The limit to rarefaction wave with vacuum for 1D compressible fluids with temperature-dependent viscosities

Mingjie Li*, Teng Wang†, and Yi Wang‡

*College of Science, Minzu University of China, Beijing 100081, P. R. China
† Institute of Applied Mathematics, AMSS, CAS, Beijing 100190, P. R. China
‡ Institute of Applied Mathematics and NCMIS, AMSS, CAS, Beijing 100190, P. R. China

Abstract

In this paper we study the zero dissipation limit of the one-dimensional full compressible Navier-Stokes (CNS) equations with temperature-dependent viscosity and heat-conduction coefficient. It is proved that given a rarefaction wave with one-side vacuum state to the full compressible Euler equations, we can construct a sequence of solutions to the full CNS equations which converge to the above rarefaction wave with vacuum as the viscosity and the heat conduction coefficient tend to zero. Moreover, the uniform convergence rate is obtained. The main difficulty in our proof lies in the degeneracies of the density, the temperature and the temperature-dependent viscosities at the vacuum region in the zero dissipation limit.

Keywords: compressible Navier-Stokes equations, temperature-dependent viscosities, zero dissipation limit, rarefaction wave, vacuum.

*The research of M. Li was supported by the NSFC Grant No. 11201503. E-mail: lmjmath@gmail.com.
†E-mail: tengwang@amss.ac.cn.
‡The research of Y. Wang was supported by the NSFC Grant No. 11171326 and the CAS Program for Cross & Cooperative Team of the Science & Technology Innovation. E-mail: wangyi@amss.ac.cn.
1 Introduction and main result

We consider the zero dissipation limit of the one-dimensional compressible Navier-Stokes equations with heat-conduction in Eulerian coordinates which read

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= (\epsilon \mu(\theta) u)_x, \\
[\rho(e + \frac{1}{2} u^2)]_t + [\rho u(e + \frac{1}{2} u^2) + up]_x &= (\epsilon \kappa(\theta) \theta_x + \epsilon \mu(\theta) uu_x)_x,
\end{align*}
\]

(1.1)

where \(\rho(x, t) \geq 0\), \(u(x, t)\), \(p(x, t)\), \(e(x, t) \geq 0\) and \(\theta(x, t) \geq 0\) represent the mass density, velocity, pressure, internal energy and absolute temperature of the gas, respectively, and \(\epsilon \mu(\theta)\) and \(\epsilon \kappa(\theta)\) denote the viscosity and heat-conduction coefficient, respectively, with \(\epsilon > 0\) being positive constant and

\[
\mu(\theta) = \mu_1 \theta^\alpha, \quad \text{and} \quad \kappa(\theta) = \kappa_1 \theta^\alpha
\]

(1.2)

for positive constants \(\mu_1, \kappa_1\) and \(\alpha > 0\). Without loss of generality, it is assumed that \(\mu_1 = \kappa_1 = 1\). Here we consider the ideal polytropic gas, that is, the pressure \(p\) and the internal energy \(e\) are given respectively by

\[
p = R\rho \theta = A \rho^\gamma \exp(\gamma - 1) \frac{\gamma - 1}{R} S, \quad e = \frac{R\theta}{\gamma - 1},
\]

(1.3)

satisfying the second law of thermodynamics

\[
de = \theta dS + \frac{p}{\rho^2} d\rho.
\]

(1.4)

In the state equations (1.3) and (1.4), \(S_S(x, t)\) denotes the entropy of the gas and \(\gamma > 1\) is the adiabatic exponent and \(A, R\) are both positive constants. For simplicity, it is normalized that

\[
A = R = \gamma - 1.
\]

The compressible Navier-Stokes equations (1.1) with the temperature-dependent viscosities (1.2) can be derived exactly from the Chapman-Enskog expansion for the Boltzmann equation with respect to the Knudsen number, one can refer to [2], [13] for the details. Formally, as \(\epsilon \to 0^+\), the system (1.1) tends to the corresponding inviscid Euler equations

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= 0, \\
[\rho(e + \frac{1}{2} u^2)]_t + [\rho u(e + \frac{1}{2} u^2) + up]_x &= 0.
\end{align*}
\]

(1.5)
The Euler system (1.5) is a strictly hyperbolic one for $\rho > 0$ whose first and third characteristic fields are genuinely nonlinear and second characteristic field is linearly degenerate, that is, in the equivalent system

$$
\begin{pmatrix}
\rho \\
u \\
S
\end{pmatrix}_t + \begin{pmatrix}
u & \rho & 0 \\
\frac{\rho u}{\rho} & u & \frac{\rho S}{\rho} \\
0 & 0 & u
\end{pmatrix} \begin{pmatrix}
\rho \\
u \\
S
\end{pmatrix}_x = 0,
$$

the Jacobi matrix

$$
\begin{pmatrix}
u & \rho & 0 \\
\frac{\rho u}{\rho} & u & \frac{\rho S}{\rho} \\
0 & 0 & u
\end{pmatrix}
$$

has three distinct eigenvalues

$$
\lambda_1(\rho, u, \theta) = u - \sqrt{p\rho(\rho, S)}, \quad \lambda_2(\rho, u, \theta) = u, \quad \lambda_3(\rho, u, \theta) = u + \sqrt{p\rho(\rho, S)}
$$

with corresponding right eigenvectors

$$
r_1(\rho, u, S) = (-\rho, \sqrt{p\rho(\rho, S)}, 0)^t, \quad r_2(\rho, u, S) = (pS, 0, -p\rho)^t, \quad r_3(\rho, u, S) = (\rho, \sqrt{p\rho(\rho, S)}, 0)^t,
$$

such that

$$
r_i(\rho, u, S) \cdot \nabla(\rho, u, S)\lambda_i(\rho, u, S) \neq 0, \quad i = 1, 3,
$$

and

$$
r_2(\rho, u, S) \cdot \nabla(\rho, u, S)\lambda_2(\rho, u, S) = 0.
$$

Thus the two $i$-Riemann invariants ($i = 1, 3$) can be defined by (cf. [20])

$$
\Sigma_i^{(1)} = u + (-1)^{i-1} \int^\rho \frac{\sqrt{p\rho(z, S)}}{z} dz, \quad \Sigma_i^{(2)} = S,
$$

such that

$$
\nabla(\rho, u, S)\Sigma_i^{(j)}(\rho, u, S) \cdot r_i(\rho, u, S) \equiv 0, \quad i = 1, 3, \quad j = 1, 2.
$$

The study of the limiting process of viscous flows when the viscosity tends to zero is one of the important problems in the theory of the compressible fluid. When the solution of the inviscid flow is smooth, the zero dissipation limit can be solved by classical scaling method. However, the inviscid compressible flow contains singularities such as shock, contact discontinuity and the vacuum in general. Therefore, determining how to justify the zero dissipation limit to the Euler equations with basic wave patterns and the vacuum states is a natural and difficult problem.

There have been many results on the zero dissipation limit of the compressible fluid with basic wave patterns without vacuum. For the system of the hyperbolic conservation laws with artificial viscosity

$$
u_t + f(u)_x = \varepsilon u_{xx},
$$
Goodman-Xin [4] first verified the viscous limit for piecewise smooth solutions separated by non-interacting shock waves using a matched asymptotic expansion method. Later Yu [25] proved it for the corresponding hyperbolic conservation laws with both shock and initial layers. In 2005, important progress made by Bianchini-Bressan[1] justifies the vanishing viscosity limit in BV space even though the problem is still unsolved for the physical system such as the compressible Navier-Stokes equations.

For the isentropic compressible Navier-Stokes equations with constant viscosity where the conservation of energy in (1.1) is neglected, Hoff-Liu [5] first proved the vanishing viscosity limit for piecewise constant shock even with initial layer. Later Xin [23] obtained the zero dissipation limit for rarefaction waves without vacuum for both rarefaction wave data and well-prepared smooth data. Then Wang [21] generalized the result of Goodmann-Xin [4] to the isentropic Navier-Stokes equations. Recently, Chen-Perepelitsa [3] proved the vanishing viscosity to the compressible Euler equations for the isentropic compressible Navier-Stokes equations (1.1) with constant viscosity by compensated compactness method for the case that the far field of the initial values of Euler system has no vacuums.

For the full Navier-Stokes equations (1.1) with constant viscosity, there are also many results on the zero dissipation limit to the corresponding full Euler system with basic wave patterns without vacuum. We refer to Jiang-Ni-Sun [11] and Xin-Zeng [24] for the rarefaction wave, Wang [22] for the shock wave, Ma [17] for the contact discontinuity, Huang-Wang-Yang [9] and Huang-Jiang-Wang [7] for the superposition of two rarefaction waves and a contact discontinuity, Huang-Wang-Yang [10] for the superposition of one shock and one rarefaction wave and Zhang-Pan-Wang-Tan [26] for the superposition of two shock waves with the initial layer. Recently, Huang-Wang-Wang-Yang [8] succeed in justifies the vanishing viscosity limit of compressible Navier-Stokes equations in the setting of Riemann solutions for the superposition of shock wave, rarefaction wave and contact discontinuity.

It is well-known that the vacuum states are generic in inviscid compressible Euler equations (1.5) since the vacuum states may occur in the Riemann solutions instantaneously as \( t > 0 \) even if the initial Riemann data are non-vacuum states on both sides at \( t = 0 \). Therefore, vacuum states are important physical states in gas dynamics and often yield degeneracies and certain singularities in the physical system, which cause some essential analytical difficulties. For example, the velocity can not even be defined in the vacuum region. In the setting of Riemann solutions, as pointed out by Liu-Smoller [15], among the three elementary hyperbolic waves, i.e., shock and rarefaction waves and contact discontinuities, to the one-dimensional isentropic compressible Euler equations (1.5), only the rarefaction wave can be connected to the vacuum states. There are some mathematical results on the time-asymptotic stability and the vanishing viscosity limit to the rarefaction wave with the vacuum. Perepelitsa [19] consider the time-asymptotic stability of solutions to 1-d isentropic compressible Navier-Stokes equations with
fixed and constant viscosity toward rarefaction waves connected to vacuum in Lagrangian co-
ordinate. Then Jiu-Wang-Xin [12] study the large time asymptotic behavior toward rarefaction wave with vacuum for solution to the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, which can be viewed as the compressible Navier-Stokes equations (1.1) with fixed $\epsilon = 1$ in isentropic regime. Note that in isentropic regime, there is no energy equation (1.1)$_3$ and the viscosity coefficient in the momentum equation (1.1)$_2$ transfer to depend on the density. More recently, Huang-Li-Wang [6] justified the vanishing viscosity limit of one-dimensional isentropic compressible Navier-Stokes equations with constant viscosity to the rarefaction wave with one-side vacuum state to the corresponding compressible Euler equations. However, for the full compressible Navier-Stokes equations (1.1) with temperature-dependent viscosities, as far as we know, there is no any result on the zero dissipation limit to the rarefaction wave with the vacuum due to various difficulties mentioned below.

Now we give a description of the rarefaction wave connected to the vacuum to the full compressible Euler equations (1.5); see also the reference [20]. For definiteness, 3-rarefaction wave will be considered. If we investigate the compressible Euler system (1.5) with the Riemann initial data

\[
\begin{align*}
\rho(0, x) &= 0, \quad x < 0, \\
(\rho, u, \theta)(0, x) &= (\rho_+, u_+, \theta_+), \quad x > 0,
\end{align*}
\]  

(1.7)

where the left side is the vacuum state and $\rho_+ > 0, u_+, \theta_+ > 0$ are prescribed constants on the right state, then the Riemann problem (1.5), (1.7) admits a 3–rarefaction wave connected to the vacuum on the left hand side. By the fact that along the 3–rarefaction wave curve, 3–Riemann invariant $\Sigma_3^{(i)}(\rho, u, \theta), (i = 1, 2)$ defined in (1.6) keeps constant in $(x, t)$, one can get the velocity

\[ u_- = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+) \]

being the speed of the gas coming into the vacuum from the 3-rarefaction wave. On the other hand, in the vacuum region, the absolute temperature $\theta$ also becomes zero due to the state equation (1.3), that is, $\theta = \rho^{\gamma-1}e^S$ and the fact that the entropy $S$ keeps constant along the 3-rarefaction wave. Correspondingly, the main difficulty of the present paper lies in how to deal with the degeneracies of the temperature-dependent viscosities in the vacuum region in the zero dissipation limit process.

As described above, the 3–rarefaction wave connecting the vacuum state $\rho = 0$ to $(\rho_+, u_+, \theta_+)$ is the self-similar solution $(\rho^r_3, u^r_3, \theta^r_3)(\xi), (\xi = \frac{x}{t})$ of (1.5) defined by

\[
\begin{align*}
\lambda_3(\rho^r_3(\xi), u^r_3(\xi), \theta^r_3(\xi)) = \begin{cases} 
\rho^r_3(\xi) & \text{if } \xi < \lambda_3(0, u_-, 0) = u_-, \\
\xi & \text{if } u_- \leq \xi \leq \lambda_3(\rho_+, u_+, \theta_+), \\
\lambda_3(\rho_+, u_+, \theta_+) & \text{if } \xi > \lambda_3(\rho_+, u_+, \theta_+),
\end{cases}
\end{align*}
\]  

(1.8)

\[
\Sigma_3^{(1)}(\rho^r_3(\xi), u^r_3(\xi), \theta^r_3(\xi)) = \Sigma_3^{(1)}(0, u_-, 0) = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+),
\]  

(1.9)
and

$$\Sigma_3^{(2)} = S^r_3 = S_+ := -(\gamma - 1) \log \rho_+ + \log \theta_+.$$  \hfill (1.10)

Thus one can define the momentum $m^r_3$ and the total internal energy $n^r_3$ of 3-rarefaction wave by

$$m^r_3(\xi) := \begin{cases} 
\rho^r_3(\xi) u^r_3(\xi), & \text{if } \rho^r_3 > 0, \\
0, & \text{if } \rho^r_3 = 0,
\end{cases}$$  \hfill (1.11)

and

$$n^r_3(\xi) := \begin{cases} 
\rho^r_3(\xi) \theta^r_3(\xi), & \text{if } \rho^r_3 > 0, \\
0, & \text{if } \rho^r_3 = 0.
\end{cases}$$  \hfill (1.12)

In the present paper, we want to construct a sequence of global-in-time solutions $(\rho^\epsilon, m^\epsilon := \rho^\epsilon u^\epsilon, n^\epsilon := \rho^\epsilon \theta^\epsilon)(x, t)$ to the compressible Navier-Stokes equations (1.1) with temperature-dependent viscosities, which converge to the 3-rarefaction wave $(\rho^r_3, m^r_3, n^r_3)(x/t)$ with vacuum defined above as $\epsilon$ tends to zero. The effects of initial layers will be ignored by choosing the well-prepared initial data (3.1) depending on the viscosity for the Navier-Stokes equations.

As mentioned before, the main novelty and difficulty here is determining how to control the degeneracies caused by the vacuum in the rarefaction wave. To overcome this difficulty, we first cut off the 3-rarefaction wave with vacuum along the rarefaction wave curve suitably ($\nu$ is the cut-off parameter and the details can be seen in Section 2), and use the fact that the viscosity $\epsilon$ can control the degeneracies caused by the vacuum in rarefaction waves by choosing suitably $\nu = \nu(\epsilon)$. In fact, we choose $\nu = \epsilon^a |\ln \epsilon|$ with $a$ defined in (1.14) in the present paper. The other observation is that we can carry out the energy estimates under the a priori assumptions (3.10)-(3.11) such that the perturbation is suitably small in $L^\infty(\mathbb{R})$ norm with some decay rate with respect to $\epsilon$.

On the other hand, compared with the previous works [6] for the isentropic Navier-Stokes equations with the constant viscosity case, some new difficulties occur for the full Navier-Stokes equations (1.1) with temperature-dependent viscosities considered in the present paper. Firstly, in order to overcome the difficulties caused by the non-isentropic regime, the relative entropy-entropy flux pair $(\eta, q)$ defined in (3.17) is used as in [16]. Secondly, since in the vacuum region, the temperature becomes zero due to the entropy keeps constant by the structure of rarefaction wave. Therefore, not only the density but also the temperature cause the degeneracies in the vacuum region and so the temperature-dependent viscosities do. Thus, to deal with the terms for the temperature-dependent viscosities, such as (3.35), becomes subtle. In fact, the derivative estimates of the perturbation of the density depends on the second order derivative estimates of velocity with some degenerate coefficients (see (3.50)), which is quite different from the constant viscosity case in [6] and [14]. By choosing the convergence rate $a$ suitably as in (1.14) and then the parameters $\nu, \delta$ as in (3.12) closes the a priori estimates and yields the desired result.

Now we state our main result as follows.
Theorem 1.1. Let \((\rho^r, m^r, n^r)(x/t)\) be the 3-rarefaction wave with one-side vacuum state defined by (1.8)-(1.12). Then there exists a small positive constant \(\epsilon_0\) such that for any \(\epsilon \in (0, \epsilon_0)\), we can construct a family of global smooth solutions \((\rho^\epsilon, m^\epsilon, n^\epsilon, x/t)\) to the compressible Navier-Stokes equation (1.1) satisfying the following properties.

(1) \((\rho^\epsilon - \rho^r, m^\epsilon - m^r, n^\epsilon - n^r), (\rho^\epsilon, m^\epsilon, n^\epsilon) \in C(0, +\infty; L^2(\mathbb{R})), \quad u^\epsilon, \theta^\epsilon \in L^2(0, +\infty; L^2(\mathbb{R}))).\)

(2) As the viscosity \(\epsilon \to 0\), \((\rho^\epsilon, m^\epsilon, n^\epsilon)(x, t)\) converges to \((\rho^r, m^r, n^r)(x/t)\) pointwisely except the original point \((0, 0)\). Furthermore, for any given positive constant \(l\), there exists a constant \(C_l > 0\), independent of \(\epsilon\), such that

\[
\sup_{t \geq l} \|\rho^\epsilon(\cdot, t) - \rho^r(\cdot)\|_{L^\infty} \leq C_l \epsilon^a |\ln \epsilon|, \\
\sup_{t \geq l} \|m^\epsilon(\cdot, t) - m^r(\cdot)\|_{L^\infty} \leq C_l \epsilon^a |\ln \epsilon|, \\
\sup_{t \geq l} \|n^\epsilon(\cdot, t) - n^r(\cdot)\|_{L^\infty} \leq C_l \epsilon^a |\ln \epsilon|.
\]

with the positive constant \(a\) given by

\[
a = \frac{1}{18\gamma + 12\alpha(\gamma - 1)}. \tag{1.14}
\]

Remark 1.2. From (1.13) and (1.14), one can see that the decay rate \(\epsilon^a |\ln \epsilon|\) in Theorem 1.1 decreases monotonically with respect to \(\alpha\), which is consistent to the observation that the viscosity becomes weaker as \(\alpha\) becomes larger due to the vacuum in the rarefaction wave, although the convergence rates in (1.13) and (1.14) may not be optimal.

Remark 1.3. By using some ideas in the present paper, Li-Wang [14] generalize Huang-Li-Wang [6]'s result to the non-isentropic compressible Navier-Stokes equations (1.1) with constant viscosity.

The rest of the paper is organized as follows. In section 2, we construct a smooth 3-rarefaction wave profile which approximates the cut-off rarefaction wave for the Euler equations based on the inviscid Burgers equation. Then the global-in-time solution to CNS (1.1) is obtained around the smooth 3-rarefaction wave profile and finally the proof the Theorem 1.1 is shown in Section 3.

Throughout this paper, \(H^k(\mathbb{R}), k = 0, 1, 2, \ldots,\) denotes the \(l\)-th order Sobolev space with its norm

\[
\|f\|_k = \left(\sum_{j=0}^{k} \|\partial_y^j f\|^2\right)^{\frac{1}{2}}, \quad \text{and} \quad \|\cdot\| := \|\cdot\|_{L^2(dz)},
\]

while \(L^2(dz)\) means the \(L^2\) integral over \(\mathbb{R}\) with respect to the Lebesgue measure \(dz\), and \(z = x\) or \(y\). For simplicity, we also write \(C\) as generic positive constants which are independent of time \(t\) or \(\tau\) and viscosity \(\epsilon\) unless otherwise stated.


2 Rarefaction waves

Since there is no exact rarefaction wave profile for the Navier-Stokes equations (1.1), the following approximate rarefaction wave profile satisfying the Euler equations was motivated by Matsumura-Nishihara [18] and Xin [23].

Consider the Riemann problem for the inviscid Burgers equation:

\[
\begin{aligned}
    w_t + w w_x &= 0, \\
    w(x, 0) &= \begin{cases} 
        w_-, & x < 0, \\
        w_+, & x > 0.
    \end{cases}
\end{aligned}
\] (2.1)

If \( w_- < w_+ \), then the Riemann problem (2.1) admits a rarefaction wave solution \( w^r(x, t) = w^r(x/t) \) given by

\[
\begin{aligned}
w^r(x/t) &= \begin{cases} 
    w_-, & \frac{x}{t} \leq w_-, \\
    \frac{x}{t}, & w_- \leq \frac{x}{t} \leq w_+, \\
    w_+, & \frac{x}{t} \geq w_+.
\end{cases}
\end{aligned}
\] (2.2)

Motivated by [18, 23], the approximate rarefaction wave to the compressible Navier-Stokes equations (1.1) can be constructed by the Burgers equation

\[
\begin{aligned}
w_t + w w_x &= 0, \\
w(0, x) &= \delta(x) = w(x/\delta) = \frac{w_+ - w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\delta},
\end{aligned}
\] (2.3)

where \( \delta > 0 \) is a small parameter to be determined and the hyperbolic tangent function \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \). In fact, we choose \( \delta = \epsilon^a \) in (3.12) with \( a \) given by (1.14). Note that the solution \( w^r_\delta(t, x) \) of the problem (2.3) can be given explicitly by

\[
w^r_\delta(t, x) = w_\delta(x_0(t, x)), \quad x = x_0(t, x) + w_\delta(x_0(t, x))t.
\] (2.4)

And \( w^r_\delta(t, x) \) has the following properties:

Lemma 2.1. ([23, 6]) The problem (2.3) has a unique smooth global solution \( w^r_\delta(x, t) \) for each \( \delta > 0 \) such that

1. \( w_- < w^r_\delta(x, t) < w_+, \partial_x w^r_\delta(x, t) > 0, \) for \( x \in \mathbb{R}, t \geq 0, \delta > 0. \)

2. The following estimates hold for all \( t > 0, \delta > 0 \) and \( p \in [1, \infty] \):

\[
\begin{aligned}
    \| \partial_x w^r_\delta(\cdot, t) \|_{L^p} &\leq C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p}, \\
    \| \partial^2_x w^r_\delta(\cdot, t) \|_{L^p} &\leq C(\delta + t)^{-1}\delta^{-1+1/p}, \\
    \left| \frac{\partial^2 w^r_\delta(x, t)}{\partial x^2} \right| &\leq \frac{4}{\delta} \frac{\partial w^r_\delta(x, t)}{\partial x}.
\end{aligned}
\] (2.5) (2.6) (2.7)
(3) There exist a constant $\delta_0 \in (0, 1)$ such that for $\delta \in (0, \delta_0), t > 0$,

$$\|w^{\nu}_3(\cdot, t) - w^{\nu}(\cdot, t)\|_{L^\infty} \leq C\delta t^{-1}\left[\ln(1 + t) + |\ln \delta|\right].$$

Note that Lemma 2.1 is a little different from the one in [23] as stated in [6]. For the detailed proof of Lemma 2.1, one can refer to [23] and [6] and we omit it here for brevity.

As mentioned in the introduction, we will first cut off the 3-rarefaction wave with vacuum along the wave curve in order to overcome the degeneracies caused by the vacuum. More precisely, for any constant $\nu > 0$ to be determined, we can get a state $(\rho, u, \theta) = (\nu, u_\nu, e^{\bar{S}\nu^{-1}})$ belonging to the 3-rarefaction wave curve, where $\bar{S} = S_{\pm} = -(\gamma - 1)\log \rho_+ + \log \theta_+$. From the fact that 3-Riemann invariant $\Sigma_3^{(i)}(\rho, u, \theta), (i = 1, 2)$ is constant along the 3-rarefaction wave curve, $u_\nu$ can be computed explicitly by

$$u_\nu = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+) + 2\sqrt{\frac{\gamma}{\gamma - 1}}\nu^{-1}e^{\bar{S}_+}.$$  

Now we get a new cut-off 3-rarefaction wave $(\rho^{\nu}_3, u^{\nu}_3, \theta^{\nu}_3)(\xi), (\xi = x/t)$ connecting the state $(\nu, u_\nu, e^{\bar{S}\nu^{-1}})$ to the state $(\rho_+, u_+, \theta_+)$ which can be expressed explicitly by

$$\lambda_3(\rho^{\nu}_3, u^{\nu}_3, \theta^{\nu}_3)(\xi) = \begin{cases} 
\lambda_3(\nu, u_\nu, e^{\bar{S}\nu^{-1}}), & \xi < \lambda_3(\nu, u_\nu, e^{\bar{S}\nu^{-1}}), \\
\xi, & \lambda_3(\nu, u_\nu, e^{\bar{S}\nu^{-1}}) \leq \xi \leq \lambda_3(\rho_+, u_+, \theta_+), \\
\lambda_3(\rho_+, u_+, \theta_+), & \xi > \lambda_3(\rho_+, u_+, \theta_+). 
\end{cases} \quad (2.8)$$

and

$$\Sigma_3^{(1)}(\rho^{\nu}_3, u^{\nu}_3, \theta^{\nu}_3) = \Sigma_3^{(1)}(\nu, u_\nu, e^{\bar{S}\nu^{-1}}) = \Sigma_3^{(1)}(\rho_+, u_+, \theta_+). \quad (2.9)$$

Correspondingly, we can define the momentum function and the total internal energy $m^{\nu}_3 := \rho^{\nu}_3u^{\nu}_3$ and $n^{\nu}_3 := \rho^{\nu}_3\theta^{\nu}_3$ respectively. It is easy to show that the cut-off 3-rarefaction wave $(\rho^{\nu}_3, m^{\nu}_3, n^{\nu}_3)(x/t)$ converges to the original 3-rarefaction wave with vacuum $(\rho^3, m^3, n^3)(x/t)$ in sup-norm with the convergence rate $\nu$ as $\nu$ tends to zero. More precisely, it holds that

**Lemma 2.2.** There exist a constant $\nu_0 \in (0, 1)$ such that for $\nu \in (0, \nu_0], t > 0$,

$$\|(\rho^{\nu}_3, m^{\nu}_3, n^{\nu}_3)(\cdot/t) - (\rho^3, m^3, n^3)(\cdot/t)\|_{L^\infty} \leq C\nu,$$

where the positive constant $C$ is independent of $\nu$.

Now the approximate rarefaction wave $(\bar{\rho}_{\nu, \delta}, \bar{u}_{\nu, \delta}, \bar{\theta}_{\nu, \delta})(x, t)$ of the cut-off 3-rarefaction wave $(\rho^{\nu}_3, u^{\nu}_3, \theta^{\nu}_3)(\xi)$ to compressible Euler equations (1.5) can be defined by

$$\begin{align*}
w_+ &= \lambda_3(\rho_+, u_+, \theta_+), \
w_- &= \lambda_3(\nu, u_\nu, e^{\bar{S}\nu^{-1}}), \\
w^{\nu}_3(x, t) &= \lambda_3(\bar{\rho}_{\nu, \delta}, \bar{u}_{\nu, \delta}, \bar{\theta}_{\nu, \delta})(x, t), \\
\Sigma_3^{(1)}(\bar{\rho}_{\nu, \delta}, \bar{u}_{\nu, \delta}, \bar{\theta}_{\nu, \delta})(x, t) &= \Sigma_3^{(1)}(\rho_+, u_+, \theta_+) = \Sigma_3^{(1)}(\nu, u_\nu, e^{\bar{S}\nu^{-1}}),
\end{align*} \quad (2.10)$$
where \( w^*_0(\bar{x}, t) \) is the solution of Burger’s equation (2.3) defined in (2.4). From now on, the subscription of \((\bar{\rho}, \bar{u}, \bar{\theta})(x, \bar{t})(x, t)\) will be abbreviated as \((\bar{\rho}, \bar{u}, \bar{\theta})(x, t)\) for simplicity. Then the approximate 3-rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})\) defined above satisfies

\[
\begin{align*}
\bar{\rho}_t + (\bar{\rho}\bar{u})_x &= 0, \\
(\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + \bar{p})_x &= 0, \\
[\bar{\rho}(\bar{e} + \frac{\bar{u}^2}{2})]_t + [\bar{\rho}\bar{u}(\bar{e} + \frac{\bar{u}^2}{2}) + \bar{u}\bar{p}]_x &= 0,
\end{align*}
\]

(2.11)

where

\[
\bar{p} = R\bar{\rho}\bar{\theta} = A\bar{\rho}\gamma \exp\left(\frac{\gamma - 1}{R}\bar{S}\right), \quad \text{and} \quad \bar{e} = \frac{R}{\gamma - 1}\bar{\theta}.
\]

The properties of the approximate rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})\) is listed without proof in the following Lemma.

**Lemma 2.3.** The approximate cut-off 3-rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})\) defined in (2.10) satisfies the following properties:

(i) \( \bar{u}_x(x, t) = \frac{2}{\gamma + 1}(w^*_0)_x > 0 \), for \( x \in \mathbb{R}, \ t \geq 0 \),

\[
\bar{\rho}_x = \frac{1}{\gamma(\gamma - 1)e^{\bar{\rho}}} \bar{p}^{\frac{3-\gamma}{2}} \bar{u}_x, \quad \text{and} \quad \bar{\rho}_{xx} = \frac{1}{\gamma(\gamma - 1)e^{\bar{\rho}}} \bar{p}^{\frac{3-\gamma}{2}} \bar{u}_{xx} + \frac{3-\gamma}{\gamma(\gamma - 1)e^{\bar{\rho}}} \bar{p}^{\frac{3-\gamma}{2}} \bar{u}_x^2,
\]

\[
\bar{\theta}_x = \sqrt{\frac{2-\gamma}{\gamma}} \bar{\theta}^{\frac{1}{2}} \bar{u}_x, \quad \text{and} \quad \bar{\theta}_{xx} = \frac{2-\gamma}{\gamma} \bar{\theta}^{\frac{1}{2}} \bar{u}_{xx} + \frac{2}{\gamma} \bar{\theta}^{\frac{1}{2}} \bar{u}_x^2.
\]

(ii) The following estimates hold for all \( \ t > 0, \ \delta > 0 \) and \( p \in [1, \infty] \):

\[
\|\bar{u}_x(\cdot, t)\|_{L^p} \leq C(w_+ - w_-)^{1/p}(\delta + t)^{-1+1/p},
\]

\[
\|\bar{u}_{xx}(\cdot, t)\|_{L^p} \leq C(\delta + t)^{-1}\delta^{-1+1/p}.
\]

(iii) There exist a constant \( \delta_0 \in (0, 1) \) such that for \( \delta \in (0, \delta_0], \ t > 0, \)

\[
\| (\bar{\rho} - \rho^*_0, \bar{u} - u^*_0, \bar{\theta} - \theta^*_0)(\cdot, t) \|_{L^\infty} \leq C\delta t^{-1} [\ln(1 + t) + |\ln \delta|].
\]

The proof of Lemma 2.3 can be got similarly as in [6] and will be omitted for brevity.

### 3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we construct the global smooth solution \((\rho^\epsilon, u^\epsilon, \theta^\epsilon)\) as the perturbation around the approximate rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})\). Consider the Cauchy problem (1.1) with the smooth initial data

\[
(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(x, t = 0) = (\bar{\rho}, \bar{u}, \bar{\theta})(x, 0).
\]

(3.1)
Then we introduce the perturbation

\[(\phi, \psi, \zeta)(y, \tau) = (\rho^\epsilon, u^\epsilon, \theta^\epsilon)(x, t) - (\bar{\rho}, \bar{u}, \bar{\theta})(x, t), \quad (3.2)\]

where \(y, \tau\) are the scaled variables as

\[y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \quad (3.3)\]

and \((\rho^\epsilon, u^\epsilon, \theta^\epsilon)\) is assumed to be the solution to the problem (1.1). For the simplicity of the notation, we will omit the superscription of \((\rho^\epsilon, u^\epsilon, \theta^\epsilon)\) as \((\rho, u, \theta)\) from now on if there is no confusion of the notation. Substituting (3.2) and (3.3) into the system (1.1) and using the equations for \((\bar{\rho}, \bar{u}, \bar{\theta}),\) one can obtain

\[
\left\{
\begin{array}{l}
\phi_\tau + \rho \psi_y + u \phi_y = -f, \\
\rho \psi_\tau + \rho u \psi_y + (\gamma - 1)(\theta \phi_y + \rho \zeta_y) - \mu(\bar{\theta}) \psi_{yy} = -g + \mu(\bar{\theta}) u_y + ((\mu(\bar{\theta}) - \mu(\bar{\theta})) u_y)_y, \\
\rho \zeta_\tau + \rho u \zeta_y + (\gamma - 1)\rho \theta \psi_y - \kappa(\bar{\theta}) \zeta_{yy} = -h + \kappa(\bar{\theta}) u_y + ((\kappa(\bar{\theta}) - \kappa(\bar{\theta})) \theta_y)_y + \mu(\bar{\theta}) u_y^2,
\end{array}
\right. \quad (3.4)
\]

with the initial data

\[(\phi, \psi, \zeta)(y, 0) = 0, \quad (3.5)\]

where

\[
\left\{
\begin{array}{l}
f = \bar{u}_y \phi + \bar{\rho}_y \psi, \\
g = -\mu(\bar{\theta}) \bar{u}_y + \rho \psi \bar{u}_y + (\gamma - 1)(\bar{\rho}_y \phi - \bar{\rho}_\gamma \phi),
\end{array}
\right. \quad (3.6)
\]

We seek a global-in-time solution \((\phi, \psi, \zeta)\) to the problem (3.4) – (3.6). To this end, the solution space for (3.4) – (3.6) is defined by

\[
\chi(0, \tau_1(\epsilon)) = \{(\phi, \psi, \zeta) | (\phi, \psi, \zeta) \in C([0, \tau_1(\epsilon)]; H^1(\mathbb{R})), \quad \phi_y \in L^2(0, \tau_1(\epsilon); L^2(\mathbb{R})),
\]

\[
\psi_y, \zeta_y \in L^2(0, \tau_1(\epsilon); H^1(\mathbb{R}))\}
\]

with \(0 < \tau_1(\epsilon) \leq +\infty\).

**Theorem 3.1.** There exist positive constants \(\epsilon_1\) and \(C\) independent of \(\epsilon\), such that if \(0 < \epsilon \leq \epsilon_1\), then the problem (3.4) – (3.5) admits a unique global-in-time solution \((\phi, \psi, \zeta) \in \chi(0, +\infty)\) satisfying

\[
\sup_{\tau \in [0, +\infty)} \int_{\mathbb{R}} \left( \bar{\rho}^{\gamma - 2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{2 - \gamma} \zeta^2 \right)(\tau, y) dy
\]

\[+ \int_0^{+\infty} \int_{\mathbb{R}} \left[ \bar{u}_y \left( \bar{\rho}^{\gamma - 2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{2 - \gamma} \zeta^2 \right) + \bar{\theta}^{\alpha - 1} \zeta_y \phi_y^2 \right] dy d\tau \leq \epsilon^{\frac{1}{2}}, \quad (3.7)
\]

\[
\sup_{\tau \in [0, +\infty)} \int_{\mathbb{R}} \frac{\bar{\theta}^{\alpha + 1}}{\rho^2} \phi_y^2 dy + \int_0^{+\infty} \int_{\mathbb{R}} \frac{\bar{\theta}^{\alpha + 1}}{\rho^2} \phi_y^2 d\tau \leq \epsilon^{\frac{1}{3} - 3\gamma} |\ln \epsilon|^{-3\gamma}, \quad (3.8)
\]
and
\[
\sup_{\tau \in [0, +\infty)} \int_{\mathbb{R}} (\psi_y^2 + \zeta_y^2) \, dy + \int_0^{+\infty} \int_{\mathbb{R}} \left[ \bar{u}_y(\psi_y^2 + \zeta_y^2) + \frac{\bar{\theta}^a}{\bar{\rho}} (\psi_{yy}^2 + \zeta_{yy}^2) \right] \, dy \, d\tau \leq \epsilon^{\frac{1}{2}}, \tag{3.9}
\]
where \( a \) is given by (1.14).

In what follows, the analysis is always carried out under the a priori assumptions
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \| \phi(\cdot, \tau) \|_{L^\infty} \leq \epsilon^a, \quad \sup_{\tau \in [0, \tau_1(\epsilon)]} \| \zeta(\cdot, \tau) \|_{L^\infty} \leq \epsilon^{(\gamma - 1)a}, \tag{3.10}
\]
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \| \psi(\cdot, \tau) \|_{L^\infty} \leq \epsilon^a, \quad \sup_{\tau \in [0, \tau_1(\epsilon)]} \| (\psi_y, \zeta_y)(\cdot, \tau) \| \leq 1, \tag{3.11}
\]
where \( a \) is given by (1.14), \([0, \tau_1(\epsilon)]\) is the time interval in which the solution exists and \( \tau_1(\epsilon) \) may depend on \( \epsilon \).

Take
\[
\nu = \epsilon^a |\ln \epsilon|, \quad \delta = \epsilon^a, \tag{3.12}
\]
in the sequel. Then it follows that \( \nu \geq C \epsilon^a \) with \( C \geq \max\{2, (2e^{-S})^{\frac{1}{\gamma - 1}}\} \) if \( \epsilon \ll 1 \). Under the a priori assumption (3.10), one can get
\[
\frac{\bar{\rho}}{2} \leq \rho \leq \frac{3\bar{\rho}}{2}, \quad \text{and} \quad \frac{\bar{\theta}}{2} \leq \theta \leq \frac{3\bar{\theta}}{2}. \tag{3.13}
\]

In fact, if \( \epsilon \ll 1 \), then one has
\[
\rho = \bar{\rho} + \phi \geq \bar{\rho} - \| \phi \|_{L^\infty} \geq \bar{\rho} - \epsilon^a \geq \bar{\rho} - \frac{1}{2} \nu \geq \frac{\bar{\rho}}{2},
\]
and
\[
\rho = \bar{\rho} + \phi \leq \bar{\rho} + \| \phi \|_{L^\infty} \leq \bar{\rho} + \epsilon^a \leq \bar{\rho} + \frac{1}{2} \nu \leq \frac{3\bar{\rho}}{2}.
\]

Similarly, note that \( \bar{\theta} = \bar{\rho}^{\gamma - 1} \epsilon^S \geq \nu^{\gamma - 1} \epsilon^S \) by the definition of the rarefaction wave profile defined in (2.10), it holds that
\[
\theta = \bar{\theta} + \zeta \geq \bar{\theta} - \| \zeta \|_{L^\infty} \geq \bar{\theta} - \epsilon^a (\gamma - 1) \geq \bar{\theta} - \frac{\epsilon^S}{2} \nu^{\gamma - 1} \geq \bar{\theta} - \frac{\bar{\theta}}{2} = \frac{\bar{\theta}}{2},
\]
and
\[
\theta = \bar{\theta} + \zeta \leq \bar{\theta} + \| \zeta \|_{L^\infty} \leq \bar{\theta} + \epsilon^a (\gamma - 1) \leq \bar{\theta} + \frac{\epsilon^S}{2} \nu^{\gamma - 1} \leq \bar{\theta} + \frac{\bar{\theta}}{2} = \frac{3\bar{\theta}}{2}.
\]

Since the proof for the local existence of the solution to (3.4) – (3.6) is standard, we omit it for brevity. Note that in order to get the convergence rate of the local solution with respect to \( \epsilon \) as in (3.10), the local existence time interval, denoted by \([0, \tau_0]\) where \( \tau_0 \) may depend on \( \epsilon \), that is, \( \tau_0 = \tau_0(\epsilon) \). The next step for the proof of Theorem 3.1 is to extend the local solution to the global solution in \([0, \infty)\) for small but fixed viscosity coefficient and heat conduction coefficient \( \epsilon \). To do so, it is sufficient to show the following a priori estimates for fixed \( \epsilon \) with \( 0 < \epsilon \ll 1 \).
Lemma 3.2. (A priori estimates) Let \((\phi, \psi, \zeta) \in \chi(0, \tau_1(\epsilon))\) be a solution to the problem (3.4) – (3.6), where \(\tau_1(\epsilon)\) is the maximum existence time of the solution satisfying a priori assumptions (3.10)-(3.11). Then there exists a positive constant \(\epsilon_2\) such that if \(0 < \epsilon \leq \epsilon_2\), then

\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \left( \bar{\rho}^{-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{-2} \zeta^2 \right)(\tau, y) dy + \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \left[ \bar{u}_y \left( \bar{\rho}^{-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{-2} \zeta^2 \right) + \bar{\theta} \psi_y^2 + \bar{\theta}^{-1} \zeta_y^2 \right] dy d\tau \leq \epsilon^3, \tag{3.14}
\]

and

\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \bar{\rho}^{2\alpha} \phi_y^2 dy + \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \bar{\theta}^{\alpha+1} \phi_y^2 dy d\tau \leq \epsilon^{\frac{1}{2} - 3\alpha\gamma} |\ln \epsilon|^{-3\gamma}, \tag{3.15}
\]

and

\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \left( \psi_y^2 + \zeta_y^2 \right) dy + \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \left[ \bar{v}_y \left( \psi_y^2 + \zeta_y^2 \right) + \frac{\bar{\theta}}{\rho^{2\alpha}} \left( \psi_y^2 + \zeta_y^2 \right) \right] dy d\tau \leq \epsilon^{\frac{3}{4}}. \tag{3.16}
\]

Proof of Lemma 3.2: The proof of Lemma 3.2 consists of the following steps.

Step 1. First, as in [16], one can define the entropy-entropy flux pair \((\eta, q)\) as

\[
\begin{align*}
\eta &= -\bar{\theta} \{ \rho S - \bar{\rho} \bar{S} - \nabla x (\rho S)|_{x = \bar{X}} \cdot (X - \bar{X}) \}, \\
q &= -\bar{\theta} \{ \rho u S - \bar{\rho} \bar{u} \bar{S} - \nabla x (\rho S)|_{x = \bar{X}} \cdot (Y - \bar{Y}) \},
\end{align*} \tag{3.17}
\]

where

\[
\begin{align*}
X &= \left( \rho, \rho u, \rho \left( \frac{1}{2} |u|^2 + \theta \right) \right)^t, \\
Y &= \left( \rho u, \rho u^2 + (\gamma - 1) \rho \theta, \rho \left( \frac{1}{2} |u|^2 + \gamma \theta \right) \right)^t.
\end{align*} \tag{3.18}
\]

Since

\[
\begin{align*}
(\rho S)_p &= S + \frac{|u|^2}{2\bar{\theta}} - \gamma, \\
(\rho S)_m &= -\frac{u}{\bar{\theta}}, \\
(\rho S)_E &= \frac{1}{\bar{\theta}},
\end{align*} \tag{3.19}
\]

with \(m = \rho u\) and \(E = \rho (\theta + \frac{|u|^2}{2})\), we get

\[
\begin{align*}
\eta &= \rho \theta - \bar{\theta} \rho S + \rho \left( [S - \gamma] \bar{\theta} + \frac{1}{2} |u - \bar{u}|^2 \right) + (\gamma - 1) \bar{\rho} \bar{\theta}, \\
\quad &= (\gamma - 1) \rho \bar{\theta} \Phi \left( \frac{\bar{\rho}}{\rho} \right) + \frac{\rho \psi^2}{2} + \bar{\rho} \Phi \left( \frac{\theta}{\bar{\theta}} \right), \\
q &= w \eta + (\gamma - 1) (u - \bar{u}) (\rho \theta - \bar{\rho} \bar{\theta}),
\end{align*} \tag{3.20}
\]

where

\[
\Phi(\eta) = \eta - \ln \eta - 1. \tag{3.21}
\]

Direct computations yield

\[
\eta_r + q_y + \bar{u}_y H = \psi (\mu(\theta) u_y)_y + \frac{\zeta}{\bar{\theta}} \mu(\theta) u_y^2 + \frac{\zeta}{\bar{\theta}} (\kappa(\theta) \theta_y)_y, \tag{3.22}
\]

13
where

\[ H = \rho(u - \bar{u})^2 + (\gamma - 1)\rho\bar{\theta}\left(\frac{\theta}{\bar{\theta}}\right) + (\gamma - 1)^2\rho\bar{\theta}\left(\frac{\rho}{\bar{\rho}}\right) + \sqrt{\frac{\gamma - 1}{\gamma}}\bar{\theta}\frac{1}{2}\rho(u - \bar{u})\left((\gamma - 1)\log\frac{\bar{\rho}}{\rho} + \log\frac{\theta}{\bar{\theta}}\right) \]

\[ \geq (1 - \beta)\rho(u - \bar{u})^2 + (\gamma - 1)\rho\bar{\theta}\left[\Phi\left(\frac{\theta}{\bar{\theta}}\right) + (\gamma - 1)\Phi\left(\frac{\rho}{\bar{\rho}}\right) - \frac{1}{4\beta\gamma}\left((\gamma - 1)\log\frac{\bar{\rho}}{\rho} + \log\frac{\theta}{\bar{\theta}}\right)\right] \]

(3.23)

with \(0 < \beta < 1\) is the positive constant to be determined. Let

\[ x_1 = \frac{\theta}{\bar{\theta}}, \quad x_2 = \frac{\rho}{\bar{\rho}}, \]

under the a priori assumptions (3.10), one has \(x_1, x_2 \sim 1\) as \(\epsilon \to 0\). Consider the following function

\[ f^\beta(x_1, x_2) = x_1 - \log x_1 - 1 + (\gamma - 1)(x_2 - \log x_2 - 1) - \frac{1}{4\beta\gamma}\left((\gamma - 1)\log x_2 + \log x_1\right)^2. \]

It is easy to check that

\[ f^\beta(1, 1) = f^\beta_{x_1}(1, 1) = f^\beta_{x_2}(1, 1) = 0, \]

and the Hessian matrix of \(f^\beta\) at point (1,1) is

\[
\nabla^2 f^\beta(1, 1) = \begin{pmatrix}
1 - \frac{1}{2\beta\gamma} & -\frac{\gamma - 1}{2\beta\gamma} \\
-\frac{\gamma - 1}{2\beta\gamma} & (\gamma - 1)(1 - \frac{\gamma - 1}{2\beta\gamma})
\end{pmatrix}.
\]

Thus the determinant of \(\nabla^2 f^\beta(1, 1)\) is

\[ \det \nabla^2 f^\beta(1, 1) = (\gamma - 1)(1 - \frac{1}{2\beta}). \]

Take \(\beta = \frac{3}{4}\), it is easy to see that \(\nabla^2 f^\beta(1, 1)\) is definitely positive near the point (1, 1). So one has

\[ f^\beta(x_1, x_2) \geq C_\beta(x_1^2 + x_2^2), \quad \text{as} \ x_1, x_2 \sim 1. \]

Therefore, under the a priori assumptions (3.10) and take \(\beta = \frac{3}{4}\) in (3.23), one can get

\[ H \geq C \left[\bar{\rho}\psi^2 + \frac{\bar{\theta}}{\bar{\rho}}\phi^2 + \frac{\bar{\rho}}{\bar{\theta}}\zeta^2\right]. \]

(3.24)

Similarly, by the facts \(\Phi(1) = \Phi'(1) = 0\), and \(\Phi''(1) = 1 > 0\), one has

\[ \eta \geq C \left[\bar{\rho}\psi^2 + \frac{\bar{\theta}}{\bar{\rho}}\phi^2 + \frac{\bar{\rho}}{\bar{\theta}}\zeta^2\right] \]
under a priori assumptions (3.10). The right hand-side of (3.22) can be rewritten as

$$[\mu(\theta)\psi\psi_y + \frac{\kappa(\theta)}{\theta} \xi_y]_y - \mu(\theta)\frac{\partial}{\partial \theta} \psi_y^2 - \frac{\kappa(\theta)}{\theta^2} \psi_y^2 + \mu(\theta)\psi\mu_y + \frac{\kappa(\theta)}{\theta} \xi_y$$

$$+ \mu(\theta)\xi(2\psi\mu_y + \mu_y^2) + \frac{\kappa(\theta)}{\theta^2} \xi_y \xi_y + \psi \mu(\theta)\mu_y + \frac{1}{\theta} \kappa(\theta)\xi_y \xi_y.$$  

(3.25)

Then integrating the equation (3.22) over $\mathbb{R}^1 \times [0, \tau]$ and using (3.13), (3.22)-(3.25) imply

$$\int_0^\tau \left( \rho^{\gamma-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{2-\gamma} \xi^2 \right)(\tau, y) dy$$

$$+ \int_0^\tau \int_0^{\bar{\tau}} \left[ \bar{u}_y \left( \rho^{\gamma-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{2-\gamma} \xi^2 \right) + \bar{\theta}^{\alpha-1} \psi^2 + \bar{\theta}^{\alpha-1} \xi^2 \right] d\tau dy$$

$$\leq C \int_0^\tau \int_0^{\bar{\tau}} |\bar{\theta}^{\alpha-1} \xi| |(\bar{\theta}_{yy}, \bar{u}_y^2)| + |\bar{\theta}^{\alpha-1} \psi \bar{u}_{yy}| + |\bar{\theta}^{\alpha-1} \psi \bar{u}_y^2| + |\bar{u}_y(\bar{\theta}^{\alpha-1} \xi \psi_y, \bar{\theta}^{\alpha-1} \xi \psi_y, \bar{\theta}^{\alpha-1} \xi \psi_y)| dyd\tau$$

$$:= \sum_{i=1}^4 I_i.$$  

(3.26)

By Sobolev inequality and Lemma 2.3, we can obtain

$$I_1 = \int_0^\tau \int_0^{\bar{\tau}} |\bar{\theta}^{\alpha-1} \xi| |(\bar{\theta}_{yy}, \bar{u}_y^2)|| dyd\tau \leq C \nu^{1-\gamma} \int_0^\tau \| (\bar{\theta}_{yy}, \bar{u}_y^2) \|_{L^1} \| \xi \|_{1/2} \| \psi \|_{1/2} d\tau$$

$$\leq C \nu^{\frac{-(4+\alpha)(\gamma-1)}{4}} \int_0^\tau \left( \frac{1}{\tau + \delta/\epsilon} \right)^{\frac{1}{2}} \| \xi \|_{1/2} \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_{1/2} d\tau$$

$$\leq \frac{1}{16} \int_0^\tau \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 d\tau + C \nu^{\frac{-(4+\alpha)(\gamma-1)}{3}} \int_0^\tau \| \xi \| \left( \frac{1}{\tau + \delta/\epsilon} \right)^{\frac{1}{2}} d\tau$$

$$\leq \frac{1}{16} \int_0^\tau \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 d\tau + C \nu^{\frac{-(4+\alpha)(\gamma-1)+1}{3}} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 + C \nu^{\frac{-(4+\alpha)(\gamma-1)+1}{2}} \left( \frac{\epsilon}{\delta} \right)^{1/2}$$

$$\leq \frac{1}{16} \int_0^\tau \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 d\tau + \frac{1}{16} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 + C \nu^{\frac{-(4+\alpha)(\gamma-1)+1}{2}} \left( \frac{\epsilon}{\delta} \right)^{1/2}$$

$$\leq \frac{1}{16} \int_0^\tau \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 d\tau + \frac{1}{16} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \sqrt{\bar{\theta}^{\alpha-1} \xi_y} \|_2^2 + \frac{1}{3},$$

where we used the fact that

$$C \nu^{\frac{-(4+\alpha)(\gamma-1)+1}{2}} \left( \frac{\epsilon}{\delta} \right)^{1/2} = C \epsilon \frac{1-\alpha(4+\alpha)(\gamma-1))}{2} \ln \left| \frac{\epsilon}{\delta} \right| \leq \epsilon^{1/4}, \quad \text{if} \quad \epsilon \ll 1.$$
Similarly, it holds that

\[
I_2 = \int_0^\tau \int_{\mathbb{R}} |\hat{\Theta}^a \psi \bar{u}_y| dy d\tau \leq C \int_0^\tau \|\bar{u}_y\|_{L^1} \|\psi\|^{1/2} \|\psi_y\|^{1/2} d\tau
\]

\[
\leq C \nu^{-\frac{\alpha(\gamma-1)}{4}} \int_0^\tau \frac{1}{\tau + \delta/\epsilon} \|\psi\|^{1/2} \|\sqrt{\Theta^a} \psi_y\|^{1/2} d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \|\sqrt{\Theta^a} \psi_y\|^2 d\tau + C \nu^{-\frac{\alpha(\gamma-1)}{3}} \int_0^\tau \left( \frac{1}{\tau + \delta/\epsilon} \right)^{\frac{3}{4}} \|\psi\|^{2/3} d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \|\sqrt{\Theta^a} \psi_y\|^2 d\tau + \frac{1}{16} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \|\sqrt{\rho \psi}\|^2 + C \nu^{-\frac{\alpha(\gamma-1)+1}{2}} \left( \frac{\epsilon}{\delta} \right)^{1/2}
\]

\[
\leq \frac{1}{16} \int_0^\tau \|\sqrt{\Theta^a} \psi_y\|^2 d\tau + \frac{1}{16} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \|\sqrt{\rho \psi}\|^2 + \epsilon^4,
\]

where we used the fact that

\[
C \nu^{-\frac{\alpha(\gamma-1)+1}{2}} (\frac{\epsilon}{\delta})^{1/2} = C e^{1-\frac{\alpha(\gamma-1)+2}{2}} \|\ln \epsilon\|^{-\frac{\alpha(\gamma-1)+1}{2}} \leq \epsilon^4, \quad \text{if } \epsilon \ll 1.
\]

By using Lemma 2.3 yields

\[
I_3 = \int_0^\tau \int_{\mathbb{R}} |\hat{\Theta}^{-1/2} \psi \bar{u}_y|^2 dy d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \bar{u}_y \bar{\rho} \psi^2 dy d\tau + C \nu^{-\gamma} \int_0^\tau \|\bar{u}_y\|^3 d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \bar{u}_y \bar{\rho} \psi^2 dy d\tau + C \nu^{-\gamma} \int_0^\tau \left( \frac{1}{\tau + \delta/\epsilon} \right)^2 d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \bar{u}_y \bar{\rho} \psi^2 dy d\tau + C \nu^{-\gamma} (\frac{\epsilon}{\delta}).
\]

And

\[
I_4 \leq C \int_0^\tau \int_{\mathbb{R}} \bar{\rho}^{-\frac{1}{2}} \bar{\Theta}^{-\frac{1}{2}} |\bar{u}_y| (\sqrt{\bar{\rho}^{-\gamma} \zeta_y} |\bar{\Theta}^2 \psi_y|, \sqrt{\bar{\rho}^{-\gamma} \zeta_y} |\bar{\Theta}^{-\frac{1}{2}} \zeta_y|, \sqrt{\bar{\rho} \psi} |\bar{\Theta}^{-\frac{1}{2}} \zeta_y|) dy d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \left( \bar{\Theta}^a \psi^2_y + \bar{\Theta}^{-1} \zeta_y^2 \right) dy d\tau + C \frac{\epsilon}{\nu \gamma \delta} \int_0^\tau \int_{\mathbb{R}} \bar{u}_y \left( \bar{\rho}^{-\gamma} \zeta^2 + \bar{\rho} \psi^2 \right) dy d\tau
\]

\[
\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \left( \bar{\Theta}^a \psi^2_y + \bar{\Theta}^{-1} \zeta_y^2 + \bar{u}_y \bar{\rho}^{-\gamma} \zeta^2 + \bar{u}_y \bar{\rho} \psi^2 \right) dy d\tau,
\]

where we used the fact that

\[
C \frac{\epsilon}{\nu \gamma \delta} = C e^{1-a-\alpha} |\ln \epsilon|^{-\gamma} \leq C \epsilon^\frac{1}{2} |\ln \epsilon|^{-\gamma} \leq \frac{1}{16}, \quad \text{if } \epsilon \ll 1.
\]

Combining (3.27)-(3.30) and recalling (3.12) yield that

\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \left( \bar{\rho}^{-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{-\gamma} \zeta^2 \right) (\tau, y) dy
\]

\[
+ \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \left[ \bar{u}_y \left( \bar{\rho}^{-2} \phi^2 + \bar{\rho} \psi^2 + \bar{\rho}^{-\gamma} \zeta^2 \right) + \bar{\Theta}^a \psi^2 + \bar{\Theta}^{-1} \zeta^2 \right] dy d\tau \leq \epsilon^4.
\]
Step 2. Next we derive the estimation of $\phi_y$. Differentiating (3.4) with respect to $y$ and then multiplying the resulted equation by $\mu(\bar{\theta})\phi_y/\rho^3$ imply that

$$
(\mu(\bar{\theta})\frac{\phi_y^2}{2\rho^3})_\tau + (\mu(\bar{\theta})\frac{u\phi_y^2}{2\rho^3})_y + \mu(\bar{\theta})\frac{\psi_{yy}\phi_y}{\rho^2} = -\mu(\bar{\theta})\frac{\phi_y}{\rho^3}(\bar{u}_{yy}\phi + \bar{\rho}_{yy}\psi + 2\bar{\rho}_y\psi_y) + \frac{\phi_y^2}{2\rho^3}(\mu(\bar{\theta})_\tau + \mu(\bar{\theta})_y u).$$

(3.32)

Multiplying (3.4) by $\mu(\bar{\theta})\phi_y/\rho^2$ gives

$$
(\mu(\bar{\theta})\frac{\psi\phi_y}{\rho})_\tau - (\mu(\bar{\theta})\frac{\psi\phi_y}{\rho})_y - \mu(\bar{\theta})\phi_y^2 + (\gamma - 1) \left( \frac{\mu(\bar{\theta})\theta\phi_y^2}{\rho^2} + \mu(\bar{\theta})\frac{\phi_y\zeta y}{\rho^2} \right) - \mu(\bar{\theta})\psi_{yy}\phi_y + \mu(\bar{\theta})\psi_{y}\phi_y \right) = \mu(\bar{\theta})\mu(\bar{\theta}) \frac{u_y\phi_y}{\rho^2} \\
+ \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho}.

(3.33)

Combining (3.32) and (3.33) together, then integrating the resulted equation over $\mathbb{R}^1 \times [0, \tau]$ imply

$$
\int_\mathbb{R} \left( \mu^2(\bar{\theta})\frac{\phi_y^2}{2\rho^3} + \mu(\bar{\theta})\frac{\psi\phi_y}{\rho} \right) dy + (\gamma - 1) \int_0^\tau \int_\mathbb{R} \left( \mu(\bar{\theta})\frac{\theta\phi_y^2}{\rho^2} + \mu(\bar{\theta})\frac{\phi_y\zeta y}{\rho^2} \right) dy d\tau \\
= \int_0^\tau \int_\mathbb{R} \left\{ \mu(\bar{\theta})\psi_{y}^2 + \mu(\bar{\theta}) \left( \bar{u}_{yy}\phi + \bar{\rho}_{yy}\psi + 2\bar{\rho}_y\psi_y \right) - \mu(\bar{\theta})g\frac{\phi_y}{\rho^2} \right\} dy d\tau \\
+ \int_0^\tau \int_\mathbb{R} \left\{ \frac{\phi_y^2}{2\rho^2} \left( \mu^2(\bar{\theta})_\tau + \mu(\bar{\theta})_y u \right) + \mu(\bar{\theta})\mu(\bar{\theta}) \frac{u_y\phi_y}{\rho^2} + \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} - \mu(\bar{\theta}) \frac{\psi\phi_y}{\rho} \\
- \mu(\bar{\theta}) \bar{\rho}_y \frac{\psi\phi_y}{\rho^2} \left( \bar{\phi}_y + \bar{\rho}_y \right) + \left( \mu(\bar{\theta}) - \mu(\bar{\theta}) \right) \frac{u_y\phi_y}{\rho^2} \right\} dy d\tau.

(3.34)

Combining (3.31) with (3.34), and using the equation (3.4)_1 and the fact that

$$
\left( (\mu(\bar{\theta}) - \mu(\bar{\theta})) u_y \right)_y = \left( (\mu(\bar{\theta}) - \mu(\bar{\theta})) \psi_{yy} + \alpha(\theta^{-1}\psi_y - \bar{\theta}\bar{\psi}_y) \right) = \left( (\mu(\bar{\theta}) - \mu(\bar{\theta})) \psi_{yy} + \alpha \left[ \theta^{-1}\psi_y + \bar{\psi}_y \bar{u}_y \right] \right) + \left( \theta^{-1} - \bar{\theta} \right) \bar{\psi}_y \bar{\psi}_y + (\theta^{-1} - \bar{\theta}_y) \bar{\psi}_y \bar{u}_y,

(3.35)
it holds that
\[
\int_{\mathbb{R}} \left( \mu^2(\theta) \frac{\phi_y^2}{\rho^2} + \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 + \tilde{\rho}^2 - \gamma^2 \right)(\tau, y) dy
\]
\[+ \int_0^\tau \int_{\mathbb{R}} \left[ \tilde{u}_y \left( \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 + \tilde{\rho}^2 - \gamma^2 \right) + \tilde{\theta} \alpha \psi_y^2 + \tilde{\theta}^{\alpha - 1} \gamma^2 + \frac{\tilde{\theta}^{\alpha + 1}}{\rho^2} \phi_y^2 \right] dy d\tau \]
\[\leq C \left| \int_0^\tau \int_{\mathbb{R}} \left( \mu(\theta) \tilde{u}_y \frac{\psi_y \phi}{\rho} + \mu(\theta) \tilde{u}_y \frac{\psi \phi_y}{\rho^2} - 2 \mu^2(\theta) \frac{\phi_y \psi_y}{\rho^3} \psi \right) dy d\tau \right|
\]
\[+ C \left| \int_0^\tau \int_{\mathbb{R}} \mu^2(\theta) \frac{\phi_y \psi_y}{\rho^2} dy d\tau \right| + C \left| \int_0^\tau \int_{\mathbb{R}} \frac{\phi_y^2}{2 \rho^2} \left( \mu^2(\theta) + \mu^2(\tilde{\theta}) \right) u dy d\tau \right|
\]
\[+ C \left| \int_0^\tau \int_{\mathbb{R}} \left\{ \mu(\theta) \tilde{u}_y \frac{\psi \phi_y}{\rho^2} + \mu(\theta) \frac{\psi \phi_y}{\rho^2} \right\} \psi \left( \mu(\theta) - \mu(\tilde{\theta}) \right) \tilde{u}_y \phi_y + \tilde{\theta} \phi_y \right| dy d\tau \]
\[+ C \left| \int_0^\tau \int_{\mathbb{R}} \left( \mu(\theta) \frac{\phi_y \psi_y}{\rho^2} - \mu(\theta) \frac{\phi_y \psi_y}{\rho^2} \right) \frac{\phi_y}{\rho} dy d\tau \right|
\]
\[+ C \left| \int_0^\tau \int_{\mathbb{R}} \mu(\theta) \frac{\phi_y \psi_y}{\rho^2} dy d\tau \right| + C \left| \int_0^\tau \int_{\mathbb{R}} \theta^{\alpha - 1} \gamma^2 \psi \left( \mu(\theta) - \mu(\tilde{\theta}) \right) \phi_y \right| dy d\tau \]
\[+ C \left| \int_0^\tau \int_{\mathbb{R}} \theta^{\alpha - 1} \gamma^2 \psi \phi_y \psi_y dy d\tau \right| + C \left| \int_0^\tau \int_{\mathbb{R}} \tilde{u}_y \left( \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 \right) dy d\tau \right|
\]
\[+ C \left| \int_{\mathbb{R}} \left( \phi_y \frac{\phi_{yy}}{\rho^2} \right) \left( \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 \right) dy \right| + C \left| \int_0^\tau \int_{\mathbb{R}} \tilde{u}_y \left( \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 \right) dy d\tau \right|
\]
\[+ C e^{\frac{\gamma}{2}} := \sum_{i=1}^9 J_i + C e^{\frac{\gamma}{2}}.
\]

The terms on the right-hand side of (3.36) will be estimated one by one as follows. By Lemma 2.3, (3.13), and Cauchy inequality, it holds that
\[
J_1 \leq C \left| \int_0^\tau \int_{\mathbb{R}} \left\{ \tilde{\rho}^{\alpha(\gamma - 1) - \gamma} \tilde{u}_y \left( ||\sqrt{\tilde{\theta} \alpha} \psi_y|| \sqrt{\rho}^{-2} \phi, ||\sqrt{\tilde{\theta}^{\alpha + 1}} \phi_y|| \sqrt{\rho} \right) \right\} dy d\tau \right|
\]
\[+ \rho^{\alpha(\gamma - 1) - \gamma} \tilde{u}_y \left( ||\sqrt{\tilde{\theta} \alpha} \psi_y|| \sqrt{\rho}^{-2} \phi, ||\sqrt{\tilde{\theta}^{\alpha + 1}} \phi_y|| \sqrt{\rho} \right) \right\} dy d\tau \right|
\[\leq \frac{1}{16} \int_0^\tau \int_{\mathbb{R}} \left( \tilde{\rho}^{\alpha + 1} \phi_y^2 + \tilde{\theta} \alpha \psi_y^2 \right) dy d\tau + C \frac{\epsilon}{\sqrt{\gamma} \delta} \int_0^\tau \int_{\mathbb{R}} \tilde{u}_y \left( \tilde{\rho}^{-2} \phi^2 + \tilde{\rho} \psi^2 \right) dy d\tau,
\]
Recalling (2.7) from Lemma 2.1 and the fact (i) in Lemma 2.3, one can arrive at
\[
|\tilde{\rho}_{xx}| \leq C \left( \rho^{\frac{\gamma - 1}{\delta}} \tilde{\rho}_x + \tilde{\rho}^{2 - \gamma} \tilde{\rho}_x^2 \right).
\]
Thus one has

\begin{align}
J_2 & \leq C \left| \int_0^\tau \int_\mathbb{R} \rho \frac{2\alpha(\gamma-1)-2\gamma}{\delta} \bar{u}_y \sqrt{\frac{\theta^{\alpha+1}}{\rho^2}} \phi_y \left( |\sqrt{\rho^{\gamma-2}} \phi | + |\sqrt{\rho^2} \psi | + |\sqrt{\rho^{2-\gamma}} \zeta | \right) dyd\tau \right| \\
& \leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y dyd\tau + C \left( \frac{\epsilon}{\nu^2 \delta} \right)^3 \int_0^\tau \int_\mathbb{R} \bar{u}_y (\rho^{\gamma-2} \phi^2 + \rho^2 \psi^2 + \rho^{2-\gamma} \zeta^2) dyd\tau. 
\end{align}

(3.39)

And

\begin{align}
J_3 = C \left| \int_0^\tau \int_\mathbb{R} \mu(\bar{\theta}) g \frac{\phi_y}{\rho^2} dyd\tau \right| & \leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y dyd\tau + C \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha-1}}{\rho^{2-\gamma}} \zeta^2 dyd\tau \tag{3.40}
\end{align}

Recalling (3.6), (3.13) and (i) in Lemma 2.3, one can get

\begin{align}
|g| & \leq C (\bar{\theta}^\alpha |\bar{u}_{yy}| + |\bar{\rho} \bar{u}_y \psi| + |\bar{\rho}_y \bar{\zeta}| + |\bar{\rho}_y \bar{\rho}^{\gamma-2} \phi|) \\
& \leq C (\bar{\theta}^\alpha |\bar{u}_{yy}| + \bar{u}_y (|\bar{\rho} \psi| + |\bar{\rho}^{\frac{1}{2-\gamma}} \zeta| + |\bar{\rho}^{\frac{1}{2-\gamma}} \phi|)). 
\end{align}

(3.41)

Thus the last term in (3.40) can be estimated by

\begin{align}
\left| \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha-1}}{\rho^{2-\gamma}} \zeta^2 dyd\tau \right| & \leq C \nu^{-1-\gamma} \int_0^\tau \| \bar{u}_{yy} \|^2 d\tau + C \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha-1}}{\rho^{2-\gamma}} \bar{u}_y (\rho^{\gamma-2} \phi^2 + \rho^2 \psi^2 + \rho^{2-\gamma} \zeta^2) dyd\tau \\
& \leq C \nu^{-1-\gamma} \left( \frac{\epsilon}{\delta} \right)^2 + C \frac{\epsilon}{\nu^2 \delta} \int_0^\tau \int_\mathbb{R} \bar{u}_y (\rho^{\gamma-2} \phi^2 + \rho^2 \psi^2 + \rho^{2-\gamma} \zeta^2) dyd\tau. 
\end{align}

(3.42)

By Cauchy inequality, it holds that

\begin{align}
J_4 & \leq C \left| \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y \frac{\theta^{\alpha-\frac{1}{2}}}{\rho^{2-\gamma}} \bar{u}_y dyd\tau \right| \leq C \nu^{\frac{1+3\gamma}{2}} \frac{\epsilon}{\delta} \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y dyd\tau \\
& \leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y dyd\tau \text{ if } \epsilon \ll 1. 
\end{align}

(3.43)

Recalling (3.13) and (i) in Lemma 2.3, one can get

\begin{align}
J_5 & \leq C \left| \int_0^\tau \int_\mathbb{R} \left\{ \rho \frac{2\alpha(\gamma-1)-2\gamma}{\delta} \bar{u}_y \sqrt{\frac{\theta^{\alpha+1}}{\rho^2}} \phi_y \left| |\sqrt{\rho^{\gamma-2}} \phi | + |\sqrt{\rho^{\gamma-2}} \psi | + |\sqrt{\rho^{2-\gamma}} \zeta | \right| \\
& + \rho \frac{2\alpha(\gamma-1)-2\gamma}{\delta} \bar{u}_y \left| |\sqrt{\rho^{\gamma-2}} \phi | + |\sqrt{\rho^{\gamma-2}} \psi | + |\sqrt{\rho^{2-\gamma}} \zeta | \right| \right\} dyd\tau \right| \\
& \leq \left( \frac{1}{32} + C \frac{\epsilon}{\nu^2 \delta} \right) \int_0^\tau \int_\mathbb{R} \frac{\theta^{\alpha+1}}{\rho^2} \phi^2_y dyd\tau + C \frac{\epsilon}{\nu^2 \delta} \int_0^\tau \int_\mathbb{R} \left( \rho^{\gamma-2} \phi^2 + \rho^{2-\gamma} \zeta^2 + \bar{u}_y \rho \psi^2 + \bar{u}_y \rho^{2-\gamma} \zeta^2 \right) dyd\tau. 
\end{align}

(3.44)
Similarly, $J_6$ can be estimated as

$$
J_6 \leq C\left| \int_0^\tau \int_\mathbb{R} \tilde{\theta}^{\alpha /2-1} \frac{1}{\rho} \bar{u}_y |\sqrt{\rho} \psi| dy d\tau \right| \leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \nu^{1-2\gamma} \frac{\epsilon}{\delta} \int_0^\tau \int_\mathbb{R} \bar{u}_y \bar{\rho} \psi^2 dy d\tau.
$$

(3.45)

Recalling (3.13) and Lemma 2.3, one can get

$$
J_7 \leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \int_0^\tau \int_\mathbb{R} \bar{\theta}^{\alpha-2} \frac{1}{\rho} \bar{u}_y^4 dy d\tau \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \nu^{-2\gamma} \int_0^\tau \|\bar{u}_y\|^4_{L^4} d\tau \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C(\frac{\epsilon}{\nu^\gamma})^2.
$$

(3.46)

Recalling (3.11) and Cauchy inequality, it holds that

$$
J_8 = C\left| \int_0^\tau \int_\mathbb{R} \mu(\tilde{\theta}) \frac{\phi_y}{\rho^2} (\mu(\theta) - \mu(\tilde{\theta})) \psi_{yy} dy d\tau \right| \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha}}{\rho} \psi_{yy}^2 \bar{\theta}^{2\alpha-3} \frac{1}{\rho} dy d\tau \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \nu^{2-2\gamma} \sup_{[0,\tau]} \|\psi_{yy}\| \sup_{[0,\tau]} \|\zeta_{yy}\| \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha}}{\rho} \psi_{yy}^2 dy d\tau \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \nu^{1-3\gamma} \sup_{[0,\tau]} \|\sqrt{\rho^{2-2\gamma}} \zeta_{yy}\| \sup_{[0,\tau]} \|\zeta_{yy}\| \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha}}{\rho} \psi_{yy}^2 dy d\tau \\
\leq \frac{1}{16} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha+1}}{\rho^2} \phi_y^2 dy d\tau + C \nu^{1-3\gamma} \frac{1}{\bar{u}_y} \int_0^\tau \int_\mathbb{R} \frac{\tilde{\theta}^{\alpha}}{\rho} \psi_{yy}^2 dy d\tau.
$$

(3.47)
Similarly,

\[ J_y = C|\alpha| \int_0^T \int_\mathbb{R} \theta^{\alpha-1} \zeta_y \psi_y \mu(\theta) \frac{\phi_y}{\rho^2} dy d\tau \]

\[ \leq C \nu^{-\frac{2\gamma}{4}} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ \leq C \nu^{-\frac{2\gamma+\alpha(\gamma-1)}{4}} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} \nu^{-\frac{1}{2}} dy d\tau \]

\[ \leq \nu^{2\alpha(\gamma-1)} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ + C \nu^{-\frac{2\gamma+3\alpha(\gamma-1)}{4}} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ \leq \nu^{2\alpha(\gamma-1)} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ + C \nu^{-2\gamma-3\alpha(\gamma-1)} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ \leq \nu^{2\alpha(\gamma-1)} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

\[ + C \nu^{-2\gamma-4\alpha(\gamma-1)} |\ln \epsilon|^{-1} \int_0^T \int_0^T \left( \sqrt{\frac{2\alpha}{\rho^3}} \phi_y \psi_y \right) \left( \sqrt{\theta^{\alpha-1}} \zeta_y \psi_y \right)^{1/2} dy d\tau \]

where we have used the fact that

\[ C \nu^{-2\gamma-4\alpha(\gamma-1)} |\ln \epsilon| = C \epsilon^{1-2\gamma-4\alpha(\gamma-1)} \| \ln \epsilon \|^{1-2\gamma-4\alpha(\gamma-1)} \leq \epsilon^{\frac{1}{2}}, \text{ if } \epsilon \ll 1. \]

Combining (3.37)-(3.48) yields that

\[
\sup_{\tau \in [0, \tau(\epsilon)]} \int_\mathbb{R} \left( \frac{\theta^{2\alpha}}{\rho^3} \phi_y^2 + \rho^{-\gamma} \phi_y^2 + \rho^{-2} \psi_y^2 + \rho^{-2-\gamma} \zeta_y^2 \right) d\tau + \int_0^{\tau(\epsilon)} \int_\mathbb{R} \left[ \tilde{u}_y \left( \rho^{-\gamma} \phi_y^2 + \rho^{-2} \psi_y^2 + \rho^{-2-\gamma} \zeta_y^2 \right) + \phi_y \psi_y + \phi_y \zeta_y - \frac{\theta^{\alpha+1}}{\rho^2} \phi_y^2 \right] d\tau d\tau \]

\[
\leq C \nu^{2\alpha(\gamma-1)} |\ln \epsilon|^{-1} + \nu^{1-3\gamma} \epsilon^{\frac{1}{2}} \int_0^{\tau(\epsilon)} \int_\mathbb{R} \frac{\theta^{\alpha}}{\rho} \psi_y^2 dy d\tau + C \epsilon^{\frac{1}{2}}.
\]
In particular, it holds that

\[ \sup_{\tau \in [0, \tau(\epsilon)]} \int_0^\tau \frac{\bar{\theta}^{2\alpha}}{\bar{\rho}^3} \phi^2_y dy + \int_0^\tau \frac{\bar{\theta}^{\alpha+1}}{\bar{\rho}^2} \phi^2_y dyd\tau \leq C \left( \mu^{2\gamma(\gamma-1)} |\ln \epsilon|^{-1} + \nu^{1-3\gamma} \epsilon^{\frac{1}{6}} \right) \int_0^\tau \int_0^{\tau(\epsilon)} \frac{\bar{\theta}^\alpha}{\bar{\rho}} \psi^2_y dyd\tau + C \epsilon^{\frac{1}{3}}. \]  \hspace{1cm} (3.50)

**Step 3.** In the following, we estimate \( \sup_\tau \| \psi_y \| \). For this, rewrite (3.42) as

\[ \rho \psi_x + \rho u \psi_y + (\gamma - 1) (\theta \phi_y + \rho \zeta_x) - \mu(\theta) \psi_y = -\bar{g} + \mu(\theta) u_y, \]  \hspace{1cm} (3.51)

where

\[ \bar{g} = -\mu(\theta) \bar{u} y_y + \rho \psi \bar{u}_y + (\gamma - 1) (\bar{\rho} \zeta - \bar{\rho} \theta \phi / \bar{\rho}). \]  \hspace{1cm} (3.52)

Multiplying (3.51) by \( -\psi_y / \rho \) gives

\[ \left( \frac{\psi^2_y}{2} \right)_\tau - (\psi_x \psi_y + u \frac{\psi^2_y}{2}) y_y + \frac{1}{2} \bar{u} y \psi^2_y + \frac{\mu(\theta)}{\rho} \psi^2_y \]

\[ = -\frac{\psi^3_y}{2} + (\gamma - 1) (\theta \phi_y + \rho \zeta_x) \frac{\psi_y}{\rho} + \bar{g} \frac{\psi_y}{\rho} - \mu(\theta) u_y \frac{\psi_y}{\rho}. \]

Integrating the above equation over \( \mathbb{R}^1 \times [0, \tau] \) yields

\[ \int_\mathbb{R} \frac{\psi^2_y}{2} dy + \int_0^\tau \int_\mathbb{R} \left( \frac{\bar{u} y \psi^2_y}{2} + \frac{\mu(\theta)}{\rho} \psi^2_y \right) dyd\tau \]

\[ = \int_0^\tau \int_\mathbb{R} \left\{ \bar{g} \frac{\psi_y}{\rho} - \frac{\psi^3_y}{2} + (\gamma - 1) (\theta \phi_y + \rho \zeta_x) \frac{\psi_y}{\rho} - \mu(\theta) u_y \frac{\psi_y}{\rho} \right\} dyd\tau. \]

First, it follows from (3.42) that

\[ \left| \int_0^\tau \int_\mathbb{R} \frac{\bar{g} \psi_y}{\rho} dyd\tau \right| \]

\[ \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \bar{\theta}^\alpha \psi^2_y dyd\tau + C \left| \int_0^\tau \int_\mathbb{R} \bar{\theta}^{-\alpha} \bar{\rho} \bar{g}^2 dyd\tau \right| \]

\[ \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \bar{\theta}^\alpha \psi^2_y dyd\tau + C \nu^{-2\alpha(\gamma-1)} \int_0^\tau \int_\mathbb{R} \bar{\theta}^{\alpha-1} \bar{\rho}^{-1} \bar{g}^2 dyd\tau \]

\[ \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \bar{\theta}^\alpha \psi^2_y dyd\tau + C \nu^{-2\alpha(\gamma-1)-\gamma} \frac{\epsilon^\delta}{\delta} \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \bar{\theta}^\alpha \psi^2_y dyd\tau + \epsilon^\frac{1}{\delta}, \]

where in the last inequality we used the fact that

\[ C \nu^{-2\alpha(\gamma-1)-\gamma} \frac{\epsilon^\delta}{\delta} = C \epsilon^{1-\alpha(\gamma+2\alpha(\gamma-1)+1)} |\ln \epsilon|^{-2\alpha(\gamma-1)-\gamma} \leq C \epsilon^{\frac{1}{2}} |\ln \epsilon|^{-2\alpha(\gamma-1)-\gamma} \leq 1, \quad \text{if} \quad \epsilon \ll 1. \]
Furthermore, we can compute that

\[
\left| \int_0^\tau \int_0^\infty \frac{\psi_y^3}{2} dyd\tau \right| \leq C \int_0^\tau \|\psi_y\| \|\psi\|^2 d\tau \leq C \tau^{\alpha(\gamma-1)/4} \int_0^\tau \|\bar{\psi}_y\| \|\psi\|^2 d\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + C \tau^{-\alpha(\gamma-1)/3} \int_0^\tau \|\psi_y\|^4 d\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + C \tau^{-2\alpha(\gamma-1)} \sup_{\tau \in [0,\tau_1]} \|\psi_y\|^4 \int_0^\tau \int_0^\infty \bar{\psi}^2_{yy} dyd\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + C \tau^{-2\alpha(\gamma-1)} \epsilon^{1/3},
\]  

(3.56)

where in the last inequality we have used the a priori assumptions (3.11). By Cauchy inequality, one has

\[
\left| \int_0^\tau \int_0^\infty (\dot{\theta} \phi_y + \rho \zeta_y) \frac{\psi_{yy}}{\rho} dyd\tau \right| \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + C \int_0^\tau \sqrt{\frac{\bar{\psi}_{yy}}{\rho} + \bar{\psi}^2_{yy}} \int_0^\tau \sqrt{\frac{\bar{\psi}_{yy}}{\rho} + \bar{\psi}^2_{yy}} dyd\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + C \tau^{-2\alpha(\gamma-1)} \int_0^\tau \int_0^\infty \bar{\psi}^2_{yy} dyd\tau + C \tau^{-2\alpha(\gamma-1)} \epsilon^{1/3},
\]  

(3.57)

Finally, one has

\[
\left| \int_0^\tau \int_0^\infty \frac{\mu(\theta) u_y}{\psi_{yy}} dyd\tau \right| \\
\leq C \int_0^\tau \int_0^\infty \bar{\psi}^{-1} \bar{\rho}^{-1} |\psi_{yy}| (|\psi_y\zeta_y| + \bar{u}_y(\bar{\psi}_{y}^{1/2}|\psi_y| + |\zeta_y|) + \bar{\psi}_{y}^{1/2}|\zeta_y|) dyd\tau := \sum_{i=1}^3 K_i.
\]  

(3.58)

The terms \(K_i\) (\(i = 1, 2, 3\)) will be estimated as follows,

\[
K_1 = C \int_0^\tau \int_0^\infty \bar{\psi}^{-1} \bar{\rho}^{-1} |\psi_{yy}| \psi_y \zeta_y dyd\tau \leq C \tau^{-\gamma/2} \int_0^\tau \sqrt{\bar{\psi}_{yy}} \sqrt{\bar{\psi}_{yy}} \sqrt{\sqrt{\bar{\psi}_{yy}}} \sqrt{\psi_y} dyd\tau \\
\leq C \tau^{-2\gamma+\alpha(\gamma-1)/4} \int_0^\tau \|\bar{\psi}_{yy}\|^2 \sqrt{\sqrt{\bar{\psi}_{yy}}} \sqrt{\psi_y} dyd\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} + C \tau^{-2\gamma+\alpha(\gamma-1)} \int_0^\tau \|\bar{\psi}_{yy}\|^4 \sqrt{\sqrt{\bar{\psi}_{yy}}} \sqrt{\psi_y} dyd\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} + C \tau^{-1-3\gamma+\alpha(\gamma-1)} \sup_{[0,\tau_1]} |\zeta_y|^2 \sup_{[0,\tau_1]} \|\psi_y\|^2 \int_0^\tau \int_0^\infty \bar{\psi}^2_{yy} dyd\tau \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} + C \tau^{-1-3\gamma+\alpha(\gamma-1)} \epsilon^{1/3} \\
\leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} + C \tau^{-1-3\gamma+\alpha(\gamma-1)} \epsilon^{1/3} \leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} + C \tau^{-1-3\gamma+\alpha(\gamma-1)} \epsilon^{1/3} \leq \frac{1}{8} \int_0^\tau \int_0^\infty \frac{\bar{\psi}^2_{yy}}{\rho} dyd\tau + \epsilon^{1/3},
\]  

(3.59)
where we used the fact that
\[ \nu^{1-3\gamma-2\alpha(\gamma-1)} \epsilon^\delta = \epsilon^a |\ln \epsilon|^{1-3\gamma-2\alpha(\gamma-1)} \leq \epsilon^a, \quad \text{if } \epsilon \ll 1. \]

By Cauchy inequality, it holds that
\[
K_2 = C \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + C \int_0^\tau \int_R (\bar{\psi}_{yy} + \bar{\psi}_{yy}^2) \psi_y \bar{\psi}_{yy} dar{\psi}_{yy}
\]
\[
\leq \frac{1}{8} \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + C \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + C \nu^{-\gamma} (\frac{\epsilon}{\delta})^2 \epsilon^\frac{a}{2} \leq \frac{1}{8} \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + \epsilon^\frac{a}{2}. \tag{3.60}
\]

Recalling Lemma 2.3, one can get
\[
K_3 = C \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + C \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + C \nu^{-\gamma} (\frac{\epsilon}{\delta})^2 \epsilon^\frac{a}{2} \leq \frac{1}{8} \int_0^\tau \int_R \bar{\psi}_{yy} \psi_y dar{\psi}_{yy} + \epsilon^\frac{a}{2}. \tag{3.61}
\]

Substituting (3.55)-(3.61) into (3.54), it holds that
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_R \bar{\psi}_{yy} \psi_y d\bar{\psi}_{yy} + \int_0^{\tau_1(\epsilon)} \int_R \left( \bar{u}_y \psi_y^2 + \bar{\psi}_{yy} \psi_y^2 \right) d\bar{\psi}_{yy} \leq C \nu^{-2\alpha(\gamma-1)} \int_0^{\tau_1(\epsilon)} \int_R \bar{\psi}_{yy} \psi_y d\bar{\psi}_{yy} + \epsilon^\frac{a}{2}. \tag{3.62}
\]

Then substituting (3.62) into (3.50), one can get
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_R \bar{\psi}_{yy} \phi_y^2 d\bar{\psi}_{yy} + \int_0^{\tau_1(\epsilon)} \int_R \bar{\psi}_{yy} \phi_y^2 d\bar{\psi}_{yy}
\]
\[
\leq (\nu^{2\alpha(\gamma-1)} |\ln \epsilon|^{-1} + C \nu^{1-3\gamma} \epsilon^{\frac{1}{\delta}}) \int_0^{\tau_1(\epsilon)} \int_R \bar{\psi}_{yy} \psi_y d\bar{\psi}_{yy} + C \epsilon^\frac{1}{2} \tag{3.63}
\]
\[
\leq C (|\ln \epsilon|^{-1} + \epsilon^a) \int_0^{\tau_1(\epsilon)} \int_R \bar{\psi}_{yy} \phi_y^2 d\bar{\psi}_{yy} + C \nu^{1-3\gamma} \epsilon^{\frac{1}{2}+a} + C \epsilon^\frac{1}{2}.
\]

Note that
\[ \nu^{2\alpha(\gamma-1)} \epsilon^{\frac{1}{\delta}+a} |\ln \epsilon| = \epsilon^{\frac{1}{\delta}-3\alpha\gamma+a} |\ln \epsilon|^{2\alpha(\gamma-1)-1} \leq \epsilon^{\frac{1}{\delta}-3\alpha\gamma} |\ln \epsilon|^{\gamma}, \quad \text{if } \epsilon \ll 1, \]

and
\[ C \nu^{1-3\gamma} \epsilon^{\frac{1}{\delta}+a} = C \epsilon^{\frac{1}{\delta}-3\alpha\gamma+2a} |\ln \epsilon|^{1-3\gamma} \leq \epsilon^{\frac{1}{\delta}-3\alpha\gamma} |\ln \epsilon|^{-3\gamma}, \quad \text{if } \epsilon \ll 1, \]

and
\[ C \nu^{1-3\gamma} \epsilon^{\frac{1}{\delta}+a} = C \epsilon^{\frac{1}{\delta}-3\alpha\gamma+2a} |\ln \epsilon|^{1-3\gamma} \leq \epsilon^{\frac{1}{\delta}-3\alpha\gamma} |\ln \epsilon|^{-3\gamma}, \quad \text{if } \epsilon \ll 1, \]

and
Finally, we estimate \( \sup \int_{[0, \tau_1(\epsilon)]} \) that
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \frac{\bar{\partial}^2 \phi_y^2}{\rho^2} dy + \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \frac{\bar{\partial}^{\alpha+1} \phi_y^2}{\rho^2} dy d\tau \leq \epsilon^{\frac{1}{\delta} - 3\alpha\gamma} |\ln \epsilon|^{-3\gamma}, \quad \text{if } \epsilon \ll 1. \tag{3.64}
\]

Meanwhile, it holds that
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int_{\mathbb{R}} \psi^2_y dy + \int_{0}^{\tau_1(\epsilon)} \int_{\mathbb{R}} \left( \bar{u}_y \psi^2_y + \frac{\bar{\partial}^\alpha}{\rho} \psi^2_y \right) dy d\tau \leq C\nu^{-2\alpha(\gamma-1)} \epsilon^{\frac{1}{\delta} - 3\alpha\gamma} |\ln \epsilon|^{-3\gamma} + \epsilon^{\frac{1}{\delta} + \alpha} = C\epsilon^{\frac{1}{\delta}} \epsilon^{-3\gamma - 2\alpha(\gamma-1)} + \epsilon^{\frac{1}{\delta} + \alpha} \leq \epsilon^\tau. \tag{3.65}
\]

**Step 4.** Finally, we estimate \( \sup \|\zeta_y\| \). For this, rewrite (3.4) as
\[
\rho \zeta_x + \rho u \zeta_y + (\gamma - 1) \rho \theta \psi_y - \kappa(\theta) \zeta_{yy} = -\bar{h} + \kappa(\theta) \bar{\theta} \theta_y + \mu(\theta) u_y^2, \tag{3.66}
\]
where
\[
\bar{h} = -\kappa(\theta) \bar{\theta} \theta_y + \rho \psi \bar{\theta} \theta_y + (\gamma - 1) \rho \zeta \bar{u}. \tag{3.67}
\]

Multiplying (3.66) by \(-\zeta_{yy}/\rho\) gives
\[
\left( \frac{\zeta_y^2}{2} \right)_\tau - (\zeta_x \zeta_y + \frac{u \zeta_y^2}{2})_y + \frac{1}{2} \bar{u}_y \zeta_y^2 + \frac{\kappa(\theta)}{\rho} \zeta_{yy}^2 = -\bar{h} \zeta_y + (\gamma - 1) \rho \theta \psi \zeta_y - \frac{\psi_y \zeta_y^2}{2} - (\kappa(\theta) \theta_y + \mu(\theta) u_y^2) \frac{\zeta_{yy}}{\rho}. \tag{3.68}
\]

Integrating the above equation over \( \mathbb{R}^3 \times [0, \tau] \) yields
\[
\int_{\mathbb{R}} \frac{\zeta_y^2}{2} dy + \int_{0}^{\tau} \int_{\mathbb{R}} \left( \bar{u}_y \zeta_y^2 + \frac{\kappa(\theta)}{\rho} \zeta_{yy}^2 \right) dy d\tau = \int_{0}^{\tau} \int_{\mathbb{R}} \left\{ h \zeta_{yy} + (\gamma - 1) \theta \psi \zeta_y - \frac{\psi_y \zeta_y^2}{2} - (\kappa(\theta) \theta_y + \mu(\theta) u_y^2) \frac{\zeta_{yy}}{\rho} \right\} dy d\tau. \tag{3.69}
\]

First,
\[
\int_{0}^{\tau} \int_{\mathbb{R}} \bar{h} \zeta_{yy} \rho dy d\tau \leq \frac{1}{8} \int_{0}^{\tau} \int_{\mathbb{R}} \bar{\partial}^\alpha \zeta_{yy}^2 dy d\tau + C \int_{0}^{\tau} \int_{\mathbb{R}} \bar{\partial}^{-\alpha} \rho^{-1} |\bar{h}|^2 dy d\tau. \tag{3.70}
\]

Recalling (3.13), (3.67), and (i) in Lemma 2.3, one can get
\[
|\bar{h}| \leq C(\bar{\partial}^\alpha |\bar{\theta}_y| + |\bar{\rho} \bar{\theta} \psi| + |\bar{u}_y |) \leq C \left\{ \bar{\partial}^\alpha (\bar{\theta}^2 |\bar{u}_{yy}| + |\bar{u}_y|^2) + \bar{u}_y (|\theta^2 \bar{\rho} \psi| + |\bar{\rho} \psi|) \right\}. \tag{3.71}
\]

So the last term in (3.70) can be estimated by
\[
\int_{0}^{\tau} \int_{\mathbb{R}} \bar{\partial}^{-\alpha} \rho^{-1} |\bar{h}|^2 dy d\tau \leq C\nu^{-1} \int_{0}^{\tau} (|\bar{u}_{yy}|^2 + ||\bar{u}_y||_{L^4}) d\tau + C \int_{0}^{\tau} \bar{u}_y (\bar{\rho} \psi^2 + \rho^2 \zeta^2) \bar{\theta}^{-\alpha} \bar{u}_y dy d\tau \leq C\nu^{-1} (\frac{\epsilon}{\delta})^2 + C\nu^{-\alpha(\gamma-1)} \frac{\epsilon^{\frac{1}{\delta}} \epsilon^{-3}}{\epsilon^{\frac{1}{\delta}}} \leq \epsilon^\frac{1}{3}. \tag{3.72}
\]
By Cauchy inequality, it holds that

\[
\left| \int_0^\tau \int_\mathbb{R} \theta \psi_y \zeta_{yy} dy \right| \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \zeta_{yy}^2 + C \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \psi_y^2 \zeta_{yy}^2 dy \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \zeta_{yy}^2 + C \nu^{-2\alpha(\gamma-1)} \epsilon^{1/3}. \tag{3.73}
\]

By Sobolev inequality and (3.62), it holds that

\[
\left| \int_0^\tau \int_\mathbb{R} \frac{\psi_y^2}{2} dy \right| \leq C \int_0^\tau \left\| \psi_y \right\|_{L^2}^2 d\tau \leq C \int_0^\tau \left\| \psi_y \right\|_{H^1} \left\| \zeta_y \right\|_{L^2}^2 d\tau \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \zeta_{yy}^2 dy \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \psi_y^2 dy + C \nu^{-4\alpha(\gamma-1)/3} \epsilon^{1/3}. \tag{3.74}
\]

where we have used the a priori assumptions (3.11), and in the last inequality we have used the fact that

\[
C \nu^{-4\alpha(\gamma-1)/3} \epsilon^{1/3} = C \epsilon^{1-4\alpha(\gamma-1)/3} \ln \epsilon^{1-4\alpha(\gamma-1)/3} \leq \epsilon^{1/3}, \quad \text{if } \epsilon \ll 1.
\]

By Cauchy inequality, we have

\[
\left| \int_0^\tau \int_\mathbb{R} \left( \kappa(\theta) \theta_y + \mu(\theta) u_y^2 \right) \zeta_{yy} dy d\tau \right| \leq C \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \left| \zeta_{yy} \right| (\zeta_{yy}^2 + \psi_y^2 + u_y^2) dy d\tau := \sum_{i=1}^3 L_i. \tag{3.75}
\]

Now we estimate the terms on the right-hand side of (3.75) one by one. By Sobolev inequality, it holds that

\[
L_1 = \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \zeta_{yy}^2 dy d\tau \leq C \nu^{1/2-\gamma} \int_0^\tau \left\| \frac{\partial}{\partial \rho} \zeta_{yy} \right\|_{L^2}^2 \leq C \nu^{1-4\gamma-\alpha(\gamma-1)} \int_0^\tau \left\| \frac{\partial}{\partial \rho} \zeta_{yy} \right\|_{L^4}^4 \leq \frac{1}{8} \int_0^\tau \int_\mathbb{R} \frac{\partial}{\partial \rho} \zeta_{yy}^2 dy d\tau
\]

where in the last inequality we have used the fact that

\[
C \nu^{2-4\gamma-2\alpha(\gamma-1)} \epsilon^{1/3} = C \epsilon^{1/3 + a(2-4\gamma-2\alpha(\gamma-1))} \ln \epsilon^{2-4\gamma-2\alpha(\gamma-1)} \leq \epsilon^{1/3}, \quad \text{if } \epsilon \ll 1.
\]

26
Similarly, one has

\[
L_2 = \int_0^T \int_{\mathbb{R}} \tilde{\theta}^{\alpha} \rho^{-1} |\zeta_{yy} \psi^2_y| dy d\tau \leq C \nu^{-\frac{1}{2}} \int_0^T \| \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy} \| \| \psi_y \|_{L^2}^2 d\tau
\]

\[
\leq C \nu^{-\frac{1}{2}} \int_0^T \left\| \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy} \right\| \| \psi_y \|_{L^2}^2 d\tau \leq C \nu^{-2+\alpha(\gamma-1)} \int_0^T \| \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy} \| \| \frac{\tilde{\theta}^{\alpha}}{\rho} \psi_{yy} \|_{L^2}^2 \| \psi_y \|_{L^2}^2 d\tau
\]

\[
\leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + \frac{1}{8} \left( \frac{\tilde{\theta}^{\alpha}}{\rho} \right)^2 \psi_{yy}^2 dy d\tau + C \nu^{-2-2\alpha(\gamma-1)} \int_0^T \| \psi_y \|_{L^2}^4 d\tau
\]

\[
\leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + \frac{1}{8} \left( \frac{\tilde{\theta}^{\alpha}}{\rho} \right)^2 \psi_{yy}^2 dy d\tau + C \nu^{-2-2\alpha(\gamma-1)} \int_0^T \| \psi_y \|_{L^2}^4 d\tau
\]

\[
\leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + \frac{1}{8} \left( \frac{\tilde{\theta}^{\alpha}}{\rho} \right)^2 \psi_{yy}^2 dy d\tau + C \nu^{-2-2\alpha(\gamma-1)} \int_0^T \| \psi_y \|_{L^2}^4 d\tau
\]

(3.77)

Recalling Lemma 2.3 and using Cauchy inequality, it holds that

\[
L_3 = \int_0^T \int_{\mathbb{R}} \tilde{\theta}^{\alpha} \rho^{-1} |\zeta_{yy} | u_y^2 dy d\tau \leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + C \int_0^T \int_{\mathbb{R}} \tilde{\theta}^{\alpha} \rho^{-1} u_y^4 dy d\tau
\]

\[
\leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + C \nu^{-1} \int_0^T \| u_y \|_{L^2}^4 d\tau \leq \frac{1}{8} \int_0^T \int_{\mathbb{R}} \frac{\tilde{\theta}^{\alpha}}{\rho} \zeta_{yy}^2 dy d\tau + \nu^\gamma.
\]

(3.78)

Substituting (3.70), (3.74)-(3.78) into (3.69), it holds that

\[
\sup_{\tau \in [0, T)} \int_{\mathbb{R}} \zeta_y^2 dy + \int_0^{\tau_1(\epsilon)} \int_{\mathbb{R}} \left( \tilde{u}_y \psi_y^2 + \tilde{\theta}^{\alpha} \zeta_{yy}^2 \right) dy d\tau \leq \epsilon^\frac{1}{2}.
\]

(3.79)

Therefore, (3.14), (3.15) and (3.16) can be derived directly from (3.31), (3.64), (3.65) and (3.79).

It follows from (3.14)-(3.16) that if \( \epsilon \) is suitably small, then

\[
\sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \phi(\cdot, \tau) \|_{L^\infty} \leq \sqrt{2} \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \phi(\cdot, \tau) \|_{L^1}^{1/2} \| \phi_y(\cdot, \tau) \|_{L^1}^{1/2} \leq C \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \left( \nu^{-\gamma-2\alpha(\gamma-1)} \int_{\mathbb{R}} \tilde{\rho}^{\gamma-2} \phi^2 dy \int_{\mathbb{R}} \frac{\tilde{\theta}^{2\alpha}}{\rho^\gamma} \phi_y^2 dy \right)^{1/2} \leq C \left( \nu^{-\gamma-2\alpha(\gamma-1)} \epsilon^\frac{1}{2} \cdot \epsilon^{\frac{1}{2} - 3\alpha(\gamma-1)} \ln |\epsilon| \right)^{1/4} \leq C \epsilon^{\frac{4\alpha+3\alpha(\gamma-1)}{2(18\gamma+12\alpha(\gamma-1))}} \leq \epsilon^{\frac{1}{18\gamma+12\alpha(\gamma-1)}} \leq \epsilon^\alpha,
\]
Thus the a priori assumptions (3.10)-(3.11) are verified and the proof of Lemma 3.2 is completed.

It is note that the a priori estimates (3.14)-(3.16) are better than the a priori assumptions (3.10)-(3.11) in the time interval $[0, \tau_1(\epsilon)]$ with $\tau_1(\epsilon)$ being the maximum existence time. Based on these a priori estimates, we can claim $\tau_1(\epsilon) = \infty$. In fact, if $\tau_1(\epsilon) < \infty$, then by again using the local existence at time $\tau = \tau_1(\epsilon)$, we can find another time $\tau_2(\epsilon) > \tau_1(\epsilon)$ so that the solution satisfies the assumptions (3.10)-(3.11) in the time interval $[0, \tau_2(\epsilon)]$ which contradicts the assumption that $\tau_1(\epsilon)$ is the maximum existence time. Therefore we extend the local solution to the global one in $[0, \infty)$ for small but fixed $\epsilon$.

**Proof of Theorem 1.1:** It remains to prove (1.13) with $a$ given in (1.14). From Lemma 2.2, Lemma 2.3 (iii), (3.7)-(3.9) and recalling that $\nu = \epsilon^a |\ln \epsilon|$, $\delta = \epsilon^a$ in (3.12), it holds that for any given positive constant $l$, there exists a constant $C_l > 0$ which is independent of $\epsilon$ such that

\[
\sup_{t \geq l} \|\rho(\cdot, t) - \rho^\gamma_a(t)\|_{L^\infty} \leq \sup_{0 \leq \tau \leq \tau_1(\epsilon)} \|\phi(\cdot, \tau)\|_{L^\infty} + \sup_{t \geq l} \|\tilde{\rho}(\cdot, t) - \rho^\gamma_a(\cdot)\|_{L^\infty} + \sup_{t \geq l} \|\rho^\gamma_a(\cdot) - \rho^\gamma_a(\cdot)\|_{L^\infty} \leq C_l (\epsilon^a + |\ln \delta| + \nu) \leq C_l \epsilon^a |\ln \epsilon|. \]
About the convergence of the momentum, we have
\[
\sup_{t \geq l} \| m(\cdot, t) - m^{\tau_3}(\cdot, t) \|_{L^\infty} \\
\leq C \sup_{\tau \in [0, +\infty)} \left( \| \psi(\cdot, \tau) \|_{L^\infty} + \| \phi(\cdot, \tau) \|_{L^\infty} \right) + \sup_{t \geq l} \left( \| \bar{m}(\cdot, t) - m^{\tau_3}(\cdot, t) \|_{L^\infty} + \| m^{\tau_3}(\cdot, t) - m^{\tau_3}(\cdot, t) \|_{L^\infty} \right) \\
\leq C_l (\epsilon^a + \delta |\ln \delta| + \nu) \leq C_l \epsilon^a |\ln \epsilon|.
\]

About the convergence of the total energy, we can get
\[
\sup_{t \geq l} \| n(\cdot, t) - n^{\tau_3}(\cdot, t) \|_{L^\infty} \\
\leq C \sup_{\tau \in [0, +\infty)} \left( \| \zeta(\cdot, \tau) \|_{L^\infty} + \| \phi(\cdot, \tau) \|_{L^\infty} \right) + \sup_{t \geq l} \left( \| \bar{\rho}(\cdot, t) - n^{\tau_3}(\cdot, t) \|_{L^\infty} + \| n^{\tau_3}(\cdot, t) - n^{\tau_3}(\cdot, t) \|_{L^\infty} \right) \\
\leq C_l (\epsilon^a + \delta |\ln \delta| + \nu) \leq C_l \epsilon^a |\ln \epsilon|,
\]
where we have used the fact that
\[
\sup_{\tau \in [0, +\infty)} \| \zeta(\cdot, \tau) \|_{L^\infty} \leq C \epsilon^{\frac{1}{2} - \frac{\alpha}{2}} |\ln \epsilon|^{-\frac{1}{2}} \leq \epsilon^a.
\]
Then the proof of Theorem 1.1 is completed. \qed

References

[1] S. Bianchini, A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. (2), 161 (2005), pp. 223-342.

[2] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, 3rd edition, 1990.

[3] G. Q. Chen, M. Perepelitsa, Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow, Comm. Pure Appl. Math., 63, (2010), pp. 1469-1504.

[4] J. Goodman, Z. P. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, Arch. Ration. Mech. Anal., 121 (1992), pp. 235-265.

[5] D. Hoff, T. P. Liu, The inviscid limit for the Navier-Stokes equations of compressible, isentropic flow with shock data, Indiana Univ. Math. J., 38 (1989), pp. 861-915.

[6] F. Huang, M. Li, Y. Wang, Zero dissipation limit to rarefaction wave with vacuum for one-dimensional compressible Navier-Stokes equations, SIAM J. Math. Anal., 44 (2012), pp. 1742-1759.
[7] F.M. Huang, S. Jiang and Y. Wang, Zero dissipation limit of full compressible Navier-Stokes equations with a Riemann initial data, Preprint.

[8] F. Huang, Y. Wang, Y. Wang, T. Yang, The Limit of the Boltzmann Equation to the Euler Equations for Riemann Problems, SIAM J. Math. Anal., 45 (2013), no. 3, pp.1741-1811.

[9] F. M. Huang, Y. Wang and T. Yang, Fluid Dynamic Limit to the Riemann Solutions of Euler Equations: I. Superposition of rarefaction waves and contact discontinuity, Kinetic and Related Models, 3 (2010), pp. 685-728.

[10] F. M. Huang, Y. Wang and T. Yang, Vanishing viscosity limit of the compressible Navier-Stokes equations for solutions to Riemann problem, Arch. Ration. Mech. Anal., 203 (2012), pp. 379-413.

[11] S. Jiang, G. X. Ni and W. J. Sun, Vanishing viscosity limit to rarefaction waves for the Navier-Stokes equations of one-dimensional compressible heat-conducting fluids, SIAM J. Math. Anal., 38 (2006), pp. 368-384.

[12] Q. S. Jiu, Y. Wang and Z. P. Xin, Vacuum behaviors around rarefaction waves to 1D compressible Navier-Stokes equations with density-dependent viscosity, Preprint, 2011.

[13] S. Kawashima, A. Matsumura, T. Nishida, On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation, Comm. Math. Phys., 70 (1979), no. 2, pp. 97-124.

[14] M. Li, T. Wang, Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier-Stokes equations, J. Differential Equations, 248 (2010), pp. 95–110.

[15] T. P. Liu, J. Smoller, On the vacuum state for the isentropic gas dynamics equations, Adv. in Appl. Math., 1 (1980), pp. 345-359.

[16] T. Liu, T. Yang, S. H. Yu and H. J. Zhao, Nonlinear stability of rarefaction waves for the Boltzmann equation, Arch. Rat. Mech. Anal., 181 (2006), 333–371.

[17] S. X. Ma, Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier-Stokes equations, J. Differential Equations, 248 (2010), pp. 95–110.

[18] A. Matsumura, K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math., 3 (1986), pp. 1-13.

[19] M. Perepelitsa, Asymptotics toward rarefaction waves and vacuum for 1-D compressible Navier-Stokes equations, SIAM J. Math. Anal., 42 (2010), pp. 1404-1412.
[20] J. Smoller, Shock Waves and Reaction-Diffusion Equations. 2nd ed. Grundlehren der Mathematischen Wissenschaften. 258. New York: Springer-Verlag, xxii, 1994.

[21] H. Y. Wang, Viscous limits for piecewise smooth solutions of the p-system, J. Math. Anal. Appl., 299 (2004), pp. 411-432.

[22] Y. Wang, Zero dissipation limit of the compressible heat-conducting Navier-Stokes equations in the presence of the shock, Acta Mathematica Scientia, 28B (2008), pp. 727-748.

[23] Z. P. Xin, Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases, Comm. Pure Appl. Math., 46 (1993), pp. 621-665.

[24] Z. P. Xin, H. H. Zeng, Convergence to the rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations, J. Diff. Eqs., 249 (2010), pp. 827-871.

[25] S. H. Yu, Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws, Arch. Ration. Mech. Anal., 146 (1999), pp. 275-370.

[26] Y. Zhang, R. Pan, Y. Wang, Z. Tan, Zero dissipation limit with two interacting shocks of the 1D non-isentropic Navier-Stokes equations, to appear in Indiana Univ. Math. J.