Hydrodynamics of massive particles with spin $1/2$

Wojciech Florkowski

Institute of Theoretical Physics, Jagiellonian University
ul. prof. St. Łojasiewicza 11, 30-348 Kraków, Poland

Abstract

The formulation of relativistic hydrodynamics for massive particles with spin $1/2$ is shortly reviewed. The proposed framework is based on the Wigner function treated in a semi-classical approximation or, alternatively, on a classical treatment of spin $1/2$. Several theoretical issues regarding the choice of the energy-momentum and spin tensors used to construct the hydrodynamic framework with spin are discussed in parallel.

Keywords:

1. Introduction

Non-central heavy-ion collisions make it possible that large amount of the initial orbital angular momentum is transferred to produced systems. Some part of such an angular momentum can be subsequently shifted from the orbital part to the spin part. The latter can be reflected in the spin polarization of produced hadrons such as $\Lambda$ and $\bar{\Lambda}$ hyperons [1, 2, 3, 4, 5]. The spin polarization of $\Lambda$’s and $\bar{\Lambda}$’s has been indeed measured by the STAR Collaboration at BNL [6, 7] and the data shows global, out-of-plane polarization, which reminds us of the Einstein – de Haas and Barnett effects [8, 9].

The phenomenon of global polarization has been successfully explained by the hydrodynamic models, which directly identify spin polarization effects with the so-called thermal vorticity [10]. The latter is defined by the rank-two antisymmetric tensor $\omega_{\mu\nu} = \frac{1}{2}(\partial_{\mu} \beta_{\nu} - \partial_{\nu} \beta_{\mu})$, where $\beta_{\mu}$ is the ratio of the hydrodynamic flow $u_{\mu}$ and local temperature $T$, $\beta_{\mu} = u_{\mu}/T$ [11, 12, 13]. There exist, however, problems regarding description of the longitudinal polarization, since a theoretically predicted longitudinal polarization of $\Lambda$’s [14] has an opposite sign of the dependence on the azimuthal angle of emitted particles, as compared to the data [15].

Using general thermodynamic arguments one can argue that there are situations where the spin polarization effects (quantified by the tensor $\omega_{\mu\nu}$, dubbed below the spin polarization tensor) are independent of the thermal vorticity $\omega_{\mu\nu}$ [16]. Associated with this generalization is an extension of the concept of local thermodynamic equilibrium (where $\omega_{\mu\nu} = \omega_{\mu\nu}$) to a local spin-thermodynamic equilibrium (where we allow for $\omega_{\mu\nu} \neq \omega_{\mu\nu}$). This idea was originally proposed in Ref. [17] and developed in Refs. [18, 19, 20] (see the recent review [21] and related works [22, 23, 24]). Within this approach, the space-time evolution of...
polarization is determined by the conservation law for total angular momentum — for particles with spin, the latter turns out to have a non-trivial form.

Incorporation of spin polarization into the hydrodynamic framework is an interesting challenge, as the present works say little about the changes of the spin polarization during the heavy-ion collision process. As the fluid dynamics represents now the basic element of heavy-ion models (for recent developments see \[25, 26\]), it is even more demanding to have spin effects included into the hydrodynamic picture of heavy-ion collisions. So far, relatively little work has been done in this direction, although the studies of fluids with spin have a rather long history that started in 1940s \[27, 28, 29\].

2. Local spin-thermodynamic equilibrium

The local spin-thermodynamic equilibrium is described, besides the standard hydrodynamic quantities such as temperature \( T(x) \), flow four-vector \( \mu^\mu(x) \), and chemical potential \( \mu(x) = \xi(x)T(x) \), by the spin chemical potential \( \Omega_{\mu \nu}(x) = \omega_{\mu \nu}(x)T(x) \). One uses these quantities to construct the density operator \( \rho_{\text{LEQ}} \) and to obtain the expectation values of the energy-momentum tensor \( T^{\mu \nu} \), the spin tensor \( S^{\mu \nu} \), and the baryon current \( j^\mu \):

\[
T^{\mu \nu} = \text{tr} \left( \rho_{\text{LEQ}} \, \bar{T}^{\mu \nu} \right), \quad S^{\mu \nu} = \text{tr} \left( \rho_{\text{LEQ}} \, \bar{S}^{\mu \nu} \right), \quad j^\mu = \text{tr} \left( \rho_{\text{LEQ}} \, \bar{J} \right).
\]

In this way we obtain constitutive equations:

\[
T^{\mu \nu} = T^{\mu \nu}[\beta, \omega, \xi], \quad S^{\mu \nu} = S^{\mu \nu}[\beta, \omega, \xi], \quad j^\mu = j^\mu[\beta, \omega, \xi].
\]

With dissipation effects neglected, one can assume that the density operator \( \rho_{\text{LEQ}} \) is constant, which leads to the following equations:

\[
\partial_\mu T^{\mu \nu} = 0, \quad \partial_\mu S^{\lambda \mu \nu} = T^{\mu \nu} - T^{\nu \mu}, \quad \partial_\mu j^\mu = 0.
\]

These are eleven equations for eleven unknown functions (temperature, three independent components of the fluid four-velocity, chemical potential, and six independent components of the tensor \( \omega_{\mu \nu} \), which becomes now a new hydrodynamic quantity). We note that in general only the total angular momentum is conserved, which leads to the middle equation in (3).

3. Semiclassical kinetic equation

Combining the concepts presented in \[30\] with the idea of spin-thermodynamic equilibrium, one can introduce equilibrium Wigner functions. Using a semi-classical formalism worked out in \[31\], we write

\[
\mathcal{W}_{\text{eq}}(x, k) = \frac{1}{2} \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \delta^{(3)}(k - p) \mu'(p) \bar{u}'(p) f^s_+(x, p),
\]

where \( k \) is the four-momentum and \( f^s_+(x, p) \) is the spin-dependent phase-space density matrix of particles [32], which in equilibrium depends on \( \beta^\mu, \omega_{\mu \nu}, \text{and} \xi \). A similar expression can be introduced for antiparticles. Any Wigner function can be furthermore expressed as a linear combination of the generators of the Clifford algebra \[33, 34, 35, 36, 57, 38\],

\[
\mathcal{W}(x, k) = \frac{1}{4} \left[ \mathcal{F}^+(x, k) + i\gamma_5 \mathcal{P}^+(x, k) + \gamma^\mu \mathcal{V}^\mu_+(x, k) + \gamma_5 \gamma^\mu \mathcal{A}^\mu_+(x, k) + \Sigma^{\mu \nu} S^{\mu \nu}_+(x, k) \right],
\]

which is a very useful expression for studying a semiclassical limit of the quantum kinetic equation,

\[
\left( \gamma^\mu K^\mu - m \right) \mathcal{W}(x, k) = C[\mathcal{W}(x, k)].
\]
Here $K^\mu$ is the operator defined by the expression
\[ K^\mu = k^\mu + \frac{i\hbar}{2} \partial^\mu \]  
(6)
and $C[W(x,k)]$ is the collision term. In global or local equilibrium, the collision term vanishes and one can study only the left-hand side of (5) that should be equal to zero in this case.

At this point it is important to distinguish between the global and local equilibrium. In the case of global equilibrium the Wigner function $W(x,k)$ exactly satisfies the equation
\[ (\gamma^\mu K^\mu - m)W(x,k) = 0. \]  
(7)
From the leading and next-to-leading terms of the expansion of $W$ in powers of $\hbar$, one finds that $\xi = \text{const.}$, $\omega_{\mu\nu} = \text{const.}$, and $\partial_\mu \beta_\nu - \partial_\nu \beta_\mu = 0$. The last equation is known as the Killing equation that (in the flat space-time) has a solution $\beta^\mu = b^\mu_0 + \omega^{\mu}_{\nu} x^\nu$ with $b^\mu_0 = \text{const.}$ and $\omega_{\mu\nu} = -\omega_{\nu\mu} = \text{const.}$ Interestingly, the tensors $\omega_{\mu\nu}$ and $\omega^{\mu}_{\nu} = \omega_{\nu\mu}$ might be different.

In the case of local equilibrium, one assumes that only specific moments of Eq. (7) vanish. This point has been discussed in more detail in Ref. [20], where it is shown that this procedure leads to the following equations:
\[ \partial_\mu S^\lambda_{\mu\nu}^{\text{GLW}}(x) = 0, \quad \partial_\alpha T^\alpha_{\mu\nu}^{\text{GLW}}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\text{GLW}}(x) = 0. \]  
(8)
This form is consistent with the general scheme of the hydrodynamics with spin discussed above, however, the forms of the tensors appearing in Eqs. (8) are different from those used in a phenomenological approach of Ref. [17]. As a matter of fact, these forms agree with the expressions used by de Groot, van Leeuwen, and van Weert in Ref. [31]. This fact is stressed by the use of the GLW labels in Eqs. (8). We note that the first numerical solutions of Eqs. (8) have been obtained recently in Ref. [39]. We also note that Eqs. (8) can be derived from an approach where the spin is treated in a classical way [21].

4. Pseudo-gauge symmetry

It is important to emphasize that one can also use the canonical versions of the energy-momentum and spin tensors to construct the hydrodynamic equations with spin. It turns out, that the canonical and GLW forms are connected by a pseudo-gauge transformation. Indeed, if we introduce a tensor $\Phi^\lambda_{\text{can}}$ defined by the relation
\[ \Phi^\lambda_{\text{can}} \equiv S^{\mu,\nu}_{\text{GLW}} - S^{\nu,\mu}_{\text{GLW}}, \]  
(9)
one can check that
\[ S^{\lambda,\mu\nu}_{\text{can}} = S^{\lambda,\mu\nu}_{\text{GLW}} - \Phi^\lambda_{\text{can}}, \quad T^\mu_{\text{can}} = T^\mu_{\text{GLW}} + \frac{1}{2} \partial_\lambda \left( \Phi^\lambda_{\text{can}} + \Phi^\mu_{\text{can}} + \Phi^\nu_{\text{can}} \right). \]  
(10)
The pseudo-gauge transformation given above is similar to the Belinfante construction but it does not eliminate the spin tensor which can be used to describe spin degrees of freedom.

One can check that the canonical and GLW hydrodynamic equations are the same, however, the forms of the energy-momentum and spin tensors are different. In particular, in the canonical case the energy-momentum tensor is not symmetric and, consequently, the divergence of the canonical spin tensor does not vanish. The GLW version seems to be a convenient rearrangements of the terms used to define $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}_{\text{can}}$, which leads to a symmetric $T^{\mu\nu}$ and a conserved $S^{\lambda,\mu\nu}_{\text{can}}$. We must admit, however, that it is not clear if such a rearrangement is possible if one goes beyond the semi-classical description discussed in this contribution.
5. Conclusions

The results presented in this contribution describe dynamics of a perfect fluid consisting of massive particles with spin $1/2$. The main challenge for next developments is the proper inclusion of dissipation (for example, a calculation of kinetic coefficients related to spin observables). First steps in this direction have been made, for example, in Ref. [40, 41]. In the closest future, it would be interesting to examine more closely the relation of the results presented in Ref. [40] to the formalism discussed herein. It is also mandatory to study in more detail the relation between spin polarization and thermal vorticity. An effect describing convergence of the spin polarization tensor to the thermal vorticity should be included in the complete formalism of viscous hydrodynamics with spin.

I thank F. Becattini, S. Bhadury, B. Friman, A. Jaiswal, A. Kumar, E. Speranza, R. Ryblewski for very fruitful collaboration and numerous illuminating discussions. This work was supported in part by the Polish National Science Center Grant No. 2016/23/B/ST2/00717.

References

[1] S. A. Voloshin, nucl-th/0410089
[2] Z. T. Liang and X. N. Wang, Phys. Rev. Lett. 94 (2005) 102301 Erratum: [Phys. Rev. Lett. 96 (2006) 039901]
[3] B. Betz, M. Gyulassy and G. Torrieri, Phys. Rev. C 76 (2007) 044901.
[4] J. H. Gao, Z. T. Liang, Q. Wang and X. N. Wang, Phys. Rev. C 77 (2008) 044902.
[5] S. A. Voloshin, EPJ Web Conf. 17 (2018) 100701.
[6] L. Adamczyz et al. [STAR Collaboration], Nature 548 (2017) 62.
[7] J. Adam et al. [STAR Collaboration], Phys. Rev. C 98 (2018) 014910.
[8] A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17 (1915) 152.
[9] S. J. Barnett, Rev. Mod. Phys. 7 (1935) 129.
[10] I. Karpenko and F. Becattini, Eur. Phys. J. C 77 (2017) 213.
[11] F. Becattini and F. Piccinini, Annals Phys. 323 (2008) 2452.
[12] F. Becattini and L. Tinti, Annals Phys. 325 (2010) 1566.
[13] F. Becattini, I. Karpenko, M. Lisa, I. Upsal and S. Voloshin, Phys. Rev. C 95 (2017) 054902.
[14] F. Becattini and I. Karpenko, Phys. Rev. Lett. 120 (2018) 012302.
[15] T. Niida [STAR Collaboration], Nucl. Phys. A 982 (2019) 511.
[16] F. Becattini, W. Florkowski and E. Speranza, Phys. Lett. B 789 (2019) 419.
[17] W. Florkowski, B. Friman, A. Jaiswal and E. Speranza, Phys. Rev. C 97 (2018) no.4, 041901.
[18] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski and E. Speranza, Phys. Rev. D 97 (2018) 116017.
[19] W. Florkowski, E. Speranza and F. Becattini, Acta Phys. Polon. B 49 (2018) 1409.
[20] W. Florkowski, A. Kumar and R. Ryblewski, Phys. Rev. C 98 (2018) 044906.
[21] W. Florkowski, R. Ryblewski and A. Kumar, Prog. Part. Nucl. Phys. 108 (2019) 103709.
[22] Y. Sun and C. M. Ko, Phys. Rev. C 99 (2019) 014903.
[23] N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, Phys. Rev. D 100 (2019) no.5, 056018.
[24] J. H. Gao and Z. T. Liang, Phys. Rev. D 100 (2019) 056021.
[25] W. Florkowski, M. P. Heller and M. Spalinski, Rept. Prog. Phys. 81 (2018) 046001.
[26] P. Romatschke, U. Romatschke, “Relativistic Fluid Dynamics In and Out of Equilibrium” (Cambridge University Press, 2019).
[27] J. Weyssenhoff and A. Raabe, Acta Phys. Polon. 9 (1947) 7.
[28] D. Bohm, J. P. Vigier, Phys. Rev. 109 (1958) 1882.
[29] F. Halbwachs, “The Relativistic Theories of Spinning Fluids” (Elsevier, New York, 1960).
[30] F. Becattini, V. Chandra, L. Del Zanna and E. Grossi, Annals Phys. 338 (2013) 32.
[31] S. R. De Groot, W. A. Van Leeuwen and C. G. Van Weert, (Amsterdam, North-Holland, 1980).
[32] E. Leader, “Spin in Particle Physics” (Cambridge University Press, 2001).
[33] H. T. Elze, M. Gyulassy and D. Vasko, Phys. Lett. B 177 (1986) 402.
[34] H. T. Elze, M. Gyulassy and D. Vasko, Nucl. Phys. B 276 (1986) 706.
[35] D. Vasak, M. Gyulassy and H. T. Elze, Annals Phys. 173 (1987) 462.
[36] P. Zhuang and U. W. Heinz, Annals Phys. 245 (1996) 311.
[37] W. Florkowski, J. Hufner, S. P. Klevansky and L. Neise, Annals Phys. 245 (1996) 445.
[38] J. H. Gao, S. Pu and Q. Wang, Phys. Rev. D 96 (2017) 016002.
[39] W. Florkowski, A. Kumar, R. Ryblewski and R. Singh, Phys. Rev. C 99 (2019) 044910.
[40] K. Hattori, M. Hongo, X. G. Huang, M. Matsuo and H. Taya, Phys. Lett. B 795 (2019) 100.
[41] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar and R. Ryblewski, arXiv:2002.03937 [hep-ph].