Higher rho numbers and the mapping of analytic surgery to homology

Paolo Piazza, Thomas Schick and Vito Felice Zenobi
May 29, 2019

Abstract
Let $\Gamma$ be a finitely generated discrete group and let $\tilde{M}$ be a Galois $\Gamma$-covering of a smooth compact manifold $M$. Let $u: X \to B\Gamma$ be the associated classifying map. Finally, let $S^\Gamma_\ast(\tilde{M})$ be the analytic structure group, a K-theory group appearing in the Higson-Roe analytic surgery sequence $\cdots \to S^\Gamma_\ast(\tilde{M}) \to K_\ast(M) \to K_\ast(C^*\Gamma) \to \cdots$. Under suitable assumptions on the group $\Gamma$ we construct two pairings, first between $S^\Gamma_\ast(\tilde{M})$ and the delocalized part of the cyclic cohomology of the complex group ring of $\Gamma$, and secondly between $S^\Gamma_\ast(\tilde{M})$ and the relative cohomology $H^\ast(M \to B\Gamma)$. Both are compatible with known pairings associated with the other terms in the Higson-Roe sequence. In particular, we define higher rho numbers associated to the rho class $\rho(\tilde{D}) \in S^\Gamma_\ast(\tilde{M})$ of an invertible $\Gamma$-equivariant Dirac type operator on $\tilde{M}$. This applies, for example, to the rho class $\rho(g)$ of a positive scalar curvature metric $g$ on $M$, if $M$ is spin. Regarding the first pairing we establish in fact a more general result, valid without additional assumptions on $\Gamma$: indeed, we prove that it is possible to map the Higson-Roe analytic surgery sequence to the long exact sequence in even/odd-graded noncommutative de Rham homology

\[ \cdots \to H_{-1}(A\Gamma) \xrightarrow{j} H^\text{del}_{-1}(A\Gamma) \xrightarrow{\delta} H^\ast(A\Gamma) \xrightarrow{j} \cdots \]

with $A\Gamma$ a dense holomorphically closed subalgebra of $C^\ast_r\Gamma$ and $H^\text{del}_\ast(A\Gamma)$ and $H^\ast(A\Gamma)$ denoting versions of the delocalized homology and the homology localized at the identity element, respectively.

Contents

1 Introduction 2

2 The relation between the structure algebra $D^\ast(\tilde{M})^\Gamma$ and $\Psi^0_\Gamma(\tilde{M})$ 6

2.1 Realizing the surgery sequence with pseudodifferential operators 7

3 Rho classes 8

3.1 Analytic properties of $\pi_{\geq}(\tilde{D})$ 8

3.2 Rho classes and K-theory of pseudodifferential operators 8

4 Non-commutative de Rham homology and Chern characters 9

4.1 Non-commutative de Rham homology 9

4.2 Chern characters 10

4.3 The homology exact sequence of group algebras 14

5 Pseudodifferential operators and smooth subalgebras 15

5.1 Lott’s isomorphisms 15

5.2 Pseudodifferential operators with coefficients in differential forms on $C^\ast_r\Gamma$ 17

5.3 MF calculi associated to dense holomorphically closed subalgebras of $C^\ast_r\Gamma$ 17
1 Introduction

Let $\Gamma$ be a discrete group and $\tilde{M} \rightarrow M$ a Galois $\Gamma$-covering of a smooth compact manifold $M$. In this context Higson and Roe [12] employ large scale index theory to construct a long exact sequence of K-theory groups

$$\cdots \rightarrow K_{*+1}(C^*(\tilde{M})^\Gamma) \rightarrow K_{*+1}(D^*(\tilde{M})^\Gamma) \rightarrow K_{*+1}(D^*(\tilde{M})^\Gamma/C^*(\tilde{M})^\Gamma) \rightarrow \cdots$$

We call this sequence the Higson-Roe exact sequence or the Higson-Roe analytic surgery sequence.

This sequence organizes and captures a large amount of primary and secondary higher index information about geometric operators defined on $M$ and $\tilde{M}$. In particular, in a canonical way

$$K_*(C^*(\tilde{M})^\Gamma) \cong K_*(\text{C}_r\text{-alg.}^\Gamma),$$

the K-theory of the reduced C*-algebra of $\Gamma$, and

$$K_{*+1}(D^*(\tilde{M})^\Gamma/C^*(\tilde{M})^\Gamma) \cong K_*(M),$$

the K-homology of $M$ (this latter isomorphism is called Paschke duality). Moreover, under these isomorphisms the connecting homomorphism of the long exact sequence $K_{*+1}(D^*(\tilde{M})^\Gamma/C^*(\tilde{M})^\Gamma) \rightarrow K_*(C^*(\tilde{M})^\Gamma)$ becomes the Baum-Connes assembly map $K^\Gamma_*(\tilde{M}) \rightarrow K_*(\text{C}_r\text{-alg.}^\Gamma)$. Setting

$$S^\Gamma_*(\tilde{M}) := K_{*+1}(D^*(\tilde{M})^\Gamma),$$

we can rewrite the Higson-Roe surgery sequence as

$$\cdots \rightarrow K_{*+1}(\text{C}_r\text{-alg.}^\Gamma) \rightarrow S^\Gamma_*(\tilde{M}) \rightarrow K_*(M) \xrightarrow{\partial} K_*(\text{C}_r\text{-alg.}^\Gamma) \rightarrow \cdots$$

In geometric applications, the group $K_*(\text{C}_r\text{-alg.}^\Gamma)$ is the home of index invariants associated to $\Gamma$-equivariant geometric operators $\tilde{M}$. For example, if $M$ has a spin structure, the C*-algebraic index of the associated spin Dirac operator on $\tilde{M}$ lives in $K_*(\text{C}_r\text{-alg.}^\Gamma)$; similarly, if $M$ is oriented the C*-algebraic analytic signature of $M$, defined in terms of the signature operator on $\tilde{M}$, is an element in this group.

The group $S^\Gamma_*(\tilde{M})$ is called the analytic structure group by Higson and Roe. It is the home of secondary invariants. In particular, if $M$ is a spin manifold with a positive scalar curvature metric $g$, one obtains
a secondary invariant \( \rho(g) \in S^\Gamma_{\dim M}(\bar{M}) \), called the rho class of \( g \), which captures information about the positive scalar curvature metric \( g \), and allows often to distinguish different bordism classes of such metrics. Similarly, if \( N \xrightarrow{f} M \) is an oriented homotopy equivalence, then we have a rho class \( \rho(f) \in S^\Gamma_{\dim M} (\bar{M}) \) and this class can distinguish triples \( N' \xrightarrow{f'} M \) and \( N \xrightarrow{f} M \) that are not h-cobordant or, put it differently, triples that define distinct elements in the manifold structure set \( S(M) \).

More generally, the rho class can be defined for any \( L^2 \)-invertible \( \Gamma \)-equivariant Dirac operator \( \tilde{D} \) on \( \bar{M} \), \( \rho(\tilde{D}) \in S^\Gamma_{\dim M} (\bar{M}) \); in fact, we can allow for invertible smoothing perturbations of such operators, see [31].

As already mentioned, there are two main domains where the Higson-Roe exact sequence is used: signature invariants of orientable manifolds, as part of surgery theory to classify smooth manifolds, and classifications of metrics of positive scalar curvature. The first domain is organized in the celebrated surgery exact sequence of Browder, Novikov, Sullivan and Wall, and the connection between topology and analysis is made systematic by the construction of a map of exact sequences, from the surgery exact sequence in topology to the Higson-Roe analytic surgery sequence. This is carried out originally in [12–14] using analytic homological algebra. A more direct construction of this map, based on certain higher index theorems, is given in [31]. This second approach is in fact inspired by the precursor work [30], where the Stolz exact sequence for positive scalar curvature metrics (compare [30] Proposition 1.27) is mapped to the Higson-Roe exact sequence. The Stolz exact sequence is the analogue, in the world of positive scalar curvature metrics, of the surgery exact sequence in differential topology; it organizes questions about existence and (bordism) classification of Riemannian metrics of positive scalar curvature in a long exact sequence of groups, but where the main terms are somewhat hard to analyze directly.

The results of Piazza-Schick about the Stolz surgery sequence [30] were established in odd dimensions. Subsequently, Xie and Yu gave a different treatment of these results, valid in all dimensions and based on Yu’s localization algebras (see [39], [37]). The even dimensional case of the results of Piazza-Schick was later settled in [41], using suspension. Yet different treatments of these results were given by Zeidler in [40] and by Zenobi in [43].

Both for the surgery sequence in topology and for the Stolz sequence for positive scalar curvature metrics, the map to the Higson-Roe exact sequence allows to apply all the tools from \( C^* \)-algebraic index theory and \( C^* \)-algebra \( K \)-theory to obtain information about the geometrically defined sequences. In this toolbox we find, in particular, homological methods. This has a long history for primary index invariants: in the seminal paper of Connes and Moscovici [7], under favourable assumptions on the group \( \Gamma \), for example if \( \Gamma \) is Gromov hyperbolic, a pairing between the group \( K_*(C_{red}^*(\Gamma)) \) and the cohomology of \( \Gamma \) was defined. When applied to the index class of a Dirac operator this pairing produced numbers called the higher indeces of the operator. The pairing employs in a crucial way the transition from the group cohomology of \( \Gamma \) to the cyclic cohomology of \( C\Gamma \). The higher index theorem of Connes and Moscovici gives an explicit geometric formula for these numbers.

Lott, inspired by the work of Bismut on the family index theorem, gave a different treatment of the Connes-Moscovici theorem, both for defining the higher indeces and for computing them. This approach employs in a crucial way the Chern character homomorphism of Karoubi, from \( K_*(C_{red}^*(\Gamma)) \equiv K_*(A\Gamma) \) to the noncommutative de Rham homology groups \( H_*(A\Gamma) \), with \( A\Gamma \) a suitable smooth subalgebra of \( C_{red}^*(\Gamma) \). Lott establishes an explicit formula for the Chern character of the index class and this formula is valid without any assumption on \( \Gamma \); only in a second stage, under the same assumptions of Connes and Moscovici, for example if \( \Gamma \) is Gromov hyperbolic, the Chern character of the index class is paired with the \( C\Gamma \)-cyclic cocycles associated to group cocycles for \( \Gamma \).

We extract the following information from the above discussion: by mapping the index class first to noncommutative de Rham homology and then pairing the result with cyclic cohomology (this second step under additional assumption on \( \Gamma \)) it has been possible to define higher indices for a \( \Gamma \)-equivariant Dirac operator on \( \bar{M} \).

One goal of this work is to give an answer to the following question: can one define higher rho numbers associated to the rho class \( \rho(\tilde{D}) \in S^\Gamma_{\dim M} (\bar{M}) \) of an invertible \( \Gamma \)-equivariant Dirac operator?
We shall give a positive answer to this question in two ways. First, following [4], denote by $HC^*(\mathcal{C}^*; \langle x \rangle)$ the summand of the cyclic cohomology of $\mathcal{C}^*$ supported on a conjugancy class $\langle x \rangle$ of $\Gamma$. If $\Gamma$ is Gromov hyperbolic, we prove that there is a pairing 
\[
\langle \cdot, \cdot \rangle : HC^*(\mathcal{C}^*; \langle x \rangle) \times S^1_x(M) \to \mathbb{C} \quad \forall x \neq e.
\]  
(1.1)

Secondly, under different assumptions on $\Gamma$ that are for example satisfied by groups that are Gromov hyperbolic or of polynomial growth, we prove that there is a well defined pairing 
\[
\langle \cdot, \cdot \rangle_{rel} : H^*(M) \to \mathbb{C}.
\]  
(1.2)

We shall now put these two results in context. The first result should be thought of as the analogue of the Connes-Moscovici pairing but done here for the group $S^1_x(M)$, which is the recipient of secondary invariants, instead of $K_*(C_{red}^*\Gamma)$, which is the recipient of primary invariants. Following Lott’s approach to the Connes-Moscovici higher index theorem, we shall in fact solve a much more general problem. Indeed, without any additional assumption on the group $\Gamma$ we shall map the whole Higson-Roe analytic surgery sequence to an homological sequence in non-commutative de Rham homology: 
\[
\cdots \to H_{s-1}(\mathcal{A}\Gamma) \xrightarrow{j} H_{s-1}^{del}(\mathcal{A}\Gamma) \xrightarrow{\delta} H^*_s(\mathcal{A}\Gamma) \xrightarrow{j} \cdots
\]  
(1.3)

with $\mathcal{A}\Gamma$ a dense holomorphically closed subalgebra of $C^*_red\Gamma$. In a second stage, building on work of Puschnigg and also Meyer, we pair the “delocalized parts” of the cyclic cohomology of $\mathcal{C}^*$, $HC_{del}^*(\mathcal{C}^*; \langle x \rangle)$, with elements in $H^*_s(\mathcal{A}\Gamma)$, the “delocalized part” of the noncommutative de Rham homology of $\mathcal{A}\Gamma$ of a (suitable) smooth subalgebra $\mathcal{A}\Gamma$ of $C^*_red\Gamma$. It is only at this point that we need to make additional assumptions on the group $\Gamma$, for example that $\Gamma$ is Gromov hyperbolic. In this case the smooth subalgebra $\mathcal{A}\Gamma$ is the one constructed by Puschnigg in [42]. Thus, under this additional assumption we show that there is a well defined pairing 
\[
\langle \cdot, \cdot \rangle : HC^*(\mathcal{C}^*; \langle x \rangle) \times H^*_s(\mathcal{A}\Gamma) \to \mathbb{C} \quad \forall e \neq x \in \Gamma
\]  
(1.4)

and we can therefore define the higher rho number associated to the rho class $\rho(\tilde{D})$ and a class $\tau \in HC_{del}^*(\mathcal{C}^*; \langle x \rangle)$ as 
\[
\rho_{\tau}(\tilde{D}) := \langle \tau, Ch^{del}(\rho(\tilde{D})) \rangle.
\]  
(1.5)

As an easy application of our analysis we prove that by pairing the rho class $\rho(\tilde{D})$ with the delocalized trace associated to a non-trivial conjugacy class $< g >$, 
\[
\tau_{< g >}(\sum_{\gamma} \alpha_\gamma) := \sum_{\gamma \in < g >} \alpha_\gamma,
\]  
we obtain precisely the delocalized eta invariant of Lott (24, [32]): 
\[
\langle \tau_{< g >}, Ch^{del}(\rho(\tilde{D})) \rangle = \eta_{< g >}(\tilde{D}).
\]  
More generally, using our results, the delocalized Atiyah-Patodi-Singer index theorem in K-theory [4] and the higher APS index theorem in noncommutative de Rham homology [18] we are able to show (if $M$ is a boundary) that 
\[
Ch^{del}(\rho(\tilde{D})) = -\vartheta_{Lott}(\tilde{D}) \quad \text{in} \quad H_*(\mathcal{A}\Gamma)^{del}
\]
where on the right hand side appears the higher rho invariant of Lott.

The mapping of the Higson-Roe analytic surgery sequence to an homological sequence was a program that had been already proposed in [32]. One might wonder why it took so long to achieve it. The problem is that the algebra $D^*(\tilde{M})^T$ is “very large”: what made the difference for the present work is the description given by the third author of the surgery sequence through the adiabatic groupoid [43], involving much smaller algebras. Here we use a slight variation of this description, see [42]. In [42] the compatibility between the Higson-Roe approach and the approach via groupoids is carefully discussed.
The mapping of the Higson-Roe surgery sequence to homology already appears in the work of Deeley and Goffeng, see [8]. However, their solution passes through the geometric realisation of the Higson-Roe sequence, where geometric means à la Baum-Douglas; more precisely, in the work of Deeley-Goffeng one maps out of the Higson-Roe surgery sequence by inverting the isomorphisms going from the the geometric realisation of the Higson-Roe sequence to the true Higson-Roe surgery sequence. Inverting explicitly these isomorphisms, especially for the part involving \( S^1_*(\tilde{M}) \), is a difficult problem. Our Chern characters, on the other hand, are computable by direct calculations. It remains an interesting problem to show the full compatibility of the two results.

We finally come to the second paring, the one with the relative cohomology \( H^*(M \xrightarrow{u} B\Gamma) \). There is certainly a pairing, obtained via the Chern character

\[
H^*(M) \times K_*(M) \to \mathbb{C}.
\]  

Let us identify \( H^*(\Gamma) = H^*(B\Gamma) \). As already pointed out, if \( \Gamma \) is Gromov hyperbolic or of polynomial growth, we have the Connes-Moscovici pairing, rewritten as

\[
H^*(B\Gamma) \times K_*(C^*_red\Gamma) \to \mathbb{C}
\]

Writing the Higson-Roe analytic surgery sequence with Paschke duality

\[
\cdots \to S^1_*\left(\tilde{M}\right) \to K^1_*(\tilde{M}) \xrightarrow{\rho} K_*(C^*_red\Gamma) \to S^{\nu}_{\nu-1}(\tilde{M}) \cdots
\]

and comparing it with the long exact sequence

\[
\cdots \to H^{n+1}(M \xrightarrow{u} B\Gamma) \to H^n(M) \to H^n(B\Gamma) \to H^n(M \xrightarrow{u} B\Gamma) \to \cdots
\]

we understand that \( S^n_*(\tilde{M}) \) should be paired with \( H^n(M \xrightarrow{u} B\Gamma) \) with \( * = n \mod 2 \). This is precisely what we achieve in the second part of this article. Crucial for this result is an Alexander-Spanier description of \( H^*(M \xrightarrow{u} B\Gamma) \), the new description by Zenobi of the group \( S^n_*(\tilde{M}) \), together with arguments directly inspired by the work of Connes and Moscovici. We show that the pairing \((1.2)\) exists if \( \Gamma \) satisfies the Rapid Decay (RD) condition and has a combing of polynomial growth (for example, if \( \Gamma \) is Gromov hyperbolic).

The paper is organized as follows. In Section 2 and Section 3 we recall the alternative description, due to the third author, of the Higson-Roe surgery sequence and the realization of the rho class of an invertible Dirac operator in this new context. This new realisation of the surgery sequence employs \( \Gamma \)-equivariant pseudodifferential operators on \( \tilde{M} \) in an essential way. In Section 4.1 we recall the definition of Karoubi’s Chern character and we extend it to the relative case. In Section 4.2 we discuss results of Lott on pseudodifferential operators on \( \tilde{M} \) with coefficients \( \Omega_*(C\Gamma) \). Section 5 is devoted to one of the the main result of this paper, the mapping of the analytic surgery sequence to homology. As already remarked, this mapping is valid for any finitely generated discrete group \( \Gamma \). In this section we also compare the delocalized Chern character of the \( \rho \) class associated to an invertible Dirac operator \( \tilde{D} \) to Lott’s class \( \alpha_{\text{out}}(\tilde{D}) \) and show that at least if \( M \) is a boundary \( \text{Ch}^{\text{del}}(\rho(\tilde{D})) = -\alpha_{\text{out}}(\tilde{D}) \in H^4_{\text{del}}(\mathcal{M}) \). In Section 6, assuming the group \( \Gamma \) to be Gromov hyperbolic, we finally define the higher \( \rho \) numbers associated to classes in \( HC^*_\text{red}(C\Gamma; x) \). In Section 7 we discuss carefully how to realize the relative cohomology groups \( H^*(M \xrightarrow{u} B\Gamma) \) in terms of locally zero \( \Gamma \)-equivariant Alexander-Spanier cocycles on \( \tilde{M} \); we use this description and the pseudodifferential realization of \( S^n_*(\tilde{M}) \) in order to define, under suitable assumptions on \( \Gamma \), the pairing \( H^*(M \xrightarrow{u} B\Gamma) \times S^n_*(\tilde{M}) \to \mathbb{C} \).

As this article is rather long, we would like to summarize at this point our main results:

1. The development of a relative Chern character in the setting of relative non-commutative de Rham homology, see Section 4.2.
2. The mapping of the analytic surgery exact sequence to non-commutative de Rham homology for a general discrete group \( \Gamma \), see Theorem 6.33.
(3) the isomorphism $H_{\lambda,\text{pol}}^\ast(\Gamma;\langle x \rangle) \cong H_{\lambda}^\ast(\Gamma;\langle x \rangle)$ for $\Gamma$ hyperbolic, which says that the cyclic cohomology of the group ring can be calculated using cochains of polynomial growth along each conjugacy class, see Corollary 7.12.

(4) the construction of the pairing $HC^\ast(\Gamma;\langle x \rangle) \times S_0^\ast(\widehat{M}) \to \mathbb{C}$, $\forall \langle e \rangle \neq \langle x \rangle \in (\Gamma)$ for hyperbolic groups, see Theorem 7.15 and the definition of higher $\bar{\varrho}$-numbers given in Definition 7.16.

(5) the description of $H^\ast(M \xrightarrow{\varrho} B\Gamma)$ using suitable Alexander-Spanier cocycles, via the isomorphism $H^\ast_{\ast,0,\Gamma}(\widehat{M}) \xrightarrow{\cong} H^\ast(M \xrightarrow{\varrho} B\Gamma)$, see Proposition 8.11.

(6) the construction of the pairing $H^\ast(M \xrightarrow{\varrho} B\Gamma) \times S_0^\ast(\widehat{M}) \to \mathbb{C}$ for discrete groups with property RD and with a combing of polynomial growth, see Theorem 8.33 and the corresponding definition of higher $\bar{\varrho}$-numbers given in Definition 8.34.

The results of this paper relative to hyperbolic groups were announced in a seminar at Fudan University in January 2019. At the same time the article by Chen, Wang, Xie and Yu [5] appeared, giving, among other things, a different treatment of some of the same results. The two articles have a non-trivial intersection, and the techniques employed for the common results are completely different.

Acknowledgements.
We are indebted to Nigel Higson for suggesting the pairing [1,2] and Proposition 8.11 (which we therefore call “Higson’s Lemma”). We thank Michael Puschnigg for very useful discussions about hyperbolic groups. We also had interesting correspondence and discussions with John Lott, Ralf Meyer and Ryszard Nest and we take this opportunity to thank them all.

Part of this work was performed while the second author was visiting Sapienza Università di Roma, with a 3-months visiting professorship. There were also several short-time visits of Schick and Zenobi to Sapienza. We are grateful to Sapienza Università di Roma for the financial support that made all these visits possible.

2 The relation between the structure algebra $D^\ast(\overline{M})^\Gamma$ and $\Psi_0^\Gamma(\overline{M})$

Let $\Psi_0^{\Gamma,\text{loc}}(\widehat{M})$ be the algebra of $\Gamma$-equivariant 0th-order pseudodifferential operators on $\widehat{M}$ with Schwartz kernel of $\Gamma$-compact support. Let $\Psi_0^\Gamma(\widehat{M})$ denote the $C^*$-closure of $\Psi_0^{\Gamma,\text{loc}}(\widehat{M})$ in the bounded operators of $L^2(\widehat{M})$. Recall the algebra $D_0^\ast(\widehat{M})^\Gamma$ of finite propagation pseudolocal bounded operators on $L^2(\widehat{M})$ and its $C^*$-closure $D^\ast(\widehat{M})^\Gamma$. We don’t repeat the details of the definitions of these algebras, as we are actually not using them. Directly from the definition of $D_0^\ast(\widehat{M})^\Gamma$ and the properties of pseudodifferential operators, we understand that if $P \in \Psi_0^{\Gamma,\text{loc}}(\widehat{M})$ then $P \in D_0^\ast(\widehat{M})^\Gamma$. One might then wonder what is the precise relationship between the algebra $\Psi_0^\Gamma(\widehat{M})$ and the algebra $D^\ast(\widehat{M})^\Gamma$, at least as far as K-theory is concerned. The next subsection, based on [12], gives an answer to this question.

We shall consider the mapping cone $C^*$-algebra of a $\ast$-homomorphism $\varphi: A \to B$. We will use both notations $K_\ast(A \xrightarrow{\varphi} B)$ and $K_\ast(\varphi)$ for denoting the K-theory of this mapping cone. This is also called the relative K-theory of $\varphi$. In the realization of $K_\ast(\varphi)$ as in [1, Section 2.3], elements of $K_0(\varphi)$ are given by homotopy classes of triples $[p_0, p_1, q_i]$ with $p_0, p_1$ projections over $A$ and $q_i$ a path of projections over $B$ such that $\varphi(p_i) = q_i$ for $i = 0, 1$. Furthermore, elements in $K_1(\varphi)$ are couples $[u, f]$ with $u$ a unitary over $A$ and $f$ a path of unitary over $B$ joining $\varphi(u)$ and the identity.
2.1 Realizing the surgery sequence with pseudodifferential operators.

Consider the classic short exact sequence of pseudodifferential operators

$$0 \to C^*_r(\tilde{X} \times \Gamma, \tilde{X}) \to \Psi_0^e(\tilde{M}) \xrightarrow{\partial} C(S^*M) \to 0$$

where we recall that $\Psi_0^e(\tilde{M})$ denotes the $C^*$-closure of $\Psi_0^e(\tilde{M})$ in the bounded operators of $L^2(\tilde{M})$. Consider the natural inclusion $C(M) \xrightarrow{m} \Psi_0^e(\tilde{M})$, with $m$ denoting the map that associates to $f \in C(M)$ the multiplication operator by the equivariant lift $\tilde{f}$ of $f$. This induces a long exact sequence in K-theory for the mapping cones $C^*$-algebras:

$$\cdots \xrightarrow{\partial} K_*(0 \to C^*_r(\tilde{X} \times \Gamma, \tilde{X})) \xrightarrow{\iota} K_*(C(M) \xrightarrow{m} \Psi_0^e(\tilde{M})) \xrightarrow{\partial} K_*(C(M) \xrightarrow{\pi} C(S^*M)) \xrightarrow{\partial} \cdots$$

where $\pi: S^*M \to M$ is the bundle projection of the cosphere bundle. Observe that $K_*(0 \to C^*_r(\tilde{X} \times \Gamma, \tilde{X}))$ is nothing but $K_*(C^*_r(\tilde{M} \times \Gamma, \tilde{M}) \otimes C_0([0,1]))$. In [42] the author proves the existence of a canonical isomorphism of long exact sequences

$$\xymatrix{ & K_{*+1}(C^*(\tilde{M})^\Gamma) \ar[d] \ar[r] & K_{*+1}(D^*(\tilde{M})^\Gamma) \ar[d] \ar[r] & K_{*+1}(D^*(\tilde{M})^\Gamma/C^*(\tilde{M})^\Gamma) \ar[d] \ar[r] & \cdots \ar[d] \\
K_*(C^*_r(\tilde{M} \times \Gamma, \tilde{M}) \otimes C_0([0,1])) & K_*(C(M) \xrightarrow{m} \Psi_0^e(\tilde{M})) \ar[r] & K_*(C(M) \xrightarrow{\pi} C(S^*M)) \ar[r] & \cdots }$$

(2.1)

We remark that the mapping cone $C^*$-algebra $C_\pi$, that is $C(M) \xrightarrow{\pi^*} C(S^*M)$, is isomorphic to $C_0(T^*M)$. Indeed, if the $f: X \to Y$ is a continuous map of topological spaces, then the mapping cone of $f$ is the quotient of $X \times [0,1] \sqcup Y$ by the relation $(x,1) \sim f(x)$ and $(x,0) \sim (x',0)$. It is easy to see that the algebra of continuous functions on the mapping cone of $f$ vanishing at $[X \times \{0\}]$ is isomorphic to the mapping cone $C^*$-algebra of $f^*: C(Y) \to C(X)$. Since the mapping cone of $\pi$ is the one-point compactification of $T^*X$, one has that the mapping cone $C^*$-algebra associated to $\pi^*$ is isomorphic to $C_0(T^*M)$.

Let us consider the pair sequence of $m$

$$\cdots \xrightarrow{\partial} K_*(C(M) \otimes C_0([0,1])) \xrightarrow{\iota} K_*(\Psi_0^e(\tilde{M}) \otimes C_0([0,1])) \xrightarrow{\iota} K_*(m) \xrightarrow{(ev_{\iota})_*} K_*(C(M)) \xrightarrow{\partial} \cdots$$

where the map $ev_\iota$ is given by the following composition

$$K_*(m) \xrightarrow{\iota} K_*(\pi^*) \xrightarrow{\sim} K_*(C_0(T^*M)) \xrightarrow{j_*} K_*(C(D^*M)) \xrightarrow{i_*} K_*(C(M)) .$$

(2.2)

Here, $\pi^*: C(M) \to C(S^*M)$ is the pull-back map through the bundle projection, $j: T^*M \to D^*M$ is given by a diffeomorphism between the cotangent bundle and the open codisk bundle, and $i: M \to D^*M$ is the inclusion. If the Euler characteristic of $M$ is zero, then $j^*$ is the 0 map and, consequently, so is $(ev_\iota)_*$. To prove that $j^* = 0$ consider the following exact sequence

$$\cdots \xrightarrow{j_*} K_*(C_0(T^*M)) \xrightarrow{ev_\iota^*} K_*(C(D^*M)) \xrightarrow{i_*} K_*(C(S^*M)) \xrightarrow{\partial} \cdots$$

(2.3)

Then it is easy to see that its boundary map is given by the composition of the suspension isomorphism $K_*(C(S^*M)) \to K_{*+1}(C_0(T^*M \setminus M))$ and the inclusion $i: C_0(T^*M \setminus M) \to C_0(T^*M)$. Indeed, if the Euler characteristic of $M$ is zero, the cotangent bundle of $M$ admits a nonzero section $\xi$. Then take a diffeomorphism $\varphi$ between $T^*M$ and the open disc bundle $D^*M$ and a diffeomorphism $\psi$ between $D^*M$ and $U_\xi$, a tubular neighbourhood of the image of $\xi$. The morphism $h = \psi \circ \varphi$ induces a morphism $h_*: C_0(T^*M) \to
$C_0(T^*M \setminus M)$ such that $\iota_* \circ h_* = \text{Id} : K_*(C_0(T^*M)) \to K_*(C_0(T^*M))$. This means that $\iota_*$ is surjective and, by exactness of $[2,3]$, that $j_*$ is zero. This tells us that $K_*(\Psi^0_\Gamma(\widetilde{M})) \to K_{+1}(m)$ is surjective and then any class in $K_{+1}(m)$ can be lifted, up to Bott periodicity, to $K_*(\Psi^0_\Gamma(\widetilde{M}))$. Observe that this is always true for $n + 1 = \dim M$ odd. Thus, when the Euler characteristic of $M$ is zero, we have the following surjective morphism of exact sequences:

$$
\cdots \to K_{+1}(C^*_\Gamma(\widetilde{M} \times \widetilde{M})) \to K_{+1}(\Psi^0_\Gamma(\widetilde{M})) \to K_{+1}(C(S^*M)) \to \cdots \\
\cdots \to K_*(C^*_\Gamma(\widetilde{M} \times \widetilde{M}) \otimes C_0(0,1)) \to K_*(C(M) \otimes \Psi^0_\Gamma(\widetilde{M})) \to K_*(C(M) \otimes C(S^*M)) \to \cdots
$$

(2.4)

We shall use this information in order to simplify some of our formulas.

### 3 Rho classes

#### 3.1 Analytic properties of $\pi_\geq(\widetilde{D})$

Assume that $\widetilde{D}$ is a Dirac type operator on $\widetilde{M}$ which is $L^2$-invertible. Let $\pi_\geq : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[0, \infty)$. Functional calculus then allows us to define $\pi_\geq(\widetilde{D})$ as a bounded operator on $L^2(\widetilde{M})$. We shall need more precise properties of this operator and these are given in the next two propositions.

**3.1 Proposition.** The bounded operator $\pi_\geq(\widetilde{D})$ is an element in the $C^*$-closure $\Psi^0_\Gamma(\widetilde{M})$ of $\Psi^0_{\Gamma,c}(\widetilde{M})$.

**Proof.** First recall that there is a chopping function $f : \mathbb{R} \to [-1, 1]$ which is odd, monotonically increasing, smooth, “quasi-Schwarz”, with $\lim_{t \to \infty} f(t) = 1$ and such that its Fourier transform has compact support. Note that this Fourier transform has a singularity at zero and is smooth outside. Then $f(\widetilde{D})$ is a pseudodifferential operator of order zero, which is $\Gamma$-compactly supported because of finite propagation speed and the compact support of $f$. By the mapping properties of $f$, $\pi_\geq(\widetilde{D}) = \pi_\geq(f(\widetilde{D}))$. Because $C^*$-algebras are closed under continuous functional calculus and because $f(\widetilde{D})$ is invertible, $\pi_\geq(f(\widetilde{D}))$ is contained in the $C^*$-closure of $\Psi^0_{\Gamma,c}(M)$.

#### 3.2 Proposition. If $\widetilde{D}$ is $L^2$-invertible, then $\pi_\geq(\widetilde{D})$ belongs to any holomorphically closed $*$-algebra $\mathcal{A}$ such that

$$\Psi^0_{\Gamma,c}(\widetilde{M}) \subset \mathcal{A} \subset \Psi^0(\widetilde{M}).$$

**Proof.** With the choice of $f$ as in the previous proof, so that $f(\widetilde{D}) \in \Psi^0_{\Gamma,c}(\widetilde{M})$, we observe that $\pi_\geq(f(\widetilde{D}))$ is obtained by holomorphic functional calculus; indeed for any $T \in \Psi^0_{\Gamma,c}(\widetilde{M})$ invertible in $\Psi^0_{\Gamma}(\widetilde{M})$ we have

$$\pi_\geq(T) = \frac{T \circ (\sqrt{T^*T})^{-1} + 1}{2}.$$ 

It then follows that $\pi_\geq(f(\widetilde{D}))$ belongs to any algebra closed under holomorphic functional calculus which contains $\Psi^0_{\Gamma,c}(\widetilde{M})$.

#### 3.3 Remark. The same reasoning applies to the operator defined by the sign function, $\text{sign}(\widetilde{D})$. 

3.2 Rho classes and K-theory of pseudodifferential operators

We want to give the new realisation of the ρ classes in the setting of relative K-theory, see [42]. Let $\tilde{M}$ be a Galois $\Gamma$-covering of a closed smooth manifold $M$. Let $\tilde{D}$ be an $L^2$-invertible $\Gamma$-equivariant Dirac operator on $M$.

3.4 Definition. • If dim $M$ is even, then $\rho(\tilde{D})$ is defined as the class in $K_0(m)$ given by

$$
\left[ \begin{array}{ccc}
1_+ & 0 & 0 \\
0 & 0 & 0 \\
0 & 1_+ & 0
\end{array} \right] \cdot \left( \begin{array}{ccc}
\cos^2(\frac{\pi}{2}t)1_+ & \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t)\chi(\tilde{D})_+ \\
\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t)\chi(\tilde{D})_- & \sin^2(\frac{\pi}{2}t)1_-
\end{array} \right)
$$

(3.5)

where $1_+$ is the projection which defines the $C(M)$-module $C(M, E_+)$ of continuous sections of the even part $E_+$ of the Clifford bundle $E$, entering into the definition of $\tilde{D}$, and $1_-$ is defined correspondingly using the odd part of $E$.

• If dim $tM$ is odd then $\rho(\tilde{D})$ is defined as the class in $K_1(m)$ given by $[1, 1 + \pi_\geq(\tilde{D})(z - 1)]$, with $z \in S^1 \subset \mathbb{C}$.

It is proved in [42] that these definitions are compatible with the original ones, as given in [30], through the isomorphism [22]. Notice that in the odd dimensional case we know that $K_1(\Psi^0_1(M) \otimes C_0(0, 1)) \to K_1(m)$ is surjective. In fact, the ρ class we have just defined is the image, up to suspension isomorphism, of the class in $K_0(\Psi^0_1(M))$ given by $[\pi_\geq]$. Put it differently, $[\pi_\geq] \in K_0(\Psi^0_1(M))$ is a natural lift of $\rho(\tilde{D}) \in K_1(m)$.

In the sequel, we shall be interested in pairing $\rho(\tilde{D}) \in K_1(m)$ with suitable delocalized cyclic cocycles. Since the kernel of $K_1(\Psi^0_1(M) \otimes C_0(0, 1)) \to K_1(m)$ is $K_0(C(M))$ (up to suspension isomorphism) and since this group pairs trivially with such delocalized cocycles, we conclude that in the odd dimensional case we can focus solely on $K_0(\Psi^0_1(M))$ and on $[\pi_\geq] \in K_0(\Psi^0_1(M))$.

3.6 Example. Let $g$ be a $\Gamma$-equivariant metric with positive scalar curvature on a spin $\Gamma$-covering $\tilde{M}$ of $M$. Then we will denote by $\rho(g) \in K_*(m)$ the ρ-class associated to the spin Dirac operator $\tilde{D}$ defined on the spinor bundle. Recall that it is $L^2$-invertible because the scalar curvature is positive.

4 Non-commutative de Rham homology and Chern characters

4.1 Non-commutative de Rham homology

For the material in this section we refer the reader to [10]. Let $A$ be a unital Fréchet algebra. A Fréchet differential graded algebra (DGA) $(\Omega_*, d)$, with $\Omega_* = \oplus \Omega_i$, is said to be a DGA over $A$ if $\Omega_0 = A$. Let $\Omega_{ab,*}$ denote the vector space quotient of $\Omega_*$ by the closure of the graded commutators $[\Omega_i, \Omega_j^*]$. Notice that $d$ induces a differential on $\Omega_{ab,*}$ since $d[\omega_i, \omega_j] = [d\omega_i, \omega_j] + [\omega_i, d\omega_j]$. Define the de Rham homology $H_*(\Omega_*, d)$ as the $i$-th homology group of the complex $(\Omega_{ab,*}, d)$. To be able to apply standard homological algebra, we use non-reduced homology, i.e. we quotient by the image of the differential, and not its closure. Here, our conventions differ from part of the literature. However, there is the canonical map from unreduced to reduced homology, and our invariants are lifts of the previously constructed ones (where applicable), and thus contain a priori even more information.

4.1 Example. Let $\tilde{\Omega}_*(A)$ be the universal differential graded algebra of $A$, constructed using the projective tensor product. Then set

$$
H_*(A) := H_*(\tilde{\Omega}_*(A)).
$$

Observe that, by universality, we always have a map of complexes $\theta : \tilde{\Omega}_*(A) \to \Omega_*$ for any Fréchet DGA $\Omega_*$ over $A$. Consequently, one always has a map $\theta_* : H_*(A) \to H_*(\Omega_*)$. 


Now consider a homomorphism of Fréchet algebras $\varphi: A \to B$. We will work with relative non-commutative de Rham homology in this context and work out the basics for this now.

**4.2 Definition.** A pair of DGAs over $\varphi$ is given by a homomorphism of DGAs $\Phi_*: \Omega_*^A \to \Omega_*^B$ with $(\Omega^A, d^A)$ a DGA over $A$, $(\Omega^B, d^B)$ a DGA over $B$ and $\Phi_0 = \varphi$. The universal pair over $\varphi$ is given by the homomorphism of DGAs $\Omega_*(\varphi): \Omega_*(A) \to \Omega_*(B)$, obtained by the universal property of $\Omega_*(A)$.

By general principles, we can associate to a pair over $\varphi$ the mapping cone complex

$$C(\Phi)_* := \Omega^A_{ab} \star_1 \oplus \Omega^B_{ab} \star$$

which is equipped with the differential

$$d^\Phi := \begin{pmatrix} d^A & 0 \\ -\Phi_{*+1} & d^B \end{pmatrix}.$$ 

**4.3 Definition.** Define the relative non-commutative de Rham homology of the pair $\Phi_*: \Omega_*^A \to \Omega_*^B$ as the homology of the mapping cone complex and denote it by $H_*$. The universal relative non-commutative de Rham homology of $\varphi$ is the homology of the mapping cone complex of $\Omega_*(\varphi)$ and is denoted by $H_*(\varphi)$.

**4.2 Chern characters**

Let $(\Omega_*, d)$ be a DGA over $A$. We are going to recall the definition of the Chern character

$$\text{Ch}: K_0(A) \to H_{\text{even}}(\Omega_*, d).$$

(4.4)

For this, consider a class $[p] \in K_0(A)$, where $p$ is an $n \times n$ matrix over $A$, and denote by $E$ the finitely generated projective module $pA^n$. Then $\nabla := p \circ d \circ p: E \to E \otimes_A \Omega_1$ is a connection and its curvature $\Theta := \nabla^2$ is an $A$-linear endomorphism of $E$ with coefficients in $\Omega_2$ explicitly given by $pd(pd)$.

**4.5 Definition.** The degree $2k$ part of the Chern character of $[p] \in K_0(A)$ associated to the DGA $(\Omega_*, d)$ is defined as the class of

$$\text{Ch}_{2k}(p) := \frac{1}{k!} \text{Tr}(\Theta^k) \in H_{2k}(\Omega_*, d).$$

To be rigorous we should keep track of $\Omega_*$ in the notation for the Chern character, but we will avoid to do that in order to lighten the notation.

We want to define a relative version of the Chern character. To this end we need to prove a transgression formula for the absolute Chern character. Let $p_t$ be a smooth path of projections in $M_n(A)$, with $t \in [0, 1]$. We regard it as a projection $\tilde{p}$ over $C^\infty[0, 1] \otimes A$. Moreover, if $(\Omega_*, d)$ is a Fréchet DGA over $A$, then we can canonically associate to it the following Fréchet DGA over $C^\infty[0, 1] \otimes A$

$$\left( \left( C^\infty[0, 1] \oplus C^\infty[0, 1] dt \right) \otimes \Omega_* \right),$$

$$D := dt \wedge \frac{\partial}{\partial t} \otimes \text{id} + \text{id} \otimes d.$$ 

The connection $\tilde{p} \circ D \circ \tilde{p}$ is equal to

$$p_t \circ d \circ p_t + p_t \circ dt \wedge \frac{\partial}{\partial t} \circ p_t$$

(4.6)

and its curvature $(\tilde{p} \circ D \circ \tilde{p})^2$ is equal to

$$\Theta_t + \Xi_t \wedge dt$$

for some path of 1-forms $\Xi_t$ and with $\Theta_t$ being the curvature of the connection $p_t \circ d \circ p_t$.

**4.7 Lemma.** The path $\Xi_t$ of $A$-linear morphisms with coefficient in $\Omega_1$ is given by $p_t dp_t \tilde{p}_t + p_t \tilde{p}_t dp_t$. 

Paolo Piazza, Thomas Schick and Vito Felice Zenobi
and the result follows.

Proof. First recall that the equality \( p_t = p_t^2 \) implies that \( \dot{p}_t(1-p_t) = p_t \dot{p}_t \). It follows that \( p_t \dot{p}_t p_t = 0 \) and that \( 0 = d(p_t \dot{p}_t p_t) = d(p_t \dot{p}_t p_t) = p_t \dot{p}_t dp_t \). By a direct calculation we see that

\[
(p_t \circ D \circ \dot{p}_t)^2 = p_t \circ D \circ \dot{p}_t \circ dt \wedge \frac{\partial}{\partial t} \circ p_t + \frac{\partial}{\partial t} \circ p_t \circ \dot{p}_t \circ dt \wedge \frac{\partial}{\partial t} \circ p_t \circ \dot{p}_t \circ dt = \]

\[
= (p_t \circ D \circ \dot{p}_t \circ p_t + p_t \circ D \circ \dot{p}_t \circ p_t + \dot{p}_t \circ D \circ p_t \circ \dot{p}_t \circ dt) \wedge dt. \quad (4.8)
\]

Now apply the coefficient of \( dt \) in the last line to an element \( x_t \in \tilde{p}(C^\infty[0,1] \otimes A^n) \). Then

\[
p_t d(p_t \frac{\partial}{\partial t}(p_t x_t)) + p_t \frac{\partial}{\partial t}(p_t d(p_t x_t)) =
\]

\[
= p_t d(p_t \dot{p}_t x_t + p_t \dot{x}_t) + p_t \frac{\partial}{\partial t}(p_t d(p_t x_t) + p_t dx_t) =
\]

\[
= p_t \dot{p}_t d(p_t x_t + p_t \dot{x}_t) + p_t d(p_t \dot{p}_t x_t + p_t \dot{x}_t) + p_t d(p_t x_t + p_t \dot{x}_t) + p_t d(p_t x_t) =
\]

\[
= (p_t \dot{p}_t \dot{p}_t + p_t \dot{p}_t \dot{p}_t) x_t,
\]

and the result follows.

It follows that the \( 2k \)-degree component of the Chern character of \( \dot{p} \) is of the following form

\[
\text{Ch}_{2k}(\dot{p}) = \text{Ch}_{2k}(p_t) + \text{Ch}_{2k-1}(p_t) \wedge dt. \quad (4.10)
\]

We call \( \text{Ch}_{2k-1}(p_t) \) in the above formula the transgression Chern character.

4.11 Lemma. The transgression Chern character is explicitly given by

\[
\text{Ch}_{2k-1}(p_t) = \frac{1}{k!} \text{Tr} \left( \Theta_t^{k-1} \Omega_i \right) \quad (4.12)
\]

and satisfies the transgression formula

\[
\frac{\partial}{\partial t} \text{Ch}_{2k}(p_t) = -d \text{Ch}_{2k-1}(p_t). \quad (4.13)
\]

It is functorial for algebra maps \( f : A \rightarrow B \) covered by a DGA map \( F : \Omega^A \rightarrow \Omega^B \).

Proof. The first statement is given by a direct calculation of \( \text{Ch}_{2k}(\dot{p}) \) as follows

\[
\frac{1}{k!} \text{Tr} \left( (\Theta_t + \Omega_i \wedge dt)^k \right) =
\]

\[
= \frac{1}{k!} \text{Tr} \left( \Theta_t^k + \sum_{j=0}^{k-1} \Theta_t^j (\Omega_i \wedge dt) \Theta^{k-j-1} \right) =
\]

\[
= \frac{1}{k!} \text{Tr} (\Theta_t^k) + \frac{1}{(k-1)!} \text{Tr} (\Omega_i \Theta_t^{k-1}) \wedge dt.
\]

Here we used the fact that \( \text{Tr} \) is linear and zero on graded commutators. Moreover, since \( \text{Ch}_{2k}(\dot{p}) \) is closed with respect to the differential \( D \) and \( \text{Ch}_{2k}(p_t) \) is closed with respect to \( d \), applying \( D \) to \( 4.10 \) we obtain \( 4.13 \). Functoriality follows directly from \( 4.12 \).

4.14 Corollary. For the smooth path of projections \( p_t \), we have

\[
\text{Ch}_{2k}(p_0) - \text{Ch}_{2k}(p_1) = d \int_0^1 \text{Ch}_{2k-1}(p_t) dt. \quad (4.15)
\]
Now we are ready to define the relative Chern character. Recall that the relative K-group $K_0(\varphi)$ is defined via homotopy classes of triples $(p_0, p_1; q_t)$ where $p_0$ and $p_1$ are projections over $A$ and $q_t$ is a path of projections over $B$ such that $q_0 = \varphi(p_0)$ and $q_1 = \varphi(p_1)$. The equality (4.15) allows to give the following definition.

**4.16 Definition.** The degree $2k - 1$ part of the relative Chern character of $[p_0, p_1; q_t] \in K_0(\varphi)$ associated to a pair of DGAs $\Phi^* : \Omega^0_\bullet \to \Omega^B_\bullet$ over $\varphi$ is defined as

$$\text{Ch}_{2k-1}^\tau([p_0, p_1; q_t]) := \left(\text{Ch}_{2k}(p_0) - \text{Ch}_{2k}(p_1), \int_0^1 \overline{\text{Ch}}_{2k-1}(q_t) \, dt\right) \in H_{2k}(\Phi). \quad (4.17)$$

**4.18 Proposition.** The relative Chern character of a class in $K$-theory does not depend on the choice of a representative.

*Proof.* Let $(p_0^t, p_1^t; q^t)$ be a smooth path of relative K-cycles for $\varphi$. We can see $q^t_1$ as a projection $\tilde{q}$ over $C^\infty(0, 1)^2 \otimes B$. Consider the Frechet DGA $\left(C^\infty[0, 1]^2 \otimes (1 \otimes dt \otimes ds \otimes dt \wedge ds)\right) \otimes \Omega^B_\bullet$, with differential $D := ds \wedge \frac{\partial}{\partial s} + dt \wedge \frac{\partial}{\partial t} + d^B$.

Let $E$ be the module associated to $\tilde{q}$ and let us equip $E$ with the connection $\tilde{q} D \tilde{q}$. Its curvature is given by

$$\Theta = (q^t_1 ds \wedge \frac{\partial}{\partial s} q^t_1 + q^t_1 dt \wedge \frac{\partial}{\partial t} q^t_1 + q^t_1 d^B q^t_1)^2$$

which is the sum of four terms

$$\Xi = \Theta_{t,s} + \Xi_{t,s} \wedge ds + \Xi_{t,s} \wedge ds + \Psi_{t,s} \wedge dt \wedge ds.$$

Then the degree $2k$ component of the Chern character of $\tilde{q}$ is given by

$$\text{Ch}_{2k}(\tilde{q}) = \text{Ch}_{2k}(q_1^t) + \text{Ch}_{2k-1}^\tau(q_1^t) \wedge dt + \text{Ch}_{2k-2}^\tau(q_1^t) \wedge ds + \overline{\text{Ch}}_{2k-2}(q_1^t) \wedge dt \wedge ds \quad (4.19)$$

where $\text{Ch}_{2k-1}$ and $\text{Ch}_{2k-1}^\tau$ are the transgression Chern characters in $4.10$ with respect to the parameter $t$ and $s$, respectively, whereas $\text{Ch}_{2k-2}(q_1^t)$ is a form in $\Omega^B_{2k-2}$. Now, applying $D$ to (4.19), we obtain that $0$ on the left side is equal to

$$0 = D \text{Ch}_{2k}(\tilde{q}) = \left(ds \wedge \frac{\partial}{\partial s} + dt \wedge \frac{\partial}{\partial t}\right) \text{Ch}_{2k}(q_1^t) + d^B \left(\text{Ch}_{2k-1}^\tau(q_1^t) \wedge dt + \text{Ch}_{2k-2}^\tau(q_1^t) \wedge ds\right)$$

$$+ \left(-\frac{\partial}{\partial s} \text{Ch}_{2k-1}(q_1^t) + \frac{\partial}{\partial t} \text{Ch}_{2k-1}^\tau(q_1^t) + d^B \overline{\text{Ch}}_{2k-2}(q_1^t)\right) \wedge dt \wedge ds. \quad (4.20)$$

In particular, $\frac{d}{ds} \text{Ch}_{2k-1}^\tau(q_1^t) + \frac{d}{dt} \text{Ch}_{2k-1}^\tau(q_1^t) + d^B \overline{\text{Ch}}_{2k-2}(q_1^t) = 0$. So we have that

$$\int_0^1 \left(\text{Ch}_{2k-1}^\tau(q_1^t) - \overline{\text{Ch}}_{2k-1}^\tau(q_1^t)\right) dt = \int_0^1 \int_0^1 \frac{\partial}{\partial s} \text{Ch}_{2k-1}(q_1^t) dt \wedge ds =$$

$$- \int_0^1 \int_0^1 \left(\frac{\partial}{\partial t} \text{Ch}_{2k-1}(q_1^t) + d^B \overline{\text{Ch}}_{2k-2}(q_1^t)\right) dt \wedge ds = - \Phi_{2k-1} \left(\int_0^1 \left(\text{Ch}_{2k-1}^\tau(p_0^t) - \text{Ch}_{2k-1}^\tau(p_1^t)\right) ds\right) + d^B \omega$$

$$= - \Phi_{2k-1} \left(\int_0^1 \left(\text{Ch}_{2k-1}^\tau(p_0^t) - \text{Ch}_{2k-1}^\tau(p_1^t)\right) ds\right) + d^B \omega \quad (4.21)$$
for some ω ∈ Ω^B_{2k−1}. This imply that
\[ \text{Ch}_{2k}^{rel}(p_0^0; p_1^0; q_1^0) - \text{Ch}_{2k}^{rel}(p_0^1; p_1^1; q_1^1) = \]
\[ \left( \left( \text{Ch}_{2k}(p_0^0) - \text{Ch}_{2k}(p_0^1) \right) - \left( \text{Ch}_{2k}(p_1^0) - \text{Ch}_{2k}(p_1^1) \right) \right) \int_0^1 \left( \text{Ch}_{2k-1}(q_1^0) - \text{Ch}_{2k-1}(q_1^1) \right) dt = \]
(4.22)
\[ \left( d^A \int_0^1 \left( \text{Ch}_{2k-1}^{rel}(p_0^0) - \text{Ch}_{2k-1}^{rel}(p_1^1) \right) ds, - \Phi_{2k-1} \left( \int_0^1 \left( \text{Ch}_{2k-1}(p_0^0) - \text{Ch}_{2k-1}(p_1^1) \right) ds \right) + d^B \omega \right) \]
where we have applied (4.15) to the first entry. The last term is a boundary in the mapping cone complex. This, together with fact that evidently \text{Ch}_{2k}^{rel} is unchanged under stabilizations, concludes the proof. □

4.23 Remark. Notice that the definition of \( \overline{\text{Ch}}(p_t) \) is coherent with the formula [21 (1.46)], since up to commutators
\[ \Xi, \Omega = (p_d p_t p_t + p_t p_d p_t)(p_d p_t p_t)^{k-1} = \]
\[ = (p_d p_t p_t)^{k-1} p_d p_t p_t + p_t p_d p_t (p_d p_t p_t)^{k-1} = \]
\[ = (p_d p_t p_t)^{k-1} p_d p_t + (1 - p_t)(p_d p_t p_t)^{k-1} p_t = \]
\[ = (1 - p_t)(p_d p_t p_t + p_t p_d (p_d p_t p_t)^{k-1}) = \]
\[ = ((p_t - 1)p_t p_t + p_t p_t)(p_d p_t p_t)^{k-1} = \]
\[ = (2p_t - 1)p_t p_t (p_d p_t p_t)^{k-1} \]
which is exactly the integrand in [21 (1.46)].

Let us now consider the odd case. Let \([u] \) be a class in \( K_1(A) \). By suspension, it corresponds to a class \([p_t] \in K_0(A \otimes C^\infty(0, 1)) \). Then we have the following definition.

4.25 Definition. The \( 2k + 1 \)-th Chern character of \([u] \in K_1(A) \) associated to the DGA \( \Omega^A_* \) is defined as the class
\[ \text{Ch}_{2k+1}(u) := \int_0^1 \overline{\text{Ch}}_{2k+1}(p_t) dt \in H_{2k+1}(\Omega^A). \]
(4.26)

4.27 Remark. Of course, there is also a direct description of \( \text{Ch}_{2k+1}(u) \) in terms of the invertible element \( u \). We use suspension because it is more convenient for our treatment of relative K-theory.

Let \([u; v_1] \) be a relative class in \( K_1(\varphi) \); thus \( v_1 \) is a path of invertible elements from \( u \) to the identity. Then by suspension, it corresponds to a class \([p_t^0, p_t^1; q_1^0] \) in the relative K-theory of \( \varphi \otimes \text{id}: A \otimes C^\infty(0, 1) \to B \otimes C^\infty(0, 1) \), where these projectors are explicitly given in [29 pag. 322].

4.28 Definition. The degree \( 2k \) part of the relative Chern charater of \([u, v_1] \in K_1(\varphi) \) associated to a pair of DGAs \( \Phi_*: \Omega^A_* \to \Omega^B_* \) over \( \varphi \) is defined as the relative class
\[ \text{Ch}_{2k}(u; v_1) := \left( \int_0^1 \overline{\text{Ch}}_{2k+1}(p_t^0) dt - \int_0^1 \overline{\text{Ch}}_{2k+1}(p_t^1) dt, \int_0^1 \int_0^1 \overline{\text{Ch}}_{2k}(q_{t,s}) \wedge ds \wedge dt \right) \in H_{2k}(\Phi). \]
(4.29)

The fact that the odd absolute and relative Chern characters are well-defined and functorial is proved exactly as in the proof of the Proposition 4.18.

We note that relative and absolute Chern character are compatible to each other in the following sense:

4.30 Proposition. For an algebra homomorphism \( \phi: A \to B \) and a homomorphism of DGAs \( \Phi_*: \Omega^A_* \to \Omega^B_* \) covering \( \phi \), the following diagram is commutative:

\[
\begin{array}{ccc}
K_*(\phi) & \longrightarrow & K_*(A) \\
\downarrow \text{Ch}_{2k+1}^{rel} & & \downarrow \text{Ch}_{2k}^{rel} \\
H_{2k+1*}(\Phi) & \longrightarrow & H_{2k+1*}(\Omega^A)
\end{array}
\]
such that

Let us denote by \( \Omega \)

and we will keep the same notation for the morphism induced between the abelianized DGAs.

\[
\int_0^1 \int_0^1 \text{Ch}_{2k-2}(p \otimes \beta) dt \wedge ds = \text{Ch}_{frm-ek}(p) \in H_{2k}(A) \quad \text{for all } [p] \in K_0(A).
\]

Proof. First notice that \( H_{\ast}(S((0, 1) \times (0, 1))) \) is isomorphic to \( \mathbb{R} \) if \( \ast = 2 \) and is 0 otherwise. In particular the integration of forms

\[
\int_0^1 \int_0^1 : H_{\ast}(S((0, 1) \times (0, 1))) \rightarrow \mathbb{R}
\]

gives the explicit isomorphism and \( \int_0^1 \int_0^1 \text{Tr}(\beta d\beta d\beta) = 1. \) Then one has the following calculations

\[
\text{Ch}_{2k}(p) = \text{Ch}_{2k}(p) \otimes 1 = \text{Ch}_{2k}(p) \otimes \int_0^1 \int_0^1 \text{Tr}(\beta d\beta d\beta) = \int_0^1 \text{Ch}_{2k}(p) \otimes \text{Tr}(\beta d\beta d\beta) = \int_0^1 \int_0^1 \text{Ch}_{2k}(p \otimes \beta) dt \wedge ds.
\]

In the last equality we used formula \((4.19)\). For the last but one equality we observe that

\[
\text{Ch}_{2k}(p \otimes \beta) = \text{Tr}(p \otimes \beta d(p \otimes \beta)d(p \otimes \beta)) = \text{Tr}(p \otimes \beta(dp \otimes \beta + p \otimes d\beta)^k);
\]

thus the equality follows because all the terms in the integral which involve higher powers of \( d\beta d\beta \) disappear.

4.3 The homology exact sequence of group algebras

We refer the reader to [8, Section 4.1]. Let \( A \Gamma \) a \( * \)-algebra Fréchet completion of \( C \Gamma \). The inclusion \( C \Gamma \hookrightarrow A \Gamma \) induces the following morphism of DGAs

\[
j : \Omega_{\ast}(C \Gamma) \rightarrow \Omega_{\ast}(A \Gamma)
\]

and we will keep the same notation for the morphism induced between the abelianized DGAs.

4.33 Definition. Let us denote by \( \Omega^{\text{c}}_{\ast}(C \Gamma) \) the sub-DGA of \( \Omega_{\ast}(C \Gamma) \) generated by those forms \( g_0dg_1 \ldots dg_n \) such that \( g_0g_1 \ldots g_n = e \in \Gamma \). Moreover, denote by \( \Omega^{\text{del}}_{\ast}(C \Gamma) \) the complementary space to \( \Omega^{\text{c}}_{\ast}(C \Gamma) \) in \( \Omega_{\ast}(C \Gamma) \) spanned by \( g_0dg_1 \ldots dg_k \) with \( g_0g_1 \ldots g_k = e \in \Gamma \).

We define the closed subcomplex \((\hat{\Omega}^{\text{c}}_{\ast}(A \Gamma)_{ab}, d)\) of \((\hat{\Omega}_{\ast}(A \Gamma)_{ab}, d)\) as the closure of \( j(\Omega^{\text{c}}_{\ast}(C \Gamma)_{ab}) \), and we denote the associated homology group by \( H^{\text{c}}_{\ast}(A \Gamma) \). Consider the resulting short exact sequence of complexes

\[
0 \longrightarrow \hat{\Omega}^{\text{c}}_{\ast}(A \Gamma)_{ab} \xrightarrow{j} \hat{\Omega}_{\ast}(A \Gamma)_{ab} \xrightarrow{q} \hat{\Omega}^{\text{del}}_{\ast}(A \Gamma)_{ab} \longrightarrow 0 \quad (4.34)
\]
where $\hat{\Omega}_{\epsilon}^{dcl}(A\Gamma)_{ab} := \hat{\Omega}_{\epsilon}(A\Gamma)_{ab}/\hat{\Omega}_{\epsilon}(A\Gamma)_{ab}$. We then have the associated long exact sequence of homology groups
\[
\ldots \rightarrow H_*^{s}(A\Gamma) \rightarrow H_*^{dcl}(A\Gamma) \rightarrow \hat{\Omega}_{\epsilon}^{*}(A\Gamma) \rightarrow H_{*+1}^{s}(A\Gamma) \rightarrow \ldots \tag{4.35}
\]
Whereas $\hat{\Omega}_{\epsilon}^{*}(\mathbb{C}\Gamma)_{ab}$ is a direct summand of $\Omega_{\epsilon}^{*}(\mathbb{C}\Gamma)_{ab}$, it is not clear and in general false that $\hat{\Omega}_{\epsilon}^{*}(A\Gamma)_{ab}$ is a chain complex direct summand of $\hat{\Omega}_{\epsilon}^{*}(A\Gamma)_{ab}$. Therefore, in general we can’t expect that (4.35) splits into short exact sequences. To talk of the “localized” and “delocalized” parts of the homology is therefore not really appropriate (although frequently done).

5 Pseudodifferential operators and smooth subalgebras

In this section we shall give a detailed recapitulation of classical facts about pseudodifferential calculus in the Mishchenko-Fomenko setting and holomorphically closed algebras of pseudodifferential operators, mostly following the work of John Lott.

5.1 Lott’s isomorphisms

We want to work rather explicitly with the space of sections of the Mishchenko bundle and with certain classes of pseudodifferential operators on it. For this, it is useful and necessary to consider a couple of identifications. We use the following setup:

- a compact smooth manifold $M$ without boundary;
- a $\Gamma$-principal bundle $\tilde{M} \to M$, where $\Gamma$ acts from the right and we denote by $R_\gamma : \tilde{M} \to \tilde{M}$ the (right) action map for $\gamma \in \Gamma$;
- a smooth vector bundle $E \to M$ and its equivariant lift $\tilde{E}$ to $\tilde{M}$;
- this gives rise to several Mishchenko bundles, e.g. $V_{\mathbb{C}\Gamma} := \tilde{M} \times_\Gamma \mathbb{C}\Gamma$, the first a flat bundle of $\mathbb{C}\Gamma$-right modules (fiberwise free of rank one) on $M$, the latter such a bundle of $\mathbb{C}\Gamma^r$-right modules; more generally if $\Gamma$ acts on an algebra $A$, we will denote by $V_A$ the bundle $\tilde{M} \times_\Gamma A$;
- $\Gamma$ acts from the right on the space of functions $C^\infty(\tilde{M})$ via $f \cdot \gamma := R_{\gamma^{-1}}^* f$, i.e. $(f \cdot \gamma)(x) = f(x\gamma^{-1})$;
- $\Gamma$ acts from the left on $C^\infty(\tilde{M}; \tilde{E} \otimes \mathbb{C}\Gamma)$ via $\gamma \cdot \sum_{g \in \Gamma} f_g g := \sum_{g \in \Gamma} R_{\gamma^{-1}}^* f_g \gamma g$. This commutes with the right $\Gamma$-action which just multiplies on the right in the argument.

5.1 Lemma ([23]). There is a canonical identification of $\mathbb{C}\Gamma$-modules
\[
L : C^\infty_c(\tilde{M}; \tilde{E}) \cong C^\infty(\tilde{M}; \tilde{E} \otimes \mathbb{C}\Gamma)^\Gamma \cong C^\infty(M; E \otimes V_{\mathbb{C}\Gamma}).
\]

The second isomorphism is a tautological one, just rewriting what a section of an associated bundle is in explicit terms. The first isomorphism is given by
\[
s \mapsto \sum_{g \in \Gamma} (R_g^* s) g.
\]

We shall now pass to pseudodifferential operators, concentrating first on the smoothing ones. Recall that, by definition, the smoothing operators $\Psi^{-\infty}(M, V_{\mathbb{C}\Gamma})$ are the smooth sections of the bundle
\[
\text{Hom}_{\mathbb{C}\Gamma}(pr_2^* V_{\mathbb{C}\Gamma}, pr_1^* V_{\mathbb{C}\Gamma}) \to M \times M.
\]
This bundle is an associated bundle for the obvious right action of $\Gamma \times \Gamma$ on $\tilde{M} \times \tilde{M}$ and the left representation given by $\Gamma \times \Gamma \to \text{End}_{\mathbb{C}}(\mathbb{C})$ (with right $\mathbb{C}$-module structure)

$$(\gamma_1, \gamma_2) \cdot \phi = L_{\gamma_1} \circ \phi \circ L^{-1}_{\gamma_2}.$$  

Recalling that $\text{End}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}$ by means of the map $\phi \mapsto \phi(e)$, the action simply becomes on $\mathbb{C}$

$$(\gamma_1, \gamma_2) \cdot \gamma = \gamma_1 \gamma_2^{-1}. $$  

To justify and explain our claim about $\text{Hom}_{\mathbb{C}}(pr_2^*V_{\mathbb{C}}, pr_1^*V_{\mathbb{C}}) \to M \times M$, we observe that $\text{Hom}_{\mathbb{C}}(\mathcal{E}, \mathcal{F}) = \mathcal{F} \otimes_{\mathbb{C}} \mathcal{E}^*$ for two free finitely generated right $\mathbb{C}$-modules, where $\mathcal{E}^* = \text{Hom}_{\mathbb{C}}(\mathcal{E}, \mathbb{C})$. For us, $\mathcal{E} = \mathbb{C}$ which has an additional $\Gamma$-action (by multiplication from the left), and $\gamma \in \Gamma$ acts on the left on $\mathcal{E}^*$ via $$(\gamma \cdot f)(x) := f(\gamma^{-1} x)$$ for $x \in \mathcal{E}$ and $f \in \mathcal{E}^*$ inherits a canonical action of $\gamma \in \Gamma$ by left multiplication in the argument by $\gamma^{-1}$. Under the identification $\mathbb{C} \cong \text{Hom}_{\mathbb{C}}(\mathcal{E}^*, \mathbb{C})$, this action becomes multiplication by $\gamma^{-1}$ on the right and this finally gives the description of the associated bundle as claimed. Then we have the following identifications

$$\Psi^{-\infty}(M, V_{\mathbb{C}}) \cong C^\infty(M \times M, \text{Hom}(pr_2^*V, pr_1^*V)) \cong C^\infty(\tilde{M} \times \tilde{M}, \text{End}_{\mathbb{C}}(\mathbb{C}))^{\Gamma \times \Gamma} \cong C^\infty(M \times M, \mathbb{C})^{\Gamma \times \Gamma}.$$  

If $k_1, k_2 \in C^\infty(\tilde{M} \times \tilde{M}, \mathbb{C})^{\Gamma \times \Gamma}$, where $k_1 = \sum_{g \in \Gamma} (k_1)_g g$ and $f \in C^\infty(\tilde{M}, \mathbb{C})^\Gamma$ is of the form $\sum_{g \in \Gamma} f_g g$, the action is given by

$$(k_1 \ast k_2)_g(x, z) = \sum_{h \in \Gamma} \int_{\tilde{M}/\Gamma} (k_1)_h^{-1}(x, y) \cdot (k_2)_h(y, z) \, dy; \quad (k_1 f)_g(x) = \sum_{h \in \Gamma} \int_{\tilde{M}/\Gamma} (k_1)_h^{-1}(x, y) f_h(y) \, dy$$

where we use that the expressions, as functions of $y$, are $\Gamma$-invariant by the invariance properties we assume. Of course, the integral over $\tilde{M}/\Gamma$ can also be written as an integral over a fundamental domain.

Let us also fix the following identification $\Psi_{\mathcal{C}, \mathbb{C}}^{-\infty}(\tilde{M}) \cong C^\infty(\tilde{M} \times \tilde{M})$, which associates to a smoothing $\Gamma$-equivariant operator with compact support its kernel, which is a smooth $\Gamma$-equivariant function on $\tilde{M} \times \tilde{M}$ with respect to the diagonal action of $\Gamma$ and with $\Gamma$-compact support.

**5.2 Proposition.** [23] Proposition 6] We have the following isomorphism of $^\ast$-algebras

$$C_c^\infty(\tilde{M} \times \Gamma) \cong C^\infty(\tilde{M} \times \tilde{M}, \mathbb{C})^{\Gamma \times \Gamma}, \quad k \mapsto \sum_{\gamma \in \Gamma} R^*_\gamma k \gamma$$  

(5.3)

where $R^*_\gamma k(x, y) := k(xy^{-1}, y\gamma^{-1})$. It is given by $Ad_L$, the adjoint of the canonical isomorphism

$$L: C_c^\infty(\tilde{M}) \to C^\infty(\tilde{M}, \mathbb{C})^\Gamma.$$  

All of this applies in a straightforward way if we add another auxiliary $\Gamma$-equivariant vector bundle. Moreover $L$ clearly extends to an isomorphism of $C^\ast_{\mathbb{C}} \Gamma$-modules and the isomorphism (5.3) extends to an isomorphism of $C^\ast$-algebras

$$Ad_L: C^\ast(\tilde{M}, L^2(\tilde{M}, \tilde{E}))^\Gamma \to \mathbb{K}(\mathcal{E}^{MF}_{C^\ast\Gamma}(\tilde{M}, \tilde{E}))$$  

(5.4)

where $\mathcal{E}^{MF}_{C^\ast\Gamma}(\tilde{M}, \tilde{E})$ is the $C^\ast_{\mathbb{C}} \Gamma$-module of the $L^2$-sections of $E \otimes V_{C^\ast\Gamma}$ and $C^\ast(\tilde{M}, L^2(\tilde{M}, \tilde{E}))^\Gamma$ is the Roe algebra associated to the $\Gamma$-equivariant $C_0(\tilde{M})$-module $L^2(\tilde{M}, \tilde{E})$.

Now let us consider the $^\ast$-algebra $\Psi_{\mathcal{C}, \mathbb{C}}^0(M, \tilde{E})$ of $\Gamma$-equivariant $C^\ast$-algebras $\mathcal{C}(\tilde{M}, \tilde{E})$ with compact support. We can extend $Ad_L$ to an isomorphism between the multiplier algebra of $C^\ast(M, L^2(\tilde{M}, \tilde{E}))^\Gamma$ and the multiplier algebra of $\mathbb{K}(\mathcal{E}^{MF}_{C^\ast\Gamma}(\tilde{M}, \tilde{E}))$, which is $\mathbb{B}(\mathcal{E}^{MF}_{C^\ast\Gamma}(\tilde{M}, \tilde{E}))$, see [Lance, Section 2]. This extension maps isomorphically the $^\ast$-algebra $\Psi_{\mathcal{C}, \mathbb{C}}^0(M, \tilde{E})$, which is certainly contained in the multiplier algebra of $C^\ast(M, L^2(\tilde{M}, \tilde{E}))^\Gamma$, to its image in $\mathbb{B}(\mathcal{E}^{MF}_{C^\ast\Gamma}(\tilde{M}, \tilde{E}))$.  

---

**Paolo Piazza, Thomas Schick and Vito Felice Zenobi**
5.2 Pseudodifferential operators with coefficients in differential forms on $\mathbb{C}\Gamma$

All we did in the previous section can be extended to the situation where coefficients in non-commutative differential forms are taken into account.

Let $\Omega_*(\mathbb{C}\Gamma)$ denote the universal differential algebra associated to $\mathbb{C}\Gamma$. Recall that it has basis as a vector space over $\mathbb{C}\{g_0dg_1\ldots dg_k\}$ and the multiplication is given by the following formula

$$(g_0dg_1\ldots dg_k)(g_{k+1}dg_{k+2}\ldots dg_n) = \sum_{j=1}^{k} (-1)^{n-j} g_0dg_1\ldots d(g_jg_{j+1})\ldots dg_n + (-1)^n g_0g_1dg_2\ldots dg_n$$

whereas the differential is given by

$$d(g_0dg_1\ldots dg_k) = dg_0dg_1\ldots dg_k \quad \text{and} \quad de = 0$$

First we fix some notation:

- if $\omega = g_1 \otimes \cdots \otimes g_n$, then we set $d\omega := dg_1dg_2\ldots dg_n \in \Omega_*(\mathbb{C}\Gamma)$ and $\pi(\omega) := g_1g_2\ldots g_n \in \Gamma$.
- Observe that the elements of the form $\pi(\omega)^{-1}d\omega$ constitute a basis of $\Omega_*(\mathbb{C}\Gamma)$ over $\mathbb{C}\Gamma$.
- Let $\mathcal{E}^{MF}_{\Gamma}(\tilde{M},\tilde{E})$ denote the right $\mathbb{C}\Gamma$-module of sections $C^\infty(\tilde{M},\tilde{E} \otimes \mathbb{C}\Gamma)^\Gamma$.
- Let $\mathcal{E}^{MF}_{\Omega_*(\mathbb{C}\Gamma)}(\tilde{M},\tilde{E})$ denote the right $\Omega_*(\mathbb{C}\Gamma)$-module $C^\infty(\tilde{M},\tilde{E} \otimes \Omega_*(\mathbb{C}\Gamma))^\Gamma$.

5.7 Definition. Let us define a subalgebra $\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\tilde{M},\tilde{E})$ of the bounded $\Omega_*(\mathbb{C}\Gamma)$-linear operators on the module $\mathcal{E}^{MF}_{\Omega_*(\mathbb{C}\Gamma)}(\tilde{M},\tilde{E})$ in the following way. Elements in $\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\tilde{M},\tilde{E})$ are given by sums of the following type

$$T = \sum_{\lambda,\omega} R^*_{(\lambda-1,e)}T_\omega \otimes \lambda \pi(\omega)^{-1}d\omega$$

where $T_\omega$ are the distributional kernels of $\Gamma$-equivariant $\Gamma$-compactly supported 0-order pseudodifferential operators on $\tilde{M}$ and where the second factor acts by right multiplication in the $\Omega_*(\mathbb{C}\Gamma)$ tensor factor in $C^\infty(\tilde{M},\tilde{E} \otimes \Omega_*(\mathbb{C}\Gamma))^\Gamma$.

5.8 Remark. Consider an element $T$ in $\Psi^0_{\Omega_*(\mathbb{C}\Gamma)}(\tilde{M},\tilde{E})$, let us highlight that the only coefficients $R^*_{(\lambda-1,e)}T_\omega$ with pseudodifferential singularities on the diagonal of $\tilde{M} \times \tilde{M}$ are those of $\lambda \pi(\omega)^{-1}d\omega \in \Omega_*(\mathbb{C}\Gamma)$ which, in our basis, is equivalent to say that $\lambda \neq e \in \Gamma$.
5.3 MF calculi associated to dense holomorphically closed subalgebras of $C^*_\Gamma$

If now $\mathcal{A}_\Gamma$ is a dense and holomorphically closed *-subalgebra of $C^*_\Gamma$, then $\Psi_{M\mathcal{F}}(M, E \otimes \mathcal{V}_\mathcal{A}_\Gamma)$ is dense and holomorphically closed in $\Psi_{M\mathcal{F}}^\infty(M, E \otimes \mathcal{V}_\mathcal{A}_\Gamma) \subset \mathcal{E}^{MF}_{C^*_\Gamma}(\tilde{M}, \tilde{E})$. This follows from [23, Section 6]. Of course, this implies that $\Psi_{M\mathcal{F}}^\infty(M, E \otimes \mathcal{V}_\mathcal{A}_\Gamma)$ is dense and holomorphically closed in $\mathcal{E}^{MF}_{C^*_\Gamma}(\tilde{M}, \tilde{E})$.

5.9 Definition. We define $\Psi_{\tilde{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E})$ as the *-subalgebra

$$\Psi_{\tilde{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E}) = \Psi_{M\mathcal{F}}^0(\tilde{M}, \tilde{E}) + \Psi_{M\mathcal{F}}^\infty(M, E \otimes \mathcal{V}_\mathcal{A}_\Gamma).$$

Analogously, let $\Psi_{\hat{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E})$ denote the *-subalgebra

$$\Psi_{\hat{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E}) = \Psi_{M\mathcal{F}}^0(\tilde{M}, \tilde{E}) + \Psi_{M\mathcal{F}}^\infty(M, E \otimes \mathcal{V}_{\hat{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E})).$$

5.10 Remark. Recall that the notation $\hat{\Omega}_*$ for the universal differential algebra of a Fréchet algebra $\mathcal{A}$ means that we are using the projective tensor product in its definition, see for instance [23, Section 2].

5.11 Proposition. The *-subalgebra $Ad_{-1}(\Psi_{\tilde{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E}))$ is dense and holomorphically closed in $\Psi_{M\mathcal{F}}^0(\tilde{M}, \tilde{E})$.

The corresponding result is true for operators with coefficients in forms.

Proof. Consider the algebras $\mathcal{J} := C^*_\Gamma(\tilde{M}, L^2(\tilde{M}, \tilde{E}))$, $\mathcal{B} := \Psi_{\tilde{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E})$, $\mathcal{I} := Ad_{-1}(\Psi_M^\infty(M, E \otimes \mathcal{V}_{\mathcal{A}_\Gamma}))$ and $\mathcal{A} := Ad_{-1}(\Psi_{\hat{\mathcal{A}}_\Gamma}(\tilde{M}, \tilde{E}))$. All the hypotheses of [17, Theorem 4.2] are verified and the result follows. Notice that [17, Equation (4.4)] has a misprint. It should contain $\mathcal{A}$ instead of $\mathcal{B}$. □

6 Mapping analytic surgery to homology

6.1 Lott’s connection and delocalized traces

Let $h: \tilde{M} \to [0, 1]$ be a cut-off function for the $\Gamma$-action on $\tilde{M}$, namely a smooth function such that

$$\sum_{g \in \Gamma} h_g(x) = 1 \quad \forall x \in \tilde{M}.$$ 

Notice that, since the action of $\Gamma$ is cocoompany, $h$ can be chosen with compact support.

6.1 Definition. Lott’s connection $\nabla^{Lott}: \mathcal{E}^{MF}_{C^*_\Gamma}(\tilde{M}, \tilde{E}) \to \mathcal{E}^{MF}_{\hat{\mathcal{A}}_\Gamma(\tilde{M}, \tilde{E})}$ is defined by

$$\nabla^{Lott} \left( \sum_{\lambda \in \Gamma} R^*_\lambda f \lambda \right) := \sum_{\lambda, g \in \Gamma} h_{\lambda g} \cdot R^*_\lambda f \otimes \lambda g^{-1} dg.$$  

6.2 Remark. Notice that this definition is coherent with the original definition of Lott [23, Equation (41)], which is given by

$$\sum_{g \in \Gamma} h_{g^{-1}} f \otimes dg \quad \text{for} \quad f \in \mathcal{E}^{MF}_{C^*_\Gamma}(\tilde{M}, \tilde{E}).$$  

In Definition 6.1, we have written the expression (6.3) in our basis $\{ g^{-1} dg \}$ of $\Omega_1(\mathcal{G})$ instead of the basis $\{dg\}$. It is shown in [23, Proposition 9] that $\nabla^{Lott}$ is indeed a connection.

By standard arguments we can extend Lott’s connection to $\mathcal{E}^{MF}_{\hat{\mathcal{A}}_\Gamma(\tilde{M}, \tilde{E})}$ and we keep the notation $\nabla^{Lott}$ for the extended connection. The curvature of Lott’s connection $\Theta := (\nabla^{Lott})^2$ is an element in $\text{Hom}(\mathcal{E}^{MF}_{C^*_\Gamma}(\tilde{M}, \tilde{E}), \mathcal{E}^{MF}_{\hat{\mathcal{A}}_\Gamma(\tilde{M}, \tilde{E})})$. One has naturally a connection on $\Psi_{\hat{\mathcal{A}}_\Gamma(\tilde{M}, \tilde{E})}$ defined in the following way

$$\nabla^{Lott}(T) = [\nabla^{Lott}, T]$$ 

and it is immediate to see $(\nabla^{Lott})^2(T) = \Theta T - T \Theta$ and that $\nabla^{Lott}(\Theta) = 0$. 

6.4 Remark. Let now $A\Gamma$ be any Fréchet completion of $C\Gamma$ contained in $C_0^\ast\Gamma$. The proof of [23 Proposition 9] applies using $A\Gamma$, therefore $\nabla^{\text{Lott}}$ is a well-defined connection on $E_{\text{M}}^{MF}(\tilde{M}, \tilde{E})$ and everything we saw so far still holds in this setting.

6.5 Definition. In [23, Section II], Lott defines a far still holds in this setting.

Let $\Omega_\ast(\Gamma)$ be any Fréchet completion of $\Omega_\ast(\Gamma)$ contained in $C_0^\ast\Gamma$, hence $\text{Tr}$ is a well-defined connection on $E_{\text{M}}^{MF}(\tilde{M}, \tilde{E})$. The proof of [23, Proposition 9] applies using $\text{Tr}$ is the fiberwise trace of vector bundle homomorphisms.

In this section, we will define in a similar fashion a delocalized trace on $\Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$. Recall that, as a vector space, $\Omega_\ast(\mathbb{C}\Gamma)$ decomposes into the direct sum $\Omega_\ast(\mathbb{C}\Gamma) \oplus \Omega_\ast(\mathbb{C}\Gamma)$. In turn, always as a vector space, $\Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$ splits into the direct sum $\Psi_{\Omega_\ast}(\tilde{M}, \tilde{E}) \oplus \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$. Then we have a projection $\pi_{\text{del}} : \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E}) \to \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$. Recall that, thanks to Remark 5.8, if we consider an element in the image of $\pi_{\text{del}}$ then the restriction to the diagonal of its kernel is smooth. Hence we can give the following definition.

6.6 Definition. Let $T$ be an element of $\Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$. Define

$$\text{TR}_0^{\text{del}}(T) := \int_{\tilde{F}} \text{Tr}(\pi_{\text{del}}(T)(\tilde{x}, \tilde{x})) d\text{vol}(\tilde{x}) \in \Omega_\ast(\mathbb{C}\Gamma)_{ab},$$

where $\tilde{F}$ is a fundamental domain for the action of $\Gamma$ on $\tilde{M}$.

Unrolling the definition, we obtain $\text{TR}_0^{\text{del}}(T) = \sum_{\lambda, \omega} \int_{\tilde{F}} \text{Tr}(R_{(\lambda^{-1}, \omega)} T_{\omega}(\tilde{x}, \tilde{x})) d\text{vol}(\tilde{x}) \lambda \text{vol}(\tilde{x})^{-1} d\omega$, where it is important to notice that the sum is finite. Furthermore it is straightforward, as for $\text{TR}$, to see that $\text{TR}_0^{\text{del}}$ is a trace and that Definition 6.6 does not depend on the choice of the fundamental domain $\tilde{F}$.

We now want to extend the definition of the delocalized trace to $\Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$. To this end, let us consider a smooth function $\chi$ on $\tilde{M} \times \tilde{M}$ such that is equal to 1 on the diagonal and is properly supported, namely the projections $\pi_1, \pi_2 : \tilde{M} \times \tilde{M} \to \tilde{M}$ are proper on the support of $\chi$. Then an element $T = \sum_{\lambda, \omega} R_{(\lambda^{-1}, \omega)} T_{\omega} \lambda \text{vol}(\tilde{x})^{-1} d\omega \in \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$ decomposes as $T = T_0 + T_\infty$, where

$$T_0 = \sum_{\lambda, \omega} R_{(\lambda^{-1}, \omega)} (\chi T_{\omega}) \lambda \text{vol}(\tilde{x})^{-1} d\omega.$$

Observe that $T_0 \in \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$ and that $T_\infty \in \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E})$.

6.7 Definition. We define $\text{TR}_0^{\text{del}} : \Psi_{\Omega_\ast}(\tilde{M}, \tilde{E}) \to \Omega_\ast(\tilde{M}, \tilde{E})$ by

$$\text{TR}_0^{\text{del}}(T) := \text{TR}_0^{\text{del}}(T_0) + \text{TR}(T_\infty).$$

6.8 Remark. It is straightforward to see that this definition does not depend on the choice of the properly supported function $\chi$. Moreover, notice that $\text{TR}_0^{\text{del}}$ is a well-defined map with values in $\Omega_\ast(\tilde{M}, \tilde{E})$, but it is a trace only if we pass to the quotient $\Omega_\ast(\tilde{M}, \tilde{E})_{ab}$. 

Higher rho numbers, surgery and homology

19
6.9 Proposition. If $T \in \Psi_{\Omega,\langle \Gamma \rangle}^0 (\tilde{M}, \tilde{E})$, then
\[ d\text{TR}^{\text{det}}_0 (T) = \text{TR}^{\text{det}}_0 (|\nabla_{\text{Lott}}^{\text{Lott}}, T|) \in \Omega_\ast (\Gamma)^{\text{det}}_{\text{ab}}. \]

Proof. Let $T$ be given by the sum $\sum_{\omega, \gamma} R^\ast_{\lambda-1, e} T^\ast_{\omega} \lambda (\omega)^{-1} d\omega$. A direct calculation gives
\[ |\nabla_{\text{Lott}}^{\text{Lott}}, T| = \sum_{\omega, \gamma} R^\ast_{\lambda-1, e} T^\ast_{\omega} \gamma^{-1} d\gamma \cdot \pi (\omega)^{-1} d\omega + \sum_{\omega, \gamma} T^\ast_{\omega} \pi (\omega)^{-1} d\omega - \sum_{\omega, \gamma} T^\ast_{\omega} R^\ast_{\gamma} \pi (\omega)^{-1} d\omega \cdot \gamma^{-1} d\gamma. \]

Passing to the Mishchenko-Fomenko context and applying $\text{TR}^{\text{det}}_0$ we obtain
\[ \text{TR}^{\text{det}}_0 (|\nabla_{\text{Lott}}^{\text{Lott}}, T|) = \int_{\tilde{E}} \sum_{\lambda \neq e, \omega, \gamma} \text{Tr} \left( R^\ast_{\lambda-1, e} h (x) R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) \lambda \gamma^{-1} d\gamma \cdot \pi (\omega)^{-1} d\omega 
+ \int_{\tilde{E}} \sum_{\lambda \neq e, \omega} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) \lambda d\pi (\omega)^{-1} d\omega 
- \int_{\tilde{E}} \sum_{\lambda \neq e, \omega, \gamma} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) R^\ast_{\gamma} h (x) \right) \lambda \pi (\omega)^{-1} d\omega \cdot \gamma^{-1} d\gamma. \tag{6.10} \]

Remember that this expression takes place in $\Omega_\ast (\Gamma)^{\text{det}}_{\text{ab}}$, where
\[ \lambda \pi (\omega)^{-1} d\omega \cdot \gamma^{-1} d\gamma = \pi (\omega)^{-1} d\omega \cdot \gamma^{-1} d\gamma \cdot \lambda = \pi (\omega)^{-1} d\omega \cdot (\gamma^{-1} d(\gamma \lambda) - d\lambda) = (\gamma^{-1} d(\gamma \lambda) - d\lambda) \cdot \pi (\omega)^{-1} d\omega. \]

Then the third term of (6.10) is equal to
\[ - \int_{\tilde{E}} \sum_{\lambda \neq e, \omega, \gamma} \text{Tr} \left( R^\ast_{\lambda-1, e} h (x) R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) \gamma^{-1} d(\gamma \lambda) \cdot \pi (\omega)^{-1} d\omega + \sum_{\lambda \neq e, \omega} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) d\lambda \cdot \pi (\omega)^{-1} d\omega \]
that in turn, after changing the first summation over $\gamma$ to the summation over $\mu = \gamma \lambda$, is equal to
\[ - \int_{\tilde{E}} \sum_{\lambda \neq e, \omega, \mu} \text{Tr} \left( R^\ast_{\mu-1, e} h (x) R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) \lambda \mu^{-1} d\mu \cdot \pi (\omega)^{-1} d\omega + \sum_{\lambda \neq e, \omega} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) d\lambda \cdot \pi (\omega)^{-1} d\omega. \tag{6.11} \]

Now, observing that the first term in (6.11) is equal to the opposite of the first term in (6.10), it follows that (6.10) becomes
\[ \text{TR}^{\text{det}}_0 (|\nabla_{\text{Lott}}^{\text{Lott}}, T|) = \int_{\tilde{E}} \sum_{\lambda \neq e, \omega} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) \lambda d\pi (\omega)^{-1} d\omega + \sum_{\lambda \neq e, \omega} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) d\lambda \cdot \pi (\omega)^{-1} d\omega = 
= \int_{\tilde{E}} \sum_{\lambda \neq e} \text{Tr} \left( R^\ast_{\lambda-1, e} T^\ast_{\omega} (x, x) \right) d(\lambda \cdot \pi (\omega)^{-1} d\omega) 
= d\text{TR}^{\text{det}}_0 (T). \tag{6.12} \]

Notice that, due to the fact that the support of the kernel of $T^\ast_{\omega}$ is $\Gamma$-compact, the sums over $\lambda$ are finite. \(\square\)

6.13 Proposition. Let $T \in \Psi_{\Omega,\langle \Gamma \rangle}^{-\infty} (\tilde{M}, \tilde{E})$, then
\[ d\text{TR} (T) = \text{TR} (|\nabla_{\text{Lott}}^{\text{Lott}}, T|) \in \tilde{\Omega}_\ast (\Gamma)^{\text{det}}_{\text{ab}}. \]

Proof. The proof is analogous to the one of Proposition 6.9. \(\square\)

6.14 Corollary. If $T \in \Psi_{\Omega,\langle \Gamma \rangle}^0 (\tilde{M}, \tilde{E})$, then
\[ d\text{TR}^{\text{det}} (T) = \text{TR}^{\text{det}} (|\nabla_{\text{Lott}}^{\text{Lott}}, T|) \in \tilde{\Omega}_\ast (\Gamma)^{\text{det}}_{\text{ab}}. \]
6.2 Results in relative K-theory and homology

Let us fix the following algebraic setting: \( A \) is a Fréchet algebra, dense and holomorphically closed in a C*-algebra \( A \); moreover let \( I \subset A \) be a closed ideal and \( B \subset A \) a closed subalgebra such that \( I \cap B = \{0\} \).

6.15 Remark. Because of the condition \( I \cap B = \{0\} \) we have a split \( \alpha \) for the following exact sequence

\[
0 \longrightarrow I \longrightarrow I + B \overset{\alpha}{\longrightarrow} B \longrightarrow 0,
\]

which gives an isomorphism \( K_\ast(I) \cong K_\ast(I + B)/\alpha_\ast(K_\ast(B)) \).

Let \( \partial : K_\ast(B \rightarrow A/I) \to K_{\ast + 1}(0 \to I) \) be the boundary map associated to the following short exact sequence

\[
0 \longrightarrow (0 \to I) \longrightarrow (B \to A) \longrightarrow (B \to A/I) \longrightarrow 0,
\]

where \( (B \to A) \) and \( (B \to A/I) \) denote the mapping cone algebras and \( (0 \to I) \) is a way of writing the suspension of \( I \) as a mapping cone.

6.16 Lemma. The following diagram is commutative

\[
\begin{array}{ccccccccc}
\cdots & K_{\ast - 1}(I) & \overset{\iota_\ast \circ S}{\longrightarrow} & K_\ast(B \to A) & \longrightarrow & K_\ast(B + I \to A) & \overset{\partial'}{\longrightarrow} & K_\ast(I) & \overset{S}{\longrightarrow} & \cdots \\
S & & \downarrow{\text{id}} & & & & \downarrow{q_\ast} & \cong & & \downarrow{S} \\
\cdots & K_\ast(0 \to I) & \overset{\iota_\ast}{\longrightarrow} & K_\ast(B \to A) & \longrightarrow & K_\ast(B \to A/I) & \overset{\partial}{\longrightarrow} & K_{\ast + 1}(0 \to I) & \longrightarrow & \cdots
\end{array}
\]

where \( q : (B + I \to A) \to (B \to A/I) \) is the quotient by \( (I \to I) \), \( S \) is the suspension isomorphism and \( \partial' \) is the following composition

\[
K_\ast(B + I \to A) \longrightarrow K_\ast(B + I) \longrightarrow K_\ast(B + I)/\alpha_\ast(K_\ast(B)) \cong K_\ast(I).
\]

The top row could be interpreted as the long exact K-theory sequence of a hypothetical extension \( 0 \to (B \to A) \to (B + I \to A) \to (I \to 0) \to 0 \). Of course, as \( B \) is not assumed to be an ideal in \( B + I \), this is not an extension of C*-algebras. Nonetheless, the lemma uses the split \( \alpha \) to obtain the associated K-theory sequence, and gives explicitly the relevant maps.

Proof. We only need to check the commutativity of the third square and this consists in chasing the following diagram.
Here $K_{+1}\left(B \to A\right)$ denotes the K-theory of the double mapping cone $((B \to A) \to (B \to A/I))$ (the same holds for the other similar K-groups). The map $i_*$ is given by the composition of the suspension isomorphism and the natural inclusion; $\lambda$ is induced by the obvious map $K_*(B + I) \to K_{+1}(B \to B + I)$ and it is well defined on the quotient by $\alpha_* K_*(B)$ because its restriction to $\alpha_* K_*(B)$ factors through $K_{+1}(B \to B) = 0$; $\mu$ denotes the quotient by $\begin{pmatrix} 0 \to A \\ 0 \to A \end{pmatrix}$ composed with the Bott periodicity; $\nu$ is induced by the inclusion of $I$ into $A$. The only part which is not clearly commutative is the right hand-side trapezoid, but this can be easily proved using [43, Lemma 3.6]. We leave the details to the reader, knowing that $A, B$ and $C$ in the notation of [43, Lemma 3.6] are here represented by $\begin{pmatrix} 0 \to A \\ I \to A \end{pmatrix}, \begin{pmatrix} 0 \to A \\ 0 \to A/I \end{pmatrix}$ and $(0 \to I)$ and that $\partial_B$ is the isomorphism given by the composition of the Bott periodicity and the inverse in K-theory of the inclusion of $I$ in $(A \to A/I)$ and $\partial_C$ is just the suspension isomorphism.

Let $\Omega_I, \Omega_B$ and $\Omega_A$ be DGAs over $I, B$ and $A$, respectively, equipped with DGAs morphisms over the natural inclusions of Fréchet algebras. Moreover, assume that $\Omega_I \cap \Omega_B = \{0\} \subset \Omega_A$ and that $\Omega_I + \Omega_B \hookrightarrow \Omega_A$ is an inclusion of DGAs over the inclusion $I + B \hookrightarrow A$. As for K-theory in Remark 6.15 we have a split $\alpha$ of the sequence

$$\begin{array}{cccccc}
0 & \longrightarrow & \Omega_I & \longrightarrow & \Omega_I + \Omega_B & \overset{\alpha}{\longrightarrow} & \Omega_B & \longrightarrow & 0,
\end{array}$$

which is compatible with the Chern character. In particular, we have a map $\delta': H_{+1}(\Omega_I + \Omega_B \to \Omega_A) \to H_*(\Omega_I)$, defined analogously to $\delta''$ by

$$H_{+1}(\Omega_I + \Omega_B \to \Omega_A) \longrightarrow H_*(\Omega_B + \Omega_I) \longrightarrow H_*(\Omega_B + \Omega_I)/\alpha_*(H_*(\Omega_B)) \overset{\cong}{\longrightarrow} H_*(\Omega_I). \quad (6.18)$$

**6.19 Remark.** In contrast to the K-theory situation, it would not have been necessary to make this complicated definition of $\delta'$: we can form the quotient complex whenever we have a subcomplex (leaving the world of DGAs). And indeed, in the case at hand $\delta'$ is induced by the quotient map in the following short exact sequence of complexes

$$\begin{array}{cccccc}
0 & \longrightarrow & (\Omega_B \to \Omega_A) & \longrightarrow & (\Omega_I + \Omega_B \to \Omega_A) & \longrightarrow & (\Omega_I \to 0) & \longrightarrow & 0.
\end{array} \quad (6.20)$$

The induced boundary map is given by the composition of the suspension of chain complexes and the inclusion of $(0 \to \Omega_I)$ into $(\Omega_B \to \Omega_A)$. We use the more complicated description to simplify the comparison of the K-theory and the homology side.

By the definition of the boundary maps in homology, the following diagram

$$\begin{array}{cccccc}
\cdots & \longrightarrow & H_{+1}(\Omega_B \to \Omega_A) & \longrightarrow & H_{+1}(\Omega_I + \Omega_B \to \Omega_A) & \overset{\delta'}{\longrightarrow} & H_*(\Omega_I) & \longrightarrow & \cdots, \\
\end{array} \quad (6.21)$$

is commutative. Here $\Omega_A/\Omega_I$ is the quotient DGA over $A/I$ and $\delta$ is the boundary map associated to the exact sequence of DGAs

$$\begin{array}{cccccc}
0 & \longrightarrow & (0 \to \Omega_I) & \longrightarrow & (\Omega_B \to \Omega_A) & \longrightarrow & (\Omega_B \to \Omega_A/\Omega_I) & \longrightarrow & 0.
\end{array} \quad (6.22)$$

**6.22 Theorem.** The following diagram is commutative

$$\begin{array}{cccccc}
\cdots & \longrightarrow & K_{+1}(I) & \overset{i_* + S}{\longrightarrow} & K_*(B \to A) & \longrightarrow & K_*(B + I \to A) & \overset{\delta''}{\longrightarrow} & K_*(I) & \longrightarrow & \cdots, \\
\text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch} & \text{Ch}.
\end{array} \quad (6.23)$$
Proof. We have different definitions of the Chern character for even and odd K-theory, therefore, we have to prove the commutativity of 6 different squares. First, consider the diagram

\[
\begin{array}{ccc}
K_{s+1}(I) & \xrightarrow{S} & K_s(0 \to I) \\
\downarrow \text{Ch} & & \downarrow \text{Ch}^{rel} \\
H_{s+1}(\Omega_I) & \equiv & H_{s+1}(0 \to \Omega_I)
\end{array}
\]

(6.24)

For \( \ast \) even, \( \text{Ch}: K_1(I) \to H_{\text{odd}}(\Omega_I) \) is defined via the suspension isomorphism and \( \text{Ch}^{rel} \), which also enters \( \text{Ch}^{rel} \). In this case, \( \ast \) is an immediate consequence of the definitions that the left square of (6.24) commutes. If \( \ast \) is odd, by definition \( \text{Ch}^{rel} \circ S: K_0(I) \to H_{\text{even}}(0 \to \Omega_I) \) is computed from the double suspension. Explicitly, \( \ast \) is a projector over \( I \), then \( \text{Ch}^{rel} \circ S(p) = [0, \frac{1}{0} \int_0^1 \text{Ch}(p \otimes \beta) \, dt \wedge ds] \in H_{\text{even}}(0 \to \Omega_I) \), where \( \beta \) is the Bott projector as in Lemma 4.3. By this lemma, this class equals \( \text{Ch}(p) \) under the identification \( H_s(\Omega_I) \equiv H_s(0 \to \Omega_I) \). Therefore, the left square of (6.24) always commutes. The right square of (6.24) commutes because of naturality of the relative Chern character, Lemma 4.11. This also gives the commutativity of the middle square of (6.23). Naturality also implies the commutativity of the rightmost square of (6.23), as \( \delta' \) is just obtained by the composition of induced maps. Finally, (6.24) is a factorisation of the leftmost square of (6.23) which therefore also commutes.

6.3 Mapping the analytic surgery sequence to noncommutative de Rham homology

In this section we shall use the tools developed in Section 4 to map the analytic surgery exact sequence to the non-commutative de Rham homology exact sequence (4.35). We shall use ideas of Connes, Gorokhovsky and Lott, see [3,9].

Notice that the pair \( (\Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}), \nabla^{\text{Lott}}) \) is not a complex, because the square of Lott’s connection \( \nabla^{\text{Lott}} \) is not zero. Nevertheless, the triple

\[
(\Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}), \nabla^{\text{Lott}}, \Theta)
\]

does verify the hypotheses of [6] Lemma 9, III.3], which produces in a canonically way the following complex.

6.25 Definition. Let us define \( \Omega_*(\Psi_{\text{AG}}^0(\tilde{M}, \tilde{E}), d) \) as the following complex over \( \Psi_{\text{AG}}^0(\tilde{M}, \tilde{E}) \):

- as a vector space, \( \Omega_*(\Psi_{\text{AG}}^0(\tilde{M}, \tilde{E})) \) is

\[
\Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}) \oplus X \cdot \Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}) \oplus \Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}) \cdot X \oplus X \cdot \Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}) \cdot X
\]

where \( X \) is a degree 1 auxiliary variable.

- The algebra structure is defined by setting \( X^2 = \Theta \) and \( T_1 XT_2 = 0 \) for all \( T_i \in \Psi_{\Omega,\text{(AG)}}^0(\tilde{M}, \tilde{E}) \).

- The differential \( d \) is defined by \( dX = 0 \) and \( dT = \nabla^{\text{Lott}} T + X T + (-1)^{\deg T} T X \). One easily checks that \( d^2 = 0 \).

The same construction applies to \( (\Psi_{\Omega,\text{(AG)}}^{\infty}(\tilde{M}, \tilde{E}), \nabla^{\text{Lott}}) \) and we obtain a subcomplex \( \Omega_*(\Psi_{\text{AG}}^{\infty}(\tilde{M}, \tilde{E})) \) of \( \Omega_*(\Psi_{\text{AG}}^0(\tilde{M}, \tilde{E})) \).

Now we want to extend form valued traces to these new complexes.
6.26 Definition. Let us consider an element of \( \overline{\Omega}_* (\Psi^0_{\text{At}}(\widetilde{M}, \widetilde{E})) \), it is of the form \( T_{11} + T_{12} X + X T_{21} + X T_{22} X \). Then set

\[
\overline{\text{TR}}^{\text{det}} (T_{11} + T_{12} X + X T_{21} + X T_{22} X) := \text{TR}^{\text{det}} (T_{11}) - (-1)^{\text{deg} T_{22}} \text{TR}^{\text{det}} (T_{22} \Theta) \in \Omega_* (\text{At})^{\text{det}}.
\]  

The extension \( \overline{\text{TR}} : \overline{\Omega}_* (\Psi^{-\infty}_{\text{At}}(\widetilde{M}, \widetilde{E})) \to \Omega_* (\text{At})^{\text{det}} \) of \( \text{TR} \) is defined analogously.

6.28 Lemma. The map \( \overline{\text{TR}}^{\text{det}} : \overline{\Omega}_* (\Psi^0_{\text{At}}(\widetilde{M}, \widetilde{E}))_{\text{ab}} \to \Omega_*^{\text{det}} (\text{At})_{\text{ab}} \), defined in (6.27), is a morphism of complexes. The same is true for \( \overline{\text{TR}} : \overline{\Omega}_* (\Psi^{-\infty}_{\text{At}}(\widetilde{M}, \widetilde{E}))_{\text{ab}} \to \Omega_* (\text{At})^{\text{det}}_{\text{ab}} \).

Proof. Let \( T_{11} + T_{12} X + X T_{21} + X T_{22} X \) be an element of \( \overline{\Omega}_* (\Psi^0_{\text{At}}(\widetilde{M}, \widetilde{E})) \), then by Proposition 6.9 we have

\[
\overline{\text{TR}}^{\text{det}} (d(T_{11} + T_{12} X + X T_{21} + X T_{22} X)) =
\]

\[
= \text{TR}^{\text{det}} (\nabla \text{Lott}, T_{11}) - (-1)^{\text{deg} T_{22}} \text{TR}^{\text{det}} (\nabla \text{Lott}, T_{22} \Theta) =
\]

\[
= \text{TR}^{\text{det}} (\nabla \text{Lott}, T_{11}) - (-1)^{\text{deg} T_{22}} \text{TR}^{\text{det}} (\nabla \text{Lott}, T_{22} \Theta) =
\]

\[
= d \text{TR}^{\text{det}} (T_{11} + T_{12} X + X T_{21} + X T_{22} X) =
\]

and hence the lemma follows. The proof for \( \overline{\text{TR}} \) is identical. \( \square \)

6.30 Lemma. Let \( \overline{\Omega}_* (C^\infty(M, E)) \) be the subcomplex of \( \overline{\Omega}_* (\Psi^0_{\text{At}}(\widetilde{M}, \widetilde{E})) \) generated by \( C^\infty(M, E) \), then we have that

(1) \( \overline{\Omega}_* (C^\infty(M, E)) \cap \overline{\Omega}_* (\Psi^{-\infty}_{\text{At}}(\widetilde{M}, \widetilde{E})) = \{0\} \),

(2) \( \overline{\text{TR}}^{\text{det}} \) sends \( \overline{\Omega}_* (C^\infty(M, E)) \) to zero.

Proof. First observe that \( \nabla \text{Lott} f \) is zero for any smooth function \( f \) on \( M \). Then notice that the support of any element in \( \overline{\Omega}_* (C^\infty(M, E)) \) is contained in the diagonal of \( \tilde{M} \times \tilde{M} \) and hence the lemma follows immediately. \( \square \)

We are now in the position to define our maps from the analytic surgery exact sequence to (4.35). This will employ Definitions 4.16 and 4.28 for the relative Chern characters.

We first observe that there is a canonical isomorphism of long exact sequences

\[
\cdots \to K_*(0 \to \Psi^\infty_{\text{At}}(\tilde{M})) \xrightarrow{\partial} K_*(C^\infty(M)) \xrightarrow{\sim} \Psi^0_{\text{At}}(\tilde{M}) \xrightarrow{\sigma^\ast} K_*(C^\infty(M) \to \Psi^0_{\text{At}}(\tilde{M})) \xrightarrow{\sigma} \cdots
\]

\[
\cdots \to K_*(0 \to C^\infty_*(\tilde{X} \times_{G} \tilde{X})) \xrightarrow{\partial} K_*(C^\infty(M)) \xrightarrow{\sim} \Psi^0_{\text{At}}(\tilde{M}) \xrightarrow{\sigma^\ast} K_*(C^\infty(M) \to \Psi^0_{\text{At}}(\tilde{M})) \xrightarrow{\sigma} \cdots
\]

The first two vertical arrows are isomorphism because they are induced by inclusions of dense holomorphically closed subalgebras. The third homomorphism is induced by the symbol map and it is an isomorphism because of the Five Lemma.

We can think to the exact sequence

\[
\cdots \to K_*(0 \to \Psi^\infty_{\text{At}}(\tilde{M})) \xrightarrow{\partial} K_*(C^\infty(M)) \xrightarrow{\sim} \Psi^0_{\text{At}}(\tilde{M}) \xrightarrow{\sigma^\ast} K_*(C^\infty(M) \to \Psi^0_{\text{At}}(\tilde{M})) \xrightarrow{\sigma} \cdots
\]

as a smooth version of the analytic surgery sequence.
6.31 **Definition.** (1) Let $x$ be a class in $K_*(0 \to \Psi^{-\infty}_{\mathcal{A}^c}(\widetilde{M}))$. Then define

$$\text{Ch}_r(x) := \mathcal{TR}_{[s-1]} \left( \text{Ch}^{rel}(x) \right) \in H_{[s-1]}(\mathcal{A}^c).$$

Here

$$\text{Ch}^{rel}: K_*(0 \to \Psi^{-\infty}_{\mathcal{A}^c}(\widetilde{M})) \to H_{[s-1]} \left( 0 \to \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c}(\widetilde{M})) \right),$$

and $\mathcal{TR}_*$ is a compact notation for

$$(0, \mathcal{TR}_*): H_* \left( 0 \to \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c}(\widetilde{M})) \right) \to H_*(\mathcal{A}^c).$$

(2) Let $y$ be a class in $K_*(C^{\infty}(M) \to \Psi^0_{\mathcal{A}^c}(\overline{M}))$. Then define

$$\text{Ch}_r^{det}(y) := \mathcal{TR}_{[s-1]}^{det} \left( \text{Ch}^{rel}(y) \right) \in H_{[s-1]}^{det}(\mathcal{A}^c).$$

Here

$$\text{Ch}^{rel}: K_*(C^{\infty}(M) \to \Psi^0_{\mathcal{A}^c}(\overline{M})) \to H_{[s-1]} \left( \overline{\Omega}(C^{\infty}(M)) \to \overline{\Omega}(\Psi^0_{\mathcal{A}^c}(\overline{M})) \right)$$

and $\mathcal{TR}_{*}^{det}$ is a compact notation for

$$(0, \mathcal{TR}_{*}^{det}): H_* \left( \overline{\Omega}(C^{\infty}(M)) \to \overline{\Omega}(\Psi^0_{\mathcal{A}^c}(\overline{M})) \right) \to H_*^{det}(\mathcal{A}^c).$$

(3) Let $z$ be a class in $K_*(C^{\infty}(M) \to \Psi^0_{\mathcal{A}^c}(\overline{M})/\Psi^{-\infty}_{\mathcal{A}^c}(\overline{M}))$. Recall from Lemma 6.16 that

$$q_*: K_*(C^{\infty}(M) + \Psi^{-\infty}_{\mathcal{A}^c}(\overline{M}) \to \Psi^0_{\mathcal{A}^c}(\overline{M}) \to K_*(C^{\infty}(M) \to \Psi^0_{\mathcal{A}^c}(\overline{M})/\Psi^{-\infty}_{\mathcal{A}^c}(\overline{M}))$$

is an isomorphism. Then define

$$\text{Ch}_r(z) := j^{-1} \circ (\mathcal{TR}, \mathcal{TR}_{*}^{det})_{[s-1]} \left( \text{Ch}^{rel}(q_*^{-1}(z)) \right) \in H_{[s]}^{det}(\mathcal{A}^c).$$

Here

$$\text{Ch}^{rel}: K_*(C^{\infty}(M) + \Psi^{-\infty}_{\mathcal{A}^c}(\overline{M}) \to \Psi^0_{\mathcal{A}^c}(\overline{M})) \to H_* \left( \overline{\Omega}(C^{\infty}(M)) + \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c}(\overline{M})) \to \overline{\Omega}(\Psi^0_{\mathcal{A}^c}(\overline{M})) \right)$$

and

$$\left( \mathcal{TR}, \mathcal{TR}_{*}^{det} \right): \left( \overline{\Omega}(C^{\infty}(M)) + \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c}(\overline{M})) \to \overline{\Omega}(\Psi^0_{\mathcal{A}^c}(\overline{M})) \right) \to \left( \hat{\Omega}(\mathcal{A}^c) \to \hat{\Omega}^{det}(\mathcal{A}^c) \right)$$

is the morphism of mapping cone complexes; the isomorphism $j$ (which is of degree $-1$) is induced by the inclusion of $\hat{\Omega}^{c}(\mathcal{A}^c)$ into $\hat{\Omega}(\mathcal{A}^c) \to \hat{\Omega}^{det}(\mathcal{A}^c)$.

Observe that we can extend the map of complexes $\mathcal{TR}$ to $\overline{\Omega}(C^{\infty}) + \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c})$ by setting it to be zero on $\overline{\Omega}(C^{\infty})$. Indeed, although it is not a direct sum of DGA, $\overline{\Omega}(C^{\infty}) + \overline{\Omega}(\Psi^{-\infty}_{\mathcal{A}^c})$ is a direct sum of complexes and then this extension makes sense, using Lemma 6.28.

The following theorem is one of the main results of this article.

6.33 **Theorem.** The following diagram, with vertical maps as in Definition 6.31, is commutative:

$$\cdots \xrightarrow{\delta} K_*(0 \to \Psi^{-\infty}_{\mathcal{A}^c}(\overline{M})) \xrightarrow{\iota_*} K_*(C^{\infty}(M) \xrightarrow{\mathcal{m}} \Psi^0_{\mathcal{A}^c}(\overline{M})) \xrightarrow{\pi_*} K_*(C^{\infty}(M) \xrightarrow{\mathcal{p}} \Psi^0_{\mathcal{A}^c}(\overline{M})/\Psi^{-\infty}_{\mathcal{A}^c}(\overline{M})) \xrightarrow{\partial} \cdots \xrightarrow{\text{Ch}_r} H_{[s-1]}(\mathcal{A}^c) \xrightarrow{\text{Ch}_r^{det}} H_{[s-1]}^{det}(\mathcal{A}^c) \xrightarrow{\delta} H_{[s]}^{det}(\mathcal{A}^c) \xrightarrow{\text{Ch}_r^{det}} \cdots$$

(6.34)
Proof. In the following, we write $C^\infty$, $\Psi_{\text{alg}}^0$ and $\Psi_{\text{rel}}^-$ as shorthands for $C^\infty(\Omega)$, $\Psi_{\text{alg}}^0(\tilde{M})$ and $\Psi_{\text{rel}}^-(\tilde{M})$, respectively. By Lemma 6.16, we have the following isomorphism of long exact sequences

$$
\cdots \to K_*^e(\Psi_{\text{alg}}^0) \xrightarrow{S} K_*^e(C^\infty) \to \Psi_{\text{alg}}^0(\tilde{M}) \to K_*^e(\Psi_{\text{alg}}^0) \xrightarrow{\delta'} \cdots
$$

and by Theorem 6.22 the following diagram is commutative:

$$
\cdots \to H_{[\delta+1]}(\Pi(\Psi_{\text{alg}}^-)) \xrightarrow{S} H_{[\delta+1]}(\Pi(C^\infty)) \to \Pi(\Psi_{\text{alg}}^0) \to H_{[\delta+1]}(\Pi(\Psi_{\text{alg}}^0)) \xrightarrow{\delta'} \cdots
$$

Furthermore the following commutative diagram of complexes

$$
\begin{align*}
0 \to \Pi_*^e(C^\infty) & \xrightarrow{\delta_{\text{rel}}} \Pi_*^e(C^\infty) + \Pi_*^e(\Psi_{\text{alg}}^-) & \xrightarrow{\delta_{\text{rel}}} \Pi_*^e(\Psi_{\text{alg}}^-) & \xrightarrow{\delta_{\text{rel}}} 0 \\
0 \to \hat{\Omega}_{[\Gamma]}^e & \xrightarrow{\delta_{\text{rel}}} \hat{\Omega}_{[\Gamma]}^e & \xrightarrow{\delta_{\text{rel}}} \hat{\Omega}_{[\Gamma]}^e & \xrightarrow{\delta_{\text{rel}}} 0
\end{align*}
$$

induces a map of long exact sequences in homology. Finally observe that we have a further isomorphism of exact sequences given by

$$
\cdots \to H_*^e(\hat{\Omega}_{[\Gamma]}^e) \xrightarrow{\delta_{\text{rel}}} H_{*+1}^e(\hat{\Omega}_{[\Gamma]}^e) \to H_{*+1}^e(\hat{\Omega}_{[\Gamma]}^e) \xrightarrow{\delta_{\text{rel}}} \cdots
$$

where the arrow in the middle is induced by the inclusion of $\hat{\Omega}_{[\Gamma]}^e$ into $\hat{\Omega}_{[\Gamma]}^e$. As the mapping of exact sequences in (6.34) is given by the composition of the inverse of (6.35) followed by (6.36), (6.37) and the inverse of (6.38), the result follows.

6.4 The algebraic index and the localized Chern character $\text{Ch}_\Gamma^e$

In the next section we will deal with the delocalized Chern character $\text{Ch}_\Gamma^{\text{del}}$. In the current subsection, we want to recognize the localized $\text{Ch}_\Gamma^e$ as a well known homomorphism involving the algebraic index.

6.39 Remark. Before proceeding with this section, we need to recall some facts about relative algebraic K-theory, without entering into the details. Observe that the equivalence relation used in the algebraic setting between two idempotent $e_0$ and $e_1$ over $A$ is to be stably conjugated by an invertible element $z$ over $A$. Hence the elements in the relative K-group $K_{0\text{alg}}^e(A, z, B)$ are given by triples $[e_0, e_1, z]$, where $e_0, e_1$ are idempotents over $A$ and $z$ is an invertible element over $B$ which conjugate $\varphi(e_0)$ into $\varphi(e_1)$. If $A$ is a dense holomorphically closed subalgebra of a $C^*$-algebra, this definition of relative K-theory is equivalent to the one used so far.

In the algebraic setting, given two idempotent $e_0, e_1$ which are conjugated by an invertible element $z$, they are homotopic through a path $f_t$ defined as in (6.35). This is sufficient to define $\text{Ch}_{0\text{alg}}^{\text{rel}}, K_{0\text{alg}}^e(A \to \text{Ch}_\Gamma^e(A))$.
$B \to H_{\text{even}}(\Omega(A) \to \Omega(B))$ exactly as in Definition 4.16 notice indeed that $\int_0^1 \text{Ch}(f_t)dt$ can be explicitly calculated and it is given by a completely algebraic expression in $e_0, e_1$ and $z$.

We now recall the definition of the algebraic index, see for instance [7, Definition 5.2]. Set $\Sigma := \Psi^{0,\text{At}}(M)/\Psi^{0,\infty}(M) \simeq \Psi^\infty_{\text{CR}}(M)/\Psi^{\infty}_{\text{CR}}(M)$, the algebra of complete symbols. The isomorphism is obtained by cutting Schwartz kernels in a neighbourhood of the diagonal.

The algebraic index

\[ \text{Ind}_{\text{CR}}: K_0(C^\infty(M) \to \Sigma) \to K_0(\Psi^\infty_{\text{CR}}(M)) \]

is defined in the following way. A class in $K_0(C^\infty(M) \to \Sigma)$ is given by a triple $[e_0, e_1; \sigma]$, where $e_i \in M_n(C^\infty(M))$ is the projection associated to the vector bundle $E_i$, and where $\sigma$ is an invertible complete symbol such that $e_0 = \sigma^{-1} e_1 \sigma$, where we identify $e_i$ with its image in the complete symbol algebra. Let then $P$ and $Q$ be two pseudodifferential operators in $\Psi^{0,\text{At}}(M; E_0, E_1)$ and $\Psi^\infty_{\text{CR}}(M; \hat{E}_0, \hat{E}_1)$, respectively, corresponding to $\sigma$ and $\sigma^{-1}$. Concretely, $P = e_1 A e_0$ and $Q = e_0 B e_1$ where $A, B \in \Psi^0_{\text{At}}(M, \mathbb{C}^n)$ with symbols $\sigma, \sigma^{-1}$. Then $S = 1_{E_0} - Q P , T = 1_{E_1} - P Q \in \Psi^{-\infty}_{\text{CR}}(M, \hat{E}_0 \oplus \hat{E}_1)$ and $\text{Ind}_{\text{CR}}([e_0, e_1; \sigma])$ is given by $[CM(P, Q)] - [\text{diag}(0, 1_{E_i})] \in K_0(\Psi^\infty_{\text{CR}}(M))$ where

\[ CM(P, Q) := \left( \frac{S^2}{TP}, \frac{S(1_{E_0} + S)Q}{1_{E_1} - T^2} \right) \in \mathbb{M}_2(\Psi^{-\infty}_{\text{CR}}(M; \hat{E}_0 \oplus \hat{E}_1)^+) \quad (6.40) \]

6.41 Lemma. In the above situation, as we can make our choices so that $CM(P, Q)$ is almost local and since the class does not depend on choices, we see that the image of $\text{TR}_* \circ \text{Ch} \circ \text{Ind}_{\text{CR}}$ belongs to $H_{\text{even}}(\Omega^*(\mathbb{C}T)) \subset H_{\text{even}}(\Omega_*(\mathbb{C}T))$.

Using the lemma, we define

6.42 Definition. $\text{Ch}^e_{\text{alg}} := \text{TR}_* \circ \text{Ch} \circ \text{Ind}_{\text{CR}}: K_0(C^\infty(M) \to \Sigma) \to H^e_{\text{even}}(\mathbb{C}T)$.

In fact, we can say much more:

6.43 Proposition. Assume that $[e_0, e_1; \sigma] \in K_0(C^\infty(M) \to \Sigma)$ is a Dirac class. Then

\[ \text{TR}_* \circ \text{Ch} \circ \text{Ind}_{\text{CR}}([e_0, e_1; \sigma]) = \left[ \int_M \text{AS}(M, \sigma) \wedge \omega_{\text{Lott}} \right] \in H^e_{\text{even}}(\mathbb{C}T), \]

where $\text{AS}(M, \sigma)$ is the corresponding Atiyah-Singer integrand and where $\omega_{\text{Lott}}(M) \in \Omega^*(M) \otimes \Omega_*(\mathbb{C}T)$ is Lott’s bi-form, a bi-form which is closed in both arguments and that appears naturally in Lott’s treatment of the Connes-Moscovici higher index theorem.

Proof. This follows from Lott’s higher index theorem for operators of Dirac type. For this particular statement we employ the version of Lott’s theorem given in [9, Section 5] (just take $B$ equal to a point there), see also [10, Appendix A]. In this references the left hand side is paired with a cyclic cocycle; however, as explained in [20], see in particular Definition 11 and Theorem 4 (1) there, one can give a statement directly at the level of non-commutative de Rham classes.

The formula before (6.40) for $\text{Ind}_{\text{CR}}$ is an explicit implementation of the composition of $K_0(C^\infty(M) \to \Sigma) \to K_0(\Psi^0_{\text{CR}}(M) \to \Sigma)$ and the excision isomorphism, given by the inverse of $K_0(\Psi^\infty_{\text{CR}}(M)) \xrightarrow{\sim} K_0(\Psi^0_{\text{CR}}(M) \to \Sigma)$. Equivalently, $\text{Ind}_{\text{CR}}$ is the boundary map for the extension of pairs of algebras $0 \to (0 \to \Psi^\infty_{\text{CR}}(M)) \to (C^\infty(M) \to \Psi^0_{\text{CR}}(M)) \to (C^\infty(M) \to \Sigma) \to 0$, composed with the suspension isomorphism. It is then a direct consequence of Theorem 6.22 that the following square is commutative, where we use that the top row is the K-theory $\theta'$ of Theorem 6.22

\[ \begin{array}{ccc}
K_0(C^\infty + \Psi^\infty_{\text{CR}} \to \Psi^0_{\text{CR}}) & \xrightarrow{\sim} & K_0(C^\infty \to \Sigma) \\
\text{Ch}^{\theta'} & \xrightarrow{\text{Ind}_{\text{CR}}} & K_0(\Psi^0_{\text{CR}}) \\
\text{H}_{\text{odd}}(\Omega_+(C^\infty + \Psi^\infty_{\text{CR}})) & \xrightarrow{\text{Ind}_{\text{CR}}} & \text{H}_{\text{even}}(\Omega_+(\Psi^0_{\text{CR}}))
\end{array} \]

\[ (6.44) \]
6.45 Remark. Before stating the main proposition of this section, it is worth to notice that, as explained in Remark 6.39, the following square is commutative:

\[
\begin{array}{ccc}
K_0^{alg}(C_\infty + \Psi_{\infty}^- \rightarrow \Psi_0^-) & \xrightarrow{\iota_\ast} & K_0(C_\infty + \Psi_{\infty}^- \rightarrow \Psi_0^+) \\
\text{Ch}_{\text{rel}}^{\ast} & & \text{Ch}_{\text{rel}}^{\ast} \\
H_{\text{odd}}(\Pi(C_\infty) + \Pi(\Psi_{\infty}^-) \rightarrow \Pi(\Psi_0^-)) & \xrightarrow{\iota_\ast} & H_{\text{odd}}(\Pi(C_\infty) + \Pi(\Psi_{\infty}^-) \rightarrow \Pi(\Psi_0^+))
\end{array}
\]

where the horizontal maps are induced by the inclusions. Moreover, by naturality also the following diagram commutes:

\[
\begin{array}{ccc}
H_{\text{odd}}(\Pi(C_\infty) + \Pi(\Psi_{\infty}^-) \rightarrow \Pi(\Psi_0^-)) & \xrightarrow{(\text{TR}, \text{TR}^{del})_\ast} & H_{\text{odd}}(\Omega(\Sigma) \rightarrow \Omega^{del}(\Sigma)) \xleftarrow{\iota_\ast} \text{H}_{\text{even}}(\Omega^f(\Sigma)) \\
\text{H}_{\text{odd}}(\Pi(C_\infty) + \Pi(\Psi_{\infty}^-) \rightarrow \Pi(\Psi_0^-)) & \xrightarrow{(\text{TR}, \text{TR}^{del})_\ast} & H_{\text{odd}}(\hat{\Omega}(A) \rightarrow \hat{\Omega}^{del}(A)) \xleftarrow{1} \text{H}_{\text{even}}(\Omega^f(A))
\end{array}
\]

and it follows that we have the following algebraic characterization of \( \text{Ch}_\Gamma^{rel} \):

\[ \text{Ch}_\Gamma^{rel} = j_\ast \circ (\iota_\ast)^{-1} \circ (\text{TR}, \text{TR}^{del})_\ast \circ \text{Ch}_{\text{rel}} \circ (i_\ast)^{-1}. \]

Now we are in the position of proving the following result.

6.46 Proposition. Let \( j: \Omega^f_\ast(\Sigma) \hookrightarrow \Omega^f_\ast(A) \) be the natural inclusion. The localized Chern character factors through the algebraic one, specifically

\[ \text{Ch}_\Gamma^{rel} = j_\ast \circ \text{Ch}_{\text{rel}}^{alg}; K_0(C_\infty(M) \rightarrow \Sigma) \rightarrow \text{H}_{\text{even}}(\Omega^f(\Sigma)). \]

Moreover, if \([e_0, e_1; \sigma] \in K_0(C_\infty(M) \rightarrow \Sigma)\) is defined by a Dirac-type operator, then

\[ \text{Ch}_\Gamma^{rel}([e_0, e_1; \sigma]) = j_\ast \left[ \int_M \text{AS}(M, \sigma) \wedge \omega_{\text{Lott}} \right] \]

with \( \text{AS}(M, \sigma) \) equal to the Atiyah-Singer integrand corresponding to \( \sigma \) and with \( \omega_{\text{Lott}}(M) \in \Omega^\ast(M) \otimes \Omega_\ast(\Sigma) \) equal to Lott’s bi-form.

Since \( K_0(C_\infty(M) \rightarrow \Sigma) \) is generated modulo torsion by Dirac classes and since the range of \( \text{TR}, \text{TR} \) is a vector space, Proposition 6.46 gives a complete description of the homomorphism \( \text{Ch}_\Gamma^{rel} \).

Proof. We only have to prove (6.47), given that (6.48) follows directly from Proposition 6.43. The main difficulty is that \( \text{Ch}_{\text{rel}}^{alg} \) involves taking the K-theoretic index map, whereas \( \text{Ch}_\Gamma^{rel} \) doesn’t. However, the commutativity of (6.44) is the commutativity of the top rectangle in the following diagram. It follows then immediately that it commutes:

\[
\begin{array}{ccc}
K_0(C_\infty + \Psi_{\infty}^- \rightarrow \Psi_0^-) & \xrightarrow{\sim} & K_0(C_\infty \rightarrow \Sigma) \xrightarrow{\text{Ind}_{\Gamma}} K_0(\Psi_{\infty}^-) \\
\text{Ch}_{\text{rel}}^{\ast} & & \text{Ch} \\
H_{\text{odd}}(\Pi(C_\infty) + \Pi(\Psi_{\infty}^-) \rightarrow \Pi(\Psi_0^-)) & \xrightarrow{\delta'^{\text{pr}}} & \text{H}_{\text{even}}(\Pi(\Psi_{\infty}^-)) \\
\text{H}_{\text{even}}(\Omega^f(\Sigma)) \xrightarrow{\sim} H_{\text{odd}}(\Omega(\Sigma) \rightarrow \Omega^{del}(\Sigma)) \xrightarrow{\text{S}_{\text{pr}}} \text{H}_{\text{even}}(\Omega(\Sigma))
\end{array}
\]

Here \( \delta'^{\text{pr}} \) is defined in (6.18), i.e. indeed is induced by the projection onto \( \Pi(\Psi_{\infty}^- \rightarrow 0) \) composed with suspension to \( \text{H}_{\text{even}}(\Pi(\Psi_{\infty}^-)) \). Consequently, the composition of the lower row is the inclusion \( \text{H}_{\text{even}}(\Omega^f(\Sigma)) \hookrightarrow \text{H}_{\text{even}}(\Omega(\Sigma)) \). Equality (6.47) then follows from Remark 6.45. \( \square \)
6.5 Comparison with Lott's $\varrho$ form

Let $\tilde{D}_M$ an $L^2$-invertible $\Gamma$-equivariant Dirac operator on a Galois $\Gamma$ covering $\tilde{M} \to M$. This subsection is devoted to a comparison of our class $\text{Ch}_{\text{red}}^\text{del}(\varrho(\tilde{D}_M))$ with the higher rho invariant of Lott. For simplicity we assume that $M$ is odd dimensional.

Under the additional assumption that $\Gamma$ is of polynomial growth, the higher rho invariant of Lott appeared for the first time in [24], as the delocalized part of its higher eta invariant. Lott’s higher eta invariant, defined in [24] under the same assumptions, is an element

$$\eta_{\text{Lott}}(\tilde{D}_M) \in \hat{\Omega}_*(\mathcal{A}\Gamma)_{ab},$$

with $\mathcal{A}\Gamma \subset C^*_\text{red}\Gamma$ the algebra of rapidly decreasing functions on $\Gamma$. It was conjectured in [24] that the higher eta invariant was the boundary correction term in a higher Atiyah-Patodi-Singer index theorem. The conjecture was settled in [18]. Later, following an idea of Lott [25], the higher eta invariant and the higher Atiyah-Patodi-Singer index theorem were established for any finitely generated discrete group, see [19]. In this generality the algebra $\mathcal{A}\Gamma \subset C^*_\text{red}\Gamma$ can be taken to be the Connes-Moscovici algebra, using crucially that the latter is the projective limit of involutive Banach algebras with unit. Wahl extended these results even further, allowing $\mathcal{A}\Gamma \subset C^*_\text{red}\Gamma$ to be any projective limit of involutive Banach algebras with unit. Thus, following Wahl, in the sequel we shall assume that we have a projective system of involutive Banach algebras with units $(\mathcal{A}_j, i_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j)_{j \in \mathbb{N}}$ satisfying the following conditions:

(i) $\mathcal{A}_0 = C^*_\text{red}\Gamma$;
(ii) the map $i_{j+1,j} : \mathcal{A}_{j+1} \to \mathcal{A}_j$ is injective for any $j$;
(iii) the induced map $i_j$ from the projective limit $\mathcal{A}_\infty$ to $\mathcal{A}_j$ has dense image;
(iv) for any $j > 0$ the algebra $\mathcal{A}_j$ is holomorphically closed in $\mathcal{A}_0 \equiv C^*_\text{red}\Gamma$;
(v) $\mathcal{A}\Gamma$ is a subalgebra of each $\mathcal{A}_j$.

Under these assumptions we can consider $\mathcal{A}\Gamma := \mathcal{A}_\infty$, an involutive $m$-convex Fréchet algebra with unit. Wahl defines the higher eta invariant as the element $\eta_{\text{Lott}}(\tilde{D}_M) \in \hat{\Omega}_*(\mathcal{A}\Gamma)_{ab}$. As already remarked, the Connes-Moscovici algebra $\mathcal{B}_\infty$ satisfies these hypothesis. If $\Gamma$ is Gromov hyperbolic then a particular algebra $\mathcal{A}\Gamma$ defined by Puschnigg in [32] and discussed in detail in Section 7.2 also satisfies these hypothesis. By extending to the non-commutative context the arguments given in Chapter 10 of [3], one can prove that because of the invertibility assumption on $\tilde{D}$ the following equality holds:

$$d\eta_{\text{Lott}}(\tilde{D}_M) = \int_M \text{AS}(M) \wedge \omega_{\text{Lott}}(M) \quad \text{in} \quad \hat{\Omega}_*(\mathcal{A}\Gamma)_{ab}$$

with $\text{AS}(M)$ the local Atiyah-Singer integrand and $\omega_{\text{Lott}}(M) \in \hat{\Omega}^*(M) \otimes \hat{\Omega}_* (\mathcal{C}\Gamma)$ denoting again Lott’s biform. Following [33], we define Lott’s higher rho invariant $\{\eta_{\text{Lott}}(\tilde{D}_M)\}$ as the image of $\eta_{\text{Lott}}(\tilde{D}_M)$ in the quotient $\hat{\Omega}^\text{del}_(\mathcal{A}\Gamma)_{ab} = \hat{\Omega}_*(\mathcal{A}\Gamma)_{ab}/\hat{\Omega}^\text{red}_*(\mathcal{A}\Gamma)_{ab}$. Since $\omega_{\text{Lott}}(M)$ is localized at the identity element by [23],

$$d\{\eta_{\text{Lott}}(\tilde{D}_M)\} = 0 \quad \text{in} \quad \hat{\Omega}^\text{del}_*(\mathcal{A}\Gamma)_{ab}$$

and we obtain

$$\varrho_{\text{Lott}}(\tilde{D}) := [\{\eta_{\text{Lott}}(\tilde{D}_M)\}] \in H^\text{del}(\mathcal{A}\Gamma).$$

(From now on we shall omit the curly brackets from the notation.)

We want to compare $\varrho_{\text{Lott}}(\tilde{D})$ with our class $\text{Ch}^\text{del}_{\text{red}}(\varrho(\tilde{D}_M))$, which are both elements of the group $H^\text{del}(\mathcal{A}\Gamma)$. For simplicity, we carry this out in the bounding case, where we can use deep results from higher APS index theory. Thus, we assume that there exists a cocompact $\Gamma$-covering with boundary $\tilde{W} \to W$ with product metric near the boundary, and a $\Gamma$-equivariant odd $\mathbb{Z}_2$-graded Dirac operator $\tilde{D}_W$ on $W$ such that $\partial \tilde{W} = \tilde{M}$ and such that the boundary operator of $\tilde{D}_W$ is equal to $\tilde{D}_M$. Under this additional assumption, our strategy is the following: using the delocalized Chern character, we wish to connect the delocalized APS index theorem in K-theory, proved in [30], with the delocalized APS index theorem in noncommutative de Rham homology. The latter is obtained from the following higher index formula, proved in [18] and stated below for the benefit of the reader:
6.49 Proposition. There exists a well defined APS index class $\text{Ind}_{\Gamma,b}(\tilde{D}_W) \in K_0(A\Gamma)$ such that

$$\text{Ch}(\text{Ind}_{\Gamma,b}(\tilde{D}_W)) = \left[ \int_W \text{AS}(W) \wedge \omega_{\text{Lott}}(W) - \eta_{\text{Lott}}(\tilde{D}_M) \right] \in H_{\text{even}}(A\Gamma).$$

(6.50)

In this formula, $\omega_{\text{Lott}}(W)$ is an explicit bi-form, which still is concentrated at the identity element of the group.

By applying $q_* : H_{\text{even}}(A\Gamma) \to H_{\text{even}}(A\Gamma)^{\text{del}}$ to both sides and using that $\omega_{\text{Lott}}(W)$ is concentrated at the identity element, we obtain the delocalized APS index theorem in noncommutative de Rham homology:

$$q_*(\text{Ch}(\text{Ind}_{\Gamma,b}(\tilde{D}_W))) = -\rho_{\text{Lott}}(\tilde{D}_M) \quad \text{in} \quad H_{\text{even}}^{\text{del}}(A\Gamma).$$

(6.51)

We now want to show that, on the other hand,

$$q_*(\text{Ch}(\text{Ind}_{\Gamma,b}(\tilde{D}_W))) = \text{Ch}_{\Gamma}^{\text{del}}(\tilde{g}(\tilde{D}_M)) \quad \text{in} \quad H_{\text{even}}^{\text{del}}(A\Gamma)$$

(6.52)

thus obtaining the desired comparison, viz.

$$\text{Ch}_{\Gamma}^{\text{del}}(\tilde{g}(\tilde{D}_M)) = -\rho_{\text{Lott}}(\tilde{D}_M).$$

We would like to obtain (6.52) by employing the delocalized APS index theorem in K-Theory proved in [30]. However, since we are using a different description of the analytic surgery sequence, we shall first need to reformulate the delocalized APS index theorem in K-theory in our new setting. To this end, we recall and slightly modify results due to the third author, see [43].

We start by observing that in the above geometric setting we have the following $b$-groupoid,

$$\tilde{W} \times_{\Gamma} \tilde{W} \cup \tilde{M} \times_{\Gamma} \tilde{M} \times R \cong W.$$  

This is a Lie groupoid given, in a more rigorous way, by the blow-up of $\tilde{M} \times_{\Gamma} \tilde{M}$ in $\tilde{W} \times_{\Gamma} \tilde{W}$. In order to lighten the notation we will denote by $G(M)$ the groupoid $\tilde{M} \times_{\Gamma} \tilde{M}$, by $G(W,M)$ the $b$-groupoid, and by $G(W)$ the groupoid $\tilde{W} \times_{\Gamma} \tilde{W}$.

We have the following commutative diagram of K-theory groups

$$
\begin{array}{cccccc}
K_0(C^*_\Gamma(G(\tilde{W}))) & \xrightarrow{S} & K_1(C^*_\Gamma(G(\tilde{W}) \otimes C_0(0,1))) \\
\downarrow{j_*} & & \downarrow{j_*} & & \downarrow{i_*} \\
K_0(\Psi^0_{\Gamma}(\tilde{W})) & \xrightarrow{\sigma} & K_1(C_0(\tilde{W}) \rightarrow \Psi^0_{\Gamma}(\tilde{W})) & \cong & K_1(C^*_\Gamma(G(W)_{0,1})) \\
\uparrow{\sigma^*} & & \uparrow{\sigma^*} & & \uparrow{\sigma^*} \\
K_0(\Psi^0_{\Gamma}(\tilde{M})) & \xrightarrow{S} & K_1(\Psi^0_{\Gamma,R}(\tilde{M} \times R)) & \rightarrow & K_0(C(M) \rightarrow \Psi^0_{\Gamma,R}(\tilde{M} \times R)) & \cong & K_0(C^*_\Gamma((G(M) \times R)_{0,1})))
\end{array}
$$

(6.53)

In this diagram:

- $S$ denotes the suspension isomorphism;
- $i$ is the natural inclusion of $C^*_\Gamma(G(\tilde{W}) \otimes C_0(0,1))$ into $C^*_\Gamma(G(W)_{0,1})$;
- $j$ and $j^c$ are the natural inclusions of $C^*_\Gamma(G(\tilde{W}))$ into $\Psi^0_{\Gamma}(\tilde{W})$ and of $C^*_\Gamma(G(\tilde{W}) \otimes C_0(0,1))$ into the mapping cone of $C_0(\tilde{W}) \rightarrow \Psi^0_{\Gamma}(\tilde{W})$ respectively.
Next we define the homomorphisms $\partial^\Psi$, $\partial^c$, $\partial^{ad}$ and $S^\Psi$, thus completing the description of the above diagram.

For the homomorphisms $\partial^\Psi$, $\partial^c$, $\partial^{ad}$, we consider $\Psi_{\Gamma,b}^0(\overline{W})$, the $C^*$-closure of the 0-order $b$-pseudodifferential operators on $W$, which is the same as the $C^*$-closure of the compactly supported 0-order pseudodifferential operators on $G(W,M)$. Then the restriction to the boundary gives rise to the following short exact sequences

$$0 \longrightarrow \Psi_{\Gamma,b}^0(\overline{W}) \longrightarrow \Psi_{\Gamma,b}^0(\overline{W}) \longrightarrow \Psi_{\Gamma,b}^0(\overline{M} \times \mathbb{R}) \longrightarrow 0,$$

(6.54)

where $\Psi_{\Gamma,b}^0(\overline{M} \times \mathbb{R})$ are the suspended operators on $\overline{M}$, namely operators on $\overline{M} \times \mathbb{R}$ which are translation invariant on $\mathbb{R}$. Then $\partial^\Psi$ is the boundary morphism associated to (6.54). Analogously, one defines $\partial^c$ and $\partial^{ad}$. Finally, the map $S^\Psi$ in (6.53) is given by the composition of the suspension isomorphism $S$ followed by the homomorphism $\iota_*$ induced by the inclusion $\iota : \Psi_{\Gamma}^0(\overline{M}) \otimes C_0(\mathbb{R}) \rightarrow \Psi_{\Gamma,b}^0(\overline{M} \times \mathbb{R})$ given by Fourier transform in the $\mathbb{R}$ direction.

Let us now consider the $\Gamma$-equivariant Dirac $b$-operator $\tilde{D}_W$ on $\overline{W}$. Since $\tilde{D}_M$, the operator on the boundary, is $L^2$-invertible we know that $\tilde{D}_W$ is fully elliptic and there is a well-defined $b$-index class $\text{Ind}_{\Gamma,b}(\tilde{D}_W) \in K_0(C^*_r(G(\overline{W})))$. This index class precisely corresponds to the one appearing in (6.50) through the isomorphisms $K_0(C^*_r(G(\overline{W}))) = K_0(C^*_r(\Gamma)) = K_0(\mathcal{A}^\Gamma)$.

In this context, Zenobi [43, Equation (3.3)] has defined the adiabatic rho class $\partial^{ad}(\tilde{D}_M)$ as an element in $K_0(C^*_r(G(M^{[0,1]}_{ad}))$. The delocalized APS index theorem (in K-theory) in the groupoid framework, [43, Theorem 3.7], states that

$$i_*(S(\text{Ind}_{\Gamma,b}(\tilde{D}_W))) = \partial^{ad}(\tilde{D}_M).$$

(6.55)

We want to use this fundamental equality in order to prove the identity (6.52). To this end we consider the $b$-groupoid $G([0,1],\{\})$. We have the following commutative diagram:

$$\begin{array}{cccccc}
0 & \longrightarrow & \Psi_{\Gamma}^0(\overline{M}) \otimes C^*_r(\{(0,1) \times (0,1)\}) & \longrightarrow & \Psi_{\Gamma,b}^0(\overline{W}) & \longrightarrow & \Psi_{\Gamma,b}^0(\overline{M} \times \mathbb{R}) & \longrightarrow & 0 \\
& & \downarrow \iota' & & \downarrow \iota'' & & \downarrow \iota & & \\
0 & \longrightarrow & \Psi_{\Gamma,b}^0(\overline{W}) & \longrightarrow & \Psi_{\Gamma,b}^0(\overline{W}) & \longrightarrow & \Psi_{\Gamma,b}^0(\overline{M} \times \mathbb{R}) & \longrightarrow & 0
\end{array}$$

(6.56)

with $\iota$ from above and where $\iota''$ is defined by the fact that over a collar neighbourhood of the boundary $G(W,M)$ is isomorphic to $G(M) \times G([0,1],\{\})$. Finally, $\iota'$ is the restriction of $\iota$. We denote by $\text{id} \otimes \partial$ the K-theory boundary map of the first row. Notice that $\partial$ is the isomorphism which sends the generator of $K_1(C_0(\mathbb{R}))$ to the generator of $K_0(C^*_r((0,1) \times (0,1)))$, namely the class of a rank one projector $e \in \mathcal{K}(L^2(0,1))$. Now consider $x \in K_0(\Psi_{\Gamma,b}^0(\overline{M}))$, then by naturality and by the above remarks

$$\partial^\Psi(S^\Psi(x)) = \partial^\Psi(\iota_*(Sx)) = \iota'_*(\text{id} \otimes \partial(Sx)) = \iota'_*(x \otimes e).$$

Consider now the class $[\pi_\geq(\tilde{D}_M)] = : [\pi_\geq] \in K_0(\Psi_{\Gamma,b}^0(\overline{M}))$ given by the projection on the positive spectrum of the Dirac operator $\tilde{D}_M$ on $\overline{M}$. We know from [42, Section 5.3] that $[\pi_\geq]$ is sent to the adiabatic rho class $\partial^{ad}(\tilde{D}_M) \in K_0(C^*_r(G(M^{[0,1]}_{ad})))$ by the composition of the homomorphisms appearing in the bottom row in (6.53). By using the delocalized APS index theorem in the groupoid framework, i.e. the equality $i_*(S(\text{Ind}_{\Gamma,b}(\tilde{D}_W))) = \partial^{ad}(\tilde{D}_M))$, and by a simple diagram chase in (6.53), we see that the difference

$$j_*(\text{Ind}_{\Gamma,b}(\tilde{D}_W)) - \partial^\Psi(S^\Psi([\pi_\geq]))$$

is in the image of $K_0(C_0(\overline{W}))$ and is therefore local in the sense that its Chern character lies in the image of $H^*(\mathcal{A}^\Gamma)$. 

Higher rho numbers, surgery and homology
We then have the following sequence of equalities

\[ q_* (\text{Ch}_1^{\text{del}}(\mathbb{D}_W)) = \text{Ch}_1^{\text{del}}(j_*(\text{Ind}_\Gamma,\delta(\mathbb{D}_W))) = \text{Ch}_1^{\text{del}}(\partial \Phi([\pi_\geq])) = \text{Ch}_1^{\text{del}}([\pi_\geq] \otimes e) = \text{Ch}_1^{\text{del}}([\pi_\geq]) \]

(6.57)

where the second equality employs the fact that \( \text{Ch}_1^{\text{del}} \) is zero on local terms whereas the last equality is given by Corollary 4.32 or, explicitly, by the fact that the trace of the tensor product is the product of the traces and that the trace of \( e \) is 1. Summarizing:

\[ q_* (\text{Ch}_1(\text{Ind}_\Gamma,\delta(\mathbb{D}_W))) = \text{Ch}_1^{\text{del}}([\pi_\geq]) \]

(6.58)

which is the equality we wanted to show. The following proposition is then the combination of (6.58) and (6.51).

6.59 Proposition. If \( \tilde{M} = \partial \tilde{W} \) and \( \tilde{D}_M = \tilde{D}_W^{+,\partial} \) as above, then

\[ \text{Ch}_1^{\text{del}}(\rho(\tilde{D}_M)) = -\varrho_{\text{Lott}}(\tilde{D}_M) \quad \text{in} \quad H_\text{even}^{\text{del}}(A \Gamma). \]

At least in the bounding case this answers a question raised by Lott in [24, Remark 4.11.3]. We leave the general non-bounding case to future investigations; we expect the techniques in [9, Section 5] to play a crucial role.

7 Higher \( \varrho \)-numbers from delocalized higher cocycles on \( \mathbb{C} \Gamma \)

7.1 Cyclic cohomology of group algebras

Let \( \Gamma \) be a discrete group. First, let us recall the definition of the cyclic set \( Z \Gamma \), for more background see for instance [35 Section 9.7]. Here \( Z_n \Gamma := \Gamma^{n+1} \) and the degeneracies, the face maps, and the cyclic structures are defined in the following way:

\[
\partial_i((g_0, \ldots, g_n)) := \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_n), & \text{for } i < n \\ (g_n g_0, g_1, \ldots, g_{n-1}), & \text{for } i = n \end{cases}
\]

\[
\sigma_i((g_0, \ldots, g_n)) := (g_0, \ldots, g_i, e, g_{i+1}, \ldots, g_n),
\]

\[
l((g_0, \ldots, g_n)) := (g_n, g_0, \ldots, g_{n-1}).
\]

Now, for \( x \in \Gamma \), let \( Z_n(\Gamma, x) \) denote the subset of \( Z_n \Gamma \) consisting of all \( (g_0, \ldots, g_n) \) such that the product \( g_0 \cdots g_n \) is conjugate to \( x \). As \( n \) varies these subsets form a cyclic subset \( Z(\Gamma, x) \) of \( Z \Gamma \).

The Hochschild cohomology \( HH^*(\mathbb{C} \Gamma) \) is given by the cohomology of the cochain complex \( C^*(Z \Gamma) \) associated to the simplicial object underlying \( Z \Gamma \). In particular, we have that

\[ HH^*(\mathbb{C} \Gamma) = \prod_{(x) \in (\Gamma)} HH^*(\mathbb{C} \Gamma; (x)) \]

(7.1)

which corresponds to the decomposition \( H^*(Z \Gamma) = \prod_{(x) \in (\Gamma)} H^*(Z(\Gamma, x)) \). Here, \( (\Gamma) \) is the set of conjugacy classes in \( \Gamma \), and \( (x) \) denotes the conjugacy class of \( x \).

7.2 Definition. Define \( C_{\text{pol}}^*(\mathbb{C} \Gamma; (x)) \) as the subcomplex of the Hochschild complex \( C^*(\mathbb{C} \Gamma; (x)) \) associated to \( Z(\Gamma, x) \) whose elements are cochains of polynomial growth, i.e., functions \( f : Z(\Gamma, x) \to \mathbb{C} \) with \(|f(g_0, \ldots, g_n)| \leq C(1 + l(g_0) \cdots l(g_n))^{N} \) for suitable \( C, N \). Denote by \( HH_{\text{pol}}^*(\mathbb{C} \Gamma; (x)) \) its cohomology.
Let $\Gamma_x := \{g \in \Gamma \mid gx = xg\}$ be the centralizer of $x$ in $\Gamma$. By [35 Proposition 9.7.4], for all $x \in \Gamma$ the inclusion $\Gamma_x \hookrightarrow \Gamma$ induces a homotopy equivalence $\iota: Z(\Gamma_x, x) \to Z(\Gamma, x)$ of cyclic sets with homotopy inverse $\rho: Z(\Gamma, x) \to Z(\Gamma_x, x)$, explicitly defined in the following way. First, choose representatives of minimal word length for the right $\Gamma_x$-coset in $\Gamma$. Given $(g_0, \ldots, g_n) \in \mathbb{Z}_n(G, x)$, let $y_i$ the chosen coset representative such that $y_i(g_{i+1} \cdots g_{n}g_0 \cdots g_i)y_i^{-1} = x$ and set

$$\rho(g_0, \ldots, g_n) := (y_0g_0y_0^{-1}, y_0g_1y_1^{-1}, \ldots, y_{n-1}g_ny_n^{-1}).$$

Moreover, the simplicial homotopy between the identity map of $Z(\Gamma, x)$ and $\iota \circ \rho$ is given by

$$h_j(g_0, \ldots, g_n) := (y_0g_0y_0^{-1}, y_0g_1y_1^{-1}, \ldots, y_{j-1}g_jy_j^{-1}, y_j, g_{j+1}, \ldots, g_n); \quad j \in \{0, \ldots, n\}.$$ 

7.5 Remark. Observe that the simplicial cochain complex associated to $Z(\Gamma_x, x)$ is exactly the standard bar complex which defines the group cohomology $H^*(\Gamma_x, \mathbb{C})$. Hence we have that the group cohomology $H^*(\Gamma_x, \mathbb{C})$ is the simplicial set cohomology $H^*(Z(\Gamma_x, x))$, which in turn is isomorphic to $HH^*(\mathbb{C}; \langle x \rangle) := HH^*(Z(\Gamma, x))$ by means of the pull-back map $\rho^*$.

Let us assume from now on that $\Gamma$ is hyperbolic. Then $\Gamma_x$ is hyperbolic for all $x \in \Gamma$, see [11 Section 8.5.M] We fix a finite symmetric set of generators $S$ of $\Gamma$ and let $l$ be the associated word-length function. We assume that the Cayley-graph $G(\Gamma, S)$ is $\delta$-hyperbolic. For every element $x \in \Gamma$ we choose a word $w(\gamma)$ of minimal length representing it and fix an element $\sigma(\gamma)$ of minimal word length $l(\gamma)$ in the conjugacy class $\langle \gamma \rangle$. We recall the statement of [32 Lemma 4.1].

7.6 Lemma. For given $R > 0$ there exists a constant $C(R) > 0$ such that the following holds. Let $\gamma \in \Gamma$ be written as a word $w(\gamma)$ in the generators $S$ and assume that $\min_{g \in \langle \gamma \rangle} l(g) \leq R$. Then some cyclic permutation of $w(\gamma)$ represents an element of length less than $C(R)$ in $\Gamma$.

As a consequence, we have the following result.

7.7 Lemma. If $\Gamma$ is hyperbolic, the pull-back of cochains through $\rho$ sends cochains of polynomial growth on $Z(\Gamma_x, x)$ to cochains of polynomial growth on $Z(\Gamma, x)$. The same is true for the simplicial homotopy $\{h_j\}$.

Proof. Start with $g_0, \ldots, g_n \in \Gamma$ such that $g_0 \cdots g_n \in \langle x \rangle$. Abbreviate $G_i := g_{i+1} \cdots g_ng_0 \cdots g_i$. The construction of $\rho$ and of the simplicial homotopy is based on the elements $y_0, \ldots, y_n$ which depends on $g_0, \ldots, g_n$. We show that their length is bounded linearly in the length of the $g_i$, which immediately implies the statement.

First, observe that each $G_i$ is conjugated to $x$. Using Lemma 7.6 there is $C > 0$ (depending only on $l(x)$) such that we find a subword $a_i$ of $G_i$ and $l(a_iG_i^{-1}a_i^{-1}) \leq C$. Note that, as a subword of a product of the $g_i$, $l(a_i) \leq \sum_{k=0}^n l(g_k)$. Secondly, the set of elements of $\langle x \rangle$ of length $\leq C$ is finite. Therefore, there is $D > 0$ depending only on $C$ and $x$ and $b_i$ with $l(b_i) \leq D$ and such that $b_ia_iG_i^{-1}a_i^{-1}b_i^{-1} = x$. Now, also $y_iG_i^{-1}y_i^{-1} = x$, therefore $b_ia_i$ and $y_i$ belong to the same $\Gamma_x$-coset. By minimality,

$$l(y_i) \leq l(b_ia_i) \leq D + \sum_{k=0}^n l(g_k).$$

Let now $f: \Gamma^{n+1} \to \mathbb{C}$ be a cochain of polynomial growth, i.e. $|f(g_0, g_1, \ldots, g_n)| \leq C(1 + l(g_0) \cdots l(g_n))^N$ for some $C, N$. It follows now immediately from the definition of $\rho$ that also $\rho^*f$ has polynomial growth, as

$$|\rho^*f(g_0, \ldots, g_n)| = |f(y_0g_0y_0^{-1}, \ldots, y_{n-1}g_ny_n^{-1})| \leq C(1 + (l(y_0) + l(g_0)) \cdots (l(y_{n-1}) + l(g_{n-1}) + l(g_n)))^N$$

which, using our preparation, is bounded by a polynomial in $l(g_0), \ldots, l(g_n)$ (even of unchanged degree). The same argument applies to the chain homotopy and the result follows.


7.8 Proposition. Let $\Gamma$ be a hyperbolic group. Then the inclusion of complexes $C^*_\text{pol}(\Gamma; \langle x \rangle) \hookrightarrow C^*(\Gamma; \langle x \rangle)$ induces an isomorphism $HH^*_\text{pol}(\Gamma; \langle x \rangle) \cong HH^*(\Gamma; \langle x \rangle)$.

Proof. By [28] Corollary 5.3, since hyperbolic groups admit a combing of polynomial growth, the natural inclusion of complexes $C^*_\text{pol}(\Gamma_x; \mathbb{C}) \hookrightarrow C^*(\Gamma_x; \mathbb{C})$ induces an isomorphism between $H^*_\text{pol}(\Gamma_x; \mathbb{C})$ and $H^*(\Gamma_x; \mathbb{C})$; here $C^*_\text{pol}(\Gamma_x; \mathbb{C})$ is the subcomplex of the bicomplex $C^*(\Gamma_x; \mathbb{C})$ whose elements are cochains with polynomial growth on $\Gamma_x$. Now, by Lemma 7.7 $\rho^*$ preserve the polynomial growth of cochains and we have the following commutative square

\[
\begin{array}{c}
H^*_\text{pol}(\Gamma_x; \mathbb{C}) \cong H^*(\Gamma_x; \mathbb{C}) \\
\downarrow \rho^* \downarrow \rho^* \\
HH^*_\text{pol}(\Gamma; \langle x \rangle) \cong HH^*(\Gamma; \langle x \rangle).
\end{array}
\]  

(7.9)

Since the simplicial homotopy $\{h_j\}$ preserves polynomial growth, $\rho^*$ induces an isomorphism on both sides of the square and the result follows.

---

The cyclic cohomology of $\Gamma$, denoted by $HC^*(\Gamma)$, is by definition the cohomology of the total complex of the bicomplex $B^*(\Gamma)$, see [22] Section 2.4.3. Recall that the first column of $B^*(\Gamma)$ is exactly the Hochschild complex of $\Gamma$ (and the other columns are shifted copies of that complex). Recall that projection onto the first column induces the Connes periodicity exact sequence

\[
\cdots \rightarrow HH^n(\Gamma) \xrightarrow{B} HC^{n-1}(\Gamma) \xrightarrow{S} HC^{n+1}(\Gamma) \xrightarrow{I} HH^{n+1}(\Gamma) \xrightarrow{B} \cdots
\]  

(7.10)

Observe now that, since $Z\Gamma = \bigcup Z(\Gamma, x)$ is a disjoint union of cyclic subset, we have that also $B^*(\Gamma)$ decomposes as $\prod_{(x) \in \Gamma} B^*(\Gamma; \langle x \rangle)$. This decomposition is compatible with the construction of (7.10).

7.11 Proposition. Let $\Gamma$ be a hyperbolic group. Let $B^*_\text{pol}(\Gamma; \langle x \rangle)$ be the subcomplex of $B^*(\Gamma; \langle x \rangle)$ whose elements are cochains of polynomial growth. Then the natural inclusion induces an isomorphism

$HC^*_\text{pol}(\Gamma; \langle x \rangle) \cong HC^*(\Gamma; \langle x \rangle)$.

Proof. Consider the commutative diagram

\[
\begin{array}{c}
HH^n(\Gamma; \langle x \rangle) \xrightarrow{B} HC^{n-1}(\Gamma; \langle x \rangle) \xrightarrow{S} HC^{n+1}(\Gamma; \langle x \rangle) \xrightarrow{I} HH^{n+1}(\Gamma; \langle x \rangle) \xrightarrow{B} HC^n \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
HH^n(\Gamma; \langle x \rangle) \xrightarrow{B} HC^{n-1}(\Gamma; \langle x \rangle) \xrightarrow{S} HC^{n+1}(\Gamma; \langle x \rangle) \xrightarrow{I} HH^{n+1}(\Gamma; \langle x \rangle) \xrightarrow{B} HC^n \\
\end{array}
\]

By induction, starting with the trivial cases $n = -2$ and $n = -1$, the 5-lemma and Proposition 7.8 for $HH$ imply the isomorphism for $HC^n$ for all $n \geq -2$.

Finally, let $C^*_\Lambda(\Gamma)$ be the subcomplex of $C^*(\Gamma)$ given by cyclic cochains, originally used by Connes to define cyclic cohomology. It is known that the inclusion of $C^*_\Lambda(\Gamma)$ into the first column of $B^*(\Gamma)$ induces an isomorphism $H^*_\Lambda(\Gamma) \cong HC^*(\Gamma)$. One can check that this isomorphism is compatible with the decomposition with respect to conjugancy classes and that it restricts to the subcomplexes of cochains with polynomial growth. As a consequence, we have the following result.

7.12 Corollary. If $\Gamma$ is an hyperbolic group, then $H^*_\Lambda(\Gamma; \langle x \rangle) \cong H^*_\Lambda(\Gamma; \langle x \rangle)$.  

7.2 Extendig cocycles for hyperbolic groups

Let $\Gamma$ be a Gromov hyperbolic group. We want to use cyclic cohomology to study primary and secondary invariants of Dirac operators. This is possible due to a dense and holomorphically closed subalgebra of $C^*_{\lambda}(\Gamma)$, that we will denote by $\mathcal{A}\Gamma$, constructed by Puschnigg in [32]. For this algebra we thus have that $K_*(\mathcal{A}\Gamma) \cong K_*(C^*_{\lambda}(\Gamma))$. Moreover, Puschnigg proves that any delocalized trace $\tau$ on $\mathcal{A}\Gamma$, which is supported on a finite number of conjugacy classes, extends to $\mathcal{A}\Gamma$. We will show that this is true also for higher delocalized cocycles. Recall from Corollary 7.12 that the summand $H\mathcal{C}^*(\mathcal{A}\Gamma; \langle x \rangle)$ supported on $\langle x \rangle$ is computed by $C^a_{\lambda, pol}(\mathcal{A}\Gamma; \langle x \rangle)$, the subcomplex of cyclic cochains of polynomial growth supported on $CC_n(\mathcal{A}\Gamma; \langle x \rangle)$.

7.13 Proposition. If $\tau$ is a delocalized cyclic $n$-cocycle over $\mathcal{A}\Gamma$ which is supported on $\langle x \rangle$ is polynomial growth, then it extends to a cyclic cocycle over $\mathcal{A}\Gamma$.

Proof. By [32, Proposition 5.6] we have the following commutative diagram

$$
\begin{array}{ccc}
CC_n(\mathcal{A}\Gamma) \otimes C^a_n(\mathcal{A}\Gamma; \langle x \rangle) & \xrightarrow{\pi(x) \otimes \text{id}} & CC_n(\mathcal{A}\Gamma; \langle x \rangle) \otimes C^a_n(\mathcal{A}\Gamma; \langle x \rangle) \\
\downarrow c_{\text{alg}} & & \downarrow \pi(x) \otimes \text{id} \\
CC_n(\mathcal{A}\Gamma) \otimes C^a_n(\mathcal{A}\Gamma; \langle x \rangle)) & & CC_n(\mathcal{A}\Gamma; \langle x \rangle) \otimes C^a_n(\mathcal{A}\Gamma; \langle x \rangle)) \\
\end{array}
$$

(7.14)

A crucial property of $\mathcal{A}\Gamma$, established in [32, Proposition 5.6, a)] is the existence of the (continuous) projection $\pi(x)$, $CC_*(\mathcal{A}\Gamma) \to CC_*(\mathcal{A}\Gamma; \langle x \rangle)$, extending the corresponding one for $\mathcal{A}\Gamma$. This gives the diagonal arrow $\pi(x)$. In [32, Proposition 5.6, b)] the map $\iota$ is constructed and shown to be an isomorphism.

By the very definition of $l_{\text{RD}}\Gamma$, the canonical pairing $c_{\text{alg}}$ between $CC_n(\mathcal{A}\Gamma)$ and $C^a_n(\mathcal{A}\Gamma; \langle x \rangle)$ (which factors through $CC_n(\mathcal{A}\Gamma; \langle x \rangle)$) extends by continuity uniquely to a well defined pairing $c_{\text{RD}}: CC_n(l_{\text{RD}}\Gamma; \langle x \rangle) \otimes C^a_n(\mathcal{A}\Gamma; \langle x \rangle)) \to \mathbb{C}$. In other words, the diagram without the dashed arrow commutes. The dashed arrow (the extension of the pairing sought for) is defined as the composition of $\pi(x) \otimes \text{id}$ and $\iota^{-1} \circ c_{\text{RD}}$. \hfill $\square$

7.3 Higher $\rho$ numbers

In this subsection, let $\Gamma$ be an hyperbolic group. Recall from the proof of Proposition 7.13 that Puschnigg constructs a topological projection from $\Omega_*(\mathcal{A}\Gamma)$ to the part localized at the conjugacy class of the neutral element, $\Omega_n(l_{\text{RD}}\Gamma)$. Using Proposition 7.13 and Corollary 7.12 we obtain immediately the following result.

7.15 Theorem. Let $\Gamma$ be an hyperbolic group and let $\tilde{M}$ be a Galois covering of a smooth compact manifold $M$ with covering group $\Gamma$. Then there exists a well-defined pairing

$$
K_{*+1}(\Gamma \to \Psi_0^B(\tilde{M})) \times H\mathcal{C}^*(\mathcal{A}\Gamma; \langle x \rangle) \to \mathbb{C}
$$

given by

$$
(\xi, [\tau]) \mapsto \langle \text{Ch}_{\text{det}}(\xi), \tau \rangle
$$

for $\xi \in K_{*+1}(\Gamma \to \Psi_0^B(\tilde{M}))$ and $\tau \in C^a_*(\mathcal{A}\Gamma; \langle x \rangle)$.
7.16 Definition. Let $\tilde{D}$ be a generalized Dirac operator which is $\Gamma$-equivariant on the Galois $\Gamma$-covering $\tilde{M}$ of a compact smooth manifold $M$. Suppose that $\tilde{D}$ is $L^2$-invertible. Then define its higher $\varrho$-number associated to $[\tau] \in HC^*(C\Gamma; \langle x \rangle)$ as

$$\varrho_{\tau}(\tilde{D}) := \langle Ch^!_\Gamma (\varrho(\tilde{D})), \tau \rangle \in \mathbb{C},$$

where $\varrho(\tilde{D})$ is defined as in Definition 3.4.

7.17 Example. Let $g$ be a metric with positive scalar curvature on a spin manifold $M$ of dimension $n$ with fundamental group $\Gamma$; then define the higher $\varrho$-number associated to $[\tau] \in HC^{*+1}(C\Gamma; \langle x \rangle)$ as

$$\varrho_{\tau}(g) := \langle Ch^!_\Gamma (\varrho(g)), \tau \rangle \in \mathbb{C},$$

where $\varrho(g) \in K_* \left( C(M) \to \Psi^0(\tilde{M}) \right)$ is the $\varrho$-class of the spin Dirac operator for the metric $g$.

Recall from (2.4) that, if the dimension of $M$ is odd, we have a surjective map

$$K_0(\Psi^0_{At}(\tilde{M})) \to K_1(C(M) \to \Psi^0_{At}(\tilde{M})).$$

given by the composition of the suspension isomorphism and the natural inclusion. Moreover, by Lemma 4.32 the triangle

$$\xymatrix{ K_0(\Psi^0_{At}(\tilde{M})) \ar[r] & K_1(C(M) \to \Psi^0_{At}(\tilde{M})) \ar[d]^{Ch^!_{\Gamma}} \ar[r]_{\text{Tr}^* \circ \text{Ch}} & H^*_{\text{even}}(\tilde{M} \Gamma) \ar[d]^{\text{Ch}^!_{\Gamma}} \ar[l]_{\text{Ch}^!_{\Gamma}} }$$

is commutative. Therefore, given a metric with positive scalar curvature $g$ on an odd dimensional manifold $M$, we can express $Ch^!_{\Gamma}(\varrho(g))$ by $\text{Tr}^* \circ \text{Ch}(\varrho(\pi_\geq))$, as in Section 3.2. In order to calculate the higher $\varrho$-number associated to a even dimensional cyclic cocycle $\tau$, we then proceed as in the following proposition.

7.18 Proposition. Let $\tau$ be a cyclic cocycle of degree $2k$ of polynomial growth with support contained in a conjugacy class $\langle x \rangle$ of $\Gamma$. Then $\rho_{\tau}(g)$ is given by the following expression

$$\sum_{g_0, \ldots, g_{2k} \in \langle x \rangle} \frac{1}{k!} \int_{\tilde{M} \times \cdots \times \tilde{M}} \text{Tr}(\chi(x_0)P(x_0g_0, x_1)h(x_1)P(x_1, x_2, \ldots, x_{2k})P(x_{2k}g_{2k}, x_0)) \tau(g_0, \ldots, g_{2k}) \, dx_0 \cdots dx_{2k}$$

(7.19)

where $P$ is the Schwartz kernel of $\pi_\geq(\tilde{D}_0)$ and where $\chi$ is the characteristic function of a fundamental domain $F$ for the action of $\Gamma$ on $\tilde{M}$.

Proof. First observe, using the notation of Section 6 that $(PdPdP)^2 = PdPdPdP$ and that $PdP = P[\nabla^{\text{Lott}}, P][\nabla^{\text{Lott}}, P] + P \Theta P$. Moreover, using that $P[\nabla^{\text{Lott}}, P] = 0$,

$$P[\nabla^{\text{Lott}}, P][\nabla^{\text{Lott}}, P] + P \Theta P = P[\nabla^{\text{Lott}}, P][\nabla^{\text{Lott}}, P] + P(\nabla^{\text{Lott}})^2 P + P[\nabla^{\text{Lott}}, P] P \nabla^{\text{Lott}}$$

$$= P[\nabla^{\text{Lott}}, P] \nabla^{\text{Lott}} P + P(\nabla^{\text{Lott}})^2 P$$

$$= P \nabla^{\text{Lott}} P \nabla^{\text{Lott}} P$$

(7.20)

Hence we have that $Ch^!_{\Gamma, 2k}(\pi_\geq)$ is given by

$$\text{Tr}^* \circ \text{Ch}( [P \nabla^{\text{Lott}} P]^{2k}).$$

In order to obtain an explicit formula for this we use the version of $\nabla^{\text{Lott}}$ given by (6.3). A direct calculation then gives (7.19). \qed
The same result applies to any $L^2$-invertible $\Gamma$-equivariant Dirac type operator on an odd dimensional manifold $M$.

7.21 Example. Let $\tau$ be a 0-cocycle supported on the conjugacy class $\langle x \rangle$ in $\Gamma$. Then

$$
\varphi(y) = \sum_{\gamma \in \langle x \rangle} \frac{1}{2} \int_{\mathcal{F}} \frac{D}{|D|} (x\gamma, x) \, d\text{vol}(x) \tau(\gamma)
$$

which is exactly the delocalized $\eta$-invariant of Lott obtained in [24, Section 4.10.1].

8 Higher $\varrho$ numbers from relative cohomology $H^*(M \to B\Gamma)$

When looking at the Higson-Roe sequence $\cdots \to K_*(M) \to K_*(C^*\Gamma) \to S^1_{-1}(\widetilde{M}) \to K_{*-1}(M) \to \cdots$ we realize that, at least morally, $K(C^*\Gamma)$ is closely related to $K(B\Gamma)$. Consequently, there should be a second source of information on the relative term (the structure set $S^1_{-1}(\widetilde{M})$) in terms of the difference between $M$ and $B\Gamma$. In this section, we make this idea precise: we construct a pairing between the analytic structure set $S^1_{-1}(\widetilde{M})$ and the relative cohomology $H^*(M \to B\Gamma; \mathbb{R})$, at least for suitable groups. To achieve this and to arrive at explicit formulas, we start with a new and convenient cocycle model for $H^*(M \to B\Gamma; \mathbb{R})$, using suitable Alexander-Spanier cochains on $\widetilde{M}$.

8.1 Alexander-Spanier cochains and singular cohomology

8.1 Definition. Let $C^q_{AS}(\widetilde{M})^\Gamma$ be the vector space of all (not necessarily continuous) functions $\varphi: \widetilde{M}^{q+1} \to \mathbb{R}$ which are invariant with respect to the diagonal action of $\Gamma$ on $\widetilde{M}^{q+1}$. Define $\delta: C^q_{AS}(\widetilde{M})^\Gamma \to C^{q+1}_{AS}(\widetilde{M})^\Gamma$ by the standard formula

$$
(\delta \varphi)(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1})
$$

which makes $\{C^*_{AS}(\widetilde{M})^\Gamma, \delta\}$ a cochain complex.

An important subcomplex of $C^*_{AS}(\widetilde{M})^\Gamma$ is given by locally zero cochains. An element $\varphi \in C^*_{AS}(\widetilde{M})^\Gamma$ is said to be locally zero if there is an open neighborhood of the diagonal $\widetilde{M} \subset \widetilde{M}^{q+1}$ on which $\varphi$ vanishes. We denote by $C^q_{AS,0}(\widetilde{M})^\Gamma$ the vector space of all such functions, forming a subcomplex of $C^*_{AS}(\widetilde{M})^\Gamma$.

We thus obtain the following exact sequence of cochain complexes

$$
0 \longrightarrow C^*_{AS,0}(\widetilde{M})^\Gamma \longrightarrow C^*_{AS}(\widetilde{M})^\Gamma \longrightarrow C^*_{AS}(\widetilde{M})^\Gamma \longrightarrow 0
$$

where $C^*_{AS}(\widetilde{M})^\Gamma$ is the quotient complex. Let us denote in the following way the associated cohomology groups, forming the associated long exact sequence,

$$
\cdots \longrightarrow H^*_{AS,0,\Gamma}(\widetilde{M}) \longrightarrow H^*_{AS,\Gamma}(\widetilde{M}) \longrightarrow H^*_{AS, \Gamma}(\widetilde{M}) \longrightarrow \cdots
$$

We first identify $H^*_{AS,\Gamma}(\widetilde{M})$ with $H^*(\Gamma)$. In order to do that fix a point $z$ in $\widetilde{M}$ and consider the inclusion $i: \Gamma \hookrightarrow \widetilde{M}$ given by $i(\gamma) = z\gamma$. Observe that $i$ is $\Gamma$-equivariant.

8.4 Proposition. The map $i^*: C^*_{AS}(\widetilde{M})^\Gamma \to C^*_{AS}(\Gamma)^\Gamma$ induces an isomorphism in cohomology, indeed is a chain homotopy equivalence.
8.7 Remark. We have the following variant of the cochain homotopy in Section 6.1. Let \( \mu : \hat{M} \to \Gamma \) be the map with \( \mu(x) = \gamma \) for \( x \in \mathcal{F}_\gamma \), and notice that \( \mu \) is \( \Gamma \)-equivariant. Observe that \( \mu \circ i = \text{id}_\Gamma \) and \( r := i \circ \mu \) is the map which sends \( x \in \mathcal{F}_\gamma \) to \( z_\gamma \). To prove that \( \ast \) induces an isomorphism in cohomology we construct a cochain homotopy \( K : C^k_{\text{AS}}(\hat{M})^\Gamma \to C^k_{\text{AS}}(\hat{M})^\Gamma \) between \( \text{id} \) and \( r^* \), setting

\[
K(\varphi)(x_0, \ldots, x_{k-1}) := \sum_{j=0}^{k-1} (-1)^{j+1} \varphi(r(x_0), \ldots, r(x_j), x_j, \ldots, x_{k-1})
\]

for \( \varphi \in C^k_{\text{AS}}(\hat{M})^\Gamma \). Of course, since \( r \) is \( \Gamma \)-equivariant, so is \( K \). We have

\[
\delta K(\varphi)(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^i K(\varphi)(x_0, \ldots, x_i, x_{i+1}, \ldots, x_k) = \\
= \sum_{i=0}^{k} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^{j+1} \varphi(r(x_0), \ldots, r(x_j), x_j, \ldots, x_{i-1}, x_i, \ldots, x_k) \right) + \sum_{j=i+1}^{k} (-1)^j \varphi(r(x_0), \ldots, r(x_i), x_i, \ldots, x_j, x_j, \ldots, x_k)
\]

\[
K(\delta \varphi)(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^{i+1} \delta \varphi(r(x_0), \ldots, r(x_i), x_i, \ldots, x_k) = \\
= \sum_{j=0}^{k} (-1)^{i+1} \left( \sum_{i=0}^{j-1} (-1)^i \varphi(r(x_0), \ldots, r(x_i), x_i, \ldots, x_{j-1}, x_j, x_j, \ldots, x_k) \right) + \sum_{i=j}^{k} (-1)^{i+1} \varphi(r(x_0), \ldots, r(x_i), x_i, \ldots, x_j, x_j, \ldots, x_k)
\]

Comparing the terms of the two expressions we see that they cancel out perfectly, except for the term for \( i, j = 0 \) from the second sum which gives \( -\varphi(x_0, \ldots, x_1) \) and the term for \( i, j = k \) from the second sum which gives \( \varphi(r(x_0), \ldots, r(x_k)) \), which means that \( \delta K + K \delta = r^* - \text{id} \). The proposition follows. \( \square \)

8.8 Remark. Notice that \( C^\bullet_{\text{AS}}(\Gamma)^\Gamma \) is precisely the standard cobar complex used to define \( H^\bullet(\Gamma; \mathbb{R}) \).
constructions, we get a commutative diagram of cochain complexes

\[
\begin{array}{cccc}
C^\bullet_{AS}(\Gamma)^F & \xrightarrow{=} & C^\bullet_{AS}(\Gamma)^F \\
\sim & & \sim \\
C^\bullet_{AS}(ET)^F & \xrightarrow{\tilde{\mu}^*} & C^\bullet_{AS}(\tilde{M})^F \\
\downarrow f_E & & \downarrow f_M \\
\tilde{C}^\bullet_{AS}(ET)^F & \xrightarrow{\bar{\mu}} & \tilde{C}^\bullet_{AS}(\tilde{M})^F
\end{array}
\]

The first vertical maps are induced by the inclusions $\Gamma \to \tilde{M}$ and $\Gamma \to ET$ given by $g \mapsto zg$ and $g \mapsto \tilde{\mu}(z)g$ respectively. They are both chain homotopy equivalences by Proposition 8.4. Therefore also $\tilde{\mu}^*$ is a chain homotopy equivalence. For the quotients by the locally zero cochains, we note in addition that the pullback $\mu$ respectively. They are both chain homotopy equivalences by Proposition 8.4. Therefore also $\tilde{\mu}^*$ is a chain homotopy equivalence. For the quotients by the locally zero cochains, we note in addition that the pullback defines canonical vertical isomorphisms

\[
\begin{array}{cccc}
\tilde{C}^\bullet_{AS}(B\Gamma) & \xrightarrow{\bar{\mu}} & \tilde{C}^\bullet_{AS}(M) \\
\cong & & \cong \\
\tilde{C}^\bullet_{AS}(ET)^F & \xrightarrow{\bar{\mu}} & \tilde{C}^\bullet_{AS}(\tilde{M})^F
\end{array}
\]

Now, $f_M$ is surjective with kernel $C^\bullet_{AS,0}(\tilde{M})^F$. We get a sequence of cochain maps

\[
C^\bullet_{AS,0}(\tilde{M})^F \xrightarrow{\cong} C^\bullet(f_M) \xleftarrow{\tilde{\mu}^*} C^\bullet(f_M \circ \tilde{\mu}^*) = C^\bullet(\tilde{\mu}^* \circ f_E) = C^\bullet(\bar{\mu}) \cong C^\bullet(\bar{\mu}) =: C^\bullet_{AS}(\tilde{M} \to B\Gamma)
\]

The first map is induced by the inclusion and is the standard chain homotopy equivalence between the mapping cone of a surjection and the kernel. If the extension of chain complexes is $0 \to A^\bullet \xrightarrow{j} B^\bullet \xrightarrow{\mu} C^\bullet \to 0$, it sends $a \in A^n$ to $(j(a), 0) \in B^n \oplus C^{n-1} = C^n(q)$, which is the mapping cone complex. The second map is the standard map induced by $g$ from the mapping cone of a composition $A^\bullet \xrightarrow{j} B^\bullet \xrightarrow{\mu} C^\bullet$ to the one of $f: C^n(g \circ f) = A^n \oplus C^{n-1} \ni (a, c) \mapsto (j(a), c) \in B^n \oplus C^{n-1} = C^n(g)$. It is standard and easy to see that this is a chain homotopy equivalence or homology isomorphism if and only if $f: A^\bullet \to B^\bullet$ is. Because of Proposition 8.4 the map induced by $\tilde{\mu}^*$ here is a chain homotopy equivalence (and implicitly, we have in mind to use a chain homotopy inverse of it). The next map, $f_E$, is obtained the same way. The isomorphism at the end implements the isomorphism (8.9). Finally, we use the definition of the relative cohomology of $M \to B\Gamma$ defined via Alexander-Spanier cochains. Because $M$ is a manifold and $B\Gamma$ a CW-complex, this is canonically isomorphic to the relative singular cohomology.

8.11 Proposition. (Higson’s Lemma) The sequence of cochain maps of (8.10) defines a canonical isomorphism

\[
H^\bullet_{AS,0,\Gamma}(\tilde{M})^F \xrightarrow{\cong} H^\bullet(M \xrightarrow{\mu} B\Gamma).
\]

Proof. It only remains to prove that $f_E: \tilde{C}^\bullet_{AS}(ET)^F \to \tilde{C}^\bullet_{AS}(ET)^F$ is a cohomology isomorphism. Note that the cohomology in both cases is the cohomology of $\Gamma$ in a canonical way, and therefore there already is a canonical isomorphism between the two cohomology groups. We will explain this, and will show that $f_E$ induces precisely this canonical isomorphism. This also makes the identification asserted in the proposition truly canonical.

We start with the free simplicial set generated by the vertices $\Gamma$, considered as a (discrete) set. Being free, this is contractible, and the associated chain complex $\ldots \to Z[\Gamma^3] \to Z[\Gamma^2] \to Z[\Gamma] \to Z$ is acyclic. Note that it consists of free $\mathbb{Z}[\Gamma]$-modules, therefore is a free resolution of the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$. Applying the functor $\text{Hom}_{\mathbb{Z}[\Gamma]}(\cdot, \mathbb{R})$ yields the complex $C^\bullet_{AS}(ET)^F$, whose homology therefore canonically is $H^\bullet(\Gamma; \mathbb{R})$. 
Note that our cochain equivalence with the bar complex $C^\bullet_{\text{AS}}(\Gamma)^\Gamma$ is obtained from a chain equivalence of the underlying free resolutions, and also $\hat{\mu}^*$ is such a cochain equivalence.

To get the canonical identification of $\overline{\text{H}}^\bullet_{\text{AS}}(ET)^\Gamma$ with $H^\bullet(\Gamma)$, we use singular homology as intermediate step. Let $\Delta^\bullet(X)$ the singular cochain complex of a space $X$. Let $\Delta^\bullet_*(X)$ the subcomplex of cochains $\phi$ which are locally 0, i.e. by definition for which an open covering $\{U_i\}$ of $X$ exists such that $\phi$ vanishes on all singular simplices $\sigma : \Delta^\bullet \to X$ whose image is contained in one of the $U_i$. Define $\overline{\Delta}^\bullet(X) := \Delta^\bullet(X)/\Delta^\bullet_*(X)$.

There is a canonical map $C^\bullet_{\text{AS}}(X) \to \Delta^\bullet(X)$ dual to the map between the singular chain complex $\Delta^\bullet(X)$ and the Alexander-Spanier chain complex $\mathbb{Z}[X^\bullet+1]$ which sends a singular simplex to the tuple of its endpoints. It induces $C^\bullet_{\text{AS}}(X) \to \overline{\Delta}^\bullet(X)$.

It is a standard result, compare [26, Section 8.8], that for spaces with a nice local topology, in particular for CW-spaces like our $B\Gamma$ or $M$, these maps are chain homotopy equivalences, which we combine here with the projection induced isomorphisms with equivariant cochains on $ET$

$$
\begin{align*}
\overline{C}^\bullet_{\text{AS}}(B\Gamma)^\Gamma & \xrightarrow{\sim} \overline{\Delta}^\bullet(B\Gamma)^\Gamma & \Delta^\bullet(B\Gamma)^\Gamma & \xleftarrow{\sim} \overline{\Delta}^\bullet(B\Gamma)^\Gamma \\
\overline{C}^\bullet_{\text{AS}}(ET)^\Gamma & \xrightarrow{\sim} \overline{\Delta}^\bullet(ET)^\Gamma & \Delta^\bullet(ET)^\Gamma & \xleftarrow{\sim} \overline{\Delta}^\bullet(ET)^\Gamma 
\end{align*}
$$

Finally, as $ET$ is contractible, the singular chain complex of $ET$ provides another resolution of $\mathbb{Z}$ (by free $\mathbb{Z}\Gamma$-modules), and the canonical map to the Alexander-Spanier chain complex (which is a $\mathbb{Z}\Gamma$-module map) is automatically a chain equivalence. After taking $\text{Hom}_{\mathbb{Z}\Gamma}(\cdot, \mathbb{R})$ this induces, once again, the standard cohomology isomorphism for different ways to compute group cohomology. Combining this information with (8.12) we obtain the diagram

$$
\begin{align*}
C^\bullet_{\text{AS}}(ET)^\Gamma & \xrightarrow{f_E} \Delta^\bullet(ET)^\Gamma \\
\overline{C}^\bullet_{\text{AS}}(ET)^\Gamma & \xrightarrow{\sim} \overline{\Delta}^\bullet(ET)^\Gamma 
\end{align*}
$$

It follows that our map $f_E$ is a chain equivalence, and, because all the other maps induce the canonical isomorphisms, that it induces the canonical isomorphism between different ways to compute group cohomology.

8.13 Proposition. In [8.2], the inclusion of the subcomplexes of smooth and antisymmetric cochains induce cohomology isomorphism. Here, a cochain $\varphi$ is antisymmetric if $\varphi(x_0, \ldots, x_n) = \text{sgn}(\sigma)\varphi(x_{\sigma(0)}, \ldots, x_{\sigma(n)})$, for all the permutations $\sigma \in \mathfrak{S}_{n+1}$.

Proof. By [7, Lemma 1.1 and Lemma 1.4], for calculating $\overline{\text{H}}^\bullet_{\text{AS}}(\widetilde{M})$ one can use the subcomplex of smooth antisymmetric Alexander-Spanier cochains $\varphi$. Analogously $H^\bullet(\Gamma)$ can be calculated by means of the subcomplex of antisymmetric group cochains $c$. This implies by the five lemma the same for $H^\bullet_{\text{AS},0,\Gamma}(\widetilde{M})$. □

8.14 Definition. Define the subcomplex $C^\bullet_{\text{AS},0,\text{pol}}(\widetilde{M})^\Gamma$ of $C^\bullet_{\text{AS},0}(\widetilde{M})^\Gamma$ consisting of smooth skew-symmetric cochains of polynomial growth, namely cochains $\varphi$ such that

$$
|\varphi(x_0, \ldots, x_n)| \leq K \prod(1 + d(x_i, x_{i+1}))^k
$$

for some $K > 0$ and all $k \in \mathbb{N}$. Set

$$
H^\star_{\text{pol}}(M \to B\Gamma) := H^\star(C^\bullet_{\text{AS},0,\text{pol}}(\widetilde{M})^\Gamma).
$$

8.15 Proposition. In our situation, the canonical inclusion induced map

$$
H^\star_{\text{pol}}(M \to B\Gamma) \to H^\star(M \to B\Gamma)
$$

(8.16)
is an isomorphism if and only if $H^*_{pol}(\Gamma) \to H^*(\Gamma)$ is an isomorphism. In particular, $\text{AS}_{pol}^* (\Gamma)$ is an isomorphism if $\Gamma$ is a group with a polynomial combing, e.g. a hyperbolic group or more generally an automatic group, or if $\Gamma$ has polynomial growth.

Proof. First, our proof of Proposition 8.4 works also for the subcomplexes of polynomial growth. This implies that $\text{pol}^* (\Gamma)$ is an isomorphism. The same holds groups of polynomial growth by [27]. This concludes the proof.

8.2 Lifting $H^*(M \to B\Gamma)$ to relative cyclic cocycles

We continue to fix a smooth compact manifold $M$ and a $\Gamma$-covering $\tilde{M} \to M$.

8.17 Lemma. Given $\chi \in C^k_{\text{AS},0} (\tilde{M})^{\Gamma}$ smooth and antisymmetric and $A_0, \ldots, A_k \in \Psi^0_{\Gamma,c}(\tilde{M}, \tilde{E})$, then the (formal) expression

$$
\tau_\chi (A_0, \ldots, A_k) = \int_F \text{Tr} \left( \int_{\tilde{M}^k} A_0(x_0, x_1) A_1(x_1, x_2) \ldots A_k(x_k, x_0) \chi(x_0, \ldots, x_k) dx_1 \ldots dx_k \right) dx_0 \quad (8.18)
$$

where $A_j(x, y)$ is the Schwartz kernel of $A_j$ and $\mathcal{F}$ is a measurable fundamental for $\tilde{M} \to M$

(1) makes sense and gives a well defined value in $\mathbb{C}$ (independent of $\mathcal{F}$),

(2) vanishes if all the $A_j$ are local operators, i.e. the support of the Schwartz kernel is supported on the diagonal (e.g. if the $A_j$ are the operators of multiplication with a smooth function),

(3) is antisymmetric under cyclic permutations of the $A_j$,

(4) satisfies $\tau_\alpha \chi = b r_\chi$ where $\partial$ is the Alexander-Spanier differential and $b$ the cyclic differential.

Proof. To establish (1), by assumption, there exists $\delta > 0$ such that $\chi(x_0, \ldots, x_k) = 0$ if $d(x_i, x_j) < \delta$ for $i, j = 0, 1, \ldots, k$. Choose a smooth $\Gamma$-equivariant function $\alpha : \tilde{M} \times \tilde{M} \to [0, 1]$ which is 1 in a neighborhood of the diagonal and 0 outside the $\delta/2$-neighborhood of the diagonal. Set $\beta := 1 - \alpha$. To simplify the notation we assume $k = 1$, it will be clear how the argument generalizes.

We observe that $B_0(x, y) := A_0(x, y) \cdot \beta(x, y)$ is smooth and therefore a smoothing operator. The composition of $B_0$ with $A_1$ therefore is well defined and a smoothing operator with smooth kernel

$$(B_0 \circ A_1)(x_0, x_2) = \int_{\tilde{M}} A_0(x_0, x_1) A_1(x_1, x_2) \beta(x_0, x_1) dx_1.$$ 

Note that the composition makes sense and the integral is over a compact subset of $\tilde{M}$ due to the $\Gamma$-compactness of the supports of $A_0$ and $A_1$. If we multiply with $f(x)g(x_1)$ for smooth functions $f, g$, the
kernel \((x_0, x_2) \mapsto \int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\beta(x_0, x_1)f(x_0)g(x_1)\, dx_1\) is (by definition) the Schwartz kernel of the composition of \(m_1, B_0, m_2, A_1\) which again is smoothing and indeed depends continuously on the function \((x, y) \mapsto f(x)g(y)\). Note that by the \(\Gamma\)-compact support condition, for fixed \(x_0, x_2\) only the values of \(f(x)g(y)\) on a compact subset of \(\tilde{M} \times \tilde{M}\) enter. As the span of functions of the form \((x, y) \mapsto f(x)g(y)\) is dense in all continuous functions of two variables and similarly for three variables. By continuity we therefore also define \(\int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\beta(x_0, x_1)\chi(x_0, x_1, x_2)\, dx_1\). The operator \(C_0 := A_0 - B_0\) has Schwartz kernel \(A_0(x, y)\alpha(x, y)\) with support in the \(\delta/2\)-neighborhood of the diagonal. Define \(B_1(x, y) := A_1(x, y)\beta(x, y)\), again a smoothing kernel. As before, the function \((x_0, x_2) \mapsto \int_{\tilde{M}} C_0(x_0, x_1)A_1(x_1, x_2)\beta(x_1, x_2)\chi(x_0, x_1, x_2)\, dx_1\) is well defined and smooth, obtained as Schwartz kernel of a composition which involves the smoothing operator \(B_1\). Finally, observe that \(\alpha(x_0, x_1)\alpha(x_1, x_2)\chi(x_0, x_1, x_2)\) is identically 0, as the product of the first two factors has support on the \(\delta/2\)-neighborhood of the diagonal where \(\chi\) vanishes identically. Using
\[
\beta(x_0, x_1) - \beta(x_0, x_1)\beta(x_1, x_2) + \alpha(x_0, x_1)\alpha(x_1, x_2) + \beta(x_1, x_2) = \alpha(x_1, x_2) + \beta(x_1, x_2) = 1,
\]
we therefore define
\[
\int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\chi(x_0, x_1, x_2)\, dx_1 := \\
\int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\chi(x_0, x_1, x_2)\, dx_1 + \int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\beta(x_1, x_2)\chi(x_0, x_1, x_2)\, dx_1 - \\
- \int_{\tilde{M}} A_0(x_0, x_1)A_1(x_1, x_2)\beta(x_1, x_2)\chi(x_0, x_1, x_2)\, dx_1.
\]
The support condition on \(\chi\) implies that this expression is independent of the chosen function \(\alpha\). By \(\Gamma\)-invariance, the final integral appearing in (8.18) is independent of \(\mathcal{F}\) (indeed, the function descends to a smooth function of \(x_0 \in M\)).

Concerning (2) the formula clearly gives 0 if all the Schwartz kernels have support on the diagonal. To prove (3), the reduction as in the proof of (1) shows that the expression which defines \(\tau_\chi\) reduces to the \(L^2\)-trace (the integral over \(\mathcal{F}\) over the diagonal) of a composition of a smoothing \(\Gamma\)-equivariant operator with bounded \(\Gamma\)-invariant operators. We can then use the trace property of the \(L^2\)-trace \([2]\) and get
\[
\tau_\chi(A_k, A_0, \ldots, A_{k-1}) = \int_{\mathcal{F}} \operatorname{Tr} \left( \int_{\tilde{M}} A_k(x_0, x_1)A_0(x_1, x_2)\ldots A_{k-1}(x_k, x_0)\chi(x_0, \ldots, x_k)\, dx_0 \cdots dx_k \right) \, dx_0
\]
where we note that commuting the operators involves commuting the contribution of \(\chi\). Using the skew-adjointness \(\chi(x_k, x_0, \ldots, x_{k-1}) = (-1)^k \chi(x_0, \ldots, x_k)\) gives the desired formula.

The last property (4) is checked by a standard straightforward calculation, compare \([7]\) Lemma 2.1. 

**8.19 Corollary.** Let \(m_\ast : C^\infty(M) \to \Psi^0_{\Gamma}(\tilde{M})\) be the inclusion as \(\Gamma\)-equivariant multiplication operators. Formula (8.18) defines a homomorphism
\[
H^\ast(M) \to B\Gamma \xrightarrow{m_\ast} H^\ast_{AS, 0, \Gamma}(\tilde{M}) \to HC^\ast(C^\infty(M) \xrightarrow{m_\ast} \Psi^0_{\Gamma}(\tilde{M})
\]
given by \([\chi] \mapsto ([0, \tau_\chi])\).

**Proof.** By Lemma 8.17 (2) and (4), \(m_\ast(\tau_\chi) = 0\), i.e. \((0, \tau_\chi)\) indeed is a relative cyclic cocycle, provided \(\chi\) is a de Rham cocycle, and \((0, \tau_{\partial_\chi}) = b(0, \tau_\chi)\). It follows that the assignment gives a well defined map between the homology groups. 

\[\square\]
8.3 Rapid decay

We next need a different kind of smooth subalgebra of pseudodifferential operators. In this section we shall assume that $\Gamma$ has property (RD). We recall the definition below. Consider the Lie groupoid $G := \tilde{M} \times_{\Gamma} \tilde{M} \rightrightarrows M$. Let us recall some basic definition.

8.20 Definition. A length function on $G$ is a proper and continuous function $l: G \to \mathbb{R}_+$ such that $l(\gamma \gamma') \leq l(\gamma) + l(\gamma')$, $l(\gamma) = l(\gamma^{-1})$, and $l(x) = 0$ for all $x \in M$ and $\gamma, \gamma' \in G$ composable.

8.21 Example. Let $d$ be any proper $\Gamma$-invariant distance function on $\tilde{M}$. Let $[\tilde{x}, \tilde{y}]$ be an element of $G$. Then $l([\tilde{x}, \tilde{y}]) := d(\tilde{x}, \tilde{y})$ is a proper length function on $G$.

Let us fix a non-zero length function $l$ on $G$. Define for $m \in \mathbb{N}$ the following norms on $C^\infty_c(\tilde{M} \times_{\Gamma} \tilde{M})$:

$$
\|f\|_{2,m,l}^2 = \max \left\{ \sup_{x \in M} \int_{s^{-1}(x)} |f(\tilde{x}, \tilde{y})|^2 |1 + l(\tilde{x}, \tilde{y})|^{2m}, \sup_{y \in M} \int_{r^{-1}(y)} |f(\tilde{x}, \tilde{y})|^2 |1 + l(\tilde{x}, \tilde{y})|^{2m} \right\}.
$$

(8.22)

8.23 Definition. Set $S^2_l(\tilde{M} \times_{\Gamma} \tilde{M}) := \text{invlim}_{m \in \mathbb{N}} L_{m,l}$, where $L_{m,l}$ is the completion of $C^\infty_c(\tilde{M} \times_{\Gamma} \tilde{M})$ with respect to the norm $\|\cdot\|_{m,l}$. We say that $G$ has property (RD) with respect to $l$ if there exists $m \in \mathbb{N}$ and $C > 0$ such that

$$
\|f\|_{\text{red}} \leq C \|f\|_{l,m} \quad \forall f \in C^\infty_c(\tilde{M} \times_{\Gamma} \tilde{M}),
$$

where $\|\cdot\|_{\text{red}}$ denotes the norm of the reduced groupoid C*-algebra.

Recall that, by definition, the finitely generated group $\Gamma$ itself has property (RD) if it has property (RD) as a groupoid over $\{\ast\}$, i.e. if $G = \Gamma \times_{\Gamma} \Gamma \rightrightarrows \{\ast\}$ has. In other words, there must exist $C > 0$ and $m \in \mathbb{N}$ such that $\|f\|_{\text{red}}^2 \leq C \sum_{g \in \Gamma} |f(g)|^2 (1 + l(g))^{2m}$ for all $f \in \mathbb{C}[\Gamma]$, where $l(\cdot)$ is the word length function associated to a finite set of generators.

8.24 Proposition. If the finitely generated group $\Gamma$ has property (RD) then $\tilde{M} \times_{\Gamma} \tilde{M}$ is also (RD) with respect to the length function $l: [\tilde{x}, \tilde{y}] \mapsto d(\tilde{x}, \tilde{y})$, where the metric $d$ on $\tilde{M}$ is induced from a Riemannian metric on $M$ via pullback.

Proof. Let $\mathcal{F}$ be a bounded measurable fundamental domain for the action of $\Gamma$ on $\tilde{M}$.

Fix $u \in L^2(\tilde{M})$ and $f \in C^\infty_c(\tilde{M} \times_{\Gamma} \tilde{M})$. We also use the letter $f$ for its $\Gamma$-equivariant pullback to $\tilde{M} \times \tilde{M}$.
Then

\[
|f * u|^2_{L^2(\tilde{M})} = \sum_{g \in \Gamma} \int_{\mathcal{F}} \left| \sum_{h \in \Gamma} \int_{\mathcal{F}} f(x, y) u(\tilde{y}) \, d\tilde{y} \right|^2 \, dx \\
\leq \sum_{g \in \Gamma} \int_{\mathcal{F}} \left| \sum_{h \in \Gamma} f(xg, \tilde{y}h) \right| \left| u(\tilde{y}h) \right| \, d\tilde{y} \\
\leq \int_{\mathcal{F}} \sum_{g \in \Gamma} \left( \int_{\mathcal{F}} f(xg, \tilde{y}h) \, d\tilde{y} \right)^2 \left( \int_{\mathcal{F}} \left| u(\tilde{y}h) \right|^2 \, d\tilde{y} \right)^{1/2} \, dx \\
= \int_{\mathcal{F}} \sum_{g \in \Gamma} |vf(x, \cdot) * vu(\cdot)(g)|^2 \\
\leq \int_{\mathcal{F}} \|vf(x, \cdot)\|^2_{\text{red}} \cdot \sum_{h \in \Gamma} |vu(h)|^2 \, dx \\
\leq C \int_{\mathcal{F}} \sum_{g \in \Gamma} |vf(xg, g)|^2 (1 + l(g))^{2m} \cdot \left( \sum_{h \in \Gamma} \int_{\mathcal{F}} \left| u(\tilde{y}h) \right|^2 \, d\tilde{y} \right) \\
\leq CC' \int_{\mathcal{F}} \sum_{g \in \Gamma} \int_{\mathcal{F}} |f(xg, \tilde{y}g)|^2 (1 + d(\tilde{x}, \tilde{y}g))^{2m} d\tilde{y} \cdot |u|^2_{L^2(\tilde{M})} \\
\leq CC' \text{vol}(M) \sup_{x \in M} \int_{s^{-1}(x)} \left| f(\tilde{x}, \tilde{y}) \right|^2 (1 + d(\tilde{x}, \tilde{y}))^{2m} \, d\tilde{y} \cdot \|u\|^2_{L^2(\tilde{M})}
\]

The first inequality uses the elementary fact that the $L^2$-norm is unconditional (i.e. the norm of a convolution is majorized if the functions are replaced by their absolute value functions). The second inequality is the Cauchy-Schwartz inequality for the integral over $\mathcal{F}$.

Then we use that

\[
h \mapsto vu(h) = \int_{\mathcal{F}} \left( \left| u(\tilde{y}h) \right|^2 \, d\tilde{y} \right)^{1/2} \in L^2(\Gamma); \quad g \mapsto vf(x, g) = \left( \int_{\mathcal{F}} \left| f(x, \tilde{y}g^{-1}) \right|^2 \, d\tilde{y} \right)^{1/2} \in C[\Gamma].
\]

The forth inequality is property (RD) for $\Gamma$. Finally, we use the Milnor-Svarc result that the map $\Gamma \to \tilde{M}$ given by $g \mapsto \tilde{x}g$ is a quasi-isometry, with constants uniform for $x$ in the compact subset $\mathcal{F}$, to obtain the fifth inequality. As $C, C'$ do not depend on $f$ and $u$, we have proved that $\tilde{M} \times_\Gamma \tilde{M}$ has property (RD).

**8.25 Proposition.** The $^\ast$-algebra $S^2_t(\tilde{M} \times_\Gamma \tilde{M})$ is dense and holomorphically closed in $C^*_r(\tilde{M} \times_\Gamma \tilde{M})$.

**Proof.** The proof of the corresponding result [15] Theorem 4.2] for $r$-discrete groupoids also works in this case, replacing sums over the $r$-fibers by integrals. This follows an idea which already appeared in [7], and more concretely in [36] Section 4.

**8.26 Definition.** Set

\[
S^2_{t,\infty} := \{ f \in S^2_t : f \text{ is smooth and } \|f\|_{m, l, \alpha, \beta} < \infty \quad \forall m, \alpha, \beta \},
\]
B and hence it is sufficient to prove the following Lemma.

**Proof.** It is immediate that $S^2_\tau$ is a dense semi-ideal in $S^2_\tau$. By the polynomial growth of $\chi$, $\tau$ indeed it is sufficient to prove that it extends to $S^2_\tau$. Then we use [17, Prop 3.3] to conclude.

**8.27 Proposition.** The $*$-algebra $S^2_\tau$ is dense and holomorphically closed in $C^*_r(\tilde{M} \times \Gamma \tilde{M})$.

**Proof.** The proof of this proposition follows verbatim the proof of [17, Corollary 7.9]. It is enough to replace $\mathcal{G}, \Psi^0(\mathcal{G}), \mathcal{S}(\mathcal{G}, \phi)$ and $\Psi^0_\Gamma(\mathcal{G})$ there with $\tilde{X} \times_\Gamma \tilde{X}, \Psi^0_\Gamma(\mathcal{M}), S^2_{\tau,\infty}(\mathcal{M} \times \Gamma \tilde{M})$ and $\Psi^0_\Gamma(\tilde{M})$, respectively.

**8.4 Extending higher cyclic cocycles associated to classes in $H^*_p(M \to BT\Gamma)$**

Let $\chi: \tilde{M}^{k+1} \to \mathbb{R}$ by a smooth anti-symmetric $k$-dimensional delocalized Alexander-Spanier cocycle of polynomial growth. Take the corresponding cyclic cocycle $\tau_\chi$ as in Lemma 8.17 and Corollary 8.19 defined on the algebra $\Psi^0_{\Gamma, c}(\mathcal{M})$. In this section we are going to prove that $\chi$ extends continuously to the algebra $\Psi^0_{\Gamma, rd}(\mathcal{M})$. Recall that this algebra is defined as the sum of $\Psi^0_{\Gamma, c}(\mathcal{M})$ and $S^2_{\tau,\infty}(\mathcal{M} \times \Gamma \tilde{M})$ of Definition 8.26. Hence it is sufficient to prove the following Lemma.

**8.29 Lemma.** The cyclic cocycle $\tau_\chi$ extends continuously to the algebra $S^2_{\tau,\infty}(\tilde{M} \times \Gamma \tilde{M})$.

**Proof.** Indeed it is sufficient to prove that it extends to $S^2_\tau(\tilde{M} \times \Gamma \tilde{M})$. To this end we have to show that $\tau_\chi$ is continuous with respect to some seminorm of the type 8.22. We shall apply the method used in [7, Proposition 6.5]. By the polynomial growth of $\chi$ we know that there exists a constant $C > 0$ and an integer $n \geq 0$ such that

$$|\chi(x_0, x_1, \ldots, x_k)| \leq C(1 + d(x_0, x_1))^{2n} \ldots (1 + d(x_{k-1}, x_k))^{2n}$$

for all $x_0, x_1, \ldots, x_k \in \tilde{M}$.

Let $A_0, \ldots, A_k$ be elements in $C^\infty_c(\tilde{M} \times \Gamma \tilde{M})$ and put $B_i(x, y) := |A_i(x, y)|(1 + d(x, y))^{2n}$ for $i = 0, \ldots, k-1$ and $B_k(x, y) := |A_k(x, y)|$. Then

$$|\tau_\chi(A_0, \ldots, A_k)| = \left| \int_M Tr \left( \int_{\tilde{M}^k} A_0(x_0, x_1) \cdots A_k(x_k, x_k) \chi(x_0, x_1, \ldots, x_k) dx_0 \cdots dx_k \right) dx_0 \right| \leq C \int_M \left( \int_{\tilde{M}^k} B_0(x_0, x_1)B_1(x_1, x_2) \cdots B_k(x_k, x_k) dx_0 \right) dx_0 = C \int_M (B_0 * B_1 * \cdots * B_k)(x_0, x_0) dx_0 \leq C \sup_{x \in \tilde{M}} |B_0 * B_1 * \cdots * B_k(x, x)| \leq C'' \|B_0 * B_1 * \cdots * B_k\|_{red} \leq C'' \|B_0\|_{red} \|B_1\|_{red} \cdots \|B_k\|_{red}$$

where in the last but one step we use the fact that the map $C^*_r(G) \to C_0(G(0))$, given by the restriction to the units, is a continuous conditional expectation of $C^*$-algebras.
Then, since the groupoid has the property (RD), we have that
\[
|\tau_k(A_0, \ldots, A_k)| \leq C'' m \cdot \|B_0\|_{l,m} \cdot \|B_1\|_{l,m} \cdots \|A_k\|_{rd}
\leq C'' m \cdot \|A_0\|_{l,m+n} \cdot \|A_1\|_{l,m+n} \cdots \|A_k\|_{l,m+n}
\]
which gives the desired result.

Propositions 8.11 and 8.15, Lemma 8.29 and Corollary 8.19 now give the following result.

**8.33 Theorem.** Let \( \Gamma \) be a discrete group with property (RD). Let \( \tilde{M} \to M \) be a \( \Gamma \)-covering of a compact manifold \( M \). Then there is a well-defined homomorphism
\[
\Xi : H^*_\text{pol}(M \to \Gamma) \to H^*\text{pol}(\tilde{M}) \to \Psi^0_\Gamma,\text{rd}(\tilde{X})
\]
Using Proposition 8.28 we then get a well defined pairing
\[
H^*_\text{pol}(M \to \Gamma) \times K_{s+1}(C(M) \to \Psi^0_\Gamma(\tilde{M})) \to \mathbb{C},
\]
explicitly given by associating to \((\alpha, x)\) the number \(\langle \Xi(\alpha), x \rangle\), where the pairing between cyclic cohomology and K-theory has been used.

If, in addition, \( H^*_\text{pol}(\Gamma) \to H^*\text{pol}(\Gamma) \) is an isomorphism, we can replace \( H^*_\text{pol}(M \to \Gamma) \) by the usual relative cohomology \( H^*(M \to \Gamma) \). Examples of groups which satisfy both conditions are hyperbolic groups or groups of polynomial growth \([27][28]\).

**8.34 Definition.** Let \( \tilde{D} \) be a generalized Dirac operator which is \( \Gamma \)-equivariant on the Galois \( \Gamma \)-covering \( \tilde{M} \) of a compact smooth manifold \( M \). Suppose that \( \tilde{D} \) is \( L^2 \)-invertible. Then define its higher \( \varrho \)-number associated to \([\alpha] \in H^{s-1}(M \to \Gamma)\) as
\[
\varrho_\alpha(\tilde{D}) := \langle \Xi(\alpha), \varrho(\tilde{D}) \rangle \in \mathbb{C},
\]
where \( \varrho(\tilde{D}) \) is defined as in Definition 3.4.

**8.35 Example.** Let \( g \) be a metric with positive scalar curvature on a closed spin manifold \( M \) of dimension \( n \) with fundamental group \( \Gamma \); we define the higher \( \varrho \)-number associated to \( g \) and to \( \alpha \in H^{s-1}(M \to \Gamma) \) as
\[
\varrho_\alpha(g) := \langle \Xi(\alpha), \varrho(g) \rangle \in \mathbb{C},
\]
where \( \varrho(g) \in K_n(\tilde{M}) \) is the \( \varrho \)-class of the spin Dirac operator for the metric \( g \).

**8.36 Remark.** In Sections 6 and 7, we use the “difference” between \( M \) and \( \Gamma \) to pair with the analytic structure group \( S^*_\Gamma(M) \). In Sections 8 and 9, we use the delocalized group cohomology, coming from the difference between \( \Gamma \) and \( C^* \Gamma \). This reflects a two-step approach to the fact that \( S^*_\Gamma(M) \) is the relative term in the Higson-Roe exact sequence, the other two terms determined by \( M \) and \( C^* \Gamma \).

We leave it as a challenge to combine these two approaches into one “universal” target: one should construct a relative (cyclic homological) group between the homology of \( M \) and the cyclic homology of a smooth subalgebra \( \mathcal{A} \) of \( C^* \Gamma \), and construct a Chern character from the Higson-Roe exact sequence to a suitably defined “universal” exact sequence in cyclic homology, involving precisely the homology of \( M \), of \( \mathcal{A} \) and the relative term, with pairings with the dual cohomology groups.

**8.37 Remark.** Our construction of the delocalized Chern character in Section 6 depends on Zenobi’s description (recalled in Section 2) of the analytic structure group with algebras of pseudodifferential operators. Using the original definition of the structure group with Roe algebras, one knows that \( S^*_\Gamma(M) \) is functorial in \( M \). Indeed, one uses this to define a universal analytic structure group \( S_\text{u}(\Gamma) \) which replaces \( S^*_\Gamma(\Gamma) \) and is defined as the colimit of \( S^*_\Gamma(M_i) \) for a sequence of manifolds \( M_1 \to M_2 \to \ldots \) whose homotopy colimit is \( \Gamma \). Note that, in general, \( \dim(M_i) \to \infty \). It is now a challenge, which we leave for future research to
(1) give an workable description of functoriality of $S^*_{\Gamma}$ in terms of the description of Section 2

(2) with the goal to prove that our delocalized Chern character (with fixed target) is natural for this functoriality of $S^*_{\Gamma}$.

In particular, this would allow us to define the delocalized Chern character on the universal delocalized structure group.

References

[1] Paolo Antonini, Sara Azzali, and Georges Skandalis, Flat bundles, von Neumann algebras and K-theory with R/Z-coefficients, J. K-Theory 13 (2014), no. 2, 275–303, DOI 10.1017/is014001024jkt253. MR3189427

[2] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Colloque “Analyse et Topologie” en l’Honneur de Henri Cartan (Orsay, 1974), Soc. Math. France, Paris, 1976, pp. 43–72. Astérisque, No. 32-33. MR0420729

[3] Robin J. Deeley and Magnus Goffeng, Realizing the analytic surgery group of Higson and Roe geometrically part III: higher rho invariants, Math. Ann. 366 (2016), no. 3-4, 1513–1559, DOI 10.1007/s00208-016-1365-6. MR3563244

[4] Alexander Gorokhovsky and John Lott, Local index theory over foliation groupoids, J. Inst. Math. Jussieu 5 (2006), no. 2, 413–447, DOI 10.1017/is008001021jkt051. MR2496452

[5] Alain Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994. MR1303779

[6] Max Karoubi, Homologie cyclique et K-théorie, Astérisque 149 (1987), 147 (French, with English summary). MR913964

[7] Alexander Gorokhovsky and John Lott, Local index theory over étale groupoids, J. Reine Angew. Math. 560 (2003), 151–198. DOI 10.1515/crll.2003.054. MR1992804

[8] Eric Leichtnam and Paolo Piazza, The b-pseudodifferential calculus on Galois coverings and a higher Atiyah-Patodi-Singer index theorem, Mém. Soc. Math. Fr. (N.S.) 68 (1997), iv+121 (English, with English and French summaries). MR1488084

[9] Robert Lauter, Bertrand Monthubert, and Victor Nistor, Spectral invariance for certain algebras of pseudodifferential operators, J. Inst. Math. Jussieu 4 (2005), no. 3, 405–442, DOI 10.1017/S1474748005000125. MR2197064

[10] Eric Leichtnam and Paolo Piazza, The b-pseudodifferential calculus on Galois coverings and a higher Atiyah-Patodi-Singer index theorem, Mém. Soc. Math. Fr. (N.S.) 68 (1997), iv+121 (English, with English and French summaries). MR1488084

[11] Eric Leichtnam and Paolo Piazza, Homotopy invariance of twisted higher signatures on manifolds with boundary, Bull. Soc. Math. France 127 (1999), no. 2, 307–331 (English, with English and French summaries). MR1708639

[12] Eric Leichtnam and Paolo Piazza, Étale groupoids, eta invariants and index theory, J. Reine Angew. Math. 587 (2005), 169–233, DOI 10.1515/crll.2005.2005.587.169. MR2186978

[13] Matthias Lesch, Henri Moscovici, and Markus J. Pflaum, Relative pairing in cyclic cohomology and divisor flows, J. K-Theory 3 (2009), no. 2, 359–407, DOI 10.1017/s1474748009990021. MR2496452

[14] Jean-Louis Loday, Cyclic homology, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1992. Appendix E by Marla O. Ronco. MR1217970
[23] J. Lott, Superconnections and higher index theory, Geom. Funct. Anal. 2 (1992), no. 4, 421–454, DOI 10.1007/BF01896662. MR1191568 \(\uparrow 15, 16, 18, 19, 29\)

[24] John Lott, Higher eta-invariants, K-Theory 6 (1992), no. 3, 191–233, DOI 10.1007/BF00961464. MR1189276 \(\uparrow 14, 29, 32, 37\)

[25] William S. Massey, Homology and cohomology theory, Marcel Dekker, Inc., New York-Basel, 1978. An approach based on Alexander-Spanier cochains; Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46. MR0488016 \(\uparrow 40\)

[26] Paolo Piazza and Thomas Schick, Rho-classes, index theory and Stolz' positive scalar curvature sequence, J. Topol. 7 (2014), no. 4, 965–1004, DOI 10.1112/jtopol/jtt048. MR3286895 \(\uparrow 34, 41, 46\)

[27] Michael Puschnigg, New holomorphically closed subalgebras of \(C^*\)-algebras of hyperbolic groups, Geom. Funct. Anal. 20 (2010), no. 1, 243–259, DOI 10.1007/s00039-010-0062-y. MR2647141 \(\uparrow 29\)

[28] Zhizhang Xie and Guoliang Yu, Positive scalar curvature, higher rho invariants and localization algebras, Adv. Math. 262 (2014), 823–866, DOI 10.1016/j.aim.2014.06.001. MR3228443 \(\uparrow 3\)

[29] Vito Felice Zenobi, Mapping the surgery exact sequence for topological manifolds to analysis, J. Topol. Anal. 9 (2017), no. 2, 329–361, DOI 10.1142/S17935431750011X. MR3622237 \(\uparrow 3\)