THE CHROMATIC SPLITTING CONJECTURE AT \( n = p = 2 \)

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Abstract. We show that the strongest form of Hopkins’ chromatic splitting conjecture, as stated by Hovey, cannot hold at chromatic level \( n = 2 \) at the prime \( p = 2 \). More precisely, for \( V(0) \) the mod 2 Moore spectrum, we prove that \( \pi_k L_1 L_{K(2)} V(0) \) is not zero when \( k \) is congruent to 5 modulo 8. We explain how this contradicts the decomposition of \( L_1 L_{K(2)} S \) predicted by the chromatic splitting conjecture.

CONTENTS

1. Introduction \hspace{1cm} 1
2. The \( E(1) \)-local Duality Resolution Spectral Sequence \hspace{1cm} 4
3. The homotopy of \( L_1 (E^{hG_{24}} \wedge V(0)) \) and \( L_1 (E^{hC_2} \wedge V(0)) \) \hspace{1cm} 8
4. Some Elements in \( \pi_* L_1 L_{K(2)} V(0) \) \hspace{1cm} 10
References \hspace{1cm} 13

1. Introduction

Fix a prime \( p \). Let \( S \) be the \( p \)-local sphere spectrum and \( L_n S \) be the Bousfield localization of \( S \) at the Johnson-Wilson spectrum \( E(n) \). Let \( K(n) \) be Morava \( K \)-theory. There is a homotopy pull-back square

\[
\begin{array}{ccc}
L_n S & \longrightarrow & L_{K(n)} S \\
\downarrow & & \downarrow \\
L_{n-1} S & \leftarrow & L_{n-1} L_{K(n)} S.
\end{array}
\]

Let \( F_n \) be the fiber of the map \( L_n S \rightarrow L_{K(n)} S \). Note that \( F_n \) is weakly equivalent to the fiber of \( \iota \). It was shown in [12] that \( F_n \) is weakly equivalent to the function spectrum \( F(L_{n-1} S, L_n S) \). Hopkins’ chromatic splitting conjecture, as stated by Hovey [12], stipulates that \( \iota \) is the inclusion of a wedge summand, so that

\begin{equation}
L_{n-1} L_{K(n)} S \simeq L_{n-1} S \vee \Sigma F_n.
\end{equation}

It also gives an explicit decomposition of \( \Sigma F_n \) as a wedge of suspensions of spectra of the form \( L_i S \) for \( 0 \leq i < n \).

The conjectured decomposition comes from the connection between the \( K(n) \)-local category and the cohomology of the Morava stabilizer group \( G_n \). Let \( S_n \) be the group of automorphisms of the formal group law of \( K(n) \) over \( \mathbb{F}_{p^n} \). Let \( G_n \) be the extension of \( S_n \) by the Galois group \( \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \). Let \( \mathcal{W} \) be the Witt vectors
on $\mathbb{F}_p \rho$. There is a spectral sequence
\begin{equation}
H_{c}^{*}(\mathbb{G}_n, (E_n)_t) \Rightarrow \pi_{t-s} L_{K(n)} S.
\end{equation}
Note that $\mathbb{W}$ sits naturally in $(E_n)_0 \cong \mathbb{W}[[u_1, \ldots, u_n]]$. The inclusion induces a map
$$H_{c}^{*}(\mathbb{G}_n, \mathbb{W}) \to H_{c}^{*}(\mathbb{G}_n, (E_n)_0).$$
The chromatic splitting conjecture stipulates that there is an exterior algebra $E(x_1, \ldots, x_n)$ in $H^{*}(\mathbb{G}_n, \mathbb{W})$, and that the non-zero classes $x_{i_1} \cdots x_{i_j}$ survive in (1.2) to homotopy classes $\pi_{-2((\sum i_k) + j)} L_{K(n)} S$. Further, it states that there is a factorization
\begin{equation}
\begin{array}{cccc}
S^{-2((\sum i_k) + j)} & \longrightarrow & L_{n-\max(i_k)} S^{-2((\sum i_k) + j)} & \\
\downarrow & & \downarrow & \\
L_{K(n)} S & \longrightarrow & \Sigma F_n,
\end{array}
\end{equation}
and that these maps decompose $\Sigma F_n$ as
\begin{equation}
\Sigma F_n \cong \bigvee_{1 \leq j \leq n} \bigvee_{1 \leq i_1 < \cdots < i_j \leq n} L_{n-\max(i_k)} S^{-2((\sum i_k) + j)}.
\end{equation}

The chromatic splitting conjecture has been shown for $n \leq 2$ and for all primes $p$, except in the case $p = n = 2$. For $n = 1$, it follows from Adams and Mahowald’s computation of $L_1 S$. At $n = 2$ and $p \geq 5$, it follows from Shimomura and Yabe’s computations in [20]. The proof can be found in Behrens’s account of their computations in [4]. At $n = 2$ and $p = 3$, the conjecture was proved recently by Goerss, Henn and Mahowald in [10].

In this paper, we show that the chromatic splitting conjecture as stated above cannot hold for $n = p = 2$. At $n = 2$, (1.1) and (1.3) imply that
\begin{equation}
L_1 L_{K(2)} S \cong L_1 S \vee L_1 S^{-1} \vee L_0 S^{-3} \vee L_0 S^{-4}.
\end{equation}
We show that the right hand side of (1.4) has too few homotopy groups for the equivalence to hold. However, our methods do not contradict the possibility that $\iota$ is this inclusion of a wedge summand. Giving an alternative description for the fiber in this case is work in progress.

That our methods might disprove (1.4) was first suggested to the author by Paul Goerss. Further, the computations of Shimomura and Wang ([18] and [19]) already suggest that the right hand side of (1.4) is too small.

**Statement of the results.** Let $V(0)$ be the cofiber of multiplication by $p$ on $S$. Note that for any $p$-local spectrum $X$, there is a cofiber sequence
$$X \xrightarrow{p} X \to X \wedge V(0).$$
Since Bousfield localization of spectra preserves exact triangles (Lemma 1.10 of [6]), it follows that
$$L_{E} V(0) \simeq L_{E} S \wedge V(0)$$
for any spectrum $E$. This has the following consequence.

**Proposition 1.5.** The chromatic splitting conjecture at $n = 2$ implies that
$$L_1 L_{K(2)} V(0) \simeq L_1 V(0) \vee L_1 \Sigma^{-1} V(0).$$
We now fix our attention to the case when $p = 2$. The homotopy groups of $L_1 V(0)$ were computed by Mahowald. They do not form a ring as $V(0)$ is not a ring spectrum. However, they are a module over $\pi_* L_1 S$, and we give a description which reflect this fact, and is depicted in Figure 1. Namely,

$$\pi_* L_1 V(0) = \left( \mathbb{Z}_2[\eta, \beta, \zeta_1]/(2\eta, 2\beta, 2\zeta_1, \zeta_1^2) \right) / \{ 1, 2 \cdot 1 = \eta^2 \cdot 1 \},$$

where $\eta \in \pi_1$ is the Hopf map, $\beta \in \pi_8$ is the $v_1$-self-map detected by $v_4^4$ and $\zeta_1 \in \pi_{-1}$ is detected by the determinant in $H^1(G_1; \mathbb{Z}_2)$. The element $1 \in \pi_0$ represents the inclusion of the bottom cell $S \hookrightarrow V(0)$ and $\eta \in \pi_2$ is defined by

$$
\begin{array}{c}
\pi \\
\downarrow \Sigma \eta \\
V(0) \xrightarrow{\pi} S^3 \xrightarrow{2} S^1,
\end{array}
$$

using the fact that $2\eta = 0$ in $\pi_1 S$.

The following result is a consequence of Proposition 1.5.

**Corollary 1.6.** The chromatic splitting conjecture implies that $\pi_k L_1 L_{K(2)} V(0)$ is zero when $k \equiv 5$ modulo 8.

However, in this paper, we prove the following result.

**Theorem 1.7.** There are non-trivial homotopy classes $e\beta^t$ in $\pi_{8t-3} L_1 L_{K(2)} V(0)$ and $e\rho_2 \beta^t$ in $\pi_{8t-4} L_1 L_{K(2)} V(0)$.

This has the following immediate consequence.

**Theorem 1.8.** The homotopy group $\pi_k L_1 L_{K(2)} V(0)$ is non-zero when $k \equiv 5$ modulo 8. Therefore, the decomposition (1.4) of the chromatic splitting conjecture does not hold when $n = 2$ and $p = 2$.

The broad strokes of the proof of Theorem 1.7 when $t = 0$ are as follows. Let $G_{24} \cong Q_8 \rtimes C_3$ be a representative of the unique conjugacy class of maximal finite subgroups of $S_2$. Let $C_6$ be its subgroup of order 6. Let $S_2^1$ be the norm one subgroup so that $S_2 \cong S_2^1 \times \mathbb{Z}_2$. It follows from the duality resolution techniques of Goerss, Henn, Mahowald and Rezk and the work of Bobkova in [5] that, for $X$ finite, there is a spectral sequence

$$E_1^{p,t} = \pi_t (E_p \wedge X) \Longrightarrow \pi_{t-p} (E_2^{hS_2^1} \wedge X),$$
where $E_p$ are spectra such that $\pi_*E_p \cong \pi_*E^{G_{24}}_p$ if $p = 0$, $\pi_*E_p \cong \pi_*E^{hG_2}_p$ if $p = 1, 2$ and $\pi_*E_p \cong \pi_*\Sigma^4 E^{hG_{24}}_2$ if $p = 3$. Localizing at $E(1)$, we obtain spectral sequences

$$E^{p,t}_1 = \pi_1 L_1(E_p \wedge X) \xrightarrow{t} \pi_{t-p} L_1(E^{hG_2}_p \wedge X).$$

At chromatic level $n = 1$, Mahowald and Miller have shown for any spectrum $X$ with a $v_1$ self-map, $L_1X \cong v_1^{-1}X$. We use this to show that the homotopy groups $\pi_1(L(E^{hG_{24}}_2 \wedge V(0)))$ are 24-periodic with periodicity generator $\Delta$. The class

$$\Delta^{-2} \in E^{3,0}_1 = \pi_0 L_1(\Sigma^4 E^{hG_{24}}_2 \wedge V(0))$$

is a permanent cycle in (1.9). It is Galois invariant and represents a class $e$ in $\pi^{-3}L_1(E^{hG_2}_2 \wedge V(0))$. In the fiber sequence

$$L_1 L_{K(2)} V(0) \rightarrow L_1(E^{hG_2}_2 \wedge V(0)) \rightarrow L_1(E^{hG_2}_2 \wedge V(0)),$$

this class gives rise to classes $e \in \pi_{-3}L_1L_{K(2)}V(0)$ and $\rho_2 e \in \pi_{-4}L_1L_{K(2)}V(0)$.

We give a quick comparison to the situation at the prime 3. There are duality resolution spectral sequences

$$E^{p,t}_1 = \pi_1 L_1(\mathcal{F}_p \wedge X) \xrightarrow{t} \pi_{t-p} L_1(E^{hG_2}_p \wedge X).$$

with $\pi_*\mathcal{F}_p \cong \pi_*E^{hG_{24}}_p$ if $p = 0$, $\pi_*\mathcal{F}_p \cong \pi_*\Sigma^8 E^{hSD_{16}}_2$ if $p = 1, 2$ and $\pi_*\mathcal{F}_p \cong \pi_*\Sigma^4 E^{hG_{24}}_2$ if $p = 3$ (see [9] for the definitions of $SD_{16}$ and $G_{24}$). For $X = V(0)$, it follows from the computations of Henn, Karamanov and Mahowald in [11] that the class $\Delta^{-2} \in E^{3,0}_1$ is not a permanent cycle. Therefore, it does not give rise to a class in $\pi_{-3}L_1L_{K(2)}V(0) \cong v_1^{-1} \pi_{-3}L_1L_{K(2)}V(0)$. Note however that there is a spectral sequence $E^{p,t}_1 = \pi_1 \mathcal{F}_p \Rightarrow \pi_{t-p}E^{hG_2}_p$ and a permanent cycle $3\Delta^{-2} \in E^{3,0}_1$. This class gives rise to an element in $\pi_{-3}L_0L_{K(2)}S$, which Goerss, Henn and Mahowald call $e$ in [10]. This highlights the difference between the prime 2 and the prime 3.

**Organization of the paper.** In Section 2, we specialize to the case $n = 2$ and $p = 2$ and describe the duality resolution spectral sequence and its $E(1)$-localization. In Section 3, we compute the $E_1$-page of this spectral sequence for $V(0)$. In Section 4, we use the previous computations to prove Theorem 1.7.

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## 2. The $E(1)$-local Duality Resolution Spectral Sequence

We take the point of view that, at height 2, the Honda formal group law may be replaced by the formal group law of a super singular elliptic curve. This was carefully explained in [3]. (The reader who wants to ignore this subtlety may take $S_C$, $G_C$ and $E_C$ to mean $S_2$, $G_2$ and $E_2$ respectively.)

Let $S_C$ be the group of automorphisms of the formal group law of the super singular elliptic curve

$$C : y^2 + y = x^3$$

of height two over $\mathbb{F}_4$ (see [3] for the comparison.) It admits an action of the Galois group $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Define

$$G_C = S_C \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$
THE CHROMATIC SPLITTING CONJECTURE AT $n = p = 2$

Let $E_C$ be the spectrum which classifies the deformations of the formal group law of $C$ over $\mathbb{F}_4$ as described in [17]. It can be chosen to be a complex oriented ring spectrum with

$$(E_C)_* = \mathbb{W}[[u_1]][u^\pm 1]$$

for $|u_1| = 0$, $|u| = -2$, whose formal group law is the formal group law of the curve

$$C_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3.$$  

It admits an action of $G_C$ and for any finite spectrum $X$,

$$L_{K(2)}X \simeq E_C^{hG_C} \wedge X \simeq (E_C \wedge X)^{hG_C}.$$  

The group of automorphisms $\text{Aut}(C)$ of $C$ is of order 24 and injects into $S_C$. We let $G_{24}$ denote the image of $\text{Aut}(C)$. We note that $G_{24} \cong Q_8 \rtimes C_3$, where $Q_8$ is a quaternion subgroup and $C_3$ a cyclic group of order 3. The group $S_C$ contains a central subgroup of order 2, which we denote by $C_2$. We define

$$C_6 = C_2 \times C_3.$$  

The groups $S_C$ and $G_C$ admit a norm. We let $\mathcal{S}_C^1$ and $\mathcal{G}_C^1$ be the kernel of that norm, and note that the torsion of $S_C$ and $G_C$ is contained in $\mathcal{S}_C^1$ and $\mathcal{G}_C^1$ respectively. Further,

$$(2.2) \quad S_C \cong \mathcal{S}_C^1 \rtimes \mathbb{Z}_2, \quad G_C \cong \mathcal{G}_C^1 \rtimes \mathbb{Z}_2.$$  

We will need the following theorem of Irina Bobkova, which is Theorem 1.1 of [5]. We restate it here using our notation for convenience. To avoid technicalities about convergence and homotopy fixed points, we assume that $X$ is a finite spectrum.

**Theorem 2.3** (Goerss, Henn, Mahowald, Rezk, Bobkova). There is a resolution of spectra in the $K(2)$-local category given by

$$E_C^{hG_{24}} \rightarrow E_C^{hC_6} \rightarrow E_C^{hC_6} \rightarrow E_C^{hC_6} \rightarrow E_3$$

where $\pi_*E_3 \simeq \pi_*\Sigma^{48}E_C^{hG_{24}}$. Further, for any finite spectrum $X$, the resolution gives rise to a tower of fibrations spectral sequence

$$(2.4) \quad E_{p,t}^0 = \pi_t(E_p \wedge X) \xrightarrow{SS_1} \pi_{t-p}(E_C^{hG_{24}} \wedge X)$$

with differentials $d_r : E_{p,t}^r \rightarrow E_{p+r,t-r-1}$.

We call the resolution of Theorem 2.3 the **duality resolution**. Let $\pi$ generate $\mathbb{Z}_2$ in the decompositions (2.2) and let $G'_{24} = \pi G_{24} \pi^{-1}$. Recall from [2] or [3] that there is also an **algebraic duality resolution**

$$(2.5) \quad \mathbb{Z}_2[[S_C/G'_{24}]] \rightarrow \mathbb{Z}_2[[S_C/C_6]] \rightarrow \mathbb{Z}_2[[S_C/C_6]] \rightarrow \mathbb{Z}_2[[S_C/G_{24}]] \rightarrow \mathbb{Z}_2$$

where $\mathcal{E}_3 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0$. 
Resolving (2.5) into a double complex of projective $\mathbb{S}_1$-modules and applying the functor $\text{Hom}_{\mathbb{Z}_2[[\mathbb{S}_1]]}(-, (E_C)_tX)$ gives rise to a spectral sequence

\[
E_1^{p,q} = \text{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_1]]}(\mathcal{E}_p, (E_C)_tX) \xrightarrow{SS_2} H^{p+q}(\mathbb{S}_1, (E_C)_tX)
\]

with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$. Further, in each fixed degree $p$, there are descent spectral sequences

\[
E_1^{s,t} = \text{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_1]]}(\mathcal{E}_p, (E_C)_tX) \xrightarrow{SS_3} \pi_{t-s}(\mathcal{E}_p \wedge X)
\]

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. Finally, there is also a descent spectral sequence

\[
E_1^{s,t} = H^s(\mathbb{S}_1, (E_C)_tX) \xrightarrow{SS_3} \pi_{t-s}(E^{h\mathbb{S}_1}_C \wedge X)
\]

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. Therefore, we obtain a diagram of spectral sequences

\[
\begin{array}{ccc}
\text{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_1]]}(\mathcal{E}_p, (E_C)_tX) & \xrightarrow{SS_2} & H^{p+q}(\mathbb{S}_1, (E_C)_tX) \\
\downarrow{SS_1} & & \downarrow{SS_4} \\
\pi_{t-s}(\mathcal{E}_p \wedge X) & \xrightarrow{SS_3} & \pi_{t-(p+q)}(E^{h\mathbb{S}_1}_C \wedge X).
\end{array}
\]

For elements of Adams-Novikov filtration $(s, t) = (0, t)$ in $E_1^{p,q}$ of the duality resolution spectral sequence $SS_1$, the differentials $d_1$ are induced by the $d_1$-differentials in the algebraic duality resolution spectral sequence $SS_2$. This is made precise in the following result.

**Proposition 2.10.** Let $x$ in $E_1^{p,t} = \pi_t(\mathcal{E}_p \wedge X)$ in the spectral sequence of Theorem 2.3 be detected by a class $\tilde{x}$ of degree $(p, 0, t)$ in

\[
\text{Ext}^0_{\mathbb{Z}_2[[\mathbb{S}_1]]}(\mathcal{E}_p, (E_C)_tX) \cong \text{Hom}_{\mathbb{Z}_2[[\mathbb{S}_1]]}(\mathcal{E}_p, (E_C)_tX).
\]

The differential $d_1 : E_1^{p,t} \to E_1^{p+1,t}$ on $x$ in the duality resolution spectral sequence (2.4) is induced by the differential $d_1 : E_1^{p,0} \to E_1^{p+1,0}$ on $\tilde{x}$ in the algebraic duality resolution spectral sequence (2.6).

**Proof.** The differential $d_1 : \pi_t(\mathcal{E}_p \wedge X) \to \pi_t(\mathcal{E}_{p+1} \wedge X)$ is induced by the map $\delta_{p+1} \wedge 1_X : \mathcal{E}_p \wedge X \to \mathcal{E}_{p+1} \wedge X$, where

\[
\delta_{p+1} : \mathcal{E}_p \to \mathcal{E}_{p+1}
\]

is the map in the duality resolution of Theorem 2.3. Let

\[
h : \pi_t(\mathcal{E}_p \wedge X) \to (E_C)_t(\mathcal{E}_p \wedge X)
\]

be the Hurewicz homomorphism. Let $\partial_{p+1} : \mathcal{E}_{p+1} \to \mathcal{E}_p$ be the map in the algebraic duality resolution (2.5). As in [9, Proposition 2.4 and (2.7)], for $X$ finite and $G$ a closed subgroup of $G_L$, there are isomorphisms of Morava modules

\[
(E_C)_t(E_C^G \wedge X) \cong \text{Hom}^e(G, (E_C)_tX) \cong \text{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2[[G_{/G}]], (E_C)_tX).
\]

Therefore, $\partial_{p+1}$ induces a map

\[
\partial_{p+1}^* : (E_C)_t(\mathcal{E}_p \wedge X) \to (E_C)_t(\mathcal{E}_{p+1} \wedge X).
\]
Further, there is a natural map
\[ \text{Hom}_{\text{Z}_2[[\mathbb{F}_p]]}(E_p, (E_C)_t X) \to (E_C)_t (E_p \wedge X) \]
obtained from the isomorphisms (2.11). The map \( \delta_{p+1} \) is constructed so that the following diagram commutes,
\[
\begin{array}{ccc}
\text{Hom}_{\text{Z}_2[[\mathbb{F}_p]]}(E_p, (E_C)_t X) & \xrightarrow{\partial_{p+1}} & \text{Hom}_{\text{Z}_2[[\mathbb{F}_p]]}(E_{p+1}, (E_C)_t X) \\
(E_C)_t (E_p \wedge X) & \xrightarrow{\partial_{p+1}} & (E_C)_t (E_{p+1} \wedge X) \\
\pi_t (E_p \wedge X) & \xrightarrow{\pi_t (\delta_{p+1} \wedge 1_X)} & \pi_t (E_{p+1} \wedge X)
\end{array}
\]
(see the proof of [5, Theorem 1.1]). The top map induces the differential
\[ d_1 : E_1^{p,0} \to E_1^{p+1,0} \]
in the algebraic duality resolution spectral sequence (2.6).

Now, recall that the Telescope conjecture holds at height \( n = 1 \). This was proved at odd primes by Miller [15] and at \( p = 2 \) by Mahowald [13]. In particular, we have the following result. For any spectrum \( X \) (not necessarily finite) which admits a \( v_1 \)-self map, let
\[ v_1^{-1} X := \text{colim} \left( \ldots \xrightarrow{v_1^k} \Sigma^{2k} X \xrightarrow{v_1^k} X \xrightarrow{v_1^k} \ldots \right). \]
Similarly, we let
\[ v_1^{-1} \pi_s X = \text{colim} \left( \ldots \xrightarrow{v_1^k} \pi_{s-2k} X \xrightarrow{v_1^k} \pi_s X \xrightarrow{v_1^k} \ldots \right). \]

**Theorem 2.12** (Mahowald). Let \( X \) admit a \( v_1 \) self-map \( v_1^k : \Sigma^{2k} X \to X \). Then
\[ L_1 X \simeq L_{K(1)} X \simeq v_1^{-1} X. \]

**Corollary 2.13.** For any finite type 1 spectrum \( X \), with self map \( v_1^k : \Sigma^{2k} X \xrightarrow{v_1^k} X \), there is a diagram of strongly convergent spectral sequences
\[
\begin{array}{ccc}
v_1^{-1} \text{Ext}^2_{\text{Z}_2[[\mathbb{F}_p]]}(E_p, (E_C)_t X) & \xrightarrow{L_1 S S_2} & v_1^{-1} H^{p+q}(S^1_{\mathbb{C}}, (E_C)_t X) \\
\pi_{t-q} L_1 (E_p \wedge X) & \xrightarrow{L_1 S S_1} & \pi_{t-(p+q)} L_1 (E_C^{\mathbb{H} S^1} \wedge X)
\end{array}
\]
Furthermore, in each spectral sequence (2.9), one may invert \( v_1^k \) at the \( E_r \)-term for all \( r \).

**Proof.** The spectra \( E_p \wedge X \) have \( v_1 \)-self maps if \( X \) is of type 1. Hence,
\[ v_1^{-1} \pi_s (E_p \wedge X) \cong \pi_s L_1 (E_p \wedge X). \]
Localization with respect to \( v_1 \) is exact. Therefore, the localized spectral sequences will converge if they have horizontal vanishing lines at the \( E_\infty \)-term. The spectral
sequences $SS_1$ and $SS_2$ have a vanishing line at $p = 4$ for all $r \geq 1$. As noted in the proof of [7, Lemma 3.5], for any closed subgroup $G$, it follows from the fact that $E^h_G$ is $K(n)_*$-local $E_n$-nilpotent, (see [8, Proposition A.3]) that the descent spectral sequences $SS_3$ and $SS_4$ have horizontal vanishing lines. Therefore, the spectral sequences $L_1SS_1$ exist and converge.

3. The homotopy of $L_1(E^h_{C_G}(V(0))$ and $L_1(E^h_{C} \wedge V(0))$

The spectrum $V(0)$ has a self map

$$\beta : \Sigma^4 V(0) \xrightarrow{v^4} V(0).$$

In this section, we give the $E_1$-term for

$$E_1^{p,q} = \pi_q(\partial \wedge V(0)) \xrightarrow{SS_1} \pi_{q-p}(E^h_{C} \wedge V(0)).$$

In order to do so, we must compute $\pi_* L_1(E^h_{C_G}(V(0))$ and $\pi_* L_1(E^h_{C} \wedge V(0))$. We do this using the descent spectral sequences

$$v_1^{-1} H^*(G,(E_{C})_\ast V(0)) \Rightarrow \pi_* L_1(E^h_{G} \wedge V(0)).$$

**Notation 3.1.** We use the following conventions. First,

$$v_1 = u_1 u^{-1},$$

$$v_2 = u^{-3},$$

$$j_0 = u_1^3.$$

The element $\Delta$ is the discriminant of $C_U$, and hence is given by

$$\Delta = 27v_2(v_1^3 - v_2^3) \equiv v_2(v_1^3 + v_2)^3 \mod (2),$$

and

$$c_4 = 9v_1^4 + 72v_1 v_2 \equiv v_1^4 \mod (2).$$

The $j$-invariant is

$$j = c_4^3 \Delta^{-1} \equiv v_1^{12} \Delta^{-1} \mod (2).$$

We let

$$h_1 = \delta(v_1),$$

where $\delta$ is the Bockstein associated to

$$(E_{C})_\ast \xrightarrow{2} (E_{C})_\ast \rightarrow (E_{C})_\ast V(0).$$

Parameters $a_1$ and $a_3$ are often used in the literature. The correspondence is given by

$$a_1 = 3v_1,$$

$$a_3 = v_1^3 - v_2.$$

**Proposition 3.2.** There are isomorphisms

$$v_1^{-1} H^*(G_{C_4},(E_{C})_\ast V(0)) \cong \mathbb{F}_4[[j]][v_1^\pm 1, \Delta^\pm 1, h_1]/(\Delta^{-1} v_1^{12} = j)$$

and

$$v_1^{-1} H^*(C_6; (E_{C})_\ast V(0)) \cong \mathbb{F}_4[[j_0]][v_1^\pm 1, v_2^\pm 1, h_1]/(v_1^3 = v_2 j_0).$$

The degrees $(s,t)$ (for $s$ the cohomological grading, and $t$ the internal grading) are given by $|h_1| = (1,2)$, $|v_2| = (0,6)$, $|\Delta| = (0,24)$, $|v_1| = (0,2)$, $|j| = (0,0)$ and $|j_0| = (0,0)$.
Proof. This follows from Theorem 4.11 and Lemma 4.12 of [3] after inverting \( v_1 \). \( \square \)

**Proposition 3.3.** Let
\[
\mathcal{R} = (\mathbb{W}[j][\eta, \beta, \Delta^\pm]/(\beta^4 = j\Delta, 2\eta, \eta^2, 2\beta, 2\Delta, 2j)) [\beta^{-1}],
\]
where the Adams-Novikov filtration \((s, t)\) of \( \eta \) is \((1, 2)\), that of \( \beta \) is \((0, 8)\), that of \( \Delta \) is \((0, 24)\) and that of \( j \) is \((0, 0)\). Then
\[
\pi_*L_1(E^{hG_{24}}_C \wedge V(0)) = \mathcal{R}\{1, \eta\}/(2 \cdot 1, 2 \cdot \eta - \eta^2 \cdot 1),
\]
where the Adams-Novikov filtration \((s, t)\) of \( 1 \) is \((0, 0)\) and that of \( \eta \) is \((0, 2)\). (See Figure 2.)

Proof. The differentials \( d_4(v_1^1\Delta^k) = h_3^2\Delta^k \) and \( d_3(v_1^1\Delta^k) = v_1^3h_3\Delta^k \) together with the fact that the differentials are \( v_1^1 \) linear determine all \( d_4 \)-differentials. The \( E_4 \)-term has a horizontal vanishing line at \( s = 3 \). Therefore, there cannot be any higher differentials. The extension \( 2 \cdot \eta = \eta^2 \cdot 1 \) is already present in \( \pi_2V(0) \). \( \square \)

**Proposition 3.4.** Let
\[
\mathcal{S} = (\mathbb{W}[j_0][\eta, \beta, \gamma, \Delta^\pm]/(\gamma^3 = j_0\Delta, 2\eta, \eta^2, 2\beta, 2\gamma, 2\Delta, 2j_0)) [\beta^{-1}],
\]
where the Adams-Novikov filtration \((s, t)\) of \( \eta \) is \((1, 2)\), that of \( \beta \) is \((0, 8)\), that of \( \Delta \) is \((0, 24)\) and that of \( j_0 \) is \((0, 0)\) and that of \( \gamma \) is \((0, 8)\). Then
\[
\pi_*L_1(E^{hC_6}_C \wedge V(0)) = \mathcal{S}\{1, \eta\}/(2 \cdot 1, 2 \cdot \eta - \eta^2 \cdot 1),
\]
where the Adams-Novikov filtration \((s, t)\) of \( 1 \) is \((0, 0)\) and that of \( \eta \) is \((0, 2)\). The element \( \gamma \) is detected by \( v_1v_2 + v_1^4 \). (See Figure 2.)

**Figure 2.** The homotopy groups \( \pi_*L_1(E^{hG_{24}}_C \wedge V(0)) \) (top) and the homotopy groups \( \pi_*L_1(E^{hC_6}_C \wedge V(0)) \) (bottom). A \( \circ \) denotes a copy of \( \mathbb{F}_4[\beta^\pm] \), so that not all \( \mathbb{F}_4 \)-generators appear.

**Proof.** Let \( TMF(\Gamma_0(3)) \), \( M(\Gamma_0(3)) \) and \( \omega \) be as in [14]. There is a map of spectral sequences
\[
H^*(\mathcal{M}(\Gamma_0(3)), \omega^\otimes) \longrightarrow H^*(C_6, (E_C)_s)^{\Gal(\mathbb{F}_4/\mathbb{F}_2)}
\]
\[
\pi_*TMF(\Gamma_0(3)) \longrightarrow \pi_*E^{hC_6}_C)^{h\Gal(\mathbb{F}_4/\mathbb{F}_2)}.
\]
This implies that differentials in the spectral sequence of [14] induce differentials in
\[
H^*(C_6, (E_C)_s) \Longrightarrow \pi_*E^{hC_6}_C.
\]
Further, the pinch map \( p : V(0) \to S^1 \) induces a map of spectral sequences

\[
H^\ast(C_6, (E_C) \ast V(0)) \longrightarrow H^\ast(C_6, (E_C) \ast S^1) \quad (3.6)
\]

Therefore, differentials \( d_r(p(x)) = p(y) \) induce differentials \( d_r(x) = y \) if all classes are defined at the \( E_r \)-term.

We use (3.5) and (3.6) to compute the differentials in

\[
v_1^{-1}H^\ast(C_6, (E_C) \ast V(0)) \Longrightarrow \pi_\ast L_1(E^h_{C_6} \ast V(0)).
\]

Given our choice of curve \( C_U \), we have \( a_1 = 3v_1, a_3 = v_1^3 - v_2 \). Then (3.5) induces the following differentials. We note the classic differentials

\[
d_3(v_1^2) = h_1^3, \quad d_3(v_1) = h_1^2 v_1.
\]

From Section 4 of [14], letting \( \zeta = v_1^{-1}h_1 \), we obtain that

\[
d_3(h_1) = d_3(v_1v_2) = d_3(\zeta v_3^2) = 0, \quad d_3(v_3^2) = \zeta^2 h_1 v_2^2, \quad d_3(\zeta a_2) = \zeta^3 h_1 a_2.
\]

It follows from (3.6) that

\[
d_3(v_3^2) = 0, \quad d_3(v_2) = \zeta^2 h_1 a_2.
\]

This determines all \( d_3 \)-differentials. The \( E_4 \)-term has a horizontal vanishing line at \( s = 3 \), hence the spectral sequence collapses. \( \square \)

**Corollary 3.7.** There are isomorphisms

\[
\pi_\ast L_1(E^h_{C_6} \ast V(0)) \cong \pi_\ast L_1(\Sigma^{24} E^h_{G_{24}} \ast V(0))
\]

and

\[
\pi_\ast L_1(E^h_{S^1} \ast V(0)) \cong \pi_\ast L_1(\Sigma^{24} E^h_{G_{24}} \ast V(0)).
\]

**Proof.** This follows from Proposition 3.3 and Proposition 3.4, noting that the homotopy groups of \( L_1(E^h_{G_{24}} \ast V(0)) \) and \( L_1(E^h_{C_6} \ast V(0)) \) are 24-periodic, with periodicity generator \( \Delta \). \( \square \)

4. Some Elements in \( \pi_\ast L_1 L_{K(2)} V(0) \)

We now turn to examining the spectral sequence

\[
E_1^{p,q} = \pi_q L_1(\mathcal{E}_p \ast V(0)) \xrightarrow{L_{1,SS}} \pi_{q-p} L_1(E^h_{S^1} \ast V(0)).
\]

Note that \( \mathcal{E}_p \cong E^h_G \) for \( G \) one of \( G_{24}, G_{24}' \) or \( C_6 \). For

\[
x \in \pi_q L_1(E^h_G \ast V(0)) \cong \pi_q L_1(\mathcal{E}_p \ast V(0))
\]

we denote the element it represents in \( E_1^{p,q} \) by \( x[p] \).

**Theorem 4.1.** The spectral sequence

\[
v_1^{-1}H^\ast(S^1, (E_C) \ast V(0)) \Longrightarrow \pi_\ast L_1(E^h_{S^1} \ast V(0))
\]

collapses at the \( E_2 \)-term and, as an \( \mathbb{F}_4[v_1^{-1}, h_1] \)-module, the \( E_\infty \)-term has rank 6 on generators \( 1[0], 1[1], \gamma_1[1], 1[2], \gamma_2[2] \) and \( 1[3] \), where

\[
\widetilde{\gamma}_p = v_1 v_2 \lambda_p
\]

for a unit \( \lambda_i \in \mathbb{F}_4[[v_0]] \).
Proof. By Corollary 2.13, we may localize the spectral sequence
\[ E_1^{p,q} = \text{Ext}^q_{Z_2[[\Sigma]]}(\mathcal{C}_p, (E_C V(0))_i) \xrightarrow{SS_2} H^{p+q}(S^1_C, (E_C)_i V(0)) \]
at the $E_2$-term. By Theorem 1.8 of [3],
\[ v_1^{-1} E_2^{p,q} = F_4[v_1^{\pm 1}, h_1\{1[0], 1[1], \tilde{\gamma}_1[1], 1[2], \tilde{\gamma}_1[2], 1[3]\}], \]
where $1[0], 1[1], 1[2], 1[3]$ are detected by $a_0, b_0, c_0$ and $d_0$ in [3]), and $\tilde{\gamma}_1[1]$ and $\tilde{\gamma}_2[1]$ are detected by $v_1b_1$ and $v_1c_1$ in [3]. Since the differentials $d_r$ are $v_1$ and $h_1$-linear, the spectral sequence must collapse.

**Corollary 4.2.** Let $\gamma_p[p]$ be the class detected by $\tilde{\gamma}_p[p]$ in $\pi_* L_1(\mathcal{C}_p \wedge V(0))$. Let
\[ B = (W_2[\eta, \beta^\pm 1]/(2\eta, \eta^3, 2\beta)) \{1, \overline{\eta}\}/(2 \cdot 1, 2 \cdot \overline{\eta} = \eta^2 \cdot 1). \]
The $E_2$-term of the spectral sequence
\[ E_1^{p,q} = \pi_q L_1(\mathcal{C}_p \wedge V(0)) \xrightarrow{SS_2} \pi_{q-p} L_1(E_C^{h_{S^1}} \wedge V(0)) , \]
is given by
\[ E_1^{p,*} = \begin{cases} B\{1[0]\} & p = 0 \\ B\{1[1], \gamma_1[1]\} & p = 1 \\ B\{1[2], \gamma_2[2]\} & p = 2 \\ B\{1[3]\} & p = 3. \end{cases} \]
(The $E_2$-term is depicted in Figure 3.)

**Proof.** By Proposition 2.10, the differentials $d_1$ on classes in $E_1^{p,0}$ are determined by the differentials in the spectral sequence
\[ v_1^{-1} \text{Ext}^q_{Z_2[[\Sigma]]}(\mathcal{C}_p, (E_C)_i V(0)) \xrightarrow{SS_2} v_1^{-1} H^{p+q}(S^1_C, (E_C)_i V(0)). \]
It follows from Theorem 4.1 that $E_2^{p,0}$ is as claimed. Now note that the differentials are $\eta$-linear. Since all classes in $E_1^{p,q}$ for $q > 0$ are in the image of $\eta$, the claim follows from $\eta$-linearity.

**Proposition 4.3.** The group $\pi_k L_1(E_C^{h_{S^1}} \wedge V(0))$ is $F_4$ if $k \equiv 5$ modulo 8.

**Proof.** The element $\beta'[3] \in E_2^{3,8t}$ is a permanent cycle. Further, it cannot be a boundary since $E_2^{1,8t-1}$ and $E_2^{2,8t-2}$ are both zero.

**Corollary 4.4.** The group $\pi_k L_1(E_C^{h_{S^1}} \wedge V(0))$ is $F_2$ if $k \equiv 5$ modulo 8.

**Proof.** This follows from Proposition 4.3 by taking fixed points with respect to the Galois group $\text{Gal}(F_4/F_2)$.

**Definition 4.5.** We define the class $e \in \pi_{-3} L_1(E_C^{h_{S^1}} \wedge V(0))$ to be the non zero generator.
Figure 3. The $E_2$-term of the spectral sequence $E_1^{p,q} = \pi_q L_1(\mathbb{F}_2 \wedge V(0))$ converging to $\pi_{q-p} L_1(E_C^{hG^1} \wedge V(0))$. Horizontal rows correspond to $E_2^{p,*}$, where the bottom row corresponds to $E_2^{0,*}$. The horizontal grading in the $p$-th row corresponds to $q$, while the vertical grading is the Adams-Novikov grading. The $p$-th row is drawn with a shift so that the total columns corresponds to a homotopy groups. A • denotes a copy of $\mathbb{F}_2$ (as opposed the a o in Figure 2, which is why the $E_2$-term appears to contain more classes than the $E_1$-term at first glance). Lines of slope $(1,1)$ denote mutliplication by $\eta$ and vertical lines denote multiplication by 2.

Recall that

$$G_C \cong G_C^1 \rtimes \mathbb{Z}_2.$$ Let $\pi$ be a topological generator of the subgroup $\mathbb{Z}_2$ in $G_C$. There is a fiber sequence

$$L_K(2)S \to E_C^{hG^1} \xrightarrow{\pi^{-1}} E_C^{hG^1}.$$ We can now prove our main result.

Proof of Theorem 1.7. Since $L_K(2)S \wedge V(0) \simeq L_K(2)V(0)$ and localization preserves exact triangles (Lemma 1.10 of [6]), the fiber sequence (4.6) gives rise to a fiber sequence

$$L_1 L_K(2)V(0) \to L_1(E_C^{hG^1} \wedge V(0)) \xrightarrow{\pi^{-1}} L_1(E_C^{hG^1} \wedge V(0)).$$ There are no non-trivial representations of $\mathbb{F}_2$. Therefore, in the long exact sequence on homotopy groups, the class $e$ is in the kernel of $\pi - 1$, and the image of $e$ under the map induced by $L_1(E_C^{hG^1} \wedge V(0)) \to \Sigma L_1 L_K(2)V(0)$ is non-zero. We denote it by $\rho_2 e$.

Definition 4.5 is motivated by the fact that $e$ is in the image of

$$H^3(S_1^1, \mathbb{F}_2) \to H^3(G^1_C, (E_C)_*V(0)).$$
In fact, $e$ can be lifted to $H^3(S^1_p, \mathbb{Z}_2)$ where it generates a copy of $\mathbb{Z}_2$. The name $e$ is chosen in analogy to the class $e \in H^3(G_2, (E_2)_0)$ of [10] (see Theorem 4.3). At $p = 3$, the class $e$ corresponds to the class which Hovey calls $\rho$ in $H^3(S_2, \mathbb{Z}_p)$ for $p \geq 5$. The class $e$ at $p = 3$ (resp. $\rho$ at $p \geq 5$) survives to a map $e : S^3 \rightarrow L_{K(2)} S$ such that the composite $S^{-3} \rightarrow L_{K(2)} S \rightarrow \Sigma F_2$ factors through $L_0 S^{-3}$.

Note that the name $\rho$ already has a meaning at the prime 2. The class $\rho_2$ in [16] is one of the classes in $H^1(S_C, F_2)$ which comes from the determinant. There is another class coming from the determinant in $H^1(S_C, F_2)$ called $\zeta_2$. Although $\rho_2$ is not detected in $H^1(S^1_p, F_2)$, the class $\zeta_2$ is detected in the spectral sequence

$$E_1^{p,q} = \text{Ext}^2_{\mathbb{Z}_2[[\mathbb{Z}_2]]}(F_p, F_2) \Rightarrow H^{p+q}(S^1_C, F_2)$$

by the class $1[1] \in E_1^{1,0}$. Note that $\zeta_2$ is non-zero in $H^2(S^1_p, F_2)$ and is detected by $1[2] \in E_1^{2,0}$. Further, $\zeta_2$ can be lifted to a class of order 2 in $H^2(S^1_p, \mathbb{Z}_2)$.

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