The Spectrum of Strings on BTZ Black Holes and Spectral Flow in the SL(2,R) WZW Model

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Abstract

We study the spectrum of bosonic string theory on rotating BTZ black holes, using a SL(2,R) WZW model. Previously, Natsuume and Satoh have analyzed strings on BTZ black holes using orbifold techniques. We show how an appropriate spectral flow in the WZW model can be used to generated the twisted sectors, emphasizing how the spectral flow works in the hyperbolic basis natural for the BTZ black hole. We discuss the projection condition which leads to the quantization condition for the allowed quantum numbers for the string excitations, and its connection to the anomaly in the corresponding conserved Noether current.

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1 Introduction

One of the obstacles in testing the AdS/CFT correspondence \cite{1, 2} in its strong form has been gaps in understanding string theory in AdS backgrounds. In the case of AdS$_3$, the spacetime can be identified as the $SL(2, R)$ group manifold, and string theory can be formulated as a $SL(2, R)$ WZW model \cite{3}. It was recently understood \cite{4} that the model has a spectral flow symmetry which can be used to understand better the string spectrum, and to study the consistency of the model \cite{5}.

One interesting extension is to study Lorentzian orbifolds of the AdS$_3$ spacetime. Conical defects in AdS$_3$ spacetimes, or point masses in AdS$_3$ \cite{6}, for a discrete set of deficit angles or masses can be constructed as AdS$_3/Z_N$ orbifolds; and black holes in AdS$_3$ (the Bañados-Teitelboim-Zanelli (BTZ) black holes \cite{7}) can be constructed as AdS$_3/Z$ orbifolds. Apart from being interesting in their own right, they appear in near-horizon geometries of (spinning) six-dimensional black strings, five-dimensional black holes, or extremal four-dimensional black holes \cite{3, 7}. More precisely, the near-horizon geometries are (fibered) BTZ×$S^3$ geometries. Since strings on $S^3$ can be studied with $SU(2)$ WZW models, it might be possible to formulate string theory in the above near-horizon geometries as a combination of $SL(2, R)$ and $SU(2)$ WZW models, in the spirit of \cite{9}. Moreover, it has been emphasized recently that conical defects in AdS$_3$ and BTZ black holes give a simple setting where to study formation of black holes in string theory \cite{10}.

Strings on AdS$_3/Z_N$ orbifolds were recently investigated in the context of spectral flow \cite{11, 12}. Among other things, it was shown how the twisted sectors can be thought to be generated by spectral flow. String theory on BTZ black holes was investigated by Natsuume and Satoh \cite{13}, using a WZW model based on the $\tilde{SL}(2, R)/Z$ orbifold interpretation \cite{14, 15}. They constructed the twisted sector vertex operators, and analyzed the properties of the spectrum. In the light of current understanding \cite{4}, ref. \cite{13} focused on the properties of short strings. At that time the understanding of the string spectrum was incomplete. There appeared to be ghosts in the spectrum, unless the spectrum was truncated by hand. But such a truncation implied an upper bound on the mass of the string states. These puzzles were clarified in \cite{4}, where it was found that the spectrum must also include long strings, generated by spectral flow from tachyonic untwisted strings.

In this paper, we reinterpret the string spectrum on rotating BTZ black holes, in particular the twisted sector, as being generated by spectral flow. We show explicitly how the spectral flow is realized, and discuss the projection needed for the level matching condition. Following \cite{3, 13}, we interpret the problem of finding the projection generator as a Noether current ambiguity, and discuss that in some more detail than in \cite{13}. We do not address the problem of ghosts, or modular invariance. We believe it will be straightforward to generalize the corresponding results from the work in pure AdS$_3$ to the
present set-up. Here the focus has been in setting up the dictionary.

2 BTZ Black Hole and the WZW model

We begin from the WZW action:

$$S = \frac{k}{8\pi} \int d^2\sigma \text{Tr} \left( g^{-1} \partial_a g g^{-1} \partial^a g \right) + \frac{ik}{12\pi} \int \text{Tr} \left( g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right).$$  

(1)

An element $g$ of the group $SL(2, R)$ can be parametrized as

$$g = \frac{1}{\ell^2} \begin{pmatrix} x_1 + x_2 & x_3 + x_0 \\ x_3 - x_0 & x_1 - x_2 \end{pmatrix}$$  

(2)

$$\ell^2 = x_0^2 + x_1^2 - x_2^2 - x_3^2$$  

(3)

The latter equation shows that the manifold $SL(2, R)$ is a three-dimensional hyperboloid embedded in $R^{2,2}$. In fact, this is exactly same for $AdS_3$, so they are the same manifold. The dimensional parameter $\ell$ is then the $AdS_3$ radius.$^3$ In contrast to the $SU(2)$ WZW model, the level $k$ is not required to be an integer since the cohomology class $H^3$ vanishes for $SL(2, R)$.

We introduce a new coordinate system in $AdS_3$ in order to obtain the BTZ black hole. The $AdS_3$ manifold is divided into three regions $^7$, that describe the regions outside the outer horizon, between the outer and the inner horizon, and inside the inner horizon. In this paper we will focus on strings in the exterior region of the black hole, and leave the issue of including all the regions across the horizon(s) for further study. The exterior region, outside the outer horizon ($\hat{r}^2 > 1$), is described by the following coordinate patch:

$$x_1 = \hat{r} \cosh \hat{\phi}$$  

(4)

$$x_2 = \hat{r} \sinh \hat{\phi}$$  

(5)

$$x_0 = \sqrt{\hat{r}^2 - 1} \sinh \hat{t}$$  

(6)

$$x_3 = \sqrt{\hat{r}^2 - 1} \cosh \hat{t}$$  

(7)

The corresponding group elements can be expressed using the generators of $SL(2, R)$ in the hyperbolic basis $^{13}$:

$$g = e^{\imath \sigma_3} e^{\imath \sigma_1} e^{-\imath \sigma_3}$$  

(8)

$^3$We will henceforth work in units where $\ell = 1.$
where we have introduced new coordinates \( u, v \) and \( \rho \):

\[
\begin{align*}
\hat{r} &= \cosh \rho \\
\hat{t} &= u + v \\
\hat{\phi} &= u - v
\end{align*}
\]

With this parametrization the action looks like

\[
S = \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{h} \left( h^{\alpha\beta} G_{\mu\nu} + i e^{\alpha\beta} B_{\mu\nu} \right) \partial\alpha X^\mu \partial\beta X^\nu,
\]

which is the Polyakov action for a string propagating in a spacetime in the presence of a background field \( B_{\mu\nu} \). The spacetime metric of this action reads

\[
ds^2 = \alpha' k \left\{ - \left( \hat{r}^2 - 1 \right) d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2 - 1} + \hat{r}^2 d\hat{\phi}^2 \right\}
\]

and the antisymmetric background field is

\[
B = \alpha' k \hat{r}^2 d\hat{\phi} \wedge d\hat{t}.
\]

A further change of variables

\[
\begin{align*}
\hat{r}^2 &= \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right) \\
\begin{pmatrix} \hat{t} \\ \hat{\phi} \end{pmatrix} &= \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} t \\ \phi \end{pmatrix}
\end{align*}
\]

brings the metric into a form where the black hole is easily recognized:

\[
ds^2 = \alpha' k \left\{ - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 (d\phi - \frac{r_+ r_-}{r^2} dt)^2 \right\}.
\]

The antisymmetric tensor is

\[
B = \alpha' k r^2 d\phi \wedge dt,
\]

up to an exact two form. After the periodic identification of the coordinate

\[
\phi \sim \phi + 2\pi,
\]

one obtains the BTZ black hole. In the coordinates \((\hat{t}, \hat{\phi})\), the periodicity condition reads as

\[
(\hat{t}, \hat{\phi}) \sim (\hat{t} - 2\pi r_-, \hat{\phi} + 2\pi r_+).
\]
The allowed values for the coordinates \((17)\) are \(-\infty < t < +\infty, 0 \leq r < +\infty, \) and \(0 \leq \phi < 2\pi\). By allowing the time coordinate to have all real values, we have moved to the universal cover \(\tilde{SL}(2, R)\) of \(SL(2, R)\). The positive constants \(r_{\pm}\) denote the radii of the outer and the inner horizon. The mass \(M_{BH}\) and the angular momentum \(J_{BH}\) of the black hole can be read off from the metric \((17)\):

\[
M_{BH} = \left( r_{+}^2 + r_{-}^2 \right) \tag{21}
\]

\[
J_{BH} = 2r_{+}r_{-} \tag{22}
\]

### 3 Geodesic equations of the BTZ black hole

The geodesic equations for the BTZ metric have been derived in [16]:

\[
 r^2 \dot{r}^2 = -m^2 \left( r^4 - r^2 M_{BH} + \frac{J_{BH}^2}{4} \right) + \left( E^2 - L^2 \right) r^2 + L^2 M_{BH} - ELJ_{BH} \tag{23}
\]

\[
 \dot{\phi} = \frac{(r^2 - M_{BH}) L + \frac{1}{2} EJ_{BH}}{r^4 - r^2 M_{BH} + \frac{1}{4}J_{BH}^2} \tag{24}
\]

\[
 \dot{t} = \frac{E r^2 - \frac{1}{2} L J_{BH}}{r^4 - r^2 M_{BH} + \frac{1}{4}J_{BH}^2} \tag{25}
\]

The constants \(E\) and \(L\) are associated with the Killing vectors of the BTZ metric, \(\partial_t\) and \(\partial_\phi\).

For \(m^2 > 0\) (= 0, < 0), the solutions describe motions of timelike (lightlike, spacelike) particles. As explained in [16], spectral flow will stretch the timelike geodesics to worldsheets of short strings, and spacelike geodesics to worldsheets of long strings. Incidentally, it should be noted that the spacetime of a rotating black hole is geodesically complete.

The stress tensor remains constant on geodesics. The left- and rightmoving components of the stress tensor can be written in the form

\[
 T_{L,R} = -2\alpha' \operatorname{Tr} (J_{L,R} J_{L,R}) , \tag{26}
\]

where \(\alpha'\) is the string slope. Substituting the currents (see eqn. (8)) and the group elements (8), we obtain

\[
 T_{R,L} = \frac{\alpha' k^2}{4} \left\{ (\partial_\alpha \dot{\phi})^2 \cosh^2 \rho - (\partial_\alpha \dot{t})^2 \sinh^2 \rho + (\partial_\alpha \rho)^2 \right\} \tag{27}
\]

where \(\partial_\alpha = \partial_+ (\partial_-)\) for \(T_R (T_L)\). Then, for example, substituting a radial geodesic in a non-rotating black hole,

\[
 \dot{\hat{r}}^2 = -m^2 (\hat{r}^2 - 1) + \hat{E}^2
\]

\[
 \dot{\hat{t}} = \frac{\hat{E}}{\hat{r}^2 - 1} \tag{28}
\]
we obtain
\[ T_L = T_R = -\frac{\alpha'k^2m^2}{4} \equiv \pm \frac{1}{4}k\alpha^2 , \] (29)
which is of the same form as in [4], after identifying \( \alpha^2 = |km|^2 \). The components \( T_{R,L} \) are negative (positive) for timelike (spacelike) geodesics.

4 Strings on BTZ

The WZW model has a chiral \( \tilde{SL}(2, R)_R \times \tilde{SL}(2, R)_L \) symmetry, which implies the existence of conserved currents:
\[ J_R(x^+) = \frac{ik}{2}\partial_+ g g^{-1}, \quad J_L(x^-) = \frac{ik}{2}g^{-1}\partial_- g \] (30)
The generators of \( SL(2, R) \) are \( \tau^0 = -\frac{i}{2}\sigma_2, \quad \tau^1 = \frac{i}{2}\sigma_1 \) and \( \tau^2 = \frac{i}{2}\sigma_3 \). With the hyperbolic parametrization (8), the components
\[ J^a_{L,R} = -2 \text{ Tr } (\tau^a J_{L,R}) \] (31)
read as
\[
\begin{align*}
J^2_R &= k(\partial_+ u - \cosh 2\rho \partial_+ v) \\
J^\pm_R &= k(\pm \partial_\rho - \sinh 2\rho \partial_\pm v) e^{\mp 2u} \\
J^2_L &= k(-\partial_- v + \cosh 2\rho \partial_- u) \\
J^\pm_L &= k(\pm \partial_\rho - \sinh 2\rho \partial_\mp u) e^{\mp 2v}
\end{align*}
\] (32-35)
where \( J^\pm \) are defined as \( J^\pm = J^0 \pm J^1 \). This is contrast to the elliptic parametrization, where the \( J^\pm \) are defined as linear combinations of the non-compact generators \( J^1, J^2 \).

The Hilbert space of the WZW model decomposes into products of irreducible unitary representations of the \( \tilde{SL}_k(2, R) \) current algebras. Using the mode expansions
\[ J^a_R = \sum_{n=-\infty}^{\infty} J^a_n e^{-inx^+}; \quad J^a_L = \sum_{n=-\infty}^{\infty} \bar{J}^a_n e^{-inx^-}, \] (36)
the non-trivial current commutation relations become
\[
\begin{align*}
[J^2_n, J^\pm_m] &= \pm iJ^\pm_{n+m} \\
[J^+, J^-] &= -2iJ^2_{n+m} - kn\delta_{n+m,0} \\
[J^2_n, J^2_m] &= \frac{k}{2}n\delta_{n+m,0}
\end{align*}
\] (37-39)
and similarly for the antiholomorphic sector. The currents and the Virasoro generators are found to satisfy the commutation relations

\[
\begin{align*}
[L_n, J^a_m] &= -mJ^a_{n+m} \quad (40) \\
[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} . \quad (41)
\end{align*}
\]

The currents and the group elements satisfy the OPE’s

\[
J^a(z)g(w, \bar{w}) \sim -\tau^a g \frac{z - w}{z - \bar{w}} , \quad \bar{J}^a(\bar{z})g(w, \bar{w}) \sim -g\tau^a \frac{\bar{z} - \bar{w}}{\bar{z} - w} \quad (42)
\]

which yields the commutation relations

\[
\begin{align*}
[J^a_n, g] &= -\tau^a g w^n ; \quad \left[\bar{J}^a_n, g\right] = -g\tau^a \bar{w}^n . \quad (43)
\end{align*}
\]

Using the parametrization (8) and the relations (43), one can derive the time translation and space rotation generators, satisfying

\[
\begin{align*}
\delta_t g &= i\delta t [Q_t, g] \quad (44) \\
\delta_\phi g &= i\delta \phi [Q_\phi, g] , \quad (45)
\end{align*}
\]

to be the following combinations of zero modes\(^4\) of \(J^2, \bar{J}^2\):

\[
\begin{align*}
Q_t &= \Delta_- J^2_0 - \Delta_+ \bar{J}^2_0 \quad (46) \\
Q_\phi &= \Delta_- J^2_0 + \Delta_+ \bar{J}^2_0 . \quad (47)
\end{align*}
\]

In [13], Natsuume and Satoh discussed the spectrum of strings on BTZ black holes. They first constructed the Kac-Moody primaries transforming under the unitary irreducible representations of the global \(SL(2, R)_R \times \bar{SL}(2, R)_L\) symmetries, using the hyperbolic basis. The vertex operators for the primaries are of the form

\[
V_{j_R, j_L}^{j_0, 0} = D_{j_R, j_L}^j(g)e^{-iJ^{0}_{R}u + iJ^{0}_{L}v} \quad (48)
\]

where \(D_{j_R, j_L}^j(g)\) are matrix elements of the unitary irreps of the \(\bar{SL}(2, R)\) group. The representations that appear in the Hilbert space are the principal discrete representations (short strings) and the principal continuous representations (long strings). A subtlety, that we will discuss at more length below, is that the representations are now expressed in the hyperbolic basis which diagonalizes the non-compact generators \(J^2_0, \bar{J}^2_0\).

To describe the black hole, one needs to incorporate the periodicity of the angular coordinate \(\phi\). So, one needs to twist the WZW model with respect to the discrete action

\(^4\)Note however that the definition of the generators has a Noether ambiguity, to be discussed in section 5.
of $Q_\phi$ which generates the periodic identifications. After constructing the twist fields $W_n(z, \bar{z})$ [17] which create twisting with winding number $n$, one obtains the generic twisted sector primary fields

$$V_{j_R, J_L}^{j,n}(z, \bar{z}) = V_{j_R, J_L}^{j,0}(z, \bar{z}) W_n(z, \bar{z}).$$

(49)

The Kac-Moody primaries are then the states

$$|j, J, n\rangle = V_{j,n}^{j,0} J_R, J_L |0\rangle |0\rangle.$$  (50)

### 4.1 Representations in the hyperbolic basis

The unitary irreducible representations of $SL(2, R)$ are typically discussed in the elliptic basis which diagonalizes the non-compact direction. When the commutation relations are written in the hyperbolic basis,

$$\left[ J_0^+, J_0^- \right] = -2i J_0^0; \quad \left[ J_0^0, J_0^\pm \right] = \pm i J_0^\pm$$

(51)

they appear first slightly puzzling, because the latter equation seems to contradict the hermiticity of $J_0^0$. Let us introduce [18] a basis of eigenvectors of $J_0^2$:

$$J_0^2 |\lambda\rangle = \lambda |\lambda\rangle; \quad \langle \lambda |\lambda'\rangle = \delta(\lambda - \lambda') ,$$

(52)

where $\lambda \in R$ since $J_0^2$ is Hermitean. Then, it appears that the state $J_0^+ |\lambda\rangle$ is an eigenstate of $J_0^2$ with eigenvalue $\lambda + i$. The solution to the puzzle is the same as for the Heisenberg algebra $[p, q] = -i$ in quantum mechanics: the basis vectors $|\lambda\rangle$ do not represent normalizable vectors in the Hilbert space. A normalizable vector is a linear combination

$$|\phi\rangle = \int_{-\infty}^{\infty} d\lambda \ \phi(\lambda)|\lambda\rangle$$

(53)

where the wavefunction $\phi(\lambda)$ must satisfy

$$||\phi|| = \int_{-\infty}^{\infty} d\lambda |\phi(\lambda)|^2 < \infty.$$  (54)

Each of the generators $J_0^2, J_0^\pm$ has its corresponding domain of vectors $|\phi\rangle$ on which it is defined. For example, for $J_0^+$ the wavefunction $\phi(\lambda)$ determines via analytic continuation a unique new wavefunction

$$J_0^+ |\phi\rangle = J_0^+ \int_{-\infty}^{\infty} d\lambda \ \phi(\lambda)|\lambda\rangle = \int_{-\infty}^{\infty} d\lambda \ f(\lambda)\phi(\lambda - i)|\lambda\rangle$$

(55)

such that one obtains a normalizable state:

$$\int_{-\infty}^{\infty} d\lambda \ |f(\lambda)\phi(\lambda - i)|^2 < \infty.$$  (56)
One could formally introduce states labeled as $|\lambda+i\rangle$ to rewrite the relation (55). However, these states should be understood to be expanded in the original basis $\{|\lambda\rangle\}$ where the $J_0^2$ eigenvalues are real. Similarly, the operator $J_0^-$ acts as

$$J_0^-|\phi\rangle = J_0^- \int_{-\infty}^{\infty} d\lambda \, \phi(\lambda)|\lambda\rangle = \int_{-\infty}^{\infty} d\lambda \, g(\lambda + i)\phi(\lambda + i)|\lambda\rangle .$$

(57)

The functions $f(\lambda), g(\lambda)$ play the role of matrix elements of $J_0^+, J_0^-$. The commutation relation (51) requires that

$$f(\lambda)g(\lambda) - g(\lambda + i)f(\lambda - i) = -2i\lambda .$$

(58)

Now, to construct the irreducible representations of $SL(2, R)$ using the hyperbolic basis, one views them as reducible representations with respect to the subalgebra generated by $J_0^2, J_0^+$. Irreducible representations of the latter algebra are given in the basis $|\lambda\rangle$. So one needs to find out how these representations are added together to form the irreducible representations of $SL(2, R)$. In particular, one needs to identify which one of the known irreducible representations is being constructed. This is done by checking that the spectrum of the eigenvalues of the compact generator $J_0^0 = (J_0^+ + J_0^-)/2$ corresponds to that of the continuous or discrete series representations. Note that both of $J_0^\pm$ are hermitian operators, instead of conjugates of one another, so it turns out that one of the matrix element functions, e.g. $f(\lambda)$, can be chosen freely, with the other one $g(\lambda)$ being determined by the commutation relation (58). Particular choices of $f(\lambda)$ then turn out to correspond to a discrete or continuous spectrum of $J_0^0$, and yield the discrete or continuous representations of $SL(2, R)$ [18]. This will also determine the multiplicity of the representations of the subalgebra.

So the upshot is that in the hyperbolic basis, the states of the irreducible representations of $SL(2, R)$ are of the form

$$|\lambda, r\rangle ; \quad J_0^2|\lambda, r\rangle = \lambda|\lambda, r\rangle$$

(59)

where $r$ is an index that enumerates the multiplicities of the representations of the $J_0^2, J_0^+$ subalgebra.

### 4.2 Representations of the current algebra

After having obtained the representations of $\widehat{SL}(2, R)$, or the Kac-Moody primaries, the rest of the states in the current algebra (the representation of $\widehat{SL}_k(2, R)$) are obtained by acting on the states with the generators $J_{n}^{2+}, \ n \geq 1$:

$$J_N|J_R, r\rangle \bar{J}_N|J_L, r\rangle$$

(60)
where $J_N, \bar{J}_N$ denote generic products of $J^a_n$'s and $\bar{J}^a_n$'s and we are back to using the labels $J_R, J_L$ for the continuous real eigenvalues of $J^2_0, \bar{J}^2_0$. However, to keep the $J^2_0$ eigenvalues real, one must restrict the number of $J^+_{-n}$'s in $J_N$ to be equal to the number of $J^-_{-n}$'s. One then considers only states of the form

$$K_N|J_R, r\rangle \bar{K}_N|J_L, r\rangle$$

(61)

where $K_N$ is a generic product of operators $K^a_{-n}$ defined by

$$K^2_{-n} \equiv J^2_{-n} ; \quad K^+_n \equiv J^+_n J^-_0 ; \quad K^-_n \equiv J^-_n J^+_0.$$ 

(62)

They satisfy the commutation rules

$$[J^2_0, K^a_{-n}] = 0 ; \quad [L_0, K^\pm_{-n}] = nK^\pm_{-n}. \quad (63)$$

### 4.3 Spectral flow

Next we consider the spectral flow. We take a solution of the equations of motion for the WZW model and generate a new one using a specific coordinate transformation. In the hyperbolic basis, the spectral flow for $SL(2, R)$ model reads:

$$g \rightarrow e^{-iw_+x^+\tau^2} g e^{iw_-x^-\tau^2}$$ 

(64)

so the coordinates $u, v$ transform as $u \rightarrow u + \frac{1}{2}w_+x^+, \quad v \rightarrow v + \frac{1}{2}w_-x^-$. In particular, the time and angular coordinates transform as

$$\hat{t} \rightarrow \hat{t} + \frac{1}{2}[(w_+ + w_-)\tau + (w_+ - w_-)\sigma] \quad (65)$$

$$\hat{\phi} \rightarrow \hat{\phi} + \frac{1}{2}[(w_+ + w_-)\sigma + (w_+ - w_-)\tau]. \quad (66)$$

So, after the periodic identifications which make a BTZ black hole, the spectral flow parameters $w_\pm$ are allowed to have the discrete values

$$w_\pm = (r_+ \mp r_-)n \equiv \Delta_\pm n. \quad (67)$$

Under spectral flow, the components of the currents transform:

$$J^2_R \rightarrow \tilde{J}^2_R \equiv J^2_R + \frac{k}{2}w_+ ; \quad J^\pm_R \rightarrow \tilde{J}^\pm_R \equiv J^\pm_R e^{\mp w_+x^+} \quad (68)$$

$$J^2_L \rightarrow \tilde{J}^2_L \equiv J^2_L - \frac{k}{2}w_- ; \quad J^\pm_L \rightarrow \tilde{J}^\pm_L \equiv J^\pm_L e^{\mp w_-x^-} \quad (69)$$
and the stress tensor transforms as

\[
T_R \rightarrow \tilde{T}_R = T_R + w_+ J^2_R + \frac{k}{4}(w_+)^2, \quad (70)
\]

\[
T_L \rightarrow \tilde{T}_L = T_L - w_- J^2_L + \frac{k}{4}(w_-)^2. \quad (71)
\]

After spectral flow in the hyperbolic basis, the \(J^2\) component of the current is still periodic:

\[
J^2(x^+2\pi) = J^2(x^+) \quad (72)
\]

leading to the expansion

\[
J^2(z) = \sum_{s \in \mathbb{Z}} z^{-s-1} J^2_s. \quad (73)
\]

The \(J^\pm\) components become quasiperiodic:

\[
J^\pm(x^+ + 2\pi) = e^{\pm 2\pi w_+} J^\pm(x^+) \quad (74)
\]

giving rise to the expansions

\[
J^\pm(z) = \sum_{s \in \mathbb{Z} \pm iw_+} z^{-s-1} J^\pm_s. \quad (75)
\]

The antiholomorphic current satisfies the same properties and equations, with \(w_+\) replaced by \(w_-\). The nonvanishing commutator relations of the flowed current generators are the same as in (37)-(39), with the mode labels \(n, m\) replaced by complex labels \(r, s \in \mathbb{Z} \pm iw_+\).

From above, we see that under spectral flow the components of \(J^2, \bar{J}^2\) transform as:

\[
J^2_n \rightarrow \tilde{J}^2_n \equiv J^2_n + \frac{k}{2} w_+ \delta_{n,0} ; \quad J^\pm_n \rightarrow \tilde{J}^\pm_n \equiv J^\pm_{n \pm iw_+} \quad (76)
\]

\[
\bar{J}^2_n \rightarrow \bar{\tilde{J}}^2_n \equiv \bar{J}^2_n - \frac{k}{2} w_- \delta_{n,0} ; \quad \bar{J}^\pm_n \rightarrow \bar{\tilde{J}}^\pm_n \equiv \bar{J}^\pm_{n \pm iw_-}. \quad (77)
\]

We also find the following transformation rules for the Virasoro generators:

\[
L_n \rightarrow L_n + w_+ J^2_n + \frac{k}{4} w_+^2 \delta_{n,0} \quad (78)
\]

\[
\bar{L}_n \rightarrow \bar{L}_n - w_- \bar{J}^2_n + \frac{k}{4} w_-^2 \delta_{n,0} \quad (79)
\]

The commutation relations of the current algebra and the Virasoro algebra remain invariant under spectral flow. Further, the commutation relations (33) also remain invariant under spectral flow. The generators \(K^a_n\) transform under spectral flow as

\[
K^2_n \rightarrow \tilde{K}^2_n = \bar{J}^2_{-n} ; \quad K^\pm_n \rightarrow \tilde{K}^\pm_n = \bar{J}^\pm_{-n} + \bar{J}^\mp_{0} = J^\pm_{n \pm iw_+}, J^\mp_{n \pm iw_+}. \quad (80)
\]
and it is easy to check that
\[ [\tilde{J}_0^2, \tilde{K}_n^a] = 0 ; [\tilde{L}_0, \tilde{K}_n^\pm] = n\tilde{K}_n^\pm . \quad (81) \]

The Virasoro constraints and the spectrum are similar to those in [11], except for the allowed values of \( w_\pm \). The \( L_0 \) constraint for the holomorphic part is
\[ (L_0 - 1) |\tilde{j}, \tilde{J}_R\rangle = \left( -\frac{j(j+1)}{k-2} + h + \tilde{N} + w_+\tilde{J}_R - \frac{k}{4}w_+^2 - 1 \right) |\tilde{j}, \tilde{J}_R\rangle = 0 \quad (82) \]
and for the antiholomorphic part:
\[ (\bar{L}_0 - 1) |\tilde{j}, \tilde{J}_L\rangle = \left( -\frac{j(j+1)}{k-2} + \bar{h} + \tilde{N} - w_-\tilde{J}_L - \frac{k}{4}w_-^2 - 1 \right) |\tilde{j}, \tilde{J}_L\rangle = 0 . \quad (83) \]

These lead to the following allowed values for \( \tilde{J}_R, \tilde{J}_L \):
\[ \tilde{J}_R = \frac{k}{4}w_+ - \frac{1}{w_+} \left( -\frac{j(j-1)}{k-2} + h + \tilde{N} - 1 \right) \quad (84) \]
\[ \tilde{J}_L = -\frac{k}{4}w_- + \frac{1}{w_-} \left( -\frac{j(j-1)}{k-2} + \bar{h} + \tilde{N} - 1 \right) \quad (85) \]

The result agrees (up to conventions) with [13]. The level matching condition requires that
\[ \tilde{N}_{tot} - \tilde{N}_{tot} = w_+\tilde{J}_R + w_-\tilde{J}_L - \frac{k}{4}(w_+^2 - w_-^2) = \text{integer} , \quad (86) \]
where \( \tilde{N}_{tot}, \tilde{N}_{tot} \) are the total level numbers. Substituting the allowed discrete values (87) for the spectral flow parameters, leads to the conditions
\[ \tilde{N}_{tot} = \tilde{N}_{tot} + nl \quad (87) \]
\[ l = \Delta_-\tilde{J}_R + \Delta_+\tilde{J}_L + \frac{k}{2}nJ_{BH} \quad (88) \]
where \( n \) is the winding number and \( l \in \mathbb{Z} \) is the angular momentum. With the latter condition, the vertex operators can be shown to be periodic [13].

Let us finally comment on the space-time energy spectrum of the string. The short strings correspond to the principal discrete series representations. They are parameterized by the spin \( j \) which can take any real value \( j > 0 \). The eigenvalue of the compact generator \( J_0^0 \) is of the form \( j + q \), where \( q \) is an integer. For strings in AdS\(_3\) vacuum, one uses the basis where \( J_0^0 \) is diagonalized, and it is possible to solve for \( j \) as a function of the level of the current algebra, the level number of string excitations, and the spectral flow parameter \( w \) which takes discrete values. So one finds that only a discrete set of spin
values $j$ are realized in the string spectrum. In other words, a discrete set of principal discrete representations are realized in the spectrum of short string excitations. Thus, the spacetime energy spectrum of short strings is discrete.

However, in the BTZ black hole background, the representations are expressed in the hyperbolic basis. The short strings again correspond to the principal discrete series representation, with discrete $J_0^0$ eigenvalues $j + q$. However, now it is the non-compact operator $J_0^2$ which has been diagonalized. The equations (84) and (85) relate the eigenvalues of the non-compact operators to the level $k$, the level numbers $N, \bar{N}$ and the spin $j$. But these are not related to the $J_0^0$ eigenvalues. So there is no equation to solve for the spin $j$ as a function of just the discrete parameters $k, N, w$. Hence we conclude that all principal discrete representations $j > 0$ are realized in the string spectrum. Then, the $J_0^0, \bar{J}_0^0$ eigenvalues can take any real value $J_L, J_R$. The two continuous parameters are related by one constraint equation (88), leaving one free continuous parameter. The spacetime energies of short string excitations are related to the eigenvalues of the time translation generator $Q_t$ given by (46). So the energy spectrum

$$E = \Delta - J_R - \Delta + J_L$$

is continuous, as opposed to the discrete spectrum in the AdS$_3$ case. But a continuous spectrum is just what we expected to find in the black hole case.

5 Invariant subspace

The physical spectrum consists of states that are invariant under the periodic identification $\phi \sim \phi + 2\pi$. This imposes a condition on the allowed quantum numbers. The projection is realized by the operator

$$P = \exp (i2\pi Q) ,$$

(89)

where $Q$ is the operator that generates translations in $\phi$. The states that are invariant under the action by $P$ are to be retained in the spectrum. This requires the eigenvalues $l$ of $Q$ to be integers on the states that are retained. The eigenvalue $l$ corresponds to angular momentum.

However, there is a subtlety in finding out the correct generator $Q$. There is a well known ambiguity in defining the Noether current in field theory, since one is always free to add to it a divergence of an antisymmetric tensor without affecting the conservation law. However, this may give an additional contribution to the conserved charge in a topologically non-trivial sector. This was discussed in the context of WZW coset models e.g. in [1, 13]. The conserved current must be such that the charge that appears in
the projection operator yields a projection condition consistent with the level matching condition.

Naively, one might consider the operator \( Q_\phi = \Delta_- J_0^2 + \Delta_+ \tilde{J}_0^2 \) to be the generator to be used in the projection operator. This would yield the quantization condition

\[
\Delta_- \tilde{J}_R + \Delta_+ \tilde{J}_L + k n J_{BH} = l \in \mathbb{Z} \tag{90}
\]
on the states which are kept under the projection. However, if the black hole is rotating, \( J_{BH} \neq 0 \), this is clearly inconsistent with the level matching condition (88) – there is a mismatch by a factor of 2 in the last term, for strings with non-zero winding number \( n \).

Another way to derive the conserved current under rotations in \( \phi \) is to use the sigma model action (12). Varying with respect to \( \phi \) gives rise to the conserved current with the time component

\[
J_\tau = \frac{k}{2\pi} \{ r^2 \partial_\tau \phi - \frac{1}{2} J_{BH} \partial_\tau t - (r^2 - c) \partial_\sigma t \} . \tag{91}
\]

Here \( c \) is an arbitrary constant, reflecting the freedom to add an exact two-form to \( B \).

Let us compare this with the current \( \Delta_- J_R^2 + \Delta_+ J_L^2 \) using the expressions (32), (34):

\[
J_\phi \equiv \Delta_- J_R^2 + \Delta_+ J_L^2 = k \{ r^2 \partial_\tau \phi - \frac{1}{2} J_{BH} \partial_\tau t - (r^2 - M_{BH}) \partial_\sigma t - \frac{1}{2} J_{BH} \partial_\sigma \phi \} .
\]

This agrees with (91), up to the last total derivative term. The last term plays a role in the winding sector, where \( \partial_\sigma \phi = n \neq 0 \). In the non-winding sector the currents agree, and give rise to the same symmetry generator

\[
Q_{action} = \int_0^{2\pi} d\sigma J_\tau = Q_\phi = \Delta_- J_0^2 + \Delta_+ \tilde{J}_0^2 = \int_0^{2\pi} \frac{d\sigma}{2\pi} J_\phi . \tag{92}
\]

However, in the topologically non-trivial winding sector, the current ambiguity plays a role, and contributes to the current generator. The correct generator which agrees with the level matching condition turns out to be \( Q_{action} \). From above, we see that generator \( Q \) to use in \( P \) is

\[
Q = Q_{action} = \int_0^{2\pi} \frac{d\sigma}{2\pi} (J_\phi + \frac{k}{2} J_{BH} \partial_\sigma \phi)
\]

\[
= \Delta_- J_0^2 + \Delta_+ \tilde{J}_0^2 + \frac{k}{2} n J_{BH} , \tag{93}
\]

leading to the quantization condition

\[
\Delta_- \tilde{J}_R + \Delta_+ \tilde{J}_L + \frac{k}{2} n J_{BH} = l \in \mathbb{Z} , \tag{94}
\]
in agreement with (88).

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