On the isoperimetric inequality and surface diffusion flow for multiply winding curves

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Joint work with Shinya Okabe (Tohoku University)
1. Introduction (2 pages)
2. Isoperimetric Inequality (6 pages)
3. Surface Diffusion Flow (6 pages)
Section 1: Introduction

1. Introduction (2 pages)
   ▶ Main characters
   ▶ Outline
Multiply winding curves:
- Immersed closed curves $\gamma$ in $\mathbb{R}^2$ of rotation number $\geq 2$.

Isoperimetric Inequality (Iso Ineq):
- Classical geometric inequality. For a closed plane curve $\gamma$,

$$\text{Length}(\gamma)^2 \geq 4\pi \text{Area}(\gamma).$$

Surface Diffusion Flow (SDF):
- 4th order geometric evolution equation. For closed plane curves,

$$V = -\partial_{ss} \kappa,$$

where $V$ normal velocity, $s$ arclength, $\kappa$ curvature.
Stationary solutions:

- **SDF**’s stationary solution satisfies $0 = \partial_{ss} \kappa$.
- Since the curve is closed, curvature is constant.
- Must be a circle $C_N$, of an arbitrary rotation number $N \geq 1$.

**Stability** ($N = 1$):

- A singly winding circle $C_1$ is “dynamically stable”.
- **Iso Ineq** $L^2 \geq 4\pi A$ comes into play in a variational proof.

**Stability** ($N \geq 2$):

- Multiply-winding circles $C_N$ are “not stable”.
- Lacking is **Iso Ineq** of the form $L^2 \geq 4\pi N A$ (equality for $C_N$).

**Main results**:

- **Iso Ineq**: $L^2 \geq 4\pi N A$ under rotational symmetry.
- **SDF**: Stability of $C_N$ ($N \geq 2$) for rotationally symmetric perturbations.
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1. Isoperimetric Inequality (6 pages)
   ▶ Basic definitions
   ▶ Isoperimetric ratio
   ▶ Rotational symmetry
   ▶ Main theorem I: Isoperimetric Inequality
   ▶ Idea of the proof
Let $\gamma : S^1 \to \mathbb{R}^2$, where $S^1 := \mathbb{R}/\mathbb{Z}$, be smooth and regular $|\partial_x \gamma(x)| > 0$.

- **Length:**
  
  $$L(\gamma) := \int_{\gamma} ds,$$

  where $s$ denotes the arclength parameter.

- **Signed area:** counterclockwise = positive.

  $$A(\gamma) := -\frac{1}{2} \int_{\gamma} \gamma \cdot \nu ds,$$

  where $\nu := R_{\frac{\pi}{2}} \partial_s \gamma$ and $R_{\theta}$: $\theta$-rotation matrix.

- **Rotation number:**

  $$N(\gamma) := \frac{1}{2\pi} \int_{\gamma} \kappa ds \in \mathbb{Z},$$

  where $\kappa = \partial_s^2 \gamma \cdot \nu$. 
Isoperimetric ratio:

\[ I(\gamma) := \begin{cases} 
\frac{L(\gamma)^2}{4\pi A(\gamma)} & (A(\gamma) > 0), \\
\infty & (A(\gamma) \leq 0).
\end{cases} \]

Remark:

- In general, \( I \geq 1 \) holds.
- For \( N \)-times covered circle \( C_N \), we have \( I(C_N) \geq N \).
- If \( \gamma_R \) consists of two circles of radii 1 and \( R \geq 1 \), then

\[ I(\gamma_R) = \frac{(2\pi + 2\pi R)^2}{4\pi(\pi + \pi R^2)} = 1 + \frac{2R}{1 + R^2}. \]

The value \( I(\gamma_R) \) decreases from 2 to 1 as \( R : 1 \to \infty \).

Goal: Find a class \( X \) for which \( \inf_X I = N \) holds. (\( N \): rotation number.)
Rotational symmetry

**Class** $A(n, m)$:

- $A(n, m) := \{ \gamma \in \text{Sym}(m) \mid N(\gamma) = n \}$.

$m$-th rotational symmetry:

- $\gamma \in \text{Sym}(m) \iff \exists i \in \{1, \ldots, m\}$ such that $\gamma \in \text{Sym}(m, i)$.
- $\gamma \in \text{Sym}(m, i) \iff \gamma(x + \frac{1}{m}) = R_{\frac{2\pi i}{m}} \gamma(x)$ holds for every $x \in S^1$.

Remark: The index $i$ is characterized by $n, m$.

- $\gamma \in A(n, m) \implies \gamma \in \text{Sym}(m, i_{n,m})$, where $i_{n,m} := n + m - m \left\lfloor \frac{n}{m} \right\rfloor$.
- The index $i_{n,m}$ is a unique element of $\{1, \ldots, m\} \cap (n + m\mathbb{Z})$.
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\[ A(1, 3) \quad A(2, 3) \quad A(4, 3) \quad A(5, 3) \]
Theorem 1 (M.-Okabe)

Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{>0}$. Recall $A(n, m) := \{ \gamma \in \text{Sym}(m) \mid N(\gamma) = n \}$. Then

$$\inf_{\gamma \in A(n, m)} I(\gamma) = i_{n,m}. $$

The infimum is attained iff $1 \leq n \leq m \ (\Leftrightarrow \ i_{n,m} = n)$ and $\gamma$ is an $n$-circle.

Corollary 2 (Isoperimetric Inequality)

If $1 \leq n \leq m$, then $I(\gamma) \geq n$ for $\gamma \in A(n, m)$. Equality only by an $n$-circle.

Remark:

- Corollary 2 has been known if $\gamma$ is in addition locally convex.
  [Epstein-Gage’87] ($1 \leq n \leq m/2$), [Chou’03], [Süssmann’11], [Wang-Li-Chao’17].
Main theorem I: Isoperimetric Inequality

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Idea of the proof

Direct method for closed curves?:

- Take a min seq \( \{\gamma_j\} \subset A(n, m) \) such that \( I(\gamma_j) \to \inf_{A(n, m)} I \).
- By compactness, up to subseq, \( \gamma_j' \to \exists \tilde{\gamma} \) in certain first order sense.
- If \( \tilde{\gamma} \in A(n, m) \), then \( \tilde{\gamma} \) attains \( \inf \) by lower semicontinuity of \( I \).
- BUT \( N[\tilde{\gamma}] = n \) may not hold in general! \( N \) is of second order.

Change the strategy:

- Just look at one period of \( \gamma \in A(n, m) \).
- Direct method for open curves.
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Free boundary problem for open curves:

- $X_\theta := \{ \gamma \in \text{Lip}([0, 1]; \mathbb{R}^2) \mid (\text{Boundary Condition}) \}$.
- (BC) $\frac{\gamma(0)}{|\gamma(0)|} = (1, 0), \frac{\gamma(1)}{|\gamma(1)|} = (\sin \theta, \cos \theta)$, and $|\gamma(0)| = |\gamma(1)| > 0$.
- All zeroth order. Direct method applicable.

**Theorem 3**

For $\theta \in (0, 2\pi]$, $\min_{X_\theta} I(\gamma) = \theta / 2\pi$. Equality only by a circular arc of angle $\theta$.

Original inequality:

- One period $\gamma|_m$ of $\gamma \in A(n, m)$ lives in $X_\theta$ for $\theta := \frac{2\pi i_{n,m}}{m}$.
- Since $I(\gamma) = mI(\gamma|_m)$, we get $I(\gamma) \geq m \cdot \theta / 2\pi = i_{n,m}$. 
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Section 3: Surface Diffusion Flow

3. Surface Diffusion Flow (6 pages)
   ► Quick review
   ► Global existence: Singly winding case \( (N = 1) \)
   ► Global existence: Multiply winding case \( (N \geq 2) \)
   ► Main theorem II: Global existence for SDF
   ► Sketch of the proof
Quick review

Surface Diffusion Flow: Given a smooth initial data $\gamma_0 : S^1 \to \mathbb{R}^2$, consider

$$\begin{cases}
\partial_t \gamma = (-\partial_s^2 \kappa) \nu & \text{on } S^1 \times [0, T), \\
\gamma(\cdot, 0) = \gamma_0,
\end{cases}$$

where $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ is a family of immersed curves.

- $T \in (0, \infty]$: maximal existence time. ($T > 0$ by parabolicity.)

Problem: Which initial curve $\gamma_0$ admits a global solution ($T = \infty$)?

Basic facts$^1$:

- Along the flow, $\frac{d}{dt} L \leq 0$ and $\frac{d}{dt} A = 0$. Hence, $I$ is non-increasing.
- If $T = \infty$, then $\gamma$ converges to an $N(\gamma_0)$-circle as $t \to \infty$.
- $\exists$ initial curve $\gamma_0$ with finite time blowup ($T < \infty$).
- If $T < \infty$, then $L^2$-blowup of curvature $\int \kappa^2 ds \gtrsim (T - t)^{-1/4}$.
- Convexity and embeddedness are not necessarily preserved.

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$^1$ Cf. Giga-Ito’98,’99, Dziuk-Kuwert-Schätzle’02, Chou’03, Wheeler (arXiv:2004.08494).
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Global existence: Singly winding case ($N = 1$)

Finite time blowup:
- **Example**: $N(\gamma_0) = 1$ and $A(\gamma_0) \leq 0$.
  - If $T = \infty$, then the solution would converge to a counterclockwise circle. However, this is impossible by the area-preserving property.

Global existence:
- If $\gamma_0$ is close to a circle, then $T = \infty$.
  - [Elliott-Garcke’97], [Escher-Mayer-Simonett’98], [Wheeler’13]

Major open problems: (not addressed in this talk)
- Finite time blowup for $\gamma_0$ embedded? (or $A(\gamma_0) > 0$?)
- Giga’s conjecture: If $\gamma(\cdot, t)$ embedded for all $t \in [0, T)$, then $T = \infty$?
- Chou’s conjecture: Concerning classification of “Type I” singularity.
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- Chou’s conjecture: Concerning classification of “Type I” singularity.
Global existence: Multiply winding case ($N \geq 2$)

Finite time blowup:

- Example: $I(\gamma_0) < N(\gamma_0)$ [Chou ’03].
- The above occurs even if $\gamma_0$ is close to an $N$-circle.

![Graphs showing finite time blowup](image-url)
Global existence:

- Symmetric global solutions are known numerically.

![Graphs showing the evolution of a geometric figure over time. The figure transitions from a rose-like shape to a closed circle.](image)

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[Escher-Mayer-Simonett '98]
Theorem 4 (M.-Okabe)  

Let $1 \leq n \leq m$. If $\gamma_0 \in A(n, m)$, and if $\gamma_0$ is \textit{“$H^2$-close” to an $n$-circle}, then $\gamma_0$ admits a global solution to SDF, i.e., $T = \infty$.

Remark:

- The proof crucially relies on our isoperimetric inequality $I(\gamma(t)) \geq n$.
- Key point: No a priori convexity along SDF, but our isoperimetric inequality does not assume convexity!
Sketch of the proof:

- **Goal:** Prove $L^2$-boundedness of curvature $\implies T = \infty$.
- **Wheeler’s estimate:** Let $K_n^* := \frac{2\pi}{3} (\sqrt{1 + 3\pi n^2} - \sqrt{3\pi n^2})$. Then, as long as

$$K_{osc}(\gamma(t)) := \frac{1}{L(\gamma(t))} \int_{\gamma(t)} (\kappa - \bar{\kappa})^2 ds \leq 2K_n^*,$$

the curvature oscillation is more precisely controlled:

$$K_{osc}(\gamma(t)) \leq K_{osc}(\gamma_0) + 4\pi^2 n^2 \log \frac{L(\gamma_0)^2}{L(\gamma(t))^2}.$$ 

- By our isoperimetric inequality "$L(\gamma(t))^2 \geq 4\pi n A(\gamma(t))$",

$$\frac{L(\gamma_0)^2}{L(\gamma(t))^2} \leq \frac{L(\gamma_0)^2}{4\pi n A(\gamma(t))} = \frac{L(\gamma_0)^2}{4\pi n A(\gamma_0)} = \frac{I(\gamma_0)}{n}.$$ 

- If $K_{osc}(\gamma_0) \leq K_n^*$ and $\frac{I(\gamma_0)}{n} \leq \exp\left(\frac{K_n^*}{8\pi^2 n^2}\right)$, then

$$\sup_{t \in [0, T)} K_{osc}(\gamma(t)) \leq \frac{3}{2} K_n^*.$$
Summary:

- **Isoperimetric Inequality** for rotationally symmetric curves:
  \[ 1 \leq n \leq m, \quad \gamma \in A(n, m) \quad \implies L^2 \geq 4\pi n A. \]

- **Surface Diffusion Flow** admits rotationally symmetric global solutions:
  \[ 1 \leq n \leq m, \quad \gamma_0 \in A(n, m), \quad \gamma_0 \text{ nearly circular} \quad \implies T = \infty. \]

Future directions:

- **Iso Ineq**: How about curved ambient spaces?
- **SDF**: More precise understanding of singularities.

– Thank you very much!