Doomsdays in a modified theory of gravity: A classical and a quantum approach

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Abstract
By far cosmology is one of the most exciting subject to study, even more so with the current bulk of observations we have at hand. These observations might indicate different kinds of doomsdays, if dark energy follows certain patterns. Two of these doomsdays are the Little Rip (LR) and Little Sibling of the Big Rip (LSBR). In this work, aside from proving the unavoidability of the LR and LSBR in the Eddington-inspired-Born-Infeld (EiBI) scenario, we carry out a quantum analysis of the EiBI theory with a matter field, which, from a classical point of view would inevitably lead to a universe that ends with either LR or LSBR. Based on a modified Wheeler-DeWitt equation, we demonstrate that such fatal endings seems to be avoidable.

Keywords: Quantum cosmology, Modified theories of gravity, Dark energy doomsdays, Palatini type of theories

1. Introduction
The scrutiny of extensions on General Relativity (GR) is a well motivated topic in cosmology. Some phenomena, such as the current red accelerating expansion of the universe or gravitational singularities like the big bang, would presage extensions of GR in the infra-red as well as in the ultra-violet limits. Among these extensions, the EiBI theory \cite{1}, which is constructed on a Palatini formalism, is an appealing model in the sense that it is inspired by the Born-Infeld electrodynamics \cite{2} and the big bang singularity can be removed through a regular stage with a finite physical curvature \cite{1}. Various important issues of the EiBI theory have been addressed such as cosmological solutions \cite{3-9}, compact objects \cite{10-15}, cosmological perturbations \cite{16-18}, parameter constraints \cite{19-21}, and the quantization of the theory \cite{22,23}. However, some possible drawbacks of the theory were discovered in Ref. \cite{24}. Finally, some interesting generalizations of the theory were proposed in Refs. \cite{25-28}.

As is known, the cause of the late time accelerating expansion of the universe can be resorted to phantom dark energy, which violates the null energy condition (at least from a phenomenological point of view) while remains consistent with observations so far. red Nonetheless, the phantom energy may induce more cosmological singularities in GR. In particular there are three kinds of behaviors intrinsic to phantom models, which can be characterized by the behaviors of the scale factor $a$, the Hubble rate $H = \dot{a}/a$, and its cosmic derivatives $\dot{H}$ near the singular points: (a) The big rip singularity (BR) happens at a finite cosmic time $t$ when $a \to \infty$, $H \to \infty$, and $\dot{H} \to \infty$ \cite{29-38}. (b) the LR happens at $t \to \infty$ when $a \to \infty$, $H \to \infty$ and $\dot{H} \to \infty$ \cite{39-47}. (c) the LSBR happens at $t \to \infty$ when $a \to \infty$, $H \to \infty$, while $\dot{H} \to \text{constant}$ \cite{48-50}. All these three scenarios would lead to the universe to rip itself as all the structures in the universe would be destroyed no matter what kind of binding energy is involved.

Interestingly, even though the EiBI theory can cure the big bang, in Refs. \cite{5,6} it was found that the BR and LR are unavoidable in the EiBI setup, hinting that the EiBI theory is still not complete and some quantum treatments near these singular events may be necessary. In this paper, we will extend the investigations ....
in Ref. [22] where we showed that the BR in the EiBI phantom model is expected to be cured in the context of quantum geometrodynamics. We will carry an analysis to encompass the rest of truly phantom dark energy abrupt events; i.e. the LR and LSBR.

2. The EiBI model: The LR and LSBR

The EiBI action proposed in [1] is (from now on, we assume $8\pi G = c = 1$)

$$S_{EiBI} = \frac{1}{2} \int d^4x \left[ \sqrt{\det(g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma))} - \lambda \sqrt{|g|} \right] + S_m(g),$$

(1)

where $|g_{\mu\nu} + \kappa R_{\mu\nu}|$ is the determinant of the tensor $g_{\mu\nu} + \kappa R_{\mu\nu}$. The parameter $\kappa$ characterizes the theory and $\lambda$ quantifies the effective cosmological constant. $S_m$ is the matter Lagrangian. The field equations are obtained by varying (1) with respect to $g_{\mu\nu}$ and the connection $\Gamma$. In a flat, homogeneous and isotropic (FLRW) universe filled with a perfect fluid whose energy density and pressure are $\rho$ and $p$, respectively, the Friedmann equations of the physical metric $g_{\mu\nu}$ and of the auxiliary metric compatible with $\Gamma$ are [6]

$$\kappa H^2 = \frac{8}{3} \left[ \frac{\dot{\rho} + 3\bar{\rho} - 2}{(1 - \bar{\rho})(4 + \bar{\rho} - 3\bar{\rho}) + 3\frac{\ddot{\rho}}{\dot{\rho}}(1 + \bar{\rho})(\bar{\rho} + \bar{\rho})^2} \right],$$

(2)

and

$$\kappa H_q^2 = \kappa \left( \frac{1}{b} \frac{d\,db}{dt} \right)^2 = \frac{1}{3} + \frac{\bar{\rho} + 3\bar{\rho} - 2}{6\sqrt{1 + \bar{\rho}(1 - \bar{\rho})}},$$

(3)

where $\bar{\rho} \equiv \kappa \rho$ and $\bar{\rho} \equiv \kappa p$. On the above equations $a$ and $b$ are the scale factor of the physical and auxiliary metrics, respectively. $\bar{t}$ is a rescaled time such that the auxiliary metric can be written in a FLRW form.

In GR, the LR and LSBR can be driven (separately) by two phantom energy models as follows [44, 48]

$$\rho_{LR} = -\rho_{LR} - A_{LR} \sqrt{p_{LR}}, \quad \rho_{LSBR} = -\rho_{LSBR} - A_{LSBR},$$

where $A_{LR}$ and $A_{LSBR}$ are positive constants. Therefore,

$$\rho_{LR} = \rho_{LR} = \left( \frac{3A_{LR}}{2 \sqrt{\rho_0}} \ln(a/a_0) + 1 \right)^2, \quad \rho_{LSBR} = 3A_{LSBR} \ln(a/a_0) + \rho_0,$$

(4)

where we take $\rho_{LR} = \rho_{LSBR} = \rho_0$ when $a = a_0$ [44, 48]. The abrupt events happen at an infinite future where $\rho$ and $\rho$ diverge. Inserting these phantom energy contents into the EiBI model, i.e., Eqs. (2) and (3), and considering the large $a$ limit (for $\rho$ given in Eqs. (3)), we have

$$\kappa H^2 \approx \frac{\bar{\rho}}{3} \rightarrow \infty, \quad \kappa H_q^2 \approx \frac{1}{3},$$

(5)

and

$$\dot{H} \approx \begin{cases} \frac{A_{LR}}{2} \sqrt{p_{LR}}, & \text{LR} \\ \frac{A_{LSBR}}{2}, & \text{LSBR} \end{cases}$$

(6)

for these two phantom energy models. Therefore, the LR and LSBR of the physical metric are unavoidable within the EiBI model while the auxiliary metric behaves as a de-Sitter phase at late time.

3. The EiBI quantum geometrodynamics: The LR and LSBR minisuperspace model

The deduction of the WDW equation of the EiBI model is based on the construction of a classical Hamiltonian that is promoted to a quantum operator. As shown in [22], this can be achieved more straightforwardly by considering an alternative action which is dynamically equivalent to the EiBI action (1):

$$S_a = \lambda \int d^4x \sqrt{-q} R(q) - \frac{2}{3} \frac{\lambda}{\kappa} \left( \frac{1}{\kappa} \left( q_{ab} q_{ab} - 2 \frac{\sqrt{q}}{q} \right) \right) S_m(g).$$

(7)

In Ref. [8] it has been shown that the field equations obtained by varying the action (7) with respect to $g_{\mu\nu}$ and the auxiliary metric $g_{\mu\nu}$ are the same to those derived from the action (1). Starting from action (7) and inserting the FLRW ansatz, the Lagrangian of this model in which matter field is described by a perfect fluid can be written as (see Ref. [22])

$$\mathcal{L} = \lambda M b^3 \left[ \frac{6b^2}{M^2 b^3} + \frac{2}{k} \frac{N^2}{M^2} + \frac{3a^2}{b^2} - \frac{3Na^3}{M b^3} - 6 \rho N a^3, \right]$$

where $N$ and $M$ are the lapse functions of $g_{\mu\nu}$ and $q_{\mu\nu}$, respectively.

3.1. Classical Hamiltonian

The classical Hamiltonian can be obtained in a standard way via $\mathcal{H}_t = b p_b - \mathcal{L}$ where $p_b = \partial \mathcal{L}/\partial \dot{b}$. After choosing a gauge to fix the lapse function $N$, we obtain the Hamiltonian to construct the WDW equation [22, 35]

$$\mathcal{H}_t = M \left[ \frac{p_b^2}{244b^3} + \frac{2 \lambda^2}{k} b^3 + \frac{1}{\kappa \lambda} (\lambda + \kappa p_0) \frac{s^2}{b^3} - \frac{3 \lambda}{k} b_0^3 \right].$$

(8)
where $\rho(a)$ is given in Eqs. (4). As there is no singularity at $b = 0$ for this model, we can safely rescale the Hamiltonian as

$$b^3\mathcal{H}_2 = M \left[ -\frac{b^2 p^2_b}{24\lambda} + \frac{2\lambda^2}{\kappa} b^6 + 1 \frac{(\lambda + \kappa \rho(a))^2}{\kappa} a^6 - \frac{3\lambda}{\kappa} a^2 b^4 \right] = 0.$$  

(9)

3.2. Wheeler-DeWitt equation: factor ordering 1

In order to prove that our results are independent of the factor ordering, we make two choices of it. First, we choose the following factor ordering:

$$b^2 p^2_b = -\hbar^2 \left( \frac{\partial}{\partial y} \right) \left[ (\lambda + \kappa \rho(a))^2 \delta \right]^{1/2}.$$  

(10)

where $x = \ln(\sqrt{ab})$, and the WDW equation reads:

$$\left[ \frac{\partial^2}{\partial x^2} + V_1(a,x) \right] \Psi(a,x) = 0,$$  

(11)

where

$$V_1(a,x) = \frac{24}{kh^2} \left[ 2e^{2x} - 3a^2 e^{2x} + (\lambda + \kappa \rho(a))^2 \delta \right].$$  

(12)

Next, we rewrite the potential $V_1(a,x)$ as

$$V_1(a,x) = \frac{24}{kh^2} e^{2x} \left[ 2 - 3\delta + (\lambda + \kappa \rho(a))^2 \delta \right].$$  

(13)

where $\delta = a^2 e^{-2x}$. Near the classical singularity where $a \to \infty$, the behavior of the potential can be classified as follows:

- If $a^2$ diverges slower than $e^{2x}$, i.e., $\delta \to 0$, the second term in the bracket in (13) is negligible compared with the first term. The remaining two terms are positive and blow up when $a$ and $x$ go simultaneously to infinity.

- If $a^2$ diverges faster than $e^{2x}$, i.e., $\delta \to \infty$, the potential can be approximated as

$$V_1(a,x) \approx \frac{24}{kh^2} (\lambda + \kappa \rho(a))^2 a^6,$$  

(14)

when $a$ goes to infinity.

- If $a^2$ diverges comparably with $e^{2x}$, the potential can also be approximated as in Eq. (14) because the phantom energy density blows up when $a \to \infty$.

Therefore, we find that the potential $V_1(a,x)$ goes to positive infinity when $a \to \infty$ for all values of $x$. With the help of the first order WKB approximation (see the appendix), it can be easily proven that the wave function of the WDW equation (11) vanishes when $a \to \infty$ for all values of $x$.

3.3. Wheeler-DeWitt equation: factor ordering 2

From the Hamiltonian (8) we can choose another factor ordering

$$\frac{p^2_b}{b} = -\hbar^2 \left( \frac{1}{\sqrt{b}} \frac{\partial}{\partial b} \right) \left( \frac{1}{\sqrt{b}} \frac{\partial}{\partial b} \right).$$  

(15)

and rewrite the corresponding WDW equation by introducing a new variable $y \equiv (\sqrt{ab})^{3/2}$ as follows

$$\left[ \frac{\partial^2}{\partial y^2} + V_2(a,y) \right] \Psi(a,y) = 0,$$  

(16)

where

$$V_2(a,y) = \frac{32}{3kh^2} y^2 \left[ 2 - 3\eta + (\lambda + \kappa \rho(a))^2 \eta \right].$$  

(17)

$\eta \equiv a^2 y^{-4/3}$ and $\rho(a)$ is given in Eqs. (4). Before proceeding further, we highlight that this quantization is based on the Laplace-Beltrami operator which is the Laplacian operator in minisuperspace [51]. This operator depends on the number of degrees of freedom involved. For the case of a single degree of freedom, it can be written as in Eq. (15).

The behavior of $V_2(a,y)$ given in (17) near the classical singularity where $a \to \infty$ can be classified as follows:

- If $a^2$ diverges slower than $y^{4/3}$, i.e., $\eta \to 0$, the second term of $V_2(a,y)$ is negligible compared with the first term. The remaining two terms are positive and blow up when $a$ and $y$ go simultaneously to infinity.

- If $a^2$ diverges faster than $y^{4/3}$, i.e., $\eta \to \infty$, the potential can be approximated as

$$V_2(a,y) \approx \frac{32}{3kh^2} (\lambda + \kappa \rho(a))^2 \eta \theta a^3,$$  

(18)

when $a$ goes to infinity.

- If $a^2$ diverges comparably with $y^{4/3}$, the potential can also be approximated as in Eq. (18) because the phantom energy density blows up when $a \to \infty$.

Therefore, we find that the potential $V_2(a,y)$ goes to positive infinity when $a \to \infty$ for all values of $y$. Then, using again the WKB approximation (cf. the appendix), we can claim that the wave function $\Psi(a,x)$ and $\Psi(a,y)$ vanish when $a \to \infty$. 

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3.4. Wheeler-DeWitt equation: without a WKB approximation

In addition, by assuming the validity of one of the field equations, which implies the dependence between $a$ and $b$, we can obtain an approximated solution for the wave function without relying on a WKB approximation. Near the LR singular event, the energy density $\rho$ behaves as $\rho \propto (\ln a)^2$. On that regime, the dependence between the auxiliary scale factor $b$ and $a$ is $b \propto a \ln a$. On the other hand, near the LSBR event the energy density behaves as $\rho \propto \ln a$ and $b$ behaves as $b \propto a \sqrt{\ln a}$. For both cases, the WDW equation \([11]\) can be written as

\[
\left( \frac{d^2}{dx^2} + \frac{48}{k^2} e^{6x} \right) \Psi(x) = 0, \tag{19}
\]

where $x$ and $a$ go to infinity. The wave function reads \([52]\)

\[
\Psi(x) = C_1 J_0(A_1 e^{3x}) + C_2 Y_0(A_1 e^{3x}), \tag{20}
\]

and consequently when $x \to \infty$, its asymptotic behavior reads \([52]\)

\[
\Psi(x) \approx \sqrt{\frac{3}{\pi A_1}} e^{-3x/2} \left[ C_1 \cos \left( A_1 e^{3x} - \frac{\pi}{4} \right) + C_2 \sin \left( A_1 e^{3x} - \frac{\pi}{4} \right) \right], \tag{21}
\]

where

\[
A_1 \equiv \frac{4}{\sqrt{3k\hbar^2}}. \tag{22}
\]

Here $J_n(x)$ and $Y_n(x)$ are Bessel function of the first kind and second kind, respectively.

Similarly, in both cases (LR and LSBR) the WDW equation \([16]\) can be written as

\[
\left( \frac{d^2}{dy^2} + \frac{64}{3k\hbar^2} y^2 \right) \Psi(y) = 0, \tag{23}
\]

when $a$ and $y$ approach infinity. The solution of the previous equation read \([52]\)

\[
\Psi(y) \approx C_1 \sqrt{3} J_{1/4}(A_1 y^2) + C_2 \sqrt{3} Y_{1/4}(A_1 y^2), \tag{24}
\]

and when $y \to \infty$, therefore, \([52]\)

\[
\Psi(y) \approx \sqrt{\frac{2}{\pi A_1 y}} \left[ C_1 \cos \left( A_1 y^2 - \frac{3\pi}{8} \right) + C_2 \sin \left( A_1 y^2 - \frac{3\pi}{8} \right) \right]. \tag{25}
\]

Consequently, the wave functions approach zero when $a$ goes to infinity. According to the DeWitt criterion for singularity avoidance \([53]\), the LR and LSBR is expected to be avoided independently of the factor orderings considered in this work.

3.5. Expected values

We have shown that the DeWitt criterium of singularity avoidance is fulfilled hinting that the universe would escape the LR and LSBR in the EiBI model once the quantum effects are important. We next estimate the expected value of the scale factor of the universe $a$ by estimating the expected value of $b$. The reason we have to resort to the expected value of $b$ rather than $a$ is that in the classical theory \([19]\) that we have quantized the dynamics is only endowed to the scale factor $b$. We remind at this regard that when approaching the LR and LSBR, $b \propto a \ln a$ and $b \propto a \sqrt{\ln a}$, respectively, at least within the classical framework. Therefore if the expected value of $b$, which we will denote as $\mathbf{b}$, is finite, then we expect that the expected value of $a$; i.e. $\mathbf{a}$ would be finite as well. Therefore, non of the cosmological and geometrical divergences present at the LR and LSBR would take place.

We next present a rough estimation for an upper limit of $\mathbf{b}$ for the two quantization procedures presented on the previous subsection.

- **Factor ordering I:**

  The expected value of $b$ at late-time can be estimated as follows:

  \[
  \mathbf{b} = \int_{x_1}^{\infty} \Psi^*(x) \frac{e^x}{\sqrt{\lambda}} \Psi(x) dx, \tag{26}
  \]

  where $x_1$ is large enough to ensure the validity of the approximated potential in \([19]\), i.e., $\delta \to 0$. In this limit, we can use the asymptotic behavior for the wave function c.f. Eq. \((21)\). Then, it can be shown that the approximated value of $\mathbf{b}$ is bounded as

  \[
  \int_{x_1}^{\infty} \Psi^*(x) \frac{e^x}{\sqrt{\lambda}} \Psi(x) dx < \frac{|C_1|^2 + |C_2|^2}{\pi A_1} \frac{e^{-2x_1}}{\sqrt{\lambda}}. \tag{27}
  \]

  Therefore, we can conclude that $\mathbf{b}$ has an upper finite limit. Consequently, the LR and LSBR are avoided.

- **Factor ordering II:**

  In this case the expected value of $b$ can be written as

  \[
  \mathbf{b} = \int_{y_1}^{\infty} \Psi^*(y) \frac{e^y}{\sqrt{\lambda}} \Psi(y) f(y) dy, \tag{28}
  \]

  where $y_1$ is large enough to ensure the validity of the approximated potential in \([23]\), i.e., $\eta \to 0$. In addition, we have introduced a phenomenological weight $f(y)$ such that the norm of the wave function is well defined and finite for large $y$ \([54][56]\).
In fact, we could as well choose \( f(y) = y^{-\alpha} \) with \( 2/3 < \alpha \). After some simple algebra, we obtain
\[
\mathbf{b} < \frac{2 \left( |C_1|^2 + |C_2|^2 \right)}{\pi A_1 \sqrt{A}} \int_{y_1}^{y_\infty} y^{-\frac{2}{3}} f(y). \tag{29}
\]
Consequently, we get
\[
\mathbf{b} < \frac{2 \left( |C_1|^2 + |C_2|^2 \right)}{\pi A_1 \sqrt{A} \left( \alpha - 2/3 \right)} y_1^{-\alpha}. \tag{30}
\]
Once again, we reach the conclusion that \( \mathbf{b} \) is finite. Therefore, the LR and LSBR are avoided.

4. Conclusions

Singularities seem inevitable in most theories of gravity. It is therefore natural to ask whether by including quantum effects would the singularities be removed. In the case of the EiBI scenario, while the big bang singularity can be removed, the intrinsic phantom dark energy doomsday remains inevitable [31]. We solved the modified Wheeler-DeWitt equation of the EiBI model for a homogeneous and isotropic universe whose matter content corresponds to two kinds of perfect fluid. Those fluids within a classical universe would unavoidably induce LR or LSBR. We show that within the quantum approach we invoked, the DeWitt criterion is fulfilled and it leads toward the potential avoidance of the LR and LSBR. Our conclusion appears unaffected by the choice of factor ordering.

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Appendix: Validity of the WKB approximation

Given the following differential equation
\[
\left[ \frac{d^2}{du^2} + g(u) \right] \Psi(u) = 0, \tag{31}
\]
the first order WKB approximation reads \([38, 49, 57]\)
\[
\Psi(u) \approx \left| g(u) \right|^{-\frac{1}{4}} \left( C_1 e^{\int g^{1/2} \sqrt{g(dg/du)}} + C_2 e^{-\int g^{1/2} \sqrt{g(dg/du)}} \right), \tag{32}
\]
where \( C_1 \) and \( C_2 \) are constants. This method is valid in the regions where the following inequality is fulfilled
\[
\left| \frac{5 \left| g'(u) \right|^2 - 4 \left| g''(u) \right| \left| g(u) \right|}{16 \left| g(u) \right|^3} \right| < 1, \tag{33}
\]
where prime stands for derivatives with respect to the variable \( u \).

The first order WKB approximation is a useful tool which can give hints about the convergence (or divergence) of the wave function. However, this analysis must be applied to a regime where the approximation itself is valid. In this work, we have used two factor orderings, each of them induce an effective potential which distinguishes two different regions. As we next present, in all cases, the WKB method is valid close to the LR and LSBR abrupt events. Therefore, the convergence of the wave functions based on these approximated solutions is fully justified.

- Factor ordering I: For this factor ordering the corresponding effective potential is given in Eq. (15).
  If \( \delta \to 0 \) the potential can be approximated as
  \[
  V_I(a, x) \approx \frac{48}{k \hbar^2} e^{\delta x}, \tag{34}
  \]
  therefore, the corresponding WKB solution is valid if
  \[
  \left| \frac{3 k \hbar^2}{64} e^{-6x} \right| \ll 1, \tag{35}
  \]
  which is fulfilled for large values of \( x \). On the other hand, if \( \delta \) is close to unity or we are in the regime \( \delta \to \infty \), the potential can be approximated as in Eq. (14) since the energy density blows up for large scale factors. Note that in this case there is no dependence on \( x \), therefore, the WKB approximation is valid since the derivatives of the potential with respect to \( x \) vanish.
- Factor ordering II: For this factor ordering the corresponding effective potential is given in Eq. (17).
If $\eta \to 0$, the potential can be approximated as
\[
V_2(a, y) \approx \frac{64}{3k\hbar^2} y^2.
\]
(36)

therefore, the corresponding WKB solution is valid if
\[
\left| \frac{9k\hbar^2}{256} \right| \ll 1,
\]
(37)

which is indeed fulfilled for large values of $y$. On the other hand, if $\eta \approx \eta^3 \sim 1$ or $\eta \to \infty$, the potential can be approximated as in Eq. (18) because the energy density diverges when the scale factor reaches very large values. Eq. (18) can be rewritten in terms of $a$ and $y$ as
\[
V_2(a, y) \approx \frac{32}{3k\hbar^2} (\lambda + \kappa \rho(a))^2 a^6 y^{-2}.
\]
(38)

Therefore, the first order WKB approximation is valid if
\[
\left| \frac{3k\hbar^2}{228} \frac{a^{-6}}{(\lambda + \kappa \rho(a))} \right| \ll 1.
\]
(39)

Note that there is no dependence on $y$ in the inequality. Since the energy density diverges when the scale factor reaches very large values the condition is rigorously fulfilled.

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