Herglotz’ generalized variational principle and contact type Hamilton-Jacobi equations

Piermarco Cannarsa, Wei Cheng, Kaizhi Wang and Jun Yan

Abstract We develop an approach for the analysis of fundamental solutions to Hamilton-Jacobi equations of contact type based on a generalized variational principle proposed by Gustav Herglotz. We also give a quantitative Lipschitz estimate on the associated minimizers.

1 Introduction

The so called generalized variational principle was proposed by Gustav Herglotz in 1930 (see [31] and [32]). It generalizes classical variational principle by defining the functional, whose extrema are sought, by a differential equation. More precisely, the functional is defined in an implicit way by an ordinary differential equation

\[ \dot{u}(s) = F(s, \xi(s), \dot{\xi}(s), u(s)), \quad s \in [0, t], \]

1

Piermarco Cannarsa
Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica 1, 00133 Roma, Italy, e-mail: cannarsa@mat.uniroma2.it

Wei Cheng
Department of Mathematics, Nanjing University, Nanjing 210093, China, e-mail: chengwei@nju.edu.cn

Kaizhi Wang
School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China, e-mail: kzwang@sjtu.edu.cn

Jun Yan
School of Mathematical Sciences, Fudan University and Shanghai Key Laboratory for Contemporary Applied Mathematics, Shanghai 200433, China, e-mail: yanjun@fudan.edu.cn
with \( u(t) = u_0 \in \mathbb{R} \), for \( t > 0 \), a function \( F \in C^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}) \) and a piecewise \( C^1 \) curve \( \xi : [0,t] \rightarrow \mathbb{R}^n \). Here, \( u(\xi, s) \) can be regarded as a functional, on a space of paths \( \xi(\cdot) \). The generalized variational principle of Herglotz is as follows:

\[
\text{Let the functional } u = u(\xi, t) \text{ be defined by (1) with } \xi \text{ in the space of piecewise } C^1 \text{ functions on } [0,t]. \text{ Then the value of the functional } u(\xi, t) \text{ is an extremal for the function } \xi \text{ such that the variation } \frac{d}{d\varepsilon} u(\xi + \varepsilon \eta, t) = 0 \text{ for arbitrary piecewise } C^1 \text{ function } \eta \text{ such that } \eta(0) = \eta(t) = 0.
\]

Herglotz reached the idea of the generalized variational principle through his work on contact transformations and their connections with Hamiltonian systems and Poisson brackets. His work was motivated by ideas from S. Lie, C. Carathéodory and other researchers. An important reference on the generalized variational principle is the monograph [30]. The variational principle of Herglotz is important for many reasons:

- The solutions of the equations (1) determine a family of contact transformations, see [30, 11, 21, 28];
- The generalized variational principle gives a variational description of energy-nonconservative processes even when \( F \) in (1) is independent of \( t \);
- If \( F \) has the form \( F = -\lambda u + L(x,v) \), then the relevant problems are closely connected to the Hamilton-Jacobi equations with discount factors (see, for instance, [19, 18, 9, 34, 35, 37, 29, 36]). As an extension to nonlinear discounted problems, various examples are discussed in [14].
- Even for a energy-nonconservative process which can be described with the generalized variational principle, one can systematically derive conserved quantities as Noether’s theorems such as [26, 27];
- The generalized variational principle provides a link between the mathematical structure of control and optimal control theories and contact transformation (see [25]);
- There are some interesting connections between contact transformations and equilibrium thermodynamics (see, for instance, [39]).

In this note, we will clarify more connections between the generalized variational principle of Herglotz and Hamilton-Jacobi theory motivated by recent works in [41, 42] under a set of Tonelli-like conditions. We will begin with generalized variational principle of Herglotz in the frame of Lagrangian formalism different from the methods used in [41, 42]. Throughout this paper, let \( L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \) be a function of class \( C^2 \) such that the following standing assumptions are satisfied:

(L1) \( L(x,r,\cdot) > 0 \) is strictly convex for all \((x,r) \in \mathbb{R}^n \times \mathbb{R}\).

(L2) There exist two superlinear nondecreasing function \( \overline{\theta}_0, \theta_0 : [0, +\infty) \rightarrow [0, +\infty) \), \( \theta_0(0) = 0 \) and \( c_0 > 0 \), such that

\[
\overline{\theta}_0(|v|) \geq L(x,0,v) \geq \theta_0(|v|) - c_0, \quad (x,v) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

(L3) There exists \( K > 0 \) such that

\[
|L_x(x,r,v)| \leq K, \quad (x,r,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.
\]
Remark 1. For each \( r \in \mathbb{R} \), from the conditions (L2) and (L3) we could take
\[
\theta_r := \theta_0 + K|r|, \quad \bar{\theta}_r := \bar{\theta}_0, \quad c_r := c_0 + K|r|,
\]
such that
\[
\bar{\theta}_r(|v|) \geq L(x, r, v) \geq \theta_r(|v|) - c_r, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.
\] (2)
Obviously, \( \bar{\theta}_r \) and \( \theta_r \) are both nonnegative, superlinear and nondecreasing functions, \( c_r > 0 \).

It is natural to introduce the associated Hamiltonian
\[
H(x, r, p) = \sup_{v \in \mathbb{R}^n} \{ \langle p, v \rangle - L(x, r, v) \}, \quad (x, r, p) \in \mathbb{R}^n \times \mathbb{R} \times (\mathbb{R}^n)^*.
\]

Let \( x, y \in \mathbb{R}^n, t > 0 \) and \( u_0 \in \mathbb{R} \). Set
\[
\Gamma^t_{x,y} = \{ \xi \in W^{1,1}([0,t], \mathbb{R}^n) : \xi(0) = x, \xi(t) = y \}.
\]
We consider a variational problem
\[
\text{Minimize} \quad u_0 + \inf_{\xi} \int_0^t L(\xi(s), u_\xi(s), \dot{\xi}(s)) \, ds,
\] (3)
where the infimum is taken over all \( \xi \in \Gamma^t_{x,y} \) such that the Carathéodory equation
\[
\dot{u}_\xi(s) = L(\xi(s), u_\xi(s), \dot{\xi}(s)), \quad a.e. \ s \in [0,t],
\] (4)
admits an absolutely continuous solution \( u_\xi \) with initial condition \( u_\xi(0) = u_0 \). It is already known that the variational problem (3) with subsidiary conditions (4) is closely connected to the Hamilton-Jacobi equations in the form
\[
H(x, u(x), Du(x)) = c.
\] (5)

The readers can refer to [28] for a systematic approach of Hamilton-Jacobi equations in the form (5), especially in the context of contact geometry.

In [41, 42], a weak KAM type theory on equations (5) was developed on compact manifolds under the aforementioned Tonelli-like conditions. Problem (4) is understood as an implicit variational principle (41) and, by introducing the positive and negative Lax-Oleinik semi-groups, an existence result for weak KAM type solutions of (5) was obtained provided \( c \) in the right side of equation (5) belongs to the set of critical values (42). The same approach adapts to the evolutionary equations in the form
\[
D_t u + H(x, u, Du) = 0.
\] (6)

Unlike the methods used in [41, 42], in this note, our approach of the equations (5) and (6) is based on the the variational problem (3) under subsidiary conditions (4). We give all the details of such a Tonelli-like theory and its connection to viscosity solutions of (5) and (6).
In view of Proposition 1 below, the infimum in (3) can be achieved. Suppose that \( \xi \in \Gamma_{a,b}^\prime \) is a minimizer for (3) where \( u_\xi \) is uniquely determined by (3) with initial condition \( u_\xi(0) = u_0 \). Then we call such \( \xi \) an extremal. Due to Proposition 1 below, each extremal \( \xi \) and associated \( u_\xi \) are of class \( C^2 \) and satisfy the Herglotz equation (Generalized Euler-Lagrange equation by Herglotz)

\[
\frac{d}{ds}L_v(\xi(s), u_\xi(s), \dot{\xi}(s)) = L_x(\xi(s), u_\xi(s), \dot{\xi}(s)) + L_u(\xi(s), u_\xi(s), \dot{\xi}(s))L_v(\xi(s), u_\xi(s), \dot{\xi}(s)) .
\]

Moreover, let \( p(s) = L_v(\xi(s), u_\xi(s), \dot{\xi}(s)) \) be the so-called dual arc. Then \( p \) is also of class \( C^2 \) and we conclude that \( (\xi, p, u_\xi) \) satisfies the following Lie equation

\[
\begin{align*}
\dot{\xi}(s) &= H_p(\xi(s), u_\xi(s), p(s)); \\
p(s) &= -H_x(\xi(s), u_\xi(s), p(s)) - H_u(\xi(s), u_\xi(s), p(s))p(s); \\
u_\xi(s) &= p(s) \cdot \dot{\xi}(s) - H(\xi(s), u_\xi(s), p(s)),
\end{align*}
\]

where the reader will recognize the classical system of characteristics for (5).

The paper is organized as follows: In Section 2, we afford a detailed and rigorous treatment of (3) under subsidiary conditions (4). In Section 3, we study the regularity of the minimizers and deduce the Herglotz equation (7) and Lie equation (8) as well. In Section 4, we show that the two approaches between (41, 42) and ours are equivalent. We also sketch the way to move Herglotz’ variational principle to manifolds.

2 Existence of minimizers in Herglotz’ variational principle

Fix \( x_0, x \in \mathbb{R}^n, t > 0 \) and \( u_0 \in \mathbb{R} \). Let \( \xi \in \Gamma_{x_0,t}^\prime \), we consider the Carathéodory equation

\[
\begin{align*}
\dot{u}_\xi(s) &= L(\xi(s), u_\xi(s), \dot{\xi}(s)), \quad \text{a.e. } s \in [0,t] , \\
u_\xi(0) &= u_0 .
\end{align*}
\]

We define the action functional

\[
J(\xi) := \int_0^t L(\xi(s), u_\xi(s), \dot{\xi}(s)) \, ds,
\]

where \( \xi \in \Gamma_{x_0,t}^\prime \) and \( u_\xi \) is defined in (30) by Proposition 7 in Appendix. Notice that Carathéodory’s theorem (Proposition 7) is just a local result, but the existence and uniqueness of the solution of (30) holds on \([0,t]\) since condition (L3) and that \( \xi \in \mathcal{A} \). Our purpose is to minimize \( J(\xi) \) over

\[
\mathcal{A} = \{ \xi \in \Gamma_{x_0,t}^\prime : (30) \text{ admits an absolutely continuous solution } u_\xi \}.
\]
Notice that $\mathcal{A} \neq \emptyset$ because it contains all piecewise $C^1$ curves connecting $x_0$ to $x$. It is not hard to check that, for each $a \in \mathbb{R}$, 

$$
\mathcal{A} = \mathcal{A}^t := \{ \xi \in \Gamma_{a,x}^t : \text{the function } s \mapsto L(\xi(s), a, \dot{\xi}(s)) \text{ belongs to } L^1([0,t]) \}.
$$

Indeed, if $\xi \in \mathcal{A}$, then $L(\xi(s), u_\xi(s), \dot{\xi}(s))$ is integrable on $[0,t]$ and $u_\xi$ is bounded. Thus $\dot{\xi} \in \mathcal{A}^t$ since 

$$
|L(\xi, 0, \dot{\xi})| \leq |L(\xi, u_\xi, \dot{\xi})| + K|u_\xi|.
$$

On the other hand, if $\xi \in \mathcal{A}^t$, then 

$$
\dot{u}_\xi \leq L(\xi, 0, \dot{\xi}) + K|u_\xi|.
$$

Therefore, $\dot{\xi} \in \mathcal{A}$.

For the following estimate, we define $L_0(x, v) := L(x, 0, v)$.

**Lemma 1.** Let $x_0, x \in \mathbb{R}^n$, $t > 0$, $u_0 \in \mathbb{R}$. Given $\xi \in \Gamma_{a,x}^t$ such that (30) admits an absolutely continuous solution, then we have that 

$$
|u_\xi(s)| \leq \exp(Ks)(|u_0| + c_0s) \quad (11)
$$

if $u_\xi(s) < 0$. In particular, we have 

$$
\dot{u}_\xi(s) \geq -\exp(Ks)(|u_0| + c_0s), \quad s \in [0,t]. \quad (12)
$$

**Proof.** Let $x_0, x \in \mathbb{R}^n$, $t > 0$, $u_0 \in \mathbb{R}$ and $\xi \in \mathcal{A}$. Suppose that $u_\xi(s_0) < 0$, $s_0 \in (0,t]$. We define $E = \{ s \in [0,s_0] : u_\xi(s) \geq 0 \}$ and 

$$
a = \begin{cases} 
0 & E = \emptyset, \\
\sup E & E \neq \emptyset.
\end{cases}
$$

Then, we have that $u_\xi(s) \leq 0$ for all $s \in [a,s_0]$ and $u_\xi(a) = 0$ if $E \neq \emptyset$. Now, we are assuming that $E \neq \emptyset$. For any $s \in [a,s_0]$ we have that 

$$
-|u_\xi(s)| = u_\xi(s) = u_\xi(a) + \int_a^s L(\xi(\tau), u_\xi(\tau), \dot{\xi}(\tau)) \, d\tau 
\geq -|u_\xi(a)| + \int_a^s \Theta(\xi(\tau), \dot{\xi}(\tau)) \, d\tau - K\int_a^s |u_\xi(\tau)| \, d\tau \, d\tau 
\geq -|u_\xi(a)| + \int_a^s \Theta(\xi(\tau), \dot{\xi}(\tau)) \, d\tau - c_0(s-a) - K\int_a^s |u_\xi(\tau)| \, d\tau 
\geq -|u_\xi(a)| - c_0s - K\int_a^s |u_\xi(\tau)| \, d\tau.
$$

Then, we have that
\[ |u_\xi(s)| \leq (|u_0| + c_0s) + K \int_a^s |u_\xi(\tau)| \, d\tau, \quad s \in [a, s_0]. \]

Then Gronwall inequality implies
\[ |u_\xi(s)| \leq \exp(K(s-a))(|u_0| + c_0s) \leq \exp(Ks)(|u_0| + c_0s), \quad s \in [a, s_0]. \]

If \( E = \emptyset \), then \( a = 0 \) and the proof is the same. This leads to (11) and (12). \( \square \)

In view to Lemma (I), we conclude that \( \inf_{\xi \in \mathcal{A}} J(\xi) \) is bounded below. Now, for any \( \varepsilon > 0 \), set
\[ \mathcal{A}_\varepsilon = \{ \xi \in \mathcal{A} : \inf_{\eta \in \mathcal{A}} J(\eta) + \varepsilon \geq u_\xi(t) - u_0 \}. \]

**Lemma 2.** Suppose \( x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R} \) and \( |x - x_0| \leq R \). Let \( \varepsilon > 0 \) and \( \xi \in \mathcal{A}_\varepsilon \). Then we have that
\[ u_\xi(t) - u_0 \leq t(\kappa(R/t) + K|u_0|) \exp(Kt) + \varepsilon, \]
with \( \kappa(r) = \overline{\theta}(r) + 2c_0 \). Moreover, there exist two nondecreasing and superlinear functions \( F, G : [0, +\infty) \to [0, +\infty) \) such that
\[ |u_\xi(t)| \leq tF(R/t) + G(t)|u_0| + \varepsilon, \quad (13) \]
where \( F(r) = \max\{ \kappa(r), c_0 \exp(Kr) \} \) and \( G(r) = \max\{ rK \exp(Kr) + 1, \exp(Kr) \} \).

**Proof.** Suppose \( x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R} \) and \( |x - x_0| \leq R \). Let \( \varepsilon > 0 \) and \( \xi \in \mathcal{A}_\varepsilon \). First, notice that
\[ |L_0(x, v)| \leq L_0(x, v) + 2c_0 \leq \overline{\theta}(|v|) + 2c_0, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n. \quad (14) \]
Set \( \kappa(r) = \overline{\theta}(r) + 2c_0 \).

Define \( \xi_0(s) = x_0 + s(x - x_0)/t \) for any \( s \in [0, t] \), then \( \xi_0 \in \mathcal{A} \). Then, for any \( s \in [0, t] \), we have that
\[ |u_{\xi_0}(s) - u_0| \leq \int_0^s |L_0(\xi_0, \xi_0)\, d\tau + K \int_0^s |u_{\xi_0}| \, d\tau \]
\[ \leq t\kappa(R/t) + K \int_0^s |u_{\xi_0} - u_0| \, d\tau + tK|u_0|. \]

Due to Gronwall inequality, we obtain
\[ |u_{\xi_0}(s) - u_0| \leq t(\kappa(R/t) + K|u_0|) \exp(Kt), \quad s \in [0, t]. \quad (15) \]
Together with Lemma (I) this completes the proof. \( \square \)

**Lemma 3.** Suppose \( x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R} \) and \( |x - x_0| \leq R \). Let \( \varepsilon > 0 \) and \( \xi \in \mathcal{A}_\varepsilon \). Then there exist two continuous functions \( F_1, F_2 : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) depending on \( R \), with \( F_i(r_1, \cdot) \) being nondecreasing and superlinear and \( F_i(\cdot, r_2) \) being nondecreasing for any \( r_1, r_2 \geq 0, i = 1, 2 \), such that
\[ |u_\xi(s)| \leq t F_1(t, R/t) + C_1(t)(\varepsilon + |u_0|), \quad s \in [0, t] \] (16)

and
\[ \int_0^t |L(\xi, u_\xi, \dot{\xi})| \, d\tau \leq t F_2(t, R/t) + C_2(t)(\varepsilon + |u_0|), \] (17)

where \( C_i(t) > 0 \) for \( i = 1, 2 \).

**Proof.** Suppose \( x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R} \) and \(|x - x_0| \leq R\). Let \( \varepsilon > 0 \) and \( \xi \in \mathcal{A}_\varepsilon \).

If \( u_\xi(t) \geq 0 \), we define \( E_+ = \{ s \in [0, t] : u_\xi(s) > u_\xi(t) \} \). If \( E_+ = \emptyset \), then we have that \( u_\xi(s) \leq u_\xi(t) \) for all \( s \in [0, t] \). Now, we suppose that \( E_+ \neq \emptyset \). It is known that \( E_+ \) is the union of a countable family of open intervals \( \{(a_i, b_i)\} \) which are mutually disjoint (It is possible that \( a_i = 0 \) and this case can be dealt with separately but similarly). For any \( \tau \in E_+ \), there exists an open interval \( (a, b) \), a component of \( E_+ \) containing \( s \), such that \( u_\xi(\tau) > u_\xi(t) \geq 0 \) for all \( \tau \in (a, b) \) and \( u_\xi(b) = u_\xi(t) \). Therefore, for almost all \( s \in [a, b] \), we have that
\[ \dot{u}_\xi(s) = L(\xi(s), u_\xi(s), \dot{\xi}(s)) \geq L_0(\xi(s), \dot{\xi}(s)) - Ku_\xi(s). \]

Invoking condition (L2), it follows that, for all \( s \in [a, b] \),
\[ e^{Kb}u_\xi(b) - e^{Ks}u_\xi(s) \geq \int_s^b e^{K\tau}L_0(\xi(\tau), \dot{\xi}(\tau)) \, d\tau \geq -c_0(b - s)e^{Kb} \]

Thus we obtain that
\[
\begin{align*}
|u_\xi(s)| &\leq c_0(e^{K(b-s)} + e^{K(b-s)}u_\xi(t)) \\
&\leq c_0e^{Kt} + e^{Ks}[(tK(R/t) + K|u_0|)e^{Kt} + \varepsilon + |u_0|] \quad s \in [0, t],
\end{align*}
\]

(18)

where \( F_1(r_1, r_2) := e^{Kr_1}(c_0 + K(r_2)) \) and \( G_1(r) = e^{Kr}Ke^{Kr} + 1 \).

If \( u_\xi(t) < 0 \), define \( v_\xi(s) = u_\xi(s) - u_\xi(t) \), then \( v_\xi(s) \) satisfies the Carathéodory equation
\[ v_\xi(s) = L(\xi(s), v_\xi(s) + u_\xi(t), \dot{\xi}(s)), \quad s \in [0, t] \]

with initial condition \( v_\xi(0) = u_0 - u_\xi(t) \). Similarly, We define \( F_+ = \{ s \in [0, t] : v_\xi(s) > v_\xi(t) \} \). If \( F_+ = \emptyset \), then we have that \( v_\xi(s) \leq v_\xi(t) = 0 \) for all \( s \in [0, t] \).

Now, we suppose that \( F_+ \neq \emptyset \) and \( F_+ \) is the union of a countable family of open intervals \( \{(c_i, d_i)\} \) which are mutually disjoint. For any \( s \in F_+ \), there exists an open interval, say \( (c, d) \), such that \( v_\xi(s) > v_\xi(t) = 0 \) for all \( s \in (c, d) \) and \( v_\xi(d) = v_\xi(t) \). Therefore, for almost all \( s \in [c, d] \), we have that
\[ v_\xi(s) \geq L_0(\xi(s), \dot{\xi}(s)) - Ku_\xi(t) - Ku_\xi(s). \]

It follows that, for all \( s \in [c, d] \),
\[ e^{Kd}v_\xi(d) - e^{Ks}v_\xi(s) \geq \int_s^d e^{K\tau}L_0(\xi(\tau), \dot{\xi}(\tau)) \, d\tau - Kt|u_\xi(t)|e^{Kt}, \]

and this gives rise to

\[ v_\xi(s) \leq c_0 te^{K(d-s)} + K|u_\xi(t)|e^{K(t-s)} + e^{K(d-s)}v_\xi(d) \leq c_0 te^{Kt} + K|u_\xi(t)|e^{Kt}, \]

since \( v_\xi(d) = 0 \). It follows that, for all \( s \in [0,t] \),

\[
\begin{align*}
\|u_\xi(s)\| &\leq c_0 te^{Kt} + Kr|u_\xi(t)| + u_\xi(t) \leq c_0 te^{Kt} + (Kr + 1)|u_\xi(t)| \\
&\leq c_0 te^{Kt} + (Kr + 1)(tF_2(R/t) + G_2(t)|u_0| + \varepsilon)
\end{align*}
\]

(19)

with \( F_2 \) and \( G_2 \) determined by Lemma 2. By combining (18) and (19) and setting

\[ F_3(r_1, r_2) = \max\{F_1(r_1, r_2), c_0 e^{Kr_1} + F_2(r_2)(Kr_1 + 1)\}, \quad C_1(r) = \max\{G_1(r), G_2(r)(Kr_1 + 1)\}, \quad C_2(r) = \max\{C_1(r), e^{Kr}c_0\}, \]

we conclude that

\[
\begin{align*}
\|u_\xi(s)\| &\leq tF_3(t, R/t) + C_1(t)(|u_0| + \varepsilon), \\
\|u_\xi(s)\| &\leq tF_3(t, R/t) + C_2(t)(|u_0| + \varepsilon).
\end{align*}
\]

(20) \quad (21)

This leads to the proof of (16) together with Lemma 1.

Now, by (13), Lemma 2 and (21), we have that

\[
\int_0^s |L_0(\xi, \dot{\xi})| \, d\tau \leq \int_0^s (L_0(\xi, \dot{\xi}) + 2c_0) \, d\tau \leq 2c_0s + u_\xi(s) - u_0 + K \int_0^s |u_\xi| \, d\tau
\]

\[
\leq 2c_0t + tF_2(t, R/t) + C_3(t)(|u_0| + \varepsilon) + |u_0|
\]

\[
+ t^2KF_2(t, R/t) + tKC_2(t)(|u_0| + \varepsilon)
\]

\[
\leq tF_3(t, R/t) + C_3(t)(|u_0| + \varepsilon).
\]

Therefore, (17) follows from the estimate below

\[
\int_0^t |L(\xi, u_\xi, \dot{\xi})| \, d\tau \leq \int_0^t |L_0(\xi, \dot{\xi})| \, d\tau + K \int_0^t |u_\xi| \, d\tau
\]

\[
\leq tF_3(t, R/t) + C_3(t)(|u_0| + \varepsilon) + tK(tF_3(t, R/t) + C_2(t)(|u_0| + \varepsilon))
\]

\[
=tF_3(t, R/t) + C_4(t)(|u_0| + \varepsilon).
\]

We relabel the function \( F_i \) and this completes our proof. \( \square \)

Remark 2. Now, fix any \( \varepsilon \in (0,1) \) and any \( \xi \in \mathcal{A}_\varepsilon \subset \mathcal{A}_1 \). The definition of (31) can be replaced by \( L_0 = L(x, \phi(u), v) \) with \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) a bounded nondecreasing smooth function such that \( \phi(u) = u \) for \( |u| \leq tF_1(t, R/t) + C_1(t)(1 + |u_0|) \) and \( \phi(u) \equiv u^* \), a suitably selected real number, for \( |u| \geq tF_1(t, R/t) + C_1(t)(1 + |u_0|) + 1 \), where
$F_1(t, R/t)$ and $C_1(t)$ are determined by (16) in Lemma 3 and $F_1$ and $C_1$ are both independent of $\varepsilon$. Therefore, to minimize $J$ defined in (31), we can suppose that $\sup_{\xi \in \mathcal{A}_\varepsilon} \{L(\hat{\xi}(s), u, \hat{\xi}(s))\}$ is bounded by an integrable function $f \in L^1([0, t])$. In this situation, (30) is indeed a Carathéodory equation which admits a unique solution by Proposition 7.

**Corollary 1.** For any $\varepsilon \in (0, 1)$, the set $\{u_\varepsilon : \xi \in \mathcal{A}_\varepsilon\}$ is relatively compact in $C^0([0, t], \mathbb{R})$.

**Proof.** Suppose $x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R}$ and $|x - x_0| \leq R$. For any $\varepsilon \in (0, 1)$ and $\xi \in \mathcal{A}_\varepsilon$. Recall that $u_\varepsilon$ is the unique solution of (30) by Remark 2, it follows from Lemma 3. Invoking Ascoli-Arzela theorem, we get our conclusion. □

**Lemma 4.** Suppose $x_0 \in \mathbb{R}^n, t, R > 0, u_0 \in \mathbb{R}$ and $|x - x_0| \leq R$. Let $\varepsilon \in (0, 1)$ and $\xi \in \mathcal{A}_\varepsilon$. Then there exist a continuous function $F = F_{u_0} : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, $F(r_1, \cdot)$ is nondecreasing and superlinear and $F(\cdot, r_2)$ is nondecreasing for any $r_1, r_2 \geq 0$, such that

$$\int_0^t |\hat{\xi}(s)| ds \leq tf(t, R/t) + \varepsilon.$$

Moreover, the family $\{\hat{\xi}\}_{\xi \in \mathcal{A}_\varepsilon}$ is equi-integrable.

**Proof.** Let $\varepsilon > 0$ and $\xi \in \mathcal{A}_\varepsilon$. Then, by (L2) we obtain

$$u_\varepsilon(t) - u_0 = \int_0^t L(\hat{\xi}(s), u_\varepsilon(s), \hat{\xi}(s)) ds \geq \int_0^t \{L(\hat{\xi}(s), 0, \hat{\xi}(s)) - K|u_\varepsilon(s)|\} ds$$

$$\geq \int_0^t \{\theta_0(|\hat{\xi}(s)|) - c_0 - K|u_\varepsilon(s)|\} ds$$

$$\geq \int_0^t \{\hat{\xi}(s)| - K|u_\varepsilon(s)| - (c_0 + \theta_0(1))\} ds. \quad (22)$$

In view of Lemma 2, Lemma 3 and (22), we obtain that

$$\int_0^t |\hat{\xi}(s)| ds \leq \int_0^t K|u_\varepsilon(s)| ds + t(c_0 + \theta_0(1)) + u_\varepsilon(t) - u_0$$

$$\leq tK(F_1(t, R/t) + C_1(t)(\varepsilon + |u_0|)) + t(c_0 + \theta_0(1))$$

$$+ tF_2(t, R/t) + \varepsilon := tF_3(t, R/t) + \varepsilon.$$

Now we turn to proof of the equi-integrability of the family $\{\hat{\xi}\}_{\xi \in \mathcal{A}_\varepsilon}$. Since $\theta_0$ is a superlinear function, then for any $\alpha > 0$ there exists $C_\alpha > 0$ such that $r \leq \theta_0(r)/\alpha$ for $r > C_\alpha$. Thus, for any measurable subset $E \subset [0, t]$, invoking (L2), (L3) and Lemma 3, we have that
\[ \int_{E \cap \{|\xi| > C_\alpha\}} |\dot{\xi}| ds \leq \frac{1}{\alpha} \int_{E \cap \{|\xi| > C_\alpha\}} \theta_0(|\xi|) ds \leq \frac{1}{\alpha} \int_{E \cap \{|\xi| > C_\alpha\}} (L_0(\xi, \dot{\xi}) + c_0) ds \]

\[ \leq \frac{1}{\alpha} \int_{E \cap \{|\xi| > C_\alpha\}} (L(\xi, u_\xi, \dot{\xi}) + K|u_\xi| + c_0) ds \]

\[ \leq \frac{1}{\alpha} (u_\xi(t) - u_0 + K(tF_1(t, R/t) + C_1(t)(\varepsilon + |u_0|)) + tc_0) \]

\[ \leq \frac{1}{\alpha} (tF_2(t, R/t) + 1 + K(tF_1(t, R/t) + C_1(t)(1 + |u_0|)) + tc_0) \]

\[ := \frac{1}{\alpha} F_4(t, R/t). \]

Therefore, we conclude that

\[ \int_E |\dot{\xi}| ds \leq \int_{E \cap \{|\xi| > C_\alpha\}} |\dot{\xi}| ds + \int_{E \cap \{|\xi| \leq C_\alpha\}} |\dot{\xi}| ds \leq \frac{1}{\alpha} F_4(t, R/t) + |E|C_\alpha. \]

Then, the equi-integrability of the family \( \{\dot{\xi}\}_{\xi \in \mathcal{C}_\alpha} \) follows since the right-hand side can be made arbitrarily small by choosing \( \alpha \) large and \( |E| \) small, and this proves our claim.

**Proposition 1.** The functional

\[ \Gamma'_{x_0} \ni \xi \mapsto J(\xi) = \int_{t_0}^t L(\xi(s), u_\xi(s), \dot{\xi}(s)) \, ds, \]

where \( u_\xi \) is determined by \[30\] with initial condition \( u_\xi(0) = u_0 \), admits a minimizer in \( \Gamma'^{t_0}_{x_0} \).

**Proof.** Fix \( x_0, x \in \mathbb{R}^n \), \( t > 0 \) and \( u_0 \in \mathbb{R} \). Consider any minimizing sequence \( \{\xi_k\} \) for \( J \), that is, a sequence such that \( J(\xi_k) \rightarrow \inf \{ J(\xi) : \xi \in \Gamma'^{t_0}_{x_0} \} \) as \( k \rightarrow \infty \). We want to show that this sequence admits a cluster point which is the required minimizer. Notice there exists an associated sequence \( \{u_{\xi_k}\} \) given by \[30\] in the definition of \( J(\xi_k) \). The idea of the proof is standard but a little bit different.

First, notice that Lemma \[4\] implies that the sequence of derivatives \( \{\dot{\xi}_k\} \) is equi-integrable. Since the sequence \( \{\xi_k\} \) is equi-integrable, by the Dunford-Pettis Theorem there exists a subsequence, which we still denote by \( \{\xi_k\} \), and a function \( \eta^* \in L^1([0, t], \mathbb{R}^n) \) such that \( \dot{\xi}_k \rightarrow \eta^* \) in the weak-\( L^1 \) topology. The equi-integrability of \( \{\xi_k\} \) implies that the sequence \( \{\xi_k\} \) is equi-continuous and uniformly bounded. Invoking Ascoli-Arzelà theorem, we can also assume that the sequence \( \{\xi_k\} \) converges uniformly to some absolutely continuous function \( \xi_\infty \in \Gamma'^{t_0}_{x_0} \). For any test function \( \varphi \in C_0^\infty([0, t], \mathbb{R}^n) \),

\[ \int_0^t \varphi \eta^* \, ds = \lim_{k \rightarrow \infty} \int_0^t \varphi \dot{\xi}_k \, ds = \lim_{k \rightarrow \infty} \int_0^t \varphi \dot{\xi}_k \, ds = \int_0^t \varphi \xi_\infty \, ds. \]

By the fundamental lemma in calculus of variation (see, for instance, \[10\] Lemma 6.1.1), we can conclude that \( \xi_\infty = \eta^* \) almost everywhere. Similarly, Corollary \[1\]
implies \( \{ u_{\xi_k} \} \) is relatively compact in \( C^0([0,t], \mathbb{R}) \). Therefore \( \{ u_{\xi_k} \} \) converges uniformly to \( u_{\xi_k} \) by taking a subsequence if necessary.

We recall a classical result (see, for instance, [3, Theorem 3.6] or [2, Section 3.4]) on the sequentially lower semicontinuous property on the functional

\[
L^1([0,t], \mathbb{R}^m) \times L^1([0,t], \mathbb{R}^m) \ni (\alpha, \beta) \mapsto F(\alpha, \beta) := \int_0^t L(\alpha(s), \beta(s)) \, ds.
\]

One has that if (i) \( L \) is lower semicontinuous; (ii) \( L(\alpha, \cdot) \) is convex on \( \mathbb{R}^n \), then the functional \( F \) is sequentially lower semicontinuous on the space \( L^1([0,t], \mathbb{R}^m) \times L^1([0,t], \mathbb{R}^m) \) endowed with the strong topology on \( L^1([0,t], \mathbb{R}^m) \) and the weak topology on \( L^1([0,t], \mathbb{R}^m) \). Now, let \( L(\alpha_k(s), \beta_k(s)) := L(\xi_k(s), u_{\xi_k}(s), \eta_{\xi_k}(s)) \) with \( \alpha_k(s) = (\xi_k(s), u_{\xi_k}(s)) \) and \( \beta_k(s) = \eta_{\xi_k}(s) \), then we have

\[
\liminf_{k \to \infty} \int_0^t L(\xi_k(s), u_{\xi_k}(s), \eta_{\xi_k}(s)) \, ds \geq \int_0^t L(\xi_m(s), u_{\xi_m}(s), \eta_{\xi_m}(s)) \, ds.
\]

Therefore, \( \xi_m \in \Gamma_{x_0,t} \) is a minimizer of \( J \) and this completes the proof of the existence result.

**Corollary 2.** Suppose \( x_0 \in \mathbb{R}^n \), \( t, R > 0 \), \( u_0 \in \mathbb{R} \) and \( |x - x_0| \leq R \). If \( \xi \in \Gamma_{x_0,t} \) is a minimizer for \( J \), then, there exists a continuous function \( F = F_{x_0,t} : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \), \( F(r_1, \cdot) \) is nondecreasing and superlinear and \( F(\cdot, r_2) \) is nondecreasing for any \( r_1, r_2 \geq 0 \), such that

\[
\int_0^t |\xi(s)| \, ds \leq t F(t, R/t).
\]

Moreover, we conclude that

\[
\text{ess inf}_{s \in [0,t]} |\xi(s)| \leq F(t, R/t), \quad \sup_{s \in [0,t]} |\xi(s) - x_0| \leq t F(t, R/t).
\]

**Proof.** The first assertion is direct from Lemma 4. The last two inequality follows from the relations

\[
\text{ess inf}_{s \in [0,t]} |\xi(s)| \leq \frac{1}{t} \int_0^t |\xi(s)| \, ds, \quad \text{and} \quad |\xi(s) - x_0| \leq \int_0^t |\xi(s)| \, ds,
\]

together with the first assertion.
3 Necessary conditions and regularity of minimizers

3.1 Lipschitz estimate of minimizers

To obtain the regularity properties of a minimizers $\xi$ of (31), we need an a priori Lipschitz estimate of $\xi$. For such an estimate, a key point is to verify the Ermann condition, which is standard for classical autonomous Tonelli Lagrangians. Our proof is a modification of the original one by Francis Clarke (see, [15] or [16]).

**Proposition 2.** Suppose $x_0 \in \mathbb{R}^n$, $t, R > 0$, $u_0 \in \mathbb{R}$ and $|x - x_0| \leq R$. If $\xi \in \Gamma_{x_0,t}$ is a minimizer for (31), then, there exists a function $F = F_{u_0,R} : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$, $F(r_1, r_2)$ is nondecreasing and superlinear and $F(\cdot, r_2)$ is nondecreasing for any $r_1, r_2 \geq 0$, such that

$$\text{ess sup}_{s \in [0,t]} |\dot{\xi}(s)| \leq F(t, R/t).$$

**Proof.** Let $\xi \in \Gamma_{x_0,t}$ be a minimizer of (31) with $u_\xi$ determined by (30) and $u_\xi(0) = u_0$. Let $\alpha : [0, t] \rightarrow [1/2, 3/2]$ be a measurable function satisfying $\int_0^t \alpha(s) \, ds = t$ (the set of all such functions $\alpha$ is denoted by $\Omega$), we define

$$\tau(s) = \int_0^s \alpha(r) \, dr, \quad s \in [0, t],$$

then $\tau : [0, t] \rightarrow [0, t]$ is a bi-Lipschitz map and its inverse $s(\tau)$ satisfies

$$s'(\tau) = \frac{1}{\alpha(s(\tau))}, \quad a.e. \tau \in [0, t].$$

Now we define a reparametrization $\eta$ by $\eta(\tau) = \xi(s(\tau))$. It follows that $\dot{\eta}(\tau) = \dot{\xi}(s(\tau))/\alpha(s(\tau))$. Let $u_\eta$ be the unique solution of (30) with initial condition $u_\eta(0) = u_0$, then we have

$$J(\xi) \leq J(\eta) = \int_0^t L(\eta(\tau), u_\eta(\tau), \dot{\eta}(\tau)) \, d\tau = \int_0^t L(s(\xi), u_\xi, \alpha(s), \xi(s)/\alpha(s)) \alpha(s) \, ds$$

where $u_{\xi, \alpha}$ solves

$$\dot{u}_{\xi, \alpha}(s) = L(s(\xi), u_{\xi, \alpha}(s), \dot{\xi}(s)/\alpha(s)) \alpha(s).$$

Notice that
By measurable selection theorem our problem by minimizing where $K$ of close-valued measurable set-valued maps. Since $\alpha \in \{0, 1\}$, we can not apply the standard measurable selection theorem (see, for instance, [18 Corollary 6.23]) directly. But, one can apply the theorem to an increasing sequence of close-valued measurable set-valued maps.

$$u_{\xi, \alpha}(s) - u_\xi(s) = \int_0^s \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha - \mathcal{L}(\xi, u_\xi, \dot{\xi}) \, dr$$

$$\leq K \int_0^s |\alpha u_{\xi, \alpha} - u_\xi| \, dr + \int_0^s \left( \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha - \mathcal{L}(\xi, u_\xi, \dot{\xi}) \right)_+ \, dr,$$

where $f_+ = \max\{f, 0\}$ for any measurable function $f$. By Proposition 4 below, $u_{\xi, \alpha}(s) = u_\eta(s) \preceq u_\xi(s)$ for all $s \in [0, t]$. Invoking Gronwall’s inequality, we obtain

$$|u_{\xi, \alpha}(s) - u_\xi(s)| \leq \exp(K_1 t) \int_0^s \left( \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha - \mathcal{L}(\xi, u_\xi, \dot{\xi}) \right)_+ \, dr,$$

where $K_1 = \frac{\lambda}{2} K \geq K \|\alpha\|_{L^\infty}$. Therefore, we have that

$$J(\xi) \leq \int_0^t \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha \, dr + \int_0^t K_1 |u_{\xi, \alpha}(s) - u_\xi(s)| \, ds$$

$$\leq \int_0^t \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha \, ds + t K_1 \exp(K_1 t) \int_0^t \left( \mathcal{L}(\xi, u_{\xi, \alpha}, \dot{\xi} / \alpha) \alpha - \mathcal{L}(\xi, u_\xi, \dot{\xi}) \right)_+ \, ds.$$

For all $\alpha \in [1/2, 3/2]$, we define

$$\Phi(s, \alpha) = \mathcal{L}(\xi(s), u_\xi(s), \dot{\xi}(s) / \alpha)$$

$$+ t K_1 \exp(K_1 t) \left( \mathcal{L}(\xi(s), u_\xi(s), \dot{\xi}(s) / \alpha) \alpha - \mathcal{L}(\xi(s), u_\xi(s), \dot{\xi}(s)) \right)_+$$

and

$$\Lambda(\alpha) := \int_0^t \Phi(s, \alpha(s)) \, ds.$$

It is clear that

$$J(\xi) = \Lambda(1) \leq \Lambda(\alpha), \quad \alpha \in \Omega.$$

For almost all $s$, by continuity, there exists $\delta(s) \in (0, 1/2]$ such that

$$-1 \leq \Phi(s, \alpha) - \Phi(s, 1) \leq 1, \quad \forall \alpha \in [1 - \delta(s), 1 + \delta(s)].$$

By measurable selection theorem we can assume $\delta(\cdot)$ is measurable. Let $S \subset L^\infty([0, t], \mathbb{R})$ be the set of the measurable functions $\alpha : [0, t] \to [1/2, 3/2]$ such that $\alpha(s) \in [1 - \delta(s), 1 + \delta(s)]$. It is obvious that $S$ is convex. Now, we can formulate our problem by minimizing

$$\Lambda(\alpha) \text{ over } S \subset L^\infty([0, t], \mathbb{R}) \quad (23)$$

\[1\] Since $\delta(s) \in (0, 1/2]$, we can not apply the standard measurable selection theorem (see, for instance, [18 Corollary 6.23]) directly. But, one can apply the theorem to an increasing sequence of close-valued measurable set-valued maps.
where $S$ satisfies the equality constraint
\[ h(\alpha) = \int_0^1 (\alpha(s) - 1) \, ds = 0. \]

We remark that $\Lambda$ is convex in $S$ (we take $\Lambda(\alpha) = +\infty$ if $\alpha$ does not lie in $S$), and $\alpha^* \equiv 1$ solves (23).

The next step is to write the Lagrange multiplier rule for the problem (23) and its solution. By Kuhn-Tucker Theorem (see [16, Theorem 9.4]), we obtain a nonzero vector $(\lambda_1, \lambda_2)$ in $\mathbb{R}^2$ (with $\lambda_1 = 0$ or 1) such that
\[ \lambda_1 \Lambda(\alpha) + \lambda_2 h(\alpha) \geq \lambda_1 \Lambda(\alpha^*), \quad \forall \alpha \in S. \]

It is clear that $\lambda_1 = 1$. Indeed, if $\lambda_1 = 0$, then one can take $\alpha \in S$ such that $h(\alpha) < 0$ which is absurd. Therefore, we have, for any $\alpha$ in $S$, the inequality
\[ \int_0^1 \Phi(s, \alpha(s)) + \lambda_2\alpha(s) \, ds \geq \int_0^1 \Phi(s, 1) + \lambda_2 \, ds. \]

Invoking [15] Proposition 1.2, we deduce that, for almost every $s$ the function
\[ \alpha \mapsto \Phi(s, \alpha) + \lambda_2\alpha \]
attains a minimum over the interval $[1 - \delta(s), 1 + \delta(s)]$ at the interior point $\alpha = 1$. For such a value of $s$, let $E(s) = L(\xi(s), u_\xi(s), \xi'(s)) - \langle L_\nu(\xi(s), u_\xi(s), \xi'(s)), \xi(s) \rangle$. Therefore, by the calculus of the subdifferentials of the convex function $\Phi(s, \alpha)$ (see, for instance, [33 Corollary 4.3.2]), there exists $\mu(s) \in [0, 1]$ such that
\[-\lambda_2 = E(s) + tK_1 \exp(K_1t)\mu(s)E(s). \]

Thus, we conclude the Erdmann condition\footnote{In fact a conservative energy here is $E_1(s) := e^{-\int |\xi(\xi(s), u_\xi(s), \xi'(s))| \, d\tau} \cdot |L_\nu(\xi(s), u_\xi(s), \xi'(s)), \xi(s) \rangle - L(\xi(s), u_\xi(s), \xi(s))|$ which is constant.}
\[ |E(s)| \leq \frac{|\lambda_2|}{1 + tK_1 \exp(K_1t)\mu(s)} \leq |\lambda_2|, \quad \text{a.e. } s \in [0, t]. \quad (24) \]

Finally, let $s$ be such that $\xi(s)$ exists, and such that (24) holds. By convexity, we have that
\[ L(\xi(s), u_\xi(s), \xi'(s))/(1 + |\xi(s)|) - L(\xi(s), u_\xi(s), \xi'(s)) \geq ((1 + |\xi(s)|)^{-1} - 1) \cdot \langle L_\nu(\xi(s), u_\xi(s), \xi'(s)), \xi(s) \rangle \]
\[ \geq (1 + |\xi(s)|)^{-1} - 1) \cdot (L(\xi(s), u_\xi(s), \xi'(s)), \xi'(s) + |\lambda_2|). \]

It follows that
Let \( C = \sup_{s \in [0, t], \|v\| \leq 1} L(\xi(s), u_\xi(s), \dot{\xi}(s)) \leq L(\xi(s), u_\xi(s), \dot{\xi}(s)/(1 + |\dot{\xi}(s)|))(1 + |\dot{\xi}(s)|) + |\lambda_2||\dot{\xi}(s)|. \)

Let \( C = \sup_{s \in [0, t], \|v\| \leq 1} L(\xi(s), u_\xi(s), v) \), then \( C \) is finite (by Corollary 2 and Lemma 3) and we have that

\[
L(\xi(s), u_\xi(s), \dot{\xi}(s)) \leq C + (|\lambda_2|)|\dot{\xi}(s)|.
\]

Therefore, invoking Lemma 3, we obtain that

\[
(C + |\lambda_2| + 1)|\dot{\xi}(s)| - (\Theta_0(C + |\lambda_2| + 1) + c_0)
\]

\[
\leq \Theta_0(|\dot{\xi}(s)|) - c_0 \leq L(\xi(s), 0, \dot{\xi}(s)) \leq L(\xi(s), u_\xi(s), \dot{\xi}(s)) + K|u_\xi(s) |
\]

\[
\leq C + (C + |\lambda_2|)|\dot{\xi}(s)| + (|u_0| + F_1(t, R/t) K).
\]

This leads to

\[
|\dot{\xi}(s)| \leq (\Theta_0(C + |\lambda_2| + 1) + c_0) + C + (|u_0| + F_1(t, R/t) K) := F_2(t, R/t),
\]

which completes the proof.

### 3.2 Regularity of minimizers - Herglotz equations - Lie equations

Let \( \xi \in \Gamma'_{0, \lambda} \) be a minimizer of (31) where \( u_\xi \) is determined uniquely by (30). For any \( \lambda \in \mathbb{R} \) and any Lipschitz function \( \eta \in \Gamma'_{0, \lambda} \), we denote \( \xi_\lambda(s) = \xi(s) + \lambda \eta(s) \). It is clear that \( \xi_\lambda \in \Gamma'_{0, \lambda} \) and \( J(\xi) \leq J(\xi_\lambda) \). Let \( u_{\xi_\lambda} \) be the associated unique solution of (30) with respect to \( \xi_\lambda \) and the initial condition \( u_0 \). Notice that

\[
\frac{\partial}{\partial \lambda} J(\xi_\lambda)|_{\lambda = 0} = \frac{\partial}{\partial \lambda} u_{\xi_\lambda}(t)|_{\lambda = 0} = 0.
\]

Now for any \( s \in [0, t] \) we set

\[
\Delta_\lambda(s) = \frac{u_{\xi_\lambda}(s) - u_\xi(s)}{\lambda} = \frac{1}{\lambda} \int_0^s L(\xi_\lambda, u_{\xi_\lambda}, \dot{\xi}_\lambda) - L(\xi, u_\xi, \dot{\xi}) \, d\tau,
\]

and

\[
f_1^\lambda(s) = \frac{1}{\lambda} \int_0^s L(\xi_\lambda, u_{\xi_\lambda}, \dot{\xi}_\lambda(s)) - L(\xi_\lambda, u_{\xi_\lambda}, \dot{\xi}) \, d\tau,
\]

\[
f_2^\lambda(s) = \frac{1}{\lambda} \int_0^s L(\xi_\lambda, u_{\xi_\lambda}, \dot{\xi}) - L(\xi, u_\xi, \dot{\xi}) \, d\tau.
\]

Then \( f_1^\lambda \) and \( f_2^\lambda \) are all absolutely continuous functions on \([0, t]\), and it follows

\[
\Delta_\lambda(s) = f_1^\lambda(s) + f_2^\lambda(s) + \frac{1}{\lambda} \int_0^s L(\dot{\xi}_\lambda, (u_{\xi_\lambda} - u_\xi)) \, d\tau, \quad s \in [0, t],
\]
where
\[
\overline{L}_\lambda(t) = \int_0^1 L_\lambda(\xi(\tau), u_\xi(\tau) + \theta(u_\xi(\tau) - u_\xi(\tau)), \dot{\xi}(\tau)) \, d\theta, \quad \tau \in [0, t].
\]

Thus, we conclude that for almost all \(s \in [0, t]\), the following Carathéodory equation holds:
\[
\Delta_\lambda(s) = f_1^\lambda(s) + f_2^\lambda(s) + \overline{L}_\lambda(s) \cdot \Delta_\lambda(s)
\]  
(25)
with initial condition \(\Delta_\lambda(t) = a_\lambda\). Notice that \(\lim_{\lambda \to 0} \Delta_\lambda(t)\) exists and \(\lim_{\lambda \to 0} a_\lambda = \frac{\partial}{\partial \tau} u_{\xi}(t)\big|_{\lambda=0}\) since \(\xi\) is a minimizer of \(J\). It is not difficult to solve (25), we obtain that
\[
\Delta_\lambda(s) = a_\lambda e^{\int_0^s L_\lambda(r) \, dr} + e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot \left( f_1^\lambda(r) + f_2^\lambda(r) \right) \, dr.
\]

Since \((\xi_\lambda(s), \dot{\xi}_\lambda(s), u_{\xi}(s))\) tends \((\xi(s), \dot{\xi}(s), u_\xi(s))\) as \(\lambda \to 0\) for almost all \(s \in [0, t]\), together with Proposition 2 and Corollary 1 it follows that, for all \(s \in [0, t]\), we have
\[
f(s) := \frac{\partial}{\partial \lambda} u_{\xi}(s)\big|_{\lambda=0} = e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot \left( f_1^\lambda(r) + f_2^\lambda(r) \right) \, dr.
\]
(26)

where \(g = L_\eta \cdot \eta + L_\xi \cdot \dot{\eta}\) and \(h = L_\alpha\) which are both measurable and bounded. This implies that equality (26) becomes
\[
0 = \int_0^t g(s) + h(s) \cdot e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot \left( f_1^\lambda(r) + f_2^\lambda(r) \right) \, dr \, ds
\]
\[
= \int_0^t g(s) \, ds + e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot \left( \int_0^s f_1^\lambda(r) \, dr \cdot g(r) \, dr \right) _0^s
\]
\[- \int_0^t e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot g(s) \, ds
\]
\[
e^{\int_0^t \overline{L}_\lambda(r) \, dr} \cdot \left( \int_0^s e^{\int_0^s \overline{L}_\lambda(r) \, dr} \cdot g(s) \, ds \right).
\]

It follows that
\[
0 = \int_0^t e^{-\int_0^r h(s) \, ds} \cdot g(s) \, ds = \int_0^t e^{-\int_0^r h(s) \, ds} \cdot (L_\alpha \cdot \eta + L_\xi \cdot \dot{\eta})(s) \, ds.
\]

The equality above can be also obtained by using the boundary condition \(f(0) = 0\). Invoking the fundamental lemma in calculus of variation (see, for instance, Lemma 6.1.1 in [10]), we obtain that, for almost all \(s \in [0, t]\),
\[
\frac{d}{ds} e^{-\int_0^r h(s) \, ds} L_\alpha(u_\xi(s), u_\xi(s), \dot{\xi}(s)) = e^{-\int_0^r h(s) \, ds} L_\alpha(u_\xi(s), u_\xi(s), \dot{\xi}(s)).
\]
This leads to the so called Herglotz equation (for almost all \(s \in [0, t]\))
\[
\frac{d}{ds} L_v(\xi(s), u_\xi(s), \dot{\xi}(s)) = \left\{ L_t(\xi(s), u_\xi(s), \dot{\xi}(s)) + L_u(\xi(s), u_\xi(s), \dot{\xi}(s)) \right\} L_v(\xi(s), u_\xi(s), \dot{\xi}(s)).
\]

(27)

Since \( L \) is of class \( C^2 \) and \( L(x, u, \cdot) \) is strictly convex, then by the standard argument as in \([10\, Section 6.2]\), we conclude that:

**Theorem 1.** Under our standing assumptions, we have the following regularity properties for any minimizer \( \xi \) for \( (31) \):

1. Both \( \xi \) and \( u_\xi \) are of class \( C^2 \) and \( \xi \) satisfies Herglotz equation (27) for all \( s \in [0, t] \) where \( u_\xi \) is the unique solution of (30);
2. Let \( p(s) = L_v(\xi(s), u_\xi(s), \dot{\xi}(s)) \) be the dual arc. Then \( p \) is also of class \( C^2 \) and we conclude that \( (\xi, p, u_\xi) \) satisfies Lie equation (8).

**Remark 3.** If \( L \) is only of class \( C^1 \) satisfying (L1), (L2) and (L3), then all the results in previous sections still hold true. Indeed, we only use the the \( C^1 \) regularity property of \( L \). More precisely, Problem (31) under the subsidiary condition (30) admits a minimizer \( \xi \in \Gamma_s \), which is of class \( C^1 \). Moreover, \( L_v(\xi(\cdot), \dot{\xi}(\cdot)) \) is absolutely continuous on \([0, t]\) and \( \xi \) together with \( u_\xi \) uniquely determined by (30) satisfy Herglotz equation (27) for almost all \( s \in [0, t] \).

**Proof.** We first need to show that \( \xi \) is of class \( C^1 \). Let \( N \) be the set of zero Lebesgue measure where \( \dot{\xi} \) does not exist. For \( \bar{t} \in [0, t] \), choose a sequence \( \{t_k\} \in [0, T] \setminus N \) such that \( t_k \to \bar{t} \). Then \( \dot{\xi}(t_k) \to \bar{v} \) for some \( \bar{v} \in \mathbb{R} \) (up to subsequences) and

\[
L_v(\xi(\bar{t}), u_\xi(\bar{t}), \dot{\xi}(\bar{t})) = \lim_{k \to \infty} L_v(\xi(t_k), u_\xi(t_k), \dot{\xi}(t_k)) = \int_0^\bar{t} \left\{ L_t(\xi(s), u_\xi(s), \dot{\xi}(s)) + L_u(\xi(s), u_\xi(s), \dot{\xi}(s)) \right\} ds
\]

by (27). From the strict convexity of \( L \) it follows that the map \( v \mapsto L_v(\xi(s), u_\xi(s), v) \) is a diffeomorphism. This implies that \( \bar{v} \) is uniquely determined, i.e.,

\[
\lim_{\|v\| \to 0} \frac{L_v(\xi(s, u_\xi(s), v)}{v} = \bar{v}.
\]

Now, by Lemma 6.2.6 in \([10]\), \( \dot{\xi}(\bar{t}) \) exists and \( \lim_{\|v\| \to 0, s \to \bar{s}} \dot{\xi}(s) = \dot{\xi}(\bar{t}) \). It follows \( \xi \) is of class \( C^1 \). In view of (30), \( u_\xi \) is also of class \( C^1 \).

In view of (7), by setting

\[
F(s) = \int_0^s \left\{ L_t(\xi, u_\xi, \dot{\xi}) + L_u(\xi, u_\xi, \dot{\xi}) \right\} d\tau,
\]

we have that

\[
\left. \{L_v(\xi(s), u_\xi(s), v) - F(s)\}\right|_{v=\xi(s)} = L_v(\xi(0), u_\xi(0), \dot{\xi}(0)).
\]
Proposition 4. Let \( x \in \mathbb{R}^n \), \( t \in [0,1] \), and \( A(x, \xi, 0) = u_0 \), being a minimizer.

Then \( \xi(t) \) is a solution of (8) with conditions \( \xi(0) = x, \xi(1) = x \) and \( |\xi(t)| \) defined by

\[
A(t, x, \xi, 0) = u_0 + \inf \int_0^1 L(t, s, \xi(s), \dot{\xi}(s)) \, ds
\]

Proposition 3 (41). Let \( M \) be a \( C^1 \)-closed manifold and let \( L : C^1 \times M \to \mathbb{R} \) satisfy conditions (41) for \( M \) instead of \( \mathbb{R} \) here. Given any \( x \in M \) and \( u_0 \in \mathbb{R} \), there exists a unique continuous function \( h_{u_0}(x) \) defined on \( (0, \infty) \times M \), satisfying

\[
h_{u_0}(t, x) = u_0 + \inf \int_0^t L(t, s, h_{u_0}(s, x), \dot{h}_{u_0}(s, x)) \, ds
\]

The rest of the proof is standard and we omit.
Herglotz’ generalized variational principle

\[ u_\xi(s) = h_{x_0, u_0}(s, \xi(s)), \quad s \in [0, t]. \]  

(29)

Remark 4. The relation (29) holds only when \( \xi \) is a minimizer of (31).

Proof. Suppose \( x_0, x \in \mathbb{R}^n, t > 0 \) and \( u_0 \in \mathbb{R} \). Let \( \xi \in \Gamma^t_{x, \xi} \) be a minimizer of (31) and \( u_\xi(s) = u_\xi(s, u_0) \) be the unique solution of (30) with \( u_\xi(0) = u_0 \).

Now, let \( 0 < t' < t \). Let \( \xi_1 \in \Gamma^t_{x, \xi(t')} \) and \( \xi_2 \in \Gamma^{t-t'}_{\xi(t'), y} \) be the restriction of \( \xi \) on \([0, t']\) and \([t', t]\) respectively. Then, we have that

\[
\begin{align*}
    u_\xi(t', u_0) &= u_0 + \int_0^{t'} L(\xi_1(s), u_{\xi_1}(s), \dot{\xi}_1(s)) \, ds, \\
    u_\xi(t, u_0) - u_\xi(t', u_0) &= \int_{t'}^t L(\xi_2(s), u_{\xi_2}(s), \dot{\xi}_2(s)) \, ds.
\end{align*}
\]

Then both \( \xi_1 \) and \( \xi_2 \) are minimal curve for (31) restricted on \([0, t']\) and \([t', t]\) respectively by summing up the equalities above and the assumption that \( \xi \) is a minimizer of (31). In particular, (28) follows. The next assertion is direct from the relation

\[
u_\xi(s_1 + s_2, u_0) = u_\xi(s_2, u_\xi(s_1)), \quad \forall s_1, s_2 > 0, \ s_1 + s_2 \leq t,
\]

since \( u_\xi \) solves (30). The last assertion is a direct application of Gronwall’s inequality. Indeed, we know that for all \( s \in [0, t] \),

\[
\begin{align*}
u_\xi(s) &= u_0 + \int_0^s L(\xi(r), u_\xi(r), \dot{\xi}(r)) \, dr, \\
h_{x_0, u_0}(s, \xi(s)) &= u_0 + \int_0^s L(\xi(r), h_{x_0, u_0}(r, \xi(r)), \dot{\xi}(r)) \, dr.
\end{align*}
\]

By condition (L3), it follows that

\[
|h_{x_0, u_0}(s, \xi(s)) - u_\xi(s)| \leq K \int_0^s |h_{x_0, u_0}(r, \xi(r)) - u_\xi(r)| \, dr.
\]

Our conclusion is a consequence of Gronwall’s inequality.

### 4.2 Herglotz’ generalized variational principle on manifolds

Now, we try to explain how to move the Herglotz’ generalized variational principle to any connected and closed smooth manifold \( M \).

Fix \( x, y \in M, t > 0 \) and \( u \in \mathbb{R} \). Let \( \xi \in \Gamma^t_{x, \xi}(M) \), we consider the Carathéodory equation

\[
\begin{align*}
    \dot{u}_\xi(s) &= L(\xi(s), u_\xi(s), \dot{\xi}(s)), \quad a.e. \ s \in [0, t], \\
    u_\xi(0) &= u.
\end{align*}
\]

(30)

We define the action functional
where \( \xi \in \Gamma^t_{\alpha}(M) \) and \( u_\xi \) is defined in (30). Our purpose is to minimize \( J(\xi) \) over
\[
\mathcal{A}(M) = \{ \xi \in \Gamma^t_{\alpha}(M) : (30) \text{ admits an absolutely continuous solution } u_\xi \}.
\]
Notice that \( \mathcal{A}(M) \neq \emptyset \) because it contains all piecewise \( C^1 \) curves connecting \( x \) to \( y \). In view of the remark before Lemma 1, for each \( a \in \mathbb{R} \),
\[
\mathcal{A}(M) = \{ \xi \in \Gamma^t_{\alpha}(M) : \text{the function } s \mapsto L(\xi(s), a, \dot{\xi}(s)) \text{ belongs to } L^1([0,t]) \}.
\]
We begin with the case when \( M = \mathbb{R}^n \). Fix \( \kappa > 0 \). Suppose 0 < \( t \leq 1 \), \( x, y \in \mathbb{R}^n \) such that \( |x - y| \leq \kappa t \). Suppose \( \eta \in \mathcal{A}(\mathbb{R}^n) \) is a minimizer of the action functional \( \eta \mapsto J(\eta) \). Invoking the aforementioned \textit{a priori} estimates, \( \eta \) is as smooth as \( L \).
Moreover, there exist constants \( C_1(\kappa) > 0, C_2(u, t, \kappa) > 0 \) such that
\[
\sup_{s \in [0,t]} |\eta(s)| \leq C_1(\kappa), \quad \sup_{s \in [0,t]} |\eta(s) - x| \leq C_1(\kappa)t, \quad \sup_{s \in [0,t]} |u_\eta(s)| \leq C_2(u, t, \kappa).
\]
Let \( D_1 = B_{\mathbb{R}^n}(x, \kappa t) \) and \( D_2 = B_{\mathbb{R}^n}(x, (C_1(\kappa) + 1)t) \), where the subscript is used for the ball in \( \mathbb{R}^n \). Then, \( D_1 \subset D_2 \) since \( \kappa \leq C_1(\kappa) \). By denoting
\[
\mathcal{B}(\mathbb{R}^n) = \{ \eta \in \mathcal{A}(\mathbb{R}^n) : \eta(s) \in D_2 \text{ for all } s \in [0,t] \}.
\]
Therefore we can claim that for any \( x \in \mathbb{R}^n \) and \( y \in D_1 \) the following problems are equivalent:
\[
\inf_{\mathcal{A}(\mathbb{R}^n)} J(\xi) = \inf_{\mathcal{B}(\mathbb{R}^n)} J(\xi)
\]
They admit the same minimizers.

Now we move to the manifold case. Let \( \{(B_t, \Phi_t)\} \) be a local chart for the \( C^2 \) closed manifold \( M \). We can suppose that \( \{B_t\}_{t=1}^N \) is a finite open cover of \( M \) and \( \Phi_t : B_t \to D_2 \subset \mathbb{R}^n \) is a \( C^2 \)-diffeomorphism for each \( i = 1, \ldots, N \) and \( \Phi_j^{-1} \circ \Phi_i : B_i \cap B_j \to B_i \cap B_j \) is a \( C^2 \)-diffeomorphism for each \( i \neq j = 1, \ldots, N \).

Fix \( i \), let \( B = B_i \) and \( \Phi = \Phi_i : B \to D_2 \) be a local coordinate. Let \( L : TM \times \mathbb{R} \to \mathbb{R} \) be a Lagrangian satisfying (L1)-(L3). Then
\[
(\Phi, d\Phi) : TB \to D_2 \times \mathbb{R}^n
\]
defines a local trivialization of \( TB \). Let \( L_\Phi : D_2 \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be defined by
\[
L_\Phi(\bar{x}, u, \bar{v}) = L(\Phi^{-1}(\bar{x}), u, d\Phi^{-1}(\bar{x})\bar{v}), \quad (\bar{x}, \bar{v}) \in D_2 \times \mathbb{R}^n, \ u \in \mathbb{R}.
\]
Therefore, Herglotz’ generalized variational principle for \( L \) restricted to \( TB \times \mathbb{R} \) is equivalent to the one for \( L_\Phi \) on \( D_2 \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) since \( \Phi \) is a bi-Lipschitz homeomorphism and a \( C^2 \)-diffeomorphism.
Proposition 5. Fix $\kappa > 0$, $0 < t \leq 1$. Then there exist a local chart $\{(B_i, \Phi_i)\}_{i=1}^N$ and a constant $C_2(\kappa) > 0$ such that each $B_i \subset B_M(x, C_2(\kappa)t)$, and for any $x, y \in B_i$ and $u \in \mathbb{R}$, the following points on the Herglotz’s generalized variational principle hold:

(a) The functional

$$\mathcal{A}(B_i) \ni \xi \mapsto J(\xi) = \int_0^t L(\xi(s), u_\xi(s), \dot{\xi}(s)) \, ds,$$

where $u_\xi$ is determined by (30) with initial condition $u_\xi(0) = u$, admits a minimizer in $\mathcal{A}(M)$.

(b) Suppose $x, y \in B_i$. Let $\xi \in \mathcal{A}(B_i)$ be a minimizer of $J$. Then there exists a function $F = F_{B_i} : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, with $F(\cdot, r)$ being nondecreasing for any $r \geq 0$, such that

$$|u_\xi(s)| \leq tF(t, \kappa) + C(t)|u|, \quad s \in [0, t]$$

where $C(t) > 0$ is also nondecreasing in $t$.

(c) Suppose $x, y \in B_i$. Let $\xi \in \mathcal{A}(B_i)$ be a minimizer of $J$. Then, there exists a function $F = F_{a,B_i} : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, with $F(\cdot, r)$ is nondecreasing for any $r \geq 0$, such that

$$\text{ess sup}_{s \in [0, t]} |\dot{\xi}(s)| \leq F(t, \kappa).$$

(d) We have the following regularity properties for any minimizer $\xi$ for (31):

1) Both $\xi$ and $u_\xi$ are of class $C^2$ and $\xi$ satisfies Herglotz equation (27) in local charts for all $s \in [0, t]$ where $u_\xi$ is the unique solution of (30);

2) Let $p(s) = L_r(\xi(s), u_\xi(s), \dot{\xi}(s))$ be the dual arc. Then $p$ is also of class $C^2$ and we conclude that $(\xi, p, u_\xi)$ satisfies Lie equation (8) in local charts for all $s \in [0, t]$.

Thus, by using $L_{\Phi_i}$, it is not difficult to see that there exists a finer open cover, which we also denote by $\{(B_i, \Phi_i)\}_{i=1}^N$, such that the Herglotz’ generalized variational principle can be applied in the case when $x, y \in B_i$ and $0 < t \leq 1$ ($i = 1, \ldots, N$) since $\{\Phi_i\}_{i=1}^N$ is equi-bi-Lipschitz.

Now, let us recall the standard “broken geodesic” argument. Pick any $x, y \in M$, $t > 0$ and $u \in \mathbb{R}$. Let $\{(B_i, \Phi_i)\}_{i=1}^N$ be the local chart in the proposition above. We suppose without loss of generality that $x \in B_1$ and $y \in B_N$. Let $\xi \in \mathcal{A}(M)$. Then there exists a partition $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = t$ such that $z_j = \xi(t_j)$ and $z_{j+1} = \xi(t_{j+1})$ are contained in the same $B_j$. For each $j$, we define

$$h_{\xi}^{j+1}(t_{j+1} - t_j, z_j, z_{j+1}, u_j) = \inf_{\xi_j} \int_{t_j}^{t_{j+1}} L(\xi_j(s), u_{\xi_j}(s), \dot{\xi}_j(s)) \, ds,$$
where $\xi_j$ is an absolutely continuous curve constrained in $B_i$ connecting $z_j$ to $z_{j+1}$ and $u_{\xi_j}$ is uniquely determined by (30) with initial condition $u_j$. Now we consider the problem

$$g(t,x,y,u) := \inf \sum_{j=1}^{k} h^L_j(t_{j+1} - t_j, z_j, z_{j+1}, u_j),$$

(32)

where the infimum is taken over any partition $0 = t_0 < t_1 < t_2 < \cdots < t_k = t$, $z_j, z_{j+1} \in M$ contained in the same $B_i$ and $u_j \in \mathbb{R}$. Due to Proposition 5(b), $\{u_j\}$ can be constrained in a compact subset of $\mathbb{R}$ depending on $u, x, y$ and $t$. Therefore the infimum in (32) can be attained. Thanks to the local semiconcavity of the fundamental solution $h^L_j$, $h^L_j$ is differentiable at each minimizer which leads to the fact

$$h^L(t,x,y,u) = g(t,x,y,u).$$

Proposition 6. Proposition 5 holds for any connected and closed $C^2$ manifold $M$.

4.3 Further remarks

Comparing to the method used in [41, 42], one can see more from our approach as follows:

- We can derive the generalized Euler-Lagrange equations in a modern and rigorous way which does not appear in both [41, 42];
- There should be an extension of the main results of this paper under much more general conditions (like Osgood type conditions) to guarantee the existence and uniqueness of the solutions of the associated Carathéodory equation (30);
- Along this line, the quantitative semiconcavity and convexity estimate of the associated fundamental solutions have been obtained in [7] recently, which is useful for the intrinsic study of the global propagation of singularities of the viscosity solutions of [5] and [6] ([8, 4, 5, 6]);
- When the Lagrangian has the form $L(x,v) - \lambda u$, by solving the associated Carathéodory equation (30) directly, one gets the representation formula for the associated viscosity solutions immediately ([19, 40, 43, 13]). The representation formula bridges the PDE aspects of the problem with the dynamical ones;
- Consider a family of Lagrangians in the form $\{L(x,v) + \sum_{i=1}^{k} \alpha_i u_i\}$, a problem of Herglotz’ variational principle in the vector form is closely connected to certain stochastic model of weakly coupled Hamilton-Jacobi equations (see, for instance, [20, 23, 33]).

Acknowledgements This work is partly supported by Natural Scientific Foundation of China (Grant No. 11631006, No. 11771283 and No.11790272), and the National Group for Mathematical Analysis, Probability and Applications (GNAMPA) of the Italian Istituto Nazionale di Alta Matematica “Francesco Severi”. Kaizhi Wang is also supported by China Scholarship Council (Grant No. 201706235019). The authors acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. The
Herglotz’ generalized variational principle

authors are grateful to Qinbo Chen, Cui Chen and Kai Zhao for helpful discussions. This work was motivated when the first two authors visited Fudan University in June 2017. The authors also appreciate the anonymous referee for helpful suggestion to improve the paper.

Appendix

Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open set. A function \( f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is said to satisfy Carathéodory condition if

- for any \( x \in \mathbb{R}^n \), \( f(\cdot, x) \) is measurable;
- for any \( t \in \mathbb{R} \), \( f(t, \cdot) \) is continuous;
- for each compact set \( U \) of \( \Omega \), there is an integrable function \( m_U(t) \) such that

\[
|f(t, x)| \leq m_U(t), \quad (t, x) \in U.
\]

A classical problem is to find an absolutely continuous function \( x \) defined on a real interval \( I \) such that \((t, x(t)) \in \Omega \) for \( t \in I \) and satisfies the following Carathéodory equation

\[
\dot{x}(t) = f(t, x(t)), \quad a.e., t \in I.
\]  \( (33) \)

Proposition 7 (Carathéodory). If \( \Omega \) is an open set in \( \mathbb{R}^{n+1} \) and \( f \) satisfies the Carathéodory conditions on \( \Omega \), then, for any \((t_0, x_0)\) in \( \Omega \), there is a solution of \( (33) \) through \((t_0, x_0)\). Moreover, if the function \( f(t, x) \) is also locally Lipschitzian in \( x \) with a measurable Lipschitz function, then the uniqueness property of the solution remains valid.

For the proof of Proposition 7 and more results related to Carathéodory equation \( (33) \), the readers can refer to [17, 24].

References

1. Arnol’d, V. I.: Mathematical methods of classical mechanics. Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, (1989)
2. Buttazzo, G.: Semicontinuity, relaxation and integral representation in the calculus of variations. Pitman Research Notes in Mathematics Series, 207. Longman, (1989)
3. Buttazzo, G., Giaquinta, M., Hildebrandt, S.: One-dimensional variational problems. An introduction. Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press, Oxford University Press, (1998)
4. Cannarsa, P., Cheng, W.: Generalized characteristics and Lax-Oleinik operators: global theory. Calc. Var. Partial Differential Equations 56, 56:125, (2017)
5. Cannarsa, P., Cheng, W., Fathi, A.: On the topology of the set of singularities of a solution to the Hamilton-Jacobi equation. C. R. Math. Acad. Sci. Paris 355, 176–180, (2017)
6. Cannarsa, P., Cheng, W., Mazzola, M., Wang, K., Global generalized characteristics for the Dirichlet problem for Hamilton-Jacobi equations at a supercritical energy level, preprint, arXiv:1803.01591, 2018.
7. Cannarsa, P., Cheng, W., Yan, J.: *Regularity properties of the fundamental solutions of Hamilton-Jacobi equations of contact type*, preprint, 2017.

8. Cannarsa, P., Cheng, W., Zhang, Q.: *Propagation of singularities for weak KAM solutions and barrier functions*. Comm. Math. Phys. 331, 1–20, (2014)

9. Cannarsa, P., Quincampoix, M.: *Vanishing discount limit and nonexpansive optimal control and differential games*. SIAM J. Control Optim. 53, 1789–1814, (2015)

10. Cannarsa, P., Sinestrari, C.: *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*. Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser Boston, Inc., Boston, MA, (2004)

11. Carathéodory, C.: *Calculus of variations and partial differential equations of the first order*, second edition, Chelsea Publishing, New York, (1989)

12. Chen, C., Cheng, W.: *Lasry-Lions, Lax-Oleinik and generalized characteristics*. Sci. China Math. 59, 1737–1752, (2016)

13. Chen, C., Cheng, W., Zhang, Q.: *Lasry-Lions approximations for discounted Hamilton-Jacobi equations*. J. Differential Equations 265, 719–732, (2018)

14. Chen, Q., Cheng, W., Ishii, H., Zhao, K.: *On the vanishing contact structure for viscosity solutions of contact type Hamilton-Jacobi equations II: stationary equations*, preprint, (2018)

15. Clarke, F.: *A Lipschitz regularity theorem*. Ergodic Theory Dynam. Systems 27, 1713–1718, (2007)

16. Clarke, F.: *Functional analysis, calculus of variations and optimal control*. Graduate Texts in Mathematics, 264. Springer, (2013)

17. Codlington, E. A., Levinson, N.: *Theory of ordinary differential equations*. McGraw-Hill, (1955)

18. Davini, A.; Fathi, A.; Iturriaga, R., Zavidovique, M.: *Convergence of the solutions of the discounted equation: the discrete case*. Math. Z. 284, 1021–1034, (2016)

19. Davini, A.; Fathi, A.; Iturriaga, R., Zavidovique, M.: *Convergence of the solutions of the discounted Hamilton-Jacobi equation*. Invent. Math. 206, 29–55, (2016)

20. Davini, A., Zavidovique, M.: *Aubry sets for weakly coupled systems of Hamilton-Jacobi equations*. SIAM J. Math. Anal. 46, 3361–3389, (2014)

21. Eisenhart, L. P.: *Continuous groups of transformations*. Dover Publications, New York, (1961)

22. Fathi, A.: *Weak KAM theorem in Lagragian dynamics*, to be published by Cambridge University Press.

23. Figalli, A., Gomes, D.; Marcon, D.: *Weak KAM theory for a weakly coupled system of Hamilton-Jacobi equations*. Calc. Var. Partial Differential Equations 55, Art. 79, (2016)

24. Filippov, A. F.: *Differential equations with discontinuous righthand sides*. Mathematics and its Applications (Soviet Series), 18. Kluwer Academic Publishers Group, (1988)

25. Furuta, K., Sano, A., Atherton, D.: *State Variable Methods in Automatic Control*, John Wiley, New York, (1988)

26. Georgieva, B., Guenther, R.: *First Noether-type theorem for the generalized variational principle of Herglotz*. Topol. Methods Nonlinear Anal. 20, 261–273, (2002)

27. Georgieva, B., Guenther, R.: *Second Noether-type theorem for the generalized variational principle of Herglotz*. Topol. Methods Nonlinear Anal. 26, 307–314, (2005)

28. Giaquinta, M., Hildebrandt, S.: *Calculus of variations. II. The Hamiltonian formalism*. Grundlehren der mathematischen Wissenschaften, 311. Springer-Verlag, (1996)

29. Gomes, D. A.: *Generalized Mather problem and selection principles for viscosity solutions and Mather measures*. Adv. Calc. Var. 1, 291–307, (2008)

30. Guenther, R., Gottsch, A., Guenther, C.: *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*. Schauder Center For Nonlinear Studies, Copernicus University, (1995)

31. Herglotz, G.: *Berührungstransformationen*, Lectures at the University of Göttingen, Göttingen, (1930)

32. Herglotz, G.: *Gesammelte Schriften* (H. Schwerdtfeger, ed.), Vandenhoeck & Ruprecht, Göttingen, (1979)

33. Hiriart-Urruty, J.-B.; Lemarchal, C.: *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, (2001)
34. Ishii, H., Mitake, H., Tran, H. V.: *The vanishing discount problem and viscosity Mather measures. Part 1: The problem on a torus*. J. Math. Pures Appl. (9) **108**, 125–149, (2017)

35. Ishii, H., Mitake, H., Tran, H. V.: *The vanishing discount problem and viscosity Mather measures. Part 2: Boundary value problems*. J. Math. Pures Appl. (9) **108**, 261–305, (2017)

36. Iturriaga, R., Sánchez-Morgado, H. *Limit of the infinite horizon discounted Hamilton-Jacobi equation*. Discrete Contin. Dyn. Syst. Ser. B **15**, 623–635, (2011)

37. Marò, S., Sorrentino, A.: *Aubry-Mather theory for conformally symplectic systems*. Comm. Math. Phys. **354**, 775–808, (2017)

38. Mitake, H., Siconolfi, A., Tran, H. V., Yamada, N.: *A Lagrangian approach to weakly coupled Hamilton-Jacobi systems*. SIAM J. Math. Anal. **48**, 821–846, (2016)

39. Mrugała, R.: *Contact transformations and brackets in classical thermodynamics*. Acta Phys. Polon. A **5**, 19–29, (1980)

40. Su, X., Wang, L., Yan, J.: *Weak KAM theory for Hamilton-Jacobi equations depending on unknown functions*. Discrete Contin. Dyn. Syst. **36**, 6487–6522, (2016)

41. Wang, K., Wang, L., Yan, J.: *Implicit variational principle for contact Hamiltonian systems*. Nonlinearity **30**, 492–515, (2017)

42. Wang, K., Wang, L., Yan, J.: *Variational principle for contact Hamiltonian systems and its applications*, to appear in Journal de Mathématiques Pures et Appliquées.

43. Zhao K., Cheng, W.: *On the vanishing contact structure for viscosity solutions of contact type Hamilton-Jacobi equations I: Cauchy problem*. preprint, arXiv:1801.06088, (2018)