DEFECT OF AN EXTENSION, KEY POLYNOMIALS AND
LOCAL UNIFORMIZATION

by

Jean-Christophe SAN SATURNINO

Abstract. — For all simple and finite extension of a valued field, we prove that its
defect is the product of the effective degrees of the complete set of key polynomials
associated. As a consequence, we obtain a local uniformization theorem for valuations
of rank 1 centered on an equicharacteristic quasi-excellent local domain satisfying
some inductive assumptions of lack of defect.

0. Introduction

In order to obtain a local uniformization theorem, F.-V. Kuhlmann systematized
the study of the defect of an extension of valued fields. We know that this defect is
a power of \( p \), the characteristic of the residual field of the valuation. Another way to
approaches the local uniformization problem for a valuation of rank 1 is studying the
set of key polynomials associated to the valuation. The key polynomials have been
developed in [10] and [3], a link of the two approach is given in [7]. When the set
of key polynomials do not have limit key polynomials, we proved in [9] that we can
obtain a local uniformization theorem for valuation of rank 1. In particular, we also
proved that, if the residual field of the valuation is of characteristic zero, we have
local uniformization of the valuation, result known since Zariski.

In [11], M. Vaquié found a link between the defect and the key polynomials:
the jump of the valuation. In the language of [3], this is the degree of a limit key
polynomial over the degree of a previous key polynomial whose degree is the same
of an infinite family of key polynomials being between this polynomial and the limit
key polynomial.

In this paper, we want to study more precisely the link between the defect and
the key polynomials. We will express the defect as a product of effective degrees and
generalize the result of M. Vaquié. The effective degree is closely related with the
Newton polygon associated to a key polynomial. We will use this result to prove a local uniformization theorem for valuations of rank 1 centered on an equicharacteristic quasi-excellent local domain satisfying some inductive assumptions of defectless of the quotient field.

In the first section, we recall some definitions about the center of a valuation and the graded algebras associated to a valuation.

In the second section we define the defect of a valuation, roughly speaking, for a valuation $\nu$ of a field $K$ having a unique extension $\mu$ of a field $L$, the defect is the number who gives the equality in this inequality:

$$[L : K] \leq [\Gamma_\mu : \Gamma_\nu][k_\mu : k_\nu],$$

where $\Gamma_\nu$ (resp. $\Gamma_\mu$) is the value group of $\nu$ (resp. $\mu$) and $k_\nu$ (resp. $k_\mu$) is the residual field of $\nu$ (resp. $\mu$). We know that is a power of $p$, the characteristic of $k_\nu$.

In the third section we first recall the definitions of key polynomials given in [3] or [9]. With this definitions, we give some characterizations of a complete set of key polynomials, this is very useful to prove if a set of polynomials is a complete or a 1-complete set of key polynomials. For a key polynomial $Q$, we also define the effective degree of a polynomial $P$ corresponding to this key polynomial. When we write the standard expansion of $P$ in terms of $Q$, the greatest power of $Q$ where the valuation of the monomials in $Q$ is maximal is the effective degree of $P$. We prove after that some graded algebras associated to a valuation are euclidean for the effective degree. In the end of this section, we recall that the sequence of effective degrees is decreasing and so, there exist a constant value. If this value is 0 or 1, the set of key polynomial considered is 1-complete.

In the fourth section, we first prove that the defect of an extension of a valuation is the product of the effective degree of the limit key polynomials provided by the valuation. The proof uses the notion of key polynomials of W. Mahboub (see [8]), so it is true for valuation of any rank. The principle idea is that the Newton polygon has only one face if the field is henselian. We compute after some defects with four examples of M. Vaquié, W. Mahboub and S.D. Cutkosky.

In the next section we prove that there is no limit key polynomials for valuation of rank one over a defectless field.

In the final section we generalize the results of [9] and prove a local uniformization theorem for a valuation of rank 1 with some inductive assumptions of local uniformization in lower dimension and no defect for a well defined extension. With this we can prove again the local uniformization for a valuation of rank 1 in characteristic zero.

I want to thank M. Vaquié who suggested me to study the precise link between the defect and the limit key polynomial and B. Teissier for his advice on clarification and structuration of the notion of key polynomials. I also thank W. Mahboub for his ability to produce examples of key polynomial which verify or not my ideas. Finally, I want to thank M. Spivakovsky for our discussions and his careful reading.
Notation. Let \( \nu \) be a valuation of a field \( K \). We write \( R_\nu = \{ f \in K \mid \nu(f) \geq 0 \} \), this is a local ring whose maximal ideal is \( m_\nu = \{ f \in K \mid \nu(f) > 0 \} \). We then denote by \( k_\nu = R_\nu/m_\nu \) the residue field of \( R_\nu \) and \( \Gamma_\nu = \nu(K^*) \).

For a field \( K \), we will denote by \( K \) an algebraic closure of \( K \), by \( K_{sep} \) a separable closure of \( K \) and by \( \text{Aut}(K/K) \) the group of automorphisms of \( K/K \).

If \( R \) is a ring and \( I \) an ideal of \( R \), we will denote by \( \hat{R}^I \) the \( I \)-adic completion of \( R \).

When \( (R, m) \) is a local ring, we will say the completion of \( R \) instead of the \( m \)-adic completion of \( R \) and we will denote it by \( \hat{R} \).

Finally, if \( R \) is a domain, we denote by \( \text{Frac}(R) \) its quotient field.

1. Center of a valuation, graded algebra associated and saturation

In this section, \( \Gamma \) will be a totally ordered commutative group.

**Definition 1.1.** — Let \( R \) be a ring and \( P \) a prime ideal. A valuation \( \nu : R \to \Gamma \cup \{ \infty \} \) centered on \( P \) is the data of a minimal prime ideal \( P_\infty \) of \( R \) included in \( P \) and a valuation of the quotient field of \( R/P_\infty \) centered on \( P/P_\infty \). The ideal \( P_\infty \) is the support of the valuation, that is \( P_\infty = \nu^{-1}(\infty) \).

If \( R \) is a local ring with maximal ideal \( m \), we will say that \( \nu \) is centered on \( R \) to say that \( \nu \) is centered on \( m \).

**Definition 1.2.** — Let \( R \) be a ring and \( \nu : R \to \Gamma \cup \{ \infty \} \) a valuation centered on a prime ideal of \( R \). For all \( \alpha \in \nu(R \setminus \{ 0 \}) \), we define the ideals:

\[
P_\alpha = \{ f \in R \mid \nu(f) \geq \alpha \};
\]

\[
P_\alpha^+ = \{ f \in R \mid \nu(f) > \alpha \}.
\]

Then we define the graded algebra of \( R \) associated to \( \nu \) by:

\[
gr_\nu(R) = \bigoplus_{\alpha \in \nu(R \setminus \{ 0 \})} P_\alpha/P_\alpha^+.
\]

The algebra \( gr_\nu(R) \) is a domain. For \( f \in R \setminus \{ 0 \} \), we write \( \text{in}_\nu(f) \) its image in \( gr_\nu(R) \).

If \( R \) is a local domain, we define another graded algebra by:
**Definition 1.3.** — Let $R$ be a local domain, $K = \text{Frac}(R)$ and $\nu : K^* \to \Gamma \cup \{\infty\}$ a valuation of $K$ centered on $R$. For all $\alpha \in \Gamma$, we define the $R_\nu$-submodules of $K$ following:

\[
P_\alpha = \{ f \in K \mid \nu(f) \geq \alpha \} \cup \{0\};
\]

\[
P_{\alpha,+} = \{ f \in K \mid \nu(f) > \alpha \} \cup \{0\}.
\]

We define then the graded algebra associated at $\nu$ by:

\[
G_\nu = \bigoplus_{\alpha \in \Gamma} P_\alpha/P_{\alpha,+}.
\]

For $f \in K^*$, we write $\text{in}_\nu(f)$ his image in $G_\nu$.

**Remark 1.4.** — We have the natural injective function:

\[
gr_\nu(R) \hookrightarrow G_\nu.
\]

**Definition 1.5.** — Let $G$ be a graded algebra without zero divisors. We called saturation of $G$ the graded algebra $G^*$ define by:

\[
G^* = \left\{ \frac{f}{g} \mid f, g \in G, g \text{ homogène, } g \neq 0 \right\}.
\]

We say that $G$ is saturated if $G = G^*$.

**Remark 1.6.** — For all graded algebra $G$, we have:

\[
G^* = (G^*)^*.
\]

In particular, $G^*$ is always saturated.

**Example 1.7.** — Let $\nu$ be a valuation centered on a local ring $R$. Then:

\[
G_\nu = (\text{gr}_\nu(R))^*.
\]

In particular, $G_\nu$ is saturated.

2. Defectless fields

**Definition 2.1.** — ([11], Définition 1.2) A valued field $(K,\nu)$ is said to be henselian if for all algebraic extension $L|K$, there exits a unique valuation $\mu$ of $L$ which extends $\nu$.

We call henselization of $(K,\nu)$ all extension $(K^h,\mu)$ of $(K,\nu)$ such that $(K^h,\mu)$ is henselian and for all henselian valued field $(L,\nu')$ and all immersion $\sigma : (K,\nu) \hookrightarrow (L,\nu')$, there exists a unique immersion $\sigma' : (K^h,\mu) \hookrightarrow (L,\nu')$ which extends $\sigma$.

\[
\begin{array}{c}
(K,\nu) \\
\sigma \downarrow \\
(K^h,\mu) \leftrightsquigarrow (L,\nu')
\end{array}
\]
Remark 2.2. — All the henselizations of a given valued field are isomorphic, it is its smaller henselian extensions. Moreover, the henselization $(K^h, \mu)$ of $(K, \nu)$ is an immediate extension, that is $\Gamma_\nu = \Gamma_\mu$ and $k_\nu = k_\mu$ (for a proof, we can see Theorem 7.42 of [6]).

The henselization of a given valued field $(K, \nu)$ can be constructed explicitly, we give here the construction proposed by Kuhlmann in [6], Chapter 7. Consider $\mu$ to be an extension of $\nu$ on $\overline{K}$. Write:

$$G^d(\overline{K}/K, \mu) = \{ \sigma \in \text{Aut}(\overline{K}/K) | \forall \alpha \in \overline{K}, \mu(\sigma(\alpha)) = \mu(\alpha) \},$$

$$K^{h(\mu)} = \{ \alpha \in K^{sep} | \forall \sigma \in G^d(\overline{K}/K, \mu), \sigma(\alpha) = \alpha \}.$$

$(K^{h(\mu)}, \mu)$ is an henselization of $(K, \nu)$, it is an algebraic separable immediate extension of $(K, \nu)$.

Definition 2.3. — Let $K$ be a field with a valuation $\nu$. Consider $L$ a finite extension of $K$ and write $\mu_1, \ldots, \mu_g$ the valuations of $L$ which extend $\nu$. For $1 \leq i \leq g$, choose a valuation $\pi_i$ of $K$ whose restriction on $L$ is $\mu_i$. We write $K^{h(\pi_i)}$ the field constructed previously, it is an henselization of $(K, \nu)$. Moreover, it is a subfield of $L^{h(\pi_i)} := L.K^{h(\pi_i)}$ which is itself an henselization of $(L, \mu_i)$. We define the defect of the extension $L|K$ in $\mu_i$ by:

$$d_{L|K}(\mu_i, \nu) = \frac{[L^{h(\pi_i)} : K^{h(\pi_i)}]}{e_i f_i},$$

where $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ and $f_i = [k_{\mu_i} : k_\nu]$.

Remark 2.4. — By a theorem of Ostrowski (Lemma 11.17 of [6]), we can show that $d_{L|K}(\mu_i, \nu) = p^{a_i}$ where $a_i \in \mathbb{N}$ and $p = \text{char}(k_\nu)$.

Proposition 2.5. — ([6], Lemma 7.46) Let $K$ be a field with a valuation $\nu$. Consider $L$ a finite extension of $K$ and write $\mu_1, \ldots, \mu_g$ the valuations of $L$ which extend $\nu$. Then:

$$[L : K] = \sum_{i=1}^{g} d_i e_i f_i,$$

where $d_i = d_{L|K}(\mu_i, \nu)$, $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ and $f_i = [k_{\mu_i} : k_\nu]$.

Definition 2.6. — Let $K$ be a field with a valuation $\nu$. Consider $L$ a finite extension of $K$ and write $\mu_1, \ldots, \mu_g$ the valuation of $L$ which extend $\nu$. We call global defect of the extension $L|K$ on $\nu$ the quotient:

$$d_{L|K}(\nu) = \frac{[L : K]}{\sum_{i=1}^{g} e_i f_i},$$

where $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ et $f_i = [k_{\mu_i} : k_\nu]$.

We say that $L$ is a defectless extension of $K$ if:

$$d_{L|K}(\nu) = 1.$$
Definition 3.1. — If the extension $L|K$ is normal, then the global defect is a power of $p$, where $p = \text{char}(k_{\nu})$, because $d_{L|K}(\nu) = d_{L|K}(\mu_i, \nu)$, for all $i \in \{1, \ldots, g\}$ (see [6], Lemma 11.3); otherwise it is a rational number.

Remark 2.8. — $d_{L|K}(\nu) = 1 \iff \forall i \in \{1, \ldots, g\}$, $d_{L|K}(\mu_i, \nu) = 1$.

Definition 2.9. — A field $K$ is said to be defectless if all finite extension of $K$ is defectless.

Proposition 2.10. — ([6], Theorem 11.23) All field with a valuation $\nu$ such that $\text{char}(k_{\nu}) = 0$ is a defectless field.

3. Key polynomials and effective degree

Let $K \hookrightarrow K(x)$ be a simple transcendental extension of fields. Let $\mu'$ be a valuation of $K(x)$, write $\mu := \mu'_K$. Write $G$ the group of values of $\mu'$ and $G_1$ the smallest non-zero isolated subgroup of $G$. Suppose that the rank of $\mu$ is 1 and $\mu'(x) > 0$. Finally, for $\beta \in G$, recall that:

$P'_\beta = \{f \in K(x) | \mu'(f) \geq \beta \} \cup \{0\}$;

$P'_{\beta,+} = \{f \in K(x) | \mu'(f) > \beta \} \cup \{0\}$;

$G_{\mu'} = \bigoplus_{\beta \in G} P'/P'_{\beta,+}$;

and $\text{in}_{\mu'}(f)$ is the image of $f \in K(x)$ in $G_{\mu'}$.

Definition 3.1. — A complete set of key polynomials for $\mu'$ is a well ordered collection:

$Q = \{Q_i\}_{i \in \Lambda} \subset K[x]$

such that, for all $\beta \in G$, the additive group $P'_\beta \cap K[x]$ is generated by the products of the form $a \prod_{j=1}^{s} Q_{i_j}^{\gamma_j}$, $a \in K$, such that $\sum_{j=1}^{s} \gamma_j \mu'(Q_{i_j}) + \mu(a) \geq \beta$.

The set is said to be 1-complete if the precedent condition occurs for all $\beta \in G_1$ and if for all $i \in \Lambda$, $\mu'(Q_i) \in G_1$.

Theorem 3.2. — ([3], Theorem 7.11) There exists a collection $Q = \{Q_i\}_{i \in \Lambda}$ which is a 1-complete set of key polynomials.

By the Theorem 3.2, we know that there exists a 1-complete set of key polynomials $Q = \{Q_i\}_{i \in \Lambda}$ and that the order type of $\Lambda$ is at most $\omega \times \omega$. If $K$ is defectless, we will see that the order type of $\Lambda$ is at most $\omega$ at so, there is no limit key polynomials, this is particularly the case if $\text{char}(k_{\mu}) = 0$. For all $i \in \Lambda$, write $\beta_i = \mu'(Q_i)$.

Let $l \in \Lambda$, write:

$\alpha_i = d_{Q_{i-1}}^\circ(Q_i)$, $\forall i \leq l$;

$\alpha_{i+1} = \{\alpha_i\}_{i \leq l}$;

$Q_{l+1} = \{Q_i\}_{i \leq l}$.
We also use the notation \( \gamma_{l+1} = \{ \gamma_i \}_{i \leq l} \) where the \( \gamma_i \) are all zero except for a finite number, \( Q_{l+1}^\gamma = \prod_{i \leq l} Q_i^{\gamma_i} \).

**Definition 3.3.** — A multi-index \( \gamma_{l+1} \) is said to be standard with respect to \( \alpha_{l+1} \) if \( 0 \leq \gamma_i < \alpha_{i+1} \), for \( i \leq l \).

An \( l \)-standard monomial in \( Q_{l+1} \) is a product of the form \( \alpha_{l+1} Q_{l+1}^{\gamma_{l+1}} \), where \( \alpha_{l+1} \in K \) and \( \gamma_{l+1} \) is standard with respect to \( \alpha_{l+1} \).

An \( l \)-standard expansion not involving \( Q_i \) is a finite sum \( \sum_{\beta} S_{\beta} \) of \( l \)-standard monomials not involving \( Q_i \), where \( \beta \) ranges over a certain finite subset of \( G_+ \) and \( S_{\beta} = \sum_j d_{\beta,j} \) is a sum of standard monomials of value \( \beta \) such that \( \sum j \mu'_i(d_{\beta,j}) \neq 0 \).

**Definition 3.4.** — Let \( f \in K[x] \) and \( i \leq l \), an \( i \)-standard expansion of \( f \) is an expression of the form:

\[
f = \sum_{j=0}^{s_i} c_{j,i} Q_i^j,
\]

where \( c_{j,i} \) is an \( i \)-standard expansion not involving \( Q_i \).

**Remark 3.5.** — Such expansion exists by Euclidean division and is unique in the sense that the \( c_{j,i} \in K[x] \) are unique. More precisely, if \( i \in \mathbb{N} \), we can prove by induction that the \( i \)-standard expansion is unique.

**Definition 3.6.** — Let \( f \in K[x], i \leq l \) and \( f = \sum_{j=0}^{s_i} c_{j,i} Q_i^j \) an \( i \)-standard expansion of \( f \). We define the \( i \)-truncation of \( \mu' \), denoted by \( \mu'_i \), as being the pseudo-valuation:

\[
\mu'_i(f) = \min_{0 \leq j \leq s_i} \{ j \mu'(Q_i) + \mu'(c_{j,i}) \}.
\]

**Remark 3.7.** — We can prove that \( \mu'_i \) is a valuation. Moreover, we have:

\[
\forall f \in K[x], i \in \Lambda, \mu'_i(f) \leq \mu'(f).
\]

**Proposition 3.8.** — Let \( \{ Q_i \}_{i \in \Lambda} \subseteq K[x] \). Write \( H = \{ f \in K[x] | \mu'(f) \notin G_1 \} \) and, for a graded algebra \( G \), denote by \( G^* \) its saturation (see Definition 1.5). 

Consider the following assertions:

1. The set \( \{ Q_i \}_{i \in \Lambda} \) is 1-complete;
2. \( \forall f \in K[x] \setminus H, \exists i \in \Lambda, \mu'_i(f) = \mu'(f) \).
3. For all \( i \in \Lambda, Q_i \notin H \) and there exists \( h \in H \) monic or zero such that the set \( \{ Q_i \}_{i \in \Lambda} \cup \{ h \} \) is complete;
4. There exists \( h \in H \) monic or zero such that \( G_{\mu'} = (G_{\mu} | \{ \mu'_i(Q_i) \}_{i \in \Lambda}, in_{\mu'}(h))^* \).

Then 1. \( \iff \) 2. \( \iff \) 3. and 1. \( \implies \) 4.

**Proof:** Show that \( \{ Q_i \}_{i \in \Lambda} \) is 1-completed if and only if for all \( f \in K[x] \setminus H \) there exists \( i \in \Lambda, \mu'_i(f) = \mu'(f) \).

Suppose that \( \{ Q_i \}_{i \in \Lambda} \) is 1-completed and consider \( f \in K[x] \) such that \( \mu'(f) \in G_1 \).
Write $\beta = \mu'(f)$, then $P'_\beta \cap K[x]$ is generated by the products of the form $a \sum_{j=1}^s Q_{i,j}^{\gamma_j}$, $a \in K$, with $\sum_{j=1}^s \gamma_j \mu'(Q_{i,j}) + \mu(a) \geq \beta = \mu'(f)$, thus $\mu'_i(f) \geq \mu'(f)$. Moreover, by definition, for all $i \in \Lambda$, $\mu'_i(f) \leq \mu'(f)$. We conclude that $\mu'_i(f) = \mu'(f)$. Reciprocally, if all $f \in K[x]$ such that $\mu'(f) \in \mathcal{G}_1$ is as $\mu'_i(f) = \mu'(f)$, for all $i \in \Lambda$, then, $f$ writes such a sum of monomials in $Q_{i+1}$ of value at least $\mu'(f)$ \geq \beta$. So $f$ is in the ideal generated by all this monomials. We can show in the same way that $\mu'_i(f) = \mu'(f)$.

Remark 3.9. — Note that 4. \neq 1., consider the example 2.2 of [5]:

Let $k$ a field and $\iota : k(u,v) \hookrightarrow k[[t]]$ a monomorphism defined by $\iota(v) = t$ and $\iota(u) = \sum_{i \geq 1} c_i t^i$ where $c_i \in k^*$ and $Q_\infty = u - \sum_{i \geq 1} c_i v^i$ is transcendantal over $k(u,v)$.

Denote by $\mu'$ the valuation of $k(u,v)$ induced by the $t$-adic valuation of $k[[t]]$ through $\iota$ and $\mu$ by the restriction of $\mu'$ on $K = k(v)$. The set of key polynomials associated to the extension $K \hookrightarrow K(u)$ is $\{Q_i\}_{i \geq 1}$ with $Q_1 = u$ and $Q_i = u - \sum_{j=1}^{i-1} c_i v^i$ while $G_{\mu'} = k_{\mu'}[\iota_{\mu'}(u)]$.

For the next definition, we use the terminology of [12] following the notations of [3].

Definition 3.10. — Let $h \in K[x]$, consider its $i$-standard expansion $h = \sum_{j=0}^{s_i} c_{j,i} Q_i^j$. We call by the $i$-th effective degree of $h$ the natural number:

$$\delta_i(h) = \max \{j \in \{0, ..., s_i\} \mid j \beta_i + \mu'(c_{j,i}) = \mu'_i(h)\}.$$
By convention, \( \delta_i(0) = -\infty \).

**Remark 3.11.** — If we denote by:

\[
in_i(h) = \sum_{j \in S_i(h, \beta_i)} in_{\mu'}(c_{j,i})X^j \in G_{\mu'}[in_{\mu'}(Q_i), X]
\]

where \( S_i(h, \beta_i) = \{ j \in \{0, \ldots, s_i \} \mid j \beta_i + \mu'(c_{j,i}) = \mu'_i(h) \} \), then \( \delta_i(h) = d_X(\ln_i(h)) \).

Moreover, if \( in_{\mu'}(h) = in_{\mu'}(g) \) then \( \delta_i(h) = \delta_i(g) \). The definition of the effective degree extends naturally to \( gr_{\mu'}(K[x]) \).

**Remark 3.12.** — Remind that, by Proposition 5.2 of [3], for \( l \in \Lambda \) an ordinal number, the sequence \( (\delta_{l+i}(h))_{i \in \mathbb{N}^*} \) decreases. Thus there exists \( i_0 \in \mathbb{N}^* \) such that \( \delta_{l+i_0}(h) = \delta_{l+i_0+1}(h) \), for all \( i \geq 1 \) and we denote this common value by \( \delta_{l+i_0}(h) \) or \( \delta_{l+i_0} \) if no confusion is possible. Note that \( \delta_{l+i_0} \) may be 0.

The next three lemmas generalize the lemmas 2.12, 2.13 and 2.14 of [2], the proofs are quasi the same.

**Lemma 3.13.** — Let \( l \in \Lambda \) and \( f, g \in K[x] \):

\[
\delta_l(fg) = \delta_l(f) + \delta_l(g).
\]

**Proof:** It is an immediate consequence of the Remark 3.11.

**Lemma 3.14.** — For all \( h \in K[x] \), \( \delta_l(h) = 0 \) if and only if \( in_{\mu'}(h) \) is a unit of \( gr_{\mu'}(K[x]) \).

**Proof:** If \( \delta_l(h) = 0 \) then \( \mu'_i(h) = \mu'(h) \) and \( in_{\mu'}(h) = in_{\mu'}(c_{0,i}) \) in \( gr_{\mu'}(K[x]) \). As the polynomial \( Q_i \) is irreducible in \( K[x] \) and \( d^\circ(c_{0,i}) < d^\circ(Q_i) \), the polynomials \( Q_i \) and \( c_{0,i} \) are coprime in \( K[x] \). Then, there exists \( U, V \in K[x] \) such that \( UV_i + V_{c_{0,i}} = 1 \). We deduce that \( \mu'_i(Vc_{0,i}) = \mu'_i(1) < \mu'_i(UQ_i) \) and \( h \) is a unit of \( gr_{\mu'}(K[x]) \).

Reciprocally, it is sufficient to apply the Lemma 3.13 and the Remark 3.11.

**Lemma 3.15.** — For all \( i \in \Lambda \), the ring \( gr_{\mu'}(K[x]) \) is euclidean for \( \delta_l \), ie:

\[
\forall g, h \in K[x], h \neq 0, \exists Q, R \in K[x], g = hQ + R \text{ in } gr_{\mu'}(K[x]) \text{ and } 0 \leq \delta_l(R) < \delta_l(h).
\]

**Proof:** Write \( h = \sum_{j=0}^{s_i} c_{j,i}Q_j^i \), we can suppose without loss of generalities then \( c_{j,i} = 0 \) for \( j > \delta_l(h) \) because \( \mu'(c_{j,i}Q_j^i) > \mu'_i(c_{j,i}Q_j^i) \) and so \( in_{\mu'} \left( \sum_{j=\delta_l(h)+1}^{s_i} c_{j,i}Q_j^i \right) = 0 \). We can also suppose that \( c_{\delta_l(h),i} = 1 \) by Lemma 3.14. Note that \( d^\circ(h) = \delta_l(h).d^\circ(Q_i) \).

By euclidean division in \( K[x] \), there exists \( P, Q \in K[x] \) such that \( g = hQ + P \) with
Thus, to prove that $R/P$ is archimedean, we deduce that there exists a non-trivial isolated subgroup of $\Gamma$.

Remark 3.17. — Recall the Definition 1.1, the data of a valuation $\nu$ centered on a local ring $(R, m)$ is the data of a minimal prime ideal $P_\infty$ of $R$ (the support of the valuation) and of a valuation $\nu'$ of the quotient field of $R/P_\infty$ such that $R/P_\infty \subseteq R_{\nu'}$ and $m/P_\infty \subseteq R/P_\infty \cap m_{\nu'}$.

Proof: Let $(\beta_i)_{i \geq 1}$ an increasing infinite sequence of $\nu(R \setminus P_\infty)$ bounded by $\beta$. This sequence corresponds to a decreasing infinite sequence of ideals of $R/P_\beta$. It is sufficient to prove that $R/P_\beta$ have finite length. Write $m$ the maximal ideal of $R$, $\nu(m) = \min \{\nu(R \setminus P_\infty) \setminus \{0\}\}$ and $\Gamma$ the value group of $\nu$. Note that the group $\nu(R \setminus P_\infty)$ is archimedean. Indeed, by absurd, if $\nu(R \setminus P_\infty)$ is not archimedean, there exists $\alpha, \beta \in \nu(R \setminus P_\infty)$ such that, for all $n \geq 1$, $n\beta \leq \alpha$. Particularly, the set:

$$\{\gamma \in \Gamma \mid \exists n \in \mathbb{N} \setminus \{0\}, -n\beta < \gamma < n\beta\}$$

is a non-trivial isolated subgroup of $\Gamma$.

We deduce that there exists $n \in \mathbb{N}$ such that:

$$\beta \leq n\nu(m).$$

Thus, $m^n \subseteq P_\beta$ and there exists a surjective map:

$$R/m^n \rightarrow R/P_\beta.$$
Proposition 3.18. — ([3], Proposition 3.30) Suppose that we have construct an infinite set of key polynomials $Q_ω = \{Q_i\}_{i \geq 1}$ such that, for all $i \in N^*$, $β_i \in G_1$. Suppose further that the sequence $\{β_i\}_{i \geq 1}$ is not bounded in $G_1$. Then, the set of key polynomials $Q_ω$ is 1-complete.

Proof: It is sufficient to prove that, for all $β \in G_1$ and for all $h \in K[x]$ such that $µ'(h) = β$, $h$ belongs to the $R_µ$-submodule of $K[x]$ generated by all the monomials of the form $a \prod_{j=1}^s Q_{i_j}^{γ_j}$, $a \in K$, such that $µ'\left(a \prod_{j=1}^s Q_{i_j}^{γ_j}\right) ≥ β$.

Therefore consider $h \in K[x]$ such that $µ'(h) \in G_1$. Write $h = \sum_{j=0}^d h_jx^j$, we can suppose, without loss of generalities, that:

$$\forall j \in \{0, ..., d\}, µ(h_j) ≥ 0.$$ 

Indeed, otherwise it is sufficient to multiply $h$ by an element of $K$ appropriately chosen.

Since the sequence $\{β_i\}_{i \geq 1}$ is not bounded in $G_1$, there exists $i_0 \in N^*$ such that:

$$µ'(h) < β_{i_0}.$$ 

Then denote by $h = \sum_{j=0}^{i_0} c_{j,i_0}Q_{i_0}^{j}$, the $i_0$-standard expansion of $h$. This expansion is obtain by euclidean division, seen the choice made on the coefficients of $h$ and, since the sequence $\left\{\frac{β_j}{d^{γ_j}}\right\}_{i \geq 1}$ is strictly increasing (it is sufficient to take the $(i - 1)$-standard expansion of $Q_i$), we prove easily that:

$$\forall j \in \{0, ..., s_{i_0}\}, µ(c_{j,i_0}) ≥ 0.$$ 

Recall that, by construction of the key polynomials, for $j \in \{0, ..., s_{i_0}\}$, $µ'_{i_0}(c_{j,i_0}) = µ'(c_{j,i_0})$. We then deduce that:

$$\forall j \in \{1, ..., s_{i_0}\}, µ'(c_{j,i_0}Q_{i_0}^{j}) = µ'_{i_0}(c_{j,i_0}Q_{i_0}^{j}) > µ'(h).$$

Thus, $µ'(h) = µ'(c_0,i_0)$ and so, $h$ is a sum of monomials in $Q_{i_0+1}$ of valuation at least $µ'(h)$ (in particular, $µ'_{i_0}(h) = µ'(h))$.

We consider now two cases:

1. $\#\{i \geq 1 \mid α_i > 1\} = +∞$;
2. $\#\{i \geq 1 \mid α_i > 1\} < +∞$.

In case (1), with the Proposition 3.19, we can prove that the infinite set of key polynomials is always 1-complete, independently of the characteristic of $k_µ$. In case (2), if the effective degree is always 1 and if the set of key polynomials $Q_ω = \{Q_i\}_{i \geq 1}$ is not complete, we will prove in the Proposition 3.20 that the sequence $\{β_i\}_{i \geq 1}$ is never bounded. In this case, with the Proposition 3.18, we deduce that the set of key polynomials $Q_ω = \{Q_i\}_{i \geq 1}$ is also 1-complete.
Proposition 3.19. — ([3], Corollary 5.8) Suppose that we have construct an infinite set of key polynomials $Q_\omega = \{Q_i\}_{i \geq 1}$ such that, for all $i \in \mathbb{N}^*$, $\beta_i \in G_1$. More, suppose that the set $\{i \geq 1 \mid \alpha_i > 1\}$ is infinite. Then, $Q_\omega$ is a 1-complete set of key polynomials.

Proof: Let $h \in K[x]$, as in the proof of the Proposition 3.18, it is sufficient to show that $\mu'_i(h) = \mu'(h)$ for a $i \geq 1$. But, if we write:

$$\delta_i(h) = \max S_i(h, \beta_i),$$

where:

$$S_i(h, \beta_i) = \{ j \in \{0,...,s_i\} \mid j \beta_i + \mu'(c_{j,i}) = \mu'_i(h) \},$$

$$h = \sum_{j=0}^{s_i} c_{j,i} Q_i^j,$$

by (1) of Proposition 5.2 of [3], we have:

$$\alpha_{i+1} \delta_{i+1}(h) \leq \delta_i(h), \forall i \geq 1.$$

We deduce that if $\delta_i(h) > 0$ and $\alpha_{i+1} > 1$:

$$\delta_{i+1}(h) < \delta_i(h), \forall i \geq 1.$$

The set $\{i \geq 1 \mid \alpha_i > 1\}$ is infinite and the previous inequality does not occur infinitely, we conclude that there exists $i_0 \geq 1$ such that $\delta_{i_0}(h) = 0$ and so $\mu'_{i_0}(h) = \mu'(h)$.

From now, we suppose that we have construct an infinite set of key polynomials $Q_\omega = \{Q_i\}_{i \geq 1}$ such that $\alpha_i = 1$, for all $i$ sufficiently large. Thus for this $i$, we have:

$$Q_{i+1} = Q_i + z_i,$$

where $z_i$ is an $i$-standard homogeneous expansion, of value $\beta_i$, not containing $Q_i$. More we suppose that $\beta_i \in G_1$ and there exists $h \in K[x]$ such that, for all $i \geq 1$:

$$\mu'_i(h) < \mu'(h).$$

Write $Q_\omega$ the monic polynomial of smaller degree possible satisfying the previously inequality. As in the proof of the Proposition 3.19, write:

$$\delta_i(Q_\omega) = \max S_i(Q_\omega, \beta_i),$$

where:

$$S_i(Q_\omega, \beta_i) = \{ j \in \{0,...,s_i\} \mid j \beta_i + \mu'(c_{j,i}) = \mu'_i(Q_\omega) \},$$

$$Q_\omega = \sum_{j=0}^{s_i} c_{j,i} Q_i^j.$$

By (1) of Proposition 5.2 of [3], we have:

$$\alpha_{i+1} \delta_{i+1}(Q_\omega) \leq \delta_i(Q_\omega), \forall i \geq 1.$$

Since $\alpha_i = 1$ for $i$ sufficiently large, there exists $\delta_\omega \in \mathbb{N}^*$ such that $\delta_\omega = \delta_i(Q_\omega)$, for $i$ sufficiently large.
Proposition 3.20. — ([3], Proposition 6.8) Under the previously assumptions, if \( \delta_\omega = 1 \) then the sequence \( \{ \beta_i \}_{i \geq 1} \) is not bounded in \( G_1 \).

4. Defect and effective degree

In this section, we do not suppose any assumption about the rank of the valuation. We use the construction of key polynomials of W. Mahboub (see [8]). Recall that the order type of the set of key polynomials is at most \( \omega \times \omega \).

Lemma 4.1. — Let \((K, \mu) \to (L, \mu')\) be a finite and simple valued field. Let \( \{Q_i\}_{i \in \Lambda} \) be the well ordered set of key polynomials associated to this extension. Then there exists \( n_0 \in \mathbb{N}^\ast \) such that \( \Lambda \leq \omega n_0 \), i.e \( \{Q_i\}_{i \in \Lambda} \) have a finite number of limit key polynomials.

Proof: Let \( P \in K[x] \) monic and irreducible such that \( L = K[x]/(P) \). Denote also by \( \mu' \) the pseudo-valuation of \( K[x] \) for which the set of key polynomials is associated, thus for all \( Q \in (P) \), \( \mu'(Q) = \infty \). Suppose that the set \( \{Q_{\omega i}\}_{i \geq 1} \) is infinite. Such \( \alpha_{\omega i} > 1 \), for all \( i \geq 1 \), the sequence \( \{d^\circ(Q_{\omega i})\}_{i \geq 1} \) is strictly increasing. Then there exists \( n_0 \in \mathbb{N}^\ast \) such that \( d^\circ(P) \leq d^\circ(Q_{\omega i}) \), for all \( i \geq n_0 \).

Let \( i \geq n_0 \), by euclidean division there exists \( S_i, R_i \in K[x] \) such that \( Q_{\omega i} = PS_i + R_i \) with \( R_i = 0 \) or \( 0 \leq d^\circ(R_i) < d^\circ(P) \leq d^\circ(Q_{\omega i}) \). If \( R_i \neq 0 \) then \( \mu'(Q_{\omega i}) = \mu'(R_i) \) which is absurd by minimality of the degree of a limit key polynomial. If \( R_i = 0 \) then \( Q_{\omega i} \in (P) \) and the \( Q_{\omega i} \) are not key polynomials for \( i > n_0 \).

Lemma 4.2. — Let \((K, \mu)\) be an henselian field and \( L \) be a finite and simple extension of \( K \). By definition, \( \mu \) extends uniquely to \( L \) and this extension corresponds to a (pseudo-)valuation in \( K[x] \) denoted by \( \mu' \). Consider \( \{Q_i\}_{i \in \Lambda} \) the well ordered set of key polynomials associated to \( \mu' \) and \( n_0 \in \mathbb{N}^\ast \) the smallest possible such that \( \Lambda \leq \omega n_0 \). Then there exists an index \( i_0 \in \Lambda \) having a predecessor, such that:

\[
[L : K] = d^\circ(Q_{\omega(n_0-1)+i_0})d_{\omega n_0}
\]

where:

\[
d_{\omega n_0} = \begin{cases} \delta_{\omega n_0} & \text{si } \Lambda = \omega n_0 \text{ et } \sharp\{i \geq 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} = +\infty \vspace{1em} \cr 1 & \text{si } \Lambda < \omega n_0 \text{ ou } \Lambda = \omega n_0 \text{ et } \sharp\{i \geq 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} < +\infty \end{cases}
\]

Proof: By assumption, \( L = K[x]/(P(x)) \) with \( P \in K[x] \) irreducible and monic. Suppose that \( \Lambda \) is an ordinal having an immediate predecessor, to fix ideas denote it by \( \omega(n_0-1) + n, n \in \mathbb{N}^\ast \). By definition, such \( \Lambda < \omega n_0 \). By construction of key polynomials, \( P = Q_{\omega(n_0-1)+n} \). Thus, \( i_0 = n \) and:

\[
[L : K] = d^\circ(P) = d^\circ(Q_{\omega(n_0-1)+i_0})d_{\omega n_0}.
\]

Suppose now that \( \Lambda \) is a limit ordinal, denote it by \( \omega n_0 \).

If \( \sharp\{i \geq 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} < +\infty \), then \( \delta_{\omega n_0} = 1 \). As in the proof of Proposition 3.19, there exists \( i_0 \in \mathbb{N}^\ast \) minimal such that \( \mu'_{\omega(n_0-1)+i_0}(P) = \mu'(P) \). Thus, if we write the \( \omega(n_0-1) + i_0 \)-standard expansion of \( P \), we note that \( Q_{\omega(n_0-1)+i_0} \) belongs to the center of the valuation \( \mu' \) which is the ideal \( (P) \). Note that, by definition and
construction of the key polynomials and by choice of \( i_0 \), \( d^\circ(P) \leq d^\circ(Q_{\omega(n_0-1)+i_0}) \).
Since \( Q_{\omega(n_0-1)+i_0} \in (P) \), we conclude that \( Q_{\omega(n_0-1)+i_0} = cP, c \in K^* \) and so:
\[
[L : K] = d^\circ(P) = d^\circ(Q_{\omega(n_0-1)+i_0})d_{\omega n_0}.
\]
If \( \sharp \{ i \geq 1 \mid \alpha_{\omega(n_0-1)+i} = 1 \} = +\infty \), then \( d_{\omega n_0} = \delta_{\omega n_0} \). In this case, \( P = Q_{\omega n_0} \).
Take \( i_0 \in \mathbb{N}^* \) sufficiently large such that \( \alpha_{\omega(n_0-1)+i} = 1 \) and \( \delta_{\omega(n_0-1)+i}(P) = \delta_{\omega(n_0-1)+i+1}(P) \) for all \( i \geq i_0 \). Recall that this common value is denoted by \( \delta_{\omega n_0} \).
By Proposition 2.12 of [12], since \( K \) is henselian, the Newton polygon \( \Delta_{\omega(n_0-1)+i}(P) \) have a unique side of slope \( \beta_{\omega(n_0-1)+i} \). This is equivalent to:
\[
\mu'(c_{0,\omega(n_0-1)+i}) = \mu'(c_{\omega(n_0-1)+i},\omega(n_0-1)+i) + s_{\omega(n_0-1)+i}\beta_{\omega(n_0-1)+i}
\]
with \( P = \sum_{j=0}^{s_{\omega(n_0-1)+i}} c_{j,\omega(n_0-1)+i} Q_{\omega(n_0-1)+i}^j \) and \( 0 \leq j \leq s_{\omega(n_0-1)+i} \). By Corollary 3.23 of [3], \( \beta_{\omega(n_0-1)+i} \) determine always a side of \( \Delta_{\omega(n_0-1)+i}(P) \) which only have one, so:
\[
\mu'_{\omega(n_0-1)+i}(P) = \mu'(c_{0,\omega(n_0-1)+i}) = \mu'(c_{\omega(n_0-1)+i},\omega(n_0-1)+i) + s_{\omega(n_0-1)+i}\beta_{\omega(n_0-1)+i}.
\]
We deduce, by definition of \( \delta_{\omega(n_0-1)+i}(P) \), that:
\[
\delta_{\omega(n_0-1)+i}(P) = s_{\omega(n_0-1)+i}.
\]
But, for \( i \geq i_0 \), \( d^\circ(Q_{\omega(n_0-1)+i}) = d^\circ(Q_{\omega(n_0-1)+i_0}) \) and \( \delta_{\omega(n_0-1)+i}(P) = \delta_{\omega n_0} \). Thus:
\[
d_{\omega n_0} = s_{\omega(n_0-1)+i} = \frac{d^\circ(P)}{d^\circ(Q_{\omega(n_0-1)+i_0})}.
\]

**Corollary 4.3.** — With the assumptions of Lemma 4.2, we have:
\[
\prod_{j=1}^{n_0} d_{\omega j} = d_{L|K}(\mu', \mu).
\]

**Proof:** Recall that we denote by \( \alpha_{\omega j} = d^\circ_{Q_{\omega j}(j-1)+i_0}(Q_{\omega j}) \) with \( i_0 \) a \( \omega j \)-inessentiel index and \( j \in \mathbb{N}^* \). In [11], this number corresponds to the jump of order \( j \) denote by \( s(j) \).
By Proposition 3.4.4 of [8], we can always suppose that \( \delta_{\omega j} = \alpha_{\omega j} \), for all \( j < n_0 \). By Proposition 2.9 of [11]:
\[
d^\circ(Q_{\omega(n_0-1)+i_0}) = [\Gamma_{\mu', \omega(n_0-1)+i_0} : \Gamma_{\mu}] [k_{\mu'}^n : k_{\mu}] \prod_{j=1}^{n_0-1} \alpha_{\omega j}.
\]
We verified after that if \( \Lambda = \omega(n_0-1) + i_0 \) or if \( \Lambda = \omega n_0 \), we always have:
\[
[\Gamma_{\mu'} : \Gamma_{\mu}] = [\Gamma_{\mu', \omega(n_0-1)+i_0} : \Gamma_{\mu}],
\]
\[
[k_{\mu'} : k_{\mu}] = [k_{\mu'}^n : k_{\mu}].
\]
Thus, applying Lemma 4.2, we obtain:

\[ [L : K] = \left[ \Gamma_{\mu'} : \Gamma_{\mu} \right] [k_{\mu'} : k_{\mu}] \prod_{j=1}^{n_0-1} \alpha_{\omega_j} \] \ d_{\omega_{n_0}}.  

Since \( d_{\omega_j} = \delta_{\omega_j} = \alpha_{\omega_j} \), we concluded that:

\[ d_{L|K}(\mu', \mu) = \left( \prod_{j=1}^{n_0-1} \alpha_{\omega_j} \right) d_{\omega_{n_0}} = \left( \prod_{j=1}^{n_0-1} d_{\omega_j} \right) d_{\omega_{n_0}} = \prod_{j=1}^{n_0} d_{\omega_j}. \]

**Corollary 4.4.** — With the assumptions of Lemma 4.2, we have:

\[ \delta_{\omega_{n_0}} = \alpha_{\omega_{n_0}}. \]

Proof: By Corollary 2.10 of [11], the defect is the total jump, applying Corollary 4.3 we have the equality:

\[ \prod_{j=1}^{n_0} \alpha_{\omega_j} = n_0 \prod_{j=1}^{n_0} d_{\omega_j}. \]

Such \( \delta_{\omega_j} = \alpha_{\omega_j} \neq 0 \), for all \( j < n_0 \), we deduced that:

\[ \delta_{\omega_{n_0}} = \alpha_{\omega_{n_0}}. \]

**Corollary 4.5.** — Let \( (K, \mu) \) be a valued field and \( L \) be a finite and simple extension of \( K \). Write \( \mu^{(1)}, ..., \mu^{(g)} \) the different extensions of \( \mu \) on \( L \), it corresponds to a (pseudo-)valuation of \( K[x] \) denoted by the same way. Consider \( \{Q^{(i)}_{\mu} \}_{i \in \Lambda^{(i)}} \) the set of key polynomials associated to \( \mu^{(i)} \) and \( n_0^{(i)} \in \mathbb{N}^* \) the smallest possible such that \( \Lambda^{(i)} \leq \omega_{n_0^{(i)}} \), \( 1 \leq i \leq g \). Then:

\[ d_{L|K}(\mu^{(i)}, \mu) = \prod_{j=1}^{n_0^{(i)}} d_{\omega_j}^{(i)}. \]

We deduced that:

\[ [L : K] = \sum_{i=1}^{g} e_i f_i d_{\omega_j}^{(i)} ... d_{\omega_{n_0^{(i)}}^{(i)}}. \]

where \( e_i = [\Gamma_{\mu^{(i)}} : \Gamma_{\mu}] \) and \( f_i = [k_{\mu^{(i)}} : k_{\mu}] \).

Proof: By Proposition 2.5, we know that:

\[ [L : K] = \sum_{i=1}^{g} e_i f_i d_i, \]

where \( d_i = d_{L|K}(\mu^{(i)}, \mu), e_i = [\Gamma_{\mu^{(i)}} : \Gamma_{\mu}] \) and \( f_i = [k_{\mu^{(i)}} : k_{\mu}] \). Applying Corollary 4.3 to the fields \( L^{h(\mu^{(i)})} \) and \( K^{h(\mu^{(i)})} \) we have the equalities.
**Corollary 4.6.** — Under the assumptions of Corollary 4.5 and denoting by $p = \text{car}(k_\mu)$, then, for $1 \leq i \leq g$ and $1 \leq j \leq n_0^{(i)}$, there exists $e_{\omega j}^{(i)} \in \mathbb{N}$ such that:

$$b_{\omega j}^{(i)} = p^{e_{\omega j}^{(i)}}.$$

**Proof:** It is an immediate consequence of Corollary 4.5 and Remark 2.4.

**Example 4.7.** — We study the example 3.2. of [11]. Let $k$ be an algebraic closed field of characteristic 0. Consider $K = k(y)$ equipped with its $y$-adic valuation denote by $\mu$. The polynomial:

$$P = x^4 + y^2x^3 + y^3(y^2 - 2)x^2 - y^5x + y^6$$

is irreducible in $K[x]$. The valuation extends only on two valuations $\mu^{(1)}$ and $\mu^{(2)}$ over the field $L = K[x]/(P)$ given by the sequences of key polynomials $\{Q_i^{(1)}\}_{i \in \Lambda^{(1)}}$ and $\{Q_i^{(2)}\}_{i \in \Lambda^{(2)}}$ such that:

- $Q_1^{(1)} = x$ \quad $\beta_1^{(1)} = 3/2$
- $Q_2^{(1)} = x^2 - y^3$ \quad $\beta_2^{(1)} = 7/2$
- $Q_i^{(1)} = Q_2^{(1)} + (y^2 - u_{i-3})x$ \quad $\beta_i^{(1)} = 3/2 + i$; $i \geq 3$

and:

- $Q_1^{(2)} = x$ \quad $\beta_1^{(2)} = 3/2$
- $Q_2^{(2)} = x^2 - y^3$ \quad $\beta_2^{(2)} = 9/2$
- $Q_i^{(2)} = Q_2^{(2)} + u_{i-2}x$ \quad $\beta_i^{(2)} = 3/2 + (i + 1)$; $i \geq 3$

where $u_0 = 0$, $u_l = \sum_{j=1}^{l} c_j y^{j+2}$, for all $l \geq 1$, $c_1 = c_2 = 1$ and $c_j = 4^{j-2} \frac{3^{(2j-3)} - 2^{(2j)}}{6^{(2j)}}$ for all $j \geq 3$. Following the notations of the Corollary 4.5, with $\Lambda^{(1)} = \Lambda^{(2)} = \omega$, and $\alpha_i^{(1)} = \alpha_i^{(2)} = 1$ for all $i \geq 2$, we deduced that:

$$d_{\omega}^{(1)} = d_{\omega}^{(2)} = 1,$$

since $p = 1$ if $\text{car}(k) = 0$. Moreover, since a field of characteristic zero is defectless, we found the fact that:

$$d_{L/K}(\mu^{(1)}, \mu) = 1 = d_{\omega}^{(1)}$$

and:

$$d_{L/K}(\mu^{(2)}, \mu) = 1 = d_{\omega}^{(2)}.$$

Finally, since $\Gamma_\mu = \mathbb{Z}$, $\Gamma_{\mu^{(1)}} = \Gamma_{\mu^{(2)}} = (1/2)\mathbb{Z}$, and $k_\mu = k_{\mu^{(1)}} = k_{\mu^{(2)}} = k$ we deduced that:

$$e_1 = e_2 = 2 \text{ et } f_1 = f_2 = 1.$$

We still find the result of the Corollary 4.5, that is to say:

$$4 = [L : K] = e_1 f_1 d_{\omega}^{(1)} + e_2 f_2 d_{\omega}^{(2)} = 2 \times 1 \times 1 + 2 \times 1 \times 1.$$
Example 4.8. — We study the example of W. Mahboub given in [7]. Let $k$ be a field of characteristic $p > 2$. Denote by $\nu$ the $z$-adic valuation of $k(z)$ and by $\mu$ the valuation of $K = k(z, y)$ defined by the set of key polynomials $\{Q_{y, i}\}_{i \geq 1}$ given in [7] extending $\nu$. For $e \in \mathbb{N}^*$, write:

$$f = x^{p^e} - y^2 - z.$$ 

We want to find all the valuations of $L = K[x]/(f)$ whose extends $\mu$. Consider the valuation $\mu'$ of $L$ define by the set of key polynomials $\{Q_i\}_{i \in \Lambda}$:

$$Q_1 = x, \quad \beta_1 = 1 - \frac{1}{4p};$$
$$Q_i = x^{p^e - 1} - \sum_{j=0}^{i} h_j, \quad \beta_i = 1 - \frac{1}{2^{2j}p^3}; \quad i \geq 2$$

where $h_0 = 0$ and $h_j = \frac{Q_j^2}{z^{2^{2j}p^3 - 1}}$, for all $j \geq 1$. We deduced that, for all $i \geq 1$:

$$f = (Q_i + \sum_{j=0}^{i} h_j)^p - y^2 - z = Q_i^p + \sum_{j=0}^{i} h_j^p - Q_{y, 2} = Q_i^p + \frac{Q_{y, 2i}}{z^{2^{2i} - 2p - p}}.$$

But $\mu(h_i^p) = \mu\left(\frac{Q_{y, 2i}}{z^{2^{2i} - 2p - p}}\right)$, so:

$$in_i(f) = X^p - in_\mu(h_i^p) = (X - in_\mu(h_i))^p.$$ 

Thus, for all $i \geq 1$, $\delta_i(f) = p$ and so $\delta_\omega = p$. Moreover we are in the situation where $\Lambda = \omega$ and $\alpha_i = 1$, for all $i \geq 3$. We deduced, with the Corollary 4.5, that:

$$[k_{\mu'} : k_\mu] = p^{e-1};$$
$$d_{L/K}(\mu', \mu) = d_\omega = \delta_\omega = p.$$ 

Since $[L : K] = p^e$, by the Corollary 4.5, we deduced that $\mu'$ is the unique valuation extending $\mu$ on $L$.

Example 4.9. — Let $k$ be a field of characteristic $p > 0$. We denote by $K = k((y, z))$. Consider the valuation $\nu : K^* \rightarrow \mathbb{Z}_p^2$, trivial on $k$, defined by $\nu(z) = (1, 0)$ and $\nu(y) = (0, 1)$. Consider the polynomial $P \in K[w]$ defined by:

$$P(w) = w^p - z^{p^2 - p}w + y^p z^{p^2}.$$ 

Write $L = K[w]/(P)$, we want to find all the valuations $\nu^{(i)}$ of $L$ extending $\nu$ and all the corresponding sets of key polynomials $\{Q_j^{(i)}\}_{j \in \Lambda^{(i)}}$.

Find a valuation of $L$ extending $\nu$ is equivalent to find a pseudo-valuation of $K[w]$ of kernel $P$. Let $\mu$ a such pseudo-valuation. The only possibilities values of $w$ are:

$$\mu(w) = \begin{cases} (p, p) \\ (p, 1) \\ (p, 0) \end{cases}$$
But if \( \mu(w) = (p,1) \) then \( \mu(wp) = \mu(y^p z^{p^2}) = (p^2,p) > \mu(z^{p^2 - p} w) = (p^2,1) \), the monomial \( wp \) is not of minimal value in \( P \). The two only values possible of \( w \) are:

\[
\mu(w) = \begin{cases} 
(p,p) \\
(p,0)
\end{cases}
\]

1) Suppose that \( \mu(w) = (p,p) \).

We are in the situation of the example 5.3.2 of [8]. In this case, we prove that the key polynomials are:

\[
\begin{align*}
Q_1^{(1)} &= w, \\
Q_j^{(1)} &= w - z^p u_j,
\end{align*}
\]

where \( u_j = \sum_{i=1}^j y^p \) and \( \mu(Q_j^{(1)}) = (p,p^j) \), for all \( j \geq 2 \). Moreover the polynomial \( P \) is a limit key polynomial. Finally, the polynomial \( in_j(P) \) is irreducible for all \( j \geq 1 \), the valuation extends uniquely since \( \mu(w) = (p,p) \).

Thus, a first way to extend \( \nu \) over \( L \) is to consider the valuation \( \nu^{(1)} \) defined by the key polynomials \( Q_j^{(1)}, j \geq 1 \) and \( Q_0^{(1)} = P \).

Write \( u = \sum_{i \geq 1} y^p \) the limit of the sequence \( (u_j)_j \) in \( k[[y, z]] \). Remark that \( u \) is solution of the equation \( X - X^p = y^p \). The other solutions of the equation are the \( u - l \) with \( l \in \{1, ..., p-1\} \). Writing \( \alpha_l = z^p(u - l) \), for \( l \in \{0, ..., p-1\} \), we can prove that:

\[
\prod_{l=0}^{p-1} (w - \alpha_l) = w^p - z^{p^2 - p} w + y^p z^{p^2}.
\]

So, the roots of \( P \) are \( \alpha_0, ..., \alpha_{p-1} \) and if we write \( Q_\infty^{(1)} = w - \alpha_0 = w - z^p u \) then:

\[
P(w) = Q_\infty^{(1)}(w) \prod_{l=1}^{p-1} (w - \alpha_l).
\]

Finally, since \( in_j(P) = -z^{p^2 - p}(X - y^p z^p) \), for all \( j \geq 1 \) and \( \Lambda^{(1)} = \omega \), we deduced that:

\[
d_{L|K}(\nu^{(1)}, \nu) = d^{(1)}_\omega = \delta^{(1)}_\omega = 1.
\]

2) Suppose that \( \mu(w) = (p,0) \).

Denote by \( Q_1^{(2)} = w \), then:

\[
in_1(P) = X \left( X^{p-1} - z^{p^2 - p} \right).
\]

Write \( Q_2^{(2)} = \left( Q_1^{(2)} \right)^{p-1} - z^{p^2 - p} \). Then:

\[
P = Q_1^{(2)} Q_2^{(2)} + y^p z^{p^2}.
\]

The unique choice for the valuation of \( Q_2^{(2)} \) is:

\[
\mu(Q_2^{(2)}) = (p^2 - p, p) > (p,0) = \mu(Q_1^{(2)}).
\]
In this case, the polynomial $i_{n_2}(P)$ is:

$$i_{n_2}(P) = i_{\mu}(Q_{1}^{(2)}) \left( X + \frac{in_{\mu}(y^p z^2)}{in_{\mu}(Q_{1}^{(2)})} \right).$$

Remark that $\mu(u) = \mu(y^p) = (0, p)$ then $\mu(\alpha_0) = \mu(z^p u) = (p, p)$. Writing $h = \sum_{m=1}^{p-1} \alpha_0^m w^{m-1}$, we remark that $h \in K[w]$ represents $\frac{in_{\mu}(y^p z^2)}{in_{\mu}(Q_{1}^{(2)})}$ because $\mu(h) = \mu(w^{p-2} \alpha_0) = (p^2 - p, p)$. Then denote by:

$$Q_{3}^{(2)} = Q_{2}^{(2)} + h.$$ But, $(w - \alpha_0)h = \alpha_0(w^p - 1 - \alpha_0^p)$. So:

$$(w - \alpha_0)Q_{3}^{(2)} = w^p - z^p - w + \alpha_0 z^p - \alpha_0^p.$$ But, $\alpha_0 z^p - \alpha_0^p = u z^p - w^p z^p = (u - w^p) z^p = y^p z^p$ we deduced that:

$$Q_{3}^{(2)} = (w - \alpha_0)Q_{3}^{(2)} = P.$$ The sequence of key polynomials stop. The second way to extend $\nu$ on $L$ is to consider the valuation $\nu^{(2)}$ defined with the key polynomials $Q_{1}^{(2)}$, $Q_{2}^{(2)}$ and $Q_{3}^{(2)}$. Finally, since $\Lambda^{(2)} = 3 < \omega$, we deduced that:

$$d_{L/K}(\nu^{(2)}, \nu) = d_{\omega}^{(2)} = 1.$$ Then we obtain:

$$p = [L : K] = d^o(P) = d^o(Q_{1}^{(1)}d_{\omega}^{(1)} + d^o(Q_{2}^{(2)}d_{\omega}^{(2)}) = 1 + (p - 1).$$

**Example 4.10.** — We will study an example proposed by S.D. Cutkosky in [1]. Let $k$ a field of characteristic $p > 0$. Denote by $K = k(u, v)$ and by $L = K[y]/(f)$ where:

$$f = y^{p^2 + 1} + y^{p^2} - y^p + u - v^p.$$ We define a valuation $\nu$ over $K$ such that $\nu(u) = p$ with:

$$R_{\nu} = \bigcup_{i \geq 0} A_i,$$ where $A_i = k[u_i, v_i](u_i, v_i)$ and:

$$\left\{ \begin{array}{l}
  u_0 = u \\
  v_0 = v
\end{array} \right\} \left\{ \begin{array}{l}
  u_{pi+j} = \frac{u_{pi}}{v_{pi+j}} , 1 \leq j < p \\
  v_{pi+j} = v_{pi}
\end{array} \right\}, 1 \leq j < p;$$

$$\left\{ \begin{array}{l}
  u_{p(i+1)} = v_{pi} \\
  v_{p(i+1)} = v_{pi} + \gamma_{p(i+1)} ; \gamma_{p(i+1)} \in k*. \end{array} \right\}$$

Then we have $\nu(v_{pi}) = \frac{1}{p^i}$ and $\nu(u_{pi}) = \frac{1}{p^{i-1}}$. Remark that, for $i \geq 0$:

$$v_{p(i-1)} - \gamma_{p(i+1)} v_{pi} = v_{pi} v_{p(i+1)},$$ where we write $v_{-p} = u$. Then:

$$\nu(v_{p(i-1)} - \gamma_{p(i+1)} v_{pi}) = \frac{1}{p^{i-1}} - \frac{1}{p^{i+1}}.$$
We want to extend \( \nu \) on \( L \). There exist a first valuation denoted by \( \nu^* \) defined by the sequence of key polynomials \( \{Q_l\}_{l \in \Lambda} \):

\[
Q_1 = y, \quad \beta_1 = \nu^*(Q_1) = \frac{1}{p}; \quad Q_l = Q_{l-1} - \gamma^*_l \prod_{j=0}^{l-2} v_{p(2j)}^{(2j+1)}, \quad \beta_l = \nu^*(Q_l) = \sum_{j=0}^{l-1} \frac{1}{p^{2j+1}}.
\]

Remark that \( \lim_{l \to +\infty} \beta_l = \frac{p}{p^2 - 1} \), so there exists a first limit key polynomial which is:

\[
Q_\omega = y - v, \quad \beta_\omega = \nu^*(Q_\omega) = 1.
\]

We continue to define the set of key polynomials, for \( i \geq 1 \), by:

\[
Q_{\omega+i} = Q_{\omega+i-1} + \gamma^*_i \prod_{j=0}^{i-1} v_{p(2j)}^{(2j+1)}, \quad \beta_{\omega+i} = \nu^*(Q_{\omega+i}) = \sum_{j=0}^{i} \frac{1}{p^{2j}}.
\]

Remark that \( \lim_{i \to +\infty} \beta_{\omega+i} = 1 + \frac{1}{p^2 - 1} \), so there exists a second limit key polynomial which is:

\[
Q_{\omega2} = y^2 - u - v^p = Q_\omega^p + u, \quad \beta_{\omega2} = \nu^*(Q_{\omega2}) = p + \frac{1}{p}.
\]

We end the set of key polynomials with, for \( n \geq 1 \):

\[
Q_{\omega+2n} = Q_{\omega+2n-1} + (-1)^n uy^n, \quad \beta_{\omega+2n} = \nu^*(Q_{\omega+2n}) = p + \frac{n + 1}{p}.
\]

The last and third limit key polynomial is \( Q_\omega3 = f \) and \( \beta_{\omega3} = +\infty \). Remark that for all \( n \geq 0 \):

\[
f = (y + 1)Q_{\omega2+n} + (-1)^{n+1} uy^{n+1}.
\]

So, in \( K [[y]] \), \( f = (y + 1)Q_\infty \) where \( Q_\infty = y^p + u - v^p - uy \sum_{n\geq1}(-1)^n y^n \). Remark that for \( l \geq 1 \):

\[
y^p - v = Q_l^p - v_{p(2(l-1))} \prod_{j=0}^{l-2} v_{p(2j+1)}^{(2j+1)},
\]

then:

\[
in_l(y^p - v) = X^p - in_l \left( \prod_{j=0}^{l-2} v_{p(2j+1)}^{(2j+1)} \right) = \left( X - in_l \left( \prod_{j=0}^{l-2} v_{p(2j+1)} \right) \right)^p.
\]

We deduce that for all \( l \geq 1 \):

\[
\alpha_l = 1,
\]

\[
d_{\omega} = \delta_{\omega} = \delta_l(y^p - v) = p = \alpha_{\omega}.
\]

In the same way, for \( i \geq 1 \):

\[
y^{p^2} + u - v^p = Q_{\omega+i}^p + u = Q_{\omega+i}^p - v_{p(2i-1)} \prod_{j=0}^{i-2} v_{p(2j)}^{(2j+1)},
\]
Proposition 5.1. —

Proof

\[ \alpha_{\omega+i} = 1, \]
\[ d_{\omega+2} = \delta_{\omega+2} = \delta_{\omega+2+i}(y^p + u - v^p) = p = \alpha_{\omega+2}. \]

Finally, for all \( n \geq 0 \):
\[ i n_{\omega+n}(f) = X + i n_\nu \left((-1)^{n+1} u y^{n+1}\right). \]

We deduce that for all \( n \geq 1 \):
\[ \alpha_{\omega+n} = 1, \]
\[ d_{\omega+n} = \delta_{\omega+n}(f) = 1. \]

By the Corollary 4.3:
\[ d_{L|K}(\nu^*, \nu) = d_{\nu}d_{\omega}d_{\omega+n} = p \times p \times 1 = p^2. \]

Remark that \([L : K] = p^2 + 1\), so there exists a second valuation \( \nu^{(2)} \) of \( L \) which extends \( \nu \) and defectless. By the equality \( f = (y + 1)Q_\infty \) in \( K[[y]] \), the valuation verifies \( \nu^{(2)}(y) = 0 \), so this is the trivial valuation.

5. Key polynomials and defectless fields

Consider \( K \hookrightarrow K(x) \) a simple transcendental field extension. Let \( \mu' \) be a valuation of \( K(x) \), write \( \mu := \mu'_{|K} \). We denote by \( G \) the value group of \( \mu' \) and \( G_1 \) the smallest isolated non zero subgroup of \( G \). We suppose that \( \mu \) is of rank 1, \( \mu'(x) > 0 \).

**Proposition 5.1.** — If \( K \) is defectless, there exists a 1-complete set of key polynomials \( \{Q_i\}_{i \in \Lambda} \) such that \( \Lambda \) is a finite set or \( \mathbb{N}^* \). In particular, there is no limit key polynomial for valuations of rank 1 over defectless fields.

**Proof:** Apply the process of [3]. If there exists \( i_0 \in \mathbb{N} \), such that \( \beta_{i_0} \notin G_1 \), we write \( \Lambda = \{1, ..., i_0-1\} \) and, by definition, \( \{Q_i\}_{i \in \Lambda} \) is 1-completed. Otherwise, for all \( i \in \mathbb{N} \), \( \beta_i \in G_1 \) and we write \( \Lambda = \mathbb{N}^* \). If \( \{i \geq 1 \mid \alpha_i > 1\} = +\infty \), by Proposition 3.19, the set \( \{Q_i\}_{i \in \Lambda} \) is 1-completed. If \( \{i \geq 1 \mid \alpha_i > 1\} < +\infty \), write \( Q_\omega \) the monic polynomial of smallest degree such that, for all \( i \geq 1 \):
\[ \mu'_i(Q_\omega) < \mu'(Q_\omega). \]

Since \( K \) is defectless, the extension \( K \hookrightarrow L = K[x]/(Q_\omega) \) is defectless. Consider \( \mu' \) as the composition of a valuation \( \mu^{(1)} \) of value group \( G_1 \) centered on \( K[x]/(Q_\omega) \) and a valuation \( \theta \) of value group \( G/G_1 \) centered on \( K[x]_{(Q_\omega)} \). The set of key polynomials for \( \mu^{(1)} \) is the same as the set of key polynomials for \( \mu' \) except that \( \mu^{(1)}(Q_\omega) = \infty \). Thus, \( d_{L|K}(\mu^{(1)}, \mu) = 1 \). By Corollary 4.5, we deduce that:
\[ \delta_{\omega} = d_{L|K}(\mu^{(1)}, \mu) = 1. \]
Applying Proposition 3.20, we conclude that the sequence \( \{ \beta_i \}_{i \geq 1} \) is unbounded in \( G_1 \) and so, by Proposition 3.18, the set \( \{ Q_i \}_{i \in \Lambda} \) is a 1-completed set of key polynomials.

**Corollary 5.2.** — If \( \text{car}(k_{\mu}) = 0 \), there exists a 1-completed set of key polynomials \( \{ Q_i \}_{i \in \Lambda} \) such that \( \Lambda \) is a finite or \( \mathbb{N}^* \). In particular, there is no limit key polynomials for valuations of rank 1 whose residual field is of characteristic zero.

**Proof:** Apply Proposition 2.10 and Proposition 5.1.

---

### 6. Local uniformization of quasi-excellent local domain without defect

We extend here the results of the section 8 of [9] for a valuation satisfying some inductive assumption about defect. More precisely, in [9], to obtain a theorem of monomialization, the valuation needs to have a complete set of key polynomials without limit key polynomial: this is the case if the valuation is defectless. As a corollary, we find the local uniformization in characteristic zero.

Let \( (R, m, k) \) be a local complete regular equicharacteristic ring of dimension \( n \) with \( m = (u_1, ..., u_n) \). Let \( \nu \) be a valuation of \( K = \text{Frac}(R) \), centered on \( R \), of value group \( \Gamma \) and \( \Gamma_1 \) the smallest non-zero isolated subgroup of \( \Gamma \). Write:

\[
H = \{ f \in R \mid \nu(f) \notin \Gamma_1 \}.
\]

\( H \) is a prime ideal of \( R \) (see the proof of Theorem 6.4). Moreover suppose that:

\[
n = e(R, \nu) = \text{emb.dim} \ (R/H),
\]

that is to say:

\[
H \subseteq m^2.
\]

Write \( r = r(R, u, \nu) = \text{dim}_Q \left( \sum_{i=1}^{n} Q\nu(u_i) \right) \).

The valuation \( \nu \) is unique if \( \text{ht}(H) = 1 \); it is the composition of the valuation \( \mu : L^* \to \Gamma_1 \) of rank 1 centered on \( R/H \), where \( L = \text{Frac}(R/H) \), with the valuation \( \theta : K^* \to \Gamma/\Gamma_1 \), centered on \( R_H \), such that \( k_{\theta} \simeq \kappa(H) \).

By abuse of notation, for \( f \in R \), we will denote by \( \mu(f) \) instead of \( \mu(f \mod H) \). By the Cohen’s theorem, we can suppose that \( R \) is of the form:

\[
R = k [[u_1, ..., u_n]].
\]

For \( j \in \{ r + 1, ..., n \} \), write \( \{ Q_{j,i} \}_{i \in \Lambda_j} \), the set of key polynomials of the extension \( k((u_1, ..., u_{j-1})) \hookrightarrow k((u_1, ..., u_{j-1}))(u_j) \), \( Q_{j,i} = \{ Q_{j,i'} \mid i' \in \Lambda_j, i' < i \} \), \( \Gamma^{(j)} \) the value group of \( \nu_{k((u_1, ..., u_{j-1})))} \) and \( \nu_{j,i} \) the \( i \)-truncation of \( \nu \) for this extension.

For the definition of local framed sequences, one may consult Définition 7.1 and the sections 4.1 and 4.2 of [9].

**Theorem 6.1.** — Suppose that, for \( R_{n-1} = k [[u_1, ..., u_{n-1}]] \) we have:
1. (a) If \( H \cap R_{n-1} \neq (0) \) and there exists a local framed sequence \( (R_{n-1}, u) \to (R', u') \) such that:
\[
eq (R_{n-1}, \nu);
\]
(b) If \( H \cap R_{n-1} = (0) \) and for all \( f \in R_{n-1} \), there exists a local framed sequence \( (R_{n-1}, u) \to (R', u') \) such that \( f \) is a monomial in \( u' \) times a unit of \( R' \).

Moreover suppose that \( k(\{(u_1, \ldots, u_{n-1})\}) \to k(\{(u_1, \ldots, u_n)\}) / H \) is defectless. Then the assumptions 1. and 2. are true with \( R \) instead of \( R_{n-1} \).

\textbf{Proof:} The proof is the same as the proofs on Theorem 5.1 and 7.2 of [9]. With the assumptions of the Theorem 6.1, we can use the Proposition 5.2 of [9]. Then \( H \) is generated by a irreducible monic polynomial in \( u_n \). Since \( k(\{(u_1, \ldots, u_{n-1})\}) \to k(\{(u_1, \ldots, u_n)\}) / H \) is defectless, by Proposition 5.1, the set of key polynomials \( \{Q_{j,i}\}_{i \in \Lambda_i} \) has not limit key polynomial. To conclude it is sufficient to apply Theorem 7.2 of [9].

Consequently, we obtain the local uniformization of a valuation of rank 1 centered on a local quasi-excellent equicharacteristic domain satisfying some assumptions of lack of defect. The proof uses the notion of implicit prime ideal, for more details see [5] or section 4.3 of [9].

\textbf{Definition 6.2.} — For a local noetherian ring \( (R, \mathfrak{m}) \), with \( \mathfrak{m} = (u) = (u_1, \ldots, u_n) \) and \( f_1, \ldots, f_s \in \mathfrak{m} \), we call the \textit{monomial property for} \( R \) and \( f_1, \ldots, f_s \) this three statements:

1. \( R \) is regular;
2. For \( 1 \leq j \leq s \), \( f_j \) is a monomial in \( u \) times a unit of \( R \);
3. For \( 1 \leq j \leq s \), \( f_1 \) divides \( f_j \) in \( R \).

\textbf{Definition 6.3.} — Let \( (S, \mathfrak{m}, k) \) be a local domain of quotient field \( L \) and \( \mu \) be a valuation of \( L \) of rank 1 and of value group \( \Gamma_1 \), centered on \( S \).

Write \( u = (u_1, \ldots, u_n) \) a minimal set of generators of \( \mathfrak{m} \) and \( \overline{\mathfrak{m}} \) the implicit prime ideal of \( \overline{S} \).

Write \( u = (y, x) \) such that \( x = (x_1, \ldots, x_l) \), \( l = \text{emb.dim} \left( \overline{S} / \overline{H} \right) \) and the images of \( x_1, \ldots, x_l \) in \( \overline{S} / \overline{\mathfrak{m}} \) induce a minimal set of generators of \( (\mathfrak{m} \overline{S}) / \overline{\mathfrak{m}} \).

Let \( f_1, \ldots, f_s \in \mathfrak{m} \) such that \( \mu(f_1) = \min_{1 \leq j \leq s} \{ \mu(f_j) \} \).

1. We say that \( S \) and \( f_1, \ldots, f_s \) have the \textit{quotient local uniformization property of dimension} 1 if there exists a local framed sequence:
\[
(S, u, k) = (S^{(0)}, u^{(0)}, k^{(0)}) \xrightarrow{\rho_0} (S^{(1)}, u^{(1)}, k^{(1)}) \xrightarrow{\rho_1} \ldots \xrightarrow{\rho_{i-1}} (S^{(i)}, u^{(i)}, k^{(i)}) ,
\]
such that \( S^{(i)} / \overline{H^{(i)}} \) and \( \overline{f}_1, \ldots, \overline{f}_s \) have the monomial property, where \( \overline{H^{(i)}} \) is the implicit prime ideal of \( S^{(i)} \) and \( \overline{f}_j \) are the images of \( f_j \) mod \( \overline{H^{(i)}} \), \( 1 \leq j \leq s \).
2. We say that \( S \) and \( f_1, \ldots, f_s \) have the local uniformization property of dimension \( l \) if there exists a local framed sequence:

\[
(S, u, k) = (S^{(0)}, u^{(0)}, k^{(0)}) \xrightarrow{\rho_0} (S^{(1)}, u^{(1)}, k^{(1)}) \xrightarrow{\rho_1} \ldots \xrightarrow{\rho_{l-1}} (S^{(l)}, u^{(l)}, k^{(l)})
\]

such that \( S^{(i)} \) and \( \overline{f_1}, \ldots, \overline{f_s} \) have the monomial property.

**Theorem 6.4.** — Let \((S, \mathfrak{m}, k)\) be a local noetherian equicharacteristic domain of quotient field \( L \) and \( \mu \) be a valuation of \( L \) of rank 1 with value group \( \Gamma_1 \), centered on \( S \).

Write \( u = (u_1, \ldots, u_n) \) a minimal set of generators of \( \mathfrak{m} \) and \( \overline{H} \) the implicit prime ideal of \( \widehat{S} \).

Write \( u = (y, x) \) such that \( x = (x_1, \ldots, x_l) \), \( l = \text{emb.dim} \left( \frac{\widehat{S}}{\overline{H}} \right) \) and the images of \( x_1, \ldots, x_l \) in \( S/\overline{H} \) induce a minimal set of generators of \( (\mathfrak{m}S)/\overline{H} \).

Suppose that the quotient local uniformization property of dimension \( l-1 \) is true for all local domain and \( k ((x_1, \ldots, x_{l-1})) \hookrightarrow k ((x_1, \ldots, x_{l-1}))[x_l]/\overline{H} \) is defectless.

Let \( f_1, \ldots, f_s \in \mathfrak{m} \) such that \( \mu (f_i) = \text{min}_{1 \leq j \leq l} \{\mu (f_j)\} \). Then \( S \) and \( f_1, \ldots, f_s \) have the quotient local uniformization property of dimension \( l \).

**Proof:** By Theorem 2.1 of [5], \( \mu \) extends uniquely in a valuation \( \widehat{\mu} \) centered on \( \widehat{S}/\overline{H} \). By the Cohen’s structure theorem, we know that there exists a complete regular local ring of characteristic zero \( R \) and a surjective morphism \( \varphi \):

\[
\varphi : R \twoheadrightarrow \widehat{S}/\overline{H}.
\]

Write \( H = \ker \varphi \), since \( \overline{H} \) is a prime ideal (Theorem 2.1 of [5]), \( H \) is a prime ideal of \( R \). Choose \( R \) such that \( \dim(R) = l \). Write \( K \) the quotient field of \( R \), \( K \) is of the form \( k ((x_1, \ldots, x_l)) \).

Let \( \theta \) a valuation of \( K \), centered on \( R_H \), such that \( k_\theta = k(\mathfrak{H}) \).

If we look \( \widehat{\mu} \) as a valuation centered on \( R/\mathfrak{H} \) by the morphism \( \varphi \), we can consider the valuation \( \nu = \widehat{\mu} \circ \theta \) centered on \( R \) and of value group \( \Gamma_1 \). Then, \( \Gamma_1 \) is the non-zero smallest isolated subgroup of \( \Gamma \) and:

\[
H = \{ f \in R \mid \nu(f) \not\in \Gamma_1 \}.
\]

Let \( T = \varphi^{-1}(\sigma(S)) \), this is a local subring \( R \) of maximal ideal \( \varphi^{-1}(\sigma(\mathfrak{m})) = \mathfrak{m} \cap T \).

Then, \( T \) contains \( x_1, \ldots, x_l \) and:

\[
T/(\mathfrak{m} \cap T) \simeq k.
\]

Since \( k ((x_1, \ldots, x_{l-1})) \hookrightarrow k ((x_1, \ldots, x_{l-1}))[x_l]/\mathfrak{H} \), we can apply Theorem 6.1. We end the proof in the same way that the proof of Theorem 8.1 of [9].

**Theorem 6.5.** — Let \((S, \mathfrak{m}, k)\) be an equicharacteristic quasi-excellent local domain of quotient field \( L \) and \( \mu \) be a valuation of \( L \) of rank 1 and of value group \( \Gamma_1 \), centered on \( S \).

Write \( u = (u_1, \ldots, u_n) \) a minimal set of generators of \( \mathfrak{m} \) and \( \overline{H} \) the implicit prime ideal of \( \widehat{S} \).
Write $u = (y, x)$ such that $x = (x_1, \ldots, x_l)$, $l = \text{emb.dim} \left( \frac{\hat{S}}{H} \right)$ and the images of $x_1, \ldots, x_l$ in $\hat{S}/\overline{H}$ induce a minimal set of generators of $(m\hat{S})/\overline{H}$. Suppose that the local uniformization property of dimension $l-1$ is true for all local domain and $k ((x_1, \ldots, x_{l-1})) \twoheadrightarrow k ((x_1, \ldots, x_{l-1})) [x_l]/\overline{H}$ is defectless. Let $f_1, \ldots, f_s \in m$ such that $\mu (f_1) = \min_{1 \leq j \leq s} \{ \mu (f_j) \}$. Then $S$ and $f_1, \ldots, f_s$ have the local uniformization property of dimension $l$.

In other words, $\mu$ admits an embedded local uniformization in the sense of Property 2.11 of [9].

Proof: Take the notations of Theorem 6.4. We saw that there exists a surjective morphism:

$$\psi : \hat{S} \rightarrow \frac{\hat{S}}{\overline{H}} \simeq R/H.$$ 

Since $k ((x_1, \ldots, x_{l-1})) \twoheadrightarrow k ((x_1, \ldots, x_{l-1})) [x_l]/\overline{H}$ is defectless, by Theorem 6.4, after an auxiliary local framed sequence, we can suppose that $\hat{S}/\overline{H}$ is regular and so that $R/H \simeq k [[x_1, \ldots, x_l]]$. The end of the proof is the same as the proof of Theorem 8.3 of [9].

Corollary 6.6. — Let $(S, m, k)$ be a quasi-excellent local domain of quotient field $L$ and $\mu$ be a valuation of $L$ of rank 1 centered on $S$, such that $\text{car} (k_\mu) = 0$. Then $\mu$ admits an embedded local uniformization in the sense of Property 2.11 of [9].

Proof: Write $u = (u_1, \ldots, u_n)$ a minimal set of generators of $m$ and $\overline{H}$ the implicit prime ideal of $\hat{S}$. Write $u = (y, x)$ such that $x = (x_1, \ldots, x_l)$, $l = \text{emb.dim} \left( \frac{\hat{S}}{H} \right)$ and the images of $x_1, \ldots, x_l$ in $\hat{S}/\overline{H}$ induce a minimal set of generators of $(m\hat{S})/\overline{H}$. By Theorem 2.1 of [5], $\mu$ etends uniquely in a valuation $\hat{\mu}$ centered on $\frac{\hat{S}}{\overline{H}}$ and, since $\text{car} (k_\mu) = 0$, then $\text{car} (k_{\hat{\mu}}) = 0$. We saw that there exists a surjective morphism:

$$\psi : \hat{S} \rightarrow \frac{\hat{S}}{\overline{H}} \simeq R/H,$$

where $H = \ker \psi$. The quotient field of $R$ is of the form $k ((x_1, \ldots, x_l))$. By Proposition 2.10, we deduce that the valued fields $(k ((x_1, \ldots, x_{j-1})) \hat{\mu} | _{k ((x_1, \ldots, x_{j-1}))})$ are defectless. To conclude, it is sufficient to apply recursively on $j \in \{1, \ldots, n\}$ the Theorem 6.5 for $(k ((x_1, \ldots, x_{j-1})) \hat{\mu} | _{k ((x_1, \ldots, x_{j-1}))})$.

References

[1] Steven Dale Cutkosky. Ramification of valuations and counterexamples to local monomialization in positive characteristic, 2014. preprint math.AG/arXiv:1404.7459.

[2] Charles Favre and Mattias Jonsson. The valuative tree, volume 1853 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2004.

[3] F. J. Herrera Govantes, W. Mahboub, M. A. Olalla Acosta, and M. Spivakovsky. Key polynomials for simple extensions of valued fields, 2014. preprint math.AG/arXiv:1406.0657.
[4] F. J. Herrera Govantes, M. A. Olalla Acosta, and M. Spivakovsky. Valuations in algebraic field extensions. J. Algebra, 312(2):1033–1074, 2007.
[5] F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, and B. Teissier. Extending a valuation centered in a local domain to the formal completion. Proc. London Math. Soc., 105(3):571–621, 2012.
[6] Franz-Viktor Kuhlmann. Valuation Theory. http://math.usask.ca/ fvk/Fvkbook.htm, 2011.
[7] Wael Mahboub. Key Polynomials. J. Pure Appl. Algebra, 217(6):989–1006, 2013.
[8] Wael Mahboub. Une construction explicite de polynômes-clés pour des valuations de rang fini. Thèse de Doctorat, Institut de Mathématiques de Toulouse, 2013.
[9] Jean-Christophe San Saturnino. Uniformisation locale des schémas quasi-excellents de caractéristique nulle. arXiv:1311.3525, 2013. Soumis.
[10] Michel Vaquié. Extension d’une valuation. Trans. Amer. Math. Soc., 359(7):3439–3481 (electronic), 2007.
[11] Michel Vaquié. Famille admissible de valuations et défaut d’une extension. J. Algebra, 311(2):859–876, 2007.
[12] Michel Vaquié. Extensions de valuation et polygone de Newton. Ann. Inst. Fourier (Grenoble), 58(7):2503–2541, 2008.