Iterative data-driven inference of nonlinearity measures
via successive graph approximation

Tim Martin and Frank Allgöwer*

Abstract—In this paper, we present an iterative data-driven approach to derive guaranteed bounds on nonlinearity measures of unknown nonlinear systems. In this connection, nonlinearity measures quantify the strength of the nonlinearity of a dynamical system by the distance of its input-output behaviour to a set of linear models. First, the computation of a guaranteed bound of these measures by given input-output data based on a non-parametric data-based model of the unknown system and local inferences of nonlinearity measures is presented. Second, we propose an iterative scheme to improve this bound iteratively by further samples of the unknown input-output behaviour.

I. INTRODUCTION

Deriving a model-based controller for complex systems requires a sufficiently precise model. However, modelling such systems is often difficult and more time consuming than the controller design itself. Hence, a data-driven controller design is desired, where a controller is received without identifying a precise model of the system. [1] gives an overview of such approaches.

One approach is examined in [2], where control-theoretic system properties, as \( L_2 \)-gain and conic relations, are learned from a large set of given input-output samples. By the knowledge of these properties, insight into the open-loop system is gained and well-known feedback theorems can be applied. Analogously, given input-output samples are considered in [3] to derive a linear surrogate model that minimizes the maximal deviation to an unknown nonlinear system. By the knowledge of the linear model and its approximation error, well-studied techniques from robust control theory can be applied to determine a controller with closed-loop guarantees. Furthermore, the approximation error corresponds to the nonlinearity measures from [4] and [5], which quantify the nonlinearity of dynamical systems.

One disadvantage of the approach in [2] and [3] is the assumed access of a large number of input-output samples. Contrary, iterative approaches require to perform sequentially experiments on the plant to identify the control-theoretic properties. These algorithms provide an (optimal) decision what experiment should be applied next to improve the estimation of the system property. For example, the \( L_2 \)-gain and a linear surrogate model are computed in [6] and [7], respectively, for linear time-invariant systems (LTI) based on solving optimization problems using gradient-based methods.

To deduce an iterative scheme for nonlinear systems, we exploit a non-parametric data-based model. Instead of a statistical approach as in Gaussian process regression [8], we consider a non-parametric model where an envelope of the graph of a Lipschitz function is obtained directly from given input-output data. Such Lipschitz approximation techniques are investigated, e.g., in set-membership identification [9] and in Kinky inferences for nonlinear model predictive control [10], [11]. In both applications, the system dynamics is approximated by a function that minimizes the maximal error with respect to some function norms.

In this paper, we determine a guaranteed upper bound on nonlinearity measures by means of an envelope that contains the unknown input-output behaviour of the dynamical system. Especially, the bound is obtained from the maximal distance of a linear approximation model to all realizations of mappings which are contained in the envelope. A bound on this distance is calculated by local inferences of the nonlinearity measure. This approach is the first contribution of this paper and constitutes an alternative to [2] and [3] for deriving guaranteed bounds on system-theoretic properties from given input-output samples. Compared to [2] and [3], the presented approach does not require the computation of the covering radius and can be considered for noisy output measurements. Moreover, this approach can be extended to an iterative scheme, the second contribution of this paper, to reduce the derived upper bound on the nonlinearity measure by further sampling. Here, we ensure that the computational complexity of this algorithm does not increase with further iterations, while the convergence to the true nonlinearity measure is guaranteed. In contrast to set-membership identification, we approximate the nonlinear system by a linear model that is mostly not contained in the envelope. Instead, the linear model is a projection of the envelope on a set of linear models. Therefore, the results of [9] are not applicable. Moreover, our goal differs from system identification in the sense that we approximate the behaviour of a complex (nonlinear) system by a simple (linear) model. Hence, the linear model is a surrogate model of the nonlinear system.

The paper is organized as follows. First, we introduce nonlinearity measures. Second, we state the problem of estimating nonlinearity measures via graph approximation. Third, we solve this problem by means of local inference of nonlinearity measures and propose the iterative scheme to improve the estimation of the nonlinearity measure by successive sampling. The paper concludes with a numerical example, where the iterative scheme is compared with the offline approach [3].

* T. Martin and F. Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart. This work was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC 2075 - 390740016. For correspondence, mailto: tim.martin@ist.uni-stuttgart.de.
II. Problem setup and definition of a nonlinearity measure

Let the input-output behaviour of a discrete-time nonlinear SISO system be described by the mapping

\[ N : U \subset \mathbb{R}^n \to \mathcal{Y} \subset \mathbb{R}^n \]

of input on output trajectories of length \( n \) with \( N(0) = 0 \), without loss of generality. Moreover, let the input set \( U \) be spanned by an orthonormal basis of signals \( v_i \in \mathbb{R}^n, i = 1, \ldots, \mu \leq n \), i.e.,

\[ U = \{ u \in \mathbb{R}^n : u = [v_1, \ldots, v_\mu] \bar{u}, \bar{u} \in \bar{U} \subset \mathbb{R}^\mu \} \tag{1} \]

where the amplitudes \( \bar{u} \) are bounded by the box constraint \( \bar{U} = [\bar{u}_1, \bar{\alpha}_1] \times \cdots \times [\bar{u}_\mu, \bar{\alpha}_\mu] \). This compact input set is also supposed in \([2]\) and \([3]\). In practice, the basis of signals is often chosen as Fourier basis or Legendre polynomials. This assumption will play a crucial role for the iterative scheme, as the algorithm decides the input applied to the plant. Hence, we can ensure that all these inputs are experimentally admissible by a suitable choice of basis signals. Note that, each input \( u \in U \) corresponds to an unique amplitude \( \bar{u} \in \bar{U} \), since \( v_i \in \mathbb{R}^n, i = 1, \ldots, \mu \) is an orthogonal basis and \( \mu \leq n \). Therefore, we can exchange \( u \) by its corresponding amplitude \( \bar{u} \) and vice versa.

Furthermore, we suppose that \( \mathcal{Y} \) is a compact set to ensure the well-definitionness of the following definition of nonlinearity measures from \([4]\).

**Definition 1.** The nonlinearity of a dynamical system \( N : U \subset \mathbb{R}^n \to \mathcal{Y} \subset \mathbb{R}^n \) is quantified by the additive error nonlinearity measure (AE-NLM)

\[ \Phi_{\text{AE}}^{U,G} := \inf_{G \in \mathcal{G}} \sup_{u \in U \setminus \{0\}} \frac{\|N(u) - G(u)\|}{\|u\|}, \tag{2} \]

where \( G : U \to \mathcal{Y} \) is an element of a set \( \mathcal{G} \) of LTI systems and \( \| \cdot \| \) denotes the Euclidean vector norm.

Solving the optimization problem \([2]\) yields the ‘best’ linear approximation \( G^* \) that minimizes the gain of the error system \( \Delta := N - G^* \) with respect to the Euclidean norm. Thus, the nonlinear system can be written as the interconnection of the linear model \( G^* \) and the error model \( \Delta \), which gain corresponds to the AE-NLM. Therefore, techniques from robust control theory for linear systems can be applied once \( G^* \) and the nonlinearity measure are known. Furthermore, \( G^* \) can be seen as the projection of the nonlinear system \( N \) on the set of linear systems \( \mathcal{G} \), as illustrated in Figure 1. As shown in \([3]\), the AE-NLM is related to the \( \ell_2 \)-gain and the conic relations from \([12]\) by the special choice \( \mathcal{G} = \{0\} \) and \( \mathcal{G} = \{G = cI : c \in \mathbb{R}\} \) with \( I : u \mapsto u \), respectively. For further reading on nonlinearity measures, we refer to \([3]\), where, amongst other things, parameterizations of \( \mathcal{G} \) are proposed and a characterization of stability for feedback interconnections using nonlinear measures is derived via the concept of graph separation.

III. A data-based non-parametric model for Lipschitz mappings

In this section, we introduce the data-based non-parametric model for Lipschitz mappings from \([9]\). Subsequent, we state the problem setup to calculate an upper bound for the AE-NLM from this non-parametric model.

To conclude on the input-output behaviour of the unknown nonlinear system \( N \), we assume the access of input-output trajectories

\[ U_D \times \mathcal{Y}_D := \{(u_1, y_1), \ldots, (u_D, y_D)\} \subset U \times \mathcal{Y} \]

of the nonlinear system, i.e., \( y_i = N(u_i), i = 1, \ldots, D \). Moreover, let \( U_D \) denote the set of amplitudes \( \bar{u}_i, i = 1, \ldots, D \) corresponding to \( u_i, i = 1, \ldots, D \). Since the set of mappings generating \( U_D \times \mathcal{Y}_D \) is unbounded, the rate of variation of \( N \) is restricted in an additional assumption.

**Assumption 1.** The input-output mapping \( N : U \to \mathcal{Y} \) is Lipschitz continuous with \( L > 0 \), i.e.,

\[ \|N(u) - N(u')\| \leq L \|u - u'\|, \quad \forall u, u' \in U, \]

and the Lipschitz constant \( L \) is known.

In general, the Lipschitz constant \( L \) is not known beforehand. However, different methods were developed to estimate \( L \) from data, e.g., Strongins estimator \([13]\) and POKI \([14]\). Under the prior knowledge of \( U_D \times \mathcal{Y}_D \) and Assumption 1, we can conclude that the graph of the mapping \( N \) is contained in the envelope

\[ E(U_D \times \mathcal{Y}_D) := \{(u, y) \in U \times \mathcal{Y} : \|y - y_i\| \leq L \|u - u_i\|, \quad i = 1, \ldots, D\}. \]

An illustration of this envelope can be found in Figure 2 for \( n = 1 \) and in \([9]\) for higher dimensions. Since the envelope \( E(U_D \times \mathcal{Y}_D) \) is defined through the input-output samples \( U_D \times \mathcal{Y}_D \), the envelope establish a data-based non-parametric model of \( N \).

In the sequel of this paper, we exploit the envelope \( E(U_D \times \mathcal{Y}_D) \) to determine an upper bound on the AE-NLM. Since the graph of \( N \) is a subset of \( E(U_D \times \mathcal{Y}_D) \),

\[ \inf_{G \in \mathcal{G}} \max_{u \in U, ||u|| \geq \epsilon} \frac{||N(u) - G(u)||}{||u||} \leq \inf_{G \in \mathcal{G}} \max_{(u, y) \in E(U_D \times \mathcal{Y}_D), ||u|| \geq \epsilon} \frac{||y - G(u)||}{||u||} =: \Phi_{\text{AE}}^{E(U_D \times \mathcal{Y}_D), \mathcal{G}}. \tag{3} \]

First, observe that, we exclude small inputs \( ||u|| < \epsilon \) in \([5]\) similar to \([2]\) and \([3]\). Otherwise, the upper bound \( \Phi_{\text{AE}}^{E(U_D \times \mathcal{Y}_D), \mathcal{G}} \) would be at least \( L \) regardless of the data set.
Fig. 2. Envelope of an one dimensional input-output mapping $y = N(u)$.

$\mathcal{U}_D \times \mathcal{Y}_D$. Indeed, there exists a neighbourhood of $(u, y) = (0, 0)$ with $E(\{(0, 0)\}) = E(\mathcal{U}_D \times \mathcal{Y}_D)$ and

$$\inf_{(u, y) \in E(\{(0, 0)\})} \sup_{u \neq 0} \frac{||y - G(u)||}{||u||} = \sup_{u \neq 0} \frac{||y||}{||u||} = L.$$ 

Here, the first equality holds due to the optimal approximation of Lipschitz functions from [9]. Second, note that, there always exist inputs that solves the left- and right-hand side of (3), as the input set is compact by assumption. However, the solution is not necessarily unique.

**Remark 2.** To solve the optimization problem of $\Phi_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}$ in (5), we could follow [15]. Thus, we apply the S-procedure to derive a semi-definite programming for the right-hand side of (3). However, due to the relaxation of the S-procedure, the problem emerges that the optimal linear model is zero regardless of the data set and the relaxation exhibits a data-inefficient estimation for (3) as well as for other quadratic system properties, e.g., passivity.

**IV. AN ITERATIVE APPROACH FOR THE INFERENCE OF NONLINEARITY MEASURE**

As mentioned in Remark 2 the relaxation of the S-procedure for the right-hand side of (3) exhibits a conservative estimation. Therefore, we establish in Section IV-A an approach to solve this optimization problem for a given linear approximation model and given input-output samples. In Section IV-B we extend this approach to an iterative procedure to reduce the upper bound of the AE-NLM by iterative sampling.

**A. Local inference of the AE-NLM**

In this section, we solve for a given linear surrogate model $G$ the optimization problem

$$\Phi_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G) = \max_{(u, y) \in E(\mathcal{U}_D \times \mathcal{Y}_D), ||u|| \geq \varepsilon} \frac{||y - G(u)||}{||u||}, \tag{4}$$

which corresponds to the right-hand side of (3) with $G = \{G\}$. Similar to [11], we obtain the global inference (4) by means of local inferences of the AE-NLM. Subsequent, we suggest a nonconvex relaxation to calculate properly the local AE-NLM inferences.

According to localised Kinky inference from [11], we consider a partition of $\mathcal{U}$ by hyperrectangles $\mathcal{U}_{H_1}, \ldots, \mathcal{U}_{H_n}$. $\mathcal{U}_{H_1}, \ldots, \mathcal{U}_{H_n}$ denote the resulting partition of $\mathcal{U}$. Since the partition covers the whole input set, the solution of (4) is obtained by solving

$$\Phi_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D), \mathcal{U}_{H_i}}(G) = \max_{(u, y) \in E(\mathcal{U}_D \times \mathcal{Y}_D), \mathcal{U}_{H_i}, ||u|| \geq \varepsilon} \frac{||y - G(u)||}{||u||}, \tag{5}$$

for each subset $\mathcal{U}_{H_i}, i = 1, \ldots, h$ and taking the maximum over all $\Phi_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D), \mathcal{U}_{H_i}}(G), i = 1, \ldots, h$. Due to the increase of number of constraints in (5) from the envelope $E(\mathcal{U}_D \times \mathcal{Y}_D)$ with number of samples, we define the notion of local inference of the AE-NLM analogously to localised Kinky inference.

**Definition 2.** For each subset $\mathcal{U}_{H_1}, \ldots, \mathcal{U}_{H_h}$, we define the local AE-NLM inference

$$\Phi_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D), \mathcal{U}_{H_i}}(G) = \max_{(u, y) \in E((u_{H_i}, y_{H_i}), (u_{H_i}^\prime, y_{H_i}^\prime)), \mathcal{U}_{H_i}, ||u|| \geq \varepsilon} \frac{||y - G(u)||}{||u||}, \tag{6}$$

where the two samples $(u_{H_i}, y_{H_i}), (u_{H_i}^\prime, y_{H_i}^\prime) \in \mathcal{U}_D \times \mathcal{Y}_D$ are chosen such that the inputs $u_{H_i}$ and $u_{H_i}^\prime$ are the closest samples of $\mathcal{U}_D$ to $\mathcal{U}_{H_i}$.

Note that, the local inference (6) is only a upper bound of the global inference (5) as $E((u_{H_i}, y_{H_i}), (u_{H_i}^\prime, y_{H_i}^\prime)) \not\subseteq E(\mathcal{U}_D \times \mathcal{Y}_D)$. However, the number of constraints in (6), is reduced significant compared to (3) and regardless on the number of samples in $\mathcal{U}_D$. For each subset $\mathcal{U}_{H_i}$, the consideration of the two closest data samples in (3) is reasonable as these samples mostly generate active constraints in (5). This choice of data samples is also motivated by the case $n = 1$, where this choice implies equivalence of the global inference (5) and the local inference (6).

In the following theorem, we present a nonconvex relaxation based on geometrical arguments to reduce the complexity of calculating the local inference (6).

**Theorem 3.** Let two input-output samples $\mathcal{U}_D \times \mathcal{Y}_D = \{(u_1, y_1), (u_2, y_2)\}$ be given. Then, the local inference of the AE-NLM (5) is bounded from above by

$$\max\{\alpha_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G), \beta_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G), \gamma_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G)\} \tag{7}$$

with

$$\alpha_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G) = \max_{u \in \mathcal{U}_{H_i}, ||u|| \geq \varepsilon} \frac{||M(u) - G(u)|| + d(u)}{||u||},$$

$$\beta_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G) = \max_{u \in \mathcal{U}_{H_i}, ||u|| \geq \varepsilon} \frac{||y_1 - G(u)|| + r_1(u)}{||u||},$$

$$\gamma_{AE}^{E(\mathcal{U}_D \times \mathcal{Y}_D)}(G) = \max_{u \in \mathcal{U}_{H_i}, ||u|| \geq \varepsilon} \frac{||y_2 - G(u)|| + r_2(u)}{||u||},$$

where $M(u)$ is a Lipschitz function from $[9]$.
and the geometric variables
\[ r = ||y_1 - y_2||, r_1(u) = L||u - u_1||, r_2(u) = L||u - u_2||, \]
\[ d(u) = \frac{1}{2r} \sqrt{(r_2^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - r_2^2)}, \]
\[ M(u) = y_2 + \sqrt{L^2||u - u_2||^2 - d(u)^2}, \]
\[ G(u) \quad (\gamma \in \Upsilon_D). \]

Proof. Let \( \gamma \in \Upsilon = \{\gamma \in \Upsilon_D : \text{G(u)} \} \) for an input \( u \in \mathcal{U} \) on \( \gamma \), i.e., the set of possible outputs for input \( u \in \mathcal{U} \) included in \( \gamma \). Since \( \gamma \) contains two data samples, \( \gamma \) corresponds to the intersection of two \( n \) \(-\) dimensional spheres with center \( y_1 \) and \( y_2 \), respectively, and radius \( r_1(u) \) and \( r_2(u) \), respectively. Due to the Lipschitz continuity of \( \gamma \), \( \gamma \) is non-empty and contains a \( n \) \(-\) dimensional sphere with diameter \( 2d(u) \) and center \( M(u) \). To calculate the upper bound \( \gamma \), we bound the distance of \( G(u) \) to all outputs in \( \gamma \). We distinguish between two cases depending on the location of the center \( M(u) \).

In the first case, the center \( M(u) \) lies between \( y_1 \) and \( y_2 \), as depicted in Figure 3. Note that, the distance of \( M(u) \) to any point in \( \gamma \) is less than or equal to the half of the diameter \( 2d(u) \). Hereby, the triangle inequality yields
\[ \max_{y \in \gamma} ||y - G(u)|| \leq ||M(u) - G(u)|| + d(u), \]
which corresponds to \( \beta_{AE}(\gamma_D, G). \)

The second case is depicted in Figure 4. where the center \( M(u) \) does not lie between \( y_1 \) and \( y_2 \) and, potentially, one sphere is completely included in the other. This case is characterized by \( r_2(u)^2 = \gamma r_1(u)^2 + r_2(u)^2 \) if \( y_1 \) lies in \( \gamma \) (by \( r_1(u)^2 \geq r_2(u)^2 + r_2(u)^2 \) if \( y_2 \) lies in \( \gamma \)), as follows from the orange triangle in Figure 4. Since \( y_1 \) \( y_2 \) lies in \( \gamma \),
\[ \max_{y \in \gamma} ||y - G(u)|| \leq ||y_1 - G(u)|| + ||y_2 - G(u)|| + d(u), \]

Based on the triangle inequality. This results in \( \beta_{AE}(\gamma_D, G) \).

Though the relaxation from Theorem 3 is nonconvex, the complexity of its optimization problems is significant lower compared to \( \gamma \), as the optimization over \( y \in \mathbb{R}^n \) is avoided. Note that, the relaxation \( \gamma \) requires the optimization of \( \mu \) variables because the input set \( \mathcal{U} \) is spanned by a \( \mu \) \(-\) dimensional orthonormal basis (1).

So far the output trajectories \( \gamma_D \) are assumed to be measured without noise. However, we can adapt the envelope \( E(\mathcal{U}_D \times \gamma_D) \) and the presented relaxation to provide a guaranteed upper bound on the AE-NLM for noisy measurements, as shown in the next remark.

Remark 4. If the measured output \( \tilde{y} \) of the system \( N(u) \) is corrupted by additive and bounded noise \( v \), i.e.,
\[ \tilde{y} = N(u) + v, \text{ } v^Tv \leq \delta^2, \]
then the Lipschitz continuity implies
\[ ||N(u') - \tilde{y}|| \leq L||u' - u|| + \delta. \]

Hence, we increase the radii \( r_1(u) \) and \( r_2(u) \) in Theorem 3 by \( \delta \) to ensure a guaranteed upper bound on the AE-NLM. Similar, if the noise has a signal-to-noise ration of \( \delta \), i.e., \( v^Tv \leq \delta^2 y^Tv \), then the Lipschitz continuity and the assumption \( N(0) = 0 \) imply
\[ ||v||^2 \leq \delta^2 ||N(u)||^2 \leq \delta^2 L^2 ||u||^2 \leq \delta^2 L^2 ||u_H||^2, \]
where \( u_H \) denotes the largest input of the considered subset of the partition with respect to the Euclidean norm. Thereby, we increase the radii \( r_1(u) \) and \( r_2(u) \) by \( \delta L ||u_H|| \).

B. Iterative scheme for AE-NLM inference

In the previous section, local inferences and a nonconvex relaxation are studied to derive the (global) inference of the AE-NLM for given data samples. To improve the guaranteed upper bound, further experiments can be evaluated iteratively on the plant. We establish in the following such an iterative sampling procedure.

Algorithm 5 (Iterative scheme for AE-NLM inference)

1) Suppose a set of input-output samples are given. Initially, compute the linear approximation model \( G \), that minimizes the maximal distance to the data samples, according to the semidefinite-program in (3). Moreover, define a
partition of \( \overline{U} \) and \( U \), respectively, and compute the local AE-NLM inference \( \Phi_{AE}^{(0)} \) and its maximizing ‘worst-case’ input for all hyperrectangles \( \overline{U}_{H_i}, i = 1, \ldots, h \). Set the number of iterations \( k \) to zero.

2) Identify the hyperrectangle \( \overline{U}_{H^*} \) with the largest local AE-NLM inference \( \Phi_{AE}^{(k)} \) and add this to the sequence \( \Phi_{AE}^{(0)}, \Phi_{AE}^{(1)}, \ldots, \Phi_{AE}^{(k)} := \Phi_{AE}^{(k)} \). Moreover, add the corresponding ‘worst-case’ amplitudes \( \overline{u}^*(0), \ldots, \overline{u}^*(k) := \overline{u}^* \) to the sequence \( \overline{u}^*(0), \ldots, \overline{u}^*(k) := \overline{u}^* \).

3) Divide hyperrectangle \( \overline{U}_{H^*} \) into two hyperrectangles \( \overline{U}_{H_1}^* \) and \( \overline{U}_{H_2}^* \) by a \( \mu - 1 \) dimensional hyperplane, which is orthogonal to one dimension and divides the largest edge of \( \overline{U}_{H^*} \). Moreover, the hyperplane contains

3a) \( \overline{u}^* \) if \( 1/\alpha < \text{vol}(\overline{U}_{H_1}^*)/\text{vol}(\overline{U}_{H_2}^*) < \alpha \) for some \( \alpha > 0 \);

3b) else the middle point of \( \overline{U}_{H^*} \).

4) Determine the output of the plant for

4a) \( \overline{u}^* \) in case of 3a);

4b) the middle point of \( \overline{U}_{H^*} \) in case of 3b).

5) Compute the local inference of the AE-NLM for hyperrectangles \( \overline{U}_{H_1}^* \) and \( \overline{U}_{H_2}^* \). Saturate these local AE-NLM inferences by the local AE-NLM inference of \( \overline{U}_{H^*} \).

6) Set \( h := h + 1 \) and \( k := k + 1 \). Go to Step 2).

An illustration of Algorithm 5 is depicted in Figure 5. In the sequel, we comment on Algorithm 5 and on some properties of Algorithm 5 more thoroughly.

In step 3), we suggest one, among others, proceedings of dividing the hyperrectangle \( \overline{U}_{H^*} \). In particular, the decision between 3a) and 3b) is required to prove convergence of the sequence \( \Phi_{AE}^{(0)}, \Phi_{AE}^{(1)}, \ldots, \) to the true AE-NLM in Theorem 5. Furthermore, the new evaluated input from step 4) is taken into account in the computation of the local AE-NLM inferences in step 5), since its distance to the new hyperrectangles is zero.

First property of Algorithm 5 is that its complexity does not increase with further iterations, as the local AE-NLM inference \( \Phi_{AE}^{(0)} \) of two hyperrectangles is computed in each iteration. Hence, stopping the iteration after some iterations due to increased computation time won’t occur, once the algorithm can be initialized. Second, the sequence of global inferences \( \Phi_{AE}^{(k)} \) from step 2) is monotone decreasing, as the local AE-NLM inferences of the hyperrectangles \( \overline{U}_{H_1}^* \) and \( \overline{U}_{H_2}^* \) are saturated by the local inference of \( \overline{U}_{H^*} \). Otherwise, the sequence could increase as the envelopes considered for \( \overline{U}_{H_1}^* \) and \( \overline{U}_{H_2}^* \), respectively, are not necessarily a subset of the envelope considered for \( \overline{U}_{H^*} \). Third, the sequence of global inferences \( \Phi_{AE}^{(k)} \) converges to the true AE-NLM as proven in the following theorem.

**Theorem 6.** The sequence of global inferences of the AE-NLM \( \Phi_{AE}^{(k)} \) from step 2) of Algorithm 5 converges to the solution of the left-hand side of (5), \( \Phi_{AE}^{(k)} \), i.e.,

\[
\lim_{k \to \infty} \Phi_{AE}^{(k)}(u) = \Phi_{AE}^{(k)}(u).
\]

**Proof.** The sequence \( \Phi_{AE}^{(k)}(\cdot) \) is lower bounded by zero and non-increasing which implies its convergence. We suppose the sequence \( \Phi_{AE}^{(k)}(\cdot) \) does not converge to \( \Phi_{AE}^{(k)}(\cdot) \). Thus, the sequence of corresponding ‘worst-case’ inputs \( u^*(\cdot) \in U \) from step 2) does not converge to the set of inputs \( U^* \subset U \), which solve the left-hand side of (3), but to a subset \( U^* \neq U^* \), which we don’t fixed yet. Due to the convergence to \( U^* \), the distinction of 3a) and 3b), and the sampling in step 4), we can choose \( U^c \) such that the radii \( r_1(u) \) and \( r_2(u) \) from relaxation (7) for each subset \( U_{H_i} \subseteq U_{H^*} \) are bounded by an arbitrary small \( \rho > 0 \), i.e.,

\[
\max\{r_1(u), r_2(u)\} < \rho, \quad \forall u \in U_{H_i},
\]

for all subsets \( U_{H_i} \subseteq U_c \). Therefore, the local inference of the AE-NLM (4) using the relaxation (7) is bounded from above by

\[
\lim_{\rho \to 0} \max_{u \in U_{H_i}, ||u|| \geq \rho} \frac{||y_{i} - G(u)|| + \rho}{||u||} \leq \frac{||y - G(u)||}{||u||}, \quad i = 1, 2.
\]

implies that \( \rho \) can be chosen small enough such that (8) is for all \( U_{H_i} \subseteq U_c \) less than

\[
\frac{||N(u^*) - G(u^*)||}{||u^*||} \leq \frac{||y - G(u)||}{||u||},
\]

which leads to a contradiction for the convergence of the sequence \( u^*(\cdot) \in U \) to \( U^* \neq U^* \). Hence, the sequence \( u^*(\cdot) \) converges to \( U^* \), and therefore \( \Phi_{AE}^{(k)}(\cdot) \) to \( \Phi_{AE}^{(k)}(\cdot) \). \( \square \)

**V. NUMERICAL EXAMPLE**

In this section, we apply Algorithm 5 to conclude on the AE-NLM of the input-output behaviour of the SISO system

\[
\begin{align*}
x_1 &= -3x_1 + 4x_2 + 2x_2^3 - 0.2\sin(3x_1) + u, \\
x_2 &= -x_1 + 0.6x_2 - 0.5x_1^3, \\
y &= x_1.
\end{align*}
\]

The discrete-time input-output trajectories are generated based on simulations for 30 time steps by Euler integration with \( \Delta t = 0.2 \text{s} \). Similar to (3), the input set is spanned by

\[
U = \{u \in \mathbb{R}^n : u = \sum_{i=1}^{2} \alpha_i \frac{\bar{v}_i}{||\bar{v}_i||}, (\alpha_1, \alpha_2) \in \{0, 4, 0\} \}.
\]
where $v_1, i = 1, 2$ denote the stacked time-samplings of the basis signals $v_1(t) = \sin(\pi/3t)$ and $v_2(t) = \sin(\pi t)$. The Lipschitz constant is estimated to $L = 1.04$. Algorithm 5 is initialized by the computation of a linear approximation model, described by a lower Toeplitz matrix, for 20 samples.

Figure 6 shows the sequence of NLM-AE inferences $\Phi_{AE}^{U,G}(\cdot)$ of Algorithm 5. If the output is corrupted by additive noise with a signal-to-noise ratio of 10%, then Algorithm 5 has a similar convergence rate as in the noise-free case and still exhibits an upper bound of the AE-NLM as described in Remark 4. Furthermore, Figure 6 shows a lower bound on the AE-NLM derived from the collected data, as shown in [3]. The knowledge of the distance of the guaranteed lower and upper bound on the AE-NLM in each iteration constitutes a reasonable termination criterion for the iteration. Indeed, this distance measures the potential improvement of the AE-NLM estimation by further sampling. Figure 7 demonstrates the division of the input set $U$ into hyperrectangles by Algorithm 5. Based on the observation that the iterative scheme samples small inputs more often, we conclude that the ‘worst-case’ input lies close to zero. Due to the iterative approach, Algorithm 5 is more data-efficient than the offline approach [3] that requires 10000 data samples to conclude an upper bound of 0.48 for the AE-NLM.

VI. CONCLUSIONS

In this paper, we exploited a non-parametric data-based model of the input-output mapping of an unknown nonlinear system to derive a conclusion on its strength of nonlinearity. To conclude on the nonlinearity measure from given input-output samples by an optimization problem including this non-parametric model, we investigated local inference of the nonlinearity measure and proposed a relaxation to reduce the complexity of the corresponding optimization problem. Subsequent, an iterative scheme was presented to decrease the guaranteed upper bound of the AE-NLM by further performed experiments, while the complexity of the algorithm does not increase with iterations and the convergence to the true nonlinearity measure is guaranteed. In a numerical example, the presented algorithm was more data efficient than the approach in [3]. In a future work, other dissipativity properties could be studied, the iterative scheme could be extended by an alternating optimization of the linear approximation model, the estimation of the local inference could be improved by considering local Lipschitz constants.

REFERENCES

[1] Z. S. Hou and Z. Wang. From model-based control to data-driven control: Survey, classification and perspective. Information Sciences, 235, 3-35, 2013.
[2] J. M. Montenbruck and F. Allgöwer. Some Problems Arising in Controller Design from Big Data via Input-Output Methods. In Proc. 55th IEEE Conf. on Decision and Control, pp. 6525-6530, 2016.
[3] T. Martin and F. Allgöwer. Nonlinearity measures for data-driven system analysis and control. In Proc. 58th IEEE Conf. on Decision and Control, pp. 3605-3610, 2019.
[4] F. Allgöwer. Definition and Computation of a Nonlinearity Measure. In Proc. 3rd IFAC Nonlinear Control Syst. Design Symp., Tahoe City, CA, pp. 257-262, 1995.
[5] T. Schweickhardt and F. Allgöwer. On System Gains, Nonlinearity Measures, and Linear Models for Nonlinear Systems. IEEE Trans. Automatic Control, 54(1):62-78, 2009.
[6] B. Wahlberg, M. B. Syberg and H. Hjalmarsson. Non-parametric methods for $L_2$-gain estimation using iterative experiments. Automatica, 46(8):1376-1381, 2010.
[7] T. Martin, A. Koch and F. Allgöwer. Data-driven surrogate models for LTI systems via saddle-point dynamics. In Proc. 21st IFAC World Congress, 2020.
[8] C. E. Rasmussen and C. K. I. Williams. Gaussian Processes for Machine Learning. The MIT Press, 2006.
[9] M. Milanese and C. Novara. Set Membership identification of nonlinear systems. Automatica, 40(6):957975, 2004.
[10] J. P. Calliess. Conservative decision-making and inference in uncertain dynamical systems. University of Oxford, 2014.
[11] A. BlaaS, J. M. Manzano, D. Limon and J. Calliess. Localised Kinky Inference. In Proc. 18th European Control Conference (ECC), pp. 983992, 2019.
[12] G. Zames. On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems. Part I: Conditions Derived Using Concepts of Loop Gain, Conicity, and Positivity. IEEE Trans. Automat. Control, 11(2):228-238, 1966.
[13] R. G. Strongin. On the convergence of an algorithm for finding a global extremum. Engineering in Cybernetics, 1973.
[14] J. P. Calliess. Lipschitz optimisation for Lipschitz interpolation. In Proc. American Control Conference (ACC), pp. 31413146, 2017.
[15] S. H. Nair, M. Bujarbaruah and F. Borrelli. Modeling of Dynamical Systems via Successive Graph Approximations. arXiv preprint, arXiv:1910.03719.
[16] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. Foundations and Trends in Machine Learning, 3(1):1122, 2010.