LIMITS OF DIHEDRAL GROUPS

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ABSTRACT. We give a characterization of limits of dihedral groups in the space of finitely generated marked groups. We also describe the topological closure of dihedral groups in the space of marked groups on a fixed number of generators.

1. Introduction

The space of marked groups is a topological setting for expressing approximation among groups and algebraic limit processes in terms of convergence. A convenient definition consists in topologizing the set of normal subgroups of a given free group \( F \) (we hence topologize its marked quotients) with the topology induced by the product topology of \( \{0,1\}^F \). The idea of topologizing the set of subgroups of a given group goes back to Chabauty’s topology on the closed subgroups of a locally compact group [Cha50]. At the end of its celebrated paper “Polynomial growth and expanding maps” [Gro81], Gromov sketched what could be a topology on finitely generated groups and put it into practice in the purpose of growth results. The space of marked groups on \( m \) generators is properly defined in [Gri84] where the study of the neighborhood of the first Grigorchuk group turned out to be fruitful. This compact, totally disconnected and metrizable space has been used to prove the existence of infinite groups with unexpected or rare properties [Ste96, Cha00, Sha00]. In [CG05] Champetier and Guirardel propose a new approach of Sela’s limit groups [Sel01] (which appeared to coincide with the long-studied class of finitely generated fully residually free groups, see definition below) in the topological framework of the space of marked groups. They give indeed the first proof not relying on the finite presentability of limit groups [KM98a, KM98b, Sel01, Gui04] that limit groups are limit of free groups in the space of marked groups. Among others, they provide a simple proof of the fact that a finitely generated group is a limit group if and only if it has the same universal theory as a free group of rank two. They hence relate topology and logic. In the present paper, we use these links to tackle the easier case of limits of dihedral groups. Our motivation is:

Problem. [dlH00, CG05] Describe the topological closure of finite groups in the space of marked groups.

We carry out such a description for the most elementary finite groups: cyclic and dihedral finite groups.

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Let $n$ be in $\{1, 2, \ldots\} \cup \{\infty\}$. We define the dihedral group

$$D_{2n} := \langle a, b \mid a^2 = b^n = 1, a^{-1}ba = b^{-1} \rangle,$$

where we omit the relation $b^n = 1$ when $n = \infty$. If $n$ is finite, then $D_{2n}$ is a finite group of order $2n$. The groups $D_2$ (which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$) and $D_4$ (which is isomorphic to the Vierergruppe $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) are the only abelian dihedral groups. If $3 \leq n < \infty$, then $D_{2n}$ is isomorphic to the group of Euclidean isometries of a regular $n$-gon $P_n$ (any function that maps $a$ to a reflection and $b$ to a rotation with angle $\frac{2\pi}{n}$, reflection and rotation both preserving $P_n$, extends to a unique isomorphism from $D_{2n}$ to $\text{Isom}(P_n)$). In this case we can identify $\{1, b, \ldots, b^{n-1}\}$ with the set of rotations of $P_n$ and $\{a, ab, \ldots, ab^{n-1}\}$ with the set of reflections of $P_n$. If $n \geq 3$ the center $Z(D_{2n})$ has two elements when $n$ is even (1 and $b^{n/2}$) and $Z(D_{2n})$ is trivial when $n$ is odd. For all $n < \infty$, the subgroup $\langle b \rangle$ generated by $b$ is a cyclic normal subgroup of order $n$ on which $\langle a \rangle$ acts by conjugation. Thus $D_{2n}$ is isomorphic to the semidirect product $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/n\mathbb{Z}$ is multiplying by $-1$. The infinite dihedral group $D_\infty$ is centerless and is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. All proper quotients of $D_\infty$ are finite dihedral groups and any such group is a quotient of $D_\infty$.

We denote by $\mathcal{M}$ the space of all finitely generated marked groups (see Section 2 for a definition). Unless otherwise stated, limits of groups considered are limits in $\mathcal{M}$.

Let $P$ be a group theoretic property. A group $G$ is fully residually $P$ if for any finite subset $F$ of $G \setminus \{1\}$ there is a group $H$ with $P$ and a homomorphism from $G$ to $H$ that maps no element of $F$ to the trivial element. We address the reader to Section 2 for the definitions required in our first theorem (convergence in $\mathcal{M}$, universal theory $\text{Th}_U(G)$ of $G$, ultrafilter and ultraproduct). The only abelian limits in $\mathcal{M}$ of dihedral groups are easily seen to be the marked groups abstractly isomorphic to $D_2$ or $D_4$ (see Section 2). We give in Section 4 the following characterization of the non abelian limits:

**Theorem A** (Th. 4.1). Let $G$ be a non abelian finitely generated group. The following conditions are equivalent:

- (lim) $G$ is a limit of dihedral groups;
- (res) $G$ is fully residually dihedral;
- (iso) $G$ is isomorphic to a semidirect product $A \rtimes \mathbb{Z}/2\mathbb{Z}$ where $A$ is a limit of cyclic groups on which $\mathbb{Z}/2\mathbb{Z}$ acts by multiplication by $-1$;
- (Th$_U$) $\text{Th}_U(G) \supset \bigcap_{n \geq 3} \text{Th}_U(D_{2n})$;
- (Π/𝒰) $G$ is isomorphic to a subgroup of $\left(\prod_{n \geq 3} D_{2n}\right)/\mathfrak{U}$ for some ultra-filter $\mathfrak{U}$ on $\mathbb{N}$.

Proposition 4.2 shows that limits of cyclic groups are finitely generated abelian groups with cyclic torsion subgroup.

We denote by $\omega$ the smallest infinite ordinal, i.e. the set $\mathbb{N}$ of positive integers endowed with its natural order. In Section 5 we finally describe the set of limits of dihedral groups on $m$ generators:

**Theorem B** (Th. 5.2). The topological closure $\mathcal{D}_m$ of dihedral marked groups in $\mathcal{M}_m$ is homeomorphic to $\omega^{m-1}(2^m - 1) + 1$ endowed with the order topology.
In other words, \( \mathcal{D}_m \) is the disjoint union of \( 2^m - 1 \) copies of \( \overline{\mathbb{N}}^{m-1} \) where \( \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) is the Alexandroff compactification of \( \mathbb{N} \). We use a theorem of Mazurkiewicz and Sierpinski [MS20] on Cantor-Bendixson invariants of countable compact spaces to prove our last result. For comparison, the set of abelian marked groups on \( m \) generators is homeomorphic to \( \omega^m + 1 \). This fact can be easily derived from the proof of Theorem [13].

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2. Convergence and logic

Here, we give preliminary definitions and results that we need for the proofs of the main theorems. We first define marked groups and the topology (called Chabauty’s topology, Cayley’s topology or weak topology) on the set of marked groups. Second, we relate convergence in the space of marked groups to universal theory and ultraproducts (see [CG05] for a more full-bodied exposition).

**Definition 2.1.** The pair \((G, S)\) is a marked group on \( m \) generators if \( G \) is a group and \( S = (g_1, \ldots, g_m) \) is an ordered system of generators of \( G \). We call also \( S \) a generating \( G \)-vector of length \( n \) (or simply a marking of \( G \) of length \( m \)) and we denote by \( V(G, m) \) the set of these \( G \)-vectors.

We denote by \( r(G) \) the rank of \( G \), i.e. the smallest number of generators of \( G \). Two marked groups \((G, S)\) and \((G', S')\) (with \( S = (g_1, \ldots, g_m) \) and \( S' = (g'_1, \ldots, g'_m) \)) are equivalent if there is an isomorphism \( \phi : G \to G' \) such that \( \phi(g_i) = g'_i \) for \( i = 1, \ldots, m \).

**Nota Bene 2.2.** We denote also by \((G, S)\) the equivalence class of \((G, S)\) and we call this class a marked group.

Let \( \mathbb{F}_m \) be the free group with basis \((e_1, \ldots, e_m)\) and let \( \mathbb{F}_\infty \) the free group with basis \((e_i)_{i \geq 1}\). Let \( G \) and \( G' \) be two groups and let \( p : \mathbb{F}_m \to G \) and \( p' : \mathbb{F}_m \to G' \) be two epimorphisms. The epimorphisms \( p \) and \( p' \) are equivalent if there is an isomorphism \( \phi \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{F}_m & \xrightarrow{p} & G \\
\downarrow & & \downarrow \phi \\
 & G' & \\
\end{array}
\]

Using the universal property of \( \mathbb{F}_m \), we establish a natural bijection from marked groups on \( m \) generators to equivalence classes of epimorphisms with source \( \mathbb{F}_m \). The last set is clearly in one-one correspondence with the set \( \mathcal{N}(\mathbb{F}_m) \) of all normal subgroups of \( \mathbb{F}_m \). We endow \( \mathcal{N}(\mathbb{F}_m) \) with the topology induced by the Tychonoff topology on the product \( \{0, 1\}^{\mathbb{F}_m} \) which identifies with the set of all subsets of \( \mathbb{F}_m \). We easily check that \( \mathcal{N}(\mathbb{F}_m) \) is a closed subspace of \( \{0, 1\}^{\mathbb{F}_m} \). We hence define a compact, metrizable and totally discontinuous topology on the corresponding set \( \mathcal{M}_m \) of marked groups on \( m \) generators. We have for
instance $\left(\mathbb{Z}/k\mathbb{Z}, [1]_k\right) \rightarrow (\mathbb{Z}, 1)$ in $M_1$ and $\left(\mathbb{D}_{2k}, (a, b)\right) \rightarrow (\mathbb{D}_\infty, (a, b))$ in $M_2$ with respect to the Chabauty’s topology. Let $G$ be a finitely generated group. We define $\mathcal{N}(G)$ as the set of normal subgroups of $G$ endowed with the topology induced by $\{0, 1\}^G$. Let $p : \mathbb{F}_m \rightarrow G$ be an epimorphism and let $p^* : \mathcal{N}(G) \rightarrow \mathcal{N}(\mathbb{F}_m)$ be defined by $p^*(N) = p^{-1}(N)$. The map $p^*$ is clearly continuous and injective. When $G$ is finitely presented, we have:

**Lemma 2.3.** [CG05, Lemma 2.2] The map $p^*$ is an open embedding.

Thus, for any marking $S$ of a finitely presented group $G$ in $M_m$, there is a neighborhood of $(G, S)$ containing only quotients of $G$ (the quotient map is induced by the natural bijection between the markings). Namely, this is the set corresponding to the image of $p^*$ where $p$ is the epimorphism defined by means of $S$. More generally, $p : G \rightarrow H$ induces an open embedding $p^* : \mathcal{N}(H) \hookrightarrow \mathcal{N}(G)$ if and only if $\ker p$ is finitely generated as a normal subgroup of $G$.

**Remark 2.4.** It follows from Lemma 2.3 that finite groups are isolated in $M$. Isolated groups are characterized in [Gri05, CGP07]. Since nilpotent groups are finitely presented, we can also derive of Lemma 2.3 that the set of nilpotent groups of a given class $c$ is open in $M$.

From the spaces $M_m$, we build up the space $M$ of all finitely generated marked groups. Observe first that the map $(G, (g_1, \ldots, g_m)) \mapsto (G, (g_1, \ldots, g_m, 1))$ defines a continuous and open embedding $i_n : M_m \hookrightarrow M_{m+1}$. The inductive limit $M$ of the of the system $\{i_n : M_m \hookrightarrow M_{m+1}\}_{m \geq 1}$ is a metrizable locally compact and totally discontinuous space. Observe that a convergence in $M$ boils down to a convergence in $M_m$ for some $m$ and the converse is obvious. The space $M$ can be viewed as the set of (equivalence classes) of groups marked with an infinite sequence of generators which are eventually trivial. Let $v_1, \ldots, v_k, w_1, \ldots, w_l$ be elements of $\mathbb{F}_\infty$ and let $(\Sigma)$ be the system

$$\begin{cases} v_1 = 1, \ldots, v_k = 1 \\ w_1 \neq 1, \ldots, w_l \neq 1 \end{cases}.$$ 

We denote by $O_\Sigma$ the set of marked groups $(G, S)$ of $M$ for which $S$ (possibly completed by trivial elements) is a solution of $(\Sigma)$ in $G$. The family of sets $O_\Sigma$ defines a countable basis of open and closed subsets of $M$.

**Remark 2.5.** Let $P$ be group theoretic property stable under taking subgroups and let $G$ be a fully residually $P$ group. Using the previous basis, we easily deduce that for any marking $S$ of $G$, $(G, S)$ is the limit in $M$ of a sequence of marked groups with $P$. The converse is also true when $G$ is finitely presented because of Lemma 2.3.

**Lemma 2.6** (Subgroup and convergence [CG05]). Let $(G_n, S_n)_n$ be convergent sequence in $M$ with limit $(G, S)$. Let $H$ be a finitely generated subgroup of $G$. Then, for any marking $T$ of $H$ there is a sequence of marked groups $(H_n, T_n)_n$ which converges in $M$ to $(H, T)$ and such that $H_n$ is a subgroup of $G_n$.

It is important to note that being a limit in $M$ of marked groups with a given property $P$ does not depend on the marking:
Lemma 2.7. [Cha00, dCGP07] Let \((G, S)\) and \((H, T)\) be in \(\mathcal{M}\). Assume that \(G\) and \(H\) are abstractly isomorphic. Then we can find a neighborhood \(U\) of \((G, S)\), a neighborhood \(V\) of \((H, T)\), and a homeomorphism \(\phi : U \rightarrow V\) mapping \((G, S)\) onto \((H, T)\). Moreover, \(\phi\) preserves the isomorphism relation.

Thus, being a limit of finite (or equally free, cyclic, dihedral) groups is a group property that doesn’t depend on the marking. As \(\mathbb{D}_\infty\) is a limit of finite dihedral groups, limits of dihedral groups are all limits of finite ones.

Let \(G\) be a finite group. For all \(m \geq r(G)\), there are only finitely many marked groups isomorphic to \(G\) in \(\mathcal{M}_m\) (there are only finitely many equivalence classes of epimorphisms \(p : F_m \rightarrow G\)). Hence any compact subset of \(\mathcal{M}\) contains only finitely many marked groups isomorphic to \(G\). Since the property of being abelian is open in \(\mathcal{M}\) (use Lemma 2.3 for instance), abelian limits of dihedral groups are limits of (finite) abelian dihedral groups \(\mathbb{D}_2\) or \(\mathbb{D}_4\). These limits are consequently isomorphic either to \(\mathbb{D}_2\) or \(\mathbb{D}_4\). For this reason, we consider only non abelian limits.

We turn now to relations between convergence and logic. We fix a countable set of variables \(\{x_1, x_2, \ldots\}\) and the set of symbols \(\{\land, \lor, \neg, (,), \cdot, ^{-1}, =, 1\}\) that stand for the usual logic and group operations. We consider the set \(Th_\forall\) of universal sentences in group theory written with the variables, i.e. all formulas \(\forall x_1 \ldots \forall x_k \phi(x_1, \ldots, x_k)\), where \(\phi(x_1, \ldots, x_k)\) is a quantifier free formula built up from the variables and the available symbols (see [BS69, CG05] for a precise definition). For instance, \(\forall x \forall y (xy = yx)\) is a universal sentence that is true in any abelian group while \(\forall x (x = 1 \lor x^2 \neq 1)\) expresses that there is no 2-torsion in a group. If a universal sentence \(\sigma\) is true in \(G\) (we also say that \(G\) satisfies \(\sigma\)), we write \(G \models \sigma\). We denote by \(Th_\forall(G)\) the set of universal sentences which are true in \(G\). Let \((A_n)_n\) be a sequence of subsets of \(Th_\forall\). We set \(\limsup A_n := \bigcap_n \bigcup_{k \geq n} A_k\) and \(\liminf A_n := \bigcup_n \bigcap_{k \geq n} A_k\). The set of marked groups of \(\mathcal{M}\) satisfying a given family of universal sentences define a closed subset. More precisely, quoting Proposition 5.2 of [CG05] 5.3 and reformulating slightly Proposition 5.3, we have:

**Proposition 2.8 (Limits and universal theory).** Let \(G\) be a finitely generated group and let \((G_n)_n\) be a sequence of finitely generated groups.

(i) Assume that \((G, S)\) is the limit in \(\mathcal{M}\) of \((G_n, S_n)_n\) for some ordered generating set \(S\) of \(G\) and some ordered generating set \(S_n\) of \(G_n\). Then \(Th_\forall(G) \supset \limsup Th_\forall(G_n)\).

(ii) Assume \(Th_\forall(G) \supset \bigcap_n Th_\forall(G_n)\). Then, for any ordered generating set \(S\) of \(G\), there is some integer sequence \((n_k)_k\) such that \((G, S)\) is the limit in \(\mathcal{M}\) of some sequence \((H_k, T_k)\) satisfying \(H_k \leq G_{n_k}\).

It directly follows that a variety of groups (see [Neu67] for a definition) defines closed subspaces of \(\mathcal{M}_m\) and \(\mathcal{M}\). For example, limits of dihedral groups are metabelian groups (2-solvable groups) because dihedral groups are metabelian. It can be easily proved, by using Lemma 2.3 that a variety of groups defines an open subspace of \(\mathcal{M}_m\) if and only if its free group on \(m\) generators is finitely presented.
Convergence of a sequence \((G_n)_n\) in \(\mathcal{M}\) can also be related to ultraproducts of the \(G_n\)'s that we define now.

**Definition 2.9 (Ultrafilter).** An ultrafilter \(\mathcal{U}\) on \(\mathbb{N}\) is a finitely additive measure with total mass 1 which takes values in \(\{0, 1\}\). In other words, it is map from \(\mathcal{P}(\mathbb{N})\) (the set of all subsets of \(\mathbb{N}\)) to \(\{0, 1\}\) satisfying \(\mathcal{U}(\mathbb{N}) = 1\) and such that, for all disjoint subsets \(A\) and \(B\) of \(\mathbb{N}\), we have \(\mathcal{U}(A \cup B) = \mathcal{U}(A) + \mathcal{U}(B)\).

Let \(\mathcal{U}\) be an ultrafilter on \(\mathbb{N}\) and let \((G_n)_n\) be sequence of groups. There is a natural relation on the cartesian product \(\prod_n G_n\): \((g_n)_n\) and \((h_n)_n\) are equal \(\mathcal{U}\)-almost everywhere if \(\mathcal{U}(\{n \in \mathbb{N} \mid g_n = h_n\}) = 1\).

**Definition 2.10 (Ultraproduct).** The ultraproduct of the sequence \((G_n)_n\) relatively to \(\mathcal{U}\) is the quotient of \(\prod_n G_n\) by the equivalence relation of equality \(\mathcal{U}\)-almost everywhere. We denote it by \((\prod_n G_n)/\mathcal{U}\).

Let \((A_n)_n\) be a sequence of subsets of \(Th_\varphi\). We set \(\lim\mathcal{U} A_n := \{\phi \in Th_\varphi \mid \phi \text{ belongs to } A_n \text{ for } \mathcal{U} - \text{almost every } n\}\). We have:

**Theorem 2.11 (Łos [BS69]).** \(Th_\varphi((\prod_n G_n)/\mathcal{U}) = \lim\mathcal{U} Th_\varphi(G_n)\).

An ultrafilter is said to be principal if it is a Dirac mass. The following proposition relates convergence in \(\mathcal{M}\) to ultraproducts:

**Proposition 2.12.** (Limits and ultraproducts [CG05, Prop. 6.4]) Let \((G, S)\) be in \(\mathcal{M}\) and let \((G_n, S_n)_n\) be sequence in \(\mathcal{M}\).

1. If \((G_n, S_n)_n\) accumulates on \((G, S)\) in \(\mathcal{M}\), then \(G\) embeds isomorphically into \((\prod_n G_n)/\mathcal{U}\) for some non principal ultrafilter \(\mathcal{U}\).
2. If \((G_n, S_n)_n\) converges to \((G, S)\) in \(\mathcal{M}\), then \(G\) embeds isomorphically into \((\prod_n G_n)/\mathcal{U}\) for all non principal ultrafilter \(\mathcal{U}\).
3. Let \(H\) be a finitely generated group. If \(H\) embeds isomorphically into \((\prod_n G_n)/\mathcal{U}\) for some non principal ultrafilter \(\mathcal{U}\), then for all ordered generating set \(T\) of \(H\), we can find a sequence of integers \((n_k)_k\) and a sequence \((h_k, t_k)_k\) that converges to \((H, T)\) in \(\mathcal{M}\) and such that \(H_k \leq G_{n_k}\) for all \(k\).

3. **Cantor-Bendixson invariants**

This section is devoted to the basics of the Cantor-Bendixson analysis we use in Theorem 5.2. Let \(X\) be a topological space. We denote by \(X'\) the set of accumulation points of \(X\). Let \(X^{(0)} := X\). We define by transfinite induction the \(\alpha\)-th derived set of \(X\): \(X^{(\alpha)} = (X^{\alpha-1})'\) if \(\alpha\) is a successor and \(X^{(\alpha)} = \bigcap_{\beta < \alpha} X^\beta\) if \(\alpha\) is a limit ordinal. We denote by \(\omega\) the set \(\mathbb{N}\) of integers endowed with its natural order. We use the following topological classification theorem:

**Theorem** (Mazurkiewicz-Sierpinski Theorem [MS20]). For any given pair \((\alpha, n)\) where \(\alpha\) is countable ordinal number and \(n\) belongs to \(\mathbb{N}\), there is (up to homeomorphism) a unique countable compact space \(X\) such that \(X^{(\alpha)}\) has exactly \(n\) points: the set \(\omega^n n + 1\) endowed with the order topology.
The pair \((\alpha, n)\) is the characteristic system of \(X\). For example, the Alexandroff compactification \(\mathbb{N}\) of \(\mathbb{N}\) is homeomorphic to \(\omega + 1\). Its characteristic system is then \((1, 1)\). Similarly, the characteristic system of \(\mathbb{N}^k\) is \((k, 1)\). The Cantor-Bendixson rank of a point \(x\) in \(X\) is the smallest ordinal number \(\alpha\) such that \(x\) doesn’t belong to \(X^{(\alpha)}\). A countable compact space has the characteristic system \((\alpha, n)\) if and only if the set of points of maximal Cantor-Bendixson rank (i.e. rank \(\alpha\)) has cardinal \(n\). Observe that the Cantor-Bendixson rank of \((G, S)\) in \(M_m\) is the Cantor-Bendixson rank of \((G, S)\) in \(M\) and does not depend on \(S\) because of Lemma 2.7.

### 4. Characterization of limits

**Theorem 4.1.** Let \(G\) be a non abelian finitely generated group. The following conditions are equivalent:

- \((\lim)\) \(G\) is a limit of dihedral groups;
- \((\res)\) \(G\) is fully residually dihedral;
- \((\iso)\) \(G\) is isomorphic to a semi-direct product \(A \rtimes \mathbb{Z}/2\mathbb{Z}\) where \(A\) is a limit of cyclic groups on which \(\mathbb{Z}/2\mathbb{Z}\) acts by multiplication by \(-1\);
- \((\Th)\) \(\Th(G) \supset \bigcap_{n \geq 3} \Th(D_{2n})\);
- \((\Pi/\mathfrak{U})\) \(G\) is isomorphic to a subgroup of \((\prod_{n \geq 3} \mathbb{D}_{2n}) / \mathfrak{U}\) for some ultra-filter \(\mathfrak{U}\) on \(\mathbb{N}\).

We first give a characterization of limits of cyclic groups that we use in the proof of theorem 4.1.

**Proposition 4.2.** Let \(G\) be a finitely generated group. The following conditions are equivalent:

- \((\lim)_c\) \(G\) is a limit of cyclic groups;
- \((\res)_c\) \(G\) is fully residually cyclic;
- \((\iso)_c\) \(G\) is isomorphic to an abelian group with cyclic torsion subgroup;
- \((\Th)_c\) \(\Th(G) \supset \bigcap_{n \geq 1} \Th(\mathbb{Z}/n\mathbb{Z})\);
- \((\Pi/\mathfrak{U})_c\) \(G\) is isomorphic to a subgroup of \((\prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}) / \mathfrak{U}\) for some ultra-filter \(\mathfrak{U}\) on \(\mathbb{N}\).

We now show Proposition 4.2 and then Theorem 4.1.

**Proof of Proposition 4.2.** Here is the logical scheme of the proof:

![Logical Scheme](attachment:image.png)

We begin with the right triangle of implications.

\((\lim)_c \implies (\Pi/\mathfrak{U})_c\) : we first assume that there is a sequence \((\mathbb{Z}/n\mathbb{Z}, S_n)_n\) in \(\mathcal{M}\) which accumulates on \((G, S)\) for some ordered generating set \(S\) of \(G\). Then \(G\) embeds isomorphically into \((\prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}) / \mathfrak{U}\) for some non principal
ultrafilter $\mathcal{U}$ by Proposition 2.12(i). If $G$ is the limit of stationary sequence of finite cyclic groups, then $G$ is a finite cyclic group isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for some $k \geq 1$. We then set $\mathcal{U}$ as the dirac mass in $k$.

$(\Pi/\mathcal{U})_c \rightarrow (Th\nu)_c$: as $G$ embeds in $\prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z} / \mathcal{U}$ for some ultrafilter $\mathcal{U}$, we get then $Th\nu(G) \supset Th\nu(\prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z} / \mathcal{U})$.

By Los's theorem, we have $Th\nu(\prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z} / \mathcal{U}) = \lim_{\mathcal{U}} Th\nu(\mathbb{Z}/n\mathbb{Z})$.

We then complete the proof by showing: $(Th\nu) \Rightarrow (iso)$. This last step relies on specific sentences that can be found in the universal theory of all non abelian dihedral groups. We use the following lemma:
Lemma 4.3. The following sentences are true in any non abelian dihedral group:

\begin{enumerate}[(P_1)]
\item $\forall x \forall y (x^2 \neq 1 \land y^2 \neq 1) \Rightarrow xy = yx$ (rotations commute);
\item $\forall x \forall y \forall z (x \neq 1 \land x^2 = 1 \land y^2 \neq 1 \land xz \neq xz) \Rightarrow x^{-1}yx = y^{-1}$ (conjugation of a rotation by a reflection reverses its angle);
\item $\forall x \forall y \forall z \forall u (xz \neq yz \land yt \neq ty \land x^2 = 1 \land y^2 = 1 \land (xy)^2 = 1) \Rightarrow (xy)u = u(xy)$ (the product of two commuting reflections is central);
\item $\forall x \forall y \forall z \forall t (x \neq 1 \land x^2 = 1 \land y \neq 1 \land y^2 = 1 \land z^2 \neq 1 \land t^2 \neq 1 \land xz = zx \land yt = ty) \Rightarrow x = y$ (there is at most one central element of order 2).
\end{enumerate}

Proof of Lemma 4.3. There are two kinds of symmetries in a dihedral group $D_{2n}$ ($n \geq 3$): the rotations (positive isometries of the Euclidean plane or the Euclidean line) and the reflections (negative isometries). The non central rotations $x$ of $D_{2n}$ are characterized by the inequation $x^2 \neq 1$. All reflections have order 2 and there is possibly one central rotation of order 2. All sentences can be readily shown by writing $D_{2n}$ as the semidirect product $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

In fact, the three first sentences are true in any generalized dihedral group $Dih(A) := A \rtimes \mathbb{Z}/2\mathbb{Z}$ where $A$ is abelian and $\mathbb{Z}/2\mathbb{Z}$ acts on $A$ by multiplication by $-1$.

End of the proof of Theorem 4.1.

By assumption, all sentences of Lemma 4.3 are true in $G$. We denote by $A$ the subgroup of $G$ generated by the set $\{x \in G \mid x^2 \neq 1\} \cup Z(G)$. We show the following claims:

(i) $A$ is an abelian subgroup of index 2 in $G$;
(ii) $G$ is isomorphic to the semidirect product $A \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts on $A$ by taking the inverse;
(iii) there is at most one element of order 2 in $A$;
(iv) $A$ is a limit of cyclic groups.

Let us prove (i). By (P_1) of Lemma 4.3, $A$ is generated by a set of pairwise commuting elements. Hence $A$ is abelian. As $G$ is not abelian, the sentence (P_1) implies that $G$ has at least one non central element of order 2. Let $s$ be such an element. We show that $G = A \cup sA$. Let $x$ be in $G \setminus A$. There are two cases:

(case 1) $(sx)^2 \neq 1$. Then $sx$ belongs to $A$;
(case 2) $(sx)^2 = 1$. Since $s^2 = x^2 = 1$, $s$ and $x$ commute. Hence $s$ and $x$ are non central commuting elements of order 2. We deduce from (P_3) that $sx$ belongs to $Z(G) \subset A$.

Thus $x$ belongs to $sA$ in both cases, which shows that $A$ has index 2 in $G$. As $G$ is finitely generated, so is $A$.

By (P_2), central elements of $G$ have order at most 2. The sentence (P_3) shows that the conjugation by $s$ of an element of $A$ consists in taking its inverse. Hence (ii) is proved.
Let us prove \((iii)\). Using \((P_4)\), we deduce that the center of \(G\) has at most two elements. We now show that elements of \(A\) which have order 2 are central in \(G\). Let \(a\) be in \(A\) and such that \(a^2 = 1\). Write \(a = yz\) with \(y\) in the subgroup of \(G\) generated by \(\{x \in G \mid x^2 \neq 1\}\) and \(z\) in \(Z(G)\). By \((P_2)\), we have \(s^{-1}as = y^{-1}z\). Since \(a^2 = y^2 = 1\), we deduce that \(s^{-1}as = a\). Thus \(a\) is central, which completes the proof of \((iii)\).

We now prove \((iv)\). We set \(A^2 := \{a^2 \mid a \in A\}\) and \(\mathbb{D}_{2n}^2 := \{g^2 \mid g \in \mathbb{D}_{2n}\}\).

We first show that \(A^2\) is a limit of cyclic groups. By proposition \(\[1,2\]\), it suffices to show that \(Th_v(A^2) \supset \bigcap_n Th_v(\mathbb{Z}/n\mathbb{Z})\). Let \(\phi(x_1, \ldots, x_k)\) be a quantifier free formula in variables \(x_1, \ldots, x_k\) and consider the sentences \((P)\forall x_1 \ldots \forall x_k \phi(x_1, \ldots, x_k)\) and \((P^2)\forall x_1 \ldots \forall x_k \phi(x_1^2, \ldots, x_k^2)\). We observe the following equivalences:

\[
\mathbb{D}_{2n} \models P^2 \iff \mathbb{D}_{2n}^2 \models P \text{ and } A \models P^2 \iff A^2 \models P.
\]

Assume \(\mathbb{Z}/n\mathbb{Z} \models P\) for all \(n \geq 1\). Then \(\mathbb{D}_{2n}^2 \models P\) for all \(n \geq 3\) because \(\mathbb{D}_{2n}^2\) is a finite cyclic group. It follows that \(\mathbb{D}_{2n} \models P^2\) for all \(n \geq 3\). By assumption, \(G \models P^2\), hence \(A \models P^2\). Consequently, \(A^2 \models P\). We deduce that \(A^2\) is a limit of cyclic group, hence \(A^2\) is isomorphic to \(\mathbb{Z}^n \times \mathbb{Z}/k\mathbb{Z}\) for some \(n \geq 0, k \geq 1\) by Proposition \(\[1,2\]\). By \((P_4)\), there is at most one element of order 2 in \(A\). We deduce that \(A\) is isomorphic to \(\mathbb{Z}^n \times \mathbb{Z}/k\mathbb{Z}\) with \(k\) in \(\{k, 2k\}\). \(\square\)

5. The Space of Limits of Dihedral Groups on \(m\) Generators

Let \(G\) be a non abelian limit of dihedral groups that is generated by two of its (necessarily non trivial) elements, say \(x\) and \(y\). It follows from \((P_1)\) of Lemma \(\[1,3\]\) that either \(x\) or \(y\) has order 2. Assume then \(x^2 = 1\). By \((P_3)\) of the same lemma, we have either \(y^2 = 1\) or simultaneously \(y^2 \neq 1\) and \(x^{-1}yx = y^{-1}\). Thus \(G\) is an homomorphic image of \(\mathbb{D}_\infty\) (both \(\langle x, y \mid x^2 = y^2 = 1 \rangle\) and \(\langle x, y \mid x^2 = 1, x^{-1}yx = y^{-1} \rangle\) are presentations of \(\mathbb{D}_\infty\)). Hence \(G\) is a dihedral group. This shows that 2-generated limits of dihedral groups are dihedral groups (more generally, the rank of a limit \(G = A \times \mathbb{Z}/2\mathbb{Z}\) of dihedral groups is \(r(A) + 1\)). Consequently:

**Corollary 5.1.** The space of dihedral marked groups on 2 generators is a closed and open subspace of \(\mathcal{M}_2\).

**Proof.** It only remains to show that this subspace is open. As a quotient of a dihedral group is a dihedral group, the result follows from Lemma \(\[2,3\]\) \(\square\)

We have a complete topological description of the space of dihedral groups on two generators. For each \(n \neq 2\), there are exactly three distinct marked groups on two generators which are isomorphic to \(\mathbb{D}_{2n}\):

\[
\begin{align*}
A_{2n} & := \langle a,b \mid a^2 = b^2 = (ab)^n = 1 \rangle, \\
B_{2n} & := \langle a,b \mid a^2 = b^n = 1, a^{-1}ba = b^{-1} \rangle, \\
\overline{B}_{2n} & := \langle a,b \mid b^2 = a^n = 1, b^{-1}ab = a^{-1} \rangle.
\end{align*}
\]

There is a unique marked group of \(\mathcal{M}_2\) isomorphic to \(\mathbb{D}_4\): \(A_4 = B_4 = \overline{B}_4\). The only accumulation points in the space of dihedral groups on two generators are the three distinct marked infinite dihedral groups \(A_\infty, B_\infty\) and \(\overline{B}_\infty\) (see
Figure 1 below). This last fact is proved in Proposition 5.4. The remaining statements can be readily adapted from this proposition.

We can carry out such an analysis in the space of marked groups on $m$ generators by using Cantor-Bendixson invariants defined in Section 3. We denote by $\omega$ the smallest infinite ordinal, i.e. the set $\mathbb{N}$ of positive integers endowed with its natural order.

**Theorem 5.2.** The topological closure $D_m$ of dihedral marked groups in $M_m$ is homeomorphic to $\omega^{m-1}(2^m - 1) + 1$ endowed with the order topology.

Because of the Mazurkiewicz-Sierpinski Theorem (Section 3), it suffices to show that the $(m - 1)$-th derived set $D_m^{(m-1)}$ of $D_m$ contains $2^m - 1$ points. This is carried out with the two following propositions:

**Proposition 5.3.** Let $G = A \rtimes \mathbb{Z}/2\mathbb{Z}$ be a limit of dihedral groups on $m$ generators. Then the Cantor-Bendixson rank of $G$ in $D_m$ is the free rank of $A$.

Hence, the only remaining marked groups in $D_m^{(m-1)}$ are marked groups abstractly isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We now count them:

**Proposition 5.4.** Let $G = \mathbb{Z}/2\mathbb{Z}^m$ with $m \geq 2$. In $M_m$, there are exactly $2^m - 1$ marked groups which are abstractly isomorphic to $G$.

We show Propositions 5.3 and 5.4. Theorem 5.2 then directly follows.

Let $A \subset M$ be the set of abelian marked groups, let $D \subset M$ be the set of dihedral marked groups and let $\widehat{D} \subset M$ be the set of generalized dihedral marked groups. Let $C$ be the topological closure of cyclic marked groups in $M$. We define $Dih(A, S)$ with $S = (a_1, a_2, \ldots)$ as the marked group $(Dih(A), S')$ with $S' = (a, a_1, a_2, \ldots)$ where $a$ denotes the generator of $\mathbb{Z}/2\mathbb{Z}$.
Lemma 5.5. The map \( Dih : A \rightarrow \tilde{D} \) is a continuous and open embedding. Moreover, \( Dih(C) = D \).

Proof of Lemma 5.5. We fix words \( v_1, \ldots, v_k, w_1, \ldots, w_l \) in \( F_{m+1} \) for which the exponent sum of \( e_1 \) is zero. We then define the system

\[
(\Sigma) : \begin{cases} 
  v_1 = 1, \ldots, v_k = 1, \\
  w_1 \neq 1, \ldots, w_l \neq 1.
\end{cases}
\]

Let

\[
D := \langle e_1, \ldots, e_{m+1} \mid e_1^2 = 1, e_1 e_i e_1^{-1} = e_i^{-1}, [e_i, e_j] = 1, i = 2, \ldots, m + 1 \rangle.
\]

We reduce the words \( v_i, w_j \) in \( D \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, l \) to get words without symbols \( e_i \). We then shift the indices on the left (\( e_i \) becomes \( e_{i-1} \)) to obtain words \( v'_i, w'_j \) in \( F_m \). We define \( (\Sigma') \) by replacing \( v_i \) by \( v'_i \) and \( w_j \) by \( w'_j \) in \( (\Sigma) \). We consider the elementary open sets \( O_{\Sigma'} \subset A \) and \( O_{\Sigma} \subset D \). Let \( (A, S) \) be in \( A \). It is trivial to check that \( Dih(A, S) \in O_{\Sigma} \iff (A, S) \in O_{\Sigma'} \). Hence \( Dih \) is a continuous and open map.

Assume \( Dih(A, S) = Dih(B, T) \) with \( (A, S), (B, T) \) in \( A \). There is an isomorphism \( \phi : Dih(A) \rightarrow Dih(B) \) such that \( \phi \cdot S = T \). Observe that for any abelian group \( C \), \( C \) is the characteristic subgroup of \( Dih(C) \) generated by the set \( \{ c \in Dih(C) \mid c^2 \neq 1 \text{ or } c \text{ is central in } Dih(C) \} \). This shows that \( \phi \) induces an isomorphism from \( A \) onto \( B \). Hence \( (A, S) = (B, T) \) which proves the injectivity of \( Dih \).

Lemma 5.6. Let \( A \) be a finitely generated abelian group (respectively a limit of cyclic groups). Then the Cantor-Bendixson rank of \( A \) in \( M \) (respectively in \( C \)) is the free rank of \( A \).

Proof of Lemma 5.6. Consider a finitely generated abelian group \( A \). Since \( A \) is finitely presented, the Cantor-Bendixson rank of \( A \) in \( M \) is the Cantor-Bendixson rank of \( \{ 0 \} \) in \( \mathcal{N}(A) \) by Lemma 2.3. The proof is an induction on the free rank \( r \) of \( A \). We first show that the Cantor-Bendixson rank of \( \{ 0 \} \) is not less than \( r \). If \( r = 0 \), then \( A \) is finite. It follows that \( \mathcal{N}(A) \) is a finite discrete space in which \( \{ 0 \} \) is obviously isolated. Assume \( r \geq 1 \). Consider an infinite cyclic subgroup \( \langle z \rangle \) of \( A \). For all \( n \) in \( \mathbb{N} \), the Cantor-Bendixson rank of \( \langle z^n \rangle \) in \( \mathcal{N}(A) \) is the Cantor-Bendixson rank of \( \{ 0 \} \) in \( \mathcal{N}(A/\langle z^n \rangle) \) by Lemma 2.3. By the induction hypothesis, this rank is at least \( r - 1 \). Since \( \langle z^n \rangle \) tends to \( \{ 0 \} \) in \( \mathcal{N}(A) \) as \( n \) tends to infinity, the Cantor-Bendixson of \( \{ 0 \} \) is at least \( r \). We now show that the Cantor-Bendixson rank of \( A \) is not greater than \( r \). It is clear if \( r = 0 \). Assume \( r \geq 1 \). Consider the set \( V \) of subgroups of \( A \) whose intersection with \( Tor(A) \) is trivial. Then \( V \) is an open neighborhood of \( \{ 0 \} \) in \( \mathcal{N}(A) \). Let \( B \neq \{ 0 \} \) be in \( V \). By the induction hypothesis, the Cantor-Bendixson rank of \( \{ 0 \} \) in \( \mathcal{N}(A/B) \) is at most \( r - 1 \). Since this is also the Cantor-Bendixson rank of \( B \) in \( \mathcal{N}(A) \), the Cantor-Bendixson rank of \( \{ 0 \} \) in \( \mathcal{N}(A) \) is at most \( r \).

If \( A \) is in \( C \), we then consider the set \( \mathcal{N}_C(A) \) of subgroups \( B \) of \( A \) such that \( A/B \) is a limit of cyclic groups. For any infinite cyclic factor \( \langle z \rangle \) of \( A \), the subgroup \( \langle z^n \rangle \) belongs to \( \mathcal{N}_C(A) \) for all \( n \geq 1 \) coprime with \( |Tor(A)| \). We can hence apply the reasoning above to such an \( A \).
Proof of Proposition 5.3. Let \( G = A \times \mathbb{Z}/2\mathbb{Z} \) be a limit of dihedral groups on \( m \) generators. As \((G, S)\) is in the image of \( Dih \) for a suitable ordered generating set \( S \), its Cantor-Bendixson rank in \( \mathcal{D}_m \) is the Cantor-Bendixson rank of \( A \) in \( C \) by Lemma 5.5. This is the free rank of \( G \). \[ \square \]

We fix \( m \geq 2 \) and \( G = \mathbb{Z}^{m-1} \rtimes \mathbb{Z}/2\mathbb{Z} \). We denote by \( a \) the generating element of the subgroup \( \mathbb{Z}/2\mathbb{Z} \) and we use the additive notation in the normal subgroup \( \mathbb{Z}^{m-1} \) of \( G \). Let \( \phi \) be in \( Aut(G) \). Since \( \phi(a) \) has order 2, we can write \( \phi(a) = v(\phi)a \) with \( v(\phi) \) in \( \mathbb{Z}^{m-1} \). We denote by \( \mathbb{Z}^{m-1} \rtimes GL_{n-1}(\mathbb{Z}) \) the semidirect product where \( GL_{n-1}(\mathbb{Z}) \) acts on \( \mathbb{Z}^{m-1} \) in the standard way. We denote by \( (e_i)_{1 \leq i \leq n-1} \) the canonical basis of \( \mathbb{Z}^{m-1} \).

Lemma 5.7. The map
\[
\Phi : Aut(G) \longrightarrow \mathbb{Z}^{m-1} \rtimes GL_{m-1}(\mathbb{Z})
\]
\[
\phi \longmapsto (v(\phi), \phi|_{\mathbb{Z}^{m-1}})
\]
is an isomorphism.

Proof of Lemma 5.7. Since \( \mathbb{Z}^{m-1} \) is the subgroup of \( G \) generated by elements of infinite order, \( \mathbb{Z}^{m-1} \) is a characteristic subgroup. Hence \( \Phi \) is well defined. Checking that \( \Phi \) is a homomorphism is routine. Consider \( v \) in \( \mathbb{Z}^{m-1} \) and \( f \) in \( GL_{m-1}(\mathbb{Z}) \). The group \( G \) has the presentation \( \langle a, e_1, \ldots, e_{m-1} \mid a^2 = 1, ae_i a^{-1} = e_i^{-1}, [e_i, e_j] = 1, i, j = 1, \ldots, m-1 \rangle \). We use then Von Dyck’s Theorem to show that there is unique automorphism \( \phi \) of \( G \) such that \( \phi(a) = va \) and \( \phi|_{\mathbb{Z}^{m-1}} = f \). \[ \square \]

Proof of Proposition 5.4. The set of marked groups which are isomorphic to \( G \) in \( \mathcal{M}_m \) corresponds bijectively to the the set of \( Aut(G) \)-orbits of \( V(G, m) \) of the diagonal action. For \( S = (g_1, \ldots, g_m) \) in \( V(G, m) \), we define \( I(S) \subset \{1, \ldots, m\} \) as the set of indices \( i \) satisfying \( g_i^2 = 1 \). Let \( P \) be a non empty subset of \( \{1, \ldots, m\} \). We denote by \( V(P) \), the set of all generating \( G \)-vectors \( S \) of length \( m \) such that \( I(S) = P \). Clearly, the sets \( V(P) \) are pairwise disjoint \( Aut(G) \)-invariant sets and there are \( 2^m - 1 \) such sets. We now prove that the action of \( Aut(G) \) is transitive on \( V(P) \) for all \( P \). Starting with the generating vector \( S_0 = (e_1, \ldots, e_{m-1}, a) \) and using elementary Nielsen transformations, we can get a generating system \( S' \) such that \( I(S') = P \) for any non empty \( P \). Since the actions of the Nielsen group \( Aut(F_m) \) on \( V(G, m) \) commutes with the action of \( Aut(G) \), it suffices to prove that \( Aut(G) \cdot S_0 = V(\{m\}) \). Let \( S = (f_1, \ldots, f_{m-1}, va) \) be in \( V(\{m\}) \). It is easy to check that any word on the elements of \( S \) that belongs to \( \mathbb{Z}^{m-1} \) can actually be generated by the elements \( f_1, \ldots, f_{m-1} \) only. Consequently, \( (f_1, \ldots, f_{m-1}) \) is a basis of \( \mathbb{Z}^{m-1} \) and there is \( f \) in \( GL_{m-1}(\mathbb{Z}) \) such that \( f(e_i) = f_i \) for \( i = 1, \ldots, m-1 \). By Lemma 5.7, there is \( \phi \) in \( Aut(G) \) such that \( \phi(a) = va \) and \( \phi|_{\mathbb{Z}^{m-1}} = f \). Thus we have \( \phi \cdot S_0 = S \). \[ \square \]

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