PARAMETRIZED AND EQUIVARIANT HIGHER ALGEBRA

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ABSTRACT. We develop the rudiments of a theory of parametrized \( \infty \)-operads, including parametrized generalizations of monoidal envelopes, Day convolution, operadic left Kan extensions, results on limits and colimits of algebras, and the symmetric monoidal Yoneda embedding.

Contents

1. Introduction 1
   1.1. Summary of results 2
   1.2. Related work 2
   1.3. Acknowledgements 3
   2. Parametrized \( \infty \)-operads 3
      2.1. First definitions 3
      2.2. Morphisms of operads 7
      2.3. Parametrized Segal condition 8
      2.4. Examples 11
      2.5. \( T \)-operadic nerve 15
      2.6. Model structures 17
      2.7. Big \( T \)-\( \infty \)-operads 19
      2.8. Monoidal envelopes 21
      2.9. Subcategories and localization 22
   3. Parametrized Day convolution 23
      3.1. \( \mathcal{O} \)-promonoidal \( T \)-\( \infty \)-categories and \( T \)-Day convolution 23
      3.2. \( \mathcal{O} \)-monoidality of the \( T \)-Day convolution 27
      3.3. Pointwise \( \mathcal{O} \)-monoidal structure 34
   4. Parametrized operadic left Kan extensions 35
      4.1. Generalized \( T \)-\( \infty \)-operads 36
      4.2. \( T \)-operadic join 37
      4.3. Construction of \( T \)-operadic left Kan extensions 39
   5. \( T \)-\( \infty \)-categories of \( \mathcal{O} \)-algebras 42
      5.1. Parametrized (co)limits in general 43
      5.2. Units and initial objects 45
      5.3. Indexed coproducts in the \( T \)-symmetric monoidal case 49
   6. \( T \)-symmetric monoidal structure on \( T \)-presheaves 55
   References 59

1. Introduction

The goal of this paper is to lay foundations for a theory of parametrized \( \infty \)-operads. To explain the concept, suppose \( G \) is a finite group and let us first recall the concept of a \( G \)-symmetric monoidal \( \infty \)-category, after Hill–Hopkins [HH16], Blumberg–Hill [BH20], and Bachmann–Hoyois [BH21]. Let \( \mathbf{F}_G \) be the category of finite \( G \)-sets, \( \text{Span}(\mathbf{F}_G) \) the \((2,1)\)-category of spans of finite \( G \)-sets, and \( \text{Cat} \) the (huge) \( \infty \)-category of (large) \( \infty \)-categories.
1.0.1. **Definition.** A $G$-symmetric monoidal $\infty$-category is a product-preserving functor
\[ \mathcal{E}^G : \text{Span}(\mathbf{F}_G) \rightarrow \text{Cat}. \]

For example, the $\infty$-category $\text{Sp}^G$ of genuine $G$-spectra extends to a $G$-symmetric monoidal $\infty$-category $(\text{Sp}^G)^\otimes$ whose value on $G/H$ is equivalent to $\text{Sp}^H$ and whose covariant functoriality encodes the symmetric monoidal structures on $\{\text{Sp}^H\}_{H \leq G}$ as well as the Hill–Hopkins–Ravenel norm functors $f_\otimes : \text{Sp}^H \rightarrow \text{Sp}^K$ associated to maps of $G$-orbits $f : G/H \rightarrow G/K$ (cf. [BH21, §9]). More generally, one can substitute other base $\infty$-categories apart from $\mathbf{F}_G$ as needed for other applications; in particular, in the motivic context Bachmann and Hoyois work with spans over certain categories of schemes and have extensively investigated the properties of such normed symmetric monoidal $\infty$-categories and their algebras in [BH21].

Just as the theory of symmetric monoidal $\infty$-categories admits a generalization to a theory of $\infty$-operads, we will see that the theory of $G$-symmetric monoidal $\infty$-categories admits a corresponding sort of generalization. Roughly speaking, a simplicial $G$-operad should consist of the data of a space of multimorphisms associated to every map of finite $G$-sets, with a composition law then associated to every composite of maps of finite $G$-sets.\(^1\) In fact, just as an $\infty$-operad is really the $\infty$-categorical counterpart of a simplicial colored operad (i.e., a simplicial multicategory), our theory of $G$-$\infty$-opers will encompass both $G$-symmetric monoidal $\infty$-categories and simplicial colored $G$-operads via a suitably defined coherent nerve construction. Abstractioning away from the equivariant situation, we will be able to make this idea work under the following hypotheses on our base $\infty$-category, which were first articulated in the first author’s work [Nar16] on parametrized stability.

1.0.2. **Definition** ([Nar16, Def. 4.1]). Let $\mathcal{T}$ be a small $\infty$-category. We say that $\mathcal{T}$ is orbital if its finite coproduct completion admits all pullbacks. We say that $\mathcal{T}$ is atomic if it has no non-trivial retracts, so that every map with a left inverse is an equivalence.

1.0.3. **Example.** The orbit category $\mathbf{O}_G$ of a finite group is atomic orbital. Some other examples are enumerated in [Nar16, Ex. 4.2].

1.0.4. **Remark.** The condition for an $\infty$-category to be atomic orbital is a highly restrictive one; for example, if $\mathcal{T}$ is atomic orbital and admits a terminal object, then $\mathcal{T}$ is equivalent to the nerve of a $1$-category $T$ (Proposition 2.5.1).

At this point, the reader should examine the definition of a simplicial colored $T$-operad (Definition 2.5.4) to get a conceptual handle on the forthcoming definition of a $\mathcal{T}$-$\infty$-operad.

1.1. **Summary of results.** After some preliminaries on the $\mathcal{T}$-$\infty$-category $\mathcal{E}_{\mathcal{T},*}$ of pointed finite $\mathcal{T}$-sets (Definition 2.1.2), we give the definition of $\mathcal{T}$-$\infty$-operad as Definition 2.1.7 and algebras therein as Definition 2.2.1. We explicate the parametrized Segal condition (Theorem 2.3.3) and show how the definition of a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category recovers Definition 1.0.1 (Theorem 2.3.9). We then study parametrized generalizations of three essential constructions in the theory of $\infty$-operads: monoidal envelopes (Definition 2.8.4), Day convolution (Definition 3.1.6), and operadic left Kan extension (Definition 4.3.5). Finally, we study $\mathcal{T}$-(co)limits in $\mathcal{T}$-$\infty$-categories of $\mathcal{O}$-algebras, first in the context of a general $\mathcal{T}$-$\infty$-operad $\mathcal{O}^\otimes$ (Theorem 5.1.3 and Theorem 5.1.4) and then in the special case of the $\mathcal{O}^\otimes = \mathcal{E}_{\mathcal{T},*}$ (Theorem 5.3.7), and we establish a $\mathcal{T}$-symmetric monoidal refinement of the universal property of $\mathcal{T}$-presheaves (Corollary 6.0.12).

1.2. **Related work.** This paper is part of a larger body of work on parametrized higher category theory and higher algebra [BDG+16a, BDG+16b, Sha21a, Sha21b, Nar16, Nar17]. In particular, all of the conventions, terminology, and notation from [Sha21b] are in force in this paper, and the reader should at least skim the introduction and §2 of [Sha21b] before reading this work. Furthermore, the definition of a $\mathcal{T}$-$\infty$-operad was developed in joint work with Barwick, Dotto, and Glasman circa 2016 and has previously appeared in the first author’s thesis [Nar17, §3.1]. On the other hand, this paper doesn’t otherwise expand on [Nar17, §3]; for instance, we will not recapitulate the first author’s work on tensor products of $\mathcal{T}$-presentable $\mathcal{T}$-$\infty$-categories.

As this paper is intended to play a foundational and supporting role in the literature, we don’t discuss many interesting examples or applications here. Horev [Hor19] has used these results in his development of a theory of genuine equivariant factorization homology (see also [HHK +20]), and he in particular discusses the example of the $G$-$\infty$-operad $\mathcal{E}_V$ associated to a finite-dimensional real $G$-representation $V$. The second

\(^1\)Beware that this isn’t the notion of $G$-operad that appears in the work of Blumberg-Hill [BH15].
author and Quigley have applied these results in their study of the parametrized Tate construction [QS21a] and real cyclotomic spectra [QS21b]. Hilman has introduced similar ideas in his study of parametrized

In a different direction, the theory of $G$-operads in their various guises has a long history that we don’t attempt to summarize here; some recent references are [BH15, GW18, Rub21, BP21, MMO21, GMMO18]. In terms of the relationship to the $N\infty$-operads of Blumberg–Hill, we discuss $I$-indexing systems $I$ in our framework in Definition 2.4.8, the corresponding commutative $I\infty$-operad $\com I$ in Definition 2.4.10, and how they identify with $G$-indexing systems in the sense of Blumberg–Hill when $I = O_G$ in Remark 2.4.12. It should be possible to adapt ideas of Hinich from [Hin15] to establish a formal comparison between the $\infty$-category of $\com I$-algebras in our sense and those in the sense of [BH15], but we do not attempt to do this now.

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2. Parametrized $\infty$-Operads

2.1. First definitions. We begin by introducing the basic definitions of parametrized higher algebra in parallel to Lurie’s development of the foundations of $\infty$-operads [Lur17, §2.1]. Let $I$ be an atomic orbital $\infty$-category, whose objects we refer to as orbits, and let $F_I$ be its finite coproduct completion, which we refer to as the $\infty$-category of finite $I$-sets.

2.1.1. Definition. For every orbit $V \in I$, let

$$F_{I/V} := (F_I)^V = \operatorname{Ar}(F_I) \times_{F_I} \{V\}$$

(thus fixing a preferred choice of finite coproduct completion of $I^V$), and let

$$F_{I/V,*} := (F_{I/V})^{id_V/} = (F_I)^{V/V}$$

be the $\infty$-category of finite pointed $I^V$-sets.

Using that $I$ is orbital, we could then define the $I\infty$-category of finite $I$-sets $\mathcal{F}_I$ as the full $I$-subcategory of $\spc I$ spanned by the finite $I^V$-sets in each fiber $(\spc I)_V$ over an orbit $V$, so that as a cocartesian fibration, $\mathcal{F}_I$ is classified by the assignment $V \mapsto (F_I)^V$ with functoriality that given by pullback. Similarly, we could define a pointed variant $\mathcal{F}_{I,*}$ as the full $I$-subcategory of $\spc I$ classified by the assignment $V \mapsto (F_I)^{V/V}$.

However, although conceptually transparent, these definitions of $\mathcal{F}_I$ and $\mathcal{F}_{I,*}$ are ill-suited to writing down arbitrary morphisms that may interpolate between different fibers. Instead, we will follow the first author’s work in [Nar16, §4] and instead define $\mathcal{F}_I$ and $\mathcal{F}_{I,*}$ as certain $\infty$-categories of spans, along the lines of the construction of the dual cocartesian fibration in [BGN18] as well as the span description of finite pointed sets in terms of finite sets and partially defined maps (cf. [Nar16, 4.11]).

2.1.2. Definition. Let

$$\mathcal{E}_I := \operatorname{Ar}(F_I) \times_{F_I} I,$$

so that the functor $ev_1 : \mathcal{E}_I \rightarrow I$ given by evaluation at the target is a cartesian fibration classified by $V \mapsto (F_I)^V$. Labeling an arbitrary morphism $[\phi : f \rightarrow g]$ of $\mathcal{E}_I$ as

$$U \xrightarrow{h} X \xleftarrow{f} Y \xrightarrow{g} V,$$

we define wide subcategories

$$((\mathcal{E}_I)^\text{ideg}, (\mathcal{E}_I)^\text{st}, (\mathcal{E}_I)^\text{cart}) \subset \mathcal{E}_I^V$$

Explicitly, we could take $F_I \subset P(I)$ to be the full subcategory spanned by finite coproducts of representables. However, any equivalent choice will suffice.
as containing those morphisms $\phi$ such that $k$ is degenerate, $U \to X \times_Y V$ is a summand inclusion, and $U \to X \times_Y V$ is an equivalence, respectively.\(^3\) Then the triples
\[
(F^V_r; (F^V_r)^{cart}, (F^V_r)^{tdeg}) \quad \text{and} \quad (F^V_r; (F^V_r)^{si}, (F^V_r)^{tdeg})
\]
are adequate in the sense of [Bar17, 5.2].\(^4\) Consequently, we may form the associated span $\infty$-categories\(^5\)
\[
F_T := \text{Span}(F^V_r; (F^V_r)^{cart}, (F^V_r)^{tdeg}) \quad \text{and} \quad F_{T, *} := \text{Span}(F^V_r; (F^V_r)^{si}, (F^V_r)^{tdeg}).
\]

We regard $F_T$ and $F_{T, *}$ as $T$-$\infty$-categories via the structure map $ev_1$ given by evaluation at the target, so that a morphism $\psi$ (for either $\infty$-category)
\[
\begin{array}{c}
U \leftarrow Z \\
\downarrow \ \\
V \leftarrow Y
\end{array}
\]

is $ev_1$-cocartesian if and only if $m : Z \to X$ is an equivalence and $Z \to U \times_Y V$ is an equivalence (cf. [Nar16, Lem. 4.9 and Def. 4.12]). The canonical inclusion $F_T \subset F_{T, *}$ of span $\infty$-categories is thus the inclusion of a $T$-subcategory. We also have an ‘identity’ cocartesian section $I : T^{op} \to F_T$ that sends $V$ to $[V = V]$.

2.1.3. Definition. In the notation of Definition 2.1.2, we declare a morphism $\psi$ in $F_{T, *}$ to be inert if $m : Z \to X$ is an equivalence and active if $Z \to U \times_Y V$ is an equivalence. Note that a morphism $\psi$ in $F_{T, *}$ is both inert and active if and only if $\psi$ is $ev_1$-cocartesian.

2.1.4. Remark. Note that $F_T$ is by definition the dual cocartesian fibration to $F^V_T$ in the sense of [BGN18, Def. 3.4]. As such, for any orbit $V$ we have an equivalence
\[
(F^V_T)_V = F_{T/V} \xrightarrow{\cong} \text{Span}(F_{T/V}; (F_{T/V})^\cong, F_{T/V}) = (F_T)_V
\]
implemented by inclusion.

We next describe $F_{T, *}$, Let $F^{si}_T$ denote the wide subcategory on the summand inclusions in $F_T$, so $(F^V_T)_V = F^{si}_{T/V}$. As was noted in [Nar16, Lem. 4.14], for any orbit $V$ we have an equivalence
\[
(F_{T, *})_V = \text{Span}(F_{T/V}; F^{si}_{T/V}, F_{T/V}) \xrightarrow{\cong} F_{T/V,*},
\]
under which an object $[U \xleftarrow{\alpha} V]$ is sent to $[U \cup V \xrightarrow{f \cup id} V]$ pointed at $V$, and a span
\[
U \leftarrow W \xrightarrow{\beta} U',
\]
with $\alpha$ given by the summand inclusion $W \subset W \cup W' \cong U$, is sent to the pointed map
\[
U \cup V \xrightarrow{\gamma} U' \cup V
\]
with $\gamma|_W = \beta$ and $\gamma|_{W'} = \text{const}_V$. Consequently, we will often refer to an object $f = [U \to V]$ of $F_{T, *}$ as $f^+ = [U^+ \to V]$ to emphasize the implicit presence of the basepoint. We will also denote the canonical inclusion $F_T \subset F_{T, *}$ of span $\infty$-categories by
\[
(-)_+ : F_T \hookrightarrow F_{T, *}
\]
and refer to this as the pointing $T$-functor. By [Nar16, Lem. 4.14], $( - )_+$ has a ‘forgetful’ right $T$-adjoint which sends $[U^+ \to V]$ to $[U \cup V \to V]$. Note also that a morphism $\psi$ in $F_{T, *}$ is active if and only if it is in the image of $( - )_+$.

2.1.5. Remark. For an orbit $V \in T$, we obtain from Definition 2.1.3 a definition for inert and active edges in $(F_{T, *})_V$ by restriction to the fiber. Under the equivalence $(F_{T, *})_V \simeq F_{T/V,*}$, of Remark 2.1.4, a pointed map $f : U \cup V \to U' \cup V$ is then inert if and only if its pullback along $U' \subset U' \cup V$ is an equivalence, and is active if and only if $f \simeq g_+$ for some $g : U \to U'$ in $F_{T/V}$.

\(^3\)Note that the “target degenerate” morphisms are a subclass of $ev_1$-cocartesian edges (for which more generally the map $k$ is an equivalence) and the “cartesian” morphisms are exactly the $ev_1$-cartesian edges.

\(^4\)Note that we have swapped the order of the wide subcategories from that of [Bar17], so that the first subcategory will indicate the backward facing arrows and the second will indicate the forward facing arrows when forming the span $\infty$-category.

\(^5\)In [Bar17] and [Nar16], the term “effective Burnside $\infty$-category” $A^{eff}$ is used as a synonym for “span $\infty$-category”.}
2.1.6. **Definition.** Suppose \( f_+ = [U_+ \to V] \) is an object of \( \mathbf{E}_\mathcal{T},_* \). Let \( \text{Orbit}(U) \) be the set of orbits of \( U \), so that we have an equivalence
\[
U \simeq \prod_{W \in \text{Orbit}(U)} W
\]
in \( \mathbf{F}_\mathcal{T} \) with each \( W \) an object in \( \mathcal{T} \). Given \( W \in \text{Orbit}(U) \), the **characteristic morphism**
\[
\chi_{[W \subset U]} : f_+ \to I(W)_+
\]
is defined to be
\[
\begin{align*}
U & \ni W \xrightarrow{=} W \\
V & \ni W \xrightarrow{=} W.
\end{align*}
\]
Here we make essential use of our assumption that \( \mathcal{T} \) is atomic to ensure that \( W \to U \times_V W \) is a summand inclusion. Clearly, \( \chi_{[W \subset U]} \) is inert.

2.1.7. **Definition.** A \( \mathcal{T}-\infty \)-operad is a pair \((\mathcal{C}^\otimes, p)\) consisting of a \( \mathcal{T}-\infty \)-category \( \mathcal{C}^\otimes \) along with a \( \mathcal{T} \)-functor \( p : \mathcal{C}^\otimes \to \mathbf{E}_\mathcal{T},_* \), which is a categorical fibration and satisfies the following additional conditions:

1. For every inert morphism \( \psi : f_+ \to g_+ \) of \( \mathbf{E}_\mathcal{T},_* \) and every object \( x \in \mathcal{C}^\otimes_{f_+} \), there is a \( p \)-cocartesian edge \( x \to y \) in \( \mathcal{C}^\otimes \) covering \( \psi \).
2. For any object \( f_+ = [U_+ \to V] \) of \( \mathbf{E}_\mathcal{T},_* \), the \( p \)-cocartesian edges lying over the characteristic morphisms
\[
\{ \chi_{[W \subset U]} : f_+ \to I(W)_+ \ | \ W \in \text{Orbit}(U) \}
\]
together induce an equivalence
\[
\prod_{W \in \text{Orbit}(U)} (\chi_{[W \subset U]})_! : \mathcal{C}^\otimes_{f_+} \xrightarrow{\simeq} \prod_{W \in \text{Orbit}(U)} \mathcal{C}^\otimes_{I(W)_+},
\]
3. For any morphism
\[
\psi : f_+ = [U_+ \to V] \to g_+ = [U'_+ \to V']
\]
of \( \mathbf{E}_\mathcal{T},_* \), objects \( x \in \mathcal{C}^\otimes_{f_+} \) and \( y \in \mathcal{C}^\otimes_{g_+} \), and any choice of \( p \)-cocartesian edges
\[
\{ y \to y_W \ | \ W \in \text{Orbit}(U') \}
\]
lying over the characteristic morphisms
\[
\{ \chi_{[W \subset U']} : g_+ \to I(W)_+ \ | \ W \in \text{Orbit}(U') \},
\]
the induced map
\[
\text{Map}^\psi_{\mathcal{C}^\otimes}(x, y) \xrightarrow{\simeq} \prod_{W \in \text{Orbit}(U')} \text{Map}^{\chi_{[W \subset U'] \otimes \psi}}_{\mathcal{C}^\otimes}(x, y_W)
\]
is an equivalence.

We will typically omit the structure map \( p \) and simply refer to \( \mathcal{C}^\otimes \) as a \( \mathcal{T}-\infty \)-operad. Given a \( \mathcal{T}-\infty \)-operad \( \mathcal{C}^\otimes \), its **underlying \( \mathcal{T}-\infty \)-category** is the fiber product
\[
\mathcal{C} := \mathcal{T}^{\text{op}} \times_{I(-), \mathcal{E}_\mathcal{T},_*} \mathcal{C}^\otimes.
\]

2.1.8. **Definition.** Suppose \((\mathcal{C}^\otimes, p)\) is a \( \mathcal{T}-\infty \)-operad. Then an edge of \( \mathcal{C}^\otimes \) is **inert** if it is \( p \)-cocartesian over an inert edge of \( \mathbf{E}_\mathcal{T},_* \), and is **active** if it factors as a \( p \)-cocartesian edge followed by an edge lying over a fiberwise active edge in \( \mathbf{E}_\mathcal{T},_* \). We let \( \mathcal{C}^\otimes_{\text{inert}} \) be the wide \( \mathcal{T} \)-subcategory of \( \mathcal{C}^\otimes \) on the inert edges, and \( \mathcal{C}^\otimes_{\text{active}} \) the wide \( \mathcal{T} \)-subcategory of \( \mathcal{C}^\otimes \) on the active edges.

2.1.9. **Remark.** Let \( \mathcal{C}^\otimes \) be a \( \mathcal{T}-\infty \)-operad and \( U \in \mathbf{F}_\mathcal{T} \). Note that for any orbit \( V \) and morphism \( f : U \to V \), we have an equivalence
\[
\mathcal{C}^\otimes_{f_+} \simeq \mathcal{C}_U = \prod_{W \in \text{Orbit}(U)} \mathcal{C}_W.
\]
We will often write objects \( x \in \mathcal{C}^\otimes_{f_+} \) as tuples \( (x_W) \).
2.1.10. **Remark** (Simplified condition on mapping spaces). In Definition 2.1.7, in view of the inert-fiberwise active factorization system on a $\mathcal{T}$-$\infty$-operad (Example 2.8.1) we may replace (3) by the following apparently weaker condition:

$(3')$ Let $\alpha : [U \xrightarrow{f} V] \to [U' \xrightarrow{\alpha_+} V]$ be a morphism in $F_{\mathcal{T}, V}$, which defines an active edge $\alpha_+$ in $(\mathcal{E}_{\mathcal{T}, V})$. Let $x \in \mathcal{C}_{i,x}^{\oplus}$, $y \in \mathcal{C}_{y}$ be objects, and for each $W \in \text{Orbit}(U')$ let $y \to y_W$ be a p-cocartesian edge lifting the characteristic morphism $\chi_{[W \subset U']}$. For every $W \in \text{Orbit}(U')$ we have a commutative square

$$
\begin{array}{c}
[U_+ \to V] \xrightarrow{\alpha_+} [U'_+ \to V] \\
\downarrow^{\rho_W} \quad \quad \quad \downarrow^{\chi_{[W \subset U']}} \\
[(U \times_{U'}, W)_+ \to W] \xrightarrow{(\alpha_W)_+} [W_+ \to W]
\end{array}
$$

where $\rho_W$ is the inert edge corresponding to the summand inclusion $U \times_{U'} W \to U \times_{U'} W$ and $\alpha_W : U \times_{U'} W \to W$ is the pullback of $\alpha : U \to U'$ along $W \subset U'$. (Note that the lower composition is the inert-fiberwise active factorization of the upper composition $\chi_{[W \subset U']} \circ \alpha_+$.) Let

$$
\{x \to x_W \mid W \in \text{Orbit}(U')\}
$$

be any choice of p-cocartesian edges lying over the morphisms $\rho_W$. Then the induced map

$$
\text{Map}_{\mathcal{C}_{\oplus}}^{\alpha_+}(x, y) \xrightarrow{\sim} \prod_{W \in \text{Orbit}(U')} \text{Map}_{\mathcal{C}_{\oplus}}^{(\alpha_W)_+}(x_W, y_W)
$$

is an equivalence.

2.1.11. **Remark** (Spaces of multimorphisms and operadic composition). Suppose $\mathcal{C}$ is a $\mathcal{T}$-$\infty$-operad, $\alpha : U \to U'$ is a morphism in $F_{\mathcal{T}}$, and $x \in \mathcal{C}_U$, $y \in \mathcal{C}_{U'}$ are tuples of objects in $\mathcal{C}$. For every $W \in \text{Orbit}(U')$, let

$$
\alpha_W : U_W = U \times_{U'} W \to W
$$

be the pullback of $\alpha$ along the summand inclusion $W \subset U'$. Consider the component $y_W$ of $y$ as an object in $\mathcal{C}_{(W)_+}$ and the sub-tuple $x_W \in \mathcal{C}_{U_W}$ of $x$ as an object in $\mathcal{C}_{(\alpha_W)_+}$. Let

$$
\text{Mul}_C^\alpha(x, y) := \prod_{W \in \text{Orbit}(U')} \text{Map}_{\mathcal{C}_{\oplus}}^{(\alpha_W)_+}(x_W, y_W)
$$

be the space of $(\alpha; x, y)$-multimorphisms encoded by $\mathcal{C}$. Then for any choice of map $U' \to V$ in $F_{\mathcal{T}}$ down to an orbit $V$, we have the canonical equivalence

$$
\text{Map}_{\mathcal{C}_{\oplus}}^{\alpha_+}(x, y) \simeq \text{Mul}_C^\alpha(x, y)
$$

of (2.1.1) (compare Remark 2.1.9).

These spaces of multimorphisms are interrelated by the structure of $\mathcal{C}_{\oplus}$. For instance, for every composite morphism $U_0 \xrightarrow{\alpha_1} U_1 \xrightarrow{\alpha_2} U_2$ in $F_{\mathcal{T}}$ and $x_i \in \mathcal{C}_{U_i}$, $i \in \{0, 1, 2\}$, any choice of map $\rho : U_2 \to V$ to an orbit $V$ yields a map

$$
\circ : \text{Mul}_C^\alpha(x_0, x_1) \times \text{Mul}_C^\beta(x_1, x_2) \to \text{Mul}_C^{\alpha_2}(x_0, x_2)
$$

defined by the composition in $\mathcal{C}_{\oplus}$, and one may check that this map is independent of the choice of $\rho$. Likewise, for every composition of pullback squares in $F_{\mathcal{T}}$

$$
\begin{array}{ccc}
X & \xrightarrow{f_\alpha} & X' \\
\Downarrow & & \Downarrow \\
U & \xrightarrow{\alpha} & U'
\end{array} \quad \quad \begin{array}{ccc}
& & W \\
\Downarrow & & \Downarrow \\
& & f
\end{array}
$$

with $V, W$ orbits, and objects $x \in \mathcal{C}_U$, $y \in \mathcal{C}_{U'}$, one has a base-change map

$$
f^* : \text{Mul}_C^\alpha(x, y) \to \text{Mul}_C^{f_\alpha}(f^* x, f^* y)
$$

induced by the cocartesian pushforward in $\mathcal{C}_{\oplus}$ along $f$ in $\mathcal{T}_{\text{op}}$. Note that these maps extend the functoriality on the underlying $\mathcal{T}$-$\infty$-category $\mathcal{C}$.

Altogether, these maps satisfy homotopy coherent unitality, associativity, and base-change compatibility constraints as encapsulated by $\mathcal{C}_{\oplus}$. 

2.2. Morphisms of operads. We next introduce morphisms of $\mathcal{T}$-$\infty$-operads and algebras over $\mathcal{T}$-$\infty$-operads.

2.2.1. Definition. Suppose $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ are two $\mathcal{T}$-$\infty$-operads. A morphism of $\mathcal{T}$-$\infty$-operads is a $\mathcal{T}$-functor $A : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ over $\mathbf{F}_\mathcal{T}$, that carries inert morphisms to inert morphisms. Conceptually, $A$ is a $\mathcal{C}$-algebra valued in $\mathcal{D}$.

If $A$ is moreover a categorical fibration, then we call $A$ a fibration of $\mathcal{T}$-$\infty$-operads. Given fibrations of $\mathcal{T}$-$\infty$-operads $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ and $q : \mathcal{D}^\otimes \to \mathcal{O}^\otimes$, we let

$$\mathbf{Alg}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$$

denote the full subcategory of $\mathbf{Fun}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by the morphisms of $\mathcal{T}$-$\infty$-operads, and

$$\mathbf{Alg}_{\mathcal{T}}^\ast(\mathcal{C}, \mathcal{D})$$

the corresponding full $\mathcal{T}$-subcategory of the $\mathcal{T}$-$\infty$-category $\mathbf{Fun}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ [Sha21b, Notn. 4.7].

If $p$ is the identity on $\mathcal{O}^\otimes$, then we will also denote $\mathbf{Alg}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{Alg}_{\mathcal{T}}(\mathcal{D})$. If $\mathcal{O}^\otimes = \mathbf{F}_\mathcal{T}$, then we will also denote $\mathbf{Alg}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{Alg}_{\mathcal{T}}(\mathcal{D})$. Combining these two cases, if $\mathcal{C}^\otimes = \mathcal{O}^\otimes = \mathbf{F}_\mathcal{T}$, then we will also denote $\mathbf{Alg}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ as $\mathbf{CAlg}_{\mathcal{T}}(\mathcal{D})$, the $\mathcal{T}$-category of $\mathcal{T}$-commutative algebras in $\mathcal{D}$.

2.2.2. Warning. In the case $\mathcal{T} = \ast$, our notation for $\mathcal{O}$-categories of algebras conflicts with that of Lurie in [Lur17, Def. 2.1.3.1].

2.2.3. Definition. Suppose $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a fibration of $\mathcal{T}$-$\infty$-operads in which $p$ is moreover a cocartesian fibration. In this case, we call $\mathcal{C}^\otimes$ a $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category. If $\mathcal{O}^\otimes = \mathbf{F}_\mathcal{T}$, we also call $\mathcal{C}^\otimes$ a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category. We also refer to $\mathcal{C}$ as $\mathcal{O}$-monoidal if the additional structure $(\mathcal{C}^\otimes, p)$ is understood from context.

2.2.4. Notation. Let $\mathcal{C}^\otimes$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category. For an active morphism $f : x \to y$ in $\mathcal{O}^\otimes$, we typically denote the cocartesian pushforward functor associated to $f$ by $f_\otimes : \mathcal{C}_x^\otimes \to \mathcal{C}_y^\otimes$ and refer to it as the norm functor for $f$. If $\mathcal{O}^\otimes = \mathbf{F}_\mathcal{T}$, then for any morphism $f : U \to V$ of finite $\mathcal{T}$-sets with $V$ an orbit, we have a norm functor $f_\otimes : \mathcal{C}_U \to \mathcal{C}_V$ associated to $f_\ast : [U_\ast] \to [V_\ast]$.

More generally, if $V$ is a finite $\mathcal{T}$-set with orbit decomposition $\bigsqcup_{i=1}^n V_i$ so that $f = \bigsqcup_{i=1}^n (f_i : U_i \to V_i)$, then we let $f_\otimes$ be the product of the functors $\{(f_i)_\otimes\}_{i=1}^n$. (We will also describe in Section 2.7 how to dispense with the orbit restriction in the formalism by passing to ‘big’ $\mathcal{T}$-$\infty$-operads.)

2.2.5. Definition. Given two $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-categories $p, q : \mathcal{C}^\otimes, \mathcal{D}^\otimes \to \mathcal{O}^\otimes$, a $\mathcal{T}$-functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is lax $\mathcal{O}$-monoidal if it is a morphism of $\mathcal{T}$-operads, and is (strict) $\mathcal{O}$-monoidal if it carries $p$-cocartesian edges to $q$-cocartesian edges. We let

$$\mathbf{Fun}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

the corresponding $\mathcal{T}$-subcategory of $\mathbf{Fun}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by the $\mathcal{O}$-monoidal $\mathcal{T}$-functors, and

$$\mathbf{Fun}_{/\mathcal{O}^\otimes}^\ast(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

and speak of lax and strict $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-functors.

In the situation of a cocartesian fibration $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ over a $\mathcal{T}$-$\infty$-operad $\mathcal{O}^\otimes$, we have the following simplification of the conditions for $\mathcal{C}^\otimes$ to be a $\mathcal{T}$-$\infty$-operad and hence $\mathcal{O}$-monoidal.

2.2.6. Proposition. Let $(\mathcal{O}^\otimes, q)$ be a $\mathcal{T}$-$\infty$-operad and let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of $\mathcal{T}$-$\infty$-categories. Then $(\mathcal{C}^\otimes, q \circ p)$ is a $\mathcal{T}$-$\infty$-operad if and only if for every $f_\ast : [U_\ast] \to [V_\ast]$ in $\mathbf{F}_\mathcal{T}$, and $x \in \mathcal{O}^\otimes_{f_\ast}$, the inert edges $\{x \to x_W \mid W \in \text{Orbit}(U)\}$ in $\mathcal{O}^\otimes$ together induce an equivalence

$$\mathcal{C}_x^\otimes \simeq \bigsqcup_{W \in \text{Orbit}(U)} \mathcal{C}_{x_W}.$$
\begin{proof}
The proof is exactly analogous to that of [Lur17, Prop. 2.1.2.12], so we will omit it. \qedhere
\end{proof}

Lastly, we state the evident notions of \(T\)-suboperad and 0-monoidal \(T\)-subcategory.

2.2.7. \textbf{Definition.} Let \((C^\otimes, p)\) be a \(T\)-\(\infty\)-operad and let \(D^\otimes\) be a \(T\)-subcategory of \(C^\otimes\) with inclusion \(T\)-functor \(i\). We say that \(D^\otimes\) is a \(T\)-\textit{suboperad} of \(C^\otimes\) if \(p \circ i\) exhibits \(D^\otimes\) as a \(T\)-\(\infty\)-operad and \(i\) is a morphism of \(T\)-\(\infty\)-operads. If \(C^\otimes\) is moreover an 0-monoidal \(T\)-\(\infty\)-category via \(q : C^\otimes \to O^\otimes\), then \(D^\otimes\) is a \(0\)-\textit{monoidal} \(T\)-\textit{subcategory} of \(C^\otimes\) if \(D^\otimes \subset C^\otimes\) is stable under \(q\)-cocartesian edges, so that \(q \circ i\) exhibits \(D^\otimes\) as an 0-monoidal \(T\)-\(\infty\)-category and \(i\) is an 0-monoidal functor.

2.3. \textbf{Parametrized Segal condition.} We next want to interpret condition (2) of Definition 2.1.7 as an equivalence of \(T/V/\infty\)-categories (i.e., a \(T\)-Segal condition). First, we extend our notation for the \(T\)-fibers of \(\mathcal{T}\).

2.3.1. \textbf{Notation.} Let \(F : X \to C\) be a \(T\)-functor and let \(\sigma : \Delta^n \to C\) be a \(n\)-simplex of \(C\). Define the \(T\)-fiber of \(X\) over \(\sigma\) to be

\[
\mathcal{X}_\sigma := \Delta^n \times_{\sigma, C, ev_0} \text{Ar}^{\text{cocart}}(C) \times_{\text{ev}_1, C, F} X.
\]

2.3.2. \textbf{Construction.} For any \(T\)-\(\infty\)-category \(X\) and edge \(f : x \to y\) in \(X\), we can construct a \(T\)-functor

\[
\phi : \Delta^1 \times y = \Delta^1 \times (\Delta^0 \times y, x, ev_0 \text{Ar}^{\text{cocart}}(X)) \to \Delta^1 \times f, x, ev_0 \text{Ar}^{\text{cocart}}(X)
\]

which fits into the commutative diagram

\[
\begin{array}{ccc}
\{0\} \times y & \xrightarrow{\phi} & \Delta^1 \times y \\
\downarrow & & \downarrow \\
\Delta^1 \times y & \xrightarrow{\phi} & \Delta^1 \times f, x, ev_0 \text{Ar}^{\text{cocart}}(X) \\
\{1\} \times y & \xrightarrow{=} & \{1\} \times y
\end{array}
\]

where \(f^* : y \to x\) is the \(T\)-functor defined in [Sha21a, Rem. 12.11], which sends a cocartesian edge \([e : y \to z]\) to the cocartesian edge \([f^*(e) : x \to z']\) given by the factorization of \([e \circ f]\) as the composite of \(f^*(e)\) and a fiberwise edge \(\phi(e)_1\).

Explicitly, let \(h : \Delta^1 \times \Delta^1 \to X\) be given by

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & = & \downarrow \\
y & \xrightarrow{f} & y
\end{array}
\]

and let

\[
\mathcal{M} = \Delta^1 \times_{h, \text{Fun}^\Delta_1(\Delta^1 \times \Delta^1, X \times \Delta^1), ev_0} \text{Fun}^\Delta_1(\Delta^1 \times \Delta^1, \text{Ar}^{\text{cocart}}(X) \times \Delta^1) \times_{\text{ev}_1, \text{Fun}^\Delta_1(\Delta^1 \times \Delta^1, \text{Top}^\Delta \times \Delta^1), I} (\text{Top}^\Delta \times \Delta^1),
\]

where \(I\) denotes the identity section, so that \(\mathcal{M} \to \Delta^1 \times \text{Top}^\Delta\) is a \(T\)-correspondence with

\[
\begin{align*}
\mathcal{M}_0 &= \{0\} \times_f \text{Ar}(X), ev_0 \text{Ar}(\text{Ar}^{\text{cocart}}(X)) \times_{\text{ev}_1, \text{Ar}(\text{Top}), I} \text{Top}^\Delta, \\
\mathcal{M}_1 &= \{1\} \times_{\text{id}_y, \text{Ar}(X), ev_0 \text{Ar}(\text{Ar}^{\text{cocart}}(X)) \times_{\text{ev}_1, \text{Ar}(\text{Top}), I} \text{Top}^\Delta.
\end{align*}
\]

We have a zig-zag of \(T\)-functors over \(\Delta^1 \times \text{Top}^\Delta\)

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\pi} & \mathcal{M} \\
\xrightarrow{\rho} \Delta^1 \times f, x, ev_0 \text{Ar}^{\text{cocart}}(X) & \xrightarrow{\rho} & \mathcal{M}
\end{array}
\]

where \(\pi\) restricts to the trivial fibrations \(\mathcal{M}_0 \to y \times \{0\}\), \(\mathcal{M}_1 \to y \times \{1\}\) of [Sha21a, Lem. 12.10] and \(\rho\) restricts to \(\mathcal{M}_0 \to \mathcal{M}_1, \mathcal{M}_1 \to \mathcal{M}\). Thus \(\pi\) is a trivial fibration and we may choose a section \(\tau\) which fixes \(\mathcal{M} \times \{1\} \subset \mathcal{M}_1\). Then we let \(\phi = \rho \circ \tau\).
2.3.3. **Theorem.** Let $\mathcal{C} \to \mathcal{O}$ be a fibration of $\mathcal{T}$-$\infty$-operads. Let $x \in \mathcal{O}$ be an object over $[f : U \to V] \in \mathcal{F}_{U \to V}$. Let $U \simeq U_1 \coprod \cdots \coprod U_n$ be an orbit decomposition, let $f_i : U_i \to V$ denote the induced morphisms, and let $e_i : x \to x_i$ be inert edges in $\mathcal{O}$ lifting the characteristic morphisms $\chi_{[U_i \subset U]}$. Then we have an equivalence of $\mathcal{T}/V$-$\infty$-categories

$$
\mathcal{C} \cong \prod_{f} \left( \prod_{1 \leq i \leq n} e_{x_i} \right) \cong \prod_{1 \leq i \leq n} \left( \prod_{f_i} e_{x_i} \right). \tag{8}
$$

**Proof.** Let $h_i : \Delta^1 \times x_i \to \Delta^1 \times e_i \circ \ar \text{cocart}(\mathcal{O}) \to \mathcal{O}$ be the homotopy associated to the edge $e_i$ as defined in Construction 2.3.2. Because $h_i$ lands in $\mathcal{O}_{nc}^\circ$, the pullback $\mathcal{C} \times_{\mathcal{O}} (\Delta^1 \times x_i) \to \Delta^1 \times U_i$ is a cocartesian fibration. This corresponds to a $\mathcal{T}/U_i$-functor $\rho^i : f_i^* (\mathcal{C}^\circ) \to \mathcal{C}_{x_i}$. Taking the coproduct of the $\rho^i$ and taking the adjoint of that, we get a comparison $\mathcal{T}/V$-functor

$$
\rho : \mathcal{C} \to \prod_{f} \left( \prod_{1 \leq i \leq n} e_{x_i} \right).
$$

We claim that $\rho$ is a equivalence of $\mathcal{T}/V$-$\infty$-categories. We will check that for every object $[g : V' \to V]$, the fiber $\rho_g$ is an equivalence. Consider the pullback square

$$
\begin{array}{ccc}
U' & \xrightarrow{g'} & U \\
\downarrow f' & & \downarrow f \\
V' & \xrightarrow{g} & V.
\end{array}
$$

Let $[U \to V] \to [U_+ \to V']$ be the corresponding inert morphism in $\mathcal{F}_{t \to s}$ and let $x \to x'$ be an inert lift of that morphism to $\mathcal{O}^\circ$. Also let $U' \simeq U'_1 \coprod \cdots \coprod U'_m$ be an orbit decomposition and let $e'_j : x' \to x'_j$ be inert morphisms lifting the characteristic morphisms $\chi_{[U'_j \subset U]}$. Note that

$$(\mathcal{C}^\circ)_{g} \cong (\mathcal{C}_{x_i}^\circ)_{g} \cong \mathcal{C}_{x'_j}^\circ,
$$

and

$$
\left( \prod_{f} \left( \prod_{1 \leq i \leq n} e_{x_i} \right) \right)_{g} \cong \left( \prod_{f'} \left( (g')^* \left( \prod_{1 \leq i \leq n} e_{x_i} \right) \right) \right)_{f'} \cong \left( (g')^* \left( \prod_{1 \leq i \leq n} e_{x_i} \right) \right)_{f'} \cong \left( \prod_{1 \leq j \leq m} e_{x'_j} \right)_{f'} \cong \prod_{1 \leq j \leq m} e_{x'_j}.
$$

A diagram chase then shows that the functor $\rho_g : \mathcal{C}_{x_i} \to \prod_{1 \leq j \leq m} \mathcal{C}_{x'_j}$ implements the equivalence of condition (2) in Definition 2.1.7. $\square$

---

In this expression, each $\mathcal{C}_{x_i}$ is a $\mathcal{T}/U_i$-$\infty$-category, their coproduct is a cocartesian fibration over $(\mathcal{T}/U)^{op} = \mathcal{U} \cong \coprod_{1 \leq i \leq n} U_i$, the righthand product is taken in $\mathcal{T}/V$-$\infty$-categories, and the indexed product $\prod_{f}$ denotes the right adjoint to pullback along the induced functor $U \to V$.\[8\]
2.3.4. Corollary (\(\mathcal{T}\)-Segal condition). Let \(C^\circ \to \mathbf{E}_{\mathcal{T},*}\) be a \(\mathcal{T}\)-\(\infty\)-operad. Then for every object \([U_+ \overset{f}{\to} V]\) in \(\mathbf{F}_{\mathcal{T},*}\), we have an equivalence of \(\mathcal{T}/V\)\-\(\infty\)-categories

\[
\mathcal{C}^\circ|_U \simeq \text{Fun}_{\mathcal{T}/V}(U, C_V).
\]

Proof. In view of Theorem 2.3.3, we only need to note that \(\text{Fun}_{\mathcal{T}/V}(U, \cdot) \simeq \prod_i f_i^*\) as endofunctors of \(\text{Cat}_{\mathcal{T}/V}\) and that for an orbit decomposition \(U \simeq U_1 \coprod \cdots \coprod U_n\), \(\prod_{i \leq n} C_{U_i} \simeq C_{U_1} \simeq U \times V\). \(\square\)

2.3.5. Example. Let \(C^\circ\) be a \(\mathcal{T}\)-symmetric monoidal \(\mathcal{T}\)\-\(\infty\)-category. Then for every morphism \(f : U \to V\) of finite \(\mathcal{T}\)-sets, the norm functor \(f^\circ : C_U \to C_V\) of Notation 2.2.4 canonically refines to a norm \(\mathcal{T}/V\)-functor \(\text{Fun}_{\mathcal{T}/V}(U, C_V) \to C_V\). Indeed, if \(V\) is an orbit this is encoded by the cocartesian fibration \(C^\circ \to \mathbf{F}_{\mathcal{T},*}\), in view of Corollary 2.3.4, and one extends to general \(V\) by taking coproducts.

We also have a reformulation of condition (3) in Definition 2.1.7 (or rather (3') in Remark 2.1.10), whose proof is the same as that of Theorem 2.3.3. Recall the notion of \(\mathcal{T}\)-mapping spaces from [Sha21a, §11].

2.3.6. Notation. Let \(p : X \to B\) be a \(\mathcal{T}\)-fibration, let \(\alpha : a \to b\) be a morphism in a fiber \(B_V\), and let \(x, y \in X\) so that \(p(x) = a\) and \(p(y) = b\). Then we define as a pullback of \(\mathcal{T}/V\)-spaces

\[
\text{Map}^\alpha_{C^\circ}(x, y) := \alpha \times_{\text{Map}^\alpha_B(a, b)} \text{Map}_X(x, y).
\]

2.3.7. Proposition. Let \(C^\circ\) be a \(\mathcal{T}\)-\(\infty\)-operad and let notation be as in Remark 2.1.10. Then we have an equivalence of \(\mathcal{T}/V\)-spaces

\[
\text{Map}^\alpha_{C^\circ}(x, y) \xrightarrow{\sim} \prod_{g} \left( \prod_{W \in \text{Orbit}(U')} \text{Map}^{(\alpha W)}_{C^\circ}(x_W, y_W) \right).
\]

Let us now apply the \(\mathcal{T}\)-Segal condition to characterize \(\mathcal{T}\)-symmetric monoidal \(\mathcal{T}\)\-\(\infty\)-categories as \(\mathcal{T}\)-commutative monoids in \(\text{Cat}_\mathcal{T}\).

2.3.8. Remark. Under the equivalences (given by straightening and [Sha21a, Prop. 3.10], respectively)

\[
\text{Cat}^\text{corcart}_{\mathcal{E}_{\mathcal{T},*}} \simeq \text{Fun}(\mathbf{E}_{\mathcal{T},*}, \text{Cat}) \simeq \text{Fun}_\mathcal{T}(\mathbf{F}_{\mathcal{T},*}, \text{Cat}_\mathcal{T})
\]

we see that \(\mathcal{T}\)-symmetric monoidal \(\mathcal{T}\)\-\(\infty\)-categories \(C^\circ\) correspond to \(\mathcal{T}\)-commutative monoids \(M\), i.e., \(\mathcal{T}\)-semiadditive \(\mathcal{T}\)-functors [Nar16, Def. 5.3], since by Corollary 2.3.4, \(M\) transforms

\[
[U_+ \to V] \simeq \prod_{W \in \text{Orbit}(U)} \prod_{W' \to V} I(W)_+ \in \mathbf{F}_{\mathcal{T}/V,*}
\]

into

\[
\prod_{W \in \text{Orbit}(U)} \prod_{W' \to V} C_W \simeq \prod_{W \in \text{Orbit}(U)} \text{Fun}_{\mathcal{T}/V}(W, C_V) \simeq \text{Fun}_{\mathcal{T}/V}(U, C_V) \in \text{Cat}_{\mathcal{T}/V}.
\]

Furthermore, in [Nar16, Thm. 6.5] the first author identified \(\mathcal{T}\)-commutative monoids with \(\mathcal{T}\)-Mackey functors. Using this, we can relate our notion of \(\mathcal{T}\)-\(\infty\)-categories with that which appears in [BH21], wherein the norm functors of Notation 2.2.4 appear as the covariant part of categorical Mackey functors.

2.3.9. Theorem. We have a canonical equivalence of \(\infty\)-categories

\[
\text{Cat}^\circ_{\mathcal{T}} \simeq \text{Fun}^\times(\text{Span}(\text{F}_\mathcal{T}), \text{Cat})
\]

between the \(\infty\)-category \(\text{Cat}^\circ_{\mathcal{T}}\) of \(\mathcal{T}\)-symmetric monoidal \(\mathcal{T}\)\-\(\infty\)-categories and the \(\infty\)-category of product-preserving functors from \(\text{Span}(\text{F}_\mathcal{T})\) to \(\text{Cat}\).

Proof. In [Nar16, Thm. 6.5], the first author proved that given a \(\mathcal{T}\)\-\(\infty\)-category \(\mathcal{D}\) with finite \(\mathcal{T}\)-limits, precomposition by the inclusion \(\mathbf{F}_{\mathcal{T},*} \to \text{Span}(\text{F}_\mathcal{T}) := \text{Span}(\text{F}_\mathcal{T})_\mathcal{T}^{(\text{cat})}\) induces an equivalence of \(\mathcal{T}\)\-\(\infty\)-categories

\[
\text{Fun}^\times(\text{Span}(\text{F}_\mathcal{T}), \mathcal{D}) \xrightarrow{\sim} \text{CMon}^\times(\mathcal{D}).
\]

In particular, if \(\mathcal{D} = \text{Cat}_\mathcal{T}\), then as we just observed \(\text{CMon}^\times(\mathcal{D}) \simeq \text{Cat}^\circ_{\mathcal{T}}\), and passing to cocartesian sections we obtain

\[
\text{Fun}^\times(\text{Span}(\text{F}_\mathcal{T}), \text{Cat}_\mathcal{T}) \simeq \text{Cat}^\circ_{\mathcal{T}}.
\]
Under the equivalence 
\[ \text{Fun}_T(\text{Span}(F_T), \text{Cat}) \simeq \text{Fun}(\text{Span}(F_T), \text{Cat}) \]
let Fun'(Span(F_T), Cat) denote the image of Fun^{\simeq}_T(\text{Span}(F_T), \text{Cat}). Explicitly, functors in Fun' send cartesian edges to equivalences and fiberwise products to products, where cartesian edges in Span(F_T) are given by

\[
\begin{array}{ccc}
    U & \xleftarrow{=} & U \\
    \downarrow & & \downarrow \\
    W & \xleftarrow{f} & V \\
    \downarrow & & \downarrow \\
    & V & \\
\end{array}
\]

in view of the adjunction
\[ f^* : \text{Span}(F_{T/\ast}) \xrightarrow{\simeq} \text{Span}(F_{T/\ast}) : f. \]

Let
\[ s : \text{Span}(F_T) \rightarrow \text{Span}(F_T) \]
denote the source map. Then we claim that precomposition by s induces an equivalence
\[ \text{Fun}^{\simeq}(\text{Span}(F_T), \text{Cat}) \xrightarrow{\simeq} \text{Fun}'(\text{Span}(F_T), \text{Cat}) \]
with inverse given by right Kan extension.

First note that given a product preserving functor \( G : \text{Span}(F_T) \rightarrow \text{Cat}, \)
\( s^*G \) evidently sends cartesian edges to equivalences and fiberwise products to products. Conversely, suppose we have a functor \( F : \text{Span}(F_T) \rightarrow \text{Cat} \) in Fun'. Let \( X \in \text{Span}(F_T) \) be an object. Note that \( (F_T)^{X/} \rightarrow \text{Span}(F_T)^{X/} \) is an initial functor as it admits a right adjoint which sends a span \( X \leftarrow Z \rightarrow Y \) to \( X \leftarrow Z \). Pulling back, we thereby obtain an initial functor
\[ (F_T)^{X/} \rightarrow (F_T)^{X/} \rightarrow \text{Span}(F_T)^{X/} \]
and we are interested in computing the limit of the functor
\[ F' = F \circ \text{pr} : (F_T)^{X/} \rightarrow (F_T)^{X/} \rightarrow \text{Span}(F_T)^{X/} \rightarrow \text{Cat}. \]

By our assumption on \( F, F' \) is the right Kan extension of its restriction to \( T^{op} \times F_T^{op} (F_T)^{X/} \) (where \( T^{op} \rightarrow (F_T)^{op} \) is the identity section). Indeed, given an object \( I = [V \leftarrow U \rightarrow X] \) and an orbit decomposition \( U \simeq \prod_{i=1}^n U_i, \) the \( n \) projection maps \( I \rightarrow I_i = [U_i = U_i \rightarrow X] \) induce an equivalence
\[ F'(I) = F([U \rightarrow V]) \simeq \prod_{i=1}^n F(I_i) = \prod_{i=1}^n F([U_i = U_i]). \]

We conclude that the limit of \( F' \) is \( \prod_{i=1}^n F((X_i = X_i)) \) for some orbit decomposition \( X \simeq \prod_{i=1}^n X_i, \) so \( s_*F(X) \simeq \prod_{i=1}^n F(id_{X_i}). \) Using this pointwise formula and a simple argument regarding the morphisms, we see that \( s_*F \) preserves products and the counit and unit maps are equivalences.

\( \square \)

2.4. Examples. In this subsection, we discuss some basic examples of \( T \)-\( \infty \)-operads.

2.4.1. Example (/co)cartesian \( T \)-symmetric monoidal structures. Let \( \mathcal{C} \) be a \( T \)-\( \infty \)-category and let \( \pi : \mathcal{C}^X \rightarrow F_T \) be the cartesian fibration defined as in [Sha21a, Prop. 5.12], so \( (\mathcal{C}^X)_{id} \simeq \prod_{W \in \text{Orbit}(U)} \mathcal{C}_W \) and the functoriality is given by restriction. Suppose \( \mathcal{C} \) admits finite \( T \)-coproducts. Then by [Sha21a, Prop. 5.12], \( \pi \) is a Beck–Chevalley fibration with cartesian functoriality given by the coinduction functors, and by Barwick’s unfolding construction [Bar17, §11], \( \pi \) straightens to a product-preserving functor \( \text{Span}(F_T) \rightarrow \text{Cat}. \) Let
\[ \mathcal{C}^{\Pi} \rightarrow F_{T,*} \]
denote the resulting \( T \)-symmetric monoidal \( T \)-\( \infty \)-category under the equivalence of Theorem 2.3.9. We call \( \mathcal{C}^{\Pi} \) the \( T \)-categorical \( T \)-symmetric monoidal structure on \( \mathcal{C}. \)

Dually, suppose that \( \mathcal{C} \) admits finite \( T \)-products. Then by the dual of [Sha21a, Prop. 5.12], the vertical opposite \( \pi^{op} : (\mathcal{C}^X)^{op} \rightarrow F_T \) is a Beck–Chevalley fibration with cartesian functoriality given by the (opposite of the) induction functors, and thus we obtain a product-preserving functor \( \text{Span}(F_T) \rightarrow \text{Cat}. \) After postcomposing by the opposite automorphism of \( \text{Cat}, \) let
\[ \mathcal{C}^{\Pi} \rightarrow F_{T,*} \]
denote the resulting $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category under the equivalence of Theorem 2.3.9. We call $\mathcal{O}^\otimes$ the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\mathcal{C}$.

2.4.2. Example ($G$-spectra). Let $\text{Gpd}_{\text{fin}}$ be the $(2,1)$-category of finite groupoids and let

$$\text{SH}^\otimes : \text{Span}(\text{Gpd}_{\text{fin}}) \to \text{CAlg}(\text{Cat}^{\text{fsh}})$$

denote the (restriction of the) functor of [BH21, §9.2]. Let $G$ be a finite group and let

$$\omega_G : \text{Span}(F_G) \to \text{Span}(\text{Gpd}_{\text{fin}})$$

be the action groupoid functor. Let $(\text{Sp}^G)^\otimes$ be the $G$-symmetric monoidal $G$-$\infty$-category associated to $\text{SH}^\otimes \circ \omega_G$ under Theorem 2.3.9. Then $(\text{Sp}^G)^\otimes$ is the $G$-symmetric monoidal structure on $\text{Sp}^G$ that encodes the Hill–Hopkins–Ravenel norm functors.

2.4.3. Example (Trivial $\mathcal{T}$-$\infty$-operad). Let $\text{Triv}_{\mathcal{T}}^\otimes \subset \text{Fun}_{\mathcal{T},*}$ be the wide subcategory on the inert edges. Then $\text{Triv}_{\mathcal{T}}^\otimes$ is a $\mathcal{T}$-suboperad of $\text{Fun}_{\mathcal{T},*}$ such that the identity cocartesian section $I^+_*: \mathcal{T}^\text{op} \to \text{Fun}_{\mathcal{T},*}$ restricts to a fully faithful functor into $\text{Triv}_{\mathcal{T}}^\otimes$ and an equivalence onto $\text{Triv}_{\mathcal{T}}$. We call $\text{Triv}_{\mathcal{T}}^\otimes$ the trivial $\mathcal{T}$-$\infty$-operad.

We claim that given any $\mathcal{T}$-$\infty$-operad $\mathcal{C}^\otimes$, we have an equivalence

$$\text{Alg}_{\text{Triv}_{\mathcal{T}}, \mathcal{T}}(\mathcal{C}) \xrightarrow{\cong} \mathcal{C}$$

implemented by restriction along $I^+_*$. To show this, we need the following lemma.

2.4.4. Lemma. Let $(\mathcal{O}^\otimes, p)$ be a $\mathcal{T}$-$\infty$-operad and let

$$\mathcal{O}^\otimes_{\text{ne}} := \text{Triv}_{\mathcal{T}}^\otimes \times_{\text{Fun}_{\mathcal{T},*}} \mathcal{O}^\otimes$$

be the wide subcategory on the inert edges. Let $F^\text{ne}_O : \text{Triv}_{\mathcal{T}}^\otimes \to \text{Cat}$ be the functor classifying the cocartesian fibration $p|_{\mathcal{O}^\otimes_{\text{ne}}}$. Then $F^\text{ne}_O$ is the right Kan extension of its restriction $F_O$ along $I^+_* : \mathcal{T}^\text{op} \to \text{Triv}_{\mathcal{T}}^\otimes$ (which classifies the underlying $\mathcal{T}$-$\infty$-category $\mathcal{O}$).

Proof. Let $f^+_* = [U^+_* \to V]$ be any object in $\text{Triv}_{\mathcal{T}}^\otimes$ and let $J^\text{op} = \mathcal{T}^\text{op} \times_{\text{Triv}_{\mathcal{T}}^\otimes} (\text{Triv}_{\mathcal{T}}^\otimes)^{f^+_*/}$. We need to show that the natural map

$$F^\text{ne}_O(f^+_*) \to \text{lim}(\theta : J^\text{op} \to \mathcal{T}^\text{op} \xrightarrow{\mathcal{F}_O} \text{Cat})$$

is an equivalence. If we view $\text{Orbit}(U)$ as a discrete category, then we have a functor $\phi : \text{Orbit}(U) \to J^\text{op}$ that sends $W$ to $\{W, \chi_{[W \subset U]})$, and by definition we have an equivalence

$$F^\text{ne}_O(f^+_*) \sim \text{lim}(\theta \circ \phi).$$

Consequently, it suffices to show that $\phi$ is right cofinal. Since $\text{Triv}_{\mathcal{T}}^\otimes \simeq ((\text{Fun}_{\mathcal{T}}^\text{f})^\text{st})^\text{op}$ under the inclusion into $\text{Fun}_{\mathcal{T},*} = \text{Span}(\text{Fun}_{\mathcal{T}}^\text{f}; (\text{Fun}_{\mathcal{T}}^\text{st})^\text{st}, (\text{Fun}_{\mathcal{T}}^\text{tdeg})$, we may equivalently show that

$$\phi^\text{op} : \text{Orbit}(U)^{\text{op}} \to J \simeq \mathcal{T} \times (\text{Fun}_{\mathcal{T}}^\text{st})^\text{st}/J$$

is left cofinal. For this, we will apply Quillen’s Theorem A. Let

$$\overline{\alpha} = \left( \begin{array}{ccc} X & \to & U \\ \downarrow & \downarrow & \downarrow \\ X & \to & V \end{array} \right)$$

be any object in $J$. Then since $\text{Orbit}(U)$ is discrete, we have an equivalence

$$\text{Orbit}(U) \times_J J^{\overline{\alpha}} \simeq \coprod_{W \in \text{Orbit}(U)} \text{Map}_J(\overline{\alpha}, \chi_{[W \subset U]}).$$

If we let $W$ be the orbit in $U$ that $\alpha$ factors through, then for all other $W' \in \text{Orbit}(X)$, we have

$$\text{Map}_J(\overline{\alpha}, \chi_{[W' \subset U]} = \emptyset.$$
To compute the remaining mapping space, observe that we have a homotopy pullback square

\[
\begin{array}{ccc}
\text{Map}_\mathcal{F}(\mathcal{F}, \chi|_{W \subseteq U'}) & \longrightarrow & \text{Map}(\mathcal{F}_\tau, \chi|_{W \subseteq U'}) \\
\pi \downarrow & & \downarrow \\
\text{Map}(\mathcal{F}_\tau, \chi, f) & \simeq & \text{Map}(\mathcal{F}_\tau, \chi, f)
\end{array}
\]

where for the righthand equivalences we use our assumption that \( \mathcal{T} \) is orbital. But the righthand vertical map identifies with

\[
\text{Map}_{\mathcal{F}_\tau}(X, W) \longrightarrow \text{Map}_{\mathcal{F}_\tau}(X, U) \times \text{Map}_{\mathcal{F}_\tau}(X, V) \text{Map}_{\mathcal{F}_\tau}(X, V) \simeq \text{Map}_{\mathcal{F}_\tau}(X, U)
\]

and is hence an equivalence. We conclude that \( \text{Orbit}(U) \times_\mathcal{T} \mathcal{T} \) is an equivalence of \( \infty \)-categories between \( \mathcal{T} \)-\( \infty \)-operads and for any \( \mathcal{T} \)-\( \infty \)-operad \( \mathcal{O}_\mathcal{T} \), we have an equivalence of \( \mathcal{T} \)-\( \infty \)-categories

\[
\text{Alg}_\mathcal{T}(\mathcal{T}(\mathcal{O}_\mathcal{T}), 0) \xrightarrow{\sim} \text{Fun}_{\mathcal{T}}(\mathcal{O}_\mathcal{T}, 0)
\]

implemented by restriction along \( \mathcal{I}_+ \).

**Proof.** By Lemma 2.4.4, we have an equivalence of \( \infty \)-categories between \( \mathcal{T} \)-monoidal \( \mathcal{T} \)-\( \infty \)-categories and the full subcategory of \( \text{Fun}(\mathcal{T}(\mathcal{O}_\mathcal{T}), \mathcal{O}(\mathcal{T}_\tau))^\circledast \) spanned by those functors right Kan extended from \( \mathcal{T}(\mathcal{O}_\mathcal{T}) \) along \( \mathcal{T}(\mathcal{T}(\mathcal{O}_\mathcal{T})) \). Then \( \mathcal{T}(\mathcal{O}_\mathcal{T}) \) is a \( \mathcal{T} \)-\( \infty \)-operad and for any \( \mathcal{T} \)-\( \infty \)-operad \( \mathcal{O}_\mathcal{T} \), we have an equivalence of \( \mathcal{T} \)-\( \infty \)-categories

\[
\text{Alg}_\mathcal{T}(\mathcal{T}(\mathcal{O}_\mathcal{T}), 0) \simeq \text{Fun}_{\mathcal{T}}(\mathcal{O}_\mathcal{T}, 0)
\]

is an equivalence of \( \infty \)-categories. But this follows immediately from Lemma 2.4.4. \( \square \)

**2.4.5. Example.** By Corollary 2.4.5, the \( \mathcal{T} \)-suboperads of the trivial \( \mathcal{T} \)-\( \infty \)-operad are in bijective correspondence with sieves of \( \mathcal{T} \). For instance, we have that the initial \( \mathcal{T} \)-\( \infty \)-operad \( \mathcal{T}(\mathcal{O}_\mathcal{T}) \) corresponds to \( \emptyset \subseteq \mathcal{T} \) and identifies with the full subcategory of \( \mathcal{E}_{\mathcal{T}, \ast} \) on objects \( [\emptyset, \longrightarrow V] \).

**2.4.6. Example.** By Corollary 2.4.5, the \( \mathcal{T} \)-suboperads of the trivial \( \mathcal{T} \)-\( \infty \)-operad are in bijective correspondence with sieves of \( \mathcal{T} \). For instance, we have that the initial \( \mathcal{T} \)-\( \infty \)-operad \( \mathcal{T}(\mathcal{O}_\mathcal{T}) \) corresponds to \( \emptyset \subseteq \mathcal{T} \) and identifies with the full subcategory of \( \mathcal{E}_{\mathcal{T}, \ast} \) on objects \( [\emptyset, \longrightarrow V] \).

In the parametrized setting, we may further define a family of \( \mathcal{T} \)-suboperads of \( \mathcal{E}_{\mathcal{T}, \ast} \) so as to encode different flavors of parametrized commutativity. First, define the minimal \( \mathcal{T} \)-commutative \( \mathcal{T} \)-\( \infty \)-operad \( \mathcal{C}_{\mathcal{T}, \ast} \) to be the wide subcategory of \( \mathcal{E}_{\mathcal{T}, \ast} \) containing all morphisms

\[
\begin{array}{ccc}
U & \leftarrow & Z \\
\downarrow & & \downarrow \\
V & \leftarrow & Y
\end{array}
\]

where \( m \) is a coproduct of fold maps (including possibly empty fold maps). In other words, if we let \( \nabla \) denote the collection of fold maps and \( (\mathcal{E}_{\mathcal{T}})_{\nabla}^{\ast} \subset \mathcal{E}_{\mathcal{T}} \) the wide subcategory on morphisms with source in \( \nabla \) and target degenerate, then

\[
\mathcal{C}_{\mathcal{T}, \ast} = \text{Span}(\mathcal{E}_{\mathcal{T}}^{\ast} - (\mathcal{E}_{\mathcal{T}}^{\ast})^{\ast}) \subset \mathcal{E}_{\mathcal{T}}^{\ast}
\]

where we use that \( \nabla \) is stable under pullback by summand inclusions, and \( \mathcal{C}_{\mathcal{T}, \ast} \) is classified by the functor \( \mathcal{T} \longrightarrow \text{Cat} \) that sends an orbit \( V \) to \( \text{Span}(\mathcal{F}_{\mathcal{T}, \ast}, \mathcal{F}_{\mathcal{T}, \ast}^{\ast}, \mathcal{F}_{\mathcal{T}, \ast}^{\ast}) \). Since \( \mathcal{C}_{\mathcal{T}, \ast} \) contains all inert edges in \( \mathcal{C}_{\mathcal{T}, \ast} \) and \( \mathcal{C}_{\mathcal{T}, \ast} \simeq \mathcal{T} \), to verify that \( \mathcal{C}_{\mathcal{T}, \ast} \) is a \( \mathcal{T} \)-suboperad it only remains to check condition (3') of Remark 2.1.10. But this condition is satisfied since a map \( \alpha : U \longrightarrow U' \) of finite \( \mathcal{T} \)-sets is a fold map if and only if for all \( W \in \text{Orbit}(U') \), the pullback \( \alpha_W : U \times U' W \longrightarrow W \) is a fold map.

---

Using [Sha21a, Ex. 2.26], we could give a definition of \( \mathcal{T}(\mathcal{O}_\mathcal{T}) \) at the level of marked simplicial sets, without passing through straightening and unstraightening.
Now suppose that we want to define a $\mathcal{T}\text{-}\infty$-operad $\mathcal{O}^\otimes$ such that we have $\mathcal{T}$-operadic inclusions
\[
\text{Com}_\mathcal{T}^\otimes \subseteq \mathcal{O}^\otimes \subseteq \text{Com}_\mathcal{T}^\otimes.
\]
Since $\text{Com}_\mathcal{T}^\otimes = \text{Com}_\mathcal{T} \simeq \mathcal{T}^{\text{op}}$, the only constraint on $\mathcal{O}^\otimes$ to be a $\mathcal{T}$-suboperad arises from condition (3'). In other words, to specify $\mathcal{O}^\otimes$ we may as well specify the morphisms $\alpha : U \to V$ with $V$ an orbit that we wish to be active. We already have that all fold maps are active, so in particular all summand inclusions are active. Furthermore, as observed in Remark 2.1.11, for any orbit $W \subseteq U$, the composite map $W \subseteq U \xrightarrow{\alpha} V$ yields an operadic composition map
\[
\text{Mul}^W_0 \times \text{Mul}^D_0 \simeq * \times \text{Mul}^D_0 \to \text{Mul}_0^{\alpha \mid W},
\]
so if $\alpha$ is active, we must have that $\alpha \mid W$ is active for all $W \in \text{Orbit}(U)$. The converse holds by a similar argument since $\alpha$ factors as
\[
U \xrightarrow{\sim} \bigsqcup_{W \in \text{Orbit}(U)} W \xrightarrow{\| \alpha \mid W} \bigsqcup_{W \in \text{Orbit}(U)} V \xrightarrow{\nabla} V.
\]
We thereby reduce to specifying whether or not morphisms $\alpha : W \to V$ are active for both $W$ and $V$ orbits. Moreover, by examining Remark 2.1.11 again we see that the only constraints are:

1. The active morphisms contain all equivalences and are closed under composition, so assemble to a subcategory $\mathcal{I} \subseteq \mathcal{T}$ such that $\mathcal{I}$ contains the maximal subgroupoid $\mathcal{T}^\otimes$ of $\mathcal{T}$.
2. The active morphisms are closed under base-change, in the sense that for any commutative square
\[
\begin{array}{ccc}
W' & \xrightarrow{\alpha'} & W \\
\downarrow & & \downarrow \\
V' & \xrightarrow{\alpha} & V
\end{array}
\]
such that the map $W' \to V' \times_V W$ is a summand inclusion, if $\alpha$ is in $\mathcal{I}$ then $\alpha'$ is in $\mathcal{I}$.

2.4.8. **Definition.** A $\mathcal{T}$-indexing system is a subcategory $\mathcal{I}$ of $\mathcal{T}$ that satisfies the above conditions (1) and (2).

2.4.9. **Remark.** An indexing system $\mathcal{I}$ is the same data as a subcategory $\overline{\mathcal{I}} \subseteq \text{F}_\mathcal{T}$ such that:

1. $\overline{\mathcal{I}}$ contains the maximal subgroupoid $\text{F}_\mathcal{T}^\otimes$.
2. Morphisms in $\overline{\mathcal{I}}$ are closed under base-change and binary coproducts.
3. $\overline{\mathcal{I}}$ contains all fold maps (and hence all summand inclusions).

Indeed, the assignment $\mathcal{I} \rightsquigarrow \mathcal{I} = \overline{\mathcal{I}} \times_{\text{F}_\mathcal{T}} \mathcal{T}$ is seen to identify the two notions, with inverse given by taking the finite coproduct completion of $\mathcal{I}$.

2.4.10. **Definition.** Let $\mathcal{I}$ be a $\mathcal{T}$-indexing system and let $(\text{F}_\mathcal{T}^\otimes)^{\mathcal{T}\text{-indeg}} \subseteq \text{F}_\mathcal{T}^\otimes$ be the wide subcategory on morphisms with source in $\overline{\mathcal{I}}$ and target degenerate. We then define the $\mathcal{I}$-commutative $\mathcal{T}\text{-}\infty$-operad to be
\[
\text{Com}_\mathcal{I}^\otimes := \text{Span}(\text{F}_\mathcal{T}^\otimes, (\text{F}_\mathcal{T}^\otimes)^{\mathcal{T}\text{-indeg}}).
\]
More generally, we define a commutative $\mathcal{T}\text{-}\infty$-operad to be any $\mathcal{T}$-suboperad of $\text{Com}_\mathcal{T}^\otimes$ containing $\text{Com}_\mathcal{T}^\otimes$.

The above analysis confirms the following proposition.

2.4.11. **Proposition.** The assignment $\mathcal{I} \rightsquigarrow \text{Com}_\mathcal{I}^\otimes$ implements an inclusion-preserving bijection between $\mathcal{T}$-indexing systems and commutative $\mathcal{T}\text{-}\infty$-operads.

2.4.12. **Remark.** Let $\mathcal{T} = \mathcal{O}_G$. A $G$-indexing system $I$ in the sense of Blumberg-Hill [BH15, Def. 3.22] as reformulated by Rubin [Rub21, Def. 2.12] is a collection $\{ I(H) \}$ of finite $H$-sets for every subgroup $H \leq G$ such that $I(H)$ contains all finite $H$-sets with trivial $H$-action and satisfies the following closure properties:

1. If $U \in I(H)$ and $U' \equiv U$, then $U' \in I(H)$.
2. For every subgroup $K \leq H$, if $U \in I(H)$ then $\text{res}_K^H U \in I(K)$.
3. For every conjugate $H' = gHg^{-1}$ of $H$, if $U \in I(H)$ then its conjugate $U' \in I(H')$.
4. If $U \in I(H)$ and $U' \subset U$, then $U' \in I(H)$.
5. If $U, U' \in I(H)$, then $U \bigsqcup U' \in I(H)$.
6. For any subgroup $K \leq H$, if $U \in I(K)$ and $H/K \in I(H)$ then $\text{ind}_K^H U \in I(H)$. 


In [BH15, Thm. 3.17], it was shown that $G$-indexing systems are in bijective correspondence with subcategories of $F_G$ satisfying the conditions of Remark 2.4.9 (and thus with the $O_G$-indexing systems of Definition 2.4.8). For the convenience of the reader, we review this correspondence. Note that under the equivalences $F_H \simeq F_G^{(G/H)}$, given a subgroup $K \leq H$, induction corresponds to postcomposition by $G/K \to G/H$. Therefore, given a $G$-indexing system $I$, we may define a wide subcategory $J \subset O_G$ to be the subcategory whose morphisms $f : V \to W \cong G/H$ are such that $f \in I(H)$. The enumerated conditions then imply that $J$ is an $O_G$-indexing system, using closure under restriction and inclusion to validate the base-change condition.

Conversely, suppose $J$ is an $O_G$-indexing system and let $\overline{J} \subset F_G$ be the subcategory generated by $J$ as in Remark 2.4.9. Let $I(H)$ be the subset of objects of $F_H$ given by morphisms in $\overline{J}$ with target $G/H$ under $F_H \simeq F_G^{(G/H)}$. Then one sees that $I$ is a $G$-indexing system and these assignments are mutually inverse – note that condition (5) holds since given $U, U' \to G/H$, the coproduct in $F_H$ is given by the composition $U \coprod U' \to G/H \coprod G/H \to G/H$.

Consequently, by the work of Bonventre–Pereira [BP21], Gutiérrez–White [GW18], and Rubin [Rub21], we see that the commutative $G$-$\infty$-operads are in bijection with the $N_{\infty}$-operads of Blumberg–Hill.

A straightforward adaptation of the proof of Theorem 2.3.9 shows the following.

2.4.13. Definition. Given a $T$-indexing system $J$, let $\text{Cat}_J^\otimes$ be the $\infty$-category of $J$-symmetric monoidal $T$-$\infty$-categories and $J$-symmetric monoidal $T$-functors thereof.

2.4.14. Theorem. Let $J$ be a $T$-indexing system. We then have a canonical equivalence

$$\text{Cat}_J^\otimes \simeq \text{Fun}^\otimes(\text{Span}(F_T; F_T, \overline{J}), \text{Cat}).$$

2.4.15. Corollary. For the minimal indexing system $J = T^\omega$, we have a canonical identification of $T^\omega$-symmetric monoidal $T$-$\infty$-categories with $T^\omega$-cocartesian families of symmetric monoidal $\infty$-categories ([Lur17, Def. 4.8.3.1]).

2.5. $T$-operadic nerve. In this subsection, we suppose that $T$ is equivalent to the nerve of a 1-category $T$. For example, by the following proposition we could take $T$ to be any atomic orbital $\infty$-category that admits a final object.

2.5.1. Proposition. Suppose $T$ is an atomic orbital $\infty$-category that admits a final object $*$. Then $T$ is equivalent to the nerve of a 1-category.

Proof. By [Lur09, Prop. 2.3.4.18] it suffices to show that the mapping spaces of $T$ are 0-truncated, or equivalently that the essentially unique maps $V \to *$ are 0-truncated for all $V \in T$. By [Lur09, Lem. 5.5.6.15], this occurs if and only if the diagonal $\delta : V \to V \times V$ in $F_T$ is $(-1)$-truncated. But since $T$ is atomic and $\delta$ is split by either projection to $V$, it follows that $\delta$ is a summand inclusion and hence a monomorphism. \qed

Correspondingly, let $F_T$ denote the subcategory of the category of $\text{Set}$-valued presheaves on $T$ spanned by the finite coproducts of representables, so that $F_T \simeq N(F_T)$.

2.5.2. Remark. If $T$ is a 1-category, then the a priori $(2,1)$-category

$$E_{T,*} := \text{Span}(F_T^{\Delta^1} \times T, (F_T^{\Delta^1})^{\text{tdg}}, (F_T^{\Delta^1})^{*i})$$

is enriched in setoids and therefore equivalent to a 1-category. Indeed, the only automorphisms of spans of the form

$$
\begin{array}{ccc}
U & \xleftarrow{\delta} & \bar{U} \to U' \\
\downarrow & & \downarrow \\
V & \xleftarrow{\delta} & V' \to V'
\end{array}
$$

where the left square is in $(F_T^{\Delta^1})^{*i}$ are identities. In what follows, we will implicitly make a choice of 1-categorical model for $E_{T,*}$ by picking a representative for every equivalence class of morphisms.

Our goal is to indicate how to prescribe the data of a $T$-$\infty$-operad in terms of the stricter data of a simplicial colored $T$-operad, which will be defined along the lines suggested by Remark 2.1.11. To concisely state its definition, we first need to introduce some notation.
2.5.3. **Notation.** Let $\mathbf{AF}_T \subseteq \mathbf{Ar}(\mathbf{F}_T)$ denote the wide subcategory of the arrow category whose morphisms are cartesian squares. Suppose that $\text{ob}O \rightarrow \mathbf{F}_T$ is a Grothendieck fibration fibered in sets, and let $\text{ob}O(U)$ denote the fiber of $\text{ob}O$ over $U \in \mathbf{F}_T$. We then write

$$A^O\mathbf{F}_T := \mathbf{AF}_T \times (\mathbf{F}_T \times \mathbf{F}_T) (\text{ob}O \times \text{ob}O)$$

for the category whose objects are triples $(f : U \rightarrow V, x \in \text{ob}O(U), y \in \text{ob}O(V))$ and whose morphisms are cartesian squares

$$
\begin{array}{ccc}
U' & \xrightarrow{\phi} & U \\
\downarrow^{f'} & & \downarrow^{f} \\
V' & \xrightarrow{\psi} & V
\end{array}
$$

such that $x' = \phi^*x$ and $y' = \psi^*y$. We have functors

$$1 : \text{ob}O \rightarrow A^O\mathbf{F}_T, \quad (x \in \text{ob}O(U)) \mapsto (\text{id}_U, x, x),$$

$$\circ : A^O\mathbf{F}_T \times_{\text{ob}O} A^O\mathbf{F}_T \rightarrow A^O\mathbf{F}_T, \quad (f : U \rightarrow V, g : V \rightarrow W, x, y, z) \mapsto (gf : U \rightarrow W, x, z).$$

2.5.4. **Definition.** A (fibrant) simplicial colored $T$-operad $O$ is the data of

1. A ‘$T$-set of colors’, given as a Grothendieck fibration fibered in sets

$$\text{ob}O \rightarrow \mathbf{F}_T$$

classified by a functor $\mathbf{F}_T^{\text{op}} \rightarrow \mathbf{Set}$ preserving finite products.

2. A collection of spaces of multimorphisms, packaged into a functor

$$\text{Mul}_O : (A^O\mathbf{F}_T)^{\text{op}} \rightarrow \mathbf{sSet}, \quad (f : U \rightarrow V, x \in \text{ob}O(U), y \in \text{ob}O(V)) \mapsto \text{Mul}_O^f(x, y)$$

that preserves finite products and is valued in Kan complexes.

3. A distinguished ‘identity’ for $\text{Mul}_O$, given by a natural transformation

$$1 : * \rightarrow \text{Mul}_O^{\text{id}_U}(x, x)$$

of functors $(\text{ob}O)^{\text{op}} \rightarrow \mathbf{sSet}$, where the right hand side is the composition

$$(\text{ob}O)^{\text{op}} \xrightarrow{1} A^O\mathbf{F}_T^{\text{op}} \xrightarrow{\mathbf{Mul}} \mathbf{sSet}.$$  

4. A ‘composition law’ for $\text{Mul}_O$, given by a natural transformation

$$\circ : \text{Mul}_O^f(x, y) \times \text{Mul}_O^g(y, z) \rightarrow \text{Mul}_O^h(x, z)$$

of functors $(A^O\mathbf{F}_T \times_{\text{ob}O} A^O\mathbf{F}_T)^{\text{op}} \rightarrow \mathbf{sSet}$, where the left hand side is the composition

$$(A^O\mathbf{F}_T \times_{\text{ob}O} A^O\mathbf{F}_T)^{\text{op}} \rightarrow (A^O\mathbf{F}_T \times A^O\mathbf{F}_T)^{\text{op}} \xrightarrow{\text{Mul} \times \text{Mul}} \mathbf{sSet} \times \mathbf{sSet} \xrightarrow{\cong} \mathbf{sSet},$$

and the right hand side is the composition

$$(A^O\mathbf{F}_T \times_{\text{ob}O} A^O\mathbf{F}_T)^{\text{op}} \xrightarrow{\circ} A^O\mathbf{F}_T^{\text{op}} \xrightarrow{\mathbf{Mul}} \mathbf{sSet}.$$  

These data are required to satisfy the following compatibilities:

- **Unitality:** The compositions

$$\text{Mul}_O^f(x, y) \xrightarrow{(\text{id}, 1_x)} \text{Mul}_O^f(x, y) \times \text{Mul}_O^{\text{id}_V}(y, y) \xrightarrow{\circ} \text{Mul}_O^f(x, y)$$

and

$$\text{Mul}_O^f(x, y) \xrightarrow{(1_x, \text{id})} \text{Mul}_O^{\text{id}_V}(x, x) \times \text{Mul}_O^f(x, y) \xrightarrow{\circ} \text{Mul}_O^f(x, y)$$

are the identity natural transformation.

- **Associativity** The following diagram is commutative

$$
\begin{array}{ccc}
\text{Mul}_O^f(x, y) \times \text{Mul}_O^g(y, z) \times \text{Mul}_O^h(z, t) & \xrightarrow{(\text{id}, \text{id})} & \text{Mul}_O^g(y, z) \\
\downarrow^{(\text{id}, \text{id})} & & \downarrow^{\circ} \\
\text{Mul}_O^f(x, y) \times \text{Mul}_O^g(y, t) & \xrightarrow{\circ} & \text{Mul}_O^h(x, t)
\end{array}
$$

\[\text{If } V \text{ is an orbit, we thus obtain a } T^/V\text{-space } \text{Mul}_O^f(x, y) : (T^/V)^{\text{op}} \rightarrow \mathbf{sSet} \text{ by precomposing } \text{Mul}_O \text{ with the functor } T^/V \rightarrow A^O\mathbf{F}_T \text{ over } \mathbf{F}_T \text{ determined by } (f, x, y) – \text{ indeed, one has an equivalence } (A^O\mathbf{F}_T)^{(f,x,y)} \simeq (\mathbf{F}_T)^{/V}.\]
2.5.5. Construction. From the data of Definition 2.5.4 we will build a simplicial category $O^\otimes$ over $F_{T,\ast}$ as follows. We let the objects of $O^\otimes$ be the pairs $([U_+ \rightarrow V], x)$, where $[U_+ \rightarrow V]$ is an object of $F_{T,\ast}$ and $x \in \text{ob}O(U)$, and we define as mapping simplicial sets
\[
\text{Map}_{O^\otimes}(([U_+ \rightarrow V], x), ([U'_+ \rightarrow V'], x')) := \bigsqcup_{U^0 \subseteq U_0 \rightarrow U''} \text{Mul}^f(i^*x, x')
\]
where the coproduct is indexed by the set of all maps $[U_+ \rightarrow V] \rightarrow [U'_+ \rightarrow V']$ in $F_{T,\ast}$. The identity of $([U_+ \rightarrow V], x)$ is given by $1_{(U,x)} \in \text{Mul}^{id_U}(x, x)$. If
\[
\begin{array}{c}
\text{Uo} \\
\downarrow f \\
\text{U'} \\
\downarrow \\
\text{U''}
\end{array}
\]
is a diagram representing a composition in $F_{T,\ast}$, composition over it is given by
\[
\text{Mul}^f(i^*x, x') \times \text{Mul}^g(j^*x', x'') \rightarrow \text{Mul}^f((j')^*i^*x, j^*x') \times \text{Mul}^g(j^*x', x'') \rightarrow \text{Mul}^{g'}((ij')^*x, x'').
\]
Verifying that this satisfies associativity and unitality is left as an exercise for the reader.

2.5.6. Proposition. The map $N(O^\otimes) \rightarrow N(F_{T,\ast}) \simeq F_{T,\ast}$ is a $T$-\text{-\textbeta-}operad.

Proof. Using [Lur09, Prop. 2.4.1.10] we see that the above map is an inner fibration and that its restriction over the subcategory of inert edges is a cocartesian fibration. Then remaining properties are true because $OB$ and Mul preserve finite products.

2.6. Model structures. In this subsection, we introduce a model structure for $T$-\text{-\textbeta-}operads by means of Lurie’s theory of categorical patterns ([Lur17, Def. B.0.19]). For a $T$-\text{-\textbeta-}operad $O^\otimes$, let $Ne \subset (O^\otimes)_1$ denote the subset of inert morphisms.

2.6.1. Definition. We define categorical patterns $P_T$ and $P_{O^\otimes}$ on $F_{T,\ast}$ as follows. For each collection of morphisms $\alpha = \{\alpha_i : U_i \rightarrow V\}_{i=1}^n$ in $T$, let $\alpha : U = \bigsqcup_{i=1}^n U_i \rightarrow \bigsqcup_{i=1}^n V$ and define a morphism $f_\alpha : (n^\otimes)^2 \rightarrow F_{T,\ast} (Ne)$ (for $n = \{1, \ldots, n\}$ regarded as a discrete category) that sends the cone point $v$ to $[U_+ \rightarrow V]$, $i \in n$ to $[U_i \rightarrow U_i]$, and $v \rightarrow i$ to the characteristic morphism $\chi_{[\{i\} \subseteq U]}$. Then let $A$ be the set of the $\alpha$ and let
\[
P_T = (Ne, \text{All}, \{f_\alpha : n^\otimes \rightarrow F_{T,\ast} (Ne)\}_{\alpha \in A}),
\]
\[
P_{O^\otimes} = (\text{All}, \text{All}, \{f_\alpha : n^\otimes \rightarrow F_{T,\ast} (Ne)\}_{\alpha \in A}).
\]
Furthermore, for any $T$-\text{-\textbeta-}operad $O^\otimes$, we define categorical patterns $P_O$ and $P_{O^\otimes}$ on $O^\otimes$ by the construction of [Lur17, Rem. B.1.5]. In other words, let $B$ denote the set of pairs $(\alpha, \overline{f_\alpha})$ where $\overline{f_\alpha} : (n^\otimes)^2 \rightarrow (O^\otimes, Ne)$ is any lift of $f_\alpha$, and let
\[
P_O = (Ne, \text{All}, \{\overline{f_\alpha} : n^\otimes \rightarrow O^\otimes\}_{(\alpha, \overline{f_\alpha}) \in B}),
\]
\[
P_{O^\otimes} = (Ne, \text{All}, \{\overline{f_\alpha} : n^\otimes \rightarrow O^\otimes\}_{(\alpha, \overline{f_\alpha}) \in B}).
\]

2.6.2. Theorem-Construction. The $T$-\text{-\textbeta-}operadic model structure on the category $sSet/F_{T,\ast} (Ne)$ is that defined by the categorical pattern $P_T$ of Definition 2.6.1 according to [Lur17, Thm. B.0.20]. The $T$-\text{-\textbeta-}operadic model structure is left proper, combinatorial, simplicial, and has the following properties:

(1) The cofibrations are precisely the monomorphisms.
(2) A marked map $X \rightarrow Y$ over $(F_{T,\ast} (Ne))$ is a weak equivalence if for any $T$-\text{-\textbeta-}operad $O^\otimes$, the induced map
\[
\text{Map}_{F_{T,\ast} (Ne)}(Y, (O^\otimes, Ne)) \rightarrow \text{Map}_{F_{T,\ast} (Ne)}(X, (O^\otimes, Ne))
\]
is a weak equivalence.
(3) An object is fibrant if it is of the form \((\Oop^\circ, N_e)\) for some \(\iCat\) operad \(\Oop^\circ\).

(4) The fibrations between fibrant objects in \(sSet^+_{/(\Oop^\circ, N_e)}\) are exactly given by the fibrations of \(\iCat\)-operads.

Furthermore, for any \(\iCat\) operad \(\Oop^\circ\), we define the \(\iCat\)-operadic model structure on the category \(sSet^+_{/(\Oop^\circ, N_e)}\) via the categorical pattern \(\mathcal{P}_\Oop\), and this coincides with the model structure induced from the \(\iCat\)-operadic model structure on \(sSet^+_{/(\Oop^\circ, N_e)}\) by slicing over \((\Oop^\circ, N_e)\).

Finally, we also define the \(\iCat\)-monoidal model structure on the category \(sSet^+_{/\Oop^\circ}\) via the categorical pattern \(\mathcal{P}_\Oop^\circ\). This construction has the same formal properties as with the \(\iCat\)-operadic model structure, but where the fibrant objects are precisely the \(\Oop\)-monoidal \(\iCat\)-categories with the cocartesian edges marked.

**Proof.** The construction of the \(\iCat\)-operadic model structure on \(sSet^+_{/(\Oop^\circ, N_e)}\) and the first three claims about it follows immediately from [Lur17, Thm. B.0.20] and the definition of a \(\iCat\)-operad. The fourth claim and the assertion about the model structure on \(sSet^+_{/(\Oop^\circ, N_e)}\) follow from [Lur17, Prop. B.2.7]. Finally, the analogous assertions about the \(\iCat\)-monoidal model structure all follow in the same way (cf. [Lur17, Var. 2.1.4.13]). □

### 2.6.3. Definition

We define the \(\infty\)-category of (small) \(\iCat\)-operads

\[
\Op_\iCat := N((sSet^+_{/(\E_op, N_e)})^f)
\]

to be the simplicial nerve of the full simplicial subcategory of \(sSet^+_{/(\E_op, N_e)}\) spanned by the fibrant objects in the \(\iCat\)-operadic model structure. We further let \(\Cat^\circ_{\iCat}\) denote the subcategory of \(\Op_\iCat\) spanned by the (small) \(\iCat\)-symmetric monoidal \(\iCat\)-categories and \(\iCat\)-symmetric monoidal functors thereof, or equivalently, the simplicial nerve \(N((sSet^+_{/(\E_op, N_e)})^f)\) taken with respect to the \(\iCat\)-monoidal model structure.

For a small \(\iCat\)-operad \(\Oop^\circ\), we then let

\[
\Op_{\Oop, \iCat} := N((sSet^+_{/(\Oop^\circ, N_e)})^f), \quad \Cat^\circ_{\Oop, \iCat} := N((sSet^+_{/(\Oop^\circ)}))^f).
\]

Note that \(\Op_{\Oop, \iCat} \simeq (\Op_\iCat)^\Oop\) and \(\Cat^\circ_{\Oop, \iCat}\) includes as the subcategory of \(\Op_{\Oop, \iCat}\) on the \(\Oop\)-monoidal \(\iCat\)-categories and morphisms thereof.

### 2.6.4. Corollary

For any small \(\iCat\)-operad \(\Oop^\circ\), the \(\infty\)-categories \(\Op_{\Oop, \iCat}\) and \(\Cat^\circ_{\Oop, \iCat}\) are presentable.

**Proof.** Since the \(\iCat\)-operadic model structure on \(sSet^+_{/(\Oop^\circ, N_e)}\) and the \(\iCat\)-monoidal model structure on \(sSet^+_{/(\Oop^\circ)}\) are combinatorial and simplicial by Theorem-Construction 2.6.2, it follows from [Lur09, Prop. A.3.7.6] that \(\Op_{\Oop, \iCat}\) and \(\Cat^\circ_{\Oop, \iCat}\) are presentable. □

Using the theory of categorical patterns, it is easy to construct cotensors in the \(\infty\)-category of \(\iCat\)-operads fibered over a given base.

### 2.6.5. Construction

Let \(\Oop^\circ\) be a \(\iCat\)-operad and let \(K\) be a marked simplicial set. By [Lur17, Prop. B.1.9] applied to the trivial categorical pattern on \(sSet^+\) and \(\Oop^\circ\) on \(sSet^+_{/(\Oop^\circ, N_e)}\), the functor

\[
(- \times K) : sSet^+_{/(\Oop^\circ, N_e)} \to sSet^+_{/(\Oop^\circ, N_e)}, \quad A \mapsto A \times K
\]

is left Quillen. We denote its right adjoint on fibrant objects by \((\C^\Oop, p) \mapsto ((\C^\Oop, p)^K, p^K)\) and the underlying \(\iCat\)-category by \((\C, p)^K\). Since this adjunction is also simplicial, for \((\C, p)\) and \((\D^\Oop, q)\) fibrations of \(\iCat\)-operads over \(\Oop^\circ\) we obtain equivalences of \(\infty\)-categories

\[
Alg_{\Oop, \iCat}(\C, (D, q)^K) \simeq Fun_{/(\Oop^\circ, N_e)}(K \times (\C^\Oop, N_e), (\D^\Oop, N_e)) \simeq Fun(K, Alg_{\Oop, \iCat}(\C, D)^\sim)
\]

where \((-)^\sim\) means we take the marking given by the equivalences. In other words, we have constructed the cotensor of \(\Op_{\Oop, \iCat}\) over \(\Cat\) at the level of marked simplicial sets. Note also that a fibrant replacement of \(K \times (\C^\Oop, N_e)\) computes the tensor. Repeating this analysis with the categorical pattern \(\mathcal{P}_\Oop\), we see that if \(\C^\Oop\) and \(\D^\Oop\) are \(\Oop\)-monoidal, then \((\D^\Oop, p)^K\) is moreover \(\Oop\)-monoidal, and we have equivalences of \(\infty\)-categories

\[
Fun_{\Oop, \iCat}(\C, (D, q)^K) \simeq Fun_{/(\Oop^\circ)}(K \times (\C^\Oop)^2, (\D^\Oop)^2) \simeq Fun(K, Fun_{\Oop, \iCat}(\C, D)^\sim).
\]
Moreover, note that if \( F : \mathcal{O} \to \textbf{Cat} \) denotes the functor classifying \( \mathcal{D} \to \mathcal{O} \), then \( (\mathcal{D}, q)^K \to \mathcal{O} \) is classified by the functor \( \mathcal{O} \to \textbf{Cat} \) given by applying \( \text{Fun}(K, (-)^\omega) \) fiberwise to \( F \). We may thus consider \( (\mathcal{D}^\omega, p)^K \) to be a construction of the pointwise \( \mathcal{O} \)-monoidal structure on \( (\mathcal{D}, q)^K \).

2.7. Big \( \mathcal{T} \)-\( \omega \)-operads. For certain arguments, it is technically inconvenient that the base \( \mathcal{T} \) does not admit pullbacks. We will thus need to consider an equivalent definition of a \( \mathcal{T} \)-\( \omega \)-operad.

2.7.1. Definition. Let \( \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{deg}}, \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{si}} \) denote the wide subcategories of \( \text{Ar}(\mathcal{F}_\mathcal{T}) \) on morphisms

\[
\sigma = \left( \begin{array}{ccc}
U & \to & X \\
\downarrow & & \downarrow \\
V & \to & Y
\end{array} \right)
\]

such that \( \text{ev}_1 : V \to Y \) is a degenerate edge, resp. the induced morphism \( U \to V \times_Y X \) is a summand inclusion. Then \( (\text{Ar}(\mathcal{F}_\mathcal{T}); \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{si}}, \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{deg}}) \) is a disjunctive triple, and we define

\[
\mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} := \text{Span}(\text{Ar}(\mathcal{F}_\mathcal{T}); \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{si}}, \text{Ar}(\mathcal{F}_\mathcal{T})^{\text{deg}}).
\]

Evaluation at the target defines a structure map \( \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \to \mathcal{F}_\mathcal{Op}^\sigma \), which is a cocartesian fibration.

2.7.2. Definition. We define a categorical pattern \( \overline{\text{P}}_{\mathcal{F}_\mathcal{T}} \) on \( \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \) as follows. Let \( \phi : U \to V \) be a morphism in \( \mathcal{F}_\mathcal{T} \) and \( \Sigma = \{\sigma_1, \ldots, \sigma_n\} \) be a collection of commutative squares in \( \mathcal{F}_\mathcal{T} \)

\[
\sigma_i = \left( \begin{array}{ccc}
U_i & \to & U \\
\downarrow & & \downarrow \\
V_i & \to & V
\end{array} \right)
\]

such that \( \alpha_i \) is a summand inclusion and the induced map \( U_i \to V_i \times_Y U \) is also a summand inclusion. Let \( \chi_{\sigma_i} : \Delta^1 \to (\text{Ar}(\mathcal{F}_\mathcal{T})^{\text{si}})^\text{op} \subset \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \) be the morphism corresponding to \( \sigma_i \).

Suppose moreover that the summand inclusions \( \alpha_i \) combine to yield an equivalence \( \prod_{1 \leq i \leq n} U_i \simeq U \). Let

\[
f_{\phi, \Sigma} : n^\sigma \to \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}}
\]

denote the functor which selects the \( n \) morphisms \( \chi_{\sigma_1}, \ldots, \chi_{\sigma_n} \). We then let

\[
\overline{\text{P}}_{\mathcal{F}_\mathcal{T}} = (\text{Ne}, \text{All}, \{f_{\phi, \Sigma}\})
\]

where \( \phi \) and \( \Sigma \) range over all possible choices.

2.7.3. Definition. The \( \mathcal{T} \)-\( \omega \)-operadic model structure on the category \( \text{sSet}^{+}_{/\overline{\text{P}}_{\mathcal{F}_\mathcal{T}, \text{Ne}}} \) is that defined by the categorical pattern \( \overline{\text{P}}_{\mathcal{F}_\mathcal{T}} \) of Definition 2.7.2 according to [Lur17, Thm. B.0.20]. We call the fibrant objects big \( \mathcal{T} \)-\( \omega \)-operads.

For any big \( \mathcal{T} \)-\( \omega \)-operad \( \mathcal{O}^\omega \to \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \), we then define the \( \mathcal{T} \)-\( \omega \)-operadic model structure on \( \text{sSet}^{+}_{/\overline{\text{P}}_{\mathcal{F}_\mathcal{T}, \text{Ne}}} \) via the categorical pattern

\[
\overline{\text{P}}_{\mathcal{O}} = (\text{Ne}, \text{All}, \{f_{x, \phi, \Sigma}\}),
\]

where the \( f_{x, \phi, \Sigma} : n^\sigma \to \mathcal{O}^\omega \) range over all cocartesian sections of the \( f_{\phi, \Sigma} \), and the \( x \) in the notation denotes the value of \( f_{x, \phi, \Sigma} \) on the cone point \( \nu \).

Given a big \( \mathcal{T} \)-\( \omega \)-operad \( \mathcal{O}^\omega \to \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \to \mathcal{F}_{\mathcal{T}, *}, \mathcal{T}^{\text{op}} \), we will let \( \mathcal{O}^\omega \to \mathcal{F}_{\mathcal{T}, *}, \mathcal{T}^{\text{op}} \) denote its pullback along the inclusion \( \mathcal{T}^{\text{op}} \subset \mathcal{F}_{\mathcal{T}, *}, \mathcal{T}^{\text{op}} \). Clearly, \( \mathcal{O}^\omega \) is a \( \mathcal{T} \)-\( \omega \)-operad.

2.7.4. Proposition. Let \( \mathcal{O}^\omega \) be a big \( \mathcal{T} \)-\( \omega \)-operad over \( \mathcal{F}_{\mathcal{T}, *, \sigma}^{\text{big}} \) and consider the span

\[
(\mathcal{O}^\omega, \text{Ne}) \xrightarrow{\text{ev}_0} (\text{Ar}(\mathcal{N}^\omega(\mathcal{O}^\omega)) \times_{\mathcal{O}^\omega} (\mathcal{O}^\omega, \text{Ne}) \xrightarrow{\text{pr}_{\mathcal{O}^\omega}} (\mathcal{O}^\omega, \text{Ne}).
\]

Then the adjunction

\[
(\text{pr}_{\mathcal{O}^\omega} : (\text{ev}_0)^*) : \text{sSet}^{+}_{/(\mathcal{O}^\omega, \text{Ne})} \rightleftarrows \text{sSet}^{+}_{/(\mathcal{O}^\omega, \text{Ne})} : (\text{ev}_0)_* (\text{pr}_{\mathcal{O}^\omega})^*
\]

is a Quillen equivalence.
Proof. We first show that \((\text{pr}_{\odot})_!(\text{ev}_0)\ast - (\text{ev}_0)_!(\text{pr}_{\odot})\ast\) is a Quillen adjunction. For this, it suffices to show that \((\text{ev}_0)_!(\text{pr}_{\odot})\ast\) preserves fibrant objects. Examining the proof of [Sha21b, Prop. 3.5(1)], we see that it implies
\[
\text{ev}_0 : \text{Ar}^{\ast//}(\odot) \times_{\odot} \odot \to \odot
\]
is a cartesian fibration, because any fiberwise active edge with target in \(\odot\) necessarily has source in \(\odot\). Therefore, the hypotheses of [Lur17, Thm. B.4.2] are satisfied excluding those involving the maps \(f_{x,\phi,\Sigma}\).

We deduce that \((\text{ev}_0)_!(\text{pr}_{\odot})\ast\) sends fibrant objects to fibrations over \(\odot\) which are cocartesian over the inert edges. Given a \(\mathcal{F}\)-\(\infty\)-operad \(\mathcal{C}\), let \(\tilde{\mathcal{C}}^{\ast} = (\text{ev}_0)_!(\text{pr}_{\odot})\ast(\mathcal{C}^{\ast})\). It remains to show that for every \(f_{x,\phi,\Sigma} : n \to \odot\),

(i) the functor \(n \to \mathbf{Cat}\) classifying the cocartesian fibration \(n \times_{\odot} \odot\) is a limit diagram.

(ii) For every cocartesian section \(n \to n \times_{\odot} \odot\), the composite \(n \to \tilde{\mathcal{C}}^{\ast}\) is a \(f\)-limit diagram for \(f : \tilde{\mathcal{C}}^{\ast} \to \odot\).

In fact, we only need to consider \(f_{x,\phi,\Sigma}\) where \(\Sigma\) is given by squares
\[
\begin{array}{ccc}
U_i & \longrightarrow & U \\
\downarrow & & \downarrow \phi \\
U_i & \longrightarrow & V
\end{array}
\]
with \(U_i\) an orbit, so we will suppose this in the remainder of the argument.

For (i), recall from [Sha21a, Ex. 2.26] that the right Kan extension of a functor \(\mathcal{C} \to \mathbf{Cat}\) along \(\mathcal{C} \to \mathcal{D}\) is modeled at the level of cocartesian fibrations by the push-pull construction involving the span
\[
\mathcal{D} \leftarrow \text{Ar}(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C} \to \mathcal{C}.
\]
Therefore, \(\tilde{\mathcal{C}}^{\ast}_n\) is the right Kan extension of \(\mathcal{C}^{\ast}_n\) along \(\odot^{\ast}_n \subset \odot^{\ast}_n\). In particular, given \(x \in \odot\) over \([f_+ : U_+ \to V] \in \mathcal{F}^{\text{big},\ast}_{x+}\), and an orbit decomposition \(V \simeq V_1 \coprod \ldots \coprod V_m\), let \(x \to x_j\) be inert morphisms lifting the cocartesian morphisms \([U_+ \to V] \to [(U \times V)_{+} \times V_j]\). Then we have an equivalence \(\tilde{\mathcal{C}}^{\ast}_n \simeq \coprod_{1 \leq j \leq m} \mathcal{C}^{\ast}_{x_j}\), and postcomposing with the further decompositions of \(\mathcal{C}^{\ast}_{x_j}\) given by orbit decompositions of \(U \times V\) verifies (i).

For (ii), it suffices to prove that for every fiberwise active edge \(\alpha : x \to y \in \odot\) over \([U'_+ \to V] \to [U_+ \to V]\), objects \(\overline{x}, \overline{y} \in \mathcal{C}^{\ast}\) over \(x, y\), and identification \(\overline{y} \simeq (\overline{y}_1, \ldots, \overline{y}_n)\) induced by \(f_{y,\phi,\Sigma}\),
\[
\text{Map}_{\mathcal{C}^{\ast}}^{\mathcal{C}^{\ast}}(\overline{x}, \overline{y}) \simeq \coprod_{1 \leq i \leq n} \text{Map}_{\mathcal{C}^{\ast}}^{\mathcal{C}^{\ast}}(\overline{x}_i, \overline{y}_i).
\]
where \(\alpha_i\) is the composition \(x \to y \to y_i\).

However, for an orbit decomposition \(V \simeq V_1 \coprod \ldots \coprod V_m\) and corresponding inert morphisms \(\overline{x} \to \overline{x}_j\), \(\overline{y} \to \overline{y}_j\), we have that
\[
\text{Map}_{\mathcal{C}^{\ast}}^{\mathcal{C}^{\ast}}(\overline{x}, \overline{y}) \simeq \coprod_{1 \leq j \leq m} \text{Map}_{\mathcal{C}^{\ast}}^{\mathcal{C}^{\ast}}(\overline{x}_j, \overline{y}_j)
\]
where \(\alpha \simeq (\alpha^j : x_j \to y_j)\) under the same decomposition for mapping spaces in \(\odot\). Using the known decompositions of mapping spaces in \(\mathcal{C}^{\ast}\) then yields the claim.

Finally, it is easy to see that the induced adjunction of \(\infty\)-categories
\[
\left(\text{Op}^{\text{big}}_{\mathcal{F}}\right)_{/\odot} \rightleftarrows (\text{Op}_{\mathcal{F}})_{/\odot}
\]
is an equivalence because the unit and counit transformations are equivalences. Hence the Quillen adjunction is a Quillen equivalence. \(\square\)

2.7.5. Corollary. Let \(\odot\) be a big \(\mathcal{F}\)-\(\infty\)-operad over \(\mathcal{F}^{\text{big}}_{\mathcal{F}}\), and let \(i : \odot \to \odot\) denote the inclusion. Then we have a Quillen equivalence
\[
i_! : \text{sSet}^{+}_{/\mathcal{F}_{/\odot, \mathcal{N}\mathcal{E}}} \rightleftarrows \text{sSet}^{+}_{/\mathcal{F}_{/\odot, \mathcal{N}\mathcal{E}}} : i^*
\]
2.8. Monoidal envelopes. In this subsection, we apply the theory of parametrized factorization systems [Sha21b, §3] to construct the O-monoidal envelope of any fibration of \( \mathcal{T}\)-\(\infty\)-operads \( C^\otimes \to O^\otimes \). First recall the notion of \( \mathcal{T}\)-factorization system from [Sha21b, Def. 3.1] and the associated ‘total’ factorization system of [Sha21b, Def. 3.2].

2.8.1. Example. We have the inert-active \( \mathcal{T}\)-factorization system on \( F_{\mathcal{T},*} \) given fiberwise by the inert-active factorization system on \( F_{\mathcal{T}/V,*} \) as in Remark 2.1.5. Note that the definition of (possibly non-fiberwise) inert and active edges in \( F_{\mathcal{T},*} \) given initially in Definition 2.1.3 then matches that of [Sha21b, Def. 3.2]. More generally, we have the inert-active \( \mathcal{T}\)-factorization system on any \( \mathcal{T}\)-\(\infty\)-operad \( O^\otimes \). From this, we obtain the inert-fiberwise active factorization system on \( O^\otimes \) itself.

2.8.2. Remark. The inert edges in \( F_{\mathcal{T},*} \) and \( F_{\mathcal{T},*}^\text{big} \) satisfy the following right cancellation property: if we have a 2-simplex

\[
\begin{array}{c}
  f \\
  x_0 \\
  x_1 \\
  x_2 \\
  \hline
  h \\
  g
\end{array}
\]

such that \( f \) is inert, then \( g \) is inert if and only if \( h \) is inert. The ‘only if’ direction is clear. To see the converse, note that by factoring \( f \) as a cocartesian edge followed by a fiberwise inert edge, we may suppose that \( f \) is of either form. Then by examining the composition of spans, we see that the claimed assertion reduces to the two-out-of-three property for equivalences in \( (F_{\mathcal{T}})^{/U} \), where \( x_2 \) lies over \( U \).

In contrast, the active edges do not satisfy the left cancellation property, because cocartesian edges do not. However, fiberwise active edges do satisfy the left cancellation property, just as they do in the theory of \( \infty\)-operads.

2.8.3. Notation. Given a \( \mathcal{T}\)-\(\infty\)-operad \( O^\otimes \), let \( Ar^{\text{act}}(O^\otimes) \) denote the full \( \mathcal{T}\)-subcategory of \( Ar(O^\otimes) \) on the active morphisms, and let

\[
Ar^{\text{act}}(O^\otimes) = \mathcal{T}^{\text{op}} \times_{Ar(\mathcal{T}^{\text{op}})} Ar^{\text{act}}(O^\otimes).
\]

2.8.4. Definition. Given a fibration of \( \mathcal{T}\)-\(\infty\)-operads \( p : C^\otimes \to O^\otimes \), the \( O\)-monoidal envelope of \( p \) is

\[
\text{Env}_{O,\mathcal{T}}(C^\otimes) := C^\otimes \otimes O^\otimes Ar^{\text{act}}(O^\otimes) \to O^\otimes.
\]

If \( O^\otimes = F_{\mathcal{T},*} \), we will abbreviate \( \text{Env}_{O,\mathcal{T}}(C^\otimes) \) as \( \text{Env}_{\mathcal{T}}(C^\otimes) \) and refer to it as the \( \mathcal{T}\)-symmetric monoidal envelope of \( C^\otimes \).

2.8.5. Remark. For a \( \mathcal{T}\)-\(\infty\)-operad \( C^\otimes \), the underlying \( \mathcal{T}\)-\(\infty\)-category of \( Env_{\mathcal{T}}(C^\otimes) \) is \( C^{\otimes}_{\text{act}} \).

2.8.6. Proposition. Let \( p : C^\otimes \to O^\otimes \) be a fibration of \( \mathcal{T}\)-\(\infty\)-operads. Then \( \text{Env}_{O,\mathcal{T}}(C^\otimes) \) is a \( O\)-monoidal \( \mathcal{T}\)-\(\infty\)-category.

Proof. We need to show that

\[
ev_1 : C^\otimes \otimes_{O^\otimes} Ar^{\text{act}}(O^\otimes) \to O^\otimes
\]

is a cocartesian fibration of \( \mathcal{T}\)-\(\infty\)-operads. By \([\text{Sha21b}, \text{Prop. 3.5(2)}]\), \( \ev_1 \) is a cocartesian fibration. We now seek to verify the criterion of Proposition 2.2.6 to finish the proof. Because \( O^\otimes \) is a \( \mathcal{T}\)-\(\infty\)-operad, for any object \( y \in O^\otimes_{[U_i \to V]} \), orbit decomposition \( U \simeq U_1 \sqcup \cdots \sqcup U_n \), and inert edges \( \rho^i : y \to y_i \) lifting the characteristic morphisms \( \chi_{[U_i \subset U]} \), the \( \rho^i \) induce an equivalence

\[
((O^\otimes_{\text{act}})^y U_1) \simeq \prod_{1 \leq i \leq n} ((O^\otimes_{\text{act}})^y U_i).
\]

Using that \( C^\otimes \to O^\otimes \) is a fibration of \( \mathcal{T}\)-\(\infty\)-operads, we have the further equivalence

\[
(C^\otimes_{\text{act}})^y \times (O^\otimes_{\text{act}})^y U_1) \simeq \prod_{1 \leq i \leq n} (C^\otimes_{\text{act}} U_i) \times (O^\otimes_{\text{act}} U_i) ((O^\otimes_{\text{act}})^y U_i).
\]

Proof. We need to show that

\[
ev_1 : C^\otimes \otimes_{O^\otimes} Ar^{\text{act}}(O^\otimes) \to O^\otimes
\]

is a cocartesian fibration of \( \mathcal{T}\)-\(\infty\)-operads. By \([\text{Sha21b}, \text{Prop. 3.5(2)}]\), \( \ev_1 \) is a cocartesian fibration. We now seek to verify the criterion of Proposition 2.2.6 to finish the proof. Because \( O^\otimes \) is a \( \mathcal{T}\)-\(\infty\)-operad, for any object \( y \in O^\otimes_{[U_i \to V]} \), orbit decomposition \( U \simeq U_1 \sqcup \cdots \sqcup U_n \), and inert edges \( \rho^i : y \to y_i \) lifting the characteristic morphisms \( \chi_{[U_i \subset U]} \), the \( \rho^i \) induce an equivalence

\[
((O^\otimes_{\text{act}})^y U_1) \simeq \prod_{1 \leq i \leq n} ((O^\otimes_{\text{act}})^y U_i).
\]

Using that \( C^\otimes \to O^\otimes \) is a fibration of \( \mathcal{T}\)-\(\infty\)-operads, we have the further equivalence

\[
(C^\otimes_{\text{act}})^y \times (O^\otimes_{\text{act}})^y U_1) \simeq \prod_{1 \leq i \leq n} (C^\otimes_{\text{act}} U_i) \times (O^\otimes_{\text{act}} U_i) ((O^\otimes_{\text{act}})^y U_i).
\]
Using that the fiberwise active edges are left cancellative, we identify the lefthand side with \((\mathcal{C}^0 \times_{\mathcal{O}^0} \text{Ar}^c_{\mathcal{E}}(\mathcal{O}^0))_y\), and similarly for the righthand side. The stated equivalence is then the one induced by the Segal maps, and we conclude that \(\text{ev}_1 : \mathcal{C}^0 \times_{\mathcal{O}^0} \text{Ar}^c_{\mathcal{E}}(\mathcal{O}^0) \to \mathcal{O}^0\) is a cocartesian fibration of \(\mathcal{T}_{\infty}\)-operads.

2.8.7. **Proposition.** Let \(p : \mathcal{C}^0 \to \mathcal{O}^0\) be a fibration of \(\mathcal{T}_{\infty}\)-operads and let \(q : \mathcal{D}^0 \to \mathcal{O}^0\) be a cocartesian fibration of \(\mathcal{T}_{\infty}\)-operads. Let \(i : \mathcal{C}^0 \subset \text{Env}_{\mathcal{O}, \tau}(\mathcal{O}^0)\) denote the inclusion of \(\mathcal{C}^0\) into its \(\mathcal{O}\)-monoidal envelope.

\[
\begin{align*}
(1) & \quad \text{Precomposition by } i \text{ yields an equivalence} \\
& \quad i^* : \text{Fun}^\otimes_{\mathcal{O}, \tau}(\text{Env}_{\mathcal{O}, \tau}(\mathcal{E}), \mathcal{D}) \to \text{Alg}_{\mathcal{O}, \tau}(\mathcal{C}, \mathcal{D}).
\end{align*}
\]

(2) We have an adjunction

\[
\begin{align*}
i_! : \text{Alg}_{\mathcal{O}, \tau}(\mathcal{C}, \mathcal{D}) & \rightleftarrows \text{Alg}_{\mathcal{O}, \tau}(\text{Env}_{\mathcal{O}, \tau}(\mathcal{E}), \mathcal{D}) : i^*
\end{align*}
\]

where \(i_!\) is the fully faithful inclusion of \(\text{Fun}^\otimes_{\mathcal{O}, \tau}(\text{Env}_{\mathcal{O}, \tau}(\mathcal{E}), \mathcal{D})\) under the equivalence of (1).

**Proof.** This follows immediately from [Sha21b, Thm. 3.6], using the inert-active \(\mathcal{T}\)-factorization system on \(\mathcal{O}^0\).

2.8.8. **Corollary.** Let \(\mathcal{O}^0\) be a \(\mathcal{T}_{\infty}\)-operad. We have an adjunction

\[
\begin{align*}
\text{Env}_{\mathcal{O}, \tau} & : \text{Op}_{\mathcal{O}, \tau} \rightleftarrows \text{Cat}_{\mathcal{O}, \tau} : \text{U}.
\end{align*}
\]

2.9. **Subcategories and localization.** Let \(\mathcal{C}^0 \to \mathcal{O}^0\) be a fibration of \(\mathcal{T}_{\infty}\)-operads. Given a full \(\mathcal{T}\)-subcategory \(\mathcal{D} \subset \mathcal{C}\), let \(\mathcal{D}^0\) be the full \(\mathcal{T}\)-subcategory of \(\mathcal{C}^0\) on the objects of \(\mathcal{D}\) (using the Segal decompositions of the fibers of \(\mathcal{C}^0\)). Clearly, \(\mathcal{D}^0 \to \mathcal{O}^0\) is a fibration of \(\mathcal{T}_{\infty}\)-operads, and the inclusion \(\mathcal{D}^0 \to \mathcal{C}^0\) is a morphism of \(\mathcal{T}_{\infty}\)-operads over \(\mathcal{O}^0\). In this subsection, we state conditions under which \(\mathcal{D}^0\) inherits an \(\mathcal{O}\)-monoidal structure from \(\mathcal{C}^0\). Our presentation of these results parallels and extends those of [Lur17, §2.2.1].

2.9.1. **Proposition.** Let \(\mathcal{C}^0 \to \mathcal{O}^0\) be a fibration of \(\mathcal{T}_{\infty}\)-operads and let \(\mathcal{D} \subset \mathcal{C}\) be a full \(\mathcal{T}\)-subcategory.

Suppose that for every fiberwise active edge \(\alpha : x \to y\) in \(\mathcal{O}_V^0\) with \(y \in \mathcal{O}_V\), the pushforward functor \(\otimes_{\alpha} : \mathcal{C}_x^0 \to \mathcal{C}_y\) restricts to a functor \(\mathcal{D}_x^0 \to \mathcal{D}_y\). Then \(\mathcal{D}^0 \to \mathcal{O}^0\) is a cocartesian fibration and the inclusion \(\mathcal{D}^0 \to \mathcal{C}^0\) is an \(\mathcal{O}\)-monoidal \(\mathcal{T}\)-functor.

**Proof.** This is immediate in light of the inert-fiberwise active factorization system on \(\mathcal{C}^0\) (Example 2.8.1).

We say that a \(\mathcal{T}\)-functor \(L : \mathcal{C} \to \mathcal{C}\) is a \(\mathcal{T}\)-localization if for every object \(V \in \mathcal{T}\), \(L_V\) is a localization functor. If we let \(\mathcal{D}\) denote the essential image of \(L\), then by [Lur17, 7.3.2.6] we have a \(\mathcal{T}\)-adjunction

\[
\begin{align*}
L : \mathcal{C} & \rightleftarrows \mathcal{D} : R
\end{align*}
\]

where \(R : \mathcal{D} \to \mathcal{C}\) is the inclusion. Given a \(\mathcal{T}\)-localization \(L : \mathcal{C} \to \mathcal{C}\), a morphism in \(\mathcal{C}\) is an \(L\)-equivalence if it lies in a fiber \(\mathcal{C}_V\) and is a \(L_V\)-equivalence. Similarly, given a \(\mathcal{T}_{\infty}\)-operad \(\mathcal{C}^0\), a morphism in \(\mathcal{C}^0\) is an \(L\)-equivalence if it lies entirely in a fiber \(\mathcal{C}_{\mathcal{O}}(\mathcal{O}_V, \mathcal{V})\) and is a product of \(L\)-equivalences under the Segal decomposition of that fiber.

2.9.2. **Theorem.** Let \(\mathcal{O}^0\) be a \(\mathcal{T}_{\infty}\)-operad and \(\mathcal{C}^0 \to \mathcal{O}^0\) an \(\mathcal{O}\)-monoidal \(\mathcal{T}_{\infty}\)-category. Let \(L : \mathcal{C} \to \mathcal{C}\) be a \(\mathcal{T}\)-localization and let \(\mathcal{D} \subset \mathcal{C}\) be its essential image. Suppose that for every fiberwise active edge \(\alpha : x \to y\) with \(y \in \mathcal{O}_V\), the pushforward functor \(\otimes_{\alpha} : \mathcal{C}_x^0 \to \mathcal{C}_y\) preserves \(L\)-equivalences. Then we have a relative adjunction over \(\mathcal{O}^0\)

\[
\begin{align*}
L^\otimes : \mathcal{C}^0 & \rightleftarrows \mathcal{D}^0 : R^\otimes
\end{align*}
\]

with \(L^\otimes\) an \(\mathcal{O}\)-monoidal functor (i.e., preserving cocartesian edges over \(\mathcal{O}^0\)) and \(R^\otimes\) a lax \(\mathcal{O}\)-monoidal functor (i.e., a morphism of \(\mathcal{T}_{\infty}\)-operads), which prolongs the \(\mathcal{T}\)-adjunction \(L : \mathcal{C} \rightleftarrows \mathcal{D} : R\).

**Proof.** This is immediate from the inert-fiberwise active factorization system on \(\mathcal{C}^0\) (Example 2.8.1) together with the criterion of [BH21, Prop. D.7].

2.9.3. **Remark.** In the case where \(\mathcal{O}^0 = \mathcal{E}_{\mathcal{T}, \tau}\), the criterion of Theorem 2.9.2 amounts to

(1) For every object \(V \in \mathcal{T}\) and \(Z \in \mathcal{C}_V\), the functor

\[
- \otimes Z : \mathcal{C}_V \to \mathcal{C}_V
\]

preserves \(L_V\)-equivalences.
For every morphism $f : V \to W$ in $\mathcal{T}$, the norm functor

$$f_\otimes : \mathcal{E}_V \to \mathcal{E}_W$$

sends $L_V$-equivalences to $L_W$-equivalences.

3. Parametrized Day convolution

In this section, we construct a (partially-defined) internal hom for $\mathcal{T}$-$\infty$-operads fibered over an arbitrary base $\mathcal{T}$-$\infty$-operad $\mathcal{O}^\otimes$: the $\mathcal{T}$-Day convolution. We first introduce the notion of an $\mathcal{O}$-promonoidal $\mathcal{T}$-$\infty$-category $\mathcal{C}^\otimes$, which is the analogue of a flat categorical fibration\(^{11}\) in the context of $\mathcal{T}$-$\infty$-operads. The $\mathcal{O}$-promonoidal condition ensures the existence of the $p$-operadic coinduction functor (Corollary 3.1.5), and given a $\mathcal{O}$-promonoidal $\mathcal{T}$-$\infty$-category $p : \mathcal{E}^\otimes \to \mathcal{O}^\otimes$ and any fibration $\mathcal{E}^\otimes \to \mathcal{O}^\otimes$ of $\mathcal{T}$-$\infty$-operads, we may then use the $p$-operadic coinduction on the pullback of $\mathcal{E}^\otimes$ over $\mathcal{O}^\otimes$ to construct the $\mathcal{T}$-Day convolution (Definition 3.1.6)

$$\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{E}, \mathcal{E})^\otimes \to \mathcal{O}^\otimes.$$

We then state conditions on $\mathcal{E}^\otimes$ under which $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{E}, \mathcal{E})^\otimes$ is $\mathcal{O}$-monoidal – these amount to the existence of certain $\mathcal{T}$-colimits as well as $\mathcal{T}$-distributivity of the tensor product (Theorem 3.2.6).

3.1. $\mathcal{O}$-promonoidal $\mathcal{T}$-$\infty$-categories and $\mathcal{T}$-Day convolution.

3.1.1. Definition. Let $p : \mathcal{E}^\otimes \to \mathcal{O}^\otimes$ be a fibration of $\mathcal{T}$-$\infty$-operads. We say that $\mathcal{E}^\otimes$ is $\mathcal{O}$-promonoidal if for every $V \in \mathcal{T}$, the functor $p_\ast : (\mathcal{E}^\otimes)_\ast \to (\mathcal{O}^\otimes)_\ast$ is a flat categorical fibration.

3.1.2. Example. Suppose $(\mathcal{E}^\otimes, p)$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category, so that $p$ is a cocartesian fibration. $\mathcal{E}^\otimes$ is then $\mathcal{O}$-promonoidal since cocartesian fibrations are flat [Lur17, Ex. B.3.4].

3.1.3. Remark. To understand the relevance of the $\mathcal{O}$-promonoidal condition, the reader may find it useful to first review the nonparametrized story from [Sha21b, §10]. To our knowledge, the correct definition of an $\mathcal{O}$-promonoidal $\infty$-category first appeared in Hinich’s work [Hin20] (and was misstated in [BGS20]).

For technical reasons, to construct the $\mathcal{T}$-Day convolution we will first work in the setting of big $\mathcal{T}$-$\infty$-operads (i.e., so that the base is $\mathcal{F}_\mathcal{T}^{\text{op}}$ in place of $\mathcal{F}_\mathcal{T}$). Let $\text{Ar}^{\text{nc}}(\mathcal{O}^\otimes)$ be notation for the full subcategory of $\text{Ar}(\mathcal{O}^\otimes)$ on the inert edges.

3.1.4. Theorem. Let $\mathcal{O}^\otimes$ and $\mathcal{E}^\otimes$ be big $\mathcal{T}$-$\infty$-operads over $\mathcal{F}_\mathcal{T}^{\text{big}}$, and let $p : \mathcal{E}^\otimes \to \mathcal{O}^\otimes$ be a fibration of big $\mathcal{T}$-$\infty$-operads such that the restriction $p' : \mathcal{E}^\otimes \to \mathcal{O}^\otimes$ is $\mathcal{O}$-promonoidal. Consider the span of marked simplicial sets

$$(\mathcal{O}^\otimes, \text{Ne}) \xleftarrow{\text{ev}_0} (\text{Ar}^{\text{nc}}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes, \text{Ne}) \xrightarrow{\text{pr}_{\mathcal{E}^\otimes}} (\mathcal{E}^\otimes, \text{Ne})$$

where by the middle marking $\text{Ne}$ we mean those edges in $\text{Ar}^{\text{nc}}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes$ whose source in $\mathcal{O}^\otimes$ is inert and whose projection to $\mathcal{E}^\otimes$ is inert. Then the functor

$$(\text{ev}_0)_\ast \circ (\text{pr}_{\mathcal{E}^\otimes})^\ast : \text{sSet}_+/_{(\mathcal{O}^\otimes, \text{Ne})} \to \text{sSet}_+/_{(\mathcal{E}^\otimes, \text{Ne})}$$

is right Quillen with respect to the $\mathcal{T}$-operadic model structures of Definition 2.7.3.

Proof. We verify the hypotheses of [Lur17, Thm. B.4.2].

1. $\text{ev}_0$ is flat by [Sha21b, Lem. 10.1] applied to the inert-fiberwise active factorization system on $\mathcal{O}^\otimes$ (Example 2.8.1), noting that products of flat fibrations are flat in order to promote the flatness condition on $(p')_{\ast\ast} \to pf_{\ast\ast}$.
2. It is obvious that inert edges are closed under composition and contain the equivalences.
3. Vacuously true since the categorical patterns we are looking at contain all 2-simplices.

\(^{11}\)Some authors also call this an exponentiable fibration to highlight its key property: a categorical fibration $\pi : \mathcal{E} \to \mathcal{D}$ is said to be flat if the right adjoint $\pi_+$ to the pullback functor $\pi^+ : \mathcal{Cat}_{/\mathcal{D}} \to \mathcal{Cat}_{/\mathcal{C}}$ exists.
(4) Suppose $e : x_0 \to y_0$ is an inert edge in $\tilde{O}^\otimes$. Then as we saw in [Sha21b, Prop. 3.5(1)], given an inert edge $y_0 \to y_1$ the cartesian lift of $e$ to an edge $e' : [x_0 \to x_1] \to [y_0 \to y_1]$ in $\text{Ar}^{nc}(\tilde{O}^\otimes)$ has $ev_1(e') : x_1 \to y_1$ an equivalence. It follows that given a further lift of $y_0 \to y_1$ to an object $(y_0 \to y_1, c \in \tilde{C}_y^\otimes)$ in $\text{Ar}^{nc}(\tilde{O}^\otimes) \times \tilde{C}_y^\otimes$, $e$ admits a cartesian lift to an edge $e'' : (x_0 \to x_1, e') \to (y_0 \to y_1, c)$ (with $e' \to c$ an equivalence).

(5) Let $f_{x, \phi, \Sigma} : n^\triangleleft \to \tilde{O}^\otimes$ be in the categorical pattern defining the $T$-operadic model structure on $\text{sSet}^+_n(\tilde{O}^\otimes, \text{Ne})$. We claim that the pullback

$$
\pi : n^\triangleleft \times \tilde{O}^\otimes \times \text{Ar}(\tilde{O}^\otimes) \times \tilde{C}^\otimes \to n^\triangleleft
$$

is a cocartesian fibration. Because inert edges are right cancellative (Remark 2.8.2) and $\tilde{O}^\otimes$ is cocartesian over the inert edges of $\tilde{O}^\otimes$, it suffices to show that

$$
n^\triangleleft \times \tilde{O}^\otimes \times \text{Ar}(\tilde{O}^\otimes) \to n^\triangleleft$$

is cocartesian, where $\tilde{O}^\otimes \subset \tilde{O}^\otimes$ is the wide subcategory with morphisms restricted to the inert edges. In fact, we will prove the stronger assertion that

$$
ev_0 : \text{Ar}(\tilde{O}^\otimes) \to \tilde{O}^\otimes_{nc}
$$

is cocartesian. For this, by [Lur09, 6.1.1.1], it suffices to show that $\tilde{O}^\otimes_{nc}$ admits pushouts. Since the inert edges in $\tilde{O}^\otimes$ are defined to be the cocartesian lifts of inert edges in $E_{T, s}^{\text{big}}$, we may reduce to the case $\tilde{O}^\otimes = E_{T, s}^{\text{big}}$. It then suffices to show that $\text{Ar}(\text{F}_T)^\dagger$ (as in Definition 2.7.1) admits pullbacks. So suppose we have a commutative cube

\[
\begin{array}{ccc}
W \times_U X & \to & X \\
| & | \\
W & \to & U \\
| & | \\
Z \times_V Y & \to & Y \\
| & | \\
Z & \to & V
\end{array}
\]

We want to show that if $W \to Z \times_V U$ is a summand inclusion, then $W \times_U X \to Z \times_Y X$ is a summand inclusion. To see this, consider the diagram

\[
\begin{array}{ccc}
W \times_U X & \to & Z \times_Y X & \to & X \\
| & | & | & | \\
W & \to & Z \times_V U & \to & U
\end{array}
\]

The right square and outer rectangle are both pullback squares, so the left square is as well. Since summand inclusions are stable under pullback, the desired conclusion follows.

(6) Let $s : n^\triangleleft \to n^\triangleleft \times \tilde{O}^\otimes \times \tilde{C}^\otimes$ be a cocartesian section of $\pi$ defined as above. Suppose that $s(\{v\}) = (x \xrightarrow{c} y \in \tilde{O}^\otimes_{nc_\phi}, c \in \tilde{C}^\otimes_y)$ and $e$ is a cocartesian lift of the inert morphism in $E_{T, s}^{\text{big}}$ defined by the square

\[
\begin{array}{ccc}
U' & \to & U \\
\downarrow^{\phi'} & & \downarrow^{\phi} \\
V' & \to & V
\end{array}
\]
in $F_\tau$. If $\Sigma = \{\sigma_1, ..., \sigma_n\}$ is given by squares $\sigma_i$

\[
\begin{array}{ccc}
U_i & \longrightarrow & U \\
\downarrow & & \downarrow^\phi \\
V_i & \longrightarrow & V
\end{array}
\]

then let $\Sigma' = \{\sigma'_1, ..., \sigma'_n\}$ be given by the collection of squares $\sigma'_i$

\[
\begin{array}{ccc}
U_i \times_U U' & \longrightarrow & U' \\
\downarrow & & \downarrow^\phi' \\
V_i \times_V V' & \longrightarrow & V'.
\end{array}
\]

Because summand inclusions are stable under pullback, the morphisms $U_i \times_U U' \longrightarrow U'$ are summand inclusions, and clearly induce an equivalence $\coprod_{1 \leq i \leq n} U_i \times_U U' \simeq U'$. Therefore, the data of $(c, \phi', \Sigma')$ defines a morphism $f_{c, \phi', \Sigma'} : n^{-} \longrightarrow \tilde{C}^\otimes$ which is part of the categorical pattern defining the $\mathcal{T}$-operadic model structure on $\mathbf{sSet}^+_{/ (\tilde{C}^\otimes, \text{Ne})}$. Moreover, by the analysis done in (5) we may identify the composite map

\[
n^{-} \longrightarrow n^{-} \times_{\tilde{O}^\otimes} \text{Ar}^{nc}(\tilde{O}^\otimes) \times_{\tilde{O}^\otimes} \tilde{C}^\otimes \longrightarrow \tilde{C}^\otimes
\]

with $f_{c, \phi', \Sigma'}$.

(7) We check the following: suppose we have a commutative diagram

\[
\begin{array}{ccc}
x_0 & \longrightarrow & x_1 & \longrightarrow & x_2 \\
\downarrow & & \downarrow & & \downarrow \\
y_0 & \longrightarrow & y_1 & \longrightarrow & y_2
\end{array}
\]

in $\text{Ar}^{nc}(\tilde{O}^\otimes)$ and $c_0 \longrightarrow c_1 \longrightarrow c_2$ in $\tilde{C}^\otimes$ that covers $y_0 \longrightarrow y_1 \longrightarrow y_2$, such that $c_1 \longrightarrow c_2$ is an equivalence (so $y_1 \longrightarrow y_2$ is an equivalence), $x_1 \longrightarrow x_2$ is inert, $x_0 \longrightarrow x_1$ is an equivalence. Then $c_0 \longrightarrow c_1$ is inert if and only if $c_0 \longrightarrow c_2$ is inert. But this is clear from the definitions.

(8) It suffices to check the following: suppose we have a commutative diagram

\[
\begin{array}{ccc}
x_0 & \longrightarrow & x_1 & \longrightarrow & x_2 \\
\downarrow & & \downarrow & & \downarrow \\
y_0 & \longrightarrow & y_1 & \longrightarrow & y_2
\end{array}
\]

in $\text{Ar}^{nc}(\tilde{O}^\otimes)$ and $c_0 \longrightarrow c_1 \longrightarrow c_2$ in $\tilde{C}^\otimes$ that covers $y_0 \longrightarrow y_1 \longrightarrow y_2$, such that $x_0 \longrightarrow x_1$, $y_0 \longrightarrow y_1$, and $c_0 \longrightarrow c_1$ are inert. Then $\{x_1 \longrightarrow x_2, y_1 \longrightarrow y_2, c_1 \longrightarrow c_2\}$ are inert if and only if $\{x_0 \longrightarrow x_2, y_0 \longrightarrow y_2, c_0 \longrightarrow c_2\}$ are inert. But this follows from the right cancellativity of inert morphisms (Remark 2.8.2).

□

Having passed to the big $\mathcal{T}$-$\infty$-operads to construct the $\mathcal{T}$-operadic coinduction and verify its properties, we now pass back to the usual formulation of $\mathcal{T}$-$\infty$-operads.

3.1.5. **Corollary.** Let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-$\infty$-operad and let $(\mathcal{C}^\otimes, p)$ be an $\mathcal{O}$-promonoidal $\mathcal{T}$-$\infty$-category. Consider the span diagram of marked simplicial sets

\[
(\mathcal{O}^\otimes, \text{Ne}) \xleftarrow{\text{ev}_0} (\text{Ar}^{nc}(\mathcal{O}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes, \text{Ne}) \xrightarrow{\text{pr}_{\mathcal{C}^\otimes}} (\mathcal{C}^\otimes, \text{Ne}).
\]

Then the functor

\[
(\text{ev}_0)_* \circ (\text{pr}_{\mathcal{C}^\otimes})^* : \mathbf{sSet}^+_{/(\mathcal{C}^\otimes, \text{Ne})} \longrightarrow \mathbf{sSet}^+_{/(\mathcal{O}^\otimes, \text{Ne})}
\]

is right Quillen with respect to the $\mathcal{T}$-operadic model structures.

**Proof.** Combine Theorem 3.1.4, Proposition 2.7.4, and Corollary 2.7.5. □
3.1.6. **Definition.** In the situation of Corollary 3.1.5, given a fibration of $\mathcal{T}\text{-}\infty$-operads $\mathcal{E}^\oplus \to \mathcal{C}^\oplus$, define the $p$-operadic coinduction or $p$-norm of $\mathcal{E}^\oplus$ to be the $\mathcal{T}\text{-}\infty$-operad
\[(\text{Norm}_p\mathcal{E})^\oplus := (\text{ev}_0)_*(\text{pr}_{\mathcal{E}^\oplus})^*(\mathcal{E}^\oplus, \text{Ne}),\]
given as a marked simplicial set fibered over $(\mathcal{O}^\oplus, \text{Ne})$.

Given a fibration of $\mathcal{T}\text{-}\infty$-operads $\mathcal{E}^\oplus \to \mathcal{O}^\oplus$, define the *Day convolution* $\mathcal{T}\text{-}\infty$-operad to be
\[\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{E})^\oplus := (\text{Norm}_p \mathcal{E}^\oplus)^\oplus.\]

If $\mathcal{O}^\oplus = \mathcal{E}_{\mathcal{T},p}$, we will also denote $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{E})^\oplus$ as $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{E})^\oplus$.

3.1.7. **Proposition.** Let $p, q : \mathcal{D}^\oplus \to \mathcal{O}^\oplus$ be fibrations of $\mathcal{T}\text{-}\infty$-operads. Then the functor
\[\iota : (\mathcal{D}^\oplus \times_{\mathcal{O}^\oplus} \mathcal{O}^\oplus, \text{Ne}) \to (\mathcal{D}^\oplus \times_{\mathcal{O}^\oplus} \text{Ar}^{nc}(\mathcal{O}^\oplus) \times_{\mathcal{O}^\oplus} \mathcal{C}^\oplus, \text{Ne})\]
induced by the identity section is a homotopy equivalence in $\text{sSet}_f^{\mathcal{T}/(\mathcal{O}^\oplus, \text{Ne})}$. Consequently, for $\mathcal{O}$-promonoidal $(\mathcal{C}^\oplus, p)$, $\iota^*$ induces an equivalence of $\mathcal{T}\text{-}\infty$-categories
\[\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{D}, \text{Norm}_p \mathcal{E}) \xrightarrow{\sim} \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{D} \times_\mathcal{O} \mathcal{C}, \mathcal{E})\]
and an equivalence of $\infty$-categories
\[\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{D}, \text{Norm}_p \mathcal{E}) \xrightarrow{\sim} \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{D} \times_\mathcal{O} \mathcal{C}, \mathcal{E}).\]

**Proof.** Let $P : \mathcal{D}^\oplus \times_{\mathcal{O}^\oplus} \text{Ar}^{nc}(\mathcal{O}^\oplus) \to \mathcal{D}^\oplus$ be a cocartesian pushforward chosen so that $P|_{\mathcal{D}^\oplus} = \text{id}$, and let
\[P' = P \times \text{id}_{\mathcal{C}^\oplus} : \mathcal{D}^\oplus \times_{\mathcal{O}^\oplus} \text{Ar}^{nc}(\mathcal{O}^\oplus) \times_{\mathcal{O}^\oplus} \mathcal{C}^\oplus \to \mathcal{D}^\oplus \times_{\mathcal{O}^\oplus} \mathcal{C}^\oplus.\]

Then $P'$ respects the given markings, and as in the proof of [Sha21a, Lem. 3.2(2)] we may construct an explicit homotopy $h : \text{id} \to \iota \circ P'$ such that $h$ sends objects to fiberwise marked edges. This shows that $P$ is a marked homotopy inverse to $\iota$ and hence $\iota$ is a homotopy equivalence in $\text{sSet}_f^{\mathcal{T}/(\mathcal{O}^\oplus, \text{Ne})}$. The consequences then follow from the definition of the left adjoint to $\text{Norm}_p$. \(\square\)

3.1.8. **Corollary.** The Quillen adjunction of Corollary 3.1.5 descends to an adjunction of $\infty$-categories
\[p^* : (\text{Op}_{\mathcal{T}})/\mathcal{O}^\oplus \rightleftarrows (\text{Op}_{\mathcal{T}})/\mathcal{C}^\oplus : \text{Norm}_p.\]
In particular, if $(\mathcal{C}^\oplus, p)$ is $\mathcal{O}$-promonoidal, then the right adjoint to $p^*$ exists and is computed by $\text{Norm}_p$.

3.1.9. **Proposition.** The underlying $\mathcal{T}\text{-}\infty$-category of $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})^\oplus$ is the $\mathcal{T}$-pairing construction $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})^\oplus$ of [Sha21a, Constr. 9.1]. In particular, if $\mathcal{O}^\oplus = \mathcal{E}_{\mathcal{T},p}$, the underlying $\mathcal{T}\text{-}\infty$-category of $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})^\oplus$ is $\text{Fun}_{\mathcal{T}}(\mathcal{E}, \mathcal{C})$.

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
\text{Ar}^{\text{cocart}}(\mathcal{O}) \times_\mathcal{O} \mathcal{C} & \to & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O} \times_\mathcal{O} \text{Ar}^{nc}(\mathcal{O}^\oplus) \times_\mathcal{O} \mathcal{C}^\oplus & \to & \text{Ar}^{nc}(\mathcal{O}^\oplus) \times_\mathcal{O} \mathcal{C}^\oplus \\
\downarrow & & \downarrow \\
\mathcal{O} \times_\mathcal{O} \mathcal{O}^\oplus & \to & \mathcal{O}^\oplus
\end{array}
\]
where $\iota$ is defined using the inclusions $\mathcal{O} \subset \mathcal{O}^\oplus$ and $\mathcal{C} \subset \mathcal{C}^\oplus$. Unwinding the definitions, to prove the claim it suffices to show that the map
\[\text{Ar}^{\text{cocart}}(\mathcal{O}) \times_\mathcal{O} \mathcal{C} \to \mathcal{O} \times_\mathcal{O} \text{Ar}^{nc}(\mathcal{O}^\oplus) \times_\mathcal{O} \mathcal{O} \times_\mathcal{O} \mathcal{C}\]
is a homotopy equivalence (in $\text{sSet}_f^{\mathcal{T}/\mathcal{O}}$, via the target map). But this is clear, since the inert edges in $\mathcal{O}^\oplus$ with source and target in $\mathcal{O}$ are, up to equivalence, precisely the cocartesian edges in $\mathcal{O}$. \(\square\)
3.2. $\mathcal{O}$-monoidality of the $\mathcal{I}$-Day convolution. We now establish $\mathcal{O}$-monoidality of the $\mathcal{I}$-Day convolution $\mathcal{I}$-$\infty$-operad given appropriate conditions on the input $\mathcal{I}$-$\infty$-operads. For this, we will use repeatedly the following criterion for when a fibration of $\mathcal{I}$-$\infty$-operads is locally cartesian or cocartesian.

3.2.1. Lemma. Let $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ be a fibration of $\mathcal{I}$-$\infty$-operads. Suppose that for every fiberwise active edge $e : x \longrightarrow y$ in $\mathcal{O}^\otimes$ with $y \in \mathcal{O}$, and $c \in \mathcal{O}^\otimes$ with $p(c) = x$, there exists a locally cartesian edge $f : c \longrightarrow c'$ over $e$. Then $p$ is a locally cocartesian fibration.

Furthermore, suppose that for every composition of fiberwise active edges $x \xrightarrow{e} y \xrightarrow{e'} z$ in $\mathcal{O}^\otimes$ with $z \in \mathcal{O}$, and $c \in \mathcal{O}^\otimes$ with $p(c) = x$, locally cocartesian lifts of $e$ to $f : c \longrightarrow c'$ and $e'$ to $f' : c' \longrightarrow c''$ compose to yield a locally cocartesian edge $f'' : c \longrightarrow c''$. Then $p$ is a cocartesian fibration.

Proof. For the first assertion, using the inert-fiberwise active factorization system on $\mathcal{O}^\otimes$ and that $p$ admits cocartesian lifts over inert edges, we reduce to checking that $p$ admits locally cocartesian lifts over fiberwise active edges. Then given any fiberwise active edge $e : x \longrightarrow y$ in $\mathcal{O}^\otimes$, we may use that $\mathcal{O}^\otimes$ is a $\mathcal{I}$-$\infty$-operad to obtain a decomposition of $e$ as $(e_i : x_i \longrightarrow y_i)_{i \in I}$ for $y_i \in \mathcal{O}$ under the identifications $\mathcal{O}^\otimes \simeq \prod_{i \in I} \mathcal{O}^\otimes_{y_i,}$, $\mathcal{O}^\otimes \simeq \prod_{i \in I} \mathcal{O}^\otimes_{x_i},$ and $\text{Map}_{\mathcal{O}^\otimes}(x, y) \simeq \prod_{i \in I} \text{Map}_{\mathcal{O}^\otimes}(x_i, y_i)$. Suppose $c \in \mathcal{C}^\otimes$ over $x$. Because $p$ is a fibration of $\mathcal{I}$-$\infty$-operads, we get $c_i \in \mathcal{C}^\otimes$ over $x_i$, and we may take the product of locally cocartesian lifts $c_i \longrightarrow c'_i$ over the $e_i$ to obtain the desired locally cocartesian lift $c \longrightarrow c'$ of $e$.

For the second assertion, we need to check that locally cocartesian edges compose to yield again a locally cocartesian edge. Note that for any commutative diagram in $\mathcal{C}^\otimes$

\[
\begin{array}{ccc}
c & \xrightarrow{f} & d \\
\downarrow{g} & & \downarrow{g'} \\
c' & \xrightarrow{f'} & d'
\end{array}
\]

with $f$ locally cocartesian over a fiberwise active edge and $g, g'$ inert, then $f'$ is necessarily locally cocartesian. Using this, we can reduce to checking that locally cocartesian edges over fiberwise active edges compose. Then as before, we can further reduce to supposing that the last object lies in $\mathcal{O}$.

The next proposition allows us to understand $\mathcal{I}$-Day convolution as a locally cocartesian fibration over the base $\mathcal{I}$-$\infty$-operad $\mathcal{O}^\otimes$.

3.2.2. Proposition. Let $(\mathcal{C}^\otimes, p)$ be an $\mathcal{O}$-promonoidal $\mathcal{I}$-$\infty$-category and let $(\mathcal{E}^\otimes, q)$ be an $\mathcal{O}$-monoidal $\mathcal{I}$-$\infty$-category in which for every object $x \in \mathcal{O}_V$, the parametrized fiber $\mathcal{E}^\otimes_x$ is $\mathcal{I}^/V$-cocomplete.

1. The $\mathcal{I}$-Day convolution $\mathcal{F}_{\mathcal{O}, \mathcal{I}} = \mathcal{F}_{\mathcal{C}, \mathcal{E}} : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ is a locally cocartesian fibration, and for every $x \in \mathcal{O}_V$, its parametrized fiber $\mathcal{F}^\otimes_x$ is $\mathcal{I}^/V$-cocomplete.

2. Suppose $\mathcal{O}$ is a fiberwise active edge in $\mathcal{F}_{\mathcal{O}, \mathcal{I}} = \mathcal{F}_{\mathcal{C}, \mathcal{E}} : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$, which in turn lifts $f : U \longrightarrow V$ in $\mathcal{F}_{\mathcal{I}}$ (identified with the fiberwise active edge $\{U \rightarrow V\} \longrightarrow \{V \rightarrow V\}$ in $\mathcal{F}_{\mathcal{I}, \ast}$). The data of $\mathcal{O}$ is given by a commutative diagram of $\mathcal{I}^/\infty$-categories

\[
\begin{array}{ccc}
\{x\} \times_{\mathcal{O}^\otimes} \text{Ar}_{ne}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) & \xrightarrow{\mathcal{F}} & \mathcal{E}^\otimes_x \\
\downarrow{i} & & \downarrow{q_V} \\
\Delta^1 \times_{\mathcal{O}^\otimes} \text{Ar}_{ne}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) & \longrightarrow & \mathcal{O}^\otimes_x
\end{array}
\]

where we use the projections of the lefthand $\infty$-categories to $\{V\} \times_{\mathcal{O}^\otimes} \text{Ar}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \simeq (\mathcal{I}^/V)^{\text{op}}$ in order to pullback to the base $(\mathcal{I}^/V)^{\text{op}}$.

Then $\mathcal{O}$ is a locally cocartesian edge if and only if $H$ is a weak $q_\mathcal{E}^{\mathcal{I}/V}$-left Kan extension of $F$ in the sense of [Sha21b, Def. 6.1].

3. Note that we have inclusions of full $\mathcal{I}$-subcategories

\[
\begin{align*}
\mathcal{C}^\otimes_\mathcal{C}^\otimes & := \{x\} \times_{\mathcal{O}^\otimes} \text{Ar}_{\text{cocart}}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \subset \{x\} \times_{\mathcal{O}^\otimes} \text{Ar}_{ne}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \\
\mathcal{C}^\otimes_\mathcal{E}^\otimes & := \Delta^1 \times_{\mathcal{O}^\otimes} \text{Ar}_{\text{cocart}}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \subset \Delta^1 \times_{\mathcal{O}^\otimes} \text{Ar}_{ne}(\mathcal{O}^\otimes, \mathcal{C}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes.
\end{align*}
\]
Choose a cocartesian pushforward \( P_\alpha : \mathcal{E}^\otimes_\mathcal{A} \to \mathcal{E}_y \) and consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}^\otimes_\mathcal{A} & \xrightarrow{F} & \mathcal{E}^\otimes_\mathcal{A} \\
\downarrow i & & \downarrow \alpha \circ P_\alpha \\
\mathcal{E}^\otimes_\mathcal{A} & \xrightarrow{P_\alpha} & (\mathcal{T}/\mathcal{V})^{op}
\end{array}
\]

(where we abusively continue to write \( H \) and \( F \) for the canonical lifts of those functors to have codomains \( \mathcal{E}^\otimes_\mathcal{A} \) and \( \mathcal{E}^\otimes_\mathcal{A} \subset \mathcal{E}^\otimes \) respectively). Then \( \alpha \circ P_\alpha \) is a locally cocartesian edge if and only if \( P_\alpha \circ H \) is a \( \mathcal{T}/\mathcal{V} \)-left Kan extension of \( P_\alpha \circ F \).

Furthermore, if we let \( G = (P_\alpha \circ H)|_{\mathcal{E}^\otimes_\mathcal{A}} : \mathcal{E}^\otimes_\mathcal{A} \to \mathcal{E}_y \), then we have the data of a diagram

\[
\begin{array}{ccc}
\mathcal{E}^\otimes_\mathcal{A} & \xrightarrow{F} & \mathcal{E}^\otimes_\mathcal{A} \\
\downarrow \eta & & \downarrow \alpha \circ P_\alpha \\
\mathcal{E}^\otimes_\mathcal{A} & \xrightarrow{G} & (\mathcal{T}/\mathcal{V})^{op}
\end{array}
\]

in which the natural transformation \( \eta \) exhibits \( G \) as a \( \mathcal{T}/\mathcal{V} \)-left Kan extension of \( \alpha \circ F \) along \( \alpha \).

**Proof.** (2): By definition, \( \alpha \circ P_\alpha \) is a locally cocartesian edge if and only if \( H \) is initial in the space of all such fillers. But if \( H \) is a weak \( \mathcal{T}^\otimes \)-\( \mathcal{T}/\mathcal{V} \)-left Kan extension of \( F \), then it is in particular initial. Conversely, provided that we know a weak \( \mathcal{T}^\otimes \)-\( \mathcal{T}/\mathcal{V} \)-left Kan extension of \( F \) exists, then it necessarily coincides with the filler defined by \( \alpha \).

(1): We now show existence of these weak \( \mathcal{T}^\otimes \)-\( \mathcal{T}/\mathcal{V} \)-left Kan extensions. Let

\[
K := \{ x \} \times_{\mathcal{O}^\otimes} \mathcal{A} \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes_\mathcal{A}
\]

\[
L := \Delta^1 \times_{\mathcal{O}^\otimes} \mathcal{A} \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes_\mathcal{A}
\]

Note that any object in \( L \) that is not in \( K \) is either of the form \( (y \to z = p(c) \in \mathcal{A} \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes_\mathcal{A}) \) or \( (y \to z = p(c) \in \mathcal{A} \times_{\mathcal{O}^\otimes} \mathcal{E}^\otimes_\mathcal{A}) \) with \( z \) over \( \theta_+ \to W \). Let \( L_0 \subset L \) denote the full \( \mathcal{T} \)-subcategory excluding the second type of objects. Because the fiber of \( \mathcal{E}^\otimes_\mathcal{A} \) over any object \( \theta_+ \to W \) is contractible, we may replace \( L \) with \( L_0 \) and instead consider fillers \( H_0 : L_0 \to \mathcal{E}^\otimes_\mathcal{A} \). Moreover, note that there are no inert edges in \( L_0 \) not either cocartesian over \( \mathcal{T}^{op} \) or in \( K \), so any extension \( H_0 \) necessarily defines an edge of \( \text{Fun}_{\mathcal{T},\mathcal{T}^{op}}(\mathcal{E},\mathcal{E})^\otimes_\mathcal{A} \).

Because both the additional objects in \( L_0 \) and morphisms between those objects lie over \( y \to \mathcal{O} \subset \mathcal{O}^\otimes \), by [Sha21b, Thm. 6.2] in conjunction with [Sha21b, Prop. 5.8] it suffices to have \( \mathcal{E}_y \) be \( \mathcal{T}/\mathcal{V} \)-cocomplete for a weak \( \mathcal{T}^\otimes \)-\( \mathcal{T}/\mathcal{V} \)-left Kan extension of \( F \) to exist. This is ensured by our hypothesis.

By Lemma 3.2.1, we see that we just considered suffices to show that \( \text{Fun}_{\mathcal{T},\mathcal{T}^{op}}(\mathcal{E},\mathcal{E})^\otimes_\mathcal{A} \) is a locally cocartesian fibration. In addition, by Proposition 3.1.9 and [Sha21a, Prop. 9.7], for every \( x \in \mathcal{O}_V \) we have an equivalence

\[
\text{Fun}_{\mathcal{T},\mathcal{T}^{op}}(\mathcal{E},\mathcal{E})^\otimes_\mathcal{A} \simeq \text{Fun}_{\mathcal{T}^{op}}(\mathcal{E}^\otimes_\mathcal{A}, \mathcal{E}_x^\otimes_\mathcal{A})
\]

and the latter \( \mathcal{T}/\mathcal{V} \)-category is \( \mathcal{T}/\mathcal{V} \)-cocomplete by the pointwise computation of \( \mathcal{T}/\mathcal{V} \)-colimits in \( \mathcal{T}/\mathcal{V} \)-functor categories.

(3): Observe that for \( l = (y \to z, c) \in \mathcal{E}^\otimes_\mathcal{A} \subset L, K \times_L L/l^L \simeq \mathcal{E}^\otimes_\mathcal{A} \times_{\mathcal{O}^\otimes} (\mathcal{E}^\otimes_\mathcal{A})^L/l^L \). By the pointwise formula for \( \mathcal{T}/\mathcal{V} \)-left Kan extensions, the first part of the claim follows. The second part then follows by [Sha21b, Rem. 2.14].

As with ordinary Day convolution, we need an additional distributivity hypothesis on the target for \( \mathcal{T} \)-Day convolution to be \( \mathcal{O} \)-monoidal. We first recall the definition of a distributive functor (as originally formulated by the first author).

**3.2.3. Definition** ([Sha21b, Def. 8.18]). Let \( f : U \to V \) be a map of finite \( \mathcal{T} \)-sets, let \( \mathcal{E} \) be a \( \mathcal{T}^U \)-\( \mathcal{T} \)-category, and let \( \mathcal{D} \) be a \( \mathcal{T}/\mathcal{V} \)-\( \mathcal{T} \)-category. Let \( F : \prod_{j} \mathcal{E} = f_* \mathcal{E} \to \mathcal{D} \) be a \( \mathcal{T}/\mathcal{V} \)-functor. Then we say that
$F$ is distributive if for every pullback square

$$
\begin{array}{ccc}
U' & \xrightarrow{f'} & V' \\
\downarrow{g'} & & \downarrow{g} \\
U & \xrightarrow{f} & V
\end{array}
$$

of finite $\mathcal{T}$-sets and $\mathcal{T}/U'$-colimit diagram $\mathcal{F} : \mathcal{K}^{(n)} \rightarrow g^* \mathcal{C}$, the $\mathcal{T}/V'$-functor

$$(f'_* \mathcal{F})(\mathcal{K}) \cong f'_* (g^* \mathcal{F}) \xrightarrow{f'_* g} g^* f_* \mathcal{C} \cong g^* f'_* \mathcal{C} \xrightarrow{g^* \alpha} g^* \mathcal{D}$$

is a $\mathcal{T}/V'$-colimit diagram.

3.2.4. Definition. Let $\mathcal{C}^\otimes$ be a $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category and suppose that for all $y \in \mathcal{O}_V$, $\mathcal{C}^\otimes_y$ is $\mathcal{T}/V'$-cocomplete. We say that $\mathcal{C}^\otimes$ is distributive if for every fiberwise active edge $\alpha : x \rightarrow y$ lifting $[U_+ \rightarrow V] \rightarrow [V_+ \rightarrow V]$ (corresponding to $f : U \rightarrow V$ in $\mathbf{F}_\mathcal{C}$), the associated pushforward $\mathcal{T}/V'$-functor

$$\alpha^\otimes : \mathcal{C}^\otimes_y \rightarrow \mathcal{C}^\otimes_x$$

distributes. Here, for this condition to be sensible, we use that for an orbit decomposition $U \simeq U_1 \coprod \cdots \coprod U_n$ and $n$ cocartesian morphisms $x \rightarrow x_i$ lifting the characteristic maps $\chi_{U_i \subset U} : [U_+ \rightarrow V] \rightarrow [(U_i)_+ \rightarrow U_i]$, the parametrized fiber $\mathcal{C}^\otimes_{x_i}$ is identified with $\prod f \left( \mathcal{C}_{x_1} \coprod \cdots \coprod \mathcal{C}_{x_n} \right)$ by the $\mathcal{T}$-Segal condition (cf. Example 2.3.5).

Also note that in particular, for each morphism $\alpha : x \rightarrow y$ in $\mathcal{O}_V$, the condition that the pushforward $\mathcal{T}/V'$-functor $\alpha^\otimes : \mathcal{C}^\otimes_y \rightarrow \mathcal{C}^\otimes_x$ is distributive is equivalent to $\alpha^\otimes$ strongly preserving all small $\mathcal{T}/V'$-colimits.

The following proposition furnishes some examples of distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-categories.

3.2.5. Proposition. Let $\mathcal{C}$ be a cocomplete $\infty$-category with finite products such that the products commute with colimits separately in each variable. Let $f : U \rightarrow U'$ be a morphism of finite $\mathcal{T}$-sets. Then the product $\mathcal{T}/U'$-functor

$$\mu : f^*_* \mathcal{C}_{\mathcal{T}/U'} \rightarrow \mathcal{C}_{\mathcal{T}/U'}$$

is distributive. Consequently, the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\mathcal{C}_{\mathcal{T}}$ is $\mathcal{T}$-distributive.

Proof. By the universal property of the category of $\mathcal{T}/U'$-objects ([Sha21a, Prop. 3.10]), $\mu$ can be identified with a functor

$$\mu^! : f^*_* \mathcal{C}_{\mathcal{T}/U'} \rightarrow \mathcal{C}$$

such that its restriction to the fiber over $[W \rightarrow U'] \in \mathcal{T}/U'$ is the functor

$$\prod_{V \in \text{Orbit}(U \times_U W)} \text{Fun}(V, \mathcal{C}) \xrightarrow{\prod_{V \in \text{Orbit}(U \times_U W)} \chi_x} \prod_{V \in \text{Orbit}(U \times_U W)} \mathcal{C} \xrightarrow{x} \mathcal{C}.$$
3.2.6. **Theorem.** In the situation of Proposition 3.2.2, suppose moreover that \( q : \mathcal{E}^\otimes \to \mathcal{O}^\otimes \) is distributive. Then \( \text{Fun}_{\mathcal{O}, \tau}(\mathcal{E}, \mathcal{E})^\otimes \to \mathcal{O}^\otimes \) is a cocartesian fibration and \( \text{Fun}_{\mathcal{O}, \tau}(\mathcal{E}, \mathcal{E})^\otimes \) is distributive.

**Proof.** Let

\[
\begin{array}{ccc}
\alpha & \nearrow & \beta \\
x & & z \\
\gamma & \searrow & \\
y & & \\
\end{array}
\]

be a 2-simplex \( \sigma \) of fiberwise active edges in \( \mathcal{O}_W^\otimes \) with \( z \in \mathcal{O}_W \), which covers

\[
\begin{array}{ccc}
V & & \mathcal{O}_W \\
\nearrow & f & \searrow g \\
U & \downarrow & W \\
\end{array}
\]

in \( \mathcal{F}_\tau \) (viewed as a 2-simplex in \( (\mathcal{F}_{\tau, \ast})^\otimes_{W} \)). We need to verify that given a lift of \( \sigma \) to a 2-simplex \( \sigma' \)

\[
\begin{array}{ccc}
\pi & \nearrow & \pi' \\
\gamma & \searrow & \\
\tau & \downarrow & \tau' \\
\end{array}
\]

in \( \text{Fun}_{\mathcal{O}, \tau}(\mathcal{E}, \mathcal{E})^\otimes \), if \( \pi \) and \( \pi' \) are locally cocartesian edges then \( \pi' \) is a locally cocartesian edge.

Suppose that \( V \simeq \coprod_{1 \leq i \leq n} V_i \) is an orbit decomposition of \( V \) with respect to which \( \alpha \) decomposes as \( \{\alpha_i : x_i \to y_i\}_{1 \leq i \leq n} \). Then the locally cocartesian edge \( \pi \) corresponds to \( n \) commutative diagrams of \( \mathcal{T}/V_i \)-\( \infty \)-categories

\[
\begin{array}{ccc}
\mathcal{E}_{x_i}^\otimes & \xrightarrow{F_{x_i}} & \mathcal{E}_{y_i}^\otimes \\
\downarrow & & \downarrow \\
\mathcal{E}_{\alpha_i}^\otimes & \xrightarrow{F_{\alpha_i}} & \\
\end{array}
\]

in which \( F_{\alpha_i} \) is a \( \mathcal{T}/V_i \)-left Kan extension of \( F_{x_i} \). From this, we obtain the commutative diagram of \( \mathcal{T}/W \)-\( \infty \)-categories

\[
\begin{array}{ccc}
\Pi_g(\coprod_{1 \leq i \leq n} \mathcal{E}_{x_i}^\otimes) & \xrightarrow{F_x} & \Pi_g(\coprod_{1 \leq i \leq n} \mathcal{E}_{y_i}^\otimes) \\
\downarrow & & \downarrow \\
\Pi_g(\coprod_{1 \leq i \leq n} \mathcal{E}_{\alpha_i}^\otimes) & \xrightarrow{\beta_{\alpha} \circ F_{\alpha}} & \mathcal{E}_z^\otimes \\
\end{array}
\]

in which \( \beta_{\alpha} \circ F_{\alpha} \) is a \( \mathcal{T}/W \)-left Kan extension of \( \beta_{\alpha} \circ F_{x} \), invoking the hypothesis that \( \mathcal{E}^\otimes \) is distributive.

On the other hand, the locally cocartesian edge \( \beta \) corresponds to a commutative diagram of \( \mathcal{T}/W \)-\( \infty \)-categories

\[
\begin{array}{ccc}
\mathcal{E}_{y}^\otimes & \xrightarrow{F_{y}} & \mathcal{E}_{z}^\otimes \\
\downarrow & & \downarrow \\
\mathcal{E}_{\beta}^\otimes & \xrightarrow{F_{\beta}} & \\
\end{array}
\]
in which $F_y$ is the restriction of $\beta \circ F_\alpha$ along the inclusion $C^\otimes \subseteq C^\otimes_\varnothing$ and $F_\beta$ is a $\mathcal{T}/W$-left Kan extension of $F_y$. Combining these two diagrams, we obtain a commutative diagram of $\mathcal{T}/W$-$\infty$-categories

\[
\begin{array}{c}
\Delta^2 \times \sigma, \mathcal{O} \to \text{Ar}^{cocart}(\mathcal{O}^\otimes) \times \mathcal{O} \times C^\otimes;
\end{array}
\]

where $C^\otimes := \Delta^2 \times \sigma, \mathcal{O} \to \text{Ar}^{cocart}(\mathcal{O}^\otimes) \times \mathcal{O} \times C^\otimes$. Because $C^\otimes$ is $\mathcal{O}$-promonoidal and $\sigma$ is a 2-simplex of fiberwise active edges, the lefthand square is a pushout square of $\mathcal{T}/W$-$\infty$-categories, and the dotted $\mathcal{T}/W$-functor $F_\sigma$ is obtained from gluing together $\beta \circ F_\alpha$ and $F_\beta$. Indeed, $F_\sigma$ corresponds to the 2-simplex $\sigma$ in $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(C^\otimes, \mathcal{E})^\otimes$. By Lemma 3.2.7, $F_\sigma$ is a $\mathcal{T}/W$-left Kan extension of $\beta \circ F_\alpha$. By transitivity of $\mathcal{T}/W$-left Kan extensions, $F_\sigma$ is a $\mathcal{T}/W$-left Kan extension of $\beta \circ F_x$, and the restriction $F_\gamma$ of $F_\sigma$ to $C^\otimes \subseteq C^\otimes_\varnothing$ is also a $\mathcal{T}/W$-left Kan extension of $\beta \circ F_x$. But this exactly means that $\sigma$ is a locally cocartesian edge. Finally, by Lemma 3.2.1 this suffices to show that $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(C^\otimes, \mathcal{E})^\otimes \to \mathcal{O}^\otimes$ is a cocartesian fibration.

To show that $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(C^\otimes, \mathcal{E})^\otimes$ is distributive, we check the definition. So suppose that $\alpha : x \to y$ is a fiberwise active edge in $\mathcal{O}^\otimes_\varnothing$ with $y \in \mathcal{O} \mathcal{V}$, which lifts $f : U \to V$ in $F_\sigma$. Let $U \simeq U_1 \coprod \ldots \coprod U_n$ be an orbit decomposition and suppose we have $\mathcal{T}/U_i$-colimits

\[
\begin{array}{c}
K_i \xrightarrow{p_i} \text{Fun}_{\mathcal{T}/U_i}(C^\otimes_{x_i}, \mathcal{E}^\otimes_{x_i}) \xrightarrow{q_i} (\mathcal{T}/U_i)^{\text{op}}
\end{array}
\]

(Here and throughout we suppress the data of the natural transformations.) By [Sha21a, Prop. 9.17] and its proof, these are adjoint to $\mathcal{T}/U_i$-left Kan extensions

\[
\begin{array}{c}
K_i \times (\mathcal{T}/U_i)^{\text{op}} \xrightarrow{p_i'} C^\otimes_{x_i} \xrightarrow{q_i'} \mathcal{E}^\otimes_{x_i}
\end{array}
\]

Let $K = \coprod_{1 \leq i \leq n} K_i$, $p = \coprod_{1 \leq i \leq n} p_i$, and $q = \coprod_{1 \leq i \leq n} q_i$, and the same for $K'$, $p'$, $q'$. We need to show that

\[
\begin{array}{c}
\Pi_f K \xrightarrow{\Pi_f p} \text{Fun}_{\mathcal{O}, \mathcal{T}}(C^\otimes, \mathcal{E})^\otimes \xrightarrow{\alpha \otimes} \text{Fun}_{\mathcal{T}/V}(C^\otimes_{x}, \mathcal{E}^\otimes_{y}) \xrightarrow{(\mathcal{T}/V)^{\text{op}}} \alpha \otimes \Pi_f q
\end{array}
\]

is a $\mathcal{T}/V$-colimit. Equivalently, we need to show that in the diagram
the $\mathcal{V}/\mathcal{V}$-functor $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $(\alpha \circ \Pi_f p)'$. Here we change the domain from $\mathcal{C} \otimes \mathcal{Y}$ to $\mathcal{C} \otimes \alpha$ because we are not supposing that $\mathcal{C} \otimes \mathcal{Y}$ is $\mathcal{O}$-monoidal. Note that $(\alpha \circ \Pi_f q)'$ is by definition a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$.

Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$. Because $\mathcal{E} \otimes \mathcal{Y}$ is distributive, we have that $\Pi_f \mathcal{K} \times (\mathcal{V}/\mathcal{V}) \mathcal{P} \mathcal{O} \mathcal{T} \mathcal{Y}$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension. By definition, $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$, so it suffices to show that $(\alpha \circ \Pi_f q)'$ is a $\mathcal{V}/\mathcal{V}$-left Kan extension of $\alpha \circ (\Pi_f p)'$.
by sending an object \( U = \coprod_i U_i \) (decomposed as a disjoint union of orbits) to \( \coprod_i \Delta^1 \), and a morphism \( f : \coprod_i U_i \to \coprod_i V_i \), \( \phi : I \to J \) contravariantly to the restriction functor \( \phi^* : \prod_J \Delta^1 \to \prod_I \Delta^1 \) and covariantly to the product over \( J \) of

\[
\min : \coprod_{I_j} \Delta^1 \to \Delta^1, \ (x_i) \mapsto \min(x_i)
\]

if \( I_j \) is nonempty, and \( 1 : \Delta^0 \to \Delta^1 \) otherwise. (One easily verifies the base-change condition, so this indeed defines a functor.)

Let \( \left( \Delta^1 \times \mathcal{T}^{op} \right) \) denote the resulting \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category under the equivalence of Theorem 2.3.9. Let \( \mathcal{C}^{op} \) be a distributive \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category. Then by Theorem 3.2.6, we have that \( \mathbf{Fun}_T(\Delta^1 \times \mathcal{T}^{op}, \mathcal{C})^{op} \) is a distributive \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category. Moreover, the fiberwise tensor products admit a simple description because the underlying \( \mathcal{T} \)-category of the source is constant. Namely, for the fold map \( \nabla : U \coprod U \to U \), the tensor product

\[
\otimes : \mathbf{Fun}(\Delta^1, \mathcal{C}_U) \times \mathbf{Fun}(\Delta^1, \mathcal{C}_U) \to \mathbf{Fun}(\Delta^1, \mathcal{C}_U)
\]

is given by taking the cartesian product into \( \mathbf{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}_U \times \mathcal{C}_U) \), postcomposition by \( \otimes : \mathcal{C}_U \times \mathcal{C}_U \to \mathcal{C}_U \), and then left Kan extension along \( \min : \Delta^1 \to \Delta^1 \) (which is computed by taking colimits fiberwise because \( \min \) is a cocartesian fibration).

Now suppose in addition that \( \mathcal{C}^{op} \) is \( \mathcal{T} \)-cartesian \( \mathcal{T} \)-symmetric monoidal, so the tensor product on the fibers of \( \mathbf{Fun}_T(\Delta^1 \times \mathcal{T}^{op}, \mathcal{C}) \) is the pushout product. Let \( \mathcal{C}_* \) denote the full \( \mathcal{T} \)-subcategory of \( \mathbf{Fun}_T(\Delta^1 \times \mathcal{T}^{op}, \mathcal{C}) \) given over an object \( U \in \mathcal{T} \) by those functors \( \Delta^1 \to \mathcal{C}_U \) which take 0 to a final object \( * \) of \( \mathcal{C}_U \). The inclusion \( \mathcal{C}_* \subset \mathbf{Fun}_T(\Delta^1 \times \mathcal{T}^{op}, \mathcal{C}) \) admits a left adjoint \( L \), which over an object \( U \in \mathcal{T} \) takes a functor \( F : \Delta^1 \to \mathcal{C}_U \) to

\[
L(F) : \Delta^1 \to \mathcal{C}_U, \ [0 \to 1] \mapsto [* \to F(1)/F(0)].
\]

\( L \) is a \( \mathcal{T} \)-localization functor, so to descend the cartesian \( \mathcal{T} \)-symmetric monoidal structure on \( \mathcal{C} \) to the smash product on \( \mathcal{C}_* \), we can check the criterion of Theorem 2.9.2 (or rather, Remark 2.9.3):

1. For the fold map \( \nabla : U \coprod U \to U \) and an object \([z_0 \to z_1] \in \mathbf{Fun}(\Delta^1, \mathcal{C}_U)\),

\[
\otimes [z_0 \to z_1] : \mathbf{Fun}(\Delta^1, \mathcal{C}_U) \to \mathbf{Fun}(\Delta^1, \mathcal{C}_U)
\]

preserves \( L_U \)-equivalences. Indeed, let

\[
\begin{array}{ccc}
x_0 & \to & y_0 \\
\down & & \down \\
x_1 & \to & y_1
\end{array}
\]

be an \( L_U \)-equivalence, i.e. \( x_1/x_0 \to y_1/y_0 \) is an equivalence in \( \mathcal{C}_U \). We have an equivalence

\[
(x_1 \times z_1) / (x_0 \times z_1 \cup x_0 \times z_0 \times x_1 \times z_0) \simeq \left( \frac{x_1 \times z_1}{x_0 \times z_1} \right) / \left( \frac{x_1 \times z_0}{x_0 \times z_0} \right).
\]

Using that we have a diagram of pushout squares

\[
\begin{array}{ccc}
x_0 \times z_j & \to & z_j \\
\down & & \down \\
x_1 \times z_j & \to & x_1/x_0 \times z_j \to (x_1 \times z_j) / (x_0 \times z_j)
\end{array}
\]

(and similarly for \( y \)) by cartesian closedness, we deduce that

\[
\left( \frac{x_1 \times z_1}{x_0 \times z_1} \right) / \left( \frac{x_1 \times z_0}{x_0 \times z_0} \right) \to \left( \frac{y_1 \times z_1}{y_0 \times z_1} \right) / \left( \frac{y_1 \times z_0}{y_0 \times z_0} \right)
\]

is an equivalence, as desired.

2. For a map \( f : U \to V \) in \( \mathcal{T} \),

\[
f_\otimes : \mathbf{Fun}(\Delta^1, \mathcal{C}_U) \to \mathbf{Fun}(\Delta^1, \mathcal{C}_V)
\]

sends \( L_U \)-equivalences to \( L_V \)-equivalences: to show this, suppose

\[
\theta : [x_0 \to x_1] \to [y_0 \to y_1]
\]
is a $L_U$-equivalence in $\text{Fun}(\Delta^1, \mathcal{C}_U)$. Equivalently, we have a left Kan extension

$$\Lambda^2_0 \times \Delta^1 \longrightarrow \mathcal{C}_U$$

where restriction to the cone point is sent to an equivalence $x_1/x_0 \cong y_1/y_0$. Using distributivity, we get a $T/V$-left Kan extension diagram

$$\Pi_f((\Lambda^2_0 \times \Delta^1) \times (T/U)^{\text{op}}) \longrightarrow \Pi_f \mathcal{C}_U \longrightarrow \Pi_f \mathcal{C}_V$$

The vertical arrow factors as

$$\prod_f((\Lambda^2_0 \times \Delta^1) \times (T/U)^{\text{op}}) \longrightarrow (\Lambda^2_0 \times \Delta^1) \times (T/V)^{\text{op}} \longrightarrow ((\Lambda^2_0)^{\text{op}} \times \Delta^1) \times (T/V)^{\text{op}}$$

where the first arrow is induced by the symmetric monoidal structure on $\Delta^1 \times T^{\text{op}}$. Therefore, the left Kan extension corresponding to $f_{G}(\theta)$

$$\Lambda^2_0 \times \Delta^1 \longrightarrow \mathcal{C}_V$$

restricts on the cone point to an equivalence, so $f_{G}(\theta)$ is a $L_V$-equivalence.

In particular, suppose that $T = O_G$ and $\mathcal{C} = \text{Spc}_{G^{\ast}}$. Then we obtain the smash product $G$-symmetric monoidal structure on pointed $G$-spaces $\text{Spc}_{G^{\ast}}$, and given a map of orbits $f : G/H \longrightarrow G/K$ corresponding to an inclusion of subgroups $K \subset H$ and a real $K$-representation $R$, the norm functor $f_{\otimes}$ sends the representation sphere $S^R$ to the representation sphere $S^{\text{Ind}_K^H R}$.

3.3. Pointwise $O$-monoidal structure. In this brief subsection, we indicate how to adapt the construct of the $T$-Day convolution so as to construct the cotensor of $O_G$-operads. In other words, given a fibration of $T$-$\infty$-operads $p : D^{\otimes} \longrightarrow O^{\otimes}$ and a $T$-$\infty$-category $\mathcal{K}$, we have a $T$-$\infty$-operad $\text{Fun}_{O_G,T}(\mathcal{K} \times_{T^{\text{op}}} O, D)^{\otimes}$ that satisfies the universal mapping property

$$\text{Alg}_{O_G,T}(\mathcal{C}, \text{Fun}_{O_G,T}(\mathcal{K} \times_{T^{\text{op}}} O, D)) \simeq \text{Fun}_T(\mathcal{K}, \text{Alg}_{O_G,T}(\mathcal{C}, D))$$

for all fibrations of $\infty$-operads $q : \mathcal{C}^{\otimes} \longrightarrow O^{\otimes}$.

For the following construction, observe the isomorphism

$$\text{Ar}^{nc}(O^{\otimes}) \times_{ev_1, O^{\otimes}, pr} (O^{\otimes} \times_{T^{\text{op}}} \mathcal{K}) \simeq \text{Ar}^{nc}(O^{\otimes}) \times_{T^{\text{op}}} \mathcal{K}.$$

3.3.1. Theorem-Construction. Let $\mathcal{O}^{\otimes}$ be a $T$-$\infty$-operad and $\mathcal{K}$ a $T$-$\infty$-category. Consider the span diagram of marked simplicial sets

$$(\mathcal{O}^{\otimes}, \text{Ne}) \xleftarrow{\text{ev}_1} (\text{Ar}_{nc}(\mathcal{O}^{\otimes}) \times_{T^{\text{op}}} \mathcal{K}, \text{Ne}) \xrightarrow{\text{ev}_1} (\mathcal{O}^{\otimes}, \text{Ne}).$$

where by the middle marking $\text{Ne}$ we mean those edges whose source and target in $\mathcal{O}^{\otimes}$ are inert and whose projection to $\mathcal{K}$ is a cocartesian edge. Then the functor

$$(\text{ev}_0)_{\ast} \circ (\text{ev}_1)^{\ast} : \text{SSet}_{/(\mathcal{O}^{\otimes}, \text{Ne})} \longrightarrow \text{SSet}_{+(\mathcal{O}^{\otimes}, \text{Ne})}$$

is right Quillen with respect to the $T$-operadic model structures. For a fibration of $\infty$-operads $p : \mathcal{C}^{\otimes} \longrightarrow O^{\otimes}$, we let

$$\text{Fun}_{O_G,T}(\mathcal{K} \times_{T^{\text{op}}} O, \mathcal{C})^{\otimes} := (\text{ev}_0)_{\ast}((\mathcal{C}^{\otimes}, \text{Ne})).$$

If $\mathcal{O} \simeq T^{\text{op}}$, then we more simply write

$$\text{Fun}_T(\mathcal{K}, \mathcal{C})^{\otimes} := (\text{ev}_0)_{\ast}((\mathcal{C}^{\otimes}, \text{Ne})).$$
This construction satisfies the universal property (3.3.1) and its underlying \( \mathcal{T}\)-\( \infty \)-category is as the notation indicates.

\textit{Proof.} This follows along the same lines as our proofs of the analogous results for \( \mathcal{T}\)-Day convolution in Section 3.1. \qed

3.3.2. \textbf{Remark.} In the situation of Theorem-Construction 3.3.1, a fibrant replacement of \( (\mathcal{O}^\otimes, \mathcal{C}) \times_{\mathcal{T}\times \mathcal{O}} \mathcal{K} \) in the \( \mathcal{T}\)-operadic model structure on \( \mathsf{sSet}^+_{\mathcal{O}^\otimes, \mathcal{C}} \) computes the \textit{tensor} of \( \mathcal{O}^\otimes \)-monoidal structure on \( \mathsf{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \). See also [CH21].

If we then suppose that \( \mathcal{C}^\otimes \) is an \( \mathcal{O}\)-monoidal \( \mathcal{T}\)-\( \infty \)-category, we obtain the \textit{pointwise} \( \mathcal{O}\)-monoidal structure on \( \mathsf{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \). In contrast to the \( \mathcal{T}\)-Day convolution, we don’t need to impose any further hypotheses on \( \mathcal{C}^\otimes \) for this to exist.

3.3.3. \textbf{Theorem-Construction.} Let \( \mathcal{O}^\otimes \) be a \( \mathcal{T}\)-\( \infty \)-operad and \( \mathcal{K} \) a \( \mathcal{T}\)-\( \infty \)-category. Consider the span diagram of marked simplicial sets

\[
(\mathcal{O}^\otimes)^\sharp \xleftarrow{ev_0} \mathsf{Ar}^{ne}(\mathcal{O}^\otimes)^\sharp \times_{\mathcal{T}\times \mathcal{O}} \mathcal{K} \xrightarrow{ev_1} (\mathcal{O}^\otimes)^\sharp.
\]

Then the functor

\[
(ev_0)_* \circ (ev_1)^* : \mathsf{sSet}^+_{\mathcal{O}^\otimes} \rightarrow \mathsf{sSet}^+_{\mathcal{O}^\otimes}
\]

agrees with the construction of Theorem-Construction 3.3.1 on underlying simplicial sets and is right Quillen with respect to the \( \mathcal{T}\)-monoidal model structures. Given any \( \mathcal{O}\)-monoidal \( \mathcal{T}\)-\( \infty \)-categories \( \mathcal{O}^\otimes, \mathcal{D}^\otimes \), we then have a natural equivalence

\[
\mathsf{Fun}^\otimes_{\mathcal{T}}(\mathcal{C}, \mathsf{Fun}_{\mathcal{T}}(\mathcal{K} \times_{\mathcal{T}\times \mathcal{O}} \mathcal{D}, \mathcal{D})) \simeq \mathsf{Fun}_{\mathcal{T}}(\mathcal{K}, \mathsf{Fun}^\otimes_{\mathcal{T}}(\mathcal{C}, \mathcal{D})).
\]

\textit{Proof.} That \( (ev_0)_* \circ (ev_1)^* \) is right Quillen follows by [Lur17, Thm. B.4.2] as in the proof of Theorem 3.1.4. The only differences to note are regarding conditions (4) and (7). For (4), by [Sha21b, Prop. 3.5(1)] the \( ev_0 \)-cartesian edges in \( \mathsf{Ar}^{ne}(\mathcal{O}^\otimes) \) are fiberwise active in the target, hence \( ev_0 : \mathsf{Ar}^{ne}(\mathcal{O}^\otimes) \times_{\mathcal{T}\times \mathcal{O}} \mathcal{K} \rightarrow \mathcal{O}^\otimes \) is a cartesian fibration whose cartesian edges project to equivalences in \( \mathcal{K} \). (7) then follows by inspection.

The universal mapping property then follows by restricting the equivalence (3.3.1). \qed

3.3.4. \textbf{Example.} Suppose \( \mathcal{O}^\otimes = \mathcal{E}_{\mathcal{K}, \mathcal{T}} \), let \( \mathcal{C}^\otimes \) be a \( \mathcal{T}\)-symmetric monoidal \( \mathcal{T}\)-\( \infty \)-category, and let \( \mathcal{K} \) be a \( \mathcal{T}\)-\( \infty \)-category. Then we may unwind the pointwise \( \mathcal{T}\)-symmetric monoidal structure on \( \mathsf{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C}) \) as follows:

\((*) \) Let \( f : U \rightarrow V \) be a map of finite \( \mathcal{T}\)-sets. Then the norm functor

\[
f_\otimes : \mathsf{Fun}_{\mathcal{T}/U}(\mathcal{K}_U, \mathcal{C}_U) \rightarrow \mathsf{Fun}_{\mathcal{T}/V}(\mathcal{K}_V, \mathcal{C}_V)
\]

sends a \( \mathcal{T}/U \)-functor \( F : \mathcal{K}_U \rightarrow \mathcal{C}_U \) to the \( \mathcal{T}/V \)-functor

\[
\mathcal{K}_U \xrightarrow{\eta_U} \prod_f \mathcal{K}_U \simeq \mathsf{Fun}_{\mathcal{T}/V}(U, \mathcal{K}_V) \xrightarrow{\prod_f F} \prod_f \mathcal{C}_U \simeq \mathsf{Fun}_{\mathcal{T}/V}(U, \mathcal{C}_V) \xrightarrow{f_\otimes} \mathcal{C}_V
\]

where \( \eta \) is the diagonal \( \mathcal{T}/V \)-functor (i.e., the unit of the restriction-coinduction adjunction) and the norm \( \mathcal{T}/V \)-functor \( f_\otimes \) is defined as in Example 2.3.5.

In particular, note that if \( \mathcal{C}^\otimes \) is distributive, then \( \mathsf{Fun}_{\mathcal{T}}(\mathcal{K}, \mathcal{C})^\otimes \) is also distributive.

4. Parametrized operadic left Kan extensions

In this section, we construct \( \mathcal{T}\)-operadic left Kan extensions, implementing in the operadic context the strategy that the second author used to construct \( \mathcal{T}\)-left Kan extensions in [Sha21a, §§9-10].

4.0.1. \textbf{Remark.} The strategy of our construction of \( \mathcal{T}\)-operadic left Kan extensions will be to first show that the \( \mathcal{T}\)-colimit of a lax \( \mathcal{O}\)-monoidal functor canonically inherits a \( \mathcal{O}\)-algebra structure, and to then reduce to this case via the \( \mathcal{O}\)-monoidal envelope. If we let \( \mathcal{T} = \Delta^\otimes \), this gives a new construction of Lurie’s operadic left Kan extension ([Lur17, §3.1.2]). See also [CH21].

We first begin with some necessary preliminaries on generalized \( \mathcal{T}\)-\( \infty \)-operads and the \( \mathcal{T}\)-operadic join.
4.1. **Generalized \( \mathcal{T} \)-∞-operads.** Given a map of finite \( \mathcal{T} \)-sets \( \phi : U \to V \), let \( \sigma_1, \sigma_2 \) be two squares in \( \mathbf{F}_\mathcal{T} \)

\[
\begin{array}{c}
U_i \xrightarrow{\alpha_i} U \\
\phi_i \downarrow \phi \\
V_i \xrightarrow{\beta_i} V
\end{array}
\]

such that \( \alpha_i \) is a summand inclusion, the induced map \( U_i \to V_i \times_V U \) is also a summand inclusion, and moreover \( (\alpha_1, \alpha_2) : U_1 \sqcup U_2 \to U \) is an epimorphism. Let \( U_{12} = U_1 \times_U U_2 \) and \( V_{12} = V_1 \times_V V_2 \). Then we have an induced map

\[
g_{\phi, \sigma_1, \sigma_2} : (\Lambda_2^3)^{\Delta} \to \mathbf{F}^{\text{big}}_{\mathcal{T},*}
\]

which selects the square

\[
[U_+ \to V] \quad [ (U_1)_+ \to V_1] \\
\downarrow \quad \downarrow \\
[(U_2)_+ \to V_2] \quad [(U_{12})_+ \to V_{12}]
\]

in which every morphism is inert.

We define the **generalized \( \mathcal{T} \)-operadic model structure** on \( \mathbf{sSet}^+_{/(\mathbf{F}^{\text{big}}_{\mathcal{T},*}, Ne)} \) to be the model structure defined using the categorical pattern \( (\text{Ne}, \text{All}, \{g_{\phi, \sigma_1, \sigma_2}\}) \) on \( \mathbf{F}^{\text{big}}_{\mathcal{T},*} \), letting the \( \phi \) and \( \{\sigma_1, \sigma_2\} \) range over all possible choices. We call the fibrant objects for this model structure **generalized \( \mathcal{T} \)-∞-operads**. Note that \( \mathcal{T} \)-∞-operads are fibrant in this model structure, so are examples of generalized \( \mathcal{T} \)-∞-operads. However, the converse is not true. Indeed, let \( \sigma_0 : \mathbf{F}_\mathcal{T}^{op} \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \) be the cocartesian section which selects \( [\emptyset_+ \to V] \) in each fiber and define \( \mathbf{C}_0 := \mathbf{F}_\mathcal{T}^{op} \times_{\sigma_0} \mathbf{F}^{\text{big}}_{\mathcal{T},*} \). Then if \( \mathbf{C}^0 \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \) is a generalized \( \mathcal{T} \)-∞-operad, \( \mathbf{C}_0 \) is not necessarily the terminal \( \mathcal{T} \)-∞-category.

We then say that \( \mathbf{C}^0 \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \) is a **generalized \( \mathcal{T} \)-∞-operad** if it is the pullback of a generalized \( \mathcal{T} \)-∞-operad \( \mathbf{C}^0 \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \) under the inclusion \( \mathbf{F}^{\text{op}}_{\mathcal{T},*} \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \). Let \( \mathbf{C}_0 \) be the corresponding pullback of \( \mathbf{C}_0 \). The \( \mathcal{T} \)-functor \( \mathcal{T}^{op} \times \Delta^1 \to \mathbf{F}_\mathcal{T},* \), which selects the inert edge \( [V_+ \to V] \to [\emptyset_+ \to V] \) in each fiber induces a \( \mathcal{T} \)-functor \( \mathbf{C} \to \mathbf{C}_0 \).

**4.1.1. Lemma.** Suppose \( \mathbf{C}^0 \to \mathbf{F}^{\text{big}}_{\mathcal{T},*} \) is a generalized \( \mathcal{T} \)-∞-operad. Let \( f : U \to V \) be a morphism in \( \mathbf{F}_\mathcal{T} \) and suppose that we have an orbit decomposition \( U \simeq \sqcup_{1 \leq i \leq n} U_i \). Abbreviate \( U_i \times_V U_j \) as \( U_{ij} \) and let \( f_i : U_i \to V \), \( f_{ij} : U_{ij} \to V \) denote the induced maps. Then we have an equivalence of \( \mathcal{T}^{/V} \)-∞-categories

\[
\mathbf{C}^{\otimes}_{/[U_+ \to V]} \cong \prod_{f_1} \mathbf{C}_{U_1} \times \prod_{f_2} \mathbf{C}_{U_{12}} \times \prod_{f_3} \mathbf{C}_{U_{13}} \times \cdots \times \prod_{f_n} \mathbf{C}_{U_{1n}}
\]

**Proof.** Note that \( U_{ij} \) need not be an orbit, so the notation \( (\emptyset_0)_{U_{ij}} \) means \( \prod_{0 \leq i \leq n} (\emptyset_0)_{U_{ij}} \) for any orbit decomposition of \( U_{ij} \). Without loss of generality, we may suppose \( n = 2 \). We have a pullback square in \( \mathbf{F}_\mathcal{T} \)

\[
\begin{array}{c}
U_{12} \xrightarrow{g_2} U_1 \\
\downarrow g_1 \\
U_2 \xrightarrow{f_2} V.
\end{array}
\]

The unit maps for the restriction-coinduction adjunction yield \( \mathcal{T}^{/V} \)-functors

\[
\prod_{f_1} \mathbf{C}_{U_1} \longrightarrow \prod_{f_{12}} \mathbf{C}_{U_{12}} \longleftarrow \prod_{f_1} \mathbf{C}_{U_1}
\]

and postcomposing with

\[
\prod_{f_{12}} \mathbf{C}_{U_{12}} \to \prod_{f_{12}} (\emptyset_0)_{U_{12}}
\]

we obtain the \( \mathcal{T}^{/V} \)-functors which define the pullback. Using the ‘fiberwise’ definition of generalized \( \mathcal{T} \)-∞-operad, we can use the same argument as in the proof of Theorem 2.3.3 to accomplish the proof. \( \square \)
4.2. \(\mathcal{T}\)-operadic join.

4.2.1. Definition. Let \(\widetilde{C}\), \(\widetilde{D}\), \(\widetilde{O}\) be generalized \(\mathcal{T}\)-\(\infty\)-operads over \(\mathbf{F}_{\mathcal{T},*}\) and let \(p : \widetilde{C} \to \widetilde{O}\) be a categorical fibration preserving the inert edges. Then the \(\mathcal{T}\)-operadic join of \(p\) and \(q\) is defined to be

\[
(\widetilde{C} \star \widetilde{O}) := \widetilde{O} \leftarrow \widetilde{C} \leftarrow \widetilde{C} \circlearrowleft \widetilde{O}.
\]

Similarly, given \(C\), \(D\), \(O\) fibrations of generalized \(\mathcal{T}\)-\(\infty\)-operads over \(\mathbf{F}_{\mathcal{T},*}\), we define the \(\mathcal{T}\)-operadic join \((C \star O)^\otimes\) to be \(C \circlearrowleft O \to O\).

Note that the underlying \(\mathcal{T}\)-\(\infty\)-category of a \(\mathcal{T}\)-operadic join \((C \star O)^\otimes\) is the \(\mathcal{O}\)-join of the underlying \(\mathcal{T}\)-\(\infty\)-categories of the factors (by the base-change property [Sha21a, Lem. 4.4]).

4.2.2. Proposition. \((\widetilde{C} \star \widetilde{O})^\otimes\) is a generalized \(\mathcal{T}\)-\(\infty\)-operad and the structure map to \(\widetilde{O}\) is a fibration of generalized \(\mathcal{T}\)-\(\infty\)-operads.

Proof. By the proof of [Sha21a, Prop. 4.7(2)], \(\pi : (\widetilde{C} \star \widetilde{O})^\otimes \to \mathbf{F}_{\mathcal{T},*}\) admits cocartesian lifts over the inert edges. Moreover, an edge \(\Delta^1 \to (\widetilde{C} \star \widetilde{O})^\otimes\) is \(\pi\)-cocartesian if and only if it factors through either \(\widetilde{C}\) or \(\widetilde{O}\) and is inert there.

The fiber of \((\widetilde{C} \star \widetilde{O})^\otimes\) over an object \([U_+ \to V]\) of \(\mathbf{F}_{\mathcal{T},*}\) is given by \(\overline{C}^\otimes_{[U_+ \to V]} \to \overline{D}^\otimes_{[U_+ \to V]}\). Moreover, the relative join is functorial in the following sense: given commutative diagrams

\[
\begin{array}{ccc}
X & \to & X' \\
\downarrow & & \downarrow \\
B & \to & B'
\end{array}
\quad
\begin{array}{ccc}
Y & \to & Y' \\
\downarrow & & \downarrow \\
B & \to & B'
\end{array}
\]

we have an induced map \(X \star_B Y \to X' \star_{B'} Y'\) covering \(B \times \Delta^1 \to B' \times \Delta^1\). From our explicit description of the \(\pi\)-cocartesian edges, we see that the Segal maps for \((\widetilde{C} \star \widetilde{O})^\otimes\) are obtained in this way. Consequently, it is clear that they are equivalences.

It remains to check that for all of the defining maps \(g_{\phi,\sigma_1,\sigma_2} : (\Lambda_2^2)^\otimes \to \mathbf{F}_{\mathcal{T},*}\), we have that for every lift \(G : (\Lambda_2^2)^\otimes \to (\widetilde{C} \star \widetilde{O})^\otimes\) where all edges are sent to \(\pi\)-cocartesian edges, \(G\) is a \(\pi\)-limit diagram. For this, there are two cases to consider. Either \(G\) factors through \(\widetilde{C}\), in which case the assertion obviously follows from \(G\) being a \(\pi\)-\(\infty\)-limit diagram, or \(G\) factors through \(\widetilde{O}\), in which case the assertion is a consequence of Lemma 4.2.3.

Finally, the second assertion is obvious given the first. \(\square\)

4.2.3. Lemma. Let \(K\), \(X\), \(Y\) and \(B\) be \(\infty\)-categories, let \(f_1, f_2 : X, Y \to B\) be categorical fibrations and let \(p : K \to Y\) be a functor. Also let \(p\) denote the composition \(K \overset{p}{\to} Y \subset X \star_B Y\). Then we have an equivalence of \(\infty\)-categories over \(B/\{f_1, f_2\} \times \Delta^1\)

\[
(X \star_B Y)^{/p} \simeq (X \times_B B/\{f_1, f_2\}) \star_{B/\{f_1, f_2\}} Y^{/p}.
\]

Consequently, if \(\overline{p} : K^\otimes \to Y\) is a \(f_2\)-limit diagram, then \(\overline{p} : K^\otimes \to Y \subset X \star_B Y\) is a \(f\)-limit diagram (where \(f\) denotes the structure map \(X \star_B Y \to B\)).

Proof. Let \(q\) denote \(K \to X \star_B Y \to B \times \Delta^1\) and note that \(q\) factors as \(K \overset{f_2}{\to} B \times \{1\} \subset B \times \Delta^1\). We first place \((X \star_B Y)^{/p}\) into the diagram of pullback squares

\[
\begin{array}{ccc}
(X \star_B Y)^{/p} & \to & Z \\
\downarrow & & \downarrow \\
\{p\} & \to & \text{Fun}_{/B}(K,Y) \\
\downarrow & & \downarrow \\
\{q\} & \to & \text{Fun}(K, X \star_B Y)
\end{array}
\quad
\begin{array}{ccc}
\text{Fun}(K^\otimes, X \star_B Y) & \to & \text{Fun}(K, X \star_B Y) \\
\downarrow & & \downarrow \\
\text{Fun}(K, B \times \Delta^1).
\end{array}
\]
In addition, using that the composition $\text{Fun}(K^{<}, X_{*} B Y) \to \text{Fun}(K, X_{*} B Y) \to \text{Fun}(K, B \times \Delta^{1})$ agrees with $\text{Fun}(K^{<}, X_{*} B Y) \to \text{Fun}(K^{<}, B \times \Delta^{1}) \to \text{Fun}(K, B \times \Delta^{1})$, $Z$ fits into the diagram of pullback squares

\[
\begin{array}{ccc}
Z & \to & \text{Fun}(K^{<}, X_{*} B Y) \\
\downarrow & & \downarrow \\
(B \times \Delta^{1})/q & \to & \text{Fun}(K^{<}, B \times \Delta^{1}) \cong \text{Fun}(K^{<}, B \times \text{Fun}(K^{<}, \Delta^{1})) \\
\downarrow & & \downarrow \\
\{q\} = \{f_{p}\} \times \{\text{const}_{1}\} & \to & \text{Fun}(K, B \times \Delta^{1}) \cong \text{Fun}(K, B \times \text{Fun}(K, \Delta^{1})).
\end{array}
\]

Because $(\Delta^{1})/\text{const}_{1} \cong \Delta^{1}$, we get that $(B \times \Delta^{1})/q \cong B/\Delta^{1}$. Consequently,

\[
(X_{*} B Y)/p \cong \{p\} \times_{\text{Fun}_{/B}(K,Y)} \left(\text{Fun}(K^{<}, X_{*} B Y) \times_{\text{Fun}(K^{<}, B \times \Delta^{1})} B/\Delta^{1}\right).
\]

Let $A \to B/\Delta^{1}$ be any functor. We will identify $\text{Fun}_{/(B/\Delta^{1})}(A, (X_{*} B Y)/p)$ with $\text{Fun}_{/B}(A_{0}, X) \times \text{Fun}_{/(B/\Delta^{1})}(A_{1}, Y/p)$ and thereby prove the claim. Let

\[
A \times K^{<} \to B/\Delta^{1} \times K^{<} \to B \times \Delta^{1}
\]

be the composite of the given map and the map adjoint to $B/\Delta^{1} \to \text{Fun}(K^{<}, B \times \Delta^{1})$. Note that $(A \times K^{<})_{0} \cong A_{0}$ and $(A \times K^{<})_{1} \cong (A \times K) \cup_{A_{1} \times K} A_{1} \times K^{<}$. We then have the chain of equivalences

\[
\begin{align*}
\text{Fun}_{/(B/\Delta^{1})}(A, (X_{*} B Y)/p) & \cong \{p \circ \text{pr}_{K}\} \times_{\text{Fun}_{/(A \times K, X_{*} B Y)}} \text{Fun}_{/(B \times \Delta^{1})}(A \times K^{<}, X_{*} B Y) \\
& \cong \{p \circ \text{pr}_{K}\} \times_{\text{Fun}_{/(A \times K, X_{*} B Y)}} \left(\text{Fun}_{/B}(A_{0}, X) \times \text{Fun}_{/B}((A \times K) \cup_{A_{1} \times K} A_{1} \times K^{<}, Y)\right) \\
& \cong \text{Fun}_{/B}(A_{0}, X) \times \left(\{p \circ \text{pr}_{K}\} \times_{\text{Fun}_{/(A \times K, Y)}} \text{Fun}_{/B}(A \times K, Y) \times_{\text{Fun}_{/B}(A_{1} \times K^{<}, Y)} \text{Fun}_{/B}(A_{1} \times K^{<}, Y)\right) \\
& \cong \text{Fun}_{/B}(A_{0}, X) \times \left(\{p \circ \text{pr}_{K}\} \times_{\text{Fun}_{/(A_{1} \times K, Y)}} \text{Fun}_{/B}(A_{1} \times K^{<}, Y)\right)
\end{align*}
\]

and finally

\[
\{p \circ \text{pr}_{K}\} \times_{\text{Fun}_{/B}(A_{1} \times K, Y)} \text{Fun}_{/B}(A_{1} \times K^{<}, Y) \cong \text{Fun}_{/(B/\Delta^{1})}(A_{1}, Y/p)
\]

because both sides compute the total fiber of the punctured cube

\[
\begin{array}{ccc}
\Delta^{0} & \to & \text{Fun}(A_{1} \times K, Y) \\
\downarrow & & \downarrow \\
\Delta^{0} & \to & \text{Fun}(A_{1} \times K^{<}, B) \\
\downarrow & & \downarrow \\
\Delta^{0} & \to & \text{Fun}(A_{1} \times K, B)
\end{array}
\]

For the last assertion, we need to show that

\[
(X_{*} B Y)/\mathcal{P} \to (X_{*} B Y)/p \\
\downarrow & \downarrow \\
B/\mathcal{P} & \to B/p
\]
is a homotopy pullback square. But by the first part of the lemma this is equivalent to
\[
(X \times_B B^{/\mathcal{T}}) *_{B^{/\mathcal{T}}} Y^{/\mathcal{P}} \longrightarrow (X \times_B B^{/\mathcal{P}}) *_{B^{/\mathcal{P}}} Y^{/\mathcal{P}}
\]
being homotopy pullback squares (the second by our hypothesis).

4.3. Construction of $\mathcal{T}$-operadic left Kan extensions. We now turn towards constructing $\mathcal{T}$-operadic left Kan extensions. We will need a variant of the $\mathcal{T}$-Day convolution for our proofs, where we allow the source $\mathcal{T}$-$\infty$-operad to be generalized.

4.3.1. Variant. Suppose that $\mathcal{C}^\otimes$ is a generalized $\mathcal{T}$-$\infty$-operad. Then the proofs of Theorem 3.1.4 and Corollary 3.1.5 still go through to show that
\[
\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes
\]
is a (non-generalized) $\mathcal{T}$-$\infty$-operad. Moreover, if $\mathcal{C}^\otimes, \mathcal{E}^\otimes \longrightarrow \mathcal{O}^\otimes$ are cocartesian fibrations, then the proof of Proposition 3.2.2 goes through to show that $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{C}, \mathcal{E})^\otimes \longrightarrow \mathcal{O}^\otimes$ is a locally cocartesian fibration. However, the proof of Theorem 3.2.6 doesn’t directly apply because if $\mathcal{C}^\otimes$ is generalized, we have a different formula for $\mathcal{C}^\otimes$ involving fiber products instead of products (cf. Lemma 4.1.1).

For the following results, let $\mathcal{O}^\otimes$ be a $\mathcal{T}$-$\infty$-operad, $p : \mathcal{C}^\otimes \longrightarrow \mathcal{O}^\otimes$ an $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category and $q : \mathcal{E}^\otimes \longrightarrow \mathcal{O}^\otimes$ a distributive $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category.

4.3.2. Lemma. Consider the commutative diagram
\[
\begin{array}{ccc}
\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{E} * \mathcal{O}, \mathcal{E})^\otimes & \longrightarrow & \mathcal{E}^\otimes \\
\downarrow^{\lambda} & & \downarrow^{q} \\
\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{E}, \mathcal{E})^\otimes & \longrightarrow & \mathcal{O}^\otimes.
\end{array}
\]
where $\lambda$ is given by restriction along $\mathcal{C}^\otimes \subset (\mathcal{C} * \mathcal{O})^\otimes$, $\rho$ is given by restriction along $\mathcal{O}^\otimes \subset (\mathcal{C} * \mathcal{O})^\otimes$ followed by the equivalence $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}, \mathcal{E})^\otimes \simeq \mathcal{E}^\otimes$ induced by precomposition with the identity section $\iota : \mathcal{O}^\otimes \longrightarrow \text{Ar}^{in}(\mathcal{O}^\otimes)$ (cf. Proposition 3.1.7), and $\pi$ is the structure map.

1. An edge $e$ in $\widetilde{\text{Fun}}_{\mathcal{O}, \mathcal{T}}(\mathcal{E} * \mathcal{O}, \mathcal{E})^\otimes$ is locally $\pi\lambda$-cocartesian if and only if $\lambda(e)$ is locally $\pi$-cocartesian and $\rho(e)$ is locally $q$-cocartesian. Consequently, $\pi\lambda$ is cocartesian.

2. $\lambda$ is $\mathcal{O}^\otimes$-cartesian and $\rho$ is cocartesian.

Proof. (1): $\rho$ and $\lambda$ are fibrations of $\mathcal{T}$-$\infty$-operads by the functoriality of the Day convolution, hence preserve inert edges. By Lemma 3.2.1, it suffices to consider a locally $\pi\lambda$-cocartesian edge $e$ over a fiberwise active edge $f : U \longrightarrow V \in F_\mathcal{T}$ (identified with $[U_* \longrightarrow V] \longrightarrow [V_* \longrightarrow V]$ in $F_{\mathcal{T}, u}$). Suppose $e$ covers $\alpha : x \longrightarrow y$ in $\mathcal{O}^\otimes_V$. Then by Proposition 3.2.2(3), $e$ corresponds to a $\mathcal{T}/V$-left Kan extension
\[
\begin{array}{ccc}
(C^\otimes_v) & \longrightarrow & (\mathcal{T}/V)^{op} \\
(\alpha^\otimes \downarrow) & \searrow & \\
(C^\otimes_x) & \longrightarrow & (\mathcal{T}/V)^{op}.
\end{array}
\]
Examining the pointwise formula defining $\mathcal{T}/V$-left Kan extensions, we see that $G$ is a $\mathcal{T}/V$-left Kan extension of $\alpha \circ F$ along $\alpha \circ \text{id}$ if and only if $G|_{C^\otimes_v}$ is a $\mathcal{T}/V$-left Kan extension of $\alpha \circ F|_{C^\otimes_v}$ along $\alpha$ and $G|_{V} \simeq \alpha \circ F|_{V}$. 
(for the latter, using that the inclusion of the right $\mathcal{T}$-cone point is fiberwise cofinal and [Sha21a, Thm. 6.7]). This implies the first part of the claim. For the consequence, we only need to check that the composition of locally cocartesian edges is again locally cocartesian, and this is clear using the claim and Theorem 3.2.6.

(2): Taking the fiber over an object $y \in \mathcal{O}_V$, we get a bifibration

$$\text{Fun}_{\mathcal{T}/}(\mathcal{E}_y, \mathcal{E}_y) \rightarrow \text{Fun}_{\mathcal{T}/}(\mathcal{E}_y, \mathcal{E}_y) \times \mathcal{E}_y.$$ Combining this observation with the product decomposition over a general object $x \in \mathcal{O}^\otimes$ obtained by the $\mathcal{T}$-Segal condition, we deduce that $\lambda$ is fiberwise cartesian and $\rho$ is fiberwise cocartesian (over $\mathcal{O}^\otimes$). It remains to show that for $\lambda$ the cocartesian pushforward of fiberwise cartesian edges remain fiberwise cartesian, and the dual statement for $\rho$. This is obvious over inert edges, so it suffices to consider a fiberwise active edge $\alpha : x \rightarrow y$ in $\mathcal{O}^\otimes_y$. Also, without loss of generality suppose $y \in \mathcal{O}_V$. Let $\theta : F_0 \rightarrow F_1$ be an edge in $\text{Fun}_{\mathcal{T}/}(\mathcal{E}, \mathcal{E})_x$, which corresponds to a natural transformation

$$\theta : \Delta^1 \times \mathcal{E}_x \rightarrow \mathcal{E}_x.$$ A fiberwise cartesian edge in $\text{Fun}_{\mathcal{T}/}(\mathcal{E} \odot \mathcal{O}, \mathcal{E})_x$ over $\theta$ is given by

$$\theta' : \Delta^1 \times (\mathcal{E}_x \odot \mathcal{E}_y) \rightarrow \mathcal{E}_x$$

which restricts to $\theta$ and is a constant natural transformation when restricted to the right $\mathcal{T}$-cone point. The cocartesian pushforward $\alpha \theta'$ is given by the $\mathcal{T}^\vee$-left Kan extension of $\alpha \otimes \theta'$ along $id \times \alpha \otimes$. Clearly, this is still a constant natural transformation when restricted to the right $\mathcal{T}$-cone point, which proves that $\alpha \theta'$ is a fiberwise cartesian edge lifting $\alpha \theta$. A similar argument handles the fiberwise cocartesian edges. $\square$

The next proposition is a very general and parametrized form of the following observation: the colimit of a lax symmetric monoidal functor canonically inherits the structure of a commutative algebra.

4.3.3. Proposition. Let $F : \mathcal{E}^\otimes \rightarrow \mathcal{E}^\otimes$ be a lax $\mathcal{O}$-monoidal $\mathcal{T}$-functor and let $\sigma_F : \mathcal{O}^\otimes \rightarrow \text{Fun}_{\mathcal{T}/}(\mathcal{E}, \mathcal{E})^\otimes$ be the associated section (which is an $\mathcal{O}$-algebra map). Then there exists a $\mathcal{O}$-algebra lift of $\sigma_F$ to

$$\sigma^\mathcal{T} : \mathcal{O}^\otimes \rightarrow \text{Fun}_{\mathcal{T}/}(\mathcal{E} \odot \mathcal{O}, \mathcal{E})^\otimes$$

such that the resulting $\mathcal{O}$-algebra

$$A = \mathcal{F}|_{\mathcal{O}^\otimes} : \mathcal{O}^\otimes \rightarrow \mathcal{E}^\otimes$$

has underlying section $A|_{\mathcal{O}} : \mathcal{O} \rightarrow \mathcal{E}$ computed as the $q|_{\mathcal{E}} \mathcal{T}$-left Kan extension of $F|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}$ along the inclusion $\mathcal{E} \rightarrow \mathcal{E} \odot \mathcal{O}$.

Proof. Factoring $\sigma_F$ through the $\mathcal{O}$-monoidal envelope $\text{Ar}^\mathcal{T}(\mathcal{O}^\otimes)$ of id : $\mathcal{O}^\otimes \rightarrow \mathcal{O}^\otimes$, we obtain a pullback square of $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-categories and (strong) $\mathcal{O}$-monoidal $\mathcal{T}$-functors

$$\begin{array}{ccc}
X & \rightarrow & \text{Fun}_{\mathcal{T}/}(\mathcal{E} \odot \mathcal{O}, \mathcal{E})^\otimes \\
\lambda_F & \downarrow & \downarrow \lambda \\
\text{Ar}^\mathcal{T}(\mathcal{O}^\otimes) & \rightarrow & \text{Fun}_{\mathcal{T}/}(\mathcal{E}, \mathcal{E})^\otimes.
\end{array}$$

We proceed to identify the fibers of $\lambda_F$. By definition, for any object $x \in \mathcal{O}^\otimes$, $\sigma_F(x)$ is given by the functor $F_{\mathbf{x}} : \mathcal{E}_x^\otimes \rightarrow \mathcal{E}_x^\otimes$, and for any fiberwise active edge $\alpha : x \rightarrow y$, $\tau_F(\alpha)$ is given by the cocartesian pushforward $\alpha F_{\mathbf{x}} : \mathcal{E}_x^\otimes \rightarrow \mathcal{E}_y^\otimes$. If $\alpha$ decomposes via the $\mathcal{T}$-$\infty$-operad axioms as $(\alpha_i : x_i \rightarrow y_i)_{1 \leq i \leq n}$ for $y_i \in \mathcal{O}_V$, (induced by an orbit decomposition $V \simeq V_1 \sqcup \ldots \sqcup V_n$ if $y_i$ covers $[V_+ \rightarrow \mathcal{E}]$ in $\mathcal{E}_{\mathcal{T},*}$), then $\alpha_i F_{\mathbf{x}}$ may be explicitly identified as the collection of $\mathcal{T}^\vee$-left Kan extensions $((\alpha_i)_{\mathbf{x}})_{\otimes} F_{\mathbf{x}}$ of $(\alpha_i)_{\otimes} F_{\mathbf{x}} : \mathcal{E}_x^\otimes \rightarrow \mathcal{E}_y^\otimes$. Therefore, the fiber of $\lambda_F$ over $\{\alpha\}$ is given by

$$\prod_{1 \leq i \leq n} \mathcal{E}^{((\alpha_i)_{\mathbf{x}})_{\otimes} F_{\mathbf{x}}}_{\mathcal{T}^\vee}.$$ Because each $\mathcal{E}_y^\otimes$ is $\mathcal{T}^\vee$-cocomplete by assumption, these fibers all have initial objects, which are moreover preserved by the cocartesian edges over $\mathcal{T}^{op}$ (i.e., by restriction). We claim that restricting $\lambda_F$ to the full $\mathcal{T}$-subcategory $X_0$ on these initial objects yields a trivial Kan fibration $\lambda'_F : X_0 \rightarrow \text{Ar}^\mathcal{T}(\mathcal{O}^\otimes)$. By (2) of
Lemma 4.3.2, $\lambda$ is $\mathcal{O}^{\otimes}$-cartesian, hence the pulled back map $\lambda_F$ is $\mathcal{O}^{\otimes}$-cartesian. Therefore, it suffices to check that the cocartesian edges in $X$ over $\mathcal{O}^{\otimes}$ preserve initial objects. This is obvious for cocartesian edges over inert edges in $\mathcal{O}^{\otimes}$, so it suffices to consider the case of a fiberwise active edge $\beta : y \to z$ in $\mathcal{O}^{\otimes}_W$ with $z \in \mathcal{O}_W$. Then for our fiberwise active edge $\alpha : x \to y \in \mathcal{O}^{\otimes}_W$ above, $\beta(\alpha) = \beta \circ \alpha : x \to z$ computes the cocartesian pushforward in $\Lambda_{\mathcal{O}_W}^{act}(\mathcal{O}^{\otimes})$. As we have seen, an initial object in $X$ covering $\alpha$ corresponds to a collection of $\mathcal{T}/W$-colimits of $(\alpha_i)_{i \in I}$.

![Diagram](image)

By our assumption that $\mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ is distributive, applying $\prod_\beta$ and postcomposing with $\otimes_{\beta} : \mathcal{E}^{\otimes}_{\mathcal{Z}} \to \mathcal{E}_{\mathcal{Z}}$ yields a $\mathcal{T}/W$-colimit diagram.

![Diagram](image)

Factoring $\mathcal{E}^{\otimes}_{\mathcal{Y}} \to (\mathcal{T}/W)^{op}$ through $\mathcal{E}_{\mathcal{Z}}$ and using the transitivity of $\mathcal{T}/W$-left Kan extensions, this further implies that the diagram

![Diagram](image)

is a $\mathcal{T}/W$-colimit diagram, where inspecting the definitions reveals that the top horizontal arrow may be identified with $(\beta \circ \alpha)|_{\mathcal{F}_{\mathcal{Z}}}$. This is an initial object covering $\beta \circ \alpha$, as desired.

Choosing a section of $\lambda_F$ and postcomposing by the map $X_0 \to \overline{\text{Fun}}_{\mathcal{T}/\mathcal{O}}(\mathcal{E} \star \mathcal{O}, \mathcal{E})^{\otimes}$, we obtain the desired extension $\sigma_{\mathcal{F}}$. Finally, the assertion about $A|_\mathcal{O}$ is clear from the construction if we consider only those objects $x$, $\alpha = \text{id}_x$, and edges $\beta$ entirely in $\mathcal{O}$.

We can then promote Proposition 4.3.3 to a global existence result.

4.3.4. **Theorem.** We have $\mathcal{O}^{\otimes}$-adjunctions

$$\overline{\text{Fun}}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}, \mathcal{E})^{\otimes} \xrightarrow{e_{\mathcal{E}^{\otimes}}} \overline{\text{Fun}}_{\mathcal{T}/\mathcal{O}}(\mathcal{E} \star \mathcal{O}, \mathcal{E})^{\otimes} \xleftarrow{e_{\mathcal{E}}} \mathcal{E}^{\otimes}.$$  

Consequently, on passing to $\infty$-categories of $\mathcal{O}$-algebras, we obtain the adjunction

$$p_* : \text{Alg}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}, \mathcal{E}) \rightleftarrows \text{Alg}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}) : p^*,$$

where $p_*$ is computed as in Proposition 4.3.3 and $p^*$ is restriction along $p$.

**Proof.** The only subtlety involves the first $\mathcal{O}^{\otimes}$-adjunction. We may invoke [Lur17, 7.3.2.12] because the second condition there is ensured by distributivity in $\mathcal{E}^{\otimes}$, using the same argument as in the proof of Proposition 4.3.3. We can then extract the adjunction involving $\infty$-categories of $\mathcal{O}$-algebra maps by pulling back along the structure map $e_{\mathcal{E}} : \Lambda_{\mathcal{T}}^{act}(\mathcal{O}^{\otimes}) \to \mathcal{O}^{\otimes}$ of the $\mathcal{O}$-monoidal envelope and taking cocartesian sections.

Now suppose that $p : \mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ is only a fibration of $\mathcal{T}$-$\infty$-operads and consider the factorization of $p$ through its $\mathcal{O}$-monoidal envelope $\text{Env}_{\mathcal{T}/\mathcal{O}}(\mathcal{E})^{\otimes}$. In view of Proposition 2.8.7 and Theorem 4.3.4, we have the composite adjunction

$$p_* : \text{Alg}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}, \mathcal{E}) \rightleftarrows \text{Alg}_{\mathcal{T}/\mathcal{O}}(\text{Env}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}), \mathcal{E}) \rightleftarrows \text{Alg}_{\mathcal{T}/\mathcal{O}}(\mathcal{E}) : p^*.$$
4.3.5. **Definition.** Given a $O$-algebra map $F : \mathcal{C}^O \to \mathcal{E}^O$, we define the $T$-operadic left Kan extension of $F$ to be $p_!F$.

4.3.6. **Remark.** Given a fibration of $T$-$\infty$-operads $O^O \to \mathcal{P}^O$ and $\mathcal{E}^O \to \mathcal{P}^O$ a distributive $\mathcal{P}$-monoidal $T$-$\infty$-category, we will also speak of the $T$-operadic left Kan extension of a $T$-algebra map $F : \mathcal{C}^O \to \mathcal{E}^O$ along $p$ in the obvious way (note that distributivity is stable under pullback). In other words, we also have an adjunction

$$p_! : \text{Alg}_{\mathcal{P},T}(\mathcal{C}, \mathcal{E}) \rightleftarrows \text{Alg}_{\mathcal{P},T}(O, \mathcal{E}) : p^*.$$

Note that the underlying $T$-functor $p(F)|_O : O \to \mathcal{E}$ is computed by first extending $F$ to

$$i_!F : \text{Env}_{O,T}(\mathcal{C})^O \to \mathcal{E}^O$$

and then taking the $T$-left Kan extension of $i_!F|_{\text{Env}_{O,T}(\mathcal{C})}$ along the structure map to $O$.

4.3.7. **Example.** Suppose that $\mathcal{E}^\otimes = \text{Triv}^\otimes = (E_{\Gamma,*})_{\text{act}}$ and $O^\otimes = E_{\Gamma,*}$. Then the $T$-symmetric monoidal envelope of $\text{Triv}^\otimes$ is $(\text{Triv}^\otimes)_{\text{act}} = (E_{\Gamma,*})_{\text{coact}}$, the maximal sub-left fibration of $E_{\Gamma,*} \to \mathcal{T}^\text{op}$ obtained by taking the wide subcategory spanned by the cocartesian edges.

In the case that $T = O_G$ is the orbit category of a finite group, we can identify this with something familiar. Namely, let $\Sigma_n$ be the symmetric group on $n$ letters, and let $O_{G \times \Sigma_n, \Gamma_n}$ be the full subcategory of the orbit category of $G \times \Sigma_n$ on the $\Sigma_n$-free transitive $G \times \Sigma_n$-sets. (Recall that every object in this subcategory is isomorphic to an orbit $G \times \Sigma_n / \Gamma$, where $\Gamma$ is the graph of a homomorphism $\phi : H \to \Sigma_n$ for some subgroup $H$ of $G$.) Define a functor

$$- / \Sigma_n : O_{G \times \Sigma_n, \Gamma_n} \to O_G, \ U \mapsto U / \Sigma_n.$$

This is left adjoint to restriction along the projection $G \times \Sigma_n \to G$ and sends $(G \times \Sigma_n) / \Gamma_\phi$ to $G / H$. Then $(- / \Sigma_n)^\text{op}$ is a left fibration and exhibits $O_{G \times \Sigma_n, \Gamma_n}^\text{op}$ as a $G$-$\infty$-category. In fact, $O_{G \times \Sigma_n, \Gamma_n}^\text{op}$ is a model for the $G$-space $B_G \Sigma_n$ which classifies $G$-equivariant principal $\Sigma_n$-bundles ([QS21a, Rem. 3.17]).

Now define a $G$-functor $F : O_{G \times \Sigma_n, \Gamma_n}^\text{op} \to (E_G)_{\text{coact}}$ which sends an object $U$ to the morphism of $G$-sets $(U \times n) / \Sigma_n \to U / \Sigma_n$ and a morphism $U \to V$ to

$$\xymatrix{ (U \times n) / \Sigma_n \ar[r] & (V \times n) / \Sigma_n \ar[d] \ar[r]^= & (U \times n) / \Sigma_n \ar[d] \\
U / \Sigma_n \ar[r] & V / \Sigma_n \ar[r]^= & V / \Sigma_n }$$

where we note that the left square is a pullback square of $G$-sets. (Note that this suffices to define a functor since $E_G$ is equivalent to a 1-category.) It follows from $\Sigma_n$-freedom and an elementary argument that $F$ is fully faithful. Moreover, taking the disjoint union over all $n \geq 0$ and postcomposing with $(-)_+$, we obtain an equivalence of $G$-$\infty$-categories

$$\prod_{n \geq 0} O_{G \times \Sigma_n, \Gamma_n}^\text{op} \xrightarrow{\sim} (E_G)_{\text{coact}} \xrightarrow{\sim} (E_G)_{\text{coact}}.$$

Therefore, for a $G$-symmetric monoidal $\infty$-category $\mathcal{C}^\otimes$, the free $G$-commutative algebra on an object $x : O_G^\otimes \to \mathcal{C}$ is computed by the $G$-colimit of the induced functor

$$\prod_{n \geq 0} O_{G \times \Sigma_n, \Gamma_n}^\text{op} \to \mathcal{C}.$$
5.1. Parametrized (co)limits in general. In this section, let $\mathcal{C}^\circ \longrightarrow \mathcal{O}^\circ$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category, let $\mathcal{P}^\circ \longrightarrow \mathcal{O}^\circ$ be a fibration of $\mathcal{T}$-$\infty$-operads, and let $\mathcal{K} = \{K_V : V \in \mathcal{T}\}$ be a collection of classes $K_V$ of small $\mathcal{T}/V$-$\infty$-categories closed with respect to base-change in $\mathcal{T}$. We are interested in criteria for when the $\mathcal{T}$-$\infty$-category $\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C})$ of algebras strongly admits $\mathcal{K}$-indexed $\mathcal{T}$-limits and colimits. To solve this problem, we will first need to understand how to compute $\mathcal{T}$-limits and $\mathcal{T}$-colimits in an indexed product.

5.1.1. Lemma. Let $f : \mathcal{T}_0 \longrightarrow \mathcal{T}_1$ be a categorical fibration of $\infty$-categories, let $\mathcal{C}$ be a $\mathcal{T}_0$-$\infty$-category, and let $\mathcal{K}$ be a $\mathcal{T}_1$-$\infty$-category. Let

$$f^* : \text{Cat}_{\mathcal{T}_1} \longrightarrow \text{Cat}_{\mathcal{T}_0} : f_*$$

denote the restriction-coinduction adjunction.

1. Let $(F : \mathcal{C} \longrightarrow \mathcal{D} : G)$ be a $\mathcal{T}_0$-adjunction. Then $(f_*F : f_*\mathcal{C} \longrightarrow f_*\mathcal{D} : f_*G)$ is a $\mathcal{T}_1$-adjunction.

2. We have a canonical equivalence

$$\text{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{C}) \simeq f_*\text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C})$$

under which $\delta_{\mathcal{K}} \simeq f_*\delta_{f^*\mathcal{K}}$ as $\mathcal{T}_1$-functors with common domain $f_*\mathcal{C}$, where $\delta_{\mathcal{K}}$, resp. $\delta_{f^*\mathcal{K}}$ is the constant $\mathcal{K}$-diagram $\mathcal{T}_1$-functor, resp. constant $f^*\mathcal{K}$-diagram $\mathcal{T}_0$-functor.

3. Let $p : \mathcal{K} \longrightarrow f_*\mathcal{C}$ be a $\mathcal{T}_1$-functor and let $q : f^*\mathcal{K} \longrightarrow \mathcal{C}$ be its adjoint $\mathcal{T}_0$-functor. Then $p$ admits a $\mathcal{T}_1$-limit if and only if $q$ admits a $\mathcal{T}_0$-limit, and moreover an extension $\overline{p} : \mathcal{K} \longrightarrow f_*\mathcal{C}$ is a $\mathcal{T}_1$-limit diagram if and only if the adjoint extension $\overline{q} : f^*\mathcal{K} \longrightarrow \mathcal{C}$ is a $\mathcal{T}_0$-limit diagram. The analogous statements also hold for parametrized colimits.

Proof. (1): It suffices to show that for all $t \in \mathcal{T}_1$, $(f_t, \mathcal{C}) \dashv (f_t, \mathcal{D})$ is an adjunction. In fact, since $(f_*\mathcal{C})_t \simeq \text{Fun}_{\mathcal{T}_1}(\mathcal{T}_1^t \circ f, \mathcal{C})$, we can more generally verify that $\text{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{C}) \dashv \text{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{D})$ is an adjunction for every $\mathcal{T}_1$-$\infty$-category $\mathcal{K}$. But this holds since

$$\text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{F}) : \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C}) \longrightarrow \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{D}) : \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{G})$$

is an adjunction by [Sha21a, Prop. 8.2].

(2): We check the claimed equivalence at the level of representable functors:

$$\text{Fun}_{\mathcal{T}_1}(\mathcal{L}, \text{Fun}_{\mathcal{T}_1}(\mathcal{K}, f_*\mathcal{C})) \simeq \text{Fun}_{\mathcal{T}_1}(\mathcal{L} \times_{\mathcal{T}_0^p} \mathcal{K}, f_*\mathcal{C}) \simeq \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{L} \times_{\mathcal{T}_0^p} f^*\mathcal{K}, \mathcal{C})$$

$$\simeq \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{L}, \text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C})) \simeq \text{Fun}_{\mathcal{T}_0}(\mathcal{L}, f_*\text{Fun}_{\mathcal{T}_0}(f^*\mathcal{K}, \mathcal{C}))$$

The assertion about constant functors is shown in a similar manner.

(3): This follows from combining (1) and (2).

5.1.2. Corollary. Let $\mathcal{C}^\circ \longrightarrow \mathcal{O}^\circ$ be a fibration of $\mathcal{T}$-$\infty$-operads, let $f : U \simeq \prod_{1 \leq i \leq n} U_i \longrightarrow V$ be a morphism in $\mathcal{F}_\mathcal{T}$ with $U_i$ and $V$ orbits, let $x \in \mathcal{O}^\circ_{f_*^1}$, and let $e_i : x \longrightarrow x_i$ be inert edges in $\mathcal{O}^\circ$ lifting the characteristic morphisms $\chi_{U_i, U_i}^1$ in $\mathcal{E}_{\mathcal{T}_*}$. Suppose $e_{x_i}$ admits all $\mathcal{K}_{U_i}$-indexed $\mathcal{T}/U_i$-(co)limits for each $1 \leq i \leq n$. Then $\mathcal{E}_{\mathcal{T}}$ admits all $\mathcal{K}_{U}$-indexed $\mathcal{T}/U$-(co)limits.

Proof. Combine Lemma 5.1.1(3) and the Segal equivalence $\mathcal{E}_{\mathcal{T}}^\varnothing \simeq \prod_{1 \leq i \leq n} \prod_f e_{x_f}$ of Theorem 2.3.3.

We may now prove our main result on parametrized limits.

5.1.3. Theorem. Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category and let $\mathcal{P}^\circ \longrightarrow \mathcal{O}^\circ$ be a fibration of $\mathcal{T}$-$\infty$-operads. Let $\mathcal{K} = \{K_V : V \in \mathcal{T}\}$ be a collection of classes $K_V$ of small $\mathcal{T}/V$-$\infty$-categories closed with respect to base-change in $\mathcal{T}$ (e.g., we could take $K_V$ to be all small $\mathcal{T}/V$-$\infty$-categories for each $V \in \mathcal{T}$). Suppose for all $x \in \mathcal{O}_{\mathcal{T}}$ that $\mathcal{E}_{\mathcal{T}}^x$ admits all $\mathcal{K}_V$-indexed $\mathcal{T}/V$-limits. Then:

1. Both $\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C})$ and $\text{Fun}_{\mathcal{O}^\circ, \mathcal{T}}(\mathcal{P}^\circ, \mathcal{C}^\circ)$ strongly admit all $\mathcal{K}$-indexed $\mathcal{T}$-limits, and the inclusion

$$\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C}) \subset \text{Fun}_{\mathcal{O}^\circ, \mathcal{T}}(\mathcal{P}^\circ, \mathcal{C}^\circ)$$

strongly preserves all $\mathcal{K}$-indexed $\mathcal{T}$-limits.

2. $\text{Fun}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C})$ strongly admits all $\mathcal{K}$-indexed $\mathcal{T}$-limits, and the forgetful $\mathcal{T}$-functor

$$U : \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C}) \longrightarrow \text{Fun}_{\mathcal{O}, \mathcal{T}}(\mathcal{P}, \mathcal{C})$$

strongly creates all $\mathcal{K}$-indexed $\mathcal{T}$-limits.
Proof. By [Sha21b, Cor. 4.17] and Corollary 5.1.2, \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C}) \) strongly admits all \( \mathcal{K} \)-indexed \( \mathcal{T} \)-limits. Moreover, by the explicit formula for parametrized limits in \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C}) \) given in [Sha21b, 4.16(3)], we see the remaining claims follow from the observation that for every fiberwise inert edge \( e : x \to x' \) in \( \mathcal{O}_V^\circ \), the associated pushforward functor \( e_! : \mathcal{C}_x^\circ \to \mathcal{C}_{x'}^\circ \) is identified with projection to a subset of factors in a fiber product under the Segal equivalence of Theorem 2.3.3, so in particular preserves all \( \mathcal{K}_V \)-indexed \( \mathcal{T}^V \)-limits. This proves (1). (2) then follows by invoking [Sha21b, 4.16(3)] once more.

Next, we handle the case of parametrized colimits, for which we will need an additional distributivity assumption on \( \mathcal{C}_x^\circ \to \mathcal{O}_x^\circ \). Let \( \mathcal{K}_V \) be the class \( \mathcal{K}_V^{\text{inh}} \) of \( \mathcal{T}^V \)-sifted \( \mathcal{T}^V \)-colimits and write \( \mathcal{K}^{\text{inh}} \mathcal{K} = \mathcal{K} \).

5.1.4. Theorem. Suppose \( \mathcal{E} \) is a distributive \( \mathcal{O} \)-monoidal \( \mathcal{T} \)-\( \infty \)-category (Definition 3.2.4), and let \( \mathcal{P}^\circ \to \mathcal{O}^\circ \) be a fibration of \( \mathcal{T} \)-\( \infty \)-operads. Then:

1. Both \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) and \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C}) \) strongly admit all \( \mathcal{K}^{\text{inh}} \)-indexed \( \mathcal{T} \)-colimits, and the inclusion

\[
\mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \subseteq \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}^\circ, \mathcal{C}^\circ)
\]

strongly preserves all \( \mathcal{K}^{\text{inh}} \)-indexed \( \mathcal{T} \)-colimits.

2. \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C}) \) is \( \mathcal{T} \)-cocomplete and the forgetful \( \mathcal{T} \)-functor

\[
U : \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \to \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C})
\]

strongly creates all \( \mathcal{K}^{\text{inh}} \)-indexed \( \mathcal{T} \)-colimits.

3. \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) is \( \mathcal{T} \)-cocomplete.

4. Suppose in addition that \( \mathcal{E} \) is fiberwise presentable. Then \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) is fiberwise presentable.

Proof. (1) and (2): To show the claim for \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}^\circ, \mathcal{C}^\circ) \), we verify the criterion of [Sha21b, Thm. 4.16(4)]. Since a fiberwise morphism \( \alpha : x \to y \) in \( \mathcal{O}^\circ_V \) factors as the composite of a fiberwise inert edge and a fiberwise active edge, and the pushforward functor associated to a fiberwise inert edge is a projection, we may suppose \( \alpha \) is fiberwise active. Moreover, using again the Segal equivalence of Theorem 2.3.3, we may suppose that \( \alpha \) covers a fiberwise active edge \( f_+ : [U_+ \to V] \to [V_+ \to V] \) in \( \mathcal{F}_{\tau,*} \) defined by a map \( f : U \to V \) of finite \( \mathcal{T} \)-sets. Then using the distributive hypothesis on \( \mathcal{C}^\circ \) together with [Sha21b, Prop. 8.19], we have that the pushforward \( \mathcal{T}^V \)-functor \( \alpha_! = \otimes_\alpha : \mathcal{C}_x^\circ \to \mathcal{C}_y^\circ \) preserves all \( \mathcal{K}_V^{\text{inh}} \)-indexed colimits, so \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}^\circ, \mathcal{C}^\circ) \) strongly admits all \( \mathcal{K}^{\text{inh}} \)-indexed \( \mathcal{T} \)-colimits. Similarly, we see that \( \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{P}, \mathcal{C}) \) is \( \mathcal{T} \)-cocomplete. The remaining claims then follow as in the proof of Theorem 5.1.3, now using [Sha21b, Thm. 4.16(4)].

(3): By part (1) and [Sha21a, Cor. 12.15] (or [Sha21b, Thm. 8.6]), it suffices to check that \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) admits finite \( \mathcal{T} \)-coproducts. For this, we employ the same strategy as in the proof of [Lur17, Prop. 3.2.3.3]. Pulling back \( \mathcal{C}^\circ \to \mathcal{O}^\circ \) via \( \mathcal{P}^\circ \to \mathcal{O}^\circ \), we may suppose \( \mathcal{P}^\circ = \mathcal{O}^\circ \) without loss of generality. Now let \( \mathcal{O}_{ne}^\circ = \mathcal{O}_{ne}^\circ \times_{\mathcal{F}_{\tau,*}} \text{Triv}^\circ \) and note that by Lemma 2.4.4 and Corollary 2.4.5, we have that \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{O}_{ne}^\circ, \mathcal{E}) \simeq \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{O}, \mathcal{E}) \). By Theorem 4.3.4, we thus obtain the free-forgetful \( \mathcal{T} \)-adjunction

\[
F : \text{Fun}_{/\mathcal{O}, \tau}(\mathcal{O}, \mathcal{E}) \rightleftarrows \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{E}) : \mathcal{U}
\]

as an instance of \( \mathcal{T} \)-operadic left Kan extension along \( \mathcal{O}_{ne}^\circ \to \mathcal{O}^\circ \). This implies that each fiber \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{E})_V \) admits finite coproducts of free objects, and for each morphism \( \alpha : V \to W \) in \( \mathcal{T} \), the putative left adjoint \( \alpha_! \) to the restriction functor \( \alpha^* : \mathcal{Alg}_{\mathcal{O}, \tau}(E)_W \to \mathcal{Alg}_{\mathcal{O}, \tau}(E)_V \) is at least defined on the full subcategory of free objects, using the pointwise criterion for the existence of an adjoint. By part (2) and the observation that \( U \) is fiberwise conservative, the assumptions of [Lur17, Prop. 4.7.3.14] are satisfied for each of the adjunctions \( F_V \dashv U_V \), so for each object \( A \in \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{E})_V \), there exists a simplicial object \( A_n \) such that each \( A_n \) is free and \( A \simeq |A_n| \). It follows that the requisite finite coproducts and left adjoints exist for \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{E}) \). Finally, verification of the base-change condition also reduces to free objects in the same way.

(4): Upon replacing \( \mathcal{T} \) by \( \mathcal{T}^V \) this amounts to showing that \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) is presentable. Given (3), it remains to show that \( \mathcal{Alg}_{\mathcal{O}, \tau}(\mathcal{P}, \mathcal{E}) \) is accessible. But since all the pushforward functors \( \alpha_! : \mathcal{C}_x^\circ \to \mathcal{C}_{x'}^\circ \) indexed by morphisms \( \alpha : x \to x' \in \mathcal{O}^\circ \) preserve sifted colimits (which reduces to the aforementioned assertion for fiberwise active \( \alpha \) given the inert-fiberwise active factorization on \( \mathcal{O}^\circ \)), this follows from [Lur09, Prop. 5.4.7.11] in exactly the same way as in the proof of [Lur17, Cor. 3.2.3.5].
5.1.5. **Corollary.** Suppose $C$ is a distributive $O$-monoidal $\mathcal{T}$-$\infty$-category. Then the free-forgetful $\mathcal{T}$-adjunction

$$F : \text{Fun}_{/O, \mathcal{T}}(O, C) \leftrightarrow \text{Alg}_{O, \mathcal{T}}(C) : U$$

of Theorem 4.3.4 applied to $O^\otimes \subset \mathcal{O}^\otimes$ is fiberwise monadic.

**Proof.** We verify the hypotheses of the Barr–Beck–Lurie Theorem [Lur17, Thm. 4.7.3.5]. After Theorem 5.1.4(2), it only remains to note that $U$ is fiberwise conservative, which is immediate from the definitions. □

5.2. **Units and initial objects.** In this subsection, we identify $\mathcal{T}$-initial objects in $\text{Alg}_{O, \mathcal{T}}(C)$ in the case where $\mathcal{O}^\otimes$ is a *unital* $\mathcal{T}$-$\infty$-operad and $\mathcal{C}^\otimes$ is any $O$-monoidal $\mathcal{T}$-$\infty$-category.

5.2.1. **Definition.** Let $O^\otimes$ be a $\mathcal{T}$-$\infty$-operad. We say that $O^\otimes$ is *unital* if for all orbits $V \in \mathcal{T}$ and objects $x \in O_V$, the space of multimorphisms $\text{Mul}_O(\emptyset, x)$ is contractible.

For example, $E_{\mathcal{T}, *}$ is unital and $\text{Triv}^\otimes$ is not unital. We next introduce the minimal $\mathcal{T}$-suboperad of $E_{\mathcal{T}, *}$ which remains unital.

5.2.2. **Definition.** Let $E_{0, \mathcal{T}} \subset E_{\mathcal{T}, *}$ be the $\mathcal{T}$-suboperad given by the wide subcategory on those morphisms

$$U \leftarrow Z \overset{m}{\longrightarrow} X$$

$$V \leftarrow Y \overset{y}{\longrightarrow} Y$$

for which $m$ is a summand inclusion.

Given any $\mathcal{T}$-$\infty$-operad $O^\otimes$, we will then write $O^\otimes_0$ for the pullback $E_{0, \mathcal{T}} \times E_{*}, O^\otimes$ in this subsection. Note that the inclusion $E_{0, \mathcal{T}} \subset E_{\mathcal{T}, *}$ is stable under equivalences and is thus a fibration of $\mathcal{T}$-$\infty$-operads, and the same is true for the pullback $O^\otimes_0 \subset O^\otimes$.

5.2.3. **Remark.** Let $O^\otimes$ be a $\mathcal{T}$-$\infty$-operad and for each orbit $V \in \mathcal{T}$, let $*_{V}$ be a choice of object in the fiber $O^\otimes[\emptyset \rightarrow V] \simeq *$, which is unique up to contractible choice. We note that $*_{V}$ is a final object in $O^\otimes_{V}$. Indeed, if $O^\otimes = F_{\mathcal{T}, *}$, then $[\emptyset \rightarrow V]$ is a zero object in $(F_{\mathcal{T}, *})_{V} \simeq F_{\mathcal{T}, V, *}$, and the general case follows by the definition of a $\mathcal{T}$-$\infty$-operad. Since for each $a : V \longrightarrow W \in \mathcal{T}$ we have that $a^{*}(\ast_{W}) \simeq \ast_{V}$, the $\ast_{V}$ assemble to define a $\mathcal{T}$-final object $\ast : \mathcal{T}^{op} \longrightarrow O^\otimes$.

Now suppose $O^\otimes$ is unital. Then by the same reasoning, we see that $\ast_{V}$ is a zero object in $O^\otimes_{V}$ and hence $\ast$ is a $\mathcal{T}$-zero object. In this case, we will also write $0_{V}$ for $\ast_{V}$ and $0$ for $\ast$.

5.2.4. **Definition.** We define the $\mathcal{T}$-functor $\omega : \Delta^{1} \times \mathcal{T}^{op} \longrightarrow E_{\mathcal{T}, *}$ to be the unique homotopy from $0$ to $I(-)_{+}$. For a unital $\mathcal{T}$-$\infty$-operad $O^\otimes$, we then define the $\mathcal{T}$-functor $\omega_{0}$ lying in the commutative diagram

$$\begin{array}{ccc}
O^\otimes & \longrightarrow & O^\otimes_0 \\
\downarrow & & \downarrow \\
\Delta^{1} \times \mathcal{T}^{op} & \longrightarrow & E_{0, \mathcal{T}} \longrightarrow E_{\mathcal{T}, *}
\end{array}$$


to be the unique $\mathcal{T}$-functor extending the inclusion $O \subset O^\otimes_0$ of the underlying $\infty$-category, whose restriction along the cone inclusion $\mathcal{T}^{op} \subset O^\otimes$ is 0.

For an $O$-monoidal $\mathcal{T}$-$\infty$-category $p : C^\otimes \longrightarrow O^\otimes$, we define the $\mathcal{T}$-functor $\tilde{\omega}_{C}$ lying in the commutative diagram

$$\begin{array}{ccc}
O^\otimes & \longrightarrow & C^\otimes \\
\downarrow & & \downarrow \omega \\
O^\otimes_0 & \longrightarrow & O^\otimes
\end{array}$$


to be the unique lift of $\omega_{0}$ so that $\tilde{\omega}_{C}$ sends the edges $(0_{V} \longrightarrow x) \in O^\otimes_{V}$ to $p$-cocartesian edges (and hence all edges to $p$-cocartesian edges).

We then define the *unit* of $(C^\otimes, p)$ to be the $\mathcal{T}$-functor $1 := \tilde{\omega}_{C}|_{0} : O \longrightarrow C$. 
5.2.5. Remark. The \( \mathcal{T} \)-functors \( \omega, \omega_\circ \), and \( \omega_c \) of Definition 5.2.4 may be defined rigorously as follows. For \( \omega \), choose a section \( \sigma \) of the trivial fibration \( \mathcal{T}^{\op} \times_{0, s_0, \ev_0} \mathcal{A}_T(\mathcal{E}^\circ_{0, \mathcal{T}}) \xrightarrow{\simeq} \mathcal{E}^\circ_{0, \mathcal{T}} \) of Lemma 5.2.6 and define \( \omega \) to be the adjoint to the composite \( \omega^\perp = \sigma \circ \delta \circ I_+ : \mathcal{T}^{\op} \longrightarrow \mathcal{A}_T(\mathcal{E}^\circ_{0, \mathcal{T}}) \).

Then for \( \omega_\circ \), choose a section \( \sigma_\circ \) of the trivial fibration \( \mathcal{T}^{\op} \) of Lemma 5.2.6 applied to \( 0^\circ \longrightarrow \mathcal{E}^\circ_{0, \mathcal{T}} \), let \( \rho : \mathcal{O} \longrightarrow \mathcal{T}^{\op} \) and \( i : \mathcal{O} \subset 0^\circ \) denote the structure map and inclusion, and define the \( \mathcal{T} \)-functor
\[
\omega_\circ^\perp = \sigma \circ (\omega^\perp \rho, i) : \mathcal{O} \longrightarrow (\mathcal{T}^{\op} \times_{0^\circ, \ev_0} \mathcal{A}_T(\mathcal{E}^\circ_{0, \mathcal{T}})) \times_{\mathcal{E}^\circ_{0, \mathcal{T}}} 0^\circ \longrightarrow \mathcal{T}^{\op} \times_{0^\circ, \ev_0} \mathcal{A}_T(\mathcal{O}^\circ).
\]
By [Sha21a, Cor. 4.27] and [Sha21a, Prop. 4.30], for any \( \mathcal{T} \)-\( \infty \)-category \( \mathcal{D} \) and cocartesian section \( \phi : \mathcal{T}^{\op} \longrightarrow \mathcal{D} \) we have natural equivalences
\[
\mathcal{D}_{(\phi, \mathcal{T})} = \mathcal{D}_{(\phi, \mathcal{T})} \cong \mathcal{T}^{\op} \times_{\mathcal{D}} \mathcal{A}_T(\mathcal{D}),
\]
and by [Sha21a, Prop. 4.25] for any \( \mathcal{T} \)-\( \infty \)-category \( \mathcal{A} \) we have the natural \( \mathcal{T} \)-join and slice equivalence
\[
\mathcal{F}_n(A, \mathcal{D}_{(\phi, \mathcal{T})}) \cong \mathcal{F}_n(A, \mathcal{D}_{(\phi, \mathcal{T})}),
\]
We may thus adjoin \( \omega_\circ^\perp \) to define \( \omega_\circ \) so that it fits into the indicated commutative diagram over \( \omega \).

Finally, to define \( \omega_c \), first let \( \ast : \mathcal{T}^{\op} \longrightarrow 0^\circ \) be a lift of 0 to the \( \mathcal{T} \)-final object of Remark 5.2.3, and let \( \sigma_\circ : 0^\circ \times 0^\circ \mathcal{A}_T(0^\circ) \xrightarrow{\simeq} \mathcal{A}_T^{\text{cart}}(0^\circ) \) be a choice of section for the trivial fibration. Then we have the composite
\[
\mathcal{T}^{\op} \times_{0^\circ} \mathcal{A}_T(0^\circ) \ast_{\text{id}} \mathcal{E}^\circ_{0, \mathcal{T}} \times_{0^\circ} \mathcal{A}_T(0^\circ) \xrightarrow{\simeq} \mathcal{A}_T^{\text{cart}}(0^\circ) \subset \mathcal{A}_T(0^\circ),
\]
which restricts to
\[
f : \mathcal{T}^{\op} \times_{0^\circ} \mathcal{A}_T(0^\circ) \longrightarrow \mathcal{T}^{\op} \times_{\ast_{0^\circ}} \mathcal{A}_T(0^\circ).
\]
The adjoint of \( f \circ \omega_\circ^\perp \) then defines the lift \( \omega_c \) of \( \omega_\circ \).

We also verify the uniqueness assertion for \( \omega_c \) and leave that for \( \omega_\circ \) as an exercise for the reader. By Lemma 5.2.7, the functor given by restriction along the \( \mathcal{T} \)-cone point
\[
\mathcal{F}_n(0^\circ, \mathcal{E}^\circ_{0, \mathcal{T}}) \longrightarrow \mathcal{F}_n(\mathcal{T}^{\op}, \mathcal{E}^\circ_{0, \mathcal{T}})
\]
is an equivalence, and since \( \mathcal{E}^\circ_{0, \mathcal{T}} \mathcal{T}^{\op} \simeq \mathcal{T}^{\op} \), we see that the righthand side is contractible, which shows the claim (and gives another construction of \( \omega_c \)).

5.2.6. Lemma. Suppose \( q : \mathcal{D} \longrightarrow \mathcal{B} \) is a \( \mathcal{T} \)-fibration and \( 0 : \mathcal{T}^{\op} \longrightarrow \mathcal{D} \) is a \( \mathcal{T} \)-functor such that 0 and \( q \circ 0 \) are \( \mathcal{T} \)-initial objects. Then the \( \mathcal{T} \)-functor
\[
\psi : (\mathcal{T}^{\op} \times_{0, \mathcal{D}, \ev_0} \mathcal{A}_T(\mathcal{D})) \longrightarrow (\mathcal{T}^{\op} \times_{0, \mathcal{B}, \ev_0} \mathcal{A}_T(\mathcal{B})) \times_{\mathcal{B}, q} \mathcal{D}
\]
is a trivial fibration.

Proof. Since \( q \) is a categorical fibration, it follows that \( \psi \) is as well. It thus suffices to prove that \( \psi \) is a categorical equivalence, for which we may suppose that \( \mathcal{B} = \mathcal{T}^{\op} \) by the two-out-of-three property of equivalences. The claim then follows by our hypothesis that \( 0_V \) is initial for all \( V \in \mathcal{T} \).

5.2.7. Lemma. Let \( q : \mathcal{D} \longrightarrow \mathcal{B} \) be a \( \mathcal{T} \)-cocartesian fibration and let \( i : \mathcal{T}^{\op} \longrightarrow \mathcal{B} \) be a \( \mathcal{T} \)-initial object. Then the \( \mathcal{T} \)-functor
\[
i^* : \mathcal{F}_n^{\text{cart}}(\mathcal{B}, \mathcal{D}) \longrightarrow \mathcal{F}_n(\mathcal{T}^{\op}, \mathcal{D})
\]
is an equivalence.

Proof. It suffices to check this assertion fiberwise. Replacing \( \mathcal{T} \) by \( \mathcal{T}^{\mathcal{T}} \), we may further suppose that \( \mathcal{T} \) has a final object \( \ast \), and we reduce to showing that \( i^* : \mathcal{F}_n^{\text{cart}}(\mathcal{B}, \mathcal{D}) \longrightarrow \mathcal{F}_n(\mathcal{T}^{\mathcal{T}}, \ast, \mathcal{D}) \) is an equivalence of \( \infty \)-categories, or equivalently, that \( (\mathcal{T}^{\mathcal{T}})^\mathcal{T} \longrightarrow \mathcal{B}^{\mathcal{T}} \) is a cocartesian equivalence in \( \text{sSet}^{\mathcal{T}} \). But the inclusion of the initial object \( \ast \in \mathcal{T}^{\mathcal{T}} \) is a cocartesian equivalence to both \( (\mathcal{T}^{\mathcal{T}})^\mathcal{T} \) and \( \mathcal{B}^{\mathcal{T}} \), so by the two-out-of-three property of the cocartesian equivalences we are done.

We may canonically endow the unit of an \( \mathcal{O} \)-monoidal \( \mathcal{T} \)-\( \infty \)-category with the structure of an \( \mathcal{O} \)-algebra in the following way.

5.2.8. Proposition. Let \( \mathcal{O}^\circ \) be a unital \( \mathcal{T} \)-\( \infty \)-operad and let \( p : \mathcal{E}^\circ \longrightarrow \mathcal{O}^\circ \) be an \( \mathcal{O} \)-monoidal \( \mathcal{T} \)-\( \infty \)-category. Then there is a unique cocartesian section \( 1^\circ \) of \( p \) such that \( 1^\circ \) extends the unit \( 1 : \mathcal{O} \longrightarrow \mathcal{E} \).
Proof. Since $\mathcal{O}^\otimes$ has the $\mathcal{T}$-initial object $0$ by assumption, the claim follows from Lemma 5.2.7.

We next identify $\mathcal{O}_0^\otimes$-monoidal $\mathcal{T}$-$\infty$-categories and $\mathcal{O}_0$-algebras therein in more familiar terms.

5.2.9. Proposition. Let $\mathcal{O}^\otimes$ be a unital $\mathcal{T}$-$\infty$-operad.

1. Suppose $(\mathcal{C}^\otimes, p)$ is a $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category and let $F^\omega_0 : \mathcal{O}_0^\otimes \to \mathbf{Cat}$ be the functor classifying the cocartesian fibration $p_{|[\mathcal{O}_0^\otimes]}$. Then $F^\omega_0$ is the right Kan extension of its restriction $f_{\omega_0}$ along $\omega_0$.

2. Let $f : \mathcal{O}_0^\otimes \to \mathbf{Cat}$ be a functor whose restriction along the cone inclusion $\mathcal{T}^\text{op} \subset \mathcal{O}^\otimes$ is constant at the final object, and let $F$ be the right Kan extension of $f$ along $\omega_0$. The cocartesian fibration $q : \mathcal{D} \to \mathcal{O}_0^\otimes$ classified by $F$ is then an $\mathcal{O}_0$-monoidal $\mathcal{T}$-$\infty$-category.

3. The assignment $(\mathcal{C}^\otimes, p) \mapsto (\ast \to f_{\omega_0})$ (where $f_{\omega_0}$ denotes the restriction of $f_{\omega}$ to $\mathcal{C}$, regarded as pointed in $\text{Fun}(\mathcal{O}, \mathbf{Cat})$ via the unit $1 : \mathcal{O} \to \mathcal{C}$) implements an equivalence of $\infty$-categories

\[ \mu : \text{Cat}_{\mathcal{O}_0}^\otimes \cong \text{Fun}(\mathcal{O}, \text{Cat})^{\ast/} \]

and an equivalence of $\mathcal{T}$-$\infty$-categories

\[ \text{Cat}_{\mathcal{O}_0}^\otimes \cong \text{Fun}(\mathcal{O}, (\text{Cat}_{\mathcal{T}})^{(\ast/)}) \rightarrow \mathcal{T}/. \]

4. For any two $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-categories $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$, $\mu$ induces an equivalence of $\infty$-categories

\[ \text{Fun}_{\mathcal{O}_0, \mathcal{T}}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \cong \text{Nat}_*(f_{\omega_0}, f_{\omega}). \]

Proof. We first analyze right Kan extension along $\omega_0$ in general. Let $x \in \mathcal{O}_0^\otimes[U_{[x \to V]}]$, let $U \simeq \prod_{i=1}^n U_i$ be an orbit decomposition, and let $\rho_i : x \to x_i$ be cocartesian edges lifting the characteristic morphisms $\chi_{[U_i \subset U]}$. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{O}_0^\otimes(\mathcal{O}_0^\otimes)^{x/}$ be the full subcategories on objects over

\[ (U \leftarrow Z \rightarrow X) \]

such that $Z = \emptyset$ and $Z \neq \emptyset$, respectively, and note that $\mathcal{O}_0^\otimes(\mathcal{O}_0^\otimes)^{x/}$ decomposes as the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. Let $\phi = \phi_\mathcal{A} \cup \phi_\mathcal{B} : \{a\} \sqcup \text{Orbit}(U) \to \mathcal{A} \cup \mathcal{B}$ be the functor that sends $a$ to $(x \to 0_V)$ and $U_i$ to $\rho_i$.

We claim that $\phi$ is right cofinal. By the same argument as in the proof of Lemma 2.4.4, $\phi$ is right cofinal, so it only remains to show $(x \to 0_V)$ is an initial object in $\mathcal{A}$. Since $\mathcal{O}_0^\otimes \times \mathcal{O}_0^\otimes(\mathcal{O}_0^\otimes)^{x/} \overset{\omega_0}\to \mathcal{O}_0^\otimes \overset{\mathcal{T}^\text{op}}\to$ is a cocartesian fibration, it suffices to show that for all morphisms $\alpha : W \to V \in \mathcal{T}$, $\alpha^*((x \to 0_V)) \simeq (x \to 0_W)$ is an initial object in $\mathcal{B}_W$. We first check that for objects $(x \to y) \in \mathcal{B}$ covering $\gamma : [U_{+} \to V] \overset{I(W)_+}{\to} W$, the mapping space $\text{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y)$ is contractible. This mapping space fits into the commutative diagram

\[ \text{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y) \longrightarrow \text{Map}_{\mathcal{O}_0^\otimes}(x \to 0_W, x \to y) \longrightarrow \ast \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ \ast \simeq \text{Map}_{\mathcal{O}_0^\otimes}(0_W, y) \longrightarrow \text{Map}_{\mathcal{O}_0^\otimes}(0_W, y) \quad \text{via} \quad (x \to 0_W)^* \longrightarrow \text{Map}_{\mathcal{O}_0^\otimes}(x, y), \]

where the lower horizontal composite selects the inert-fiberwise active factorization of $x \to y$ in the connected component $\text{Map}_{\mathcal{O}_0^\otimes}(x, y)$. In fact, since $\text{Map}_{\mathcal{O}_0^\otimes}(0_W, y) \simeq \ast$, this map is an equivalence onto that connected component, and the contractibility of $\text{Map}_{\mathcal{B}_W}(x \to 0_W, x \to y)$ follows. Finally, the argument for $(x \to 0_W)$ itself proceeds the same way.

We conclude that given a functor $f : \mathcal{O}_0^\otimes \to \mathbf{Cat}$, the right Kan extension $(\omega_0)_! f$ sends $x$ to $f(0_V) \times \prod_{i=1}^n f(x_i)$. This shows (1) – more precisely, the unit map $F^\omega_0 \cong (\omega_0)_! (\omega_0)^* F^\omega_0$ is seen to be an equivalence. By Proposition 2.2.6, this also shows (2). Moreover, we see that $(\omega_0)^* (\omega_0)_! f \overset{\ast}{\to} f$ if and only if $f|_{\mathcal{T}^\text{op}}$ is constant at $\ast \in \mathbf{Cat}$. We deduce that the adjunction $(\omega_0)^* \dashv (\omega_0)_*$ restricts to an adjoint equivalence

\[ (\omega_0)^* : \text{Cat}_{\mathcal{O}_0}^\otimes \cong \text{Fun}'(\mathcal{O}_0^\otimes, \mathbf{Cat}) : (\omega_0)_* \]

where we take the right-hand side to consist of the full subcategory on functors $\mathcal{O}_0^\otimes \to \mathbf{Cat}$ that restrict to $\ast$ on $\mathcal{T}^\text{op}$. Note that since the inclusion of a final object is fully faithful, we have an equivalence

\[ \text{Fun}'(\mathcal{O}_0^\otimes, \mathbf{Cat}) \simeq \Delta^0 \times_\ast \text{Fun}(\mathcal{T}^\text{op}, \mathbf{Cat}) \text{Fun}(\mathcal{O}_0^\otimes, \mathbf{Cat}), \]
and under the equivalence $\text{Fun}_\mathcal{T}(\mathcal{O}_\equiv, \mathbf{Cat}) \simeq \text{Fun}_\mathcal{T}(\mathcal{O}_\equiv, \mathbf{Cat}_\tau)$ of [Sha21a, Prop. 3.9], this yields an equivalence

$$\text{Fun}_\mathcal{T}(\mathcal{O}_\equiv, \mathbf{Cat}_\tau) \simeq \text{Fun}_{\mathcal{T}^{op}/\mathcal{T}}(\mathcal{O}_\equiv, \mathbf{Cat}_\tau),$$

to the $\infty$-category of $\mathcal{T}$-functors $\mathcal{O}_\equiv \to \mathbf{Cat}_\tau$ that restrict on $\mathcal{T}^{op}$ to the $\mathcal{T}$-final object of $\mathbf{Cat}_\tau$. By the $\mathcal{T}$-join and slice adjunction of [Sha21a, Prop. 4.25] (together with [Sha21a, Cor. 4.27]), we have an equivalence

$$\text{Fun}_{\mathcal{T}/\mathcal{T}}(\mathcal{O}_\equiv, \mathbf{Cat}_\tau) \simeq \text{Fun}_\mathcal{T}(\mathcal{O}, (\mathbf{Cat}_\tau^{\{*\}})/, \mathcal{T}^{op}/\mathcal{T}),$$

which we claim yields an equivalence $\text{Fun}_{\mathcal{T}/\mathcal{T}}(\mathcal{O}_\equiv, \mathbf{Cat}_\tau) \simeq \text{Fun}(\mathcal{O}, \mathbf{Cat})^{*/}$ upon passage to cocartesian sections – for this, just examine the pullback square of $\infty$-categories

$$\begin{array}{ccc}
\text{Fun}_\mathcal{T}(\mathcal{O}, (\mathbf{Cat}_\tau)^{\{*,\mathcal{T}\}}) & \\ & \searrow & \\
& & \text{Fun}_\mathcal{T}(\mathcal{O}, \mathbf{Cat})
\end{array}$$

This shows the first part of (3), and the second follows since we have a comparison $\mathcal{T}$-functor that we just proved is an equivalence fiberwise. The claim of (4) (i.e., that $\mu$ promotes to an equivalence of $(\infty, 2)$-categories) then follows since $\mu$ clearly respects cotensors by $\infty$-categories (cf. Construction 2.6.5).

### 5.2.10. Theorem

Suppose $\mathcal{O}^\otimes$ is a unital $\mathcal{T}_{\infty}$-operad and $\mathcal{C}^\otimes$ is a $\mathcal{V}$-monoidal $\mathcal{T}_{\infty}$-category. Then we have a canonical equivalence of $\mathcal{T}_{\infty}$-categories

$$\text{Alg}_{\mathcal{T}^{\infty}}(\mathcal{O}_0, \mathcal{C}) \simeq \text{Fun}_{\mathcal{T}_{\infty}^{op}/\mathcal{T}}(\mathcal{O}_0, \mathcal{C})^{(1,\mathcal{T})}/.$$

**Proof.** By the usual reduction, it will suffice to prove the statement without the ‘underlining’. Our strategy is to replace $\mathcal{O}_0^\otimes$ by its $\mathcal{O}_0$-monoidal envelope and then invoke Proposition 5.2.9(4). We first identify $\text{Env}_{\mathcal{O}_0, \mathcal{T}}(\mathcal{O}_0)^\otimes = \text{Ar}_\mathcal{T}\mathcal{O}(\mathcal{O}_0^\otimes)$ in simpler terms. Let

$$\lambda : \mathcal{O} \times \Delta^1 \to \text{Ar}_\mathcal{T}^{\text{act}}(\mathcal{O}_0^\otimes) \times_{\mathcal{O}_0^\otimes} \mathcal{O}$$

be the $\mathcal{T}$-functor which for $x \in \mathcal{O}_V$ sends $(x, 0) \mapsto (x, 1)$ to the evident morphism $[0_V \to x] \to \text{id}_x$ of active arrows. More precisely, we may define the adjoint of $\lambda$ projecting to $\text{Ar}_\mathcal{T}^{\text{act}}(\mathcal{O}_0^\otimes)$ as the composite

$$\mathcal{O} \times \Delta^1 \times \Delta^1 \xrightarrow{s} \mathcal{O} \to \mathcal{O}_0^\otimes,$$

where to define $h$, we regard $\mathcal{O} \times (\Delta^1 \times \Delta^1)$ as fibered over $\mathcal{T}^{op} \times \Delta^1$ via the structure map $\pi$ for $\mathcal{O}$ and $(i, j) \mapsto \max(i, j)$, and let $h$ be given by $(\pi, \text{pr}_0)$ under the defining universal property of the $\mathcal{T}$-join (cf. [Sha21a, Prop. 4.3]). We then let

$$\overline{\lambda} : \text{Ar}_\mathcal{T}(\mathcal{O}) \times \Delta^1 \to \text{Ar}_\mathcal{T}^{\text{act}}(\mathcal{O}_0^\otimes) \times_{\mathcal{O}_0^\otimes} \mathcal{O}$$

be the induced morphism of $\mathcal{T}$-cocartesian fibrations over $\mathcal{O}$ extending $\lambda$ under the equivalence of [Sha21b, Ex. 3.8]. We claim that $\overline{\lambda}$ is an equivalence, for which it suffices to check fiberwise. But for every $x \in \mathcal{O}_V$, we have that $\overline{\lambda}_x$ is essentially surjective in view of the unital assumption on $\mathcal{O}^\otimes$ (since the active edges must be of the form $[f : x' \to x]$ in $\mathcal{O}_V$ or $[0_V \to x]$ factoring through some $f$), and an easy computation of mapping spaces shows that $\overline{\lambda}_x$ is also fully faithful.

By similar reasoning, we also see that the composition

$$\mathcal{O} \times \{0\} \subset \mathcal{O} \times \Delta^1 \to \text{Ar}_\mathcal{T}(\mathcal{O}) \times \Delta^1 \xrightarrow{\overline{\lambda}} \text{Ar}_\mathcal{T}^{\text{act}}(\mathcal{O}_0^\otimes) \times_{\mathcal{O}_0^\otimes} \mathcal{O}$$

identifies with the unit map for $\text{Env}_{\mathcal{O}_0, \mathcal{T}}(\mathcal{O}_0)^\otimes$. Now let $\mathcal{E} : \mathcal{O} \to \mathbf{Cat}$ be the functor obtained by straightening $\text{ev}_1 : \text{Ar}_\mathcal{T}(\mathcal{O}) \to \mathcal{O}$. We have shown that under the correspondence of Proposition 5.2.9, $\text{Env}_{\mathcal{O}_0, \mathcal{T}}(\mathcal{O})^\otimes$ straightens to $\Delta^1 \times \mathcal{E}$. In the notation of that proposition, consider the pullback square

$$\begin{array}{ccc}
\text{Nat}_* (\Delta^1 \times \mathcal{E}, f e) & \\ & \searrow & \\
& & \text{Nat}(\Delta^1 \times \mathcal{E}, f e)
\end{array}$$

Using the universal property of the free $\mathcal{T}$-cocartesian fibration, we deduce an equivalence

$$\text{Nat}_* (\Delta^1 \times \mathcal{E}, f e) \simeq \text{Fun}_{\mathcal{T}^{op}/\mathcal{T}}(\mathcal{O}, \mathcal{E})^{(1,\mathcal{T})}/.$$
5.2.11. Theorem. Let $\mathcal{O}$ be a unital $\mathcal{T}$-$\infty$-operad and let $p : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be an $\mathcal{O}$-monoidal $\mathcal{T}$-$\infty$-category.

(1) The $\mathcal{O}$-algebra $\mathcal{O}^\otimes$ of Proposition 5.2.8 is an initial object of $\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{C})$.

(2) $\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{C})$ admits a $\mathcal{T}$-initial object given fiberwise by $(1^\otimes)_W$.

Proof. Since units are stable under base-change, it suffices to prove the first assertion. Let $1_{\mathcal{un}}^\otimes : \mathcal{O}^\otimes_0 \to \mathcal{C}^\otimes$ be the unique morphism of $\mathcal{T}$-$\infty$-operads over $\mathcal{O}^\otimes$ that sends all edges in $\mathcal{O}^\otimes_0$ to $p$-cocartesian edges in $\mathcal{C}^\otimes$ (so $1_{\mathcal{un}}^\otimes$ extends $* : \mathcal{T}^{\text{op}} \to \mathcal{C}^\otimes$ under the equivalence of Lemma 5.2.7). Then $1_{\mathcal{un}}^\otimes$ corresponds to $1 \in \text{Fun}_{/\mathcal{O}, \mathcal{T}}(\mathcal{O}, \mathcal{C})^{(1, \mathcal{T})}$ under the equivalence of Theorem 5.2.10 and is hence an initial object of $\text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0, \mathcal{C})$.

Let $i : \mathcal{O}^\otimes_0 \to \mathcal{O}^\otimes$ denote the inclusion. It will suffice to show that the left adjoint $i^!$ to the forgetful functor $i^* : \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{C}) \to \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0, \mathcal{C})$ is defined on $1_{\mathcal{un}}^\otimes$ and sends $1_{\mathcal{un}}^\otimes$ to $1^\otimes$. Consider the factorization of $i$ through its $\mathcal{O}$-monoidal envelope

$$\mathcal{O}^\otimes_0 \xrightarrow{i} \text{Env}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0)^\otimes = \mathcal{O}^\otimes_0 \times \mathcal{O}^\otimes \text{Ar}_{\mathcal{T}}^\otimes(\mathcal{O}^\otimes) \xrightarrow{i^!} \mathcal{O}^\otimes$$

and the resulting sequence of adjunctions

$$\begin{align*}
& \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0, \mathcal{C}) \simeq \text{Fun}_{/\mathcal{O}, \mathcal{T}}(\text{Env}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0), \mathcal{C}) \\
& \quad \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0, \mathcal{C}) \xrightarrow{i^!} \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}, \mathcal{C}) \\
& \quad \text{Alg}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}, \mathcal{C}) \xrightarrow{\text{ev}_1} \mathcal{C}
\end{align*}$$

where the dotted left adjoints are not necessarily defined. Observe that for every orbit $V \in \mathcal{T}$, the fiber $\text{Env}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0)^\otimes_V$ admits an initial object given by $\text{id}_{\mathcal{O}_0}$, and these assemble to define a $\mathcal{T}$-initial object of $\text{Env}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0)^\otimes$. By Lemma 5.2.7, the $\mathcal{O}$-monoidal $\mathcal{T}$-functor $1^\otimes \circ \text{ev}_1$ restricts to $1_{\mathcal{un}}^\otimes$ as they both send all edges to $p$-cocartesian edges and extend $*$, so $i^!=(1_{\mathcal{un}}^\otimes) \simeq 1^\otimes \circ \text{ev}_1$. Now observe that for every $x \in \mathcal{O}^\otimes_0$, the unique map $0_V \to x$ is active and is an initial object in the fiber $\text{Env}_{\mathcal{O}, \mathcal{T}}(\mathcal{O}_0)^\otimes \times \mathcal{O}^\otimes \{x\}$. Consequently, the ordinary left Kan extension of $1^\otimes \circ \text{ev}_1$ along $\text{ev}_1$ exists and is computed by $1^\otimes$ itself. Since the ordinary left Kan extension is an $\mathcal{O}$-algebra in this case, we conclude that $(\text{ev}_1)_!(1^\otimes \circ \text{ev}_1) \simeq 1^\otimes$ and hence $i^!=(1_{\mathcal{un}}^\otimes) \simeq 1^\otimes$.

5.3. Indexed coproducts in the $\mathcal{T}$-symmetric monoidal case. We identify finite $\mathcal{T}$-indexed coproducts with tensor products in the case of a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category $\mathcal{C}$, following the strategy of [Lur17, §3.2.4]. To precisely articulate this identification, we first discuss how to equip $\text{CAAlg}_{\mathcal{C}}(\mathcal{C})$ with a $\mathcal{T}$-symmetric monoidal structure.

5.3.1. Construction. We define the smash product $\mathcal{T}$-functor

$$\land : \mathcal{E}_{\mathcal{T}^*} \times \mathcal{T}^{\text{op}} \mathcal{E}_{\mathcal{T}^*} \to \mathcal{E}_{\mathcal{T}^*}, \quad (\{U_+ \to V\}, \{U'_+ \to V\}) \mapsto (U \times V, U'_+ \to V)$$

as follows. Recall that given an $\infty$-category $\mathcal{D}$ with finite products, we can define the smash product functor $\land : \mathcal{D}_* \times \mathcal{D}_* \to \mathcal{D}_*$ as the composition of functors

$$\mathcal{D}_* \times \mathcal{D}_* \subset \mathcal{D}^{\Delta^1} \times \mathcal{D}^{\Delta^1} \xrightarrow{\text{min}} \mathcal{D}^{\Delta^1 \times \Delta^1} \xrightarrow{\text{colim}} \mathcal{D}^{(\Delta^2)^\vee} \xrightarrow{\text{ev}(z)} \mathcal{D}$$

provided the functors $\text{min}$ and $\text{colim}$ exist pointwise on objects ($* \to x, * \to y$) and ($* \leftarrow x \times y \to x \times y$), as they then extend to partially defined left adjoints so that the composition exists. If we then have a presheaf $\mathcal{D}_* : \mathcal{T}^{\text{op}} \to \text{Cat}$ such that for every $\alpha : V \to W$ in $\mathcal{T}$, $\alpha^* : \mathcal{D}_W \to \mathcal{D}_V$ preserves finite products, wedge sums, and cofibers $(x \times y)/(x \vee y)$, it follows from the existence theorem for relative adjunctions ([Lur17, Prop. 7.3.2.6]) and [Lur17, Prop. 7.3.2.11]) that we obtain a $\mathcal{T}$-functor

$$\land : \mathcal{D}_* \times \mathcal{T}^{\text{op}} \mathcal{D}_* \to \mathcal{D}_*$$

given fiberwise by formation of smash products, where $\mathcal{D}_*$ is the unstraightening of the pointed presheaf $\mathcal{D}_{**}$. 

\[\text{We may now conclude using Proposition 5.2.9(4).}\]
5.3.2. **Definition.** Let $\mathcal{O}^\oplus, \mathcal{P}^\oplus, \mathcal{Q}^\oplus$ be $\mathcal{T}$-$\infty$-operads. We say that a $\mathcal{T}$-functor $F : \mathcal{O}^\oplus \times_{\mathcal{T}} \mathcal{P}^\oplus \to \mathcal{Q}^\oplus$ is a bifunctor of $\mathcal{T}$-$\infty$-operads if it sends pairs of inert edges to inert edges and the diagram

\[
\begin{array}{ccc}
\mathcal{O}^\oplus \times_{\mathcal{T}} \mathcal{P}^\oplus & \xrightarrow{F} & \mathcal{Q}^\oplus \\
\downarrow & & \downarrow \\
\mathcal{E}_{\mathcal{T}, \bullet} \times_{\mathcal{T}} \mathcal{E}_{\mathcal{T}, \bullet} & \xrightarrow{\Lambda} & \mathcal{E}_{\mathcal{T}, \bullet}
\end{array}
\]

is homotopy commutative.\(^\text{12}\)

5.3.3. **Variant.** By the same construction as in Construction 5.3.1, we have a smash product $\mathcal{F}_{\mathcal{T}}^\text{op}$-functor

\[
\Lambda : \mathcal{F}_{\mathcal{T}, \bullet}^\text{big} \times_{\mathcal{T}} \mathcal{F}_{\mathcal{T}, \bullet}^\text{big} \to \mathcal{F}_{\mathcal{T}, \bullet}^\text{big}
\]

that can be chosen to extend the smash product on $\mathcal{F}_{\mathcal{T}, \bullet}$. Likewise, we have the analogous definition of a bifunctor of big $\mathcal{T}$-$\infty$-operads, and the datum of one determines the other essentially uniquely under the correspondence of Corollary 2.7.5.

5.3.4. **Theorem-Construction.** Suppose $F : \mathcal{O}^\oplus \times_{\mathcal{T}} \mathcal{P}^\oplus \to \mathcal{Q}^\oplus$ is a bifunctor of $\mathcal{T}$-$\infty$-operads and let the $\mathcal{T}$-functor

\[
P : \mathcal{O}^\oplus \times_{\mathcal{T}} \mathcal{P}^\oplus \mathcal{A}(\mathcal{T}) \to \mathcal{O}^\oplus, \quad (x, V \xrightarrow{\beta} W) \mapsto \beta^*(x)
\]

be a choice of cocartesian pushforward. Consider the spans of marked simplicial sets

\[
\begin{array}{rcl}
(\mathcal{O}^\oplus, \mathcal{N}) & \xleftarrow{\pi} & (\mathcal{O}^\oplus, \mathcal{N}) \times_{\mathcal{T}} \mathcal{A}(\mathcal{T})^\text{op} \times_{\mathcal{T}} (\mathcal{P}^\oplus, \mathcal{N}) \xrightarrow{G} (\mathcal{Q}^\oplus, \mathcal{N}), \\
(\mathcal{O}^\oplus)^{\text{op}} & \xleftarrow{\pi} & (\mathcal{O}^\oplus)^{\text{op}} \times_{\mathcal{T}} \mathcal{A}(\mathcal{T})^\text{op} \times_{\mathcal{T}} (\mathcal{P}^\oplus, \mathcal{N}) \xrightarrow{G} (\mathcal{Q}^\oplus)^{\text{op}},
\end{array}
\]

where $G = F \circ (P \times \text{id}_{\mathcal{P}^\oplus})$ and $\pi$ is the projection to $\mathcal{O}^\oplus$. Then these spans determine Quillen adjunctions

\[
G_{\mathcal{P}}^\pi : \mathcal{S}\text{Set}_{/(\mathcal{O}^\oplus, \mathcal{N})}^{\text{op}} \rightleftarrows \mathcal{S}\text{Set}_{/(\mathcal{Q}^\oplus, \mathcal{N})}^{\text{op}} : \pi_* G^*, \quad G_{\mathcal{P}}^\pi : \mathcal{S}\text{Set}_{/(\mathcal{Q}^\oplus)}^{\text{op}} \rightleftarrows \mathcal{S}\text{Set}_{/(\mathcal{Q}^\oplus)}^{\text{op}} : \pi_* G^*
\]

with respect to the $\mathcal{T}$-operadic model structures and $\mathcal{T}$-monoidal model structures. Given a fibration $\mathcal{O}^\oplus \to \mathcal{Q}^\oplus$ of $\mathcal{T}$-$\infty$-operads, we then let

\[
\text{Alg}_{\mathcal{T}}(\mathcal{P}, \mathcal{B})^\oplus \to \mathcal{O}^\oplus
\]

denote the resulting fibration of $\mathcal{T}$-$\infty$-operads given by $\pi_* G^*(\mathcal{B}^\oplus, \mathcal{N})$.\(^\text{13}\)

If $\mathcal{B}^\oplus$ is $\mathcal{T}$-monoidal, then $\text{Alg}_{\mathcal{T}}(\mathcal{P}, \mathcal{B})^\oplus$ is $\mathcal{T}$-monoidal, and has cocartesian edges marked as in $\pi_* G^*(\mathcal{B}^\oplus)$.

**Proof.** Note that the underlying simplicial sets of $\pi_* G^*(-)$ are the same regardless of whether we work over $(\mathcal{Q}^\oplus, \mathcal{N})$ or $(\mathcal{Q}^\oplus)^{\text{op}}$. We first establish the assertion on the Quillen adjunction between $\mathcal{T}$-operadic model structures. For the proof, it will be convenient to pass to big $\mathcal{T}$-$\infty$-operads (cf. Variant 5.3.3). Let

\[
\bar{F} : \bar{\mathcal{F}}^\oplus \times_{\mathcal{F}_{\mathcal{T}}^\text{op}} \bar{\mathcal{P}}^\oplus \to \bar{\mathcal{Q}}^\oplus
\]

be the bifunctor of big $\mathcal{T}$-$\infty$-operads extending $F$, let the $\mathcal{F}_{\mathcal{T}}$-functor

\[
\bar{P} : \bar{\mathcal{F}}^\oplus \times_{\mathcal{F}_{\mathcal{T}}^\text{op}} \mathcal{A}(\mathcal{F}_{\mathcal{T}}^\text{op}) \to \bar{\mathcal{Q}}^\oplus
\]

be a choice of cocartesian pushforward over $\mathcal{F}_{\mathcal{T}}^\text{op}$ extending $P$, let $\bar{G} = \bar{F} \circ (\bar{P} \times \text{id}_{\mathcal{P}^\oplus})$, and consider the span of marked simplicial sets

\[
\begin{array}{rcl}
(\bar{\mathcal{O}}^\oplus, \mathcal{N}) & \xleftarrow{\pi} & (\bar{\mathcal{O}}^\oplus, \mathcal{N}) \times_{\mathcal{F}_{\mathcal{T}}^\text{op}} \mathcal{A}(\mathcal{F}_{\mathcal{T}}^\text{op})^\text{op} \times_{\mathcal{F}_{\mathcal{T}}^\text{op}} (\bar{\mathcal{P}}^\oplus, \mathcal{N}) \xrightarrow{\bar{G}} (\bar{\mathcal{Q}}^\oplus, \mathcal{N}).
\end{array}
\]

We claim that this span satisfies the hypotheses of [Lur17, Thm. B.4.2] with respect to the categorical patterns $\bar{\mathcal{P}}_\mathcal{O}$ and $\bar{\mathcal{P}}_\mathcal{Q}$ of Definition 2.7.3. (2) is clear and (3) is vacuous. By [Lur09, Cor. 2.4.7.17], the source functor $\text{ev}_\mathcal{O} : \mathcal{A}(\mathcal{F}_{\mathcal{T}}^\text{op}) \times_{\mathcal{F}_{\mathcal{T}}^\text{op}} \bar{\mathcal{F}}^\oplus \to \mathcal{F}_{\mathcal{T}}^\text{op}$ is a cartesian fibration, so the pullback $\bar{\pi}$ is a cartesian fibration.\(^\text{12}\) If $\mathcal{T} = \ast$, then the smash product for $\text{Fin}_{\mathcal{T}}$ can be defined without the ambiguity of a contractible space of choices. Therefore, one can choose the square to strictly commute in $\mathcal{S}\text{Set}_{/(\text{Fin}_{\mathcal{T}}^\oplus, \mathcal{N})}^{\text{op}}$ for the non-parametrized definition of a bifunctor of $\mathcal{O}^\oplus$-operads as in [Lur17, Def. 2.2.5.3].\(^\text{13}\) Beware that the notation $\text{Alg}_{\mathcal{T}}(\mathcal{P}, \mathcal{B})^\oplus$ hides the dependence of the structure map $\mathcal{F}_{\mathcal{T}}^\text{op} \to \mathcal{Q}^\oplus$ on the choice of parameter in $\mathcal{O}^\oplus$.
This proves (1) and (4). Moreover, an edge in $\tilde{O}^\otimes \times_{F^p_T} \mathcal{Ar}(F^p_T)^\otimes \tilde{P}^\otimes$ is $\pi$-cartesian if and only if its projection to $\tilde{P}^\otimes$ is an equivalence, which implies (7).

Now let $f_{x,\phi,\Sigma} : n^\triangleleft \to \tilde{O}^\otimes$ be as in Definition 2.7.3, so that $\phi : U \to V$ is a morphism in $F_T$, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a collection of commutative squares

\[
\begin{array}{ccc}
U_i & \xrightarrow{\alpha_i} & U \\
\downarrow \phi_i & & \downarrow \phi \\
V_i & \xrightarrow{\beta_i} & V
\end{array}
\]

such that $\alpha_i$ and $U_i \to V_i \times_V U$ are summand inclusions and $U \simeq \bigsqcup_{1 \leq i \leq n} U_i$, $f(v) = x$, and each morphism $\chi_i := f(v_i) : x \to x_i := f(i)$ is an inert edge covering $\chi_{\sigma_i} : [U_+ \to V] \to [(U_i)_+ \to V_i]$ in $\Xi^{big}_P$. We first prove (5) by showing that the restriction

$\mathcal{F}_T \leftarrow \mathcal{F}_T \otimes n^\triangleleft \mathcal{Ar}(F^p_T)^\otimes \tilde{P}^\otimes \mathcal{F}_T \mathcal{F}_T \to n^\triangleleft$

of $\pi$ along $f_{x,\phi,\Sigma}$ is a cocartesian fibration. In fact, by [Lur17, Lem. 6.1.1.1], for any cocartesian fibration $X \to F^p_T$, the source functor $ev_0 : \mathcal{F}(F^p_T) \times_{F^p_T} X \to F^p_T$ is a cocartesian fibration, with an edge

\[
\left(\begin{array}{c}
V \\
W
\end{array}\right) \leftarrow \left(\begin{array}{c}
V' \\
W'
\end{array}\right) \to \left(\begin{array}{c}
x \\
y
\end{array}\right)
\]

cocartesian if and only if the square in $F_T$ is a pullback square and $x \to y$ is a cocartesian edge. Next, let

$s : n^\triangleleft \to n^\triangleleft \mathcal{Ar}(F^p_T)^\otimes \tilde{P}^\otimes$

be a cocartesian section determined by $s(v) = (V \leftarrow W, y \in \tilde{P}^\otimes_{U'[\to W']})$, so that $s(i) = (V_i \leftarrow W_i := W \times_V V_i, y_i)$ for $y \to y_i$ an inert edge lifting $W_i \to W$. Let $z = \tilde{F}(\gamma^* x, y)$, let $\sigma'_i$ be

\[
\begin{array}{ccc}
U_i \times_V U' & \to & U \times_V U' \\
\downarrow \phi' & & \downarrow \phi' \\
W_i & \to & W,
\end{array}
\]

and let $\Sigma' = \{\sigma'_1, \ldots, \sigma'_n\}$. Then a chase of the definitions shows that the composition $\tilde{F} \circ \tilde{P} \circ f_{x,\phi,\Sigma}$ is of the form $f_{z,\phi',\Sigma'}$, which shows (6). Finally, (8) follows from the right cancellation property of inert edges. This completes the verification of the hypotheses of [Lur17, Thm. B.4.2]. We then deduce the theorem in question by means of Proposition 2.7.4 and Corollary 2.7.5. Finally, repeating this analysis for the other span proves the assertions regarding the monoidality of the construction. \[\square\]

5.3.5. Proposition. Let $F : \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} \mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ be a bifunctor of $\mathcal{T}$-\textsuperscript{\infty}-operads and let $\mathcal{C}^\otimes \to \mathcal{Q}^\otimes$ be a fibration of $\mathcal{T}$-\textsuperscript{\infty}-operads. We then have the following properties of the construction $\mathcal{Alg}_{2,T}(\mathcal{P}, \mathcal{C})^\otimes \to \mathcal{O}^\otimes$:

(1) For every object $x \in \mathcal{O}$ over $V \in \mathcal{T}^{op}$, the parametrized restriction

$F_x : x \times_{\mathcal{T}^{op}} \mathcal{P}^\otimes \to (\mathcal{Q}^\otimes)_V$

is a morphism of $\mathcal{T}/V$-\textsuperscript{\infty}-operads, and we obtain a canonical equivalence of $\mathcal{T}/V$-\textsuperscript{\infty}-categories

$\mathcal{Alg}_{2,T}(\mathcal{P}, \mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} x \simeq \mathcal{Alg}_{\mathcal{T}/V}(\mathcal{P}_V, \mathcal{C}_V)$.

Similarly, for every cocartesian section $\tau : \mathcal{T}^{op} \to \mathcal{O}$, we have a canonical equivalence of $\mathcal{T}$-\textsuperscript{\infty}-categories

$\mathcal{Alg}_{2,T}(\mathcal{P}, \mathcal{C})^\otimes \times_{\mathcal{O}^\otimes} \tau \simeq \mathcal{Alg}_{2,T}(\mathcal{P}, \mathcal{C})$.

(2) For every object $y \in \mathcal{P}$ over $V \in \mathcal{T}^{op}$, the parametrized restriction

$F_y : \mathcal{O}^\otimes \times_{\mathcal{T}^{op}} y \to (\mathcal{Q}^\otimes)_V$
is a morphism of $\mathcal{T}/V$-$\infty$-operads, and ‘evaluation at $y$’ furnishes a commutative square of $\mathcal{T}/V$-$\infty$-operads

$$
\begin{array}{ccc}
(\text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes})_V & \xrightarrow{\text{ev}_y} & (\mathcal{C}^{\otimes})_V \\
\downarrow & & \downarrow \\
(\mathcal{O}^{\otimes})_V \simeq \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\text{op}}} y & \xrightarrow{F_\tau} & (\mathcal{Q}^{\otimes})_V.
\end{array}
$$

Similarly, for every cocartesian section $\tau : \mathcal{T}^{\text{op}} \to \mathcal{P}$, we have a morphism of $\mathcal{T}$-$\infty$-operads

$$
\text{ev}_\tau : \text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}
$$

covering $F_\tau : \mathcal{O}^{\otimes} \to \mathcal{Q}^{\otimes}$ and given fiberwise by evaluation at $\tau(V)$.

(3) If $\mathcal{C}^{\otimes}$ is $\mathcal{Q}$-monoidal (so that $\text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes}$ is $\mathcal{Q}$-monoidal), then $\text{ev}_y$ and $\text{ev}_\tau$ preserve cocartesian edges.

Proof. (1): We prove the assertion about $x$ – that for $\tau$ will hold by the same reasoning. We have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}^{\otimes} \times_{\mathcal{T}^{\text{op}}} \mathcal{P}^{\otimes} & \xrightarrow{\text{ev}_y} & \mathcal{Q}^{\otimes} \\
\downarrow & & \downarrow \\
(\mathcal{T}/V)^{\text{op}} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*} & \xrightarrow{\text{ev}_y} & \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*}
\end{array}
$$

where the outer square induces the $\mathcal{T}/V$-functor $F_\tau$. The definition of a bifunctor of $\mathcal{T}$-$\infty$-operads then immediately shows that $F_\tau$ is a morphism of $\mathcal{T}/V$-$\infty$-operads. Next, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}^{\otimes} \times_{\mathcal{T}^{\text{op}}} \mathcal{P}^{\otimes} & \xrightarrow{\text{pr}} & \mathcal{P}^{\otimes} \\
\downarrow & & \downarrow \\
(\mathcal{T}/V)^{\text{op}} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*} & \xrightarrow{\text{pr}} & \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*}
\end{array}
$$

where the functor $\rho$ is the trivial fibration used in the definition of the cocartesian pushforward. Using the variant of [Sha21b, Lem. 4.8] for algebra maps, after marking appropriately this diagram induces a comparison $\mathcal{T}/V$-functor

$$
\text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{P} \to \text{Alg}_{\mathcal{O},\mathcal{T}/V}(\mathcal{P}_V, \mathcal{C}_V),
$$

which by [Sha21a, Lem. 2.27] is an equivalence.

(2): By the same logic as in (1), $F_y$ is a morphism of $\mathcal{T}/V$-$\infty$-operads. Using the compatibility of the construction $\text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ with base-change in $\mathcal{T}$, without loss of generality we may replace $\mathcal{T}/V$ with $\mathcal{T}$ and suppose $y \in \mathcal{P}$ lies over a final object in $\mathcal{T}$. Choosing a section $\sigma$ of the trivial Kan fibration $\mathcal{P} \to \mathcal{T}$, let $j = (\text{id}, \iota, \sigma) : \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\text{op}}} \text{Ar}(\mathcal{T}^{\text{op}}) \times_{\mathcal{T}^{\text{op}}} \mathcal{P}^{\otimes}$ and consider the morphism of spans

$$
\begin{array}{ccc}
\mathcal{O}^{\otimes} & \xrightarrow{\text{ev}_y} & \mathcal{O}^{\otimes} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\otimes} \times_{\mathcal{T}^{\text{op}}} \text{Ar}(\mathcal{T}^{\text{op}}) \times_{\mathcal{T}^{\text{op}}} \mathcal{P}^{\otimes} & \xrightarrow{\text{ev}_y} & \mathcal{O}^{\otimes} \times_{\mathcal{T}^{\text{op}}} \mathcal{P}^{\otimes}
\end{array}
$$

Noting that $j$ respects markings for the first span in Theorem-Construction 5.3.4, we then see that $j$ induces the desired morphism of $\mathcal{T}$-$\infty$-operads $\text{Alg}_{\mathcal{O},\mathcal{T}}(\mathcal{P}, \mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}$.

(3): This follows from the proof of (2) since the functor $j$ also respects markings for the second span in Theorem-Construction 5.3.4. □

We now specialize to the bifunctor $\wedge : \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*} \to \mathbf{E}_{\mathcal{T},*}$. Fix a choice of cocartesian pushforward $P : \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \text{Ar}(\mathcal{T}^{\text{op}}) \to \mathbf{E}_{\mathcal{T},*}$ and also write

$$
\wedge : \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*} \overset{\text{lax}}{\to} \mathbf{E}_{\mathcal{T},*} \times_{\mathcal{T}^{\text{op}}} \text{Ar}(\mathcal{T}^{\text{op}}) \times_{\mathcal{T}^{\text{op}}} \mathbf{E}_{\mathcal{T},*} \to \mathbf{E}_{\mathcal{T},*}
$$
for the composition $\land \circ (P \times \text{id})$.

5.3.6. **Construction.** Let $C^\otimes$ be a $T$-symmetric monoidal $T$-$\infty$-category. We construct a $T$-symmetric monoidal $T$-functor

$$(-)^{\text{can}} : \mathbf{CAlg}_T(C^\otimes) \to \mathbf{CAlg}_T(\mathbf{CAlg}_T(C^\otimes))$$

that is split by the ‘forgetful’ evaluation $T$-functor $U$ of Proposition 5.3.5(2). First observe that we have a commutative diagram of marked simplicial sets

\[
\begin{array}{ccc}
\mathbf{F}_{T,s}^+ \times_{\mathbf{F}_{T,s}^+, Ne} \mathbf{F}_{T,s}^+ & \xrightarrow{\land \times_{\mathbf{F}_{T,s}^+, Ne} \mathbf{F}_{T,s}^+} & \mathbf{F}_{T,s}^+ \\
\text{pr}_1 \downarrow & & \uparrow \text{pr}_1 \\
\mathbf{F}_{T,s}^+ \times_{\mathbf{F}_{T,s}^+, Ne} \mathbf{F}_{T,s}^+ & \xrightarrow{\land} & \mathbf{F}_{T,s}^+ \\
\text{pr}_1 \downarrow & & \uparrow \text{pr}_1 \\
\mathbf{F}_{T,s}^+ & \xrightarrow{\land} & \mathbf{F}_{T,s}^+
\end{array}
\]

in which the square is a pullback (here, $\text{pr}_{12}$ denotes projection away from the third factor). Also write $\land$ for the upper horizontal composite. Then we have that (cf. [Sha21a, Lem. 2.26])

$$(\text{pr}_1)_* \land^* \cong (\text{pr}_1)_* \land^* (\text{pr}_1)_* \land^* : \mathbf{sSet}_{/\mathbf{F}_{T,s}^+} \to \mathbf{sSet}_{/\mathbf{F}_{T,s}^+} : x^\otimes \to \mathbf{CAlg}_T(\mathbf{CAlg}_T(C^\otimes)).$$

Now consider the morphisms of spans

\[
\begin{array}{ccc}
\mathbf{F}_{T,s}^+ \xleftarrow{\text{pr}_1} \mathbf{F}_{T,s}^+ \times_{\mathbf{F}_{T,s}^+, Ne} \mathbf{F}_{T,s}^+ & \xrightarrow{\land} & \mathbf{F}_{T,s}^+ \\
\text{pr}_1 \downarrow & & \uparrow \text{id} \\
\mathbf{F}_{T,s}^+ \times_{\mathbf{F}_{T,s}^+, Ne} \mathbf{F}_{T,s}^+ & \xrightarrow{\land} & \mathbf{F}_{T,s}^+. \\
\text{pr}_1 \downarrow & & \uparrow \text{id} \\
\mathbf{F}_{T,s}^+ & \xrightarrow{\land} & \mathbf{F}_{T,s}^+.
\end{array}
\]

Then the lower vertical arrow defines $(-)^{\text{can}}$, and since the upper vertical arrow induces $U$ and the composite is homotopic to the identity, we see that $U \circ (-)^{\text{can}} \simeq \text{id}$.

5.3.7. **Theorem.** Let $C^\otimes$ be a $T$-symmetric monoidal $T$-$\infty$-category. Then $\mathbf{CAlg}_T(C^\otimes)$ has all finite $T$-coproducts. Moreover, for any map of finite $T$-sets $f : U \to V$, we have a canonical equivalence

$$\coprod_f \simeq f^\circ : \mathbf{CAlg}_T(C^\otimes)_U \to \mathbf{CAlg}_T(C^\otimes)_V,$$

where $f^\circ$ is furnished by the $T$-symmetric monoidal structure on $\mathbf{CAlg}_T(C^\otimes)$ of Theorem-Construction 5.3.4.

**Proof.** Since the base-change condition for left adjoints to the restriction functors $\{f^*\}$ of $\mathbf{CAlg}_T(C^\otimes)$ to furnish finite $T$-coproducts is already satisfied by the maps $\{f^\circ\}$, it will suffice to construct unit and counit maps exhibiting $f^\circ$ as left adjoint to $f^*$. Without loss of generality, we may suppose $V$ is an orbit, and after replacing $T$ by $T/V$, we may suppose $V = *$ is the final object of $T$ so that $\mathbf{CAlg}_T(C^\otimes)_V \simeq \mathbf{CAlg}_T(C^\otimes)$. In addition, by Theorem 5.2.11, $\mathbf{CAlg}_T(C^\otimes)$ admits an initial object given by the unit $1$, which is $f^\circ(*)$ for $f : \emptyset \to *$. We may thus suppose that $U$ is nonempty. Using Construction 5.3.6, given a $T$-commutative algebra $A$ we get

$$A^\text{can} : \mathbf{F}_{T,s}^+ \to \mathbf{CAlg}_T(C^\otimes),$$

and we then get a map $f^\circ f^* A \to A$ by applying $A^\text{can}$ to $f^* : [U+ \to *] \to [+_+ \to *]$ and factoring the resulting map $f^* A \to A$ through a cocartesian lift over $f_+$ in the base. Using the naturality of this procedure in $A$, we then obtain a candidate for the counit transformation $\epsilon : f^\circ f^* \to \text{id}$. To define the
unit transformation \( \eta : \text{id} \rightarrow f^*f_{\otimes} \), consider the pullback square
\[
\begin{array}{ccc}
U \times U & \rightarrow & U \\
\downarrow^{pr_2} & & \downarrow^{f} \\
U & \rightarrow & * \\
\end{array}
\]
and the associated equivalence \( f^*f_{\otimes} \simeq (pr_1)_{\otimes}(pr_2)^* \). The summand inclusion \( \delta : U \rightarrow U \times U \) yields a natural transformation
\[
\epsilon_{pr_2}(-) : \delta_{\otimes} \simeq \delta_{\otimes}(\delta^* pr_2^*) \rightarrow pr_2^*,
\]
which on objects \( B \in \text{CAlg}_\mathcal{C}(\mathcal{C})_U \) may be described as follows: if we write \( U \times U \simeq U \sqcup U \) and \( g = pr_2|_{U'} \), then in terms of the decomposition \( \text{CAlg}_\mathcal{T}(\mathcal{C})_{U \times U} \simeq \text{CAlg}_\mathcal{T}(\mathcal{C})_U \times \text{CAlg}_\mathcal{T}(\mathcal{C})_{U'} \), we have that
\[
\epsilon_{pr_2}B : \delta_{\otimes}(B) = (B, 1_U) \rightarrow pr_2^*(B) = (B, g*B)
\]
is given by the identity on the first factor and the unique map out of the initial object on the second factor. We then let
\[
\eta = (pr_1)_{\otimes}(\epsilon_{pr_2}(-)) : \text{id} \simeq (pr_1)_{\otimes}\delta_{\otimes} \rightarrow (pr_1)_{\otimes}pr_2^* \simeq f^*f_{\otimes}.
\]
It remains to verify the triangle identities. Let \( U \simeq \prod_{i=1}^n U_i \) be an orbit decomposition of \( U \), let \( \iota_i : U_i \rightarrow U \) denote the inclusion, and let \( f_i = f \circ \iota_i \). Observe that after pullback to \( U_i \), the map \( f \) acquires a canonical section, i.e., for all \( 1 \leq i \leq n \) we have a factorization of the identity map
\[
\text{id} : U_i \xrightarrow{(\text{id}, \iota_i)} U_i \times U \xrightarrow{p_i} U_i,
\]
where \( p_i \) denote the projection. Using \((-)^{\otimes} \), this furnishes a factorization
\[
\text{id} : f^*_i A \rightarrow (p^i)_{\otimes}(p^i)^*(f^*_i A) \simeq (p^i)_{\otimes}(pr_2)^*f^*A \rightarrow f^*_i A,
\]
where we use the commutative square
\[
\begin{array}{ccc}
U_i \times U & \rightarrow & U \\
\downarrow^{p_i} & \uparrow^{f_i} & \downarrow^{f} \\
U_i & \rightarrow & * \\
\end{array}
\]
for the middle equivalence. To express this in more familiar terms, note that if we write \( U_i \times U \simeq U_i \sqcup U_i' \) and \( q^i = p^i|_{U_i} \), then we may identify this as
\[
f^*_i A \simeq f^*_i A \otimes 1_{U_i} \xrightarrow{\text{id} \otimes 1} f^*_i A \otimes (q^i)_{\otimes}(q^i)^* A \xrightarrow{\text{id} \otimes q^i} f^*_i A \otimes f^*_i A \rightarrow f^*_i A,
\]
where \( \otimes \) denotes the fiberwise tensor product on \( \text{CAlg}_\mathcal{T}(\mathcal{C})_{U_i} \) induced by the fold map \( \nabla : U_i \sqcup U_i' \rightarrow U_i \).

Now regarding the composition \( f^*A \xrightarrow{\nabla} f^*f_{\otimes}f^*A \xrightarrow{\eta} f^*A \) by an elementary diagram chase we see that after pullback along \( \iota_i \) this identifies with the factorization of \( f^*_i A \) given above, which validates this half of the triangle identities.

Finally, we consider the composition \( f_{\otimes}B \xrightarrow{\eta_{\otimes}} f_{\otimes}f^*f_{\otimes}B \xrightarrow{\epsilon^{\otimes}_{B}} f_{\otimes}B \). By the \( \mathcal{T} \)-symmetric monoidality of \((-)^{\otimes} \), we get a canonical equivalence \( f_{\otimes}(B^{\otimes} \mathcal{C}) \simeq (f_{\otimes}B)^{\otimes} \mathcal{C} \) in \( \text{CAlg}_\mathcal{T}(\mathcal{C}) \). If we then write \( B = (A_1, ..., A_n) \) under the decomposition \( \text{CAlg}_\mathcal{T}(\mathcal{C})_{U} \simeq \prod_{i=1}^n \text{CAlg}_\mathcal{T}(\mathcal{C})_{U_i} \), it follows that we obtain an equivalence
\[
f_{\otimes}f^*f_{\otimes}B \simeq f_{\otimes}((p^1)_{\otimes}(p^1)^* A_1, ..., (p^n)_{\otimes}(p^n)^* A_n)
\]
under which \( \epsilon_{f_{\otimes}B} \simeq f_{\otimes}(\epsilon_{A_1}, ..., \epsilon_{A_n}) \) and the composite \( f_{\otimes}B \circ f_{\otimes}\eta_B \) identifies with \( f_{\otimes} \) of the composite defined factorwise by the map \( A_i \rightarrow (p^i)_{\otimes}(p^i)^* A_i \rightarrow A_i \) induced from \( \text{id} : U_i \xrightarrow{(\text{id}, \iota_i)} U_i \times U \xrightarrow{p_i} U_i \). Since these all compose as identities, we deduce the other half of the triangle identities.

\[ \square \]

**5.3.8. Corollary.** Let \( \mathcal{C}^{\otimes} \) be a \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category. Then the \( \mathcal{T} \)-symmetric monoidal structure on \( \text{CAlg}_\mathcal{T}(\mathcal{C}) \) of Theorem Construction 5.3.4 is cocartesian.

Next, we consider the more general case of an arbitrary \( \mathcal{T} \)-indexing system \( \mathcal{I} \) (Definition 2.4.8).
5.3.9. **Theorem.** Let \( \mathcal{C}^\otimes \) be a \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category. Then for all orbits \( V \in \mathcal{T} \), the fiber \( \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})_V \) admits finite coproducts, and for all morphisms \( f : V \rightarrow W \in \mathcal{T} \), the restriction functor \( \bar{f}^* : \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})_W \rightarrow \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})_V \) preserves finite coproducts. Moreover, finite coproducts are computed as tensor products in terms of the symmetric monoidal structures on the fibers \( \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})_V \) constructed via Theorem-Construction 5.3.4 applied to the bifunctor
\[
\Lambda_\mathcal{T} : \text{Com}_{\mathcal{T}}^\otimes \times_{\mathcal{T}^\otimes} \text{Com}_{\mathcal{T}}^\otimes \rightarrow \text{Com}_{\mathcal{T}}^\otimes
\]
obtained by restriction of \( \Lambda : \mathbf{F}_{\mathcal{T},*} \times_{\mathcal{T}^\otimes} \mathbf{F}_{\mathcal{T},*} \rightarrow \mathbf{F}_{\mathcal{T},*} \).

**Proof.** The proof is exactly analogous to that of Theorem 5.3.7, where in place of Construction 5.3.6 we instead use the composition of the span involving \( \Lambda_\mathcal{T} \) with that involving \( \Lambda_\mathcal{T}^\otimes \) to define the \( \mathcal{T}^\otimes \)-symmetric monoidal \( \mathcal{T} \)-functor
\[
\mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})^\otimes \rightarrow \mathbf{CAlg}_{\mathcal{T}}(\mathbf{CAlg}(\mathcal{C}))^\otimes,
\]
and also the identification of Corollary 2.4.15. \( \square \)

5.3.10. **Corollary.** Let \( \mathcal{C}^\otimes \) be a distributive \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category such that \( \mathcal{C} \) is fiberwise presentable, let \( \mathcal{T} \) be a \( \mathcal{T} \)-indexing system, and write \( \mathcal{C}^\otimes = \text{Com}_{\mathcal{T}}^\otimes \times_{\mathcal{E}_\mathcal{T}} \mathcal{C} \). Let
\[
U : \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C}) \rightarrow \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C}_\mathcal{T})
\]
be the forgetful \( \mathcal{T} \)-functor implemented by restriction along \( \text{Com}_{\mathcal{T}}^\otimes \subset \mathbf{F}_{\mathcal{T},*} \). Then for all \( V \in \mathcal{T} \), \( U_V \) is a conservative functor of presentable \( \infty \)-categories that preserves all small limits and colimits, and is hence comonadic. In particular, \( \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C})_V \) is comonadic over \( \mathbf{CAlg}(\mathcal{C}_V) \).

**Proof.** To reduce notational clutter, let us replace \( \mathcal{T} \) by \( \mathcal{T}/V \) so that \( V = * \) is a terminal object of \( \mathcal{T} \). By Theorem 5.1.4(4), both \( \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C}) \) and \( \mathbf{CAlg}_{\mathcal{T}}(\mathcal{C}_\mathcal{T}) \) are presentable. Since the forgetful functor to \( \mathcal{C}_\mathcal{T} \) is conservative for a reduced \( \mathcal{T} \)-\( \infty \)-operad, \( U_* \) is conservative as well. By Theorem 5.1.3(2), \( U_* \) preserves all small limits. By Theorem 5.1.4(2), \( U_* \) preserves all sifted colimits. By Theorem 5.3.7 and Theorem 5.3.9, \( U_* \) preserves all finite coproducts. It follows that \( U_* \) preserves all small colimits and hence admits a right adjoint by the adjoint functor theorem, so we are entitled to ask about the comonadicity of \( U_* \). The conclusion then follows from the Barr–Beck–Lurie Theorem [Lur09, Thm. 4.7.3.5]. \( \square \)

6. **\( \mathcal{T} \)**-**SYMMETRIC MONOIDAL STRUCTURE ON \( \mathcal{T} \)**-**PRESHEAVES**

Let \( \mathcal{C} \) be a \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category. In this section, we construct a \( \mathcal{T} \)-symmetric monoidal structure on the \( \mathcal{T} \)-\( \infty \)-category of \( \mathcal{T} \)-presheaves \( \mathbf{P}_{\mathcal{T}}(\mathcal{C}) \) such that for any \( \mathcal{T} \)-distributive \( \mathcal{T} \)-symmetric monoidal \( \mathcal{T} \)-\( \infty \)-category \( \mathcal{D} \), the universal mapping property
\[
\text{Fun}_{\mathcal{T}}(\mathbf{P}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})
\]
of [Sha21a, Thm. 11.5] refines to an equivalence
\[
\text{Fun}_{\mathcal{T}}(\mathbf{P}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D});
\]
cf. Corollary 6.0.12. This result has also been achieved by Hilman as [Hil22a, Thm. 2.3.6].

We first extend our discussion of universal constructions from [Sha21b] by proving the parametrized analogue of [Lur09, Prop. 5.3.6.2].\(^{14}\) Let \( \mathcal{K} = \{ \mathcal{K}_U : U \in \mathcal{T} \} \) be a collection of classes \( \mathcal{K}_U \) of small \( \mathcal{T}/U \)-\( \infty \)-categories such that for each morphism \( f : U \rightarrow V \in \mathcal{T} \), \( f^*(\mathcal{K}_V) \subset \mathcal{K}_U \); we call such a collection closed. For each \( V \in \mathcal{T} \), let \( \mathcal{K}_V = \{ \mathcal{K}_U : [f : U \rightarrow V] \in \mathcal{T} \} \). Recall the following definition from [Sha21b, Def. 2.8]

6.0.1. **Definition.** Let \( \mathcal{C} \) be a \( \mathcal{T} \)-\( \infty \)-category. We say that \( \mathcal{C} \) strongly admits \( \mathcal{K} \)-indexed \( \mathcal{T} \)-colimits if for all \( U \in \mathcal{T} \), \( \mathcal{E}_U \mathcal{K}_U \) admits \( \mathcal{K}_U \)-indexed \( \mathcal{T}/U \)-colimits. Likewise, for any \( V \in \mathcal{T} \), we may refer to \( \mathcal{C}_V \) strongly admitting \( \mathcal{K}_V \)-indexed \( \mathcal{T}/V \)-colimits.

Given a \( \mathcal{T} \)-functor \( F : \mathcal{C} \rightarrow \mathcal{D} \), we say that \( F \) strongly preserves \( \mathcal{K} \)-indexed \( \mathcal{T} \)-colimits if for all \( U \in \mathcal{T} \), the \( \mathcal{T}/U \)-functor \( \mathcal{F}_U : \mathcal{E}_U \mathcal{K}_U \rightarrow \mathcal{D}_U \) preserves all \( \mathcal{K}_U \)-indexed \( \mathcal{T}/U \)-colimits. Likewise, for any \( V \in \mathcal{T} \), we may refer to \( \mathcal{F}_V \) strongly preserving \( \mathcal{K}_V \)-indexed \( \mathcal{T}/V \)-colimits. We then let
\[
\text{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})
\]

\(^{14}\)For this, we need not suppose that \( \mathcal{T} \) is atomic.
be the full $\mathcal{T}$-subcategory spanned in each fiber over $V \in \mathcal{T}$ by those $\mathcal{T}^{U/\cdot}$-functors that strongly preserve $K|_{\mathcal{V}}$-indexed $\mathcal{T}^{U/\cdot}$-colimits. Note that the global sections $\text{Fun}_K^V(\mathcal{C}, \mathcal{D})$ of $\text{Fun}_K^V(\mathcal{C}, \mathcal{D})$ is then the full subcategory of $\text{Fun}_\mathcal{T}(\mathcal{C}, \mathcal{D})$ spanned by those $\mathcal{T}$-functors that strongly preserve $K$-indexed $\mathcal{T}$-colimits.

Suppose given a collection $\mathcal{R} = \{\mathcal{R}_U : U \in \mathcal{T}\}$ of classes $\mathcal{R}_U = \{p_\alpha : K_\mathcal{V}^\alpha \to \mathcal{C}_\mathcal{U}\}$ of $\mathcal{T}^{U/\cdot}$-diagrams in $\mathcal{C}_\mathcal{U}$ (which are not necessarily $\mathcal{T}^{U/\cdot}$-colimit diagrams), such that for each morphism $f : U \to V$ in $\mathcal{T}$, $f^*(\mathcal{R}_V) \subset \mathcal{R}_U$. Then a $\mathcal{T}$-functor $F : \mathcal{C} \to \mathcal{D}$ strongly preserves $\mathcal{R}$-indexed $\mathcal{T}$-colimits if for all $U \in \mathcal{T}$, $F|_U : \mathcal{C}_U \to \mathcal{D}_U$ sends each $p_\alpha$ to $\mathcal{T}^{U/\cdot}$-colimit diagram in $\mathcal{D}_U$. Likewise, for any $V \in \mathcal{T}$ we have the same notion for $F|_V$ with respect to $\mathcal{R}|_V$. We then let

$$\text{Fun}_\mathcal{T}^\mathcal{R}(\mathcal{C}, \mathcal{D}) \subset \text{Fun}_\mathcal{T}(\mathcal{C}, \mathcal{D})$$

be defined as before.

6.0.2. Proposition. Let $\mathcal{C}$ be a $\mathcal{T}$-$\infty$-category and let $\mathcal{R}$ be as in Definition 6.0.1, such that each $K_\alpha$ lies in $K_\mathcal{U}$.

Then there exists a $\mathcal{T}$-$\infty$-category $\text{Fun}_K^\mathcal{R}(\mathcal{C})$ and a $\mathcal{T}$-functor $j : \mathcal{C} \to \text{Fun}_K^\mathcal{R}(\mathcal{C})$ such that:

1. $\text{Fun}_K^\mathcal{R}(\mathcal{C})$ strongly admits $K$-indexed $\mathcal{T}$-colimits.

2. For all $\mathcal{T}$-$\infty$-categories $\mathcal{D}$ such that $\mathcal{D}$ strongly admits $K$-indexed $\mathcal{T}$-colimits, precomposition with $j$ induces an equivalence of $\mathcal{T}$-$\infty$-categories

$$j^*: \text{Fun}_\mathcal{T}(\text{Fun}_K^\mathcal{R}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}_\mathcal{T}^\mathcal{R}(\mathcal{C}, \mathcal{D})$$

which upon passage to global sections yields an equivalence of $\infty$-categories

$$j^*: \text{Fun}_\mathcal{T}(\text{Fun}_K^\mathcal{R}(\mathcal{C}), \mathcal{D}) \simeq \text{Fun}_\mathcal{T}^\mathcal{R}(\mathcal{C}, \mathcal{D}).$$

3. Suppose that all $\mathcal{T}^{U/\cdot}$-functors $p_\alpha : K_\mathcal{V}^\alpha \to \mathcal{C}_\mathcal{U}$ in $\mathcal{R}_U$ are $\mathcal{T}^{U/\cdot}$-colimit diagrams. Then $j$ is fully faithful.

Proof. The proof is essentially identical to that of [Lur09, Prop. 5.3.6.2]; we spell out a few details for the reader’s benefit. After enlarging universes, we may suppose $\mathcal{C}$ is small.$^{15}$ Let $j_0 : \mathcal{C} \to \text{Fun}_\mathcal{T}(\mathcal{C})$ be the $\mathcal{T}$-Yoneda embedding. For every $p_\alpha \in \mathcal{R}_U$, let $p_\alpha = p_\alpha|_{K_\alpha}$, let $Y_\alpha \in \text{Fun}_\mathcal{T}(\mathcal{C})_U$ be the image of the cone point (in the fiber over $U$) under $p_\alpha$, let $X_\alpha \in \text{Fun}_\mathcal{T}(\mathcal{C})_U$ be a $\mathcal{T}^{U/\cdot}$-colimit for $(j_0)^*p_\alpha : K_\alpha \to \text{Fun}_\mathcal{T}(\mathcal{C})_U$, let $s_\alpha : X_\alpha \to j_0(Y_\alpha)$ be the induced map, and let $S_U = \{s_\alpha\}$.

Note that for all morphisms $f : U \to V$ in $\mathcal{T}$, $f^*S_V \subset S_U$ by our closure hypothesis on $\mathcal{R}$. Thus, we may form the full $\mathcal{T}$-subcategory $S^{-1}\text{Fun}_\mathcal{T}(\mathcal{C}) \subset \text{Fun}_\mathcal{T}(\mathcal{C})$ given on each fiber over $U \in \mathcal{T}$ by the full subcategory of $S_U$-local objects of $\text{Fun}_\mathcal{T}(\mathcal{C})_U = \text{Fun}_\mathcal{T}(\mathcal{C})_U$. We then have a $\mathcal{T}$-left adjoint $L : \text{Fun}_\mathcal{T}(\mathcal{C}) \to S^{-1}\text{Fun}_\mathcal{T}(\mathcal{C})$ given fiberwise by the usual localization. Finally, we define $\text{Fun}_K^\mathcal{R}(\mathcal{C})$ to be the smallest full $\mathcal{T}$-subcategory of $S^{-1}\text{Fun}_\mathcal{T}(\mathcal{C})$ which contains the essential image of $L \circ j_0$ and such that for each $U \in \mathcal{T}$, $\text{Fun}_K^\mathcal{R}(\mathcal{C})$ is closed under $K_\mathcal{U}$-indexed colimits, and we let $j = L \circ j_0$ be the induced map.

Given this construction, the verification of properties (1)–(3) proceeds exactly as in the proof of [Lur09, Prop. 5.3.6.2] (with parametrized analogues of non-parametrized statements involving colimits, left Kan extensions, etc. substituted as appropriate).

6.0.3. Remark. If $K = \mathcal{A}ll$ then we may also write $\text{Fun}_K^\mathcal{R}(\mathcal{C})$ as $\text{Fun}_\mathcal{R}(\mathcal{C})$.

Now let $\text{Cat}_\mathcal{T}^\otimes \to \text{Funct}_{\mathcal{T},*}$ be the $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category given by the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\text{Cat}_\mathcal{T}^\otimes$ (Example 2.4.1). Objects of $\text{Cat}_\mathcal{T}^\otimes$ are then given by tuples

$$([f : U \to V] \in \text{Funct}_{\mathcal{T},*}, \ e_1 \in \text{Cat}_{\mathcal{T}/U_1}, \ldots, e_n \in \text{Cat}_{\mathcal{T}/U_n})$$

where $U \simeq U_1 \sqcup \ldots \sqcup U_n$ is an orbit decomposition.

To describe morphisms in $\text{Cat}_\mathcal{T}^\otimes$, consider a morphism

$$
\begin{array}{ccc}
U & \xleftarrow{Z} & X \\
\downarrow f & & \downarrow g \\
V & \xleftarrow{Y} & Y
\end{array}
$$

$^{15}$Since we suppose $\mathcal{T}$ is small, a $\mathcal{T}$-$\infty$-category $\mathcal{C}$ is small if and only if it is fiberwise small.
in $F_{T,s}$, let $U \simeq \bigsqcup_{i=1}^m U_i$, $X \simeq \bigsqcup_{j=1}^n X_j$ be orbit decompositions, and let $m_j : Z_j \simeq X_j \times_Y Z \to X_j$ be the restriction over the summand $X_j$. Let $\{c_i \in \mathbf{Cat}_{T/U_i}\}$ and $\{d_j \in \mathbf{Cat}_{T/X_j}\}$. Let $\mathcal{C} = \bigsqcup_{i=1}^m c_i$ denote the given $T/U$-$\infty$-category and let $p^*\mathcal{C}$ denote the pullback of $\mathcal{C}$ to a $T/U \times V$, $\infty$-category along the projection $U \times V \to U$. By definition, the map $Z \to U \times V$ is a summand inclusion; let $\mathcal{C}'$ denote the corresponding summand of $\mathcal{C}$ regarded as a $T/V$, $\infty$-category, and also let $\mathcal{C}_j'$ denote the $T/V$, $\infty$-category given by the further summand of $\mathcal{C}'$. Then a morphism

$$(f, \{c_i\}) \to (g, \{d_j\})$$

is given by a collection of $T/X$-functors $F_j : (m_j)_* (\mathcal{C}_j') \to \mathcal{D}_j$, or written more concisely, a $T/X$-functor $F : m_* (\mathcal{C}') \to \mathcal{D}$.

6.0.4. Definition. Let $M \subseteq \mathbf{Cat}_{\infty}$ be the subcategory defined as follows:

- $M_0 = \mathbf{Cat}_{\infty}$.
- An object $([f : U \to V], c_1 \in \mathbf{Cat}_{T/U_1}, ..., c_n \in \mathbf{Cat}_{T/U_n})$ belongs to $\mathbf{Cat}_{\infty} \times \{1\}$ if and only if each $c_i$ strongly admits $T/U_i$-colimits.
- All morphisms $(f, \{c_i\}, 0) \to (g, \{d_j\}, 1)$ belong to $M$.
- A morphism $(f, \{c_i\}, 1) \to (g, \{d_j\}, 1)$ belongs to $M$ if and only if each $F_j$ is $T/X$, $\infty$-distributive.

Let $p : M \to F_{T,s} \times \Delta^1$ denote the composite of the inclusion and the structure map.

For the following, we have implicitly extended the notion of (closed) collection $K = \{K_U\}$ to be indexed over all finite $T$-sets $U$.

6.0.5. Notation. Let $f : U \to V$ be a morphism in $F_T$ and let $\mathcal{C}$ be a $T/U$, $\infty$-category. Let $R = \{R_\alpha : [U \to U] \in F_T\}$ be a closed collection of diagrams in $\mathcal{C}$, $R_\alpha = \{p_\alpha : p_\alpha^* : \mathcal{C}_U \to \mathcal{C}_{U_i}\}$. We let $f_*, R$ denote the closed collection of diagrams in $f_*, \mathcal{C}$ specified at each morphism $\gamma : V' \to V$ as follows:

- $f' : U' \times V \to V'$ and let $f' : U' \to V'$ be the pullback of $f$. For every $T/V'$-diagram $p_{\gamma} : p_\alpha^* \to \mathcal{C}_{U'_i}$ in $R_{U'}$, we may form the $T/V'$ diagram $f'_*(\mathcal{C}_\gamma) \to f'_*(\mathcal{C}_\alpha)$. We let $(f, \mathcal{C})_\gamma$ be the set of these diagrams.

6.0.6. Notation. Given a finite $T$-set $U$ with orbit decomposition $U \simeq \bigsqcup_{i=1}^m U_i$, let $P_{T/U}(-)$ be given by the coproduct of the $P_{T/U_i}(-)$.

6.0.7. Lemma. Let $f : U \to V$ be a morphism of finite $T$-sets and let $\mathcal{C}$ be a $T/U$, $\infty$-category. Consider the $T/V$-functor

$$\phi : f_* (P_{T/U} (\mathcal{C})) \to P_{T/V} (f_* \mathcal{C})$$

given by the composite of the $T/U$-Yoneda embedding (for $f_* (P_{T/U} (\mathcal{C}))$) and restriction along $f_*$ of the $T/V$-Yoneda embedding (for $\mathcal{C}$). Then $\phi$ is $T/V$, $\infty$-distributive.

Proof. Suppose $p : K \to P_{T/U} (\mathcal{C})$ is a $T/U$, $\infty$-diagram. We need to show that the $T/V$, $\infty$-colimit of the composite

$$f_*$, K \xrightarrow{f_* (p)} f_* (P_{T/U} (\mathcal{C})) \xrightarrow{\phi} P_{T/V} (f_* \mathcal{C})$$

evaluates to $f_* (\text{colim}^{T/U} p)$. It suffices to check this after evaluation at all objects of $f_*, \mathcal{C}$, and without loss of generality it suffices to consider $x \in (f_*, \mathcal{C})_V \simeq \mathcal{C}_U$ (by the usual base-change argument). A diagram chase then shows that the composite

$$f_* (P_{T/U} (\mathcal{C})) \xrightarrow{\phi} P_{T/V} (f_* \mathcal{C}) \xrightarrow{ev_x} \mathbf{Spc}_{T/V}$$

identifies with the composite

$$f_* (P_{T/U} (\mathcal{C})) \xrightarrow{f_* ev_x} f_* \mathbf{Spc}_{T/U} = f_* f^* \mathbf{Spc}_{T/V} \xrightarrow{m} \mathbf{Spc}_{T/V}$$

where $m$ is the multiplication given by the $T$-distributive $T$-cartesian $T$-symmetric monoidal structure on $\mathbf{Spc}_T$. Thus, $m$ is $T/V$, $\infty$-distributive. \qed
6.0.8. **Lemma.** Let \( f : U \rightarrow V \) be a morphism of finite \( T \)-sets, let \( C \) be a \( T/\mathcal{U} \)-\( \infty \)-category, and let \( D \) be a \( T/\mathcal{U} \)-completes \( T/\mathcal{U} \)-\( \infty \)-category. Then a \( T/\mathcal{V} \)-functor \( F : f_! \mathcal{P}_{T/\mathcal{U}}(C) \rightarrow D \) is \( T/\mathcal{V} \)-distributive if and only if it is the \( T/\mathcal{V} \)-left Kan extension of its restriction along \( f_* : f_* C \subset f_! \mathcal{P}_{T/\mathcal{U}}(C) \).

**Proof.** For the ‘if’ direction, it suffices to consider the universal example given by the \( T/\mathcal{V} \)-functor \( \phi \) of Lemma 6.0.7, which we showed to be \( T/\mathcal{V} \)-distributive. For the ‘only if’ direction, let \( F' = (f_* j)(f_* j)^* F \) and consider the comparison map \( \theta : F' \Rightarrow F \). Without loss of generality, it suffices to show that for all \( x \in \{ f_* \mathcal{P}_{T/\mathcal{U}}(C) \}_{\mathcal{V}} \simeq \{ \mathcal{P}_{T/\mathcal{U}}(C) \}_{\mathcal{V}}, \theta_x \) is an equivalence. On the one hand, by the pointwise formula for \( T/\mathcal{V} \)-left Kan extensions, we have that \( F'(x) \) is given by the \( T/\mathcal{V} \)-colimit of \( p : (f_* C)/x \rightarrow f_* C \rightarrow D \). On the other hand, since \( f_* : \text{Cat}_{T/\mathcal{U}} \rightarrow \text{Cat}_{T/\mathcal{V}} \) preserves cotensors and limits, we have a natural equivalence \( f_* \mathcal{C}/x \simeq f_*(C/x) \) such that \( [(f_* C)/x \rightarrow f_* C] \simeq [f_*(C/x) \rightarrow C] \), and using that \( F \) is \( T/\mathcal{V} \)-distributive, we get that \( F(x) \) is also given by the \( T/\mathcal{V} \)-colimit of \( p \). The naturality of all operations considered shows further that \( \theta_x \) implements this equivalence. \( \square \)

In the following corollary, we disambiguate our terminology for distributive functors by referring to the morphism of finite \( T \)-sets and not just the target.

6.0.9. **Corollary.** Let \( U \xrightarrow{f} V \xrightarrow{g} W \) be a composite of morphisms of finite \( T \)-sets, let \( C \) be a \( T/\mathcal{U} \)-\( \infty \)-category, and let \( D \) be a \( T/\mathcal{W} \)-\( \infty \)-category. Consider the composite \( T/\mathcal{W} \)-functor \( j : g_* f_* C \xrightarrow{j_*} g_* \mathcal{P}_{T/\mathcal{V}}(f_* C) \rightarrow g_* \mathcal{P}_{T, f, \text{Alt}}(f_* C) \)

Then a \( T/\mathcal{W} \)-functor \( F : g_* \mathcal{P}_{T, f, \text{Alt}}(f_* C) \rightarrow D \) is \( g \)-distributive if and only if \( j^* F \) is \( g f \)-distributive. Consequently, we have an equivalence \( \mathcal{P}_{T, g f, \text{Alt}}((g f)_* C) \simeq \mathcal{P}_{g_* \text{Alt}}(g_* \mathcal{P}_{T, f, \text{Alt}}(f_* C)) \).

**Proof.** The first claim follows immediately by restricting the equivalence of Lemma 6.0.8, noting that by definition \( g_* \mathcal{P}_{T, f, \text{Alt}}(f_* C) \) is a localization of \( g_* \mathcal{P}_{T/\mathcal{V}}(f_* C) \) at the relevant class of morphisms. The equivalence then follows by universal property. \( \square \)

6.0.10. **Proposition.** The map \( p \) is a cocartesian fibration.

**Proof.** We adapt the proof of [Lur17, Prop. 4.8.1.3] to the parametrized context. We first show that \( p \) is a locally cocartesian fibration. This is clear if we restrict to the fiber over 0. For the other cases, first suppose \( (f : U \rightarrow V, \{ C_i \}) \in \mathcal{M}_0 \) and let \( (f, 0) \rightarrow (g, 1) \) be a morphism in \( \mathcal{E}_{T, *}, T/\mathcal{X} \), with notation as above. Let \( D_f = \mathcal{P}_{T, f, (m_j)}((m_j)_* (C_i)) \) and take \( F_f \) to be the identity. We then have a morphism \( (f, \{ C_i \}, 0) \rightarrow (g, \{ D_f \}, 1) \) which is a locally cocartesian edge by the universal property of the \( T/\mathcal{X} \)-presheaves.

Next, suppose \( (f : U \rightarrow V, \{ C_i \}) \in \mathcal{M}_1 \) and let \( (f, 1) \rightarrow (g, 1) \) be a morphism in \( \mathcal{E}_{T, *}, T/\mathcal{X} \), with notation as above. Let \( \mathcal{R} \) be the closed collection of parametrized colimit diagrams in \( C_i \), i.e., for each morphism \( \alpha : Z' \rightarrow Z \in \text{F}_T \), \( \mathcal{R}_\alpha \) is the collection of \( T/\mathcal{Z}' \)-colimit diagrams in \( C_i \). We let \( \mathcal{D} = \mathcal{P}_{T, \mathcal{R}}(m_* C_i) \) for \( f = j : m_* C_i \rightarrow D \) be the \( T/\mathcal{X} \)-functor as in Proposition 6.0.2. We then have that the morphism \( j : (f, C_i, 1) \rightarrow (g, D_1, 1) \) lies in \( \mathcal{M} \) by definition. Moreover, it is a locally cocartesian edge in view of the universal property supplied by Proposition 6.0.2.

To then see that \( p \) is a cocartesian fibration, we need to see that the composite of locally cocartesian edges is again locally cocartesian. We already know the restriction over 0 is a cocartesian fibration. If the first edge lies over \( 0 \rightarrow 1 \), we may apply the parametrized analogue of [Lur09, Prop. 5.3.6.11]; since this step is straightforward we leave the details to the reader. If both edges lie over 1, then without loss of generality we may suppose both edges are fiberwise active as edges over \( \mathcal{E}_{T, *}, \) in which case the claim follows from the transitivity property established in Corollary 6.0.9. \( \square \)

Let \( p_0 \) and \( p_1 \) denote the two fibers of \( p \) over \( 0, 1 \in \Delta^1 \).

6.0.11. **Corollary.** The maps \( p_0 \) and \( p_1 \) exhibit \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) as \( T \)-symmetric monoidal \( T/\infty \)-categories.

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\( ^{16} \)cf. [https://math.stackexchange.com/questions/4004599/when-left-kan-extension-preserve-colimits](https://math.stackexchange.com/questions/4004599/when-left-kan-extension-preserve-colimits) for a description of this standard reduction, which also works in the parametrized context.
Proof. We already have that the map $p_0$ is the structure map of $\mathbf{Cat}^0_T$. As for $p_1$, since it is a cocartesian fibration it remains to verify the parametrized Segal condition. But in the definition of $\mathcal{M}$, all the inert morphisms in $\mathbf{Cat}^0_T$ continue to lie in $\mathcal{M}$, so we see that $\mathcal{M}_1$ inherits the parametrized Segal condition from $\mathbf{Cat}^0_T$. \hfill \square

Now write $(\mathbf{Cat}_T^L)^0 = \mathcal{M}_1$. The cocartesian fibration $p$ classifies the $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-functor

$$\mathbf{P}_T^{\mathcal{M}_1} : \mathbf{Cat}_T \to (\mathbf{Cat}_T^L)^0$$

whose underlying $\mathcal{T}$-functor is given by the usual $\mathcal{T}$-presheaf construction $\mathbf{P}_T^{\mathcal{M}_1}$. Since $\mathbf{P}_T^{\mathcal{M}_1}$ admits a right $\mathcal{T}$-adjoint given by the forgetful $\mathcal{T}$-functor $U$, $U$ canonically inherits a lax $\mathcal{T}$-symmetric monoidal structure. Passing to $\mathcal{T}$-commutative algebra objects and considering the unit of the adjunction, we obtain:

6.0.12. Corollary. Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category. Then $\mathbf{P}_T(\mathcal{C})$ and the $\mathcal{T}$-Yoneda embedding $j : \mathcal{C} \to \mathbf{P}_T(\mathcal{C})$ inherit a $\mathcal{T}$-symmetric monoidal structure such that:

1. $\mathbf{P}_T(\mathcal{C})$ is $\mathcal{T}$-distributive.
2. For every $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category $\mathcal{D}$, restriction along $j$ yields an equivalence

$$\text{Fun}_{\mathcal{T}}^{\infty}((\mathbf{P}_T(\mathcal{C})), \mathcal{D}) \cong \text{Fun}_{\mathcal{T}}^{\infty}(\mathcal{C}, \mathcal{D}).$$

6.0.13. Remark. Let $\mathcal{C}$ be a $\mathcal{T}$-symmetric monoidal $\mathcal{T}$-$\infty$-category. Consider the $\mathcal{T}$-distributive $\mathcal{T}$-symmetric monoidal structure on $\mathbf{P}_T(\mathcal{C}) = \text{Fun}_{\mathcal{T}}^{\infty}(\mathcal{C}, \mathbf{Spc}_T)$ given by $\mathcal{T}$-Day convolution (Theorem 3.2.6), where we have the $\mathcal{T}$-symmetric monoidal structure on $\mathcal{C}$ induced by the opposite automorphism on $\mathbf{Cat}$ under the equivalence of Theorem 2.3.9 and the $\mathcal{T}$-cartesian $\mathcal{T}$-symmetric monoidal structure on $\mathbf{Spc}_T$. Then one may show directly that the full $\mathcal{T}$-subcategory $\mathcal{C} \subset \mathbf{P}_T(\mathcal{C})$ is closed under this $\mathcal{T}$-symmetric monoidal structure. Using Corollary 6.0.12(2), it then follows that the $\mathcal{T}$-Day convolution $\mathcal{T}$-symmetric monoidal structure agrees with that defined above.

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