Normalization of Polynomials in Algebraic Invariants of Three-Dimensional Orthogonal Geometry

Hongbo Li
KLMM, AMSS, Chinese Academy of Sciences, Beijing 100190, China
hli@mmrc.iss.ac.cn

ABSTRACT
In classical invariant theory, the Gröbner base of the ideal of syzygies and the normal forms of polynomials of invariants are two core contents. To improve the performance of invariant theory in symbolic computing of classical geometry, advanced invariants are introduced via Clifford product [5]. This paper addresses and solves the two key problems in advanced invariant theory: the Gröbner base of the ideal of syzygies among advanced invariants, and the normal forms of polynomials of advanced invariants. These results beautifully extend the straightening of Young tableaux to advanced invariants.

Keywords
Invariant theory; Clifford algebra; bracket algebra; noncommutative Gröbner base; straightening of Young tableaux.

1. INTRODUCTION
In traditional analytic approach to classical geometry, coordinates are introduced to represent points in the geometric space, and equations of the coordinates are used to define constraints among the points, forming a representation of higher dimensional objects such as curves, surfaces, etc. Basic manipulations of coordinates include addition and multiplication, resulting in polynomials in the coordinates. Since the coordinates of generic points are independent, and the multiplication of coordinate variables are commutative, normalization of the polynomials in the coordinates is very easy. The normal forms of the polynomials are required in many manipulations, e.g., division among polynomials.

Another analytic approach to classical geometry, dating back to Euclid, is to use geometric invariants such as lengths, angles, areas, etc. A typically algebraic system of geometric invariants is a polynomial ring generated by basic invariants. In such a system, a vector variable in a linear space is used to represent a point or direction in classical geometry, the inner product of a vector with itself represents the squared length of the vector, the inner product of two unit vectors represents the cosine of the angle between them, etc. Such operators among vectors generate a set of basic invariants, and the polynomials in these basic invariants are advanced invariants.

Although the multiplication of invariants are commutative, the basic invariants generated by generic vector variables of the linear space are not independent, and there are polynomial relations among them, called syzygy relations. The dependency is largely caused by the dimension constraint of the linear space upon vectors. While the dimension constraint can be easily reflected by the number of coordinates introduced to represent a point and the independency among the coordinates, for basic invariants generated by the points, fully representing the dimension constraint is by no means trivial. Classical invariant theory studies the generators of invariants, the syzygy relations among the basic invariants, and the normal forms of advanced invariants as polynomials in the basic ones [11], [12].

In symbolic geometric computing, both the coordinate approach and the basic invariant approach encounter the difficulty of very big polynomial size, in particular in the middle of symbolic manipulations. In [5], a recipe to alleviate the difficulty is proposed, called long geometric product, BREEFS, and Clifford factorization, among which the long geometric product (or Clifford product) is the foundation. The idea is to convert polynomials of basic invariants into advanced invariants, converse to the approach of classical invariant theory, by means of an associative and multilinear product among the vector variables representing points. The associativity of the product and the symmetries within a long bracket provide powerful manipulations that cannot be done with basic invariants, nor with coordinates. This is a top-down approach to advanced invariants [6], while the classical invariant theory is a bottom-up approach.

Dealing with the syzygy relations among advanced invariants and finding the normal forms of polynomials in advanced invariants are two fundamental tasks in such “advanced invariant theory”. The 2D case is easy, while higher dimensional cases are difficult. Little advance has been achieved in six years since the publication of [5] in 1997.

In this paper, the two fundamental problems are solved for the advanced invariant theory of 3D orthogonal geometry: the Gröbner base of the syzygy ideal of “long brackets”, and
the normal forms of Clifford bracket polynomials. It turns out that the normal forms of such bracket polynomials are surprisingly “beautiful”. The description is the following.

In classical invariant theory for \((n-1)D\) projective geometry, the basic invariants are brackets of length \(n\), or in coordinate form, the \(n \times n\) determinants formed by the homogeneous coordinates of \(n\) vector variables. A bracket polynomial is in normal form if when each term is up to coefficient written in Young tableau form, the entries in each row are increasing, while the entries in each column are non-decreasing \([13]\). For example for vector variables \(v_1 < v_2 < \ldots < v_m\), a bracket monomial \([v_{i_1} v_{i_2} \cdots v_{i_{1c}}] [v_{i_{1c}} v_{i_{1c+1}} \cdots v_{i_{2c}}] \cdots [v_{i_{1c}} v_{i_{2c}} \cdots v_{i_{rc}}]\), where the \(i_{jc}\) are repetitive selections of the \(m\) vector variables, is normal if and only if in

\[
\begin{bmatrix}
v_{i_{11}} & v_{i_{12}} & \cdots & v_{i_{1c}} \\
v_{i_{21}} & v_{i_{22}} & \cdots & v_{i_{2c}} \\
\vdots & \vdots & \ddots & \vdots \\
v_{i_{rc}} & v_{i_{12}} & \cdots & v_{i_{rc}}
\end{bmatrix},
\]

\(v_{i_{11}} < v_{i_{21}} < \cdots < v_{i_{rc}},\) while \(v_{i_{1c}} \preceq v_{i_{2c}} \preceq \cdots \preceq v_{i_{rc}}\).

In the advanced invariant theory for 3D orthogonal geometry, each “elementary” advanced invariant is a bracket of length \(> 1\), whose entries are vector variables representing 3D points. In a bracket monomial, different brackets may have different lengths, and a bracket monomial is in normal form if and only if not only the entries in each row are increasing, the entries in each column are non-decreasing, but all the entries in the tableau after removing the first column, are non-decreasing. For example if \([11]\) is normal in this setting, then the sequence \(v_{i_{11}} v_{i_{12}} \cdots v_{i_{1c}} v_{i_{22}} v_{i_{2c}} \cdots v_{i_{1}} v_{i_{2}} \cdots v_{i_{rc}}\) is non-decreasing, where \(v_{i_{k}}\) denotes that \(v_{i_{k}}\) does not occur in the sequence.

This paper is organized as follows. Section 2 introduces orthogonal geometric invariants by quaternions, Clifford algebra and bracket algebra. Section 3 introduces the main results in \([7]\) on vector-variable polynomials and basics of “advanced bracket algebra”. Section 4 provides the Gröbner base and normal forms of long brackets in the multilinear case. Section 6 extends the results to general case, by means of the square-free vector-variable polynomials introduced in Section 5. Section 7 proposes a normalization algorithm for bracket polynomials.

### 2. QUATERNIONS, CLIFFORD ALGEBRA, AND BRACKET ALGEBRA

In the vector algebra over \(\mathbb{R}^3\), there are three multilinear products among vectors: (i) the inner product of two vectors, (ii) the cross product of two vectors, (iii) the hybrid product of three vectors. None of them can be extended to include more vectors while preserving the associativity.

The quaternionic product, on the other hand, is associative while still multilinear. Let \(q\) represent the quaternionic conjugate of quaternion \(q\). Among quaternions, a vector \(v\) refers to a pure imaginary quaternion, i.e., \(v = -v_s\), and a scalar \(s\) refers to a real quaternion, i.e., \(s = s\). All vectors span a 3D real inner-product space with metric diag\((-1, -1, -1)\), denoted by \(\mathbb{R}^{-3}\).

We always use juxtaposition of elements to denote their quaternionic product. The inner product of two vectors \(v_i, v_j\) is defined by

\[
[v_i v_j] := (v_i v_j + v_j v_i)/2.
\]

The cross product of two vectors \(v_i, v_j\) is defined by

\[
v_i \times v_j := (v_i v_j - v_j v_i)/2.
\]

The result is a vector, so its inner product with a third vector \(v_k\) is a scalar. Define the hybrid product of three vectors \(v_i, v_j, v_k\) by

\[
[v_i v_j v_k] := (v_i v_j v_k - v_k v_j v_i)/2.
\]

Then \([v_i v_j v_k] = [(v_i \times v_j) v_k]\).

The vector algebra over \(\mathbb{R}^{-3}\) is included in the quaternions. The latter is equipped with a powerful product, the quaternionic product, making it possible to use quaternions to represent 3D orthogonal transformations \([1]\).

The magnitude of a quaternion \(q\) is \(\sqrt{q q^*}\). A quaternion \(q\) is said to be unit if \(q q^* = 1\). Let \(q\) be a unit quaternion, and \(v\) be a vector. The conjugate adjoint action of \(q\) on \(v\) is defined by

\[
Ad_q(v) := q v q^*.
\]

Since \([Ad_q(v_1) Ad_q(v_2)] = [v_1 v_2]\) for any two vectors \(v_1, v_2\), \(Ad_q\) realizes an orthogonal transformation in \(\mathbb{R}^{-3}\). A classical result states that in fact all orthogonal transformations in \(\mathbb{R}^{-3}\) are realized in this way, and two different unit quaternions realize the same orthogonal transformation if and only if they differ by sign.

For a quaternion \(Q\), the bracket \([Q]\) is its scalar part:

\[
[Q] := (Q + Q^*)/2.
\]

The axis of \(Q\) is the vector part of \(Q\):

\[
A(Q) := (Q - Q^*)/2.
\]

In particular, \(A(v_1 v_2) = v_1 \times v_2\). We interpret them in geometrical terms below.

For a unit vector \(v_1\), \(Ad_{v_1}\) realizes the reflection with respect to the plane normal to \(v_1\). In general, for unit vectors \(v_1, v_2, \ldots, v_{2k+1}\), \(Ad_{v_1 v_2 \cdots v_{2k+1}}\) realizes the reflection with respect to the plane normal to axis \(A(v_1 v_2 \cdots v_{2k+1})\), if the latter is nonzero.

For two unit vectors \(v_1, v_2\) that are linearly independent, \(Ad_{v_1 v_2}\) realizes the rotation about the axis \(v_1 \times v_2\): in the plane spanned by \(v_1, v_2\), the rotation is from \(v_1\) to the reflection of \(v_1\) with respect to \(v_2\), i.e., the angle of rotation is \(\theta = 2\angle(v_1, v_2)\). Furthermore, \([v_1 v_2] = \cos(\theta/2)\). When we say “rotation \(v_1 v_2\)”, we mean the one induced by \(Ad_{v_1 v_2}\).

In general, for unit vectors \(v_1, v_2, \ldots, v_{2k}\), \(Ad_{v_1 v_2 \cdots v_{2k}}\) realizes the rotation about the axis \(A(v_1 v_2 \cdots v_{2k})\), if the latter is nonzero. The rotation is the composition of \(k\) rotations \(v_1 v_2, v_3 v_4, \ldots, v_{2k-1} v_{2k}\). If the angle of rotation is \(\theta\), then

\[
[v_1 \cdots v_{2k}] = \cos(\theta/2), \quad |A(v_1 \cdots v_{2k})| = |\sin(\theta/2)|.
\]
Let \( A(v_1v_2 \cdots v_k) \neq 0 \). By \( [v_1v_2 \cdots v_{2k}] = [A(v_1v_2 \cdots v_{2k})v_{2k+1}] \), we get
\[
[v_1v_2 \cdots v_{2k+1}] = \cos \angle(A(v_1v_2 \cdots v_{2k}), v_{2k+1}) \sin(\theta/2),
\]
where \( \theta \) is the angle of rotation \( v_1v_2 \cdots v_{2k} \). In particular when \( k = 1 \), for linearly independent unit vectors \( v_1, v_2 \), \( \sin(\theta/2) = \sin(\angle(v_1, v_2)) = \text{the area of the parallelogram spanned by } v_1, v_2 \), and \( \cos(\angle(v_1 \times v_2, v_3)) = \text{the height from the end of unit vector } v_3 \) to the plane spanned by \( v_1, v_2, v_3 \), so \([v_1v_2v_3] = \text{the volume of the parallelepiped spanned by } v_1, v_2, v_3\).

In classical invariant theory, an algebraic invariant is a polynomial whose variables are basic invariants. In 3D orthogonal geometry, there are two kinds of basic invariants: \([v_i v_j]\) and \([v_i v_j v_k]\) for all vector variables \( v_i, v_j, v_k \). Given \( n \) vector variables \( v_1, \ldots, v_n \), the brackets \([v_1 v_2 \cdots v_m]\) for arbitrary \( 1 < m < \infty \) and arbitrary repetitive selection of elements \( v_1, v_2, \ldots, v_m \) from the \( n \) variables, form an infinite set of advanced algebraic invariants. That each is a polynomial of the \([v_i v_j]\) and \([v_i v_j v_k]\) is guaranteed by the following Caianiello expansion formulas [2, 6]: let \( V_m = v_1 v_2 \cdots v_{m+1} \), then
\[
\begin{align*}
[v_2 & ] = \sum_{l=2}^{2l} (-1)^l [v_1 v_2] [v_3 v_4 \cdots v_{2l}], \\
A(V_{2l-1}) & = \sum_{m=(2l-2,2)} \left[ v_{2l+1} (1) \right] [V_{2l-1}(2)]; \\
A(V_{2l}) & = \sum_{m=(2l-2,2l-1)} \left[ v_{2l+1} (1) \right] [V_{2l-1}(2)]; \\
[V_{2l+1}] & = \sum_{m=(2l-3,2l+1)} \left[ v_{2l+1} (1) \right] [V_{2l+1}(2)],
\end{align*}
\]
where (i) \((h, m - h) = V_m\) is a bipartition of the \( m \) elements in the sequence \( V_m \) into two subsequences \( V_{m(1)} \) and \( V_{m(2)} \) of length \( h \) and \( m - h \) respectively; (ii) in \([V_{m(1)}]\), the product of the \( h \) elements in the subsequence is denoted by the same symbol \( V_{m(1)} \); (iii) the summation \( \sum_{(h, m-h)} V_m \) is over all such bipartitions of \( V_m \), and the sign of permutation of the new sequence \( V_{m(1)} \), \( V_{m(2)} \) is assumed to be carried by the first factor \([V_{m(1)}]\) of the addend.

While quaternions are sufficient for describing orthogonal transformations in 3D, they cannot be generalized to higher dimensions directly. In quaternions, the hybrid product \([v_i v_j v_k]\) is a scalar. To make high-dimensional generalization this requirement must be removed, at the same time the property that this element be in the center of the algebra needs to be preserved. If we denote the quaternions by \( \mathbb{Q} \), then the above revision leads to a new algebra \( \mathbb{Q} \otimes \mathcal{Q} \) of dimension 8, where \( \mathcal{Q} = [v_1 v_2 v_3] \) for three fixed vector variables that are linearly independent. This algebra is the Clifford algebra over \( \mathbb{R}^{3-} \).

The formal definition of the Clifford algebra \( \text{Cl}(\mathcal{V}^n) \) over an \( n \)-dimensional \( \mathbb{K} \)-linear space \( \mathcal{V}^n \), where the characteristic of \( \mathbb{K} \) is 2, is the quotient of the tensor algebra \( \otimes \mathcal{V}^n \) over the ideal generated by elements of the form \( v \otimes v - Q(v) \) where \( Q \) is a \( \mathbb{K} \)-quadratic form. The product induced from the tensor product is called the Clifford product, also denoted by juxtaposition of elements [4, 8].

When \( \mathcal{V}^n = \mathbb{R}^{3-} \), the quaternionic product of vectors is the image of their Clifford product under the homomorphism induced by mapping \( i \) to a nonzero scalar. In Clifford algebra, \( i \) is not a scalar, but called a pseudoscalar because it not only commutes with everything, but spans a 1D real space containing all hybrid products. The concept quaternionic conjugate is replaced by the Clifford conjugate, which is the linear extension of the following operation: for any vectors \( v_1, v_2, \ldots, v_k \), let \( V_k = v_1 v_2 \cdots v_k \), then
\[
\nabla_k := (-1)^k V_k^*, \quad \text{where } V_k^* := v_k v_{k-1} \cdots v_1 \text{ is the reversion of } V_k.
\]

With the Clifford conjugate, we can define the magnitude of \( V_k \), the conjugate adjoint action \( \text{Ad}_V \), the bracket \([V_i, V_j]\), the axis \( A(V_k) \), together with the concepts not involving conjugate: the inner product \([V_i, V_j]\), the cross product \( v_i \times v_j \), and the hybrid product \([v_i v_j v_k]\), just the same as in the case of quaternions. The only difference is that since \([v_i v_j v_k]\) is now a pseudoscalar, while \( A(V_{2l}) \) is not, but a pseudovector. Geometrically, when \( A(V_{2l}) \neq 0 \), it represents the plane spanned by vectors \( v_1, v_2, \) or equivalently, the invariant plane of rotation \( \text{Ad}_V v_2 \) supporting \( v_1, v_2 \).

We see that unlike the quaternions where there are only two kinds of objects of different dimensions: scalars which are usually called 0-D objects, and vectors which represent 1-D directions and so are usually called 1-D objects, in Clifford algebra \( \text{Cl}(\mathbb{R}^{3-}) \) there are four kinds of objects of different dimensions. Besides scalars and vectors, there are pseudovectors which represent 2-D directions (planes), and pseudoscalars which represent 3-D orientations (spaces). This is the reason why \( \mathbb{R}^{3-} \) can be extended to higher dimensions by being capable of discerning objects of different dimensions.

To represent algebraic invariants in 3D orthogonal geometry, using the quaternionic product or the Clifford product in the brackets does not make any difference. The geometric interpretations of \( \mathcal{Q} \) and the Caianiello expansion formulas are identical for both products. This justifies the use of juxtaposition of elements to represent both products.

The two kinds of basic invariants \([v_i v_j]\) and \([v_i v_j v_k]\) form a commutative ring, called inner-product bracket algebra. Formally, given a set of \( n \) symbols \( M = \{v_1, \ldots, v_n\} \), two kinds of new symbols can be defined as following: (1) all \( 2t \)-tuples of symbols selected repetitively from \( M \), by requiring that each \( 2t \)-tuple be symmetric with respect to its two entries; such a \( 2t \)-tuple is denoted by \([v_i v_j]\). (2) All \( 3t \)-tuples of symbols selected repetitively from \( M \), by requiring that each \( 3t \)-tuple is anti-symmetric with respect to its three entries, and in particular, if there are identical entries in a \( 3t \)-tuple, setting the \( 3t \) to be zero; such a \( 3t \)-tuple is denoted by \([v_i v_j v_k]\).

The two kinds of symbols must satisfy the following dimension-three constraints:

\[
\begin{align*}
[v_1 v_2] & = 0, \\
[v_1 v_2 v_3] & = 0,
\end{align*}
\]

(2.11)
For any six symbols $v_1, \ldots, v_6$,

$$[v_{11} v_{12} v_{13}] [v_{44} v_{55} v_{66}] = - [v_{13} v_{14}] [v_{12} v_{15}] [v_{13} v_{16}] .$$

(2.12)

The inner-product bracket algebra is the commutative ring generated by the above two kinds of symbols, satisfying the symmetry requirements and the dimension-three constraints.

To include brackets of longer length, the concept quaternionic bracket algebra or Clifford bracket algebra needs to be introduced. As explained before, there is no need to distinguish between the two concepts in the setting of 3D orthogonal geometry, so we simply call it bracket algebra. To distinguish from the concept of the same name arising from Grassmann-Cayley algebra [14], we call that in classical bracket algebra.

Formally, besides the above 2-tuples and 3-tuples, a hierarchy of infinitely many new symbols can be defined: for any length $l > 3$, there are all $l$-tuples of symbols selected repetitively from $\mathcal{M}$, with the requirement that the first and the last equalities in Caianiello expansion (2.1) are satisfied; such an $l$-tuple is denoted by $[v_{j1} v_{j2} \cdots v_{jk}]$. By setting $[1] = 1$ (0-tuple) and $[v_i] = 0$ (1-tuple) for all $i$, we get a full hierarchy of new symbols marked by brackets, with arbitrary length $l \geq 0$. The new symbols together with their specific requirements, form a a commutative ring called the bracket algebra over 3D inner-product space. This is the bottom-up approach to defining bracket algebra. The concept quaternionic product or Clifford product is not needed.

3. VECTOR-VARIABLE POLYNOMIALS AND BRACKET POLYNOMIALS

In this paper, we use (1) bold-faced digital numbers and bold-faced lower-case letters to denote vector variables, e.g., $v_1$; (2) bold-faced upper-case letters to denote monic vector-variable monomials, e.g., $A; \textbf{X}$; (3) Roman-styled lower-case letters to denote polynomials, e.g., $f, g$; (4) Greek letters to denote $\mathbb{K}$-coefficients, e.g., $\lambda, \mu$.

Although the background is real orthogonal geometry, the algebraic manipulations under investigation are independent of the real field. In fact, only the following coefficients occur in computing: $\pm 2^k$ for $k \in \mathbb{Z}$. We set the base field $\mathbb{K}$ to be of characteristic $\neq 2$.

Now start from quaternions. Let $v_1, v_2, \ldots, v_n$ be symbols. What properties determine that the multilinear associative product among the symbols is the quaternionic one, and that these symbols represent vectors of a 3D real inner-product space with metric diag($-1, -1, -1$)? [14] gives a rather simple answer.

The inner product of two vectors $v_i, v_j$ is a scalar, so it commutes with a third vector $v_k$: $[v_i v_j v_k] = [v_k v_i v_j]$. For three vectors $v_i, v_j, v_k$, since $[v_i v_j v_k] = v_i [v_j v_k]$, for a fourth vector $v_l$, the commutativity $[v_i v_j v_k v_l] = [v_l v_i v_j v_k]$ holds. The two commutativities are all that characterize the equal-
A polynomial of vector variables is a \(K\)-linear combination of monic monomials. The leading term of a polynomial \(f\) is the term of highest order, denoted by \(\text{lt}(f)\). The degree of a polynomial is that of its leading term. The leading terms of all elements in a subset \(S\) of polynomials are denoted by \(\text{lt}(S)\). When specifying the field \(K_2\) or \(K_3\), we can get the corresponding concepts quaternionic polynomial and Clifford polynomial.

Fix a multiset of vector variables \(\mathcal{M}\) composed of \(m \geq 3\) symbols \(v_1, v_2, \ldots, v_n\). Let \(n\) be the number of different elements in \(\mathcal{M}\), where \(3 \leq n \leq m\). In the \(K\)-tensor algebra \(\bigotimes(v_1, v_2, \ldots, v_n)\) generated by the \(n\) symbols taken as vectors, a tensor monomial is up to coefficient to the tensor product of finitely many such vectors. The \(K\)-tensor algebra over \(M\), denoted by \(\bigotimes[M]\), is the \(K\)-subspace of \(\bigotimes(v_1, v_2, \ldots, v_n)\) spanned by tensor monomials whose vector variables by counting multiplicity are in \(\mathcal{M}\), equipped with the tensor product that is undefined if the result is no longer in \(\bigotimes[M]\).

When the product among the elements in \(\mathcal{M}\) is the vector-variable product, we have the corresponding concept of 3D vector-variable polynomial ring over multiset \(\mathcal{M}\), denoted by \(Q[\mathcal{M}]\). Each element in \(Q[\mathcal{M}]\) is a polynomial whose multiset of vector variables in each term is a submultiset of \(\mathcal{M}\). The syzygy ideal \(I[\mathcal{M}]\) of \(Q[\mathcal{M}]\) is still defined by (6.1).

The concepts of Gröbner base and normal form are defined in \(Q[\mathcal{M}]\) just as in \(\bigotimes(v_1, v_2, \ldots, v_n)\). For two monomials \(h_1, h_2\) in \(\bigotimes(v_1, v_2, \ldots, v_n)\), \(h_1\) is said to be reduced with respect to \(h_2\), if \(h_2\) is not a factor of \(h_1\), or \(h_1\) is not a multiplier of \(h_2\), i.e., there do not exist monomials \(l, r\), including elements of \(K\), such that \(h_1 = lh_2r\). For two polynomials \(f\) and \(g\), \(f\) is said to be reduced with respect to \(g\), if the leading term of \(f\) is reduced with respect to that of \(g\). The term “non-reduced” means the opposite.

Let \(\{f_1, f_2, \ldots, f_k\}\) be a set of vector-variable polynomials. Another set of vector-variable polynomials \(\{g_1, g_2, \ldots, g_k\}\) is said to be a reduced Gröbner base of the ideal \(I := (f_1, f_2, \ldots, f_k)\) generated by the \(f_i\) in \(Q[\mathcal{M}]\), if (1) \(g_1, g_2, \ldots, g_k = I\), (2) the leading term of any element in \(I\) is a multiplier of the leading term of some \(g_i\), (3) the \(g_i\) are pairwise reduced with respect to each other.

The reduction of a polynomial \(f\) with respect to a reduced Gröbner base \(g_1, g_2, \ldots, g_k\) is the repetitive procedure of dividing the highest-order non-reduced term \(L\) of \(f\) by some \(g_i\), whose leading term is a factor of \(L\), then updating \(f\) by replacing \(L\) with its remainder, until all terms of \(f\) are reduced. The result is called the normal form of \(f\) with respect to the Gröbner base. Two polynomials are equal if and only if they have identical normal forms.

In [7], two theorems are established for the Gröbner base and normal forms of 3D vector-variable polynomials, one for the multilinear case where each element in multiset \(\mathcal{M}\) has multiplicity 1, the other for the general case \(Q[\{v_1, \ldots, v_n\}]\).

**Theorem 2.** [7] Let \(I[\{v_1, \ldots, v_n\}]\) be the syzygy ideal of the multilinear polynomial ring \(Q[\{v_1, \ldots, v_n\}]\) in \(n\) different vector variables \(v_1 < v_2 \ldots < v_n\).

(1) **[Gröbner base]** The following are a reduced Gröbner base of \(I[\{v_1, \ldots, v_n\}]\): for all \(1 \leq i_1 < i_2 < \ldots < i_j \leq n\),

\[
\text{G3: } [v_{i_1}v_{i_2}v_{i_1} - [v_1v_{i_2}v_{i_1}], \text{ and } [v_{i_1}v_{i_2}v_{i_1} - [v_2v_{i_2}v_{i_1}]]
\]

\[
\text{Gj: } [v_{i_1}v_{i_2}v_{i_1}v_{i_5} \cdots v_{i_1}v_{i_3}v_{i_1}], \text{ for all } j > 3.
\]

(2) **[Normal form]** In a normal form, every term is up to coefficient of the form \(V_{Y_1}V_{Y_2}v_{z_1} \cdots V_{Y_k}V_{z_k}v_{z_k}\), where

(i) \(k \geq 0\),

(ii) \(v_{z_1}v_{z_2} \cdots v_{z_k}\) is ascending,

(iii) every \(V_{Y_i}\) is an ascending monomial of length \(\geq 1\),

(iv) \(V_{Y_1}V_{Y_2} \cdots V_{Y_k}\) (or \(V_{Y_1}V_{Y_2} \cdots V_{Y_k}\) if \(V_{Y_k+1}\) occurs) is ascending,

(v) for every \(i \leq k\), if \(v_{t_i}\) is the trailing variable of monomial \(V_{Y_i}\), then \(v_{t_i} < v_{t_i}\).

**Theorem 3.** [2] Let \(I[\{v_1, \ldots, v_n\}]\) be the syzygy ideal of the polynomial ring \(Q[\{v_1, \ldots, v_n\}]\) in \(n\) different vector variables \(v_1 < v_2 \ldots < v_n\).

(1) **[Gröbner base]** The following are a reduced Gröbner base of \(I[\{v_1, \ldots, v_n\}]\):

G3, Gj:

for all \(3 < j < \infty\), and all \(1 \leq i_1 < i_2 < i_3 < i_4 \leq \ldots \leq i_{j-1} < i_j \leq n\);  

EG2:

for all \(i_1 < i_2\),

\[
V_{i_2}v_{i_2}v_{i_1} - v_{i_1}v_{i_2}v_{i_2}.
\]

EGj:

\[
[v_{i_1}v_{i_2}v_{i_3}v_{i_4} \cdots v_{i_1}v_{i_1}v_{i_1}], \text{ for all } 2 < j < \infty, \text{ and all } 1 \leq i_1 < i_2 < i_3 < i_4 \leq \ldots \leq i_{j-1} < i_j \leq n.
\]

(2) **[Normal form]** In a normal form, every term is up to coefficient of the form \(V_{Y_1}V_{Y_1}v_{z_1} \cdots V_{Y_k}v_{h_k}v_{z_k}\) or \(V_{Y_1}V_{h_1}v_{z_1} \cdots V_{Y_k}v_{h_k}v_{z_k}v_{z_k}\), where

(i) \(k \geq 0\),

(ii) \(v_{z_1}v_{z_2} \cdots v_{z_k}\) is non-descending,

(iii) \(v_{h_1}v_{h_2} \cdots v_{h_k}\) is non-descending,

(iv) every \(V_{Y_i}\) is a non-descending monomial of length \(\geq 0\),

(v) \(V_{Y_1}V_{h_1}V_{z_2} \cdots V_{Y_k}v_{h_k}\) (or \(V_{Y_1}V_{h_1}V_{z_2} \cdots V_{Y_k}v_{h_k}V_{z_k}\) if \(V_{Y_k+1}\) occurs) is non-descending,

(vi) for every \(i \leq k\), if the length of \(V_{Y_i}\) is nonzero, let \(v_{t_i}\) be the trailing variable of \(V_{Y_i}\), then \(v_{t_i} < v_{t_i}\).

For a general multiset \(\mathcal{M}\) in which the different vector variables are \(v_1, v_2, \ldots, v_n\), the Gröbner base of the syzygy ideal \(I[\mathcal{M}]\) is the restriction of the Gröbner base of \(I[\{v_1, \ldots, v_n\}]\) to \(Q[\mathcal{M}]\), denoted by \(G[\mathcal{M}]\). In \(Q[\mathcal{M}]\), a polynomial is said to be \(I\)-normal if its leading term is reduced with respect to the Gröbner base. The procedure of deriving the normal form of a polynomial is called \(I\)-reduction.

Now that any vector-variable polynomial has a normal form by \(I\)-reduction, so does a bracket polynomial whenever
Consider the following simple example: for a single bracket
\[ 2[v_1v_2\cdots v_m] = v_1v_2\cdots v_m + (-1)^m v_m\cdots v_2v_1, \]
the I-reduction goes as follows: for \(0 \leq j < m\), if we define
\[ V_{m-j} = v_{j+1}v_{j+2}\cdots v_m, \]
then when \(m \geq 3\),
\begin{align*}
V_m & \overset{I}{=} v_1V_{m-1} + V_{m-1}v_1 - v_1(V_{m-1} + (-1)^{m-1}V_{m-1}^\dagger) \\
& \overset{I}{=} v_1V_{m-1} + V_{m-1}v_1 - v_1(v_2V_{m-2} + V_{m-2}v_2) \\
& \quad + v_1v_2(V_{m-2} + (-1)^{m-2}V_{m-2}^\dagger) \\
& = v_2V_{m-2}v_1 - v_1V_{m-2}v_2 \\
& \quad + v_1v_2(V_{m-2} + (-1)^{m-2}V_{m-2}^\dagger). \tag{3.4}
\end{align*}
From this recursive formula, we get for \(m \geq 1\),
\[ 2[v_1v_2\cdots v_m] \overset{I}{=} 1 + \frac{(-1)^m}{2} v_1v_2\cdots v_m \\
+ \sum_{i=1}^{m-1} (-1)^{i+1}(v_1\cdots v_i\cdots v_m)v_i. \tag{3.5}
\]
So the normal form of the simplest bracket \([v_1v_2\cdots v_m]\) is composed of up to \(m\) terms. What is worse is that only when the terms are summed up can they represent a single algebraic invariant (the bracket), while missing a single term destroys the invariance of the whole expression.

From the appearance, the bracket symbol only hides half of a binomial. There are much more behind this appearance.

By definition, for a sequence \(A\) of \(a > 1\) vector variables, its bracket is \([A] = 2^{-1}A + (-1)^a 2^{-1}A^\dagger\). Monomial \(A\) is called the representative of bracket \([A]\). Later on, when we write \([A]\), we always assume that monomial \(A\) is the representative of the bracket.

The definition of a bracket endows with the symbol the reverse symmetry (or equivalently, the conjugate symmetry) up to sign: \([v_{i_1}\cdots v_{i_a}] = (-1)^a[v_{i_a}\cdots v_{i_1}]\). By \([3.3]\), the bracket symbol also has shift symmetry. So up to sign a bracket of \(a\) vector variables has the symmetry group \(D_{2a}\) (dihedral group).

The above analysis is only for a single bracket. For bracket polynomials, there are a lot of polynomial identities, or syzygies, among them. These complexities justify the separation of bracket algebra from vector-variable polynomial ring in symbolic manipulations of algebraic invariants. Finding the \(\vph\) base of the syzygies and then characterizing the normal forms of bracket polynomials are the main goal of this paper.

The following are some terminology on brackets. The representative of a bracket polynomial is the the vector-variable polynomial whose terms are each the product of the coefficient with the representatives of the bracket factors in the same term. The representative of a bracket polynomial is allowed to contain brackets. For example, \([v_1v_2[v_3v_4]]\) is taken as bracket binomial \(2^{-1}[v_1v_2(v_3v_4 + v_4v_3)]\); its representative is the content \(v_1v_2[v_3v_4]\) within the outer bracket.

The lexicographic ordering of bracket polynomials is that of their representatives. The leading variable of a bracket refers to that of its representative. The leading term of a bracket polynomial is always under the lexicographic ordering. For example, if \(v_1 < v_2\), then \([v_1v_2] < [v_2v_1]\), and \([v_1v_2][v_2v_1] < [v_2v_1][v_1v_2]\).

The leader (or expanded leading term) of a bracket polynomial, refers to the leading term of the bracket polynomial when taken as a vector-variable one, i.e., the corresponding vector-variable polynomial obtained from expanding each bracket into two terms by definition. For example, the leader of bracket \([A]\) refers to the one of higher order between \(2^{-1}A\) and \((-1)^a 2^{-1}A^\dagger\).

Among all the bracket polynomials that are equal to the same bracket polynomial, there are two that have strong features: the first is the one whose representative is the lowest, the second is the one whose leader is the lowest. The second is unique but the first is not. To make the first unique we introduce the following concept.

In \(\mathbb{Q}[\mathcal{M}]\), where the number of elements in multiset \(\mathcal{M}\) is \(m\), a uni-bracket monomial refers to a single bracket of length \(m\). A uni-bracket polynomial is a \(\mathbb{K}\)-linear combination of uni-bracket monomials. All uni-bracket polynomials form a \(\mathbb{K}\)-linear space, denoted by \(\mathbb{Q}[\mathcal{M}]\). The \(\mathbb{K}\)-linear space of the representatives of elements in \(\mathbb{Q}[\mathcal{M}]\) is just the \(\mathbb{K}\)-linear space of degree-\(m\) vector-variable polynomials, denoted by \(\mathbb{Q}_m[\mathcal{M}]\). Obviously, \(\mathbb{Q}[\mathcal{M}]\) is a linear subspace of \(\mathbb{Q}_m[\mathcal{M}]\).

When taken as a vector-variable polynomial, a uni-bracket is a binomial. In appearance, a uni-bracket is a monomial. To distinguish between the two understandings, we need a device to get rid of the bracket symbol and extract the representative of the uni-bracket. This can be done by taking \([\mathbb{Q}][\mathcal{M}]\) as the quotient of \(\mathbb{Q}_m[\mathcal{M}]\) modulo the ideal \(\mathcal{J}[\mathcal{M}] := \mathbb{Z}[\mathcal{M}] + [\mathbb{Z}][\mathcal{M}]\), \(\text{(3.6)}\)

where \([\mathbb{Z}][\mathcal{M}]\) is composed of the vector parts of degree-\(m\) polynomials, i.e., the \(\mathbb{K}\)-linear span of elements of the form
\[ R : v_{i_1}v_{i_2}\cdots v_{i_m} - (-1)^m v_{i_m}\cdots v_{i_2}v_{i_1}, \tag{3.7}\]
for all permutations of the \(m\) elements in \(\mathcal{M}\).

The modulo-\([\mathbb{Z}][\mathcal{M}]\) operation identifies a uni-bracket with its representative, or equivalently, identifies any degree-\(m\) vector-variable polynomial with the uni-bracket polynomial it serves as the representative. This operation retains the bracket symbols of all the brackets of length \(< m\), while removing the bracket symbols from all brackets of length \(m\).

4. GRÖBNER BASE AND NORMAL FORM FOR MULTILINEAR UNI-BRACKET POLYNOMIALS

From this section on, we use bold-faced digital numbers to denote vector variables, and use bold-faced capital letters to denote monic monomials of vector variables.

In this section, the multiset \(\mathcal{M}\) is composed of \(m \geq 3\) different vector variables, and the modulo-\([\mathbb{Z}][\mathcal{M}]\) operation is always assumed. Then \([\mathbb{Q}][\mathcal{M}]\) and \(\mathbb{Q}_m[\mathcal{M}]\) are identical, and
a uni-bracket no longer has the outer bracket symbol. Ideal 
$[I][M]$ is called the uni-bracket removal ideal in $Q[M]$, and ideal 
$J[M]$ is called the syzygy ideal of $[Q][M]$ in $\mathcal{M}[M]$. Below we compute the Gröbner base of $J[M]$ and characterize the normal forms of uni-bracket polynomials.

The following are elements of $J[M]$: for all monomials $A$ to $F$ such that $A1B, 1C, 1D2, 1E2F$ are of length $m$ and $E$ has length $e > 0$,

$$S_1 : A1B - 1BA, \quad S_1^\circ : \text{the } S_1 \text{ in which } A1B \text{ is } I\text{-normal,}$$

$$R_1 : 2(1[C]) = 1C - (-1)^m1C^I, \quad R_1^* : 2(1[D2]) = 1D2 - (-1)^m12D^I, \quad R_{12} : 2(1[E2[F]]) = 1E2[F] - (-1)^m12E^I[F].$$

Since the reduced Gröbner base of $I[M]$ is $G[M]$, we only need to consider the elements of type $R$ in $J[M]$, as they span $[I][M]$. By

$$A1B - (-1)^mB^1A^1 = (A1B - 1BA) + (1BA - (-1)^m1A^1B^1) - (-1)^m(B^1A^1 - 1A^1B^1),$$

we get

**Lemma 4.** $R$ is a subset of the ideal $(S_1, R_1)$. For any type-$R$ element $f$ but not of type $R_1$, $lt(f) \in lt(S_1)$.

**Lemma 5.** $S_1$ is a subset of $(S_1^\circ, R_1) + I[M]$. For any type-$S_1$ element $f$ but not of type $S_1^\circ$, $lt(f) \in lt(I[M])$.

**Proof.** Consider a general type-$S_1$ element $f = A1B - 1BA$, where $A$ is not empty. If $A1B$ is $I$-normal, then $f \in S_1^\circ$. If $I$-reductions are carried out to $A, B$, say $A = A^N + I$ and $B = B^N + I$, where $A^N, B^N$ are both $I$-normal, then $f = A^N1B^N - 1B^N A^N + I$, and $A^N1B^N$ is the new leading term. So by $I$-reductions we can assume that both $A$ and $B$ are $I$-normal.

Assume that $A1B$ is not $I$-normal. Further assume that any type-$S_1$ element $g < f$ is in $(S_1^\circ, R_1) + I[M]$. We prove the conclusion for $f$ by reduction on the order. There are three possibilities to apply Gröbner base elements of $I[M]$ to make reduction to $A1B$: (i) $G3$ at the end of $A1$, (ii) $G1$ for $i > 3$ at the end of $A1$, (iii) $G3$ on 1 and two variables from $A, B$ respectively.

Case (i). Let $A1B = Cuv1B$, where $u \succ v > 1$. Then

$$f \overset{G3}{\Rightarrow} -Cuv1B + C1(uv + vu)B - 1BCuv \overset{induction}{\Rightarrow} -1BC(vu + uv) + 1(uv + vu)BC \overset{\tau}{\Rightarrow} 0.$$

Case (ii). Let $A1B = CuvD1B$, where $1 \prec v \prec u \prec \prec \prec$ all variables of $D$. Let the length of $D$ be $d > 0$. Let the lengths of $B, C$ be $b, c$ respectively. Then $b + c = m - d - 3$.

$$f \overset{G3}{\Rightarrow} C(vD1 + (-1)^d1D^v)uB - (-1)^dCu1D^vB - 1BCuvD \overset{induction}{\Rightarrow} 1\{(BuBC - BuC^Dv)D + (-1)^dD^v(uBuBC - BuC^Dv)\} \overset{R_1}{\Rightarrow} (-1)^dD^v\{(1 - (-1)^b+c)B^d + BuBC\} + u((-1)^d1D^v + BuBC) \overset{\tau}{\Rightarrow} 0.$$
(III) \( A_1A_2Y_2z_2Y_3z_3 \cdots Y_iz_i \), where \( k \geq 2 \), \( A \) and the \( Y_i, z_i \) are as in (II), and for some \( 2 \leq i \leq k \), if \( l_i \) is the leading variable of \( Y_i \), then \( l_i \prec z_i \).

\[ \text{Proof.} \quad \text{There are several steps.} \]

**Step 1.** We need to prove that \( R_{12} \) is a subset of the ideal \( (R_{12}^N + I[M]) \). Once this is done, then since the leader of any element of \( R_{12}^N \) is \( I \)-normal and cannot be cancelled by the leader of any other element of type \( R_{12}^N \), the \( G_i \) and \( R_{12}^N[j] \) form a reduced Gröbner base of \( (R_{12}^N + I[M]) \). By this and the previous three lemmas, we get conclusion (1) of the theorem.

Once conclusion (1) holds, then any \( I \)-normal monomial of length \( m \) with leading variable \( l \) is the representative of a uni-bracket in normal form if and only if it is not the leader of an \( R_{12}^N \)-typed element. Conclusion (2) follows.

**Step 2.** The idea of proving the statement in Step 1 is to use \( G_i[M] \) to decrease the order of the leader of any element of type \( R_{12}^N \) at \( I \), so that the same time keep the reduction result to be within the \( I \)-linear space spanned by elements of type \( R_{12}^N \). Then ultimately all the leaders of these elements become \( I \)-normal.

We start with the \( I \)-reduction on the leading term of a general \( R_{12}^N \)-typed element \( f = A_1A_2 - (\cdots)^{N}2A_1^N \), where the length of \( A_1 \) is \( m - 2 \). If by \( I \)-reduction, \( A = A^N + I \), then \( f = A_1^N - (\cdots)^{N}2A_1^N + I \). So we can assume that \( A \) is \( I \)-normal.

If \( A_2 \) is \( I \)-normal, then \( f \) is just \( R_{12}^N[1] \). When \( A_2 \) is not \( I \)-normal, if \( A_2 \) is non-reduced with respect to \( G_3 \), let \( A_2 = Buv_2 \), where \( u \succ v \succ 2 \), then

\[
1Buv_2 = -1Bvu_2 + 2\{B_2[uv]\}. \tag{4.3}
\]

The result consists of the leading terms of one \( R_{12}^N \)-typed element and one \( R_{12}^N[\cdot] \)-typed element. The leaders of both terms are lower than \( f \).

If \( A_2 \) is non-reduced with respect to \( G_i \) for some \( i > 3 \), let \( A_2 = BuvC_2 \), where \( u \succ v \succ 2 \), and \( uC \) is ascending, and the length of \( C \) is \( c \geq 0 \). Then

\[
1BuvC_2 \equiv 1Bu + (\cdots)^{N}2C]\preceq
= \begin{cases} 1Bu\geq2u + 1Bu((-1)^{N}v + uvC) & \text{if } D \text{ contains more than one variable, then } u \succ l_D, \tag{4.3} \\ 1Bu + 1Bu((-1)^{N}v + uvC) & \text{if } D \text{ contains one variable, then } u \succ l_D, \tag{4.3} \\ 1Bu\geq2u + 1Bu((-1)^{N}v + uvC) & \text{if } D \text{ contains more than one variable, then } u \succ l_D, \tag{4.3} \\ 1Bu + 1Bu((-1)^{N}v + uvC) & \text{if } D \text{ contains one variable, then } u \succ l_D, \tag{4.3} \\ \end{cases}
\]

The result consists of the leading terms of two \( R_{12}^N \)-typed elements and one \( R_{12}^N[\cdot] \)-typed element. The leaders of the three terms are lower than \( f \).

By (4.3) and (4.4), a monomial that is non-reduced with respect to a \( G_i \) for some \( i \geq 3 \) must contain a subsequence of the form \( uDv \), where (i) the length of \( D \) is \( d \geq 2 \); (ii) \( u \succ v \); (iii) if \( l_D \) is the leading variable of \( D \), then \( u \succ l_D \); (iv) if \( D \) contains more than one variable, then \( u \succ v \). (4.3) and (4.4) can be written in the following unified form:

\[
uDv \equiv 2(u[D]v) + 2(v[uD]) - Duv. \tag{4.5}
\]

It is called the fundamental \( I \)-reduction formula.

**Step 3.** Consider \( I \)-reductions on the leader of a general \( R_{12}^N[\cdot] \)-typed element \( f = A_1A_2[1] - (\cdots)^{N}2A_1^N[A_2] \), where the length of \( B \) is \( b > 0 \).

Since \( [B] = (\cdots)^{N}[B] \), henceforth we assume that in any \( R_{12}^N[\cdot] \)-typed element to be normalized, the leading variable of any bracket has higher order than the trailing variable of the bracket. Then the leader of the bracket is always its representative.

In this step, we consider \( I \)-reduction to the representative \( B \) of \([B] \). Let \([B] = [CuDvE] \). Substituting (4.5) into it, we get

\[
[CuDvE] = 2[CuvE][D] + 2[CvE][uD] - [CDuvE]. \tag{4.6}
\]

**Step 4.** Consider \( I \)-reductions on the leader \( 1A_2B \) of \( f = A_1A_2B - (\cdots)^{N}2A_1^N[A_2] \) involving both the tail part of \( A \) and the head part of \( B \), where the leading variable of \( B \) is assumed to be higher than the trailing variable.

As \( 2 \) is lower than any element of \( A, B \), the only possible reduction is by \( G_3 \). Let \( 1A_2B = 1Ca_2bD \), where \( C \) may be empty but \( D \) is not. Assume \( a \succ b \succ t_D \), where \( t_D \) is the trailing variable of \( D \). Let the length of \( D \) be \( d \). It is easy to prove that applying \( G_3 \) to \( a_2b_D \) in vector-variable binomial \( 1Ca_2b_D \) is equivalent to the following absorption of bracket:

\[
1Ca_2b_D \equiv 1C[bD]a_2 = 2^{-1}(1ChDa_2) + (\cdots)^{d-1}1CD\{ba_2\}. \tag{4.7}
\]

Each term in the result is a leading term of an \( R_{12}^N \)-typed element lower than \( f \).

**Step 5.** In Step 3, we have seen that a single bracket after \( I \)-reduction, may be split into two brackets. The split can continue and we gradually get expressions of the form

\[
R_{12}[j] : 1E_2F_2F_3 \cdots F_j - (\cdots)^{N}2E_2F_3 \cdots F_j \tag{4.8}
\]

where the length of \( E \) is \( e \geq 0 \), and the length of \( 1E_2F_2F_3 \cdots F_j \) is \( m \). \( R_{12}[j] \) is a \( \omega \)-linear combination of elements of type \( R_{12}[\cdot] \) if all but one bracket are each expanded into two terms.

Consider \( I \)-reductions of \( R_{12}[j] \) involving more than two bracket factors, and \( I \)-reductions involving \( 1E_2 \) and more than one bracket factor. Since \( 2 \) is lower than all elements of \( E \) and \( F_i \), \( G_3 \) is the only possible Gröbner base element that may apply to \( 2 \) and its neighbors on both sides simultaneously. \( G_3 \) can involve only \( [F_2] \) among the brackets.

In \( [F_2][F_3] \cdots [F_j] \), only \( G_i \) where \( i > 3 \) can involve more than two brackets. However, since \( G_i \) is of the form \( uDv \) where \( D \) is ascending, if the product of the leaders of three brackets is non-reduced with respect to some \( G_i \), then the middle bracket must be composed of a subsequence of \( D \) of length \( \geq 2 \), contradicting with the assumption that the leading variable in the middle bracket be higher than the trailing variable.
So each $I$-reduction of $R_{12}[j]$ by a single $G_i$ where $i \geq 3$, can involve at most two bracket factors, or the 1E2 and one bracket factor.

Step 6. Consider $I$-reductions on $[F_1][F_2]$, where the leading variable in each bracket is higher than the trailing variable. If the leading variable $I_{F_1}$ of $F_1$ is higher than the leading variable $I_{F_2}$ of $F_2$, then an $I$-reduction commuting the two brackets reduces the order of their product. Below we always assume $I_{F_1} < I_{F_2}$.

For $G_3$, there are two possibilities to involve both $F_1, F_2$ in the leader $F_1, F_2$: two variables at the end of $F_1$ and the third at the beginning of $F_2$, or one variable at the end of $F_1$ and the other two at the beginning of $F_2$. The latter is impossible because $I_{F_1} < I_{F_2}$. For $G_i$ where $i > 3$, there are also two possibilities: two variables at the end of $F_1$ and the rest at the beginning of $F_2$, or one variable at the end of $F_1$ and the rest at the beginning of $F_2$. The latter is also impossible due to $I_{F_1} < I_{F_2}$.

Case G3. Let $[F_1][F_2] = [Buv][wCd]$, where $u \succ v$, and $u \succ w \succ d$. Let the length of $C$ be $c \geq 0$, and let the leading variable of $Buv$ be $1$. Then $w \succ v \succ u$. Applying G3 to $uvw$ is equivalent to the following absorption of the second bracket:

$$
[Buv][wCd] = [BwCd]uv + (-1)^2(wCd)[Bw].
$$

The leader of each bracket monomial in the result has lower order than the leader of $[F_1][F_2]$.

Case G1. Let $[F_1][F_2] = [Buv][aDwC]$, where $aD$ is ascending, and $a \succ u \succ v \succ w$. Let the lengths of $B, C, D$ be $b, c, d$ respectively. Let the leading variable of $Buv$ be $1$, and let the trailing variable of $wC$ be $t$. Then $a \succ v \succ w$. Applying $G(d+4)$ to $uvw$, we get

$$
4[Buv][aDwC] = BuvaDwC + (-1)^{c+d}BuvaDwC+1d + (-1)^{d+t}BuvaDwC+1 + (-1)^{d+t}BuvaDwC+1t + (-1)^{d+t}BuvaDwC+1u + (-1)^{d+t}BuvaDwC+1v + (-1)^{d+t}BuvaDwC+1w.
$$

Step 7. Consider a bracket of the form $h = [a_1B_1c_1a_2B_2c_2 \cdots a_kB_kc_k]$, where $(1) > k \geq 1$, $(2) a_1 \succ c_1$ for every $i$, $(3) a_i \succ c_j$ for all $1 \leq j \leq k$. Let the length of $B_i$ be $b_i$.

When $k = 2$,

$$
[a_1B_1c_1a_2B_2c_2] = -\sum (-1)b_1^i b_2^j \cdot \sum (-1)^{b_1^i + b_2^j}B_1^i c_1^i t_1 B_2^j c_2^j t_2.
$$

The leader of each bracket monomial in the result has lower order than the leader of $[F_1][F_2]$.

For $k > 2$,

$$
[a_1B_1c_1 \cdots a_kB_kc_k] = -\sum (-1)^{b_1^i + b_2^j}B_1^i c_1^i t_1 B_2^j c_2^j t_2 \cdots B_{k-1}^i c_{k-1}^i t_{k-1} B_k^j c_k^j t_k.
$$

The $[F_1][F_2]$ can be used to split a long bracket whose representative is $I$-normal. It can also be used in the converse direction, to concatenate short brackets into a long one.

Step 8. So far we have proved that for any $R_{12}$-typed element $1E2$ or any $R_{12}[j]$-typed element $1[A2][F_2][F_3] \cdots [F_j]$, as long as the leader is not $I$-normal, $I$-reductions can always be carried out to change $1E2$ or $1A2[F_2][F_3] \cdots [F_j]$ into the following form:

$$
1 = \sum_{\alpha} \lambda_{\alpha} 1E_\alpha 2 + \sum_{\beta} \mu_{\beta} 1A_\beta 2[F_\beta_2][F_\beta_3] \cdots [F_\beta_j] + \sum_{\gamma} \tau_{\gamma} 1D_{\gamma 1} [D_{\gamma 2}] \cdots [D_{\gamma k}],
$$

where the leading variable in each bracket is higher than the trailing variable, the $1E_\alpha 2$ and $1A_\beta 2F_\beta_2F_\beta_3 \cdots F_\beta_j$ are all $I$-normal.

Since any $I$-normal form is of type $Y_1 z_1^{\alpha_1} Y_2 z_2^{\alpha_2} \cdots Y_k z_k^{\alpha_k}$, it must be that
(i) \(1 \mathbf{E}_0, 2 = Y_1z_1\), i.e., \(\mathbf{E}_0 = 34 \cdots m\).

(ii) \(1A_{\beta} F_{\beta_2} F_{\beta_3} \cdots F_{\beta_j} = Y_1z_1 \cdots Y_1z_j\), and

\[
\begin{aligned}
1A_{\beta} F_{\beta_2} &= Y_1z_1, \\
F_{\beta_2} &= Y_2z_2, \\
\vdots \\
F_{\beta_j} &= Y_jz_j.
\end{aligned}
\]

(i) is obvious. In (ii), the trailing variable of each bracket must be some \(z_i\). If an \(F_{\beta_i} = Y_1z_1 \cdots Y_{h+p}z_{h+p}\) for some \(p > 0\), let the leading variable of \(Y_s\) be \(l_s\), then \(l_s > z_{h+p} > z_{h+p-1} > \cdots > z_1\). By (1.11), \([F_{\beta_j}]\) is split into \(2^n[Y_1z_1] \cdots [Y_{h+p}z_{h+p}]\) plus some bracket monomials of lower leader. Then \(I\)-reductions continue to the terms involving such bracket monomials. Ultimately each bracket is of the form \([Y_1z_1]\).

In (1.12), \(1 \mathbf{E}_0, 2 = (-1)^m 1 \mathbf{E}_0, 2^1\) by \(R_2(1, 1)\, 1 \mathbf{A}_{\beta} 2 \, [F_{\beta_2}] \cdots [F_{\beta_j}] = (-1)^m 1 \mathbf{A}_{\beta} 2 \, [F_{\beta_2}] \cdots [F_{\beta_j}] \) by \(R_2(1, 1)\) and \(12 \mathbf{D}_{\gamma_1} \cdots [D_{\gamma_k}] = (-1)^m 1 \mathbf{D}_{\gamma_1} \cdots [D_{\gamma_k}] \) by \(I[M]\). So \(1T - (-1)^m 1T^1\) is reduced to zero by \(I[M]\) and \(R_{12}\).

5. SQUARE-FREE VECTOR-VARIABLE POLYNOMIAL RING

When \(M\) is a general multiset of vector variables, a square in \(Q[M]\) refers to the product of a vector with itself. Denote \(v_i^2 := v_i v_i\). It commutes with everything in \(Q[M]\).

Proposition 8. In \(Q[M]\), let \(f = g v_i^2 h\) be a multiplier of \(v_i^2\). If \(f\) is not \(I\)-normal, then by doing \(I\)-reduction to \(g, h\), together with rearranging the position of \(v_i^2\) in each term, \(f\) can become \(I\)-normal.

Proof. Suppose \(g, h\) are \(I\)-normal. There are three cases for \(f\) to be non-reduced with respect to the Gröbner basis \(g[M]\):

(1) If \(f\) contains as a factor the leader of \(G_k\) for \(k > 3\), or \(EG_j\) for \(j > 1\) involving both of \(v_i^2\), then \(v_i^2\) is preserved by the reduction with the Gröbner basis element.

(2) If \(g v_i\) is non-reduced, then switch the element of \(g[M]\) with respect to which \(g v_i\) is non-reduced:

Case EG2: Let \(g v_i = \text{Auv}_i v_i\) or \(\text{Auv}_i v_i\), where \(u \succ v_i\). Then \(\text{Auv}_i v_i^2 h = \text{Auv}_i v_i^2 h = \text{Auv}_i v_i^2 h\).

Case G3: Let \(g v_i = \text{Au}_v v_i\) where \(u \succ w\) and \(u \succ v_i\). Then \(\text{Au}_w v_i^2 h = \text{Au}_w v_i^2 h\).

Case Gk for \(k > 3\) or \(EG_j\) for \(j > 2\): Let \(g v_i = \text{CuwD}_v\) where \(u \succ w \succ v_i\). Then \(\text{CuwD}_v v_i^2 h = \text{CuwD}_v v_i^2 h\).

(3) If \(v_i h\) is non-reduced, then switch the element of \(g[M]\) with respect to which \(v_i h\) is non-reduced:

Case EG2: Let \(v_i h = v_i v_i z B\) or \(v_i z z B\), where \(v_i \succ z\). Then \(g v_i^2 v_i z B = g v_i^2 v_i z B\) or \(g v_i^2 z z B = g z v_i^2 z B\).

Case G3: Let \(v_i h = v_i y z B\) where \(v_i \succ y\) and \(v_i \succ z\). Then \(g v_i^2 y z B = g z v_i^2 z B\).

Case Gk for \(k > 3\) or \(EG_j\) for \(j > 2\): Let \(v_i h = v_i y z D\) where \(v_i \succ y \succ z\). Then \(g v_i^2 y z D = g v_i^2 y z D\).

In all the cases, the order of \(f\) is decreased while preserving \(v_i^2\). By induction on the order we get the conclusion. □

Proposition 8 suggests a “square-free normalization” of vector-variable polynomials, by moving all squares to a set free of any reduction operation, and maintaining the set of squares in a normal form.

In a vector-variable monoid, let the set of squares be separated from the remainder of the monomial by a symbol “□”, such that all elements on the right side of the symbol are squares. Two things need to be established before such a symbol can be used in algebraic manipulations: (1) algebraic structure of the new symbolic system, (2) connection with the canonical system based on V2, V3, V4.

Let \(S\) be a commutative monoid. All elements in \(S\) span a \(K\)-vector space whose dimension equals the number of elements in \(S\). The product in the vector space is the multilinear extension of the product in \(S\). The vector space equipped with this product forms a commutative \(K\)-algebra, called the \(K\)-algebra extension of monoid \(S\), denoted by \(K[S]\).

For a \(K\)-algebra \(A\), when \(S\) is a subset of the center of \(A\), then \(A\) is not only a module over the ring \(K[S]\), but a multilinear algebra over \(K[S]\), called a \(K[S]\)-algebra.

For \(K\)-tensor algebra \(\otimes[M]\), let

\[
\otimes[M] := \otimes[M]/(V^2).
\]

It is easy to see that when setting \(S\) to be generated by elements of the form \(v_i^2 := v_i \otimes v_i\), for all \(v_i \in M\), then \(\otimes[M]\) is a \(K[S]\)-algebra, called the \(K[S]\)-tensor algebra over monoid \(M\), or the square-free tensor algebra over \(M\). The product in \(\otimes[M]\) is induced from the tensor product. For brevity we still denote the product by “□”, but denote the commutative product in \(S\) by juxtaposition of elements.

In \(\otimes[M]\), for all \(q \in \otimes[M]\) and \(s \in S\), we introduce the notations

\[
\begin{aligned}
q \sqsubset s &:= q \otimes s \in \otimes[M], \\
q \sqsupset &:= q \in \otimes[M], \\
\sqsubset s &:= s \in S.
\end{aligned}
\]

Then

\[
\begin{aligned}
(q_1 \sqsubset s_1) \otimes (q_2 \sqsubset s_2) &= q_1 \otimes q_2 \sqsubset s_1 s_2.
\end{aligned}
\]

Formally, an element \(q \in \otimes[M]\) is taken as \(q \sqsubset 1\), and an element \(s \in K[S]\) is taken as \(1 \sqsubset s\). In other words, factor \(\sqsubset 1\) (or \(\sqsupset\)) in \(q \sqsubset 1\) (or \(q\)) is usually omitted. So

\[
\begin{aligned}
q \sqsubset s &= q \otimes (\sqsubset s) = (\sqsubset s) \otimes q, \\
\sqsubset st &= (\sqsubset s) \otimes (\sqsubset t) = (\sqsubset t) \otimes (\sqsubset s).
\end{aligned}
\]

That \(S\) is generated by squares can be succinctly expressed by the following identity:

\[
v_i \otimes v_i = \sqsubset v_i^2.
\]

making left multiplication with \(f\) and right multiplication with \(g\) on both sides of the identity, we get \(f \otimes (v_i \otimes v_i) \otimes g = \cdots \).
\( f \otimes g \Box v_2^2 \). It includes \( V_2 \) as a special case.

The degree, or length, of a monomial in \( \Box^2 [M] \) is the degree of the monomial when taken as an element in \( \Box [M] \). The left degree or left length of a monomial refers to the degree of the monomial on the left side of the square symbol. For a monomial \( f \in \Box^2 [M] \), its canonical form in \( \Box [M] \) is defined to be the monomial of lowest lexicographic order among all monomials equal to \( f \) modulo \( V_2 \). The order of \( f \) is that of its canonical form. This ordering is still called the lexicographic ordering.

The canonical form of \( f = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \Box v_{j_1} v_{j_2}^2 \cdots v_{j_l}^2 \), where \( v_{j_1} < v_{j_2} < \cdots < v_{j_l} \), can be obtained as follows:

1. Set \( g = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \).
2. For \( p \) from 1 to \( l \), let \( v_{i_p} \) be the first variable in the sequence of \( g \) such that \( v_{i_p} \succ v_{j_p} \geq v_{i_{p-1}} \). Insert \( v_{i_p} \otimes v_{j_p} \otimes \cdots \otimes v_{j_l} \) to the position before \( v_{i_p} \) in \( g \), and update \( g \).
3. Output \( g \).

The vector-variable polynomial ring \( Q[M] \) when taken as the quotient of \( \Box^2 [M] \) modulo the two-sided ideal \( \Box^2 [M] \) generated by \( V_3 \), \( V_4 \), is a \( K \)-algebra, called the square-free polynomial ring, denoted by \( \Box^2 [M] \). The product in \( Q^2 [M] \) is still denoted by juxtaposition of elements. \( \Box^2 [M] \) is called the syzygy ideal of \( Q^2 [M] \).

Theorem 8 has the following square-free version for multisets \( M \):

**Theorem 9.** Let \( M \) be a multiset of \( m \) symbols, among which \( m \) are different ones: \( v_1 \prec v_2 \prec \ldots \prec v_n \), and let \( \Box^2 [M] \) be the syzygy ideal of the square-free polynomial ring \( Q^2 [M] \).

1. **[Gröbner base]** The following are a reduced Gröbner base of \( \Box^2 [M] \): G3, and

\[
\begin{align*}
Gj & : \text{for all } 3 \leq j < n + 1, \text{ and } i_1 < i_2 < \cdots < i_j, \\
& \quad [v_{i_3} v_{i_2} v_{i_1} v_{i_5} \cdots v_{i_j} v_{i_1}] - [v_{i_2} v_{i_4} v_{i_5} \cdots v_{i_j} v_{i_1} v_{i_3}], \\
EGk & : \text{for all } 3 \leq k \leq n + 1, \text{ and } i_1 < i_2 < \cdots < i_k, \\
& \quad [v_{i_3} v_{i_2} v_{i_1} v_{i_4} \cdots v_{i_k} v_{i_1} v_{i_3}] - [v_{i_2} v_{i_3} v_{i_4} \cdots v_{i_k} v_{i_1} v_{i_3}].
\end{align*}
\]

The above Gröbner base is denoted by \( G\Box^2 [M] \).

2. **[Normal form]** In a normal form, every term is up to coefficient of the form \( V_{Y_1} V_{x_1} V_{Y_2} V_{x_2} \cdots V_{Y_k} v_{x_k} \Box v_{y_1}^a \cdots v_{y_k}^a \), where

(i) \( k \geq 0 \),
(ii) \( v_{x_1} \cdots v_{x_k} \) is non-descending,
(iii) every \( V_{Y_i} \) is an ascending monomial of length \( > 0 \),
(iv) \( V_{Y_1} V_{x_1} \cdots V_{Y_k} \) (or \( V_{Y_1} \cdots V_{Y_k} V_{y_i} \) if \( Y_{k+1} \) occurs) is non-descending,
(v) for every \( i \leq k \), let \( v_{y_i} \) be the trailing variable of \( V_{Y_i} \),
then \( v_{y_i} \succ v_{x_i} \).

(vi) \( s \) is either 1 or the product of several squares.

In \( Q^2 [M] \), a polynomial is said to be \( \Box^2 \)-normal if its leading term is reduced with respect to the Gröbner base \( G\Box^2 [M] \).

A monic square-free uni-bracket monomial is of the form \( [A] \Box s \), where \( A \) is either 1 or a monomial of length \( > 1 \), \( s \) is a product of squares, and the length of \( A \Box s \) is \( m \). A square-free uni-bracket polynomial is a \( K \)-linear combination of square-free uni-bracket monomial. The space of square-free uni-bracket polynomials is denoted by \( [Q] \Box^2 [M] \).

The \( K \)-linear space of degree-\( m \) square-free polynomials is denoted by \( Q_m [M] \). The space \( [Q] \Box^2 [M] \) can be taken as the quotient of \( Q_m [M] \) modulo the ideal

\[
\Box^2 [M] : = \Box^2 [M] + [I] \Box^2 [M],
\]

where \( [I] \Box^2 [M] \) is composed of the vector parts of degree-\( m \) square-free polynomials, i.e., the \( K \)-linear span of elements of the form

\[
R^2 : A \Box s - (-1)^a A \Box 0 \Box s,
\]

where \( A \) is a monomial of length \( a > 0 \) and contains no square, \( s \) is a product of squares, and the length of \( A \Box s \) is \( m \).

The modulo-\( [I] \Box^2 [M] \) operation identifies a square-free uni-bracket with its representative. It removes the outer bracket symbol on the left side of the “\( \Box \)” symbol from every square-free uni-bracket, disregarding the length of the bracket. Ideal \( [I] \Box^2 [M] \) is called the uni-bracket removal ideal in \( Q^2 [M] \), and ideal is called the syzygy ideal of \( [Q] \Box^2 [M] \) in \( \Box^2 [M] \).

6. **Gröbner Base and Normal Form For Uni-Bracket Polynomials**

In this section, we extend Theorem 8 to the case of general multiset \( M \) with \( m \geq 3 \) different vector variables. The modulo-\( [I] \Box^2 [M] \) operation is always assumed, i.e., \( [Q] \Box^2 [M] \) and \( Q_m [M] \) are identical, and a uni-bracket does not have the outer bracket symbol on the left side of the “\( \Box \)” symbol.

The following are elements of \( \Box^2 [M] \):

\[
\begin{align*}
R^2 (k) & : (K - (-1)^a K^a) \Box s, \\
S_1 (k) & : (i A B - b_1 B A) \Box s, \quad \text{for } i \neq b_1, \\
S_10 (k) & : (i C_1 - b_1 C) \Box s, \quad \text{for } i \neq b_1, \\
S_11 (k) & : (b_1 A_1 B - b_1 B_1 A) \Box s, \\
S_{1k} (k) & : (b_1 C_1 - C b_1^2) \Box s, \\
S_{1k} (s) & : (b_1 E B_1 - E B_1^2) \Box s, \\
R_{1k} (k) & : b_1 D \Box s, \\
R_{11} (k) & : b_1 C_1 \Box s, \\
R_{1k} (s) & : b_1 E B_1 \Box s, \\
R_{12} (k) & : b_1 C_2 \Box s, \\
R_{1k} (s) & : b_1 E B_2 \Box s, 
\end{align*}
\]

where

(a) \( s \) has length \( m - k \), and the left length of each expression is \( k > 0 \);
(b) \( b_1, b_2 \) are respectively the variables of the lowest order and the second lowest order on the left side of the square symbol;
(c) in $S_1$, either $A$ or $B$ can be empty, while in $S_{11}$, both $A$ and $B$ are non-empty;
(d) in $R_1$, $D$ is either empty (i.e., $D = 1$), or of length $> 1$;
(e) in $S_{10}$ and $R_{11}$, $C$ is non-empty;
(f) in $S_{11}^*[s](k)$ and $R_{11}[s]$, $E, F$ are non-empty, and $F$ does not contain $b_1$;
(g) in $R_{12}$, $C$ is non-empty and does not contain $b_1$;
(h) in $R_{12}[s]$, $E, F$ are non-empty and do not contain $b_1$, and $F$ does not contain $b_2$.

In bracket $[1A1]$, we have $[1A1] = [A]B^2$. That the leading variable has higher order than the trailing variable is always possible. This is taken as a postulate for all the brackets in $(6.1)$.

Consider a general element $f = (K - (-1)^k K^l) \sqsubseteq s$ of type $R^k(k)$:
1. If $b_1$ occurs in $K$ both as the leading variable and trailing variable, then $f \in (S_{11}^*[k], R^k(k-2))$.
2. If $b_1$ occurs in $K$ at only one end, then $f \in (R_3(k), S_1(k))$.
3. If $b_1$ occurs at the interior of $K$, set $K = Ab_1B$, where $A, B$ are both non-empty, and $b_1$ does not occur at any end of $A$ or $B$. By $(1)$, $f \in (R_3(k), S_1(k))$.

By induction on $k$, we get

**Lemma 10.**

$$R^k(k) \subseteq \bigcup_{h \leq k} (\sum_{h \leq k} ((S_1(h)) + (S_{11}(h)) + (R_1(h))).$$

In $R_1(k)$, when $D$ contains $b_1$, let $D = Ab_1B$, where the lengths of $A, B$ are respectively $a, k - a - 2$, then

$$b_1[Ab_1B] \sqsubseteq s \begin{cases} b_1[Ab_1B] \sqsubseteq s = \langle b_1, BA \sqsubseteq b_1^2 \rangle \sqsubseteq b_1^2 \qquad \text{for } i \geq 1 \text{ and } h \geq 1, \\
+ \langle b_1, B \sqsubseteq \square b_1^2 \rangle \sqsubseteq b_1^2 \qquad \text{for } i = 0 \text{ and } h = 1. \end{cases}$$

By induction on $k$, we get that both $b_1[Ab_1B] \sqsubseteq s$ and $b_1[BAb_1B] \sqsubseteq s$ are equivalent to $S_{11}^*[k]$:

**Lemma 11.** $R^k(k)$ is a subset of the ideal

$$\bigcup_{h \leq k} (\sum_{h \leq k} ((S_1(h)) + \langle R_1(h) \rangle + (R_1(1)) + \mathcal{I}^k[M]).$$

**Theorem 12.** Let $M$ be a multiset of $m > 2$ symbols, among which $n \geq 2$ are different ones: $1 < 2 < \cdots < n$, and let $\mathcal{G}^k[M]$ be the square-free vector-arbitrary polynomial ring over $M$. Let $[Q]^k[M]$ be the space of square-free uni-bracket polynomials, and let $\mathcal{J}^k[M]$ be its syzygy ideal in $\mathcal{G}^k[M]$.

**Proof.** There are several steps.

**Step 1.** We need to prove by induction on $k$ that $S_1(k)$, $R_{11}(k), R_{12}(k)$ are all in the ideal

$$\sum_{\text{left length} \leq k} (S_{11}^*[k] + I^k[M]),$$

where the asterisk stands for the $(j, l)$.

Once this is done, then since the leader of any element of type $S_{11}^*[k]$ is $I^k$-normal and cannot be cancelled by the leader of any other element of type $S_{10}^*[k]$ or $R_{10}[k]$, the $S_{11}^*[k], R_{10}[k]$ and $\mathcal{G}^k[M]$ must be reduced $\mathcal{G}^k[M]$ base of $(S_1, R_{11}, R_{12}, R_1(1)) + I^k[M]$. By Lemma 11 this ideal is just $\mathcal{J}^k[M]$. This proves conclusion (1), and conclusion (2) follows.
Step 2. \( S_1(k) \) requires \( k > 1 \); \( R_{11}(k) \) and \( R_{12}(k) \) both require \( k > 2 \). When \( k = 2 \), \( S_1(2) = (b_2b_1 - b_1b_2)\square \) is in \( S_1^2 \).

When \( k = 3 \), \( R_{11}(3) = (b_1b_2b_2 - b_1b_1b_2)\square = R_{11}^{[1,0]} \), and \( R_{12}(3) = b_1[b_1b_2]\square = R_{11}^{[0,1]} \), where \( b_1 > b_2 \).

Consider \( S_1(3) \). There are 3 elements led by variable \( b_2 \):

\[
(b_2b_2b_2 - b_1b_2b_2)\square, \quad (b_2b_2b_2 - b_1b_1b_2)\square, \quad (b_2b_1b_1 - b_1b_1b_2)\square.
\]

They all belong to \( S_1^3 \). There are two other elements in \( S_1(3) \):

\[
(b_2b_1b_2 - b_1b_2b_2)\square \quad \text{and} \quad (b_2b_1b_2 - b_1b_1b_2)\square.
\]

By

\[
\begin{align*}
& b_1b_1b_2 - b_1b_2b_1 \equiv_3 b_1b_2b_1 - b_2b_1b_1, \\
& b_1b_1b_2 - b_1b_2b_1 \equiv_3 (b_2b_1b_1 - b_1b_1b_2) + (b_2b_1b_2 - b_1b_1b_2),
\end{align*}
\]

both are in \( (S_1^2) + Z^2[M] \).

So the statement in Step 1 holds for \( k \leq 3 \). Assume that it holds for all \( k < h \). When \( k = h \), we need to make \( Z^2 \)-reduction to the leaders of the elements of any of the types

\[
S_1(h), \quad R_{11}(h), \quad R_{12}(h), \quad R_{11}[s](h), \quad R_{12}[s](h), \quad (6.6)
\]

at the same time keep the reduction result to be within the \( \mathbb{K} \)-linear space spanned by elements of the types \( (6.6) \) but where the left length \( \ell \) is replaced by all \( i \leq h \). Then ultimately all the leaders of these elements become \( Z^2 \)-normal.

Step 3. Consider types \( R_{11}(h), R_{11}[s](h), R_{12}(h), R_{12}[s](h) \). Let there be an \( R_{11}(h) \)-typed element \( f = 2(b_1[Ab_1]\square) \), and an \( R_{12}(h) \)-typed element \( g = 2(b_2[BB_2]\square) \), where \( B \) does not contain \( b_1 \). In the following we omit the factor \( \square \).

Do \( Z^2 \)-reductions to \( A, B \), and assume that the results are

\[
\begin{align*}
A & \equiv Z^2 C_A b_1 + b_1D_A + b_1E_A b_1 + A^N, \\
B & \equiv Z^2 C_B b_2 + b_2D_B + b_2E_B b_2 + B^N,
\end{align*}
\]

where (i) none of the terms in \( C_A, D_A, E_A, A^N \) has \( b_1 \) at any end;
(ii) none of the terms in \( C_B, D_B, E_B, B^N \) has \( b_2 \) at any end;
(iii) any of the four terms in each result may not occur;
(iv) the component on the right side of the square symbol in each term, together with the symbol itself, are omitted, as they do not affect the analysis below;
(v) in the extreme case, \( A^N \) or \( B^N \) may be in \( \mathbb{K} \), if all vector variables in the term form squares and are moved to the right side of the square symbol.

Substituting the reduction results into \( f, g \), we get

\[
\begin{align*}
f & \equiv \langle \langle (D_A b_1 - (-1)^k D_A b_1)\square b_1^2 \rangle \rangle, \quad \langle S_1(h - 2) , \\
& \quad + (b_1C_A - (-1)^k b_1C_A)\square b_1^2 \rangle \rangle, \quad \langle R_1(h - 2) \rangle, \\
& \quad + \langle E_A b_1 - (-1)^k b_1E_A b_1\rangle\square b_1^2 \rangle \rangle, \quad \langle R_1(h - 2) \rangle, \\
& \quad + \langle 2(b_1[AN_b]b_1) \rangle \rangle, \quad \langle R_{11}(h) \rangle, \\
g & \equiv \langle \langle 2(b_1[CB_1]b_2b_2)\square \rangle \rangle, \quad \langle R_1(h - 2) \rangle, \langle R_1(h - 2) \rangle, \\
& \quad + \langle 2(b_1E_B b_2)\square \rangle \rangle, \quad \langle R_1(h - 2) \rangle, \langle R_{12}(h - 2) \rangle, \\
& \quad + \langle 2(b_1[B^N]b_2) \rangle \rangle, \quad \langle R_{12}(h) \rangle.
\end{align*}
\]

Notice that the left lengths indicated on the right column are the maximal possible ones for the corresponding types. So by induction hypothesis, we can assume that in \( f, g \), monomials \( A = A^N, B = B^N \) and both are \( Z^2 \)-normal.

The \( Z^2 \)-reduction to the leaders of \( f, g \) are much the same with the procedure in the proof of Theorem (7) starting from Step 3 there to Step 8, with negligible revisions. Formula (6.11) can also be used to split the leader of factor \( b_1[Y_1 b_1 \cdots Y_1 b_1] \) in a type-R11[*, s] element, and the leader of factor \( b_1[Y_1 b_1 \cdots Y_1 b_1] \) in a type-R12[*, s] element.

By induction on the order of the leader, we get that \( R_{11}(h), R_{12}(h), S_1(h), S_1[s](h), R_{11}[s](h), R_{12}[s](h) \) are all in \( (6.7) \) where \( k = h \).

Step 4. Consider a general type-\( S_{10}(h) \) element \( f = (Ab_1 - b_1A)\square \). Let the \( Z^2 \)-reduction result of \( A \) be as in \( (6.7) \). Then if omitting “\( \square \)”,

\[
f \equiv \langle \langle C_A b_1^2 - b_1C_A b_1 \rangle \rangle \quad \langle S_1(h) \rangle, \\
+ \langle b_1D_A b_1 - C_A b_1^2 \rangle \rangle, \quad \langle S_1(h) \rangle, \\
+ \langle b_1E_A b_1 - E_A b_1^2 \rangle \rangle, \quad \langle S_1(h - 2) \rangle, \\
+ \langle A^N b_1 - b_1A^N \rangle \rangle, \quad \langle S_1(h) \rangle.
\]

So we can assume that in \( f = (Ab_1 - b_1A)\square \), monomial \( A = A^N \) and \( Z^2 \) is \( Z^2 \)-normal.

The \( Z^2 \)-reduction to the leading term of \( f \) is much the same with the procedure in the proof of Lemma (5) starting from Case (i) there to Case (ii). By induction on the order of the leading term, we get that \( S_1(h) \) is in \( (6.7) \) where \( k = h \).

Step 5. Consider a general type-\( S_{11}(h) \) element \( g = (Ab_1 - b_1A)b_2 \). Let the \( Z^2 \)-reduction results of \( A, B \) be as in \( (6.7) \), where every \( b_2 \) is replaced by \( b_1 \). Then if omitting “\( \square \)”,

\[
g \equiv \langle \langle C_A b_2 b_1 - b_1C_A b_1 - b_2C_A b_1 \rangle \rangle \quad \langle S_1(h) \rangle, \\
+ \langle C_A b_1 b_2 - b_1C_A b_2 \rangle \rangle, \quad \langle S_1(h) \rangle, \\
+ \langle C_A b_1 b_2 - b_1C_A b_2 \rangle \rangle, \quad \langle S_1(h - 2) \rangle, \\
+ \langle A^N b_1 - b_1A^N \rangle \rangle, \quad \langle S_1(h - 2) \rangle.
\]

In the above result, the lines that do not belong to the ideal \( (S_1(h), S_1(h - 2), S_1(h - 2), S_1(h - 4)) \) are

\[
\begin{align*}
b_1D_A b_1 b_2 - b_1A^N b_1 b_2, & \quad \langle S_1(h), S_1(h) \rangle, \\
+ \langle b_1C_A b_1 b_2 - b_1C_A b_2 \rangle \rangle, \quad \langle S_1(h) \rangle, \\
+ \langle b_1E_A b_1 b_2 - b_1E_A b_2 \rangle \rangle, \quad \langle S_1(h) \rangle, \\
+ \langle b_1A^N b_1 b_2 - b_1A^N b_2 \rangle \rangle, \quad \langle S_1(h) \rangle.
\end{align*}
\]

Some remarks on (6.8) are necessary. The first line of (6.8), if
nonzero, is $S_{11}(h)$ when $B^N \notin \mathbb{K}$, and $S_{11}(h)$ otherwise. By $A^N b_1 C b_1 b_1 - b_1 C b_1 b_1 A^N = (A^N b_1 C b_1 b_1 - b_1 A^N b_1 C b_1) + (b_1 A^N b_1 C b_1 - b_1 C b_1 b_1 A^N)$, the second line of (6.3) is a $\mathbb{K}$-linear combination of an element of type $S_{10}(h)$ and another element of type $S_{11}(h)$.

Consider a general type-$S_{11}(h)$ element $p = (b_1 A b_1 B - b_1 B b_1 A) \sqsubseteq s$, where the length of $A$ is $a > 0$. When omitting $\sqsubseteq s$,

$$p \equiv b_1 B (b_1 A - (1)^a b_1 b_1) + ((1)^a A) B b_1 B - b_1 B b_1 A = (1)^a A) B b_1 B - b_1 B b_1 A = (1)^a (A^1 b_1 B - b_1 A b_1) \sqsubseteq B_{s_1(h)} = (1)^a (A^1 b_1 B - b_1 A b_1) \sqsubseteq B_{s_1(h)} \in (S_1(h - 2)).$$

So $S_{11}(h)$ is in (6.5) where $k = h$.

By (6.8), we can assume that in $g = (A b_1 B - b_1 B A) \sqsubseteq s$, monomials $A = A^N B = B^N$ and both are $\mathcal{I}^2$-normal. The $\mathcal{I}^2$-reduction to the leading term of $g$ is much the same with that in the proof of Lemma 6 starting from Case (i) there to Case (iii). By induction on the order of the leading term, we get that $S_{11}(h)$ is in (6.5) where $k = h$.

Consider a bracket polynomial $f$ whose multiset of variables is $\mathcal{M}$. In any term of $f$, when all the brackets but one are expanded into two terms by definition, $f$ is changed into a uni-bracket polynomial $g$. Using the Gröbner base $BG[\mathcal{M}]$ to make reduction to $g$ results in a uni-bracket polynomial $h$, where each term is $\mathcal{I}^2$-normal. $h$ must have the lowest order lexicographically among all uni-bracket polynomials equal to $f$. It is called the lowest-representative normal form of $g$, or the uni-bracket normal form of $f$.

Remark. In the above definition of normal forms, we only considered square-free ones. Of course any normal form can be converted to a canonical one, where all representatives are $\mathcal{I}$-normal instead of $\mathcal{I}^2$-normal. Later on, we consider only square-free ones.

Let the size of $\mathcal{M}$ be $m$. Given any partition $(i_1, \ldots, i_k)$ of integer $m$, where each $i_j > 1$, there is a Caianiello expansion of uni-bracket polynomials into bracket polynomials where each term is composed of $k$ brackets of length $i_1, \ldots, i_k$ respectively. Each expansion produces a normal form. It is not clear if such a normal form is of any value.

7. NORMALIZATION OF BRACKET POLYNOMIALS

In this section, we do NOT remove bracket symbols from uni-brackets in multiset of variables $\mathcal{M}$.

Consider attaching an additional vector variable $v_0 \notin \mathcal{M}$ to $\mathcal{M}$ to form a bigger multiset $\mathcal{M}$. Let $v_0 \prec \mathcal{M}$ in $\mathcal{M}$. In the procedure of obtaining $R_{\mathcal{I}}^2$ by the $\mathcal{I}^2$-reduction of $b_1[A]$ in the proof of Theorem 12 or in more details, in the proof of Theorem 4 if the input is $v_0[A]$ where $A \in Q_m[\mathcal{M}]$, then among the Gröbner base of $\mathcal{I}^2[\mathcal{M}]$, only those elements in $G^{\mathcal{I}}[\mathcal{M}]$ are needed in $\mathcal{I}^2$-reduction.

The reduction of $v_0[A]$ is a procedure of recursively doing $\mathcal{I}^2$-reductions to the leader of the bracket polynomial obtained from previous $\mathcal{I}^2$-reductions to $[A]$. At any instance, the representative of a bracket monomial in reduction is the leader of the bracket monomial. The reduction results in a $\mathbb{K}$-linear combination of monomials of the form $v_0[Y, z_1] \cdot \ldots \cdot [Y, z_k]$, where $z_i$ is ascending for every $i$. After removing $v_0$ from the result, we get another normal form of uni-bracket $[A]$ in $\mathcal{M}$.

**Theorem 13.** Let $f$ be a square-free bracket polynomial in multiset of variables $\mathcal{M}$. Do the following to $f$:

1. Always select the leader of a bracket as its representative.
2. Rearrange the order of the brackets in the same term, so that the leading variables of the brackets are non-descending.
3. Use (4.3) to normalize the interior of a bracket; it also splits a bracket into two.
4. Use (4.9) to absorb a bracket into the one ahead of it.
5. Use (4.10) to decrease the order of the product of two brackets by lifting a lower-order variable from the second bracket to the first.
6. Use (4.11) to segment a long bracket of type $Y_1 z_1 Y_2 z_2 \ldots Y_m z_m$ into short ones.
7. Once the representative of the leading term of $f$ is $\mathcal{I}^2$-normal, output the leading term, and continue the above $\mathcal{I}^2$-normalization to the remainder of $f$.

The output, called the lowest-leader normal form, or leader-normal form is a bracket polynomial where the representative of each term is up to coefficient of the form $[Y_1 z_1] \cdot \ldots \cdot [Y_1 z_k] \sqsubseteq s$, where each $z_i Y_i$ is ascending, $Y_i Y_2 \ldots Y_m$ and $z_i z_2 \ldots z_k$ are both non-descending, and the leading variable $l_1$ of each $Y_i$ satisfies $l_1 \succ z_i$. Two bracket polynomials are equal if and only if their leader-normal forms are identical.

**Proof.** After operations 1 and 2, for any bracket monomial in the reduction procedure, its representative is also its leader, so that the representative of the leading term of bracket polynomial $f$ is the leader of $f$. Once the leader of $f$ is $\mathcal{I}^2$-normal, it is the leading term of the normal form of vector-variable polynomial $f$ with respect to the Gröbner base $G^2[\mathcal{M}]$. By induction on the order of the output terms from the highest down, we get the uniqueness of the leader-normal form for $f$.

For two equal bracket polynomials, they have identical uni-bracket normal forms, and so have identical leader-normal forms.

From the above proof, we see that the leader-normal form of a bracket polynomial $f$ has the following properties: (1) the representative of any term is the leader of the term; (2) the representative of the leading term is the leading term of the $\mathcal{I}^2$-normal form of vector-variable polynomial $f$.

By Theorem 12 the Gröbner base $BG[\mathcal{M}]$ of $\mathcal{I}^2[\mathcal{M}]$ is composed of $G^2[\mathcal{M}]$ and the $R_{\mathcal{I}}^2[\mathcal{M}] = v_0[\mathcal{M}]$ for all monomials $g$ in the leader-normal forms of bracket polynomials in $\mathcal{M}$, such that $g$s has length $m$. This phenomenon is easy to understand: the $\mathbb{K}$-linear subspace of $[\mathcal{I}^2][\mathcal{M}]$ composed of polynomials whose terms are led by variable $v_0$, is the space
of degree-$(m + 1)$ polynomials of the form $v_0 f \{c\}$, for all bracket polynomials $f$ in $\mathcal{M}$ such that $f_s$ has length $m$. The leader-normal forms are a basis of the $K$-linear space of length-$m$ square-free bracket polynomials in $\mathcal{M}$.

In a leader-normal form, if we commute $Y_i$ and $z_i$ in $[Y, z_i]$, we get another normal form whose terms are up to coefficient of the form $[z_1 Y_1] \cdots [z_k Y_k]$. If we write the left side of the square symbol in the following tableau form, where $Y_{i1} = y_{i1} y_{i2} \cdots y_{ikt}$, we get

$$
\begin{bmatrix}
  z_1 & y_{11} & \cdots & \cdots & y_{1t_1} \\
  z_2 & y_{21} & \cdots & \cdots & y_{2t_2} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  z_k & y_{k1} & \cdots & \cdots & y_{kt_k}
\end{bmatrix}
$$

where (1) each row does not need to have equal length, and it is not required that the length be non-increasing as in Young tableau;
(2) each row is an ascending sequence of variables;
(3) each column is a non-descending sequence of variables;
(4) $y_{it_i} \preceq y_{(i+1)t_i}$ for $1 \leq i < k$.

Such a normal form is called the straight form. Feature (4) above makes this definition stronger than the straight form (or standard form) of Young tableau. In comparison, in classical bracket algebra a bracket monomial is in straight form if and only if the entries are ascending along each row, and non-descending along each column.

The procedure of deriving the straight form of a bracket polynomial is called straightening. Among the formulas used in Theorem 13 for straightening, (1.10) is highly nontrivial and requires further investigation.

Set $Bu$, $aD$ in (1.10) to be new $A, B$ respectively, and let the lengths of $B, C$ be $b, c$.

Then (1.10) can be written succinctly as follows:

$$
\begin{align*}
[Av][BwC] &= [AwCv][B] - (1)^6[AwB^1v][C] \\
&\quad - (1)^6[wCv][AB^1] + (1)^6[wBv][AC] \\
&\quad - [Aw][CvB] \\
\end{align*}
$$

It is called the shuffle formula for bracket normalization.

**Proposition 14.** For any two monomials $A, B$ of length $a, b$ respectively,

$$
\begin{align*}
[AB + BA] &= [AB] + (1)^a([A][B] - [A][B]) \\
[AB - (1)^{a+b}[A][B]] &= (1)^b([A][B] - [A][B]).
\end{align*}
$$

**Proof.**

$$
\begin{align*}
AB + BA &= 2[AB] - (1)^{a+b}[A][B] + 2[B][A] - (1)^b[B][A] \\
&= 2([AB] + [B][A] - (1)^b[B][A]).
\end{align*}
$$

The fundamental $L$-reduction formula (7.2) is a direct consequence of the first identity in (7.2) for $\text{AB} = uD$. The shuffle formula (7.4) is a consequence of the following identity by making left multiplication with $A$ and then applying the bracket operator to both sides of the identity:

$$
\begin{align*}
[v][BwC] + w[CvB] &= (-1)^6[wCv][B^1] - [wCv][B^1] \\
&\quad - wB^1[v][C] + wB^1[v][C].
\end{align*}
$$

The identity can be obtained as follows: by the second identity of (7.2) from right to left, the right side of (7.3) equals

$$
\begin{align*}
[2^a &\{(BwC) + vBwC\} - (1)^{b+c} (B^1vCw + wB^1vC)] \\
&= [vBwC] - (1)^{b+c} [vCwB] \\
&\quad - (1)^{b+c} [wB^1vC] + w[CvB] \\
&= v[BwC] + w[CvB].
\end{align*}
$$

The shuffle formula can be further generalized. In monomial $[Av][BwC]$, let the leading variable and trailing variable of any sequence $F$ be $t_F$ and $t_F$ respectively. Assume $I_A \preceq I_B$, and $I_A \succ t_D$, and $I_B \succ t_C$. Further assume $I_A \succ v \succ w$. Then $Z^a$-reductions can be made to $[Av][BwC]$ to decrease its leader, leading to the following result:

**Proposition 15.** Let the lengths of monomials $A, B, C, D$ be $a, b, c, d$ respectively. Then

$$
\begin{align*}
[Av][BwC] &= [vDBw][AC] - (1)^{b+c}[vDCw][AB] \\
&\quad - (1)^d[Av][D^1vBC] - [vD][Av][BC] \\
&\quad - (1)^{d+2}[Av][B^1][C][D] + [Av][CvD][B].
\end{align*}
$$

**Proof.**

$$
\begin{align*}
4\text{[Av][BwC]} &= A^x DBwC - (1)^{b+c} A^x DC^i wB^j \\
&\quad - (1)^{a+b+c+d} D^j vA^1 wC^i wB^j \\
&\quad - \{(vDBw + (1)^{d+2} wB^1D^1v)[AC] \\
&\quad - (1)^{b+c} ((1)^d vDC^i w((1)^d vCD^1v) AB) \\
&\quad - (1)^{b+d} AwB^1vC + (1)^{b+c} Aw^1D^1vB \\
&\quad - (1)^{a+d} [vBwC - (1)^{b+c} C^i vB^j A] \\
&\quad + (1)^{a+d} (BCD^i v - D^1vCB) wA^i \\
&\quad + (1)^{b+d} Aw(BC^1vB^j - B^1D^1vC^j) \\
\end{align*}
$$

By (7.2),

$$
\begin{align*}
(1)^d (BCD^i v - D^1vCB) &= (1)^d BCD^i v - (1)^{b+c} vDC^i B^j \\
&\quad + (1)^{d+2} (D^1v (C^i B^j + C^i B^j)),
\end{align*}
$$

we get from (7.5) the following:

$$
\begin{align*}
4\text{[Av][BwC]} &= 4\{vDBw][AC] - (1)^{b+c}[vDCw][AB] \\
&\quad + (1)^{b+c} (BCD^i v - (1)^{b+c} vDC^i B^j \\
&\quad + (1)^{d+2} D^j vB[C^i] + (1)^{d+2} D^j vC^i B^j) - (1)^{b+c} wA^i \\
&\quad + (1)^d Aw - (1)^b vDC^i B^j + (1)^b B^i C^j vD \\
&\quad + (1)^d C^j vB^j - (1)^b B^i D^1vC^j
\end{align*}
$$
we get the following shuffle formula, also called product as follows. For any four vectors $v\wedge w\wedge c\wedge b$, we have another formula for straightening. By
\begin{align*}
0 &= (av)\vee (dbwc) \\
&= [avb][wbc] - [avb][dwc] + [awv][dbc] \quad (7.7) \\
&+ [adb][vwc] - [adb][vbc] + [abw][vde],
\end{align*}
where “$\vee$” is the dual of the exterior product called the meet product \cite{14}, we get the following shuffle formula, also called von der Waerden relation:
\begin{align*}
\begin{bmatrix}
  a & v & d \\
  b & w & c
\end{bmatrix} &= \begin{bmatrix}
  a & v & b \\
  d & w & c
\end{bmatrix} + \begin{bmatrix}
  a & v & w \\
  b & d & c
\end{bmatrix}
\end{align*}
(1) The first line commutes $d$ of the first row and one of the first two vectors of the second row; (2) the second line commutes $v$ of the first row and one of the first two vectors of the second row; (3) the last line commutes $vd$ of the first row and the first two vectors of the second row.

For straightening in classical bracket algebra, there are other operations besides $\vee$. When $a \succ b$, we only need to commute the two brackets. When $d \succ c$ while $a \preceq b$ and $v \preceq w$, we have another formula for straightening. By
\begin{align*}
0 &= (av)\vee (dbwc) \\
&= [avb][wbc] - [avb][dwc] + [awv][dbc] - [avc][vde],
\end{align*}
we get the following Grassmann-Plücker relation:
\begin{align*}
\begin{bmatrix}
  a & v & d \\
  b & w & c
\end{bmatrix} &= \begin{bmatrix}
  a & v & b \\
  d & w & c
\end{bmatrix} + \begin{bmatrix}
  a & v & w \\
  b & d & c
\end{bmatrix} + \begin{bmatrix}
  a & v & c \\
  b & w & d
\end{bmatrix}.
\end{align*}
The last vector $d$ of the first row commutes in turn with every vector of the second row.

8. CONCLUSION

In the bottom-up approach to manipulating brackets, long brackets are expanded into basic ones by Caianiello expansion, in the end only brackets of length 2 and 3 are left for further algebraic manipulations. This approach proves to be inefficient in practice, despite the fact that there are straightening algorithms for polynomials of basic invariants \cite{3, 10}.

Uni-bracket polynomials provide a top-down approach to manipulating brackets. Given a bracket polynomial, by “ungrading”, each bracket but one in every term is expanded into a vector-variable binomial, and the bracket polynomial is changed into a uni-bracket one. Algebraic manipulations of uni-bracket polynomials can take full advantage of the associativity of the vector-variable product and the symmetries within a uni-bracket. The Gröbner base $\mathcal{G}([\mathcal{M}])$ provided by this paper further fulfills the arsenal of symbolic manipulations on uni-bracket polynomials.

The last section of this paper suggests a third approach to manipulating brackets by algebraic manipulations directly upon the input brackets. To establish this approach there are many research topics ahead: division among bracket polynomials, properties of principal ideals, bracket polynomial factorization, and simplification by reducing the number of terms, etc. This seems to be a promising approach.
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