VANISHING OF NIL-TERMS AND NEGATIVE $K$-THEORY FOR
ADDITIVE CATEGORIES

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Abstract. We extend the notion of regular coherence from rings to additive
categories and show that well-known consequences of regular coherence for
rings also apply to additive categories. For instance the negative $K$-groups and
all twisted Nil-groups vanish for an additive category, if it is regular coherent.
This will be applied to nested sequences of additive categories, motivated by
our ongoing project to determine the algebraic $K$-theory of the Hecke algebra
of a reductive $p$-adic group.

1. Introduction

Background. The Bass-Heller-Swan Theorem gives isomorphisms

$$K_n R[t, t^{-1}] \cong K_{n-1}(R) \oplus K_n(R) \oplus \tilde{\text{Nil}}_{n-1}(R) \oplus \tilde{\text{Nil}}_{n-1}(R)$$

for $K$-theory of rings. For regular rings all Nil groups $\tilde{\text{Nil}}_n(R)$, $n \in \mathbb{Z}$ and neg-
ative $K$-groups $K_n(R)$, $n \in \mathbb{Z}_{<0}$ vanish and this simplifies the Bass-Heller-Swan
formula. Waldhausen [22, 23] proved far reaching extensions of the Bass-Heller-
Swan formula for other group rings. He also introduced regular coherence for rings
and proved generalizations of the above vanishing results for regular coherent rings.
Waldhausen’s motivation was that some group rings are regular coherent (but not regular)
and this allowed him to bootstrap $K$-theory computations for group rings.
The Bass-Heller-Swan Theorem is also an important ingredient in $K$-theory compu-
tations via the Farrell-Jones conjecture. If $R$ is regular, then so is $R[t, t^{-1}]$, but we
do not know, whether the same inheritance statement holds for regular coherence.
This is one reason, why we will not only concentrate on regular coherence here, but
also on regularity.

The goal of this paper is to extend the notions of regularity and regular coherence
from rings to additive categories and to extend the vanishing results in $K$-theory to
additive categories. The basic strategy will be to embed a given additive category
$\mathcal{A}$ in the category of $\mathbb{Z}A$-modules. The latter category is abelian. This mimics the
additive subcategory of finitely generated free $R$-modules of the abelian category of
all $R$-modules and allows the extension of arguments and definitions from rings to
additive categories. This is a standard construction and has been used for a long
time, for example to define Noetherian additive categories and global dimension for
additive categories.

We also extract intrinsic characterizations on the level of additive categories. For
instance, we call a sequence $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ in $\mathcal{A}$ exact at $A_1$, if $f_1 \circ f_0 = 0$ and for
every object $A$ and morphism $g: A \to A_1$ with $f_1 \circ g = 0$ there exists a morphism
$\mathcal{G}: A \to A_0$ with $f_0 \circ \mathcal{G} = g$, see Definition 5.9. We show in Lemma 6.6(iv) that
an idempotent complete additive category $\mathcal{A}$ is regular coherent, if and only if for
every morphism \( f_1: A_1 \to A_0 \) we can find a sequence of finite length in \( A \)
\[
0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0,
\]
which is exact at \( A_i \) for \( i = 1, 2, \ldots, n \). It is \( l \)-uniformly regular coherent if the
number \( n \) can be arranged to satisfy \( n \leq l \) for every morphisms \( f_1 \).

Our motivation. In our experience it is often more convenient to work with ad-
dictive categories in place of rings in connection with \( K \)-theory. Sometimes a minor
drawback is that results for \( K \)-theory of rings have not been fully developed for
additive \( K \)-theory, although often they are really no more complicated. This paper
takes care of the extension of regular coherence from rings to additive categories
that we expect to be helpful.

More concretely, we rely on the present paper in our ongoing work aimed at
the computation of the \( K \)-theory of Hecke algebras of reductive \( p \)-adic groups.
There we apply regular coherence and the \( K \)-theory vanishing results to certain
additive categories. Namely, we consider a decreasing nested sequence of additive
subcategories \( A_\ast = (A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots) \), see Definition 13.1 and associate
to it the additive category \( S(A_\ast) \), called sequence category, and a certain quotient
additive category \( L(A_\ast) \), called limit category, see Definition 13.3 An object in
\( S(A_\ast) \) is a sequence \( A = (A_m)_{m \geq 0} \) of objects in \( A_0 \) such that for every \( l \in \mathbb{N} \)
almost all \( \phi_m \) lie in \( A_0 \). A morphism \( \phi: A \to A' \) in \( S(A_\ast) \) consists of a sequence
of morphisms \( \phi_m: A_m \to A'_m \) in \( A_\ast \) such that for every \( l \in \mathbb{N} \) almost all \( \phi_m \) lie in
\( A_0 \). If the system \( A_\ast \) is constant, i.e., \( A_m = A_0 \), then \( S(A_\ast) = \prod_{m \in \mathbb{N}} A_0 \)
and \( L(A_\ast) \) is the quotient additive category \( \prod_{m \in \mathbb{N}} A_0 / \bigoplus_{m \in \mathbb{N}} A_0 \).

Typically each \( A_m \) will be regular, but this does not imply that \( S(A_\ast) \) or \( L(A_\ast) \) is
regular as well, see Remark 11.3. The problem is that the property Noetherian does
not pass to infinite products of additive categories, see Example 13.15 Therefore
we have to discard the condition Noetherian in our considerations.

Main results. As mentioned above we discuss various regularity properties, which
are well-known for rings and extend them to additive categories. As long as we are
concerned with the notion regular or Noetherian, we follow the standard proof for
rings, which carry over to additive categories. This is done for the convenience of
the reader.

As described above, we need to discard the property Noetherian and stick to reg-
ular coherence and the new notion of uniform regular coherence. These notions do
pass to infinite products of additive categories, see Lemma 11.3 and more generally
under a certain exactness condition about \( A_\ast \), to the additive categories \( S(A_\ast) \) and
\( L(S_\ast) \), see Lemma 13.10 We remark that algebraic \( K \)-theory does commute with
infinite products for additive categories, see [4] and also [7, Theorem 1.2], but not
with infinite products of rings.

We will show the vanishing of twisted Nil-terms and of the negative \( K \)-theory
for regular coherent additive categories in Sections 7 and Section 12.

The for us most valuable result is the technical Theorem 14.1 whose proof relies
on the vanishing of twisted Nil-terms. It will be a key ingredient in our project
to extend the \( K \)-theoretic Farrell-Jones Conjecture for discrete groups to reductive
\( p \)-adic groups, notably, when we want to reduce the family of subgroups, which
map with a compact kernel to \( \mathbb{Z} \), to the family of compact open subgroups. For
discrete groups there is a well-known similar reduction relying also on regularity
conditions. However, in the discrete case it typically suffices to use regularity for

\footnote{These categories come in our situation from controlled algebra; typically control conditions
get more restrictive with \( m \to \infty \).}
rings, while our approach to $K$-theory of reductive $p$-adic groups necessitates the use of the weaker regularity conditions introduced in the present paper.

**Acknowledgements.** This paper is funded by the ERC Advanced Grant “KL2MG-interactions” (no. 662400) of the second author granted by the European Research Council, by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 - SFB 1442, as well as under Germany’s Excellence Strategy - GZ 2047/1, Projekt-ID 390685813, Hausdorff Center for Mathematics at Bonn, and EXC 2044 - 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

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2. Z-categories, additive categories, and idempotent completions

2.a. Z-categories. A Z-category is a category $\mathcal{A}$ such that for every two objects $A$ and $A'$ in $\mathcal{A}$ the set of morphisms $\text{mor}_{\mathcal{A}}(A, A')$ has the structure of a $\mathbb{Z}$-module, for which composition is a $\mathbb{Z}$-bilinear map. Given a ring $R$, we denote by $\mathbb{R}$ the $\mathbb{Z}$-category with precisely one object, whose $\mathbb{Z}$-module of endomorphisms is given by $R$ with its $\mathbb{Z}$-module structure and composition is given by the multiplication in $R$.

2.b. Additive categories. An additive category is a $\mathbb{Z}$-category such that for any two objects $A_1$ and $A_2$ in $\mathcal{A}$ together with morphisms $i_k: A_k \to A$ for $k = 1, 2$ such that for every object $B$ in $\mathcal{A}$ the $\mathbb{Z}$-map

$$\text{mor}_{\mathcal{A}}(A, B) \xrightarrow{\sim} \text{mor}_{\mathcal{A}}(A_1, B) \times \text{mor}_{\mathcal{A}}(A_2, B), \quad f \mapsto (f \circ i_0, f \circ i_1)$$

is bijective.

Given a ring $R$, the category $R$-$\text{MOD}_{\text{fgf}}$ of finitely generated free left $R$-modules carries an obvious structure of an additive category.

An equivalence $F$: $\mathcal{A} \to \mathcal{A}'$ of $\mathbb{Z}$-categories or of additive categories respectively is a functor of $\mathbb{Z}$-categories or of additive categories respectively such that for all objects $A_1, A_2$ in $\mathcal{A}$ the induced map $F_{A_1, A_2}: \text{mor}_{\mathcal{A}}(A_1, A_2) \xrightarrow{\sim} \text{mor}_{\mathcal{A}'}(F(A_1), F(A_2))$ sending $f$ to $F(f)$ is bijective, and for any object $A'$ in $\mathcal{A}'$ there exists an object $A$ in $\mathcal{A}$ such that $F(A)$ and $A'$ are isomorphic in $\mathcal{A}'$. This is equivalent to the existence of a functor $F$: $\mathcal{A}' \to \mathcal{A}$ of $\mathbb{Z}$-categories or of additive categories respectively such that both composites $F \circ F'$ and $F' \circ F$ are naturally equivalent as such functors to the identity functors.

Given a $\mathbb{Z}$-category, let $\mathcal{A}_{\oplus}$ be the associated additive category, whose objects are finite tuples of objects in $\mathcal{A}$ and whose morphisms are given by matrices of morphisms in $\mathcal{A}$ (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [14, Section 1.3].

Let $R$ be a ring. Then we can consider the additive category $\mathbb{R}_{\oplus}$. The obvious inclusion of additive categories

$$\theta_{\text{fgf}}: \mathbb{R}_{\oplus} \xrightarrow{\sim} R$-$\text{MOD}_{\text{fgf}}$$

is an equivalence of additive categories. Note that $\mathbb{R}_{\oplus}$ is small, in contrast to $R$-$\text{MOD}_{\text{fgf}}$. 

References
2.c. **Idempotent completion.** Given an additive category $\mathcal{A}$, its idempotent completion $\text{Idem}(\mathcal{A})$ is defined to be the following additive category. Objects are morphisms $p: A \rightarrow A$ in $\mathcal{A}$ satisfying $p \circ p = p$. A morphism $f$ from $p_1: A_1 \rightarrow A_1$ to $p_2: A_2 \rightarrow A_2$ is a morphism $f: A_1 \rightarrow A_2$ in $\mathcal{A}$ satisfying $p_2 \circ f \circ p_1 = f$. The structure of an additive category on $\mathcal{A}$ induces the structure of an additive category on $\text{Idem}(\mathcal{A})$ in the obvious way. The identity of an object $(A, p)$ is given by the morphism $p: (A, p) \rightarrow (A, p)$. A functor of additive categories $F: \mathcal{A} \rightarrow \mathcal{A}'$ induces a functor $\text{Idem}(F): \text{Idem}(\mathcal{A}) \rightarrow \text{Idem}(\mathcal{A}')$ of additive categories by sending $(A, p)$ to $(F(A), F(p))$.

There is a obvious embedding
\[ \eta(\mathcal{A}): \mathcal{A} \rightarrow \text{Idem}(\mathcal{A}) \]

sending an object $A$ to $\text{id}_A: A \rightarrow A$ and a morphism $f: A \rightarrow B$ to the morphism given by $f$ again. An additive category $\mathcal{A}$ is called idempotent complete, if $\eta(\mathcal{A}): \mathcal{A} \rightarrow \text{Idem}(\mathcal{A})$ is an equivalence of additive categories, or, equivalently, if for every idempotent $p: A \rightarrow A$ in $\mathcal{A}$ there exists objects $B$ and $C$ and an isomorphism $f: A \xrightarrow{\sim} B \oplus C$ in $\mathcal{A}$ such that $f \circ p \circ f^{-1}: B \oplus C \rightarrow B \oplus C$ is given by $\begin{pmatrix} \text{id}_B & 0 \\ 0 & 0 \end{pmatrix}$. The idempotent completion $\text{Idem}(\mathcal{A})$ of an additive category $\mathcal{A}$ is idempotent complete.

For a ring $R$, let $R\text{-MOD}_{fgp}$ be the additive category of finitely generated projective $R$-modules. We get an equivalence of additive categories $\text{Idem}(R\text{-MOD}_{fgp}) \xrightarrow{\sim} R\text{-MOD}_{fgp}$ by sending an object $(F, p)$ to $p(1)$. It and the functor of (2.1) induce an equivalence of additive categories
\[ (2.2) \quad \theta_{fgp}: \text{Idem}(R_{fgp}) \xrightarrow{\sim} R\text{-MOD}_{fgp}. \]

Note that $\text{Idem}(R_{fgp})$ is small, in contrast to $R\text{-MOD}_{fgp}$.

2.d. **Twisted finite Laurent category.** Let $\mathcal{A}$ be an additive category. Let $\Phi: \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism of additive categories.

**Definition 2.3** (Twisted finite Laurent category $\mathcal{A}_\phi[t, t^{-1}]$). Define the $\Phi$-twisted finite Laurent category $\mathcal{A}_\phi[t, t^{-1}]$ as follows. It has the same objects as $\mathcal{A}$. Given two objects $A$ and $B$, a morphism $f: A \rightarrow B$ in $\mathcal{A}_\phi[t, t^{-1}]$ is a formal sum $f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i$, where $f_i: \Phi^i(A) \rightarrow B$ is a morphism in $\mathcal{A}$ from $\Phi^i(A)$ to $B$ and only finitely many of the morphisms $f_i$ are non-trivial. If $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j$ is a morphism in $\mathcal{A}_\phi[t, t^{-1}]$ from $B$ to $C$, we define the composite $g \circ f: A \rightarrow C$ by
\[ g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{i, j \in \mathbb{Z}, i + j = k} g_j \circ \Phi^j(f_i) \right) \cdot t^k. \]

The direct sum and the structure of a $\mathbb{Z}$-module on the set of morphism from $A$ to $B$ in $\mathcal{A}_\phi[t, t^{-1}]$ are defined in the obvious way using the corresponding structures of $\mathcal{A}$. We sometimes also write $\mathcal{A}_\phi[\mathbb{Z}]$ instead of $\mathcal{A}_\phi[t, t^{-1}]$.

**Example 2.4.** Let $R$ be a ring with an automorphism $\phi: R \xrightarrow{\cong} R$ of rings. Let $R_{\phi}[t, t^{-1}]$ be the ring of $\phi$-twisted finite Laurent series with coefficients in $R$. We obtain from $\phi$ an automorphism $\Phi: R \xrightarrow{\cong} R$ of $\mathbb{Z}$-categories. There is an obvious isomorphism of $\mathbb{Z}$-categories
\[ (2.5) \quad R_{\phi}[t, t^{-1}] \xrightarrow{\cong} R_{\phi}[t, t^{-1}] \]
We obtain equivalences of additive categories
\[ (R_Φ)\Phi[t, t^{-1}] \cong R_Φ[t, t^{-1}]-\text{MOD}_{fgf}; \]
\[ \text{Idem}((R_Φ)\Phi[t, t^{-1}]) \cong R_Φ[t, t^{-1}]-\text{MOD}_{fgp}. \]

**Definition 2.6** \((A_Φ[t] \text{ and } A_Φ[t^{-1}])\). Let \(A_Φ[t]\) and \(A_Φ[t^{-1}]\) respectively be the additive subcategory of \(A_Φ[t, t^{-1}]\), whose set of objects is the set of objects in \(A\) and whose morphism from \(A\) to \(B\) are given by finite formal Laurent series \(\sum_{i\in\mathbb{Z}} f_i \cdot t^i\) with \(f_i = 0\) for \(i < 0\) and \(i > 0\) respectively.

### 3. The Algebraic K-theory of \(\mathbb{Z}\)-categories

Let \(A\) be an additive category. One can interpret it as an exact category in the sense of Quillen or as a category with cofibrations and weak equivalence in the sense of Waldhausen and obtains the connective algebraic K-theory spectrum \(K(A)\) by the constructions due to Quillen [17] or Waldhausen [24]. A construction of the non-connective K-theory spectrum \(K^\infty(A)\) of an additive category can be found for instance in [13] or [16].

**Definition 3.1** (Algebraic K-theory of \(\mathbb{Z}\)-categories). We will define the algebraic K-theory spectrum \(K^\infty(A)\) of the \(\mathbb{Z}\)-category \(A\) to be the non-connective algebraic K-theory spectrum of the additive category \(A_\oplus\). Define for \(n \in \mathbb{Z}\)

\[ K_n(A) := \pi_n(K^\infty(A)). \]

The connective algebraic K-theory spectrum \(K(A)\) is defined to be the connective algebraic K-theory spectrum of the additive category \(A_\oplus\).

If \(A\) is an additive category and \(i(A)\) is the underlying \(\mathbb{Z}\)-category, then there is a canonical equivalence of additive categories \(i(A)_\oplus \to A\). Hence there are canonical weak homotopy equivalences \(K(i(A)) \to K(A)\) and \(K^\infty(i(A)) \to K^\infty(A)\).

A functor \(F: A \to A'\) of \(\mathbb{Z}\)-categories induces a map of spectra

\[ K^\infty(F): K^\infty(A) \to K^\infty(A'). \]

We call a full additive subcategory \(A\) of \(A'\) cofinal, if for any object \(A'\) in \(A'\) there is an object \(A\) in \(A\) together with morphisms \(i: A' \to A\) and \(r: A' \to A\) satisfying \(r \circ i = \text{id}\).

**Lemma 3.3.** Let \(I: A \to A'\) be the inclusion of a full cofinal additive subcategory.

(i) The induced map
\[ \pi_n(K(I)): \pi_n(K(A)) \to \pi_n(K(A')) \]

is bijective for \(n \geq 1\);

(ii) The induced map
\[ K^\infty(I): K^\infty(A) \to K^\infty(A') \]

is a weak homotopy equivalence.

**Proof.** (i) This is proved for \(A' = \text{Idem}(A)\) in [21 Theorem A.9.1.]. Now the general case follows from the observation that \(\text{Idem}(A) \to \text{Idem}(A')\) is an equivalence of additive categories.

(ii) This follows from assertion (i) and [13 Corollary 3.7].
4. The Bass-Heller-Swan decomposition for additive categories

Denote by $\text{Add-Cat}$ the category of additive categories. Let us consider the group $\mathbb{Z}$ as a groupoid with one object and denote by $\text{Add-Cat}^\mathbb{Z}$ the category of functors $\mathbb{Z} \to \text{Add-Cat}$, with natural transformations as morphisms. Note that an object of this category is a pair $(\mathcal{A}, \Phi)$ consisting of an additive category together with an automorphism $\Phi : \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories. We recall from [13] Theorem 0.1] using the notation of this paper here and in the sequel:

**Theorem 4.1** (The Bass-Heller-Swan decomposition for non-connective $K$-theory of additive categories). Let $\Phi : \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories. Then there exists for $\mathcal{A}$ an additive category, which is idempotent complete.

(i) There exists a weak homotopy equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$a^\infty \vee b^\infty \simeq T_{K^\infty(\Phi^{-1})} \vee NK^\infty(A[t]) \vee NK^\infty(A[t^{-1}]) \xrightarrow{\cong} K^\infty(A[t, t^{-1}])$$

where $T_{K^\infty(\Phi^{-1})}$ is the mapping torus of $K^\infty(\Phi^{-1}) : K^\infty(\mathcal{A}) \to K^\infty(\mathcal{A})$ and $NK^\infty(A[t])$ is the homotopy fiber of the map $K^\infty(A[t]) \to K^\infty(A)$ given by evaluation $t = 0$.

(ii) There exist a functor $E^\infty : \text{Add-Cat}^\mathbb{Z} \to \text{Spectra}$ and weak homotopy equivalences of spectra, natural in $(\mathcal{A}, \Phi)$,

$$\Omega NK^\infty(A[t]) \xrightarrow{\simeq} E^\infty(A, \Phi)$$

$$\left. K^\infty(A) \vee E^\infty(A, \Phi) \xrightarrow{\simeq} K^\infty_{\text{Nil}}(A, \Phi) \right|$$

where $K^\infty_{\text{Nil}}(A, \Phi)$ is the non-connective $K$-theory of a certain Nil-category $\text{Nil}(A, \Phi)$.

**Theorem 4.2** (Fundamental sequence of $K$-groups). Let $\mathcal{A}$ be an additive category. Then there exists for $n \in \mathbb{Z}$ a split exact sequence, natural in $\mathcal{A}$

$$(4.3) \quad 0 \to K_n(\mathcal{A}) \xrightarrow{(k_+)_* \oplus (k_-)_*} K_n(A[t]) \oplus K_n(A[t^{-1}]) \xrightarrow{(l_+)_* \oplus (l_-)_*} K_n(A[t, t^{-1}]) \xrightarrow{\delta_n} K_{n-1}(\mathcal{A}) \to 0,$$

where $(k_+)_*$, $(k_-)_*$, $(l_+)_*$, and $(l_-)_*$ are induced by the obvious inclusions $k_+$, $k_-$, $l_+$, and $l_-$ and $\delta_n$ is the composite of the inverse of the (untwisted) Bass-Heller-Swan isomorphism

$$K_n(\mathcal{A}) \oplus K_{n-1}(\mathcal{A}) \oplus NK_n(A[t]) \oplus NK_n(A[t^{-1}]) \xrightarrow{\cong} K_n(A[t, t^{-1}]),$$

see Theorem 4.1 with the projection onto the summand $K_{n-1}(\mathcal{A})$.

**Proof.** This follows directly from the untwisted version of Theorem 4.1.

There is also a version for the connective $K$-theory spectrum $K$. Denote by $\text{Add-Cat}_{\text{id}} \subset \text{Add-Cat}$ the full subcategory of idempotent complete additive categories.

**Theorem 4.4** (The Bass-Heller-Swan decomposition for connective $K$-theory of additive categories). Let $\mathcal{A}$ be an additive category, which is idempotent complete. Let $\Phi : \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

(i) Then there is a weak equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$a \vee b_+ \vee b_- : T_{K^\infty(\Phi^{-1})} \vee NK(A[t]) \vee NK(A[t^{-1}]) \xrightarrow{\cong} K(A[t, t^{-1}])$$

where $T_{K^\infty(\Phi^{-1})}$ is the mapping torus of $K(\Phi^{-1}) : K(\mathcal{A}) \to K(\mathcal{A})$ and $NK(A[t])$ is the homotopy fiber of the map $K(A[t]) \to K(\mathcal{A})$ given by evaluation $t = 0$;
(ii) There exist a functor \( E : (\text{Add-Cat}_w)^2 \to \text{Spectra} \) and weak homotopy equivalences of spectra, natural in \((\mathcal{A}, \Phi)\),

\[
\Omega \text{NK}(\mathcal{A}_\Phi[I]) \cong E(\mathcal{A}, \Phi); \quad \text{K}(\mathcal{A}) \vee E(\mathcal{A}, \Phi) \cong \text{K}(\text{Nil}(\mathcal{A}, \Phi)),
\]

where \( \text{K}(\text{Nil}(\mathcal{A}, \Phi)) \) is the connective \( K \)-theory of a certain Nil-category \( \text{Nil}(\mathcal{A}, \Phi) \).

The purpose of the following sections is to find properties of \( \mathcal{A} \), which imply for any automorphism \( \Phi \) the vanishing of the Nil-terms above and are hopefully inherited by the passage from \( \mathcal{A} \) to \( \mathcal{A}[t, t^{-1}] \).

5. \( \mathcal{Z}A \)-modules and the Yoneda Embedding

5.A. Basics about \( \mathcal{Z}A \)-modules. Let \( \mathcal{A} \) be a \( \mathcal{Z} \)-category. We denote by \( \mathcal{Z}A \)-\text{MOD} and \( \text{MOD}-\mathcal{Z}A \) respectively the abelian category of covariant or contravariant respectively functors of \( \mathcal{Z} \)-categories \( \mathcal{A} \) to \( \text{Z-MOD} \). The abelian structure comes from the abelian structure in \( \mathcal{Z} \)-\text{MOD}. For instance, a sequence \( F_0 \xrightarrow{\tau_1} F_1 \xrightarrow{\tau_2} F_2 \) in \( \text{MOD}-\mathcal{Z}A \) is declared to be exact, if for each object \( A \in \mathcal{A} \) the evaluation at \( A \) yields an exact sequence of \( \mathcal{Z} \)-modules \( F_0(A) \xrightarrow{\tau_1(A)} F_1(A) \xrightarrow{\tau_2(A)} F_2(A) \). The cokernel and kernel of a morphism \( T : F_0 \to F_1 \) are defined by taking for each object \( A \in \mathcal{A} \) the kernel or cokernel of the morphism \( T(A) : F_0(A) \to F_1(A) \) in \( \text{MOD}-\mathcal{Z} \).

In the sequel \( \mathcal{Z}A \)-modules means contravariant \( \mathcal{Z}A \)-module, unless specified explicitly differently.

Given an object \( A \) in \( \mathcal{A} \), we obtain an object \( \text{mor}_A(\cdot, A) \) in \( \text{MOD}-\mathcal{Z}A \) by assigning to an object \( B \) the \( \mathcal{Z} \)-module \( \text{mor}_A(B, A) \) and to a morphism \( g : B_0 \to B_1 \) the \( \mathcal{Z} \)-homomorphism \( g^* : \text{mor}_A(B_1, A) \to \text{mor}_A(B_0, A) \) given by precomposition with \( g \).

The elementary proof of the following lemma is left to the reader.

Lemma 5.1 (Yoneda Lemma). For each object \( A \) in \( \mathcal{A} \) and each object \( M \) in \( \text{MOD}-\mathcal{Z}A \), we obtain an isomorphism of \( \mathcal{Z} \)-modules

\[
\text{mor}_{\text{MOD}-\mathcal{Z}A}(\text{mor}_A(\cdot, A), M(\cdot)) \xrightarrow{\cong} M(A), \quad T \mapsto (T)(\text{id}_A).
\]

We call a \( \mathcal{Z}A \)-module \( M \) free, if it is isomorphic as \( \mathcal{Z}A \)-module to \( \bigoplus_I \text{mor}_A(\cdot, A_i) \) for some collection of objects \( \{A_i \mid i \in I\} \) in \( \mathcal{A} \) for some index set \( I \). A \( \mathcal{Z}A \)-module \( M \) is called projective, if for any epimorphism \( p : N_0 \to N_1 \) of \( \mathcal{Z}A \)-modules and morphism \( f : M \to N_1 \) there is a morphism \( \bar{f} : M \to N_0 \) with \( p \bar{f} = f \). A \( \mathcal{Z}A \)-module \( M \) is \textit{finately generated}, if there exists a collection of objects \( \{A_i \mid j \in J\} \) in \( \mathcal{A} \) for some finite index set \( J \) and an epimorphism of \( \mathcal{Z}A \)-modules \( \bigoplus_{j \in J} \text{mor}_A(\cdot, A_j) \to M \). Equivalently, \( M \) is finitely generated, if there exists a finite collection of objects \( \{A_i \mid j \in J\} \) in \( \mathcal{A} \) together with elements \( x_j \in M(A_j) \) such that for any object \( A \) and any \( x \in M(A) \) there are morphisms \( \varphi : A \to A_j \) such that \( x = \sum_j M(\varphi_j)(x_j) \).

(The \( x_j \) are the images of \( \text{id}_{A_j} \) under the above epimorphism.) Given a collection of objects \( \{A_i \mid i \in I\} \) in \( \mathcal{A} \) for some index set \( I \), the free \( \mathcal{Z}A \)-module \( \bigoplus_I \text{mor}_A(\cdot, A_i) \) is finitely generated, if and only if \( I \) is finite. A \( \mathcal{Z}A \)-module \( M \) is \textit{finitely presented}, if there are finitely generated free \( \mathcal{Z}A \)-modules \( F_1 \) and \( F_0 \) and an exact sequence \( F_1 \to F_0 \to M \to 0 \). We say that a \( \mathcal{Z}A \)-module has \textit{projective dimension} \( \leq d \), denoted by \( \text{pd} \mathcal{Z}A(M) \leq d, \) for a natural number \( d, \) if there exists an exact sequence \( 0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \) such that each \( \mathcal{Z}A \)-module \( P_i \) is projective. If we replace projective by free, we get an equivalent definition, if \( d \geq 1. \) We call a \( \mathcal{Z}A \)-module of type FP or of type FL respectively, if there exists an exact sequence of finite length \( 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \) such
Let each \( \mathbb{Z}A \)-module \( F_i \) be finitely generated free or finitely generated projective respectively.

**Remark 5.2.** Note the setting in this paper is different from the one appearing in [8], since here a \( \mathbb{Z}A \)-module \( M \) satisfies \( M(f + g) = M(f) + M(g) \) for two morphisms \( f, g: A \to B \), which is not required in [8]. Nevertheless many of the arguments in [8] carry over to the setting of this paper because of the Yoneda Lemma [5.1] which replaces the corresponding Yoneda Lemma in Subsection 9.16 on page 167.

However, the next result has no analogue in the setting of [8].

**Lemma 5.3.** Let \( A \) be an additive category. For an \( \mathbb{Z}A \)-module \( M \) and objects \( A_1, A_2, \ldots, A_n \), we obtain a natural isomorphism

\[
\bigoplus_{i=1}^{n} M(p_i) : \bigoplus_{i=1}^{n} M(A_i) \xrightarrow{\cong} M\left(\bigoplus_{i=1}^{n} A_i\right),
\]

where \( p_i : \bigoplus_{i=1}^{n} A_i \to A_j \) is the canonical projection for \( j = 1, 2, \ldots, n \).

**Proof.** One easily checks, using the fact that the functor \( M \) is compatible with the \( \mathbb{Z} \)-module structures on the morphisms, that the inverse is given

\[
M\left(\bigoplus_{i=1}^{n} A_i\right) \to \bigoplus_{i=1}^{n} M(A_i), \quad x \mapsto (M(k_i)(x)),
\]

where \( k_j : A_j \to \bigoplus_{i=1}^{n} A_i \) is the inclusion of the \( j \)-th summand for \( j = 1, 2, \ldots, n \).

**Lemma 5.4.** Let \( A \) be a \( \mathbb{Z} \)-category.

(i) Every free \( \mathbb{Z}A \)-module is projective;

(ii) Let \( 0 \to M \to M' \to M'' \to 0 \) be an exact sequence of \( \mathbb{Z}A \)-modules. If both \( M \) and \( M'' \) are free or projective respectively, then \( M' \) is free or projective respectively;

(iii) Let \( 0 \to M \to M' \to M'' \to 0 \) be an exact sequence of \( \mathbb{Z}A \)-modules. If two of the \( \mathbb{Z}A \)-modules \( M \), \( M' \) and \( M'' \) are of type FL or FP respectively, then all three are of type FL or FP respectively;

(iv) Let \( C_* \) be a projective \( \mathbb{Z}A \)-chain complex i.e., a \( \mathbb{Z}A \)-chain complex, all whose chain modules \( C_n \) are projective. Then the following assertions are equivalent:

(a) Consider a natural number \( d \). Let \( B_d(C_*) \) be the image of \( c_{d+1} : C_{d+1} \to C_d \) and \( j : B_d(C) \to C_d \) be the inclusion. There is a \( \mathbb{Z}A \)-submodule \( C_d^+ \) such that for the inclusion \( i : C_d^+ \to C_d \) the map \( i \oplus j : C_d^+ \oplus B_d(C) \to C_d \) is an isomorphism. Moreover, the following chain map from a \( d \)-dimensional projective \( \mathbb{Z}A \)-chain complex to \( C_* \) is a \( \mathbb{Z}A \)-chain homotopy equivalence

\[
\cdots \to 0 \xrightarrow{c_{d+2}} C_{d+1} \xrightarrow{c_{d+1}} C_d \xrightarrow{c_d} \cdots \xrightarrow{c_1} C_0 \to 0 \xrightarrow{c_{d+1}} C_{d+1} \xrightarrow{c_d} \cdots \xrightarrow{c_1} C_0;
\]

(b) \( C_* \) is a \( \mathbb{Z}A \)-chain homotopy equivalent to a \( d \)-dimensional projective \( \mathbb{Z}A \)-chain complex;

(c) \( C_* \) is dominated by \( d \)-dimensional projective \( \mathbb{Z}A \)-chain complex \( D_* \), i.e., there are \( \mathbb{Z}A \)-chain maps \( i : C_* \to D_* \) and \( r_* : D_* \to C_* \) satisfying \( r_* \circ i_* \simeq \text{id}_{C_*} \).
(d) $H_d(C_*)$ is a direct summand in $C_d$ and $H_i(C_*) = 0$ for $i \geq d + 1$;
(e) $H^{d+1}_{Z^A}(C_*; M) := H^{d+1}(\text{hom}_{Z^A}(C_*, M))$ vanishes for every $Z^A$-module $M$ and $H_i(C_*) = 0$ for all $i \geq d + 1$;
(v) Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $Z^A$-modules.
If $\text{pdim}_{Z^A}(M), \text{pdim}_{Z^A}(M') \leq d$, then $\text{pdim}_{Z^A}(M'') \leq d$;
If $\text{pdim}_{Z^A}(M), \text{pdim}_{Z^A}(M') \leq d$, then $\text{pdim}_{Z^A}(M'') \leq d + 1$;
If $\text{pdim}_{Z^A}(M') \leq d$, $\text{pdim}_{Z^A}(M'') \leq d + 1$, then $\text{pdim}_{Z^A}(M) \leq d$;
(vi) Suppose that $A$ is an additive category. For two objects $A_0$ and $A_1$ in $A$ together with a choice of a direct sum $i_k : A_k \to A_0 \oplus A_1$ for $k = 0, 1$, the induced $Z$-map

$$i_0 \oplus i_1 : \text{mor}_A(?, A_0) \oplus \text{mor}_A(?, A_1) \cong \text{mor}_A(?, A_0 \oplus A_1)$$

is an isomorphism. In particular each finitely generated free $Z^A$-module is isomorphic to $Z^A$-module of the shape $\text{mor}_A(?, A)$ for an appropriate object $A$ in $A$.

Proof. (i) This follows from the Yoneda Lemma 5.1.

(ii) This is obviously true.

(iii) The proof is analogous to the one of [8] Lemma 11.6 on page 216.

(iv) The proof is analogous to the one of [8] Proposition 11.10 on page 221.

(v) This follows from (iv) for the projective dimension using the long exact (co)homology sequence associated to a short exact sequence of (co)chain complexes, since every $Z^A$-module has a free resolution by the Yoneda Lemma 5.1.

(vi) This is obvious and hence the proof of Lemma 5.4 is finished. \qed

If $M$ and $N$ are $Z^A$-modules, then $\text{hom}_{Z^A}(M, N)$ is the $Z$-module of $Z^A$-homomorphisms $M \to N$. Given a contravariant or covariant $Z^A$-module $M$ and a $Z$-module $T$, we obtain a covariant or contravariant $Z^A$-module $\text{hom}_{Z^A}(M, T)$ by sending an object $A$ to $\text{hom}_{Z^A}(M(A), T)$. Given a contravariant $A$-module $M$ and covariant $Z^A$-module $N$, their tensor product $M \otimes_{Z^A} N$ is the $Z$-module given by $\bigoplus_{A \in \text{ob}(A)} M(A) \otimes_{Z} N(A)/T$. Here $T$ is the $Z$-submodule of $\bigoplus_{A \in \text{ob}(A)} M(A) \otimes_{Z} N(A)$ generated by elements of the form $m f \otimes n - m \otimes f n$ for a morphism $f : A \to B$ in $A$, $m \in M(A)$ and $n \in N(B)$, where $m f := M(f)(m)$ and $f n = N(f)(n)$. It is characterized by the property that for any $Z$-module $T$, there are natural adjunction isomorphisms

\begin{align}
\text{hom}_{Z^A}(M \otimes_{Z^A} N, T) &\cong \text{hom}_{Z^A}(M, \text{hom}_{Z^A}(N, T)); \\
\text{hom}_{Z^A}(M, \text{hom}_{Z^A}(N, T)) &\cong \text{hom}_{Z^A}(N, \text{hom}_{Z^A}(M, T)).
\end{align}

Let $F : A \to B$ be a functor of $Z$-categories. Then the restriction functor

$$F^* : \text{MOD-Z}^B \to \text{MOD-Z}^A$$

is given by precomposition with $F$. The induction functor

$$F_* : \text{MOD-Z}^A \to \text{MOD-Z}^B$$

sends a contravariant $Z^A$-module $M$ to $M(?) \otimes_{Z^A} \text{mor}_B(?, F(?))$. We get for a $Z^B$-module an identification $F^* N = \text{hom}_{Z^B}(\text{mor}_A(?, F(?)), N(??))$ from the Yoneda Lemma 5.1. We conclude from (5.20)

\begin{equation}
\text{hom}_{Z^B}(F, M, N) \cong \text{hom}_{Z^A}(M, F^* N)
\end{equation}

for a $Z^A$-module $M$ and a $Z^B$-module $N$. The counit $\beta(N) : F_* F^* (N) \to N$ of the adjunction (5.7) is the adjoint of $\text{id}_{F_* N}$ and sends the equivalence class of $x \otimes j$ for $x \in N(F(A))$ and $f \in \text{mor}_B(B, F(A))$ to $x f = N(f)(x)$. The unit $\alpha(M) : M \to \text{hom}_{Z^B}(F, M, N) \cong \text{hom}_{Z^A}(M, F^* N)$ is given by $f \mapsto \sum_{f \in \text{mor}_B(B, F(A))} (\text{id}_{F_* N})^{-1}[x f]$.
Lemma 5.11. The functor $F^*$ is flat. The functor $F_*$ is compatible with direct sums over arbitrary index sets, is right exact, see [23, Theorem 2.6.1. on page 51], and $F_* \text{mor}_{\text{Z}A}(?, C)$ is $\text{ZB}$-isomorphic to $\text{mor}_{\text{ZB}}(?, F(C))$. In particular $F_*$ respect the properties finitely generated, free, and projective.

5.B. The Yoneda embedding. The Yoneda embedding is the following covariant functor

(5.8) $\iota: \mathcal{A} \to \text{MOD-Z}A$.

It sends an object $A$ to $\iota(A) = \text{mor}_{\mathcal{A}}(? , A)$ and a morphism $f: A_0 \to A_1$ to the transformation $\iota(f): \text{mor}_{\mathcal{A}}(?, A_0) \to \text{mor}_{\mathcal{A}}(?, A_1)$ given by composition with $f$. Let $\text{MOD-Z}A_\mathcal{A}$ be the full subcategory of $\text{MOD-Z}A$ consisting of $\mathcal{A}$-modules $\text{mor}_{\mathcal{A}}(?, A)$ for any object $A$ in $\mathcal{A}$. Let $\text{MOD-Z}A_{\text{fg}}$ be the full subcategory of $\text{MOD-Z}A$ consisting of finitely generated free $\mathcal{A}$-modules.

Definition 5.9. Let $\mathcal{A}$ be a $\mathcal{Z}$-category. We call a sequence $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ in $\mathcal{A}$ exact at $A_1$, if $f_1 \circ f_0 = 0$ and, for every object $A$ and morphism $g: A \to A_1$ with $f_1 \circ g = 0$, there exists a morphism $\overline{g}: A \to A_0$ with $f_0 \circ \overline{g} = g$.

Lemma 5.10. If $\mathcal{A}$ is a $\mathcal{Z}$-category, the Yoneda embedding (5.8) induces an equivalence of $\mathcal{Z}$-categories denoted by the same symbol

$\iota: \mathcal{A} \to \text{MOD-Z}A_\mathcal{A}$.

If $\mathcal{A}$ is an additive category, the Yoneda embedding (5.8) induces an equivalence of additive categories denoted by the same symbol

$\iota: \mathcal{A} \to \text{MOD-Z}A_{\text{fg}}$.

Both functors are faithfully flat.

Proof. This follows directly from the Yoneda Lemma 5.1 and Lemma 5.4 (vi) \hfill \square

The gain of Lemma 5.10 is that we have embedded $\mathcal{A}$ as a full subcategory of the abelian category MOD-ZA and we can now do certain standard homological constructions in MOD-ZA, which a priori make no sense in $\mathcal{A}$.

The elementary proof of the following lemma based on Lemma 5.10 is left to the reader.

Lemma 5.11. An additive category $\mathcal{A}$ is idempotent complete, if and only if every finitely generated projective $\mathcal{Z}A$-module is a finitely generated free $\mathcal{Z}A$-module.

6. Regularity properties of additive categories

6.A. Definition of regularity properties in terms of the Yoneda embedding. Recall the following standard ring theoretic notions:

Definition 6.1 (Regularity properties of rings). Let $R$ be a ring and let $l$ be a natural number.

(i) We call $R$ Noetherian, if any $R$-submodule of a finitely generated $R$-module is again finitely generated;

(ii) We call $R$ regular coherent, if every finitely presented $R$-module $M$ is of type FP;

(iii) We call $R l$-uniformly regular coherent, if every finitely presented $R$-module $M$ admits an $l$-dimensional finite projective resolution, i.e., there exist an exact sequence $0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ such that each $P_i$ is finitely generated projective;
(iv) We call $R$ von Neumann regular, if for any element $r \in R$ there exists an element $s \in R$ with $r = rsr$;
(v) We call $R$ regular, if it is Noetherian and regular coherent;
(vi) We call $R$ $l$-uniformly regular, if it is Noetherian and $l$-uniformly regular coherent;
(vii) We say that $R$ has global dimension $\leq l$, if each $R$-module $M$ has projective dimension $\leq l$.

The notion von Neumann regular should not be confused with the notion regular. It stems from operator theory. A ring is von Neumann regular, and only if it is 0-uniformly regular coherent. For more information about von Neumann regular rings, see for instance [9, Subsection 8.2.2 on pages 325-327].

Let $\mathcal{A}$ be an additive category. Then we define analogously:

**Definition 6.2** (Regularity properties of additive categories). Let $\mathcal{A}$ be an additive category and let $l$ be a natural number.

(i) We call $\mathcal{A}$ Noetherian if the category $\text{MOD}_{\mathcal{A}}$ is Noetherian in the sense that any $\mathcal{A}$-submodule of a finitely generated $\mathcal{A}$-module is again finitely generated, see [13, p.18];
(ii) We call $\mathcal{A}$ regular coherent, if every finitely presented $\mathcal{A}$-module $M$ is of type FP;
(iii) We call $\mathcal{A}$ $l$-uniformly regular coherent, if every finitely presented $\mathcal{A}$-module $M$ possesses an $l$-dimensional finite projective resolution, i.e., there exist an exact sequence $0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ such that each $P_i$ is finitely generated projective;
(iv) We call $\mathcal{A}$ regular, if it is Noetherian and regular coherent;
(v) We call $\mathcal{A}$ $l$-uniformly regular, if is Noetherian and $l$-uniformly regular coherent;
(vi) We say that $\mathcal{A}$ has global dimension $\leq l$, if each $\mathcal{A}$-module $M$ has projective dimension $\leq l$, see [13, page 42].

### 6.3. The definitions of the regularity properties for rings and additive categories are compatible.

**Lemma 6.3.** Let $R$ be a ring. The functor

$$F: \text{R-MOD} \to \text{MOD}_{\mathcal{A}}$$

sending $M$ to $\text{hom}_R(\theta_{\text{id}}(-), M)$ is an equivalence of additive categories, is faithfully flat, and respects each of the properties finitely generated, free and projective, where the equivalence $\theta_{\text{id}}$ has been defined in (2.1)

**Proof.** In the sequel we denote by $[n]$ the $n$-fold direct sum in $\mathcal{A}_\oplus$ of the unique object in $\mathcal{A}$. Notice that $\theta([n]) = R^n$. Define

$$G: \text{MOD}_{\mathcal{A}} \to \text{R-MOD}$$

by sending $M$ to $M(\theta(1))$. There is a natural equivalence $G \circ F \to \text{id}_{\text{R-MOD}}$ of functors of additive categories, its value on the $R$-module $M$ is given by evaluating at $1 \in R = \theta([1])$,

$$G \circ F(M) = \text{hom}_R(\theta([1]), M) \xrightarrow{\cong} M.$$  

Next we construct an equivalence $S: F \circ G \to \text{id}_{\text{R-MOD}}$ of functors of additive categories. For a $\mathcal{A}$-module $N$ and objects $A_1, \ldots, A_n$, we obtain from Lemma 5.3

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2In [13, page 18] this is called left Noetherian; one obtains right Noetherian by working with $\mathcal{A}$-MOD in place of $\text{MOD}_{\mathcal{A}}$. 
a natural isomorphism
\[
\bigoplus_{i=1}^n N(\text{pr}_i) : \bigoplus_{i=1}^n N(A_i) \xrightarrow{\sim} N\left(\bigoplus_{i=1}^n A_i\right),
\]
where \(\text{pr}_j : \bigoplus_{i=1}^n A_i \to A_j\) is the canonical projection for \(j = 1, 2, \ldots, n\).

Recall that \([n]\) is the \(n\)-fold direct sum of copies of \([1]\), in other words, we have an identification \([n] = \bigoplus_{i=1}^n [1]\). It induces an isomorphism
\[
\bigoplus_{i=1}^n \theta([1]) \xrightarrow{\sim} \theta([n]).
\]

Given an object \([n]\) in \(R\) and an \(R\)-module \(M\), we define \(S(M)([n])\) by the following composite of \(R\)-isomorphisms
\[
F \circ G(M)([n]) = \text{hom}_R(\theta([n]), M(\theta(1))) \xrightarrow{\sim} \text{hom}_R\left(\bigoplus_{k=1}^n \theta([1]), M(\theta(1))\right)
\]
\[\xrightarrow{\sim} \bigoplus_{k=1}^n \text{hom}_R(\theta([1]), M(\theta(1))) = \bigoplus_{k=1}^n \text{hom}_R(R, M(\theta(1)))\]
\[\xrightarrow{\sim} \bigoplus_{k=1}^n M(\theta(1)) \xrightarrow{\sim} M\left(\bigoplus_{i=1}^n \theta([1])\right) = M(\theta([n])).\]

The functor \(F\) is faithfully exact, since for any object \([n]\) in \(R\) there is an \(R\)-isomorphism \(\bigoplus_{i=1}^n M \xrightarrow{\sim} F(M)([n])\), natural in \(M\). Since \(F\) is compatible with direct sums over arbitrary index sets and sends \(R\) to \(\text{hom}_R(\theta(-), R) = \text{mor}_R(?, [1])\), it respects the properties finitely generated, free and projective.

The following lemma implies in particular that the inclusion \(i : A \to \text{Idem}(A)\) induces equivalences
\[
\text{MOD-Z-A} \xrightarrow{i^*} \text{MOD-Z-Idem(A)}.
\]

**Lemma 6.4.** Let \(i : A \to A'\) be an inclusion of an additive subcategory \(A\) of the additive subcategory \(A'\), which is full and cofinal, for instance \(A \to A' = \text{Idem}(A)\). Then:

(i) If \(M\) is a \(ZA\)-module, then the adjoint
\[
\alpha(M) : M \xrightarrow{\sim} i^* i_* M
\]
of \(i_{1, M}\) under the adjunction (5.7) is an isomorphism of \(ZA\)-modules, natural in \(M\);

(ii) The restriction functor \(i^* : \text{MOD-Z-A} \to \text{MOD-Z-A}\) is faithfully flat. It sends a finitely generated \(ZA\)-module to a finitely generated \(ZA\)-module and a projective \(ZA\)-module to a projective \(ZA\)-module;

(iii) The induction functor \(i_* : \text{MOD-Z-A} \to \text{MOD-Z-A'}\) is faithfully flat. It sends a finitely generated \(ZA\)-module to a finitely generated \(ZA\)-module and a projective \(ZA\)-module to a projective \(ZA\)-module;

(iv) If \(M'\) is a \(ZA'\)-module, then the adjoint
\[
\beta(M') : i_* i^* M' \xrightarrow{\sim} M'
\]
of \(i_{1, M'}\) under the adjunction (5.7) is an isomorphism of \(ZA'\)-modules, natural in \(M'\);

(v) \(A\) is Noetherian, if and only \(A'\) is Noetherian;
Lemma 5.1. Since 

This implies that 

(i) The faithful flatness follows from assertions (i) and (ii). Since 

\[ M(?) \otimes_{\mathbb{Z},A} \text{mor}_{\mathbb{Z},A}(i(?)', i(?)) = M(?) \otimes_{\mathbb{Z},A} \text{mor}_{\mathbb{Z},A}(?, ?) \xrightarrow{\cong} M(?) \]

\[ x \otimes \phi \mapsto x\phi = M(\phi)(x). \]

(ii) Obviously \( i^* \) is flat.

Consider a sequence of \( \mathbb{Z},A' \)-modules \( M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \) such that restriction with \( i \) yields the exact sequence of \( \mathbb{Z},A \)-modules \( i^* M_0 \xrightarrow{i^* f_0} i^* M_1 \xrightarrow{i^* f_1} i^* M_2 \). We have to show for any object \( A' \in A' \) that the sequence of \( R \)-modules \( M_0(A') \xrightarrow{f_0(A')} M_1(A') \xrightarrow{f_1(A')} M_2(A') \) is exact. Since \( A \) is by assumption cofinal in \( A' \), we can find an object \( A \) in \( A \) and and morphisms \( j: A' \to i(A) \) and \( r: i(A) \to A' \) in \( A' \) satisfying \( r \circ i = \text{id}_{A'} \). We obtain the following commutative diagram of \( R \)-modules

\[
\begin{array}{c}
M_0(A') \xrightarrow{f_0(A')} M_1(A') \xrightarrow{f_1(A')} M_2(A') \\
\downarrow M_0(j) \quad \downarrow M_1(j) \quad \downarrow M_2(j) \\
M_0(i(A)) \xrightarrow{f_0(i(A))} M_1(i(A)) \xrightarrow{f_1(i(A))} M_2(i(A)) \\
\downarrow M_0(r) \quad \downarrow M_1(r) \quad \downarrow M_2(r) \\
M_0(A') \xrightarrow{f_0(A')} M_1(A') \xrightarrow{f_1(A')} M_2(A')
\end{array}
\]

such that the composite of the two vertical arrows appearing in each of the three columns is the identity. Since the middle horizontal sequence is exact, an easy diagram chase shows that the upper horizontal sequence is exact. This shows that \( i^* \) is faithfully flat.

Consider an object \( A' \) in \( A' \). Since \( A \) is by assumption cofinal in \( A \), we can find an object \( A \) in \( A \) and and morphism \( j: A' \to i(A) \) and \( q: i(A) \to A' \) in \( A' \) satisfying \( q \circ j = \text{id}_{A'} \). Composition with \( q \) and \( j \) yield maps of \( \mathbb{Z},A' \)-modules \( j: \text{mor}_{\mathbb{Z},A'}(?, ?, A') \to \text{mor}_{\mathbb{Z},A}(?, i(A)) \) and \( Q: \text{mor}_{\mathbb{Z},A}(?, i(A)) \to \text{mor}_{\mathbb{Z},A}(?, A') \) satisfying \( Q \circ j = \text{id}_{\text{mor}_{\mathbb{Z},A}(?, A')} \). If we apply \( i^* \), we obtain homomorphisms of \( \mathbb{Z},A' \)-modules \( i^* J: i^* \text{mor}_{\mathbb{Z},A'}(?, A') \to i^* \text{mor}_{\mathbb{Z},A}(?, i(A)) \) and \( i^* Q: i^* \text{mor}_{\mathbb{Z},A}(?, i(A)) \to i^* \text{mor}_{\mathbb{Z},A}(?, A') \) satisfying \( i^* Q \circ i^* J = \text{id}_{i^* \text{mor}_{\mathbb{Z},A}(?, A')} \). Since \( i^* \text{mor}_{\mathbb{Z},A}(?, i(A)) = \text{mor}_{\mathbb{Z},A}(i(?), i(A)) = \text{mor}_{\mathbb{Z},A}(?, A) \), the \( \mathbb{Z},A \)-module \( i^* \text{mor}_{\mathbb{Z},A}(?, A') \) is a direct summand in \( \text{mor}_{\mathbb{Z},A}(?, A) \) and hence a finitely generated projective \( \mathbb{Z},A \)-module.

Let \( M' \) be a finitely generated \( \mathbb{Z},A' \)-module. Fix an epimorphism \( \text{mor}_{\mathbb{Z},A}(?, A') \to M' \) for some object \( A' \) in \( A' \). We conclude that the \( \mathbb{Z},A \)-module \( i^* M \) is a quotient of \( \text{mor}_{\mathbb{Z},A}(?, A) \) for some object \( A \) in \( A \) and hence finitely generated. Hence \( i^* \) respects the property finitely generated.

Let \( P \) be a projective \( \mathbb{Z},A' \)-module. Then we can find a collection of objects \( \{ A'_k \mid k \in K \} \) together with an epimorphism \( \bigoplus_{k \in K} \text{mor}_{\mathbb{Z},A}(?, A'_k) \to P \) by the Yoneda Lemma. Since \( P \) is projective, \( P \) is a direct summand in \( \bigoplus_{k \in K} \text{mor}_{\mathbb{Z},A}(?, A'_k) \). This implies that \( i^* P \) is a direct summand in the direct sum \( \bigoplus_{k \in K} i^* \text{mor}_{\mathbb{Z},A}(?, A'_k) \) of projective \( \mathbb{Z},A \)-modules and hence itself a projective \( \mathbb{Z},A \)-module. Hence \( i^* \) respects the property projective.

(iii) The faithful flatness follows from assertions (i) and (ii). Since \( i_4 \text{mor}_{\mathbb{Z},A}(?, A) =
mor_\mathcal{A}(?, i(A)) holds for any object A in \mathcal{A}, the functor i_* respects the properties finitely generated and projective.

**(iv)** We begin with the case \( M = \text{mor}_\mathcal{A}(? ', i(A)) = i_* \text{mor}_\mathcal{A}(?, A) \) for some object A in \mathcal{A}. Then the claim follows from assertion (i) applied to the \( \mathbb{Z} \mathcal{A} \)-module \( \text{mor}_\mathcal{A}(?, A) \), since in this case \( \beta(M) = i_* \alpha(M) \). Consider an object \( A' \) in \( \mathcal{A}' \). Since \( \mathcal{A} \) is by assumption cofinal in \( \mathcal{A} \), we can find an object A in \( \mathcal{A} \) and a morphism \( j: A' \to i(A) \) and \( q: i(A) \to A' \) in \( \mathcal{A} \) satisfying \( q \circ j = \text{id}_{A'} \). Composition with j and q yield maps of \( \mathbb{Z} \mathcal{A}' \)-modules \( J: \text{mor}_\mathcal{A}(?, A') \to \text{mor}_\mathcal{A}(?, i(A)) \) and \( Q: \text{mor}_\mathcal{A}(?, i(A)) \to \text{mor}_\mathcal{A}(?, A') \) satisfying \( Q \circ J = \text{id}_{\text{mor}_\mathcal{A}(?, A')} \). Hence we get a commutative diagram of \( \mathbb{Z} \mathcal{A}' \)-modules

\[
\begin{array}{ccc}
\text{mor}_\mathcal{A}(?, A') & \xrightarrow{\beta(\text{mor}_\mathcal{A}(?, A'))} & \text{mor}_\mathcal{A}(?, A') \\
\downarrow{\text{mor}^*(J)} & & \downarrow{\text{mor}^*(J)} \\
\text{mor}_\mathcal{A}(?, i(A)) & \xrightarrow{\beta(\text{mor}_\mathcal{A}(?, i(A)))} & \text{mor}_\mathcal{A}(?, i(A)) \\
\downarrow{\text{mor}^*(Q)} & & \downarrow{\text{mor}^*(Q)} \\
\text{mor}_\mathcal{A}(?, A') & \xrightarrow{\beta(\text{mor}_\mathcal{A}(?, A'))} & \text{mor}_\mathcal{A}(?, A')
\end{array}
\]

such that the composite of the two vertical maps in each of the two columns is the identity and the middle arrow is an isomorphism. Hence the upper arrow is an isomorphism.

For any \( \mathbb{Z} \mathcal{A}' \)-module \( M' \) we can collect a collection of objects \( \{ A'_l \mid l \in L \} \) in \( \mathcal{A}' \) together with an epimorphism \( f_0: F_0 := \bigoplus_{l \in L} \text{mor}_\mathcal{A}(?, A'_l) \to M' \) by the Yoneda Lemma 5.1. Repeating this construction for \( \ker(f_0) \) instead of \( M \), we obtain another collection \( \{ A'_l \mid k \in K \} \) of objects in \( \mathcal{A}' \) together with a map \( f_1: F_1 := \bigoplus_{k \in K} \text{mor}_\mathcal{A}(?, A'_k) \to F_0 \) whose image is \( \ker(f_1) \). We obtain from assertions (ii) and (iii) a commutative diagram of \( \mathbb{Z} \mathcal{A}' \)-modules with exact rows

\[
\begin{array}{cccccc}
i_* i^* F_1 & \xrightarrow{i_* i^* f_1} & i_* i^* F_0 & \xrightarrow{i_* i^* f_0} & i_* i^* M' & \to 0 \\
\downarrow{\beta(F_1)} & & \downarrow{\beta(F_0)} & & \downarrow{\beta(M)} & \\
F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M' & \to 0.
\end{array}
\]

Since \( \beta \) is compatible with direct sums over arbitrary index sets, the maps \( \beta(F_1) \) and \( \beta(F_0) \) are isomorphisms. Hence \( \beta(M') \) is an isomorphism.

**(v)** (vi) and (vii) They follow now directly from assertions (i), (ii), (iii), and (iv).

We conclude from Lemma 6.3 and Lemma 6.4 (v), (vi) and (vii).

**Corollary 6.5.** Let \( R \) be a ring and let \( l \) be a natural number. Then the following assertions are equivalent:

(i) The ring \( R \) is Noetherian, regular coherent, \( l \)-uniformly regular coherent, regular, uniformly \( l \)-regular, or of global dimension \( \leq l \) in the sense of Definition 6.1 respectively;

(ii) The additive category \( \text{Idem}(R) \) is Noetherian, regular coherent, \( l \)-uniformly regular coherent, regular, uniformly \( l \)-regular, or of global dimension \( \leq d \) in the sense of Definition 6.2 respectively;

(iii) The additive category \( \text{Idem}(R) \) is Noetherian, regular coherent, \( l \)-uniformly regular coherent, regular, uniformly \( l \)-regular, or of global dimension \( \leq l \) in the sense of Definition 6.2 respectively.
6.c. Intrinsic definitions of the regularity properties. One can give an intrinsic definition of the regularity properties above without referring to the Yoneda embedding. The situation is quite nice for regular coherent and $l$-uniformly regular coherent for an idempotent complete additive category as explained below.

Lemma 6.6 (Intrinsic Reformulation of regular coherent). Let $\mathcal{A}$ be an idempotent complete additive category.

(i) Let $l \geq 2$ be a natural number. Then $\mathcal{A}$ is $l$-uniformly regular coherent, if and only if for every morphism $f_1: A_1 \to A_0$ we can find a sequence of length $l$ in $\mathcal{A}$

$$0 \to A_1 \xrightarrow{f_1} A_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact at $A_i$ for $i = 1, 2, \ldots, n$;

(ii) $\mathcal{A}$ is 1-uniformly regular coherent, if and only if for every morphism $f: A_1 \to A_0$ we can find a factorization $A_1 \xrightarrow{f_1} A \xrightarrow{f_0} A_0$ of $f$ such that $f_1$ is surjective and $f_0$ is injective;

(iii) The following assertions are equivalent:

(a) $\mathcal{A}$ is 0-uniformly regular coherent;

(b) For every morphism $f_1: A_1 \to A_0$ there exists a morphism $f_0: A_0 \to A_{l-1}$ such that $A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{l-1} \to 0$ is exact;

(c) For every morphism $f: A_1 \to A_0$ there exists a morphism $g: A_0 \to A_1$ satisfying $f \circ g \circ f = f$;

(iv) $\mathcal{A}$ is regular coherent, if and only if for every morphism $f_1: A_1 \to A_0$ we can find a sequence of finite length in $\mathcal{A}$

$$0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact at $A_i$ for $i = 1, 2, \ldots, n$.

Proof. It suffices to prove that the following statements are equivalent:

(a) For any morphisms $f_1: P_1 \to P_0$ of finitely generated projective $\mathbb{Z}\mathcal{A}$-modules, we can find finitely generated projective $\mathbb{Z}\mathcal{A}$-modules $P_2, P_3, \ldots, P_l$ and an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to P_1 \xrightarrow{f_1} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0;$$

(b) For any finitely presented $\mathbb{Z}\mathcal{A}$-module $M$, there exists finitely generated projective $\mathbb{Z}\mathcal{A}$-modules $P_0, P_1, \ldots, P_l$ and an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to P_1 \xrightarrow{f_1} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0.$$

The implication (b) $\implies$ (a) is obvious, since $\text{cok}(f_1)$ is a finitely presented $\mathbb{Z}\mathcal{A}$-module. It remains to prove the implication (a) $\implies$ (b). Let $f_1: P_1 \to P_0$ be a $\mathbb{Z}\mathcal{A}$-homomorphism of finitely generated projective $\mathbb{Z}\mathcal{A}$-modules. By assumption we can find an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to Q_l \xrightarrow{c_l} Q_{l-1} \xrightarrow{c_{l-1}} \cdots \xrightarrow{c_2} Q_1 \xrightarrow{c_1} Q_0 \xrightarrow{c_0} \text{cok}(f_1) \to 0.$$

Let $P_*$ be the $l$-dimensional $\mathbb{Z}\mathcal{A}$-chain complex whose first differential is $f_1$. Let $Q_*$ be the $l$-dimensional $\mathbb{Z}\mathcal{A}$-chain complex whose $i$th chain module is $Q_i$, for $0 \leq i \leq l$ and whose ith differential is $c_i: Q_i \to Q_{i-1}$ for $1 \leq i \leq l$. One easily constructs a $\mathbb{Z}\mathcal{A}$-chain map $u_*: P_* \to Q_*$ such that $H_l(u_*)$ is an isomorphism. Let $\text{cone}(u_*)$ be the mapping cone. We conclude $H_i(\text{cone}(u_*)) = 0$ for $i \neq 2$ from the long exact homology sequence associated to the exact sequence $0 \to P_* \xrightarrow{i_*} \text{cyl}(u_*) \xrightarrow{p_*} \text{cone}(u_*) \to 0$ and the fact that the canonical projection $q_*: \text{cyl}(u_*) \to Q_*$ is a $\mathbb{Z}\mathcal{A}$-chain homotopy equivalence with $q_* \circ i_* = u_*$. Let $D_* \subseteq \text{cone}(u_*)$ be the $\mathbb{Z}\mathcal{A}$-subchain complex, whose $i$-th chain module is $\text{cone}(u_*)$ for $i \geq 3$, the kernel
of the second differential of cone($u_*$) for $i = 2$ and $\{0\}$ for $i = 0, 1$. Then $D_i$ is finitely generated projective for $i \geq 0$ and the inclusion $k_*: D_* \to \text{cone}(u_*)$ induces isomorphisms on homology groups. Define the $\mathbb{Z}A$-chain complex $C_*$ by the pullback

$$
\begin{array}{ccc}
C_* & \xrightarrow{\iota_*} & D_* \\
\downarrow{k_*} & & \downarrow{k_*} \\
\text{cyl}(u_*) & \xrightarrow{p_*} & \text{cone}(u_*)
\end{array}
$$

This can be extended to a commutative diagram of $\mathbb{Z}A$-chain complexes with exact rows

$$
\begin{array}{ccc}
0 & \rightarrow & P_* & \xrightarrow{\iota_*} & C_* & \xrightarrow{\iota_*} & D_* & \rightarrow & 0 \\
\downarrow{\text{id}} & & \downarrow{k_*} & & \downarrow{k_*} & & \downarrow{k_*} & & \downarrow{\text{id}} \\
0 & \rightarrow & P_* & \rightarrow & \text{cyl}(u_*) & \rightarrow & \text{cone}(u_*) & \rightarrow & 0
\end{array}
$$

Then $C_*$ is an $l$-dimensional $\mathbb{Z}A$-chain complex whose $\mathbb{Z}A$-chain modules are finitely generated projective. Since $D_i = 0$ for $i = 0, 1$, we can identify $P_1 = C_1$ and $P_0 = C_0$ and the first differentials of $P_*$ and $C_*$. Since $k_*$ induces isomorphisms on homology, the same is true for $k_*$. Hence $C_*$ yields the desired extension of $f_1$ to an exact sequence

$$
0 \rightarrow C_1 \rightarrow C_{-1} \rightarrow \cdots \rightarrow C_2 \rightarrow P_1 \xrightarrow{f_1} P_0
$$

This finishes the proof of assertion [i].

(ii) Suppose that $\mathcal{A}$ is 1-uniformly regular coherent. Consider a morphism $f: A_1 \rightarrow A_0$. Let $M$ be the finitely presented $\mathbb{Z}A$-module given by the cokernel of the $\mathbb{Z}A$-homomorphism $\iota(f): \iota(A_1) \rightarrow \iota(A_0)$. By assumption we can find an exact sequence $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of $\mathbb{Z}A$-modules, where $P_1$ and $P_0$ are finitely generated projective. We conclude from Lemma 5.10(iv) that we obtain a short exact sequence $0 \rightarrow \iota(A_1) \rightarrow \iota(A_0) \rightarrow M \rightarrow 0$ such that $\iota(f)$ is a finitely generated projective $\mathbb{Z}A$-module, $f_1$ is surjective, and $f_0$ is injective. We conclude from Lemma 5.11 and Lemma 5.11 that $\text{im}(f)$ can be identified with $\iota(B)$ for some object $B$ in $\mathcal{A}$ and there are morphisms $f_1: A_1 \rightarrow B$ and $f_0: B \rightarrow A_1$ such that $f_1 = f_1 \circ f_0$ and $f_0 = f_1$. Moreover, $f_1$ is surjective, $f_0$ is injective and $f = f_0 \circ f_1$.

Suppose that for every morphism $f: A_1 \rightarrow A_0$ we can find a factorization $A_1 \xrightarrow{f_1} B \xrightarrow{f_0} A_0$ such that $f_1$ is surjective and $f_0$ is injective. Consider any finitely presented $\mathbb{Z}A$-module $M$. We conclude from Lemma 5.10 that there is a morphism $f: A_1 \rightarrow A_0$ in $\mathcal{A}$ and a morphism $p: \iota(A_0) \rightarrow M$ of $\mathbb{Z}A$-modules such that the sequence $\iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{p} M \rightarrow 0$ is exact. Choose a factorization $f = f_1 \circ f_0$ such that $f_1$ is surjective and $f_0$ is injective. Let $B$ be the domain of $f_1$. We conclude from Lemma 5.10 that we obtain a short exact sequence $0 \rightarrow \iota(B) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{p} M \rightarrow 0$. This is a 1-dimensional finite projection $\mathbb{Z}A$-resolution of $M$. This finishes the proof of assertion [ii].

(iii) We first show [iii]a $\Rightarrow$ [iii]c. Consider a morphism $f: A_1 \rightarrow A_0$. Let $M$ be the finitely presented $\mathbb{Z}A$-module given by the cokernel of $\iota(f): \iota(A_1) \rightarrow \iota(A_0)$. We obtain an exact sequence of $\mathbb{Z}A$-modules $\iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{p} M \rightarrow 0$. By assumption $M$ is a finitely generated projective $\mathbb{Z}A$-module. Let $\iota(f): \iota(A_1) \xrightarrow{p} \text{im}(\iota(f)) \xrightarrow{j} \iota(A_0)$ be the obvious factorization of $\iota(f)$. Since $M$ is projective, $\text{im}(f)$
is a direct summand in $\iota(A_0)$. We conclude from Lemma 5.10 and Lemma 5.11 that we can identify $\text{im}(\iota(f))$ with $\iota(B)$ for an appropriate object $B$ in $\mathcal{A}$ and can find morphisms $r: A_0 \to B$ and $s: B \to A_1$ in $\mathcal{A}$ such that $\iota(r) \circ f = \iota(s)$ and $q \circ s = \iota(R)$. Define $g: A_0 \to A_1$ by $g = s \circ r$. One easily checks that $\iota(f) \circ g \circ f = \iota(f)$. Hence $f \circ g \circ f = f$.

Next we show $\text{(iii)c} \implies \text{(iii)b}$ Let $f: A_1 \to A_0$ be a morphism in $\mathcal{A}$. Choose a morphism $h: A_0 \to A_1$ with $f \circ h \circ f = f$. Then $f \circ h: A_0 \to A_0$ is an idempotent. Since $\mathcal{A}$ is idempotent complete, we can find objects $A_{-1}$ and $A_1$ and an isomorphism $u: A_0 \cong A_{-1} \oplus A_1$ in $\mathcal{A}$ such that $u \circ (\text{id}_{A_0} - f \circ h) \circ u^{-1}$ is $\begin{pmatrix} \text{id}_{A_{-1}} & 0 \\ 0 & 0 \end{pmatrix}$.

Define $g: A_0 \to A_{-1}$ by the composite $A_0 \xrightarrow{u} A_{-1} \oplus A_1 \xrightarrow{\text{pr}_{A_{-1}}} A_{-1}$. One easily checks that the sequence $A_1 \xrightarrow{\text{id}} A_0 \xrightarrow{\text{id}} A_{-1} \to 0$ is exact.

Finally we show $\text{(iii)b} \implies \text{(iii)a}$ Consider a finitely presented $\mathbb{Z}\mathcal{A}$-module $M$. We conclude from Lemma 5.10 and Lemma 5.11 that we can find a morphism $f_1: A_1 \to A_0$ together with an exact sequence of $\mathbb{Z}\mathcal{A}$-modules $\iota(A_1) \xrightarrow{(f)} \iota(A_0) \xrightarrow{f_0} M \to 0$. Choose a morphism $f_0: A_0 \to A_{-1}$ such that the sequence $A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{-1} \to 0$ is exact in $\mathcal{A}$. Then we obtain an exact sequence of $\mathbb{Z}\mathcal{A}$-modules $\iota(A_1) \xrightarrow{(f_1)} \iota(A_0) \xrightarrow{(f_0)} \iota(A_{-1}) \to 0$ by Lemma 5.10. This implies that $M$ is $\mathbb{Z}\mathcal{A}$-isomorphic to $\iota(A_{-1})$ and hence finitely generated projective. This finishes the proof of assertion (iii).

(iv) This follows from assertion (iii). This finishes the proof of Lemma 6.6. $\square$

Next we deal with the property Noetherian. Consider two morphisms $f: A \to B$ and $f': A' \to B$. We write $f \subseteq f'$ if there exists a morphism $g: A \to A'$ with $f = f' \circ g$. Obviously we have

$$(6.7) \quad f \subseteq f' \iff \text{im}(f_\ast: \text{mor}_{\mathcal{A}}(?,A) \to \text{mor}_{\mathcal{A}}(?,?,B)) \subseteq \text{im}(f'_\ast: \text{mor}_{\mathcal{A}}(?,?,A') \to \text{mor}_{\mathcal{A}}(?,?,B)).$$

**Lemma 6.8 (Intrinsic Reformulation of Noetherian).** Let $\mathcal{A}$ be an additive category. Then the following assertions are equivalent:

(i) $\mathcal{A}$ is Noetherian;

(ii) Each object $A$ has the following property: Consider a sequence of morphisms $f_n: A_n \to A$ with fixed target $A$ and $f_n \subseteq f_{n+1}$ for $n \geq 0$. Then there exists $n_0$ such that $f_n \subseteq f_{n_0}$ holds for all $n \in \mathbb{N}$ with $n \geq n_0$.

**Proof.** Let $N$ be a finitely generated $\mathbb{Z}\mathcal{A}$-module and $M \subseteq N$ a $\mathbb{Z}\mathcal{A}$-submodule. Then there exists an object $A$ in $\mathcal{A}$ together with an epimorphism of $\mathbb{Z}\mathcal{A}$-module $u: \text{mor}_{\mathcal{A}}(\?,A) \to M$, see Lemma 5.3. Obviously $M$ is finitely generated if $u^{-1}(M)$ is finitely generated. Hence $\mathcal{A}$ is Noetherian if and only if for any object $A$ in $\mathcal{A}$ the $\mathbb{Z}\mathcal{A}$-module $\text{mor}_{\mathcal{A}}(\?,A)$ is Noetherian, i.e., any $\mathbb{Z}\mathcal{A}$-submodule $M$ of $\text{mor}_{\mathcal{A}}(\?,A)$ is finitely generated. By the usual argument $\text{mor}_{\mathcal{A}}(\?,A)$ is Noetherian if and only if for any nested sequence $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of finitely generated $\mathbb{Z}\mathcal{A}$-submodules of $\text{mor}_{\mathcal{A}}(\?,A)$ there exists a natural number $n_0$ such that $M_n \subseteq M_{n_0}$ holds for all $n \geq n_0$, see for instance [15] Proposition 0.2.17 on page 18.

Consider finitely generated $\mathbb{Z}\mathcal{A}$-submodules $M_0, M_1, M_2, \ldots$ of $\text{mor}_{\mathcal{A}}(\?,A)$. We can find for every natural number $n$ an object $A_n$ together with an epimorphism $\text{mor}_{\mathcal{A}}(\?,A_n) \to M_n$, see Lemma 5.3. By the Yoneda Lemma 5.1 there is a morphism $f_n: A_n \to A$ such that the image of $(f_n)_\ast: \text{mor}_{\mathcal{A}}(\?,A_n) \to \text{mor}_{\mathcal{A}}(\?,A)$ is $M_n$. Hence we get $M_n \subseteq M_0$ if and only if $(f_m)_\ast \subseteq (f_n)_\ast$ holds. Now Lemma 6.8 follows. $\square$
Lemma 6.9. Let $\mathcal{A}$ be a full additive subcategory of the additive category $\mathcal{B}$. If $\mathcal{B}$ is Noetherian, 0-uniformly regular coherent, or 0-uniformly regular, then $\mathcal{A}$ has the same property.

Proof. This follows from Lemma 6.3 [V] Lemma 6.6 [iii] and Lemma 6.8. □

7. Vanishing of Nil-terms

7.A. Nil-categories. The next definition is taken from [14] Definition 7.1.

Definition 7.1 (Nilpotent morphisms and Nil-categories). Let $\mathcal{A}$ be an additive category and $\Phi$ be an automorphism of $\mathcal{A}$.

(i) A morphism $f: \Phi(A) \to A$ of $\mathcal{A}$ is called $\Phi$-nilpotent, if for some $n \geq 1$, the $n$-fold composite

$$f^{(n)} := f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f): \Phi^n(A) \to A.$$ is trivial;

(ii) The category $\operatorname{Nil}(\mathcal{A}, \Phi)$ has as objects pairs $(A, \phi)$ where $\phi: \Phi(A) \to A$ is a $\Phi$-nilpotent morphism in $\mathcal{A}$. A morphism from $(A, \phi)$ to $(B, \mu)$ is a morphism $u: A \to B$ in $\mathcal{A}$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\Phi(A) & \xrightarrow{\phi} & A \\
\downarrow{\Phi(u)} & & \downarrow{u} \\
\Phi(B) & \xrightarrow{\mu} & B.
\end{array}$$

The category $\operatorname{Nil}(\mathcal{A}, \Phi)$ inherits the structure of an exact category from $\mathcal{A}$, a sequence in $\operatorname{Nil}(\mathcal{A}, \Phi)$ is declared to be exact if the underlying sequence in $\mathcal{A}$ is split exact.

Let $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ be an automorphism of an additive category $\mathcal{A}$. It induces an automorphism $\Phi^{-1*}: \operatorname{MOD-}\mathbb{Z}\mathcal{A} \xrightarrow{\cong} \operatorname{MOD-}\mathbb{Z}\mathcal{A}$ of abelian categories by precomposition with $\Phi^{-1}: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$. It sends $\operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf}$ to itself, since $\Phi^{-1*} \operatorname{mor}_{\mathcal{A}}(?, \Phi(A))$ is isomorphic to $\operatorname{mor}_{\mathcal{A}}(?, \Phi(A))$. Thus we obtain an automorphism of additive categories $\Phi^{-1*}: \operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf} \xrightarrow{\cong} \operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf}$

Lemma 7.2. There is an equivalence of exact categories

$$\iota: \operatorname{Nil}(\mathcal{A}; \Phi) \xrightarrow{\cong} \operatorname{Nil}(\operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf}; \Phi^{-1*})$$

Proof. The desired functors $\iota$ sends an object $(A, f)$ in $\operatorname{Nil}(\mathcal{A}; \Phi)$ given by a morphism $f: \Phi(A) \to A$ to the object in $\operatorname{Nil}(\operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf}; \Phi^{-1*})$ given by the composite

$$\Phi^{-1*} \operatorname{mor}_{\mathcal{A}}(?, A) = \operatorname{mor}_{\mathcal{A}}(\Phi^{-1}(?, A)) \xrightarrow{\Phi} \operatorname{mor}_{\mathcal{A}}(?, \Phi(A)) \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f)} \operatorname{mor}_{\mathcal{A}}(?, A).$$

A morphism $u: (A, f) \to (A', f')$ in $\operatorname{Nil}(\mathcal{A}; \Phi)$, which given by a morphism $u: A \to A'$ in $\mathcal{A}$ satisfying $f' \circ \Phi(u) = \Phi(u) \circ f$, is sent to the morphism in $\operatorname{Nil}(\operatorname{MOD-}\mathbb{A}_{\mathbb{Z}gf}; \Phi^{-1*})$ given by the morphism $u_{\Phi}: \operatorname{mor}_{\mathcal{A}}(?, A) \to \operatorname{mor}_{\mathcal{A}}(?, A')$. It defines indeed a morphism from $\iota(A, f)$ to $\iota(A', f')$ by the commutativity of the following diagram

$$\begin{array}{ccc}
\Phi(\Phi^{-1}(?, A)) & \xrightarrow{\Phi} & \Phi(?, \Phi(A)) \\
\downarrow{\Phi(\Phi^{-1}(?, u))} & & \downarrow{\Phi(?, \Phi(u))} \\
\Phi(\Phi^{-1}(?, A')) & \xrightarrow{\Phi} & \Phi(?, \Phi(A'))
\end{array}$$

$$\begin{array}{ccc}
\operatorname{mor}_{\mathcal{A}}(?, \Phi(A)) & \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f)} & \operatorname{mor}_{\mathcal{A}}(?, A) \\
\downarrow{\operatorname{mor}_{\mathcal{A}}(?, \Phi(u))} & & \downarrow{\operatorname{mor}_{\mathcal{A}}(?, u)} \\
\operatorname{mor}_{\mathcal{A}}(?, \Phi(A')) & \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f')} & \operatorname{mor}_{\mathcal{A}}(?, A')
\end{array}$$

It is an equivalence of additive categories by Lemma 5.10. □
7.3. Connective $K$-theory.

**Lemma 7.3.** Let $A$ be an idempotent complete additive category. Suppose that $A$ is regular coherent. Let $\Phi: A \xrightarrow{\cong} A$ be any automorphism of additive categories. Denote by $J: A \to \text{Nil}(A, \phi)$ the inclusion sending an object $A$ to the object $(A, 0)$.

Then the induced map on connective $K$-theory

$$K(J): K(A) \to K(\text{Nil}(A, \Phi))$$

is a weak homotopy equivalence.

**Proof.** We abbreviate $\Psi = \Phi^{-1}$. We have the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{J} & \text{Nil}(A, \Phi) \\
\downarrow{\iota} & & \downarrow{\iota} \\
\text{MOD-}ZA_{fgf} & \xrightarrow{J} & \text{Nil}(\text{MOD-}ZA_{fgf}, \Psi)
\end{array}
$$

where the vertical arrows are equivalences of exact categories given by Yoneda embeddings, see Lemma 5.10 and Lemma 7.2, and the lower horizontal arrow is the obvious analogue of the upper horizontal arrow. Hence it suffices to show that the map

$$K(J): K(\text{MOD-}ZA_{fgf}) \to K(\text{Nil}(\text{MOD-}ZA_{fgf}, \Psi))$$

is a weak homotopy equivalence.

Denote by $\text{MOD-}ZA_{FL}$ the full subcategory of $\text{MOD-}ZA$ consisting of $ZA$-modules which are of type FL, i.e., possess a finite dimensional resolution by finitely generated free $ZA$-modules.

Consider the following commutative diagram

$$
\begin{array}{ccc}
K(\text{MOD-}ZA_{fgf}) & \xrightarrow{} & K(\text{Nil}(\text{MOD-}ZA_{fgf}, \Psi)) \\
\downarrow & & \downarrow \\
K(\text{MOD-}ZA_{FL}) & \xrightarrow{} & K(\text{Nil}(\text{MOD-}ZA_{FL}, \Psi))
\end{array}
$$

where all arrows are induced by the obvious inclusions of categories.

The left vertical arrow in the diagram (7.4) is a weak homotopy equivalence by the Resolution Theorem, see [20, Theorem 4.6 on page 41].

Next we show that the lower horizontal arrow in the diagram (7.4) is a weak homotopy equivalence. Consider an object $(M, f)$ in $\text{Nil}(\text{MOD-}ZA_{FL}, \Psi)$.

Recall that nilpotent means that for some natural number $n \geq 0$ the composite

$$f^{(n)}: \Psi^n(M) \xrightarrow{\Psi^{n-1}(f)} \Psi^{n-1}(M) \xrightarrow{\Psi^{n-2}(f)} \cdots \xrightarrow{\Psi(f)} \Psi(M) \xrightarrow{f} M$$

is trivial. We get a filtration of $(M, f)$ by subobjects

$$(M, f) \supset (\text{im}(f), f|_{\text{im}(f)}) \supset (\text{im}(f^2), f|_{\text{im}(f^2)}) \supset \cdots \supset (\text{im}(f^{n-1}), f|_{\text{im}(f^{n-1})}) \supset (\text{im}(f^n), f|_{\text{im}(f^n)}) = (\{0\}, \text{id}_{\{0\}}).$$

Here we consider $\Psi(\text{im}(f^{(i)}))$ as a $ZA$-submodule of $\Psi(M)$ by the injective map $\Psi(\text{im}(f^{(i)})) \to \Psi(M)$, which is obtained by applying $\Psi$ to the inclusion $\text{im}(f^{(i)}) \to M$. We get exact sequences of $ZA$-modules

$$0 \to \text{im}(f^{(i)}) \to M \to M/\text{im}(f^{(i)}) \to 0;$$

$$0 \to \text{im}(f^{i+1}) \to \text{im}(f^i) \to \text{im}(f^i)/\text{im}(f^{i+1}) \to 0.$$
Since $M$ is finitely presented and $\text{im}(f^i)$ is finitely generated, $M/\text{im}(f^{(i)})$ is finitely presented. Since $A$ is regular coherent and idempotent complete by assumption, $M$ and $M/\text{im}(f^{(i)})$ for all $i$ are of type FL. We conclude by induction over $i = 0, 1, \ldots$ from Lemma 5.4 (iii) that $\text{im}(f^{(i)})$ and $\text{im}(f^{(i)})/\text{im}(f^{(i+1)})$ belong to $\text{MOD}-Z_{A_{FL}}$ again. The quotient of $(\text{im}(f^{(i)}), f|_{\Psi(\text{im}(f^{(i)}))})$ by $(\text{im}(f^{(i+1)}), f|_{\Psi(\text{im}(f^{(i+1)}))})$ is given by $(\text{im}(f^{(i)})/\text{im}(f^{(i+1)}), 0)$, and hence belongs to $\text{MOD}-Z_{A_{FL}}$ for all $i$. Now the lower horizontal arrow in diagram (7.4) is a weak homotopy equivalence by the Devissage Theorem, see [10] Theorem 4.8 on page 42.

Next we show that the right vertical arrow in the diagram (7.4) induces split injections on homotopy groups. For this purpose we consider the following commutative diagram of exact categories

\[
\begin{array}{ccc}
\text{Nil}(\text{MOD}-Z_{A_{gf}}, \Psi) & \xrightarrow{I_1} & \text{HNil}(\text{Ch}(\text{MOD}-Z_{A_{gf}}), \Psi) \\
\downarrow{I_2} & & \downarrow{I_4} \\
\text{HNil}(\text{Ch}_{\text{res}}(\text{MOD}-Z_{A_{gf}}), \Psi) & \xrightarrow{H_0} & \text{Nil}(\text{MOD}-Z_{A_{FL}}, \Psi)
\end{array}
\]

The category $\text{HNil}(\text{Ch}(\text{MOD}-Z_{A_{gf}}), \Psi)$ is given by finite-dimensional chain complexes $C_i$ over $\text{MOD}-Z_{A_{gf}}$ (with $C_i = 0$ for $i \leq -1$) together with chain maps $\phi: C_i \to C_{i-1}$, which are homotopy nilpotent, and $\text{HNil}(\text{Ch}_{\text{res}}(\text{MOD}-Z_{A_{gf}}), \Psi)$ is the full subcategory of $\text{HNil}(\text{Ch}(\text{MOD}-Z_{A_{gf}}), \Psi)$ consisting of those chain complexes, for which $H_i(C_*) = 0$ for $i \geq 1$. The maps $I_k$ for $k = 1, 2, 3, 4$ are the obvious inclusions, the functor $H_0$ is given by taking the zeroth homology group. The upper horizontal arrow induces a weak homotopy equivalence on connective $K$-theory by [14] page 173. The functor $H_0$ induces a weak homotopy equivalence on connective $K$-theory by the Approximation Theorem of Waldhausen, see for instance [14] Theorem 4.18. Hence the map induced by $I_3$ on connective $K$-theory, which is the right vertical arrow in the diagram (7.4), induces split injections on homotopy groups.

We conclude that all arrows appearing in the diagram (7.4) induce weak homotopy equivalences on connective algebraic $K$-theory. This finishes the proof of Lemma (7.3).

\[\text{Theorem 7.5 (The connective $K$-theory of additive categories).} \] Let $A$ be an additive category, which is idempotent complete and regular coherent. Consider any automorphism $\Phi: A \xrightarrow{\cong} A$ of additive categories.

Then we get a map of connective spectra

\[a: T_{K(\Phi^{-1})} \to K(A_\Phi[T, t^{-1}])\]

such that $\pi_n(a)$ is bijective for $n \geq 1$.

\[\text{Proof.} \] This follows from Theorem 4.3 since Lemma 5.3 implies $\pi_n(E(R, \Phi)) = 0$ for $n \geq 0$ and hence $\pi_n(NK(A_\Phi[t])) = \pi_n(NK(A_\Phi[t^{-1}])) = 0$ for all $n \geq 1$.

We will need later the following consequence of Lemma 7.3 where we can drop the assumption that $A$ is idempotent complete.

\[\text{Lemma 7.6.} \] Let $A$ be an additive category. Suppose that $A$ is regular coherent. Let $\Phi: A \xrightarrow{\cong} A$ be any automorphism of additive categories. Denote by $J: A \to \text{Nil}(A, \phi)$ the inclusion sending an object $A$ to the object $(A, 0)$. 

Then the induced map
\[ \pi_n(K(J)) : \pi_n(K(A)) \rightarrow \pi_n(K(\text{Nil}(A, \Phi))) \]
is bijective for \( n \geq 1 \).

**Proof.** We have the obvious commutative diagram coming from the inclusion \( A \rightarrow \text{Idem}(A) \).

\[ \begin{array}{ccc}
\pi_n(K(A)) & \longrightarrow & \pi_n(K(\text{Nil}(A, \Phi))) \\
\downarrow & & \downarrow \\
\pi_n(K(\text{Idem}(A))) & \longrightarrow & \pi_n(K(\text{Nil}(\text{Idem}(A)), \text{Idem}(\Phi))).
\end{array} \]

The left vertical arrow is bijective for \( n \geq 1 \) by Lemma 3.3 (i). The lower horizontal arrow is bijective for \( n \geq 1 \) by Lemma 7.3 since \( \text{Idem}(A) \) is regular coherent by Lemma 6.4 (vi). Hence we have to show that the right vertical arrow is bijective for \( n \geq 1 \). For this purpose it suffices to show because of Lemma 3.3 (i) that \( \text{Nil}(A, \Phi) \) is a cofinal full subcategory of \( \text{Nil}(\text{Idem}(A), \text{Idem}(\Phi)) \). This follows from the fact that \( A \) is a cofinal full subcategory of \( \text{Idem}(A) \). □

**7.c. Non-connective \( K \)-theory.** In the sequel define \( A[\mathbb{Z}^m] \) inductively over \( m \) by \( A[\mathbb{Z}^m] := A[\mathbb{Z}^{m-1}][t, t^{-1}] \), where \( A[\mathbb{Z}^{m-1}][t, t^{-1}] \) is the (untwisted) finite Laurent category associated to \( A[\mathbb{Z}^{m-1}] \) and the automorphism given by the identity, see Subsection 2.1.

**Lemma 7.7.** Let \( A \) be an additive category. Suppose that \( A[\mathbb{Z}^m] \) is regular coherent for every \( m \geq 0 \). Consider any automorphism \( \Phi : A \rightarrow A \) of additive categories. Denote by \( J : A \rightarrow \text{Nil}(A, \Phi) \) the inclusion sending an object \( A \) to the object \( (A, 0) \).

Then the induced map on non-connective \( K \)-theory
\[ K^\infty(J) : K^\infty(A) \rightarrow K^\infty_{\text{Nil}}(\text{Nil}(A, \Phi)) \]
is a weak homotopy equivalence.

**Proof.** Fix \( n \in \mathbb{Z} \). We have to show that \( \pi_n(K^\infty(J)) \) is bijective. This follows from Lemma 7.6 for \( n \geq 1 \) and is proved in general as follows.

From the definitions and the construction in [13, Section 6], one obtains for every \( n \in \mathbb{Z} \) a commutative diagram

\[ \begin{array}{ccc}
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{\text{Nil}}(A, \Phi)) \\
\downarrow i & & \downarrow j \\
\pi_{n+1}(K^\infty(A[\mathbb{Z}])) & \longrightarrow & \pi_{n+1}(K^\infty_{\text{Nil}}(A[\mathbb{Z}], \Phi[\mathbb{Z}]))) \\
\downarrow r & & \downarrow s \\
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{\text{Nil}}(A, \Phi))
\end{array} \]

where \( r \circ i = \text{id} \) and \( j \circ s = \text{id} \) and these maps are part of the corresponding (untwisted) Bass-Heller-Swan decompositions. Iterating this, one obtains for every
m \geq 0 a commutative diagram

\[
\begin{array}{ccc}
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{Nil}(A, \Phi)) \\
\downarrow s & & \downarrow j \\
\pi_{n+m}(K^\infty(A[Z^m])) & \longrightarrow & \pi_{n+m}(K^\infty_{Nil}(A[Z^m], \Phi[Z^m])) \\
\downarrow r & & \downarrow s \\
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{Nil}(A, \Phi))
\end{array}
\]

where \( r \circ i = id \) and \( j \circ s = id \) holds. Now choose \( m \) such that \( n + m \geq 1 \) holds. Then the middle horizontal arrow can be identified by construction with its connective version

\[
\pi_{n+m}(K(A[Z^m])) \rightarrow \pi_{n+m}(K(\text{Nil}(A[Z^m], \Phi[Z^m]))).
\]

Since this map is a bijection by Lemma \( \text{6.3} \) the upper horizontal arrow is a retract of an isomorphism and hence itself an isomorphism.

\[ \square \]

**Theorem 7.8** (The non-connective \( K \)-theory of additive categories). Let \( A \) be an additive category. Suppose that \( A[Z^m] \) is regular coherent for every \( m \geq 0 \). Consider any automorphism \( \Phi : A \xrightarrow{\sim} A \) of additive categories.

Then we get a weak homotopy equivalence of non-connective spectra

\[
a^\infty : T_{K^\infty(\Phi^{-1})} \xrightarrow{\simeq} K^\infty(A[t, t^{-1}]).
\]

**Proof.** This follows from Theorem \( \text{4.4} \) since Lemma \( \text{7.7} \) implies \( \pi_n(E^\infty(R, \Phi)) = 0 \) and hence \( \pi_n(NK^\infty(R, \Phi)) = 0 \) for all \( n \in \mathbb{Z} \).

\[ \square \]

8. **Noetherian additive categories**

**Theorem 8.1** (Hilbert Basis Theorem for additive categories).

Consider an additive category \( A \) together with an automorphism \( \Phi : A \xrightarrow{\sim} A \).

(i) If the additive category \( A \) is Noetherian, then the additive categories \( A[t], A_{\Phi}[t], A_{\Phi}[t^{-1}], A_{\Phi}[t, t^{-1}] \) are Noetherian.

(ii) If the additive category \( A_{\Phi}[t] \) is Noetherian, then the additive category \( A_{\Phi}[t, t^{-1}] \) is Noetherian.

**Proof.** \( \square \) We only treat \( A_{\Phi}[t] \), the proof for \( A_{\Phi}[t^{-1}] \) is analogous. For \( A_{\Phi}[t, t^{-1}] \) the claim will follow then from \( \square \).

We translate the usual proof of the Hilbert Basis Theorem for rings to additive categories. Consider a finitely generated \( \mathbb{Z}A_{\Phi}[t]-\)module \( N \) and a \( \mathbb{Z}A_{\Phi}[t]-\)submodule \( M \subseteq N \). We have to show that \( M \) is finitely generated. Lemma \( \text{5.4}(\text{vi}) \) implies that there is an epimorphisms \( \phi : \text{mor}_{A_{\Phi}[t]}(\?; A) \rightarrow N \) for some object \( A \). If \( \phi^{-1}(M) \) is finitely generated, then \( M \) is finitely generated, since \( f \) induces an epimorphism \( f^{-1}(M) \rightarrow M \). Hence we can assume without loss of generality \( N = \text{mor}_{A_{\Phi}[t]}(\?; A) \).

Fix an object \( Z \) in \( A \). Consider a non-trivial element \( f : Z \rightarrow A \) in \( N(Z) \). We can write it as a finite sum \( \sum_{k=0}^{d(f)} f_k : k^l \), where \( f_k : \Phi^k(Z) \rightarrow A \) is a morphism in \( A \) and \( f_d(f) \neq 0 \). We call the natural number \( d(f) \) the **degree of** \( f \) and \( R(f) = f_d(f) : \Phi^d(f)(Z) \rightarrow A \) the **leading coefficient of** \( f \). We put \( d(0) : Z \rightarrow A = -\infty \) and \( R(0) : Z \rightarrow A = 0 \).

We define now \( I_d \) as the \( \mathbb{Z}A \)-submodule of \( \text{mor}_{A}(\?; A) \) that is generated by all \( R(f) \) with \( f \in M(Z) \) and \( d(f) = d \) for some object \( Z \) from \( A \). We have \( I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \) and define \( I \) to be the \( \mathbb{Z}A \)-submodule \( \bigcup_{d \geq 0} I_d \). As \( A \) is by assumption Noetherian, \( I \) and all the \( I_d \) are finitely generated. Therefore we find a finite collection of morphisms \( f_i \in M(Z_i) \subseteq \text{mor}_{A_{\Phi}[t]}(Z_i, A) \) such that the \( R(f_i) \) generate \( I \). We abbreviate \( d_i := d(f_i) \). Since each \( f_i \) lies in of the \( I_d \)-s, we can find
a natural number $d_0$ such that $I = I_{d_0} = I_d$ holds for $d \geq d_0$. Hence we can also arrange for the $f_i$ to have the following property: for each $d$ the $R(f_i)$ with $d_0 \leq d$ generate $d_d$. We record that $R(f_i) \in I_d(\Phi^d(Z_i)) \subseteq \text{mor}_A(\Phi^d(Z_i), A)$.

We will show that the $f_i$ generate $M$. Let $f \in M(Z)$, $f \neq 0$. We abbreviate $d := d(f)$. We have $R(f) \in I_d(\Phi^d(Z)) \subseteq \text{mor}_A(\Phi^d(Z), A)$. We can write

$$R(f) = \sum_i R(f_i) \circ \varphi_i$$

with $\varphi_i \in \text{mor}_A(\Phi^d(Z), \Phi^d(Z_i))$ and $\varphi_i = 0$ whenever $d(f_i) > d(f)$. Set $\bar{\varphi}_i := \Phi^{-d_i}(\varphi_i) \cdot t^{d_i-d_i} \in \text{mor}_{A_0[t]}(Z, Z_i)$.

Then

$$R\left(\sum_i f_i \circ \bar{\varphi}_i\right) = \sum_i R(f_i) \circ \Phi^{-d_i}(\varphi_i) = \sum_i R(f_i) \circ \varphi_i.$$  

Thus $d(f - \sum_i f_i \circ \bar{\varphi}_i) < d$. Now we can repeat the argument for $f' := f - \sum_i f_i \circ \bar{\varphi}_i$. By induction on $d(f)$ we now find that $f$ belongs to the submodule of $\text{mor}_{A_0[t]}(Z, A)$ generated by the $f_i$. Hence $M$ is a finitely generated $Z_0[t]$-module.

[(iii)] It suffices to show for a $A_0[t, t^{-1}]$-module $M$ of $\text{mor}_{A_0[t, t^{-1}]}(Z, A)$ that $M$ is finitely generated as $A_0[t, t^{-1}]$-module. For $Z \in A$ we have $\text{mor}_{A_0[t]}(Z, A) \subseteq \text{mor}_{A_0[t, t^{-1}]}(Z, A)$. We define the $Z_0A_0[t]$-module $M'$ by

$$M'(Z) := M(Z) \cap \text{mor}_{A_0[t]}(Z, A).$$

Since $A_0[t]$ is Noetherian, we find a finite collection of morphisms $f_i \in M'(Z_i) \subseteq M(Z_i) \subseteq \text{mor}_{A_0[t, t^{-1}]}(Z_i, A)$ that generate $M'$ as a $Z_0A_0[t]$-module. We claim that the $f_i$ also generate $M$ as a $Z_0A_0[t, t^{-1}]$-module. For $d \geq 0$ we have $\text{id}_{Z} \cdot t^d \in \text{mor}_{A_0[t, t^{-1}]}(\Phi^{-d}(Z), Z)$. For sufficiently large $d$ we have $f \circ (\text{id}_{Z} \cdot t^d) \in \text{mor}_{A_0[t]}(\Phi^{-d}(Z), A) \cap M(\Phi^{-d}(Z)) = M'(\Phi^{-d}(Z))$. Thus $f \circ (\text{id}_{Z} \cdot t^d)$ belongs to the $Z_0A_0[t, t^{-1}]$-submodule of $M$ generated by the $f_i$. As $(\text{id}_{Z} \cdot t^d)$ is an isomorphism in $A_0[t, t^{-1}]$, $f$ also belongs to the $Z_0A_0[t, t^{-1}]$-submodule of $M$ generated by the $f_i$.

\[ \square \]

### 9. Additive categories with finite global dimension

Let $\Phi: A \to A$ be an automorphism of the additive category $A$. Let $\Phi^*: A_0[t] \to A_0[t]$ be the automorphism of additive categories induced by $\Phi$, which sends the morphisms $\sum_{k=0}^\infty f_k \cdot t^k : A \to B$ to the morphism $\sum_{k=0}^\infty (\Phi(f_k)) \cdot t^k : \Phi(A) \to \Phi(B)$. Denote by $i: A \to A_0[t]$ the inclusion sending $f: A \to B$ to $(f \cdot t^0): A \to B$.

Obviously we have $\Phi \circ i = i \circ \Phi$.

#### 9.A. The characteristic sequence.

Consider a $Z_0A_0[t]$-module $M$. Let

$$\alpha: i_*i^*M \to M$$

be the morphism, which is the adjoint of the $Z_0A_0$-homomorphism $\id: i^*M \to i^*M$ under the adjunction $(\mathbf{5.7})$. We get for every object $A$ in $A$ a morphism $\id_{\Phi(A)} \cdot t: A \to \Phi(A)$ in $A_0[t]$. It induces a $Z_0A_0[t]$-morphism $\id_{\Phi(A)} \cdot t: \Phi(A) \to M(A)$. Since for a morphisms $u: A \to B$ in $A$ we have

$$(\id_{\Phi(B)} \cdot t) \circ i(u) = (\id_{\Phi(B)} \cdot t) \circ (u \cdot t^0) = \Phi(u) \cdot t = (\Phi(u) \cdot t^0) \circ (\id_{\Phi(A)} \cdot t) = i(\Phi(u)) \circ (\id_{\Phi(A)} \cdot t),$$

we obtain a morphism of $Z_0A$-modules

\[ (9.1) \alpha^*: \Phi^*i_*M \xrightarrow{\cong} i_*i^*M. \]
By applying $i_*$ we obtain a morphism of $\mathbb{Z}\mathcal{A}_\Phi[t]$-modules
\[ \alpha: i_*\Phi^*i^*M \rightarrow i_*i^*M. \]
The morphism $\text{id}_{\Phi(A)} \cdot t: A \rightarrow \Phi(A)$ in $\mathcal{A}_\Phi[t]$ induces also a $\mathbb{Z}$-map
\[ \beta(A): i_*\Phi^*i^*M(A) \rightarrow i_*i^*M(A). \]
Since for any morphism $v = \sum_{k=0}^{\infty} f_k \cdot t^k: A \rightarrow B$ in $\mathcal{A}_\Phi[t]$ we have
\[ (\text{id}_{\Phi(B)} \cdot t) \circ v = (\text{id}_{\Phi(B)} \cdot t) \circ \left( \sum_{k=0}^{\infty} f_k \cdot t^k \right) \]
\[ = \sum_{k=0}^{\infty} (\text{id}_{\Phi(B)} \cdot t) \circ (f_k \cdot t^k) \]
\[ = \sum_{k=0}^{\infty} \Phi(f_k) \cdot t^{k+1} \]
\[ = \sum_{k=0}^{\infty} (\Phi(f_k) \cdot t^k) \circ (\text{id}_{\Phi(A)} \cdot t) \]
\[ = \Phi \left( \sum_{k=0}^{\infty} f_k \cdot t^k \right) \circ (\text{id}_{\Phi(A)} \cdot t) \]
\[ = \Phi(v) \circ (\text{id}_{\Phi(A)} \cdot t), \]
we get a $\mathbb{Z}\mathcal{A}_\Phi[t]$-homomorphism denoted by
\[ \beta: i_*\Phi^*i^*M \rightarrow i_*i^*M. \]
Define the so called characteristic sequence of $\mathbb{Z}\mathcal{A}_\Phi[t]$-modules by
\[ 0 \rightarrow i_*\Phi^*i^*M \xrightarrow{\alpha - \beta} i_*i^*M \xrightarrow{\epsilon} M \rightarrow 0. \]
Given an object $A \in \mathcal{A}$, $(\alpha - \beta)(A)$ is explicitly given by
\[ M(\Phi(?)) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}_\Phi[t]}(A, ?) \rightarrow M(?) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}_\Phi[t]}(A, ?), \]
\[ x \otimes (f_k \cdot t^k: A \rightarrow ?) \mapsto M(\text{id}_{\Phi(?)} \cdot t: ?) \rightarrow \Phi(?)(x) \otimes (f_k \cdot t^k: A \rightarrow ?) \]
\[ - x \otimes (\Phi(f_k) \cdot t^{k+1}: A \rightarrow \Phi(?)), \]
and $\epsilon(A)$ is explicitly given by
\[ M(?) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}_\Phi[t]}(A, ?) \rightarrow M(A), \quad x \otimes (u: A \rightarrow ?) \mapsto M(u)(x) = xu. \]

**Lemma 9.3.** The characteristic sequence (9.2) is natural in $M$ and exact.

**Proof.** It is obviously natural in $M$. To prove exactness, it suffices to prove the exactness of the sequence of $\mathbb{Z}\mathcal{A}$-modules
\[ 0 \rightarrow i^*i_*\Phi^*i^*M \xrightarrow{\alpha - \beta} i^*i_*i^*M \xrightarrow{\epsilon} i^*M \rightarrow 0. \]
Let $N$ be a $\mathbb{Z}\mathcal{A}$-module. We obtain a $\mathbb{Z}\mathcal{A}$-isomorphism
\[ S(N): \bigoplus_{k=0}^{\infty} \Phi^k(N) \xrightarrow{\cong} i^*i_*N, \]
which is defined for an object $A$ in $\mathcal{A}$ by the $\mathbb{Z}$-isomorphism
\[ S(N)(A): \bigoplus_{k=0}^{\infty} N(\Phi^k(A)) \xrightarrow{\cong} i^*i_*N(A) = N(?) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}_\Phi}(i(A), ?) \].
sending \((x_k)_{k \geq 0}\) to \(\sum_{k=0}^{\infty} x \otimes (\text{id}_{\Phi^k(A)} \cdot t^k : A \to \Phi^k(A))\). The inverse of \(S(N)(A)\) sends \(y \otimes \left(\sum_{k=0}^{\infty} f_k \cdot t^k : A \to \Phi(A)\right)\) to \(\sum_{k=0}^{\infty} N(f_k)(y)\). Applying this to \(N = i^* i^*N = i^* i^*M\) and \(N = i^* M\), we get identifications

\[
i^* i^* i^* \Phi M = \bigoplus_{k=1}^{\infty} (\Phi^k)^* i^* M;
\]

\[
i^* i^* i^* M = \bigoplus_{k=0}^{\infty} (\Phi^k)^* i^* M.
\]

Consider natural numbers \(m\) and \(n\) with \(m \geq n\). For an object \(A\) let the map

\[
s_{m,n}(A) : (\Phi^m)^* M(A) \to (\Phi^n)^* M(A)
\]

be the map obtained by applying \(M\) to the morphism \(\text{id}_{\Phi^m(A)} : (\Phi^n)(A) \to (\Phi^m)(A)\) in \(\mathcal{A}_\Phi[t]\). This yields a map of \(\mathcal{A}\)-modules.

Under these identifications the \(\mathcal{A}\)-sequence \((\ref{sequence})\) becomes the sequence

\[
0 \to \bigoplus_{m=1}^{\infty} (\Phi^m)^* i^* M \xrightarrow{(\text{id} \ s_{1,0} \ s_{2,0} \cdots)} \bigoplus_{n=0}^{\infty} (\Phi^n)^* i^* M \to i^* M \to 0.
\]

Since \(s_{m,n} \circ s_{l,m} = s_{l,n}\) for \(l \geq m \geq n\) and \(s_{m,m} = \text{id}\) hold, this sequence is split exact, with a splitting given by

\[
\bigoplus_{m=1}^{\infty} (\Phi^m)^* M \xleftarrow{\bigoplus_{n=0}^{\infty} (\Phi^n)^* M} \bigoplus_{n=0}^{\infty} (\Phi^n)^* i^* M \to i^* M \to 0.
\]

\(\Box\)

### 9.3. Localization.

**Definition 9.6 (Local module).** We call a \(\mathcal{A}_\Phi[t]\)-module \(M \) local, if for any object \(A\) in \(\mathcal{A}\) and any natural number \(k \in \mathbb{N}\) the map

\[
M(\text{id}_{\Phi^k(A)} \cdot t^k) \colon M(A) \to M(\Phi^k(A))
\]

induced by the morphism \(\text{id}_{\Phi^k(A)} \cdot t^k : A \to \Phi^k(A)\) in \(\mathcal{A}_\Phi[t]\) is bijective.

Let \(j : \mathcal{A}_\Phi[t] \to \mathcal{A}_\Phi[t, t^{-1}]\) be the inclusion.

**Lemma 9.7.** A \(\mathcal{A}_\Phi[t]\)-module \(M\) is local, if and only if there is a \(\mathcal{A}_\Phi[t, t^{-1}]\)-module \(N\) such that \(M\) and \(j^* N\) are isomorphic as \(\mathcal{A}_\Phi[t]\)-modules.

**Proof.** Since the morphism \(\text{id}_{\Phi^k(A)} \cdot t^k : A \to \Phi^k(A)\) in \(\mathcal{A}_\Phi[t]\) becomes invertible when considered in \(\mathcal{A}_\Phi[t, t^{-1}]\), a \(\mathcal{A}_\Phi[t]\)-module \(M\) is local, if there is a \(\mathcal{A}_\Phi[t, t^{-1}]\)-module \(N\) such that \(M\) and \(j^* N\) are isomorphic as \(\mathcal{A}_\Phi[t, t^{-1}]\)-modules.
Now consider a local $\mathcal{Z}A_\Phi[t]$-module $M$. We have to explain how the $\mathcal{Z}A_\Phi[t]$-structure extends to a $\mathcal{Z}A_\Phi[t, t^{-1}]$-structure. Consider a morphism $u: A \to B$ in $A_\Phi[t, t^{-1}]$. Then we can choose a natural number $m$ such that the composite $A \xrightarrow{u} B \xrightarrow{\text{id}_m \circ \Phi^m} \Phi^m(B)$ is a morphism in $A_\Phi[t]$. Hence we have the $\mathcal{Z}$-map $M((\text{id}_m \circ \Phi^m) \circ u): M(\Phi^m(B)) \to M(A)$. Since $M$ is local, the $\mathcal{Z}$-map $M(\text{id}_m \circ \Phi^m): M(\Phi^m(B)) \to M(B)$ is an isomorphism. Now define

$$
M(u): M(B) \xrightarrow{M(\text{id}_m \circ \Phi^m)} M(\Phi^m(B)) \xrightarrow{M((\text{id}_m \circ \Phi^m) \circ u)} M(A).
$$

We leave it to the reader to check that the definition of $M(u)$ is independent of the choice of $m$ and that we obtain the desired $\mathcal{Z}A_\Phi[t, t^{-1}]$-structure on $M$ extending the given $\mathcal{Z}A_\Phi[t]$-structure. □

Let $M$ be a $\mathcal{Z}A_\Phi[t]$-module. We want to assign to it a $\mathcal{Z}A_\Phi[t, t^{-1}]$-module $S^{-1}M$ as follows. Consider an object $A$ in $\mathcal{A}$. Define the abelian group

$$
S^{-1}M(A) := \{(l, x) \mid l \in \mathbb{Z}, x \in M(\Phi^l(A))\}/\sim
$$

for the equivalence relation $\sim$, where $(l_0, x_0)$ and $(l_1, x_1)$ are equivalent, if and only if there is an integer $l \in \mathbb{Z}$ with $l \leq l_0, l_1$ such that the elements $M(\text{id}_{\Phi^l}(A)^{(t^{-l})}(x_0))$ and $M(\text{id}_{\Phi^l}(A)^{(t^{-l})}(x_0))$ of $M(\Phi^l(A))$ agree. Given a morphism $u: A \to B$ in $A_\Phi[t, t^{-1}]$, we can choose a natural number $m$ such that the composite $A \xrightarrow{u} B \xrightarrow{\text{id}_m \circ \Phi^m} \Phi^m(B)$ is a morphism in $A_\Phi[t]$. Define $S^{-1}(M)(u): S^{-1}(M)(B) \to S^{-1}(M)(A)$ by sending $(l, x)$ to the class of $(l - m, M(\Phi^{-m}(\text{id}_m \circ \Phi^m) \circ u)(x))$. This is independent of the choice of the representative of $(l, x)$, since we get for the different representative $(l - 1, M(\text{id}_m \circ \Phi^1) \circ u)(x))$

$$
S^{-1}(M)([l - 1, M(\text{id}_m \circ \Phi^1) \circ u](x))
$$

This is independent of the choice of $m$ by the following calculation

$$
[l - (m + 1), M(\Phi^{-(m+1)}(\text{id}_m \circ \Phi^m) \circ u)](x)
$$

This is independent of the choice of $m$ by the following calculation

$$
[l - (m + 1), M(\Phi^{-(m+1)}(\text{id}_m \circ \Phi^m) \circ u)](x)
$$
We leave it to the reader to check that $S^{-1}M(v \circ u) = S^{-1}M(u) \circ S^{-1}M(v)$ holds for any two composable morphisms $u: A \to B$ and $v: B \to C$ in $\mathcal{A}_\Phi[t, t^{-1}]$ and $S^{-1}M(1_A) = 1_{S^{-1}M(A)}$ holds for any object $A$ in $\mathcal{A}$. Note that the $\mathcal{A}_\Phi[t]$-module $j^*S^{-1}M$ is local by Lemma 9.7.

There is a natural map of $\mathcal{A}_\Phi[t]$-modules

$$I: M \to j^*S^{-1}M,$$

which is given for an object $A$ of $\mathcal{A}$ by the map $I(A): M(A) \to S^{-1}M(A)$ sending $x$ to $(0, x)$. We claim that $I$ is a localization in the sense that for any local $\mathcal{A}_\Phi[t]$-module $N$ and any $\mathcal{A}_\Phi[t]$-homomorphism $f: M \to N$ there exists precisely one $\mathcal{A}_\Phi[t]$-homomorphism $S^{-1}f: S^{-1}M \to N$.

Firstly we explain that there is at most one such map $S^{-1}f$ with these properties. Namely, consider an object $A \in \mathcal{A}$ and an element $[m, x] \in S^{-1}(M)(A)$. If $m \geq 0$, then we compute

$$S^{-1}f(A)([m, x]) = S^{-1}(A)([0, M(id_{\Phi^n(A)} \cdot t^m)(x)]) = S^{-1}(A) \circ I(A) \circ M(id_{\Phi^n(A)} \cdot t^m)(x) = f(A) \circ M(id_{\Phi^n(A)} \cdot t^m)(x).$$

Suppose $m \leq 0$. Since we have $S^{-1}(M)(id_A \cdot t^{-m})([m, x]) = [0, x]$, we compute for $[m, x] \in S^{-1}(M)(A)$

$$S^{-1}(N)(id_A \cdot t^{-m}) \circ S^{-1}f(A)([m, x]) = S^{-1}(f(\Phi^m(A)) \circ S^{-1}(M)(id_A \cdot t^{-m})([m, x]) = S^{-1}f(\Phi^m(A))([0, x]) = S^{-1}(f(\Phi^m(A)) \circ I(A)(x)) = f(\Phi^m(A))(x).$$

Since the locality of $N$ implies that $S^{-1}(N)(id_{\Phi^n(A)} \cdot t^m)$ is an isomorphism, we conclude

$$S^{-1}f(A)([m, x]) = S^{-1}(N)(id_A \cdot t^{-m})^{-1} \circ f(\Phi^m(A))(x).$$

Hence $S^{-1}f(A)$ is determined by the equations (9.8) and (9.9). We leave it to the reader to check that it makes sense to define the desired $\mathcal{Z}_\mathcal{A}[t]$-homomorphism $S^{-1}f(A)$ by the equations (9.8) and (9.9).

The adjoint of $I: M \to j^*S^{-1}M$ under the adjunction (5.7) is denoted by

$$\alpha: j_*M \to S^{-1}M.$$ (9.10)

The adjoint of $id_{j_*M}$ under the adjunction (5.7) is the $\mathcal{Z}_\mathcal{A}_\Phi[t]$-homomorphism

$$\lambda: M \to j^*j_*M,$$ (9.11)

which is explicitly given by $M(?) \to \text{mor}_{\mathcal{A}_\Phi[t, t^{-1}]}(?, ?) \otimes_{\mathcal{Z}_\mathcal{A}_\Phi[t]} M(?)$ sending $u \in M(?)$ to $id_{id} \otimes u$. Given an $\mathcal{Z}_\mathcal{A}_\Phi[t, t^{-1}]$-module $N$, the adjoint of $id_{j_*N}$ under the adjunction (5.7) is the $\mathcal{Z}_\mathcal{A}_\Phi[t, t^{-1}]$-homomorphism

$$\rho: j_*j^*N \to N,$$ (9.12)

which is explicitly given by $N(?) \otimes_{\mathcal{Z}_\mathcal{A}_\Phi[t]} \text{mor}_{\mathcal{A}_\Phi[t, t^{-1}]}(?, ?) \to N(??)$ sending $x \otimes u$ to $N(u)(x) = xu$.

Lemma 9.13. (i) The $\mathcal{Z}_\mathcal{A}_\Phi[t]$-homomorphism $\lambda: M \to j^*j_*M$ of (9.11) is a localization;

(ii) The $\mathcal{Z}_\mathcal{A}_\Phi[t, t^{-1}]$-homomorphism $\alpha: j_*M \to S^{-1}M$ of (9.10) is an isomorphism, which is natural in $M$;

(iii) Let $N$ be a $\mathcal{Z}_\mathcal{A}_\Phi[t, t^{-1}]$-module. Then the $\mathcal{Z}_\mathcal{A}_\Phi[t, t^{-1}]$-map $\rho: j_*j^*N \to N$ of (9.12) is an isomorphism.
Proof. (i) Let \( f: M \to N \) be a \( \mathbb{Z}_A[t] \)-map with a local \( \mathbb{Z}_A[t] \)-module as target. Because of Lemma [9.14] there is a \( \mathbb{Z}_A[t, t^{-1}] \)-module \( N' \) and a \( \mathbb{Z}_A[t] \)-isomorphism \( u: N \to j^*N' \). Let the \( \mathbb{Z}_A[t, t^{-1}] \)-map \( v_l: J^*_lM \to N \) be the adjoint of \( u \circ f \) under the adjunction \([5.7]\). Because of the naturality of the adjunction \([5.7]\) we get for the composite \( J^*_lM \to J^*N' \) \( \alpha \mapsto u_\lambda \) that \( \overline{f} \circ \alpha = f \) holds. We conclude that \( \overline{f} \) is uniquely determined by \( \overline{f} \circ \alpha = f \) from the explicit description of \( \lambda \) and from the fact that for any morphism \( u: A \to B \) in \( \mathbb{Z}_A[t, t^{-1}] \) there is a natural number such that the composite of \( u \) with \( \id_{\Phi^m(B)} \) \( \Phi^m(M) \to \Phi^m(B) \) lies in \( \mathbb{A}_\Phi[t] \).

(ii) Obviously \( \alpha \) is natural in \( M \). The naturality of the adjunction \([5.7]\) implies

\[ j^*\alpha \circ \lambda = I. \]

Since both \( J: M \to j^*S^{-1}M \) and \( \lambda: M \to j^*_lM \) are localizations, \( j^*\alpha \) and hence \( \alpha \) are bijective.

(iii) It suffices to show that \( j^*\rho: j^*j_\ast N \to j^*N \) is bijective. Assertion \([i]\) applied to \( j^*N \) and the naturality of the adjunction \([5.7]\) imply that \( j^*\rho: j^*N \to j^*j_\ast N \) is a localization. Since \( \id_{j^*N} : j^*N \to j^*N \) is a localization, \( j^*\rho \) is an isomorphism. \( \square \)

Lemma 9.14. The functor \( j_*: \text{MOD-}\mathbb{Z}_A[t] \to \text{MOD-}\mathbb{Z}_A[t, t^{-1}] \) is flat.

Proof. Because of the adjunction \([5.7]\) the functor \( j_* \) is right exact by a general argument, see [25, Theorem 2.6.1. on page 51]. Hence it remains to show that \( S^{-1}i: S^{-1}M \to S^{-1}N \) is injective. Consider an object \( A \) in \( \mathbb{A} \) and an element \([l, x]\) in the kernel of \( S^{-1}(i)(A) \). Since \( S^{-1}(l)(x) = [l, i(\Phi^l(A))((x))] \), there is a natural number \( m \leq l \) such that \( N(\id_{\Phi^l(A)} t^{m-l}) = i(\Phi^l(A))((x)) = 0 \). Since \( N(\id_{\Phi^l(A)} t^{m-l}) \circ i(\Phi^l(A)) = i(\Phi^{m-l}(A)) M(\id_{\Phi^l(A)} t^{m-l}) \) and \( i(\Phi^{m-l}(A)) \) is by assumption injective, \( M(\id_{\Phi^l(A)} t^{m-l})(x) = 0 \). This implies \([l, x] = 0 \). \( \square \)

9.c. Global dimension. Recall that an additive category \( \mathbb{A} \) has global dimension \( \leq d \), if the abelian category \( \text{MOD-}\mathbb{Z}_A \) has global dimension \( \leq d \), i.e., if every \( \mathbb{Z}_A \)-module has a projective \( d \)-dimensional resolution, see Definition \([62]\).

Theorem 9.15 (Global dimension and the passage from \( \mathbb{A} \) to \( \mathbb{A}_\Phi[t] \)). Let \( \mathbb{A} \) be an additive category and \( \Phi: \mathbb{A} \to \mathbb{A} \) be an automorphism of additive categories.

(i) Let \( M \) be a \( \mathbb{Z}_A[t] \)-module. If \( \pdim_A(\Phi^d) \leq d \), then \( \pdim_{\mathbb{A}} M(\Phi^d) \leq d+1 \);

(ii) If \( \mathbb{A} \) has global dimension \( \leq d \), then \( \mathbb{A}_\Phi[t] \) has global dimension \( \leq (d+1) \).

Theorem \([9.15]\) is a version of the Hilbert syzygy theorem. Its proof is not much different from the classical syzygy Theorem for rings. For a more general version see [15, Corollary 31.1 on page 119].

Proof of Theorem \([9.15]\). Obviously \( \Phi^*: \text{MOD-}\mathbb{Z}_A[t] \to \text{MOD-}\mathbb{Z}_A \) is faithfully flat and is compatible with direct sums over arbitrary index sets. Next we show that \( \Phi^* \) sends projective \( \mathbb{Z}_A[t] \)-modules to projective \( \mathbb{Z}_A \)-modules. It suffices to show that \( \Phi^* \operatorname{mor}_A(?, A) \cong \Phi^* \operatorname{mor}_A(?, A) \) is free as a \( \mathbb{Z}_A \)-module for any object \( A \). This follows from the \( \mathbb{Z}_A \)-isomorphism \([9.15]\), since \( \Phi^* \operatorname{mor}_A(?, A) \cong \operatorname{mor}_A(?, \Phi^k(A)) \).

The functor \( \Phi^*: \text{MOD-}\mathbb{Z}_A \to \text{MOD-}\mathbb{Z}_A[t] \) is compatible with direct sums over arbitrary index sets, is right exact and sends \( \operatorname{mor}_A(?, A) \) to \( \operatorname{mor}_{\mathbb{Z}_A[t]}(?, A) \). In particular \( \Phi^* \) respects the properties finitely generated, free, and projective. Next we want to show that \( \Phi^* \) is faithfully flat. For this purpose it suffices to show that \( \Phi^* \circ \Phi^* \) is faithfully flat. This is obvious since \( \Phi^* \circ \Phi^* \) is the functor sending a morphism \( f: M \to N \) to the morphism \( \bigoplus_{k \in \mathbb{N}} (\Phi^k)^*(f): \bigoplus_{k \in \mathbb{N}} (\Phi^k)^*(M) \to \bigoplus_{k \in \mathbb{N}} (\Phi^k)^*(f) \) under the identification \([0.3] \).
Now consider a $\mathbb{Z}_\Phi[t]$-module $M$ with $\text{pd}_A(i^*M) \leq d$. Since the $\mathbb{Z}_A$-modules $i^*M$ and $\Phi^*i^*M$ are isomorphic, see [9.1], we get $\text{pd}_A(\phi^*i^*M) \leq d$. Since $i_*$ is faithfully flat and respects projective modules, we conclude $\text{pd}_{A_{\Phi}[t]}(i_*i^*M) \leq d$ and $\text{pd}_{A_{\Phi}[t]}(i_*\Phi^*i^*M) \leq d$. Now Lemma 5.3(v) and Lemma 9.3 together imply $\text{pd}_{A_{\Phi}[t]}(M) \leq (d+1)$.

(i) This follows directly from assertion (i).

Theorem 9.16 (Global dimension and the passage from $A_{\Phi}[t]$ to $A_{\Phi}[t, t^{-1}]$). Let $A$ be an additive category and let $\Phi: A \to A$ be an automorphism of additive categories.

(i) Let $M$ be a $\mathbb{Z}_A[t, t^{-1}]$-module. If we have $\text{pd}_A(t^*M) \leq d$, then we get $\text{pd}_{A[t, t^{-1}]}(M) \leq d$.

(ii) If $A_{\Phi}[t]$ has global dimension $\leq d$, then $A_{\Phi}[t, t^{-1}]$ has global dimension $\leq d$.

Proof. (i) Let $M$ be a $\mathbb{Z}_A[t, t^{-1}]$-module satisfying $\text{pd}_A(t^*M) \leq d$. The functor $j_*: \text{MOD-}\mathbb{Z}_A[t] \to \text{MOD-}\mathbb{Z}_A[t, t^{-1}]$ is flat by Lemma 9.14. Since it respects the property projective, we get $\text{pd}_{A_{\Phi}[t, t^{-1}]}(j_*t^*M) \leq d$. Lemma 9.13 (iii) implies $\text{pd}_{A_{\Phi}[t, t^{-1}]}(M) \leq d$.

(ii) This follows from assertion (i).

10. Regular additive categories

Regularity for additive categories $A$ requires finite resolutions of finitely presented modules, but not for arbitrary modules. In particular, regularity has no consequence for global dimension and we cannot use Theorem 9.15 in the following result.

Theorem 10.1 (Regularity and the passage from $A$ to $A_{\Phi}[t]$). Let $A$ be an additive category $A$ and let $\Phi: A \to A$ be an automorphism of additive categories. Let $l$ be a natural number.

(i) Suppose that $A$ is regular or $l$-uniformly regular respectively. Then $A_{\Phi}[t]$ is regular or $(l+2)$-uniformly regular respectively;

(ii) Suppose that $A[t]$ is regular or $l$-uniformly regular respectively. Then $A_{\Phi}[t, t^{-1}]$ is regular or $l$-uniformly regular respectively.

Proof. (i) We know already that $A_{\Phi}[t]$ is Noetherian because of Theorem 8.1. Let $M$ be a finitely generated $A_{\Phi}[t]$-module. We have to show that it has a finitely generated projective resolution, which is finite-dimensional or $(l+1)$-dimensional. Since $A_{\Phi}[t]$ is Noetherian, there exists a finitely generated projective resolution of $M$, which may be infinite-dimensional. We conclude from Theorem 5.4(iv) that it suffices to show the projective dimension of $M$ is finite or bounded by $(l+1)$ respectively. As $M$ is finitely generated, we find a finite collection of elements $x_j \in M(Z_j)$ with objects $Z_j$ from $A$ such that the $x_j$ generate $M$ as a $\mathbb{Z}_A[t]$-module. For $d \geq 0$ consider the morphism $\text{id}_{Z_j} \cdot t^d: \Phi^{-d}(Z_j) \to Z_j$ in $A_{\Phi}[t]$ and set $x_j[d] := M(\text{id}_{Z_j} \cdot t^d)(x_j) \in M(\Phi^{-d}(Z_j))$. Let $M_n$ be the $\mathbb{Z}_A$-submodule of $i^*M$ generated by all $x_j[d]$ with $d \leq n$. We obtain an increasing subsequence $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of $\mathbb{Z}_A$-submodules of $i^*M$ with $i^*M = \bigcup_{n \geq 0} M_n$. Let $T_n: i^*M \to \Phi^n i^*M$ be the following $\Phi$-morphism. For an object $Z$ from $A$ consider $\text{id}_{\Phi^n(Z)} \cdot t^n \in \text{mor}_{A_{\Phi}[t]}(Z, \Phi^n(Z))$ and define $T_Z: i^*M(Z) = M(Z) \to \Phi^n i^*M(Z) = M(\Phi^n(Z))$ to be $M(\text{id}_{\Phi^n(Z)} \cdot t^n)$. Let $\text{pr}_n: (\Phi^n)^*(M_n) \to (\Phi^n)^*(M_n)/(\Phi^n)^*(M_{n-1})$ be the projection. The composite

$$f_n: M_0 \xrightarrow{\text{pr}_n} (\Phi^n)^*(M_n) \xrightarrow{\text{pr}_n} (\Phi^n)^*(M_n)/(\Phi^n)^*(M_{n-1})$$
is surjective and we write $K_n$ for its kernel. We obtain an increasing sequence of $\mathcal{A}$-submodules $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of $M_0$. Since $\mathcal{A}$ is Noetherian and $M_0$ is finitely generated, there exists an integer $n_0 \geq 1$ such that $K_n = K_{n_0}$ holds for $n \geq n_0$. Define $g_n : (\Phi^n_0)^* M_{n}/(\Phi^n_0)^* M_{n_0-1} \to (\Phi^n_0)^* M_{n}/(\Phi^n_0)^* M_{n_0-1}$ for $n \geq n_0$ to be the map induced by $\Phi^n(T_{n-n_0})$ for $n \geq n_0$. We obtain for every natural number $n$ with $n \geq n_0$ a commutative diagram of $\mathbb{Z}$-$\mathcal{A}$-modules with exact rows

$$
\begin{array}{cccccc}
0 & \to & K_{n_0} & \to & M_0 & \to \to (\Phi^n_0)^* M_{n_0}/(\Phi^n_0)^* M_{n_0-1} & \to 0 \\
 & & \downarrow{f_{n_0}} & & \downarrow{\cong} & & \downarrow{\text{id}_{M_0}} & \\
0 & \to & K_n & \to & M_0 & \to \to (\Phi^n_0)^* M_n/(\Phi^n_0)^* M_{n-1} & \to 0
\end{array}
$$

Hence $g_n$ is an isomorphism of $\mathbb{Z}$-$\mathcal{A}$-module $n \geq n_0$. As $\Phi^*$ is an isomorphism we have

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n-1} = \text{pdim}_{\mathcal{A}}(\Phi^*)^* (M_n/M_{n-1}) = \text{pdim}_{\mathcal{A}}(\Phi^*)^* M_n/(\Phi^*)^* M_{n_0-1}).$$

Thus for $n \geq n_0$ we have $\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n-1} = \text{pdim}_{\mathcal{A}}(\Phi^*)^* M_n/M_{n_0-1})$. We have the short exact sequence $0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0$ and hence get from Lemma 5.3 (v)

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n} \leq \sup\{\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n_1}, \text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n_0}/M_{n_0-1}\}.$$

This implies by induction over $n \geq n_0$

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n} \leq \sup\{\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n_1}, \text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n_0}/M_{n_0-1}\}.$$

Put

$$D := \sup\{\sup\{\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{k} \mid k = 0, 1, \ldots, n_0 - 1\}, \text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n_0}/M_{n_0-1}\}.$$

Note that $D < \infty$, if $\mathcal{A}$ is regular, and $D \leq l$, if $\mathcal{A}$ is uniformly $l$-regular. We get

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/(\bigoplus_{n \in \mathbb{N}} M_n) \leq \sup\{\sup\{\text{pdim}_{\mathbb{Z}} \mathcal{A}/M_{n} \mid n \geq 0\} \leq D.$$

We have the short exact sequence of $\mathcal{A}$-modules

$$0 \to \bigoplus_{n \in \mathbb{N}} M_n \to \bigoplus_{n \in \mathbb{N}} M_n \to i^* M \to 0,$$

where the first map is given by $(x_n)_{n \geq 0} \mapsto (x_0, x_1 - x_0, x_2 - x_1, \ldots)$ and the second by $(x_n)_{n \geq 0} \mapsto \sum_{n \geq 0} x_n$. We conclude from Lemma 5.3 (v)

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/(i^* M) \leq D + 1.$$

Now Theorem 9.15 (ii) implies

$$\text{pdim}_{\mathbb{Z}} \mathcal{A}/(i^* M) \leq D + 1.$$

This finishes the proof if assertion 0[1]

[1] We know already that $\mathcal{A}/[t, t^{-1}]$ is Noetherian because of Theorem 8.1 (ii). Let $M$ be a finitely generated $\mathcal{A}/[t, t^{-1}]$-module. We can find a finitely generated free $\mathbb{Z} \mathcal{A}/[t]$-module $F_0$ and a free $\mathbb{Z} \mathcal{A}/[t]$-module $F_1$ together with an exact sequence of $\mathbb{Z} \mathcal{A}/[t, t^{-1}]$-modules $j_* F_0 \cong j_* F_1 \to M \to 0$. Here we write $j_*$ for the inclusion $\mathcal{A}/[t] \to \mathcal{A}/[t, t^{-1}]$. By composing $f$ with an appropriate automorphism of $j_* F_0$ one can arrange that $f = j_* g$ for some $\mathbb{Z} \mathcal{A}/[t]$-homomorphism $g : F_0 \to F_1$. The cokernel of $g$ is a finitely generated $\mathbb{Z} \mathcal{A}/[t]$-module $N$ and there is an obvious exact sequence of $\mathbb{Z} \mathcal{A}/[t]$-modules $F_1 \cong F_1 \to M \to 0$. Since the functor $j_*$ is flat by Lemma 9.14 and respects the property projective, we obtain an $\mathbb{Z} \mathcal{A}/[t, t^{-1}]$-isomorphism $j_* N \cong M$ and have $\text{dim}_{\mathbb{Z} \mathcal{A}/[t, t^{-1}]}(j_* N) \leq \text{dim}_{\mathbb{Z} \mathcal{A}/[t]}(N)$. Hence we get $\text{dim}_{\mathbb{Z} \mathcal{A}/[t, t^{-1}]}(M) \leq \text{dim}_{\mathbb{Z} \mathcal{A}/[t]}(N)$. This finishes the proof of Theorem 10.1. □
Remark 10.2. We do not know whether Theorem 10.1 remains true if we replace regular by regular coherent. To our knowledge it is an open problem, whether for a regular coherent ring $R$ the rings $R[t]$ or $R[t,t^{-1}]$ are regular coherent again.

11. Directed union and infinite products of additive categories

A functor of additive categories $F: \mathcal{A} \to \mathcal{B}$ is called flat, if for every exact sequence $A_0 \xrightarrow{i_0} A_1 \xrightarrow{i} A_2$ in $\mathcal{A}$ the sequence in $F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(i)} F(A_2)$ in $\mathcal{B}$ is exact. It is called faithfully flat, if a sequence $A_0 \xrightarrow{i_0} A_1 \xrightarrow{i} A_2$ in $\mathcal{A}$ is exact, if and only if the sequence in $F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(i)} F(A_2)$ in $\mathcal{B}$ is exact.

Lemma 11.1. Let $i: \mathcal{A} \to \mathcal{A'}$ and $j: \mathcal{B} \to \mathcal{B'}$ be inclusions of cofinal full additive subcategories. Suppose that the following diagram of functors of additive categories commutes

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\downarrow i & & \downarrow j \\
\mathcal{A'} & \xrightarrow{F'} & \mathcal{B'}
\end{array}
\]

Then

(i) The inclusion $i: \mathcal{A} \to \mathcal{A'}$ is faithfully flat;

(ii) $F'$ is flat or faithfully flat respectively if and only if $F'$ is flat or faithfully flat respectively.

Proof. We first show that $F'$ is exact or faithfully exact respectively, provided that $F$ is exact or faithfully exact respectively.

Consider morphisms $f': A'_0 \to A'_1$ and $g': A'_1 \to A'_2$ in $\mathcal{A'}$. Choose objects $A_k$ in $\mathcal{A}$ and morphisms $i_k: A'_k \to A_k$ and $r_k: A_k \to A'_k$ in $\mathcal{A'}$ satisfying $r_k \circ i_k = \text{id}_{A_k}$ for $k = 0, 1, 2$. Define $f: A_0 \to A_1$ and $g: A_1 \to A_2$ by $f = i_1 \circ f' \circ r_0$ and $g = i_2 \circ g' \circ r_1$. Then the following diagram of morphisms in $\mathcal{A'}$ commutes

\[
\begin{array}{ccc}
A'_0 & \xrightarrow{f'} & A'_1 & \xrightarrow{g'} & A'_2 \\
\downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 \oplus 0 \\
A_0 & \xrightarrow{f} & A_1 & \xrightarrow{g \oplus (\text{id}_{A_1} - i_1 \circ r_1)} & A_2 \oplus A_1 \\
\downarrow r_0 & & \downarrow r_1 & & \downarrow r_2 \oplus 0 \\
A'_0 & \xrightarrow{f'} & A'_1 & \xrightarrow{g'} & A'_2
\end{array}
\]

Next we check that the middle row is exact in $\mathcal{A}$, if and only if the upper row is exact in $\mathcal{A'}$. Suppose that the middle row is exact in $\mathcal{A}$. Consider a morphism $v': B' \to A'_1$ in $\mathcal{A'}$ such that $g' \circ v' = 0$. Choose an object $B$ in $\mathcal{A}$ and maps $j: B' \to B$ and $s: B \to B'$ with $s \circ j = \text{id}_{B'}$. Then we have the morphism $i_1 \circ v' \circ s: B \to A_1$ whose composite with $g \oplus (\text{id}_{A_1} - i_1 \circ r_1): A_1 \to A_2 \oplus A_1$ is zero. Hence we can find a morphism $u_0: B \to A_0$ with $f \circ u_0 = i_1 \circ v' \circ s$. Define $u': B' \to A'_0$ by the composite $r_0 \circ u_0 \circ j$. One easily checks that $f' \circ u' = v'$. Hence the upper row is exact in $\mathcal{A'}$.

Suppose that the upper row is exact in $\mathcal{A'}$. Consider a morphism $v: B \to A_1$ in $\mathcal{A}$ such that $g \oplus (\text{id}_{A_1} - i_1 \circ r_1) \circ v = 0$. Then $g \circ v = 0$ and $v = i_1 \circ r_1 \circ v$. We conclude

\[g' \circ (r_1 \circ v) = r_2 \circ i_2 \circ g' \circ r_1 \circ v = r_2 \circ g \circ i_1 \circ r_1 \circ v = r_2 \circ g \circ v = r_2 \circ 0 = 0.\]
Since the upper row is exact, we can find \( u' : B \to A'_0 \) satisfying \( f' \circ u'_0 = r_1 \circ v \). Define \( u : B \to A_0 \) by \( i_0 \circ u' \). Then
\[
f \circ u = f \circ i_0 \circ u' = i_1 \circ f' \circ u' = i_1 \circ r_1 \circ v = v.
\]
Hence the middle row is exact.

If we apply \( F' \) and put \( i'_k = F'(i_k) \) and \( r'_k = F'(r_k) \), we get \( r'_k \circ i'_k = \text{id}_{F'(A'_0)} \)
and the commutative diagram

\[
\begin{array}{ccc}
F'(A'_0) & \overset{F'(f')}{\longrightarrow} & F'(A'_1) \\
\downarrow i'_0 & & \downarrow i'_1 \\
F(A_0) & \overset{F(f)}{\longrightarrow} & F(A_1) \\
\downarrow r'_0 & & \downarrow r'_1 \\
F'(A'_0) & \overset{F'(f')}{\longrightarrow} & F'(A'_1)
\end{array}
\]

and, by the same argument as above, the middle row is exact in \( B \), if and only if the upper row is exact in \( B' \). We conclude that the functor \( F' \) is exact or faithfully exact respectively, provided that \( F \) is exact or faithfully exact respectively.

Since both \( \text{id}_{A} \) and \( \text{id}_{B} \) are faithfully flat, this special case shows that both \( i : A \to A' \) and \( j : B \to B' \) are faithfully flat.

Suppose that \( F' \) is flat or faithfully flat respectively. Then \( j \circ F = F' \circ i \) is flat or faithfully flat respectively. This implies that \( F \) is flat or faithfully flat respectively. This finishes the proof of Lemma 11.1.

**Lemma 11.2.** Let \( A = \bigcup_{i \in I} A_i \) be the directed union of additive subcategories \( A_i \) for an arbitrary directed set \( I \).

(i) The idempotent completion \( \text{Idem}(A) \) is the directed union of the idempotent completions \( \text{Idem}(A_i) \).

(ii) Consider \( l \geq 1 \).

Suppose that \( A_i \) is regular coherent or \( l \)-uniformly regular coherent respectively for every \( i \in I \) and for every \( i, j \in I \) with \( i \leq j \) the inclusion \( A_i \to A_j \) is flat. Then the inclusion \( \text{Idem}(A_i) \to \text{Idem}(A_i) \) is flat for every \( i, j \in I \) with \( i \leq j \) and both \( A \) and \( \text{Idem}(A) \) are regular coherent or \( l \)-uniformly regular coherent respectively.

(iii) Suppose that \( A_i \) is 0-uniformly regular coherent respectively for every \( i \in I \).

Then both \( A \) and \( \text{Idem}(A) \) are 0-uniformly regular coherent respectively.

**Proof.** (i) This is obvious.

(ii) If the inclusion \( A_i \to A_j \) is flat, then also the inclusion \( \text{Idem}(A_i) \to \text{Idem}(A_j) \) is flat by Lemma 11.1. In view of Lemma 6.3(vi) and assertion (i) we can assume without loss of generality that each \( A_i \) and \( A \) are idempotent complete. Hence we can use the criterion for regular coherent given in Lemma 6.6 in the sequel. We treat only the case \( l \geq 2 \), the case \( l = 1 \) is proved analogously.

Consider a morphism \( f_1 : A_1 \to A_0 \) in \( A \). Choose an index \( i \) such that \( f_1 \) belongs to \( A_i \). Then we can find a sequence of morphisms
\[
0 \to A_n \xrightarrow{f_0} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0
\]
which is in \( A_i \) exact at \( A_k \) for \( k = 1, 2, \ldots, n \). It remains to show that this sequence is exact at \( A_k \) for \( k = 1, 2, \ldots, n \). Fix \( k \in \{1, 2, \ldots, n\} \). It remains to show to any object \( A \in A \) and morphism \( g : A \to A_k \) with \( f_k \circ g = 0 \) that there exists a morphism \( f : A \to A_{k+1} \) with \( f_{k+1} \circ f = g \). We can choose \( j \in I \) with \( i \leq j \) such that \( g \) belongs to \( A_j \). Since \( A_{k+1} \) is exact in \( A_i \), we conclude...
from the assumptions that it is also exact in $\mathcal{A}_j$ and hence we can construct the desired lift $\overline{\varphi}$ already in $\mathcal{A}_j$.

(iii) In view of Lemma 6.3 [vi] and assertion [i] we can assume without loss of generality that each $\mathcal{A}_i$ and $\mathcal{A}$ are idempotent complete. Now the claim follows from the equivalence $(iii) \iff (iii')$ appearing in Lemma 6.6 [iii].

Lemma 11.3. Let $l$ be a natural number. Let $\mathcal{A} = \{ \mathcal{A}_i \mid i \in I \}$ be a collection of additive categories $\mathcal{A}_i$ for an arbitrary index set $I$. Let $l$ be a natural number.

(i) Suppose that each $\mathcal{A}_i$ is $l$-uniformly regular coherent. Then $\prod_{i \in I} \text{Idem}(\mathcal{A}_i)$ is $l$-uniformly regular coherent;

(ii) Suppose that each $\mathcal{A}_i$ is $l$-uniformly regular coherent, $l$-uniformly regular, regular coherent, regular, or Noetherian. Then $\bigoplus_{i \in I} \mathcal{A}_i$ has the same property.

Proof. Obviously $\prod_{i \in I} \mathcal{A}_i$ inherits the structure of an additive category. Recall that $\bigoplus_{i \in I} \mathcal{A}_i$ is the full additive subcategory of $\prod_{i \in I} \mathcal{A}_i$ consisting of those objects $A_i \mid i \in I \}$, for which only finitely many of the objects $A_i$ are different from zero.

Obviously

$$\text{Idem}(\bigoplus_{i \in I} \mathcal{A}_i) \cong \bigoplus_{i \in I} \text{Idem}(\mathcal{A}_i);$$

$$\text{Idem}(\prod_{i \in I} \mathcal{A}_i) \cong \prod_{i \in I} \text{Idem}(\mathcal{A}_i).$$

Lemma 6.3 implies that $\bigoplus_{i \in I} \text{Idem}(\mathcal{A}_i)$ and $\prod_{i \in I} \text{Idem}(\mathcal{A}_i)$ are $l$-uniformly regular coherent, if each $\text{Idem}(\mathcal{A}_i)$ is $l$-uniformly regular coherent. We conclude from Lemma 6.3 [vi] that $\bigoplus_{i \in I} \mathcal{A}_i$ and $\prod_{i \in I} \mathcal{A}_i$ are $l$-uniformly regular coherent, if each $\mathcal{A}_i$ is $l$-uniformly regular coherent.

Analogously one shows that $\bigoplus_{i \in I} \mathcal{A}_i$ is regular coherent if each $\mathcal{A}_i$ is regular coherent.

Next we show that $\bigoplus_{i \in I} \mathcal{A}_i$ is Noetherian if each $\mathcal{A}_i$ is Noetherian. Consider any object $A$ in $\bigoplus_{i \in I} \mathcal{A}_i$. Choose a finite index set $J \subseteq I$ such that $A$ belongs to $\bigoplus_{i \in J} \mathcal{A}_i$. Let $\text{pr}: \bigoplus_{i \in J} \mathcal{A}_i \rightarrow \bigoplus_{i \in J} \mathcal{A}_i$ be the projection. Consider two morphisms $f_0: A_0 \rightarrow A$ and $f_1: A_1 \rightarrow A$ in $\bigoplus_{i \in I} \mathcal{A}_i$. Then $f_0 = f_1$ holds if and only if $\text{pr}(f_0) = \text{pr}(f_1)$ holds. Hence $f_0 \subseteq f_1$ holds in $\bigoplus_{i \in I} \mathcal{A}_i$ if and only if $\text{pr}(f_0) \subseteq \text{pr}(f_1)$ holds in $\bigoplus_{i \in I} \mathcal{A}_i$. We conclude from Lemma 6.8 that it suffices to show that $\bigoplus_{i \in J} \mathcal{A}_i$ is Noetherian for any finite subset $J \subseteq I$. But this is an easy consequence of Lemma 6.8 again.

Lemma 11.3 [i] will be generalized in Lemma 13.10.

Remark 11.4 (Advantage of the notion $l$-uniformly regular coherent). The decisive advantage of the notion $l$-uniformly regular coherent is that it satisfies both Lemma 11.2 and Lemma 11.3. Lemma 11.2 and Lemma 11.3 [vi] are not true, if one replaces $l$-uniformly regular coherent by any of the properties regular coherent, $l$-uniformly regular, regular or Noetherian, unless $I$ is finite.

12. Vanishing of negative $K$-groups

Theorem 12.1 (Vanishing of negative $K$-groups). Let $\mathcal{A}$ be an additive category, such that $\mathcal{A}[t_1, t_2, \ldots, t_m]$ is regular coherent for every $m \geq 0$.

Then $K_n(\mathcal{A}) = 0$ holds for all $n \leq -1$.

Proof. For an additive category $\mathcal{B}$ define $G'_0(\mathbb{Z}\mathcal{B})$ to be the abelian group with isomorphism classes $[M]$ of finitely presented $\mathbb{Z}\mathcal{B}$-modules $M$ as generators such that for each exact sequence of finitely presented $\mathbb{Z}\mathcal{B}$-modules $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$...
we have the relation $[M_0] - [M_1] + [M_2] = 0$. Define $K_0(\mathbb{Z}B)$ analogously but with finitely presented replaced by finitely generated projective. A functor of additive categories $F: B \rightarrow B'$ induces a homomorphism $F_*: K_0(\mathbb{Z}B) \rightarrow K_0(\mathbb{Z}B')$ by sending $[M]$ to $[F_*M]$. It induces a homomorphism $F_*: G_0'(\mathbb{Z}B') \rightarrow G_0'(\mathbb{Z}B)$ by sending $[M]$ to $[F_*M]$, if $F_*: \text{MOD-}\mathbb{Z}A_B \rightarrow \text{MOD-}\mathbb{Z}A_B'$ is flat. There is the forgetful functor $U: K_0(\mathbb{Z}B) \rightarrow G_0'(\mathbb{Z}B')$. If $B$ is regular coherent, then $U$ is a bijection by the Resolution Theorem, see 20 Theorem 4.6 on page 41. The Yoneda embedding induces an isomorphism $K_0(B) \xrightarrow{\cong} K_0(\mathbb{Z}B)$, natural in $B$.

Suppose $A[t]$ is regular coherent. We show that $A[t,t^{-1}]$ is regular coherent and $K_{-1}(A) = 0$. The functor $j_*: \text{MOD-}\mathbb{Z}A[t] \rightarrow \text{MOD-}\mathbb{Z}A[t,t^{-1}]$ is flat by Lemma 9.14. Let $M_*$ be a finitely presented $\mathbb{Z}A[t,t^{-1}]$-module. Then we can find a morphism $f: A \rightarrow A'$ in $A[t,t^{-1}]$ together with an exact sequence of $\mathbb{Z}A[t,t^{-1}]$-modules

$$\text{mor}_{\mathbb{Z}A[t,t^{-1}]}(?,A) \xrightarrow{j_*} \text{mor}_{\mathbb{Z}A[t,t^{-1}]}(?,A') \rightarrow M \rightarrow 0.$$ 

Choose a natural number $s$ and a morphism $g: A \rightarrow A'$ in $A[t]$ such that $(\text{id}_{A'} \cdot t^s) \circ f = g$ holds in $A[t,t^{-1}]$. Since $\text{id}_{A'}: A' \xrightarrow{\cong} A'$ is an isomorphism in $A[t,t^{-1}]$, we obtain an exact sequence of $\mathbb{Z}A[t,t^{-1}]$-modules

$$j_* (\text{mor}_{\mathbb{Z}A[t]}(?),A)) \xrightarrow{j_* (g_*)} j_* (\text{mor}_{\mathbb{Z}A[t]}(?),A')) \rightarrow M \rightarrow 0.$$ 

Let $N$ be the finitely presented $\mathbb{Z}A[t]$-module, which is the cokernel of the $\mathbb{Z}A[t]$-homomorphism $g_*: \text{mor}_{\mathbb{Z}A[t]}(?),A) \rightarrow \text{mor}_{\mathbb{Z}A[t]}(?),A')$. Since $j_*$ is flat and in particular right exact, we obtain an isomorphism of finitely presented $\mathbb{Z}A[t,t^{-1}]$-modules $j_* N \xrightarrow{\cong} M$. This implies that the homomorphism $j_*: G_0'(\mathbb{Z}A[t]) \rightarrow G_0'(\mathbb{Z}A[t,t^{-1}])$ is surjective and that $A[t,t^{-1}]$ is regular coherent since $A[t]$ is regular coherent by assumption.

Hence we obtain a commutative diagram

$$
\begin{array}{ccc}
K_0(A[t]) & \xrightarrow{\cong} & K_0(A[t,t^{-1}]) \\
\downarrow & & \downarrow \\
K_0(\mathbb{Z}A[t]) & \xrightarrow{\cong} & K_0(\mathbb{Z}A[t,t^{-1}]) \\
\downarrow & & \downarrow \\
G_0'(\mathbb{Z}A[t]) & \xrightarrow{\cong} & G_0'(\mathbb{Z}A[t,t^{-1}]),
\end{array}
$$

whose vertical arrows are bijections and whose lowermost horizontal arrow is surjective. Hence the uppermost horizontal arrow is surjective. We conclude from Theorem 11.2 that $K_{-1}(A)$ vanishes.

Next we show by induction for $n = 1, 2, \ldots$ that $K_{-m}(A)$ vanishes for $m = 1, 2, \ldots, n$. The induction beginning $n = 1$ has been taken care of above. The induction step from $n \geq 1$ to $n + 1$ is done as follows. One shows using the claim above by induction for $i = 1, 2, \ldots, n$ that $A[Z^i][t_{i+1}, \ldots, t_{n+1}]$ is regular coherent. In particular $A[Z^i][t_{n+1}]$ is regular coherent.

We conclude from the $n$-times iterated Bass-Heller-Swan isomorphism, see Theorem 11.1 that $K_{-n-1}(A)$ is a direct summand in $K_{-1}(A[Z^n])$. Hence it suffices to show that $K_{-1}(A[Z^n])$ is trivial. This follows from the induction beginning applied to $A[Z^n]$.

We conclude from Theorem 11.1 and Theorem 12.1.

**Corollary 12.2** (Vanishing of negative $K$-groups of regular additive categories). Let $A$ be an additive category which is regular.

Then $K_n(A) = 0$ holds for all $n \leq -1$. 

13. Nested sequences, the associated categories, and their $K$-theory

13.1. Nested sequences and the associated categories.

Definition 13.1 (Nested sequences of additive categories). A nested sequence of additive categories $A_*$ is a decreasing sequence of additive subcategories

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots.$$  

We have two notions of morphisms.

Definition 13.2 (Pro-morphisms of nested sequences of additive categories). A morphism of nested sequences of additive categories $F: A_* \rightarrow A'_*$ is a sequence of functors of additive categories $F_m: A_m \rightarrow A'_m$ for $m \in \mathbb{N}$ such that $F_m$ restricted to $A_{m+1}$ is $F_{m+1}$.

A pro-morphism of nested sequences of additive categories $F: A_* \rightarrow A'_*$ is a functor of additive categories $F: A_0 \rightarrow A'_0$ such that there is a function $N: \mathbb{N} \rightarrow \mathbb{N}$ with the property that for every $m, n \in \mathbb{N}$ with $m \geq N(n)$ we have $F(A_m) \subseteq A'_n$.

Obviously a morphism $F$, defines a pro-morphisms by taking $F_0$, but not every pro-morphism comes from a morphism in this way. The composite of two morphisms and of two pro-morphisms is defined in the obvious way.

Note that a pro-automorphism $\phi: A_* \rightarrow A_*$ is the same as an automorphism of additive categories $\Phi: A_0 \rightarrow A_0$ such that there is a function $N: \mathbb{N} \rightarrow \mathbb{N}$ with the property that for every $m, n \in \mathbb{N}$ with $m \geq N(n)$ we have $\Phi(A_m) \subseteq A_n$ and $A_m \subseteq \Phi(A_n)$.

Definition 13.3 (The sequence category $S(A_*)$ and the limit category $L(A_*)$). Define the additive category $S(A_*)$, called sequence category, associated to the nested sequence of additive categories $A_*$ as follows:

- An object in $S(A_*)$ is a sequence $\overline{A} = (A_m)_{m \in \mathbb{N}}$ of objects in $A_0$ such that there exists a function (depending on $\overline{A}$) $L: \mathbb{N} \rightarrow \mathbb{N}$ with the property that $A_m$ belongs to $A_l$ for $l, m \in \mathbb{N}$ with $m \geq L(l)$;
- A morphism $\overline{\varphi}: \overline{A} \rightarrow \overline{A}'$ in $S(A_*)$ consists of a sequence of morphisms $\varphi_m: A_m \rightarrow A'_m$ in $A_0$ such that there exists a function $L: \mathbb{N} \rightarrow \mathbb{N}$ with the property that $\varphi_m: A_m \rightarrow A'_m$ belongs to $A_l$ for $m, l \in \mathbb{N}$ with $m \geq L(l)$;
- Composition and the structure of an additive category on $S(A_*)$ comes from the corresponding structure on $A_0$.

Let $T(A_*)$ be the full subcategory of $S(A_*)$ consisting of objects $\overline{A}$, for which there exists a natural number $M$ (depending on $\overline{A}$) with $A_m = 0$ for $m \geq M$.

The additive category $L(A_*)$, called limit category, is defined to be the additive quotient category $S(A_*)/T(A_*)$. Recall that for a full additive subcategory $U \subseteq B$ of an additive category $B$ the additive quotient category $B/U$ has the same objects as $B$ and that morphisms in $B/U$ are equivalence classes of morphisms in $B$, where two morphisms $\varphi, \varphi'$ are identified, if their difference $\varphi - \varphi'$ can be factorized.
through an object of \( U \). For \( \mathcal{L}(A_\ast) \) this means that morphisms \( \varphi, \varphi' \) from \( S(A_\ast) \) are identified in \( \mathcal{L}(A_\ast) \), if and only if \( \varphi_m = \varphi'_m \) for all but finitely many \( m \).

Obviously \( S(A_\ast) \) is an additive subcategory of \( \prod_{m \in \mathbb{N}} A_m \) and is equal to it, if the nested sequence is constant, i.e., \( A_0 = A_m \) for \( m \in \mathbb{N} \). Obviously \( T(A_\ast) \) can be identified with \( \bigoplus_{m \in \mathbb{N}} A_m \). Controlled categories appear for instance in proofs of the Furell-Jones Conjecture. In for us important cases controlled categories correspond to nested sequences of additive categories, where one should think of the control condition to become sharper the larger the index \( m \) gets.

For the notion of a Karoubi filtration and the associated weak homotopy fibration sequence we refer for instance to [3, and §3. Definition 5.4] or [11, Section 12.2].

One easily checks

**Lemma 13.4.** The inclusion \( T(A_\ast) \subseteq S(A_\ast) \) is a Karoubi filtration and we have the weak homotopy fibration sequence

\[
\mathcal{K}^\infty(T(A_\ast)) \to \mathcal{K}^\infty(S(A_\ast)) \to \mathcal{K}^\infty(\mathcal{L}(A_\ast)).
\]

Let \( F: A_\ast \to A'_\ast \) be a pro-morphisms. It induces functors of additive categories

\[
\begin{align*}
&\mathcal{S}(F): S(A_\ast) \to S(A'_\ast); \\
&\mathcal{T}(F): T(A_\ast) \to T(A'_\ast); \\
&\mathcal{L}(F): \mathcal{L}(A_\ast) \to \mathcal{L}(A'_\ast),
\end{align*}
\]

as follows. We begin with \( S(F) \). It sends an object \( \underline{A} = (A_m)_{m \in \mathbb{N}} \) to the object \( S(F)(\underline{A}) = (F(A_m))_{m \in \mathbb{N}} \). We have to check that the latter collection defines an element in \( S(A'_\ast) \). Recall that there is a function \( L: \mathbb{N} \to \mathbb{N} \) with the property that \( A_m \) belongs to \( A_l \) for \( l, m \in \mathbb{N} \) with \( m \leq L(l) \) and that there is a function \( N: \mathbb{N} \to \mathbb{N} \) with the property that for every \( m, n \in \mathbb{N} \) with \( m \geq N(n) \) we have \( F(A_m) \subseteq A_n \). Consider the function \( L \circ N: \mathbb{N} \to \mathbb{N} \). For \( m, n \in \mathbb{N} \) with \( m \geq L \circ N(n) \) we conclude \( A_m \in A_{N(n)} \) and hence \( F(A_m) \in F(A_{N(n)}) \subseteq A_n \). Hence \( (F(A_m))_{m \in \mathbb{N}} \) is a well-defined object in \( S(A_\ast) \).

Given a morphism \( \varphi: \underline{A} \to \underline{A}' \) in \( S(A_\ast) \), we can define \( \mathcal{S}(F)(\varphi): S(F)(\underline{A}) \to S(F)(\underline{A}') \) in \( S(A'_\ast) \) by the collection \( (F(\varphi_m))_{m \in \mathbb{N}} \). Obviously \( \mathcal{S}(F \circ \alpha) = \mathcal{S}(F) \circ \mathcal{S}(\alpha) \). By construction \( \mathcal{S}(F) \) induces a functor of additive categories \( \mathcal{T}(F): \mathcal{T}(A_\ast) \to \mathcal{T}(A'_\ast) \). By passing to the quotients we also get a functor of additive categories \( \mathcal{L}(F): \mathcal{L}(A_\ast) \to \mathcal{L}(A'_\ast) \).

Note that a pro-automorphism \( \Phi: A_\ast \to A_\ast \) induces automorphisms of additive categories

\[
\begin{align*}
&\mathcal{S}(\Phi): S(A_\ast) \xrightarrow{\cong} S(A_\ast); \\
&\mathcal{T}(\Phi): T(A_\ast) \xrightarrow{\cong} T(A_\ast); \\
&\mathcal{L}(\Phi): \mathcal{L}(A_\ast) \xrightarrow{\cong} \mathcal{L}(A_\ast).
\end{align*}
\]

**Definition 13.8.** We call a function \( I: \mathbb{N} \to \mathbb{N} \) admissible, if it has the following properties

\[
\begin{align*}
I(m) &\leq m \quad \text{for } m \in \mathbb{N}; \\
\lim_{m \to \infty} I(m) &= \infty.
\end{align*}
\]

Let \( \mathcal{I} \) be the set of admissible functions. It becomes a directed set by defining for \( I, J \in \mathcal{I} \)

\[
I \leq J \iff I(m) \geq J(m) \quad \text{for all } m \in \mathbb{N}.
\]

Note that \( \mathcal{I} \) is indeed directed. For \( I, J \in \mathcal{I} \) define \( K: \mathbb{N} \to \mathbb{N} \) by \( K(m) = \min\{I(m), J(m)\} \). Then \( K \in \mathcal{I} \) and \( I, J \leq K \) holds.
Lemma 13.9. (i) Let \( \phi \) be a morphism in \( S(A_\ast) \). Then there exists an admissible function \( I \in I \) such that \( \phi_m \in A_{I(m)} \) holds for all \( m \in \mathbb{N} \); (ii) Let \( \phi \) be a sequence of morphisms \( \phi_m : A_m \to A'_m \) in \( A_0 \). Suppose that there exists an admissible function \( I \in I \) such that \( \phi_m \in A_{I(m)} \) holds for all \( m \in \mathbb{N} \). Then \( \phi \) belongs to \( S(A_\ast) \).

Proof. (i) Choose a function \( L : \mathbb{N} \to \mathbb{N} \) such that \( \phi_m \) belongs to \( A_l \) if \( m \geq L(l) \). Define a new function \( I' : \mathbb{N} \to \mathbb{N} \) by

\[
I'(m) = \max\{i \in \{0, 1, \ldots, m\} \mid \phi_m \in A_i\}.
\]

It satisfies

\[
\begin{align*}
I'(m) &\leq m & \text{for } m \in \mathbb{N}; \\
l &\leq I'(m) & \text{for } l, m \in \mathbb{N}, m \geq L(l), m \geq l; \\
\phi_m &\in A_{I'(m)} & \text{for } m \in \mathbb{N}.
\end{align*}
\]

Define the function \( I : \mathbb{N} \to \mathbb{N} \) by

\[
I(m) = \min\{I(j) \mid j \in \mathbb{N}, m \leq j\}.
\]

Then we get for all \( n \in \mathbb{N} \)

\[
\begin{align*}
I(m) &\leq m & \text{for } m \in \mathbb{N}; \\
I(m) &\leq I(m + 1) & \text{for } m \in \mathbb{N}; \\
l &\leq I(m) & \text{for } l, m \in \mathbb{N}, m \geq L(l), m \geq l; \\
\phi_m &\in A_{I(m)} & \text{for } m \in \mathbb{N}.
\end{align*}
\]

The first three properties imply \( I \in I \).

(ii) Suppose that there exists \( I \in I \) satisfying \( \phi_m \in A_{I(m)} \) for all \( m \in \mathbb{N} \). Define the desired function \( L : \mathbb{N} \to \mathbb{N} \) by \( L(l) = \min\{m \in \mathbb{N} \mid l \leq I(m)\} \).

\[\square\]

13.3. Uniform regular coherence.

Lemma 13.10. Consider the nested sequence \( A_\ast \) of additive categories \( A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \). Suppose that for the natural number \( l \geq 2 \) each of the additive categories \( A_m \) is \( l \)-uniformly regular coherent and that the inclusion \( A_m \to A_{m+1} \) is flat for \( m \in \mathbb{N} \).

Then \( S(A_\ast) \) and \( L(A_\ast) \) are \( l \)-uniformly regular coherent.

Proof. We first treat \( S(A_\ast) \). Let \( \phi^1 : A^1 \to A^0 \) be a morphism in \( S(A_\ast) \). Because of Lemma 13.9(i) we can choose \( I \in I \) with \( A^1, A^0, \phi^1 \in A_{I(m)} \). By assumption we can find for each \( m \in \mathbb{N} \) an exact sequence

\[
0 \to A^1 \xrightarrow{\phi^1_m} A^0_{m-1} \xrightarrow{\phi^1_{m-1}} \cdots \xrightarrow{\phi^1_2} A^1 \xrightarrow{\phi^1_1} A^0 
\]

in \( A_{I(m)} \). We conclude from Lemma 13.9(ii) that the collection of these sequences for \( m = 0, 1, 2 \ldots \) defines a sequence in \( S(A_\ast) \)

\[
0 \to A^1 \xrightarrow{\phi^1} A^{l-1} \xrightarrow{\phi^1} \cdots \xrightarrow{\phi^1} A^1 \xrightarrow{\phi^1} A^0.
\]

Finally we show that the sequence \( 0 \to A^1 \xrightarrow{\phi^1} A^{l-1} \xrightarrow{\phi^1} \cdots \xrightarrow{\phi^1} A^1 \xrightarrow{\phi^1} A^0 \) is exact as a sequence in \( S(A_\ast) \). We have to solve for every \( j \in \{1, \ldots, l\} \) the following lifting problem in \( S(A_\ast) \).

\[
A^{j-1} \xrightarrow{\phi} A^j \xrightarrow{\phi} A^{j-1} \xrightarrow{\phi} A^j \xrightarrow{\phi} A^0.
\]
Because of Lemma \[13.9\] (i) we can choose \( J \in \mathcal{I} \) such that \( B_m, \mu_m \in A_{J(m)} \) holds. Choose \( K \in \mathcal{I} \) with \( I, J \leq K \). Now consider the following lifting problem in \( \mathcal{A}_K(m) \)

\[
A_j^{j+1} \xrightarrow{\phi_j^{j+1}} A_j^j \xrightarrow{\phi_j^j} A_j^{j-1}
\]

As the inclusion \( \mathcal{A}_I(m) \to \mathcal{A}_K(m) \) is flat by assumption, and the sequence \( A_j^{j+1} \xrightarrow{\phi_j^{j+1}} A_j^j \xrightarrow{\phi_j^j} A_j^{j-1} \) is by construction exact at \( A_j^j \), when considered in \( \mathcal{A}_I(m) \), it is exact at \( A_j^j \), when considered in \( \mathcal{A}_K(m) \). Hence \((13.13)\) has a solution \( \nu_m : B_m \to A^{j+1} \) when considered in \( \mathcal{A}_K(m) \). We conclude from Lemma \[13.9\] (iii) that the collection of the morphisms \( \nu_m \) yields a morphism \( \nu : \mathcal{B} \to \dot{\mathcal{A}}^{j+1} \) in \( \mathcal{S}(A_\star) \). Therefore \( \mathcal{B} \) is a solution to the lifting problem \((13.12)\) in \( \mathcal{S}(A_\star) \). We conclude that \((13.11)\) is an exact sequence in \( \mathcal{S}(A_\star) \). This finishes the proof of Lemma \[13.10\] for \( \mathcal{S}(A_\star) \).

The proof for \( \mathcal{L}(A_\star) \) is the following modification of the one for \( \mathcal{S}(A_\star) \). Let \( \tilde{\phi} : \dot{\mathcal{A}} \to \dot{\mathcal{A}}^0 \) be a morphism in \( \mathcal{L}(A_\star) \). Choose a representative \( \phi : \mathcal{B} \to \dot{\mathcal{A}}^0 \) in \( \mathcal{S}(A_\star) \). Now we proceed as above and constructs the sequence \((13.11)\) in \( \mathcal{S}(A_\star) \). However, instead of solving the lifting problem \((13.12)\) in \( \mathcal{S}(A_\star) \) we have to solve the lifting problem

\[
\dot{\mathcal{A}}^{j+1} \xrightarrow{\tilde{\phi}^{j+1}} \dot{\mathcal{A}}^j \xrightarrow{\tilde{\phi}^j} \dot{\mathcal{A}}^{j-1}
\]

in \( \mathcal{L}(A_\star) \). Choose a representative \( \mu \) for \( \tilde{\phi} \). There is a natural number \( M \) such that \( \phi_m \circ \mu_m = 0 \) holds for \( m \geq M \). We can change the representative \( \mu \) by putting \( \mu_m = 0 \) for \( m < M \) and by leaving \( \mu_m \) unchanged for \( m \geq M \). Now we choose a solution \( \nu_m \) to the lifting problem \((13.13)\) in \( \mathcal{A}_K(m) \) for \( m \geq M \). Put \( \nu_m = 0 \) for \( m < M \). Then we get a morphism \( \nu : \mathcal{B} \to \dot{\mathcal{A}}^{j+1} \) in \( \mathcal{S}(A_\star) \) such that its class \( \mathcal{B} : \mathcal{B} \to \dot{\mathcal{A}}^{j+1} \) in \( \mathcal{L}(A_\star) \) is a solution to the lifting problem \((13.14)\). This finishes the proof of Lemma \[13.10\]. \( \square \)

**Example 13.15** (The property Noetherian does not pass to the sequence category). The analogue of Lemma \[13.10\] for the properties Noetherian, regular, or \( l \)-uniformly regular instead of uniformly \( l \)-regular coherent does not hold as the following example shows. Suppose that none of the \( A_m \) is the trivial additive category. Consider an object \( \dot{\mathcal{A}} \) of \( \mathcal{S}(A_\star) \) such that \( A_m \neq \{0\} \) for \( m \in \mathbb{N} \), and the \( \mathbb{Z}\mathcal{S}(A_\star) \)-module

\[
F = \text{mor}_{\mathcal{S}(A_\star)}(\dot{\mathcal{A}}, \underline{A}).
\]

Define a \( \mathbb{Z}\mathcal{S}(A_\star) \)-submodule \( V \) of \( F \) by

\[
V(\tilde{\phi}) = \{ \phi \in F(\tilde{\phi}) \mid \exists M(\phi) \in \mathbb{N} \text{ with } \phi_m = 0 \text{ for } m \geq M(\phi) \}.
\]

Suppose that there exists an object \( \mathcal{B} \) and an epimorphism \( f : \text{mor}_{\mathcal{S}(A_\star)}(\dot{\mathcal{A}}, \underline{\mathcal{B}}) \to V \). If we write \( f(\text{id}(\mathcal{B})) = \tilde{\psi} \in V(\tilde{\mathcal{B}}) \), then there must be a natural number \( M \) with \( \psi_m = 0 \) for \( m \geq M \). This implies that for any \( \psi \in V(\tilde{\mathcal{B}}) \) we have \( \phi_m = 0 \) for \( m \geq M \). This is a contradiction, since \( M \) does not depend on \( \phi \). Hence \( V \) is not finitely generated and \( \mathcal{S}(A_\star) \) is not Noetherian. This construction yields also a counterexample for \( \mathcal{L}(A_\star) \).
13.c. The algebraic $K$-theory of the sequence categories $\mathcal{S}(A_*)$, $\mathcal{T}(A_*)$ and $\mathcal{L}(A_*)$. Given $I \in \mathcal{I}$, define a subcategory $\mathcal{S}(A_*)_I$ of $\mathcal{S}(A_*)$ as follows. An object $A_i$ in $\mathcal{S}(A_*)$ belongs to $\mathcal{S}(A_*)_I$, if $A_m \in A_I(m)$ holds for $m \in \mathbb{N}$. A morphism $\phi: A_i \to A_j$ in $\mathcal{S}(A_*)$ belongs to $\mathcal{S}(A_*)_I$, if and only if $\phi_m \in A_I(m)$ holds for $m \in \mathbb{N}$.

Next we show

\begin{align}
(13.16) & \quad \mathcal{S}(A_*)_I \subseteq \mathcal{S}(A_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \\
(13.17) & \quad \mathcal{S}(A_*) = \bigcup_{I \in \mathcal{I}} \mathcal{S}(A_*)_I.
\end{align}

The first equation is obvious since $A_i \subseteq A_j$ for $i \geq j$. The second follows from Lemma 13.19.

Define analogously the subcategory $\mathcal{T}(A_*)_I$ of $\mathcal{T}(A_*)$. Then we get

\begin{align}
(13.18) & \quad \mathcal{T}(A_*)_I \subseteq \mathcal{T}(A_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \\
(13.19) & \quad \mathcal{T}(A_*) = \bigcup_{I \in \mathcal{I}} \mathcal{T}(A_*)_I.
\end{align}

One easily checks that the inclusion $\mathcal{T}(A_*)_I \subseteq \mathcal{S}(A_*)_I$ is Karoubi filtration. Define

\begin{align}
(13.20) & \quad \mathcal{L}(A_*)_I = \mathcal{S}(A_*)_I/\mathcal{T}(A_*)_I.
\end{align}

Then we get

\begin{align}
(13.21) & \quad \mathcal{L}(A_*)_I \subseteq \mathcal{L}(A_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \\
(13.22) & \quad \mathcal{L}(A_*) = \bigcup_{I \in \mathcal{I}} \mathcal{L}(A_*)_I.
\end{align}

Lemma 13.23.

(i) We get for $I, J \in \mathcal{I}$ with $I \leq J$ functors

\begin{align*}
\mathcal{S}(A_*)_I & \to \mathcal{L}(A_*)_J; \\
\mathcal{S}(A_*)_I & \to \mathcal{L}(A_*),
\end{align*}

and analogously for $\mathcal{T}$ and $\mathcal{L}$;

(ii) The functors appearing in assertion (i) induce weak homotopy equivalences, natural in $A_*$

\begin{align*}
\hocolim_{I \in \mathcal{I}} K^\infty(\mathcal{S}(A_*)_I) & \xrightarrow{\simeq} K^\infty(\mathcal{S}(A_*)); \\
\hocolim_{I \in \mathcal{I}} K^\infty(\mathcal{T}(A_*)_I) & \xrightarrow{\simeq} K^\infty(\mathcal{T}(A_*)); \\
\hocolim_{I \in \mathcal{I}} K^\infty(\mathcal{L}(A_*)_I) & \xrightarrow{\simeq} K^\infty(\mathcal{L}(A_*)).
\end{align*}

Proof. \(\Box\) The desired functors come from (13.16), (13.18), and (13.21).

This follows from of (13.17), (13.19), (13.22) and [13] Corollary 7.2.

Given $I \in \mathcal{I}$, we define $\bigoplus_{m \in \mathbb{N}} A_{I(m)}$ to be the full subcategory of $\prod_{m \in \mathbb{N}} A_{I(m)}$ consisting of those objects $\bigoplus_{m \in \mathbb{N}} A_{I(m)}$ for which there exists a natural number $M$ (depending on $A$) satisfying $A_m = 0$ for $m \geq M$. Let $(\prod_{m \in \mathbb{N}} A_{I(m)}) / (\bigoplus_{m \in \mathbb{N}} A_{I(m)})$ be the quotient additive category.

Lemma 13.24. Fix $I \in \mathcal{I}$. There are weak homotopy equivalences, natural in $A_*$,

\begin{align*}
K^\infty(\prod_{m \in \mathbb{N}} A_{I(m)}) & \xrightarrow{\simeq} \prod_{m \in \mathbb{N}} K^\infty(A_{I(m)}); \\
\bigvee_{m \in \mathbb{N}} K^\infty(A_{I(m)}) & \xrightarrow{\simeq} K^\infty(\bigoplus_{m \in \mathbb{N}} A_{I(m)}),
\end{align*}

\(\Box\)
and
\[ \text{hocofib}(K^\infty(\bigoplus_{m \in \mathbb{N}} A_{I(m)}) \to K^\infty(\prod_{m \in \mathbb{N}} A_{I(m)})) \]
\[ \cong K^\infty \left( \prod_{m \in \mathbb{N}} A_{I(m)} / \bigoplus_{m \in \mathbb{N}} A_{I(m)} \right). \]

Proof. The first one is weak homotopy equivalence by [4], see also [7, Theorem 1.2], since the non-connective algebraic K-theory spectrum is indeed an Ω-spectrum. The second one is a weak homotopy equivalence, since \( \bigoplus_{m \in \mathbb{N}} A_{m} \) is the union of the subcategories \( \bigoplus_{m=0}^{n} A_{i} \) and hence we get a weak homotopy equivalence \( \hocolim_{n \to \infty} K^\infty(\bigoplus_{m=0}^{n} A_{I(m)}) \cong K^\infty(\bigoplus_{m \in \mathbb{N}} A_{I(m)}) \), and the natural map
\[ \bigvee_{m=0}^{n} K(A_{I(m)}) \cong K^\infty(\bigoplus_{m=0}^{n} A_{I(m)}) \]
is a weak homotopy equivalence. The third map is a weak homotopy equivalence, since the inclusion \( \bigoplus_{m \in \mathbb{N}} A_{I(m)} \subseteq \prod_{m \in \mathbb{N}} A_{I(m)} \) is a Karoubi filtration. □

Lemma 13.25. Given \( I \in I \), there are weak homotopy equivalences, natural in \( A^* \),
\[ K^\infty(S(A_*)_I) \cong \prod_{m \in \mathbb{N}} K^\infty(A_{I(m)}); \]
\[ \bigvee_{m \in \mathbb{N}} K^\infty(A_{I(m)}) \cong K^\infty(T(A_*)_I); \]
\[ \text{hocofib}(K^\infty(\bigoplus_{m \in \mathbb{N}} A_{I(m)}) \to K^\infty(\prod_{m \in \mathbb{N}} A_{I(m)})) \cong K^\infty(L(A)_I). \]

Proof. The are obvious identifications
\[ \prod_{m \in \mathbb{N}} A_{I(m)} = S(A_*)_I; \]
\[ \bigoplus_{m \in \mathbb{N}} A_{I(m)} = T(A_*)_I; \]
\[ \prod_{m \in \mathbb{N}} A_{I(m)} / \bigoplus_{m \in \mathbb{N}} A_{I(m)} = L(A_*)_I. \]
Now the claim follows from Lemma 13.24 □

As a consequence of Lemma 13.23(ii) and Lemma 13.25, we get

Lemma 13.26 (K-groups of \( S(A_*) \), \( T(A_*) \), and \( L(A_*) \)). There are zigzags of weak homotopy equivalences of spectra, natural in \( A_* \),
\[ \hocolim_{I \in I} \prod_{m \in \mathbb{N}} K^\infty(A_{I(m)}) \cong K^\infty(S(A_*)); \]
\[ \hocolim_{I \in I} \bigvee_{m \in \mathbb{N}} K^\infty(A_{I(m)}) \cong K^\infty(T(A_*)); \]
\[ \hocolim_{I \in I} \left( \text{hocofib}(K^\infty(\bigoplus_{m \in \mathbb{N}} A_{I(m)}) \to K^\infty(\prod_{m \in \mathbb{N}} A_{I(m)})) \right) \cong K^\infty(L(A)). \]
In particular we get from Lemma [13.24] for every \( n \in \mathbb{Z} \) an isomorphism

\[
(13.27) \quad \text{colim}_{i \in I} \left( \prod_{m \in \mathbb{N}} K_n(A_f(m)) \bigg/ \bigoplus_{m \in \mathbb{N}} K_n(A_{f(m)}) \right) \cong K_n(\mathcal{L}(A_*)).
\]

14. The main technical result

14.1. The statement of the main technical result. Fix a natural number \( r \) and a nested sequence of additive categories \( A_* \). We have defined the additive category \( \mathcal{L}(A_*) \) in Definition 13.3. Suppose that each \( A_* \) comes with a \( \mathbb{Z}^r \)-action \( \Phi : \mathbb{Z}^r \to \text{aut}(\mathcal{L}(A_*)) \) by \( \text{pro-automorphisms} \) in the sense of Definition 13.2. Then we obtain a \( \mathbb{Z}^r \)-action \( \Phi : \mathbb{Z}^r \to \text{aut}(\mathcal{L}(A_*)) \) on \( \mathcal{L}(A_*) \); see (13.7). We obtain a covariant functor, see for instance [1, Section 9],

\[
K_{\mathcal{L}(A_*)}^\infty : \text{Or}(\mathbb{Z}^r) \to \text{Spectra}.
\]

It determines a \( \mathbb{Z}^r \)-homology theory \( H^r_n(\mathbb{Z}^r; \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \) with the property that for every subgroup \( H \subseteq \mathbb{Z}^r \) and \( n \in \mathbb{Z} \) we have the natural isomorphisms

\[
H^r_n(\mathbb{Z}^r/H; \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \cong K_n(\mathcal{L}(A_*) \rtimes \varphi[H] \mathbb{Z}^r)
\]

as explained for instance in [1 Section 9]. The nested sequence of additive categories \( A_* \) yields for any natural number \( d \) another nested sequence of additive categories \( A_*[\mathbb{Z}^d] \) by \( A_0[\mathbb{Z}^d] \supseteq A_1[\mathbb{Z}^d] \supseteq A_2[\mathbb{Z}^d] \supseteq \cdots \), where \( A_m[\mathbb{Z}] \) is the untwisted case of Definition 13.3 and we define inductively \( A_m[\mathbb{Z}^d] = (A_m[\mathbb{Z}^{d-1}])[\mathbb{Z}] \). Moreover, \( \mathcal{L}(A_*[\mathbb{Z}^d]) \) inherits a \( \mathbb{Z}^r \)-action \( \Phi[\mathbb{Z}^d] : \mathbb{Z}^r \to \text{aut}(\mathcal{L}(A_*[\mathbb{Z}^d])) \).

The main result of this section is

Theorem 14.1. Suppose:

(i) For every natural number \( d \) there exists a natural number \( l(d) \) such that for any natural number \( m \) in the additive category \( A_m[\mathbb{Z}^d] \) is \( l(d) \)-uniformly regular coherent;

(ii) The inclusion \( A_{m+1}[\mathbb{Z}^d] \rightarrow A_m[\mathbb{Z}^d] \) is exact for any natural numbers \( d \) and \( m \).

Then the map induced by the projection \( E\mathbb{Z}^r \to \text{pt} \)

\[
H^r_n(\mathbb{Z}^r; \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \rightarrow H^r_n(\text{pt}; \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \cong K_n(\mathcal{L}(A_*) \rtimes \varphi \mathbb{Z}^r)
\]

is bijective for all \( n \in \mathbb{Z} \).

14.B. Reduction to the case \( r = 1 \).

Lemma 14.2. If Theorem 14.1 holds for \( r = 1 \), it is true for all \( r \geq 1 \).

Proof. The Farrell–Jones Conjecture is known to be true for \( \mathbb{Z}^r \) and implies that the map induced by the projection \( E\mathbb{V}_{\text{cyc}}(\mathbb{Z}^r) \to \text{pt} \)

\[
H^r_n(E\mathbb{V}_{\text{cyc}}(\mathbb{Z}^r); \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \rightarrow H^r_n(\text{pt}; \mathbb{K}_{\mathcal{L}(A_*)}^\infty)
\]

is an isomorphism for all \( n \in \mathbb{Z} \), where \( E\mathbb{V}_{\text{cyc}}(\mathbb{Z}^r) \) is the classifying space of the family \( \mathbb{V}_{\text{cyc}} \) of virtually cyclic subgroups of \( G \), see for instance [10]. We also have the map induced by the up to \( \mathbb{Z}^r \)-homotopy unique \( \mathbb{Z} \)-map \( E\mathbb{Z}^r \rightarrow E\mathbb{V}_{\text{cyc}}(\mathbb{Z}^r) \).

\[
(14.3) \quad H^r_n(E\mathbb{Z}^r; \mathbb{K}_{\mathcal{L}(A_*)}^\infty) \rightarrow H^r_n(\mathbb{V}_{\text{cyc}}(\mathbb{Z}^r); \mathbb{K}_{\mathcal{L}(A_*)}^\infty).
\]

Obviously it suffices to show that (14.3) is bijective for all \( n \in \mathbb{Z} \). By the Transitivity Principle, see for instance [12, Theorem 65 on page 742], this boils down to show that for any non-trivial virtually cyclic subgroup \( V \) of \( \mathbb{Z}^n \) the map induced by the projection \( EV \rightarrow \text{pt} \)

\[
H^r_n(EV; \mathbb{K}_{\mathcal{L}(A_*)}^\infty|_V) \rightarrow H^r_n(\text{pt}; \mathbb{K}_{\mathcal{L}(A_*)}^\infty|_V)
\]
is bijective for all \(n \in \mathbb{Z}\). Since any non-trivial virtually cyclic subgroup \(V\) of \(\mathbb{Z}^r\) is isomorphic \(\mathbb{Z}\), we have reduced the proof of Theorem 14.1 to the special case \(r = 1\).

14.c. **Strategy of proof of Theorem 14.1.** We first explain, why the proof of Theorem 14.1 is more difficult than we had expected and the reader may anticipate. There are the following reasons.

- There is no proof of the Devissage Theorem for non-connective \(K\)-theory.
- We have used the Devissage Theorem for connective \(K\)-theory in the proof of Lemma 7.3. If Devissage would hold also in the non-connective setting, the proof of Lemma 7.3 could be extended to the non-connective \(K\)-theory spectrum and hence Lemma 7.7 and Theorem 7.8 would still be true, if we replace the assumption that \(A[\mathbb{Z}^m]\) is regular coherent for every \(m \geq 0\) by the weaker assumption that that \(A\) is regular coherent.
- Then the stronger version of Theorem 14.1, where one demands the conditions only for \(d = 0\), would follow from the version of Theorem 7.8 mentioned above, and Lemma 13.10 and Lemma 14.2.

Since the known proofs of the Devissage Theorem for non-connective \(K\)-theory make strong assumptions on the underlying categories, which are not at all true in our case, we cannot argue like this. The consequence is that we need to make regularity assumptions about \(A_m[\mathbb{Z}^d]\) for all \(d \geq 0\) and not only for \(d = 0\).

- If the additive category \(A\) is regular coherent, it is unknown whether \(A[\mathbb{Z}]\) is regular coherent.
- It is an open well-known problem, whether for a regular coherent ring \(R\) the ring \(R[\mathbb{Z}^d]\) is again regular coherent. Hence we do not know, whether for a regular coherent additive category \(A\), the additive category \(A[\mathbb{Z}^d]\) is regular coherent again.

- The additive category \(\mathcal{L}(A_\ast)\) is not Noetherian and hence not regular.
- We have shown that for a regular additive category \(A\), the additive category \(A[\mathbb{Z}^d]\) is regular again, see Theorem 10.1. However, the additive category \(\mathcal{L}(A_\ast)\) is not Noetherian and hence not regular, see Example 13.15.

- The canonical inclusion \(\mathcal{L}(A_\ast)[\mathbb{Z}^d] \to \mathcal{L}(A_\ast[\mathbb{Z}^d])\) is not an equivalence of additive categories for \(d \geq 4\).

The assumptions on the \(A_m\) imply that \(\mathcal{L}(A_\ast[\mathbb{Z}^d])\) is regular coherent, but we have no information about the ring \(\mathcal{L}(A_\ast)[\mathbb{Z}^d]\). For this reason we cannot use the fact (coming from the Bass-Heller-Swan decomposition) that \(K_n(\mathcal{L}(A_\ast)\) is a direct summand of \(K_{n+d}(\mathcal{L}(A_\ast))[\mathbb{Z}^d]\) to deduce Theorem 14.1 for all \(n\) from the case \(n > 0\). On the other hand for \(\mathcal{L}(A_\ast[\mathbb{Z}^d])\) we do not have a Bass-Heller-Swan decomposition. But we will exhibit \(K_n(\mathcal{L}(A_\ast)\)) also as a direct summand of \(K_{n+d}(\mathcal{L}(A_\ast[\mathbb{Z}^d]))\) and this will be the main work in the remainder of this paper. To this end we will check that enough of the arguments used in 14 to construct the Bass-Heller-Swan decomposition can also be applied to \(\mathcal{L}(A_\ast[\mathbb{Z}^d])\) and yield the desired direct summand.

14.d. **Twisted Laurent categories.** Recall the notion of a (twisted) finite Laurent category from Definition 7.3. Given a sequences of additive subcategories \(A_\ast\), define the new sequences of additive subcategories \(A_\ast[t^\pm]\) and \(A_\ast[t, t^\pm]\) by replacing \(A_m\) by the associated untwisted finite Laurent categories \(A_m[t^\pm]\) and \(A_m[t, t^\pm]\).

\(^3\)The inclusion is bijective on objects, but not on morphisms.

\(^4\)It seems not obvious how a Bass-Heller-Swan decomposition for \(\mathcal{L}(A_\ast[\mathbb{Z}^d])\) should look like and what the Nil terms should be.
A pro-automorphism \(\Phi : A_* \to A_*\) defines pro-automorphisms \(\Phi[t^\pm] : A_*[t^\pm] \to A_*[t^\pm]\) and \(\Phi[t, t^{-1}] : A_*[t, t^{-1}] \to A_*[t, t^{-1}]\). Define

\[
\begin{align*}
C(\Phi) &= S(A_*)_{S(\Phi)[\mathbb{Z}]}; \\
C[t^\pm](\Phi) &= S(A_*[t^\pm])_{S(\Phi[t^\pm])[\mathbb{Z}]}; \\
C[t, t^{-1}](\Phi) &= S(A_*[t, t^{-1}])_{S(\Phi[t, t^{-1}])[\mathbb{Z}]}.
\end{align*}
\]

The passage from \(\Phi\) to \(C(\Phi)\), \(C[t^\pm](\Phi)\), and \(C[t, t^{-1}](\Phi)\) is natural. Often we omit \(\Phi\) from the notation. The set of objects of the categories \(C(\Phi)\), \(C[t^\pm](\Phi)\), and \(C[t, t^{-1}](\Phi)\) can and will be identified with the set of objects of \(S(A_*)\) and hence are independent of \(\Phi\).

14.E. Induction functors. Next we define a commutative square of additive categories

\[
\begin{array}{ccc}
C & \xrightarrow{i_+} & C[t] \\
\downarrow i_- & & \downarrow j_+ \\
C[t^{-1}] & \xrightarrow{j_-} & C[t, t^{-1}]
\end{array}
\]

(14.4)

where the functors \(i_0, i_+, i_-, j_-, j_+\) induce the identity on the set of objects. For each additive category \(A_m\), we have a diagram of functors of untwisted Laurent categories, where all functors are the canonical ones, see [14] Section 1.4,

\[
\begin{array}{ccc}
A_m & \xrightarrow{i_+(A_m)} & A_m[t] \\
i_-(A_m) & \downarrow i_0(A_m) & \downarrow j_+(A_m) \\
A_m[t^{-1}] & \xrightarrow{j_-(A_m)} & A_m[t, t^{-1}].
\end{array}
\]

This construction is natural with respect to the inclusions \(A_{m+1} \to A_m\). Hence we obtain a commutative diagram of additive categories

\[
\begin{array}{ccc}
S(A_*) & \xrightarrow{S(i_+(A_*))} & S(A_*[t]) \\
S(i_-(A_*)) & \downarrow S(i_0(A_*)) & \downarrow S(j_+(A_*)) \\
S(A_*)[t^{-1}] & \xrightarrow{j_-(A_*))} & S(A_*[t, t^{-1}]).
\end{array}
\]

Since it is compatible with the automorphisms of additive categories \(S(\Phi)\) of \(S(A_*)\), \(S(\Phi[t^\pm])\) of \(S(A_*[t^\pm])\), and \(S(\Phi[t, t^{-1}])\) of \(S(A_*[t, t^{-1}])\), it yield the desired diagram \((14.4)\) in the obvious way.

Note that the diagram diagram \((14.4)\) is natural in \(\Phi : A_* \to A_*\).

14.F. Formally adjoining infinite direct sums. In [14] Section 1.3] a functorial extension \(A \subseteq A^\infty\) is constructed for every additive category \(A\) such that \(A \subseteq A^\infty\) is an inclusion of additive categories and in \(A^\infty\) the direct sums over a collection of objects over a countable set is defined. Now define for a nested sequence of additive categories a functor of additive categories \(A_*\) given by \(A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots\) the nested sequence of additive categories \(A^\infty_*\) by \(A^\infty_0 \supseteq A^\infty_1 \supseteq A^\infty_2 \supseteq \cdots\). The given pro-automorphism \(\phi : A_* \to A_*\) yields a pro-automorphism \(\phi^\infty : A^\infty_* \to A^\infty_*\) by passing
14. Restriction functors. In [14, Section 1.5] restriction functors are defined for each additive category \( A_m \)

\[
\begin{align*}
\kappa^0(\mathcal{A}_m) : \mathcal{A}_m [t, t^{-1}]^\kappa & \to \mathcal{A}_m; \\
\kappa^\pm(\mathcal{A}_m) : \mathcal{A}_m [t^{\pm}]^\kappa & \to \mathcal{A}_m,
\end{align*}
\]

such that they come with adjunctions \( (\kappa^0(\mathcal{A}_m))^\kappa, \kappa^0(\mathcal{A}^\kappa_m)), \kappa(\mathcal{A}_m)^\kappa, \kappa^+ \kappa(\mathcal{A}_m)^\kappa \), and \( \kappa(\mathcal{A}_m)^\kappa \). Moreover, everything is natural and in particular compatible with the automorphisms \( \Phi(t, t^{-1}]^\kappa: A_0[t, t^{-1}]^\kappa \to A_0[t, t^{-1}]^\kappa \) and \( \Phi[\kappa^\pm]: A_0[\kappa^\pm] \to A_0[\kappa^\pm] \). Hence these data define analogously to the construction of Subsection 14.3 restriction functors

\[
\begin{align*}
\kappa^0 : \mathcal{C} & \to \mathcal{C}^\kappa; \\
\kappa^\pm : \mathcal{C}[t^{\pm}] & \to \mathcal{C}^\kappa,
\end{align*}
\]

such that we have adjunctions \( ((\kappa^0)^\kappa, \kappa^0), ((\kappa^+)\kappa, \kappa^+), \) and \( ((\kappa)^\kappa, \kappa^-) \). Everything is natural in \( \Phi \).

14.8. Truncation functors. Put

\[ \mathbb{Z} := \mathbb{Z} \cap \{ -\infty, \infty \}. \]

Notation 14.8 (Truncation for objects). Let \( \mathbf{A} = (A_m)_{m \in \mathbb{N}} \) and \( \mathbf{A}' = (A'_m)_{m \in \mathbb{N}} \) be objects in \( \mathcal{S}(A) \). Consider elements \( \mathbf{a} = (a_m)_{m \in \mathbb{N}} \) and \( \mathbf{b} = (b_m)_{m \in \mathbb{N}} \) in \( \prod_{m \in \mathbb{N}} \mathbb{Z} \). Define an object in \( \mathcal{C}^\kappa \) by

\[
\mathbf{A}[\mathbf{b}, \mathbf{b}'] = \left\{ \bigoplus_{k, a, m} A_m \bigg| m \in \mathbb{N} \right\},
\]

where \( \bigoplus_{k, a, m} A_m \) is to be defined to be zero if \( a_i > b_i \) or if we have \( a_i = b_i \) and \( b_i \notin \{ -\infty, \infty \} \). Note that the direct sum \( \bigoplus_{k, a, m} A_m \) has as entry for \( k_i \) always the same object, namely \( A_m \). Since we are working in \( \mathcal{C}^\kappa \), this definition makes sense also in the case where \( a_i = -\infty \) or \( b_i = \infty \).

Given an element \( c \in \mathbb{Z} \), denote by \( \mathbf{c} \) the element in \( \prod_{m \in \mathbb{N}} \mathbb{Z} \), whose value at every \( m \in \mathbb{N} \) is \( c \). Then we get for any object \( \mathbf{A} \) in \( \mathcal{C}[t, t^{-1}] \), which is the same as an object in \( \mathcal{S}(A) \),

\[ i^0 \mathbf{A} = \mathbf{A}[\mathbf{-\infty, \infty}]. \]
Given a morphism \( f : A \to A' \) in \( C[t, t^{-1}] \) and \( a, b, a', b' \) in \( \prod_{m \in \mathbb{N}} \mathbb{Z} \), define the \( C^\infty \)-morphism \( f[: A[a, b] \to A'[a', b'] \) to be the composite
\[
\tilde{f}[: A[a, b] \to A[\infty, \infty] = i^0 A \mapsto A'[\infty, \infty] = \tilde{f}[: A[\infty, \infty] \to A'[\infty, \infty],
\]
where \( \tilde{f} \) is the obvious inclusion and \( \tilde{p} \) is the obvious projection in \( C^\infty \).

The morphism \( f[: A = A[\infty, \infty] \to A'[\infty, \infty] \) agrees with \( i^0 \tilde{f} \) for a morphism \( f : A \to A' \) in \( C[t, t^{-1}] \). If \( f \) belongs to \( C[t^\pm] \), we abbreviate \((i \pm f)[:] \) by \( f[:] \) again.

Note that \((a \circ \tilde{f})[:] \) is in general not equal to \( g[:] \circ f[:] \) and \( \text{id}[:] \) is in general not the identity. As a typical example, let \( f^+: A \to A \) be the morphism \( \text{id} + t \) and \( f^- : A \to A \) be the morphism \( \text{id} \circ t^{-1} \). Then
\[
(f^+ \circ f^-)[:] : A[0, \infty] \to A[0, \infty]
\]
is the identity. The map
\[
f^-[:] : A[0, \infty] \to A[0, \infty]
\]
is given at each \( m \in \mathbb{N} \) by the map \( \bigoplus_{k=0}^\infty A_m \to \bigoplus_{k=0}^\infty A_m \) sending \( \{u_k | k = 0, 1, 2, \ldots \} \) to \( \{u_{k+1} | k = 0, 1, 2, \ldots \} \). Since \( f^-[:] \) is not injective, we do not have \( f^[:] \circ f^-[:] = (f^+ \circ f^-)[:]. \)

As another example,
\[
\text{id} A[:] : A[\infty, \infty] \to A[0, 0]
\]
is given at every \( m \in \mathbb{N} \) by the projection onto the 0th summand \( \bigoplus_{k=-\infty}^\infty A_m \to A_m \).

**Notation 14.9** (Truncation for chain complexes). If \( C^+ \) is a \( C[t] \)-chain complex and \( a, b \in \prod_{m \in \mathbb{N}} \mathbb{Z} \), then we obtain a \( C^\infty \)-chain complex \( C^+ [a, b] \) by defining the \( n \)-th chain object to be \( C^+_n [a, b] \) and the \( n \)-th differential to be \( c_n[:] : C^+_n [a, b] \to C^+_{n-1} [a, b] \), if \( c_n \) is the differential of \( C^+ \). (One has to check that \( c_n[:] \circ c_{n+1}[:] = 0 \).)

A chain map \( f : C^+ \to D^+ \) of \( C[t] \)-chain complexes induces a \( C^\infty \)-chain map denoted by \( f[:] : C^+[a, b] \to D^+[a, b] \), provided that \( a' \leq a \), i.e., \( a'_n \leq a_n \) for all \( m \in \mathbb{N} \), and \( b' \leq b \) hold.

If \( C^- \) is an \( C[t^{-1}] \)-chain complex and \( a, b \in \prod_{m \in \mathbb{N}} \mathbb{Z} \), define the \( C^\infty \)-chain complex \( C^- [a, b] \) analogously. A chain map \( f : C^- \to D^- \) of \( C[t^{-1}] \)-chain complexes induces a \( C^\infty \)-chain map denoted by \( f[:] : C^- [a, b] \to D^- [a, b] \), provided that \( b' \geq b \), i.e., \( a'_m \geq a_m \) for all \( m \in \mathbb{N} \), and \( b' \geq b \) hold.

Note that Notation 14.9 (in contrast to Notation 14.8) does in this generality not make sense for chain complexes in \( C[t, t^{-1}] \), e.g., \( c_n[:] \circ c_{n+1}[:] = 0 \) does not hold anymore.

14.1. **Some basic tools for non-connective \( K \)-theory.** Recall that we have defined the negative \( K \)-theory of an additive category using the delooping construction based on the Bass-Heller-Swan decomposition of [13]. In this section we present another definition based on the non-connective \( K \)-theory spectrum associated to appropriate Waldhausen categories due to Bunke-Kasprowski-Winge [2].

The next definition is taken from [2, Definition 2.1].

**Definition 14.10.**

(i) The Waldhausen category \( W \) admits factorizations, if every morphism in \( W \) can be factorized into a cofibration followed by a weak equivalence; no functoriality of this factorization is assumed;
(ii) The Waldhausen category $\mathcal{W}$ is homotopical, if it admits factorizations and the weak equivalences satisfy the two-out-of-six property, i.e., if for composable morphisms $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3$ in $\mathcal{W}$ both $f_2 \circ f_1$ and $f_3 \circ f_2$ are weak equivalences, then also $f_1$, $f_2$, $f_3$, and $f_3 \circ f_2 \circ f_1$ are weak equivalences.

Let $\text{Wald}^{ho}$ be the category of homotopical Waldhausen categories. In the sequel we denote by

$$K^{\infty,W} : \text{Wald}^{ho} \to \text{Spectra}$$

the non-connective $K$-theory functor constructed in [2, Definition 2.37].

**Remark 14.12.** Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ becomes a Waldhausen category, if we define the weak equivalences to be the isomorphisms and the cofibration to be the morphisms $f : A \to B$, for which there exists an object $A^+$ and an isomorphism $u : A \oplus A^+ \xrightarrow{\sim} B$ such that the composite of $u$ with the canonical inclusion $A \to A \oplus A^+$ is $f$. Note that this Waldhausen category is not homotopical, as it does not satisfy factorization. So we cannot apply (14.11) to the Waldhausen category $\mathcal{A}$.

Let $\text{Ch}(\mathcal{A})$ be the Waldhausen category of bounded chain complexes over $\mathcal{A}$, where a cofibration $f_* : C_* \to D_*$ is a chain map such that $f_n : C_n \to D_n$ is a cofibration in $\mathcal{A}$ and the weak equivalences are the chain homotopy equivalences. Then $\text{Ch}(\mathcal{A})$ is homotopical thanks to the mapping cylinder construction. Hence we can apply (14.11) to the Waldhausen category $\text{Ch}(\mathcal{A})$ and can consider its non-connective $K$-theory spectrum $K^{\infty,W}(\text{Ch}(\mathcal{A}))$.

More generally, if $\mathcal{A}$ is an exact category, then the Waldhausen category $\text{Ch}(\mathcal{A})$ can be defined analogously and is homotopical.

Suppose that $W$ is a category with cofibrations and that $W$ is equipped with two categories of weak equivalences, one finer than the other, $vW \subseteq wW$. Thus $W$ becomes a Waldhausen category in two ways. Suppose that in both cases $W$ is a homotopical Waldhausen category. Let $W^w$ denote the full subcategory of $W$ given by the objects $C$ in $W$ having the property that the map $C \to \text{pt}$ belongs to $vW$. Then $W^w$ inherits two Waldhausen structures, if we put $vW^w = W^w \cap vW$ and $wW^w = W^w \cap wW$. Both yield homotopical Waldhausen categories.

**Theorem 14.13** (Fibration Theorem). Under the assumptions above we get a weak homotopy fibration of spectra

$$K^{\infty,W}(W^w, vW^w) \to K^{\infty,W}(W, vW) \to K^{\infty,W}(W, wW).$$

**Proof.** This follows from [2, Theorem 2.35].

**Theorem 14.14** (Cisinski’s Approximation Theorem). Let $F : W_0 \to W_1$ be an exact functor of homotopical Waldhausen categories. Assume:

(i) An arrow in $W_0$ is a weak equivalence in $W_0$, if and only if its image in $W_1$ is a weak equivalence in $W_1$;

(ii) Given any object $C_0$ in $W_0$ and any map $f : F(C_0) \to C_1$ in $W_1$, there exists a commutative diagram in $W_1$

$$\begin{array}{ccc}
F(C_0) & \xrightarrow{f} & C_1 \\
\downarrow F(u) & & \downarrow v \\
F(D_0) & \xrightarrow{w} & D_1
\end{array}$$

for a morphism $u : C_0 \to D_0$ in $W_0$ and weak equivalences $v : C_1 \to D_1$ and $w : F(D_0) \to D_1$ in $W_1$. 


Then the map of spectra $\mathbf{K}^{\infty,W}(F): \mathbf{K}^{\infty,W}(W_0) \to \mathbf{K}^{\infty,W}(W_1)$ is a weak homotopy equivalence.

Proof. This follows from [2, Theorem 2.16]. □

**Theorem 14.15** (Cofinality Theorem). Let $I: W_0 \to W_1$ be the inclusion of a full homotopical Waldhausen subcategory $W_0$ into a homotopical Waldhausen category $W_1$. Assume:

(i) The functor $F$ admits a mapping cylinder argument, i.e., for every morphism $f: C_0 \to C_1$ in $W_1$ such that $C_0$ belongs to $W_0$ and $C_1$ is the target of a weak equivalence with some object in $W_0$ as source, there is a factorization in $W_1$

$$C_0 \xrightarrow{f'} C' \xrightarrow{f''} C_1$$

such that $C''$ belongs to $W_0$ and $f''$ is a weak equivalence;

(ii) The category $W_1$ is dominated by $W_0$, i.e., for any object $C_1$ in $W_1$ there exists an object $C_0$ in $W_0$ and an object $C_1'$ in $W_1$ and morphisms $r: C_0 \to C_1$ and $i: C_1' \to C_0$ such that $r \circ i$ is a weak equivalence.

Then $\mathbf{K}^{\infty,W}(I): \mathbf{K}^{\infty,W}(W_0) \to \mathbf{K}^{\infty,W}(W_1)$ is a weak homotopy equivalence.

Proof. This follows from [2, Theorem 2.30] and the fact that on the level of stable $\infty$-categories non-connective $K$-theory is inverting the passage to the idempotent completion. □

For the proof of the following result we refer to [11, Theorem 18.17].

**Theorem 14.16** (The weak homotopy fibration sequence of a stable Karoubi filtration for $K$-theory in the setting of Waldhausen categories). Let $\mathcal{A}$ be a additive category and $i: \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory. If the additive category $\mathcal{A}$ is stably $\mathcal{U}$-filtered, then

$$\mathbf{K}^{\infty,W}(\text{Ch}(\mathcal{U})) \to \mathbf{K}^{\infty,W}(\text{Ch}(\mathcal{A})) \to \mathbf{K}^{\infty,W}(\text{Ch}(\mathcal{A}/\mathcal{U}))$$

is a weak homotopy fibration of non-connective spectra, where $\mathbf{K}^{\infty,W}$ has been defined in [11,11].

The proof of the next result can be found in [11, Theorem 18.33].

**Theorem 14.17** (Gillet-Waldhausen zigzag for non-connective $K$-theory). There is a zigzag of weak homotopy equivalences, natural in $\mathcal{A}$, from the non-connective $K$-theory spectrum $\mathbf{K}^{\infty,W}(\text{Ch}(\mathcal{A}))$ in the sense of Bunke-Kasprowski-Winges [2] to the non-connective $K$-theory spectrum $\mathbf{K}^{\infty}(\mathcal{A})$ in the sense of Lück-Steimle [13].

### 14.1. The projective line.

**Definition 14.18** (Projective line). We define the projective line $\mathcal{X}$ to be the following additive category. Objects are triples $(C^+, f, C^-)$ consisting of objects $C^+$ in $\mathcal{C}[t^+]$ and $C^-$ in $\mathcal{C}[t^-]$ and an isomorphism $f: j_+(C^+) \to j_-(C^-)$ in $\mathcal{C}[t, t^{-1}]$. A morphism $(\varphi^+, \varphi^-): (C^+, f, C^-) \to (D^+, g, D^-)$ in $\mathcal{X}$ consists of morphisms $\varphi^+: C^+ \to D^+$ in $\mathcal{C}[t]$ and $\varphi^-: C^- \to D^-$ in $\mathcal{C}[t^{-1}]$ such that the following diagram commutes in $\mathcal{C}[t, t^{-1}]$

$$\begin{array}{ccc}
j_+(C^+) & \xrightarrow{f} & j_-(C^-) \\
j_+(\varphi^+) & & j_-(\varphi^-) \\
j_+(D^+) & \xrightarrow{g} & j_-(D^-) \end{array}$$
Let
\[ k^\pm : \mathcal{X} \to \mathcal{C}[t^{\pm 1}] \]
be the functor sending \((C^+, f, C^-)\) to \(C^\pm\).

The category \(\mathcal{X}\) is naturally an exact category by declaring a sequence to be exact, if and only if becomes (split) exact both after applying \(k^+\) and \(k^-\). With this exact structure we obtain the structure of a homotopical Waldhausen category on \(\text{Ch}(\mathcal{X})\). Hence \(\mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{X}))\) is defined, in contrast to \(\mathcal{K}_{\infty}(\mathcal{X})\) which does not make sense, since \(\mathcal{X}\) is neither an additive category nor a homotopical Waldhausen category.

The next theorem will be a main ingredient in the proof of Theorem 14.31.

**Theorem 14.20** (The algebraic \(K\)-theory of the projective line). Consider the following (not necessarily commutative) diagram of spectra

\[
\begin{array}{ccc}
\mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{X})) & \xrightarrow{\mathcal{K}_{\infty,W}(\text{Ch}(k^-))} & \mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{C}[t^{\pm 1}])), \\
\mathcal{K}_{\infty,W}(\text{Ch}(k^+)) & \downarrow & \mathcal{K}_{\infty,W}(\text{Ch}(j_-)) \\
\mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{C}[t])) & \xrightarrow{\mathcal{K}_{\infty,W}(\text{Ch}(j_+))} & \mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{C}[t,t^{-1}])),
\end{array}
\]

There is a natural equivalence of functors \(T : j_+ \circ k^+ \cong j_- \circ k^-\), which is given on an object \((A^+, f, A^-)\) by \(f\). It induces a preferred homotopy \(\mathcal{K}_{\infty,W}(\text{Ch}(j_+)) \circ \mathcal{K}_{\infty,W}(\text{Ch}(k^+)) \cong \mathcal{K}_{\infty,W}(\text{Ch}(j_-)) \circ \mathcal{K}_{\infty,W}(\text{Ch}(k^-))\).

Then the diagram above is a weak homotopy pullback, i.e., the canonical map from \(\mathcal{K}_{\infty,W}(\mathcal{X})\) to the homotopy pullback of

\[
\begin{array}{ccc}
\mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{C}[t^{-1}])) & \downarrow & \\
\mathcal{K}_{\infty,W}(\text{Ch}(j_-)) & \mathcal{K}_{\infty,W}(\text{Ch}(j_+)) & \mathcal{K}_{\infty,W}(\text{Ch}(\mathcal{C}[t,t^{-1}])),
\end{array}
\]

is a weak homotopy equivalence.

For the proof of Theorem 14.20 we can use the ideas of [13] Section 5] taking into account that meanwhile the basic tools appearing in [13] Section 4] have been proved also for non-connective \(K\)-theory.

In the first step of the proof of Theorem 14.20 we replace the additive category \(\mathcal{C}[t]\) by a larger exact category \(\mathcal{Y}\) with equivalent \(K\)-theory. It is defined as follows: An object of \(\mathcal{Y}\) is a triple \((A^+, f, A^-)\) consisting of objects \(A^+\) and \(A^-\) of \(\mathcal{C}[t,t^{-1}]\) and an isomorphism \(f : j_+ A^+ \to A \in \mathcal{C}[t,t^{-1}]\). A morphism from \((A^+, f, A)\) to \((B^+, g, B)\) is a morphism \(\varphi : A^+ \to B^+\) in \(\mathcal{C}[t]\) and a morphism \(\varphi : A \to B \in \mathcal{C}[t,t^{-1}]\) such that diagram in \(\mathcal{C}[t,t^{-1}]\)

\[
\begin{array}{ccc}
j_+(A^+) & \xrightarrow{f} & A \\
j_+(\varphi^+) & \downarrow & \varphi \\
j_+(B^+) & \xrightarrow{g} & B
\end{array}
\]

commutes. Note that \(\varphi\) is determined already by \(\varphi^+\). The category \(\mathcal{Y}\) is exact in the same way as \(\mathcal{X}\) is. Note that \(\mathcal{X}\) and \(\mathcal{Y}\) have the same set of objects but \(\mathcal{Y}\) contains more morphisms, since in contrast to \(\mathcal{X}\) we do not require that \(\varphi\) belongs to \(\mathcal{C}[t^{-1}]\).
Lemma 14.21. The functors
\[ a: \mathcal{C}[t] \to \mathcal{Y}, \quad A \mapsto (A, \text{id}, j_A) \]
and
\[ b: \mathcal{Y} \to \mathcal{C}[t], \quad (A^+, f, A^-) \mapsto A^+ \]
are exact. The composite $b \circ a$ is the identity and the composite $a \circ b$ is naturally isomorphic to the identity functor. In particular, they induce homotopy equivalences on non-connective $K$-theory, homotopy inverse to each other,
\[ K^\infty_{\mathcal{W}}(\text{Ch}(a)): K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{C}[t])) \xrightarrow{\simeq} K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{Y})); \]
\[ K^\infty_{\mathcal{W}}(\text{Ch}(b)): K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{Y})) \xrightarrow{\simeq} K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{C}[t])). \]

Proof. It is clear that the functors are exact. Obviously $b \circ a$ is the identity. The composite $a \circ b$ is naturally isomorphic to the identity functor: the isomorphism in $\mathcal{Y}$ at the object $(A^+, f, A)$ is given by $(\text{id}, f)$: $(A^+, \text{id}, j_A A^+) \xrightarrow{\simeq} (A^+, f, A)$. This implies $K(a) \circ K(b) \simeq \text{id}$. □

Denote by
\[ k': \mathcal{X} \to \mathcal{Y} \]
the inclusion functor, and define
\[ j': \text{Ch}(\mathcal{Y}) \to \text{Ch}(\mathcal{C}[t, t^{-1}]), \quad (A^+, f, A) \mapsto A. \]

Then the square
\[ \begin{array}{ccc}
\text{Ch}(\mathcal{X}) & \xrightarrow{\text{Ch}(k^-)} & \text{Ch}(\mathcal{C}[t^{-1}]) \\
\text{Ch}(k') \downarrow & & \downarrow \text{Ch}(j^-) \\
\text{Ch}(\mathcal{Y}) & \xrightarrow{\text{Ch}(j')} & \text{Ch}(\mathcal{C}[t, t^{-1}])
\end{array} \]
is strictly commutative, and we are going to show that it induces a weak homotopy pullback after applying $K^\infty_{\mathcal{W}}$. To show that the square is a weak homotopy pullback on non-connective $K$-theory, we are going to show that the horizontal homotopy fibers of $K^\infty_{\mathcal{W}}(\text{Ch}(k^-))$ and $K^\infty_{\mathcal{W}}(\text{Ch}(j'))$ agree up to weak homotopy equivalence.

Let $w\text{Ch}(\mathcal{X})$ be the subcategory of $\text{Ch}(\mathcal{X})$ consisting of all chain maps, which become after applying $\text{Ch}(k^-)$ weak equivalences in $\text{Ch}(\mathcal{C}[t^{-1}])$. Let $\text{Ch}(\mathcal{X})^w$ be the full subcategory of $\text{Ch}(\mathcal{X})$ of all objects, which are $w$-acyclic. In other words, an object $(C^+, f, C^-)$ belongs to $\text{Ch}(\mathcal{X})^w$, if and only if $C^-$ is contractible as an $C[t^{-1}]$-chain complex. Similarly, denote by $w\text{Ch}(\mathcal{Y})$ the subcategory of all morphisms $f$ such that $\text{Ch}(j')(f)$ is a chain homotopy equivalence in $\text{Ch}(\mathcal{C}[t, t^{-1}])$, and adopt the notation $\text{Ch}(\mathcal{Y})^w$ for the $w$-acyclic objects.

Lemma 14.22. The maps
\[ K^\infty_{\mathcal{W}}(\text{Ch}(k^-)): K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{X}), w) \to K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{C}[t^{-1}])); \]
\[ K^\infty_{\mathcal{W}}(\text{Ch}(j')): K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{Y}), w) \to K^\infty_{\mathcal{W}}(\text{Ch}(\mathcal{C}[t, t^{-1}])), \]
are homotopy equivalences.

Proof. We want to apply the Cisinki’s Approximation Theorem 14.14. We give the details only for $K^\infty_{\mathcal{W}}(\text{Ch}(k^-))$, the analogous proof for $K^\infty_{\mathcal{W}}(\text{Ch}(j'))$ is left to the reader. We have to verify the conditions (i) and (ii) appearing in Cisinki’s Approximation Theorem 14.14.
A morphism $f$ in $\text{Ch}(\mathcal{X})$ is by definition in $w\text{Ch}(\mathcal{X})$, if and only if $\text{Ch}(k^-)(f)$ is a chain homotopy equivalence in $\text{Ch}(\mathcal{C}[t^{-1}])$. This takes care of condition (i) for $\mathbf{K}^{\infty}_{\mathcal{W}}(\text{Ch}(k^-))$.

Next we deal with condition (ii). Consider an object $(C^+, f, C^-)$ in $\text{Ch}(\mathcal{X})$ and a morphism $\varphi^- : C^- \to D^-$ in $\text{Ch}(\mathcal{C}[t^{-1}])$. We will extend $\varphi^-$ to a morphism

$$\varphi = (\varphi^+, \varphi^-) : (C^+, f, C^-) \to (D^+, g, D^-)$$

in $\text{Ch}(\mathcal{X})$. If we have achieved this, we are done by the following argument. Note that $\varphi = (\varphi^+, \varphi^-)$ is a morphism in $\text{Ch}(\mathcal{X})$ projecting to $\varphi^-$ under $\text{Ch}(k^-)$. Then, factorizing $\varphi = \mu \circ \psi$ into a cofibration $\psi$ followed by a weak equivalence $\mu$ (using the mapping cylinder), we can write $\varphi^- = \mu^- \circ \text{Ch}(k^-)(\psi)$, where $\psi$ is a cofibration and $\mu^-$ is a weak equivalence, as required in condition (ii).

The construction of $D^+$ and $\varphi^+$ require the following preparation.

Consider an object $A \in S(A_*)$ and an element $k = \{k_m \mid m \in \mathbb{N}\} \in \prod_{m \in \mathbb{N}} \mathbb{Z}$. Define a morphism $t^k : A \to A$ in $S(A_*[t, t^{-1}])$ by $(t^{k_m})_{m \in \mathbb{N}}$, where $t^{k_m} : A_m \to A_m$ is the obvious morphism in $A_m[t, t^{-1}]$ determined by $t^{k_m}$. We can and will regard $t^k : A \to A$ also as a morphism in $\mathcal{C}[t, t^{-1}]$. One easily checks for any morphism $f : A \to B$ in $\mathcal{C}[t, t^{-1}]$ that $t^k \circ f = f \circ t^k$ holds. Moreover we have $t^{k+k'} = t^k \circ t^{k'}$ for $k, k' \in \mathbb{Z}$, $k \geq k'$ defined by the componentwise addition in $\prod_{m \in \mathbb{N}} \mathbb{Z}$ and $\mathbf{id} = \text{id}$ for $0 \in \prod_{m \in \mathbb{N}} \mathbb{Z}$ given by the element which has zero as entry for the component for $m \in \mathbb{N}$. The following property is important for us. Given a morphism $f : A \to B$ in $\mathcal{C}[t, t^{-1}]$, there exists $k_f \in \prod_{m \in \mathbb{N}} \mathbb{Z}$ such that for every $k \in \prod_{m \in \mathbb{N}} \mathbb{Z}$ with $k \geq k_f$ we have $t^k \circ f \in \mathcal{C}[t]$, where $k \geq k_f$ is to be understood componentwise.

Choose a natural number $N$ such that $C^+_m = D^-_m = 0$ for $|m| > N$. Consider $k \in \prod_{m \in \mathbb{N}} \mathbb{Z}$. For any integer $n$ define $n \cdot k$ to be the element, whose $i$-th entry is $n \cdot k_m$. Then we obtain the following commutative diagram in $\mathcal{C}[t, t^{-1}]$.
Theorem 14.23. There are weak homotopy fibration sequences
\[ K^\infty,W(\text{Ch}(X)^w) \to K^\infty,W(\text{Ch}(X)) \to K^\infty,W(\mathcal{C}[t^{-1}]); \]
\[ K^\infty,W(\text{Ch}(Y)^w) \to K^\infty,W(\text{Ch}(Y)) \to K^\infty,W(\mathcal{C}[t, t^{-1}]). \]

Proof. We give the details only for the first sequence, the analogous proof for the second one is left to the reader.

We apply the Fibration Theorem 14.13 in the case \( W = \text{Ch}(X) \), \( w \) as described above and \( v \) the structure of weak equivalences coming from chain homotopy equivalences. The necessary conditions appearing in the Fibration Theorem 14.13 are
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satisfied by [13] Lemma 3.1 and Lemma 4.12. Thus we obtain a weak homotopy fibration of spectra

$$K\infty,W(\text{Ch}(X)^w) \to K\infty,W(\text{Ch}(X)) \to K\infty,W(\text{Ch}(X), w).$$

Because of Lemma [14, Lemma 3.1 and Lemma 4.12] we obtain a weak homotopy fibration

$$K\infty,W(\text{Ch}(X)^w) \to K\infty,W(\text{Ch}(X)) \to K\infty,W(\text{Ch}(C[t^{-1}])),\]$$

\[\square\]

**Lemma 14.24.** The functor $k'$ induces a weak homotopy equivalence

$$K\infty,W(\text{Ch}(X)^w) \xrightarrow{\simeq} K\infty,W(\text{Ch}(Y)^w).$$

**Proof.** Again we will use the Cisinki’s Approximation Theorem [14,14]. We have to verify the conditions (i) and (ii) appearing in Cisinki’s Approximation Theorem [14,14].

Let

$$j_+ C^+ \xrightarrow{f} j_- C^-$$

$$j_+ \varphi^+ \downarrow \varphi^- \downarrow j_- D^-$$

represent a morphism in $\text{Ch}(X)^w$, which maps to a weak equivalence in $\text{Ch}(Y)^w$. Then $\varphi^+$ is a chain homotopy equivalence in $C[t]$ and $\varphi^-$ is a chain homotopy equivalence in $C[t, t^{-1}]$. By assumption, $C^-$ and $D^-$ are contractible in $C[t, t^{-1}]$, so $\varphi^-$ has to be an equivalence in $C[t, t^{-1}]$. It follows that the morphism given by (14.25) is a weak equivalence in $\text{Ch}(X)^w$ already. This takes care of condition (i).

It remains to check condition (ii). Suppose now that

$$j_+ C^+ \xrightarrow{f} C^-$$

$$j_+ \varphi^+ \downarrow \varphi^- \downarrow D^-$$

represents a morphism in $\text{Ch}(X)^w$ satisfying $(C^+, f, C^-)$ in $\text{Ch}(X)^w$. We have to factor this morphism through a map in $\text{Ch}(X)^w$ (which we may then replace by a cofibration using the mapping cylinder) and a weak equivalence in $\text{Ch}(Y)^w$.

Note that the morphism $\varphi^-$ is a chain homotopy equivalence in $C[t, t^{-1}]$, as both $C^-$ and $D^-$ are contractible in that category by assumption. We conclude from [14, Lemma 3.1 (ix)] that there is a chain isomorphism of the shape

$$\left(\begin{array}{c} \varphi^- \\ x \\ z \end{array} \right): C^- \oplus E \xrightarrow{\cong} D^- \oplus E',$$

where $E$ and $E'$ are elementary chain complexes in $C[t, t^{-1}]$, or even in $C$, since both categories have the same objects.

For appropriate $k \in \prod_{m \in \mathbb{N}} \mathbb{Z}$, the commutative diagram

$$\begin{array}{c}
j_+ C^+ \xrightarrow{f} C^- \\
j_+ \varphi^+ \downarrow \varphi^- \downarrow j_- D^-
\end{array}$$

$$\begin{array}{c}
\left(\begin{array}{c} j_+ \varphi^+ \\
j_+ D^+ \oplus i_0 E' \end{array} \right) \xrightarrow{\left(\begin{array}{cc} g^{-1} \circ \varphi^- & g^{-1} \circ y \\ \Delta \circ x & \Delta \circ z \end{array} \right)^{-1}} C^- \oplus i_0 E \\
\left(\begin{array}{c} 1 \\ 0 \end{array} \right) \xrightarrow{\left(\begin{array}{c} 1 \\ 0 \end{array} \right)} \left(\begin{array}{c} \varphi^- \\ y \end{array} \right) \\
j_+ D^+ \xrightarrow{g} D^-
\end{array}$$
provides the desired factorization of \((14.26)\). \(\square\)

**Proof of Theorem [14.20]** Theorem [14.20] follows from Lemma [14.21], Lemma [14.23] and Lemma [14.24]. \(\square\)

14.k. **The global section functor.** Recall that we have defined truncation functors in Subsection [14.h]

**Definition 14.27** (Global section functor). The global section functor  
\[ \Gamma : \text{Ch}(\mathcal{X}) \to \text{Ch}(\mathcal{C}^n) \]

sends an object \((C^+, f, C^-)\) to the \(\mathcal{C}^n\)-chain complex  
\[ \Sigma^{-1} \text{cone}(f[]) : C^+[0, \infty] \to C^-[-1, \infty]). \]

A morphism \((\varphi^+, \varphi^-) : (C^+, f, C^-) \to (D^+, g, D^-)\) of \(\text{Ch}(\mathcal{X})\) is sent to the morphism in \(\text{Ch}(\mathcal{A}^n)\) obtained from the commutative diagram (using the trivial homotopy)

\[
\begin{array}{ccc}
C^+[0, \infty] & \xrightarrow{f[]} & C^-[-1, \infty] \\
\varphi^[] & \downarrow & \varphi^-[] \\
D^+[0, \infty] & \xrightarrow{g[]} & D^-[-1, \infty].
\end{array}
\]

Note that \(f[]\) is indeed a chain map, as it is the composite of the three chain maps  
\[ C^+[0, \infty] \xrightarrow{\id} C^+[-\infty, \infty] \xrightarrow{\iota f = f} C^-[-\infty, \infty] \xrightarrow{\id} C^-[-1, \infty]. \]

Let \(\text{Ch}^{bf}(\mathcal{C}) \subseteq \text{Ch}(\text{Idem}(\mathcal{C}^n))\) be the full subcategory of homotopy finite chain complexes over \(\text{Idem}(\mathcal{C}^n)\), i.e., chain complexes over \(\text{Idem}(\mathcal{C}^n)\), which are homotopy equivalent to a bounded chain complex over \(\text{Idem}(\mathcal{C})\).

It follows from [14] Lemma 3.5) that this category is closed under pushouts along a cobifibration, so it is a Waldhausen subcategory of \(\text{Ch}(\text{Idem}(\mathcal{C}^n))\). The Approximation Theorem [14.13] of Cisinski shows that the inclusion \(I(\mathcal{C}) : \text{Ch}(\text{Idem}(\mathcal{C})) \to \text{Ch}^{bf}(\mathcal{C})\) induces an equivalence  
\[ \text{K}^{\infty, W}(I(\mathcal{C})) : \text{K}^{\infty, W}(\text{Ch}(\text{Idem}(\mathcal{C}))) \xrightarrow{\sim} \text{K}^{\infty, W}(\text{Ch}^{bf}(\mathcal{C})) \]

on non-connective \(K\)-theory.

**Lemma 14.29.**

(i) The functor \(\Gamma\) is Waldhausen exact (for the canonical Waldhausen structures);

(ii) For any object \((C^+, f, C^-)\) of \(\text{Ch}(\mathcal{X})\), the chain complex given by the projective line \(\Gamma(C^+, f, C^-) \in \text{Ch}(\mathcal{C}^n) \subseteq \text{Ch}(\text{Idem}(\mathcal{C}^n))\) is chain homotopy equivalent to an object in \(\text{Ch}(\text{Idem}(\mathcal{C}))\).

Thus, \(\Gamma\) defines a Waldhausen exact functor  
\[ \Gamma(\mathcal{C}) : \text{Ch}(\mathcal{X}) \to \text{Ch}^{bf}(\mathcal{C}). \]

**Proof.** It is not hard to check using [14] Section 4.1] that the two functors \(k^\pm : \text{Ch}(\mathcal{X}) \to \text{Ch}(\mathcal{C}[t^\pm])\) are Waldhausen exact. The restriction functors from \(\text{Ch}(\mathcal{C}[t])\), \(\text{Ch}(\mathcal{C}[t^{-1}])\) to \(\text{Ch}(\mathcal{C})\) are defined on the level of additive categories and hence are Waldhausen exact. Taking cones and suspensions is also Waldhausen exact.

Write the morphism \(f_n^{-1} : C^+_n \to C^-_n \in \mathcal{C}[t, t^{-1}]\) as a sum  
\[ \sum_{n=0}^{m} f_n[l_n] \cdot s^l_n \]

for appropriate morphisms \(f_n[l_n] : \mathcal{S}(\Phi[t, t^{-1}])^n(C^-_n) \to C^+_n\) in \(\mathcal{S}(\mathcal{A}_n[t, t^{-1}])\). Note that \(f_n[l_n] = (f_n[l_n]_{m})_{m \in \mathbb{N}}\) for appropriate morphisms \(f_n[l_n] : \Phi^{\mathbb{N}}((C^-_n)_m) \to (C^+_n)_m\) in \(\mathcal{A}_n[t, t^{-1}]\). Now choose for \(n \in \mathbb{Z}, l_n \in \mathbb{Z}\), and \(m \in \mathbb{N}\) a natural number \(k_n[l_n]_m\) such that we can write  
\[ f_n[l_n]_{m} = \sum_{j_n,l_n,m = -k_n[l_n]_m}^{k_n[l_n]_m} f_n[l_n]_{m}[j_n,l_n,m] \cdot l^j_n \cdot s^{l_n} \]

for appropriate morphisms \(f_n[l_n][j_n,l_n,m] : \Phi^{\mathbb{N}}((C^-_n)_m) \to (C^+_n)_m\) in \(\mathcal{A}_n\). Since
$C^+$ and $C^-$ are bounded, there exists a natural number $N$ such that $C^-_n = 0$ and $C^+_n = 0$ holds for $|n| > N$. Hence we get for every $n \in \mathbb{Z}$ with $|n| > N$ and every $m \in \mathbb{N}$ that $(C^-_n)_m = 0$ and $(C^+_n)_m = 0$ hold. Now define for $m \in \mathbb{N}$ a natural number $k_m$ by

$$k_m = 1 + \max\{k_n[l_n]_m \mid -N \leq n \leq N, a_n \leq l_n \leq b_n\}.$$ 

This definition makes sense since the set $\bigcup_{n=-N}^N \{l_n \in \mathbb{Z} \mid a_n \leq l_n \leq b_n\}$ is finite. Note that $k_m$ depends only on $m$, in contrast to $k_n[l_n]_m$. We conclude that $f_n[l_n]_m = \sum_{j_{n, l_n, m} = -k_m}^{k_m - 1} f_n[l_n]_m(j_{n, l_n, m})$ holds for every $m \in \mathbb{N}$.

Define $k \in \prod_{m \in \mathbb{N}} \mathbb{Z}$ by the collection $\{k_m \mid m \in \mathbb{N}\}$. Then $f^{-1}[1] : C^-[-\infty, \infty] \rightarrow C^+[k, \infty]$ factorizes as

$$C^-[-\infty, \infty] \xrightarrow{f^{-1}[1]} C^+[k, \infty] \xrightarrow{\text{id}[\cdot]} C^-[-\infty, \infty]$$

and the composite

$$C^+[k, \infty] \xrightarrow{f^{-1}[1]} C^-[-\infty, \infty] \xrightarrow{\text{id}[\cdot]} C^+[k, \infty]$$

is the identity map.

Hence over $\text{Idem}(\mathcal{C})$, the chain complex $C^-[1, \infty]$ splits as

$$C^-[1, \infty] \cong C^+[k, \infty] \oplus R,$$

where $R_n$ is given by $(C^-_n[1, \infty], \text{id} - p_n)$ for the projection

$$p_n : C^-[1, \infty] \xrightarrow{f^{-1}[1]} C^+[k, \infty] \xrightarrow{\text{id}[\cdot]} C^-[1, \infty].$$

Since the composite

$$C^-[2 \cdot k, \infty] \xrightarrow{f^{-1}[1]} C^+[k, \infty] \xrightarrow{f^{-1}[1]} C^-[2 \cdot k, \infty]$$

is the identity, $\text{id} - p_n$ restricted to $C^-[2 \cdot k, \infty]$ is trivial. Hence we can find a projection $q_n : C^-_n[1, 2 \cdot k - 1, \infty] \rightarrow C^-_n[1, 2 \cdot k - 1, \infty]$ such that $R_n$ and $(C^-_n[1, 2 \cdot k - 1, \infty], q_n)$ are isomorphic. Since $C^-_n[1, 2 \cdot k - 1, \infty]$ belongs to $\mathcal{C}$, $R_n$ is isomorphic to an object in $\text{Idem}(\mathcal{C})$ for every $n \in \mathbb{Z}$. This implies that the bounded $\text{Idem}(\mathcal{C}^\ast)$-chain complex $R$ is isomorphic to a bounded $\text{Idem}(\mathcal{C})$-chain complex. We obtain an exact sequence of $\text{Idem}(\mathcal{C}^\ast)$-chain complexes

$$0 \rightarrow C^+[k, \infty] \rightarrow C^+[0, \infty] \rightarrow C^+[0, k - 1, \infty] \rightarrow 0$$

$$0 \rightarrow C^+[k, \infty] \rightarrow C^-[1, \infty] \rightarrow C^-[0, k - 1] \rightarrow 0$$

where $r_n : C^-_n[1, \infty] \rightarrow R_n$ is the canonical projection given by $\text{id} - p_n$ and $g$ is the induced map on the quotients. We conclude that $\Sigma^{-1} \text{cone}(- f[\cdot]) \cong \Sigma^{-1} \text{cone}(g)$, which is isomorphic to a chain complex in $\text{Idem}(\mathcal{C})$. Hence $\Gamma(C^+, f, C^-)$ belongs to $\text{Ch}^{\text{fr}}(\mathcal{C})$. \hfill $\square$
14.1. **Embedding lower-K-theory in higher K-theory.** For \( k \in \mathbb{Z} \) and a spectrum \( E \), let \( \Sigma^k E \) be the spectrum obtained by shifting, i.e., \((\Sigma^k E)_n = E_{n-k}\). Let \( (14.30) \)

\[
\text{d}(E) : \Sigma^{-1} S^1_+ \land E \to E
\]

be the weak homotopy equivalence, natural in \( E \), which is given in dimension \( n \) by the \( n \)-th structure map of \( E \). Let \( i : S^1 \to S^1 \lor S^1 \) be the pinching map. For a spectrum \( E \), define a map of spectra

\[
\nabla : \Sigma^{-1} S^1_+ \land E \to (\Sigma^{-1} S^1_+ \land E) \lor (\Sigma^{-1} S^1_+ \land E),
\]

natural in \( E \), by

\[
(\Sigma^{-1} S^1_+ \land E)_n = S^1_+ \land E_{n+1} \xrightarrow{i_n \land E_{n+1}} (S^1 \lor S^1)_+ \land E_{n+1}
\]

\[
\cong (S^1_+ \land E_{n+1}) \lor (S^1_+ \land E_{n+1}) = (\Sigma^{-1} S^1_+ \land E)_n \lor (\Sigma^{-1} S^1_+ \land E)_n.
\]

The next theorem essentially says that \( S^1_+ \land K^{\infty,W}(\text{Ch}(\mathcal{C})) \) is up to weak homotopy equivalence a retract of \( K^{\infty,W}(\text{Ch}(\mathcal{C}[t,t^{-1}])) \).

**Theorem 14.31.** There is a diagram of spectra

\[
\begin{array}{ccc}
S^1_+ \land K^{\infty,W}(\text{Ch}(\mathcal{C})) & \xrightarrow{\cong} & \text{Ch}(\mathcal{C}[t,t^{-1}]) \\
\text{d}(S^1_+ \land K^{\infty,W}(\text{Ch}(\mathcal{C}))) & \xrightarrow{i} & S^1_+ \land (\Sigma^{-1} S^1_+ \land K^{\infty,W}(\text{Ch}(\mathcal{C}))), \\
\text{H}(\mathcal{C}) & \xrightarrow{r} & \text{Ch}(\mathcal{C}[t,t^{-1}])
\end{array}
\]

such that the triangle commutes up to a preferred homotopy \( h \) and the maps marked with \( \cong \) are weak homotopy equivalences. Moreover, everything including the preferred homotopy \( h \) is natural in \( \Phi \).

For its proof we need the next Lemma 14.33.

We define for \( j = 0, 1 \) a functor of additive categories

\[
(14.32) \quad L_j : \mathcal{C} \to \mathcal{X}
\]

as follows. The functor \( L_j(\mathcal{C}) \) sends an object \( C \) in \( \mathcal{C} \) to the object \( (C, t^j \mathcal{C}, C) \) in \( \mathcal{X} \), where \( j \in \prod_{m \in \mathbb{Z}} \mathbb{Z} \) is given by the constant function \( I \to \mathbb{Z} \) with value \( j \). A morphism \( u : C \to C' \) in \( \mathcal{C} \) is sent to the morphism from \( (C, t^j \mathcal{C}, C) \) to \( (C', t^j \mathcal{C}, C') \) in \( \mathcal{X} \) given by the morphism \( j_u : C \to C \) in \( \mathcal{C}[t] \) and the morphism \( j_{-u} : C \to C \) in \( \mathcal{C}[t^{-1}] \). Let \( J(\mathcal{C}) : \mathcal{C} \to \text{Idem}(\mathcal{C}) \) and \( I(\mathcal{C}) : \text{Ch}(\text{Idem}(\mathcal{C})) \to \text{Ch}^{hf}(\mathcal{C}) \) be the obvious inclusions and \( 0 : \text{Ch}(\mathcal{C}) \to \text{Ch}^{hf}(\mathcal{C}) \) be the constant functor with value the chain complex, all of whose chain objects are 0.

**Lemma 14.33.**

(i) There is a natural equivalence of functors \( \text{Ch}(\mathcal{C}) \to \text{Ch}^{hf}(\mathcal{C}) \)

\[
T_0 : I(\mathcal{C}) \circ \text{Ch}(J(\mathcal{C})) \xrightarrow{\cong} \Gamma \circ \text{Ch}(L_0),
\]

which is natural in \( \Phi \);
(ii) There is a natural equivalence of functors \( \text{Ch}(\mathcal{C}) \to \text{Ch}^{\text{hf}}(\mathcal{C}) \)

\[
T_1 : 0 \xrightarrow{\simeq} \Gamma \circ \text{Ch}(L_1),
\]

which is natural in \( \Phi \);

Proof. (i) Fix an object \( C \) in \( \text{Ch}(\mathcal{C}) \). Then \( \Gamma(L_0(C)) \) is by definition

\[
\Sigma^{-1} \text{cone}(\text{id}[: C[0, \infty] \to C[1, \infty]]).
\]

We have the obvious split exact sequence in \( \text{Ch}(C^*) \)

\[
0 \to C[0, 0] \xrightarrow{\text{id}:} C[0, \infty] \xrightarrow{\text{id}} C[1, \infty] \to 0.
\]

It induces a chain homotopy equivalence in \( \text{Ch}^{\text{hf}}(\mathcal{C}) \)

\[
T_0(C) : C \xrightarrow{\simeq} \Gamma(L_0(C)).
\]

(ii) Fix an object \( C \) in \( \text{Ch}(\mathcal{C}) \). Then \( \Gamma(L_1(C)) \) is by definition

\[
\Sigma^{-1} \text{cone}(\text{id}[: C[0, \infty] \to C[1, \infty]]).
\]

The chain map \( \text{id}[: C[0, \infty] \to C[1, \infty]] \) is a chain isomorphism, its inverse is \( \Sigma^{-1} : C[1, \infty] \to C[0, \infty] \). Hence \( \Gamma(L_1(C)) \) is contractible and we obtain a chain homotopy equivalence in \( \text{Ch}^{\text{hf}}(\mathcal{C}) \)

\[
T_1(C) : 0 \xrightarrow{\simeq} \Gamma(L_0(C)).
\]

\[\Box\]

Proof of Theorem \( \text{(14.31)} \). Denote by * the trivial spectrum. Consider the following diagram of non-connective spectra

\[
\begin{array}{ccc}
S \wedge K^{\infty}_W(\text{Ch}(\mathcal{C})) & \xrightarrow{\vee} & \text{Ch}(\mathcal{C}) \\
\downarrow & & \downarrow \\
(S \wedge K^{\infty}_W(\text{Ch}(\mathcal{C}))) \lor (S \wedge K^{\infty}_W(\text{Ch}(\mathcal{C}))) & \xrightarrow{a} & S \wedge K^{\infty}_W(\text{Ch}(\mathcal{C}[t]))
\end{array}
\]

where we abbreviate \( S = \Sigma^{-1} S^1 \), \( u : S^1 \to S^1 \) sends \( z \) to \( z^{-1} \), and the maps \( a \) and \( b^\pm \) are given by

\[
a := (\Sigma^{-1} \text{id}_{S^1} \wedge K(\text{Ch}(L_0))) \lor (\Sigma^{-1} u \wedge K^{\infty}_W(\text{Ch}(L_1)));
\]

\[
b^{\pm} := \Sigma^{-1} \text{id}_{S^1} \wedge K^{\infty}_W(\text{Ch}(k^{\pm})).
\]

Note that \( u \) is a base point preserving selfmap of \( S^1 \) of degree \(-1\). Fix a pointed nullhomotopy for the composite \( S^1 \xrightarrow{1} S^1 \lor S^1 \xrightarrow{\text{id}_{S^1} \lor u} S^1 \). Since \( k^{\pm} \circ L_0 \) and \( k^{\pm} \circ L_1 \) agree, both are equal to with \( i^{\pm} \), the upper left square and the upper right square commute up to a preferred homotopy, natural in \( \Phi \). The lower left and the lower right square commute. The homotopy pushout of the upper row is \( S^1 \wedge (\Sigma^{-1} S^1 \wedge K^{\infty}_W(\text{Ch}(\mathcal{C}))) \) and of the lower row is \( S^1 \wedge (\Sigma^{-1} S^1 \wedge K^{\infty}_W(\text{Ch}^{\text{hf}}(\mathcal{C}))). \)
Let $\text{HP}(\mathcal{C})$ be the homotopy pushout of the middle row. Then we get from the data above maps of spectra

\[
s: S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\mathcal{C}))) \to \text{HP}(\mathcal{C});
\]
\[
r: \text{HP}(\mathcal{C}) \to S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(Ch^{hf}(\mathcal{C}))).
\]

The canonical inclusion $J(\mathcal{C}) : \mathcal{C} \to \text{Idem}(\mathcal{C})$ induces a weak homotopy equivalence $K^{\infty,W}(\text{Ch}(J(\mathcal{C}))) : K^{\infty,W}(\text{Ch}(\mathcal{C})) \xrightarrow{\cong} K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C})))$. This follows from Theorem $[14.15]$ where the elementary proof that the conditions appearing in Theorem $[14.15]$ are satisfied is left to the reader. Define the weak homotopy equivalence

\[
i =: S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\mathcal{C}))) \xrightarrow{S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C}))))} S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C}))))
\]

\[
\xrightarrow{id S^1_+ \wedge \Sigma^{-1} \text{id}_{S^1_+} \wedge K^{\infty,W}(J(\mathcal{C}))} S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}^{hf}(\mathcal{C}))),
\]

where $K^{\infty,W}(J(\mathcal{C}))$ is the weak homotopy equivalence of $[14.28]$. The composite

\[
r \circ s: S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\mathcal{C}))) \to S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}^{hf}(\mathcal{C})))
\]

is homotopic to $i$ by Lemma $[14.33]$. From the commutative diagram

\[
\begin{array}{ccc}
\Sigma^{-1} \text{id}_{S^1_+} \wedge K^{\infty,W}(\text{Ch}(t)) & \xrightarrow{d} & K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C})))
\\
\Sigma^{-1} \text{id}_{S^1_+} \wedge K^{\infty,W}(\text{Ch}(t^-)) & \xrightarrow{d} & K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C})))
\\
\Sigma^{-1} \text{id}_{S^1_+} \wedge K^{\infty,W}(\text{Ch}(t^+)) & \xrightarrow{d} & K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C})))
\\
\Sigma^{-1} \text{id}_{S^1_+} \wedge K^{\infty,W}(\text{Ch}(t^-)) & \xrightarrow{d} & K^{\infty,W}(\text{Ch}(\text{Idem}(\mathcal{C})))
\end{array}
\]

and Theorem $[14.20]$ we obtain a weak homotopy equivalence of spectra, natural in $\Phi$,

\[
f: \text{HP}(\mathcal{C}) \xrightarrow{\cong} K^{\infty,W}(\text{Ch}(\mathcal{C}[t,-1])).
\]

We get from the weak homotopy equivalence $d(K^{\infty,W}(\text{Ch}(\mathcal{C})))$ of $[14.30]$ a weak homotopy equivalence

\[
id S^1_+ \wedge d(K^{\infty,W}(\mathcal{C}))) : S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\mathcal{C}))) \xrightarrow{\cong} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\mathcal{C})).
\]

This finishes the proof of Theorem $[14.31]$.

Next we repeat the same construction in the easy case, where we do not take the automorphisms $\Phi$ into account. So, given $\mathcal{A}_*$, we define additive categories

\[
\tilde{\mathcal{C}} = S(\mathcal{A}_*)
\]
\[
\tilde{\mathcal{C}}[t^\pm] = S(\mathcal{A}_*[t^\pm])
\]
\[
\tilde{\mathcal{C}}[t, t^{-1}] = S(\mathcal{A}_*[t, t^{-1}]),
\]

induction functors fitting in a commutative diagram
additive categories

\[
\begin{align*}
\hat{C}^\kappa &= S(A_\kappa^+); \\
\hat{C}[t^\pm]^\kappa &= S(A_\kappa[t^\pm]^\kappa); \\
\hat{C}[t,t-1]^\kappa &= S(A_\kappa[t,t-1]^\kappa),
\end{align*}
\]

and restriction functors

\[
\begin{align*}
\tilde{\iota}_0 : \hat{C}[t,t-1]^\kappa &\rightarrow \hat{C}^\kappa; \\
\tilde{\iota}_\pm : \hat{C}[t^\pm]^\kappa &\rightarrow \hat{C}^\kappa,
\end{align*}
\]

such that we have adjunctions \((\tilde{\iota}_0)^\kappa, \tilde{\iota}_0\), \((\tilde{\iota}_+)^\kappa, \tilde{\iota}_+\), and \((\tilde{\iota}_-)^\kappa, \tilde{\iota}_-\). There is an obvious definition of the projective line \(\hat{\mathcal{X}}\) and of the global section functor \(\hat{\Gamma} : \hat{\mathcal{X}} \rightarrow \text{Ch}^{\text{hf}}(\hat{C})\).

Now analogously to the proof of Theorem 14.31 one can show

\textbf{Theorem 14.34.} There is a diagram of spectra

\[
\begin{align*}
S^1_+ \wedge K^{\infty,W}(\text{Ch}(\hat{C})) \\
\simeq \begin{array}{c}
S^1_+ \wedge S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\hat{C}))) \\
\hat{r} \\
\simeq \hat{r}
\end{array} \\
\text{HP}(\hat{C}) \\
\simeq \begin{array}{c}
S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge K^{\infty,W}(\text{Ch}(\hat{C}))) \\
\hat{f}
\end{array}
\end{align*}
\]

such that the triangle commutes up to a preferred homotopy \(\hat{h}\) and the maps marked with \(\simeq\) are weak homotopy equivalence. Moreover, everything including the preferred homotopy \(\hat{h}\) is natural in \(\Phi\).

There are obvious inclusions \(l : \hat{C} \rightarrow C, l[t^\pm] : \hat{C}[t^\pm] \rightarrow C[t^\pm]\), and \(l[t,t^{-1}] : \hat{C}[t,t^{-1}] \rightarrow C[t,t^{-1}]\). They induce a map denoted by \(l\) from the diagram appearing in Theorem 14.34 to the one appearing in Theorem 14.31.

The pro-automorphism \(\Phi : A_+ \rightarrow A_+\) induces in the obvious way an automorphism of the diagram appearing in Theorem 14.34 denoted by \(t\). Taking the mapping torus in each entry of the diagram appearing in Theorem 14.34 yields a
diagram of the form
\[(14.35)\]
\[
\begin{array}{c}
S^1_+ \wedge T_{K^{\infty,w}(Ch(S(\Phi))]} \\
\cong \text{id}_{S^1_+} \wedge Ad(T_{K^{\infty,w}(Ch(S(\Phi))])} \\
S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge T_{K^{\infty,w}(Ch(S(\Phi))])} \\
\cong \sigma \quad T_{\varphi} \quad \cong \tau \\
T_{HP(S(\Phi))} \\
\cong T_{\varphi} \\
T_{K^{\infty,w}(Ch(S(\Phi[t,t^{-1}])})}
\end{array}
\]

It is not true that \(I \circ t\) agrees with \(I\) again. However, this is true up to preferred homotopy, and therefore we get a map from the diagram \((14.35)\) to the diagram appearing in Theorem \((14.31)\). These homotopies are all induced by obvious natural transformations of functors. For instance, for the automorphism \(\hat{\Phi} = S(\Phi) : \hat{\mathcal{C}} = S(A_\ast) \simeq_1 S = S(A_\ast)\) and the inclusion \(l : \hat{\mathcal{C}} = S(A_\ast) \to \mathcal{C} = S(A_\ast)\) we get a natural transformation \(l \to l \circ \hat{\Phi}\), if we assign an object \(\hat{\mathcal{A}}\) in \(\hat{\mathcal{C}} = S(A_\ast)\) the morphism \(\mathcal{A} \to \hat{\mathcal{A}}\) in \(\mathcal{C} = S(A_\ast)\) given by \(id_{\hat{\mathcal{A}}_\ast} \circ t\). From these data we get a map from the diagram \((14.35)\) to the diagram appearing in Theorem \((14.31)\).

Now we apply the functor \(\pi_n\) to the diagram \((14.35)\), the diagram appearing in Theorem \((14.31)\) and the map between them constructed above. Taking into account that some of the arrows appearing in the diagram \((14.35)\) and the diagram Theorem \((14.31)\) are weak equivalences, we get for every \(n \in \mathbb{Z}\) a commutative diagram
\[(14.36)\]
\[
\begin{array}{c}
\pi_{n-1}(T_{K^{\infty,w}(Ch(S(\Phi))])} \\
\cong \pi_{n-1}(K^{\infty,w}(Ch(S(A_\ast)\ast\Phi[Z])]
\end{array}
\]

such that the composite of the left two vertical arrows and the composite of the right two vertical arrows are isomorphisms.

Next we explain, how we get the corresponding diagram for \(L\) instead of \(S\)
\[(14.37)\]
\[
\begin{array}{c}
\pi_{n-1}(T_{K^{\infty,w}(Ch(L(\Phi))])} \\
\cong \pi_{n-1}(K^{\infty,w}(Ch(L(A_\ast)L(\Phi[Z])]
\end{array}
\]

such that the composite of the left two vertical arrows and the composite of the right two vertical arrows are isomorphisms. Note that the upper horizontal arrow
Lemma 14.40. For every $n \in \mathbb{Z}$ there exists a commutative diagram

$$
\begin{array}{ccc}
H^*_n(\mathbb{E}Z, K^\infty_{\mathcal{L}(A_i)}) & \rightarrow & H^*_n(\text{pt}, K^\infty_{\mathcal{L}(A_i)}) \\
| & & | \\
H^*_n(\mathbb{E}Z, K^\infty_{\mathcal{L}(A_i)}) & \rightarrow & H^*_n(\text{pt}, K^\infty_{\mathcal{L}(A_i)}) \\
\end{array}
$$

such that the composite of the two vertical arrows appearing in the left and in the right column are isomorphisms and the horizontal arrows are induced by the projection $E\mathbb{Z} \to \text{pt}$.

In particular the upper horizontal arrow is bijective, if the middle horizontal arrow is bijective.
Proof. Because of Lemma 14.30 it suffices to proof Lemma 14.40 in the case, where we replace $K^\infty(\mathcal{L}(A_*))$ by $K^\infty_{\text{Ch}}(\mathcal{L}(A_*))$ and $K^\infty_{\text{CH}}(\mathcal{L}(A_*|t^{-1}))$ by $K^\infty_{\text{CH}}(\mathcal{L}(\mathcal{A}_*,|t^{-1}))$.

Now one easily checks unravelling the definitions that the rows of the diagram above this replacement agree with the rows of the diagram (14.37). Now the claim follows from the diagram (14.37). □

14.M. Reduction to the connective case.

Theorem 14.41 (Reduction to the connective case). Fix a natural number $n_0$. Suppose that for every natural number $d$ and every natural number $n$ satisfying $n \geq n_0$ the map
\[
H_n^Z(EZ, K^\infty_{\text{CH}}(\mathcal{L}(A_*,[Z^d]))) \to H_n^Z(\text{pt}, K^\infty_{\text{CH}}(\mathcal{L}(A_*,[Z^d]))) = K_n(\mathcal{L}(A_*,[Z^d]) \times \mathcal{L}(\Phi[Z^d], Z))
\]
is an isomorphism.

Then for every $n \in \mathbb{Z}$ the map
\[
H_n^Z(EZ, K^\infty_{\text{CH}}(\mathcal{L}(A_*))) \to H_n^Z(\text{pt}, K^\infty_{\text{CH}}(\mathcal{L}(A_*))) = K_n(\mathcal{L}(A_*) \times \mathcal{L}(\Phi, Z))
\]
is an isomorphism.

Proof. One can iterate the passage from $A_*$ to $A_*|t|\cdot t^{-1} = A_*|Z$ and obtain a passage from $A_*$ to $A_*|Z^d| = (A_*|Z^d|)\cdot t^{-1}$ for every natural number $d$. Now Theorem 14.41 follows from Lemma 14.40. □

14.N. Proof of Theorem 14.41. Now we are ready to finalize the proof of Theorem 14.41.

Thanks to Lemma 14.2, we can assume without loss of generality that $r = 1$. Because of Theorem 14.41 it suffices to show that for every natural number $d$ and every natural number $n$ satisfying $n \geq 2$ the map
\[
H_n^Z(EZ, K^\infty_{\text{Ch}}(\mathcal{L}(A_*,[Z^d]))) \to H_n^Z(\text{pt}, K^\infty_{\text{Ch}}(\mathcal{L}(A_*,[Z^d]))) = K_n(\mathcal{L}(A_*|Z^d) \times \mathcal{L}(\Phi[Z^d], Z))
\]
is an isomorphism.

Let $K_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))$ be the connective version of $K^\infty_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))$. Since we have $\dim(EZ) \leq 1$, we conclude from Lemma 6.5 that the vertical arrows appearing in the commutative diagram
\[
\begin{array}{ccc}
H_n^Z(EZ, K_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) & \xrightarrow{\cong} & H_n^Z(\text{pt}, K_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) \\
\downarrow & & \downarrow \\
H_n^Z(EZ, K^\infty_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) & \xrightarrow{\cong} & H_n^Z(\text{pt}, K^\infty_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) \\
\downarrow & & \downarrow \\
H_n^Z(EZ, K^\infty_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) & \xrightarrow{\cong} & H_n^Z(\text{pt}, K^\infty_{\text{idem}}(\mathcal{L}(A_*,[Z^d])))
\end{array}
\]
are bijective for $n \geq 2$. Hence it suffices show that for every natural number $d$ the map
\[
H_n^Z(EZ, K_{\text{idem}}(\mathcal{L}(A_*,[Z^d]))) \to H_n^Z(\text{pt}, K_{\text{idem}}(\mathcal{L}(A_*,[Z^d])))
\]
is bijective for $n \in \mathbb{Z}, n \geq 2$.

By assumption the category $A_*/[Z^d]$ is uniformly $l(d)$-regular coherent and the inclusion $A_*/[Z^d] \to A_*/[Z^d]$ is flat for every $m \geq 0$. We conclude from Lemma 13.10 that $\mathcal{L}(A_*/[Z^d])$ is uniformly $l(d)$-regular coherent. We conclude from Lemma 6.4 (vi) that $\text{idem}(\mathcal{L}(A_*/[Z^d]))$ is uniformly $l(d)$-regular coherent. Hence $\text{idem}(\mathcal{L}(A_*/[Z^d]))$ is idempotent complete and regular coherent. Now the bijectivity of (14.42) for $n \in \mathbb{Z}, n \geq 2$ follows from Theorem 7.5 applied to $\text{idem}(\mathcal{L}(A_*/[Z^d]))$, since for
the map $a$ appearing there the homomorphism $\pi_n(a)$ can be identified with the map (14.42) for $n \geq 2$. This finishes the proof of Theorem 14.1.

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