Two and three electrons
in a quantum dot: \( 1/|J| \) - expansion

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Abstract

We consider systems of two and three electrons in a two-dimensional parabolic quantum dot. A magnetic field is applied perpendicularly to the electron plane of motion. We show that the energy levels corresponding to states with high angular momentum, \( J \), and a low number of vibrational quanta may be systematically computed as power series in \( 1/|J| \). These states are relevant in the high-\( B \) limit.
1 Introduction

Recently, few-electron systems in nearly two-dimensional semiconductor structures has been a subject of intense theoretical and experimental research [1]. In the present paper, we continue a programme, initiated in [2, 3], aimed at providing reliable analytical estimates for the energy levels of model two-dimensional systems on the basis of a simple physical picture for the states.

The idea is to use the inverse of angular momentum as an expansion parameter in Schrödinger equation, thus obtaining nonperturbative series for the energy and the wave function. Variants of this method have been widely used in Atomic and Nuclear Physics [4, 5].

The static exciton in a magnetic field was considered in [2]. Energy levels were computed as a function of the mass ratio and the magnetic field. Comparison with estimates obtained from two-point Pade approximants [6] yielded excellent results at any magnetic field strength.

In paper [3], a model of three electrons in a quantum dot with \( \frac{1}{r^2} \) repulsion, considered previously in [7], was studied. The \( \frac{1}{|J|} \)-method provided analytical estimates for the energy and a set of approximate quantum numbers to label the states. We will make use below of many of the results obtained in [3].

Systems of two and three electrons in a two-dimensional parabolic quantum dot are to be studied in the present paper. A magnetic field is applied perpendicularly to the dot’s plane. In this problem, there is only one relevant parameter, \( \beta = (E_{\text{coul}}/\hbar \Omega)^{1/6} \), where \( E_{\text{coul}} \) is the characteristic Coulomb energy, and \( \Omega \) is expressed in terms of the dot and the cyclotron frequencies as \( \Omega = \sqrt{\omega_0^2 + \omega_c^2/4} \). \( \beta \to 0 \) will be called the oscillator limit, and \( \beta \to \infty \) – the Wigner limit, suggesting that we are dealing with a few-electron version of the Wigner solid. In typical quantum dot experiments, \( \beta \sim 1 \).

The physical picture emerging from the \( \frac{1}{|J|} \)-expansion is the following. In the leading approximation, the electrons are located at the vertices of a regular polygon, which side minimises the total classical energy (centrifugal plus potential). The first corrections take account of small (harmonic) vibrations around the equilibrium configuration. Higher corrections come from anharmonic oscillations. This picture is supported by numerical calculations for few-electron systems not only in the strong coupling regime [8], but in the region \( \beta \sim 1 \) as well [9].

A special comment deserves paper [9], in which the few-electron problem in Eckardt coordinates is treated quasiclassically. Although the starting
points are far apart, our results are very similar up to the harmonic approximation. The main difference between paper [9] and ours is the expansion parameter, which is $\hbar$ in [9], and $1/|J|$ in our work. Higher order of the quasiclassical expansion are not reported in [9].

The plan of the paper is as follows. In section 2, we present the $1/|J|$-method. Results for two and three electrons are shown in sections 3 and 4. Concluding remarks are given in the last section.

## 2 The $1/|J|$-expansion

We start with the dimensionless hamiltonian governing the two-dimensional motion of $N$ electrons in a quantum dot of energy $h\omega_0$, and in a magnetic field normal to the plane of motion,

$$H_{\hbar\Omega} = \frac{1}{2} \sum_{k=1}^{N} (p_k^2 + r_k^2) + \beta^3 \sum_{k<l} \frac{1}{r_{kl}} + \frac{\omega_c}{2\Omega} \sum_{k=1}^{N} J_k + \frac{g\omega_c}{2\Omega} S_z. \quad (1)$$

The conventions are as follows. $\mu$ is the electron effective mass, $\omega_c = eB/\mu c$ is the cyclotron frequency, $g$ is the effective gyromagnetic factor, $S_z$ is the $z$ component of the total spin of the system, $\Omega = \sqrt{\omega_0^2 + \omega_c^2/4}$ is the effective dot frequency. The length unit is $\sqrt{\hbar/(\mu\Omega)}$. The parameter $\beta$ is given by $\beta^3 = \sqrt{\frac{\mu e^4}{\kappa^2\hbar^2}}/\hbar\Omega$, where $\kappa$ denotes the dielectric constant.

The actual parameter governing the problem is $\beta$. When $\beta << 1$, the levels are small perturbations around oscillator energies. On the other hand, when $\beta \to \infty$, the energy minimum is reached in a classical configuration which can be seen as a few-body model of a Wigner crystal. Both limiting situations condue to exactly solvable problems.

Through the introduction of Jacobi coordinates, the center-of-mass motion is separated from the internal motion,

$$\frac{H}{\hbar\Omega} = \frac{H_{\text{cm}}}{h\Omega} + \frac{H_{\text{int}}}{h\Omega}, \quad (2)$$

where

$$\frac{H_{\text{cm}}}{h\Omega} = - \left( \frac{\partial^2}{\partial \rho_{\text{cm}}^2} + \frac{1}{\rho_{\text{cm}}^2} \frac{\partial}{\partial \rho_{\text{cm}}} \right) + \frac{J_{\text{cm}}^2}{\rho_{\text{cm}}^2} + \frac{1}{4} \rho_{\text{cm}}^2 + \frac{\omega_c}{2\Omega} J_{\text{cm}}, \quad (3)$$
\[
\frac{H_{\text{int}}}{\hbar \Omega} = -\sum_{k=1}^{N-1} \left( \frac{\partial^2}{\partial \rho_k^2} + \frac{1}{\rho_k} \frac{\partial}{\partial \rho_k} \right) - \sum_{k=1}^{N-2} \left( \frac{1}{\rho_k^2} + \frac{1}{\rho_{k+1}} \right) \frac{\partial^2}{\partial \theta_k^2} + 2 \sum_{k=1}^{N-3} \frac{1}{\rho_{k+1}^2} \frac{\partial^2}{\partial \theta_k \partial \theta_{k+1}} + \frac{2iJ}{N-1} \sum_{k=1}^{N-2} \left( \frac{1}{\rho_k^2} - \frac{1}{\rho_{k+1}} \right) \frac{\partial}{\partial \theta_k} + \sum_{k=1}^{N-1} \left( \frac{J^2}{(N-1)^2 \rho_k^2} + \frac{1}{4} \rho_k^2 \right) + \beta^3 \sum_{k<l} \frac{1}{\rho_k \rho_l} + \frac{\omega_c}{2\Omega} J + g \frac{\omega_c}{2\Omega} S_z, \quad (4)
\]

We have written \( H_{\text{int}} \) in terms of rotational invariant coordinates, i.e., the moduli of the Jacobi vectors and the angles between them. The Jacobi vectors were defined as,

\[
\vec{\rho}_k = \sqrt{\mu_k/\mu_1} \left\{ \vec{r}_{k+1} - \frac{1}{k} \sum_{l=1}^{k} \vec{r}_l \right\}, \quad k = 1, 2, ..., N - 1, \quad (5)
\]

where the reduced masses are \( \mu_k = k/(k + 1) \). The angle between \( \vec{\rho}_k \) and \( \vec{\rho}_{k+1} \) is denoted \( \theta_k \). \( J \) labels the total internal angular momentum.

The eigenvalues of \( H_{\text{cm}} \) are simply,

\[
\frac{E_{cm}}{\hbar \Omega} = 2n_{cm} + |J_{cm}| + 1 + \frac{\omega_c}{2\Omega} J_{cm}, \quad (6)
\]

We shall obtain approximate expressions for the eigenvalues of \( H_{\text{int}} \). We consider states with high \( |J| \) values, which are the relevant states at high magnetic fields. It is intuitively evident that at high \( |J| \), \( E_{\text{int}} \sim |J|, < \rho_k^2 > \sim |J| \). The solution of the Schrodinger equation may be organised as a power series in \( 1/|J| \). A scaling of dimensions such that \( \rho^2 \to |J|\rho^2 \) makes evident the dependence of each term on \( |J| \). The scaled hamiltonian is written as

\[
\hbar = \frac{1}{|J|} \left\{ \frac{H_{\text{int}}}{\hbar \Omega} - \frac{\omega_c}{2\Omega} J - \frac{g \omega_c}{2\Omega} S_z \right\}
\]

\[
= \sum_{k=1}^{N-1} \left( \frac{1}{(N-1)^2 \rho_k^2} + \frac{1}{4} \rho_k^2 \right) + \beta^3 \sum_{k<l} \frac{1}{\rho_k \rho_l} + \frac{J^2}{(N-1)^2} \sum_{k=1}^{N-1} \left( \frac{\partial^2}{\partial \rho_k^2} + \frac{1}{\rho_k} \frac{\partial}{\partial \rho_k} \right) + \sum_{k=1}^{N-2} \left( \frac{1}{\rho_k^2} + \frac{1}{\rho_{k+1}} \right) \frac{\partial^2}{\partial \theta_k^2} + \frac{2}{J(N-1)} \sum_{k=1}^{N-2} \left( \frac{1}{\rho_k^2} - \frac{1}{\rho_{k+1}} \right) \frac{\partial}{\partial \theta_k}, \quad (7)
\]
where we have introduced the “renormalised” coupling constant $\tilde{\beta}^3 = \beta^3 / |J|^{3/2}$.

When taking the formal limit $|J| \to \infty$, $\tilde{\beta}^3$ is kept fixed to take account of Coulomb repulsion nonperturbatively.

In what follows, we consider only the two- and three-electron problems. The only term surviving in the r.h.s of (7) when $|J| \to \infty$ is the effective potential. Its absolute minimum gives a classical contribution to the energy. It is reached in a configuration where the particles sit at the corners of a regular polygon. In this configuration, the effective potential is a function of $\rho_1$ only,

$$U_{\text{eff}} = \frac{2 \sin^2 \pi/N}{N \rho_1^2} + \frac{N \rho_1^2}{8 \sin^2 \pi/N} + \frac{\tilde{\beta}^3 \sin \pi/N}{\rho_1} \sum_{i<j} \frac{1}{|\sin \theta_{ij}/2|},$$

(8)

where $\theta_{ij}$ is the angle between particles $i$ and $j$, measured from the c.m. Minimisation of $U_{\text{eff}}$ leads to an equilibrium value for $\rho_1$, $\rho_{10}$. The equilibrium values of the others coordinates, $\rho_k$, $\theta_k$, are determined from geometry.

Higher contributions to the energy come from relaxing the equilibrium configuration, $\rho_k = \rho_{k0} + y_k/|J|^{1/2}$, $\theta_k = \theta_{k0} + z_k/|J|^{1/2}$. The r.h.s. of (7) may, thus, be expanded as

$$h = h_0 + h_2 \frac{1}{|J|} + h_3 \frac{1}{|J|^{3/2}} + h_4 \frac{1}{|J|^2} + \ldots,$$

(9)

where $h_0 = U_{\text{eff}}(\rho_{10})$, $h_2$ describes in general harmonic oscillations in fictitious magnetic fields, and $h_3$, $h_4$, etc account for anharmonicities.

For the spatial wave function and the scaled energy (related to $h$), we write series like (9)

$$\psi = \psi_0 + \psi_1 \frac{1}{|J|^{1/2}} + \psi_2 \frac{1}{|J|} + \ldots,$$

$$\epsilon = \epsilon_0 + \epsilon_2 \frac{1}{|J|} + \epsilon_4 \frac{1}{|J|^2} + \ldots.$$

(10)

(11)

These expressions are substituted into Schrödinger equation, leading to the chain of uncoupled equations,

$$h_0 = \epsilon_0,$$

$$h_2 \psi_0 = \epsilon_2 \psi_0, \text{ etc.}$$

(12)

(13)
Higher corrections to the energy are obtained from ordinary perturbation theory, where $h_3$, $h_4$, etc. are interpreted as perturbations.

Below, we present results for two and three electrons.

### 3 Two electrons

For two particles, there is only one Jacobi coordinate, $\bar{\rho}_1 = \bar{r}_2 - \bar{r}_1$. It’s equilibrium value (the modulus) satisfies the equation

$$0 = -2 + \frac{1}{2} \rho_{10}^4 - \tilde{\beta}^3 \rho_{10}. \tag{14}$$

Writing $\rho_1 = \rho_{10} + y_1/|J|^{1/2}$, and substituting into the r.h.s. of (7), we obtain the operator coefficients $h_2$, $h_3$, etc. In particular

$$h_2 = -\frac{d^2}{dy_1^2} + \frac{1}{2} U''(\rho_{10}) y_1^2,$$

$$= \omega_1 (a_1^\dagger a_1 + \frac{1}{2}), \tag{15}$$

where $\omega_1 = \sqrt{2U''(\rho_{10})} = \sqrt{3 + 4/\rho_{10}^4}$, $a_1 = \omega_{1/2} y_1/2 + ip_1/\omega_{1/2}$, $a_1^\dagger = \omega_{1/2} y_1/2 - ip_1/\omega_{1/2}$. Note that in the oscillator limit $\omega_1 \to 2$, whereas in the Wigner limit $\omega_1$ approaches the classical result $\sqrt{3}$ [10].

To this order, the spatial wave function is written as $e^{iJ\theta} \psi_0(y_1)$, where $\psi_0 \sim (a_1^\dagger)^n|0>$, and $\theta$ is the polar angle associated to the vector $\bar{\rho}_1$. Under a permutation of particles $\theta$ changes by $\pi$, thus even $|J|$ correspond to unpolarised spin states, $S = 0$, and odd $|J|$ — to polarised states, $S = 1$.

The operators $h_k$, $k \geq 3$, can be written in general as

$$h_k = \frac{(-1)^k}{\rho_{10}^{k-2}} \left\{ \left( \frac{k - 1}{\rho_{10}^4} + \frac{1}{2} \right) y_1^k + y_1^{k-3} \frac{\partial}{\partial y_1} \right\}. \tag{16}$$

Their matrix elements can be straightforwardly computed. Let us write the results for the first nonzero terms of the series (11) in the present case,

$$\epsilon_0 = \frac{3}{4} \rho_{10}^2 - \frac{1}{\rho_{10}^2}, \tag{17}$$

$$\epsilon_2 = \omega_1 (n + \frac{1}{2}), \tag{18}$$
\[
\begin{align*}
\epsilon_4 &= \frac{3}{\rho_{10}^2 \omega_1^2} \left( \frac{3}{\rho_{10}^4} + \frac{1}{2} \right) (2n^2 + 2n + 1) - \frac{1}{4 \rho_{10}^4} \\
&\quad - \frac{1}{\rho_{10}^2 \omega_1^2} \left( \frac{2}{\rho_{10}^4} + \frac{1}{2} \right)^2 (30n^2 + 30n + 11),
\end{align*}
\]
\begin{align*}
\epsilon_6 &= -\frac{15}{8 \rho_{10}^4 \omega_1^8} (4/\rho_{10}^4 + 1)^4 (2n + 1)(47n^2 + 47n + 31) \\
&\quad + \frac{9}{2 \rho_{10}^4 \omega_1^8} (4/\rho_{10}^4 + 1)^2 (6/\rho_{10}^4 + 1)(2n + 1)(25n^2 + 25n + 19) \\
&\quad - \frac{1}{2 \rho_{10}^4 \omega_1^8} (6/\rho_{10}^4 + 1)^2 (2n + 1)(87n^2 + 87n + 86) \\
&\quad + \frac{10}{\rho_{10}^2 \omega_1^5} (2n + 1)(14n^2 + 14n + 13) \\
&\quad + \frac{1}{\rho_{10}^3 \omega_1^3} (2n + 1) \{ (10/\rho_{10}^4 + 1)(5n^2 + 5n + 9) - 9/\rho_{10}^4 \} \\
&\quad - \frac{3}{4 \rho_{10}^4 \omega_1} (2n + 1), \quad \text{etc.}
\end{align*}
\]
4 Three electrons

In the three electron system, there are two distances \( \rho_1, \rho_2 \), and one angle, \( \theta_1 \). The equilibrium configuration is an equilateral triangle (\( \rho_{10} = \rho_{20}, \theta_{10} = \pm \pi/2 \)), which side satisfies the equation

\[
0 = -1 + \rho_{10}^4 - 3 \beta^3 \rho_{10}. \tag{21}
\]

At its minimum, \( U_{\text{eff}}(\rho_{10}) = 3\rho_{10}^2/2 - \rho_{10}^{-2}/2 \), thus providing a leading approximation to the energy

\[
\epsilon_0 = \frac{3}{2} \rho_{10}^2 - \frac{1}{2\rho_{10}^2}. \tag{22}
\]

Expanding around one of the two equivalent configurations, i.e. fixing \( \theta_{10} = \pi/2 \) for example, we obtain a series like (9) in which the first operator coefficients are given by

\[
h_2 = - \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{2}{\rho_{10}^2} \frac{\partial^2}{\partial z_1^2} \right) - \frac{2i}{\rho_{10}^2} \text{sign}(J)(y_1 - y_2) \frac{\partial}{\partial z_1} - \frac{1}{4} \left( \frac{3}{\rho_{10}^4} + 1 \right) (y_1^2 + y_2^2)
\]

\[
+ \frac{1}{16} \left( 1 - \frac{1}{\rho_{10}^4} \right) (5y_1^2 + 6y_1y_2 + 5y_2^2 + 3\rho_{10}^2 z_1^2), \tag{23}
\]

\[
h_3 = - \frac{1}{\rho_{10}} \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right) + \frac{2}{\rho_{10}^2} (y_1 + y_2) \frac{\partial^2}{\partial z_1^2}
\]

\[
+ \frac{3i}{\rho_{10}^2} \text{sign}(J) (y_1^2 - y_2^2) \frac{\partial}{\partial z_1} - \frac{1}{\rho_{10}^2} (y_1^3 + y_2^3)
\]

\[
- \frac{1}{64\rho_{10}} \left( 1 - \frac{1}{\rho_{10}^4} \right) (19y_1^3 + 3y_1^2y_2 + 33y_1y_2^2 + 9y_2^3)
\]

\[
- 9\rho_{10}^2 y_1 z_1^2 + 21\rho_{10}^2 y_2 z_1^2), \tag{24}
\]

\[
h_4 = \frac{1}{\rho_{10}^2} \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) - \frac{3}{\rho_{10}^4} (y_1^2 + y_2^2) \frac{\partial^2}{\partial z_1^2}.
\]
\[- \frac{4i}{\rho_{10}^6} \text{sign}(J)(y_1^3 - y_2^3) \frac{\partial}{\partial z_1} + \frac{5}{4\rho_{10}^6}(y_1^4 + y_2^4) \]
\[+ \frac{1}{256\rho_{10}^2} \left( 1 - \frac{1}{\rho_{10}^2} \right) \left( \frac{329}{4} y_1^4 - 25y_1^3y_2 + \frac{123}{2} y_1^2y_2^2 + 135y_1y_2^3 + \frac{9}{4} y_2^4 \right) \]
\[+ \frac{99}{2} \rho_{10}^2 y_1^2 z_1^2 + 27 \rho_{10}^2 y_1y_2 z_1^2 + \frac{141}{2} \rho_{10}^4 z_1^2 + \frac{41}{4} \rho_{10}^4 z_1^4 \), etc (25)\]

The hamiltonian $h_2$, which is to be taken as zeroth order hamiltonian, describes a harmonic oscillator of frequency $\omega_1 = \sqrt{3 + 1/\rho_{10}^4}$ in the variable $y_s = (y_1 + y_2)/\sqrt{2}$, plus a combination of harmonic oscillators and a fictitious magnetic field in the variables $y_m = (y_1 - y_2)/\sqrt{2}$ and $z_m = \rho_{10} z_1/\sqrt{2}$. Note that the “magnetic field” comes from the coupling between the angular momentum and the variable $\theta_1$. We may use a symmetric gauge to describe the fictitious field. The hamiltonian is transformed according to $h'_2 = e^{if} he^{-if}$, where $f = \text{sign}(J)y_mz_m/(2\rho_{10}^2)$. $h'_2$ takes the symmetric form

\[h'_2 = - \frac{\partial^2}{\partial y_s^2} + \frac{1}{4} \left( 3 + \frac{1}{\rho_{10}^4} \right) y_s^2 - \left( \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial^2}{\partial \alpha^2} \right) \]
\[- i \frac{\text{sign}(J)}{\rho_{10}^2} \frac{\partial}{\partial \alpha} + \frac{1}{8} \left( 3 - \frac{1}{\rho_{10}^4} \right) \xi^2, \quad (26)\]

where $\xi = \sqrt{y_m^2 + z_m^2}$ and $\alpha = \arctan(z_m/y_m)$. The corresponding eigenvalues and eigenfunctions are

\[\epsilon_2 = \omega_1(n_s + 1/2) + \omega_2(2n + |m| + 1) + \text{sign}(J)n\omega_3, \quad (27)\]
\[\psi_0 \sim H_{n_s}(\sqrt{\omega_1/2}y_s)e^{-\omega_1y_s^2/4} \xi^{|m|} e^{-\omega_2 \xi^2/4} L_n^{|m|}(\omega_2 \xi^2/2) e^{i\omega_3}, \quad (28)\]

in which $\omega_2 = \sqrt{3/2 - \rho_{10}^{-4}/2}$, $\omega_3 = 1/\rho_{10}^2$, and $n$ and $m$ are respectively principal and magnetic quantum numbers in the fictitious field. In the oscillator limit: $\omega_1 \to 2$, $\omega_2$, $\omega_3 \to 1$, whereas in the Wigner limit: $\rho_{10} \to \infty$, $\omega_1 \to \sqrt{3}$, $\omega_2 \to \sqrt{3/2}$, and $\omega_3 \to 0$. At finite $\tilde{\beta}$ there is no degeneracy between the states of $h_2$, thus we may use nondegenerate perturbation theory to compute corrections.

We shall identify which of the levels $(J, n_s, n, m)$ may describe electron states. This set of quantum numbers will serve as approximate quantum
numbers for the three-electron problem. At intermediate couplings, a small mixing between the states of \( h'_2 \) will be induced by the corrections \( h'_3 \) and \( h'_4 \).

Let us recall the symmetry requirements for three-electron wave functions. A spatially antisymmetric wave function, \( \Psi_0 = e^{iJ\Xi}\psi_0 \), where \( \Xi \) accounts for global rotations, corresponds to a spin-polarised state, \( S = 3/2 \), whereas a mixed-symmetry \( \Psi_0 \) is related to an unpolarised spin state, \( S = 1/2 \). Symmetry transformations shall be expressed in terms of the variables we are using, i.e. \( y_s, \xi \) and \( \alpha \). Additionally, to identify physical states, \( \Psi_0 \) shall be compared with oscillator functions at \( \tilde{\beta} \to 0 \). A detailed analysis is presented in the Appendix of paper [3], to which we refer the reader.

The main conclusion is that in the \((n, m)\) plane, the spatially symmetric and antisymmetric states occupy the lines \( m = J + 3k \), where \( k \) is an integer. For example, let us consider the first states with \( J = -3 \) when \( \tilde{\beta} \ll 1 \). The lowest state is \((n_s, n, m) = (0, 0, 0)\). It corresponds to both one antisymmetric and one symmetric state. In our scheme, as tunnelling effects are not included, they are degenerated in energy. Of course, only the antisymmetric state may correspond to a state of three electrons. Very near to the lowest state there is the mixed-symmetry doublet, \((0, 0, 1)\). At excitation energies near 2, there are the levels \((1, 0, 0)\), \((1, 0, 1)\), \((0, 0, -1)\), \((0, 1, 0)\) and \((0, 1, 1)\). Symmetric and antisymmetric states have \( m = 0 \). The next set of states is at excitation energies near 4, etc.

Up to this order, the energy computed from \( \epsilon = \epsilon_0 + \epsilon_2 / |J| \) leads to results very similar to that reported in [9]. That is, when \( \beta \sim 1 \) the relative error of the lowest state with a given \( |J| \) represents only a few percents of the total energy at \( |J| = 3 \), and decreases considerably as \( |J| \) is increased. The reader may look at figures 3 and 4 of paper [9] in quality of examples.

Let us study \( \epsilon \) when \( \beta \) is varied from 0 to \( \infty \). As in the two-electron problem, the higher corrections, \( \epsilon_4 / |J|^2 + \ldots \), go to zero in both \( \beta \to 0 \) and \( \beta \to \infty \) limits. However, for certain levels, a rearrangement of the spectrum may take place. For example, let us consider spatially antisymmetric states with \( J = -3k \), where \( k \) is positive and high enough. The lowest state in this sector has quantum numbers \((0, 0, 0)\). We show in figure 3 the excitation energies of three states with quantum numbers \((0, 0, 3)\), \((1, 0, 0)\) and \((0, 1, 0)\). Their energies are \( 3(\omega_2 - \omega_3) \), \( \omega_1 \) and \( 2\omega_2 \) respectively. Transitions among these states induced by \( h'_3 \), \( h'_4 \), etc. may occur at \( \tilde{\beta} \geq 0.6 \), thus resulting in a rearrangement of levels. To compute their energies at the intermediate values of \( \beta \) where the intersections occur, we shall resolve the degeneracy by means of degenerate perturbation theory. On the other hand, there are
levels, in particular those with numbers (0,0,0), for which we can apply the $1/|J|$-expansion continuously from $\tilde{\beta} = 0$ to $\infty$.

Let us compute the coefficient $\epsilon_4$ for a state with quantum numbers (0,0,0). We assume $J < 0$ and $|J|$ large enough. These are the relevant states at high magnetic fields. We shall compute the matrix elements entering the expression

$$\epsilon_4 = <0,0,0|h'_4|0,0,0> - \sum_{n_s,n,m} \frac{<0,0,0|h'_3|n_s,n,m><n_s,n,m|h'_3|0,0,0>}{n_s\omega_1 + (2n + |m|)\omega_2 - m\omega_3}. \quad (29)$$

Taking into account the explicit form of $h'_3$, i.e.

$$h'_3 = A \frac{\partial}{\partial y_s} + By_s + Cy^3_s + D, \quad (30)$$

where

$$A = -\sqrt{\frac{2}{\rho_{10}}}, \quad (31)$$

$$B = \frac{\sqrt{2}}{\rho_{10}} \left( \frac{\partial^2}{\partial z_m^2} + 2i \frac{\text{sign}(J)}{\rho_{10}^2} y_m \frac{\partial}{\partial z_m} - \frac{1}{4\rho_{10}^2} y_m^2 \right) - \frac{3}{16} \left( 1 - \frac{1}{\rho_{10}^4} \right) (y_m^2 + z_m^2), \quad (32)$$

$$C = -\frac{\sqrt{2}}{4\rho_{10}} \left( 1 + \frac{1}{\rho_{10}^4} \right), \quad (33)$$

$$D = -\frac{5\sqrt{2}}{32\rho_{10}} \left( 1 - \frac{1}{\rho_{10}^4} \right) (y_m^3 - 3y_m z_m^2), \quad (34)$$

we obtain that the sum over intermediate states in (29) contains only a few terms. The following result for $\epsilon_4$ arises,

$$\epsilon_4 = <0,0,0|h'_4|0,0,0> - \frac{C^2}{3\omega_1} <3|y^3_s|0>^2 - \frac{1}{3} <0,3|D|0,0>^2 \left( \frac{1}{\omega_2 - \omega_3} + \frac{1}{\omega_2 + \omega_3} \right) + \frac{A^2}{\omega_1} <1|\frac{\partial}{\partial y_s}|0>^2$$
\[ - \frac{2C}{\omega_1} < 1|y_s|0 > < 1|y_s|^2|0 > < 0, 0|B|0, 0 > \\
- \frac{C^2}{\omega_1} < 1|y_s|^2|0 >^2 - \frac{1}{\omega_1} < 1|y_s|0 >^2 < 0, 0|B|0, 0 >^2 \\
- < 1|y_s|0 >^2 \left\{ < 0, 2|B|0, 0 >^2 \frac{1}{\omega_1 + 2(\omega_2 - \omega_3)} + < 0, -2|B|0, 0 >^2 \frac{1}{1 + 2(\omega_2 + \omega_3)} \right\}, \tag{35} \]

where

\[ < 0, 0, 0|h'_4|0, 0, 0 > = -\frac{1}{\rho_{10}^2} - \frac{3}{\omega_2 \rho_{10}^2} \left( \frac{1}{\omega_2} + \frac{1}{\omega_1} \right) + \frac{15}{8 \rho_{10}^2} \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right)^2 \]
\[ + \frac{3}{8 \rho_{10}^2} \left( 1 + \frac{\omega_2}{\omega_1} + \frac{1}{\omega_1 \omega_2 \rho_{10}^2} + \frac{1}{\omega_2 \rho_{10}^2} \right) \]
\[ + \frac{3}{64 \rho_{10}^2} \left( 1 - \frac{1}{\rho_{10}^4} \right) \left( \frac{13}{\omega_1^2} + \frac{12}{\omega_1 \omega_2} + \frac{16}{\omega_2^2} \right), \tag{36} \]

\[ < 0, 3|D|0, 0 > = -\frac{5\sqrt{6}}{16 \rho_{10}^3 \omega_2^{3/2}} \left( 1 - \frac{1}{\rho_{10}^4} \right), \tag{37} \]

\[ < 0, 0|B|0, 0 > = -\frac{\sqrt{2}\omega_2}{2 \rho_{10}}, \tag{38} \]

\[ < 0, \pm 2|B|0, 0 > = \pm \frac{1}{\rho_{10}^2} - \frac{\omega_2}{4 \rho_{10}} - \frac{1}{4 \omega_2 \rho_{10}^3}, \tag{39} \]

and the matrix elements involving \( y_s \) are ordinary oscillator matrix elements, that is \( < 1|y_s|0 > = 1/\sqrt{\omega_1} \), etc. Note that, as mentioned, \( \epsilon_4 \to 0 \) in both the \( \beta \to 0 \) and \( \beta \to \infty \) limits.

In figure 4 we compare the \( 1/|J| \)-expansion and the two-point Pade approximant \( P_{5,4} (\beta) \) of [11] for the lowest antisymmetric state with \( J = -3 \). As in the two-electron case, the relative error is of the same order of the incertitude of the Pade estimate, i.e. a few parts in \( 10^3 \). We show in figure 5 the relative weight of \( \epsilon_4 \) in \( \epsilon = \epsilon_0 + \epsilon_2/|J| + \epsilon_4/|J|^2 \). For states with still higher values of \(|J|\), the relative magnitude of the contribution of \( \epsilon_4 \) decreases considerably.
5 Conclusions

We have computed the energy levels of two and three electrons in an ideal parabolic two-dimensional quantum dot in the presence of a magnetic field. States with high angular momentum and a low number of excitation (vibrational) quanta were considered. The energy was found as a series in the “small” parameter $1/|J|$. The series exhibits good convergence properties even at $|J| = 3$.

At still lower values of the angular momentum the method is not applicable. However, we can obtain reliable analytic estimations by means of an alternative procedure: the two-point Padé approximants, which are extremely simply constructed for the low-lying states [11]. $1/|J|$-expansions and Padé approximants were shown to give similar results at intermediate $|J|$ values, i.e. at $|J| = 3$.

Other problems are currently being treated along these lines. For example, the effects of anyonic statistics on the energy levels of two quasiparticles in a quantum dot may be straightforwardly studied by means of the $1/|J|$-method [12].

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Figure Captions

Figure 1.
   a) The energy of the $|J| = 3, n = 2$ state of two electrons.
   b) The relative weight of $\epsilon_6$ in $\epsilon$.

Figure 2. Relative difference between $|J|\epsilon$ and the $P_{6,5}$ Padé approximant for two electrons in a state with $|J| = 3, n = 0$.

Figure 3. Excitation energies of three states with $J = -3k$, $k$ - positive and high enough. Quantum numbers are indicated.

Figure 4. Relative difference between $|J|\epsilon$ and the $P_{5,4}$ Padé approximant for three electrons. The lowest antisymmetric state with $J = -3$ is studied.

Figure 5. Relative weight of $\epsilon_4$ in $\epsilon$. The same state as in figure 4 is considered.
$\epsilon_6/(|J|^3 \epsilon)$

$\beta$
\[ \frac{|P_{6.5} - |J|\epsilon|}{P_{6.5}} \]
\frac{|P_{5,4} - J|\epsilon}{P_{5,4}}
