Time-frequency Analysis of two-wavelet theory in Weinstein setting
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Abstract
In this paper, we introduce the notion of Weinstein two-wavelet and we define the
two-wavelet localization operators in the setting of the Weinstein theory. Then we give
a host of sufficient conditions for the boundedness and compactness of the two-wavelet
localization operator on \( L^p_\alpha(\mathbb{R}^{d+1}) \) for all \( 1 \leq p \leq \infty \), in terms of properties of the
symbol \( \sigma \) and the functions \( \varphi \) and \( \psi \). In the end, we study some typical examples of
the Weinstein two-wavelet localization operators.

Keywords. Weinstein operator; Weinstein wavelet transform; Weinstein two-wavelet transform; Time-frequency Analysis; localization operators.

Mathematics Subject Classification. Primary 44A05; Secondary 42B10

1 Introduction
The Weinstein operator \( \Delta_{W,\alpha}^d \) defined on \( \mathbb{R}^{d+1} = \mathbb{R}^d \times (0, \infty) \), by
\[
\Delta_{W,\alpha}^d \varphi = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2} x_j + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} \varphi = \Delta_d + L_\alpha, \quad \alpha > -1/2,
\]
where \( \Delta_d \) is the Laplacian operator for the \( d \) first variables and \( L_\alpha \) is the Bessel operator
for the last variable defined on \( (0, \infty) \) by
\[
L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}}.
\]
The Weinstein operator \( \Delta_{W,\alpha}^d \) has several applications in pure and applied mathematics,
especially in fluid mechanics [6, 39].

Very recently, many authors have been investigating the behaviour of the Weinstein transform (2.5) with respect to several problems already studied for the classical Fourier transform. For instance, Heisenberg-type inequalities [29], Littlewood-Paley g-function [31], Shapiro and Hardy\-L"{o}wner type inequalities [28, 30], Paley-Wiener theorem [21], Uncertainty principles [24, 32, 37], multiplier Weinstein operator [33], wavelet and
continuous wavelet transform [15, 23], Wigner transform and localization operators [34, 36],
and so forth...

In the classical setting, the notion of wavelets was first introduced by Morlet in connection
with his study of seismic traces and the mathematical foundations were given by Grossmann and Morlet [18]. Later, Meyer and many other mathematicians recognized many classical
results of this theory \[20\], \[26\]. Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics \[11\], \[16\], \[19\].

Recently, the theory of wavelets and continuous wavelet transform has been extended and generalized in the context of differential-difference operators \[15\], \[22\], \[23\], \[25\]. Wavelet analysis has attracted attention for its ability to analyze rapidly changing transient signals. Any application using the Fourier like transform can be formulated using wavelets to provide more accurately localized temporal and frequency information. The reason for the extension from one wavelet to two wavelets comes from the extra degree of flexibility in signal analysis and imaging when the localization operators are used as time-varying filters. One of the aims of the continuous wavelet transform, is the study of their localization operators.

The time-frequency representations required for localization operators wish have been object of study in quantum mechanics, in PDE and signal analysis recently. In engineering, a natural language is given by time-frequency analysis. Localization operators arise from pure and applied mathematics in connection with various areas of research. They were initiated by Daubechies \[9\], \[10\], \[12\], and before she highlighted the role of these operators to localize a signal simultaneously in time and frequency.

Nowadays, these operators have found many applications to time-frequency analysis, the theory of differential equations, quantum mechanics. Depending on the field of application, these operators are known under the names of Wick, anti-Wick or Toeplitz operators, as well as wave packets, Gabor or short time Fourier transform multipliers. Arguing from these point of view, many works were done on them, we refer, for instance \[1\], \[2\], \[8\], \[13\], \[17\], \[22\].

Using the harmonic analysis associated with the Weinstein operator (generalized translation operators, generalized convolution, Weinstein transform, ...) and the same idea as for the classical case, we study the localisations operators associated with the Weinstein two-wavelet \[35\] and we prove that under suitable conditions on the symbols and two Weinstein wavelets, the boundedness and compactness of these localization operators. Our main results for the boundedness and compactness of the Weinstein two wavelet localization operators, with different symbols and windows, are summarized in the following table.

| Symbol $\sigma$ | Windows $\varphi$ | $\psi$ | Localization Operator $L_{\varphi,\psi}(\sigma)$ |
|-----------------|------------------|--------|-----------------------------------------------|
| $L_0^1(\mathcal{X})$ | $L_0^\infty(\mathbb{R}^{d+1}_+)$ | $L_0^1(\mathbb{R}^{d+1}_+)$ | $B(L_0^\infty(\mathbb{R}^{d+1}_+))$ |
| $L_0^1(\mathcal{X})$ | $L_0^1(\mathbb{R}^{d+1}_+)$ | $L_0^\infty(\mathbb{R}^{d+1}_+)$ | $B(L_0^1(\mathbb{R}^{d+1}_+))$ |
| $L_0^1(\mathcal{X})$ | $L_0^1 \cap L_0^\infty(\mathbb{R}^{d+1}_+)$ | $B(L_0^1(\mathbb{R}^{d+1}_+))$, $p \in [1, \infty)$ |
| $L_0^1(\mathcal{X})$ | $L_0^1(\mathbb{R}^{d+1}_+)$ | $L_0^\infty(\mathbb{R}^{d+1}_+)$ | $B(L_0^\infty(\mathbb{R}^{d+1}_+))$, $p \in [1, \infty)$ |
| $L_0^1(\mathcal{X})$, $r \in [1, 2]$ | $L_0^1 \cap L_0^1 \cap L_0^\infty(\mathbb{R}^{d+1}_+)$ | $B(L_0^p(\mathbb{R}^{d+1}_+))$, $p \in [r, r']$ |

Table 1: Boundedness and compactness of localization operators

This paper is organized as follows. In Section 2, we recall some properties of harmonic analysis for the Weinstein operators and Weinstein two-wavelet theory. In Section 3, we give a host of sufficient conditions for the boundedness and compactness of the two-wavelet localization operator on $L_p^0(\mathbb{R}^{d+1}_+)$ for all $1 \leq p \leq \infty$, in terms of properties of the symbol $\sigma$ and the functions $\varphi$ and $\psi$. In the end, we study some typical examples of the Weinstein two-wavelet localization operators.
2 Preliminaires

2.1 Harmonic analysis associated with the Weinstein operator

For all $\lambda = (\lambda_1, \ldots, \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system

$$\begin{align*}
\frac{\partial^2 u}{\partial x_j^2}(x) &= -\lambda_j^2 u(x), \quad \text{if } 1 \leq j \leq d \\
L_\alpha u(x) &= -\lambda_{d+1}^2 u(x),
\end{align*}$$

(2.1)

$$u(0) = 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \leq j \leq d$$

has a unique solution denoted by $\Lambda^d_\alpha(\lambda, \cdot)$, and given by

$$\Lambda^d_\alpha(\lambda, x) = e^{-i<x',\lambda'>}j_\alpha(x_{d+1}\lambda_{d+1})$$

(2.2)

where $x = (x', x_{d+1})$, $x' = (x_1, x_2, \ldots, x_d)$, $\lambda = (\lambda', \lambda_{d+1})$, $\lambda'_d = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $j_\alpha$ is the normalized Bessel function of index $\alpha$ defined by

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Gamma(\alpha + k + 1)}.$$

The function $(\lambda, x) \mapsto \Lambda^d_\alpha(\lambda, x)$ is called the Weinstein kernel and has a unique extension to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$, and satisfied the following properties.

(i) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda^d_\alpha(\lambda, x) = \Lambda^d_\alpha(x, \lambda).$$

(ii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda^d_\alpha(\lambda, -x) = \Lambda^d_\alpha(-\lambda, x).$$

(iii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we get

$$\Lambda^d_\alpha(\lambda, 0) = 1.$$

(iv) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}$ and $\lambda \in \mathbb{C}^{d+1}$ we have

$$\left| D^\nu_{\lambda} \Lambda^d_\alpha(\lambda, x) \right| \leq ||\nu|| ||x|| e^{||x|| \Im \lambda}$$

where $D^\nu_{\lambda} = \partial^\nu / (\partial \lambda_1^{\nu_1} \ldots \partial \lambda_{d+1}^{\nu_{d+1}})$ and $|\nu| = \nu_1 + \ldots + \nu_{d+1}$. In particular, for all $(\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, we have

$$\left| \Lambda^d_\alpha(\lambda, x) \right| \leq 1.$$

(2.3)

In the following we denote by

(i) $-\lambda = (-\lambda', \lambda_{d+1})$

(ii) $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
(iii) \( S_\alpha(\mathbb{R}^{d+1}) \), the space of the \( C^\infty \) functions, even with respect to the last variable, and rapidly decreasing together with their derivatives.

(iv) \( S_\alpha(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}) \), the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \) even with respect to the last two variables.

(v) \( D_\alpha(\mathbb{R}^{d+1}) \), the space of \( C^\infty \)-functions on \( \mathbb{R}^{d+1} \) which are of compact support, even with respect to the last variable.

(vi) \( L^p_\alpha(\mathbb{R}^{d+1}) \), \( 1 \leq p \leq \infty \), the space of measurable functions \( f \) on \( \mathbb{R}^{d+1} \) such that

\[
\|f\|_{\alpha,p} = \left( \int_{\mathbb{R}^{d+1}} |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p} < \infty, \quad p \in [1, \infty),
\]

\[
\|f\|_{\alpha,\infty} = \text{ess sup}_{x \in \mathbb{R}^{d+1}} |f(x)| < \infty,
\]

where \( d\mu_\alpha(x) \) is the measure on \( \mathbb{R}^{d+1} = \mathbb{R}^d \times (0, \infty) \) given by

\[
d\mu_\alpha(x) = \frac{x_2^{2\alpha+1}}{(2\pi)^{2d/2}\Gamma(\alpha + 1)} \, dx.
\]

For a radial function \( \varphi \in L^1_\alpha(\mathbb{R}^{d+1}) \) the function \( \tilde{\varphi} \) defined on \( \mathbb{R}_+ \) such that \( \varphi(x) = \tilde{\varphi}(|x|) \), for all \( x \in \mathbb{R}_+^{d+1} \), is integrable with respect to the measure \( r^{2\alpha+d+1} dr \), and we have

\[
\int_{\mathbb{R}_+^{d+1}} \varphi(x) d\mu_\alpha(x) = a_\alpha \int_0^\infty \tilde{\varphi}(r) r^{2\alpha+d+1} dr,
\]

where

\[
a_\alpha = \frac{1}{2^{\alpha+d/2}\Gamma(\alpha + d/2 + 1)}.
\]

The Weinstein transform generalizing the usual Fourier transform, is given for \( \varphi \in L^1_\alpha(\mathbb{R}_+^{d+1}) \) and \( \lambda \in \mathbb{R}_+^{d+1} \), by

\[
\mathcal{F}_W(\varphi)(\lambda) = \int_{\mathbb{R}_+^{d+1}} \varphi(x) \Lambda_\alpha^d(x,\lambda) d\mu_\alpha(x),
\]

We list some known basic properties of the Weinstein transform are as follows. For the proofs, we refer [3, 27].

(i) For all \( \varphi \in L^1_\alpha(\mathbb{R}_+^{d+1}) \), the function \( \mathcal{F}_W(\varphi) \) is continuous on \( \mathbb{R}_+^{d+1} \) and we have

\[
\|\mathcal{F}_W\varphi\|_{\alpha,\infty} \leq \|\varphi\|_{\alpha,1}.
\]

(ii) The Weinstein transform is a topological isomorphism from \( S_\alpha(\mathbb{R}^{d+1}) \) onto itself. The inverse transform is given by

\[
\mathcal{F}_W^{-1}\varphi(\lambda) = \mathcal{F}_W\varphi(-\lambda), \text{ for all } \lambda \in \mathbb{R}_+^{d+1}.
\]
(iii) For all \( f \in D_+(\mathbb{R}^{d+1}) \) (resp. \( S_+(\mathbb{R}^{d+1}) \)), we have the following relations

\[
\forall \lambda \in \mathbb{R}_+^{d+1}, \quad \mathcal{F}_W(\varphi)(\lambda) = \mathcal{F}_W(\tilde{\varphi})(\lambda), \tag{2.8}
\]

\[
\forall \lambda \in \mathbb{R}_+^{d+1}, \quad \mathcal{F}_W(\tilde{\varphi})(\lambda) = \mathcal{F}_W(\varphi)(-\lambda), \tag{2.9}
\]

where \( \tilde{\varphi} \) is the function defined by

\[
\forall \lambda \in \mathbb{R}_+^{d+1}, \quad \tilde{\varphi}(\lambda) = \varphi(-\lambda).
\]

(iv) Parseval’s formula: For all \( \varphi, \phi \in S_+(\mathbb{R}^{d+1}) \), we have

\[
\int_{\mathbb{R}_+^{d+1}} \varphi(x)\overline{\phi(x)}d\mu_\alpha(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W(\varphi)(x)d\mathcal{F}_W(\phi)(x)d\mu_\alpha(x). \tag{2.10}
\]

(v) Plancherel’s formula: For all \( \varphi \in L^2_\alpha(\mathbb{R}_+^{d+1}) \), we have

\[
\|\mathcal{F}_W\varphi\|_{\alpha, 2} = \|\varphi\|_{\alpha, 2}. \tag{2.11}
\]

(vi) Plancherel Theorem: The Weinstein transform \( \mathcal{F}_W \) extends uniquely to an isometric isomorphism on \( L^2_\alpha(\mathbb{R}_+^{d+1}) \).

(vii) Inversion formula: Let \( \varphi \in L^1_\alpha(\mathbb{R}_+^{d+1}) \) such that \( \mathcal{F}_W\varphi \in L^1_\alpha(\mathbb{R}_+^{d+1}) \), then we have

\[
\varphi(\lambda) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W\varphi(x)\Lambda^d_\alpha(-\lambda, x)d\mu_\alpha(x), \text{ a.e. } \lambda \in \mathbb{R}_+^{d+1}. \tag{2.12}
\]

Using relations (2.6) and (2.11) with Marcinkiewicz’s interpolation theorem \cite{38} we deduce that for every \( \varphi \in L^p_\alpha(\mathbb{R}_+^{d+1}) \) for all \( 1 \leq p \leq 2 \), the function \( \mathcal{F}_W(\varphi) \in L^q_\alpha(\mathbb{R}_+^{d+1}) \), \( q = \frac{p}{(p-1)} \), and

\[
\|\mathcal{F}_W\varphi\|_{\alpha, q} \leq \|\varphi\|_{\alpha, p}. \tag{2.13}
\]

**Definition 2.1.** The translation operator \( \tau_x^\alpha \), \( x \in \mathbb{R}_+^{d+1} \) associated with the Weinstein operator \( \Delta_{W, \alpha}^d \), is defined for a continuous function \( \varphi \) on \( \mathbb{R}_+^{d+1} \), which is even with respect to the last variable and for all \( y \in \mathbb{R}_+^{d+1} \) by

\[
\tau_x^\alpha \varphi(y) = C_\alpha \int_0^\pi \varphi(x' + y', \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos \theta})(\sin \theta)^{2\alpha}d\theta,
\]

with

\[
C_\alpha = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)}.
\]

By using the Weinstein kernel, we can also define a generalized translation, for a function \( \varphi \in S_+(\mathbb{R}^{d+1}) \) and \( y \in \mathbb{R}_+^{d+1} \) the generalized translation \( \tau_x^\alpha \varphi \) is defined by the following relation

\[
\mathcal{F}_W(\tau_x^\alpha \varphi)(y) = \Lambda^d_\alpha(x, y)\mathcal{F}_W(\varphi)(y). \tag{2.14}
\]

In the following proposition, we give some properties of the Weinstein translation operator:
Proposition 2.2. The translation operator $\tau_x^\alpha$, $x \in \mathbb{R}_+^{d+1}$ satisfies the following properties.

i). For $\varphi \in \mathcal{C}_s(\mathbb{R}_+^{d+1})$, we have for all $x, y \in \mathbb{R}_+^{d+1}$

$$\tau_x^\alpha \varphi(y) = \tau_y^\alpha \varphi(x) \quad \text{and} \quad \varphi \in \mathcal{L}.$$  \hspace{1cm} (2.15)

ii). Let $\varphi \in L_0^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq \infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $\tau_x^\alpha \varphi$ belongs to $L_0^p(\mathbb{R}_+^{d+1})$ and we have

$$||\tau_x^\alpha \varphi||_{a,p} \leq ||\varphi||_{a,p}.$$  \hspace{1cm} (2.16)

Proposition 2.3. Let $\varphi \in L_1^1(\mathbb{R}_+^{d+1})$. Then for all $x \in \mathbb{R}_+^{d+1}$,

$$\int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha \varphi(y)d\mu_\alpha(y) = \int_{\mathbb{R}_+^{d+1}} \varphi(y)d\mu_\alpha(y).$$  \hspace{1cm} (2.17)

Proof. The result comes from combination identities (2.12) and (2.14). \hfill \Box

By using the generalized translation, we define the generalized convolution product $\varphi * \psi$ of the functions $\varphi, \psi \in L_1^1(\mathbb{R}_+^{d+1})$ as follows

$$\varphi * \psi(x) = \int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha \varphi(-y) \psi(y)d\mu_\alpha(y).$$  \hspace{1cm} (2.18)

This convolution is commutative and associative, and it satisfies the following properties.

Proposition 2.4. i) For all $\varphi, \psi \in L_1^1(\mathbb{R}_+^{d+1})$, (resp. $\varphi, \psi \in \mathcal{S}_s(\mathbb{R}_+^{d+1})$), then $\varphi * \psi \in L_1^1(\mathbb{R}_+^{d+1})$, (resp. $\varphi * \psi \in \mathcal{S}_s(\mathbb{R}_+^{d+1})$) and we have

$$\mathcal{F}_\alpha(\varphi * \psi) = \mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi).$$  \hspace{1cm} (2.19)

ii) Let $p, q, r \in [1, \infty]$, such that $1 + \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\varphi \in L_0^p(\mathbb{R}_+^{d+1})$ and $\psi \in L_0^q(\mathbb{R}_+^{d+1})$ the function $\varphi * \psi$ belongs to $L_0^r(\mathbb{R}_+^{d+1})$ and we have

$$||\varphi * \psi||_{a,r} \leq ||\varphi||_{a,p} \ ||\psi||_{a,q}.$$  \hspace{1cm} (2.20)

iii) Let $\varphi, \psi \in L_1^2(\mathbb{R}_+^{d+1})$. Then

$$\varphi * \psi = \mathcal{F}_\alpha^{-1} (\mathcal{F}_\alpha(\varphi) \mathcal{F}_\alpha(\psi)).$$  \hspace{1cm} (2.21)

iv) Let $\varphi, \psi \in L_0^2(\mathbb{R}_+^{d+1})$. Then $\varphi * \psi$ belongs to $L_0^2(\mathbb{R}_+^{d+1})$ if and only if $\mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi)$ belongs to $L_0^2(\mathbb{R}_+^{d+1})$ and we have

$$\mathcal{F}_\alpha(\varphi * \psi) = \mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi).$$  \hspace{1cm} (2.22)

v) Let $\varphi, \psi \in L_0^2(\mathbb{R}_+^{d+1})$. Then

$$||\varphi * \psi||_{a,2} = ||\mathcal{F}_\alpha(\varphi)\mathcal{F}_\alpha(\psi)||_{a,2},$$  \hspace{1cm} (2.23)

where both sides are finite or infinite.
2.2 Weinstein two-wavelet theory

In the following, we denote by 
\[ X = \{(a, x) : x \in \mathbb{R}^{d+1} \text{ and } a > 0\}. \]

\[ L_p^\alpha(X), \ p \in [1, \infty] \]
the space of measurable functions \( \varphi \) on \( X \) such that
\[ ||\varphi||_{L_p^\alpha(X)} = \left( \int_X |\varphi(a, x)|^p d\mu_\alpha(a, x) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \]
\[ ||\varphi||_{L_\infty^\alpha(X)} = \text{ess sup}_{(a, x) \in X} |\varphi(a, x)| < \infty, \]
where the measure \( \mu_\alpha(a, x) \) is defined on \( X \) by
\[ d\mu_\alpha(a, x) = \frac{d\mu_\alpha(x)da}{a^{2\alpha+d+\delta}}. \]

**Definition 2.5.** [15] A classical wavelet on \( \mathbb{R}^{d+1} \) is a measurable function \( \varphi \) on \( \mathbb{R}^{d+1} \) satisfying for almost all \( \xi \in \mathbb{R}^{d+1} \), the condition
\[ 0 < C_\varphi = \int_0^\infty |F_W(\varphi)(a\xi)|^2 \frac{da}{a} < \infty. \]  
(2.24)

We extend the notion of the wavelet to the two-wavelet in Weinstein setting as follows.

**Definition 2.6.** Let \( \varphi \) and \( \psi \) be in \( L_2^\alpha(\mathbb{R}^{d+1}) \). We say that the pair \( (\varphi, \psi) \) is a Weinstein two-wavelet on \( \mathbb{R}^{d+1} \) if the following integral
\[ C_{\varphi,\psi} = \int_0^\infty F_W(\psi)(a\xi)\overline{F_W(\varphi)(a\xi)} \frac{da}{a} \]  
(2.25)
is constant for almost all \( \xi \in \mathbb{R}^{d+1} \) and we call the number \( C_{\varphi,\psi} \) the Weinstein two-wavelet constant associated to the functions \( \varphi \) and \( \psi \).

It is to highlight that if \( \varphi \) is a Weinstein wavelet then the pair \( (\varphi, \psi) \) is a Weinstein two-wavelet, and \( C_{\varphi,\psi} \) coincides with \( C_\varphi \).

Let \( a > 0 \) and \( \varphi \) be a measurable function. We consider the function \( \varphi_a \) defined by
\[ \forall x \in \mathbb{R}^{d+1}, \quad \varphi_a(x) = \frac{1}{a^{2\alpha+d+2}} \varphi \left( \frac{x}{a} \right). \]  
(2.26)

**Proposition 2.7.** 1. Let \( a > 0 \) and \( \varphi \in L_p^\alpha(\mathbb{R}^{d+1}), \ p \in [1, \infty] \). The function \( \varphi_a \) belongs to \( L_p^\alpha(\mathbb{R}^{d+1}) \) and we have
\[ ||\varphi_a||_{L_p^\alpha} = a^{(2\alpha+d+2)(\frac{1}{p}-1)} ||\varphi||_{L_p^\alpha}. \]  
(2.27)

2. Let \( a > 0 \) and \( \varphi \in L_1^\alpha(\mathbb{R}^{d+1}) \cup L_2^\alpha(\mathbb{R}^{d+1}). \) Then, we have
\[ F_W(\varphi_a)(\xi) = F_W(\varphi)(a\xi), \quad \xi \in \mathbb{R}^{d+1}. \]  
(2.28)

For \( a > 0 \) and \( \varphi \in L_2^\alpha(\mathbb{R}^{d+1}) \), we consider the family \( \varphi_{a,x}, \ x \in \mathbb{R}^{d+1} \) of Weinstein wavelets on \( \mathbb{R}^{d+1} \) in \( L_2^\alpha(\mathbb{R}^{d+1}) \) defined by
\[ \forall y \in \mathbb{R}^{d+1}, \quad \varphi_{a,x} = a^{\alpha+\frac{d}{2}} F_{x,a} \varphi_a(y). \]  
(2.29)
Remark 2.8. 1. Let $\varphi$ be a function in $L^2_\alpha(\mathbb{R}^{d+1}_+)$, then we have
\[ \forall(a,x) \in \mathcal{X}, \quad \|\varphi_{a,x}\|_{a,2} \leq \|\varphi\|_{a,2}. \] (2.30)

2. Let $p \in [1, \infty]$ and $\varphi$ be a function in $L^p_\alpha(\mathbb{R}^{d+1}_+)$, then we have
\[ \forall(a,x) \in \mathcal{X}, \quad \|\varphi_{a,x}\|_{a,p} \leq a^{(2a+d+2)(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{a,p}. \] (2.31)

Definition 2.9. [23] Let $\varphi$ be a Weinstein wavelet on $\mathbb{R}^{d+1}_+$ in $L^2_\alpha(\mathbb{R}^{d+1}_+)$. The Weinstein continuous wavelet transform $\Phi^W_{\varphi}$ on $\mathbb{R}^{d+1}_+$ is defined for regular functions $f$ on $\mathbb{R}^{d+1}_+$ by
\[ \forall(a,x) \in \mathcal{X}, \quad \Phi^W_{\varphi}(f)(a,x) = \int_{\mathbb{R}^{d+1}_+} f(y)\varphi_{a,x}(y) d\mu_\alpha(y) = \langle f, \varphi_{a,x}\rangle_{a,2}. \] (2.32)

This transform can also be written in the form
\[ \Phi^W_{\varphi}(f)(a,x) = a^{\alpha+1+\frac{d}{2}} \hat{f} * \hat{\varphi}_a(x). \] (2.33)

Remark 2.10. 1. Let $\varphi$ be a function in $L^p_\alpha(\mathbb{R}^{d+1}_+)$, and Let $f$ be a function in $L^q_\alpha(\mathbb{R}^{d+1}_+)$, with $p \in [1, \infty]$, we define the Weinstein continuous wavelet transform $\Phi^W_{\varphi}(f)$ by the relation (2.33).

2. Let $\varphi$ be a Weinstein wavelet on $\mathbb{R}^{d+1}_+$ in $L^2_\alpha(\mathbb{R}^{d+1}_+)$. Then from the relations (2.30) and (2.32), we have for all $f \in L^2_\alpha(\mathbb{R}^{d+1}_+)$
\[ \|\Phi^W_{\varphi}(f)\|_{a,\infty} \leq \|f\|_{a,2} \|\varphi\|_{a,2}. \] (2.34)

3. Let $\varphi$ be a function in $L^p_\alpha(\mathbb{R}^{d+1}_+$, with $p \in [1, \infty]$, then from the inequality (2.21) and the identity (2.33), we have for all $f \in L^q_\alpha(\mathbb{R}^{d+1}_+)$
\[ \|\Phi^W_{\varphi}(f)\|_{a,\infty} \leq \|f\|_{a,q} \|\varphi\|_{a,p}. \] (2.35)

Theorem 2.11. (Parseval’s formula) [23] Let $(\varphi, \psi)$ be a Weinstein two-wavelet. Then for all $\varphi$ and $\psi$ in $L^2_\alpha(\mathbb{R}^{d+1}_+)$, we have the following Parseval type formula
\[ \int_{\mathcal{X}} \Phi^W_{\varphi}(f)(a,x)\Phi^W_{\psi}(g)(a,x) d\mu_\alpha(a,x) = C_{\varphi,\psi} \int_{\mathbb{R}^{d+1}_+} f(x)g(x) d\mu_\alpha(x), \] (2.36)
where $C_{\varphi,\psi}$ is the Weinstein two-wavelet constant associated to the functions $\varphi$ and $\psi$ given by the identity (2.22).

Corollary 2.12. [33] Let $(\varphi, \psi)$ be a Weinstein two-wavelet. Then we have the following assertion: If the Weinstein two-wavelet constant $C_{\varphi,\varphi}$ is 0, then $\Phi^W_{\varphi}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ and $\Phi^W_{\psi}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ are orthogonal.

Theorem 2.13. (Inversion formula) [33] Let $(\varphi, \psi)$ be a Weinstein two-wavelet. For all $f \in L^1_\alpha(\mathbb{R}^{d+1}_+$) (resp. $L^2_\alpha(\mathbb{R}^{d+1}_+$)) such that $\mathcal{F}_W(f)$ belongs to $f \in L^1_\alpha(\mathbb{R}^{d+1}_+$) (resp. $L^2_\alpha(\mathbb{R}^{d+1}_+$) $\cap L^\infty_\alpha(\mathbb{R}^{d+1}_+$), we have
\[ f(y) = \frac{1}{C_{\varphi,\psi}} \int_0^\infty \int_{\mathbb{R}^{d+1}_+} \Phi^W_{\varphi}(f)(a,x)\psi_{a,x}(y) d\mu_\alpha(a,x), \] (2.37)
where for each $y \in \mathbb{R}^{d+1}_+$ both the inner integral and the outer integral are absolutely convergent, but eventually not the double integral.
3 The Weinstein two-wavelet localization operators

In this section, we will give a host of sufficient conditions for the boundedness and compactness of the two-wavelet localization operator $L_{\varphi,\psi}(\sigma)$ on $L^p_{\alpha}(\mathbb{R}^{d+1})$ for all $1 \leq p \leq \infty$, in terms of properties of the symbol $\sigma$ and the functions $\varphi$ and $\psi$.

**Definition 3.1.** Let $\varphi, \psi$ be measurable functions on $\mathbb{R}^{d+1}$, $\sigma$ be measurable function on $\mathcal{X}$, we the two-wavelet localization operator noted by $L_{\varphi,\psi}(\sigma)$, on $L^p_{\alpha}(\mathbb{R}^{d+1})$, $1 \leq p \leq \infty$, by

$$L_{\varphi,\psi}(\sigma)(f)(y) = \int_{\mathcal{X}} \sigma(a,x)\phi^W_\varphi(f)(a,x)\psi_{a,x}(y)d\mu_\alpha(a,x), \ y \in \mathbb{R}^{d+1}. \quad (3.1)$$

In accordance with the different choices of the symbols $\sigma$ and the different continuities required, we need to impose different conditions on the functions $\varphi$ and $\psi$, and then we obtain a two-wavelet localization operator on $L^p_{\alpha}(\mathbb{R}^{d+1})$. It is often more practical to interpret the definition of the localization operator in a weak sense as following: for all $f$ in $L^p_{\alpha}(\mathbb{R}^{d+1})$, $1 \leq p \leq \infty$, and $f$ in $L^q_{\alpha}(\mathbb{R}^{d+1})$

$$\langle L_{\varphi,\psi}(\sigma)(f), g \rangle_{\alpha,2} = \int_{\mathcal{X}} \sigma(a,x)\phi^W_\varphi(f)(a,x)\overline{\phi^W_\psi(g)(a,x)}d\mu_\alpha(a,x), \ y \in \mathbb{R}^{d+1}. \quad (3.2)$$

**Proposition 3.2.** Let $1 \leq p \leq \infty$. Then the adjoint of the two-wavelet localization operator

$$L_{\varphi,\psi}(\sigma) : L^p_{\alpha}(\mathbb{R}^{d+1}) \rightarrow L^q_{\alpha}(\mathbb{R}^{d+1}),$$

is $L_{\psi,\varphi}(\overline{\sigma}) : L^q_{\alpha}(\mathbb{R}^{d+1}) \rightarrow L^p_{\alpha}(\mathbb{R}^{d+1})$.

**Proof.** Let $f$ in $L^p_{\alpha}(\mathbb{R}^{d+1})$ and $g$ in $L^q_{\alpha}(\mathbb{R}^{d+1})$. Then we have from the relation (3.2)

$$\langle L_{\varphi,\psi}(\sigma)(f), g \rangle_{\alpha,2} = \int_{\mathcal{X}} \sigma(a,x)\phi^W_\varphi(f)(a,x)\overline{\phi^W_\psi(g)(a,x)}d\mu_\alpha(a,x)$$

$$= \int_{\mathcal{X}} \sigma(a,x)\phi^W_\varphi(f)(a,x)\phi^W_\psi(g)(a,x)d\mu_\alpha(a,x)$$

$$= \langle L_{\psi,\varphi}(\overline{\sigma})(g), f \rangle_{\alpha,2}$$

$$= \langle f, L_{\psi,\varphi}(\overline{\sigma})(g) \rangle_{\alpha,2}. \quad (3.3)$$

Thus,

$$L_{\varphi,\psi}^*(\sigma) = L_{\psi,\varphi}(\overline{\sigma}). \quad \Box$$

3.1 $L^p_{\alpha}$-Boundedness of $L_{\varphi,\psi}(\sigma)$

For $1 \leq p \leq \infty$, put $\sigma \in L^1_{\alpha}(\mathcal{X}), \psi \in L^p_{\alpha}(\mathbb{R}^{d+1})$ and $\varphi \in L^q_{\alpha}(\mathbb{R}^{d+1})$. In this subsection, we are going to show that the Weinstein two-wavelet localization operator $L_{\varphi,\psi}(\sigma)$ is a bounded operator on $L^p_{\alpha}(\mathbb{R}^{d+1})$.

**Proposition 3.3.** Let $\sigma \in L^1_{\alpha}(\mathcal{X}), \varphi \in L^\infty_{\alpha}(\mathbb{R}^{d+1})$ and $\psi \in L^1_{\alpha}(\mathbb{R}^{d+1})$, then the localization operator $L_{\varphi,\psi}(\sigma)$ is bounded and linear from $L^1_{\alpha}(\mathbb{R}^{d+1})$ onto itself and we have

$$\|L_{\varphi,\psi}(\sigma)\|_{B(L^1_{\alpha}(\mathbb{R}^{d+1}))} \leq \|\varphi\|_{a,\infty} \|\psi\|_{a,1} \|\sigma\|_{L^1_{\alpha}(\mathcal{X})}.$$
Proof. Let $f$ be a function in $L^1(\mathcal{X})$. From the definition of the two-wavelet localization operator (3.3) and according to relations (2.29), (2.33), (2.20) and (2.16), we have

$$
\|L_{\phi, \psi}(\sigma)(f)\|_{a,1} \leq \int_{\mathbb{R}^d_{+}} \int_{\mathcal{X}} |\sigma(a, x)| |\Phi^W_\phi(f)(a, x)| |\psi_{\sigma}(y)| d\mu_a(a, x) d\mu_a(y)
$$

$$
\leq \int_{\mathbb{R}^d_{+}} \int_{\mathcal{X}} |\sigma(a, x)| \|f\|_{a,1} |\phi\|_{a,\infty} |\tau^\alpha_x \psi(y)| d\mu_a(x) \frac{da}{a} d\mu_a(y)
$$

$$
\leq \int_{\mathbb{R}^d_{+}} \int_{\mathcal{X}} |\sigma(a, x)| \|f\|_{a,1} \|\phi\|_{a,\infty} \|\tau^\alpha_x \psi(y)| d\mu_a(x) \frac{da}{a} d\mu_a(y)
$$

$$
\leq \|f\|_{a,1} \|\phi\|_{a,\infty} \int_{\mathcal{X}} |\sigma(a, x)| \left[ \int_{\mathbb{R}^d_{+}} |\tau^\alpha_x \psi(y)| d\mu_a(y) \right] d\mu_a(x) \frac{da}{a} d\mu_a(y)
$$

$$
\leq \|f\|_{a,1} \|\phi\|_{a,\infty} \int_{\mathcal{X}} |\sigma(a, x)| \left[ \int_{\mathbb{R}^d_{+}} |\psi(y)| d\mu_a(y) \right] d\mu_a(x) \frac{da}{a} d\mu_a(y)
$$

$$
\leq \|f\|_{a,1} \|\phi\|_{a,\infty} \|\psi\|_{a,1} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

Therefore,

$$
\|L_{\phi, \psi}(\sigma)\|_{B(L^1_\alpha(\mathcal{X}))} \leq \|\phi\|_{a,\infty} \|\psi\|_{a,1} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

\[\square\]

Proposition 3.4. Let $\sigma \in L^1_\alpha(\mathcal{X})$, $\phi \in L^1_\alpha(\mathbb{R}^d_{+})$ and $\psi \in L^\infty_\alpha(\mathbb{R}^d_{+})$, then the localization operator $L_{\phi, \psi}(\sigma)$ is bounded and linear from $L^\infty_\alpha(\mathbb{R}^d_{+})$ onto itself and we have

$$
\|L_{\phi, \psi}(\sigma)\|_{B(L^\infty_\alpha(\mathbb{R}^d_{+}))} \leq \|\phi\|_{a,1} \|\psi\|_{a,\infty} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

Proof. Let $f$ be a function in $L^\infty_\alpha(\mathcal{X})$. From the definition of the two-wavelet localization operator (3.3) and as above, according to relations (2.29), (2.33), (2.20) and (2.16), we have for all $y \in \mathbb{R}^d_{+}$

$$
|L_{\phi, \psi}(\sigma)(f)(y)| \leq \int_{\mathcal{X}} |\sigma(a, x)| |\Phi^W_\phi(f)(a, x)| |\psi_{\sigma}(y)| d\mu_a(a, x)
$$

$$
\leq \|f\|_{a,\infty} \|\phi\|_{a,1} \|\psi\|_{a,\infty} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

Thus,

$$
\|L_{\phi, \psi}(\sigma)\|_{B(L^\infty_\alpha(\mathbb{R}^d_{+}))} \leq \|\phi\|_{a,\infty} \|\psi\|_{a,1} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

\[\square\]

By interpolations of the results of Propositions 3.3 and 3.4, we get the following result.

Theorem 3.5. Let $\phi$ and $\psi$ be functions in $L^1_\alpha(\mathbb{R}^d_{+}) \cap L^\infty_\alpha(\mathbb{R}^d_{+})$. Then for all $\sigma \in L^1_\alpha(\mathcal{X})$, there exists a unique bounded linear operator $L_{\phi, \psi}(\sigma)$ from $L^p_\alpha(\mathbb{R}^d_{+})$ onto itself with $1 \leq p \leq \infty$, such that

$$
\|L_{\phi, \psi}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}^d_{+}))} \leq \|\phi\|^{\frac{1}{p}}_{a,1} \|\psi\|^{\frac{1}{p}}_{a,\infty} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$

In the following proposition, we generalize and we improve Proposition 3.3.

Proposition 3.6. Let $\sigma \in L^1_\alpha(\mathcal{X})$, $\psi \in L^p_\alpha(\mathbb{R}^d_{+})$ and $\phi \in L^q_\alpha(\mathbb{R}^d_{+})$, for $1 < p \leq \infty$ then the localization operator $L_{\phi, \psi}(\sigma)$ is bounded linear operator from $L^p_\alpha(\mathbb{R}^d_{+})$ onto itself and we have

$$
\|L_{\phi, \psi}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}^d_{+}))} \leq \|\phi\|_{a,q} \|\psi\|_{a,p} \|\sigma\|_{L^1_\alpha(\mathcal{X})}.
$$
Proof. Let \( f \) be a function in \( L^p_a(\mathbb{R}^{d+1}_+) \). We consider the linear functional
\[
\mathcal{J}_f : L^p_a(\mathbb{R}^{d+1}_+) \rightarrow \mathbb{C} \\
g \mapsto (g, \mathcal{L}_{\varphi,\psi}(\sigma)(f))_{a,2}.
\]
According to relation (3.2), we have
\[
|\langle \mathcal{L}_{\varphi,\psi}(\sigma)(f), g \rangle_{a,2}| \leq \int_{\mathbb{R}^d} |\sigma(a, x)| |\Phi_{\varphi}^W(f)(a, x)| |\Phi_{\psi}^W(g)(a, x)| d\mu_a(a, x)
\]
\[
\leq \|\Phi_{\varphi}^W(f)\|_{L^\infty_{\alpha}(\mathcal{X})} \|\Phi_{\psi}^W(g)\|_{L^\infty_{\alpha}(\mathcal{X})} \|\sigma\|_{L^1_{\alpha}(\mathcal{X})}.
\]
Next, using the relations (2.33) and (2.20), we get
\[
|\langle \mathcal{L}_{\varphi,\psi}(\sigma)(f), g \rangle_{a,2}| \leq \|\sigma\|_{L^1_{\alpha}(\mathcal{X})} \|\varphi\|_{a,q} \|\psi\|_{a,p} \|f\|_{a,p} \|g\|_{a,q}.
\]
Next, it is easy to see that
\[
\|\mathcal{L}_{\varphi,\psi}(\sigma)(f)\|_{a,p} = \|\mathcal{J}_f\|_{B(L^p_a(\mathbb{R}^{d+1}_+))} \leq \|\sigma\|_{L^1_{\alpha}(\mathcal{X})} \|\varphi\|_{a,q} \|\psi\|_{a,p} \|f\|_{a,p},
\]
which completes the proof. \( \square \)

By combining the results obtained in Propositions 3.3 and 3.6 we have the following version of the \( L^p_a \)-boundedness result.

Theorem 3.7. Let \( \sigma \in L^1_{\alpha}(\mathcal{X}), \ \psi \in L^p_a(\mathbb{R}^{d+1}_+) \) and \( \varphi \in L^p_{\alpha}(\mathbb{R}^{d+1}_+), \) for \( 1 \leq p \leq \infty, \) then the localization operator \( \mathcal{L}_{\varphi,\psi}(\sigma) \) is a bounded and linear from \( L^p_a(\mathbb{R}^{d+1}_+) \) onto itself, and we have
\[
\|\mathcal{L}_{\varphi,\psi}(\sigma)\|_{B(L^p_a(\mathbb{R}^{d+1}_+))} \leq \|\sigma\|_{L^1_{\alpha}(\mathcal{X})} \|\varphi\|_{a,q} \|\psi\|_{a,p} \|f\|_{a,p}.
\]

According to Schur technique, we can obtain an \( L^p_a \)-boundedness result as in the previous Theorem with crude estimate of the norm \( \|\mathcal{L}_{\varphi,\psi}(\sigma)\|_{B(L^p_a(\mathbb{R}^{d+1}_+))}.\)

Theorem 3.8. Let \( \sigma \in L^1_{\alpha}(\mathcal{X}), \ \varphi \) and \( \psi \) in \( L^1_{\alpha}(\mathbb{R}^{d+1}_+) \cap L^\infty_{\alpha}(\mathbb{R}^{d+1}_+) \). Then there exists a unique bounded linear operator \( \mathcal{L}_{\varphi,\psi}(\sigma) \) from \( L^p_a(\mathbb{R}^{d+1}_+) \) onto itself with \( 1 \leq p \leq \infty \) and we have
\[
\|\mathcal{L}_{\varphi,\psi}(\sigma)\|_{B(L^p_a(\mathbb{R}^{d+1}_+))} \leq \max \left( \|\varphi\|_{a,1}, \|\psi\|_{a,\infty}, \|\varphi\|_{a,\infty}, \|\psi\|_{a,1} \right) \|\sigma\|_{L^1_{\alpha}(\mathcal{X})}.
\]

Proof. We put the function \( \mathcal{R} \) defined on \( \mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+ \) by
\[
\mathcal{R}(y, z) = \int_{\mathbb{R}^d} \sigma(a, x) \varphi_{a,x}(z) \psi_{a,x}(y) d\mu_a(a, x).
\]
Then the Weinstein two-wavelet localization operator can be written in terms of \( \mathcal{R}(y, z) \) as follows
\[
\mathcal{L}_{\varphi,\psi}(\sigma)(f)(y) = \int_{\mathbb{R}^{d+1}_+} \mathcal{R}(y, z) f(z) d\mu_a(z).
\]
Next, it is easy to see that
\[
\int_{\mathbb{R}^{d+1}_+} |\mathcal{R}(y, z)| d\mu_a(y) \leq \|\varphi\|_{a,\infty} \|\psi\|_{a,1} \|\sigma\|_{L^1_{\alpha}(\mathcal{X})}, \quad z \in \mathbb{R}^{d+1}_+,
\]
bounded and linear from $L^p_\alpha(\mathbb{R}^{d+1})$ onto itself for all $1 \leq p \leq \infty$, and we have

$$\|L_{\varphi,\psi}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}^{d+1}))} \leq \max\left(\|\varphi\|_{a,\infty}, \|\psi\|_{a,\infty}, \|\varphi\|_{a,1}, \|\psi\|_{a,1}\right) \|\sigma\|_{L^*_\alpha(\mathcal{X})}.$$ 

The previous Theorem tells us that the unique bounded linear operator on the spaces $L^p_\alpha(\mathbb{R}^{d+1}), 1 \leq p \leq \infty$, obtained in Theorem 3.8 is in fact the integral operator on $L^p_\alpha(\mathbb{R}^{d+1}), 1 \leq p \leq \infty$ with kernel $\mathcal{R}$.

Subsequently, we can now state and prove the main result in this subsection.

**Theorem 3.9.** Let $\sigma \in L^*_\alpha(\mathcal{X}), r \in [1, 2]$, and $\varphi, \psi$ in $L^1_\alpha(\mathbb{R}^{d+1}) \cap L^2_\alpha(\mathbb{R}^{d+1}) \cap L^\infty_\alpha(\mathbb{R}^{d+1})$. Then there exists a unique bounded linear operator $L_{\varphi,\psi}(\sigma)$ from $L^p_\alpha(\mathbb{R}^{d+1})$ onto itself for all $p \in [r, r']$ and we have

$$\|L_{\varphi,\psi}(\sigma)\|_{B(L^p_\alpha(\mathbb{R}^{d+1}))} \leq K_1^t K_2^{1-t} \|\sigma\|_{L^*_\alpha(\mathcal{X})},$$

where

$$K_1 = \left(\|\varphi\|_{a,\infty} \|\psi\|_{a,1}\right)^{\frac{2}{r}} \left(\sqrt{C_\varphi C_\psi} \|\varphi\|_{a,2} \|\psi\|_{a,2}\right)^{\frac{1}{r}},$$

$$K_2 = \left(\|\varphi\|_{a,1} \|\psi\|_{a,\infty}\right)^{\frac{2}{r'}} \left(\sqrt{C_\varphi C_\psi} \|\varphi\|_{a,2} \|\psi\|_{a,2}\right)^{\frac{1}{r'}},$$

and

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

**Proof.** Consider the linear functional

$$\mathcal{J} : (L^1_\alpha(\mathcal{X}) \cap L^2_\alpha(\mathcal{X}) \times (L^1_\alpha(\mathbb{R}^{d+1}) \cap L^2_\alpha(\mathbb{R}^{d+1})) \subset (\sigma, f) \mapsto L^1_\alpha(\mathbb{R}^{d+1}) \cap L^2_\alpha(\mathbb{R}^{d+1})$$

According to Proposition 3.3, we obtain

$$\|\mathcal{J}(\sigma, f)\|_{a,1} \leq \|\varphi\|_{a,\infty} \|\psi\|_{a,1} \|f\|_{a,1} \|\sigma\|_{L^*_\alpha(\mathcal{X})}$$

and from [23] Theorem 3.1, we have

$$\|\mathcal{J}(\sigma, f)\|_{a,2} \leq \left(\sqrt{C_\varphi C_\psi} \|\varphi\|_{a,2} \|\psi\|_{a,2}\right)^{\frac{1}{2}} \|f\|_{a,2} \|\sigma\|_{L^*_\alpha(\mathcal{X})}.$$ 

Therefore, by the multi-linear interpolation theory [7, Section 10.1], we obtain a unique bounded linear operator

$$\mathcal{J} : L^r_\alpha(\mathcal{X}) \times L^r_\alpha(\mathbb{R}^{d+1}) \rightarrow L^r_\alpha(\mathbb{R}^{d+1})$$

such that

$$\|\mathcal{J}(\sigma, f)\|_{a,r} \leq K_1 \|f\|_{a,r} \|\sigma\|_{L^*_\alpha(\mathcal{X})}, \quad (3.4)$$

where

$$K_1 = \left(\|\varphi\|_{a,\infty} \|\psi\|_{a,1}\right)^{\theta} \left(\sqrt{C_\varphi C_\psi} \|\varphi\|_{a,2} \|\psi\|_{a,2}\right)^{\frac{1-t}{2}}.$$
and
\[ \frac{\theta}{1} + \frac{1 - \theta}{2} = \frac{1}{r}. \]

By the definition of the linear functional \( J \), we have
\[ \| L_{\varphi, \psi}(\sigma) \|_{L^1_{\alpha}(X)} \leq (\| \varphi \|_{\alpha, \infty} \| \psi \|_{\alpha, 1})^{\frac{2}{p}} \left( \sqrt{C_{\varphi} C_{\psi}} \| \varphi \|_{\alpha, 2} \| \psi \|_{\alpha, 2} \right)^{\frac{1}{p}} \| \sigma \|_{L^p_{\alpha}(X)}. \] (3.5)

Like the adjoint of \( L_{\varphi, \psi}(\sigma) \) is \( L_{\psi, \varphi}(\sigma) \), therefore \( L_{\varphi, \psi}(\sigma) \) is a bounded linear map on \( L^p_{\alpha}(\mathbb{R}^{d+1}_+) \) with its operator norm
\[ \| L_{\varphi, \psi}(\sigma) \|_{B(L^p_{\alpha}(\mathbb{R}^{d+1}_+))} = \| L_{\psi, \varphi}(\sigma) \|_{B(L^p_{\alpha}(\mathbb{R}^{d+1}_+))} \leq K_2 \| \sigma \|_{L^p_{\alpha}(X)}, \] (3.6)

where
\[ K_2 = (\| \varphi \|_{\alpha, 1} \| \psi \|_{\alpha, \infty})^{\frac{2}{p}} \left( \sqrt{C_{\varphi} C_{\psi}} \| \varphi \|_{\alpha, 2} \| \psi \|_{\alpha, 2} \right)^{\frac{1}{p}}. \]

Finally, by interpolation of (3.5) and (3.6), we obtain that, for all \( p \in [r, r'] \),
\[ \| L_{\varphi, \psi}(\sigma) \|_{B(L^p_{\alpha}(\mathbb{R}^{d+1}_+))} \leq K_1^t K_2^{1-t} \| \sigma \|_{L^p_{\alpha}(X)}, \]

with
\[ \frac{t}{r} + \frac{1 - t}{r'} = \frac{1}{p}. \]

\[ \square \]

### 3.2 \( L^p_{\alpha} \)-Compactness of \( L_{\varphi, \psi}(\sigma) \)

In this subsection, we establish the compactness of the Weinstein two-wavelet localization operators \( L_{\varphi, \psi}(\sigma) \) on \( L^p_{\alpha}(\mathbb{R}^{d+1}_+) \), \( 1 \leq p \leq \infty \). Let us start with the following proposition.

**Proposition 3.10.** Under the same assumptions of Theorem 3.3, the Weinstein two-wavelet localization operator
\[ L_{\varphi, \psi}(\sigma) : L^1_{\alpha}(\mathbb{R}^{d+1}_+) \rightarrow L^1_{\alpha}(\mathbb{R}^{d+1}_+) \]

is compact.

**Proof.** Let \( (f_n)_{n \in \mathbb{N}} \in L^1_{\alpha}(\mathbb{R}^{d+1}_+) \) a sequence of functions that converge weakly to 0 in \( L^1_{\alpha}(\mathbb{R}^{d+1}_+) \) as \( n \) converge to \( \infty \). To show the compactness of these localization operators, it is enough to prove that
\[ \lim_{n \to \infty} \| L_{\varphi, \psi}(\sigma)(f_n) \|_{\alpha, 1} = 0. \]

We have
\[ \| L_{\varphi, \psi}(\sigma)(f_n) \|_{\alpha, 1} \leq \int_{\mathbb{R}^{d+1}_+} \int_X |\sigma(a, x)||\langle f_n, \varphi_{a,x} \rangle_{\alpha, 2}||\psi_{a,x}(y)|d\mu_{\alpha}(a, x)d\mu_{\alpha}(y). \] (3.7)

Using the fact that \( (f_n)_{n \in \mathbb{N}} \) converge weakly to 0 in \( L^1_{\alpha}(\mathbb{R}^{d+1}_+) \) as \( n \) converge to \( \infty \), we deduce
\[ \forall a > 0, \forall x, y, \in \mathbb{R}^{d+1}_+, \lim_{n \to \infty} |\sigma(a, x)||\langle f_n, \varphi_{a,x} \rangle_{\alpha, 2}||\psi_{a,x}(y)| = 0. \] (3.8)

Moreover, as \( f_n \to 0 \) weakly in \( L^1_{\alpha}(\mathbb{R}^{d+1}_+) \), then there exists a positive constant \( C \) such that \( \| f_n \|_{\alpha, 1} \leq C \). Hence,
\[ \forall a > 0, \forall x, y, \in \mathbb{R}^{d+1}_+, \ |\sigma(a, x)||\langle f_n, \varphi_{a,x} \rangle_{\alpha, 2}||\psi_{a,x}(y)| \]

\[ \leq C \]

\[ \| f_n \|_{\alpha, 1} \leq C. \]

Hence,
\[ \forall a > 0, \forall x, y, \in \mathbb{R}^{d+1}_+, \ |\sigma(a, x)||\langle f_n, \varphi_{a,x} \rangle_{\alpha, 2}||\psi_{a,x}(y)| \leq C \].
\[ \leq C|\sigma(a,x)||\varphi||_{\alpha,\infty}|\tau^a_x\psi(y)|. \] (3.9)

On the other hand, from Fubini’s theorem and relation (2.16), we obtain
\[
\int_{\mathbb{R}^{d+1}} \int_{X} |\sigma(a,x)||\langle f_n, \varphi_{a,x} \rangle_{\alpha,2}||\psi_{a,x}(y)||d\mu_\alpha(a,x)d\mu_\alpha(y)
\leq C\|\varphi\|_{\alpha,\infty}\int_{X} |\sigma(a,x)||\tau^a_x\psi(y)||d\mu_\alpha(a,x)
\leq C\|\varphi\|_{\alpha,\infty}\int_{X} |\sigma(a,x)||\psi_{a}(y)||d\mu_\alpha(a,x)
\leq C\|\varphi\|_{\alpha,\infty}\|\psi\|_{\alpha,1}\|\sigma\|_{L^1_\alpha(X)} < \infty. \] (3.10)

Thus, according to the relations (3.7)−(3.10) and the Lebesgue dominated convergence theorem we deduce that
\[ \lim_{n \to \infty} \|L_{\varphi,\psi}(\sigma)(f_n)||_{\alpha,1} = 0 \]
which completes the proof.

**Theorem 3.11.** Under the same assumptions of Theorem 3.5, the Weinstein two-wavelet localization operator
\[ L_{\varphi,\psi}(\sigma) : L^p_\alpha(\mathbb{R}^{d+1}) \longrightarrow L^p_\alpha(\mathbb{R}^{d+1}) \]
is compact for all \( p \in [1, \infty] \).

**Proof.** We only need to show that the result holds for \( p = \infty \). in fact, the operator
\[ L_{\varphi,\psi}(\sigma) : L^{\infty}_\alpha(\mathbb{R}^{d+1}) \longrightarrow L^{\infty}_\alpha(\mathbb{R}^{d+1}), \] (3.11)
is the adjoint of the operator
\[ L_{\psi,\varphi}(\sigma) : L^1_\alpha(\mathbb{R}^{d+1}) \longrightarrow L^1_\alpha(\mathbb{R}^{d+1}), \]
which is compact by the previous Proposition. Therefore by the duality property, the operator given by (3.11) is compact. Finally, by an interpolation of the compactness on \( L^1_\alpha(\mathbb{R}^{d+1}) \) and on \( L^{\infty}_\alpha(\mathbb{R}^{d+1}) \) like the one given on [1] Theorem 2.9], the proof is complete.

In the following Theorem, we state a compactness result for the Weinstein two-wavelet localization operator analogue of Theorem 3.9.

**Theorem 3.12.** Under the same assumptions of Theorem 3.9, the Weinstein two-wavelet localization operator
\[ L_{\varphi,\psi}(\sigma) : L^p_\alpha(\mathbb{R}^{d+1}) \longrightarrow L^p_\alpha(\mathbb{R}^{d+1}) \]
is compact for all \( p \in [r, r'] \).

**Proof.** By the same manner in the proof of the previous Theorem, the result is an immediate consequence of an interpolation of [23 Corollary 4.2] and Proposition 3.10 like the one given on [1] Theorem 2.9].

Using similar ideas as above we can prove the following result.

**Theorem 3.13.** Under the same assumptions of Theorem 3.7, the Weinstein two-wavelet localization operator
\[ L_{\varphi,\psi}(\sigma) : L^p_\alpha(\mathbb{R}^{d+1}) \longrightarrow L^p_\alpha(\mathbb{R}^{d+1}) \]
is compact for all \( p \in [1, \infty] \).
3.3 Examples

In this subsection, as in the paper of Wong [40], we study some typical examples of the Einstein two-wavelet localization operators. We show that on the space

\[ X = \{ (a, x) : x \in \mathbb{R}_{+}^{d+1} \text{ and } a > 0 \} \]

the localization operators associated to admissible Einstein wavelets \( \varphi \) and \( \psi \) and separable symbols \( \sigma \) are paracommutators and we show that if the symbol is a function of \( x \) only, then the localization operator can be expressed in terms of a paraproduct. We show in the end if the symbol is a function of \( a \) only, then the localization operator \( L_{\varphi, \psi}(\sigma) \) is a Einstein multiplier.

3.3.1 Paracommutators

Let \( \sigma \) be a separable function on \( X \) given by

\[ \sigma(a, x) = \chi(a) \zeta(x), \]

where \( \chi \) and \( \zeta \) are suitable functions, respectively, on \( (0, \infty) \) and \( \mathbb{R}_{+}^{d+1} \). Then, according to Parseval’s formula (2.10) and Fubini’s theorem, we have for all \( f, g \in L_{2, \alpha}^{2}(\mathbb{R}_{+}^{d+1}) \)

\[ \langle L_{\varphi, \psi}(\sigma)(f), g \rangle_{\alpha, 2} = \int_{\mathcal{X}} \sigma(a, x) \Phi_{\varphi}^{W}(f)(a, x) \Phi_{\psi}^{W}(g)(a, x) d\mu_{\alpha}(a, x) \]

\[ = \int_{0}^{\infty} \chi(a) \int_{\mathbb{R}_{+}^{d+1}} \int_{\mathbb{R}_{+}^{d+1}} \tau_{\eta}^{a} \mathcal{F}_{W}(\zeta)(-\xi) \mathcal{F}_{W}(f)(\xi) \mathcal{F}_{W}(\varphi)(a\xi) \]

\[ \times \mathcal{F}_{W}(g)(\eta) \mathcal{F}_{W}(\psi)(a\xi) d\mu_{\alpha}(\eta) d\mu_{\alpha}(\xi) \frac{d\alpha}{a} \]

\[ = \int_{\mathbb{R}_{+}^{d+1}} \int_{\mathbb{R}_{+}^{d+1}} K(\xi, \eta) \tau_{\eta}^{a} \mathcal{F}_{W}(\zeta)(-\xi) \mathcal{F}_{W}(f)(\xi) \mathcal{F}_{W}(g)(\eta) \]

\[ d\mu_{\alpha}(\eta) d\mu_{\alpha}(\xi), \]

where

\[ K(\xi, \eta) = \int_{0}^{\infty} \chi(a) \mathcal{F}_{W}(\varphi)(a\xi) \mathcal{F}_{W}(\psi)(a\eta) \frac{d\alpha}{a}, \quad \forall \xi, \eta \in \mathbb{R}_{+}^{d+1}. \]

Thus, the localization operator \( L_{\varphi, \psi}(\sigma) \) is a paracommutator with Einstein kernel \( K \) and symbol \( \zeta \).

3.3.2 Paraproduct

In this example, we specialize to the case when the symbol \( \sigma \) is a function of \( x \) only, i.e.

\[ \forall (a, x) \in X, \quad \sigma(a, x) = \zeta(x), \]

where \( \zeta \) is a suitable function on \( \mathbb{R}_{+}^{d+1} \). Indeed, using Parseval’s formula (2.10) and Fubini’s theorem as in the preceding example, we get for all \( f, g \in L_{2, \alpha}^{2}(\mathbb{R}_{+}^{d+1}) \)

\[ \langle L_{\varphi, \psi}(\sigma)(f), g \rangle_{\alpha, 2} = \int_{\mathcal{X}} \sigma(a, x) \Phi_{\varphi}^{W}(f)(a, x) \Phi_{\psi}^{W}(g)(a, x) d\mu_{\alpha}(a, x) \]

\[ = \int_{\mathbb{R}_{+}^{d+1}} \left[ \int_{0}^{\infty} (\zeta(\Theta_{a} * f) * \psi_{a})(x) \frac{d\alpha}{a} \right] g(x) d\mu_{\alpha}(x), \quad (3.12) \]
where \( \Theta(x) = \overline{\varphi(-x)} \). Therefore, we deduce that

\[
\mathcal{L}_{\varphi, \psi}(\sigma)(f)(x) = \int_{0}^{\infty} (\tilde{\zeta}(\alpha \ast f) \ast \psi_{\alpha})(x) \frac{d\alpha}{\alpha}, \quad \forall x \in \mathbb{R}^{d+1}.
\]

This formula for the Weinstein two-wavelet localization operator is an interesting formula in its own right. Further analysis of (3.12) using Fubini’s theorem gives

\[
\langle \mathcal{L}_{\varphi, \psi}(\sigma)(f), g \rangle_{2} = \int_{\mathbb{R}^{d+1}} (\zeta(x)p_{\varphi, \psi}(f, g)(x) \frac{d\alpha}{\alpha}, \quad \forall x \in \mathbb{R}^{d+1},
\]

where

\[
p_{\varphi, \psi}(f, g)(x) = \int_{0}^{\infty} (\tilde{\Theta}_{\alpha \ast f}(\alpha \ast g)(x) \frac{d\alpha}{\alpha}, \quad \forall x \in \mathbb{R}^{d+1},
\]

with \( \Upsilon(x) = \overline{\psi(-x)} \).

Several versions of paraproducts exist in the literature. It should be remarked that the notion of a paraproduct is rooted in Bony’s work [5] on linearization of nonlinear problems. The paraproduct connection (3.12) and the fact that the Weinstein two-wavelet localization operators associated to symbols \( \sigma \) in \( L^{\infty}_{\alpha}(X) \) are bounded linear operators (see [23, Corollary 3.2]) such that

\[
\|\mathcal{L}_{\varphi, \psi}(\sigma)\|_{\mathcal{S}_{\infty}} \leq \sqrt{C_{\varphi}C_{\psi}}\|\sigma\|_{L^{\infty}_{\alpha}(X)},
\]

allow us to give an \( L^{1}_{\alpha} \)-estimate on the paraproduct \( p_{\varphi, \psi}(f, g) \), where \( \varphi \) and \( \psi \) are the Weinstein wavelets, and \( f \) and \( g \) are functions in \( L^{2}_{\alpha}(\mathbb{R}^{d+1}) \). First, we need the following Lemma.

**Lemma 3.14.** Let \( \varphi \) and \( \psi \) two Weinstein wavelets such that \((\varphi, \psi)\) be a Weinstein two-wavelet. Then we have for all \( f \) and \( g \) in \( L^{2}_{\alpha}(\mathbb{R}^{d+1}) \)

\[
\int_{\mathbb{R}^{d+1}} p_{\varphi, \psi}(f, g)(x) d\mu_{\alpha}(x) = C_{\varphi, \psi}(f, g)_{\alpha, 2},
\]

**Proof.** According to relation (2.19), Parseval’s formula (2.10) and Fubini’s theorem, we obtain

\[
\int_{\mathbb{R}^{d+1}} p_{\varphi, \psi}(f, g)(x) d\mu_{\alpha}(x) = \int_{\mathbb{R}^{d+1}} \int_{0}^{\infty} (\tilde{\Theta}_{\alpha \ast f}(\alpha \ast g)(x) \frac{d\alpha}{\alpha} d\mu_{\alpha}(x)
\]

\[
= \int_{0}^{\infty} \int_{\mathbb{R}^{d+1}} F_{W}(f)(\xi)\overline{F_{W}(\Theta)(a\xi)F_{W}(\xi)} d\mu_{\alpha}(x)
\]

\[
= \left[ \int_{0}^{\infty} F_{W}(\Theta)(a\xi)\overline{F_{W}(\xi)} \frac{d\alpha}{\alpha} \right] F_{W}(f)(\xi) d\mu_{\alpha}(x)
\]

\[
= C_{\varphi, \psi}(f, g)_{\alpha, 2}.
\]

An \( L^{1}_{\alpha} \)-estimate for the paraproduct \( p_{\varphi, \psi}(f, g) \) in terms of the \( L^{2}_{\alpha} \)-norms of \( f \) and \( g \) is given in the following theorem.

**Theorem 3.15.** Let \( \varphi \) and \( \psi \) two Weinstein wavelets such that

\[
\|\varphi\|_{\alpha, 2} = \|\psi\|_{\alpha, 2} = 1.
\]

Then we have for all \( f \) and \( g \) in \( L^{2}_{\alpha}(\mathbb{R}^{d+1}) \)

\[
\|p_{\varphi, \psi}(f, g)\|_{\alpha, 1} \leq \sqrt{C_{\varphi}C_{\psi}} \|f\|_{\alpha, 2} \|g\|_{\alpha, 2}.
\]
Proof. From \[23\] Corollary 3.2, we know that
\[
\|L_{\varphi,\psi}(\sigma)\|_{S_\infty} \leq \sqrt{C_{\varphi}C_{\psi}} \|\sigma\|_{L_\infty^\infty(X)}.
\]

Therefore, by relation (3.13) and Cauchy-Schwarz inequality, we get
\[
\frac{1}{\sqrt{C_{\varphi}C_{\psi}}} \left| \int_{\mathbb{R}^{d+1}_+} \hat{\varphi}(x)p_{\varphi,\psi}(f,g)(x)d\mu_\alpha(x) \right| \leq \|\varphi\|_{\alpha,\infty}\|f\|_{\alpha,2}\|g\|_{\alpha,2}.
\]

Since \(\frac{1}{\sqrt{C_{\varphi}C_{\psi}}} p_{\varphi,\psi}(f,g)\) belongs to \(L_\alpha^1(\mathbb{R}^{d+1})\), it follows from the Hahn-\v{A}šBanach theorem that is in the dual \((L_\alpha^\infty(\mathbb{R}^{d+1}))^*\) of \(L_\alpha^\infty(\mathbb{R}^{d+1})\) and we have
\[
\frac{1}{\sqrt{C_{\varphi}C_{\psi}}} \|p_{\varphi,\psi}(f,g)\|_{\alpha,1} \leq \|f\|_{\alpha,2}\|g\|_{\alpha,2}.
\]

### 3.3.3 Weinstein Multipliers

We see in this section that if the symbol \(\sigma\) is a function of \(a\) only, then the Weinstein two-wavelet localization operator is a Weinstein multiplier \(T^W_m\) with symbol \(m\) defined on \(L_\alpha^2(\mathbb{R}^{d+1})\) as follow
\[
T^W_m = F_W^{-1}(mF_W(f))
\]

**Proposition 3.16.** Let \(\sigma\) be a function on \(X\) given by
\[
\sigma(a,x) = \chi(a), \quad \forall (a,x) \in X,
\]
where \(\chi\) is a suitable function on \((0,\infty)\). Then, we have
\[
L_{\varphi,\psi}(\sigma) = T^W_m,
\]
where \(T^W_m\) is the Weinstein multiplier with symbol \(m\) given by
\[
m(\xi) = \int_0^\infty \chi(a)\overline{F_W(\varphi)(a\xi)}\overline{F_W(\psi)(a\xi)}\frac{da}{a}, \quad \forall \xi \in \mathbb{R}^{d+1}.
\]

**Proof.** For any positive integer \(m\), we define \(I_m\) by
\[
I_m = \int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} \int_X e^{-\frac{|b|^2}{2m}} \chi(a)\overline{F_W(\varphi)(a\xi)}\overline{F_W(\psi)(a\eta)}\Lambda^d_\alpha(ib,\xi)\Lambda^d_\alpha(-ib,\eta)
\]
\[
\times \overline{F_W(f)(\xi)}\overline{F_W(g)(\eta)}d\mu_\alpha(b)d\mu_\alpha(\eta)d\mu_\alpha(\xi)\frac{da}{a}.
\]

Then using Fubini’s theorem and the fact that the Weinstein transform of the function \(h(x) = e^{-\frac{|x|^2}{2}}\) is equal to itself for all \(x \in \mathbb{R}^{d+1}\) (see \[15\] Example 1), we get
\[
I_m = \int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} \int_0^\infty \tau_{\alpha,\eta}^m h_m(\xi)\chi(a)\overline{F_W(\varphi)(a\xi)}\overline{F_W(\psi)(a\eta)}
\]
\[
\times \overline{F_W(f)(\xi)}\overline{F_W(g)(\eta)}d\mu_\alpha(\eta)d\mu_\alpha(\xi)\frac{da}{a}
\]
\[
= \int_{\mathbb{R}^{d+1}_+} \int_0^\infty \chi(a)\overline{F_W(\varphi)(a\xi)}\left(F_W(\psi_a)\overline{F_W(g)} \ast h_m\right)(\xi)\overline{F_W(f)(\xi)}d\mu_\alpha(\xi)\frac{da}{a}.
\]

(3.15)
On the other hand, it’s easy to see that
\[ F_{\psi}(\psi_\alpha) F_{\psi}(\psi) * h_m \to F_{\psi}(\psi_\alpha) F_{\psi}(\psi) \] (3.16)
in \( L^2_\alpha(\mathbb{R}^{d+1}_+) \) and almost everywhere on \( \mathbb{R}^{d+1}_+ \) as \( n \to \infty \). Therefore, according to relations (3.15) and (3.16), we obtain
\[ I_m \to \int_{\mathbb{R}^{d+1}_+} \int_0^\infty \chi(a) F_{\psi}(\psi)(a\xi) F_{\psi}(\psi)(\xi) d\mu (\xi) \frac{da}{\alpha}, \] (3.17)
as \( n \to \infty \). Subsequently, using Lebesgue’s dominated convergence theorem, we see that
\[ I_m \to \langle L_{\varphi,\psi}(\sigma)(f), g \rangle_{\alpha,2}. \]

Therefore,
\[ \langle L_{\varphi,\psi}(\sigma)(f), g \rangle_{\alpha,2} = \langle T_m^W f, g \rangle_{\alpha,2}, \]
for all \( f \) and \( g \) in \( L^2_\alpha(\mathbb{R}^{d+1}_+) \), where \( T_m^W \) is the Weinstein multiplier with symbol \( m \) given by
\[ m(\xi) = \int_0^\infty \chi(a) F_{\psi}(\psi)(a\xi) F_{\psi}(\psi)(\xi) \frac{da}{\alpha}, \quad \forall \xi \in \mathbb{R}^{d+1}_+. \]

\[ \square \]

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