Role of guiding centre in Landau level system and mechanical and pseudo orbital angular momenta

Yoshio Kitadono\textsuperscript{a,}\textsuperscript{*}, Masashi Wakamatsu\textsuperscript{a,b}, Liping Zou\textsuperscript{a}, Pengming Zhang\textsuperscript{a}

\textsuperscript{a}Institute of Modern Physics (IMP), Chinese Academy of Sciences (CAS), 509, Nanchang Road, Lanzhou, 730000, People’s Republic of China

\textsuperscript{b}Theory Center, Institute of Particle and Nuclear Studies (IPNS), High Energy Accelerator Research Organization (KEK), 1-1, Oho, Tsukuba, Ibaraki, 305-0801, Japan

Abstract

There is an interesting but not so popular quantity called pseudo orbital angular momentum (OAM) in the Landau level system, besides the well-known mechanical OAM. The pseudo OAM can be regarded as a gauge invariant extension of the canonical OAM, which is formally gauge invariant and reduces to the canonical OAM in a certain gauge. Since both of the pseudo OAM and the mechanical OAM are gauge invariant, it is impossible to judge which of those is superior to the other solely from the gauge principle. However, these two OAMs have different physical meanings. The mechanical OAM shows the manifest observability and the clear correspondence with the classical OAM of the cyclotron motion. On the other hand, we shall demonstrate that the standard canonical OAM as well as the pseudo OAM are the concepts which crucially depend on the choice of the origin of the coordinate system. We show the relation between the pseudo OAM and the mechanical OAM as well as their observability by paying special attention to the role of guiding centre operator.

Keywords: Landau level, Gauge theory, Pseudo angular momentum, Angular momentum

PACS: 71.70.Di, 03.65.-w, 11.15.-q, 11.30.-j

\textsuperscript{*}Corresponding author

Email address: kitadono@impcas.ac.cn (Yoshio Kitadono)
1. Introduction: two gauge invariant angular momenta

The quantum mechanics of the electron in a uniform and constant magnetic field plays central roles to discuss the fundamental property of the electron not only in matter physics, but also in high energy physics. In particular, the electron's energy level of this system is quantised and called the Landau level [1]. To describe this system in quantum mechanics, one needs to introduce the vector potential \( A \) which gives the magnetic field \( B \) through the relation, \( B = \nabla \times A \).

As is known, there are two different momenta for the electron under the influence of the magnetic field. The first one is the canonical momentum, \( \hat{p} = -i\nabla \), and the second one is the mechanical (or kinetic) momentum, \( \hat{\Pi} = \hat{p} + eA \). Obviously, the canonical momentum is not gauge invariant operator and the mechanical momentum is gauge invariant one and hence the canonical one is usually believed not to describe physical observables due to its gauge dependence [2]. One can generalise this concept to other operators so that physical observables should not depend on a choice of gauge. This is called the ”gauge principle” and works well. We can construct other operators, like orbital angular momentum (OAM), from these canonical and mechanical momenta. Consequently, the canonical OAM, \( \mathbf{x} \times \hat{p} \), depends on a gauge and the mechanical OAM, \( \mathbf{x} \times \hat{\Pi} \), does not depend on the gauge.

Recently, differences between the canonical OAM and the mechanical OAM for the electron in a uniform magnetic field and related topics are intensively studied in the context of a recent development of the vortex electron beam [3, 4, 5] and angular momenta of quarks and gluon for the nucleon spin decomposition problem [6, 7]. Although the purposes to study OAMs are different, there are similarities and common interests between these two different research fields, namely, what is the difference between the canonical OAM and the mechanical OAM in quantum theory from the viewpoint of gauge theory?

According to these recent developments on experiment and theory, it is a good time to consider OAMs in gauge theory again. We discussed several angular momenta in this Landau level system [8] by using DeWitt’s gauge invariant formalism of quantum electrodynamics (QED) [9] which enables us to solve the eigen equation without fixing a gauge, but with a choice of path. We focused on three OAMs: 1) the canonical OAM, 2) the mechanical OAM, and 3) ”pseudo” OAM [10, 11], where the mechanical and pseudo OAMs are gauge invariant. One can regard this pseudo OAM as a ”gauge invariant
extension of the canonical OAM”, because it is formally gauge invariant and it reduces to the canonical OAM in a particular gauge [6, 7].

We showed in the previous paper that the mechanical OAM has several favourable feature for that it is regarded as the electron’s physical OAM. It’s expectation value is independent of the gauge choice. It is related to the electron’s familiar OAM describing its cyclotron motion in classical theory. Furthermore, it is related to an observable of the system, i.e. the Landau energy in conformity with the gauge principle. On the contrary, although the pseudo OAM is gauge invariant and conserved, it does not correspond to any observable in the Landau system.

There are two questions left in our previous conclusion. If there is a gauge invariant extension of a gauge-dependent operator like the pseudo OAM, the observability of this quantity is not quite clear, because the gauge principle does not forbid the observability of the pseudo OAM which is formally gauge invariant. Moreover, now we have two gauge invariant OAMs, namely, the mechanical OAM and the pseudo OAM in the Landau level system. What is the difference between these two gauge invariant OAMs ? Can the pseudo OAM be an observable ? The Landau level system, as it is a solvable one, is a good testing ground to answer these questions.

The purpose of this paper is to clarify the difference among three OAMs in a new perspective. We compare the three OAMs defined with respect to the origin and the three OAMs defined with respect to the guiding centre both in classical theory and quantum theory. This enables us to understand the physical meaning as well as the difference of these three OAMs in the clearest fashion.

This paper is organised as follows. First, in Sec. 2 we discuss basics of Landau level system, i.e. wave functions in the Landau gauge and in symmetric gauge, the energy level, the gauge transformation between different gauges, and expectation values of these three OAMs. Next, in Sec. 3 we address some important properties of the concept of the guiding centre in classical theory and quantum theory, which is the key concept of this paper. We point out that the time-averaged mechanical OAM in classical theory corresponds to the well-known electron’s OAM of the cyclotron motion and hence we use this property as the guideline to distinguish two gauge invariant OAMs (mechanical OAM and pseudo OAM) in quantum theory. In Sec. 4, we discuss the expectation values for the three OAMs in quantum theory to address our questions: 1) what is the difference between the mechanical and pseudo OAMs ? and 2) can the pseudo OAM be an observable ?
2. Mechanical and pseudo orbital angular momenta of electron in a uniform magnetic field

In this section, we briefly review the well known basics of the Landau level system and we mention some aspects expected from the viewpoint of gauge theory. In addition, we review the main characteristics of the three OAMs, i.e. the canonical, mechanical, and the pseudo OAM in this system.

2.1. Landau level system

As is well known, the energy level for the charged particle (electron in this paper) moving in a two-dimensional plane with the electric charge $-e$ ($e > 0$) and mass $m_e$ in a uniform magnetic field is described by the Landau Hamiltonian:

$$\hat{H} = \frac{\hat{\Pi}^2}{2m_e} = \left(\hat{p} + e\mathbf{A}\right)^2, \quad \hat{H}\psi(x,y) = E\psi(x,y),$$

where $\hat{p} = -i\nabla$ is the canonical momentum, $\hat{\Pi} = \hat{p} + e\mathbf{A}$ is the mechanical momentum, and $\mathbf{A}$ is the gauge potential satisfying $\mathbf{B} = \nabla \times \mathbf{A}$. Here, we take $z$-axis as the direction of the uniform and constant magnetic field, namely, $\mathbf{B} = (0, 0, B)$. We use the natural unit, $\hbar = c = 1$, in this paper.

The quantised energy $E \equiv E_n$ is given by

$$E_n = \omega \left(n + \frac{1}{2}\right), \quad \omega = \frac{eB}{m_e},$$

which is called Landau level [1] and $n = 0, 1, 2, \cdots$ is the Landau quantum number. Although the expression of the Hamiltonian depends on a choice of gauge, the Landau level does not depend on the gauge, as expected.

However, solutions of the eigen equation depend on a choice of gauges. For example, the typical choice of gauges are:

$$\mathbf{A}^{(L_1)} = B(-y, 0, 0), \quad \mathbf{A}^{(L_2)} = B(0, x, 0), \quad \mathbf{A}^{(S)} = \frac{B}{2} (-y, x, 0).$$

We call them the first Landau gauge ($L_1$), the second Landau gauge ($L_2$), and the symmetric gauge (S), respectively. The wave functions in the first and in the second Landau gauges specified for the two-dimensional motion can be easily obtained [12]:

$$\psi_{n,k_x}^{(L_1)}(x,y) = N_n e^{ik_x x} e^{-\frac{\xi^2}{2}} H_n(\xi), \quad \xi \equiv \frac{y - y_0}{l_B}, \quad y_0 = +l_B^2 k_x,$$
and
\[ \psi_{n,k_y}(L^2_{1}) (x,y) = N_n e^{ik_y y} e^{-\frac{x^2}{2}} H_n(\eta), \quad \eta \equiv \frac{x-x_0}{l_B}, \quad x_0 = -l_B^2 k_y, \quad (5) \]

where \( l_B = \frac{1}{\sqrt{eB}} \) is the magnetic length, \( N_n = \frac{1}{\sqrt{\sqrt{\pi^2 n! l_B}}} \) is the normalisation factor, and \( H_n(z) \) is the \( n \)-th order Hermite polynomial. We can introduce a length \( L_x(y) \) for \( x(y) \)-direction in the first (second) Landau gauge, if we need the box normalisation. For instance, \( e^{ik_x x}/\sqrt{2\pi} \) in the above wave function is replaced by \( e^{ik_x x}/\sqrt{L_x} \) for \( \psi_{n,k_x}^{(L_1)}(x,y) \). We do not consider the motion along \( z \)-axis because it is the plane wave and trivial. These wave functions are the eigen states of the canonical momenta, i.e. \( \hat{p}_x \psi_{n,k_x}^{(L_1)} = k_x \psi_{n,k_x}^{(L_1)} \), and \( \hat{p}_y \psi_{n,k_y}^{(L_2)} = k_y \psi_{n,k_y}^{(L_2)} \), respectively.

On the other hand, the wave function in the symmetric gauge is given in [12]:
\[ \psi_{n,m}^{(S)} (r, \phi) = N_{n,m} \frac{e^{im\phi}}{\sqrt{2\pi}} e^{\frac{-\rho}{2\pi L^m_{n,m}}} \]
\[ \rho \equiv \frac{r^2}{2l_B^2}, \quad N_{n,m} = \frac{1}{l_B} \sqrt{\frac{(n-m|\pm_m|)!}{(n+|m|-m)!}}, \quad (6) \]

where \( L^m_{n}(z) \) is the \( n \)-th order associated Laguerre polynomial, and the sign factor of \( N_{n,m} \) in this paper is different from the one in Ref. [8]. This wave function is the eigen state of the canonical OAM, \( L^\text{can}_z = -i \frac{\partial}{\partial \phi}, \) i.e. \( \hat{L}^\text{can}_z \psi_{n,m}^{(S)} = m \psi_{n,m}^{(S)} \) with \( m \leq n \). It is worthy of notice that the canonical OAM commutes with the Hamiltonian only in this symmetric gauge and does not commute with the Hamiltonian in the Landau gauges. Hence, this conservation law depends on the gauge choice. Note that these wave functions are not related to each other through a gauge transformation. By this reason, the comparison of expectation values of some quantities between two different gauges sometimes shows an inconsistency. For example, the comparison of the expectation values for the mechanical OAM in two different gauges shows a discrepancy, in spite of the gauge invariant nature of the mechanical OAM. This is overcome by taking into account the degeneracy of wave functions in each gauge [8].
2.2. Gauge transformations

The relation between the wave functions in different gauges in the Landau level system is a delicate problem, which requires careful consideration. In a general gauge theory, if we change the set of wave function and the gauge field, \((\psi, \mathbf{A})\) to a new set, \((\psi', \mathbf{A}')\), these functions should be related to each other through the following gauge transformation:

\[
\psi'(x, y) = e^{-ie\chi(x,y)}\psi(x, y), \quad \mathbf{A}'(x, y) = \mathbf{A}(x, y) + \nabla \chi(x, y),
\]

where \(\chi(x, y)\) is a gauge transformation connecting two gauges. The question is whether we can find explicit forms of gauge transformations connecting the wave functions given in Eqs. (4), (5), and (6). This is not a trivial task because of the degeneracies of the Landau levels. The answer to this question is given as follows.

For simplicity, we first discuss the relation between the eigen functions in the first Landau gauge and those in the second Landau gauge [13]. As is well known, the vector potentials in these two gauges are related by the following gauge transformation,

\[
\mathbf{A}^{(L_2)}(x, y) = \mathbf{A}^{(L_1)}(x, y) + \nabla \chi(x, y),
\]

with \(\chi(x, y) = B xy\). Because of the degeneracies of the Landau levels, what is related with the eigen functions in the Landau gauge is a particular linear combination of the eigen functions in the second Landau gauge. Namely, we have the relation,

\[
\psi^{(L_2)}_{n,k_x}(x, y) = \int_{-\infty}^{\infty} dk_y U_n(k_x, k_y) \psi^{(L_1)}_{n,k_y}(x, y).
\]

Here the weight function \(U_n(k_x, k_y)\) of the linear combination turns out to be the matrix element of the gauge function \(e^{-ieBxy}\) between the eigen functions in the second Landau gauge and those in the first Landau gauge:

\[
U_n(k_x, k_y) = \langle \psi^{(L_2)}_{n,k_x} | e^{-ieBxy} | \psi^{(L_1)}_{n,k_y} \rangle = C_n e^{-il^2B k_x k_y},
\]

with \(C_n = \frac{1}{\sqrt{2\pi}n}\).

Similarly, the relation between the eigen functions in the symmetric gauge and those in the second Landau gauge is given in Refs. [8, 14] as

\[
\psi^{(L_2)}_{n,m}(x, y) = e^{-ie\chi(x,y)}\psi^{(S)}_{n,m}(x, y),
\]

\[
\mathbf{A}^{(L_2)}(x, y) = \mathbf{A}^{(S)}(x, y) + \nabla \chi(x, y),
\]
where $\chi(x,y) = \frac{B_{xy}}{2}$. Here, $\psi^{(L_2)}_{n,m}(x,y)$ is a particular linear combination of the eigen functions in the second Landau gauge, i.e.

$$\psi^{(L_2)}_{n,m}(x,y) = \int_{-\infty}^{\infty} dk_y U_{n,m}(k_y) \psi^{(L_2)}_{n,k_y}(x,y).$$

(14)

The weight function $U_{n,m}(k_y)$ of the linear combination is the matrix element of the gauge function $e^{-\frac{i}{2} eB_{xy}}$ between the eigen function in the second Landau gauge and those in the symmetric gauge:

$$U_{n,m}(k_y) \equiv \langle \psi^{(L_2)}_{n,k_y} | e^{-\frac{i}{2} eB_{xy}} | \psi^{(S)}_{n,m} \rangle = C_{n,m} H_{n-m}(-l_B k_y) e^{-\frac{i}{2} l_B k^2},$$

(15)

with

$$C_{n,m} = (-1)^{n+m+|m|} \sqrt{\frac{l_B}{2 \pi 2^{n-m}(n-m)!}}.$$  

(16)

Here the sign factor in $C_{n,m}$ is slightly different from that used in our previous paper [8].

2.3. Gauge invariant pseudo OAM and its observability

Using the canonical momentum or the mechanical momentum, one can construct two OAMs, that is, the canonical OAM, $\hat{L}_{\text{can}} = \mathbf{r} \times \hat{\mathbf{p}}$, and the mechanical OAM, $\hat{L}_{\text{mech}} = \mathbf{r} \times \hat{\mathbf{H}}$, where the canonical OAM is not gauge invariant, while the mechanical one is invariant under the local gauge transformation. It is usually believed that the physical OAM is the mechanical one according to the gauge principle and classical correspondence [15]. However, we have another gauge invariant OAM in this system, i.e. the pseudo OAM, alternatively called the gauge invariant canonical OAM. An intricate question is whether this pseudo OAM also corresponds to an observable or not. At least, it seems to be that the gauge principle does not forbid the observation of the pseudo OAM.

The pseudo OAM in a constant magnetic field discussed recently in Refs. [10, 11] is defined by:

$$\hat{L}_{z}^{\text{ps}} \equiv \hat{L}_{z}^{\text{mech}} - \frac{e B}{2} r^2 = x (\hat{p}_y + eA_y) - y (\hat{p}_x + eA_x) - \frac{e B}{2} r^2,$$

(17)

where $r^2 = x^2 + y^2$. This OAM is apparently gauge invariant. Besides, it commutes with the Hamiltonian, $[\hat{L}_{z}^{\text{ps}}, \hat{H}] = 0$, in arbitrary gauges. Furthermore, it reduces to the canonical OAM in the symmetric gauge. Hence this
gives an example of the so-called gauge invariant canonical OAM, and at the time it is a conserved quantity. Actually, we noticed that this pseudo OAM was already discussed by Johnson and Lippmann many years ago for a different purpose in Refs. [16, 17]. They used this conserved operator and its eigen equation for the ground state to obtain wave functions for a higher Landau-level.

Our previous study [8] based on the DeWitt’s gauge invariant method shows the following expectation values for the canonical, mechanical, and pseudo OAMs:

$$\langle \hat{L}_z^{\text{can}} \rangle = m, \quad \langle \hat{L}_z^{\text{mech}} \rangle = 2n + 1, \quad \langle \hat{L}_z^{\text{ps}} \rangle = m, \quad (18)$$

where $m$ is the eigen value of the canonical and pseudo OAM with respect to the origin. The readers who are not familiar with the gauge invariant method can check Sec. III in Ref. [8] for the technical detail.

The above result for the expectation value $\langle \hat{L}_z^{\text{ps}} \rangle$ of the pseudo OAM should be compared with the expectation value $\langle \hat{L}_z^{\text{can}} \rangle$ of the canonical OAM. Although the pseudo OAM is a gauge invariant and conserved OAM, its expectation value coincides with that of the canonical OAM, which is undoubtedly a gauge-variant quantity. This seems only natural, however, because the pseudo OAM is a gauge-invariant extension of the canonical OAM and it reduces to the canonical OAM in a particular gauge. An immediate natural question is then as follows. Is there any physical significance in the formal gauge symmetry of the pseudo OAM regarded as a gauge-invariant extension of the gauge-variant canonical OAM? Only from the gauge theoretical viewpoint, both of the mechanical OAM and the pseudo OAM appear to have qualification for observables, since they are both gauge invariant.

Still, the comparison above of the expectation values of the three OAMs indicates that the pseudo OAM is not an observable just like the ordinary canonical OAM is not, at least in the Landau-level system. After all, the comparison solely based on the gauge invariance argument does not tell us which of the mechanical OAM and the pseudo OAM is superior from the observational viewpoint. In the following, we will discuss the guiding centre of the electron cyclotron motion in the Landau level system as a new touchstone to judge relative advantages of the mechanical OAM and pseudo OAM.
3. Guiding centre in classical theory and quantum theory

In this section, we consider the role of the guiding centre and the pseudo OAM for the electron moving in the uniform magnetic field both in classical theory and quantum theory. First, we begin our discussion of the guiding centre and the pseudo OAM with the well-known cyclotron motion in classical theory. Next, we clarify the meaning and the role of guiding centre and pseudo OAM in quantum theory by using the analogy with classical picture.

3.1. Guiding centre and pseudo OAM in classical theory

The motion of the electron with the charge \(-e\) \((e > 0)\) and the mass \(m_e\) in the classical theory is determined by equation of motion (EOM):

\[
m_e \ddot{\mathbf{v}}(t) = -e(\mathbf{v}(t) \times \mathbf{B}),
\]

where the dot means the time derivative and \(\mathbf{v}(t) \equiv \dot{\mathbf{x}}(t)\). As is well-known, the general solutions for the electron’s orbit \((x(t), y(t), 0)\) in the two-dimension plane is given by,

\[
\begin{align*}
  x(t) &= X + \frac{1}{\omega_c} y(t), \quad X = x_0 - \frac{v_{x0}}{\omega_c}, \\
  y(t) &= Y - \frac{1}{\omega_c} v_x(t), \quad Y = y_0 - \frac{v_{y0}}{\omega_c},
\end{align*}
\]

where \(v_x(t) = v_0 \cos(\omega_c t + \alpha)\), \(v_y(t) = v_0 \sin(\omega_c t + \alpha)\), \(v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}\), \(\tan \alpha = \frac{v_{y0}}{v_{x0}}\). The solution satisfies the initial conditions, \(x(0) = x_0, v_x(0) = v_{x0}, y(0) = y_0, v_y(0) = v_{y0}\), and \(\omega_c = \frac{eB}{me}\) is the cyclotron frequency. In the classical mechanics, the coordinate \((X, Y)\) is called the guiding centre which has the clear meaning, i.e. the centre of the cyclotron motion. Importantly, this guiding centre \((X, Y)\) is time independent, \(\dot{X} = \dot{Y} = 0\). The above solution indicates the following constants in time, namely, conserved quantities:

\[
\begin{align*}
  r_c^2 &\equiv (x(t) - X)^2 + (y(t) - Y)^2 = \frac{m_e v_0^2}{e^2 B^2}, \\
  R^2 &\equiv X^2 + Y^2,
\end{align*}
\]

where \(r_c\) is the cyclotron radius and \(R\) is the distance between the coordinate origin and the centre of cyclotron motion. In addition to these conserved
quantities, we can find other conserved momentum and angular momentum called the pseudo momentum and the pseudo OAM \cite{10, 11} from EOM in Eq.(19):

\[
\frac{d}{dt} \mathbf{P}^\text{ps} \equiv \frac{d}{dt} \left[ m_e \mathbf{v}(t) + e \mathbf{x}(t) \times \mathbf{B} \right] = 0, \tag{22}
\]

\[
\frac{d}{dt} L_z^\text{ps} \equiv \frac{d}{dt} \left[ \mathbf{x}(t) \times m_e \mathbf{v}(t) - \frac{e}{2} r^2(t) \mathbf{B} \right]_z = 0, \tag{23}
\]

where we used the constant and uniform nature of the magnetic field \( \mathbf{B} \) and the squared distance \( r^2(t) = x^2(t) + y^2(t) \) which is the perpendicular to the direction of the magnetic field. We can show that these conserved quantities, \( \mathbf{P}^\text{ps} \) and \( L_z^\text{ps} \), do not correspond to the well known electron’s classical momentum and OAM. For later discussion in quantum theory, we consider the two cases, \((X, Y) = (0, 0)\) and \((X, Y) \neq (0, 0)\), separately. (In classical theory, this is not absolutely necessary, since the first case is simply obtained by taking the limit \( X \rightarrow 0 \) and \( Y \rightarrow 0 \). In quantum theory, however, the situation becomes a little more complicated. This is because the guiding centre becomes a q-number operator, the expectation value of which cannot be fixed from the outset.)

We first point out that there is one subtlety in discussing OAM, namely, we have to specify a reference axis for OAM and this point is different from the discussion on momenta. In addition, the canonical momenta (OAM) does not appear in the classical EOM of the Lorentz force, although those temporarily appear in the middle step of the derivation of EOM based on the Hamilton’s canonical formalism. Hence we only consider the mechanical and pseudo OAM in classical theory.

We consider two reference-axes for the mechanical OAM and the pseudo OAM, namely, the origin and the guiding centre which is different from the origin, for each OAM. Hence we define the following four OAMs in total:

\[
L^{\text{mech}}_z \equiv \left[ \mathbf{x}(t) \times m_e \mathbf{v}(t) \right]_z, \tag{24}
\]

\[
L^\text{ps}_z \equiv \left[ \mathbf{x}(t) \times m_e \mathbf{v}(t) - \frac{e}{2} r^2(t) \mathbf{B} \right]_z, \tag{25}
\]

\[
\mathcal{L}^{\text{mech}}_z \equiv \left[ (\mathbf{x}(t) - \mathbf{R}) \times m_e \mathbf{v}(t) \right]_z, \tag{26}
\]

\[
\mathcal{L}^\text{ps}_z \equiv \left[ (\mathbf{x}(t) - \mathbf{R}) \times m_e \mathbf{v}(t) - \frac{e}{2} (\mathbf{x}(t) - \mathbf{R})^2 \mathbf{B} \right]_z, \tag{27}
\]

where \( \mathbf{R} = (X, Y, 0) \) is the coordinate vector to the guiding centre from the origin and \( \mathbf{x}(t) = (x(t), y(t), 0) \) is the coordinate vector to the electron’s
position from the origin. The first two OAMs \( L_{\text{mech}}^z, L_{\text{ps}}^z \) express the mechanical and the pseudo OAM with respect to the origin, while the last two OAMs \( L_{\text{mech}}^z, L_{\text{ps}}^z \) express the mechanical and pseudo OAM with respect to the guiding centre \( \mathbf{R} = (X, Y) \).

3.1.1. OAM in the case A: \( (X, Y) = (0, 0) \)

First, we set \((X, Y) = (0, 0)\). This choice of the initial conditions corresponds to the case (A) of Fig. 1. Substituting the solution to the definition of the pseudo momenta, the pseudo momenta are reduced to

\[
\begin{align*}
P_{\text{ps}}^x &= m_e v_x(t) + eB_y(t) = 0, \\
P_{\text{ps}}^y &= m_e v_y(t) + eB_x(t) = 0.
\end{align*}
\]

It is obvious that the above momenta (null results) cannot describe the electron’s physical momenta, since the direction of the electron in the uniform magnetic field changes in time.

Similarly, after one substitutes the solutions of the EOM into each definition, the mechanical and pseudo OAMs for \((X, Y) = (0, 0)\) are reduced to

\[
\begin{align*}
L_{\text{mech}}^z &= L_x^z = xm_e v_y(t) - ym_e v_x(t) \\
&= r_c m_e v_0, \\
L_{\text{ps}}^z &= L_z^z = xm_e v_y(t) - ym_e v_x(t) - \frac{eB}{2} (x^2(t) + y^2(t)) \\
&= r_c m_e v_0 \frac{eB}{2}.
\end{align*}
\]

Here, it is obvious that \( L_{\text{mech}}^z \) gives the well-known conserved and classical OAM, \( r_c m_e v_0 \), of the cyclotron motion with respect to the origin. On the other hand, \( L_{\text{ps}}^z \) gives only half of the mechanical OAM.

3.1.2. OAM in the case B: \((X, Y) \neq (0, 0)\)

Next, we set \((X, Y) \neq (0, 0)\). This choice corresponds to the case (B) of Fig. 1. The pseudo momenta are reduced to

\[
\begin{align*}
P_{\text{ps}}^x &= +eB Y, \\
P_{\text{ps}}^y &= -eB X,
\end{align*}
\]

where the above results again cannot correspond to the electron’s momenta in any sense because of the same reason in the case (A).
Next, we consider the two sets of OAMs, namely, \((L_{\text{mech}}^z, L_{\text{ps}}^z)\) defined in Eqs. (24), (25) and \((L_{\text{mech}}^z, L_{\text{ps}}^z)\) defined in Eqs. (26), (27), for \((X, Y) \neq (0, 0)\). These four OAMs reduce to:

\[
L_{\text{mech}}^z = m_e r_c v_0 + m_e [X v_y(t) - Y v_x(t)], \quad (32)
\]
\[
L_{\text{ps}}^z = \frac{1}{2} r_c m_e v_0 - \frac{e B}{2} [X^2 + Y^2], \quad (33)
\]
\[
L_{\text{mech}}^z = r_c m_e v_0, \quad (34)
\]
\[
L_{\text{ps}}^z = \frac{1}{2} r_c m_e v_0. \quad (35)
\]

We find that \(L_{\text{mech}}^z\) consists of two terms, i.e. the first time-independent term and the second time-dependent term. The first term just coincides with the well-known angular momentum corresponding to the classical cyclotron motion, while the second term vanishes if we take the time-average over one period of the cyclotron motion. Namely, we have

\[
\langle L_{\text{mech}}^z \rangle_T \equiv \frac{1}{T} \int_0^T dt L_{\text{mech}}^z = r_c m_e v_0, \quad (36)
\]

where \(T \equiv 2\pi/\omega_c\).

\(L_{\text{ps}}^z\) also consists of two terms. The first term is just one half of the OAM corresponding to the cyclotron motion, while the second term is a function of the square of the distance from the origin to the guiding centre. Different from the above two OAMs defined with respect to the origin, the corresponding two OAMs \(L_{\text{mech}}^z\) and \(L_{\text{ps}}^z\) defined with respect to the guiding centre turn out to be both time-independent and also independent of the guiding centre coordinates \(X\) and \(Y\). As one sees, \(L_{\text{mech}}^z\) coincides with the well-known OAM corresponding to the classical cyclotron motion, whereas \(L_{\text{ps}}^z\) is just one half of it.

3.2. Pseudo OAM and guiding centre in quantum theory

In quantum theory, the mechanical momentum is replaced by the operator, \(\hat{\Pi} = -i \nabla + eA\), and hence the guiding centre is also replaced by the operator:

\[
\hat{X} = x - \frac{1}{eB} \hat{\Pi}_y = x - \frac{1}{eB} [-i \partial_y + eA_y], \quad (37)
\]
\[
\hat{Y} = y + \frac{1}{eB} \hat{\Pi}_x = y + \frac{1}{eB} [-i \partial_x + eA_x]. \quad (38)
\]
Figure 1: The cyclotron motion of the electron at $z = 0$ plane in classical mechanics: the case (A) for $(X, Y) = (0, 0)$ and the case (B) for $(X, Y) \neq (0, 0)$. The orbit of the electron is expressed by $(x(t), y(t))$ as the function of time.

It is important to recognise that the guiding centre in quantum theory is a $q$-number and we cannot freely set $(\hat{X}, \hat{Y})$ zero, like a $c$-number in classical theory. In this sense, the meaning of the guiding centre operator in the quantum theory is less intuitive.

Very interestingly, the guiding centre operator $(\hat{X}, \hat{Y})$ is time independent even in quantum theory. In fact, we can easily check the following well-known commutation relations, $[\hat{X}, \hat{H}] = [\hat{Y}, \hat{H}] = [\hat{R}^2, \hat{H}] = 0$, and $[\hat{X}, \hat{Y}] = il_B^2$, where $\hat{H}$ stands for the quantised Hamiltonian and $\hat{R}^2 \equiv \hat{X}^2 + \hat{Y}^2$. The last commutation relation indicates the Heisenberg’s uncertainty relation between $\hat{X}$ and $\hat{Y}$, that is, we cannot exactly specify the position of $\hat{X}$ and $\hat{Y}$ simultaneously.

By analogy with classical theory, we can define four types of OAM in quantum theory:

$$\hat{L}_{z}^{\text{mech}} \equiv [\mathbf{x} \times \hat{\Pi}]_z,$$  \hspace{1cm} (39)

$$\hat{L}_{z}^{\text{ps}} \equiv [\mathbf{x} \times \hat{\Pi} - \frac{e}{2} (x^2 + y^2) \mathbf{B}]_z,$$  \hspace{1cm} (40)

$$\hat{\mathcal{L}}_{z}^{\text{mech}} \equiv [(\mathbf{x} - \hat{\mathbf{R}}) \times \hat{\Pi}]_z,$$  \hspace{1cm} (41)

$$\hat{\mathcal{L}}_{z}^{\text{ps}} \equiv [(\mathbf{x} - \hat{\mathbf{R}}) \times \hat{\Pi} - \frac{e}{2} (\mathbf{x} - \hat{\mathbf{R}})^2 \mathbf{B}]_z,$$  \hspace{1cm} (42)

where $\mathbf{x} = (x, y, 0)$ and $\hat{\mathbf{R}} = (\hat{X}, \hat{Y}, 0)$ are the coordinate for the electron’s
position and the quantum guiding centre operator, respectively. The first two OAMs, \((\hat{L}^{\text{mech}}_z, \hat{L}^{\text{ps}}_z)\), are the quantum mechanical OAM and pseudo OAM with respect to the origin. On the other hand, the last two OAMs, \((\hat{L}^{\text{mech}}_z, \hat{L}^{\text{ps}}_z)\), are the quantum mechanical OAM and pseudo OAM with respect to the quantum guiding centre. In addition, in quantum theory, the canonical OAMs are given by:

\[
\hat{L}^{\text{can}}_z \equiv [\mathbf{x} \times \hat{\mathbf{p}}]_z, \\
\hat{L}^{\text{can}}_z \equiv \left[\left(\mathbf{x} - \hat{\mathbf{R}}\right) \times \hat{\mathbf{p}}\right]_z,
\]

(43)

where \(\hat{L}^{\text{can}}_z, \hat{L}^{\text{can}}_z\) stand for the canonical OAM with respect to the origin and with respect to the quantum guiding centre, respectively.

As is well known, we cannot describe an orbit of the electron as a function of time in quantum theory. To be more specific to our Landau problem, the guiding centre has a meaning as a centre of cyclotron motion. Nevertheless, a delicate point is that the guiding centre in quantum theory is a q-number operator and there is an inherent uncertainty in its position (see Figs. 1,2 of Ref. [18], and Fig. 7 in Ref. [19]). Still, we need to specify the coordinate system, in particular, the origin, for solving the Schrödinger equation. Then, the standard procedure is first to choose the origin of the coordinate artificially, i.e. on no account of the guiding centre concept. After that, we choose a gauge potential which reproduces the uniform magnetic field. The most convenient choice of the vector potential for our discussion is the symmetric gauge potential and the associated wave functions. Alternatively, we can use the wave functions derived by the DeWitt’s gauge-invariant method [8].

First, we consider three OAMs, namely, the mechanical, the pseudo, and the canonical OAMs with respect to the origin. We have already given these results earlier. They are given by:

\[
\langle \hat{L}^{\text{can}}_z \rangle = m, \quad \langle \hat{L}^{\text{mech}}_z \rangle = 2n + 1, \quad \langle \hat{L}^{\text{ps}}_z \rangle = m.
\]

(45)

Next, we consider the three OAMs with respect to the guiding centre. Although the three OAMs \(\hat{L}^{\text{can}}_z, \hat{L}^{\text{mech}}_z, \) and \(\hat{L}^{\text{ps}}_z\) defined with respect to the origin do not reduce to simple forms, the OAMs \(\hat{L}^{\text{can}}_z, \hat{L}^{\text{mech}}_z, \) and \(\hat{L}^{\text{ps}}_z\) defined with respect to the quantised guiding centre reduce to simple forms:

\[
\hat{L}^{\text{can}}_z = \frac{2}{\omega} \hat{H} - \frac{1}{2} \hat{L}^{\text{can}}_z - \frac{eB}{4} r^2, \quad \hat{L}^{\text{mech}}_z = \frac{2}{\omega} \hat{H}, \quad \hat{L}^{\text{ps}}_z = \frac{1}{\omega} \hat{H}.
\]

(46)
In particular, we find that $\hat{L}_{\text{mech}}$ and $\hat{L}_{\text{ps}}$ are proportional to the Hamiltonian $\hat{H}$ and hence the conservation of the expectation values for $\hat{L}_{\text{mech}}$ and $\hat{L}_{\text{ps}}$ are obvious from the above relations. We emphasise that the quantised guiding centre plays important roles in these relations. We found that the expectation values of these three OAMs defined with respect to the quantum guiding centre are given by:

$$\langle \hat{L}_{\text{can}} \rangle = \frac{2n+1}{2}, \quad \langle \hat{L}_{\text{mech}} \rangle = 2n + 1, \quad \langle \hat{L}_{\text{ps}} \rangle = \frac{2n+1}{2}. \quad (47)$$

We point out that the above results for the expectation values of the mechanical OAM and the pseudo OAM are just consistent with the classical theory. The expectation value of the mechanical OAM with respect to the guiding centre gives the well-known value which is consistent with the cyclotron motion around it, while the expectation value of the pseudo OAM gives only half of that of the mechanical OAM. In addition, the reasonable point is that the expectation value of the gauge-invariant pseudo OAM just coincides with that of the canonical OAM with respect to the guiding centre. What is non-trivial here is that the expectation values of the pseudo OAM and also that of the canonical OAM defined with respect to the guiding centre is entirely different from those of the corresponding operators defined with respect to the coordinate origin. At any rate, the reference-axis independence observed in the expectation value of the mechanical OAM appears to be a distinguishing feature of it as compared with the other two OAMs, i.e. the canonical OAM and the pseudo OAM, which are thought to be physically equivalent.

4. Discussion and Conclusion

We have carried out a comparative analysis of three OAMs, i.e. the canonical, mechanical, and the pseudo OAMs in the Landau level system. Our main objective is to answer the following questions: 1) what is the difference between two gauge invariant OAMs, i.e. the mechanical OAM and the pseudo OAM ? and 2) does the pseudo OAM correspond to an observable, because of its gauge-invariant nature ?

First, we compared four OAMs of the electron. They are the mechanical OAM $L_{z}^{\text{mech}}$ and the pseudo OAM $L_{z}^{\text{ps}}$ defined with respect to a chosen coordinate origin and the mechanical OAM $L_{z}^{\text{mech}}$ and the pseudo OAM $L_{z}^{\text{ps}}$
defined with respect to a suitably chosen coordinate origin and the mechanical OAM $L_{z}^{\text{mech}}$ and the pseudo OAM $L_{z}^{\text{ps}}$ defined with respect to the centre of the cyclotron motion which we call the guiding centre. In classical theory, we found that the time-average of the above four OAMs are given by Eqs. \((32)-(35)\). The remarkable point here is that the time-averaged mechanical OAM defined with respect to the origin and that defined with respect to the guiding centre precisely coincide each other and both reproduce the well-known answer as expected from the picture of the classical cyclotron motion. This is an strong indication of the physical nature of the mechanical OAM.

On the other hand, the time-averaged pseudo OAMs defined with respect to the two different reference points, i.e. the origin and the guiding centre, do not coincide with each other. The reason can be explained as follows. Since the magnetic field in the Landau level system spreads uniformly over the whole plane, there is no special or preferential point in the plane. This means that the choice of the origin of the coordinate system is totally arbitrary. The concept of the pseudo OAM is therefore vitally depends on the choice of the coordinate system. As a consequence, the definition of the pseudo OAM itself depends inherently on the choice of the coordinate system and origin. In fact, one sees that the pseudo OAM defined with respect to the arbitrary chosen coordinate origin depends on the guiding centre $(X,Y)$ in the chosen coordinate system (see Eq. \((33)\)). On the other hand, the pseudo OAM defined with respect to the guiding centre is a quantity, which is free from the choice of the coordinate system (see Eq. \((35)\)). We found that this latter quantity is time-independent and just a half of the time-averaged mechanical OAM. (Actually, the mechanical OAM defined with respect to the guiding centre also turns out to be time-independent.)

When going to quantum theory, some additional delicacies appear. That is, the guiding centre in quantum theory is a q-number and there is an inherent uncertainty in its position. Still, we can consider two different types of OAM, i.e. the OAM defined with respect to the coordinate origin and the OAM defined with respect to the quantum guiding centre. In addition, one can consider two types of the canonical OAM defined with respect to the origin and the guiding centre. Thus, we totally consider the six OAMs defined in Eqs. \((39)-(44)\) in quantum theory.

Using the well-known wave function in the symmetric gauge, the expectation values for these six OAMs are given in Eqs. \((45),(47)\). First, we point out that the expectation values of $L_{z}^{\text{mech}}$ and $L_{z}^{\text{mech}}$ exactly coincide with each other in perfect conformity with the classical consideration, which gives
the equality \( \langle \mathcal{L}_z^{\text{mech}} \rangle_T = \langle \mathcal{L}_z^{\text{mech}} \rangle_T \) for the time-averaged two OAMs. The expectation value in quantum theory is expressed by the so-called Landau quantum number \( n \) characterising the eigen energies of the Landau level system. This is again interpreted as the physical nature of the mechanical OAM. We also confirm the relations, \( \langle \hat{L}_z^{\text{ps}} \rangle = \langle \hat{L}_z^{\text{can}} \rangle \) and \( \langle \hat{L}_z^{\text{ps}} \rangle = \langle \hat{L}_z^{\text{can}} \rangle \). This is only natural, because the pseudo OAMs reduce to the canonical OAMs in the symmetric gauge aside from the unphysical gauge degree of freedom. From the physical viewpoint, the pseudo OAM and canonical OAM are essentially the same quantity. This also implies that the pseudo OAM, just like the ordinary canonical OAM, need not correspond to an observable even though it is formally gauge invariant.

Another non-trivial point here is the relation that \( \langle \hat{L}_z^{\text{ps}} \rangle \neq \langle \hat{L}_z^{\text{ps}} \rangle \) and \( \langle \hat{L}_z^{\text{can}} \rangle \neq \langle \hat{L}_z^{\text{can}} \rangle \). This again reminds us of that the concept of the pseudo OAM or canonical OAM crucially depends on the choice of the coordinate system. The expectation values of the pseudo OAM and the canonical OAM with respect to the origin depend on the quantum number \( m \). This quantum number \( m \) is a conserved quantity reflecting the rotational (or axial) symmetry of the Hamiltonian. This symmetry is not independent of the choice of the coordinate system and the vector potential, both of which have large arbitrariness, because of the complete uniformity of the magnetic field in the Landau level system. The dependence on the choice of the coordinate-origin must be the origin of non-observability of the quantum number \( m \) in the system.

On the contrary, the expectation value of the pseudo OAM and/or the canonical OAM defined with respect to the guiding centre turns out to be expressed with the Landau quantum number \( n \), which means that they are related to an observable. Note that, since these OAMs are defined with respect to the guiding centre, which has a meaning of the centre of the cyclotron motion, they are independent of the choice of the coordinate origin. Undoubtedly, this is the reason why they are related to the observable through the Landau quantum number \( n \).

Summarising our point again, whether the quantity in question is an observable or not has little to do with its gauge invariance or gauge non-invariance property. In fact, as we have shown, in spite that the pseudo OAM defined with respect to the origin and that defined with respect to the guiding centre are both gauge invariant quantities, only the latter are related to observables. The reason of non-observability of the former quantity is its dependence on the choice of the coordinate origin, which breaks an important
physical principle, i.e. the coordinate-choice independence of observables. This should be contrasted with the pseudo OAM defined with respect to the guiding centre, which has a clear meaning independently of the choice of the coordinate system.

To conclude, the so-called gauge principle claims that observables must be gauge invariant, and this fundamental principle of physics is widely believed to be correct. However, we must be careful about the fact that the converse of this theorem is not necessarily true. As we have shown through several concrete examples, the gauge invariance of some quantities does not always ensure its observability. In such an occasion, the gauge symmetry is thought of as just a redundancy without any physical significance.

Acknowledgement

Y. K., L.P. Z., and P.M. Z. are supported by the National Natural Science Foundation of China (Grant No. 11575254 and 11805242). This work is partly supported by Chinese Academy of Sciences President’s International Fellowship Initiative (No. 2018VMA0030 and 2018PM0028).

References

[1] L.D. Landau, Z. Phys. 64 (1930) 629.
[2] J. J. Sakurai, J. Napolitano, Modern Quantum Mechanics, second edition, (Addison-Wesley, San Francisco, 2011).
[3] P. Schattschneider et al., Nature Comm. 5 (2014) 4586.
[4] K. Y. Bliokh, P. Schattschneider, J. Verbeeck, and F. Nori, Phys. Rev. X2 (2012) 041011.
[5] C. Greenshields, R. L. Stamps, S. Franke-Arnold, and S. M. Barnett, Phys. Rev. Lett. 113 (2014) 240404.
[6] E. Leader, C. Lorcé, Phys. Rep. 541 (2014) 163.
[7] M. Wakamatsu, Int. J. Mod. Phys. A29 (2014) 1430012.
[8] M. Wakamatsu, Y. Kitadono, and P.-M. Zhang, Ann. Phys. 392 (2018) 287.
[9] B. S. DeWitt, Phys. Rev. 125 (1962) 2189.

[10] G. Konstantinou, K. Moulopoulos, Eur. J. Phys. 37 (2016) 065401.

[11] G. Konstantinou, K. Moulopoulos, Int. J. Theor. Phys. 56 (2017) 1484.

[12] L. D. Landau, E. M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Course of Theoretical Physics, Vol. 3 (Pergamon, New York, 1977).

[13] R. J. Swenson, Am. J. Phys. 57 (1989) 381.

[14] T. Haugset, J. Aa. Ruud, F. Ravndal, Phys. Scr. 47 (1993) 715.

[15] H. Murayama, 221A Lecture Notes: Landau Level, http://hitoshi.berkeley.edu/221a/landau.pdf.

[16] M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, No. 6 (1949) 828.

[17] M. H. Johnson and B. A. Lippmann, Phys. Rev. 77, No. 5 (1950) 702.

[18] C.-F. Li, Q. Wang, Physica B 269 (1999) 22.

[19] I. D. Vagner, V. M. Gvozdikov, and P. Wyder, Quantum mechanics of electrons in strong magnetic field, HIT Journal of Science and Engineering, Vol. 3, Issue 1, 5 (Holon Institute of Technology, Holon, 2006).