Abstract

The question if conserved currents can be sensibly defined in supersymmetric minisuperspaces is investigated in this essay. The objective is to employ exclusively the differential equations obtained directly from the Lorentz and supersymmetry quantum constraints. The “square-root” structure of N=1 supergravity is the motivation to contemplate this tempting idea. However, it is shown that such prospect is not feasible but for some very simple scenarios. Otherwise, conserved currents (and consistent probability densities) can be derived from subsequent Wheeler-DeWitt like equations obtained from the supersymmetric algebra of constraints.

Philosophers, theologians and scientists have long been pondering on the origin, evolution and purpose of our universe [1]. Einstein’s theory of general relativity, quantum mechanics and particle physics models represent overwhelming breakthroughs towards providing answers to those longstanding questions [2]-[4]. A natural development would be a theory of quantum gravity, which constitutes one of the foremost aspirations in theoretical physics.

The purpose of quantum gravity is to apply the principles of quantum mechanics to the entire universe. Basically, one has to adjoin definite laws of initial conditions with suitable laws governing its evolution. Several approaches have been provided [5]–[8] and their conceivable aim is to allow for a complete explanation of all cosmological observations.

The inclusion of supersymmetry may also yield significant benefits. Supersymmetry is a transformation which relates bosons and fermions [9, 10]. Its promotion to a gauge symmetry resulted in an elegant field theory: supergravity [9, 10]. Supersymmetry may play an important role when dealing divergences in quantum gravity [11] and removing Planckian masses induced by wormholes [12, 13]. Furthermore, it would be adequate to consider bosons and their fermionic partners on an equivalent level when studying the very early universe.

N=1 supergravity [9, 10] constitutes a “square-root” [14] of gravity: it is sufficient to just solve the Lorentz and supersymmetry constraints [15, 16]. The algebra of constraints implies that a physical wave functional Ψ will consequently obey the Hamiltonian constraints[7]. The supersymmetry and Lorentz constraints lead to differential equations, which are of first-order in the bosonic variables. Such relation temptingly suggests that the possibility to derive sensible...
conserved currents (and positive-definite probability densities) should be explored, similarly to the procedure intertwining the Klein-Gordon and Dirac equations \[15\].

The Wheeler-DeWitt equation has associated with it a conserved current \[13\] \( J \sim \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \). It satisfies \( \nabla J = 0 \), where \( \nabla \) is the Laplacian in superspace \[19\]. A conserved probability can be defined from \( J \) but it can be afflicted with negative values \[3, 4, 13, 21\]. A possible solution requires a suitable choice of hypersurfaces. Such procedure works only within a semiclassical minisuperspace approximation\[7, 21, 22\].

A closed FRW supersymmetric minisuperspace model with a scalar multiplet will be used in this essay. Several improvements concerning previous results in ref. \[23, 24, 25\] are included as well. The action of the more general theory of N=1 supergravity in the presence of gauged supermatter (see eq. (25.12) in ref. \[10\]) will be employed. The metric variables are represented by the tetrad: \( e_{\alpha \mu} = \text{diag}(N(\tau), a E_{\alpha i}) \). Here \( \hat{a} \) and \( i \) run from 1 to 3 and \( E_{\alpha i} \) is a basis of left-invariant 1-forms on the unit 3-sphere with volume \( \sigma^2 = 2\pi^2 \). This reduces the number of degrees of freedom provided by \( e_{AA'\mu} \), the spinorial form of the tetrad. A suitable choice for the gravitino fields, \( \tilde{\psi}^A \) and \( \tilde{\psi}^{A'} \), is then required. Hence \( \tilde{\psi}^A_0 \) and \( \tilde{\psi}^{A'}_0 \) will be functions of time only and \( \tilde{\psi}^A_i = e^{\Lambda A' i} \tilde{\psi}_{A'}(t) \), \( \tilde{\psi}^{A'}_i = e^{\Lambda A' i} \tilde{\psi}_{A}(t) \), where the new spinors \( \tilde{\psi}_{A} \) and their Hermitian conjugate \( \tilde{\psi}_{A}^{\dagger} \) are introduced \[17, 23, 24, 27\]. The scalar supermultiplet, consisting of a complex scalar field \( \phi, \bar{\phi} \) and spin-\( \frac{1}{2} \) fields \( \chi_A, \bar{\chi}_A \), are chosen to depend only on time. The remaining fields are taken henceforth to be zero.

The analysis becomes simpler if the fermionic fields, \( \chi_A, \psi_A \), are redefined as follows: \( \hat{\chi}_A = \frac{\sigma a^2}{2\pi} (1 + \phi \phi) \chi_A \), \( \hat{\psi}_A = \frac{\sqrt{2}}{2\pi} \sigma a^2 \psi_A \), and similarly for their Hermitian conjugates. In addition, unprimed spinors will be used: \( \tilde{\psi}_A = 2n_{A B'} \tilde{\psi}_{B'} \), \( \tilde{\chi}_A = 2n_{A B'} \tilde{\chi}_{B'} \). The coordinates of the configuration space are chosen to be \( (\chi_A, \psi_A, \phi, \bar{\phi}) \) and \( (\tilde{\chi}_A, \tilde{\psi}_A, \pi_a, \pi_\phi, \tilde{\pi}_\bar{\phi}) \) form the momentum operators in this representation. Quantum mechanically (with \( \hbar = 1 \)):

\[
\begin{align*}
\hat{\chi}^A &\rightarrow -\frac{\partial}{\partial \chi^A}, \\
\hat{\psi}_A &\rightarrow \frac{\partial}{\partial \psi_A}, \\
\pi_a &\rightarrow \frac{\partial}{\partial a}, \\
\pi_\phi &\rightarrow -i \frac{\partial}{\partial \phi}, \\
\tilde{\pi}_\bar{\phi} &\rightarrow -i \frac{\partial}{\partial \bar{\phi}}.
\end{align*}
\] (1)

The Lorentz constraint \( J_{AB} = 0 \) takes the form

\[
J_{AB} = \psi(A \tilde{\psi}_B) - \chi(A \tilde{\chi}_B) = 0.
\] (2)

This constraint implies that the most general form for the wave function of the universe is

\[
\Psi = A + B \psi^C \psi_C + C \psi^C \chi_C + D \chi^C \chi_C + E \psi^C \psi_C \chi^D \chi_D
\] (3)

where \( A, B, C, D, E \) are functions of \( a, \phi, \bar{\phi} \) only.

The following eight equations result from the application of the supersymmetry constraints \( S_A, \bar{S}_A \) on \( \Psi \) given by (3) (see ref. [24]):

\[
\begin{align*}
\frac{a}{2\sqrt{6}} \frac{\partial A}{\partial a} + \sqrt{\frac{3}{2}} \sigma^2 a^2 A &= 0, \\
\frac{a}{\sqrt{6}} \frac{\partial E}{\partial a} - \sqrt{6} \sigma^2 a^2 E &= 0, \\
\frac{\partial A}{\partial r} - i \frac{\partial A}{r \partial \theta} &= 0, \\
\frac{\partial E}{\partial r} + i \frac{\partial E}{r \partial \theta} &= 0.
\end{align*}
\] (4,5)

\(^2\)This is probably sufficient for all practical purposes. In the semiclassical case, the wave function is of the WKB form \( \Psi \sim C e^{-I} \), where \( I \) and \( C \) are both complex, \( I = I_R - iS \) and \( |\nabla S| \gg |\nabla I_R| \). I.e., \( S \) will be an approximate solution of the Lorentzian Hamilton-Jacobi equation. In particular, \( \nabla J = 0 \) is obtained with \( J \sim e^{-I_R} |C^2| \nabla S \) [3, 19].
(1 + \phi \bar{\phi}) \frac{\partial B}{\partial \phi} + \frac{1}{2} \phi B + \frac{a}{4 \sqrt{3}} \frac{\partial C}{\partial a} - \frac{7}{4 \sqrt{3}} C + \frac{\sqrt{3}}{2} \sigma^2 a^2 C = 0 , \quad (6)
\frac{a}{\sqrt{3}} \frac{\partial B}{\partial a} - 2 \sqrt{3} \sigma^2 a^2 B - \sqrt{3} B + (1 + \phi \bar{\phi}) \frac{\partial C}{\partial \phi} + \frac{3}{4} \sigma^2 C = 0 , \quad (7)
\frac{a}{\sqrt{3}} \frac{\partial D}{\partial a} + 2 \sqrt{3} \sigma^2 a^2 D - \sqrt{3} D - (1 + \phi \bar{\phi}) \frac{\partial C}{\partial \phi} - \frac{3}{4} \sigma^2 C = 0 , \quad (8)
(1 + \phi \bar{\phi}) \frac{\partial D}{\partial \phi} + \frac{1}{2} \phi D - \frac{a}{4 \sqrt{3}} \frac{\partial C}{\partial a} + \frac{7}{4 \sqrt{3}} C + \frac{\sqrt{3}}{2} \sigma^2 a^2 C = 0 . \quad (9)

Notice that \( r^2 \equiv \phi \bar{\phi} \) with \( \phi \equiv re^{i\theta} \) was employed in eq. (9).

Eq. (4), (5) constitute decoupled equations for \( A \) and \( E \). Eq. (8) and (9) constitute coupled equations between \( B \) and \( C \), while eq. (6), (7) are coupled equations between \( C \) and \( D \). It can be shown that eq. (8)-(9) imply \( C = 0 \) \[24, 25, 26\]. A two-dimensional spherically symmetric Kähler geometry \[10\] has been chosen here but this result seems independent of that choice \[24\].

The following steps are now followed. Multiply the first equation in (4) by \( \bar{\lambda} \), then the second by \( A \) and subtract them. Employing now \( C = 0 \), multiply eq. (6) by \( \bar{D} \) and eq. (8) by \( B \). Then add them. Finally, multiply eq. (3) by \( D \), eq. (4) by \( B \) and subtract them. The overall result is written as

\[
\frac{\partial (A \cdot E)}{\partial a} + \frac{\partial (A \cdot E)}{\partial \theta} - i r \left( \frac{\partial E}{\partial r} A - \frac{\partial A}{\partial r} E \right) = 0 , \quad (10)
\]
\[
D_a (B \cdot D) + \frac{\partial (B \cdot D)}{\partial \theta} - i r \left( \frac{\partial B}{\partial r} D - \frac{\partial D}{\partial r} B \right) = 0 , \quad (11)
\]

with the generalized derivative \( D_a \equiv \partial_a - \frac{\sigma \partial \theta}{\sigma_r} \).

From eq. (4), (5) and eq. (3)-(4) with \( \phi = re^{i\theta} \) the quantum state corresponding to a \( k = 1 \) FRW supersymmetric model with scalar supermultiplets is given by

\[
\Psi = c_1 r^{\lambda_1} e^{-i\lambda_1 \theta} e^{-3\sigma^2 a^2} + c_2 a^{3r^3} e^{-i\lambda_3 \theta} (1 + \epsilon^2)^{\frac{3}{2}} e^{3\sigma^2 a^2} \psi C \psi C + c_4 a^2 r^{\lambda_4} e^{i\lambda_4 \theta} (1 + r^2)^{\frac{3}{2}} e^{-3\sigma^2 a^2} \chi^2 \chi C + c_5 r^{\lambda_5} e^{i\lambda_5 \theta} e^{3\sigma^2 a^2} \psi C \psi C \chi^2 \chi^2 , \quad (12)
\]

where \( \lambda_1...\lambda_4 \) and \( c_1...c_4 \) are constants. The exponential factors \( e^{\pm 3\sigma^2 a^2} \) in (12) are to be viewed as \( e^I \), where \( I \) is the Euclidean action for a classical solution without matter outside or inside a three-sphere with radius \( a \) (see ref. [23]). In the absence of matter, the Hartle-Hawking state [3] for this model is therefore given by \( \Psi_{HH} = \psi A \psi^A e^{3\sigma^2 a^2} \). A solution \( \Psi_{WH} = e^{-3\sigma^2 a^2} \) bears quantum wormhole properties [13, 20, 21]. However, the full physical interpretation of the bosonic coefficients in (12) is less clear. \( C = 0 \) seems to imply that while a fermionic state \( \chi^A \psi_A \) is allowed by Lorentz invariance, supersymmetry effectively rules it out since the spin-\( \frac{1}{2} \) \( \chi^A \), \( \psi_A \) fields have different roles. In addition, the scalar field dependence is different from the expressions in non-supersymmetric quantum FRW models with complex scalar fields (cf. ref. [28]).

The presence of the last term in both eq. (10), (11) clearly prevent us to associate them with an equation of the type of \( \nabla J = 0 \). Notice that eq. (3) and (4) lead directly to

\[
\frac{\partial (A \cdot E)}{\partial \theta} - i r \left( \frac{\partial E}{\partial r} A - \frac{\partial A}{\partial r} E \right) = 0 \quad \text{and} \quad \frac{\partial (B \cdot D)}{\partial \theta} - i r \left( \frac{\partial B}{\partial r} D - \frac{\partial D}{\partial r} B \right) = 0 , \quad \text{respectively. This feature is}
\]

\footnote{Using \( \phi, \bar{\phi} \) does not allow to find the explicit dependence of \( \Psi \) (see ref. [14, 24, 26, 25]). An alternative approach is to write \( \phi = re^{i\theta} \) and hence to effectively decouple the two degrees of freedom associated with the complex scalar field. It should be stressed that this procedure has not yet been employed directly in the supersymmetry constraints but rather on the Hamiltonian constraints [3, 28].}
Moreover, eq. (14) is basically translated into the last two terms in eq. (10), (11). Hence, non-invariant terms in (13) and (14) are also a direct consequence of the Lorentz and supersymmetry constraints. A relation as \( \nabla A \psi_0 + \nabla A \bar{\psi}_0 = 0 \) can only be achieved for simple cases of pure N=1 supergravity, where eq. (10) become reduced to just (4). Consequently, we obtain from the supersymmetry constraints equations. Only for very simple scenarios does this becomes possible. Our results should then be compared with the assertions present in ref. [29, 30, 31]. A Wheeler-DeWitt–like equation becomes mandatory when supersymmetry implies a mixing between the fermionic sectors in \( \Psi \). Additionally, it should also be stressed that the wave functional

\[
\pi_r = 2 \frac{\partial r}{\partial t} \left( 2(1 + r^2) \frac{e^{-i\theta}}{\sqrt{2(1 + r^2)^2}} \left( \chi^A \psi_0 + 3n_{AA'} \chi^A \bar{\psi}^A \right) \right)
\]

and notice that the usual procedure \( \mathcal{H} \sim p \dot{q} - L \) involves a term like \( \sigma^2 a^3 \left[ \left( \frac{\partial r}{\partial t} \right)^2 + i r^2 \left( \frac{\partial \theta}{\partial t} \right)^2 \right] \). It is precisely the last two terms in (13) and (14) that allow the supersymmetry constraints to be obtained explicitly from the coefficients in \( \psi_0, \bar{\psi}_0^A \) in the Hamiltonian \( \mathcal{H} \). But the last two terms in both (13) and (14) are also a direct consequence of the non-invariant terms in the action (10). Moreover, eq. (14) is basically translated into the last two terms in eq. (10). Hence, \( \theta \) no longer being a cyclical coordinate implies that a relation as \( \nabla J = 0 \) cannot be sensibly defined. Furthermore, this fact is inherited from local supersymmetry being now a feature of the reduced model. Thus, it seems then that there is close relation between the absence of cyclical coordinates, conserved currents from \( \Psi \) and the presence of supersymmetry.

No conserved currents are possible to obtain in our supersymmetric minisuperspace directly from the Lorentz and supersymmetry constraints. A relation as \( \nabla J = 0 \) can only be achieved for the simple case of pure N=1 supergravity, where eq. (10)–(14) are reduced to just (4). Consequently, we obtain \( 2(\sigma A B) \frac{\partial \psi_0}{\partial t} = 0 \) as expected.

Overall, the final message in this essay is the following. Conserved currents do not seem possible to obtain directly from the supersymmetry constraints equations. Only for very simple scenarios does this becomes possible. Our results should then be compared with the assertions present in ref. [29, 30, 31]. A Wheeler-DeWitt–like equation becomes mandatory when supersymmetry implies a mixing between the fermionic sectors in \( \Psi \). Additionally, it should also be stressed that the wave functional...
\[ \Psi(e_{\mu}^{AA^\prime}; \psi_{\mu}; \ldots) \] for \( N=1 \) quantum supergravity is a Grassman-algebra-valued functional and thus quite different from relativistic quantum mechanics wave functions.

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