On meromorphic functions defined by a differential system of order 1, II

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Abstract. Given a nonzero germ $h$ of holomorphic function on $(\mathbb{C}^n, 0)$, we study the condition: "the ideal $\text{Ann}_\mathcal{D} 1/h$ is generated by operators of order 1". When $h$ defines a generic arrangement of hypersurfaces with an isolated singularity, we show that it is verified if and only if $h$ is weighted homogeneous and $-1$ is the only integral root of its Bernstein-Sato polynomial. When $h$ is a product, we give a process to test this last condition. Finally, we study some other related conditions.

1 Introduction

Let $h \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ be a nonzero germ of holomorphic function such that $h(0) = 0$. We denote by $\mathcal{O}[1/h]$ the ring $\mathcal{O}$ localized by the powers of $h$. Let $\mathcal{D} = \mathcal{O}\langle \partial_1, \ldots, \partial_n \rangle$ be the ring of linear differential operators with holomorphic coefficients and $F_\bullet \mathcal{D}$ its filtration by order. In [28], we study the following condition on $h$:

$$\mathbf{A}(1/h) : \text{The left ideal } \text{Ann}_\mathcal{D} 1/h \subset \mathcal{D} \text{ of operators annihilating } 1/h \text{ is generated by operators of order one.}$$

This property is very natural when one considers sections of $\mathcal{O}[1/h]/\mathcal{O}$ with an algebraic viewpoint, see [20]. On the other hand, it seems to be linked to the topological property \textbf{LCT}(h): \textit{the de Rham complex } $\Omega^\bullet[1/h]$ \textit{of meromorphic forms with poles along } $h = 0$ \textit{is quasi-isomorphic to its subcomplex of logarithmic forms.} In particular, \textbf{LCT}(h) implies \textbf{A}(1/h) for free germs $\mathbf{S}$ (in the sense of K. Saito [20]). The study of this condition \textbf{LCT}(h) was initiated in [9] by F.J. Castro Jiménez, D. Mond and L. Narváez Macarro (see

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for a survey). In this paper, we pursue the study of the condition $A(1/h)$, and more precisely when $h$ is a reducible germ. Our motivation is to deepen the link between $\text{LCT}(h)$ and $A(1/h)$.

Let us recall that this last condition is closely linked to the following ones:

$H(h)$: The germ $h$ belongs to the ideal of its partial derivatives.

$B(h)$: $-1$ is the smallest integral root of the Bernstein polynomial of $h$.

$A(h)$: The ideal $\text{Ann}_D h^s$ is generated by operators of order one.

Indeed, condition $H(h)$ seems to be necessary in order to have $A(1/h)$, see [29]. Moreover, condition $A(1/h)$ always implies $B(h)$ ([28], Proposition 1.3). This last condition has the following algebraic meaning: the $D$-module $\mathcal{O}[1/h]$ is generated by $1/h$ (see below). On the other hand, one can easily check that:

If conditions $H(h)$, $B(h)$ and $A(h)$ are verified, then so is $A(1/h)$. (1)

Our first part is devoted to condition $B(h)$. For testing this condition, it seems natural to avoid the full determination of the Bernstein polynomial of $h$, denoted by $b(h^s, s)$. Here we give such a trick when $h$ is not irreducible, using Bernstein polynomials associated with sections of holonomic $D$-modules.

Given a nonzero germ $f \in \mathcal{O}$ and an element $m \in \mathcal{M}$ of a holonomic $D$-module without $f$-torsion, we recall that there exists a functional equation:

$$b(s)mf^s = P(s) \cdot mf^{s+1}$$

in $(Dm) \otimes \mathcal{O}[1/f, s]f^s$, where $P(s) \in D[s] = D \otimes \mathbb{C}[s]$ and $b(s) \in \mathbb{C}[s]$ are nonzero [17]. The Bernstein polynomial of $f$ associated with $m$, denoted by $b(mf^s, s)$, is the monic polynomial $b(s) \in \mathbb{C}[s]$ of smallest degree which verifies such an equation. When $f$ is not a unit and $m \in f^{-1}\mathcal{M} - f^\ast\mathcal{M}$ with $r \in \mathbb{N}^*$, it is easy to check that $-r$ is a root of $b(mf^s, s)$. Thus we consider the following condition:

$B(m, f)$: $-1$ is the smallest integral root of $b(mf^s, s)$

for $m \in \mathcal{M} - f\mathcal{M}$; this extends our previous notation when $m = 1 \in \mathcal{O} = \mathcal{M}$. By generalizing a well known result due to M. Kashiwara, this condition means: the $D$-module $(Dm)[1/f]$ is generated by $m/f$ (see Proposition [25]). Hence we get:

**Proposition 1.1** Let $h_1, h_2 \in \mathcal{O}$ be two nonzero germs without common factor and such that $h_1(0) = h_2(0) = 0$.

(i) We have: $B(h_1h_2) \Rightarrow B(1/h_1, h_2) \Rightarrow B(\hat{1}/h_1, h_2)$ where $\hat{1}/h_1 \in \mathcal{O}[1/h_1]/\mathcal{O}$.

(ii) If $B(h_1)$ is verified, then $B(h_1h_2) \Leftrightarrow B(1/h_1, h_2)$.

(iii) If $B(h_2)$ is verified, then $B(1/h_1, h_2) \Leftrightarrow B(\hat{1}/h_1, h_2)$.
Of course, the equivalence in (ii) just means: \((\mathcal{O}[1/h_1])[1/h_2] = \mathcal{O}[1/h_1 h_2]\). Let us insist on the condition \(B(1/h_1,h_2)\). Indeed, the polynomial \(b((1/h_1)h_2,s)\) may be considered as a Bernstein polynomial of the function \(h_2\) in restriction to the hypersurface \((X_1,0) \subset (\mathbb{C}^n,0)\) defined by \(h_1\), see [26]. In particular, \(b((1/h_1)h_2^s,s)\) coincides with the (classical) Bernstein Sato polynomial of \(h_2|_{X_1} : (X_1,0) \to (\mathbb{C},0)\) if \(h_1\) defines a smooth germ \((X_1,0)\) (Corollary [24]); thus this trick is very relevant when \(h\) has smooth components. As an application, we prove that \(B(h)\) is true when \(h\) defines a hyperplane arrangement \((\text{Proposition 2.7})\), by using the classical principle of ‘Deletion-Restriction’.

This result was first obtained by A. Leykin [31], and more recently by M. Saito [22].

What about the condition \(A(1/h)\) when \(h = h_1 \cdot h_2\) is a product with \(h_1(0) = h_2(0) = 0\) and \(h_1,h_2\) have no common factor? It is also natural to consider the ideal \(\text{Ann}_D(1/h_1)h_2^s\) and the Bernstein polynomial \(b((1/h_1)h_2^s,s)\). Indeed \(B(1/h_1,h_2)\) is a weaker condition than \(B(h_1 h_2)\) \((\text{Proposition 1.1})\) and we have an analogue of (1). Of course, it is difficult to verify if \(\text{Ann}_D(1/h_1)h_2^s\) is - or not - generated by operators of order one. Meanwhile, this may be done under strong assumptions on the components of \(h\), by using the characteristic variety of \(D(1/h_1)h_2^s\) which may be explicited in terms of the one of \(D(1/h_1)\) [14]. Let us give a definition.

**Definition 1.2** A reduced germ \(h \in \mathcal{O}\) defines a generic arrangement of hypersurfaces with an isolated singularity if it is a product \(\prod_{i=1}^p h_i\), \(p \geq 2\), of germs \(h_i\) which defines an isolated singularity, and such that, for any index \(2 \leq k \leq \min(p,n)\), the morphism \((h_{i_1},\ldots,h_{i_k}) : (\mathbb{C}^n,0) \to (\mathbb{C}^k,0)\) defines a complete intersection with an isolated singularity at the origin.

In the second part, we give a full characterization of \(A(1/h)\) for such a type of germ.

**Theorem 1.3** Let \(h = \prod_{i=1}^p h_i \in \mathcal{O}\), \(p \geq 2\), define a generic arrangement of hypersurfaces with an isolated singularity. Then the ideal \(\text{Ann}_D 1/h\) is generated by operators of order one if and only if the following conditions are verified:

1. the germ \(h\) is weighted homogeneous;
2. \(-1\) is the only integral root of the Bernstein polynomial of \(h\).

We recall that a nonzero germ \(h\) is weighted homogeneous of weight \(d \in \mathbb{Q}^+\) for a system \(\alpha \in (\mathbb{Q}^+)^n\) if there exists a system of coordinates in which \(h\) is a linear combination of monomials \(x_1^{\gamma_1} \cdots x_n^{\gamma_n}\) with \(\sum_{i=1}^n \alpha_i \gamma_i = d\).
This result generalizes the case of a hypersurface with an isolated singularity [26]. Moreover, the condition $B(h)$ is also explicit when $p = 2$, $h$ weighted homogeneous (Corollary 3.6), and the trick above for testing $B(h)$ may be generalized for $p \geq 3$ (Proposition 2.5). On the other hand, these conditions on the components of $h$ are strong and they are not verified in general. To illustrate this limitation, we end this part by studying the condition $A(1/h)$ for $h = (x_1 - x_2x_3)g$ when $g \in C[x_1, x_2]$ is a weighted homogeneous polynomial.

**Proposition 1.4** Let $g \in C[x_1, x_2]$ be a weighted homogeneous reduced polynomial of multiplicity greater or equal to 3. Let $h \in C[x_1, x_2, x_3]$ be the polynomial $(x_1 - x_2x_3)g$.

(i) If $g$ is not homogeneous, then the condition $A(1/h)$ does not hold for $h$.

(ii) If $g$ is homogeneous of degree 3, then $A(1/h)$ holds for $h$.

Here $H(h)$ are $B(h)$ are verified (see Lemma 3.7) whereas $A(h)$ fails. We mention that this family of surfaces was intensively studied by the Sevilian group in order to understand the condition $\text{LCT}(h)$ [4], [6], [10], [12], [13].

In the last part, we give some results on conditions closely linked to $A(1/h)$. First, we show how the Sebastiani-Thom process allows to construct germs $h$ which verify the condition $A(h)$. Then, we do some remarks on a natural generalization of condition $A(1/h)$. We end this note with some remarks on the holonomy of a particular $\mathcal{D}$-module which appears in the study of $\text{LCT}(h)$.

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2 The condition $B(h)$ for reducible germs

2.1 Preliminaries

In this paragraph, we recall some results about Bernstein polynomials of a germ $f \in \mathcal{O}$ associated with a section $m$ of a holonomic $\mathcal{D}$-module $\mathcal{M}$ without $f$-torsion. As they appear in [24] (unpublished), we recall some proofs for the convenience of the reader.

**Lemma 2.1** Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Let $m$ be a germ of holonomic $\mathcal{D}$-module $\mathcal{M}$ without $f$-torsion. Let $P(s) \in \mathcal{D}[s]$ be a differential operator such that $P(j)mf^j \in \mathcal{M}[1/f]$ is zero for a infinite sequence of integers
Proof. We have the following identity:

\[ P(s)mf^s = \left( \sum_{i=0}^{d} m_i s^i \right) f^{s-N} \]  \hspace{1cm} (3)

in \( \mathcal{M}[1/f, s]f^s \), where \( m_i \in \mathcal{M} \) and \( N \in \mathbb{N} \) denotes the order of \( P \). By assumption, there exists some integers \( j_0 < \cdots < j_d \) such that \( \sum_{i=0}^{d} (j_i)i m_i = 0 \) in \( \mathcal{M} \) for \( 0 \leq k \leq d \). Since the Gram matrix of the integers \( j_0, \ldots, j_d \) is invertible, the previous identities imply that \( m_i = 0 \) for \( 0 \leq i \leq d \). We conclude with \( \Box \).

**Lemma 2.2** Let \( f \in \mathcal{O} \) be a nonzero germ such that \( f(0) = 0 \). Let \( m \in \mathcal{M} \) be a nonzero section of a holonomic \( \mathcal{D} \)-module without \( f \)-torsion.

(i) If \( g \in \mathcal{O} \) is such that \( g \cdot m = 0 \), then \( b(mf^s, s) \) coincides with \( b(m(f+g)^s, s) \).

(ii) If \( m \in \mathcal{M} - f\mathcal{M} \), then \( (s+1) \) divides \( b(mf^s, s) \).

(iii) For all \( p \in \mathbb{N}^s \), \( b(mf^s, s) \) divides the \( \prod_{i=0}^{p-1} b(mf^s, ps+i) \), and the polynomial lcm \( (m(f^s), ps, \ldots, (m(f^s), ps+p-1)) \) divides \( b(mf^s, s) \). In particular, these polynomials have the same roots.

Proof. In order to prove the first point, we just have to check that the polynomial \( b(m(f+g)^s, s) \) is a multiple of \( b(mf^s, s) \) for any \( g \in \text{Ann}_\mathcal{O} m \), and to apply this fact with \( \tilde{f} = f+g \), \( \tilde{g} = -g \).

Let \( P(s) \in \mathcal{D}[s] \) be a differential operator which realizes the Bernstein polynomial of \( m(f+g)^s \). In particular, \( R(s) = b(m(f+g)^s, s) - P(s)f \) belongs to \( \text{Ann}_\mathcal{D}[s] m(f+g)^s \). As \( (f+g)^s \cdot m = f^s \cdot m \) for all \( i \in \mathbb{N} \), the operator \( R(s) \) annihilates \( m(f+g)^s \) by Lemma 2.1. Thus the polynomial \( b(mf^s, s) \) divides \( b(m(f+g)^s, s) \).

Now, we prove (ii). Let \( R \in \mathcal{D} \) be the remainder in the division of \( P(s) \) by \( (s+1) \) in a nontrivial identity \( \Box \). Thus \( R \cdot mf^{s+1} = (R \cdot mf^s) + (s+1)af^s \) where \( a \in \mathcal{M}[1/f, s] \). From \( \Box \), we get \( b(-1)m = fR(m) \). Hence \( b(-1) = 0 \) since \( m \not\in f\mathcal{M} \).

The last point is an easy exercise. \( \Box \)

**Proposition 2.3** Let \( X \subset \mathbb{C}^n \) be an analytic subvariety of codimension \( p \) passing through the origin. Let \( i : X \hookrightarrow \mathbb{C}^n \) denote the inclusion and let \( h_1, \ldots, h_p \in \mathcal{O} \) be local equations of \( i(X) \). Let \( f \in \mathcal{O} \) be a germ such that \( f \circ i \) is not constant and let \( \mathcal{M} \) be a holonomic \( \mathcal{D}_{X,0} \)-module without \((f \circ i)\)-torsion.

If \( m \in \mathcal{M} \) is nonzero, then \( b(m(f \circ i)^s, s) \) coincides with the polynomial \( b(i_+(m) f^s, s) \) where \( i_+(m) \in \mathcal{M} \otimes (\mathcal{O}[1/h_1 \cdots h_p]/\sum_{i=1}^{p} \mathcal{O}[1/h_1 \cdots \hat{h}_i \cdots h_p]) \) denotes the element \( 1/h_1 \cdots h_p \).
**Proof.** Up to a change of coordinates, we can assume that \( h_i = x_i, 1 \leq i \leq p \). Then the remainder \( \tilde{f} \in \mathbb{C}\{x_{p+1}, \ldots, x_n\} \) in the division of \( f \) by \( x_1, \ldots, x_p \) defines the germ \( f \circ i \). Thus we have \( b(i_+ (m)f^s, s) = b(i_+ (m)\tilde{f}^s, s) \) by using Lemma [2.2](#). Let us prove that \( b(i_+ (m)\tilde{f}^s, s) \) coincides with \( b(m\tilde{f}^s, s) \). Firstly, it easy to check that a functional equation for \( b(m\tilde{f}^s, s) \) induces an equation for \( b(i_+ (m)\tilde{f}^s, s); \) thus \( b(i_+ (m)\tilde{f}^s, s) \) divides \( b(m\tilde{f}^s, s) \). On the other hand, we consider the following equation:

\[
b(i_+ (m)\tilde{f}^s, s)i_+ (m)\tilde{f}^s = P \cdot i_+ (m)\tilde{f}^{s+1}
\]

(4)

where \( P \in \mathcal{D}[s] \). It may be written \( P = \sum_{i=1}^P Q_i x_i + R \) where \( Q_i \in \mathcal{D}[s] \) and the coefficients of \( R \in \mathcal{D}[s] \) do not depend on \( x_1, \ldots, x_p \); in particular, we can change \( P \) by \( R \) in (4). Let \( \tilde{R} \in \mathcal{D}_{X,0}[s] = \mathbb{C}\{x_{p+1}, \ldots, x_n\}\{\partial_{p+1}, \ldots, \partial_n\}[s] \) denote the constant term of \( R \) as an operator in \( \partial_1, \ldots, \partial_p \) with coefficients in \( \mathcal{D}_{X,0}[s] \). Obviously we can change \( R \) by \( \tilde{R} \) in (4). As the annihilator of \( i_+ (m)\tilde{f}^s \) in \( \mathcal{D}_{X,0}[s] \) coincides with the one of \( m\tilde{f}^s \), we deduce that \( b(i_+ (m)\tilde{f}^s, s) \) is a multiple of \( b(m\tilde{f}^s, s) \). This completes the proof. \( \square \)

**Corollary 2.4** Let \( h_1, h_2 \in \mathcal{O} \) be two nonzero germs without common factor and such that \( h_1(0) = h_2(0) = 0 \). Assume that \( h_1 \) defines a smooth germ \( (X_1, 0) \subset (\mathbb{C}^n, 0) \). Then \( b((1/h_1)h_2^s, s) \) coincides with the (classical) Bernstein Sato polynomial of \( h_2\big|_{X_1} : (X_1, 0) \rightarrow (\mathbb{C}, 0) \).

**Proposition 2.5** Let \( f \in \mathcal{O} \) be a nonzero germ such that \( f(0) = 0 \). Let \( m \) be a section of a holonomic \( \mathcal{D} \)-module without \( f \)-torsion, and \( \ell \in \mathbb{N}^* \). The following conditions are equivalent:

1. The smallest integral root of \( b(mf^s, s) \) is strictly greater than \(-\ell - 1\).

2. The \( \mathcal{D} \)-module \( (\mathcal{D}m)[1/f] \) is generated by \( mf^{-\ell} \).

3. The following morphism is an isomorphism:

\[
\frac{\mathcal{D}[s]mf^s}{(s + \ell)\mathcal{D}[s]mf^s} \longrightarrow (\mathcal{D}m)[1/f]
\]

\[
P(s)mf^s \mapsto P(-\ell) \cdot mf^{-\ell}.
\]

This is a direct generalization of a well known result due to M. Kashiwara and J.E. Björk for \( m = 1 \in \mathcal{O} = \mathcal{M} \) (see [16] Proposition 6.2, [2] Propositions 6.1.18, 6.3.15 & 6.3.16).
2.2 Is $-1$ the only integral root of $b(h^s, s)$?

First of all, let us prove Proposition 1.1.

**Proof of Proposition 1.1** Assume that condition $B(h_1 h_2)$ is verified. From Proposition 2.5 this means $\mathcal{D}1/h_1 h_2 = \mathcal{O}[1/h_1 h_2]$. In particular, we have $(\mathcal{D}1/h_1)[1/h_2] \subset \mathcal{D}1/h_1 h_2$; thus, by using Proposition 2.5 with $m = 1/h_1$, condition $B(1/h_1, h_2)$ is verified. The second relation in (i) is clear since a functional equation realizing $b((1/h_1) h_2^s, s)$ induces a functional equation for $b((1/h_1) h_2^s, s)$.

The second point is clear, since it just means $(\mathcal{O}[1/h_1])[1/h_2] = \mathcal{O}[1/h_1 h_2]$ (using three times Proposition 2.5). Now, given $P \in \mathcal{D}$ and $\ell \in \mathbb{N}$, let us prove that $(P \cdot 1/h_1) \otimes 1/h_2^\ell$ belongs to $\mathcal{D}1/h_1 h_2$ when $B(1/h_1, h_2)$ and $B(h_2)$ are verified. From Proposition 2.5 there exists an operator $Q \in \mathcal{D}$ such that $(P \cdot 1/h_1) \otimes 1/h_2^\ell = Q \cdot 1/h_1 \otimes 1/h_2$ in $(\mathcal{O}[1/h_1]/\mathcal{O})[1/h_2]$. Hence we have $(P \cdot 1/h_1) \otimes 1/h_2^\ell = Q \cdot 1/h_1 h_2 + a/h_2^\ell$, where $a \in \mathcal{O}$ and $N \in \mathbb{N}^*$. As condition $B(h_2)$ is verified, there exists $R \in \mathcal{D}$ such that $R \cdot 1/h_2 = a/h_2^\ell$. Thus we get $(P \cdot 1/h_1) \otimes 1/h_2^\ell = (Q + Rh_1) \cdot 1/h_1 h_2$. In consequence, the condition $B(1/h_1, h_2)$ is also verified. \(\square\)

The following examples show that there is no other relation between $B(h_1 h_2)$, $B(1/h_1, h_2)$, $B(1/h_1, 1/h_2)$ and $B(h_1)$, $B(h_2)$.

**Example 2.6** (i) If $h_1 = x_1$ and $h_2 = x_1 + x_2 x_3 + x_4 x_5$, then $b(h_1^s, s) = b(h_2^s, s) = s + 1$ but $B((1/h_1) h_2^s, s) = b((x_2 x_3 + x_4 x_5)^s, s) = (s + 1)(s + 2)$ by using Corollary 2.3.

(ii) If $h_1 = x_1 x_2 + x_3 x_4$ and $h_2 = x_1 x_2 + x_3 x_5$, then $b(h_1^s, s) = b(h_2^s, s) = (s + 1)(s + 2)$, but $b((h_1 h_2)^s, s)$ is equal to $(s + 1)(s + 3/2)$ by using Macaulay 2 [15, 18]. Moreover, if $h_3 = x_1$, then condition $B(h_1 h_3)$ is also true, since $b((h_1 h_3)^s, s) = (s + 1)(s + 3/2)$ using Macaulay 2. Hence condition $B(h_1 h_2)$ does not depend in general of the conditions $B(h_1)$ and $B(h_2)$.

(iii) Assume that $h_1 = x_1$ and $h_2 = x_1^2 + x_2^4 + x_3^4$. Then $b(h_1^s, s) = s + 1$ and condition $B(1/h_1, h_2)$ is true, since $b((1/h_1) h_2^s, s) = b((x_1^2 + x_2^4 + x_3^4)^s, s)$ by Corollary 2.4. But a direct computation using [24] shows that condition $B(1/h_1, h_2)$ is false.

(iv) Assume that $h_1 = x_1 x_2 x_3 + x_4 x_5$ and $h_2 = x_1$. Then $b((1/h_1) h_2^s, s) = b((1/h_1) h_2^s, s) = b((x_4 x_5)^s, s) = (s + 1)^2$, using Proposition 2.9 and Proposition 1. On the other hand, $(s + 1)(s + 2)$ divides $b((h_1 h_2)^s, s)$ and $b(h_1^s, s)$, by the semi-continuity of the Bernstein polynomial (since when $u$ is a unit, we have $b((u x_2 x_3 + x_4 x_5)^s, s) = (s + 1)(s + 2)$). Thus $B(1/h_1, h_2)$ does not imply $B(h_1 h_2)$ in general.

As an application of Proposition 1.1 we obtain a new proof of the following result.
Proposition 2.7 ([31], [22]) Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ be the product of nonzero linear forms (distinct or not). Then the Bernstein polynomial of $h$ has only $-1$ as integral root.

Proof. Let $h$ be the product $l_1^{p_1} \cdots l_r^{p_r}$ where $r, p_1, \ldots, p_r \in \mathbb{N}^*$ are positive integers, and $l_i \in (\mathbb{C}^n)^*$ are distinct. We prove the result by induction on $r$. If $r = 1$, this is a direct consequence of the following identity:

$$
\frac{1}{p^r} \left( \frac{\partial}{\partial x} \right)^p (x^p)^{s+1} = (s + \frac{1}{p})(s + \frac{2}{p}) \cdots (s + \frac{p-1}{p})(s + 1)(x^p)^s
$$

for $p \in \mathbb{N}^*$. Now, we assume that the assertion is true for any germ as above with at most $N \geq 1$ distinct irreducible components. Let $h$ be such a germ with $r = N$. Let $l \in (\mathbb{C}^n)^*$ be a nonzero form which is not a factor of $h$, and $p \in \mathbb{N}^*$. In particular, $-1$ is the only integral root of the Bernstein polynomial of $l, l^p$ and $h$. Let us remark that the assertion for $h \cdot l$ implies the assertion for $h \cdot l^p$. Indeed, using Lemma 2.2 it is easy to check that $B(1/h, l)$ implies $B(1/h, l^p)$. We conclude with the help of Proposition 2.4 (ii).

In order to prove $B(h \cdot l)$, we just have to check that $-1$ is the only integral root of $b((\hat{1}/l)h^*, s)$ (Proposition 2.4 (iii)). But this is true by induction on $N$ since this last polynomial coincides with the Bernstein polynomial of $h|_{t=0}$ (Corollary 2.4). This completes the proof. □

When $h$ has more than two components, the following result provides a generalized criterion for the condition $B(h)$.

Proposition 2.8 Let $h_1, \ldots, h_p \in \mathcal{O}$ be nonzero germs without common factor, and such that $h_1(0) = \cdots = h_p(0) = 0$.

(i) Assume that $2 \leq p \leq n$ and that $(h_1, \ldots, h_p)$ defines a complete intersection. If $B(h_1 \cdots \hat{h}_j \cdots h_p)$, $1 \leq j \leq p$, are verified, then $B(\delta, h_1)$ implies $B(h_1 \cdots h_p)$ where $\delta = 1/h_2 \cdots h_p \in \mathcal{O}[1/h_2 \cdots h_p]/\sum_{i=2}^{p} \mathcal{O}[1/h_2 \cdots h_i \cdots h_p]$.

(ii) Assume that $p = n$ and $(h_1, \ldots, h_n)$ defines the origin. If the conditions $B(h_1 \cdots \hat{h}_j \cdots h_n)$, $1 \leq j \leq n$, are verified, then so is $B(h_1 \cdots h_n)$.

(iii) Assume that $p \geq n + 1$. If the conditions $B(h_{i_1} \cdots h_{i_n})$ are verified for $1 \leq i_1 < \cdots < i_n \leq p$ then so is $B(h_1 \cdots h_p)$.

Proof. We start with the first assertion. From Proposition 2.4 we just have to prove $B(1/h_2 \cdots h_p, h_1)$ (since $B(h_2 \cdots h_p)$ is verified). Thus, given $P \in \mathcal{D}$ and $\ell \in \mathbb{N}$, let us prove that $(P \cdot 1/h_2 \cdots h_p) \otimes 1/h_1^\ell$ belongs to $\mathcal{D}1/h_1 \cdots h_p$. Using condition $B(\delta, h_1)$, we have

$$(P \cdot \frac{1}{h_2 \cdots h_p}) \otimes \frac{1}{h_1^\ell} = R \cdot \frac{1}{h_1 \cdots h_p} + \sum_{2 \leq i \leq p} \frac{q_i}{h_1^{\ell_1} \cdots \hat{i} \cdots \hat{i_{i-1}} \cdots \hat{i_{i+1}} \cdots h_p^{\ell_p}}$$
with $q_i \in \mathcal{O}$ and $\ell_{i,j} \in \mathbb{N}$. We conclude by using that $\mathcal{O}[1/h_1 \cdots h_i \cdots h_p]$ is generated by $1/h_1 \cdots h_i \cdots h_p$ for $2 \leq i \leq p$ by assumption.

In order to prove (ii), we have to check that $B(\delta, h_1)$ is verified when $p = n$. Firstly, we notice that the $D$-module $\mathcal{O}[1/h_2 \cdots h_p]/\sum_{i=2}^{p} \mathcal{O}[1/h_2 \cdots \hat{h}_i \cdots h_p]$ is generated by $\delta$ (using condition $B(h_2 \cdots h_p)$). Thus $N = (D\delta)[1/h_1]/D\delta$ is isomorphic to the module of local algebraic cohomology with support in the origin; in particular, any nonzero section generates $N$. We deduce easily that $(D\delta)[1/h_1]$ is generated by $\delta \otimes 1/h_1$. From Proposition 2.5, the condition $B(\delta, h_1)$ is verified.

The last point is a direct consequence of the following fact, proved by A. Leykin [31], Remark 5.2: if the condition $B(h_{i_1} \cdots h_{i_{k-1}})$ is verified for $1 \leq i_1 < \cdots < i_{k-1} \leq k$ with $k \geq n + 1$, then so is $B(h_1 \cdots h_k)$. □

Example 2.9 Let $n = 3$, $p \geq 3$ and $h_i = a_{i,1}x_1^2 + a_{i,2}x_2^3 + a_{i,3}x_3^4$ where the vector $a_i = (a_{i,1}, a_{i,2}, a_{i,3})$ belongs to $\mathbb{C}^3$ and the rank of $(a_{i,1}, a_{i,2}, a_{i,3})$ is maximal for $1 \leq i_1 < i_2 < i_3 \leq p$. Thus the polynomial $h = h_1 \cdots h_p$ defines a generic arrangement of hypersurfaces with an isolated singularity. By using the closed formulas for $b(h_i^s, s)$ and $b((1/h_i)h_j^s, s)$, $1 \leq i \neq j \leq p$, (see [32], [23]), it is easy to check that the conditions $B(h_i)$ and $B(1/h_i, h_j)$ are verified; thus so is $B(h)$.

3 The condition A(1/h) for a generic arrangement of hypersurfaces with an isolated singularity

In this part, we characterize the condition A(1/h) when $h \in \mathcal{O}$ defines a generic arrangement of hypersurfaces with an isolated singularity. Then we study this condition for a particular family of free germs (3.3).

3.1 A convenient annihilator

This paragraph is devoted to the determination of an annihilator which will allow us to characterize A(1/h).

Notation 3.1 Let $h = (h_1, \ldots, h_r) : \mathbb{C}^n \rightarrow \mathbb{C}^r$, $1 \leq r < n$, be an analytic morphism. For any $K = (k_1, \ldots, k_{r+1}) \in \mathbb{N}^{r+1}$ where $1 \leq k_1, \ldots, k_{r+1} \leq n$ and $k_i \neq k_j$ for $i \neq j$, let $\Delta_K^h \in D$ denote the vector field:

$$\sum_{i=1}^{r+1} (-1)^i m_{K(i)}(h) \partial_{k_i} = \sum_{i=1}^{r+1} (-1)^i \partial_{k_i} m_{K(i)}(h)$$
where $K(i) = (k_1, \ldots, k_i, \ldots, k_{r+1}) \in \mathbb{N}^r$ and $m_{K(i)}(h)$ is the determinant of the $r \times r$ matrix obtained from the Jacobian matrix of $h$ by deleting the $k$-th columns with $k \notin \{k_1, \ldots, k_i, \ldots, k_{r+1}\}$.

**Proposition 3.2** Assume that $n \geq 3$. Let $h = \prod_{i=1}^{p} h_i \in \mathcal{O}$, $p \geq 2$, define a generic arrangement of hypersurfaces with an isolated singularity, and let $\tilde{h}$ be the product $\prod_{i=2}^{p} h_i$. Then the ideal $\text{Ann}_D(1/\tilde{h})h_1^r$ is generated by the operators:

$$\Delta_{h_1, \ldots, h_i} \prod_{i \neq i_1, \ldots, i_r} h_i$$

with $1 \leq r \leq \min(n-1, p)$ and $1 = i_1 < \cdots < i_r \leq p$.

**Proof.** Let $I \subset D$ be the left ideal generated by the given operators, and let $\mathcal{I} \subset \mathcal{O}[\xi_1, \ldots, \xi_n]$ denote the ideal generated by their principal symbols. We will just prove that $\text{Ann}_D(1/\tilde{h})h_1^r \subset I$, since the reverse inclusion is obvious. Let us study $\text{char}_D D(1/\tilde{h})h_1^r \subset T^*\mathbb{C}^n$ the characteristic variety of $D(1/\tilde{h})h_1^r$.

Given an analytic subspace $X \subset \mathbb{C}^n$, we denote by $W_{h_1} X$ the closure in $T^*\mathbb{C}^n$ of the set $\{(x, \xi + \lambda h_1(x)) : \lambda \in \mathbb{C}, (x, \xi) \in T^*_X \mathbb{C}^n\}$.

**Assertion 1.** The characteristic variety of $D(1/\tilde{h})h_1^r$ is the union of the subspaces $W_{h_1}$ and $W_{h_1} \cap T^*_X \mathbb{C}^n$, $2 \leq i_1 < \cdots < i_r \leq p$, $1 \leq r \leq \min(n-1, p)$, where $X_{i_1, \ldots, i_r} \subset \mathbb{C}^n$ is the complete intersection defined by $h_1, \ldots, h_{i_r}$.

**Proof.** Under our assumption, $(\tilde{h}^{-1}(0), x)$ is a germ of a normal crossing hypersurface for any $x \in \tilde{h}^{-1}(0)/\{0\}$ close enough to the origin. In particular, $D1/\tilde{h}$ coincides with $D[1/h_{i_1} \cdots h_{i_r}]$ on a neighborhood of such a point, where $\{i_1, \ldots, i_r\} = \{i \mid h_i(x) = 0, 2 \leq i \leq p\}$. Hence, the components of the characteristic variety of $D1/\tilde{h}$ which are not supported by $h_1 = 0$ are $T^*_X \mathbb{C}^n$ and the conormal spaces $T^*_X \mathbb{C}^n$, with $2 \leq i_1 < \cdots < i_r \leq p$ and $1 \leq r \leq \min(n-1, p)$. The assertion follows from a result of V. Ginzburg ([13] Proposition 2.14.4).

We recall that the relative conormal space$^2$ $W_{h_1} \subset T^*\mathbb{C}^n$ is defined by the polynomials $\sigma(\Delta_{k_1, k_2} h_1, h_2) = h_2(x) \xi_{k_2} - h_1(x) \xi_{k_1}$, $1 \leq k_1 < k_2 \leq n$ (see [32] for example). One can also determine explicitly the defining ideal of the spaces $W_{h_1} \cap T^*_X \mathbb{C}^n$.

**Assertion 2 ([32]).** The conormal space $W_{h_1} \cap T^*_X \mathbb{C}^n$ is defined by $h_{i_1}, \ldots, h_{i_r}$ and by the principal symbol of the vector fields $\Delta_{K} h_{i_1, \ldots, i_r}$ (when $r < n-1$), where $K = (k_1, \ldots, k_{r+2}) \in \mathbb{N}^r+2$ with $1 \leq k_1 < \cdots < k_{r+2} \leq n$.

Now we can determine the equations of $\text{char}_D D(1/\tilde{h})h_1^r$. 

---

$^2$See [33]
Assertion 3. The defining ideal of $\mathcal{D}(1/\tilde{h})h^s_1$ is included in $\mathcal{I}$.

Proof. Let $A \in \mathcal{O}[\xi] = \mathcal{O}[\xi_1, \ldots, \xi_n]$ be a polynomial which is zero on the characteristic variety of $\mathcal{D}(1/\tilde{h})h^s_1$. We will prove the result when $p \geq n$ - the case $p \leq n - 1$ is analogous.

Using the inclusion $W_{h_1} X_{i_1, \ldots, i_{n-1}} \subset \text{char} \mathcal{D}(1/\tilde{h})h^s_1$ and Assertion 2, we have: $A \in (h_{i_1}, \ldots, h_{i_{n-1}})\mathcal{O}[\xi]$ for $2 \leq i_1 < \cdots < i_{n-1} \leq p$. By an easy induction on $p \geq n$, one can check that:

$$\bigcap_{2 \leq i_1 < \cdots < i_{n-1} \leq p} (h_{i_1}, \ldots, h_{i_{n-1}})\mathcal{O} = \sum_{2 \leq i_1 < \cdots < i_{n-2} \leq p} \prod_{i \neq 1, i_1, \ldots, i_{n-2}} h_i \mathcal{O}$$

using that every sequence $(h_{i_1}, \ldots, h_{i_n})$ is regular. Thus $A$ may be written as a sum $\sum_{2 \leq i_1 < \cdots < i_{n-2} \leq p} A^{(0)}_{i_1, \ldots, i_{n-2}} (\prod_{i \neq 1, i_1, \ldots, i_{n-2}} h_i)$ for some $A^{(0)}_{i_1, \ldots, i_{n-2}} \in \mathcal{O}[\xi]$.

Now let us fix $i_1 < \cdots < i_{n-2}$ a family of index as above. From the inclusion $W_{h_1} X_{i_1, \ldots, i_{n-2}} \subset \text{char} \mathcal{D}(1/\tilde{h})h^s_1$ and Assertion 2, $A$ belongs to the ideal $\mathcal{I}_{1, i_1, \ldots, i_{n-2}} = (h_{i_1}, \ldots, h_{i_{n-2}})\mathcal{O}[\xi] + \sum_K \sigma(\Delta_K^{h_i}) \mathcal{O}[\xi]$. On the other hand, let us remark that $h_i$ is $\mathcal{O}[\xi]/\mathcal{I}_{1, i_1, \ldots, i_{n-2}}$-regular for $i \neq 1, i_1, \ldots, i_{n-2}$ [by the principal ideal theorem, using that $\mathcal{I}_{1, i_1, \ldots, i_{n-2}}$ defines the irreducible space $W_{h_1} X_{i_1, \ldots, i_{n-2}}$ of pure dimension $n + 1$]. Thus we have $A^{(0)}_{i_1, \ldots, i_{n-2}} \in \mathcal{I}_{1, i_1, \ldots, i_{n-2}}$, and $A$ may be written: $A = U + \sum_{2 \leq i_1 < \cdots < i_{n-3} \leq p} A^{(1)}_{i_1, \ldots, i_{n-3}} (\prod_{i \neq 1, i_1, \ldots, i_{n-3}} h_i)$ where $A^{(1)}_{i_1, \ldots, i_{n-3}} \in \mathcal{O}[\xi]$ and $U \in \mathcal{I}$. Up to a division by $\mathcal{I}$, we can assume that $U = 0$. After iterating this process with $W_{h_1} X_{i_1, \ldots, i_r}$, $1 \leq r \leq n - 2$, we deduce that $A - A^{(n-2)}h_i$ belongs to $\mathcal{I}$. Hence, using that $W_{h_1} \subset \text{char} \mathcal{D}(1/\tilde{h})h^s_1$, we have: $A^{(n-2)} \in \sum_{1 \leq k_1 < k_2 \leq n} \sigma(\Delta_K^{h_{1}}) \mathcal{O}[\xi]$. In particular, $A^{(n-2)}h_i$ belongs to $\mathcal{I}$, and we conclude that $A \in \mathcal{I}$. □

Now let us prove the proposition. Let $P \in \text{Ann}_{\mathcal{D}} (1/\tilde{h})h^s_1$ be a nonzero operator of order $d$. In particular, $\sigma(P)$ is zero on $\text{char} \mathcal{D}(1/\tilde{h})h^s_1$, and by Assertion 3: $\sigma(P) \in \mathcal{I}$. In other words, there exists $Q \in I$ such that $\sigma(Q) = \sigma(P)$. Thus, the operator $P - Q \in \text{Ann}_{\mathcal{D}} (1/\tilde{h})h^s_1 \cap F_{d-1} \mathcal{D}$ belongs to $\mathcal{I}$, and so does $P$ (by induction on the order of operators). □

Remark 3.3 We are not able to determine $\text{Ann}_{\mathcal{D}} h^s$ when $h$ defines a generic arrangement of hypersurfaces with an isolated singularity. In particular, we do not know if the condition $A(h)$ (or $W(h)$) is - or not - verified (see [19]).

Given a germ $h \in \mathcal{O}$ such that $h(0) = 0$, let us denote by $\text{Der}(- \log h)$ the coherent $\mathcal{O}$-module of logarithmic derivations relative to $h$, that is, vector fields which preserve $h\mathcal{O}$ (see [19]).

Corollary 3.4 Let $h = \prod_{i=1}^{n} h_i \in \mathcal{O}$, $p \geq 2$, define a generic arrangement of hypersurfaces with an isolated singularity. Assume that $n \geq 3$ and that $h$...
is a weighted homogeneous polynomial. Then $\text{Der}(- \log h)$ is generated by the Euler vector field $\chi$ such that $\chi(h) = h$ and the vector fields

$$
\left[ \prod_{i \neq i_1, \ldots, i_r} h_i \right] \Delta^{h_{11}, \ldots, h_{ir}}_K
$$

where $1 \leq r \leq \min(n - 1, p)$ and $1 = i_1 < \cdots < i_r \leq p$.

**Proof.** We denote by $\tilde{h} \in \mathcal{O}$ the product $h_2 \cdots h_p$. Let $v$ be a logarithmic vector field; in particular, $v(h) = ah$. As $h = h_1h$, it is easy to check that $v(h_1) = a_1h_1$ and $v(\tilde{h}) = \tilde{a}h_1$ for $a_1, \tilde{a} \in \mathcal{O}$ such that $a_1 + \tilde{a} = a$. In particular, $v \cdot (1/\tilde{h})h_1^s = (a_1s - \tilde{a})(1/\tilde{h})h_1^s$. Thus $v + \tilde{a} - a_1 \chi$ belongs to $\text{Ann}_D (1/\tilde{h})h_1^s$, and by using the proof of the previous result, we have:

$$
v = -\tilde{a} + a_1 \chi + \sum_{r=1}^{\min(n-1,p)} \sum_{1 \leq i_1 < \cdots < i_r \leq p} \lambda_{i_1, \ldots, i_r} \Delta_{K}^{i_1, \ldots, i_r} \prod_{i \neq i_1, \ldots, i_r} h_i
$$

where $\lambda_{i_1, \ldots, i_r} \in \mathcal{O}$ for $1 \leq i_1 < \cdots < i_r \leq p$. As $v$ is a vector field, we get $v = a_1 \chi + \sum_r \sum_{1 \leq i_1 < \cdots < i_r \leq p} \lambda_{i_1, \ldots, i_r} [\prod_{i \neq i_1, \ldots, i_r} h_i] \Delta_{K}^{i_1, \ldots, i_r}$ and the assertion follows. $\square$

### 3.2 The expected characterization

The proof of Theorem 1.3 is an easy consequence of the following result

**Proposition 3.5** Let $h = \prod_{i=1}^p h_i \in \mathcal{O}$, $p \geq 2$, define a generic arrangement of hypersurfaces with an isolated singularity. Assume that $n \geq 3$ and that the origin is a critical point of $h_1$. Let $\hat{h}$ denote the product $\prod_{i=2}^p h_i$. Then the ideal $\text{Ann}_D 1/h$ is generated by operators of order one if and only if the following conditions are verified:

1. the germ is weighted homogeneous;
2. $-1$ is the smallest integral root of the Bernstein polynomial $b((1/\hat{h})h_1^s, s)$.

**Proof.** We can assume that $h$ does not define a normal crossing divisor. Indeed, the conditions $A(1/h)$, 1 and 2 are obviously verified for a normal crossing divisor. In particular, the constant term with the coefficient on the right side of any operator in $\text{Ann}_D (1/\hat{h})h_1^s$ is not a unit (see Proposition 3.2).

Firstly, we prove that conditions 1 & 2 imply $\text{A}(1/h)$. By an Euclidean division, we have a decomposition

$$
\text{Ann}_D[1/h] h_1^s = D[s](-\tilde{q} - v) + D[s]\text{Ann}_D[1/h] h_1^s
$$
where \( v \) denotes the Euler vector field such that \( v(h_1) = h_1 \) and \( v(\tilde{h}) = \tilde{q} \tilde{h} \) with \( \tilde{q} \in Q^{++} \). Moreover, with the condition 2, the ideal \( \text{Ann}_D 1/(\tilde{h} h_1) \) is obtained by fixing \( s = -1 \) in a system of generators of \( \text{Ann}_{D[s]}(1/\tilde{h}) h_1^s \) (see [20] Proposition 3.1). From Proposition 3.2 the condition \( A(1/\tilde{h}) \) is therefore verified.

Now, we prove the reverse. Let us assume that \( \text{Ann}_D 1/\tilde{h} \) is generated by the operators \( Q_1, \ldots, Q_w \in F_1 D \). From Proposition 1.3 in [28], \( B(h) \) is verified, and so\(^3\) is condition 2 by Proposition 1. Hence, we just have to check that \( h \) is necessarily weighted homogeneous. Let \( q_i \) be the germ \( Q_i(1) \in \mathcal{O} \) and \( Q_i' \) the vector field \( Q_i - q_i \). In particular, we have \( Q_i'(h) = q_i h \) for \( 1 \leq i \leq w \).

As \( h = h_1 \tilde{h} \), it is easy to deduce that \( Q_i'(\tilde{h}) = \tilde{q}_i \tilde{h} \) and \( Q_i'(h_1) = q_{i,1} h_1 \) where \( \tilde{q}_i, q_{i,1} \in \mathcal{O} \) verify

\[
\tilde{q}_i + q_{i,1} = q_i, \quad 1 \leq i \leq w.
\]

On the other hand, we have the following fact:

**Assertion 1.** There exists a differential operator \( R \) in \( \text{Ann}_D(1/\tilde{h}) h_1^s \) such that \( R = 1 + \sum_{i=1}^w A_i q_{i,1} \) with \( A_i \in D \).

**Proof.** The proof is analogous to the one of [26] Lemme 3.3. From [14] p 351 or [21], there exists a ‘good’ operator \( R_0(s) \) of degree \( N \geq 1 \) in \( \text{Ann}_{D[s]}(1/\tilde{h}) h_1^s \), that is \( R_0(s) = s^N + \sum_{k=0}^{N-1} s^k P_k \) with \( P_k \in F_{N-k} D \), \( 0 \leq k \leq N - 1 \). By Euclidean division, we have \( R_0(s) = (s+1)S(s) + R_0(-1) \) where \( S(s) \) is monic in \( s \) of degree \( N-1 \) and \( R_0(-1) \in \text{Ann}_D 1/\tilde{h} \). Thus, there exists \( A_1, \ldots, A_w \in D \) such that \( R_0(-1) = \sum_{i=1}^w A_i Q_i \). From the relations above, we get

\[
(s + 1) S(s)\frac{1}{\tilde{h}} h_1^s + (s + 1) \sum_{i=1}^w A_i q_{i,1} \frac{1}{\tilde{h}} h_1^s = 0.
\]

Hence \( R_1(s) = S(s) + \sum_{i=1}^w A_i q_{i,1} \) belongs to \( \text{Ann}_{D[s]}(1/\tilde{h}) h_1^s \). By iteration, we can assume that \( S(s) = 1 \). \( \square \)

In particular, at least one of the \( q_{i,1} \) is a unit (see the very beginning of the proof.)

**Assertion 2.** If \( q_{i,1} \) is a unit, then so is \( q_i \).

**Proof.** As the assertion is clear if \( \tilde{q}_i \) is not a unit, we can assume that \( \tilde{q}_i \) is a unit. Let \( \chi_i \) denote the vector field \( q_{i,1}^{-1} Q_i' \) in particular \( \chi_i(h_1) = h_1 \). As \( h_1 \) defines an isolated singularity, a famous result due to K. Saito [14] asserts that, up to a change of coordinates, \( \chi_i \) is an Euler vector field \( \sum_{k=1}^n \alpha_k x_k \partial_k \) with \( \alpha_k \in Q^{++} \). Hence, the relation \( \chi_i(\tilde{h}) = q_{i,1}^{-1} \tilde{q}_i \tilde{h} \) implies that the constant \( (q_{i,1}^{-1} \tilde{q}_i)(0) \) belongs

\(^3\)In fact, the same proof shows directly that condition \( A(1/\tilde{h}) \) implies \( B(1/\tilde{h}, h_1) \).
to $\mathbb{Q}^+$ [consider the initial part of $q_i^{-1}\tilde{q}_i h$ relative to $\alpha_1, \ldots, \alpha_n$]. In particular, $q_i^{-1}\tilde{q}_i + 1$ is a unit, and so is $q_i = \tilde{q}_i + q_{i,1}$.

We recall that a formal power series $g \in \mathbb{C}[[x_1, \ldots, x_n]]$ is weakly weighted homogeneous of type $(\beta_0, \beta_1, \ldots, \beta_n) \in \mathbb{C}^{n+1}$ if for all monomial $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with a nonzero coefficient in the power expansion of $g$, we have $\beta_1 \gamma_1 + \cdots + \beta_n \gamma_n = \beta_0$.

Let us pursue the proof. We have proved that there exists an Euler vector field $\chi_i$ such that $q_i^{-1}\chi_i(h) = h$ (in particular, $q_i(0) > 0$). From [19], Corollary 3.3, there exists a formal change of coordinates $\phi$ such that $h \circ \phi$ is weakly weighted homogeneous of type $(1, \alpha_1 q_i^{-1}(0), \ldots, \alpha_n q_i^{-1}(0))$. As the $\alpha_k q_i^{-1}(0)$ are strictly positive, $h \circ \phi$ is in fact weighted homogeneous, and according to a theorem of Artin [1], a convergent change of coordinates exists. This completes the proof.

Proof of Theorem 1.3. The case $n = 2$ is done in [20], Theorem 1.2. We just have to check that the condition 2 in the previous statement may be replaced by $B(h)$. Indeed, condition A $(1/h)$ always implies $B(h)$ (28 Proposition 1.3), and on the other hand, $B(h)$ is stronger than $B(1/\tilde{h}, h_1)$ (Proposition 11).

Of course, we can use [22] to test if condition $B(h)$ is verified. In the particular case $p = 2$ and $h$ weighted homogeneous, we obtain the following characterization:

**Corollary 3.6** Let $h_1, h_2 \in \mathbb{C}[x_1, \ldots, x_n]$ be two weighted homogeneous polynomial of degree $d_1, d_2$ for a system $\alpha \in (\mathbb{Q}^+)^n$, defining hypersurfaces with an isolated singularity at the origin and without common components. Let $K \subset \mathcal{O}$ be the ideal generated by the maximal minors of the Jacobian matrix of $(h_1, h_2)$. Then the annihilator of $1/h_1 h_2$ is generated by operators of order 1 if and only if for $j = 1$ or 2, there is no weighted homogeneous element in $\mathcal{O}/h_j \mathcal{O} + K$ whose weight belongs to the set $\{d_j \times k - \sum_{i=1}^n \alpha_i ; k \in \mathbb{N} & k \geq 2\}$.

This relies on the existence of closed formulas for $b((1/\tilde{h})h_1^s, s)$ under these assumptions [20].

### 3.3 About a family of free germs

In this part, we prove Proposition 1.4. As the two parts are quite distinct, we will prove them successively.

**Lemma 3.7** Let $g \in \mathbb{C}\{x_1, x_2\}$ be a nonzero reduced germ of plane curve such that $g(0) = 0$. Then $-1$ is the only integral root of the Bernstein polynomial of $(x_1 - x_2 x_3)g(x_1, x_2)$.
Let us write $g$.

**Proof.** As $g$ is a reduced germ of plane curve, $B(g)$ is verified \[30\], \[21\]. Thus, by using Proposition \[\text{[1]}\] the three conditions $B((x_1 - x_2 x_3)g(x_1, x_2))$, $B(1/x_1 - x_2 x_3, g)$ and $B(1/x_1 - x_2 x_3, g)$ are equivalent. Let us prove the last one. From Corollary \[\text{[2.4]}\], we have $b((1/x_1 - x_2 x_3)g^*, s) = b((g(x_2 x_3, x_2))^*, s)$. Let us write $g(x_2 x_3, x_2) = x_2^\ell \tilde{g}(x_2, x_3)$, where $\tilde{g} \in \mathbb{C}\{x_2, x_3\} - x_2 \mathbb{C}\{x_2, x_3\}$ is reduced and $\ell \in \mathbb{N}^*$. If $\tilde{g}$ is a unit, then $B(g(x_2 x_3, x_2))$ is verified and so is $B((x_1 - x_2 x_3)g(x_1, x_2))$. Now we assume that $\tilde{g}$ is not a unit. As it is reduced, $B(\tilde{g})$ is verified and $B(\tilde{g} x_2^\ell)$ is equivalent to $B(1/\tilde{g}, x_2^\ell)$. Using Lemma \[\text{[2.2]}\] it is easy to check that $B(1/\tilde{g}, x_2)$ implies $B(1/\tilde{g}, x_2^\ell)$. Thus we just have to prove $B(1/\tilde{g}, x_2)$. As condition $B(\tilde{g})$ is verified, the conditions $B(1/\tilde{g}, x_2)$, $B(\tilde{g} x_2)$ and $B(1/x_2, \tilde{g})$ are equivalent (Proposition \[\text{[1]}\]). Both of them are verified since $b((1/x_2)\tilde{g}^*, s) = b((\tilde{g}(0, x_3))^*, s)$ from Corollary \[\text{[2.4]}\] where $\tilde{g}(0, x_3) = ux_3^N$ with $u \in \mathbb{C}\{x_3\}$ is a unit. This completes the proof. $\square$

We recall that a nonzero germ $h \in \mathcal{O}$ defines a germ of free divisor if the module of logarithmic derivations relative to $h$ is $\mathcal{O}$-free \[\text{[20]}\]. Moreover, such a germ defines a Koszul-free divisor if there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $\text{Der}(-\log h)$ such that the sequence of principal symbols $(\sigma(\delta_1), \ldots, \sigma(\delta_n))$ is $\text{gr}F\mathcal{D}$-regular.

**Lemma 3.8** Let $g \in \mathbb{C}[x_1, x_2]$ be a weighted homogeneous and reduced polynomial whose multiplicity is greater or equal to 3. Let $h \in \mathbb{C}[x_1, x_2, x_3]$ denote the polynomial $(x_1 - x_2 x_3)g(x_1, x_2)$.

(i) The polynomial $h$ defines a free divisor and verifies the condition $\text{H}(h)$.

(ii) The polynomial $h$ defines a Koszul-free divisor if and only if the weighted homogeneous polynomial $g$ is not homogeneous.

**Proof.** (i) It is enough to remark that the following vector fields verify Saito’s criterion \[\text{[20]}\] for $h$:

\[
\begin{align*}
\delta_1 &= \alpha_1 x_1 \partial_1 + \alpha_2 x_2 \partial_2 + (\alpha_1 - \alpha_2) x_3 \partial_3 \\
\delta_2 &= g'_{x_2} \partial_1 - g'_{x_1} \partial_2 + (x_3 u - v) \partial_3 \\
\delta_3 &= (x_1 - x_2 x_3) \partial_3
\end{align*}
\]

where $(\alpha_1, \alpha_2) \in (\mathbb{Q}^+)^2$ is a system of weights for $g$, and $u \in \mathbb{C}[x_1, x_2, x_3]$, $v \in \mathbb{C}[x_2, x_3]$ are the polynomials of degree in $x_3$ less or equal to 1 uniquely defined by the relation

\[x_3 g'_{x_1}(x_1, x_2) + g'_{x_2}(x_1, x_2) = u(x_1, x_2, x_3)x_1 - v(x_2, x_3)x_2\]

(we use that $g'_{x_1}, g'_{x_2} \in (x_1, x_2)\mathbb{C}[x_1, x_2]$ under our assumptions.)

(ii) As the sequence $(\sigma(\delta_1), \sigma(\delta_2), \xi_3)$ is regular, the germ $h$ is Koszul-free if and only if the sequence $(\sigma(\delta_1), \sigma(\delta_2), x_1 - x_2 x_3)$ is $\mathcal{O}[\xi]$-regular. By division
by \(x_1 - x_2x_3\), this condition may be rewritten: the polynomials

\[
\begin{align*}
\gamma_1 &= \alpha_1 x_2x_3 \xi_1 + \alpha_2 x_2 \xi_2 + (\alpha_1 - \alpha_2)x_3 \xi_3 \\
\gamma_2 &= g'_{x_2}(x_2x_3, x_2) \xi_1 - g'_{x_1}(x_2x_3, x_2) \xi_2 + (x_3 u(x_2x_3, x_2) - v(x_2, x_3)) \xi_3
\end{align*}
\]

have no common factor. Let us notice that \(x_2\) is the only (irreducible) common factor of \(g'_{x_2}(x_2x_3, x_2)\) and \(g'_{x_2}(x_2x_3, x_2)\) [since \(g \in \mathbb{C}[x_1, x_2]\) defines an isolated singularity.] Thus, when \(\gamma_1\) and \(\gamma_2\) have a common factor, this factor is \(x_2\) (up to a multiplicative constant). As \(g\) belongs in \((x_1, x_2)^3\mathbb{C}[x_1, x_2]\), we have \(g'_{x_1}, g'_{x_2} \in (x_1, x_2)^2\mathbb{C}[x_1, x_2]\); thus \(u, v \in (x_1, x_2)\mathbb{C}[x_1, x_2, x_3]\). In particular, \(x_2\) is a factor of \(\gamma_2\), and \(\gamma_1, \gamma_2\) have no common factor if and only if \(\alpha_1 \neq \alpha_2\). This completes the proof. \(\square\)

Of course, for \(g = x_1x_2(x_1 + x_2)\), \(h\) is the example of F.J. Calderón-Moreno in \(\text{[4]}\) and it is not Koszul-free.

Proof of Proposition \(\text{[4]}\) part (i). Without loss of generality, we will assume that \(\delta_1(h) = h\). Let us take \(\delta_2' = \delta_2 - u \cdot \delta_1\) and \(\delta_3' = \delta_3 + x_2\delta_1\); in particular, \(\{\delta_1, \delta_2', \delta_3'\}\) is a basis of \(\text{Der}(\log h)\) such that \(\delta_2'(h) = \delta_3'(h) = 0\).

From the characterization of condition \(A(1/h)\) for Koszul-free germs (see Corollary 1.8), it is enough to check that condition \(A(h)\) fails, that is, the sequence \((x_1 - x_2x_3, \sigma(\delta_2'), \sigma(\delta_3'))\) is not regular. As \(g\) belongs to \((x_1, x_2)^3\mathbb{C}[x_1, x_2]\), we have \(\sigma(\delta_2'), \sigma(\delta_3') \in (x_1, x_2)\mathcal{O}[\xi]\). By division by \(x_1 - x_2x_3\), we deduce that the sequence is not regular. \(\square\)

Notation 3.9 Given a homogeneous polynomial \(g \in \mathbb{C}[x_1, x_2] - \mathbb{C}\) of degree \(p \geq 1\), we denote by \(g_1, g_2 \in \mathbb{C}[x_1, x_2, x_3]\) the quotient of the division of \(g'_{x_1}, g'_{x_2}\) by \(x_1 - x_2x_3\). In particular:

\[
g'_{x_i} = (x_1 - x_2x_3)\tilde{g}_i + x_2^{p-1} g'_{x_2}(x_3, 1), \quad i \in \{1, 2\}.
\]

Lemma 3.10 Let \(g \in \mathbb{C}[x_1, x_2]\) be a homogeneous reduced polynomial of degree \(p \geq 3\). Then the characteristic variety of \(\mathcal{D}(1/x_1 - x_2x_3)g^s\) is defined by the following polynomials: \((x_1 - x_2x_3)\xi_3, g'_{x_2}(x_3, 1)\xi_1 - g'_{x_1}(x_3, 1)\xi_2 + px_2^{p-2}g(x_3, 1)\xi_3,\text{ and }\]

\[
[x_2g'_{x_2}(x_3, 1)\xi_1 - x_2g'_{x_1}(x_3, 1)\xi_2 + pg(x_3, 1)\xi_3] \xi_3\]

Proof. Using Proposition 2.14.4, the characteristic variety of the \(\mathcal{D}\)-module \(\mathcal{D}(1/x_1 - x_2x_3)g^s\) is the union of the conormal spaces \(W_g\) and \(W_{g|x_1=x_2x_3}\). It is easy to check that they are defined by the ideals \(I_1 = \langle \xi_3, g'_{x_2}(x_3, 1)\xi_1 - g'_{x_1}(x_3, 1)\xi_2 \rangle \mathcal{O}[\xi]\) and \(I_2 = \langle x_1 - x_2x_3, x_2g'_{x_2}(x_3, 1)\xi_1 - x_2g'_{x_1}(x_3, 1)\xi_2 + pg(x_3, 1)\xi_3 \rangle \mathcal{O}[\xi]\) respectively. Clearly, the ideal \(I\) generated by the given polynomials is contained in \(I_1 \cap I_2\). Thus, we just have to prove the reverse relation.

Let \(A, B, C, D \in \mathcal{O}[\xi]\) be such that

\[
A(x_1 - x_2x_3) + B(x_2g'_{x_2}(x_3, 1)\xi_1 - x_2g'_{x_1}(x_3, 1)\xi_2 + pg(x_3, 1)\xi_3) = C\xi_3 + D(g'_{x_2}(x_3, 1)\xi_1 - g'_{x_1}(x_3, 1)\xi_2).
\]
Proof of Proposition 1.4, part (ii). We will prove that $\text{Ann}(\alpha_{x}) - \text{we know that we have the decomposition Ann}(1 - \alpha_{x})$ such that:

\[ A - D(\tilde{g}_{2}\xi_{1} - \tilde{g}_{1}\xi_{2})(x_{1} - x_{2}x_{3}) + (pB\tilde{g}(x_{3}, 1) - C)\xi_{3} \]

\[ + (B - Dx_{2}^{2})x_{2}(g'_{x_{2}}(x_{3}, 1)\xi_{1} - g'_{x_{1}}(x_{3}, 1)\xi_{2}) = 0 \]

Since the sequence $(x_{1} - x_{2}x_{3}, \xi_{3}, x_{2}(g'_{x_{2}}(x_{3}, 1)\xi_{1} - g'_{x_{1}}(x_{3}, 1)\xi_{2}))$ is $\mathcal{O}[\xi]$-regular, there exist $U, V, W \in \mathcal{O}[\xi]$ such that

\[ \begin{cases} A - D(\tilde{g}_{2}\xi_{1} - \tilde{g}_{1}\xi_{2}) = U\xi_{3} + Wx_{2}(g'_{x_{2}}(x_{3}, 1)\xi_{1} - g'_{x_{1}}(x_{3}, 1)\xi_{2}) \\ B - Dx_{2}^{2} = -V\xi_{3} - W(x_{1} - x_{2}x_{3}) \end{cases} \]

Thus one can notice that the first part of the first identity belongs to $I$, that is, $I$ is the defining ideal of $W_{g} \cup W_{g|x_{1}=x_{2}x_{3}}$. □

Lemma 3.11 Let $g \in \mathcal{C}[x_{1}, x_{2}]$ be a homogeneous reduced polynomial of degree 3. Then the annihilator of $(1/x_{1} - x_{2}x_{3})g^{*}$ is generated by the following differential operators:

\[ (x_{1} - x_{2}x_{3})\partial_{1} - x_{2}, \quad g'_{x_{2}}\partial_{1} - g'_{x_{1}}\partial_{2} + 3x_{2}g(x_{3}, 1)\partial_{3} + x_{3}\tilde{g}_{1} + \tilde{g}_{2} \quad \text{and} \]

\[ [x_{2}g'_{x_{2}}(x_{3}, 1)\partial_{1} - x_{2}g'_{x_{1}}(x_{3}, 1)\partial_{2} + 3g(x_{3}, 1)\partial_{3} + \tilde{g}_{2}\partial_{1} - \tilde{g}_{1}\partial_{2} + 3g'_{x_{1}}(x_{3}, 1)\partial_{3} + u'_{1}] \]

where $u = x_{3}\tilde{g}_{1} + \tilde{g}_{2}$.

Proof. Let us denote by $I \subset \mathcal{D}$ the ideal generated by the given operators $S_{1}, S_{2}, S_{3}$. It is not hard to check the inclusion $I \subset \text{Ann}_{\mathcal{D}}(1/x_{1} - x_{2}x_{3})g^{*}$. Let us prove that the reverse inclusion by induction on the order of operators.

Let $P \in \text{Ann}_{\mathcal{D}}(1/x_{1} - x_{2}x_{3})g^{*}$ be an operator of order $d$. As $d = 0$ implies $P = 0$, we can assume $d \geq 1$. Then $\sigma(P)$ is zero on the characteristic variety of $\mathcal{D}(1/x_{1} - x_{2}x_{3})g^{*}$. From the previous result, there exists $A_{1} \in \mathcal{O}[\xi]$ (resp. $A_{2}, A_{3}$) zero or homogeneous in $\xi$ of degree $d - 1$ (resp. $d - 1, d - 2$) such that $\sigma(P) = \sum_{i=1}^{3} A_{i}\sigma(S_{i})$. If $\hat{A}_{i} \in \mathcal{D}$, $1 \leq i \leq 3$, are such that $\sigma(A_{i}) = A_{i}$ for $1 \leq i \leq 3$, then $P - \sum_{i=1}^{3} \hat{A}_{i}S_{i}$ belongs to $\mathcal{D}d_{-1}$ and annihilates $(1/x_{1} - x_{2}x_{3})g^{*}$. By induction, it belongs to $I$ and so does $P$. □

Proof of Proposition 1.4 part (ii). We will prove that $\text{Ann}_{\mathcal{D}}1/h$ is generated by the operators $\tilde{\delta}_{1} = \delta_{1} + 4$, $\tilde{\delta}_{2} = \delta_{2} + u$, $\tilde{\delta}_{3} = \delta_{3} - x_{2}$ (with the notations introduced in the proof of Lemma 3.8 with $\alpha_{1} = \alpha_{2} = 1$). From Lemma 3.7 we know that $-1$ is the smallest integral root of $b((1/x_{1} - x_{2}x_{3})g^{*}, s)$. Thus we have the decomposition $\text{Ann}_{\mathcal{D}}1/h = \mathcal{D}\tilde{\delta}_{1} + \text{Ann}_{\mathcal{D}}(1/x_{1} - x_{2}x_{3})g^{*}$, and the assertion is a direct consequence of the previous result and of the following relation in $\mathcal{D}$:

\[ [g'_{x_{2}}(x_{3}, 1)x_{2}\partial_{1} - g'_{x_{1}}(x_{3}, 1)x_{2}\partial_{2} + 3g(x_{3}, 1)\partial_{3} + 3g'_{x_{1}}(x_{3}, 1)](\partial_{3}\tilde{\delta}_{1} - \partial_{1}\tilde{\delta}_{3}) \]

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\[ + [\partial_2 + x_3\partial_1](\partial_3\tilde{\delta}_2 + (\tilde{g}_2\partial_1 - \tilde{g}_1\partial_2)\tilde{\delta}_3) = -2S_3 + \partial_1\tilde{\delta}_2 - (\tilde{g}_2\partial_1 - \tilde{g}_1\partial_2 + u'_{x_1})\tilde{\delta}_1 \]

where \( S_3 \) is the operator of order 2 which appears in the given system of generators of \( \text{Ann}_D(1/x_1 - x_2x_3)g^s \). □

4 Some other conditions

In this part, \( h \in \mathcal{O} \) denotes a nonzero germ such that \( h(0) = 0 \).

4.1 The condition \( A(h) \) for Sebastiani-Thom germs

We recall that the condition \( A(h) \) on the ideal \( \text{Ann}_D h^s \) may be considered almost as a geometric condition. Indeed the following condition implies \( A(h) \):

\( W(h) \): The relative conormal space \( W_h \) is defined by linear equations in \( \xi \). Since \( W_h = \{(x, \lambda dh) | \lambda \in C\} \subset T^*\mathbb{C}^n \) is the characteristic variety of \( D h^s \) (16). For example, \( W(h) \) is true for hypersurfaces with an isolated singularity \( \mathbb{G} \) and for locally weighted homogeneous free divisors \( \mathbb{G} \). This condition also means that the kernel of the morphism of graded \( \mathcal{O} \)-algebras:

\[
\mathcal{O}[X_1, \ldots, X_n] \longrightarrow \mathcal{R}(\mathcal{J}_h)
\]

is generated by homogeneous elements of degree 1, where \( \mathcal{J}_h \) denotes the Jacobian ideal \( (h'_{x_1}, \ldots, h'_{x_n})\mathcal{O} \) and \( \mathcal{R}(\mathcal{J}_h) \) is the Rees algebra \( \bigoplus_{d \geq 0} \mathcal{J}_h^{d+1} \). Following a terminology due to W.V. Vasconcelos, one says that \( \mathcal{J}_h \) is of linear type (see [6] for more details). Finally, let us give a third condition trapped between \( A(h) \) and \( W(h) \):

\( G(h) \): The graded ideal \( \text{gr}^F \text{Ann}_D h^s \) is generated by homogeneous polynomials in \( \xi \) of degree 1.

Remark 4.1 (i) We do not know if the conditions \( A(h), G(h) \) and \( W(h) \) are - or not - equivalent.

(ii) These conditions are not stable by multiplication by a unit.

It seems uneasy to find sufficient conditions on \( h \) for \( A(h) \) or \( W(h) \). Thus, it is natural to study if the class of germs \( h \) which verify \( A(h) \) or \( W(h) \) is - or not - stable by Thom-Sebastiani sums. Here we give a positive answer in a particular case.

Proposition 4.2 Let \( g \in \mathcal{O} \) be a nonzero germ such that \( g(0) = 0 \) and which verifies the condition \( W(g) \). Let \( f \in \mathcal{C}\{z_1, \ldots, z_p\} \) be a nonzero germ which defines an isolated singularity at the origin. Then \( h = g + f \) verifies the condition \( W(h) \).
 Proposition 4.3 Let $g \in \mathcal{O}$ be a nonzero germ such that $g(0) = 0$, and $\Upsilon_1, \ldots, \Upsilon_w \in \mathcal{O}[\xi]$ be homogeneous polynomials which generate the defining ideal of $W_g$.

Let $f \in \mathbb{C}\{z_1, \ldots, z_p\}$ be a nonzero germ which defines an isolated singularity and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_p$ denote the conormal coordinates on $T^*\mathbb{C}^n \times \mathbb{C}^p$. Then the relative conormal space $W_{g+f} \subset T^*\mathbb{C}^n \times \mathbb{C}^p$ is defined by the polynomials $f'_i \eta_j - f'_j \eta_i$, $1 \leq i < j \leq p$, $g'_{x_k} \eta_i - f'_x \xi_k$, $1 \leq i \leq p$, $1 \leq k \leq n$, and $\Upsilon_1, \ldots, \Upsilon_w$.

Proof. Let us denote by $E \subset \mathbb{C}\{z_1, \ldots, z_p\}$ a $\mathbb{C}$-vector space of finite dimension isomorphic to $\mathbb{C}\{z_1, \ldots, z_p\}/(f'_i, \ldots, f'_p)$ by projection, and by $\mathcal{C}\{x, z\}$ the ring $\mathbb{C}\{x, z, x_1, \ldots, z_p\}$. In particular, any germ $p \in \mathcal{C}\{x, z\}$ may be written in a unique way: $p = \tilde{p} + r$ where $\tilde{p} \in E \otimes \mathcal{O} \subset \mathcal{C}\{x, z\}$ and $r \in (f'_i, \ldots, f'_p)\mathcal{C}\{x, z\}$.

We denote by $I_{f+g} \subset \mathcal{C}\{x, z\}[\xi, \eta]$ the ideal generated by the given operators, and by $I_g \subset \mathcal{C}\{x, z\}[\xi, \eta]$ (resp. $I_f$) the ideal generated by $\Upsilon_1, \ldots, \Upsilon_w$ (resp. $f'_i \eta_j - f'_j \eta_i$, $1 \leq i < j \leq p$). Obviously, any element of $I_{g+f}$ is zero on $W_{g+f}$. Let us prove the reverse relation.

Let $P \in \mathcal{C}\{x, z\}[\xi, \eta]$ be a homogeneous polynomial of degree $N \in \mathbb{N}^*$ in $(\xi, \eta)$ which is zero on $W_{g+f}$.

Assertion 1. There exists $\tilde{P}(\xi, \eta) \in \mathcal{C}\{x, z\}[\xi, \eta]$ such that $P - \tilde{P}(\xi, \eta)$ belongs to $I_{g+f}$, and it is of the form:

$$\tilde{P}(\xi, \eta) = Q(\eta) + \sum_{|\gamma| \leq N-1} \tilde{P}_\gamma(\xi) \eta_1^{\gamma_1} \cdots \eta_p^{\gamma_p}$$

where $\gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{N}^p$, $\tilde{P}_\gamma(\xi) \in (E \otimes \mathcal{O})[\xi]$ are zero or homogeneous in $\xi$ of degree $N - |\gamma|$, $Q(\eta) \in \mathcal{C}\{x, z\}[\eta]$ is zero or homogeneous of degree $N$.

Proof. Let us write: $P = \sum_{|\beta + \gamma| = N} p_{\beta, \gamma} \eta^\gamma \xi^\beta$ with $p_{\beta, \gamma} \in \mathcal{O}$. For all $\beta \in \mathbb{N}^n$, $|\beta| = N$, the germ $p_{\beta, 0} \in \mathcal{O}$ may be written in a unique way $p_{\beta, 0} = \tilde{p}_{\beta, 0} + r_{\beta, 0}$ with $\tilde{p}_{\beta, 0} \in E \otimes \mathcal{O}$ and $r_{\beta, 0} = \sum_{i=1}^p r_{\beta, 0,i} f'_i$ for some $r_{\beta, 0,i} \in \mathcal{C}\{x, z\}$. As $|\beta| \geq 1$, there exists an index $k$ such that $\beta_k \neq 0$. Thus

$$r_{\beta, 0} \xi_1^{\beta_1} \cdots \xi_n^{\beta_n} - \sum_{i=1}^p r_{\beta, 0,i} g'_{x_k} \eta_i \xi_1^{\beta_1} \cdots \xi_k^{\beta_k-1} \cdots \xi_n^{\beta_n} \in I_{g+f}$$

and we fix $\tilde{P}_0(\xi) = \sum_{|\beta| = N} \tilde{p}_{\beta, 0} \xi^\beta$. By iterating this process for increasing $|\gamma|$, we get a decomposition $P = Q(\eta) + \sum_{|\gamma| \leq N-1} \tilde{P}_\gamma(\xi) \eta^\gamma + R$ where $R \in I_{g+f}$. □

Assertion 2. The polynomials $\tilde{P}_\gamma(\xi)$ belong to $I_g$. 19
Proof. We prove it by induction on $\gamma$, using the lexicographical order on $\mathbb{N}^p$. As $\tilde{P}(g_{x_1}, \ldots, g_{x_n}, f_{z_1}, \ldots, f_{z_p}) = 0$, we have $\tilde{P}_0(g_{x_1}, \ldots, g_{x_n}) \in (f_{z_1}, \ldots, f_{z_p})\mathcal{C}\{x, z\}$. Thus $\tilde{P}_0(\xi)$ belongs to $I_g$ (since $\tilde{P}_0(\xi) \in (E \otimes \mathcal{O})[\xi]$ and $g \in \mathcal{O}$). Now, let us assume that $\tilde{P}_\gamma(\xi) \in I_g$ for all $\gamma' < \gamma$. Then, we deduce that $\tilde{P}_\gamma(g_{x_1}, \ldots, g_{x_n}) = 0$ and $\tilde{P}_\gamma'(g_{x_1}, \ldots, g_{x_n}) = 0$ for $\gamma' < \gamma$, we have:

$$\tilde{P}_\gamma(g_{x_1}, \ldots, g_{x_n})f_{z_1}^r \cdots f_{z_p}^r \in (f_{z_1}^r, f_{z_2}^r, \ldots, f_{z_p}^r)\mathcal{C}\{x, z\} + Q(f_{z_1}^r, \ldots, f_{z_p}^r)\mathcal{C}\{x, z\} \subset (f_{z_1}^r, \ldots, f_{z_p}^r)\mathcal{C}\{x, z\}$$

since the degree of $Q(\eta)$ is strictly greater than $|\gamma|$. From this identity, we deduce that $\tilde{P}_\gamma(g_{x_1}, \ldots, g_{x_n}) \in (f_{z_1}^r, \ldots, f_{z_p}^r)\mathcal{C}\{x, z\}$ using that $(f_{z_1}^r, \ldots, f_{z_p}^r)$ is a $\mathcal{C}\{x, z\}$-regular sequence. Thus $\tilde{P}_\gamma(\xi)$ belongs to $I_g$ as above. $\Box$

In particular, the polynomial $P - Q(\eta)$ belongs to $I_{g+f}$. As $P$ is zero on $W_{g+f}$, we have $Q(f_{z_1}^r, \ldots, f_{z_p}^r) = 0$. Thus $Q(\eta)$ belongs to $I_f$ (since $(f_{z_1}^r, \ldots, f_{z_p}^r)$ is $\mathcal{C}\{x, z\}$-regular). We conclude that $P \in I_{g+f}$, and this completes the proof. $\Box$

Remark 4.4 Let us recall that the reduced Bernstein polynomial of the germ $h = g(x) + z^N$ has no integral root for $N$ ‘generic’ [21]. In particular, our result allows to construct some examples of weighted homogeneous polynomials $h$ which verify condition $A(1/h)$ [with the help of identity [11] of the Introduction].

4.2 The condition $A_{log}(1/h)$

Let us recall how the condition $A(1/h)$ appears in the study of the so-called logarithmic comparison theorem. If $D$ is a free divisor, F.J. Calderón-Moreno and L. Narváez-Macarro [3] have obtained a differential analogue of the condition $\text{LCT}(D)$; in particular, it implies that the natural $\mathcal{D}$-linear morphism $\varphi_D : \mathcal{D}_X \otimes \mathcal{V}_D^0 \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(\ast D)$ is an isomorphism. Here $\mathcal{O}_X(D)$ denotes the $\mathcal{O}_X$-module of meromorphic functions with at most a simple pole along $D$, and $\mathcal{V}_D^0 \subset \mathcal{D}_X$ is the sheaf of ring of logarithmic differential operators, that is, $P \in \mathcal{D}_X$ such that $P \cdot (h_D)^k \subset (h_D)^k \mathcal{O}$ for any $k \in \mathbb{N}$, where $h_D$ is a (local) defining equation of $D$. Locally, we have $\mathcal{O}_X(D) = \mathcal{V}_D^0 \cdot (1/h_D)$, thus $\varphi_D$ is given by

$$\mathcal{D}/\mathcal{D}\text{Ann}_{\mathcal{V}_D^0} 1/h_D \rightarrow \mathcal{O}[1/h_D]$$

$$P \mapsto P \cdot \frac{1}{h_D}$$
where \( \text{Ann}_{\mathcal{V}^D} 1/h_D \subset \mathcal{V}^D_0 \) is the ideal of logarithmic operators which annihilate \( 1/h_D \). From the structure theorem of logarithmic operators associated with a free divisor \([4]\), we have \( \mathcal{V}^D_0 = \mathcal{O}_X[\text{Der}(\log h_D)] \); hence the ideal \( \text{Ann}_{\mathcal{V}^D_0} 1/h_D \) is locally generated by \( v_i + a_i, 1 \leq i \leq n \), where \( \{v_1, \ldots, v_n\} \) is a basis of \( \text{Der}(\log h_D) \) and \( a_i \in \mathcal{O} \) is defined by \( v_i(h_D) = a_i h_D, 1 \leq i \leq n \). In particular, the injectivity of \( \varphi_D \) means that the condition \( A(1/h) \) is verified.

Let us notice that the following condition may also be considered:

\( A_{\log}(1/h) \): The ideal \( \text{Ann}_{\mathcal{D}} 1/h \) is generated by logarithmic operators.

In this paragraph, we compare these two conditions. Firstly, it is easy to see that the condition \( A(1/h) \) always implies \( A_{\log}(1/h) \). On the other hand, we do not know if these conditions are distinct or not. Meanwhile, we have the following result:

**Lemma 4.5** Let \( h \in \mathcal{O} \) be a nonzero germ such that \( h(0) = 0 \). Assume that one of the following conditions is verified:

1. the ring \( \mathcal{V}^D_0 \) coincides with \( \mathcal{O}[\text{Der}(\log h)] \), the \( \mathcal{O} \)-subalgebra of \( \mathcal{D} \) generated by the logarithmic derivations relative to \( h \).

2. the conditions \( A(h) \) and \( H(h) \) are verified.

Then the conditions \( A(1/h) \) and \( A_{\log}(1/h) \) are equivalent.

**Proof.** Assume that condition 1 is verified, and let \( P \in \mathcal{V}^D_0 \cap \text{Ann}_{\mathcal{D}} 1/h \) be a nonzero logarithmic operator annihilating \( 1/h \). By assumption, it may be written as a sum \( \sum_{|\gamma| \leq d} p_\gamma v_1^{\gamma_1} \cdots v_N^{\gamma_N} \) where \( p_\gamma \in \mathcal{O} \) and \( v_1, \ldots, v_N \) is a generating system of \( \text{Der}(\log h) \). As \( \text{Der}(\log h) \) is stable by Lie brackets, we have

\[
P = \sum_{|\gamma| \leq d} p_\gamma (v_1 + a_1)^{\gamma_1} \cdots (v_N + a_N)^{\gamma_N} + \sum_{|\gamma| < d} r_\gamma v_1^{\gamma_1} \cdots v_N^{\gamma_N} + R
\]

where \( r_\gamma \in \mathcal{O} \), and \( a_i \in \mathcal{O} \) is defined by \( v_i(h) = a_i h, 1 \leq i \leq N \); in particular, \( R \) belongs to \( \mathcal{V}^D_0 \cap \text{Ann}_{\mathcal{D}} 1/h \). By induction, we conclude that \( P \) belongs to the ideal \( \mathcal{D}(v_1 + a_1, \ldots, v_N + a_N) \); thus \( A_{\log}(1/h) \) implies the condition \( A(1/h) \).

Now we assume that the conditions \( A_{\log}(1/h) \), \( A(h) \) and \( H(h) \) are verified. From Proposition \([4,7]\), the condition \( B(h) \) is also verified. Thus so is \( A(1/h) \) (see \([\Pi]\) in the Introduction). This completes the proof. \( \square \)

In particular, these conditions coincides for weighted homogeneous polynomials which define an isolated singularity.
Remark 4.6 Some criterions for condition 1 are given by M. Schulze in [23].

Finally, this condition $A_{\log}(1/h)$ always implies $B(h)$ (as $A(1/h)$ does.)

Proposition 4.7 Let $h \in O$ be a nonzero germ such that $h(0) = 0$. If the ideal $\text{Ann}_h 1/h$ is generated by logarithmic operators, then $-1$ is the only integral root of the Bernstein polynomial of $h$.

Proof. The proof is analogous to the one of [20], Proposition 1.3. We need the following fact.

Assertion. If $Q$ is a logarithmic operator relative to $h$, then $Q \cdot h^s = q(s)h^s$ with $q(s) \in O[s]$.

Proof. We have $Q \cdot h^s = a(s)h^{s-N}$ with $a(s) = \sum_{i=0}^{N} a_i s^i$, $a_i \in O$, and $N$ is the degree of $Q$. Thus we just have to prove that $a(s) \in h^N O[s]$. As $Q$ is logarithmic, $Q \cdot h^k$ belongs to $h^k O$ for $k \geq 1$; in particular $\sum_{i=0}^{N} a_i k^i \in h^N O$ for $1 \leq k \leq N + 1$. By solving this system, we get $a_i \in h^N O$, $0 \leq i \leq N$, that is, $a(s) \in h^N O[s]$.

Let $Q_1, \ldots, Q_w$ be a generating system of logarithmic operators which annihilate $1/h$. For $1 \leq i \leq w$, we have $Q_i \cdot h^s = q_i(s)h^s$ with $q_i(s) \in O[s]$. As $Q_i$ annihilates $1/h$, the polynomial $q_i(s)$ belongs to $(s + 1)O[s]$ and we denote $\tilde{q}_i(s) \in O[s]$ the quotient of $q_i(s)$ by $(s + 1)$. Let us suppose that the Bernstein polynomial of $h$, denoted by $b(s)$, has an integral root strictly smaller than $-1$. We denote by $k \leq -2$, the greatest integral root of $b(s)$ verifying this condition. Using a Bernstein equation which gives $b(s)$, we get:

$$b(s) \cdots b(s-k-2)h^s = P(s)h^{s-k-1}$$

where $P(s) \in D[s]$. Thus $P(k)$ annihilates $1/h$ and it may be written $\sum_{i=1}^{w} A_i Q_i$ with $A_i \in D$, $1 \leq i \leq w$. If $P'(s) \in D[s]$ is the quotient of $P(s)$ by $s - k$, the previous equation becomes:

$$\underbrace{b(s) \cdots b(s-k-2)}_{c(s)} h^s = (s-k) \left[ P'(s) + \sum_{i=1}^{w} A_i \tilde{q}_i \right] h^{-k-2} \cdot h^{s+1}$$

where $-k-2 \geq 0$ and the multiplicity of $k$ in $c(s)$ is the same in $b(s)$. Hence, by division by $(s-k)$, we get a Bernstein functional equation such that the polynomial in the left member is not a multiple of $b(s)$. But this is not possible, because $b(s)$ is the Bernstein polynomial of $h$. Hence we have the result. □
4.3 The condition $M(h)$

Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. In this paragraph, we study the following condition

$M(h)$: The $\mathcal{D}$-module $\tilde{\mathcal{M}}_h = \mathcal{D}/\tilde{\mathcal{I}}_h$ is holonomic

where $\tilde{\mathcal{I}}_h \subset \mathcal{D}$ is the left ideal generated by the operators of order 1 which annihilate $1/h$. This condition only depends on the ideal $h\mathcal{O}$ (since the right multiplication by a unit $u \in \mathcal{O}$ induces an isomorphism of $\mathcal{D}$-modules from $\tilde{\mathcal{M}}_h$ to $\tilde{\mathcal{M}}_{uh}$).

Let us recall that this condition and this ‘logarithmic’ $\mathcal{D}$-module - introduced by F.J Castro-Jiménez and J.M. Ucha in [11] - are very natural in this topic. Indeed, the condition $A(1/h)$ always implies $M(h)$, since $A(1/h)$ means that the morphism $\tilde{\mathcal{M}}_h \to \mathcal{O}[1/h]$ defined by $P \mapsto P \cdot 1/h$ is an isomorphism. Moreover, the condition $\text{LCT}(D)$ needs locally $M(h_D)$ for a free divisor $D$ (see the beginning of the previous paragraph).

Here, we link the condition $M(h)$ with some other conditions introduced in this topic (see §4.1). Firstly, let us consider the following one:

$L(h)$: The ideal in $\mathcal{O}_{T^* \mathbb{C}^n}$ generated by $\pi^{-1}\text{Der}(-\log h)$ defines an analytic space of (pure) dimension $n$

where $\pi$ denotes the canonical map $T^* \mathbb{C}^n \to \mathbb{C}^n$. In K. Saito’s language, one says that the irreducible components of the logarithmic characteristic variety are holonomic; moreover, this is equivalent to the local finiteness of the logarithmic stratification associated with $h$ (see [20], §3). For a free germ, this is exactly the notion of Koszul-free germ (see [20], Proposition 6.3; [3], Corollary 1.9).

**Proposition 4.8.** Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$.

(i) The condition $L(h)$ implies $M(h)$.
(ii) The condition $A(h)$ implies $M(h)$.
(iii) The condition $G(h)$ implies $L(h)$.
(iv) If $h$ defines a locally weighted homogeneous divisor, then the condition $L(h)$ is verified.

**Proof.** The first point is clear since $\pi^{-1}\text{Der}(-\log h) \subset \text{gr} \tilde{I}_h$. Let us prove (ii).

By assumption, the ideal $J = \text{Ann}_{\mathcal{D}} h^*$ is included $\tilde{I}$. On the other hand, it is obvious that the operators $h\partial_i + h' x_i$, $1 \leq i \leq n$, belong to $\tilde{I}$. Hence, we have the following inclusion: $\text{gr}^F J + (h\xi_1, \ldots, h\xi_n)\mathcal{O}[\xi] \subset \text{gr}^F \tilde{I}$. We notice that

$$\text{gr}^F J + (h\xi_1, \ldots, h\xi_n)\mathcal{O}[\xi] = (\text{gr}^F J, h)\mathcal{O}[\xi] \cap (\xi_1, \ldots, \xi_n)\mathcal{O}[\xi]$$
since \( \text{gr}^F J \subset (\xi_1, \ldots, \xi_n)\mathcal{O}[\xi] \). Thus the characteristic variety of \( \tilde{\mathcal{M}}_h \) is included in \( V(\text{gr}^F J, h) \cup V(\xi_1, \ldots, \xi_n) \subset T^*\mathbb{C}^n \). Let us recall that the characteristic variety of \( Dh^* \) is the closure \( W_h \subset T^*\mathbb{C}^n \) of the set \( \{(x, \lambda dh(x)) \mid \lambda \in \mathbb{C}\} \) \[1\]; in particular, \( W_h \) is irreducible of pure dimension \( n + 1 \). From the principal ideal theorem, \( W_h \cap \{h = 0\} = V(\text{gr}^F J, h) \) has a pure dimension \( n \). Hence \( \mathcal{M}_h \) is holonomic.

The proof of (iii) is the very same, since the ideal generated by the principal symbol of the elements in \( \text{Der}(\log h) \) contains \( \text{gr}^F J + (h\xi_1, \ldots, h\xi_n)\mathcal{O}[\xi] \).

Let us prove (iv), by induction on dimension. Let \( D \subset \mathbb{C}^n \) denote the hypersurface defined by \( h \), and let \( L \) be the associated logarithmic characteristic variety. If \( n = 2 \), then \( W(h) \) is verified and so is \( L(h) \) by (iii). Now, we assume that \( n \geq 3 \). From Proposition 2.4 in [9], there exists a neighborhood \( U \) of the origin such that, for each point \( w \in U \cap D, w \neq 0 \), the germ of pair \( (\mathbb{C}^n, D, w) \) is isomorphic to a product \( (\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0,0)) \) where \( D' \) is a locally weighted homogeneous divisor of dimension \( n - 2 \). Up to this identification, \( \text{Der}(\log h)_w \) is generated by the elements in \( \text{Der}(\log h_{D'}) \) and \( \partial/\partial z \), where \( z \) is the last coordinate on \( \mathbb{C}^{n-1} \times \mathbb{C} \); in particular, the induction hypothesis applied to \( D' \) implies the result for \( \mathbb{C} \times D' \). Hence, the dimension of \( L \cap \pi^{-1}(U - \{0\}) = L - T^*_{\{0\}}\mathbb{C}^n \) is \( n \). Let \( C \subset L \) be an irreducible component of \( L \). If \( \pi(C) = \{0\} \), then \( C \) coincides with \( T^*_{\{0\}}\mathbb{C}^n \) since \( \dim C \) is at most equal to \( n \) (see [2], Proposition 1.14 (i)). Now, if \( \pi(C) \) is not the origin, then \( \dim C = \dim(C - T^*_{\{0\}}\mathbb{C}^n) = \dim(L - T^*_{\{0\}}\mathbb{C}^n) = n \). We conclude that \( L \) has dimension \( n \). \( \square \)

We recall that K. Saito proved that the condition \( L(h) \) is verified for any hyperplane arrangements [20], Example 3.14. The point (iv) may be considered as a generalization of this fact. On the other hand, it generalizes also the fact that locally weighted homogeneous free divisors are Koszul-free [4] (of course, our proof is similar).

The following diagram summarizes the previous relations:
Let us notice that the reverse relations are false. Firstly, if $h$ is the germ $(x_1 - x_2x_3)(x_1x_2^3 + x_1^2x_2)$ then $L(h)$ and $A(h)$ are not verified but $A(1/h)$ holds \cite{20, 5, 6, 10, 28}. On the other hand, if $h = (x_1 - x_2x_3)(x_1^3 + x_2^4)$ then it defines a Koszul-free germ (see Lemma \cite{3, 8} for instance); in particular, $L(h)$ is verified where as $A(h)$ and $A(1/h)$ fail (see the proof of Proposition \cite{1, 2} (i)). Finally, L. Narváez-Macarro and F.J Calderón-Moreno prove in \cite{8} that the free divisor defined by $h = (x_1 - x_2x_3)(x_1^5 + x_2^4 + x_1^4x_2)$ is not of Spencer type$^4$. In fact, the condition $M(h)$ is no more verified, since all elements of a system of generators of $	ilde{I}$ belongs to $D(x_1, x_2)$, see \cite{8} §5.

References

\cite{1} Artin M., On the solution of analytic equations, Invent. Math. (1968) 277–291

\cite{2} Björk J.E., Analytic D-Modules and Applications, Kluwer Academic Publishers 247, 1993.

\cite{3} Bruce F.W., Roberts R.M., Critical points of functions on analytic varieties, Topology 27 (1988) 57–90.

\cite{4} Calderón-Moreno F.J., Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor, Ann. Sci. École Norm. Sup. 32 (1999) 577–595.

$^4$This is a necessary condition on a free divisor $D$ for verifying $\text{LCT}(D)$, see \cite{8}.
[5] Calderón-Moreno F.J., Castro-Jiménez F.J., Mond D., Narváez-Macarro L., Logarithmic cohomology of the complement of a plane curve, Comment. Math. Helv. 77 (2002) 24–38.

[6] Calderón-Moreno F.J., Narváez-Macarro L., The module $\mathcal{D}f^*$ for locally quasi-homogeneous free divisors, Compos. Math. 134 (2002) 59–74.

[7] Calderón-Moreno F.J., Narváez-Macarro L., Locally quasi-homogeneous free divisors are Koszul-free, Tr. Mat. Inst. Steklova 238 (2002) 81–85.

[8] Calderón-Moreno F.J., Narváez-Macarro L., Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres, Ann. Inst. Fourier (Grenoble) 55 (2005) 47–75.

[9] Castro-Jiménez F.J., Mond D., Narváez-Macarro L., Cohomology of the complement of a free divisor, Trans. Amer. Math. Soc. 348 (1996) 3037–3049.

[10] Castro-Jiménez F.J., Ucha-Enríquez J.M., Explicit comparison theorems for $\mathcal{D}$-modules, J. Symbolic Comput. 32 (2001) 677–685.

[11] Castro-Jiménez F.J., Ucha J.M., Free divisors and duality for $\mathcal{D}$-modules, Tr. Mat. Inst. Steklova 238 (2002) 97–105.

[12] Castro-Jiménez F.J., Ucha Enríquez J.M., Testing the Logarithmic Comparison Theorem for Spencer free divisors, Experiment. Math. 13 (2004) 441–449.

[13] Castro-Jiménez F.J., Ucha Enríquez J.M., Logarithmic comparison theorem and some Euler homogeneous free divisors, Proc. Amer. Math. Soc. 133 (2005) 1417–1422.

[14] Ginsburg V., Characteristic varieties and vanishing cycles, Invent. Math. 84 (1986) 327–402.

[15] Grayson D., Stillman M., Macaulay2: A Software System for Research in Algebraic Geometry, available from World Wide Web (http://www.math.uiuc.edu/Macaulay2), 1999.

[16] Kashiwara M., $B$-functions and holonomic systems, Invent. Math. 38 (1976) 33–53.

[17] Kashiwara M., On the holonomic systems of differential equations II, Invent. Math. 49 (1978) 121–135.
[18] Leykin A., Tsai H., D-Module Package for Macaulay 2, available from World Wide Web (http://www.math.cornell.edu/~htsai), 2001.

[19] Saito K., Quasihomogene isolierte Singularitäten von Hyperflächen, Invent. Math. 14 (1971) 123–142.

[20] Saito K., Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo 27 (1980) 265–291.

[21] Saito M., On microlocal b-function, Bull. Soc. Math. France 122 (1994) 163–184.

[22] Saito M., Bernstein-Sato polynomials of hyperplane arrangements, arXiv:math.AG/0602527.

[23] Schulze M., A criterion for the logarithmic differential operators to be generated by vector fields, arXiv:math.CV/0406023.

[24] Torrelli T., Équations fonctionnelles pour une fonction sur un espace singulier, Thèse, Université de Nice-Sophia Antipolis, 1998.

[25] Torrelli T., Équations fonctionnelles pour une fonction sur une intersection complète quasi homogène à singularité isolée, C. R. Acad. Sci. Paris 330 (2000) 577–580.

[26] Torrelli T., Polynômes de Bernstein associés à une fonction sur une intersection complète à singularité isolée, Ann. Inst. Fourier 52 (2002) 221-244.

[27] Torrelli T., Bernstein polynomials of a smooth function restricted to an isolated hypersurface singularity, Publ. RIMS, Kyoto Univ. 39 (2003) 797-822.

[28] Torrelli T., On meromorphic functions defined by a differential system of order 1, Bull. Soc. Math. France 132 (2004) 591–612.

[29] Torrelli T., Logarithmic comparison theorem ans D-modules: an overview, arXiv:math.CV/0510430.

[30] Varchenko A.N., Asymptotic Hodge structure in the vanishing cohomology, Math. USSR Izvestià 18 (1982)

[31] Walther U., Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements, Compos. Math. 141 (2005) 121–145.
[32] YANO T., *On the theory of $b$-functions*, Publ. R.I.M.S. Kyoto Univ. 14 (1978) 111–202.