Group theory/Topology

Unbounded asymmetry of stretch factors

Asymétrie non bornée des facteurs d'étirement

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Abstract

A result of Handel–Mosher guarantees that the ratio of logarithms of stretch factors of any fully irreducible automorphism of the free group $F_N$ and its inverse is bounded by a constant $C_N$. In this short note, we show that this constant $C_N$ cannot be chosen independent of the rank $N$.

Résumé

Un énoncé de Handel–Mosher certifie que le rapport des logarithmes des facteurs d'étirement d'un automorphisme irréductible du groupe libre $F_N$ et de son inverse est borné par une constante $C_N$. Nous montrons dans la présente courte note que cette constante $C_N$ ne peut pas être choisie indépendante de $N$.

Let $F_N$ be the free group of rank $N \geq 2$. An outer automorphism $\varphi \in \text{Out}(F_N)$ is said to be fully irreducible if no power of $\varphi$ preserves the conjugacy class of any proper free factor of $F_N$. In this case $\varphi$ has a well defined stretch factor $\lambda(\varphi)$, which, for any non-$\varphi$-periodic conjugacy class $\alpha$ in $F_N$ and a free basis $X$ of $F_N$, is given by

$$\lambda(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \| \varphi^n(\alpha) \|_X,$$

where $\| \cdot \|_X$ denotes cyclically reduced word length with respect to $X$. As was observed in [2] (see also [7]), there exist fully irreducible elements $\varphi \in \text{Out}(F_N)$ with the property that $\varphi$ and $\varphi^{-1}$ have different stretch factors:

$$\lambda(\varphi) \neq \lambda(\varphi^{-1}).$$

However, the following result from [6] describes the extent to which they can differ. To state their result precisely, let $N \geq 2$ be an integer and set

$$C_N = \sup_{\varphi} \frac{\log(\lambda(\varphi))}{\log(\lambda(\varphi^{-1}))},$$

where $\varphi$ ranges over all fully irreducible elements of $\text{Out}(F_N)$.
Theorem 1. (Handel–Mosher [6].) For every integer $N \geq 2$, we have $C_N < \infty$.

An alternate proof of this result was more recently given by Algom-Kfir and Bestvina [1]. While the proofs of this theorem appeal to the fact that $N$ is fixed, it is not clear that this dependence is necessary. In this note, we prove that in fact it is.

Theorem 2. With $\{C_N\}_{N \geq 2}$ defined as above, $\limsup_{N \to \infty} C_N = \infty$.

Proof. The proof will appeal to a construction and analysis carried out in [4] and [5]. To that end, let $F_3 = \langle a, b, c \rangle$ and consider the element $\varphi \in \text{Aut}(F_3)$ defined by
\[
\varphi(a) = b, \quad \varphi(b) = b^{-1}a^{-1}bac, \quad \varphi(c) = a.
\]
It was shown in [4, Example 2.19] that $\varphi$ is fully irreducible. Next, let
\[
G = F_3 \times_{\varphi} \mathbb{Z} = \langle a, b, c, r \mid r^{-1} rx = \varphi(x) \text{ for all } g \in F_3 \rangle
\]
be the free-by-cyclic group determined by $\varphi$, and let $u_0 : G \to \mathbb{Z}$ in $\text{Hom}(G; \mathbb{R}) = H^1(G; \mathbb{R})$ be the associated homomorphism obtained by sending $r$ to 1 and all other generators to 0.

In [4], we construct a cone $\mathcal{A} \subset H^1(G; \mathbb{R})$ containing $u_0$ with the property that every other primitive integral element $u \in \mathcal{A}$ has kernel $\ker(u)$ a finitely generated free group.

The action of $u(G) = \mathbb{Z}$ on $\ker(u)$ is generated by a monodromy automorphism $\varphi_u \in \text{Aut}(\ker(u))$ determining an expression of $G$ as a semidirect product $G \cong \ker(u) \rtimes_{\varphi_u} \mathbb{Z}$ with associated homomorphism $u$. One of the main results of [4] is that all such $\varphi_u$ are fully irreducible.

In [5], we construct a strictly larger open, convex cone $\mathcal{A} \subset H^1(G; \mathbb{R})$ and a function $\tilde{\gamma} : S \to \mathbb{R}$ that is convex, real analytic, and homogeneous of degree $-1$ (i.e., $\tilde{\gamma}(tu) = \frac{1}{t} \tilde{\gamma}(u)$) such that
\[
\log(\lambda(\varphi_{u})) = \tilde{\gamma}(u)
\]
for any primitive integral class $u \in \mathcal{A}$. In fact, this holds for all primitive integral $u \in S$ with the appropriate interpretation of $\lambda(\varphi_u)$. We also show that $S$ is the cone on the component of the BNS-invariant $\Sigma(G)$ [3] containing $u_0$ [5, Theorem 1] and that $\mathcal{A}$ lies over the symmetrized BNS-invariant (that is, both $\mathcal{A}$ and $-\mathcal{A}$ project into $\Sigma(G)$) [5, Corollary 13.7]. In fact, a key result of Bieri–Neumann–Strebel is that an integral class $u \in \text{Hom}(G; \mathbb{Z})$ has $\ker(u)$ finitely generated if and only if both $u$ and $-u$ lie in the $\Sigma(G)$ [3].

The homomorphism $-u_0$ has $\ker(-u_0) = \ker(u_0) = F_N$ and associated monodromy $\varphi^{-1}$, thus expressing $G$ as $F_N \rtimes_{\varphi^{-1}} \mathbb{Z}$. Since $\varphi^{-1}$ is also fully irreducible, the main result of [5] provides another open, convex cone $S_\mathcal{A} \subset H^1(G; \mathbb{R})$ containing $-u_0$ and a corresponding convex, real analytic, homogeneous of degree $-1$ function $\tilde{\gamma}_\mathcal{A} : S_\mathcal{A} \to \mathbb{R}$. Since $\mathcal{A}$ projects into $\Sigma(G)$ and $S_\mathcal{A}$ is the cone on the component of $\Sigma(G)$ containing $-u_0$, we see that $-\mathcal{A} \subset S_\mathcal{A}$. Thus $\tilde{\gamma}_\mathcal{A}$ calculates the inverse stretch factors
\[
\tilde{\gamma}_\mathcal{A}(-u) = \log(\lambda(\varphi_{u}^{-1}))
\]
for all primitive integral $u \in \mathcal{A}$.

Example 8.3 of [5] exhibits a primitive integral class $u_1 \in S$ which lies on the boundary of $\mathcal{A}$ (see [5, Fig. 8]) for which $\ker(u_1)$ is not finitely generated. It follows that $-u_1$ is not in the BNS-invariant. The key observation is that $-u_1$ then necessarily lies on the boundary of $S_\mathcal{A}$ (since $-u_1 \notin -\mathcal{A} \subset S_\mathcal{A}$ but $-u_1 \notin S_\mathcal{A}$).

Let $\{u_n\}_{n=2}^\infty \subset \mathcal{A}$ be primitive integral classes progressively converging to $u_1$. That is, there exists $\{t_n\}_{n=2}^\infty \subset \mathbb{R}$ so that $\lim_{n \to \infty} t_n u_n = u_1$. Since this convergence occurs inside $S$ it follows that
\[
\lim_{n \to \infty} \tilde{\gamma}(t_n u_n) = \tilde{\gamma}(u_1) < \infty.
\]
On the other hand, since $\lim_{n \to \infty} -t_n u_n = -u_1 \in \partial S_\mathcal{A}$, it follows from [5, Theorem F] that
\[
\lim_{n \to \infty} \tilde{\gamma}_\mathcal{A}(-t_n u_n) = \infty.
\]
Therefore, appealing to the homogeneity of $\tilde{\gamma}$ and $\tilde{\gamma}_\mathcal{A}$, we have:
\[
\lim_{n \to \infty} \frac{\log(\lambda(\varphi_{u_n}^{-1}))}{\log(\lambda(\varphi_{u_n}))} = \lim_{n \to \infty} \frac{\tilde{\gamma}(u_n)}{\tilde{\gamma}(u_n)} = \lim_{n \to \infty} \frac{\tilde{\gamma}_\mathcal{A}(-t_n u_n)}{\tilde{\gamma}(-t_n u_n)} = \infty. \quad \square
\]

Acknowledgements

The first author was partially supported by the NSF postdoctoral fellowship, NSF MPSRF No. 1204814. The second author was partially supported by the NSF grant DMS-1405146 and by the Simons Foundation Collaboration grant No. 279836. The third author was partially supported by the NSF grant DMS-1207183. The third author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “GEAR Network”.


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