QUADRATIC-LIKE DYNAMICS OF CUBIC POLYNOMIALS

ALEXANDER BLOKH, LEX OVERSTEEGEN, ROSS PTACEK, AND VLADLEN TIMORIN

Abstract. A small perturbation of a quadratic polynomial with a non-repelling fixed point gives a polynomial with an attracting fixed point and a Jordan curve Julia set, on which the perturbed polynomial acts like angle doubling. However, there are cubic polynomials with a non-repelling fixed point, for which no perturbation results into a polynomial with Jordan curve Julia set. Motivated by the study of the closure of the Cubic Principal Hyperbolic Domain, we describe such polynomials in terms of their quadratic-like restrictions.

1. Introduction

In this paper, we study topological dynamics of complex cubic polynomials. We denote the Julia set of a polynomial $f$ by $J(f)$ and the filled Julia set of $f$ by $K(f)$. Let us recall classical facts about quadratic polynomials. The Mandelbrot set $M_2$, perhaps the most well-known mathematical set outside of the mathematical community, can be defined as the set of all complex numbers $c$ such that the sequence

$$c, \ c^2 + c, \ (c^2 + c)^2 + c, \ldots$$

is bounded. The numbers $c$ label polynomials $z^2 + c$. Every quadratic polynomial can be reduced to this form by an affine coordinate change.

By definition, $c \in M_2$ if the orbit of 0 under $z \mapsto z^2 + c$ is bounded. What is so special about the point 0? It is the only critical point of the
polynomial $z^2 + c$ in $\mathbb{C}$. A critical point of a complex polynomial has a meaning in the realm of topological dynamics. Namely, this is a point that does not have a neighborhood on which the map is one-to-one. Generally, the behavior of critical orbits to a large extent determines the dynamics of other orbits. E.g., by a classical theorem of Fatou and Julia, $c \in \mathcal{M}_2$ if and only if the filled Julia set of $z^2 + c$

$$K(z^2 + c) = \{ z \in \mathbb{C} \mid z, z^2 + c, (z^2 + c)^2 + c, \cdots \not\to \infty \}$$

is connected. If $c \notin \mathcal{M}_2$, then the set $K(z^2 + c)$ is a Cantor set.

The Mandelbrot set has a complicated fractal shape. However, one can see many components of the interior of $\mathcal{M}_2$ that are bounded by real analytic curves (in fact, ovals of real algebraic curves). The central part of the Mandelbrot set, the so called Principal Hyperbolic Domain $\text{PHD}_2$, is bounded by a cardioid (a curve, whose shape resembles that of a heart, thus the name). This cardioid is called the Main Cardioid. By definition, the Principal Hyperbolic Domain $\text{PHD}_d$ consists of all parameter values $c$ such that the polynomial $z^2 + c$ is hyperbolic, and the set $K(z^2 + c)$ is a Jordan disk (a polynomial of any degree is said to be hyperbolic if the orbits of all its critical points converge to attracting cycles). Equivalently, $c \in \text{PHD}_2$ if and only if $z^2 + c$ has an attracting fixed point. The closure of $\text{PHD}_2$ consists of all parameter values $c$ such that $z^2 + c$ has a non-repelling fixed point. Our aim is to study a similar situation for cubic polynomials.

Complex numbers $c$ are in one-to-one correspondence with affine conjugacy classes of quadratic polynomials (throughout we call affine conjugacy classes of polynomials classes of polynomials). Thus a natural higher-degree analog of the set $\mathcal{M}_2$ is the degree $d$ Mandelbrot set $\mathcal{M}_d$ defined as the set of classes of degree $d$ polynomials $f$, all of whose critical points do not escape, or, equivalently, whose Julia set $J(f)$ is connected. The Principal Hyperbolic Domain $\text{PHD}_d$ of $\mathcal{M}_d$ is defined as the set of classes of hyperbolic degree $d$ polynomials with Jordan curve Julia sets. Equivalently, the class $[f]$ of a degree $d$ polynomial $f$ belongs to $\text{PHD}_d$ if all critical points of $f$ are in the immediate attracting basin of the same attracting (or super-attracting) fixed point.

If the class of a cubic polynomial $f$ belongs to $\text{PHD}_3$ then $f$ must have a non-repelling fixed point (indeed, as we approximate $f$ with polynomials $g$, whose classes belong to $\text{PHD}_3$, the attracting fixed points of $g$ converge to some non-repelling fixed point of $f$). In general if we perturb a cubic polynomial $f$ with a non-repelling fixed point so that the resulting polynomial $g$ has an attracting fixed point, then $g$ restricted to the basin of attraction $A(g)$ of that point is either two-to-one or three-to-one. In this paper we study polynomials $f$ with a non-repelling fixed
point such that \( f \notin \text{PHD}_3 \), i.e. for all maps \( g \) sufficiently close to \( f \) and with a corresponding fixed attracting point, the corresponding basin of attraction is such that the map on it is two-to-one.

Let \( \mathcal{F} \) be the space of all polynomials

\[
    f_{\lambda,b} = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.
\]

An affine change of variables reduces any cubic polynomial \( f \) to the form \( f_{\lambda,b} \). Note that 0 is a fixed point for every polynomial in \( \mathcal{F} \). The set of all polynomials \( f \in \mathcal{F} \) such that 0 is non-repelling for \( f \) is denoted by \( \mathcal{F}_{nr} \). Define the \( \lambda \)-slice \( \mathcal{F}_\lambda \) of \( \mathcal{F} \) as the space of all polynomials \( g \in \mathcal{F} \) with \( g'(0) = \lambda \). Let us assume that \( J(f) \) is connected. In [McM94], the notion of \( J \)-stability was introduced for any holomorphic family of rational functions: a map is \( J \)-stable if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the given family. We say that \( f \in \mathcal{F}_\lambda \) is stable if it is \( J \)-stable with respect to \( \mathcal{F}_\lambda \) with \( \lambda = f'(0) \), otherwise we say that \( f \) is unstable. The set \( \mathcal{F}^{st}_\lambda \) of all stable polynomials \( f \in \mathcal{F}_\lambda \) is an open subset of \( \mathcal{F}_\lambda \).

**Theorem A.** Suppose that \( f \in \mathcal{F}_{nr} \). If, in some neighborhood of \( f \), there is no polynomial \( g \in \mathcal{F} \) such that \( |g'(0)| < 1 \) and \( [g] \in \text{PHD}_3 \), then one of the following holds:

1. The map \( f : U^* \to V^* \) is quadratic-like for some Jordan domains \( U^* \) and \( V^* \) so that 0 belongs to its filled Julia set; this map is hybrid equivalent to a quadratic polynomial \( z^2 + c \) with \( c \in \text{PHD}_2 \);
2. The polynomial \( f \) is stable, has connected Julia set, has neither repelling periodic cutpoints in \( J(f) \) nor neutral periodic points distinct from 0, and either a critical point of \( f \) is eventually mapped to a Siegel disk containing 0, or the Julia set of \( f \) has positive measure and carries an invariant line field.

There is no loss of generality in that we consider only perturbations of \( f \) in \( \mathcal{F} \): instead, we could consider small perturbations \( g \) of \( f \) such that, arbitrarily close to 0, the map \( g \) has an attracting fixed point. Theorem A generalizes some results from [BuHe01, Zak99]. If there are Jordan domains \( U^* \) and \( V^* \) containing 0 such that \( f : U^* \to V^* \) is a quadratic-like map, we say that \( f \) admits quadratic-like dynamics. The filled quadratic-like Julia set \( K^* \) is then defined [DH85] as the set of all points, whose orbits never leave \( U^* \), and \( J^* \) is defined as the boundary \( \text{Bd}(K^*) \) of \( K^* \). Hence \( 0 \in K^* \), i.e. either \( 0 \in J^* \) or 0 belongs to some bounded complementary component of \( J^* \).

We need the definition of hybrid equivalence: two polynomial-like maps are said to be hybrid equivalent if their restrictions to sufficiently
small neighborhoods of their filled Julia sets are quasi-conformally con-
jugate, and a quasi-conformal conjugacy \( \varphi \) can be chosen to satisfy 
\( \overline{\partial} \varphi = 0 \) on the filled Julia set. The map \( \varphi \) is called a straightening map. By the Straightening Theorem [DH85], there exists a quadratic
polynomial \( z^2 + c \) hybrid equivalent to \( f : U^* \to V^* \). It is easy to see
that under hybrid equivalence repelling periodic points cannot corre-
spond to non-repelling periodic points (see e.g., Lemma 3.5). Thus, the
point 0, a non-repelling fixed point of \( f \), corresponds to a non-repelling
fixed point of \( z^2 + c \) with \( c \in \overline{\text{PHD}}_2 \), and in case (1) it suffices to prove
that \( f \) admits quadratic-like dynamics.

The next result follows from Theorem A. Define the extended closure of
the cubic Principal Hyperbolic Domain as follows. Suppose that
\( |\lambda| \leq 1 \) and that there is a component \( W \) of the set \( \mathcal{F}_{\lambda}^{st} \) such that 
\( [\text{Bd}(W)] \subset \overline{\text{PHD}}_3 \), i.e. \( [f] \in \overline{\text{PHD}}_3 \) for all \( f \in \text{Bd}(W) \). Then we add
the entire \( [W] \) to \( \overline{\text{PHD}}_3 \). If we do this wherever possible, we will in the
end construct the extended closure of the cubic Principal Hyperbolic
Domain \( \overline{\text{PHD}}_3 \).

**Theorem B.** Suppose that \( f \) is a cubic polynomial with a non-repelling
fixed point. If \( [f] \notin \overline{\text{PHD}}_3 \) then \( f : U^* \to V^* \) is a quadratic-like map for
some \( U^* \) and \( V^* \). Moreover, this quadratic-like map is hybrid equivalent
to a quadratic polynomial \( z^2 + c \) with \( c \in \overline{\text{PHD}}_2 \).

We conjecture that \( \overline{\text{PHD}}_3 = \overline{\text{PHD}}_3 \).

**Notation and Preliminaries:** we write \( \overline{A} \) for the closure of a subset \( A \) of a topo-
logical space and \( \text{Bd}(A) \) for the boundary of \( A \); the \( n \)-th iterate of a map \( f \) is
denoted by \( f^n \). We identify the unit circle \( S \) with \( \mathbb{R}/\mathbb{Z} \) and denote by \( \beta \gamma \) the
chord with endpoints \( \beta, \gamma \in S \). The \( d \)-tupling map of the unit circle is denoted by
\( \sigma_d \). We let \( \mathbb{C} \) stand for the complex plane, \( \mathbb{C}^* \) for the Riemann sphere, \( \mathbb{D} \) for
the open unit disk consisting of all complex numbers \( z \) with \( |z| < 1 \), and \( S = \text{Bd}(\mathbb{D}) \) for
the unit circle. We also assume knowledge of basic notions from complex dynamics,
such as Green function, dynamic rays (of specific argument), Böttcher coordinate,
Fatou domain, repelling, attracting, neutral periodic points, parabolic, Siegel, Cre-
mmer periodic points, impressions, principal sets, and the like (see, e.g., [McM94]).

We will talk about principal sets of arbitrary continuous paths \( \gamma : (0, \infty) \to \mathbb{C} \)
such that \( \lim_{t \to \infty} \gamma(t) = \infty \), not necessarily external rays. The principal set of \( \gamma \) is
defined as \( \bigcap_{\varepsilon > 0} \gamma(0, \varepsilon) \).

2. **Polynomials, whose special perturbations are not in \( \overline{\text{PHD}}_3 \)**

Throughout Section 2, we consider a cubic polynomial \( f \in \mathcal{F}_{nr} \) such that,
in some neighborhood of \( f \) in \( \mathcal{F} \), there is no polynomial \( g \) with
\(|g'(0)| < 1\) and \([g] \in \text{PHD}_3\). In other words, we assume that \(f\) satisfies the assumptions of Theorem A. We define the principal critical point \(\omega_1(f)\). We also define the sets \(Z(f)\), \(X(f)\) associated with \(f\) and construct a holomorphic motion of these sets.

2.1. The principal critical point of \(f\). Fix \(f\) as above.

**Lemma 2.1.** The polynomial \(f\) has two distinct critical points.

**Proof.** Assume that \(\omega(f)\) is the only critical point of \(f\) (then it has multiplicity two). Let \(C\) be the space of all polynomials \(g \in \mathcal{F}\) with a multiple critical point \(\omega(g)\). This is an algebraic curve in \(\mathcal{F}\) passing through \(f\). The map taking \(g \in C\) to \(g'(0)\) is a non-constant holomorphic function. Hence there are polynomials \(g \in C\) arbitrarily close to \(f\), for which \(|g'(0)| < 1\). The class of any such polynomial \(g\) belongs to \(\text{PHD}_3\) as the immediate basin of 0 with respect to \(g\) must contain the multiple critical point \(\omega(g)\), contradicting our assumption on \(f\). \(\square\)

Define \(A\) as the set of all cubic polynomials \(g \in \mathcal{F}\) with \(|g'(0)| < 1\). For \(g \in A\), we write \(A(g)\) for the immediate basin of attraction of 0 with respect to \(g\). We consistently approximate \(f\) by polynomials from \(A\). If \(f \in A\), it itself serves as its own approximation. By our assumption, there is a neighborhood of \(f\) in \(\mathcal{F}\) in which there is no polynomial \(g \in A\) with \([g] \in \text{PHD}_3\).

By Lemma 2.1 there are two critical points of \(f\). A critical point \(c\) of \(f\) is said to be principal if there is a neighborhood \(U\) of \(f\) in \(\mathcal{F}\) and a holomorphic function \(\omega_1 : U \to \mathbb{C}\) defined on this neighborhood such that \(c = \omega_1(f)\), and, for every \(g \in U \cap A\), the point \(\omega_1(g)\) is a critical point of \(g\) contained in \(A(g)\).

**Theorem 2.2.** There exists a unique principal critical point of \(f\).

**Proof.** By Lemma 2.1 the two critical points of \(f\) are different. Then there are two holomorphic functions, \(\omega_1\) and \(\omega_2\), defined on a convex neighborhood \(U\) of \(f\) in \(\mathcal{F}\), such that \(\omega_1(g)\) and \(\omega_2(g)\) are the critical points of \(g\) for all \(g \in U\). Suppose that neither \(\omega_1(f)\), nor \(\omega_2(f)\) is principal. Then, arbitrarily close to \(f\), there are cubic polynomials \(g_1\) and \(g_2 \in A\) with \(\omega_2(g_1) \notin A(g_1)\) and \(\omega_1(g_2) \notin A(g_2)\). Since \(A(g_i)\) contains a critical point for \(i = 1, 2\), we must have that \(\omega_1(g_i) \in A(g_i)\).

The set \(A\) is convex. Therefore, the intersection \(U \cap A\) is also convex, hence connected. Let \(O_i, i = 1, 2\), be the subset of \(U \cap A\) consisting of all polynomials \(g\) with \(\omega_i(g) \in A(g)\). By the preceding paragraph, \(g_1 \in O_1\) and \(g_2 \in O_2\). We claim that \(O_i\) is open. Indeed, if \(g \in O_i\), then there exists a Jordan disk \(U \subset A(g)\) with \(g(U)\) compactly contained in \(U\), and \(\omega_i(g) \in U\). If \(g \in U \cap A\) is sufficiently close to \(g\), then \(g(U)\) is
still compactly contained in $U$, and $\omega_i(\tilde{g})$ is still in $U$, by continuity. It follows that $U \subset A(\tilde{g})$, in particular, $\omega_i(\tilde{g}) \in A(\tilde{g})$. Thus, $\mathcal{O}_i$ is open. Since $\mathcal{O}_1, \mathcal{O}_2$ are open and non-empty, the set $U \cap A$ is connected, and

$$U \cap A = \mathcal{O}_1 \cup \mathcal{O}_2,$$

the intersection $\mathcal{O}_1 \cap \mathcal{O}_2$ is nonempty. Note that $\mathcal{O}_1 \cap \mathcal{O}_2$ consists of polynomials, whose classes are in PHD$_3$. Since $U$ can be chosen arbitrarily small, it follows that $f$ can be approximated by maps $g \in A$ with $[g] \in$ PHD$_3$, a contradiction.

The existence of a principal critical point of $f$ is thus proved. The uniqueness follows immediately from our assumption on $f$. □

Denote by $\omega_1(f)$ the principal critical point of $f$. For $g \in \mathcal{F}_{nr}$ sufficiently close to $f$, the point $\omega_1(g)$ is a holomorphic function of $g$.

2.2. Holomorphic motion. Let $\Lambda$ be a Riemann surface, and $Z \subset \mathbb{C}^*$ any (!) subset. A holomorphic motion of the set $Z$ is a map $\mu : Z \times \Lambda \to \mathbb{C}^*$ with the following properties:

- for every $z \in Z$, the map $\mu(z, \cdot) : \{z\} \times \Lambda \to \mathbb{C}^*$ is holomorphic;
- for $z \neq z'$ and every $\nu \in \Lambda$, we have $\mu(z, \nu) \neq \mu(z', \nu)$;
- there is a point $\nu_0$ such that $\mu(z, \nu_0) = z$ for all $z \in Z$.

We will use the following crucial $\lambda$-lemma of Mañé, Sad and Sullivan [MSS83]: a holomorphic motion of a set $Z$ extends to a unique holomorphic motion of the closure $\overline{Z}$; moreover, this extension is a continuous function in two variables such that for every $\nu \in \Lambda$ the map $\varphi : \overline{Z} \to \mathbb{C}^*$ defined as $\varphi(z) = \mu(z, \nu)$ is quasi-symmetric. There have been useful generalizations of this result, but we will only need the original version.

We will now define a countable set $Z(f)$ of iterated preimages of the principal critical point $\omega_1(f)$. By definition, a point $z \in \mathbb{C}$ belongs to $Z(f)$ if there exists an open convex neighborhood $U_z$ of $f$ in $\mathcal{F}$ and a holomorphic function $\zeta : U_z \to \mathbb{C}$ with the following properties:

- $\zeta(f) = z$;
- we have $g^n(\zeta(g)) = \omega_1(g)$ for all $g \in U_z$ and for some $n \geq 0$ independent of $g$;
- we have $\zeta(g) \in A(g)$ for all $g \in U_z \cap A$.

A holomorphic function $\zeta : U_z \to \mathbb{C}$ like above is called a deformation of $z \in Z(f)$. As it will always be clear what kind of deformation we consider, in what follows we will suppress the subscript in the notation for $U$.

Recall that the $\lambda$-slice $\mathcal{F}_\lambda$ of $\mathcal{F}$ is the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$. Let $\Lambda$ be a Jordan neighborhood of $f$ in $\mathcal{F}_\lambda$ such that
Lemma 2.3. Let $f$ be as above.

1. The critical point $\omega_1(f)$ is not eventually mapped to $\omega_2(f)$.
2. The set $Z(f)$ contains no critical values of $f$.

Proof. Suppose first that $\omega_1(f)$ is eventually mapped to $\omega_2(f)$, say, $f^m(\omega_1(f)) = \omega_2(f)$, and the number $m$ is the minimal positive integer with this property. Consider the set $C$ of all $g \in U$ such that $g^m(\omega_1(g)) = \omega_2(g)$. This set is a piece of an algebraic curve. The function $g \mapsto g'(0)$ is a complex analytic function on $C$. Since the value of this function at $f$ lies in $D$, there are maps $g \in C$ arbitrarily close to $f$ such that $|g'(0)| < 1$. The class of any such $g$ must belong to PHD$_3$. Indeed, the attracting basin $A(g)$ must contain the principal critical point $\omega_1(g)$ by definition of the principal critical point. Since $\omega_1(g)$ is eventually mapped to $\omega_2(g)$, the critical point $\omega_2(g)$ is also contained in $A(g)$. We arrive at a contradiction with our assumption on $f$.

Suppose now that $v \in Z(f)$ is a critical value. Let $\zeta : U \to \mathbb{C}$ be a deformation of $v$. Consider the set $C$ of all $g \in U$ such that $\zeta(g)$ is a critical value. This set is a piece of an algebraic curve. Take a sequence $g_n \in C \cap A$ that converges to $f$. Since $\zeta(g_n) \in A(g_n)$ is a critical value with at least two $g_n$-preimages in $A(g_n)$, counting multiplicities, the set $A(g_n)$ must contain a critical point $d$ with $g_n(d) = \zeta(g_n)$. The fact that $\omega_1(g_n)$ is not periodic implies that $d \neq \omega_1(g_n)$. Thus, both critical points of $g_n$ are contained in $A(g_n)$, and so $[g_n] \in$ PHD$_3$. We again arrive at a contradiction with our assumption on $f$. \hfill \Box

Lemma 2.4. For every $z \in Z(f)$, there are exactly two points of $Z(f)$ that are mapped to $z$ under $f$.

Proof. The proof is similar to that of Theorem 2.2. Let $\zeta : U \to \mathbb{C}$ be a deformation of $z$. Since the set $Z(f)$ cannot contain a critical value of $f$, there are three holomorphic functions $\zeta_1, \zeta_2, \zeta_3$ defined on $U$ and such that $g(\zeta_i(g)) = \zeta(g)$ (we may need to pass to a smaller neighborhood $U$ to arrange this).

The intersection $U \cap A$ is convex, hence connected. For any 2-element subset $\{i, j\} \subset \{1, 2, 3\}$, define a subset $O_{ij} \subset U \cap A$ as the set of all polynomials $g \in U \cap A$ such that $\zeta_i(g) \in A(g)$ and $\zeta_j(g) \in A(g)$. All three sets $O_{12}, O_{23}$ and $O_{13}$ are open (cf. the proof of Theorem 2.2). On the other hand, we have

$$A \cap U = \bigcup_{i=1}^{3} O_{i}.$$




Hence either only one of the sets $O_{ij}$ is nonempty, or at least two of the sets $O_{ij}$ intersect. In the latter case, $\zeta_i(g) \in A(g)$ for some $g \in A \cap U$ and all $i = 1, 2, 3$. It follows that $[g] \in \text{PHD}_3$. Since the neighborhood $U$ can be chosen to be arbitrarily small, it follows that $f$ can be approximated by polynomials in $A$, whose classes are in $\text{PHD}_3$, a contradiction. The contradiction shows that only one of the sets $O_{ij}$ is nonempty, for a suitable choice of the neighborhood $U$. Assume that $i = 1$ and $j = 2$; then $\zeta_1(f), \zeta_2(f) \in Z(f)$ but $\zeta_3(f) \notin Z(f)$.

The proof of Lemma 2.4 implies a stronger claim below.

**Corollary 2.5.** Let $\zeta_i$ be holomorphic functions introduced in the proof of Lemma 2.4. Suppose that $\zeta_1(f), \zeta_2(f) \in Z(f)$. Then there is a neighborhood $U$ of $f$ in $F$ such that $\zeta_3(g) \notin A(g)$ for all $g \in U \cap A$.

**Proposition 2.6.** For every $z \in Z(f)$, there is a holomorphic function $\zeta : \Lambda \to \mathbb{C}$ such that $\zeta(h) \in Z(h)$ for all $h \in \Lambda$ and $\zeta(f) = z$.

**Proof.** The function $\zeta$ with these properties is defined at least on some open neighborhood of $f$ in $\Lambda$, by definition of the set $Z(f)$. Assume by induction that the statement of the proposition holds for the point $f(z)$, i.e. there is a holomorphic function $\eta : \Lambda \to \mathbb{C}$ such that $\eta(h) \in Z(h)$ for all $h \in \Lambda$ and $\eta(f) = f(z)$. It follows that there is an integer $n$ such that $h^{\omega(n-1)}(\eta(h)) = \omega_1(h)$ for all $h \in \Lambda$. Consider the multivalued analytic function $h \mapsto h^{-1}(\eta(h))$. If this function has no branch points in $\Lambda$, then we can define the holomorphic function $\zeta$ as the branch of this function such that $\zeta(f) = z$. Suppose that there is a branch point $h_0$ of the multivalued function $h \mapsto h^{-1}(\eta(h))$. Then the point $\eta(h_0)$ must be a critical value of $h_0$, a contradiction with Lemma 2.3.

Thus we have defined the holomorphic function $\zeta : \Lambda \to \mathbb{C}$ such that $h(\zeta(h)) = \eta(h)$, and $\zeta(f) = z$. Moreover, $\zeta(h) \in Z(h)$ for all $h \in \Lambda$ sufficiently close to $f$. It suffices to prove that $\zeta(h) \in Z(h)$ for all $h \in \Lambda$. To this end, we will prove that the set of polynomials $h \in \Lambda$ such that $\zeta(h) \in Z(h)$ is open and closed in $\Lambda$. The openness is obvious. Consider a sequence $h_n \in \Lambda$ converging to some polynomial $h \in \Lambda$, and suppose that $\zeta(h_n) \in Z(h_n)$ but $\zeta(h) \notin Z(h)$. Therefore, there are two other holomorphic functions $\zeta_1, \zeta_2$ defined on some neighborhood of $h$ such that $\zeta_i(h) \in Z(h)$, $i = 1, 2$. It follows that $\zeta_i(h_n) \in Z(h_n)$ for sufficiently large $n$. But then all three points $\zeta_1(h_n), \zeta_2(h_n)$ and $\zeta(h_n)$ are preimages of $\eta(h_n)$ in $Z(h_n)$. This contradicts Lemma 2.4.

Proposition 2.6 and Lemma 2.3 imply the following theorem.
Theorem 2.7. There exists a holomorphic motion \( \mu : Z(f) \times \Lambda \rightarrow \mathbb{C} \) that is equivariant in the sense that for every \( h \in \Lambda \), and for every \( z \in Z(f) \setminus \{ \omega_1(f) \} \), we have \( h(\mu(z, h)) = \mu(f(z), h) \).

By the \( \lambda \)-lemma, the holomorphic motion \( \mu \) gives rise to the holomorphic motion \( \overline{\mu} : \overline{Z(f)} \times \Lambda \rightarrow \mathbb{C} \). Since \( \mu \) is equivariant, the holomorphic motion \( \overline{\mu} \) is also equivariant, i.e. it preserves the dynamics.

2.3. The set \( X(f) \). Let \( Z_n(f) \) be the subset of \( Z(f) \) consisting of all preimages of \( \omega_1(f) \) mapped to \( \omega_1(f) \) in \( n \) steps, i.e. \( z \in Z_n(f) \) if \( f^{\circ n}(z) = \omega_1(f) \). Define the set \( X(f) \) as the limit of the sets \( Z_n(f) \), i.e.

\[
X(f) = \bigcap_{n \geq 0} \bigcup_{n \geq m} Z_n(f)
\]

Theorem 2.7 and the \( \lambda \)-lemma imply that the sets \( X(h) \) move holomorphically for \( h \in \Lambda \). Clearly, \( X(h) \) is forward invariant under \( h \).

Lemma 2.8. Every point \( x \in X(f) \) has at least two preimages in \( X(f) \), counting multiplicities.

Proof. This follows immediately from Lemma 2.4.

Lemma 2.9. The set \( X(f) \) is a subset of the Julia set \( J(f) \).

Proof. The set \( X(f) \) is contained in the accumulation set of the backward orbit of \( \omega_1(f) \). The backward orbit of a point can accumulate in the Fatou set only if the point lies in a Siegel disk. However \( \omega_1(f) \) cannot lie in a Siegel disk as a Siegel disk contains no critical points.

Recall that by the \( \lambda \)-lemma \( \overline{\mu} : \overline{Z} \times \Lambda \rightarrow \mathbb{C} \) is continuous. In particular, if a sequence \( z_n \in \overline{Z} \) converges to \( z \in \overline{Z} \), then \( \mu(z_n, h) \) converges to \( \mu(z, h) \), for every \( h \in \Lambda \).

Lemma 2.10. The set \( X(f) \) contains no neutral periodic points different from 0.

Proof. If \( X(f) \) contains a periodic neutral point \( x \neq 0 \), then, clearly, the point \( x \) is the limit of some sequence \( z_n \in Z(f) \). Let \( r \) be the minimal period of \( x \). Since the holomorphic motion \( \overline{\mu} \) is equivariant, \( \overline{\mu}(x, h) = \overline{\mu}(f^{\circ r}(x), h) = h^{\circ r}(\overline{\mu}(x, h)) \) which proves that \( \overline{\mu}(x, h) = x(h) \) is a periodic point of \( h \) of period \( r \), for every \( h \in \Lambda \).

The holomorphic function \( h \mapsto (h^{\circ r})'(x(h)) \) is non-constant on the multiplier slice \( \mathcal{F}_\lambda \). It follows that \( x(h) \) is an attracting periodic point with respect to \( h \), for some polynomials \( h \) in arbitrarily small neighborhood of \( f \). Since \( x = \lim_{n \rightarrow \infty} z_n \), we must also have \( x(h) = \overline{\mu}(x, h) = \lim_{n \rightarrow \infty} \overline{\mu}(z_n, h) \). However, by definition of \( Z(f) \) this is impossible if \( x(h) \) is attracting.
Theorem 2.11 explicitly summarizes the results of this section.

**Theorem 2.11.** Suppose that $f \in \mathcal{F}_\lambda$ is a cubic polynomial satisfying the assumptions of Theorem A. Then there is a neighborhood $\Lambda$ of $f$ in $\mathcal{F}_\lambda$ and an equivariant holomorphic motion $\overline{\tau} : X(f) \times \Lambda \to \mathbb{C}$. The set $X(f)$ is a forward invariant subset of $J(f)$. It contains no neutral periodic points different from 0. Every point of $X(f)$ has at least two preimages in $X(f)$ counting multiplicities.

In the rest of the paper we adopt the following approach. First we establish several types of conditions sufficient for the existence of domains $U$ and $V$ such that $f : U \to V$ is quadratic-like. Then we verify that these conditions are fulfilled for various cubic polynomials. In the end this leads to the proofs of our results.

3. Dynamics on $X(f)$ and quadratic-like maps

In this section we establish the existence of quadratic-like Julia sets in $J(f)$ in the case when $f$ is unstable (in a certain sense).

3.1. Parabolic dynamics. Suppose that 0 is a parabolic point of $f$ with rotation number $p/q$, i.e. we have $f'(0) = \exp(2\pi i p/q)$. It follows that $f^q(z) = z + az^{m+1} + O(z^{m+2})$ for small $z$ and some non-zero coefficient $a$, where $m = q$ or $2q$. A repelling vector is defined as a vector $v$ such that $av^m$ is a positive real number. Repelling vectors define $m$ straight rays originating at 0. These rays divide the plane into $m$ open attracting sectors. Let $S$ be an attracting sector, and $D$ a small round disk centered at 0. The map $z \mapsto z^{-m}$ is defined on $S \cap D$ and takes $S \cap D$ into a subset of the plane containing the half-plane $\text{Re}(w) < -M$ for some big $M > 0$. Moreover, the map $z \mapsto z^{-m}$ conjugates $f^q|_{S \cap D}$ with a map $F$ asymptotic to $w - ma$ as $w \to \infty$. If $M$ is big enough, then $F$ takes the half-plane $\text{Re}(w) < -M$ into itself. An attracting petal $P$ of $f$ at 0 is defined as the closure of the pullback of this half-plane to $S$. An attracting petal depends on the choice of an attracting sector $S$ and on the choice of the number $M$.

The following properties of attracting petals are immediate from the definition:

- any attracting petal $P$ is a compact subset of the plane such that $tP \subset P$ for $t \in [0, 1]$;
- if $P$ is an attracting petal, then the map $f^q : P \to \mathbb{C}$ is univalent, and we have $f^q(P) \subset P$;
- the set $f(P)$ lies in some attracting petal of $f$;

In what follows, given $f \in \mathcal{F}$ and small $\varepsilon > 0$, we define $g_{f, \varepsilon} \in \mathcal{F}$ as the cubic polynomial affinely conjugate to $(1 - \varepsilon)f$. For the most part
we consider such maps for $f \in \mathcal{F}_{nr}$; if it is clear what we mean we will write $g_{\epsilon}$ rather than $g_{f, \epsilon}$.

Lemma 3.1. Let $P$ be an attracting petal of $f$. If $\epsilon > 0$ is sufficiently small, and $g = g_{f, \epsilon}$, then $P$ is contained in $A(g)$.

Proof. Let us show that an attracting petal $\tilde{P}$ of $f$ is contained in $A((1 - \delta) f)$ for any $0 < \delta < 1$. Assume that there are attracting petals $\tilde{P}_0 = \tilde{P}, \tilde{P}_1, \ldots, \tilde{P}_{q-1}$ with $f(\tilde{P}_i) \subset \tilde{P}_{i+p \mod q}$. It follows from the first property of attracting petals that the map $(1 - \delta) f$ takes $\tilde{P}_i$ to a subset of $\tilde{P}_{i+q \mod q}$. Hence, $\tilde{P} \subset A((1 - \delta) f)$. Now, a conjugacy between $(1 - \epsilon) f$ and $g$ is given by the map $z \mapsto (1 - \epsilon)^{1/2} z$, and we may choose $\epsilon$ so small that the set $(1 - \epsilon)^{-1/2} P$ is contained in some (slightly bigger) attracting petal $\tilde{P}$ of $f$. By the proven in the beginning of this paragraph, $(1 - \epsilon)^{-1/2} P \subset A((1 - \epsilon) f)$, hence $P$ lies in $A(g)$. \hfill \Box

Now we can prove Corollary 3.2.

Corollary 3.2. Suppose that $f \in \mathcal{F}_{nr}$ satisfies the assumptions of Theorem A. If $0$ is a parabolic fixed point of $f$ and $c$ is a critical point of $f$ belonging to a Fatou domain $U$ of $f$ at $0$, then $c = \omega_1(f)$. Thus, there is only one cycle of parabolic domains at $0$.

Proof. Let $P \subset U$ be an attracting petal. Take an integer $m > 0$ such that $f^{om}(c)$ belongs to the interior of $P$. Connect $c$ and $f^{om}(c)$ by an arc $I \subset U$. Then it is easy to see (using compactness of $I$ and the fact that all points of $U$ eventually enter $P$ and stay there) that there exists
By way of contradiction, assume that $f^{oN}(I)$ is contained in the interior of $P$. Let $\varepsilon > 0$ be a small real number, and set $g = g_{f,\varepsilon} \in \mathcal{F}$. Let $c_g$ be the critical point of $g$ close to $c$. By Lemma 3.1, $P \subset A(g)$. By continuity, $g^{oN}(I) \subset P$. Since $I$ is an arc which contains points of $P \subset A(g)$, it follows that $I \subset A(g)$. Hence $c_g = \omega_1(g)$ and $c = \omega_1(f)$. □

3.2. Main results of Section 3. First, we need a general Lemma 3.3.

Lemma 3.3. Suppose that $f$ is a cubic polynomial such that $Y$ is an $f$-invariant compact set, a critical point $c$ of $f$ belongs to $Y$ while the other critical point $d$ of $f$ does not belong to $Y$, and the restriction $f : Y \to Y$ is two-to-one for all points except that $f^{-1}(f(c)) \cap Y = \{c\}$. Then there is a neighborhood $V$ of $Y$ such that for any point $x \in V \setminus Y$, the image $f(x)$ does not belong to $Y$ (in particular, if $Y$ is connected then $Y$ coincides with a component of $f^{-1}(f(Y))$).

Proof. Assume that there is a sequence $x_n \in \mathbb{C} \setminus Y$ with $f(x_n) \in Y$ converging to a point $x \in Y$. We may assume that $f(x)$ has exactly two preimages $x, x'$ in $Y$ counted with multiplicities. Then for a large $n$ the two preimages of $f(x_n)$ in $Y$ are close to $x$ and $x'$ while $x_n$ is close to $x$. Hence $x$ is a critical point, and so $x = c$. Since $f^{-1}(f(c)) \cap Y = \{c\}$, then $x = x'$ implying that $f$ is three-to-one near $x = c$, a contradiction. □

Let us now recall the definition of a quasi-symmetric map.

Definition 3.4. A homeomorphism $f : X \to Y$ of metric spaces is said to be quasi-symmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for any triple $x, y, z$ of distinct points in $X$, we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)}\right).$$

It is well-known that the inverse of a quasi-symmetric homeomorphism is quasi-symmetric.

Lemma 3.5. Suppose that $g$ and $h$ are polynomials, $Y \subset J(g)$ and $Z \subset J(h)$ are compact invariant sets, and $\varphi : Y \to Z$ is a quasi-symmetric conjugacy between $g|_Y$ and $h|_Z$. Suppose that a point $y \in Y$ is $g$-fixed, a point $z \in Z$ is $h$-fixed, and $\varphi(y) = z$. Finally, suppose that $y$ is not isolated in $Y$ (hence also $z$ is not isolated in $Z$). Then $y$ and $z$ are either both neutral or both repelling. Moreover, if $y$ and $z$ are neutral then either they both are parabolic or they both are Cremer.

Proof. By way of contradiction, assume that $y$ is repelling such that $|g'(y)| = \lambda > 1$ and $z$ is neutral. Fix a number $\delta$ such that $\frac{1}{\lambda} < \delta < 1$. 12
Let $\eta$ be the function from Definition 3.4 as applies to $\varphi$. Then
\[
\left| \varphi(x) - \varphi(y) \right| \leq \eta \left( \frac{|x - y|}{|g^{\circ N}(x) - y|} \right)
\]
for every $x$ and $N$, or, equivalently,
\[
\left| \varphi(x) - z \right| \leq \eta \left( \frac{|x - y|}{|g^{\circ N}(x) - y|} \right).
\]

Since $z$ is neutral, for any $N$ and every $x$ sufficiently close to $y$ the point $\varphi(x)$ will be so close to $z$ that $\eta \left( \frac{|x - y|}{|g^{\circ N}(x) - y|} \right) > \frac{1}{2}$ which implies that $\eta \left( \frac{|x - y|}{|g^{\circ N}(x) - y|} \right) > \frac{1}{2}$. On the other hand, we may assume that $\frac{|x - y|}{|g^{\circ N}(x) - y|} < \delta^N$. This contradicts the fact that $\eta$ is a homeomorphism. Since the inverse of $\varphi$ is also quasi-symmetric, we are done with the first claim.

To prove the second claim observe that if, say, $y$ is a Cremer point then by [Per94, Per97] in any neighborhood of $y$ there exists an infinite $g$-invariant compact set. On the other hand, if $z$ is parabolic then it is known that for any small neighborhood $U$ of $z$ and any point $x \in U \cap J(h)$ different from $z$ there exists a number $n$ such that $g^{\circ n}(x) \notin U$. This proves the last claim of the lemma. \(\square\)

Sometimes the dynamics on $X(f)$ implies that the restriction of $f$ to a suitable Jordan neighborhood of $X(f)$ is quadratic-like. Given a compact set $Y \subset \mathbb{C}$, define the topological hull $\text{TH}(Y)$ of $Y$ as the union of $Y$ with all bounded components of $\mathbb{C} \setminus Y$.

**Theorem 3.6.** Suppose that $f$ has an invariant connected set $Y \subset J(f)$ such that if there exists a parabolic periodic point in $Y$ then it is fixed and such that there is only one cycle of Fatou domains at that point. Suppose that $f|_Y$ is quasi-symmetrically conjugate to the restriction of a quadratic polynomial to its Julia set, and $\omega_2(f) \notin \text{TH}(Y)$. Then there exists a quadratic-like map $f : U^* \to V^*$, whose Julia set coincides with $Y$. In particular, this holds if $Y = X(f)$ is such that $f|_X(f)$ is quasi-symmetrically conjugate to the restriction of a quadratic polynomial to its Julia set, and $\omega_2(f) \notin \text{TH}(X(f))$.

The proof uses some ideas communicated by M. Lyubich to the fourth named author.

**Proof.** Set $K^* = \text{TH}(Y)$. Let us show that if $y \in Y$ is parabolic, then parabolic domains at $y$ are contained in $K^*$. Denote by $\varphi$ the quasi-symmetric conjugacy between $f|_Y$ and the restriction of a quadratic polynomial $h$ to its Julia set $Z$. Then, by Lemma 3.2, the point $\varphi(y)$ is parabolic. Since the entire $J(h)$ is the $\varphi$-image of $Y$, then $\text{TH}(Y)$
must contain all parabolic domains at $y$. Moreover, by the assumption these domains form a cycle.

Let $\phi : \mathbb{D} \to \mathbb{C}^* \setminus K^*$ be a Riemann map. By Lemma 3.3, a point $x \not\in K^*$ close to $K^*$ cannot map into $K^*$. Hence we can choose $\varepsilon > 0$ so that the map $F = \phi^{-1} \circ f \circ \phi$ is defined and holomorphic in the annulus $A_\varepsilon = \{ z : 1 - \varepsilon < |z| < 1 \}$. Moreover, the map $F$ extends continuously to the unit circle $\{ |z| = 1 \}$. Indeed, the map $\phi$ induces a homeomorphism $\hat{\phi}$ between the set of prime ends of $\mathbb{C}^* \setminus K^*$ and the unit circle. Note that $f$ induces a continuous map $\hat{f}$ on the prime ends of $\mathbb{C}^* \setminus K^*$. The continuous extension of $F$ is obtained by conjugating the map $\hat{f}$ by the homeomorphism $\hat{\phi}$.

By the Schwartz reflection principle, we can extend the map $F$ to a holomorphic map of the annulus $1 - \varepsilon < |z| < (1 - \varepsilon)^{-1}$ to $\mathbb{C}$ preserving $S$ (hence taking this annulus to another annulus around $S$). By a theorem of Mañé [Mañ85], if $F$ has no attracting or parabolic periodic points on $S$, and no critical points on $S$, then $F$ is expanding, i.e. $|(F^n)'(z)| \geq C \mu^n$ for some $C > 0$ and $\mu > 1$.

Since $F$ takes $A_\varepsilon$ to a subset of the disk $|z| < 1$, it has no critical points on the unit circle. Suppose that $F$ has an attracting or a parabolic periodic point $z$ on the unit circle of period $r$. In both cases, there is a convex Jordan domain $\tilde{E}$ such that $F^{\text{or}}(\tilde{E}) \subset \tilde{E}$, the closure of $\tilde{E}$ contains $z$, and all points of $\tilde{E}$ converge to $z$ under the iterations of $F^{\text{or}}$. Note that $\tilde{E}$ and the unit disk intersect over a convex Jordan domain $E$. By definition, $F^{\text{or}}(E) \subset E$, and all points in $E$ converge to $z$ under the iterations of $F^{\text{or}}$.

Set $B = \phi(E)$. Then $f^{\text{or}}(B) \subset B$. By the Denjoy–Wolff theorem, all points of $B$ converge under the iterations of $f^{\text{or}}$ to a fixed point $x \in \text{Bd}(B)$. This fixed point must belong to $Y$, and is either attracting or parabolic (as it attracts an open set of points). Since there are no attracting points in $Y$, the point $x$ is parabolic, and by our assumptions $x$ is fixed. However, by the first paragraph of the proof all parabolic domains at $x$ are contained in $K^*$, a contradiction.

Thus, the map $F$ is expanding on the unit circle and $\varepsilon$ can be chosen so that the $F$-pullback of $A_\varepsilon$ is compactly contained in $A_\varepsilon$. Let $V^*$ be the Jordan domain bounded by the $\phi$-image of the curve $|z| = 1 - \varepsilon$. Set $U^*$ to be the component of $f^{-1}(V^*)$ that contains $Y$. Then $\overline{U^*} \subset V^*$, and $f : U^* \to V^*$ is a quadratic-like map. Let us show that the Julia set $J^*$ of this quadratic-like map is $Y$. Indeed, since $Y \subset J(f)$ is forward invariant, $Y \subset J^*$. On the other hand, $f|_Y$ is topologically conjugate to the restriction of a quadratic polynomial to its Julia set, hence $Y = J^*$. 


To see that this applies to \( Y = X(f) \) we can use Lemma 2.10 and Corollary 3.2.

4. Laminations

In this section, we introduce combinatorial techniques that we will use to prove that certain polynomials admit quadratic-like restrictions. The fact that finding quadratic-like restrictions can be reduced to a combinatorial question is explained in Theorem 5.1.

Given a continuum \( Q \), define a finest map (of \( Q \) onto a locally connected continuum \( Z \)) as a map \( \phi : Q \to Z \) such that for any monotone map \( t : Q \to T \) onto a locally connected continuum \( T \) there exists a map \( h : Z \to T \) such that \( t = h \circ \phi \). It is easy to see that \( \phi \) and \( Z \) are defined up to a homeomorphism, and we can talk about the finest map \( \phi = \phi \in [BCO11] \). Given a point \( y \in Q \), define the \( \sim \)-class generated by \( y \) as the \( \sim \)-class in \( S \) consisting of angles which map under \( p_{\sim} \) to the point \( \phi(y) \). The set \( p_{\sim}^{-1}(\phi(y)) \) is called the fiber of \( y \) (or the fiber of \( \phi(y) \), or the fiber of the \( \sim \)-class generated by \( y \)).

If \( Q = J(f) \) is the connected Julia set of a polynomial \( f \) of degree \( d \), this construction is compatible with the dynamics of \( \sigma_d \). Set \( \sim_{J(f)} = \sim_f \) and call it the topological Julia set.

**Theorem 4.1** ([BCO11] [Kiw04]). The polynomial \( f \) and the topological polynomial \( f_{\sim} \) are semiconjugate by \( \phi_f \). The map \( \phi_f \) is one-to-one outside the filled Julia set, on Fatou domains whose boundaries are not collapsed by \( \phi_f \), and on the set of all periodic points \( x \in J(f) \) such that the \( \sim_f \)-class generated by \( x \) is finite. Moreover, if \( U \) is a Fatou domain

...
of $f$ that eventually maps onto an attracting or parabolic Fatou domain then $\varphi_f(U)$ is a non-degenerate Jordan disk.

Note the following immediate corollary from Theorem 4.1 if angles $\alpha$ and $\beta$ belong to the same finite $\sim_f$-class, then the rays $R_f(\alpha)$, $R_f(\beta)$ land at the same point.

The map $\varphi_f$ sends external rays of $J(f)$ to topological external rays to $J_\sim(f)$ all of which land. For a point $x \in J_\sim(f)$ all arguments of rays $\varphi_f(R_f(\alpha))$ landing at $x$ form one $\sim_f$-class. By [RCO11], the impression of $R_f(\alpha)$ is contained in $\varphi_f^{-1}(x)$ if and only if $\alpha$ belongs to the $\sim_f$-class corresponding to $x$ and is otherwise disjoint from $\varphi_f^{-1}(x)$. For a leaf $\tau \in \mathcal{L}_f$, set $\varphi_f^{-1}(\tau) = \varphi_f^{-1}(p_{\sim_f}(\tau))$. Extend $\sigma_d$ linearly over all leaves in $\mathcal{L}_f$ so that each leaf $\beta \gamma \in \mathcal{L}_f$ maps onto the leaf $\sigma_d(\beta) \sigma_d(\gamma)$. The map $\sigma_d$ can be extended onto $\overline{B}$ using the barycentric construction [Thu85].

By [BL02] any infinite $\sim_f$-class $g$ is preperiodic or periodic. If $\tilde{g}$ is $n$-periodic then $S = p_\sim(g)$ is a Jordan curve such that $f^{\sim_n}|_S$ is conjugate to $\sigma_k$ with some $k > 1$. The convex hull of $g$ in $\overline{B}$ is called a Fatou gap (of period $n$ and of degree $k$). Otherwise such gaps appear in $\mathcal{L}_f$ as follows. By Theorem 4.1 if $f$ has a cycle $\mathcal{I}$ of Fatou domains of period $n$, and for $U \in \mathcal{I}$, the map $f^{\sim_n}|_U$ is a branched covering of degree $k > 1$, then $\varphi_f(Bd(U)) = S \subset J_\sim(f)$ is a Jordan curve and $f^{\sim_n}|_S$ is conjugate to $\sigma_k$. Let $G' \subset S$ be the set of arguments of topological external rays to $J_\sim(f)$ landing in $S$. The convex hull $G$ of $G'$ in $\overline{B}$ is called the Fatou gap (of degree $k > 1$) associated to $U$. Then $\sigma_d(G)$ is the Fatou gap associated to $f(U)$, and $G$ is periodic of period $n$.

Let us define a general puzzle-piece. Loosely, for a polynomial $f$ with connected Julia set a general puzzle-piece is a subcontinuum $X$ of $K(f)$ carved in it by exit continua $E_1, \ldots, E_n$ located on $\text{Bd}(X)$, cf. [BFMOT12]. The definition below is a little less general than that in [BFMOT12].

**Definition 4.2.** Let $E_1, \ldots, E_n$ be a finite (perhaps empty) set of continua in $J(f)$ each containing principal sets of more than one external ray. Denote the union of $E_i$ with these external rays by $E_i$. Suppose that there is a component $T$ of $\mathbb{C} \setminus \bigcup E_i$ whose boundary intersects all $E_1, \ldots, E_n$. Continua $E_i$ are called exit continua. Clearly, $Y = [T \cap K(f)] \cup (\bigcup E_i)$ is a continuum. Any non-separating continuum $X \subset Y, X \supset \bigcup E_i$ is called a general puzzle-piece. For each $i$ let $W_i$ be the component of $\mathbb{C} \setminus E_i$ containing $T$.

**Theorem 4.3 ([BFMOT12]).** Let $f$ be a polynomial with connected Julia set and $X$ be a general puzzle-piece. Suppose that (1) each $E_i$ is...
either a fixed point or maps forward in such a way that $f(E_i) \subset W_i$ ($E_i$ is mapped “towards $T$”), and (2) $f(X) \setminus X$ is disjoint from $T$ (“$X$ can only grow through $E_i$’s”). Then $X$ contains either a fixed Cremer point, or a fixed Siegel point, or an invariant attracting or parabolic Fatou domain, or a fixed repelling or parabolic point at which several rays land so that $f$ rotates them non-trivially.

Theorem 4.3 applies to invariant non-separating continua in $K(f)$ (with empty collection of exit continua).

**Proposition 4.4.** A periodic fiber $Q$ of $f$ that corresponds to an infinite $\sim_f$-class must contain a Cremer or a Siegel periodic point.

**Proof.** Apply Theorem 4.3 to $Q$. Let $Q$ contain neither periodic Cremer points nor periodic Siegel disks. For a Fatou domain $U$, by Theorem 4.1 the set $Q$ is either disjoint from $U$, or contains $\overline{U}$. Moreover, if $U$ is eventually mapped to an attracting or a parabolic Fatou domain, then $\phi_f(Bd(U))$ is a Jordan curve, in particular $\overline{U}$ cannot be in $Q$.

Hence by Theorem 4.3 the set $Q$ contains a fixed point $x_0$ at which several rays land and rotate. Suppose that the period of the rays is $m_0$. Choose a component $Q_1$ of $Q \setminus \{x_0\}$ and consider $f_{\sim_m}$ on this component united with $x_0$. Then $Q_1 \cup \{x_0\}$ is a general puzzle-piece, and the argument can be repeated. Thus, $Q$ contains infinitely many periodic cutpoints while by [BCO11, Proposition 40], the fiber $Q$ can only contain a finite collection of periodic cutpoints. 

**5. Major cuts of periodic type and quadratic-like maps**

In what follows we resume studying a cubic polynomial $f = f_{\lambda,b}$ with connected Julia set satisfying the assumptions of Theorem A. Various objects introduced in the previous sections (such as $\omega_1(f), X(f), \mathcal{F}_\lambda$ etc) are defined for $f$. In this section we prove that in some cases there exist Jordan domains $U, V$ such that $f : U \rightarrow V$ is a quadratic-like map with connected filled Julia set containing 0 and $\omega_1$.

Let $A(f) = A$ be an invariant Fatou domain of $f$ on which $f$ is two-to-one (a Fatou domain $U$ of period $k$ with $f^k|_U$ two-to-one is called quadratic). By Theorem 4.1 $\sim_f$ is non-degenerate ($S$ is not collapsed to one $\sim_f$-class). If $\sim_f$ is trivial (all $\sim_f$-classes are degenerate), then it follows that $f$ has an invariant domain on which $f$ is three-to-one, a contradiction. Hence using $\sim_f$ we associate to $A$ an invariant gap $G(f) = G$ on which the map is two-to-one (a periodic Fatou gap of period $k$ mapping onto itself by $\sigma_3^k$ two-to-one is called quadratic). In [BOPT13] invariant quadratic gaps are studied independently of their origins, so now we will quote some results of [BOPT13].
Let $G$ be an invariant quadratic gap (i.e., $\sigma_{3}\vert_{G'}$ is two-to-one). The gap $G$ has a unique edge $M(f) = M$ such that the arc $H(M)$ (called the major hole of $G = G_M$) which is the component of $\mathbb{S} \setminus M$ containing no points of $G' = G \cap \mathbb{S}$ is of length at least $\frac{1}{3}$ and at most $\frac{1}{2}$. Moreover, major holes of distinct quadratic invariant gaps do not contain each other. Then $M$ is called the major (leaf) of $G$ and all other edges of $G = G_M$ map to $M$ in finitely many steps. The set $G'$ consists of all points of $\mathbb{S}$ which never enter $H(M)$.

There are two types of majors. First, a major $M$ (and the corresponding gap $G = G_M$) can be of regular critical type. Then $M = \theta_1\theta_2$ is critical meaning that $\theta_2 - \theta_1$ is $\frac{1}{3}$. Also, $G$ can be of periodic type with its major $M = \theta_1\theta_2$ being a periodic edge of $G$ of period $k$. Call such $M$ a major (leaf) of periodic type. Clearly, $\sigma_3\vert_{H(M)}$ wraps around the circle. On the other hand, for every $i$ such that $1 \leq i \leq k - 1$, the map $\sigma_3$ is one-to-one on the circle arc with endpoints $\sigma^o_i(\theta_1), \sigma^o_i(\theta_2)$ that is disjoint from $G$. Then inside $H(M)$ there are points $\alpha, \beta$ such that $N = \alpha\beta$ has the same image as $M$; we call $N$ a sibling leaf of $M$.

One can construct a gap $V_M$ with edges $M$ and $N$ consisting of all $x$ such that for every $n \geq 0$, the point $\sigma_3^n(x)$ belongs to the closure of the arc with the same endpoints as $\sigma_3^n(M)$ disjoint from $G$. Then $V_M$ is a quadratic gap. If $M \neq \mathbb{1}_{\frac{1}{2}}$, then either $0 \in H(M)$ is the only fixed angle in $H(M)$ and the only angle which stays in $H(M)$ forever, or the same holds for $\frac{1}{2}$.

A curve $\Gamma$ in the dynamic plane of $f$ consisting of dynamic rays $R_f(\theta_1), R_f(\theta_2)$ with $\theta_1 \neq \theta_2$ and their common landing point $x$ is called a cut; $x$ is called the vertex of $\Gamma$. Depending on the type of the vertex, a cut is called repelling or parabolic. If $\theta_1\theta_2 = M$ is a major of periodic type of period $k$, then we call $\Gamma$ a major cut of periodic type. The wedges in the plane bounded by images of $\Gamma$ and not containing external rays with arguments from $G_M$ are called the outer wedges corresponding to $\Gamma$. Clearly, closures of these components are disjoint (except perhaps for their vertices). The outer wedge which corresponds to $\Gamma$ is called the major outer wedge. The outer wedge corresponding to $f(\Gamma)$ is called the minor outer wedge. Let $f$ have a major cut of periodic type of period $k$ with outer wedges $W_0, \ldots, W_{k-1}$ at vertices $z_0, \ldots, z_{k-1}$. Say that $f$ repels in outer wedges if for each $i$ there exists a neighborhood $U_i$ of $z_i$ so that $f^{ok}$ repels in $U_i \cap W_i$ away from $z_i$.

In Theorem 5.1 which serves as a useful tool in what follows we relate major cuts of periodic type with renormalization.

**Theorem 5.1.** Suppose that a polynomial $f$ has a major cut of periodic type $\Gamma$ such that the union of all outer wedges contains $\omega_2(f)$ but
contains neither $\omega_1(f)$ nor 0. If $f$ repels in outer wedges, then there exist Jordan domains $U^*$, $V^*$ containing 0 such that $f : U^* \to V^*$ is a quadratic-like map with connected filled Julia set containing 0.

Proof. Let $W_0, \ldots, W_{k-1}$ be the outer wedges at vertices $z_0, \ldots, z_{k-1}$ bounded by the cuts $\Gamma, f(\Gamma), \ldots, f^{c_k-1}(\Gamma)$, respectively. Let $\tilde{W}$ be the union of the outer wedges. The set $\tilde{W}$ contains $\omega_2(f)$, and its complement contains $\omega_1(f)$ and 0. Choose a tight equipotential of $f$ enclosing a closed Jordan disk $\Delta$. Let $\hat{\Delta}$ be the set $\Delta \setminus \tilde{W}$.

Let us thicken $\hat{\Delta}$ using a thickening procedure from [Mil00b] (we only sketch it here). Around each point $z_i$ draw a tiny disk $D_i$ chosen so that its boundary intersects any outer wedge $W_i$ at $z_i$ along an arc $Y_i$ (if $z_i = 0$ then there may be more than one outer wedge at $z_i$). By the assumptions we can do it so that under $f^{c_k}$ the arc $Y_i$ maps out within $W_i$. Take two external rays inside $W_i$ close to its boundary rays. Their sub-rays from infinity to their first points of intersection with $Y_i$ united with the subarc of $Y_i$ connecting these points form a cut $C_i$ inside $W_i$. Do this for all outer wedges. The cuts $C_i$ carve a closed Jordan disk $V^* \supset \hat{\Delta}$ from $\Delta$; the domain $V^*$ is only a little bigger than $\hat{\Delta}$. By the above, the new cuts can be chosen consistently so that each cut maps outside of $V^*$. Clearly, $\omega_1(f) \in V^*$ and $0 \in V^*$. Now, set $U^*$ to be the pullback of $V^*$ contained in $V^*$. As in [Mil00b] it follows that $f : U^* \to V^*$ is a quadratic-like map.

To use Theorem 5.1 we now need to show the existence of major cuts of periodic type for various maps.

6. MAJOR CUTS OF PERIODIC TYPE IN THE DISCONNECTED CASE

In this section, we will prove the existence of major cuts of periodic type for certain polynomials with disconnected Julia sets.

We need [BCLOS10] Section 6], based in part on [LP96]; we discuss it only as it applies to a polynomial $g \in A$ with disconnected $J(g)$ (we fix such $g$ in the rest of Section 6). Set $U^\infty = U^\infty(J(g))$ to be the basin of attraction of infinity of $g$. The equipotential of a point $z \in U^\infty$ is the closure of the union of all preimages $g^{-on}(g^{on}(z))$, $n = 1, 2, \ldots$. Then $U^\infty$ is foliated by equipotentials. The polynomial $g$ has a unique escaping critical point $\omega_2(g)$. Denote by $\Omega$ the set of all preimages $g^{-on}(\omega_2(g))$, $n = 0, 1, \ldots$ of $\omega_2(g)$. A component of an equipotential is a smooth curve if and only if it does not contain a point of $\Omega$.

Let us modify the notion of an external ray for $g$. An external ray $R_{\theta}(\theta)$ of $g$ with argument $\theta$ is an unbounded curve $R$, such that either $R$ is smooth, crosses every equipotential orthogonally and terminates in
$J(g)$ (then $R$ is called smooth), or $R$ is a one-sided limit of smooth rays (then $R$ is called non-smooth or one-sided). An external ray is smooth if and only if it is disjoint from $\Omega$. Every point of $U^\infty$ belongs to an external ray, and smooth external rays are dense in $U^\infty$. An external ray, whether smooth or not, accumulates inside one component of $J(g)$.

The argument $\theta$ of $R_g(\theta)$ is defined as the angle at which $R(\theta)$ goes asymptotically to infinity. If the ray is non-smooth, then there is precisely one more (non-smooth) external ray with the same argument; this will not cause ambiguity, because we will speak about external rays, not their arguments. A ray is periodic if and only if the argument of the ray is periodic (cf. [GM93, LP96]).

The next lemma goes back to Douady and Hubbard [DH8485].

**Lemma 6.1 (Lemma B.1 [GM93]).** Let $f$ be a polynomial, and $z$ be a repelling periodic point of $f$. If a smooth ray $R_f(\theta)$ lands at $z$, then, for every polynomial $g$ sufficiently close to $f$, the ray $R_g(\theta)$ is also smooth, lands at a repelling periodic point $w$ close to $z$, and $w$ depends holomorphically on $g$.

Lemma 6.1 holds for polynomials with disconnected Julia sets as stated. However, it can be generalized by allowing for non-smooth rays [BCLOS10]. E.g., suppose that $f \in \mathcal{F}_{nr}$ has a repelling periodic cutpoint $z_f$ of $J(f)$ so that external rays with arguments from a set $\Theta$ with more than one element land at $z_f$. Then any close by polynomial $g$ (with connected or disconnected Julia set) has a repelling periodic point $z_g$ at which external rays of $g$ with arguments from $\Theta$ land as well. In particular, $[f] \notin \PHD$, i.e. $f$ satisfies the assumptions of Theorem A.

Consider $g \in \mathcal{A}$, and let $K^*$ be the component of $K(g)$ containing 0. By [BuHe01] there are Jordan domains $U, V$ such that $g : U \to V$ is a quadratic-like map with filled Julia set $K^*$ which is hybrid equivalent to a quadratic polynomial $h$ with an attracting fixed point [DHS5]. Fix a corresponding straightening map $\psi$. External rays of $h$ are smooth. For an external ray $R_h$ of $h$ of argument $\tau$, its $\psi^{-1}$-image $l_\tau := \psi^{-1}(R_h)$ in $V$ is called the polynomial-like ray of argument $\tau$. Clearly, $K^* = \overline{A(g)}$ is a Jordan disk.

Call an external ray of $g$ with principal set in $K^*$ an external ray to $K^*$. Since $K^*$ is locally connected, all polynomial-like rays to $K^*$ land (at each point of $\text{Bd}(K^*)$ exactly one polynomial-like ray lands). Theorem 6.2 is a particular case of the much more general Theorem 6.9 of [BCLOS10].

\[20\]
Theorem 6.2 (BCLOS10, Theorem 6.9). For each external ray $R$ to $K^*$ there is exactly one polynomial-like ray $l = \lambda(R)$ with the same principal set such that $l$ and $R$ are homotopic in $\mathbb{C} \setminus K^*$ among curves with the same limit set. Moreover, $\lambda : R \mapsto l$ maps the set of external rays to $K^*$ onto the set of polynomial-like rays, and is “almost injective” in the following sense: if $\lambda^{-1}(l) = \{R_1, R_2, \ldots, R_k\}$ with $k > 1$ then either

(i) $k = 2$ and both rays $R_1, R_2$ are non-smooth and share a common arc to $K$, or

(ii) there is a (pre)periodic point $z$ at which all rays $R_i$ land and at least two of the rays $R_1, \ldots, R_k$ are disjoint.

Choose for each $y \in \text{Bd}(K^*)$ the greatest (by inclusion) wedge $W_y$ formed by external rays landing at $y$ and disjoint from $K^*$. Let $\alpha_y, \beta_y$ be the arguments of the two external rays $R_g(\alpha_y), R_g(\beta_y)$ forming the boundary of $W_y$ (if there is only one external ray landing at $y$, then we set $\alpha_y = \beta_y$ to be the argument of this ray). Consider the set $G'(g)$ of all angles $\alpha_y, \beta_y$ taken over all points $y \in \text{Bd}(K^*)$, and then the convex hull $G(g)$ of the set $G'(g)$. Observe that no point of $\text{Bd}(K^*)$ is critical. Since wedges described above map onto each other, $G(g)$ is an invariant quadratic gap with some major $M = \overline{\alpha' \alpha''}$.

Lemma 6.3. If $G(g)$ has a major $M = \overline{\alpha' \alpha''}$ of periodic type then there are periodic external (possibly non-smooth) rays $R_g(\alpha'), R_g(\alpha'')$ which land at the same periodic point $z_g \in \text{Bd}(A(g))$.

Proof. Suppose that the rays $R_g(\alpha'), R_g(\alpha'')$ land at different points $x', x''$. Clearly, $x', x'' \in \text{Bd}(K^*)$. Then an arc $I$, one of the two arcs in $\text{Bd}(K^*)$ between $x'$ and $x''$, will be such that no external ray can land at its points (the points of $I$ will be shielded by the rest of $K^*$ from the external rays with arguments from $(\alpha'', \alpha')$ and cannot be landing points of external rays with arguments from $(\alpha', \alpha'')$ by definition of $G(g)$). This contradicts Theorem 6.2, hence the rays $R_g(\alpha'), R_g(\alpha'')$ land at the same point $z_g$. Clearly, $z_g$ is periodic.

Call $R_g(\alpha') \cup R_g(\alpha'') \cup \{z_g\}$ a major cut of periodic type, although the rays $R_g(\alpha'), R_g(\alpha'')$ do not need to be smooth in this case.

7. Major cuts of periodic type in the connected case

In this section we use notation and terminology from Section 5. We need the following topological lemma.
Lemma 7.1 (Corollary 7.5.4 [BFMOT12], Lemma 37 [BCO11]). If $I$ is a union of finitely many one-sided impressions of fixed external rays of a polynomial $f$ such that either 

1. the set $\text{TH}(I)$ contains no Siegel or Cremer points, or 
2. the set $I$ is disjoint from impressions of all other angles,

then $I$ is a singleton.

We also need some results of [BOPT13]; we will use the notation from Lemma 7.2 in the rest of the paper.

Lemma 7.2 ([BOPT13]). Let $G_M$ be a quadratic invariant gap of periodic type generated by a major $M = \theta_1\theta_2$ of period $k$. Suppose that $M$ is a leaf of the geo-lamination $\mathcal{L}_{\sim}$ generated by a lamination $\sim$. Then the $\sim$-class of $M$ is either $\{\theta_1, \theta_2\}$, or the basis of a periodic gap $T \subset G_M$ with $M$ as its edge, or the set $V_M'$ (then $\theta_1, \theta_2$ are the only angles of period $k$ in $V_M'$). The only gap with edge $M$ located on the same side of $M$ as $V_M$ must coincide with $V_M$.

Let a polynomial $f$ have a periodic point $x$ at which some external rays land. Let $\text{Ar}_f'(x)$ be the set of arguments of all rays landing at $x$ and $\text{Ar}_f(x)$ the convex hull of $\text{Ar}_f'(x)$. Say that $\text{Ar}_f'(x)$ is compatible with a quadratic invariant gap $G$ if the convex hulls of the sets from the forward orbit of $\text{Ar}_f'(x)$ do not cross edges of $G$ inside $D$ unless they coincide with those edges or contain those edges in their boundaries. If $f$ is fixed we may omit the subscript from the notation.

Lemma 7.3 ([BOPT13]). For a repelling/parabolic cutpoint $x$ there exists at most one quadratic invariant gap $G$ compatible with $\text{Ar}_f'(x)$. If such $G$ exists, then for some major $M$ of periodic type we have $G = G_M$, and for some $i \geq 0$ the set $\sigma_i^3(\text{Ar}(x))$ separates $M$ from $N$ where $N$ is the sibling leaf of $M$ disjoint from $G$.

Lemma 7.4 deals with invariant finite gaps with two cycles of edges.

Lemma 7.4 ([BOPT13]). Suppose that $G$ is an invariant quadratic gap and $T \subset G$ is an invariant gap with two cycles of edges. Then one cycle of edges of $T$ includes the major edge of $G$.

We need the following simple lemma.

Lemma 7.5. Suppose that $h_n \to h$ is an infinite sequence of polynomials of degree $d$ and $\{\alpha, \beta\}$ is a pair of periodic arguments such that the external rays $R_{h_n}(\alpha), R_{h_n}(\beta)$ land at the same repelling periodic point $x_n$ of $h_n$. If $R_h(\alpha), R_h(\beta)$ do not land at the same periodic point of $h$, then one of these two rays must land at a parabolic point of $h$. 
In particular, suppose that $f \in F_{nr}$ and there are maps $g_n \to f$, $g_n \in A$ with $G(g_n) = G_M$ for all $n$, where $M = \theta_1 \theta_2$ is a major leaf of periodic type. Suppose that maps $g_n$ have major cuts of periodic type corresponding to $M$. Then, if $f$ does not have a major cut of periodic type corresponding to $M$ then $f$ must have a parabolic periodic point at which one of the rays $R_f(\theta_1), R_f(\theta_2)$ lands.

**Proof.** We may assume that $x_n$ converge to an $h$-periodic point $x$. If both rays $R_h(\alpha), R_h(\beta)$ land at distinct repelling periodic points, then by Lemma 6.1 we get a contradiction with the fact that $R_{h_n}(\alpha), R_{h_n}(\beta)$ land at $x_n$ and $x_n \to x$. Hence one of the rays $R_h(\alpha), R_h(\beta)$ must land at a parabolic periodic point. $\square$

**Corollary 7.6.** Suppose that $h_n \to h$ is an infinite sequence of polynomials of degree $d$. Suppose that there exists an infinite set of pairs of periodic angles $\alpha^i, \beta^i$ such that for all $n$ the rays $R_{h_n}(\alpha^i), R_{h_n}(\beta^i)$ land at the same repelling periodic point $x_n^i$. Then $J(h)$ has infinitely many repelling periodic cutpoints.

**Proof.** Suppose that $h$ has finitely many repelling periodic cutpoints. Since $h$ has at most finitely many parabolic points, it follows from Lemma 7.3 that at some repelling or parabolic periodic point infinitely many rays of $h$ land, a contradiction. $\square$

Observe that, in general, $\alpha \sim_f \beta$ does not imply that the rays $R_f(\alpha)$ and $R_f(\beta)$ land at the same point, even if $\alpha$ and $\beta$ are periodic.

**Theorem 7.7.** If $f \in F_{nr}$ with connected $J(f)$ has a major $M$ of periodic type in $\mathcal{L}_f$, then $f$ has a major cut of periodic type corresponding to $M$.

Set $D_i = \overline{0^2}$. Let $\text{FG}_a$ be the invariant quadratic gap located above $D_i$ (the set $\text{FG}_a \cap S$ consists of all points of $S$ with orbits above $D_i$). Let $\text{FG}_b$ be the similar gap $\text{FG}_a$ located below $D_i$. To prove Theorem 7.7 by way of contradiction we assume that $M = \theta_1\theta_2$ is a major of periodic type of period $k$ in $\mathcal{L}_f$ but $f$ has no major cut corresponding to $M$, i.e. the rays $R_f(\theta_1), R_f(\theta_2)$ do not land at the same point.

If $J(f)$ is locally connected, then $\varphi_f$ is a conjugacy. It follows that $J(f)$ is not locally connected. By Lemma 7.2 the $\sim_f$-class $m$ of $M$ is finite, or is the basis of a periodic Fatou gap $T$ which has $M$ as its edge. If $m$ is finite, then by Theorem 4.1 the landing points of $R_f(\theta_1), R_f(\theta_2)$ coincide. Thus $m = T'$ for some periodic Fatou gap of period $m$, and $M$ is an edge of $T$.

**Standing Assumption.** We assume that the following holds.
(1) The rays \( R_f(\theta_1), R_f(\theta_2) \) do not land at the same point, and \( J(f) \) is not locally connected. The \( \sim_f \)-class of \( M \) is the basis of a periodic Fatou gap \( T \) of period \( m \) which has \( M \) as its edge. The fiber \( Q \) of \( M \) is non-degenerate. By Proposition 4.4 there is a Siegel or Cremer point \( \zeta \) in \( Q \).

(2) Recall that \( k \) is the period of \( M \). If \( k > 1 \), then the gap \( G_M \) is well-defined. If \( k = 1 \) and \( M = \text{Di} \), choose \( G_M \) from \( \text{FG}_a, \text{FG}_b \) so that \( G_M \) does not coincide with \( T \). The major hole of \( G_M \) is denoted by \( H \).

(3) Set \( \hat{Q} = Q \cup R_f(\theta_1) \cup R_f(\theta_2) \); denote by \( \hat{H} \) the component of \( C \setminus \hat{Q} \) containing rays with argument from \( H \), and by \( \hat{H}^* \) the other component of \( C \setminus \hat{Q} \).

Lemma 7.8. If \( T = G_M \), then \( 0 \in Q \), otherwise \( 0 \in \hat{H}^* \).

Proof. If 0 is attracting or parabolic, let \( Y \) be the closure of the invariant attracting Fatou domain or of the invariant union of parabolic domains containing 0. Then \( \varphi_f(Y) \) corresponds to an invariant Fatou gap or an invariant union of Fatou gaps. If 0 is a Siegel point and the corresponding invariant Siegel domain \( U \) is not collapsed by \( \varphi_f \), then we set \( Y = \overline{U} \). Otherwise (i.e., if 0 is a Siegel point and its Siegel domain is collapsed by \( \varphi_f \), or if 0 is a Cremer point) set \( Y \) to be the fiber of \( \varphi_f(0) \). In all of the above cases \( Y \) contains impressions of infinitely many rays. Moreover, \( \varphi_f(Y) \) contains a unique fixed point \( \varphi_f(0) \).

Assume that \( T = G_M \). Then \( k > 1 \) as otherwise (for \( k = 1 \)) we choose \( G_M \) different from \( T \) by the Standing Assumption, part (2). In particular, \( M \neq \text{Di} \). If \( \varphi_f(0) = \varphi_f(Q) \) then \( 0 \in Q \) and we are done. So we may assume that the point \( \varphi_f(Q) \) does not belong to \( \varphi_f(Y) \). By the previous paragraph, the arguments of the topological rays landing in \( \varphi_f(Y) \) belong to the same hole \( I \) of \( G_M \). On the other hand, this family of rays is \( \sim_f \)-invariant because \( \varphi_f(Y) \) is \( \sim_f \)-invariant. It is easy to see then that the unique topological ray landing in \( \varphi_f(Y) \) is either of argument 0 or \( \frac{1}{2} \) (depending on \( G_M \)), a contradiction as there are infinitely many topological rays landing in \( \varphi_f(Y) \). So if \( T = G_M \) then \( 0 \in Q \). Assume that \( T \neq G_M \) but \( k > 1 \). Then the period of \( Q \) is greater than 1 and so \( 0 \notin Q \). The same argument shows then that \( 0 \in \hat{H}^* \).

Now let \( k = 1 \) and \( T = \text{FG}_a \). By our assumptions, \( R_f(0) \) and \( R_f(1/2) \) land at distinct points. By Proposition 4.4 \( Q \) contains a Siegel or a Cremer fixed point, hence \( Q \) contains all fixed points of \( f \), and by Theorem 3.2 \( \text{GM93} \) \( f \) cannot have any invariant Fatou components. Thus, \( \text{FG}_b \) is not a gap of \( \sim_f \) and leaves of \( \sim_f \) approach \( M \) from below.
Choose a class $e$ with a leaf $\ell$ that is below $M$ very close to $M$; let $E$ be the union of the fiber of $e$ and all rays with arguments from $e$. Take the union $\hat{E}$ of $E$ with the component of $\mathbb{C} \setminus E$ containing rays with arguments located below $\ell$. By Theorem 4.3 there exists a fixed point of $f$ in $\hat{E}$, a contradiction.

□

Lemma 7.9. Under the Standing Assumption, we have $T = G_M$. In particular, if $T \neq G_M$ then $f$ has a major cut of periodic type corresponding to $M$.

Proof. Suppose $T \neq G_M$. By Lemma 7.8 then $0 \in \hat{H}^*$ and $0 \neq \zeta$. By [Kiw00], there exists a periodic cut $\Gamma$ with vertex $z$ separating $\zeta$ from $0$ (if $0$ is not parabolic), or, for every parabolic domain $U$ at $0$, there exists a cut $\Gamma_U$ separating $\zeta$ from $U$. We consider both cases.

Case 1. The point $0$ is not parabolic. Then by the Fatou-Shishikura inequality, $z$ is repelling. Clearly, $A_r(z) \subset S \setminus H$ and $A_r(z)$ cannot contain both $\theta_1$ and $\theta_2$. Take $g \in A$ close to $f$. Clearly, $G(g)$ is separated by $A_r(z)$ from $M$ (perhaps containing one endpoint of $M$ but not both) and the major hole of $G(g)$ strictly contains $H$. By [BOPT13] (see Section 5), this is impossible.

Case 2. The point $0$ is parabolic. If there is one cycle of rays landing at $0$, then we can choose the Fatou domain $\hat{U}$ at $0$ not separated from $\zeta$ by the rays landing at $0$. By [Kiw00] there exists a periodic cut $\Gamma_U$ with repelling vertex $z$ separating $\hat{U}$ from $\zeta$. Repeating the arguments from Case 1 we get a contradiction. Hence there are two cycles of rays landing at $0$. Then Lemma 7.4 implies the desired.

□

Lemma 7.10. If $V_M = V$ is a gap of $\sim_f$ then $f$ has a major cut of periodic type corresponding to $M$.

Proof. Clearly, $V'$ is not one $\sim_f$-class as otherwise it would be united with $G'_M$. Hence there exists a Fatou domain $\hat{V}$ of $f$ corresponding to $V$. Consider periodic rays $R_f(\alpha_n)$ where $\alpha_n \to \theta_1$ are periodic angles from $V'$ landing at points $y_n \in \text{Bd}(\hat{V})$ (by Proposition 4.3 we may always choose angles $\alpha_n \in V'$ with degenerate impressions). Since by [RY08], the boundary of $\hat{V}$ is a Jordan curve on which $f^{ok}$ is topologically conjugate to the angle-doubling map, we may assume that $y_n \to y$ where $y \in \text{Bd}(\hat{V})$ is a unique $f^{ok}$-fixed point in $\text{Bd}(\hat{V})$. Let $R_f(\theta_1)$ land at a point $x \neq y$. It follows that the one-sided impression $I$ of $\theta_1$ taken from the side of $V$ contains $x$ and $y$. Suppose that $I$ contains a periodic Cremer or Siegel point $\zeta'$. By [Kiw00] there exists a periodic cut $\Sigma$ with vertex $w$ separating $\hat{V}$ from $\zeta'$. Let $W$ be the wedge created by $\Sigma$ and containing $\zeta'$. By our assumption, $y \neq w$. Thus, $I$ contains $\zeta', w$
and \( y \). By definition of a one-sided impression, this is impossible (the rays which accumulate and in the limit create \( I \) all come from either within \( W \) or without \( W \)). Hence no Cremer (Siegel) point belongs to \( I \); by Lemma 7.1 the set \( I \) is a point; hence \( x = y \). Repeating these arguments for the angle \( \theta_2 \), we obtain the desired. \( \square \)

**Lemma 7.11.** Suppose that there is a repelling periodic cutpoint in \( \hat{H} \). Then the rays \( R_f(\theta_1), R_f(\theta_2) \) land at the same point.

**Proof.** Let \( z_f \in \hat{H} \) be a repelling periodic cutpoint. Choose \( g \in \mathcal{A} \) close to \( f \). By Lemma 6.1 the map \( g \) has a repelling periodic cutpoint \( z_g \) such that \( Ar'(z_g) = Ar'(z_f) \). By Lemma 7.3 we have \( G(g) = G_M \). Thus, there exists a sequence \( g_n \to f, g_n \in \mathcal{A} \) with \( G(g_n) = G_M \) for all \( n \).

By Lemma 6.3 (in the disconnected case) or Lemma 7.9 (in the connected case), both rays \( R_{g_n}(\theta_1) \), \( R_{g_n}(\theta_2) \) land at a \( g_n \)-periodic point \( x_n \), and Lemma 7.5 applies. We may assume that \( R_f(\theta_2) \) lands at a parabolic point \( y \); since \( \theta_2 \in T' \), we have \( y \in Q \). Let \( \hat{U} \) be a parabolic domain at \( y \). Then by Theorem 4.1 the domain \( \hat{U} \) corresponds to a Fatou gap \( U \). Since \( T = G_M \), the gap \( U \) is located on the other side of \( M \) than \( G_M \). Note that \( U \) is located between \( M \) and its sibling leaf \( N \) disjoint from \( G_M \). Then since \( H \) is the only hole of \( G_M \) of length greater than or equal to \( \frac{1}{3} \) it is easy to see that \( \sigma_3^i(U) \) is separated from \( G_M \) by \( \sigma_3^i(M) \) for all \( i \). Since \( U \) is periodic, it follows that \( U \subset V \). By Lemma 7.2 we have \( U = V \), and Lemma 7.10 implies the desired. \( \square \)

**Proof of Theorem 7.7.** By Lemma 7.2 \( T = G_M \). By Lemma 7.8 it follows that \( 0 \in Q \). Consider parts of the lamination \( \sim_f \) outside \( G_M \). If there exists a gap \( F \) of \( \sim_f \) outside \( G_M \) with an edge \( M \), then \( F \subset V \) because the images of \( F \) must be disjoint from \( G_M \). Since \( V \) is quadratic, \( F = V \), and Lemma 7.10 implies the desired.

Thus, we may assume that the leaf \( M \) is approached from outside of \( G_M \) by edges \( \ell_n = \alpha_n\beta_n \) of \( \sim_f \)-classes \( \mathcal{I}_n \) chosen so that there are no points of \( \mathcal{I}_n \) between \( \ell_n \) and \( M \) and so that the angles \( \theta_1, \theta_2 \) are not in \( (\alpha_n, \beta_n) \). Let \( \hat{L}_n \) be the fiber associated with \( \mathcal{I}_n \). By the properties of laminations there are \( \sim_f \)-classes \( \mathcal{I}_n \) also located between \( M \) and \( N \) and such that \( \sigma_3(\hat{L}_n) = \sigma_3(\mathcal{I}_n) \). Clearly, \( \sim_f \)-classes \( \mathcal{I}_n \) have edges \( \tilde{\alpha}_n\tilde{\beta}_n \) with the same images as \( \alpha_n\beta_n \). Let \( \tilde{L}_n \) be the fiber associated with \( \mathcal{I}_n \). Denote by \( \tilde{S}_n \) the component of

\[
\mathbb{C} \setminus [\hat{L}_n \cup R_f(\alpha_n) \cup R_f(\beta_n) \cup \tilde{L}_n \cup R_f(\tilde{\alpha}_n) \cup R_f(\tilde{\beta}_n)]
\]
that is disjoint from $Q$ and whose boundary intersects both $\hat{L}_n$ and $\tilde{L}_n$.

By Theorem 4.3 for sufficiently large $n$, the set $\hat{S}_n \cap K(f)$ contains a $f^ok$-fixed set $Y$ that is a repelling/parabolic point at which non-fixed rays land, or a Cremer point, or a Fatou domain.

**Case 1:** the set $Y$ is a repelling periodic point. Then the statement follows from Lemma 7.11.

**Case 2:** the set $Y$ is an attracting or parabolic domain. Then by Theorem 4.4 there is a gap $G_Y$ of $\sim f$ corresponding to $Y$. Since $G_Y$ is $k$-periodic, we have $G_Y = V$, a contradiction with the existence of $\sim f$-classes $I_n$.

**Case 3:** The set $Y$ is a Siegel domain or a Cremer point. Then by [Kiw00] there exists a periodic cut $\Gamma$ with vertex $z$ separating 0 from $Y$. By the Fatou-Shishikura inequality $z$ must be repelling. Now, if $\text{Ar}'(z) \subset H$ then by Lemma 7.11 we are done. If $\text{Ar}'(z) \subset G'_M$, then we can use the same argument as in Case 1 in the proof of Lemma 7.9. Hence the only remaining possibility is $\text{Ar}(z) = M$ as desired.

**Case 4:** The set $Y$ is a parabolic point, and rays landing at $Y$ are rotated by $f^ok$. Parabolic domains at $Y$ correspond to Fatou gaps of $\sim f$ contained in $V$. The collapse of edges of $V$ semiconjugates $\sigma_3^{ok}|_{V'}$ with $\sigma_2$ and produces several Fatou gaps rotating around a finite gap. The dynamics of these gaps and their mutual location are the same as for a quadratic map from the Main Cardioid. Hence the critical gap among them “faces” $M$. By [Kiw00] the corresponding Fatou domain must be separated from 0, this separation is done by a periodic cut, and as in Case 3 this leads to the existence of a major cut of periodic type.

Now we prove the existence of quadratic-like Julia sets in some cases.

**Lemma 7.12.** Suppose that $f$ satisfies conditions of Theorem A and is such that one of the following holds:

1) 0 is parabolic and there is a repelling periodic cutpoint $z$ of $J(f)$;
2) $\omega_2(f)$ lies in a periodic attracting basin.

Then $f$ has a major cut of periodic type and there exist Jordan domains $U^*$, $V^*$ containing 0 such that $f : U^* \to V^*$ is a quadratic-like map with filled Julia set $K^*$ containing 0 (and therefore connected). Moreover, in (2) we have $K^* = \text{TH}(X(f))$.

**Proof.** (1) If $\text{Ar}(0)$ has two cycles of edges, the first claim follows from Lemma 7.4. Let $\text{Ar}(0)$ have one cycle of edges. Then $f$ has one cycle of parabolic domains attached at 0. By Theorem 4.4 $\sim f$ has a cycle of Fatou gaps which share edges with $\text{Ar}(0)$. These edges are isolated in $L_f$ which implies that their preimages are isolated too. Now let us
follow [BOPT11] and remove all these isolated edges from \( L_f \) together with all their pullbacks. The remaining leaves still form a closed family because we only remove isolated leaves. By [BOPT11] this family is a geo-lamination corresponding to a non-degenerate lamination \( \approx \) (not all leaves of \( L_f \) are pullbacks of edges of \( \text{Ar}(0) \) because edges of the \( \sim_f \)-class associated to \( z \) cannot be mapped to edges of \( \text{Ar}(0) \)). By the construction, \( \approx \) has an infinite invariant quadratic gap \( G \) (containing the now removed \( \text{Ar}(0) \)). As \( G \) is compatible with \( \text{Ar}(z) \), by Lemma [7.3] \( G = G_M \) for some major \( M \) of periodic type which proves that \( M \) is a leaf of \( \sim_f \). By Theorem [7.1] the map \( f \) has a major cut \( \Gamma \) of periodic type with vertex \( y \) corresponding to \( M \).

Clearly, \( \omega_2(f) \) is separated from 0 by \( \Gamma \). If there were a parabolic domain \( U \) at \( y \), this would imply the existence of a gap \( V_M \) in \( L_f \) contradicting the existence of \( z \). Thus \( U \) does not exist and \( f \) repels in outer wedges. By Theorem [5.1] the rest of the lemma follows.

(2) First we prove that \( f \) has a major cut of periodic type. Let 0 be parabolic. Then \( f \) is of finite type, and, by [DH8485], the Julia set of \( f \) is locally connected. If \( \text{Ar}(0) \) has two cycles of edges, the claim follows from Lemma [7.4]. Suppose that \( \text{Ar}(0) \) has one cycle of edges. As cuts with vertex at 0 cannot separate all parabolic domains at 0 from attracting Fatou domains of \( f \), by [Kiw00], the map \( f \) has a periodic repelling cutpoint. By (1) the existence of a major cut of periodic type follows.

Now, if 0 is not parabolic then by [Kiw00] there is a repelling periodic cutpoint \( z \). If 0 is attracting, then \( \sim_f \) has an invariant quadratic gap \( G \); by Lemma [7.3] \( G = G_M \) must be of periodic type with major leaf of periodic type \( M \) depending only on \( \text{Ar}_f(z) \). By Theorem [7.1] we obtain a major cut of periodic type. Suppose now that 0 is not attracting. Choose a sequence of maps \( g_n \to f, g_n \in \mathcal{A} \). By the above, \( G(g_n) = G_M \). Since by the Fatou-Shishikura inequality \( f \) does not have parabolic points, by Lemma [7.3] it has a major cut of periodic type.

Now, applying the arguments similar to the ones in the last paragraph of the proof of (1) we see that \( f \) repels in outer wedges and, by Theorem [5.1] admits quadratic-like dynamics. To complete the proof, choose a periodic repelling point \( x \) on the boundary of the attracting Fatou domain \( U \) containing \( \omega_2(f) \) and then choose a point \( x' \in \text{Bd}(U) \) with \( f(x') = f(x) \). Choose two external rays \( R \) and \( R' \) landing at \( x \) and \( x' \), respectively, so that \( f(R) = f(R') \). Then the major cut of periodic type \( C_f \) of \( f \) separates the union \( T_f = R \cup R' \cup U \) from 0. Denote by \( W_f \) the component of \( \mathbb{C} \setminus T_f \) containing 0. Since the set \( T_f \) is stable with respect to small perturbations, then, if \( g \) is a polynomial close to \( f \) and
such that 0 is attracting, then $A(g) \subset W_f$. By definition, this implies that $X(f) \subset W_f$ and hence, by Lemma 2.8 the map $f|_{X(f)}$ is two-to-one (except for $f(\omega_1(f))$ which has one preimage in $X(f)$). Similarly, the quadratic-like Julia set $J^*$ whose existence was established above, is also contained in $W_f$ and is also such that $f|_{J^*}$ is two-to-one. This implies that $J^* = X(f)$ as desired. □

8. Unstable maps and the proof of Theorem B

In this section, we assume that $f$ is unstable. We establish some further properties of $X(f)$. Recall that $\Lambda$ is a Jordan neighborhood of $f$ in $F_\lambda$ such that $h \mapsto \omega_1(h)$ is a holomorphic function of $h \in \Lambda$, and all polynomials in $\Lambda$ satisfy the assumptions of Theorem A.

Lemma 8.1. There are polynomials $h \in \Lambda$ arbitrarily close to $f$ such that $\omega_2(h)$ is periodic.

Proof. Suppose otherwise. Consider the set $\Omega_2(f)$ of all iterated preimages of $\omega_2(f)$. By Lemma 2.3 for no map $h \in \Lambda$ can $\omega_1(h)$ eventually map to $\omega_2(h)$. If there exists a neighborhood $\Lambda_1 \subset \Lambda$ such that for every $h \in \Lambda_1$ the point $\omega_2(h)$ is not periodic, then the set $\Omega_2(f)$ admits a holomorphic and equivariant motion in $\Lambda_1$. As $\Omega_2(f) \supset J(f)$, by the $\lambda$-lemma we see that $J(f)$ admits an equivariant holomorphic motion over $\Lambda_1$, a contradiction with the assumption that $f$ is unstable. □

Theorem 8.2. There exist a neighborhood $U$ of $f$ in $\Lambda$, Jordan neighborhoods $U^* \subset V^*$ of $X(f)$ containing 0, and a quadratic polynomial $z^2 + c$ with $c \in \text{PHD}_2$ such that for all polynomials $g \in U$, the map $g : U^* \to V^*$ is quadratic-like with connected Julia set $X(g)$, hybrid equivalent to $z^2 + c$, and $\omega_2(g) \notin X(g)$.

Proof. First we prove the theorem for $f$. By Lemma 8.1, choose a polynomial $h \in \Lambda$ with $\omega_2(h)$ periodic. It follows from Lemma 7.12 that $X(h)$ is connected, 0 $\in \text{TH}(X(h))$, and the map $h : X(h) \to X(h)$ is quasi-symmetrically conjugate to the restriction of a quadratic polynomial to its Julia set. Let us now use the holomorphic motion $\nu$ of the set $X(f)$ whose existence is established in Theorem 2.11. This theorem implies that the map $h : X(h) \to X(h)$ is quasi-symmetrically conjugate to the map $f : X(f) \to X(f)$. Thus, $X(f)$ is connected, 0 $\in \text{TH}(X(f))$, and the map $f : X(f) \to X(f)$ is quasi-symmetrically conjugate to the restriction of a quadratic polynomial, say, $z^2 + c$, to its connected Julia set. As 0 is non-repelling, $c \in \text{PHD}_2$.

To prove that $\omega_2(f) \notin X(f)$ assume, to the contrary, that $\omega_2(f)$ coincides with the limit $z$ of a sequence $z_n \in Z(f)$. Set $z(h) = \nu(z, h)$ and
\( z_n(h) = \mu(z_n, h) \). By Lemma 8.1, the function \( z(h) \) is not identically equal to \( \omega_2(h) \) because if \( \omega_2(h) \) is periodic, it cannot be equal to \( z(h) \). Hence we may suppose that \( z(h) \) and \( \omega_2(h) \) are different holomorphic functions of \( h \). Let \( \Delta \) be a sufficiently small disk in \( \Lambda \) centered at \( f \). As \( h \) makes a full turn around the boundary of \( \Delta \), the point \( z(h) - \omega_2(h) \) makes one or more turns around \( 0 = z(f) - \omega_2(f) \). Since the functions \( z_n(h) \) converge uniformly to \( z(h) \), the point \( z_n(h) - \omega_2(h) \) also makes one or more turns around \( 0 \), for sufficiently large \( n \). Thus, the function \( h \mapsto z_n(h) - \omega_2(h) \) must vanish somewhere inside \( \Delta \), that contradicts the fact that the point \( \omega_2(h) \) is not in \( Z(h) \). By Theorem 3.6 this implies the desired. It remains to observe that the claim for maps \( g \) close to \( f \) follows from Theorem 2.11 and continuity. \( \square \)

**Lemma 8.3.** If \( f \) is stable then it has no periodic parabolic points different from 0.

**Proof.** Suppose that \( x \) is a periodic parabolic point of \( f \). By Lemma 3.5 there exists a neighborhood \( U \) of \( f \) in the corresponding slice \( \mathcal{F}_\lambda \) such that all maps from \( U \) have a parabolic point corresponding to \( x \), a contradiction. \( \square \)

**Proof of Theorem B.** Let \( W \) be a component of \( \mathcal{F}_x^{\text{st}} \) such that there exists a polynomial \( f \in \text{Bd}(W) \), whose class does not belong to \( \text{PHD}_3 \). Let us show that then, for every \( h \in W \), there are Jordan domains \( U^*, V^* \) such that \( h : U^* \to V^* \) is a quadratic-like map. Indeed, by Theorem 8.2 there is a polynomial \( g \in W \) (very close to \( f \)) which satisfies conditions of Theorem 3.6. Now, \( h \) and \( g \) belong to \( W \) and are stable. Let us show that then \( h \) satisfies conditions of Theorem 3.6 too.

Indeed, since \( g, h \in W \) we may consider the set \( Y \subset J(h) \) corresponding to \( X(g) \) in the sense of the equivariant holomorphic motion used in the definition of stable maps. Then \( h|_Y \) is quasi-symmetrically conjugate to the restriction of a quadratic polynomial from \( \text{PHD}_2 \) on its Julia set. By Lemma 2.10 if \( X(g) \) contains a neutral periodic point then it must be 0, and by Corollary 3.2, then there is one cycle of Fatou domains of \( g \) at 0. By Lemma 3.3 if \( Y \) contains a parabolic periodic point \( y \) then \( y = 0 \). Hence \( h \) satisfies conditions of Theorem 3.6 so that the corresponding set \( Y \) contains 0 in its topological hull, and there are Jordan domains \( U^* \subset V^* \), \( 0 \in U^* \) such that \( h : U^* \to V^* \) is a quadratic-like map. \( \square \)

**9. Proof of Theorem A**

If \( J(f) \) is disconnected, Theorem A follows from [BuHe01].
Lemma 9.1 ([BuHe01]). Suppose that $J(f)$ is disconnected. Then the component $E$ of $J(f)$ containing 0 is a quadratic-like Julia set.

Now, suppose that $f$ is stable. For such maps there are several cases according to the position of the critical point $\omega_2(f)$:

**Case 1:** $\omega_2(f)$ lies in a periodic attracting basin but not in $A(f)$.
**Case 2:** $|\lambda| < 1$, and $\omega_2(f)$ is eventually mapped to $A(f)$.
**Case 3:** $\lambda$ is a root of unity, and $\omega_2(f)$ is eventually mapped to a periodic Fatou domain at 0.
**Case 4:** $\omega_2(f)$ is eventually mapped to the Siegel disk around 0.
**Case 5:** $\omega_2(f)$ lies in the Julia set of $f$.

Cases 1–5 cover all stable cases. Call maps $f$ for which Cases 1, 2 or 3 hold maps of type 1, 2 or 3. Since maps of type 2 and 3 are geometrically finite, their Julia sets are locally connected [DH8485].

We are ready to prove Theorem A.

**Proof of Theorem A.** By Theorem 8.2, it suffices to assume that $f$ is stable. If $f$ is of type 1, then, by Lemma 7.12, it admits quadratic-like dynamics. Let us prove the same fact for maps of type 2 or 3. First we show that if $f$ is of type 2 then it has a major cut of periodic type. Indeed, $A(f)$ is not of regular critical type (if it is, $\omega_2(f)$ cannot be attracted to 0). Hence $G(f) = G_M$ for some major $M$ of periodic type. Since $J(f)$ is locally connected, $M$ corresponds to a major cut of periodic type. The construction implies that all other conditions of Theorem 5.1 hold. Hence $f$ of type 2 admits quadratic-like dynamics.

Let us now show that $f$ of type 2 has infinitely many periodic repelling cutpoints. Let $y \in J(f)$ be the point corresponding to $M$. If $V_M$ is a gap of $\sim_f$ then $\omega_2(f)$ must belong to the corresponding Fatou domain and cannot be eventually mapped to $A(f)$. Hence $V_M$ is not a gap of $\sim_f$. By Lemma 7.2 this implies that $M$ is approximated by $\sim_f$-classes $g_i \to M$ from outside of $G_M$. Let $x_i$ be the corresponding points of $J(f)$. Denote by $U$ the critical gap of $\sim_f$ which eventually maps onto $G(f)$ (the critical point $\omega_2(f)$ belongs to the Fatou domain corresponding to $U$). We may assume that $f^{\circ n}(\omega_2(f)) \in A(f)$ and $f^{\circ n}(U) = G(f)$.

Choose the edge $\ell$ of $U$ which “faces” $G(f)$ and let $l \in J(f)$ be the corresponding point. Then choosing big $i$ we see that $f^{\circ n}$ maps $x_i$ in the direction of $l$ and maps $l$ in the direction of $x_i$. Hence Theorem 4.3 applies to the component of $K(f)$ between $x_i$ and $l$. By Theorem 4.3 there exists a repelling periodic cutpoint $z_1$ of $J(f)$ in that component. We now can choose $x_j$ much closer to $y$ and repeat the same argument for the component of $K(f)$ between $z_1$ and $x_j$. By Theorem 4.3 this
yields the existence of a periodic repelling cutpoint $z_2 \neq z_1$ in that component. Repeating this argument we prove the claim.

Let us now consider a polynomial $f$ of type 3. Approximate $f$ by cubic polynomials $g_n \to f$, $g_n \in \mathcal{A}$. Then $\omega_1(g_n)$ is close to $\omega_1(f)$ and $\omega_2(g_n)$ is close to $\omega_2(f)$ (clearly $\omega_1(g_n)$, $\omega_2(g_n)$ are defined), $\omega_1(g_n) \in A(g_n)$ and $\omega_2(g_n) \notin A(g_n)$ (all this follows from the fact that $f$ satisfies assumptions of Theorem A). By Lemma 3.1 choose $g_n$ so that $g_n^m(\omega_2(g_n)) \in A(g)$ for some $m = m_f$ and any $n$. Thus, maps $g_n$ are of type 2. Since $m_f$ is determined by $f$, there are only finitely many topological types of $J(g_n)$. We may assume that $g_n|_{J(g_n)}$ are all pairwise topologically conjugate (recall that $J(g_n)$ is locally connected), have the same lamination, and are such that their common Fatou gap $G = G(g_n)$ is of periodic type and has a major $M$ (the gap $G$ cannot be of regular critical type because $\omega_2(g_n)$ is mapped to $A(g_n)$). By the above all maps $g_n$ have infinitely many periodic repelling cutpoints at which rays with the same arguments land. Hence by Corollary 7.6 the map $f$ has infinitely many periodic repelling cutpoints. By Lemma 7.12 we obtain that $f$ admits quadratic-like dynamics. We conclude that, for maps of types 1–3, case (1) of Theorem A takes place.

The remaining stable cases are Case 4 and Case 5. We claim that in Case 5, the Julia set of $f$ has positive measure and carries a measurable invariant line field. If $f$ has a Siegel disk around 0, this is proved in [Zak99, Theorem 3.4]. More precisely, it is proved there that, if $f$ with a Siegel disk around 0 is stable but neither Case 1 nor Case 4 holds, then $J(f)$ has positive measure and carries an invariant line field. The proof is based on the fact that the holomorphic motion of the Julia set admits an equivariant extension to the entire Riemann sphere. This is easy if there are no Siegel disks, and most difficult if there is a Siegel disk, whose boundary is not locally connected. Thus [Zak99, Theorem 3.4] works out only the difficult case. Note that it suffices to consider a Siegel disk around 0, since the existence of other Siegel disks contradicts the stability of $f$. This completes considering stable cases and the proof of Theorem A.

Indeed, let $f$ satisfy conditions of Theorem A but not admit quadratic-like dynamics. By the above this means that $f$ is not a map of type 1, 2 or 3. If $f$ is stable then the only remaining possibilities for $f$ are covered by Case 4 and Case 5. As explained in the previous paragraph, this means that either a critical point of $f$ is eventually mapped to a Siegel disk containing 0, or the Julia set of $f$ has positive measure and carries an invariant line field. It remains to prove that $f$ has no periodic repelling cutpoints and no neutral periodic points different from 0.

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Assume the contrary. Let $z_f$ be a repelling periodic cutpoint of $J(f)$. Consider maps $g_n \in A$ converging to $f$. By Lemma 6.1 we may assume that all maps $g_n$ have periodic repelling points $z_n$ close to $z_f$ such that $\text{Ar}^i(z_n) = \text{Ar}^i(z_f)$. Since gaps $G(g_n)$ are all compatible with $\text{Ar}^i(z_f)$, by Lemma 7.3 we have $G(g_n) = G_M$ where $M = \alpha' \alpha''$ is a major of periodic type. By Lemma 7.5, if $f$ does not have a major cut of periodic type, then one of the rays $R_f(\alpha'), R_f(\alpha'')$ lands at a parabolic point $y$. If $y = 0$ then, by Lemma 7.12, the map $f$ admits quadratic-like dynamics. On the other hand, if $y \neq 0$, then by Lemma 8.3 the polynomial $f$ is unstable, a contradiction.

Now, let $f$ be stable with a non-repelling periodic point $x \neq 0$. Then the multiplier of $x$ must be constant on the slice $\mathcal{F}_\lambda$ containing $f$, a contradiction. $\square$

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(Alexander Blokh, Lex Oversteegen and Ross Ptacek) Departmen of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294

(Vladlen Timorin) Faculty of Mathematics, Laboratory of Algebraic Geometry and its Applications, Higher School of Economics, Vavilova St. 7, 112312 Moscow, Russia

(Vladlen Timorin) Independent University of Moscow, Bolshoy Vlasievskiy Pereulok 11, 119002 Moscow, Russia

E-mail address, Alexander Blokh: ablokh@math.uab.edu
E-mail address, Lex Oversteegen: overstee@math.uab.edu
E-mail address, Ross Ptacek: rptacek@uab.edu
E-mail address, Vladlen Timorin: vtimorin@hse.ru

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