Backstepping Density Control for Large-Scale Heterogeneous Nonlinear Stochastic Systems

Tongjia Zheng\(^1\), Qing Han\(^2\) and Hai Lin\(^1\)

Abstract—This work studies the problem of controlling the probability density of large-scale stochastic systems, which has applications in various fields such as swarm robotics. Recently, there is a growing amount of literature that employs partial differential equations (PDEs) to model the density evolution and uses density feedback to design control laws which, by acting on individual systems, stabilize their density towards a target profile. In spite of its stability property and computational efficiency, the success of density feedback relies on assuming the systems to be homogeneous first-order integrators (plus white noise) and ignores higher-order dynamics, making it less applicable in practice. In this work, we present a backstepping design algorithm that extends density control to heterogeneous and higher-order stochastic systems in strict-feedback forms. We show that the strict-feedback form in the individual level corresponds to, in the collective level, a PDE (of densities) distributedly driven by a collection of heterogeneous stochastic systems. The presented backstepping design then starts with a density feedback design for the PDE, followed by a sequence of stabilizing design for the remaining stochastic systems. We present a candidate control law with stability proof and apply it to nonholonomic mobile robots. A simulation is included to verify the effectiveness of the algorithm.

Index Terms—Density control, backstepping, stochastic systems

I. INTRODUCTION

The recent years have witnessed an enormous growth of research on the control of large-scale stochastic systems, and specific topics appear in different forms such as deployment of sensors and transportation of autonomous vehicles. In this work, we study the problem of controlling the probability density of a large group of heterogeneous nonlinear systems.

Control problems of large-scale systems have been extensively studied by a wide range of methodologies, such as graph theoretic design\(^1\) and game theoretic formulation (especially potential games\(^2\) and mean-field games\(^3\)). We pursue a strategy that controls directly the probability density of the systems. This control strategy shares similar philosophies like mean-field games and mean-field type control\(^4\) in the usage of the mean-field density. However, unlike mean-field games/control where the mean-field density usually appears in a secondary place to approximate the collective effect of the whole population, we aim at the direct control of this mean-field density.

Density control has been studied using discrete- and continuous-state models. The former relies on a partition of the state space and boils down to designing transition matrices for Markov chain models\(^5\), which usually suffers from the state explosion issue. Continuous-state models result in a control problem of PDEs that describe the time evolution of the density function. Early efforts on the density control of PDEs tend to adopt an optimal control formulation, which relies on expensive numerical computation of the optimality conditions and usually only generates open-loop control\(^6\),\(^7\). Closed-loop optimal density control is studied in\(^8\),\(^9\) by establishing a link between density control and Schrödinger Bridge problems. However, except for the linear case which adopts closed-form solutions, numerically solving the associated Schrödinger Bridge problem also suffers from the curse of dimensionality. More recent efforts have sought to use the real-time density as feedback to design closed-loop and closed-form control\(^10\)–\(^13\). Density feedback laws are able to guarantee closed-loop stability and can be efficiently computed on board. However, the success of density feedback design relies on the assumption that the systems are homogeneous first-order integrators (with white noise). This assumption makes the density control strategy less applicable for many systems in practice, such as wheeled mobile robots. Stochastic systems with heterogeneous and higher-order dynamics are different to handle and, to the best of our knowledge, have not been studied so far.

In this work, we aim to extend density feedback design to heterogeneous and nonlinear (in particular, strict-feedback) stochastic systems. The control objective is to design control laws to stabilize the probability density of states of a collection of strict-feedback stochastic systems. The strict-feedback form is not a restrictive requirement, because many mobile vehicle robots satisfy this form and some nonlinear stochastic systems can be converted to strict-feedback forms through coordinate transformation\(^14\). We will show that the strict-feedback form in the individual level corresponds to, in the collective level, a PDE (of densities) distributedly driven by a collection of heterogeneous stochastic systems. Our key idea is to perform a backstepping design which starts with a density feedback design for the PDE, followed by a sequence of stabilizing design for the remaining stochastic systems.

We note that backstepping design for stochastic systems has been widely studied; see, e.g.,\(^15\)–\(^17\). Unlike these works where the control objective is to stabilize each system in the individual level, our control goal is to stabilize the probability density of these systems, meaning that each system does not necessarily exhibit equilibrium behaviors in the individual

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\(^2\)Tongjia Zheng and Hai Lin are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA. tzengl1@nd.edu, hlin1@nd.edu.

\(^3\)Qing Han is with the Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA. Qing.Han.7@nd.edu.
level. This is a non-classical control problem and requires new backstepping design algorithms. In summary, our contribution includes: 1) presenting a backstepping design algorithm for the density control problem of large-scale heterogeneous and nonlinear stochastic systems, 2) providing specific control laws with stability analysis, and 3) applying the design algorithm to nonholonomic mobile robots as an illustration.

The rest of the paper is organized as follows. Section II introduces some preliminaries. Problem formulation is given in Section III. Section IV is our main results in which we present the backstepping design algorithm and provide specific control laws. Section V provides an example using nonholonomic mobile robots. Section VI presents an agent-based simulation to verify the effectiveness.

II. PRELIMINARIES

A. Notations

For a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), its Euclidean norm is denoted by \( \|x\| := (\sum_{i=1}^n |x_i|^2)^{1/2} \). Let \( E \subset \mathbb{R}^n \) be a measurable set. For \( f : E \to \mathbb{R} \), its \( L^2 \)-norm is denoted by \( \|f\|_{L^2(E)} := \left( \int_E |f(x)|^2 \, dx \right)^{1/2} \). We will omit \( E \) in the notation when it is clear. The gradient and Laplacian of a scalar function \( f \) are denoted by \( \nabla f \) and \( \Delta f \), respectively. The divergence of a vector field \( F \) is denoted by \( \nabla \cdot F \).

B. Input-to-state stability

Define the following classes of comparison functions:

\[ \mathcal{P} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous, } \gamma(0) = 0, \text{ and } \gamma(r) > 0 \text{ for } r > 0 \} \]

\[ \mathcal{K} := \{ \gamma \in \mathcal{P} | \gamma \text{ is strictly increasing} \} \]

\[ \mathcal{K}_\infty := \{ \gamma \in \mathcal{P} | \gamma \text{ is unbounded} \} \]

\[ \mathcal{V}_{\mathcal{K}_\infty} := \{ \alpha \in \mathcal{K}_\infty | \alpha \text{ is convex} \} \]

\[ \mathcal{C}_{\mathcal{K}_\infty} := \{ \alpha \in \mathcal{K}_\infty | \alpha \text{ is concave} \} \]

\[ \mathcal{L} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0 \} \]

\[ \mathcal{KL} := \{ \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ | \beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0, \beta(r, \cdot) \in \mathcal{L}, \forall r > 0 \} \]

Let \((X, \| \cdot \|_X)\) and \((U, \| \cdot \|_U)\) be the state and input space, endowed with norms \( \| \cdot \|_X \) and \( \| \cdot \|_U \), respectively. Denote \( U_c = PC(\mathbb{R}_+; U) \), the space of piecewise right-continuous functions from \( \mathbb{R}_+ \) to \( U \), equipped with the sup-norm.

We introduce the ISS concept applicable for both finite- and infinite-dimensional deterministic systems [13]. Consider a control system \( \Sigma = (X, U_c, \phi) \) where \( \phi : \mathbb{R}_+ \times X \times U_c \to X \) is a transition map. Let \( x(t) = \phi(t, x_0, u) \).

**Definition 1**: \( \Sigma \) is called input-to-state stable (ISS), if \( \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K} \), such that

\[
\|x(t)\|_X \leq \beta(\|x_0\|_X, t) + \gamma \left( \sup_{0 \leq s \leq t} \|u(s)\|_U \right),
\]

\( \forall x_0 \in X, \forall u \in U_c \) and \( \forall t \geq 0 \).

**Definition 2**: A continuous function \( V : \mathbb{R}_+ \times X \to \mathbb{R}_+ \) is called an ISS-Lyapunov function for \( \Sigma \), if \( \exists \psi_1, \psi_2 \in \mathcal{K}_\infty, \chi \in \mathcal{K} \), and \( W \in \mathcal{P} \), such that:

(i) \( \psi_1(\|x\|_X) \leq V(t, x) \leq \psi_2(\|x\|_X), \forall x \in X \)

(ii) \( \forall x \in X, \forall u \in U_c \) with \( u(0) = \xi \in U \) it holds:

\[
\|x\|_X \geq \chi(\|\xi\|_U) \Rightarrow V(t, x) \leq -W(\|x\|_X).
\]

**Theorem 1**: If \( \Sigma \) admits an ISS-Lyapunov function, then it is ISS.

Now we introduce ISS for stochastic systems [19]. Consider an \( n \)-dimensional stochastic differential equation (SDE):

\[ dx = f(x, t, u) \, dt + g(x, t, u) \, dW_t, \quad t \in \mathbb{R}_+, \]

\[ x(0) = x_0 \in \mathbb{R}^n, \]

where \( f : X \times \mathbb{R}_+ \times U \to \mathbb{R}^n, g : X \times \mathbb{R}_+ \times U \to \mathbb{R}^{n \times m} \) are sufficiently smooth, and \( W_t \) is an \( m \)-dimensional Wiener process. For any \( V(x, t) \in C^2(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \), define the differential operator \( L \):

\[
LV(t, x) = V_t + V_x f + \frac{1}{2} \text{Tr}(g^T V_{xx} g).
\]

**Definition 3**: System (1) is called \( p \)-th moment ISS (p-ISS) if \( \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \) such that

\[
E[\|x(t)\|^p] \leq \beta(\|x_0\|^p, t) + \gamma \left( \sup_{0 \leq s \leq t} \|u(s)\|_U \right),
\]

\( \forall x_0 \in X, \forall u \in U_c \) and \( \forall t \geq 0 \).

**Definition 4**: \( V \in C^2(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \) is called an ISS-Lyapunov function for system (1), if \( \exists \psi_1 \in \mathcal{V}_{\mathcal{K}_\infty}, \psi_2 \in \mathcal{K}_\infty, \chi \in \mathcal{K}_\infty, W \in \mathcal{K}_\infty \), such that:

(i) \( \psi_1(\|x\|^p) \leq V(x, t) \leq \psi_2(\|x\|^p), \forall x \in X \)

(ii) \( LV(x, t) \leq -W(\|x\|^p + \chi(\|\cdot\|_U)) \).

The following theorem is based on Theorem 3.1 in [19].

**Theorem 2**: If system (1) admits an p-ISS-Lyapunov function, then it is p-ISS.

III. PROBLEM FORMULATION

We study the density control problem for a family of \( N \) (heterogeneous) strict-feedback stochastic systems given by:

\[
dx_i = v_i(x_i, t) \, dt + g_1(x_i, t) \, dW_{i1}^1, \quad i = 1, \ldots, N
\]

\[ dv_i = u_i \, dt + g_2(x_i, v_i, t) \, dW_i^1,
\]

where

\( x_i, v_i \in \mathbb{R}^n \): states of the \( i \)-th system;

\( W_i^1 \in \mathbb{R}^m \): standard Wiener processes independent across \( i \);

\( u_i \): control input of the \( i \)-th system;

\( g_1, g_2 \in \mathbb{R}^{n \times m} \): matrix-valued \( C^2 \) and bounded functions.

Throughout this work, the superscription \( i \) is reserved to represent the \( i \)-th system. We note that \( g_2^i \) are allowed to be different/heterogeneous for different systems. We only require the upper system of \( \Sigma \) to be homogeneous. The control objective is to design \( u_i \) to stabilize the probability density of \( \{x^i\} \) towards a target density.

**Remark 1**: For clarity, we have restricted our attention to the case of two stages in \( \Sigma \). However, the backstepping design
algorithm to be presented later generalizes easily to systems with more stages:
\[
\begin{align*}
\frac{dx_i}{dt} &= v_i^1(x^i, t)dt + g_i(x^i, t)dW^i_t, \quad i = 1, \ldots, N \\
\frac{dv_i}{dt} &= v_i^{l+1} + g_i^{l+1}(x^i, v_i^l)dt + dW^i_t, \quad l = 1, \ldots, M_i \\
\frac{dv_{M_i+1}}{dt} &= v_{M_i+1}dt + g_{M_i+1}(x^i, v_{M_i+1}, t)dW^i_t,
\end{align*}
\]
(3) (4) (5)

where \( v_i^l = [(v_i^1)^T, \ldots, (v_i^l)^T]^T \) and \( M_i + 1 \) is the length of stages of the \( i \)-th system. The objective is to design \( u^i \) to stabilize the probability density of \( \{x^i\} \) (the states of the first stage). Again, only (3) is required to be homogeneous. The remaining stages (4–5) can be heterogeneous with different length. This is because in our backstepping design, only the first step is identical for all systems. The remaining steps are performed independently for different systems. The generalization will be made clear later.

Resume our discussion on (2). We treat the collection of states \( \{x^i\} \) as being driven by the same continuous vector field \( v(x, t) \) such that \( v(x^i(t), t) = v^i(x^i(t), t) \) for all \( t \). In other words, when projecting onto the trajectory of the \( i \)-th system, \( v \) coincides with \( v^i \). In this case, the lower equation of (2) is understood as the time differential of \( v^i(x^i(t), t) \) along the trajectory \( x^i(t) \). Note that we implicitly assume that \( v^i(x^i(t), t) = v^j(x^j(t), t) \) when \( x^i(t) = x^j(t) \) for all \( i \neq j \) and all \( t \). This is a mild assumption. Notice that \( x^i \) are all stochastic processes and the probability of \( x^i = x^j \) for some \( j \) is 0. Hence, even if the above assumption is violated, it will not cause any problem to the analysis.

We confine \( \{x^i\} \) within a bounded domain \( \Omega \in \mathbb{R}^n \) with boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0, T) \), \( x = [x_1, \ldots, x_n]^T \), and \( v = [v_1, \ldots, v_n]^T \). Define
\[
\sigma := [\sigma_{jk}]_{n \times n} = \frac{1}{2} g_1 g_1^T,
\]
\[
\nabla \sigma := \left[ \sum_{k=1}^n \partial_{x_k} (\sigma_{jk} p), \ldots, \sum_{k=1}^n \partial_{x_k} (\sigma_{nk} p) \right]^T.
\]

Let \( p(x, t) \) be the probability density of \( \{x^i\} \), which is known to satisfy the Fokker-Planck-Kolmogorov equation:
\[
\begin{aligned}
\partial_t p &= -\sum_{j=1}^n \partial_{x_j} (v_j p) + \sum_{j,k=1}^n \partial_{x_j} \partial_{x_k} (\sigma_{jk} p) \quad \text{in} \ \Omega_T \\
&\quad = -\nabla \cdot (v p - \nabla \sigma p), \\
p &= p_0 \quad \text{on} \ \Omega \times \{0\}, \\
\mathbf{n} \cdot (v p - \nabla \sigma p) &= 0 \quad \text{on} \ \partial \Omega \times (0, T),
\end{aligned}
\]
(6)

where \( p_0 \) is the initial density and \( \mathbf{n} \) is the outward normal to \( \partial \Omega \). The last equation is the reflecting boundary condition to preserve the mass.

Our control problem is stated as follows.

Problem 1 (Density control): Given a target density \( p_* \), design \( u_i \) such that \( p \to p_* \).

IV. BACKSTEPPING DENSITY CONTROL

In backstepping design, a sequence of stabilizing functions is recursively constructed [20]. Backstepping design for stochastic systems has been widely studied [15–17]. The major novelty of this work is that instead of stabilizing each \( x^i \) in the individual level, we aim to stabilize \( p \), the density of \( \{x^i\} \), in the macroscopic level. This non-classical control objective requires new backstepping design algorithms beyond [15–17].

The first step is identical for all systems. Given a smooth target density \( p_*(x) > c > 0 \) with \( c \) a constant, define \( \tilde{p}(x, t) = p(x, t) - p_*(x) \). Let \( v_d \) be a stabilizing control law for (6) (or the upper system of (2)) and \( V_1(\tilde{p}) = \int_\Omega \phi(\tilde{p})dx \) be a Lyapunov function such that
\[
\frac{dV_1}{dt} \bigg|_{v=v_d} = \int_\Omega -W(\tilde{p})dx
\]
(7)

where \( \phi \) is \( C^1 \) and \( W \) is positive definite. The design of \( v_d \) is the subject of density control problems which are increasingly studied in recent years [10–13]. A particular example of \( (\nu_1, \phi) \) will be presented later.

The remaining steps of the backstepping design proceed independently for different systems. Along the trajectory \( x^i(t) \), define \( \tilde{v}^i(x^i(t), t) = v^i(x^i(t), t) - v_d(x^i(t), t) \). We can extend \( \tilde{v}^i \) to be a constant function on the \( x \)-domain by defining \( \tilde{v}^i(x, t) := \tilde{v}^i(x^i(t), t) \) for \( \forall x \in \Omega \). Using \( \tilde{p} = p - p_* \), we rewrite (6) as
\[
\partial_t \tilde{p} = -\nabla \cdot (v_d p - \nabla \sigma p + \tilde{v}^i),
\]
(8)

where \( v(x^i(t), t) = \tilde{v}^i(x^i(t), t) \). By Itô’s lemma, we have, along \( x^i(t), i = 1, \ldots, N, \)
\[
\begin{align*}
d\tilde{v}^i &= (u^i - \partial_i v_d - \partial_x v_d u^i - G)dt \\
&\quad + (g^i_2 - \partial_x v_d g_1)dt,
\end{align*}
\]
(9)

where
\[
\partial_x v_d = \left[ \frac{\partial v_d}{\partial x_k} \right]_{n \times n}, \quad G = \left[ \frac{1}{2} \text{Tr} \left( g_1 \frac{\partial^2 v_d}{\partial x_k \partial x_j} \right)_{n \times n} g_1 \right].
\]

Equations (8) and (9) constitute a composite system where the PDE (6) is distributedly driven by a collection of SDEs (9). To determine the control inputs \( u_i \), consider the following augmented Lyapunov function:
\[
V_2(\tilde{p}, \tilde{v}^i) = \int_\Omega \phi(\tilde{p}) + \frac{1}{4} \| \tilde{v}^i \|^4 dx.
\]

We have
\[
LV_2 = \int_\Omega (\phi' \tilde{p} - \partial_t \tilde{p} + \| \tilde{v}^i \|^2 (\tilde{v}^i)^T (u^i - \partial_i v_d - \partial_x v_d u^i - G) + \frac{1}{2} \text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (2\tilde{v}^i (\tilde{v}^i)^T + \| \tilde{v}^i \|^2 (g^i_2 - \partial_x v_d g_1)) \right) dx,
\]
where for the first term, by the divergence theorem and Young’s inequality, we have
\[
\int_{\Omega} \phi'(\tilde{p}) \partial_t p dx = \int_{\Omega} \phi'(\tilde{p}) \nabla \cdot (v_d p - \nabla \sigma p + \tilde{v}^i p) dx = \int_{\Omega} \nabla \phi'(\tilde{p}) \cdot (v_d p - \nabla \sigma p + \tilde{v}^i p) dx \leq \int_{\Omega} -W(\tilde{p}) + (\tilde{v}^i)^T p \nabla \phi'(\tilde{p}) dx \leq \int_{\Omega} -W(\tilde{p}) + \frac{1}{4} \| (\tilde{v}^i)^T p \nabla \phi'(\tilde{p}) \|^4 \frac{4}{4} + \frac{4}{4} \epsilon_1(t)^{4/3} dx \leq \int_{\Omega} -W(\tilde{p}) + \frac{\| \tilde{v}^i \|^4 \| p \nabla \phi'(\tilde{p}) \|^4}{4 \epsilon_1(t)^{4/3}} + \frac{3 \epsilon_1(t)^{4/3}}{4} dx,
\]
for any function \( \epsilon_1(t) > 0 \), and for the last term, by Young’s inequality, we have for all t ≥ 0,
\[
\begin{align*}
\text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (2\tilde{v}^i (\tilde{v}^i)^T (g^i_2 - \partial_x v_d g_1)) \right) = 3\| \tilde{v}^i \|^2 \text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (g^i_2 - \partial_x v_d g_1) \right) \\
\leq \frac{3}{2} \left( \frac{\| \tilde{v}^i \|^4}{\epsilon_2(t)^2} \text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (g^i_2 - \partial_x v_d g_1) \right)^2 + \epsilon_2(t)^2 \right),
\end{align*}
\]
for any function \( \epsilon_2(t) > 0 \). Hence,
\[
LV_2 \leq \int_{\Omega} -W(\tilde{p}) - k\| \tilde{v}^i \|^4 + \frac{3}{4} \left( \epsilon_1(t)^{4/3} + \epsilon_2(t)^4 \right) dx,
\]
where \( k > 0 \) is a selected constant. The above procedure has proved the following theorem.

**Theorem 3:** Consider the collection of systems (3). Let \( v_d \) be a density control law for (6) and \( V_1(\tilde{p}) = \int_{\Omega} \phi(\tilde{p}) dx \) be positive definite such that
\[
\frac{dV_1}{dt} \bigg|_{v=v_d} = \int_{\Omega} -W(\tilde{p}) dx,
\]
where \( \phi \) is a \( C^1 \) function and \( W \) is a positive definite function. Then with the feedback control laws:
\[
u^i = -k \tilde{v}^i + \partial_t v_d + \partial_x v_d \tilde{v}^i + G - \frac{\| p \nabla \phi'(\tilde{p}) \|^4}{4 \epsilon_1^2} \tilde{v}^i \leq \frac{3 \epsilon_1^2}{4 \epsilon_2^2} \text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (g^i_2 - \partial_x v_d g_1) \right)^2
\]
the Lyapunov function
\[
V_2(\tilde{p}, \tilde{v}^i) = \int_{\Omega} \phi(\tilde{p}) + \frac{1}{4} \| \tilde{v}^i \|^4 dx
\]
satisfies
\[
LV_2 \leq \int_{\Omega} -W(\tilde{p}) - k\| \tilde{v}^i \|^4 + \frac{3}{4} \left( \epsilon_1(t)^{4/3} + \epsilon_2^2 \right) dx.
\]
This theorem presents a design algorithm for the density control problem of (3). To compute \( u^i \) for the \( i \)-th system, simply substitute \( x = x'(t) \) into (11). The ultimate form of \( u^i \) depends on the selection of \( v_d \) and \( \phi \). We can choose \( \epsilon_1(t) \) and \( \epsilon_2(t) \) to be small to reduce the convergence error, but this will potentially result in large control inputs. Hence, (11) is essentially a high-gain control.

**Remark 2:** We believe that the property of the Lyapunov function (12) implies ISS in certain probabilistic sense for the solution \((\tilde{p}, \tilde{v})\). Unfortunately, we are unaware of an ISS-Lyapunov function theorem that is directly applicable for (12). The development of such a theorem is left as future work. We are, however, able to prove ISS properties for a special choice of \((v_d, \phi)\) presented in (13). The proof takes advantage of the fact that the closed-loop system resembles a cascade system of a deterministic PDE distributedly driven by a collection of SDEs.

**Theorem 4:** Let \( u^i \) be given by (11) and let
\[
\begin{align*}
v_d &= -\frac{\alpha \nabla (p - p_\ast) - \nabla \sigma p}{p}, \quad (13) \\
\phi &= \frac{1}{2} \tilde{p}^2,
\end{align*}
\]
where \( \alpha > 0 \) is a constant. Then \( \| \tilde{p} \|_L^2 \) is 1-th ISS and \( \| v^i \|_L^2 \) is 4-th ISS, both with respect to \( \epsilon_1(t) \) and \( \epsilon_2(t) \).

**Proof:** Substituting (13) and (11) into (8) and (9) respectively, we obtain the closed-loop system:
\[
\begin{align*}
\partial_t \tilde{p} &= \nabla \cdot (\alpha \nabla \tilde{p}) - \nabla \cdot (\tilde{v}^i), \\
\dot{\tilde{v}}^i &= -\frac{3 \epsilon_1^2}{4 \epsilon_2^2} \text{Tr} \left( (g^i_2 - \partial_x v_d g_1)^T (g^i_2 - \partial_x v_d g_1) \right)^2 + \frac{\| p \nabla \tilde{p} \|^4}{4 \epsilon_1^2} + k \tilde{v}^i + (g^i_2 - \partial_x v_d g_1) dW_i.
\end{align*}
\]
Consider a Lyapunov function \( V_1 = \int_{\Omega} \frac{1}{2} \tilde{p}^2 dx \) for (15). By the divergence theorem and Poincaré’s inequality, we have
\[
\frac{dV_1}{dt} = \int_{\Omega} -\alpha \nabla (\tilde{p})^2 + p \nabla \tilde{p} \cdot \dot{v} dx \leq \int_{\Omega} -\alpha \nabla (\tilde{p})^2 + \| \nabla \tilde{p} \| \| p \tilde{v} \|_L^2 dx \leq -\alpha (1 - \theta) \| \nabla \tilde{p} \|_L^2 - \alpha \theta \| \nabla \tilde{p} \|_L^2 \| \tilde{v} \|_L^2 + \| \nabla \tilde{p} \|_L^2 \| p \tilde{v} \|_L^2 \leq -c_1^2 \alpha (1 - \theta) \| \tilde{p} \|_L^2 + \| \nabla \tilde{p} \|_L^2 \| \tilde{v} \|_L^2 \leq -c_2 \alpha (1 - \theta) \| \tilde{p} \|_L^2 + \| \nabla \tilde{p} \|_L^2 \| \tilde{v} \|_L^2 \leq -c_2 \| \tilde{p} \|_L^2 \| \tilde{v} \|_L^2 ,
\]
where \( \theta \in (0, 1) \) and \( c_1 > 0 \) is the constant from the Poincaré inequality. By the maximum principle for (14), the solution \( \tilde{p} \) is upper bounded by a positive constant \( c_2 \) and we have \( \| \tilde{p} \|_L^2 \leq c_2 \| \tilde{v} \|_L^2 \). Hence, if
\[
\| \tilde{p} \|_L^2 \geq \frac{c_2}{c_1 \alpha \theta} \| \tilde{v} \|_L^2 ,
\]
then we have
\[
\frac{dV_1}{dt} \leq -c_2 \alpha (1 - \theta) \| \tilde{p} \|_L^2.
\]
This implies that there exist constants $\lambda_1, \kappa_1 > 0$ such that
\[
\|\tilde{p}(t)\|_{L^2} \leq \|\tilde{p}(0)\|_{L^2} e^{-\lambda_1 t} + \kappa_1 \|\tilde{\theta}(t)\|_{L^2}.
\]
Taking expectation on both sides and using Jensen’s inequality for concave functions, we have
\[
\begin{align*}
\mathbb{E}[\|\tilde{p}(t)\|_{L^2}] &\leq \|\tilde{p}(0)\|_{L^2} e^{-\lambda_1 t} + \kappa_1 (\int_0^t \mathbb{E}[\|\tilde{\theta}(s)\|_{L^2}]^2 ds) + e_1 \|\tilde{\theta}\| L^2 \|\Omega\|. \\
&\leq \|\tilde{p}(0)\|_{L^2} e^{-\lambda_1 t} + \kappa_1 |\Omega|^{\frac{1}{2}} \max_i \mathbb{E}[\|\tilde{\theta}_i(t)\|_{L^2}^2] + e_1 |\Omega|.
\end{align*}
\] (17)

Now we study (16). For any fixed $x \in \Omega$, consider a Lyapunov function $V_3 = \frac{1}{2} \|\tilde{\theta}\|^2$. Using (16), we have
\[
LV_3 = -\frac{3}{4} \frac{1}{c_1^2} \sum_i \left( (g_i^2 - \partial_x v_d g_1)^2 + \frac{\|p\| \|\tilde{\theta}\|}{4c_1^2} + k \right) \|\tilde{\theta}\|^2
\]
\[
- \frac{3}{2} \frac{1}{2} \|\tilde{\theta}\|^2 \sum_i \left( \frac{\|p\| \|\tilde{\theta}\|}{4c_1^2} + k \right) \|\tilde{\theta}\|^2 + \frac{3}{4} \sigma^2.
\]
This implies that there exist constants $\lambda_2, \kappa_2 > 0$ such that for all $t$, along $x_i(t)$,
\[
\mathbb{E}[\|\tilde{\theta}(t)\|_{L^2}] \leq \mathbb{E}[\|\tilde{\theta}(0)\|_{L^2}] e^{-\lambda_2 t} + \kappa_2 |\Omega|^\frac{1}{2}.
\] (18)

Combining (17) and (18) and using the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, we obtain the desired result.

Remark 3: Control laws like (13) are called density feedback because they use the real-time density $p$ as feedback. More related work on this topic can be found in [10]-[13]. $p$ can be estimated by classical density estimation algorithms like kernel density estimation, or the recently proposed density filtering algorithms [21, 22].

Remark 4: The design algorithm can be easily generalized to systems in the form (3)-(5) by augmenting (12) as
\[
\begin{align*}
V_{M_t} &= \int_\Omega \phi(\tilde{p}) + \sum_{l=1}^{M_t} \frac{1}{4} \|v_i - v_{i,l}\|^2 dx,
\end{align*}
\]
where $v_{i,l}, 1 \leq l \leq M_t$ is a sequence of stabilizing functions. Starting from the third step ($l \geq 2$), the design objective is similar with the classic backstepping design for stochastic systems studied in [13]-[17], where the presented algorithms can potentially be borrowed to address the remaining steps.

V. DENSITY CONTROL OF NONHOLONOMIC MOBILE ROBOTS

As an example, we apply backstepping density control to a group of heterogeneous nonholonomic mobile robots.

The position of a robot is specified by $q_i = [x_i^1, x_i^2, \theta_i]^T$, where $x_i^1, x_i^2$ are the coordinates and $\theta_i$ is the orientation. After adding white noise, the complete motion equations are given by [23]
\[
dq_i = S_i(q_i)v_i dt + f_1(q_i, t) dW_i^1,
\]
\[
dM_i(q_i) v_i dt = (-V_m(q_i, q_i^*) v_i - F_i(v_i) + \tau_i) dt + f_2(q_i, v_i, t) dW_i^2,
\]
where $M_i \in \mathbb{R}^{2 \times 2}$ are symmetric positive definite inertia matrices, $V_m^i \in \mathbb{R}^{2 \times 2}$ are the centripetal and coriolis matrices, $F_i \in \mathbb{R}^{2 \times m}$ are the surface frictions, $W_i^1 \in \mathbb{R}^{m}$, $f_1 \in \mathbb{R}^{2 \times m}$, $f_2 \in \mathbb{R}^{2 \times m}$ are as in (3), $\tau_i \in \mathbb{R}^{2}$ are the inputs, $d^i$ are related to geometric structures, and $S_i$ are given by
\[
\begin{align*}
S_i(q_i) &= \begin{bmatrix}
\cos \theta_i & -d_i \sin \theta_i \\
\sin \theta_i & d_i \cos \theta_i
\end{bmatrix}.
\end{align*}
\]

All the states are assumed to be available to the controller.

In the density control problem, we are interested in the density of only $x_i := [x_i^1, x_i^2]^T$, the positions of the robots. So $\theta_i$ will be treated as known parameters. Let $u^i$ be an auxiliary input. By applying the nonlinear feedback [23]
\[
\tau_i = M_i(q_i) u^i + V_m^i(q_i, q_i^*) v_i + F_i(v_i)
\]
and removing the third row from the kinematics equations, we obtain
\[
\begin{align*}
&dx_i = T_i(\theta_i) v_i dt + g_1(x_i, \theta_i) dW_i^1,
\end{align*}
\]
\[
\begin{align*}
du_i = u^i dt + g_2(x_i, v_i, \theta_i) dW_i^2.
\end{align*}
\] (20)

where $T_i$ consists of the first two rows of $S_i$, which is invertible, $g_1$ consists of the first two rows of $f_1$, and $g_2 = (M_i)^{-1} f_2$. The above equations are in the same form of (2). Hence, the stabilizing density feedback is given by
\[
v_d = (T_i)^{-1} \left[-\frac{\alpha(x, t) \nabla(p(x) - p_\star) - \nabla_x p}{p}\right],
\]
where $\alpha > 0$ can be used by individual robots to adjust their velocity magnitude. The auxiliary inputs $u^i$ can be computed according to (11), which then generate the actual input $u_i$ for each robot according to (19). By following $\tau_i$, the density of the robots’ positions converge towards a target density.

VI. SIMULATION STUDY

An agent-based simulation using 600 nonholonomic mobile robots is performed on Matlab to verify the proposed control law. We set $\Omega = (0, 1)^2$. Each robot is simulated according to (20) where $u^i$ is given by (11) and the parameters are given by $\alpha = 0.003$, $k = 0.008$, $\epsilon_1(t) = \epsilon_2(t) = 2$. Their initial positions are drawn from a uniform distribution. The target density $p_\star(x)$ is illustrated in Fig.2. We discretize $\Omega$ into a 30 × 30 grid, and the time difference is 0.02s. We use KDE (in which we set $h = 0.04$) to estimate the real-time density $p$. Simulation results are given in Fig.1. It is seen that the swarm is able to evolve towards the desired density. The convergence error $\|p - p_\star\|_{L^2}$ is given in Fig.2 which converges and remains bounded.
VII. CONCLUSION

This work studied the density control problem of large-scale heterogeneous strict-feedback stochastic systems. We converted it to a control problem of a PDE which is distributedly driven by a collection of heterogeneous SDEs and presented a backstepping design algorithm that extends the density feedback technique. The presented backstepping design is suitable for many nonlinear stochastic systems, including mobile robots. We applied the algorithm to nonholonomic mobile robots and included a simulation to verify the effectiveness. Our future work is to study the performance when the density is estimated using the density filters we recently reported \cite{21}.

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