Laplace Variational Method for System of Partial Differential Equations

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Abstract. The dynamics of complex biological systems can be analyzed with the aid of mathematical models. These mathematical models are mostly based on systems of coupled linear or nonlinear partial differential equations. The semi-analytic technique Laplace Variational Method has been presented in this article and is successfully applied to different mathematical models. The method simplifies the basic application procedure of Variational Iteration Method by applying the Laplace transformation. We have confirmed this advantage of this method over other methods with the help of examples and their comparative analysis.

1. Introduction

In the present era, many semi-analytical and numerical methods have been frequently used in the literature to find the solution of PDEs represents the nonlinear dynamics, such as Laplace Transform Method (LTM) [1], Parameter Expansion Method [2], Variational Iteration Method (VIM) [3], Perturbation techniques [4], Modified Homotopy Perturbation Method [5] and Homotopy Analysis Method [6, 7]. LTM is a very effective tool not only for the nonlinear differential equations with constant coefficients but also for the differential equations with variable coefficients [1]. Similarly, variational iteration method is the very strong method because it has widely been used to find the solution of various nonlinear PDEs [8, 9, 10].

Notable attention has been paid in combining two or more mathematical techniques to obtain the solution of complex nonlinear problems. In this article, a new semi-analytic technique is proposed to solve the above mentioned problem in the continuation of already available techniques such as Variational Iteration Method (VIM) [3], Laplace Transform Method (LTM) [1]. VIM was first introduced by He [8] and was optimally applied to solve different linear and nonlinear problems [9, 10]. In the present article semi-analytical technique Laplace Variational Method (LVM) obtained by combining LTM and VIM has been proposed to achieve the better approximate solution of nonlinear system of partial differential equations.

In order to obtain the recursive relations in LVM, first we take Laplace transform of every equation in the system and multiply it with the Lagrange's multipliers. Comparison have been made in between VIM and LVM. Afterwards, restrictions of variational theory are used to identify the
Lagrange's multipliers optimally. These relations neither involves the integral evaluations nor the convolutions.

We have validated the method by presenting trivial examples. After the successful implication of the technique, we have deployed it on a system of coupled nonlinear pde's. The system we have selected for analysis, governs the tumour induced angiogenesis. The results obtained are in close agreement with the results available in literature. With the aid of this new semi-analytic technique, we can solve highly nonlinear coupled systems governing complex biological phenomena very swiftly.

In this article, we have used two semi-analytical techniques LVM and VIM on advection model which have applications in flood wave propagation, simulations of advection, dispersion and chemical reactions of constraints in groundwater systems [11, 12], the nonlinear WBK equations which have applications in different nonlinear problems arises in shallow water theory [13, 14]. Again, we employ LVM and VIM on tumor angiogenesis model governed by the system of partial differential which have application in the augmentation of new blood vessels that tumors need to grow. This process is originated by the release of chemicals by the tumor and by host cells close to the tumor [15].

2. Variational Iteration Method (VIM) for System of Equations
In this section, we are using the generalised system of nonlinear partial differential equations

\[
L_1 \phi(x,t) + N_1(\phi(x,t), \psi(x,t)) = f(x,t)
\]

\[
L_2 \phi(x,t) + N_2(\phi(x,t), \psi(x,t)) = g(x,t)
\]

(2.1)

where \( L_1 \) and \( L_2 \) are linear and \( N_1, N_2 \) are nonlinear operators, \( f(x,t) \) , \( g(x,t) \) are some given functions. By using the variational iteration method, we can make the correctional functional where \( \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers which can be evaluated optimally via variational theory. The second terms in both the equations of system Eqs.(2.1) are called correction and \( n \) is the \( nth \) order approximation. \( \tilde{\phi} \) and \( \tilde{\psi} \) are considered as restricted variation. We can suppose that the above correctional functionals are stationary, i.e \( \delta \tilde{\phi}_n = 0 \) and \( \delta \tilde{\psi}_n = 0 \), then the Lagrange multipliers are estimated.

Example: 1 In this example, system of nonlinear Partial Differential Equations is solved by Variational Iteration Method, i.e

\[
\phi_i(x,t) = \frac{\partial}{\partial x} \left\{ D\phi_i(x,t) - \chi \phi(x,t) \psi_i(x,t) \right\} + (1+\nu(x,t)) - \rho \phi(x,t) \psi_i(x,t),
\]

(2.2)

\[
\psi_i(x,t) = \beta \phi(x,t) - \gamma \psi(x,t) \psi_i(x,t),
\]

(2.3)

\[
\nu_i(x,t) = -\eta \phi(x,t) \psi_i(x,t).
\]

(2.4)

Where, \( \phi = \phi(x,t) \); \( \psi = \psi(x,t) \) and \( \nu = \nu(x,t) \), where \( \chi, \rho, \beta, \gamma, \eta \) are the constants along with the conditions

\[
\phi(0,t) = 0 = \phi(\pi,t),
\]

\[
\psi(0,t) = 0 = \psi(\pi,t),
\]

\[
\nu(0,t) = 0 = \nu(\pi,t),
\]

\[
\phi(x,0) = \psi(x,0) = \nu(x,0) = \sin x.
\]

To solve the system with the given initial conditions by VIM, we make the correctional functional in t-direction as
\[
\varphi_{n+1}(x,t) = \phi_n(x,t) + \int_0^t \lambda_1(t)\{D(\phi_n(x,t))\} - \frac{\partial}{\partial x} \{D(\phi_n(x,t))\} + \frac{\partial}{\partial x} \left\{ \frac{\chi\phi_n(x,t)(\nu_n(x,t))}{1 + \nu_n(x,t)} \right\} + \frac{\partial}{\partial x} \left\{ \rho\phi_n(x,t)(\psi_n(x,t)) \right\} dt,
\]
(2.5)
\[ \nu_{n+1} = \nu_n - \int_0^t \left( (\nu_n(x,t))_x + \eta \phi_n(x,t) \nu_n(x,t) \right) dt, \]  

(2.11)

Using the initial conditions

\[ \phi(x,0) = \phi_0 = \psi(x,0) = \psi_0 = \nu(x,0) = \nu_0 = \sin x, \]

we get the first approximation

\[ \phi_1 = \sin x - \frac{t}{2(1 + \sin x)} \{ \chi + 2 \cos x + (2 \rho + \chi) \cos 2x + \rho \sin x - 2 \chi \sin x + \sin 2x + \rho \sin 3x \} \]

\[ \psi_1 = \sin x - t(\gamma \sin^2 x - \beta \sin x) \]

\[ \nu_1 = \sin x - t\eta \sin^2 x \]

3. Laplace Variational Method for System of equations

To understand the methodology, we consider the non-homogeneous system of PDEs

\[ L_1(\phi) + N_1(\phi, \psi, \nu) = f_1, \]

\[ L_2(\phi) + N_2(\phi, \psi, \nu) = f_2, \]

\[ L_3(\phi) + N_3(\phi, \psi, \nu) = f_3, \]

(3.1)

subject to the initial conditions

\[ \phi(x,0) = \phi_0; \quad \psi(x,0) = \psi_0; \quad \nu(x,0) = \nu_0, \]

(3.2)

where \( L_i \)s are linear and \( N_i \)s are nonlinear operators. \( f_i \)s are the known functions. The recurrence relations after applying the Laplace transform will becomes

\[ \phi_{n+1}(x,s) = \phi_n(x,s) + \lambda \mathcal{L} \left\{ L_1 \phi_n(x,t) + N_1(\phi_n(x,t), \psi_n(x,t), \nu_n(x,t)) - f_1(x,t) \right\}, \]

\[ \psi_{n+1}(x,s) = \psi_n(x,s) + \lambda \mathcal{L} \left\{ L_2 \psi_n(x,t) + N_2(\phi_n(x,t), \psi_n(x,t), \nu_n(x,t)) - f_2(x,t) \right\}, \]

\[ \nu_{n+1}(x,s) = \nu_n(x,s) + \lambda \mathcal{L} \left\{ L_3 \nu_n(x,t) + N_3(\phi_n(x,t), \psi_n(x,t), \nu_n(x,t)) - f_3(x,t) \right\}. \]

(3.3)

Now taking variation to identify the Lagrange multipliers \( \lambda \) by using the optimality conditions

\[ \frac{\delta \phi_{n+1}}{\delta \phi_n} = 0; \quad \frac{\delta \psi_{n+1}}{\delta \psi_n} = 0; \quad \frac{\delta \nu_{n+1}}{\delta \nu_n} = 0, \]

implies that \( \lambda = \frac{1}{s} \) by assuming that \( \delta \phi_n = 0, \delta \psi_n = 0, \delta \nu_n = 0 \). Substituting the values of \( \lambda \) and taking inverse Laplace transform on both sides of Eqs.(3.3), we get

\[ \phi_{n+1} = \phi_n - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ L_1 \phi_n + N_1(\phi_n, \psi_n, \nu_n) - f_1 \right] \right\}, \]

(3.4)

\[ \psi_{n+1} = \psi_n - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ L_2 \psi_n + N_2(\phi_n, \psi_n, \nu_n) - f_2 \right] \right\}, \]

(3.5)

\[ \nu_{n+1} = \nu_n - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ L_3 \nu_n + N_3(\phi_n, \psi_n, \nu_n) - f_3 \right] \right\}. \]

(3.6)

On substituting \( n = 0, 1, 2, \ldots \), we get the successive pair of approximations \((\phi_1, \psi_1, \nu_1), (\phi_2, \psi_2, \nu_2), \ldots\)
Example: 2

In this example, we are solving the following system of nonlinear partial differential equations by using Laplace variational method, i.e

\[
\phi_t(x,t) = \frac{\partial}{\partial x} \{ D\phi_x(x,t) - \frac{\chi\phi(x,t)\nu_x(x,t)}{1 + \nu(x,t)} - \rho\phi(x,t)\psi_x(x,t) \}, \tag{3.7}
\]

\[
\psi_t(x,t) = \beta\phi(x,t) - \gamma\phi(x,t)\psi(x,t), \tag{3.8}
\]

subject to the conditions

\[
\phi(0,t) = 0 = \phi(\pi,t),
\]

\[
\psi(0,t) = 0 = \psi(\pi,t),
\]

\[
\nu(0,t) = 0 = \nu(\pi,t),
\]

\[
\phi(x,0) = \psi(x,0) = \nu(x,0) = \sin x.
\]

The recurrence relations corresponding to the given system of partial differential equations after applying Laplace transform on both sides, we obtain

\[
\phi_{n+1}(x,s) = \phi_n(x,s) + \lambda_1 \{ s\phi_n(x,s) - \phi_0(x,0) \} + \lambda_2 \mathcal{L} \{ -D \frac{\partial}{\partial x} (\phi_n(x,t))_x \}
\]

\[
+ \lambda_3 \{ (\phi_n(x,t)(\nu_n(x,t))_t \} + \rho \frac{\partial}{\partial x} (\phi_n(x,t)(\psi_n(x,t))_x), \tag{3.10}
\]

\[
\psi_{n+1}(x,s) = \psi_n(x,s) + \lambda_2 \{ s\psi_n(x,s) - \psi_0(x,0) \} + \lambda_2 \mathcal{L} \{ -\phi_n(x,t) + \gamma \phi_n(x,t)\psi_n(x,t) \}, \tag{3.11}
\]

\[
\nu_{n+1}(x,s) = \nu_n(x,s) + \lambda_3 \{ s\nu_n(x,s) - \nu_0(x,0) \} + \lambda_3 \mathcal{L} \{ -\eta \phi_n(x,t)\nu_n(x,t) \}. \tag{3.12}
\]

Applying the variation, we have

\[
\delta\phi_{n+1}(x,t) = \delta\phi_n(x,t) + \lambda_1 \delta \{ s\phi_n(x,s) - \phi_0(x,0) \} + \lambda_2 \delta \mathcal{L} \{ -D \frac{\partial}{\partial x} (\phi_n(x,t))_x \}
\]

\[
+ \lambda_3 \delta \{ (\phi_n(x,t)(\nu_n(x,t))_t \} + \rho \frac{\partial}{\partial x} (\phi_n(x,t)(\psi_n(x,t))_x), \tag{3.13}
\]

\[
\delta\psi_{n+1}(x,t) = \delta\psi_n(x,t) + \lambda_2 \delta \{ s\psi_n(x,s) - \psi_0(x,0) \} + \lambda_2 \delta \mathcal{L} \{ -\beta \phi_n(x,t) + \gamma \phi_n(x,t)\psi_n(x,t) \}, \tag{3.14}
\]

\[
\delta\nu_{n+1}(x,t) = \delta\nu_n(x,t) + \lambda_3 \delta \{ s\nu_n(x,s) - \nu_0(x,0) \} + \lambda_3 \delta \mathcal{L} \{ -\eta \phi_n(x,t)\nu_n(x,t) \}. \tag{3.15}
\]

By using the optimality conditions, we get
and assuming that \( \delta \phi_n = 0, \delta u_n = 0, \delta v_n = 0 \) implies that \( \lambda_1 = \lambda_2 = \lambda_3 = \frac{-1}{s} \). Substituting the values in Eq.(3.10), Eq.(3.11) & Eq.(3.12), we have

\[
\psi_{n+1}(x,t) = \psi_n(x,t) - \frac{1}{s} \mathcal{L}\left\{ \frac{\partial \psi_n(x,t)}{\partial t} - \beta \phi_n(x,t) + \rho \phi_n(x,t)\psi_n(x,t) \right\},
\]

(3.14)

\[
\nu_{n+1}(x,t) = \nu_n(x,t) - \frac{1}{s} \mathcal{L}\left\{ \frac{\partial \nu_n(x,t)}{\partial t} + \eta \phi_n(x,t)\nu_n(x,t) \right\}.
\]

(3.15)

Taking inverse Laplace transform, we end up with the following recurrence relations

\[
\phi_{n+1}(x,t) = \phi_n(x,t) - \frac{1}{s} \mathcal{L}^{-1}\left\{ \frac{\partial \phi_n(x,t)}{\partial t} - D \frac{\partial}{\partial x} (\phi_n(x,t))_x + \chi \frac{\partial}{\partial x} \left( \frac{\phi_n(x,t)(\nu_n(x,t))}{1 + \nu_n(x,t)} \right) + \rho \frac{\partial}{\partial x} (\phi_n(x,t)\psi_n(x,t)) \right\},
\]

\[
\psi_{n+1}(x,t) = \psi_n(x,t) - \frac{1}{s} \mathcal{L}^{-1}\left\{ \frac{\partial \psi_n(x,t)}{\partial t} - \beta \phi_n(x,t) + \gamma \phi_n(x,t)\psi_n(x,t) \right\},
\]

\[
\nu_{n+1}(x,t) = \nu_n(x,t) - \frac{1}{s} \mathcal{L}^{-1}\left\{ \frac{\partial \nu_n(x,t)}{\partial t} + \eta \phi_n(x,t)\nu_n(x,t) \right\}.
\]

For \( n = 0, 1, 2, \cdots \), we get the successive pair of approximations \( (\phi_1, \psi_1, \nu_1), (\phi_2, \psi_2, \nu_2), \cdots \), where the initial approximation is \( \phi_0 = \psi_0 = \nu_0 = \sin x \). The first pair of approximations is

\[
\phi_1 = \sin x - \frac{t}{2(1 + \sin x)} \{ \chi + 2 \cos x + (2 \rho + \chi) \cos 2x - \rho \sin x - 2 \chi \sin x + \sin 2x + \rho \sin 3x \},
\]

\[
\psi_1 = \sin x - t(-\beta \sin x + \gamma \sin^2 x),
\]

\[
\nu_1 = \sin x - t\eta \sin^2 x.
\]

4. Conclusion

The Laplace Variational Method can be easily understand with only the basic knowledge of Advanced Calculus, even the reader has no background of calculus of variations in pure mathematics. The novel method (Laplace Variational Method) is simple and easy to apply in comparison with the traditional Variational Iteration Method. The advantage of this extended variational method is the Laplace
The transform helps to expedite the computational cost and can easily be applied to nonlinear biological systems using user friendly softwares such as Mathematica™ and Maple™.

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