Generalized Radar 4-Coordinates and Equal-Time Cauchy Surfaces for Arbitrary Accelerated Observers.

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Abstract

All existing 4-coordinate systems centered on the world-line of an accelerated observer are only locally defined like it happens for Fermi coordinates both in special and general relativity. As a consequence, it is not known how non-inertial observers can build equal-time surfaces which a) correspond to a conventional observer-dependent definition of synchronization of distant clocks; b) are good Cauchy surfaces for Maxwell equations. Another type of coordinate singularities generating the same problems are those connected to the relativistic rotating coordinate systems used in the treatment of the rotating disk and the Sagnac effect.

We show that the use of Hamiltonian methods based on 3+1 splittings of space-time allows to define as many observer-dependent globally defined radar 4-coordinate systems as nice foliations of space-time with space-like hyper-surfaces admissible according to Møller (for instance only differentially rotating relativistic coordinate system, but not the rigidly rotating ones of non-relativistic physics, are allowed). All these conventional notions of an instantaneous 3-space for an arbitrary observer can be empirically defined by introducing generalizations of Einstein convention for clock synchronization in inertial frames. Each admissible 3+1 splitting has two naturally associated congruences of time-like observers: as a consequence every 3+1 splitting gives rise to non-rigid non-inertial frames centered on anyone of these observers. Only for the Eulerian observers the simultaneity leaves are orthogonal to the observer world-line.

When there is a Lagrangian description of an isolated relativistic system, its reformulation as a parametrized Minkowski theory allows to show that all the admissible synchronization conventions are gauge equivalent, as it also happens in canonical metric and tetrad gravity, where, however, the chrono-geometrical structure of space-time is dynamically determined.

The framework developed in this paper is not only useful for a consistent description of the rotating disk, but is also needed for the interpretation of the future ACES experiment on the synchronization of laser cooled atomic clocks and for the synchronization of the clocks on the three LISA spacecrafts.

Key words: synchronization of clocks, accelerated observers, radar coordinates, one-way velocity of light, space navigation

March 24, 2022
I. INTRODUCTION

A physical observer is mathematically described by a future-oriented time-like world-line $\gamma$, carrying an ideal standard clock measuring proper time and a tetrad field [1]. Both in the Minkowski space-time of special relativity and in the set of globally hyperbolic space-times, i.e. admitting a global time function, compatible with Einstein general relativity, the Lorentz signature of the 4-metric allows the observer to identify the local light-cone in each point of $\gamma$ and to speak of events space-like with respect to a point of $\gamma$. In the case of special relativity an inertial observer can use the clock moving along the straight-line $\gamma$ to verify the validity of the two independent postulates of the theory: in every inertial system the round-trip or two-ways velocity of light A) is the same ($c$) and B) is isotropic.

However, every physical observer has no natural notion of instantaneous 3-space (of a present in colloquial terms) at each point of $\gamma$ with all the clocks synchronized with the one on $\gamma$ allowing the introduction of an associated notion of spatial distance and an associated definition of one-way velocity of light from $\gamma$ to every other time-like world-line whose clock has been synchronized with the one on $\gamma$.

In special relativity, usually, one considers only global rigid inertial reference frames associated with an inertial observer whose world-line $\gamma$ is a straight-line playing the role of the time axis and with a point taken as origin of Cartesian 4-coordinates $x^\mu$ for Minkowski space-time. By means of an adiabatic slow transport of clocks, one identifies the $x^\alpha = \text{const.}$ space-like hyper-planes as the instantaneous 3-space with all the clocks synchronized [2]. The same notion of instantaneous 3-space is arrived at if the inertial observer $\gamma$ sends rays of light to another time-like observer $\gamma_1$, who reflects them back towards $\gamma$. Given the emission ($\tau_i$) and adsorption ($\tau_f$) times on $\gamma$, the point $P$ of reflection on $\gamma_1$ is assumed to be simultaneous with the point $Q$ on $\gamma$ where $\tau_P \overset{def}{=} \tau_Q = \tau_i + \frac{1}{2} (\tau_f - \tau_i) = \frac{1}{2} (\tau_i + \tau_f)$. With this so-called Einstein’s $\frac{1}{2}$ convention for the synchronization of distant clocks [3] again the instantaneous 3-space is the space-like hyper-plane $x^\alpha = \text{const.}$ orthogonal to $\gamma$, the point $Q$ is the midpoint between the emission and adsorption points and, since $\tau_P - \tau_i = \tau_f - \tau_P$, the one-way velocity of light between $\gamma$ and every $\gamma_1$ is isotropic and equal to the round-trip velocity of light $c$.

The relativity principle then leads to the identification of the kinematical Poincare’ group as the set of transformations connecting all the possible inertial observers with the appearance of the standard effects of time dilation and length contraction. As a consequence, the
theoretical relevance of the inertial frames seems to suggest that space-like hyper-planes orthogonal to $\gamma$ are a natural notion of *instantaneous 3-space*. However, as we shall see, there are many other geometrical possibilities which simply do not correspond to rigid inertial frames but to non-rigid non-inertial ones (the only ones existing in general relativity due to a global interpretation of the equivalence principle). Therefore, each observer has to *stipulate some convention* defining a possible notion of instantaneous 3-space and to study the transformation rules from a convention to every other possible one. As a consequence, the notion of instantaneous 3-space is both *observer-dependent and conventional*.

Inertial frames are a limiting notion and, also disregarding general relativity, all the observers on the Earth are non-inertial. According to the IAU 2000 Resolutions [4], for the physics in the solar system one can consider the *Solar System Barycentric Celestial Reference Frame* (with the axes identified by fixed stars (quasars) of the Hyppparcos catalog) as a quasi-inertial frame. Instead the *Geocentric Celestial Reference Frame*, with origin in the center of the geoid, is a non-inertial frame whose axes are *non-rotating* with respect to the Solar Frame. Every frame fixed on the surface of the Earth is both non-inertial and rotating.

Therefore, we need a definition of *an instantaneous 3-space* for a non-inertial time-like observer, first in special relativity and then in general relativity, and of what can be a non-rigid non-inertial reference frame. In particular we have to define an *accelerated coordinate system* and to face the problem of rotating frames.

Traditionally, given an arbitrary time-like observer with world-line $\gamma$ and unit 4-velocity $u^{\mu}(\tau)$ ($\tau$ is the proper time measured by the clock of the observer), one introduces the *Fermi coordinates* [5, 6, 7] by considering as *instantaneous 3-space* the space-like hyper-planes orthogonal to the 4-velocity in each point of $\gamma$ [8]. While the proper time $\tau$ labels the hyper-planes, the spatial coordinates are defined by using three orthogonal 3-geodesics emanating from $\gamma$ on the hyper-planes. However, this nice local geometric construction defines 4-coordinates only locally in a world-tube around $\gamma$, whose extension is determined by the acceleration radii [9] where hyper-planes corresponding to different values of $\tau$ intersect. Also Martzke-Wheeler [12] and Pauri-Vallisneri [13] attempts, trying to generalize Einstein convention to non-inertial observers (and then to observers in general relativity), has the same singularities of Fermi coordinates as shown in Ref.[14].

To this type of coordinate-singularities we have to add the singularities shown by all the
rotating coordinate systems (the problem of the rotating disk): the 4-metric expressed in these coordinates has pathologies at the distance $R$ from the rotation axis where $\omega R = c$ with $\omega$ being the constant angular velocity of rotation [15]. Again, given the unit 4-velocity field of the points of the rotating disk, there is no notion of an instantaneous 3-space orthogonal to the associated congruence of time-like observers, due to the non-zero vorticity of the congruence [16]. Moreover, an attempt to use Einstein convention to synchronize the clocks on the rim of the disk fails and one finds a synchronization gap (see Ref.[20] and the bibliography of Ref.[21] for these problems and for the Sagnac effect).

In conclusion this so-called 1+3 point of view (threading splitting) of the accelerated observer is not able to build good simultaneity hyper-surfaces, which, besides defining a conventional instantaneous 3-space, are also good Cauchy surfaces for Maxwell equations: only in this way we can find meaningful solutions of these equations and to control the conservation laws.

The problems quoted till now are not academic, but are becoming relevant for measurements of one-way time transfer between Earth and satellites with laser cooled high precision clocks [22], for future space navigation [23], for the Sagnac corrections in the Global Positioning System [24], for the interpretation of the future measurements of gravito-magnetism [25, 26] from Gravity Probe B, for LISA [27].

In this paper we show that the 3+1 point of view (slicing splitting) used in the Hamiltonian treatment of dynamics, allows to find a general solution of the previous problems.

First of all, in Section II we shall remind which are the admissible 4-coordinate transformations in Minkowski space-time (frame-preserving diffeomorphisms) according to Møller [28]: with suitable restrictions at spatial infinity they allow the definition of general notions of simultaneity as nice foliations with space-like leaves, which replace the rigid inertial reference frames with non-rigid non-inertial frames. Note that for non-inertial frames there is no relativity principle and no kinematical transformation group, like it happens in general relativity, where the principle of general covariance implies the replacement of the Poincare’ group with the full diffeomorphism group. In special relativity the frame-preserving diffeomorphisms are playing the same role in replacing the Poincare’ group.

Then we consider an arbitrary time-like observer with world-line $\gamma$ ($x^\mu(\tau)$ are the coordinates in an inertial frame and $\tau$ is the observer proper time), which intersects the leaves of every admissible 3+1 splitting just in a point but in general is not orthogonal to any leaf.
Given anyone of the admissible 3+1 foliations, the world-line $\gamma$ is used to build an observer-dependent, globally defined, Lorentz-scalar radar 4-coordinate system by using the observer proper time $\tau$ to label the leaves $\Sigma_\tau$ of the foliation and by means of curvilinear 3-coordinates $\sigma^r$ [29], with origin on the world-line $\gamma$, on the leaves. The inverse of the coordinate transformation $z^\mu \mapsto \sigma^A(z) = (\tau, \vec{\sigma})$, namely $\sigma^A \mapsto z^\mu(\tau, \vec{\sigma})$, gives the embedding of the leaves $\Sigma_\tau$ in Minkowski space-time as seen from the inertial frame. The hyper-surfaces $\Sigma_\tau$ are possible instantaneous 3-spaces, i.e. they are globally defined simultaneity hyper-surfaces with all the clocks synchronized and also Cauchy surfaces.

Let us remark that when we have a Lagrangian description of an isolated relativistic system, we can arrive to a Lagrangian depending also on the embedding $z^\mu(\tau, \vec{\sigma})$ and being explicitly reparametrization invariant under frame-preserving diffeomorphisms (the parametrized Minkowski theories of Refs.[30] and of the Appendix of Ref.[31]; see Subsection C of Section II). Therefore, in this way we get a special relativistic notion of general covariance, which is a restriction of the general covariance under arbitrary diffeomorphisms of general relativity to the frame-preserving ones.

This implies that the transition from an admissible 3+1 splitting to another one, i.e. the change of the convention of synchronization of distant clocks and of instantaneous 3-space, can be rephrased as a gauge transformation.

Let us also note that many 3+1 splittings can be shown to agree with the locality principle: an accelerated observer at each instant along its world-line is physically equivalent to an otherwise identical momentarily comoving inertial observer, namely a non-inertial observer passes through a continuous infinity of hypothetical momentarily comoving inertial observers [32].

As a byproduct of the study of Møller admissibility conditions it will be shown that global rigid rotations are not allowed in special (and also general) relativity: on simultaneity and Cauchy surfaces we must always have differential rotations. Instead global rigid translational accelerations are allowed. We also give the simplest set of rotating 4-coordinates without pathologies, which can be used in the treatment of the one-way time delay of signals between an Earth station and a satellite (see Section VID of Ref.[21]).

Moreover, in Appendix A, we solve the inverse problem of finding admissible 3+1 splittings associated to a given unit 4-velocity field with non-zero vorticity, like it happens with the rotating disk. This allows to define genuine instantaneous 3-spaces with synchronized clocks for a rotating disk, so that it is possible to give a description of the Sagnac effect
without synchronization gap (see Sections VIB and C of Ref.[21]).

In conclusion the 3+1 point of view allows to find an infinite number of admissible conventions for the definition of an instantaneous 3-space, which can be used as a good Cauchy surface, of an arbitrary accelerated observer. The pathologies of the Fermi coordinates are avoided because the simultaneity surfaces, with all the clocks synchronized, are not hyperplanes orthogonal to the world-line of the observer.

Moreover, in Section III we will show that the name radar coordinates is justified, because they correspond to generalizations of Einstein $\frac{1}{2}$ convention [3, 28] [34]. In this Section we also outline the inverse problem of how to build operationally this type of coordinates by using a cluster of spacecrafts like the one used in the Global Positioning System (GPS) [24].

Then we make some concluding remarks regarding the extension of these results to general relativity (see Ref.[40] for the status of the Hamiltonian formulation of metric and tetrad gravity), where each solution of Einstein’s equations dynamically determines which 3+1 splittings of the globally hyperbolic space-time can be associated to it, and some comments on which problems have still to be clarified at the post-Newtonian level.

We refer to Ref.[21] for an extended discussion and a rich bibliography on all these problems, for a treatment of the rotating disk and of the Sagnac effect and for the determination of the time-delay of signals from the Earth to a satellite (one-way velocity of light is required) within the 3+1 point of view and, finally, for Maxwell equations in non-inertial frames.
II. THE 3+1 POINT OF VIEW AND ACCELERATED OBSERVERS.

Let us consider the 3+1 splittings of Minkowski space-time associated to its foliations with arbitrary space-like hyper-surfaces and not only with space-like hyper-planes. Each of these hyper-surfaces is both a simultaneity surface and a Cauchy surface for the equations of motion of the relativistic systems of interest. Having given a notion of simultaneity, there will be associated notions of instantaneous 3-space, synchronization of distant clocks, spatial length and one-way velocity of light. After the choice of a foliation, i.e. of a notion of simultaneity, we can determine, as we shall see, which are the non-inertial observers compatible with that notion of simultaneity.

First of all we must find which 3+1 splittings of Minkowski space-time are geometrically allowed as nice foliations whose leaves are space-like hyper-surfaces. This leads to Møller admissible coordinate transformations (Subsection A).

Then let us consider an arbitrary time-like observer whose world-line $\gamma$ intersects each leaf of an admissible 3+1 splitting in a point. The world-line $\gamma$ can be used as a centroid to define observer-adapted Lorentz-scalar radar 4-coordinates (Subsection B). As a consequence, each admissible 3+1 splitting may be chosen as a conventional notion of instantaneous 3-space for an accelerated observer. In Subsection C we show that every isolated relativistic system with a Lagrangian description can be reformulated as a parametrized Minkowski theory in which all the conventions are gauge equivalent.

In Subsection D we identify the 3+1 splittings which agree with the locality hypothesis, namely in which it is evident that the accelerated observer can be visualized as a sequence of comoving inertial observers. Here we state which is the form assumed by Møller admissibility conditions for such 3+1 splittings. As a byproduct we show that, while we can have rigid non-inertial frames with arbitrary translational acceleration, rigidly rotating relativistic frames do not exist: globally defined rotating frames must necessarily have differential rotations. In Subsection E we identify the simplest global relativistic rotating frames as a family of 3+1 splittings with space-like hyper-planes on which there are differentially rotating coordinates.

A. Møller Admissible Coordinates.

Given an inertial system with Cartesian 4-coordinates $x^\mu$ in Minkowski space-time and with the $x^0 = \text{const.}$ simultaneity hyper-planes, Møller, in Chapter VIII, Section 88 of Ref.[28] (see also Hilbert [41] and Havas [39]), defines the admissible coordinates transfor-
motions $x^\mu \mapsto y^\mu = f^\mu(x)$ [with inverse transformation $y^\mu \mapsto x^\mu = h^\mu(y)$] as those transformations whose associated metric tensor $g_{\mu\nu}(y) = \frac{\partial h^{\alpha}(y)}{\partial y^\mu} \frac{\partial h^{\beta}(y)}{\partial y^\nu} \eta_{\alpha\beta}$ satisfies the following conditions (it can be shown that the inverse metric $g^{\mu\nu}(y)$ satisfies the same conditions)

$$\epsilon g_{oo}(y) > 0,$$

$$\epsilon g_{ii}(y) < 0, \quad \left| \begin{array}{cc} g_{ii}(y) & g_{ij}(y) \\ g_{ji}(y) & g_{jj}(y) \end{array} \right| > 0, \quad \epsilon \det [g_{ij}(y)] < 0,$$

$$\Rightarrow \det [g_{\mu\nu}(y)] < 0.$$  \hspace{1cm} (2.1)

These are the necessary and sufficient conditions for having $\frac{\partial h^{\mu}(y)}{\partial y^\rho}$ behaving as the velocity field of a relativistic fluid, whose integral curves, the fluid flux lines, are the world-lines of time-like observers. Eqs. (2.1) say:

i) the observers are time-like because $\epsilon g_{oo} > 0$;

ii) that the hyper-surfaces $y^o = f^o(x) = \text{const.}$ are good space-like simultaneity surfaces.

Moreover we must ask that $g_{\mu\nu}(y)$ tends to a finite limit at spatial infinity on each of the hyper-surfaces $y^o = f^o(x) = \text{const.}$ If, like in the ADM canonical formulation of metric gravity [31, 42], we write $g_{oo} = \epsilon (N^2 - g_{ij} N^i N^j)$, $g_{oi} = g_{ij} N^j$ introducing the lapse $(N)$ and shift $(N^i)$ functions, this requirement says that the lapse function (i.e. the proper time interval between two nearby simultaneity surfaces) and the shift functions (i.e. the information about which points on two nearby simultaneity surfaces are connected by the so-called evolution vector field $\frac{\partial h^{\mu}(y)}{\partial y^\rho}$) must not diverge at spatial infinity. This implies that at spatial infinity on each simultaneity surface there is no asymptotic either translational or rotational acceleration [43] and the asymptotic line element is $ds^2 = g_{\mu\nu}(y) dy^\mu dy^\nu \rightarrow \text{spatial infinity} \quad \epsilon \left( F^2(y^o)(dy^o)^2 + 2 G_i(y^o)(dy^o dy^i - d\vec{y}^2) \right)$. But this would break manifest covariance unless $F(y^o) = 1$ and $G_i(y^o) = 0$. As a consequence, the simultaneity surfaces must tend to space-like hyper-planes at spatial infinity.

In this way all the admissible notions of simultaneity of special relativity are formalized as 3+1 splittings of Minkowski space-time by means of foliations whose leaves are space-like hyper-surfaces tending to hyper-planes at spatial infinity. Let us remark that admissible coordinate transformations $x^\mu \mapsto y^\mu = f^\mu(x)$ constitute the most general extension of the Poincare’ transformations $x^\mu \mapsto y^\mu = a^\mu + \Lambda^\mu_\nu x^\nu$ compatible with special relativity.
An important sub-group of Møller admissible transformations consists the *frame-preserving* diffeomorphisms: \( x^\alpha \mapsto y^\alpha = f^\alpha(x^\alpha, \vec{x}), \) \( \vec{x} \mapsto \vec{y} = \vec{f}(\vec{x}) \), with inverse transformations \( x^\alpha = h^\alpha(y^\alpha, \vec{y}), \) \( \vec{x} = \vec{h}(\vec{y}) \). Let us remark that the asymptotic conditions at spatial infinity restrict Møller admissible transformations to those which have the behavior \( \frac{\partial h^\alpha}{\partial y^\alpha} \to 1, \frac{\partial h^\alpha}{\partial y^i} \to 0, \frac{\partial h^i}{\partial y^j} \to \delta^i_j \) at spatial infinity.

**B. From Møller Admissible Coordinates to Radar 4-Coordinates adapted to an Arbitrary Accelerated Observer.**

It is then convenient to describe [30, 31] the simultaneity surfaces of an admissible 3+1 splitting of Minkowski space-time with *adapted Lorentz-scalar admissible coordinates* \( x^\mu \mapsto \sigma^A = (\tau, \vec{\sigma}) = f^A(x) \) [with inverse \( \sigma^A \mapsto x^\mu = z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma}) \)] such that:

i) the scalar time coordinate \( \tau \) labels the leaves \( \Sigma_\tau \) of the foliation (\( \Sigma_\tau \approx R^3 \));

ii) the scalar curvilinear 3-coordinates \( \vec{\sigma} = \{\sigma^r\} \) on each \( \Sigma_\tau \) are defined with respect to the world-line \( \gamma \) of an arbitrary time-like centroid \( x^\mu(\tau) \) chosen as their origin;

iii) if \( y^\mu = f^\mu(x) \) is any admissible coordinate transformation describing the same foliation, i.e. if the leaves \( \Sigma_\tau \) are also described by \( y^\alpha = f^\alpha(x) = \text{const.} \), then, modulo reparametrizations, we must have \( y^\mu = f^\mu(z(\tau, \vec{\sigma})) = \tilde{f}^\mu(\tau, \vec{\sigma}) = A^\mu_A \sigma^A \) with \( A^\alpha_A = \text{const.}, A^\alpha_r = 0, \) so that we get \( y^\alpha = \text{const.}, \tau, y^i = A^i_A(\tau, \vec{\sigma}) \sigma^A \). Therefore, modulo reparametrizations, the \( \tau \) and \( \vec{\sigma} \) adapted admissible coordinates are *intrinsic coordinates*, which are mathematically allowed as charts in an enlarged atlas for Minkowski space-time taking into account the extra structure of the admissible 3+1 splittings. They are called *radar-like 4-coordinates* (see Section III for the justification of this name) and, probably, they were introduced for the first time by Bondi [44]. The use of these Lorentz-scalar adapted coordinates allows to make statements depending only on the foliation but not on the 4-coordinates \( y^\mu \) used for Minkowski space-time.

*If we identify the centroid \( x^\mu(\tau) \) with the world-line \( \gamma \) of an arbitrary time-like observer and \( \tau \) with the observer proper time, we obtain as many globally defined observer-dependent Lorentz-scalar radar 4-coordinates for an accelerated observer as admissible 3+1 splittings of Minkowski space-time and each 3+1 splitting can be viewed as a conventional choice of an instantaneous 3-space and of a synchronization prescription for distant clocks. The world-line \( \gamma \) is not in general orthogonal to the simultaneity leaves and Einstein \( \frac{1}{2} \) convention is suitably generalized (see Section III).*
The simultaneity hyper-surfaces $\Sigma_{\tau}$ are described by their embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ in Minkowski space-time $[(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma}), \mathbb{R}^3 \mapsto \Sigma_{\tau} \subset M^4]$ and the induced metric is $g_{AB}(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) z^\nu_B(\tau, \vec{\sigma}) \eta_{\mu\nu}$ with $z^\mu_A = \partial z^\mu / \partial \sigma^A$ [45]. Since the vector fields $z^\mu_r(\tau, \vec{\sigma})$ are tangent to the surfaces $\Sigma_{\tau}$, the time-like vector field of normals $l^\mu(\tau, \vec{\sigma})$ is proportional to $\epsilon^{\alpha\beta\gamma} z^\alpha_1(\tau, \vec{\sigma}) z^\beta_2(\tau, \vec{\sigma}) z^\gamma_3(\tau, \vec{\sigma})$. Instead the time-like evolution vector field is $z^\mu_{\tau}(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma}) + N^r(\tau, \vec{\sigma}) z^\mu_r(\tau, \vec{\sigma})$, so that we have $dz^\mu(\tau, \vec{\sigma}) = z^\mu_{\tau}(\tau, \vec{\sigma}) d\tau + z^\mu_r(\tau, \vec{\sigma}) d\sigma^r = N(\tau, \vec{\sigma}) d\tau l^\mu(\tau, \vec{\sigma}) + [N^r(\tau, \vec{\sigma}) d\tau + d\sigma^r] z^\mu_r(\tau, \vec{\sigma})$.

Since the 3-surfaces $\Sigma_{\tau}$ are equal time 3-spaces with all the clocks synchronized, the spatial distance between two equal-time events will be $dl_{12} = \int_1^2 dl \sqrt{g_{rs}(\tau, \vec{\sigma}(l)) \frac{d\sigma^r(l)}{dl} \frac{d\sigma^s(l)}{dl}} (\vec{\sigma}(l))$ is a parametrization of the 3-geodesic $\gamma_{12}$ joining the two events on $\Sigma_{\tau}$). Moreover, by using test rays of light we can define the one-way velocity of light between events on different $\Sigma_{\tau}$’s.

Therefore the accelerated observer plus one admissible 3+1 splitting with the observer-dependent radar 4-coordinates define a non-rigid non-inertial reference frame whose time axis is the world-line $\gamma$ of the observer and whose instantaneous 3-spaces are the simultaneity hyper-surfaces $\Sigma_{\tau}$.

The main property of each admissible foliation with simultaneity surfaces is that the embedding of the space-like leaves of the foliation automatically determine two time-like vector fields and therefore two congruences of (in general) non-inertial time-like observers to be used to define non-inertial frames with the given simultaneity surfaces:

i) The time-like vector field $l^\mu(\tau, \vec{\sigma})$ of the normals to the simultaneity surfaces $\Sigma_{\tau}$ (by construction surface-forming, i.e. irrotational), whose flux lines are the world-lines of the so-called (in general non-inertial) Eulerian observers. The simultaneity surfaces $\Sigma_{\tau}$ are (in general non-flat) Riemannian 3-spaces in which the physical system is visualized and in each point the tangent space to $\Sigma_{\tau}$ is the local observer rest frame of the Eulerian observer through that point. This 3+1 viewpoint is called hyper-surface 3+1 splitting.

ii) The time-like evolution vector field $z^\mu_{\tau}(\tau, \vec{\sigma}) \sqrt{g_{\tau\tau}(\tau, \vec{\sigma})}$, which in general is not surface-forming (i.e. it has non-zero vorticity like in the case of the rotating disk). The observers associated to its flux lines have the local observer rest frames, the tangent 3-spaces orthogonal to the evolution vector field, not tangent to $\Sigma_{\tau}$: there is no notion of 3-space for these observers (1+3 point of view or threading splitting) and no visualization of the physical system in large. However these observers can use the notion of simultaneity associated to the embedding $z^\mu(\tau, \vec{\sigma})$, which determines their 4-velocity. In this way we get non-inertial frames centered on these observers, whose world-lines are not orthogonal to the
simultaneity surfaces. This 3+1 viewpoint is called *slicing 3+1 splitting*.

**C. Parametrized Minkowski Theories.**

As said in the Introduction, when we have a Lagrangian description of an isolated system, it can be extended to a *parametrized Minkowski theory* [30, 31]. In this approach, besides the configuration variables of the isolated system, there are the embeddings $z^{\mu}(\tau, \vec{\sigma})$ as extra *gauge configuration* variables in a suitable Lagrangian determined in the following way. Given the Lagrangian of the isolated system in the Cartesian 4-coordinates of an inertial system, one makes the coupling to an external gravitational field and then replaces the external 4-metric with $g_{\mu \nu} = z^{\mu}_{A} \eta_{\mu \nu} z^{\nu}_{B}$.

Therefore the resulting Lagrangian depends on the embedding through the associated metric $g_{AB}$. It can be shown that, due to the presence of a *special-relativistic type of general covariance* (reparametrization invariance of the action under frame-preserving diffeomorphisms), the transition from a foliation to another one (i.e. a change of the notion of simultaneity) is a *gauge transformation* of the theory. Therefore, in parametrized Minkowski theories the *conventionalism of simultaneity* is rephrased as a *gauge problem* (in a way different from Refs. [36, 38]), i.e. as the *arbitrary choice of a gauge fixing selecting a well defined notion of simultaneity among those allowed by the gauge freedom*. Moreover, for every isolated system there is a preferred notion of simultaneity, namely the one associated with the 3+1 splitting whose leaves are the *Wigner hyper-planes* orthogonal to the conserved 4-momentum of the system: *this preferred simultaneity, intrinsically selected by the isolated system, identifies the intrinsic inertial rest frame centered on an inertial observer having the system global 4-velocity and leads to the Wigner-covariant rest-frame instant form of dynamics* [30].

Besides scalar and spinning positive-energy particles, Klein-Gordon, electro-magnetic, Yang-Mills and Dirac fields have been reformulated as parametrized Minkowski theories [30, 31].

This same state of affairs is also present in Hamiltonian metric and tetrad gravity: the change of the notion of simultaneity is a *gauge transformation* [31, 40, 46, 47]. The distinguishing property of general relativity is that the solutions of Einstein's equations determine *dynamically* which notions of simultaneity are allowed [47].
D. Admissible 4-Coordinates and the Locality Hypothesis: Non-Existence of Rigid Rotating Reference Frames.

Let us now consider a class of 4-coordinate transformations which implements the idea of accelerated observers as sequences of comoving observers (the locality hypothesis) [48] and let us determine the conditions on the transformations to get a set of admissible 4-coordinates. From now on we shall use Lorentz-scalar radar-like 4-coordinates $\sigma^A = (\tau; \vec{\sigma})$ adapted to the foliation, whose simultaneity leaves are denoted $\Sigma_\tau$.

As we have said, the admissible embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$ [inverse transformations of $x^\mu \mapsto \sigma^A(x)$], defined with respect to a given inertial system, must tend to parallel space-like hyper-planes at spatial infinity. If $l^\mu = l^\mu_\infty = \epsilon^\mu_\tau, l^\mu_\tau = \epsilon$, is the asymptotic normal, let us define the asymptotic orthonormal tetrad $\epsilon^\mu_A, A = \tau, 1, 2, 3$, by using the standard Wigner boost for time-like Poincare' orbits $L^\mu_\nu(l_\infty^\tau, l^\tau_\infty), l^\tau_\infty = (1; \vec{0})$: $\epsilon^\mu_A \equiv L^\mu_\nu(l^\tau_\infty, l^\tau_\infty), \eta_{AB} = \epsilon^\mu_A \eta_{\mu\nu} \epsilon^\nu_B$.

Then a parametrization of the asymptotic hyper-planes is $z^\mu = x^\mu_0 + \epsilon^\mu_A \sigma^A = x^\mu(\tau) + \epsilon^\mu \sigma^r$ with $x^\mu(\tau) = x^\mu_0 + \epsilon^\mu_\tau \tau$ a time-like straight-line (an asymptotic inertial observer). Let us define a family of 3+1 splittings of Minkowski space-time by means of the following embeddings [49]

$$z^\mu(\tau, \vec{\sigma}) = x^\mu_0 + \Lambda^\mu_\nu(\tau, \vec{\sigma}) \epsilon^\nu_A \sigma^A = \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}), \quad F^\mu(\tau, \vec{0}) = 0,$$

$$\tilde{x}^\mu(\tau) = x^\mu_0 + \Lambda^\mu_\nu(\tau, \vec{0}) \epsilon^\nu_\tau \tau,$$

$$F^\mu(\tau, \vec{\sigma}) = [\Lambda^\mu_\nu(\tau, \vec{\sigma}) - \Lambda^\mu_\nu(\tau, \vec{0})] \epsilon^\nu_\tau \tau + \Lambda^\mu_\nu(\tau, \vec{\sigma}) \epsilon^\nu_\sigma^r,$$

$$\Lambda^\mu_\nu(\tau, \vec{\sigma}) \to_{|\vec{\sigma}| \to \infty} \delta^\mu_\nu, \quad \Rightarrow \quad z^\mu(\tau, \vec{\sigma}) \to_{|\vec{\sigma}| \to \infty} x^\mu_0 + \epsilon^\mu_A \sigma^A = x^\mu(\tau) + \epsilon^\mu_\sigma^r,$$

where $\Lambda^\mu_\nu(\tau, \vec{\sigma})$ are Lorentz transformations ($\Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta = \eta_{\alpha\beta}$) belonging to the component connected with the identity of $SO(3, 1)$. While the functions $F^\mu(\tau, \vec{\sigma})$ determine the form of the simultaneity surfaces $\Sigma_\tau$, the centroid $\tilde{x}^\mu(\tau)$, corresponding to an arbitrary time-like observer chosen as origin of the 3-coordinates on each $\Sigma_\tau$, determines how these surfaces are packed in the foliation.
Since the asymptotic foliation with parallel hyper-planes, having a constant vector field \( l^\mu = \epsilon^\mu_\tau \) of normals, defines an inertial reference frame, we see that the foliation (2.2) with its associated non-inertial reference frame is obtained from the asymptotic inertial frame by means of point-dependent Lorentz transformations. As a consequence, the integral lines, i.e. the non-inertial Eulerian observers and (non-rigid) non-inertial reference frames associated to this special family of simultaneity notions, are parametrized as a continuum of comoving inertial observers as required by the locality hypothesis [50].

Let us remark that when an arbitrary isolated system is described by a Minkowski parametrized theory, in which the embeddings \( z^\mu(\tau, \vec{\sigma}) \) are gauge configuration variables, the transition from the description of dynamics in one of these non-inertial reference frames compatible with the locality hypothesis to another arbitrary allowed reference frame, like the one of footnote 10, is a gauge transformation: therefore in this case the locality hypothesis can always be assumed valid modulo gauge transformations.

An equivalent parametrization of the embeddings of this family of reference frames is

\[
z^\mu(\tau, \vec{\sigma}) = x^\mu_o + \epsilon^\mu_B \Lambda^B_A(\tau, \vec{\sigma}) \sigma^A = x^\mu_o + U^\mu_A(\tau, \vec{\sigma}) \sigma^A = \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}),
\]

\[
\tilde{x}^\mu(\tau) = x^\mu_o + U^\mu_\tau(\tau, \vec{0}) \tau,
\]

\[
F^\mu(\tau, \vec{\sigma}) = [U^\mu_\tau(\tau, \vec{\sigma}) - U^\mu_\tau(\tau, \vec{0})] \tau + U^\mu_\tau(\tau, \vec{\sigma}) \sigma^r,
\]  

(2.3)

where we have defined:

\[
\Lambda^B_A(\tau, \vec{\sigma}) = \epsilon^B_\mu \Lambda^\mu_\nu(\tau, \vec{\sigma}) \epsilon^\nu_A, \quad U^\mu_A(\tau, \vec{\sigma}) \eta_{\mu\nu} U^\nu_B(\tau, \vec{\sigma}) = \epsilon^\mu_A \eta_{\mu\nu} \epsilon^\nu_B = \eta_{AB},
\]

\[
U^\mu_A(\tau, \vec{\sigma}) = \epsilon^\mu_B \Lambda^B_A(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} \epsilon^\mu_A,
\]  

(2.4)

where \( \epsilon^B_\mu = \eta_{\mu\nu} \eta^{BA} \epsilon^\nu_A \) are the inverse tetrads.

A slight generalization of these embeddings allows to find Nelson’s [51] 4-coordinate transformation (but extended from \( \vec{\sigma} \)-independent Lorentz transformations \( \Lambda^\mu_\nu = \Lambda^\mu_\nu(\tau) \) to \( \vec{\sigma} \)-dependent ones!) implying Møller rotating 4-metric [52]
\[ z^\mu(\tau, \vec{\sigma}) = x_0^\mu + \epsilon^\mu_A \left[ \Lambda^A_B(\tau, \vec{\sigma}) \sigma^B + V^A(\tau, \vec{\sigma}) \right], \]

\[ V'(\tau, \vec{\sigma}) = \int_0^\tau d\tau_1 \Lambda^r_\tau(\tau_1, \vec{\sigma}) - \Lambda^r_\tau(\tau, \vec{\sigma}) \tau, V'(\tau, \vec{\sigma}) = \int_0^\tau d\tau_1 \Lambda^r_\tau(\tau_1, \vec{\sigma}) - \Lambda^r_\tau(\tau, \vec{\sigma}) \tau. \]

(2.5)

However, in general these 3+1 splittings do not produce a metric satisfying Eqs.(2.1).

Let us study the conditions imposed by Eqs.(2.1) on the foliations of the type (2.3) (for the others it is similar) to find which ones correspond to admissible 4-coordinate transformations. We shall represent each Lorentz matrix \( \Lambda \) as the product of a Lorentz boost \( B \) and a rotation matrix \( R \) to separate the translational from the rotational effects (\( \vec{\beta} = \vec{v}/c \) are the boost parameters, \( \gamma(\vec{\beta}) = 1/\sqrt{1-\vec{\beta}^2}, \vec{\beta}^2 = (\gamma^2-1)/\gamma^2, B^{-1}(\vec{\beta}) = B(-\vec{\beta}); \alpha, \beta, \gamma \) are three Euler angles and \( R^{-1} = R^T \))

\[ \Lambda(\tau, \vec{\sigma}) = B(\vec{\beta}(\tau, \vec{\sigma})) R(\alpha(\tau, \vec{\sigma}), \beta(\tau, \vec{\sigma}), \gamma(\tau, \vec{\sigma})), \]

\[ B^A_B(\vec{\beta}) = \left( \begin{array}{cc} \gamma(\vec{\beta}) & \gamma(\vec{\beta}) \beta^s \\ \gamma(\vec{\beta}) \beta^r & \delta^{rs} + \frac{\gamma(\vec{\beta}) \beta^r \beta^s}{\gamma(\vec{\beta})+1} \end{array} \right), \quad R^A_B(\alpha, \beta, \gamma) = \left( \begin{array}{cc} 1 & 0 \\ 0 & R_s^r(\alpha, \beta, \gamma) \end{array} \right), \]

\[ R(\alpha, \beta, \gamma) = \]

\[ = \left( \begin{array}{ccc} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \beta \cos \gamma \\ -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \beta \sin \gamma \\ \cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \end{array} \right). \]

Then we get

\[ z^\mu(\tau, \vec{\sigma}) = U^\mu_\tau(\tau, \vec{\sigma}) + \partial_\tau U^\mu_A(\tau, \vec{\sigma}) \sigma^A = \]

\[ = U^\mu_\tau(\tau, \vec{\sigma}) + U^\mu_B(\tau, \vec{\sigma}) \Omega^B_A(\tau, \vec{\sigma}) \sigma^A, \]

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\[ z_\nu^\mu(\tau, \vec{\sigma}) = U_\nu^\mu(\tau, \vec{\sigma}) + \partial_\tau U_\nu^\mu(\tau, \vec{\sigma}) \sigma^A = \]

\[ = U_\nu^\mu(\tau, \vec{\sigma}) + U_B^\mu(\tau, \vec{\sigma}) \Omega_B^A(\tau, \vec{\sigma}) \sigma^A, \]

\[ l^\mu(\tau, \vec{\sigma}) = \frac{1}{\sqrt{\left| \det g_{rs}(\tau, \vec{\sigma}) \right|}} \epsilon^{\mu \alpha \beta \gamma} [z_1^\alpha z_2^\beta z_3^\gamma](\tau, \vec{\sigma}), \]

(normal to the simultaneity surfaces), \hspace{1cm} (2.7)

where we have introduced the following matrices

\[ \Omega_A^B = (\Lambda^{-1} \partial_\tau \Lambda)^A_B = (\mathcal{R}^{-1} \partial_\tau \mathcal{R} + \mathcal{R}^{-1} B^{-1} \partial_\tau B \mathcal{R})^A_B = (\Omega_R + \mathcal{R}^{-1} \Omega_B \mathcal{R})^A_B, \]

\[ \Omega_R = \mathcal{R}^{-1} \partial_\tau \mathcal{R} = \begin{pmatrix} 0 & 0 \\ 0 & \Omega_R = R^{-1} \partial_\tau R \end{pmatrix}, \]

\[ \Omega_B = B^{-1}(\vec{\beta}) \partial_\tau B(\vec{\beta}) = -\partial_\tau B(-\vec{\beta}) B^{-1}(-\vec{\beta}) = \]

\[ = \begin{pmatrix} 0 & \gamma (\partial_\tau \beta^u + \frac{\gamma^2 \beta^u \partial_\tau \beta^u}{\gamma + 1}) \\ \gamma (\partial_\tau \beta^r + \frac{\gamma^2 \beta^r \partial_\tau \beta^r}{\gamma + 1}) & -\frac{\gamma^2}{\gamma + 1} (\beta^r \partial_\tau \beta^s - \partial_\tau \beta^r \beta^s) \end{pmatrix}, \]

\[ \Omega = \begin{pmatrix} 0 & \gamma (\partial_\tau \beta^u + \frac{\gamma^2 \beta^u \partial_\tau \beta^u}{\gamma + 1}) R^u_s \\ R^u_u \gamma (\partial_\tau \beta^u + \frac{\gamma^2 \beta^u \partial_\tau \beta^u}{\gamma + 1}) \Omega^u_s - \frac{\gamma^2}{\gamma + 1} R^u_u (\beta^u \partial_\tau \beta^v - \partial_\tau \beta^u \beta^v) R^v_s \end{pmatrix}, \]

\[ \Omega_{(v)B} = (\Lambda^{-1} \partial_\tau \Lambda)^A_B = (\mathcal{R}^{-1} \partial_\tau \mathcal{R} + \mathcal{R}^{-1} B^{-1} \partial_\tau B \mathcal{R})^A_B = (\Omega_R(v) + \mathcal{R}^{-1} \Omega_B(v) \mathcal{R})^A_B = \]

\[ = \begin{pmatrix} 0 & \gamma (\partial_\tau \beta^u + \frac{\gamma^2 \beta^u \partial_\tau \beta^u}{\gamma + 1}) R^u_s \\ R^u_u \gamma (\partial_\tau \beta^u + \frac{\gamma^2 \beta^u \partial_\tau \beta^u}{\gamma + 1}) \Omega^u_s - \frac{\gamma^2}{\gamma + 1} R^u_u (\beta^u \partial_\tau \beta^v - \partial_\tau \beta^u \beta^v) R^v_s \end{pmatrix}, \hspace{1cm} (2.8) \]
assumed to vanish at spatial infinity, $\Omega^A B(\tau, \vec{\sigma}), \Omega^A_B(\tau, \vec{\sigma}) \to |\vec{\sigma}| \to \infty 0$. The matrix $\Omega_B$ describes the translational velocity ($\vec{\beta}$) and acceleration ($\partial_\tau \vec{\beta}$), while the matrix $\Omega_R$ the rotational angular velocity.

The $z^\mu_A$'s and the associated 4-metric are

$$z^\mu_\tau(\tau, \vec{\sigma}) = \left( [1 + \Omega^\tau_\tau \sigma^\tau] U^\mu_\tau + \Omega^A_\tau \sigma^A U^\mu_\tau \right)(\tau, \vec{\sigma}),$$

$$z^\mu_r(\tau, \vec{\sigma}) = \left( \Omega^r(\tau)_s \sigma^s U^\mu_\tau + [\delta^s_r + \Omega^s_\tau \sigma^A] U^\mu_s \right)(\tau, \vec{\sigma}),$$

and

$$g_{\tau\tau}(\tau, \vec{\sigma}) = \left( z^\mu_\tau \eta_{\mu\nu} z^\nu_\tau \right)(\tau, \vec{\sigma}) = \epsilon \left\{ [1 + \Omega^\tau_\tau \sigma^\tau]^2 - \sum_r [\Omega^r_\tau \sigma^A]^2 \right\} (\tau, \vec{\sigma}),$$

$$g_{r\tau}(\tau, \vec{\sigma}) = \left( z^\mu_r \eta_{\mu\nu} z^\nu_r \right)(\tau, \vec{\sigma}) = \epsilon \left( \Omega^r(\tau)_s \sigma^s [1 + \Omega^r_u \sigma^u] - \sum_s \Omega^s_\tau \sigma^A [\delta^s_r + \Omega^s_\tau \sigma^A] \right)(\tau, \vec{\sigma}),$$

$$g_{rs}(\tau, \vec{\sigma}) = \left( z^\mu_r \eta_{\mu\nu} z^\nu_s \right)(\tau, \vec{\sigma}) = \epsilon \left\{ - \delta_{rs} - \Omega^r_\tau A s + \Omega^s_\tau A r \right\} \sigma^A +$$

$$+ \Omega^r(\tau)_u \Omega^r(s)_v \sigma^u \sigma^v - \sum_u \Omega^u_\tau A \Omega^u(s)_A \sigma^A \sigma^B \right)(\tau, \vec{\sigma}).$$

Eqs.(2.1) are complicated restrictions on the parameters $\vec{\beta}(\tau, \vec{\sigma}), \alpha(\tau, \vec{\sigma}), \beta(\tau, \vec{\sigma}), \gamma(\tau, \vec{\sigma})$ of the Lorentz transformations, which say that translational accelerations and rotational frequencies are not independent but must balance each other if Eqs.(2.3) describe the inverse of an admissible 4-coordinate transformation.

Let us consider two extreme cases.

A) Rigid non-inertial reference frames with translational acceleration exist. An example are the following embeddings, which are compatible with the locality hypothesis only for $f(\tau) = \tau$ (this corresponds to $\Lambda = B(\vec{0}) \mathcal{R}(0,0,0)$, i.e. to an inertial reference frame)
\[ z^\mu(\tau, \vec{\sigma}) = x_0^\mu + \epsilon_\tau^\mu f(\tau) + \epsilon_{\tau}^\mu \sigma^\tau, \]

\[ g_{\tau\tau}(\tau, \vec{\sigma}) = \epsilon \left( \frac{df(\tau)}{d\tau} \right)^2, \quad g_{\tau\tau}(\tau, \vec{\sigma}) = 0, \quad g_{rs}(\tau, \vec{\sigma}) = -\epsilon \delta_{rs}. \quad (2.11) \]

This is a foliation with parallel hyper-planes with respect to a centroid \( x^\mu(\tau) = x_0^\mu + \epsilon_\tau^\mu f(\tau) \) (origin of 3-coordinates). The hyper-planes have translational acceleration \( \ddot{x}^\mu(\tau) = \epsilon_\tau^\mu \dot{f}(\tau) \), so that they are not uniformly distributed like in the inertial case \( f(\tau) = \tau \).

B) On the other hand rigid rotating reference frames do not exist. Let us consider the embedding (compatible with the locality hypothesis) with \( \Lambda = B(\vec{0}) R(\alpha(\tau), \beta(\tau), \gamma(\tau)) \) and

\[ x^\mu(\tau) = x_0^\mu + \epsilon_\tau^\mu \tau \]

\[ z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + \epsilon_\tau^\mu R^s(\tau) \sigma^s, \]

\[ \dot{z}^\mu(\tau, \vec{\sigma}) = \dot{x}^\mu(\tau) + \epsilon_\tau^\mu \dot{R}^s(\tau) \sigma^s, \quad \ddot{z}^\mu(\tau) = \epsilon_\tau^\mu R^s(\tau), \]

\[ g_{\tau\tau}(\tau, \vec{\sigma}) = \epsilon \left( \ddot{x}^2(\tau) + 2 \dot{x}_\mu(\tau) \epsilon_\tau^\mu \dot{R}^s(\tau) \sigma^s - \epsilon \dot{R}^u(\tau) \dot{R}^v(\tau) \sigma^u \sigma^v \right), \]

\[ g_{\tau\tau}(\tau, \vec{\sigma}) = \epsilon \left( \dot{x}_\mu(\tau) \epsilon_\tau^\mu \dot{R}^s(\tau) - \epsilon \dot{R}^u(\tau) \dot{R}^v(\tau) \sigma^u \right), \]

\[ g_{rs}(\tau, \vec{\sigma}) = -\epsilon R^u(\tau) R^s(\tau), \quad (2.12) \]

which corresponds to a foliation with parallel space-like hyper-planes with normal \( l^\mu = \epsilon_\tau^\mu \).

It can be verified that it is not the inverse of an admissible 4-coordinate transformation, because the associated \( g_{\tau\tau}(\tau, \vec{\sigma}) \) has a zero at [53]

\[ \sigma = \sigma_R = \frac{1}{\Omega(\tau)} \left[ -\dot{x}_\mu(\tau) b^{\mu}_{\tau}(\tau) \dot{(\sigma \times \Omega(\tau)})^\tau + \sqrt{\ddot{x}^2(\tau) + \left[ \dot{x}_\mu(\tau) b^{\mu}_{\tau}(\tau) \dot{(\sigma \times \Omega(\tau))}^\tau \right]^2} \right]. \quad (2.13) \]
with \( \sigma_R \to \infty \) for \( \Omega \to 0 \). At \( \sigma = \sigma_R \) the time-like vector \( z^\mu(\tau, \vec{\sigma}) \) becomes light-like (the horizon problem), while for an admissible foliation with space-like leaves it must always remain time-like.

This pathology (the so-called horizon problem) is common to most of the rotating coordinate systems (see Subsection D of the Introduction of Ref. [21] for a partial list of the existing options). Let us remark that an analogous pathology happens on the event horizon of the Schwarzschild black hole. Also in this case we have a coordinate singularity where the time-like Killing vector of the static space-time becomes light-like. For the rotating Kerr black hole the same coordinate singularity happens already at the boundary of the ergosphere [54]. Also in the existing theory of rotating relativistic stars [55], where differential rotations are replacing the rigid ones in model building, it is assumed that in certain rotation regimes an ergosphere may form [56]: again, if one uses 4-coordinates adapted to the Killing vectors, one gets a similar coordinate singularity.

In the next Subsection we shall consider the minimal modification of Eq. (2.12) so to obtain the inverse of an allowed 4-coordinate transformation to a rotating coordinate system.

E. The Simplest Notion of Simultaneity when Rotations are Present.

Let us look for the simplest embedding \( x^\mu = z^\mu(\tau, \vec{\sigma}) \), inverse of an admissible 4-coordinate transformation \( x^\mu \mapsto \sigma^A \) compatible with the locality hypothesis, which contains a rotating reference frame, with also translational acceleration, of the type of Eq. (2.12). The minimal modification of Eq. (2.12) is to replace the rotation matrix \( R(\tau) \) with \( R(\tau, |\vec{\sigma}|) \), namely the rotation varies as a function of some radial distance \( |\vec{\sigma}| \) (differential rotation) from the arbitrary time-like world-line \( x^\mu(\tau) \), origin of the 3-coordinates on the simultaneity surfaces. Since the 3-coordinates \( \sigma^r \) are Lorentz scalar we shall use the radial distance \( \sigma = |\vec{\sigma}| = \sqrt{\delta_{rs} \sigma^r \sigma^s} \), so that \( \sigma^r = \sigma \hat{\sigma}^r \) with \( \delta_{rs} \hat{\sigma}^r \hat{\sigma}^s = 1 \). Therefore let us replace Eq. (2.12) with the following embedding.
\[ z^\mu(\tau, \sigma) = x^\mu(\tau) + \epsilon^\mu_r R^r_s(\tau, \sigma) \sigma^s \equiv x^\mu(\tau) + b^\mu_r(\tau, \sigma) \sigma^r, \]

\[ R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^r_s, \quad \partial_{\sigma} R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} 0, \]

\[ b^\mu_s(\tau, \sigma) = \epsilon^\mu_r R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \epsilon^\mu_s; \quad [b^\mu_r \eta_{\mu\nu} b^\nu_s](\tau, \sigma) = -\epsilon \delta_{rs}. \quad (2.14) \]

Since \( z^\mu_r(\tau, \sigma) = \epsilon^\mu_s \partial_{\sigma} [R^r_s(\tau, \sigma) \sigma^u] \), it follows that the normal to the simultaneity surfaces is \( l^\mu = \epsilon^\mu_r \), namely the hyper-surfaces are parallel space-like hyper-planes. These hyper-planes have translational acceleration \( \ddot{x}^\mu(\tau) \) and a rotating 3-coordinate system with rotational frequency

\[ \Omega^r(\tau, \sigma) = -\frac{1}{2} \epsilon^{r uv} \left[R^{-1}(\tau, \sigma) \frac{\partial R(\tau, \sigma)}{\partial \tau}\right]^{uv} \rightarrow_{\sigma \rightarrow \infty} 0, \]

\[ \downarrow \]

\[ \frac{\partial b^\mu_s(\tau, \sigma)}{\partial \tau} = \epsilon^\mu_r \frac{\partial R^r_s(\tau, \sigma)}{\partial \tau} = -\epsilon^{suw} \Omega^u(\tau, \sigma) b^\mu_v(\tau, \sigma), \]

\[ \Omega^1(\tau, \sigma) = \left[\partial_\tau \beta \sin \gamma - \partial_\tau \alpha \sin \beta \cos \gamma\right](\tau, \sigma), \]

\[ \Omega^2(\tau, \sigma) = \left[\partial_\tau \beta \cos \gamma + \partial_\tau \alpha \sin \beta \sin \gamma\right](\tau, \sigma), \]

\[ \Omega^3(\tau, \sigma) = \left[\partial_\tau \gamma + \partial_\tau \alpha \cos \beta\right](\tau, \sigma). \quad (2.15) \]

In the last three lines we used Eqs.(2.6) to find the angular velocities. Moreover we can define

\[ \Omega^u_{(r)}(\tau, \sigma) = \left[R^{-1} \partial_{\sigma} R\right](\tau, \sigma) = 2\delta^r \left[R^{-1} \frac{\partial R}{\partial \sigma}\right](\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} 0, \]

\[ \Omega^u_{(r) v}(\tau, \sigma) \sigma^v = \Phi^u_{uv}(\tau, \sigma) \frac{\sigma^r}{\sigma} \sigma^v, \quad \Phi_{uv} = -\Phi_{vu}. \quad (2.16) \]
As a consequence we have

\[ \dot{x}^\mu(\tau) = \epsilon \left( [\dot{x}_\nu(\tau) l^\nu] l^\mu - \sum_r [\dot{x}_\nu(\tau) \epsilon_r^\nu] e_r^\mu \right), \]

\[ z_r^\mu(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) l^\mu + N^r(\tau, \vec{\sigma}) z_r^\mu(\tau, \vec{\sigma}) = \]

\[ = \dot{x}^\mu(\tau) - \epsilon^{suv} \Omega^u(\tau, \sigma) b_v^\mu(\tau, \sigma) \sigma^s = \]

\[ = \dot{x}^\mu(\tau) - (\vec{\sigma} \times \vec{\Omega}(\tau, \sigma))^r b_r^\mu(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \dot{x}^\mu(\tau), \]

\[ z_r^\mu(\tau, \vec{\sigma}) = e_r^\mu \left[ R^s(\tau, \sigma) + \partial_r R^s(\tau, \sigma) \sigma^u \right] = \]

\[ = b^\mu_s(\tau, \sigma) \left[ \delta^s_r + \Omega^s(\tau, \sigma) \sigma^u \right] \rightarrow_{\sigma \rightarrow \infty} e_r^\mu, \quad (2.17) \]

and then we obtain

\[ g_{\tau\tau}(\tau, \vec{\sigma}) = \dot{x}^2(\tau) - 2 \dot{x}_\mu(\tau) b^\mu_r(\tau, \sigma) (\vec{\sigma} \times \vec{\Omega}(\tau, \sigma))^r - \epsilon (\vec{\sigma} \times \vec{\Omega})^2 = \]

\[ = \left[ N^2 - g_{rs} N^r N^s \right](\tau, \vec{\sigma}), \]

\[ g_{\tau r}(\tau, \vec{\sigma}) = \left[ g_{rs} N^s \right](\tau, \vec{\sigma}) = \]

\[ = \dot{x}_\mu(\tau) b^\mu_r(\tau, \sigma) \left[ \delta^s_r + \Omega^s(\tau, \sigma) \sigma^u \right] + \epsilon (\vec{\sigma} \times \vec{\Omega}(\tau, \sigma))^s \left[ \delta^s_r + \Omega^s(\tau, \sigma) \sigma^u \right], \]

\[ -\epsilon g_{rs}(\tau, \vec{\sigma}) = \delta_{rs} + \left( \Omega^s(\tau, \sigma) + \Omega^s(\tau, \sigma) \right) \sigma^u + \sum_w \Omega^u_s(\tau, \sigma) \Omega^w_s(\tau, \sigma) \sigma^u \sigma^v \cdot \quad (2.18) \]

The requirement that \( g_{\tau\tau}(\tau, \vec{\sigma}) \) and \( g_{\tau r}(\tau, \vec{\sigma}) \) tend to finite limits at spatial infinity puts the restrictions
\[ |\tilde{\Omega}(\tau, \sigma)|, \quad |\Omega_{(\tau, \sigma)}^n| \rightarrow_{\sigma \rightarrow \infty} O(\sigma^{-(1+\eta)}), \quad \eta > 0, \]

\[
\downarrow
\]

\[ \partial_A R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} O(\sigma^{-(1+\eta)}), \Rightarrow R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} O(\sigma^{-(1+\eta)}), \]

\[ z^\mu_r(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} \tilde{x}^\mu(\tau) + O(\sigma^{-\eta}), \]

\[ b^\mu_r(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \epsilon^\mu_r + O(\sigma^{-(1+\eta)}), \quad z^\mu_r(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} \epsilon^\mu_r + O(\sigma^{-(1+\eta)}), \]

\[ N^r(\tau, \bar{\sigma}) z^\mu_r(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} -\epsilon \tilde{x}_\nu(\tau) \tilde{\epsilon}^\nu_r + O(\sigma^{-(1+2\eta)}), \]

\[ N^r(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} -\epsilon \delta^{rs} \tilde{x}_\nu(\tau) \tilde{\epsilon}_s^\nu + O(\sigma^{-\eta}), \]

\[ N(\tau, \bar{\sigma}) \quad l^\mu = [z^\mu_r - N^r z^\mu_r](\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} \epsilon [\tilde{x}_\nu(\tau) l^\nu] l^\mu + O(\sigma^{-\eta}), \]

\[ g_{\tau\tau}(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} \bar{x}^2(\tau) + O(\sigma^{-2\eta}), \]

\[ g_{\tau r}(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} \bar{x}_\mu(\tau) \epsilon^\mu_r + O(\sigma^{-\eta}), \]

\[ g_{rs}(\tau, \bar{\sigma}) \rightarrow_{\sigma \rightarrow \infty} -\epsilon \delta_{rs} + O(\sigma^{-\eta}). \quad (2.19) \]

Let us look for a family of rotation matrices \( R^r_s(\tau, \sigma) \) satisfying the condition \( \epsilon g_{\tau\tau}(\tau, \bar{\sigma}) > 0 \) of Eqs.(2.1).

Let us make the ansatz that the Euler angles of \( R(\alpha, \beta, \gamma) \) have the following factorized dependence on \( \tau \) and \( \sigma \)

\[ \alpha(\tau, \sigma) = F(\sigma) \tilde{\alpha}(\tau), \quad \beta(\tau, \sigma) = F(\sigma) \tilde{\beta}(\tau), \quad \gamma(\tau, \sigma) = F(\sigma) \tilde{\gamma}(\tau), \quad (2.20) \]
with

\[ F(\sigma) > 0, \quad \frac{dF(\sigma)}{d\sigma} \neq 0, \quad F(\sigma) \to \sigma \to \infty O(\sigma^{-(1+\eta)}). \]  \hspace{1cm} (2.21)

We get

\[ \Omega^1(\tau, \sigma) = F(\sigma) \left( \dot{\hat{\gamma}}(\tau) \sin [F(\sigma) \ddot{\gamma}(\tau)] - \dot{\hat{\beta}}(\tau) \sin [F(\sigma) \ddot{\beta}(\tau)] \cos [F(\sigma) \ddot{\gamma}(\tau)] \right), \]

\[ \Omega^2(\tau, \sigma) = F(\sigma) \left( \dot{\hat{\gamma}}(\tau) \cos [F(\sigma) \ddot{\gamma}(\tau)] + \dot{\hat{\beta}}(\tau) \sin [F(\sigma) \ddot{\beta}(\tau)] \sin [F(\sigma) \ddot{\gamma}(\tau)] \right), \]

\[ \Omega^3(\tau, \sigma) = F(\sigma) \left( \ddot{\gamma}(\tau) + \dot{\hat{\beta}}(\tau) \cos [F(\sigma) \ddot{\beta}(\tau)] \right). \]

\[ \downarrow \]

\[ \Omega^r(\tau, \sigma) = F(\sigma) \hat{\Omega}(\tau, \sigma) \hat{n}^r(\tau, \sigma), \quad \hat{n}^2(\tau, \sigma) = 1, \]

\[ 0 < \hat{\Omega}(\tau, \sigma) \leq 2 \max \left( \dot{\hat{\alpha}}(\tau), \dot{\hat{\beta}}(\tau), \dot{\hat{\gamma}}(\tau) \right) = 2 M_1. \]  \hspace{1cm} (2.22)

Since \( l^\mu = \epsilon^\mu r \) and \( b^\mu_r(\tau, \sigma) \) form an orthonormal tetrad \( [b^\mu_A(\tau, \sigma) \eta_{\mu\nu} b^\nu_B(\tau, \sigma) = \eta_{AB}] \), let us decompose the future time-like 4-velocity \( \dot{x}^\mu(\tau) \) on it (\( \nu_1(\tau) \) is the asymptotic lapse function)

\[ \dot{x}^\mu(\tau) = \nu_1(\tau) l^\mu - \sum_r v_r(\tau, \sigma) b^\mu_r(\tau, \sigma) \]

\[ \nu_1(\tau) = \epsilon \dot{\nu}_1(\tau) l^\mu > 0, \quad v_r(\tau, \sigma) = \epsilon \dot{\nu}_r(\tau) b^\mu_r(\tau, \sigma), \]

\[ \epsilon \dot{x}^2(\tau) = \nu_1^2(\tau) - \sum_r v_r^2(\tau, \sigma) > 0, \Rightarrow \sum_r v_r^2(\tau, \sigma) = \bar{\nu}^2(\tau, \sigma) \equiv \nu^2(\tau) < \nu_1^2(\tau), \]  \hspace{1cm} (2.23)

We add the condition

\[ |\bar{\nu}(\tau)| \leq \frac{\nu_1(\tau)}{K}, \quad K > 1. \]  \hspace{1cm} (2.24)
This condition is slightly stronger than the last of Eqs. (2.23), which does not exclude the possibility that the observer in $\sigma = 0$ has a time-like 4-velocity $\dot{x}^\mu(\tau)$ which, however, becomes light-like at $\tau = \pm \infty$ [57]. The condition (2.24) excludes this possibility. In other words the condition (2.24) tells us that the observer is without even t-horizon, namely he can explore all the Minkowski space-time by light-signal.

Then the condition $\epsilon g_{\tau\tau}(\tau, \bar{\sigma}) > 0$ becomes

$$
\epsilon g_{\tau\tau}(\tau, \bar{\sigma}) =
$$

$$
= \epsilon \dot{x}^2(\tau) - 2 \sigma F(\sigma) \bar{\Omega}(\tau, \sigma) \sum \limits_r v_r(\tau, \sigma) \left[ \dot{\bar{\sigma}} \times \hat{n}(\tau, \sigma) \right]_r - \sigma^2 \bar{\Omega}^2(\tau, \sigma) F^2(\sigma) \left[ \dot{\bar{\sigma}} \times \hat{n}(\tau, \sigma) \right]^2
$$

$$
= c^2(\tau) - 2 b(\tau, \bar{\sigma}) X(\tau, \sigma) - a^2(\tau, \bar{\sigma}) X^2(\tau, \sigma) > 0
$$

(2.25)

where we have defined

$c^2(\tau) = \epsilon \dot{x}^2(\tau) = v^2_l(\tau) - \bar{v}^2(\tau) > 0$,  \hspace{1cm} c^2(\tau) \geq \frac{K^2 - 1}{K^2} v^2_l(\tau),$

$b(\tau, \bar{\sigma}) = \sum \limits_r v_r(\tau, \sigma) \left[ \dot{\bar{\sigma}} \times \hat{n}(\tau, \sigma) \right]_r,$

$|b(\tau, \bar{\sigma})| \leq |\bar{v}(\tau)| < v_l(\tau)$,  \hspace{1cm} or  \hspace{1cm} $|b(\tau, \bar{\sigma})| \leq \frac{v_l(\tau)}{K}, \hspace{1cm} K > 1,$

$a^2(\tau, \bar{\sigma}) = \left[ \dot{\bar{\sigma}} \times \hat{n}(\tau, \sigma) \right]^2 > 0$,  \hspace{1cm} $a^2(\tau, \bar{\sigma}) \leq 1$,  \hspace{1cm} $b^2(\tau, \bar{\sigma}) + a^2(\tau, \sigma) c^2(\tau) > 0,$

$X(\tau, \sigma) = \sigma F(\sigma) \bar{\Omega}(\tau, \sigma).$  \hspace{1cm} (2.26)

The study of the equation $a^2 X^2 + 2 b X - c^2 = A^2 (X - X_+)(X - X_-) = 0$, with solutions $X_\pm = \frac{1}{a^2} (-b \pm \sqrt{b^2 + a^2 c^2})$, shows that $\epsilon g_{\tau\tau} > 0$ implies $X_- < X < X_+$; being $X_- < 0$ and $X > 0$ [see Eq. (2.22)], we have that a half of the conditions ($X_- < X$) is always satisfied. We have only to discuss the condition $X < X_+$.

Since $-v_l/K \leq b \leq v_l/K$, when $b$ increases in this interval $X_+$ decrease with $b$. This
implies
\[ X_+ > \frac{1}{a^2} \left( -\frac{v_i}{K} + \sqrt{\frac{v_i^2}{K^2} + a^2 c^2} \right), \]
so that \( c^2 \geq \frac{K^2-1}{K^2} v_i^2 \) implies that we will have \( g_{\tau\tau} > 0 \) if \( 0 < X < \frac{v_i}{K a^2} (\sqrt{1 + (K^2 - 1) a^2} - 1) \), namely if the function \( F(\sigma) \) satisfies the condition
\[
|F(\sigma)| < \frac{v_i(\tau)}{K \sigma a^2(\tau, \bar{\sigma}) \Omega(\tau, \sigma)} \left( \sqrt{1 + (K^2 - 1) a^2(\tau, \bar{\sigma})} - 1 \right) = \frac{v_i(\tau)}{K \Omega(\tau, \sigma)} g(a^2). 
\]

Since \( a^2 \leq 1 \) and \( g(x) = (1/x)(\sqrt{1 + (K^2 - 1)x} - 1) \) is decreasing for \( x \) increasing in the interval \( 0 < x < 1 \) \((K > 1)\), we get \( g(a^2) > g(1) = K - 1 \) and the stronger condition
\[
|F(\sigma)| < \frac{v_i(\tau)}{K \Omega(\tau, \sigma)} (K - 1).
\]

The condition (2.22) on the Euler angles and the fact that Eq.(2.24) implies \( \min v_i(\tau) = m > 0 \) lead to the following final condition on \( F(\sigma) \) [59]

\[
0 < F(\sigma) < \frac{m}{2 K M_1 \sigma} (K - 1) = \frac{1}{M \sigma}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0,
\]

or
\[
|\partial_\tau \alpha(\tau, \sigma)|, |\partial_\tau \beta(\tau, \sigma)|, |\partial_\tau \gamma(\tau, \sigma)| < \frac{m}{2 K \sigma} (K - 1),
\]

or
\[
|\Omega^r(\tau, \sigma)| < \frac{m}{K \sigma} (K - 1). \tag{2.27}
\]

This means that, while the linear velocities \( \dot{x}^\mu(\tau) \) and the translational accelerations \( \ddot{x}^\mu(\tau) \) are arbitrary, the allowed rotations \( R(\alpha, \beta, \gamma) \) on the leaves of the foliation have the rotational frequencies, namely the angular velocities \( \Omega^r(\tau, \sigma) \), limited by an upper bound proportional to the minimum of the linear velocity \( v_i(\tau) = \dot{x}_\mu(\tau) l^\mu \) orthogonal to the parallel hyper-planes.

Instead of checking the conditions (2.1) on \( g_{rs}(\tau, \bar{\sigma}) \), let us write
\[ z^\mu(\tau, \vec{\sigma}) = \xi_l(\tau, \vec{\sigma}) l^\mu - \sum_r \xi_r(\tau, \vec{\sigma}) e_r^\mu, \]

\[ \xi_l(\tau, \vec{\sigma}) = e z_\mu(\tau, \vec{\sigma}) l^\mu = e x_\mu(\tau) l^\mu = x_l(\tau), \]

\[ \xi_r(\tau, \vec{\sigma}) = e z_\mu(\tau, \vec{\sigma}) e_r^\mu = e x_\mu(\tau) e_r^\mu + R^r_s(\tau, \sigma) \sigma^s = x_{l r}(\tau) + R^r_s(\tau, \sigma) \sigma^s, \quad (2.28) \]

so that we get

\[ \partial_\tau \xi_l(\tau, \vec{\sigma}) = \dot{x}_l(\tau) = v_l(\tau), \quad \partial_\tau \xi_l(\tau, \vec{\sigma}) = 0, \]

\[ \partial_u \xi_r(\tau, \vec{\sigma}) = R^r_u(\tau, \sigma) + \partial_u R^r_s(\tau, \sigma) \sigma^s = \]

\[ = R^r_v(\tau, \sigma) \left[ \delta^u_v + \omega^v_{(u)w}(\tau, \sigma) \sigma^w \right] = \]

\[ = R^r_v(\tau, \sigma) \left[ \delta^u_v + \Phi_{uv}(\tau, \sigma) \frac{\sigma^u \sigma^v}{\sigma} \right] \overset{\text{def}}{=} \left( R(\tau, \sigma) M(\tau, \vec{\sigma}) \right)_{\nu u}, \quad (2.29) \]

and let us show that \( \sigma^A = (\tau, \vec{\sigma}) \mapsto (\xi_l(\tau, \vec{\sigma}), \xi_r(\tau, \vec{\sigma})) \) is a coordinate transformation with positive Jacobian [60]. This will ensure that these foliations with parallel hyper-planes are defined by embeddings such that \( \sigma^A \mapsto x^\mu = z^\mu(\tau, \vec{\sigma}) \) is the inverse of an admissible 4-coordinate transformation \( x^\mu \mapsto \sigma^A \).

Therefore we have to study the Jacobian

\[ J(\tau, \vec{\sigma}) = \begin{pmatrix} \frac{\partial \xi_l(\tau, \vec{\sigma})}{\partial \tau} & \frac{\partial \xi_l(\tau, \vec{\sigma})}{\partial \sigma^r} \\ \frac{\partial \xi_r(\tau, \vec{\sigma})}{\partial \tau} & \frac{\partial \xi_r(\tau, \vec{\sigma})}{\partial \sigma^r} \end{pmatrix} = \begin{pmatrix} v_l(\tau) \\ 0_r \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_l(\tau, \vec{\sigma})}{\partial \tau} & \frac{\partial \xi_l(\tau, \vec{\sigma})}{\partial \sigma^r} \\ \frac{\partial \xi_r(\tau, \vec{\sigma})}{\partial \tau} & \frac{\partial \xi_r(\tau, \vec{\sigma})}{\partial \sigma^r} \end{pmatrix}, \]

\[ \det J(\tau, \vec{\sigma}) = v_l(\tau) \det R(\tau, \sigma) \det M(\tau, \vec{\sigma}) = v_l(\tau) \det M(\tau, \vec{\sigma}). \quad (2.30) \]
To show that \( \det M(\tau, \vec{\sigma}) \neq 0 \), let us look for the null eigenvectors \( W_r(\tau, \vec{\sigma}) \) of the matrix \( M(\tau, \vec{\sigma}), M_{rs}(\tau, \vec{\sigma}) W_s(\tau, \vec{\sigma}) = 0 \) or \( W_r(\tau, \vec{\sigma}) - \Phi_{uv}(\tau, \sigma) \frac{a^u}{\sigma} \sigma^s W_s(\tau, \vec{\sigma}) = 0 \) [see Eq.(2.16)]. Due to \( \Phi_{uv} = -\Phi_{vu} \), we get \( \sigma^s W_s(\tau, \vec{\sigma}) = 0 \) and this implies \( W_r(\tau, \vec{\sigma}) = 0 \), i.e. the absence of null eigenvalues. Therefore \( \det M(\tau, \vec{\sigma}) \neq 0 \) and an explicit calculation shows that \( \det J(\tau, \vec{\sigma}) = v_l(\tau) > 0 \). Therefore, \( x^\mu \mapsto \sigma^A \) is an admissible 4-coordinate transformation.

Let us remark that the congruence of time-like world-lines associated to the constant normal \( l^\mu \) defines an inertial reference frame: each inertial observer is naturally endowed with the orthonormal tetrad \( b^\mu_A = (l^\mu; \epsilon^\mu_r) \).

Let us consider the second skew congruence, whose observer world-lines are \( x^\mu_{\vec{\sigma}}(\tau) = z^\mu(\tau, \vec{\sigma}) \), and let us look for an orthonormal tetrad \( V^\mu_A(\tau, \vec{\sigma}) = (z^\mu(\tau, \vec{\sigma})/\sqrt{g_{rr}(\tau, \vec{\sigma})}; V^\mu(\tau, \vec{\sigma})) \) to be associated to each of its time-like observers. Due to the orthonormality we have \( V^\mu_A(\tau, \vec{\sigma}) = \Lambda^\mu_{\nu=A}(\tau, \vec{\sigma}) \) with \( \Lambda(\tau, \vec{\sigma}) \) a Lorentz matrix. Therefore we can identify them with \( SO(3,1) \) matrices parametrized as the product of a pure boost with a pure rotation as in Eqs. (2.6). If we introduce

\[
\hat{E}_r(\tau, \vec{\sigma}) = \{ E^k_r(\tau, \vec{\sigma}) \} = R^k_{\nu=k}(\alpha(\tau, \sigma), \beta_m(\tau, \sigma), \gamma_m(\tau, \sigma))
\]

\[
\Rightarrow \frac{\partial \hat{E}_r(\tau, \vec{\sigma})}{\partial \tau} = \omega_{m}(\tau) \times \hat{E}_r(\tau, \vec{\sigma}),
\]

\[
B^{jk}(\vec{\beta}_m(\tau, \vec{\sigma})) = \delta^{ij} + \frac{\gamma^2(\vec{\beta}_m(\tau, \sigma))}{\gamma(\vec{\beta}_m(\tau, \sigma)) + 1} \beta^i_m(\tau, \sigma) \beta^j_m(\tau, \sigma),
\]

we can write

\[
V^\mu_A(\tau, \vec{\sigma}) = \Lambda^\mu_{\nu=A}(\tau, \vec{\sigma}) = \begin{pmatrix}
\frac{1}{\sqrt{1-\beta^2_m(\tau, \vec{\sigma})}} & \frac{\vec{\beta}_m(\tau, \vec{\sigma}) \cdot \hat{E}_r(\tau, \vec{\sigma})}{\sqrt{1-\beta^2_m(\tau, \vec{\sigma})}} \\
\frac{\beta^i_m(\tau, \vec{\sigma})}{\sqrt{1-\beta^2_m(\tau, \vec{\sigma})}} & \frac{\beta^j_m(\tau, \vec{\sigma})}{\sqrt{1-\beta^2_m(\tau, \vec{\sigma})}}
\end{pmatrix} B^{jk}(\vec{\beta}_m(\tau, \vec{\sigma})) E^k_r(\tau, \vec{\sigma}).
\]

We stress that for every observer \( x^\mu_{\vec{\sigma}}(\tau) \) the choice of the \( V^\mu_A(\tau, \vec{\sigma}) \)'s, and therefore also of the \( \hat{E}_r(\tau, \vec{\sigma}) \)'s, is arbitrary. As a consequence the angular velocity \( \omega_{m}(\tau) \) defined by the second of the Eqs.(2.31) is in general not related with the angular velocity (2.15) defined by the embedding. On the contrary, the parameter \( \beta^i_m(\tau, \vec{\sigma}) \) is related to the embedding by the relation \( \beta^i_m(\tau, \vec{\sigma}) = z^i_r(\tau, \vec{\sigma})/z^o_r(\tau, \vec{\sigma}) \).
For every observer $x^\mu_\sigma(\tau)$ of the congruence, endowed with the orthonormal tetrad $E^\mu_\sigma A(\tau) = V^\mu_A(\tau, \bar{\sigma})$, we get

\[
\frac{dE^\mu_\sigma A(\tau)}{d\tau} = A_{\sigma AB}(\tau) V^\mu_B(\tau),
\]

\[
\Rightarrow A_{\sigma AB}(\tau) = -A_{\sigma BA}(\tau) = \frac{dE^\mu_\sigma A(\tau)}{d\tau} \eta_{\mu\nu} E^\nu_B(\tau), \tag{2.33}
\]

Using the (2.32) we obtain

\[
\gamma(\tau, \bar{\sigma}) = \frac{1}{\sqrt{1 - \vec{\beta}^2(\tau, \bar{\sigma})}}, \quad \dot{\vec{\beta}}(\tau, \bar{\sigma}) = d\vec{\beta}(\tau, \bar{\sigma})/d\tau
\]

\[
a_{\bar{\sigma} r}(\tau) = A_{\bar{\sigma} rr}(\tau) = \left[ -\gamma(\vec{\beta} \cdot \vec{E}_r) - \frac{\gamma^3}{\gamma + 1} (\vec{\beta} \cdot \vec{\beta})(\vec{\beta} \cdot \vec{E}_r) \right](\tau, \bar{\sigma})
\]

\[
\Omega_{\bar{\sigma} r}(\tau) = \frac{1}{2} \epsilon_{r uv} A_{\bar{\sigma} uv}(\tau) = \left[ -\vec{\omega} \cdot \vec{E}_r - \frac{\gamma^2}{\gamma + 1} \epsilon^{snu}(\vec{\beta} \cdot \vec{E}_s)(\vec{\beta} \cdot \vec{E}_u) \right](\tau, \bar{\sigma}) \tag{2.34}
\]

Therefore the acceleration radii (see the Introduction) of these observers are

\[
I_1 = \bar{\Omega}^2_{\bar{\sigma}} - \bar{a}^2_{\bar{\sigma}} = \left[ \vec{\omega}^2 + 2 \gamma^2 \vec{\omega} \cdot (\vec{\beta} \times \vec{\beta}) + \gamma^2(\gamma - 2) \vec{\beta}^2 - \frac{\gamma^6}{\gamma + 1} (\vec{\beta} \cdot \vec{\beta})^2 \right](\tau, \bar{\sigma})
\]

\[
I_2 = \bar{a}_{\bar{\sigma}} \cdot \vec{\Omega}_{\bar{\sigma}} = \left[ \gamma(\vec{\beta} \cdot \vec{\omega}) + \frac{\gamma^3}{\gamma + 1} (\vec{\beta} \cdot \vec{\beta})(\vec{\beta} \cdot \vec{\omega}) \right](\tau, \bar{\sigma}) \tag{2.35}
\]

Let us remark that, even if we have finite acceleration radii, our radar 4-coordinates are globally defined, differently from Fermi coordinates.

The non-relativistic limit of the embedding (2.14) can be obtained by choosing $\epsilon^\mu_\sigma = (0; e^i_\sigma)$. We obtain a generalization of the standard translating and rotating 3-coordinate systems on the hyper-planes of constant absolute Newtonian time

\[
t'(\tau) = t(\tau),
\]

\[
z'(\tau, \bar{\sigma}) = x^i(\tau) + e^i_\sigma R^s_\sigma(\tau, \sigma) \sigma^s, \tag{2.36}
\]

without any restriction on rotations, namely with $R = R(\tau)$ allowed.
III. THE MODIFICATION OF EINSTEIN $\frac{1}{2}$ CONVENTION ASSOCIATED TO ADMISSIBLE RADAR COORDINATES AND THEIR EMPIRICAL DETERMINATION.

In the previous Section we have seen how it possible to build as many conventional observer-dependent globally defined radar 4-coordinate systems as admissible 3+1 splittings of space-time. Many of them have the equal-time leaves, describing the conventional instantaneous present, not orthogonal to the world-line of the accelerated observer.

In this Section we show that each admissible 3+1 splitting of space-time leads to a different generalization (see also Havas [39]) of Einstein $\frac{1}{2}$ convention for the synchronization of distant clocks by means of light signals.

Then we show how such generalizations can lead to an operational method for an empirical determination of admissible radar 4-coordinates around the world-line of a non-inertial observer simulated by a spacecraft belonging to a cluster of spacecrafts (or satellites) like the one used in the Global Positioning System.

A. A Cluster of Spacecrafts like in the Global Positioning System.

In Eqs.(2.14) we gave a family of embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$ defining possible notions of simultaneity, i.e. admissible 3+1 splittings of Minkowski space-time with foliations with space-like hyper-planes $\Sigma_\tau$ as leaves, to be associated to the world-line $x^\mu(\tau)$ of an arbitrary time-like observer $\gamma$, chosen as origin of the 3-coordinates on each simultaneity leaf $\Sigma_\tau$, i.e. $x^\mu(\tau) = z^\mu(\tau, \vec{0})$ (with this definition in general $\tau$ is not the proper time of the observer, but $(\tau, \vec{\sigma})$ are a good set of observer-dependent radar coordinates). The space-like hyper-planes $\Sigma_\tau$ are not orthogonal to $\gamma$: if $l^\mu = \epsilon^\mu_\tau$ is the normal to $\Sigma_\tau$ we have $l_\mu \frac{z^\mu(\tau)}{\sqrt{\epsilon z^2(\tau)}} \neq \epsilon$ except in the limiting case of an inertial observer with 4-velocity proportional to $l^\mu$.

If $\tau$ is not the proper time of the observer, the proper time $T_\gamma$ of the standard atomic clock $C$ of $\gamma$ will be defined by $dT_\gamma = \sqrt{\epsilon g_{\tau\tau}(\tau, \vec{0})} d\tau$ $[x^\mu(\tau) = \tilde{x}^\mu(T_\gamma)]$. This defines $T_\gamma = F_\gamma(\tau)$ as a monotonic function of $\tau$, whose inverse will be denoted $\tau = G(T_\gamma)$. Moreover, we make an arbitrary conventional choice of a tetrad $E^\mu_A(\tau)$ associated to $\gamma$ with $\langle \gamma \rangle E^\mu_A(\tau) = \frac{\tilde{x}^\mu(\tau)}{\sqrt{\epsilon \tilde{z}^2(\tau)}}$.

Let us consider a set of $N$ arbitrary time-like world-lines $x^\mu_i(\tau)$, $i = 1, \ldots, N$, associated to observers $\gamma_i$, so that $\gamma$ and the $\gamma_i$'s can be imagined to be the world-lines of $N+1$ spacecrafts (like in GPS [24]) with $\gamma$ chosen as a reference world-line (the accelerated observer). Each
of the world-lines $\gamma_i$ will have an associated standard atomic clock $C_i$ and a conventional tetrad $(\gamma_i) E_\lambda^\mu(\tau)$.

To compare the distant clocks $C_i$ with $C$ in the chosen notion of simultaneity, we define the 3-coordinates $\bar{\eta}_i(\tau)$ of the $\gamma_i$

$$x_i^\mu(\tau) \overset{\text{def}}{=} z^\mu(\tau, \bar{\eta}_i(\tau)).$$

(3.1)

Then the proper times $T_{\gamma_i}$ of the clocks $C_i$ will be expressed in terms of the scalar coordinate time $\tau$ of the chosen simultaneity as

$$dT_{\gamma_i} = \sqrt{\epsilon \left[ g_{\tau\tau}(\tau, \bar{\eta}_i(\tau)) + 2 g_{\tau r}(\tau, \bar{\eta}_i(\tau)) \dot{\eta}_r^e(\tau) + g_{r s}(\tau, \bar{\eta}_i(\tau)) \dot{\eta}_r^e(\tau) \dot{\eta}_s^e(\tau) \right]} \ d\tau, \quad (3.2)$$

so that with this notion of simultaneity the proper times $T_{\gamma_i}$ are connected to the proper time $T_\gamma$ by the following relations

$$dT_{\gamma_i} = \left. \sqrt{\epsilon \left[ g_{\tau\tau}(\tau, \bar{\eta}_i(\tau)) + 2 g_{\tau r}(\tau, \bar{\eta}_i(\tau)) \dot{\eta}_r^e(\tau) + g_{r s}(\tau, \bar{\eta}_i(\tau)) \dot{\eta}_r^e(\tau) \dot{\eta}_s^e(\tau) \right]} \right|_{\tau = g(\tau_i)} dT_\gamma. \quad (3.3)$$

This determines the synchronization of the $N + 1$ clocks once we have expressed the 3-coordinates $\bar{\eta}_i(\tau)$ in terms of the given world-lines $x^\mu(\tau)$, $x_i^\mu(\tau)$ and of an admissible embedding. For the embedding (2.14) from the definition

$$x_i^\mu(\tau) = z^\mu(\tau, \bar{\eta}_i(\tau)) = x^\mu(\tau) + \epsilon^e_r R^r_s(\tau, |\bar{\eta}_i(\tau)|) \eta_i^s(\tau), \quad (3.4)$$

we get $|\bar{\eta}_i(\tau)| \overset{\text{def}}{=} \sqrt{\delta_{r s} \eta_i^r(\tau) \eta_i^s(\tau)}$, $\eta_i^e(\tau) = |\bar{\eta}_i(\tau)| \dot{\eta}_i^e(\tau)$, $\delta_{r s} \dot{\eta}_i^r(\tau) \dot{\eta}_i^s(\tau) = 1$

$$\eta_i^\nu(\tau) = - \sum_w [R^{-1}(\tau, |\bar{\eta}_i(\tau)|)]^\nu_w \epsilon^\nu_w [x_{i\nu}(\tau) - x_{\nu}(\tau)]. \quad (3.5)$$

Then, if we put the solution

$$|\bar{\eta}_i(\tau)| = F_i \left[ \epsilon^\mu_r (x_{i\mu}(\tau) - x_{\mu}(\tau)) \right], \quad (3.6)$$

of the equations
|\vec{\eta}_i(\tau)|^2 = \delta_{rs} \sum_{mn} [R^{-1}(\tau, |\vec{\eta}_i(\tau)|)]^r_m [R^{-1}(\tau, |\vec{\eta}_i(\tau)|)]^s_n \\
e^\mu_m [x_{i\mu}(\tau) - x_\mu(\tau)] e^\nu_n [x_{i\nu}(\tau) - x_\nu(\tau)], \quad (3.7)

into Eqs.(3.5), we obtain the looked for expression of the 3-coordinates \(\vec{\eta}_i(\tau)\)

\[\vec{\eta}_i(\tau) = \sum_{m} \left[R^{-1}(\tau, F_i [e^a_w (x_{ia}(\tau) - x_a(\tau))]) \right]^u_m e^\nu_m [x_{i\nu}(\tau) - x_\nu(\tau)]. \quad (3.8)\]

**B. The Modification of Einstein \(1/2\) Convention for a Møller Admissible Radar Coordinate System.**

Let us now consider an admissible embedding \(x^\mu = z^\mu(\tau, \vec{\sigma})\) of the family (2.14) (but the discussion applies to every admissible embedding), where \((\tau, \vec{\sigma})\) are the radar coordinates adapted to the accelerated observer with world-line \(\gamma\). On the simultaneity leave \(\Sigma_\tau\) having the point \(Q\) of 4-coordinates \(x^\mu(\tau)\) on \(\gamma\) as origin of the 3-coordinates \(\vec{\sigma}\), let us consider a point \(P\) on \(\Sigma_\tau\) with coordinates \(z^\mu(\tau, \vec{\sigma})\) (for \(\vec{\sigma} = \vec{\eta}_i(\tau)\) it corresponds to the spacecraft \(\gamma_i\)). We want to express the observer-dependent radar 4-coordinates \(\tau = \tau(P), \vec{\sigma} = \vec{\sigma}(P)\) of \(P\) in terms of data on the world-line \(\gamma\) corresponding to the emission of a light signal in \(Q_-\) at \(\tau_- < \tau\) and to its reception in \(Q_+\) at \(\tau_+ > \tau\) after reflection at \(P\).

Let \(x^\mu(\tau_-)\) be the intersection of the world-line \(\gamma\) with the past light-cone through \(P\) and \(x^\mu(\tau_+)\) the intersection with the future light-cone through \(P\). To find \(\tau_\pm\) we have to solve the equations \(\Delta^2_\pm = [x^\mu(\tau_\pm) - z^\mu(\tau, \vec{\sigma})]^2 = 0\) with \(\Delta^\mu_\pm = x^\mu(\tau_\pm) - z^\mu(\tau, \vec{\sigma})\). We are interested in the solutions \(\Delta^\mu_- = |\vec{\Delta}_-|\) and \(\Delta^\mu_+ = -|\vec{\Delta}_-|\). Let us remark that on the simultaneity surfaces \(\Sigma_\tau\) we have \(x^\alpha(\tau) \neq z^\alpha(\tau, \vec{\sigma})\) for the Cartesian coordinate times.

Let us show that the \(\gamma\)-dependent radar coordinates \(\tau\) and \(\vec{\sigma}\) of the event \(P\), with Cartesian 4-coordinates \(z^\mu(\tau, \vec{\sigma})\) in an inertial system, can be determined in terms of the *emission scalar time* \(\tau_-\) of the light signal, the *emission unit 3-direction* \(\hat{\eta}(\tau_-)(\theta(\tau_-), \phi(\tau_-))\) [so that \(\Delta^\mu_- = |\vec{\Delta}_-| (-e_i \hat{\eta}(\tau_-))\)] and the *reception scalar time* \(\tau_+\) registered by the observer \(\gamma\) with world-line \(x^\mu(\tau)\) [61]. These data are usually given in terms of the proper time \(T(\tau)\) of the observer \(\gamma\) by using \(dT = \sqrt{g_{\tau\tau}(\tau, \vec{0})} d\tau\).

Let us introduce the following parametrization by using Eqs.(2.14)
\[ z^\mu(\tau, \bar{\sigma}) = x^\mu(\tau) + \epsilon^\mu_\nu R^r_s(\tau, \sigma) \sigma^s \overset{\text{def}}{=} \]
\[ = \xi_l(\tau, \bar{\sigma}) l^\mu + \xi^r(\tau, \bar{\sigma}) \epsilon^\mu_r = [x_l(\tau) l^\mu + \sum_r [x^r_\epsilon(\tau) + \zeta^r(\tau, \bar{\sigma})] \epsilon^\mu_r], \]
\[ \xi_l(\tau, \bar{\sigma}) = \epsilon z_\mu(\tau, \bar{\sigma}) l^\mu = \epsilon x_\mu(\tau) l^\mu = x_l(\tau), \]
\[ \xi^r(\tau, \bar{\sigma}) = \epsilon z_\mu(\tau, \bar{\sigma}) \epsilon^\mu_r = x^r_\epsilon(\tau) + \zeta^r(\tau, \bar{\sigma}), \]
\[ x^r_\epsilon(\tau) = \epsilon x_\mu(\tau) \epsilon^\mu_r, \quad \zeta^r(\tau, \bar{\sigma}) = R^r_s(\tau, \sigma) \sigma^s \rightarrow \sigma \rightarrow \infty \sigma^r. \quad (3.9) \]

Then the two equations \( \Delta^2_\pm = [x^\mu(\tau_\pm) - z^\mu(\tau, \bar{\sigma})]^2 = \epsilon ([x_l(\tau_\pm) - x_l(\tau)]^2 - [x^r_\epsilon(\tau_\pm) - x^r_\epsilon(\tau) - \tilde{\zeta}(\tau, \bar{\sigma})]^2) = 0 \) can be rewritten in the form

\[ x_l(\tau_+) = x_l(\tau) + |\tilde{\Delta}_+| = x_l(\tau) + |x^r_\epsilon(\tau_+) - \tilde{\xi}(\tau, \bar{\sigma})|, \]
\[ |\tilde{\Delta}_+| = |x^r_\epsilon(\tau_+) - \tilde{\xi}(\tau, \bar{\sigma})| \rightarrow \sigma \rightarrow \infty |x^r_\epsilon(\tau_+) - x^r_\epsilon(\tau) - \tilde{\xi}(\tau, \bar{\sigma})|, \]
\[ x_l(\tau_-) = x_l(\tau) - |\tilde{\Delta}_-| = x_l(\tau) - |x^r_\epsilon(\tau_-) - \tilde{\xi}(\tau, \bar{\sigma})|, \]
\[ |\tilde{\Delta}_-| = |x^r_\epsilon(\tau_-) - \tilde{\xi}(\tau, \bar{\sigma})| \rightarrow \sigma \rightarrow \infty |x^r_\epsilon(\tau_-) - x^r_\epsilon(\tau) - \tilde{\xi}(\tau, \bar{\sigma})|. \quad (3.10) \]

It can be shown [62] that, if no observer is allowed to become a Rindler observer [58], then each equation admits a unique [63] solution \( \tau_\pm = T_\pm(\tau, \bar{\sigma}) \).

Therefore the following four data measured by the observer \( \gamma \)

\[ \tau_\pm = T_\pm(\tau, \bar{\sigma}), \]

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\[ \hat{n}_{(\tau_+)}(\theta_{(\tau_+)}), \phi_{(\tau_+)})) = (\sin \theta_{(\tau_+)} \sin \phi_{(\tau_+}), \sin \theta_{(\tau_+)} \cos \phi_{(\tau_+)}), \cos \theta_{(\tau_+)} \),

\[ = \frac{\vec{\Delta}_+}{|\Delta_+|} = \frac{\vec{x}_e(\tau_+) - \vec{x}_e(\tau) - \vec{\zeta}(\tau, \sigma)}{|\vec{x}_e(\tau_+) - \vec{x}_e(\tau) - \vec{\zeta}(\tau, \sigma)|}_{|\tau_\pm = \tau_\pm(\tau_\sigma)}, \quad \hat{\tau}(\tau, \vec{\sigma}) = \hat{n}(\tau, \sigma), \quad (3.11) \]
can be inverted to get the adapted coordinates \( \tau(P), \vec{\sigma}(P) \) of the event \( P \) with 4-coordinates \( z^\mu(\tau, \vec{\sigma}) \) in terms of the data (Einstein’s convention for the radar time would be \( \mathcal{E} = \frac{1}{2} \))

\[ \tau(P) = \tau_\pm = \tau_\pm(\tau_\sigma), \quad \tau = \frac{\tau_+ + \tau_-}{2} = \frac{1}{2} (\tau_+ + \tau_-), \quad \mathcal{E} = \frac{1}{2}, \]

\[ \sigma = |\vec{\zeta}(\tau, \sigma)| = \frac{1}{2} (\tau_+ - \tau_-), \quad \vec{\zeta}(\tau, \sigma) = -\frac{1}{2} (\tau_+ - \tau_-) \hat{n}_e(\tau_\sigma), \]

\[ \sigma^r = \mathcal{G}^r = \frac{1}{2} (\tau_+ - \tau_-) (R^{-1})^s_{\tau} \left( \frac{\tau_+ + \tau_-}{2}, \frac{\tau_+ - \tau_-}{2} \right) \hat{n}_e(\tau_\sigma); \]

\[ \text{i) for } z^\mu(\tau) = \tau \Gamma^\mu (\text{inertial observer with world-line orthogonal to } \Sigma_\tau; \vec{x}_e(\tau) = \vec{0}) \text{ we get the Einstein’s convention for the radar time, because we have}

\[ \tau_\pm = \tau \pm |\vec{\zeta}(\tau, \sigma)|, \quad \tau = \tau_\pm + \frac{1}{2} (\tau_+ - \tau_-) = \frac{1}{2} (\tau_+ + \tau_-), \quad \mathcal{E} = \frac{1}{2}, \]

\[ \sigma = |\vec{\zeta}(\tau, \sigma)| = \frac{1}{2} (\tau_+ - \tau_-), \quad \vec{\zeta}(\tau, \sigma) = -\frac{1}{2} (\tau_+ - \tau_-) \hat{n}_e(\tau_\sigma), \]

\[ \sigma^r = \mathcal{G}^r = \frac{1}{2} (\tau_+ - \tau_-) (R^{-1})^s_{\tau} \left( \frac{\tau_+ + \tau_-}{2}, \frac{\tau_+ - \tau_-}{2} \right) \hat{n}_e(\tau_\sigma); \]

\[ \text{ii) for } z^\mu(\tau) = \tau \left[ \Gamma^\mu + \epsilon_\mu^a \vec{a}^r \right] (\text{inertial observer with world-line non-orthogonal to } \Sigma_\tau; \vec{x}_e(\tau) = \tau \vec{a}), \text{ after some straightforward calculations, we get}

\[ \tau_\pm = \tau \pm \frac{1}{1 - \vec{a}^2} \left[ -\vec{a} \cdot \vec{\zeta}(\tau, \sigma) \pm \sqrt{(\vec{a} \cdot \vec{\zeta}(\tau, \sigma))^2 + (1 - \vec{a}^2) \sigma^2} \right], \]

\[ \tau = \frac{1}{2} \left[ \tau_+ + \tau_- + \frac{\tau_+ - \tau_-}{1 - \vec{a}^2} \sqrt{\frac{-\vec{a}^2 + \vec{a} \cdot \hat{n}(\tau_\sigma)}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2 \vec{a}^2) \vec{a} \cdot \hat{n}(\tau_\sigma)}} \right], \]

\[ \mathcal{E} = \frac{1}{2} \left[ 1 + \frac{1}{1 - \vec{a}^2} \sqrt{-\vec{a}^2 + \vec{a} \cdot \hat{n}(\tau_\sigma)} \right] \]
\[ \sigma = |\zeta(\tau, \vec{\sigma})| = \frac{1}{2} (\tau_+ - \tau_-) \sqrt{\frac{1 + \vec{a}^2 + 2 \vec{a} \cdot \hat{n}_r(\tau_-)}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2 \vec{a}^2) \vec{a} \cdot \hat{n}_r(\tau_-)}}, \]

\[ \zeta^r(\tau, \vec{\sigma}) = - \frac{\sqrt{\alpha^2 + \vec{a} \cdot \hat{n}_r(\tau_-)}}{\sqrt{1 + \vec{a}^2 + 2 \vec{a} \cdot \hat{n}_r(\tau_-)}} \frac{\alpha^r + \hat{n}_r^r(\tau_-)}{1 - \vec{a}^2}, \]

\[ \sigma^r = \mathcal{G}^r = - \frac{1}{2} (\tau_+ - \tau_-) \left( 1 + \sqrt{\frac{\alpha^2 + \vec{a} \cdot \hat{n}_r(\tau_-)}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2 \vec{a}^2) \vec{a} \cdot \hat{n}_r(\tau_-)}} \right) \]

\[ (R^{-1})_s^r \left( \frac{1}{2} \left[ \tau_+ + \tau_- + \frac{\tau_+ - \tau_-}{1 - \vec{a}^2} \sqrt{\frac{\alpha^2 + \vec{a} \cdot \hat{n}_r(\tau_-)}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2 \vec{a}^2) \vec{a} \cdot \hat{n}_r(\tau_-)}} \right] \right), \]

\[ \frac{1}{2} (\tau_+ - \tau_-) \sqrt{\frac{1 + \vec{a}^2 + 2 \vec{a} \cdot \hat{n}_r(\tau_-)}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2 \vec{a}^2) \vec{a} \cdot \hat{n}_r(\tau_-)}}, \]

\[ \frac{\alpha^s + \hat{n}_r^s(\tau_-)}{1 - \vec{a}^2}. \]

iii) for non-inertial trajectories \( x^\mu(\tau) = f(\tau) \ell^\mu + \epsilon^\mu_r g^r(\tau) [\epsilon [j^2(\tau) - \sum_r \hat{g}(\tau) \hat{g}^r(\tau)] > 0] \)

the evaluation of \( \mathcal{E} \) and \( \mathcal{G} \) cannot be done analytically, but only numerically.

In conclusion, each admissible 3+1 splitting of Minkowski space-time together with an accelerated observer of world-line \( \gamma \) defines four functions \( \mathcal{E}(\tau_-, \hat{n}_r(\tau_-), \tau_+), \mathcal{G}(\tau_-, \hat{n}_r(\tau_-), \tau_+) \) describing the modification of the Einstein \( \frac{1}{2} \) convention associated to the observer-dependent radar coordinates describing the simultaneity surfaces of the foliation. Since the admissible foliation of Minkowski space-time is well defined at spatial infinity, the functions \( \mathcal{E} \) and \( \mathcal{G} \) will have a finite limit for \( \tau_s \to \pm \infty \), i.e. at spatial infinity on \( \Sigma_r \).

Let us add a remark on the one-way velocity of light associated to these modifications of Einstein convention for the synchronization of distant clocks.

Given an admissible embedding \( z^\mu(\tau, \vec{\sigma}) \) and its associated 4-metric \( g_{AB}(\tau, \vec{\sigma}) \), the instantaneous spatial distance between two events \( P \) and \( Q \) on the same hyper-surface \( \Sigma_r \) is given by the line element on \( \Sigma_r \): \( dl(\tau, \vec{\sigma}) = \sqrt{\epsilon g_{rs}(\tau, \vec{\sigma}) d\sigma^r d\sigma^s} \). If \( P \) and \( Q \) have a finite separation, their spatial distance is obtained by integrating this line element along a geodesic on \( \Sigma_r \) connecting them, \( \Gamma(P, Q) \): \( D_{\tau}(P, Q) = \int_{\Gamma(P, Q)} dl(\tau, \vec{\sigma}) \).

If \( Q \) is on the observer world-line \( \gamma \), with coordinates \( (\tau, \vec{0}) \), and the nearby \( P \) has coordinates \( (\tau, \vec{d}) \), then we have \( dl(\tau, \vec{0}) = \sqrt{\epsilon g_{rs}(\tau, 0) d\sigma^r d\sigma^s} \) (\( g_{rr}(\tau, 0) \) defines the relation between the \( \tau \) and the observer proper time \( T_{\gamma} \)). Then the coordinates of \( P \) can be parametrized in the form \( d\sigma^r = \hat{n}^r d\tau_{\gamma}(P, Q) \), where \( \hat{n}^r \) are the components of a unit 3-vector \( [\epsilon g_{rs}(\tau, 0) \hat{n}^r \hat{n}^s = -1] \).
We have now all the elements to define the instantaneous velocity of light for a ray emitted by the observer $\gamma$ at $Q$ (on $\Sigma_\tau$) in the direction $\hat{n}^r$. If we assume that the ray of light is received at the event $P$ with radar coordinates $\left(\tau + d\tau_P, d\sigma^r = \hat{n}^r d\ell_r(P, Q)\right)$, the instantaneous velocity of light is $c_- = \frac{d\ell_r(P, Q)}{d\tau_P}$, if the infinitesimal time difference is given by the future pointing solution of the equation $g_{\tau\tau}(\tau, 0)(d\tau_P)^2 + 2g_{\tau r}(\tau, 0)\hat{n}^r d\tau_P - [d\ell_r(P, Q)]^2 = 0$. Therefore, we obtain the following non-isotropic, synchronization-dependent velocity of light

$$c_- = g_{\tau r}(\tau, 0)\hat{n}^r + \sqrt{(g_{\tau r}(\tau, 0)\hat{n}^r)^2 + g_{\tau\tau}(\tau, 0)}.$$  \hspace{1cm} (3.13)

Following Synge [64] and using the functions $T_{\pm}(\tau, \vec{\sigma})$ of Eqs.(3.11), we can show that we have $[d\ell_r(P, Q)]^2 = \frac{1}{g_{\tau\tau}(\tau, 0)} \left( \frac{\partial T_+}{\partial \tau}(\tau, 0) \frac{\partial T_-}{\partial \sigma^r}(\tau, 0) d\sigma^r + g_{\tau r}(\tau, 0) \right)$. As a consequence, an observer choice of the synchronization convention, i.e. of $T_{\pm}(\tau, \vec{\sigma})$, determines both the infinitesimal spatial distance and the instantaneous velocity of light.

Instead, in Section 4 of Ref.[37], there is an attempt to define a mean spatial distance and mean one-way light velocities, when the observer $\gamma$ emits a ray of light at $\tau_-\ (\text{event } Q_-)$ and reabsorbs it at $\tau_+\ (\text{event } Q_+)$ after a reflection at $P$. If we call $|\vec{\Delta}_-|$ the light distance of $Q_-\text{ on } \gamma$ to $P$ and $|\vec{\Delta}_+|$ the light distance of $P$ to $Q_+\text{ on } \gamma$ (these are mean distances) we get the following two mean one-way velocities of light (with $c = 1$) in coordinates adapted to the given notion of simultaneity

$$c_- = \frac{|\vec{\Delta}_-|}{\tau - \tau_-} = \frac{|\vec{\Delta}_-|}{\mathcal{E}(\tau_+ - \tau_-)} = \frac{2 \eta |\vec{\Delta}|}{\mathcal{E}(\tau_+ - \tau_-)}, \hspace{1cm} \text{from } Q_-\text{ to } P,$$

$$c_+ = \frac{|\vec{\Delta}_+|}{\tau_+ - \tau} = \frac{|\vec{\Delta}_+|}{(1 - \mathcal{E})(\tau_+ - \tau_-)} = \frac{2 (1 - \eta) |\vec{\Delta}|}{(1 - \mathcal{E})(\tau_+ - \tau_-)}, \hspace{1cm} \text{from } P\text{ to } Q_+,$$

$$|\vec{\Delta}| \overset{\text{def}}{=} \frac{1}{2} (|\vec{\Delta}_+| + |\vec{\Delta}_-|), \quad \eta \overset{\text{def}}{=} \frac{|\vec{\Delta}_-|}{|\vec{\Delta}_+| + |\vec{\Delta}_-|}.$$  \hspace{1cm} (3.14)

If $c_{\tau} = \frac{2|\vec{\Delta}|}{\tau_+ - \tau_-}$ is the isotropic average round-trip $\tau$-coordinate velocity of light, we get $c_+ = \frac{1}{1 - \mathcal{E}} c_\tau$, $c_- = \frac{\eta}{\mathcal{E}} c_\tau$.

If $x^\mu(\tau)$ is a straight-line (inertial observer) we can adopt Einstein’s convention $\mathcal{E} = \frac{1}{2}$,
i.e. \( \tau(P) = \frac{1}{2}(\tau_+ + \tau_-) \) and \( |\vec{\sigma}| = |\vec{\sigma}| = \frac{1}{2}(\tau_+ - \tau_-) \) (hyper-planes orthogonal to the observer world-line). This implies \( |\vec{\Delta}_+| = |\vec{\Delta}_-| \) and \( \eta = \frac{1}{2} \).

Instead, if we ask \( c_\tau = c_+ = c_- \), i.e. isotropy of light propagation, we get \( E = \eta \). The conclusion of Ref. [37] is that once we have made a convention on two of the quantities spatial distance, one-way speed of light and simultaneity, the third one is automatically determined. We have seen that at the level of exact, not mean, quantities the convention about simultaneity and the associated geodesic spatial distance on \( \Sigma_\tau \) are enough to determine the one-way velocity of light.

C. The Inverse Problem and the Empirical Determination of a Set of Radar Coordinates.

We can now formulate an inverse problem: which are the restrictions on four functions \( \mathcal{E}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+), \vec{G}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+) \), so that they describe the modification of Einstein \( \frac{1}{2} \) convention for an accelerated observer using the leaves of an admissible 3+1 splitting of Minkowski space-time as simultaneity surfaces?

Let us consider an infinitesimal displacement \( \delta z^\mu = z^\mu(\tau + \delta \tau, \vec{\sigma} + \delta \vec{\sigma}) - z^\mu(\tau, \vec{\sigma}) \) of \( P \) on \( \Sigma_\tau \) to \( P' \) on \( \Sigma_{\tau + \delta \tau} \). The event \( P' \) will receive light signals from the event \( Q(\tau_+ + \delta \tau_-) \) on \( \gamma \) and will reflect them towards the event \( Q(\tau_+ + \delta \tau_+) \) on \( \gamma \). Now, using \( \Delta_\pm^2 = 0 \), we have \( \Delta'_\pm = \Delta^\mu_\pm + \dot{x}^\mu(\tau_\pm) \delta \tau_\pm - \delta z^\mu \) and \( \Delta'^2_\pm = 2 \Delta^\mu_\pm [\dot{x}^\mu(\tau_\pm) \delta \tau_\pm - \delta z^\mu] + (\text{higher order terms}) \). As a consequence we get (see Ref.[13])

\[
\frac{\partial \tau_\pm}{\partial z^\mu} = \frac{\Delta^\mu_\pm}{\Delta_\pm \cdot \dot{x}(\tau_\pm)},
\]

with \( \epsilon \Delta_+ \cdot \Delta_- < 0, \quad \epsilon \dot{x}(\tau_+) \cdot \Delta_+ > 0, \quad \epsilon \dot{x}(\tau_-) \cdot \Delta_- < 0. \) (3.15)

Since \( \frac{\partial \tau(P)}{\partial z^\mu} \) is a time-like 4-vector orthogonal to \( \Sigma_\tau \), it must be proportional to the normal \( l^\mu \) to the space-like hyper-planes of the foliation (2.14) till now considered. For a general admissible foliation we have (from \( \Delta_+^2 = 0 \) we get \( \Delta_- \cdot \frac{\partial \Delta_-}{\partial z^\mu} = 0 \) and then \( \Delta_+^\mu \frac{\partial \hat{n}_{\tau_+}}{\partial z^\mu} = 0; \) instead in general \( \Delta_+^\mu \frac{\partial \hat{n}_{\tau_+}}{\partial z^\mu} \neq 0 \)
\[
\frac{\partial \tau(P)}{\partial z^\mu} = \left[ E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \right] \frac{\partial \tau_+}{\partial z^\mu} + \\
+ \left[ 1 - E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_-} \right] \frac{\partial \tau_-}{\partial z^\mu} + \\
+ (\tau_+ - \tau_-) \frac{\partial E}{\partial \hat{n}(\tau_-)} \frac{\partial \hat{n}(\tau_-)}{\partial z^\mu} = \\
= \left[ E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \right] \frac{\Delta_+ + \mu}{\Delta_+ \cdot \hat{x}(\tau_+)} + \\
+ \left[ 1 - E + (\tau_+ - \tau_-) \left( \frac{\partial E}{\partial \tau_+} + \frac{\partial E}{\partial \hat{n}(\tau_-)} \frac{\partial \hat{n}(\tau_-)}{\partial \tau_-} \right) \right] \frac{\Delta_- - \mu}{\Delta_- \cdot \hat{x}(\tau_-)} + \\
+ (\tau_+ - \tau_-) \frac{\partial E}{\partial \hat{n}(\tau_-)} \frac{\partial \hat{n}(\tau_-)}{\partial z^\mu},
\]

\[
\epsilon \left( \frac{\partial \tau(P)}{\partial z^\mu} \right)^2 = \epsilon \frac{\Delta_+ \cdot \Delta_-}{\Delta_+ \cdot \hat{x}(\tau_+) \Delta_- \cdot \hat{x}(\tau_-)} \left[ E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \right] \\
+ \left[ 1 - E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_-} \right] + (\tau_+ - \tau_-)^2 \left( \frac{\partial E}{\partial \hat{n}(\tau_-)} \right)^2 \left( \frac{\partial \hat{n}(\tau_-)}{\partial z^\mu} \right)^2 + \\
+ 2(\tau_+ - \tau_-) \left[ E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \right] \frac{\partial E}{\partial \hat{n}(\tau_-)} \frac{\Delta_+ \cdot \hat{n}(\tau_-)}{\Delta_+ \cdot \hat{x}(\tau_+)} \frac{\Delta_- - \mu}{\Delta_- \cdot \hat{x}(\tau_-)} > 0,
\]

for every \( \tau_-, \theta(\tau_-), \phi(\tau_-), \tau_+ \).

(3.16)

This is the condition on the function \( E(\tau_-, \hat{n}(\tau_-), \tau_+) \) to have an admissible foliation.

Since \( \epsilon \frac{\Delta_+ \cdot \Delta_-}{\Delta_+ \cdot \hat{x}(\tau_+) \Delta_- \cdot \hat{x}(\tau_-)} > 0 \), in the special case \( \frac{\partial E}{\partial \hat{n}(\tau_-)} = 0 \) it must be

\[
\left[ E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \right] \left[ 1 - E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_-} \right] > 0,
\]

\[
\Downarrow
\]

\[
E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_+} \leq 0, \quad 1 - E + (\tau_+ - \tau_-) \frac{\partial E}{\partial \tau_-} \leq 0.
\]

(3.17)

If \( x^\mu = z^\mu(\tau, \bar{\sigma}) \) is the admissible embedding generating the four functions \( E \) and \( \tilde{G} \), the inverse transformation \( x^\mu \mapsto (\tau(x); \bar{\sigma}(x) = \tilde{G}(x)) \) allows to define the inverse metric.
\( g^{AB}(\tau, \vec{\sigma}) = \frac{\partial \sigma^A(x)}{\partial x^\mu} \frac{\partial \sigma^B(x)}{\partial x^\nu} \eta^{\mu\nu} \), also satisfying Eqs.(2.1), and the condition (3.16) on \( \mathcal{E} \) turns out to be nothing else that \( \epsilon g^{\tau\tau}(\tau, \vec{\sigma}) > 0 \). As a consequence the four functions \( \mathcal{E} \) and \( \vec{G} \) generating an admissible 3+1 splitting of Minkowski space-time must be such that \( \mathcal{E} \) satisfies Eq.(3.16) and \( \vec{G} \) generates an inverse 3-metric \( g^{rs}(\tau, \vec{\sigma}) = \frac{\partial G^r(x)}{\partial x^\mu} \frac{\partial G^s(x)}{\partial x^\nu} \eta^{\mu\nu} \) which satisfies the conditions

\[
\epsilon g^{\tau\tau}(\tau, \vec{\sigma}) < 0, \quad \left| \frac{g^{rr} g^{rs}}{g^{ss}} \right| (\tau, \vec{\sigma}) > 0, \quad \epsilon \det |g^{rs}(\tau, \vec{\sigma})| < 0,
\]

\[
\Rightarrow \quad \det |g^{AB}(\tau, \vec{\sigma})| < 0. \tag{3.18}
\]

Moreover, for \( \sigma \to \infty \) the quantities \( \left( \frac{\partial \tau(x)}{\partial x^\mu}, \frac{\partial \tau'(x)}{\partial x^\mu} \right) \) must tend to a constant limit \( \epsilon^A_\mu = (\epsilon^r_\mu; \epsilon^\gamma_\mu) \), where the asymptotic cotetrad \( \epsilon^A_\mu \) is dual to the asymptotic tetrad \( \epsilon^\mu_A \) appearing in the embedding of the leaves of the admissible 3+1 splitting.

Therefore, given the world-line \( x^\mu(\tau) \) of an observer \( \gamma \) and four functions \( 0 < \mathcal{E}(\tau_-, \hat{n}(\tau_-), \tau_+) < 1 \) and \( \vec{\mathcal{G}}(\tau_-, \hat{n}(\tau_-), \tau_+); \to \tau_- \to \tau_+ 0 \), with a finite limit for \( \tau_\pm \to \pm \infty \), with \( \mathcal{E}(\tau_-, \hat{n}(\tau_-), \tau_+) \) satisfying Eq.(3.16) and with \( \vec{\mathcal{G}}(\tau_-, \hat{n}(\tau_-), \tau_+) \) satisfying Eqs.(3.18), we can build the admissible observer-dependent 4-coordinates \( \tau, \vec{\sigma} \) of a \( \gamma \)-dependent notion of simultaneity, because Eqs.(3.16) and (3.18) ensure that the surfaces \( \Sigma_\tau \) are the space-like leaves of an admissible 3+1 splitting.

As a consequence, to give four admissible functions \( \mathcal{E}(\tau_-, \hat{n}(\tau_-), \tau_+) \), \( \vec{\mathcal{G}}(\tau_-, \hat{n}(\tau_-), \tau_+) \) is equivalent to define an operational method to build a grid of radar 4-coordinates associated with the arbitrarily given time-like world-line \( x^\mu(\tau) = z^\mu(\tau, \vec{0}) \) of the spacecraft \( \gamma \), i.e. of an accelerated observer. The reconstruction of the admissible embedding \( z^\mu(\tau, \vec{\sigma}) \) (we have used Eq.(2.14) as an example) is done locally [65] by the observer \( \gamma \) with a suitable computer software, which, starting from the four functions, builds the associated simultaneity surfaces on which the clocks of the spacecrafts \( \gamma_i \) are synchronized. This procedure simulates the use of light signals emitted by \( \gamma \) and reflected towards \( \gamma \) from the other spacecrafts \( \gamma_i \). This justify the name radar 4-coordinates.

In general relativity on globally hyperbolic space-times, we can define in a similar way the admissible, dynamically determined [40, 47], global notions of simultaneity and the admissible one-way velocities of test light. Then the knowledge of the functions \( \mathcal{E} \) and \( \vec{\mathcal{G}} \), associated to an admissible notion of simultaneity, will allow an operational determination
of the 4-coordinates \((\tau, \vec{\sigma})\) adapted to the chosen notion of simultaneity with simultaneity surfaces \(\tau = \text{const.}\) as radar coordinates. This is a step towards implementing the operational definition of space-time proposed in Refs. [46, 47]. The lacking ingredient is an operational confrontation of the tetrads \((\gamma_i)\epsilon^\mu_A(\tau)\) with the tetrad \((\gamma)\epsilon^\mu_A(\tau)\) of the reference world-line: this would allow a determination of the 4-metric in the built radar 4-coordinates on a finite region of space-time around the \(N + 1\) spacecrafts of the GPS type, whose trajectories are supposed known (for instance determined with the standard techniques of space navigation [23] controlled by a station on the Earth). See Refs.[66] for other approaches to GPS type coordinates.
IV. CONCLUSIONS.

Traditionally, due to the relativity principle, special relativity is presented in inertial frames and the role of the instantaneous 3-space of every inertial observer is taken by the $x^o = \text{const.}$ hyper-planes, on which all the clocks are assumed to be synchronized. This choice implies the orthogonality of the simultaneity surfaces to the straight world-lines of the inertial observers and Einstein $\frac{1}{2}$ convention for the synchronization of distant clocks.

However, actual observers are always non inertial and rotating. Till now all the efforts to define an instantaneous 3-space for an accelerated observer tried to preserve the two properties of orthogonality of the simultaneity surfaces to the observer world-line and of Einstein synchronization. The consequence of these requirements was the appearance of coordinate singularities: Fermi coordinates are only a local coordinate chart defined in a suitable world-tube around the accelerated observer. Analogously, in extended rotating relativistic systems like the rotating disk there is no accord on which can be the disk instantaneous 3-space and the use of Einstein convention, starting from the rotation axis, leads to synchronization gaps and discontinuities (see Ref.[21] and its bibliography). Again there are coordinate singularities signalled by the pathologies of the rotating 4-metrics.

With only a local coordinate chart instead of an equal-time Cauchy surface with a good atlas of coordinates, the accelerated observer cannot integrate Maxwell equations and check the validity of the conservation laws.

These problems become worse in general relativity, even in the case of globally hyperbolic space-times. Fermi or Martzke-Wheeler 4-coordinates again constitute only local coordinate charts around the (either geodesic or accelerated) world-line of a time-like observer and the use of Einstein convention [67] for the definition of an instantaneous 3-space (identified with a tangent space) has a limited range of validity.

In this paper we have emphasized that, given an arbitrary accelerated observer, the definition of an instantaneous 3-space is both observer-dependent and conventional and that there are as many possibilities as admissible 3+1 splittings of space-time, namely of nice foliations with space-like hyper-surfaces satisfying Møller and asymptotic (at spatial infinity) admissibility conditions, implying for instance the non-existence of relativistic rigidly rotating frames (only differential rotations are admissible). Each admissible 3+1 splitting allows to find an observer-adapted globally defined radar 4-coordinate system for non-inertial frames centered on anyone of the time-like observers of the two naturally associated congru-
ences and a globally defined notion of instantaneous 3-space which is also a good Cauchy surface for the equations of motion. Besides the Eulerian observers with world-lines orthogonal to the simultaneity surfaces, each admissible 3+1 splitting identifies a non-orthogonal congruence of time-like observers with non-zero vorticity, whose unit 4-velocity field (the evolution vector field of the foliation) simulates a rotating disk. In this way a foliation, whose leaves are genuine instantaneous 3-spaces for a rotating disk, may be identified with standard notions of spatial distances and one-way velocity of light based on a well defined modification of Einstein convention.

These notions of instantaneous 3-space for an accelerated observer are globally defined and the synchronization of distant clocks is done with a generalization of Einstein convention (following an old suggestion of Havas [39]). We have also suggested an empirical method for a local construction of this type of radar coordinates around an accelerated observer (a spacecraft of a cluster like in GPS).

Finally, when we have a Lagrangian description of an isolated system in special relativity, it can be generalized to a parametrized Minkowski theory in which the embeddings of the simultaneity surfaces are configuration gauge variables. The restricted general covariance (frame-preserving diffeomorphisms) of these theories implies that all the admissible conventions about the instantaneous 3-space, to be also used as Cauchy surface for the equations of motion of the system, are gauge equivalent. Each gauge choice identifies a non-rigid non-inertial frame (i.e. a physical extended non-inertial laboratory)) with a notion of instantaneous 3-space and the gauge transformations are nothing else that the coordinate transformations among such frames.

In general relativity in globally hyperbolic, asymptotically flat at spatial infinity space-times without super-translations, we have the same pattern if we use the Hamiltonian treatment of metric and tetrad gravity developed in Refs.[46, 47]. As shown in Refs.[3, 21, 39] the definition of an instantaneous 3-space is again observer-dependent and conventional and all the admissible conventions (3+1 splittings) are gauge equivalent. But now, since the chrono-geometrical structure of a general relativistic space-time is dynamical, each solution of Hamilton equations (i.e. of Einstein’s equations) identifies on-shell a dynamical notion of simultaneity in each coordinate system admissible for that solution (see Refs.[40, 47] for more details). In other words, in each Einstein’s space-time there is a dynamical emergence of a notion of instantaneous 3-space in each coordinate system, namely a dynamical
definition of the non-rigid non-inertial frame using that coordinate system (i.e. a physical extended laboratory) [70].

Till now all these problems have been considered academic and all space experiments around the Earth (GPS included) have been done by replacing the conventions for the choice of an instantaneous 3-space with a set of empirical (often semi-Galilean) transformation rules and with ad hoc Sagnac corrections of the rotating clocks to match descriptions in different accelerated local coordinate systems (one of them is always a rotating Earth-fixed one).

However, the development of high precision laser cooled clocks and their synchronization with one-way light signals [22], space navigation in the solar system [23], GPS and Galileo system [24], VLBI [71], LISA [27] are pointing towards the necessity to rephrase relativistic phenomena in the 3+1 framework of this paper by using conventional globally defined observer-dependent radar coordinates not influenced by the rotation of the Earth. They would correspond to physical space laboratories (non-rigid non-inertial reference frames) with the transformation rules among them given by suitable frame preserving diffeomorphisms. In the post-Newtonian approximation to the gravitational field of the geoid these laboratories are further restricted by the requirement of reproducing the (dynamically determined) post-Newtonian 4-metric around the geoid.

Let us finish with a list of problems to be treated in the next future:

1) Write explicitly the coordinate transformations between the admissible radar coordinate systems used in Section VID of Ref.[21] for the one-way time delay of the signals between Earth and a satellite and the (locally defined) rotating laboratories fixed on the Earth.

2) Since the observer-dependent radar coordinates are Lorentz scalars, study how the phenomena of length contraction and time dilation (evident in inertial Cartesian coordinates) are reformulated in this language.

3) Include the (dynamically determined) post-Newtonian effects of the gravitational field of the solar system, like it happens with the prescribed IAU 4-metrics for the Solar and Geocentric Celestial Frames [4], which however have fixed axes identified by the fixed stars like in Minkowski space-time.

3A) If we insist that these 4-metrics live in Minkowski space-time, find the post-Newtonian Møller admissible simultaneity surfaces (they cannot be hyper-planes $x^o = \text{const.}$).
3B) If the IAU frames live in general relativistic globally hyperbolic space-times in which the Sun or the Earth are accelerated observers, find which theoretical type (harmonic, 3-orthogonal, ..) of 4-coordinates is approximated by the IAU 4-metrics and fixed axes and which are the dynamically allowed simultaneity surfaces.

4) In particular, try to understand to which theoretical type of 4-coordinates are associated the empirical NASA coordinates used for the orbits of satellites [72]. We have to identify some experiment, whose post-Newtonian description in different theoretical coordinates is possible, to be used to make a theoretical calibration of the empirical coordinates. For instance the (coordinate- and fixed star- dependent) Shapiro time delay of the geoid to be measured by the future ESA ACES mission on the synchronization of laser cooled clocks located on the Earth and on the space station [22]. Also the synchronization of the three clocks on the LISA spacecrafts will require the framework developed in this paper.

5) Try to translate the astronomical, astrophysical and cosmological definitions of distance of stars and galaxies from the Earth in a notion of simultaneity, i.e. in a convention on the synchronization of distant clocks, since, differently from experiments inside the solar system with spacecrafts with synchronized clocks, here the clock on the star is replaced by a definition of distance and a hypothesis on the one-way velocity of light. Do all these distances correspond to Einstein $\frac{1}{2}$ convention, like it is probably true for the parallax distance?
APPENDIX A: NOTION OF SIMULTANEITY ASSOCIATED TO ROTATING REFERENCE FRAMES.

In this Section we consider the inverse problem of finding a foliation of Minkowski spacetime with simultaneity surfaces associated to a given arbitrary reference frame with non-zero vorticity, namely to a time-like vector field whose expression in Cartesian 4-coordinates in an inertial system is \( \tilde{u}^\mu(x) \) with \( \tilde{u}^2(x) = \epsilon \). In other words we are looking for embeddings \( z^\mu(\tau, \vec{\sigma}) \), inverse of an admissible 4-coordinate transformation, such that we have \( \tilde{u}^\mu(z(\tau, \vec{\sigma})) = u^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma})/\sqrt{\epsilon g_{\tau\tau}(\tau, \vec{\sigma})} \). Let us remark that if the vorticity is zero, the vector field \( \tilde{u}^\mu(x) \) is surface-forming, there is a foliation whose surfaces have the normal field proportional to \( u^\mu(\tau, \vec{\sigma}) \) and these surfaces automatically give an admissible foliation with space-like hyper-surfaces of Minkowski space-time.

Let us first show that, given an arbitrary time-like vector field \( \tilde{u}^\mu(x) \), the looked for foliation exists. Let us consider the equation

\[
\tilde{u}^\mu(x) \frac{\partial s(x)}{\partial x^\mu} = 0, \tag{A1}
\]

where \( s(x) \) is a scalar field. This equation means that \( s(x) \) is constant along the integral lines \( x^\mu(s) [dx^\mu(s)/ds = \tilde{u}^\mu(x(s))] \) of the vector field, i.e. it is a comoving quantity, since

\[
\frac{ds(x(s))}{ds} = \tilde{u}^\mu(x(s)) \frac{\partial s}{\partial x^\mu}(x(s)) = 0. \tag{A2}
\]

Since Eq.(A1) has three independent solutions \( s^{(r)}(x), r = 1, 2, 3, \) they can be used to identify three coordinates \( \sigma^r(x) = s^{(r)}(x) \). Moreover the three 4-vectors \( \frac{\partial \sigma^r(x)}{\partial x^\mu} \) are space-like by construction.

Since Minkowski space-time is globally hyperbolic, there exist time-functions \( \tau(x) \) such that i) \( \tau(x) = \text{const.} \) defines space-like hyper-surfaces; ii) \( \frac{\partial \tau(x)}{\partial x^\mu} \) is a time-like 4-vector.

As a consequence we can build an invertible 4-coordinate transformation \( x^\mu \mapsto \sigma^A(x) = (\tau(x), \sigma^r(x)) \), with inverse \( \sigma^A = (\tau, \sigma^r) \mapsto x^\mu = z^\mu(\tau, \vec{\sigma}) \) for every choice of \( \tau(x) \). It can be shown that we get always a non-vanishing Jacobian [73]

\[
J = \det \left( \frac{\partial \tau(x)}{\partial x^\mu}, \frac{\partial \sigma^r(x)}{\partial x^\mu} \right) \neq 0. \tag{A3}
\]

By using
\[
\frac{\partial \sigma^A(x)}{\partial x^\nu} \frac{\partial x^\mu}{\partial \sigma^A(\sigma(x))} = \eta^\mu_{\nu},
\] 
\[(A4)\]

and Eq. (A1) we get the desired result

\[
\tilde{u}^\mu(x) = \tilde{u}^\nu(x) \frac{\partial \sigma^A(x)}{\partial x^\nu} \frac{\partial x^\mu}{\partial \sigma^A(\sigma(x))} = \left( \tilde{u}^\nu(x) \frac{\partial \tau(x)}{\partial x^\nu} \right) \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \tau} = \frac{z^\mu(\tau, \vec{\sigma})}{\sqrt{\epsilon g_{\tau\tau}(\tau, \vec{\sigma})}}.
\] 

\[(A5)\]

Given a unit time-like vector field \( \tilde{u}^\mu(x) = u^\mu(\tau, \vec{\sigma}) \) such that \( u^\mu(\tau, \vec{\sigma}) \to |\vec{\sigma}| \to \infty n^\mu(\tau) \) and \( \frac{\partial u^\mu(\tau, \vec{\sigma})}{\partial \sigma^r} \to |\vec{\sigma}| \to \infty 0 \), to find the embeddings \( z^\mu(\tau, \vec{\sigma}) \) we must integrate the equation

\[
\frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \tau} = f(\tau, \vec{\sigma}) u^\mu(\tau, \vec{\sigma}), \quad u^2(\tau, \vec{\sigma}) = \epsilon,
\] 

\[(A6)\]

where \( f(\tau, \vec{\sigma}) \) is an integrating factor.

Since Eq. (A6) implies \( \epsilon g_{\tau\tau}(\tau, \vec{\sigma}) = f^2(\tau, \vec{\sigma}) > 0 \), the only restrictions on the integrating factor are:

i) it must never vanish;
ii) \( f(\tau, \vec{\sigma}) \to |\vec{\sigma}| \to \infty f(\tau) \) finite.

The integration of Eq. (A2) gives

\[
z^\mu(\tau, \vec{\sigma}) = g^\mu(\vec{\sigma}) + \int_0^\tau d\tau_1 f(\tau_1, \vec{\sigma}) u^\mu(\tau_1, \vec{\sigma}),
\]

\[
\downarrow
\]

\[
z^\mu(\tau, \vec{\sigma}) = \partial_\tau g(\vec{\sigma}) + \int_0^\tau d\tau_1 \partial_\tau \left[ f(\tau_1, \vec{\sigma}) u^\mu(\tau_1, \vec{\sigma}) \right],
\]

\[
g_{\tau\tau}(\tau, \vec{\sigma}) = f(\tau, \vec{\sigma}) u_\mu(\tau, \vec{\sigma}) \left[ \partial_\tau g(\vec{\sigma}) + \int_0^\tau d\tau_1 \partial_\tau \left[ f(\tau_1, \vec{\sigma}) u^\mu(\tau_1, \vec{\sigma}) \right] \right]
\]

\[
\to |\vec{\sigma}| \to \infty f(\tau) n_\mu(\tau) \left[ \lim_{|\vec{\sigma}| \to \infty} \partial_\tau g(\vec{\sigma}) \right],
\] 

\[(A7)\]

where \( g(\vec{\sigma}) \) is arbitrary and we have assumed that the integrating factor satisfies \( \partial_\tau f(\tau, \vec{\sigma}) \to |\vec{\sigma}| \to \infty 0 \).
For the sake of simplicity let us choose $g(\vec{\sigma}) = \epsilon^\mu_r \sigma^r$ with the constant 4-vectors $\epsilon^\mu_A$ belonging to an orthonormal tetrad $\epsilon^\mu_A$. Then $g_{\tau r}(\tau, \vec{\sigma})$ has the finite limit $f(\tau) n_\mu(\tau) \epsilon^\mu_r$.

With this choice for $g(\vec{\sigma})$ we get

$$z^\mu_\tau(\tau, \vec{\sigma}) = [\delta_{rs} + \alpha_{rs}(\tau, \vec{\sigma})] \epsilon^\mu_s + \beta_r(\tau, \vec{\sigma}) \epsilon^\mu_r,$$

$$\alpha_{rs}(\tau, \vec{\sigma}) = \int^\tau_0 d\tau_1 \partial_r [f(\tau_1, \vec{\sigma}) \epsilon_{s\mu} \mu_{\tau}(\tau_1, \vec{\sigma})],$$

$$\beta_r(\tau, \vec{\sigma}) = \int^\tau_0 d\tau_1 \partial_r [f(\tau_1, \vec{\sigma}) \epsilon_{r\mu} \mu_{\tau}(\tau_1, \vec{\sigma})].$$

(A8)

Since $u_{\mu}(\tau, \vec{\sigma})$ and $\epsilon^\mu_r$ are future time-like $[\epsilon_\nu u^\nu(\tau, \vec{\sigma}) > 0, \epsilon_\nu \epsilon^\nu_r > 0]$, we have $u_{\mu}(\tau, \vec{\sigma}) = \epsilon a(\tau, \vec{\sigma}) \epsilon^\mu_r + b_r(\tau, \vec{\sigma}) \epsilon^\mu_r$ with $a(\tau, \vec{\sigma}) > 0$ and without zeroes.

Let us determine the integrating factor $f(\tau, \vec{\sigma})$ by requiring $\beta_r(\tau, \vec{\sigma}) = 0$ as a consequence of the equation

$$0 = \epsilon \partial_r [f(\tau, \vec{\sigma}) \epsilon_{r\mu} \mu_{\tau}(\tau, \vec{\sigma})] = f(\tau, \vec{\sigma}) \partial_r a(\tau, \vec{\sigma}) + \partial_r f(\tau, \vec{\sigma}) a(\tau, \vec{\sigma}),$$

\[\downarrow\]

$$f(\tau, \vec{\sigma}) = e^{c(\tau)} a(\tau, \vec{\sigma}),$$

$$z^\mu_\tau(\tau, \vec{\sigma}) = [\delta_{rs} + \alpha_{rs}(\tau, \vec{\sigma})] \epsilon^\mu_s,$$

$$\alpha_{rs}(\tau, \vec{\sigma}) = \int^\tau_0 d\tau_1 e^{c(\tau_1)} \partial_r [a(\tau_1, \vec{\sigma}) \epsilon_{s\mu} \mu_{\tau}(\tau_1, \vec{\sigma})],$$

$$g_{rs}(\tau, \vec{\sigma}) = -\epsilon \left( \delta_{rs} + \alpha_{rs}(\tau, \vec{\sigma}) + \alpha_{sr}(\tau, \vec{\sigma}) + \sum_u \alpha_{ru}(\tau, \vec{\sigma}) \alpha_{su}(\tau, \vec{\sigma}) \right).$$

(A9)
Let us choose the arbitrary function $C(\tau) = e^{\epsilon(\tau)}$ so small that $|\alpha_{rs}(\tau, \sigma)| << 1$ for every $r, s, \tau, \sigma$, so that all the conditions on $g_{rs}(\tau, \sigma)$ from Eqs.(2.1) are satisfied.

In conclusion given an arbitrary congruence of time-like world-lines, described by a vector field $\tilde{u}^\mu(x)$, an embedding defining a good notion of simultaneity is 

$$[x^\mu(\tau) \overset{\text{def}}{=} z^\mu(\tau, 0)]$$

$$z^\mu(\tau, \sigma) = \epsilon^\mu_\nu \sigma^\nu + \int_{\tau}^{\sigma} d\tau_1 C(\tau_1) \epsilon_{\tau \nu} u^\nu(\tau_1, \sigma) u^\mu(\tau_1, \sigma) = x^\mu(\tau) + \epsilon^\mu_\nu \sigma^\nu + \int_{\tau}^{\sigma} d\tau_1 C(\tau_1) \epsilon_{\tau \nu} \left[ u^\nu(\tau_1, \sigma) u^\mu(\tau_1, \sigma) - u^\nu(\tau, 0) u^\mu(\tau, 0) \right],$$

(A10)

for sufficiently small $C(\tau)$. Here $\epsilon^\mu_A$ is an arbitrary orthonormal tetrad.

As a consequence, given any congruence associated to a rotating disk, we can find admissible 3+1 splittings of Minkowskki space-time, with the space-like simultaneity leaves not orthogonal to the rotation axis of the disk, which allow to define genuine instantaneous 3-spaces with synchronized clocks for every rotating disk. See Ref.[21] (Sections VIB and C) for the 3+1 treatment of the rotating disk and the Sagnac effect along these lines.
The time-like axis is given by observer's unit 4-velocity, while the spatial axes correspond to a conventional choice of the orientation and transport of three mutually orthogonal gyroscopes. The Minkowski metric has signature $\epsilon (\pm - - -)$ with $\epsilon = \pm 1$ according to the particle physics ($\epsilon = 1$) or general relativity ($\epsilon = -1$) convention.

For an inertial observer $x^\alpha$ coincides with the proper time $\tau$ of the clock on $\gamma$ since the inertial frame is the rest frame.

A.Einstein, *On the Electrodynamics of Moving Bodies*, in *The Principle of Relativity* (Dover, New York, 1962), pp.37-65 [originally published in Ann.Phys.(Leipzig) 17, 891 (1905)]. *Relativity, the Special and General Theory* (Methuen, London, 1920).

M.Soffel, S.A.Klioner, G.Petit, P.Wolf, S.M.Kopeikin, P.Bretagnon, V.A.Brumberg, N.Capitaine, T.Damour, T.Fukushima, B.Guinot, T.Huang, L.Lindegren, C.Ma, K.Nordtvedt, J.Ries, P.K.Seidelmann, D.Vokrouhlický, C.Will and Ch.Xu, *The IAU 2000 Resolutions for Astrometry, Celestial Mechanics and Metrology in the Relativistic Framework: Explanatory Supplement* (astro-ph/0303376).

B.Guinot, *Application of General Relativity to Metrology*, Metrologia 34, 261 (1997).

E.Fermi, *Sopra i fenomeni che avvengono in vicinanza di una linea oraria*, Atti Acad.Naz. Lincei Rend. Cl.Sci.Fiz.Mat.Nat. 31, 184-187 and 306-309 (1922).

F.K.Manasse and C.W.Misner, *Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry*, J.Math.Phys. 4, 735 (1963).

W.T.Ni and M.Zimmermann, *Inertial and Gravitational Effects in the Proper Reference Frame of an Accelerated, Rotating Observer*, Phys.Rev. D17, 1473 (1978).

W.Q.Li and W.T.Ni, *Coupled Inertial and Gravitational Effects in the Proper Reference Frame of an Accelerated, Rotating Observer*, J.Math.Phys. 20, 1473 (1979); *Expansions of the Affinity, Metric and Geodesic Equations in Fermi Normal Coordinates about a Geodesic*, J.Math.Phys. 20, 1925 (1979).

K.P.Marzlin, *On the Physical Meaning of Fermi Coordinates*, Gen.Rel.Grav. 26, 619 (1994); *Fermi Coordinates for Weak Gravitational Fields*, Phys.Rev. D50, 888 (1994); *What is the Reference Frame of an Accelerated Observer?*, Phys.Lett. A215, 1 (1996).

B.Mashhoon, *On Tidal Phenomena in a Strong Gravitational Field*, Astrophys.J. 197, 705 (1975); *Tidal Radiation*, Astrophys.J. 216, 591 (1977).

J.L.Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1964).

C.Chicone and B.Mashhoon, *Ultrarelativistic Motion: Inertial and Tidal Effects in Fermi Coordinates* (gr-qc/0409017); *Significance of $c/\sqrt{2}$ in Relativistic Physics* (gr-qc/0406118); *Black Holes and Ultrarelativistic Particles* (astro-ph/0406005); *Tydal Dynamics of Relativistic Flows Near Black Holes* (astro-ph/0404170); *Dynamics of Relativistic Flows*, Int.J.Mod.Phys. D13, 945 (2004) (astro-ph/0308421); *The Generalized Jacobi Equation*, Class.Quantum Grav. 19, 4231 (2002).

Here there is an identification of a tangent 3-space with a 3-manifold.

$\mathcal{L} = \frac{c^2}{a} \frac{\alpha}{\Omega}$ for an observer with translational acceleration $a$; $\mathcal{L} = \frac{c^2}{\Omega}$ for an observer rotating with frequency $\Omega$. See Refs. [10, 11].
Theory and General Relativity, in Black Holes: Theory and Observation, Lecture Notes in Physics 514, ed. F.W.Hehl, C.Kiefer and R.J.K.Metzler (Springer, Heidelberg, 1998), p.269. Acceleration-Induced Nonlocality, in Advances in General Relativity and Cosmology, ed. G.Ferrarese (Pitagora, Bologna, 2003) (gr-qc/0301065).

B.Mashhoon and U.Muench, Length Measurement in Accelerated Systems, Ann.Phys. (Leipzig) 11, 532 (2002).

[12] R.F.Marzke and J.A.Wheeler, Gravitation as Geometry- I: the Geometry of the Space-Time and the Geometrodynamical Standard Meter, in Gravitation and Relativity, eds. H.Y.Chiu and W.F.Hoffman (Benjamin, New York, 1964).

C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation (Freeman, New York, 1973).

[13] M.Pauri and M.Vallisneri, Marzke-Wheeler Coordinates for Accelerated Observers in Special Relativity, Found.Phys.Lett. 13, 401 (2000) (gr-qc/0006095).

[14] D.Bini, L.Lusanna and B.Mashhoon, Limitations of Radar Coordinates for Accelerated Observers, Int.J.Mod.Phys. D14, 1413 (2005) (gr-qc/0409052).

[15] This is the horizon problem: a time-like 4-vector becomes light-like, even if there is no real horizon like it happens for Schwartzschild black holes.

[16] A reference frame \( l^\mu \), i.e. a time-like vector field \( l^\mu(x) \frac{\partial}{\partial x^\mu} \) with its congruence of time-like world-lines and its associated 1+3 splitting of \( TM^4 \), admits the decomposition \[ P_{\mu\nu}(x) = \eta_{\mu\nu} - \epsilon_{\mu\nu\alpha\beta} l^\alpha(x) l^\beta(x) \] is the 3-metric in the rest-frame in the point \( x^\mu \), i.e. in the tangent 3-plane orthogonal to \( l^\mu(x) \); \( D^{(n)}(x) \) is the Levi-Civita covariant derivative on Minkowski space-time

\[
D^{(n)}_\mu l^\nu = l^\mu a_\nu + \frac{1}{3} \Theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu},
\]

\[
a^\mu = l^\nu D^{(n)}_\nu l^\mu,
\]

\[
\Theta = D^{(n)}_\mu l^\mu,
\]

\[
\sigma_{\mu\nu} = \frac{1}{2} (a_\mu l^\nu + a_\nu l^\mu) + \frac{1}{2} (D^{(n)}_\mu l^\nu + D^{(n)}_\nu l^\mu) - \frac{1}{3} \Theta P_{\mu\nu},
\]

with magnitude \( \sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \),

\[
\omega_{\mu\nu} = -\omega_{\nu\mu} = \epsilon_{\mu\nu\alpha\beta} \omega^\alpha l^\beta = \frac{1}{2} (a_\mu l^\nu - a_\nu l^\mu) + \frac{1}{2} (D^{(n)}_\mu l^\nu - D^{(n)}_\nu l^\mu),
\]

\[
\omega^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} \omega_{\alpha\beta} l^\gamma,
\]

where \( a^\mu \) is the 4-acceleration, \( \Theta \) the expansion (it measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence), \( \sigma_{\mu\nu} \) the shear (it measures how an initial sphere in the tangent space to the given world-line, which is Lie transported along \( l^\mu \), is distorted towards an ellipsoid with principal axes given by the eigenvectors of \( \sigma^\mu_{\nu} \), with rate given by the eigenvalues of \( \sigma^\mu_{\nu} \)) and \( \omega_{\mu\nu} \) the twist or vorticity or rotation (it measures the rotation of the nearby world-lines infinitesimally surrounding the given one); \( \sigma_{\mu\nu} \) and \( \omega_{\mu\nu} \) are purely spatial (\( \sigma_{\mu\nu} l^\nu = \omega_{\mu\nu} l^\nu = 0 \)). Due to the Frobenius theorem, the congruence is (locally) hyper-surface orthogonal, i.e. locally synchronizable [17], if and only if \( \omega_{\mu\nu} = 0 \).

[17] R.K.Sachs and H.Wu, General Relativity for Mathematicians (Springer, Berlin, 1977).

[18] M.H.Soffel, Relativity in Astrometry, Celestial Mechanics and Geodesy (Springer, Berlin, 1989).
[19] W.A.Rodrigues jr and M.Sharif, *Equivalence Principle and the Principle of Local Lorentz Invariance*, Found.Phys. 31, 1785 (2001) [erratum Found.Phys. 32, 811 (2002)].
W.A.Rodrigues jr and M.Sharif, *Rotating Frames in SRT: the Sagnac Effect and Related Issues*, Found.Phys. 31, 1767 (2001).

[20] G.Rizzi and M.L.Ruggiero eds., *Relativity in Rotating Frames. Relativistic Physics in Rotating Reference Frames*. (Kluwer, Dordrecht, 2003).
G.Rizzi and A.Tartaglia, *Speed of Light on Rotating Platforms*, Found.Phys. 28, 1663 (1998) (gr-qc/9805089); *On Local and Global Measurements of the Speed of Light on Rotating Platforms*, Found.Phys.Lett. 12, 179 (1999).
A.Tartaglia, *Lengths on Rotating Platforms*, Found.Phys.Lett. 12, 17 (1999).
G.Rizzi, M.L.Ruggiero and A.Serafini, *Synchronization Gauges and the Principles of Special Relativity* (gr-qc/0409105).
G.E.Stedman, *Ring-Laser Tests of Fundamental Physics and Geophysics*, Rep.Prog.Phys. 60, 615 (1997).
E.J.Post, *Sagnac Effect*, Rev.Mod.Phys. 39, 475 (1967).

[21] D.Alba and L.Lusanna, *Simultaneity, Radar 4-Coordinates and the 3+1 Point of View about Accelerated Observers in Special Relativity* (gr-qc/0311058).

[22] L.Blanchet, C.Salomon, P.Teyssandier and P.Wolf, *Relativistic Theory for Time and Frequency Transfer to Order 1/c3*, Astron.Astrophys. 370, 320 (2000).
A.Spallicci, A.Brillet, G.Busca, G.Catastini, I.Pinto, I.Roxburgh, C.Salomon, M.Soffel and C.Veillet, *Experiments on Fundamental Physics on the Space Station*, Class.Quant.Grav. 14, 2971 (1997).
P.Lemonde, P.Laurent, G.Santarelli, M.Abgrall, Y.Sortais, S.Bize, C.Nicolas, S.Zhang, A.Clairen, N.Dimarcq, P.Petit, A.Mann, A.Luiten, S.Chang and C.Salomon, *Cold Atom Clocks on Earth and Space*, in *Frequency Measurement and Control, Advanced Techniques and Future Trends*, ed.A.N.Luiten (Springer, Berlin, 2001).
S.Weyers, U.Hübner, R.Schröder, C.Tamm and A.Bausch, *Uncertainty Evaluation of the Atomic Caesium Fountain CSFI of the PTB*, Metrologia 38, 343 (2001).
S.Bize, Y.Sortais, M.S.Santos, C.Mandache, A.Clairen and C.Salomon, *High-Accuracy Measurement of the 87Rb Ground-State Hyperfine Splitting in an Atomic Fountain*, Europhys.Lett. 45, 558 (1999).
R.Holzwarth, Th.Udem, T.W.Hänsch, J.C.Knoght, W.J.Wadsworth and P.St.J.Russell, *Optical Frequency Synthesizer for Precision Spectroscopy*, Phys.Rev.Lett. 85, 2264 (2000).
Th.Udem, S.A.Diddams, K.R.Vogel, C.W.Oates, E.A.Curtis, W.D.Lee, W.M.Itano, R.E.Drullinger, J.C.Bergquist and L.Hollberg, *Absolute Frequency Measurements of the Hg+ and Ca Optical Clock Transitions with a Femtosecond Laser*, Phys.Rev.Lett. 86, 4996 (2001).

[23] T.B.Bahder, *Fermi Coordinates of an Observer Moving in a Circle in Minkowski Space: Apparent Behaviour of Clocks* (gr-qc/9811009); *Navigation in Curved Space-Time*, Am.J.Phys. 69, 315 (2001); *Relativity of GPS Measurement* (gr-qc/0306076).
S.G.Turyshev, *Relativistic Navigation: A Theoretical Foundation*, NASA/JPL No 96-013 (gr-qc/9606063).

[24] N.Ashby and J.J.Spilker, *Introduction to Relativistic Effects on the Global Positioning System*, in *Global Positioning System: Theory and Applications*, Vol.1, eds. B.W.Parkinson and J.J.Spilker (American Institute of Aeronautics and Astronautics, 1995).
N.Ashby, *Relativity in the Global Positioning System*, Living Reviews in Relativity (http://www.livingreviews.org).
[25] H. Thirring, *On the Effect of Rotating Distant Masses in Einstein's Theory of Gravitation*, Phys. Z. **19**, 33 (1918); *Correction to "On the Effect of Rotating Distant Masses in Einstein's Theory of Gravitation"*, Phys. Z. **22**, 29 (1921).

J. Lense and H. Thirring, *On the Influence of the Proper Rotation of Central Bodies on the Motion of Planets and Moons According to Einstein's Theory of Gravitation*, Phys. Z. **19**, 156 (1918) [English translation in B. Mashhoon, F. W. Hehl and D. S. Theiss, *On the Gravitational Effects of Rotating Masses: The Thirring-Lense Papers*, Gen. Rel. Grav. **16**, 711 (1984)].

[26] For Gravity Probe B see *Nonlinear Gravitodynamics. The Lense-Thirring Effect*, eds. R. Ruffini and C. Sigismondi (World Scientific, Singapore, 2003).

I. Ciufolini, *A Comprehensive Introduction to the LAGEOS Gravitomagnetic Experiment: from the Importance of the Gravitational Field in Physics to Preliminary Error Analysis and Error Budget*, Int. J. Mod. Phys. **A4**, 3083 (1989). *The 1995-99 Measurements of the Lense-Thirring Effect using Laser-Ranged Satellites*, Class. Quant. Grav. **17**, 2369 (2000). *Test of General Relativity: 1995-2002 Measurement of Frame Dragging*, presented at Physics in Collision, Stanford 2002 (gr-qc/0209109).

B. Mashhoon, F. Gronwald and H. M. Lichtenegger, *Gravitomagnetism and the Clock Effect*, in *Gyros, Clocks, Interferometers: Testing Relativistic Gravity in Space*, eds. C. Lämmerzahl, C. W. F. Everitt and F. W. Hehl, Lecture Notes Phys. **562**, 83 (Springer, Berlin, 2001) (gr-qc/9912027).

R. J. Jantzen, P. Carini and D. Bini, *The Many Faces of Gravito-Magnetism*, Ann. Phys. (N.Y.) **215**, 1 (1992) (gr-qc/0106043).

D. Bini and R. T. Jantzen, *Circular Holonomy, Clock Effects and Gravito-Magnetism: Still Going around in Circles after All These Years?*, Proc. of the 9th ICRA Workshop on Fermi and Astrophysics, 2001, eds. R. Ruffini and C. Sigismondi (World Scientific, 2002) (gr-qc/0202085).

D. Bini, R. T. Jantzen and B. Mashhoon, *Gravitomagnetism and Relative Observer Clock Effects*, Class. Quant. Grav. **18**, 653 (2001) (gr-qc/0012065).

[27] P. Bender and the LISA Study Team, *LISA: A Cornerstone Mission for the Observation of Gravitational Waves*, System and Technology Study Report ESA-SCI (2000) **11**, 2000.

K. Danzmann and A. Rudiger, *LISA Technology - Concept, Status, Prospects*, Class. Quantum Grav. **20**, S1-S9 (2003).

M. Tinto, F. B. Estabrook and J. W. Armstrong, *Time-Delay Interferometry for LISA*, Phys. Rev. **D65**, 082003 (2002).

N. J. Cornish and R. W. Hellings, *The Effects of Orbital Motion on LISA Time Delay Interferometry* (gr-qc/0306096).

A. Pai, K. Rajesh Nayak, S. V. Dhandhar and J. Y. Vinet, *Time Delay Interferometry and LISA Optimal Sensitivity* (gr-qc/0306057).

D. A. Shaddock, *Operating LISA as a Sagnac Interferometer* (gr-qc/0306125).

M. Tinto, F. B. Estabrook and J. W. Armstrong, *Time Delay Interferometry with Moving Spacecraft Arrays* (gr-qc/0310017).

[28] C. M. Moller, *The Theory of Relativity* (Oxford Univ. Press, Oxford, 1957).

[29] We use the vector notation $\hat{\sigma}$ for notational simplicity.

[30] L. Lusanna, *The N- and 1-Time Classical Descriptions of N-Body Relativistic Kinematics and the Electromagnetic Interaction*, Int. J. Mod. Phys. **A12**, 645 (1997).

F. Bigazzi and L. Lusanna, *Spinning Particles on Spacelike Hypersurfaces and their Rest Frame Description*, Int. J. Mod. Phys. **A14**, 1429 (1999) (hep-th/9807052).
Observables, Int. J. Mod. Phys. A14, 1877 (1999) (hep-th/9807054).

L. Lusanna, Towards a Unified Description of the Four Interactions in Terms of Dirac-Bergmann Observables, invited contribution to the book Quantum Field Theory: a 20th Century Profile, of the Indian National Science Academy, ed. A.N. Mitra, forewords by F.J. Dyson (Hindustan Book Agency, New Delhi, 2000) (hep-th/9907081).

[31] L. Lusanna, The Rest-Frame Instant Form of Metric Gravity, Gen. Rel. Grav. 33, 1579 (2001) (gr-qc/0101048).

[32] While this hypothesis (implying that rods and clocks must be assumed not to be influenced by acceleration; see for instance Ref. [28]) is verified in Newtonian mechanics and in those relativistic cases in which a phenomenon can be reduced to point-like coincidences of classical point particles and light rays (geometrical optic approximation), its validity is questionable in presence of electro-magnetic waves. As emphasized by Mashhoon [10, 11], in this case we can trust the locality hypothesis only when the wave-length $\lambda$ of the electro-magnetic wave is much shorter of the acceleration length $L$ of the observer, describing the degree of variation of its state, i.e. when $\lambda << L$. When $\lambda << L$ holds, so that the period of the wave satisfies $\frac{1}{c} << \frac{L}{c}$, the observer state does not change appreciably on the time scale needed to detect a few oscillations of the wave and to measure its frequency. Instead in the case of the electromagnetic waves radiated by an accelerating charged particle with acceleration length $L$, we have $\lambda \approx L$. In this case it is highly problematic to consider the particle momentarily equivalent to an identical comoving inertial particle. This fact is confirmed by the causality problems (pre-acceleration, runaway solutions) of the classical Abraham - Lorentz - Dirac equation of motion of the particle (see for instance Ref. [33]), which depend on the time derivative of the acceleration.

[33] C. Itzykson and J. B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 3rd printing 1987).

[34] This convention was already criticized in Refs. [35, 36, 37, 38, 39].

[35] H. Reichenbach, Axiomatik der relativistischen Raum-Zeit-Lehre (F. Vieweg and Sohn, Braunschweig, 1924). English translation: Axiomatization of the Theory of Relativity (University of California Press, Berkeley/Los Angeles, 1969). The Philosophy of Space and Time (Dover, New York, 1958).

A. Grünbaum, Amer. J. Phys. 23, 450 (1955). Philosophical Problems of Space and Time (A. A. Knopf, New York, 1963; 2nd edition Reidel, Dordrecht, 1973).

[36] R. Anderson and G. E. Stedman, Distance and the Conventionality of Simultaneity in Special Relativity, Found. Phys. Lett. 5, 199 (1992).

R. Anderson, I. Vetharaniam and G. E. Stedman, Conventionality of Synchronization, Gauge Dependence and Test Theories of Relativity, Phys. Rep. 295, 93 (1998).

[37] M. M. Capria, On the Conventionality of Simultaneity in Special Relativity, Found. Phys. 31, 775 (2001).

[38] E. Minguzzi, On the Conventionality of Simultaneity, Found. Phys. Lett. 15, 153 (2002). Simultaneity and Generalized Connections in General Relativity, Class. Quantum Grav. 20, 2443 (2003) (gr-qc/0204063).

E. Minguzzi and A. Macdonald, Universal One-Way Light Speed from a Universal Light Speed over Closed Paths, Found. Phys. Lett. 16, 587 (2003) (gr-qc/0211091).

[39] P. Havas, Simultaneity, Conventionism, General Covariance and the Special Theory of Relativity, Gen. Rel. Grav. 19, 435 (1987).

[40] L. Lusanna, The Chrono-Geometrical Structure of Special and General Relativity: A Re-
Visitation of Canonical Geometrodynamics, lectures at 42nd Karpacz Winter School of Theoretical Physics: Current Mathematical Topics in Gravitation and Cosmology, Ladek, Poland, 6-11 Feb 2006 (gr-qc/0604120).

D.Alba and L.Lusanna, The York Map as a Shanmugadhasan Canonical Transformation in Tetrad Gravity and the Role of Non-Inertial Frames in the Geometrical View of the Gravitational Field (gr-qc/0604086).

[41] D.Hilbert, Gott.Nachr. Math.-Phys. Kl. 53 (1917).
[42] R.Arnowitt, S.Deser and C.W.Misner, Canonical Variables for General Relativity, Phys.Rev. 117, 1595 (1960).
The Dynamics of General Relativity, in Gravitation: an Introduction to Current Research, ch. 7, ed.L.Witten (Wiley, New York, 1962).

[44] H.Bondi, Assumption and Myth in Physical Theory (Cambridge Univ.Press, Cambridge, 1967).
R.D’Inverno, Introducing Einstein Relativity, (Oxford Univ.Press, Oxford, 1992).
C.E.Dolby, Simultaneity and the Concept of Particle, to appear in the Proceedings of Time and Matter: An International Colloquium on the Science of Time (TAM 2002), Venice, Italy, 11-17 Aug 2002 (gr-qc/0305097).

[45] Their inverse $z^A_{\mu}(\tau, \vec{\sigma})$ are flat cotetrad fields over Minkowski space-time: $\tilde{z}^A_\mu g_{AB} z^B_\nu = \eta_{\mu\nu}$.

[46] L.Lusanna and S.Russo, A New Parametrization for Tetrad Gravity, Gen.Rel.Grav. 34, 189 (2002)(gr-qc/0102074).
R.De Pietri, L.Lusanna, L.Martucci and S.Russo, Dirac's Observables for the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge, Gen.Rel.Grav. 34, 877 (2002) (gr-qc/0105084).

[47] L.Lusanna, The Chrono-Geometrical Structure of Special and General Relativity: Towards a Background-Independent Description of the Gravitational Field and Elementary Particles, invited paper for the book Progress in General Relativity and Quantum Cosmology Research, (gr-qc/0404122).
L.Lusanna and M.Pauri, The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - I: Dynamical Synchronization and Generalized Inertial Effects, Gen.Rel.Grav. 38, 187 (2006) (gr-qc/0403081).
L.Lusanna and M.Pauri, The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - II : Dirac versus Bergmann Observables and the Objectivity of Space-Time, Gen.Rel.Grav. 38, 229 (2006) (gr-qc/0407007).
M.Pauri and M.Vallisneri, Ephemeral Point-Events: is there a Last Remnant of Physical Objectivity?, essay for the 70th birthday of R.Torretti, Dialogos 79, 263 (2002) (gr-qc/0203014).
L.Lusanna and M.Pauri, Explaining Leibniz Equivalence as Difference of Non-Inertial Appearances: Dis-solution of the Hole Argument and Physical Individuation of Point-Events, talk at the Oxford Conference on Spacetime Theory (2004), to appear in Studies in History and Philosophy of Modern Physics (gr-qc/0604087).

[48] In Ref.[39] Havas proposed the following two examples (the second one is a time-dependent transformation) of simultaneity foliations associated with admissible 4-coordinates, i.e. whose
The second non-inertial and non-surface-forming congruence of observers associated with these embeddings, with vector field \( \vec{\omega} \) comoving frame; \( \vec{x} \) comoving frame; \( \hat{\Omega} = \Omega \hat{\Omega}, \dot{\hat{\Omega}} = \hat{\Omega}^2 = 1, \Omega^u = -\frac{1}{2} \epsilon^{urs} (\hat{R} R^{-1})^r_s, \)

and with associated 4-metric

\[
g_{\mu\nu}(y) = -c^2 f^2, \quad g_{\mu i}(y) = -y^\mu c^2 f \frac{\partial f}{\partial y^i}, \quad g_{ij}(y) = \delta_{ij} - (y^\mu)^2 c^2 \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}
\]

Both of them are examples of admissible 4-coordinate systems not interpretable in terms of comoving observers as required by the locality hypothesis.

A variant are the embeddings

\[
z^\mu(\tau, \vec{\sigma}) = x^\mu_\tau(\tau) + \Lambda^{\mu}_{\nu}(\tau, \vec{\sigma}) \epsilon^A_{\nu} \sigma_A \rightarrow|\vec{\sigma}| \rightarrow \infty x^\mu_\tau(\tau) + \epsilon^\mu_\tau \tau + \epsilon^\mu_r \sigma_r = x^\mu(\tau) + \epsilon^\mu_r \sigma_r,
\]

with \( x^\mu(\tau) = x^\mu_\tau(\tau) + \epsilon^\mu_r \tau = \epsilon^\mu_r [\tau + f(\tau)] \) an arbitrary time-like straight-line (an inertial observer) not parametrized in terms of the proper time. The only difference now is that the asymptotic hyper-planes are no more uniformly spaced like in the case \( x^\mu(\tau) = x^\mu_\tau + \epsilon^\mu_r \tau \) \( (z^\mu = \epsilon^\mu_r \rightarrow z^\mu_\tau(\tau) = \epsilon^\mu_r [1 + f(\tau)]).\)

The second non-inertial and non-surface-forming congruence of observers associated with these embeddings, with vector field \( \vec{z}^\mu(\tau, \vec{\sigma}) \rightarrow|\vec{\sigma}| \rightarrow \infty \epsilon^\mu_r = l^\mu_r(\infty) \) if \( \partial_\tau \Lambda^{\mu}_{\nu}(\tau, \vec{\sigma}) \rightarrow|\vec{\sigma}| \rightarrow \infty 0.\)

R.A.Nelson, *Generalized Lorentz Transformation for an Accelerated, Rotating Frame of Reference*, J.Math.Phys. 28, 2379 (1987) [erratum J.Math.Phys. 35, 6224 (1994)].

\[
g_{\mu\nu} = \epsilon(((1 + \frac{\vec{a} \vec{x}}{c^2})^2 - (\omega \times \vec{x})^2), g_{\mu i} = -\epsilon \frac{1}{c} (\vec{\omega} \times \vec{x})^i, g_{ij} = -\epsilon \delta_{ij}, \text{ where } \vec{a} \text{ is the time-dependent acceleration of the observer’s frame of reference relative to the comoving inertial frame and } \vec{\omega} \text{ is the time-dependent angular velocity of the observer’s spatial rotation with respect to the comoving frame; } \vec{x} \text{ is the position vector of a spatial point with respect to the origin of the observer’s accelerated frame.}
\]

We use the notations \( \vec{\sigma} = \sigma \hat{\sigma}, \sigma = |\vec{\sigma}|, \vec{\Omega} = \Omega \hat{\Omega}, \dot{\vec{\Omega}} = \hat{\Omega}^2 = 1, \Omega^u = -\frac{1}{2} \epsilon^{urs} (\hat{R} R^{-1})^r_s, \)

R.M.Wald, *General Relativity* (Chicago Univ. Press, Chicago, 1984).

M.Heusler, *Black Hole Uniqueness Theorems* (Cambridge Univ. Press, Cambridge, 1996); *Stationary Black Holes: Uniqueness and Beyond*, Living Reviews in Relativity 1998 (www.livingreviews.org/Articles/Volume1/1998-Heusler).

N.Stergioulas, *Rotating Stars in Relativity*, Living Reviews in Relativity 2003 (www.livingreviews.org/lrr-2003-3).
As said in Subsections C and D of Section VI of Ref. [21], the choice \( W.Rindler, \)\footnote{J.M.Bardeen and R.Wagoner, \textit{Relativistic Disks. I. Uniform Rotation}, Ap.J. \textbf{167}, 359 (1971). E.M.Butterworth and J.R.Ipser, \textit{On the Structure and Stability of Rapidly Rotating Fluid Bodies in General Relativity. I. The Numerical Method for Computing Structure and its Application to Uniformly Rotating Homogeneous Bodies}, Ap.J. \textbf{204}, 200 (1976). N.Comins and B.F.Schutz, \textit{On the Ergoregion Instability}, Proc. R. Soc. London \textbf{A364}, 211 (1978); \textit{On the Existence of Ergoregions in Rotating Stars}, Mon.Not.R.astr.Soc. \textbf{182}, 69 (1978). J.L.Friedman, \textit{Ergosphere Instability}, Commun.Math.Phys. \textbf{63}, 243 (1978).}

\( K\text{ruskal Space and the Uniformly Accelerated Frame} \)\footnote{This is the case of a (non-time-like) Rindler observer with uniform 4-acceleration, see Ref.[58]. W.Rindler, \textit{Kruskal Space and the Uniformly Accelerated Frame}, Am.J.Phys. \textbf{34}, 1174 (1966).}

This is the case of a (non-time-like) Rindler observer with uniform 4-acceleration, see Ref.[58].

\( J.M.Bardeen and R.Wagoner, \)\footnote{W.Rindler, \textit{Kruskal Space and the Uniformly Accelerated Frame}, Am.J.Phys. \textbf{34}, 1174 (1966).}

\( \textit{Relativistic Disks. I. Uniform Rotation} \)\footnote{As said in Subsections C and D of Section VI of Ref.[21], the choice \( F(\sigma) = \frac{1 + \frac{R^2 - \sigma^2}{c^2}}{1 + \frac{\omega^2 R^2}{c^2}} \) < \( \frac{2}{1 + \frac{\omega^2 R^2}{c^2}} \)}

\( \rightarrow c \rightarrow \infty \) \( 1 + \omega^2 (R^2 - \sigma^2) + O\left(\frac{1}{c^2}\right) \), replaces the rigid rotation \( \Omega(\sigma) = \omega \) for \( \sigma < R \) of a rotating disk of radius \( R \) (with \( \omega R < c \)) with an admissible differential rotation \( \Omega(\sigma) = \omega F(\sigma) \). By varying the admissible functions \( F(\sigma) \) (a gauge transformation in parametrized Minkowski theories) we can approximate the step function \( \Omega(\sigma) = \omega \) for \( \sigma < R \), \( \Omega(\sigma) = 0 \) for \( \sigma > R \), as much as we wish without violating Eq.(2.27).

Let us remark that the conditions (2.1) may be replaced by the single condition \( \epsilon g_{\mu \nu}(y) > 0 \) plus the requirement of the existence of the inverse transformation, namely the existence of a positive Jacobian.

The emission unit 3-direction \( \hat{n}_{(\tau^-)}(\theta_{(\tau^-)}, \phi_{(\tau^-)}) \) could be replaced with a reception unit 3-direction \( \hat{n}_{(\tau^+)}(\theta_{(\tau^+)}, \phi_{(\tau^+)}) \) if more convenient.

\( P.Dombrowski, J.Kuhlmann and U.Proff, \)\footnote{P.Dombrowski, J.Kuhlmann and U.Proff, \textit{On the Spatial Geometry of a Non-Inertial Observer in Special Relativity}, in \textit{Global Riemannian Geometry}, eds. T.J.Willmore and N.J.Hitchin (Horwood, Wiley, New York, 1984).}

\( \textit{On the Spatial Geometry of a Non-Inertial Observer in Special Relativity} \)\footnote{If we introduce the function \( g_\pm(y) = x_i(y) - x_i(\tau) \pm |\vec{x}_e(y) - \vec{\xi}(\tau, \vec{\sigma})| \), Eqs. (3.10) are equivalent to \( g_\pm(y) = 0 \). The solution is unique because the functions \( g_\pm(y) \) are decreasing in \( y \), since we have \( \frac{dg_\pm(y)}{dy} = -v_l(y) \pm \sum_r v_r(y) \frac{x_r(y) - \xi^r(\tau, \vec{\sigma})}{|\vec{x}_e(y) - \vec{\xi}(\tau, \vec{\sigma})|} \). Using Eq.(2.24) in the form \( \sum_r v_r(y) \frac{x_r(y) - \xi^r(\tau, \vec{\sigma})}{|\vec{x}_e(y) - \vec{\xi}(\tau, \vec{\sigma})|} \leq |\vec{v}(y)| < v_l(y), \) we get \( \frac{dg_\pm(y)}{dy} < 0 \), since \( v_l(y) > 0 \).}

\( \textit{On the Spatial Geometry of a Non-Inertial Observer in Special Relativity} \)\footnote{This procedure replaces the determination of the 3-geodesics needed to build the Fermi normal coordinates.}

\( J.L.Synge, \)\footnote{J.L.Synge, \textit{Relativity: The General Theory} p.112 (North-Holland, Amsterdam, 1964).}

\( \textit{Relativity: The General Theory} \)\footnote{This procedure replaces the determination of the 3-geodesics needed to build the Fermi normal coordinates.}

\( C.Rovelli, \)\footnote{C.Rovelli, \textit{GPS Observables in General Relativity}, e-print 2001 (gr-qc/0110003).}

\( \textit{GPS Observables in General Relativity} \)\footnote{M.Blagojevic, J.Garecki, F.W.Hehl and Yu.N.Obukhov, \textit{Real Null Coframes in General Relativity and GPS Type Coordinates}, e-print 2001 (gr-qc/0110078).}

\( M.Blagojevic', J.Garecki, F.W.Hehl and Yu.N.Obukhov, \)\footnote{See for instance the Landau-Lifschitz 3-metric [68] and its criticism in Ref.[69] and in Section II of Ref.[21].}

\( \textit{Real Null Coframes in General Relativity and GPS Type Coordinates} \)\footnote{See for instance the Landau-Lifschitz 3-metric [68] and its criticism in Ref.[69] and in Section II of Ref.[21].}

\( \textit{Real Null Coframes in General Relativity and GPS Type Coordinates} \)
[68] L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley, Cambridge, 1951).

[69] R. N. Henriksen and L. A. Nelson, *Clock Synchronization by Accelerated Observers: Metric Construction for Arbitrary Congruences of Worldlines*, Can. J. Phys. 63, 1393 (1985).

[70] In absence of matter the dynamical admissible 3+1 splittings of Minkowski space-time considered as a special solution of Einstein equations (and not as an autonomous theory, special relativity) must have the simultaneity hyper-surfaces 3-conformally flat due to the vanishing of the Cotton-York tensor [31, 46]. This restriction does not exist in special relativity, which, therefore, admits a bigger family of non-dynamical admissible non-inertial frames.

[71] O. J. Sovers and J. L. Fanselow, *Astrometry and Geodesy with Radio Interferometry: Experiments, Models, Results*, Rev. Mod. Phys. 70, 1393 (1998).

[72] T. D. Moyer, *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation* (John Wiley, New York, 2003).

[73] Let us show that the equations

\[ \alpha \frac{\partial \tau(x)}{\partial x^\mu} + \beta_r \frac{\partial \sigma^r(x)}{\partial x^\mu} = 0 \]

implies \( \alpha = \beta_r = 0 \). If we multiply for \( \tilde{u}^\mu(x) \), we get \( \alpha \tilde{u}^\mu(x) \frac{\partial \tau(x)}{\partial x^\mu} = 0 \). But \( \frac{\partial \tau(x)}{\partial x^\mu} \) and \( \tilde{u}^\mu(x) \) are both time-like with \( \tilde{u}^\mu(x) \frac{\partial \tau(x)}{\partial x^\mu} \neq 0 \), so that we get \( \alpha = 0 \). We remain with the equations \( \beta_r \frac{\partial \sigma^r(x)}{\partial x^\mu} = 0 \), which imply \( \beta_r = 0 \) since the \( \frac{\partial \sigma^r(x)}{\partial x^\mu} \) are independent by construction.