\Phi_H}([x],[y]) \) such that (a) the sub-chain obtained by throwing away the first \( i+2 \) arcs of \( \chi^{(i+1)} \) is either empty (if \( i = N - 1 \)), a single arc (if \( i = N - 2 \)), or a \( (\delta, T + 3/4) \)-chain; and (b) for all \( j \in \{1, \ldots, i+1\} \):

\[
\gamma^{(i+1)}_{j+1}(\tau^{(i+1)}_{j+1}), \gamma^{(i+1)}_{j}(\tau^{(i+1)}_{j}) \notin [V'].
\]

Hence by induction we obtain a chain \( \chi \in Ch^{\epsilon,T}_{\Phi_H}([x],[y]) \) satisfying \( \gamma_i(\tau_i), \gamma_i(\tau_{i+1}) \notin [V'] \) for every arc of \( \chi \). This shows that there exists an \( (\epsilon, T) \)-special chain from \([x]\) to \([y]\) and completes the proof of Step 1.

**Step 2:** Fix \( \epsilon > 0 \) and define \( W' := \eta(I) \cup \pi_0(Z \times [0, \frac{1}{2}]) \subseteq I' \). Since \( W' \) is compact and \( \pi|_{W'} \) is injective, \( \pi|_{W'}: W' \to [W'] = \Sigma_H \setminus [V'] \) is a homeomorphism of compact metric spaces [Lee10, Lem 4.50.d]. It follows that the inverse homeomorphism \( (\pi|_{W'})^{-1}: \Sigma_H \setminus [V'] \to W' \) is uniformly continuous, so there exists \( \delta > 0 \) such that \( d_{\Sigma_H}([z],[w]) < \delta \) implies that \( d_{W'}(z,w) < \epsilon \) for all \( z, w \in W' \).

Now fix \( \epsilon, T > 0 \), let \( \delta > 0 \) as in the above paragraph, and let \( x, y \in I \) be such that \([x],[y] \in [I] \subseteq [W']\) are Conley related for \( \Phi_H \). By Step 1, there exists a \( (\delta, T) \)-special chain \( \chi = (N, \tau, \eta, \gamma) \) from \([x]\) to \([y]\). Since \( \pi|_{W'} \) is a homeomorphism onto its image, we can define a sequence of “lifted and reset-subdivided” continuous arcs \( \tau_j \) in \( I' \) by first lifting each component of \( \gamma^{-1}_{i}([W']) \) via \( (\pi|_{W'})^{-1} \), then extending each lifted component terminating at a point \( \pi_0(z, \frac{1}{2}) \) in the boundary of \( W' \) via concatenation with \( \left( t \in [0, \frac{1}{2}] \mapsto \pi_0(z, t + \frac{1}{2}) \right) \). Since \( d_{\Sigma_H}((\gamma_i(\tau_{i+1}), \gamma_{i+1}|_{\tau_{i+1}})) < \delta \) for each \( i \), it follows from our choice of \( \delta \) that the resulting family \( \tau_j \) of arcs yields an \( (\epsilon, T) \)-chain for \( H' \). This shows that \( x, y \in I \) are Conley related for \( H' \). By Lemma 6, this shows that \( x, y \in I \) are also Conley related for \( H \) and completes the proof.

6.4. **Proofs of Theorems 1 and 2.** We are now in a position to prove our main theorems, which we restate for convenience.

**Theorem 1** (Conley’s decomposition theorem for MHS). Let \( H = (I, F, Z, \varphi, r) \) be a deterministic metric hybrid system. Assume that \( I \) is compact and that \( Z \) is a trapping guard. Further suppose that, for every \( x \in I \), there is an infinite or Zeno execution starting at \( x \). Then the hybrid chain recurrent set \( R(H) \) admits a Conley decomposition:

\[
R(H) = \bigcap \{A \cup A^* | A \text{ is an attracting set for } H\}.
\]

Furthermore, \( x, y \in I \) are chain equivalent if and only if either \( x, y \in A \) or \( x, y \in A^* \) for every attracting-repelling pair \((A,A^*)\).

**Proof.** Let \( H' = (I', F', Z', \varphi', r') \) and \((\Sigma_H, \Phi_H)\) be the relaxed hybrid system and hybrid suspension of Definitions 14 and 15 (equipped with any compatible extended metrics), \( \iota: I \to I' \) be the obvious embedding, and \( \pi: I' \to \Sigma_H \) be the quotient map of Definition 15. Letting \( R(\Phi_H) \) denote the chain recurrent set for \( \Phi_H \), the Conley decomposition theorem for semiflows [Hur95, Thm 2] and Example 2 imply that

\[
R(\Phi_H) = \bigcap \{B \cup B^* | B \text{ is an attracting set for } \Phi_H\},
\]

and that \( \pi \circ \iota(x) \) is chain equivalent to \( \pi \circ \iota(y) \) if and only if either \( \pi \circ \iota(x), \pi \circ \iota(y) \in B \) or \( \pi \circ \iota(x), \pi \circ \iota(y) \in B^* \) for every attracting-repelling pair \((B,B^*)\) for \( \Phi_H \). Proposition 5 implies that \( R(H) = (\pi \circ \iota)^{-1}(R(\Phi_H)) \) and, furthermore, that the chain equivalence classes of \( H \) are precisely the \((\pi \circ \iota)^{-1}\)-preimages of chain equivalence classes for \( \Phi_H \). Hence to complete the proof it would suffice to show that \((A,A^*)\) is an attracting-repelling pair for \( H \) if and only if \((A,A^*) = ((\pi \circ \iota)^{-1}(B), (\pi \circ \iota)^{-1}(B^*))\) for some attracting-repelling pair \((B,B^*)\) for \( \Phi_H \), but this is the content of Proposition 4. This completes the proof.

We now prove

\[\footnote{For this step of the proof it is crucial that our definition of \((\epsilon,T)\)-chains (Definition 4) allows for “double jumps,” as illustrated in Figure 2.}\]
Theorem 2 (Conley’s fundamental theorem for MHS). Let $H = (I, F, Z, \varphi, r)$ be a deterministic metric hybrid system. Assume that $I$ is compact and that $Z$ is a trapping guard. Further suppose that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Then there exists a complete Lyapunov function for $H$.

Proof. Let $H' = (I', F', Z', \varphi', r')$ be the relaxed hybrid system of Definition 14 and $\iota: I \to I'$ the obvious embedding, and let $(\Sigma_H, \Phi_H)$ be the hybrid suspension of Definition 15 with quotient $\pi: I' \to \Sigma_H$.

As discussed in Definition 15, Lemma 4 implies that $\Sigma_H$ is compact and metrizable. Since $\Sigma_H$ is compact, the Conley relation is independent of the choice of compatible extended metric on $\Sigma_H$ (c.f. Remark 5). Hence (after equipping $\Sigma_H$ with any compatible extended metric and appealing to Example 2) we may apply the fundamental theorem of dynamical systems for semiflows [Pat11, Thm 1.1] to conclude that there exists a complete Lyapunov function $V: \Sigma_H \to \mathbb{R}$ for $\Phi_H$. Letting $R(\Phi_H)$ denote the chain recurrent set for $\Phi_H$, this means that $V$ is a continuous function such that (i) $t \mapsto V(\Phi_H(\pi(x)))$ is strictly decreasing for all $\pi(x) \not\in R(\Phi_H)$, (ii) $V(R(\Phi_H))$ is nowhere dense in $\mathbb{R}$, and (iii) for all $c \in V(R(\Phi_H))$, $V^{-1}(c)$ is a chain equivalence class.

Define $L: I \to \mathbb{R}$ via $L := V \circ \pi \circ \iota$; $L$ is continuous since $L$ is a composition of continuous functions. By construction, we have that (i) for every $x \in F \setminus ((\pi \circ \iota)^{-1}(R(\Phi_H)))$, $t > 0$, and $y \in \chi_x(t)$, $L(y) < L(x)$; and (ii) if $x \in Z \setminus ((\pi \circ \iota)^{-1}(R(\Phi_H)))$, then $L(r(x)) < L(x)$. Proposition 5 implies that $(\pi \circ \iota)(R(H)) \subseteq R(\Phi_H)$, so $L(R(H)) = V \circ \pi \circ \iota(L(R(H))) \subseteq V(R(\Phi_H))$; therefore, $L(R(H))$ is nowhere dense in $\mathbb{R}$. It remains only to show that, for each $c \in L(R(H))$, $L^{-1}(c)$ is a chain equivalence class for $H$. But $V^{-1}(c)$ is a $\Phi_H$ chain equivalence class if $c \in L(R(H)) \subseteq V(R(\Phi_H))$, and Proposition 5 implies that every $(\pi \circ \iota)$-preimage of a $\Phi_H$ chain equivalence class is an $H$ chain equivalence class, so $L^{-1}(c) = (\pi \circ \iota)^{-1}(V^{-1}(c))$ is an $H$ chain equivalence class if $c \in L(R(H))$. This completes the proof. \hfill $\square$

7. Conclusion

Using the language of hybrid systems, we have obtained a simultaneous generalization (Theorem 2) of both the continuous-time and discrete-time versions of Conley’s fundamental theorem [Con78, Fra88]. As in the classical setting, our theorem asserts the existence of a globally-defined complete Lyapunov function (Definition 12). We have also proved a result (Theorem 1) generalizing Conley’s decomposition theorem, which asserts that the chain recurrent set (Definition 6) is the intersection of all attracting-repelling pairs (Definition 10).

While this unification of the continuous and discrete is pleasingly parsimonious, our motivation is not merely parsimony for its own sake. Our long-term aim, motivated particularly by applications in robotics and biomechanics (e.g., legged locomotion [HFKG06, RK15, SKR+17]), is to continue advancing the program of developing hybrid dynamical systems theory to the same footing as its more mathematically mature parents, the theories of continuous-time and discrete-time dynamical systems. For example, beyond the constructive applications of Conley’s theorems discussed in the introduction, practitioners might well choose to use their appearance as a kind of litmus test against which to judge the relative merits of the many different hybrid systems models that have appeared in the literature. Models that do not possess such a decomposition into chain-recurrent and gradient-like parts might be subject to greater scrutiny—their questionably disorderly behavior only tolerated in consequence of expressing some physical property essential to the phenomena of interest. \hfill $\square$ Thus, by the same token, we hope that our presentation of sufficient conditions for a Conley theory of hybrid systems may encourage more theorists to help determine which properties are necessary. Indeed, despite the ubiquity of hybrid systems in engineering, mathematicians have mostly avoided them, perhaps due to the lack of a single concise definition. In this light, one of the contributions of the current paper is a parsimonious definition partially generalizing a physically important [JBK16] class of hybrid systems (Definition 1), which we hope may be more inviting to the mathematically inclined reader.

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23 This program has contributions from many investigators. We only mention a few: [BGM93, Gue95, YMH98, AHLP00, SJSL00, SJLS01, SSJL02, LJS+03, SJLS05, HTP05, BR15, GST09, Ler16, JBK16, BSKR16, BLC18, BC18, CBC19, CB20, LS20].

24 See the second paragraph of §A.1 for a relevant discussion.
Norton \cite{Nor95} emphasized that the Fundamental Theorem of Dynamical Systems is not the end of the theory but the beginning. Just as the Fundamental Theorems of Arithmetic, Algebra, and Calculus provide the most basic tools of their respective fields, the Fundamental Theorem of Dynamical Systems indicates that the coarsest building blocks of dynamical systems are the countable components of the steady state (the “chain-recurrent”) set and their basins (adding in the components of the “gradient-like” sets that lead to them). The primary results of this paper show that these same building blocks fit together in the same way to describe a broad, physically important class of hybrid dynamical systems.

\section*{Appendix A. Relation with selected prior work}

\subsection*{A.1. Relation of Definition 1 to \cite{JBK16}.} Our definition of THS strictly generalizes \cite[Def. 2]{JBK16}, modulo our added regularizing assumption requiring that the union of guard sets be closed. Since any disjoint union of smooth (paracompact) manifolds with corners is metrizable, our definition of MHS similarly strictly generalizes \cite[Def. 2]{JBK16}, modulo the closed guard assumption and the choice of a compatible extended metric on state space. However, we note that our Theorems 1 and 2 impose two additional conditions on MHS that are not assumed in \cite{JBK16}: our theorems require that (i) state space is compact, and (ii) the trapping guard condition (Definition 11) is satisfied. (We also add that \cite{JBK16} consider continuations of Zeno executions past the stop time, while we do not; c.f. Remark 2.) Regarding (i) we note that, as discussed in Remark 5, the specific choice of compatible extended metric is immaterial for the majority of our purposes since Theorems 1 and 2 require that state space is compact.\footnote{For the interested reader, we briefly mention that specific metrizations of hybrid systems are discussed in \cite{BGV+15}. However, we caution that, e.g., the pseudometric defined in \cite[Sec. III.A]{BGV+15} is not generally an extended metric compatible with the topology on state space (under certain assumptions it defines a metric on a certain quotient of state space, the \textit{hybrifold} discussed in \S A.4.1), so it is not generally an admissible extended metric making a metrizable THS into an MHS.}

Regarding (ii), the hybrid systems model of \cite{JBK16} allows for the possibility that no hybrid suspension semiflow (Definition 15) exists which is continuous-in-state, thereby precluding the trapping guard condition as shown in Appendix C which, in turn, may compromise the necessity of a Conley decomposition and Lyapunov function (e.g., see Examples 3 and 5 for one view of the gap between the sufficiency and the necessity of this condition). For other classes of physical models, continuity can fail for different reasons. While our Definition 1 and \cite[Def. 2]{JBK16} require reset maps to be continuous, parsimonious hybrid models of certain physical systems may fail even to have, e.g., continuous reset maps (though in many applications it might be acceptable to insure continuity—e.g., one might smooth down the model of an exterior wall’s outer corner so as to insure that balls bounce off it in a continuous manner). However, the discontinuities of behavior allowed by the \cite{JBK16} and other hybrid systems models may play a key role in other problem settings, such as legged leaping as explored in \cite[Fig. 7, Sec. III.b]{BDJK15}, \cite{JK13}. Clearly, more theoretical work is needed to understand the prospect for achieving Conley-style results in these settings, while, at the same time, more empirical work is needed to understand how the phenomena of interest should be formally represented and intuitively understood in their absence.

\subsection*{A.2. Relation of Definition 1 to \cite{AS05}.} Our definition of THS is particularly similar to the definition of “classical hybrid system” in \cite[p. 92]{AS05}. However, there are some differences. First, we ignore any underlying graph structure of the hybrid system, although Remark 1 explains that this is immaterial. Second, the definition in \cite[p. 92]{AS05} amounts, using our notation, to requiring a \textit{flow} $\Phi$ be defined on $I$; in contrast, we only require a \textit{semiflow} be defined on $F \subseteq I$. Finally, we impose the regularizing requirement that the guard $Z \subseteq I$ be closed; this requirement is not made in \cite[p. 92]{AS05}.

\subsection*{A.3. Relation of Definition 4 to \cite{CGKS19}.} Our definition of $(\epsilon, T)$-chains (Definition 4) is closest to that of \cite[Def. 2.18]{CGKS19}. While our presentations differ, the only mathematical difference is our requirement that an $(\epsilon, T)$-chain contain at least two arcs. If this were not the case, then the Conley relation (as defined in Definition 5) would be the trivial relation in which every pair of points are related. It is clear that every $(\epsilon, T)$-chain in our sense is also an $(\epsilon, T)$-chain in the sense of \cite[Def. 2.18]{CGKS19}, but not vice versa.

\subsection*{A.4. Relation of the relaxed hybrid system and hybrid suspension to prior work.}
A.4.1. **Generalized hybrifolds.** The **hybrifold** of a hybrid system was introduced in [SJSL00, SJLS05] for a class of hybrid systems satisfying various smoothness assumptions: e.g., state space is required to be a disjoint union of manifolds with “piecewise-smooth boundary,” and reset maps are required to be diffeomorphisms onto their images. Our classes THS and MHS of hybrid systems do not assume any such smoothness nor injectivity properties, but we can still give a definition analogous to that of the hybrifold in our setting. We will refer to this analogous, but (formally) more general, construction as the **generalized hybrifold.**

Let $H := (I, F, Z, \varphi, r)$ be a THS. Using the notation “$M_H$” of [SJSL00, SJLS05], we define the **generalized hybrifold** $M_H$ of $H$ to be the topological space obtained by gluing points $z \in Z \subseteq I$ to $I$ along the reset $r$:

$$M_H := I / \{ z \sim r(z) \}.$$ 

Assuming that $H$ is deterministic, satisfies the trapping guard condition, and has only infinite or Zeno maximal executions, we define the **generalized hybrifold semiflow** $\Psi_{M_H}$ to be the unique semiflow on $M_H$ such that the quotient $\pi_{M_H} : I \to M_H$ sends $H$ executions to $\Psi_{M_H}$ trajectories while preserving time. Using Lemma 2 and the universal property of the quotient topology [Lee10, Thm 3.70], it can be shown that these hypotheses are **sufficient** to ensure that $\Psi_{M_H}$ is well-defined and continuous. A cartoon depiction of a generalized hybrifold (including a trajectory of $\Psi_{M_H}$) is shown in the bottom-left panel of Figure 9.

Lemma 4 implies that, if $I$ is metrizable and the reset $r$ is a closed map with compact fibers, then $M_H$ is metrizable. In particular, if $I$ is compact and metrizable, $M_H$ is metrizable.

However, the generalized hybrifold $M_H$ of a compact MHS can not be used to prove Theorems 1 and 2 for multiple reasons. For example (as pointed out in Remarks 17 and 19), omega-limit sets, attracting-repelling pairs, and chain recurrence for $(M_H, \Psi_H)$ are not generally compatible with the corresponding notions for $H$.

Furthermore, even if these compatibility issues were not present for a **specific** MHS $H$, a complete Lyapunov function for $\Psi_{M_H}$ will not generally pull back to a complete Lyapunov function for $H$. More explicitly, if $V : M_H \to \mathbb{R}$ is a complete Lyapunov function for $\Psi_{M_H}$, then the function $L := V \circ \pi_{M_H}$ will not generally be a complete Lyapunov function for $H$, because it will not satisfy the second condition of Definition 12 ($L$ will not decrease across resets). Thus, the technique used in the proof of Theorem 2 would still fail if the hybrifold was used instead of the hybrid suspension.

A.4.2. **Relaxed hybrid system.** As mentioned in §6.2, the relaxed hybrid system $H'$ (Definition 14) formalizes the idea of requiring that executions of the hybrid system $H$ “wait” one time unit after impacting the guard before resetting. A cartoon depicting the relaxed system is shown in the top-right panel of Figure 9. The relaxed system is essentially an example of a **temporal relaxation** in the sense of [JELS99], where it was used to regularize Zeno executions, although we give the definition for THS which are (formally speaking) more general than the specific examples considered in [JELS99, Sec. 3–4] (e.g., the local semiflows for THS are not assumed to be generated by vector fields and, furthermore, the state space of a THS is a general topological space rather than any sort of manifold). While we recover this Zeno regularization in our setting (Remark 13), our primary motivation for the relaxed system is to use it as an intermediate alternative, this generalization has also been referred to as a “colimit” in [AS05, p. 94].

26The terminology hybrifold is unfortunately no longer appropriate since no manifolds are involved. As one possible alternative, this generalization has also been referred to as a “colimit” in [AS05, p. 94].

27We emphasize that these conditions—in particular, the trapping guard condition—are only **sufficient** to ensure that a well-defined and continuous generalized hybrifold semiflow exists. The simple example of a THS $H = (I, F, Z, \varphi, r)$ with $I = [0, 1], F = (0, 1], Z = \emptyset, r(0) = 1$, and $\varphi$ generated by the vector field $-x(1-x)\frac{\partial}{\partial x}$ shows that the trapping guard is **not necessary** for the generalized hybrifold semiflow $\Psi_{M_H}$ to be well-defined and continuous. The reader may wish to contrast this with the converse statement of Corollary 5 in Appendix C for the hybrid suspension semiflow $\Phi_H$.

28Engineers might be unimpressed by the apparently slight formal gain of generality. Applications generally present models with smooth manifolds carrying (at least piecewise) smooth vector fields. In contrast, classical Conley theory is rooted in the tools of topological dynamics whose framework we have thus found it natural to adopt here. Furthermore, we hope that the imperative to eliminate smoothness assumptions from the spaces carrying these dynamics may be intuitively apparent when considering the (pinched and creased non-manifold) **topological spaces** that inevitably arise as depicted, for example, in the hybrid suspension $\Sigma_H$ of Fig. 8.
Hybrid system $H = (I, F, Z, \varphi, r)$

Relaxed system $H'$ associated to $H$

(Generalized) hybrifold $M_H$ of $H$

Hybrid suspension $\Sigma_H$ ($= M_{H'}$) of $H$

Figure 9. Comparison of the constructions from §6.2 depicted in Figure 8 with the generalized hybrifold $M_H$ of $H$ discussed in §A.4.1. Top left: a THS $H = (I, F, Z, \varphi, r)$. Top right: its relaxed version $H' = (I', F', Z', \varphi', r')$; see Definition 14. Bottom right: hybrid suspension $(\Sigma_H, \Phi_H)$ of $H$; see Definition 15. Bottom left: the generalized hybrifold $M_H$ of $H$; $M_H$ is formed by gluing $Z$ directly to $r(Z)$ along $r$, without first embedding $I$ in a larger space (unlike $\Sigma_H = M_{H'}$). We mention $M_H$ only for purposes of comparison, i.e., we do not use $M_H$ in this paper. We remark that $\Sigma_H$ coincides with the generalized hybrifold $M_{H'}$ of $H'$ (but not with $M_H$). Additionally, we emphasize that $(\Sigma_H, \Phi_H)$ strictly generalizes the classical suspension of a discrete-time dynamical system discussed in Appendix B; indeed, if $I = Z$ (c.f. Example 1) our construction reduces to the classical one.

step in constructing the hybrid suspension (Definition 15), which has better properties than those of the generalized hybrifold discussed in §A.4.1.

A.4.3. Prior work related to the hybrid suspension. The technique we used to prove Theorems 1 and 2 involves showing that a THS satisfying the trapping guard condition and certain other assumptions is, in a certain sense (Propositions 3, 4, and 5), no different from a certain continuous-time (semi-)dynamical system. As discussed in §A.4.1 and Remarks 17 and 19, this continuous-time system is not the generalized hybrifold (local) semiflow; it is the hybrid suspension semiflow constructed in Definition 15. We choose to use the terminology “suspension” because the hybrid suspension strictly generalizes the classical suspension [Sma67, BS02, p. 797, pp. 21–22] of a discrete-time dynamical system; indeed, if $I = Z$ our construction
of hybrid systems which are not as general as THS. The hybrid suspension appeared in the literature under different names, although (to the best of our knowledge) only for classes in Appendix B.

We finally note that, in the terminology introduced in §A.4.1, the hybrid suspension $\Sigma_H$ of a THS $H$ coincides with the generalized hybritfold $H_F'$ (where $H'$ is the relaxed hybrid system) but not with the hybritfold $M_H$. See Figure 9.

APPENDIX B. CLASICAL SUSPENSION OF A DISCRETE-TIME DYNAMICAL SYSTEM

The purpose of this appendix is to explain, in a self-contained way, (i) the classical suspension of a discrete-time dynamical system and (ii) how the hybrid suspension (Definition 15) strictly generalizes the classical notion.

The classical suspension is often considered in the context of a $C^{r \geq 1}$ diffeomorphism of a $C^r$ manifold [Sma67, PT77, PjdM82, Rob99, p. 797, pp. 343–345, p. 111, p. 173]. However, relevant for us is the more general context of a discrete-time (semi-)dynamical system defined by a continuous map of a topological space; we now describe the classical suspension in this context, following roughly [BS02, pp. 21–22].

Let $X$ be a topological space, $f: X \to X$ be a continuous map defining a discrete-time (semi-)dynamical system $(n,x) \mapsto f^n(x)$, and $c: X \to (0, \infty)$ be a continuous function bounded away from zero. Consider the quotient space

$$X_c := \{(x,t) \in X \times [0,\infty) : 0 \leq t \leq c(x)\} / \sim,$$

where $\sim$ is the equivalence relation $(x,c(x)) \sim (f(x),0)$. $X_c$ is called the **suspension with ceiling (or roof) function** $c$. Letting $[(x,s)] \in X_c$ denote the equivalence class of $(x,s)$, the **suspension semiflow (with ceiling function)** $c$ is the semiflow $\phi_c: [0,\infty) \times X_c \to X_c$ given by $\phi^t([x,s]) = [(f^t(x),n')]$, where $n \in \mathbb{N}$ and $n' \geq 0$ satisfy

$$\sum_{i=0}^{n-1} c(f^i(x)) + n' = t + s, \quad 0 \leq n' \leq c(f^n(x)).$$

It is common to simply take the ceiling function $c$ to be $c(x) \equiv 1$ [Sma67, Rob99, PT77, PjdM82, Rob99, p. 797, pp. 343–345, p. 111, p. 173], and in this case it is common to simply refer to $(X_1, \phi_1)$ as “the” suspension of (the discrete-time semi-dynamical system defined by) $f$. Our **hybrid suspension** defined in Definition 15 strictly generalizes “the” suspension of $f$, and this can be seen as follows. Define a THS $H = (I,F,Z,\varphi,r)$ by setting $I = Z = X$, $F = \emptyset$, $r = f$, and (viewed set-theoretically) $\varphi = \emptyset$ (c.f. Example 1). Then the hybrid suspension $(\Sigma_H, \Phi_H)$ coincides precisely with $(X_1, \phi_1)$.

APPENDIX C. CONTINUOUS HYBRID SUSPENSION SEMIFLOW IMPLIES THE TRAPPING GUARD CONDITION

Let $H = (I,F,Z,\varphi,r)$ be a deterministic THS having only infinite or Zeno maximal executions. Under the assumption that $H$ satisfies the trapping guard condition (Definition 11), in §6.2 (Definitions 14 and 15) we defined

$$I' := \frac{I \cup (Z \times [0,1])}{z \sim (z,0)}, \quad \pi_0: I \cup (Z \times [0,1]) \to I', \quad \pi: I \to \Sigma_H,$$

\[\text{(30)}\]
constructed the suspension semiflow $\Phi_H : [0, \infty) \times \Sigma_H \to \Sigma_H$, and showed that $\Phi_H$ is continuous. It is immediate from the definitions that $\Phi_H$ satisfies the following two properties.

1. $\Phi_H(\pi \circ \pi_0(z, s)) = \pi \circ \pi_0(z, t + s)$ for all $z \in Z$, $s \in [0, 1]$, and $t \in [0, 1 - s]$.
2. For all $(t, x) \in \text{dom}(\varphi)$, $\pi \circ \iota(\varphi^0(x)) = \Phi_H(\pi \circ \iota(x))$.

While for convenience of exposition we only defined the quantities in (30) under all of the above assumptions (in particular, assuming the trapping guard condition), the definitions in (30) make sense verbatim for any THS. Thus, for an arbitrary THS $H$, it makes sense to ask the following question: under what circumstances does there exist a well-defined “suspension semiflow” $\Phi$ on $\Sigma_H$ for $H$ in the sense that $\Phi$ satisfies Conditions 1 and 2 (stated above for $\Phi_H$)?

In this appendix we prove a result (Proposition 6) which implies (Corollary 5) that, if $H = (I, F, Z, \varphi, r)$ is a deterministic THS with Hausdorff $I$ satisfying the trapping guard condition and having only infinite or Zeno maximal executions, then there exists a continuous suspension semiflow $\Phi : [0, \infty) \times \Sigma_H \to \Sigma_H$ in the above sense if and only if $H$ satisfies the trapping guard condition.

We state Proposition 6 after first establishing the following preliminary result.

**Lemma 8.** Let $H = (I, F, Z, \varphi, r)$ be a THS. Define $I'$, $\Sigma_H$, $\pi_0$, $\iota$, and $\pi$ as in (30). Then

$$\pi \circ \pi_0|_{Z \times [0, \frac{1}{2}]} : Z \times [0, \frac{1}{2}] \to \Sigma_H$$

are homeomorphisms onto their images.

**Proof.** We first show that $\pi|_{\iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}] )}$ is a closed map. Define $Z' := \pi_0(Z \times \{1\})$ and $r' : Z' \to I'$ via $r'(\pi_0(z, 1)) := \pi_0(r(z))$, and let $C \subseteq \iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}])$ be an arbitrary closed set. We compute

$$\pi^{-1}(\pi(C)) = C \cup (r')^{-1}(C) \cup r'(C \cap Z').$$

Since $r'$ is continuous, the right side of (31) is the union of three closed sets (the third is empty since $\iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}])$ is disjoint from $Z'$). By the definition of the quotient topology, it follows that $\pi(C)$ is closed in $\Sigma_H$, so $\pi|_{\iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}])}$ is indeed a closed map.

Since $\pi|_{\iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}])}$ is a closed map,

$$\pi \circ \pi_0|_{Z \times [0, \frac{1}{2}]} = \pi|_{\iota(I) \cup \pi_0(Z \times [0, \frac{1}{2}])} \circ \pi_0|_{Z \times [0, \frac{1}{2}]}$$

are closed maps by composition, since $\pi_0$ and $\iota$ are closed maps. That $\pi_0$ is closed follows by repeating the proof of Lemma 5 verbatim, and $\iota$ is closed since $\pi_0^{-1}(\iota(D)) = D \cup ((D \cap Z) \times \{0\})$ is closed in $I \cup (Z \times [0, 1])$ for any closed set $D \subseteq I$.

It is immediate from the definitions that both maps in the statement of the lemma are continuous and injective. Since we have shown that they are also closed, it follows that they are homeomorphisms onto their images [Lee10, Ex. 2.29].

**Proposition 6.** Let $H = (I, F, Z, \varphi, r)$ be a deterministic THS such that $I$ is Hausdorff and $I = F \cup Z$. Define $I'$, $\Sigma_H$, $\pi_0$, $\iota$, and $\pi$ as in (30), and suppose that $\Phi : [0, \infty) \times \Sigma_H \to \Sigma_H$ is a continuous semiflow satisfying the following conditions.

1. $\Phi^t(\pi \circ \pi_0(z, s)) = \pi \circ \pi_0(z, s + t)$ for all $z \in Z$, $s \in [0, 1]$, and $t \in [0, 1 - s]$.
2. For all $(t, x) \in \text{dom}(\varphi)$, $\pi \circ \iota(\varphi^0(x)) = \Phi^t(\pi \circ \iota(x))$.

Then $H$ satisfies the trapping guard condition.

**Corollary 5.** Let $H = (I, F, Z, \varphi, r)$ be a deterministic THS such that $I$ is Hausdorff. Assume that, for every $x \in I$, there is an infinite or Zeno execution starting at $x$. Define $I'$, $\Sigma_H$, $\pi_0$, $\iota$, and $\pi$ as in (30). Then there exists a continuous “suspension semiflow” $\Phi : [0, \infty) \times \Sigma_H \to \Sigma_H$ for $H$—in the sense that $\Phi$ satisfies conditions 6.1 and 6.2 of Proposition 6—if and only if $H$ satisfies the trapping guard condition.

**Proof of Corollary 5.** If $H$ satisfies the trapping guard condition, then the continuous semiflow $\Phi_H$ constructed in Definition 15 is such a semiflow.
Conversely, assume that a continuous semiflow $\Phi$ satisfying Conditions 6.1 and 6.2 exists. The assumption that all maximal executions are infinite or Zeno implies that $I = F \cup Z$ (by Remark 3), so the hypotheses of Proposition 6 are satisfied. By the conclusion of Proposition 6, $H$ satisfies the trapping guard condition.

Proof of Proposition 6. For purposes of readability, we define $f := \pi \circ \iota: I \to \pi \circ \iota(I)$, $B := Z \times [0, \frac{1}{2}]$, and $g := (\pi \circ \pi_0)|_B: B \to \pi \circ \pi_0(B)$. By Lemma 8, $f$ and $g$ are homeomorphisms.

Letting $\mu: I \to [0, +\infty]$ be the maximum flow time defined in (1), we first show that, for any $x \in F \cap \mu^{-1}([0, \infty))$,

$$\ell(x) := \lim_{t \to \mu(x)} \varphi^t(x) \in Z.$$  \hspace{1cm} (32)

That the limit $\ell(x)$ exists follows from continuity of $f$, $f^{-1}$, and $\Phi$ since Condition 6.2 implies that $\varphi^t(x) = f^{-1}(\Phi^t(f(x)))$ for all $t \in \{t \mid (t, x) \in \text{dom}(\varphi)\}$, and the properties of a local semiflow imply that $\{t \mid (t, x) \in \text{dom}(\varphi)\} = [0, \mu(x))$ for any $x \in F$ [HS06, Sec. 1.3]. Furthermore, it cannot be the case that $\ell(x) \in F$, because the trajectory image $\varphi^{[0, \mu(x)]}(x)$ would then have compact closure in $F$, and this in turn would imply that $\mu(x)$ is infinite [HS06, Sec. 1.3], a contradiction. Since we have also assumed that $I = F \cup Z$, it follows that $\ell(x) \in Z$.

Next, define $\tilde{U} := \Phi^{-\frac{1}{2}}(g(B))$, $U := f^{-1}(\tilde{U})$, and the continuous maps $h: g(B) \to [0, \frac{1}{2}]$ and $\nu: U \to [0, \frac{1}{2}]$ via $h(g(z, t)) := t$ and $\nu := \frac{1}{2} - h \circ \Phi^{-\frac{1}{2}} \circ f |_U$. By the definition of $\nu$ and Condition 6.1 it follows that $\nu^{-1}(0) = Z$ and $\Phi^\nu(x)(f(x)) \in f(Z)$ for all $x \in U$. We will now show that $\mu|_U = \nu$. Since $\varphi$ is $F$-valued but $\Phi^\nu(x)(f(x)) \in f(Z)$, it follows from Condition 6.2 and the fact that $F \cap Z = \emptyset$ (since $H$ is deterministic) that $(\nu(x), x) \notin \text{dom}(\varphi)$ for any $x \in U$. Since $\{t \mid (t, x) \in \text{dom}(\varphi)\} = [0, \mu(x))$ for any $x \in F$ [HS06, Sec. 1.3], it follows that $\mu|_{U \cap \nu} \leq \nu|_{U \cap \nu}$, and therefore $\mu|_U \leq \nu$ since $\mu|_U = \nu|_U = 0$. We now show the reverse inequality. It follows from 6.1 that, if $q \in f(Z)$, then $\Phi^q(q) \notin f(Z)$ for all $t \in (0, 1)$. Since $\nu \leq \frac{1}{2}$ and $\Phi^\nu(x)(f(x)) \in f(Z)$ for all $x \in U$, it therefore follows that $\Phi^\nu(f(x)) \notin f(Z)$ for all $t \in [0, \nu(x))$. Therefore, 6.2 implies that $\lim_{s \to t-} \varphi^s(x) = f^{-1}(\Phi^s(f(x))) \notin Z$ for any $x \in U$ and $t \in [0, \nu(x))$. From this and (32) it follows that $\mu|_U \geq \nu$. Since we have already shown that $\mu|_U \leq \nu$, this establishes that $\mu|_U = \nu$.

Since $\tilde{U} \cap f(I) = \Phi^{-\frac{1}{2}}(g(B)) \cap f(I) = \Phi^{-\frac{1}{2}}(g(Z \times [0, 1])) \cap f(I)$ is a neighborhood of $f(Z)$ in $f(I)$, $U = f^{-1}(\tilde{U})$ is a neighborhood of $Z$ in $I$. We now define $\tilde{\varphi} : \text{cl}(\text{dom}(\varphi)) \cap ([0, \infty) \times U) \to I$ and $\rho: U \to Z$ via

$$\tilde{\varphi}^\nu(x) := f^{-1} \circ \Phi^\nu \circ f(x), \hspace{1cm} \rho(x) := \varphi^\nu(x).$$  \hspace{1cm} (33)

Condition 6.2 implies that $\tilde{\varphi}$ is a continuous extension of $\varphi_{\text{dom}(\varphi) \cap ([0, \infty) \times U)}$, which satisfies Equation (2) of Definition 11 since $\nu = \mu|_U$, and this extension is automatically unique since $I$ is Hausdorff. The map $\rho$ is a continuous retraction by (33) and the fact that $\Phi^\nu(x)(f(x)) \in f(Z)$ for all $x \in U$ (as noted in the previous paragraph). Since $\mu|_U = \nu$ is continuous, it follows that all conditions of Definition 11 are satisfied. Hence $H$ satisfies the trapping guard condition. \hfill $\square$

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