Single-mode nonclassicality measure from Simon-Peres-Horodecki criterion

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Nonclassicality of a single-mode field is equivalent to the rank of two-mode entanglement this field generates at the output of a beam-splitter [Phys. Rev. A 89, 052302 (2014)]. We derive a measure for the nonclassicality of a single-mode field using the two-mode inseparability measure of Simon-Peres-Horodecki. Degree of the single-mode nonclassicality requires the knowledge of only \( \langle \hat{a}^2 \rangle \) and \( \langle \hat{a} \hat{a}^\dagger \rangle \). This condition/measure is both a necessary and sufficient condition for Gaussian single-mode states. We show that this measure increases with squeezing strength and display a jump at the critical coupling for a superradiant phase transition. Additionally, we derive a simple analytical condition, \( |\langle \hat{a}^2 \rangle| > \langle \hat{a} \hat{a}^\dagger \rangle \), for nonclassicality from Duan-Giedke-Cirac-Zoller criterion. In a forthcoming study, we use the derived measure to show the following. Non-causal linear optical response of an optomechanical cavity emerges at the same critical cavity-mechanical coupling where output field becomes nonclassical.

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I. INTRODUCTION

Achieving nonclassical states of a single-mode light is attractive for realization of genuine quantum optical phenomena. A single-mode nonclassical field can generate entangled photon pairs at the output of a beam-splitter [2,4] which are vital for quantum information and quantum teleportation [5–7]. Additionally, single-mode nonclassical field, e.g. squeezed states, can be utilized to perform measurements below the standard quantum limit [8–11].

In the detection of the single-mode nonclassicality usually the negativity of quasiprobability distributions, such as Wigner function [12,13] or Sudarshan-Glauber P distribution [14,15], are evaluated [16,17]. Just a decade ago Asboth et al. [18] brought a new perspective on the concept of nonclassicality. They associated the degree of the nonclassicality of a single-mode field with its ability to produce two-mode entanglement [19] at the output of a beam-splitter. Recently, Vogel and Sperling [19] demonstrated an exciting relation. They showed that the rank of the two-mode entanglement a single-mode light generates through a beam-splitter is equal to the rank of the expansion of this nonclassical state in terms of classical coherent states. More explicitly, a nonclassical single-mode state (of annihilation operator \( \hat{a} \))

\[
|\psi_N\rangle = \sum_{i=1}^r \kappa_i |\alpha_i\rangle,
\]

expressed in terms of coherent states \( |\alpha_i\rangle \), generates a two-mode entangled state \( |\psi_{\text{Ent}}\rangle \)

\[
|\psi_{\text{Ent}}\rangle = \sum_{i=1}^r \lambda_i |\alpha_1^{(i)}\rangle \otimes |\alpha_2^{(i)}\rangle
\]

with the entanglement rank equal to \( r \). Here, \( |\alpha_1^{(i)}\rangle \) and \( |\alpha_2^{(i)}\rangle \) represent the states of output modes, \( \hat{a}_1 \) and \( \hat{a}_2 \) modes, of the beam-splitter and \( \kappa_i \), \( \lambda_i \) are expansion coefficients.

In a recent study [21] we demonstrate an additional connection. Rank of the inseparability of \( N \) identical 2-level particles

\[
|\psi_N\rangle = \sum_{i=1}^r \kappa_i |\alpha_{ACS}^{(i)}\rangle,
\]

is equivalent to the rank of the nonclassicality its excitations (quasiparticles) \( \hat{a} \) have [in Eq. (1)]. \( \alpha_{ACS}^{(i)} \) are atomic coherent states (ACS) [22]. Hence, i) entanglement of \( N \) identical 2-level particles, ii) two-mode entanglement of these particles, iii) nonclassicality of the single-mode quasiparticle excitation \( \hat{a} \) and iv) two-mode entanglement this single-mode field generates at the output of a beams-splitter are all equivalent. We used this analogy [21] to obtain nonclassicality conditions using the limits of Holstein-Primakoff (HP) transformation [23,24].

However, HP approach is limited since two major criteria, Simon-Peres-Horodecki (SPH) [27,29] and Duan-Giedke-Cirac-Zoller (DGCZ) [30], include terms which do not preserve the number of particles, e.g. \( \hat{c}_g \hat{c}_c \). This avoids the use of Holstein-Primakoff transformation. \( \hat{c}_g \) and \( \hat{c}_c \) are operators annihilating an identical particle (see Appendix 1A in Ref. [31]) at the ground and excited levels, respectively.

In this paper, we use beam-splitter approach [24] to obtain a nonclassicality measure \( E_N \) from the two-mode entanglement criterion of Simon-Peres-Horodecki (SPH) [27,29]. SPH criterion is based on the negativity of partial transpose of a bipartite system and is both a necessary and sufficient criterion for Gaussian states which are proper for weakly interacting optomechanical systems [32,34].

SPH criterion also provides a measure [28,29,32] based on the logarithmic negativity of symplectic eigenvalues of noise matrix. We adopt this two-mode entanglement measure for quantifying the degree of single-
mode nonclassicality, using the beam-splitter transformations \cite{21} and the concept of equivalence mentioned in Ref.\cite{11} \cite{15} \cite{20}. Nonclassicality measure $E_N$ requires the knowledge of only $\langle \hat{a}^2 \rangle$ and $\langle \hat{a}^2 \rangle$. We present the behaviour of $E_N$ for squeezed single-mode state (Fig. 1) and in a superradiant phase transition (Fig. 2). In a forthcoming paper \cite{35}, we use this measure to determine the threshold for cavity-mirror coupling strength ($g_c$) at which nonclassical light production emerges. Interestingly, the coupling thresholds ($g_c$) for the nonclassical cavity output and noncausal behavior of the refractive index —replacing the optomechanical cavity— come out to be the same.

We also derive a simple nonclassicality condition, that is $|\langle \hat{a}^2 \rangle| > (\hat{a}^2 \hat{a})$, using the Duan-Giedke-Cirac-Zoller \cite{30} two-mode entanglement criterion. This criterion is also a necessary and sufficient condition for entanglement of Gaussian states. Hence, as expected, the condition deduced from DGCZ criterion \cite{30} provides a stronger condition compared to the one obtained in Ref.\cite{21}, that is $|\langle \hat{a} \rangle| > (\hat{a}^2 \hat{a})^{1/2}$, from Hillery & Zubairy criterion \cite{30} which is not a necessary one.

II. NONCLASSICALITY MEASURE FROM SPH CRITERION

In this section, we introduce the two-mode entanglement criterion \cite{27} and measure \cite{28} \cite{29} of Simon-Peres-Horodecki. Next, we relate the variance matrix of SPH criterion to single-mode noises $\langle \hat{a}^2 \rangle$ and $\langle \hat{a}^2 \rangle$ using the beam-splitter transformations \cite{21} \cite{4}. We obtain a measure for quantifying the degree of nonclassicality $E_N(\langle \hat{a}^2 \rangle, \langle \hat{a}^2 \rangle \hat{a})$. Finally, we demonstrate the behavior of the measure in two examples, single-mode squeezed states and superradiant phase transition.

A. Two-mode entanglement measure of SPH

A bipartite Gaussian state can be completely characterized by its covariance (correlation) matrix \cite{28} \cite{29} \cite{32} \cite{57}

$$V_{ij} = \frac{1}{2} (\hat{Y}_i \hat{Y}_j + \hat{Y}_j \hat{Y}_i) ,$$

(4)

where $\hat{Y} = [\hat{x}_1 , \hat{p}_1 , \hat{x}_2 , \hat{p}_2]$ operators with $\hat{x}_i = (\hat{a}_i^\dagger + \hat{a}_i)/\sqrt{2}$ and $\hat{p}_i = (i(\hat{a}_i^\dagger - \hat{a}_i))/\sqrt{2}$. First order moments in variances \cite{4} are absent, since they can be adjusted to zeros without affecting the entanglement features \cite{28}. $V$ matrix can be written as

$$B = \begin{bmatrix} \langle \hat{x}_2^2 \rangle & \langle \hat{x}_2 \hat{p}_2 + \hat{p}_2 \hat{x}_2 \rangle/2 \\ \langle \hat{x}_2 \hat{p}_2 + \hat{p}_2 \hat{x}_2 \rangle/2 & \langle \hat{p}_2^2 \rangle \end{bmatrix} ,$$

(7)

$$C = \frac{1}{2} \begin{bmatrix} \langle \hat{x}_1 \hat{x}_2 + \hat{p}_1 \hat{p}_2 \rangle & \langle \hat{x}_1 \hat{p}_1 + \hat{p}_1 \hat{x}_2 \rangle \\ \langle \hat{p}_1 \hat{x}_2 + \hat{x}_2 \hat{p}_1 \rangle & \langle \hat{p}_1 \hat{p}_2 + \hat{p}_2 \hat{p}_1 \rangle \end{bmatrix} ,$$

(8)

with $C$ denoting the correlation noise.

A bipartite Gaussian system is separable if the symplectic eigenvalues of the partial transposed state satisfy

$$2\eta^- > 1 \quad \text{or} \quad 2\eta^+ > 1 .$$

(9)

Symplectic eigenvalues can be calculated from

$$\eta^\pm \equiv \frac{1}{\sqrt{2}} \left( \sigma(V) \pm \{ |\sigma(V)|^2 - 4 \det(V) \}^{1/2} \right)^{1/2} ,$$

(10)

where $\sigma(V) = \det(A) + \det(B) - 2 \det(C)$. Therefore, the strength of violation of Eq. \cite{9} can be handled \cite{28} \cite{29} as a measure of the entanglement between $\hat{a}_1$ and $\hat{a}_2$ modes. Since $\eta^-$ is smaller compared to $\eta^+$, one defines the logarithmic negativity of $\ln(2\eta^-)$ as the entanglement measure,

$$E_N = \max(0, -\ln(2\eta^-)) .$$

(11)

$E_N$ is zero if $2\eta^- > 1$ and increases as symplectic eigenvalue becomes $2\eta^- < 1$.

Besides the symplectic eigenvalues, Eq. \cite{9}, one can equivalently check \cite{27} if

$$\lambda_{\text{simon}} = \det(A)\det(B) + \left( \frac{1}{4} - |\det(C)| \right)^{2} - \text{tr}(A J C J B J C^T J) - \frac{1}{4} (\det(A) + \det(B)) < 0$$

(12)

to determine the presence of two-mode entanglement with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

(13)

However, $\lambda_{\text{simon}}$ is not an entanglement measure.

B. Single-mode nonclassicality measure

A single-mode field $\hat{a}$ generates the two output modes $\hat{a}_1$ and $\hat{a}_2$ passing through a beam-splitter. The nonclassicality of $\hat{a}$ mode is measured by the rank of entanglement it can generate with linear beam-splitter operations \cite{18}. This rank is equivalent to the number of terms required for expanding a nonclassical state in terms of classical coherent states \cite{1} \cite{20}, see Eq. (eq:coheexpansion). It is also equal to the rank of $N$ particle entanglement of identical 2-level atoms, where $\hat{a}$ is the quasiparticle excitation of the system \cite{21}. 

$$\lambda_{\text{simon}} = \det(A)\det(B) + \left( \frac{1}{4} - |\det(C)| \right)^{2} - \text{tr}(A J C J B J C^T J) - \frac{1}{4} (\det(A) + \det(B)) < 0$$

(12)

to determine the presence of two-mode entanglement with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

(13)

However, $\lambda_{\text{simon}}$ is not an entanglement measure.
Our beam-splitter approach is as follows. i) We evaluate the variance terms, like $\hat{a}_1^2$ and $\hat{a}_1 \hat{a}_2$, in terms of $\hat{a}^2$ and $\hat{a}^\dagger \hat{a}$ using the beam-splitter transformations, see Eqs. (14a), (14b). Next, we evaluate the elements of the covariance matrix, Eqs. (4), (5) and (6). Thereby, $v_{ij}$ depends only on the $\hat{a}^2$ and $\hat{a}^\dagger \hat{a}$ values and transmission ($t$) and reflection ($r$) parameters of the beam-splitter. iii) Finally, we calculate the $E_N$ from $V_{ij}$ matrix elements using Eqs. (10) and (11). In order to obtain the maximum possible two-mode entanglement at the beam-splitter output, we maximize $E_N(t, \phi)$ with respect to $t = [0, 1]$ and $\phi = [0, 2\pi]$.

The two-mode state, generated at the output of a beam-splitter from a single-mode input state $|\psi_n\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, can be calculated using the transformed operators

$$\hat{a}_1(\xi) = \hat{B}^\dagger(\xi) \hat{a}_1 \hat{B}(\xi) = t e^{i\phi} \hat{a}_1 + r \hat{a}_2 , \hspace{1cm} (14a)$$

$$\hat{a}_2(\xi) = \hat{B}^\dagger(\xi) \hat{a}_2 \hat{B}(\xi) = -r \hat{a}_1 + t e^{-i\phi} \hat{a}_2 , \hspace{1cm} (14b)$$

where

$$\hat{B}(\xi) = e^{\xi \hat{a}_1^\dagger - \xi^* \hat{a}_1}$$

is the beam-splitter operator. In calculating the expectation values, one uses $|\psi_n\rangle_1 \otimes |0\rangle_2$ as the initial state of the two-mode output, where input state $|\psi_n\rangle$ is placed into the state of the first mode ($\hat{a}_1$). This is equivalent to Schrödinger picture where two-mode state is determined as $|\psi_{12}\rangle = f(\mu_1 \hat{a}_1^\dagger + \mu_2 \hat{a}_2^\dagger)|0\rangle_1 \otimes |0\rangle_2$ with $f(\hat{a})$ is the expansion of the input mode $|\psi_n\rangle = \sum_{n=0}^{\infty} d_n (\hat{a}^\dagger)^n$. To give an explicit example, working in the Heisenberg picture, one can calculate the expectation value as

$$\langle \hat{a}_1 \hat{a}_2 \rangle = 2 \langle 0 \rangle \otimes 1 \langle \psi_n | \hat{B}^\dagger(\xi) \hat{a}_1 \hat{a}_2 \hat{B}(\xi) |\psi_n\rangle_1 \otimes |0\rangle_2$$

(16)

which is equal to

$$\langle \hat{a}_1 \hat{a}_2 \rangle = 2 \langle 0 \rangle \otimes 1 \langle \psi_n | (te^{i\phi} \hat{a}_1 + r \hat{a}_2) (-r \hat{a}_1 + t e^{-i\phi} \hat{a}_2) |\psi_n\rangle_1 \otimes |0\rangle_2$$

(17)

using the transformations (14a) and (14b). Since $\langle 1 | \psi_n | \hat{a}_1^2 |\psi_n\rangle_1 = \langle \psi_n | \hat{a}^2 |\psi_n\rangle_1 \equiv \langle \hat{a}^2 \rangle$, Eq. (17) becomes

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -r e^{i\phi} \langle \hat{a}^2 \rangle .$$

(18)

Similarly, matrix elements of $V_{ij}$ can be calculated as

$$A = \begin{pmatrix} t^2 [\cos(\theta + 2\phi)v_a + n_a] + \frac{1}{2} & t^2v_a \sin(\theta + 2\phi) \\ t^2v_a \sin(\theta + 2\phi) & t^2 [\cos(\theta + 2\phi)v_a + n_a] + \frac{1}{2} \end{pmatrix},$$

(19)

$$B = \begin{pmatrix} r^2 [\cos(\theta)v_a + n_a] + \frac{1}{2} & r^2v_a \sin \theta \\ r^2v_a \sin \theta & r^2 [\cos(\theta)v_a + n_a] + \frac{1}{2} \end{pmatrix},$$

(20)

$$C = \text{tr} \begin{pmatrix} -\sin(\theta + \phi)v_a + \sin \phi n_a & -\sin(\theta + \phi)v_a + \sin \phi n_a \\ -\sin(\theta + \phi)v_a + \sin \phi n_a & \cos(\theta + \phi)v_a - \sin \phi n_a \end{pmatrix},$$

(21)

where $\langle \hat{a}^2 \rangle = v_a e^{i\theta}$ with $v_a$ is real and positive, and $\langle \hat{a}^\dagger \hat{a} \rangle = n_a$.

Therefore, given the $\langle \hat{a}^2 \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$ for a single-mode state, one can calculate the degree of two-mode entanglement $E_N$ this input mode generates through a beam-splitter. One inserts Eqs. (19)-(21) into Eq. (10) to determine $\eta^-$ and uses Eq. (11) to obtain the measure of entanglement $E_N$. This calculated measure of $E_N$ is also the degree of nonclassicality of the single-mode state $|\psi_n\rangle$ [20]. We note that, a maximization of $E_N(t, \phi)$ over $t$ and $\phi$ is carried out, with reflection $r^2 = 1 - t^2$ is already constrained.

C. Examples on the Behaviour of $E_N$

1. Single-mode squeezed state

Nonclassical squeezed states can be generated by applying the operator $\exp[\beta \hat{a}^\dagger - \beta^* \hat{a}^2]$ to coherent states $|\alpha\rangle$ or to vacuum $|0\rangle$. Here $\beta = re^{i\theta}$ determined the strength ($r$) and the phase ($\theta$) of the quadrature squeezing. For a squeezed coherent state

$$\langle \hat{a} \rangle = C \alpha - S e^{i\theta} \alpha^*,$$

(22)

$$\langle \hat{a}^2 \rangle = C^2 \alpha^2 S^2 e^{2i\theta} - C S e^{i\theta} (2|\alpha|^2 + 1) ,$$

(23)

$$\langle \hat{a}^\dagger \hat{a} \rangle = C^2 |\alpha|^2 + S^2 (1 + |\alpha|^2) - C S e^{i\theta} \alpha^2 + e^{-i\theta} \alpha^2,$$

where $C \equiv \cosh r$ ans $S \equiv \sinh r$. The variances of the field, to use in Eqs. (19)-(21), can be calculated from $\langle \delta \hat{a}^\dagger \delta \hat{a} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle - |\langle \hat{a} \rangle|^2$. 

$$\langle \hat{a}^\dagger \hat{a} \rangle - |\langle \hat{a} \rangle|^2 .$$
In Fig. 1 we plot the dependence of the degree of nonclassicality \( (E_N) \) with respect to squeezing strength \( r \). It is a well known phenomenon that emergence and the degree of two-mode entanglement depends on the choice of the phase of the parametrized pump \( \beta = r e^{i \theta} \), see Ref. [37] and Refs [38,40]. Hence, in Fig. 1 we give two plots. Solid-line is for a fixed squeezing angle \( \theta = 0 \). In the calculation of the dotted-line we search over \( \theta \) values which maximizes the \( E \) for a given squeezing strength \( r \). We observe that by tuning the squeezing angle one can obtain larger degrees of nonclassicality which is a common phenomenon occurring also in two-mode entanglement in parametric amplifiers.

2. Superradiant phase-transition

When an ensemble of 2-level atoms are pumped above a critical intensity, the coupling of matter and light induces a new phase. This is called as Dicke phase-transition [23, 41–43] or superradiant phase where all atoms radiate collectively. Such a phase-transition is accompanied by a jump in the entanglement properties of the system [44].

The Hamiltonian for this system can be written as [24]

\[
\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \omega_{eg} \hat{S}_z + \hbar \frac{g}{\sqrt{N}} (\hat{S}_+ \hat{a} + \hat{S}_- \hat{a}^\dagger),
\]

where \( \hat{S}_+ \) and \( \hat{S}_- \) are the collective raising/lowering and energy level operators for the \( N \) identical 2-level atom system. \( \omega_{eg} \) is the level spacing of a single 2-level atom. \( \omega \) is the frequency of the light and \( \hat{a} \) is the annihilation of this single-mode field. \( g \) is the interaction strength between a single atom and photon. The superradiant phase transition occurs above \( g > g_c = \sqrt{\omega_{eg} \omega} \).

Similar to Ref. [14], we numerically find the ground state of Hamiltonian (25) by choosing \( N = 80 \) and 142 dimensions for the field mode \( \hat{a} \). Since the particles are identical, the dimension of the atomic system is only 2\( S+1 \) \( =N+1 = 80 \), not \( 2^N \). Because, atoms can only occupy the symmetric Dicke states [45] which is in the maximum cooperation set.

In Fig. 2, we plot the mean photon occupation of the ground state of Hamiltonian (25) with respect to the atom-photon coupling strength \( g \). We observe that for \( g < g_c \), both atoms and field energetically favour to stay in the not-excited state. For \( g > g_c \), atoms move to excited state and photon occupancy emerges in the single-mode field \( \hat{a} \). [23]

Fig. 2, shows that this quantum phase transition is accompanied with a jump in the in the degree of nonclassicality \( E_N \). Even though \( E_N \) display oscillations above \( g > g_c \), in Fig. 2, the Simon criterion [27] \( \lambda_{\text{simon}} \) exhibits a smooth behaviour signalling (not measuring) the emergence of stronger nonclassicality. \( \lambda_{\text{simon}} \) is defined in

\[ \lambda = -\log(N) \]
Eq. \([12]\) and evaluated using matrices \([19]-[21]\). Here, we remind that both \(E_N\) and \(\lambda_{\text{simon}}\) are necessary and sufficient conditions only for Gaussian states. Since in a quantum jump (like a superradiant phase transition) field need not stay in Gaussian form, the abrupt behaviour of \(E_N\) is natural to occur. We plot the absolute value of \(\lambda_{\text{simon}}\) in logarithmic scale for a better visualization.

### III. NONCLASSICALITY CONDITION FROM DGCZ CRITERION

Duan-Giedke-Cirac-Zoller (DGCZ) derived a criterion for detecting the presence of two-mode entanglement \([30]\). This criterion becomes necessary and sufficient for Gaussian bipartite states, similar to Simon-Peres-Horodecki \([27]\). A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied. A bipartite system is inseparable if the condition \([27]\) is satisfied.

We arrange sign(\(c\)) and \(e^{i\phi}\) freely to be able to satisfy Eq. \([32]\) with the minimum requirements for the magnitude of \(\langle \hat{a}\rangle\) on order to be able to obtain a stronger condition. Hence, we obtain the simple condition

\[
\langle \hat{a}^\dagger \hat{a}\rangle < |\langle \hat{a}^2\rangle| \quad (33)
\]

for the presence of nonclassicality. For a Gaussian single-mode state Eq. \([33]\) is both a necessary and sufficient condition.

We note that this condition, Eq. \([33]\), is stronger compared to the one

\[
\langle \hat{a}^\dagger \hat{a}\rangle < |\langle \hat{a}^2\rangle| \quad (34)
\]

which is obtained from Hillery & Zubairy criterion \([36]\) since \(|\langle \hat{a}^2\rangle| \leq |\langle \hat{a}\rangle|^2\) due to Schwartz inequality. This is as it would be expected, since Hillery & Zubairy \([36]\) condition \([\text{Eq. (34)}]\) is only a sufficient condition for nonclassicality.

### IV. SUMMARY AND CONCLUSIONS

The rank of nonclassicality can be defined as the number of terms needed to expand a single-mode states as a superposition of coherent states \([11]-[20]\). This rank is also equal to the rank of two-mode entanglement the single-mode field creates at the output of a beamsplitter. Duan-Giedke-Cirac-Zoller (DGCZ) derived a criterion from Simon-Peres-Horodecki (SPH) \([27]-[29]\) inseparability. We relate the noises of of two output modes to the noise of input single-mode field. By applying the two-mode entanglement measures, we obtain a measure for the rank of single-mode nonclassicality \((E_N)\).

The degree of nonclassicality, obtained from SPH inseparability, increases with the squeezing strength \((r)\) of the single-mode input, see Fig. 4. Degree and emergence of nonclassicality depends on the choice of the phase of the parametrized pump as it is also for two-mode entanglement, see Eq. (13) in Ref. \([37]\) and Refs. \([38-40]\). We also present the behaviour of \(E_N\) near the critical coupling strength for a superradiant phase transition \([23]-[41]\), see Fig. 2. \(E_N\) displays an abrupt change above the critical coupling \(g > g_c\) for phase transition. Above the phase transition, nonclassicality condition \(\lambda_{\text{simon}}\) (not a measure) exhibits a smooth behaviour, unlike \(E_N\). We note that \(E_N\) and \(\lambda_{\text{simon}}\) are necessary conditions only for Gaussian states. Phase transitions change the nature of the quantum states.

In addition to \(E_N\), we derive a simple analytical condition, \(|\langle \hat{a}^2\rangle| > |\langle \hat{a}\rangle|\), for single-mode nonclassicality also from Duan-Giedke-Cirac-Zoller (DGCZ) entanglement criterion \([30]\). This condition is both necessary and sufficient for Gaussian states. We show that this condition is stronger compared to the one deduced \([21]\) from Hillery & Zubairy (HZ) criterion \([36]\), that is \(|\langle \hat{a}\rangle|^2 > |\langle \hat{a}\rangle|\). This is already expected since HZ criterion is only a sufficient condition.

In a forthcoming study \([33]\), we use the introduced nonclassicality measure \([\text{see Eqs. (11), (10) and (19)-(21)}]\) to investigate the nonclassicality of the output field in an optomechanical system. Interestingly, we find that noncausal behaviour of refractive index \(n(\omega)\) – replacing the optomechanical cavity – occurs at the same critical
Following the emergence of nonclassicality, here, \( g_c = 2\sqrt{\frac{\gamma_m}{\gamma_c} \omega_m} \) following the emergence of nonclassicality. Here, \( \gamma_m (\gamma_c) \) is the mechanical (cavity) damping rate and \( \omega_m \) is the resonance of mechanical oscillator.

One shall gather the i) result of the previous paragraph [35] with ii) nonclassicality of a single-mode photon (quasi-particle) field is equivalent to entanglement of 2-level identical particles [21], iii) the mechanism of inseparability is equivalent to a wormhole connecting the entangled particles [10] [46–48], and iv) nonclassical behavior of a refractive index may arise due to superluminal communication [49] of source and potentials (see Eq. (6.42) in Ref. [54], Ref. [55] and Sec. IV in Ref. [35]) even though it may be an uncontrolled one [70]. Hence it becomes natural to raise the question; if nonclassicality is the entanglement of vacuum generating the photons as quasiparticles [58] [59].

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