COMBINATORICS OF DISCRIMINANTAL ARRANGEMENTS

SIMONA SETTEPANELLA AND SO YAMAGATA

Abstract. In 1985 Crapo introduced in [4] a new mathematical object that he called geometry of circuits. Four years later, in 1989, Manin and Schechtman defined in [14] the same object and called it discriminantal arrangement, the name by which it is known now a days. Subsequently in 1997 Bayer and Brandt (see [3]) distinguished two different type of those arrangements calling very generic the ones having intersection lattice of maximum cardinality and non very generic the others. Results on the combinatorics of very generic arrangements already appear in Crapo [4] and in 1997 in Athanasiadis [2] while the first known result on non very generic case is due to Libgober and the first author in 2018. In their paper [13], they studied the combinatorics of non very generic case in rank 2. In this paper we further develop their result providing a sufficient condition for the discriminantal arrangement to be non very generic which holds in rank \( r \geq 2 \).

1. Introduction

In 1989, Manin and Schechtman ([14]) introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the discriminantal arrangements (p.209 [14]). Such an arrangement \( \mathcal{B}(n,k,\mathcal{A}) \), \( n,k \in \mathbb{N} \) for \( k \geq 2 \) depends on a choice \( \mathcal{A}^0 = \{H_0^0, \ldots, H_n^0\} \) of a collection of hyperplanes in general position in \( \mathbb{C}^n \), i.e., such that \( \dim \bigcap_{i \in K \subseteq k} H_i^0 = 0 \). It consists of parallel translate of \( H_0^0, \ldots, H_n^0, (t_1, \ldots, t_n) \in \mathbb{C}^n \) which fail to form a general position arrangement in \( \mathbb{C}^k \). \( \mathcal{B}(n,k,\mathcal{A}) \) can be viewed as a generalization of the braid arrangement ([16]) with which \( \mathcal{B}(n,1,\mathcal{A}) \) coincides.

These arrangements have several beautiful relations with diverse problems such as the Zamolodchikov equation with its relation to higher category theory (see Kapranov-Voevodsky [10], see also [8],[9]), the vanishing of cohomology of bundles on toric varieties ([17]), the representations of higher braid groups (see [11]) and, naturally, with combinatorics. The latter is the connection we are mainly interested in and it goes from matroids to special configurations of points, from fiber polytopes to higher Bruhat orders.

Manin and Schechtman introduced discriminantal arrangements as higher braid arrangements in order to introduce higher Bruhat orders which model the set of minimal path through a discriminantal arrangement. Even if Ziegler showed (see Theorem 4.1 in [20]) in 1991 that we have to choose a cyclic arrangement instead of discriminantal arrangement for this, few years later, in a subsequent work (see [7]) Felsner and Ziegler reintroduced the combinatorics of discriminantal arrangement in the study of higher Bruhat orders (this connection uses fiber polytopes as observed by Falk in [6]). From a different perspective, unknown in the literature of discriminantal arrangement until Athanasiadis pointed it out in 1999 (see [2]), Crapo introduced for the first time in 1985 (see [4]) what he called geometry of circuits and which is the matroid \( M(n,k,C) \) of circuits of the configuration \( C \) of \( n \) generic points in \( \mathbb{R}^k \).

The circuits of the matroid \( M(n,k,C) \) are the hyperplanes of \( \mathcal{B}(n,k,\mathcal{A}) \), \( \mathcal{A}^0 \) arrangement of \( n \) hyperplanes in \( \mathbb{R}^k \) orthogonal to the vectors joining the origin with the \( n \) points in \( C \) (for further development see [5]).

Both Manin-Schechtman ([14]) and Crapo ([4]) were mainly interested in arrangements \( \mathcal{B}(n,k,\mathcal{A}) \) for which the intersection lattice is constant when \( k \) varies within a Zariski open set \( Z \) in the space of general position arrangements. Crapo shows that, in this case, the matroid \( M(n,k) \) is isomorphic to the Dilworth completion of the \( k \)-th lower truncation of the Boolean algebra of rank \( n \). More recently in [2], Athanasiadis proved a conjecture by Bayer and Brandt (see [3]) providing a full description of combinatorics of \( \mathcal{B}(n,k,\mathcal{A}) \) when \( \mathcal{A}^0 \) belongs to \( Z \). Following [2] (more precisely Bayer and Brandt), we call arrangements \( \mathcal{A}^0 \) in \( Z \) very generic, non very generic otherwise.

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However Manin and Schechtman do not describe the set $\mathcal{Z}$ of very generic arrangements explicitly, which, in time, led to the misunderstanding that the combinatorial type of $\mathcal{B}(n, k, \mathcal{A}_0)$ was independent from the arrangement $\mathcal{A}_0$ (see for instance, [15], sect. 8, [16] or [12]). Neither Crapo in [4] provided a description of $\mathcal{Z}$ even if he presented the first known example of a non very generic arrangement: 6 lines in generic position in $\mathbb{R}^2$ which admit translated that are respectively sides and diagonals of a quadrilateral as in Figure 1 (Crapo calls it a quadrilateral set). Few years later in 1994, Falk provided an higher dimensional example of non very generic arrangement of 6 planes in $\mathbb{R}^3$ (see [6]). Similar to Crapo’s example, Falk’s example too turned out to be related to a special configuration of lines, this time in projective plane (see [18], [19]).

![Figure 1. Central generic arrangement of 6 lines in $\mathbb{R}^2$, its generic translation on the left and its non (very) generic translation on the right.](image)

In 2018 the first general result on non very generic arrangements is provided. In [13] Libgober and the first author described a sufficient geometric condition on the arrangement $\mathcal{A}_0$ to be non very generic. This condition ensures that $\mathcal{B}(n, k, \mathcal{A}_0)$ admits codimension 2 strata of multiplicity 3 which do not exist in very generic case. It is given in terms of the notion of dependency for the arrangement $\mathcal{A}_\infty$ in $\mathbb{P}^{k-1}$ of hyperplanes $H_{0,1}, \ldots, H_{0,n}$ which are the intersections of projective closures of $H_1^\infty, \ldots, H_n^\infty \in \mathcal{A}_0$ with the hyperplane at infinity. Their main result shows that $\mathcal{B}(n, k, \mathcal{A}_0), k > 1$ admits a codimension two stratum of multiplicity 3 if and only if $\mathcal{A}_\infty$ is an arrangement in $\mathbb{P}^{k-1}$ admitting a restriction which is a dependent arrangement. This construction generalizes Falk’s example which corresponds to the case $n = 6, k = 3$ and which has been object of study in two subsequent papers, ([18], [19]) by Sawada and the first and second authors. In those papers the authors proved how the arrangement $\mathcal{A}_0$ of 6 planes in $\mathbb{R}^3$ (resp. $\mathbb{C}^3$) for which the rank 2

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1Here restriction is the standard restriction of arrangements to subspaces as defined in [16].
intersections of $B(6,2,\mathcal{A})$ are in minimal number corresponds to Pappus’s ( resp. Hesse’s ) configuration providing a main example of what conjectured by Crapo that the intersection lattice of discriminantal arrangement represents a combinatorial way to encode special configurations of points in the space. Notice that in [13] the authors connected the non very generic arrangements $\mathcal{A}$ of $n$ planes in $\mathbb{C}^3$ to well defined hypersurfaces in Grassmannian $Gr(3,n)$. In this paper we advance the study of non very generic arrangements and generalize the dependency condition given in [13] providing a sufficient condition for the existence in rank $r \geq 2$ of non very generic intersections, i.e. intersections which doesn’t exist in $B(n,k,\mathcal{A})$, $\mathcal{A} \subseteq \mathbb{Z}$. In particular we call simple an intersection of $r$ hyperplanes in $B(n,k,\mathcal{A})$ which satisfies the property that if the arrangement $\mathcal{A}$ is very generic then all simple intersections of multiplicity $r$ have rank $r$ (that is they are $r$ hyperplanes intersecting transversally). Then we provide both geometric and algebraic necessary and sufficient conditions for existence of simple intersections of multiplicity $r$ in rank strictly lower than $r$, i.e. simple non very generic intersections. This result firstly connect configurations of non very generic points to special families of graphs (called $K_T$-configurations) which help to understand $B(n,k,\mathcal{A})$ for $\mathcal{A} \notin \mathbb{Z}$ (as conjectured by Crapo in [4]). Secondly it reduces the geometric problem of the existence of special (non very generic) configurations of points to a combinatorial problem on the numerical properties that $r$ subsets of indices $L_i \subseteq \{1,\ldots,n\}, i=1,\ldots,r$ of cardinality $k+1$ have to satisfy in order for the $K_T$-configuration, $T=\{L_1,\ldots,L_r\}$, to give rise to a simple non very generic intersection. The latter problem is left open together with the problem of necessary and sufficient conditions for existence of intersections in $B(n,k,\mathcal{A})$ which are nor simple nor very generic. The content of the paper is the following. In Section [2] we recall the definition of discriminantal arrangement and basic properties of the intersection lattice of discriminantal arrangement in very generic case. We also give the definition of simple intersection. In Section [3] we introduce the notion of $K_T$-translated and $K_T$-configuration associated to a generic arrangement $\mathcal{A}$ providing a geometric condition for $\mathcal{A}$ to be non very generic (Theorem 3.9). Finally we define the $K_T$-vector condition. In Section [4] we prove that the existence of a finite number of sets of vectors which satisfy the $K_T$-vector condition is sufficient condition for $\mathcal{A}$ to be non very generic (Theorem 4.5). In the last section we provide examples of non very generic arrangements obtained by imposing the condition stated in Theorem [4.5].

2. Preliminaries

2.1. Discriminantal arrangement. Let $H^0_i, i=1,\ldots,n$ be a central arrangement in $\mathbb{C}^k, k < n$ which is generic[2] i.e. any $m$ hyperplanes intersect in codimension $m$ at any point except for the origin for any $m \leq k$. We will call such an arrangement a central generic arrangement. Space of parallel translates $\mathbb{S}(H^0_1,\ldots,H^0_n)$ (or simply $\mathbb{S}$ when dependence on $H^0_n$ is clear or not essential) is the space of $n$-tuples of translates $H_1,\ldots,H_n$ such that either $H_i \cap H^0_i = \emptyset$ or $H_i = H^0_i$ for any $i=1,\ldots,n$.

One can identify $\mathbb{S}$ with $n$-dimensional affine space $\mathbb{C}^n$ in such a way that $(H^0_1,\ldots,H^0_n)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$ (see [13]). Given a central generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i=1,\ldots,n$ the trace at infinity, denoted by $\mathcal{A}_\infty$, is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}^0_i \cap H_\infty$ in the space $H_\infty \approx \mathbb{P}^{k-1}(\mathbb{C})$, where $\bar{H}^0_i$ are projective closures of affine hyperplanes $H^0_i$ in the compactification $\mathbb{P}^k(\mathbb{C})$ of $\mathbb{C}^k \approx \mathbb{P}^k(\mathbb{C}) \backslash H_\infty$. Notice that condition of genericity is equivalent to $\bigcup_i H^0_{\infty,i}$ being a normal crossing divisor in $\mathbb{P}^{k-1}(\mathbb{C})$, i.e. $\mathcal{A}_\infty$ is a generic arrangement.

The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of $n$-tuples of hyperplanes in $\mathbb{P}^k$). Fixed a generic central arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This subset of $\mathbb{S}$ is a union of hyperplanes $D_L \subset \mathbb{S}$ (see [14]). Each hyperplane $D_L$ corresponds to a subset $L = \{l_1,\ldots,l_k+1\} \subset [n] := \{1,\ldots,n\}$ and it consists of $n$-tuples of translates of hyperplanes $H^0_{l_1},\ldots,H^0_{l_k}$ in which translates of $H^0_{l_k+1}$ fail to form a generic arrangement. The arrangement $B(n,k,\mathcal{A})$ of hyperplanes $D_L$ is called discriminantal arrangement and has been introduced by Manin and Schechtman in [14][3]. Notice that $B(n,k,\mathcal{A})$ depends on the trace at infinity $\mathcal{A}_\infty$ hence it is sometimes more properly denoted by $B(n,k,\mathcal{A}_\infty)$.

Notice that, in general, generic, referred to an arrangement of hyperplanes, has a slightly different meaning. With an abuse of notation, we use the word generic in this case since the defined property is equivalent to the existence of a translated of the given central arrangement which is generic in the classical sense.

Notice that Manin and Schechtman defined the discriminantal arrangement starting from a generic arrangement instead of its central translated as we do in this paper. For our purpose the latter is a more convenient choice.
2.2. Very generic and non very generic discriminantal arrangements. It is well known (see, among others [4], [14]) that there exists an open Zarisky set \( \mathcal{Z} \) in the space of (central) generic arrangements of \( n \) hyperplanes in \( \mathbb{C}^k \), such that the intersection lattice of the discriminantal arrangement \( \mathcal{B}(n, k, \mathcal{A}) \) is independent from the choice of the arrangement \( \mathcal{A} \in \mathcal{Z} \). Bayer and Brandt in [3] call the arrangements \( \mathcal{A} \in \mathcal{Z} \) very generic and the ones which are not in \( \mathcal{Z} \), non very generic.

We will use their terminology in the rest of this paper. The name very generic comes from the fact that in this case the cardinality of the intersection lattice of \( \mathcal{B}(n, k, \mathcal{A}) \) is the largest possible for any (central) generic arrangement \( \mathcal{A} \) of \( n \) hyperplanes in \( \mathbb{C}^k \).

In [4] Crapo proved that the intersection lattice of \( \mathcal{B}(n, k, \mathcal{A}), \mathcal{A} \in \mathcal{Z} \) is isomorphic to the Dilworth completion of the k-times lower-truncated Boolean algebra \( \mathcal{B}_n \) (see Theorem 2. page 149). A more precise description of this lattice is due to Athanasiadis who proved in [2] a conjecture by Bayer and Brandt which stated that the intersection lattice of a central generic arrangement in very generic case is isomorphic to the collection of all sets \( \{S_1, \ldots, S_m\}, S_i \subset [n] = \{1, \ldots, n\}, |S_i| \geq k + 1 \), such that

\[
| \bigcup_{i \in I} S_i | > k + \sum_{i \in I} (| S_i | - k) \quad \text{for all } I \subset [m] = \{1, \ldots, m\}, |I| \geq 2 .
\]

The isomorphism is the natural one which associate to the set \( S_i \) the space \( D_{S_i} = \bigcap_{L \subset S_i} D_L \), \( D_L \in \mathcal{B}(n, k, \mathcal{A}) \) of all translated of \( \mathcal{A} \) having hyperplanes indexed in \( S_i \) intersecting in a not empty space. In particular \( \{S_1, \ldots, S_m\} \) will correspond to the intersection \( \bigcap_{i=1}^m D_{S_i} \).

If \( \mathcal{A} \) is very generic and the condition in equation (1) is satisfied, this implies that the subspaces \( D_{S_i}, i = 1, \ldots, m \) intersect transversally (Corollary 3.6 in [2]) or, equivalently, since rank \( D_{S_i} = |S_i| - k \), that

\[
\text{rank} \bigcap_{i=1}^m D_{S_i} = \sum_{i=1}^m (| S_i | - k)
\]

Notice that the condition in equation (1) is satisfied (see also (2)) if

\[
\bigcap_{i \in I} D_{S_i} \neq \bigcap_{S \supset I} \bigcup_{i \in I} D_{S_i} \quad \text{for any } I \subset [r] = \{1, \ldots, r\}, |I| \geq 2 .
\]

The fact that for all sets \( \{S_1, \ldots, S_m\} \) which satisfy condition in equation (3) the condition in equation (2) is also satisfied corresponds to the definition provided by Crapo in [4] for discriminantal arrangement in very generic case (which he called geometry of circuits). From those considerations we can get the following remark.

**Remark 2.1.** Let \( \mathcal{A} \) be a translated of a central generic arrangement \( \mathcal{A} \), such that the hyperplanes in \( \mathcal{A} \) indexed in subsets \( L_i \subset [n] \), \( |L_i| = k + 1, i = 1, \ldots, r \) intersect at the points \( P_i = \bigcap_{L \not\in L_i} H_p \neq \emptyset \) and satisfy \( \bigcap_{i \in L \not\in [t]} H_p = \emptyset \) for any \( t \not\in L_i \). Then \( \mathcal{A} \) is an element in the intersection \( \bigcap_{i=1}^r D_{L_i} \) of \( \mathcal{A} \) for any \( S_i \supset L_i \), i.e. \( \bigcap_{i \in I} D_{L_i} \neq \emptyset \) for any \( I \subset [r] = \{1, \ldots, r\}, |I| \geq 2 \). In particular if \( \mathcal{A} \) is very generic then, by equation (2), \( D_{L_i} \) are \( r \) hyperplanes intersecting transversally, i.e. rank \( \bigcap_{i=1}^r D_{L_i} = r \).

Contrary to the very generic case, very few is known about the non very generic case. In recent papers the first author (see [13]) and the first and second authors (see [18], [19]) showed that non very generic arrangements are arrangements which hyperplanes give rise to special configurations (e.g. Pappus's configuration or Hesse configuration). Following this direction, in the rest of the paper we further develop the result in [13] providing a geometric and an algebraic condition for a central generic arrangement \( \mathcal{A} \) to be non very generic. In order to do that we will use Remark 2.1 which essentially states that if \( \mathcal{A} \) is a central generic arrangement of \( n \) hyperplanes in \( \mathbb{C}^k \) for which there exist \( r \) subsets \( L_i \subset [n], |L_i| = k + 1, i = 1, \ldots, r \) such that

1. for any translated \( \mathcal{A}' \in \bigcap_{i=1}^r D_{L_i} \) of \( \mathcal{A} \) the hyperplanes in \( \mathcal{A}' \) satisfy the additional condition \( \bigcap_{p \in L \not\in [t]} H_p = \emptyset \) for any \( t \not\in L_i \),
2. \( \text{rank} \bigcap_{i=1}^r D_{L_i} < r \)

then \( \mathcal{A} \) is non very generic. The following definition will be very useful in the rest of the paper.

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4Here Crapo followed the preference of his advisor Rota who rarely used the name matroid.
Definition 2.2. An element $X$ in the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is said to be simple intersection if $X = \bigcap_{i=1}^{r} D_{L_i}$, $|L_i| = k + 1$ and $\bigcap_{i\in I} D_{L_i} \neq D_{S_i}$, $|S| > k + 1$ for any $I \subset [r]$, $|I| \geq 2$. We call multiplicity of the simple intersection $X$ the number $r$ of hyperplanes intersecting in $X$.

By the above considerations we can conclude that if the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ contains a simple intersection of rank strictly less than its multiplicity, then $\mathcal{A}$ is non very generic. This will play an important role in the rest of the paper since we will focus on a necessary and sufficient condition for the existence of such a simple intersection.

2.3. Motivating examples. In this subsection we provide the two main examples given by Crapo (see [4]) and Falk (see [5] and [6]) of simple but non very generic intersections. Those two examples inspired the rest of the content of this paper.

Crapo example is illustrated in Figure 1. In this case the arrangement $\mathcal{A}$ on the right of Figure 1 is an element in the simple intersection $X = \bigcap_{i=1}^{4} D_{L_i}$ with $L_1 = \{1, 2, 3\}$, $L_2 = \{1, 4, 5\}$, $L_3 = \{2, 4, 6\}$, $L_4 = \{3, 5, 6\}$. On the other hand, the only rank 4 intersection of $\mathcal{B}(6, 2, \mathcal{A})$ is given by $D_{[6]}$, the space of all central translated of $\mathcal{A}$. Since $\mathcal{A}$ is not central, this implies that $X \neq D_{[6]}$ and hence rank $X < 4$, that is $X$ is a simple intersection of multiplicity 4 and rank $3 < 4$. That is the arrangement in Figure 1 is non very generic.

Falk example is illustrated in Figure 2. Let $\mathcal{B}(6, 3, \mathcal{A})$ be the discriminantal arrangement associated to a generic arrangement $\mathcal{A}$ of 6 hyperplanes $H_i$ in $\mathbb{R}^3$ which satisfy the condition that $H_i \cap H_{i+1} \cap H_\infty$, $i = 1, 3, 5$ span a line at infinity (see Figure 2). In [13] the authors proved that such an arrangement $\mathcal{A}$ admits a translation $\mathcal{A}'$ which belongs to the simple intersection $X = \bigcap_{i=1}^{3} D_{L_i}$ with $L_1 = \{1, 2, 3, 4\}$, $L_2 = \{1, 2, 5, 6\}$, $L_3 = \{3, 4, 5, 6\}$, that is, in particular, $\mathcal{A}'$ is not a central arrangement. On the other hand the only element in rank 3 in $\mathcal{B}(6, 3, \mathcal{A})$ is $D_{[6]}$ the space of all central translated of $\mathcal{A}$ hence rank $X < 3$, that is $X$ is a simple intersection of multiplicity 3 and rank $2 < 3$, i.e. $\mathcal{A}$ is non very generic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Figure of non very generic arrangement with 6 hyperplanes in $\mathbb{R}^3$.}
\end{figure}
3. A geometric condition for non very genericity

In this section we provide a necessary and sufficient condition on a central generic arrangement \( \mathcal{A}^0 \) in \( \mathbb{C}^k \) for the existence of a simple intersection \( X \) of multiplicity \( r \) and rank \( X < r \) in the intersection lattice of the discriminantal arrangement \( \mathcal{B}(n,k,\mathcal{A}^0) \). As noticed in previous section, this is a sufficient condition for \( \mathcal{A}^0 \) to be non very generic.

**Notation 3.1.** To begin with let us fix some notations we will use throughout the paper.

- \( \mathcal{A}^0 \) is a central generic arrangement of \( n \) hyperplanes in \( \mathbb{C}^k \)
- For each subset \( L \) of \( \{1, \ldots, n\} \) with \( |L| = k + 1 \), \( D_L \subset \mathbb{C}^k \) will denote the hyperplane in \( \mathcal{B}(n,k,\mathcal{A}^0) \) corresponding to the subset \( L \).
- Fixed a set \( \mathcal{T} = \{L_1, \ldots, L_r\} \) of subsets \( L_i \subset [n] \), \( |L_i| = k + 1 \), for any arrangement \( \mathcal{A} = \{H_1, \ldots, H_n\} \) translated of \( \mathcal{A}^0 \) we will denote by \( P_i = \bigcap_{p \in L_i} H_p \) and \( H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \). Notice that \( P_i \) is a point if and only if \( \mathcal{A} \in D_L \), it is empty otherwise.

3.1. \( K_T \)-translated and \( K_T \)-configurations. Let \( \mathcal{A}^0 = \{H^0_1, \ldots, H^0_n\} \) be a central generic arrangement in \( \mathbb{C}^k \), \( \mathcal{T} = \{L_1, \ldots, L_r\} \) fixed as in Notation 3.1 and such that the conditions

\[
\bigcup_{i=1}^r L_i = \bigcup_{i \in [r], j = r-1} L_i \quad \text{and} \quad L_i \cap L_j \neq \emptyset
\]

are satisfied for any subset \( I \subset [r], |I| = r - 1 \) and any two indices \( 1 \leq i < j \leq r \). In the rest of the paper a set \( \mathcal{T} \) which satisfies those properties will be called an \( r \)-set.

A translated \( \mathcal{A} = \{H_1, \ldots, H_n\} \) of \( \mathcal{A}^0 \) will be called \( K_T \) or \( K_T \)-translated if \( \bigcap_{p \in L_i} H_p \neq \emptyset \) and \( \bigcap_{p \in L_i \cap L_j} H_p = \emptyset \) for any \( i \neq L_i \) and \( t \in L_i \). The complete graph (as depicted in Figure 3) having the points \( P_i = \bigcap_{p \in L_i} H_p \) as vertices and the vectors \( P_i P_j \) joining \( P_i \) and \( P_j \) as edges will be called \( K_T \)-configuration and denoted by \( K_T(\mathcal{A}) \) (examples of graphs \( K_T(\mathcal{A}) \) for \( |\mathcal{T}| = 3, 4, 5 \) are represented in Figure 4). Notice that \( P_i P_j \in H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \neq \emptyset \) for any \( 1 \leq i < j \leq r \).

\( \mathcal{A} \) will be called almost- \( K_T \) if it is \( K_T \) but for one hyperplane \( H^0_i \) and a set \( S_i \), i.e. if there exists a hyperplane \( H_i \in \mathcal{A}, i \in \bigcup_{i=1}^r L_i \setminus \bigcap_{i=1}^r L_i \), and \( S_i \subset \{L_i \in \mathcal{T} \mid i \in L_i\} \) such that \( \bigcap_{p \in L_i} H_p = \emptyset \) for any \( L_i \in S_i \) and \( \bigcap_{p \in L_i} H_p \neq \emptyset \) for any \( L_j \in \mathcal{T} \setminus S_i \). Notice that since \( \mathcal{A}^0 \) is a central generic arrangement and \( |L_i| = k + 1 \), then \( \bigcap_{p \in L_i \cap L_j} H_p \neq \emptyset \) for any \( L_i \subset S_i \). Moreover by condition in equation (4) if the set \( \{L_i \in \mathcal{T} \mid i \in L_i\} \) is not empty then its cardinality is \( |\{L_i \in \mathcal{T} \mid i \in L_i\}| \geq 2 \). If we keep the notation \( P_i = \bigcap_{p \in L_i \setminus L_i} H_p, L_i \subset S_i, P_i = \bigcap_{p \in L_i} H_p, L_i \notin S_i \), the complete graph having \( P_i \) as vertices and \( P_i P_j \) as edges will be called almost \( K_T \)-configuration and denoted by \( K_{T,S_i}(\mathcal{A}) \).

![Figure 3. K_T-configuration for |\mathcal{T}| = r](image-url)
Remark 3.2. Notice that as soon as \( L_i \cap L_j \neq \emptyset \) then \( H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \neq \emptyset \) for any translated \( \mathcal{A} \) of \( \mathcal{A}' \). Indeed \( H_{i,j} \) is an affine space having as underlying vector space \( H_{i,j}^0 = \bigcap_{p \in L_i \cap L_j} H_p^0 \) of codimension the cardinality \( | L_i \cap L_j | \) of \( L_i \cap L_j \). In particular for any translated \( \mathcal{A} \) such that \( \mathcal{P}_1 = \bigcap_{p \in L_i} H_p \neq \emptyset \) there is one and only one vector \( v_{i,j} \in H_{i,j}^0 \) such that \( v_{i,j} \) applied to the point \( P_i \) has exactly \( P_j \) as ending point. This is very relevant fact as this essentially provide, in non very generic case, what Gale called affine dependency.

Remark 3.3. Similarly to the \( K_{\mathcal{T}} \)-configuration we could define the \( \Delta_{\mathcal{T}} \)-configuration as the simplicial complex having as \( t \)-face \( p_1 \ldots p_{s+1} \in \bigcap_{l=1}^{s+1} L_l \cap \mathcal{T} \). Notice that in general, the intersection \( \bigcap_{l=1}^{s+1} L_l \cap \mathcal{T} \) can be empty, that is \( \Delta_{\mathcal{T}} \) is not a simplex. As pointed out by Crapo in [4], this simplicial complex may play a fundamental role in the study of non very generic arrangements.

With the notations introduced above, we provide the following main definition.

Definition 3.4. A central generic arrangement \( \mathcal{A}' \) of \( n \) hyperplanes in \( \mathbb{C}^k \) is called \((r,s)\)-dependent if there exist an \( r \)-set \( \mathcal{T} = \{L_1, \ldots, L_r \} \), an index \( l \in \bigcup_{i=1}^{r} L_i \setminus \bigcap_{i=1}^{s} L_i \) and a subset \( S \subseteq \{L_i \mid l \in L_i \} \), \( |S| = s \), such that any almost \( K_{\mathcal{T}} \)-configuration \( K_{\mathcal{T}}(\mathcal{A}) \) gives rise to a \( K_{\mathcal{T}} \)-configuration \( K_{\mathcal{T}}(\mathcal{A}') \) with the \( K_{\mathcal{T}} \)-translated \( \mathcal{A}' \) obtained from the almost \( K_{\mathcal{T}} \)-translated \( \mathcal{A} \) by a suitable translation of the hyperplane \( H_l \in \mathcal{A} \). If \( s = 2 \), then we call \( \mathcal{A}' \) \( r \)-dependent.

Example 3.5 (Crapo’s example). Let us consider the Crapo’s example in Subsection 2.3. Looking at Figure 5, we can easily check that if we choose \( l = 5 \), then the almost \( K_{\mathcal{T}} \)-configuration given by \( \mathcal{P}_1 = H_1 \cap H_2 \cap H_3 \), \( \mathcal{P}_2 = H_1 \cap H_4 \), \( \mathcal{P}_3 = H_3 \cap H_4 \), \( \mathcal{P}_4 = H_3 \cap H_5 \) becomes a \( K_{\mathcal{T}} \)-configuration by the translation of \( H_5 \) such that \( \mathcal{P}_2 \cap H_5 \neq \emptyset \) (as depicted in Figure 5). That is the arrangement \( \mathcal{A}' \) depicted in Figure 7 is \((3,2)\)-dependent. Remark that if \( \mathcal{A} \in \bigcap_{l=1}^{r} D_{l,v} \), then \( \mathcal{A} \in D_{l,v} \).

The definition of \((r,s)\)-dependency generalizes the definition of dependency given in [13]. Indeed we have the following remark which, in particular, applies to Falk’s example (see Example 3.7).

Remark 3.6 (Dependency and 3-dependency). Let’s focus on the case in which \( \mathcal{T} = \{L_1, L_2, L_3 \} \) is a set of cardinality 3 to show that, the definition of \( r \)-dependency is a generalization of the definition of dependency given in [13]. In order to do that it is enough to show that both conditions, i.e. 3-dependency and dependency, are equivalent to the condition that the space \( H_{i,j} \) is a subspace of \( H_{i,k} + H_{k,j} \).

Dependency. Recall that an arrangement \( \mathcal{A}' \) of \( s \) hyperplanes in \( \mathbb{C}^{2s-1} \) is \( s \)-dependent if it exists a set \( \mathcal{T} = \{L_1, L_2, L_3 \} \) of subsets \( L_i \subseteq \{3s \} \) such that \( |L_i| = 2s \), \( |L_i \cap L_j| = s \), \( |L_i \cap L_j \cap L_k| = 3s \) and spaces \( H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \) span a subspace of dimension \( 2s - 2 \) in \( \mathbb{C}^{2s-1} \). The condition \( |L_i \cap L_j| = s \) implies that \( H_{i,j} \) are spaces of codimension \( s \), that is \( s \leq 1 \) in \( \mathbb{C}^{2s-1} \). Moreover \( |L_i \cap L_j \cap L_k| = 3s \) implies that the \( \bigcup_{i=1}^{3s} L_i \) is disjoint union of the three sets \( L_i \cap L_j \), that is any two subspaces \( H_{i,j} \) are in direct sum, i.e. \( H_{i,k} \oplus H_{k,j} \) span a space of dimension \( 2s - 2 \). Hence dependency condition is equivalent to the fact that \( H_{i,j} \) belongs to the space generated by \( H_{i,k} \oplus H_{k,j} \).

3-dependency First of all notice that \( K_{\mathcal{T}} \)-configuration when \( |\mathcal{T}| = 3 \) is equivalent to the fact that \( P_iP_j = P_jP_k + P_kP_i \).
Example 3.7. [Falk’s example] Consider the Falk’s example in Subection 2.3. In this case the discriminantal arrangement is non very generic.

In particular, the equality (5) implies that the rank $X$ is non very generic.

Lemma 3.8. The discriminantal arrangement $\mathcal{A}'$ of $n$ hyperplanes in $\mathbb{C}^k$ is $(r, s)$-dependent for some $s > 1$ then $\mathcal{A}'$ is non very generic.
Let’s remark that while Theorem 3.9 provides a geometric condition which gives rise to non very genericity, it is quite hard to verify if an arrangement satisfies such condition. In the rest of the paper we will focus on providing an equivalent condition for $(r, s)$-dependency which can be computationally verified and which allows to build non very generic arrangements. In order to do this we will need to introduce the following vector condition.

Let $T$ be an $r$-set, $D([r]) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} | i + 1 < j\}$ be the set of not adjacent pairs of integers mod $r$ and for any $(i, j)$ in $D([r])$, $v_{i,j}$ vectors defined as linear combinations of the form

\[ v_{i,j} = \sum_{p=i}^{j-1} v_{p,p+1} = -\sum_{p=j}^{i-1} v_{p,p+1}, \quad v_{p,p+1} \in H_{p,p+1}^0 \]

where the right summand is intended from $j$ to the first representative $h > j$ such that $h \equiv i - 1$ (see Figure 6). Notice that up to re-order of indices, the vectors $P_p + v_{p,p+1}$, i.e. $v_{p,p+1}$ applied to points $P_p = \bigcap_{l \in L_p} H_l$, $H_l$’s translated of $H_l^0$, can be regarded as sides of an $r$-gon having as vertices the applications points $P_p$ and as edges the vectors $v_{p,p+1}$ directed counter clockwise as depicted in Figure 6.

**Definition 3.10.** We say that vectors $v_{i,j}$ satisfy the $K_T$-vector condition if there exist a hyperplane $H_{i,j}^0 \in \mathcal{A}^0, l \in \bigcup_{i=1}^r L_i \setminus \bigcap_{i=1}^r L_i$, and a subset $S_l \subseteq \{L_i \in \mathcal{T} \mid l \in L_i\}$ such that if $v_{i,j} \in H_{i,j}^0 = \bigcap_{p \in L_i \cap S_l} H_p^0, L_i \not\in S_l$, and $v_{i,j} \in \bigcap_{p \in L_i \setminus S_l} H_p^0, L_i, L_j \in S_l$ then $v_{i,j} \in H_{i,j}^0$, for any $(i, j) \in D([r])$.

By definition if $\mathcal{A}$ is a $K_T$-translated of $\mathcal{A}^0$, then each vector $v_{i,j} \in H_{i,j}^0$ is a vector that applied to the point $P_i$ has $P_j$ as ending point in the $K_T$-configuration $K_T(\mathcal{A})$, that is

\[ P_i + v_{i,j} = P_j \in H_p \]

as described in Figure 6. In particular the vector $P_i P_j$ is a translated of $v_{i,j}$ and the following lemma holds.

**Lemma 3.11.** A central generic arrangement $\mathcal{A}^0$ of $n$ hyperplanes in $\mathbb{C}^k$ is $(r, s)$-dependent if and only if it exists a set $T = \{L_1, \ldots, L_r\}$ such that any set of vectors $\{v_{i,j}\}$ defined as in equation (6) satisfies the $K_T$-vector condition for some $H_{i,j}^0 \in \mathcal{A}^0, l \in \bigcup_{i=1}^r L_i \setminus \bigcap_{i=1}^r L_i$ and a subset $S_l$ of cardinality $s$. 

![Figure 6](https://via.placeholder.com/150) 

Figure 6. Diagonal vectors $v_{i,j}$ can be written as a sum of side vectors $v_{p,p+1}$.
In the next section we will use the $K_T$-vector condition to simplify the condition for $(r, s)$-dependency. In particular we will show that it is not needed, as stated in Lemma 3.11, that any set of vectors $\{v_{ij}\}$ defined as in equation (6) satisfies the $K_T$-vector condition in order for $A^0$ to be $(r, s)$-dependent, but it is sufficient that the $K_T$-vector condition is satisfied by just a finite subset of them.

4. An algebraic condition for non very genericity

In this section $A^0 = \{H_1^0, \ldots, H_n^0\}$ will denote the translation of the central generic arrangement $A^0$ of parallel translates of $A^0$ with $C^k$ in such a way that the arrangement $A^0$ corresponds to the origin. The discriminantal arrangement $B(n, k, A^0)$ is not essential arrangement of center $D_{[n]} = \mathbb{C}^k$ given by all translated $A^0$ of $A^0$. Hence we can consider its essentialization $ess(B(n, k, A^0))$ in $\mathbb{C}^{n-k} \cong S[D_{[n]}]$. An element $A^0 \in ess(B(n, k, A^0))$ will corresponds uniquely to a translation $t \in C^0/C \cong C^{n-k}$.

Proposition 4.1. Let $A^0$ be a generic central arrangement of $n$ hyperplanes in $C^0$. Translations $A^0, \ldots, A^0 v$ of $A^0$ are linearly independent vectors in $S[D_{[n]}] \cong C^{n-k}$ if and only if $t_1, \ldots, t_d$ are linearly independent vectors in $C^n/C$.

Given $A^0, \ldots, A^0 v$, $K_T$-translated of $A^0$, we will say that the $K_T$-configurations $K_T(A^0)$ are independent if $A^0 v$, $i = 1, \ldots, d$ are.

Let’s consider vectors $\{v_{ij}\}$ introduced in equation (6) associated to an $r$-set $T$. We can remark the following three facts:

1. To each $K_T$-configuration $K_T(A^0)$ of a translated arrangement $A^0 v$, $t = (x_1, \ldots, x_n)$, corresponds a unique family $\{v'_{ij}\}$ of vectors such that $P_j^v + v'_{ij} = P_j^0 = \bigcap_{\alpha \in r} H_j^{\alpha}$. In the rest of the section we will denote by $\{v'_{ij}\}$ the family of vectors associated to $K_T(A^0)$. Notice that the converse is not uniquely defined since two different $K_T$-configurations can define the same family $\{v'_{ij}\}$.

2. By construction vectors $\{v'_{ij}\}$ satisfy the property that $v'_{k,j} = v'_{i,j} - v'_{i,k}$ (this can be easily seen looking at translated vectors $v_{i,j}$’s represented in Figure 6). Then the set $\{v'_{ij}\}$ is uniquely determined by any subset of the form $\{v'_{j,h}, v'_{i,h}\}_{h \in H}$ for a fixed index $h \in r$. For simplicity in the rest of the Section we will use the set $\{v'_{j,h}, v'_{i,h}\}_{h \in H}$ instead of $\{v'_{ij}\}$ and we will call it $K_T$-vector set.

3. Any family of vectors $\{v'_{ij}\}$ associated to a $K_T$-configuration $K_T(A^0)$ satisfies, by construction, the $K_T$-vector condition and, consequently, the family of vectors $\{v'_{ij}\}$ built via the relations in (2) from the $K_T$-vector set $\{v'_{j,h}, v'_{i,h}\}_{h \in H}$ does. Hence it is enough to say that $\{v'_{j,h}, v'_{i,h}\}_{h \in H}$ satisfies the $K_T$-vector condition since $\{v'_{ij}\}$ satisfies it and only if $\{v'_{j,h}, v'_{i,h}\}_{h \in H}$ does.

Given a $K_T$-vector set we can naturally define operation of multiplication by a scalar

$$a(v'_{j,h}, v'_{i,h})_{h \in H} := (av'_{j,h}, av'_{i,h})_{h \in H}, \quad a \in \mathbb{C}$$

and sum of two different $K_T$-vector sets

$$\{v'_{j,h}, v'_{i,h}\}_{h \in H} + \{v'_{j,h}, v'_{i,h}\}_{h \in H} = \{v'_{j,h} + v'_{j,h}, v'_{i,h} + v'_{i,h}\}_{h \in H} .$$

With above notations and operations, we have the following definition.

Definition 4.2. For a fixed set $\mathcal{T}$, $d$ different $K_T$-vector sets $\{v'_{j,h}, v'_{i,h}\}_{h=1,\ldots,d}$ are linearly independent if and only if for any $a_1, \ldots, a_d \in \mathbb{C}$ such that

$$\sum_{h=1}^d a_h v'_{j,h} v'_{i,h} = 0,$$

\[\text{(7)}\]

\[\text{It is unique in the quotient space } S[D_{[n]}] \cong C^{n-k}\]
then \( a_1 = \ldots = a_d = 0 \).

The following remark is a key point to prove the connection between linearly independence of \( K_\tau \)-configurations and linearly independence of associated \( K_\tau \)-vector sets.

**Remark 4.3.** Let \( K_\tau(\mathcal{A}) \) be the \( K_\tau \)-configuration of the arrangement \( \mathcal{A} \) translated of \( \mathcal{A}^0 \). Then for any \( c \in \mathbb{C} \), the \( K_\tau \)-configuration \( K_\tau(\mathcal{A}^0) \) is an "expansion" by \( c \) of \( K_\tau(\mathcal{A}) \), that is \( v_{i,j}^c = cv_{i,j} \) (Figure 7 shows an example of expansion in the case of \( |\mathcal{T}| = 4 \)). This is consequence of the fact that for any \( i \in [r] \) the vector \( OP_i \) joining the origin with the points \( P_i^n \) satisfies \( OP_i^n = cOP_i \) by definition of translation. Hence \( P_i^n P_j^n = cP_i^n P_j^n \), i.e.

\[
\begin{align*}
  v_{i,j}^c = c v_{i,j}.
\end{align*}
\]

Analogously we have that, if \( t_1, t_2 \in \mathbb{C}^n \) are two translations then

\[
\begin{align*}
  v_{i,j}^{t_1} + v_{i,j}^{t_2} = v_{i,j}^{t_1 + t_2}.
\end{align*}
\]

We can now prove the main lemma of this section.

**Lemma 4.4.** Let \( \mathcal{A}^0 \) be a central generic arrangement of \( n \) hyperplanes in \( \mathbb{C}^d \) and \( \mathcal{T} = \{L_1, \ldots, L_r\} \) be an \( r \)-set such that \( [n] = \bigcup_{i=1}^r L_i \). The \( K_\tau \)-translated arrangements \( \mathcal{A}^1, \ldots, \mathcal{A}^d \) of \( \mathcal{A}^0 \) are linearly independent if and only if their associated \( K_\tau \)-vector sets \( \{v_{j,0}^h, v_{j,1}^h\}_{j=0,1, \ldots, d} \) are linearly independent.

**Proof.** By definition \( \mathcal{A}^i, \ldots, \mathcal{A}^d \) are linearly independent if and only if translations \( t_1, \ldots, t_d \) are linearly independent vectors in \( \mathbb{C}^n / \mathbb{C} \). Let’s consider a linear combination \( \sum_{h=1}^d a_h t_h \) of vectors \( t_h \) and translated arrangements \( \mathcal{A}^i, \mathcal{A}^h \). By Remark 4.3 we have that \( K_\tau \)-vector sets associated to \( \mathcal{A}^{h,t} \) verify

\[
\sum_{h=1}^d a_h v_{j,0}^h v_{j,1}^h \mid_{j \neq t} = \{v_{j,0}^{\sum_{h=1}^d a_h t_h}, v_{j,1}^{\sum_{h=1}^d a_h t_h}\}_{j \neq t}.
\]

Since \( v_{i,j}^c \) is, by definition, the vector such that \( P_i^n + v_{i,j}^c = P_j^n \) then \( v_{i,j}^c = 0 \) if and only if \( t \) is a translation such that points \( P_i^n = P_j^n \) coincides. Hence the condition that \( \sum_{h=1}^d a_h v_{j,0}^h v_{j,1}^h = 0 \) is equivalent to the fact that \( \sum_{h=1}^d a_h t_h \in \mathbb{C}^n \).
C. Indeed $\sum_{h=1}^{d} a_h (v_{h,i}, v_{h,j})_{j \neq h} = \{ \sum_{h=1}^{d} a_h I_h, \sum_{h=1}^{d} a_h I_h \}_{j \neq h} = 0$ if and only if $\sum_{h=1}^{d} a_h I_h$ is a translation such that all intersection points $P_j$ coincide with the same point $P_{t_i}$, i.e. $P_{t_i} = \bigcap_{p \in \mathcal{U}_{x_i}, L} H_{v_{x_i}I_i}$. The center of the translated arrangement $\mathcal{A}^{\sum_{h=1}^{d} a_h I_h}$ of $[n] = \bigcup_{x_i=1}^{L_i}$ hyperplanes. The proof of the statement follows.

Let us remark that the assumption that the $r$-set $T = \{ L_1, \ldots, L_r \}$ satisfies the condition $\bigcup_{i=1}^{r} L_i = [n]$ in Lemma 4.4 is equivalent to consider, in the more general case in which $\bigcup_{i=1}^{r} L_i \subset [n]$, a subset $\mathcal{A}_0 \subset \mathcal{A}$ which only contains the hyperplanes indexed in $\bigcup_{i=1}^{r} L_i$. On the other hand if a (central) generic arrangement $\mathcal{A}$ contains a subarrangement $\mathcal{A}_0$ which is non very generic then $\mathcal{A}$ is non very generic (this simply comes from the fact that non very genericity is a local property on the intersection lattice of the Discriminantal arrangement). Analogously, if there exists a restriction arrangement $\mathcal{A}_0^r = \{ H \cap Y_{r}, H \in \mathcal{A}_0 \setminus \mathcal{T} \}, Y_{r} = \bigcap_{l \in \mathcal{R}} H$ of $\mathcal{A}$ which is non very generic, then $\mathcal{A}$ is non very generic and the following main theorem of this Section follows.

**Theorem 4.5.** Let $\mathcal{A}$ be a central generic arrangement of $n$ hyperplanes in $\mathbb{C}^k$. If there exists a set $T = \{ L_1, \ldots, L_r \}$ with $| \bigcup_{i=1}^{r} L_i | = m$ and rank $\bigcap_{p \in \mathcal{U}_{x_i}, L} H_p = y$, which admits $m - y - k - r'$ independent $K_{T}$-vector sets for some $r' < r$, then $\mathcal{A}$ is non very generic.

**Proof.** Let's consider the subarrangement $\mathcal{A}'$ of $\mathcal{A}$ given by hyperplanes indexed in the $\bigcup_{i=1}^{r} L_i$ and its essentialization, i.e. the restriction arrangement $\mathcal{A}'^{r'} = \bigcap_{i=1}^{r} L_i H_p$. If $y = \text{rank } \mathcal{A}$ then the arrangement $\mathcal{A}'$ is a central essential arrangement in $\mathbb{C}^{m-\gamma}, m = | \bigcup_{i=1}^{r} L_i |$. By Lemma 4.4 if $\mathcal{A}_0^{r'} = \mathcal{A}_0^{r'} \subset \mathcal{A}$ associated to the $m - y - k - r'$ independent $K_{T}$-vector sets, then $\mathcal{A}_0^{r'}, \ldots, \mathcal{A}_0^{r'+r'}$ are linearly independent in $S[\mathcal{A}'^{r'}]/D_{m|n} = \mathbb{C}^{m-\gamma-\gamma'}$. That is $\mathcal{A}_0^{r'}, \ldots, \mathcal{A}_0^{r'+r'}$ span a subspace of dimension $m - y - k - r'$. On the other hand, by construction, $\mathcal{A}_0^{r'}$ is $K_{T}$-translated, i.e. $\mathcal{A}_0^{r'} \in \mathcal{E}(X), X = \bigcap_{i=1}^{r} D_{i}$, for any $r = 1, \ldots, m - y - k - r'$, that is the space spanned by $\mathcal{A}_0^{r'}, \ldots, \mathcal{A}_0^{r'+r'}$ is included in $\mathcal{E}(X)$. This implies that the simple intersection $\mathcal{E}(X)$ has dimension $d \geq m - y - k - r' > m - y - k - r$ that is its codimension is smaller than $r$, i.e. rank $\mathcal{E}(X) < r$ and hence rank $X < r$. This implies that $\mathcal{A}$ is non very generic and hence $\mathcal{A}$ is non very generic.

Theorem 4.5 allows to build non very generic arrangements simply imposing linear dependency conditions on vectors $v_{i,j} \in H_{i,j}$ and, vice versa, to check whether an arrangement is non very generic by checking opportunely defined linear dependencies. Still two main questions are left open. One from geometric point of view and the other one combinatorial.

1. While we provided a geometric/algebraic necessary and sufficient condition for a simple intersection $X$ of multiplicity $r$ to be of rank $X < r$, it is still open the problem on non simple intersections. That is, is it possible to have intersections $\bigcap_{i=1}^{m} D_{i}, S > k$ such that $\bigcap_{i \in I} D_{i} \neq D_{k}$ for some subset $I \subset [m], |I| \geq 2$ and rank $\bigcap_{i=1}^{m} D_{i_k} < \sum_{i}(S_i - k)$? More precisely, does any such intersection contain a simple intersection of rank strictly lesser than its multiplicity?

2. Which are the numerical conditions on the sets $L_i$'s for an intersection $X$ to be simple and non very generic?

In the next section we will provide non trivial examples of how to build non very generic arrangements by means of Theorem 4.5.

### 5. Examples of non very generic arrangements

In this section we present few examples to illustrate how to use Theorem 4.5 to construct non very generic arrangements. To construct the numerical examples we used the software CoCoA-5.2.4 (see [1]).

**Example 5.1 (B(12, 8, \mathcal{A})) with an intersection of multiplicity 4 in rank 3.** Let $L_1 = [12] \setminus \{ 10, 11, 12 \}, L_2 = [12] \setminus \{ 7, 8, 9 \}, L_3 = [12] \setminus \{ 4, 5, 6 \}$ and $L_4 = [12] \setminus \{ 1, 2, 3 \}$ be subsets of $[12]$ of $k + 1 = 9$ indices. It is an easy computation that the set $T = \{ L_1, L_2, L_3, L_4 \}$ is a 4-set. Let's consider a central generic arrangement $\mathcal{A}_0$ of 12 hyperplanes in $\mathbb{C}^9$ and $\mathcal{A}$ an almost $K_{T}$-translated, i.e. $K_{T}$ but for the hyperplane $H_{0}^9$ and $S_{12} = \{ L_2, L_3, L_4 \}$. In this case $m = n = 12, y = 0$ and $m - k - r = 12 - 8 - 4 = 0$, hence, by Theorem 4.5, in order for $\mathcal{A}$ to be non very generic it is enough the existence of just one $K_{T}$-vector set $\{ v_{1,2}, v_{1,3}, v_{1,4} \}$ such that the vectors $v_{2,3} = v_{1,3} - v_{1,2} \in \bigcap_{p \in L_2 \cap L_4 \setminus [12]} H_{p}$. And
In this case, we have the $K_T$-configuration $K_T(\mathcal{A}^0)$ of $\mathcal{B}(12, 8, \mathcal{A}^0)$. $v_{i,j}$ are vectors in $H^0_{i,j}$.

Let's see a numerical example. Let us consider hyperplanes of equation $H_i^0 : \alpha_i \cdot x = 0$, with $\alpha_i$, $i = 1, \ldots, 11$ assigned as following:

$$
\alpha_1 = (0, 0, 1, 0, 1, 0, 1, 1, 1, -1, 1), \alpha_2 = (0, 0, 0, 0, 1, 0, 0, 0, 1, 1, -1), \\
\alpha_3 = (0, 0, -1, 0, 0, 0, 1, 1, 1), \alpha_4 = (0, 1, 0, 0, 1, 1, 1, 1, 1), \\
\alpha_5 = (0, 2, 0, -1, 0, 1, 1, 1, -1, 1, -1), \alpha_6 = (0, -1, 0, 2, 1, 1, -1, -1, 1, 1), \\
\alpha_7 = (1, 0, 0, 1, 0, -1, -1, 1), \alpha_8 = (-1, 0, 0, 0, 2, 1, 1, 1), \alpha_9 = (-4, 0, 0, 0, 1, -1, 1, 1), \\
\alpha_{10} = (1, 1, 1, -1, -1, -1, 1), \alpha_{11} = (1, 1, 1, 2, 2, 0, 3).
$$

In this case, we have the $K_T$-vector set

$$\{v_{1,2}, v_{1,3}, v_{1,4}\} = \{(1,0,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0),(0,0,-1,0,0,0,0,0)\}.$$  

The other vectors are obtained by means of relations

$$v_{2,3} = v_{1,3} - v_{1,2}, v_{2,4} = -(v_{1,2} + v_{1,4}), v_{3,4} = -(v_{1,3} + v_{1,4}),$$

which has to contain vectors $v_{1,2}, v_{1,3}, v_{1,4}$.

and, finally, we get $v_{12} = (-2, -2, -2, 3, 4, -5, 6, 7)$ by imposing the condition that $\alpha_{12}$ has to be orthogonal to $v_{2,3}$ and $v_{2,4}$.

**Example 5.2** ($\mathcal{B}(16, 11, \mathcal{A}^0)$ with intersection of multiplicity 4 in rank 3). Let $L_1 = [16] \setminus \{13, 14, 15, 16\}, L_2 = [16] \setminus \{9, 10, 11, 12\}, L_3 = [16] \setminus \{5, 6, 7, 8\}$ and $L_4 = [16] \setminus \{1, 2, 3, 4\}$ be subsets of $[16]$ of $k + 1 = 12$ indices. The set $\mathcal{T} = \{L_1, L_2, L_3, L_4\}$ is a 4-set. Let’s consider a central generic arrangement $\mathcal{A}^0$ of 16 hyperplanes in $\mathbb{C}^{11}$ and $\mathcal{A}$ be an almost $K_T$-translated but for the hyperplane $H^0_{16}$ and $S_{16} = (L_2, L_3, L_4)$. In this case $m = n = 16$, $y = 0$ and $m - k - r = 16 - 11 - 4 = 1$, hence, by Theorem 4.5 in order for $\mathcal{A}^0$ to be non very generic we need two linearly independent $K_T$-vector sets $\{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1\}$ and $\{v_{1,2}^2, v_{1,3}^2, v_{1,4}^2\}$ such that the vectors $v_{2,3}^k \in \bigcap_{p \in L_2 \cap L_3} H^0_p$ and $v_{2,4}^k \in \bigcap_{p \in L_2 \cap L_4} H^0_p$, $k = 1, 2$, belong to $H^0_{16}$ (see Figure 9). Notice that since $v_{3,4}^1 = v_{2,4}^1 - v_{2,3}^1 \in \bigcap_{p \in L_2 \cap L_4} H^0_p$, $v_{1,4}^1 \in H^0_{16}$ if $v_{2,3}^1, v_{2,4}^1 \in H^0_{16}$. That is all hyperplanes in $\mathcal{A}^0$ can be chosen freely, but $H^0_{16}$ which has to contain vectors $v_{2,3}^1, v_{2,4}^1$, $k = 1, 2$.

---

6Here and in the rest of this section, freely means that we only impose the condition that $\mathcal{A}^0$ is a central generic arrangement. In particular this condition is always taken as given and imposed even if not written.
In this case, we have the $K_T \cap H_p \cap H_{16}$ but for the hyperplane $H^0$ hence, by Theorem 4.5 in order for Figure 10). Notice that since in this case hyperplanes are planes, then the three vectors $v^k_i, v^l_j, v^m_k$ are vectors in $H^0_{i,j,k}$.

Let’s see a numerical example. Let us consider hyperplanes of equation $H^0_i : \alpha \cdot x = 0$, with $\alpha_i, i = 1, \ldots, 15$ assigned as following.

(10)

$\alpha_1 = (0, 0, 1, 0, 0, 1, 0, 0, 1, -1), \alpha_2 = (0, 0, 1, 0, 0, 1, 0, 0, 1, 1), \alpha_3 = (0, 0, 2, 0, 0, 1, 0, 0, 1, 1, 0), \alpha_4 = (0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, -1), \alpha_5 = (0, 0, 2, 0, 0, 1, 1, 1, 0, 1, 1, 0), \alpha_6 = (0, 1, 0, 0, 2, 0, 0, 0, 1, 0, 1, 1, 0), \alpha_7 = (0, 0, -1, 0, 0, 2, 0, 1, 1, 1, 0, 0), \alpha_8 = (0, -1, 0, 0, 2, 0, 1, 1, 1, 0, 0), \alpha_9 = (1, 0, 0, -3, 0, 0, -1, -1, 1, 1, 1, 0), \alpha_{10} = (2, 0, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0), \alpha_{11} = (3, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), \alpha_{12} = (1, 0, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0), \alpha_{13} = (1, 1, 1, -3, -3, -3, -3, -3, -2, -2, -1, -8, 1, 1), \alpha_{14} = (1, 1, 1, 0, 0, 0, -2, 1, -8, 1, 1), \alpha_{15} = (0, 0, 0, -5, -5, -5, 1, 2, -3, -4, 7).

In this case, we have the $K_T$-vector sets

\[
\begin{align*}
\{v^1_{1,2}, v^1_{1,3}, v^1_{1,4}\} & = \{(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\}, \\
\{v^2_{1,2}, v^2_{1,3}, v^2_{1,4}\} & = \{(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\}.
\end{align*}
\]

The other vectors are obtained by means of relations $v^k_{1,2,3} = v^k_{1,3} - v^k_{1,2}, v^k_{1,2,4} = -(v^k_{1,2} + v^k_{1,4}), v^k_{1,3,4} = -(v^k_{1,3} + v^k_{1,4}), k = 1, 2,$ that is

(11)

\[
\begin{align*}
v^1_{1,2} & = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
v^1_{1,3} & = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
v^1_{1,4} & = (0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
v^2_{1,2} & = (0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
v^2_{1,3} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
v^2_{1,4} & = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
\end{align*}
\]

and, finally, we get $\alpha_{16} = (1, 1, 1, -2, -2, -2, 5, 6, 7, 8, 9)$ by imposing the conditions that $\alpha_{16}$ has to be orthogonal to $v^1_{1,2}, v^2_{1,3}, v^2_{1,4}$, $k = 1, 2$.

Example 5.3 ($\mathcal{B}(10, 3, \mathcal{A}^0)$ with an intersection of multiplicity 5 in rank 4). Let $L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 8, 9\}, L_4 = \{3, 6, 8, 10\}$ and $L_5 = \{4, 7, 9, 10\}$ be subsets of $[10]$ of $k + 1 = 4$ indices. The set $\mathcal{T} = \{L_1, L_2, L_3, L_4, L_5\}$ is a 5-set. Let’s consider a central generic arrangement $\mathcal{A}^0$ of 10 hyperplanes in $\mathbb{C}^3$ and $\mathcal{A}^0$ be almost $K_T$-translated but for the hyperplane $H^0_{10}$ and $S_{10} = \{L_4, L_5\}$. In this case $m = n = 10$, $y = 0$ and $m - k - r = 10 - 3 - 5 = 2$, hence, by Theorem 4.5 in order for $\mathcal{A}^0$ to be non very generic we need three linearly independent $K_T$-vector sets $\{v^1_{1,2}, v^1_{1,3}, v^1_{1,4}, v^1_{1,5}\}, \{v^2_{1,2}, v^2_{1,3}, v^2_{1,4}, v^2_{1,5}\}$ and $\{v^3_{1,2}, v^3_{1,3}, v^3_{1,4}, v^3_{1,5}\}$ such that the vectors $v^k_{i,j}, k = 1, 2, 3$ belong to $H^0_{10}$ (see Figure 10). Notice that since in this case hyperplanes are planes, then the three vectors $v^k_{i,j}, k = 1, 2, 3$ have to be linearly dependent for any choice of indices $(i, j), i \neq j$. This additional condition forces that at most 8 hyperplanes in $\mathcal{A}^0$ can be chosen freely, while both $H^0_{10}$ and $H^0_{10}$ have to contain the dependent vectors $v^3_{1,3}$ and $v^4_{1,4}$. $k = 1, 2, 3$, respectively.
Let’s see a numerical example. Let us consider hyperplanes of equation \( H_i^0 : \alpha_i \cdot x = 0 \), with \( \alpha_i, i = 1, \ldots, 8 \) assigned as following.

\[
\begin{align*}
\alpha_1 &= (0,10,3), \\
\alpha_2 &= (20,0,-9), \\
\alpha_3 &= (2,-3,0), \\
\alpha_4 &= (3,1,0), \\
\alpha_5 &= (0,0,1), \\
\alpha_6 &= (1,-1,1), \\
\alpha_7 &= (1,2,2), \\
\alpha_8 &= (4,-1,-3).
\end{align*}
\]

In this case, we have the \( K_T \)-vector sets

\[
\begin{align*}
\{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1, v_{1,5}^1\} &= \{(1,-3,10), (9/2, 21/2, 10), (9/2, 3/2, 25/9), (-77/9, 77/9, -125/9)\}, \\
\{v_{1,2}^2, v_{1,3}^2, v_{1,4}^2, v_{1,5}^2\} &= \{(-2,6,-20), (-9,-47,-20), (-3,-2,-27), (-2/3,2,-50/3)\}, \\
\{v_{1,2}^3, v_{1,3}^3, v_{1,4}^3, v_{1,5}^3\} &= \{(-3,3,-10), (-9/2,-2391/80,-10), (-1467/1040, -489/520, -16151/1040), (-4/3,4,-71/6)\}.
\end{align*}
\]

The other vectors are obtained by means of relations \( v_{i,j}^k = v_{i,j}^1 - v_{i,j}^k \) where \( 2 \leq i < j \leq 5, k = 1,2,3, \) that is

\[
\begin{align*}
v_{2,3}^1 &= \left(\frac{7}{2}, \frac{27}{2}, 0\right), v_{2,4}^1 &= \left(\frac{7}{6}, \frac{5}{2}\right), v_{2,5}^1 &= \left(-\frac{86}{9}, \frac{86}{3}, \frac{215}{9}\right), \\
v_{3,4}^1 &= (0, -\frac{15}{2}, \frac{5}{2}), v_{3,5}^1 = \left(-\frac{235}{18}, \frac{91}{6}, -\frac{215}{9}\right), v_{4,5}^1 = \left(-\frac{235}{18}, \frac{68}{3}, -\frac{475}{18}\right), \\
v_{2,3}^2 &= (-7,-53,0), v_{2,4}^2 = (-1,-8,-7), v_{2,5}^2 = \left(\frac{4}{3}, -4, \frac{10}{3}\right), \\
v_{3,4}^2 &= (6,45,-7), v_{3,5}^2 = \left(\frac{25}{3}, \frac{49}{3}, \frac{10}{3}\right), v_{4,5}^2 = \left(\frac{7}{3}, \frac{4}{3}, \frac{31}{3}\right), \\
v_{2,3}^3 &= \left(-\frac{3}{2}, -\frac{2631}{80}, 0\right), v_{2,4}^3 = \left(-\frac{1653}{1040}, \frac{2049}{520}, -\frac{5751}{1040}\right), v_{2,5}^3 = \left(\frac{5}{3}, 1, -\frac{11}{6}\right), \\
v_{3,4}^3 &= \left(\frac{3213}{1040}, \frac{6021}{208}, \frac{5751}{1040}\right), v_{3,5}^3 = \left(\frac{19}{6}, \frac{2711}{80}, -\frac{11}{6}\right), v_{4,5}^3 = \left(\frac{241}{3120}, \frac{2569}{520}, \frac{11533}{3120}\right).
\end{align*}
\]

Finally, we get \( \alpha_9 = (314, -40, -197) \) and \( \alpha_{10} = (139, 30, -43) \) by imposing the conditions that \( \alpha_9 \) and \( \alpha_{10} \) have to be orthogonal to \( v_{3,5}^k \) and \( v_{4,5}^k \), \( k = 1,2,3. \)

**Remark 5.4.** Notice that Example 5.3 is slightly different from other examples for two reasons. Firstly, it uses a different combinatorics. In the Examples 5.1 and 5.2 the 4-sets \( \mathcal{T} = \{L_1, L_2, L_3, L_4\} \) are of the form \( L_i = [n] \setminus K_i \) with
$K_i$'s which satisfy the properties $\bigcup_{i=1}^{n-1} K_i = [n]$ and $K_i \cap K_j = \{\emptyset\}$ while in the Example 5.3 they are not. Secondly, in the Examples 5.2 and 5.4 in order to obtain non very generic arrangement we could choose all hyperplanes freely but one, while in the Example 5.3 two hyperplanes had to be fixed as a result of the need of three independent $K_i$-vector sets in two dimensional hyperplanes. Indeed this dependency condition gives rise to 27 independent equations of the form

$$v_{i,j}^3 = \alpha v_{i,j}^1 + \beta v_{i,j}^2$$

which fix the entries of the vectors $v_{i,j}^k$, $i = 3, 4, 5$ uniquely for any choice of three dependent vectors $v_{i,2}^k$, $k = 1, 2, 3$. Hence the vectors $v_{i,3}^3$ and $v_{i,4}^3$, $k = 1, 2, 3$ are determined and so are the two hyperplanes $H_{i,0}^3$ and $H_{i,0}^4$.

**Remark 5.5.** Notice that the above examples are just very special cases of r-sets $T$. How to describe the r-sets $T$ that can give rise to (simple) non very generic intersections is an open problem.

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**Email address:** s.settepanella@math.sci.hokudai.ac.jp

**Email address:** so.yamagata@math.sci.hokudai.ac.jp

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7 Notice that this is a generalization of the combinatorics used in [13].