On some non-linear projections of self-similar sets in $\mathbb{R}^3$

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Abstract. In the last years considerable attention has been paid for the orthogonal and non-linear projections of self-similar sets. In this paper we consider orthogonal transformation-free self-similar sets in $\mathbb{R}^3$, i.e. the generating IFS has the form $\{\lambda_i z + t_i\}_{i=1}^q$. We show that if the dimension of the set is strictly bigger than 1 then the projection of the set under some non-linear functions onto the real line has dimension 1. As an application, we show that the distance set of such self-similar sets has dimension 1. Moreover, the third algebraic product of a self-similar set with itself on the real line has dimension 1 if its dimension is at least $1/3$.

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1. Introduction and Statements

We call a non-empty compact set $\Lambda$ self-similar in $\mathbb{R}^d$ if there exists an iterated function system (IFS) $\Phi$ of the form

$$\Phi = \{f_i(z) = \lambda_i O_i z + t_i\}_{i=1}^q,$$

where $\lambda_i \in (0,1)$, $t_i \in \mathbb{R}^d$ and $O_i$ is an orthogonal transformation of $\mathbb{R}^d$ for every $i = 1, \ldots, q$, and $\Lambda$ is the attractor of $\Phi$, i.e. $\Lambda = \bigcup_{i=1}^q f_i(\Lambda)$. We call a measure $\mu$ self-similar if there exists IFS $\Phi$ in the form (1.1) and a probability vector $(p_1, \ldots, p_q)$ that

$$\mu = \sum_{i=1}^q p_i (f_i)_* \mu,$$

where $(f)_* \mu = \mu \circ f^{-1}$.

Let us denote the orthogonal projections from $\mathbb{R}^d$ to $\mathbb{R}^k$ by $\Pi_{d,k}$. The classical result of Marstrand [11, Corollary 9.4, Corollary 9.8] states that for any $A \subseteq \mathbb{R}^d$ Borel set $\dim_H \pi A = \min \{k, \dim_H A\}$ for almost every $\pi \in \Pi_{d,k}$, where $\dim_H$ denotes the Hausdorff dimension. Let us denote the packing dimension by $\dim_P$ and the box dimension by $\dim_B$. For the definition and basic properties of Hausdorff, packing and box dimension we refer to [2].

Hochman and Shmerkin [10] proved that if the IFS $\Phi$ satisfies the strong separation condition (SSC), i.e. $f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ for every $i \neq j$ and the orthogonal transformations of the IFS $\Phi$ satisfies a minimality assumption, that is

$$\bigcup_{n=1}^\infty \{\pi O_{i_1} \cdots O_{i_n} : i_1, \ldots, i_n = 1, \ldots, q\}$$

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is dense in $\Pi_d,k$ then $\dim_H \pi \Lambda = \min \{ k, \dim_H \Lambda \}$ for every $\pi \in \Pi_d,k$, moreover, $\dim_H g(\Lambda) = \min \{ k, \dim_H \Lambda \}$ for every $g \in C^1(\mathbb{R}^d \to \mathbb{R}^k)$ without singular points. Recently, Farkas \cite{farkas} generalized this result by omitting strong separation condition.

In this paper, we focus on orthogonal transformation-free self-similar sets (OTFSS set) in $\mathbb{R}^3$, which is $O_i = I$ for every $i = 1, \cdots, q$, where $I$ denotes the identity. Similarly, we consider orthogonal transformation-free self-similar measures (OTFSS measure).

It is well known fact that in this case the dimension may drop under some orthogonal projections. Bond, Laba and Zhal \cite{bond} showed that if $\Lambda$ is a OTFSS set with SSC on $\mathbb{R}^2$ then $\dim_H g(\Lambda) = \min \{ 1, \dim_H \Lambda \}$ for some $g : \mathbb{R}^2 \to \mathbb{R}^2$ functions, see Theorem 2.7.

Our goal is to generalize this result for OTFSS sets in $\mathbb{R}^3$, at least in the case when $\dim_H \Lambda$ is large enough.

During the paper we will have a special interest on the radial projection $P_d : \mathbb{R}^d \to S^{d-1}$, where $S^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. Precisely, $P_d(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|}$. For simplicity, denote the gradient vector of a function $g : \mathbb{R}^d \to \mathbb{R}$ at a point $\underline{x}$ by $\nabla g$.

**Theorem 1.1.** Let $\Lambda$ be an orthogonal transformation-free self-similar set in $\mathbb{R}^3$ such that $\dim_H \Lambda > 1$ and not contained in any plane (but not necessarily satisfying SSC). Suppose that $g : \mathbb{R}^3 \to \mathbb{R}$ is a $C^1$ function in a $V \supseteq \Lambda$ open set such that

1. $\|\nabla g\| \neq 0$ for every $\underline{x} \in \Lambda$,
2. $\|\nabla_t \underline{x} g \times \nabla g\| = 0$ for every $\underline{x} \in V$ and $t \in \mathbb{R}$,
3. The function $\underline{x} \mapsto P_3(\nabla g)$ is bi-Lipsitz on $P_3(\Lambda) \subseteq S^2$.

Then $\dim_H g(\Lambda) = 1$.

We apply Theorem 1.1 in two ways. First, we show a corollary for the distance set of OTFSS sets in $\mathbb{R}^3$. Let us denote the distance set of $A \subseteq \mathbb{R}^d$ by $D(A)$. That is,

$$D(A) = \left\{ \|\underline{x} - \underline{y}\| : \underline{x}, \underline{y} \in A \right\}. \tag{1.2}$$

Falconer’s distance set conjecture states that if $\dim_H A > d/2$ then $D(A)$ has positive Lebesgue measure for any measurable $A \subseteq \mathbb{R}^d$. Recently, Orponen \cite{orponen} showed that for any self-similar set $\Lambda$ in $\mathbb{R}^2$ if $\mathcal{H}^1(\Lambda) > 0$ then $\dim_H D(\Lambda) = 1$, where $\mathcal{H}^1$ denotes the Hausdorff measure. We improve Orponen’s result for OTFSS sets in $\mathbb{R}^3$ in the following way.

**Theorem 1.2.** Let $\Lambda$ be an OTFSS set in $\mathbb{R}^3$ such that $\dim_H \Lambda > 1$. Then $\dim_H D(\Lambda) = 1$.

As a second application, we consider the algebraic product of a self-similar set on the real line with itself. Let $A, B \subseteq \mathbb{R}$ and denote $A \cdot B$ the algebraic product $A$ and $B$, that is,

$$A \cdot B = \{ x \cdot y : x \in A \text{ and } y \in B \}. \tag{1.3}$$

As a consequence of the result of Bond, Laba and Zhal \cite{bond} we show that for every $\Lambda$ self-similar set on the real line

$$\dim_H \Lambda \cdot \Lambda = \min \{ 2 \dim_H \Lambda, 1 \}, \tag{1.3}$$

see Corollary 2.9. We generalize this result for $\Lambda \cdot \Lambda \cdot \Lambda$ in the following way.

**Theorem 1.3.** Let $\Lambda$ be a self-similar set in $\mathbb{R}$ such that $\dim_H \Lambda > 1/3$. Then $\dim_H \Lambda \cdot \Lambda \cdot \Lambda = 1$. 

2. Preliminaries and non-linear projections in \( \mathbb{R}^2 \)

This section is devoted to enumerate our tools to prove Theorem 1.1 and to state the projection theorems in \( \mathbb{R}^2 \). First we introduce some notations. Let \( \Phi \) an IFS in \( \mathbb{R}^d \) with contracting similitudes in the form [13]. Denote the attractor of \( \Phi \) by \( \Lambda \). Let us denote the set of symbols by \( \Sigma = \sigma^* \). Denote by \( \sigma \) the left-shift operator on \( \Sigma \). Let us define the upper and lower local dimension of a measure \( \mu \) in the usual way, for any \( i = (i_0, i_1, \ldots) \in \Sigma \)

\[
\rho(i) = \lim_{n \to \infty} f_{i_0} \circ f_{i_1} \circ \cdots \circ f_{i_n}(0),
\]

where \( 0 = (0, \ldots, 0) \in \mathbb{R}^d \). It is easy to see that \( \rho(i) = f_{i_0}(\rho(i_{\sigma(i)})) \).

Let \( p = (p_1, \ldots, p_d) \) be a probability vector with strictly positive elements. Denote the Bernoulli measure on \( \Sigma \) by \( \text{Fix}(\sigma) = \prod_{i=1}^{d} \sigma_i = (\sigma_1 \circ \cdots \circ \sigma_d) \). We call the attractor and self-similar measures of such a system an OTFSS set. Let \( \Phi = \{ f_i \}_{i=1}^{q} \) be a composition of functions \( f_{i_0} \circ \cdots \circ f_{i_{n-1}} \), we write \( f_{\tau} \), where \( \tau = (i_0, \ldots, i_{n-1}) \). We denote the fixed point of a function \( f_{\tau} \) by \( \text{Fix}(f_{\tau}) \). Denote \( [\tau] = \{ j = (j_0, j_1, \ldots) \in \Sigma : i_0 = j_0, \ldots, i_{n-1} = j_{n-1} \} \).

We denote the projection of a cylinder set by \( \Lambda_{\tau} = \rho([\tau]) = f_{\tau}(\Lambda) \), and we call it as a cylinder set of \( \Lambda \). We note that if \( \mu \) is a OTSS measure (or \( \Lambda \) is a OTFSS set) with IFS \( \Phi = \{ f_i \}_{i=1}^{q} \) in \( \mathbb{R}^d \) then for any \( \tau \in \Sigma^* \), the measure \( \mu_{\tau} := \frac{\mu|_{\Lambda_{\tau}}}{\mu(\Lambda)} \) (or respectively \( \Lambda_{\tau} \) is also a self-similar measure (or self-similar set) with IFS \( \Phi_{\tau} := \{ f_{i}(\tau) \}_{i=1}^{q} \) of \( \Phi_{\tau} \). On the other hand, for any \( \pi \in \Pi_{d,k} \) the measure \( \pi_{\mu} = \mu \circ \pi^{-1} \) is OTFSS measure (or respectively \( \pi \Lambda \) is a OTFSS set), as well, with IFS \( \pi \Phi := \{ \lambda \pi + \pi(f_{\tau}) \}_{i=1}^{q} \). We denote the nth iteration of the IFS by \( \Phi_{n} = \{ f_{\tau} \}_{\tau \in \Sigma^*} \).

Our first approach of the study of orthogonal transformation-free self-similar sets is to find a proper approximating subsystem.

**Proposition 2.1.** Let \( \Lambda \) be an OTFSS set in \( \mathbb{R}^d \) with IFS \( \Phi = \{ f_i \}_{i=1}^{q} = \{ \lambda_i x + \xi_i \}_{i=1}^{q} \). For every \( \varepsilon > 0 \), there exists an IFS \( \Phi' \) of the form \( \{ g_i \}_{i=1}^{q} = \{ \lambda_i x + \xi'_i \}_{i=1}^{q} \) with \( \lambda_i \in (0, 1) \) such that the attractor \( \Lambda' \) of \( \Phi' \) satisfies the SSC, \( \Lambda' \subseteq \Lambda \) and \( \dim_H \Lambda' > \dim_H \Lambda - \varepsilon \). Moreover, the functions of \( \Phi' \) can be written as the composition of functions in \( \Phi \).

We call the attractor and self-similar measures of such a system \( \Phi' \) as homogeneous orthogonal transformation-free self-similar set and measures (HOTFSS).

The proof is analogous to the proof of Peres and Shmerkin [13, Proposition 6], therefore we omit it. Let us denote the Hausdorff dimension of a measure \( \mu \) by \( \dim_H \mu \). That is,

\[
\dim_H \mu = \inf \{ \dim_H A : \mu(A) > 0 \}.
\]

Let us define the upper and lower local dimension of a measure \( \mu \) at a point \( x \) in the usual way by

\[
d^-_\mu(x) = \liminf_{r \to 0+} \frac{\log \mu(B_r(x))}{\log r}, \quad d^+_\mu(x) = \limsup_{r \to 0+} \frac{\log \mu(B_r(x))}{\log r},
\]

where \( B_r(x) \) is the ball of radius \( r \) centered at \( x \).
where $B_r(x)$ is the ball with radius $r$ centered at $x$. By [1] Theorem 1.2,
\[
\dim_H \mu = \mu - \operatorname{essinf}_x d_\mu(x).
\]
We say that the measure $\mu$ is exact dimensional if $d_\mu(x) = 1$ for $\mu$-a.e. $x$. By [1] Corollary 2.1, if $\mu$ is exact dimensional then
\[
\dim_H \mu = \inf\{\dim_H A : \mu(A) = 1\}.
\]
\[
(2.1)
\]
\textbf{Lemma 2.2.} Let $\mu$ and $\nu$ be Borel probability measures such that $\mu \ll \nu$ (that is, $\mu$ is absolutely continuous with respect to $\nu$) and $\nu$ is exact dimensional. Then $\dim_H \mu = \dim_H \nu$.

\textbf{Proof.} Since $\mu \ll \nu$, for any measurable set $A$, if $\mu(A) > 0$ then $\nu(A) > 0$. Thus, $\dim_H \mu \geq \dim_H \nu$. On the other hand, since $\nu$ is exact dimensional
\[
\dim_H \nu = \inf\{\dim_H A : \nu(A) = 1\} = \inf\{\dim_H A : \nu(A^c) = 0\} = \inf\{\dim_H A : \mu(A) = 1\} \geq \dim_H \mu,
\]
where $A^c$ denotes the complement of $A$.

Our second approach is to approximate the non-linear projections of OTFSS measures with SSC by orthogonal projections. Let $g: \mathbb{R}^d \mapsto \mathbb{R}$ a $C^1$ function. We denote the projection of a Borel measure $\mu$ on $\mathbb{R}^d$ by $g_*\mu = \mu \circ g^{-1}$. Let us denote the gradient of $g$ at a point $x = (x_1, \ldots, x_d)$ by $\nabla x g$, i.e.
\[
\nabla x g = \left( g'_{x_1}(x) \right) \vdots \left( g'_{x_d}(x) \right).
\]
Denote $\pi_{g,z} \in \Pi_{d,1}$ the orthogonal projection from $\mathbb{R}^d$ to the subspace spanned by $\nabla x g$, that is, $\pi_{g,z}(y) = \frac{<\nabla x g, y>}{\|\nabla x g\|}$, where $<\cdot, \cdot>$ denotes the standard scalar product on $\mathbb{R}^d$ and $\|\cdot\|$ denotes the induced norm. The next theorem is a consequence of the results of Hochman [5].

\textbf{Theorem 2.3.} Let $\mu$ be an OTFSS measure with SSC in $\mathbb{R}^d$ and let $g: \mathbb{R}^d \mapsto \mathbb{R}$ a $C^1$ function with $\|\nabla x g\| \neq 0$ for every $x \in \operatorname{spt} \mu$. Then
\[
\dim_H g_* \mu \geq \mu - \operatorname{essinf}_x \dim_H \pi_{g,z} \mu.
\]

\textbf{Proof.} Let $\mu$ be an OTFSS measure with SSC in $\mathbb{R}^d$. Then by [5] Example 4.3 the measure $\mu$ is a homogeneous uniformly scaling measure, see [5] Definition 1.5(3) and Definition 1.35. Let $P$ be the ergodic fractal distribution generated by $\mu$, see [5] Definition 1.2, Definition 1.5(1) and Proposition 1.36). For a $\pi \in \Pi_{d,1}$, let
\[
E_P(\pi) = \int \dim_H \pi dP(\nu).
\]
Applying [5] Theorem 1.23] and [5] Proposition 1.36] we have for $g: \mathbb{R}^d \mapsto \mathbb{R}$ a $C^1$ function with $\|\nabla x g\| \neq 0$
\[
\dim_H g_* \mu \geq \mu - \operatorname{essinf}_x E_P(\pi_{g,z}).
\]
By [8] Proposition 1.36] for P-a.e ν measure there exists a ball $B$ that $\mu \ll (T_B)_*\nu$, where $T_{B_\epsilon}(y) = \frac{y - z}{\epsilon}$. Hence, $\pi\mu \ll \pi(T_B)_*\nu$ for every $\pi \in \Pi_{d,1}$ and P-a.e. ν. On the other hand, by [8] Theorem 1.22] the measure $\pi\nu$ is exact dimensional for P-a.e. ν. Since $T_B$ is a bi-Lipschitz map, by Lemma 2.2] $\dim_H \pi\mu = \dim_H \pi\nu$ for every $\pi \in \Pi_{d,1}$ and P-a.e. ν, which implies that $E_F(\pi) = \dim_H \pi\mu$. \[ \square \nabla

As a consequence of Theorem [2.3] and [9] Theorem 1.8], we state here a modified version of the proposition of Bond, Laba and Zhal [1, Proposition 2.6].

**Proposition 2.4.** Let $\mu$ be a HOTFSS measure with SSC in $\mathbb{R}^2$ such that $\text{spt}\mu$ is not contained in any line. Suppose that $g : \mathbb{R}^2 \mapsto \mathbb{R}$ is a HOTFSS map such that $\|\nabla_x g\| \neq 0$ and

$$\left\| \left( g''_{xy}(x)g_y(x) - g''_{yx}(x)g_x(y) \right) \right\| \neq 0$$

for every $x \in \text{spt}\mu$. Then

$$\dim_H g_*\mu = \min \{1, \dim_H \mu\}.$$

Before we prove the proposition, we need a technical lemma.

**Lemma 2.5.** Let $\mu$ be a HOTFSS measure with SSC in $\mathbb{R}^2$ such that $\text{spt}\mu$ is not contained in any line. Then there exists a constant $c > 0$ that $\dim_H \pi\mu \geq c > 0$ for every $\pi \in \Pi_{2,1}$.

**Proof.** Let $\mu$ be a HOTFSS measure with SSC in $\mathbb{R}^2$ such that $\text{spt}\mu$ is not contained in any line and let $\{\ell_i(x) = \lambda x + l_i\}_{i=1}^d$ be the corresponding IFS and $p = (p_1, \ldots, p_d)$ the corresponding probability vector.

Since $\text{spt}\mu$ is not contained in any line, there exist three fixed points of the functions, let say $f_1, f_2$ and $f_3$, which forms a triangle. Let us denote the sides of the triangle by $a, b$ and $c$. Let $\kappa = \inf_{\pi \in \Pi_{2,1}} \max \{|\pi a|, |\pi b|, |\pi c|\} > 0$ and let $N = \lceil \frac{\log \kappa}{\log 2} \rceil$, where $\lceil \rceil$ denotes the diameter of a set. Let $x \in \pi\text{spt}\mu$ be arbitrary, and let

$$z_n(x) := \sum_{\substack{\pi \in \Pi_N \cap \Lambda \neq \emptyset \\text{s.t.} \pi(x) \cap \Lambda \neq \emptyset}} \nu(\pi),$$

It is easy to see by the definition of $N$ and $\kappa$ that there exists an $\pi \in \Sigma^*$ with $|\pi| = N$ that $B_{\tilde{x}_\lambda}(x) \cap \pi\Lambda_{\pi} = \emptyset$. Thus, $z_1(x) \leq (1 - p_{\min}^N)$, where $p_{\min} = \min \{p_1, \ldots, p_d\}$. On the other hand, for every $\pi \in \Sigma^*$ with $|\pi| = nN$ and $B_{\tilde{x}_\lambda}(x) \cap \pi\Lambda_{\pi} \neq \emptyset$ there exists a $\gamma \in \Sigma$ with $|\gamma| = N$ that $B_{\tilde{x}_\lambda}(x) \cap \pi\Lambda_{\pi} = \emptyset$. Thus,

$$\sum_{\substack{\pi \in \Pi_N \cap \Lambda \neq \emptyset \\text{s.t.} \pi(x) \cap \Lambda \neq \emptyset}} \nu(\pi) \leq 1 - p_{\min}^N. \tag{2.2}$$

We prove by induction that $z_n(x) \leq (1 - p_{\min}^N)^n$. For $n = 1$ it has already been showed. Assume that it
holds for $n$. Then by (2.2)

$$z_{n+1}(x) = \sum_{\tau \in S^{(n+1)N}} \nu([\tau]) = \sum_{\tau \in S^{nN}} \sum_{\pi \in S^{N}} \nu([\pi]) =$$

$$\sum_{\tau \in S^{nN}} \nu([\tau]) \leq (1 - p_{\min}^N) z_n(x) \leq (1 - p_{\min}^N)^{n+1}.$$ 

Hence, for any $x \in \text{spt}\mu$

$$\liminf_{r \to 0^+} \frac{\log \mu(B_r(x))}{\log r} = \liminf_{n \to \infty} \frac{\log \mu(B_{\frac{1}{\lambda}^N}(x))}{nN \log \lambda} \geq \liminf_{n \to \infty} \frac{\log z_n(x)}{nN \log \lambda} = \frac{\log(1 - p_{\min}^N)}{N \log \lambda} > 0.$$

Which implies by (2.1) that $\dim_H \pi\mu \geq \frac{\log(1 - p_{\min}^N)}{N \log \lambda} > 0$ for every $\pi \in \Pi_{2,1}$. □

**Proof of Proposition 2.4.** Let $\mu$ be a HOTFSS measure with SSC such that $\text{spt}\mu$ is not contained in any line. Since $\dim_H g_*\mu \leq \min\{1, \dim_H \mu\}$, it is enough to show the lower bound. By Theorem 2.3 we have

$$\dim_H g_*\mu \geq \mu - \text{essinf}_{x} \dim_H \pi_{g_*x}\mu.$$ 

Thus, it is enough to show that

$$\dim_H \pi_{g_*x}\mu = \min\{1, \dim_H \mu\} \quad \text{for } \mu\text{-a.e. } x. \quad (2.3)$$

If $\mu$ is a HOTFSS measure with IFS $\{\lambda x + \xi_i\}_{i=1}^q$, then for any $\pi \in \Pi_{2,1}$ the measure $\pi\mu$ is HOTFSS measure, as well, with IFS $\{\lambda x + \pi(\xi_i)\}_{i=1}^q$. By using the parametrization $\pi_\theta(x) = \langle \cos \theta, \sin \theta, x \rangle$ and (9) Theorem 1.8, it follows that

$$\dim_P \{\theta \in [0, \pi) : \dim_H \pi_\theta \mu < \min\{1, \dim_H \mu\}\} = 0.$$ 

Hence, to verify (2.3) it is enough to show that

$$\dim_H f_*\mu > 0,$$

where $f(x) = \arctan(\frac{g_{x_1}(\xi)}{g_{x_2}(\xi)})$. By our assumption $\|\nabla f\| \neq 0$ for every $x \in \text{spt}\mu$. By applying Theorem 2.3 and Lemma 2.5 we get

$$\dim_H f_*\mu \geq \mu - \text{essinf}_{x} \dim_H \pi_{f_*x}\mu \geq \inf_{\pi \in \Pi_{2,1}} \dim_H \pi\mu \geq c > 0.$$ 

□

First, we state here a consequence of Proposition 2.4 which plays an important role for further studies of the non-linear projections in $\mathbb{R}^3$. The corollary is analogous to [11, Proposition 2.5] but for measures.
Corollary 2.6. If μ is a HOTFSS measure with SSC in \( \mathbb{R}^2 \) such that \( 0 \notin \text{spt} \mu \) and \( \text{spt} \mu \) is not contained in any line. Then

\[
\dim_H(P_2)_*\mu = \min\{1, \dim_H \mu\}. \tag{2.4}
\]

Proof. Since \( \mu \) can be written as a convex combination of self-similar measures restricted to cylinder sets, we have

\[
\dim_H(P_2)_*\mu = \min_{\tau \in \mathcal{S}^n} \dim_H(P_2)_\tau\mu
\]

for every \( n \geq 1 \). Thus it is enough to show that for sufficiently large \( n \geq 1 \), (2.4) holds for any \( \tau \in \mathcal{S}^n \). By choosing \( n \) sufficiently large and by applying a rotation transformation, without loss of generality we may assume that \( \text{spt} \mu_\tau \) is contained in the upper half plane separated away from the \( x \)-axis.

Since the map \( h : x \mapsto (x, \sqrt{1 - x^2}) \) is bi-Lipschitz for every \( x \in (-1 + \varepsilon, 1 + \varepsilon) \), it is enough to show that for the map \( g : (x,y) \mapsto \frac{x}{\sqrt{x^2+y^2}} \), \( \dim_H g_*\mu_\tau = \min\{1, \dim_H \mu_\tau\} \). Indeed, \( g \) satisfies the assumptions of Proposition 2.4.

As another consequence of Proposition 2.4 we can state the following theorem for general self-similar sets in \( \mathbb{R}^2 \).

Theorem 2.7. Let \( \Lambda \) be an arbitrary self-similar set in \( \mathbb{R}^2 \) not contained in any line. Suppose that \( g : \mathbb{R}^2 \to \mathbb{R} \) is a \( C^2 \) map such that \( \|\nabla g\| \neq 0 \) and

\[
\left\| \begin{pmatrix} g''_{xx}(x)g_y'(x) - g''_{xy}(x)g'_x(x) \\ g''_{xy}(x)g_y'(x) - g''_{yy}(x)g'_x(x) \end{pmatrix} \right\| \neq 0
\]

for every \( x \in \Lambda \). Then

\[
\dim_H g(\Lambda) = \min\{1, \dim_H \Lambda\}.
\]

Proof. Let \( \Lambda \) be a self-similar set in \( \mathbb{R}^2 \) not contained in any line. Applying [12, Lemma 3.4], for every \( \varepsilon > 0 \) there exists a self-similar set \( \Lambda' \subseteq \Lambda \) not contained in any line such that \( \dim_H \Lambda' \geq \dim_H \Lambda - \varepsilon \) and its the attractor of IFS \( \Phi' \) satisfying SSC. If one of the functions of \( \Phi' \) contains an irrational rotation then by [10, Corollary 1.7]

\[
\dim_H g(\Lambda) \geq \dim_H g(\Lambda') = \min\{1, \dim_H \Lambda'\} \geq \min\{1, \dim_H \Lambda\} - \varepsilon.
\]

If none of the functions of \( \Phi' \) contains irrational rotation then by [12, Lemma 4.2] there exists a self-similar set \( \Lambda'' \subseteq \Lambda \) that satisfies the assumptions of generating IFS \( \Phi'' \) of \( \Lambda'' \) does not contain any rotation or reflection, i.e. it is a OTFSS set with SSC. By Proposition 2.4, there exists a HOTFSS set \( \Lambda''' \) with SSC that \( \Lambda''' \subseteq \Lambda \) and \( \dim_H \Lambda''' \geq \dim_H \Lambda - 3\varepsilon \).

Let \( \mu \) be the natural self-similar measure on \( \Lambda''' \), that is, \( \mu \) is the equidistributed self-similar measure on the cylinder sets. Hence, \( \dim_H \mu = \dim_H \Lambda''' \). By Proposition 2.4

\[
\dim_H g(\Lambda) \geq \dim_H g_*\mu = \min\{1, \dim_H \mu\} = \min\{1, \dim_H \Lambda'''\} \geq \min\{1, \dim_H \Lambda\} - 3\varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, the statement of the theorem is proven.

As a corollary of Theorem 2.7, one can prove a weaker version of Falconer’s distance set conjecture in \( \mathbb{R}^2 \). This is a little bit stronger than Orponen’s result [12, Theorem 1.2], since we only assume that \( \dim_H \Lambda \geq 1 \) and we do not need that \( \mathcal{H}^1(\Lambda) > 0 \).
Corollary 2.8. If $\Lambda$ is a self-similar set in $\mathbb{R}^2$ with $\dim_H \Lambda \geq 1$. Then
\[
\dim_H D(\Lambda) = 1,
\]
where $D(\Lambda)$ denotes the distance set of $\Lambda$ defined in (1.2).

Proof. Since if $\Lambda$ is contained in a line then $\dim_H D(\Lambda) = \dim_H \Lambda$, we may assume that $\Lambda$ is not contained in any line. Let $a$ be an arbitrary element of $\Lambda$ and let $\Lambda_T$ be a cylinder set such that $\text{dist}(a, \Lambda_T) > 0$. Then $D_a(x) = \|x - a\|$ satisfies the conditions of Theorem 2.7 with self-similar set $\Lambda_T$. Thus,
\[
\dim_H D(\Lambda) \geq \dim_H D_a(\Lambda) \geq \dim_H D_a(\Lambda_T) = \min \{1, \dim_H \Lambda_T\} = \min \{1, \dim_H \Lambda\} = 1.
\]

Another corollary of Theorem 2.7 is (1.3).

Corollary 2.9. If $\Lambda$ is a self-similar set in $\mathbb{R}$ then
\[
\dim_H \Lambda \cdot \Lambda = \min \{2 \dim_H \Lambda, 1\}.
\]

Proof. Let $\Lambda$ be an arbitrary self-similar set on $\mathbb{R}$. Without loss of generality, we may assume that $\Lambda$ is not a singleton. Then there exists a cylinder set $\Lambda_T$ of $\Lambda$ that every element in $\Lambda_T$ is either strictly positive or strictly negative.

By [13, Proposition 6], for every $\varepsilon > 0$ there exists a self-similar set $\Lambda_T' \subseteq \Lambda_T$ such that $\dim_H \Lambda_T' \geq \dim_H \Lambda - \varepsilon$ and its the attractor of IFS $\Phi$ satisfying SSC and has the form
\[
\Phi = \{f_i(x) = \lambda x + t_i\}_{i=1}^q.
\]

Let $g(x, y) = xy$. Then
\[
\|\nabla g\| = \sqrt{y^2 + x^2} \neq 0 \quad \text{and} \quad \left\|\left(\frac{g''_x(x)g'_y(x) - g''_y(x)g'_x(x)}{g''_x(x)g_y(x) - g''_y(x)g'_x(x)}\right)\right\| = \sqrt{x^2 + y^2} \neq 0
\]
for any $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Thus by Theorem 2.7
\[
\dim_H \Lambda \cdot \Lambda \geq \dim_H \Lambda' \cdot \Lambda' = \dim_H g(\Lambda' \times \Lambda') = \min \{1, \dim_H \Lambda' \times \Lambda'\} \geq \min \{1, 2 \dim_H \Lambda\} - 2\varepsilon,
\]
where we used that $\dim_H \Lambda' \times \Lambda' = 2 \dim_H \Lambda'$. Since $\varepsilon > 0$ was arbitrary, the proof is complete.\]
The proof is similar and uses the method of [6 Proposition 4.3]. Before we prove the proposition, we need a lemma on transversality.

Let \( \Phi = \{ f_i(x) = \lambda x + L \}_{i=1}^q \). We parametrize the projections \( \Pi_{i,j} \) as follows
\[
\pi_{u,v}(x, y, z) = \cos(u) \cos(v)x + \cos(u) \sin(v)y + \sin(u)z,
\]
where \( u \in (-\pi/2, \pi/2) \) and \( v \in [0, 2\pi] \). For every \( (u, v) \in (-\pi/2, \pi/2) \times [0, 2\pi] \), let
\[
\Phi_{u,v} = \{ f_i^{u,v}(x) = \lambda x + \pi_{u,v}(L_i) \}_{i=1}^q.
\]
For any \( (u, v) \in \mathbf{U}_m \),
\[
f^{u,v}_\tau(0) = \sum_{n=0}^{\lfloor r \rfloor - 1} \lambda^n \pi_{u,v}(L_n) \quad \text{and let} \quad \Delta_{\tau,j}(u, v) = f^{u,v}_\tau(0) - f^{u,v}_\tau(0) = f^{u,v}_\tau(\rho(i)) - f^{u,v}_\tau(\rho(j)).
\]
For any \( i, j \in \Sigma \) let \( \Delta_{i,j}(u, v) = \pi_{u,v}(\rho(i)) - \pi_{u,v}(\rho(j)) \).

**Lemma 2.11.** Let \( \Phi = \{ f_i(x) = \lambda x + L \}_{i=1}^q \) be an IFS on \( \mathbb{R}^3 \) with \( \lambda \in (0, 1) \) and assume that satisfies the SSC. Let \( \mathbf{U}_m = [-\pi/2 + 1/m, \pi/2 - 1/m] \times [0, 2\pi] \) for \( m \geq 1 \). Then there exists a \( \delta = \delta(m) > 0 \) such that for every \( n \geq 1 \) and \( \tau \neq j \in \mathbb{S}^n \) and \( (u, v) \in \mathbf{U}_m \)
\[
\max \left\{ \left| \Delta_{\tau,j}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial u}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial v}(u, v) \right| \right\} > \delta \lambda^n.
\]

**Proof.** It is enough to show that there exists a \( \delta > 0 \) that for every \( \tau, j \in \mathbb{S}^n \) with \( i_0 \neq j_0 \) and \( (u, v) \in \mathbf{U}_m \)
\[
\max \left\{ \left| \Delta_{\tau,j}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial u}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial v}(u, v) \right| \right\} > \delta.
\]

Suppose that it is not true. That is for every \( \delta > 0 \) there exist \( n \geq 1 \), \( \tau, j \in \mathbb{S}^n \) with \( i_0 \neq j_0 \) and \( (u, v) \in \mathbf{U}_m \) that
\[
\max \left\{ \left| \Delta_{\tau,j}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial u}(u, v) \right|, \left| \frac{\partial \Delta_{\tau,j}}{\partial v}(u, v) \right| \right\} \leq \delta.
\]
Since \( \Sigma, \Lambda \) and \( \mathbf{U}_m \) are compact, we can find \( i, j \in \Sigma \) with \( i_0 \neq j_0 \) and \( (u_0, v_0) \in \mathbf{U}_m \) that
\[
\Delta_{i,j}(u_0, v_0) = \frac{\partial \Delta_{i,j}}{\partial u}(u_0, v_0) = \frac{\partial \Delta_{i,j}}{\partial v}(u_0, v_0) = 0.
\]
Since \( \pi_{u,v} \) linear we get that
\[
\begin{pmatrix}
\cos(u_0) \cos(v_0) & \cos(u_0) \sin(v_0) & \sin(u_0) \\
-\sin(u_0) \cos(v_0) & -\sin(u_0) \sin(v_0) & \cos(u_0) \\
-\cos(u_0) \sin(v_0) & \cos(u_0) \cos(v_0) & 0
\end{pmatrix}
\begin{pmatrix}
\rho(i) - \rho(j)
\end{pmatrix} = 0.
\]
Since the IFS satisfies the SSC and \( i_0 \neq j_0 \), \( \rho(i) - \rho(j) \neq 0 \). On the other hand, the determinant of the matrix on the left hand side is equal to \( -\cos(u_0) \neq 0 \) if \( (u_0, v_0) \in \mathbf{U}_m \), which is a contradiction. \( \square \)
Proof of Proposition 2.10. Since the exceptional set in (2.5) can be covered by \([-\pi/2] \times [0, 2\pi] \cup \{\pi/2\} \times [0, 2\pi] \cup \bigcup_{m=1}^{\infty} U_m\), it is enough to show that
\[
\dim_P \{ (u, v) \in U_m : \dim_H \pi_{u,v}\mu < \min \{1, \dim_H \mu\} \} \leq 1
\]
for every \(m \geq 1\).

By [9, Theorem 1.1], if \(\dim_H \pi_{u,v}\mu < \min \{1, \dim_H \mu\}\) then \(\lim_{n \to \infty} \frac{-1}{n} \log \Delta_n(u, v) = \infty\), where \(\Delta_n(u, v) = \min_{\tau \neq \gamma \in S^n} |\Delta_{\tau,\gamma}(u, v)|\). Thus, the set \(E_m = \{ (u, v) \in U_m : \dim_H \pi_{u,v}\mu < \min \{1, \dim_H \mu\} \}\) can be covered by
\[
E_m \subseteq U_m \cap \bigcap_{n>N} \bigcup_{\tau \neq \gamma \in S^n} \Delta_{\tau,\gamma}^{-1}(-\varepsilon^n, \varepsilon^n).
\]

It is easy to see that for any \(\tau \neq \gamma \in S^n\) set \(\Delta_{\tau,\gamma}^{-1}(0)\) is either a graph of a smooth function from \([0, \pi/2]\) to \((-\pi/2, \pi/2)\) or \(A \times (-\pi/2, \pi/2)\), where \(A\) is a finite set with at most 2 elements. By Lemma 2.11, \(\Delta_{\tau,\gamma}^{-1}(-\varepsilon^n, \varepsilon^n)\) is contained in a \(C(\varepsilon/\lambda)^n\) neighbourhood of \(\Delta_{\tau,\gamma}^{-1}(0)\), and thus can be covered by \(C'\lambda^n\) balls with radius \((\varepsilon/\lambda)^n\). Hence, \(U_m \cap \bigcap_{n>N} \bigcup_{\tau \neq \gamma \in S^n} \Delta_{\tau,\gamma}^{-1}(-\varepsilon^n, \varepsilon^n)\) can be covered by at most \(C'|S|^2n\lambda^n\) balls with radius \((\varepsilon/\lambda)^n\), and therefore,
\[
\dim_P E_m \leq \lim_{\varepsilon \to 0^+} \sup_N \frac{\dim_B \left( U_m \cap \bigcap_{n>N} \bigcup_{\tau \neq \gamma \in S^n} \Delta_{\tau,\gamma}^{-1}(-\varepsilon^n, \varepsilon^n) \right)}{N} \leq 1.
\]

Finally, we state here the dimension conservation phenomena for OTFSS measures, first showed by Furstenberg [7] and generalized by Falconer and Jin [3].

**Theorem 2.12.** Let \(\mu\) be an OTFSS measure with SSC in \(\mathbb{R}^d\) and let \(\pi \in \Pi_{d,k}\) arbitrary. Then
\[
\dim_H \pi\mu + \dim_H \mu_{\pi^{-1}(z)} = \dim_H \mu \text{ for } \pi\mu\text{-a.e. } z \in \mathbb{R}^k,
\]
for the proof of the theorem we refer to [3, Theorem 1.37].
3. Radial projection in $\mathbb{R}^3$

The critical point of our study is the examination of the radial projection. Unfortunately, we cannot prove the analogue of Corollary 2.6 in general. However, we prove that a OTFSS set with dimension larger than 1 contains a HOTFSS measure which radial projection is strictly larger than 1. This section is devoted to prove the following statement.

**Theorem 3.1.** Let $\Lambda$ be a OTFSS set in $\mathbb{R}^3$ such that $\dim_H \Lambda > 1$ and not contained in any plane. Then there exists a $\mu$ HOTFSS measure such that $\text{spt}\mu \subseteq \Lambda$ and $\dim_H \mu \geq \dim_H(P_3)\mu > 1$.

Let us denote the closed double cone with vertex $x \in \mathbb{R}^3$ and with angle $\alpha$ and with axis $\varphi \in \mathbb{R}^3$ with $\|\varphi\| = 1$ by $C_{\alpha,\varphi}(x)$. That is,

$$C_{\alpha,\varphi}(x) = \left\{ \varphi \in \mathbb{R}^3 : \langle \varphi - y, \varphi \rangle \geq \left| \cos(\alpha) \right| \|x - y\| \right\}.$$

In other words, the angle between $x - y$ and $\varphi$ is less than or equal to $\alpha$. First, we show the following lemma.

**Lemma 3.2.** Let $\Lambda$ be a HOTFSS set in $\mathbb{R}^3$ such that it is not contained in any plane. Then for every vector $x \in \mathbb{R}^3$ with $\|x\| = 1$ and $y \in \Lambda$ there exists an $\pi/2 > \alpha > 0$ such that for every $r > 0$

$$\text{int}(B_r(x) \cap C_{\alpha,\varphi}(x)) \cap \Lambda \neq \emptyset,$$

where $\text{int}(A)$ denotes the interior of a set $A$.

**Proof.** We argue by contradiction. Assume that there exists a vector $x \in \mathbb{R}^3$ with $\|x\| = 1$ and $y \in \Lambda$ that for every $\pi/2 > \alpha > 0$ there exists an $r = r(\alpha) > 0$ that

$$\text{int}(B_r(x) \cap C_{\alpha,\varphi}(x)) \cap \Lambda = \emptyset.$$

Let $\Phi = \{ f_i(x) = \lambda x + t_i \}_{i=1}^n$ be the corresponding IFS and let $n(r) = \min \{ n : \lambda^n < r \}$.

For a $\pi/2 > \alpha > 0$ if $x \in \Lambda_\tau$ with $|\tau| \geq n(r(\alpha))$ then $\Lambda_\tau \subseteq \text{int}(B_{r(\alpha)}(y)) \cap \Lambda$. Thus by our assumption $\Lambda_\tau \cap \text{int}(C_{\alpha,\varphi}(x)) = \emptyset$. Since $\Phi$ does not contain any orthogonal transformation.

$$\Lambda \cap \text{int}(C_{\alpha,\varphi}(f^{-1}_\tau(x))) = f^{-1}_\tau(\Lambda \cap \text{int}(C_{\alpha,\varphi}(x))) = \emptyset.$$

Thus, for every $\pi/2 > \alpha > 0$ there exists an $N \geq 1$ that for every $\tau = \rho(i) = f(\rho(\sigma^{\pi/2}))$ and $|\tau| \geq N$

$$\Lambda \cap \text{int}(C_{\alpha,\varphi}(f^{-1}_\tau(x))) = \emptyset.$$

Let $y$ be a density point of the sequence $\{ f^{-1}_\tau(x) \}$. Since $\Lambda$ is compact $y \in \Lambda$ and $\Lambda \cap \text{int}C_{\alpha,\varphi}(y) = \emptyset$. But $\alpha$ was arbitrary, thus $\Lambda$ must be contained in a plane with normal vector $\varphi$ and containing $y$ which is a contradiction. $\square$

Denote by $V_\pi$ the subspace to which $\pi \in \Pi_{3,2}$ projects and denote the normal vector of $V_\pi$ by $\varphi_\pi \in \mathbb{R}^3$ with $\|\varphi_\pi\| = 1$. For a projection $\pi \in \Pi_{3,2}$ let

$$\sin(\varepsilon_\pi) = \inf_{x \neq y \in \Lambda} \frac{\|\varphi_\pi \times (x - y)\|}{\|x - y\|}. \quad (3.1)$$
Since $\Lambda$ is compact, if $\pi \Lambda$ satisfies the SSC then $\sin(\varepsilon_\pi) > 0$. Let $\Gamma_\pi$ be as follows

$$
\Gamma_\pi = \left\{ (x, y) \in \Lambda \times \Lambda : x \neq y \ & \ & \sin(\varepsilon_\pi) = \frac{\|n_\pi \times (x - y)\|}{\|x - y\|} \right\}.
$$

Since $\Phi = \{f_i\}_{i=1}^q$ is orthogonal transformation free,

$$
\sin(\varepsilon_\pi) = \min_{i \neq j} \inf \frac{\|n_\pi \times (x - y)\|}{\|x - y\|}.
$$

Moreover, $\Lambda$ is compact, thus there exist $i \neq j$ and $x_i \in f_i(\Lambda), y_j \in f_j(\Lambda)$ that $(x_i, y_j) \in \Gamma_\pi \neq \emptyset$. By definition

$$
\int \text{C} \in \Lambda = \emptyset \text{ for every } x \in \Lambda. \quad (3.2)
$$

**Lemma 3.3.** Let $\Lambda$ be a HOTFSS set not contained in any plane and suppose that $\pi \Lambda$ satisfies the SSC. If $(x, y) \in \Gamma_\pi$ then $l_{x,y} \cap \Lambda \setminus \{x, y\} = \emptyset$, where $l_{x,y}$ is the line containing $x$ and $y$. In other words, if $(x, y), (x, z) \in \Gamma_\pi$ then $(y, z) \notin \Gamma_\pi$.

**Proof.** Let us suppose that $(x, y), (x, z), (y, z) \in \Gamma_\pi$. It is easy to see that $x, y, z$ must be contained in a line. Indeed, $z$ must be a common element of the boundary of the cones $C_{\varepsilon_\pi, \Lambda_+}(x)$ and $C_{\varepsilon_\pi, \Lambda_+}(y)$, see Figure 1.
Without loss of generality, assume that \( \pi \) is between \( \varphi \) and \( y \). Let \( V \) be the common tangent plane of the cones \( C_{\pi \varphi \varphi} \) and \( C_{\pi \varphi \varphi} \), and let \( v \) be its normal vector. Applying Lemma 3.2 there exists an \( \pi/2 > \alpha > 0 \) that \( \text{int}(B_{\pi}(\varphi) \cap C_{\alpha \varphi \varphi}) \cap \Lambda \neq \emptyset \), for every \( r > 0 \). Let \( r > 0 \) be sufficiently small that \( \text{int}(B_{\pi}(\varphi) \cap C_{\alpha \varphi \varphi}) \subseteq \text{int}(C_{\pi \varphi \varphi}(\varphi)) \). Then
\[
\emptyset \neq \text{int}(B_{\pi}(\varphi) \cap C_{\alpha \varphi \varphi}(\varphi)) \cap \Lambda \subseteq \text{int}(C_{\pi \varphi \varphi}(\varphi)) \cap \Lambda.
\]
But by (3.2), \( \text{int}(C_{\pi \varphi \varphi}(\varphi)) \cap \Lambda = \emptyset \), which is a contradiction. \( \square \)

**Proposition 3.4.** Let \( \Lambda \) be a OTFSS set in \( \mathbb{R}^3 \) such that \( \dim_H \Lambda > 1 \) and not-contained in any plane. Then there exists an orthogonal projection \( \pi \in \Pi_{\mathbb{R}^3} \) and a self-similar measure \( \mu \) such that \( \text{spt}\mu \subseteq \Lambda \), \( \text{spt}\mu \) is not contained in any plane, \( \pi \mu \) satisfies the SSC and \( \dim_H \mu > \dim_H \pi \mu > 1 \).

**Proof.** Let \( \Lambda \) be a OTFSS set satisfying the assumptions. By Marstrand’s projection theorem [11] Corollary 9.4, Corollary 9.8 there exists a \( \pi \in \Pi_{\mathbb{R}^3} \) such that \( \dim_H \pi \Lambda = \min \{2, \dim_H \Lambda \} \). The set \( \pi \Lambda \) is a OTFSS set in \( \mathbb{R}^2 \). Applying Proposition 2.1, there exists a HOTFSS set \( \Lambda_1 \) and an IFS \( \Phi_1 = \{ f_1(\varphi) = \lambda_1 \varphi + L_1 \}, f \in \Sigma \) such that \( \Lambda_1 \subseteq \Lambda \), \( \pi \Lambda_1 \) satisfies the SSC and \( 2 > \dim_H \Lambda_1 = \dim_H \pi \Lambda_1 > 1 \).

Let \( \varepsilon_\pi \) be as defined in (3.1) and let \( i \neq j \) and \( \varphi_i \in f_i(\Lambda_1), \varphi_j \in f_j(\Lambda_1) \) such that \( (\varphi_i, \varphi_j) \in \Gamma_\pi \), arbitrary, but fixed. Denote the projection onto the subspace with normal vector \( \frac{x_i - x_j}{\|x_i - x_j\|} \) by \( \pi \). Then by Lemma 3.3, the projection \( \pi \) is 2 to 1 on \( \Lambda \). Thus, \( \dim_H \Lambda_1 = \dim_H \pi \Lambda_1 \), but the SSC does not hold (however, it satisfies the open set condition (OSC)).

Let \( \delta > 0 \) be sufficiently small that \( \dim_H \Lambda_1 - 3\delta > 1 \). Let us fix \( i, j \in \Sigma \) such that \( |i| = |j| \), \( \varepsilon_i \in f_i(\Lambda_1), \varepsilon_j \in f_j(\Lambda_1) \) and choose \( m := |i| = |j| \) sufficiently large that the attractor \( \Lambda \) of IFS \( \Phi := \{ f_{i_0} \circ \cdots \circ f_{i_{m-1}} \}_{i_0, \cdots, i_{m-1} = 1} \{ f_i, f_j \} \) satisfies \( \dim_H \Lambda \geq \dim_H \Lambda_1 - \delta \). Since \( \pi \) is still at most 2 to 1 on a smaller set \( \Lambda \) we have \( \dim_H \Lambda = \dim_H \pi \Lambda \).

Let \( \mu \) be the natural OTFSS measure on \( \Lambda \). By Theorem 2.12, the function \( \pi \mapsto \dim_H \pi \mu \) is lower semi-continuous at \( \pi \). Hence, \( \pi \mapsto \dim_H \pi \Lambda \) is lower semi-continuous at \( \pi \). Let \( \beta > 0 \) sufficiently small that for every projection \( \pi \in \Pi_{\mathbb{R}^3} \), with \( \|x_i - x_j\| < |\sin(\beta)| \)
\[
\dim_H \pi \Lambda \geq \dim_H \pi \Lambda - \delta = \dim_H \Lambda - \delta.
\]

Since the fixed points of the iterates of the functions are dense in \( \Lambda \), we may find \( h_1, h_2 \in \Sigma \) with \( |h_1| = |h_2| \) that
\[
\| \text{Fix}(f_{h_1}) - \text{Fix}(f_{h_2}) \times (\varepsilon_i - y_j) \| < |\sin(\beta)|.
\]

Denote the projection onto the subspace with normal vector \( \| \text{Fix}(f_{h_1}) - \text{Fix}(f_{h_2}) \| \times (\varepsilon_i - y_j) \times \pi' \) by \( \pi' \). Applying Proposition 2.1 again, there exists a HOTFSS set \( \tilde{\Lambda} \) and an IFS \( \tilde{\Phi}_1 \) with \( \text{spt}\mu \subseteq \tilde{\Lambda} \), \( \pi' \tilde{\Lambda} \) satisfies the SSC and \( \dim_H \tilde{\Lambda} = \dim_{H \pi'} \tilde{\Lambda} > \dim_H \pi' \tilde{\Lambda} - \delta \).

There exist a \( m, k \geq 1 \) that the IFS \( \big( \pi' \tilde{\Phi}_1 \big)^m \cup \bigg\{ \left. \pi' f_{h_1} \circ \cdots \circ \pi' f_{h_1} =: \pi' f_{h_1}^k \right\} \) satisfies the SSC and homogeneous. Since the system \( \pi' \tilde{\Phi}_1 \) satisfies SSC and homogeneous, then for every \( m \geq 1 \) \( \big( \pi' \tilde{\Phi}_1 \big)^m \) still
By definition of conditional measures

Proof. By changing the coordinates, without loss of generality we may assume that the projection in Proposition 3.4 satisfies SSC and homogeneous. By Proposition 2.1 the contraction ratio of \( \tilde{\Phi}_1 \) is \( \lambda_i \) for an \( l \geq 1 \). On the other hand, the contraction ratio of \( f_{\ell n_1} \) is \( \lambda^{\ell n_1 \cdot l} \). Now, let us fix the ratio \( k/m = l/|\ell n_1| \). Since the original system \( \pi \tilde{\Phi} \) satisfied OSC, by choosing \( k \) sufficiently large, the SSC holds.

Let \( \Phi := (\pi \tilde{\Phi})^m \cup \{ f_{\ell n_1}^k, f_{\ell n_2}^k \} \) and its attractor \( \Lambda' \). We claim that \( \pi' \) and the natural self-similar measure of \( \Phi' \) satisfy the prescribed properties in the statement of the proposition.

Observe, that by definition \( \pi'((\text{Fix}(f_{\ell n_1})) = \pi'(\text{Fix}(f_{\ell n_2}))) \), thus there functions \( \pi' f_{\ell n_1} \equiv \pi' f_{\ell n_1}^k \), i.e. there are exact overlaps. Hence, \( \pi' \Phi' = (\pi' \tilde{\Phi})^m \cup \{ f_{\ell n_1}^k \} \) and therefore, satisfies SSC.

Let \( \Lambda' \) be the attractor of \( \Phi' \). Then

\[
\dim_H \pi' \Lambda' \geq \dim_H \pi' \tilde{\Lambda} \geq \dim_H \pi' \tilde{\Lambda} - \delta \geq \dim_H \pi \tilde{\Lambda} - 2\delta = \dim_H \tilde{\Lambda} - 2\delta \geq \dim_H \Lambda - 3\delta > 1.
\]

Let \( \mu' \) be the HOTFSS measure on \( \Lambda' \) with weights \( \frac{1}{2(\hat{\Phi}_1)^{k+1}} \) for the functions in \( \tilde{\Phi}_1^k \). Thus, \( \pi' \mu' \) is the natural self-similar measure on \( \pi' \Lambda' \) and therefore, \( 1 < \dim_H \pi' \Lambda' \geq \dim_H \pi' \mu' \). Because of the exact overlap and the fact that \( \text{spt} \pi' \mu' = \pi' \Lambda' \) cannot be contained in a line, \( \text{spt} \pi' \mu' \) cannot be contained in a plane. The exact overlap and \( \dim_H \mu' \leq \dim_H \Lambda < 2 \) imply \( \dim_H \mu' > \dim_H \pi' \mu' \), which had to be proven.

By changing the coordinates, without loss of generality we may assume that the projection in Proposition 3.4 is a coordinate projection \( \pi : (x, y, z) \mapsto (x, y) \). Moreover, since the measure \( \mu \) in Proposition 3.4 cannot be contained in any plane, we may assume that \( \text{spt} \mu \) is contained in an octant, separated away from the \( z \) axis by restricting \( \mu \) to a cylinder set.

Let us denote the projection along geodesics on \( S^2 \) to \( S^1 \) by \( \gamma \). We note that \( \gamma \) is well defined except on the poles. On the other hand, \( \gamma \circ P_3 = P_2 \circ \pi \).

Let \( \nu := (P_3)_* \mu \). Thus, \( \gamma_* \nu = (P_2)_* \pi_* \mu \). By convenience, we use the cylindrical coordinates in \( \mathbb{R}^3 \) and the radial coordinates on \( S^2 \). That is, for \( \mathbb{R}^3 \ni \underline{x} = (r, \varphi, z), \pi(\underline{x}) = (r, \varphi), P_2(\pi(\underline{x})) = \varphi = \gamma(P_3(\underline{x})) \). Let us denote the conditional measures of \( \mu \) on \( \pi^{-1}(r, \varphi) \) by \( \mu_{\pi^{-1}(r, \varphi)} \), the conditional measures of \( \pi \mu \) on \( P_2^{-1}(r) \) by \( \pi \mu_{P_2^{-1}(r)} \) and the conditional measures of \( \nu \) on \( \gamma^{-1}(\varphi) \) by \( \nu_{\gamma^{-1}(\varphi)} \), see Figure 2.

**Lemma 3.5.** For \( \gamma_* \nu \)-almost every \( \varphi \in S^1 \), \( \dim_H \nu_{\gamma^{-1}(\varphi)} \geq \dim_H \mu - \dim_H \pi \mu > 0 \).

**Proof.** By definition of conditional measures \( \nu = \int \nu_{\gamma^{-1}(\varphi)} d\gamma \nu(\varphi) \). On the other hand, \( \pi \mu = \int \pi \mu_{P_2^{-1}(\varphi)} d\gamma \nu(\varphi) \) and thus, \( \mu = \int \mu_{\pi^{-1}(r, \varphi)} d\pi \mu(\varphi) = \int \mu_{\pi^{-1}(r, \varphi)} d\pi \mu_{P_2^{-1}(\varphi)}(r) d\gamma \nu(\varphi) \). Hence,

\[
\nu = (P_3)_* \mu = \int \int (P_3)_* \mu_{\pi^{-1}(r, \varphi)} d\pi \mu_{P_2^{-1}(\varphi)}(r) d\gamma \nu(\varphi).
\]

Since the conditional measures are uniquely defined up to a zero measure set

\[
\nu_{\gamma^{-1}(\varphi)} = \int (P_3)_* \mu_{\pi^{-1}(r, \varphi)} d\pi \mu_{P_2^{-1}(\varphi)}(r) \quad \text{for} \quad \gamma_* \nu \text{-almost every} \ \varphi.
\]

Let us observe that for any compact line segment \( I \subset \mathbb{R}^3 \) which is not contained in any 1 dimensional subspace of \( \mathbb{R}^3 \) the map \( P_3 : I \mapsto S^2 \) is bi-Lipsitz. Hence, by Theorem 2.12 and Proposition 3.4,

\[
\dim_H \mu_{\pi^{-1}(r, \varphi)} = \dim_H \mu_{\pi^{-1}(r, \varphi)} = \dim_H \mu - \dim_H \pi \mu > 0 \quad \text{for} \ \pi \mu \text{-a.e.} \ (r, \varphi).
\]
By using the definition of Hausdorff dimension, let $A_{\varphi,n}$ be the set such that $\nu_{\gamma^{-1}(\varphi)}(A_{\varphi,n}) > 0$ and $\dim_H \nu_{\gamma^{-1}(\varphi)} \geq \dim_H A_{\varphi,n} - \frac{1}{n}$. Thus, by (3.3) for $\gamma_* \nu$-a.e. $\varphi$ there exists a set $B_{\varphi,n}$ that $\pi\mu_{P_2^{-1}(\varphi)}(B_{\varphi,n}) > 0$ and for $\pi\mu_{P_3^{-1}(\varphi)}$-a.e. $r \in B_{\varphi,n}$

$$(P_3)_r \mu_{\pi^{-1}(r,\varphi)}(A_{\varphi,n}) > 0.$$ 

Hence,

$$\dim_H \nu_{\gamma^{-1}(\varphi)} + \frac{1}{n} \geq \dim_H A_{\varphi,n} \geq \dim_H (P_3)_r \mu_{\pi^{-1}(r,\varphi)} = \dim_H \mu - \dim_H \pi\mu > 0$$ for $\gamma_* \nu$-a.e. $\varphi$.

Since $n$ was arbitrary, the proof is complete.

\textbf{Proof of Theorem 3.1.} Let $\mu$ and $\pi$ as in Proposition 3.4. Since $\pi\mu$ is a HOTFSS measure satisfying SSC, we can apply Corollary 2.6 and therefore,

$$\dim_H \gamma_* \nu = \dim_H (P_2)_* \pi\mu = \min \{1, \dim_H \pi\mu\} = 1.$$ 

By Lemma 3.5

$$\dim_H (P_3)_* \mu_{\gamma^{-1}(\varphi)} = \dim_H \nu_{\gamma^{-1}(\varphi)} \geq \dim_H \mu - \dim_H \pi\mu > 0.$$ 

Thus, by [8, Lemma 6.13]

$$\dim_H (P_3)_* \mu = \dim_H \nu \geq \dim_H \gamma_* \nu + \dim_H \nu_{\gamma^{-1}(\varphi)} > 1.$$ 


4. Proof of main theorems

In this section we show the remaining proofs.

Proof of Theorem 1.1. Let $\Lambda$ be an OTFSS set in $\mathbb{R}^3$ that it is not contained in any plane and $\dim_H \Lambda > 1$. Moreover, let $g : \mathbb{R}^3 \to \mathbb{R}$ be satisfying the assumptions. Since $\Lambda$ is compact, there exists an open neighbourhood of $\Lambda$ that $\| \nabla_2 g \| > 0$ on the neighbourhood. By considering a sufficiently small cylinder of $\Lambda$ we may assume that there exists a ball $B$ that $\Lambda \subseteq B$ and $\| \nabla_2 g \| > 0$ for every $x \in B$. Let $f : x \in B \mapsto \nabla_2 g$. By assumptions (2) $f(x) = f(\lambda x)$ for every $x \in B$. Thus, by assumption (3), for any $\mu$ OTFSS measure with $\text{spt} \mu \subseteq \Lambda$

$$\dim_H f_* \mu = \dim_H (P_3)_* \mu.$$ 

It is enough to show the lower bound. Let $\mu$ be the HOTFSS measure as in Theorem 3.1. Then by Theorem 2.3

$$\dim_H g(\Lambda) \geq \dim_H g_* \mu \geq \mu - \text{essinf}_x \dim_H \pi_{g, x} \mu,$$

where we recall that $\pi_{g, x}(y) = \frac{\langle \nabla_2 g, y \rangle}{\| \nabla_2 g \|}$. By Proposition 2.10

$$\dim_H \{ n \in S^2 : \dim_H \pi_n \mu < 1 \} \leq 1.$$ 

But by Theorem 3.1 $\dim_H f_* \mu = \dim_H (P_3)_* \mu > 1$, thus,

$$f_* \mu(\{ n \in S^2 : \dim_H \pi_n \mu < 1 \}) = 0.$$ 

And therefore $\mu - \text{essinf}_x \dim_H \pi_{g, x} \mu = 1$. 

Proof of Theorem 1.2. If $\Lambda$ is contained in a plane then we refer to Corollary 2.8 or [12, Theorem 1.2]. So we may assume that $\Lambda$ is not contained in any plane.

By shifting $\Lambda$ we may assume that $0 \in \Lambda$. Let $\Lambda_T$ be a cylinder set such that $\text{dist}(0, \Lambda_T) > 0$. Then $g(x) := \| x \|$ satisfies the conditions of Theorem 1.1 with self-similar set $\Lambda_T$. Thus,

$$\dim_H D(\Lambda) \geq \dim_H g(\Lambda) \geq \dim_H g(\Lambda_T) = \min \{ 1, \dim_H \Lambda_T \} = \min \{ 1, \dim_H \Lambda \} = 1.$$ 

Proof of Theorem 1.3. Let $\Lambda$ be an arbitrary self-similar set on $\mathbb{R}$ that $\dim_H \Lambda > 1/3$ with IFS $\{ \lambda_i x + t_i \}_{i=1}^q$, where $\lambda_i \in (-1, 1)$. By applying [13] Proposition 6 there exists a self-similar set $\Lambda' \subseteq \Lambda$ in $\mathbb{R}$ that $\dim_H \Lambda' > 1/3$ with IFS $\{ \lambda x + t' \}_{i=1}^q$, where $\lambda \in (0, 1)$. Since $\Lambda'$ is not a singleton, there exists a cylinder set $\Lambda'_a$ that $0 \notin \Lambda'_a$.

It is easy to see that $\Lambda'_a \times \Lambda'_a \times \Lambda'_a$ is an OTFSS set in $\mathbb{R}^3$ separated away from planes determined by the axes. Thus it is contained in one of the octants. Moreover, $\dim_H \Lambda'_a \times \Lambda'_a \times \Lambda'_a > 1$.

Let $g(x, y, z) = xyz$. Then

$$\nabla_2 g = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}.$$
It is easy to see that $\nabla_\mathbb{R} g$ satisfies the assumptions (1) and (2) of Theorem 1.1 on $\Lambda'_i \times \Lambda'_i \times \Lambda'_i$. On the other hand, since $x \mapsto \nabla_\mathbb{R} g$ is invertible on every open octant and $\det(H_x g) = 2xyz \neq 0$ for any $(x, y, z) \in \Lambda'_i \times \Lambda'_i \times \Lambda'_i$ compact set contained in an octant, where $H_x g$ denotes the Hesse matrix of $g$ we get that assumption (3) is satisfied. Thus, by Theorem 1.1

$$\dim_H \Lambda \cdot \Lambda \cdot \Lambda \geq \dim_H \Lambda'_i \cdot \Lambda'_i \cdot \Lambda'_i = \dim_H g(\Lambda'_i \times \Lambda'_i \times \Lambda'_i) = 1.$$  

Remark 1. Unfortunately, our method does not allow us to prove similar statements if $\dim_H \Lambda \leq 1$. The method depends on dimension of the exceptional directions of orthogonal projections from $\mathbb{R}^3$ to $\mathbb{R}$. By using Hochman’s result Theorem 2.3

$$\dim_H g_* \mu \geq \mu - \text{essinf}_x \dim_H \pi_{g_1} \mu.$$  

On the other hand, in the case self-similar sets

$$\dim_H \{ \pi \in \Pi_{3, 1} : \dim_H \pi \Lambda < \min \{1, \dim_H \Lambda \} \} \leq 1,$$

see Theorem 2.10. Hence, to prove that the dimension does not drop, it is enough to show that $\dim_H f_* \mu > 1$, where $f : \mathbb{R} \mapsto \mathbb{R}^3(\nabla_\mathbb{R} g)$. However, it is not possible if $\dim_H \Lambda \leq 1$ and in particular if $\dim_H \mu \leq 1$.

Remark 2. Conditions (2) and (3) in Theorem 1.1 implies that we have to check only that $\dim_H (P_3)_* \mu > 1$. This conditions seems rather technical, and we conjecture that can be replaced by some more natural condition.

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