Training Linear Neural Networks: Non-Local Convergence and Complexity Results

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Abstract

Linear networks provide valuable insight into the workings of neural networks in general.

In this paper, we improve the state of the art in [1] by identifying conditions under which gradient flow successfully trains a linear network, in spite of the non-strict saddle points present in the optimization landscape.

We also improve the state of the art for computational complexity of training linear networks in [2] by establishing non-local linear convergence rates for gradient flow.

Crucially, these new results are not in the lazy training regime, cautioned against in [3, 4].

Our results require the network to have a layer with one neuron, which corresponds to the popular spiked covariance model in statistics, and subsumes the important case of networks with a scalar output. Extending these results to all linear networks remains an open problem.

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1 Introduction and Overview

Consider the training samples and their labels \((x_i, y_i)_{i=1}^m \subset \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}\), respectively. By concatenating \(\{x_i\}_i\) and \(\{y_i\}_i\), we form the matrices

\[
X \in \mathbb{R}^{d_x \times m}, \quad Y \in \mathbb{R}^{d_y \times m}.
\]

A linear network is a neural network where the non-linear activation functions are replaced with the identity map.

To be specific, consider a linear network with \(N\) layers and the corresponding weight matrices \(\{W_i\}_{i=1}^N\). This network is characterized by the linear map

\[
\begin{align*}
\mathbb{R}^{d_x} & \to \mathbb{R}^{d_y} \\
x & \to Wx,
\end{align*}
\]

and the matrix \(W \in \mathbb{R}^{d_y \times d_x}\) in (2) is specified with the (often over-parametrized) map

\[
\begin{align*}
\mathbb{R}^{d_1 \times d_0} \times \cdots \times \mathbb{R}^{d_N \times d_N-1} & \to \mathbb{R}^{d_y \times d_x} \\
(W_1, \ldots, W_N) & \to W := W_N \cdots W_1.
\end{align*}
\]

Above, we set \(d_0 = d_x\) and \(d_N = d_y\) for consistency.

In foregoing the full generality of nonlinear neural networks, linear networks afford us a level of insight and technical rigor that is out of the reach for the nonlinear networks, at least with our current technical tools [5, 6, 7, 8, 9, 10, 11, 12].

Indeed, despite the absence of activation functions, matrix \(W\) in linear network (2,3) is a nonlinear function of \(\{W_i\}_i\), and training this linear network thus involves solving a nonconvex optimization problem in \(\{W_i\}_i\), which shares many interesting features of the nonlinear neural networks.

Simply put, we cannot claim to understand neural networks in general without understanding linear networks.

Training the linear network (2,3) with the data \((X, Y)\) in (1) can be cast as the optimization problem

\[
\begin{align*}
& \min_{W_1, \ldots, W_N} \frac{1}{2} \|Y - W_N W_{N-1} \cdots W_1 X\|_F^2 \\
& \text{subject to } W_j \in \mathbb{R}^{d_{j} \times d_{j-1}} \quad \forall j \in [N],
\end{align*}
\]

which is nonconvex when \(N \geq 2\), and \([N] = \{1, \ldots, N\}\).

Let us introduce the shorthand

\[
W_N := (W_1, \ldots, W_N) \in \mathbb{R}^{d_1 \times d_0} \times \cdots \times \mathbb{R}^{d_N \times d_{N-1}} :=: \mathbb{R}^{d_N},
\]

which allows us to rewrite problem (4) more compactly as

\[
\min_{W_N} L_N(W_N) \text{ subject to } W_N \in \mathbb{R}^{d_N},
\]

where \(L_N(W_N) := \frac{1}{2} \|Y - W_N \cdots W_1 X\|_F^2\). With this setup and before turning to the details, let us highlight the contributions of this paper, in the order of appearance.

- Theorem 8 in Section 2 provides a new analysis for the optimization landscape of linear networks, where we identify a clear link to the celebrated Eckart-Young-Mirsky theorem and the geometry of the principal component analysis (PCA).

- Theorem 16 in Section 3 improves the state of the art in [1] by identifying conditions under which gradient flow successfully trains a linear network, despite the
presence of non-strict saddle points in the optimization landscape.

Theorem 16 appears to be the first such result outside of the lazy training regime, as detailed later.

Theorem 16 is applicable to linear networks that have a layer with a single neuron, see Assumption 14. This case corresponds to the popular spiked covariance model in statistics, and subsumes the important case of networks with a scalar output. Extension of Theorem 16 to all linear networks remains a challenging open problem.

- Theorem 20 in Section 4 improves the state of the art in [2] by establishing non-local convergence rates for gradient flow, and by quantifying the benefits of network depth for faraway convergence.

Theorem 20 again appears to be the first such result outside the lazy training regime. Theorem 20 also corresponds to the spiked covariance (6).

The landscape of nonconvex program (6) has been variation of Proposition 32 in [1] with an additional assumption.

Theorem 20 appears to be the first such result for faraway convergence.

The state of the art for training linear networks, under this new assumption.

To begin, let us concretely define the notion of optimality for problem (6).

Definition 1. (first-order stationarity for (6)) We say that $\nabla L_N(W_N) = 0$, (7)

where $\nabla L_N(W_N)$ is the gradient of $L_N$ at $W_N$.

Definition 2. (second-order stationarity for (6)) We say that $W_N \in \mathbb{R}^{d_N}$ is a second-order stationary point (SOSP) of problem (6) if, in addition to (7), it holds that

$$\nabla^2 L_N(W_N) |_{\Delta N} \succeq 0, \quad \forall \Delta N \in \mathbb{R}^{d_N},$$

where $\nabla^2 L_N(W_N) |_{\Delta N}$ is the second derivative of $L_N$ at $W_N$ along the direction $\Delta N$.

Definition 3. (strict saddles of (6)) Any FOSP of problem (6), which is not an SOSP, is a strict saddle point of (6).

Any SOSP of problem (6) is either a local minimizer of (6), or a non-strict saddle point of problem (6). Unlike a non-strict saddle point, there always exists a descent direction to escape from a strict saddle point [22]. To continue, let

$$r := \min_{j \in N} d_j,$$

denote the smallest width of the linear network (2,3). As shown in Appendix A, we can reformulate problem (6) as

$$\min_{W_N} L_N(W_N) = \min_{\substack{W \in \mathbb{R}^{d_N} \quad \text{subject to} \quad \text{rank}(W) \leq r}} \frac{1}{2} \|Y - WX\|_F^2 := L_1(W),$$

subject to $\text{rank}(W) \leq r$

$$= \min_{P,Q} \frac{1}{2} \|Y - PQX\|_F^2 := L_2(P,Q),$$

subject to $P \in \mathbb{R}^{d_x \times r}$, $Q \in \mathbb{R}^{r \times d_y}$.

In particular, the notion of optimality for problem (10b) is defined similarly to Definitions 1-3.

There is a correspondence between the stationary points of problems (6) and (10b), proved in Appendix B.

**Lemma 4.** Any FOSP $W_N = (W_1, \ldots, W_N)$ of problem (6) corresponds to an FOSP $(\overline{P}, \overline{Q})$ of problem (10b), provided that $\overline{W} = W_N \ldots \cup W_1$ is rank-$r$.

Moreover, any SOSP $W_N$ of problem (6) corresponds to an SOSP $(\overline{P}, \overline{Q})$ of problem (10b), provided that $\text{rank}(\overline{W}) = r$.

Let $P_X := X^\top X$ and $P_{X^\perp} := I_m - P_X$ denote the orthogonal projections onto the row span of $X$ and its orthogonal complement, respectively. Here, $X^\dagger$ is the pseudo-inverse of $X$ and $I_m \in \mathbb{R}^{m \times m}$ is the identity matrix.

Using the decomposition $Y = YP_X + YP_{X^\perp}$, we can in turn rewrite problem (10b) as

$$\min_{P,Q} \frac{1}{2} \|YP_X - PQ\|_F^2 + \min_{P,Q,Q} \frac{1}{2} \|YP_{X^\perp} - PQ\|_F^2$$

subject to $Q' = QX$

$$\geq \frac{1}{2} \|YP_X\|_F^2 + \frac{1}{2} \|YP_{X^\perp}\|_F^2 \geq \frac{1}{2} \min_{P,Q} \frac{1}{2} \|YP_{X^\perp} - PQ\|_F^2 = L_2(P,Q).$$

The relaxation above is tight, and there is a correspondence between the stationary points, proved in Appendix C.

**Lemma 5.** Suppose that $X$ has full column-rank, namely, $XX^\top$ is invertible. Then it holds that

$$-\frac{1}{2} \|YP_X\|_F^2 + \min_{P,Q} L_2(P,Q)$$

subject to $Q' = QX$

$$= \frac{1}{2} \min_{P,Q} \frac{1}{2} \|YP_X - PQ\|_F^2 = L_2(P,Q).$$

Any FOSP $(\overline{P}, \overline{Q})$ of problem (10b) corresponds to an FOSP $(\overline{P}, \overline{Q})$ of problem (12).

Moreover, any SOSP $(\overline{P}, \overline{Q})$ of problem (10b) corresponds to an SOSP $(\overline{P}, \overline{Q})$ of problem (12).
Note that solving problem (12) involves finding a best rank-\(r\) approximation of \(YP_X\) or, equivalently, finding \(r\) leading principal components of \(YP_X\) [23].

By combining Lemmas 4 and 5, we immediately reach the following conclusion.

**Lemma 6.** Suppose that \(X\) has full column-rank. Then any FOSP \(\overline{W}_N\) of problem (6) corresponds to an FOSP \((\overline{P}, \overline{Q})\) of problem (12), provided that \(\overline{W} = \overline{W}_N \cdots \overline{W}_1\) is rank-\(r\).

Moreover, any SOSP \(\overline{W}_N\) of problem (6) corresponds to an SOSP \((\overline{P}, \overline{Q})\) of (12), provided that \(\text{rank}(\overline{W}) = r\).

We next recall a variant of the EYM theorem [20, 21, 24], which specifies the landscape of the PCA problem (12).

**Theorem 7.** Any SOSP \((\overline{P}, \overline{Q})\) of the PCA problem (12) is also a global minimizer of problem (12).

With Lemma 6 at hand, we invoke Theorem 7 to uncover the landscape of problem (6), see Appendix D for the proof.

**Theorem 8.** Suppose that \(X\) has full column-rank and that \(\text{rank}(Y^tX) \geq r\).

Then any SOSP \(\overline{W}_N = (\overline{W}_1, \ldots, \overline{W}_N)\) of problem (6) is a global minimizer of problem (6), provided that \(\overline{W}_N \cdots \overline{W}_1\) is rank-\(r\).

In words, Theorem 8 identifies certain SOSP\'s of problem (6) which are also global minimizers of problem (6).

Crucially, any “rank-degenerate” SOSP of problem (6) is excluded from Theorem 8. For example, the zero matrix is an spurious SOSP of problem (6) when the network depth \(N \geq 2\), see Proposition 33 in [1], or [7, 9].

By “rank-degenerate” SOSP \(\overline{W}_N = (\overline{W}_1, \ldots, \overline{W}_N)\) above, we mean that \(\text{rank}(\overline{W}_N \cdots \overline{W}_1) < r\). The landscape of problem (6) in general is thus more complicated than the special case of \(N = 2\), specified in Theorem 7.

Theorem 8 is a variation of Proposition 32 in [1] with an assumption on \(YP_X\), which will be necessary in Section 3. Similar assumptions have been used in the PCA, for example [25].

The proof of Theorem 8 here and in Appendix D establishes a pairwise correspondence with the stationary points and the geometry of the PCA problem which had not been explored in the literature to our knowledge.

Since \(\text{rank}(YP_X) \leq m\), Theorem 8 poses the mild requirement of \(m \geq r\) on the sample size, common in the literature, see for example [26].

For completeness, we also prove Theorem 8 using Proposition 32 in [1] as the starting point, see Appendix E.

### 3 Convergence of Gradient Flow

In view of Theorem 8 above, even though nonconvex, the landscape of problem (6) has certain favourable properties.

On the other hand, problem (6) fails to satisfy the strict saddle property that enables first-order algorithms to avoid saddle points [27, 28]. Indeed, as mentioned earlier, the zero matrix is a non-strict saddle point of problem (6) when the network depth \(N \geq 2\).

Against this mixed background, it is natural to ask when first-order algorithms can successfully train linear neural networks, see the references in Sections 1 and 2.

The state of the art here, Theorem 36(a) in [1], guarantees the convergence of gradient flow to a minimizer of \(L_N\), when restricted to one of few regions in \(\mathbb{R}^{d_N}\).

While these regions are known in advance, it is not known which region would contain the limit point of gradient flow, for a given initialization.

That is, Theorem 36(a) cannot guarantee the convergence of gradient flow to a global minimizer of problem (6), and the limit point of gradient flow might be an spurious SOSP of problem (6), such as the zero matrix.

Indeed, outside the lazy training regime exemplified by [2], it is not known when gradient flow can successfully solve problem (6). We will review this literature later in Section 4.

This section answers the above question in an important setting. Let us begin with the necessary preparations.

Consider gradient flow applied to program (6), specified as

\[
\dot{W}_j(t) = -\frac{dW_j(t)}{dt} = -\nabla_{W_j} L_N(W_N(t)),
\]

\(\forall j \in [N], \forall t \geq 0,\) \text{ (gradient flow) (13)}

and initialized at \(W_{N,0} \in \mathbb{R}^{d_N}\). Above, \(\nabla_{W_j} L_N\) is the gradient of \(L_N\) with respect to \(W_j\), the weight matrix for the \(j\)th layer of the linear network, see (4,5,6).

A consequence of the Lojasiewicz’ theorem is the following convergence result for gradient flow. See Appendix F for the proof, similar to Theorem 11 in [1].

**Lemma 9.** If \(X\) has full column-rank, then gradient flow (13) converges.

Moreover, the limit point is an SOSP \(\overline{W}_N \in \mathbb{R}^{d_N}\) of problem (6), for almost every initialization \(W_{N,0}\) with respect to the Lebesgue measure in \(\mathbb{R}^{d_N}\).

To study this limit point \(\overline{W}_N\), we focus here on a common initialization technique for linear networks [14, 13, 2, 5].

**Definition 10.** (balanced initialization) For gradient flow (13), we call \(W_{N,0} = (W_{1,0}, \ldots, W_{N,0}) \in \mathbb{R}^{d_N}\) a balanced initialization if

\[
W_{j+1,0}^T W_{j+1,0} = W_{j,0} W_{j,0}^T, \quad \forall j \in [N-1].
\]

(14)

Claim 4 in [2] underscores the necessity of a (nearly) balanced initialization. More generally, see [29] for the importance of initialization in deep networks.

The main result of this section, Theorem 16, thus requires a balanced initialization. A useful observation is that, if the initialization is balanced, gradient flow remains balanced afterwards, see for example Lemma 2 in [1].

More formally, gradient flow (13) satisfies

\[
W_{j+1,0}^T W_{j+1,0} = W_{j,0} W_{j,0}^T, \quad \forall j \in [N-1]
\]

\[
\Rightarrow W_{j+1}(t)^T W_{j+1}(t) = W_j(t) W_j(t)^T,
\]

(15)
for every \( j \in [N-1] \) and every \( t \geq 0 \). Above, the weight matrix \( W_j(t) \) is the \( j \)th component of \( W_N(t) \), see (5).

Alongside gradient flow (13), it is convenient to introduce another flow [1, 2]. Concretely, for a matrix \( W \in \mathbb{R}^{d_x \times d_z} \), consider the linear operator \( \mathcal{A}_W \) specified as

\[
\mathcal{A}_W : \mathbb{R}^{d_x \times d_z} \rightarrow \mathbb{R}^{d_x \times d_z}, \quad \Delta \rightarrow \sum_{j=1}^{N} (W W^\top)^{\frac{r_j}{2}} \Delta (W^\top W)^{\frac{1-r_j}{2}}.
\]

For a balanced initialization \( W_{N,0} = (W_{1,0}, \ldots, W_{N,0}) \), gradient flow (13) in \( \mathbb{R}^{d_N} \) induces a flow in \( \mathbb{R}^{d_x \times d_z} \), initialized at \( W_0 = W_{N,0} \), specified as

\[
\dot{W}(t) = -\mathcal{A}_W(t) (\nabla L_1(W(t))) \quad \forall t \geq 0,
\]

\[
= -\mathcal{A}_W(t) W(t) XX^\top - YX^\top \quad \text{(see (10a))}
\]

(17)

(18)

We will refer to (17) as the induced flow.

It is known that induced flow (17) admits an analytic singular value decomposition (SVD), see for example Lemma 1 and Theorem 3 in [30] or [31]. More specifically, it holds that

\[
W(t) \overset{\text{SVD}}{=} \tilde{U}(t) \tilde{S}(t) \tilde{V}(t)^\top, \quad \forall t \geq 0,
\]

(19)

provided that the network depth \( N \geq 2 \).

In (19), \( \tilde{U}(t), \tilde{S}(t) \), and \( \tilde{V}(t) \) are analytic functions of \( t \) [32]. Moreover, \( \tilde{U}(t) \in \mathbb{R}^{d_x \times d_u}, \tilde{V}(t) \in \mathbb{R}^{d_x \times d_v} \) are orthonormal bases, and \( \tilde{S}(t) \in \mathbb{R}^{d_x \times d_z} \) contains the singular values of \( W(t) \) in no specific order.

The evolution of the singular values of \( W(t) \) in (19) is also known [30, 33]. In particular, the following byproduct about the rank of \( W(t) \) is important for us, see Appendix G for the proof.

**Lemma 11.** For induced flow (17), \( \text{rank}(W(t)) = \text{rank}(W_0) \) for all \( t \geq 0 \), provided that \( X \) has full column-rank and the network depth \( N \geq 2 \).

Let us henceforth assume that \( X \) has full column-rank, and that gradient flow (13) is initialized at \( W_{N,0} \in \mathcal{M}_{N,r} \), where

\[
\mathcal{M}_{N,r} := \left\{ W_N : \text{rank}(W_N \cdots W_1) = r \right\} \subset \mathbb{R}^{d_N}, \quad (20)
\]

see (5). We make the following observations about \( \mathcal{M}_{N,r} \), proved in Appendix H.

**Lemma 12.** 1. \( \mathcal{M}_{N,r} \) is not a closed subset of \( \mathbb{R}^{d_N} \).
2. The complement of \( \mathcal{M}_{N,r} \) in \( \mathbb{R}^{d_N} \) has Lebesgue measure zero. (In particular, \( \mathcal{M}_{N,r} \) is a dense subset of \( \mathbb{R}^{d_N} \).)

In view of Lemma 12, almost every initialization \( W_{N,0} \in \mathbb{R}^{d_N} \) of gradient flow (13) falls into the set \( \mathcal{M}_{N,r} \), namely,

\[
W_{N,0} \in \mathcal{M}_{N,r}, \quad \text{almost surely.} \quad (21)
\]

Moreover, once initialized in \( \mathcal{M}_{N,r} \) with a balanced initialization, induced flow (17) remains rank-\( r \) at all times by (18,20) and Lemma 11. Consequently, gradient flow (13) remains in \( \mathcal{M}_{N,r} \) at all times, see (18,20). We combine this last observation with (21) to obtain that

\[
W(t) \in \mathcal{M}_{N,r}, \quad \forall t \geq 0, \quad \text{almost surely,} \quad (22)
\]

over the choice of balanced initialization \( W_{N,0} \in \mathbb{R}^{d_N} \).

Despite (22), the limit point \( W_N \) of gradient flow (13) might not belong to \( \mathcal{M}_{N,r} \) because \( \mathcal{M}_{N,r} \) is not closed, see Lemma 12.

Therefore, even though the limit point \( W_N \) of gradient flow is almost surely an SOSP of problem (6) by Lemma 9, we cannot apply Theorem 8 and \( W_N \) might be an spurious SOSP of problem (6), such as the zero matrix.

Indeed, Remark 39 in [1] constructs an example where \( W_N \notin \mathcal{M}_{N,r} \), see also [6]. To avoid this, it is necessary to impose additional assumptions.

Our first assumption is that the data is statistically whitened, which is common in the analysis of linear networks, see for example [2, 13].

**Definition 13. (whitened data)** We say that the data matrix \( X \in \mathbb{R}^{d_x \times m} \) is whitened if

\[
\frac{X X^\top}{m} = \frac{1}{m} \sum_{i=1}^{m} x_i x_i^\top = I_{d_x}, \quad (23)
\]

where \( I_{d_x} \in \mathbb{R}^{d_x \times d_x} \) is the identity matrix.

Our second assumption is that \( r = 1 \) in (9). This case is significant as it corresponds to the popular spiked covariance model in statistics and signal processing [34, 35, 36, 37, 38], to name a few.

Moreover, \( r = 1 \) subsumes the important case of networks with a scalar output.

Lastly, the case \( r = 1 \) appears to be the natural building block for the case \( r > 1 \) via a deflation argument [39, 40]. Indeed, gradient flow (13) moves orthogonal to the principal directions that it has previously discovered or “peeled”. Extending our results to the case \( r > 1 \) remains a challenging open problem.

From (10a) with \( r = 1, \) recall that problem (6) for training a linear neural network is closely related to the problem

\[
\min_{W} \frac{1}{2} \| YP_X - W X \|_F^2 \quad \text{subject to} \quad \text{rank}(W) \leq r = 1
\]

\[
= \min_{W} \frac{m}{2} \| Z - W \|_F^2 \quad \text{subject to} \quad \text{rank}(W) \leq 1, \quad (24)
\]

where the second line above is obtained using (23), and

\[
Z := \frac{Y X^\top}{m}. \quad (25)
\]

We are now ready to collect all the assumptions.

**Assumption 14.** In this section, we assume that the linear network (2,3) has depth \( N \geq 2 \), and one of the layers has only one neuron, namely, \( r = 1 \) in (9).

Moreover, the data matrix \( X \) in (1) is whitened as in (23), and \( Z = \frac{1}{m} Y X^\top \) in (25) satisfies

\[
\text{rank}(Z) \geq r = 1, \quad \gamma Z := \frac{Z_{22}}{Z_{11}} < 1, \quad (26)
\]
where $s_Z$ and $s_{Z,2}$ are the two largest singular values of $Z$. Lastly, we assume that the initialization of gradient flow (13) is balanced, see Definition 10.

In view of (26), let us define
\[ Z_1 = u_Z \cdot s_Z \cdot v_Z^\top \]  (27)
to be the best rank-1 approximation of $Z$, obtained via SVD. Here, $\|u_Z\|_2 = \|v_Z\|_2 = 1$, and $s_Z$ appeared in (26).

Note that $Z_1$ is the unique solution of problem (24), because $Z$ has a nontrivial spectral gap in (26), see for example Section 1 of [41].

Let us fix $\alpha \in [\gamma Z, 1)$. To exclude the zero matrix as the limit point of gradient flow (13), the key is to restrict our attention to a particular subset of the feasible set of problem (6) with $r = 1$, specified as
\[ N_{N,\alpha} := \{ W_N = (W_1, \cdots, W_N) : W_N \cdots W_1 \xrightarrow{\text{tSVD}} u_{W} \cdot s_{W} \cdot v_{W}, \]  
\[ s_{W} > (\alpha - \gamma Z)s_{Z}, u_{W}^\top Z_1 v_{W} > \alpha s_{Z} \} \subset \mathbb{R}^{dN}, \]  (28)
where $s_Z, \gamma Z$ were defined in (26). Above, tSVD stands for the thin SVD. A simple observation is that the set $N_{N,\alpha}$ has infinite Lebesgue measure in $\mathbb{R}^{dN}$.

Such restriction of the feasible set of problem (6) is necessary given the negative example constructed in Remark 39 of [1]. Crucially, note that the end-to-end matrices in $N_{N,\alpha}$ are positively correlated with $Z_1$, and also bounded away from the origin.

An important observation is that, once initialized in $N_{N,\alpha}$, gradient flow (13) avoids the zero matrix, see Appendix I.

**Lemma 15.** For gradient flow (13) initialized at
\[ W_{N,0} \in N_{N,\alpha}, \]  the limit point exists and satisfies
\[ W_N \in M_{N,1}. \]

Above, $\alpha \in (\gamma Z, 1)$, and Assumption 14 and its notation are in force, see also (20,28).

Combining Lemma 15 with Lemma 9, we find that the limit point $\bar{W}_N \in M_{N,1}$ of gradient flow (13) is an SOSP of problem (6), for every balanced initialization $W_{N,0} \in N_{N,\alpha}$ outside a subset with Lebesgue measure zero.

We finally invoke Theorem 8 to conclude that this SOSP $\bar{W}_N \in M_{N,1}$ is in fact a global minimizer of $L_N$ in $\mathbb{R}^{dN}$. This conclusion is summarized below.

**Theorem 16.** Gradient flow (13) converges to a global minimizer of problem (6) from every balanced initialization in $N_{N,\alpha} \subset \mathbb{R}^{dN}$, outside of a subset with Lebesgue measure zero, see (28).

Above, $\alpha \in (\gamma Z, 1)$, and Assumption 14 and its notation are in force.

A few important remarks are in order.

Under Assumption 14, Theorem 16 improves over Theorem 36 in [1] which cannot guarantee the convergence of gradient flow (13) to a solution of problem (6), see our earlier discussion.

Outside the lazy training regime exemplified by [2], Theorem 16 appears to be the first result to identify when gradient flow can successfully train a linear network. We will review this literature in Section 4. Also relevant is the many-particle limit approach of [42].

Crucially, Theorem 16 restricts the initialization of gradient flow (13) to the stable set $N_{N,\alpha}$ in (28). Such a restriction is indeed necessary as discussed after (28).

Let us also examine the content of Assumption 14.

The case $r = 1$ in (9) corresponds to the spiked covariance model popular in statistics, and covers the important case of networks with a scalar output. Lastly, $r = 1$ appears to be the natural building block for extension to $r > 1$, which remains an open problem, see the discussion after (23).

The assumption of whitened data in (23) is commonly used in the context of linear networks, see for example [2, 13]. Moreover, the requirement that $\text{rank}(Z) = \text{rank}(YP_X) \geq r = 1$ in Assumption 14 is clearly necessary to avoid the limit point of zero.

Finally, the induced flow for an unbalanced initialization deviates rapidly from its balanced counterpart in (17), see Lemma 2 in [1].

It is therefore not clear if an unbalanced flow would provably avoid rank-degenerate limit points. However, one might expect any disadvantages of an unbalanced initialization to disappear asymptotically as the network depth $N$ grows larger, see Equation 8 in [1].

4 Convergence Rate of Gradient Flow

It is natural to ask how fast we can train a linear network with gradient flow. Theorem 16 above is notably silent about the convergence rate of gradient flow (13) to a solution of problem (6).

Indeed, along its way to a global minimizer of problem (6), gradient flow (13) might visit the vicinity of one or more strict saddle points of problem (6). In the worst case, gradient flow would require exponential time to escape each saddle point, see Section 6 in [22].

Despite this negative observation, is it still possible for gradient flow to efficiently solve problem (6)? Several works have contributed to our understanding here, including [43, 13, 44, 45, 46, 45], and [47, 48, 49, 30, 50, 51] in the related area of implicit regularization.

For our purposes, [2] exemplifies the current state of the art and its shortcomings.

Theorem 1 in [2] loosely-speaking states that, when the initial loss is small, gradient flow (13) solves problem (6) to an accuracy of $\epsilon > 0$ in the order of
\[ C^{-(1-\frac{1}{s})} \log(1/\epsilon) \]  (29)
time units; $C$ is independent of the network depth $N$.

Theorem 1 in [2] might to some degree disappoint the researchers. For one, (29) suggests that increasing the network depth $N$ only marginally speeds up the training.

More importantly, [2] requires a close initialization, which was not necessary for convergence, see Theorem 16.

Indeed, Theorem 1 in [2] hinges on a perturbation argument, and Definition 2 therein requires the initial-
ization $W_{N,0}$ of gradient flow (13) to satisfy
\[ L_N(W_{N,0}) = \text{sufficiently small. (see (6))} \tag{30} \]

In this sense, Theorem 1 in [2] joins the growing body of literature that quantifies the behavior of neural networks when the learning trajectory is short [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 45, 64, 65, 66, 67, 68, 69], and many more.

To be sure, restricting the initialization is necessary for successful training [29]. Gradient flow would stall when initialized at a saddle point, for example.

However, while this line of research is valuable, it is widely-believed that first-order algorithms can successfully train neural networks far beyond the lazy training regime considered by [2] and others.

Indeed, the learning trajectory of neural networks is in general not short, and the learning is often not local. We refer to [3, 4] for a detailed critique of lazy training, see also Appendix J.

Let us call this more general regime non-local training. Understanding the non-local convergence rate of linear networks appears to be a vital step towards understanding the non-local training of nonlinear neural networks.

In an important setting, this section quantifies the non-local training of linear networks, and addresses both of the shortcomings of Theorem 1 in [2].

Moreover, Theorem 20 establishes that the faraway convergence rate of gradient flow (13) to a solution of problem (6), even when (30) is violated.

To quantify the convergence rate of induced flow (17) to the global minimizer $Z_1$ of problem (24), we next write down the evolution of the loss function $L_{1,1}$ in (31) as
\[
\frac{dL_{1,1}(W_t)}{dt} = \mathcal{A}_{W_t}(W_t - Z_1). \tag{34}
\]

Loosely speaking, $T_{1,t}$ above gauges the error in estimating the (only) nonzero singular value $s_Z$ of the target $Z_1$, whereas $T_{2,t}$ gauges the misalignment between $W_t$ and $Z_1$. Both $T_{1,t}, T_{2,t}$ are nonnegative for all $t \geq 0$, see (27,32).

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\[
dL_{1,1}(W_t) = \frac{d}{dt} L_{1,1}(W_t) \tag{36}
\]

where $u_t \in \mathbb{R}^{d_y}$ and $v_t \in \mathbb{R}^{d_z}$ have unit-norm, and $s_t > 0$ is the only nonzero singular value of $W_t$.

A simple calculation using (32), deferred to Appendix L, upper bounds the loss function $L_{1,1}$ in (31) as
\[
L_{1,1}(W_t) \leq \left( \frac{T_{1,t}}{T_{2,t}} \right)^2 \left( s_t - u_t^T Z_1 v_t \right)^2 + s_Z \left( s_z - u_t^T Z_1 v_t \right). \tag{33}
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\]
where \( s_Z, \gamma_Z \) were defined in \((26)\).

The necessity of such a restriction was discussed after \((28)\), and the (new) upper bound on \( s_W \) in \((36)\) controls the (unwanted) positive terms in \((35)\). Note that \( N_{\alpha, \beta}(Z_1) \) is a neighborhood of \( Z_1 \), namely, \( Z_1 \in N_{\alpha, \beta}(Z_1) \) by \((27)\).

Once initialized in \( N_{\alpha, \beta}(Z_1) \), induced flow \((17)\) remains in \( N_{\alpha, \beta}(Z_1) \), see Appendix N, closely related to Lemma 15.

**Lemma 19.** Fix \( \alpha \in \{\gamma_Z, 1\} \) and \( \beta > 1 \). For induced flow \((17)\), \( W_0 \in N_{\alpha, \beta}(Z_1) \) implies that \( W_t \in N_{\alpha, \beta}(Z_1) \) for all \( t \geq 0 \).

Above, Assumption 14 and the notation therein are in force.

In view of Lemma 19, we can now use \((36)\) to bound \( s_t \) and \( u_t^T Z_1 v_t \) in \((35)\). We can then distinguish two regimes (fast and slow convergence) in the dynamics of the loss function in \((35)\) depending on the dominant term on the right-hand side of \((33)\). The remaining technical details are deferred to Appendix O and we finally arrive at the following result.

**Theorem 20.** With Assumption 14 and its notation in force, fix \( \alpha \in \{\gamma_Z, 1\} \) and \( \beta > 1 \). Suppose that the inverse spectral gap \( \gamma_Z \) is small enough so that the exponents below are both negative.

Consider gradient flow \((13)\) with the balanced initialization \( W_{N, 0} = (W_{1, 0}, \ldots, W_{N, 0}) \in \mathbb{R}^{d_N \times N} \) such that \( W_0 := W_{N, 0} \cdots W_{1, 0} \in \mathbb{R}^{d_N \times d_N} \) satisfies

\[
\text{rank}(W_0) = 1, \quad W_0 \overset{\text{SVD}}{=} u_0 s_0 v_0^T.
\]

\[
(\alpha - \gamma_Z)s < s_0 < \beta s_Z, \quad u_0^T Z_1 v_0 > \alpha s_Z. \tag{37}
\]

Let \( W_N(t) = (W_1(t), \ldots, W_N(t)) \) be the output of gradient flow \((13)\) at time \( t \), and set \( W(t) := W_{N, 0} \cdots W_{1, 0}(t) \), which satisfies \( \text{rank}(W(t)) = 1 \) for every \( t \geq 0 \).

Let \( \tau \geq 0 \) be the first time when \( s(\tau) \leq \sqrt{6} s_N \), where \( s(\tau) \) is the (only) nonzero singular value of \( W(\tau) \). Then the distance to the target matrix \( Z_1 \) evolves as

\[
\forall t \leq \tau, \quad \|Z_1 - W(t)\|_F^2 \leq \|Z_1 - W_0\|_F^2 \tag{38}
\]

\[
\quad \cdot e^{-m_s N_T z^2 \frac{\gamma_Z}{2} (\alpha - \gamma_Z)^2 z^2 - 2\gamma_N z^2 \frac{\gamma_Z}{2} t^2},
\]

\[
\forall t \geq \tau, \quad \|Z_1 - W(t)\|_F^2 \leq \|Z_1 - W(\tau)\|_F^2 \tag{39}
\]

\[
\cdot e^{-m_s N_L z^2 \frac{\gamma_Z}{2} (\alpha - \gamma_Z)^2 z^2 - 2\gamma_N N_T z^2 \frac{\gamma_Z}{2} (t - \tau)}.
\]

The remarks after Theorem 16 apply here about Assumption 14 and the case \( \tau > 1 \). Some new remarks will follow.

Under Assumption 14, Theorem 20 states that gradient flow successfully trains a linear network with linear rate whenever the end-to-end initialization matrix in \((37)\) is positively correlated with the target matrix \( Z_1 \), and away from the origin, see after \((28)\) for the necessity of such initialization.

Note that \((37)\) should not be seen as an initialization scheme, but as a theoretical stable set for training linear networks.

The faraway convergence rate improves with network depth \( N \), whereas the nearby convergence rate does not appear to benefit from increasing \( N \), see \((38,39)\). In fact, for the exponent in \((39)\) to be negative, the inverse spectral gap \( \gamma_Z \) must be sufficiently small, namely, \( O(N^{-1}) \). Since \((39)\) is an upper bound, it is difficult to infer any theoretical trade-offs about the network depth from Theorem 20.

Note also the scale-dependence of the convergence rate, as predicted on page 8 of [30].

With the initialization \( W_{N, 0} = (W_{1, 0}, \ldots, W_{N, 0}) \), Theorem 1 in [2] uses a perturbation argument in their Claim 1, requiring that

\[
\|Z - W_{N, 0} \cdots W_{1, 0}\|_F < s_{\text{min}}(Z), \tag{40}
\]

which is impossible if \( \text{rank}(Z) > r \). Indeed, the network architecture forces that \( \text{rank}(W_{N, 0} \cdots W_{1, 0}) \leq r \), see \((9)\).

In fact, we are not aware of any previous results when the linear network has a hidden layer smaller than the input/output layers, see Section 3.2.1 in [2].

In contrast, Theorem 20 applies even when \((40)\) is violated, as it does away entirely with the local perturbation argument.

Qualitatively speaking, Theorem 20 thus ventures beyond the reach of the local or lazy training in [2], prevalent also in the over-parametrization literature, see Appendix J.

Thorough numerics for linear networks are abound, see for example [1, 2, 5], and we refrain from lengthy simulations. Let us only provide a simple numerical example in Figure 1 of the supplementary material to visualize the (gradual) change of regimes from fast to slow convergence, see \((38,39)\). This example also suggests new research questions about linear networks.

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A Derivation of (10a,10b)

To show (10a,10b), it suffices to show that the map
\[ \Pi(d_N) : \mathbb{R}^{d_N} \to M_{d_{N+1} \times d_0}^{d_N} \]
\[ W_N = (W_1, \ldots, W_N) \to W = W_N \cdots W_1, \tag{41} \]
is surjective, which we now set out to do. Above, \(d_N = (d_0, \ldots, d_N)\) and \(\mathbb{R}^{d_N} = \mathbb{R}^{d_0} \times \cdots \times \mathbb{R}^{d_N}\) is the domain of the function. Also, \(M_{d_{N+1} \times d_0}^{d_N} \subset \mathbb{R}^{d_{N+1} \times d_0}\) is the set of all \(d_{N+1} \times d_0\) matrices of rank at most \(r\). As a side note, \(M_{d_{N+1} \times d_0}^{d_N}\) is the closure of the manifold of rank-\(r\) matrices. Lastly, the network architecture dictates that satisfies
\[ \min_j d_j = r. \tag{42} \]

The proof of this surjective property is by induction. The base of induction for \(N = 1\) is trivial because \(\Pi(d_0, d_1)\) is simply the identity map by (41) and thus surjective, in particular for any pair of integers \((d_0, d_1)\) that satisfies (42).

For the step of induction, suppose that \(\Pi(d_N)\) is surjective for every tuple \(d_N = (d_0, \ldots, d_N)\) that satisfies (42). For an arbitrary integer \(d_{N+1}\), consider also an arbitrary matrix
\[ W \in M_{d_{N+1} \times d_0}^{d_N+1}, \tag{43} \]
with the SVD
\[ W = \text{SVD} \tilde{U} \tilde{S} \tilde{V}^\top =: \tilde{U} \cdot \tilde{Q}, \tag{44} \]
where \(\tilde{U} \in \mathbb{R}^{d_{N+1} \times d_{N+1}}\) and \(\tilde{V} \in \mathbb{R}^{d_0 \times d_0}\) are orthonormal bases, and \(\tilde{S} \in \mathbb{R}^{d_{N+1} \times d_0}\) contains the singular values of \(W\).

In particular, note that \(\tilde{Q} \in \mathbb{R}^{d_{N+1} \times d_0}\). Note also that (43) implies that
\[ \text{rank}(\tilde{Q}) = \text{rank}(W) \leq r, \tag{45} \]
because \(\tilde{U}\) is an orthonormal basis.

Combining (42) and (45), we reach
\[ \text{rank}(\tilde{Q}) \leq r \leq d_N. \tag{46} \]

In view of (46), it is therefore possible (by padding with zero columns or removing some columns from \(\tilde{U}\) and the corresponding rows from \(\tilde{Q}\)) to create \(\tilde{U}'\) and \(\tilde{Q}'\) such that
\[ W = \tilde{U}' \cdot \tilde{Q}'^\top, \]
where \(\tilde{U}' \in \mathbb{R}^{d_{N+1} \times d_N}, \tilde{Q}' \in \mathbb{R}^{d_N \times d_0}. \tag{47} \]

In this construction, \(\text{rank}(\tilde{Q}') = \text{rank}(\tilde{Q}) \leq r\) and \(\tilde{Q}' \in \mathbb{R}^{d_N \times d_0}\). Consequently, the step of induction guarantees the existence of \(W_N = (W_N, \cdots, W_1) \in \mathbb{R}^{d_N}\) such that
\[ \tilde{Q}' = W_N \cdots W_1. \tag{48} \]

It follows that
\[ W = \tilde{U}' \cdot \tilde{Q}' \quad (\text{see (47)}) \]
\[ = \tilde{U}' W_N \cdots W_1. \quad (\text{see (48)}) \tag{49} \]
That is,
\[ W = \Pi(d_0, \ldots, d_{N+1})[W_1, \ldots, W_N, \tilde{U}'], \]  
(50)
which completes the induction. We thus proved that \( \Pi(d_N) \) is a surjective map for every tuple \( d_N \) that satisfies (42).

## B Proof of Lemma 4

Let \( W_N = (W_1, \ldots, W_N) \in \mathbb{R}^{d_N} \) be an FOSP of problem (6). For an infinitesimally small perturbation \( \Delta_N = (\Delta_1, \ldots, \Delta_N) \in \mathbb{R}^{d_N} \), we can expand \( L_N \) in (6) as

\[
L_N(W_N + \Delta_N) = L_N(W_N) + \nabla L_N(W_N)[\Delta_N] + \frac{1}{2} \nabla^2 L_N(W_N)[\Delta_N] + o
\]

(51)
where \( o \) represents (negligible) higher order terms, and the second identity above holds because \( W_N \) is assumed to be an FOSP in Lemma 4, see Definition 1. Above, \( \nabla^2 L_N(W_N)[\Delta_N] \) contains all second order terms in the variables \( \Delta_N \).

Let \( j_0 \) correspond to a layer with the smallest width \( r \in \mathbb{N} \) (2,3), namely,

\[
r = \min_{j \leq N} d_j \quad (\text{see (9)})
\]

(52)
We also set

\[
P := W_N \cdots W_{j_0+1} \in \mathbb{R}^{d_N \times r},
\]

\[
Q := W_{j_0} \cdots W_1 \in \mathbb{R}^{r \times d_e},
\]

(53)
for short, and note that

\[
W := P \cdot Q = W_N \cdots W_1,
\]

\[
\text{rank}(W) = \text{rank}(P) = \text{rank}(Q) = r,
\]

(54)
where the second line above holds by the assumption of Lemma 4.

Indeed, \( W = P \cdot Q \) implies that

\[
\text{min}(\text{rank}(P), \text{rank}(Q)) \geq r.
\]

(55)
Note also that \( P \) has \( r \) columns and \( Q \) has \( r \) rows, thus

\[
\text{max}(\text{rank}(P), \text{rank}(Q)) \leq r.
\]

(56)
Together, (55) and (56) give the second line of (54).

On the one hand, for an arbitrary \( (\Delta_P, \Delta_Q) \), we can relate the perturbation of \( W_N \) to the perturbation of \( (P, Q) \) as

\[
(W_N + \Delta_N) \cdots (W_1 + \Delta_1) = (P + \Delta_P)(Q + \Delta_Q),
\]

(57)
where

\[
\Delta_1 = W_1 Q^\dagger \Delta_Q,
\]

\[
\Delta_1 = 0, \quad 2 \leq i \leq N - 1,
\]

\[
\Delta_N = \Delta_P P^\dagger W_N,
\]

(58)
and \( \dagger \) denotes the pseudo-inverse, and we used the second identity in (54).

Indeed, for the choice of \( \Delta_N \) in (58), it holds that

\[
(W_N + \Delta_N) \cdots (W_1 + \Delta_1)
\]

\[
= (W_N + \Delta_P P^\dagger W_N)W_{N-1} \cdots \nabla^2 W_1 (I_{d_e} + \Delta_P P^\dagger Q^\dagger \Delta_Q)
\]

(see (58))

\[
= (I_{d_e} + \Delta_P P^\dagger)W_N W_{N-1} \cdots \nabla^2 W_1 (I_{d_e} + Q^\dagger \Delta_Q)
\]

(see (54))

\[
= (P + \Delta_P)(Q + \Delta_Q),
\]

(59)
which agrees with (57). The last line above uses the second identity in (54), namely, \( \text{rank}(P) = \text{rank}(Q) = r \).

Above, \( I_{d_e} \in \mathbb{R}^{d_e \times d_e} \) is the identity matrix.

On the other hand, we can expand \( L_2 \) in (10a) as

\[
L_2(P + \Delta_P, Q + \Delta_Q)
\]

\[
= L_2(P, Q) + \nabla L_2(P, Q)[\Delta_P, \Delta_Q] + \nabla^2 L_2(P, Q)[\Delta_P, \Delta_Q] + o,
\]

(60)
where \( o \) again higher order terms. Above, \( \nabla L_2(P, Q)[\Delta_P, \Delta_Q] \) collects all first order terms in the variables \( (\Delta_P, \Delta_Q) \). Likewise, \( \nabla^2 L_2(P, Q)[\Delta_P, \Delta_Q] \) contains all second order terms in \( (\Delta_P, \Delta_Q) \).

For convenience, let us define the map

\[
L_1 : \mathbb{R}^{d_n \times d_e} \rightarrow \mathbb{R}
\]

\[
W \rightarrow \frac{1}{2}\|Y - WX\|_F^2,
\]

(61)
and note that

\[
L_2(P, Q) = L_1(PQ),
\]

(see (10b))

\[
L_N(W_N) = L_1(W_N \cdots W_1),
\]

(see (6))
for every \( P, Q, W_N \).

In view of (62), we now write that

\[
L_2(P + \Delta_P, Q + \Delta_Q)
\]

\[
= L_1((I_{d_e} + \Delta_P)(Q + \Delta_Q))
\]

(see (62))

\[
= L_1((W_N + \Delta_N) \cdots (W_1 + \Delta_1))
\]

(see (57))

\[
= L_N(W_N + \Delta_N),
\]

(see (62))
for \( \Delta_N = (\Delta_1, \ldots, \Delta_N) \) specified in (58).

As a result of (63), the expansions in (51) and (60) must match. That is, for an arbitrary \( (\Delta_P, \Delta_Q) \) and the corresponding choice of \( \Delta_N \) in (57), it holds that

\[
\nabla L_2(P, Q)[\Delta_P, \Delta_Q]
\]

\[
= \nabla L_N(W_N)[\Delta_N] = 0,
\]

(64)
and

\[
\nabla^2 L_2(P, Q)[\Delta_P, \Delta_Q]
\]

\[
= \nabla^2 L_N(W_N)[\Delta_N] = 0.
\]

(65)
It follows from (64) that \( (P, Q) \) is an FOSP of problem (10a) if \( W_N \) is an FOSP of problem (6).
Moreover, if \( \mathcal{W}_N \) is an SOSP of problem (6), then
the last line of (65) is nonnegative, see Definition 2. That is,
\[
\nabla^2 L_2(\mathcal{P}, \mathcal{Q})[\Delta_P, \Delta_Q] \\
= \nabla^2 L_2(\mathcal{W}_N)(\Delta_N) \geq 0,
\]
(66)
for an arbitrary \((\Delta_P, \Delta_Q)\) and the corresponding choice
of \(\Delta_N\) in (57). Therefore, \((\mathcal{P}, \mathcal{Q})\) is an SOSP of
problem (10a) if \(\mathcal{W}_N\) is an SOSP of problem (6). This
completes the proof of Lemma 4.

\section*{C Proof of Lemma 5}

Recall that \(\mathcal{P}_X\) and \(\mathcal{P}_{X^\perp}\) denote the orthogonal pro-
jections onto the row span of \(X\) and its complement, respectively.

Using the decomposition \(Q' = Q'\mathcal{P}_X + Q'\mathcal{P}_{X^\perp}\), the last program in (12) can be written as
\[
\min_{P',Q'} \frac{1}{2}\|YP_X - PQ'\|_F^2
\]
\[
= \min_{P',Q'} \frac{1}{2}\|YP_X - PQ'\mathcal{P}_X\|_F^2 + \min_{P',Q'} \frac{1}{2}\|PQ'\mathcal{P}_{X^\perp}\|_F^2.
\]
(67)
From the above decomposition, it is evident that the minimum above is achieved when the last term in (67) vanishes. This observation allows us to write that
\[
\min_{P',Q'} \frac{1}{2}\|YP_X - PQ'\|_F^2
\]
\[
= \min_{P',Q',Q''} \frac{1}{2}\|YP_X - PQ''\|_F^2
\]
\[
= \min_{P',Q',Q''} \frac{1}{2}\|YP_X - PQ''\|_F^2 \quad \text{(subject to } Q'' = Q'\mathcal{P}_X) \\
= \min_{P',Q'} \frac{1}{2}\|YP_X - PQ''\|_F^2 \quad \text{(subject to row span } (Q'')) \subseteq \text{row span}(X) \\
= \min_{P',Q'} \frac{1}{2}\|YP_X - PQ''\|_F^2 \quad \text{(subject to } Q'' = QX) \\
= \min_{P',Q'} \frac{1}{2}\|YP_X - PQX\|_F^2,
\]
(68)
which proves the tight relaxation claimed in (12). The third identity above uses the fact that the map
\[
\mathbb{R}^{r \times d_x} \rightarrow \text{row span}(X) \\
Q' \rightarrow Q'' = Q'\mathcal{P}_X
\]
is surjective.

To prove the second claim in Lemma 5, let \((\mathcal{P}, \mathcal{Q})\) be
an FOSP of problem (10b), which satisfies
\[
0 = (Y - \mathcal{P} \cdot \mathcal{Q} X)X^\top \mathcal{Q}^\top, \\
0 = \mathcal{P}^\top (Y - \mathcal{P} \cdot \mathcal{Q} X)X^\top.
\]
(70)
After setting
\[
\mathcal{Q}' = \mathcal{Q} X,
\]
(71)
the above identities read as
\[
0 = (Y - \mathcal{P} \cdot \mathcal{Q} X)X^\top \mathcal{Q}^\top \quad \text{(see (70))} \\
0 = \mathcal{P}^\top (Y - \mathcal{P} \cdot \mathcal{Q} X)X^\top \quad \text{(see (70))}
\]
(72)
and
\[
0 = \mathcal{P}^\top (Y - \mathcal{P} \cdot \mathcal{Q} X)X^\top \quad \text{(see (70))} \\
= \mathcal{P}^\top (Y \mathcal{P}_X - \mathcal{P} \cdot \mathcal{Q} X)X^\top.
\]
(73)
Recall that
\[
\text{row span}(\mathcal{Q}') \subseteq \text{row span}(X).
\]
(74)
With this in mind, (73) implies that
\[
0 = \mathcal{P}^\top (Y \mathcal{P}_X - \mathcal{P} \cdot \mathcal{Q} X)X^\top \quad \text{(see (73))} \\
= \mathcal{P}^\top (Y \mathcal{P}_X - \mathcal{P} \cdot \mathcal{Q} X), \quad \text{(see (74))}
\]
(75)
where we also used the assumption that \(X\) has full column-rank.

By combining (72,75), we conclude that \((\mathcal{P}, \mathcal{Q}')\) is an
FOSP of problem (12) if \((\mathcal{P}, \mathcal{Q})\) is an FOSP of
problem (10b).

To prove the last claim of Lemma 5, let \((\mathcal{P}, \mathcal{Q})\) be an
FOSP of problem (10b), which satisfies
\[
\frac{1}{2}\|\Delta_P \mathcal{Q} X + \mathcal{P} \Delta Q X\|_F^2 + \langle \mathcal{P} \cdot \mathcal{Q} X - Y, \Delta_P \Delta Q X\rangle
\]
\[
\geq 0, \quad \forall (\Delta_P, \Delta_Q X).
\]
(76)
Let us set \(\mathcal{Q}' = \mathcal{Q} X\) as before, and also note that the map
\[
\mathbb{R}^{r \times d_x} \rightarrow \text{row span}(X) \\
\Delta_Q \rightarrow \Delta Q' = \Delta Q X
\]
is evidently surjective. Then we may rewrite (76) as
\[
\frac{1}{2}\|\Delta_P \mathcal{Q}' + \mathcal{P} \Delta Q'\|_F^2 + \langle \mathcal{P} \cdot \mathcal{Q}' - Y \mathcal{P}_X, \Delta_P \Delta Q' \rangle \geq 0,
\]
\[
\forall (\Delta_P, \Delta Q') \in \mathbb{R}^{d_y \times r} \times \text{row span}(X),
\]
(78)
On the other hand, recall again (74). When
\[
\Delta Q' \perp \text{row span}(X),
\]
(79)
we have that
\[
\frac{1}{2}\|\Delta_P \mathcal{Q}' + \mathcal{P} \Delta Q'\|_F^2 + \langle \mathcal{P} \cdot \mathcal{Q}' - Y \mathcal{P}_X, \Delta_P \Delta Q' \rangle
\]
\[
= \frac{1}{2}\|\Delta_P \mathcal{Q}'\|_F^2 + \|\mathcal{P} \Delta Q'\|_F^2 \geq 0, \\
\forall (\Delta_P, \Delta Q') \in \mathbb{R}^{d_y \times r} \times \text{row span}(X),
\]
(80)
where the identity above uses (74,79). By combining
(78,80), we reach
\[
\frac{1}{2}\|\Delta_P \mathcal{Q}' + \mathcal{P} \Delta Q'\|_F^2 + \langle \mathcal{P} \cdot \mathcal{Q}' - Y \mathcal{P}_X, \Delta_P \Delta Q' \rangle
\]
\[
\geq 0, \quad \forall (\Delta_P, \Delta Q').
\]
(81)
It is evident from (81) that \((\mathcal{P}, \mathcal{Q}')\) is an SOSP of
problem (12) if \((\mathcal{P}, \mathcal{Q})\) is an SOSP of problem (10b).
This completes the proof of Lemma 5.
D Proof of Theorem 8

We begin with a technical lemma below, proved with the aid of EYM Theorem 7. This result is standard but a proof is included for completeness.

**Lemma 21.** If \( \text{rank}(YP_X) \geq r \), then any SOSP \( (\bar{P}, \bar{Q}') \) of problem (12) is a global minimizer of problem (12) and satisfies
\[
\text{rank}(\bar{P}) = \text{rank}(\bar{Q}') = \text{rank}(\bar{W}) = r, \tag{82}
\]
where \( \bar{W} = \bar{P} \cdot \bar{Q}' \).

Before proving the above lemma in the next appendix, let us show how it can be used to prove Theorem 8.

Let us assume that \( \text{rank}(YP_X) \geq r \), so that Lemma 21 is in force. Then any SOSP \( (\bar{P}, \bar{Q}') \) of problem (12) is a global minimizer of problem (12) and satisfies (82).

Let us also assume that \( X \) has full column-rank, so that Lemma 6 is in force. Lemma 6 then implies that any SOSP \( \bar{W}_N \) of problem (6) corresponds to an SOSP \( (\bar{P}, \bar{Q}') \) of problem (12), provided that \( \bar{W} = \bar{W}_N \cdots \bar{W}_1 \) is rank-\( r \). The relationship between these quantities is
\[
\bar{W}_N \cdots \bar{W}_1 X = \bar{W} X = \bar{P} \cdot \bar{Q}'. \tag{83}
\]

In light of the preceding paragraph, we observe that any SOSP \( \bar{W}_N \) of problem (12) corresponds to a global minimizer \( (\bar{P}, \bar{Q}') \) of problem (12), provided that \( \bar{W} \) is rank-\( r \).

Using the decomposition \( Y = YP_X + YP_{X\perp} \), we can therefore write that
\[
\begin{align*}
\frac{1}{2} \left\| Y - \bar{W}_N \cdots \bar{W}_1 X \right\|_F^2 &= \frac{1}{2} \left\| YP_X - \bar{W}_N \cdots \bar{W}_1 X \right\|_F^2 + \frac{1}{2} \left\| YP_{X\perp} \right\|_F^2 \\
&= \frac{1}{2} \left\| YP_X - \bar{P} \cdot \bar{Q}' \right\|_F^2 + \frac{1}{2} \left\| YP_{X\perp} \right\|_F^2 \quad \text{(see (83))} \\
&= \frac{1}{2} \left\| YP_X - \bar{P}Q \right\|_F^2 + \frac{1}{2} \left\| YP_{X\perp} \right\|_F^2 \\
&= \frac{1}{2} \left\| Y - \bar{P}Q \right\|_F^2 \quad \text{(see (12))} \\
&= \min_{\bar{w}_1, \ldots, \bar{w}_N} \frac{1}{2} \left\| Y - \bar{W}_N \cdots \bar{W}_1 X \right\|_F^2. \quad \text{(10b)}
\end{align*}
\]

That is, any SOSP \( \bar{W}_N \) of problem (6) is a global minimizer of problem (6), provided that \( \bar{W} \) is rank-\( r \). This completes the proof of Theorem 8.

D.1 Proof of Lemma 21

We conveniently assume that
\[
\text{rank}(YP_X) = r, \tag{85}
\]
but the same argument is valid also when \( \text{rank}(YP_X) > r \). Let
\[
YP_X \equiv \text{SVD} \tilde{U} \tilde{S} \tilde{V} \tag{86}
\]
denote the thin SVD of \( YP_X \), where \( \tilde{U} \in \mathbb{R}^{d_x \times r} \) has orthonormal columns, \( \tilde{V} \in \mathbb{R}^{r \times m} \) has orthonormal rows, and the diagonal matrix \( \tilde{S} \in \mathbb{R}^{r \times r} \) contains the singular values of \( YP_X \).

By the way of contradiction, suppose that \( (\bar{P}, \bar{Q}') \) is an SOSP of problem (12) such that
\[
\text{rank}(\bar{P} \cdot \bar{Q}') < r. \tag{87}
\]

Without loss of generality, we can in fact replace (87) with
\[
\text{rank}(\bar{P}) = \text{rank}(\bar{Q}') = \text{rank}(\bar{P} \cdot \bar{Q}') < r. \tag{88}
\]

(Indeed, for example if \( \text{rank}(\bar{P}) < \text{rank}(\bar{Q}') < r \), then \( (\bar{P} \bar{P}_S, \bar{P}_S \bar{Q}') \) takes the same objective value in problem (12) as \( (\bar{P}, \bar{Q}') \). Here, \( \bar{P}_S \) is the orthogonal projection onto the subspace \( \bar{S} = \text{row span}(\bar{P}) \cap \text{column span}(\bar{Q}') \).

On the other hand, by EYM Theorem 7, the SOSP \( (\bar{P}, \bar{Q}') \) is in fact a global minimizer of problem (12). Therefore, \( (\bar{P} \bar{P}_S, \bar{P}_S \bar{Q}') \) too is a global minimizer of problem (12) and a fortiori an SOSP of problem (12). We can thus replace \( (\bar{P}, \bar{Q}') \) with the SOSP \( (\bar{P} \bar{P}_S, \bar{P}_S \bar{Q}') \) which satisfies \( \text{rank}(\bar{P} \bar{P}_S) = \text{rank}(\bar{P}_S \bar{Q}') < r \). That is, the assumption made in (88) indeed does not reduce the generality of the following argument.)

Assuming (88), next note that \( (\bar{P}, \bar{Q}') \) satisfies
\[
\bar{P} = \begin{bmatrix} US_P & 0_{d_x \times 1} \end{bmatrix} \in \mathbb{R}^{d_x \times r},
\bar{Q}' = \begin{bmatrix} S_Q V \\ 0_{1 \times d_x} \end{bmatrix} \in \mathbb{R}^{r \times m}, \tag{89}
\]
where \( U \in \mathbb{R}^{d_x \times (r-1)} \) and \( V \in \mathbb{R}^{(r-1) \times m} \) correspond to those left and right singular vectors of \( YP_X \) that might be present in \( \bar{P} \cdot \bar{Q}' \), see for example Lemma 5.1 (Item 5) in [24].

In (89), \( S_P, S_Q' \) are (not necessarily diagonal) matrices, and we note that \( U \) and \( V \) are column and row submatrices of \( \tilde{U} \) and \( \tilde{V} \), respectively.

In view of (85, 86, 87), there exists a unique pair \( (u, v) \) of left and right singular vectors of \( YP_X \) that is absent from (89), namely,
\[
U^T u = 0, \quad V^T v = 0. \tag{90}
\]

To match the representation in (86), note that \( u \) above is a column-vector whereas \( v \) is a row-vector. In particular,
\[
\tilde{U} = \begin{bmatrix} U & u \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V \\ v \end{bmatrix},
\]
\[
YP_X = \tilde{U} \tilde{S} \tilde{V} = USV + usv, \quad \text{(see (86))} \tag{91}
\]
where \( S \in \mathbb{R}^{r-1} \) and \( s \in \mathbb{R} \) collect the singular values corresponding to \( (\tilde{U}, V) \) and \( (u, v) \), respectively.

To proceed, consider infinitesimally small scalars \( \delta_u \) and \( \delta_v \). Consider also an infinitesimally small perturbation \( (\Delta_P, \Delta_Q') \) in \( (\bar{P}, \bar{Q}') \), specified as
\[
\Delta_P + P = \begin{bmatrix} US_P & \delta_u u \end{bmatrix},
\Delta_Q' + Q' = \begin{bmatrix} S_Q V \\ \delta_v v \end{bmatrix}, \tag{92}
\]

...
It immediately follows that
\[
(\mathcal{P} + \Delta P)(\mathcal{Q} + \Delta Q) = U \Sigma P \delta u, V \delta v \Sigma V = \mathcal{P} \cdot \mathcal{Q} + \delta u \delta v \Sigma V.
\]
(see (89))

From (90), it is evident that the perturbation in (93) is orthogonal to \((\mathcal{P}, \mathcal{Q})\).

To continue, let us define the orthogonal projections \(\mathcal{P}_U = U U^T\) and \(\mathcal{P}_u = u u^T\), and define \(\mathcal{P}_V, \mathcal{P}_v\) similarly. In particular, we can decompose \(Y \mathcal{P}_X\) as
\[
Y \mathcal{P}_X = (\mathcal{P}_U + \mathcal{P}_u)(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v)
\]
where the cross terms above vanish by properties of the SVD, see (86,90). Indeed, for example,
\[
\mathcal{P}_U(Y \mathcal{P}_X) \mathcal{P}_v = U U^T(u u^T + u u^T)u^T v = U_u^T u v = 0,
\]
where \(S\) and \(s\) collect the corresponding singular values for the singular vectors collected in \((U, V)\) and \((u, v)\), respectively. We will use the decomposition (94) immediately below.

Under the perturbation in (92), the objective function of problem (12) becomes
\[
\frac{1}{2} ||Y \mathcal{P}_X - (\mathcal{P} + \Delta P)(\mathcal{Q} + \Delta Q)||_F^2
\]
\[
= \frac{1}{2} ||Y \mathcal{P}_X - (\mathcal{P} + \delta u \delta v)(\mathcal{Q} + \delta v)||_F^2
\]
\[
= \frac{1}{2} ||(\mathcal{P}_U + \mathcal{P}_u)(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - (\mathcal{P}_U \mathcal{P}_X \mathcal{P}_V + \mathcal{P}_u \mathcal{P}_v)||_F^2
\]
\[
\quad + ||(\mathcal{P}_u(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - \delta u \delta v)||_F^2
\]
\[
= \frac{1}{2} ||(\mathcal{P}_U + \mathcal{P}_u)(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - (\mathcal{P}_U \mathcal{P}_X \mathcal{P}_V + \mathcal{P}_u \mathcal{P}_v)||_F^2
\]
\[
+ \frac{1}{2} ||(\mathcal{P}_u(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - \delta u \delta v)||_F^2
\]
\[
= \frac{1}{2} ||\mathcal{P}_U(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - \delta u \delta v||_F^2
\]
(96)

Indeed, we can upper bound the last line of (97) as
\[
\frac{1}{2} ||Y \mathcal{P}_X - (\mathcal{P} + \delta u)(\mathcal{Q} + \delta v)||_F^2
\]
\[
= \frac{1}{2} ||(\mathcal{P}_U + \mathcal{P}_u)(Y \mathcal{P}_X)(\mathcal{P}_V + \mathcal{P}_v) - \delta u \delta v||_F^2
\]
\[
+ \frac{1}{2} ||u^T(Y \mathcal{P}_X)v^T - \delta u \delta v||_F^2
\]
(97)

where we chose \(\delta u, \delta v\) above such that \(\text{sign}(\delta u, \delta v) = \text{sign}(u^T(Y \mathcal{P}_X)v^T)\).

Note that (98) contradicts the assumption that \((\mathcal{P}, \mathcal{Q})\) is an SOSP of problem (12), see Definition 2. In fact, \((\mathcal{P}, \mathcal{Q})\) is a strict saddle point of problem (12) because \((\Delta P, \Delta Q)\) is a descent direction, see Definition 3.

Provided that \(\text{rank}(Y \mathcal{P}_X) \geq r\), we conclude that any SOSP \((\mathcal{P}, \mathcal{Q})\) of problem (12) satisfies
\[
\text{rank}(W) = r,
\]
where \(W = \mathcal{P} \cdot \mathcal{Q}\).

We can in fact replace the conclusion in (99) with
\[
\text{rank}(\mathcal{P}) = \text{rank}(\mathcal{Q}) = \text{rank}(W) = r.
\]
(100)

Indeed, \(\max(\text{rank}(\mathcal{P}), \text{rank}(\mathcal{Q})) \leq r \) because \(\mathcal{P}\) has \(r\) columns and \(\mathcal{Q}\) has \(r\) rows, see (12). On the other hand, \(\min(\text{rank}(\mathcal{P}), \text{rank}(\mathcal{Q})) \geq r \) because of (99). These two observations imply that \(\text{rank}(\mathcal{P}) = \text{rank}(\mathcal{Q}) = r\), as claimed in (100).

Lastly, by EYM Theorem 7, any SOSP of the PCA problem (12) is also a global minimizer of problem (12). This completes the proof of Lemma 21.

**E Another Proof of Theorem 8**

Here, we establish Theorem 8 with Proposition 32 in [1] as the starting point.

First, let us recall from (10b,12) that
\[
\min_{W_N} L_N(W_N) = \min_{W_N} \frac{1}{2} ||Y - W_N \cdots W_1 X||_F^2
\]
\[
= \min_{W_N} \frac{1}{2} ||Y - P Q X||_F^2
\]
\[
= \frac{1}{2} ||Y P X - P Q X||_F^2 + \min_{P,Q} \frac{1}{2} ||Y P X - P Q' X||_F^2
\]
\[
= \frac{1}{2} ||Y P X - P Q X||_F^2 + \min_{P,Q} \frac{1}{2} ||Y P X - P Q' X||_F^2
\]
\[
= \frac{1}{2} ||Y P X - P Q X||_F^2 + \min_{P,Q} \frac{1}{2} ||Y P X - P Q X||_F^2
\]
(101)

where the last line above is from (12).

In view of (101), a global minimizer \(W_N^* = (W_1^*, \cdots, W_N^*)\) of problem (6) (the first program in (101)) corresponds to a global minimizer \((P^o, Q^o)\) of problem (12) (the last program in (101)) such that
\[
W_N^* \cdots W_1^* =: W^*, \quad W^* X = P^o Q^o
\]
(102)

By assumption of Theorem 8, it holds that \(\text{rank}(Y \mathcal{P}_X) \geq r\). We can therefore invoke Lemma 21 in
Appendix D to find that
\[ \text{rank}(W_0^\circ \cdots W_0^\circ) = \text{rank}(W^\circ) \quad (\text{see (102)}) \]
\[ = r. \quad (\text{see Lemma 21}) \quad (103) \]

It is convenient to rewrite (103) as
\[ (W_0^\circ, \ldots, W_0^\circ) \in \mathcal{M}_{N,r}, \quad (104) \]
where
\[ \mathcal{M}_{N,r} := \left\{ W_N = (W_N, \ldots, W_1) : \text{rank}(W_N \cdots W_1) = r \right\} \subset \mathbb{R}^{d_N}. \quad (105) \]

On the other hand, with \( k = r \), Proposition 32 in [1] states that an SOSP \( W_N \in \mathcal{M}_{N,r} \) of problem (6) is almost surely a global minimizer of \( L_N \) restricted to the set \( \mathcal{M}_{N,r} \). In view of (104), we see that \( W_N \) is in fact a global minimizer of \( L_N \) in \( \mathbb{R}^{d_N} \). This completes our alternative proof for Theorem 8.

**F Proof of Lemma 9**

On the one hand, note that the objective function \( L_N(W_N) \) of problem (6) is analytic in \( W_N \).

On the other hand, recall the assumption that \( X \) has full column-rank. Then, regardless of initialization, gradient flow (13) is bounded regardless of initialization, see Step 1 in the proof of Theorem 11 in [1].

We can now invoke the Lojasiewicz’ theorem, see for example Theorem 10 in [1] or [70, 71], to conclude that gradient flow converges to an FOSP \( \overline{W}_N \) of problem (6), regardless of initialization.

Lastly, gradient flow (13) avoids strict saddle points of \( L_N \) for almost every initialization \( W_N,0 \), with respect to the Lebesgue measure in \( \mathbb{R}^{d_N} \), see Theorem 4.1 in [27].

We conclude that the limit point \( \overline{W}_N \) of gradient flow (13) is in fact an SOSP of problem (6), for almost every initialization with respect to the Lebesgue measure in \( \mathbb{R}^{d_N} \). This completes the proof of Lemma 9.

Part of this argument is identical to the one in Theorem 11 of [1].

**G Proof of Lemma 11**

In the SVD of \( W(t) \) in (19), we let \( \{s_i(t)\}_{i=1}^{\min(d_r, d_s)} \) denote the singular values of \( W(t) \) in no particular order, with the corresponding left and right singular vectors denoted by \( \{u_i(t), v_i(t)\}_{i=1}^{\min(d_r, d_s)} \), for every \( t \geq 0 \).

On the one hand, the evolution of the singular values of \( W(t) \) in (19) is described by Theorem 3 in [30] as
\[ \dot{s}_i(t) = -\lambda_{s_i}(t)^2 \frac{2}{|\lambda_{s_i}(t)|} \cdot u_i(t)^\top \nabla L_1(W(t)) v_i(t), \quad (106) \]
for every \( t \geq 0 \), where \( L_1 \) was defined in (10a). Moreover, since the network depth \( N \geq 2 \), \( \{s_i(t)\}_{i=1}^{\min(d_r, d_s)} \) remain nonnegative for every \( t \geq 0 \).

On the other hand, the singular values of \( W(t) \) are bounded, namely, \( \max_{i \leq d_s} s_i(t) < \infty \).

Indeed, if \( X \) has full column-rank, then gradient flow (13) is bounded regardless of initialization, see Step 1 in the proof of Theorem 11 in [1]. Recall also that gradient flow (13) and induced flow (17) are related through the map
\[ \mathbb{R}^{d_N} \to \mathbb{R}^{d_s \times d_t} \]
\[ W_N = (W_1, \ldots, W_N) \to W = W_N \cdots W_1. \quad (107) \]
Consequently, induced flow (17) and a fortiori its singular values too are bounded.

We finally apply Lemma 4 in [30] to (106) and find that each \( s_i(t) \) is either zero for all \( t \geq 0 \), or positive for all \( t \geq 0 \), provided that the network depth \( N \geq 2 \).

In other words, \( \text{rank}(W(t)) \) is invariant with \( t \), namely,
\[ \text{rank}(W(t)) = \text{rank}(W_0), \quad \forall t \geq 0, \quad (108) \]
which completes the proof of Lemma 11.

**H Proof of Lemma 12**

It is easy to see that \( \mathcal{M}_{N,r} \) is not a closed set for any integer \( r \). For example, one can construct a sequence of rank-1 matrices that converge to the zero matrix.

For the second claim in Lemma 12, the proof is by induction over the depth \( N \) of the linear network.

For the base of induction, when \( N = 1 \), note that \( W_N = W_1 \in \mathbb{R}^{d_1 \times d_c} \), see (3). It now follows from (9) that
\[ \min(d_0, d_1) = r. \quad (\text{see (9)}) \quad (109) \]

In turn, it follows from (109) that almost every \( W_1 \) is rank-\( r \), with respect to the Lebesgue measure in \( \mathbb{R}^{d_N} \).

For the step of induction, suppose that
\[ \text{rank}(W) = r, \quad \text{with } W = W_N \cdots W_1, \quad (110) \]
for almost every \( W_N = (W_1, \cdots, W_1) \), with respect to the Lebesgue measure in \( \mathbb{R}^{d_N} \). In particular, it follows from (110) that range(\( W \)) is almost surely an \( r \)-dimensional subspace in \( \mathbb{R}^{d_N} \), namely,
\[ \dim(\text{range}(W)) = r, \quad \text{almost surely}. \quad (111) \]

Consider a generic matrix \( W_{N+1} \), with respect to the Lebesgue measure in \( \mathbb{R}^{d_{N+1} \times d_N} \). We distinguish two cases.

In the first case, suppose that \( d_{N+1} \geq d_N \). Then, \( W_{N+1} \in \mathbb{R}^{d_{N+1} \times d_N} \) has a trivial null space almost surely. With null standing for null space of a matrix, it follows that
\[ (W_{N+1}W) = (W), \quad (112) \]
and, consequently,
\[ \text{rank}(W_{N+1}W \cdots W_1) = \text{rank}(W_{N+1}W) \quad (\text{see (110)}) \]
\[ = d_0 - \dim(W_{N+1}W) \]
\[ = d_0 - \dim(W) \quad (\text{see (112)}) \]
\[ = \text{rank}(W) = r, \quad (113) \]
That is, \( A_{\alpha} \) is a generic subspace in \( \mathbb{R}^{d_{N+1}} \) that satisfies (115), it almost surely holds that
\[
\text{range}(W) \cap (W_{N+1}) = \{0\}. \tag{116}
\]

Consequently,
\[
(W_{N+1}W) = (W), \quad \text{(see (116))} \tag{117}
\]
and it follows identically to (113) that
\[
\text{rank}(W_{N+1} \cdots W_1) = r, \tag{118}
\]
almost surely with respect to the Lebesgue measure in \( \mathbb{R}^{d_{N+1}} \).

We conclude from (113,118) that the induction is complete, and this in turn completes the proof of the second and final claim in Lemma 12.

\section{Proof of Lemma 15}

Recall that the initialization of gradient flow (13) is balanced by Assumption 14 and consider induced flow (17). Let us define the set
\[
\mathcal{N}_\alpha(Z_1) := \left\{ W \overset{\text{SVD}}{=} u_W \cdot s_W \cdot v_W^\top : \right. \nonumber
\]
\[
s_W > (\alpha - \gamma_Z)s_Z, \quad u_W^\top Z_1 v_W > \alpha s_Z \right\} \subset \mathbb{R}^{d_y \times d_z}, \tag{119}
\]
for \( \alpha \in [\gamma_Z, 1) \). Once initialized in \( \mathcal{N}_\alpha(Z_1) \), induced flow remains there, as detailed in the next technical lemma.

\begin{lemma}
For induced flow (17) and \( \alpha \in [\gamma_Z, 1) \), \( W_0 \in \mathcal{N}_\alpha(Z_1) \) implies that \( W_t \in \mathcal{N}_\alpha(Z_1) \) for all \( t \geq 0 \).
That is,
\[
W_0 \in \mathcal{N}_\alpha(Z_1) \implies W_t \in \mathcal{N}_\alpha(Z_1), \quad \forall t \geq 0. \tag{120}
\]
Above, Assumption 14 and the notation therein are in force.
\end{lemma}

Before proving Lemma 22 in the next appendix, we show how it helps us prove Lemma 15.

Indeed, from Lemma 22 and the balanced initialization of gradient flow (13), it follows that
\[
W_{N,0} \in \mathcal{N}_{\alpha,0} \implies W_{N,t} \in \mathcal{N}_{\alpha,0}, \quad \forall t \geq 0, \tag{121}
\]
under Assumption 14 and for \( \alpha \in [\gamma_Z, 1) \), where we used the definition of \( \mathcal{N}_{\alpha,0} \) in (28).

Recall that the limit point \( W_N \) of gradient flow (13) exists by Lemma 9, since \( X \) has full-column rank by Assumption 14.

A byproduct of (121) about the limit point \( W_N \) of gradient flow (13) is that
\[
W_{N,0} \in \mathcal{N}_{\alpha,0} \implies W_N \in \text{closure}(\mathcal{N}_{\alpha,0}) \subset \mathcal{M}_{1,1}, \tag{122}
\]
where the set inclusion above holds true provided that \( \alpha \in (\gamma_Z, 1) \), see (20,28,121). In words, (122) indicates that gradient flow does not converge to the zero matrix.

This completes the proof of Lemma 15.

\subsection{Proof of Lemma 22}

The proof relies on the following technical lemma, which roughly-speaking states that the (rank-1) induced flow (17) always points in a similar direction as the (rank-1) target matrix \( Z_1 \).

\begin{lemma}
Under Assumption 14 and for \( \alpha \in [\gamma_Z, 1) \), \( u_0^\top Z_1 v_0 > \alpha s_Z \) implies that \( u_t^\top Z_1 v_t > \alpha s_Z \) for every \( t \geq 0 \). That is,
\[
u^\top Z_1 v > \alpha s_Z \implies u^\top Z_1 v > \alpha s_Z, \quad \forall t \geq 0. \tag{123}
\]
Above, \( W_t \overset{\text{SVD}}{=} u_t s_t v_t^\top \) is the rank-1 induced flow in (17,32), and \( s_Z, \gamma_Z \) were defined in (26).
\end{lemma}

Before proving Lemma 23 in the next appendix, let us see how Lemma 23 can be used to prove Lemma 22.

Let us fix \( \alpha \in [\gamma_Z, 1) \). If \( W_0 \overset{\text{SVD}}{=} u_0 s_0 v_0^\top \in \mathcal{N}_{\alpha}(Z_1) \), then \( u_0^\top Z_1 v_0^\top > \alpha s_Z \) by definition of \( \mathcal{N}_\alpha(Z_1) \) in (119). Lemma 23 then implies that
\[
u^\top Z_1 v > \alpha s_Z, \quad \forall t \geq 0. \tag{124}
\]
To prove Lemma 22, by the way of contradiction, let \( \tau > 0 \) be the first time that the induced flow (17) leaves the set \( \mathcal{N}_\alpha(Z_1) \). It thus holds that
\[
s_t = \alpha s_Z - s_{Z,2} \quad (\text{see (119)}) \nonumber
\]
\[
< u_t^\top Z_1 v_t - s_{Z,2}, \quad (\text{see (124)}) \tag{125}
\]
where the first line above uses the continuity of \( s_t \) as a function of time \( t \). Indeed, we know \( s_t \) to be an analytic function of \( t \), see (32).

On the other hand, let us recall the evolution of the nonzero singular value of flow (17) from (106), which we repeat here for convenience:
\[
\dot{s}_t = -N s_t^2 \frac{\alpha - \gamma_Z}{2} \cdot u_t^\top \nabla L_1(W_t)v_t. \tag{106}
\]
Recalling the definition of \( L_1 \) from (10a) and the whitened data assumption in (23), we simplify the above gradient as
\[
\nabla L_1(W_t) = W_t XX^\top - YX^\top = m(W_t - Z). \quad (\text{see (23,25)}) \tag{127}
\]
Substituting (127) back into (126) and using the thin SVD of $W_t$ in (32), we write at

\[ s_t = -mNs_t^2 \cdot u_t^\top(W_t - Z)v_t \quad \text{(see (126,127))} \]
\[ = -mNs_t^2 \cdot (s_t - u_t^\top Z)v_t \quad \text{(see (32))} \]
\[ > -mNs_t^2 \cdot (u_t^\top Z_1v_t - s_{Z,2} - u_t^\top Z_1v_t) \quad \text{(see (125))} \]
\[ = -mNs_t^2 \cdot (u_t^\top Z_1v_t - s_{Z,2} - u_t^\top Z_1v_t) \quad \text{(see (170))} \]
\[ = -mNs_t^2 \cdot (-s_{Z,2} - u_t^\top Z_1v_t) \geq 0, \quad \text{(128)} \]

which pushes the singular value up and thus pushes the induced flow back into $N_\alpha(Z_1)$. That is, the induced flow cannot escape from $N_\alpha(Z_1)$.

In the last line of (128), we used the fact that $s_{Z,2}$ is the second largest singular value of $Z$ and hence the largest singular value of the residual matrix $Z_1^+$, see (171). In the same line, we also used the fact that $u_t, v_t$ are unit-length vectors by construction, so that $u_t^\top Z_1v_t \geq -s_{Z,2}$. This completes the proof of Lemma 22.

I.2 Proof of Lemma 23

From (32), recall the thin SVD of induced flow (17), namely,

\[ W_t \overset{\text{SVD}}{=} u_t s_t v_t^\top, \quad \forall t \geq 0, \quad \text{(129)} \]

where

\[ \|u_t\|_2^2 = \|v_t\|_2^2 = 1, \quad \text{(130)} \]

and the only nonzero singular value is $s_t > 0$.

By taking the derivative of the identities in (130) with respect to $t$, we find that

\[ u_t^\top u_t = 0, \quad v_t^\top v_t = 0, \quad \forall t \geq 0. \quad \text{(131)} \]

By taking derivative of both sides of the thin SVD (129), we also find that

\[ \hat{W}_t = \hat{u}_t s_t v_t^\top + u_t \hat{s}_t v_t^\top + u_t s_t \hat{v}_t^\top, \quad \forall t \geq 0. \quad \text{(132)} \]

Let $U_t \in \mathbb{R}^{d_x \times (d_x - 1)}$ with orthonormal columns be orthogonal to $u_t$. By multiplying both sides of (132) by $U_t^\top$, we find that

\[ U_t^\top \hat{W}_t = U_t^\top \hat{u}_t s_t v_t^\top, \quad \forall t \geq 0, \quad \text{(133)} \]

which after rearranging yields that

\[ U_t^\top \hat{u}_t = s_t^{-1} U_t^\top \hat{W}_tv_t, \quad \forall t \geq 0. \quad \text{(134)} \]

Combining (131,134) yields that

\[ u_t = s_t^{-1} \mathcal{P}_{U_t} \hat{W}_tv_t, \quad \forall t \geq 0, \quad \text{(135)} \]

where $\mathcal{P}_{U_t} = U_t U_t^\top$ is the orthogonal projection onto the subspace orthogonal to $u_t$.

Similarly, let $V_t \in \mathbb{R}^{d_x \times (d_x - 1)}$ with orthonormal columns be orthogonal to $v_t$. As before, by multiplying both sides of (132) by $V_t$, we find that

\[ \hat{W}_t V_t = u_t s_t u_t^\top V_t, \quad \forall t \geq 0, \quad \text{(136)} \]

which after rearranging yields

\[ V_t^\top \hat{v}_t = s_t^{-1} V_t^\top \hat{W}_tu_t, \quad \forall t \geq 0. \quad \text{(137)} \]

Then, combining (131,137) leads us to

\[ \hat{v}_t = s_t^{-1} \mathcal{P}_{V_t} \hat{W}_tu_t, \quad \forall t \geq 0, \quad \text{(138)} \]

where $\mathcal{P}_{V_t} = V_t V_t^\top$.

Both expressions (135,138) involve $\hat{W}_t$. Under the assumption of whitened data in (23), we express $\hat{W}_t$ as

\[ \hat{W}_t = -\mathcal{A}_W(W_tX^\top - YX^\top) \quad \text{(see (17))} \]
\[ = -m\mathcal{A}_W(W_t - Z) \quad \text{(see (23,25))} \]
\[ = -mNs_t^{-1} \cdot u_t^\top Zv_t W_t \quad \text{(see (173))} \]
\[ + ms_t^{-1} \mathcal{P}_{U_t} u_t \cdot Zv_t, \quad \forall t \geq 0, \quad \text{(139)} \]

where $\mathcal{P}_{U_t} = u_t u_t^\top$ and $\mathcal{P}_{v_t} = v_t v_t^\top$. The last identity above invokes the first part of Lemma 24, which collects some basic properties of the operator $\mathcal{A}_W$.

Substituting $\hat{W}_t$ back into (135,138), we reach

\[ u_t = ms_t^{-1} \mathcal{P}_{U_t} Zv_t, \quad \text{(see (139))} \]
\[ v_t = ms_t^{-1} \mathcal{P}_{V_t} Z^\top u_t, \quad \forall t \geq 0. \quad \text{(140)} \]

It immediately follows from the first identity in (140) that

\[ u_t^\top u_t = ms_t^{-1} u_t^\top Zv_t \quad \text{and} \quad v_t^\top v_t = ms_t^{-1} v_t^\top Zu_t \quad \text{see (140)} \]
\[ = ms_t^{-1} u_t^\top \mathcal{P}_{U_t} (u_t s_t v_t^\top + Z_1^+) v_t \quad \text{see (170)} \]
\[ = ms_t^{-1} u_t^\top \mathcal{P}_{U_t} u_t \cdot v_t + Z_1^+ v_t \]
\[ + ms_t^{-1} s_t \mathcal{P}_{U_t} Z_1^+ v_t, \quad \forall t \geq 0. \quad \text{(141)} \]

To bound the last term above, note that

\[ \|u_t^\top \mathcal{P}_{U_t} Z_1^+ v_t\|_2 \leq \|\mathcal{P}_{U_t} u_t\|_2 \cdot \|Z_1^+ v_t\|_2 \quad \text{Cauchy-Schwarz ineq.} \]
\[ \leq \|\mathcal{P}_{U_t} u_t\|_2 \cdot \|\mathcal{P}_{v_t} v_t\|_2 \quad \text{see (170,171)} \quad \text{(142)} \]

where $s_{Z,2}$ is the second largest singular value of $Z$ and thus the largest singular value of the residual matrix $Z_1^+$. Above, $V_t \in \mathbb{R}^{d_x \times (d_x - 1)}$ with orthonormal columns is orthogonal to $v_t$, see (170,171).

Similarly, it follows from the second identity in (140) that

\[ v_t^\top v_t = ms_t^{-1} u_t^\top \mathcal{P}_{U_t} Z_1^+ u_t \quad \text{see (140)} \]
\[ = ms_t^{-1} u_t^\top \mathcal{P}_{U_t} (u_t s_t v_t^\top + Z_1^+) u_t \quad \text{see (170)} \]
\[ = ms_t^{-1} s_t \mathcal{P}_{U_t} Z_1^+ u_t + ms_t^{-1} s_t \mathcal{P}_{U_t} Z_1^+ v_t \quad \text{see (140)} \]
\[ \geq 0. \quad \text{(143)} \]
To bound the last term above, we write that
\[
|v^T Z u| \leq \|v^T Z u\|_2 \cdot \|Z u\|_2 \leq \|P v^T Z u\|_2 \cdot \|Z u\|_2, \tag{144}
\]
where the residual \(R\) where the residual
\[
= 2 ms_Z s_t (a_t^2 + b_t^2 - a_t b_t) \leq ms_Z s_t (1 - a_t^2) (1 - b_t^2). \tag{149}
\]
and the second above uses the fact that
\[
|a_t| = |u^T Z u| \leq \|u\|_2 \cdot \|Z\|_2 \leq 1, \tag{150}
\]
for every \(t \geq 0\), and similarly \(|b_t| \leq 1\). The third line in (149) uses the inequality
\[
\sqrt{(1 - a_t^2)(1 - b_t^2)} = \sqrt{1 - a_t^2 - b_t^2 + a_t^2 b_t^2} \leq \sqrt{1 - 2a_t b_t + a_t^2 b_t^2} = 1 - a_t b_t, \tag{151}
\]
where the last line above again uses (150).

The residual \(R_t\) is small when the spectral gap of \(Z\) is large. Indeed, note that
\[
|R_t| \leq 2 ms_Z s_t (1 - a_t b_t) \leq 2 ms_Z s_t (1 - a_t b_t), \tag{152}
\]
where the last line above holds provided that
\[
a_t b_t > \frac{s_Z^2}{s_Z} = \gamma_Z. \tag{153}
\]
We continue and bound the last line of (152) as
\[
|R_t| < 2 ms_Z s_t (a_t b_t (1 - a_t b_t)) \leq 2 ms_Z s_t (a_t b_t - a_t^2 b_t^2) \leq 2 ms_Z s_t \left(\frac{a_t^2 + b_t^2 - a_t b_t}{2}\right) = ms_Z s_t (a_t^2 + b_t^2 - 2a_t b_t) = ms_Z s_t \left((1 - a_t^2) b_t^2 + a_t b_t (1 - b_t^2)\right). \tag{154}
\]
By comparing the above bound on the residual \(R_t\) with (148), for a fixed time \(t\), we conclude that
\[
\frac{d(u^T Z v)}{dt} > 0, \tag{155}
\]
provided that
\[
u_t^T Z v_t = s_Z \cdot u_t^T v_t \tag{170}
\]
and similarly for every \(t \geq 0\), which completes the proof of Lemma 23.

## J Lazy Training

For completeness, here we verify that Theorem 1 in [2] suffers from lazy training [3].

For the sake of clarity, let us assume that \(d_x = m\) and \(X = \sqrt{m} I_{d_x}\), which satisfies the whitened requirement in (23) and [2]. Here, \(I_{d_x} \in \mathbb{R}^{d_x \times d_x}\) is the identity matrix.

Recalling the loss function \(L_N\) in (6) and the initialization \(W_{N,0} \in \mathbb{R}^{d_N}\) of gradient flow (13), we write that
\[
L_N(W_{N,0}) = \frac{1}{2} \|Y - W_{N,0} \cdots W_1 X\|^2_F \tag{6}
\]
and
\[
= \frac{1}{2} \|Y - \sqrt{m} W_{N,0} \cdots W_1\|^2_F \quad (X = \sqrt{m} I_{d_x})
\]
\[
= \frac{m}{2} \|Y^T m - W_{N,0} \cdots W_1\|^2_F \quad (X = \sqrt{m} I_{d_x})
\]
\[
= \frac{m}{2} \|Z - W_{N,0} \cdots W_1\|^2_F. \tag{25}
\]

Definition 2 in [2] requires the last line above and, consequently, \(L_N(W_{N,0})\) to be small. In turn, \(L_N(W_{N,0})\) appears in Equation (1) in [3]. Definition 2 in [2] thus requires the factor \(\kappa\) in Equation (1) in [3] to be small, which is how the authors define the lazy training regime there.
Proof of Lemma 17

From Theorem 16, recall that gradient flow (13) converges to a solution of (6) from almost every balanced initialization in the set $\mathcal{N}_{U,\alpha}$. That is,

$$\lim_{t \to \infty} \frac{1}{2} \|Y - W_N(t) \cdots W_1(t)X\|^2_F = \min_{w_1, \ldots, w_N} \frac{1}{2} \|Y - W_N \cdots W_1X\|^2_F.$$  \hspace{1cm} (159)

On the other hand, recall that gradient flow (13) induces the flow (17) under the surjective map $\mathbb{R}^{d_N} \to \mathcal{M}_{1,\ldots,r}$

$$W_N = (W_1, \ldots, W_N) \to W = W_N \cdots W_1,$$  \hspace{1cm} (160)

where $\mathcal{M}_{1,\ldots,r}$ is the set of all $d_N \times d_2$ matrices of rank at most $r$, see Appendix A for the proof of the surjective property.

In view of (159), induced flow (17) therefore satisfies

$$\lim_{t \to \infty} \frac{1}{2} \|Y - W(t)X\|^2_F = \min_{\text{rank}(W) \leq \frac{r}{2}} \frac{1}{2} \|Y - WX\|^2_F,$$  \hspace{1cm} (161)

where $W(t) = W_N(t) \cdots W_1(t)$.

Let $\mathcal{P}_X = X^TX$ and $\mathcal{P}_{X^\perp} = I_m - \mathcal{P}_X$ denote the orthogonal projections onto the row span of $X$ and its orthogonal complement, respectively. We can decompose $Y$ as

$$Y = Y \mathcal{P}_X + Y \mathcal{P}_{X^\perp}.$$  \hspace{1cm} (162)

Using this decomposition, we can rewrite (161) as

$$\lim_{t \to \infty} \frac{1}{2} \|Y \mathcal{P}_X - W(t)X\|^2_F = \frac{1}{2} \|Y\mathcal{P}_X - WX\|^2_F.$$  \hspace{1cm} (163)

That is, in words, a linear network can only learn the component of $Y$ within the row span of $X$.

Under Assumption 14, the data matrix $X$ is whitened, so that $\mathcal{P}_X = X^TX = \frac{1}{m} X^TX$, see (23). We can therefore revise (163) as

$$\lim_{t \to \infty} \frac{m}{2} \|Z - W(t)\|^2_F = \min_{\text{rank}(W) \leq \frac{r}{2}} \frac{m}{2} \|Z - WX\|^2_F,$$  \hspace{1cm} (164)

where above we also used the definition of $Z$ in (25). To prove Lemma 17, we continue by setting $r = 1$ in (164).

Recall also from Assumption 14 and specifically (26) that $Z$ has a nontrivial spectral gap, namely, $s_Z > s_{Z,2}$. Therefore, $Z_1 = u_Z s_Z v_Z^\top$ is the unique solution of the optimization problem in (164), where the vectors $u_Z, v_Z$ are the corresponding leading left and right singular vectors of $Z_1$, see for example Section 1 in [41]. In view of this, it now follows from (164) with $r = 1$ that

$$\lim_{t \to \infty} \|Z_1 - W(t)\|^2_F = 0,$$  \hspace{1cm} (165)

which completes the proof of Lemma 17.

Derivation of (33)

From Appendix M, we will use the orthonormal bases $U_t = [u_t, U_t]$ and $V_t = [v_t, V_t]$. We decompose the loss function in these two bases as

$$L_{1,1}(W_t) = \frac{1}{2} \|W_t - Z_1\|^2_F$$  \hspace{1cm} (see (31))

$$= \frac{1}{2} \|P_{u_t}(W_t - Z_1)P_{v_t}\|^2_F$$

$$+ \frac{1}{2} \|P_{u_t}(W_t - Z_1)P_{v_t}\|^2_F$$

$$+ \frac{1}{2} \|P_{u_t}(W_t - Z_1)P_{v_t}\|^2_F$$

$$+ \frac{1}{2} \|P_{u_t}(W_t - Z_1)P_{v_t}\|^2_F,$$  \hspace{1cm} (166)

where $P_{u_t} = u_t u_t^\top$ is the orthogonal projection onto the span of $u_t$, and the remaining projection operators above are defined similarly.

Recalling the thin SVD $W_t = u_t s_t v_t^\top$ from (32) allows us to simplify (166) as

$$L_{1,1}(W_t) = \frac{1}{2} (s_t - u_t^\top Z_1 v_t)^2$$

$$+ \frac{1}{2} \|P_{u_t} Z_1 P_{v_t}\|^2_F$$

$$+ \frac{1}{2} \|P_{u_t} Z_1 P_{v_t}\|^2_F$$

$$+ \frac{1}{2} \|P_{u_t} Z_1 P_{v_t}\|^2_F.$$  \hspace{1cm} (167)

Using the thin SVD $Z_1 = u_Z s_Z v_Z^\top$ from (170) simplifies the last line above as

$$\|Z_1\|^2_F - \|P_{u_t} Z_1 P_{v_t}\|^2_F = s_Z^2 - s_Z^2 (u_t^\top u_Z)^2 (v_t^\top v_Z)^2.$$  \hspace{1cm} (168)

Substituting the above identity back into (167) yields that

$$L_{1,1}(W_t) = \frac{1}{2} (s_t - u_t^\top Z_1 v_t)^2$$

$$+ \frac{s_Z^2}{2} (1 - (u_t^\top u_Z)^2 (v_t^\top v_Z)^2)$$  \hspace{1cm} (see (167,168))

$$= \frac{1}{2} (s_t - u_t^\top Z_1 v_t)^2$$

$$+ \frac{1}{2} s_Z^2 - (u_t^\top Z_1 v_t)^2$$  \hspace{1cm} (see (170))

$$= \frac{1}{2} (s_t - u_t^\top Z_1 v_t)^2$$

$$+ \frac{1}{2} (s_Z + u_t^\top Z_1 v_t) (s_Z - u_t^\top Z_1 v_t)$$

$$\leq \frac{1}{2} (s_t - u_t^\top Z_1 v_t)^2$$

$$+ s_Z (s_Z - u_t^\top Z_1 v_t),$$  \hspace{1cm} (169)

where the second identity and the inequality above use again the thin SVD of $Z_1$. The inequality above also uses $u_t^\top Z_1 v_t \leq s_Z$ twice, which holds true because $u_t, v_t$ are unit-length vectors by construction and $s_Z$ is the only nonzero singular of $Z_1$, see (170). This completes the derivation of (33).
\textbf{M Proof of Lemma 18}

To begin, recall from (27) that \( Z_1 \) is the leading rank-1 component of \( Z \), and let \( Z_{1+} = Z - Z_1 \) denote the corresponding residual. We thus decompose \( Z \) as

\[
Z = Z_1 + Z_{1+} = \underbrace{u_Z \cdot s_Z \cdot v_Z^\top}_\text{SVD} + \underbrace{U_Z S_Z V_Z^\top}_\text{remaining},
\]

where \( U_Z, V_Z \) contain the remaining left and right singular vectors of \( Z \), and \( S_Z \) contains the remaining singular values of \( Z \). In particular, let us repeat that

\[
s_Z = \|Z\| = \|Z_1\|, \quad s_{Z,2} = \|Z_{1+}\|, \quad \text{(see (26))}
\]

where \( \| \cdot \| \) stands for spectral norm.

In this appendix, we compute the evolution of loss function \( L_{1,1} \) with time, which we recall from (34) as

\[
d_{L_{1,1}}(W_{1,1}) = \frac{d}{dt} \|Z_\text{SVD} - \tilde{U} \tilde{S} \tilde{V}^\top\|^2
\]

where the SVD \( \text{SVD} \) contains the singular values of \( Z \). For the first inner product in the last line of (172),

\[
\langle W_t - Z_1, A_{W_t}(W_t - Z_1) \rangle
\]

where the second line above uses the SVD (see (175))

\[
W_t = \underbrace{\tilde{U}_t \tilde{S}_t \tilde{V}_t^\top}_\text{SVD},
\]

for every \( t \geq 0 \), where 0 above is the \((d_y - 1) \times (d_x - 1)\) zero matrix. Using (176), we simplify (174) to read

\[
\langle W_t - Z_1, A_{W_t}(W_t - Z_1) \rangle
\]

\[
= N s_t^{2-\frac{2}{p}} \|s_t - u_t^\top Z_1 v_t\|^2
\]

\[
+ s_t^{2-\frac{2}{p}} \|u_t^\top Z_1 v_t\|^2
\]

\[
+ s_t^{2-\frac{2}{p}} \|U_t^\top Z_1 v_t\|^2.
\]

The two norms above can be further simplified. Let us set

\[
a_t := u_t^\top u_z, \quad b_t := v_t^\top v_z,
\]

noting (178) for short. Then we expand the first norm in (177) as

\[
\|u_t^\top Z_1 v_t\|^2 = s_z^2 \|u_t^\top u_z v_t^2\|_2 \quad \text{(see (170))}
\]

\[
= s_z^2 \|u_t^\top u_z\| \cdot \|v_t^\top v_z\|_2
\]

\[
= s_z^2 a_t (1 - b_t^2), \quad \text{(see (178))}
\]

where the last line above follows because \( V_t \) spans the orthogonal complement of \( v_t \) Likewise, the second norm in (177) is expanded as

\[
\|U_t^\top Z_1 v_t\|^2 = s_z^2 \|U_t^\top a_z\| \cdot \|v_t^\top v_z\|
\]

\[
= s_z^2 b_t (1 - a_t^2). \quad \text{(see (178))}
\]

In particular, by combining (179,180), we find that

\[
\|u_t^\top Z_1 v_t\|^2 + \|U_t^\top Z_1 v_t\|^2
\]

\[
= s_z^2 (a_t^2 (1 - b_t^2) + (1 - b_t^2) a_t^2) \quad \text{(see (178,190))}
\]

\[
= s_z^2 (a_t^2 + b_t^2 - 2 a_t^2 b_t^2)
\]

\[
\geq s_z^2 (2 a_t b_t - 2 a_t^2 b_t^2)
\]

\[
= 2 s_z^2 a_t b_t (1 - a_t b_t),
\]

(181)

where the penultimate line above uses the inequality \( a_t^2 + b_t^2 \geq 2 a_t b_t \). Plugging (181) back into (177), we arrive at

\[
\langle W_t - Z_1, A_{W_t}(W_t - Z_1) \rangle
\]

\[
\geq N s_t^{2-\frac{2}{p}} (s_t - u_t^\top Z_1 v_t)^2
\]

\[
+ 2 s_t^{2-\frac{2}{p}} s_z^2 a_t b_t (1 - a_t b_t).
\]

(182)

For the second inner product in the last line of (172), we invoke the first identity in (173) to write that

\[
\langle W_t - Z_1, A_{W_t}(W_t - Z_1) \rangle
\]

\[
= N s_t^{2-\frac{2}{p}} (s_t - u_t^\top Z_1 v_t)^2
\]

\[
+ 2 s_t^{2-\frac{2}{p}} s_z^2 a_t b_t (1 - a_t b_t),
\]

(182)

For the second inner product in the last line of (172), we invoke the first identity in (173) to write that

\[
\langle W_t - Z_1, A_{W_t}(Z_{1+}) \rangle
\]

\[
= N s_t^{2-\frac{2}{p}} (s_t - u_t^\top Z_1 v_t) (u_t^\top Z_{1+} v_t)
\]

\[
- s_t^{2-\frac{2}{p}} \langle u_t^\top Z_1 v_t, u_t^\top Z_{1+} v_t \rangle
\]

\[
- s_t^{2-\frac{2}{p}} \langle U_t^\top Z_1 v_t, U_t^\top Z_{1+} v_t \rangle,
\]

(183)
and, consequently,
\[ |(W_t - Z, A_{W_t}(Z_1))| \]
\[ \leq \frac{N}{s_t} s_{Z,2} |s_t - u_t^\top Z_1 v_t| (1 - a_t b_t) \]
\[ + s_{Z,2}^2 \left( |s_t - u_t^\top Z_1 v_t| \left( 1 - a_t b_t \right) \right) \]
\[ \leq \frac{N}{s_t} s_{Z,2} |s_t - u_t^\top Z_1 v_t| (1 - a_t b_t) \]
\[ + 2 s_{Z,2}^2 \left( |s_t - u_t^\top Z_1 v_t| (1 - a_t b_t) \right), \]  (184)
where the last line above follows from the chain of inequalities
\[ (a_t + b_t) \sqrt{1 - a_t^2} \sqrt{1 - b_t^2} \]
\[ \leq 2 \sqrt{1 - a_t^2 - b_t^2 + a_t^2 b_t^2} \]  (see (178))
\[ \leq 2 \sqrt{1 - 2 a_t b_t + a_t^2 b_t^2} \]  (see (178))  (189)

Above, in the second and last lines, we used the fact that \( a_t \leq 1 \) and \( b_t \leq 1 \), see their definition in (178).

By combining (182,188), we can upper bound the evolution of the loss function as
\[ \frac{dL_{n,1}(W_t)}{dt} \]
\[ \leq -mn s_t^{2 - \frac{2}{\Delta}} \left( |s_t - u_t^\top Z_1 v_t| \right)^2 \]
\[ - 2mn s_t^{2 - \frac{2}{\Delta}} s_{Z,2}^2 (1 - a_t b_t) \]
\[ + mn s_t^{2 - \frac{2}{\Delta}} s_{Z,2}^2 |s_t - u_t^\top Z_1 v_t| (1 - a_t b_t) \]
\[ + 2mn s_t^{2 - \frac{2}{\Delta}} s_{Z,2}^2 (1 - a_t b_t) \]
\[ = 2(1 - a_t b_t). \]  (see (178))  (189)

where the identity above uses the fact that \( a_t b_t = u_t^\top Z_1 v_t \), see (170,178). The inverse spectral gap \( \gamma_Z = s_{Z,2}/s_Z \) was introduced in (26). This completes the proof of Lemma 18.

M.1 Proof of Lemma 24

Let \( W^\text{SVD} = \tilde{U} \tilde{S} \tilde{V}^\top \) denote the SVD of \( W \), where \( \tilde{U}, \tilde{V} \) are orthonormal bases, and \( \tilde{S} \) contains the singular values of \( W \).

Using the definition of \( A_W \), we write that
\[ A_W(\Delta) = \sum_{j=1}^N (\tilde{W} \tilde{W}^\top)^{\frac{a_t}{\Delta}} \Delta(\tilde{W} \tilde{W}^\top) \frac{b_t}{\Delta} \]  (see (16))
\[ = \sum_{j=1}^N (\tilde{U} \tilde{S} \tilde{V}^\top)^{\frac{a_t}{\Delta}} \Delta(\tilde{V} \tilde{S} \tilde{V}^\top) \frac{b_t}{\Delta} \]
\[ = \sum_{j=1}^N (\tilde{S} \tilde{S}^\top)^{\frac{a_t}{\Delta}} \tilde{U}^\top \Delta \tilde{V}(\tilde{S} \tilde{S}^\top) \frac{b_t}{\Delta} \tilde{V}^\top \]
\[ =: N \sum_{j=1}^N A_{W,j}(\Delta), \]  (191)
which proves the first claim in Lemma 24.

For every \( j \in \mathbb{N} \), it also holds that
\[ \langle \Delta, A_{W,j}(\Delta) \rangle = \left\| (\tilde{S} \tilde{S}^\top)^{\frac{a_t}{\Delta}} \tilde{U}^\top \Delta \tilde{V}(\tilde{S} \tilde{S}^\top)^{\frac{b_t}{\Delta}} \tilde{V}^\top \right\|^2 \]  (192)
where the last line uses the fact that \( \tilde{U}, \tilde{V} \) are orthonormal bases. The proof of Lemma 24 is complete after summing up the above identity over \( j \).
The proof is similar to that of Lemma 22. Let us fix $\alpha \in [\gamma_Z, 1]$ and $\beta > 1$. If
$$W_0^{eVD} = u_0s_0v_0^\top \in \mathcal{N}_{\alpha, \beta}(Z_1),$$
then $u_0^\top Z_1v_0 > \alpha s_Z$ by definition of $\mathcal{N}_{\alpha, \beta}(Z_1)$ in (36). Lemma 23 then implies that
$$u_t^\top Z_1v_t > \alpha s_Z, \quad \forall t \geq 0.$$  
(194)

To prove Lemma 19, by the way of contradiction, let $\tau > 0$ be the first time that induced flow (17) leaves the set $\mathcal{N}_{\alpha, \beta}(Z_1)$. It thus holds that
$$s_\tau = \alpha s_Z - s_Z, \quad (195)$$
or
$$s_\tau = \beta s_Z, \quad (196)$$
where both of the identities above use the continuity of $s_t$ as a function of $t$. Indeed, we know $s_t$ to be an analytic function of time $t$, see (32).

The case where (195) happens is handled identically to the proof of Lemma 22. We therefore focus on when the second case happens, namely, (196).

Recalling the second identity in (128), we bound the evolution of the singular value of induced flow (17) as
$$s_t = -mNs_\tau - \frac{\beta}{mN} \cdot (s_\tau - u_t^\top Zv_t), \quad (198)$$
which pushes the singular value down and thus pushes the induced flow back into $\mathcal{N}_{\alpha, \beta}(Z_1)$. That is, the induced flow cannot escape from $\mathcal{N}_{\alpha, \beta}(Z_1)$.

In the third line of (197), we used the fact that $s_Z$ is the leading singular value of $Z$, and $u_t,v_t$ are unit-norm vectors, see (170,129), thus $u_t^\top Zv_t \leq s_Z$. The last line in (197) holds because Lemma 19 assumes that $\beta > 1$. This completes the proof of Lemma 19.

## O Proof of Theorem 20

Let us fix $\alpha \in [\gamma_Z, 1)$ and $\beta > 1$. In view of Lemma 19, we assume henceforth that induced flow (17) is initialized within $\mathcal{N}_{\alpha, \beta}(Z_1)$ and thus remains there forever, namely,
$$W_t \in \mathcal{N}_{\alpha, \beta}(Z_1), \quad \forall t \geq 0.$$  
(198)

Using the definition of $\mathcal{N}_{\alpha, \beta}(Z_1)$ in (36) and (198), we can update (35) as
$$\frac{dL_{1,1}(W_t)}{dt} \leq -ms^2 - \frac{\alpha}{N}L_{1,1}(W_t) + 2mN(\alpha - \gamma_Z)^2 s_Z(s_t - u_t^\top Zv_t) + 2m(\beta s_Z)^2 s_Z(s_t - u_t^\top Zv_t).$$
(199)

Recalling the upper bound on the loss function in (33), we can distinguish two regimes in the dynamics of (199), depending on the dominant term on the right-hand side of (33), as detailed next.

### (Fast convergence)

When
$$\frac{1}{2}(s_t - u_t^\top Zv_t)^2 \geq s_Z(s_t - u_t^\top Zv_t),$$
the loss function can be bounded as
$$L_{1,1}(W_t) \leq (s_t - u_t^\top Zv_t)^2, \quad (see (33,200)) \quad (201)$$
and the evolution of loss in (199) thus simplifies to
$$\frac{dL_{1,1}(W_t)}{dt} \leq -ms^2 - \frac{\alpha}{N}L_{1,1}(W_t).$$
(202)

### (Slow convergence)

On the other hand, when
$$\frac{1}{2}(s_t - u_t^\top Zv_t)^2 \leq s_Z(s_t - u_t^\top Zv_t),$$
the loss function can be bounded as
$$L_{1,1}(W_t) \leq 2s_Z(s_t - u_t^\top Zv_t), \quad (see (33,203)) \quad (204)$$
and the evolution of loss in (199) simplifies to
$$\frac{dL_{1,1}(W_t)}{dt} \leq -ms^2 - \frac{\alpha}{N}L_{1,1}(W_t).$$
(205)

In view of (200,203), the key transition between fast and slow convergence rates happens when
$$T_t := T_{1,t} - s_ZT_{2,t} + \frac{1}{2}(s_t - u_t^\top Zv_t)^2 - s_Z(s_t - u_t^\top Zv_t) \quad (206)$$
changes sign. Above, we used the definition of $T_{1,t}$ and $T_{2,t}$ in (33).

Instead of the first time such a sign change happens, it is convenient to consider the more conservative choice of time $\tau \geq 0$ when
$$s_\tau \leq \sqrt{6}s_Z \quad (207)$$
for the first time. Indeed, if (207) does not hold, then $T_\tau > 0$ and thus the fast convergence is in force. This claim is verified in Appendix O.2 for completeness.

With the definition of $\tau$ at hand from (207), we can combine (202) and (205) to obtain that
$$\frac{dL_{1,1}(W_t)}{dt} \leq -ms^2 - \frac{\alpha}{N}L_{1,1}(W_t)$$
$$\left\{ \begin{array}{ll} N \left( (\alpha - \gamma_Z)^2 - \frac{\alpha}{N} \right) - 2\gamma_ZN\beta^2 - \frac{\alpha}{N} \right) t \leq \tau \\ (\alpha - \gamma_Z)^2 - \frac{\alpha}{N} - 2\gamma_ZN\beta^2 - \frac{\alpha}{N} \right) t \geq \tau. \quad (208)$$

Suppose that inverse spectral gap $\gamma_Z$ is small enough such that the right-hand side of (208) is negative, see (26). Using this observation that $L_{1,1}(W_t)$ is decreasing in $t$ and by applying the Gronwalls inequality to (208), we arrive at Theorem 20.
O.1 Derivation of (202,205)

We begin with the detailed derivation of (202). Let us repeat (199) for convenience:

\[
\frac{dL_{1,1}(W_t)}{dt} \leq -mN((\alpha - \gamma_Z)s)z^2 - \hat{\beta}(s_t - u_t^\top Z_1 v_t)^2
\]

- \(2\alpha_m s_Z((\alpha - \gamma_Z)sz^2 - \hat{\beta}(s_t - u_t^\top Z_1 v_t))
\]

+ \(mN(\beta s_Z)z^2 - \hat{\beta} \gamma_Z(s_t - u_t^\top Z_1 v_t)\)

+ \(2m(\beta s_Z)^2 - \hat{\beta} s_Z(3z - u_t^\top Z_1 v_t). \) (see (199))

Recall also that (200) is in force. By ignoring the nonpositive term in the third line above, we arrive at

\[
\frac{dL_{1,1}(W_t)}{dt} \leq -mN((\alpha - \gamma_Z)s)z^2 - \hat{\beta}(s_t - u_t^\top Z_1 v_t)^2
\]

+ \(mN(\beta s_Z)z^2 - \hat{\beta} \gamma_Z(s_t - u_t^\top Z_1 v_t)\)

\cdot \sqrt{s_Z(s_t - u_t^\top Z_1 v_t)}

+ 2m(\beta s_Z)^2 - \hat{\beta} \gamma_Z s_Z(3z - u_t^\top Z_1 v_t). \]

To obtain the first inequality in (209), note that \(W_t \in \mathcal{N}_{\alpha,\beta}(Z_1)\) by (198) and, in particular,

\[
u_t^\top Z_1 v_t \geq 0, \quad \forall t \geq 0, \quad (210)
\]

by definition of \(\mathcal{N}_{\alpha,\beta}(Z_1)\) in (36). In turn, (210) implies that \(s_Z - u_t^\top Z_1 v_t \leq s_Z\).

We continue to bound the right-hand side of (209) as

\[
\frac{dL_{1,1}(W_t)}{dt} \leq -mN((\alpha - \gamma_Z)s)z^2 - \hat{\beta}(s_t - u_t^\top Z_1 v_t)^2
\]

+ \(mN(\beta s_Z)z^2 - \hat{\beta} \gamma_Z(s_t - u_t^\top Z_1 v_t)\)

\cdot \sqrt{s_Z(s_t - u_t^\top Z_1 v_t)}

+ 2m(\beta s_Z)^2 - \hat{\beta} \gamma_Z s_Z(3z - u_t^\top Z_1 v_t).
\]

(see (210)) (213)

To obtain the first inequality above, note that

\[
\frac{1}{2}(s_t - u_t^\top Z_1 v_t)^2 \leq s_Z(s_t - u_t^\top Z_1 v_t) \quad (\text{see (203))}
\]

\[
\leq s_Z^2 \quad (\text{see (210))}
\]

which, after rearranging, reads as

\[
|s_t - u_t^\top Z_1 v_t| \leq \sqrt{2} s_Z. \quad (212)
\]

The last inequality in (211) uses (204).

O.2 Derivation of (207)

In the slow convergence regime in (203), it holds that

\[
\frac{1}{2}(s_t - u_t^\top Z_1 v_t)^2 \leq s_Z(s_t - u_t^\top Z_1 v_t) \quad (\text{see (203))}
\]

\[
\leq s_Z^2. \quad (\text{see (210))} \quad (213)
\]

On the other hand, we can also lower bound the first term in (213) as

\[
s_Z^2 \geq \frac{1}{2}(s_t - u_t^\top Z_1 v_t)^2 \quad (\text{see (213)})
\]

\[
\geq \frac{s_t^2}{4} - \frac{1}{2}(u_t^\top Z_1 v_t)^2
\]

\[
\geq \frac{s_t^2}{4} - \frac{s_a^2}{2}, \quad (214)
\]

where the penultimate line above uses the inequality \((a - b)^2 \geq \frac{a^2}{2} - b^2\) for scalars \(a, b\). The last inequality above holds because \(u_t, v_t\) are both unit-norm vectors and \(s_Z\) is the only nonzero singular value of \(Z_1\), see (129,170), and thus \(|u_t^\top Z_1 v_t| \leq s_Z\).

By rearranging (214), we find that \(s_t > \sqrt{6} s_Z\) implies the fast convergence regime in (200).
Figure 1: Suppose that the sample size is $m = 50$, and consider a randomly-generated whitened training dataset $(X, Y) \in \mathbb{R}^{d_x \times m} \times \mathbb{R}^{d_y \times m}$, with $d_x = 5$ and $d_y = 1$. For this dataset, the above figure depicts the distance from induced flow (17) to the target matrix $Z_1 = Z = YX^\top / m$ in (25,27), plotted versus time $t$, for training a linear network with $d_x$ inputs and $d_y$ outputs, as the network depth $N$ varies. The direction of the initial end-to-end vector $W_0 \in \mathbb{R}^{d_y \times d_x}$ is obtained by randomly rotating the direction of the target vector $Z_1$ by about 30 degrees. We also set $\|W_0\|_2 = 10\|Z\|_2$. Instead of induced flow (17), we implemented the discretization of (17) obtained from the explicit (or forward) Euler method with a step size of $10^{-6}$ with $10^5$ steps. This simple numerical example visualizes the (gradual) slow-down in the convergence rate of gradient flow with time, see (38,39), and also shows the faster faraway convergence rate for deeper networks, see Theorem 20. The above figure also suggests that the nearby convergence rate of gradient flow (13) might actually be slower for deeper networks. It is however difficult to verify this theoretically from Theorem 20, because (39) is an upper bound for the nearby error. The precise nearby convergence rates of linear networks (and any trade-offs associated with the network depth) thus remain as open questions. Note also that the local analysis of [2] cannot be applied here, as discussed after Theorem 20. The code will also be made publicly available for any interested readers.