On the crack-tip asymptotic fields

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Abstract. Analytical expression between the vectorial stress intensity factor (SIF) and displacement discontinuity intensity factor (DDIF) is derived for an arbitrary shaped plane crack with a smooth crack boundary. The crack is placed in an elastic anisotropic medium with monoclinic symmetry (syngonium). The analysis is based on constructing the outer and inner asymptotes for the corresponding surface stress and displacement discontinuity fields at the crack boundary. The closed form operator of the class -1 in Hörmander 1/2-space of the theory of cracks along with the associated amplitudes and symbols are constructed. Both strain and displacement fields assumed to be infinitesimal.

1. Introduction

A closed form relation between two intensity factors, the vectorial stress intensity factor (SIF) and the displacement discontinuity intensity factor (DDIF), is derived by solving a specially constructed pseudodifferential equation connecting asymptotes of the outer stress field (out of plane crack) evaluated at a particular point at the crack boundary, with the inner asymptote of the displacement discontinuity field evaluated in the same point belonging to the crack boundary. It is assumed that the displacement discontinuity field satisfies a tensorial integro-differential (pseudodifferential) equation for the linearly elastic anisotropic medium possessing monoclinic symmetry (syngonium). It is also assumed that the plane supporting crack region belongs to the plane of elastic symmetry. The displacement field assumed to be infinitesimal, thus equations of the linear theory of anisotropic elasticity can be applied.

The presented analysis is developed for an arbitrary shaped plane crack with smooth crack boundary, residing in a medium with arbitrary elastic monoclinic syngonium, meaning anisotropy with a single plane of elastic symmetry. In the considered case the plane of elastic symmetry assumed to coincide with the crack plane.

The crack occupying plane region $\Omega$ with smooth Lyapunov (or Hölder) type boundary $\partial\Omega \in C^{1,\alpha}$, $\alpha > 1$, is shown in Fig. 1, where $\mathbf{n}$ defines the unit outward normal to $\Omega$, and dashed arrows indicate directions for evaluating SIF and DDIF. The plane with the unit normal $\mathbf{v}$ supporting $\Omega$ is defined by $\Pi_\mathbf{v}$. Additionally, the crack region is assumed to be simply connected ensuring absence of points of intersection. Note, that Lyapunov condition imposed on the crack boundary excludes fractal cracks with boundary containing infinite number of irregularities, and herein a more rigorous condition $\partial\Omega \in C^{1,\alpha}$, $\alpha > 1$ is implied.
Definition 1.1. The following formulae define respectively SIF (S) and DDIF (D),

\[ T_{x_0} = (2\pi)^{1/2} \lim_{x' \to x_0} \left( |x' \cdot n_{x_0} - x_0|^{1/2} t_v(x') \right) \]

\[ D_{x_0} = (2\pi)^{-1/2} \lim_{x' \to x_0} \left( |x' \cdot n_{x_0} - x_0|^{-1/2} b(x') \right), \quad (1.1) \]

where \( x_0 \in \partial \Omega \) is a particular point belonging to the crack boundary; \( x' \) defines spatial variables belonging to the \( \Pi_v \)-plane; \( t_v \) is the surface stress field acting on the \( \Pi_v \)-plane:

\[ t_v = \mathbf{v} \cdot \mathbf{t}, \quad (1.2) \]

herein, \( \mathbf{t} \) is the stress tensor. Vector \( \mathbf{b} \) in (1.1) defines the displacement discontinuity field, the latter is defined by the limits taken along normal directions to the \( \Pi_v \)-plane:

\[ b(x') = \lim_{x \to x'} u(x) - \lim_{x \to x'} u(x), \quad (1.3) \]

In (1.3) \( u \) is the displacement field; and \( x' \in \Pi_v \) defines the orthogonal projection of a point \( x \) onto \( \Pi_v \)-plane.

Note, that definition (1.1)_1 is a slightly more restrictive than used in the theory of cracks; see [1, 2]; that is because of expression (1.1), defining SIF only along normal to \( \partial \Omega \) directions. Thus, definition (1.1)_1 yields no dependence of SIF upon direction of approaching, while a more general definition allows for different approaching directions.

The discussed relation between \( T_{x_0} \) and \( D_{x_0} \) reads as follows

\[ T_{x_0} = M(n_{x_0}) \cdot D_{x_0}, \quad (1.4) \]

where \( M(n_{x_0}) \) is a specially constructed \( 3 \times 3 \) matrix depending solely on elastic properties of the medium and direction of the normal \( n_{x_0} \) to the crack boundary \( \partial \Omega \). The structure of the matrix \( M(n_{x_0}) \) is discussed later on.

In deriving relation (1.4) the asymptotic analysis of a specially constructed operator of the class -1 in Hörmander 1/2-space of the theory of cracks [3, 4] and the notion of the wave front (WF). For the general definition of the wave front (WF) [5, 6], are used.

2. Single and double layer potentials

Displacement field produced by the crack discontinuity can be represented by the following double-layer potential [7, 8]

\[ u(x) = \int_{\Omega} b(y') \cdot T(y', v_y) E(x - y') d\gamma, \quad (2.1) \]
where \( y' \in \Omega; \) \( b \) is the crack discontinuity field; \( E \) is the Green tensor in \( R^3 \), while in case of arbitrary elastic monoclinic syngonium analytical expressions for \( E \) are not known [9], the final expression for matrix \( M \) in (1.4) will not contain \( E \). In Eq. (2.1) \( T \) is the surface stress operator:

\[
T(\nabla_y, \nu_y) \equiv \nu_y \cdot C \cdot \nabla y,
\]

(2.2)

where \( C \) is the fourth-order elasticity tensor, assumed to be strongly elliptic [7].

Now, surface stress field on the \( \Pi_y \)-plane can be defined by applying operator (2.2) to the potential (2.1) and transition to the non-tangential limits to \( \Pi_y \) (actually, it is sufficient to consider limits along normal directions to the \( \Pi_y \)-plane):

\[
\lim_{x \to x', \Omega} T(\nabla_x, -\nu_x') \int b(y') \cdot T(\nabla_y, \nu_y) E(x - y') dy', \quad x' \in \Pi_y
\]

(2.3)

The last formula is correctly defined due to the Lyapunov – Tauber theorem for elastic potentials [9].

3. Operator of the class -1 in Hörmander 1/2-space of the theory of cracks

Applying integral Fourier transform to expression (2.3), yields the amplitude [4] of the corresponding integro-differential operator in the form

\[
Z^{-}(\xi) = (2\pi)^2 \nu_y \cdot C \cdot \xi \otimes E^{-}(\xi) \otimes \xi \cdot \nu_x
\]

(3.1)

where symbol \( \sim \) stands for the integral Fourier transform; and \( \nu_x, \nu_y \) correspond to the same unit normal \( \nu \).

Restricting amplitude (3.1) on \( \Pi_y \)-plane yields the (principal) symbol of the desired operator of the crack theory depending on \( \xi' \in \Pi_y \) variable only:

\[
Z_0^{-}(\xi') = (2\pi)^2 \text{F.P.} \int_{-\infty}^{\infty} Z^{-}(\xi) d\xi''
\]

(3.2)

where \( \xi \in R^3, \xi' \equiv \text{Pr}_{\Pi_y} \xi = \xi - (\xi \cdot \nu) \nu \). In Eq. (3.2) F.P. stands for the Finite Part of the divergent improper integral, this can be evaluated by a regularization technique; see [4]. The following properties of the symbol \( Z_0^{-} \) immediately follow from its definition:

**Proposition 3.1.**

a) Symbol \( Z_0^{-} \) is symmetric, as tensorial symbol (with respect to permutation of its indices) and strongly elliptic:

\[
a \cdot Z_0^{-} \cdot a > 0
\]

(3.3)

For any vector \( a \neq 0 \), note that herein vector \( a \in R^3 \).

b) Symbol \( Z_0^{-} \) is positively homogeneous of degree 1 with respect to \( |\xi'| \).

Now, in view of Proposition 3.1.b, symbol \( Z_0^{-} \) can be written in the form

\[
Z_0^{-}(\xi') = -2\pi|\xi'|^2 K^{-}(\xi')
\]

(3.4)

where symbol \( K^{-}(\xi') \) is of the order -1, and hence it corresponds to a smoothing integral operator in the considered plane case \( R^2 \).

Applying integral Fourier transform inversion to Eq. (3.4), yields

\[
Z_0(x') = \Delta_{x'} \circ K(x'), \quad x' \in \Pi_y
\]

(3.5)

where \( \Delta_{x'} \) is the Laplace operator in \( \Pi_y \) and \( K(x') \) is the homogeneous \( 3 \times 3 \)-matrix kernel of degree -1 (with respect to \( |x'| \)). Kernel \( K(x') \) corresponds to the matrix integral operator in \( \Pi_y \). In (3.5) symbol \( \circ \) defines composition of two operators.

Now, applying decomposition (3.5), allows one to write
\[ \mathbf{t} = \left( \Lambda_{x'} \circ \mathbf{K} \right) \mathbf{b} \quad (3.6) \]

Equation (3.6) will be further restricted to the crack domain.

Restriction of the operators in (3.6) to the crack domain \( \Omega \), yields
\[ \mathbf{t}_\Omega = \left( \Lambda_{x'} \circ \mathbf{K}_\Omega \right) \cdot \mathbf{b} \quad (3.7) \]

Now, inverting the latter equation yields
\[ \mathbf{b} = \left( r_\Omega \circ \mathbf{K}_\Omega^{-1} \right) \mathbf{t}_\Omega \quad (3.8) \]

where \( r_\Omega \) is the Green function for the Dirichlet problem in \( \Omega \) with homogeneous boundary conditions on \( \partial \Omega \); similarly, \( \mathbf{K}_\Omega^{-1} \) defines restriction to \( \Omega \) of the inverse operator \( \mathbf{K}^{-1} \), in view of (3.5) the latter is the operator of the class -1 in Hörmander 1/2-space and its symbol is of order +1.

4. SIF and DDIF: a closed form relation

Henceforth it is assumed that both fields \( \mathbf{t}_v \) and \( \mathbf{b} \) have no singularities except those located on \( \partial \Omega \). The wave front (WF) of the surface stress field \( \mathbf{t}_v \) acting on the \( \Pi_v \)-plane at a particular point \( x_0 \in \partial \Omega \) can be written in the form [5]
\[ \text{WF}(\mathbf{t}_v) = (x_0, \Psi_{\mathbf{t}_v \cdot \mathbf{n}_{x_0}}), \quad x_0 \in \partial \Omega, \quad (4.1) \]

where, as before \( \mathbf{n}_{x_0} \) defines the (outward) unit normal to \( \partial \Omega \) at \( x_0 \in \partial \Omega \), and \( \Psi_{\mathbf{t}_v \cdot \mathbf{n}_{x_0}} \) is a closed cone in \( \mathbb{R}^3 \) containing at least straight line \( (\xi' \cdot \mathbf{n}_{x_0}) \); see [10].

Similarly, the displacement discontinuity field \( \mathbf{b} \) in view of Eq. (3.6) can have no other singularities, except belonging to \( \partial \Omega \), thus the WF of \( \mathbf{b} \) takes the form
\[ \text{WF}(\mathbf{b}) = (x_0, \Phi_{\mathbf{b} \cdot \mathbf{n}_{x_0}}), \quad x_0 \in \partial \Omega, \quad (4.2) \]

where \( \Phi \) is a (closed) cone in \( \mathbb{R}^3 \) containing at least straight line \( (\xi' \cdot \mathbf{n}_{x_0}) \) [11].

Noting that the asymptotic expansion of the surface stress field at the crack boundary admits the following decomposition along line \( \xi' \cdot \mathbf{n}_{x_0} \) [12]:
\[ \mathbf{t}_v \sqsubset \mathbf{T}_{x_0} r^{-1/2} + o\left(r^{-1/2}\right), \quad r \equiv \left(\mathbf{x}' \cdot \mathbf{n}_{x_0} - x_0\right) \to +0, \quad (4.3) \]

and in view of Eqs. (3.8), (4.3) the displacement discontinuity field \( \mathbf{b} \) admits the following asymptotic expansion at the crack boundary [13]:
\[ \mathbf{b} \sqsubset \mathbf{D}_{x_0} r^{+1/2} + o\left(r^{+1/2}\right), \quad r \equiv \left(\mathbf{x}' \cdot \mathbf{n}_{x_0} - x_0\right) \to -0. \quad (4.4) \]

Now, Fourier integral transform of the (inner) leading asymptote of the field \( \mathbf{b} \) at \( x_0 \), yields [14]
\[ \left[ \left( (\mathbf{x}' - \mathbf{x}_0) \cdot \mathbf{n}_{x_0} \right)^{+1/2} \mathbf{D}_{x_0} \right]^{\sim} = \Gamma\left(\frac{3}{2}\right) 2\pi \xi' \cdot \mathbf{n}_{x_0}^{3/2} \exp\left(\frac{3}{4} \pi i - 2\pi i \xi' \cdot \mathbf{x}_0'\right) \delta\left(\xi' - (\xi' \cdot \mathbf{n}_{x_0}) \mathbf{n}_{x_0}\right) \mathbf{D}_{x_0}, \quad (4.5) \]

where
\[ \left( (\mathbf{x}' - \mathbf{x}_0) \cdot \mathbf{n}_{x_0} \right)^{\sim} = \begin{cases} 0, & (\mathbf{x}' - \mathbf{x}_0) \cdot \mathbf{n}_{x_0} > 0 \\ (\mathbf{x}' - \mathbf{x}_0) \cdot \mathbf{n}_{x_0}, & (\mathbf{x}' - \mathbf{x}_0) \cdot \mathbf{n}_{x_0} < 0 \end{cases} \quad (4.6) \]

At the same time the wave front of the surface stress field at the same point \( x_0 \in \partial \Omega \) in view of Eq. (3.6) takes the form
\[ \text{WF}(\mathbf{t}_v) = \text{WF}(\mathbf{Z}_0 \mathbf{b}). \quad (4.7) \]
Multiplying both sides of Eq. (4.5) by symbol $\sim$ 0 $Z_0$ yields the desired asymptote for the outer surface stress field

$$t \sim (4\pi)^{-1} (\mathbf{x}' \cdot \mathbf{n}_{x_0} - x_0)^{-1/2} Z_0^- (\mathbf{n}) \cdot D_{x_0} + o \left((\mathbf{x}' \cdot \mathbf{n}_{x_0} - x_0)^{-1/2}\right).$$

(4.8)

Equation (4.8) yields the desired equation between SIF and DDIF:

$$T_{x_0} = 2^{-1} (2\pi)^{-1/2} Z_0^- (\mathbf{n}) \cdot D_{x_0}. \quad (4.9)$$

Finally, combining Eqs. (1.4) and (4.9), matrix $M(\mathbf{n})$ takes the form

$$M(\mathbf{n}) = 2^{-1} (2\pi)^{-1/2} Z_0^- (\mathbf{n}). \quad (4.10)$$

Now, it remains to note that the symbol $\sim$ 0 $Z_0$ depends solely on elastic properties of the medium, due to Eqs. (3.1) and (3.2).

Presumably, the most interesting consequence flowing out from the obtained expressions (4.9), (4.10) lies in the field of Dugdale – Barenblatt cracks with the vanishing angle of the crack opening at the crack tip [3, 4, 15]. According to Eq. (4.9) and accounting zero degree angle at the crack tip for Dugdale – Barenblatt crack, the corresponding SIF should also vanish:

$$T_{x_0} = 2^{-1} (2\pi)^{-1/2} Z_0^- (\mathbf{n}) \cdot D_{x_0} \rightarrow 0. \quad (4.11)$$

The sketch of the Dugdale – Barenblatt crack is shown in Fig. 2.

Figure 2. Dugdale – Barenblatt zero-angle crack-tip opening

While the last fact was known for more than 50 years, the direct proof of vanishing SIF based on the PDO operator technique [8], is obtained, apparently for the first time.

5. Concluding remarks
A closed form relation between vectorial stress intensity factor (SIF) and displacement discontinuity at the crack boundary, called displacement discontinuity intensity factor (DDIF), is found by applying method of the integro-differential Hörmander-type operators of the class -1, connecting the inner asymptote of the displacement discontinuity field with the outer asymptote of the (outer) stress field at the crack boundary. The constructed relation is valid for the arbitrary shaped plane crack in a medium with arbitrary elastic monoclinic syngonium; see [15 – 18].

Moreover, relations (4.9), (4.10) reveal that the vector-valued SIF ($T_{x_0}$) and the corresponding DDIF ($D_{x_0}$) are connected up to the scalar multiplier $2^{-1} (2\pi)^{-1/2}$ (irrelevant with respect to the monoclinic symmetry) and by the strongly elliptic, homogeneous of degree 1, and symmetric matrix (tensorial) symbol $\sim$ 0 $Z_0$ (\mathbf{n}), which depends only upon elastic properties of the considered monoclinic medium along with direction of the unit normal to the crack boundary; see Eqs. (3.1) and (3.2). It should also be noted that the constructed symbol need not be inverted in terms of the integral Fourier transform, since due to Eq. (4.9) the symbol determines the desired relation between SIF and DDIF.
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