On mixtures of copulas and mixing coefficients

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Abstract

We show that if the density of the absolute continuous part of a copula is bounded away from zero on a set of Lebesgue measure 1, then it generates “lower $\psi$”-mixing stationary Markov chains. This conclusion implies $\phi$-mixing, $\rho$-mixing, $\beta$-mixing and “interlaced $\rho$-mixing”. We also provide some new results on the mixing structure of Markov chains generated by mixtures of copulas.

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1 Introduction

The importance of mixing coefficients in the theory of central limit theorems for weakly dependent random sequences is an established fact. In the recent years, many researchers have been trying to provide sufficient conditions for mixing at various rates. These efforts have invited many people to investigate the properties of copulas, because they capture the dependence structure of stationary Markov chains.

We are providing here an improvement of one of our previous results. Namely, we have shown in Longla and Peligrad (2012), that Markov chains generated by a copula are $\phi$-mixing, when the density of its absolute continuous part is bounded away from 0. This paper provides the proof, that under this condition, the Markov chains are lower $\psi$-mixing. Taking into account Theorem 1.2 and 1.3 of Bradley (1997), this conclusion is equivalent to “interlaced $\rho$-mixing”.

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, and $(X_n, n \in \mathbb{N})$ is a stationary Markov chain generated by a copula $C(x, y)$ on this space. Recall that a copula is a joint cumulative distribution on $[0, 1]^2$ with uniform marginals, and a copula-based Markov chain is nothing but a stationary Markov chain represented by the joint distribution of its consecutive states. The Sklar’s theorem ensures uniqueness of this representation for continuous random variables. Let $c(x, y)$ be the density of the copula $C(x, y)$ and $c_n(x, y)$ the density of the absolute continuous part of the $n^{th}$ fold product of $C(x, y)$. For definitions, check Longla and Peligrad (2012) or Longla (2013). Using the Lebesgue measure as invariant distribution after rescaling the effect of the marginal distribution, taking into account the notations from Longla and Peligrad (2012), the mixing coefficients of interest for an absolutely continuous copula can be defined as follows:

Definition 1 Let $\mathcal{R}$ denote the Borel $\sigma$-algebra on $[0, 1]$ and $\sigma(X_k, k \in S)$ - the $\sigma$-algebra generated by the random variables indexed by $S$. Let $\lambda$ be the Lebesgue measure on $[0, 1]$, and $B^c$ be the
Theorem 2 Let \((X_k, k \in \mathbb{N})\) be a stationary Markov chain generated by a copula \(C(x, y)\). If the density of the absolute continuous part of the copula \(c(x, y)\) is bounded away from zero on a set of Lebesgue measure 1, then the Markov chain is “lower \(\psi\)-mixing.” Therefore, it is \(\phi\)-mixing, \(\rho\)-mixing, exponential \(\beta\)-mixing and “interlaced \(\rho^*\)-mixing.”

Theorem 3 (Mixtures) Let \(A = (X_k, k \in \mathbb{N})\) be a Stationary Markov chain generated by a convex combination of copulas \(C_k(x, y), 1 \leq k \leq n\). The following statements hold:

1. If one of the copulas in the combination generates \(\rho\)-mixing stationary Markov chains, then the Markov chain \(A\) is \(\rho\)-mixing.
2. If one of the copulas in the combination generates stationary Markov chains with \(\rho^*_1 < 1\), then the Markov chain \(A\) is \(\rho^*\)-mixing.
3. If one of the copulas in the combination generates \(\psi'\)-mixing stationary Markov chains, then the Markov chain \(A\) is \(\psi'\)-mixing.
4. If one of the copulas in the combination generates ergodic stationary Markov chains with \(\psi_1 < 1\), then the Markov chain \(A\) is \(\psi'\)-mixing.
5. If one of the copulas in the combination generates ergodic stationary Markov chains with \(\phi_1 < 1\), then the Markov chain \(A\) is \(\phi\)-mixing.
2 Background and proof

2.1 Theorem 2
This condition was first investigated by Beare (2010), who showed that it implies \( \rho \)-mixing. Then, Longla and Peligrad (2012) have shown that the condition implies \( \phi \)-mixing. During a discussion at the conference on Stochastic Processes and Application (SPA 2013), Bradley suggested that this condition could imply \( \psi^2 \)-mixing. The following well known result is provided in Remark 1.1 of Bradley (1997).

Remark 4 A stationary Markov chain is exponentially \( \psi' \)-mixing, if for some integer \( m \), \( \psi'_m > 0 \).

He also showed, based on this result, that if for some \( m \), \( \psi'_m > 0 \), then the Markov chain is \( \rho^* \)-mixing with exponential decay rate. We will use this result in the context of copula-based Markov chains to establish the proof of Theorem 2.

We shall first show that if the density of the absolute continuous part is bounded away from zero on a set of Lebesgue measure 1 \( (c(x,y) \geq c \ a.s.) \), then

\[
P(A \cap B) \geq c\lambda(A)\lambda(B), \quad \text{for all } A, B \in \mathcal{R}.
\]

(1)

For any copula \( C \), there exists a unique representation \( C(x,y) = AC(x,y) + SC(x,y) \), where \( AC(x,y) \) is the absolute continuous part of \( C(x,y) \). \( AC \) induces on \([0,1]^2\) a measure \( P_c \) defined on borel sets by \( P_c(A \times B) = \int_A \int_B c(x,y)dydx \). \( SC(x,y) \) is the singular part of the copula. It induces a singular measure on \([0,1]^2\). If we keep the notation \( SC \) for this singular measure, then

\[
P(A \cap B) = P_c(A \times B) + SC(A \times B).
\]

Thus, \( P(A \cap B) \geq P_c(A \times B) = \int_A \int_B c(x,y)dydx \). Taking into account the fact that \( c(x,y) \geq c \ a.s. \), we obtain \( P(A \cap B) \geq c\lambda(A)\lambda(B) \) for all \( A, B \in \mathcal{R} \). Therefore, formula (1) holds. This inequality leads to \( \psi'_1 \geq c > 0 \). Therefore, by Remark 4 the Markov chains generated by \( C(x,y) \) are \( \psi' \)-mixing and \( \rho^* \)-mixing. It is also well known that \( \psi' \)-mixing implies \( \phi \)-mixing, which implies \( \rho \)-mixing and exponential \( \beta \)-mixing, confirming the results of previous studies.

2.2 Theorem 3
We present here a review of mixing for mixtures of copulas. Mixtures of distributions are very popular in modeling, it is important to answer the question on mixing coefficients for various models. They allow estimation of functions of the random variables or inference on model parameters. We have initially shown in Longla and Peligrad (2012) a result on mixtures of copulas for absolute regularity. A second result on \( \rho \)-mixing for mixtures of copulas was provided by Longla (2013). Here, we extend the results to other mixing coefficients, that are not less important. Let \( C(x,y) = \sum_{k=1}^n a_k C_k(x,y), \quad a_k \geq 0, \quad a_1 \neq 0, \quad \sum_{k=1}^n a_k = 1 \) and assume in each of the cases, that \( C_1(x,y) \) is the copula that generates stationary Markov chains with the given mixing property.

1. The proof of this point of Theorem 3 is similar to the one provided by Longla (2012) in Theorem 5. It is solely based on the fact that we need only to show that \( \rho_1 < 1 \). The fact that the coefficients of the convex combination add up to 1 and \( \rho_n \leq 1 \) helps in the conclusion.

2. The definition of \( \rho^*_n \) being based on that of \( \rho_n \), the proof of this fact is just a repeat of the similar conclusion for \( \rho \)-mixing, using Theorem 7.5 of Bradley (2007).
3. This point of the theorem follows from Remark 4 and the fact that the given convex combination generates stationary Markov chains for which $\psi' > 0$. Notice that, if we denote $\psi'_*\psi$ the first $\psi$-mixing coefficient of the Markov chain generated by $C_1(x,y)$ and $C(x,y) = \sum_{k=1}^n a_k C_k(x,y)$, then $\psi'_1 \geq \inf_{A,B \in \mathbb{R}, \lambda(A), \lambda(B) > 0} \frac{\int_A \int_B c_1(x,y) dxdy}{\lambda(A) \lambda(B)} \geq a_1 \inf_{A,B \in \mathbb{R}, \lambda(A), \lambda(B) > 0} \frac{\int_A \int_B c(x,y) dxdy}{\lambda(A) \lambda(B)} = a_1 \psi'_*$. Thus, $\psi'_1 \geq a_1 \psi'_* > 0$.

4. By the comment before Corollary 2 of Bradley (1999), $\psi_1 < 1$ implies exponential $\psi$-mixing for Stationary Markov chains generated by $C_1(x,y)$. $\psi$-mixing implies $\psi'$-mixing, and by the above proof, any convex combination of copulas containing $C_1(x,y)$ generates $\psi'$-mixing.

5. $C_1(x,y)$ generates stationary $\phi$-mixing Markov chains. $\phi$-mixing implies $\rho$-mixing. Thus, Markov chains generated by any mixture of copulas that contain $C_1(x,y)$ are geometrically $\rho$-mixing. Ergodicity follows. By Theorem 22.3 of Bradley (2007), these Markov chains are $\phi$-mixing, if $\phi_1 < 1$. If we denote $\phi_k$ the first $\phi$-mixing coefficient of stationary Markov chains generated by $C_k(x,y)$, then we also have $\phi_k < 1$ and $\phi_1 \leq \sum_{k=1}^n a_k \phi_k < 1$. Therefore, Markov chains generated by $C(x,y)$ are $\phi$-mixing, by Theorem 22.3 of Bradley (2007).

3 Examples

Example 5 Any convex combination of copulas that contains the independence copula has the density of its absolute continuous part bounded away from 0, and Theorem 2 applies to it.

1. An example of copula family of the above kind is the Frechet family of copulas, defined as follows: $C_{a,b}(x,y) = aW(x,y) + bM(x,y) + (1 - a - b)P(x,y)$, where $P(x,y) = xy$ - is the independence copula, $W(x,y) = \max(x + y - 1, 0)$ - is the Hoeffding lower bound and $M(x,y) = \min(x,y)$ - is the Hoeffding upper bound. $P$ is the copula of independent random variable, $W$ is a singular probability measure with support $A = \{(x,y) \in [0,1]^2 : x + y = 1\}$ and the graph of $M$ is above the graph of any copula. Copulas from this family satisfy Theorem 2 for $a + b \neq 1$.

2. The Mardia family of copulas, as a class of copulas from the Frechet family of copulas, satisfies Theorem 2. Copulas from the Mardia family are defined by $C_\theta(x,y) = \frac{\theta^2(1+\theta)}{2} M(x,y) + (1 - \theta^2)P(x,y) + \frac{\theta^2(1-\theta)}{2}$. For $\theta^2 \neq 1$, the copula generates lower $\psi$-mixing.

In fact, for any convex combination of copulas containing the independence copula, the density of the absolute continuous part is greater than the coefficient of the independence copula in the combination. These examples where shown to generate $\phi$-mixing by Longla (2013b).

Example 6 The Marshall-Olkin family of copulas defined by $C_{a,b}(x,y) = \min(xy^{1-a}, yx^{1-b})$ with $0 \leq a, b \leq 1$ satisfies Theorem 2 for values of $a, b$ such that $a \neq 1$ and $b \neq 1$.

In fact, for copulas from this family, we have $c_{a,b}(x,y) \geq \min(1 - a, 1 - b)$, except for $y^a = x^b$, which is a set a Lebesgue measure 0. Longla and Peligrad (2012) have shown that this family of copulas generates $\phi$-mixing, which is a stronger condition than what we have shown here.
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