RIGIDITY OF STEINER’S INEQUALITY FOR THE ANISOTROPIC PERIMETER

MATTEO PERUGINI

Abstract. The aim of this work is to study the rigidity problem for Steiner’s inequality for the anisotropic perimeter, that is, the situation in which the only extremals of the inequality are vertical translations of the Steiner symmetral that we are considering. Our main contribution consists in giving conditions under which rigidity in the anisotropic setting is equivalent to rigidity in the Euclidean setting. Such conditions are given in terms of a restriction to the possible values of the normal vectors to the boundary of the Steiner symmetral (see Corollary 1.17 and Corollary 1.18).

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1. Introduction

1.1. Overview. The characterization of the geometric properties of minimizers of variational problems can be in general a delicate thing to achieve. The study of perimeter inequalities under symmetrisation, and in particular the study of rigidity for such inequalities, is a good way...
to possibly provide tools in order to show symmetries of the minimizers of the problem under consideration. Steiner’s symmetrisation is a classical and powerful example of symmetrisation, that has been often used in the analysis of geometric variational problems. For instance, De Giorgi in his proof of the very celebrated Euclidean isoperimetric theorem \[16\], used Steiner’s inequality (see (1.3)) to show that the minimum for the Isoperimetric Problem is a convex set. After De Giorgi, in the seminal paper \[11\], Chlebík, Cianchi and Fusco discussed Steiner’s inequality in the natural framework of sets of finite perimeter and provided sufficient condition for the rigidity of equality cases. In our context, by rigidity of equality cases we mean the situation in which equality cases are solely obtained in correspondence of translations of the Steiner’s symmetral. Then, the characterization of the rigidity of equality cases was resumed by Cagnetti, Colombo, De Philippis and Maggi in their work presented in \[8\]. There, they managed to fully characterize the equality cases for the Steiner’s inequality and obtain further important results for the rigidity problem.

Concerning rigidity of equality cases, let us mention two results that were obtained in different settings from the one just described. First, in the framework of Gaussian perimeter, again Cagnetti, Colombo, De Philippis and Maggi managed to prove a complete characterization result of rigidity of equality cases for Ehrhard’s symmetrisation inequality (see \[7\]). The other result, is the characterization of rigidity for the Euclidean perimeter inequality under the spherical symmetrisation (see \[9\]). For a recent survey on rigidity results for perimeter inequality under symmetrisation see also \[6\].

The main goal of this work is to characterize rigidity of the equality cases of the Steiner’s inequality for the anisotropic perimeter. Our main contribution is presented in Proposition 1.16 that provides sufficient conditions to get that rigidity of equality cases of the Steiner’s inequality in the Euclidean setting coincides with rigidity of equality cases of the Steiner’s inequality in the anisotropic setting (see Corollary 1.17 and Corollary 1.18). In the remaining part of this introduction, we will introduce some notation and state our main results (see in particular Section 1.7).

The remaining part of the paper is organized as follows: In Section 2 we recall some basic notions of geometric measure theory and we introduce some useful notation. In Section 3 we focus our attention on the properties of the surface tension \(\phi_K\) (see (1.19)). In particular, we characterize the cases of additivity for the function \(\phi_K\) (see Proposition 3.21), and we prove other intermediate results that will be used in the proof of our main results about rigidity. In Section 4 we prove a characterization result for the anisotropic total variation (see Definition 3.11). Such result (see Theorem 4.1) will play an important role in Section 5. In Section 5 we prove a formula to compute the anisotropic perimeter for some classes of sets \(E \subset \mathbb{R}^n\) having finite perimeter, and whose vertical sections are segments (see Corollary 5.11 and Corollary 5.12). With these results at hands, in Section 6 we prove the first of our main results, namely the characterization of equality cases for the anisotropic perimeter inequality under Steiner’s symmetrisation (see Theorem 1.8). Lastly, in Section 7 we prove the other main results about rigidity, namely Theorem 1.10, Proposition 1.13 and Corollary 1.15.

1.2. Basic notions on sets of finite perimeter. For every \(r > 0\) and \(x \in \mathbb{R}^n\), we denote by \(B(x, r)\) the open ball of \(\mathbb{R}^n\) with radius \(r\) centred at \(x\). In the special case \(x = 0\), we set \(B(r) := B(0, r)\). Let \(E \subset \mathbb{R}^n\) be a measurable set, and let \(t \in [0, 1]\). We denote by \(E^{(t)}\) the set of points of density \(t\) of \(E\), given by

\[
E^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{\rho \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, \rho))}{\omega_n \rho^n} = t \right\},
\]
where here and in the following \( \mathcal{H}^k, k \in \mathbb{N} \) with \( 0 \leq k \leq n \), stands for \( k \)-dimensional Hausdorff measure, and \( \omega_n = \mathcal{H}^n(B(1)) \). We then define the essential boundary of \( E \) as
\[
\partial^E E := E \setminus (E^{(1)} \cup E^{(0)}).
\]

Let \( G \subset \mathbb{R}^n \) be any Borel set. We define the perimeter of \( E \) relative to \( G \) as the extended real number given by
\[
P(E; G) := \mathcal{H}^{n-1}(\partial^E E \cap G) \in [0, \infty],
\]
and the perimeter of \( E \) as \( P(E) := P(E; \mathbb{R}^n) \). When \( E \) is a set with smooth boundary, it turns out that \( \partial^E E = \partial E \), and the perimeter of \( E \) agrees with the usual notion of \((n - 1)\)-dimensional measure of \( \partial E \). If \( P(E; C) < \infty \) for every compact set \( C \subset \mathbb{R}^n \), \( E \) is called a set of locally finite perimeter and we can define the reduced boundary \( \partial^* E \) of \( E \). This has the property that \( \partial^* E \subset \partial^E E \), \( \mathcal{H}^{n-1}(\partial^* E \setminus \partial^E E) = 0 \), and is such that for every \( x \in \partial^* E \) there exists the measure theoretic outer unit normal \( \nu^E(x) \) to \( E \) at \( x \) (see Section 2).

1.3. Steiner’s inequality. We decompose \( \mathbb{R}^n, n \geq 2 \), as the Cartesian product \( \mathbb{R}^{n-1} \times \mathbb{R} \). Then, for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we will write \( x = (px, qx) \), where \( px = (x_1, \ldots, x_{n-1}) \), and \( qx = x_n \) are the ’horizontal’ and ’vertical’ projections, respectively. Given a Lebesgue measurable function \( v : \mathbb{R}^{n-1} \to [0, \infty] \), we say that a Lebesgue measurable set \( E \subset \mathbb{R}^n \) is \( v \)-distributed if, denoting by \( E_z \) its vertical section with respect to \( z \in \mathbb{R}^{n-1} \), that is
\[
E_z := \{ t \in \mathbb{R} : (z, t) \in E \}, \quad z \in \mathbb{R}^{n-1},
\]
we have that
\[
v(z) = \mathcal{H}^1(E_z), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1}.
\]

Among all \( v \)-distributed sets, we denote by \( F[v] \) the only one (up to \( \mathcal{H}^n \) negligible modifications) that is symmetric by reflection with respect to \( \{ qx = 0 \} \), and whose vertical sections are segments, that is
\[
F[v] := \left\{ x \in \mathbb{R}^n : |qx| < \frac{v(px)}{2} \right\}.
\]

If \( E \) is a \( v \)-distributed set, we define the Steiner symmetrisal \( E^s \) of \( E \) as \( E^s := F[v] \). Note that, if \( v \) if Lebesgue measurable, then \( F[v] \) is a Lebesgue measurable set. Furthermore, by Fubini’s Theorem, Steiner symmetrisation preserves the volume, that is, if \( E \) is a \( v \)-distributed set such that \( \mathcal{H}^n(E) < \infty \), then \( \mathcal{H}^n(E) = \mathcal{H}^n(F[v]) \). A very important fact is that Steiner symmetrisation acts monotonically on the perimeter. More precisely, Steiner’s inequality holds true (see for instance [20] Theorem 14.4): if \( E \) is a \( v \)-distributed set then
\[
P(E; G \times \mathbb{R}) \geq P(F[v]; G \times \mathbb{R}) \quad \text{for every Borel set } G \subset \mathbb{R}^{n-1}.
\]

The next two results give the minimal regularity assumptions needed to study inequality (1.3) (see [11] Lemma 3.1 and [3] Proposition 3.2, respectively).

**Lemma 1.1.** (Chlebík, Cianchi and Fusco) Let \( E \) be a \( v \)-distributed set of finite perimeter in \( \mathbb{R}^n \), for some measurable function \( v : \mathbb{R}^{n-1} \to [0, \infty] \). Then, one and only one of the following two possibilities is satisfied:

i) \( v(x') = \infty \) for \( \mathcal{H}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1} \) and \( F[v] \) is \( \mathcal{H}^n\)-equivalent to \( \mathbb{R}^n \);

ii) \( v(x') < \infty \) for \( \mathcal{H}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1} \), \( \mathcal{H}^n(F[v]) < \infty \), and \( v \in BV(\mathbb{R}^{n-1}) \), where \( BV(\mathbb{R}^{n-1}) \) denotes the space of functions of bounded variation in \( \mathbb{R}^{n-1} \) (see Section 2).
Lemma 1.2. Let \( v : \mathbb{R}^{n-1} \to [0, \infty) \) be measurable. Then, we have \( 0 < \mathcal{H}^n(F[v]) < \infty \) and \( P(F[v]) < \infty \) if and only if
\[
v \in BV(\mathbb{R}^{n-1}), \quad \text{and} \quad 0 < \mathcal{H}^{n-1}(\{v > 0\}) < \infty.
\] (1.4)

1.4. Rigidity for Steiner’s inequality. Given \( v \) as in (1.4) we set:
\[
\mathcal{M}(v) := \{ E \subset \mathbb{R}^n : E \text{ is } v\text{-distributed and } P(E) = P(F[v]) \}.
\] (1.5)

We say that rigidity holds true for Steiner’s inequality if the only elements of \( \mathcal{M}(v) \) are (\( \mathcal{H}^n \)-equivalent to) vertical translations of \( F[v] \), namely:
\[
E \in \mathcal{M}(v) \iff \mathcal{H}^n(E\Delta(F[v] + te_n)) = 0 \quad \text{for some } t \in \mathbb{R},
\] (RS)
where \( \Delta \) stands for the symmetric difference between sets, and \( e_1, \ldots, e_n \) are the elements of the canonical basis of \( \mathbb{R}^n \).

A natural step in order to understand when (RS) holds true, is to study the set \( \mathcal{M}(v) \). The characterization of equality cases in (1.3) was first addressed by Ennio De Giorgi in [16], where he showed that any set \( E \in \mathcal{M}(v) \) is such that
\[
E_z \text{ is } \mathcal{H}^1\text{-equivalent to a segment, for } \mathcal{H}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1},
\] (1.6)
(see also [26, Theorem 14.4]). After that, further information about \( \mathcal{M}(v) \) was given by Chlebík, Cianchi and Fusco (see [11, Theorem 1.1]). The study of equality cases in Steiner’s inequality was then resumed by Cagnetti, Colombo, De Philippis and Maggi in [8], where the authors give a complete characterization of elements of \( \mathcal{M}(v) \) (see Theorem 1.4 below). In order to explain their result, let us observe that any \( v \)-distributed set \( E \) satisfying (1.6) is uniquely determined by the barycenter function \( b_E : \mathbb{R}^{n-1} \to \mathbb{R} \), defined as:
\[
b_E(z) = \begin{cases} \frac{1}{\nu(z)} \int_{E_z} t \, d\mathcal{H}^1(t) & \text{if } 0 < v(z) < \infty, \\ 0 & \text{otherwise}. \end{cases}
\] (1.7)

Note that, if \( E \) satisfies (1.6), for every \( z \in \{0 < v < \infty\} \), \( b_E \) represents the midpoint of \( E_z \). In general, \( b_E \) may fail to be a \( BV \), or even an \( L^1_{\text{loc}} \) function, even if \( E \) is a set of finite perimeter (see [8, Remark 3.5]). The optimal regularity for \( b_E \), when \( E \) satisfies (1.6), is given by the following result (see [8, Theorem 1.7]).

Theorem 1.3. Let \( v \) be as in (1.4), and let \( E \) be a \( v \)-distributed set of finite perimeter satisfying (1.6). Then,
\[
b_\delta = 1_{\{v > \delta\}} b_E \in GBV(\mathbb{R}^{n-1}),
\]
for every \( \delta > 0 \) such that \( \{v > \delta\} \) is a set of finite perimeter, where \( 1_{\{v > \delta\}} \) stands for the characteristic function of the set \( \{v > \delta\} \). Moreover, \( b_E \) is approximately differentiable \( \mathcal{H}^{n-1} \)-a.e. on \( \mathbb{R}^{n-1} \), namely the approximate gradient \( \nabla b_E(x) \) (see Section 3) exists for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \mathbb{R}^{n-1} \). Finally, for every Borel set \( G \subset \{v^\prime > 0\} \) the following coarea formula holds:
\[
\int_{\mathbb{R}} \mathcal{H}^{n-2}(G \cap \partial^c \{b_E > t\}) dt = \int_G |\nabla b_E|d\mathcal{H}^{n-1} + \int_{G \cap S_{b_E}} |b_E|d\mathcal{H}^{n-2} + |D^c b_E|^+(G),
\] (1.8)
where \( |D^c b_E|^+ \) is the Borel measure on \( \mathbb{R}^{n-1} \) defined by
\[
|D^c b_E|^+(G) := \lim_{\delta \to 0^+} \sup_{\delta > 0} |D^c b_\delta|(G), \quad \forall G \subset \mathbb{R}^{n-1}.
\]

Here $GBV$ is the space of functions of generalized bounded variation, $v^\vee$ and $v^\wedge$ are the approximate limsup and approximate liminf of $v$ respectively, $[b_E]:=[b_E^+]-[b_E^-]$ is the jump of $b_E$, $S_{b_E}$ is the jump set of $b_E$, and $D^c b_\delta$ is the Cantor part of the distributional derivative $Db_\delta$ of $b_\delta$ (for more details see Section 2). Starting from this result it is possible to establish a formula for the perimeter of $E$ in terms of $v$ and $b_E$ (see [8, Corollary 3.3]). With such formula at hands, as shown in the next result (see [8, Theorem 1.9]), a full characterization of $\mathcal{M}(v)$ can be given. Below, we set $\tau_M(s):=\max\{-M,\min\{M,s\}\}$ for every $s \in \mathbb{R}$, and $M \geq 0$, that is

$$\tau_M(s):=\begin{cases} -M & \text{if } s \leq -M, \\ s & \text{if } -M < s < M, \\ M & \text{if } s \geq M. \end{cases}$$

**Theorem 1.4.** Let $v$ be as in (1.3), and let $E$ be a $v$-distributed set of finite perimeter. Then, $E \in \mathcal{M}(v)$ if and only if

i) $E_z$ is $H^1$-equivalent to a segment; for $H^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$,

ii) $\nabla b_E(z) = 0$, for $H^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$;

iii) $\{b_E\} \leq \{v\}$, $H^{n-2}$-a.e. on $\{v^\wedge > 0\}$;

iv) there exists a Borel function $f : \mathbb{R}^{n-1} \rightarrow [-1/2, 1/2]$ such that

$$D^c(\tau_M(b_\delta))(G) = \int_{G \cap \{v > \delta\} \cap \{\{b_E\} < M\}} f d(D^c v),$$

for every bounded Borel set $G \subset \mathbb{R}^{n-1}$ and $M > 0$, and for $H^1$-a.e. $\delta > 0$. In particular, if $E \in \mathcal{M}(v)$ then

$$2|D^c b_E|^+(G) \leq |D^c v|(G), \quad \text{for every Borel set } G \subset \mathbb{R}^{n-1},$$

and, if $K$ is a concentration set for $D^c v$ and $G$ is a Borel subset of $\{v^\wedge > 0\}$, then

$$\int_{\mathbb{R}} H^{n-2}(G \cap \partial^c \{b_E > t\})dt = \int_{G \cap S_{b_E} \cap S_v} [b_E]dH^{n-2} + |D^c b_E|^+(G \cap K).$$

Theorem 1.3 and Theorem 1.4 play a key role in the study of rigidity. Indeed, (RS) holds true if and only if the following condition is satisfied:

$$E \in \mathcal{M}(v) \iff b_E \text{ is } H^{n-1}\text{-a.e. constant on } \{v > 0\}. \quad (1.15)$$

Based on the previous results, several rigidity results are given in [8], depending of the regularity assumptions on $v$ (see [8, Theorems 1.11-1.30]). In particular, a complete characterization of rigidity is given when $v$ is a special function of bounded variation with locally finite jump set (see [8, Theorem 1.29]).

1.5. **Anisotropic perimeter.** Let us start by recalling some basic notions. A function $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ is said to be 1-homogeneous if

$$\phi(x) = |x|\phi\left(\frac{x}{|x|}\right) \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (1.16)$$

If $\phi$ is 1-homogeneous, then we say that it is coercive if there exists $c > 0$ such that

$$\phi(x) \geq c|x| \quad \forall x \in \mathbb{R}^n. \quad (1.17)$$
In the following, we will assume that
\[ K \subset \mathbb{R}^n \text{ is open, bounded, convex and contains the origin.} \]  
(1.18)

Given \( K \) as in (1.18), one can define a one-homogeneous, convex and coercive function \( \phi_K : \mathbb{R}^n \to [0, \infty) \) in this way:
\[
\phi_K(x) := \sup \{ x \cdot y : y \in K \},
\]  
(1.19)

see Figure 1.1. By homogeneity, convexity of \( \phi_K \) is equivalent to \textit{subadditivity} (see for instance [26, Remark 20.2]), namely
\[
\phi_K(x_1 + x_2) \leq \phi_K(x_1) + \phi_K(x_2), \quad \forall x_1, x_2 \in \mathbb{R}^n.
\]  
(1.20)

Let us notice that there is a one to one correspondence between open, bounded and convex sets \( K \) containing the origin and one-homogeneous, convex and coercive functions \( \phi : \mathbb{R}^n \to [0, \infty) \). Indeed, given a one-homogeneous, convex and coercive function \( \phi : \mathbb{R}^n \to [0, \infty) \), then the set
\[
K = \bigcap_{\omega \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot \omega < \phi(\omega) \},
\]  
(1.21)
satisfies (1.18), where \( \mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \), and
\[
\phi(x) = \sup \{ x \cdot y : y \in K \} = \phi_K(x),
\]
where \( \phi_K \) is given by (1.19). Let \( E \subset \mathbb{R}^n \) be a set of finite perimeter and let \( G \subset \mathbb{R}^n \) be a Borel set. Given \( K \subset \mathbb{R}^n \) as in (1.18), we define the \textit{anisotropic perimeter}, with respect to \( K \), of \( E \) relative to \( G \), as
\[
P_K(E; G) = \int_{\partial^* E \cap G} \phi_K(\nu^E(x)) d\mathcal{H}^{n-1}(x),
\]
and the anisotropic perimeter \( P_K(E) \) of \( E \) as \( P_K(E; \mathbb{R}^n) \). Observe that in the special case \( \phi_K(x) = |x| \), this notion of perimeter agrees with the one of Euclidean perimeter which corresponds to \( K = B(1) \). Note that, in general, \( \phi_K \) is not a norm, unless \( \phi_K(x) = \phi_K(-x) \) for every
$x \in \mathbb{R}^n$.

In the applications, the anisotropic perimeter can be used to describe the surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains \cite{25, 31, 32}, and represents the basic model for surface energies in phase transitions \cite{23}. These applications motivate the study of the the Wulff problem (or anisotropic isoperimetric problem):

$$\inf \left\{ \int_{\partial^*E} \phi_K(\nu^E(x))dH^{n-1}(x) : E \subset \mathbb{R}^n, \mathcal{H}^n(E) = \mathcal{H}^n(K) \right\}. \quad (1.22)$$

This name comes from the Russian crystallographer Wulff, who was the first one to study (1.22) and who first conjectured that $K$ is the unique (modulo translations and scalings) minimizer of (1.22) (see \cite{32}). Indeed the anisotropic perimeter inequality holds true (see for instance \cite{26, Chapter 20}):

$$P_K(K) \leq P_K(E) \quad \text{for every } E \subset \mathbb{R}^n \text{ with } \mathcal{H}^n(E) = \mathcal{H}^n(K), \quad (1.23)$$

with equality if and only if $\mathcal{H}^n(K\Delta(E+x)) = 0$ for some $x \in \mathbb{R}^n$. The proof of the uniqueness was then given by Taylor (see \cite{30}) and later, with a different method, by Fonseca and Müller (see \cite{20}). We usually refer to $K$ as the Wulff shape for the surface tension $\phi_K$.

### 1.6. Steiner’s inequality for the anisotropic perimeter

Note that the analogous of inequality (1.3) for the anisotropic perimeter in general fails. Indeed, choose $K$ as in (1.18) such that

$$\inf_{x \in \mathbb{R}^n} \mathcal{H}^n(K\Delta(K^* + x)) > 0,$$

where $K^*$ denotes the Steiner symmetral of $K$. Then, by uniqueness of the solution for (1.22), we have that

$$P_K(K) < P_K(K^*).$$

The above considerations show that, for an inequality as in (1.3) to hold true in the anisotropic setting, one should at least consider the perimeter $P_{K^*}$ with respect to the Steiner symmetral $K^*$ of $K$.

**Remark 1.5.** Let us observe that since $K$ is a convex set, by properties of Steiner symmetrisation, $K^*$ is convex too. This general property of Steiner symmetrisation can be summed up in the following statement. Let $v$ be as in (1.1), such that $F[v]$ is not a convex set, then every $v$-distributed set $E$ satisfying (1.6) cannot be convex. To prove this, let us consider two points $x, y \in F[v]$ such that the segment joining $x$ and $y$, namely $\overline{xy}$ is not fully contained in $F[v]$. It is not restrictive to make the following assumptions on $x, y$, namely $px \neq py$, $qx > 0$, and also $qy > 0$. Let us call by $x^-$, $y^-$ the two points of $F[v]$ obtained by reflecting $x$, and $y$ with respect to $\{x_n = 0\}$, namely $x^- = (px, -qx)$, $y^- = (py, -qy)$. Let us consider the quadrilateral $Q$ with vertices in $x, y, y^-, x^-$. By symmetry of $F[v]$, and since we assumed that $\overline{xy}$ is not fully contained in $F[v]$, there exists $z \in \overline{pxpy} \setminus \{px, py\}$ such that

$$\mathcal{H}^1(Q_z) > v(\bar{z}), \quad (1.24)$$

where we recall $Q_z$ is defined in (1.1). Let us now consider any $v$-distributed set $E$ satisfying condition (1.7), and let $x_1, y_1, x^-_1, y^-_1$ be the four points obtained from $x, y, x^-, y^-$ in the following way: $x_1 = (px, qx + b_E(px)), y_1 = (py, qr + b_E(py)), x^-_1 = (px, -qx + b_E(px)), and y^-_1 = (py, -qy + b_E(py))$. Observe that by construction $x_1, y_1, x^-_1, y^-_1 \in E$. Let us call $Q^1$ the
quadrilateral with vertexes in those four points. By construction of \(Q, Q^1\), and since Steiner symmetrisation preserves vertical distances, we have that \(H^1(Q_z) = H^1((Q^1)_z)\) for every \(z \in \overline{pxpy}\). In particular, recalling (1.24), we get
\[
H^1((Q^1)_z) > v(\bar{z}).
\]

As a direct consequence of the above inequality, we get that the quadrilateral \(Q^1\), with vertexes in \(E\), is not fully contained in \(E\), and thus \(E\) is not convex. By generality of \(E\), we conclude.

**Remark 1.6.** Let us observe that since \(K^s\) is symmetric with respect to \(\{x_n = 0\}\), then \(\forall x \in \mathbb{R}^n\) we have that \(\phi_{K^s}(px, qx) = \phi_{K^s}(px, -qx)\).

What actually can be proved is the following result (for its proof see [13, Theorem 2.8]).

**Theorem 1.7.** Let \(K \subset \mathbb{R}^n\) be as (1.18), let \(K^s\) be its Steiner symmetrical, and let \(v\) as in (1.4). Then, for every \(E \subset \mathbb{R}^n\) \(v\)-distributed we have
\[
P_{K^s}(E; G \times \mathbb{R}) \geq P_{K^s}(F[v]; G \times \mathbb{R}) \quad \text{for every Borel set } G \subset \mathbb{R}^{n-1}.
\]

**AS**

1.7. **Statement of the main results.** We are now ready to state our main results. Given \(v\) as in (1.4), and \(K \subset \mathbb{R}^n\) satisfying (1.18) we denote by
\[
\mathcal{M}_{K^s}(v) := \{E \subset \mathbb{R}^n : E \text{ is } v\text{-distributed and } P_{K^s}(E) = P_{K^s}(F[v])\},
\]
the family of sets achieving equality in (AS). In this context, we say that *rigidity* holds true for (AS) if the only elements of \(\mathcal{M}_{K^s}(v)\) are vertical translations of \(F[v]\), namely
\[
E \in \mathcal{M}_{K^s}(v) \iff H^n(E \Delta (F[v] + te_n)) = 0 \text{ for some } t \in \mathbb{R}.
\]

As done for the study of (RS), we start by characterizing the set \(\mathcal{M}_{K^s}(v)\). Note that, in the anisotropic setting, the conditions given in Theorem 1.4 do not give a characterization of equality cases of (AS). In particular, let us show with an example in dimension 2, that condition (1.10) fails to be necessary. Let \(K^s, E,\) and \(E^s\) be as in Figure 1.2. Observe that, although \(\forall b_E = b'_E = \tan(\beta) \neq 0\) we have \(P_{K^s}(E) = P_{K^s}(E^s)\), if \(0 < \beta \leq \pi/4\). Indeed, in this case, setting \(h = H^1(AD) = H^1(BC) = H^1(RU) = H^1(ST)\), and \(l = H^1(RS) = H^1(TU)\) we get
\[
P_{K^s}(E) = \phi_{K^s}(v^E_{AB})H^1(AB) + \phi_{K^s}(v^E_{CD})H^1(CD) + \phi_{K^s}(v^E_{AD})h + \phi_{K^s}(v^E_{BC})h
\]
\[
= 2\cos(\beta)H^1(AB) + 2h = 2\cos(\beta)\frac{l}{\cos(\beta)} + 2h = 2l + 2h = P_{K^s}(E^s).
\]

Interestingly, if \(\pi/4 < \beta < \pi/2\) one can see that \(P_{K^s}(E) > P_{K^s}(E^s)\).

We will see that this simple example carries some important features of the general case. In order to characterize \(\mathcal{M}_{K^s}(v)\) we start by proving a formula that allows to calculate \(P_{K}(E)\) in terms of \(b_E\) and \(v\) whenever \(E\) is a \(v\)-distributed set satisfying (1.6) (see Corollary 5.11). After that, we need to carefully study under which conditions equality holds true in (1.20), see Proposition 3.21.

Before stating our results, let us give some definitions. If \(K \subset \mathbb{R}^n\) is as in (1.18), we define the gauge function \(\phi^*_K : \mathbb{R}^n \to [0, \infty)\) as
\[
\phi^*_K(x) := \sup\{x \cdot y : \phi_K(y) < 1\}.
\]

(1.26)
Figure 1.2. Suppose that $0 < \beta \leq \pi/4$. By definition of $\phi_{K^*}$, one can check that the length of the segment in bolt equals $\phi_{K^*}(\nu_{AB}) = \phi_{K^*}(\nu_{CD}) = \cos(\beta)$. As a consequence, we have $P_{K^*}(E) = P_{K^*}(E^*)$, even if $\beta' = \tan \beta \neq 0$.

It turns out that $\phi_{K^*}$ is one-homogeneous, convex and coercive on $\mathbb{R}^n$ (see Proposition 3.4). Let now $x_0 \in \partial K$ and let $\partial \phi_{K^*}(x_0)$ denote the sub-differential of $\phi_{K^*}^*$ at $x_0$ (see Definition 3.8). We define the positive cone generated by $\partial \phi_{K^*}^*(x_0)$, as

$$C_{K^*}^*(x_0) := \{\lambda y : y \in \partial \phi_{K^*}^*(x_0), \lambda \geq 0\},$$

(1.27)

see Figure 1.3. In the following, if $\mu$ is an $\mathbb{R}^n$-valued Radon measure in $\mathbb{R}^{n-1}$, we denote by $|\mu|_K$ the anisotropic total variation (with respect to $K$) of $\mu$, see Definition 3.11.

Figure 1.3. On the left $K^*$ and a pictorial idea of the sub-differential $\partial \phi_{K^*}^*((0,1))$, whereas on the right a pictorial representation of $C_{K^*}^*((0,1))$. 
Our first result gives a complete characterization of $\mathcal{M}_{K^*}(v)$, and can be considered as the anisotropic version of Theorem 1.4. Note that, in particular, this extends [13, Theorem 2.9], where necessary conditions for a set to belong to $\mathcal{M}_{K^*}(v)$ were given.

**Theorem 1.8.** Let $v$ be as in (1.4), let $K \subset \mathbb{R}^n$ satisfy (1.18), and let $E$ be a $v$-distributed set of finite perimeter. Then, $E \in \mathcal{M}_{K^*}(v)$ if and only if

i) $E_x$ is $\mathcal{H}^1$-equivalent to a segment, for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$; 

ii) for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ there exists $z(x) \in \partial K^*$ s.t.

\[
\left\{ \left( -\frac{1}{2} \nabla v(x) + t \nabla b_E(x), 1 \right) : t \in [-1,1] \right\} \subset C_{K^*}^*(z(x));
\]  

(1.28)

iii) for $\mathcal{H}^{n-2}$-a.e. $x \in \{v^\wedge > 0\}$ we have that

\[
2|b_E|(x) \leq |v|(x);
\]  

(1.29)

iv) There exists a Borel function $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that

\[
D^c(\tau_M b_0)(G) = \int_{\mathbb{R}^{n-1} \setminus \{v > \delta\} \cap \{|b_E| < M\}} g(x) d|D^c v/2,0|_{K^*}(x),
\]

for every bounded Borel set $G \subset \mathbb{R}^{n-1}$, every $M > 0$, and $\mathcal{H}^1$-a.e. $\delta > 0$. Moreover, $g$ satisfies the following property: for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$ there exists $z(x) \in \partial K^*$ s.t.

\[
\{(h(x) + tg(x),0) : t \in [-1,1]\} \subset C_{K^*}^*(z(x)),
\]  

(1.30)

where

\[
h(x) := \frac{-dD^c v/2}{d|(D^c v/2,0)|_{K^*}}(x),
\]  

(1.31)

is the derivative of $-D^c v/2$ with respect to the anisotropic total variation $|(D^c v/2,0)|_{K^*}$ in the sense of Radon measures.

**Remark 1.9.** In Figure 1.4 we give a pictorial idea of condition (1.28) for the example of Figure 1.2.

If the first step we did, in order to study the rigidity problem in the anisotropic setting, was the characterization of the set $\mathcal{M}_{K^*}(v)$, the second step consists in the understanding of the relation between the sets $\mathcal{M}_{K^*}(v)$ and $\mathcal{M}(v)$. To get how important this is for our goal, let us observe the following fact. Let $v$ be as in (1.4), let $K \subset \mathbb{R}^n$ satisfy (1.18), and let us assume that (RAS) holds true. Then, $\mathcal{M}_{K^*}(v) \subset \mathcal{M}(v)$. Indeed, consider $E \in \mathcal{M}_{K^*}(v)$, i.e. $E$ is a $v$-distributed set of finite perimeter such that $P_{K^*}(E) = P_{K^*}(F[v])$. By (RAS), we know that $\mathcal{H}^n(E \Delta (F[v] + t e_n)) = 0$ for some $t \in \mathbb{R}$. As a direct consequence we get that $P(E) = P(F[v])$, and so $E \in \mathcal{M}(v)$. So, we have just proved that a necessary condition in order to have rigidity in the anisotropic setting is to require that $\mathcal{M}_{K^*}(v) \subset \mathcal{M}(v)$ holds true. Let us remark that the opposite inclusion, namely $\mathcal{M}(v) \subset \mathcal{M}_{K^*}(v)$ is always verified. This is true because in general, the conditions given in Theorem 1.4 are more stringent than those appearing in Theorem 1.8. So, to sum up all the previous observations we obtained, a necessary condition to require in order to get rigidity of equality cases in the anisotropic setting is that $\mathcal{M}_{K^*}(v) = \mathcal{M}(v)$.

Therefore, to study the rigidity problem in the anisotropic setting, it is crucial to understand when the non trivial inclusion $\mathcal{M}_{K^*}(v) \subset \mathcal{M}(v)$ holds true. To this aim, given $K \subset \mathbb{R}^n$ as in (1.18) and $y \in \mathbb{R}^n$, we set

\[
\mathcal{Z}_K(y) := \{z \in \partial K : y \in C_{K^*}^*(z)\}.
\]  

(1.32)
Note that $\emptyset \neq \mathcal{Z}_K(y) = \mathcal{Z}_K(\lambda y)$ for ever $y \in \mathbb{R}^n$ and for every $\lambda > 0$ (see for instance relation (3.17) in Lemma 3.23). The following two conditions will play an important role in the understanding of rigidity.

**R1:** $\forall y \in \mathbb{R}^n$, for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, and $\forall z \in \mathcal{Z}_K^*\left(\left(-\frac{1}{2}\nabla v(x), 1\right)\right)$,

$$
\left(-\frac{1}{2}\nabla v(x), 1\right) + y, \left(-\frac{1}{2}\nabla v(x), 1\right) - y \in C_{K^*}(z) \implies y = \lambda \left(-\frac{1}{2}\nabla v(x), 1\right),
$$

for some $\lambda \in [-1, 1]$.

**R2:** $\forall y \in \mathbb{R}^n$, for $|D^c v|$-a.e. $x \in \{v^\land > 0\}$, and $\forall z \in \mathcal{Z}_K^*\left((h(x), 0)\right)$,

$$(h(x), 0) + y, (h(x), 0) - y \in C_{K^*}(z) \implies y = \lambda (h(x), 0), \quad \text{for some } \lambda \in [-1, 1],$$

where $h$ has been defined in (1.31). Next result shows the importance of conditions R1 and R2. We anticipate that although conditions R1 and R2 may seem quite complicated, they can be characterized in a simple way in terms of the possible value of the normal vectors to $\partial^* F[v]$ (see Proposition 1.13 and Remark 1.14).

**Theorem 1.10.** Let $v$ be as in (1.4) and let $K \subset \mathbb{R}^n$ be as in (1.18). In addition, let us assume that R1 and R2 hold true. Then, $\mathcal{M}_{K^*}(v) \subset \mathcal{M}(v)$. As an immediate consequence, (RS) and (RAS) are equivalent.

**Remark 1.11.** The above result can be seen as a generalization of [13, Theorem 2.10].

Thanks to Theorem 1.10, all the characterization results for (RS) proved in [8], also hold true in the anisotropic setting, provided conditions R1 and R2 are satisfied. In particular, as a direct consequence of Theorem 1.10, we have the following result.
Theorem 1.12. Let \( v \) be as in (1.4) and let \( K \subset \mathbb{R}^n \) be as in (1.18) such that \( R1 \) and \( R2 \) are satisfied. Then, the following results from [8] hold true, provided rigidity is substituted with (RAS) and \( \mathcal{M}(v) \) is substituted with \( \mathcal{M}_{K^s}(v) \): [8, Theorem 1.11], [8, Theorem 1.13], [8, Theorem 1.16], [8, Theorem 1.20], [8, Theorem 1.29], and [8, Theorem 1.30].

To check whether conditions \( R1, R2 \) hold true might be difficult in general. Thus, using well known concepts of convex analysis such as the definition of extreme point and of exposed point (see Definition 3.30 and Definition 3.29 respectively), we give simple necessary and sufficient conditions for \( R1 \) and \( R2 \) to hold true (see Proposition 1.13, Figure 1.5, and Figure 1.6 below).

Indeed, next proposition shows that \( R1 \) and \( R2 \) can be expressed in a clear geometric way, by comparing the set of normal vectors to \( \partial^* F[v] \) to the set of normal vectors to \( \partial^* K^s \). Roughly speaking, conditions \( R1 \) and \( R2 \) are both satisfied if and only if the first of these two sets is contained in the closure of the second one (see also Corollary 1.17). For the proofs of this and other results about rigidity, we refer to Section 7.

To state our next result, we need another definition. If \( K \subset \mathbb{R}^n \) is as in (1.18) we define the following set:

\[
V_{K^s} := \{ \nu_{K^s}(x) : x \in \partial^* K^s \}. \tag{1.33}
\]

We indicate with \( \overline{V_{K^s}} \) the topological closure of \( V_{K^s} \).

Proposition 1.13. Let \( v \) be as in (1.4) and let \( K \subset \mathbb{R}^n \) be as in (1.18). Then, the following statements are equivalent:

i) conditions \( R1, R2 \) hold true;

ii) \( \exists S \subset \{ v^\wedge > 0 \} \) such that \( \mathcal{H}^{n-1}(S) = |D^c v|(S) = 0 \), and

\[
\nu_{F[v]} \left( z, \frac{1}{2} v(z) \right) \in \overline{V_{K^s}} \quad \forall z \in \{ v^\wedge > 0 \} \setminus S. \tag{1.34}
\]

See Figure 1.5 and Figure 1.6 for a pictorial idea of condition ii) in Proposition 1.13.

Remark 1.14. If \( K^s \) is crystalline (i.e. \( K^s \) is polyhedral) (see Figure 1.5), or in case \( K^s \) has \( C^1 \) boundary, then \( \overline{V_{K^s}} \) is closed and so in (1.34) we can substitute \( \overline{V_{K^s}} \) with \( V_{K^s} \).

Figure 1.5. In this case conditions \( R1 \) and \( R2 \) are satisfied because the set of possible normals to \( \partial^* F[v] \) is a subset of \( V_{K^s} \) (in fact coincides with it). See also Remark 1.14.
Let us stress that asking $K^s$ to be polyhedral does not automatically imply that $R_1$ and $R_2$ hold true. Indeed, by definition, the validity of such conditions depends on both $K^s$, and the function $v$. The importance of the relation between $K^s$ and the function $v$ for the validity of $R_1$ and $R_2$ is made even more explicit in Proposition 1.13 there, it is given an operative characterization of conditions $R_1$ and $R_2$ in terms of the relation that have to occur between the normal vectors to $\partial^* F[v]$ and the normal vectors to $\partial^* K^s$ (see indeed condition ii) in Proposition 1.13). An example of $R_1$ not being satisfied is indeed presented in Figure 1.2, where despite $K^s$ is a polyhedron, it is clear by construction that $\nu v\in V_{K^s}$, and so condition ii) of Proposition 1.13 is not verified, implying that condition $R_1$ fails to be true. Nonetheless, in Figure 1.5, choosing the same $K^s$ as in the previous example, but a different $v$, we show that condition ii) of Proposition 1.13 holds true.

Different conclusions can be made if instead we have that $K^s$ has $C^1$ boundary. In that case, we have the following result.

**Corollary 1.15.** Let $v$ be as in (1.4) and let $K \subset \mathbb{R}^n$ be as in (1.18). In addition, assume that $K^s$ has $C^1$ boundary. Then, conditions $R_1$, $R_2$ hold true.

To conclude this section, we combine the results obtained in Proposition 1.13, and Theorem 1.10 to obtain the following proposition that can be considered the main contribution of the present work.

**Proposition 1.16.** Let $v$ be as in (1.4) and let $K \subset \mathbb{R}^n$ be as in (1.18). Let us assume in addition that there exists $S \subset \{v^\wedge > 0\}$ such that $H^{n-1}(S) = |D^e v|(S) = 0$, and

$$\nu v\left(z, \frac{1}{2}v(z)\right) \in \overline{V_{K^s}} \quad \forall z \in \{v^\wedge > 0\} \setminus S.$$  

Then, $M_{K^s}(v) \subset M(v)$. As a consequence, (RS) and (RAS) are equivalent.

A simplified version of the above result is the following.

**Corollary 1.17.** Let $v$ be as in (1.4) and let $K \subset \mathbb{R}^n$ be as in (1.18). Let us assume in addition that the set of outer unit normal vectors to $\partial^* F[v]$ is contained in the closure of the set of outer...
unit normal vectors to $\partial^*K^s$, namely that

$$\nu_r(x) \in \nabla K^r \quad \forall x \in \partial^*F$$

Then, $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$. As a consequence, $(RS)$ and $(RAS)$ are equivalent.

Finally, we combine Theorem 1.10 and Corollary 1.15 to obtain the following result.

**Corollary 1.18.** Let $v$ be as in (1.14) and let $K \subset \mathbb{R}^n$ be as in (1.18). Let us assume in addition that $K^s$ has $C^1$ boundary. Then, $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$. As a consequence, $(RS)$ and $(RAS)$ are equivalent.

It would be actually interesting checking whether conditions $R1$ and $R2$ are also necessary in order to get $\mathcal{M}_{K^s}(v) \subset \mathcal{M}(v)$. This seems quite a delicate problem, which we think is worth further investigation.

## 2. Basic notions of Geometric Measure Theory

The aim of this section is to introduce some tools from Geometric Measure Theory that will be largely used in the article. For more details the reader can have a look in the monographs [2] and [23]. Note that even if part of the notations we will use, has been already presented across the Introduction, we briefly restate it in the next lines, in such a way that the reader can easily access to them. For $n \in \mathbb{N}$, we denote with $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, i.e.

$$\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \},$$

and we set $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{ 0 \}$. For every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define $px = (x_1, \ldots, x_{n-1})$, and $q x = x_n$ are the "horizontal" and "vertical" projections respectively, so that $x = (px, qx)$. We denote by $e_1, \ldots, e_n$ the canonical basis in $\mathbb{R}^n$, and for every $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the standard scalar product in $\mathbb{R}^n$ between $x$ and $y$. For every $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B(x, r)$ the open ball of $\mathbb{R}^n$ with radius $r$ centred at $x$. In the special case $x = 0$, we set $B(r) := B(0, r)$. For every $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the standard scalar product in $\mathbb{R}^n$ between $x$ and $y$. We denote the $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$ of center $z \in \mathbb{R}^{n-1}$ and radius $r > 0$ as

$$D_{z, r} = \{ \eta \in \mathbb{R}^{n-1} : |\eta - z| < r \}.$$  

For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, we will denote by $H^+_x\nu$ and $H^-_x\nu$ the closed half-spaces whose boundaries are orthogonal to $\nu$:

$$H^+_x\nu := \{ y \in \mathbb{R}^n : (y - x) \cdot \nu \geq 0 \}, \quad H^-_x\nu := \{ y \in \mathbb{R}^n : (y - x) \cdot \nu \leq 0 \}.$$  

If $1 \leq k \leq n$, we denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\{ E_h \}_{h \in \mathbb{N}}$ is a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ with finite volume, and $E \subset \mathbb{R}^n$ is also measurable with finite volume, we say that $\{ E_h \}_{h \in \mathbb{N}}$ converges to $E$ as $h \to \infty$, and write $E_h \to E$, if $\mathcal{H}^n(E_h \Delta E) \to 0$ as $h \to \infty$. In the following, we will denote by $\chi_E$ the characteristic function of a measurable set $E \subset \mathbb{R}^n$.

### 2.1. Density points

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $x \in \mathbb{R}^n$. The upper and lower $n$-dimensional densities of $E$ at $x$ are defined as

$$\theta^*(E, x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, r))}{\omega_n r^n}, \quad \theta_*(E, x) := \liminf_{r \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, r))}{\omega_n r^n},$$
Remark 2.1. Then, we define a

\[ \theta(E, x) := \lim_{r \to 0^+} \frac{\mathcal{H}^n(E \cap B(x, r))}{\omega_n r^n}, \]

is defined for \( \mathcal{H}^n \)-a.e. \( x \in \mathbb{R}^n \), and \( x \mapsto \theta(E, x) \) is a Borel function on \( \mathbb{R}^n \). Given \( t \in [0, 1] \), we set

\[ E^{(t)} := \{ x \in \mathbb{R}^n : \theta(E, x) = t \}. \]

By the Lebesgue differentiation theorem, the pair \( \{E^{(0)}, E^{(1)}\} \) is a partition of \( \mathbb{R}^n \), up to a \( \mathcal{H}^n \)-negligible set. The set \( \partial^* E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}) \) is called the essential boundary of \( E \).

2.2. Rectifiable sets and sets of finite perimeter. Let \( 1 \leq k \leq n \), \( k \in \mathbb{N} \). If \( A, B \subset \mathbb{R}^n \) are Borel sets we say that \( A \subset_{\mathcal{H}^k} B \) if \( \mathcal{H}^k(B \setminus A) = 0 \), and \( A =_{\mathcal{H}^k} B \) if \( \mathcal{H}^k(A \Delta B) = 0 \), where \( \Delta \) denotes the symmetric difference of sets. Let \( M \subset \mathbb{R}^n \) be a Borel set. We say that \( M \) is countably \( \mathcal{H}^k \)-rectifiable if there exist Lipschitz functions \( f_h : \mathbb{R}^k \to \mathbb{R}^n \) (\( h \in \mathbb{N} \)) such that \( M \subset_{\mathcal{H}^k} \bigcup_{h \in \mathbb{N}} f_h(\mathbb{R}^k) \). Moreover, we say that \( M \) is locally \( \mathcal{H}^k \)-rectifiable if it is countably \( \mathcal{H}^k \)-rectifiable and \( \mathcal{H}^k(M \cap K) < \infty \) for every compact set \( K \subset \mathbb{R}^n \), or, equivalently, if \( \mathcal{H}^k,M \) is a Radon measure on \( \mathbb{R}^n \). Given a \( \mathbb{R}^m \)-valued Radon measure \( \mu \) on \( \mathbb{R}^n \), we define its total variation \( |\mu| \) as

\[ |\mu|(\Omega) = \sup \left\{ \int_{\mathbb{R}^n} \varphi(x) \cdot d\mu(x) : \varphi \in C_c^\infty(\Omega; \mathbb{R}^m), \ |\varphi| \leq 1 \right\}, \quad \forall \Omega \subset \mathbb{R}^n \text{ open.} \quad (2.3) \]

If we consider a generic Borel set \( B \subset \mathbb{R}^n \) then

\[ |\mu|(B) = \inf \{|\mu|(\Omega) : B \subset \Omega, \ \Omega \subset \mathbb{R}^n \text{ open set}\}. \]

Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \), let \( 1 \leq p < \infty \) and \( m \geq 1 \) with \( m \in \mathbb{N} \). The vector space \( L^p(\mathbb{R}^n, \mu; \mathbb{R}^m) \) is defined as

\[ L^p(\mathbb{R}^n, \mu; \mathbb{R}^m) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^m : f \text{ is } \mu\text{-measurable}, \ \int_{\mathbb{R}^n} |f|^p d\mu < \infty \right\}, \]

equipped with the norm

\[ ||f||_{L^p(\mathbb{R}^n, \mu; \mathbb{R}^m)} = \left( \int_{\mathbb{R}^n} |f|^p d\mu \right)^{\frac{1}{p}}. \]

If \( p = \infty \) then \( L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m) \) is defined as

\[ L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^m : f \text{ is } \mu\text{-measurable}, \ \sup_{\mathbb{R}^n} f < \infty \right\}, \]

where

\[ \sup_{\mathbb{R}^n} f := \inf \{ c > 0 \ : \ \mu(\{|f| > c\}) = 0 \}. \]

We equip this space with the norm

\[ ||f||_{L^\infty(\mathbb{R}^n, \mu; \mathbb{R}^m)} = \sup_{\mathbb{R}^n} f. \]

We say that \( f \in L^p_{\text{loc}}(\mathbb{R}^n, \mu; \mathbb{R}^m), 1 \leq p \leq \infty \) if \( f \in L^p(C, \mu; \mathbb{R}^m) \) for every compact set \( C \subset \mathbb{R}^n \).

Remark 2.1. Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \) and let \( f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu; \mathbb{R}^m) \) with \( m \geq 1 \), \( m \in \mathbb{N} \). Then, we define a \( \mathbb{R}^m \)-valued Radon measure on \( \mathbb{R}^n \) by setting

\[ f \mu(B) = \int_B f(x) \, d\mu(x) \quad \forall \text{ Borel set } B \subset \mathbb{R}^n. \]
Its total variation is then defined as
\[ |f\mu|(B) = \int_B |f(x)|d\mu(x) \quad \forall \text{ Borel set } B \subset \mathbb{R}^n. \]

For more details see [26, Example 4.6, Remark 4.8].

A Lebesgue measurable set \( E \subset \mathbb{R}^n \) is said of \textit{locally finite perimeter} in \( \mathbb{R}^n \) if there exists a \( \mathbb{R}^n \)-valued Radon measure \( \mu_E \), called the Gauss–Green measure of \( E \), such that
\[ \int_E \nabla \varphi(x) \, dx = \int_{\mathbb{R}^n} \varphi(x) \, d\mu_E(x), \quad \forall \varphi \in C^1_c(\mathbb{R}^n), \]
where \( C^1_c(\mathbb{R}^n) \) denotes the class of \( C^1 \) functions in \( \mathbb{R}^n \) with compact support. The relative perimeter of \( E \) in \( A \subset \mathbb{R}^n \) is then defined by setting \( P(E; A) := |\mu_E|(A) \) for any Borel set \( A \subset \mathbb{R}^n \). The perimeter of \( E \) is then defined as \( P(E) := P(E; \mathbb{R}^n) \). If \( P(E) < \infty \), we say that \( E \) is a set of \textit{finite perimeter} in \( \mathbb{R}^n \). The \textit{reduced boundary} of \( E \) is the set \( \partial^* E \) of those \( x \in \mathbb{R}^n \) such that
\[ \nu^E(x) = \frac{d\mu_E}{d|\mu_E|}(x) = \lim_{r \to 0^+} \frac{\mu_E(B(x, r))}{\mu_E(\{ f > t \})} \]
exists and belongs to \( S^{n-1} \),
where \( \frac{d\mu_E}{d|\mu_E|} \) indicates the derivative of \( \mu_E \) with respect its total variation \( |\mu_E| \) in the sense of Radon measure. The Borel function \( \nu^E : \partial^* E \to S^{n-1} \) is called the \textit{measure-theoretic outer unit normal} to \( E \). If \( E \) is a set of locally finite perimeter, it is possible to show that \( \partial^* E \) is a locally \( \mathcal{H}^{n-1} \)-rectifiable set in \( \mathbb{R}^n \) [26, Corollary 16.1], with \( \mu_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E \), and
\[ \int_E \nabla \varphi(x) \, dx = \int_{\partial^* E} \varphi(x) \nu^E(x) \, d\mathcal{H}^{n-1}(x), \quad \forall \varphi \in C^1_c(\mathbb{R}^n). \]

Thus, \( P(E; A) = \mathcal{H}^{n-1}(A \cap \partial^* E) \) for every Borel set \( A \subset \mathbb{R}^n \). If \( E \) is a set of locally finite perimeter, it turns out that
\[ \partial^* E \subset E^{(1/2)} \subset \partial E. \]
Moreover, \textit{Federer’s theorem} holds true (see [24 Theorem 3.61] and [26 Theorem 16.2]):
\[ \mathcal{H}^{n-1}(\partial^* E \setminus \partial^* E) = 0, \]
thus implying that the essential boundary \( \partial^* E \) of \( E \) is locally \( \mathcal{H}^{n-1} \)-rectifiable in \( \mathbb{R}^n \).

2.3. \textbf{General facts about measurable functions.} Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Lebesgue measurable function. We define the \textit{approximate upper limit} \( f^\vee(x) \) and the \textit{approximate lower limit} \( f^\wedge(x) \) of \( f \) at \( x \in \mathbb{R}^n \) as
\begin{align*}
f^\vee(x) &= \inf \left\{ t \in \mathbb{R} : x \in \{ f > t \}^{(0)} \right\}, \quad \text{(2.4)} \\
f^\wedge(x) &= \sup \left\{ t \in \mathbb{R} : x \in \{ f < t \}^{(0)} \right\}. \quad \text{(2.5)}
\end{align*}
We observe that \( f^\vee \) and \( f^\wedge \) are Borel functions that are defined at \textit{every} point of \( \mathbb{R}^n \), with values in \( \mathbb{R} \cup \{ \pm \infty \} \). Moreover, if \( f_1 : \mathbb{R}^n \to \mathbb{R} \) and \( f_2 : \mathbb{R}^n \to \mathbb{R} \) are measurable functions satisfying \( f_1 = f_2 \mathcal{H}^n \text{-a.e. on } \mathbb{R}^n \), then \( f_1^\vee = f_2^\vee \) and \( f_1^\wedge = f_2^\wedge \) everywhere on \( \mathbb{R}^n \). We define the \textit{approximate discontinuity set} \( S_f \) of \( f \) as
\[ S_f := \{ f^\wedge < f^\vee \}. \quad \text{(2.6)} \]
Note that, by the above considerations, it follows that \( \mathcal{H}^n(S_f) = 0 \). Although \( f^\wedge \) and \( f^\vee \) may take infinite values on \( S_f \), the difference \( f^\vee(x) - f^\wedge(x) \) is well defined in \( \mathbb{R} \cup \{ \pm \infty \} \) for every
The validity of the limit relation (2.7) can be easily checked noticing that

\[ f^\vee(x) - f^\wedge(x), \quad \text{if } x \in S_f, \]
\[ 0, \quad \text{if } x \in \mathbb{R}^n \setminus S_f. \]

The approximate average of \( f \) is the Borel function

\[ \bar{f}(x) = \frac{f^\vee(x) + f^\wedge(x)}{2}, \quad \text{if } x \in \mathbb{R}^n \setminus \{f^\wedge = -\infty, f^\vee = +\infty\}, \]
\[ 0, \quad \text{if } x \in \{f^\wedge = -\infty, f^\vee = +\infty\}. \]

It also holds the following limit relation

\[ \bar{f}(x) = \lim_{M \to \infty} \tau_M f(x) = \lim_{M \to \infty} \frac{\tau_M(f^\vee) + \tau_M(f^\wedge)}{2}, \quad \forall x \in \mathbb{R}^n, \quad (2.7) \]

that we want to be true for every Lebesgue measurable function \( f : \mathbb{R}^n \to \mathbb{R} \), where, here and in the rest of the work,

\[ \tau_M(s) = \max\{-M, \min\{M, s\}\}, \quad s \in \mathbb{R} \cup \{\pm\infty\}. \quad (2.8) \]

By definition, \( \tau_M \) is equivalently defined as

\[ \tau_M(s) = \begin{cases} M & s > M \\
 s & -M \leq s \leq M \\
 -M & s < -M \end{cases} \]

and the following properties can be easily proved

\[ \tau_M(s_2) \geq \tau_M(s_1) \quad \forall s_2 \geq s_1, \text{ provided } M > 0. \quad (2.9) \]
\[ \tau_M(s_2) \geq \tau_M(s_1) \quad \forall M_2 \geq M_1 \geq 0, \text{ provided } s \geq 0. \quad (2.10) \]
\[ \tau_M(s_2) \leq \tau_M(s_1) \quad \forall M_2 \geq M_1 \geq 0, \text{ provided } s \leq 0. \quad (2.11) \]
\[ \tau_{M_2}(s_2) - \tau_{M_1}(s_1) \geq (\tau_{M_2} - \tau_{M_1})(s_1) \quad \forall s_2 \geq s_1, \text{ provided } M_2 \geq M_1 \geq 0. \quad (2.12) \]
\[ \tau_{M_2}(s_2) - \tau_{M_1}(s_1) \geq \tau_{M_1}(s_2) - \tau_{M_1}(s_1) \quad \forall M_2 \geq M_1 \geq 0, \text{ provided } s_2 \geq s_1. \quad (2.13) \]

The validity of the limit relation (2.7) can be easily checked noticing that

\[ \tau_M(f)^\wedge = \tau_M(f^\wedge), \quad \tau_M(f)^\vee = \tau_M(f^\vee), \quad \bar{\tau_M(f)}(x) = \frac{\tau_M(f^\vee) + \tau_M(f^\wedge)}{2}, \quad \forall x \in \mathbb{R}^n. \]

Using these above definitions, the validity of the following properties can be easily deduced. For every Lebesgue measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) and for every \( t \in \mathbb{R} \) we have that

\[ \{f^\wedge < t\} = \{-t < f^\vee\} \cap \{f^\vee < t\}, \quad (2.14) \]
\[ \{f^\vee < t\} \subset \{f < t\}^{(1)} \subset \{f^\vee \leq t\}, \quad (2.15) \]
\[ \{f^\wedge > t\} \subset \{f > t\}^{(1)} \subset \{f^\wedge \geq t\}. \quad (2.16) \]

Furthermore, if \( f, g : \mathbb{R}^n \to \mathbb{R} \) are Lebesgue measurable functions and \( f = g \mathcal{H}^n\text{-a.e. on a Borel set } E \), then

\[ f^\vee(x) = g^\vee(x), \quad f^\wedge(x) = g^\wedge(x), \quad [f](x) = [g](x), \quad \forall x \in E^{(1)}. \quad (2.17) \]
Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set. We say that $t \in \mathbb{R} \cup \{\pm \infty\}$ is the approximate limit of $f$ at $x$ with respect to $A$, and write $t = \text{aplim}(f, A, x)$, if

\[
\theta\left(\{|f - t| > \varepsilon\} \cap A; x\right) = 0, \quad \forall \varepsilon > 0, \quad (t \in \mathbb{R}), \tag{2.18}
\]

\[
\theta\left(\{f < M\} \cap A; x\right) = 0, \quad \forall M > 0, \quad (t = +\infty), \tag{2.19}
\]

\[
\theta\left(\{f > -M\} \cap A; x\right) = 0, \quad \forall M > 0, \quad (t = -\infty). \tag{2.20}
\]

We say that $x \in S_f$ is a jump point of $f$ if there exists $\nu \in \mathbb{S}^{n-1}$ such that

\[
f^\vee(x) = \text{aplim}(f, H^+_x, x) > f^\wedge(x) = \text{aplim}(f, H^-_x, x).
\]

If this is the case, we say that $\nu f(x) := \nu$ is the approximate jump direction of $f$ at $x$. If we denote by $J_f$ the set of approximate jump points of $f$, we have that $J_f \subset S_f$ and $\nu f : J_f \rightarrow \mathbb{S}^{n-1}$ is a Borel function.

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ Lebesgue measurable, then we say that $f$ is approximately differentiable at $x \in S_f^c$ provided $f^\wedge(x) = f^\vee(x) \in \mathbb{R}$ if there exists $\xi \in \mathbb{R}^n$ such that

\[
\text{aplim}(g, \mathbb{R}^n, x) = 0,
\]

where $g(y) = (f(y) - \tilde{f}(x) - \xi \cdot (y - x))/|y - x|$ for $y \in \mathbb{R}^n \setminus \{x\}$. If this is the case, then $\xi$ is uniquely determined, we set $\xi = \nabla f(x)$, and call $\nabla f(x)$ the approximate differential of $f$ at $x$. The localization property \([2.17]\) holds true also for the approximate differentials, namely if $g, f : \mathbb{R}^n \rightarrow \mathbb{R}$ are Lebesgue measurable functions, $f = g \mathcal{H}^n$-a.e. on a Borel set $E$, and $f$ is approximately differentiable $\mathcal{H}^n$-a.e. on $E$, then so it is $g \mathcal{H}^n$-a.e. on $E$ with

\[
\nabla f(x) = \nabla g(x), \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{2.21}
\]

### 2.4. Functions of bounded variation

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lebesgue measurable function, and let $\Omega \subset \mathbb{R}^n$ be open. We define the \textit{total variation} of $f$ in $\Omega$ as

\[
|Df|(\Omega) = \sup \left\{ \int_{\Omega} f(x) \text{div} T(x) \, dx : T \in C_0^1(\Omega; \mathbb{R}^n), |T| \leq 1 \right\},
\]

where $C_c^1(\Omega; \mathbb{R}^n)$ is the set of $C^1$ functions from $\Omega$ to $\mathbb{R}^n$ with compact support. We also denote by $C_0^1(\Omega; \mathbb{R}^n)$ the class of all continuous functions from $\Omega$ to $\mathbb{R}^n$. Analogously, for any $k \in \mathbb{N}$, the class of $k$ times continuously differentiable functions from $\Omega$ to $\mathbb{R}^n$ is denoted by $C_k^0(\Omega; \mathbb{R}^n)$. We say that $f$ belongs to the space of functions of bounded variations, $f \in BV(\Omega)$, if $|Df|(\Omega) < \infty$ and $f \in L^1(\Omega)$. Moreover, we say that $f \in BV_{\text{loc}}(\Omega)$ if $f \in BV(\Omega')$ for every open set $\Omega'$ compactly contained in $\Omega$. Therefore, if $f \in BV_{\text{loc}}(\mathbb{R}^n)$ the distributional derivative $Df$ of $f$ is an $\mathbb{R}^n$-valued Radon measure. In particular, $E$ is a set of locally finite perimeter if and only if $\chi_E \in BV_{\text{loc}}(\mathbb{R}^n)$. If $f \in BV_{\text{loc}}(\mathbb{R}^n)$, one can write the Radon–Nykodim decomposition of $Df$ with respect to $\mathcal{H}^n$ as $Df = D^a f + D^s f$, where $D^a f$ and $\mathcal{H}^n$ are mutually singular, and where $D^a f \ll \mathcal{H}^n$. We denote the density of $D^a f$ with respect to $\mathcal{H}^n$ by $\nabla f$, so that $\nabla f \in L^1(\Omega; \mathbb{R}^n)$ with $D^a f = \nabla f \, d\mathcal{H}^n$. Moreover, for a.e. $x \in \mathbb{R}^n$, $\nabla f(x)$ is the approximate differential of $f$ at $x$. If $f \in BV_{\text{loc}}(\mathbb{R}^n)$, then $S_f$ is countably $\mathcal{H}^{n-1}$-rectifiable. Moreover, we have $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$, $[f] \in L_\text{loc}^1(\mathcal{H}^{n-1} \setminus J_f)$, and the $\mathbb{R}^n$-valued Radon measure $D^j f$ defined as

\[
D^j f = [f] \nu f \, d\mathcal{H}^{n-1} \setminus J_f,
\]

is called the \textit{jump part} of $Df$. If we set $D^e f = D^a f - D^j f$, we have that $Df = D^a f + D^j f + D^e f$. The $\mathbb{R}^n$-valued Radon measure $D^e f$ is called the \textit{Cantorian part} of $Df$, and it is such that $|D^e f|(M) = 0$ for every $M \subset \mathbb{R}^n$ which is $\sigma$-finite with respect to $\mathcal{H}^{n-1}$. Let us recall some
Lemma 2.3. If \( f \in BV(\mathbb{R}^n) \), then \( |D^c f|(\{v^a = 0\}) = 0 \). In particular, if \( f = g \mathcal{H}^n\)-a.e. on a Borel set \( E \subset \mathbb{R}^n \), then \( D^c f \ll E(1) = D^c g \ll E(1) \).

Lemma 2.2. Details.

Theorem 3.1. Let \( f \underset{\mathcal{H}}{\rightharpoonup} g \) be a set of finite perimeter in \( \mathbb{R}^n \) and \( E \subset \mathbb{R}^n \) be a Borel set. Then, \( \nabla f = 1_E \nabla g \), \( \mathcal{H}^n\)-a.e. on \( \mathbb{R}^n \),

\[ D^c f = D^c g \ll E(1), \]

\[ S_f \cap E(1) = S_g \cap E(1). \]

A Lebesgue measurable function \( f : \mathbb{R}^n \to \mathbb{R} \), it’s called of generalized bounded variation on \( \mathbb{R}^n \), shortly \( f \in GBV(\mathbb{R}^n) \) if and only if \( \tau_M(u) \in BV_{loc}(\mathbb{R}^{n-1}) \) for every \( M > 0 \) (where \( \tau_M(s) \) has been defined in the previous subsection). It is interesting to notice that the structure theory of BV-functions holds true for GBV-functions too. Indeed, given \( f \in GBV(\mathbb{R}^n) \), then, (see [2 Theorem 4.34]) \( \{ f > t \} \) is a set of finite perimeter too for \( \mathcal{H}^1\)-a.e. \( t \in \mathbb{R} \), \( f \) is approximately differentiable \( \mathcal{H}^n\)-a.e. on \( \mathbb{R}^n \), \( S_f \) is countably \( \mathcal{H}^{n-1}\)-rectifiable and \( \mathcal{H}^{n-1}\)-equivalent to \( J_f \) and the usual coarea formula takes the form

\[ \int_{\mathbb{R}} P(\{ f > t \}; G) dt = \int_{\mathbb{R}} |\nabla f| d\mathcal{H}^n + \int_{G \cap S_f} |f| d\mathcal{H}^n - |D^c f|(G), \]

for every Borel set \( G \subset \mathbb{R}^n \), where \( |D^c f| \) denotes the Borel measure on \( \mathbb{R}^n \) defined as

\[ |D^c f|(G) = \lim_{M \to +\infty} |D^c(\tau_M(f))|(G) = \sup_{M > 0} |D^c(\tau_M(f))|(G), \]

whenever \( G \) is a Borel set in \( \mathbb{R}^n \).

3. Setting of the problems and preliminary results

We recall in here, few results that will be useful later on for the proof of [AS] (for more details see [11 Section 2 and 3]). Let us start with a version of a result by Vol’pert (see [11 Theorem G]).

Theorem 3.1. Let \( v \in BV(\mathbb{R}^{n-1}) \) such that \( \mathcal{H}^{n-1}(\{ v > 0 \}) < \infty \). Let \( E \subset \mathbb{R}^n \) be a \( v \)-distributed set of finite perimeter. Then, we have for \( L^{n-1}\)-a.e. \( z \in \mathbb{R}^{n-1} \),

\[ E_z \text{ has finite perimeter in } \mathbb{R}; \]

\[ (\partial^c E)_z = (\partial^c E)_z \cap (\partial^c E)_z = \partial^c (E_z); \]

\[ q(\nu^{E}(z,t)) \neq 0 \text{ for every } t \text{ such that } (z,t) \in \partial^c E; \]

In particular, there exists a Borel set \( G_E \subset \{ v > 0 \} \) such that \( L^{n-1}(\{ v > 0 \} \setminus G_E) = 0 \) and \( (3.1), (3.2), (3.3) \) are satisfied for every \( z \in G_E \).

The next result is a version of the Coarea formula for rectifiable sets (see [11 Theorem F]).

Theorem 3.2. Let \( E \) be a set of finite perimeter in \( \mathbb{R}^n \) and let \( g : \mathbb{R}^n \to [0, +\infty] \) be any Borel function. Then,

\[ \int_{\partial^c E} g(x) q(\nu^{E}(x)) d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dz \int_{(\partial^c E)_z} g(z,y) d\mathcal{H}^0(y). \]

Lastly, next result is a version of [11 Lemma 3.2].
Lemma 3.3. Let $v \in BV(\mathbb{R}^{n-1})$ such that $\mathcal{H}^{n-1}(\{v > 0\}) < \infty$. Let $E \subset \mathbb{R}^n$ be a $v$-distributed set of finite perimeter. Then, for $\mathcal{L}^{n-1}$-a.e. $z \in \{v > 0\}$

$$\frac{\partial v}{\partial x_i}(z) = -\int_{(\partial^* E)_z} \frac{\nu^E_i(z,y)}{|q(\nu^E_i(z,y))|} d\mathcal{H}^0(y), \quad i = 1, \ldots, n - 1,$$

In particular by (3.2) and the above relation, we get for $\mathcal{L}^{n-1}$-a.e. $z \in \{v > 0\}$

$$\frac{1}{2} \frac{\partial v}{\partial x_i}(z) = -\frac{\nu^E_i[v](z,y)}{|q(\nu^E_i[v](z,y))|} d\mathcal{H}^0(y), \quad i = 1, \ldots, n - 1, y \in (\partial^* F[z])_z.$$

3.1. Properties of the surface tension $\phi_K$. Let us start recalling some basic facts about the surface tension $\phi_K$. First of all, let us sum up some known properties of the gauge function in the following result, that can be easily deduced from [20 Proposition 20.10].

Proposition 3.4. Consider $K \subset \mathbb{R}^n$ as in (1.18). Consider $\phi_K, \phi^*_K : \mathbb{R}^n \to [0, \infty)$ the corresponding surface tension and gauge function defined in (1.19), (1.26) respectively. Then the following properties hold true.

i) The function $\phi^*_K$ is one-homogeneous, convex and coercive on $\mathbb{R}^n$ and there exist positive constants $c$ and $C$ such that

$$c|x| \leq \phi_K(x) \leq C|x|, \quad \forall x \in \mathbb{R}^n,$$

$$\frac{|x|}{C} \leq \phi^*_K(x) \leq \frac{|x|}{c}, \quad \forall x \in \mathbb{R}^n.$$

ii) The so called Fenchel inequality holds true i.e.

$$x \cdot y \leq \phi^*_K(x) \phi_K(y), \quad \forall x, y \in \mathbb{R}^n. \quad (3.5)$$

iii) The gauge function $\phi^*_K$ provides a new characterization for the Wulff shape $K$ i.e.

$$K = \{x \in \mathbb{R}^n : \phi^*_K(x) < 1\},$$

from which we can immediately derive that

$$\phi_K(x) = \sup \{x \cdot y : \phi^*_K(x) < 1\},$$

$$\phi_K(x) = (\phi^*_K)^*(x).$$

iv) If $x \in \partial^* K$ and $y \in S^{n-1}$, then equality holds in (3.3) if and only if $y = \nu_K(x)$; in particular

$$P_K(K) = n|K|. \quad (3.6)$$

Remark 3.5. By (i) of Proposition 3.4 we have that $E$ is a set of locally finite perimeter if and only if $E$ is a set of locally finite anisotropic perimeter i.e. $P_K(E; C) < \infty$ for every $C \subset \mathbb{R}^n$ compact set.

Remark 3.6. Thanks to iii) of the above proposition we have

$$K^* = \{x \in \mathbb{R}^n : \phi_K(x) < 1\},$$

from which together with (1.26) gives

$$\phi^*_K(x) = \sup \{x \cdot y : y \in K^*\} \quad \forall x \in \mathbb{R}^n.$$
For a pictorial idea of $K$ and $K^*$ see for instance Figure 3.1. Furthermore, observe that
\begin{align*}
\phi_K(x) &= 1 \quad \forall x \in \partial K^*, \quad (3.7) \\
\phi_K(x) &= 1 \quad \forall x \in \partial K. \quad (3.8)
\end{align*}

**Remark 3.7.** Let us consider $K \subset \mathbb{R}^n$ as in (1.18). According to Proposition 3.4, iii) another way to define the Wulff shape $K$ is
\[
K := p \left( \Sigma_{\phi_K^*} \cap \{ x_{n+1} = 1 \} \right),
\]
where $\Sigma_{\phi_K^*}$ is the epigraph of $\phi_K^*$ in $\mathbb{R}^{n+1}$ and $p: \mathbb{R}^{n+1} \to \mathbb{R}^n$ corresponds to the horizontal projection. By the one-homogeneity of $\phi_K$ we get that
\[
\phi_K(tx) = t|x|\phi_K \left( \frac{tx}{t|x|} \right) = t\phi_K(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \forall t > 0. \quad (3.9)
\]
By (3.9), we get for every constant $\lambda > 0$ that
\[
\lambda K := p \left( \Sigma_{\phi_K^*} \cap \{ x_{n+1} = \lambda \} \right).
\]
Another thing we would like to observe is that given $x, y \in \mathbb{R}^n$ with $x \in \lambda K$ and $y \in (\lambda K)^c$, (for some $\lambda > 0$) then $\phi_K^*(x) < \phi_K^*(y)$. Naturally, these considerations hold true for $K^*$ and $\phi_K$ too.

**Definition 3.8 (Sub-differential).** Let $\varphi: \mathbb{R}^n \to [0, \infty]$ be a convex function. Let us fix $x_0 \in \mathbb{R}^n$ and consider all vectors $y_0 \in \mathbb{R}^n$ such that
\[
\varphi(z) \geq \phi(x_0) + y_0 \cdot (z - x_0) \quad \forall z \in \mathbb{R}^n. \quad (3.10)
\]
The set of all vectors $y_0$ satisfying the above property is called sub-differential of $\varphi$ at $x_0$ and we indicate it by $\partial \varphi(x_0)$.

Keeping in mind Definition 1.27 we have the following remarks.

**Remark 3.9.** For every $x_0 \in \mathbb{R}^n$, the sub-differential $\partial \phi(x_0)$ is a closed and convex set of $\mathbb{R}^n$ (see [27] chapter 5). From this, it can be proved that, given $x \in \partial K$, also $C_K^*(x)$ is a convex set of $\mathbb{R}^n$, where $C_K^*(x)$ is defined as in (1.27).
Remark 3.10. Let \( \phi : \mathbb{R}^n \to [0, \infty] \) be a convex function. It is a well known result about convex functions that, \( \phi \) is differentiable in \( x_0 \in \mathbb{R}^n \) if and only if \( \partial \phi(x_0) \) consists of only one element. In that situation, we call \( \nabla \phi(x_0) \) is the only element in the sub-differential \( \partial \phi(x_0) \).

Definition 3.11. Fix an integer \( m \geq 1 \) and let \( K \subset \mathbb{R}^n \) be as in \((1.18)\). Given a \( \mathbb{R}^n \)-valued Radon measure \( \mu \) on \( \mathbb{R}^m \) and a generic Borel set \( F \subset \mathbb{R}^m \), we define the \( \phi_K \)-anisotropic total variation of \( \mu \) on \( F \) as

\[
|\mu|_K(F) = \int_F \phi_K \left( \frac{d\mu}{d|\mu|}(x) \right) d|\mu|(x).
\]

Remark 3.12. By condition i) in Proposition \( 3.4 \) we have that

\[
|\mu|_K(F) = \int_F \phi_K \left( \frac{d\mu}{d|\mu|}(x) \right) d|\mu|(x) \leq C \int_F d|\mu|(x) = C|\mu|(F).
\]

Analogously,

\[
|\mu|(F) = \int_F d|\mu|(x) \leq \frac{1}{c} \int_F \phi_K \left( \frac{d\mu}{d|\mu|}(x) \right) d|\mu|(x) = \frac{1}{c}|\mu|_K(F).
\]

Thus, \( |\mu|_K \ll |\mu| \) and \( |\mu| \ll |\mu|_K \).

Remark 3.13. Given \( f \in GBV(\mathbb{R}^{n-1}) \), motivated by \((2.25)\), for every Borel set \( G \subset \mathbb{R}^{n-1} \) we define

\[
|(D^c f, 0)|_K(G) = \lim_{M \to +\infty} |(D^c(\tau_M(f), 0))|_K(G) = \sup_{M > 0} |(D^c(\tau_M(f), 0))|(G).
\] (3.11)

The following Lemma is the anisotropic version of [2] Definition 1.4 (b).

Lemma 3.14. Fix an integer \( m \geq 1 \) and let \( K \subset \mathbb{R}^n \) be as in \((1.18)\). Given a \( \mathbb{R}^n \)-valued Radon measure \( \mu \) on \( \mathbb{R}^m \) we have

\[
|\mu|_K(G) = \sup \left\{ \sum_{h \in \mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h \in \mathbb{N}} \text{ pairwise disjoint, } \bigcup_h G_h = G \right\}, \quad \forall G \subset \mathbb{R}^m \text{ Borel},
\] (3.12)

where \( G_h \) are bounded Borel sets.

Proof. Thanks to Jensen Inequality and 1-homogeneity of \( \phi_K \) we get

\[
\phi_K(\mu(G_h)) = \phi_K \left( \int_{G_h} \frac{d\mu}{d|\mu|}(x)d|\mu|(x) \right) \leq |\mu|_K(G_h),
\]

so using that \( G_h \cap G_k = \emptyset \) \( \forall h \neq k \)

\[
|\mu|_K(G) = |\mu|_K(\bigcup_h G_h) = \sum_{h \in \mathbb{N}} |\mu|_K(G_h) \geq \sum_{h \in \mathbb{N}} \phi_K(\mu(G_h)).
\]

Taking the sup on the right hand side we proved that \( |\mu|_K(G) \) is greater or equal than the right hand side of relation \((3.12)\). We are then left to prove that

\[
|\mu|_K(G) \leq \sup \left\{ \sum_{h \in \mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h \in \mathbb{N}} \text{ pairwise disjoint, } \bigcup_h G_h = G \right\},
\]

Let \( G \subset \mathbb{R}^n \) be a bounded Borel set. Let us consider the function

\[
f(x) = \frac{d\mu}{d|\mu|}(x) \in L^\infty(\mathbb{R}^m, |\mu|; \mathbb{R}^n).
\]
For each $i \in \{1, \ldots, n\}$ we also have

$$f_i(x) = \frac{d\mu_i}{d|\mu|}(x) \in L^1_{loc}(\mathbb{R}^m, |\mu|),$$

where $\mu = (\mu_1, \ldots, \mu_n)$. Consider $\forall i \in \text{a sequence of step functions } \{f_{i,h}\}_{h \in \mathbb{N}}$ such that

$$\|f_{i,h} - f_i\|_{L^\infty(\mathbb{R}^m, |\mu|)} \to 0 \quad \text{as } h \to \infty.$$ 

As a consequence, if we set $f_h = (f_{1,h}, \ldots, f_{n,h})$ we have that $\|f_h - f\|_{L^\infty(\mathbb{R}^m, |\mu|; \mathbb{R}^n)} \to 0$ as $h \to \infty$. Fix $\epsilon > 0$, then there exists $h(\epsilon) > 0$ such that

$$\|f_h - f\|_{L^\infty(\mathbb{R}^m, |\mu|; \mathbb{R}^n)} < \epsilon \quad \forall h > h(\epsilon).$$

Since for each $i \in \{1, \ldots, n\}$ the function $f_{h,i}$ is simple, there exists $n(h) \in \mathbb{N}$ and a finite pairwise disjoint partition $\{G_{k,h}\}_{k=1,\ldots,n(h)}$ of $G$ such that $f_{h,i}$ is constant $|\mu|$-a.e. in $G_{k,h}$, $\forall k \in \{1, \ldots, n(h)\}$, namely $\exists a_{h,k} \in \mathbb{R}^n$ s.t. $f_{h,i}(x) = a_{h,k}$ for $|\mu|$-a.e. $x \in G_{k,h}$, $\forall k \in \{1, \ldots, n(h)\}$. Then thanks to the one-homogeneity and subadditivity of $\phi_K$ we get

$$\int_G \phi_K(f_h(x)) \, d|\mu|(x) = \sum_{k=1}^{n(h)} \int_{G_{k,h}} \phi_K(f_h(x)) \, d|\mu|(x) = \sum_{k=1}^{n(h)} \phi_K(a_{h,k}) |\mu|(G_{k,h})$$

$$= \sum_{k=1}^{n(h)} \phi_K \left( \int_{G_{k,h}} f(x) \, d|\mu|(x) \right) \leq \sum_{k=1}^{n(h)} \phi_K \left( \int_{G_{k,h}} (f_h - f) \, d|\mu|(x) \right) \leq \sum_{k=1}^{n(h)} \phi_K \left( \mu(G_{k,h}) \right) + \sum_{k=1}^{n(h)} \left| \int_{G_{k,h}} (f_h - f) \, d|\mu| \right| \phi_K \left( \int_{G_{k,h}} (f_h - f) \, d|\mu| \right)$$

$$= \sum_{k=1}^{n(h)} \phi_K \left( \mu(G_{k,h}) \right) + C \sum_{k=1}^{n(h)} \left| \int_{G_{k,h}} (f_h - f) \, d|\mu| \right| \leq \sum_{k=1}^{n(h)} \phi_K \left( \mu(G_{k,h}) \right) + \epsilon C \sum_{k=1}^{n(h)} |\mu|(G_{k,h})$$

$$= \sum_{k=1}^{n(h)} \phi_K \left( \mu(G_{k,h}) \right) + \epsilon C |\mu|(G) \quad \forall h > h(\epsilon),$$
where $C := \sup_{\omega \in S^{n-1}} \phi_K(\omega)$. So we proved that $\forall \epsilon > 0 \ \exists h(\epsilon) > 0$ s.t. $\forall h > h(\epsilon)$ there are $n(h) \in \mathbb{N}$ and $\{G_h^k\}_{k=1}^{n(h)}$ such that the following holds

$$
\int_G \phi_K(f_h(x)) \ d|\mu|(x) \leq \sum_{k=1}^{n(h)} \phi_K(\mu(G_h^k)) + \epsilon C|\mu|(G)
$$

$$
\leq \sup \left\{ \sum_{h \in \mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h \in \mathbb{N}} \text{ pairwise disjoint, } \bigcup_h G_h = G \right\} + \epsilon C|\mu|(G).
$$

Taking the limit as $h \to +\infty$ in the left hand side, by Lebesgue dominated theorem we get

$$
|\mu|_K(G) \leq \sup \left\{ \sum_{h \in \mathbb{N}} \phi_K(\mu(G_h)) : (G_h)_{h \in \mathbb{N}} \text{ pairwise disjoint, } \bigcup_h G_h = G \right\} + \epsilon C|\mu|(G).
$$

By the arbitrariness of $\epsilon > 0$ we conclude for $G$ bounded. Thanks to standard considerations we can extend the result also for $G$ unbounded. \qed

**Definition 3.15** (Hausdorff distance). Let $A, B \subset \mathbb{R}^n$. We define the Hausdorff distance between $A$ and $B$ as

$$
\text{dist}_H(A, B) := \max \left\{ \sup_{x \in A} d(x, B); \sup_{x \in B} d(x, A) \right\},
$$

where $d(\cdot, A)$ denotes the Euclidean distance from $A$.

**Definition 3.16** ($\epsilon$-ball property). Let $\epsilon > 0$. We say that an open bounded set $\Omega \subset \mathbb{R}^n$ satisfies the $\epsilon$-ball property if for any point $x \in \partial \Omega$ $\exists$ a unit vector $d_x \in S^{n-1}$ s.t.

$$
B(x - \epsilon d_x, \epsilon) \subset \Omega,
$$

$$
B(x + \epsilon d_x, \epsilon) \subset \mathbb{R}^n \setminus \Omega.
$$

Roughly speaking, a set satisfies the $\epsilon$-ball property if it is possible to roll two tangent balls, one in the interior and the other one in the exterior part of $\Omega$ (see for instance figure 3.2).
**Lemma 3.19.** Let $S \subset \mathbb{R}^n$ be non-empty. We say that $S$ is a $C^{1,1}$ hypersurface if for every point $x \in S$, there exists an open neighbourhood $D$ of $x$, an open set $E$ of $\mathbb{R}^{n-1}$, and a continuously differentiable bijection $\varphi : E \to D \cap S$ with $\varphi$ and its gradient $\nabla \varphi$ both Lipschitz continuous, and $J \varphi > 0$ on $E$, where $J \varphi$ stands for the Jacobian of $\varphi$.

Given $K \subset \mathbb{R}^n$ as in (1.18), we will now prove few more properties about the surface tension $\phi_K$. In particular, the main result we present is Proposition 3.21 that gives a characterization of the cases of additivity for the function $\phi_K$.

**Lemma 3.18.** Let $K \subset \mathbb{R}^n$ be as in (1.18), and let $y_1, y_2 \in \mathbb{R}^n$. Then, the following are equivalent:

1. $\phi_K(y_1) + \phi_K(y_2) = \phi_K(y_1 + y_2)$;
2. $\exists \bar{z} \in \partial K$ s.t. $\phi_K(y_1) = y_1 \cdot \bar{z}$ and $\phi_K(y_2) = y_2 \cdot \bar{z}$.

**Proof.** Assume (ii) is satisfied. Then,

$$\phi_K(y_1 + y_2) = \max_{z \in \partial K} [(y_1 + y_2) \cdot z] \geq \bar{z} \cdot (y_1 + y_2) = \phi_K(y_1) + \phi_K(y_2),$$

which gives (i). Let now (i) be satisfied and suppose, by contradiction, that

$$\exists \bar{z} \in \partial K \text{ such that } \phi_K(y_1) = y_1 \cdot \bar{z} \text{ and } \phi_K(y_2) = y_2 \cdot \bar{z}.$$  \hspace{1cm} (3.13)

Let $z_1, z_2, z_3 \in \partial K$ be such that $\phi_K(y_1) = y_1 \cdot z_1$ and $\phi_K(y_2) = y_2 \cdot z_2$, and

$$\phi_K(y_1 + y_2) = (y_1 + y_2) \cdot z_3.$$

Then,

$$y_1 \cdot z_3 \leq y_1 \cdot z_1 \quad \text{and} \quad y_2 \cdot z_3 \leq y_2 \cdot z_2.$$

Note that, in particular, from (3.13) we have that at least one of the above inequalities is strict. Thus,

$$\phi_K(y_1 + y_2) < \phi_K(y_1) + \phi_K(y_2),$$

which is a contradiction to (i). \hfill \Box

**Lemma 3.19.** Let $K \subset \mathbb{R}^n$ be as in (1.18) and consider $\phi_K$ the associated surface tension. Let $y_0 \in \mathbb{R}^n$ and let $x_0 \in \partial K$. Then,

$$\phi_K(y_0) = y_0 \cdot x_0 \iff \frac{y_0}{\phi_K(y_0)} \in \partial \phi_K^*(x_0),$$

where, we recall, $\partial \phi_K^*(x_0)$ is the sub differential of $\phi_K$ at $x_0$.

**Proof.** We divide the proof into two steps, one for each implication.

**Step 1** Suppose

$$\frac{y_0}{\phi_K(y_0)} \in \partial \phi_K^*(x_0).$$

Then, since by (3.7) we have $\phi_K^*(x_0) = 1$, we deduce that for every $z \in \mathbb{R}^n$

$$\phi_K(z) \geq \phi_K^*(x_0) - \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0) = 1 + \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0).$$

In particular, if $z \in \partial K$ we have $\phi_K^*(z) = 1$, and therefore

$$1 \geq 1 + \frac{y_0}{\phi_K(y_0)} \cdot (z - x_0), \quad \text{for every } z \in \partial K,$$
so that \(y_0 \cdot x_0 \geq y_0 \cdot z\) for every \(z \in \partial K\). Thus, \(\phi_K(y_0) = y_0 \cdot x_0\).

**Step 2** Assume that \(\phi_K(y_0) = y_0 \cdot x_0\). Then, by the Fenchel inequality, for every \(z \in \mathbb{R}^n\) we have

\[
\phi_K(y_0) \phi_K^*(z) \geq y_0 \cdot z \iff \phi_K^*(z) \geq \frac{y_0 \cdot z}{y_0 \cdot x_0} \iff \phi_K^*(z) \geq 1 + \frac{y_0 \cdot (z - x_0)}{y_0 \cdot x_0}.
\]

Recalling that \(\phi_K^*(x_0) = 1\), we conclude. \(\square\)

**Remark 3.20.** Let us observe that, given \(y_0 \in \mathbb{R}^n\) and \(x_0 \in \partial K\) then

\[
\phi_K(y_0) = y_0 \cdot x_0 \iff y_0 \in C_K^*(x_0),
\]

where \(C_K^*(x_0)\) has been defined in (1.27). Indeed, by the Lemma above and Definition 1.27, we immediately derive that if \(\phi_K(y_0) = y_0 \cdot x_0\), then \(y_0/\phi_K(y_0) \in \partial \phi_K^*(x_0)\) that implies \(y_0 \in C_K^*(x_0)\).

Whereas, if \(y_0 \in C_K^*(x_0)\) then there exists \(\lambda = \lambda(y_0) > 0\) such that \(\lambda y_0 \in \partial \phi_K^*(x_0)\) i.e.

\[
\phi_K^*(z) \geq 1 + \lambda y_0 \cdot (z - x_0) \quad \forall z \in \mathbb{R}^n.
\]

In particular, if we choose \(z \in \partial K\) we get

\[
\lambda y_0 \cdot x_0 \geq \lambda y_0 \cdot z \quad \forall z \in \partial K,
\]

that implies \(\phi_K(y_0) = y_0 \cdot x_0\).

As a direct consequence of Lemmas 3.18 and 3.19 we get the following proposition.

**Proposition 3.21.** Let \(K \subset \mathbb{R}^n\) be as in (1.18), and let \(y_1, y_2 \in \mathbb{R}^n\). Then, the following are equivalent:

(i) \(\phi_K(y_1) + \phi_K(y_2) = \phi_K(y_1 + y_2)\);

(ii) \(\exists \bar{z} \in \partial K\) s.t. \(\phi_K(y_1) = y_1 \cdot \bar{z}\) and \(\phi_K(y_2) = y_2 \cdot \bar{z}\),

(iii) \(\exists \bar{z} \in \partial K\) s.t. \(\frac{y_1}{\phi_K(y_1)} \phi_K^*(\bar{z}) = \frac{y_2}{\phi_K(y_2)} \phi_K^*(\bar{z}) \in \partial \phi_K^*(\bar{z})\).

**Remark 3.22.** By Definition 1.27 condition (iii) in the above Proposition is equivalent to say that

\[
\exists \bar{z} \in \partial K \quad \text{s.t.} \quad y_1, y_2 \in C_K^*(\bar{z}). \tag{3.14}
\]

As noticed in Remark 3.3 \(C_K^*(\bar{z})\) is a convex set and so condition (3.14) is equivalent to say that

\[
\exists \bar{z} \in \partial K \quad \text{s.t.} \quad \{\lambda y_1 + (1 - \lambda)y_2 : \lambda \in [0, 1]\} \subset C_K^*(\bar{z}). \tag{3.15}
\]

**Lemma 3.23.** Let \(K \subset \mathbb{R}^n\) be as in (1.18) and consider \(\phi_K\) the associated surface tension. Let \(x_0 \in \partial K\) then,

\[
\phi_K(y) = 1 \quad \forall y \in \partial \phi_K^*(x_0). \tag{3.16}
\]

Moreover,

\[
\bigcup_{x \in \partial K} \partial \phi_K^*(x) = \partial K^*. \tag{3.17}
\]

**Proof.** We divide the proof in two steps.

**Step 1** In this first part we prove (3.16). Let \(y \in \partial \phi_K^*(x_0)\). By definition of sub-differential, we have that

\[
\phi_K^*(z) \geq 1 + y \cdot (z - x_0) \quad \forall z \in \mathbb{R}^n.
\]
So, choosing \( z = 0 \) we get that \( y \cdot x_0 \geq 1 \). Observe that \( y \in \partial \phi_K^*(x_0) \) implies \( y \in C_K^*(x_0) \) so that \( \phi_K(y) = y \cdot x_0 \) by Remark 3.20. So, \( \phi_K(y) = y \cdot x_0 \geq 1 \). At the same time, by Lemma 3.19, the fact that \( \phi_K(y) = y \cdot x_0 \) is equivalent to say that \( y/\phi_K(y) \in \partial \phi_K^*(x_0) \). By the convexity property of the sub-differential of a convex function (see Remark 3.9), we have \( \lambda y \in \partial \phi_K^*(x_0) \) for every \( \lambda \in [1/\phi_K(y), 1] \), namely
\[
\phi_K^*(z) \geq 1 + \lambda y \cdot (z - x_0) \quad \forall z \in \mathbb{R}^n, \forall \lambda \in [1/\phi_K(y), 1].
\]
Note that choosing \( z = 0 \) we get \( \lambda \geq 1/\phi_K(y) \), while choosing \( z = 2x_0 \) we get, thanks to 1-homogeneity of \( \phi_K^* \), that \( \lambda \leq 1/\phi_K(y) \). Thus, we deduce that \( 1/\phi_K(y) = 1 \). This concludes the proof of the first step.

**Step 2** In the last step we prove (3.17). Thanks to step 1 and Remark 3.6 we have that
\[
\bigcup_{x \in \partial K} \partial \phi_K^*(x) \subseteq \partial K^*.
\]
We are left to prove the other inclusion. Let \( y \in \partial K^* \). By properties of convex sets there exists \( \nu(y) \in S^{n-1} \) such that \( K^* \subseteq H^{-}_{y, \nu(y)} \) (see relations (2.2)). So, \( \forall z \in H^{-}_{y, \nu(y)} \), and in particular \( \forall z \in K^* \) we have
\[
z \cdot \nu(y) \leq y \cdot \nu(y),
\]
that implies, recalling Remark 3.6 that \( \phi_K^*(\nu(y)) = \nu(y) \cdot y \). Thus, thanks to Lemma 3.19 recalling that \( \phi_K(y) = 1 \) we get
\[
\phi_K(\nu(y)) = \nu(y) \cdot y \quad \Leftrightarrow \quad \phi_K^* \left( \frac{\nu(y)}{\phi_K^*(\nu(y))} \right) = \frac{\nu(y)}{\phi_K^*(\nu(y))} \cdot y \quad \Leftrightarrow \quad 1 = \frac{\nu(y)}{\phi_K^*(\nu(y))} \cdot y
\]
\[
\Leftrightarrow \quad \phi_K(\nu(y)) = \frac{\nu(y)}{\phi_K^*(\nu(y))} \cdot y \quad \Leftrightarrow \quad y \in \partial \phi_K^* \left( \frac{\nu(y)}{\phi_K^*(\nu(y))} \right).
\]
Since \( \nu(y)/\phi_K^*(\nu(y)) \in \partial K \) we conclude.

**Figure 3.3.** A pictorial idea of condition (3.17) with respect to the Wulff shape \( K^* \) presented in Figure 3.1. Indeed, according to Lemma 3.23 and (3.22), we see that \( \partial \phi_K^*((0, 1)) \) is a convex subset of the boundary of \( (K^*)^* \). The fact that \( \partial \phi_K^*((0, 1)) \) actually contains the point \((0, 1)\) is just a consequence of the specific Wulff shape considered in the example.
Remark 3.24. Having in mind the definition of $C^*_K(x)$ (see (1.27), and as a consequence of (3.17), we have that

$$\bigcup_{x \in \partial K} C^*_K(x) = \mathbb{R}^n. \quad (3.18)$$

Corollary 3.25. Let $K \subset \mathbb{R}^n$ be as in (1.18) and consider $\phi_K$ the associated surface tension. Assume in addition that $\phi_K \in C^1(\mathbb{R}^n_0)$. Then,

$$\phi_K(x) = \nabla \phi_K(x) \cdot x \quad \text{and} \quad \phi^*_K(\nabla \phi_K(x)) = 1 \quad \forall x \in \mathbb{R}^n_0. \quad (3.19)$$

Proof. Firstly, let us observe it is a well known fact that the first relation in (3.19) holds true for every positive and 1-homogeneous function. So, we are left to prove the second relation in (3.19). Let $x \in \partial K^*$. As we observed in the above Lemma, by properties of convex sets there exists $\nu(x) \in \mathbb{S}^{n-1}$ such that $K^* \subset H_{x,\nu(x)}$ and $\phi_K^*(\nu(x)) = \nu(x) \cdot x$. By Lemma 3.19 having in mind Remark 3.10 we have that

$$\phi_K^*(\nu(x)) = \nu(x) \cdot x \iff \frac{\nu(x)}{\phi_K^*(\nu(x))} = \nabla \phi_K(x). \quad (3.20)$$

By the 1-homogeneity of $\phi_K$ it follows that

$$\nabla \phi_K(\lambda x) = \nabla \phi_K(x) \quad \forall \lambda > 0, \forall x \in \mathbb{R}^n_0, \quad (3.21)$$

therefore $\phi_K^*(\nabla \phi_K(x)) = 1$ for all $x \in \mathbb{R}^n_0$. This concludes the proof. □

Remark 3.26. Let $K \subset \mathbb{R}^n$ be as in (1.18), and consider $x \in \partial K$. Note that, thanks the above results we can deduce the following equivalent characterization for the subdifferential $\partial \phi_K^*(x)$, namely

$$\partial \phi_K^*(x) = \left\{ y \in \partial K^* : y \cdot \frac{x}{|x|} = \phi_K^* \left( \frac{x}{|x|} \right) \right\}. \quad (3.22)$$

Indeed, thanks to Lemma 3.23 we know that $\partial \phi_K^*(x) \subset \partial K^*$ so that $\phi_K(y) = 1$ for all $y \in \partial \phi_K(x)$. Whereas, thanks to Lemma 3.19 we have that $y \in \partial \phi_K(x)$ if and only if $1 = \phi_K^*(x) \phi_K(y) = y \cdot x$, from which, we get $y \cdot \frac{x}{|x|} = \phi_K^* \left( \frac{x}{|x|} \right)$.

The following two results will be used for the proof of Proposition 1.13.

Lemma 3.27. Let $K \subset \mathbb{R}^n$ be as in (1.18). Let $x_1, x_2 \in \partial K$ and $y \in \partial K^*$ be such that $\tilde{y} \in \partial \phi_K^*(x_1) \cap \partial \phi_K^*(x_2)$. Let us now assume that there exist $y_1, y_2 \in \partial \phi_K^*(x_2)$, with $y_1 \neq \tilde{y} \neq y_2$, such that $\tilde{y} = (1-\lambda)y_1 + \lambda y_2$ for some $\lambda \in (0,1)$. Then,

$$(1-\lambda)y_1 + \lambda y_2 \in \partial \phi_K^*(x_1) \quad \forall \lambda \in [0,1]. \quad (3.23)$$

Proof. Let us suppose by contradiction that there exists $\hat{\lambda} \in [0,1]$ such that $\tilde{y} = (1-\hat{\lambda})y_1 + \hat{\lambda}y_2 \notin \partial \phi_K^*(x_1)$. By the Fenchel inequality (3.3), (3.22), and using that $\phi_K(\tilde{y}) \leq (1-\hat{\lambda})\phi_K(y_1) + \hat{\lambda}\phi_K(y_2) \leq 1$ we get

$$\tilde{y} \cdot \frac{x_1}{|x_1|} < \tilde{y} \cdot \frac{x_1}{|x_1|} = \phi_K^* \left( \frac{x_1}{|x_1|} \right). \quad (3.24)$$

Recall that, by (1.21) applied to $K^*$ we have that

$$K^* = \bigcap_{\omega \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot \omega \leq \phi_K^*(\omega) \}. $$
By relation (3.24) we have that the continuous linear function
\[
\varphi(\lambda) := ((1 - \lambda)y_1 + \lambda y_2) \cdot \frac{x_1}{|x_1|} > \phi^*_K \left( \frac{x_1}{|x_1|} \right)
\]
for every \( \lambda \in (\bar{\lambda}, 1) \), but this is a contradiction since
\[
\{(1 - \lambda)y_1 + \lambda y_2 : \lambda \in [0, 1]\} \subset \partial \phi^*_K(x_2) \subset K^*.
\]
The case when \( \bar{\lambda} \in [\lambda, 1] \) is symmetric, and thus the proof is complete.

**Corollary 3.28.** Let \( K \subset \mathbb{R}^n \) be as in (1.18). Let \( x \in \partial K \) be such that the subdifferential of \( \phi^*_K \) in \( x \) has only one point, namely \( \partial \phi^*_K(x) = \{y\} \). Then, \( \forall z \in Z_K(y) \), where \( Z_K(y) \) is defined in (1.32), and for every \( y_1, y_2 \in C^*_K(z) \), with \( y_1/\phi_K(y_1) \neq y_2/\phi_K(y_2) \), if \( \exists \lambda \in (0, 1) \) s.t. \( y = (1 - \lambda)y_1 + \lambda y_2 \), then \( y_1 = \lambda_1 y_2 = \lambda_2 y_2 \) for some \( \lambda_1, \lambda_2 > 0 \).

**Proof.** Let us fix \( z \in Z_K(y) \) and \( y_1, y_2 \in C^*_K(z) \) and let us assume that \( y = (1 - \bar{\lambda})y_1 + \bar{\lambda}y_2 \), for some \( \bar{\lambda} \in (0, 1) \). Since \( y \in Z_K(y) \), then \( y_1, y_2 \in C^*_K(z) \), and thus
\[
\frac{y_1}{\phi_K(y_1)}, \frac{y}{\phi_K(y)} , \frac{y_2}{\phi_K(y_2)} \in \partial \phi^*_K(z) . \tag{3.25}
\]
Let us observe that \( \phi_K(y) = 1 \) since we know \( y \in \partial \phi^*_K(x) \). As a consequence of (3.25), together with the convexity of \( \partial \phi^*_K(z) \), we deduce that
\[
y \in \partial \phi^*_K(z) \cap \partial \phi^*_K(x), \tag{3.26}
y = (1 - t)\frac{y_1}{\phi_K(y_1)} + t\frac{y_2}{\phi_K(y_2)} , \tag{3.27}
\]
where \( t \in (0, 1) \). Therefore, thanks to Lemma 3.27, we have that
\[
(1 - t)\frac{y_1}{\phi_K(y_1)} + t\frac{y_2}{\phi_K(y_2)} \in \partial \phi^*_K(x) \quad \forall t \in [0, 1],
\]
but this is possible if and only if \( y_i/\phi_K(y_i) = y \) for \( i = 1, 2 \). This concludes the proof. \( \square \)

We conclude this section recalling few more definitions and a couple of results very well known in convex analysis. Such tools will play a key role in the understanding of (RAS).

**Definition 3.29.** Let \( C \subset \mathbb{R}^n \) be a convex set. We say that \( x \in C \) is an extreme point of \( C \) if and only if there is no way to express \( x \) as a convex combination \( (1 - \lambda)y + \lambda z \) such that \( y, z \in C \) and \( 0 < \lambda < 1 \), except by taking \( y = z = x \).

**Definition 3.30.** Let \( C \subset \mathbb{R}^n \) be a convex set. We say that \( x \in C \) is an exposed point of \( C \) if and only if there exists an hyperplane of the form \( H_{\nu} \), with \( \nu \in \mathbb{S}^{n-1} \), such that \( C \subset H_{\nu}^* \) and \( C \cap H_{\nu} = \{x\} \). Observe that if \( x \) is an exposed point of \( C \), then \( x \) belongs to the boundary of \( C \).

**Remark 3.31.** If \( C \subset \mathbb{R}^n \) is a closed convex set, then by [27] Theorem 18.6, the set of exposed points of \( C \) is dense in the set of extreme points of \( C \), namely, every extreme point is the limit of a sequence of exposed points (see for instance Figure 3.3).

Let us now recall a useful result about the characterization of the exposed points of a closed convex set (see for instance [27] Corollary 25.1.3)].
Figure 3.4. Given a closed convex set $C$ as in the figure above, its set of extreme points is the one that contains the parts of the boundary of $C$ that are in bold (the four points $L, F, H, G$ are included). Whereas, the set of exposed points of $C$ is the set of extreme points of $C$ without the two points $L$ and $G$.

**Lemma 3.32.** Let $C \subset \mathbb{R}^n$ be a non empty, closed, convex set, and let $g : \mathbb{R}^n \to [0, \infty)$ be any 1-homogeneous, convex function, such that

$$C = \{ z \in \mathbb{R}^n : z \cdot y \leq g(y) \ \forall \ y \in \mathbb{R}^n \}.$$ 

Then, $z \in C$ is an exposed point of $C$ if and only if there exists a point $y \in \mathbb{R}^n$ such that $g$ is differentiable at $y$ and $\nabla g(x) = z$.

4. Characterization of the anisotropic total variation

In this section we will study some properties of the anisotropic total variation (see Definition 3.11), proving also a characterization result (see Theorem 4.1). Such characterization result is already known in the literature but we decided to give a proof for the sake of completeness since we couldn’t find a precise reference. The main result of this Section 4 is the following.

**Theorem 4.1.** Let $K \subset \mathbb{R}^n$ be as in (1.18). Let $\mu$ be a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^m$, $m \geq 1$, $m \in \mathbb{N}$. Then, we have

$$|\mu|_K(\Omega) = \sup \left\{ \int_{\Omega} \varphi(x) \cdot d\mu(x) : \varphi \in C^1_c(\Omega; \mathbb{R}^n), \phi_K^*(\varphi) \leq 1 \right\} \ \forall \ \Omega \subset \mathbb{R}^m \text{ open.}$$

In order to prove Theorem 4.1 we need some intermediate results.

**Lemma 4.2.** Let $\{K_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n$, $K \subset \mathbb{R}^n$ be such that $K_h, K$ are as in (1.18) $\forall \ h \in \mathbb{N}$. Assume moreover that

i) the sequence $(K_h)_{h \in \mathbb{N}}$ is either of the form $K_h \subset K_{h+1} \subset K$, or $K \subset K_{h+1} \subset K_h$, $\forall h \in \mathbb{N}$,

ii) $\lim_{h \to +\infty} \text{dist}_H(K_h, K) = 0$.

Then, the sequence $\{\phi_{K_h}\}$ converges uniformly to $\phi_K$ in $\mathbb{S}^{n-1}$.

**Proof.** Without loss of generality we can consider the case when $K_h \subset K_{h+1} \subset K \ \forall h \in \mathbb{N}$. For every $x \in \mathbb{S}^{n-1}$ and $h \in \mathbb{N}$, let $y(x) \in \partial K$ and $y_h(x) \in \partial K_h$ be such that $\phi_K(x) = y(x) \cdot x$ and $\phi_{K_h}(x) = y_h(x) \cdot x$, respectively. Then, since $K_h \subset K$,

$$\sup_{x \in \mathbb{S}^{n-1}} |\phi_K(x) - \phi_{K_h}(x)| = \sup_{x \in \mathbb{S}^{n-1}} [x \cdot (y(x) - y_h(x))].$$
Note now that, by definition of \( y_h \), we have \(-x \cdot y_h(x) \leq -x \cdot \tilde{y} \\forall \tilde{y} \in \partial K_h\). In particular, choosing \( \tilde{y} = z(x) \in \partial K_h \) such that \( |y(x) - z(x)| = \text{dist}(y(x), \partial K_h) \), we have

\[
\sup_{x \in \mathbb{S}^{n-1}} |\phi_K(x) - \phi_{K_h}(x)| \leq \sup_{x \in \mathbb{S}^{n-1}} [x \cdot (y(x) - z(x))] \leq \text{dist}(y(x), \partial K_h) \leq \text{dist}_H(K, K_h),
\]

where in the last inequality we used the fact that \( K_h \subset K \). Passing to the limit as \( h \to +\infty \) we conclude.

\[\square\]

**Lemma 4.3.** Let \( K \subset \mathbb{R}^n \) be as in (1.18). Then there exists a sequence \( \{K_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n \) with \( K_h \) as in (1.18) for every \( h \in \mathbb{N} \), such that

i) \( K_h \) is \( C^{1,1} \), \( \forall h \in \mathbb{N} \);

ii) \( K \subset \cdots \subset K_{h+1} \subset K_h \ \forall h \in \mathbb{N} \);

iii) \( \lim_{h \to +\infty} \text{dist}_H(K_h, K) = 0 \).

**Proof.** We divide the proof in few steps. Take any \( \epsilon > 0 \) and let \( K_\epsilon = \bigcup_{x \in K} B(x, \epsilon) \) denote the \( \epsilon \)-neighbourhood of \( K \).

**Step 1** In this Step we want to prove that \( K_\epsilon \) is convex, open, bounded and it contains the origin. By construction, we need just to prove that it is convex. Consider two generic points \( x_1, x_2 \in K_\epsilon \), let us show that

\[
\lambda x_1 + (1 - \lambda)x_2 \in K_\epsilon \ \forall \lambda \in [0, 1].
\]

Observe that, since \( x_1, x_2 \in K_\epsilon \) there exist \( c_1, c_2 \in K \) such that \( |x_1 - c_1| < \epsilon \) and \( |x_2 - c_2| < \epsilon \). Thus,

\[
\lambda x_1 + (1 - \lambda)x_2 = \lambda (c_1 + (x_1 - c_1)) + (1 - \lambda)(c_2 + (x_2 - c_2))
\]

\[
= \lambda c_1 + (1 - \lambda)c_2 + \lambda (x_1 - c_1) + (1 - \lambda)(x_2 - c_2).
\]

Since \( \lambda c_1 + (1 - \lambda)c_2 \in K \) and \( |\lambda (x_1 - c_1) + (1 - \lambda)(x_2 - c_2)| < \epsilon \) we conclude the proof of step 1.

**Step 2** In this step we are going to prove that \( K_\epsilon \) satisfies the \( \epsilon \)-ball property. This is true by construction. Indeed, since \( K_\epsilon \) is as in (1.18), we can associate to it the function \( \phi_{K_\epsilon} \). So, having in mind (1.21) we know that for every \( y \in \partial K_\epsilon \) there exists \( \nu \in \mathbb{S}^{n-1} \) and an hyperplane

\( H_{\phi_{K_\epsilon}(\nu)} = \{ z \in \mathbb{R}^n : z \cdot \nu = \phi_{K_\epsilon}(\nu) \} \)

such that \( y \in H_{\phi_{K_\epsilon}(\nu)} \) and \( K_\epsilon \) lies on one side of \( H_{\phi_{K_\epsilon}(\nu)} \) (this is because \( K_\epsilon \) is a convex set). So, we can construct on the exterior of \( K_\epsilon \) a ball of arbitrary radius tangent to the hyperplane \( H_{\phi_{K_\epsilon}(\nu)} \) in the point \( y \). Let us now consider \( z \in K_\epsilon \) such that \( |z - y| = \epsilon \) in particular, \( z \in \partial K \). By construction we have that \( B(z, \epsilon) \subset K_\epsilon \) and this concludes the proof of step 2.

![Figure 4.1. A pictorial idea for the proof of Lemma 4.3](image-url)
Step 3 We have to prove that $\partial K_h$ is a hypersurface $C^{1,1}$ regular. This result is a straightforward consequence of [15, Theorem 1.8].

Step 4 We are left to prove that $\text{dist}_H(\bigcup_{x \in K} B(x, \epsilon), K) \leq \epsilon$. By definition of Hausdorff distance we have that

$$\text{dist}_H(K, K_h) = \max \left\{ \sup_{y \in K_h} d(y, K): \sup_{y \in K} d(y, K_h) \right\} = \max \{\epsilon; 0\}.$$ 

To conclude the proof of the Lemma let us observe the following. Let us fix a decreasing sequence of positive real numbers $(\epsilon_h)_{h \in \mathbb{N}}$. We can construct the sequence $(K_h)_{h \in \mathbb{N}}$ where $K_h = K_{\epsilon_h}$ is the $\epsilon_h$-neighbourhood of $K \forall h \in \mathbb{N}$. By all previous steps, the sequence $(K_h)_{h \in \mathbb{N}}$ satisfies i), ii) and iii) of the Lemma and this concludes the proof. \[
\square
\]

**Proposition 4.4.** Let $K$ be as in (1.18) and let $K^*$ be its dual. Consider $(K_h^*)_{h \in \mathbb{N}}$ a sequence as in (1.18), such that either $K_h^* \subset K_{h+1}^* \subset K^*$ or $K^* \subset K_{h+1}^* \subset K_h^*, \forall h \in \mathbb{N}$. Then, denoting with $K_h = (K_h^*)^*$ we have

$$\lim_{h \to +\infty} \text{dist}_H(K_h^*, K^*) = 0 \quad \text{if and only if} \quad \lim_{h \to +\infty} \text{dist}_H(K_h, K) = 0.$$ 

**Proof.** Let us assume that $\lim_{h \to +\infty} \text{dist}_H(K_h^*, K^*) = 0$ and, without loss of generality, that $K^* \subset K_{h+1}^* \subset K_h^*, \forall h \in \mathbb{N}$. We can apply immediately Lemma 4.2 to the sequence $(K_h^*)_{h \in \mathbb{N}}$ to obtain that $\phi_{K_h}^*$ uniformly converges to $\phi_K^*$. Consider the following quantity

$$\text{dist}_H(K_h, K) = \max \left\{ \sup_{x \in K_h} d(x, K); \sup_{x \in K} d(x, K_h) \right\}.$$ 

Now, by the way the $K_h^*$ are constructed, and having in mind iii) of Proposition 4.4 we have

$$K_h \subset K_{h+1} \subset \cdots \subset K \quad \forall h \in \mathbb{N}.$$ 

This fact immediately tells us that

$$\sup_{x \in K_h} d(x, K) = 0.$$ 

Let us focus our attention now on $\sup_{x \in K} d(x, K_h)$, thus

$$\sup_{x \in K} d(x, K_h) = \sup_{x \in \partial K} d(x, K_h) = \max_{x \in \partial K} d(x, K_h) \leq \max_{x \in \partial K} |x - x_{K_h}|,$$

where $x_{K_h} = \{tx : t > 0\} \cap \partial K_h$. By observing that $\phi_{K_h}^* (x) = \frac{|x|}{|x_{K_h}|} \phi_{K_h}^* (x_{K_h}) = \frac{|x|}{|x_{K_h}|}$, and since $|x| - |x_{K_h}| = |x - x_{K_h}|$, we get

$$|x - x_{K_h}| \frac{1}{|x_{K_h}|} = \left( \phi_{K_h}^* (x) - \phi_K^* (x) \right).$$

Thus,

$$\lim_{h \to +\infty} |x - x_{K_h}| = \lim_{h \to +\infty} |x_{K_h}| \left( \phi_{K_h}^* (x) - \phi_K^* (x) \right) = 0 \quad \forall x \in \partial K$$

thanks to the uniform convergence of $\phi_{K_h}^*$ to $\phi_K^*$. This shows that $(K_h^*)^* = K_h^* = K_h$ the proof is complete. \[
\square
\]

We can now prove Theorem 4.1.
Proof. For the sake of clarity we decided to divide the proof in several steps.

**Step 1** Assume $\Omega \subset \mathbb{R}^n$ to be an open, bounded set. We start proving
\[
\int_{\Omega} \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) d|\mu|(x) \geq \sup \left\{ \int_{\Omega} \varphi(x) \cdot d\mu(x) : \varphi \in C^1_c(\Omega; \mathbb{R}^n), \phi_K^*(\varphi) \leq 1 \right\}.
\]

Let us observe that by definition of $\phi_K$ we have
\[
|\mu|_K(\Omega) = \int_{\Omega} \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) d|\mu|(x) = \int_{\Omega} \left( \sup_{y \in K} y \cdot \frac{d\mu}{|\mu|}(x) \right) d|\mu|(x)
\]
where $\varphi \in C^1_c(\Omega; \mathbb{R}^n)$, $\phi_K^*(\varphi) \leq 1$. Passing to the sup on the right hand side we conclude the first step.

**Step 2** We want to prove the reverse inequality, namely
\[
|\mu|_K(\Omega) \leq \sup \left\{ \int_{\Omega} \varphi(x) \cdot d\mu(x) : \varphi \in C^1_c(\Omega; \mathbb{R}^n), \phi_K^*(\varphi) \leq 1 \right\},
\]
In order to do so, we consider at first the case when $\phi_K$ is in addition $C^1(\mathbb{R}^n_0)$. Recalling relations (3.19), we have
\[
|\mu|_K(\Omega) = \int_{\Omega} \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) d|\mu|(x) = \int_{\Omega} \nabla \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) \cdot \frac{d\mu}{|\mu|}(x) d|\mu|(x).
\]
Since $\nabla \phi_K \in C^0(\mathbb{R}^n_0)$, the composition $\nabla \phi_K \left( \frac{d\mu}{|\mu|}(x) \right)$ is well defined, and moreover,
\[
\nabla \phi_K \left( \frac{d\mu}{|\mu|}(\cdot) \right) \in L^1_{loc}(\Omega, |\mu|; \mathbb{R}^n),
\]
with $\phi_K^*(\nabla \phi_K \left( \frac{d\mu}{|\mu|}(x) \right)) = 1$ for $|\mu|$-a.e. $x \in \Omega$. Recall that
\[
\phi_K^* \left( \nabla \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) \right) = 1 \quad \text{implies} \quad \nabla \phi_K \left( \frac{d\mu}{|\mu|}(x) \right) \in \partial K, \quad \text{for } |\mu|$-a.e. $x \in \Omega,
Therefore, applying step 2 we get the sequence satisfies the assumptions of Lemma 4.2 so that

\[ \nabla \phi_K \left( \frac{d\mu}{|d\mu|} (x) \right) \in L^p(\Omega, |\mu|; \mathbb{R}^n). \]

By the fact that \( \Omega \) is a bounded set we have that

\[ \nabla \phi_K \left( \frac{d\mu}{|d\mu|} (x) \right) \in L^p(\Omega, |\mu|; \mathbb{R}^n) \quad \forall p \geq 1. \]

Let us call \( f := \nabla \phi_K \left( \frac{d\mu}{|d\mu|} \right) \). By [2] Remark 1.46 there exist a sequence \( (g_h)_h \in C^0_c(\Omega; \mathbb{R}^n) \) such that \( g_h \to f \) in \( L^1(\Omega, |\mu|; \mathbb{R}^n) \). Since every function in \( C^0_c \) can be uniformly approximated by functions in \( C^1_c \) we can suppose without loss of generality that the sequence \( (g_h)_h \in C^1_c(\Omega; \mathbb{R}^n) \).

Now we consider the sequence \( (\tilde{g}_h)_h \in C^0_c(\Omega; \mathbb{R}^n) \) defined as

\[ \tilde{g}_h(x) := \frac{g_h(x)}{\phi_K(g_h(x)) + 1/h} \quad \forall h \in \mathbb{N}. \]

By construction, up to a subsequence, we have that \( \tilde{g}_h \to f \) \( |\mu|\)a.e. on \( \Omega \) and, thanks to the term \( 1/h \) in the denominator, \( \tilde{g}_h(x) \in K \), so that \( \phi_K(\tilde{g}_h(x)) \leq 1 \) for every \( h \in \mathbb{N} \) and for \( |\mu|\)a.e. \( x \in \Omega \). By the continuity of the functions \( \tilde{g}_h \), for every \( h \in \mathbb{N} \) there exists \( \lambda = \lambda(h) > 0 \) such that \( 0 < \lambda(h) \leq 1 \) and \( \tilde{g}_h(x) \leq \lambda(h)K \) for every \( x \in \Omega \). Again, using the fact that \( C^1_c(\Omega; \mathbb{R}^n) \) is dense in \( C^0_c(\Omega; \mathbb{R}^n) \) we can proceed as follow: let \( (\epsilon_h)_h \in \mathbb{N} \) be such that \( \epsilon_h > 0 \) for every \( h \in \mathbb{N} \) and \( \epsilon_h \to 0 \) for \( h \to \infty \). For every \( h \in \mathbb{N} \) let \( f_h \in C^1_c(\Omega; \mathbb{R}^n) \) be such that

\[ \sup_{x \in \Omega} |f_h(x) - \tilde{g}_h(x)| < \epsilon_h. \]

Since \( \text{dist}(\partial(\lambda(h)K); \partial K) > 0 \) for every \( h \in \mathbb{N} \), choosing \( \epsilon_h \) small enough we get that \( \forall h \in \mathbb{N} \) \( f_h(x) \in K \) for every \( x \in \Omega \). Thus, by the Lebesgue dominated convergence theorem

\[
\begin{align*}
|\mu|_K(\Omega) &= \int_{\Omega} \phi_K \left( \frac{d\mu}{|d\mu|} (x) \right) d|\mu|(x) = \int_{\Omega} \lim_{h \to \infty} f_h(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x) \\
&= \lim_{h \to \infty} \int_{\Omega} f_h(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x) & \leq \sup_{h \in \mathbb{N}} \int_{\Omega} f_h(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x) \\
&\leq \sup_{\phi \in C^1_c(\Omega; \mathbb{R}^n), \phi_K(\cdot) \leq 1} \int_{\Omega} \phi(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x).
\end{align*}
\]

This concludes step 2.

**Step 3** We want now to prove the statement for a generic \( \phi_K \). Thus, thanks to Lemma 4.3 consider \( \{K_h\}_{h \in \mathbb{N}} \subset \mathbb{R}^n \) a sequence as in (1.18) with \( K_h \subset K_{h+1} \subset \cdots \subset K \) and such that the sequence satisfies the assumptions of Lemma 1.2 so that \( \phi_{K_h} \) uniformly converges to \( \phi_K \).

Therefore, applying step 2 we get

\[
|\mu|_{K_h}(\Omega) = \int_{\Omega} \phi_{K_h} \left( \frac{d\mu}{|d\mu|} (x) \right) d|\mu|(x) = \sup_{\phi \in C^1_c(\Omega; \mathbb{R}^n), \phi_{K_h}(\cdot) \leq 1} \int_{\Omega} \phi(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x) \\
\leq \sup_{\phi \in C^1_c(\Omega; \mathbb{R}^n), \phi_{K_h}(\cdot) \leq 1} \int_{\Omega} \phi(x) \cdot \frac{d\mu}{|d\mu|} (x) d|\mu|(x),
\]

where we used the fact that $\phi_K^*(\varphi) \leq 1$ as a consequence of $\phi_K^*(\varphi) \leq 1$ and of $K_h \subset K$. Now, thanks to the uniform convergence of the functions $\phi_K$ to $\phi_K^*$ we get

$$|\mu|_K(\Omega) = \int_{\Omega} \phi_K \left( \frac{d\mu}{d|\mu|} \right) d|\mu|(x) = \lim_{h \to +\infty} \int_{\Omega} \phi_K \left( \frac{d\mu}{d|\mu|} \right) d|\mu|(x)$$

$$\leq \sup_{\varphi \in C^1_b(\Omega;\mathbb{R}^n), \varphi \leq 1} \int_{\Omega} \varphi(x) \cdot \frac{d\mu}{d|\mu|}(x) d|\mu|(x).$$

This concludes the proof in the case $\Omega$ open and bounded. From standard considerations about outer measures, the extension of this result for unbounded open set follows. \qed

The following result is the anisotropic version of [8, Lemma 3.7].

**Lemma 4.5.** If $\nu$ and $\mu$ are $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^m$, then

$$2|\mu|_K(G) \leq |\mu + \nu|_K(G) + |\mu - \nu|_K(G)$$

(4.1)

for every Borel set $G \subset \mathbb{R}^m$.

**Proof.** Fix a generic partition of $G$ made by bounded Borel sets $\{G_i\}_{i \in \mathbb{N}}$, by subadditivity we have

$$\phi_K(2\mu(G_i)) = \phi_K(\mu(G_i) + \nu(G_i)) + \mu(G_i) - \nu(G_i))$$

$$\leq \phi_K(\mu(G_i)) + \phi_K(\mu - \nu(G_i)).$$

Thus,

$$\sum_{i \in \mathbb{N}} \phi_K(2\mu(G_i)) \leq \sum_{i \in \mathbb{N}} [\phi_K(\mu(G_i)) + \phi_K(\mu - \nu(G_i))].$$

Then thanks to Lemma [3.14] and passing to the sup in both sides we get

$$2|\mu|_K(G) \leq \sup_{\{G_i\} \in \mathbb{N}} \sum_{i \in \mathbb{N}} [\phi_K(\mu + \nu(G_i)) + \phi_K(\mu - \nu(G_i))]$$

$$\leq \sup_{\{G_i\} \in \mathbb{N}} \sum_{i \in \mathbb{N}} \phi_K(\mu + \nu(G_i)) + \sup_{\{G_i\} \in \mathbb{N}} \sum_{i \in \mathbb{N}} \phi_K(\mu - \nu(G_i))$$

$$= |\mu + \nu|_K(G) + |\mu - \nu|_K(G).$$

This concludes the proof. \qed

**Remark 4.6.** Let $\mu_1, \mu_2$ be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^m$. Let us observe that, by (4.1) with $\mu = \mu_1 + \mu_2$ and $\nu = \mu_1 - \mu_2$ we obtain

$$|\mu_1 + \mu_2|_K \leq |\mu_1|_K + |\mu_2|_K.$$  

(4.2)

On the other hand, let $\nu_1, \nu_2$ be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^m$. Then, by the above relation with $\mu_1 = \nu_1 + \nu_2$ and $\mu_2 = -\nu_2$ we get

$$|\nu_1 + \nu_2|_K \geq |\nu_1|_K - |\nu_2|_K.$$  

(4.3)

**Remark 4.7.** In this Remark we discuss the equality case for relation (4.1). Let us assume that

$$2|\mu|_K(G) = |\mu + \nu|_K(G) + |\mu - \nu|_K(G)$$

∀ Borel set $G \subset \mathbb{R}^m$.  

(4.4)

We immediately observe that if $|\mu|_K(G) = 0$ then $|\mu + \nu|_K(G) = |\mu - \nu|_K(G) = |\nu|_K(G) = 0$, so that

$$|\nu|_K \ll |\mu|_K.$$
Thanks to Radon-Nikodym Theorem we know that \( \exists g, h \in L^1_{\text{loc}}(\mathbb{R}^m, |\mu|_K; \mathbb{R}^n) \) s.t.

\[
\nu = g |\mu|_K \quad \text{and} \quad \mu = h |\mu|_K,
\]

thus,

\[
\mu \pm \nu = (h \pm g) |\mu|_K.
\]

Observing that

\[
|\mu \pm \nu|_K(G) = \int_G \phi_K \left( \frac{d(\mu \pm \nu)}{d|\mu \pm \nu|}(x) \right) d|\mu \pm \nu|(x) = \int_G \phi_K \left( \frac{(h \pm g)(x)}{|h \pm g|(x)} \right) |h \pm g|(x) d|\mu|_K(x),
\]

we can now rewrite (4.4) as

\[
\int_G 2 \phi_K (h(x)) d|\mu|_K(x) = \int_G \phi_K ((h + g)(x)) d|\mu|_K(x) + \int_G \phi_K ((h - g)(x)) d|\mu|_K(x).
\]

that is

\[
\int_G \phi_K (2h(x)) - \phi_K ((h + g)(x)) - \phi_K ((h - g)(x)) d|\mu|_K(x) = 0 \quad \forall G \subset \mathbb{R}^m \text{ Borel.}
\]

By subadditivity we get

\[
\phi_K (2h(x)) - \phi_K ((h + g)(x)) - \phi_K ((h - g)(x)) \leq 0 \quad |\mu|_K\text{-a.e. } x \in \mathbb{R}^m,
\]

thus,

\[
\phi_K (2h(x)) = \phi_K ((h + g)(x)) + \phi_K ((h - g)(x)) \quad |\mu|_K\text{-a.e. } x \in \mathbb{R}^m.
\]

Thus condition (4.4) is equivalent to (4.5) that is equivalent to say, thanks to Proposition 3.21, Remark 3.22 and relation (3.15) with \( y_1 = h + g \) and \( y_2 = h - g \), that for \( |\mu|_K\text{-a.e. } x \in \mathbb{R}^m \)

\[
\exists z(x) \in \partial K \text{ s.t.}
\]

\[
\{h(x) + tg(x) : t \in [-1, 1]\} \subset C^*_K(z(x)).
\]
In this section we will prove a formula for the anisotropic perimeter valid for a specific class of sets of finite perimeter. We recall that, given \( u : \mathbb{R}^{n-1} \to \mathbb{R} \), we denote by \( \Sigma_u = \{ x \in \mathbb{R}^n : q x > u(p x) \} \) and \( \Sigma^u = \{ x \in \mathbb{R}^n : q x < u(p x) \} \) the epigraph and the subgraph of \( u \), respectively. As proved in [7, Proposition 3.1], \( \Sigma_u \) is a set of locally finite perimeter if and only if \( \tau_M(u) \in BV_{loc}(\mathbb{R}^{n-1}) \) for every \( M > 0 \). Through all this section, given \( u \in BV_{loc}(\mathbb{R}^{n-1}) \) we consider \( \eta := (Du, -L^{n-1}) \) a \( \mathbb{R}^n \)-valued Radon measure on \( \mathbb{R}^{n-1} \).

**Theorem 5.1.** Let \( K \subset \mathbb{R}^n \) as in (1.18) and let \( u \in BV_{loc}(\mathbb{R}^{n-1}) \), then

\[
|\eta|_K(B) = |D1_{\Sigma^u}|_K(B \times \mathbb{R}) \quad \forall B \subset \mathbb{R}^{n-1} \text{ Borel.}
\]

**Proof.** Thanks to Theorem 4.1, the identity follows from a careful inspection of the proof of [22, Theorem 4 in (Part 4, Section 1.5)]. It is important to notice that in the present situation one should replace condition \(|\varphi| \leq 1\) with \( \phi_{K_\varepsilon}(\varphi) \leq 1\) with \( \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \). \( \square \)

We recall now an important result concerning how to determine \( \nu_{\Sigma^u} \) i.e. the outer normal to the reduced boundary of the subgraph of the function \( u \). Recall that thanks to Radon-Nykodym Theorem we have

\[ Du = D^u + D^J u + D^C u. \]

With a little abuse of notation let us call \( D^{ac} u = D^a u + D^c u \), so that

\[ D^C u = D^{ac} u \upharpoonright Z_u \]

where,

\[ Z_u = \left\{ x \in \Omega : \frac{d|D^{ac} u|}{d \mathcal{L}^{n-1}}(x) = +\infty \right\}. \]

**Theorem 5.2.** Let \( u \in BV(\Omega) \) with \( \Omega \subset \mathbb{R}^{n-1} \) open and bounded, then

i) for \( |\eta| \)-a.e. \( x \in \Omega \setminus J_u \) we have

\[
\frac{d\eta}{d|\eta|}(x) = -\nu_{\Sigma^u}(x, u(x)),
\]

ii) for \( |\eta| \)-a.e. \( x \in J_u \) we have

\[
\frac{d\eta}{d|\eta|}(x) = \left( \frac{dD^J u}{d|D^J u|}(x), 0 \right) = (\nu_u(x), 0) = -\nu_{\Sigma^u}(x, y) \quad \forall y \text{ s.t. } (x, y) \in \partial^* \Sigma^u,
\]

iii) for \( |\eta| \)-a.e. \( x \in \left( \Omega \setminus J_u \right) \cap \left\{ x \in \Omega : q \nu_{\Sigma^u}(x, u^\gamma(x)) = 0 \right\} \) we have

\[
\frac{d\eta}{d|\eta|}(x) = \left( \frac{dD^C u}{d|D^C u|}(x), 0 \right).
\]

**Proof.** Statement (i) is proved in (i) of [22, Theorem 4 in (Part 4, Section 1.5)]. Statement (ii) follows by combining (ii) of [22, Theorem 4 in (Part 4, Section 1.5)] with (ii) of [22, Theorem 3 in (Part 4, Section 1.5)]. We will give a proof of point iii). Let \( x \in \Omega \) and consider \( \rho > 0 \), and
recall (2.1), then
\[|\eta|(D_{x,\rho}) = \sup_{|f| \leq 1} \int_{D_{x,\rho}} f(y) \cdot d\eta(y)\]

\[= \sup_{|f| \leq 1} \left( \int_{D_{x,\rho}} (f_1(y), \ldots, f_{n-1}(y)) \cdot dDu(y) - \int_{D_{x,\rho}} f_n(y) dy \right)\]

\[\leq \sup_{|f| \leq 1} \int_{D_{x,\rho}} (f_1(y), \ldots, f_{n-1}(y)) \cdot dDu(y) + \sup_{|f| \leq 1} \int_{D_{x,\rho}} f_n(y) dy\]

\[= |Du|(D_{x,\rho}) + \mathcal{L}^{n-1}(D_{x,\rho}).\]

At the same time we get
\[|\eta|(D_{x,\rho}) = \sup_{|f| \leq 1} \int_{D_{x,\rho}} f(y) \cdot d\eta(y)\]

\[\geq \int_{D_{x,\rho}} (f_1(y), \ldots, f_{n-1}(y)) \cdot dDu(y),\]

so that, passing to the sup in the right hand side, it holds
\[|\eta|(D_{x,\rho}) \geq |Du|(D_{x,\rho}).\]

Putting together these two inequalities we get
\[|Du|(D_{x,\rho}) \leq |\eta|(D_{x,\rho}) \leq |Du|(D_{x,\rho}) + \mathcal{L}^{n-1}(D_{x,\rho}).\] (5.1)

Let now \(x \in Z_u\) and let \(\rho > 0\). Then,
\[\frac{\eta(D_{x,\rho})}{|\eta|(D_{x,\rho})} = \frac{|\eta|(D_{x,\rho})}{|Du|(D_{x,\rho})} \cdot \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})}\]

Since
\[\lim_{\rho \to 0^+} \frac{\eta(D_{x,\rho})}{|Du|(D_{x,\rho})} = \left( \frac{d\mathcal{D}^n u}{d|\mathcal{D}^n u|}(x), 0 \right),\]

we are left to prove that
\[\lim_{\rho \to 0^+} \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})} = 1.\] (5.2)

Thanks to (5.1) we have
\[\frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho}) + |D_{x,\rho}|} \leq \frac{|Du|(D_{x,\rho})}{|\eta|(D_{x,\rho})} \leq \frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho})} = 1.\] (5.3)

Recall that \(x \in Z_u\), so that
\[\lim_{\rho \to 0^+} \frac{|D_{x,\rho}|}{|Du|(D_{x,\rho})} = 0.\]

Thus, we can calculate the following limit for the left hand side of (5.3)
\[\lim_{\rho \to 0^+} \frac{|Du|(D_{x,\rho})}{|Du|(D_{x,\rho}) + |D_{x,\rho}|} = \lim_{\rho \to 0^+} \frac{1}{1 + \frac{|D_{x,\rho}|}{|Du|(D_{x,\rho})}} = 1.\]
Thus, by the above calculation and relation (5.3) we proved (5.2) and so we conclude the proof. □

**Proposition 5.3.** Let \( u \in BV_{loc}(\mathbb{R}^{n-1}) \) and let \( K \subset \mathbb{R}^n \) be as in (1.18). Then, for every Borel set \( B \subset \mathbb{R}^{n-1} \) we have

\[
P_K(\Sigma^n; B \times \mathbb{R}) = \int_{B \setminus (J_u \cup Z_u)} \phi_K(-\nabla u(x), 1) \, dx \\
+ \int_{B \cap J_u} [u](x) \phi_K\left(-\frac{dD^j u}{d|D^j u|}(x), 0\right) \, dH^{n-2}(x) \\
+ \int_{B \cap Z_u} \phi_K\left(-\frac{dD^\delta u}{d|D^\delta u|} (x), 0\right) \, d|D^\delta u|(x),
\]

where \( Z_u \) has been defined at the beginning of this Section.

**Proof.** Let us consider a generic Borel set \( B \subset \mathbb{R}^{n-1} \). Then, thanks to the De Giorgi structure Theorem, Theorem 5.1, and Theorem 5.2 we get

\[
P_K(\Sigma^n; B \times \mathbb{R}) = \int_{\partial^* \Sigma^n \cap (B \times \mathbb{R})} \phi_K(\nu^n(x)) \, dH^{n-1}(x) \\
= \int_{\partial^* \Sigma^n \cap (B \times \mathbb{R})} \phi_K\left(-\frac{dD_1^\Sigma^n}{d|D_1^\Sigma^n|} (x)\right) \, d|D_1^\Sigma^n|(x) \\
= \int_{B} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x).
\]

Let us split the last integral in the following way

\[
\int_{B} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x) = \int_{B \setminus (J_u \cup Z)} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x) \\
+ \int_{B \cap J_u} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x) \\
+ \int_{B \cap Z} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x).
\]

(5.5)

About the first integral on the right hand side we observe that

\[
\eta \subset \mathbb{R}^{n-1} \setminus (J_u \cup Z_u) = (D^a u, -\mathcal{L}^{n-1}) \subset \mathbb{R}^{n-1} \setminus (\nabla u, -1) \mathcal{L}^{n-1} \subset \mathbb{R}^{n-1}.
\]

Therefore, recalling Remark 2.1 we have

\[
\eta(B) = \int_B (\nabla u, -1) \, dx \quad \text{and} \quad |\eta|(B) = \int_B \sqrt{|\nabla u|^2 + 1} \, dx \quad \forall \text{ Borel set } B \subset \mathbb{R}^{n-1} \setminus (J_u \cup Z).
\]

Thus,

\[
\int_{B \setminus (J_u \cup Z)} \phi_K\left(-\frac{d\eta}{d|\eta|} (x)\right) \, d|\eta|(x) = \int_{B \setminus (J_u \cup Z)} \phi_K\left(-\frac{\nabla u(x), 1}{\sqrt{|\nabla u|^2 + 1}}\right) \sqrt{|\nabla u|^2 + 1} \, dx \\
= \int_{B \setminus (J_u \cup Z)} \phi_K(-\nabla u(x), 1) \, dx.
\]

(5.6)

Let us observe now that, thanks to (ii) of Theorem 5.2

\[
\eta \setminus J_u = (D^j u, -\mathcal{L}^{n-1}) \setminus J_u = (D^j u, 0) \setminus J_u.
\]

Thus,

\[
|\eta|(B) = |D^j u|(B) \quad \forall \text{ Borel set } B \subset J_u.
\]
Then,
\[ \int_{B \cap J_u} \phi_K \left( -\frac{d\eta}{d[\eta]}(x) \right) d[\eta](x) = \int_{B \cap J_u} \phi_K \left( -\frac{dD^j u}{d[D^j u]}(x,0) \right) d[D^j u](x) \]
\[ = \int_{B \cap J_u} \phi_K \left( -\frac{dD^j u}{d[D^j u]}(x,0) \right) [u](x) dH^{n-2}(x). \]  
(5.7)

A similar argument holds for the integral over $B \cap Z_u$, so that
\[ \int_{B \cap Z_u} \phi_K \left( -\frac{d\eta}{d[\eta]}(x) \right) d[\eta](x) = \int_{B \cap Z_u} \phi_K \left( -\frac{dD^c u}{d[D^c u]}(x,0) \right) d[D^c u](x). \]  
(5.8)

Combining equations (5.5), (5.6), (5.7) and (5.8) we conclude.

**Remark 5.4.** We can also use the notation of the anisotropic total variation to obtain a more compact formula for the perimeter,
\[ P_K(\Sigma^n; B \times \mathbb{R}) = \int_B \phi_K(-\nabla u(x),1)dx + |(-D^j u,0)|_K(B) + |(-D^c u,0)|_K(B). \]

**Remark 5.5.** Note that, since $\Sigma_u = \mathbb{R}^n \setminus \Sigma^u$, we have $\partial^* \Sigma_u = \partial^* \Sigma^u$ and $\nu^{\Sigma_u}(x) = -\nu^{\Sigma^u}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* \Sigma_u$, and so
\[ P_K(\Sigma_u; B \times \mathbb{R}) = \int_B \phi_K(\nabla u(x),-1)dx + |(D^j u,0)|_K(B) + |(D^c u,0)|_K(B) \]
for every Borel set $B \subset \mathbb{R}^n$.

Before stating the next result, we recall that given a Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, we indicate with $\tilde{f}$ the approximate average of $f$ defined as in (2.7).

**Lemma 5.6.** Let $K \subset \mathbb{R}^n$ be as in (1.18). If $u_1, u_2 \in BV_{loc}(\mathbb{R}^{n-1})$ with $u_1 \leq u_2$ and $E = \Sigma_{u_1} \cap \Sigma_{u_2}$ has finite volume, then $E$ is a set of locally finite perimeter in $\mathbb{R}^n$ and for every Borel set $B \subset \mathbb{R}^{n-1}$
\[ P_K(E; B \times \mathbb{R}) = \int_{B \cap \tilde{u}_1 < \tilde{u}_2} \phi_K(\nabla u_1(x),-1)dx + \int_{B \cap \tilde{u}_1 < \tilde{u}_2} \phi_K(\nabla u_2(x),1)dx \]
\[ + \int_{B \cap [u_1 < u_2]} \phi_K (\nabla u_1(z),0) \left( \min(u_1^*(z),u_2^*(z)) - u_1^*(z) \right) dH^{n-2}(z) \]
\[ + \int_{B \cap [u_1 > u_2]} \phi_K (\nabla u_2(z),0) \left( u_2^*(z) - \max(u_1^*(z),u_2^*(z)) \right) dH^{n-2}(z) \]
\[ + |(D^c u_1,0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}) + |(-D^c u_2,0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}). \]  
(5.9)

**Proof.** We will follow the strategy of [8, Theorem 3.1]. By [26, Theorem 16.3], if $F_1, F_2$ are sets of locally finite perimeter in $\mathbb{R}^n$, then
\[ \partial^*(F_1 \cap F_2) = \partial^*(F_1^{(1)} \cap \partial^* F_2) \cup (F_2^{(1)} \cap \partial^* F_1) \cup \left( \partial^* F_1 \cap \partial^* F_2 \cap \{\nu^{F_1} = \nu^{F_2}\} \right). \]  
(5.10)

Moreover, in the particular case of $F_1 \subset F_2$, then $\nu^{F_1} = \nu^{F_2}$ $\mathcal{H}^{n-1}$-a.e. on $\partial^* F_1 \cap \partial^* F_2$. Let us observe that $u_1 \leq u_2$ implies $\Sigma_{u_2} \subset \Sigma_{u_1}$ and that $\Sigma_{u_2} = \mathbb{R}^n \setminus \Sigma_{u_2}$ implying $\nu_{\Sigma_{u_2}} = -\nu_{\Sigma_{u_2}}$ $\mathcal{H}^{n-1}$-a.e. on $\partial^* \Sigma_{u_2}$. We thus find
\[ \nu^{\Sigma_{u_2}} = -\nu^{\Sigma_{u_1}}, \quad \mathcal{H}^{n-1}$-a.e. on $\partial^* \Sigma_{u_1} \cap \partial^* \Sigma_{u_2}. \]  
(5.11)
By (5.10) and (5.11), since $E = \Sigma_{u_1} \cap \Sigma^{u_2}$ we find
\[
\partial^* E = \mu^{n-1} \left( \partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \right) \cup \left( \partial^* \Sigma^{u_2} \cap (\Sigma_{u_1})^{(1)} \right).
\]
Recall the definition of the approximate discontinuity set of $u_i$ with $i = 1, 2$, that we denote $S_{u_i}$ (see (2.6)). Thanks to [22, Section 4.1.5] we know that $\Sigma_{u_1}$ and $\Sigma^{u_2}$ are sets of locally finite perimeter in $\mathbb{R}^n$ with
\[
\partial^* \Sigma_{u_1} \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : \tilde{u}_1(\mathbf{p}x) = \mathbf{q}x \},
\]
\[
\partial^* \Sigma^{u_1} \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : u_1^\gamma(\mathbf{p}x) < \mathbf{q}x < u_1^\gamma(\mathbf{p}x) \},
\]
\[
\Sigma_{u_1} \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : \tilde{u}_1(\mathbf{p}x) < \mathbf{q}x \},
\]
\[
(\Sigma^{u_1}) \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : u_1^\gamma(\mathbf{p}x) < \mathbf{q}x \},
\]
\[
\Sigma^{(1)} \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : \tilde{u}_1(\mathbf{p}x) < \mathbf{q}x \},
\]
\[
(\Sigma^{(2)}) \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : \tilde{u}_2(\mathbf{p}x) > \mathbf{q}x \},
\]
\[
(\Sigma^{(1)}) \cap (S_{u_1} \times \mathbb{R}) = \mu^{n-1} \{ x \in \mathbb{R}^n : u_2^\gamma(\mathbf{p}x) > \mathbf{q}x \}.
\]
We now focus on the set $\partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)}$. Observe that,
\[
P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap (B \times \mathbb{R}) \right) = P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}] \right)
\]
\[
+ P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}] \right)
\]
\[
+ P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}) \times \mathbb{R}] \right)
\]
\[
+ P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}) \times \mathbb{R}] \right).
\]
Applying (5.12) to $u_1$ and (5.16) to $u_2$ we find
\[
\left( \partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \right) \cap ((J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}) = \mu^{n-1} \{ (z, \tilde{u}_1(z)) : z \in (J_{u_1}^c \cap J_{u_2}^c), \tilde{u}_1(z) < \tilde{u}_2(z) \}.
\]
Applying (5.13) to $u_1$ and (5.16) to $u_2$ we obtain
\[
\left( \partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \right) \cap ((J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}) = \mu^{n-1} \{ (z, t) : z \in (J_{u_1} \cap J_{u_2}^c), u_1^\gamma(z) < t < \min(u_1^\gamma(z), \tilde{u}_2(z)) \}.
\]
Combining (5.13) to $u_1$ and (5.17) to $u_2$ we obtain
\[
\left( \partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \right) \cap ((J_{u_1} \cap J_{u_2}) \times \mathbb{R}) = \mu^{n-1} \{ (z, t) : z \in (J_{u_1} \cap J_{u_2}), u_1^\gamma(z) < t < \min(u_1^\gamma(z), u_2^\gamma(z)) \}.
\]
Finally, applying (5.12) to $u_1$ and (5.17) to $u_2$ we get
\[
\left( \partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \right) \cap ((J_{u_1}^c \cap J_{u_2}) \times \mathbb{R}) = \mu^{n-1} \{ (z, \tilde{u}_1(z)) : z \in (J_{u_1}^c \cap J_{u_2}), \tilde{u}_1(z) < u_2^\gamma(z) \}.
\]
Thus, thanks to Remark 5.5 and (5.18) we get
\[
P_K \left( \Sigma_{u_1} : (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1}^c \cap J_{u_2}^c) \times \mathbb{R}] \right) = \int_{\partial^* \Sigma_{u_1} \cap [(B \cap J_{u_1}^c \cap J_{u_2}^c) \cap \{ \tilde{u}_1 < \tilde{u}_2 \}) \times \mathbb{R}]} \phi_K(-\nu^{\Sigma_{u_1}}(x))d\mathcal{H}^{n-1}(x)
\]
\[
= \int_{B \cap \{ \tilde{u}_1 < \tilde{u}_2 \}} \phi_K(\nabla u_1(x), 1)dx + \int_{(D^c u_1, 0)_{K(B \cap \{ \tilde{u}_1 < \tilde{u}_2 \})}}.
\]
Using Fubini theorem and (5.19) we get
\[ P_K \left( \Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}] \right) \]
\[ = \int_{\partial^* \Sigma_{u_1} \cap \{(\Sigma^{u_2})^{(1)} \cap (B \cap J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}] \phi_K(-\nu^{\Sigma_{u_1}}(y))d\mathcal{H}^{n-1}(y) \]
\[ = \int_{\{x \in \mathbb{R}^n : px \in B \cap J_{u_1} \cap J_{u_2}^c, u_1^{\alpha}(px) < \{q \in \min(u_1^{\gamma}(px), \omega_2(px))\}} \phi_K(-\nu^{\Sigma_{u_1}}(y))d\mathcal{H}^{n-1}(y) \]
\[ = \int_{(B \cap J_{u_1} \cap J_{u_2}^c) \times \mathbb{R}} \phi_K(-\nu^{\Sigma_{u_1}}(y))1_{\{q > u_1^{\gamma}(px)\}(y)}1_{\{q \in \min(u_1^{\gamma}(px), \omega_2(px))\}(y)}d\mathcal{H}^{n-1}(y) \]
\[ = \int_{B \cap J_{u_1} \cap J_{u_2}^c} d\mathcal{H}^{n-2}(z) \int_{\mathbb{R}} \phi_K(-\nu^{\Sigma_{u_1}}(z, t))1_{\{s > u_1^{\gamma}(z)\}(z, t)}1_{\{s \in \min(u_1^{\gamma}(z), \omega_2(z))\}(z, t)}d\mathcal{H}^1(t) \]
\[ = \int_{B \cap J_{u_1} \cap J_{u_2}^c} d\mathcal{H}^{n-2}(z) \int_{\mathbb{R}} \phi_K(nu_1(z), 0)1_{\{t > u_1^{\gamma}(z)\}(z, t)}1_{\{t \in \min(u_1^{\gamma}(z), \omega_2(z))\}(z, t)}d\mathcal{H}^1(t) \]
\[ = \int_{B \cap J_{u_1} \cap J_{u_2}^c} \phi_K(nu_1(z), 0)(\min(u_1^{\gamma}(z), \omega_2(z)) - u_1^{\alpha}(z))d\mathcal{H}^{n-2}(z). \]
Observe that we could have used \(u_2^{\alpha}\) or \(u_2^{\gamma}\) instead of \(\omega_2\) since we are working in \(B \cap J_{u_1} \cap J_{u_2}^c\).
For similar arguments, using (5.20) we get that
\[ P_K \left( \Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}) \times \mathbb{R}] \right) \]
\[ = \int_{B \cap J_{u_1} \cap J_{u_2}} \phi_K(nu_1(z), 0)(\min(u_1^{\gamma}(z), u_2^{\alpha}(z)) - u_1^{\alpha}(z))d\mathcal{H}^{n-2}(z). \]
Furthermore, thanks to (5.21) we deduce that \(\mathcal{H}^{n-1}\left(\partial^* \Sigma_{u_1} \cap (\Sigma^{u_2})^{(1)} \cap (J_{u_1} \cap J_{u_2}) \times \mathbb{R}\right) = 0\).
Thus, we have that
\[ P_K \left( \Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap [(B \cap J_{u_1} \cap J_{u_2}) \times \mathbb{R}] \right) = 0. \]
Therefore,
\[ P_K \left( \Sigma_{u_1}; (\Sigma^{u_2})^{(1)} \cap (B \times \mathbb{R}) \right) = \int_{B \cap \{\tilde{u}_1 \leq \tilde{u}_2\}} \phi_K(\nabla u_1(x), -1)dx \]
\[ + \int_{B \cap J_{u_1}} \phi_K(nu_1(z), 0)(\min(u_1^{\gamma}(z), u_2^{\alpha}(z)) - u_1^{\alpha}(z))d\mathcal{H}^{n-2}(z) \]
\[ + |(D^\alpha u_1, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}). \]  
(5.23)
By symmetry, we got that
\[ P_K \left( \Sigma^{u_2}; (\Sigma_{u_1})^{(1)} \cap (B \times \mathbb{R}) \right) = \int_{B \cap \{\tilde{u}_1 < \tilde{u}_2\}} \phi_K(-\nabla u_2(x), 1)dx \]
\[ + \int_{B \cap J_{u_2}} \phi_K(nu_2(z), 0)(u_2^{\gamma}(z) - \max(u_2^{\alpha}(z), u_2^{\gamma}(z)))d\mathcal{H}^{n-2}(z) \]
\[ + |(-D^\alpha u_2, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}). \]  
(5.24)
Putting together (5.22) and (5.24) we obtain the formula for \(P_K(E; B \times \mathbb{R})\).

We now extend Lemma 5.6 to the case of GBV functions.
Theorem 5.7. Let $K \subset \mathbb{R}^n$ be as in $[1,18]$. If $u_1, u_2 \in GBV(\mathbb{R}^{n-1})$ with $u_1 \leq u_2$ and $E = \Sigma_{u_1} \cap \Sigma_{u_2}$ has finite volume, then $E$ is a set of locally finite perimeter and for every Borel set $B \subset \mathbb{R}^{n-1}$
\[
P_K(E; B \times \mathbb{R}) = \int_{B \cap \{u_1 < u_2\}} \phi_K(\nabla u_1(x), -1)dx + \int_{B \cap \{u_1 < u_2\}} \phi_K(\nabla u_2(x), 1)dx
+ \int_{B \cap \{u_1 \cap u_2\}} \phi_K(\nu_{u_1}(z), 0) \left( \min(u_1^\vee(z), u_2^\wedge(z)) - u_1^\wedge(z) \right) d\mathcal{H}^{n-2}(z)
+ \int_{B \cap \{u_2 \cap u_1\}} \phi_K(\nu_{u_2}(z), 0) \left( u_2^\vee(z) - \max(u_2^\wedge(z), u_1^\vee(z)) \right) d\mathcal{H}^{n-2}(z)
+ |(D^c u_1, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}) + |(-D^c u_2, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\})
\] (5.25)

Proof. To prove (5.25) it suffices to consider the case where $B$ is bounded since (5.25) is an identity between Borel measures on $\mathbb{R}^{n-1}$. Given $M > 0$, let $E_M = \Sigma_{\tau_M(u_1)} \cap \Sigma_{\tau_M(u_2)}$. Since $\tau_M(u_i) \in BV_{\text{loc}}(\mathbb{R}^{n-1})$ for every $M > 0$, $i = 1, 2$, by Lemma 5.6 we find that $E_M$ is a set of locally finite perimeter and that (5.9) holds true on $E_M$ with $\tau_M(u_1)$ and $\tau_M(u_2)$ in place of $u_1$ and $u_2$. To complete the proof of the theorem we are going to show the following identities
\[
P_K(E; B \times \mathbb{R}) = \lim_{M \to +\infty} P_K(E_M; B \times \mathbb{R}) \tag{5.26}
\]
\[
\int_{B \cap \{u_1 < u_2\}} \phi_K(\nabla u_1(x), -1)dx = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) < \tau_M(u_2)\}} \phi_K(\nabla \tau_M(u_1)(x), -1)dx \tag{5.27}
\]
\[
\int_{B \cap \{u_1 < u_2\}} \phi_K(\nabla u_2(x), 1)dx = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) < \tau_M(u_2)\}} \phi_K(\nabla \tau_M(u_2)(x), 1)dx \tag{5.28}
\]
\[
| (D^c u_1, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}) = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) < \tau_M(u_2)\}} \phi_K \left( \frac{dD^c \tau_M(u_1)}{d|D^c \tau_M(u_1)|}(x), 0 \right) d|D^c \tau_M(u_1)|(x) \tag{5.29}
\]
\[
| (-D^c u_2, 0)|_K(B \cap \{\tilde{u}_1 < \tilde{u}_2\}) = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) < \tau_M(u_2)\}} \phi_K \left( - \frac{dD^c \tau_M(u_2)}{d|D^c \tau_M(u_2)|}(x), 0 \right) d|D^c \tau_M(u_2)|(x) \tag{5.30}
\]
\[
\int_{B \cap \{u_1 \cap u_2\}} \phi_K(\nu_{u_1}(z), 0) \left( \min(u_1^\vee(z), u_2^\wedge(z)) - u_1^\wedge(z) \right) d\mathcal{H}^{n-2}(z) = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) \cap \tau_M(u_2)\}} \phi_K \left( \frac{dD^c \tau_M(u_1)}{d|D^c \tau_M(u_1)|}(x), 0 \right) (\min(\tau_M(u_1)^\vee(z), \tau_M(u_2)^\wedge(z)) - \tau_M(u_1)^\wedge(z)) d\mathcal{H}^{n-2}(z) \tag{5.31}
\]
\[
\int_{B \cap \{u_2 \cap u_1\}} \phi_K(\nu_{u_2}(z), 0) \left( u_2^\vee(z) - \max(u_2^\wedge(z), u_1^\vee(z)) \right) d\mathcal{H}^{n-2}(z) = \lim_{M \to +\infty} \int_{B \cap \{\tau_M(u_1) \cap \tau_M(u_2)\}} \phi_K \left( - \frac{dD^c \tau_M(u_2)}{d|D^c \tau_M(u_2)|}(x), 0 \right) (\tau_M(u_2)^\vee(z) - \max(\tau_M(u_2)^\wedge(z), \tau_M(u_1)^\vee(z))) d\mathcal{H}^{n-2}(z). \tag{5.32}
\]
Observe that by $[2$, Theorem 3.99] with $f = \tau_M$ we have for $i = 1, 2$
\[
D(\tau_M(u_i)) = 1_{\{u_i < M\}} \nabla u_i L^{n-1} + (\tau_M(u_i^\vee) - \tau_M(u_i^\wedge)) \nu_i \mathcal{H}^{n-2} \subseteq S_{u_i} + 1_{\{\tilde{u}_i < M\}} D^c u_i \tag{5.33}
\]
We divide the proof in few steps.  

**Step 1 (Jump part)** By relations (2.9)-(2.12) and relation (5.33) we get that \( \{ J_{\tau_M(u_i)} \}_{M>0} \) is a monotone increasing family of sets whose union is \( J_{u_i} \), \( i = 1, 2 \). Moreover, observing that
\[
\begin{align*}
\min (\tau_M(s); \tau_M(t)) &= \tau_M(\min(s,t)) & \forall s, t \in \mathbb{R} \\
\max (\tau_M(s); \tau_M(t)) &= \tau_M(\max(s,t)) & \forall s, t \in \mathbb{R}
\end{align*}
\]
and taking into account relation (2.12) we deduce that both
\[
\begin{align*}
(\min(\tau_M(u_1) \wedge (z), \tau_M(u_2) \wedge (z)) - \tau_M(u_1) \vee (z)), M>0, \\
(\tau_M(u_2) \wedge (z) - \max(\tau_M(u_2) \wedge (z), \tau_M(u_1) \wedge (z))) M>0
\end{align*}
\]
are increasing family of functions. Thus, the proof of (5.31) and (5.32) is completed.

**Step 2 (Cantor part)** Firstly, let us notice that by definition of approximate average (see Section 2) and relation (2.9)
\[
\\{ \tau_M(u_1) \wedge (z) - \max(\tau_M(u_2) \wedge (z), \tau_M(u_1) \wedge (z)) \}_{M>0}
\]
Thus, by relation (2.13) we deduce that both
\[
\begin{align*}
\lim_{M \to +\infty} |D^c u_i| (B \cap \{A_M\}) &= |D^c u_i|(B \cap A) = \lim_{M \to +\infty} |D^c \tau_M u_i|(B \cap A). \quad (5.34)
\end{align*}
\]
Again by the monotonicity of the family of sets \( \{A_M\}_{M>0} \) and by (5.33) we have
\[
|D^c u_i|(A) \leq |D^c \tau_M u_i|(A) \leq |D^c \tau_M u_i|(A).
\]
Thus, taking the limit for \( M \to +\infty \) in the above relation we obtain
\[
|D^c u_i|(A) \leq \lim inf_{M \to \infty} |D^c \tau_M u_i|(A) \leq \lim sup_{M \to \infty} |D^c \tau_M u_i|(A) \leq |D^c u_i|(A),
\]
proving that
\[
\lim_{M \to +\infty} |D^c \tau_M u_i|(A) = |D^c u_i|(A).
\]
Analogously, having in mind Remark 3.13 we get that
\[
\begin{align*}
|D^c u_1|_K(B \cap A) &= \lim_{M \to +\infty} |(D^c \tau_M u_1)|_K(B \cap \{A_M\}), \\
|D^c u_2|_K(B \cap A) &= \lim_{M \to +\infty} |(D^c \tau_M u_2)|_K(B \cap \{A_M\}).
\end{align*}
\]
This concludes the proof for both (5.29) and (5.30).

**Step 3 (Absolutely Continuous part)** By (5.33) we get
\[
\int_{B \cap \{\tau_M(u_1) \wedge (z) - \max(\tau_M(u_2) \wedge (z), \tau_M(u_1) \wedge (z)) \}_{M>0}} \phi_K(\tau_M(u_1)(x), -1) dx = \int_{B \cap \{\tau_M(u_1) \wedge (z) - \max(\tau_M(u_2) \wedge (z), \tau_M(u_1) \wedge (z)) \}_{M>0}} \phi_K(\nabla u_1(x), -1) dx
\]
\[
+ \int_{B \cap \{\tau_M(u_1) \wedge (z) - \max(\tau_M(u_2) \wedge (z), \tau_M(u_1) \wedge (z)) \}_{M>0}} \phi_K(0, -1) dx
\]
\[
= I^M_1 + I^M_2.
\]
Notice that
\[ |I_2^M| = \phi_K(0, -1)\mathcal{L}^{n-1}(B \cap \{\tau_M(u_1) < \tau_M(u_2)\} \cap \{|u_1| \geq M\}) \]
\[ \leq \phi_K(0, -1)\mathcal{L}^{n-1}(B \cap \{|u_1| \geq M\}). \]

By the fact that \( \{|u_1| \geq M\}_{M>0} \) is a decreasing family of sets whose intersection is \( \{|u_1| = +\infty\} \) we deduce that
\[ \lim_{M \to \infty} |I_2^M| = 0. \]

Since both \( \{|u| < M\}_{M>0} \) and \( \{\tau_M(u_1) < \tau_M(u_2)\}_{M>0} \) are increasing family of sets, we apply the monotone convergence theorem to get that
\[ \lim_{M \to \infty} I_1^M = \int_{B \cap \{|u_1| < u_2\}} \phi_K(\nabla u_1(x), -1) \, dx. \]

An analogous argument can be used for relation (5.28) and so this concludes the proof for both (5.27) and (5.28).

**Step 4 (Perimeter functional part)** Lastly, let us consider the family of sets \( E_{M_h} = E \cap \{|x_n| < M_h\} \) where the sequence of real numbers \( \{M_h\}_{h \in \mathbb{N}} \) has been chosen s.t.
\[ \lim_{h \to +\infty} \mathcal{H}^{n-1}\left(\mathcal{E}^{(1)} \cap \{|q_x| = M_h\}\right) = 0, \quad \mathcal{H}^{n-1}(\partial^f E \cap \{|q_x| = M_h\}) = 0 \quad \forall h \in \mathbb{N}. \quad (5.35) \]

Observe that the the existence of such a sequence \( \{M_h\}_{h \in \mathbb{N}} \) is guaranteed by the fact that \( |E| < \infty \) and by the fact that \( \mathcal{H}^{n-1} \subseteq \partial^f E \) is a Radon measure. Thanks to the above two relations and [26, Theorem 16.3] we have that
\[ P_K(E_{M_h}; B \times \mathbb{R}) = \int_{\partial^e E_{M_h} \cap (B \times \mathbb{R})} \phi_K(\nu_E M_h(x))d\mathcal{H}^{n-1}(x) \]
\[ = \int_{\partial^e E_{M_h} \cap (B \times \mathbb{R}) \cap \{|q_x| < M_h\}} \phi_K(\nu_E M_h(x))d\mathcal{H}^{n-1}(x) \]
\[ + \int_{\mathcal{E}^{(1)} \cap \{|q_x| = M_h\} \cap (B \times \mathbb{R})} \phi_K(\nu_E M_h(x))d\mathcal{H}^{n-1}(x). \]

Observing that,
\[ \int_{\mathcal{E}^{(1)} \cap \{|q_x| = M_h\} \cap (B \times \mathbb{R})} \phi_K(\nu_E M_h(x))d\mathcal{H}^{n-1}(x) \leq C\mathcal{H}^{n-1}(\mathcal{E}^{(1)} \cap \{|q_x| = M_h\}), \]
and considering the first relation in (5.35) we finally get
\[ \lim_{h \to +\infty} \int_{\partial^e E_{M_h} \cap (B \times \mathbb{R}) \cap \{|q_x| < M_h\}} \phi_K(\nu_E M_h(x))d\mathcal{H}^{n-1}(x) = P_K(E; B \times \mathbb{R}). \]

This concludes the proof. \( \square \)

Before stating the next result, we recall that given a Borel function \( f : \mathbb{R}^{n-1} \to \mathbb{R} \), we indicate with \( \nu_f(x) \) the approximate jump direction of \( f \) at \( x \) (see Section 2).
Lemma 5.8. If $v \in (BV \cap L^\infty)(\mathbb{R}^{n-1}; [0, \infty))$, $b \in GBV(\mathbb{R}^{n-1})$ and we set $u_1 = b - (v/2) \in GBV(\mathbb{R}^{n-1})$, $u_2 = b + (v/2) \in GBV(\mathbb{R}^{n-1})$, then for $\mathcal{H}^{n-2}$-a.e. $x \in J_v \cap J_b$ we have

If $x \in \left\{b < \left[\frac{v}{2}\right] : \nu_b = \nu_v\right\} \cup \left\{\nu_b = \nu_v\right\}$ then

$$\frac{dD^3 u_1}{d|D^3 u_1|}(x) = -\nu_v(x) \quad (5.36)$$

If $x \in \left\{b > \left[\frac{v}{2}\right] : \nu_b = \nu_v\right\}$ then

$$\frac{dD^3 u_1}{d|D^3 u_1|}(x) = +\nu_v(x) \quad (5.37)$$

If $x \in \left\{b < \left[\frac{v}{2}\right] : \nu_b = -\nu_v\right\} \cup \left\{\nu_b = \nu_v\right\}$ then

$$\frac{dD^3 u_2}{d|D^3 u_2|}(x) = +\nu_v(x) \quad (5.38)$$

If $x \in \left\{b > \left[\frac{v}{2}\right] : \nu_b = -\nu_v\right\}$ then

$$\frac{dD^3 u_2}{d|D^3 u_2|}(x) = -\nu_v(x). \quad (5.39)$$
Moreover,
\[
\text{if } x \in \left\{ [b] = \frac{1}{2} [v] : \nu_b = \nu_v \right\} \quad \text{then} \quad x \notin J_{u_1}, \tag{5.40}
\]
\[
\text{if } x \in \left\{ [b] = \frac{1}{2} [v] : \nu_b = -\nu_v \right\} \quad \text{then} \quad x \notin J_{u_2}. \tag{5.41}
\]

Proof. Firstly, let us notice that thanks to \cite[Proposition 10.5]{26} we already know that for \(H^{n-2}\text{-a.e. } x \in J_v \cap J_b\) either we have
\[
\nu_v(x) = \nu_b(x) \quad \text{or} \quad \nu_v(x) = -\nu_b(x).
\]

Let us start by proving relation (5.36). In particular, using the definition of upper and lower limits, we want to prove that when \(x \in \{ [b] < \frac{v}{2} : \nu_b = \nu_v \}\) (see Figure 5.1C) then
\[
u_u(x) = -\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x), \quad u_1^\wedge(x) = -\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x), \quad \nu_{u_1}(x) = -\nu_v(x). \tag{5.42}
\]

As said, we just need to verify if the definition of jump direction for the upper and lower limit is satisfied, namely if for every \(\epsilon > 0\) we have that
\[
\lim_{\rho \to +\infty} H^{n-1} \left( \left\{ y \in \mathbb{R}^{n-1} : \left| u_1(y) - \left( -\left(\frac{v}{2}\right)^\wedge(x) + b^\wedge(x) \right) \right| > \epsilon \right\} \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right) = 0. \tag{5.43}
\]

Let us substitute in the numerator of (5.43) \(u_1 = b - \frac{v}{2}\) and observe that by the triangular inequality we have that
\[
\left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| \right\} \subseteq \left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| + \left| \left(\frac{v}{2}\right)^\wedge(x) \right| > \epsilon \right\} = A.
\]

Consider now the following partition of \(A\),
\[
\left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| > \frac{\epsilon}{2} \right\} \cap A := A_{>\epsilon}, \tag{5.44}
\]
\[
\left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| \leq \frac{\epsilon}{2} \right\} \cap A := A_{<\epsilon}, \tag{5.45}
\]
\[
\left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| = \frac{\epsilon}{2} \right\} \cap A := A_{=\epsilon}. \tag{5.46}
\]

So, using the above partition we can estimate the quantity in the limit relation (5.43) as follows
\[
H^{n-1} \left( A \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right) \leq \frac{H^{n-1} \left( A_{>\epsilon} \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right)}{\omega_{n-1}\rho^{n-1}} \leq \frac{H^{n-1} \left( A_{>\epsilon} \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right)}{\omega_{n-1}\rho^{n-1}} + \frac{H^{n-1} \left( A_{<\epsilon} \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right) + \frac{H^{n-1} \left( A_{=\epsilon} \cap H^+_{x-v_\rho} \cap D_{x,\rho} \right)}{\omega_{n-1}\rho^{n-1}}. \tag{5.47}
\]

By relation (5.44) we have that
\[
A_{>\epsilon} \subseteq \left\{ y \in \mathbb{R}^{n-1} : \left| b(y) - b^\wedge(x) \right| > \frac{\epsilon}{2} \right\}.
\]
Thus,

\[
\lim_{\rho \to +\infty} \frac{\mathcal{H}^{n-1}(A_{>\epsilon} \cap H^\pm_{x,\nu_0} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} \leq \lim_{\rho \to +\infty} \frac{\mathcal{H}^{n-1}\left(\left\{ y \in \mathbb{R}^{n-1} : |b(y) - b^\wedge(x)| > \frac{\epsilon}{2} \right\} \cap H^+_{x,\nu_0} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} = 0,
\]

where the latter equality holds true by definition of \( b^\wedge(x) \) having in mind that \( \nu_b = \nu_v \) by assumption. Concerning \( A_{<\epsilon} \) we have that

\[
A_{<\epsilon} = \left\{ y \in \mathbb{R}^{n-1} : \left| \frac{v}{2}(y) - \left(\frac{v}{2}\right) \wedge (x) \right| > \epsilon - |b(y) - b^\wedge(x)| \geq \frac{\epsilon}{2} \right\}
\]

Thus,

\[
\lim_{\rho \to +\infty} \frac{\mathcal{H}^{n-1}(A_{<\epsilon} \cap H^+_{x,\nu_0} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} \leq \lim_{\rho \to +\infty} \frac{\mathcal{H}^{n-1}\left(\left\{ y \in \mathbb{R}^{n-1} : \left| \frac{v}{2}(y) - \left(\frac{v}{2}\right) \wedge (x) \right| > \frac{\epsilon}{2} \right\} \cap H^+_{x,\nu_0} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} = 0.
\]

Thanks to the estimate (5.47), putting together (5.48) and (5.49) we get that (5.43) holds true for every \( \epsilon > 0 \). To conclude we have to prove estimate (5.43) for \( u^\wedge_1(x) \) namely we have to prove that

\[
\lim_{\rho \to +\infty} \frac{\mathcal{H}^{n-1}\left(\left\{ y \in \mathbb{R}^{n-1} : \left| u_1(y) - \left( -\left(\frac{v}{2}\right)^\vee (x) + b^\vee(x) \right) \right| > \epsilon \right\} \cap H^-_{x,\nu_0} \cap D_{x,\rho}\right)}{\omega_{n-1}\rho^{n-1}} = 0 \quad \forall \epsilon > 0.
\]

In order to prove that, just use the same argument used for (5.43), noticing that \( H^+_{x,\nu_b} = H^-_{x,\nu_b} = H^\pm_{x,\nu_b} \). To prove the remaining statements (5.37)-(5.39), it is sufficient to consider the same argument adopted for (5.43), considering in each case the right function either \( \frac{v}{2} \) or \( b \) with which construct the partition \( A_{>\epsilon} \) and \( A_{<\epsilon} \).

Let us now prove relation (5.40). Let \( x \in \{ [b] = \frac{1}{2}[v] : \nu_b = \nu_v \} \) and let us consider the functions \( b_k, u_{1,k} \in GBV(\mathbb{R}^{n-1}), k \in \mathbb{N} \) defined as

\[
b_k(z) = \begin{cases} b(z), & \text{if } z \in H^-_{x,\nu_b(x)} \\ b(z) - \frac{1}{k}[b](x), & \text{if } z \in H^+_{x,\nu_b(x)} \end{cases} \quad u_{1,k}(z) = \begin{cases} u_1(z), & \text{if } z \in H^-_{x,\nu_b(x)} \\ u_1(z) - \frac{1}{k}[b](x), & \text{if } z \in H^+_{x,\nu_b(x)}. \end{cases}
\]

Let us note that \( u_{1,k} = b_k - \frac{1}{k}v \). Moreover, note that, \( b_k^\wedge(x) = b^\wedge(x) \), \( b_k^\vee(x) = b^\vee(x) - \frac{1}{k}[b](x) \) and so \([b_k](x) = [b](x) - \frac{1}{k}[b](x)\). In particular, we have that \( x \in \{ [b_k] < 1/2[v] : \nu_b = \nu_v \} \). Thus,
by relations (5.36) and (5.42) applied to $u_{1,k}$ we get that
\begin{align*}
  u_{1,k}^y(x) &= -\frac{1}{2}v^\wedge(x) + b_k^\wedge(x) = -\frac{1}{2}v^\wedge(x) + b^\wedge(x), \\
  u_{1,k}^\wedge(x) &= -\frac{1}{2}v^\vee(x) + b_k^\vee(x) = -\frac{1}{2}v^\vee(x) + b^\vee(x) - \frac{1}{k}[b](x) \\
  &= -\frac{1}{2}v^\vee(x) + b^\vee(x) + \left(1 - \frac{1}{k}\right)[b](x) \\
\end{align*}
(5.50)

Moreover, by (2.4) and (2.5) we have that
\begin{align*}
  u_{1,k}^y(x) &= \inf \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_{1,k} > t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} \\
  u_{1}^y(x) &= \inf \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_1 > t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} \\
  u_{1,k}^\wedge(x) &= \sup \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_{1,k} < t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} \\
  u_{1}^\wedge(x) &= \sup \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_1 < t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\}.
\end{align*}
(5.52), (5.53), (5.54), (5.55)

Observe that the sequence $(u_{1,k})_{k \in \mathbb{N}}$ is non-decreasing in $k$. Thus, we can deduce the following inclusions $\forall k > 1$
\begin{align*}
  \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_{1,k} > t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} &\subset \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_1 > t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} \\
  \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_{1,k} < t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\} &\subset \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{\mathcal{H}^{n-1}(\{u_1 < t\} \cap D_{x,\rho})}{\omega_{n-1}\rho^{n-1}} = 0 \right\}.
\end{align*}

Thanks to the above inclusions, having in mind definitions (5.52), (5.55) together with relations (5.50), (5.51) we get
\[-\frac{1}{2}v^\vee(x) + b^\vee(x) + \left(1 - \frac{1}{k}\right)[b](x) = u_{1,k}^y(x) \leq u_1^y(x) \leq u_1^\wedge(x) = u_{1,k}^\wedge(x) = -\frac{1}{2}v^\wedge(x) + b^\wedge(x).
\]

Since $-\frac{1}{2}v^\vee(x) = -\frac{1}{2}v^\wedge(x) - \frac{1}{2}[v](x)$, passing through the limit as $k \to +\infty$ in the above relation, we conclude that $u_1^y(x) = u_1^\wedge(x)$ and so $x \notin J_{u_1}$. This concludes the proof of (5.40). Using a similar argument as the one used for (5.40), we can prove (5.41). \hfill \square

**Remark 5.9.** The cases where $[b](x) = 0$ i.e. $x \in J_v \setminus J_b$ can be seen as degenerate situations in Lemma 5.8 considering in those characterizations $[b] = 0$. A similar argument can be applied to show that for $\mathcal{H}^{n-2}$-a.e. $x \in J_b \setminus J_v$ we have $\nu_{u_i} = \nu_b$, $i = 1, 2$.\hfill
Remark 5.10. Let us introduce the following compact notation.

\[ A = J_v \setminus J_b, \]

\[ B_1 = \left\{ J_v \cap J_b : \nu_v = \nu_b, [b] < \frac{1}{2} [v] \right\}, \quad B_2 = \left\{ J_v \cap J_b : \nu_v = \nu_b, [b] = \frac{1}{2} [v] \right\}, \]

\[ B_3 = \left\{ J_v \cap J_b : \nu_v = \nu_b, [b] > \frac{1}{2} [v] \right\}, \]

\[ B_4 = \left\{ J_v \cap J_b : \nu_v = -\nu_b, [b] < \frac{1}{2} [v] \right\}, \quad B_5 = \left\{ J_v \cap J_b : \nu_v = -\nu_b, [b] = \frac{1}{2} [v] \right\}, \]

\[ B_6 = \left\{ J_v \cap J_b : \nu_v = -\nu_b, [b] > \frac{1}{2} [v] \right\}, \]

\[ C = J_b \setminus J_v. \]

Note that we have

\[ J_v \cup J_b = A \cup \left( \bigcup_{i=1}^{6} B_i \right) \cup C. \quad (5.56) \]

Moreover, following the argument explained in the proof of Lemma 5.8, we can prove the following relations

if \( x \in A \) then
\[ u_1^\land(x) = -\frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} v^\lor(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} v^\lor(x) + \tilde{b}(x). \quad (5.57) \]

if \( x \in B_1 \cup B_2 \) then
\[ u_1^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} v^\lor(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} v^\lor(x) + \tilde{b}(x). \quad (5.58) \]

if \( x \in B_3 \) then
\[ u_1^\land(x) = -\frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} v^\lor(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} v^\lor(x) + \tilde{b}(x). \quad (5.59) \]

if \( x \in B_4 \cup B_5 \) then
\[ u_1^\land(x) = -\frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} v^\lor(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} v^\lor(x) + \tilde{b}(x). \quad (5.60) \]

if \( x \in B_6 \) then
\[ u_1^\land(x) = -\frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} v^\lor(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} v^\land(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} v^\lor(x) + \tilde{b}(x). \quad (5.61) \]

if \( x \in C \) then
\[ u_1^\land(x) = -\frac{1}{2} \tilde{v}(x) + \tilde{b}(x); \quad u_1^\lor(x) = -\frac{1}{2} \tilde{v}(x) + \tilde{b}(x) \]
\[ u_2^\land(x) = \frac{1}{2} \tilde{v}(x) + \tilde{b}(x); \quad u_2^\lor(x) = \frac{1}{2} \tilde{v}(x) + \tilde{b}(x). \quad (5.62) \]
Corollary 5.11. If $v \in (BV \cap L^\infty) (\mathbb{R}^n; [0, \infty))$, $b \in GBV (\mathbb{R}^{n-1})$ and
\[ W = W[v, b] = \left\{ x \in \mathbb{R}^n : |qx - b(px)| < \frac{v(px)}{2} \right\}, \tag{5.69} \]
then $u_1 = b - (v/2) \in GBV (\mathbb{R}^{n-1})$, $u_2 = b + (v/2) \in GBV (\mathbb{R}^{n-1})$, $W$ is a set of locally finite perimeter with finite volume and for every Borel set $B \subset \mathbb{R}^{n-1}$ we have
\[
P_K(W; B \times \mathbb{R}) = \int_{B \cap \{v > 0\}} \phi_K \left( \nabla \left( b - \frac{v}{2} \right), -1 \right) d\mathcal{H}^{n-1} \tag{5.70} \\
+ \int_{B \cap J_v} \min \left( v', \left( \frac{v}{2} + [b] + \max \left( \frac{v}{2} - [b], 0 \right) \right) \right) \phi_K (-\nu_v, 0) d\mathcal{H}^{n-2} \tag{5.71} \\
+ \int_{B \cap J_v} \min \left( v', \max \left( 0, [b] - \frac{v}{2} \right) \right) \phi_K (\nu_v, 0) d\mathcal{H}^{n-2} \tag{5.72} \\
+ \int_{B \cap (J_v \cup \{\nu_b\})} \min ([b], \check{v}) (\phi_K (-\nu_b, 0) + \phi_K^* (\nu_b, 0)) d\mathcal{H}^{n-2} \tag{5.73} \\
+ \left| D^c \left( b - \frac{v}{2}, 0 \right), 0 \right|_K (B \cap \{\check{v} > 0\}) \tag{5.74} \\
+ \left| -D^c \left( b + \frac{v}{2}, 0 \right), 0 \right|_K (B \cap \{\check{v} > 0\}). \tag{5.75} \]

Proof. The absolutely continuous part and the Cantor parts of the formula, namely relations (5.70), (5.74) and (5.75) are obtained directly by substitution of $u_1 = b - \frac{1}{2} v$ and $u_2 = b + \frac{1}{2} v$ in the formula (5.25). To prove the jump parts of the formula i.e. (5.71), (5.72) and (5.73) we have first to notice that (see (5.56))
\[ J_{u_1} \cup J_{u_2} = J_v \cup J_b = J_v \setminus J_b \cup (J_v \cap J_b) \cup J_b \setminus J_v = A \cup \bigcup_{i=1}^6 B_i \cup C. \]

Thanks to this relation, we can rewrite the second and third line of the formula (5.25) as
\[
\int_{B \cap (J_v \cup J_b)} \phi_K (\nu_v (z), 0) \left( \min (u_1^v (z), u_2^v (z)) - u_1^\check{v} (z) \right) \\
+ \phi_K (-\nu_v (z), 0) \left( u_2^v (z) - \max (u_2^\check{v} (z), u_1^\check{v} (z)) \right) d\mathcal{H}^{n-2}(z) \\
= \int_{B \cap (J_v \cup J_b)} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) = \int_A I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) \\
+ \sum_{i=1}^6 \int_{B_i} I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z) + \int_C I_1(z) + I_2(z) d\mathcal{H}^{n-2}(z). \]

Using then Lemma 5.8, Remark 5.9 and Remark 5.10 we deduce relations (5.71), (5.72) and (5.73). This concludes the proof. \hfill \square

Corollary 5.12. If $v$ is as in (1.4), then
\[
P_K(F[v]; G \times \mathbb{R}) = \int_{G \cap \{v > 0\}} \phi_K \left( -\frac{1}{2} \nabla (v), -1 \right) d\mathcal{H}^{n-1} + \int_{G \cap \{v > 0\}} \phi_K \left( -\frac{1}{2} \nabla (v), 1 \right) d\mathcal{H}^{n-1} \\
+ \int_{G \cap J_v} [v] \phi_K^* (-\nu_v, 0) d\mathcal{H}^{n-2} + 2 \left| \left( -\frac{1}{2} D^c v, 0 \right) \right|_K (G). \]
Proof. The proof follows by applying Corollary 5.7 with \( u_1 = -\frac{1}{2}v \) and \( u_2 = \frac{1}{2}v \), and by recalling that by Lemma 2.2 \( \left( -\frac{1}{2}D^*v, 0 \right) \right|_K (G) = \left( -\frac{1}{2}D^*v, 0 \right) \right|_K (G \cap \{ \dot{v} > 0 \}). \)

\[ \square \]

6. Characterization of equality cases for the anisotropic perimeter inequality

This section is dedicated to the proof of Theorem 1.8. This proof is on the spirit of the proof of Theorem 1.4 (see [8] Theorem 1.9). We split the proof of Theorem 1.8 in the necessary part and in the sufficient part.

Proof of Theorem 1.8: Necessary conditions. Let \( E \in \mathcal{M}_{K^*}(v) \). Condition (1.9) was already proved in [13] Theorem 2.9. As a consequence, by Theorem 1.3, we have that \( b_\delta = 1_{\{v > \delta\}} b_E \in GBV(\mathbb{R}^{n-1}) \) for every \( \delta > 0 \) such that \( \{v > \delta\} \) is a set of finite perimeter in \( \mathbb{R}^{n-1} \). Let us consider the same sets defined in [8] page 1568 namely

\[
I = \{ \delta > 0 : \{ v < \delta \} \text{ and } \{ v > \delta \} \text{ are sets of finite perimeter} \},
\]

\[
J_\delta = \{ M > 0 : \{ b_\delta < M \} \text{ and } \{ b_\delta > -M \} \text{ are sets of finite perimeter} \},
\]

where \( J_\delta \) is defined for \( \delta \in I \). Let us observe that \( \mathcal{H}^1((0, \infty) \setminus I) = 0 \) since \( v \in BV(\mathbb{R}^{n-1}) \) and that \( \mathcal{H}^1((0, \infty) \setminus J_\delta) = 0 \) for every \( \delta \in I \), as for every \( \delta \in I \) we have \( b_\delta \in GBV(\mathbb{R}^{n-1}) \). Let us fix \( \delta, L \in I \) and \( M \in J_\delta \) and set

\[
\Sigma_{\delta,L,M} = \{ \delta < v < L \} \cap \{ |b_E| < M \} = \{ |b_\delta| < M \} \cap \{ \delta < v < L \},
\]

so that \( \Sigma_{\delta,L,M} \) is a set of finite perimeter. Since \( \tau_M b_\delta \in (BV \cap L^{\infty})(\mathbb{R}^{n-1}) \), \( 1_{\Sigma_{\delta,L,M}} \in (BV \cap L^{\infty})(\mathbb{R}^{n-1}) \) and \( \tau_M b_\delta = b_\delta = b_E \) on \( \Sigma_{\delta,L,M} \), we set

\[
b_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} b_E.
\]

Note that \( b_{\delta,L,M} \in (BV \cap L^{\infty})(\mathbb{R}^{n-1}) \).

Step 1 In this step we are going to prove that for \( \mathcal{H}^{n-1} \text{-a.e. } x \in \mathbb{R}^{n-1} \) there exists \( z(x) \in \partial K^* \) such that

\[
\left\{ \left( -\frac{1}{2} \nabla v(x) + t \nabla b_{\delta,L,M}(x), 1 \right) : t \in [-1, 1] \right\} \subset C^*_K(z(x)).
\]

Indeed, let us set \( v_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} v \). Since \( v_{\delta,L,M} , b_{\delta,L,M} \in (BV \cap L^{\infty})(\mathbb{R}^{n-1}) \), we can apply Corollary 5.11 to \( W = W[v_{\delta,L,M}, b_{\delta,L,M}] \). Moreover observe that \( W[v_{\delta,L,M}, b_{\delta,L,M}] = E \cap (\Sigma_{\delta,L,M} \times \mathbb{R}) \) and thus

\[
\partial^c E \cap (\Sigma^{(1)}_{\delta,L,M} \times \mathbb{R}) = \partial^c W[v_{\delta,L,M}, b_{\delta,L,M}] \cap (\Sigma^{(1)}_{\delta,L,M} \times \mathbb{R}),
\]

and so, for every Borel set \( G \subset \Sigma^{(1)}_{\delta,L,M} \setminus (S_{v_{\delta,L,M}} \cup S_{b_{\delta,L,M}}) \) we find that

\[
P_{K^*}(E; G \times \mathbb{R}) = P_{K^*}(W[v_{\delta,L,M}, b_{\delta,L,M}]; G \times \mathbb{R})
\]

\[
= \int_G \phi_{K^*} \left( \nabla \left( b_{\delta,L,M} - \frac{v_{\delta,L,M}}{2} \right), 1 \right) + \phi_{K^*} \left( -\nabla \left( b_{\delta,L,M} + \frac{v_{\delta,L,M}}{2} \right), 1 \right) d\mathcal{H}^{n-1}
\]

\[
+ \left( D^c \left( b_{\delta,L,M} - \frac{v_{\delta,L,M}}{2} \right), 0 \right) \right|_{K^*} (G) + \left( -D^c \left( b_{\delta,L,M} + \frac{v_{\delta,L,M}}{2} \right), 0 \right) \right|_{K^*} (G),
\]
where in the first addendum of the second line we have used Remark 3.21. We can use Lemma 2.3 applied with \( \nu_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}}v \), to find that

\[
\nabla v_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}} \nabla v, \quad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{R}^{n-1},
\]

\[
D^c v_{\delta,L,M} = D^c v \mathbb{1}_{\Sigma_{\delta,L,M}^{(1)}},
\]

\[
S_{\nu_{\delta,L,M}} \cap \Sigma_{\delta,L,M}^{(1)} = S_{v} \cap \Sigma_{\delta,L,M}^{(1)}.
\]

Thus,

\[
P_{K}\ast(E; G \times \mathbb{R}) = \int_{G} \phi_{K}\ast \left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right) + \phi_{K}\ast \left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right) d\mathcal{H}^{n-1}
\]

\[+
\left| D^c \left( b_{\delta,L,M} - \frac{v}{2} \right), 0 \right|_{K}\ast (G) + \left| -D^c \left( b_{\delta,L,M} + \frac{v}{2} \right), 0 \right|_{K}\ast (G),
\]

for every Borel set \( G \subset \Sigma_{\delta,L,M}^{(1)} \setminus (S_{\nu_{\delta,L,M}} \cup S_{b_{\delta,L,M}}) \). We are assuming that \( E \in \mathcal{M}_{K}\ast(v) \) and so for every Borel set \( G \subset \mathbb{R}^{n-1} \) we have that \( P_{K}\ast(E; G \times \mathbb{R}) = P_{K}\ast(F[v]; G \times \mathbb{R}) \). In particular, having in mind the formula for \( P_{K}\ast(F[v]; G \times \mathbb{R}) \) given by Corollary 5.12 for every Borel set \( G \subset \Sigma_{\delta,L,M}^{(1)} \setminus (S_{\nu_{\delta,L,M}} \cup S_{b_{\delta,L,M}}) \) we get

\[
0 = \int_{G} \phi_{K}\ast \left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right) + \phi_{K}\ast \left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right) - 2 \phi_{K}\ast \left( -\nabla \left( \frac{v}{2} \right), 1 \right) d\mathcal{H}^{n-1}
\]

\[+
\left| D^c \left( b_{\delta,L,M} - \frac{v}{2} \right), 0 \right|_{K}\ast (G) + \left| -D^c \left( b_{\delta,L,M} + \frac{v}{2} \right), 0 \right|_{K}\ast (G) - 2 \left| -D^c \left( \frac{v}{2} \right), 0 \right|_{K}\ast (G),
\]

Let us notice that the first line in the above relation, namely (6.4), is greater than or equal to zero by the sub additivity of \( \phi_{K} \). Also the second line in the above relation, namely (6.5), is greater than or equal to zero thanks to Lemma 4.5 with \( \mu = \left( -\frac{1}{2} D^c v, 0 \right) \) and \( \nu = (D^c b_{\delta,L,M}, 0) \). Thus, we have that

\[
0 = \int_{G} \phi_{K}\ast \left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right) + \phi_{K}\ast \left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right) - 2 \phi_{K}\ast \left( -\nabla \left( \frac{v}{2} \right), 1 \right) d\mathcal{H}^{n-1}
\]

\[+
\left| D^c \left( b_{\delta,L,M} - \frac{v}{2} \right), 0 \right|_{K}\ast (G) + \left| -D^c \left( b_{\delta,L,M} + \frac{v}{2} \right), 0 \right|_{K}\ast (G) - 2 \left| -D^c \left( \frac{v}{2} \right), 0 \right|_{K}\ast (G) + 2 \phi_{K}\ast \left( -\nabla \left( \frac{v}{2} \right), 1 \right).
\]

Let us observe that the relation (6.6) is satisfied if and only if \( \mathcal{H}^{n-1}\text{-a.e. in } G \) we have

\[
\phi_{K}\ast \left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right) + \phi_{K}\ast \left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right) = 2 \phi_{K}\ast \left( -\nabla \left( \frac{v}{2} \right), 1 \right).
\]

Thanks to Proposition 3.21 the condition above is satisfied if and only if for \( \mathcal{H}^{n-1}\text{-a.e. } x \in G, \exists \tilde{z}(x) \in \partial K^{*} \) s.t.

\[
\frac{\left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right)}{\phi_{K}\ast \left( \nabla \left( b_{\delta,L,M} - \frac{v}{2} \right), 1 \right)} \cdot \frac{\left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right)}{\phi_{K}\ast \left( -\nabla \left( b_{\delta,L,M} + \frac{v}{2} \right), 1 \right)} = \partial K^{*}(\tilde{z}(x)).
\]

As we observed in Remark 3.22, and in particular using relation (3.15) with \( y_1 = \left( -\frac{1}{2} \nabla (x) + \nabla b_{\delta,L,M}(x), 1 \right) \) and \( y_2 = \left( -\frac{1}{2} \nabla (x) - \nabla b_{\delta,L,M}(x), 1 \right) \) the condition above is equivalent to say that for \( \mathcal{H}^{n-1}\text{-a.e. } x \in G, \) there exists \( \tilde{z}(x) \in \partial K^{*} \) s.t.

\[
\left\{ \left( -\frac{1}{2} \nabla (x) + t \nabla b_{\delta,L,M}(x), 1 \right) : t \in [-1, 1] \right\} \subset C_{K}^{*}(\tilde{z}(x)).
\]

(6.8)
This concludes the first step.

**Step 2** In this step we prove that there exists a Borel measurable function $g_{\delta,L,M} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that

$$D^c b_{\delta,L,M} \cap \Sigma_{\delta,L,M}^{(1)} = g_{\delta,L,M} \left[ \frac{1}{2} D^c v \right]_{K^*} \cap \Sigma_{\delta,L,M}^{(1)}.$$  

We prove also an intermediate relation for (1.30). Indeed, let us rewrite relation (6.7) as

$$\left| (-D^c v, 0) \right|_{K^*} = \left( D^c \left( b_{\delta,L,M} - \frac{v}{2} \right), 0 \right) |_{K^*} (G) + \left( -D^c \left( b_{\delta,L,M} + \frac{v}{2} \right), 0 \right) |_{K^*} (G).$$

As already observed, by calling

$$\mu = \left( -\frac{D^c v}{2}, 0 \right), \quad \nu = (D^c b_{\delta,L,M}, 0)$$

the above equality can be written as

$$2|\mu|_{K^*} = |\mu + \nu|_{K^*} + |\mu - \nu|_{K^*}.$$

Observe that we are in a case of equality in Lemma 4.5. Thus, by Remark 4.7 for $|D^c v|$-a.e. $x \in G$ we define

$$g_{\delta,L,M}(x) = \frac{dD^c b_{\delta,L,M}}{d|(-D^c v/2, 0)|_{K^*}}(x), \quad h(x) = \frac{-dD^c v/2}{d|(-D^c v/2, 0)|_{K^*}}(x),$$

and we conclude that for $|D^c v|$-a.e. $x \in G$ there exists $z(x) \in \partial K$ s.t.

$$\{(h(x) + tg_{\delta,L,M}(x), 0) : t \in [-1, 1] \} \subset C^*_K(z(x)). \quad (6.9)$$

This concludes the second step.

**Step 3** In this step we prove (1.29). We fix $\delta, L \in I$ and we define $\Sigma_{\delta,L} = \{ \delta < v < L \}$, $b_{\delta,L} = 1_{\Sigma_{\delta,L}} b_0$ and $v_{\delta,L} = 1_{\Sigma_{\delta,L}} v$. Since $\Sigma_{\delta,L}$ is a set of finite perimeter, it turns out that $b_{\delta,L} \in GBV(\mathbb{R}^{n-1})$, while, by construction, $v_{\delta,L} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$. So, we can apply the formula of Corollary 5.11 to the set $W[v_{\delta,L}, b_{\delta,L}]$. In particular, if $G \subset \Sigma_{\delta,L} \cap (S_{v_{\delta,L}} \cup S_{b_{\delta,L}})$, then

$$P_{K^*}(E; G \times \mathbb{R}) = P_{K^*}(W[v_{\delta,L}, b_{\delta,L}; G \times \mathbb{R})$$

$$= \int_{G \cap J_v} \min \left( v^\lor, \left( \left[ \frac{v}{2} \right] + [b_{\delta,L}] + \max \left( \left[ \frac{v}{2} \right] - [b_{\delta,L}], 0 \right) \right) \right) \phi_{K^*}(-\nu, 0) d\mathcal{H}^{n-2} \quad (6.10)$$

$$+ \int_{G \cap \tilde{J}_v} \min \left( v^\lor, \max \left( 0, [b_{\delta,L}] - \left[ \frac{v}{2} \right] \right) \right) \phi_{K^*}(\nu, 0) d\mathcal{H}^{n-2}$$

$$+ \int_{G \cap (J_{\delta,L} \setminus J_v)} \min \left( [b_{\delta,L}], \tilde{v} \right) \left( \phi_{K^*}(-\nu_{b_{\delta,L}}, 0) + \phi_{K^*}(\nu_{b_{\delta,L}}, 0) \right) d\mathcal{H}^{n-2},$$

where we used the fact that, thanks to (2.17)

$$\Sigma_{\delta,L}^{(1)} \cap S_{v_{\delta,L}} = \Sigma_{\delta,L}^{(1)} \cap S_{v}, \quad v_{\delta,L}^\lor = v^\lor, \quad v_{\delta,L}^\land = v^\land, \quad [v_{\delta,L}] = [v] \quad \forall x \in \Sigma_{\delta,L}^{(1)}.$$

Let us observe that, calling $I$ the argument of the integral in relation (6.10) i.e.

$$I = \min \left( v^\lor, \left( \left[ \frac{v}{2} \right] + [b_{\delta,L}] + \max \left( \left[ \frac{v}{2} \right] - [b_{\delta,L}], 0 \right) \right) \right)$$
we have that
\[ \begin{align*}
  \text{if} \quad [b_{\delta,L}] = 0 & \quad \text{then} \quad I = [v], \\
  \text{if} \quad [b_{\delta,L}] \leq \frac{1}{2}[v] & \quad \text{then} \quad I = [v], \\
  \text{if} \quad [b_{\delta,L}] > \frac{1}{2}[v] & \quad \text{then} \quad I > [v].
\end{align*} \]

Recall that
\[ P_{K^{\infty}}(F[v]; G \times \mathbb{R}) = \int_{G \cap J_v} [v] \phi_{K^{\infty}}(-\nu_v, 0) d\mathcal{H}^{n-2}. \]

Thus, since \( \phi_{K^{\infty}} \geq 0 \), imposing that \( P_{K^{\infty}}(F[v]; G \times \mathbb{R}) = P_{K^{\infty}}(E; G \times \mathbb{R}) \) and having in mind relations (6.11)-(6.13) we obtain that
\[ \min ([b_{\delta,L}], \bar{v}) = 0, \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap (S_{b_{\delta,L}} \setminus S_v) \] (6.14)
\[ \min \left( v^+, \max \left( 0, [b_{\delta,L}] - \left\lfloor \frac{v}{2} \right\rfloor \right) \right) = 0, \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap S_v \] (6.15)
\[ I = \min \left( v^+, \left\lfloor \frac{v}{2} \right\rfloor + [b_{\delta,L}] + \max \left( \left\lfloor \frac{v}{2} \right\rfloor - [b_{\delta,L}], 0 \right) \right) = [v] \quad \mathcal{H}^{n-2}\text{-a.e. in } G \cap S_v. \] (6.16)

Since \( \bar{v} \geq \delta > 0 \) in \( \Sigma_{\delta,L}^{(1)} \), from (6.14) it follows that \( S_{b_{\delta,L}} \cap \Sigma_{\delta,L}^{(1)} \subset \mathcal{H}^{n-2} S_v \). Moreover, from (6.11), (6.12) together with (6.14) and (6.15) it follows that
\[ [b_{\delta,L}] \leq \frac{[v]}{2} \quad \mathcal{H}^{n-2}\text{-a.e. } x \in G \cap S_v. \] (6.17)

By (2.17), \( [b_{\delta,L}] = [b_E] \) on \( \Sigma_{\delta,L}^{(1)} \). By taking the union of \( \Sigma_{\delta,L}^{(1)} \) on \( \delta, L \in I \) and by taking (2.15), (2.16) into account we thus find that
\[ [b_E] \leq \frac{[v]}{2} \quad \mathcal{H}^{n-2}\text{-a.e. on } \{ v^+ > 0 \} \cup \{ v^+ < \infty \}. \]

Since, by 4.5.9(3) \( \{ v^+ = \infty \} \) is \( \mathcal{H}^{n-2}\)-negligible, we have proved (1.29).

**Step 4** In this step we prove (1.28). Let \( \delta, L \in I \) and \( M \in J_\delta \). Since \( b_{\delta,L,M} = b_E \mathcal{H}^{n-1}\text{-a.e. on } \Sigma_{\delta,L,M} \) by (6.3) and by (2.21) we find that for \( \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma_{\delta,L,M} \), there exists \( z(x) \in \partial K^s \) s.t.
\[ \left\{ \left( -\frac{1}{2} \nabla v(x) + t \nabla b_E(x), 1 \right) : \ t \in [-1,1] \right\} \subset C_{K^s}^\ast(z(x)). \]

By taking a union first on \( M \in J_\delta \) and then on \( \delta, L \in I \), we find that for \( \mathcal{H}^{n-1}\text{-a.e. } x \in \{ v > 0 \} \), there exists \( z(x) \in \partial K^s \) s.t.
\[ \left\{ \left( -\frac{1}{2} \nabla v(x) + t \nabla b_E(x), 1 \right) : \ t \in [-1,1] \right\} \subset C_{K^s}^\ast(z(x)). \]

This concludes the proof of (1.28).

**Step 5** In this step we prove (1.30). Let \( \delta, L \in I \) and \( M \in J_\delta \). Since \( b_{\delta,L,M} = 1_{\Sigma_{\delta,L,M}^1} \tau_M b_{\delta} \), by Lemma 2.3 we have
\[ D_{\nu} b_{\delta,L,M} = D_{\nu} (\tau_M b_{\delta}) \mathcal{L} \Sigma_{\delta,L,M}^{(1)}. \]
Combining this fact with (6.9) we find that for every \( G \subset \Sigma^{(1)}_{\delta, L, M} \), for \( |D^c v| \)-a.e. \( x \in G \) there exists \( z(x) \in \partial K \) s.t.

\[
\{(h(x) + tg_\delta M(x), 0) : t \in [-1, 1]\} \subset C_K^*(z(x)),
\]

where for \( |D^c v| \)-a.e. \( x \in G \) the functions \( g_\delta M \) and \( h \) are given by

\[
g_\delta M(x) = \frac{dD^c(\tau M b_\delta)}{d\left|(D^c v/2)\right|_{K^*}}(x), \quad h(x) = -\frac{dD^c v/2}{d\left|(D^c v/2)\right|_{K^*}}(x).
\]

Observe now that

\[
\gamma = \frac{dD^c(\tau_M b_\delta)}{d\left|(D^c v/2)\right|_{K^*}}(x), \quad h(x) = -\frac{dD^c v/2}{d\left|(D^c v/2)\right|_{K^*}}(x).
\]

Observe that for every bounded Borel set \( G \subset \{b_\delta | < M\}(1) \cap \{v < L\}(1) \)

\[
\tau_M b_\delta = \tau_{M'} b_{\delta'} \quad \text{on} \quad \{b_\delta | < M\} \cap \{v > \delta\}.
\]

So, by Lemma 2.3 we get that

\[
D^c(\tau_M b_\delta) \{b_\delta | < M\}(1) \cap \{v > \delta\}(1) = D^c(\tau_{M'} b_{\delta'}) \{b_\delta | < M\}(1) \cap \{v > \delta\}(1),
\]

and therefore the function \( g_\delta M \) actually does not depend on \( \delta, M \). Thus, we proved that for every bounded Borel set \( G \subset \{b_\delta | < M\}(1) \cap \{v > \delta\}(1) \), for \( |D^c v| \)-a.e. \( x \in G \) there exists \( z(x) \in \partial K \) s.t.

\[
\{(h(x) + tg_\delta M(x), 0) : t \in [-1, 1]\} \subset C_K^*(z(x)).
\]

Lastly, let us notice that

\[
\tau_M b_\delta = M1_{\{b_\delta \geq M\}} - M1_{\{b_\delta \leq -M\}} + 1_{\{b_\delta | < M\} \cap \{v > \delta\}} \tau_M b_\delta, \quad \text{on} \quad \mathbb{R}^{n-1}
\]

is an identity between BV functions. Thus, thanks to [2] Example 3.97 we find that

\[
D^c\tau_M b_\delta = D^c(\tau_M b_\delta) \{G \cap \{b_\delta | < M\}(1) \cap \{v > \delta\}(1)\}
\]

i.e. the measure \( D^c\tau_M b_\delta \) is concentrated on \( \{b_\delta | < M\}(1) \cap \{v > \delta\}(1) \). Therefore, we deduce that for every bounded Borel set \( G \subset \mathbb{R}^{n-1}, \) for \( |D^c v| \)-a.e. \( x \in G \cap \{b_\delta | < M\}(1) \cap \{v > \delta\}(1) \) there exists \( z(x) \in \partial K \) s.t.

\[
\{(h(x) + tg(x), 0) : t \in [-1, 1]\} \subset C_K^*(z(x)).
\]

\( \square \)

Before entering into the details of the proof for the sufficient conditions part, we need a couple of technical results.
Proposition 6.1. Let $K \subset \mathbb{R}^n$ be as in (1.18) and let $v$ be as in (1.4). Then, if $E$ is a $v$-distributed set of finite perimeter with sections $E_z$ as segments $\mathcal{H}^{n-1}$-a.e on $\{v > 0\}$ we have that

$$ P_K(E; \{v^+ = 0\} \times \mathbb{R}) = P_K(F[v]; \{v^+ = 0\} \times \mathbb{R}) = \int_{\{v^+ = 0\}} v^+ \phi_K(-\nu_v, 0) d\mathcal{H}^{n-2}. \quad (6.21) $$

Proof. The proof of this result follows from a careful inspection of the proof of [8], Proposition 3.8, and for this reason is omitted. \hfill \Box

Lemma 6.2. If $v \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$, $b : \mathbb{R}^{n-1} \to \mathbb{R}$ is such that $\tau_M b \in (BV \cap L^\infty)(\mathbb{R}^{n-1})$ for a.e. $M > 0$ and $\mu$ is a $\mathbb{R}^{n-1}$-valued Radon measure such that

$$ \lim_{M \to \infty} |\mu - D^c \tau_M b|(G) = 0 \quad \text{for every bounded Borel set } G \subset \mathbb{R}^{n-1}, \quad (6.22) $$

then,

$$ |(D^c(b + v), 0)|_{K^*}(G) \leq |(\mu + D^c v, 0)|_{K^*}(G) \quad \text{for every bounded Borel set } G \subset \mathbb{R}^{n-1}. \quad (6.23) $$

Proof. Let $L > 0$ be such that $|v| \leq L$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$. If $f \in BV(\mathbb{R}^{n-1})$, then

$$ \tau_M f = M1_{\{f > M\}} - M1_{\{f < -M\}} + 1_{\{|f| < M\}} f \in (BV \cap L^\infty)(\mathbb{R}^{n-1}), $$

for every $M$ such that $\{f > M\}$ and $\{f < -M\}$ are of finite perimeter and thus, by [2], Theorem 3.96

$$ D^c \tau_M f = D^c \left(1_{\{|f| < M\}} f\right) = 1_{\{|f| < M\}} \left(D^c f \right) = D^c f \setminus \{|f| < M\}(1); $$

in particular,

$$ |(D^c \tau_M f, 0)|_{K^*} = |(D^c f, 0)|_{K^*} \setminus \{|f| < M\}(1) \leq |(D^c f, 0)|_{K^*}. \quad (6.24) $$

From the equality $\tau_M(\tau_{M+L}(b) + v) = \tau_M(b + v)$ and from (6.24) applied with $f = \tau_{M+L}(b) + v$ it follows that, for every Borel set $G \subset \mathbb{R}^{n-1}$,

$$ |(D^c(\tau_M(b + v)), 0)|_{K^*}(G) = |(D^c(\tau_{M+L}(b + v)), 0)|_{K^*}(G) \leq |(D^c(\tau_{M+L}(b) + v), 0)|_{K^*}(G). \quad (6.25) $$

Now observe that (6.22) implies that

$$ \lim_{M \to \infty} |-(\mu - D^c \tau_M b)|(G) = 0 \quad \text{for every bounded Borel set } G \subset \mathbb{R}^{n-1}. \quad (6.26) $$

Thanks to Remark 3.12 together with (6.22) and (6.26), for every bounded Borel set $G \subset \mathbb{R}^{n-1}$ we get

$$ \lim_{M \to \infty} |-(\mu - D^c \tau_M b, 0)|_{K^*}(G) = \lim_{M \to \infty} |(\mu - D^c \tau_M b, 0)|_{K^*}(G) = 0. \quad (6.27) $$

Since we can always write $D^c(\tau_M b) + D^c v = (D^c(\tau_M b) - \mu) + (\mu + D^c v)$ by applying relations (4.2) and (4.3) we obtain

$$ |(\mu + D^c v, 0)|_{K^*}(G) - |-(D^c(\tau_{M+L} b - \mu, 0)|_{K^*}(G) \leq |(D^c(\tau_{M+L} b + D^c v, 0)|_{K^*}(G) \quad (6.28) $$

$$ \leq |(D^c(\tau_{M+L} b - \mu, 0)|_{K^*}(G) + |(\mu + D^c v, 0)|_{K^*}(G). \quad (6.29) $$

So, by (6.27) we get

$$ \lim_{M \to \infty} |(D^c(\tau_{M+L}(b) + v), 0)|_{K^*}(G) = |(\mu + D^c v, 0)|_{K^*}(G). \quad (6.29) $$
By (6.25), we get that
\[ \limsup_{M \to \infty} |(D^c(\tau_M(b + v)), 0)|_{K^*}(G) \leq |(\mu + D^c v, 0)|_{K^*}(G), \]
so that using (3.11) we conclude the proof. \(\square\)

**Proof of Theorem 5.8: sufficient conditions.** Let \(E\) be a \(v\)-distributed set of finite perimeter satisfying (1.6), (1.28), (1.29) and (1.30). Let \(I\) and \(J_{\delta}\) be defined as in (6.1) and (6.2). Let \(\delta, S \in I\) and let us set \(b_{\delta, S} = 1_{\{\delta < v < S\}}b_{E} = 1_{\{\delta < v < S\}}b_{\delta}\). Then, for every \(M \in J_{\delta}\), we have \(\tau_M b_{\delta} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})\) and so we obtain that \(\tau_M b_{\delta, S} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})\). Let us consider the \(\mathbb{R}^{n-1}\)-valued Radon measure \(\mu_{\delta, S}\) on \(\mathbb{R}^{n-1}\) defined as
\[ \mu_{\delta, S}(G) = \int_{G \cap \{|\delta < v < S\} \cap \{|b_E|^{\gamma} < \infty\}} g(x) d\left(\frac{1}{2} D^c v, 0\right) \mid_{K^*}, \]
for every bounded Borel set \(G \subset \mathbb{R}^{n-1}\), where \(g(x)\) is the function that appears in condition (1.30), namely
\[ D^c(\tau_M(b_{\delta}))(G) = \int_{G \cap \{|b_{\delta}| < M\}^{(1)} \cap \{|v > \delta\}^{(1)} \cap \{|b_E|^{\gamma} < \infty\} \cap \{|b_E| < M\}^{(1)} \} g(x) d\left(\frac{1}{2} D^c v, 0\right) \mid_{K^*}. \]
Since \(\tau_M b_{\delta, S} = 1_{\{v < S\}} \tau_M b_{\delta}\), by Lemma 2.3 we have \(D^c(\tau_M b_{\delta, S}) = 1_{\{v < S\}} D^c(\tau_M b_{\delta})\) and thus, for every Borel set \(G \subset \mathbb{R}^{n-1}\),
\[ \lim_{M \to \infty} |\mu_{\delta, S} - D^c(\tau_M b_{\delta, S})|(G) = \lim_{M \to \infty} |\mu_{\delta, S} - D^c(\tau_M b_{\delta})|(G \cap \{v < S\}^{(1)}) \]
\[ \leq \lim_{M \to \infty} \int_{G \cap \{|\delta < v < S\} \cap \{|b_E|^{\gamma} < \infty\} \cap \{|b_E| < M\}^{(1)} \} g(x) d(|D^c v/2, 0)|_{K^*}(x) = 0, \]
where the last equality follows from the fact that \(\{|b_E| < M\}^{(1)}_{M \in I}\) is an increasing family of sets whose union is \(\{|b_E|^{\gamma} < \infty\}\). Thus, for every bounded Borel set \(G \subset \mathbb{R}^{n-1}\), we get
\[ |(-D^c(b_{\delta, S} + \frac{1}{2} v_{\delta, S}), 0)|_{K^*}(G) + \left|\left(D^c(b_{\delta, S} - \frac{1}{2} v_{\delta, S}), 0\right)\right|_{K^*}(G) \]
\[ \leq \left|\left(-\mu_{\delta, S} - \frac{1}{2} D^c v_{\delta, S}, 0\right)\right|_{K^*}(G) + \left|\left(\mu_{\delta, S} - \frac{1}{2} D^c v_{\delta, S}, 0\right)\right|_{K^*}(G) \]
\[ = |(-D^c v_{\delta, S}, 0)|_{K^*}(G), \quad (6.30) \]
where the first inequality comes from Lemma 6.2 applied to \(b_{\delta, S} - \frac{1}{2} v_{\delta, S}\) and \(-b_{\delta, S} - \frac{1}{2} v_{\delta, S}\) with \(v_{\delta, S} = 1_{\{\delta < v < S\}^{(1)}}\). (see in particular (6.23)), whereas the equality is a consequence of Lemma 4.5 applied to the two Radon measures \(\mu_{\delta, S} - \frac{1}{2} D^c v_{\delta, S}\) and \(-\mu_{\delta, S} - \frac{1}{2} D^c v_{\delta, S}\) together with Remark 4.7 having in mind (1.30). Since \(b_{\delta, S} \in GBV(\mathbb{R}^{n-1})\) and \(v_{\delta, S} \in (BV \cap L^\infty)(\mathbb{R}^{n-1})\), if \(W = W[v_{\delta, S}, b_{\delta, S}]\), then we can compute \(P_{K^*}(W; G \times \mathbb{R})\) for every Borel set \(G \subset \mathbb{R}^{n-1}\) by Corollary 5.11. In particular, if \(G \subset \{\delta < v < S\}^{(1)}\), then by \(E \cap \{\delta < v < S\} \times \mathbb{R} = W \cap \{\delta < v < S\}^{(1)}\).
and (6.37), we find that
\[ P_{K^*}(E; G \times \mathbb{R}) = P_{K^*}(W; G \times \mathbb{R}) \]
\[ \int_G \phi_{K^*} \left( \nabla \left( b_{\delta,S} - \frac{v_{\delta,S}}{2} \right), 1 \right) + \phi_{K^*} \left( -\nabla \left( b_{\delta,S} + \frac{v_{\delta,S}}{2} \right), 1 \right) d\mathcal{H}^{n-1} \]  
\[ + \int_{G \cap J_\rho} \min \left( v_{\delta,S}, \left( \left( \frac{v_{\delta,S}}{2} + [b_{\delta,S}] + \max \left( \frac{v_{\delta,S}}{2}, |b_{\delta,S}|, 0 \right) \right) \right) \phi_{K^*}(-\nu_v, 0) d\mathcal{H}^{n-2} \]
\[ + \int_{G \cap J_\rho} \min \left( v_{\delta,S}, 0 \right) \phi_{K^*}(-\nu_v, 0) d\mathcal{H}^{n-2} \]
\[ + \left| \left( D^c \left( b_{\delta,S} - \frac{v_{\delta,S}}{2} \right), 0 \right) \right|_{K^*} (G) \]  
\[ + \left| \left( -D^c \left( b_{\delta,S} + \frac{v_{\delta,S}}{2} \right), 0 \right) \right|_{K^*} (G) \]  
We can also compute \( P_{K^*}(F[v_g]; G \times \mathbb{R}) \). Taking also into account that \( F[v] \cap \{ \delta < v < S \} \times \mathbb{R} = F[v_g] \cap \{ \delta < v < S \} \times \mathbb{R} \) we obtain that
\[ P_{K^*}(F[v]; G \times \mathbb{R}) = P_{K^*}(F[v_g]; G \times \mathbb{R}) = 2 \int_G \phi_{K^*} \left( -\nabla \left( \frac{v_{\delta,S}}{2} \right), 1 \right) d\mathcal{H}^{n-1} \]
\[ + \int_{G \cap (J_\rho \setminus J_\lambda)} [v] \phi_{K^*}(-\nu_v, 0) d\mathcal{H}^{n-2} + 2 \int_G \phi_{K^*} \left( -\frac{dD^c(\nu_v, 0)}{|D^c(\nu_v, 0)|}, 0 \right) d \left| D^c \left( \frac{v_{\delta,S}}{2} \right) \right| \].
Firstly, applying (2.21) to \( b_E \) and (2.17) to \( v \) we get
\[ \nabla b_{\delta,S}(x) = \nabla b_E(x), \text{ for } \mathcal{H}^{n-1}-a.e. \ x \in \{ \delta < v < S \}, \]
\[ [v] = [v_{\delta,S}], \text{ for } \mathcal{H}^{n-2}-a.e. \text{ on } \{ \delta < v < S \}^{(1)}. \]
Putting together the above relations with the assumptions (1.28) and (1.29) we deduce that, for \( \mathcal{H}^{n-1}-a.e. \ x \in \{ \delta < v < S \} \) there exists \( z(x) \in \partial K^* \) s.t.
\[ \left\{ \left( -\frac{1}{2} \nabla v(x) + t\nabla b_E(x), 1 \right) : t \in [-1, 1] \right\} \subset C_{K^*}^*(z(x)), \]  
\[ 2[b_{\delta,S}] = 2[b_E] \leq [v] = [v_{\delta,S}], \text{ for } \mathcal{H}^{n-2}-a.e. \text{ on } \{ \delta < v < S \}^{(1)}. \]
Thanks to Proposition 3.21 and Remark 3.22 condition (6.38) is equivalent to say that we can rewrite (6.32) in the following way
\[ \int_G \phi_{K^*} \left( \nabla \left( b_{\delta,S} - \frac{v_{\delta,S}}{2} \right), 1 \right) + \phi_{K^*} \left( -\nabla \left( b_{\delta,S} + \frac{v_{\delta,S}}{2} \right), 1 \right) d\mathcal{H}^{n-1} \]
\[ \int_G \phi_{K^*} \left( \nabla \left( b_{\delta,S} - \frac{v}{2} \right), 1 \right) + \phi_{K^*} \left( -\nabla \left( b_{\delta,S} + \frac{v}{2} \right), 1 \right) d\mathcal{H}^{n-1} \]
\[ = 2 \int_G \phi_{K^*} \left( -\nabla \left( \frac{v}{2} \right), 1 \right) d\mathcal{H}^{n-1}. \]  
Furthermore, substituting (6.39) into (6.33), (6.34) and (6.35), and using (6.30) applied to (6.36) and (6.37), we find that
\[ P_{K^*}(E; \{ \delta < v < S \}^{(1)} \times \mathbb{R}) \leq P_{K^*}(F[v]; \{ \delta < v < S \}^{(1)} \times \mathbb{R}), \]
where, actually, equality holds thanks to \([\text{AS}].\) Recalling that by \([19]\ 4.5.9(3)]\) we have that 
\[ H^{n-2}(\{v^\sim = \infty\}) = 0, \]
thanks to \([2.16]\) it follows that
\[ \bigcup_{M \in I} \{ v < M \}^{(1)} = \{ v^\sim < \infty \} = H^{n-2} \mathbb{R}^{n-1}. \tag{6.42} \]
By \([2.16]\) if we consider the sequences \(\delta_h \in I\) and \(S_h \in I\) such that \(\delta_h \to 0\) and \(S_h \to 0\) as \(h \to \infty\) we get
\[ \{ v^\sim > 0 \} = \bigcup_{h \in \mathbb{N}} \{ \delta_h < v^\sim < S_h \}^{(1)}. \]
So, by the above relation together with \([6.41],\) and \([6.42]\) we get that
\[ P_{K^s}(E; \{ v^\sim > 0 \} \times \mathbb{R}) \leq P_{K^s}(F[v]; \{ v^\sim > 0 \} \times \mathbb{R}). \]
By Proposition \([6.1]\) \(P_{K^s}(E; \{ v^\sim = 0 \} \times \mathbb{R}) = P_{K^s}(F[v]; \{ v^\sim = 0 \} \times \mathbb{R})\) and thus \(P_{K^s}(E) = P_{K^s}(F[v])\). This concludes the proof. \(\Box\)

7. Rigidity of the Steiner’s inequality for the anisotropic perimeter

In this final section we will prove the main results about \([\text{RAS}].\) Let us start the section with the proof of Theorem \([1.10]\)

**Proof of Theorem \([1.10]\).** By Theorem \([1.4]\) we have to prove that conditions \([1.10]-1.12\) hold true. We divide the proof in few steps.

**Step 1** In this step we prove that \([1.10]\) holds true. Since \(E \in \mathcal{M}_{K^s}(v)\), by Theorem \([1.8]\) we have that condition \([1.28]\) holds true, namely for \(H^{n-1}\)-a.e. \(x \in \{ v > 0 \}\) there exists \(z(x) \in \partial K^s\) s.t.
\[ \left(-\frac{1}{2} \nabla v(x) + t \nabla b_E(x), 1\right) \in C_{K^s}(z(x)) \quad \forall t \in [-1, 1]. \]
By condition \(\text{R1}\) we have that for \(H^{n-1}\)-a.e. \(x \in \{ v > 0 \}\) there exists \(z(x) \in \partial K^s\) s.t. \(\forall t \in [-1, 1]\) there exists \(\lambda = \lambda(t, x) \in [0, 1]\) such that
\[ (t \nabla b_E(x), 0) = \lambda \left(-\frac{1}{2} \nabla v(x), 1\right). \]
that implies \(\nabla b_E = 0\) for \(H^{n-1}\)-a.e. \(x \in \{ v > 0 \}\), that implies \(\nabla b_E = 0\) for \(H^{n-1}\)-a.e. \(x \in \mathbb{R}^{n-1}\).

**Step 2** In this step we prove that \([1.12]\) holds true. Again, since \(E \in \mathcal{M}_{K^s}(v)\) we know that condition \([1.30]\) holds true, namely we know that for \(|D^c v|\)-a.e. \(x \in \{ v^\sim > 0 \}\) there exists \(z(x) \in \partial K^s\) s.t.
\[ (h(x) + tg(x), 0) \in C_{K^s}(z(x)), \quad \forall t \in [-1, 1]. \tag{7.1} \]
So, by condition \(\text{R2}\) we know that for \(|D^c v|\)-a.e. \(x \in \{ v^\sim > 0 \}\) there exists \(\lambda = \lambda(x) \in [-1, 1]\) such that \(g(x) = \lambda h(x)\). By definition of \(g(x)\) and \(h(x)\), for every Borel set \(G \subset \mathbb{R}^{n-1}\), every \(M > 0\), and \(H^{n-1}\)-a.e. \(\delta > 0\) we have
\[
D^c(\tau_M(b_0))(G) = \int_{\mathcal{G} \cap \{ |b_0| < M \}^{(1)} \cap \{ v > \delta \}^{(1)}} g(x) d\left(\frac{1}{2} D^c v, 0\right)_{K^s}(x)
= \int_{\mathcal{G} \cap \{ |b_0| < M \}^{(1)} \cap \{ v > \delta \}^{(1)}} \lambda(x) h(x) d\left(\frac{1}{2} D^c v, 0\right)_{K^s}(x)
= \int_{\mathcal{G} \cap \{ |b_0| < M \}^{(1)} \cap \{ v > \delta \}^{(1)}} -\frac{1}{2} \lambda(x) dD^c v(x).
\]
Since $-\frac{1}{2} \lambda(x) \in [-1/2, 1/2]$ for $|D^c v|$-a.e. $x \in \{v^\wedge > 0\}$, we conclude the proof of step 2.

**Step 3** In this step we prove that (1.13) and (1.14) holds true. By step 2 we have that (1.12) holds true. By taking the total variation in (1.12) we find that $|D^c v(\tau_M(b_\delta))| \leq |D^c v(G)|$ for every bounded Borel set $G \subset \mathbb{R}^{n-1}$. By passing to the limit for $M \to +\infty$ (in $J_\delta$) and then $\delta \to 0$ (in $I$) we prove (1.13). As observed in [3] Remark 1.10, note that (1.14) is a consequence of (1.8), taking into account (1.10), (1.12) and (1.13). This concludes the proof.

The following result provides a geometrical characterization of the validity of R1 and R2. In the following, given any set $G \subset \mathbb{R}^n$ we denote by $\overline{G}$ its topological closure. Having in mind the definitions of exposed and extreme points (see Definition 3.30 and 3.29 respectively), we can now prove the following proposition, that will be an important intermediate result in order to prove Proposition 1.13.

**Proposition 7.1.** Let $v$ be as in (1.4) and let $K \subset \mathbb{R}^n$ be as in (1.17). For $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$ let us call $\nu(x) = \left(-\frac{1}{2} \nabla v(x), 1\right)$. Then,

\[
\begin{align*}
\text{R1 holds true} & \iff \frac{\nu(x)}{\phi_{K^*}(\nu(x))} \text{ is an extreme point of } (K^*)^* \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{v > 0\}. \\
\text{R2 holds true} & \iff \frac{(h(x), 0)}{\phi_{K^*}((h(x), 0))} \text{ is an extreme point of } (K^*)^* \text{ for } |D^c v|\text{-a.e. } x \in \{v^\wedge > 0\},
\end{align*}
\]

where $h$ has been defined in (1.31).

**Proof.** Let us prove that (7.2) holds true, then statement (7.3) follows using an identical argument.

**Step 1** Let us assume that R1 holds true and suppose by contradiction that there exists $G \subset \{v > 0\}$ such that $\mathcal{H}^{n-1}(G) > 0$ and $\nu(x)/\phi_{K^*}(\nu(x))$ is not an extreme point for every $x \in G$. Note that by construction, $\nu(x)/\phi_{K^*}(\nu(x)) \in \partial (K^*)^*$. So, if $\nu(x)/\phi_{K^*}(\nu(x))$ is not an extreme point, there exist $y(x), z(x) \in \partial (K^*)^*$ with $y(x) \neq z(x)$ and $\lambda(x) \in (0, 1)$ such that

\[
\frac{\nu(x)}{\phi_{K^*}(\nu(x))} = (1 - \lambda(x))z(x) + \lambda(x)y(x)).
\]

By convexity, of $(K^*)^*$, we have that

\[
(1 - \lambda)z(x) + \lambda y(x) \in \partial (K^*)^* \quad \forall \lambda \in [0, 1],
\]

and thus there exist $\bar{x} \in \partial (K^*)^*$, and $\omega \in S^{n-1}$ such that

\[
(1 - \lambda)y(x) + \lambda z(x) \in \partial (K^*)^* \cap H_{\bar{x}, \omega} \quad \forall \lambda \in [0, 1],
\]

\[
(K^*)^* \subset H_{\bar{x}, \omega}.
\]

Thanks to the above relations, and by definition of $\phi_{K^*}$ we have that

\[
((1 - \lambda)y(x) + \lambda z(x)) \cdot \omega = \phi_{K^*}(\omega) \quad \forall \lambda \in [0, 1].
\]

Recalling (3.22) we get that

\[
(1 - \lambda)y(x) + \lambda z(x) \in \partial \phi_{K^*}(\omega) \quad \forall \lambda \in [0, 1],
\]
and thus, since $\omega \in Z_{K^*}((\nu(x)/\phi_{K^*}(\nu(x))))$, by Lemma 3.27 this implies that

$$(1 - \lambda)z(x) + \lambda y(x) \in \partial \phi_{K^*}(z) \quad \forall \lambda \in [0, 1], \forall z \in Z_{K^*}\left(\frac{\nu(x)}{\phi_{K^*}(\nu(x))}\right).$$

In particular, this implies that

$$(1 - \lambda)\phi_{K^*}(\nu(x))z(x) + \lambda \phi_{K^*}(\nu(x))y(x) \in C_{K^*}(z) \quad \forall \lambda \in [0, 1], \forall z \in Z_{K^*}\left(\frac{\nu(x)}{\phi_{K^*}(\nu(x))}\right),$$

where recall that $Z_{K^*}(\nu(x)/\phi_{K^*}(\nu(x))) = Z_{K^*}(\nu(x))$. Let us consider $z \in Z_{K^*}(\nu(x))$. Applying the above formula with $\lambda(x) \in (0, 1)$ we obtain that $\nu(x)$ belongs to the interior of $C_{K^*}(z)$, that is there exists a radius $r > 0$ such that $B(\nu(x), r) \subset C_{K^*}(z)$. Let us take $w \in B(\nu(x), r)$ such that $w \neq tv(x)$ for every $t \in \mathbb{R}$, and let us denote $\bar{w} = w - \nu(x)$. Then,

$$\bar{w} \neq t\nu(x) \quad \forall t \in \mathbb{R},$$

$$\nu(x) + \bar{w} \in C_{K^*}(z),$$

$$\nu(x) - \bar{w} \in C_{K^*}(z).$$

Relation (7.5) is true since $w \neq tv(x)$ for every $t \in \mathbb{R}$. From the choice of $w \in B(\nu(x), r)$ we get that $\nu(x) + \bar{w} = w \in B(\nu(x), r) \subset C_{K^*}(z)$. On the other hand, $\nu(x) - \bar{w} = 2\nu(x) - w$. In order to prove that $2\nu(x) - w \in B(\nu(x), r)$ let us check if $|2\nu(x) - w - v| < r$. So, $|2\nu(x) - w - v| = |\nu(x) - w| = |\bar{w}| < r$ since $w \in B(\nu(x), r)$. Thus, since (7.4) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in G$ and $\mathcal{H}^{n-1}(G) > 0$, and having in mind (7.5), (7.6), and (7.7) we reached a contradiction with R1.

**Step 2** Let us now assume that $\nu(x)/\phi_{K^*}(\nu(x))$ is an extreme point of $(K^*)^*$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{v > 0\}$, and suppose by contradiction that R1 is not verified, namely that there exists $y \in \mathbb{R}^n$, and $G \subset \{v > 0\}$ with $\mathcal{H}^{n-1}(G) > 0$ such that, for every $x \in G$ there exists $z \in Z_{K^*}(\nu(x))$ such that,

$$\nu(x) \pm y \in C_{K^*}(z) \quad \text{but} \quad y \neq \lambda \nu(x), \quad \text{for every} \ \lambda \in [-1, 1].$$

In particular, by convexity,

$$(1 - \lambda)\nu(x) + \lambda \nu(x) - y \in C_{K^*}(z), \quad \forall \lambda \in [0, 1].$$

But this implies that the projection of this segment over $\partial \phi_{K^*}(z)$ contains in its relative interior the point $\nu(x)/\phi_{K^*}(\nu(x))$, namely there exists $\lambda(x) \in (0, 1)$ such that

$$\frac{\nu(x)}{\phi_{K^*}(\nu(x))} = (1 - \lambda(x))\frac{\nu(x) + y}{\phi_{K^*}(\nu(x) + y)} + \lambda(x)\frac{\nu(x) - y}{\phi_{K^*}(\nu(x) - y)}.$$  

Since (7.8) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in G$ and $\mathcal{H}^{n-1}(G) > 0$ we contradicted our assumptions. This concludes the proof. \hfill \Box

As mentioned above, Proposition 7.1 give a characterization of conditions R1 and R2 in terms of the geometric properties of the dual Wulff shape $(K^*)^*$ we are considering. Before the proof of Proposition 1.13 we need the following lemma.

**Lemma 7.2.** Let $K \subset \mathbb{R}^n$ be as in (1.18), and consider $y \in \mathbb{R}^n$. Then, $y/\phi_{K^*}(y)$ is an extreme point of $(K^*)^*$ if and only if $y/|y| \in \mathcal{V}_{K^*}$, where $\mathcal{V}_{K^*}$ is the set defined in (1.33).

**Proof.** Step 1 We first prove the result for the exposed points of $(K^*)^*$, namely we prove that $y/\phi_{K^*}(y)$ is an exposed point of $(K^*)^*$ if and only if $y/|y| \in \mathcal{V}_{K^*}$. This first part is the direct consequence of Lemma 3.32 using $g = \phi_{K^*}$, and observing that $\partial \phi_{K^*}(x) = \{\nu^{K^*}(x)/\phi_{K^*}(\nu^{K^*}(x))\}$.
for every \( x \in \partial^* K^s \).

**Step 2** We now conclude the proof. Let \( y \in \mathbb{R}^n \) be such that \( y/\phi_{K^s}(y) \) is an extremal point of \((K^s)^\mathbb{R}\), by Remark 3.31 it implies that there exists a sequence \((\omega_h)_{h \in \mathbb{N}}\) of exposed points of \((K^s)^\mathbb{R}\) such that \( \lim_{h \to \infty} \omega_h = y/\phi_{K^s}(y) \). Observe that by definition, \( \omega_h \in \partial (K^s)^\mathbb{R} \), and so \( \phi_{K^s}(\omega_h) = 1 \) for all \( h \in \mathbb{N} \). Thanks to the first step, every \( \omega_h \) is such that \( \eta_h := \omega_h/|\omega_h| \in \mathbb{V}_{K^s} \), and so \( \phi_{K^s}(\eta_h) = 1/|\omega_h| \). Moreover, the fact that \( \omega_h \) is a converging sequence implies that there exists \( \eta \in S^{n-1} \) such that \( \lim_{h \to \infty} \eta_h = \eta \). Thus, \( \eta \in \mathbb{V}_{K^s} \), and \( y/\phi_{K^s}(y) = \eta/\phi_{K^s}(\eta) \). In particular, since \( |\eta| = 1 \), we have that \( \eta = y/|y| \in \mathbb{V}_{K^s} \). The reverse implication follows by similar argument. \( \square \)

**Corollary 7.3.** Let \( v \) be as in (1.4) and let \( K \subset \mathbb{R}^n \) be as in (1.18). Then,

R1 holds true \( \iff \exists S_1 \subset \{v^\uparrow > 0\} \) such that \( \mathcal{H}^{n-1}(S_1) = 0 \), and

\[
\nu^{F[v]} \left(x, \frac{1}{2} v(x)\right) \subset \mathbb{V}_{K^s} \quad \forall x \in \{v^\uparrow > 0\} \setminus S_1. \tag{7.9}
\]

R2 holds true \( \iff \exists S_2 \subset \{v^\uparrow > 0\} \) such that \( |D^c v|(S_2) = 0 \), and

\[
\nu^{F[v]} \left(x, \frac{1}{2} v(x)\right) \subset \mathbb{V}_{K^s} \quad \forall x \in \{v^\uparrow > 0\} \setminus S_2. \tag{7.10}
\]

**Proof.** We prove (7.9), then (7.10) follows by similar argument. Thanks to (7.2), we have just to prove that

\[
\frac{\nu(x)}{\phi_{K^s}(\nu(x))} \text{ is an extreme point of } (K^s)^\mathbb{R} \iff \exists S_1 \subset \{v^\uparrow > 0\} \text{ s.t. } \mathcal{H}^{n-1}(S_1) = 0, \quad \nu^{F[v]} \left(x, \frac{1}{2} v(x)\right) \subset \mathbb{V}_{K^s} \quad \forall x \in \{v^\uparrow > 0\} \setminus S_1, \tag{7.11}
\]

where \( \nu(x) = \left(-\frac{1}{2}\nabla v(x), 1\right) \) for \( \mathcal{H}^{n-1}\)-a.e. \( x \in \{v > 0\} \). By Lemma 7.2 we have that \( \nu(x)/\phi_{K^s}(\nu(x)) \) is an extremal point if and only if \( \nu(x)/|\nu(x)| \in \mathbb{V}_{K^s} \) for \( \mathcal{H}^{n-1}\)-a.e. \( x \in \{v > 0\} \), that is equivalent to say that there exists \( S_1 \subset \{v^\uparrow > 0\} \) s.t. \( \mathcal{H}^{n-1}(S_1) = 0 \), and \( \nu(x)/|\nu(x)| \in \mathbb{V}_{K^s} \) for every \( x \in \{v^\uparrow > 0\} \setminus S_1 \). By Theorem 5.2 with \( u = v/2 \), we deduce that \( \nu(x)/|\nu(x)| = \nu^{F[v]} \left(x, \frac{1}{2} v(x)\right) \) for \( \mathcal{H}^{n-1}\)-a.e. \( x \in \{v^\uparrow > 0\} \), and thus we conclude. \( \square \)

We are ready now to prove Proposition 1.13.

**Proof of Proposition 1.13** **Step 1** Suppose that R1 and R2 hold true. Then, by Corollary 7.3 there exist \( S_1 \subset \{v^\uparrow > 0\} \), \( S_2 \subset \{v^\uparrow > 0\} \) such that \( \mathcal{H}^{n-1}(S_1) = |D^c v|(S_2) = 0 \), and

\[
\nu^{F[v]} \left(z, \frac{1}{2} v(z)\right) \subset \mathbb{V}_{K^s} \quad \forall x \in \{v^\uparrow > 0\} \setminus S_1 \cup \{v^\uparrow > 0\} \setminus S_2.
\]

By De Morgan’s laws, calling \( S := S_1 \cap S_2 \), the above relation is equivalent to

\[
\nu^{F[v]} \left(z, \frac{1}{2} v(z)\right) \subset \mathbb{V}_{K^s} \quad \forall x \in \{v^\uparrow > 0\} \setminus S.
\]

Since \( S \subseteq S_i \) for \( i = 1, 2 \), we deduce that \( \mathcal{H}^{n-1}(S) = |D^c v|(S) = 0 \). This concludes the first step.
**Step 2** Suppose that $ii$) of Proposition 1.13 holds true, namely that $\exists S \subset \{v^+ > 0\}$ such that $\mathcal{H}^{n-1}(S) = \int \mathcal{D}^v|v|(S) = 0$, and

$$\nu^v(z, \frac{1}{2} v(z)) \in \nabla K^* \quad \forall z \in \{v^+ > 0\} \setminus S.$$  

Thanks to Corollary 7.3 with $S_1 = S_2 = S$ we deduce that R1 and R2 hold true. This concludes the proof. □

**Proof of Remark 1.14.** If $K^*$ is polyhedral, then $\nabla K^*$ coincides with the set of the outer unit normals to the facets of $K^*$. Since $K^*$ has a finite number of facets, we conclude that $\nabla K^*$ is closed. In case $K^*$ has $C^1$ boundary, we have to notice that thanks to [21 Corollary 3, Theorem 1]), every point in $\partial (K^*)^*$ is an exposed point, so by Lemma 7.2 we have that $\nabla K^*$ coincides with $S^{n-1}$, which is closed. □

**Proof of Corollary 1.15.** Thanks to Remark 1.14 we know that if $K^*$ has $C^1$ boundary, then $\nabla K^*$ coincides with $S^{n-1}$. Therefore condition (1.34), namely there exists $S \subset \{v^\lor > 0\}$ s.t. $\mathcal{H}^{n-1}(S) = \int \mathcal{D}^v|v|(S) = 0$ and $\nu^v(z, \frac{1}{2} v(z)) \in \nabla K^* = S^{n-1}$ for every $z \in \{v^\lor > 0\}$, is always verified. This concludes the proof. □

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(Matteo Perugini) Universität Münster, Applied Mathematics, Eisteinstr. 62, D-48149 Münster

Email address: matteo.perugini@uni-muenster.de