EXTENDED THEORIES OF GRAVITY IN COSMOLOGICAL AND ASTROPHYSICAL APPLICATIONS

ANETA WOJNAR

PhD Dissertation

Prof. Dr hab. Andrzej Borowiec
Institute for Theoretical Physics
Department of Physics and Astronomy
University of Wroclaw

Prof. Salvatore Capozziello
Department of Physics
University of Napoli “Federico II”
Mamie i Tacie
– Ziazia

Marcinie,
za Twoją nieskończoną (oby!) cierpliwość i miłość

BabcioM Bogusi, Marysi oraz Dziadzi Józiowi,
za bułki na parze, jabłecznik i niekończące się poczucie humoru

Pamięci Stanisława Graczyka.
1933 – 2013
ABSTRACT

The main subjects of the PhD dissertation concern cosmological models considered in Palatini $f(R)$ gravity and scalar - tensor theories. We introduce a simple generalization of the $\Lambda$CDM model which is based on Palatini modified gravity with quadratic Starobinsky term. A matter source is provided by generalized Chaplygin gas. The statistical analysis of our model is investigated as well as we use dynamical system approach to study the evolution of the Universe. The model reaches a very good agreement with the newest experimental data and yields an inflationary epoch which is caused by a singularity of the type III. The present-day accelerated expansion is also provided by the model.

We also show that the Lie and Noether symmetry approaches are very useful tools in cosmological considerations. We examine two other models of Extended Theories of Gravity (ETGs), that is, the novel hybrid metric-Palatini gravity and a minimally coupled to gravity scalar field as the simplest example of scalar-tensor theories. The first one is applied to homogeneous and isotropic model while in the scalar - tensor theory we study anisotropic universes. We use Lie and Noether symmetries in order to find unknown forms of potential and to solve classical field equations in both models. The symmetries also are very helpful in searching exact and invariant solutions of Wheeler-DeWitt equations which are a quantized version of modified Einstein’s equations.

In the last part we are interested in equilibrium configurations and stability conditions of relativistic stars in the framework of scalar - tensor theories. Firstly, we show that TOV-like form of the equilibrium equations can be obtained for a big class of ETGs if generalized energy density and pressure are defined. According to our studies, a neutron star is a stable system for the minimally coupled scalar field model.

There is a supplement including notes on symmetries as well as dynamical systems approach. The illustrative examples of applications are also provided.
STRESZCZENIE

Główne problemy rozważane w przedstawionej rozprawie doktorskiej dotyczą kosmologicznych modeli w teoriach Palatiniego oraz skalarno-tensorowych. Pierwszym badanym modelem jest proste rozszerzenie modelu ΛCDM. Badany model oparty jest na dodaniu do lagranżjana kwadratowego członu Starobinskiowego oraz założeniu, że metryka i koneksi są niezależnymi wielkościami. Źródłem materii jest uogólniony gaz Chaplygina. Statystyczna analiza pokazuje, że model zgadza się z najnowszymi danymi obserwacyjnymi. Badanie punktów osobliwych pozwoliło na określenie rodzaju osobliwości kosmologicznych. Badanie Wszechświata jako dwuwymiarowego dynamicznego układu pokazuje, że osobliwość typu III przyczynia się do nagłej przyspieszonej ekspansji, tj. inflacji, we wczesnej fazie ewolucji. Co więcej, w rozważanym modelu mamy również epokę obecnej przyspieszonej ekspansji Wszechświata.

Dwa pozostałe kosmologiczne modele są badane za pomocą narzędzi, które dostarczają symetrie Noether oraz Liego. Model hybrydowej grawitacji, łączący formalizm metryczny z formalizmem Palatiniego, jest rozważany dla jednorożnych i izotropowych wszechświata natomiast model z minimalnie sprzężonym do grawitacji polem skalarnym dotyczy anizotropowych czasoprzestrzeni. Symetrie Liego oraz Noether są używane do rozwiązywania klasycznych oraz kwantowych równań (tj. równań Wheeler-DeWitta). Rozwiązywania są dokładnie i niezmiennicze. Symetrie używane są również do znalezienia postaci potencjału pola skalarnego, którego określenie jest niezbędne do rozwiązania równań.

Ostatnia część rozprawy dotyczy relatywistycznych gwiazd, np. gwiazd neutronowych, w rozszerzonych teорia grawitacji. Pokazano, że można otrzymać równania analogiczne do równań TOV (Tolmana - Oppenheimera - Volkoffa) z uogólnionym ciśnieniem i gęstością energii. Dodatkowo zbadano także stabilność takiego układu dla modelu z minimalnie sprzężonym polem skalarnym.

Końcowa część zawiera suplement, w którym zawarto główne metody matematyczne używane w rozprawie: symetrie oraz układy dynamiczne. Znajdują się tutaj również przykłady użycia tych metod.
This PhD dissertation consists of research done at the Institute for Theoretical Physics, University of Wrocław and Department of Physics, University of Napoli "Federico II" in collaboration with Prof. Dr hab. Andrzej Borowiec and Prof. Salvatore Capozziello.

Some ideas and figures have appeared previously in the following publications:

- A. Borowiec, S. Capozziello, M. De Laurentis, F.S.N Lobo, A. Paliathanasis, M. Paolella, A. Wojnar "Invariant Solutions and Noether Symmetries in Hybrid Gravity"
  PRD 91(2015) : 023517 (arXiv: 1407.4313)

- A. Borowiec, A. Stachowski, M. Szydłowski, A. Wojnar "Inflationary cosmology with Chaplygin gas in Palatini formalism"
  JCAP 01(2016)040 (arXiv: 1512.01199)

- M. Szydłowski, A. Stachowski, A. Borowiec, A. Wojnar "Do sewn singularities falsify the Palatini cosmology?"
  pre-print (arXiv:1512.04580)

- H. Velten, A. M. Oliveira, A. Wojnar "A free parametrized TOV: Modified Gravity from Newtonian to Relativistic Stars"
  PoS(MPCS2015)025 (arXiv:1601.03000)

- A. Paliathanasis, L. Karpathopoulos, A. Wojnar, S. Capozziello "Wheeler-DeWitt equation and Lie symmetries in Bianchi scalar - field cosmology"
  EPJ C 76(4)2016, (arXiv:1601.06528)

- A. Wojnar and H. Velten "Equilibrium and stability of relativistic stars in extended theories of gravity"
  pre-print (arXiv:1604.04257)
ACKNOWLEDGEMENTS

Foremost, I would like to thank my advisors Prof. Dr hab. Andrzej Borowiec and Prof. Salvatore Capozziello. Undertaking this PhD would not have been possible to do without your guidance, patience, many stimulating discussions, and consistent encouragement that I have been receiving throughout the research work.

Very special thanks to Prof. Dr hab. Marek Szydlowski for his enthusiasm and opportunities to work on exciting projects. I would like to thank my colleagues: Dr Andronikos Paliathanasis, Dr Mariafelicia De Laurentis, Dr Francisco Lobo, Dr Hermano Velten, Mariacristina Paolella, Leonidas Karpathopoulos, Aleksander Stachowski, and Adriano Oliveira for inspiring ideas and enjoyable atmosphere during the work.

I am also indebted to my wonderful Family. Without your unconditional support and love I would never be here. Many thanks to my friends met in Wrocław, Napoli and other places who have been supporting me since the very beginning.
# CONTENTS

1 **INTRODUCTION**  

2 **$f(R)$ GRAVITY IN PALATINI FORMALISM**  
   2.1 Introduction of the model  
   2.1.1 $f(R)$ gravity as a scalar-tensor theory  
   2.1.2 FRLW cosmology in Palatini formalism  
   2.2 Palatini cosmology with Generalized Chaplygin Gas  
   2.2.1 Chaplygin Gas as a dark side of the Universe  
   2.2.2 Starobinsky’s model $f(R) = R + \gamma R^2$  
   2.2.3 Statistical analysis of the model  
   2.2.4 Cosmological singularities  
   2.2.5 Dynamical system analysis  
   2.2.6 Conclusions  

3 **HYBRID METRIC - PALATINI GRAVITY**  
   3.1 Introduction of the model  
   3.1.1 Hybrid gravity cosmology  
   3.2 Noether symmetries in cosmology  
   3.2.1 Noether symmetries of hybrid gravity model  
   3.2.2 Noether symmetries of conformal hybrid gravity Lagrangian  
   3.3 Exact and invariant solutions  
   3.3.1 Wheeler-DeWitt equation of hybrid gravity model  
   3.4 Remarks  

4 **OTHER THEORIES OF GRAVITY**  
   4.1 Bianchi cosmology in scalar - tensor theory of gravity  
   4.1.1 Invariant solutions of WDW equation and WKB approximation  
   4.1.2 Conclusions  
   4.2 Equilibrium and stability of relativistic stars  
   4.2.1 Equilibrium and stability of relativistic stars in General Relativity  
   4.2.2 Equilibrium of relativistic stars in Extended Theories of Gravity  
   4.2.3 Stability of relativistic stars in scalar - tensor theory of gravity  
   4.2.4 Remarks  

xiii
# Appendix

## A Lie symmetry method

- **A.1 One-parameter point transformations and Lie symmetries group**
- **A.2 Ordinary differential equations and Lie point symmetries**
  - **A.2.1 Reducing order of ODE’s by canonical coordinates**
  - **A.2.2 Reducing order of ODE’s by Lie invariants**
  - **A.2.3 Noether symmetries**
- **A.2.4 Linear ODE’s**
- **A.3 Lie algebra**
- **A.4 Lie point symmetries of PDE’s**
- **A.5 Handful of useful theorems**

## B ΛCDM model as a 2d dynamical system of the Newtonian type

- **B.1 Phase portraits of linear systems in \( \mathbb{R}^2 \)**
- **B.2 LCDM model as a dynamical system**

# Bibliography
INTRODUCTION

In November 2015 we celebrated 100 years of General Relativity introduced by Albert Einstein [1, 2], a theory of gravitation which has changed our thinking about the Universe and about other phenomena related to gravitational interactions. Confirmed by many observations, such as the perihelion precession of Mercury [2], the deflection of light by the Sun [3], the gravitational redshift of light [4, 5] or the expanding universe [6], General Relativity is a basic theory that we use for any gravitational phenomena that we want to explain. This year has been also very special because of another anniversary: the first exact solution of Einstein’s field equations published in 1916 by Karl Schwarzschild [7, 8]. Moreover, on 11th of February 2016, we experienced a very exciting announcement about a merger of binary black hole whose observed effects were gravitational waves [9]. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger. One should mention that the detected waveform matches the prediction of General Relativity for a gravitational wave emanating from such a particular binary system. The success of Einstein’s theory has made a lot of difficulties since there have appeared many problems in fundamental physics, astrophysics, and cosmology that one is not able to explain with the help of General Relativity. However, abandoning it, seems to be something inappropriate. Instead of that one attempts to slightly modify the theory. The first modification was done by Einstein himself by introducing a cosmological constant to the gravitational action in order to make the cosmological solutions static. He believed that the Universe does not expand. Nowadays, for cosmological purposes, one uses the equations with the cosmological constant in order to ensure that the field equations provide the scenario which is in agreement with observations indicating that the Universe undergoes the late-time acceleration [10, 11]. The most popular approach to the cosmological constant problem is included in so-called ΛCDM model (Λ Cold Dark Matter model) that is, the standard cosmological model. The model is described by Einstein’s field equations considered in Friedmann - Robertson-Lemaitre - Walker (FRLW) metric background with the cosmological constant added to the gravitational Lagrangian. The effect of the acceleration is explained by exotic fluid called dark energy represented by Λ. It has negative pressure and contributes about 68.3% of the total energy in the present-day observable
Another weird component of the standard cosmological model is dark matter [13, 14] which contributes 26.8%. The dark matter interacts only gravitationally: there is no direct observational evidence that it exists since it does not emit electromagnetic radiation. Its existence and properties are given by gravitational effects such as for instance the motions of visible matter. The problem is mainly indicated by galaxies rotation which is the discrepancy between observed galaxy rotation curves and the theoretical prediction based on the virial theorem. The amount of the ordinary baryonic matter that we are able to detect is just 4.9% of the total energy while neutrinos and photons contribute insignificantly so usually they are neglected in theoretical considerations. Besides the mentioned exotic ingredients which we do not understand, there are many other unsolved problems as inflation [15, 16], cosmological singularities (for instance Big Bang), issues related to quantum field theories in curved spacetime, non-renormalization of Einstein’s theory, unification of gravity with other interactions which has been already unified into the Standard Model, that is, electromagnetism, weak and strong interactions.

Despite these problems, the success of the $\Lambda$CDM model makes alternative theories beyond the standard model unattractive to many physicists. It seems that it is easier to accept three unknown components which are required by the $\Lambda$CDM model, that is, dark matter, dark energy and the inflaton field as an agent of the inflation phase than to look for a new model or to modify the old one. Additionally, it is assumed that the Universe is isotropic and homogeneous on large scales (Cosmological Principle), inflation happened, and that it is based on General Relativity as a correct theory to describe the Universe aside from the quantum regime [17]. The Cosmological Principle is deduced from Copernican Principle, it means that our galaxy worldline is not special so if we observe isotropy about our worldline, there is isotropy about other galaxies worldlines, too. That implies homogeneity and also leads to FRLW geometry of spacetime [17, 18]. However, it is not so sure that isotropic Cosmic Microwave Background (CMB) radiation means isotropic spacetime [19]. Such considerations indicate that we may not limit ourselves only to FRLW spacetime but also one should study anisotropic and/or inhomogeneous models.

It is believed that dark matter may correspond to weak interacting particles which have not been observed yet. There exist alternative models of gravity which explain dark matter effects without a need of the new particle, namely it is a purely gravitational effect. Similarly, there are proposals that also accelerating expansion can be described by some mechanism arising from modified theories of gravity instead of dark energy. There are two main ideas: dark energy as the cosmological constant or scalar fields, both with the feature of nega-
Introducing the $\Lambda$ component, that is, cosmological constant, just made the situation even worst since more problems appeared. Classically, it can be treated as a free parameter which can be fixed to any value that we want; particularly to the value indicated by cosmological observations. On the other hand, one usually considers $\Lambda$ as vacuum energy coming from different matter fields and the theoretical predictions on its value can be done. The estimated value of the cosmological constant is about (in reduced Planck units) $10^{-120}$ while from the Standard Model of particle physics one deals with the value 1. It also seems to be very ambiguous that its observational value is so small but it strongly dominates the Universe evolution today as well as it may contribute to the structure formation epoch.

As already mentioned, the $\Lambda$CDM model requires the inflation epoch that happened after the Big Bang. The cosmological inflation is the early Universe accelerating expansion introduced by A. Guth [16]. The inflation theory turn out to explain a lot of compelling problems such as for example cosmic size of the Universe, its large-scale structure, isotropy, homogeneity and flatness. Although its simplicity and explanations of the above issues, it also causes problems. One would like to understand what made the Universe evolution to start accelerating and then to slow down. The most popular idea is scalar field but immediately the question arises: what is a form of the potential of that field? There are also many other proposals coming from alternative theories of gravity. They are inspired by the Starobinsky proposal [15] which is in very good agreement with Planck data: he considers an extra quadratic term in the gravitational Lagrangian (see the Chapter 2).

Since the Einstein’s gravity and $\Lambda$CDM model derived from it have passed positively Solar System tests (for review see for example [20, 21]) and matched so far the observational data, one claims that Einstein’s gravity is just an effective theory. Due to that fact one looks for different approaches in order to find a good theory which is able to answer the above problems and tells us more about the Universe that we live in. Moreover, we would like to have a theory which unifies all known interactions and describes quantum effects which had appeared after Big Bang. No satisfactory result has been obtained so far (string theory, supersymmetry) that could combine particle physics and gravitation. There has been many years of research since 1915 but none of proposed models was considered as satisfactory. Due to that fact, one needs new theories of gravity which should be checked carefully with all possible tools that we possess. Each new theory should pass many theoretical and observational tests and also agree with GR in the case of weak gravitational limit. Working on Extended Theories of Gravity, specially on Palatini theories can give clues to work
on a consistent theory merging gravitation and quantum physics. It should also get closer to the answer if one needs to add scalar field in order to explain the inflation phenomena. Could the modified geometry solve the problem of the accelerated epoch at the beginning of our Universe? Are there different scenarios of the origin of the Universe indicating by the existence of singularities than the one given by the $\Lambda$CDM model, that is, Big Bang? What are exactly cosmological singularities? Can we get to know more about them without Quantum Gravity theory?

The thesis is divided into two parts: the first one consists of some examples of Extended Theories of Gravity while in the Appendix we briefly introduce mathematical tools. The Appendix A describes Lie symmetry method and show how it can be used for solving ordinary and partial differential equations. We also represent a subclass of Lie symmetries, that is, the Noether symmetries having a serious consequences in physics. In order to see how Lie symmetry method works, illustrative examples are provided. We also discuss the connection to conformal algebra of Riemannian metric which is a phase space metric of a physical system. Moreover, in the Appendix B we show how the dynamical systems theory can be used for studies of cosmological models. As some of them can be recast into two dimensional cases, we focus on phase portraits of linear systems in 2-dimensional vector space. Later on, as an example we consider $\Lambda$CDM model as a dynamical system.

In the main part we examine three models of Extended Theories of Gravity. The Chapter 2 includes a simple generalization of the $\Lambda$CDM model which is based on Palatini modified gravity with quadratic Starobinsky term and generalized Chaplygin gas as a matter source. We show that it provides inflation as well as the current accelerated expansion. The singularity which appears in the model turns out to be responsible for the inflationary epoch. We also perform statistical analysis in order to find values favored by astronomical data. Subsequently, we classify all evolutionary paths in the model phase space using dynamical system theory. The another model that we study in the Chapter 3 also have something in common with the Palatini gravity. It is a recently proposed model whose Lagrangian consists of the standard Einstein-Hilbert term considered in metric formalism, and an arbitrary function of the Palatini curvature scalar. For this investigation we use Lie and Noether symmetries in order to select $f(R)$ form. The symmetries also help us to solve the field equations for the selected model. Quantizing the model, we derive Wheeler-DeWitt equation: its invariant solution can be also given by the Lie symmetries which are determined by the methods discussed in the Appendix A.
In the last Chapter 4 we present the simplest representation of the scalar-tensor theory, it means, we focus on minimally coupled to gravity scalar field. Here, we are focused on cosmology provided by the Bianchi spacetimes. We examine anisotropic models in which the scalar field also contributes. We perform similar analysis using Lie symmetries methods as it was done for Hybrid Gravity. Additionally, we study WKB approximations in order to find classical solutions, that is, anisotropy parameters, which are given as functions of time. The further part of this chapter concerns configurations of relativistic stars. We show that for a general (not specified) form of Extended Theories of Gravity one may write the equilibrium equations in the Tolman-Oppenheimer-Volkoff-like form with new definitions of energy density and pressure. Since the stability criterion of relativistic star system must be considered case by case, we study the problem for the minimally coupled scalar field.
The most natural way (as we do not want to say the simplest one) to extend our considerations on gravity beyond General Relativity is to study some geometric modifications of the Einstein’s theory. The geometric part of the gravitational action can be changed in many different ways. One may assume that constant of Nature are not really constant values. A scalar field might be added into Lagrangian and moreover, it can be minimally or non-minimally coupled to gravity (to Ricci scalar). One proposes much more complicated functionals than the simple linear one used in GR, for example $f(R)$ gravity. The latter approach has gained a lot of interest recently as the extra geometric terms could explain not only the dark matter issue [22, 23] but also the dark energy problem because it produces the accelerated late-time effect at low cosmic densities (it means when the trace of the energy momentum tensor goes to zero). The field equations also differ from the Einstein’s ones so they could provide different behavior of the early Universe. The $f(R)$ gravity can be treated in two different ways: the metric approach and Palatini one [24]. The former arises to the fourth order differential equations which are difficult to handle and moreover, one believes that physical equations of motion should be of the second order. In contrast to the metric formalism, the Palatini $f(R)$ gravity provides second order differential equations since the connection and metric are treated as independent objects. The Riemann and Ricci tensors are constructed with the connection while for building the Ricci scalar we also use the physical metric in order to contract the indices. The Palatini approach is very important in cosmology because one may use the trace of the field equations derived with respect to the metric in order to obtain the $\mathcal{R}(a)$ dependence where $a = a(t)$ is a scale factor of the FRLW metric. Then we get the Friedmann equation which rules the dynamics of the Universe in a given model so the model might be compared with the observational data [25]. Such dynamics has a form of Newtonian one so one deals with an effective potential term depending only on the scale factor $a(t)$. Furthermore, the theory coincides with GR (with a dynamical feature that the connection is a Levi-Civita connection of the metric in comparison to GR where this is a priori assumption) if the functional is linear in $\mathcal{R}$. There also exist disadvantages of such an approach: being in conflict with the Standard Model of particle physics [24, 26, 27, 28], surface singularities of static spherically sym-
metric objects in the case of polytropic EoS \([29]\), the algebraic dependence of the post-Newtonian metric on the density \([30, 31]\), and the complications with the initial values problem in the presence of matter \([32, 33]\), although the problem was already solved in \([34]\). Another one happens at microscopic scales, that is, the theory produces instabilities in atoms which disintegrate them. However, it was shown \([35]\) that high curvature corrections do not cause such problem. What is also very promising, some of the Palatini Lagrangians avoid the Big Bang singularity. What should be also emphasized, the effective dynamics of Loop Quantum Gravity can be reproduced by the Palatini theory which gives the link to one of approaches to Quantum Gravity \([36]\). High curvature correction of the form \(f(\mathcal{R})\) changes the notion of the independent connection: in the simple Palatini \(f(\mathcal{R})\) gravity the connection is auxiliary field while in the more general Palatini theory it is dynamical without making the equations of motion second order in the fields. Furthermore, as already remarked in the Introduction, the squared curvature terms improve the renormalization \([37]\).

Palatini theories seem to be very promising and hence more investigations should be performed. We will start with the simplest representant of the Palatini theories, that is, we will study \(f(\mathcal{R})\) gravity in the mentioned formalism. We will briefly introduce the main assumptions of the theory in order to construct our model which is examined from theoretical and observational points of view.

### 2.1 Introduction of the Model

Palatini formalism of \(f(\mathcal{R})\) theory of gravity is based on the gravitation action in which not only the standard Hilbert - Einstein action \(S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} R\) is replaced by an arbitrary function of the Ricci scalar \(f(\mathcal{R})\) \([13, 21, 14, 24]\), but one uses Palatini scalar \(\mathcal{R}\) instead of the metric one \(R\): \n
\[
S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} f(\mathcal{R}) + S_m(g_{\mu\nu}, \psi),
\]

where \(\kappa = 8\pi G\) is as usually the Einstein constant. The Palatini curvature scalar \(\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}(\hat{\Gamma})\) is constructed with the metric-independent connection \(\hat{\Gamma}\). The metric \(g_{\mu\nu}\) is used for raising and lowering indices. The action \(S_m\) denotes a matter action which depends only on the metric \(g_{\mu\nu}\) and matter fields but it is independent of the connection \(\hat{\Gamma}\). One varies the action with respect to two independent objects: the metric \(g_{\mu\nu}\) and the connection \(\hat{\Gamma}\). The first variation, after applying the Palatini formula

\[
\delta \mathcal{R}_{\mu\nu} = \hat{\nabla}_\lambda \delta f^\lambda_{\mu\nu} - \hat{\nabla}_\nu \delta f^\lambda_{\mu\lambda},
\]
gives rise to the equation

\[ f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} = \kappa T_{\mu\nu}. \]  

(3)

The prime denotes the differentiation with respect to \( R \) while \( T_{\mu\nu} \) is the standard energy-momentum tensor given by the variation of the matter action with respect to \( g_{\mu\nu} \):

\[ T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta(g^{\mu\nu})}. \]  

(4)

One should mention that the energy-momentum tensor is conserved \([24, 38, 39]\) by the covariant derivative which is defined with the Levi-Civita connection of the metric \( g_{\mu\nu} \):

\[ \nabla_\mu T^{\mu\nu} = 0 \]  

but not \( \bar{\nabla}_\mu T^{\mu\nu} = 0. \)  

(5)

The consequence of the above condition is the motion of the test particles: they follow geodesics of the metric. Although, there exists another possibility, it means, that the particles follow the geodesics provided by the connection \([40, 41, 42, 43, 44]\). But from now on, we will assume that they follow the metric ones so the theory satisfies the metric postulates \([24, 20]\) and Einstein Equivalence Principle \([45, 46]\). The trace of (3) with respect to \( g^{\mu\nu} \) gives us the structural (master) equation of the spacetime which controls (3) \([47, 48, 49]\):

\[ f'(R)R - 2f(R) = \kappa T. \]  

(6)

Assuming that one has a given function \( f(R) \), we may solve (6) and express a solution as \( R(T) \). Hence, \( f(R) \) is a function of \( T \) being the trace of the energy-momentum tensor \( T = g^{\mu\nu}T_{\mu\nu} [47, 48, 49] \).

Following the approach of \([47, 48, 49]\), the generalized Einstein’s equations can be also written as

\[ \hat{R}_{\mu\nu}(\Gamma) = g_{\mu\alpha}P^\alpha_\nu, \]  

(7)

where the operator \( P^\alpha_\nu \) used above consists of two scalars \( b \) and \( c \) depending on \( \hat{R} \):

\[ p^\alpha_\nu = c \delta^\alpha_\nu + \frac{1}{b} T^\alpha_\nu, \]  

(8)

\[ b = b(\hat{R}) = f'(\hat{R}), \quad c = c(\hat{R}) = \frac{1}{2} f(\hat{R}), \]  

(9)

which will be useful later.

The variation with respect to the connection leads to the equation

\[ -\hat{\nabla}_\alpha(\sqrt{-g}f'(R)g^{\mu\nu}) + \hat{\nabla}_\sigma(\sqrt{-g}f'(R)g^{\sigma(\mu})\delta^{\nu)}_\alpha = 0, \]  

(10)
where $(\mu \nu)$ denotes a symmetrization over the indices $\mu$ and $\nu$. The trace with respect to $g$ allows us to write down the second equation of motion (10) as
\[ \hat{\nabla}_\alpha (\sqrt{-gf'}(\mathcal{R})g^{\mu \nu}) = 0 \] (11)
which stands for the Levi-Civita connection of the metric
\[ \bar{g}_{\mu \nu} = f'(\mathcal{R})g_{\mu \nu}. \] (12)
One notices that choosing $f(\mathcal{R}) = \mathcal{R}$ leads to General Relativity and from the equation (11) we get the metric-independent connection $\hat{\Gamma}$ is a Levi-Civita connection of the metric $\tilde{g}_{\mu \nu} = f'(\mathcal{R})g_{\mu \nu}$. From this equation (3) becomes Einstein’s equation. It should be noticed that in the case of General Gravity derived by Palatini formalism we deal with two equations of motion which one of them indicates that the connection $\hat{\Gamma}$ is the Levi-Civita one of the metric $g$, it means $\bar{g}_{\mu \nu} = g_{\mu \nu}$. As a contrary to GR, this is a dynamical feature, not the assumption.

Moreover, the Palatini equations of motion may be written as ones depending only on the metric and matter field [24]. As one has (12), we may use the conformal relations between Ricci tensors and scalars (recall that one uses the metric $g_{\mu \nu}$ but not $\tilde{g}_{\mu \nu}$ for raising and lowering indices):
\[ \mathcal{R}_{\mu \nu} = \tilde{R}_{\mu \nu} + \frac{3}{2} \frac{F(\mathcal{R})}{F(\mathcal{R})} \frac{F(\mathcal{R})}{F(\mathcal{R})} - \frac{1}{F(\mathcal{R})} \nabla_\mu F(\mathcal{R})_{,\nu} - \frac{1}{2} \frac{g_{\mu \nu}}{F(\mathcal{R})} \nabla_\mu F(\mathcal{R})_{,\nu}, \] (13)
in order to rewrite the equation (3) as
\[ G_{\mu \nu} = \frac{\kappa}{f'} T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \left( \mathcal{R} - \frac{f}{f'} \right) + \frac{1}{f'} (\nabla_\mu \nabla_\nu - g_{\mu \nu} \square) f' \]
\[ - \frac{3}{2} \frac{1}{F(\mathcal{R})} \left( (\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2} g_{\mu \nu} (\nabla f')^2 \right) \] (14)
which is the standard GR equation with the modified source term, where $G_{\mu \nu}$ is the Einstein tensor, it means $G_{\mu \nu} = \mathcal{R}_{\mu \nu} - \frac{1}{2} \mathcal{R} g_{\mu \nu}$.

Let us just briefly discuss perfect fluid energy-momentum tensor which will stand for the energy-momentum tensor in (3):
\[ T_{\mu \nu} = \rho u_\mu u_\nu + p h_{\mu \nu}, \] (15)
where $\rho$ and $p = p(\rho)$ are energy density and pressure of the fluid, respectively. The vector $u^\mu$ is an observer co-moving with the fluid satisfying $g_{\mu \nu} u^\mu u^\nu = -1$ and $h^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$ is a 3-projector tensor projecting 4-dimensional object on 3-dimensional hypersurface in the case when the observer $u$ is rotation-free. Hence, the trace of (15) is
\[ T = 3p - \rho. \] (16)
2.1.1 \( f(\mathcal{R}) \) gravity as a scalar-tensor theory

The theory under our consideration may be also transformed into a Brans-Dicke theory with a self-interacting potential of a scalar field. The theories (for a special choice of the parameter \( \omega \) - see below) are mathematically equivalent but one should be careful when apply physics: that is, the theories do not have to be physically equivalent (see e.g. \([41, 42, 43, 44]\)). We are going to introduce an auxiliary field \( \chi \) in order to get a dynamically equivalent theory, it means, different representations of the same theory \([24]\).

Let us introduce the field \( \chi \) and write the dynamically equivalent action:

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(\chi) + f'(\chi)(\mathcal{R} - \chi)] + S_m(g_{\mu\nu}, \psi)
\]

whose variation with respect to the field \( \chi \) gives

\[
f''(\chi)(\mathcal{R} - \chi) = 0.
\]

From the above condition it turns out that \( \chi = \mathcal{R} \) if \( f''(\chi) \neq 0 \) which obviously gives rise to the action (1). If we redefine the scalar field \( \chi \) by \( \phi = f'(\chi) \) and define the potential of the field \( \phi \) as

\[
V(\phi) = \chi(\phi)\phi - f(\chi(\phi)),
\]

then the Palatini action will take the following form

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi \mathcal{R} - V(\phi)] + S_m(g_{\mu\nu}, \psi).
\]

It is important to make a comment here that the just obtained action is not an action of the Brans-Dicke theory since \( \mathcal{R} \) is not the Ricci scalar of the metric \( g_{\mu\nu} \). But we may use the conformal relation (13) and rewrite the action which is now (skipping the boundary term)

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi \mathcal{R} + \frac{3}{2\phi} \nabla_\mu \phi \nabla^\mu \phi - V(\phi)] + S_m(g_{\mu\nu}, \psi).
\]

It is a Brans-Dicke action with B-D parameter \( \omega_0 = -\frac{3}{2} \). The variations, now taken with respect to the metric and the scalar field, are

\[
G_{\mu\nu} = \kappa T_{\mu\nu} - \frac{3}{2\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi) - \frac{V}{2\phi} g_{\mu\nu},
\]

\[
\Box \phi = \frac{\phi}{3} (\mathcal{R} - \mathcal{V}') + \frac{1}{2\phi} \nabla^\mu \phi \nabla_\mu \phi.
\]
If we take the trace of the equation (22) in order to eliminate the Ricci scalar $R$ in the equation (23), we will get

$$2V - \phi V' = \kappa T.$$  \hspace{1cm} (24)

From the obtained relation we see that the scalar field $\phi$ is algebraically related to the matter source, in means, it is not a dynamical field. Due to that fact, the Palatini $f(R)$ gravity is in conflict with the Standard Model of particle physics when we consider it as a metric theory [24, 26, 27, 29].

It should be also mentioned that performing the conformal transformation (12) (transferring the action into the Einstein frame) [50, 14, 51], but without rescaling the scalar field, the action is

$$S' = \int d^4x \sqrt{-g} \left[ \frac{\ddot{R}}{2\kappa} - U(\phi) \right] + S_m(\phi^{-1}h_{\mu\nu},\psi),$$  \hspace{1cm} (25)

where $U(\phi) = \frac{V(\phi)}{2\kappa\phi^2}$. The topic is very controversial hence we are not going to discuss it here as we will not work in Einstein frame.

2.1.2 FRLW cosmology in Palatini formalism

As the observations of the cosmic microwave background (CMB) indicate that our Universe is highly isotropic and homogeneous, it allows us to assume a perfect fluid description (15) for the matter and the Friedmann-Robertson-Lemaitre-Walker (FRLW) background metric

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{1}{1-kr^2} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right].$$  \hspace{1cm} (26)

The scalar $k = 0, 1, -1$ stands for the space curvature and $a(t)$ is a scale factor depending on cosmological time $t$. The energy-momentum tensor (15) satisfies the metric covariant conservation law $\nabla^\mu T_{\mu\nu} = 0$ which gives arise to the continuity equation

$$\dot{\rho} + 3H(p + \rho) = 0,$$  \hspace{1cm} (27)

where $H = \frac{\dot{a}}{a}$ is the Hubble constant. The relation between the pressure $p$ and energy density $\rho$, i.e. equation of state $p = p(\rho)$, leads to a dependence of $\rho$ on the scale factor $a(t)$ (by using (26) and (27)). The generalized Einstein’s field equations (7) for the FRLW metric becomes the generalized Friedmann equation [47, 48]

$$\left( \frac{a}{a_t} + \frac{b}{2b} \right)^2 + \frac{k}{a^2} = \frac{1}{2}p^1 - \frac{1}{6}p^0,$$  \hspace{1cm} (28)
where $P^1_1$ and $P^0_0$ are the components of (8). Therefore, the modified Friedmann equation may be written as [47, 48, 25, 49]:

$$
\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 + \frac{k}{a^2} = \frac{1}{2} \left(\frac{c}{b} + \frac{p}{b}\right) - \frac{1}{6} \left(\frac{c}{b} - \frac{\rho}{b}\right).
$$

(29)

2.2 Palatini Cosmology with Generalized Chaplygin Gas

2.2.1 Chaplygin Gas as a dark side of the Universe

In the standard approach to cosmology one uses the barotropic equation of state (EoS) $p = \omega \rho$ for matter filling the homogeneous and isotropic universe. The matter is represented by the perfect fluid energy-momentum tensor

$$
T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}
$$

(30)

where $\rho$ is an energy density while $p$ is pressure of the fluid considered in the model. Together with the continuity equation (27) the equation of state allows to write the energy density in terms of the scale factor $a(t)$. For the standard, barotropic equation one gets

$$
\rho = \rho(a) \propto a^{-(1+\omega)}.
$$

(31)

Dark energy is represented by the fluid with negative pressure, that is, $\omega = -1$ in the barotropic EoS while the dark matter is supposed to behave as pressure-less dust with barotropic EoS $\omega = 0$.

There exists an idea that the dark side of the Universe can be unified into single exotic fluid, so-called Chaplygin Gas which recently has gained a lot of attention in the literature [52, 53]. It was introduced by Sergey Chaplygin in 1904 in order to compute the lifting force on a wing of an airplane in aerodynamics [54] but it has also been used in cosmology [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67]. The interesting feature of Chaplygin gas is that it is the only fluid known up to now which has a super-symmetric generalization [66, 67]. Moreover, it has also a representation as tachyon field [68, 69] and it is added to matter on branes in order to stabilize them in black hole bulks [70]. Chaplygin gas gives positive and bounded square of sound velocity $v_s^2 = \frac{\dot{a}}{\rho^2}$ that is very remarkable as it is a non-trivial problem for fluids with negative pressure [55]. Due to that interesting features it seems to be a very important object to study. Therefore, we would also like to apply that idea into our cosmological considerations.
The equation of state of the pure Chaplygin Gas is
\[ p = -\frac{A}{\rho}, \]  
where \( A \) is a positive constant. Applying that relation into the continuity equation (27) one gets
\[ \rho = \sqrt{A + \frac{B}{a^6}} \]  
with \( B \) being an integration constant. From that solution one sees immediately that assuming a positive value for \( B \) the expression (33) for small \( a(t) \) (i.e. \( a^6 \ll \frac{B}{A} \)) gives rise to
\[ \rho \sim \frac{\sqrt{B}}{a^3} \]  
while for the large values of the scale factor
\[ \rho \sim \sqrt{A}, \quad p \sim -\sqrt{A}. \]

The first approximation corresponds to a universe dominated by dust-like matter whereas the second one to an empty universe with a cosmological constant \( \sqrt{A} \), that is, a de-Sitter universe \([? 70]\). For the intermediate epoch between a dust dominated universe to a de Sitter universe we have that
\[ \rho \approx \sqrt{A} + \sqrt{\frac{B}{4A}} a^{-6}, \quad p \approx -\sqrt{A} + \sqrt{\frac{B}{4A}} a^{-6} \]  
which describes the mixture of a cosmological constant with stiff matter for which the EoS is \( \omega = 1 \). It should be noticed that in a model with Chaplygin Gas one deals with a smooth evolution from the dust dominated phase to the nowadays accelerated expansion run by cosmological constant \( \sqrt{A} \). That process is reached by using only one fluid.

There is a very natural generalization of the Chaplygin gas, so-called Generalized Chaplygin Gas (GCG) whose equation of state is written as
\[ p = -\frac{A}{\rho^\alpha}, \]  
where constants \( A \) and \( \alpha \) satisfy \( A > 0 \) and \( 0 < \alpha \leq 1 \). For \( \alpha = 1 \) one deals with the original Chaplygin Gas. Substituting GCG into the conservation law (27) in FRLW spacetime we get
\[ \rho = \left( A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}, \]
where $B > 0$ is an integration constant. One notices that in the GCG model, similarly like in pure Chaplygin Gas, the early stage of the Universe is dominated by dust ($\rho \propto a^{-3}$) while at late times by cosmological constant (vacuum energy, $\rho \simeq \text{const}$).

### 2.2.2 Starobinsky’s model $f(\mathcal{R}) = \mathcal{R} + \gamma \mathcal{R}^2$

In the early 1980s A. Guth [16] and A. Starobinsky [15] proposed models that introduced inflation into cosmic evolution of our Universe. Guth studied magnetic monopoles which do not exist in our Universe; in order to explain that, he proposed inflation. In turn, Starobinsky showed that there was a link between curvature-squared corrections to the Einstein-Hilbert action and quantum corrections which were supposed to play an important role during the early Universe. That provided solutions to cosmological problems and has been showed to have a very good agreement with observational data (for example Planck [71]). The model, apart the standard gravitational Lagrangian, contains an additional term consisting of the quadratic term of the Ricci scalar $\mathcal{R}^2$, that is,

$$S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} (\mathcal{R} + \gamma \mathcal{R}^2) + S_m(g_{\mu\nu}, \psi),$$

(39)

where $\gamma$ is a small parameter with the dimension of mass. Such a choice of the Lagrangian preserves the GR effects in weak gravitational field, for example in our Solar System. The quadratic term gains an importance in the case of strong gravity (such as neutron stars or black holes) or the very early stage of the Universe. The last one will be a subject of our discussion.

Let us consider the Starobinsky ansatz $f(\mathcal{R}) = \mathcal{R} + \gamma \mathcal{R}^2$ but with the Palatini curvature scalar $\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$ instead of metric Ricci scalar. Applying it to the structural equation (6) and using the trace of the perfect fluid energy-momentum tensor (16) to the right part of the (6) we are able to find the relation $\mathcal{R} = \mathcal{R}(a)$:

$$\mathcal{R} = \left(A + B a^{-3(1+\alpha)}\right) \frac{1}{1+\alpha} \left(4A + B a^{-3(1+\alpha)}\right)$$

(40)

for the Universe filled with Generalized Chaplygin Gas [54, 57, 56]. It allows us to write the Friedmman equation (29) as a function of the scale factor $a(t)$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = M^2(a) \cdot \left[ N(a) - P(a) - \frac{K}{a^2}\right]$$

(41)
where

\[
M(a) = \frac{\rho^\alpha \left[ 2B\gamma a^{-3(1+\alpha)} + (8A\gamma + \rho^\alpha) \right]}{\left[ \frac{B^2}{\rho} \gamma a^{-6(1+\alpha)} + \frac{B}{\rho} (A(7+9\alpha)\gamma + \rho^\alpha) a^{-3(1+\alpha)} + \frac{A}{\rho}(8A\gamma + \rho^\alpha) \right]}
\]

\[
N(a) = \frac{A + 6\Lambda \gamma p + 9\Lambda^2 \gamma p^{-\alpha} + \rho^{1+\alpha}(1 + \gamma p)}{4 \left[ 6\Lambda \gamma + \rho^\alpha(1 + 2\gamma p) \right]}
\]

\[
P(a) = \frac{a^{-3(1+\alpha)} B \left[ B\gamma p^{-\alpha} a^2 + (8A\gamma p^{-\alpha} + 1) \right] - 2(-8A^2 \gamma p^{-\alpha} - 2A + \rho^{1+\alpha})}{12 \left[ 2B\gamma a^{-3(1+\alpha)} + (8A\gamma + \rho^\alpha) \right]}
\]

Since we would like to examine the model with respect to observational constraints, the Friedmann equation needs to be properly parametrized. The above form seems to be difficult to handle and for that reasons one may try to parametrize the quantities appearing in (29) (see [72]). Due to that fact we will introduce parameters \(A_s\) and \(\rho_{ch,0}\) related to physics [56, 72] instead of the theoretical ones, that is, \(A\) and \(B\). The new parameters of Generalized Chaplygin Gas are defined as

\[
\rho_{ch} = \rho_{ch,0} \left( A_s + \frac{1 - A_s}{q^3(1+\alpha)} \right)^{\frac{1}{1+\alpha}} = 3H_0^2 \Omega_{ch,0} \left( A_s + \frac{1 - A_s}{q^3(1+\alpha)} \right)^{\frac{1}{1+\alpha}}, \quad (42)
\]

where \(A_s \rho_{ch,0}^{1+\alpha} = A\), \(\rho_{ch,0}^{1+\alpha}(1 - A_s) = B\). The quantity \(\rho_{ch,0}\) corresponds to the present epoch’s value while \(H_0\) is a present value of the Hubble constant which is \(H_0 = 67.27 \frac{\text{km}}{\text{Mpc}}\) (Planck mission [71]). One also defines a new dimensionless parameter that is related to the parameter \(\gamma\)

\[
\Omega_\gamma = 3\gamma H_0^2. \quad (43)
\]

Before we will parametrize the rest of the quantities appearing in (29), let us introduce the quantity \(K\):

\[
K(a) = \frac{3A_s}{A_s + (1 - A_s)a^{-3(1+\alpha)}}, \quad (44)
\]

The function \(K(a)\) takes the values from the interval [0, 3] as the values of the scale factor lies in \(a \in [0, +\infty)\). We will use that quantity not only to write the Friedmann equation in a nicer form but it will be helpful during the further analysis of singularities that the model possesses. Moreover, it is also associated with the squared velocity of sound, that is, \(c_s^2 = \frac{\partial p}{\partial \rho} = \frac{\alpha A_s}{A_s + (1 - A_s)a^{-3(1+\alpha)}} = \frac{1}{3} \alpha K\). Similarly, like for the parameter \(\gamma\), we will need dimensionless functions
The just defined dimensionless parameters, we can write the normalized Friedmann equation as follows:

\[ \begin{align*}
\Omega_{\text{ch}} &= \Omega_{\text{ch},0} \left( \frac{\Lambda_s}{a^3(1+\alpha)} \right) + \Omega_{\text{ch},0} \left( \frac{3\Lambda_s}{K} \right), \\
\Omega_R &= \frac{\mathcal{R}}{3H_0^2} = \Omega_{\text{ch},0} \left( \frac{\Lambda_s}{a^3(1+\alpha)} \right) + \frac{1-(1-\Lambda_s)a^{-3(1+\alpha)}}{1+(1-\Lambda_s)a^{-3(1+\alpha)}} \\
&= \frac{\Omega_{\text{ch}}(K+1)}{2}, \\
\Omega_c &= \frac{c}{3H_0^2} \int \mathcal{R} = \frac{\Omega_{\text{ch}}(K+1)}{2} \left( 1 + \Omega_{\gamma} \Omega_{\text{ch}}(K+1) \right), \\
\Omega_k &= -\frac{k}{H_0^2 a^2}, \\
b &= f'(|\mathcal{R}|) = 1 + 2\Omega_{\gamma} \Omega_R = 1 + 2\Omega_{\gamma} \Omega_{\text{ch}}(K+1).
\end{align*} \]

Let us also define the another function \( d(t) := \frac{\dot{b}}{H} \), where \( \dot{b} = \frac{db}{dt} \), which one may rewrite as a function of \( K, \Omega_{\gamma}, \) and \( \Omega_{\text{ch}} \):

\[ d = 2\Omega_{\gamma} \Omega_{\text{ch}}(3-K)[\alpha(1-K) - 1]. \]

With the just defined dimensionless parameters, we can write the normalized Friedmann equation (29) as follows:

\[ \frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left( \Omega_{\gamma} \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \Omega_k \right). \]

Since the radiation, whose equation of state is \( p_r = \frac{1}{3} \rho_r \), does not contribute to the trace of the energy-momentum tensor, the structural equation (6) is the same, that is, the solution \( \mathcal{R} = \mathcal{R}(a) \) (40) does not change. That property allows as to add to the normalized Friedmann equation (51) the radiation term in a form of a dimensionless parameter \( \Omega_r = \frac{\rho_r}{3H_0^2 a^{-4}} \). In that case (51) takes the form

\[ \frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left( \Omega_{\gamma} \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \frac{\Omega_r}{b} + \Omega_k \right). \]

One may consider the coordinate transformation \( t \rightarrow \tau: \frac{|b|dt}{|b + \frac{d}{2}|} = d\tau \) and apply it to the equation (51) if it is non-singular:

\[ \frac{H^2(\tau)}{H_0^2} = \Omega_{\gamma} \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \Omega_k. \]
The new Hubble parameter is defined as \( H(\tau) = a(\tau)^{-1} \frac{da(\tau)}{d\tau} \). One should also notice that the new time \( \tau \) is a growing function of the original cosmological time \( t \). The re-parametrization of time taken under an examination whether it is a diffeomorphism or not will allow us to determine the position of the singularity \( a_{\text{sing}} \). Let us define a function \( f(K, \alpha, A_s, \Omega_\gamma) = 2b + d \) (which is just a denominator of the re-parametrization) whose the zero value indicates the singularity \( a_{\text{sing}} \):

\[
f(K(a_{\text{sing}})) = 0
\]

As the above equation cannot be solved algebraically, let us consider less complex case, that is, the case of the original Chaplygin Gas (for which the parameter \( \alpha = 1 \)):

\[
K^4 - 12K^3 + 38K^2 - \chi K + 1 = 0,
\]

where we have defined

\[
\chi = \left(12 + \frac{1}{3A_s \Omega_{\gamma}^2} \frac{\Omega_{\gamma}^2}{\Omega_{\text{ch},0}^2}\right).
\]

The quantity \( \chi \) belongs to the interval \( \chi \in [12, \infty) \). Let us recall that we are interested in the real solutions of the above algebraic equation in the interval \([0, 3]\). Hence, one finds that

\[
K_{\text{sing}} = 3 - \frac{\zeta}{\sqrt{6}} - \sqrt{\frac{16}{3} \left(1 + \frac{(9\chi - 364)}{3\xi} - 1 - \frac{27\chi^2}{2} - 3\sqrt{3(\chi - 12)^2 (27\chi^2 - 12496 - 648\chi)}\right)},
\]

where

\[
\zeta(\chi) = \sqrt{16 + \frac{2(364 - 9\chi)}{\xi}} + \frac{\xi}{4},
\]

\[
\xi(\chi) = \left(\frac{55448 - 2052\chi + \frac{27\chi^2}{2}}{2} + 3\sqrt{3(\chi - 12)^2 (27\chi^2 - 12496 - 648\chi)}\right)^{1/3}.
\]

One notices that the position of the singularity depends on the \( \chi \) parameter which for \( \Omega_\gamma \ll 1 \) is \( \chi = \left(3A_s \Omega_{\gamma}^2 \Omega_{\text{ch},0}^2\right)^{-1} \). Using the definition (44), we are able to express the scale factor for this case (remember that \( \alpha = 1 \)) as

\[
a_{\text{sing}} = \left[\frac{A_s}{1 - A_s} \left(3 - \frac{\zeta}{\sqrt{6}} - \sqrt{\frac{16}{3} + \frac{(9\chi - 364)}{3\xi} - 1 - \frac{27\chi^2}{2} - 3\sqrt{3(\chi - 12)^2 (27\chi^2 - 12496 - 648\chi)}\right)}\right]^{-\frac{1}{1 + \alpha}}.
\]
Let us consider now the case $\alpha = 0$. This implies that the matter content of our universe is the same as in $\Lambda$CDM model. It was already mentioned that the case $\alpha = \gamma = 0$ reconstructs $\Lambda$CDM model completely. However, we are considering the model with the presence of the quadratic term, that is, $\gamma \neq 0$. For $\gamma = 0$ one gets $b = 1, d = 0$ and the equation (54) has no solutions at all. Moreover, one remembers that the value $\gamma << 1$ should be very small in order to locate the singularity in an appropriate epoch and also to have a model which reproduces GR equations for weak gravitational field (e.g. our Solar System). The case $\alpha = 0$ significantly simplifies (54) and hence we are able to find singular solutions

$$K_{\text{sing}} = \frac{1}{3 + \frac{1}{3\gamma_0\Omega_{\gamma_0} A_s}}$$

and

$$a_{\text{sing}} = \left(1 - \frac{A_s}{8A_s + \frac{1}{\Omega_\gamma \Omega_{\gamma_0} A_s}}\right)^{\frac{1}{3}}.$$

In the general case ($\alpha \neq \{0, 1\}$) the re-parametrization function is

$$\frac{b^2}{(b + \frac{d}{2})^2} = \frac{(1 + 2\Omega_\gamma \Omega_{\gamma_0} (K + 1))^2}{(1 + \Omega_\gamma \Omega_{\gamma_0} (3K + \alpha(3K - 1)))^2}.$$

We should also mention that the density parameters $\alpha, \Omega_{\gamma_0}, \Omega_{k_0}, A_s, \Omega_\gamma$ are not independent as they satisfy the constraint condition

$$1 - \Omega_{\gamma_0} - \Omega_{k_0} = \frac{\Omega_\gamma \Omega_{\gamma_0} (3A_s + 1)}{2 + 4\Omega_\gamma \Omega_{\gamma_0} (3A_s + 1)} \times$$

$$\times (1 - A_s)(1 - 3\alpha A_s) \left(12 - 3\Omega_{\gamma_0} + \frac{6\Omega_\gamma \Omega_{\gamma_0}}{1 + 2\Omega_\gamma \Omega_{\gamma_0} (3A_s + 1)}\right).$$

2.2.3 Statistical analysis of the model

From that point of our consideration we will assume that the model is flat, it means $\Omega_{k_0} = 0$. There are only three parameters to estimate, that is, $A_s, \alpha$ and $\Omega_\gamma$ since the value of the parameter $\Omega_{\gamma_0}$ will be derived from the constraint condition (63). Moreover, we will also assume that the value of the today’s Hubble constant is $H_0 = 67.27\frac{\text{km}}{\text{s Mpc}}$ according to the Planck mission [71]. As it was already mentioned, the value of $\Omega_\gamma$ is small; we have also found an upper bound $\Omega_\gamma < 10^{-9}$. If the value of $\Omega_\gamma$ had been bigger than the boundary, then the epoch of the singularity would have been shifted to the epoch of recombination or later [72, 73].
Two models have been taken under statistical analysis and estimation procedure in [72]: the model with radiation (52) and with baryonic matter whose Friedmann equation is

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left( \Omega_y \Omega_{ch}^2 \frac{(K - 3)(K + 1)}{2b} + \Omega_{ch} + \Omega_{bm} + \Omega_k \right).$$  (64)

The parameter $\Omega_{bm} = \Omega_{bm,0} a^{-3}$ is related to the presence of baryonic visible matter for which the value $\Omega_{bm,0} = 0.04917$ is assumed following the Planck estimation [71]. In the following part we will consider only the model with radiation.

For the statistical analysis we have used a large set of data such as the SNIa, BAO, CMB and lensing observations, measurements of $H(z)$ for galaxies and the Alcock-Paczynski test, Union 2.1, that is, the sample of 580 supernovae [74] (see details and likelihood functions in [72]). For the estimation procedure of the model parameters the code CosmoDarkBox [72] has been used. This code uses the Metropolis-Hastings algorithm [75, 76].

The results of our statistical analysis for the Generalized Chaplygin Gas with radiation are represented in the tables 1 and 2 as well as in the figure 1. On the picture 1 there is a likelihood function with 68% and 95% confidence level. The diagram of probability density function (PDF) is presented in the figure 2.

The value of $\chi^2$ for the best fit for the model with the Generalized Chaplygin Gas and radiation is 117.722 while the value of reduced $\chi^2$ is equaled to 0.1892. Moreover, we have used the Bayesian Information Criterion (BIC) which is defined in the following way [77, 78]

$$\text{BIC} = \chi^2 + j \ln(n),$$  (65)

where $j$ is a number of parameters and $n$ is a number of data points. In our statistical analysis we have used 625 data points, hence $n = 625$. Although the number of parameters for our model is 5 ($H_0$, $\Omega_{r,0}$, $\Omega_y$, $A_s$, $\alpha$) we took $j = 3$ in computation of BIC because values for $H_0$ and $\Omega_{r,0}$ are assumed in the estimation. The value of BIC for our model with radiation is 137.036. For comparison: BIC of $\Lambda$CDM model is 125.303 (the value of $\chi^2$ is 118.866 while reduced $\chi^2$ is 0.1908). For computation of BIC of $\Lambda$CDM model we took $j = 1$ because, as previously, we assumed that the values of $H_0$, and $\Omega_{r,0}$ are already known. In consequence, the only free parameter is $\Omega_{m,0}$ representing matter. The difference between BIC of our model and $\Lambda$CDM model is $\Delta \text{BIC} = 11.733$. If the value $\Delta \text{BIC}$ is between the numbers 2 and 6 then the evidence against the model is positive in comparison to the model of the null hypothesis. If that
Table 1: The best fit and errors for model with the GCG and radiation for the case where we assume the value of $\Omega_\gamma$ from the interval $(-1.2 \times 10^{-9}, 10^{-9})$. We assume also $A_s$ from the interval $(0.67, 0.72)$, and $\alpha$ from the interval $(0, 0.06)$. The value of $\chi^2$ for the best fit is equaled 117.722.

| parameter | best fit | 68% CL | 95% CL |
|-----------|----------|--------|--------|
| $A_s$     | 0.6908   | +0.0066| +0.0104|
|           |          | -0.0069| -0.0098|
| $\alpha$  | 0.0373   | +0.0083| +0.0131|
|           |          | -0.0373| -0.0373|
| $\Omega_\gamma$ | $-1.156 \times 10^{-9}$ | $+2.156 \times 10^{-9}$ | $+2.156 \times 10^{-9}$ |
|           |          | $-0.010 \times 10^{-9}$ | $-0.015 \times 10^{-9}$ |

value is more than 6, the evidence against the model is strong [78]. Consequently, the evidence in favor $\Lambda$CDM model is strong in comparison to our model. But one should notice that all models which posses more than one parameter to be estimated will have a poor evidence in comparison to $\Lambda$CDM model.

2.2.4 Cosmological singularities

Beyond the initial singularity, there also appears another singularity in our model. It arises as the denominator of the Hubble function (52) may equal to zero (that is, the re-parametrization of time (62) will not be a diffeomorphism). Let us examine it. The picture of the function $b + \frac{4}{2} = f(A_s, \Omega_{ch,0}, \Omega_\gamma, \alpha)$ as a function of the scale factor is given in the figure 3.

As we consider singularities of FLRW models filled with perfect fluid of effective energy density $\rho_{eff}$ and pressure $p_{eff}$, we may classify them due to a well-known classification [79, 80] of finite-time, future singularities. The classification consists of four groups with respect to the behaviors of effective energy density and pressure as well as scale factors and Hubble rates. We will briefly remind their definitions.

- Type I (Big Rip): Energy density, pressure, and scale factor diverge.
- Type II (Sudden Singularity): Energy density and scale factor are finite values while pressure diverges.
Table 2: The best fit and errors for model with the GCG and radiation for the case where we assume the value of $\Omega_{\gamma}$ from the interval $(-1.2 \times 10^{-9}, 0)$. We assume also $\Lambda_s$ from the interval $(0.67, 0.72)$, and $\alpha$ from the interval $(0, 0.06)$. The value of $\chi^2$ for the best fit is equaled 117.722.

| parameter | best fit | 68% CL | 95% CL |
|-----------|----------|--------|--------|
| $\Lambda_s$ | 0.6908 | $+0.0065$ | $+0.0103$ |
|           |         | $-0.0068$ | $-0.0098$ |
| $\alpha$ | 0.0373 | $+0.0080$ | $+0.0129$ |
|           |         | $-0.0373$ | $-0.0373$ |
| $\Omega_{\gamma}$ | $-1.156 \times 10^{-9}$ | $+1.156 \times 10^{-9}$ | $+1.156 \times 10^{-9}$ |
|           |         | $-0.008 \times 10^{-9}$ | $-0.014 \times 10^{-9}$ |

Figure 1: The likelihood function of two model parameters ($\alpha$, $\Omega_{\gamma}$) with the marked 68% and 95% confidence levels for model with the GCG and radiation. We assume $H_0 = 67.27 \, \text{km s}^{-1} \text{Mpc}^{-1}$, $\Lambda_s = 0.6908$. 
Figure 2: Diagram of PDF for parameter $\alpha$ obtained as an intersection of a likelihood function for model with the GCG and radiation. Two planes of intersection likelihood function are $\Omega_\gamma = -1.15 \times 10^{-9}$ and $\Lambda_s = 0.6908$.

Figure 3: The diagram represents function $b(a) + d(a)/2$ for different values of the positive $\Omega_\gamma$ and shows that it is growing function of scale factor. Zero of this function represents a value of the scale factor for the freeze singularity $a_{\text{sing}}$. The continuous line is for $\Omega_\gamma = 10^{-10}$, the dashed line for $\Omega_\gamma = 10^{-9}$ and the dotted line is for $\Omega_\gamma = 10^{-8}$. It is assumed that $\Lambda_s = 0.7264$ and $\alpha = 0.0194$. 
• Type III (Big Freeze): Energy density and pressure diverge at a finite value of a scale factor.

• Type IV (Big Brake): Energy density and pressure go to zero at a finite value of a scale factor while higher derivatives of Hubble rate diverge.

Let us mention that the cosmological singularities classification can be enriched by adding subclasses of the above types [81, 82, 83, 84]. In our model we may express the effective quantities and equation of state $w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}}$ in terms of the potential

\begin{align}
\rho_{\text{eff}} &= -\frac{6V}{a^2}, \\
p_{\text{eff}} &= -\rho_{\text{eff}} - \frac{1}{3} \frac{d(\rho_{\text{eff}})}{d(\ln a)}, \\
w_{\text{eff}} &= -1 - \frac{1}{3} \frac{d(\ln \rho_{\text{eff}})}{d(\ln a)}
\end{align}

which diverges while the scale factor is finite. The pressure diverges too as well as \( \dot{a} \). We notice that the singularity is a singularity of acceleration because the derivative of the potential diverges. The crucial observation is that it goes to plus infinity on the left from the singular point while from the right hand side it goes to minus infinity. That behavior is represented on the picture 4 of the scale factor as a function of time: we observe the inflection point $t = t_{\text{sing}}$. One may also show the relation of the singularity as a function of the parameter $\Omega_\gamma$ which is depicted in 5. Numerical simulations showed that the singularity is sensitive on value changes of the parameter $\Omega_\gamma$ while the dependence on the parameter $\alpha$ is very weak. Therefore, for the values of $\alpha$ from the interval $(0, 1)$ the singularity does not differ.

2.2.5 Dynamical system analysis

Let us briefly discuss dynamical system analysis of the model considered in the previous sections [73]. Similarly as for $\Lambda$CDM model (see the Appendix B), our model can be investigated as a two-dimensional dynamical system of a Newtonian type [25, 85]. Since one deals with a degenerate singularity of the type III [86, 87], the phase space structure is complicated: it divides the evolutionary paths on two $\Lambda$CDM types of evolution. The full trajectories should be sewn along the singularity [88]. That singularity has an intermediate character [89] and its presence in the early evolution of the universe provides the inflationary behavior (so-called singular inflation introduced in [90]).
Figure 4: The diagram represents function $a(t)$ for positive $\Omega_\gamma$. For the scale factor of the freeze singularity, the function $a(t)$ has a vertical inflection point. The continuous line is for $\Omega_\gamma = 10^{-10}$, the dashed line is for $\Omega_\gamma = 10^{-9}$ and the dotted line is for $\Omega_\gamma = 10^{-8}$. It is assumed that $A_s = 0.7264$ and $\alpha = 0.0194$. We assume that $8\pi G = 1$ and we chose $\frac{s\, Mpc}{100 \, km}$ as a unit of time $t$. 
Figure 5: The diagram shows the relation between positive $\Omega_\gamma$ and $a_{fs}$ obtained for $A_s = 0.7264$ and $\alpha = 0.0194$. We see that this relation is a monotonic function. If $\Omega_\gamma \to 0$ then $a_{fs} \to 0$. 
The equation (53) can be seen as a Hamiltonian of the considered model with the one-dimensional potential

\[ V(a) = \frac{1}{2} a^2 \left( \Omega_y \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \Omega_k \right) \] (69)

whose motion along the energy levels \( \mathcal{H} = E = \text{const.} \) Here, the scale factor is a function of the rescaled cosmological time, that is, \( H_0 t = \tau \). In cosmology, the scale factor \( a(t) \) play a role of positional variable while a localization critical points and their type is determined by a shape of a potential. The considered dynamics is dynamics of a particle with unit mass in the potential \( V(a) \) over the energy level. Before going further, we will write down necessary notions that will be used later in the section [85, 73].

Let us consider a potential \( V(x) \) of a cosmological model. One deals with the system

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -\frac{\partial V}{\partial x}, \\
E &= \frac{y^2}{2} + V(x).
\end{align*}
\] (70)

- A static universe is represented by a critical point of the system (70). It always lies on the x-axis, that is, \( y = y_0 = 0, \ x = x_0 \).

- The point \( (x_0, 0) \) is a critical point of a Newtonian system if that is a critical point of the function of the potential \( V(x) \), it means: \( V(x) = E \), where \( E = \frac{y^2}{2} + V(x) \) is total energy of the system. Spatially flat models refer to \( y = \dot{x}; \ E = 0 \) while the ones with the spatial curvature \( k \neq 0 \) (constant) have \( E = -\frac{k}{2} \).

- A critical point \( (x_0, 0) \) is saddle one if it is a strict local maximum of the potential \( V(x) \).

- If \( (x_0, 0) \) is a strict local minimum of the analytic function \( V(x) \) then one deals with a center.

- \( (x_0, 0) \) is a cusp if it is a horizontal inflection point of the \( V(x) \).

With the above definitions one agrees that critical points of a system and their stabilities are determined by the potential which is showed in the figure 6. The configuration space is \( \{a : \ a \geq 0\} \) over the energy level \( E = 0 \) as the Hamiltonian
Figure 6: The diagram presents the potential $\tilde{V}(a)$ for $A_s = 0.7264$, $\alpha = 0.0194$ and $\Omega_\gamma = 10^{-9}$. The shaded region represents a non-physical domain forbidden for motion of a classical system for which $\dot{a}^2 \geq 0$. 

\[ V(a) \]
The dynamical system has a form $\mathcal{H}(p, a) = \frac{1}{2} p_a^2 + \tilde{V}(a) = 0$. The domain which is admissible for the universe motion is $\{ a : \tilde{V} \leq 0 \}$ with a boundary $\{ a : \tilde{V} = 0 \}$. It should be noticed that the domain $E - \tilde{V} < 0$ is forbidden for classical motion.

One may also consider different energy levels than the above one. They will correspond to different types of evolution providing different scenarios of the Universe. We classify them in the following way (see the picture 7):

1. $O_1$ — oscillating universes with initial singularities;

2. $O_2$ — ‘oscillatory solutions’ without the initial and final singularity but with the freeze singularity;

3. $B$ — bouncing solutions;

4. $E_1, E_2$ — solutions representing the static Einstein universe;

5. $A_1$ — the Einstein-de Sitter universe starting from the initial singularity and approaching asymptotically static Einstein universe;
6. $A_2$ — a universe starting asymptotically from the Einstein universe, next it undergoes the freeze singularity and approaches to a maximum size. After approaching this state it collapses to the Einstein solution $E_1$ through the freeze singularity;

7. $A_1$ — expanding universe from the initial singularity toward to the Einstein universe $E_2$ with an intermediate state of the freeze singularity;

8. EM — an expanding and emerging universe from a static $E_2$ solution (Lemaitre-Eddington type of solution);

9. IM — an inflectional model: the relation $a(t)$ possesses an inflection point. That is an expanding universe from the initial singularity undergoing the freeze type of singularity.

The last two solutions EM and IM lie above the maximum of the potential $\tilde{V}$.

It should be mentioned that the singularity appearing just after the Big Bang one which is beyond the standard classification (see [79, 80] and the discussion in the (2.2.4). The acceleration $\ddot{a}$ at the singularity point is undefined: left-hand side limit of the derivative of the potential is positive while the right-hand side limit has a negative value. We treat it as two sewn singularities: the first one is of the type III while the second one also belongs to that type but with reverse type. It will be better understood if we construct the phase portrait of the system which will also allow us to classify all evolution path of the phase space. Firstly, let us defined a new potential $V(a)$

$$V(a) = -\frac{a^2}{2} \left( \Omega_\gamma \Omega_{ch}^2 \left( \frac{(K-3)(K+1)}{2b} + \Omega_{ch} + \Omega_k \right) \right)$$  \hspace{1cm} (71)

such that $a'^2 = -2V(a)$ where we have defined the new re-parametrization of time as $' \equiv \frac{d}{d\sigma} = \frac{b+\frac{4}{b}}{b} \frac{d}{d\tau}$. The system is now

$$p = \dot{a} = x$$ \hspace{1cm} (72)

$$\ddot{a} = \dot{x} = -\frac{\partial V(a)}{\partial a} = \frac{x^2}{m} \frac{\partial m}{\partial a} - m^2 \frac{\partial V(a)}{\partial a},$$ \hspace{1cm} (73)

with the quantity $m = \frac{b}{b+\frac{4}{b}}$ for simplification. We will divide dynamics into two parts: for the scale factor $a < a_{\text{sing}}$ and $a > a_{\text{sing}}$ so it belongs to the class of the sewn dynamical systems [88]. The configuration space is glued along the
Figure 8: The diagram represents the phase portrait of the system (72-73) for positive $\Omega_\gamma$. The red trajectories represent the spatially flat universe. Trajectories under the top red trajectory and below the bottom red trajectory represent models with negative curvature. Positive curvature models are represented by the trajectories between the top and bottom red ones.
singularity $a_{\text{fsing}}$. The dynamical system (after the re-parametrization) for the first case is

$$\dot{a} = x,$$

$$\dot{x} = \frac{x^2}{m} \frac{\partial m}{\partial a} - m^2 \frac{\partial V_1(a)}{\partial a},$$

where $V_1 = V(-\eta(a - a_s) + 1)$ with respect to the new time $\sigma$ while $\eta(a)$ notes the Heaviside function while the second one $a > a_{\text{fsing}}$ is analogously given by the function $V_2 = V\eta(a - a_s)$.

Now on, we are able to draw the phase portrait of system (72-73) for positive $\Omega_{\gamma}$. It is presented on the picture 8. The case of negative values of $\Omega_{\gamma}$ can be found in [73] and will not be considered here. The red trajectories concern evolution of spatially flat models: the Universe expands, starting from the initial singularity. It goes through the freeze singularity and after the accelerating phase it goes toward the de Sitter attractor. Trajectories under the top red trajectory and below the bottom red trajectory are models with negative curvature while the ones with positive spatial curvature lie between the red trajectories. The point of sewing is located at infinity ($a = a_{\text{fsing}}$, $\dot{a} \equiv x = \infty$). Note that all trajectories of open models are passing through the freeze singularity. The phase portrait possesses the reflectional symmetry $x \to -x$. Trajectories from the domain $x < 0$ continue their evolution into domain $x > 0$. Due to this symmetry one can identify the corresponding point on the line $\{b = 0\}$ and make from the line $\{b = 0\}$ a circle $S^1$. Therefore the phase space is a cylinder. The line $\{b = 0\}$ is not shown on the picture 8.

2.2.6 Conclusions

We have been discussing a cosmological model based on the modified Lagrangian $f(R) = R + \gamma R^2$ considered in Palatini formalism. As a source we have used Generalized Chaplygin Gas. After the physical parametrization we were able to find the best fits for the density parameters $\Omega_{\gamma}, \Lambda_s, \alpha, H_0$. As expected, the value of the parameter $\Omega_{\gamma}$ must be very small and due to that fact it has an insignificant influence on the physics in our Solar System. It was shown that the model provides inflationary scenario as well as it reaches a good agreement with the present day experimental data. The inflation is given by a singularity of the type III (freeze singularity) which appeared naturally as a pole in the Newtonian potential since we had treated the dynamics of the Universe as a dynamics of a unit-mass particle in a potential. The pictures 4 and 5 show that the value of the scale factor depends strictly on the values of the density parameter $\Omega_{\gamma}$ while
the dependence on the other parameters was very weak and hence we could neglect their influences. With these properties, we could restrict the value of $\Omega_\gamma$ to the interval $[0; 10^{-9}]$ in order to have the singularity before the recombination epoch. That guaranteed the preservation of post-recombination physics of the Universe. Such construction provides the four phases of the cosmic evolution: the decelerating phase dominated by matter, an intermediate inflationary phase corresponding to III type singularity, a phase of matter domination (decelerating phase) and finally, the phase of acceleration of current universe. Moreover, the Big Bang singularity is also preserved as we incorporated the radiation term into the Friedmann equation. We call the singularity, which provides the inflation, a singularity of the type III but some comment should be given here. A freeze singularity, as well as Big Rip and Sudden singularities, has not a well-defined acceleration $\ddot{a}$; it diverges. Moreover, the left and right limits at the singular point of the acceleration differ in our case. The scale factor is finite whilst the effective energy density, pressure and the Hubble rate diverge. It suggests that we deal with a freeze singularity which are characterized by the generalized Chaplygin equation of state [91]. A new feature is the one already mentioned: the acceleration is infinite, that is, it goes to the plus infinity on the left from the singular point and to the minus infinity from the right hand side. Due to that fact (see the picture 8) one may glue the past and future trajectories in the singular point. Such type of a weak singularity has been called a "degenerate freeze singularity". The degenerate freeze singularity vanishes if $\Omega_\gamma = 0$ and therefore one deals with the standard $\Lambda$CDM model. It allows us to suppose that the presence of such singularity is related to Palatini formalism. Further studies on that topic are urgent and will be performed in the nearest future.

Furthermore, using the dynamical system approach, we were able to examine all evolutionary paths of our cosmological model. The phase space structure of dynamics is organized through the two saddle points which represent static Einstein universes. The degenerate freeze singularity lies between the saddle points on the x-axis. The set of sewn trajectories is a critical point located at the infinity ($\dot{a} = \infty, a = a_{\text{fsing}}$) and they pass through that point. Let us notice that on the phase portrait 8 we have drawn a red trajectory which represents the flat model ($k = 0$). It divides the rest of the trajectories into domains occupied by closed ($k = +1$) and open ($k = -1$) models. The closed model’s trajectories are located between the top and the bottom red ones. We see that all open models possess the degenerate freeze singularity.

One may ask a question whether the existence of singularities disqualifies Palatini $f(R)$ theories for it seems that they can be artifacts of the formalism. It could limit its application. However, we would like to notice that we have just
considered a very simple modification of General Relativity. Other forms of $f(R)$ functionals, as for example higher order terms in $R$ or its inverse, may happen not to possess singularities which we do not know how to treat on the classical level. One believes that Quantum Gravity will solve the problem of singularities.
Hybrid metric - Palatini gravity was introduced for the first time by T. Harko et al. [92] in order to avoid disadvantages of $f(R)$ gravity, both in metric and in Palatini formalisms, while its modified dynamics provides self-accelerated cosmic expansion without inserting exotic dark energy into the equations. The presented approach includes corrections to General Relativity which have a form of an arbitrary function $f(R)$ considered in Palatini formalism. It means, similarly to Palatini $f(R)$ gravity, one deals with two independent objects, that is, a metric $g_{\mu\nu}$ and a connection $\tilde{\Gamma}$ which turns out to be a Levi-Civita connection of a metric conformally related to the metric $g_{\mu\nu}$. Moreover, one may express the theory as a scalar-tensor one which makes the studies on the hybrid gravity easier to handle. Post-Newtonian analysis (weak-field and slow-motion limits) [92, 93] shows that the theory passes the Solar system constraints even for light scalar fields. The coupling of the scalar field to the local system depends on the current cosmic amplitude which is supposed to be sufficiently small. That affects the dynamics of galaxies and evolution of the Universe: the modified gravitational potential [92] seems to describe the flatness of rotational curves of galaxies very well. The Cauchy problem was also considered [94]. Moreover, some specific example of the hybrid gravity model [95] supports the stability of wormholes solutions. There are also speculations that there is no need to introduce an exotic matter violating null energy condition in order to satisfy the wormhole throat condition because such geometry can be obtained by the higher order curvature terms.

Cosmology provided by hybrid gravity has been also extensively examined [96]. By introducing a field $X = \kappa T + R$ (showing the deviation from the GR trace equation $R = -\kappa T$) one may parametrize the astrophysical scales according to the cosmic densities because it is considered as the chameleon mechanism. At short scales, as Solar System one, the variable $X \to 0$ reduces the model to Einstein one while its value grows at larges scales in order to have the effects of dark energy and dark matter. Any cosmological behavior, similarly to other ETG’s, can be achieve by an effective potential of scalar field which affects in the form of the function $f(R)$. The examination of the stability of Einstein static Universe was also done [97]. For that purpose they considered linear homogeneous perturbations of the model and showed that there exist stable solutions. Dynam-
ics of scalar perturbations was also studied for that framework and expressed in Newtonian and synchronous gauges [98]. What is interesting, they found a family of functions $f(R)$ which reproduce a cosmology that is not distinguishable from the one provided by $\Lambda$CDM model while the meaningful modifications from GR appear in the distant past what suggests a different scenario for the early Universe. Further analysis on that topic is found in [99] where they put observational data constraints on the scalar field value being in an agreement with allowed early-time deviations from GR. Dynamical system analysis was also applied to the hybrid gravity [100] where they examined the model of the form $f(R) \sim R^n$. They showed that for this case the points characterizing GR limits are unstable, it means, there is no value of parameters and initial conditions that lead to cosmic evolutions indistinguishable from GR. Furthermore, thermodynamics properties have been also studied at the apparent horizon in the FRLW background [101]. In the non-equilibrium description (continuity equation of the dark fluid coming from the modifications is not satisfied) they obtained that a new entropy production is generated violating the first law of thermodynamics. Assuming the continuity equation for the dark fluid, the law is not violated (equilibrium approach). On the other hand, the second law of thermodynamics for hybrid gravity is satisfied but with extra restrictions imposed on the cosmological even horizon.

The theory was also investigated from astrophysical point of view. The extension of the virial theorem due to hybrid gravity [102] might be a tool for observational test of that theory. One applies the theorem in order to obtain the mean density of galaxies, clusters etc., to determine the total mass of such objects as well as the stability of gravitationally bounded systems. They showed that most of the mass in a cluster is in the form of the geometric mass, that is, it is included in the extra geometric terms (scalar field) modifying Einstein’s gravity. That scalar field contribution to the gravitational energy play a role of dark matter at the galactic level. In [103] all quantities including the scalar field were expressed in terms of observable parameters opening a possibility to direct tests of hybrid gravity. Their theoretical considerations indicate that galactic rotation curves and the mass discrepancy in galaxies can be explained by the hybrid gravity field equations being perfectly consistent with observational data. Similar conclusions are given by [104], where they used the observational data of stars moving around the centre of our Galaxy.

Hybrid metric - Palatini gravity survived long enough to see its generalisation: as a brane system, that is five-dimensional hybrid gravity [105] and four-dimensional one [106] which was done by considering arbitrary functions both the metric and Palatini curvature scalars in the gravitational action. The authors
proved that such a generalisation leads to GR with two scalar fields coupled to each other when one uses the conformal transformation procedure. The model also provides the late time acceleration. Unfortunately, in [107] one shows that this gravity model possesses propagating degrees of freedom which are ghosts or tachyons. The only hybrid theory (they have also considered $f(R, R_{\mu\nu}, R^{\mu\nu})$ case) that is viable and reduces to GR is hybrid metric-Palatini one which we are also going to study. Our examinations will concern a selection procedure for viable models given by Lie and Noether symmetries. It will also allow us to solve cosmic evolutionary equations as well as to find exact solutions of Wheeler-DeWitt equations for the previously obtained models. The results of the analysis were published in [108] while the similar Lie symmetries analysis for Bianchi spacetime in the hybrid gravity framework was investigated in [109].

3.1 INTRODUCTION OF THE MODEL

We are going to consider a special form of the action introduced in [92] which consists of two parts: one deals with the standard Hilbert - Einstein action and an additional term which is constructed with arbitrary function of the Palatini curvature scalar $\mathcal{R}$:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R + f(\mathcal{R})] + S_m. \quad (76)$$

The Palatini scalar $\mathcal{R}$ is built with two independent object which are the metric $g$ (of Lorentzian signature) and the connection $\hat{\Gamma}$:

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu,\alpha} - \hat{\Gamma}^\alpha_{\mu,\alpha\nu} + \hat{\Gamma}^\alpha_{\nu,\alpha\mu} - \hat{\Gamma}^\alpha_{\mu\lambda} \hat{\Gamma}^\lambda_{\nu\alpha}. \quad (77)$$

The third term, $S_m = \int d^4x \sqrt{-g} L_m$, stands for the standard matter action and in this approach we assume that is connection independent. The Ricci scalar $\mathcal{R}$ is obtained from the metric $g_{\mu\nu}$.

The variation of the above action with respect to the metric gives the gravitational field equations (Einstein’s modified equations)

$$G_{\mu\nu} + f'(\mathcal{R}) R_{\mu\nu} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (78)$$

where $G_{\mu\nu}$ is the Einstein tensor of the metric $g$ and $f'(\mathcal{R}) = df(\mathcal{R})/d\mathcal{R}$. The matter energy - momentum tensor was defined as usual, it means

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}. \quad (79)$$
The variation with respect to the connection \( \hat{\Gamma} \) provides (as in the pure \( f(R) \) Palatini gravity) the following equation

\[
\tilde{\nabla}_\alpha (\sqrt{-g} f'(R) g^{\mu \nu}) = 0
\]  

(80)

which reveals that the independent connection is the Levi-Civita connection of the metric \( \tilde{g}_{\mu \nu} = f'(R) g_{\mu \nu} \). It means that the metrics \( g \) and \( \tilde{g} \) are conformally related and the function \( f'(R) \) is a conformal factor. It requires that \( f'(R) \) is a positive defined function. There is a well-known relation between the two curvatures [110]

\[
R_{\mu \nu} = R_{\mu \nu} + \frac{3}{2} \frac{F(R)_{,\mu} F(R)_{,\nu}}{F^2(R)} - \frac{1}{F(R)} \nabla_\mu F(R)_{,\nu} - \frac{1}{2} \frac{g_{\mu \nu} \Box F(R)}{F(R)},
\]  

(81)

where \( ;_\mu \equiv \nabla_\mu \) is the metric connection, \( \Box \equiv \nabla^\mu \nabla_\mu \) d’Alembertian operator, and we have defined \( F(R) := f'(R) \).

The trace of the modified Einstein equations (78) is called the hybrid structural equation or hybrid master equation. Assuming that \( f(R) \) has analytic solutions, one may obtain that the Palatini curvature \( R \) is expressed in terms of the variable \( X \) from the algebraic relation

\[
F(R) R - 2 f'(R) = \kappa T + R \equiv X.
\]  

(82)

The variable \( X \) measures the deviation from the General Relativity trace equation \( R = -\kappa T \) and with that definition one has \( F(R(X)) \equiv F(X) \). It is possible to reformulate the previous field equations in terms of the variable \( X \) and the function \( F(X) \) [96].

The action of the hybrid gravity theory can be also transformed into a scalar-tensor theory action in the similar manner as for the pure metric and Palatini case [13, 111]. One needs to introduce an auxiliary field \( E \) such that [92, 96]

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} [R + f(E) + f'_E (R - E)] + S_m,
\]  

(83)

where \( f_E \equiv \frac{df(E)}{dE} \). The field \( E \) is dynamically equivalent to the Palatini scalar \( R \) if \( f''(R) \neq 0 \). Let us define a scalar field and its potential

\[
\phi \equiv f'(E), \quad V(\phi) = E f'(E) - f(E).
\]  

(84)

Applying them to the above action it becomes

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} [R + \phi R - V(\phi)] + S_m.
\]  

(85)
The variation of (85) with respect to the metric, the scalar field $\phi$ and the connection provides the field equations

\[ G_{\mu\nu} + \phi R_{\mu\nu} - \frac{1}{2}(\phi R - V)g_{\mu\nu} = \kappa T, \]  

(86)

\[ R - V_{,\phi} = 0, \]  

(87)

\[ \hat{\nabla}_{\alpha}(\sqrt{-g}f'(|R|)g^{\mu\nu}) = 0. \]  

(88)

Using the conformal relation (81) between $R$ and $\mathcal{R}$ (let us remind that we are using the metric $g_{\mu\nu}$ for lowering and raising indices), that is

\[ \mathcal{R} = R + \frac{3}{2\phi^2} \nabla_{\mu} \phi \nabla^{\mu} \phi - \frac{3}{\phi} \Box \phi, \]  

(89)

the standard scalar-tensor form can be obtained:

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ (1 + \phi)R + \frac{3}{2\phi} \nabla^{\mu} \phi \nabla_{\mu} \phi - V(\phi) \right]. \]  

(90)

Applying the conformal relations to the equation (86) and (87) together with the trace of (86) we have

\[ (1 + \phi)G_{\mu\nu} - \frac{3}{4\phi} g_{\mu\nu} \nabla^{\alpha} \phi \nabla_{\alpha} \phi + \frac{3}{2\phi} \nabla_{\mu} \phi \nabla_{\nu} \phi \]

\[ + g_{\mu\nu} \Box \phi - \nabla_{\mu} \nabla_{\nu} \phi + g_{\mu\nu} V = \kappa T_{\mu\nu}, \]  

(91)

\[ -\Box \phi + \frac{1}{2\phi} \nabla^{\alpha} \phi \nabla_{\alpha} \phi + \frac{\phi}{3} (2V - (1 + \phi)V_{,\phi}) = \frac{\kappa \phi}{3} T. \]  

(92)

The equation (92) is the second-order evolution equation for the scalar field $\phi$ and can be interpreted as an effective Klein-Gordon equation. It is important to notice that in hybrid gravity theory, unlike in the Palatini case, the scalar field is dynamical (24).

### 3.1.1 Hybrid gravity cosmology

In order to examine the hybrid gravity model let us consider Friedmann - Robertson - Lemaitre - Walker (FRLW) spatially flat metric:

\[ ds^2_{\text{FRLW}} = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right), \]  

(93)

where $a(t)$ is the scale factor. One easily finds that the Ricci scalar is $R = 6(2H^2 + \dot{H})$ where one defines the Hubble rate function $H = \frac{\dot{a}(t)}{a(t)}$. The dot denotes the
The cosmic time derivative, it means $\frac{d}{dt}$. The field equations in the scalar-tensor representation (91), (92) are

$$3H^2 = \frac{1}{1 + \phi} \left[ \kappa \rho + \frac{V}{2} - 3\phi \left( H + \frac{\dot{\phi}}{4\phi} \right) \right],$$

$$2\dot{H} = \frac{1}{1 + \phi} \left[ -\kappa (\rho + p) + H\dot{\phi} + \frac{3\phi^2}{2} - \frac{\dot{\phi}}{\phi} \right],$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\phi^2}{2\phi} + \frac{\phi}{3} [2V - (1 + \phi)V_{,\phi}] = -\frac{\kappa \phi}{3} (\rho - 3p).$$

$\rho$ and $p$ are the energy density and pressure of the cosmic fluid coming from the trace of perfect fluid energy - momentum tensor:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$

$$T = g^{\mu\nu}T_{\mu\nu} = 3p - \rho,$$

where $u^\mu = (1,0,0,0)$ is a co-moving observer with the normalization condition $u^\alpha u_\alpha = -1$. The conservation of the matter energy - momentum tensor is (assuming that it is a metric theory $\nabla_\mu T^{\mu\nu} = 0$)

$$\dot{\rho} + 3H(\rho + p) = 0.$$

The above Klein - Gordon equation (96) can be written in the following form [92]:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\phi^2}{2\phi} + M_\phi^2(T) \phi = 0,$$

where $M_\phi^2(T) := m_\phi^2 - \frac{1}{3} \kappa T = \frac{1}{3} [2V - (1 + \phi)V_{,\phi} - \kappa T]$. Assuming that the scalar field $\phi$ is not rapidly changing, then $\frac{\phi^2}{2\phi} \sim 0$ and (100) represents a massive scalar field on a FRLW background. The dynamical behavior of the scalar field at late times depends on a form of the potential appearing in the Klein - Gordon equation. In [92] the authors consider two models which are consistent at Solar System and cosmological scale with asymptotically de Sitter behavior. One may also examine the deceleration parameter $[46, 96]$ defined by

$$q = \frac{\dot{H}}{H^2} - 1 = -\frac{\ddot{H}}{H^2} - 1.$$
indicates models which start from decelerating states ($q > 0$) and end in accelerated ones. Such models can be interpreted as the turning point between a structure formation epoch and dark energy [96].

Starting from now on, we will skip the matter action $S_m$ in (76) so we will always consider vacuum case ($T_{\mu \nu} = 0$) unless one indicates directly an introduction of standard matter. In the absence of matter, we may rewrite the modified Friedmann equations (94) and (95) as:

$$3H^2 = \kappa \rho_{\text{eff}},$$  \hspace{1cm} (102)

$$2\dot{H} = -\kappa (\rho_{\text{eff}} + p_{\text{eff}}),$$ \hspace{1cm} (103)

where $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are effective energy density and pressure given by

$$(1 + \phi)\kappa \rho_{\text{eff}} = -\frac{3}{4\phi} \dot{\phi}^2 + \kappa V - 3H\dot{\phi}$$  \hspace{1cm} (104)

$$(1 + \phi)\kappa p_{\text{eff}} = -\frac{3}{4\phi} \dot{\phi}^2 - \kappa V + 2H\dot{\phi} + \ddot{\phi}.$$ \hspace{1cm} (105)

Using the formulas (102) and (103) the deceleration parameter is

$$q = \frac{3}{2} \frac{\rho_{\text{eff}} + p_{\text{eff}}}{\rho_{\text{eff}}} - 1$$ \hspace{1cm} (106)

so with the definitions (104) and (105) with the Klein-Gordon equation (100) in vacuum one gets the dependence on the scalar field:

$$q = \frac{3}{2} \left( \frac{\dot{\phi}^2}{\dot{\phi}^2} + 4H\dot{\phi} + 2\kappa \phi [2V - (1 + \phi)V,\phi] \right) - 1.$$ \hspace{1cm} (107)

As mentioned above, dynamics of cosmological models may be studied with respect to the deceleration parameter. In [96] the authors showed the existence of a few viable accelerating solutions (recall that $\phi \equiv \frac{df(R)}{dR}$) for the hybrid gravity model among which there are power-law accelerating models.

From the cosmological point of view, hybrid gravity is a promising and interesting model which should be further investigated. Solutions, it means forms of the scalar field potential or functions $f(R)$, should satisfy restricted requirements. Since the field equations are quite difficult and demand a form of $V(\phi)$ (when for example one wants to solve Wheeler-DeWitt equation), we need special tools to deal with them. Due to that fact, we would like to study FRLW cosmology of the hybrid gravity model with respect to symmetries. For the further investigations we will need a point-like Lagrangian deduced from (90)

$$\mathcal{L} = 6a\dot{a}^2(1 + \phi) + 6a^2 \dot{a} \dot{\phi} + \frac{3}{2\phi} a^3 \dot{\phi}^2 + a^3 V(\phi).$$ \hspace{1cm} (108)
3.2 NOETHER SYMMETRIES IN COSMOLOGY

Using Noether symmetries approach to cosmology is nothing new but the huge number of works considering the theorem in cosmological applications (see for example [112, 113, 114, 115, 116, 117, 118]) just shows that it is a powerful tool in theoretical physics. Noether symmetries do not only allow to solve differential equations by finding integrals of motion but also, in the case of Extended Theories of Gravity, they select models which are viable due to Noether symmetries of the system. They are a physical criterion as each symmetry is related to a conserved quantity which has a physical meaning [113]. Moreover, as it was shown in [119], Noether symmetries are a selection rule to recover classical behaviors in cosmic evolution:

**Theorem 1** In the semi-classical limit of quantum cosmology and in the framework of minisuperspace approximation, the reduction procedure of dynamics, due to existence of Noether symmetries, allows to select a subset of the solution of Wheeler-DeWitt equation where oscillatory behaviors are found. As consequence, correlations between coordinates and canonical conjugate momenta emerge so that classical cosmological solutions can be recovered. Vice-versa, if a subset of the solution of WDW equation has an oscillatory behavior, due to

\[-i\partial_{\bar{j}}|\Psi\rangle = \Sigma_{j}|\Psi\rangle, \quad j = 1, \ldots, k, \quad k - number of symmetries, \tag{109}\]

where $|\Psi\rangle$ is a wave function of the Universe while $\Sigma$ is a constant of motion (first integral), conserved momenta have to exist and Noether symmetries are present. In other words, Noether symmetries select classical universes.

From the above conclusion of Capozziello and Lambiase [119] the importance of Noether symmetries in (quantum) cosmology is evident. Let us briefly discuss it.

In the canonical quantization approach (based on $3+1$ decomposition of Einstein’s field equations, also called ADM formalism [120]) to quantum gravity (see for example [121]) one deals with so-called superspace of geometrodynamics which is an infinite-dimensional configuration space where the classical dynamics takes place. It is a space of all 3-metric and matter field configurations which are defined on 3-manifold. The superspace metric is constructed with 3-metrics and matter fields, and it appears in the kinetic term of Hamiltonian constraint (see for example [122]). A quantum state of the Universe is represented by a wave functional not depending explicitly on the coordinate time $t$. Wheeler-DeWitt (WDW) equation is, roughly speaking, a quantized version of Einstein’s
field equations (together with so-called momentum constraint \([121, 122, 123]\)). It is a second-order hyperbolic functional differential equation describing the dynamical evolution of the wave function in superspace, that is, the wave function of the Universe. There are many difficulties arising from this approach. One of them is the infinite dimension of the superspace that makes WDW equations impossible to fully integrate. In order to deal with that problem one consider a toy model of quantum cosmology by imposing restrictions on the metric and matter fields engaged to a game. Such a simplified superspace is called minisuperspace. The simplest one possesses homogeneous and isotropic metrics, and matter fields. Corresponding supermetric is called minisupermetric. With finite dimension of configuration spaces one may solve WDW equations and try to interpret obtained results. The popular interpretation is given by Hartle; so-called Hartle criterion \([124, 119]\) says that strong peaks of the wave function of the Universe indicates classical trajectories (it means universes). It corresponds to the oscillatory behavior of the wave function (see the analogical situation in the non-relativistic quantum mechanics) for which the system behaves like a classical one while exponential regime of the wave function is classically forbidden.

Coming back to Noether symmetries of a minisuperspace cosmological model under consideration, one shows \([119]\) that conserved momenta are connected to oscillatory parts of the wave function in the directions of corresponding symmetries. Finding Noether symmetries and corresponding constants of motion allows not only to reduce and solve differential equations but also to specify oscillatory parts of the wave function which is an exact solution of the WDW equation.

### 3.2.1 Noether symmetries of hybrid gravity model

We are going to use Noether symmetries in order to solve classical equations of motion derived from the hybrid gravity Lagrangian \((108)\). We will follow the approach developed in \([125]\) (see notes in \((A.2.3)\) for details). One observes that the Lagrangian \((108)\)

\[
\mathcal{L} = 6a\dot{a}^2(1 + \phi) + 6a^2\dot{\phi} + \frac{3}{2\phi}a^3\dot{\phi}^2 + a^3V(\phi) \tag{110}
\]

has the standard form \(\mathcal{L} = T - V_{\text{eff}}\). The kinetic energy \(T = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\) indicates the minisuperspace metric

\[
ds_{(2)}^2 = 12a(1 + \phi)\, da^2 + 12a^2\, d\phi^2 + \frac{3}{\phi}a^3\, d\phi^2, \tag{111}
\]
while $\mathcal{V}_{\text{eff}} = -a^3 V(\phi)$ stands for effective potential. Applying the theorem 4 to the Lagrangian (108) we are able to find such forms of potential $V(\phi)$ that the hybrid gravity Lagrangian will admit Noether point symmetries.

Since the Lagrangian is time-independent, it admits the Noether symmetry $\partial_t$ with the Hamiltonian (it means the first Friedmann equation (102)) as a conservation law, that is

$$E_H = 6a (1+\phi) a^2 + 6a^2 \dot{a} + \frac{3}{2\phi} a^3 \dot{\phi}^2 - a^3 V(\phi). \quad (112)$$

Let us recall that we are considering a vacuum case so the Einstein equation $G_0^0 = 0$ being a constraint gives that $E_H = 0$. The Noether condition (342) gives us, together with $\xi = 1$, the following system of partial differential equations:

$$\eta (1+\phi) + a^2 \rho_a + a(\rho + 2(1+\phi) n_a) = 0, \quad (113)$$
$$\phi (3\eta + 4\phi n_a - a(\rho - 2 \phi \rho_a)) = 0, \quad (114)$$
$$4\phi^2 \eta, \phi + a^2 \rho_a + 2\alpha \rho (24 \eta n_a, \phi + \rho_a + n_a) = 0, \quad (115)$$
$$2(1+\phi) n_t + a \rho_t = 0, \quad (116)$$
$$2\eta_t + \frac{\alpha}{\phi} \rho_t = 0, \quad (117)$$
$$a \rho V'(\phi) + 3V(\phi) \eta = 0. \quad (118)$$

Since the potential $V(\phi)$ does not depend on the scale factor, $\eta = 0$ and hence the potential is a constant value. Applying that result into the equations (113), (114) and (116) we found that $\rho = \frac{\sqrt{\phi}}{a}$ and therefore we have just shown that there also exists an extra Noether symmetry for a constant potential $V(\phi) = V_0$:

$$X_1 = \frac{\sqrt{\phi}}{a} \partial_\phi \quad (119)$$

while the corresponding Noether integral is of the form

$$I_1 = 3 - \frac{a}{\sqrt{\phi}} (2\phi \dot{a} + a \phi). \quad (120)$$

Instead of looking for solutions of field equations in $(a, \phi)$ coordinates (which would be tough), let us perform the following coordinate transformation

$$a = u^2, \quad \phi = v^2 u^{-\frac{2}{3}}, \quad (121)$$

under which the Lagrangian of the field equations became

$$\mathcal{L}(u, v, \dot{u}, \dot{v}) = \frac{8}{3} \dot{u}^2 + 6u^2 \dot{v}^2 + V_0 u^2. \quad (122)$$
In the \((u, v)\) variables the field equations have the forms

\[
\frac{8}{3} u^2 + 6 u \dot{u} \dot{v}^2 - V_0 u^2 = 0, \tag{123}
\]
\[
\dot{u} - \frac{3}{4} u^{-\frac{1}{2}} \dot{v}^2 - \frac{3}{8} V_0 u = 0, \tag{124}
\]
\[
\ddot{v} + \frac{2}{3u} \dot{u} \dot{v} = 0, \tag{125}
\]
while we wrote the Noether integral as \(I_1 = u \dot{u} \dot{v}\). Replacing one of the velocities by \(\dot{v} = \dot{I}_1 u^{-\frac{2}{3}}\), we are able to write the general solution of the above system as

\[
\int \frac{du}{\sqrt{\frac{3}{8} V_0 u^2 - \frac{2}{4} \dot{I}_1 u^{-\frac{2}{3}}}} = \int dt. \tag{126}
\]

The solution of (126) exists but the scale factor cannot be expressed in terms of the cosmic time \(t\) in an algebraic and easy way. But we can rewrite the first Friedmann equation (126) in terms of the scale factor \(a(t)\) as the Hubble function (recall that \(H = \frac{\dot{a}}{a}\)):

\[
\frac{H^2}{H_0^2} = \left( \Omega_\Lambda + \Omega_r a^{-4} \right), \tag{127}
\]

where we have defined the density parameters of the cosmological constant and radiation as

\[
\Omega_\Lambda = \frac{V_0}{6 H_0^2}, \quad \text{and} \quad \Omega_r = \frac{\dot{I}_1^2}{H_0^4}, \tag{128}
\]

respectively. Let us just notice that in order to have a physical solution, the integral of motion has to be a complex value. Comparing the result to the \(\Lambda CDM\) model one sees that the hybrid gravity model introduces radiation term - let us underline that we are considering vacuum case from the very beginning. It means that such a geometric modification provides "matter fluids" which are a cosmological constant responsible for the late time acceleration (as the \(\Lambda CDM\) model does) and radiation. In the contrary to \(\Lambda CDM\) model, the radiation term appears "naturally" in the Friedmann equation derived from hybrid gravity while in the first case one needs to put it by hand since radiation fluid does not contribute to the trace of an energy momentum tensor.

One may also perform a similar analysis after introducing dust to the model, it means introducing \(\rho_D = \rho_{m,0} a^{-3}\) in (104). Hence, the equation (123) becomes

\[
\frac{8}{3} \ddot{u}^2 + 6 u \dot{u} \dot{v}^2 - V_0 u^2 = \rho_{m,0}. \]
The analytical solution is written as
\[ \int \frac{du}{\sqrt{\frac{3}{8} V_0 u^2 + \frac{3}{8} \rho_{m0} - \frac{9}{4} \bar{I}_1^2 u^{-\frac{3}{2}}}} = \int dt. \]

Simply, we may also write the Hubble function as
\[ \frac{H^2}{H_0^2} = (\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}), \]
where we have defined another density parameter corresponding to the dust \( \Omega_m = \frac{\rho_{m0}}{6H_0^2}. \)

3.2.2 Noether symmetries of conformal hybrid gravity Lagrangian

The examination of hybrid gravity with respect to Noether symmetries gave us a trivial solution. However, one may try to perform a conformal transformation of our Lagrangian (108) and apply results of [126, 127, 128] (see the theorem 6 and the section (A.5)). As the dynamical system provided by the Lagrangian (108) is conformally invariant \((E_H = 0)\) one is going to look for new solutions in conformal frames. We will focus on the case when the lapse function is a function of the scale factor, it means \(d\tau = N(a) dt\)

\[ ds^2 = -N^{-2} (a(\tau)) d\tau^2 + a^2 (\tau) (dx^2 + dy^2 + dz^2). \] (129)

One may also consider the lapse function which depends on a scalar field and show that hybrid gravity is conformally related to a Brans-Dicke-like scalar-tensor theory [108].

The already mentioned Lagrangian for the conformal FRLW spacetime (129) is given as:

\[ \mathcal{L} (a, \phi, a', \phi') = \frac{a^3 V(\phi)}{N(a)} + N(a) \left[ 6a(1+\phi) a'^2 + 6a^2 a'\phi' + \frac{3}{2\phi} a^3 \phi'^2 \right], \] (130)

where the prime denotes \(d/d\tau\). The conformal kinetic metric and Ricci scalar of that metric are given by

\[ ds_{(2)}^2 = N(a) \left( 12a(1+\phi) da^2 + 12a^2 d\phi d\phi^2 \right), \] (131)
and

\[ R_{(2)} = -\frac{a^2 N a a - a^2 N^2}{12 a^3 N^3}, \]

respectively. We will consider the case when \( R_{(2)} = 0 \) so the problem reduces to the dynamics of Newtonian physics \([126]\). Now the lapse function is of the form

\[ N(a) = a^{-1} e^{N_{\phi} a}. \]  \hspace{1cm} (132)

Applying the lapse function (132) into the Lagrangian (130) we may use the geometric approach developed in \([125]\). The set of differential equations, that we need to solve, coming from the Noether symmetry approach is

\[ \xi_{,a} = 0, \]  \hspace{1cm} (133)

\[ \xi_{,\phi} = 0 \]  \hspace{1cm} (134)

\[ \rho + 2(1 + \phi) \eta_{,a} + a \rho_{,a} = 0, \]  \hspace{1cm} (135)

\[ 2\phi(\eta + 2\phi \eta_{,\phi}) - a(\rho - 2\phi \rho_{,\phi}) = 0, \]  \hspace{1cm} (136)

\[ 4\phi(1 + \phi) + (a^2 \rho_{,a} + 2\phi[a + a(\rho_{,\phi} + \eta_{,a})]) = 0, \]  \hspace{1cm} (137)

\[ \rho a V''(\phi) + 4\eta V(\phi) = 0, \]  \hspace{1cm} (138)

where we have already used that \( \xi_{,t} = 0 \). The case when \( \xi_{,t} \neq 0 \) was considered in \([108]\) where computer algebra was used. Subtracting the equation (137) from (136) and using (135) one finds that \( a \eta_{,a} = 2\phi \eta_{,\phi} \). Let us examine the case for the constant \( \eta = -\frac{1}{2} \). That ansatz provides the equation (135) as \( \rho = \frac{A(\phi)}{a} \), where \( A(\phi) \) is an unknown function of the scalar field. Applying the result into (136) or (137) one gets the equation determining the function \( A(\phi) \):

\[ \phi + A(\phi) + A'(\phi) = 0 \]

with the solution \( A(\phi) = \phi + V_1 \sqrt{\phi} \), where \( V_1 \) is a constant. Using the obtained solutions to (138) we find that

\[ V(\phi) = V_0 \left( \sqrt{\phi} + V_1 \right)^4. \]  \hspace{1cm} (139)

and the extra Noether symmetry which is admitted by the conformal system has the form:

\[ X_1 = -\frac{1}{2} \partial_a + \frac{\phi + V_1 \sqrt{\phi}}{a} \partial_{\phi}. \]  \hspace{1cm} (140)

The corresponding conservation law is

\[ I_{X_1} = 6 \left( V_1 \sqrt{\phi} - 1 \right) \dot{a} + 3 \frac{a}{\sqrt{\phi}} V_1 \dot{\phi}. \]  \hspace{1cm} (141)
There exists also the second symmetry vector \([\text{108}]\) with the corresponding conservation law:

\[
X_2 = 2\tau \partial_\tau + a \left( \sqrt{\phi} V_1 + 1 \right) \partial_a - 2V_1 \sqrt{\phi} \left( \phi + 1 \right) \partial_\phi,
\]

and

\[
I_{X_2} = 12a (1 + \phi) \dot{a} + 6a^2 \left( 1 - \frac{V_1}{\sqrt{\phi}} \right) \phi,
\]

respectively, with the potential given by

\[
V(\phi) = V_0 (1 + \phi)^2 \exp \left( \frac{6}{V_1} \arctan \sqrt{\phi} \right).
\]

We have chosen \(N_0 = 0\) for both cases because in the case of \(N_0 \neq 0\), one finds that the Lagrangian \((\text{130})\) admits extra Noether symmetries only in the case of the trivial potential \(V(\phi) = 0\).

From the definition of the potential \((\text{84})\) one may try to find a form of the function \(f(\mathcal{R})\). For the first potential \((\text{139})\) the equation \((\text{84})\) is

\[
Ef'(E) - f(E) = V_0 \left( \sqrt{f'(E)} + V_1 \right)^4.
\]

Let us differentiate it with respect to \(f'(E)\):

\[
\frac{2V_0}{\sqrt{f'(E)}} \left( \sqrt{f'(E)} + V_1 \right)^3 - E = 0,
\]

and defining \(y = \sqrt{f'(E)}\), one has the polynomial equation

\[
(y + V_1)^3 - \frac{E}{2V_0} y = 0.
\]

If we put \(V_1 = 0\), the solution of the equation \((\text{147})\) is the “Starobinsky-like” ansatz:

\[
f(E) = \frac{E^2}{4V_0}.
\]

One notices that the solution \((\text{148})\) after applying to the hybrid master equation \((\text{82})\) reproduces the General Relativity trace equation, it means \(X \equiv \kappa T + R = 0\).

Similarly, for the potential \((\text{144})\) one has the following equation to solve:

\[
Ef'(E) - f(E) = V_0 \left[ 1 + f'(E) \right]^2 \exp \left( \frac{6}{V_1} \arctan \sqrt{f'(E)} \right).
\]
3.3 EXACT AND INVARIANT SOLUTIONS

As we have found the symmetries and potentials appearing in the conformal Lagrangian we may look for the exact solution of the field equations. We will focus on the potential (139) while the case of the potential (144) is considered in [108].

Symmetries are very helpful if one wants to find a suitable transformation of variables appearing in field equations. Using the symmetry (140) one may find Lie invariants by

\[-2\alpha a = \frac{\alpha d\phi}{\phi + V_1 \sqrt{\phi}}.\]

The relation between the scalar field \(\phi\) and a new variable \(v\) is

\[\phi = \left(\frac{v}{a} - V_1\right)^2.\]

Inserting the above result into the conservation law (141) and performing some simple algebra one finds the following coordinate transformation:

\[a = Cv + u, \quad \phi = \left(\frac{v}{Cv + u} - V_1\right)^2,\]

where \(C = V_1/(1 + V_1^2)\) and the new variable \(u\) is constructed with the Noether integral (141). In the new coordinates, the Lagrangian (130) becomes

\[\mathcal{L}(u, v, u', v') = 6 \left(V_1^2 + 1\right) u'^2 + \frac{6}{(V_1^2 + 1)} v'^2 + V_0 v^4.\]

Let us perform a second transformation of the form

\[x = \sqrt{12 \left(V_1^2 + 1\right)} u, \quad y = \sqrt{\frac{12}{(V_1^2 + 1)}} v,\]

which transforms the Lagrangian (153) into a much easier form

\[\mathcal{L}(x, y, x', y') = \frac{1}{2} x'^2 + \frac{1}{2} y'^2 + \bar{V}_0 y^4,\]

where \(\bar{V}_0 = \frac{V_0}{4 V_1^2} \left(V_1^2 + 1\right)^2\). The Hamiltonian of the field equations is given by

\[\mathcal{H} = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 - \bar{V}_0 y^4,\]
with the momenta \( p_x, p_y \) defined below. The Hamilton equations of the system

\[
(p'_{q_i} = -\frac{\partial \tilde{H}}{\partial q_i}, \quad q'_i = \frac{\partial \tilde{H}}{\partial p_{q_i}}, \quad q_i = (x, y))
\]

are

\[
x' = p_x, \quad y' = p_y \tag{158}
\]

\[
p_x' = 0, \quad p_y' = 4\bar{V}_0y^3. \tag{159}
\]

Since the Hamiltonian does not depend on time explicitly we may write the Hamilton-Jacobi equation as

\[
S = \bar{S}(x, y, \tilde{H}) - \tilde{H}t. \tag{160}
\]

where \( \frac{\partial S}{\partial q_i} = p_{q_i} \). From the Hamilton equations and Hamilton-Jacobi equation one gets that

\[
x = c_1\tau + c_2, \quad y' = \varepsilon \sqrt{2V_0y^4 - c_1^2}, \quad \varepsilon = \pm 1, \tag{161}
\]

\[
S = c_1x + \tilde{S}(y) + S_0, \quad \tilde{S}(y) = \varepsilon \int \sqrt{2V_0y^4 - c_1^2}dy. \tag{162}
\]

Let us write (161) as

\[
x(\tau) = c_1\tau + c_2, \tag{163}
\]

\[
\int \frac{dy}{\sqrt{2V_0y^4 - c_1^2}} = \varepsilon (\tau - \tau_0). \tag{164}
\]

and consider firstly a simply case when \( V_1 = c_2 = 0 \). For these assumptions one gets that \( a = u = (12)^{-\frac{1}{2}}x = \tilde{c}_1\tau \). From the conformal transformation of the time coordinate \( dt = a(\tau)d\tau \) the radiation solution can be obtained \( (a_0 = \sqrt{\frac{2}{\tilde{c}_1}}) \):

\[
a(t) = a_0\sqrt{t}. \tag{165}
\]

We may also perform the another simplification in order to get a more interesting solution. Let us assume that \( V_1 \neq 0 \) and \( c_1 = 0 \). Then we can solve (164)

\[
y(\tau) = -\varepsilon \frac{1}{\sqrt{2V_0}} \frac{1}{(\tau - \tau_0)} \tag{166}
\]

so now the scale factor can be expressed as

\[
a(\tau) = a_1 - a_2 \frac{1}{\tau - \tau_0}. \tag{167}
\]
3.3 Exact and Invariant Solutions

where \( a_1 = \frac{c^2}{\sqrt{12(V_1^2 + 1)}} \) and \( a_2 = \frac{12\epsilon}{\sqrt{24V_0(V_1^2 + 1)}} \). In order to write the Hubble function\(^1\) as a function of the scale factor

\[
H(\tau) = \frac{a'}{a^2} = \frac{a_2}{(a_1(\tau - \tau_0) - a_2)^2}
\]

we need to have \( \tau - \tau_0 = \frac{a_2}{a_1 - a} \) from (167) and now

\[
H(a) = a_2^{-1} (a_1 a^{-1} - 1)^2.
\]

The normalized Friedmann equation is defined as \( \frac{H^2(a)}{H_0^2} = \mathcal{F}(a) \), where \( H_0 = H(a(t) = 1) \) is a present value of the Hubble constant while \( \mathcal{F}(a) \) is a function containing density parameters which are comparable with astrophysical data. For the present time we set that \( a(t) = 1 \) and from the equation (169) we deduce \( a_2^{-1} = H_0/(|a_1| + 1)^2 \). The Hubble function is

\[
H(a) = H_0 \left( \frac{a_1 - a}{a(a_1 - 1)} \right)^2
\]

and finally, the normalized Friedman equation can be written in the following form

\[
\frac{H^2(a)}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_f a^{-1} + \Omega_\Lambda,
\]

where

\[
\Omega_f = \frac{|4a_1|}{(|a_1| + 1)^4}, \quad \Omega_\Lambda = \frac{1}{(|a_1| + 1)^4},
\]

\[
\Omega_r = \frac{|a_1|^4}{(|a_1| + 1)^4}, \quad \Omega_m = \frac{|4a_1|^3}{(|a_1| + 1)^4},
\]

\[
\Omega_k = \frac{|6a_1|^2}{(|a_1| + 1)^4}.
\]

From the above analysis arises a conclusion that each power of \( \sqrt{\Phi} \) appearing in the potential (139), that is \( V(\phi) = V_0 (\sqrt{\Phi} + V_1)^4 \), introduces into the Friedmann equation a power term of the scale factor which has a cosmological meaning. Each term represents fluid filling the Universe and they are: radiation, dust, curvature-like fluid, a dark energy fluid with equation of state \( p_f = -\frac{2}{3} \rho_f \) and a cosmological constant, respectively. Let us again recall that we have been

\(^1\) Recall that \( H = \frac{1}{a} \frac{da}{dt} = \frac{1}{a^2} \frac{da}{d\tau} \)
considering empty spacetime, that is, vacuum case of the model. Moreover, we have assumed that the spacetime is spatially flat: the curvature term which appeared in the Friedmann equation comes from hybrid gravity. We would also like to notice that for large redshift $1 + z = a^{-1}$ the Friedmann equation (171) behaves like the radiation solution which is presented in the picture 9: the hybrid gravity coincides with radiation solution in the early Universe. There are also drawn the scale factor of the standard $\Lambda$CDM cosmology and the radiation one.

3.3.1 Wheeler-DeWitt equation of hybrid gravity model

Roughly speaking, WDW equation is a quantized version of a Hamiltonian of a considered system. In the case of hybrid gravity applied to FRLW cosmology we deal with the 2-dimensional minisuperspace described by the minisuperspace metric $G_{ij} = \text{diag}(1,1)$, the WDW it has the following form

$$\Box \Psi - a^3 V(\phi) \Psi = 0,$$

(175)
where $\Box = \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial x^i} \left( \sqrt{|G|} G^{ij} \frac{\partial}{\partial x^j} \right)$ is the Laplace operator, $\Psi$ is a wave function of the Universe and $x^i = \{a, \phi\}$. Let us notice that the WDW equation is a Klein-Gordon equation with the d’Alembertian defined by the minisuperspace metric $G_{ij}$.

From the Hamiltonian (157), following canonical quantisation procedure [129, 121, 130], one may determine the WDW equation (recall that the dimension of the minisuperspace is two and the minisuperspace is flat), which has the form

$$\Psi_{xx} + \Psi_{yy} - 2\bar{V}_0 y^4 \Psi = 0,$$

where $\Psi$ is the Wave Function of the Universe [131]. We would like to solve the above equation and again we can use symmetries to do that. We could use Lie symmetries method [132, 133] in order to find a generic symmetry vector and bring it to play to lower the number of independent variables. In our case the linearized symmetry condition by which we look for symmetries is lengthy but fortunately one may follow the theorem 7 formulated in [128] which combine conformal algebras of minisuperspace with symmetries of Klein-Gordon equation. Unfortunately, the two-dimensional Riemannian space has an infinite conformal algebra but we need at least one conformal Killing vector satisfying the condition (381) to solve the WDW equation (176).

It is easy to check that the vectors $X_1 = \partial_x$ and $X_2 = \partial_y$ are Killing vectors of the minisuperspace metric $G_{ij}$, it means they satisfy the condition

$$X_{i;j} + X_{j;i} = 0.$$

(177)

There exists also the homothetic vector $X_3 = x \partial_x + y \partial_y$ with the conformal factor $\psi = 1$:

$$X_{i;j} + X_{j;i} = 2\psi G_{ij}$$

(178)

but it can be checked that only the vector $X_1$ satisfies the condition (381). That means that the generic Lie symmetry vector is

$$X = b_1 \partial_x + (b_2 \Psi + b(x, y)) \partial_y,$$

(179)

where $b(x, y)$ is a function that satisfies WDW equation (176). Let us now reduce the number of variables of the equation by the zeroth order invariants $(Y, Z)$. One gets that

$$\frac{dx}{b_1} = \frac{d\Psi}{b_2 \Psi} \quad Z = y$$

(180)

so the invariant functions coming from (180) are $\{\Psi = Ye^{\mu x}, y\}$, where $\mu \in C$ [132]. Hence the considered WDW equation reduces to the second order ordinary differential equation:

$$Y_{yy} + \left(\mu^2 - 2\bar{V}_0 y^4\right) Y = 0.$$
One recognizes the one-dimensional time-dependent oscillator. The solutions due to Lie point symmetries of such a system were considered in [134, 135, 136]. The solution of the above equation is

$$Y(y) = y_1 e^{w(y)} + y_2 e^{-w(y)},$$

where

$$w(y) = \frac{\sqrt{2}}{2} \int \sqrt{\left( 2V_0 y^4 - \mu^2 \right)} \, dy. \quad (182)$$

We are able finally to write the invariant solution of the WDW equation (176) as

$$\Psi(x, y) = \sum_{\mu} \left[ y_1 e^{\mu x + w(y)} + y_2 e^{\mu x - w(y)} \right]. \quad (183)$$

### 3.4 Remarks

Let us briefly conclude our considerations on hybrid gravity model applied to FRLW cosmology. We have used Noether and Lie point symmetries approaches which allowed us to solve classical field equations arising from Lagrangian in the first case and to find a Wave Function of the Universe which is a solution of the Wheeler-DeWitt equation being a quantized version of a cosmological Hamiltonian. The analysis was performed in the scalar-field representation of the hybrid gravity model what resulted into a system with two independent variables: the scale factor of the Universe $a(t)$ and the scalar field $\phi$. The field equations as well as WDW equations required a choice of the potential $V(\phi)$ in order to be solved, that is, coming back to the original representation of the theory, one needs to specify the gravitational Lagrangian $f(R)$. We examined two cases: first one gave as the solution with cosmological constant and radiation fluid for the constant potential $V_0$ (from the Noether symmetries procedure) while considering the conformal frame resulted in the much richer model: for the power-law potential we obtained the Universe filled with five fluids: radiation, dust, curvature-like fluid, dark energy one with EoS $p_f = -\frac{2}{3}\rho_f$ and cosmological constant. It is important to underline that we have considered spatially flat metrics ($k = 0$) and vacuum equations in both models, that is, the right-hand side of the modified Einstein’s equations is zero.

Following DeWitt [131], the solution of the WDW equation is called a Wave Function of the Universe which is related to a probability that an observed universe emerges with some initial conditions which might be specified by for example, "no boundary condition" [137] or "tunneling from nothing" [138, 139]. The oscillatory or exponential behavior depends on the signs of the variables as
x, y are functions of \( \phi \) which can be positive or negative defined. The positivity of the scalar field can be interpreted as quintessence while negative scalar field corresponds to phantom field. For the case \( \phi = 0 \) one recovers GR, that is, the function \( f(\mathcal{R}) \) turns out to be a cosmological constant. It should be also mentioned that the only power-law Lagrangian of hybrid gravity that admits Noether symmetries has a form \( f(\mathcal{R}) \sim \mathcal{R}^2 \) while in \( f(R) \) gravity in metric approach one deals with \( f(R) = R^n \), and similarly in pure Palatini one the power-law function is \( f(\mathcal{R}) = \mathcal{R}^n \).

Hybrid gravity, according to up-to-now studies, seems to be a theory worth of further considerations, especially for astrophysical objects such as neutron stars or black holes because no such examination has been performed so far. It is capable to recover the various cosmological epochs as shown in [140] and because of the scalar-tensor representation, it passes solar system tests[21, 20, 104].
Besides the cosmological constant $\Lambda$ introduced in order to explain late time accelerating expansion, a minimally coupled scalar field is one of the simplest modification of the Einstein’s field equations. Similarly as the cosmological constant, it can be treated both as the geometric modification as well as exotic fluid inserted on the right hand side of the equations. There are much more interesting but also more difficult to handle models which concern a non-minimally coupled scalar field (for example Brans-Dicke theory [141]) since their field equations have similar forms as $4+1$ decomposition of Kaluza-Klein field equations [142, 143, 144, 145]: the 5-dimensional Kaluza-Klein theory unifies gravitation and electromagnetism [146]. So far, we have investigated two non-minimally coupled scalar field models: Palatini $f(R)$ can be considered as scalar-tensor theory (although we do not treat it in this way) as well as Hybrid Gravity which we studied in scalar-tensor representation. Now on, we are going to examine minimally coupled scalar field in both cosmology and astrophysics.

In the presented chapter the first part will concern cosmological considerations. We will focus on anisotropic models in the framework of scalar-tensor theory of gravity. Mainly we will focus on Bianchi I and Bianchi II models. Later on, we will examine configurations of relativistic stars in the first part. We will briefly recall main results coming from General Relativity and then turn to Extended Theories of Gravity. Since the stability criterion must be investigated case by case, we will present, as the simplest example of ETGs, a minimally coupled scalar field.

4.1 Bianchi Cosmology in Scalar-Tensor Theory of Gravity

We have already mentioned that anisotropic models of Universe can be also very important since we do not really know if our Universe is isotropic. The simplest generalization providing anisotropy is the assumption that instead of one scale factor of the Universe one deals with three, each one for one spatial direction. Such a model is so-called Bianchi I. There are more of Bianchi spacetimes and we will briefly introduce them. After that, we will focus on Lie symmetries in Bianchi scalar-field cosmology. Similarly as it was done for the Hybrid Gravity, we will look for invariant and exact solutions of Wheeler-DeWitt equations. To
find classical solutions, we will use WKB approximation. More detailed discussion can be find in [109].

Despite the fact that on the large scale the observed Universe is homogeneous and isotropic there are visible anisotropies in the cosmic microwave background. The existence of the anisotropies means that the Universe does not expand in the same way in all directions as standard cosmological model assumes. If our Universe is considered on the large scale, the simple model described by FRLW metric is sufficient and agrees with astronomical observations (so-called LCDM standard cosmological model). As the anisotropies do not increase and are very small one supposes that anisotropic models isotropize as time approaches our epoch [147, 148]. This makes it important to study models which are not isotropic at early times and therefore the dynamics of anisotropies should be understood. There are also considerations [147, 149] that anisotropies before the inflation could be a reason for the coupling between the gravitational field and the inflaton field (scalar field minimally or non-minimally coupled to gravity). A lot of attention has been given to a scalar field in inflationary models [150, 151] but also because of a possibility that it could explain dark matter problem and the damping of cosmological constant [152]. Unfortunately, the presence of scalar fields in cosmology arises to another problem which is an unknown form of their potentials.

Anisotropic but homogeneous models of universes are described by Bianchi models. Bianchi spacetime manifolds are foliated along the time axis with 3-dimensional homogeneous hypersurfaces admitting a group of motion $G_3$. There are nine possible groups [153, 154] which gives nine possibles models which can be taken under consideration. All physical variables appearing in the models depend on time only which reduce the Einstein and other governing equations to ordinary differential equations. The line element of the Bianchi models in $3 + 1$ decomposition has a following form [153, 155]

$$ds^2 = -\frac{1}{N(t)^2} dt^2 + \bar{g}_{ij}(t) \omega^i \otimes \omega^j,$$

(184)

where $N(t)$ is the lapse function and $\{\omega^i\}$ denotes the canonical basis of 1-forms satisfying the Lie algebra

$$d\omega^i = C^i_{jk} \omega^j \wedge \omega^k$$

(185)

where $C^i_{jk}$ are the structure constants of the algebra. The spatial metric $\bar{g}_{ij}$ is diagonal and can be factorized as follows

$$g_{ij}(t) = e^{2\lambda(t)} e^{-2\beta_{ij}(t)}$$

(186)
Table 3: The Ricci scalar of the 3d hypersurfaces of the class A Bianchi spacetimes.

| Model               | $R^* (\lambda, \beta_1, \beta_2)$ |
|---------------------|-----------------------------------|
| Bianchi I           | 0                                 |
| Bianchi II          | $-\frac{1}{2} e^{(4\beta_1-2\lambda)}$ |
| Bianchi VI$0$/VII$0$| $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2(\beta_1-\sqrt{3}\beta_2)} \pm 2e^{\beta_1+\sqrt{3}\beta_2} \right)$ |
| Bianchi VIII        | $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2\beta_1} \left( e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2} \right)^2 + \right)$ |
| Bianchi IX          | $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2\beta_1} \left( e^{\sqrt{3}\beta_2} - e^{-\sqrt{3}\beta_2} \right)^2 + \right)$ + 1 |

where $e^{\lambda(t)}$ is the scale factor of the Universe and the matrix $\beta_{ij}$ is diagonal and traceless. The matrix $\beta_{ij}$ depends on two independent quantities $\beta_1, \beta_2$ which are called the anisotropy parameters \[155\]

$$\beta_{ij} = \text{diag} \left( \beta_1, -\frac{1}{2}\beta_1 + \frac{\sqrt{3}}{2}\beta_2, -\frac{1}{2}\beta_1 - \frac{\sqrt{3}}{2}\beta_2 \right) \quad (187)$$

and, in these variables, it is $\sqrt{\bar{g}} = e^{3\lambda}$. We will consider only a subclass of the Bianchi models, so-called class A, since there exist Lagrangians of field equations for them (for details, see for example \[153, 154, 109\]). Let us additionally mention that for the line element (184) together with the definitions (186) and (187), the Ricci scalar of the Bianchi class A spacetimes is

$$R = R_{(4)} + R^* \quad (188)$$

where (the dot denotes the differentiation with respect to the time $t$)

$$R_{(4)} = \frac{3}{2} N \left( 4N\dot{\lambda} + 4N\dot{\lambda}^2 + 8N\dot{\lambda}^2 + 8N\dot{\lambda}^2 + N\dot{\beta}_1^2 + N\dot{\beta}_2^2 \right) \quad (189)$$

and $R^* = R^* (\lambda, \beta_1, \beta_2)$ is the component of the three dimensional hypersurface. The exact forms of $R^*$ for some of the Bianchi models are given in the table 3.

Now on we are ready to study Bianchi models in scalar-tensor cosmology with minimally coupled scalar field. The action has a well-known form

$$S = \int d^4x \sqrt{-\bar{g}} \left( R - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + V(\phi) \right) \quad (190)$$
while the Lagrangian $L \equiv L(N, \lambda, \beta_1, \beta_2, \phi, \dot{\lambda}, \dot{\beta}_1, \dot{\beta}_2, \dot{\phi})$ is obtained from (184), (188), and (189) [150]

$$L = N(t)e^{3\lambda} \left( 6\dot{\lambda}^2 - \frac{3}{2} (\dot{\beta}_1^2 + \dot{\beta}_2^2) - \frac{1}{2}\dot{\phi}^2 \right) + \frac{e^{3\lambda}}{N(t)} (V(\phi) + R^*) \quad (191)$$

The field equations with respect to the variables $\lambda, \beta_1, \beta_2, \phi$ are

$$4\ddot{\lambda} + \left( 6\dot{\lambda}^2 + \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{1}{2}\dot{\phi}^2 \right) + \frac{N}{N} \dot{\lambda} - \frac{1}{N^2} \left( V + R^* + \frac{1}{3} \frac{\partial R^*}{\partial \lambda} \right) = 0, \quad (192)$$

$$\ddot{\beta}_{(1,2)} + \frac{N}{N} \dot{\beta}_{(1,2)} + 3\lambda \dot{\beta}_{(1,2)} + \frac{1}{3N^2} R^*_{(1,2)} = 0, \quad (193)$$

$$\ddot{\phi} + 3\lambda \dot{\phi} + \frac{N}{N} \dot{\phi} + \frac{1}{N^2} \frac{\partial V}{\partial \phi} = 0, \quad (194)$$

whilst the $00$ modified Einstein’s equation is

$$N e^{3\lambda} \left( 6\dot{\lambda}^2 - \frac{3}{2} (\dot{\beta}_1^2 + \dot{\beta}_2^2) - \frac{1}{2}\dot{\phi}^2 \right) - \frac{e^{3\lambda}}{N} (V + R^*) = 0. \quad (195)$$

Under coordinate transformations

$$(\lambda, \beta_1, \beta_2) = \left( \frac{\sqrt{3}}{6} x, \frac{\sqrt{3}}{3} y, \frac{\sqrt{3}}{3} z \right) \quad \text{and} \quad N(t) = \bar{N}(t) e^{-3\lambda}, \quad (196)$$

the equation (195) becomes

$$\frac{1}{2} \bar{N} (\dot{x}^2 - \dot{y}^2 - \dot{z}^2 - \dot{\phi}^2) - \frac{1}{\bar{N}} e^{\sqrt{3}x} (V(\phi) + R^*) = 0 \quad (197)$$

From the kinetic part of the Lagrangian (191) we notice that one deals with a flat 4-dimensional minisuperspace. Above equation, after the quantization procedure (see the sections (3.2), (3.3.1) as well as [121]) can be transformed into WDW equation

$$\Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\phi\phi} - 2e^{\sqrt{3}x} (V(\phi) + R^*) \Psi = 0 \quad (198)$$

which has a form of Klein-Gordon equation in the 4-dimensional flat space $M^4$. As we want to apply the procedure of [128, 125], which was briefly described in the section (A.5), we will need the conformal algebra of the $M^4$ spacetime. Its algebra is 15-dimensional: one may show that $C(D) \simeq O(D + 2)$, where
the dimension of the orthogonal algebra \( \text{O}(D + 2) \) is \( \frac{(D+1)(D+2)}{2} \) [156]. The considered spacetime admits ten Killing vectors:

\[
K(x) = \partial_x, \quad K(y) = \partial_y, \quad K(z) = \partial_z, \quad K(\phi) = \partial_\phi, \\
R_{(xy)} = y\partial_x + x\partial_y, \quad R_{(xz)} = z\partial_x + x\partial_z, \\
R_{(yz)} = z\partial_y - y\partial_z, \quad R_{(x\phi)} = \phi\partial_x + x\partial_\phi, \\
R_{(y\phi)} = \phi\partial_y - y\partial_\phi, \quad R_{(z\phi)} = \phi\partial_z - z\partial_\phi,
\]

one gradient homothetic Killing vector

\[
H = x\partial_x + y\partial_y + z\partial_z + \phi\partial_\phi,
\]

and four special conformal Killing vectors

\[
C_{(x)} = \frac{1}{2} (x^2 + y^2 + z^2) \partial_x + xy\partial_y + xz\partial_z + \frac{1}{2}\phi^2\partial_x + \phi x\partial_\phi, \\
C_{(y)} = xy\partial_x + \frac{1}{2} (x^2 + y^2 - z^2) \partial_y + zy\partial_z - \frac{1}{2}\phi^2\partial_y + \phi y\partial_\phi, \\
C_{(z)} = xz\partial_x + yz\partial_y + \frac{1}{2} (x^2 + z^2 - y^2) \partial_z - \frac{1}{2}\phi^2\partial_z + \phi z\partial_\phi, \\
C_{(\phi)} = x\phi\partial_x + y\phi\partial_y + z\phi\partial_z + \frac{1}{2} (x^2 + \phi^2 - y^2 - z^2) \partial_\phi,
\]

for which the conformal factors are \( \psi_{(x)} = x, \quad \psi_{(y)} = y, \psi_{(z)} = z, \quad \psi_{(\phi)} = \phi \), respectively.

Now on, we are ready to look for Lie symmetries of the WDW equation (198). Using the theorem 7 from the Appendix A we have found that the WDW equation under consideration admits Lie symmetries not only for special forms of the potential \( V(\phi) \) but also for arbitrary one. The special forms of the potential are \( V(\phi) = 0 \) for which the scalar field \( \phi \) behaves like stiff matter, \( V(\phi) = V_0 \) with \( V_0 \neq 0 \) and exponential one \( V(\phi) = V_0 e^{\mu\phi} \). The mentioned results are presented in the tables 4 and 5.

4.1.1 **Invariant solutions of WDW equation and WKB approximation**

As the first step let us examine Bianchi I spacetime. Here and further, we will assume that \( \bar{N}(t) = 1 \) which allows us to write the equation (198) in the form

\[
\Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\phi\phi} - 2e^{\sqrt{3}x}V(\phi)\Psi = 0. \quad (204)
\]

The case of zero potential gives rise to \((1 + 3)\) wave equation in \( E^3 \) which was considered in [135]. The field equations with the constant potential turn out to
Table 4: Lie symmetries of the WDW equation of the Class A Bianchi models in scalar field cosmology for $V(\phi) = 0$.

| Model  | $V(\phi) = 0$ | Lie Symmetries |
|--------|---------------|----------------|
| Bianchi I | 16 | $\Psi \partial \psi, \ K_{(x)}, \ K_{(y)}, \ K_{(z)}, \ K_{(\phi)}, \ R_{(xy)}, \ R_{(xz)}, \ R_{(yz)}, \ R_{(y\phi)}, \ R_{(z\phi)}, \ H, \ (C_{(x)} - x\Psi \partial \psi), \ (C_{(y)} - y\Psi \partial \psi), \ (C_{(z)} - z\Psi \partial \psi), \ (C_{(\phi)} - \phi\Psi \partial \psi)$ |
| Bianchi II | 7 | $\Psi \partial \psi, \ K_{(z)}, \ K_{(\phi)}, \ K_{(x)} - \frac{1}{2}K_{(y)}, \ R_{(z\phi)}$ |
| Bianchi VI$_0$/VII$_0$ | 3 | $\Psi \partial \psi, \ K_{(\phi)}, \ K_{(x)} + \frac{1}{4}K_{(y)} + \frac{\sqrt{3}}{4}K_{(z)}$ |
| Bianchi VIII/IX | 2 | $\Psi \partial \psi, \ K_{(\phi)}$ |

have a form of the ones coming from General Relativity with stiff matter and cosmological constant. Applying to the equation (204) the zeroth-order invariants of the Lie symmetries

$$\bar{X}_i = K_i + \mu_i \Psi \partial \psi, \ i = y, z, \phi$$

(205)

which form a closed Lie algebra, the WDW equation may be reduced to the linear second-order ordinary differential equation

$$\Phi'' - \left( \mu_{(y)} + \mu_{(z)} + \mu_{(\phi)} + 2V_0 e^{\sqrt{3}x} \right) \Phi = 0.$$  

(206)

The Wave Function of the Universe is now

$$\Psi = \Phi(x) \exp \left( \mu_{(y)} y + \mu_{(z)} z + \mu_{(\phi)} \phi \right).$$

The prime in (206) denotes the differentiation with respect to the variable $x$. The solution of (206) exists

$$\Phi(x) = \Phi_1 J_c \left( 3i \frac{2\sqrt{6}V_0}{\sqrt{3}} e^{\frac{\sqrt{3}}{2}x} \right) + \Phi_2 Y_c \left( i \frac{2\sqrt{6}V_0}{\sqrt{3}} e^{\frac{\sqrt{3}}{2}x} \right)$$

(207)

where $J_c, Y_c$ are the Bessel functions of the first and second kind while the constant $c = \frac{2\sqrt{3}}{3} \left( \sqrt{\mu_{(y)}^2 + \mu_{(z)}^2 + \mu_{(\phi)}^2} \right)$.

When one deals with the exponential potential in (204), it is convenient to apply the Lie invariants

$$\bar{X}_{(y)}, \bar{X}_{(z)}, \frac{\sqrt{3}}{3} \mu_{(x)} - K_{(\phi)} + \Psi \partial \psi.$$

(208)
Table 5: Lie symmetries of the WDW equation of the Class A Bianchi models in scalar field cosmology for non-zero potentials.

| Model $V(\phi) = V_0$ | # | Lie Symmetries |
|-------------------------|---|----------------|
| Bianchi I               | 7 | $\Psi\partial\psi, K(y), K(z), K(\phi), R(yz), R(y\phi), R(z\phi)$ |
| Bianchi II              | 4 | $\Psi\partial\psi, K(z), K(\phi), R(z\phi)$ |
| Bianchi VI$_0$/VII$_0$  | 2 | $\Psi\partial\psi, K(\phi)$ |
| Bianchi VIII/IX         | 2 | $\Psi\partial\psi, K(\phi)$ |

| Model $V(\phi) = V_0 e^{\mu\phi}$ | # | Lie Symmetries |
|------------------------------------|---|----------------|
| Bianchi I                          | 7 | $\Psi\partial\psi, K(y), K(z), R(yz), \frac{\sqrt{3}}{3}K(x) - K(\phi), R(y\phi) + \frac{\sqrt{3}}{3}\mu R(xy), R(z\phi) + \frac{\sqrt{3}}{3}\mu R(xz)$ |
| Bianchi II                         | 4 | $\Psi\partial\psi, K(z), K(x) - \frac{1}{2}K(y) - \frac{\sqrt{3}}{\mu}K(\phi), R(z\phi) + \frac{\sqrt{3}}{3}\mu (R(xz) - \frac{1}{2}R(yz))$ |
| Bianchi VI$_0$/VII$_0$             | 2 | $\Psi\partial\psi, K(x) - \frac{1}{2}K(y) - \frac{\sqrt{3}}{2}K(z) - \frac{\sqrt{3}}{\mu}K(\phi)$ |
| Bianchi VIII/IX                    | 1 | $\Psi\partial\psi$ |

| Model $V(\phi) = V(\phi)$         | # | Lie Symmetries |
|------------------------------------|---|----------------|
| Bianchi I                          | 4 | $\Psi\partial\psi, K(y), K(z), R(yz)$ |
| Bianchi II                         | 2 | $\Psi\partial\psi, K(z)$ |
| Bianchi VI$_0$/VII$_0$             | 1 | $\Psi\partial\psi$ |
| Bianchi VIII/IX                    | 1 | $\Psi\partial\psi$ |
The procedure gave us the WDW equation (204) reduced to
\[(3 - \mu^2) \Phi''(w) + 6\nu \Phi' - \left( \left( \mu^2_{(y)} + \mu^2_{(z)} \right) \mu^2 - 3^2 + 2V_0\mu^2 e^{\mu w} \right) \Phi = 0 \tag{209}\]
where \(\Phi' = \frac{d\Phi(w)}{dw}\) and \(w = \frac{\sqrt{3}}{\mu}x + \phi\). The wave function became
\[
\Psi(x, y, z, \phi) = \Phi(w) \exp \left( \frac{\sqrt{3}}{\mu}x + \mu_{(y)}y + \mu_{(z)}z \right).
\]

The solution of (209) depends on the value of the constant \(\mu\). For the \(\mu \neq \sqrt{3}\) one gets
\[
\Phi(w) = \exp \left( \frac{3\mu w}{\mu^2 - 3} \right) \left[ \Phi_1 J_{\tilde{c}} \left( 2\sqrt{\frac{2V_0}{\mu^2 - 3} e^{\frac{\sqrt{3}w}{\mu}}} \right) + \Phi_2 Y_{\tilde{c}} \left( 2\sqrt{\frac{2V_0}{\mu^2 - 3} e^{\frac{\sqrt{3}w}{\mu}}} \right) \right], \tag{210}
\]
where \(\tilde{c} = \frac{2}{\mu^2 - 3} \sqrt{3^2 - (\mu^2 - 3) \left( \mu^2_{(y)} + \mu^2_{(z)} \right)}\) while for the constants \(|\mu| = \sqrt{3}, \nu \neq 0\) we have
\[
\Phi(w) = \Phi_0 \exp \left[ \frac{1}{2} \left( \mu^2_{(y)} + \mu^2_{(z)} \right) - \frac{\sqrt{3}}{2} V_0 e^{\frac{\sqrt{3}w}{3}} \right]. \tag{211}
\]

Let us focus on the classical solutions of the considered Bianchi I models. We will consider WKB approximation of the equation (204) as it was performed in the section 3.3 (recall that Hamiltonian constraint is equal to zero). We simply get the Hamilton-Jacobi equation of the form
\[
\frac{1}{2} \left( \left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 - \left( \frac{\partial S}{\partial \phi} \right)^2 \right) - e^{\sqrt{3}x} V(\phi) = 0 \tag{212}
\]
where \(S = S(x, y, z, \phi)\) describes a motion of a particle in the \(M^4\) space. Hamiltonian system is
\[
\dot{x} = \frac{\partial S}{\partial x}, \quad \dot{y} = \frac{\partial S}{\partial y}, \quad \dot{z} = \frac{\partial S}{\partial z}, \quad \dot{\phi} = \frac{\partial S}{\partial \phi}. \tag{213}
\]
We easily find that for the potential \(V(\phi) = 0\) the equation (212) possesses a solution
\[
S_0(x, y, z, \phi) = c_1 y + c_2 z + c_3 \phi + \varepsilon \sqrt{c_1^2 + c_2^2 + c_3^2} x, \quad \varepsilon = \pm 1. \tag{214}
\]
Applying it to the system of equations (213) we obtain classical solutions
\[ x(t) = \epsilon \sqrt{c_1^2 + c_2^2 + c_3^2} t + x_0 \]  
\[ y(t) = -c_1 t + y_0, \quad z(t) = -c_2 t + z_0, \quad \phi(t) = -c_3 t + \phi_0. \]  

The constant potential \( V_0 \) provides the solution of (212) as
\[ S_{V_0}(x, y, z, \phi) = c_1 y + c_2 z + c_3 \phi \]
\[ + \epsilon \frac{2 \sqrt{3}}{3} \left( L(x) + \sqrt{c_{1-3}} \arctan \frac{L(x)}{\sqrt{c_{1-3}}} \right), \]  
where we have defined the function \( L(x) = \sqrt{c_1^2 + c_2^2 + c_3^2 + 2V_0 e^{3x}} \) and the constant \( c_{1-3} = c_1^2 + c_2^2 + c_3^2 \). The Hamilton equations (213) are found to be
\[ \dot{x} = L(x), \quad \dot{y} = -c_1, \quad \dot{z} = -c_2, \quad \dot{\phi} = -c_3 \]  

The classical solutions of (212) with the exponential potential \( V(\phi) = V_0 e^{\mu \phi} \) depend on the value of the constant \( \mu \), as in the case of WDW solutions. Let us just consider the solution of the Hamilton-Jacobi equation (212) for the value \( \mu = -\sqrt{3} \). In order to it, we need to perform the coordinate transformation \( \phi = \psi + x \) under which the Hamilton-Jacobi equation and the Hamiltonian system are now
\[ \frac{1}{2} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - 2 \left( \frac{\partial S}{\partial \psi} \right) \left( \frac{\partial S}{\partial y} \right) - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 \right] - V_0 e^{-\sqrt{3} \psi} = 0 \]  
\[ \dot{x} = \frac{\partial S}{\partial x}, \quad \dot{y} = -\frac{\partial S}{\partial y}, \quad \dot{z} = -\frac{\partial S}{\partial z}, \quad \dot{\psi} = -\frac{\partial S}{\partial x}. \]  

One solves the equation (221) obtaining the Hamilton action
\[ S(x, y, z, \psi) = c_1 x + c_2 y + c_3 z + \frac{(c_2^2 + c_3^2 - c_1^2)}{2c_1} \psi - \frac{\sqrt{3}V_0}{6c_1} e^{-\sqrt{3} \psi} \]
as well as the field equation
\[ \dot{x} = \frac{(c_1^2 - c_2^2 - c_3^2) - V_0 e^{\sqrt{3} \psi}}{2c_1}, \quad \dot{y} = -c_2, \quad \dot{z} = -c_3, \quad \dot{\psi} = -c_1. \quad (224) \]

The solutions of the above equations can be simply found
\[ x(t) = \frac{3}{2} c_1 t - \frac{(c_2^2 + c_3^2)}{2c_1} t + \frac{\sqrt{3} V_0}{6c_1^2} e^{-\sqrt{3} \psi_0} e^{\sqrt{3} c_1 t} + x_0, \quad (225) \]
\[ y(t) = -c_2 t + y_0, \quad z(t) = -c_3 t + z_0, \quad \psi(t) = -c_1 t + \psi_0, \quad (226) \]
where the quantities with the index 0 are constants. The details concerning the case \(|\mu| \neq \sqrt{3}\) are given in [154, 109] and hence we will not consider them here.

The procedure is similar till obtaining the field equations (213). In order to obtain analytical solutions of them, one needs to perform an extra transformation of the time variable in [154].

As an another brief example we will discuss Bianchi II models. Specifying, we will consider only the case of zero potential for which the scalar field behaves as stiff matter, that is, \(p_\phi = \rho_\phi\). It arises to the WDW equation (198)
\[ \Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\Phi \Phi} + e^{\frac{2}{\sqrt{3}}(2y + x)} \Psi = 0. \quad (227) \]
which can be solved by applying Lie invariants of the zeroth-order, similarly, as it was done for the Bianchi I models. One may get the solutions with respect to the various Lie algebras. Using for example
\( \{K_x; \frac{1}{2} K_y + \Psi \partial \psi; K_z + \mu(z) \Psi \partial \psi; K_{(\Phi)} + \mu(\Phi) \Psi \partial \psi\} \) gives the invariant solution
\[ \Psi_1(x, y, z, \Phi) = \exp \left( \frac{2}{3} (y + 2x) + \mu(z) z + \mu(\Phi) \Phi \right) \times (\Psi_1 I_\lambda (u(x, y)) + \Psi_2 K_\lambda (u(x, y))) \quad (228) \]
for which we have defined the constant \(\lambda = \frac{1}{3} \sqrt{12^2 - 9 \left( \mu_2^2(z) + \mu_2^2(\Phi) \right)}\) and the function \(u(x, y) = \exp \left( \frac{\sqrt{3}}{3} (2y + x) \right)\). The functions \(I_\lambda\) and \(K_\lambda\) denote modified Bessel functions of the first and second kind, respectively. Choosing the algebras
\( \{K_z; K_x; \frac{1}{2} K_y - \frac{1}{2} R_{(xz)} - \frac{1}{2} R_{(yz)}\} \) or
\( \{K_{(\Phi)}; K_x; \frac{1}{2} K_y - \frac{1}{2} R_{(x\Phi)} - \frac{1}{2} R_{(y\Phi)}\} \) gives also solutions in the terms of modified Bessel functions while the two Lie algebras
\[ \left\{ R_{(z\Phi)}, K_x - \frac{1}{2} K_y, R_{(x\Phi)} - \frac{1}{2} R_{(y\Phi)} \right\} \]
\[ \left\{ R_{(z\Phi)}, K_x - \frac{1}{2} K_y, R_{(xz)} - \frac{1}{2} R_{(yz)} \right\} \]
allows to solve the WDW equation and get the solution as

\[ \Psi_4(x, y, z, \phi) = \Psi_1 I_0(u(x, y)) + \Psi_2 K_0(u(x, y)). \]  

(229)

Applying the WKB approximation one may obtain classical solutions, similarly as it was done for Bianchi I.

4.1.2 Conclusions

We have discussed another example of the usefulness of the Lie symmetries method in cosmological applications. As for hybrid gravity considered for FLRW spacetime, we were able to find an unknown potentials of a scalar field for some Bianchi models. Again we treated Lie symmetries as a criterion for selection models for which we could find exact solutions of Wheeler-DeWitt equations. Moreover, as WDW equations are invariant under the action of the three dimensional Lie algebra with zero commutators, the Hamilton–Jacobi equations of the Hamiltonian system can be solved by the method of separation of variables. It means that the field equations are Liouville integrable. Such analysis can be used to construct Wave Functions of the Universe as well as conservation laws (in the case when Lie symmetries as Noether ones). Existences of symmetries gives rise to a straightforward interpretation of the Hartle criterion. It was shown [157] that symmetries generate oscillatory behaviors in a Wave Function of the Universe and then allow correlations among physical variables which gives rise to classically observable cosmological solutions. There also exists a possibility that one may use WDW equations to determine quantum potentials in the semi-classical approach of Bohmian mechanics [158, 159]. The idea should be further investigated for cosmological purposes.

4.2 Equilibrium and Stability of Relativistic Stars

In the previous parts we were focused on cosmological applications of some models of Extended Theories of Gravity. Here, we will consider astrophysical ones since there are also problematic issues concerning astrophysical objects like for instance neutron stars. Their structure and the relation between the mass and the radius are determined by equation of state (EoS) of dense matter. There are some propositions for its form, however it is still unknown. That means that the relation between density and mass is not specified and hence a radius cannot be estimated. One gets its different values depending on a model taken into account. The problem is related to maximal masses of relativistic stars since GR
predicts a maximal value for such objects. The maximal mass of neutron stars is still an open question but recent observations estimate this limit as $2M_{\odot}$: for example the pulsar PSR J1614-2230 has the limit $1.97M_{\odot}$ [160], another massive neutron star is Vela X-1 with the mass $\sim 1.8M_{\odot}$ [161]. There are also indications of the existence of more massive neutron stars with masses $\sim 2.4M_{\odot}$, for instance B1957+20 [162]. It should be also mentioned that a lot of EoS include hyperons which make the maximal mass limit for non-magnetic neutron stars significantly lower than expected $2M_{\odot}$ [163]. There are a few ways to approach the problem of "hyperon puzzle", such as hyperon-vector coupling, chiral quark-meson coupling, existence of strong magnetic field inside the star and many others. For instance, it seems that the existence of neutron stars without strong magnetic field having masses larger than two solar mass is impossible in the framework of GR [163, 164, 165]. The topic is still controversial and under debate.

As neutron stars are very peculiar objects for testing theories of matter at high density regimes, data about their macroscopic properties like mass and radius can also be used for studying possible deviations from GR. There exist suggestions [165, 166] that GR, being the only theory capable of describing strong gravitational field, is an extrapolation since the strength of gravity sourced by a star is many orders of magnitude larger that the one probed in the solar system weak field limit tests. From theoretical and experimental reasons one believes that GR should be modified when gravitational fields are strong and spacetime curvature is large [167]. Therefore, a promising route of investigation is to set a specific model of dense matter, i.e. equation of state, and then to compute macroscopic properties of neutron stars in ETGs. Indeed, the predictions of alternative theories to GR concerning the structure of compact objects is currently an active research field [168, 169, 170, 171]. The results presented in the following parts can be also found in [172, 173].

4.2.1 Equilibrium and stability of relativistic stars in General Relativity

Before we will discuss relativistic stars in theories different than General Relativity, let us present the problem in the Einstein theory [174, 175]. We will consider a spherical symmetric object whose geometry is given by the following metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2.$$ \hspace{1cm} (230)

The matter of the star is assumed to be described by the perfect-fluid energy momentum tensor

$$T_{\mu\nu} = p g_{\mu\nu} + (p + \rho)u_{\mu}u_{\nu},$$ \hspace{1cm} (231)
where $p$ and $\rho$ are pressure and the total energy density of the fluid. The four velocity $u^\mu$ of a co-moving (with the fluid) observer is normalized with the condition $u^\mu u_\mu = -1$. Additionally, in order to simplify calculations, we will make another assumption, that is, the fluid is at rest so the only non-vanishing component is $u_0 = -(-g^{00})^{-\frac{1}{2}} = -\sqrt{B(r)}$. Moreover, since the metric is time-independent and one deals with spherical symmetry, we get that pressure $p$ and energy density $\rho$ are functions only of the radial coordinate $r$.

The Einstein’s field equations ($\kappa = -8\pi G$)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$$ (232)

written for the considered system are

$$-\frac{B''}{2A} + \frac{B'}{4A} \left( \frac{A'}{B} + \frac{B'}{B} \right) - \frac{B'}{rA} = \frac{\kappa}{2}(\rho + 3p)B,$$ (233)

$$\frac{B''}{2B} - \frac{B'}{4B} \left( \frac{A'}{B} + \frac{B'}{B} \right) - \frac{A'}{rA} = \frac{\kappa}{2}(p - \rho)A,$$ (234)

$$-1 + \frac{r}{2A} \left( \frac{A'}{B} + \frac{B'}{B} \right) + \frac{1}{A} = \frac{\kappa}{2}(p - \rho)r^2,$$ (235)

where prime denotes $\frac{d}{dr}$. We have skipped the $\phi\phi$ equation as it is identical to $\theta\theta$ one. In order to find a form for $A(r)$, let us write

$$\frac{R_{rr}}{2A} + \frac{R_{00}}{2B} + \frac{R_{\theta\theta}}{r^2} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = \kappa \rho$$ (236)

which can be transformed into

$$\left( \frac{r}{A} \right)' = 1 + \kappa \rho r^2.$$ (237)

For $A(0)$ finite, the solution is

$$A(r) = \left( 1 - \frac{2G\mathcal{M}(r)}{r} \right)^{-1}$$ (238)

where one defines

$$\mathcal{M}(r) \equiv \int_0^r 4\pi\bar{r}^2 \rho(\bar{r})d\bar{r}.$$ (239)

Using the hydrostatic equilibrium $\nabla^\mu T_{\mu\nu} = 0$ which reads

$$\frac{B'}{B} = \frac{2p'}{p + \rho}$$ (240)
and the equation (238) one finds that the equation (235) is

\[ p'(r) = -\frac{GM(r)p(r)}{r^2} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \frac{4\pi r^3 p(r)}{M(r)}\right) \frac{1}{1 - \frac{2GM(r)}{r}}. \]  

(241)

The stars that we are considering are assumed to be in convective equilibrium so the entropy per nucleon is nearly constant throughout the star. Moreover, they have such a chemical composition that it is constant. Pressure \( p \) can be expressed as a function of the density \( \rho \), the entropy per nucleon \( s \), and the chemical composition, because of the equilibrium one sees that \( p(r) \) can be regarded as a function of \( \rho(r) \) alone. Due to that fact, one deals with a pair of first-order differential equation for \( \rho(r) \) and \( M(r) \):

\[ M'(r) = 4\pi r^2 \rho(r), \]  

(242)

\[ \frac{dp}{d\rho} \rho'(r) = -\frac{GM(r)p(r)}{r^2} \left(1 + \frac{p(r)}{\rho(r)}\right) \left(1 + \frac{4\pi r^3 p(r)}{M(r)}\right) \frac{1}{1 - \frac{2GM(r)}{r}}. \]

We are also equipped with an initial conditions provided by the equation (239)

\[ M(0) = 0. \]  

(243)

The above three equations, together with a given equation of state \( p(\rho) \), determine the functions \( \rho(r) \), \( M(r) \), \( p(r) \) throughout the star, once we specify the value of \( \rho(0) \). The system (242) must be integrated out the center of the star until \( p(\rho(r)) = 0 \) at some point \( r = R \). One interprets the value \( R \) as the radius of the particular star with the central density \( \rho(0) \).

The equations (242) are called Tolman-Oppenheimer-Volkoff (TOV) equations [176, 177, 178].

Finalizing that part, let us just take a look at the stability problem of the considered system of the relativistic star (for details, see for example [174]). The equilibrium state of the star represented by the equations (242) can be stable or unstable. We are interested in stable configurations. In our considerations we will need to recall the number of nucleons in the star which is defined as

\[ N = \int \sqrt{g} J_N^\mu \, dr \, d\theta \, d\phi = \int_0^R 4\pi r^2 \sqrt{A(r)B(r)} J_N^0(r) \, dr, \]  

(244)

where \( J_N^\mu \) is the conserved nucleon number current. Using the relation between \( J_N^0 \) and the nucleon number density measured in a locally inertial reference
frame at rest in the star \( n = -u_\mu J^\mu_N = \sqrt{B}J^0_N \) as well as the form of the metric (230) one gets

\[
N = \int_0^R 4\pi r^2 \left( 1 - \frac{2GM(r)}{r} \right)^{-\frac{1}{2}} n(r)\,dr. \tag{245}
\]

Similarly as for pressure, the number density \( n(r) \) is a function of the density \( \rho \), the chemical composition, and the entropy per nucleon \( s \). Hence, the quantities \( n(r) \) and \( N \) are fixed for a star for given \( \rho(0) \) together with chemical composition and constant \( s \).

The stability criterion can be express by the following theory [174]

**Theorem 2** A particular stellar configuration, with uniform entropy per nucleon and chemical composition, will satisfy the equations

\[
\mathcal{M}(r) = \int_0^R 4\pi \tilde{r}^2 \rho(\tilde{r})\,d\tilde{r},
\]

\[
p'(r) = -\frac{GM(r)\rho(r)}{r^2} \left( 1 + \frac{p(r)}{\rho(r)} \right) \left( 1 + \frac{4\pi \rho^3 p(r)}{GM(r)} \right)
\]

for equilibrium, if and only if the quantity \( \mathcal{M} \), defined by

\[
\mathcal{M} \equiv \int 4\pi r^2 \rho(r)\,dr \tag{246}
\]

is stationary with respect to all variations of \( \rho(r) \) that leave unchanged the quantity

\[
N = \int_0^R 4\pi r^2 \left( 1 - \frac{2GM(r)}{r} \right)^{-\frac{1}{2}} n(r)\,dr \tag{247}
\]

and that leave the entropy per nucleon and the chemical composition uniform and unchanged. The equilibrium is stable with respect to radial oscillations if and only if \( \mathcal{M} \) is a minimum with respect to all such variations.

The proof of the theorem can be found in [174]. It is based on the Lagrange multiplier method. Since we will perform similar calculations as presented there for a model of Extended Theories of Gravity (see the subsection 4.2.3), we will not repeat the proof of the theorem 2 here.

**4.2.2 Equilibrium of relativistic stars in Extended Theories of Gravity**

As already mentioned at the beginning of this section, the biggest challenge of modern astrophysics is the neutron stars’ equation of state. Since General Relativity provides a limit on the neutron star’s mass as not larger than \( 2M_\odot \), that
condition and recent observations require stiff nuclear equation of state \([179]\). The situation may differ in the case of Extended Theories of Gravity. Before considering any model of ETGs, one may try to understand how different modifications of the TOV equations \([242]\) contribute to the maximal mass value of a relativistic star. In \([173]\) we considered parametrized TOV equations containing five parameters \(\{\sigma, \alpha, \beta, \chi, \gamma\}\)

\[
M'(r) = 4\pi r^2 (\rho + \sigma p), \tag{248}
\]

\[
p'(r) = -\frac{G(1 + \alpha)M(r)\rho}{r^2} \left(1 + \frac{\beta p}{\rho}\right) \left(1 + \frac{\chi 4\pi r^3 p}{M(r)}\right) \left(1 - \frac{\gamma 2GM(r)}{r}\right). \tag{249}
\]

It was showed on the mass-radius diagrams how varying in parameters values shifts neutron stars’ maximal masses and changes their sizes (radii). The introduced parameters can be interpreted in the following way: \(\alpha\) is viewed as a part of the effective gravitational constant, that is, \(G_{\text{eff}} = G(1 + \alpha)\). The larger \(\alpha\), the smaller radius of the star while its maximal mass is also reduced. \(\beta\) is a coupling to the inercial pressure while \(\chi\) measures the active gravitational effects of pressure. Both extra contributions reduce the maximal mass; the latter has no effect on the radius. The parameter \(\gamma\) is an intrinsic curvature contribution (it is zero in Newtonian physics and 1 in GR). \(\sigma\) changes the way of computations of the mass function - there can appear for example some gravitational effect of pressure. The detailed discussion may be found in \([173]\). That exercise visualized that the problem of observed neutron stars’ masses bigger than predicted ones can be also explained by geometric modifications of the Einstein’s field equations. From now on, we will focus on a specific modification of Einstein’s equations which will provide generalized TOV equations.

Many gravitational models of Extended Theories of Gravity (ETGs) can be recast in the form proposed in \([180, 181, 182]\)

\[
\sigma(\Psi^i)(G_{\mu\nu} - W_{\mu\nu}) = \kappa T_{\mu\nu}. \tag{250}
\]

The tensor \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\) is the Einstein tensor, \(\kappa = -8\pi G\), the factor \(\sigma(\Psi^i)\) is a coupling to the gravity while \(\Psi^i\) represents for instance curvature invariants or other fields, like scalar ones. The symmetric tensor \(W_{\mu\nu}\) stands for additional geometric terms which may appear in a specific ETG under consideration. Non-symmetric parts coming from a considered theory could be also included in the tensor \(W_{\mu\nu}\) but then one should also add antisymmetric elements into energy-momentum tensor (for instance fermion fields). We will consider in that chapter only cases for which the tensor \(W_{\mu\nu}\) is symmetric one.
It is important to note that (250) represents a parameterization of gravitational theories at the level of field equations. The energy-momentum tensor $T_{\mu\nu}$ is treated as the one of a perfect fluid, that is $T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u_{\mu} u_{\nu}$, where $p$ and $\rho$ are the pressure and the energy density of the fluid. The four velocity $u^\mu$ of the co-moving (with the fluid) observer is normalized with the condition $u^\mu u_\mu = -1$. One could make an assumption on the equation of state for $p$ and $\rho$, but we will not do it in order to keep our considerations as general as it is possible.

It is worth noting that (250) does not include all the possible alternatives to GR at the level of the equations of motion. However, most of the main proposals like, for instance, scalar tensor theories, $f(R)$ and hybrid gravity theories (see the chapter 3), can be reshaped in this form as well as theories which have a time dependent effective gravitational coupling $\sigma \equiv \sigma(t)$ and $W_{\mu\nu} = 0$.

One may also add a coupling to the matter source (as it appears often in the so-called Einstein frame) but here we will not consider that case. From the structure of (250) one sees that GR is immediately recovered for $\sigma(\Psi^i) = 1$ in the geometric units and $W_{\mu\nu} = 0$. The modified Einstein’s field equations (250) can be written for the later convenience as

$$G_{\mu\nu} = \kappa T_{\mu\nu}^{\text{eff}} = \frac{\kappa}{\sigma} T_{\mu\nu} + W_{\mu\nu}. \tag{251}$$

We would like to note that one cannot postulate that the energy-momentum tensor of the matter $T_{\mu\nu}$ is conserved. Rather, due to the Bianchi identity $\nabla_{\mu} G^{\mu\nu} = 0$, the effective energy-momentum tensor $T_{\mu\nu}^{\text{eff}}$ is covariantly conserved i.e., $\nabla_{\mu} T_{\mu\nu}^{\text{eff}} = 0$. In some special cases of ETG [183] one deals with the conservation of the matter energy-momentum tensor but in general it does not have to be true.

The simplest configuration for a star is the static and spherically symmetric geometry

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{252}$$

From the normalization condition one has that $u_0 = -\sqrt{B(r)}$. As the metric is time independent and spherically symmetric, the pressure $p$ and energy density $\rho$ are functions of the radial coordinate $r$ only. Hence we will assume that the coupling function $\sigma$ and the geometric contributions $W_{\mu\nu}$ are also independent of the coordinates $(t, \theta, \phi)$. The symbol prime ($'$) denotes the derivative with respect to the coordinate $r$. 

4.2 Equilibrium and Stability of Relativistic Stars
We calculate in detail the components of (251). The components of the Ricci tensor read

\[
R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A} \left( \frac{A'}{B} + \frac{B'}{B} \right) - \frac{B'}{rA} = \frac{\kappa}{2\sigma} (\rho + 3p)B + W_{tt} + \frac{BW'}{2},
\]

(253)

\[
R_{rr} = -\frac{B''}{2A} - \frac{B'}{4B} \left( \frac{A'}{B} + \frac{B'}{B} \right) - \frac{A'}{rA} = \frac{\kappa}{2\sigma} (p - \rho)A + W_{rr} - \frac{AW'}{2},
\]

(254)

\[
R_{\theta\theta} = -1 + \frac{r}{2A} \left( -\frac{A'}{B} + \frac{B'}{B} \right) + \frac{1}{A} = \frac{\kappa}{2\sigma} (p - \rho)r^2 + W_{\theta\theta} - \frac{r^2W'}{2},
\]

(255)

where \( W = -B^{-1}W_{tt} + A^{-1}W_{rr} + 2r^{-2}W_{\theta\theta} \) is a trace of the tensor \( W_{\mu\nu} \). Let us notice that the \( R_{\phi\phi} \) equation is the same as the \( R_{\theta\theta} \) one multiplied by the factor \( \sin^2 \theta \) and hence we concluded that \( r^{-2} \sin^2 \theta W_{\phi\phi} = W_{\theta\theta} \). Using the above equations to write

\[
\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{2B} + \frac{R_{\phi\phi}}{r^2} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{A^2} = \frac{\kappa\rho}{\sigma} + r^2B^{-1}W_{tt}
\]

(256)

we obtain the following relation

\[
\left( \frac{r}{A} \right)' = 1 + \frac{\kappa r^2 \rho(r)}{\sigma(r)} + r^2B^{-1}(r)W_{tt}(r).
\]

(257)

Then we may solve equation (257) and write the solution as

\[
A(r) = \left( 1 - \frac{2Gm(r)}{r} \right)^{-1},
\]

(258)

where the mass function \( m(r) \) is defined here as

\[
m(r) = \int_0^r \left( 4\pi r^2 \frac{\rho(\bar{r})}{\sigma(\bar{r})} - \frac{\bar{r}^2W_{tt}(\bar{r})}{2GB(\bar{r})} \right) d\bar{r}.
\]

(259)

It is clearly different from the usual definition given by GR (242). Let us recall that the above quantity (similarly as in GR case [174]) is interpreted as total energy of a star together with the one coming from gravitational field.

For the further purposes we will also need the relations

\[
\frac{A'}{A} = \frac{1 - \frac{\kappa A}{\sigma} Q}{r},
\]

(260)

\[
\frac{B'}{B} = \frac{A - 1 - \frac{\kappa A}{\sigma} \Pi}{r},
\]

(261)
where we have defined new quantities
\[ Q(r) := \rho(r) + \frac{\sigma(r) W_{tt}(r)}{k_B(r)}, \]  
and
\[ \Pi(r) := p(r) + \frac{\sigma(r) W_{rr}(r)}{k_A(r)}. \]

The hydrostatic equilibrium \( \nabla_\mu T^\mu_{\text{eff}} = 0 \) reads then
\[ \kappa (\sigma^{-1} \nabla_\mu T^\mu_{\text{eff}} - \sigma^{-2} T^\mu_{\text{eff}} \nabla_\mu \sigma) + \nabla_\mu W^\mu_{\text{eff}} = 0, \]
or, more explicitly,
\[ \kappa \sigma^{-1} \left( p' + (p + \rho) \frac{B'}{2B} \right) - \kappa p \frac{\sigma'}{\sigma^2} - \frac{A'}{A^2} W_{rr} + A^{-1} W'_{rr} + \frac{2W_{rr}}{A r} + \frac{B'}{2B} \left( \frac{W_{rr}}{A} + \frac{W_{tt}}{B} \right) - \frac{2W_{\theta\theta}}{r^2} = 0. \]

Let us notice that from (263) and with the help of (260)
\[ \left( \frac{\Pi}{\sigma} \right)' = -\frac{Gm}{r^2} \left( \frac{Q}{\sigma} + \frac{\Pi}{\sigma} \right) \left( 1 + \frac{4\pi\tilde{r}^3 \frac{\Pi}{\sigma}}{m} \right) \left( 1 - \frac{2Gm}{r} \right)^{-1} \]
\[ + \frac{2\sigma}{kr} \left( \frac{W_{\theta\theta}}{\sigma r^2} - \frac{W_{rr}}{A} \right). \]

This equation is the basic structure for deriving the generalized hydrostatic equilibrium for stars in ETG. Together with (261) and definition (262), the equation (266) can be written as
\[ \left( \frac{\Pi}{\sigma} \right)' = -\frac{Gm}{r^2} \left( \frac{Q}{\sigma} + \frac{\Pi}{\sigma} \right) \left( 1 + \frac{4\pi\tilde{r}^3 \frac{\Pi}{\sigma}}{m} \right) \left( 1 - \frac{2Gm}{r} \right)^{-1} \]
\[ + \frac{2\sigma}{kr} \left( \frac{W_{\theta\theta}}{\sigma r^2} - \frac{W_{rr}}{A} \right). \]

The above equation (267) and
\[ m(r) = \int_0^r 4\pi\tilde{r}^2 \frac{Q(\tilde{r})}{\sigma(\tilde{r})} d\tilde{r}. \]

have a similar functional form as the standard GR result. It is worth noting that such equations determine completely the stellar equilibrium since the assumption that pressure is expressed as a function of density only, i.e., the entropy per nucleon and the chemical composition as constant throughout the star. Such assumptions will also be used in the analysis of stability of these systems.
4.2.3 Stability of relativistic stars in scalar-tensor theory of gravity

In scalar-tensor theories the gravitational interaction is mediated not only by the metric field (as in GR), but also by scalar field $\phi$. Among many realizations of scalar-tensor theories, a simple prototype is the k-essence class in which the scalar field is said to be minimally coupled to the geometric sector.

The theory can be written according to the following action

$$ S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left( R - \nabla_\mu \phi \nabla^\mu \phi - 2V(\phi) \right) + S_m[g_{\mu \nu}, \psi]. \quad (269) $$

The field equations derived from it with respect to the metric $g_{\mu \nu}$ and the scalar field $\phi$ are

$$ G_{\mu \nu} + \frac{1}{2} g_{\mu \nu} \nabla_\alpha \phi \nabla^\alpha \phi - \nabla_\mu \phi \nabla_\nu \phi + g_{\mu \nu} V(\phi) = \kappa T_{\mu \nu}, \quad (270) $$

$$ V'(\phi) - \Box \phi = 0, \quad (271) $$

respectively. From the Klein-Gordon we see that the scalar field $\phi$ depends on matter contribution ($\rho$) via the d’Alembertian operator. For the k-essence case we identify $\sigma_k = 1$ and

$$ W_{\mu \nu} = -\frac{1}{2} g_{\mu \nu} \nabla_\alpha \phi \nabla^\alpha \phi + \nabla_\mu \phi \nabla_\nu \phi - g_{\mu \nu} V(\phi) \quad (272) $$

from which we can write the following components

$$ W_{tt} = \frac{1}{2} B \nabla_\alpha \phi \nabla^\alpha \phi + BV(\phi) = B(C + 2V), \quad (273) $$

$$ W_{rr} = AC, \quad (274) $$

$$ W_{\theta \theta} = -r^2 (C + 2V). \quad (275) $$

In the above expressions we have defined $V \equiv V(\phi)$ and

$$ C \equiv C(Q, \phi, \phi') = \frac{1}{2} A^{-1} \phi'^2 - V(\phi). \quad (276) $$

Let us remind that $A$ is a function of the Q ($258$). Hence, the last term appearing in the generalized TOV equation ($267$) is $-\frac{4\sigma}{\kappa T}(C + V) = -2\sigma \frac{\phi'^2}{\kappa A T}$. Moreover, in the k-essence case, the functions Q and $\Pi$ will have the form

$$ Q = \rho(r) + \kappa^{-1} (C + 2V), \quad (277) $$

$$ \Pi = \rho(r) + \kappa^{-1} C. \quad (278) $$
4.2 Equilibrium and Stability of Relativistic Stars

Notice that the second law of thermodynamics will differ in ETG’s [184]. Let us calculate in detail the stability analysis. We then assume that the particle number \( N^\alpha = n u^\alpha \) is supposed to be conserved

\[
\nabla_\alpha (n u^\alpha) = u^\alpha \nabla_\alpha n + n \nabla_\alpha u^\alpha = 0. \tag{279}
\]

The crucial issue here is that we are dealing with effective energy-momentum tensor (from the Bianchi identities \( \nabla_\mu G^{\mu\nu} = 0 \)), therefore

\[
u_\nu \nabla_\mu T^{\mu\nu}_{\text{eff}} = \sigma^{-1} \left( u^{\mu} \nabla_\mu p - n u^{\mu} \nabla_\mu \left( \frac{p + \rho}{n} \right) + \rho u^{\mu} \nabla_\mu \sigma \right)
+ u_\nu \nabla_\mu W^{\mu\nu}, \tag{280}
\]

and

\[-nu^{\mu} \left( p \nabla_\mu \left( \frac{1}{n} \right) + \nabla_\mu \left( \frac{\rho}{n} \right) + \frac{\rho}{n} \nabla_\mu \sigma \right) + \sigma u^{\mu} W^{\mu\nu}_{\mu\nu} = 0. \tag{281}\]

As we are working with modified field equations of the specific form (270), the coupling term \( \nabla_\mu \sigma \) in the above formula vanishes. Furthermore, we will show that in the case of k-essence the tensor \( \nabla^\nu W^{\mu\nu}_{\mu\nu} = 0 \). The only non-vanishing terms that undergo infinitesimal changes with respect to the infinitesimal changes of the energy density are

\[0 = \delta \left( \frac{\rho}{n} \right) + p \delta \left( \frac{1}{n} \right), \tag{282}\]

and consequently,

\[
\delta n(r) = \frac{n(r)}{p(r) + \rho(r)} \delta \rho(r). \tag{283}\]

However, as we have already mentioned, the ETG that we are considering has a very special forms of the effective energy-momentum tensor (270). As \( W^{\mu\nu}_{\mu\nu} \) is symmetric and one also deals with the K-G equations, we notice that

\[
\nabla^{\mu} W^{\mu\nu}_{\mu\nu} = \nabla^{\mu} \phi (\nabla^{\mu} \nabla \phi - \nabla_\gamma \nabla^{\mu} \phi) = 0
\]

where we have used the K-G equation \( \Box \phi = V' \). One may also compute it explicitly for the component \( \mu = r \)

\[
\nabla_r W^r_{\gamma} = C' + (C + V)(\frac{A - 1}{r} - \kappa \alpha r \Pi + \frac{4}{r}) := C' + D, \tag{285}
\]

while the derivative \( C' = \frac{dC(\phi, \phi')}{d\tau} \) after applying K-G equation, gives rise to

\[C' = -(C + V)(\frac{A - 1}{r} - \kappa \alpha r \Pi + \frac{4}{r}) = -D.\]
Therefore, component $\mu = r$ of equation (281) resembles the GR form

$$n'(r) = n \frac{\rho'}{\rho + p}, \quad (286)$$

Now on, we are going to use the Lagrange multipliers method following the procedure presented in [174]. The nucleon number $N$ remains unchanged but we should remember that it also depends on the modified geometry (see the formula (258) and below). It reads as $N = \int_0^R 4\pi r^2 [1 - 2Gm(r)/r]^{-1/2} n(r) dr$. Then, we find

$$0 = \delta m - \lambda \delta N = \int_0^\infty 4\pi r^2 \delta Q dr$$
$$- \lambda \int_0^\infty 4\pi r^2 \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \delta n(r) dr$$
$$- \lambda G \int_0^\infty 4\pi r \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} n(r) \delta m(r) dr, \quad (287)$$

Let us notice that the integrands vanish outside the radius $R + \delta R$. It allows us to write the integration intervals as $[0, \infty]$ instead of $[0, R]$, where $R$ is a radius of a star. The variation does not change the entropy per nucleon as well as leaves the chemical composition uniform. Because of the form (293), one needs to understand the relation between $\delta \rho$ and $\delta Q$. Let us discuss it.

From the relation $\rho = Q - \kappa^{-1}(C + 2V)$ we notice that $\rho$ is a function of $Q, \phi$ and $\phi'$. Hence, we obtain

$$\delta \rho = \delta Q - \kappa^{-1}(\delta C + 2V' \delta \phi). \quad (288)$$

Since $\phi = \phi(r)$ only, we may write $\phi' = \partial_\mu \phi = \nabla_\mu \phi$. The term $\delta C$ turns out to be

$$\delta C = - \frac{1}{2} A^{-2} \phi'^2 \delta A + A^{-1} \phi' \delta \phi' - V' \delta \phi =$$
$$- \frac{G}{r} \phi'^2 \int_0^R (4\pi r^2 \delta Q dr) + A^{-1} \phi' \delta \phi' - \diamond \phi \delta \phi$$
$$= - \frac{G}{r} \phi'^2 \int_0^R (4\pi r^2 \delta Q dr) + \nabla_\mu \phi \delta \nabla_\mu \phi - \diamond \phi \delta \phi, \quad (290)$$

where we have used the KG equation $\Box \phi = V'$. Then

$$\delta \rho = \delta Q - \kappa^{-1} \left( - \frac{G}{r} \phi'^2 \int_0^R (4\pi r^2 \delta Q dr) + \nabla_\mu \phi \delta \nabla_\mu \phi + \diamond \phi \delta \phi \right). \quad (291)$$
Let us notice that \( \nabla_\mu \phi \delta \nabla^\mu \phi + \Box \phi \delta \phi = \nabla_\mu (\delta \phi \nabla^\mu \phi) \). Hence

\[
\delta \rho = \delta Q - \kappa^{-1} \left( - \frac{G}{r} \phi' \int_0^R (4 \pi r^2 \delta Q dr) + \nabla_\mu (\delta \phi \nabla^\mu \phi) \right).
\] (292)

Now we are ready to write \( \delta n \) with respect to \( \delta Q \) and \( \delta \phi \):

\[
\delta n(r) = \frac{n(r)}{p(r) + \rho(r)} \times \left( \delta Q - \kappa^{-1} \left( - \frac{G}{r} \phi' \int_0^R (4 \pi r^2 \delta Q dr) + \nabla_\mu (\delta \phi \nabla^\mu \phi) \right) \right).
\] (293)

We have used that \( \delta m'(r') = \int_0^\infty 4 \pi r^2 \delta Q dr' \). The equation (287) now reads

\[
0 = \int_0^\infty 4 \pi r^2 \delta Q dr - \lambda G \int_0^\infty 4 \pi r A_1^\frac{1}{2} n(r) \int_0^\infty (4 \pi r^2 \delta Q dr) dr
\]

\[
- \lambda \int_0^\infty 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \times \left( \delta Q - \kappa^{-1} \left( - \frac{G}{r} \phi' \int_0^\infty (4 \pi r^2 \delta Q dr) + \nabla_\mu (\delta \phi \nabla^\mu \phi) \right) \right) dr
\] (294)

Before going further, let us discuss the term

\[
\int_0^\infty 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \kappa^{-1} \nabla_\mu (\delta \phi \nabla^\mu \phi) dr.
\] (295)

We may write it as

\[
\kappa^{-1} \int_0^\infty \nabla^\mu \left( 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \delta \phi \nabla_\mu \phi \right) dr
\]

\[
- \kappa^{-1} \int_0^\infty \delta \phi \nabla_\mu \phi \nabla_\mu \left( 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \right) dr
\]

\[
= \kappa^{-1} \int_0^\infty \partial^\mu \left( 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \delta \phi \partial_\mu \phi \right) dr
\]

\[
+ \kappa^{-1} \int_0^\infty \left[ 4 \pi r^2 A_1^\frac{1}{2} \Gamma^\mu_{\mu \nu} \frac{n(r)}{p(r) + \rho(r)} - \partial_\nu \left( 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \right) \right] \delta \phi \partial^\nu \phi dr.
\]

The term \( \int_0^\infty \partial^\mu \left( 4 \pi r^2 A_1^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \delta \phi \partial_\mu \phi \right) dr \) is a constant. By choosing a suitable boundary condition for the scalar field \( \phi \), it may vanish. We will neglect it
in the further analysis. Interchanging the \(r\) and \(r'\) integrals in the equation (294) we will get the following one

\[
0 = \delta m - \lambda \delta N = \int_0^\infty 4\pi r^2 \left[ 1 - \frac{\lambda n(r)}{p(r) + \rho(r)} \right] \hat{A}^\frac{1}{2} - \lambda G \int_r^\infty 4\pi r' n(r') \hat{A}^\frac{1}{2} dr' \\
- \lambda G \hat{K}^{-1} \int_r^\infty 4\pi r \hat{A}^\frac{1}{2} \left[ \frac{n(r)}{p(r) + \rho(r)} \right] \delta Q(r) dr \\
- \lambda \hat{K}^{-1} \int_0^\infty \hat{\delta} \hat{\phi} \left[ 4\pi r^2 \hat{A}^\frac{1}{2} \hat{\Gamma}_{\mu \nu}^\mu \frac{n(r)}{p(r) + \rho(r)} \\
- \partial_\nu \left( 4\pi r^2 \hat{A}^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \right) \right] \delta \phi dr. \tag{296}
\]

In order to have the vanishing right hand side of the above equation, both terms containing the variations \(\delta Q\) and \(\delta \phi\) must vanish independently. The term with \(\delta Q\) will vanish if

\[
\frac{1}{\lambda} = \frac{n(r)}{p(r) + \rho(r)} \hat{A}^\frac{1}{2} + G \int_r^\infty 4\pi r' n(r') \hat{A}^\frac{1}{2} dr' \\
+ G \hat{K}^{-1} \int_r^\infty 4\pi r \hat{A}^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \phi'^2 dr \tag{297}
\]

while the second one with \(\delta \phi\) vanishes when

\[
4\pi r^2 \hat{A}^\frac{1}{2} \hat{\Gamma}_{\mu \nu}^\mu \frac{n(r)}{p(r) + \rho(r)} - \partial_\nu \left( 4\pi r^2 \hat{A}^\frac{1}{2} \frac{n(r)}{p(r) + \rho(r)} \right) = 0. \tag{298}
\]

We will start with (297). Deriving it with respect to \(r\) and using \(n'(r) = n \frac{\rho'}{p + \rho}\) one has:

\[
0 = -4\pi G r \hat{A} - \frac{p'}{(p + \rho)^2} + \frac{G}{p + \rho} A (4\pi r Q - \frac{m}{r^2}) \\
- 4\pi G \hat{K}^{-1} \frac{\phi'^2}{p + \rho}.
\]

Applying the following relations to the above expression:

\[
\frac{\Lambda - 1}{r} = \frac{2Gm}{r^2}, \tag{299}
\]

\[
p + \rho = \Pi_k + Q_k - 2\hat{K}^{-1} (C + V), \tag{300}
\]

\[
2(C + V) = \Lambda^{-1} \phi'^2, \tag{301}
\]

\[
\Pi'_k = p' + \hat{K}^{-1} C' = p' - \hat{K}^{-1} (C + V) \left( \frac{\Lambda - 1}{r} - \hat{K} \Pi + \frac{4}{r} \right) \tag{302}
\]
we find that
\[
\Pi' = -\frac{AGm}{r^2} (\Pi + Q) (1 + 4\pi r^3 \frac{\Pi}{m}) - 4\frac{C + V}{\kappa r}, \tag{303}
\]
which is a form of generalized TOV equation derived in the previous section for the k-essence model.

Let us come back to the equation (298). Writing the derivative with respect to \( r \) explicitly and computing the gamma term, that is, \( \Gamma_{\mu r}^{\mu} = \frac{2}{r} - \frac{1}{2} (\kappa A r (\Pi + Q)) \) we will again obtain, after applying \( n'(r) = n \frac{\rho'}{\rho + p} \) and (302)
\[
\Pi' = -\frac{AGm}{r^2} (\Pi + Q) (1 + 4\pi r^3 \frac{\Pi}{m}) - 4\frac{C + V}{\kappa r},
\]
which finally proves that the relativistic star's system provided by the k-essence model is a stable configuration.

4.2.4 Remarks

Let us here conclude our investigation. We have shown that Extended Theories of Gravity based on the phenomenological field equations (250) provide the stellar equilibrium equations for static, spherically symmetric geometries given by the equations (266) and (268). They are the analogous version of the TOV equations for any ETG. Such equations can now be further applied to specific gravitational theories. The differences between our equations and the ones provided by GR are in the definition of the mass \( m(r) \) as one deals with the coupling \( \sigma \) and the additional term \( W_{tt} \) and in the definition of pressure. Due to that fact, one needs to introduce effective quantities in order to obtain TOV-like form (242). For the particular case shown in (267) the TOV structure is preserved only if one finds a suitable theory in which \( W_{\theta \theta} = W_{tt} r^2 / \Lambda \) and regarded that we identify \( Q \) and \( \Pi \) as the effective density and effective pressure, respectively.

Concerning the stability of such systems, we argue that this analysis should be implemented case by case only, i.e., it is difficult to achieve general results without specifying the functions \( W_{\mu \nu} \) and \( \sigma(\Psi^i) \). As an example showing the applicability of our results, we worked on the specific class of k-essence theories. For this case, we generalized the stability theorem found for instance in [174] taking into account the new functions \( Q \) ans \( \Pi \). We found that the specific k-essence case leads to stable configurations.

The considered example shows that the equilibrium (267) is recovered from the Lagrange multiplier method with the reformulated stability criterion. Contrary to the standard case, even assuming uniform entropy per nucleon and
chemical composition, the interpretation of the mass function $m$ should be identified with effective energy density $Q$. The same analysis should be also applied to the definition of the nucleon number $N$.

The investigation of the stability of stellar systems in ETG and other modifications of gravity that cannot be written in the form ($250$) should be further examined. A very interesting case is the scalar-tensor gravity with non-minimally coupled scalar field. Due to that fact the equation ($280$) will have a much complex form than the GR and k-essence cases. The work is in progress.
APPENDIX
Laws of physics are often written in a form of differential equations. Solving them allows us to determine a behavior of a physical system if we know initial conditions in the case of ordinary differential equations or boundary conditions for partial ones. One immediately comes to the conclusion that the knowledge of methods whose applications result in a solution of a differential equation is particularly important for physicists. Some equations that we face are well-known differential equations with given solutions or we are just lucky to find a way to write them in a form of a known ones, already classified. The problem arises when we deal with differential equations of an unfamiliar type. Fortunately, there exist tools which may help. The ones that we are using are Lie symmetries methods.

A subclass of Lie symmetries are very well-known Noether symmetries which have reached rightful place in physics. The application of Noether theorem has been proven to have crucial importance for research in quantum and particle physics as well as in cosmology \[185, 112, 117, 126\] (see below). Unfortunately, Noether symmetries might be applied only in special cases: for systems which are modeled with a Lagrangian. Lie symmetries method unlike the Noether symmetries approach can be used in the case of differential equations which do not arise from Lagrangian of a physical system, that is, they are not obtained from variational principle. Moreover, it may happen that the system does not admit any Noether symmetries but it admits Lie ones and we are still able to solve or simplify the differential equations. In the following chapter we are going to summarize Lie symmetries methods which have been used in the thesis.

A.1 One-Parameter Point Transformations and Lie Symmetries Group

Let us consider two points P and Q living in a neighborhood \( U \) in a smooth manifold \( M, \dim M = n \), with coordinates \((x, y)\) and \((\tilde{x}, \tilde{y})\), respectively. The following transformation of the coordinates of the point P into the coordinates of Q on \( U \)

\[
\tilde{x} = \tilde{x}(x, y), \quad \tilde{y} = \tilde{y}(x, y)
\]  

(304)
is called a point transformation \([133, 186, 132]\). The functions \(\tilde{\varphi}(x, y)\) and \(\tilde{\psi}(x, y)\) are independent. They map points \((x, y)\) into points \((\tilde{x}, \tilde{y})\).

One is particularly interested in one-parameter point transformations which depend on one (or more) arbitrary parameter \(\epsilon \in \mathbb{R}\)

\[
\tilde{x} = \tilde{x}(x, y; \epsilon), \quad \tilde{y} = \tilde{y}(x, y; \epsilon) \tag{305}
\]

Moreover, we want them to be invertible and that repeated applications produce a transformation of the same family for some \(\hat{\epsilon} = \hat{\epsilon}(\tilde{\epsilon}, \epsilon)\):

\[
\hat{x} = \hat{x}(\tilde{x}, \tilde{y}; \tilde{\epsilon}) = \hat{x}(x, y; \hat{\epsilon}) \tag{306}
\]

The identity of the transformation is given by, for example, \(\epsilon = 0\):

\[
\tilde{x} = \tilde{x}(x, y; 0), \quad \tilde{y} = \tilde{y}(x, y; 0) \tag{307}
\]

The transformations \(\tilde{x}\) with the above properties form a one - parameter group of point transformations.

We call the transformation

\[
T : x \mapsto \tilde{x}(x) \tag{308}
\]

a symmetry, if it satisfies the following conditions \([133]\):

- The transformation preserves the structure.
- The transformation is a diffeomorphism.
- The transformation maps the object to itself (the symmetry condition).

Now on, let us consider an infinite set of symmetries \(T_\epsilon\) (one-parameter group of point transformations)

\[
T_\epsilon : x^s \mapsto \tilde{x}^s(x^1, \ldots, x^n; \epsilon), \quad s = 1, \ldots, n. \tag{309}
\]

We will call the set of symmetries \(T_\epsilon\) a one-parameter local Lie group if the following conditions are satisfied \([133]\)

- \(T_0\) is the trivial symmetry, so that \(\tilde{x}^s = x^s\) when \(\epsilon = 0\).
- \(T_\epsilon\) is a symmetry for every \(\epsilon\) in some neighborhood of zero.
- \(T_\epsilon T_\delta = T_{\epsilon + \delta}\) for every \(\epsilon, \delta\) sufficiently close to zero.
- Each \(\tilde{x}^s\) may be represented as a Taylor series in \(\epsilon\) in some neighborhood of \(\epsilon = 0\):

\[
\tilde{x}^s(x^1, \ldots, x^n; \epsilon) = x^s + \epsilon \xi^s(x^1, \ldots, x^n) + \mathcal{O}(\epsilon^2), \quad s = 1, \ldots, n. \tag{310}
\]
One may visualize the one-parameter group on an \(x - y\) plane. Let us consider an arbitrary point \(A = (x_0, y_0)\) on a plane with \(\epsilon = 0\). Varying the parameter \(\epsilon\), the images \((\tilde{x}_0, \tilde{y}_0)\) of the point \(A\) will move along some curve \([132]\). Let us take another initial points and repeat the procedure: one gets a family of curves. Each curve represents points which can be transformed into each other under the action of the group. That curve is called the orbit of the group. The family of the curves is characterized by a field of their tangent vectors \(X\). In order to see it, let us consider infinitesimal transformations: taking an arbitrary point \((x, y)\) and representing the transformations \((305)\) as a Taylor series

\[
\begin{align*}
\tilde{x}(x, y; \epsilon) &= x + \epsilon \xi(x, y) + \ldots = x + \epsilon Xx + \ldots, \\
\tilde{y}(x, y; \epsilon) &= y + \epsilon \eta(x, y) + \ldots = x + \epsilon Xy + \ldots
\end{align*}
\]  

One defines functions \(\xi\) and \(\eta\)

\[
\begin{align*}
\xi(x, y) &= \frac{\delta \tilde{x}}{\delta \epsilon} \bigg|_{\epsilon=0}, \\
\eta(x, y) &= \frac{\delta \tilde{y}}{\delta \epsilon} \bigg|_{\epsilon=0},
\end{align*}
\]

with the operator (tangent vector) \(X\) as

\[
X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}.
\]

The operator \(X\) is called the infinitesimal generator of the transformation.

As a simple example of a one-parameter group let us consider the rotations

\[
\begin{align*}
\tilde{x} &= x \cos \epsilon - y \sin \epsilon, \\
\tilde{y} &= x \sin \epsilon + y \cos \epsilon,
\end{align*}
\]

for which, from the definitions \((313)\) one has \(\xi(x, y) = -y\), \(\eta(x, y) = x\) so the infinitesimal generator of the rotation transformations is

\[
X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.
\]

Before starting the discussion on Lie symmetries of differential equations, we need to introduce a prolongation of the infinitesimal generator \((314)\):

**Definition 1** The prolongation up to the \(n\)th derivative of the infinitesimal generator \((314)\) of a point transformation is a vector

\[
X^{(n)} = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y'} + \ldots + \eta^{(n)} \frac{\partial}{\partial y^{(n)}},
\]

where the functions \(\eta^{(n)}(x, y, y', \ldots, y^{(n)})\) are defined as

\[
\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)} \frac{d\xi}{dx}.
\]
One should notice that the functions \( \eta^{(n)} \) are not the \( n \)th derivative of \( \eta \) but they are polynomials in the derivatives \( y', \ldots, y^{(n)} \). Since the expressions of (318) are complicated for higher \( n \), let us just write the first two steps:

\[
\begin{align*}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2, \\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 \\
&\quad - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') y'',
\end{align*}
\]

where the comma, for example in \( \eta_x \), denotes the partial derivative with respect to \( x \).

### A.2 ORINARY DIFFERENTIAL EQUATIONS AND LIE POINT SYMMETRIES

Let us start with the following theorem [132]:

**Theorem 3** We will say that an ordinary differential equation (ODE)

\[
H(x, y, y', \ldots, y^{(n)}) := y^{(n)} - \omega(x, y, y', \ldots, y^{(n-1)}) = 0,
\]

where \( y = y(x), \ y' = \frac{dy}{dx}, \ldots, y^{(n)} = \frac{d^n y}{dx^n} \), admits a group of symmetries with generator \( X \) if and only if

\[
X^{(n)} H \equiv 0, \mod H = 0
\]

is held, where \( X^{(n)} \) is the \( n \)th prolongation of \( X \).

A point transformation (305) is a symmetry transformation (a symmetry) of the \( n \)th order ODE (321) if it maps solutions into solutions. It means that the image \( \tilde{y}(\tilde{x}) \) of any solution \( y(x) \) is again a solution: (321) does not change under a symmetry transformation, so

\[
H(\tilde{x}, \tilde{y}, \tilde{y}', \ldots, \tilde{y}^{(n)}) = 0.
\]

It is important to notice that the existence of a symmetry is independent of the choice of variables that we use for expressing the ODE and its solutions. It might happen that we are dealing with a complicated looking differential equation with several symmetries found by the procedure explained below. That may mean that our differential equation is a simple one but given in unsuitable variables. Using symmetries, we can transform the equation into an easier form.
The nth order ODE (323) is valid for all values of the parameter \( \epsilon \) hence the differentiation of it with respect to \( \epsilon \) gives
\[
0 = \frac{\partial H(\tilde{x}, \tilde{y}, \tilde{y}', ..., \tilde{y}^{(n)})}{\partial \epsilon}
\]
\[
= \left( \frac{\partial H}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial \epsilon} + \frac{\partial H}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial \epsilon} + ... + \frac{\partial H}{\partial \tilde{y}^{(n)}} \frac{\partial \tilde{y}^{(n)}}{\partial \epsilon} \right) \bigg|_{\epsilon=0}
\]
and using the definitions of (318) with \( \frac{\partial H}{\partial \tilde{x}} \bigg|_{\epsilon=0} = \frac{\partial H}{\partial x} \), the condition (324) is
\[
\xi \frac{\partial H}{\partial x} + \eta \frac{\partial H}{\partial y} + \eta' \frac{\partial H}{\partial y'} + ... + \eta^{(n)} \frac{\partial H}{\partial y^{(n)}} = 0,
\] (325)
or simply
\[
X^{(n)} H = 0.
\] (326)
The ODE \( H = y^{(n)} - \omega(x, y, y', ..., y^{(n-1)}) = 0 \) is invariant under the infinitesimal transformation, it means, if \( H = 0 \) holds and it admits a group of symmetries with generators \( X \), then \( X^{(n)} H = 0 \) also holds. The converse is also true [132].

Applying the definition (317) into the symmetry condition (322) we may write it as
\[
\eta^{(n)} = X^{(n)} \omega
\]
\[
= \left( \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y'} + ... + \eta^{(n-1)} \frac{\partial}{\partial y^{(n-1)}} \right) \omega
\] (327)
with \( \eta^{(i)} \) given by (318). The n-derivative \( y^{(n)} \) appearing in \( \eta^{(n)} \) must be substituted by \( \omega \) [132, 133]. This equation reduces to a system of partial differential equations (PDE’s) after equating to zeroes terms which are multiplied by powers of \( y^{(n-1)} \), \( y^{(n-2)} \), ..., and so on because the functions \( \xi(x, y) \) and \( \eta(x, y) \) are independent of the derivatives of \( y \). The system of PDE’s determining \( \xi(x, y) \) and \( \eta(x, y) \) can usually be solved.

As an example we will consider the simplest second-order ODE \( y'' = 0 \). The linearized symmetry condition [133] is
\[
\eta^{(2)} = 0 \text{ when } y'' = 0
\]
that is,
\[
\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy} y'^3 = 0.
\] (328)
The condition (328) splits into the system of determining equations:
\[
\eta_{xx} = 0, \; 2\eta_{xy} - \xi_{xx} = 0, \; \eta_{yy} - 2\xi_{xy} = 0, \; \xi_{yy} = 0,
\] (329)
with the general solution of the last one:

\[ \xi(x, y) = A(x)y + B(x) \]

with the arbitrary functions A and B. The third equation of (329) gives (C and D are also arbitrary functions)

\[ \eta(x, y) = A'(x)y^2 + C(x)y + D(x) \]

and using these results to the remaining equations in (328) one obtains

\[ A''''(x)y^2 + C''''(x)y + D''''(x) = 0, \quad 3A''''(x)y + 2C'(x) - B''''(x) = 0. \]

Since the unknown functions in the above equations are independent of y, one equates powers of y obtaining a system of ODEs:

\[ A''''(x) = 0, \quad C''''(x) = 0, \quad D''''(x) = 0, \quad B''''(x) = 2C'(x) \]

which is easily solved. Hence, for every one-parameter Lie group of symmetries of the equation \( y''''(x) = 0 \) the function \( \xi \) and \( \eta \) are

\[ \xi(x, y) = c_1 + c_3x + c_5y + c_7x^2 + c_8xy, \]

\[ \eta(x, y) = c_2 + c_4y + c_6x + c_7xy + c_8y^2, \]

where \( c_i, \ i \in \{1, ..., 8\} \) are constants. The most general infinitesimal generator is of the form

\[ X = \sum_{i=1}^{8} c_iX_i, \]

with the vectors

\[ X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_y, \quad X_5 = y\partial_x, \]

\[ X_6 = x\partial_y, \quad X_7 = x^2\partial_x + xy\partial_y, \quad X_8 = xy\partial_x + y^2\partial_y. \]

Now on, when we are familiar with finding Lie point symmetries we may use them for simplifications of problems which come down to solving ODEs. In the next two subsections we will present two methods of reducing an order of ODE’s (for more details and examples see [133]).
A.2.1 Reducing order of ODE’s by canonical coordinates

We will say that the generator $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ can be written in its normal form $X = \partial_s$ if there exists a system of coordinates $(r(x,y), s(x,y))$ such that

$$Xr = 0, \quad Xs = 1,$$

that is,

$$\xi(x,y)r_x + \eta(x,y)r_y = 0, \quad \xi(x,y)s_x + \eta(x,y)s_y = 1,$$

$$r_x s_y - r_y s_x \neq 0.$$

The last equation is the non-degeneracy condition: it ensures that the change of coordinates is invertible in some neighborhood of $(x,y)$. The coordinates $(r(x,y), s(x,y))$ are called canonical coordinates. Using canonical coordinates allows to reduce an order of ODE as it is presented below.

Let the vector $X$ be an infinitesimal generator of a one-parameter Lie group of symmetries of the ODE

$$y^{(n)} = \omega(x,y,y',...,y^{(n-1)}), \quad n \geq 0$$

and let $(r(x,y), s(x,y))$ are canonical coordinates so that $X = \partial_s$. One may write the ODE for some function $\Omega$ in terms of canonical coordinates:

$$s^{(n)} = \Omega(r, s, \dot{s}, ..., s^{(n-1)}), \quad \dot{s} = \frac{ds}{dr}, \quad s^{(k)} = \frac{d^k s}{dr^k}.$$  \hfill (330)

But the considered ODE is invariant under the Lie group of translations in $s$ so from the symmetry condition

$$\Omega, s = 0 \quad \text{so} \quad s^{(n)} = \Omega(r, \dot{s}, s^{(n-1)}).$$

Let us introduce $v = \dot{s}$; then the above equation is an ODE of order $n - 1$:

$$v^{(n-1)} = \Omega(r, v, ..., v^{(n-2)}), \quad v^{(k)} = \frac{d^{k+1} s}{dr^{k+1}}.$$  \hfill (331)

A.2.2 Reducing order of ODE’s by Lie invariants

If a non-constant function $I(x,y,y',...,y^{(k)})$ satisfies

$$X^{(k)} I = 0,$$  \hfill (331)
where $X^{(k)}$ is the prolongation of the infinitesimal generator $X$ of a one-parameter Lie group of symmetries of the ODE (330), then we say that $I$ is a $k$th order differential invariant of the group generated by $X$. [133]. Since in canonical coordinates $X = \partial_s$, the differential invariant is of the form

$$I = F(r, s, \ldots, s^{(k)}) = F(r, v, \ldots, v^{(k-1)})$$

(332)

for some function $F$. The zeroth order differential invariant is the canonical coordinate $r(x, y)$. Moreover, all first-order invariants are functions of $r(x, y)$ and $v(x, y, y')$, and higher order invariant are functions of $r$, $v$ and derivatives of $v$ with respect to $r$. One may show [133] that the condition (331)

$$\xi I_x + \eta I_y + \ldots + \eta^{(k)} I_{y^{(k)}} = 0$$

is equivalent to

$$\frac{dx}{d\xi} = \frac{dy}{\eta} = \ldots = \frac{dy^{(k)}}{\eta^{(k)}}. \quad (333)$$

One says that $I$ is a first integral of (333). Is is worth to note that the canonical coordinate $r$ is a first integral of

$$\frac{dx}{d\xi} = \frac{dy}{\eta} \quad \quad (334)$$

and $v$ is a first integral of

$$\frac{dx}{d\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta^{(1)}}. \quad (335)$$

From the zeroth order invariant $r$ and first order invariant $v$ one may define the following differential invariants

$$\frac{dv}{dr}, \ldots, \frac{d^{n-1}v}{dr^{n-1}}, \quad \text{where} \quad \frac{dv}{dr} = \frac{v_x + v_y y' + v_{yy} y''}{u_x + u_y y'}$$

which are functions of different derivatives of $y$ appearing in (330). It allows us to rewrite (330) in terms of invariants giving us a result which is $(n-1)$th order ODE

$$\frac{d^{n-1}v}{dr^{n-1}} = \Omega \left( r, v, \frac{dv}{dr}, \ldots, \frac{d^{n-2}v}{dr^{n-2}} \right). \quad (336)$$

A.2.3 Noether symmetries

There exists a special case of Lie point symmetries which are very important in physics. They are called Noether symmetries whose first differential invariants (first integrals) have physical meaning; for example, when the symmetry is
time translation (or rotation), one deals with conservation of energy (or angular momentum).

Let

\[ S = \int_{t_1}^{t_2} L(q^k, \dot{q}^k, t) \, dt \]  

be an action of a physical system whose dynamics is described by the function \( L(q^k, \dot{q}^k, t) \) called Lagrangian. The dot denotes the derivative with respect to the time variable \( t \). The equations of motion (Euler - Lagrange equations) derived from variation principle are [187]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \]  

Now on, let us defined a Noether symmetry [132]:

**Definition 2** A Noether symmetry is a Lie point transformation that leaves the action \( S \) invariant up to an additive constant \( \hat{V}(\epsilon) \) with \( \epsilon \) being the group parameter.

Let \( \tilde{S} = S + \hat{V}(\epsilon) \) be an action obtained by mapping the action \( S \) by a point transformation and \( \hat{V}(\epsilon) = \int_{t_1}^{t_2} \frac{dV(q^k, t, \epsilon)}{dt} \, dt \). One sees that \( S \) and \( \tilde{S} \) leads to the same equations of motion (338) hence Noether symmetries leave the differential equations invariant. Expanding \( \tilde{S} \) in Taylor series with respect to \( \epsilon \) with the infinitesimal prolonged generator given by the vector field \( X^{(1)} = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} + \dot{\eta}^a \frac{\partial}{\partial \dot{q}^a} \) we have:

\[ \tilde{S} = \int \tilde{L} \, dt = \int L(\tilde{q}^k, \dot{\tilde{q}}^k, \tilde{t}) \, dt \]

\[ = \int [L(q^k, \dot{q}^k, t) + \epsilon XL + O(\epsilon^2)] \left( dt + \epsilon \frac{d\xi}{dt} \, dt + O(\epsilon^2) \right) \]

\[ = S + \epsilon \int \frac{dV(q^k, t)}{dt} \, dt + ... \]

Collecting terms linear in \( \epsilon \) one gets the condition for \( \tilde{S} = S + \hat{V}(\epsilon) \) being true; let us write it as a theorem:

**Theorem 4** The infinitesimal generator \( X \) is a Noether symmetry if there exists a function \( V = V(q^k, t) \) such that the following condition is satisfied:

\[ X^{(1)}L + \frac{d\xi}{dt}L = \frac{dV}{dt}, \]  

where \( X^{(1)} \) is the first prolongation of the generator \( X \).
Noether symmetries are called variational symmetries if $V = 0$. For every Noether symmetry there exist a first integral, it means

**Definition 3** If $X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q}$ is the generator of a Noether symmetry then

$$I_N = \xi E_H - \eta^k L_{,q^k} + V(q^k, t)$$  \hfill (343)

is a first integral that satisfies $X^{(1)} I_N = 0$. The quantity $E_H = \dot{q}^k L_{,q^k} - L$ is called Hamiltonian of the dynamical system.

One should notice that the condition (342) will not give us all possible Lie symmetries; it may happen that a system does not admit any Noether symmetries and due to that fact no conserved quantities may be used to reduce the order of the differential equations. But there can still exist Lie symmetries from which one may construct Lie invariants.

### A.2.4 Linear ODE’s

Lie point symmetries are a very useful tool for solving differential equations: one finds symmetries from the condition (327) and applying Lie invariants method we are able to reduce order of an ODE which can help to solve a differential equation. Although, they are not helpful in the case of linear ODE’s of order $n \geq 2$. It happens that one or more determining equations has the same for as the ODE that we wanted to solve. One usually needs to know the general solution of the ODE in order to find Lie point symmetries [133]:

**Theorem 5** Every homogeneous linear ODE of order $n \geq 3$ has infinitesimal generators of the form

$$X_1 = y \partial_y, \ X_2 = y_1 \partial_y, ..., \ X_{n+1} = y_n \partial_y,$$  \hfill (344)

where $\{y_1, ..., y_n\}$ is a set of functionally independent solutions of the ODE. If the ODE can be mapped into the ODE $y^{(n)} = 0$ by a point transformation, then it admits three extra infinitesimal generators:

$$X_{n+2} = \partial_x, \ X_{n+3} = x \partial_x, \ X_{n+4} = x^2 \partial_x + (n-1)xy \partial_y.$$  \hfill (345)

Let us consider the fourth order ODE:

$$\lambda^{(4)} + \frac{5}{\tau} \lambda^{\prime\prime\prime} + \left( \frac{2}{\tau^2} + \nu \right) \lambda^{\prime\prime} + \left( \frac{\nu}{\tau} - \frac{2}{\tau^3} \right) \lambda^{\prime} = 0.$$  \hfill (346)
It comes from the system of equations which gives arise when one deals with the linear approximation of Einstein’s field equations for perturbations of the spatial metric tensor in the early stages of expansion of the Universe \[188\]

\[ h_{ij} = \lambda(\tau)P_{ij} + \mu(\tau)Q_{ij}. \]

In the above \( \tau \) denotes conformal time, \( v \) is a constant, and the tensors \( P_{ij} \) and \( Q_{ij} \) are defined by a scalar function used to express the perturbation of the density of the matter filling an isotropic universe \[188\].

We will show that in the case of (346) the Lie symmetry method does not work as it has been already mentioned. As a first step let us simplify the equation by

\[ y(\tau) = \lambda' \]

\[ y''' + \frac{5}{\tau}y'' + \left( \frac{2}{\tau^2} + v \right) y' + \left( \frac{v}{\tau} - \frac{2}{\tau^3} \right) y = 0. \quad (347) \]

We will look for a vector field \( X = \xi(\tau, y)\partial_\tau + \eta(\tau, y)\partial_y \) which is a symmetry vector of (347). Before using the symmetry condition (327) for \( \omega = \frac{5}{\tau}y'' - \left( \frac{2}{\tau^2} - v \right) y' - \left( \frac{v}{\tau} + \frac{2}{\tau^3} \right) y \)

one needs to find \( \eta^{(3)} \) from (318):

\[ \eta^{(3)} = - \left\{ (y'\xi,_{yy} - \eta,_{yy} + 3\xi,_{xy}y') y' - 3\eta,_{xy}y + 3\xi,_{xxy} \right\} y' \]
\[ -3\eta,_{xy} + \xi,_{xxx} \right\} y' + y''' \left( -4y'\xi,_{y} + \eta,_{y} - 3\xi,_{x} - 3y''\xi,_{y} \right) + 3y'' \left( y' \left( -2y'\xi,_{yy} + \eta,_{yy} - 3\xi,_{xy} \right) + \eta,_{xy} - \xi,_{xx} - \eta,_{xxx} \right) \]

The symmetry conditions (327) for (347) splits into a system of partial differential equations for the functions \( \alpha(\tau), \beta(\tau), \gamma(\tau) \):

\[ \beta''' + \frac{5}{\tau}\beta'' + \left( \frac{2}{\tau^2} + v \right) \beta' + \left( \frac{v}{\tau} - \frac{2}{\tau^3} \right) \beta = 0, \quad (348) \]
\[ \gamma''' + \frac{5}{\tau}\gamma'' + \left( \frac{2}{\tau^2} + v \right) \gamma' + \left( \frac{v}{\tau} - \frac{2}{\tau^3} \right) \gamma = 0, \quad (349) \]
\[ -5\alpha + \frac{5}{\tau}\alpha' + 3\tau^2(\beta' - \alpha'') = 0, \quad (350) \]
\[ -4\frac{\alpha}{\tau^3} + \frac{2}{\tau^2}(2 + \nu\tau^2)\alpha' + \frac{1}{\tau}(10\beta' - 5\alpha'' - \tau\alpha'''') = 0, \quad (351) \]
\[ (6 - \nu\tau^2)\alpha - \tau(\nu\tau^2 - 2)\beta + 3\tau(\nu\tau^2 - 2)\alpha' = 0 \quad (352) \]

for \( \xi(\tau, y) = \alpha(\tau) \) and \( \eta(\tau, y) = \beta(\tau)y + \gamma(\tau) \).
We notice that the equations (348) and (349) have the form of the ODE (347) that we wanted to solve. The same happens when we use the method in order to solve (346).

One may try to reduce the order of (346) with the help of the theorem 5 and canonical coordinates. We know that the equation admits the symmetry vector \( X = \lambda \partial_\lambda \) from which we get that the canonical coordinates are

\[
  s = \ln \lambda, \quad r = \eta. \tag{353}
\]

If one denotes \( u = \frac{ds}{dr} \), then \( \lambda' = \lambda u \) and the ODE (346) is now the third order ODE of the form

\[
u'''' + \left( 4u + \frac{5}{\eta} \right) u'' + 3u'^2 \left( 6u^2 + v + \frac{15}{\eta} + \frac{2}{\eta^2} \right) + \left( u^3 + \frac{5}{\eta} u^2 + (v + \frac{2}{\eta^2}) u + \frac{v}{\eta} - \frac{2}{\eta^3} \right) u = 0. \tag{354}
\]

A.3 LIE ALGEBRA

It may happen the an ODE has many symmetries and some of them belong to \( m \)-parameter Lie group

\[
\begin{align*}
\tilde{x} &= \tilde{x}(x, y, \delta), \\
\tilde{y} &= \tilde{y}(x, y, \delta), \\
\delta &= e^\beta \partial_\beta, \quad \beta = 1, ..., m
\end{align*} \tag{355}
\]

with an infinitesimal generator defined as

\[
X_\beta = \xi_\beta(x, y) \partial_x + \eta_\beta(x, y) \partial_y.
\]

Symmetries belonging to an \( m \)-parameter Lie group can be regarded as a composition of symmetries from \( m \) one-parameter Lie groups.

Let \( \mathcal{L} \) denotes a set of all infinitesimal generators of one (or more)-parameter Lie group of point symmetries of an ODE of order \( n \geq 2 \). Since the linearized symmetry condition is linear in \( \xi \) and \( \eta \) one has that \( \mathcal{L} \) is a vector space

\[
X_1, X_2 \in \mathcal{L} \quad \rightarrow \quad c_1 X_1 + c_2 X_2 \in \mathcal{L}, \quad \forall c_1, c_2 \in \mathbb{R}
\]

The dimension \( m \) of the vector space is a number of arbitrary constants appearing in the general solution of the linearized symmetry condition. Every \( X \in \mathcal{L} \) may be written in the form

\[
\sum_{i=1}^{m} c_i X_i, \quad c_i \in \mathbb{R},
\]
where \( \{X_1, \ldots, X_m\} \) is a basis for \( \mathcal{L} \). Similarly, the set of point symmetries generated by all \( X \in \mathcal{L} \) forms an \( m \)-parameter Lie group.

Let \( X_1, \ldots, X_4 \in \mathcal{L} \). A first-order operator 

\[
[X_1, X_2] = X_1 X_2 - X_2 X_1
\]

is called a commutator of \( X_1 \) with \( X_2 \). It is antisymmetric, bilinear and satisfies the Jacobi identity:

\[
[X_1, X_2] = -[X_2, X_1],
\]

\[
[c_1 X_1 + c_2 X_2, X_3] = c_1 [X_1, X_3] + c_2 [X_2, X_3],
\]

\[
[X_1, c_2 X_2 + c_3 X_3] = c_2 [X_1, X_2] + c_3 [X_1, X_3],
\]

\[
0 = [X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]].
\]

Moreover, \( \mathcal{L} \) is closed under the commutator \( X_i, X_j \in \mathcal{L} \rightarrow [X_i, X_j] \in \mathcal{L} \) and the commutator of any two generators \( X_1, X_2 \in \mathcal{L} \) in the basis is a linear combinations of the basis generators

\[
[X_i, X_j] = c_{ij}^k X_k.
\]

The constants \( c_{ij}^k \) are called structure constants. From antisymmetry and the Jacobi identity one gets that

\[
c_{ij}^q = -c_{ji}^q,
\]

\[
c_{ij}^q c_{kq}^l + c_{ij}^q c_{kq}^l + c_{ij}^q c_{kq}^l = 0, \quad \forall i, j, k, l.
\]

If \( [X_i, X_j] = 0 \), \( (c_{ij}^k = 0) \), we say that the generators \( X_i \) and \( X_j \) commute.

A finite dimensional linear space \( \mathcal{L} \) with a commutator as a product on \( \mathcal{L} \) satisfying above conditions forms a Lie algebra.

One may show \([133]\) that an \( n \)th order ODE can be reduced with \( m \leq n \) Lie point symmetries, which are generated by \( \mathcal{L} \), to an ODE of order \( n - m \) (or to an algebraic equation if \( n = m \)). The considered differential equation is written in terms of the differential invariants of each generator, it means the ODE can be written in terms of functions that are invariant under all of its symmetry generators.

One should mention \([133]\) that if \( X_1 \) and \( X_2 \) generate Lie point symmetries, then so does \( [X_1, X_2] \). It can be used in order to find more Lie symmetries of the differential equation.
A.4 LIE POINT SYMMETRIES OF PDE’S

Finding Lie point symmetries of partial differential equations (PDE’s) is a very similar procedure as the case of ODE’s. Due to that fact, we will shortly give basic notions with necessary formulas for the simplest case, it means we are going to consider PDE’s with one dependent variable \( u \) and two independent ones, \( t \) and \( x \).

A point transformation \([133, 132]\) is a diffeomorphism

\[
T: (x, t, u(x, t)) \mapsto (\tilde{x}(x, t, u(x, t)), \tilde{t}(x, t, u(x, t)), \tilde{u}(x, t, u(x, t)))
\]

which maps the surface \( u = u(x, t) \) into

\[
\begin{align*}
\tilde{x} &= \tilde{x}(x, t, u(x, t)), \\
\tilde{t} &= \tilde{t}(x, t, u(x, t)), \\
\tilde{u} &= \tilde{u}(x, t, u(x, t)).
\end{align*}
\]

If \( H(x, t, u, u_x, u_t, \ldots, u_\sigma) = u_\sigma - \omega(x, t, u, u_x, u_t, \ldots) = 0 \) is an \( n \)th order PDE, where \( u_\sigma \) is one of the \( n \)th order derivatives of \( u \) and \( \omega \) is independent of \( u_\sigma \), then the point transformation \( T \) is its point symmetry if

\[ H(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}_x, \tilde{u}_t, \ldots) = 0 \text{ when } H(x, t, u, u_x, u_t, \ldots) = 0 \] holds. (361)

As for ODE’s, we are looking for one-parameter Lie groups of point symmetries \([133]\), it means, one searches for point symmetries that have the form

\[
\begin{align*}
\tilde{x} &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\
\tilde{t} &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\
\tilde{u} &= u + \epsilon \eta(x, t, u) + O(\epsilon^2),
\end{align*}
\]

with the infinitesimal generator

\[ X = \xi \partial_x + \tau \partial_t + \eta \partial_u. \] (365)

The first two prolongations of the above generator are

\[
\begin{align*}
X^{(1)} &= X + \eta^{(x)} \partial_{u_x} + \eta^{(t)} \partial_{u_t}, \\
X^{(2)} &= X^{(1)} + \eta^{(xx)} \partial_{u_{xx}} + \eta^{(xt)} \partial_{u_{xt}} + \eta^{(tt)} \partial_{u_{tt}}.
\end{align*}
\] (366, 367)
with long expressions for $\eta^{(1)}$

\[
\eta^{(x)} = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \quad (368)
\]

\[
\eta^{(t)} = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2, \quad (369)
\]

\[
\eta^{(xx)} = \eta_{xx} + (2\eta_{xx} - \xi_{xx})u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xx})u_x^2
- 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\eta_{uu} - 2\xi_{xx})u_{xx}
- 2\tau_{xu} u_x u_t - 3\xi_{u} u_x u_{xx} - \tau_{u} u_t u_{xx} - 2\tau_{u} u_x u_{xt}, \quad (370)
\]

\[
\eta^{(xt)} = \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + (\eta_{xu} - \tau_{xt})u_t - \eta_{tu} u_x^2
+ (\eta_{uu} - \xi_{xu} - \tau_{tu})u_x u_t - \tau_{xu} u_t^2 - \xi_{uu} u_x^2 u_t - \tau_{uu} u_x u_t^2
- \xi_{t} u_{xx} - \xi_{u} u_t u_{xx} + (\eta_{u} - \xi_{x} - \tau_{t})u_{xt} - 2\xi_{u} u_x u_{xt}
- 2\tau_{u} u_t u_{xt} - \tau_{x} u_{tt} - \tau_{u} u_x u_{tt}, \quad (371)
\]

\[
\eta^{(tt)} = \eta_{tt} - \xi_{tt} u_x + (2\eta_{tt} - \tau_{tt})u_t - 2\xi_{tt} u_x u_t
+ (\eta_{uu} - 2\tau_{tt} u_t^2 - \xi_{uu} u_x u_t^2 - \tau_{uu} u_t^3 - 2\xi_{tt} u_{xt}
- 2\xi_{u} u_t u_{xt} + (\eta_{u} - 2\tau_{t})u_{tt} - \xi_{u} u_x u_{tt} - 3\tau_{u} u_t u_{tt}. \quad (372)
\]

For higher terms and more general case there exist recurrence formulas [132] but one should use some computer algebra.

The linearized symmetry conditions is obtained when we differentiate the symmetry condition (361) with respect to $\epsilon$ at $\epsilon = 0$ so one has

\[
X^{(n)} H(x, t, u, u_x, u_t, ..., u_\sigma) \equiv 0, \mod H(x, t, u, u_x, u_t, ..., u_\sigma) = 0. \quad (373)
\]

The above condition gives, after eliminating $u_\sigma$, a linear system of determining equations for $\xi, \tau,$ and $\eta$.

Assuming that we have already found a Lie point symmetry of a PDE $H(x, t, u, u_x, ...) = 0$ one may use Lie invariant method as it was done for ODE’s. The difference is that one reduces a number of variables instead of reducing order of differential equation.

### A.5 handful of useful theorems

We would like to give a few extra notions and theorems which are very helpful in the case when one applies Lie symmetries method to considered problems. It has been shown [125] that when one deals with differential equations derived from a Lagrangian, one may relate Lie symmetries with conformal algebra of a metric of a Riemannian space which is a phase space of the physical system.

Let us introduce a conformal Killing vector (CKV) of a metric $G_{ij}$:

Definition 4 A vector field $u^i$ is a conformal Killing vector if it satisfies

$$\mathcal{L}_u G_{ij} = 2\psi G_{ij},$$  \hspace{1cm} (374)

where $\mathcal{L}_u$ is a Lie derivative with respect to the vector field $u^i$ and $\psi$ is called a conformal factor.

We will say that $u^i$ is

• a Killing vector if $\psi = 0$,
• a homothetic vector if $\psi_{,i} = 0$,
• a special conformal Killing vector if $\psi_{,ij} = 0$,
• a proper conformal Killing vector if $\psi_{,ij} \neq 0$.

Two metrics are called conformally related, it means one has the relation

$$\bar{G}_{ij} = N^2 G_{ij},$$

$N^2$ being a conformal factor. Moreover, if $u^i$ is a CKV of the metric $\bar{G}_{ij}$ then it is also a CKV of the conformally related metric $G_{ij}$ with the conformal factor $\psi$ defined as

$$\psi = \bar{\psi} N^2 - N N_{,i} u^i.$$

Let us again consider the action (337) with the Lagrangian of the form

$$L(q^i, \dot{q}^i) = \frac{1}{2} G_{ij} \dot{q}^i \dot{q}^j - V(q^k)$$  \hspace{1cm} (375)

which is a Lagrangian of a particle which moves under the action of the potential $V(q^k)$ in a Riemannian space with the metric $G_{ij}$. The dot represents a derivative with respect to time $t$ which is a parameter of a curve along which the particle moves. We will perform two transformations: the first one is a coordinate transformation of the form

$$d\tau = N^2(q^i) dt$$  \hspace{1cm} (376)

with the $\tau$-derivative denoted from now on as a prime $'$. The next one is a conformal transformation of the metric $G_{ij} = N^{-2} \bar{G}_{ij}$. Defining a new potential $\bar{V}(q^k) = N^{-2}(q^k)V(x^k)$ one gets that the new Lagrangian

$$\bar{L}(q^i, \dot{q}^i) = \frac{1}{2} \bar{G}_{ij} \dot{q}^{i'} \dot{q}^{i'} - \bar{V}(q^k)$$  \hspace{1cm} (377)

has the same form as the original one (375). We will say that such Lagrangians are conformally related. There exists a theorem [127] which is very useful in the case of cosmology:
Theorem 6 The Euler-Lagrange equations for two conformal Lagrangians transform covariantly under the conformal transformation relating the Lagrangians if and only if the Hamiltonian vanishes.

The equations of motion coming from the Lagrangians \( L \) in the variables \((t, q^i)\) and \( \tilde{L} \) in the variables \((\tau, q^i)\) are of the same form. In the other words, it results from the above theorem that physical systems with vanishing energy have conformally invariant equations of motion. In the case of FRLW cosmology, the Hamiltonian of the considered system is the \((0,0)\) Einstein equation being also a constraint. The more detailed discussion on the FRLW cosmology and also scalar field cosmology is given in [127].

There are two another important results concerning conformally related metrics and Lagrangians. If one considers Noether symmetries of the Lagrangian (375), they can be obtained [125] from the homothetic algebra of the metric \( G_{ij} \). The same result can be applied to the Lagrangian \( \tilde{L} \) and the metric \( \tilde{G}_{ij} \). One should also mention that the conformal algebras of the metrics \( G_{ij} \) and \( \tilde{G}_{ij} \) are spanned by the same conformal Killing vectors [189] with subalgebras of homothetic and Killing vectors different for each metric.

Let us now consider a Klein-Gordon equation

\[
\nabla w = V(q^i)w, \tag{378}
\]

where \( \nabla \) is the Laplace operator of the Riemannian space with metric \( G_{ij} \) having the following form

\[
\nabla w = \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial q^i} \left( \sqrt{|G|} G^{ij} \frac{\partial}{\partial q^j} \right) w, \tag{379}
\]

\( G \) being the determinant and \( G^{ij} \) the inverse of the metric \( G_{ij} \). It was shown [190, 128] that there exists a relation between Lie point symmetries of Klein-Gordon equation and conformal algebra:

Theorem 7 The Lie point symmetries of Klein-Gordon equation (378) are generated from the conformal Killing vectors of the metric \( G_{ij} \) which defines the Laplace operator in the following way:

- for the dimension of the Riemannian space \( n > 2 \) the generic Lie symmetry vector is

\[
X = u^i(q^k)\partial_i + \frac{2 - n}{n} \psi(q^k)w + a_0 w + b(q^k)\partial_w \tag{380}
\]
where \( u^i \) is a CKV with conformal factor \( \psi(q^k) \), the function \( b(q^k) \) is a solution of the K-G equation (378). Moreover, the following condition is satisfied for the potential \( V(q^k) \):

\[
u^k V_{,k} + 2\psi V - \frac{2-n}{n} \nabla \psi = 0; \quad (381)\]

• for \( n = 2 \) the generic Lie symmetry vector is

\[
X = u^i(q^k) \partial_i + a_\omega w + b(q^k) \partial_w
\]

where \( u^i \) is a CKV with conformal factor \( \psi(q^k) \), the function \( b(q^k) \) is a solution of the K-G equation (378). Moreover, the following condition is satisfied for the potential \( V(q^k) \):

\[
u^k V_{,k} + 2\psi V = 0; \quad (383)\]

The above theorem allows us to construct Lie point symmetries of Klein-Gordon equations from known conformal algebras and to find an unknown potential function appearing in (378) without solving determining equations. It is extremely useful in Extended Theories of Gravity when one deals with not determined actions of a theory: if one is able to transform the action into scalar-tensor action, it is possible to find the Lagrangian due to the fact that we may express unknown parts by the potential (for example \( f(R) \) gravity in metric or Palatini formalism, Hybrid Gravity etc.).

Let us consider a PDE of the form \( H(x, t, u, u_x, u_t, ..., u_\sigma) = 0 \). Applying a Lie symmetry to the considered PDE gives us a new differential equation \( \tilde{H} \) which is different than \( H \). It may happen that it also admits Lie symmetries that are not symmetries of the original equation. It has been proved [191, 192] that

**Theorem 8** If two Lie point symmetries \( X_1, X_2 \) of a PDE

\[
H(x, t, u, u_x, u_t, ..., u_\sigma) = 0
\]

commute as

\[
[X_1, X_2] = cX_1, \quad c \text{ is a constant}, \quad (384)
\]

then the reduction variables defined by \( X_1 \) will reduce \( X_2 \) to a point symmetry of the PDE (called \( H_1 \), say) obtained from \( H \) via these variables. However, the variables defined by \( X_2 \) reduce \( X_1 \) to an expression that has no relevance for the PDE (called \( H_2 \), say) obtained from \( H \) via these reduction variables.
\textbf{ΛCDM Model as a 2D Dynamical System of the Newtonian Type}

Dynamical systems theory is another approach that we use for an examination of cosmological models. In contrast to the Lie symmetries method described in the Chapter A, we are interested in the evolutionary behavior of the Universe, that is, we would like to get to know special points of the evolution such as cosmological singularities or steady states instead of looking for exact solutions of given systems [193]. The other reason for applying dynamical systems approach are difficulties in finding solutions of differential equations that describe cosmological models. It often happens in modified cosmological models that exact solutions are impossible to obtain because complicated equations appear. Such analysis provides an interesting approach to theoretical cosmology: one may examine the Universe’s evolutionary paths in a phase plane. Many authors showed that one may treat the whole Universe as a fictitious point particle moving in a one-dimensional potential well (see for instance [72, 194, 195, 10, 196, 197, 198]). Due to that fact, the considered evolution of the system reduces to a simple 2-dimensional dynamical system of the Newtonian type. The evolution for all admissible initial conditions is represented by trajectories in a phase space [25], that is, the considered potential function gives us full information about the dynamics [85].

We would like to briefly give basic notions on phase portraits and critical points. Moreover, we depict the method for the standard model of cosmology, that is, ΛCDM model, as a simple example.

\textbf{B.1 Phase Portraits of Linear Systems in } \mathbb{R}^2

Let us briefly discuss simple examples of phase portraits of linear systems in 2-dimensional vector space \mathbb{R}^2. We will consider the linear system of the form

\[
\dot{x} = A x
\]  

(385)

where \(x \in \mathbb{R}^2\) is a vector field while \(A\) is a \(2 \times 2\) matrix. One may reduce the system (385) to an uncoupled linear system by the diagonalization procedure

\[
\dot{x} = B x, \quad B = P^{-1} A P
\]  

(386)
where $P^{-1}AP = \text{diag}[\lambda_1, \lambda_2]$, $P$ is a matrix consisting of generalized eigenvectors of $A$ and $\lambda_i$, $i = \{1, 2\}$, stand for eigenvalue of the matrix $A$. One may generalize the problem to the $n$-dimensional system (see for example [199]). Let us assume that the form of the matrix $B$ is one of the following possibilities

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \quad (387)$$

Then, one may use the fundamental theorem of linear systems:

**Theorem 9** Let $B$ be an $n \times n$ matrix. Then for a given $x_0 \in \mathbb{R}^n$, the initial value problem

$$\dot{x} = Bx, \quad (388)$$
$$x(0) = x_0 \quad (389)$$

has a unique solution given by

$$\dot{x} = e^{Bt}x_0, \quad (390)$$

where

$$e^{Bt} = \sum_{k=0}^{\infty} \frac{B^k t^k}{k!}$$

is a $n \times n$ matrix which can be computed in terms of the eigenvalues and eigenvectors of $B$. It can be showed [199, 200] that the solution of the (386) with the initial condition (389) is

$$x(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} x_0, \quad x(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0, \quad x(t) = e^{\alpha t} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} x_0,$$

respectively.

Due to that solutions one may draw different types of phase portraits with respect to the eigenvalues of the matrices. We will draw phase portraits of the linear system (386); phase portraits of the linear system (385) is obtained from the drawn ones for (386) under the linear transformation of coordinates $x = Py$.

The first case

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}. \quad (391)$$
includes saddle point at the origin (see the figure 10) for $\lambda < 0 < \mu$. The arrows are reversed when $\mu < 0 < \lambda$. Considering the matrix $A$, if it has two real eigenvalues of opposite sign, $\lambda < 0 < \mu$, then the phase portrait of the system $(385)$ is linearly equivalent to the phase portrait 10: it is obtained by a linear transformation of coordinates. Separatrices of the system are the four non-zero trajectories (solution curves) approaching the equilibrium point at the origin as $t \to \pm \infty$.

If one deals with matrices of the forms

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{or} \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

(392)

with $\lambda \leq \mu < 0$ for the first one and for $\lambda < 0$ in the case of the second matrix, the phase portraits are shown in the figures 11, 12, 13. The origin is a stable node in each of these cases; the case $\lambda = \mu$ is called a proper node (Fig. 11) while in the two other cases (Fig. 12 and Fig. 13) one deals with improper nodes. Moreover, one has the arrows in the pictures 11, 12, 13 reversed if $\lambda \geq \mu > 0$ or if $\lambda > 0$ and the origins are unstable nodes. Let us notice that the stability of a node is given by the sign of the eigenvalues, that is, the node is stable if $\lambda \leq \mu < 0$ and unstable if $\lambda \geq \mu > 0$. 

Figure 10: Phase portrait of the system $\dot{x} = -x$, $\dot{y} = y$. The equilibrium point (critical point of the system, it means $\dot{x} = \dot{y} = 0$) at the origin is called a saddle point. As $\lambda < 0$, solutions along the line $y = 0$ decay to 0 (stable line) while $\mu > 0$ corresponds to the growing solutions along $x = 0$ (unstable line).
Figure 11: Phase portrait of the system $\dot{x} = -x$, $\dot{y} = -y$, that is, $\lambda = \mu < 0$. The equilibrium point is a **stable node** at the origin and in that case it is called a **proper node**. If $\lambda = \mu > 0$, the arrows are reversed and the origin is an **unstable node**.

Figure 12: Phase portrait of the system $\dot{x} = -3x$, $\dot{y} = -y$, that is, $\lambda < \mu < 0$. The equilibrium point is a **stable node** at the origin and in that case it is called a **improper node**. If $\lambda > \mu > 0$, the arrows are reversed and the origin is an **unstable node**.
Another case includes a pair of complex conjugate eigenvalues of the matrix $A$ with nonzero real part. The diagonal matrix $B$ of the matrix $A$ is then

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{with} \quad a < 0. \quad (393)$$

The phase portrait of the system (385) for $b > 0$ (counterclockwise direction) is drawn in the figure 14. The clockwise direction happens when $b < 0$. The origin is a stable focus; it will be unstable, that is, the trajectories will spiral away from the origin (with increasing $t$) if $a > 0$.

The last case refers to

$$B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}. \quad (394)$$

We say that the system (386) has a center in the origin (see the figure 17 for $b > 0$). It happens when the matrix $A$ has a pair of pure imaginary complex conjugate eigenvalues $\lambda_{\pm} = \pm ib$. The trajectories lie on circles $|x(t)| = \text{constant}$. If one or both of the eigenvalues of $A$ is zero ($\det A = 0$), the origin is called a degenerate equilibrium point of (385).

Concluding, the linear system (385) has one of the following possibilities: a saddle, a node, a focus or a center at the origin if the matrix $A$ is similar to
Figure 14: Phase portrait of the system \( \dot{x} = -\frac{1}{2}x - 3y \), \( \dot{y} = 3x - \frac{1}{2}y \), that is, \( \lambda_{\pm} = a \pm ib \) with \( a < 0 \) and \( b > 0 \). The equilibrium point is a stable focus at the origin. If \( a > 0 \), the arrows are reversed and the origin is an unstable focus.

Figure 15: Phase portrait of the system \( \dot{x} = -y \), \( \dot{y} = x \), that is, \( \lambda_{\pm} = \pm ib \) with \( b > 0 \) (counterclockwise direction). The system is said to have a center at the origin. If \( b < 0 \), the arrows are reversed (clockwise direction).
one of the matrices $B$ considered above, that is, its phase portrait is linearly equivalent to one of the phase portraits of the linear system (386). One may easily determine a kind of an equilibrium point if $\det A \neq 0$; for that purpose let us recall a theorem [199].

**Theorem 10** Let $\delta = \det A$ is a determinant of the matrix $A$ while $\tau = \text{Tr} A$ is its trace. Considering the linear system (385) one says that

- The system (385) has a saddle point at the origin if $\delta < 0$.
- If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then (385) has a node at the origin: it is stable if $\tau < 0$ and unstable if $\tau > 0$.
- If $\delta > 0$, $\tau^2 - 4\delta < 0$, and $\tau \neq 0$ then (385) has a focus at the origin which is stable if $\tau < 0$ and unstable if $\tau > 0$.
- If $\delta > 0$ and $\tau = 0$ then (385) has a center at the origin.

Moreover, we will say that a stable node or focus of (385) is a **sink** of the linear system while an unstable node or focus of (385) is called a **source** of the linear system.

In the further considerations we will be interested in Newtonian equations of motion

$$\ddot{x} = -\frac{\partial V}{\partial x},$$

(395)

with the first integral

$$\frac{\dot{x}^2}{2} + V(x) = E.$$  

(396)

Such a system describes a unit-mass particle moving in a 1-dimensional potential energy $V(x)$ with energy level $E$ on a half line $x : x \leq 0$. It can be reduced to the 2-dimensional Newtonian type dynamical system

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{\partial V}{\partial x},
\end{align*}$$  

(397)

$$E = \frac{y^2}{2} + V(x).$$

(398)

The system’s matrix $A$ after the linearization procedure becomes

$$A = \begin{bmatrix} 0 & 1 \\ \frac{\partial^2 V}{\partial x^2} & 0 \end{bmatrix}.$$  

(399)
while the characteristic equation simply gives the eigenvalues

\[ \lambda_{\pm} = \pm \frac{\partial^2 V}{\partial x^2}. \] (400)

In the following subsection we will show how the dynamical system approach may be used to examine cosmological models. We will focus on the ΛCDM model which was briefly described in the Introduction 1.

### B.2 LCDM Model as a Dynamical System

Let us recall the main ingredients of the ΛCDM model describing our Universe pretty well from radiation dominated epoch till nowadays accelerating expansion. On the large scale it is isotropic and homogeneous - that feature is described by the spatially flat \( (k = 0) \) FRLW metric (26) - and filled with pressureless substance, that is, dust. Additionally, one introduces another one possessing negative pressure, so-called cosmological constant \( \Lambda \), in order to explain the late time acceleration. The dynamics of the model consists of the following equations \((H = \dot{a}/a)\)

\[
\begin{align*}
\ddot{a} &= -\frac{1}{3} (\rho + 3p)a = -\frac{\partial V}{\partial a}, \\
\dot{\rho} &= -3H(\rho + p), \\
H^2 &= \frac{1}{3} \rho - \frac{k}{a^2},
\end{align*}
\] (401)

where the first equation (Raychaudhuri equation) comes from the Einstein’s field equations \((j, j)\), \(j = 1, 2, 3\) for the perfect fluid energy momentum tensor (15), the second one is a result of Bianchi identity (conservation of the energy-momentum tensor) while the last one is the first integral (Friedmann equation) of the two first equations. Together with the equation of state \(p_i = \omega_i \rho_i\), \(i = m, \Lambda\) one finds that the potential is of the form \(V = -\frac{1}{6}a^2 \rho\). The energy density \(\rho\) consists of two fluids: dust \((\omega_m = 0\) concluding baryonic matter and cold dark matter) and cosmological constant \((\omega_\Lambda = -1)\)

\[
\rho = \rho_m + \rho_\Lambda = \rho_{m,0}a^{-3} + \Lambda
\] (404)

and therefore the potential is

\[
V = -\frac{1}{6}(\rho_{m,0}a^{-1} + \Lambda a^2).
\] (405)
B.2 lCDM model as a dynamical system

Figure 16: The potential \( V(a) = -\frac{1}{2}(\Omega_{m,0} a^{-1} + \Omega_{\Lambda,0} a^2) \) of the \( \Lambda \)CDM model. In this example we have used the approximated values \( \Omega_{m,0} = 0.3, \ \Omega_{\Lambda,0} = 0.7 \). The shaded domain \( E - V < 0 \), \( E \) being total energy of the system, is forbidden for a classical particle.

Figure 17: The phase portrait of the \( \Lambda \)CDM model obtained from (406) and (407). The thick line represents the flat model, it means the system with the total energy \( E \sim \Omega_{k,0} = 0 \). The vertical line divides the trajectories into decelerating (left) and accelerating (right) parts.
Let us remind that quantities labeled by the index \('0'\) correspond to the present epoch values. We have neglected the radiation density parameter as it is very small today \((\Omega_{\text{red}} \sim 10^{-5})\). Having a look at the equations (401) suggests that [201] one may interpret cosmological evolutionary paths of \(\Lambda\)CDM model as a motion of a fictitious particle of unit mass. The motion takes place in configuration space \(\{a : a \geq 0\}\) in a one-dimensional potential parametrized by the scale factor \(a\). The Universe accelerates when the potential is a decreasing function of the scale factor (Fig. 16) while it decelerates when the potential grows. The extremum of the potential function corresponds to the zero acceleration case (static universe).

In order to obtain a 2-dimensional dynamical system describing the considered cosmological model (401) we need to replace all dimensional quantities by dimensionless ones. We introduce density parameters as the values \(\Omega_{i,0} = \frac{\rho_i}{3H_0^2}\). The quantity \(H_0\) is a present-day value of the Hubble’s function equaled to \(67.27 \frac{\text{km}}{\text{s Mpc}}\) [71]. Furthermore, we define a dimensionless scale factor \(x = \frac{a}{a_0}\) measuring the value of \(a\) in the units of the present value \(a_0\) and the reparametrized cosmological time \(t \rightarrow \tau: dtH_0 = d\tau\). Then we may write:

\[
\frac{dx}{d\tau} = y, \tag{406}
\]

\[
\frac{dy}{d\tau} = -\frac{\partial V}{\partial x}, \tag{407}
\]

\[
\frac{y^2}{2} + V(x) = E
\]

where now

\[
V(x) = -\frac{1}{2}(\Omega_{m,0}x^{-3} + \Omega_{\Lambda,0})x^2, \quad E = \frac{1}{2}\Omega_{k,0}.
\]

One should mention that the density parameters \(\Omega_{i,0}\) are not independent. They satisfy the constraint coming from \(\frac{H^2}{H_0^2} = 1\) for \(a = 1\) (as we assume that value for a present day). Hence, one has only one parameter to estimate, for example \(\Omega_{\Lambda,0} = 1 - \Omega_{m,0}\).

The phase portrait of the above system is drawn in the picture 17. The critical point of the system is a saddle point \([a_{\text{static}} \sim 0.6; 0]\) which represents the Einstein static universe - the extremum of the potential. A vertical line passing through the saddle point divides each trajectory into two parts: decelerating phase of the Universe (on the left from the critical point \(a_{\text{static}}\)) and accelerating one (on the right from the saddle point). It also shows that the model includes a singularity at the origin \(a_{\text{sing}}(t = 0) = 0\).
The above example shows how the dynamical system theory may be used for cosmological purposes. ΛCDM system is rather simple one while modified theories of gravity usually possess Friedmann evolutionary equations of the form much more complicated than presented here. They are first order ordinary non-linear differential equations on the scale factor \( a(t) \). Therefore, considering a geometrical structure of a phase space may help us to understand the evolution of a universe described by a model under consideration. In order to perform such analysis, one needs to represent dynamics of cosmological models in terms of dynamical system theory. Phase diagrams will allow to find critical points which correspond to extremes of the effective potentials whose diagram will provide information about the velocity of cosmic expansion and classically forbidden regions.
[1] Albert Einstein. The field equations of gravitation. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 844-847, 1915.

[2] Albert Einstein. The foundation of the general theory of relativity. *Annalen Phys.* 49, 769-822, [Annalen Phys. 14, 517(2005)], 1916.

[3] Frank W Dyson, Arthur S Eddington, and Charles Davidson. A determination of the deflection of light by the sun’s gravitational field, from observations made at the total eclipse of May 29, 1919. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 220(571-581):291–333, 1920.

[4] Robert V Pound and GA Rebka Jr. Gravitational red-shift in nuclear resonance. *Physical Review Letters*, 3(9):439, 1959.

[5] Robert V Pound and GA Rebka Jr. Apparent weight of photons. *Physical Review Letters*, 4(7):337, 1960.

[6] Edwin Hubble. A relation between distance and radial velocity among extra-galactic nebulae. *Proceedings of the National Academy of Sciences*, 15(3):168–173, 1929.

[7] Karl Schwarzschild. Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*, 1916, Seite 189-196, 1:189–196, 1916.

[8] Karl Schwarzschild. On the gravitational field of a mass point according to einstein’s theory. *arXiv preprint physics/9905030*, 1999.

[9] BP Abbott, Richard Abbott, TD Abbott, MR Abernathy, Fausto Acernese, Kendall Ackley, Carl Adams, Thomas Adams, Paolo Addesso, RX Adhikari, et al. Observation of gravitational waves from a binary black hole merger. *Physical review letters*, 116(6):061102, 2016.

[10] Edmund J Copeland, Mohammad Sami, and Shinji Tsujikawa. Dynamics of dark energy. *International Journal of Modern Physics D*, 15(11):1753–1935, 2006.
[11] Dragan Huterer and Michael S Turner. Prospects for probing the dark energy via supernova distance measurements. *Physical Review D*, 60(8):081301, 1999.

[12] Peter AR Ade, N Aghanim, MIR Alves, C Armitage-Caplan, M Arnaud, M Ashdown, F Atrio-Barandela, J Aumont, H Aussel, C Baccigalupi, et al. Planck 2013 results. i. overview of products and scientific results. *Astronomy & Astrophysics*, 571:A1, 2014.

[13] Salvatore Capozziello and Mariafelicia De Laurentis. Extended theories of gravity. *Physics Reports*, 509(4):167–321, 2011.

[14] Salvatore Capozziello and Valerio Faraoni. *Beyond Einstein gravity: A Survey of gravitational theories for cosmology and astrophysics*, volume 170. Springer Science and Business Media, 2010.

[15] Alexei A Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1):99–102, 1980.

[16] Alan H Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23(2):347, 1981.

[17] Philip Bull, Yashar Akrami, Julian Adamek, Tessa Baker, Emilio Bellini, Jose Beltrán Jiménez, Eloisa Bentivegna, Stefano Camera, Sébastien Clesse, Jonathan H Davis, et al. Beyond lcmd: Problems, solutions, and the road ahead. *arXiv preprint arXiv:1512.05356*, 2015.

[18] Chris Clarkson and Roy Maartens. Inhomogeneity and the foundations of concordance cosmology. *Classical and Quantum Gravity*, 27(12):124008, 2010.

[19] GFR Ellis, R Maartens, and SD Nel. The expansion of the universe. *Monthly Notices of the Royal Astronomical Society*, 184(3):439–465, 1978.

[20] CM Will. Theory and experiment in gravitational physics, 1981.

[21] Antonio De Felice and Shinji Tsujikawa. f(r) theories. *Living Rev. Rel.*, 13(3):1002–4928, 2010.

[22] Salvatore Capozziello, VF Cardone, and A Troisi. Dark energy and dark matter as curvature effects? *Journal of Cosmology and Astroparticle Physics*, 2006(08):001, 2006.
[23] Salvatore Capozziello, Vincenzo F Cardone, and Antonio Troisi. Low surface brightness galaxy rotation curves in the low energy limit of rn gravity: no need for dark matter. *Monthly Notices of the Royal Astronomical Society, 375*(4):1423–1440, 2007.

[24] Thomas P Sotiriou and Valerio Faraoni. f(r) theories of gravity. *Reviews of Modern Physics, 82*(1):451, 2010.

[25] Andrzej Borowiec, Michał Kamionka, Aleksandra Kurek, and Marek Szydłowski. Cosmic acceleration from modified gravity with palatini formalism. *Journal of Cosmology and Astroparticle Physics, 2012*(02):027, 2012.

[26] Eanna E Flanagan. Palatini form of 1/r gravity. *Physical review letters, 92*(7):071101, 2004.

[27] Alberto Iglesias, Nemanja Kaloper, Antonio Padilla, and Minjoon Park. How (not) to use the palatini formulation of scalar-tensor gravity. *Physical Review D, 76*(10):104001, 2007.

[28] Gonzalo J Olmo. Hydrogen atom in palatini theories of gravity. *Physical review D, 77*(8):084021, 2008.

[29] Enrico Barausse, Thomas P Sotiriou, and John C Miller. Curvature singularities, tidal forces and the viability of palatini f (r) gravity. *Classical and Quantum Gravity, 25*(10):105008, 2008.

[30] Gonzalo J Olmo. The gravity lagrangian according to solar system experiments. *Physical review letters, 95*(26):261102, 2005.

[31] Thomas P Sotiriou. The nearly newtonian regime in non-linear theories of gravity. *General Relativity and Gravitation, 38*(9):1407–1417, 2006.

[32] Marco Ferraris, Mauro Francaviglia, and Igor Volovich. The universality of vacuum einstein equations with cosmological constant. *Classical and Quantum Gravity, 11*(6):1505, 1994.

[33] Thomas P Sotiriou. f(r) gravity and scalar - tensor theory. *Classical and Quantum Gravity, 23*(17):5117, 2006.

[34] Gonzalo J Olmo and Helios Sanchis-Alepuz. Hamiltonian formulation of palatini f(r) theories à la brans-dicke theory. *Physical Review D, 83*(10):104036, 2011.
[35] Gonzalo J Olmo, Helios Sanchis-Alepuz, and Swapnil Tripathi. Dynamical aspects of generalized palatini theories of gravity. *Physical Review D*, 80(2):024013, 2009.

[36] Gonzalo J Olmo and Parampreet Singh. Covariant effective action for loop quantum cosmology à palatini. *Journal of Cosmology and Astroparticle Physics*, 2009(01):030, 2009.

[37] KS Stelle. Renormalization of higher-derivative quantum gravity. *Physical Review D*, 16(4):953, 1977.

[38] DE Barraco, E Dominguez, and R Guibert. Conservation laws, symmetry properties, and the equivalence principle in a class of alternative theories of gravity. *Physical Review D*, 60(4):044012, 1999.

[39] Tomi Koivisto. A note on covariant conservation of energy–momentum in modified gravities. *Classical and Quantum Gravity*, 23(12):4289, 2006.

[40] Salvatore Capozziello, Mariafelicia De Laurentis, M Francaviglia, and S Mercadante. From dark energy and dark matter to dark metric. *Foundations of Physics*, 39(10):1161–1176, 2009.

[41] S Capozziello, MF De Laurentis, L Fatibene, M Ferraris, and S Garruto. Extended cosmologies. *arXiv preprint arXiv:1509.08008*, 2015.

[42] Lorenzo Fatibene and Mauro Francaviglia. Mathematical equivalence vs. physical equivalence between extended theories of gravitations. *arXiv preprint arXiv:1302.2938*, 2013.

[43] S Capozziello, L Fatibene, and S Garruto. Equivalence among frames in extended gravity. *arXiv preprint arXiv:1512.08535*, 2015.

[44] L Fatibene, M Francaviglia, and G Magnano. On a characterization of geodesic trajectories and gravitational motions. *International Journal of Geometric Methods in Modern Physics*, 9(05):1220007, 2012.

[45] Albert Einstein, Anna Beck, and Peter Havas. *The Swiss Years: Writing 1900–1909:...* Princeton University Press, 1989.

[46] Charles W Misner, Kip S Thorne, and John Archibald Wheeler. *Gravitation*. Macmillan, 1973.
[47] Gianluca Allemandi, Andrzej Borowiec, Mauro Francaviglia, and Sergei D Odintsov. Dark energy dominance and cosmic acceleration in first-order formalism. *Physical Review D*, 72(6):063505, 2005.

[48] Gianluca Allemandi, Andrzej Borowiec, and Mauro Francaviglia. Accelerated cosmological models in first-order nonlinear gravity. *Physical Review D*, 70(4):043524, 2004.

[49] Gianluca Allemandi, Andrzej Borowiec, and Mauro Francaviglia. Accelerated cosmological models in ricci squared gravity. *Physical Review D*, 70(10):103503, 2004.

[50] Gianluca Allemandi, Monica Capone, Salvatore Capozziello, and Mauro Francaviglia. Conformal aspects of the palatini approach in extended theories of gravity. *General Relativity and Gravitation*, 38(1):33–60, 2006.

[51] Valerio Faraoni and Edgard Gunzig. Einstein frame or jordan frame? *International journal of theoretical physics*, 38(1):217–225, 1999.

[52] R Jackiw and AP Polychronakos. Fluid dynamical profiles and constants of motion from d-branes. *Communications in mathematical physics*, 207(1):107–129, 1999.

[53] Naohisa Ogawa. Remark on the classical solution of the chaplygin gas as d-branes. *Physical Review D*, 62(8):085023, 2000.

[54] Sergey Chaplygin. On gas jets. *Sci. Mem. Moscow Univ. Math. Phys.*, 21(1), 1904.

[55] Alexander Kamenshchik, Ugo Moschella, and Vincent Pasquier. An alternative to quintessence. *Physics Letters B*, 511(2):265–268, 2001.

[56] MC Bento, O Bertolami, and AA Sen. Generalized chaplygin gas, accelerated expansion, and dark-energy-matter unification. *Physical Review D*, 66(4):043507, 2002.

[57] Jianbo Lu. Cosmology with a variable generalized chaplygin gas. *Physics Letters B*, 680(5):404–410, 2009.

[58] Neven Bilić, Gary B Tupper, and Raoul D Viollier. Unification of dark matter and dark energy: the inhomogeneous chaplygin gas. *Physics Letters B*, 535(1):17–21, 2002.
[59] VA Popov. Dark energy and dark matter unification via superfluid chap-lygin gas. *Physics Letters B*, 686(4):211–215, 2010.

[60] J Naji, B Pourhassan, and Ali R Amani. Effect of shear and bulk viscosities on interacting modified chaplygin gas cosmology. *International Journal of Modern Physics D*, 23(02):1450020, 2014.

[61] Gilberto M Kremer and Daniele SM Alves. Palatini approach to $1/r$ gravity and its implications to the late universe. *Physical Review D*, 70(2):023503, 2004.

[62] Vittorio Gorini, Alexander Kamenshchik, and Ugo Moschella. Can the chaplygin gas be a plausible model for dark energy? *Physical Review D*, 67(6):063509, 2003.

[63] PP Avelino, K Bolejko, and GF Lewis. Nonlinear chaplygin gas cosmolo-
gies. *Physical Review D*, 89(10):103004, 2014.

[64] EO Kahya and B Pourhassan. The universe dominated by the extended chaplygin gas. *Modern Physics Letters A*, 30(13):1550070, 2015.

[65] JC Fabris, HES Velten, C Ogouyandjou, and J Tossa. Ruling out the modified chaplygin gas cosmologies. *Physics Letters B*, 694(4):289–293, 2011.

[66] Jens Hoppe. Supermembranes in four-dimensions. *arXiv preprint hep-th/9311059*.

[67] R Jackiw and AP Polychronakos. Supersymmetric fluid mechanics. *Physical Review D*, 62(8):085019, 2000.

[68] H. B. Benaoum. Accelerated universe from modified chaplygin gas and tachyonic. *arXiv preprint hep-th/0205140*.

[69] Luis P Chimento. Extended tachyon field, chaplygin gas, and solvable k-essence cosmologies. *Physical Review D*, 69(12):123517, 2004.

[70] Alexander Kamenshchik, Ugo Moschella, and Vincent Pasquier. Chaplygin-like gas and branes in black hole bulks. *Physics Letters B*, 487(1):7–13, 2000.

[71] Planck Collaboration et al. Planck 2015 results. xiii. cosmological parameters. *arXiv preprint arXiv:1502.01589*, 2015.
[72] Andrzej Borowiec, Aleksander Stachowski, Marek Szydłowski, and Aneta Wojnar. Inflationary cosmology with chaplygin gas in palatini formalism. Journal of Cosmology and Astroparticle Physics, 2016(01):040, 2016.

[73] Marek Szydlowski, Aleksander Stachowski, Andrzej Borowiec, and Aneta Wojnar. Do sewn singularities falsify the palatini cosmology? arXiv preprint arXiv:1504.02632, 2015.

[74] N Suzuki, D Rubin, C Lidman, G Aldering, R Amanullah, K Barbary, LF Barrientos, J Botyanszki, M Brodwin, N Connolly, et al. The hubble space telescope cluster supernova survey. v. improving the dark-energy constraints above z> 1 and building an early-type-hosted supernova samplebased. The Astrophysical Journal, 746(1):85, 2012.

[75] Nicholas Metropolis, Arianna W Rosenbluth, Marshall N Rosenbluth, Augusta H Teller, and Edward Teller. Equation of state calculations by fast computing machines. The journal of chemical physics, 21(6):1087–1092, 1953.

[76] W Keith Hastings. Monte carlo sampling methods using markov chains and their applications. Biometrika, 57(1):97–109, 1970.

[77] Gideon Schwarz et al. Estimating the dimension of a model. The annals of statistics, 6(2):461–464, 1978.

[78] Robert E Kass and Adrian E Raftery. Bayes factors. Journal of the american statistical association, 90(430):773–795, 1995.

[79] Shin’ichi Nojiri, Sergei D Odintsov, and Shinji Tsujikawa. Properties of singularities in the (phantom) dark energy universe. Physical Review D, 71(6):063004, 2005.

[80] Parampreet Singh and Francesca Vidotto. Exotic singularities and spatially curved loop quantum cosmology. Physical Review D, 83(6):064027, 2011.

[81] Mariusz P Dabrowski. Are singularities the limits of cosmology? arXiv preprint arXiv:1407.4851, 2014.

[82] Mariusz P Dąbrowski and Tomasz Denkiewicz. Barotropic index w-singularities in cosmology. Physical Review D, 79(6):063521, 2009.

[83] Paul H Frampton, Kevin J Ludwick, and Robert J Scherrer. The little rip. Physical Review D, 84(6):063003, 2011.
[84] Paul H Frampton, Kevin J Ludwick, and Robert J Scherrer. Pseudo-rip: Cosmological models intermediate between the cosmological constant and the little rip. Physical Review D, 85(8):083001, 2012.

[85] Marek Szydłowski. Cosmological zoo — accelerating models with dark energy. Journal of Cosmology and Astroparticle Physics, 2007(09):007, 2007.

[86] S Nojiri, SD Odintsov, and VK Oikonomou. Singular inflation from generalized equation of state fluids. Physics Letters B, 747:310–320, 2015.

[87] SD Odintsov and VK Oikonomou. Singular inflationary universe from f(r) gravity. Physical Review D, 92(12):124024, 2015.

[88] N. N. Bautin and eds. I. A. Leontovich. Methods and Techniques for Qualitative Analysis of Dynamical Systems on the Plane, volume 1. Nauka, Moscow [In Russian], 1976.

[89] Ramón Herrera, Marco Olivares, and Nelson Videla. Intermediate inflation on the brane and warped dgp models. The European Physical Journal C, 73(6):1–9, 2013.

[90] John D Barrow and Alexander AH Graham. Singular inflation. Physical Review D, 91(8):083513, 2015.

[91] Claus Kiefer. On the avoidance of classical singularities in quantum cosmology. In Journal of Physics: Conference Series, volume 222, page 012049. IOP Publishing, 2010.

[92] Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. Metric-palatini gravity unifying local constraints and late-time cosmic acceleration. Physical Review D, 85(8):084016, 2012.

[93] Salvatore Capozziello, Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. Hybrid metric-palatini gravity. Universe, 1(2):199–238, 2015.

[94] Salvatore Capozziello, Tiberiu Harko, Francisco SN Lobo, Gonzalo J Olmo, and Stefano Vignolo. The cauchy problem in hybrid metric-palatini f(x)-gravity. International Journal of Geometric Methods in Modern Physics, 11(05):1450042, 2014.

[95] Salvatore Capozziello, Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. Wormholes supported by hybrid metric-palatini gravity. Physical Review D, 86(12):127504, 2012.
[96] Salvatore Capozziello, Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. Cosmology of hybrid metric-palatini f (x)-gravity. *Journal of Cosmology and Astroparticle Physics*, 2013(04):011, 2013.

[97] Christian G Böhmer, Francisco SN Lobo, and Nicola Tamanini. Einstein static universe in hybrid metric-palatini gravity. *Physical Review D*, 88(10):104019, 2013.

[98] Nelson A Lima. Dynamics of linear perturbations in the hybrid metric-palatini gravity. *Physical Review D*, 89(8):083527, 2014.

[99] Nelson A Lima, Vanessa Smer-Barreto, and Lucas Lombriser. Constraints on decaying early modified gravity from cosmological observations. *arXiv preprint arXiv:1603.05239*, 2016.

[100] Sante Carloni, Tomi Koivisto, and Francisco SN Lobo. Dynamical system analysis of hybrid metric-palatini cosmologies. *Physical Review D*, 92(6):064035, 2015.

[101] Tahereh Azizi and Najibe Borhani. Thermodynamics in hybrid metric-palatini gravity. *Astrophysics and Space Science*, 357(2):1–9, 2015.

[102] Salvatore Capozziello, Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. The virial theorem and the dark matter problem in hybrid metric-palatini gravity. *Journal of Cosmology and Astroparticle Physics*, 2013(07):024, 2013.

[103] Salvatore Capozziello, Tiberiu Harko, Tomi S Koivisto, Francisco SN Lobo, and Gonzalo J Olmo. Galactic rotation curves in hybrid metric-palatini gravity. *Astroparticle Physics*, 50:65–75, 2013.

[104] D Borka, S Capozziello, P Jovanović, and V Borka Jovanović. Probing hybrid modified gravity by stellar motion around galactic centre. *arXiv preprint arXiv:1504.07832*, 2015.

[105] Qi-Ming Fu, Li Zhao, and Yu-Xiao Liu. Hybrid metric-palatini brane system. *arXiv preprint arXiv:1601.06546*, 2016.

[106] Nicola Tamanini and Christian G Boehmer. Generalized hybrid metric-palatini gravity. *Physical Review D*, 87(8):084031, 2013.

[107] Tomi S Koivisto and Nicola Tamanini. Ghosts in pure and hybrid formalisms of gravity theories: a unified analysis. *Physical Review D*, 87(10):104030, 2013.
[108] Andrzej Borowiec, Salvatore Capozziello, Mariafelicia De Laurentis, Francisco SN Lobo, Andronikos Paliathanasis, Mariacristina Paolella, and Aneta Wojnar. Invariant solutions and noether symmetries in hybrid gravity. Physical Review D, 91(2):023517, 2015.

[109] A Paliathanasis, L Karpathopoulos, A Wojnar, and S Capozziello. Wheeler–dewitt equation and lie symmetries in bianchi scalar-field cosmology. The European Physical Journal C, 76(4):1–14, 2016.

[110] Mariusz P Dabrowski, Janusz Garecki, and David B Blaschke. Conformal transformations and conformal invariance in gravitation. Annalen der Physik, 18(1):13–32, 2009.

[111] Gonzalo J Olmo. Palatini approach to modified gravity: f(r) theories and beyond. International Journal of Modern Physics D, 20(04):413–462, 2011.

[112] Salvatore Capozziello and Antonio De Felice. f(r) cosmology from noether’s symmetry. Journal of Cosmology and Astroparticle Physics, 2008 (08):016, 2008.

[113] Salvatore Capozziello, Ester Piedipalumbo, Claudio Rubano, and Paolo Scudellaro. Noether symmetry approach in phantom quintessence cosmology. Physical Review D, 80(10):104030, 2009.

[114] Y Kucukakca and U Camci. Noether gauge symmetry for f(r) gravity in palatini formalism. Astrophysics and Space Science, 338(1):211–216, 2012.

[115] Mahmood Roshan and Fatimah Shojai. Palatini f(r) cosmology and noether symmetry. Physics Letters B, 668(3):238–240, 2008.

[116] Yi Zhang, Yun-gui Gong, and Zong-Hong Zhu. Noether symmetry approach in multiple scalar fields scenario. Physics Letters B, 688(1):13–20, 2010.

[117] Babak Vakili. Noether symmetry in f(r) cosmology. Physics Letters B, 664 (1):16–20, 2008.

[118] Babak Vakili. Noether symmetric f(r) quantum cosmology and its classical correlations. Physics Letters B, 669(3):206–211, 2008.

[119] S Capozziello and Gen Lambiase. Selection rules in minisuperspace quantum cosmology. General Relativity and Gravitation, 32(4):673–696, 2000.
[120] Richard Arnowitt, Stanley Deser, and Charles W Misner. Dynamical structure and definition of energy in general relativity. *Physical Review*, 116(5): 1322, 1959.

[121] Claus Kiefer. *Quantum Gravity*. Oxford University Press, New York, 2007. ISBN 9780199212521.

[122] Eric Gourgoulhon. *3+1 formalism in general relativity: bases of numerical relativity*, volume 846. Springer Science & Business Media, 2012.

[123] Salvatore Capozziello, Mariafelicia De Laurentis, and Sergei D Odintsov. Hamiltonian dynamics and noether symmetries in extended gravity cosmology. *The European Physical Journal C*, 72(7):1–21, 2012.

[124] JB Hartle. Gravitation in astrophysics (eds b carter and jb hartle, 1986.

[125] Michael Tsamparlis and Andronikos Paliathanasis. Two-dimensional dynamical systems which admit lie and noether symmetries. *Journal of Physics A: Mathematical and Theoretical*, 44(17):175202, 2011.

[126] Andronikos Paliathanasis, Michael Tsamparlis, Spyros Basilakos, and Salvatore Capozziello. Scalar-tensor gravity cosmology: Noether symmetries and analytical solutions. *Physical Review D*, 89(6):063532, 2014.

[127] Michael Tsamparlis, Andronikos Paliathanasis, Spyros Basilakos, and Salvatore Capozziello. Conformally related metrics and lagrangians and their physical interpretation in cosmology. *General Relativity and Gravitation*, 45 (10):2003–2022, 2013.

[128] Andronikos Paliathanasis and Michael Tsamparlis. The geometric origin of lie point symmetries of the schrödinger and the klein–gordon equations. *International Journal of Geometric Methods in Modern Physics*, 11(04):1450037, 2014.

[129] Paul Adrien Maurice Dirac. *General theory of relativity*. Princeton University Press, 1996.

[130] Hans-Juergen Matschull. Dirac’s canonical quantization programme. *arXiv preprint quant-ph/9606031*, 1996.

[131] Bryce Seligman DeWitt and Neill Graham. *The many worlds interpretation of quantum mechanics*. Princeton University Press, 2015.
[132] Hans Stephani and Malcolm MacCallum. *Differential equations: their solution using symmetries*. Cambridge University Press, 1989.

[133] Peter E. Hydon. *Symmetry Methods for Differential Equations: A Beginner’s Guide*. Cambridge University Press, New York, USA, 1st edition, 2000.

[134] PGL Leach. The complete symmetry group of the one-dimensional time-dependent harmonic oscillator. *Journal of Mathematical Physics*, 21(2):300–304, 1980.

[135] Barbara Abraham-Shrauner, Keshlan S Govinder, and Daniel J Arrigo. Type-ii hidden symmetries of the linear 2d and 3d wave equations. *Journal of Physics A: Mathematical and General*, 39(20):5739, 2006.

[136] Andronikos Paliathanasis and Michael Tsamparlis. The reduction of the laplace equation in certain riemannian spaces and the resulting type ii hidden symmetries. *Journal of Geometry and Physics*, 76:107–123, 2014.

[137] James B Hartle and Stephen W Hawking. Wave function of the universe. *Physical Review D*, 28(12):2960, 1983.

[138] Alexander Vilenkin. Creation of universes from nothing. *Physics Letters B*, 117(1):25–28, 1982.

[139] Alexander Vilenkin. Quantum creation of universes. *Physical Review D*, 30 (2):509, 1984.

[140] Salvatore Capozziello, Tiberiu Harko, Francisco SN Lobo, and Gonzalo J Olmo. Hybrid modified gravity unifying local tests, galactic dynamics and late-time cosmic acceleration. *International Journal of Modern Physics D*, 22(12):1342006, 2013.

[141] Carl Brans and Robert H Dicke. Mach’s principle and a relativistic theory of gravitation. *Physical Review*, 124(3):925, 1961.

[142] Theodor Kaluza. Zum unitätsproblem der physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin.(Math. Phys.).*, 1921(966972):45, 1921.

[143] Oskar Klein. Quantentheorie und fünfdimensionale relativitätstheorie. *Zeitschrift für Physik*, 37(12):895–906, 1926.

[144] Oskar Klein. The atomicity of electricity as a quantum theory law. *Nature*, 118:516, 1926.
[145] LL Williams. Field equations and lagrangian for the kaluza metric evaluated with tensor algebra software. *Journal of Gravity*, 2015, 2015.

[146] Valerio Faraoni. *Cosmology in scalar-tensor gravity*, volume 139. Springer Science & Business Media, 2004.

[147] Salvatore Capozziello, Giuseppe Marmo, Claudio Rubano, and Paolo Scudellaro. Nöther symmetries in bianchi universes. *International journal of modern physics D*, 6(04):491–503, 1997.

[148] John D Barrow. Why the universe is not anisotropic. *Physical Review D*, 51(6):3113, 1995.

[149] Tony Rothman and GFR Ellis. Can inflation occur in anisotropic cosmologies? *Physics Letters B*, 180(1):19–24, 1986.

[150] M Demianski, R De Ritis, C Rubano, and P Scudellaro. Scalar fields and anisotropy in cosmological models. *Physical Review D*, 46(4):1391, 1992.

[151] Andrei D Linde. A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. *Physics Letters B*, 108(6):389–393, 1982.

[152] Larry H Ford. Cosmological-constant damping by unstable scalar fields. *Physical Review D*, 35(8):2339, 1987.

[153] Michael P Ryan and Lawrence C Shepley. *Homogeneous relativistic cosmologies*. Princeton University Press, 2015.

[154] Michael Tsamparlis and Andronikos Paliathanasis. The geometric nature of lie and noether symmetries. *General Relativity and Gravitation*, 43(6):1861–1881, 2011.

[155] Charles W Misner. Quantum cosmology. i. *Physical Review*, 186(5):1319, 1969.

[156] Asim Orhan Barut and Ryszard Raczka. *Theory of group representations and applications*, volume 2. World Scientific, 1986.

[157] Salvatore Capozziello, Mariafelicia De Laurentis, and Sergei D Odintsov. Hamiltonian dynamics and noether symmetries in extended gravity cosmology. *The European Physical Journal C*, 72(7):1–21, 2012.
[158] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. i. *Physical Review, 85*(2):166, 1952.

[159] David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. ii. *Physical Review, 85*(2):180, 1952.

[160] PB Demorest, Tim Pennucci, SM Ransom, MSE Roberts, and JWT Hessels. A two-solar-mass neutron star measured using shapiro delay. *Nature, 467* (7319):1081–1083, 2010.

[161] Meredith L Rawls, Jerome A Orosz, Jeffrey E McClintock, Manuel AP Torres, Charles D Bailyn, and Michelle M Buxton. Refined neutron star mass determinations for six eclipsing x-ray pulsar binaries. This paper includes data gathered with the 6.5 m magellan telescopes located at las campanas observatory, chile. *The Astrophysical Journal, 730*(1):25, 2011.

[162] MH Van Kerkwijk, RP Breton, and SR Kulkarni. Evidence for a massive neutron star from a radial-velocity study of the companion to the black-widow pulsar psr b1957+20. *The Astrophysical Journal, 728*(2):95, 2011.

[163] Artyom V Astashenok, Salvatore Capozziello, and Sergei D Odintsov. Magnetic neutron stars in f(r) gravity. *Astrophysics and Space Science, 355* (2):333–341, 2015.

[164] Artyom V Astashenok, Salvatore Capozziello, and Sergei D Odintsov. Maximal neutron star mass and the resolution of the hyperon puzzle in modified gravity. *Physical Review D, 89*(10):103509, 2014.

[165] Artyom V Astashenok, Salvatore Capozziello, and Sergei D Odintsov. Extreme neutron stars from extended theories of gravity. *Journal of Cosmology and Astroparticle Physics, 2015*(01):001, 2015.

[166] Kazım Yavuz Ekși, Can Güngör, and Murat Metehan Türkoğlu. What does a measurement of mass and/or radius of a neutron star constrain: Equation of state or gravity? *Physical Review D, 89*(6):063003, 2014.

[167] Emanuele Berti, Enrico Barausse, Vitor Cardoso, Leonardo Gualtieri, Paolo Pani, Ulrich Sperhake, Leo C Stein, Norbert Wex, Kent Yagi, Tessa Baker, et al. Testing general relativity with present and future astrophysical observations. *Classical and Quantum Gravity, 32*(24):243001, 2015.

[168] AM Oliveira, HES Velten, JC Fabris, and L Casarini. Neutron stars in rastall gravity. *Physical Review D, 92*(4):044020, 2015.
[169] Carlos Palenzuela and Steve Liebling. Constraining scalar-tensor theories of gravity from the most massive neutron stars. *arXiv preprint arXiv:1510.03471*, 2015.

[170] Adolfo Cisterna, Térence Delsate, and Massimiliano Rinaldi. Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling. *Physical Review D*, 92(4):044050, 2015.

[171] Adolfo Cisterna, Térence Delsate, Ludovic Ducobu, and Massimiliano Rinaldi. Slowly rotating neutron stars in the nonminimal derivative coupling sector of horndeski gravity. *Physical Review D*, 93(8):084046, 2016.

[172] Aneta Wojnar and Hermano Velten. Equilibrium and stability of relativistic stars in extended theories of gravity. *arXiv:1604.04257*, 2016.

[173] Hermano Velten, Adriano M Oliveira, and Aneta Wojnar. A free parametrized tov: Modified gravity from newtonian to relativistic stars. *Proceedings of Science (MPCS2015) 025, arXiv:1601.03000*, 2016.

[174] Steven Weinberg. *Gravitation and cosmology: principles and applications of the general theory of relativity*, volume 1. Wiley New York, 1972.

[175] Norman K Glendenning. *Compact stars: Nuclear physics, particle physics and general relativity*. Springer Science and Business Media, 2012.

[176] J Robert Oppenheimer and George M Volkoff. On massive neutron cores. *Physical Review*, 55(4):374, 1939.

[177] Richard C Tolman. Static solutions of einstein’s field equations for spheres of fluid. *Physical Review*, 55(4):364, 1939.

[178] Richard Chace Tolman. *Relativity, thermodynamics, and cosmology*. Courier Corporation, 1987.

[179] David J Nice, Eric M Splaver, Ingrid H Stairs, L Oliver, Axel Jessner, Michael Kramer, James M Cordes, et al. A 2.1m? pulsar measured by relativistic orbital decay. *The Astrophysical Journal*, 634(2):1242, 2005.

[180] Salvatore Capozziello, Francisco SN Lobo, and José P Mimoso. Energy conditions in modified gravity. *Physics Letters B*, 730:280–283, 2014.

[181] Salvatore Capozziello, Francisco SN Lobo, and José P Mimoso. Generalized energy conditions in extended theories of gravity. *Physical Review D*, 91(12):124019, 2015.
[182] José P Mimoso, Francisco SN Lobo, and Salvatore Capozziello. Extended theories of gravity with generalized energy conditions. In *Journal of Physics: Conference Series*, volume 600, page 012047. IOP Publishing, 2015.

[183] Tomi Koivisto. A note on covariant conservation of energy–momentum in modified gravities. *Classical and Quantum Gravity*, 23(12):4289, 2006.

[184] Kazuharu Bamba. Thermodynamic properties of modified gravity theories. *arXiv preprint arXiv:1604.02632*, 2016.

[185] S Capozziello, R De Ritis, C Rubano, and P Scudellaro. Nöther symmetries in cosmology. *La Rivista del Nuovo Cimento (1978-1999)*, 19(4):1–114, 1996.

[186] Andronikos Paliathanasis. Symmetries of differential equations and applications in relativistic physics. *PhD Thesis, University of Athens (2014)*, arXiv:1501.05129, 2015.

[187] Roman Stanisław Ingarden and Andrzej Jamiołkowski. *Mechanika klasyczna*. Państwowe Wydawnictwo Naukowe, 1980.

[188] D.D. Landau and E.M. Lifshitz. *The Classical Theory of Fields*. Butterworth Heinemann, 4th edition, 1994.

[189] Kentaro Yano. The theory of lie derivatives and its applications. 1957.

[190] Andronikos Paliathanasis and Michael Tsamparlis. Lie point symmetries of a general class of pdes: The heat equation. *Journal of Geometry and Physics*, 62(12):2443–2456, 2012.

[191] KS Govinder. Lie subalgebras, reduction of order, and group-invariant solutions. *Journal of mathematical analysis and applications*, 258(2):720–732, 2001.

[192] A. Paliathanasis, M. Tsamparlis, and M. T. Mustafa. Symmetry analysis of the klein–gordon equation in bianchi i spacetimes. *International Journal of Geometric Methods in Modern Physics*, 12(03):1550033, 2015.

[193] John Wainwright and George Francis Rayner Ellis. *Dynamical systems in cosmology*. Cambridge University Press, 2005.

[194] Orest Hrycyna and Marek Szydłowski. Cosmological dynamics with non-minimally coupled scalar field and a constant potential function. *Journal of Cosmology and Astroparticle Physics*, 2015(11):013, 2015.
[195] Marek Szydłowski and Orest Hrycyna. Scalar field cosmology in the energy phase-space—unified description of dynamics. *Journal of Cosmology and Astroparticle Physics*, 2009(01):039, 2009.

[196] M Sami, M Shahalam, M Skugoreva, and A Toporensky. Cosmological dynamics of a nonminimally coupled scalar field system and its late time cosmic relevance. *Physical Review D*, 86(10):103532, 2012.

[197] Alan A Coley. Dynamical systems in cosmology. *arXiv preprint gr-qc/9910074*, 1999.

[198] Sante Carloni, Peter KS Dunsby, Salvatore Capozziello, and Antonio Troisi. Cosmological dynamics of rn gravity. *Classical and Quantum Gravity*, 22(22):4839, 2005.

[199] Lawrence Perko. *Differential equations and dynamical systems*, volume 7. Springer Science and Business Media, 2013.

[200] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012.

[201] Marek Szydłowski. Generic scenarios of the accelerating universe. *Astrophysics and Space Science*, 339(2):389–399, 2012.