Energy exchange for homogeneous and isotropic universes with a scalar field coupled to matter

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Abstract

We study the late time evolution of flat and negatively curved Friedmann–Robertson–Walker (FRW) models with a perfect fluid matter source and a scalar field arising in the conformal frame of $f(R)$ theories nonminimally coupled to matter. Under mild assumptions on the potential $V$ we prove that equilibria corresponding to the non-negative local minima for $V$ are asymptotically stable, as well as horizontal asymptotes approached from above by $V$. We classify all cases of the flat model where one of the matter components eventually dominates. In particular for a nondegenerate minimum of the potential with zero critical value we prove in detail that if $\gamma$, the parameter of the equation of state, is larger than 1, then there is a transfer of energy from the fluid to the scalar field and the latter eventually dominates in a generic way.

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1. Introduction

In cosmological models containing scalar fields the exponential potential function, $V(\phi) = V_0 e^{-\lambda \phi}$, is the most popular not only because of the variety of alternative theories of gravity which predict exponential potentials, but also due to the fact that this potential has the nice property that $V \propto V$ which allows for the introduction of normalized variables according to the formalism of Wainwright et al [49]. Among many investigations with exponential potentials, we mention [12] and references therein for FRW models; [13] for spatially homogeneous Bianchi cosmologies with an exponential potential; [16, 32] for models containing both a perfect fluid of ordinary matter and a scalar field with an exponential potential, the so-called scaling cosmologies and [2] for the nonminimally coupled scalar field; [20] and references therein for scalar–tensor theories with exponential potential; [21] for inhomogeneous cosmologies; [14] for flat and open FRW models and [1] for the Bianchi type
I models with multiple fields containing as a special case that of a multi-exponential potential studied in the context of assisted inflation; [15] for an elegant mathematical generalization of multi-exponential potentials; [48] for an investigation of a class of exponential potentials, allowing exact solutions in the context of dark energy; [17] for scalar–tensor phantom cosmologies with exponential potentials; [4] for the reconstruction of the \( f(R) \) theory starting from the exponential potential in an effort to solve the puzzle of cosmic acceleration. However, a large class of potentials used in scalar-field cosmological models have a local minimum. Examples of potentials belonging to this class are the polynomial potentials of the form \( V(\phi) = \lambda \phi^{2n} \), generalized and logarithmic potentials \( V(\phi) \propto \phi^n (\ln \phi)^m \) or \( V(\phi) \propto \phi^n \exp(-\lambda \phi^m) \) studied in [41], potentials used in the study of isotropization of Bianchi-type models in scalar–tensor theories [20, 38], potentials \( V(\phi) \propto A e^{-\lambda \phi} + B e^{-\kappa \phi} \) in an effort to avoid the eternal acceleration [47], in scalar–tensor quintessence [19] (see also [8] for potentials reconstructed from observations), in phantom cosmology [9], in double scalar–tensor cosmologies [6]. Other important examples include chameleon effective potentials [33] and potentials in conformally related theories of gravity, for example

\[
V(\phi) = \frac{1}{8\alpha} (1 - e^{-\sqrt{2/3} \phi})^2
\]

which arises in the conformal frame of the \( R + \alpha R^2 \) theory [36, 39]. Multivalued potentials arising in realistic \( f(R) \) gravity theories were studied in [7]. Unbounded from below potentials may drive a flat FRW universe to recollapse [26].

Since the nature of the scalar field supposed to cause accelerated expansion is unknown, it is important to investigate the general properties shared by all FRW models with a scalar field irrespective of the particular choice of the potential. Exact solutions for flat FRW models containing only a scalar field with an arbitrary potential were obtained in [46] (see also [10] for the Bianchi type I and V models containing ordinary matter). Nevertheless, the number of papers with mathematically rigorous results is small. Flat FRW models having an arbitrary potential with a positive lower bound were studied by Foster [23]. Generalization to models including ordinary matter and scalar fields with the potentials having a zero local minimum was presented in [40]. Rendall [42] studied the asymptotic behaviour of homogeneous models with a scalar field having an arbitrary potential with a positive lower bound and showed that the no-hair theorem holds for that case (see also [43]). The Bianchi type I-VIII models were shown to isotropize in the context of k-essence in [44]. The oscillatory behaviour of scalar fields with a general potential of the form \( V(\phi) = \phi^2/2 + O(\phi^3) \) and ordinary matter was investigated in [45] for flat FRW models. Collapsing models were built using homogeneous scalar field solutions in [27] and the case where a scalar field is coupled to a perfect fluid was studied in [29].

The motivation of this investigation comes from a number of physical theories which predict the presence of a scalar field coupled to matter. For example, in string theory the dilaton field is generally coupled to matter [25]. Nonminimally coupling occurs also in scalar–tensor theories of gravity [24], in higher order gravity (HOG) theories [5] and in the models of chameleon gravity [50]. In particular, for HOG theories derived from Lagrangians of the form

\[
L = f(R)\sqrt{-g} + 2L_m(\Psi),
\]

it is well known that under the conformal transformation, \( \tilde{g}_{\mu\nu} = f'(R)g_{\mu\nu} \), the field equations reduce to the Einstein field equations with a scalar field \( \phi \) as an additional matter source, where

\[
\phi = \sqrt{\frac{3}{2}} \ln f'(R).
\]
We do not enter into the discussion about the regularity of the conformal transformation, or the equivalence issue of the two frames (see for example [18, 22, 37] and references therein). Assuming that (3) can be solved for \( R \) to obtain a function \( R(\phi) \), the potential of the scalar field is given by

\[
V(R(\phi)) = \frac{1}{2(f')^2} (Rf' - f),
\]

and the quadratic gravity with the potential (1) is a typical example. The restrictions on the potential in the papers [42–45] we used in [35] to impose conditions on the function \( f(R) \) with corresponding potential (4). The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (2) and the resulting action becomes [3]

\[
\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \{ \tilde{R} - [(\partial \phi)^2 + 2V(\phi)] + 2e^{-2\sqrt{2/3}\phi} \mathcal{L}_m(e^{-\sqrt{2/3}\phi} \tilde{g}, \Psi) \}.
\]

The variation of \( \tilde{S} \) with respect to \( \tilde{g} \) yields the field equations

\[
\tilde{G}_{\mu\nu} = T_{\mu\nu}(\tilde{g}, \phi) + \tilde{T}_{\mu\nu}(\tilde{g}, \Psi),
\]

and variation of \( \tilde{S} \) with respect to \( \phi \) yields the equation of motion of the scalar field

\[
\Box \phi - \frac{dV}{d\phi} = \frac{1}{\sqrt{6}} e^{-2\sqrt{2/3}\phi} T^\mu_{\mu}(\tilde{g}, \Psi).
\]

Note that the Bianchi identities imply that

\[
\tilde{\nabla}^\mu \tilde{T}_{\mu\nu}(\tilde{g}, \Psi) \neq 0, \quad \tilde{\nabla}^\mu T_{\mu\nu}(\tilde{g}, \phi) \neq 0,
\]

and therefore there is an energy exchange between the scalar field and ordinary matter.

In this paper we study the late time evolution of initially expanding flat and negatively curved FRW models with a scalar field having an arbitrary bounded from below potential function \( V(\phi) \). Ordinary matter is described by a barotropic fluid with the equation of state

\[ p = (\gamma - 1) \rho, \quad 0 < \gamma \leq 2. \]

The scalar field is nonminimally coupled to matter according to (5)–(7). Under general assumptions on the potential function \( V(\phi) \) (see assumption 1) we first show in section 2.1 that stable solutions are related to (possibly degenerate) local minima of \( V \) with non-negative critical value and in cases where \( V \) approaches a horizontal asymptote from above. In section 3 we focus for the sake of simplicity on the flat case and investigate the energy exchange between the fluid and the scalar field in the case of the exponential potential and in the case of a nondegenerate local minimum of \( V \). In the former case (section 3.1) the fluid energy never dominates, nevertheless the scalar energy always contributes as a nontrivial fraction of the total energy and in some cases it totally dominates over the fluid, but stable equilibria may also appear corresponding to scaling solutions, similarly to the uncoupled case treated in [16]. Concerning the case of nondegenerate minimum of \( V \) (section 3.2) we perform a careful study of the system and prove in detail that the late time qualitative behaviour is determined by the sign of the pressure of the fluid. In particular, the energy exchange between the two matter components is such that for \( \gamma > 1 \) the scalar field eventually dominates (except possibly for a particular solution, see theorem 2). To our knowledge there is no rigorous proof in the literature of this result and even for the uncoupled case studied in [45, section 4] the same conclusion that we find here, though reasonable, was only conjectured and no proof was provided.
2. Flat and negatively curved FRW with an arbitrary potential

For homogeneous and isotropic spacetimes the field equations (6) reduce to the Friedmann equation
\[ H^2 + \frac{k}{a^2} = \frac{1}{3} \left( \rho + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \]
and the Raychaudhuri equation
\[ \dot{H} = -\frac{1}{2} \dot{\phi}^2 - \frac{\gamma}{2} \rho + \frac{k}{a^2}, \]
while the equation of motion of the scalar field (7) becomes
\[ \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{4 - 3\gamma}{\sqrt{6}} \rho. \]

The Bianchi identities yield the conservation equation
\[ \dot{\rho} + 3\gamma \rho H = -\frac{4 - 3\gamma}{\sqrt{6}} \rho \dot{\phi}, \]
(see for example [3, 37]). For simplicity we drop the tilde from all quantities. We adopt the metric and curvature conventions of [49]. \( a(t) \) is the scale factor, an overdot denotes differentiation with respect to time \( t \), \( H = \dot{a}/a \) and units have been chosen so that \( c = 1 = 8\pi G \). Here, \( V(\phi) \) is the potential energy of the scalar field and \( V' = dV/d\phi \).

In the following, we study the late time behaviour of solutions of (10)–(12), which are expanding at some initial time of observation, i.e. \( H(0) > 0 \). For flat, \( k = 0 \), models the state vector of the system (10)–(12) is \( (\phi, \dot{\phi}, \rho, H) \), i.e. we have a four-dimensional dynamical system subject to the constraint (9). Defining \( y := \dot{\phi} \) and setting \( \frac{4 - 3\gamma}{\sqrt{6}} = \alpha \), we write the autonomous system as
\[
\begin{align*}
\dot{\phi} &= y, \\
\dot{y} &= -3HY - V'(\phi) + \alpha \rho, \\
\dot{\rho} &= -3\gamma \rho H - \alpha \rho y, \\
\dot{H} &= -\frac{1}{2} y^2 - \frac{\gamma}{2} \rho, \\
\end{align*}
\]
subject to the constraint
\[ 3H^2 = \rho + \frac{1}{2} y^2 + V(\phi). \]

**Remark 1.** The function \( W \) defined by
\[ W(\phi, y, \rho, H) = H^2 - \frac{1}{3} \left( \frac{1}{2} y^2 + V(\phi) + \rho \right), \]
satisfies
\[ W = -2HW. \]
By standard arguments in ordinary differential equations theory it follows that \( \text{sgn}(W) \) is invariant under the flow of (13). We deduce that solutions with \( W \) positive, null or negative represent scalar field cosmologies with \( k = -1, 0, 1 \) respectively.

**Remark 2.** Similar arguments applied to the third part of (13) show that if \( \rho > 0 \) at some initial time \( t_0 \), then \( \rho(t) > 0 \) throughout the solution. Note that in scalar field cosmologies in the context of GR, the right-hand sides (rhs) of (11) and (12) are zero. The energy exchange
between the two matter components (cf (8)) is reflected to the time derivative of the energy density of the scalar field,

$$\epsilon = \frac{1}{2}y^2 + V(\phi),$$

which contains the extra term $\alpha \rho y$. Furthermore,

$$\dot{\epsilon} + \dot{\rho} = -3H(y^2 + \gamma \rho)$$

implies that for expanding models the total energy $\epsilon + \rho$ of the system decreases.

### 2.1. Stable equilibria for open spatial topologies

The equilibria of (13) are given by $(\phi = \phi_*, y = 0, \rho = 0, H = \pm \sqrt{V(\phi_*)}/3)$ where $V'(\phi_*) = 0$, and we discuss their stability for expanding cosmologies ($H > 0$) with open spatial topology ($k = -1, 0$). Hereafter we deal with a quite general class of potentials which includes, for instance, polynomial functions with even leading term and exponential-decay functions. Essentially all we need is that $V(\phi)$ is eventually non-negative as $\phi \to \pm \infty$ and has a finite number of critical points.

**Assumption 1.** We suppose that $V(\phi) \in C^2(\mathbb{R})$ satisfies the following conditions:

(i) the (possibly empty) set $\{\phi : V(\phi) < 0\}$ is bounded;

(ii) the (possibly empty) set of critical points of $V(\phi)$ is finite.

The critical points of $V$ with negative critical value are not equilibria and they rather allow for recollapse of the model. Moreover, nondegenerate maximum points (with the non-negative critical value) for $V$ are unstable, as can be easily seen by linearizing system (13) at the corresponding equilibria and verifying the existence of at least one eigenvalue with the positive real part. If the maximum points for $V$ are degenerate, it can be shown that they are also unstable. In fact, the function $\epsilon + \rho - V(\phi_*)$ is indefinite in the intersection of the set $\{W \geq 0\}$ with an arbitrary neighbourhood of the equilibrium point and strictly decreasing under the flow of the system for expanding cosmologies (see equation (17)). In contrast, it can be proved that equilibria corresponding to the local minima of the potential with non-negative critical value are asymptotically stable. More precisely, we have the following result.

**Proposition 1.** Let $\phi_*$ a strict local minimum for $V(\phi)$, possibly degenerate, with non-negative critical value. Then, $p_* \equiv (\phi_*, y_0 = 0, \rho_0 = 0, H_* = \sqrt{V(\phi_*)}/3)$ is an asymptotically stable equilibrium point for expanding cosmologies in the open spatial topologies $k = 0$ and $k = -1$.

The proof of this proposition is given in the appendix.

Of course, we must also consider the possible stable configurations corresponding to a diverging value for the field $\phi$. In view of the assumptions, this can happen only when $V(\phi)$ has critical points ‘at infinity’ with asymptotic value $\ell \geq 0$. For simplicity, let us restrict to the case $\phi \to +\infty$. Then it can be shown that when $V(\phi) \to \ell^-$, the configurations are unstable. The proof can be done by introducing a new scalar coordinate $\psi = V(\phi) - \ell$.

---

3 The function $V$ is said to have a *degenerate* local minimum at $\phi_*$ if both $V'$ and $V''$ vanish at $\phi_*$. Moreover, if $y, \rho \to 0$ as $t \to +\infty$ and $\phi \to +\infty$, while $\lim_{t \to +\infty} H \to \pm \lim_{\phi \to +\infty} \sqrt{V(\phi)/3}$, then we call the asymptotic state $(\phi = +\infty, y = 0, \rho = 0, H = \pm \sqrt{\lim_{t \to +\infty} V(\phi)/3})$ stable equilibrium ‘at infinity’, although a more appropriate term should be *stable configuration*. 5
and writing the system in terms of \( \psi \). In this way the critical point at infinity becomes a finite local maximum (\( \psi = 0 \)). Arguing similarly as sketched before for local maxima, it is shown that when \( V(\phi) \to \ell^- \), the configuration is unstable. On the other side, when \( V(\phi) \to \ell^+ \) (e.g. as it happens for the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \)) the critical point at infinity is asymptotically stable, i.e. \( y, \rho \to 0 \), and \( \phi \to +\infty \) as \( t \to +\infty \). Moreover, since \( \lim_{\phi \to +\infty} V(\phi) = \ell^+ \), we conclude from (14) also that \( H \to \sqrt{\ell^+ / 3} \) as \( t \to \infty \). The proof follows after suitable adaptation of the arguments used in proposition 4 in [40].

3. Transition from the matter to the scalar phase

In the following we study the energy transfer from the perfect fluid to the scalar field. We are interested in studying the late time behaviour near asymptotically stable equilibrium configurations. We suppose that the initial data in the basin of attraction of this equilibrium are such that the fluid is the dominant matter component, i.e.

\[ \rho_0 > \epsilon_0. \]

and we are asking whether there is a time \( t_1 \) such that

\[ \epsilon(t) > \rho(t), \quad \forall t > t_1. \]  \tag{18}

This question is closely related to the classification of scalar field potentials with cosmological scaling solutions [34]. It is also relevant to the coincidence problem, that is, why dark energy and matter appear to have roughly the same energy density today (see [2] and references therein).

There are two cases to be examined for the limiting value of \( V(\phi) \), regardless of the critical point being finite or ‘at infinity’.

(i) \( V(\phi) \) asymptotically tends to a strictly positive limit. Then, it is easily seen that transition (18) does always occur, since the critical value of the potential behaves as an effective cosmological constant and the energy of the scalar field tends to this value whereas the energy of the fluid tends to zero\(^4\).

(ii) \( V(\phi) \) asymptotically tends to zero. This is the nontrivial case and from now on we will focus ourselves to this question, studying both the exponential potential and nondegenerate minima. For simplicity we will restrict ourselves to the flat, \( k = 0 \), case, although it seems that the same results can be stated for the general open spatial topology case [30].

3.1. Exponential potential

In the case of the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \), we already know that \( \phi \to +\infty \) and \( y, \rho \to 0 \) (see the remarks at the end of section 2.1). To investigate the late time behaviour of the energy we follow the ideas from [16], writing the system in the variables

\[ w = \sqrt{\frac{V_0}{3}} \frac{e^{-\lambda \phi}}{H}, \quad z = \frac{y}{\sqrt{6}H} \]  \tag{18}

\(^4\) In a flat FRW model where the perfect fluid at early times is that of radiation and later that of matter, dark energy domination during the radiation epoch should be avoided. Therefore, the relevant question is when the scalar dominance would start. Unfortunately, qualitative methods such as those used in this study are unable to answer this kind of questions.
and $H$, with a new time variable $\tau$ defined by $d\tau = 3Hdt$:  
\[
\frac{dw}{d\tau} = w \left( -\frac{\lambda}{\sqrt{6}} + z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right),
\]
\[
\frac{dz}{d\tau} = z \left( -1 + z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right) + \frac{1}{\sqrt{6}} (\lambda w^2 + \alpha(1 - w^2 - z^2)),
\]
\[
\frac{dH}{d\tau} = -H \left( z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right),
\]
where we have used the constraint (14) to eliminate $\rho$. This system has the nice property that the third equation decouples from the other two. Obviously the scalar field dominates when $w^2 + z^2 \to 1$. Studying the Jacobian at the equilibria of (19) one can find that there are three possible stable configurations, depending on the couple $(\lambda, \gamma)$. A summary picture is sketched in figure 1. In particular,

1. $(w, z) = (\sqrt{1 - \frac{\lambda^2}{6}}, \frac{\lambda}{\sqrt{6}})$ is a scalar field dominated solution and is the only stable asymptotic configuration when $\gamma < \frac{\lambda(4 - \sqrt{6})}{3(2 - \sqrt{6})}$;

2. $(w, z) = (0, \frac{1 - 3\gamma}{4 + \sqrt{6}\gamma})$ is a scaling solutions, where neither the scalar nor the fluid energy eventually dominates, and is the only stable asymptotic configuration when $\lambda > 2\sqrt{\frac{3}{5}}; \gamma < \frac{4(4 - \sqrt{6})}{3(2 - \sqrt{6})}$;

3. $(w, z) = \left( \frac{1 - 4\sqrt{6} + y(-2 + \sqrt{6})^{1/2}}{4 + 3y + \sqrt{6}y}, \frac{3y}{4 + 3y + \sqrt{6}y} \right)$ is another scaling solution that is the only stable asymptotic configuration when $(\lambda, \gamma)$ is on the complementary of the other two regions.

We can therefore conclude that, in the case of the equilibrium ‘at infinity’ determined by an exponential-type potential, the scalar energy $\epsilon$ generically remains as a nonzero fraction of the total energy. In the first case, $\epsilon$ totally dominates over the fluid energy $\rho$; in the other
cases both energies give a nontrivial contribution to the total energy (see also [31] for scaling solutions). Interestingly enough, these are the same conclusions of [16] in the case of a scalar field non-coupled to matter, although the ranges of the parameters $(\lambda, \gamma)$ are different.

3.2. Nondegenerate local minima of $V(\phi)$

Without loss of generality we suppose that $\phi_* = 0$ and therefore, the potential can be written in a neighbourhood of $\phi = 0$ as

$$V(\phi) = \frac{1}{2} \lambda^2 \phi^2 + O(\phi^3), \quad \lambda > 0.$$  \hfill (20)

It must be stressed that this is the most general form of an arbitrary potential near the equilibrium $\phi_*$. In fact, the first assumption does not enter into the study of the late time behaviour around a critical point $\phi_* \in \mathbb{R}$, because for that situation only the behaviour of the potential near the point is important and no further assumptions on $V$ are actually needed. For the sake of simplicity we will neglect higher order terms in the potential $V(\phi)$. Nevertheless, if we take them into account, it can be shown by lengthy calculations that the results we are going to state are not affected. As was done in the previous case we introduce new variables $w, z$:

$$w = \frac{\lambda}{\sqrt{6}} \phi \frac{H}{\lambda}, \quad z = \frac{1}{\sqrt{6}} \frac{\gamma}{\lambda} y.$$  \hfill (21)

and rewrite the system as

$$\dot{w} = \lambda z + 3H w \left( z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right),$$

$$\dot{z} = -\lambda w + 3Hz \left( z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right) + 3H \left( -z + \frac{\alpha}{\sqrt{6}} (1 - w^2 - z^2) \right),$$

$$\dot{H} = -3H^2 \left( z^2 + \frac{\gamma}{2} (1 - w^2 - z^2) \right),$$  \hfill (22)

which unfortunately does not possess the decoupling property of the corresponding system (19). Furthermore, we cannot use the time variable $\tau$ unless we lose regularity properties of the system and we cannot infer the stability of the equilibria by looking at the Jacobian of (22). Actually we will tackle this nontrivial situation by performing a qualitative study of (22) and prove the following result.

**Theorem 2.** Let $\phi_*$ be a nondegenerate minimum of $V(\phi)$ with zero critical value. Consider the solutions of (13) with $k = 0$ approaching the (asymptotically stable) equilibrium point $(\phi_*, y = 0, \rho = 0, H = 0)$. Then if $\gamma < 1$, for every such solution the fluid energy $\rho$ eventually dominates over the scalar energy $\epsilon$, whereas if $\gamma > 1$, $\epsilon$ eventually dominates over $\rho$ in a generic way, i.e. except at most for a particular solution of the system.

**Proof.** As was said before, suppose without loss of generality that $V(\phi)$ is the same as in (20), and consider the variables (21) that bring the system into the form (22). We first define the (positive) function $R$ by $R^2 = w^2 + z^2$ (of course initial data such that $R(0) < 1$ implies $R(t) < 1, \forall t > 0$) and divide the proof into some steps.

(1) If $\gamma < 1$ then $R \to 0$ for every solution: let us observe that the flow of the system eventually lies in the compact set $\{ (w, z, H) : w^2 + z^2 \leq 1, H \in [0, H_0] \}$ and that the $\omega$-limit point of every solution is such that $H = 0$. Therefore, if $(w_0, z_0, 0)$ is a limit point, then it is easily seen by (22) that the circle $w^2 + z^2 = w_0^2 + z_0^2, H = 0$ is made by limit points, which means that the solution asymptotically approaches this circle. Suppose by
contradiction the existence of a solution such that $w_0^2 + z_0^2 > 0$. Then the only possibility is that $w_0^2 + z_0^2 = 1$. Indeed, passing to the polar coordinates

$w = R \cos \theta, \quad z = R \sin \theta,$

we obtain

$\dot{\theta} + \lambda = 3H \cos \theta \left( -\sin \theta + \frac{1 - R^2}{\sqrt{6} R} \right),$

$\dot{R} = 3H (1 - R^2) \left[ R \left( \frac{\gamma}{2} - \sin^2 \theta \right) + \frac{\alpha}{\sqrt{6}} \sin \theta \right], \quad (23)$

$H = -3H^2 \left( R^2 \sin^2 \theta + \frac{\gamma}{2} (1 - R^2) \right).$

If the solution approaches $R \to R_0 > 0$, then the first equation says that $\theta$ asymptotically behaves like $-\lambda t$. This fact implies the following asymptotic estimates as $t \to +\infty$ (recall that $H \to 0^+$ monotonically):

$\int_0^t 3H \left( \frac{\gamma}{2} - \sin^2 \theta \right) d\tau \approx \log a \frac{a}{3} (\gamma - 1)/2,$

$\int_0^t 3H \sin \theta d\tau < +\infty \quad (24)$

and therefore dividing both sides of the second equation in (23) by $(1 - R^2)$ and using $R \to R_0$, the second equation can be integrated to give

$\lim_{t \to +\infty} a(t) \frac{a(t_0)}{a(t_0)} \approx \left( \frac{t}{t_0} \right)^{2/3},$

which is consistent with $R_0 > 0$ only when $R_0 = 1$. But this fact implies $\gamma \geq 1$ (when $\gamma = 1$ the argument cannot be applied, so we must include this possibility, see remark 3 below), a contradiction.

(2) If $\gamma > 1$ and $R \to 0$ then $R/H$ is eventually bounded: let us begin assuming $H/R \to 0$. Then, as above, $\theta \equiv -\lambda t$ and estimates (24) hold. Using $R \to 0$ in the third equation of (23) we see that $H(t) \approx 2/3\gamma t$ and therefore

$\left( \frac{a(t)}{a(t_0)} \right) \approx \left( \frac{t}{t_0} \right)^{\frac{2}{3 \gamma}}.$

Using again $R \to 0$ we can neglect superlinear contribution for $\dot{R}$ in (23), and plugging in the above estimate we obtain

$R(t) \approx \left( \frac{t}{t_0} \right)^{1 - \frac{1}{\gamma}} \left[ R(t_0) + c \int_{t_0}^t s^{-\frac{1}{\gamma}} \sin \theta(s) ds \right].$

for an appropriate constant $c_1$. Possibly shifting the initial instant $t_0$ we can assume that the quantity in square brackets above is strictly positive, which implies that $R(t) \to +\infty$ if $\gamma > 1$, a contradiction. Then, $H/R \not\to 0$. To show that this implies that $H/R$ is bounded away from zero, i.e. $R/H$ bounded, we set $S = H/R$ and consider system (22) in the new variables $R, H, S, X$ and $Y$, where

$w = \frac{\sqrt{6}}{3 \alpha} RX, \quad z = \frac{\sqrt{6}}{3 \alpha} RY.$

By hypothesis, the $\omega$-limit points of the solution are such that $R = 0, H = 0$ and since the $\omega$-limit set is invariant, the triple $(S, X, Y)$ of a limit point must belong to a solution of the system

$\dot{S} = -S^2 Y,$

$\dot{X} = -Y (-\lambda + SX),$

$\dot{Y} = X (-\lambda + SX),$
as it can be easily seen by writing the system in the new variables and setting \( R = H = 0 \).

If there exists an \( \omega \)-limit point of the kind \((R = 0, H = 0, S = 0, X_0, Y_0)\) then using the above system we see that every point such that \( R = 0, H = 0, S = 0, \) and \( X^2 + Y^2 = X_0^2 + Y_0^2 \) is an \( \omega \)-limit point, that is, the solution curve approaches the circle \((R = 0, H = 0, S = 0, X, Y)\) with \( X^2 + Y^2 = X_0^2 + Y_0^2 \), and in particular \( S \to 0 \).

(3) If \( \gamma > 1 \) there exists at most one solution such that \( R \to 0 \) while every other solution is such that \( R \to 1 \).

Assume by contradiction the existence of two distinct solutions \( \xi, \eta \) such that \( R(t) \to 0 \).

As mentioned before, \( H/R \) is bounded away from zero (equivalently, \( R/H \) is bounded) for both \( \xi, \eta \) and therefore we cannot exploit anymore the above argument to show \( \theta \approx -\lambda t \).

A change of coordinates is then needed in order to control the behaviour of \( \theta \) for these solutions. In the following we will derive the appropriate change of variables for a general system of the form

\[
\dot{w} = \lambda z + F_1(w, z, H), \quad (25a)
\]
\[
\dot{z} = -\lambda w + F_2(w, z, H), \quad (25b)
\]
\[
\dot{H} = F_3(w, z, H), \quad (25c)
\]

where \( F_i(w, z, H) \) are \( C^\infty \) functions such that

\[
F_i(0, 0, 0) = 0, \quad (F_i)_w(0, 0, 0) = 0, \quad (F_i)_z(0, 0, 0) = 0, \quad (26)
\]

and \( F_3 \) is a negative function for \( H > 0 \). It is easily seen that \((22)\) has the above form.

Passing to the polar coordinates \( w = R \cos \theta, z = R \sin \theta \) we obtain

\[
\dot{\theta} + \lambda \theta = \frac{F_2(R \cos \theta, R \sin \theta, H) \cos \theta - F_1(R \cos \theta, R \sin \theta, H) \sin \theta}{R}, \quad (27a)
\]
\[
R = F_1(R \cos \theta, R \sin \theta, H) \cos \theta + F_2(R \cos \theta, R \sin \theta, H) \sin \theta, \quad (27b)
\]

and \((26)\) together with \((27a)\) suggests that the condition

\[
F_i(0, 0, H) = 0, \quad i = 1, 2, \quad (28)
\]

would ensure \( \theta \approx -\lambda t \). Since \((28)\) does not hold \textit{a priori}, and in particular does not hold for the system we are considering, we look for a coordinate change

\[
w_1 = w - f(H), \quad z_1 = z - g(H),
\]

where \( f, g \) are the opportune functions of \( H \). Writing the system in the new unknown functions \( w_1, z_1, H \) and imposing \((28)\) for the new system we obtain

\[
\lambda g + F_1(f, g, H) - f'(H)F_3(f, g, H) = 0, \quad (29a)
\]
\[
-\lambda f + F_2(f, g, H) - g'(H)F_3(f, g, H) = 0. \quad (29b)
\]

This amounts to look for a solution of the original system \((25a)-(25c)\). Indeed, if \((w(t), z(t), H(t))\) is such a solution and denoting by \( t(H) \) the inverse function of \( H(t) \), we see that

\[
f(H) = w(t(H)), \quad g(H) = z(t(H)),
\]

satisfy \((29a)\) and \((29b)\).

Coming back to the original problem, we will use the solution \( \eta \) to perform the coordinate change described above and apply it to the other solution \( \xi \). Note also that, since \( R \to 0 \) and \( R/H \) is bounded, it is easy to verify that the conditions
\[
\lim_{H \to 0^+} f(H) = \lim_{H \to 0^+} g(H) = 0, \quad \text{(30a)}
\]
\[
H f'(H), \quad H g'(H) \text{ bounded near } H = 0, \quad \text{(30b)}
\]

hold for \( f, g \). Passing to the polar coordinates, the new \( \theta \) asymptotically behaves like \(-\lambda t\), and it can be found with lengthy but straightforward calculations that the new \( R := w_1^2 + z_1^2 \) satisfies the equation

\[
\dot{R} = 3HR \left\{ R^2 \sin^2 \theta + 2g R \sin \theta + g^2 + \frac{\gamma}{2} (1 - R^2 - f^2 - g^2 - 2fR \cos \theta - 2gR \sin \theta) \right. \\
+ \left[ (f + Hf'(H)) \cos \theta + (g + Hg'(H)) \sin \theta \right] \\
\times \left[ R \sin \theta + 2g \sin \theta - \frac{\gamma}{2} (R + 2f \cos \theta + 2g \sin \theta) \right] \\
- \sin^2 \theta - \frac{\alpha}{\sqrt{6}} \sin \theta (R + 2f \cos \theta + 2g \sin \theta) \right\}. \quad \text{(31)}
\]

Therefore, under the assumption that \( R \to 0 \) (note that the new \( R \) goes to zero if and only if the old \( R \) does, since \( f, g \to 0 \)) we obtain (using also (30a)–(30b))

\[
\dot{R} \approx 3HR \left( \frac{\gamma}{2} - \sin^2 \theta \right),
\]

thus, \( R \approx a^{3(\gamma-1)/2} \) which is consistent only if \( \gamma \leq 1 \) (again \( \gamma = 1 \) does not allow us to perform the above argument and so must be included, see again remark 3 below), a contradiction. Therefore, except \( \eta \) every other solution is such that \( R \) does not tend to zero, and a similar argument as in step (1) ensures that actually \( R \to 1 \). This concludes the proof. \( \square \)

Remark 3. We note that the argument of the proof relies on estimates of type (24) which are consistent when \( \gamma \neq 1 \). Therefore, we stress that the above argument fails to give information in the case \( \gamma = 1 \) when actually both situations, \( \epsilon \ll \rho \) and \( \rho \ll \epsilon \), may generically occur depending on the initial data of the problem.

4. Discussion

We analysed the late time evolution of flat and negatively curved expanding FRW models having a scalar field coupled to matter. We proved that equilibria corresponding to the nonnegative local minima of \( V \) are asymptotically stable. In case the minimum is positive, say \( V(\phi_0) > 0 \), the energy density, \( \epsilon \), of the scalar field eventually rules over the energy density of the fluid, \( \rho \), and the asymptotic state has an effective cosmological constant \( V(\phi_0) \). In case the minimum is zero and nondegenerate, \( \rho \) eventually dominates over \( \epsilon \) if \( \gamma < 1 \) and \( \epsilon \) dominates over \( \rho \) if \( \gamma > 1 \). This result could be interesting in the investigations of cosmological scenarios in which the energy density of the scalar field mimics the background energy density. For viable dark energy models, it is necessary that the energy density of the scalar field remains insignificant during most of the history of the universe and emerges only at late times to account for the current acceleration of the universe.

It must also be stressed that the energy transfer from the scalar field to the fluid showed in theorem 2 is proved \textit{generically}, i.e. even in the \( \gamma > 1 \) case there is possibly a special solution such that \( \rho \) dominates over \( \epsilon \); nevertheless this solution is unstable under perturbation of initial data of the problem. The generic behaviour of solutions is not a new feature of models involving scalar fields. We mention for example the well-known instability (with respect to bounded variation initial data) result by Christodoulou [11] about scalar fields collapsing
to a naked singularity and even in the realm of homogeneous stars (regularly matched with nonhomogeneous exteriors), genericity of collapsing solutions ending in a black hole has been discovered to be true both in GR [27] and in HOG gravity [29].

The above results were rigorously proved only assuming that critical points are finite, and that \( V(\phi) \) is eventually non-negative as \( \phi \to \pm \infty \). It must be remarked that the latter assumption plays the role to ensure that the flat plateau at infinity is unstable (as proved for the special case of potential (1) in [39]), whereas if \( V(\phi) \to \ell \) from above we have an asymptotically stable configuration. As a representative of the latter class of stable configurations we have studied the late time behaviour of the energy in the case of exponential potential, finding results similar to the non-coupled case [16]. However, our dynamical system is much harder to deal with because the evolution equation for \( H \) does not decouple as in [16] and moreover its equilibria are not hyperbolic. When the decay parameter \( \lambda \) is small, where ‘small’ depends on \( \gamma \), see region 1 in figure 1, the scalar field dominates, whereas other stable configurations correspond to scaling solutions where both energies nontrivially contribute to the total energy of the system.

One can encounter in the same potential all the situations studied so far; for instance, one can have a potential exhibiting more than one local minima, or a local minimum and an exponential decay at infinity. In that case since at least two asymptotically stable configurations for the system can be reached, the late time behaviour of the energies and their exchange necessarily depends on the initial data of the problem.

The cases studied in section 3 of course do not cover all possible situations, even under assumption 1. In particular, one can take into account degenerate minima for the potential. In this case, it is not hard to see that the counterpart of systems (22) and (23) becomes irregular near \( \omega = z = 0 \), and to infer the behaviour of the angle \( \theta \), one cannot straightforwardly extend the techniques used here. Nevertheless, research in this direction tends to confirm the same generic behaviour showed here for the nondegenerate case [30]. Therefore, it seems appropriate to conclude that energy transfer is completely driven exclusively by the sign of the fluid pressure when the scalar reaches a finite stable equilibrium, whereas for stable configurations ‘at infinity’ the potential profile becomes crucial.

Another important question that should be further investigated is the case of closed cosmologies. We believe that a closed model cannot avoid recollapse, unless the minimum of the potential is strictly positive. In that case, the asymptotic state must be the de Sitter space.

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Appendix A.

Proof of proposition 1. We will follow the ideas used in [28] in the absence of ordinary matter. We start the proof by considering the case \( V(\phi_*) > 0 \). Let \( \tilde{V} > V(\phi_*) \) be a regular value for \( V \) such that the connected component of \( V^{-1}([-\infty, \tilde{V}]) \) that contains \( \phi_* \) is a compact set in \( \mathbb{R} \). Let us denote by \( A \subseteq \mathbb{R} \) this compact set and define \( \Omega \) as

\[
\Omega = \{ (\phi, y, \rho, H) : \phi \in A, (\epsilon + \rho) \leq \tilde{V}, \rho \geq 0, W(\phi, y, \rho, H) \in [0, \tilde{W}] \}.
\]
where $\bar{W}$ is a positive constant. Then $\Omega$ is a closed set because the following facts hold.

(i) $\Omega$ is a closed set in $\mathbb{R}^4$.
(ii) $V(\phi_s) \leq V(\phi) \leq \bar{V}$. $\forall \phi \in A$.
(iii) $\frac{1}{2}y^2 + V(\phi_s) \leq \frac{1}{2}y^2 + V(\phi) + \rho \leq \bar{V}$, and therefore $y$ is bounded.
(iv) $\rho \leq \bar{V} - \frac{1}{2}y^2 - V(\phi) \leq \bar{V} - V(\phi_s)$, and therefore $\rho$ is also bounded.
(v) From (15) and the above facts, $\lim_{t \to \pm \infty} V(\phi_s) \leq \bar{H}^2 \leq \bar{W} + \bar{V}$, and so $\bar{H}$ is bounded too.

Let $\Omega_+ \subseteq \Omega$ be the connected component of $\Omega$ containing $\phi_s$. It is easy to see that $\bar{H}$ is positive on $\Omega_+$, and we claim that $\Omega_+$ is positively invariant with respect to (13), i.e., solutions, with initial data in $\Omega_+$, live in $\Omega_+$, $t > 0$. Indeed, let $\phi(t)$ be such a solution and $\bar{t} = \sup \{t > 0 : \bar{H}(t) > 0\} \in \mathbb{R} \cup \{+\infty\}$. When $t < \bar{t}$, equations (16) and (17) imply that both $W$ and $\epsilon + \rho$ decrease. Moreover, it can be proved by contradiction that

$$\phi(t) \in A, \quad \forall t < \bar{t};$$

otherwise there would exist some $t < \bar{t}$ such that $V(\phi(t)) > \bar{V}$, but then

$$\bar{V} < V(\phi(t)) \leq \frac{1}{2}y^2(t) + V(\phi(t)) + \rho(t) = \epsilon(t) + \rho(t) \leq \bar{V},$$

a contradiction. Thus, (A.1) holds. But since $W \geq 0$ along the flow (cf remark 1), it follows that

$$\bar{H}(t)^2 \geq \frac{1}{3}y(t)^2 + V(\phi(t)) + \rho(t) \geq \frac{V(\phi(t))}{3} \geq \frac{V(\phi_s)}{3},$$

where the last inequality follows from (A.1). All in all we have proved that as long as $H$ remains positive, it is strictly bounded away from zero; thus, $\bar{t} = +\infty$, and from this fact it is straightforward to conclude that $\phi(t)$ lives in $\Omega_+, \forall t > 0$.

Using the properties of $\Omega_+$, proved so far, LaSalle’s invariance theorem [51] can be applied to the functions $W$ and $(\rho + \epsilon)$ in $\Omega_+$, to show that every solution with initial data in $\Omega_+$, must be such that $\bar{H}W \to 0$ and $\bar{H}(y^2 + \gamma \rho) \to 0$ as $t \to +\infty$. Since $\bar{H}$ is strictly bounded away from zero in $\Omega_+$, both $W$ and $(y^2 + \gamma \rho)$ must tend to zero, which means $y \to 0$, $\rho \to 0$ and $\bar{H}^2 - \frac{1}{2}V(\phi) \to 0$. Now the fourth part of (13)

$$\bar{H} = -\frac{1}{2}(y^2 + \gamma \rho) - W$$

implies that $\bar{H}$ is monotone and therefore admits a limit. This means that $V(\phi)$ also admits limit, and this limit must be unavoidably $V(\phi_s)$; otherwise $V'(\phi)$ would tend to a nonzero value, and so would the right-hand side in the second equation of (13), a contradiction. Therefore, the solution approaches the equilibrium point $\phi_s$.

If $V(\phi_s) = 0$, the above argument can be easily adapted. In this case the set $\Omega_+$ is connected and we choose $\Omega_+$ to be its subset characterized by the property $H \geq 0$. The only point in $\Omega_+$ with $H = 0$ is exactly the equilibrium point $\phi_s$, and so if $H(t) \to 0$ the solution is forced to approach the equilibrium since $H$ is monotone; if by contradiction $H(t)$ had a strictly positive limit, we could argue as before to find $y \to 0$, $\rho \to 0$ and $W \to 0$ and so $\bar{H}$ must necessarily converge to zero

$\square$

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