Abstract
Non-convex optimization problems are ubiquitous in machine learning, especially in Deep Learning. While such complex problems can often be successfully optimized in practice by using stochastic gradient descent (SGD), theoretical analysis cannot adequately explain this success. In particular, the standard analyses do not show global convergence of SGD on non-convex functions, and instead show convergence to stationary points (which can also be local minima or saddle points). We identify a broad class of nonconvex functions for which we can show that perturbed SGD (gradient descent perturbed by stochastic noise—covering SGD as a special case) converges to a global minimum (or a neighborhood thereof), in contrast to gradient descent without noise that can get stuck in local minima far from a global solution. For example, on non-convex functions that are relatively close to a convex-like (strongly convex or PL) function we show that SGD can converge linearly to a global optimum.

1. Introduction
Non-convex optimization problems are ubiquitous in deep learning and computer vision (Bottou et al., 2018). The training of a neural network amounts to minimizing a non-convex loss function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$f^* = \min_{x \in \mathbb{R}^d} \left[ f(x) = \mathbb{E}_{\xi \sim D} f(x, \xi) \right] ,$$  

where stochastic gradients $\nabla f(x, \xi)$ can be evaluated on samples $\xi \sim D$ of the data distribution (this formulation covers both the online setting or empirical risk minimization on a finite set of samples). Stochastic gradient descent methods, like SGD (Robbins & Monro, 1951) or ADAM (Kingma & Ba, 2014), are core components for training neural networks. In addition to their simplicity, and almost universal applicability, the solutions obtained by stochastic methods often generalize remarkably well (see e.g. Keskar et al., 2017).

The analysis of SGD-type methods for smooth objective functions is well understood: to find an $\epsilon$-approximate stationary point, i.e. $\| \nabla f(x) \| \leq \epsilon$, SGD needs $O(\epsilon^{-4})$ gradient evaluations (Ghadimi & Lan, 2013). SGD with recursive momentum requires $O(\epsilon^{-3})$ gradient evaluations (Cutkosky & Orabona, 2019), which is optimal (Arjevani et al., 2019), and in the deterministic setting, gradient descent converges in $O(\epsilon^{-2})$ gradient evaluations (Nesterov, 2004). Still, in practice it is often possible to find approximate stationary points—and even approximate global minimizers—of nonconvex functions faster than these complexity bounds suggest. This performance gap stems from the fairly weak smoothness assumption underpinning these generic bounds. However, functions minimized in practice often admit significantly more structure, even if they are not convex.

An active line of research has started to characterize classes of functions for which gradient type methods work well, i.e. discrete methods that track the gradient flow. For instance, Ge et al. (2016) show that matrix completion exhibits “convexity-like” properties, i.e. that all local minimizers are global. In more abstract settings, Polyak (1963); Łojasiewicz (1963) study gradient dominated functions, Necoara et al. (2016) star-convex functions and Hinder et al. (2020) investigate quasar-convex functions. All these function classes have in common that the gradient flow converges to a unique minima. To show convergence of stochastic methods, it therefore suffices to control the stochastic noise, i.e. to show that the steps taken by the algorithm follow sufficiently close the true gradient direction $\mathbb{E}_{\xi} \nabla f(x, \xi) = \nabla f(x)$. This can for instance be achieved by averaging techniques (Bach & Moulines, 2011; Stich, 2019), decreasing stepsizes (Lacoste-Julien et al., 2012) or variance reduction (Johnson & Zhang, 2013; Zhang et al., 2013; Mahdavi et al., 2013; Wang et al., 2013).

However, these arguments cannot explain the success of SGD on functions with multiple local minima on which the gradient flow can get stuck in local minima that are far from a global optimal solution. To improve our understanding of the convergence of SGD on such functions we also need
to consider the effect of stochastic noise (and algorithmic randomness). Stochastic noise has been observed to have many beneficial effects in non-convex optimization: For instance, it has been proven that stochastic noise can allow SGD to escape saddle points (Ge et al., 2015; Jin et al., 2017; Daneshmand et al., 2018), and under certain conditions noise allows SGD to escape local minima (Hazan et al., 2016; Kleinberg et al., 2018). In DL, it has been observed that artificially injected noise can lead to improved generalization (Neelakantan et al., 2015; Chaudhari et al., 2017; Plappert et al., 2017), in particular in the context of large batch training (Wen et al., 2018; Haruki et al., 2019; Lin et al., 2020).

In this work, we characterize a new class of non-convex functions for which stochastic gradient methods can provably escape certain types of local minima. In particular, we characterize non-convex functions on which stochastic methods converge linearly to a global solution (in contrast, only sublinear convergence rates to local minima are known on general non-convex functions, Fang et al., 2019; Li, 2019). The class of structured functions that we study in this work, are functions $f$ that have a hidden composite structure. This structure is in general unknown to the algorithm (the algorithm can only query $\nabla f(x, \xi)$ (such as SGD) and does not have access to $g$ or $h$ separately). Concretely, we assume that $f$ is the composition of two components $g, h$: $\mathbb{R}^d \to \mathbb{R}$:

$$f(x) = g(x) + h(x).$$

As an intuitive example, suppose that $g$ satisfies the Polyak-Lojasiewicz (PL) condition (we consider other cases too). If the perturbations induced by $h$ are not too strong relative to $g$, we show that the SGD trajectory follows the gradient flow of $g$ and converges linearly to a neighborhood of the global solution. Note that proving such a statement would be impossible when just assuming smoothness of $f$, as the function can have many local minima.

**Contributions.** Our contributions can be summarized as:

- We derive new and improved complexity estimates for perturbed SGD methods—a class of randomized algorithms that perturb iterates by stochastic noise (similar to SGD) on a new class of structured non-convex functions.
- We derive worst-case complexity estimates of perturbed SGD on this function class. These estimates circumvent the lower complexity bounds that constrain the SGD analyses on general non-convex smooth functions (Arjevani et al., 2019). In particular, we characterize settings where perturbed SGD methods
  - converge linearly to the exact (or a neighborhood of) the global solution,
  - or converge sub-linearly to the exact (or a neighborhood of) the global solution.

Both these results improve over traditional analyses which only show sublinear convergence to local minima or stationary points (which can be arbitrary far from the global minima).

- Utilizing the insights developed in (Kleinberg et al., 2018), we are able to link our convergence results to the behavior of SGD and demonstrate this connection via illustrative numerical experiments.

The code for all the experiments and plots in this paper has been uploaded to the following repository: https://github.com/mlolab/perturbed-sgd-demo.

## 2. Related Works

**Benefits of Injecting Noise:** It has been observed that the noise in the gradient can help SGD to escape saddle points (Ge et al., 2015) or achieve better generalization (Hardt et al., 2016; Mou et al., 2018). This is often explained by arguing that SGD finds ‘flat’ minima with favorable generalization properties (Hochreiter & Schmidhuber, 1997; Keskar et al., 2017; Jastrzbski et al., 2017), though also ‘sharp’ minima can also generalize well (Dinh et al., 2017). These advantageous properties of SGD decrease as the batch size is increased (Keskar et al., 2017) or with variance reduction techniques (Defazio & Bottou, 2019).

Several authors proposed to artificially inject noise into the SGD process for improved generalization (Neelakantan et al., 2015; Chaudhari et al., 2017; Plappert et al., 2017), in particular in the context of large batch training (Wen et al., 2018; Haruki et al., 2019; Lin et al., 2020).

**Approximate Minima in Non-Convex Functions:** Despite their NP-hardness, several works have studied non-convex optimization problems. Standard analysis for smooth functions can guarantee convergence to a first order stationary point ($\| \nabla f(x) \| \leq \epsilon$) only (Ghadimi & Lan, 2013; 2016) at rate $O(\epsilon^{-2})$. Recently, there has been much interest in second-order stationary points, where $\epsilon$-SOSP is defined as $\| \nabla f(x) \| \leq \epsilon, \lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\epsilon}$ (Ge et al., 2015; Allen-Zhu & Li, 2018; Xu et al., 2018b). If all saddle points are strict, then all $\epsilon$-SOSP are approximate local minima (Jin et al., 2017). Thus, convergence to $\epsilon$-SOSP allows us to escape all saddle points. While SGD guarantees $O(\epsilon^{-4})$ convergence to $\epsilon$-SOSP, utilizing acceleration and second-order approximations improves it to $O(\epsilon^{-3.5})$ (Agarwal et al., 2017; Carmon et al., 2016; 2017; Jin et al., 2018b:a). Other methods, with same or slightly better rates, utilize efficient subroutines (Allen-Zhu, 2018a:b), negative curvature of the loss (Xu et al., 2018a; Yu et al., 2018; Fang et al., 2018), adaptive regularization (Xu et al., 2020; Tripathruneni et al., 2018; Nesterov & Polyak, 2006) and variance reduction (Zhou et al., 2019a; Reddi et al., 2019).
Tackling benign nonconvexity with smoothing and stochastic gradients

Table 1: Comparison to related works on non-convex optimization. Oracle complexity for finding and $\epsilon$-approximate stationary point $\|\nabla f(x)\| \leq \epsilon$, assuming Lipschitz gradients for all methods and Lipschitz Hessians for methods converging to second-order stationary points. The structural assumptions enable global convergence in certain cases.

| Output                     | Assumptions                  | Oracle                  | Method                                  | Rate                     |
|-----------------------------|------------------------------|-------------------------|-----------------------------------------|--------------------------|
| First-order stationary point | Gradient, Lipschitz          | Gradient                | (Ghadimi & Lan, 2016)                   | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Carmon et al., 2017)                   | $\tilde{O}(\epsilon^{-1.75})$ |
|                             | Hessian                      | Hessian                 | (Nesterov & Polyak, 2006)               | $\tilde{O}(\epsilon^{-1})$ |
|                             | Function, Gradient and Hessian | Gradient vector product | (Carmon et al., 2016)                   | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Agarwal et al., 2017)                  | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               | Gradient                | (Jin et al., 2017)                      | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Jin et al., 2018b)                     | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               | Stochastic Gradient     | (Zhang et al., 2017)                    | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Ge et al., 2015)                       | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Fang et al., 2019)                     | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               |                         | (Tripuraneni et al., 2018)              | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               |                         | (Allen-Zhu, 2018b)                      | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               |                         | (Reddi et al., 2018)                    | $\tilde{O}(\epsilon^{-1.75})$ |
|                             |                               | Global Minima           | (Leti et al., 2017)                     | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Allen-Zhu & Li, 2018)                  | $\tilde{O}(\epsilon^{-1})$ |
|                             |                               |                         | (Fang et al., 2018)                     | $\tilde{O}(\epsilon^{-1})$ |

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$\lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\rho d}$, where $\rho$ denotes the Lipschitz constant of the Hessian.

et al., 2018; Lei et al., 2017). Our work provides much stronger guarantees as we show convergence to a neighborhood of the global minima, in turn escaping both saddle points and local minima, with a much simpler algorithm using only first-order stochastic gradients. We provide a comprehensive comparison of these methods in Table 1.

Smoothing: Injecting artificial noise is classically also known as smoothing or convolution (le Rond d’Alembert, 1754; Domínguez, 2015) and has found countless applications in various domains and communities. In the context of optimization, smoothing has been used at least since the 1960s in (Rastrigin, 1963; Matyas, 1965; Schuler & Steiglitz, 1968). While most proofs apply to the convex setting only (Nemirovskij & Yudin, 1983; Nesterov & Spokoiny, 2017; Stich, 2014), smoothing is more prominently used in heuristic search procedures for non-convex problems (Blake & Zisserman, 1987; Hansen & Ostermeier, 2001). One of the outstanding features of the smoothing technique is that it allows to reduce the optimization complexity of non-smooth optimization problems (Duchi et al., 2012; Nesterov, 2005).

Compositional structure: Often in machine learning settings, an inherent structure $f = g + h$ is explicitly known, for instance when one term denotes a regularizer. In this case, optimization methods can be designed that exploit favorable properties of the regularizer (such as strong convexity) (Duchi et al., 2010; Nesterov, 2013). However, this is different from our approach, as these algorithms need to have explicit knowledge of the regulariser. We, instead, use the structure (2) only as an analysis tool (opposed to e.g. Chen et al., 2017), while the algorithm has only access to stochastic gradients of $f$.

Approximately convex functions: Another approach for analysis of non-convex functions investigates weaker forms of convexity. The most common formulations include PL functions (Polyak, 1963; Łojasiewicz, 1963; Karimi et al., 2016), where all minima are global minima, star-convex functions (Zhou et al., 2019b; Lee & Valiant, 2016), which are convex about the minima and approximately convex functions, which differ from convex functions by a bounded constant (Zhang et al., 2017; Jin et al., 2018a; Belloni et al., 2015). These functions are analyzed using standard techniques used for convex function, as they slightly relax the notion of convexity. Necocra et al. (2016) provide a survey of when this analysis can lead to linear convergence. The class of non-convex functions that we consider subsume most mild cases of non-convexity like PL, star-convexity or approximate convexity, by setting $h(x)$ to be bounded. Further, our framework can also be extended to stronger ones like quasar-convexity (Hinder et al., 2020; Jin, 2020), by appropriately setting the value of $g$. 

Non-convex smoothing: A theoretical connection between stochastic optimization and smoothing as been established in (Klemborg et al., 2018). They study smoothing with distributions with bounded support (while we do not make this restriction) and prove convergence under the assumption the smooth $f_{\theta}$ is star convex (Hinder et al., 2020). In (Hazan et al., 2016) a graduated smoothing technique was analyzed under the assumption the smoothed function is strongly
convex on a sufficiently large neighborhood of the optimal solution. Further, smoothing has been used in the context of derivative free optimization or in Langevin dynamics in non-convex regimes, most notably in (Jin et al., 2018a; Zhang et al., 2017; Belloni et al., 2015), however these works do not show global linear convergence in stronger paradigms of non-convexity.

3. Notation

For the reader’s convenience, we summarize here a few standard definitions (Nesterov, 2014). We say that a function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-smooth if its gradient is \( L \)-Lipschitz continuous:

\[
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| , \quad \forall x, y \in \mathbb{R}^d .
\] (3)

A function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \mu \)-strongly convex for \( \mu \geq 0 \), if

\[
\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \| x - y \|^2 , \quad \forall x, y \in \mathbb{R}^d
\]

Sometimes relaxations of this condition are considered. A function \( f : \mathbb{R}^d \to \mathbb{R} \) satisfies the Polyak-Łojasiewicz (\( \mu \)-PL) condition with respect to \( x^* \) if

\[
2\mu(f(x) - f^*) \leq \| \nabla f(x) \|^2 , \quad \forall x \in \mathbb{R}^d .
\]

Here, \( f^* = \min_{x \in \mathbb{R}^d} f(x) \). PL functions can have multiple global minima, but for strongly convex functions, \( x^* = \arg \min_{x \in \mathbb{R}^d} f(x) \) is unique. We provide additional useful standard consequences of these inequalities in Appendix A.

4. Perturbed SGD

Our main goal is to study the convergence of SGD on problem (1). The SGD algorithm is defined as

\[
x_{t+1} := x_t - \gamma \nabla f(x_t),
\]

for a constant stepsize \( \gamma \) and a uniform stochastic sample \( \xi_t \sim \mathcal{D} \). This update can equivalently be written as

\[
x_{t+1} = x_t - \gamma \nabla f(x_t) + \gamma w_t,
\]

by defining \( w_t := \nabla f(x_t) - \nabla f(x_t, \xi_t) \). Let \( w_t \sim \mathcal{W}(x_t) \), where \( \mathcal{W}(x_t) \) denotes the distribution of \( w_t \), which can depend on the iterate \( x_t \).

Standard approach. Standard analyses of SGD on non-convex \( L \)-smooth functions typically derive an upper bound on the expected one step progress (e.g. Thm. 4.8 in Bottou et al., 2018). This gives

\[
\mathbb{E} f(x_{t+1}) \leq f(x_t) - \gamma \| \nabla f(x_t) \|^2 + \frac{\gamma^2 L}{2} \text{Var}(w_t).
\]

However, following this methodology, stochastic updates can only guarantee a smaller expected one step progress than the gradient method, as the variance is always positive.

Our approach. To circumvent the aforementioned limitation, we adopt two key changes. First, by utilizing the structure (2) we study the one step progress on \( g \) and secondly, we formulate the algorithm slightly differently. Concretely, we study perturbed SGD (Algorithm 1) that we formally define as

\[
x_{t+1} = x_t - \gamma \nabla f(x_t - u_t, \xi_t) ,
\]

for a random perturbation \( u_t \sim U(x_t) \). For this method, the expected one step progress can be estimated as,

\[
\mathbb{E} g(x_{t+1}) \leq g(x_t) - \gamma \left( \mathbb{E}_{u_t, \xi_t}[\nabla f(x_t - u_t, \xi_t)] \right) + \frac{\gamma^2 L}{2} \text{Var}_{u_t, \xi_t}[\nabla f(x_t - u_t, \xi_t)].
\]

The above formulation allows us to obtain larger progress than standard analysis, by the virtue of considering \( g \) and by using an appropriate smoothing distribution \( U \). To establish convergence, we will impose appropriate conditions on terms \( \Box \) and \( \Box \) in (5), which forms the basis for our Assumptions in Section 5.2.

It is easy to see that perturbed SGD comprises SGD, for instance when \( u_t \equiv 0 \) a.s. However, there are more possibilities to trade-off the randomness in \( \xi_t \) and \( u_t \). For instance, assume for illustration that perturbed SGD can access noiseless samples of the gradient, i.e. \( \nabla f(x_t - u_t) \), and that \( f \) is quadratic function \( f \) with full rank Hessian \( A \). Then it is still possible to simulate SGD by defining \( u_t = \gamma A^{-1} w_t \), as can be seen from

\[
\nabla f(x_t - u_t) = Ax_t - Au_t = Ax_t + \gamma w_t.
\]

In Section 7, we derive more general connections between perturbed SGD and vanilla SGD.

To summarize, we introduce perturbed SGD with the purpose to study the impact of smoothing \( u \sim U \) and stochastic gradient noise \( \xi \sim \mathcal{D} \) separately. Perturbed SGD is illustrated in Algorithm 1 and implements a stochastic smoothing oracle by only accessing stochastic gradients of \( f \). For simplicity, we assume constant step length \( \gamma \).

Algorithm 1 Perturbed SGD

\begin{verbatim}
Require: \( \gamma, f(x), T, U(x), x_0 \)
1: for \( t = 0 \) to \( T - 1 \) do
2: sample \( u_t \sim U(x_t) \)  \rightarrow\text{smoothing distribution}
3: sample \( \xi_t \sim \mathcal{D} \)  \rightarrow\text{(mini-batch) data sample}
4: \( x_{t+1} = x_t - \gamma \nabla f(x_t - u_t, \xi_t) \)  \rightarrow\text{SGD update}
5: end for
\end{verbatim}
5. Setting and Assumptions

We will now introduce the main assumption on the objective function $f$ with structure (2) and give an illustrative example.

5.1. Smoothing

To formalize the notion of perturbations (i.e. the $\mathbf{u}_t$’s in Algorithm 1), we utilize the framework of smoothing (Duchi et al., 2012). Convolution-based smoothing of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as:

$$f_{\mathcal{U}}(\mathbf{x}) := E_{\mathbf{u} \sim \mathcal{U}} f(\mathbf{x} - \mathbf{u}), \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

for a probability distribution $\mathcal{U}$ (sometimes we will allow $\mathcal{U}(\mathbf{x})$ to depend on $\mathbf{x}$).

Smoothing is a linear operator $(g + h)_{\mathcal{U}} = g_{\mathcal{U}} + h_{\mathcal{U}}$ and when $f$ is convex, then $f_{\mathcal{U}}$ is convex as well. The smoothing (6) cannot be computed exactly without having access to $f$, but one can resort to a stochastic approximation in practice. For a given $f$, we can query stochastic gradients of $\nabla f_{\mathcal{U}}$ by sampling $\mathbf{u} \sim \mathcal{U}$ and evaluating $\nabla f(\mathbf{x} - \mathbf{u})$. Many works that analyze smoothing need to formulate concrete assumptions on the smoothing distribution $\mathcal{U}$, for instance that variance $\mathbb{E}_{\mathbf{u} \sim \mathcal{U}(\mathbf{x})} \|\mathbf{u}\|^2 \leq \varsigma^2$ is bounded by a parameter $\varsigma^2 > 0$. This is, for instance, satisfied for smoothing distributions with bounded support (see Duchi et al., 2012) or subgaussian noise, in particular for the normalized Gaussian kernel $\mathbf{u} \sim \mathcal{N}(0, \varsigma^2/d \mathbf{I}_d)$. In our case, we do not need to formulate such an assumption on $\mathcal{U}$ directly, instead we formulate a new assumption that jointly governs both smoothing and stochastic noise in the next section.

5.2. Main Assumptions

As mentioned earlier, these assumptions seek to improve the one step progress for perturbed SGD (Algorithm 1) by exploiting the key terms of $\mathbf{u}$, $\odot$ and $\otimes$ in (5)—in Assumptions 5.1 and 5.3 respectively.

We now list the main assumptions for the paper.

**Assumption 5.1 (Stochastic noise).** The stochastic noise is unbiased, $\mathbb{E}_{\xi \sim \mathcal{D}} f(\mathbf{x}, \xi) = f(\mathbf{x})$, the smoothing distribution is zero-mean and $\mathbb{E}_{\mathbf{u} \sim \mathcal{U}(\mathbf{x})} \mathbf{u} = 0$, and there exist parameters $\sigma^2 \geq 0$, $M' \geq 0$, such that after smoothing with $\mathcal{U}(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^d$:

$$\mathbb{E}_{\mathbf{u}, \xi} \|\nabla f(\mathbf{x} - \mathbf{u}, \xi) - \nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})\|^2 \leq \sigma^2 + M' \|\nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})\|^2.$$  

Note that $\mathbb{E}_{\xi} \nabla f(\mathbf{x} - \mathbf{u}, \xi) = \nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})$. Therefore (7) allows us to bound the variance term $\otimes$ in (5). This extends

the standard noise assumption in SGD settings (Bottou et al., 2018; Stich, 2019) which are of the form $\sigma^2 + M \|\nabla f(\mathbf{x})\|^2$ (we recover this assumption when $\mathbf{u} \equiv 0$, a.s.). While in non-convex settings this prior assumption is could be restrictive (as $\|\nabla f(\mathbf{x})\|^2$ is small for stationary points, enforcing large $\sigma^2$), in contrast, $\|\nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})\|^2$ will still be large at saddles or sharp local minima, and thus in general $\sigma^2$ in (7) can be chosen much smaller.

**Remark 5.2.** If the smoothing distribution, $\mathcal{U}(\mathbf{x})$ has variance bounded by $\varsigma^2 + Z \|\nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})\|^2$, and the variance of stochastic gradients have variance bounded as $\sigma^2 + M \|\nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x})\|^2$, for some $\sigma^2, \varsigma^2, M, Z \geq 0$, then under independence of $\mathcal{U}$ and $\mathcal{D}$ and $\mathcal{L}$-smoothness of $f$, we can choose the terms in Assumption 5.1 as $\sigma^2 := \sigma^2 + 2(L\varsigma)^2$ and $M' := M + 2(LZ)^2$.

The above remark allows us to separate the contributions of smoothing noise and stochastic noise. Further, setting the terms of smoothing ($\varsigma, Z$) to 0, we recover the standard assumptions for SGD with unbounded variance. A proof of this remark is provided in Appendix A.

We now shift our attention to the term $\otimes$ in (5). Through the next assumption, we neatly tie this to the structure of the objective function in (2).

**Assumption 5.3 (Structural properties of $g$ and $h$).** The objective function $f : \mathbb{R}^d \to \mathbb{R}$ can be written in the form (2), with $g$ being $L_g$-smooth, and there exist parameters $0 \leq m < 1$ and $\Delta \geq 0$, such that, $\forall \mathbf{x} \in \mathbb{R}^d$:

$$\|\nabla f_{\mathcal{U}(\mathbf{x})}(\mathbf{x}) - \nabla g(\mathbf{x})\|^2 \leq \Delta + m \|\nabla g(\mathbf{x})\|^2.$$  

While this function does not explicitly clarify the role of $h$, to illustrate we can split the term on LHS as $\nabla h_{\mathcal{U}(\mathbf{x})}(\mathbf{x}) + (\nabla g_{\mathcal{U}(\mathbf{x})}(\mathbf{x}) - \nabla g(\mathbf{x}))$. The difference term $(\nabla g_{\mathcal{U}(\mathbf{x})}(\mathbf{x}) - \nabla g(\mathbf{x}))$ can be bounded if $\mathcal{U}(\mathbf{x})$ has bounded variance and $g$ is smooth. The purpose of this assumption then becomes controlling $\nabla h_{\mathcal{U}(\mathbf{x})}(\mathbf{x})$, which essentially is the non-convex perturbation in $f$. Note that this allows possibly unbounded $h$, however after smoothing, $\nabla h_{\mathcal{U}(\mathbf{x})}(\mathbf{x})$ must be dominated by $\nabla g(\mathbf{x})$. This assumption is an extension of biased gradient oracles of Ajalloeian & Stich (2020).

Assumption 5.3 covers a large family of non-convex functions, including PL and convex functions trivially. The ability of $\mathcal{U}$ in reducing the non-convexity of $h$ is quantified by $m$ and $\Delta$. Setting $m = 0$, we are able to handle bounded non-convex functions $h$.

The above assumption also allows us flexibility in choosing $\mathcal{U}$. For most problems, a family of distributions satisfy this assumption, with $m$ and $\Delta$ dependent on which distribution we pick from this family. Therefore, the distribution $\mathcal{U}$ is not completely problem dependent. We describe the effects
of this Assumption and the freedom in choosing $\mathcal{U}$ using an illustrative example.

5.3. Illustrative Example

We provide an illustrative example which satisfies our assumptions while displaying a high degree of non-convexity.

Consider the following 1-dimensional function,

$$ f(x) = x^2 + ax \sin(bx), \quad (9) $$

for parameters $a, b > 0$. We can choose $g(x) = x^2$ as the convex part, while $h(x) = ax \sin(bx)$ denotes the possibly unbounded non-convex perturbation. For $ab \geq 2$, this function can have infinitely many local minima, arbitrarily far away from its global minima.

Even after smoothing with a Gaussian distribution $\mathcal{N}(0, \zeta^2)$, the non-convex perturbations do not disappear, and it cannot be convex for any $\zeta$ (for more details see Appendix C.1). However, these perturbations become smaller with respect to $g$ for larger $\zeta$, as shown in Fig. 1. This (provably) allows the function to satisfy Assumption 5.3 for $m$ and $\Delta$, which are dependent on $\zeta$, thus allowing us flexibility in the choice of distribution $\mathcal{U}$.

5.4. More Examples

Our settings also cover 'valley functions', described by Hazan et al. (2016), e.g., for $x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d$, $\alpha > 0$,

$$ f(x) = 0.5 \|x\|^2 - \alpha e^{-\frac{x_1^2}{2\alpha^2}} $$

These are non-convex functions with sharp local minima (in this case at $x = (1, 0, 0, \ldots, 0)^T$, with $\lambda$ deciding the sharpness) and resemble the loss surfaces of simple NNs.

We can also handle problems with bounded non-convexity which are common in practical learning settings. For instance, consider the training of a classifier in the presence of random label noise. A common solution approach for these problems is to modify the surrogate loss function to attain unbiased estimators—however this new optimization target might not be convex, even when starting from a convex loss function (such as least square regression). Natarajan et al. (2013, Theorem 6) prove that this non-convex optimization target is uniformly close to a convex function $g$, i.e. $h$ is bounded. The function classes we consider contains this class of problems, yet we also cover more general cases where $h$ is not uniformly bounded. We cover additional examples in detail in Appendix C.

6. Convergence Analysis

We now present the convergence analysis. All the proofs, more detailed theorem statements, and additional extensions are deferred to Appendix B.

6.1. Gradient Norm Convergence

**Theorem 6.1.** Let $f$ satisfy Assumptions 5.1 and 5.3, and assume $g$ to be $L$-smooth, then there exists a stepsize $\gamma$ such that for any $\epsilon > 0$,

$$ T = \mathcal{O}
\left(
\frac{M' + 1}{\epsilon(1 - m)} + \frac{\sigma^2}{\epsilon^2(1 - m)^2 + \Delta^2}
\right)
\text{sgn} \mathcal{G}_0
$$

iterations are sufficient to obtain

$$ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla g(x_t)\|^2] = \mathcal{O}(\epsilon + \frac{\Delta}{1 - m}), \text{ where } \mathcal{G}_t = \mathbb{E}[g(x_t)] - \min_{x \in \mathbb{R}^d} g(x). $$

This theorem shows that Algorithm 1 converges to a neighborhood of a stationary point of $g$. The size of the neighborhood depends on $\Delta$. When all stationary points of $g$ are global minima (this is for instance the case for convex, star-convex, quasar-convex or quasi-convex functions), and $\Delta = 0$, this theorem shows global convergence of Perturbed SGD. We can show convergence with faster rates under additional assumptions on $g$.

6.2. Convergence under PL Conditions

**Theorem 6.2.** Let $f$ satisfy Assumptions 5.1 and 5.3, and assume $g$ to be $\mu_g$-PL. Then there exists a stepsize $\gamma$ such...
that for any $\epsilon > 0$,

$$T = \hat{O} \left( (M' + 1) \log \frac{1}{\epsilon} + \frac{\sigma^2}{\epsilon (1 - m) \mu_g + \Delta} \right) \frac{\kappa}{1 - m}$$

iterations are sufficient to obtain $\mathcal{G}_T = \mathcal{O}(\epsilon + \frac{\Delta}{\mu_g (1 - m)})$, where $\kappa := \frac{L_g}{\mu_g}$ and $\hat{O}$ hides only log terms.

If $\sigma^2 = 0$ then this theorem shows linear convergence in $\mathcal{O}\left( \frac{\kappa}{1 - m} \log \frac{1}{\epsilon} \right)$ steps to a neighborhood of the global solution (and to the global solution when $\Delta = 0$). When $\sigma^2$ is large, the rate is dominated by the second term, $\mathcal{O}\left( \frac{\sigma^2}{\epsilon (1 - m)^2} \right)$. This matches the $\mathcal{O}\left( \frac{\kappa}{m} \right)$ convergence rate of vanilla SGD on PL functions. However, note that in our case $f$ does not need to be PL to enjoy these convergence guarantees.

### 6.3. Convergence under Strong Convexity

We now extend our results to the case when $g$ is strongly convex. Note that while Theorem 6.2 still applies (all strongly convex functions are PL), applying this result for PL case admits a weaker convergence rate by a factor proportional to $\kappa$ in contrast to the improved result in Theorem 6.4. This result is not covered in prior frameworks, as matching convergence rates were previously only derived for $m < 1/\kappa$ (Ajalloeian & Stich, 2020, Remark 7). To achieve this, we slightly refine our Assumption 5.3, ensuring we still are able to retain its expressivity.

**Assumption 6.3 (Structural properties).** The objective function $f : \mathbb{R}^d \to \mathbb{R}$ can be written in the form (2) with $g$ being $L_g$-smooth, and there exist parameters $\Delta \geq 0$, $0 \leq m < 1$ such that, for all $x \in \mathbb{R}^d$:

$$\left\| (\nabla f_t(x) - \nabla g(x)) \right\|_g \leq \Delta,$$

where $\nabla f_t(x) = \nabla f_t(x) - \nabla g(x)$.

Our main idea is to split the bound in Assumption 5.3 to its respective components. Note that we can easily verify that this is stronger than Assumption 5.3 by computing $\left\| \nabla f_t(x) \right\|_g^2$.

To ensure the same level of expressivity for both the structural assumptions, we can verify that they have similar worst-case scenarios for a biased oracle, that is, when $r(x)$ points in the opposite direction of $\nabla g$ with squared norm $m \left\| \nabla g(x) \right\|_g^2$, ignoring the constant terms of $\Delta$. Thus, our new assumption can still deal with worst-case oracles obeying Assumption 5.3 while still admitting a better analysis.

**Theorem 6.4.** Let $f$ satisfy Assumptions 5.1 and 6.3, and assume $g$ to be $L_g$-smooth and $\mu_g$-strongly-convex, then there exist non-negative weights $\left\{ w_t \right\}_{t=0}^T$, with $W_T = \sum_{t=0}^T w_t$ and stepsize $\gamma$ such that for any $\epsilon > 0$, there exist, such that

$$T = \hat{O} \left( \kappa (M' + 1) \frac{m_-}{m} \log \frac{2}{\epsilon} + \frac{2(\sigma^2 + \Delta (M' + 1))}{\mu_g \epsilon m_- + 4 \Delta} \right)$$

iterations are sufficient to obtain $\frac{1}{W_T} \sum_{t=0}^T w_t G_t = \mathcal{O}(\epsilon + \frac{\Delta}{\mu_g m_-})$, where $m_- = (1 - \sqrt{m})^2$ and $m_+ = (1 + \sqrt{m})^2$.

Comparing Theorems 6.2 and 6.4, we find that the $\kappa$ dependence is no longer present in the noise term, while our proof holds for arbitrary $m < 1$. Thus, we have addressed both the problems which we mentioned at the start of this subsection. However, this does not come for free, as the convergence rate is inversely proportional to $(1 - \sqrt{m})$, instead of $1 - m$, in the PL case and $1 - \sqrt{m} < 1 - m$. Also, we have a larger noise term $(\sigma^2 + \Delta (M' + 1))$, than with PL, which also depends on $\Delta$.

### 6.4. Discussion of Results

Our convergence results show convergence to the neighborhood of minima of $g$, $x_0^* = \arg \min_{x \in \mathbb{R}^d} g(x)$. While this does not directly imply convergence in terms of $f$, we can apply assumptions on $h$ so that it does. If $h$ is bounded, our convergence results hold for $f$ within a neighborhood defined by the bound on $h$.

For convergence in iterates we can characterize the $\| x_0^* - x^* \|$ in terms of the non-convexity $h$. The following lemma provides this bound for strongly-convex $g$.

**Lemma 6.5.** If $g$ is $\mu_g$-strongly-convex,

$$\| x_0^* - x^* \|^2 \leq \frac{2}{\mu_g} (h(x_0^*) - h(x^*)).$$

Thus, it suffices that the difference of perturbations at the global minima of $f$ and $g$, i.e. $h(x^*) - h(x_0^*)$, is bounded, in order to show convergence to a close neighborhood of $x^*$. Note that this is much weaker than assuming bounded $h$. This ensures that our Perturbed SGD converges to a neighborhood of global minima of the non-convex function $f$ in presence of local minima.

Further, our convergence results rely on the size of the neighborhood $\Delta$. This neighborhood would depend on the choice of $U$. For our toy example (9), $\zeta^2$ decides the size of this neighborhood and this is under our control. Additionally, convergence to a neighborhood of global minima allows us to escape all local minima and saddle points which are far away and have poor function value. We illustrate this further through experiments in Section 8.

### 6.5. Insights

We have derived convergence results under our novel structural assumption (2) for Perturbed SGD (Alg. 1). Our results
We explain how the analysis from the previous section is with noise \( w \). We get for free a method that enjoys much more favorable which can only converge to approximate local minima. In All convergence results depend on the joint effect of smoothing and stochastic noise, \( \sigma^2 = \sigma^2 + L^2 \xi^2 \) (see Remark 5.2). This means, that any smoothing with \( \xi^2 \leq \frac{1}{T^2} \sigma^2 \) does not worsen the convergence estimates one would get by analyzing vanilla SGD alone. Moreover, smoothing allows convergence to the minima of \( g \), and to avoid local minima of \( f \) at a linear rate. Note that this is much faster and simpler than existing methods (Zhang et al., 2017; Jin et al., 2018a) which can only converge to approximate local minima. In particular, smoothing \( f \) with the scaled gradient noise \( \frac{1}{T} D \) we get for free a method that enjoys much more favorable convergence guarantees than SGD (Ge et al., 2015). But is it even necessary to implement Perturbed SGD, or does vanilla SGD suffice? We argue in the next section that this might indeed be the case.

7. Connection to SGD

We explain how the analysis from the previous section is connected to the standard SGD algorithm. (that does not implement the smoothing perturbation \( u \sim \mathcal{U}(x) \) explicitly).

7.1. Stochastic Online Setting

This follows directly from insights in (Kleinberg et al., 2018). Let \( x_t \) be the SGD iterates as defined in (SGD), with noise \( w_t \sim \mathcal{W}(x_t) \), where \( \mathcal{W}(x_t) \) is the gradient noise distribution. Kleinberg et al. (2018) propose to study the alternate sequence \( y_t \) defined as

\[
y_{t+1} = x_t - \gamma \nabla f(x_t).
\]

Let \( z_t \) define the iterates of Algorithm 1 as defined in (perturbed SGD), with only smoothing, \( u_t \), and no gradient noise, \( \xi_t \). Let \( u_t \sim \mathcal{U}(x_t) \), where \( \mathcal{U}(x_t) \) is the smoothing distribution.

**Lemma 7.1** (Equality in Expectation, (adopted from Kleinberg et al., 2018)). For \( x_t, y_t \), and \( z_t \) as defined above, if \( z_0 = y_0 \) and \( \mathcal{U}(x_t) = \gamma \mathcal{W}(x_t) \) for all \( t \geq 0 \), then

\[
\mathbb{E}[y_t] = \mathbb{E}[y_t].
\]

The proof for this lemma relies on induction. We show this for \( t = 1 \), and refer the reader to (Kleinberg et al., 2018) for the proof. Consider \( \mathbb{E}[y_t] \),

\[
\mathbb{E}[y_t] = \mathbb{E}[x_0 - \gamma \nabla f(x_0)]
= x_0 - \mathbb{E}[w_0] - \gamma \mathbb{E}[\nabla f(y_0) - \gamma w_0]
= x_0 - \gamma \mathbb{E}[w_0] \nabla f(y_0) - \gamma w_0)
= z_0 - \gamma \mathbb{E}[w_0] \nabla f(z_0 - u_0)]
= \mathbb{E}[z_t].
\]

The first and second equation utilize the definition of \( y_t \). In the third equation, we use the fact that \( w_0 \) is zero-mean, while in the fourth equation, we substitute \( u_0 = \gamma w_0 \), since \( \mathcal{U}(x_0) = \gamma \mathcal{W}(x_0) \).

This Lemma establishes the intuition, that SGD is performing approximately gradient descent on a smooth version of \( f \). Note that we establish only a weak equivalence in expectation. However, the next lemma shows that even this weak equivalence is sufficient to use our main results from Theorem 6.4 for SGD analysis.

**Lemma 7.2.** Let \( x_t, y_t \), and \( z_t \) be as defined above. Define \( \bar{y}_{\gamma} := \frac{1}{\gamma T} \sum_{t=0}^{T-1} w_t \mathbb{E}[y_t] \) for \( \{w_t\}_{t=0}^T \) and \( W_T \) as defined in Theorem 6.4. If Lemma 7.1 holds, \( g \) is convex,

\[
g(\bar{y}_{\gamma}) - g(x_T^*) \leq \frac{1}{\gamma T} \sum_{t=0}^{T} w_t G_t.
\]
We now explain the connection between SGD and Perturbed-$\nabla f_i(x_t, \xi) = \nabla f_i(x_t)$

where $i$ is sampled uniformly at random from $[n]$. Thus, the noise in each gradient step, $\mathbf{w}_t$, is,

$$\mathbf{w}_t = \nabla f_i(x_t) - \nabla f(x_t) = \nabla f_i(x_t) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_t). \quad (10)$$

To find an equivalent smoothing distribution, we can set $U(x) = \gamma W(x)$ as described above. However, the resulting distribution would require us to compute $u_t = \gamma (\nabla f_k(x_t) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_t))$ for an uniformly at random sampled index $k$. This involves computation of a full batch gradient, rendering the resulting procedure very inefficient. To overcome this, we can define $u_t$ in the following way:

$$u_t = \gamma (\nabla f_k(x_t) - \nabla f_j(x_t)), \quad (11)$$

where $k, j$ are sampled uniformly at random from $[n]$. This results in an efficient oracle with variance

$$E_{U(x)}[||u_t||^2] = 2\gamma^2 E_{W(x)}[||\mathbf{w}_t||^2].$$

Note that this resembles the method implemented in (Haruki et al., 2019) in a distributed setting.

8. Numerical Illustrations

In this section we provide numerical illustrations to demonstrate that Perturbed SGD is able to escape local minima in contrast to gradient descent (GD) and to verify its connection to SGD.

7.2. Finite-Sum Setting

We now explain the connection between SGD and Perturbed-GD for a finite-sum objective. Note that common machine learning applications follow a finite-sum structure, where the objective function is mean of training losses on all data samples of a dataset. This formulation allows us to empirically verify the connection between SGD and Perturbed-GD for common machine learning applications like Logistic Regression and neural networks.

Consider the finite-sum objective function, $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, which is a sum of $n$ terms. For SGD, at each step $t$,

$$\nabla f(x_t, \xi) = \nabla f_i(x_t)$$

where $i$ is sampled uniformly at random from $[n]$. Thus, the noise in each gradient step, $\mathbf{w}_t$, is,

$$\mathbf{w}_t = \nabla f_i(x_t) - \nabla f(x_t) = \nabla f_i(x_t) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_t). \quad (10)$$

To find an equivalent smoothing distribution, we can set $U(x) = \gamma W(x)$ as described above. However, the resulting distribution would require us to compute $u_t = \gamma (\nabla f_k(x_t) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_t))$ for an uniformly at random sampled index $k$. This involves computation of a full batch gradient, rendering the resulting procedure very inefficient. To overcome this, we can define $u_t$ in the following way:

$$u_t = \gamma (\nabla f_k(x_t) - \nabla f_j(x_t)), \quad (11)$$

where $k, j$ are sampled uniformly at random from $[n]$. This results in an efficient oracle with variance

$$E_{U(x)}[||u_t||^2] = 2\gamma^2 E_{W(x)}[||\mathbf{w}_t||^2].$$

Note that this resembles the method implemented in (Haruki et al., 2019) in a distributed setting.

8.2. Verifying Connections to SGD

We empirically demonstrate the connections between our algorithm and SGD in two settings, when noise is-- a) independent of $x$ (Section 7.1) and b) dependent on $x$ (Section 7.2).

For our first setting (depicted in Figure 3(a)), we use our toy problem $f(x) = x^2 + 10x \sin(x)$. We fix the initial point for SGD as $x_0 = 100$ and $\zeta = 0.1$. We add a Gaussian noise sampled from $\mathcal{N}(0, \sigma^2)$ to the gradients, where $\sigma^2 = \gamma \zeta^2$. The above lemma is a straightforward application of Jensen’s inequality and equality in expectation. The complete proof is presented in Appendix A. We can now utilize the results of Thm. 6.4 for SGD iterates defined by $x_t$. The above lemma is a straightforward application of Jensen’s inequality and equality in expectation. The complete proof is presented in Appendix A. We can now utilize the results of Thm. 6.4 for SGD iterates defined by $x_t$.
For our second setting (depicted in Figure 3(b)), we consider a finite-sum objective. The stochastic noise arises from sampling one datapoint in the finite sum with replacement, and is thus dependent on \( x \). We use logistic regression with cross entropy loss on the Digits dataset (Dua & Graff, 2017) from scikit-learn (Pedregosa et al., 2011). The dataset consists of 8 \( \times \) 8 images of handwritten digits from 0 to 9, from which we use only images of 0 and 1. For SGD, \( x_0 \) is sampled uniformly from \([-0.5, 0.5]^64\). We choose the same sampling for \( U(x) \), to obtain \( U(x) = \gamma V(x) \).

For both of these cases, the mean trajectories for \( y_t \) and \( z_t \) are very close, verifying our analysis. For the uniform noise setting, the variances of the trajectories are also very similar. However, the variance for our algorithm is much smaller than SGD for the logistic regression example. Now, we illustrate this connection for deep learning examples in the next section. We also analyze our toy example under high noise settings, which are described in Appendix C.4.

### 8.3. Deep Learning Examples

We further investigate the equivalence between SGD and Perturbed SGD for a standard deep learning problem—Resnet18 (He et al., 2015) on CIFAR10 dataset (Krizhevsky, 2009). Note that in deep learning settings, our loss function is \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \), where \( f_i(x) \) is the loss, in this case cross-entropy, for the \( i^{th} \) datapoint in the dataset for network with weights given by \( x \).

We compare Perturbed SGD with mini-batch SGD with batch size 128. In Section 7.2, we describe two possible implementations for the finite-sum setting—(10) and (11). Since we require the full-batch gradient in each step of (10), we cannot use this in deep learning settings with large dataset sizes. In (11), we utilize only minibatch gradients, so we can apply it to deep learning problems. In our pytorch implementation, we break down Algorithm 1 into two steps—perturbation step which computes \( u_t \), and the gradient step which updates parameters with \( \nabla f(x_t + u_t, \xi_t) \).

To verify the equivalence of SGD and Perturbed SGD, we need to ensure the same noise levels and the number of steps for both algorithms. We briefly describe how this is achieved for finite-sum implementation of Perturbed SGD described in (11).

For (11), the perturbation step and the gradient step have 3 times the noise of SGD, as the perturbation step has 2 times the noise of SGD. To ensure the same noise levels, we set the batch size for both steps as 128 \( \times \) 3 = 384. To ensure the same number of steps as SGD in one epoch, we repeat perturbation + gradient step 3 times in each epoch.

From Fig 4, we can see that the efficient finite-sum implementation of Perturbed SGD and SGD have very similar trajectories for training accuracy, training loss and validation accuracy. This verifies our claim of equivalence of SGD and Perturbed SGD on DL examples, with the same noise levels. Moreover, the variance is higher for Perturbed SGD than SGD, despite similar gradient noise level, providing further motivation to investigate benefits of Perturbed SGD in generalization and escaping saddles (Ge et al., 2015).

### 9. Discussion and Outlook

There is a growing discrepancy between the theoretically weak complexity results for SGD and its empirically strong performance, which is often observed on non-convex DL examples. This is because the theoretical modeling of the functional class—typically smooth non-convex losses—does not reflect well the practical challenges. To break this complexity barrier, we propose a new class of functions that allow us
to justify why stochastic methods (SGD or Perturbed SGD) can provably avoid local minima and converge (at a linear rate) to a global optimal solution. However, it remains an interesting open question to prove that our structural assumptions hold for real DL tasks.

We believe that it possible to develop more advanced versions of Perturbed SGD, such as counterparts of momentum SGD, ADAM, or variance reduced methods that are specifically designed for (hidden) composite functions. Another direction could aim at proving convergence results for SGD on targets with hidden structure in a more direct way, without the detour via Perturbed SGD. Research in this direction may for example shed new light on why variance reduced methods struggle on non-convex tasks (Defazio & Bottou, 2019) and can lead to more efficient training methods for neural networks in general. An analysis of Perturbed SGD that studies its generalization properties is another promising direction (Foret et al., 2021).

References

Agarwal, N., Allen-Zhu, Z., Bullins, B., Hazan, E., and Ma, T. Finding approximate local minima faster than gradient descent. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, pp. 1195–1199, New York, NY, USA, 2017. Association for Computing Machinery. ISBN 9781450345286.

Ajalleœian, A. and Stich, S. U. Analysis of SGD with biased gradient estimators. arXiv preprint arXiv:2008.00051, July 2020.

Allen-Zhu, Z. How to make the gradients small stochastically: Even faster convex and nonconvex SGD. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018a.

Allen-Zhu, Z. Natasha 2: Faster non-convex optimization than SGD. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018b.

Allen-Zhu, Z. and Li, Y. Neon2: Finding local minima via first-order oracles. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018.

Arjevani, Y., Carmon, Y., Duchi, J. C., Foster, D. J., Srebro, N., and Woodworth, B. Lower bounds for non-convex stochastic optimization. arXiv 1912.02365, 2019.

Bach, F. R. and Moulines, É. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. In Advances in Neural Information Processing Systems 24, pp. 451–459. Curran Associates, Inc., 2011.

Belloni, A., Liang, T., Narayanan, H., and Rakhlin, A. Escaping the local minima via simulated annealing: Optimization of approximately convex functions. In Proceedings of The 28th Conference on Learning Theory, volume 40 of Proceedings of Machine Learning Research, pp. 240–265. PMLR, 03–06 Jul 2015.

Blake, A. and Zisserman, A. Visual Reconstruction. MIT press Cambridge, 1987.

Bottou, L., Curtis, F., and Nocedal, J. Optimization methods for large-scale machine learning. SIAM Review, 60(2):223–311, 2018.

Carmon, Y., Duchi, J. C., Hinder, O., and Sidford, A. Accelerated methods for non-convex optimization. arXiv preprint arXiv:1611.00756, 2016.

Carmon, Y., Duchi, J. C., Hinder, O., and Sidford, A. “Convex until proven guilty”: Dimension-free acceleration of gradient descent on non-convex functions. In Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pp. 654–663. PMLR, 06–11 Aug 2017.

Chaudhari, P., Choromanska, A., Soatto, S., LeCun, Y., Baldassi, C., Borgs, C., Chayes, J., Sagun, L., and Zecchina, R. Entropy-SGD: Biasing gradient descent into wide valleys. In International Conference on Learning Representations, 2017.

Chen, L., Zhou, S., and Zhang, Z. Stochastic variance reduction gradient for a non-convex problem using graduated optimization. arXiv preprint arXiv:1707.02727, July 2017.

Cutkosky, A. and Orabona, F. Momentum-based variance reduction in non-convex SGD. In Advances in Neural Information Processing Systems, pp. 15210–15219, 2019.

Daneshmand, H., Kohler, J., Lucchi, A., and Hofmann, T. Escaping saddles with stochastic gradients. In Proceedings of the 35th International Conference on Machine Learning, volume 80 of Proceedings of Machine Learning Research, pp. 1155–1164. PMLR, 10–15 Jul 2018.

Defazio, A. and Bottou, L. On the ineffectiveness of variance reduced optimization for deep learning. In Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019.

Dinh, L., Pascanu, R., Bengio, S., and Bengio, Y. Sharp minima can generalize for deep nets. arXiv preprint arXiv:1703.04933, 2017.

Domínguez, A. A history of the convolution operation. IEEE Pulse, 2015.

Dua, D. and Graff, C. UCI machine learning repository, 2017.

Duchi, J., Shalev-Shwartz, S., Singer, Y., and Tewari, A. Composite objective mirror descent. In Conference on Learning Theory, 2010.

Duchi, J. C., Bartlett, P. L., and Wainwright, M. J. Randomized Smoothing for Stochastic Optimization. arXiv preprint arXiv:1103.4296, April 2012.

Dinh, L., Pascanu, R., Bengio, S., and Bengio, Y. Sharp minima can generalize for deep nets. arXiv preprint arXiv:1703.04933, 2017.
Tackling benign nonconvexity with smoothing and stochastic gradients

Foret, P., Kleiner, A., Mobahi, H., and Neyshabur, B. Sharpness-aware minimization for efficiently improving generalization. In International Conference on Learning Representations, 2021.

Ge, R., Huang, F., Jin, C., and Yuan, Y. Escaping from saddle points—online stochastic gradient for tensor decomposition. In Proceedings of The 28th Conference on Learning Theory, volume 40 of Proceedings of Machine Learning Research, pp. 797–842. PMLR, 2015.

Ge, R., Lee, J. D., and Ma, T. Matrix completion has no spurious local minimum. In Advances in Neural Information Processing Systems, volume 29. Curran Associates, Inc., 2016.

Ghadimi, S. and Lan, G. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013.

Ghadimi, S. and Lan, G. Accelerated gradient methods for nonconvex and non-smooth optimization. Math. Program., 156(1):59–99, March 2016.

Hansen, N. and Ostermeier, A. Completely Derandomized Self-Adaptation in Evolution Strategies. Evolutionary Computation, 9(2):159–195, 06 2001.

Hardt, M., Recht, B., and Singer, Y. Train faster, generalize better: Stability of stochastic gradient descent. In Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pp. 1225–1234. PMLR, 2016.

Haruki, K., Suzuki, T., Hamakawa, Y., Toda, T., Sakai, R., Ozawa, M., and Kimura, M. Gradient noise convolution (GNC): Smoothing loss function for distributed large-batch SGD. arXiv preprint arXiv:1906.10822, 2019.

Hazan, E., Levy, K. Y., and Shalev-Shwartz, S. On graduated optimization for stochastic non-convex problems. In Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pp. 1833–1841. PMLR, 2016.

He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition, 2015.

Hinder, O., Sidford, A., and Sohoni, N. Near-optimal methods for minimizing star-convex functions and beyond. In Proceedings of Thirty Third Conference on Learning Theory, volume 125 of Proceedings of Machine Learning Research, pp. 1894–1938. PMLR, 09–12 Jul 2020.

Hochreiter, S. and Schmidhuber, J. Flat minima. Neural Computation, 9(1):1–42, 01 1997.

Jastrzębski, S., Kenton, Z., Arpit, D., Ballas, N., Fischer, A., Bengio, Y., and Storkey, A. Three factors influencing minima in sgd. arXiv preprint arXiv:1711.04623, 2017.

Jin, C., Ge, R., Netrapalli, P., Kakade, S. M., and Jordan, M. I. How to escape saddle points efficiently. In Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pp. 1724–1732. PMLR, 06–11 Aug 2017.

Jin, C., Liu, L. T., Ge, R., and Jordan, M. I. On the local minima of the empirical risk. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018a.

Jin, C., Netrapalli, P., and Jordan, M. I. Accelerated gradient descent escapes saddle points faster than gradient descent. In Proceedings of the 31st Conference On Learning Theory, volume 75 of Proceedings of Machine Learning Research, pp. 1042–1085. PMLR, 06–09 Jul 2018b.

Jin, J. On the convergence of first order methods for quasar-convex optimization. arXiv preprint arXiv:2010.04937, October 2020.

Johnson, R. and Zhang, T. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013.

Karimi, H., Nutini, J., and Schmidt, M. Linear Convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. arXiv preprint arXiv:1608.04636, September 2016.

Keskar, N. S., Mudigere, D., Nocedal, J., Smelyanskiy, M., and Tang, P. T. F. On large-batch training for deep learning: Generalization gap and sharp minima. In ICLR, 2017.

Kingma, D. P. and Ba, J. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

Kleinberg, B., Li, Y., and Yuan, Y. An alternative view: When does SGD escape local minima? In Proceedings of the 35th International Conference on Machine Learning, volume 80 of Proceedings of Machine Learning Research, pp. 2698–2707. PMLR, 2018.

Krizhevsky, A. Learning multiple layers of features from tiny images. Technical report, 2009.

Lacoste-Julien, S., Schmidt, M., and Bach, F. A simpler approach to obtaining an O(1/t) convergence rate for the projected stochastic subgradient method. arXiv preprint arXiv:1212.2002, December 2012.

le Rond d’Alembert, J.-B. Recherches sur différents points importants du systæme du monde. 1754.

Lee, J. C. H. and Valiant, P. Optimizing Star-Convex Functions. arXiv preprint arXiv:1511.04466, May 2016.

Lei, L., Ju, C., Chen, J., and Jordan, M. I. Non-convex finite-sum optimization via scsg methods. In Advances in Neural Information Processing Systems, volume 30. Curran Associates, Inc., 2017.

Li, Z. SSRGD: Simple stochastic recursive gradient descent for escaping saddle points. In Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019.

Lin, T., Kong, L., Stich, S. U., and Jaggi, M. Extrapolation for large-batch training in deep learning. arXiv preprint arXiv:2006.05720, 2020.

Łojasiewicz, S. Une propriété topologique des sous-ensembles analytiques réels. In Les Équations aux Dérivées Partielles, pp. 87–89, Paris, 1963. Éditions du Centre National de la Recherche Scientifique.

Mahdavi, M., Zhang, L., and Jin, R. Mixed optimization for smooth functions. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013.
Tackling benign nonconvexity with smoothing and stochastic gradients

Matyas, J. Random optimization. Automation and Remote Control, 26:246–253, 1965.

Mou, W., Wang, L., Zhai, X., and Zheng, K. Generalization bounds of sgd for non-convex learning: Two theoretical viewpoints. In Proceedings of the 31st Conference On Learning Theory, volume 75 of Proceedings of Machine Learning Research, pp. 605–638. PMLR, 06–09 Jul 2018.

Natarajan, N., Dhillon, I. S., Ravikumar, P. K., and Tewari, A. Learning with noisy labels. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013.

Necoara, I., Nesterov, Y., and Glineur, F. Linear convergence of first order methods for non-strongly convex optimization. arXiv preprint arXiv:1504.06298, August 2016.

Neelakantan, A., Vilnis, L., Le, Q. V., Sutskever, I., Kaiser, L., Kurach, K., and Martens, J. Adding gradient noise improves learning for very deep networks. arXiv preprint arXiv:1511.06807, 2015.

Nemirovskij, A. S. and Yudin, D. B. Problem complexity and method efficiency in optimization. Wiley-Interscience, 1983.

Nesterov, Y. Introductory Lectures on Convex Optimization, volume 87 of Springer Science & Business Media. Springer US, Boston, MA, 2004.

Nesterov, Y. Smooth minimization of non-smooth functions. Math. Program., 103(1):127–152, May 2005.

Nesterov, Y. Gradient methods for minimizing composite functions. Mathematical Programming, 140(1), 2013.

Nesterov, Y. Introductory Lectures on Convex Optimization: A Basic Course. Springer Publishing Company, Incorporated, 1 edition, 2014.

Nesterov, Y. and Polyak, B. Cubic regularization of Newton method and its global performance. Math. Program., 108(1):177–205, August 2006.

Nesterov, Y. and Spokoiny, V. Random Gradient-Free Minimization of Convex Functions. Found Comput Math, 17(2):527–566, April 2017.

Pedregosa, F., Varoquaux, G., Gramfort, A., Michel, V., Thirion, B., Grisel, O., Blondel, M., Prettenhofer, P., Weiss, R., Dubourg, V., et al. Scikit-learn: Machine learning in python. Journal of machine learning research, 12(Oct):2825–2830, 2011.

Plattner, M., Houthooft, R., Dhariwal, P., Sidor, S., Chen, R. Y., Chen, X., Asfour, T., Abbeel, P., and Andrychowicz, M. Parameter space noise for exploration. arXiv preprint arXiv:1706.01905, 2017.

Polyak, B. T. Gradient methods for minimizing functionals. Zh. Vychisl. Mat. Mat. Fiz., pp. 643–653, 1963.

Rastrigin, L. A. The convergence of the random search method in the extremal control of a many-parameter system. Automation and Remote Control, 24:1337–1342, 1963.

Reddi, S., Zaheer, M., Sra, S., Poczos, B., Bach, F., Salakhutdinov, R., and Smola, A. A generic approach for escaping saddle points. In Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics, volume 84 of Proceedings of Machine Learning Research, pp. 1233–1242. PMLR, 09–11 Apr 2018.

Robbins, H. and Monro, S. A Stochastic Approximation Method. The Annals of Mathematical Statistics, 22(3):400–407, September 1951.

Schumer, M. and Steiglitz, K. Adaptive step size random search. IEEE Transactions on Automatic Control, 13(3):270–276, 1968.

Stich, S. U. Convex optimization with random pursuit. PhD thesis, ETH Zurich, 2014.

Stich, S. U. Uniform optimal analysis of the (stochastic) gradient method. arXiv preprint arXiv:1907.04232, December 2019.

Tripuraneni, N., Stern, M., Jin, C., Regier, J., and Jordan, M. I. Stochastic cubic regularization for fast nonconvex optimization. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018.

Vardhan, H. and Stich, S. U. Escaping local minima with stochastic noise. In Advances in Neural Information Processing Optimization in Machine Learning Workshop (OPT), 2021.

Wang, C., Chen, X., Smola, A. J., and Xing, E. P. Variance reduction for stochastic gradient optimization. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013.

Wen, W., Wang, Y., Yan, F., Xu, C., Wu, C., Chen, Y., and Li, H. Smoothout: Smoothing out sharp minima to improve generalization in deep learning. arXiv preprint arXiv:1805.07898, 2018.

Xu, P., Roosta, F., and Mahoney, M. W. Newton-type methods for non-convex optimization under inexact Hessian information. Math. Program., 184(1):35–70, November 2020.

Xu, Y., Jin, R., and Yang, T. First-order stochastic algorithms for escaping from saddle points in almost linear time. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018a.

Xu, Y., Jin, R., and Yang, T. Neon+: Accelerated gradient methods for extracting negative curvature for non-convex optimization, 2018b.

Yu, Y., Xu, P., and Gu, Q. Third-order smoothness helps: Faster stochastic optimization algorithms for finding local minima. In Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc., 2018.

Zhang, L., Mahdavi, M., and Jin, R. Linear convergence with condition number independent access of full gradients. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013.

Zhang, Y., Liang, P., and Charikar, M. A hitting time analysis of stochastic gradient methods. In Proceedings of the 2017 Conference on Learning Theory, volume 65 of Proceedings of Machine Learning Research, pp. 1980–2022. PMLR, 07–10 Jul 2017.

Zhou, Y., Yang, J., Zhang, H., Liang, Y., and Tarokh, V. SGD converges to global minimum in deep learning via star-convex path. arXiv preprint arXiv:1901.00451, January 2019b.
A. Additional Technical Tools

We list here a few useful properties, sometimes used in the proofs. Further, we also provide missing proofs and additional analysis for Remark 5.2 and Lemma 7.1 in Section 7.

A.1. On Smooth and Convex Functions

We first provide additional definitions and formulations for smooth functions, which we will use later.

A function is $\mu$-star-convex with respect to $x^*$ if
\[
\langle \nabla f(x) - \nabla f(x^*), x - x^* \rangle \geq \mu \| x - x^* \|^2, \quad \forall x \in \mathbb{R}^d.
\] (12)

Strongly convex functions are both PL and star convex.

The smoothness assumption (3) is often equivalently written as
\[
|f(y) - f(x) + \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \| y - x \|^2, \quad \forall x, y \in \mathbb{R}^d
\] (13)

Remark A.1. Note that if a function $f$ is $L -$ smooth and has a minimizer $x^* \in \arg \min_{x \in \mathbb{R}^d} f(x)$, then it satisfies
\[
\| \nabla f(x) \|^2 \leq 2L(f(x) - r(x^*)) \quad \forall x \in \mathbb{R}^d.
\] (14)

Proof. Let $y = x - \frac{1}{L} \nabla f(x)$, then, substituting these $x$ and $y$ in above definition –
\[
\| \nabla r(x) \|^2 \leq 2L(r(x) - r(x) - \frac{1}{L} \nabla f(x)).
\]
Since $r(x - \frac{1}{L} \nabla f(x)) \geq r(x^*)$, we can substitute this in the upper bound. $\square$

Strong convexity is often written as
\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \| y - x \|^2 \quad \forall x, y \in \mathbb{R}^d.
\] (15)

A.2. Proof of Remark 5.1

To prove Remark 5.2, we first restate a more general version of the assumptions on the smoothing distribution $\mathcal{U}(x_t)$ and noise distribution $\mathcal{D}$ (in the main text we assumed $Z = 0$ for simplicity).

Assumption A.2 (Smoothing noise). For given $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the smoothing distribution $\mathcal{U}(x)$ is zero-mean ($\mathbb{E}_{u \sim \mathcal{U}(x)} u = 0$), can possibly depend on $x \in \mathbb{R}^d$ and there exists constants ($\zeta^2 \geq 0, Z^2 \geq 0$) such that the variance can be bounded as
\[
\mathbb{E}_{u \sim \mathcal{U}(x)} \| u \|^2 \leq \zeta^2 + Z^2 \| \nabla f(x) \|^2, \quad \forall x \in \mathbb{R}^d.
\] (16)

This Assumption is modeled similar to our Assumption 5.1. Further, setting $Z = 0$, we obtain a bound on the variance of the smoothing distribution, which is valid for subgaussian variables (Duchi et al., 2012).

We can use the above assumption to obtain bounds on variance of the perturbed gradient.

Lemma A.3 (Stochastic Approximation). If $f$ is $L$-smooth and Assumption A.2, the variance is bounded as
\[
\mathbb{E}_{u \sim \mathcal{U}(x)} \| \nabla f(x - u) - \nabla f_{\mathcal{U}(x)}(x) \|^2 \leq 2L^2 \zeta^2 + 2L^2 Z^2 \| \nabla f_{\mathcal{U}(x)}(x) \|^2, \quad \forall x \in \mathbb{R}^d.
\] (17)

Proof. By Jensen’s inequality and smoothness
\[
\mathbb{E}_u \| \nabla f(x - u) - \nabla f_{\mathcal{U}(x)}(x) \|^2 = \mathbb{E}_u \| \nabla f(x - u) - \mathbb{E}_{v \sim \mathcal{U}} \nabla f_{\mathcal{U}(x)}(x) \|^2
\leq \mathbb{E}_{u,v} \| \nabla f(x - u) - \nabla f(x - v) \|^2
\leq L^2 \mathbb{E}_{u,v} \| u - v \|^2 \leq 2L^2 \zeta^2 + 2L^2 Z^2 \| \nabla f_{\mathcal{U}(x)}(x) \|^2.
\] $\square$
We would like to clarify that the objective function for Figure 3(b) is of the form 
\[ f_{w}(x) = \nabla f(x) + w \]
where \( w \sim \mathcal{N}(x) \) and \( \mathcal{N}(x) \) denotes the zero-mean noise distribution, and there exist constants \( (\sigma^2 > 0, M > 0) \), such its variance can be bounded as
\[ \mathbb{E}_{w \sim \mathcal{N}(x)} ||w||^2 \leq \sigma^2 + M \||\nabla f_{\mathcal{U}(x)}(x)||^2 \|, \quad \forall x \in \mathbb{R}^d. \] (19)

Now, we are ready to present the complete the proof for Remark 5.2. We first present its extended version as a Lemma below and then prove it.

**Lemma A.5 (Extension of Remark 5.2).** If \( f \) is \( L \)-smooth, Assumptions A.2 and A.4 are satisfied, and the noise \( (W(x)) \) and smoothing distributions \( (\mathcal{U}(x)) \) are independent for \( x \), then,
\[ \mathbb{E}_{u, \xi} \| \nabla f(x - u, \xi) \| \leq \sigma^2 + (L \xi)^2 + (M + 2(LZ)^2) \||\nabla f_{\mathcal{U}(x)}(x)||^2 \|, \] (20)

Note that this is identical to Assumption 5.1, with \( \sigma'^2 = \sigma^2 + 2(L \xi)^2 \) and \( M' = M + 2(LZ)^2 \).

**Proof.** Consider the term on the left hand side,
\[ \mathbb{E}_{u, \xi} \| \nabla f(x - u, \xi) \| \leq \mathbb{E}_{w, \xi} \| \nabla f(x - u) + \nabla f_{\mathcal{U}(x)}(x), \xi ||^2 + \mathbb{E}_w ||w||^2 \]
\[ \leq (\sigma^2 + (L \xi)^2) + (M + 2(LZ)^2) \||\nabla f_{\mathcal{U}(x)}(x)||^2 \| \]
The first step is obtained by applying Assumption A.4 to separate \( w \). We can then separate terms of \( u \) and \( w \) since their distributions are independent. Then, we use Lemma A.3 and Assumption A.4 to bound the two variance terms. \( \square \)

**A.3. Additional details about Connection to SGD**
In this section, we provide missing proof for Lemma 7.2 and clarifications about Figure 3(b).

**A.3.1. Proof for Lemma 7.2**
Consider the term \( g(\bar{y}_1) - g^* \).
\[ g(\bar{y}_1) - g^* \leq \frac{1}{W_T} \sum_{t=0}^T w_t (g(\mathbb{E}[y_t]) - g^*) \]
\[ \leq \frac{1}{W_T} \sum_{t=0}^T w_t (g(\mathbb{E}[z_t]) - g^*) \]
\[ \leq \frac{1}{W_T} \sum_{t=0}^T w_t (\mathbb{E}[g(z_t)] - g^*) \]
\[ \leq \frac{1}{W_T} \sum_{t=0}^T w_t G_t. \]

For the first step, we use convexity of \( g \) with coefficients \( \left\{ \frac{w_t}{W_T} \right\}_{t=0}^T \). The second step is obtained from equality in expectation. The third step is obtained from Jensen’s inequality on convex \( g \) and the last term is the definition of \( G_t \).

**A.3.2. Clarification about Figure 3(b)**
We would like to clarify that the objective function for Figure 3(b) is of the form \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \), where \( n \) is the number of datapoints and \( f_i(x) \) is the cross-entropy loss for the \( i^{th} \) datapoint. For SGD, we sample 1 datapoint from the dataset at each step, while for the smoothing distribution, we use the formulation in (10), as described above.
B. Deferred Proofs

In this section we provide the proofs for the convergence results in Section 6.

First, we state and prove an intermediate lemma for sufficient decrease which resembles (5). Using this Lemma, we can easily prove the corresponding theorems for gradient noise, PL and strongly-convex functions. Additionally, we restate the complete theorems for these cases which contain all the details about step sizes and exact convergence rate.

B.1. One Step Progress

Lemma B.1 (One Step Progress). Let $f$ satisfy Assumptions 5.1 and 5.3 and, assume $g$ to be $L_g$-smooth and $x_t$ generated according to Algorithm 1. Then, for $\gamma \leq \frac{1}{L_g(M' + 1)}$, it holds

$$
\frac{(1 - m)}{2} \mathbb{E}[\|\nabla g(x_t)\|^2] \leq \frac{G_t - G_{t+1}}{\gamma} + \frac{\Delta}{2} + \frac{\gamma L_g}{2} \sigma^2,
$$

where $G_t$ is as defined before.

Proof. Using $L_g$-smoothness of $g$, we can write

$$
g(x_{t+1}) \leq g(x_t) + \langle \nabla g(x_t), x_{t+1} - x_t \rangle + \frac{L_g}{2} \|x_{t+1} - x_t\|^2
$$

$$
\leq g(x_t) - \gamma \langle \nabla g(x_t), \nabla g(x_t - u_t, \xi_t) + \nabla h(x_t - u_t, \xi_t) \rangle
$$

$$
+ \frac{\gamma^2 L_g}{2} \|\nabla g(x_t - u_t, \xi_t) + \nabla h(x_t - u_t, \xi_t)\|^2.
$$

Taking expectation wrt $\xi_t$ and $u_t$, and using the inequality $\mathbb{E}[\|X\|^2] = \mathbb{E}[\|X - \mathbb{E}[X]\|^2] + \mathbb{E}[\|X\|^2]$, and using the definition of smoothness we get

$$
\mathbb{E}_{\xi_t, u_t}[g(x_{t+1})] \leq g(x_t) - \gamma \langle \nabla g(x_t), \mathbb{E}_{\xi_t, u_t}[\nabla g(x_t - u_t, \xi_t) + \nabla h(x_t - u_t, \xi_t)] \rangle
$$

$$
+ \frac{\gamma^2 L_g}{2} \mathbb{E}_{\xi_t, u_t}[\|\nabla g(x_t - u_t, \xi_t) + \nabla h(x_t - u_t, \xi_t)\|^2]
$$

$$
\leq g(x_t) - \gamma \langle \nabla g(x_t), \nabla g_{U_t}(x_t) + \nabla h_{U_t}(x_t) \rangle + \frac{\gamma^2 L_g}{2} \|\nabla f_{U_t}(x_t)\|^2
$$

$$
+ \frac{\gamma^2 L_g}{2} \mathbb{E}_{\xi_t, u_t}[\|\nabla f(x_t - u_t, \xi_t) - \nabla f_{U_t}(x_t)\|^2].
$$

Using Assumption 5.1, with $\gamma \leq \frac{1}{L_g(M' + 1)}$

$$
\mathbb{E}_{\xi_t, u_t}[g(x_{t+1})] \leq g(x_t) - \gamma \langle \nabla g(x_t), \nabla g_{U_t}(x_t) + \nabla h_{U_t}(x_t) \rangle
$$

$$
+ \frac{\gamma^2 L_g(M' + 1)}{2} \|\nabla f_{U_t}(x_t)\|^2 + \frac{\gamma^2 L_g}{2} \sigma^2. \tag{21}
$$

$$
\mathbb{E}_{\xi_t, u_t}[g(x_{t+1})] \leq \frac{\gamma}{2} \left( \|\nabla g(x_t)\|^2 - \|\nabla h_{U_t}(x_t) + g_{U_t}(x_t) - g(x_t)\|^2 \right)
$$

$$
+ g(x_t) + \frac{\gamma^2 L_g}{2} \sigma^2.
$$

Now, using Assumption 5.3.

$$
\mathbb{E}_{\xi_t, u_t}[g(x_{t+1})] \leq g(x_t) - \frac{\gamma(1 - m)}{2} \|\nabla g(x_t)\|^2 + \frac{\gamma \Delta}{2} + \frac{\gamma^2 L_g}{2} \sigma^2.
$$

Taking full expectation on both sides and subtracting $\min_{x \in \mathbb{R}^d} g(x)$ from both sides, we get the required result. ■
B.2. Gradient Norm Convergence (Proof of Theorem 6.1)

We first state the extended version of Theorem 6.1.

**Extended Theorem B.2.** (Gradient Norm convergence) Under the assumptions in Lemma B.1, for stepsize \( \gamma \leq \frac{1}{L_g(M'+1)} \), after running the Algorithm 1 for \( T \) steps, it holds:

\[
\Phi_T \leq \frac{2G_0}{T \gamma(1 - m)} + \frac{\gamma L_g \sigma'^2}{1 - m} + \frac{\Delta}{1 - m},
\]

where \( \Phi_T = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla g(x_t)\|^2] \).

Further, for \( \epsilon > 0 \) and \( \gamma = \min\{ \frac{1}{L_g(M'+1)}, \frac{\epsilon(1-m)+\Delta}{2L_g \sigma'^2} \} \), then

\[ T = \mathcal{O}\left( \frac{M' + 1}{\epsilon(1-m) + \Delta} + \frac{\sigma'^2}{\epsilon^2(1-m)^2 + \Delta^2} \right) \frac{L_g}{L_g} G_0 \]

iterations are sufficient to obtain \( \Phi_T = \mathcal{O}(\epsilon + \frac{\Delta}{1-m}) \)

**Proof.** We can sum the terms of Lemma B.1 for \( t = 0 \) to \( T - 1 \), and divide both sides by \( T \), to obtain

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla g(x_t)\|^2] \leq \frac{2(G_0 - G_T)}{T \gamma(1 - m)} + \frac{\Delta}{(1 - m)} + \frac{\gamma L_g \sigma'^2}{(1 - m)} G_T + 1
\]

This proves the first part of the above Theorem. We can choose step sizes according to obtain rates in terms of \( \epsilon \). This can be found in (Ajalloeian & Stich, 2020, Lemma 3) and (Ajalloeian & Stich, 2020, Theorem 4) with different constants and notation.

B.3. Convergence for PL functions (Proof of Theorem 6.2)

We state the extended version of Theorem 6.2.

**Extended Theorem B.3.** Under Assumptions of Lemma B.1 and the additional assumption that \( g \) is \( \mu_g \)-PL, it holds for any stepsize \( \gamma \leq \frac{1}{L_g(M'+1)} \),

\[
\mathcal{G}_T \leq (1 - \gamma \mu_g(1 - m))^T G_0 + \frac{1}{2} \Xi , \quad \text{where} \quad \Xi = \frac{\Delta}{\mu_g(1 - m)} + \frac{\gamma L_g \sigma'^2}{\mu_g(1 - m)}. \]

Further, by choosing \( \gamma = \min\{ \frac{1}{L_g(M'+1)}, \frac{\epsilon(1-m)+\Delta}{L_g \mu_g} \} \), for any \( \epsilon > 0 \),

\[ T = \tilde{\mathcal{O}}\left( (M' + 1) \log \frac{1}{\epsilon} + \frac{\sigma'^2}{\epsilon(1-m) \mu_g + \Delta} \right) \frac{\kappa}{1 - m} \]

iterations are sufficient to obtain \( \mathcal{G}_T = \mathcal{O}(\epsilon + \frac{\Delta}{\mu_g(1-m)}) \), where \( \kappa := \frac{L_g}{\mu_g} \) and \( \tilde{\mathcal{O}} \) hides only log terms.

**Proof.** We use the PL condition in Lemma B.1, to obtain

\[
\mu_g \mathcal{G}_t \leq \frac{(G_t - G_{t+1})}{\gamma(1 - m)} + \frac{\Delta}{2(1 - m)} + \frac{\gamma L_g \sigma'^2}{2(1 - m)} G_T + \frac{\Delta \gamma}{2} + \frac{\gamma^2 L_g \sigma'^2}{2},
\]

\[
\mathcal{G}_{t+1} \leq (1 - \mu_g(1 - m)) \mathcal{G}_t + \frac{\Delta \gamma}{2} + \frac{\gamma^2 L_g \sigma'^2}{2}
\]

Unfolding the above recursion from \( t = 0 \) to \( t = T - 1 \), we get the first part of above Theorem. For the convergence rates in terms of \( \epsilon \), we can choose step size \( \gamma \) accordingly. This is similar to (Ajalloeian & Stich, 2020, Theorem 6) with different constants and notation.
B.4. Convergence for Strongly-convex functions (Proof of Theorem 6.4)

We first state the extended version of Theorem 6.4.

**Extended Theorem B.4.** Under Assumptions 5.1 and 6.3, if \( g \) is \( \mu_g \)-strongly convex, running Algorithm 1 for \( T \) steps, with \( \gamma \leq \frac{1-\sqrt{m}}{L_g(1+\sqrt{m})^2(M'+1)} \), there exist non-negative weights \( \{w_t\}_{t=0}^T \), with \( W_T = \sum_{t=0}^T w_t \), such that

\[
\frac{1}{W_T} \sum_{t=0}^T w_t G_t + \frac{\mu_g}{2} d_{T+1} = \mathcal{O}\left(\frac{d_0}{\gamma (1-\sqrt{m})} \exp\left(-\frac{(1-\sqrt{m})\mu_g T}{2}\right) + \Xi\right)
\]

where \( G_t \) is same as defined before, \( d_t = \mathbb{E}[\|x_t - x_g^*\|^2] \), \( x_g^* = \arg\min_{x \in \mathbb{R}^d} g(x) \), and

\[
\Xi = \gamma (\sigma^2 + \Delta(M'+1)) + \frac{2\Delta}{\mu_g (1-\sqrt{m})^2}.
\]

Further, choosing \( \gamma = \min \left\{ \frac{(1-\sqrt{m})}{L_g(M'+1)(1+\sqrt{m})^2}, \frac{\mu_g(1-\sqrt{m})^2 + 4\Delta}{2(\sigma^2 + \Delta(M'+1)(1-\sqrt{m})\mu_g)\epsilon} \right\} \),

\[
T = \tilde{O}\left(\frac{2\epsilon(M'+1)(1+\sqrt{m})^2}{(1-\sqrt{m})^2} \log \frac{1}{\epsilon} + \frac{4(\sigma^2 + \Delta(M'+1))}{\mu_g \epsilon (1-\sqrt{m})^2 + 4\Delta}\right)
\]

iterations are sufficient to obtain \( \frac{1}{W_T} \sum_{t=0}^T w_t G_t = \mathcal{O}(\epsilon + \frac{4\Delta}{\mu_g (1-\sqrt{m})^2}) \).

**Proof.** Consider \( \|x_t - x_g^*\|^2 \), and take expectations with respect to \( u_t, \xi_t \), on both sides, further use \( \mathbb{E}[\|X\|^2] = \mathbb{E}[\|X - \mathbb{E}[X]\|^2] + \mathbb{E}[\|X\|^2] \) and Assumption 5.1.

\[
\|x_{t+1} - x_g^*\|^2 = \|x_t - x_g^*\|^2 - 2\gamma \langle \nabla f(x_t - u_t, \xi_t), x_t - x_g^* \rangle \\
+ \gamma^2 \|\nabla f(x_t - u_t, \xi_t)\|^2 \\
\mathbb{E}_{u_t, \xi_t}[\|x_{t+1} - x_g^*\|^2] = \|x_t - x_g^*\|^2 - 2\gamma \langle \nabla f_{lt}(x_t) (x_t - x_g^*), x_t - x_g^* \rangle \\
+ \gamma^2 \mathbb{E}_{u_t, \xi_t}[\|\nabla f_{lt}(x_t) (x_t - x_g^*), x_t - x_g^* \|^2] \\
\leq \|x_t - x_g^*\|^2 - 2\gamma \langle \nabla g(x_t), x_t - x_g^* \rangle + \gamma^2 \sigma^2 \\
- 2\gamma \langle \nabla g_{lt}(x_t), x_t - x_g^* \rangle + \gamma^2 (M'+1) \|\nabla g_{lt}(x_t) (x_t - x_g^*), x_t - x_g^* \|^2. \tag{22}
\]

Let \( \nabla g(x_t) \) and \( \nabla g(x_t)_\perp \) be the units vector in direction of \( \nabla g(x_t) \) and perpendicular to it, respectively. For clarity of notations, let \( \langle \nabla h_{lt}(x_t), x_t - x_g^* \rangle = r(x_t) \). First, we bound the component perpendicular to \( \nabla g(x_t) \), using Assumption 5.3

\[
(r(x_t))_{g,\perp} \langle \nabla g(x_t)_{\perp}, x_t - x_g^* \rangle \leq \frac{\mu_g}{4} \|x_t - x_g^*\|^2 + \frac{1}{\mu_g} \|r(x_t)\|_{g,\perp}^2 \\
\leq \frac{\mu_g (1-\sqrt{m})}{4} \|x_t - x_g^*\|^2 + \frac{\Delta}{\mu_g (1-\sqrt{m})}. \tag{23}
\]

Now, consider the component along \( \nabla g(x_t) \) and strong convexity of \( g \) implies \( \langle \nabla g(x_t), x_t - x_g^* \rangle \geq 0 \), and using Assumption 5.3

\[
(r(x_t))_g \langle \nabla g(x_t)_g, x_t - x_g^* \rangle \geq -\frac{|r(x_t)_g|}{\|\nabla g(x_t)\|} \langle \nabla g(x_t), x_t - x_g^* \rangle \\
\geq -\sqrt{m} \langle \nabla g(x_t), x_t - x_g^* \rangle. \tag{24}
\]
Additionally, consider $\| \nabla g_{t}(x_t) + \nabla h_{t}(x_t) \|^2$ and use Assumption 6.3.

\[
\| \nabla g_{t}(x_t) + \nabla h_{t}(x_t) \|^2 \leq \| \nabla g_{t}(x_t) + \nabla h_{t}(x_t) - \nabla g(x_t) + \nabla g(x_t) \|^2 \\
\leq \| v + \nabla g(x_t) \|^2 \\
\leq \| (v)_g \nabla g(x_t) + (v)_{g\perp} \nabla g(x_t) \|^2 \\
\leq \| (v)_g \|^2 + \| (v)_{g\perp} \|^2 \\
\leq (1 + \sqrt{m})^2 \| \nabla g(x_t) \|^2 + \Delta . \tag{25}
\]

Using Eqns. (23), (24) and (25) in Eq. (22), we get

\[
\mathbb{E}_{u_t, \xi_t}[\| x_{t+1} - x^*_g \|^2] \leq \| x_t - x^*_g \|^2 (1 + \frac{\gamma \mu_g (1 - \sqrt{m})}{2}) - 2\gamma (1 - \sqrt{m}) (\nabla g(x_t), x_t - x^*_g) \\
+ \gamma^2 (M' + 1) (1 + \sqrt{m})^2 \| \nabla g(x_t) \|^2 + \gamma^2 (\sigma'^2 + \Delta(M' + 1)) \\
+ \frac{2\gamma \Delta}{\mu_g (1 - \sqrt{m})}.\]

Now, taking $\gamma \leq \frac{(1 - \sqrt{m})}{L_g (M' + 1)(1 + \sqrt{m})^2}$, taking complete expectations, and substituting $G_t = \mathbb{E}[g(x_t)] - g(x^*_g)$ and $d_t = \mathbb{E}[\| x_t - x^*_g \|^2]$.

\[
d_{t+1} \leq d_t \left( 1 - \frac{\gamma \mu_g (1 - \sqrt{m})}{2} \right) + \gamma^2 (\sigma'^2 + \Delta(M' + 1)) + \frac{2\gamma \Delta}{\mu_g (1 - \sqrt{m})} \\
- \gamma (1 - \sqrt{m}) G_t .
\]

We follow analysis in (Stich, 2019, Lemma 2) to multiply both sides by $w_t = \left( 1 - \frac{\gamma \mu_g (1 - \sqrt{m})}{2} \right)^{-(t+1)}$. If $\frac{2 \mu_g (1 - \sqrt{m})}{\gamma} < 1$,

we sum over $t = 0$ to $T$ and divide both sides by $W_T = \sum_{t=0}^{T} w_t$. We obtain the following results after performing these steps,

\[
\frac{(1 - \sqrt{m})}{W_T} \sum_{t=0}^{T} w_t G_t + \frac{w_T d_{T+1}}{\gamma W_T} \leq \frac{d_t}{\gamma W_T} + \frac{2\Delta}{\mu_g (1 - \sqrt{m})} + \gamma (\sigma'^2 + \Delta(M' + 1)) .
\]

Since $W_T \leq \frac{w_T}{(\gamma \mu_g (1 - \sqrt{m})/2)^2}$ and $W_T \geq w_T$, we obtain the first inequality

\[
\frac{1}{W_T} \sum_{t=0}^{T} w_t G_t + \frac{\mu_g d_{T+1}}{\gamma W_T} \leq \frac{d_0}{\gamma (1 - \sqrt{m})} \exp \left( - \frac{\mu_g (1 - \sqrt{m}) T}{2} \right) + \frac{2\gamma \Delta}{\mu_g (1 - \sqrt{m})^2} \\
+ \gamma (\sigma'^2 + \Delta(M' + 1)) \frac{(1 - \sqrt{m})}{(1 - \sqrt{m})} .
\]

For the second part, first let $\alpha = \sigma'^2 + \Delta(M' + 1)$ and $\beta = M' + 1$ Then, we denote the RHS of the main convergence result in terms of $\gamma$ and $T$.

\[
\Theta(\gamma, T) = \frac{d_0}{\gamma (1 - \sqrt{m})} \exp \left( - \frac{\mu_g (1 - \sqrt{m}) T}{2} \right) + \frac{\alpha}{(1 - \sqrt{m})} + \frac{2\Delta}{\mu_g (1 - \sqrt{m})^2} .
\]
We show that our bound for \( \Theta(\gamma, T) = O(\epsilon + \frac{4\Delta}{\mu_g(1-\sqrt{m})^2}) \) is achieved by \( \gamma = \min\{\gamma_1, \gamma_2\} \) and \( T = \max\{T_1, T_2\} \)

\[
\gamma_1 = \frac{1 - \sqrt{m}}{L_g M(1 + \sqrt{m})^2}, \quad \gamma_2 = \frac{\mu_g \epsilon (1 - \sqrt{m})^2 + 4\Delta}{2\alpha(1 - \sqrt{m})\mu_g},
\]

\[
T_1 = \frac{2\beta L_g (1 + \sqrt{m})^2}{\mu_g (1 - \sqrt{m})^2} \log \left( \frac{2L_g \beta d_0 (1 + \sqrt{m})^2}{\epsilon (1 - \sqrt{m})^2} \right),
\]

\[
T_2 = \frac{4\beta}{\mu_g \epsilon (1 - \sqrt{m})^2 + 4\Delta} \log \left( \frac{4d_0 \alpha \mu_g}{(\mu_g \epsilon (1 - \sqrt{m})^2 + 4\Delta)\epsilon} \right).
\]

If \( \gamma = \gamma_1 \), then \( \frac{2\alpha}{(1 - \sqrt{m})^2} \leq \frac{2\Delta}{\mu_g (1 - \sqrt{m})^2} \). Then, we can choose \( T \geq T_1 \), so that \( \Theta(\gamma, T) \leq \epsilon + \frac{4\Delta}{\mu_g (1 - \sqrt{m})^2} \).

Similarly, if \( \gamma = \gamma_2 \), then \( \frac{2\alpha}{(1 - \sqrt{m})^2} \leq \frac{2\Delta}{\mu_g (1 - \sqrt{m})^2} \). Then, we can choose \( T \geq T_2 \), so that \( \Theta(\gamma, T) \leq \epsilon + \frac{4\Delta}{\mu_g (1 - \sqrt{m})^2} \). \( \square \)

**B.5. Additional Settings**

In this subsection, we present alternative formulations to our Assumptions, namely, for bounded non-convexity and for exact smooth oracle \( \nabla f_{\hat{u}}(x) \), instead of the perturbed gradient.

**B.5.1. Convergence for Exact Smooth Oracle \( \nabla f_{\hat{u}}(x) \)**

While we have derived all results assuming we have access to \( \nabla f(x + u; \xi) \), our results can be extended to the case when we have access to \( \nabla f_{\hat{u}}(x; \xi) \). This extension is similar to extensions of SGD results to GD. This is done by setting the variance of gradients to 0, by setting \( \sigma^2 = M = 0 \). Similarly, for our case setting \( \xi^2 = Z = 0 \), yields converge rates with gradient oracle \( \nabla f_{\hat{u}}(x) \). This does not mean that the smoothing distribution \( \hat{u}(x) \) has 0 variance, just that the contribution to gradient noise due to smoothing is 0, again motivating the connection between smoothing and SGD.

**B.5.2. Non-Convexity \( h \) with Bounded Gradients**

In this section, we explore a class of non-convex functions satisfying our formulation (2), but which are easy to solve. Consider as before that \( g(x) \) and \( h(x) \) denote the convex part and non-convex perturbation of \( f(x) \), respectively. We now provide a few definitions which we will use later.

A point \( x \in \mathbb{R}^d \) is a stationary point of a differentiable function \( f : \mathbb{R}^d \to \mathbb{R} \) if

\[
\nabla f(x) = 0.
\]

Let \( \mathcal{X}^* \) denote the set of stationary points of \( f \). Additionally, let \( g^* = \min_{x \in \mathbb{R}^d} g(x) \).

A function \( h : \mathbb{R}^d \to \mathbb{R} \) has \( B_2 \)-bounded gradients if

\[
\|\nabla h(x)\|^2 \leq B_2 \quad \forall x \in \mathbb{R}^d.
\]

A function \( h : \mathbb{R}^d \to \mathbb{R} \) is \( B_1 \)-bounded if

\[
|h(x)| \leq B_1 \quad \forall x \in \mathbb{R}^d.
\]

With these definitions, we provide the below lemma, which illustrates the impact of a simple (bounded and gradient bounded) \( h \) on the stationary points of \( f \).

**Lemma B.5.** Let \( f \) satisfy structure (2) with convex part \( g \) and non-convex part \( h \).

- If \( g \) is \( \mu_g \cdot PL \) and \( h \) is \( B_2 \)-gradient bounded

\[
g^* \leq g(x) \leq g^* + \frac{B_2}{2\mu_g}, \quad \forall x \in \mathcal{X}^*.
\]
Tackling benign nonconvexity with smoothing and stochastic gradients

• If \( g \) is \( \mu_g \)-strongly convex and \( h \) is \( B_2 \)-gradient-bounded
  \[
  \|x - x^*_g\|^2 \leq \frac{B_2}{\mu_g}, \quad \forall x \in X^*.
  \]

• If \( g \) is \( \mu_g \)-PL and \( h \) is \( B_1 \)-bounded and \( B_2 \)-gradient bounded
  \[
  g^* - B_1 \leq f(x) \leq g^* + B_1 + \frac{B_2}{2\mu_g}, \quad \forall x \in X^*,
  \]
  \[
  |f(x) - f(y)| \leq 2B_1 + \frac{B_2}{2\mu_g}, \quad \forall x, y \in X^*.
  \]

Proof. Let \( y \) be a stationary point of \( f \). Then,
  \[
  \nabla g(y) = -\nabla h(y).
  \]

For the first part, since \( g \) is and \( h \) is \( B_2 \)-gradient bounded,
  \[
  2\mu_g (g(y) - g^*) \leq \|\nabla g(y)\|^2 = \|\nabla h(y)\|^2 \leq B_2.
  \]

For the second part, since \( g \) is \( \mu_g \)-strongly convex with global minima \( x^*_g \)
  \[
  g(y) \geq g^* + \frac{\mu_g}{2} \|y - x^*_g\|^2,
  \]
  and the claim follows together with the first part of this lemma (all \( \mu_g \)-strongly convex functions are also \( \mu_g \)-PL).

For the third part, assuming \( h \) is \( B_1 \)-bounded with the result from first part,
  \[
  g^* + h(y) \leq g(y) + h(y) \leq g^* + h(y) + \frac{B_2}{2\mu_g},
  \]
  \[
  g^* - B_1 \leq f(y) \leq g^* + B_1 + \frac{B_2}{2\mu_g}.
  \]

From the above lemma, we can see that if \( h \) is gradient bounded, all its stationary points are close to minima of \( g \). Thus, even GD on such a function should always end up close to the global minima. Note that Assumption 5.3 is weaker than bounded gradients for \( h \), as we allow \( h \) to have unbounded gradients and its stationary points are also not constrained to a neighborhood. This is demonstrated by our toy example \( f(x) = x^2 + ax \sin(bx) \), which we describe in detail in the next section.

C. Investigating Examples

In this section, we further investigate our toy example \( f(x) = x^2 + ax \sin(bx) \) and utilize it to compare our settings to other applications of non-convex smoothing in (Kleinberg et al., 2018; Hazan et al., 2016). Consider \( f(x) = x^2 + ax \sin(bx) \) and \( U = N(0, \zeta^2) \) as in the main text. For \( g(x) = x^2 \) and \( h(x) = ax \sin(bx) \), we observe that
  \[
  g_{\xi}(x) = x^2 + \zeta^2, \quad h_{\xi}(x) = ae^{-b^2 \zeta^2/2}(b\zeta^2 \cos(bx) + x \sin(bx))
  \]
  \[
  \nabla g_{\xi}(x) = 2x, \quad \nabla h_{\xi}(x) = abc^{-b^2 \zeta^2/2}((1 - b \zeta^2) \sin(bx) + x \cos(bx))
  \]
  \[
  \|\nabla h_{\xi}(x) + \nabla g_{\xi}(x) - \nabla g(x)\|^2 \leq a^2 b^2 e^{-b^2 \zeta^2} (x^2 + (b \zeta^2 - 1)^2)
  \]
  \[
  \leq a^2 b^2 e^{-b^2 \zeta^2} (\|\nabla g(x)\|^2 + 4(b \zeta^2 - 1)^2)
  \]

To satisfy Assumption 5.3 we can choose \( m = \frac{1}{8} a^2 b^2 e^{-b^2 \zeta^2} \) or \( \zeta = \frac{1}{b} \sqrt{2 \ln(ab) - \ln(4m)} \) (note that \( m < 1 \)) and
  \[
  \Delta = a^2 b^2 e^{-b^2 \zeta^2} (b \zeta^2 - 1)^2 = \frac{4m}{b^2} (2 \ln(ab) - \ln(4m) - b^2).
  \]

For any finite value of \( \zeta \), the function \( f_{\xi} \) is never convex. However, for every \( \zeta > \frac{1}{b} \sqrt{2 \ln(ab) - \ln(4)} \), we can always find \( m < 1, \Delta > 0 \) which satisfies our Assumption 5.3.
C.1. Toy Example is not convex after smoothing

Consider the toy example again, \( f(x) = x^2 + 10x \sin(x) \), with smoothing \( f \) with \( \mathcal{N}(0, \zeta^2) \). We obtain:

\[
    f_\mathcal{U}(x) = x^2 + \zeta^2 + a e^{-\zeta^2/2} (b \zeta^2 \cos(bx) + x \sin(bx)).
\]

(28)

According to our structure (2), we can pick \( g(x) = x^2 \) and \( h(x) = ax \sin(bx) \). We observe that smoothing reduces the non-convexity in the function and it starts resembling its convex component \( g \). This is better visualized in Figure 1, where we plot the function and its gradient for parameters \( a = 10 \) and \( b = 1 \) and \( \zeta \in \{0, 1, 2\} \), where \( \zeta = 0 \) corresponds to no smoothing.

Further, if we take our toy example again, \( f(x) = x^2 + 10x \sin(x) \), we can see that even after smoothing \( f \) with \( \mathcal{N}(0, \zeta^2) \), \( f_\mathcal{U} \) still has local minima and is not strongly-convex. To generate a concrete example, consider \( \zeta = 2 \), and denote the smoothed function with \( f_{\zeta} \) which is plotted in Figure 1(a), and for better visualization additionally in Figure 5. The smoothed function \( f_{\zeta} \) has two minima, close to \( x \approx -2.56 \) and \( x \approx 2.56 \) and an additional stationary point at \( x = 0 \). Therefore, the function \( f_{\zeta} \) is not strongly convex on a \( 3\zeta \)-ball around its minima (as each such ball contains also \( x = 0 \) and the other minima). Therefore, the example function \( f_{\zeta} \) does not satisfy the local strong convexity condition that is required for \((c, \delta)\)-nice functions, but it satisfies our Assumption 5.3 (note that \( \zeta > 2 \) satisfies the sufficient condition derived above).

![Figure 5: Function \( f(x) \) and \( f_\mathcal{U}(x) \) for \( \zeta = 2 \) (the same function as in Figure 1(a), highlighting that \( f_\mathcal{U} \) is not strongly convex in a \( 3\zeta \)-ball around its minima, as required for \((c, \zeta)\)-nice functions, but \( f_{\zeta} \) satisfies Assumption 5.3.](image)

C.2. Comparison to other Applications of Non-Convex Smoothing

In (Hazan et al., 2016), the notion of graduated optimization is utilized, by successively smoothing with decreasing \( \delta \) variance, to converge to global optima of a class of non-convex Lipschitz functions in a bounded domain \( \mathcal{X} \) ((\(c, \delta\))-nice, (Hazan et al., 2016, Definition 3.2)). Convergence of their method relies on the function becoming strongly-convex on \( \mathcal{X} \) after \( c\delta \)-smoothing. For a fixed domain, we can set \( \zeta = c\delta > \frac{1}{b} \sqrt{(2 \ln(ab) - \ln(4))} \), with appropriate \( a, b \) such that our toy example is never strongly convex in a fixed interval inside \( \mathcal{X} \), but satisfies our Assumption 5.3. Thus, their analysis fails on our example. Further, on a bounded domain, if a function is strongly-convex after smoothing, it satisfies our Assumption 5.3 for the same smoothing with \( m = \Delta = 0 \). Thus, all \((c, \delta)\)-nice functions also satisfy this assumption.

Our assumptions are weaker than those required in (Kleinberg et al., 2018). Notably, (Kleinberg et al., 2018) consider only smoothing with bounded support, while we do not have this restriction. Moreover, they need to assume that for given \( \mathcal{U} \), \( f_\mathcal{U} \) is star convex. We see from Figure 1(a) that our toy function is not star convex for all \( \zeta^2 \), while our Assumption 5.3 holds. This shows, that our setting allows more flexibility in the parameters.

C.3. Comparing to \((c, \delta)\)-Nice Functions (Hazan et al., 2016)

We consider the toy example which is \((c, \delta)\)-nice, mentioned in (Hazan et al., 2016), and show that this function can be optimized under our biased gradient assumptions as well. Consider \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \)

\[
    f(x) = 0.5 \|x\|^2 - \alpha e^{-\frac{\|x\|^2}{2\lambda^2}}
\]

This function is \((\sqrt{d}, 0.5)\)-nice for \( \lambda \leq 0.1 \) and \( \alpha \in [0, \frac{1}{200}] \). Note that, if we consider \( g(x) = x^2 \) and \( h(x) = -\alpha e^{-\frac{\|x\|^2}{2\lambda^2}} \), after smoothing with \( \mathcal{U} = \mathcal{N}(0, \zeta^2 I_d) \), we obtain

\[
    \| \nabla h_\mathcal{U}(x) + \nabla g_\mathcal{U}(x) - \nabla g(x) \|^2 \leq \frac{\alpha^2 \zeta^4}{\lambda^2 (\zeta^2 + \lambda^2)^3} (\| \nabla g(x) \|^2 + 1).
\]
Here, choosing $\zeta = k\lambda$, this function satisfies Assumption 5.3 with $\Delta = m = \frac{\alpha^2 k^4}{(k^2 + 1)^3 \lambda^3}$. For every valid $\alpha, \lambda$, we can choose $k$ such that $m < 1$.

C.4. Additional experiments on toy example

We perform additional experiments on our toy example for the same settings as Section 8. We implement Perturbed SGD with no gradient noise and different smoothing by controlling $\zeta$ and SGD, with a Gaussian gradient noise distribution, $\mathcal{W} = \mathcal{N}(0, \sigma^2)$.

From Figure 6, we can see that SGD and Perturbed SGD have similar behaviour for low noise level, as the last iterates are able to escape local minima. But, if we keep increasing the noise level, SGD starts performing poorly and its last iterates get spread out evenly over the domain. In contrast, Perturbed SGD at the same noise level concentrates around the global minima, and only at the highest noise level of $\zeta = 20$, its last iterates start spreading out. Although SGD and Perturbed SGD are equal in expectation, there are key differences especially in high noise setting which motivates further investigation.

![Figure 6: Comparison of Last iterate positions for SGD and Perturbed SGD without gradient noise for same noise levels. In each subfigure, $\zeta$ decides the noise level of both SGD and Perturbed SGD, as $\gamma$ is constant.](image)