Boundedness of dyadic maximal operators on variable Lebesgue spaces

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Abstract
We introduce three types of dyadic maximal operators and prove that under some conditions on the variable exponent $p(\cdot)$, they are bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$. Here we correct Theorem 4.2 of the paper, Szarvas and Weisz (Banach J Math Anal 13:675–696, 2019).

Keywords Variable exponent · Dyadic maximal operators · Walsh-Fourier series · Cesàro and Riesz means

Mathematics Subject Classification 42B25 · 42C10

1 Introduction

For a measurable function $p(\cdot)$, the variable Lebesgue space $L_{p(\cdot)}$ consists of all measurable functions $f$ for which $\int_0^1 |f(x)|^{p(x)} \, dx < \infty$. If $p(\cdot)$ is a constant, we get back the usual $L_p$ space. This topic needs essentially new ideas and is investigated very intensively in the literature nowadays (see e.g., Cruz-Uribe and Fiorenza [1], Diening et al. [2], Nakai and Sawano [9, 10], Jiao et al. [4–6], Liu et al. [7, 8]). Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications (see the references in Jiao et al. [4]).

Usually, we suppose that $p(\cdot)$ satisfies the Hölder continuity condition. In [4, 5, 12] as well as in this paper, we suppose a slightly more general condition. It is known (see [4, 5]) that the usual dyadic (or martingale) maximal operator is
bounded on $L_{p(\cdot)}$ with $1 < p_- \leq p_+ < \infty$. For the boundedness of the classical Hardy-Littlewood maximal operator see e.g., Cruz-Uribe and Fiorenza [1] and Diening et al. [2].

In [12], we investigated two more dyadic maximal operators denoted by $U_{\beta,s}$ and $V_{\beta}$, where $\beta$ and $s$ are positive parameters. We stated there that $U_{\beta,s}$ is bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$ and $\beta, s > 0$. However, there is a mistake in the proof, as we applied Lemma 2 in a wrong way. In this paper, we correct the proof and prove that, for $1 < p_- \leq p_+ < \infty$, $U_{\beta,s}$ is bounded on $L_{p(\cdot)}$ under the additional condition $\frac{1}{p_-} - \frac{1}{p_+} < \beta + s$. We show also that without this last additional condition, the boundedness of $U_{\beta,s}$ does not hold. Next, we verify that $V_{\beta}$ is bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$ and $\beta > 0$, without any additional condition. This was stated in [12] without proof.

In [12], we used the boundedness of the above dyadic maximal operators to prove that the maximal Cesa`ro (or $(C, \alpha)$) and Riesz operators of the Walsh-Fourier series are bounded from the variable Hardy space $H_{p(\cdot)}$ to $L_{p(\cdot)}$ if $1/(\alpha + 1) < p_- < \infty$. Here we correct the theorem and show that under this last condition, the boundedness holds if and only if $\frac{1}{p_-} - \frac{1}{p_+} < 1$.

2 Variable Lebesgue spaces

In this section, we recall some basic notations on variable Lebesgue spaces and give some elementary and necessary facts about these spaces. Our main references are Cruz-Uribe and Fiorenza [1] and Diening et al. [2].

For a constant $p$, the $L_p$ space is equipped with the quasi-norm

$$\|f\|_p := \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \quad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure $\lambda$.

We are going to generalize these spaces. A measurable function $p(\cdot) : [0, 1) \to (0, \infty)$ is called a variable exponent. For any variable exponent $p(\cdot)$ and any measurable set $A \subset [0, 1)$, we will use the notation

$$p_-(A) := \text{ess inf}_{x \in A} p(x) \quad \text{and} \quad p_+(A) := \text{ess sup}_{x \in A} p(x).$$

If $A = [0, 1)$, then the numbers $p_-(A)$ and $p_+(A)$ are denoted simply by $p_-$ and $p_+$. Denote by $\mathcal{P}$ the collection of all variable exponents $p(\cdot)$ satisfying

$$0 < p_- \leq p_+ < \infty.$$

The variable Lebesgue space $L_{p(\cdot)}$ contains all measurable functions $f$, for which

$$\|f\|_{L_{p(\cdot)}} := \inf \left\{ \rho \in (0, \infty) : \int_0^1 \left( \frac{|f(x)|}{\rho} \right)^{p(x)} \, dx \leq 1 \right\} < \infty.$$
Instead of the log-Hölder continuity condition, in [4, 12], we introduced the slightly more general condition

\[ \lambda(I)^{p_-(I) - p_+(I)} \leq C \]  

(1)

for all dyadic intervals \( I \subset [0, 1] \). By a dyadic interval, we mean one of the form \([k2^{-n}, (k + 1)2^{-n})\) for some \( k, n \in \mathbb{N}, 0 \leq k < 2^n \).

**Remark 1** There exist a lot of functions \( p(\cdot) \) satisfying (1). For concrete examples we mention the function \( a + cx \) for parameters \( a \) and \( c \) such that the function is positive \((x \in [0, 1])\). All positive Lipschitz functions with order \( 0 < \beta \leq 1 \) also satisfy (1).

The following lemma was proved in Cruze-Uribe and Fiorenza [1] and Hao and Jiao [3].

**Lemma 1** Let \( p(\cdot) \in \mathcal{P} \) satisfy (1). Then, for any dyadic interval \( I \subset [0, 1] \),

\[ \lambda(I)^{1/p_-(I)} \sim \lambda(I)^{1/p(x)} \sim \lambda(I)^{1/p_+(I)} \sim \|x\|_{p(\cdot)} \quad (\forall x \in I), \]

where \( \sim \) denotes the equivalence of the numbers.

The following lemma can be found in Jiao et al. [4, 5].

**Lemma 2** Let \( p(\cdot) \in \mathcal{P}, 1 \leq p_- \leq p_+ < \infty \), satisfy (1). Suppose that \( f \in L_{p(\cdot)} \) with \( \|f\|_{p(\cdot)} \leq 1/2 \) and \( f = f \chi_{\{|f| \geq 1\}} \). Then, for any dyadic interval \( I \subset [0, 1] \) and \( x \in I \),

\[ \left( \frac{1}{\lambda(I)} \int_I |f(t)|^p(x) \, dt \right)^{p(x)} \leq \left( \frac{C}{\lambda(I)} \int_I |f(t)|^{p(i)} \, dt \right). \]

In this paper the constants \( C \) are absolute constants and the constants \( C_{p(\cdot)} \) are depending only on \( p(\cdot) \) and may denote different constants in different contexts. For two positive numbers \( A \) and \( B \), we use also the notation \( A \lesssim B \), which means that there exists a constant \( C \) such that \( A \leq CB \).

## 3 Dyadic maximal operators

In this section, in addition to the well known Doob’s maximal operator, we introduce two new types of maximal operators. The Doob’s maximal operator is given by

\[ Mf(x) := \sup_{I \ni x} \frac{1}{\lambda(I)} \left| \int_I f \, d\lambda \right|, \]

where the supremum is taken over all dyadic intervals. The following result was proved in Jiao et al. [5]. For the boundedness of the classical Hardy–Littlewood maximal operator see e.g. Cruz-Uribe and Fiorenza [1] and Diening et al. [2].

**Theorem 1** If \( p(\cdot) \in \mathcal{P} \) satisfies (1) and \( 1 < p_- \leq p_+ < \infty \), then
For an integrable function \( f \in L_1 \), we define the second maximal operator by

\[
U_{\beta, s} f(x) := \sup_{x \in I} \left\{ \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{1}{\lambda(P_{i,j})} \left| \int_{P_i,j} f \, d\lambda \right| \right\},
\]

where \( I \) is a dyadic interval with length \( 2^{-n} \), \( \beta, s \) are positive constants and

\[
I^{ij} := I + [2^{-j-1}, 2^{-j-1} + 2^{-i}).
\]

Here \( + \) denotes the dyadic addition (see e.g., Schipp, Wade, Simon and Pál [11]). Let us define

\[
I_{k,n} := [k2^{-n}, (k + 1)2^{-n}) \quad \text{with} \quad 0 \leq k < 2^n, \ n \in \mathbb{N}.
\]

The preceding definition can be rewritten to

\[
U_{\beta, s} f := \sup_{n \in \mathbb{N}} \left\{ \sum_{k=0}^{2^n-1} \sum_{m=1}^{n} \sum_{j=0}^{m-1} \frac{1}{\lambda(P_{i,j})} \left| \int_{P_{i,j}} f \, d\lambda \right| \right\},
\]

The following theorem was proved in [12].

**Theorem 2** For all \( 1 < p < \infty \) and all \( 0 < \beta, s < \infty \), we have

\[
\| U_{\beta, s} f \|_p \leq C_p \| f \|_p \quad (f \in L_p).
\]

Now we generalize this theorem to variable Lebesgue spaces.

**Theorem 3** Let \( p(\cdot) \in \mathcal{P} \) satisfy (1), \( 1 < p_- \leq p_+ < \infty \) and \( 0 < \beta, s < \infty \). If

\[
\frac{1}{p_-} - \frac{1}{p_+} < \beta + s,
\]

then

\[
\| U_{\beta, s} f \|_{p(\cdot)} \leq C_{p(\cdot)} \| f \|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).
\]

**Proof** It is easy to see that we may suppose the conditions \( \| f \|_{p(\cdot)} \leq 1/2, \ |f| \geq 1 \) or \( f = 0 \) and

\[
\frac{1}{\lambda(P_{i,j}^{k,n})} \int_{P_{i,j}^{k,n}} |f(t)| \, dt > 1.
\]

We denote by \( I_{k,n,j,1} \) (resp. \( I_{k,n,j,2} \)) those points \( x \in I_{k,n} \) for which \( p(x) \leq p_+(I_{k,n}^{j,1}) \) (resp. \( p(x) > p_+(I_{k,n}^{j,2}) \)). Then
\[ \int_{0}^{1} |U_{\beta, s} f(x)|^{p(x)} \, dx \lesssim \sum_{l=1}^{2} \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \mathcal{I}_{I_{k,n}}(x) \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right) \]

Let \( q(x) := p(x)/p_0 > 1 \) for some \( 1 < p_0 < p_- \). Using convexity and the fact that \( q(x) \leq q_+(I_{k,n}) \) on \( I_{k,n,j,l,1} \), we get that

\[ (A) \lesssim \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \mathcal{I}_{I_{k,n}}(x) \left( \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right) \right. \]

\[ \left. \frac{\mathcal{I}_{I_{k,n,j,l,1}}(x)}{\mathcal{I}_{I_{k,n}}(x)} \left. \int_{I_{k,n}^{j,l}} |f(t)| \, dt \right)^{p_0} dx \]

\[ \lesssim \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \mathcal{I}_{I_{k,n}}(x) \left( \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right) \right. \]

\[ \left. \frac{\mathcal{I}_{I_{k,n,j,l,1}}(x)}{\mathcal{I}_{I_{k,n}}(x)} \left. \int_{I_{k,n}^{j,l}} |f(t)| \, dt \right)^{q_+(I_{k,n})} p_0 dx \]

Lemma 2 and Theorem 2 imply

\[ (A) \lesssim \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \mathcal{I}_{I_{k,n}}(x) \left( \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right) \right. \]

\[ \left. \frac{\mathcal{I}_{I_{k,n,j,l,1}}(x)}{\mathcal{I}_{I_{k,n}}(x)} \left. \int_{I_{k,n}^{j,l}} |f(t)|^{q(t)} \, dt \right)^{p_0} dx \]

\[ \lesssim \| U_{\beta,s} |f|^{q(-)} \|_{p_0}^{p_0} \lesssim \| f^{q(-)} \|_{p_0}^{p_0} \leq C. \]

Choosing \( 0 < \beta_0 < \beta \) and \( 0 < r < s + \beta_0 \), we obtain
Since
\[ |f| \geq 1 \text{ or } f = 0, \quad q(x) > q_-(I_{k,n}^{i,j}) \text{ on } I_{k,n}^{i,j}, \quad q_-(I_{k,n}^{i,j}) \leq q(t) < p(t) \text{ for all } t \in I_{k,n}^{i,j} \]
and
\[ \int_{I_{k,n}^{i,j}} |f(t)|^{q_-(I_{k,n}^{i,j})} \, dt \leq \left( \int_{I_{k,n}^{i,j}} |f(t)|^{p(t)} \, dt \right)^{1/2}, \]
we can see that
\[ (B) \lesssim \int_0^1 \left( \sup_{n \in \mathbb{Z}} \sum_{k=0}^{2^n-1} \mathcal{X}_{I_{k,n}^{i,j}}(x) \left( \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)} \beta_{j-i} 2^{(j-i) s} \right) \right)^{p_0} \]
\[ \frac{\mathcal{X}_{I_{k,n}^{i,j}}(x)}{\lambda(F_{k,n}^{i,j})} \left( \int_{F_{k,n}^{i,j}} |f(t)| q(x) \, dt \right)^{p_0} \]
\[ \lesssim \int_0^1 \left( \sup_{n \in \mathbb{Z}} \sum_{k=0}^{2^n-1} \mathcal{X}_{I_{k,n}^{i,j}}(x) \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)} (\beta_{j-i} 2^{(j-i) s} \right)

By Hölder’s inequality,
\[
(B) \lesssim \int_0^1 \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} I_{k,n}(x) \right. \\
\left. \sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+r-s)} 2^{(j-i)r q(x)} 2^{(j-i)q(x)/q_- (l_{k,n}^i)-i} \right) 2^{jrq(x)}/C_0 dx.
\]

For fixed \( k \) and \( n \) let \( J_j \) denote the dyadic interval with length \( 2^{-j} \) and \( I_{k,n} \subset J_j \). Then \( I_{k,n} \subset J_{j+2^{-j-1}} = J_j \). Inequality (1) implies that \( 2^{-j p(x)} \sim 2^{-j p_- (l_{k,n}^i)} \) for \( x \in I_{k,n} \). It is easy to check that for \( x \in I_{k,n} \),

\[
2^{j rq(x)} = 2^{jrq(x)} 2^{jq(x)} 2^{-jq(x)} \lesssim 2^{jrq(x)} 2^{jq_- (l_{k,n}^i)} 2^{-jq_- (l_{k,n}^i)}
\]

\[
< 2^{j \left( rq(x) - \frac{q(x) - q_- (l_{k,n}^i)}{q_- (l_{k,n}^i)} \right)} = 2^{j \left( rq(x) - \frac{q(x) - q_- (l_{k,n}^i)}{q_- (l_{k,n}^i)} + 1 \right)}.
\]

Furthermore,

\[
rq(x) - \frac{q(x)}{q_- (l_{k,n}^i)} + 1 \geq q(x) \left( r - \frac{1}{q_-} \right) + 1
\]

\[
\geq \begin{cases} 
1, & \text{if } r - \frac{1}{q_-} \geq 0; \\
q_+ \left( r - \frac{1}{q_-} \right) + 1, & \text{if } r - \frac{1}{q_-} < 0.
\end{cases}
\]

Let \( r_0 := \min \left( 1, q_+ \left( r - \frac{1}{q_-} \right) + 1 \right) \). Then \( r_0 > 0 \) if and only if

\[
\frac{1}{q_-} - \frac{1}{q_+} < r.
\]

Hence
(B) \leq \int_0^1 \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} X_{I,k,n}(x) \right)^p dx \\
\sum_{m=1}^{n} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} 2^{(j-i)} \left( \frac{q(x) - \frac{q(t)}{q(I_{0,n}^1)} + 1}{q(I_{0,n}^1) - q(t)} \right)^p \\
\frac{1}{\lambda(I_{k,n}^1)} \int_{I_{k,n}^1} |f(t)|^{q(t)} dt \right|_{\|f\|_{L^p}} \leq C,

whenever (3) holds. Since r can be arbitrarily near to s + \beta_0 and \beta_0 to \beta, this completes the proof.

Remark 2 Inequality (2) and Theorem 3 hold if \( p_- > \max(1/(\beta + s), 1) \).

In [12], we used the parameters \( \beta = (1 + \alpha) t - r/(r - t) > 0 \), \( s = r/(r - t) - \alpha t > 0 \) and stated that Theorem 3 holds without the condition (2). However, this is not the case.

Theorem 4 Let \( p(\cdot) \in \mathcal{P} \) satisfy (1), \( 1 < p_- \leq p_+ < \infty \) and \( 0 < \beta, s < \infty \). If

\[ \frac{1}{p_-(I_{0,n}^{1,n-1})} - \frac{1}{p_+(I_{0,n}^1)} > \beta + s \]

(4)

for all \( n \in \mathbb{N} \), then \( U_{\beta,s} \) is not bounded on \( L^{p(t)} \).

Proof Choosing \( m = n \), \( j = 0 \) and \( i = n - 1 \), we can see that

\[
\int_0^1 |U_{\beta,s} f(x)|^{p(x)} dx \\
\geq \int_0^1 X_{I_0,n}(x) \left( 2^{-n(\beta+s)} \frac{1}{\lambda(I_{0,n}^{1,n-1})} \left| \int_{I_{0,n}^{1,n-1}} f(t) dt \right| \right)^{p(x)} dx.
\]

Let

\[ f(t) := X_{I_0,n}^{1,n-1}(t) 2^{n/p_-(I_{0,n}^{1,n-1})}. \]

Lemma 1 implies that

\[ \|f\|_{L^{p(\cdot)}} = 2^{n/p_-(I_{0,n}^{1,n-1})} \|X_{I_0,n}^{1,n-1}\|_{L^{p(\cdot)}} \leq C. \]
Thus, by (1),
\[
\int_0^1 |U_{\beta,s}f(x)|^p(x) \, dx \geq \int_{I_{0,n}} 2^{-n(\beta+s)p(x)} 2^{np(x)/(p_-t_{0,n}^{n-1})} \, dx
\geq C \int_{I_{0,n}} 2^{np(x)/(p_-t_{0,n}^{n-1})-\beta-s} \, dx
= C 2^{np(x)/(p_-t_{0,n}^{n-1})-\beta-s} 2^{-n}
\]
which tends to infinity as \( n \to \infty \) if (4) holds. \( \square \)

The third dyadic maximal operator is introduced by
\[
V_{\beta,s}f(x) := \sup_{x \in I} \sum_{m=0}^{n-1} 2^{(m-n)\beta} 2^{-\frac{1}{\lambda(I_m)}} \left| \int_{I_m} f \right|,
\]
where \( f \in L_1, I \) is a dyadic interval with length \( 2^{-n} \), \( \beta \) is a positive constant and
\[
I_m := I + [0, 2^{-m}).
\]
Then
\[
V_{\beta,s}f = \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=0}^{n-1} 2^{(m-n)\beta} 2^{-\frac{1}{\lambda(I_{k,n})}} \left| \int_{I_{k,n}} f \right|.
\]
The following theorem can be found in [12].

**Theorem 5**  For all \( 1 < p < \infty \) and all \( 0 < \beta < \infty \), we have
\[
\| V_{\beta,s}f \|_p \leq C_p \| f \|_p \quad (f \in L_p).
\]

The generalization of this result reads as follows.

**Theorem 6**  If \( p(\cdot) \in \mathcal{P} \) satisfies (1), \( 1 < p_- \leq p_+ < \infty \) and \( 0 < \beta < \infty \), then
\[
\| V_{\beta,s}f \|_{p(\cdot)} \leq C_p(\cdot) \| f \|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).
\]

**Proof**  Similarly to the proof of Theorem 3, we may suppose again that \( \| f \|_{p(\cdot)} \leq 1/2 \) and
\[
\frac{1}{\lambda(I_{k,n})} \int_{I_{k,n}} |f(t)| \, dt > 1.
\]
Since \( I_{k,n} \subset I_{k,n}^m \) \((m = 0, \ldots, n-1)\), we can apply Lemma 2 and Theorem 5 to obtain
\[ \int_{0}^{1} |V_\beta f(x)|^{p(x)} \, dx \]

\[ \lesssim \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} x_{k,n}(x) \sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I_{k,n})} \int_{I_{k,n}} |f(t)| \, dt \right)^{q(x)} \, dx \]

\[ \lesssim \int_{0}^{1} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} x_{k,n}(x) \sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I_{k,n})} \int_{I_{k,n}} |f(t)| \, dt \right)^{p_0} \, dx \]

\[ \lesssim \left\| V_\beta (|f|^{q(\cdot)}) \right\|_{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0} \leq C, \]

which proves the theorem. \qed

4 The maximal Cesàro and Riesz operator on $H_{p(\cdot)}$

Using Theorems 3 and 6, we can prove the boundedness of the maximal Cesàro and Riesz operators of Walsh-Fourier series as in [12].

**Theorem 7** Let $p(\cdot) \in \mathcal{P}$ satisfy (1) and

\[ \frac{1}{p_-} - \frac{1}{p_+} < 1. \]

If $0 < \alpha \leq 1 \leq \gamma$ and $1/(\alpha + 1) < p_- < \infty$, then

\[ \left\| \sigma_\alpha^f \right\|_{p(\cdot)} + \left\| \sigma_\alpha^{q(\cdot)} f \right\|_{p(\cdot)} \lesssim \| f \|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)}). \]

The same holds for the space $H_{p(\cdot),q}$ ($0 < q \leq \infty$).

For the definitions, details and proof see [12]. We stated there that Theorem 7 holds without the condition (5). However, as we have seen in Jiao et al. [4], this is not true for $\alpha = \gamma = 1$.

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References

1. Cruz-Uribe, D.V., Fiorenza, A.: Variable Lebesgue spaces. Foundations and harmonic analysis. Birkhäuser/Springer, New York (2013)
2. Diening, L., Harjulehto, P., Hästö, P., Ružička, M.: Lebesgue and Sobolev spaces with variable exponents. Springer, Berlin (2011)
3. Hao, Z., Jiao, Y.: Fractional integral on martingale Hardy spaces with variable exponents. Fract. Calc. Appl. Anal. 18(5), 1128–1145 (2015)
4. Jiao, Y., Weisz, F., Wu, L., Zhou, D.: Variable martingale Hardy spaces and their applications in Fourier analysis. Dissertationes Math. (to appear)
5. Jiao, Y., Zhou, D., Hao, Z., Chen, W.: Martingale Hardy spaces with variable exponents. Banach J. Math 10, 750–770 (2016)
6. Jiao, Y., Zuo, Y., Zhou, D., Wu, L.: Variable Hardy–Lorentz spaces $H^{p(x)}(\mathbb{R}^n)$. Math. Nachr. 292, 309–349 (2019)
7. Liu, J., Weisz, F., Yang, D., Yuan, W.: Variable anisotropic Hardy spaces and their applications. Taiwanese J. Math. 22, 1173–1216 (2018)
8. Liu, J., Weisz, F., Yang, D., Yuan, W.: Littlewood–Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications. J. Fourier Anal. Appl. 25, 874–922 (2019)
9. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. 262(9), 3665–3748 (2012)
10. Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators. Integral Equ. Oper. Theory 77, 123–148 (2013)
11. Schipp, F., Wade, W.R., Simon, P., Pál, J.: Walsh Series: An Introduction to Dyadic Harmonic Analysis. Adam Hilger, Bristol, New York (1990)
12. Szarvas, K., Weisz, F.: The boundedness of the Cesaro- and Riesz means in variable dyadic Hardy spaces. Banach J. Math. Anal. 13, 675–696 (2019)