Possible Equilibria of Interacting Dark Energy Models

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Abstract

Interacting dark energy and the holographic principle offer a possible way of addressing the cosmic coincidence problem as well as accounting for the size of the dark energy component. The equilibrium points of the Friedmann equations which govern the evolution behavior of dark energy, matter, and curvature components can determine the qualitative behavior of the cosmological models. These possible equilibria and their behavior are examined in a general framework, and some illustrative examples are presented.

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I. INTRODUCTION

The study of type Ia supernovae indicates that we live in an accelerating Universe. This acceleration can be described by adding another component to the cosmic inventory that behaves in a similar way to the famous cosmological constant. Observations are consistent with about 70% of the density of the nearly flat universe is in the form of dark energy with the remaining 30% of ingredients is composed of matter, baryonic or non-baryonic. During the expansion of the Universe, one component tends to replace another in a small number of e-foldings, so the current closeness of these two densities is remarkable. This is known as the cosmic coincidence problem [1]. A perhaps associated question concerns why the dark energy density is so small compared to the other physical scales in particle physics [2]. These issues have led to the consideration of models that are more general than adding a simple cosmological constant. One hope is that a model based on a simple, single physical principle can account for both the coincidence and the observed size of the dark energy density.

A class of models that has been employed to explain the coincidence problem involves a nonzero interaction between dark energy and matter. In these theories, dark energy converts into matter in such a way as to create an equilibrium balance at late times. The choice of an appropriate interaction can effect a smooth and gradual transition from a matter-dominated (decelerating) Universe to a dark energy dominated (accelerating) one. These models lack a fundamental explanation for the source of the interaction. They are nevertheless an attractive way to investigate, at least at the phenomenological level, ways to account for the observed cosmological quantities such as the dark energy density parameter $\Omega_\Lambda = 0.7$ and its equation of state $w_\Lambda \approx -1$. If dark energy decays into matter, it is possible to imagine an equilibrium situation in which the tendency of the dark energy component to dominate as the Universe expands, is balanced by its decay into matter. These models represent deviations from, and are more general than, the model with a cosmological constant ($\Lambda$CDM).

The problem of the scale (size) of the dark energy component has led to the consideration of the holographic principle. This idea offers a possible mechanism for tying the scale of the dark energy to the geometric average of the Planck scale and the size of the Universe. The authors previously have used interacting dark energy in the scheme of holographic principle to find a relation between the equations governing the dark energy and matter.
densities in a flat universe. A holographic principle relates the density of dark energy to some horizon. Some examples are the apparent Hubble, future event horizon or particle horizon. It has been previously noted that, without any interaction between dark energy and matter, only the future event horizon gives results in agreement with observations. However, if we introduce an interaction, all of the above choices for the horizon might be consistent. There exists, for example, a model in which a constant interaction with Hubble horizon in a flat universe gives rise to the desired properties.

The consideration of nonzero curvature is the next obvious generalization. Although we know from the cosmic microwave background that we live in an almost flat universe, this may not have been the case at all times. Furthermore, the decay of dark energy to matter involves the conversion of a component which tends to make the universe more flat with one that has the opposite behavior. Nonzero curvature also makes the evolution equations more involved, and one can accommodate additional fixed point solutions.

We describe the equations that govern the dynamics of dark energy and curvature in Section II. Assuming that the radiation component is negligible, these are the only two independent density parameters out of the three (matter component, dark energy component, and the “curvature component”) related by \( \Omega_m + \Omega_\Lambda = 1 + \Omega_k \). To solve the evolution equations one needs to know the effective equations of state for dark energy and matter. This requires a specification of the holographic condition on dark energy, i.e. whether the length scale associated with the dark energy density is the Hubble horizon, the particle horizon, the future horizon, or some other physical scale. In addition, one needs to specify some interaction. With these two specifications, solving the resulting differential equations analytically is in general impossible, and solutions have to be done numerically. Therefore, it is desirable to arrive at some general characterization of the types of asymptotic solutions that can be obtained. One needs to find the steady-state solutions of the equations, where the variable parameters are fixed for large times in the future or past. These equilibrium solutions, and the conditions giving rise to them, are discussed in Section III. Some of these conditions have been exploited previously in the literature. To further clarify the main points of our results, we describe different examples in detail in the Section IV. It is worth mentioning that interacting models are not limited to models which employ an interaction.
between dark energy and matter. There exist models in which there are two components for dark energy: a variable cosmological constant, $\Lambda$, and a dynamical "Cosmon", $X$, with possible interaction between them. These models, called $\Lambda X$CDM models, are also capable of addressing the cosmological constant problem. See e.g. [17, 18, 19].

II. DYNAMICAL EQUATIONS FOR $\Omega_\Lambda$ AND $\Omega_k$

We start by assuming that the universe consists of matter and dark energy, nonzero curvature, and with negligible radiation. The evolution equations for dark energy and matter are,

$$\dot{\rho}_\Lambda + 3H(1+w_\Lambda)\rho_\Lambda = -Q,$$
$$\dot{\rho}_m + 3H(1+w_m)\rho_m = Q.$$  

(1)

We shall refer to the quantities $w_m$ and $w_\Lambda$ as the native equations of state. The appearance of the same $Q$ in the two equations insures the overall conservation of the energy-momentum tensor. Positive $Q$ can be interpreted as a transfer from the dark energy component to the matter component. Presumably this interaction could arise from some microscopic mechanism and might be specified more fully when the nature of the dark energy is known. It can be noted, however, that the consideration of quantum fields in curved backgrounds gives rise to particle production. However, in a cosmological scenario based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, the production rate is only at the level of the Hubble constant (and not the geometric average of the Hubble scale and the Planck scale).

It is interesting to note that the required physical scale of the interaction involves, like the dark energy itself, the size of the universe. This suggests a holographic interpretation for any such interaction.

The quantity $\rho_m$ is the matter component for which one usually takes $w_m = 0$. The quantity $\rho_\Lambda$, the density of dark energy can be related to some length scale via a holographic principle. We define this length scale $L_\Lambda$ according to the equation

$$\rho_\Lambda = \frac{3c^2 M_{Pl}^2}{8\pi L_\Lambda^2}.$$  

(2)

Here $c$ is a constant of order one (inserted here to preserve agreement with the literature) and $M_{Pl}$ is the Planck mass. Various choices considered for this length scale are the Hubble
horizon $R_H = 1/H$, the particle horizon (PH) defined by

$$R_{PH} = a \int_0^t \frac{dt}{a} = a \int_0^a \frac{da}{Ha^2}, \quad (3)$$

and the future event horizon (FH) defined by

$$R_{FH} = a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2}, \quad (4)$$

where $a(t)$ is the scale factor in the FLRW metric.

The interaction can be combined with the native equations of state, $w_\Lambda$ and $w_m$, to define effective equations of state as

$$\dot{\rho}_\Lambda + 3H(1 + w_\Lambda^\text{eff})\rho_\Lambda = 0,$$

$$\dot{\rho}_m + 3H(1 + w_m^\text{eff})\rho_m = 0. \quad (5)$$

Defining the ratio $r = \rho_m/\rho_\Lambda$ and the rate $\Gamma = Q/\rho_\Lambda$ and assuming a pressureless matter component, for which $w_m = 0$, these effective equations of states are

$$w_\Lambda^\text{eff} = w_\Lambda + \frac{\Gamma}{3H}, \quad w_m^\text{eff} = -\frac{1}{r} \frac{\Gamma}{3H}. \quad (6)$$

The density parameters

$$\Omega_\Lambda = \frac{8\pi \rho_\Lambda}{3M_P^2 H^2}, \quad \Omega_m = \frac{8\pi \rho_m}{3M_P^2 H^2}, \quad \Omega_k = \frac{k}{H^2 a^2}. \quad (7)$$

are defined so that the Friedmann equation,

$$H^2 = \frac{8\pi G}{3} (\rho_\Lambda + \rho_m) - \frac{k}{a^2}. \quad (8)$$

gives

$$\Omega_\Lambda + \Omega_m = 1 + \Omega_k. \quad (9)$$

The ratio $r$ is related to density parameters by

$$r = \frac{1 - \Omega_\Lambda + \Omega_k}{\Omega_\Lambda}, \quad (10)$$

and its time evolution is

$$\dot{r} = 3Hr \left[ w_\Lambda - w_m + \frac{1 + r}{r} \frac{\Gamma}{3H} \right] = 3Hr \left[ w_\Lambda^\text{eff} - w_m^\text{eff} \right]. \quad (11)$$
It is evident from Eq. (11) that the condition \( w_m^{\text{eff}} = w_\Lambda^{\text{eff}} \) involving the effective equations of state gives \( \dot{r} = 0 \). When this equilibrium occurs, the ratio of dark energy and dark matter densities is a constant. The Friedmann equation, Eq. (8), and the time evolution of densities can be combined to find the time evolution of Hubble parameter,

\[
\frac{1}{H} \frac{dH}{dx} = -\frac{3}{2} - \frac{1}{2} \Omega_k - \frac{3}{2} w_\Lambda \Omega_\Lambda ,
\]

where \( x = \ln(a/a_0) \) with some fixed scale factor \( a_0 \). The density parameters satisfy the differential equations,

\[
\frac{d\Omega_\Lambda}{dx} = 3\Omega_\Lambda \left[ \frac{1}{3} \Omega_k + w_\Lambda (\Omega_\Lambda - 1) - \frac{\Gamma}{3H} \right] ,
\]

\[
\frac{d\Omega_k}{dx} = \Omega_k (1 + \Omega_k + 3w_\Lambda \Omega_\Lambda) .
\]

It was shown in Ref. [3] that it is sufficient to have two physical conditions to determine the evolution of the parameters of the universe. The possibilities include a holographic condition on the dark energy \( \rho_\Lambda \), an assumption about the nature of dark energy and its native equation of state, \( w_\Lambda \) or an assumption for the form of interaction \( Q \), or equivalently \( \Gamma \). These three specifications are related by

\[
\Gamma = 3H (-1 - w_\Lambda) + 2 \frac{\dot{L}_\Lambda}{L_\Lambda} .
\]

This equation demonstrates that the interaction should generically be of the same size of the Hubble parameter, and also suggests that holographic definitions for the interaction may be useful.

Since the physical interpretation of the effective equation of state is clear, one can eliminate \( w_\Lambda \) and \( \Gamma \) in favor of \( w_\Lambda^{\text{eff}} \) and \( w_m^{\text{eff}} \) to obtain the equations,

\[
\frac{d\Omega_\Lambda}{dx} = -3\Omega_\Lambda (1 - \Omega_\Lambda)(w_\Lambda^{\text{eff}} - w_m^{\text{eff}}) + \Omega_k \Omega_\Lambda (1 + 3w_m^{\text{eff}}) ,
\]

\[
\frac{d\Omega_k}{dx} = 3\Omega_k \Omega_\Lambda (w_\Lambda^{\text{eff}} - w_m^{\text{eff}}) + \Omega_k (1 + \Omega_k)(1 + 3w_m^{\text{eff}}) .
\]

These equations are consistent with the analysis of Ref. [16], and with the substitution \( \Omega_k = 0 \), one recovers the equation for the flat case from Ref. [3]. The location of the fixed points and equilibria of these coupled differential equations will determine their asymptotic behavior. The appearance of the factors \( w_\Lambda^{\text{eff}} - w_m^{\text{eff}} \) and \( 1 + 3w_m^{\text{eff}} \) are easy to understand on
physical grounds. The first factor merely compares whether dark energy or matter comes to dominate as the universe expands. The second factor compares matter to curvature ($w_k = -\frac{1}{3}$), so it measures whether the density of matter increases or decreases as the universe expands.

III. POSSIBLE EQUILIBRIUM SOLUTIONS OF THE EVOLUTION EQUATIONS

Equilibrium solutions occur when the right hand sides of the differential equations, Eqs. (15), vanish. We wish to distinguish different kinds of equilibria. First, there are fixed points in $\Omega_\Lambda, \Omega_k$ space. These are universal in the sense that they will be present unless the effective equations of state have singularities as a density parameter vanishes. Second, some equilibria consist of one constraint on the effective equations of state and a fixed value of $\Omega_\Lambda$ or $\Omega_k$. This represents a fixed point which depends in detail on the specific form of the interaction and holographic definition of the dark energy. Finally, there is an equilibrium solution which is governed by just constraints on effective equation of states.

A procedure to identify these equilibria is the following: Set the derivatives in Eqs. (15) to zero. Taking linear combinations of the right-hand-sides, one finds solutions

\begin{align}
\Omega_\Lambda (w_\Lambda - w_m) &= 0, \\
\Omega_k (1 + 3w_m) &= 0.
\end{align}

(16)

One obtains solutions by identifying points where the density parameters vanish and/or where the effective equations of state satisfy a certain condition, namely, $w_\Lambda = w_m$ or $1 + 3w_m = 0$.

The behavior near these fixed points is important in understanding the qualitative behavior of the evolution. The fixed points can be categorized as either repellers, attractors or saddle points. Consider the two differential equations in the following form [20, 21, 22],

\[ \frac{d\Omega}{dx} = f(\Omega). \]

(17)

In this equation $\Omega = (\Omega_\Lambda, \Omega_k)$ and $f(\Omega)$ represents the right hand sides of the two differential equations of the evolutions of the density parameters. The density parameter for matter, $\Omega_m$ is then given by Eq. (9). The behavior at a specific equilibrium point depends on the
eigenvalues of the 2-by-2 matrix \( A = \frac{\partial f}{\partial \Omega} \). If the eigenvalues of this matrix are all positive, then we have a repeller fixed point. In other words, as the Universe expands and \( x \) increases, the Universe will tend to move away from the fixed point. If the eigenvalues are all negative, we have an attractor fixed point. Finally if one eigenvalue is positive and one is negative, the fixed point behaves like a saddle point.

There are three fixed points for the system of equations which occur at certain values of the density parameters. The character of these fixed points is easy to identify from the properties of the effective equations of state. The first fixed point occurs at \( \Omega = (0, -1) \). This is a negatively curved solution, and would be a future asymptotic state of the Universe if both \( w_{m}^{\text{eff}} > -\frac{1}{3} \) and \( w_{\Lambda}^{\text{eff}} > -\frac{1}{3} \). Since the Universe appears flat, we know that it is unlikely to be a viable solution. The second fixed point occurs at \( \Omega = (0, 0) \). This corresponds to a universe filled with only matter. This could be the final state of the Universe, provided \( w_{m}^{\text{eff}} < -\frac{1}{3} \) (so that the matter density grows as the Universe expands) and that \( w_{\Lambda}^{\text{eff}} > w_{m}^{\text{eff}} \) (so that matter density comes to dominate over the dark energy density). Finally, the de Sitter Universe is described by \( \Omega = (1, 0) \), a universe just filled with dark energy. This solutions pertains to the future asymptotic state of the Universe provided \( w_{\Lambda}^{\text{eff}} > -\frac{1}{3} \) (so that the matter density grows as the universe expands) and \( w_{\Lambda}^{\text{eff}} < w_{m}^{\text{eff}} \). The eigenvalues of the matrix \( A \) are shown in Table I. If one replaces the effective equations of state with their native values (\( w_{m} = 0 \) for matter and \( w_{\Lambda} = -1 \) for a cosmological constant), one recovers the usual evolution behavior associated with the \( \Lambda \)CDM. One recognizes from Table I that the behavior in a more general context is governed entirely by the values of the effective equations of state at the fixed point.

It is also worth mentioning that, for specific interactions, some of these fixed points can be eliminated. For example, if the interaction produces a \( w_{m}^{\text{eff}} \) proportional to something sufficiently singular like \( 1/\Omega_{\Lambda} \), then the point \( \Omega = (0, 0) \) will cease to be a fixed point as can be seen in Eq. (6).

The behavior of the evolution near these fixed points depends on the effective equations of state in Table I, the different possibilities for their dynamical behaviors are shown in Table II, III and IV.

The fixed points just discussed are the familiar ones that result from behavior of each density component as the universe expands. There are also fixed point, or equilibrium
| fixed point | eigenvalues of $A$ |
|-------------|------------------|
| $\Omega = (0, -1)$ | $-(1 + 3w^\text{eff}_\Lambda)$ |
| $\Omega_m = 0$ | $-(1 + 3w^\text{eff}_m)$ |
| $\Omega = (0, 0)$ | $-3(w^\text{eff}_\Lambda - w^\text{eff}_m)$ |
| $\Omega_m = 1$ | $1 + 3w^\text{eff}_m$ |
| $\Omega = (1, 0)$ | $3(w^\text{eff}_\Lambda - w^\text{eff}_m)$ |
| $\Omega_m = 0$ | $1 + 3w^\text{eff}_\Lambda$ |

TABLE I: Equilibria of the universal fixed points in $\Omega$ space. The signs of the eigenvalues are given by comparing the effective equations of state to the that of the curvature component (which is $-1/3$), or by comparing them to each other.

| $w^\text{eff}_\Lambda < -\frac{1}{3}$ | $w^\text{eff}_\Lambda > -\frac{1}{3}$ |
|--------------------------|--------------------------|
| repeller | saddle point |
| saddle point | attractor |

TABLE II: Characteristic behavior of the fixed point $\Omega = (0, -1)$, when $\Omega_m = 0$.

solutions, that can result because the effective equations of state satisfy some constraint. There are two equilibrium solutions where just one of the density parameters is determined and the other condition is a constraint on the equations of state. The first occurs when $\Omega_\Lambda = 0$ and $w^\text{eff}_m = -\frac{1}{3}$. This equilibrium would be possible only if the chosen interaction gives rise to $\Gamma_{3H} \sim \frac{1}{\Omega_\Lambda}$ around $\Omega_\Lambda = 0$. Although this doesn’t directly make the interaction $Q$

| $w^\text{eff}_\Lambda < -\frac{1}{3}$ | $w^\text{eff}_\Lambda > -\frac{1}{3}$ |
|--------------------------|--------------------------|
| attractor if $w^\text{eff}_\Lambda > w^\text{eff}_m$ | attractor |
| saddle point if $w^\text{eff}_\Lambda < w^\text{eff}_m$ | |
| repeller | repeller if $w^\text{eff}_\Lambda < w^\text{eff}_m$ |
| | saddle point if $w^\text{eff}_\Lambda > w^\text{eff}_m$ |

TABLE III: Characteristic behavior of fixed point $\Omega = (0, 0)$, when $\Omega_m = 1$. 

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The second equilibrium occurs when \( \Omega_k = 0 \) and \( w_m^{\text{eff}} = w_m^{\text{eff}} \). This solution has been employed \cite{3, 10, 23} to represent the final asymptotic state of the universe, and it has to be an attractor to be a suitable candidate. This implies that \( \dot{r} = 0 \) from Eq. (11) and the density parameters remain constant, provided this is a stable equilibrium as determined by the eigenvalues at the fixed point. The current epoch of the universe may be nearing such a fixed point. On the other hand, it is also possible that the current state of the universe is a transient one in the sense that behavior near the equilibrium point behaves like a saddle point. The eigenvalues of \( A \) are listed in Table V.

The last equilibrium condition occurs when the constraint \( w_m^{\text{eff}} = w_A^{\text{eff}} = -\frac{1}{3} \) holds. The equilibrium solution is easy to understand as both effective equations of state scale like curvature, so that the dark energy density and matter density parameters are constant. It is harder to see the behavior of this equilibrium as the derivative matrix, \( A \), includes complicated function of the derivatives of the effective equations of state. An example of a solution satisfying these constraints was presented in Ref. \cite{15}. There the model consisted

| Fixed density parameter | Constraint on effective EOS | Eigenvalues of \( A \) |
|-------------------------|-----------------------------|-----------------------|
| \( \Omega_A = 0 \)     | \( w_m^{\text{eff}} = -\frac{1}{3} \) | \( -(1 + 3w_A^{\text{eff}}) \) \( 3\Omega_k(1 + \Omega_k) \frac{\partial w_m^{\text{eff}}}{\partial \Omega_k} \) |
| \( \Omega_k = 0 \)     | \( w_m^{\text{eff}} = w_A^{\text{eff}} \) | \( -3\Omega_A(1 - \Omega_A)(\frac{\partial w_m^{\text{eff}}}{\partial \Omega_A} - \frac{\partial w_m^{\text{eff}}}{\partial \Omega_A}) \) \( 1 + 3w_m^{\text{eff}} \) |

TABLE IV: Characteristic behavior of fixed point \( \Omega = (1, 0) \), when \( \Omega_m = 0 \).

TABLE V: Equilibria specified by one fixed density parameter and one constraint.
of a holographic length scale set equal to the Hubble parameter, and a constant interaction \( \Gamma \).

These examples indicate the large variety of behavior that can result with fixed points. The behavior of the equations at these other equilibria is more involved than the universal fixed points in Table I as the eigenvalues of \( A \) depend on the derivatives of the effective equations of state as well. Some examples will be considered in next section to illustrate the different possibilities.

**IV. EXAMPLES**

In this section the behavior of the density parameters will be illustrated with some choices for the interaction. As mentioned earlier, two conditions are required to completely specify the equations. If one of the conditions is determined by a holographic principle, it will restrict the possible behaviors of the equilibria. If one chooses the length scale entering the equation of density of dark energy to be the future event horizon, defined by Eq. (4), then it can be shown that the effective equation of state for dark energy can be written as

\[
\omega_{\text{eff}}^\Lambda = -\frac{1}{3} - \frac{2}{3} \sqrt{\frac{\Omega_{\Lambda}}{c^2} - \Omega_k} .
\]  

(18)

On the other hand, if the length scale is chosen to be the particle horizon, defined by Eq. (3), with a similar procedure one can show that

\[
\omega_{\text{eff}}^\Lambda = -\frac{1}{3} + \frac{2}{3} \sqrt{\frac{\Omega_{\Lambda}}{c^2} - \Omega_k} .
\]  

(19)

In general in the FH case, \( \omega_{\text{eff}}^\Lambda \leq -\frac{4}{3} \), and for the PH case, \( \omega_{\text{eff}}^\Lambda \geq -\frac{4}{3} \). Therefore, after specification of this condition, the behavior of the fixed points will be determined by the behavior of \( \omega_{\text{eff}}^m \), which is in turn determined by the form of the interaction.

For all the examples in this section, the holographic length scale, entering in the equation of dark energy density, will be taken to be the future event horizon, \( L_\Lambda = R_{FH} \) defined in Eq. (4). The effective equation of state for dark energy is given by Eq. (18). What remains is a specification of the form of the interaction. The simplest choice is no interaction at all, for which \( \omega_{\text{eff}}^m = 0 \). In this case, there exist just the three equilibrium points listed in Table I. The behavior of the fixed points can be obtained by looking at Tables II III
FIG. 1: (a) Density parameters in a model with no interaction. (b) The associated flow diagram in the $\Omega_\Lambda, \Omega_k$ plane. The solid line represents the solution for the initial condition assumed in (a).

and considering the fact that $w_m^{\text{eff}} = 0$ and $w_\Lambda^{\text{eff}} < -\frac{1}{3}$. The fixed points at $\Omega = (0,0)$, $\Omega = (0,-1)$, and $\Omega = (1,0)$ are a repeller, a saddle point, and an attractor respectively. The behavior of the density parameters are shown in Fig. 1 where an initial state is chosen so that the Universe is filled with matter, while dark energy and curvature are very small. The interaction drives the Universe toward a final equilibrium in which the Universe is filled with dark energy. The Universe experiences a transition toward and then away from a saddle point where it is negatively curved. The physical interpretation of this behavior is similar to but not the same as $\Lambda$CDM: matter tends to create curvature in the universe, but eventually the dark energy component comes to dominate and reduces the curvature.

The next example employs an interaction between dark energy and dark matter and illustrates the appearance of a new equilibrium point. In terms of $\Gamma$, the interaction is

$$\frac{\Gamma}{3H} = \frac{b^2(1 + \Omega_k)}{\Omega_\Lambda^n},$$

which has been used in some recent works to explain the cosmic coincidence problem. When $n = 1$, the interaction term $Q = \rho_\Lambda \Gamma$ is just dependent on $H$, the Hubble parameter.
The density parameters, acting under this interaction, approach an equilibrium solution similar to their observed values at the present time. The effective equation of state for matter is

\[
    w_{\text{eff}}^m = \frac{-b^2(1 + \Omega_k)\Omega_\Lambda^{1-n}}{1 + \Omega_k - \Omega_\Lambda}.
\]

It can be seen from Eqs. (15) and the definition of \( w_{\text{eff}}^m \) that this interaction eliminates \( \Omega = (1, 0) \) as an equilibrium. However, with an appropriate choice for \( b^2 \), it produces a new attractor equilibrium at \( \Omega = (0.7, 0) \). This new fixed point arises from an equilibrium specified by \( w_{\text{eff}}^m = w_{\text{eff}}^\Lambda \) as in Table \( \text{V} \). This attractor can be employed to represent the future asymptotic state of the Universe. Although there is no theoretical justification for this interaction, it turns out to be very useful in understanding how introducing interactions can change the behavior of the density parameters. The initial values for density parameters don’t have any effect on their asymptotic behavior. Initial values, though, dictate how the density parameters evolve before reaching their final asymptotic value and how near they approach the saddle point-like equilibria.

In Fig. 2, the Universe, is taken to be filled with matter and to have very small curvature.
initially. It moves toward a saddle point where the curvature is almost $-1$ and then evolves toward its final situation, where $\Omega_\Lambda \approx 0.7$. How close the Universe approaches the intermediate saddle point depends on the initial values for $\Omega_\Lambda$ and $\Omega_k$. If those initial conditions are chosen to be smaller than the ones chosen in Fig. 2, the evolution turns around at smaller absolute values of curvature density parameter and then proceeds toward its asymptotic solution. For the last fixed point, the effective equations of states are equal, and the matter and dark energy density parameters remain at their equilibrium solutions. Any curvature is eliminated as the Universe expands under the influence of the presence of the dark energy component. This generalizes our previous result where an exactly flat initial condition was assumed [3].

A final example, involves an interaction which causes the universe to explore a transient state for a long period of time by exploiting a fixed point which is a saddle point. Weighting the interaction, Eq. (20), by a factor of $\exp(-p\Omega_k)$, we have

$$\frac{\Gamma}{3H} = \frac{b^2 e^{-p\Omega_k} (1 + \Omega_k)}{\Omega_\Lambda^n}.$$  

(22)

The exponential factor, while somewhat artificial, guarantees that evolution is driven close to a saddle point. The effective equation of state for matter is

$$w_{\text{eff}}^m = \frac{-b^2 e^{-p\Omega_k} (1 + \Omega_k)\Omega_\Lambda^{1-n}}{1 + \Omega_k - \Omega_\Lambda}.$$  

(23)

We do not ascribe any physical motivation for such an interaction, but merely employ it to show how the evolution can be driven close to a saddle point. We again choose $n = 1$ for the numerical solutions. In this case, the Universe starts from a repeller, approaches the saddle point, then is finally repelled toward the attractor. In Fig. 3 the initial value of the curvature is set very close to zero and negative, while in Fig. 4 the initial value of the curvature density parameter is set very close to $-1$. In both cases, however, the Universe is pushed by the interaction toward the saddle point where we have $\Omega_k \approx -0.05$, then goes toward the attractor located at $\Omega = (0.7, 0)$. At this point, again the condition $w_{\text{eff}}^m = w_{\text{eff}}^\Lambda$ holds, as can be seen in Fig. 4 (c).
FIG. 3: (a) Density parameters in an interacting model when $\Gamma$ is proportional to $e^{-p\Omega_k}$, when $b^2 = 0.26$ and $p = 5$. (b) The initial value of the curvature density parameter is taken to be a very small and negative number.

FIG. 4: (a) Density parameters in an interacting model with interaction, Eq. (22). The initial value of $\Omega_k$ is set close to $-1$. The interaction drives $\Omega_k$ toward zero very rapidly. (b) The associated flow diagram in the $\Omega_\Lambda, \Omega_k$ plane. The solid line represents the solution depicted in (a), for the region $\Omega_k > -0.1$. (c) Effective equations of state for dark energy and dark matter.
V. CONCLUSION

The behavior of the Universe, where there is an interaction between dark energy and dark matter, has been studied in the presence of curvature. In the literature, many different scenarios, obtained by choosing different interactions and different choices for length scale which enters the definition of dark energy density, have been explored. The present work unifies these attempts in this approach toward solving cosmic coincidence problem, at the qualitative level, by studying the possible equilibria showing up in the dynamical equations of dark energy and dark matter.

Understanding the fixed point equilibria is important, because the asymptotic behavior at these points is insensitive to initial conditions. Attempts to solve the coincidence problem in the literature usually focus on stable solutions for which $\Omega_k = 0$ and $w_{\text{eff}}^m = w_{\text{eff}}^\Lambda$. With appropriate choice of interaction, a fixed point can occur at $\Omega_\Lambda \approx 0.7$ and $\Omega_m \approx 0.3$, the observed values for these parameters. We showed that adding curvature to cosmic inventory results in additional equilibrium solutions. For example, the equilibrium point $\Omega = (0, -1)$, which corresponds to $\Omega_m = 0$, doesn’t exist when there is no curvature. The new and old equilibrium solutions, and the possible behavior of them, are studied and categorized in Section III. A particular theory with an interaction between dark energy and dark matter doesn’t typically possess all of these equilibria. For example if there is no interaction at all, there exists just the three equilibria listed in Table I. By a suitable choice of interaction, one is able to eliminate some of the equilibria and add new ones. The examples provided in Section IV illustrate how this process works.

Observation indicates that there is little curvature at the present epoch. Additional equilibria can be obtained by considering the Universe at all eras, where curvature is not necessarily small. Any saddle point equilibrium could represent a transient era (but perhaps long-lived) of Universe. Any attractor equilibrium, providing initial conditions let Universe be driven toward it, would represent the final state. In fact, no observation has confirmed that Universe is in a stable solution of its dynamical equations. So it is quite possible that Universe is currently experiencing a saddle point-like equilibrium and will ultimately be driven toward a fixed point with complete different properties.

The supernovae data requires a recent transition from deceleration to acceleration, so a
consistent solution requires that we are arriving at this equilibrium at the present epoch. The fixed point solutions represent an amelioration of the rapid transition to dark energy domination such as occurs in the ΛCDM. Any particular choice of interaction could be supported or ruled out by comparing its results with recorded history of Universe. The examples provide insight into the classification of the equilibrium points.

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