An algorithm for computing Grothendieck local residues II — general case —

Katsuyoshi Ohara and Shinichi Tajima

Abstract. Grothendieck local residue is considered in the context of symbolic computation. Based on the theory of holonomic $D$-modules, an effective method is proposed for computing Grothendieck local residues. The key is the notion of Noether operator associated to a local cohomology class. The resulting algorithm and an implementation are described with illustrations.

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1. Introduction

In this paper, we consider Grothendieck local residues from the point of view of computational algebraic analysis. In a previous paper [13], we considered the so-called shape basis cases and gave algorithm for computing Grothendieck local residues. In the present paper, we adopt the same framework given in [13] and consider the general cases. Main ingredients of our approach are local cohomology, Noether operators and holonomic $D$-modules. More precisely, we consider holonomic $D$-modules defined by annihilating left ideals in a Weyl algebra of a zero-dimensional local cohomology class, and partial differential operators associated to the Grothendieck local residue mapping

$$\varphi(x) \mapsto \text{Res}_\beta\left(\frac{\varphi \, dx}{f_1 \cdots f_n}\right) = \left(\frac{1}{2\pi \sqrt{-1}}\right)^n \int_{\Gamma(\beta)} \frac{\varphi(x) \, dx_1 \wedge \cdots \wedge dx_n}{f_1(x) \cdots f_n(x)}.$$

We compute a Noether operator, partial differential operator that describe or represent Grothendieck local residue mapping, by solving a system of partial differential equations in the Weyl algebra.

The paper is organized as follows: In Section 2, we briefly recall the notion of local residues and algebraic local cohomology groups. In Section 3, we describe theory of Noether operators associated to a primary ideal in terms of $D$-modules. In Section 4, we give a recursive method for finding annihilators of an algebraic local cohomology class that avoids the use of Gröbner bases on the Weyl algebra. In Section 5, we propose algorithms for determining Noether operator, associated to a local cohomology class, which represents the local residue mapping. The resulting algorithms have been implemented in a computer algebra system Risa/Asir [10]. An example by Risa/Asir will be shown in Section 6.

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2. Algebraic local cohomology groups

Let $K$ be a subfield of $\mathbb{C}$ and denote $K[x] = K[x_1, \ldots, x_n]$. We suppose that a polynomial sequence $F = \{f_1, \ldots, f_n\} \subset K[x]$ is regular. The polynomial ideal $I$ generated by $F$ is zero-dimensional and the zero set $V_C(I) = \{x \in \mathbb{C}^n \mid g(x) = 0, \forall g \in I\}$ consists of finite number of points.

We introduce the $n$-th algebraic local cohomology group with support on $Z = V_C(I)$ by
\[
H^n_Z(K[x]) = \lim_{k \to \infty} \Ext^n_{K[x]}(K[x]/(\sqrt{I})^k, K[x]).
\]
The algebraic local cohomology group $H^n_Z(K[x])$ can be regarded as a collection of equivalent classes of rational functions whose denominator has zero only on $Z$. Here the equivalence is given by cutting holomorphic parts of rational functions in a cohomological way. Since the local residue can be described in terms of local cohomology \([3, 20]\), we have $\Res(\frac{\varphi dx}{f_1 \cdots f_n}) = \Res(\varphi \sigma_{F} dx)$ where $\sigma_F = \left[\frac{1}{f_1 \cdots f_n}\right] \in H^n_Z(K[x])$ and $dx = dx_1 \wedge \cdots \wedge dx_n$.

Algorithms for computing primary decomposition $I = \bigcap_{\lambda=1}^{r} I_{\lambda}$ had been established (e.g. see \([2, 9]\)). Suppose that, on the ring $K[x]$, a primary decomposition algorithm can be executed. Since $I$ is zero-dimensional, the associated prime $\sqrt{I}$ is also zero-dimensional thus maximal. In other words, $K_{\lambda} = K[x]/\sqrt{I}$ is a field.

According to the primary decomposition, the zero set is written as a union of irreducible affine varieties: $Z = \bigcup_{\lambda=1}^{r} Z_{\lambda}$, where $Z_{\lambda} = V_C(\sqrt{I_{\lambda}})$. Then $H^n_{[Z]}(K[x])$ can be decomposed to a direct sum
\[
H^n_Z(K[x]) = H^n_{Z_{1}}(K[x]) \oplus \cdots \oplus H^n_{Z_{r}}(K[x]) \oplus \cdots \oplus H^n_{[Z]}(K[x]).
\]
Therefore an algebraic local cohomology class $\sigma_F \in H^n_{[Z]}(K[x])$ has a unique decomposition
\[
\sigma_F = \sigma_{F,1} + \cdots + \sigma_{F,\lambda} + \cdots + \sigma_{F,\ell},
\]
where $\sigma_{F,\lambda} \in H^n_{Z_{\lambda}}(K[x])$. Note that $\text{supp}(\sigma_{F,\lambda}) \subset Z_{\lambda}$. The decomposition above is a kind of partial fractional expansion of $\frac{1}{f_1 \cdots f_n}$ in terms of local cohomology.

Let $\beta \in Z_{\lambda}$ and $\varphi(x) \in O(U)$ where $U$ is a small neighborhood of $\beta$. Since $\sigma_{F,j}$ vanishes on $U$ for $j \neq \lambda$, we have $\Res(\varphi \sigma_{F,j} dx) = \Res(\varphi \sigma_{F,\lambda} dx)$ for $\beta \in Z_{\lambda}$.

Let $D_n = K[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ be the ring of linear partial differential operators with polynomial coefficients over $K$. Here we used the symbol $\partial_i = \frac{\partial}{\partial x_i}$. The noncommutative ring $D_n$ is called the Weyl algebra in $n$ variables.

Let $P$ be a zero-dimensional prime ideal generated by $H = \{h_1, \ldots, h_n\}$ and put $J_H = \left(\det \frac{\partial h_j}{\partial x_i}\right)_{i,j} \mod P$. An algebraic local cohomology class
\[
\delta_Y = \left[\frac{J_H}{h_1 h_2 \cdots h_n}\right] \in H^n_{[Y]}(K[x])
\]
is called the delta function with the support $Y = V_C(P)$.

Recall that an algebraic local cohomology group has a structure of $D_n$-module. It follows from $H^n_{[Y]}(K[x]) = D_n \cdot \delta_Y$ (e.g. \([13]\) Lemma 3.2) that there exists a linear differential operator $T_{F,\lambda} \in D_n$ such that $\sigma_{F,\lambda} = T_{F,\lambda}^* \cdot \delta_Z_{\lambda}$ where $T_{F,\lambda}^*$ is the formal adjoint of $T_{F,\lambda}$. Therefore
\[
\Res(\frac{\varphi dx}{f_1 \cdots f_n}) = \Res(\varphi \sigma_{F,\lambda} dx)
\]
\[
= \Res(\varphi \sigma_{F,\lambda}^* dx)
\]
\[
= \Res(\varphi \cdot (T_{F,\lambda}^* \cdot \delta_{Z_{\lambda}}) dx)
\]
\[
= \Res((T_{F,\lambda} \cdot \varphi) \cdot \delta_{Z_{\lambda}} dx)
\]
\[
= (T_{F,\lambda} \cdot \varphi)|_{x=\beta}.
\]
That is, the mapping $\varphi \mapsto \text{Res}_j(\varphi \sigma F dx)$ is determined by the differential operator $T_{F,\lambda}$. Since the set $\{(T_{F,\lambda}, Z_\lambda) \mid \lambda = 1, 2, \ldots, \ell\}$ gives the Grothendieck local residue mappings, the local residue of any meromorphic $n$-forms can be evaluated by differential operators $T_{F,\lambda}$.

### 3. Noether differential operators

In 1960’s, L. Ehrenpreis studied systems of linear partial differential equations with constant coefficients. In his theory, some differential operators with polynomial coefficients play an important role \[\text{[11]}\]. Today, these operators are called Noether operators (e.g. \[\text{[16]}\]). They are very useful to handle local cohomology classes. We give a modern definition of Noether operators in terms of $D$-modules. Throughout this section, we suppose that $J$ is a zero-dimensional primary ideal over $K[x] = K[x_1, \ldots, x_n]$. Set $M_J = D_n/D_nJ$ and $M_{\sqrt{J}} = D_n/D_n\sqrt{J}$.

**Definition 3.1.** The homomorphisms $\text{Hom}_{D_n}(M_J, M_{\sqrt{J}})$ between left $D_n$-modules is called the Noether space of $J$.

**Definition 3.2.** The ratio $m = \dim_K(K[x]/J)/\dim_K(K[x]/\sqrt{J})$ of dimensions of vector spaces is called the multiplicity of the ideal $J$.

**Lemma 3.3.** There exist $\rho_1, \rho_2, \ldots, \rho_m \in \text{Hom}_{D_n}(M_J, M_{\sqrt{J}})$ such that any $\rho \in \text{Hom}_{D_n}(M_J, M_{\sqrt{J}})$ can be written as

$$\rho = \rho_1 c_1 + \rho_2 c_2 + \cdots + \rho_m c_m$$

by unique polynomials $c_1, c_2, \ldots, c_m \in K[x]/\sqrt{J}$.

Notice that the Noether space has a structure of a right $K[x]/\sqrt{J}$-module, that is, the Noether space is an $m$-dimensional vector space over $K[x]/\sqrt{J}$.

Since $1 \in M_J$ is an equivalent class of differential operators, a representative $R_i$ of the image $\rho_i(1)$ is a differential operator, $i = 1, 2, \ldots, m$. We call the set $\mathcal{R} = \{R_1, R_2, \ldots, R_m\}$ Noether operator basis.

Let $L_i$ denotes the formal adjoint of $R_i$. Then we have the following theorem.

**Theorem 3.4.** The differential operators $L_1, L_2, \ldots, L_m$ satisfy

$$J = \{h \in K[x] \mid L_1 \cdot h, L_2 \cdot h, \ldots, L_m \cdot h \in \sqrt{J}\}, \quad (3.1)$$

Note that the differential operators above coincide with Noether operators introduced by V. P. Palamodov \[\text{[16]}\].

We describe a recursive method for computing Noether operators. We can suppose that $\text{ord}(L_1) \leq \text{ord}(L_2) \leq \cdots \leq \text{ord}(L_m)$ without loss of generality. Put $\text{NT} = \bigoplus_{i=1}^m (K[x]/\sqrt{J}) L_i$ and $\text{NT}^{(k)} = \{L \in \text{NT} \mid \text{ord}(L) \leq j\}$ for each natural number $k$. The recursion will be executed until $\dim_{K[x]/\sqrt{J}} \text{NT}^{(r)} = m$.

At first, we have $L_1 = 1$ because $\text{NT}^{(0)} = K[x]/\sqrt{J}$.

Second, a first order Noether operator can be written as $L = \sum_{i=1}^n a_i \partial_i$ where $a_i \in K[x]/\sqrt{J}$ because zeroth order terms are reduced. Let $J = \langle g_1, g_2, \ldots, g_\mu \rangle$ and $\sqrt{J} = \langle p_1, p_2, \ldots, p_\eta \rangle$. From the condition \[\text{[11]}\], for any $1 \leq j \leq \mu$, there exist polynomials $b_{j_1}, b_{j_2}, \ldots, b_{j_\eta}$ such that

$$\sum_{i=1}^n a_i \partial_i \cdot g_j + \sum_{k=1}^n b_{jk} p_k = 0.$$  \(3.2\)

Let $e_j = \frac{\partial g_j}{\partial x_i}$, as follows:

$$\sum_{i=1}^n a_i \left( \sum_{j=1}^\mu \frac{\partial g_j}{\partial x_i} e_j \right) + \sum_{j=1}^\mu \sum_{k=1}^n b_{jk} p_k e_j = 0.$$

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$$\sum_{i=1}^n a_i \partial_i \cdot g_j + \sum_{k=1}^n b_{jk} p_k = 0.$$  \(3.2\)

Let $e_j = \frac{\partial g_j}{\partial x_i}$, as follows:

$$\sum_{i=1}^n a_i \left( \sum_{j=1}^\mu \frac{\partial g_j}{\partial x_i} e_j \right) + \sum_{j=1}^\mu \sum_{k=1}^n b_{jk} p_k e_j = 0.$$
Here \( e_j \) is the \( j \)-th unit vector. Let
\[
\mathcal{N}^{(1)} = \left\{ \sum_{j=1}^n \frac{\partial g_j}{\partial x_i} e_j \mid i = 1, 2, \ldots, n \right\} \cup \{ p_k e_j \mid 1 \leq j \leq \mu, 1 \leq k \leq \mu \}.
\]

It can be regarded as a finite subset of the \( K[x] \)-module \( K[x]^{\mu} \). The solutions \((a_i; b_{jk})\) of the equation \((3.2)\) form the syzygy module with respect to \( \mathcal{N}^{(1)} \). Each solution corresponds to a first order Noether operator \( L = \sum_{i=1}^n a_i \partial_i \) if \((a_i) \neq 0\). Using a Gröbner basis of the syzygy module, we can determine a basis of \( \mathcal{N}T^{(1)} \).

Next we discuss construction of a basis of \( \mathcal{N}T^{(r)} \). Let \( \{ L_1, L_2, \ldots, L_q \} \) be a basis of \( \mathcal{N}T^{(r-1)} \) and let \( L = \sum_{1 \leq |\alpha| \leq r} a_{\alpha} \partial^\alpha \in \mathcal{N}T^{(r)} \) where the multi-index \( \alpha \) runs over \( \mathbb{N}_0^n \) and \( a_{\alpha} \in K[x] / \sqrt{\mathcal{J}} \). The commutator \([L, f] = Lf - fL\) is also an element of \( \mathcal{N}T^{(r-1)} \) if \( f \in K[x] \). Thus \([L, x_1], [L, x_2], \ldots, [L, x_n] \in \mathcal{N}T^{(r-1)} \).

**Lemma 3.5.** A differential operator \( P \in D_n \) satisfies a relation \( P \cdot J \subset \sqrt{\mathcal{J}} \) if and only if
\[
\{ P \cdot g_1, P \cdot g_2, \ldots, P \cdot g_p \} \subset \sqrt{\mathcal{J}} \quad \text{and} \quad [P, x_i] \cdot J \subset \sqrt{\mathcal{J}} \quad (i = 1, 2, \ldots, n).
\]

From the lemma above, we consider a system of membership problems \( L \cdot g_i \in \sqrt{\mathcal{J}} \) and \([L, x_j] \in \mathcal{N}T^{(r-1)} \). That is, there exist polynomials \( b_{ih}, b'_{kj}, b''_{kh} \in K[x] \) such that
\[
\sum_{1 \leq |\alpha| \leq r} a_{\alpha}(\partial^\alpha \cdot g_i) + \sum_{h=1}^\eta b_{ih} p_h = 0, \quad (1 \leq i \leq \mu) \tag{3.3}
\]
\[
\sum_{1 \leq |\alpha| \leq r} a_{\alpha}(\partial^\alpha, x_k) + \sum_{j=1}^q b'_{kj} L_j + \sum_{1 \leq |\beta| \leq r} \sum_{h=1}^\eta b''_{kh} p_h \partial^\beta = 0. \quad (1 \leq k \leq n) \tag{3.4}
\]
Here the commutator \([\partial^\alpha, x_k]\) and \( L_j \) can be expressed as \( \sum_{0 \leq |\beta| \leq r} c_{\alpha k \beta} \partial^\beta \) and \( \sum_{0 \leq |\beta| \leq r} d_{\beta \gamma} \partial^\beta \) respectively. Recall \( L_1 = 1 \). Removing zeroth order terms, the equation \((3.4)\) can be rewritten as
\[
\sum_{1 \leq |\beta| \leq r} \left( \sum_{1 \leq |\alpha| \leq r} a_{\alpha} c_{\alpha k \beta} + \sum_{j=2}^q b'_{kj} d_{j\beta} + \sum_{h=1}^\eta b''_{kh} p_h \right) \partial^\beta = 0. \quad (1 \leq k \leq n) \tag{3.5}
\]
Thus
\[
\sum_{1 \leq |\alpha| \leq r} a_{\alpha} c_{\alpha k \beta} + \sum_{j=2}^q b'_{kj} d_{j\beta} + \sum_{h=1}^\eta b''_{kh} p_h = 0. \quad (1 \leq k \leq n, 1 \leq |\beta| < r) \tag{3.5}
\]
Since the set \( \{ \beta \in \mathbb{N}_0^n \mid 1 \leq |\beta| < r \} \) has \( N := \binom{n+r-1}{n} - 1 \) elements, there exists a bijection \( \varphi : \{1, 2, \ldots, n\} \times \{ \beta \in \mathbb{N}_0^n \mid 1 \leq |\beta| < r \} \rightarrow \{ \mu + 1, \mu + 2, \ldots, \mu + nN \} \). The system \((3.3), (3.5)\) can be rewritten, as single equation on \( K[x]^{\mu+nN} \), as follows:
\[
\sum_{1 \leq |\alpha| \leq r} a_{\alpha} \left\{ \sum_{i=1}^\mu (\partial^\alpha \cdot g_i) e_i + \sum_{k=1}^n \sum_{1 \leq |\beta| < r} c_{\alpha k \beta} e_{\varphi(k, \beta)} \right\} + \sum_{j=1}^q \sum_{k=1}^n b'_{kj} \left\{ \sum_{1 \leq |\beta| < r} d_{j\beta} e_{\varphi(k, \beta)} \right\} + \sum_{h=1}^\eta f_{kh} p_h e_t = 0, \tag{3.6}
\]
where $e_t$ is the $t$-th fundamental vector of dimension $\mu + nN$. The equation (3.6) can be solved by Gröbner basis computation of the syzygy module with respect to a finite set

$\mathcal{N}^{(r)} = \left\{ \sum_{i=1}^{\mu} (\partial^\alpha \cdot g_i) e_i + \sum_{k=1}^{n} \sum_{1 \leq |\beta| < r} c_{\alpha k \beta} e_{\varphi(k, \beta)} \mid 1 \leq |\alpha| \leq r \right\}
\cup \left\{ \sum_{1 \leq |\beta| < r} d_{j \beta} e_{\varphi(k, \beta)} \mid 1 \leq j \leq q, 1 \leq k \leq n \right\}
\cup \left\{ p_h e_t \mid 1 \leq h \leq \eta, 1 \leq t \leq \mu + nN \right\}. \quad (3.7)

A solution $(a_\alpha; b'_{jk}; f_{th})$ gives an $r$-th order Noether operator $L = \sum_{\alpha} a_\alpha \partial^\alpha$ if there exists $\gamma \in \mathbb{N}_0^n$ such that $a_\gamma \neq 0$ and $|\gamma| = r$. Hence a basis of $\mathbb{N}T^{(r)}$ can be determined.

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**Algorithm 1** a basis of $\mathbb{N}T^{(1)}$

**Require:** $J = \langle g_1, g_2, \ldots, g_\mu \rangle$, $\sqrt{J} = \langle p_1, p_2, \ldots, p_\eta \rangle$

**Ensure:** $\mathcal{L}^{(1)}$: a basis of $\mathbb{N}T^{(1)}$.

$E \leftarrow$ (the unit matrix of size $\mu$)
$J \leftarrow \left[ \frac{\partial g_i}{\partial x_i} \right]_{ji}$: Jacobi matrix of size $(\mu, n)$
$M \leftarrow [J | p_1 E | \cdots | p_\eta E]$: combined matrix
$S \leftarrow$ (Gröbner basis of the syzygy module of columns vectors of $M$)
$\mathcal{L}^{(1)} \leftarrow \{1\}$

for all $(a_i; b_{jk}; f_{th}) \in S$

if $(a_1, \ldots, a_n) \neq 0$ then

$\mathcal{L}^{(1)} \leftarrow \mathcal{L}^{(1)} \cup \{ \sum_{i=1}^{n} a_i \partial_i \}$

end if

end for

return $\mathcal{L}$

---

**Algorithm 2** a basis of $\mathbb{N}T^{(r)}$

**Require:** $r \geq 2, J, \sqrt{J}, \mathcal{L}^{(r-1)}$: a basis of $\mathbb{N}T^{(r-1)}$

**Ensure:** $\mathcal{L}^{(r)}$: a basis of $\mathbb{N}T^{(r)}$.

$\mathcal{N}^{(r)} \leftarrow$ (the right hand side of the equation (3.7) by $J$, $\sqrt{J}$, $\mathcal{L}^{(r-1)}$)
$S \leftarrow$ (Gröbner basis of the syzygy module with respect to $\mathcal{N}^{(r)}$)
$\mathcal{L}^{(r)} \leftarrow \mathcal{L}^{(r-1)}$

for all $(a_\alpha; b'_{jk}; f_{th}) \in S$

$L \leftarrow \sum_{1 \leq |\alpha| \leq r} a_\alpha \partial^\alpha$

if ord($L$) = $r$ then

$\mathcal{L}^{(r)} \leftarrow \mathcal{L}^{(r)} \cup \{ L \}$

end if

end for

return $\mathcal{L}^{(r)}$
Algorithm 3 a basis of $NT$

Require: $J = \langle g_1, g_2, \ldots, g_\mu \rangle$, $\sqrt{J} = \langle p_1, p_2, \ldots, p_\eta \rangle$

Ensure: $\mathcal{L}$: a basis of $NT$

$m \leftarrow$ (the multiplicity of $J$)  
$\mathcal{L} \leftarrow$ (the result by Algorithm 1 from $J$ and $\sqrt{J}$)  
$r \leftarrow 2$

while $|\mathcal{L}| < m$

$\mathcal{L} \leftarrow$ (the result by Algorithm 2 from $r, J, \sqrt{J}$, and $\mathcal{L}$)  
$r \leftarrow r + 1$

end while

return $\mathcal{L}$

Example 3.6. Let $f_1 = (x^2 - 2)(x^4 - 4x^2 - y^4 - 5y^3 - 9y^2 - 7y + 2)$ and $f_2 = x^4 - 4x^2 - y^3 - 3y^2 - 3y + 3$. Then $J = \langle f_1, f_2 \rangle$ is a primary ideal with multiplicity 11. Using Algorithm 3 Noether operators $\mathcal{L} = \{L_1, L_2, \ldots, L_{11}\}$ can be computed as follows:

$L_1 = 1$, $L_2 = \partial_1$, $L_3 = \partial_2$, $L_4 = \partial_2^2$, $L_5 = \partial_1 \partial_2$, $L_6 = 3\partial_1^2 + 8\partial_2^3$, 
$L_7 = \partial_1 \partial_2^2$, $L_8 = 3\partial_1^2 \partial_2 + 2\partial_2^4$, $L_9 = 2\partial_1^3 - 3x\partial_1^2 + 16\partial_1 \partial_2^3$, 
$L_{10} = 15\partial_1^2 \partial_2^2 + 4\partial_2^5$, $L_{11} = 30\partial_1^4 - 60x\partial_1^3 + 480\partial_1^2 \partial_2^3 + 135\partial_1^2 + 64\partial_2^6$.

4. Annihilators

Let $\text{Ann}_{D_n}(\sigma_F)$ denote the annihilating left ideal in the Weyl algebra $D_n$ of the algebraic local cohomology class $\sigma_F = \left[ \frac{1}{f_1 \ldots f_\mu} \right]$ where $F = \{f_1, f_2, \ldots, f_\mu\}$ is a regular sequence. In general, the annihilating left ideal can be computed by using restriction algorithm that involves Gröbner basis computation in Weyl algebras [11][12]. However the cost of computation of the algorithm that outputs Gröbner basis of the ideal $\text{Ann}_{D_n}(\sigma_F)$ is quit high.

Since our approach requires only generators of the annihilating ideal, we adopt an alternative method introduced in [20] for constructing generators of $\text{Ann}_{D_n}(\sigma_F)$ for avoiding Gröbner bases computation in the Weyl algebra. In this section, we give a recursive method.

Let $\mathcal{A}^{(k)}$ denote the set of $k$-th or lower order annihilators of $\sigma_F$. Since $K[x] \subset D_n$, the polynomials can be regarded as zeroth order differential operators. From the definition of the algebraic local cohomology class $\sigma_F$, a polynomial $g(x)$ satisfies $g(x) \bullet \sigma_F = 0$ if and only if $g(x) \in I$. Thus the polynomial ideal $I$ coincides with $\mathcal{A}^{(0)}$.

Let $P \in \mathcal{A}^{(k)}$ and $f \in F$. The commutator $[P, f]$ is also an annihilator in $\mathcal{A}^{(k-1)}$. Inversely, the annihilator $P$ can be determined from commutators.

Lemma 4.1. If a differential operator $P'$ satisfies $[P', f_1], [P', f_2], \ldots, [P', f_\mu] \in \text{Ann}_{D_n}(\sigma_F)$, then there exists a polynomial $c(x)$ such that $P' + c \in \text{Ann}_{D_n}(\sigma_F)$.

Proof. We can suppose $P' \bullet \sigma_F \neq 0$ without loss of generality. Since $f_1, f_2, \ldots, f_\mu$ are zeroth order annihilators, we have

$0 = [P', f_k] \bullet \sigma_F = P' \bullet (f_k \bullet \sigma_F) - f_k \bullet (P' \bullet \sigma_F) = -f_k \bullet (P' \bullet \sigma_F)$.

Now we put

$$\left[ \begin{array}{c} -c(x) \\ f_1^{f_1} f_2^{f_2} \ldots f_\mu^{f_\mu} \end{array} \right] = P' \bullet \sigma_F$$
as a simplified expression. Here $c(x) \neq 0$. Then it follows from $f_k \bullet (P' \bullet \sigma_F) = 0$ that $\ell_k = 1$. It implies
\[
\left[ \begin{array}{c}
-c(x) \\
f_1 f_2 \cdots f_n
\end{array} \right] = P' \bullet \sigma_F.
\]
Then $P' + c(x)$ is an annihilator of $\sigma_F$.

Next, we discuss first order annihilators. Let $P = \sum_{i=1}^{n} a_i(x) \partial_i + c(x) \in \text{Ann}_{D_n}(\sigma_F)$. Since $f_k \in F$ is an annihilator, a commutator $[P, f_k] = Pf_k - f_k P$ is also a zeroth order annihilator. It implies $\sum_{i=1}^{n} a_i(x) \frac{\partial f_k}{\partial x_i} \in I$. In other words, there exist polynomials $a_i(x), b_{ij}(x) \in K[x]$ such that
\[
\sum_{i=1}^{n} a_i(x) \left( \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} e_k \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(x) f_i(x) e_j = 0,
\]
where $e_j$ is the $j$-th fundamental vector. Let
\[
M^{(1)} = \left\{ \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} e_k \mid i = 1, 2, \ldots, n \right\} \cup \left\{ f_i(x) e_j \mid i, j = 1, 2, \ldots, n \right\}.
\]
The solutions $(a_i; b_{ij})$ of the system (4.1) form the syzygy module $S$ with respect to $M^{(1)}$. When $(a_i) = 0$, the operator $P = c(x)$ is a zeroth order annihilator. Therefore we can ignore the case of $(a_i) = 0$.

Suppose $(a_i) \neq 0$, it follows from direct calculation that $\sum_{i=1}^{n} a_i \partial_i - \sum_{i=1}^{n} b_{ii} \in \text{Ann}_{D_n}(\sigma_F)$. Notice that $P + p(x)$ is also an annihilator if $p(x) \in I$. We may choose $c(x) = -\sum_{i=1}^{n} b_{ii} \mod I$ in $L$. Hence the following algorithm gives a method for computing generators of $\text{Ann}_{D_n}(\sigma_F)$.

**Algorithm 4** generators of $A^{(1)}$

Require: $F = \{f_1, \ldots, f_n\}$: a regular sequence

Ensure: $L^{(1)}$: generators of $A^{(1)}$

\[
J \leftarrow \left[ \frac{\partial f_k}{\partial x_i} \right]_{i, k} : \text{Jacobi matrix}
\]
\[
E \leftarrow \text{unit matrix of size } n
\]
\[
M \leftarrow [J \mid f_1 E \mid \cdots \mid f_n E] : \text{combined matrix}
\]
\[
S \leftarrow \text{(Gröbner basis of syzygy module of columns vectors of } M)\]
\[
L^{(1)} \leftarrow F
\]
for all $(a_i(x); b_{ij}(x)) \in S$
do
if $(a_i(x)) \neq 0$
\[
c \leftarrow -\sum_{i=1}^{n} b_{ii} \mod \langle F \rangle
\]
\[
L^{(1)} \leftarrow L^{(1)} \cup \{ \sum_{i=1}^{n} a_i(x) \partial_i + c \}
\]
end if
end for
return $L^{(1)}$

Next, we discuss $r$-th order annihilators. Let $\{P_1, P_2, \ldots, P_q\}$ be a basis of $A^{(r-1)}$ and let $P = \sum_{1 \leq |\alpha| \leq r} a_{\alpha}(x) \partial^\alpha + c(x) \in \text{Ann}_{D_n}(\sigma_F)$. If $f \in F$, the commutator $[P, f] = [P - c, f]$ is at most $(r-1)$-th order annihilator. Then there exist polynomials $b_{k1}, \ldots, b_{kq}$ such that
\[
\sum_{1 \leq |\alpha| \leq r} a_{\alpha} [\partial^\alpha, f_k] + \sum_{j=1}^{q} b_{kj} P_j = 0. \quad (1 \leq k \leq n)
\]
Using expansions $[\partial^n, f_k] = \sum_{0 \leq |\beta| < r} c_{\alpha k \beta} \partial^\beta$ and $P_j = \sum_{0 \leq |\beta| < r} d_{j \beta} \partial^\beta$, we have

$$\sum_{0 \leq |\beta| < r} \left( \sum_{1 \leq |\alpha| \leq r} a_{\alpha} c_{\alpha k \beta} + \sum_{j=1}^{q} b_{k j} d_{j \beta} \right) \partial^\beta = 0. \quad (1 \leq k \leq n)$$

Thus

$$\sum_{1 \leq |\alpha| \leq r} a_{\alpha} c_{\alpha k \beta} + \sum_{j=1}^{q} b_{k j} d_{j \beta} = 0. \quad (1 \leq k \leq n, 0 \leq |\beta| < r)$$

Set $M = \binom{n+r-1}{n}$. Using a bijection $\varphi : \{k \mid 1 \leq k \leq n\} \times \{\beta \mid 0 \leq |\beta| < r\} \to \{1, 2, \ldots, nM\}$, we can rewrite the system as follows:

$$\sum_{1 \leq |\alpha| \leq r} a_{\alpha} \left\{ \sum_{k=1}^{n} \sum_{0 \leq |\beta| < r} c_{\alpha k \beta} e_{\varphi(k, \beta)} \right\} + \sum_{j=1}^{q} \sum_{k=1}^{n} b_{k j} \left\{ \sum_{0 \leq |\beta| < r} d_{j \beta} e_{\varphi(k, \beta)} \right\} = 0, \quad (4.2)$$

where $e_{\varphi(k, \beta)}$ are fundamental vectors of dimension $nM$. Put

$$\mathcal{M}^{(r)} = \left\{ \sum_{k=1}^{n} \sum_{0 \leq |\beta| < r} c_{\alpha k \beta} e_{\varphi(k, \beta)} \left| 1 \leq |\alpha| \leq r \right\} \cup \left\{ \sum_{0 \leq |\beta| < r} d_{j \beta} e_{\varphi(k, \beta)} \left| 1 \leq j \leq q, 1 \leq k \leq n \right\} \right\} \quad (4.3)$$

Using a Gröbner basis of the syzygy module with respect to $\mathcal{M}^{(r)}$, we can solve the equation $(4.2)$. A solution $(a_{\alpha} ; b_{k j})$ gives the main part $P' = \sum_{1 \leq |\alpha| \leq r} a_{\alpha}(x) \partial^\alpha$ of an $r$-th order annihilator $P$ if there exists $\gamma \in \mathbb{N}_0^n$ such that $a_\gamma \neq 0$ and $|\gamma| = r$.

Finally, we complete the $r$-th order annihilator $P = P' + c(x)$ from the main part $P'$. Put

$$\frac{s(x)}{f_1^{\ell_1} f_2^{\ell_2} \cdots f_n^{\ell_n}} = P' \cdot \frac{1}{f_1 f_2 \cdots f_n}$$

as an irreducible fraction. From Lemma 4.1 as an algebraic local cohomology class, we have

$$\begin{bmatrix} s(x) \\ f_1^{\ell_1} f_2^{\ell_2} \cdots f_n^{\ell_n} \end{bmatrix} = \begin{bmatrix} -c(x) \\ f_1 f_2 \cdots f_n \end{bmatrix} = \begin{bmatrix} -c(x) f_1^{\ell_1-1} f_2^{\ell_2-1} \cdots f_n^{\ell_n-1} \\ f_1^{\ell_1} f_2^{\ell_2} \cdots f_n^{\ell_n} \end{bmatrix}.$$

It implies $s(x) + c(x) f_1^{\ell_1-1} f_2^{\ell_2-1} \cdots f_n^{\ell_n-1} \in \langle f_1^{\ell_1}, f_2^{\ell_2}, \ldots, f_n^{\ell_n} \rangle$. Using the syzygy method again, the polynomial $c(x)$ can be determined.
Algorithm 5: generators of $A^{(r)}$

Require: $r \geq 2$, $F = \{f_1, \ldots, f_n\}$: a regular sequence, $L^{(r-1)}$: generators of $A^{(r-1)}$

Ensure: $L^{(r)}$: generators of $A^{(r)}$.

$g \leftarrow f_1 f_2 \cdots f_n$

$M^{(r)} \leftarrow \text{the right hand side of the equation [4.3] by } r, F, L^{(r-1)}$

$S \leftarrow \text{Gröbner basis of the syzygy module with respect to } M^{(r)}$

$L^{(r)} \leftarrow L^{(r-1)}$

for all $(a_i; b_{kj}) \in S$

$P' \leftarrow \sum_{1 \leq |a_i| \leq r} a_i \partial^n$

if $\text{ord}(P') = r$ then

$h \leftarrow \text{the result by Lemma 4.11}$

$L^{(r)} \leftarrow L^{(r)} \cup \{P' - h\}$

end if

end for

return $L^{(r)}$

---

Example 4.2. Let $f_1 = (x^2 - 2)(x^4 - 4x^2 - y^4 - 5y^2 - 9y^2 - 7y + 2)$ and $f_2 = x^4 - 4x^2 - y^3 - 3y^2 - 3y + 3$. The sequence $F = \{f_1, f_2\}$ is regular. Using Algorithm 4, a basis of $A^{(1)}$ of $\sigma_F = \left[ \frac{1}{f_1 f_2} \right]$ can be computed as follows:

$$
\begin{align*}
& f_1, f_2, \quad 3(y + 1)^3 \partial_x + 4(y + 1)x(x^2 - 2) \partial_y + 22x(x^2 - 2), \\
& 3(x^2 - 2) \partial_x + 4(y + 1)x \partial_y + 34x, \quad (y + 1)^5 \partial_y - 3(x^2 - 2)^2 + (y + 1)^3(7y + 10), \\
& \{(y + 1)^3 - (x^2 - 2)^2\} \partial_y + 3(y + 1)^2, \quad (y + 1)^5 \partial_y + 4 \partial_y + 125x(2y + 1)
\end{align*}
$$

Remark that second (or higher) order annihilators are not necessary for computing the Grothendieck local residue mapping in this case. However they can be calculated by using Algorithm 5 as follows:

$$
\begin{align*}
& \{7(x^2 - 2)^2 - 4(y + 1)^3\} \partial_y^2 - 66(y + 1), \quad (y + 1)^3(x^2 - 2) \partial_y^2, \\
& (y + 1)^6 \partial_y^2, \quad 21(y + 1)^2(x^2 - 2) \partial_x \partial_y + 5x(y + 1)^3 \partial_y^2 + 604x(y + 1), \\
& (y + 1)^5 \partial_x \partial_y, \quad 21(y + 1)^4 \partial_x^2 + 16(y + 1)^3 \partial_y^2.
\end{align*}
$$

5. Differential operator $T_{F, \lambda}$ on an irreducible component

As we have discussed, the algebraic local cohomology class $\sigma_F$ can be decomposed to the direct sum

$$
\sigma_F = \sigma_{F,1} + \cdots + \sigma_{F,\lambda} + \cdots + \sigma_{F,\ell}
$$

and each $\sigma_{F,\lambda} \in H_{Z_{\lambda}}(K[x])$ can be represented as $\sigma_{F,\lambda} = T^*_{F,\lambda} \cdot \delta_{Z_{\lambda}}$. In this section, we discuss the relation between the differential operator $T^*_{F,\lambda}$ and Noether space of $I_{\lambda}$, and describe a method for computing $T^*_{F,\lambda}$ without the use of an explicit representative element of $\sigma_{F,\lambda}$.

From the definition of delta functions, the polynomial ideal $\sqrt{I_\lambda}$ annihilates $\delta_{Z_{\lambda}}$, that is, $h \delta_{Z_{\lambda}} = 0$ for any $h \in \sqrt{I_\lambda}$. Then we have the following lemma.

**Lemma 5.1.** \text{Ann}_{D_{n}}(\delta_{Z_{\lambda}}) = D_{n} \sqrt{I_{\lambda}}.

If an algebraic local cohomology class $\eta \in H_{[Z_{\lambda}]}^{\pi}(K[x])$ can be represented as $\eta = U_1^* \cdot \delta_{Z_{\lambda}}$ and $\eta = U_2^* \cdot \delta_{Z_{\lambda}}$ by using two operators $U_1^*, U_2^* \in D_{n}$, then $(U_1^* - U_2^*) \cdot \delta_{Z_{\lambda}} = 0$. It implies $U_1^* - U_2^* \in D_{n} \sqrt{I_{\lambda}}$. Hence we should identify $U_1^*$ with $U_2^*$ as an equivalent class in $(D_{n} \mod D_{n} \sqrt{I_{\lambda}})$. 

Algorithm 6 Calculation for $S^*_\lambda$

Require: $I_\lambda$: a zero-dimensional primary ideal, $G$: a Gröbner basis of $\sqrt{I_\lambda}$
Ensure: $S^*_\lambda$: a differential operator, $r$: maximal order of annihilators.

$m \leftarrow \text{(the multiplicity of } I_\lambda\text{)}$
$d \leftarrow \dim_K K_\lambda$
{$R_1 = 1, R_2, \ldots, R_m$} $\leftarrow \text{(the result by Algorithm 4)}$
$B = \{1, x^{\gamma_1}, \ldots, x^{\gamma_2}\} \leftarrow \text{(standard monomials with respect to } G\}$

for $i = 1 \text{ to } m - 1$ do
    $s_i \leftarrow c_{i,1} + c_{i,2}x^{\gamma_2} + \cdots + c_{i,d}x^{\gamma_2}$, where $c_{i,1}, c_{i,2}, \ldots, c_{i,d}$ are symbols
end for

$S^*_\lambda \leftarrow R_m + \sum_{i=1}^{m-1} R_is_i$
$L' \leftarrow \text{(generators of } A^{(1)}\text{ by Algorithm 4)}$
$L' \leftarrow \emptyset$
$E \leftarrow \emptyset$
$A \leftarrow \{\alpha \in \mathbb{N}_0^n \mid |\alpha| \leq m\}$

for $r = 1, 2, 3, \ldots$ do
    for all $L \in L \setminus L'$ do
        $\sum_{\alpha \in A} (-\delta)^\alpha u_\alpha \leftarrow L S^*_\lambda$
        $E \leftarrow E \cup \text{the coefficients of } (u_\alpha \mod \sqrt{I_\lambda}) \text{ as a polynomial in } K[B] \mid \alpha \in A$
    end for
    Solving the linear system $E$ for $U = (c_{ij})_{ij}$
    if no more free variables in $U$ then
        Assigning $U$ to $S^*_\lambda$
        return $S^*_\lambda$
    end if
    $L' \leftarrow L$
    $L \leftarrow \text{(generators of } A^{(r+1)}\text{ by Algorithm 5)}$
end for

Let $\varphi \in \text{Hom}_{D_n}(D_n, D_n)$. Consider the following diagram:

$$
\begin{array}{ccc}
\text{Ann}_{D_n}(\sigma_{F,\lambda}) & \longrightarrow & D_n \\
\downarrow & & \downarrow \varphi \\
D_n \sqrt{I_\lambda} & \longrightarrow & D_n \otimes D_n \cdot \delta_{Z_\lambda} \\
\end{array}
$$

If $P \varphi(1) \in D_n \sqrt{I_\lambda}$ for any $P \in \text{Ann}_{D_n}(\sigma_{F,\lambda})$, then we can identify $\varphi$ with an element of $\text{Hom}_{D_n}(D_n \cdot \sigma_{F,\lambda}, D_n \cdot \delta_{Z_\lambda})$. Since $\text{Hom}_{D_n}(D_n \cdot \sigma_{F,\lambda}, D_n \cdot \delta_{Z_\lambda}) \subset \text{Hom}_{D_n}(M_{I_\lambda}, M_{\sqrt{I_\lambda}})$ and $D_n \cdot \delta_{Z_\lambda} = M_{\sqrt{I_\lambda}}$, we call $\varphi$ a Noether operator associated to $\sigma_{F,\lambda}$.

Lemma 5.2 (18). Let $\varphi \in \text{Hom}_{D_n}(D_n \cdot \sigma_{F,\lambda}, D_n \cdot \delta_{Z_\lambda}) \setminus \{0\}$ and let $S^*$ be the corresponding Noether operator to $\varphi$. Then, as $K$-vector spaces, we have

$$
\text{Hom}_{D_n}(D_n \cdot \sigma_{F,\lambda}, H^n_{[Z_\lambda]}(K[x])) \simeq \text{Span}_K \{S^* h \cdot \delta_{Z_\lambda} \mid h \in K[x]/\sqrt{I_\lambda}\}.
$$

Our target $T^n_{F,\lambda}$ can be regarded as a Noether operator associated to the local cohomology class $\sigma_{F,\lambda}$ and can be written as $T^n_{F,\lambda} = S^* h$.

Let $J_F = \det \left( \frac{\partial (f_1, \ldots, f_n)}{\partial (x_1, \ldots, x_n)} \right)$ be the Jacobian of the regular sequence $F$ and let $m_\lambda$ be the multiplicity of $I_\lambda$.
Theorem 5.3 ([18] [20]). Let $\tau \in H^n_{\mathcal{Z}_\lambda}(K[x])$. If $\text{Ann}_{D_n}(\sigma_F)$ annihilates $\tau$ and $J_F \cdot \tau = m_\lambda \delta_{F,\lambda}$, then $\tau = \sigma_{F,\lambda}$ holds.

Theorem 5.4 ([18]). Let $h \in K[x]$. Then the followings are equivalent.

1. $\sigma_{F,\lambda} = S^* h \cdot \delta_{Z,\lambda}$.
2. $J_F S^* h - m_\lambda \in D_n \sqrt{I_\lambda}$.

Next we want to give an explicit algorithm for computing $T_{F,\lambda}^*$.

From the decomposition of $\sigma_F$, it follows $\text{Ann}_{D_n}(\sigma_F) \cdot \sigma_{F,\lambda} = 0$. It implies $\text{Ann}_{D_n}(\sigma_F) T_{F,\lambda}^* \subseteq D_n \sqrt{I_\lambda}$. The primary ideal $I_\lambda$ has Noether operator basis $R = \{R_1 = 1, R_2, \ldots, R_{m_\lambda}\}$, where $m_\lambda$ is the multiplicity of $I_\lambda$. It follows from Lemma 5.2 that

$$T_{F,\lambda}^* \cdot \delta_{Z,\lambda} \in \text{Span}_K \{R_i \delta_{K,\lambda} \cdot \delta_{Z,\lambda} \mid i = 1, 2, \ldots, m_\lambda\}.$$ 

Thus $T_{F,\lambda}^* \in \bigoplus_{i=1}^{m_\lambda} R_i K_\lambda$. It can be represented as

$$T_{F,\lambda}^* \equiv R_{m_\lambda} t_{m_\lambda}(x) + R_{m_\lambda-1} t_{m_\lambda-1}(x) + \cdots + R_2 t_2(x) + t_1(x) \pmod{D_n \sqrt{I_\lambda}}$$

where $t_k(x) \in K_\lambda$. We may suppose $\text{ord}(R_i) \leq \text{ord}(R_{m_\lambda})$ for any $i < m_\lambda$ without loss of generality. Then $t_m(x) \neq 0$.

Put $S_\lambda^* = T_{F,\lambda}^* \cdot t_{m_\lambda}^{-1}(x)$. Then it can be expressed as a “monic” operator:

$$S_\lambda^* \equiv R_{m_\lambda} + R_{m_\lambda-1} s_{m_\lambda-1}(x) + \cdots + R_2 s_2(x) + s_1(x) \pmod{D_n \sqrt{I_\lambda}}$$

The operator $S_\lambda^*$ satisfies the equation $\text{Ann}_{D_n}(\sigma_F) S_\lambda^* \subseteq D_n \sqrt{I_\lambda}$.

Let $d_\lambda = \dim_K K_\lambda$ be the degree of field extension $K$ over $K$ and let $G$ be a Gröbner basis of $\sqrt{I_\lambda}$. Then the finite set $B = \{x^\gamma \mid \text{uf}_G(x^\gamma) = x^\gamma\}$ is a monomial basis of the $K$-vector space $K_\lambda$. Let $B = \{1, x^{\gamma_1}, \ldots, x^{\gamma_{d_\lambda}}\}$. We utilize the method of undetermined coefficients for finding $s_i(x) \in K_\lambda$. That is, they can be written as

$$s_i(x) = c_{i,1} x^{\gamma_{1}} + c_{i,2} x^{\gamma_{2}} + \cdots + c_{i,d_\lambda} x^{\gamma_{d_\lambda}} \in K_\lambda, \quad (i = 1, 2, \ldots, m_\lambda - 1)$$

by using undetermined coefficients $U_1 = \{c_{ij} \mid 1 \leq i \leq m_\lambda, 1 \leq j \leq d_\lambda\}$.

For any $L \in \text{Ann}_{D_n}(\sigma_F)$, the product $L S_\lambda^*$ has an expansion $L S_\lambda^* = \sum_{\alpha} \partial^\alpha u_\alpha(L, x)$, where $u_\alpha(L, x)$ are polynomials written in $s_k(x)$. From $L S_\lambda^* \equiv 0 \pmod{D_n \sqrt{I_\lambda}}$, it follows a membership problem $u_\alpha(L, x) \equiv 0 \pmod{D_n \sqrt{I_\lambda}}$. By monomial reduction with respect to $G$, the polynomials $u_\alpha(L, x)$ can be expanded as

$$u_\alpha(L, x) \mod \sqrt{I_\lambda} = \sum_{\alpha} v_{\alpha,\lambda}(L) x^{\gamma_\alpha}.$$ 

Notice that the coefficients $v_{\alpha,\lambda}(L)$ are affine linear forms in undetermined coefficients $U_1$. Accordingly, the equation $L S_\lambda^* \equiv 0$ can be interpreted as linear equations $v_{\alpha,\lambda}(L) = 0$ for variables $U_1$. If the annihilating left ideal $\text{Ann}_{D_n}(\sigma_F)$ is generated by $L_1, L_2, \ldots$, then the relations $L_j S_\lambda^* \equiv 0$, $(j = 1, 2, \ldots)$ make a system of linear equations. Solving the linear system, we can find the values of the undetermined coefficients. From this procedure, we have Algorithm 6.

Next, there exists a polynomial $h(x) = t_{m_\lambda}(x)$ such that $T_{F,\lambda}^* = S_\lambda^* h(x)$. We utilize the method of undetermined coefficients for finding $h(x)$ again. It can be written as

$$h(x) \equiv c_1 + c_2 x^{\gamma_2} + \cdots + c_{d_\lambda} x^{\gamma_{d_\lambda}} \pmod{\sqrt{I_\lambda}}$$

by using undetermined coefficients $U_2 = \{c_i \mid 1 \leq i \leq d_\lambda\}$.

Put $P = J_F S_{F,\lambda}^* h(x) - m_\lambda$. The operator can be represented as $P = \sum_{\alpha} (-\partial)^\alpha u_\alpha(x)$.

From the theorem above, the operator annihilates the delta function thus $u_\alpha(x) \equiv 0 \pmod{\sqrt{I_\lambda}}$. Since the polynomials $u_\alpha(x)$ are affine linear forms in undetermined coefficients $U_2$, we have a system of linear equations. Solving the system, we can determine the polynomial $h(x)$. The procedure can be executed by Algorithm 7.
Algorithm 7 Calculation for $T_{F,\lambda}$

Require: $I_\lambda$: a zero-dimensional primary ideal, $G$: a Gröbner basis of $\sqrt{I_\lambda}$
Ensure: $T_{F,\lambda}^*$: a differential operator, $r$: maximal order of annihilators.

$m \leftarrow$ (the multiplicity of $I_\lambda$)
$d \leftarrow \dim_K K_\lambda$
$B = \{1, x_{12}^\gamma, \ldots, x_{13}^\gamma\} \leftarrow$ (standard monomials with respect to $G$)
$h \leftarrow c_1 + c_2 x_{12}^\gamma + \cdots + c_d x_{13}^\gamma$, where $c_1, c_2, \ldots, c_d$ are symbols

$S_\lambda^* \leftarrow$ (the result of Algorithm 6)
$J_F \leftarrow \det \left[ \frac{\partial f_i}{\partial x_j} \right]$: Jacobian
$J_{F,\lambda} \leftarrow J_F \mod I_\lambda$
$A \leftarrow \{\alpha \in \mathbb{N}_0^n \mid |\alpha| \leq m\}$
$\sum_{\alpha \in A} (-\partial)^\alpha u_\alpha \leftarrow J_{F,\lambda} S_\lambda^* h - m$
$E \leftarrow \emptyset$
for all $\alpha \in A$ do
  $u_\alpha \leftarrow u_\alpha \mod \sqrt{I_\lambda}$
  $E \leftarrow E \cup \{\text{coefficients of } u_\alpha\}$
end for

Solving the linear system $E$ for $(c_1, c_2, \ldots, c_d)$

$T_{F,\lambda}^* \leftarrow$ (Assigning the solution $c_i$ to $S_\lambda^* h$)
return $T_{F,\lambda}^*$

6. Local residue mapping

As we have explained, the local residue mapping can be represented as the set $\{(T_{F,\lambda}, Z_\lambda) \mid \lambda = 1, 2, \ldots, \ell\}$. The irreducible set $Z_\lambda$ is the zero set of the prime ideal $\sqrt{I_\lambda}$. In section 5 we have discussed the method for finding the differential operator $T_{F,\lambda}^*$ for the algebraic local cohomology class $\sigma_{F,\lambda}$. Remark that the computation is parallelizable for each pair $(T_{F,\lambda}, Z_\lambda)$.

Algorithm 8 Local residue mapping for $\sigma_F$

Require: $F = \{f_1, \ldots, f_n\}$: a regular sequence
Ensure: $T = \{(T_{F,\lambda}, \sqrt{I_\lambda}) \mid \lambda = 1, 2, \ldots, \ell\}$.
$I_1 \cap \cdots \cap I_\ell \leftarrow$ (primary decomposition of $I = \langle F \rangle$)
$T \leftarrow \emptyset$
for $\lambda = 1$ to $\ell$ do
  $T_{F,\lambda}^* \leftarrow$ (the result of Algorithm 7)
  $T_{F,\lambda} \leftarrow$ (the formal adjoint of $T_{F,\lambda}^*$)
  $Z_\lambda \leftarrow \sqrt{I_\lambda}$
  $T \leftarrow T \cup \{(T_{F,\lambda}, Z_\lambda)\}$
end for
return $T$

The algorithm above has been implemented on a computer algebra system Risa/Asir. First, we give an example of a local residue mapping.

Example 6.1. For simplicity, we explain a case that a regular sequence gives a primary ideal. Let $f_1 = (x^2 - 2)(x^4 - 4x^2 - y^4 - 5y^3 - 9y^2 - 7y + 2)$ and $f_2 = x^4 - 4x^2 - y^3 - 3y^2 - 3y + 3$. Then the sequence $F = \{f_1, f_2\}$ is regular and the corresponding ideal $I = \langle F \rangle$ is primary. Noether operators associated to $I$ have been given in Example 3.6 and annihilators of $\sigma_F = \left[ \frac{1}{f_1 f_2} \right]$ have been shown
in Example 4.2. For ease of implementation on a computer algebra system Risa/Asir, our program computes formal adjoints of operators in Section 5. Using Algorithm 6, the operator $S$ on $V_C(\sqrt{J})$ can be calculated as

$$S = -30\partial_x^4 + 150x\partial_x^3 - 480x^2\partial_y^3 - 135(x^2 + 3)\partial_x^2 + 720x\partial_x\partial_y^3 + \frac{1575}{2}x\partial_x - 64\partial_y^6 - 720\partial_y^3 - \frac{1575}{2}.$$ 

The leading coefficient of $T$ can be given by $h(x) = -\frac{1}{184320}x$, thus

$$T = \frac{1}{6144}x\partial_x^4 - \frac{5}{3072}\partial_x^3 + \frac{1}{384}x\partial_x^2\partial_y^3 + \frac{15}{4096}x\partial_x^2$$

$$- \frac{1}{128}\partial_x\partial_y^3 - \frac{35}{4096}\partial_x + \frac{1}{2880}x\partial_y^6 + \frac{15}{256}\partial_y^3 + \frac{35}{8192}x.$$ 

Next, we show a log of our program. The procedure residuemap is an implementation of Algorithm 8.

```asir
$ asir
[1825] load("oh_alc.rr");
[2452] V=[x,y];
[2453] F=[(x^2-2)*(x^4-4*x^2-y^4-5*y^3-9*y^2-7*y+2), x^4-4*x^2-y^3-3*y^2-3*y+3];
[2454] oh_alc.init(V,F);
[2455] oh_alc.residuemap();
[[1/6144*x*dx^4-5/3072*dx^3+1/384*x*dy^3+15/4096*x)*dx^2
+(-1/128*dy^3-35/4096)*dx+1/2880*x*dy^6+1/256*x*dy^3+35/8192*x,
[x^2-2,y+1]]
```

7. Local residues

As explained in section 2 for any holomorphic function $\varphi(x)$, the local residue at $\beta \in Z_\lambda$ can be evaluated as

$$\text{Res}_\beta(\varphi F dx) = (T_{F,\lambda} \varphi)|_{x=\beta}.$$ 

Accordingly, the local residue mapping can be represented by using a set $\{(T_{F,\lambda}, Z_\lambda) \mid \lambda = 1, 2, \ldots, \ell\}$.

Given a holomorphic function $\varphi$, it is easy to apply $T_{F,\lambda}$ to the function. The algorithm for evaluating local residues is as follows. In the result, the variable $x = (x_1, \ldots, x_n)$ expresses a zero of each component $V_C(\sqrt{T_\lambda})$.

**Algorithm 9 Local residue**

**Require:** $F = \{f_1, \ldots, f_n\}$: a regular sequence, $\varphi$: a holomorphic function

**Ensure:** $\Phi = \{(T_{F,\lambda}\varphi \mid \lambda = 1, 2, \ldots, \ell)\}.$

1. $\Phi \leftarrow \emptyset$
2. for $\lambda = 1$ to $\ell$ do
   1. $\Phi \leftarrow \Phi \cup \{(T_{F,\lambda} \varphi, \sqrt{T_\lambda})\}$$
3. return $\Phi$
8. Conclusion

We have developed an algorithm for exactly evaluating Grothendieck local residue via a representation of the local residue mapping in the general cases. Although our method is based on the theory of algebraic local cohomology groups and $D$-modules, the algorithm can be realized by using calculations on polynomial rings. The resulting algorithms have been implemented in a computer algebra system Risa/Asir.

References

[1] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley Interscience, 1970.
[2] P. Gianni, B. Trager and G. Zacharias, Gröbner bases and primary decomposition of polynomial ideals, Journal of Symbolic Computation 6 (1988), 149–167.
[3] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics 20, Springer, 1966.
[4] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, the third revised edition, North-Holland, 1990.
[5] M. Kashiwara, On the maximally overdetermined system of linear differential equations. I, Publications of the Research Institute for Mathematical Sciences 10 (1975), 563–579.
[6] M. Kashiwara, On the holonomic systems of linear differential equations. II, Inventiones Mathematicae 49 (1978), 121–135.
[7] M. Kashiwara, On holonomic systems of micro-differential equations. III — systems with regular singularities —, Publications of the Research Institute for Mathematical Sciences 17 (1981), 813–979.
[8] Y. Nakamura and S. Tajima, A method for computing holonomic systems associated with zero-dimensional algebraic local cohomology classes, RIMS Kôkyûroku 1199 (2001), 70–89. (in Japanese)
[9] M. Noro, New algorithms for computing primary decomposition of polynomial ideals, Mathematical Software — ICMS 2010, Lecture Notes in Computer Science 6327, 233–244, Springer, 2010.
[10] M. Noro et al., Risa/Asir a computer algebra system, 1994–2018. [http://www.math.kobe-u.ac.jp/Asir/](http://www.math.kobe-u.ac.jp/Asir/)
[11] T. Oaku, Algorithms for the $b$-functions, restrictions, and algebraic local cohomology groups of $D$-modules, Advances in Applied Mathematics 19 (1997), 61–105.
[12] T. Oaku and N. Takayama, Algorithms for $D$-modules — restriction, tensor product, localization, and local cohomology groups, Journal of Pure and Applied Algebra 156 (2001), 267–308.
[13] K. Ohara and S. Tajima, An algorithm for computing Grothendieck local residues I — shape base case —, submitted to Mathematics in Computer Science.
[14] OpenXM committers, OpenXM, a project to integrate mathematical software systems. 1998–2018. [http://www.openxm.org](http://www.openxm.org)
[15] F. Pham, *Singularités des Systèmes Différentiels de Gauss-Manin*, Birkhäuser, 1979.
[16] V. P. Palamodev, *Linear Differential Operators with Constant Coefficients*, Springer, 1970.
[17] S. Tajima, On Noether differential operators attached to a zero-dimensional primary ideal — a shape basis case —, Finite or Infinite Dimensional Complex Analysis and Applications, 357–366, Kyushu Univ. Press, 2005.
[18] S. Tajima, Noether differential operators and Grothendieck local residues, RIMS Kôkyûroku 1431 (2005), 123–136. (in Japanese)
[19] S. Tajima, T. Oaku, and Y. Nakamura, Multidimensional local residues and holonomic $D$-modules, RIMS Kôkyûroku 1033 (1998), 59–70. (in Japanese)
[20] S. Tajima and Y. Nakamura, Computational aspects of Grothendieck local residues, Séminaires et Congrès 10 (2005), 287–305.
Katsuyoshi Ohara
Faculty of Mathematics and Physics
Kanazawa University
Kakuma-machi, Kanazawa 920-1192, Japan
e-mail: ohara@se.kanazawa-u.ac.jp

Shinichi Tajima
Graduate School of Science and Technology
Niigata University
8050 Ikarashi 2 no-cho, Nishi-ku, Niigata 950-2181, Japan
e-mail: tajima@emeritus.niigata-u.ac.jp