SYMPLECTIC COHOMOLOGY AND A CONJECTURE OF VITERBO

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Abstract. We identify a new class of closed smooth manifolds for which there exists a uniform bound on the Lagrangian spectral norm of Hamiltonian deformations of the zero section in a unit cotangent disk bundle. This settles a well-known conjecture of Viterbo from 2007 as the special case of $T^n$, which has been completely open for $n > 1$. Our methods are different and more intrinsic than those of the previous work of the author first settling the case $n = 1$. The new class of manifolds is defined in topological terms involving the Chas–Sullivan algebra and the BV-operator on the homology of the free loop space. It contains spheres and is closed under products. We discuss generalizations and various applications, to $C^0$ symplectic topology in particular.

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1 Introduction

In this paper we prove a well-known conjecture of Viterbo from 2007 [Vit14, Conjecture 1] on a uniform bound on the spectral norm $\gamma(L', L)$ of Lagrangian submanifolds $L$ of the unit cotangent disk bundle $T^n$ of $\mathbb{R}^n$. The conjecture states that there exists a constant $C > 0$ such that $\gamma(L', L) \leq C$ for any Hamiltonian deformation $L'$ of the zero section $L$. The new class of manifolds that we introduce is defined in topological terms involving the Chas–Sullivan algebra and the BV-operator on the homology of the free loop space. It contains spheres and is closed under products. We discuss generalizations and various applications, to $C^0$ symplectic topology in particular.

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ifolds, Hamiltonian isotopic to the zero section, in the unit disk cotangent bundle $D^*L$ of the standard torus $L = T^n$, taken with respect to a fixed Riemannian metric $g$. This is a far-reaching non-linear generalization of the elementary estimate

$$\gamma(df, L) = \max f - \min f \leq ||df|| \cdot \operatorname{diam}(L, g)$$

for each function $f \in C^\infty(L, \mathbb{R})$, that despite considerable interest in the area of quantitative and $C^0$ symplectic topology [Vit14, MVZ12, MZ11, Sey13, HLS15, She22, Kha09, BC21] was completely open for all $n > 1$, after the recent work [She22] establishing the case $n = 1$. In fact, our methods yield a stronger and more general result, as they work for a large class of closed connected manifolds $L$, and apply to arbitrary exact Lagrangian submanifolds in $D^*L$. We remark that ideas and results of this paper have already found applications in Riemannian geometry [CHOS] and in Hamiltonian PDE [VR]. Section 1.2 describes applications to various aspects of symplectic topology, $C^0$ symplectic topology in particular, relying on the fact that the spectral norm is a key invariant controlling many quantitative properties of Lagrangian submanifolds. Broadly speaking, the results of this paper connect the a priori disparate fields of quantitative symplectic topology and algebraic topology of loop spaces in a new way.

The main tool of the paper is the Viterbo isomorphism [Vit99, SW06, AS10, AS14, Abo15] of BV-algebras between the symplectic cohomology of the cotangent bundle and the homology of the loop space of the base. It is combined with a quantitative study of a generalization of the operations on symplectic cohomology and Lagrangian Floer homology introduced and studied by Seidel and Solomon [SS12, Sei14] in the context of mirror symmetry, which are applied here for the first time in the altogether different context of quantitative and $C^0$ symplectic topology. Along the way, we provide new calculations of the aforementioned operations in terms of string topology.

We start by defining the new class of manifolds to which our results apply. Fix a base field $\mathbb{K}$ as a coefficient ring for homology and cohomology groups. For a closed connected smooth manifold $L$ of dimension $\dim(L) = n$, consider the constant-loop inclusion map

$$\iota : H_*(L) \rightarrow H_*(\mathcal{L}L),$$

and the evaluation map

$$ev : H_*(\mathcal{L}L) \rightarrow H_*(L),$$

between its homology and the homology of the free loop space $\mathcal{L}L$ of $L$. Given a homogeneous class $a \in H_*(\mathcal{L}L)$, let

$$m_a : H_*(\mathcal{L}L) \rightarrow H_{*-|a|+n+1}(\mathcal{L}L)$$

be the right Chas–Sullivan [CS99] string bracket $[-, a]$ with $a$. We recall that the string bracket is given in terms of the Chas–Sullivan product $\ast$, and the BV-operator

$$\Delta : H_*(\mathcal{L}L) \rightarrow H_{*+1}(\mathcal{L}L).$$
It is essentially the \( \Delta \)-differential of the product: for two homogeneous elements 
\( a, b \in H_{n-\ast}(\mathcal{L}L) \),
\[
[b, a] = (-1)^{|b|}(\Delta(b \ast a) - \Delta(b) \ast a - (-1)^{|b|}b \ast \Delta(a)).
\]
This bracket, together with the product, forms the structure of a Gerstenhaber algebra on 
\( H_{n-\ast}(\mathcal{L}L) \). In terms of these operations, the main operator that we consider in this paper is
\[
P_a : H_*(L) \to H_{*+|a|-n+1}(L)
\]
\[
P_a = ev \circ m_a \circ \iota.
\]
We introduce the following condition which we have found to be sufficient for our purposes.

**Definition 1.** We call a closed connected smooth manifold \( L \) string point-invertible over \( \mathbb{K} \) if it is \( \mathbb{K} \)-orientable and there exists a collection of classes \( a_1, \ldots, a_N \in H_*(\mathcal{L}L) \) such that the composition \( P = P_{a_N} \circ \cdots \circ P_{a_1} \) satisfies
\[
[L] = P([pt]),
\]
where \( [L] \in H_0(L) \) is the fundamental class and \( [pt] \in H_0(L) \) is the class of the point. Reformulated more abstractly, \( [L] \in P([pt]) = \{ P([pt]) \mid P \in \mathcal{P} \} \), where \( \mathcal{P} \) is the subalgebra of \( \text{Hom}(H_*(L), H_*(L)) \) generated by \( \{ P_a \mid a \in H_*(\mathcal{L}L) \} \).

**Remark 2.** We note the following three points regarding Definition 1.

i. Set \( H_*(\mathcal{L}L)^+ = \ker(ev : H_*(\mathcal{L}L) \to H_*(L)) \). A short calculation shows that we may, without loss of generality, restrict \( a_1, \ldots, a_N \) in Definition 1 to lie in \( H_*(\mathcal{L}L)^+ \) and replace \( \mathcal{P} \) with its subalgebra \( \mathcal{P}^+ \) generated by the set \( \{ P_a \mid a \in H_*(\mathcal{L}L)^+ \} \). Indeed, it is enough to consider homogeneous elements \( a \), in which case \( P_a : H_*(L) \to H_{*+|a|-n+1}(L) \) is a homogeneous operator.

Further, for all \( b \in \iota(H_*(L)) \), \( P_b = 0 \) since \( \Delta = 0 \) on \( \iota(H_*(L)) \), and \( \iota, ev \) are maps of algebras, \( H_*(L) \) being endowed with the intersection product. Hence we may correct each homogeneous element \( a \in H_*(\mathcal{L}L) \) by \( a_0 = \iota \circ ev(a) \) to obtain the homogeneous element \( a' = a - a_0 \in H_*(\mathcal{L})^+ \), with the property that \( P_{a'} = P_a \). Hence \( [L] = P_{a_N} \circ \cdots \circ P_{a_1}([pt]) \), \( a_1, \ldots, a_N \in H_*(\mathcal{L}) \) if and only if \( [L] = P_{a'_N} \circ \cdots \circ P_{a'_1}([pt]) \), with \( a'_1, \ldots, a'_N \in H_*(\mathcal{L}L)^+ \).

ii. For a class \( a \in H_*(\mathcal{L}L) \) we may consider the operator \( Q_a : H_*(L) \to H_*(L) \), given by \( Q_a = ev \circ m'_a \circ \iota \), where \( m'_a \) is the Chas–Sullivan product by \( a \). The technical arguments in this paper apply to this simpler map, however since \( ev \) and \( \iota \) are maps of algebras, and \( ev \circ \iota = \text{id} \), we observe that for \( x \in H_*(L) \),
\[
Q_a = ev \circ m'_a \circ \iota(x) = ev(a \ast \iota(x)) = ev(a) \ast ev \circ \iota(x) = ev(a) \ast x.
\]
Therefore \( Q_a \) is the multiplication operator by \( ev(a) \in H_*(L) \) with respect to the intersection
product on $H_*(L)$. In particular it does not increase degree. Therefore, while adding the operations $Q_b$, $b \in H_*(LL)$, to Definition 1 may theoretically be useful, in practice it seems to have little effect.

ii. Note that if $P_a : H_*(L) \to H_{*+|a|-n+1}(L)$ increases degree, then the homological degree of $a \in H_*(LL)$ satisfies $|a| \geq n$.

By a result of Menichi [Men09, Theorem 16] the class of string point-invertible manifolds contains spheres of odd dimension $S^{2m+1}$, $m \geq 0$, with arbitrary coefficients, and $S^2$ with coefficients in $\mathbb{F}_2$. A minor modification of the argument of Menichi for $S^2$ shows that the even-dimensional spheres $S^{2m}$, $m \geq 1$ are in this class, with $\mathbb{F}_2$ coefficients. Furthermore, we will prove the following general structural result for this class.

**Proposition 3.** The class of string point-invertible manifolds over a fixed field $\mathbb{K}$ is closed under products.

In particular, the $n$-torus $T^n$ is string point-invertible over any field $\mathbb{K}$. To verify the definition one can take (the image under coefficient change to $\mathbb{K}$ of) the sequence $a_1, \ldots, a_n$ of positive generators of $H_n(L, T^n; \mathbb{Z}) \cong \mathbb{Z}$ for free homotopy classes of loops $e_1, \ldots, e_n$ corresponding to a positively oriented basis of $\mathbb{Z}^n$. The main result of this paper is the following.

**Theorem A.** Let $L$ be string point-invertible over a field $\mathbb{K}$. Let $g$ be a Riemannian metric on $L$. Then there exists a constant $C(g, L; \mathbb{K})$ such that for all exact Lagrangian submanifolds $L_0, L_1$ contained in the unit codisk bundle $D^*_g L \subset T^*L$, the spectral norm of the pair $L_0, L_1$ satisfies

$$\gamma(L_0, L_1; \mathbb{K}) \leq C(g, L; \mathbb{K}).$$

**Remark 4.** By the triangle inequality for the spectral norm, it is enough to prove the above statement for $L_1 = L$, the zero section in $D^*_g L$.

This statement was previously known for $\mathbb{K} = \mathbb{F}_2$, and Lagrangian $L \in \{\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, S^n\}_{n \geq 1}$ by [She22], essentially in the case when $L_1$ is Hamiltonianly isotopic to the zero section, and $L_0 = L$. In particular the case of $T^n$ for $n > 1$ has remained completely open. We note that by the examples in Sect. 1.1, Theorem A is rather complementary to the result of [She22]. Furthermore, while quite a few manifolds including surfaces of higher genus are not string point-invertible, and hence Theorem A does not apply as such, the work [BC21] proves related results for bases given by arbitrary closed connected manifolds.

The strategy of the proof of Theorem A differs significantly from that of [She22], in particular in not making use of compactification arguments, and relying instead on what can be interpreted as “wrong way Lusternik–Schnirelmann inequalities”. We remark at this point that usual Lusternik–Schnirelmann methods based on products in Floer cohomology, or their generalization to “tied” cohomology classes [Vit97] appear to be insufficient for our purposes as, for instance, they are not sensitive to
the quantitative symplectic properties of the ambient domain, and they also tend not to yield the necessary information when evaluated on the point class. Furthermore, uniform bounds of this kind (see e.g. [EP03, She22]) were hitherto only obtained by route of the product structure in quantum cohomology. This requires the presence of pseudo-holomorphic disks, which are absent in the setting of cotangent bundles.

The main idea of the proof is threefold: first, a homogeneous class \( a \in H_{n-*}(LL) \) corresponds by the Viterbo isomorphism \( \text{[Vit99, SW06, AS10, AS14, Abo15]} \) to a class \( \alpha \in SH^{*}(LL) \) (the latter computed with suitable background class) with the property that \( r_{L'}(\alpha) = 0 \in HF^{*}(L', L') \) for each exact Lagrangian \( L' \subset T^*L \), where \( r_{L'} : SH^{*}(LL) \to HF^{*}(L', L') \) is the natural closed-open restriction map (by \( \text{[FSS08, Kra13, Abo12, AK18]} \) each such \( L' \) has vanishing Maslov class, is Spin relatively to the above background class, and endowed with suitable Spin structure is Floer-theoretically equivalent to the zero section \( L \)). Second, working up to \( \epsilon > 0 \), given that \( L_0, L_1 \subset D = D^*_g L \), a Liouville domain with contact boundary \( S = \partial D = S^*_g L \), a generalization and quantitative study of the work of Seidel and Solomon \( \text{[SS12]} \) gives an operation \( HF^{*}(L_0, L_1) \to HF^{*+|a|-1}(L_0, L_1) \), which raises the action filtration by no more than a symplectic-homological spectral invariant \( c(\alpha, D, S) \) corresponding to the class \( \alpha \) and the domain \( D \). Finally, using further TQFT operations for Lagrangian Floer cohomology and symplectic cohomology \( \text{[Abo15, AS10]} \), we calculate that under the Floer-theoretic equivalence with the zero section, Poincaré duality \( H^{*}(L) \cong H_{n-*}(L) \), and the Viterbo isomorphism, this operation is given by \( P_a : H_{*}(L) \to H_{*+|a|-n+1}(L) \). Therefore, in view of string point-invertibility, assuming for ease of exposition that all \( P_{a_j}, 1 \leq j \leq N \), increase degree, which tends to happen in practice, by successively writing inequalities that bound the Lagrangian spectral invariants of classes of higher homological degree in terms of those of classes of lower homological degree, we arrive to a uniform upper bound on the spectral distance \( \gamma(L_0, L_1; \mathbb{K}) \), finishing the proof.

Remark 5. A few remarks on Theorem A are in order.

i. It is not necessary that our Weinstein domain be \( D^*_g L \) for a Riemannian metric \( g \). In fact the same result holds for any bounded Weinstein domain \( D \) containing \( L \) with completion given by \( T^*L \). In this case the upper bound will be given in terms of a constant \( c(D, L; \mathbb{K}) \). For example \( D \) may be given by a Finsler metric, or an optical domain: one that is strictly fiberwise star-shaped, and has a smooth boundary.

ii. The constant \( C(g, L; \mathbb{K}) \) in Theorem A can be chosen to be equal to a certain sum of spectral invariants relative to the domain \( D \) with boundary \( S \), corresponding to any \( N \)-tuple \( a_N, \ldots, a_1 \in H_{*}(LL) \) as in Definition 1. See Equation (9). Furthermore, it is easy to see that the spectral invariants \( c(a, D, S) \) are continuous in the Banach-Mazur distance with respect to the natural \( \mathbb{R}_{>0} \)-action on \( T^*L \) \( \text{[SZ21, PRSZ20]} \) (see (2)), and hence extend for example to the non-smooth strictly fiberwise star-shaped case. As a consequence we obtain bounds
in the non-smooth fiberwise star-shaped case in terms of the extension of the spectral invariants, by approximating by domains with smooth boundaries.

iii. Let $g_0$ be the standard metric of diameter $1/2$ on $S^1 = \mathbb{R}/\mathbb{Z}$. Let $D_0 = D_{g_0}^* S^1 = [-1,1] \times S^1$. From (9), it is evident that $C(D_0, L_0; \mathbb{K}) = 1$ in this case. This upper bound is sharp, since for each $\epsilon > 0$ sufficiently small, it is easy to construct a Lagrangian $L'_0 \subset D_0$ Hamiltonian isotopic to $L_0 = S^1$ in $D_0$, with $\gamma(L'_0, L_0) > 1 - \epsilon$, and the intersection $L'_0 \cap L_0$ is transverse and consists of precisely 2 points $x, y$ of index 1 and 0 respectively. Consider now the strictly fiberwise star-shaped domain $D \subset T^*(T^n)$ given by

$$D = (D_0)^n = [-1,1]^n \times T^n.$$  

It is easy to calculate that the upper bound obtained by continuity from (9) is in this case $C(D, L; \mathbb{K}) = n$. It is seen to be sharp by noting that $L' = (L'_0)^n$ satisfies $\gamma(L', L) = n \cdot \gamma(L'_0, L_0)$, since $L' \pitchfork L$, and the only intersection point of $L'$ and $L$ indices $n$ and 0 are $(x, \ldots, x)$, and $(y, \ldots, y)$ respectively.

iv. We note that Theorem A fails for general bounded Liouville domains. For example it is false for Lagrangians Hamiltonian isotopic to $L$ in plumbings of $D^* L$ with two or more cotangent disk bundles by [Zap13].

1.1 String point-invertibility: examples and non-examples. We discuss the size of the class of string point-invertible manifolds by describing examples and non-examples, based on known calculations of the Chas–Sullivan Gerstenhaber algebra, in addition to the results of Menichi [Men09] and Proposition 3 mentioned above. These calculations turn out to be quite delicate, and to depend on the choice of coefficients, and hence so does the property of string point-invertibility.

By a result of Tamanoi [Tam06, Theorem 5-1], the complex Stiefel manifolds $V_{n+1-k}(\mathbb{C}^{n+1}) \cong SU(n+1)/SU(k)$ of orthonormal $(n+1-k)$-frames in $\mathbb{C}^{n+1}$ for all $n \geq 0$, $0 \leq k \leq n$, are string point-invertible with arbitrary coefficients. This is a direct calculation. By Menichi [Men11, Theorems 39, 41] (see also Hepworth [Hep10] for a related result), all compact connected Lie groups $G$ are string point-invertible, with characteristic zero coefficients. Indeed by these results, which rely in part on the Milnor-Moore theorem [FHT01, Theorem 21.5], as far as our purposes are concerned, the loop homology BV-algebra of the Lie group $G$ behaves like that of a product of odd-dimensional spheres.

We note that as string point-invertibility depends only on the Gerstenhaber algebra structure on $H_{n-*(L L)}$ and the evaluation and inclusion maps, at least for $L$ simply connected and coefficients of characteristic zero, by [FT08] it depends only on the quasi-isomorphism type singular cochain dg-algebra $C^*(L)$ of $L$ (see [Mal10, FMT05, Kel04, EL17, CJ02] for further context). Indeed in this case the loop space homology together with its BV-algebra structure is given in terms of the Hochschild cohomology of $C^*(L)$. This interpretation also encodes the evaluation and inclusion maps: see discussion of [FTV07, Diagrams (4),(5)].
By Menichi [Men09], this class does \textit{not} contain the even-dimensional spheres $S^{2m}$, $m \geq 1$ for coefficients of characteristic zero, for instance, and the same is true for $\mathbb{C}P^n$, $n \geq 1$, $\mathbb{H}P^n$, $n \geq 1$ and $\mathbb{O}P^2$ by results of Yang [Yan13], Chataur-Le Borgne [CL11], Hepworth [Hep09], and Cadek-Moravec [CM10]. Moreover, by Westerland [Wes07] and [CL11, Hep09, CM10] the same is true for $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$ by results of Yang [Yan13], Chataur-Le Borgne [CL11], Hepworth [Hep09], and Cadek-Moravec [CM10]. Moreover, by Westerland [Wes07] and [CL11, Hep09, CM10] the same is true for $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$ by results of Yang [Yan13], Chataur-Le Borgne [CL11], Hepworth [Hep09], and Cadek-Moravec [CM10].

1.2 Applications.

1.2.1 $C^0$ continuity of the Hamiltonian spectral norm As observed in [She22], an argument of neck-stretching around a divisor in $M \times M^-$ that makes the Lagrangian diagonal $\Delta_M \subset M \times M^-$ exact, where $(M, \omega)$ is a closed symplectic manifold such that the conclusion of Theorem A holds for $L = M$, and $M^-$ denotes the symplectic manifold $(M, -\omega)$, allows one to prove, for example in the symplectically aspherical case, that the Hamiltonian spectral norm on $\text{Ham}(M, \omega)$ is Lipschitz in the $C^0$ norm, in a $C^0$-neighborhood of the identity. We pick one instance of such an application. The $C^0$-distance between two diffeomorphisms $\phi_0, \phi_1$ of $M$ is defined as $d_{C^0}(\phi_0, \phi_1) = \max_{x \in M} d(\phi_0(x), \phi_1(x))$, the distance $d$ being taken with respect to a background Riemannian metric on $M$.

**Corollary 6.** Let $g$ be a Riemannian metric on $T^{2n}$, and $\mathbb{K}$ be a field. The spectral norm

$$\gamma : \text{Ham}(T^{2n}, \omega_{st}) \to \mathbb{R}_{\geq 0}$$

over $\mathbb{K}$ satisfies the following. There exist constants $C, \delta > 0$, such that

$$\min\{\gamma(\phi), \delta C\} \leq C \cdot d_{C^0}(\phi, 1)$$

for all $\phi \in \text{Ham}(T^{2n}, \omega_{st})$.

We refer to [She22] for a discussion of results of this kind, such as [Sey12, BHS21, She22] and their applications [LSV21, KS21, BHS21], and a proof of a similar implication [She22, Theorem C]. We observe that further such results in the setting of closed monotone symplectic manifolds that are string point-invertible as smooth manifolds are not difficult to deduce. However, for reasons of conciseness, we defer this discussion to a further publication.

1.2.2 Quasi-morphisms on the Hamiltonian group of cotangent disk bundles Similarly to [She22], Theorem A yields the existence of non-trivial homogeneous quasi-morphisms on $\text{Ham}_c(D^*_g L)$ for $L$ string point-invertible, providing new examples of quasi-morphisms and quasi-states on compactly supported Hamiltonian diffeomorphism groups of Weinstein domains. The notion of such quasi-morphisms, which are
maps to $\mathbb{R}$ that are additive up to a uniformly bounded error, played an important role in symplectic topology (see [EP03, Ent14, Lan13, BEP04, She22] for example). This in turn has applications to the geometry of the Poisson bracket of compactly supported functions in $D^*_gL$. We remind the reader that a quasi-morphism is called homogeneous if it is additive on all abelian subgroups, and non-trivial if it is not a homomorphism. We summarize this application as follows, and refer to [She22] for its deduction from Theorem A.

**Corollary 7.** Let $L$ be string point-invertible over $\mathbb{K}$. The map $\mu : \widetilde{\text{Ham}}_c(D^*L) \to \mathbb{R}$ on the universal cover of $\text{Ham}_c(D^*L)$ given by

$$\mu([\phi^t]) = \lim_{k \to \infty} \frac{1}{k} c([L], H^{(k)}),$$

for $H^{(k)}(t, x) = kH(kt, x), k \in \mathbb{Z}_{>0}$ descends to a well-defined non-zero homogeneous quasimorphism $\mu : \text{Ham}_c(D^*L) \to \mathbb{R}$. Moreover $\mu$ vanishes on each element $\phi \in \text{Ham}_c(D^*L)$ such that $\text{supp}(\phi)$ is displaceable by an element of $\text{Ham}_c(D^*L)$. For $F, G \in C^\infty_c(D^*L, \mathbb{R})$, the map $\zeta : C^\infty_c(D^*L, \mathbb{R}) \to \mathbb{R}$ by $\zeta(H) = \mu(\phi_1^tH)$, satisfies

$$|\zeta(F + G) - \zeta(F) - \zeta(G)| \leq \sqrt{2C(g, L)}||\{F, G\}||_{C^0}, \quad (1)$$

where $\{F, G\}$ is the Poisson bracket of $F, G$. In particular, whenever $\{F, G\} = 0$, we obtain

$$\zeta(F + G) = \zeta(F) + \zeta(G).$$

These maps were defined, and shown to enjoy various interesting properties in [MVZ12, Theorems 1.3 and 1.8, Propositions 1.4 and 1.9], yet the quasi-morphism property, while anticipated therein, was hitherto known only for the manifolds $L \in \{\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, S^n\}$ covered in [She22]. Moreover, in the new case of $T^n$, the quasimorphism $\mu : \text{Ham}_c(D^*T^n) \to \mathbb{R}$ in the case of $T^n$ is immediately seen to be invariant under finite coverings $T^n \to T^n$, scaled suitably, as defined in [Vit14] (see also [MVZ12]). A similar invariance holds for products $T^n \times L$ with $L$ string point-invertible, with the induced coverings, or for finite coverings $L' \to L$ with both $L, L'$ string point-invertible. It is an interesting topological question to determine whether or not the class of string point-invertible manifolds is closed with respect to finite coverings: we expect this to be the case when working with coefficients of characteristic zero. Furthermore, Theorem A provides a different proof, and indeed a strengthening, of the results of [Vit14, Section 7].

### 1.2.3 Hausdorff-continuity of the Lagrangian spectral norm

We finish with yet another application, that is proved again by a neck-stretching argument (see [She22, Theorem F]), combined with arguments related to non-trivial fundamental groups.

**Corollary 8.** Let $L$ be string point-invertible over $\mathbb{K}$. Suppose $L$ is embedded as a weakly exact $\pi_1$-injective Lagrangian submanifold in a symplectically aspherical symplectic manifold $M$ that is closed or tame at infinity. Consider the pair $(U, L)$,
for $U \subset M$ a Weinstein neighborhood of $L$ with a symplectomorphism to the pair $(D, 0_L)$, for a Weinstein domain $D \subset T^*L$ containing the zero-section $0_L \subset T^*L$. Consider $r \in (0, 1)$, and let $U^r$ be the preimage of $r \cdot D$ by the symplectomorphism. Then there exists a constant $C(D, L; \mathbb{K})$ such that if $L' \subset M$ is a Hamiltonian image of $L$ that is contained in $U^r$, then

$$\gamma(L', L; \mathbb{K}) \leq C(D, L; \mathbb{K}) \cdot r.$$ 

An example of the situation described in Corollary 8 is the torus $L = T^n$ embedded as $L_1 \times \ldots \times L_n$ inside $\Sigma_1 \times \ldots \times \Sigma_n$, where for all $1 \leq j \leq n$, the submanifold $L_j \subset \Sigma_j$ is an embedded simple closed curve in the closed oriented surface $\Sigma_j$ of genus at least 1, that does not bound a disk. The condition on the embedding holds when $\pi_2(M, L) = 0$. If the conclusions of either Corollary 6 or 8 hold, arguments following [KS21, Theorem B] show that the associated Floer-theoretic barcodes, a notion that has recently attracted much attention in symplectic topology (see [PS16] and for example [UZ16, Zha19, PSS17, PRSZ20, AKKKPRRSSZ19, Ste18, KS21, LSV21, BHS21, SZ21, Ush22, DS20, She22]), up to shift, are continuous in the Hausdorff metric on the Lagrangians considered as subsets of $M$ with respect to a background Riemannian metric (see [She22] for a discussion of this kind of result, which in particular does not follow from $C^0$ continuity of the Hamiltonian spectral norm). Moreover, one can deduce analogues of Corollary 8 for certain monotone Lagrangian submanifolds. However, for reasons of conciseness we defer this discussion to a further publication.

1.2.4 Outlook As a closing remark, we mention that it would be very interesting to see if additional algebraic structures on symplectic cohomology and string topology could be applied to extend the class of manifolds $L$ for which Viterbo’s conjecture holds. For instance, introducing suitable local systems on $L\mathcal{L}$, or considering higher operations in suitable $L_\infty$-algebras or SFT algebras, may yield further such examples.

2 Preliminaries

Throughout the paper we follow the definitions and notations of Seidel and Solomon [SS12], with one distinction: we take the opposite sign for all action functionals. Furthermore, we adopt the following convention: everywhere we argue up to $\epsilon$, and allow arbitrarily small perturbations of all Hamiltonian terms involved. For example, when the Hamiltonian perturbation data has curvature zero, it means that we may achieve regularity by a Hamiltonian term arbitrarily close to the given one, in such a way as to make $L^1$-norm of the curvature arbitrarily small.

We give further details on the definitions where required. In particular we look at exact Lagrangian submanifolds $L$ inside a Weinstein manifold $W$ with Liouville form $\theta$, and symplectic form $\omega = d\theta$. We restrict attention to the case when $W$ is the completion of a Weinstein domain $D$ with compact contact boundary $S$, and we
consider \( L \subset D \). Let \( X \) denote the Louvile vector field on \( W \) defined by \( \iota_X \omega = \theta \). It points outside of \( D \) over \( S \).

For the definition of symplectic cohomology we choose a cofinal family of Hamiltonians \( H_\lambda \) that are \( \epsilon \)-small in the \( C^2 \) norm on \( D \), that is, \( ||H_\lambda||_{C^2(D)} < \epsilon \), and are in fact non-positive Morse functions there with gradient pointing outward of \( D \) at \( S \). Furthermore outside of \( D \cup C \) for a small collar neighborhood \( C = C_\lambda \) of \( S \), \( H_\lambda = \lambda \cdot (r - r_\lambda) \), where \( r_\lambda > 1 \) is close to 1, and \( r \) is the radial coordinate on the infinite end \( ([1, \infty) \times S, d(\alpha)) \), \( \alpha = \theta|_S \), of the completion, with the property that \( \lambda \not\in \text{Spec}(\alpha, S) \), that is, it is not a period of a closed Reeb orbit of \( \alpha \). We identify \( C_\lambda \) with \( (1 - s_\lambda, 1 + s_\lambda) \times S \) by the flow of the Liouville vector field. A Reeb orbit is a smooth loop \( \gamma : \mathbb{R}/T\mathbb{Z} \to S \), \( T > 0 \), such that \( \gamma'(t) = R_\alpha \circ \gamma(t) \) for all \( t \in \mathbb{R}/T\mathbb{Z} \), and the Reeb vector field \( R_\alpha \) on \( S \) is defined by the conditions \( \iota_{R_\alpha} \alpha = 1 \), \( L_{R_\alpha} \alpha = \iota_{R_\alpha} d\alpha = 0 \). Furthermore, we require that \( 0 < \epsilon \ll \epsilon_\alpha = \min \text{Spec}(\alpha, S) \), and that \( H_\lambda \) be radial increasing and convex in \( C \). Furthermore, by an arbitrarily small time-dependent perturbation in \( D \cup C \) of \( H_\lambda \) we will assume that it is non-degenerate at all its 1-periodic orbits, which necessarily lie in \( D \cup C \). Now assume that \( \alpha = \theta|_S \) is non-degenerate, that is, if \( \xi = \ker(\alpha) \) is the contact distribution on \( S \), then for each periodic Reeb orbit \( \gamma \) of period \( T \), and point \( x \) on \( \gamma \), \( \ker(D(\phi_{R_\alpha}^t))|_\xi_x - \text{id}|_\xi_x = 0 \). This can be achieved by a \( C^\infty \)-small perturbation of \( S \), which corresponds to scaling \( \alpha \) by a positive function \( C^\infty \)-close to 1 (see [Bou09] for a relevant construction).

In this case, the perturbed \( H_\lambda \) can be chosen in such a way that its closed one-periodic orbits in \( C \) are in a 2 to 1 correspondence with the Reeb orbits of \( \alpha \) of periods in \( [\epsilon_\alpha, \lambda] \), and we choose \( C, H_\lambda \) so that for a fixed \( \delta > 0 \) independent of \( \lambda \) the \( H_\lambda \)-actions of these orbits are \( \delta \)-close to their \( \alpha \)-periods. We choose \( \delta \ll \epsilon_\alpha \).

Furthermore, we require that \( H_{\lambda_k} \leq H_{\lambda_{k+1}} \), \( k \geq 1 \), on \( W \) for a strictly increasing sequence \( \{\lambda_k\}_{k \geq 1} \), \( \lambda_k \xrightarrow{k \to \infty} \infty \), in \( \mathbb{R}_{>0} \setminus \text{Spec}(\alpha, S) \), \( r_{\lambda_k} \xrightarrow{k \to \infty} 1 \), \( C_{\lambda_k} \xrightarrow{k \to \infty} S \) in the sense that \( s_{\lambda_k} \xrightarrow{k \to \infty} 0 \), and that \( ||H_{\lambda_k}||_{C^2(D)} \xrightarrow{k \to \infty} 0 \). That these choices can be made is standard material on symplectic cohomology (see for example [GH18, Section 5]). From now on, when we write \( H_\lambda \) we assume that \( \lambda = \lambda_k \) for some \( k \geq 1 \).

Finally, for two fixed Lagrangian submanifolds \( L_0, L_1 \subset D \) we may choose \( H_{\lambda_k} \) on \( D \) so that the intersection \( \phi_{H_{\lambda_k}}^t(L_0) \cap L_1 \) is transverse for all \( k \geq 1 \). We recall that the \( \omega \)-compatible almost complex structures \( J \) that we consider are of convex type: on the infinite end of \( W \), \( J \partial_r = R_\alpha \), and \( J \) is invariant under translations in \( \rho = \log(r) \). The action functional of a Hamiltonian \( H \) evaluated on a loop \( x : S^1 \to W \) is defined as

\[
A_H(x) = -\int_0^1 H(t, x(t)) \, dt + \int_x \theta.
\]

We consider the Floer cohomology groups \( CF^*(H_\lambda) \), that as \( \mathbb{K} \)-modules have generators corresponding to 1-periodic orbits of \( H_\lambda \). The coefficient near \( x_- \) of the differential \( d_{H,J} \) evaluated on \( x_+ \), for \( J \)-generic, counts isolated solutions
$u : \mathbb{R} \times S^1 \to W$ to the Floer equation

$$\partial_s u + J_t(u)(\partial_t u - X^t H(u)) = 0,$$

with asymptotic conditions $u(s, -) \to x_\pm(-)$, as $s \to \pm \infty$, for 1-periodic orbits $x_\pm$ of $H = H_\lambda$. Here $X_H$ is the time-dependent Hamiltonian vector field of $H$ given by $\iota_{X_H} \omega = -d(H(t, -))$. Note that the critical points of $A_H$ on the loop space $\mathcal{L}W$ are precisely given by time-1 periodic orbits of the isotopy $\{\phi^t_H\}$ generated by $X_H$.

Moreover, if the coefficient of $z$ in $d_H J(y)$ is non-zero, then $A_H(y) > A_H(z)$. Finally, $CF^*(H_{\lambda_k})$ forms a direct system with respect to the natural order on $\{\lambda_k\}$, by means of Floer continuation maps: $CF^*(H_{\lambda_k}) \to CF^*(H_{\lambda_{k'}})$ for $\lambda_k \leq \lambda_{k'}$. Here it is important that $H_{\lambda_k}(t, x)$ is increasing as a function of $k$. The symplectic cohomology of $W$ is defined as

$$SH^*(W) = \lim_{\to} HF^*(H_\lambda) = \lim_{\to} HF^*(H_{\lambda_k}).$$

Its filtered version associated to $(D, S)$ is defined as

$$SH^*(W)^{<t} = \lim_{\to} HF^*(H_{\lambda_k})^{<t},$$

where $HF^*(H_{\lambda_k})^{<t}$ is the homology of the subcomplex $CF^*(H_{\lambda_k})^{<t}$ generated by 1-periodic orbits of action strictly smaller than $t$. Finally recall that when the slope $\lambda > 0$ is below the minimal period of a Reeb orbit of $\alpha$, then $HF^*(H_\lambda) \cong H^*(D)$. This gives rise to the Viterbo map

$$H^*(D) \to SH^*(D).$$

Given two exact Lagrangian submanifolds $L_0, L_1 \subset D$, we choose generic perturbation data $D = (J^{L_0, L_1}, K^{L_0, L_1})$ consisting of an almost complex structure $J^t$ that depends on time $t \in [0, 1]$, and a Hamiltonian $K^{L_0, L_1}$ that is radial outside of $D \cup C$ (for example zero there), and define the Floer complex $CF(L_0, L_1; D)$ with generators corresponding to $X_{K^{L_0, L_1}}$-chords from $L_0$ to $L_1$, the matrix coefficients $\langle d_{L_0, L_1; D}(x_+), x_- \rangle$ of whose differential $d_{L_0, L_1; D}$ count isolated solutions $u : \mathbb{R} \times [0, 1] \to W$ to the Floer equation

$$\partial_s u + J^t_{L_0, L_1}(u)(\partial_t u - X^t_{K^{L_0, L_1}}(u)) = 0,$$

with boundary conditions

$$u(\mathbb{R}, 0) \subset L_0, u(\mathbb{R}, 1) \subset L_1,$$

and uniform asymptotics

$$u(s, -) \xrightarrow{s \to \pm \infty} x_\pm(-).$$
Enhancing $L_0, L_1$ to $L_0 = (L_0, f_0), L_1 = (L_1, f_1)$ by choices of primitives $f_0 \in C^\infty(L_0, \mathbb{R}), f_1 \in C^\infty(L_1, \mathbb{R})$ of $\theta|_{L_0}$ and $\theta|_{L_1}$, we define the action functional on the space of paths $\mathcal{P}(L_0, L_1)$ in $W$ from $L_0$ to $L_1$,

$$A_{L_0, L_1; D} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$$

$$A_{L_0, L_1; D}(x) = -\int_0^1 K^{L_0, L_1}(t, x(t)) + \int_x \theta + f_1(x(1)) - f_0(x(0)).$$

The critical points of $A_{L_0, L_1; D}$ correspond to the generators of $CF(L_0, L_1; D)$, and if the coefficient of $z$ in $d\lambda = d\theta|_{L_0, L_1; D}(y)$ is non-zero then $A_{L_0, L_1; D}(y) > A_{L_0, L_1; D}(z)$.

Furthermore, as we assume that $L_0, L_1$ are connected, $A_{L_0, L_1; D}$ does not depend on the enhancements $L_0, L_1$ of $L_0, L_1$ up to an additive constant.

For a class $a \in SH^*(W) \setminus \{0\}$, its symplectic cohomology spectral invariant $c(a, D, S)$ relative to the domain $D$ with contact-type boundary $S$, is defined as

$$c(a, D, S) = \inf \{ t \in \mathbb{R} \mid a \in \text{im}(SH^*(W)^{< t} \to SH^*(W)) \},$$

where $SH^*(W)^{< t} \to SH^*(W)$ is the natural map induced by the complex $\mathcal{CF}^*(H_\lambda)^{< t} \subset \mathcal{CF}^*(H_\lambda)$. These spectral invariants (see [SZ21, Lee17, PRSZ20]) are known to satisfy the following properties.

First, unless $a$ is in the image of the Viterbo map $H^*(D) \to SH^*(D)$, $c(a, D, S)$ is given as the period $\int_\gamma \alpha$ of a certain $\alpha$-Reeb orbit $\gamma$ on $S$. In particular in this caseootnote{This will be the case of interest in all our examples.} $c(a, D, S) > 0$. It is easy to see that $c(a, D, S) = 0$ for $a$ in the image of the Viterbo map.

Second, $c(a, D, S)$ is monotone with inclusions of Liouville domains $D \subset D'$, with completion $W$ ([SZ21, [GH18, Section 8]]). Finally, for $t \in \mathbb{R}$,

$$c(a, \psi^t D, \psi^t S) = e^t c(a, D, S)$$

where $\psi^t$ is the flow of the Liouville vector field $X$ given by $\iota_X \omega = \theta$. In particular if $\psi^{-t} D \subset D' \subset \psi^t D$ and $a$ is not in the image of the Viterbo map then

$$| \log c(a, D, S) - \log c(a, D', S') | \leq t. \tag{2}$$

For a class $x \in HF^*(L_0, L_1) \setminus \{0\}$ its spectral invariant $c(x, L_0, L_1; D)$ relative to the enhancements $L_0, L_1$ and perturbation data $D$, is set to be

$$c(x, L_0, L_1; D) = \inf \{ t \in \mathbb{R} \mid x \in \text{im}(HF^*(L_0, L_1; D)^{< t} \to HF(L_0, L_1; D)) \},$$

where $HF(L_0, L_1; D)^{< t}$ is the homology of the subcomplex $\mathcal{CF}^*(L_0, L_1; D)^{< t}$ of $\mathcal{CF}^*(L_0, L_1; D)$ generated by chords $z$ of action $A_{L_0, L_1}(z) < t$. Alternatively, for chords $z_1, \ldots, z_N$ giving a base of generators, we set for $x = \sum a_j z_j, a_j \in \mathbb{K}$,

$$A_{L_0, L_1}(x) = \max_{a_j \neq 0} A_{L_0, L_1}(z_j).$$
with $\mathcal{A}_{L_0, L_1}(0) = -\infty$ by convention. This is a non-Archimedean valuation, and in particular for all $a_1, a_2 \in \mathbb{K}$, $x_1, x_2 \in CF^*(L_0, L_1; \mathcal{D})$,

$$\mathcal{A}_{L_0, L_1}(a_1 x_1 + a_2 x_2) \leq \max\{\mathcal{A}_{L_0, L_1}(x_1), \mathcal{A}_{L_0, L_1}(x_2)\}.$$  

Moreover

$$\mathcal{A}_{L_0, L_1}(d(x)) \leq \mathcal{A}_{L_0, L_1}(x)$$

for all $x \in CF^*(L_0, L_1; \mathcal{D})$. Then $CF^*(L_0, L_1; \mathcal{D})^{c_t} = \mathcal{A}_{L_0, L_1}^{-1}\{(-\infty, t)\}$, which is clearly a subcomplex by the above properties.

It is well-known (see [LZ18, She22] and references therein) that the following spectrality property holds: for $x \neq 0$, $c(x, L_0, L_1; \mathcal{D})$ is given by $\mathcal{A}_{L_0, L_1; \mathcal{D}}(z)$ for a generator $z$ of $CF^*(L_0, L_1; \mathcal{D})$, and is therefore finite. Furthermore, $c(x, L_0, L_1; \mathcal{D})$ does not depend on the almost complex structure part $J_{L_0, L_1}$ of $\mathcal{D}$, and is Lipschitz in the Hofer norm of the Hamiltonian term $K_{L_0, L_1}$ of $\mathcal{D}$, in the sense that if the Hamiltonian terms $K, K'$ of $\mathcal{D}, \mathcal{D}'$ agree outside a compact set (in our case this means that their slopes at infinity agree), then

$$|c(x, L_0, L_1; \mathcal{D}) - c(x, L_0, L_1; \mathcal{D}')| \leq \int_0^1 (\max_{W} (F_t) - \min_{W} (F_t))\, dt,$$

where $F = K' \# K$ is the Hamiltonian generating the flow $\phi_{k_t}^t \circ (\phi_{k_t}^1)^{-1}$. This allows us to extend the spectral invariant to arbitrary perturbations (even continuous ones), and in particular we define $c(x, L_0, L_1)$ as the limit of $c(x, L_0, L_1; \mathcal{D})$ as the norm of the Hamiltonian term of $\mathcal{D}$ tends to zero. Finally, we remark that if $\phi_{k_t}^t(L_0) \subset \mathcal{D} \setminus C$, for all $t \in [0, 1]$, and where $C$ is the collar neighborhood of $S$ such that $K$ is convex radial in $C$ and has slope $\lambda$ outside $D \cup C$, then $c(x, L_0, L_1; \mathcal{D})$ depends only on $K(t, x)$ for $(t, x) \in [0, 1] \times (\mathcal{D} \setminus C)$, by a suitable maximum principle. Indeed, in this case the filtered Floer complex $(CF^*(L_0, L_1; \mathcal{D}), \mathcal{A}_{L_0, L_1; \mathcal{D}})$ does not depend on $K(t, x)$ for $(t, x) \notin [0, 1] \times (\mathcal{D} \setminus C)$. In fact, for our purposes it is enough to observe that the generators and their actions only depend on $K(t, x)$ for $(t, x) \in [0, 1] \times (\mathcal{D} \setminus C)$: then by the spectrality and continuity properties the spectral invariants also depend only on these values of $K(t, x)$. In particular, if the Hamiltonian term of $\mathcal{D}_k$ is given by $H_{\lambda_k}$, then

$$c(x, L_0, L_1; \mathcal{D}_k) \xrightarrow{k \to \infty} c(x, L_0, L_1).$$

From now on, for each exact Lagrangian $L$ we fix an enhancement $\mathcal{L}$, and set $c(x, L_0, L_1; \mathcal{D}) := c(x, L_0, L_1; \mathcal{D})$, and $c(x, L_0, L_1) := c(x, L_0, L_1)$. Our results will not depend on this choice.

Signs in the count of the differentials, as well as gradings, in both kinds of Floer complexes are determined by certain background classes. We summarize these below, and refer to [SS12, AK18, Abo12, Kra13, Sei08, Abo15] for details. For grading in symplectic cohomology, we assume that $2c_1(TW) = 0$, in which case the Grassmannian Lagrangian bundle $\widetilde{Lag}(M) \to M$ admits a cover $\widetilde{Lag}(M) \to M$ with fibers
given by universal covers of the former fibers, and for signs we fix a background class \( b \in H^2(W, \mathbb{F}_2) \). In the main case we consider, \( W = T^*L \), for \( L \) a closed manifold, our assumption holds, and we set \( b = \pi^*w_2(L) \), for \( \pi: T^*L \to L \) the natural projection. The existence of the cover can be deduced by considering the section of \( \text{Lag}(M) \) given by the Lagrangian subspaces tangent to the fibres. We equip each exact Lagrangian \( L' \subset T^*L \) with the structure of a brane as follows. By [Kra13], the Maslov class of \( L' \) vanishes, whence \( \widetilde{\text{Lag}}(M)|_{L'} \) admits \( H^0(L', \mathbb{Z}) \)-worth of sections covering the Lagrangian Gauss map \( G: L' \to \text{Lag}(M)|_{L'}, \, G(x) = T_xL' \). We call these sections gradings and we pick one of them. Note that we consider connected \( L' \), hence \( H^0(L', \mathbb{Z}) = \mathbb{Z} \). By [Abo12], \( \pi_*^*w_2(L) = w_2(L') \), hence \( L' \) is relatively Spin with respect to \( b \), and furthermore out of the \( H^1(L', \mathbb{F}_2) = H^1(L, \mathbb{F}_2) \) choices of a relative Spin structure we fix one, such that \( L' \) endowed with these choices is Floer-theoretically equivalent to the zero-section \( L \) (see Theorem C) with the standard relative Spin structure and grading. Throughout the paper, when considering Lagrangians, we keep in mind such an underlying determination of a brane structure.

Cycles in Deligne–Mumford moduli spaces of disks, considered as Riemann surfaces with boundary, decorated with interior and boundary punctures, whose universal curves are equipped with choices of positive or negative (input or output type) cylindrical ends at each puncture, induce operations on the various Floer homology groups considered. Indeed, we may equip the universal curves with Floer data compatible with gluing and compactification, wherein the cylindrical ends allow one to write suitable Floer equations and asymptotic conditions on the punctures to land in the correct Floer complexes. For more details we refer to [SS12]. In our case, as we wish to consider the behavior of actions and energies in our operations, we need to make further choices. In particular, we use the notion of cylindrical strips introduced and used in [KS21, Section 2.5], whereto we refer for the technical definition. However, we note that cylindrical strips are certain maps of a strip \( \mathbb{R} \times [0, 1] \) to a Riemann surface (or a family of maps to a family of Riemann surfaces) decorated with marked points and cylindrical ends, with which they are required to be compatible. In fact, the Floer decorations for our main homological operation were already considered in [KS21] in the case of closed monotone symplectic manifolds, and their monotone Lagrangian submanifolds. We note that in contrast to the closed case, in the case of Liouville manifolds one must ensure that the images of all Floer solutions lie in a compact subset of \( W \). This is accomplished by the integrated maximum principle (see [AS10, Lemma 7.2] or [Abo15, Section 5.2.7]).

In particular, consider the moduli space of disks with a unique input interior marked point and one output boundary marked point, with the cylindrical end at the interior marked point chosen so that the asymptotic marker points towards the boundary marked point along a hyperbolic geodesic. This moduli space is a point: we can represent it by the standard disk with interior input at the origin \( 0 \in \mathbb{C} \), boundary output at \( 1 \in \mathbb{C} \) and the asymptotic marker at \( 0 \) pointing towards \( 1 \). We may equip this decorated surface with a choice of a cylindrical strip from the input
to the output. We set the Hamiltonian Floer datum to be $H_\lambda \otimes \beta$, $\beta = d(\tau) = \tau'(t) \, dt$ on the cylindrical strip, and zero elsewhere, for a smooth surjective non-decreasing function $\tau : [0, 1] \to [0, 1]$ which is constant near 0 and near 1. Furthermore, for a Lagrangian $L \subset D$ we choose $L$ as the boundary condition for the Floer solutions. This gives us, for a suitable perturbation datum with Hamiltonian part given by $H_\lambda$, an operation

$$\phi^0_L : CF^*(H_\lambda) \to CF^*(L, L; H_\lambda) \to CF^*(L, L; D),$$

which is a chain map. Here, the last map is the identification, coming from the above discussion, of $CF^*(L, L; H_\lambda)$ with $CF^*(L, L; D)$ for perturbation data $D$ with Hamiltonian part compactly supported and of $C^2$ norm $o(1)$ as $\lambda \to \infty$. This operation yields the canonical restriction map

$$r_L : SH^*(W) \to HF^*(L, L).$$

Recall that the curvature of a Hamiltonian 1-form describing the Hamiltonian part of the Floer data controls the extent to which the resulting operation shifts filtrations (see [KS21, Equations (12), (13)] for relevant action estimates). As by [KS21, Section 2.5, Remark 27], arguing up to $\varepsilon$, the Hamiltonian terms of the Floer data chosen as above have zero curvature, all perturbation data, in particular $D^{L,L}$, can be chosen to have Hamiltonian parts sufficiently small, so that this operation satisfies $A_{L,D}(\phi^0_L(x)) \leq A_{H_\lambda}(x) + 2\varepsilon$. Of course, $A_{L,D} \leq \varepsilon$ identically, so this consideration does not add new information. However, as we shall see below, it applies to more complicated situations where it is indeed non-obvious.

Now consider the moduli space of disks with three boundary marked points, two inputs and an output. The Floer equation with an arbitrarily small Hamiltonian term and boundary conditions on Lagrangians $L_0, L_1, L_2$ yields the familiar product operation

$$\mu_2 : CF^*(L_1, L_2; D) \otimes CF^*(L_0, L_1; D) \to CF^*(L_0, L_2; D).$$

If the $L^1$-norm of the curvature of the Hamiltonian term is bounded by $\varepsilon$, this operation is filtered in the sense that

$$A_{L_0,L_2}(\mu_2(y, x)) \leq A_{L_1,L_2}(y) + A_{L_0,L_1}(x) + \varepsilon.$$ 

We refer to [Sei08, Section (8g)] and [BCS21] for this and further estimates of this kind.

Furthermore, consider the moduli space of disks with two boundary marked points, an input and an output, and one interior marked point with asymptotic marker pointing towards the output. This moduli space is identified with an interval $(-1, 1) \cong \mathbb{R}$, and its Deligne–Mumford compactification is identified with a closed interval $[-1, 1] \cong \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \}$ by adding what we refer to as nodal disks at $-1$ and at $+1$. See [SS12, Figure 1] for a picture of these nodal disks: these are singular Riemann surfaces with boundary consisting of two disk components connected by
a boundary node (see also [Sei08, Chapter 9] for a slightly different case without interior marked points). In terms of Floer theory, these nodal disks together with suitable decorations yield the composition of operations given by their two disk components.

For concreteness, we may identify the moduli space with \((-1, 1)\) by observing that the universal curve over it is isomorphic to the standard disk \(D^2 \subset \mathbb{C}\) with one boundary input at \(\zeta_- = -1\), one boundary output at \(\zeta_+ = 1\), and one interior input at \(z = iy\) for \(y \in (-1, 1)\) with asymptotic marker pointing towards the output along a hyperbolic geodesic. From this point of view, it is easy to choose cylindrical ends on this universal curve accordingly (see [Sei08, Chapter (9g)]), and choose a cylindrical strip between the interior input and the boundary output, so that all extends to the compactification (see [KS21]). On this cylindrical strip, let the Hamiltonian part of the Floer datum be \(H_{\lambda} \otimes \beta\), where \(\beta = d(\tau)\) as above. This yields an operation:

\[
\phi^1_{L_0, L_1} : CF^*(H_{\lambda}) \otimes CF^*(L_0, L_1; D) \to CF^*(L_0, L_1; H_{\lambda})[-1] \to CF^*(L_0, L_1; D)[-1],
\]

where the last map is again the identification between the Floer complex with Hamiltonian term \(H_{\lambda}\) and the Floer complex with Hamiltonian term compactly supported and \(C^2\)-small. Considering the above compactification, one obtains [SS12] that \(\phi^1_{L_0, L_1}\) provides a homotopy between the two maps

\[
\mu_2(\phi^0_{L_0}(a), x),
\]

\[
(-1)^{|a||x|} \mu_2(x, \phi^0_{L_1}(a)),
\]

where \(a \otimes x \in CF^*(H_{\lambda}) \otimes CF^*(L_0, L_1)\). Furthermore, by our choice of Floer data on the cylindrical strip, whose curvature vanishes by definition, choosing the Floer data \(D = D^{L_0, L_1}\) and perturbation data for the operation to have sufficiently small Hamiltonian parts, we obtain by a standard action estimate (for example see [KS21, Equation (12)]) that for all \(a \otimes x \in CF^*(H_{\lambda}) \otimes CF^*(L_0, L_1)\),

\[
A_{L_0, L_1; D}(\phi^1_{L_0, L_1}(a, x)) \leq A_{H_{\lambda}}(a) + A_{L_0, L_1; D}(x) + 2\epsilon.
\]

Finally, let \(a \in CF^*(H_{\lambda})\) be a cycle, whose cohomology class represents \(\bar{\alpha} \in SH^*(W)\), with

\[
r_L(\bar{\alpha}) = 0 \in HF^*(L, L).
\]

Following [SS12, Definition 4.2], we call \(L\) a-equivariant with primitive \(c_L \in CF^*(L, L)\) if

\[
\phi^0_L(a) = \mu_1(c_L).
\]
As in [SS12, Equation 4.4], given a cycle \( a \in CF^k(H_\lambda) \), and two \( a \)-equivariant Lagrangians \( L_0, L_1 \subset D \), with primitives \( c_{L_0} \in CF^{k-1}(L_0, L_0) \), \( c_{L_1} \in CF^{k-1}(L_1, L_1) \), we can upgrade \( \tilde{\phi}^1_{L_0, L_1}(a, -) \) to a chain map

\[
\tilde{\phi}^1_{L_0, L_1}(a, -) : CF^*(L_0, L_1) \to CF^*(L_0, L_1)[-1 + k],
\]

by setting for homogeneous \( x \in CF^*(L_0, L_1) \),

\[
\tilde{\phi}^1_{L_0, L_1}(a, x) = \phi^1_{L_0, L_1}(a, x) - \mu_2(c_{L_0}, x) + (-1)^{(k-1)|x|} \mu_2(x, c_{L_1}). \tag{4}
\]

Of course \( \tilde{\phi}^1_{L_0, L_1}(a, x) \) depends on the choice of primitives \( c_{L_0}, c_{L_1} \) for the \( a \)-equivariant structures. We keep this dependence implicit in the notation for conciseness, and we further discuss it in Sect. 3.1.

For sufficiently \( C^2 \) Hamiltonian-small perturbation data \( \mathcal{D}^{L_0, L_0}, \mathcal{D}^{L_1, L_1} \) it is easy to see that all non-zero chains in \( CF^*(L_0, L_0) \), \( CF^*(L_1, L_1) \) are of actions \( \mathcal{A}_{L_0, L_0; \mathcal{D}}, \mathcal{A}_{L_1, L_1; \mathcal{D}} \) bounded in absolute value by \( \epsilon \). We claim that this implies that

\[
\mathcal{A}_{L_0, L_1; \mathcal{D}}(\tilde{\phi}^1_{L_0, L_1}(a, x)) \leq \mathcal{A}_{H_\lambda}(a) + \mathcal{A}_{L_0, L_1; \mathcal{D}}(x) + 3\epsilon \tag{5}
\]

for all \( a \in CF^*(H_\lambda) \), and \( x \in CF^*(L_0, L_1) \). The case \( a = 0 \) is trivial. If \( a \neq 0 \), then by our choices in the construction of symplectic cohomology \( \mathcal{A}_{H_\lambda}(a) \geq -\epsilon \), hence

\[
\mathcal{A}_{H_\lambda}(a) + \mathcal{A}_{L_0, L_1; \mathcal{D}}(x) + 3\epsilon \geq \mathcal{A}_{L_0, L_1; \mathcal{D}}(\mu_2(c_{L_0}, x))
\]

and similarly for the third term in (4). Hence (5) holds.

Finally, we recall two fundamental results on the symplectic topology of cotangent bundles. The first result, proved by Viterbo [Vit99], Abbondandolo-Schwarz [AS10], and Salamon-Weber [SW06] in the case of \( \mathbb{K} = \mathbb{F}_2 \), or for \( \text{Spin} \) manifolds and arbitrary coefficients, and by Abouzaid and Kragh in the general case (see [Abo15] and the references therein), asserts a relation between the symplectic cohomology of \( W = (T^*L, \theta_{\text{can}}) \) considered as a Weinstein manifold, and the homology of the free loop space \( \mathcal{L}L \). In general, to compare signs between the two theories, a local system on \( \mathcal{L}T^*L \) should be introduced, as mentioned above. For certain choices of \( L \) and \( \mathbb{K} \), such as \( \mathbb{K} = \mathbb{F}_2 \), or \( L \) being \( \text{Spin} \), this local system is trivial and can therefore be ignored. We note that it is important for our purposes, that this is an isomorphism of Gerstenhaber algebras over \( \mathbb{K} \), rather than simply one of \( \mathbb{K} \)-algebras. This aspect of the isomorphism is discussed in [Abo15]: in fact it is an isomorphism of BV-algebras (see [Abo15, Theorem 4.1.1, Corollary 6.1.2]). Finally, the last point of this first result follows from the \( r = 1 \) case of [Abo15, Section 4.3.3] (see also [AS10, Section 4.6]).

**Theorem B.** (Viterbo isomorphism) *Let \( W = T^*L \) with the standard Liouville structure. There exists an isomorphism

\[
\Phi : SH^*(W) \to H_{n-*}(\mathcal{L}L),
\]*
of BV-algebras over $\mathbb{K}$, where $SH^*(W)$ is endowed with the pair-of-pants product, and the BV-operator arising from the moduli space of cylinders with free asymptotic markers at infinity, while $H_{n-*}(\mathcal{L}L)$ is endowed with the Chas–Sullivan product, and the BV-operator given by suspending the $S^1$-action by loop-rotation on $\mathcal{L}L$. The map $ev : H_{n-*}(\mathcal{L}L) \to H_{n-*}(L)$ corresponds to the map $r_L : SH^*(W) \to HF^*(L)$ by the isomorphism $\Phi$ and Poincaré duality.

The second result, due to Fukaya-Seidel-Smith [FSS08] in the simply connected case, and Abouzaid [Abo12] and Kragh [Kra13], in the general case, asserts that each exact Lagrangian $L'$ in the cotangent bundle $T^*L$ is isomorphic to $L$ in the Fukaya category of $T^*L$. We state a simplified version on the level of homology that is sufficient for our purposes, referring to [AK18]. While it is not explicitly mentioned therein, it is not difficult to observe that the isomorphism from $\Phi$ is in fact an isomorphism of modules over $SH^*(T^*L)$; indeed, one knows that it is given by multiplication by continuation elements, this is a consequence of the homotopy property of $\phi^1_{L_0,L_1}$ and the associativity of the $\mu_2$ operations on the level of homology.

**Theorem C.** (Exact nearby Lagrangians are Floer-theoretically equivalent) Let $L$ and $\mathbb{K}$ be as above, and let $L'$ be an exact Lagrangian in $T^*L$. Then there exists an integer $i = i_{L'} \in \mathbb{Z}$ such that for each exact Lagrangian $K$, the $SH^*(T^*L)$-modules $HF^*(L', K)$, and $HF^*+i(L, K)$ are isomorphic, and the same is true for $HF^*(K, L')$, and $HF^*+i(K, L)$. The chain-level quasi-isomorphisms in both directions can be taken to be multiplication operators

$$
\mu_2(\cdot, x) : CF^*(L', K) \to CF^{*+i}(L, K),
$$

$$
\mu_2(\cdot, y) : CF^*(L, K) \to CF^{*-i}(L', K),
$$

respectively

$$
\mu_2(y, \cdot) : CF^*(K, L') \to CF^{*-i}(K, L),
$$

$$
\mu_2(x, \cdot) : CF^*(K, L) \to CF^{*+i}(K, L'),
$$

for certain cycles $x \in CF^i(L, L')$, $y \in CF^{-i}(L', L)$.

At this point we define the spectral norm $\gamma(L_0, L_1)$ for exact Lagrangians in $T^*L$ as follows. Choose primitives $f_0, f_1$ of the restrictions $\theta|_{L_0}, \theta|_{L_1}$ of the Liouville form $\theta$ to $L_0, L_1$ respectively. This allows us to filter $CF^*(L_0, L_1)$ by an action functional induced by $L_0 = (L_0, f_0)$, $L_1 = (L_1, f_1)$. Since $HF^*(L_0, L_1) \cong HF^*(L, L) \cong H^*(L)$, consider the classes $\mu \in HF^*(L_0, L_1)$ that correspond to the generator $\mu_L = PD([pt]) \in H^*(L)$, and the unit $1 = PD([L]) \in H^0(L)$ respectively. Recall that for a class $a \in HF^*(L_0, L_1)$ \{0\}, we defined the Lagrangian spectral invariant as

$$
c(a, L_0, L_1; D) = \inf\{t \in \mathbb{R} \mid a \in \text{im}\left(HF^*(L_0, L_1; D)^{<t} \to HF^*(L_0, L_1; D)\right)\}.
$$
These invariants are finite, and satisfy numerous useful properties, and in particular they are defined for arbitrary exact \( L_0, L_1 \), by taking the limit as the Hamiltonian term in the perturbation datum \( \mathcal{D} \) goes to zero. We set the spectral norm to be

\[
\gamma(L_0, L_1) = c(\mu, L_0, L_1) - c(e, L_0, L_1).
\]

Note that as a difference of two spectral invariants it does not depend on the choice of enhancements \( L_0, L_1 \) of \( L_0, L_1 \). Furthermore, by considering the identity \( \mu_L = \mu_L \ast 1 \) in \( H^*(L) \), one obtains the identity \( \mu = \mu_{L_1} \ast \epsilon \), under the isomorphisms \( H^*(L) \cong HF^*(L, L) \cong H^*(L_1) \) and \( H^*(L) \cong HF^*(L, L) \cong HF^*(L_0, L_1) \), from which one obtains that \( \gamma(L_0, L_1) \geq 0 \), and that the inequality is strict unless \( L_0 = L_1 \) (see [KS21]). Further properties of spectral invariants imply that \( \gamma(L_0, L_1) = \gamma(L_1, L_0) \) for all \( L_0, L_1 \) exact, and that \( \gamma(L_0, L_1) \leq \gamma(L_0, K) + \gamma(K, L_1) \) for all \( L_0, L_1, K \) exact, whence \( \gamma \) defines a metric on the space of exact Lagrangian submanifolds of \( T^*L \). Furthermore, this metric is invariant under the action of the group of Hamiltonian diffeomorphisms: for all \( H \in C^\infty_c([0, 1] \times T^*L, \mathbb{R}) \) and \( L_0, L_1 \) exact, \( \gamma(\phi L, \phi L_1) = \gamma(L, L_1) \), where \( \phi = \phi^1_H \), is the time-one map of the Hamiltonian isotopy generated by \( H \). Finally, note that we shall study the restriction of \( \gamma \) to the subspace of exact Lagrangian submanifolds in \( D \subset T^*L \), where \( D \) is a bounded Liouville domain with completion \( T^*L \).

3 Proofs

3.1 A homological calculation. We start with a new calculation of the map

\[
\bar{\phi}^1_{(L_0, L_1), a}(\cdot) = \bar{\phi}^1_{L_0, L_1}(a, -)
\]

on the level of homology in the setting of cotangent bundles. Let \( \bar{a} \in SH^k(W) \) for \( W = T^*L \) be such that \( r_{L_1}(\bar{a}) = 0 \) in \( HF^*(L, L) \). Then, in view of Theorem C, for each closed exact Lagrangian \( L_0 \) in \( W \) we have again \( r_{L_0}(\bar{a}) = 0 \) in \( HF^*(L_0, L_0) \). Now consider two such exact Lagrangians \( L_0, L_1 \) and a cycle \( a \in CF^k(H_\lambda) \) representing \( \bar{a} \). We require a compatibility assumption on the choice of \( a \)-equivariant primitives \( c_{L_0}, c_{L_1} \). It is immediate from the definition that \( c_{L_i}, i \in \{0, 1\} \), are defined uniquely up to closed chains in \( CF^{k-1}(L_i, L_i) \). Furthermore \( \bar{\phi}^1_{L_0, L_1}(a, -) \) up to chain homotopy depends only on \( c_{L_i} \) up to exact chains in \( CF^{k-1}(L_i, L_i) \). Adding cycles \( c_i \in CF^{k-1}(L_i, L_i) \) to \( c_{L_i} \), changes \( \bar{\phi}^1_{L_0, L_1}(a, x) \) on the homology level by

\[
-\mu_2([c_i], x) + (-1)^{(k-1)|x|}\mu_2(x, [c_1]).
\]

In particular, by Theorem C, and the graded-commutativity of \( HF^*(L, L) \) as an algebra, if \( L_0 = L_1 = L \), and we choose \( c_{L_0} = c_{L_1} \) exact, then we get a canonically defined operator

\[
P'_a = [\bar{\phi}^1_{(L, L), a}]
\]

on homology.
In view of the observations above, by means of Theorem C, if we are merely interested in the operator \( P'_a(L_0, L_1) = [\tilde{\phi}^1_{(L_0, L_1)}, a] \) on the homological level, and \( L_0, L_1 \) are exact Lagrangians in \( T^*L \), then we may replace both \( L_0 \), and \( L_1 \) by \( L \) in the definition. Furthermore, we may choose the primitives \( c_{L_0}, c_{L_1} \) in such a way that the corresponding operator is \( P'_a = [\tilde{\phi}^1_{(L, L)}, a] \) as defined above with respect to the same \( a \)-equivariant primitive on both copies of \( L \). This point, requires a somewhat more careful discussion of the moduli spaces involved, despite being intuitively clear. To streamline the exposition, we therefore formulate it below and defer its proof to the end of the section.

**Proposition 9.** Given \( a \in CF^k(H_\lambda) \), and \( a \)-equivariant exact Lagrangians \( L_0, L_1 \) in \( T^*L \), there exist equivariant primitives \( c_{L_0}, c_{L_1} \) for \( L_0, L_1 \) such that under the isomorphism

\[
HF^*(L_0, L_1) \cong HF^{*-iL_0-iL_1}(L, L)
\]

the operation \([\tilde{\phi}^1_{(L_0, L_1)}, a]\) corresponds to the canonical operator

\[
P'_a : HF^*(L, L) \to HF^{*-1+k}(L, L).
\]

We proceed to compute the latter operation \( P'_a \) as follows. Since this calculation is a generalization of \([SS12, Remark 4.4]\), we only outline the main steps of the argument.

**Proposition 10.** The isomorphism \( HF^*(L, L) \cong H_{n-*}(L) \) obtained by the isomorphism \( HF^*(L, L) \cong H^*(L) \) followed with the Poincaré duality isomorphism \( H^*(L) \cong H_{n-*}(L) \), identifies \( P'_a : HF^*(L, L) \to HF^*(L, L) \) and the map \( P_a : H_{n-*}(L) \to H_{n-*}(L) \).

**Proof of Proposition 10.** One way to prove this result involves first showing that \( P'_a = r_L \circ m'_a \circ \iota' \), where \( \iota' \) is a homological inclusion map \( HF^*(L, L) \cong HF_{n-*}(L, L) \to SH^*(T^*L), m'_a : SH^*(T^*L) \to SH^*(T^*L) \) is the right symplectic homology bracket with \( a \in SH^*(T^*L) \), given by the pair of pants product and the BV-operator \( \Delta' : SH^*(T^*L) \to SH^*(T^*L)[-1] \), and \( r_L : SH^*(T^*L) \to HF^*(L, L) \) is the restriction map. Consequently, one shows that each of these maps is identified with \( \iota, m_{\Phi(a)}, \) and \( ev \), respectively under the isomorphisms \( \Phi \) and \( HF^*(L, L) \cong H^*(L) \cong H_{n-*}(L) \). Theorem B takes care of identifying \( m'_a \) and \( m_{\Phi(a)} \), as it identifies the products and the BV-operators \( \Delta' \) and \( \Delta \). It also identifies \( r_L \) with \( ev \). It is left to identify \( \iota' \) with \( \iota \). The map \( \iota' \) defined below can again be seen to correspond to \( \iota \) by following the construction in \([Abo15, Section 5.3]\). Alternatively, see \([AS10, Section 4.6]\), \([AS14]\).

To show that \( P'_a = r_L \circ m'_a \circ \iota' \), we first observe that \( \iota' \) is given in \([Abo15]\) (see also \([AS10]\)) by the moduli space of disks with one interior marked point, which is an output, and boundary conditions on a cotangent fibre \( T^*_xL \) where \( x \), that we consider to be an input, is allowed to vary freely in \( L \). We choose the asymptotic marker at the output to point along a hyperbolic geodesic at an extra marked point
on the boundary introduced formally. In the quotient model this corresponds to the output being at the origin with the asymptotic marker pointing along the positive real direction. We choose the perturbation datum to be $H_\lambda \otimes dt$ on a cylindrical end by the output, and zero near the boundary. This is possible to achieve while keeping the conditions of the integrated maximum principle of Abouzaid and Seidel [AS10], that keeps the corresponding solutions to the Floer equations within $D \cup C_\lambda$.

Similarly, $r_L$ is given by the moduli space of disks with one interior marked point, which is an input, and one boundary marked point which is an output, with the asymptotic marker at the interior marked point pointing towards the boundary marked point, and with boundary conditions on $L$. It is then easy to see by gluing, that $r_L \circ \iota'$ is represented by a chain level map $\psi$ given by the moduli space of annuli of fixed modulus with boundary conditions on a cotangent fiber on one boundary component and $L$ on the other, with a boundary marked point, an output: that is in a concrete model, if the annulus is given by $[0, l] \times S^1$, the boundary marked point is $(l, \zeta)$ for $\zeta \in S^1$, and the boundary condition at $\{0\} \times S^1$ is on $T^*_xL$ for a varying $x$. Moreover it is easy to see that on the homology level $r_L \circ \iota' = \text{id}$, since $r_L$ was identified with $\text{ev}$ and $\iota'$ was identified with $\iota$. This also follows by directly constructing a chain-homotopy by means of a suitable homotopy of the Hamiltonian perturbation data and a pearly model for this operation. Hence $P'_a = P'_a \circ r_L \circ \iota'$.

In the next paragraph we will prove that it is therefore given by the compactified moduli space of annuli of a fixed modulus with one boundary component marked by a varying cotangent fibre, the other boundary component marked by $L$ and endowed with an unconstrained boundary marked point, an output, and one interior input marked point constrained to a circle separating the two boundary components of the annulus. Considering the annulus as a planar domain contained in a disk, the asymptotic marker at the interior marked point is pointing towards the output along a hyperbolic geodesic in the disk, and we use the interior marked point to plug in $a$. In the concrete quotient model, we may assume the boundary marked point is at $(l, 1) \in [0, l] \times S^1$, for $S^1 \subset \mathbb{C}$ the standard circle.

To prove the above description of $P'_a$ we proceed as follows. First we glue the moduli space of disks describing $\hat{\phi}_{L,L}^1(a, -)$ and the moduli space of annuli describing the chain level map $\psi$ (considered above) yielding $r_L \circ \iota'$ on homology. After fixing parametrization, we can fix the former to be given by the standard disk $D^2 \subset \mathbb{C}$ with two fixed marked points $\zeta_- = -1, \zeta_+ = 1$ on the boundary, and an interior marked point $z = iy$ with $\text{Re}(z) = 0$, and asymptotic marker pointing towards $\zeta_+$ along a hyperbolic geodesic. Hence the glued operation is given by a cylinder $[0, l] \times S^1$ with boundary conditions as above, and interior marked point constrained to an arc $\gamma_1$ with boundary $\{(l, \zeta_1)\} - \{(l, \zeta_0)\}$ on $\{l\} \times S^1$ separating $(l, \zeta_+)$ and $\{0\} \times S^1$, with asymptotic marker pointing towards $(l, \zeta_+)$ in the sense of considering the cylinder as a planar domain contained in a disk and pointing along hyperbolic geodesics in the disk. We shall identify, up to chain-homotopy, the composition $\nu = \nu_0 \circ \psi$ of the correction term $\nu_0 = -\mu_2(c_L, -) + (-1)^{k-1}|-| \mu_2(-, c_L)$ in $\hat{\phi}_{L,L,a}^1$, and the operator
$Q$ given by the same Riemann surface as before, now considered as an annulus with outer boundary $\{l\} \times S^1$, with the same boundary conditions and boundary marked points, with the difference that the interior marked point, used to plug in $a$, is now constrained to an arc $\gamma_2$ with boundary $-\{(l, \zeta_1)\} + \{(l, \zeta_0)\}$, which does not separate $(l, \zeta_+)$ and $\{0\} \times S^1$. After establishing this, choosing $\gamma_1$ and $\gamma_2$ suitably, and considering the family of decorated Riemann surfaces corresponding to the same annulus and the loop $\gamma_1 \# \gamma_2$ inside it, we obtain the claim. For instance, we may take $\gamma_1 = I_0 \# \gamma_1' \# I_1, \gamma_2 = T_1 \# \gamma_2' \# T_0$ to be broken arcs where $\gamma_1', \gamma_2'$ form the circle $\{l - \epsilon\} \times S^1$, and $I_0 = [l, l - \epsilon] \times \zeta_0, I_1 = [l - \epsilon, l] \times \zeta_1$ are segments from $(l, \zeta_0)$ to $p_0 = (l - \epsilon, \zeta_0)$ and from $p_1 = (l - \epsilon, \zeta_1)$ to $(l, \zeta_1)$ respectively. Here $T_i$ is the orientation reversal of $I_i, i = 0, 1$. By a transversality and dimension argument, we may assume that the contributions to the chain-level operations corresponding to the points $p_0, p_1$ vanish. Therefore the operations corresponding to $\gamma_1$ and $\gamma_2$ are given by the sum of the operations corresponding to the segments $I_0, \gamma_1', I_1$ and $T_1, \gamma_2', T_0$ respectively. Similarly, the operation for the circle $\gamma_1' \# \gamma_2'$ is given by the sum of the operations for $\gamma_1'$ and $\gamma_2'$.

To prove that $\nu$ and $Q$ are chain-homotopic, by gluing and homotopy, we describe the two summands in $\nu$ in terms of two families of decorated Riemann surfaces. Both are annuli $[0, l] \times S^1$ with boundary condition $T_x L$ at $\{0\} \times S^1, L$ at $\{l\} \times S^1$, one boundary input, which we use to plug in $c_L$, at $(l, \zeta_0)$ in the first family and at $(l, \zeta_1)$ in the second family, and one boundary output at $(l, \zeta_+)$ in both families. Note that $(l, \zeta_0)$ and $(l, \zeta_1)$ are exactly the same points which appeared in the previous paragraph. By gluing again, we see that the difference $\nu'$ between these two new operations is chain-homotopic to $\nu$. Now consider the one-parametric family of such Riemann surfaces with the boundary input at $(l, \zeta)$ with $c_L$ plugged in, as $\zeta$ varies in the circular arc $[\zeta_1, \zeta_0] \subset S^1$ not containing $\zeta_+$. This family provides a chain-homotopy between $\nu'$ and a new operation, where now $\phi_L^0 a = \mu_i^1(c_L)$ is plugged into the boundary input. By a transversality and dimension argument we may assume that in this new operation there are no contributions from the values $\zeta = \zeta_0$ and $\zeta = \zeta_1$ of the parameter. Recalling that $\phi_L^0 a$ is given by a moduli space of disks with one boundary output and one interior input with asymptotic marker pointing at the output, we can perform parametric gluing with one gluing parameter to pass up from a one-parametric family of nodal annuli with nodes at $(\zeta, l)$, for $\zeta \in [\zeta_0, \zeta_1]$ consisting of such disks and of the above Riemann surfaces to the family of decorated Riemann surfaces from the previous paragraph, which corresponds to $\gamma_2 = T_1 \# \gamma_2' \# T_0$ and defines $Q$. That is, the two chain-level operations $\nu$ and $Q$ are chain-homotopic. Note that the asymptotic marker at the interior marked point can still be made to point towards $(l, \zeta_+)$ along a hyperbolic geodesic in a disk that contains the now-smooth annulus.

Finally, it is easy to see, by gluing again, that $r_L \circ m'_L \circ \iota'$ is given by a homotopic moduli space of decorated annuli. This is immediate by gluing from the description of the string bracket [Abo15, Section 2.5.1] as the moduli space of spheres with 3
marked points, where in the model of $S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ we choose the marked points $z_0 = \infty$, $z_1 = 0$, and $z_2$ restricted to the unit circle $z_2 = e^{i\theta}$ for $\theta \in \mathbb{R}/(2\pi)\mathbb{Z}$.

The asymptotic markers at $z_0, z_1$ are chosen to be tangent to the imaginary axis, and pointing in the negative, resp. positive, direction. The asymptotic marker at $z_2 = e^{i\theta}$ is chosen to point towards the positive imaginary direction independently of $\theta$.

Using similar arguments, we now prove Proposition 9.

**Proof of Proposition 9.** We first remark that in the case that $L_0, L_1$ are Hamiltonian isotopic to the zero section, the proof is just a matter of a standard homotopy argument. Therefore we sketch the proof in the case of arbitrary exact $L_0, L_1$.

Firstly consider the homotopy inverse pair of quasi-isomorphisms $\psi : CF^*(L_0, L_1) \to CF^{*+j}(L, L)$ and $\psi' : CF^*(L, L) \to CF^{*-j}(L_0, L_1)$ for $j = iL_0 - iL_1$,

$$\psi = \mu_2(y_1, -) \circ \mu_2(-, x_0),$$
$$\psi' = \mu_2(x_1, -) \circ \mu_2(-, y_0),$$

for classes $x_0 \in CF^{iL_0}(L, L_0), y_0 \in CF^{-iL_0}(L_0, L)$ and $x_1 \in CF^{iL_1}(L, L_1), y_1 \in CF^{-iL_1}(L_1, L)$ provided by Theorem C. We observe that it is enough to prove that the map

$$P_a'' = \psi \circ \tilde{\phi}^{-1}_{(L_0, L_1), a} \circ \psi' : CF^*(L, L) \to CF^{*-1+k}(L, L)$$

for arbitrarily chosen primitives $c_{L_0}', c_{L_1}', c_L$ for the $a$-equivariant structure on $L_0, L_1$, and $L$ is chain homotopic to

$$P'_a + \mu_2(c, -)$$

for a certain cycle $c \in CF^{-1+k}(L, L)$. Indeed in this case we can choose new primitives $c_{L_0} = c_{L_0}', c_{L_1} = c_{L_1}'$ for a cycle $c_0 \in CF^{k-1}(L_0, L_0)$ whose image under the quasi-isomorphism $\mu_2(y_0, -) \circ \mu_2(-, x_0) : CF^*(L_0, L_0) \to CF^*(L, L)$ of Theorem C represents $[c]$ in homology. We proceed to prove it similarly to Proposition 10. Namely, consider the operation $P_a''$. By gluing, and up to correction by two $\mu_2$-terms coming from the definition of $\tilde{\phi}_{(L_0, L_1), a}$, it is given by a 1-dimensional family of decorated disks with five boundary inputs, one interior input, and one boundary output.

The Lagrangian labels on the boundary of the disk, which determine the boundary conditions for the Floer equation, are given in cyclic order starting with the output by $L, L_1, L, L_0, L$. The first and second boundary inputs are constrained to $y_1, x_1$, while the fourth and fifth boundary inputs are constrained to $y_0, x_0$. The interior input is constrained to an arc $\gamma$ connecting the second and fifth boundary segments $\ell_1, \ell_0$ labeled $L_1$ and $L_0$. Consider the subarcs $\gamma_0, \gamma_1$ with $\gamma_0$ from a point $z$ on $\gamma$ to $\ell_0$ and $\gamma_1$ from $z$ to $\ell_1$, so that parametrized suitably $\gamma = \gamma_1 \# \gamma_0$. Now consider arcs $\eta_1, \eta_0$ from $z$ to the third and fourth boundary segments both labeled $L$. Set $\gamma' = \eta_1 \# \eta_0, \alpha_0 = \eta_0 \# \gamma_0, \alpha_1 = \eta_1 \# \gamma_1$. 

\[ \blacksquare \]
It is not hard to see that $P'_d$ is chain-homotopic to the sum of operators given by the homotopies obtained from $\gamma', \alpha_0$ and $\alpha_1$ corrected by suitable $\mu_2$ terms (for generic perturbation data). Now, by performing homotopies of the decorated Riemann surfaces and the Floer data, as well as using the fact that $\mu_2(-, x_0), \mu_2(-, y_0)$ and $\mu_2(x_1, -), \mu_2(y_1, -)$ are homotopy-inverse pairs, this sum is in turn chain-homotopic to the sum

$$\widetilde{\phi}^1_{(L, L), a} + \mu_2 \left( \begin{array}{c} \widetilde{\phi}^1_{(L, L_1), a} (x_1), y_1 \end{array} \right) + \mu_2 \left( x_0, \widetilde{\phi}^1_{(L_0, L), a} (y_0) \right).$$

Finally, this last sum is chain-homotopic to a map of the desired form, since $L$ is exact so the left product $\mu_2(z, -)$ and the right product $\mu_2(-, z)$ on $CF^*(L, L)$ by a homogeneous cycle $z \in CF^*(L, L)$ are chain-homotopic, up to sign. Indeed, this follows from $\mu_2$ being graded-commutative on the level of homology$^2$.

### 3.2 Proof of Theorem A.

Let $a_1, \ldots, a_N \in H_{n-\ast}(LL)^+$ be such that

$$P_{a_N} \circ \ldots \circ P_{a_1}([pt]) = [L]. \quad (7)$$

Let $\mu, e \in HF^*(L_0, L_1)$ be such that $\mu$ corresponds to $[pt]$ and $e$ corresponds to $[L]$ under the isomorphism

$$HF^{\ast+i_1-i_0}(L_0, L_1) \xrightarrow{\cong} HF^{\ast-i_0}(L_0, L) \xrightarrow{\cong} HF^*(L, L) \cong H_{n-\ast}(L),$$

for suitable integers $i_0, i_1 \in \mathbb{Z}$. In this case the spectral norm $\gamma(L_0, L_1)$ is given by

$$\gamma(L_0, L_1) = c(\mu, L_0, L_1) - c(e, L_0, L_1).$$

It is therefore sufficient to prove that there exists a constant $C(g, L; K)$ such that

$$c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + C(g, L; K).$$

Let $a'_1 = \Phi^{-1}(a_1), \ldots, a'_N = \Phi^{-1}(a_N)$. Since $a_1, \ldots, a_N \in H_{n-\ast}(LL)^+$, we obtain that $L_0, L_1$ are $a_j$-equivariant for all $1 \leq j \leq N$. In view of Proposition 10, the identity (7) corresponds to the identity

$$P'_{(L_0, L_1), a'_N} \circ \ldots \circ P'_{(L_0, L_1), a'_1}(e) = \mu.$$

Set $C_j = \mathcal{A}_{H^\lambda}(\widetilde{a}'_j)$ for representatives $\widetilde{a}'_j$ of $a'_j$, and $x_j = P'_{(L_0, L_1), a'_j} \circ \ldots \circ P'_{(L_0, L_1), a'_1}(e)$ for $1 \leq j \leq N$, $x_0 = e$. We lift these elements to the chain level as follows. Let $\widetilde{x}_0 = \widetilde{e} \in CF^*(L_0, L_1)$ be a chain representative of $e$ with

$$\mathcal{A}_{L_0, L_1; D}(\widetilde{e}) \leq c(e, L_0, L_1) + \epsilon \quad (8)$$

(where $D$ is chosen to be Hamiltonian-small). Then for $1 \leq j \leq N$ we set

$$\widetilde{x}_j = \widetilde{\phi}^1_{(L_0, L_1), \widetilde{a}'_j} \circ \ldots \circ \widetilde{\phi}^1_{(L_0, L_1), \widetilde{a}'_1}.$$
In this situation we obtain by (5) that for all $0 \leq j < N$,
\[
\mathcal{A}_{L_0, L_1; \mathcal{D}}(\tilde{x}_{j+1}) \leq \mathcal{A}_{L_0, L_1; \mathcal{D}}(\tilde{x}_j) + \mathcal{A}_{H_\lambda}(a_{j+1}) + 3\epsilon.
\]
Therefore by (8), we obtain that
\[
\mathcal{A}_{L_0, L_1; \mathcal{D}}(\tilde{x}_N) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} C_j + 3(N + 1)\epsilon.
\]
As $[\tilde{x}_N] = x_N = \mu$, by definition of the spectral invariant we have
\[
c(\mu, L_0, L_1) \leq \mathcal{A}_{L_0, L_1; \mathcal{D}}(\tilde{x}_N),
\]
and this finishes the proof.

**Remark 11.** The inequality
\[
c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} C_j + 3(N + 1)\epsilon
\]
finishing the proof can be optimized as follows. First we may choose $C_j < c(a'_j, H_\lambda) + \epsilon$, to obtain
\[
c(\mu, L_0, L_1) < c(e, L_0, L_1) + \sum_{j=1}^{N} c(a'_j, H_\lambda) + 4(N + 1)\epsilon.
\]
Furthermore, as $\lambda \to \infty$, by definition of spectral invariants in filtered symplectic homology, we obtain that $c(a'_j, H_\lambda) \to c(a'_j, D, S)$. Therefore we can write
\[
c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} c(a'_j, D, S) + 4(N + 1)\epsilon,
\]
and sending $\epsilon$ to 0, we finally get the bound
\[
\gamma(L_0, L_1) \leq \sum_{j=1}^{N} c(a'_j, D, S). \tag{9}
\]

### 3.3 Proof of Proposition 3.
Let $L$, $L'$ be string point-invertible. We will show that so is $L \times L'$. Let $a_1, \ldots, a_N$, and $a'_1, \ldots, a'_N$, be sequences exhibiting the fact that $L$, respectively $L'$, are string point-invertible. Consider $b_k = a_k \otimes \iota([L'])$, for $1 \leq k \leq N$, and $b_k = \iota([L]) \otimes a'_{k-N}$, for $N < k \leq N + N'$. We claim that the sequence $b_1, \ldots, b_{N+N'}$, up to sign, is the required one for $L \times L'$. Since it is indeed enough to consider the question up to signs, we will ignore signs coming from the Koszul rule for tensor products. Now by K"unneth theorem $H_*(L \times L') \cong H_*(L) \otimes H_*(L')$ and $H_*(\mathcal{L}L \times \mathcal{L}L') \cong H_*(\mathcal{L}L) \otimes H_*(\mathcal{L}L')$. Furthermore, under changing the grading to $* - n$ everywhere, these splittings are tensor products of graded
algebras. Moreover the maps \( \iota \) and \( ev \) commute with this tensor product decomposition. The BV-operator behaves in the following way: for homogeneous elements \( x \in H_s(\mathcal{L}L), \ x' \in H_s(\mathcal{L}L') \),
\[
\Delta_{L\times L'}(x \otimes x') = \Delta_L(x) \otimes x' + (-1)^{|x|} x \otimes \Delta_{L'}(x').
\]
This implies that for \( a \in H_s(L), \ a' \in H_s(L') \),
\[
m_{a \otimes a'}(x \otimes x') = \pm m_a(x) \otimes a' \ast x' \pm a \ast x \otimes m_{a'}(x').
\]
Observe that \([pt_L \times L'] = [pt_L] \otimes [pt_{L'}], \ [L \times L'] = [L] \otimes [L']\), and furthermore for \( a' = \iota([L']) \), \( m_{a'} = 0 \). Therefore
\[
m_{b_1} \iota([pt_L] \otimes [pt_{L'}]) = m_{a_1 \otimes \iota([L'])} \iota([pt_L] \otimes [pt_{L'}]) = \pm m_{a_1}(\iota([pt_L])) \otimes \iota([pt_{L'}]).
\]
Finally, since \( ev \) is an algebra map, and it commutes with the Künneth decompositions, we obtain
\[
P_{b_1}( [pt_L \times L'] ) = \pm P_{a_1}( [pt_L] ) \otimes [pt_{L'}].
\]
The same argument shows that for all \( 1 \leq k \leq N \),
\[
P_{b_k} \circ \ldots \circ P_{b_1}( [pt_L \times L'] ) = \pm P_{a_k} \circ \ldots \circ P_{a_1}( [pt_L] ) \otimes [pt_{L'}]
\]
In particular, for \( k = N \),
\[
P_{b_N} \circ \ldots \circ P_{b_1}( [pt_L \times L'] ) = \pm [L] \otimes [pt_{L'}]. \tag{10}
\]
Similarly, we obtain that for all \( 1 \leq k \leq N' \),
\[
P_{b_{N+k}} \circ \ldots \circ P_{b_{N+1}}([L] \otimes [pt_L]) = \pm [L] \otimes P_{a_k} \circ \ldots \circ P_{a_1}([pt_{L'}]).
\]
In particular, for \( k = N' \),
\[
P_{b_{N+N'}} \circ \ldots \circ P_{b_{N+1}}([L] \otimes [pt_L]) = \pm [L] \otimes [L'] \tag{11}
\]
Hence, by (10) and (11), we obtain
\[
P_{b_{N+N'}} \circ \ldots \circ P_{b_1}( [pt_L \times L'] ) = \pm [L \times L'].
\]
If necessary, changing the sign of \( b_{N+N'} \), we finish the proof.

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