Of Course Muons Can Oscillate

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ABSTRACT

Recent theoretical claims notwithstanding, muons can and do oscillate. Muons produced in association with neutrinos (if masses and mixing angles are nonzero) exhibit a joint oscillating spatial distribution. The possible use of muon oscillations as a probe of neutrino mass and mixing parameters is discussed using very simple physical arguments. Space-time oscillations in the secondary decay vertices of muons (produced by pion decay) persist after summing over all undetected neutrinos.

1. Introduction

Classic precision measurements of \((g - 2)\) have been made for electrons[1] and muons[2-5]. For the latter case, muons are injected into a ring with a uniform applied magnetic field \(B\). The experimental quantum survival probability of the muon in the ring to decay, (via \(\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu\)) with a detected electron energy above a threshold value, has been fit to the theoretical oscillating form

\[
P_{\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e}(t) = e^{-M c^2 \Gamma t / E} \left( \frac{1 + A \cos(\Omega t + \phi)}{1 + A \cos \phi} \right).
\]

In Eq.(1.1), \(M\), \(E\), and \(\Gamma^{-1}\) represent (respectively) the mass, energy, and intrinsic lifetime of the muon; \(t\) is time in the laboratory reference frame, and \(\Omega = \left( (g - 2) eB/(2Mc) \right)\) is the experimental frequency measured in the muon survival probability due to quantum mechanical amplitude interference. Eq.(1.1) has been previously derived by us from the Dirac equation[6].

In view of the fact that the theoretical muon decay oscillations in Eq.(1.1) have been verified by some of the most precise experiments ever performed in high energy physics, it is sad to read a recent statement by Dolgov, Morozov, Okun and Schepkin[7] that “Muons do not oscillate.” The seminal muon oscillation work by Russian chemists[8] at their own institute in Moscow was ignored. Some incorrect physics in the work of Dolgov, Morozov, Okun and Schepkin[7], including (but not limited to) the fact that their equations are not dimensionally correct, is summarized in the Appendix.

Our purpose is to formulate the problem of neutrino induced muon oscillations using the notion of “quantum beats” which are discussed in recent text books[9-10] on quantum mechanics. The basic notion of a quantum mechanical decay exhibiting quantum beat
oscillations may be formulated as follows: (i) The survival amplitude at time $t$ for a normalized state $|\Psi\rangle$ is given by

$$S(t) = \langle \Psi | e^{-iHt/\hbar} | \Psi \rangle. \quad (1.2)$$

(ii) The probability density for the state to have energy $E$ is given by

$$\rho(E) = \langle \Psi | \delta(E - H) | \Psi \rangle. \quad (1.3)$$

From Eqs.(1.2) and (1.3) it follows that the survival amplitude for a state is the Fourier transform of the energy probability density

$$S(t) = \int \rho(E) e^{-iEt/\hbar} dE, \quad (1.4)$$

yielding the survival probability

$$P(t) = |S(t)|^2 = \int \int \rho(E)\rho(E') e^{-i(E-E')t/\hbar} dEdE'. \quad (1.5)$$

Eqs.(1.2)-(1.5) are exact. If the energy probability density $\rho(E)$ may be written as a sum of peaked resonances, then it follows rigorously that the survival probability $P(t)$ exhibits quantum beat oscillations in time. The proof follows directly from Eq.(1.5).

Eq.(1.5) (in a fixed reference frame) and its more general Lorentz invariant forms of expression are the basis of the discussion which follows for the theory of neutrino mass mixing induced muon decay oscillations. In Sec.2 quantum beat oscillations are discussed using the standard formalism successfully applied to atomic and molecular experimental decays. In Sec.3, the Lorentz invariant version of the theory is developed. In Sec.4 the four momentum distribution for the muon in the reaction $\pi^+ \rightarrow \mu^+ + \nu_\mu$ is discussed. In a model for which the neutrino has three possible mass eigenstates, and in which there is mixing, the (incoherent) momentum distributions obtained by us are identical to those conventionally used by others to analyse previous experiments. In Sec.5, the survival probability of the muon is computed and the interference effects (which are present if the neutrinos have both mass and flavor mixing) are discussed. In the concluding Sec.6, the rigorous relationship between the four momentum probability distribution and the survival probability is reviewed. An incoherent sum of the four momentum probability distribution may nevertheless exhibit quantum beat oscillations in the survival probability.

2. Quantum Beat Oscillations in Decays

Consider the case of quantum beats in a decay. Suppose that the initial state of the system under consideration has the form

$$|\Psi\rangle = c_1 |\Psi_1\rangle + c_2 |\Psi_2\rangle, \quad (2.1)$$

with the assumption that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ have no common decay modes. Explicitly,

$$\langle \Psi_2 | e^{-iHt/\hbar} | \Psi_1 \rangle = \sum_n \langle \Psi_2 | n, out \rangle e^{-iE_{n, out}t/\hbar} \langle n, out | \Psi_1 \rangle = 0, \quad (2.2)$$
so that
\[ \langle \Psi_2 \mid \delta(E - H) \mid \Psi_1 \rangle = 0, \quad \langle \Psi_1 \mid \delta(E - H) \mid \Psi_2 \rangle = 0. \]  
(2.3)

Eqs.(1.3), (2.1) and (2.3) imply
\[ \rho(E) = |c_1|^2 \rho_1(E) + |c_2|^2 \rho_2(E), \]  
(2.4)
where
\[ \rho_1(E) = \langle \Psi_1 \mid \delta(E - H) \mid \Psi_1 \rangle, \quad \rho_2(E) = \langle \Psi_2 \mid \delta(E - H) \mid \Psi_2 \rangle. \]  
(2.5)

Since the two states \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \) have no common decay modes, the energy probability distribution \( \rho(E) \) for the state \( |\Psi\rangle \) is a superposition of probabilities of the energy probability densities \( \rho_1(E) \) of state \( |\Psi_1\rangle \) and \( \rho_2(E) \) of state \( |\Psi_2\rangle \), including the probabilities \( |c_j|^2 = |\langle \Psi_j \mid |\Psi\rangle|^2 \).

The superposition of energy probability densities in Eq.(2.4) by no means implies that there is a superposition of survival probabilities in time. In fact, Eqs.(1.4) and (2.4) imply a superposition of survival amplitudes having the form
\[ S(t) = |c_1|^2 S_1(t) + |c_2|^2 S_2(t). \]  
(2.6)

If the two states decay into different channels according to an exponential decay law
\[ S_1(t) = e^{-\Gamma_1 t/2} e^{-iE_1 t/\hbar}, \quad S_2(t) = e^{-\Gamma_2 t/2} e^{-iE_2 t/\hbar}, \]  
(2.7)
then the survival probability
\[ P(t) = |S(t)|^2 = |c_1|^4 e^{-\Gamma_1 t} + |c_2|^4 e^{-\Gamma_2 t} + 2|c_1|^2 |c_2|^2 e^{-(\Gamma_1 + \Gamma_2)t/2} \cos(\omega_{12} t), \]  
(2.8)
where
\[ \hbar \omega_{12} = E_1 - E_2. \]
(2.9)

Eqs.(2.8) and (2.9) are fundamental equations for the quantum beat oscillations in decays which constitute (for example) commonplace experimental phenomena in atomic and molecular physics and chemistry.

3. Lorentz Invariant Space-Time Oscillations

The amplitude that a normalized state \( |\Psi\rangle \) describing a physical system in the neighborhood of the space-time origin survives a displacement to the neighborhood of the space-time point \( x \) may be defined by
\[ S(x) = \langle \Psi \mid e^{iP \cdot x/\hbar} |\Psi\rangle, \]  
(3.1)
where
\[ P = (P, H/c) \]  
(3.2)
is the operator four momentum. Note that Eq.(3.1) reduces to Eq.(1.2) for decays in the center of mass frame[11] for which the three momentum may be set to zero.
In a Lorentz covariant manner, one may start from Eq.(3.1) and write

\[ S(x) = \int \sigma(k) e^{ik \cdot x} d^4 k, \]  

(3.3)

where the four momentum distribution is given by

\[ \sigma(k) = \left\langle \Psi \right| \delta^{(4)} \left( k - \frac{P}{\hbar} \right) \left| \Psi \right\rangle. \]  

(3.4)

If the initial state is in a superposition of finitely many states,

\[ \left| \Psi \right\rangle = \sum_j c_j \left| \Psi_j \right\rangle, \]  

(3.5)

no two of which have common decay modes, then the four momenta probability densities are given by the (incoherent) superposition weighted by the probabilities \( |c_j|^2 \)

\[ \sigma(k) = \sum_j |c_j|^2 \sigma_j(k), \]  

(3.6)

where

\[ \sigma_j(k) = \left\langle \Psi_j \right| \delta^{(4)} \left( k - \frac{P}{\hbar} \right) \left| \Psi_j \right\rangle. \]  

(3.7)

Eqs.(3.6) and (3.7) are the Lorentz invariant form of Eqs.(2.4) and (2.5). The survival amplitude follows from Eqs.(3.3) and (3.6) to be of the superposition of amplitudes form

\[ S(x) = \sum_j |c_j|^2 S_j(x). \]  

(3.8)

Clearly, the survival probability exhibits quantum interference effects

\[ P(x) = |S(x)|^2 = \sum_{i,j} |c_i|^2 |c_j|^2 S_i^*(x) S_j(x), \]  

(3.9)

and quantum beat oscillations in decays have been described in a Lorentz invariant form.

4. Muon Momentum Distributions for \( \pi^+ \rightarrow \mu^+ + \nu_\mu \)

We consider a massive neutrino model with mixing and define the following parameters: For three flavors of neutrino (\( \nu_e, \nu_\mu, \) and \( \nu_\tau \)) there is a rotation matrix

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
= \begin{pmatrix}
R_{e1} & R_{e2} & R_{e3} \\
R_{\mu1} & R_{\mu2} & R_{\mu3} \\
R_{\tau1} & R_{\tau2} & R_{\tau3}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix},
\]  

(4.1)

where \( n_1, n_2, \) and \( n_3 \) denote neutrino mass states with masses \( m_1, m_2, \) and \( m_3 \) respectively. In the reaction

\[ \pi^+ \rightarrow \mu^+ + \nu_\mu, \]  

(4.2)
the four momentum distribution for the $\mu^+$ has the form of Eq.(3.6); i.e.

$$\sigma_{\mu\text{on}}(k) = \sum_{j=1}^{3} |R_{\mu j}|^2 \sigma_{\mu\text{on}}(k, m_j), \quad (4.3)$$

where $\sigma_{\mu\text{on}}(k, m_j)$ is the normalized four momentum distribution of the $\mu^+$ obtained if the original muon produced in the pion decay were to recoil against a neutrino of mass $m_j$.

Note that the four momentum probability density for the muon in Eq.(4.3), is an incoherent probability superposition. We have already summed over the unobserved neutrino final states, both flavor and momentum. The incoherent four momentum distribution for the muon has long been understood (and employed) in the analysis of experimental pion decays as a probe of possible neutrino masses[12]. Unfortunately, the previous experimental resolution for measured muon momentum distributions did not allow for the definitive detection of any finite neutrino masses. However, the resolution did allow for an improved determination[12] of the pion mass.

5. Muon Survival Probability from $\pi^+ \to \mu^+ + \nu_\mu$

From Eqs.(3.6), (3.8) and (4.3), the muon survival amplitude against $\mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e$ from muons produced in a pion decay is given by

$$S_{\mu\text{on}}(x) = \sum_{j=1}^{3} |R_{\mu j}|^2 S_{\mu\text{on}}(x, m_j), \quad (5.1)$$

where

$$S_{\mu\text{on}}(x, m_j) = \int \sigma_{\mu\text{on}}(k, m_j) e^{ik \cdot x} d^4k \quad (5.2)$$

is the survival amplitude of the $\mu^+$ obtained if the muon were to recoil against a neutrino of mass $m_j$. The survival amplitude $S_{\mu\text{on}}(x, m_j)$ is very well known; i.e.

$$S_{\mu\text{on}}(x, m_j) \approx e^{ip_j \cdot x / \hbar} e^{-\Gamma \sqrt{-x^2}/2c}. \quad (5.3)$$

In the pion rest frame $p_j = (p_j, \sqrt{M^2_\mu c^2 + |p_j|^2})$,

$$\frac{|p_j|}{c} = \frac{\sqrt{\left(M^2_\mu - (M_\mu + m_j)^2\right)\left(M^2_\mu - (M_\mu - m_j)^2\right)}}{2M_\pi}. \quad (5.4)$$

In any reference frame, the survival probability for the muon follows from Eqs.(5.1) and (5.3) to be

$$P_{\mu\text{on}}(x) = |S_{\mu\text{on}}(x)|^2 \approx e^{-\Gamma \sqrt{-x^2}/c} \sum_{j=1}^{3} \sum_{l=1}^{3} |R_{\mu j}|^2 |R_{\mu l}|^2 \cos((p_j - p_l) \cdot x / \hbar). \quad (5.5)$$
The presence of quantum beat oscillations in the muon survival probability, see Eqs.(5.4) and (5.5), is the central result of this work. The incoherent sum in the four momentum probability distribution in Eq.(4.3) exhibits quantum beat oscillations in the survival probability in Eq.(5.5).

6. Conclusion

In the pion decay \( \pi^+ \rightarrow \mu^+ + \nu_\mu \), the muon later also decays via \( \mu^+ \rightarrow e^+ + \bar{\nu}_e + \nu_\mu \). Assume that the neutrinos have both mass and flavor mixing. The four-momentum distribution \( \sigma_{\text{muon}}(k) \) of the muon produced by the pion will be a probability superposition as in Eq.(4.3). This result is well known[12], and has already been employed in the analysis of pion decay experiments searching for neutrino masses. For the muon survival amplitude, the well known Eq.(4.3) yields Eq.(5.1) and for the survival probability Eq.(5.5).

The interference effects are manifest in the survival probability Eq.(5.5). The secondary vertex from \( \mu^+ \rightarrow e^+ + \bar{\nu}_e + \nu_\mu \), distributed via \( P_{\text{muon}}(x) \), may provide a more sensitive probe of neutrino masses and mixing than does the four momentum distribution via \( \sigma_{\text{muon}}(k) \). Further details about oscillations may be found in our earlier works[13-16].

APPENDIX

In his lectures on quantum field theory, T. D. Lee[17] states “An equation in physics, say \( A = B \), must satisfy the requirement that the dimension of \( A \) be the same as that of \( B \): \( [A] = [B] \). This seemingly elementary requirement can serve the useful purpose of verifying the correctness of one’s equations, especially after a long calculation.”

Almost all the central probability equations in the work of Dolgov, Morozov, Okun and Schepkin[7], hereafter abbreviated DMOS[7], are dimensionally inconsistent and thereby fail to satisfy T. D. Lee’s elementary requirement for correct physical equations.

In Eq.(9), DMOS[7] define a dimensionless joint muon neutrino probability

\[
P_{\nu_{\alpha}}(x_\mu, x_\nu) = \beta_1^2 + \beta_2^2 + 2\beta_1^2 \beta_2^2 \cos(\phi_1 - \phi_2) \quad (\text{DMOS1})
\]

where \( \beta_{\alpha} \) are seen to be dimensionless in Eq.(11) of DMOS[7]. In Eq.(25), DMOS[7] define a single muon probability

\[
P_{\mu}^B(x_\mu) = \sum_a \int P_{\nu_{\alpha}}(x_\mu, x_\nu) dx_\nu, \quad (\text{DMOS2}).
\]

The quantity on the left hand side of Eq.(DMOS2) is alleged to be dimensionless (in fact unity) from Eq.(27) of DMOS. The right hand side of Eq.(DMOS2) has dimensions of \([\text{length}^3]\). By virtue of dimensional analysis alone, their result for the muon probability distribution is wrong. The integrations are simply incorrect. Errors evident from dimensional considerations are also present (for example) in Eq.(26) of DMOS[7].

The dimensional errors are symptomatic of a confusion in the starting wave function Eq.(6) of DMOS[7]

\[
\psi_{p_{\pi}}(x_\mu, x_\nu | x_i) = |\mu > (e^{-i\phi_1 \cos \theta}|\nu_1 > + e^{-i\phi_2 \sin \theta}|\nu_2 >). \quad (\text{DMOS3})
\]
Imagine a world in which one of the neutrinos has a large mass and the other neutrino has virtually zero mass. An energetic pion decays at CERN so that the muon “ket” which recoils from the heavy neutrino decays in Switzerland and the muon “ket” which recoils from the massless neutrino decays in France. In the DMOS[7] work the “ket” in France and the “ket” in Switzerland are identical, i.e. $|\mu>$ in Eq.(DMOS3). We prefer to think that $<\mu_{France}|\mu_{Switzerland}> = 0$. At any rate, without this physical notion there is more than a little difficulty in properly normalizing wave functions.

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