Semiclassical description of wobbling and chiral modes in triaxial nuclei

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Abstract. A time-dependent variational principle with an angular momentum coherent state as a variational state, is used to describe the dynamics associated to a triaxial rotor Hamiltonian with rigidly aligned high-\(j\) quasiparticles. Solving the variational principle within a stereographic parametrization of the coherent state, one obtains a classical energy function and a set of canonical equations of motion expressed in terms of the azimuth angle and a canonical conjugate coordinate represented by the third projection of the total angular momentum. The system’s rotational dynamics is investigated through the evolution on total angular momentum of the canonical variables as well as spherical angles corresponding to minima of the constant energy surface. The unique minimum energy conditions are spin-dependent and define phases with specific dynamic behaviour. The transition between phases is investigated for a single and two aligned quasiparticle spins. The discrete energy levels and corresponding wave-functions are obtained through a quantization procedure applied to the classical energy function. The formalism is used for numerical applications to \(^{135}\text{Pr}\) and \(^{134}\text{Pr}\) nuclei.

1. Introduction
The rigid triaxial deformation of the nuclei is a rare occurrence. It is even harder to experimentally identify rigid triaxiality, because despite its strong influence on many nuclear properties, there is not a direct experimental observable related to it. Fortunately, stable triaxial shapes are uniquely related to interesting phenomena such as specific \(\gamma\) band staggering [1, 2, 3, 4, 5], signature inversion [6], anomalous signature splitting [7], wobbling excitations [8], chiral symmetry breaking, multiple chiral doublet bands [9], whose observation is equivalent to the identification of nuclear triaxiality. Here one will present a theoretical study of the wobbling and chiral excitations based on a rigorous semiclassical treatment of a triaxial rigid rotor Hamiltonian with rigid quasiparticle alignments [11, 12, 13, 14]. The special structure of the model combines the advantages of both classical and quantum pictures. The classical picture offers a close relation with the system’s dynamics in terms of well defined and easily interpretable quantities, while its quantized counterpart makes the connection to the experimental observables and deals with quantum effects such as tunneling and phase transitions. Finally, the success of the presented theoretical formalism is evidenced through numerical applications on the chiral bands in \(^{134}\text{Pr}\) [15] and the wobbling excitations reported in \(^{135}\text{Pr}\) [16, 17].
2. Theoretical model

The obvious tool to study the wobbling and chiral bands is the quasiparticle-rotor model [1]. Consider now that we have two quasiparticles of spins \( j \) and \( j' \), whose alignment to the first and respectively second intrinsic axes of the triaxial core is rigid or frozen. In this approximation, where the quasiparticle spin projections on the body fixed principal axes are replaced with real numbers, the relevant Hamiltonian for the system’s dynamics is reduced to [11, 12, 13, 14]

\[
H_{\text{chiral}} = A_1 \hat{I}_1^2 + A_2 \hat{I}_2^2 + A_3 \hat{I}_3^2 - 2A_1 j \hat{I}_1 - 2A_2 j' \hat{I}_2,
\]

(1)

where \( \hat{I}_k \) are the operators of the total angular momentum projections on the principal axes of the intrinsic frame of reference, while \( A_k = 1/(2J_k) \) with \( J_k \) being the moments of inertia (MOI) along the same axes. For the study of wobbling motion only a single alignment is considered (\( j' = 0 \)). A time-dependent variational principle is applied to this Hamiltonian, using a coherent state for the \( SU(2) \) algebra of the angular momentum operators [10, 11, 12, 13, 14] and a couple of equations of motion for \( x, \varphi \) in a Hamilton canonical form which identify \( x \) as a generalized momentum and \( \varphi \) as a generalized coordinate. Special relations between the inertial parameters define distinct rotational phases. The separatrices defining the boundaries of each rotation phase in the parameters space of MOI, are found to be angular momentum dependent. This property allows a dynamical transition of the system from one phase to another. For example in the wobbling motion case, such a transition takes place between the transversal wobbling regime [18] of low angular momentum states towards a tilted axis wobbling motion at higher spins [11]. Similarly, the classical energy function for a system with a spontaneously broken chiral symmetry, starts to exhibit after a certain critical value of the total angular momentum, two minima in respect to the chiral variable \( x \), while \( |IKM\rangle \) are the eigenfunctions of the total angular momentum operator, and its projections on the third intrinsic (\( K \)) and laboratory (\( M \)) principal axes. The variational principle within this parametrization, leads to a classical energy function

\[
\mathcal{H}(x, \varphi) = \frac{I}{2}(A_1 + A_2) + \frac{(2I - 1)(I^2 - x^2)}{2I} (A_1 \cos^2 \varphi + A_2 \sin^2 \varphi - A_3) - 2A_1 j \sqrt{T^2 - x^2} \cos \varphi - 2A_2 j' \sqrt{T^2 - x^2} \sin \varphi,
\]

(3)

and a couple of equations of motion for \( x \) and \( \varphi \) in a Hamilton canonical form which identify \( x \) as a generalized momentum and \( \varphi \) as a generalized coordinate. Special relations between the inertial parameters define distinct rotational phases. The separatrices defining the boundaries of each rotation phase in the parameters space of MOI, are found to be angular momentum dependent. This property allows a dynamical transition of the system from one phase to another. For example in the wobbling motion case, such a transition takes place between the transversal wobbling regime [18] of low angular momentum states towards a tilted axis wobbling motion at higher spins [11]. Similarly, the classical energy function for a system with a spontaneously broken chiral symmetry, starts to exhibit after a certain critical value of the total angular momentum, two minima in respect to the chiral variable \( x \), which are associated with geometrical configurations with different handedness or chirality. Such an evolution corresponds to the change of the system’s dynamics from a chiral vibration to a static chirality character.

Besides the complete rotational dynamics extracted from the classical energy function and the equations of motion [11, 12], one can obtain the quantum wobbling or chiral energy states through a quantization procedure. For \( j' = 0 \), one speculated the fact that the classical energy function has a single minimum in the canonical variables \( (x, \varphi) \) regardless of its rotation phase, and approximated it with a quantum harmonic oscillator Hamiltonian. Although, such an approximation is very crude, it provides qualitative information in terms of wobbling frequencies expressed through simple and compact formulas. Thus, for adjacent transversal and tilted axis wobbling phases, the frequencies are

\[
\omega_{TV}(I) = \sqrt{[(2I - 1)(A_3 - A_1) + 2A_1 j][(2I - 1)(A_2 - A_1) + 2A_1 j]},
\]

(4)

\[
\omega_{TA}(I) = (2I - 1)\sqrt{(A_2 - A_3)(A_1 - A_3)} \sin \alpha,
\]

(5)
where

\[ \alpha = \text{ArcCos} \left[ \frac{2A_1j}{(2I-1)(A_1-A_3)} \right] \]  

(6)

is the tilting angle in respect to the first principal axis with the quasiparticle alignment, of the average direction of the total angular momentum vector residing in the principal plane defined by principal axes 1 and 3. While the transversal wobbling frequency recovers the result of Ref.[18] in the large angular momentum limit, the tilted axis wobbling frequency is a new result [11]. The domain of existence for the transversal wobbling phase is \( A_1S_{Ij} < A_2, A_3 \), while for the tilted axis wobbling is either \( A_3 < A_2 < A_1S_{Ij} \) or \( A_3 < A_1S_{Ij} < A_2 \), where

\[ S_{Ij} = \frac{2I-1-2j}{2I-1} \]  

(7)

is the angular momentum dependent separatrix. The critical angular momentum of the wobbling phase transition increases for more axial shapes, and the magnitude of the aligned quasiparticle spin.

The simple harmonic approximation in both variables, is no longer applicable for the chiral geometry \((j, j' \neq 0)\), where the classical energy can have two minima at high spin values. Therefore, one instead expands the classical energy function up to the second order just in respect of the variable \( \varphi \) which is the same for both minima. Quantizing then the approximated classical energy in the momentum space, one obtains a second order differential equation in \( x \) which in this particular case plays the role of a chiral variable [12]. Through a suitable change of function, the resulted differential equation can be brought to a form of a Schrödinger equation

\[ -\frac{1}{2} \int \frac{1}{\sqrt{B(x)}} \frac{d}{dx} \sqrt{B(x)} \frac{d}{dx} + V(x) \]  

\[ f(x) = Ef(x), \]  

(8)

with a coordinate-dependent effective mass \( B(x) \) and a chiral potential \( V(x) \) [12]. A similar quantum Hamiltonian was constructed from microscopic input in Ref.[24]. The advantage of the present formalism consists in the fact that it is based solely on the geometry of the system. The present chiral quantum Hamiltonian was also found to be naturally bounded by \(|x| < I\), which lead to the obvious choice of particle in the box wave-functions as diagonalization basis states. The same property is further used to construct a total wave function by a coupling of the rotational and \( x \)-vibrational motions through a weighting of the coherent state (2) with the probability distribution \( \rho_{Ip}(x) \) associated to the chiral variable

\[ |IMp⟩ = N_{Ip} \int I_p \rho_{Ip}(x)|ψ(x, ϕ_0(x))⟩. \]  

(9)

Here, \( p \) distinguish the states in respect to the excitation of \( x \), while \( ϕ_0(x) \) is the value of azimuth angle which minimizes the classical energy function for a certain value of \( x \). As was mention, this coupling is possible due to the compatibility of the coherent state’s completeness property \(|K| \leq I\) and the boundary condition \(|x| < I\) of the chiral Hamiltonian. For example, the same recipe cannot be applied to the harmonically approximated states from the wobbling excitations discussed above, because their domain of definition is infinite.

3. Numerical applications

The model was applied for the description of the observed wobbling bands in \(^{135}\text{Pr}\) and the chiral partner bands in \(^{134}\text{Pr}\). In both cases, the odd proton is of a particle type with a carried spin of \( j = 11/2 \). One also mention, that all model calculations are performed with hydrodynamic moments of inertia

\[ J_k = \frac{4}{3} J_0 \sin^2 \left( \frac{\gamma - \frac{2}{3}k\pi}{3} \right). \]  

(10)
Figure 1. Comparison between theoretical results and experimental data [16, 17] regarding wobbling bands of $^{135}$Pr. The dashed line marks the formal transition between the low spin transversal wobbling and tilted axis wobbling modes.

Table 1. The available data for the inter-band transitions in $^{134}$Pr [15] are compared to the theoretical estimations. The second index in the transition probabilities distinguish the bands presented in Fig.2.

| $I \rightarrow I'$ | Exp. | Th. | Exp. | Th. |
|-------------------|------|-----|------|-----|
| 13 $\rightarrow$ 11 | $<0.66$ | 0.208 | 0.0030(5) | 0.132 |
| 14 $\rightarrow$ 12 | $<0.24$ | 0.190 | 0.0044(7) | 0.109 |
| 15 $\rightarrow$ 13 | $<0.077$ | 0.144 | 0.025(3) | 0.087 |
| 16 $\rightarrow$ 14 | $<0.035$ | 0.101 | 0.028(2) | 0.076 |
| 17 $\rightarrow$ 15 | 0.026(3) | 0.077 | 0.105(30) | 0.071 |
| 18 $\rightarrow$ 16 | 0.038(3) | 0.062 | 0.061 |

In the chiral case, the geometry is complemented by a similar spin $j' = 11/2$ coming from the odd neutron hole. The compact expression for the wobbling frequency was used to fit the available experimental data for the yrast and the one-phonon wobbling bands. The data exhibit an anomaly in the energy spectrum which is associated with the transition from the transversal wobbling to the tilted axis wobbling. The transition’s angular momentum range identify very precisely the triaxial deformation, which is found to be $\gamma = -11.18^\circ$. Theoretical results calculated in Ref.[11] are compared with experimental data in Fig.1. With new experimental measurements of $^{135}$Pr regarding an additional two-phonon band, one has the occasion here to test the calculations of Ref.[11]. This is done by comparing the theoretical predictions with the experimental data.
extrapolated from Ref.[11] for the two-phonon band with the experimental data of Ref.[17]. As can be seen from Fig.1, the simple formalism based on harmonic wobbling frequencies reproduces quiet well even the position of the two-phonon wobbling excitations. Although the transition point is very well reproduced, there is room for improvement in what concerns the rotational behaviour of the bands. The major discrepancies between theoretical and experimental energy states are reported near the critical point, where the adopted harmonic approximation is not very good due to the system’s closeness to a separatrix. To treat consistently these states, one must go beyond the harmonic approximation by adopting for example the quantization procedure used for the chiral Hamiltonian (8). Such a program was successfully applied to the description of the chiral bands in the $^{134}$Pr nucleus [12]. As can be seen from Fig.2, the agreement with experimental energy levels is very good, despite the fact that the model calculations were performed for maximal triaxiality $\gamma = 90^\circ$ by fitting just a scale and a reference energy. Although, from the theoretical point of view on static chirality, the high angular momentum states of the chiral bands must be degenerate, experimentally this is not exactly realized. Moreover, among over three dozens of reported partner bands with possible chiral characteristics, the $^{134}$Pr is the one of the few examples when at least an approximate
degeneracy is achieved. The deviations from the theoretical picture of static chirality can have different origins: milder triaxiality, deviations from the rigid alignments due to Coriolis coupling [1], out of principal axes alignments of quasiparticles closer to the Fermi surface and even alternative scenarios disregarding the chiral geometry. The effect of the triaxial deformation on the spectral and dynamical properties of the chiral bands were discussed within the present approach in Ref.[13], while other contributing factors can be easily incorporated into the same analytical formalism. Indeed, the Coriolis coupling can be simulated by an additional spin-spin interaction $\vec{J} \cdot \vec{j}$ term [25, 26] in the original particle-rotor Hamiltonian, which will favor the alignment of quasiparticles to the core rotation axis at high values of total angular momentum. While quasiparticle alignments out of principal axes can be accommodated by introducing tilting angles degrees of freedom. Nevertheless, within the simple premise of maximal triaxiality $\gamma = 90^\circ$, the model provides also good agreement with experiment in what concerns the $E2$ transition probabilities. As can be seen from Fig.2, the in-band transitions are well described, with an especially good reproduction of transitions involving medium spins. The same interval of spins, coincides with the best description of the inter-band transitions listed in Table 1, and corresponds to the transitional region where the system undergoes a change from chiral vibration to static chirality.

4. Conclusions
The rigid alignment approximation allows a reduction in degrees of freedom, which facilitates a consistent description of the complex dynamics associated to wobbling and chiral excitations. Therefore, despite the strong character of the adopted approximation, the model provides a useful reference picture about what actually happens along the wobbling and chiral bands. This includes the dynamical transition between distinct rotational phases. Apart from the qualitative advantage of the formalism related to the ability to easily extract the dynamical behaviour of the system, it is shown that the model is able to offer also good quantitative calculations especially on energy levels and electric quadrupole transition probabilities. Examples are provided regarding the one and two phonon wobbling excitations observed in $^{135}$Pr and the chiral partner bands identified in the neighbouring nucleus $^{134}$Pr.

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