Finite temperature world-line formalism
and analytic continuation

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Abstract

In vacuum, the world-line formalism is an efficient tool for calculating observables in the presence of arbitrary constant external fields. The natural frame of this formalism is the Euclidean space. At finite temperature the analytic continuation to Minkowski space is a subtle task. We study the two point function in scalar QED, and we figure out the problem of analytic continuation giving a possible solution for it. We also show, in contrast to what was claimed in the literature, that a translationally invariant world-line Green’s function could be used at finite temperature.

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1 Introduction

The study of the effect of an external field on a thermalised system is of great importance. It is suggested [1] that a strong magnetic field could have existed during the early stages of the Universe. This would have affected the features of the electroweak phase transition. The effect of a magnetic field on a relativistic QED or QCD plasma has been investigated in the literature [2]. As this magnetic field has been generated by the phase transition, it is not expected to have overwhelmed thermal effect [2]. On the other hand an electric field is expected to have primordial importance due to its strong interaction with matter and the long period over which it last. The effect of an electric field at finite temperature has not yet been achieved.

For our study we assume that we have thermal and chemical equilibrium. We take the chemical potential to be zero.

The world-line formalism is a well adapted tool to incorporate an external magnetic field when calculating correlation functions. In vacuum, this formalism allows to obtain the n-point Green’s functions in the presence of an external field from those in the absence of such field. In this paper we show that such an advantage does not persist at finite temperature due to the problem of analytic continuation. The starting point of the world-line formalism is the Euclidean space, where the convergence of integrals is well controlled. The simple derivation of the world-line effective action in Euclidean space is balanced by the subtle analytic continuation to Minkowski space at finite temperature. This makes ‘Wick’s rotation’ very tricky. The example of the two point Green’s function in scalar QED traces the problem of analytic continuation to the use of Feynman parametrisation. The naive use of Feynman parametrization leads to wrong results. Hence the advantage of obtaining easily the two point function in the presence of an external field is lost. Therefore a direct calculation of the two point function, in the presence of an external field, should be performed.

We recall that the subtleties discussed in the paper, are not applicable to thermodynamical functions (entropy, ...). Those functions have no external momentum and hence there is no analytic continuation to be done.

The paper is organized as follows. In the first section we review the basic notations of the world-line formalism, and we show that a translationally invariant Green’s function can be used at finite temperature. Then the two point function is calculated in the world-line formalism to illustrate the relation between this formalism and Feynman parameters. Finally the problem of analytic continuation is discussed and a possible solution is proposed.
2 Notation and formalism

2.1 Zero temperature field theory

The one-loop effective action in a first quantized path integral form is well known at zero temperature [3]. A finite temperature extension can be found in the literature [4]. Recently the Hard Thermal Loop effective action was derived in the world-line formalism [5]. In this paper we use the same notation used in [4]. We do not intend here to give a complete derivation of the effective action but rather a sketch of the important steps leading to its final form.

At zero temperature we could proceed in the following way to get the effective action [6]:

- In Euclidean space, the starting point is the standard ‘Trlog(propagator operator)’ formula [7]:
  \[ e.g. \text{for scalar field theory, of mass } m \text{ and potential } V(\phi), \text{ the one loop effective action is given by} \]
  \[ \Gamma_{\text{eff}} \sim \text{Tr } \log \left( -\Box + m^2 + V''(\phi) \right). \]

- Then we use the following formula
  \[ \text{Tr } \log(A) = \text{Tr } \int \frac{dt}{t} \exp(-At) \]
  to get a form similar to the path integral obtained for the evolution operator \( \exp(-(t-t')H) \), where \( H \) is the first quantized Hamiltonian, with a kinetic term \( \sim \dot{x}^2/2 \) and a potential term.

By following these steps, the scalar QED one-loop effective action is found to be [8]

\[
\Gamma_{\text{eff}}[A] = \int_0^\infty \frac{dt}{t} e^{-m^2 t} \int_{x(t)=x(0)} Dx(\tau) \exp \left\{ -\int_0^t d\tau \left( \frac{\dot{x}^2}{4} + ie\dot{x}.A(x(\tau)) \right) \right\}. \tag{1}
\]

The n-point Green’s function can easily be derived by using the Fourier transform of the photon field \( A^\mu(x) = \sum \epsilon^\mu_i e^{ik_i x} \). For the above effective action Wick’s contraction is found to be

\[
< x | x^{\mu}(\tau_1) x^{\nu}(\tau_2) | x > = -g^{\mu\nu} \Delta(\tau_1 - \tau_2)
\]
with the convention $g^{\mu\nu} = (+, +, +, +)$. The one dimension translationally invariant Green’s function $G$ is given by:

$$G(\tau_1 - \tau_2) = \langle x| - \frac{d^2}{d\tau^2}|x> = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{t}. \quad (2)$$

Deriving the effective action twice with respect to $\epsilon$ and using Wick’s contraction, the two point function in D-dimension can easily be found:

\[
\Pi^{\mu\nu}(k) = - \frac{e^2}{2} \int_0^\infty \frac{dt}{t} e^{-m^2t} [4\pi t]^{-D/2} \int_0^t d\tau_1 \int_0^t d\tau_2 [g^{\mu\nu}k^2 - k^\mu k^\nu] \\
\times \dot{G}^2(\tau_1 - \tau_2) \exp(-G(\tau_1 - \tau_2)k^2). \quad (3)
\]

where $g^{\mu\nu}k^2 - k^\mu k^\nu$ is the standard transverse tensorial structure of the two point Green’s function.

### 2.2 Finite temperature field theory two point function

We work in scalar QED, whoever in the world-line formalism it is straightforward to obtain spinor QED from scalar QED. In addition at high temperature we can neglect the mass of the scalar particle.

At finite temperature one has to take into account the periodicity conditions. The boundary condition $x(t) = x(0)$ in the path integral of Eq. (1) changes to $x_\mu(t) = x_\mu(0) + n\beta w_\mu$, with $\beta = 1/T$ and $w_\mu = (1, 0, 0, 0)$. We have to sum also over the winding number ‘$n’ [4]. Then the effective action at finite temperature is given by

$$\Gamma_{\text{eff}}^\beta[A] = \sum_{n=-\infty}^\infty \Gamma_{\text{eff}}[A] \bigg|_{\{x(t) = x(0)\} \rightarrow \{x_\mu(t) = x_\mu(0) + n\beta w_\mu\}} \quad (4)$$

We use the Fourier transform of $A(x) = \sum_\epsilon \epsilon^e \epsilon^{ik\cdot x}$. The two point function can be obtained by deriving the effective action twice with respect to $\epsilon$. It is also necessary to integrate the so called zero mode [8] by making the change of variable: $x^\mu(\tau) = x_o^\mu + n\beta t w^\mu + y^\mu(\tau)$ where $x_o$ is $\tau$ independent. By integrating the zero mode, the two point function can be written as

\[
\Pi^{\mu\nu}(k, \beta) = - \frac{e^2}{2} \sum_{n=-\infty}^\infty \int_0^\infty \frac{dt}{t} [4\pi t]^{-2} \int_0^t d\tau_1 \int_0^t d\tau_2 \\
\exp \left(- \frac{n^2 \beta^2}{4t} + \frac{in\beta}{t} k_o(\tau_1 - \tau_2) \right) \exp(-k^2 G_{12}) \left\{ - k^\mu k^\nu \dot{G}_{12}^2 \right\}
\]
\[
+i \frac{n \beta}{t} (w^\mu k^\nu + w^\nu k^\mu) \dot{G}_{12} + g^{\mu \nu} \ddot{G}_{12} + w^\mu w^\nu \frac{n^2 \beta^2}{t^2}\]. \tag{5}
\]

Where \(G_{12} = G(\tau_1 - \tau_2)\), is the translationally invariant propagator given in Eq. (2).

The zero temperature result in Eq. (3) can be obtained from Eq. (5) by first putting \(n = 0\), then performing a partial integration to eliminate \(\ddot{G}\). Hence from now on we subtract the \(n = 0\) contribution to obtain the finite temperature part of the two point function.

To simplify the \(\tau_1\) and \(\tau_2\) integration write

\[
\Pi^{\mu \nu}(k, \beta) \equiv \sum_{n=-\infty}^{\infty} \int_0^t d\tau_1 \int_0^t d\tau_2 f_n(\tau_1 - \tau_2), \tag{6}
\]

where \(f_n(\tau_1 - \tau_2)\) is the \(n\)-dependent expression in Eq. (5). Using the symmetry properties of the propagator and its derivatives, it is easy to show that the sum over \(n\) and the integral over \(\tau_1\) and \(\tau_2\) can be replaced by:

\[
\sum_{n=-\infty}^{\infty} \int_0^t d\tau_1 \int_0^t d\tau_2 f_n(\tau_1 - \tau_2) = 2t^2 \sum_{n=-\infty}^{\infty} \int_0^1 du (1-u) f_n(ut), \tag{7}
\]

where we have used the property \(f_{-n} = f_n^*\).

Using the above transformation we obtain

\[
\Pi^{\mu \nu}(k, \beta) = -\frac{e^2}{(4\pi)^2} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{dt}{t} \int_0^1 du (1-u) \exp \left( -\frac{n^2 \beta^2}{4t} + i n \beta k_0 u - k^2 tu(1-u) \right) \left\{ -k^\mu k^\nu (1-2u)^2 \right. \\
+i \frac{n \beta}{t} (w^\mu k^\nu + w^\nu k^\mu)(1-2u) + \frac{2}{t} g^{\mu \nu} (\delta(u) - 1) + w^\mu w^\nu \frac{n^2 \beta^2}{t^2} \right\}. \tag{8}
\]

We note that Eq. (8) coincides with Eq. (11) of ref. [9].

It is important at this level to compare our derivation with the one used in [9]. The author of [9] has claimed that one could not use the translationally invariant Green’s function of Eq. (2) at finite temperature. He claimed that such a Green’s function is obtained by partial integration which will give boundary terms at finite temperature. Our derivation has shown that any expected boundary term, resulting from partial integration, cancels out.

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This allows the use of the translationally invariant Green’s function. We proceed in a way similar to [9] to eliminate the δ function in Eq. (8). Then by performing partial integration over \( u \) we obtain

\[
\Pi^{\mu\nu}(k, \beta) = -\frac{e^2}{(4\pi)^2} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \frac{dt}{t} \int_{0}^{1} du \exp\left(-\frac{n^2 \beta^2}{4t} + i\beta k_o u - k^2 tu (1 - u)\right) \left\{ (g^{\mu\nu} k^2 - k^2 k^\nu)(1 - 2u)^2 + \frac{i n \beta}{t} (w^{\mu} k^\nu + w^{\nu} k^\mu - g^{\mu\nu} w.k - \frac{k^2}{k_o} w^{\mu} w^{\nu})(1 - 2u) \right\}.
\]

(9)

It is necessary to verify gauge invariance of the above two point function. This is done in the next section.

### 2.3 Transversality

It is easy to verify transversality by contracting the two point function with \( k^\mu \). To illustrate the transversality property, let us define \( P^{\mu\nu} \) as:

\[
\frac{k^2}{k_o} P^{\mu\nu}(k) = w^{\mu} k^\nu + w^{\nu} k^\mu - g^{\mu\nu} w.k - \frac{k^2}{k_o} w^{\mu} w^{\nu}.
\]

(10)

This tensor can be related to the photon transverse and longitudinal projectors by (see appendix A for the definition of the projectors)

\[
P^{\mu\nu} = -P_{L}^{\mu\nu} - \frac{k_o}{k^2} P_T^{\mu\nu}.
\]

(11)

The transverse and the longitudinal projectors are both transverse to \( k^\mu \). Besides the transverse projector is transverse with respect to the 3-vector \( k \). Hence the two point function could be written in terms of \( P_{\mu\nu} \) and \( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \) which are transverse. Therefore the two point function is manifestly transverse.

### 3 World-line formalism and Feynman parameters

The world-line formalism and Feynman parametrization share the same property of reducing any n-point Green’s function into a one loop integration. However it is well known that the Feynman parametrization is very subtle at finite temperature. In what follows we will show that the variable \( u \) in Eq. (8) is analogous to a Feynman parameter. Then we figure out the above mentioned subtleties of Feynman parametrization and how they affect the calculation of the two point function.
3.1 Rewriting the two point function

We aim at rewriting the two point function in a form similar to a two point function evaluated using Feynman parameters. Doing partial integration on \( t \) in Eq. (9) we get rid of the \( 1/t^2 \) and we obtain

\[
\Pi_{\mu\nu}(k, \beta) = -\frac{e^2}{(4\pi)^2} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{dt}{t} \int_0^1 du (1-2u) \exp \left( -\frac{n^2 \beta^2}{4t} + in\beta k_o u - k^2 tu(1-u) \right) \left\{ -\frac{k^2 k_o^2}{k_o^2} \mathcal{P}_{\mu\nu} (1-2u) + \frac{2}{in\beta k_o} \left( \frac{k^2}{k_o^2} \right)^2 \mathcal{P}_{\mu\nu} \right\} + \frac{3m^2 k^2}{2} \mathcal{P}_{\mu\nu}. \tag{12}
\]

The last term is a boundary term resulting from partial integration. The thermal mass is defined as \( m^2 = e^2 T^2 / 3 \). The \( t \)-integration gives a Bessel function of second type \([10]\)

\[
\frac{1}{2} \int_0^\infty \frac{dt}{t} \exp \left[ -\frac{x}{2} \left( t + \frac{z^2}{t} \right) \right] \equiv K_0(xz), \tag{13}
\]

where \( x = 2k^2 u(1-u) \) and \( z = n\beta/2\sqrt{u(1-u)k^2} \). By using the transformation \( n \to -n \) for negative \( n \) in the sum we obtain

\[
\Pi_{\mu\nu}(k, \beta) = -\frac{4e^2}{(4\pi)^2} \sum_{n=1}^{\infty} \int_0^1 \frac{du}{t} (1-2u) K_0(n\beta \sqrt{u(1-u)k^2}) \times \left\{ -\cos(n\beta k_o u) \frac{k^2 k_o^2}{k_o^2} \mathcal{P}_{\mu\nu} (1-2u) + \frac{2}{n\beta k_o} \left( \frac{k^2}{k_o^2} \right)^2 \mathcal{P}_{\mu\nu} \right\} + \frac{3m^2 k^2}{2} \mathcal{P}_{\mu\nu}. \tag{14}
\]

The next step is to evaluate the sum over \( n \). This is done via the method proposed in appendix [3]. Using Eqs. (23)–(26), and performing the \( x \)-integration in the above expression, the finite temperature dependence of the two point function is found to be

\[
\Pi_{\mu\nu}(k, \beta) = \frac{3m^2 k^2}{2} \mathcal{P}_{\mu\nu} - \frac{e^2}{2\pi^2} T \sum_{m=-\infty}^{\infty} \left\{ -\frac{k^2}{k_o^2} \mathcal{P}_{\mu\nu} \sqrt{(\omega_m - k_o)^2} \right\}
\]
\[\int_0^\infty dy y^2 \int_0^1 du \frac{du}{[y^2 - k^2 u^2 + [k^2 - 2k_0 \omega_m]u + \omega_m^2]^2} \times \left[ - \frac{k^2 k_0^2}{k_0^2} P_{L}^{\mu\nu} (1 - 2u)^2 + 2u \frac{k^2}{k_0^2} P_{L}^{\mu\nu} (k^2 - 2k_0 \omega_m - 2k^2 u) \right] \] (15)

Where \( \omega_m = 2m \pi T \) are the standard Matsubara frequencies for bosons. A further simplification could be done by integrating by parts the term proportional to \( P_{L}^{\mu\nu} \) in the above equation

\[\Pi_{\beta}^{\mu\nu} (k) = \frac{e^2}{2\pi^2 T} \sum_{m=-\infty}^{\infty} \int d^3p \frac{1}{(2\pi)^3} \int_0^1 du \frac{du}{[p^2(1-u) + (k-P)^2 u]^2} \times \left\{ (p.p - \frac{(p.k)^2}{k^2}) P_{T}^{\mu\nu} + \frac{k^2}{2k^2 k_0^2} [(p-k)^2 - p^2]^2 P_{L}^{\mu\nu} \right\} \] (16)

This expression is analogous to a two point function, evaluated at one loop level, in the imaginary time formalism after the application of Feynman parameterization (the \( u \) integral). To determine the momenta of the particles in the loop we can consider \( y \) as the magnitude of a vector \( y \). Then we perform the following change of variables \( p = y + uk \). Having a \( y^2 \) dependent expression for the two point function, it is then possible to add any odd function of \( y \). Using Eq. (11) and adding the suitable odd function of \( y \) to eliminate the \( u \) dependence in the numerator we obtain

\[\Pi_{\beta}^{\mu\nu} (k) = \frac{3m^2 k^2}{2k_0^2} P_{L}^{\mu\nu} + 2e^2 T \sum_{m=-\infty}^{\infty} \int d^3p \frac{1}{(2\pi)^3} \int_0^1 du \frac{du}{[p^2(1-u) + (k-P)^2 u]^2} \times \left\{ (p.p - \frac{(p.k)^2}{k^2}) P_{T}^{\mu\nu} + \frac{k^2}{2k^2 k_0^2} [(p-k)^2 - p^2]^2 P_{L}^{\mu\nu} \right\} \] (17)

with \( P \equiv (\omega_m, p) \).

The strict application of world-line formalism indicates that we could perform the \( u \)-integration. Then we can do analytic continuation to obtain retarded or advanced two point function. This approach gives wrong results. To see this, note that the parameter \( u \) plays the role of a Feynman parameter applied for the product of two propagators of momentum \( P \) and \((P - k)\). As we will show in the next section the naive application of Feynman parametrization leads to wrong results at finite temperature. To avoid the \( u \)-integration we ‘undo’ Feynman parametrization. It is then straight-
forward to show that the two point function could be written as

$$\Pi^{\mu\nu}_\beta(k) = -\frac{3m^2}{2}g^{\mu\nu} + e^2T \sum_{m=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{4P^{\mu}P^{\nu} - k^{\mu}k^{\nu}}{P^2(P - k)^2}$$

(18)

which coincides with the expression obtained directly in the imaginary time formalism. The first term on the right hand side of Eq. (18) is the contribution of the 4-photon vertex, and the second term comes from the photon-scalar coupling. In the imaginary time formalism there is a systematic way of doing analytic continuation to obtain: retarded, advanced or Feynman two point function

$$\Pi^{\mu\nu}_R(k_o) = \Pi^{\mu\nu}_\beta(k_o + i\epsilon)$$
$$\Pi^{\mu\nu}_A(k_o) = \Pi^{\mu\nu}_\beta(k_o - i\epsilon)$$
$$\Pi^{\mu\nu}_F(k_o) = \Pi^{\mu\nu}_\beta(k_o + i\epsilon k_o).$$

(19)

However this method renders the world-line formalism useless. Since after a huge computational effort we get an expression, for the two point function, which is obtained by direct application of Feynman’s rules in the imaginary time formalism.

Therefore: is it possible to cure the analytic structure of Eq. (17) with the possibility of doing directly the $u$-integration? In the next section we try to give an answer to this question.

3.2 Feynman parameters at finite temperature

As we have shown in the previous section the world-line formalism incorporate the use of Feynman parameter method. The Feynman parameter method has a potential problem at finite temperature, and misleading results could be obtained [11]. The basic difference between the zero and finite temperature field theory is that, in the former it is possible to write any Green’s function entirely in terms of retarded or advanced propagators. At finite temperature one can not avoid the appearance of the product of retarded and advanced propagators. Hence we could have the following combination

$$\frac{1}{a + i\epsilon} \frac{1}{b - i\epsilon}$$

(20)

where $\epsilon$ is an infinitesimal positive parameter. Applying “naive” Feynman parameterization to this product will give:

$$\frac{1}{a + i\epsilon} \frac{1}{b - i\epsilon} = \mathcal{P} \int_0^1 \frac{dx}{[a(1 - x) + bx + i\epsilon(1 - 2x)]^2}$$

(21)
the imaginary part of the denominator vanishes at \( x = 1/2 \). In ref. Weldon has shown that the correct way to do a Feynman integral is:

\[
\frac{1}{a + i\epsilon} \frac{1}{b - i\epsilon} = \mathcal{P} \int_0^1 \frac{dx}{[a(1 - x) + bx + i\epsilon(1 - 2x)]^2} + \frac{4\pi\delta(a + b)}{a - b + 2i\epsilon}.
\] (22)

This modification renders the use of Feynman parameterization very tricky. To obtain the correct analytic structure one should add to the two point function a very peculiar function, resulting from the \( \delta \) term in Eq. (22).

We come now to the question asked in the previous section. Having an ‘underlying’ use of Feynman parametrization, the world line formalism suffers from the same subtleties mentioned above. Hence a naive use of the world-line formalism can not give the correct retarded or advanced two point function. To obtain the correct analytic structure of the two point function one should correct the naive \( u \)-integration in Eq. (17) by adding the same function proposed by Weldon in [12]. However this is far from being systematic, and can not be generalized to any n-point Green’s function.

4 Summary and outlook

In this paper we have studied the analytic features of the world-line formalism. We have shown that a translationally invariant Green’s function could be used at finite temperature. However this formalism has an intrinsic problem of analytic continuation. To solve this problem we can proceed in one of the following ways:

- Rewrite the two point function, derived in the world line formalism, to obtain an expression similar to the one obtained in the imaginary time formalism. Hence it does not worth using the world-line formalism, and a simpler method will be the direct application of the imaginary time formalism.

- We could correct the analytic structure of the two point function by adding an \textit{adhoc} function. However this method is not systematic. We can not implement such solution to obtain a self consistent world-line formalism at finite temperature.

Our study has shown that the naive application of the world-line formalism at finite temperature gives wrong results for the two point function. Hence a real time approach to the world-line formalism would have save us from
the acrobatic way of obtaining the correct analytic behavior of any Green’s function. However in the presence of external field the correct analytic structure could be obtained by calculating directly the two point function with this field.

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A projectors

In the rest frame of the plasma, the transverse and longitudinal projectors are given by [13]

\[ P^T_{\mu o}(k) = 0, \quad P^T_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}, \]
\[ P^L_{\rho \sigma}(k) = -P^T_{\rho \sigma}(k) + g_{\rho \sigma} - \frac{k_\rho k_\sigma}{k^2}. \] (23)

Those projectors are transverse with respect to the photon momentum \( k \)

\[ k_\mu P^T_{T,L}(k) = 0. \]

B Summation over \( n \)

The sum over the winding number \( n \) could be done using the following sum rule of the Bessel function \( K_0 \) [14]

\[
\sum_{n=1}^{\infty} \cos(n\alpha)K_0(n\phi) = \frac{1}{2} \left( \gamma_E + \ln \frac{\phi}{4\pi} \right) + \frac{\pi}{2\sqrt{\phi^2 + \alpha^2}} \]
\[ + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{\phi^2 + (2m\pi - \alpha)^2} - \frac{1}{2m\pi}} \right] \]
\[ + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{\phi^2 + (2m\pi + \alpha)^2} - \frac{1}{2m\pi}} \right] \] (24)

Where \( \gamma_E \) is the Euler constant. By substituting \( \alpha = \beta k_\alpha u \) and \( \phi = \beta \sqrt{u(1-u)k^2} \) the sum over \( n \) in the two point function could be done.
Being interested in the finite temperature part we can keep the $\beta$ dependent terms of the above sum:

\[
\sum_{n=1}^{\infty} \cos(n\alpha)K_0(n\phi)|_\beta = \frac{\pi}{2\sqrt{\phi^2 + \alpha^2}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\phi^2 + (2m\pi - \alpha)^2}} \\
+ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\phi^2 + (2m\pi + \alpha)^2}} = \frac{\pi T}{2} \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{-k^2u^2 + (k^2 - 2k_0\omega_m)u + \omega_m^2}}
\]

where $\omega_m = 2\pi mT$ are the Matsubara frequencies. A further simplification can be done if we transform the square root into a simple fraction. This can be achieved using

\[
\int_0^\infty \frac{y^2dy}{(y^2 + a)^2} = \frac{\pi}{4\sqrt{a}}.
\]

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