Brownian motion between two random trajectories

You Lv
School of science of mathematics, Beijing Normal University,
Beijing 100875, P. R. China.

Abstract: Consider the first exit time of one-dimensional Brownian motion $\{B_s\}_{s \geq 0}$ from a random passageway. We discuss a Brownian motion with two time-dependent random boundaries in quenched sense. Let $\{W_s\}_{s \geq 0}$ be another one-dimensional Brownian motion independent of $\{B_s\}_{s \geq 0}$ and let $P(\cdot | W)$ represent the conditional probability depending on the realization of $\{W_s\}_{s \geq 0}$. We show that

$$-t^{-1} \ln P_x(\forall s \leq t a + \beta W_s \leq B_s \leq b + \beta W_s | W)$$

converges to a finite positive constant $\gamma(\beta)(b-a)^{-2}$ almost surely and in $L^p$ ($p \geq 1$) if $a < B_0 = x < b$ and $W_0 = 0$. When $\beta = 1$, $a + b = 2x$, it is equivalent to the random small ball probability problem in the sense of equidistribution, which has been investigated in [4]. We also find some properties of the function $\gamma(\beta)$.

An important moment estimation has also been obtained, which can be applied to discuss the small deviation of random walk with random environment in time (see [12]).

Keywords: Brownian motion, First exit time, Random boundary, Limit theorem.

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1. Introduction

The first exit time of Brownian motion is a classic and interesting topic which has been researched by many scholars. Let us first recall a very basic result in this field. For a standard Brownian motion $\{B_t\}_{t \geq 0}$ starting from $x$, it is known that

$$\lim_{t \to +\infty} -\frac{\ln P_x(\forall s \leq t a \leq B_s \leq b)}{t} = \frac{\pi^2}{2(b-a)^2},$$

(1.1)

where $a < x < b$. (1.1) shows that the first exit time from a bounded interval has negative exponential tail distribution and the coefficient depends on the width of the bounded interval.

*Email: youlv@mail.bnu.edu.cn
A lot of further work has been done on the first exit time of Brownian motion. For the Brownian motion in high dimensional space, [9], [10] researched the first exit time $T$ from a fixed convex domain, showing that $-\ln \mathbb{P}(T > t) = O(t^\alpha)$, where $\alpha$ is a positive constant depending on the degree of dimension and the shape of the convex field. Another extension is to consider the time-dependent boundary. [14] studied the asymptotic behavior of $\mathbb{P}(\forall_0 \leq s \leq t | B_s | \leq f(t))$, where the boundaries $-f(t)$ and $f(t)$ depend on $t$. The work [18] considered the Brownian motion with two linear boundaries and calculated the distribution of the Brownian motion hitting the upper boundary before hitting the lower boundary. The model of Brownian motion with two time-dependent boundaries can be applied to many different fields such as finance (see [15]), biophysical models (see [16]) and statistical sequential analysis (see [17]).

There are several profound conclusions when the boundaries not only depend on $t$ but also a random variable. [9], [10] and [11] all discussed the probability

$$\mathbb{P}(\forall_0 \leq s \leq t \parallel B_s \parallel^p \leq 1 + \mu s^r + W(s)),$$

where $\mu > 0, r \in [0, 1), p > 1, \{B_s\}_{s \geq 0}$ is a $d$-dimensional ($d \geq 2$) standard Brownian motion and $\{W_s\}_{s \geq 0}$ is a one-dimensional Brownian motion which is independent of $\{B_s\}_{s \geq 0}$. “$\parallel \cdot \parallel$” is the Euclidean norm. That can be viewed as the first exit time $T$ of Brownian motion $\{B_s\}_{s \geq 0}$ from a random domain. Under some suitable conditions, they all showed that $t^{-\frac{p-1}{p}} \ln \mathbb{P}(T > t)$ converges to a negative constant which depends on the random domain and on the dimension as $t \to +\infty$.

What we want to discuss is the decay rate of

$$\mathbb{P}^x(\forall_{s \in [0, t]} a + \beta W_s \leq B_s \leq b + \beta W_s | W)$$

as $t \to +\infty$, where $\beta \geq 0, a < x < b$ and $\{W_t\}_{t \geq 0}$ is an other standard Brownian motion which is independent of $\{B_t\}_{t \geq 0}$. Of course, it can also be viewed as the first exit time from a random and time-dependent passageway. We obtain a kind of quenched result. We prove that the decay rate is $e^{-ct}$ almost surely and $c = (b - a)^{-2} \gamma(\beta)$. Moreover, the function $\gamma(\beta)$ is strictly increasing on $[0, +\infty)$, that is to say, although the width of the random passageway is always constant “$b - a$”, more violent fluctuation of the center will make the first exit time $T$ much shorter. We should notice that if $\beta = 1$ and $x = \frac{a+b}{2}$, by scaling property of the Brownian motion, (1.2) has the same distribution as

$$\mathbb{P}^0(\forall_{s \in [0, t]} B_s - W_s \leq \frac{b - a}{2\sqrt{t}} | W).$$

That is the random small ball probability which has been investigated in [4]. So we can see the convergence (2.2) holds in probability from [4] Theorem 6.1 if $\beta = 1$ and $x = \frac{a+b}{2}$. Motivated by the precise asymptotics of a random quantization
problem (see [3], [2] and [4] first investigated the random small ball probabilities and gave many important asymptotic estimations for the Gaussian measure \( \mu \) on a set centered at a random trajectory when the distribution of the random trajectory is also \( \mu \). In the proof of [4, Theorem 6.1], the observing of subadditivity also gives us essential inspiration. Compared with [4, Theorem 6.1], our difference is that we also consider the situation of \( \beta \neq 1 \). Moreover, we conclude that the convergence (2.2) is also almost surely and uniform for the location of the starting point and the width of the interval at the last moment. By the way, we obtain a moment estimation (see Theorem 3.1). All of these adjustments will play key role on the research of the small deviation for random walk with random environment in time (see [12]), which is a main application and motivation of this paper. One can utilize the Brownian motion between two random trajectories to approximate the random walk with random environment in time. Furthermore, the result of [12] will be a basic tool when we study the barrier problem of the branching random walk with random environment in time. The latter is a work in progress.

Another important point is that our main result can also be viewed as an extension of [13, Theorem 1.1]. Mallein and Mišoš consider the probability of a Brownian motion staying above a trajectory of another independent Brownian motion. To be more precisely, they proved

\[
- \ln \mathbb{P}(\forall 0 \leq s \leq t \beta W_s + a \leq B_s \leq \beta W_s + b, \beta W_t + a' \leq B_t \leq \beta W_t + b' | W) / \ln t \to \gamma, \; \gamma > \frac{1}{2}
\]

almost surely and in \( L^p \) (\( p \geq 1 \)). The idea of our proof is partly inspired by [13]. However, we face new difficulties when we do the moment estimation (see Theorem 3.1) since the probability of Brownian motion with two boundaries is usually smaller than the single boundary case.

The rest of this paper is organized as follows. We state the main theorem and corollaries in section 2. An important estimation of tail distribution is obtained in section 3. Based on this estimation, we give the proof of the main theorem and corollaries in section 4 and section 5 respectively.

2. Main result

Throughout this paper, we assume that real numbers \( a, b, a', b', a_0, b_0 \) meet the following basic relationship

\[
a < a_0 \leq b_0 < b, \; a \leq a' < b' \leq b.
\]  

**Theorem 2.1** Let \( B, W \) be two independent standard Brownian motions. \( W_0 \equiv 0 \).

Under the probability \( \mathbb{P}^x \), \( B_0 = x \) almost surely. Define

\[
\bar{X}_t := - \ln \inf_{x \in [a_0, b_0]} \mathbb{P}^x(\forall s \in [0, t] \beta W_s + a \leq B_s \leq \beta W_s + b, \; \beta W_t + a' \leq B_t \leq \beta W_t + b' | W);
\]
Then there exists a function $\gamma : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{t \to +\infty} \frac{X_t}{t} = \lim_{t \to +\infty} \frac{X_t}{t} = \frac{\gamma(\beta)}{(b-a)^2}, \quad \text{a.s. and in } L_p \ (p \geq 1),$$

(2.2)

where $\gamma$ is a convex and even function. Moreover, $\gamma(0) = \frac{\pi^2}{2}, \gamma(1) \leq 4\pi^2$ and for any $\beta \in \mathbb{R}$, $\gamma(\beta) \geq \frac{\pi^2(1+\beta^2)}{2}$. Hence $\gamma$ is strictly increasing on $[0, +\infty)$ and $\lim_{|\beta| \to +\infty} \gamma(\beta) = +\infty$.

**Remark 2.1** In fact, (2.2) can be strengthened to (5.2) and (5.3). Moreover, From the property of $\gamma$, we can see even though the width of the random passageway is always $\beta - a$ at every moment from 0 to $t$, the first exit time will be shorter when the random passageway has more violent fluctuation of the center (i.e., when $|\beta|$ becomes bigger).

In order to make writing more concise, we denote

$$I_{x, y}^{W_s, \beta} := [\beta W_s + x, \beta W_s + y],$$

(2.3)

where $x, y$ can be any constant or function.

**Corollary 2.1** *(Small deviation)* If $\alpha \in (0, \frac{1}{2})$, then we have, almost surely,

$$\lim_{t \to +\infty} \inf_{x \in [a_0 t^\alpha, b_0 t^\alpha]} \frac{\ln \mathbb{P}^x(\forall s \in [0, t], B_s \in I_{a_0 t^\alpha, b_0 t^\alpha}^{W_s, \beta})}{t^{1-2\alpha}} \leq -\frac{\gamma(\beta)}{(b-a)^2};$$

(2.4)

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \frac{\ln \mathbb{P}^x(\forall s \in [0, t], B_s \in [a t^\alpha + \beta W_s, b t^\alpha + \beta W_s])}{t^{1-2\alpha}} \leq -\frac{\gamma(\beta)}{(b-a)^2};$$

(2.5)

**Remark 2.2** Obviously, the “$\liminf$, $\geq$” in (2.4) and “$\limsup$, $\leq$” in (2.5) can be replaced by “$\lim$, $=$”. The same replacement can also be done in (2.6) and (2.7).

**Corollary 2.2** Let $f(s)$ and $g(s)$ be two continue functions from $[0, 1]$ to $\mathbb{R}$ such that

$$\forall s \in [0, 1], \ f(s) < g(s), \ f(0) < a_0 \leq b_0 < g(0), \ f(1) \leq a' < b' \leq g(1).$$

We have, almost surely,

$$\lim_{t \to +\infty} \inf_{x \in [a_0, b_0]} \frac{\ln \mathbb{P}^x(\forall s \in [0, t], B_s \in I_{a_0, b_0}^{W_s, \beta})}{t} \geq C_{f,g} \gamma(\beta),$$

(2.6)

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \frac{\ln \mathbb{P}^x(\forall s \in [0, t], \beta W_s + f(s) \leq B_s \leq \beta W_s + g(s))}{t} \leq C_{f,g} \gamma(\beta),$$

(2.7)

where $C_{f,g} := -\int_0^1 (g(s) - f(s))^{-2} ds$. We should notice that $C_{f,g} \in (0, +\infty)$ because of the assumption of $f(s)$ and $g(s)$.
3. The moment estimation for $\overline{X}_t$

The main tool we use to prove Theorem 2.1 is the Kingman’s subadditive ergodic theorem. For preparation, we first give an important estimation for $\overline{X}_t$ which has been defined in Theorem 2.1.

Since $\gamma$ is an even function and the distribution of the first exit time is well-known when $\beta = 0$, we will always assume $\beta > 0$ in the rest of the paper.

**Theorem 3.1** For any $t > 0, p > 0, q > 1$, we have

$$\lim_{n \to +\infty} n^p P(\overline{X}_t \geq (\ln n)^q) = 0.$$  \hspace{0.5cm} (3.1)

Thus for any $j \in \mathbb{N}$, we have

$$\mathbb{E}(\overline{X}_t^j) < +\infty.$$ \hspace{0.5cm} (3.2)

**Proof of Theorem 3.1** Notice that $\overline{X}_t$ is related to $a, b, a_0, b_0, a', b'$. Obviously, we only need to show that (3.1) holds when $a_0 \leq a' < b' \leq b$. Recalling the basic relationship (2.1), we will first prove (3.1) under the situation of

$$a_0 \leq a' < b' \leq b \quad \text{and} \quad \min\{a_0 - a, b - b_0\} > \max\{a' - a_0, b_0 - b'\}.  \hspace{0.5cm} (3.3)$$

Under this situation, we can choose $a'', b'', \delta$ such that $a' < a'' < b'' < b'$ and

$$0 < 2\beta \delta < \min\{b'' - a'', \min\{a_0 - a, b - b_0\} - \max\{a'' - a_0, b_0 - b''\}\}.  \hspace{0.5cm} (3.4)$$

We define a Markov time sequence $\{\tau_{n,\delta}\}_{n \in \mathbb{N}}$ such that

$$\tau_{0,\delta} := 0, \quad \tau_{n+1,\delta} := \inf\{s > \tau_{n,\delta} : |W_s - W_{\tau_{n,\delta}}| = \delta\}. \quad n = 0, 1, 2, \ldots$$

It is easy to see that $\{\tau_{n,\delta}\}_{n \in \mathbb{N}}$ is an i.i.d. random walk and $\tau_{1,\delta} > 0$ almost surely. We divide time $[0, t]$ into $[0, \tau_{1,\delta}], [\tau_{1,\delta}, \tau_{2,\delta}], \ldots, [\tau_{N,\delta}, t]$, where

$$N := \sup\{n : \tau_{n,\delta} < t\}.$$ 

Then by the Markov property, we have

$$\overline{X}_t \leq 1_{\{N=0\}} Z_0(W) + \sum_{i=1}^{\infty} 1_{\{N=i\}} \left( \sum_{k=0}^{i-1} Y_k(W) + Z_i(W) \right),$$

where

$$Y_k(W) := -\ln \inf_{x \in [a_0, b_0]} P(\forall_{\tau_{k,\delta} \leq s \leq \tau_{k+1,\delta}} B_s - \beta(W_s - W_{\tau_{k,\delta}}) \in [a, b],$$

$$B_{\tau_{k+1,\delta}} - \beta(W_{\tau_{k+1,\delta}} - W_{\tau_{k,\delta}}) \in [a'', b''] | W, B_{\tau_{k,\delta}} = x),$$

$$Z_k(W) := -\ln \inf_{x \in [a'', b'']} P(\forall_{\tau_{k,\delta} \leq s \leq t} B_s - (\beta W_s - \beta W_{\tau_{k,\delta}}) \in [a', b'] | W, B_{\tau_{k,\delta}} = x).$$
Let
\[ \rho_{k,\delta} := \tau_{k,\delta} - \tau_{k-1,\delta}, \quad k = 1, 2, 3, \ldots \]
By the definition of \( \tau_{k,\delta} \) we can get a further upper bound for \( Y_k(W) \) and \( Z_k(W) \), which is
\[
Y_k := -\ln \inf_{x \in [a_0, b_0]} P^x(\forall_{0 \leq s \leq \rho_{k+1,\delta}} B_s \in [a + \beta \delta, b - \beta \delta], B_{\rho_{k+1,\delta}} \in [a'' + \beta \delta, b'' - \beta \delta]|W) \geq Y_k(W),
\]
\[
Z_0 := -\ln \inf_{x \in [a'', b'']} P^x(\forall_{0 \leq s \leq t} B_s \in [a' + \beta \delta, b' - \beta \delta]) \geq Z_k(W).
\]
Note that \( Y_k \) is also depend on \( W \) but \( Z_0 \) is a non-random constant. Hence we have
\[
\bar{X}_t \leq 1_{\{N = 0\}} Z_0 + \sum_{i=1}^{\infty} 1_{\{N = i\}} (\sum_{k=0}^{i-1} Y_k + Z_0) = Z_0 + \sum_{i=1}^{\infty} 1_{\{N = i\}} \sum_{k=0}^{i-1} Y_k. \tag{3.5}
\]
Naturally, we need to estimate the upper bound of \( Y_k \). Define
\[
k(t) := \inf_{x \in [a_0, b_0]} P^x(\forall_{0 \leq s \leq t} B_s \in [a + \beta \delta, b - \beta \delta], B_t \in [a'' + \beta \delta, b'' - \beta \delta]), \tag{3.6}
\]
\[
\delta_1 := \max \{a'' + \beta \delta - a_0, b_0 - b'' + \beta \delta\}, \quad \delta_2 := \min \{a_0 - a - \beta \delta, b - \beta \delta - b_0\}.
\]
By basic calculation, we have
\[
k(t) \geq \inf_{x \in [a_0, b_0]} \left\{ P^x(B_t \in [a'' + \beta \delta, b'' - \beta \delta]) - \left[ 1 - P^x(\forall_{0 \leq s \leq t} B_s \in [a - \beta \delta, b + \beta \delta]) \right] \right\} \geq P^0(B_t \in [\delta_1, \delta_1 + b'' - a'' - 2\beta \delta]) - P^0(\sup_{s \in [0, t]} |B_s| > \delta_2).
\]
By \( (3.6) \), we know \( \delta_2 > \delta_1 \). Therefore, we can choose an \( \epsilon > 0 \) small enough such that \( \frac{(\delta_1 + \epsilon)^2}{2} \leq \frac{\delta_2^2}{2\epsilon} \) and \( \delta_1 + \epsilon \leq \delta_1 + b'' - a'' - 2\beta \delta \). That means
\[
P^0(B_t \in [\delta_1, \delta_1 + b'' - a'' - 2\beta \delta]) \geq \frac{\epsilon}{\sqrt{2\pi t}} \exp \left\{ - \frac{\epsilon^2}{2t} \right\}.
\]
Recalling the Csorgo and Revesz estimation \( [1] \) Lemma1], we know there exists a constant \( C > 0 \) such that
\[
P^0(\sup_{s \in [0, t]} |B_s| > \delta_2) \leq C \exp \left\{ - \frac{\delta_2^2}{(2 + \epsilon)t} \right\}.
\]
Then there exists a \( D > 0 \) such that for any \( t \leq D \),
\[
k(t) \geq \frac{\epsilon}{\sqrt{2\pi t}} \exp \left\{ - \frac{(\delta_1 + \epsilon)^2}{2t} \right\} - C \exp \left\{ - \frac{\delta_2^2}{(2 + \epsilon)t} \right\} \geq \exp \left\{ - \frac{(\delta_1 + \epsilon)^2}{2t} \right\}. \tag{3.7}
\]
When $t > D$, by (3.4), we can choose a $\delta_3 > 0$ such that $a' + \beta \delta + \delta_3 < b' - \beta \delta - \delta_3$. So for $t > D$, we have

$$k(t) \geq \inf_{x \in [a_0, b_0]} \mathbb{P}^x(\forall 0 \leq s \leq D B_s \in [a + \beta \delta, b - \beta \delta], B_D \in [a' + \beta \delta + \delta_3, b' - \beta \delta - \delta_3]) \times \inf_{x \in [a' + \beta \delta + \delta_3, b' - \beta \delta - \delta_3]} \mathbb{P}^x(\forall 0 \leq s \leq t - D B_s \in [a' + \beta \delta, b' - \beta \delta]).$$

(3.8)

Notice that $\inf_{x \in [a_0, b_0]} \mathbb{P}^x(\forall 0 \leq s \leq D B_s \in [a + \beta \delta, b - \beta \delta], B_D \in [a' + \beta \delta + \delta_3, b' - \beta \delta - \delta_3])$ is a positive constant. Moreover,

$$\inf_{x \in [a' + \beta \delta + \delta_3, b' - \beta \delta - \delta_3]} \mathbb{P}^x(\forall 0 \leq s \leq t - D B_s \in [a' + \beta \delta, b' - \beta \delta]) \geq \mathbb{P}(\sup_{s \in [0, t - D]} |B_s| \leq \delta_3) \geq \frac{\pi^2(t - D)}{8 \delta_3^2}. \quad (3.9)$$

Combining with (3.7)-(3.9), we conclude that there exist $C_1, C_2 > 0$ such that

$$- \ln k(t) \leq C_1 t^{-1} 1_{\{t \leq D\}} + C_2 t 1_{\{t > D\}}. \quad (3.10)$$

It implies that for any $k \in \{1, 2, \ldots, N\}$, we have

$$Y_{k-1} \leq C_1 \rho_{k, \delta}^{-1} 1_{\{\rho_{k, \delta} \leq D\}} + C_2 \rho_{k, \delta} 1_{\{\rho_{k, \delta} > D\}}. \quad (3.11)$$

Choosing $q'', q'$ such that $1 < q'' < q' < q$. When $n$ is large enough, by (3.5) and (3.11) we have

$$\mathbb{P}(X_t \geq (\ln n)^q) \leq \sum_{i=1}^{+\infty} E(1_{\{\sum_{k=0}^{i-1} Y_k + 2(\ln n)^{q'} \geq N = i\}} 1_{\{N = i\}}) + E(1_{\{Z_0 \geq (\ln n)^{q'} \}} 1_{\{N = 0\}}) \leq \sum_{i=1}^{+\infty} E(1_{\{\sum_{k=0}^{i-1} C_{k, \delta} \geq (\ln n)^{q'}, N = i\}}) \leq \sum_{i=1}^{+\infty} \left[ E(1_{\{\sum_{k=0}^{i-1} C_{k, \delta} \geq (\ln n)^{q'}, N = i\}}) + E(1_{\{\sum_{k=0}^{i-1} C_{2 \rho_{k, \delta} \geq (\ln n)^{q'}, N = i\}}) \right].$$

Notice that when $(\ln n)^{q'} \geq C_2 t$, for any $i$, we have

$$\mathbb{P}(\sum_{k=1}^{i} C_{2 \rho_{k, \delta} \geq (\ln n)^{q'}, N = i} \leq \mathbb{P}(\tau_{i, \delta} \geq \frac{(\ln n)^{q'}}{C_2}), \tau_{i, \delta} < t) = 0.$$

Hence when $n$ is large enough, denote $\varsigma_n := (\ln n)^{q''}$, it is true that

$$\mathbb{P}(X_t \geq (\ln n)^q) \leq \sum_{i=1}^{+\infty} \mathbb{P}(\sum_{k=1}^{i} C_{1 \rho_{k, \delta} \geq (\ln n)^{q'}, N = i}) \leq \sum_{i=1}^{\lfloor \varsigma_n \rfloor - 1} \mathbb{P}(\sum_{k=1}^{i} C_{1 \rho_{k, \delta} \geq (\ln n)^{q'}} + \sum_{i=\lfloor \varsigma_n \rfloor}^{+\infty} \mathbb{P}(N = i) \leq \varsigma_n \mathbb{P}(\sum_{k=1}^{\lfloor \varsigma_n \rfloor} \frac{1}{\rho_{k, \delta} \geq (\ln n)^{q'} / C_1}) + \mathbb{P}(\tau_{[\varsigma_n, 1], \delta} \leq t). \quad (3.12)$$
where \(|x|\) is the integer part of \(x\). Let \(p(t)\) be the probability density function of Markov time \(\tau_{1,\delta}\). Then the expression of \(p(t)\) is

\[
p(t) = \frac{2\delta}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{+\infty} (4n+1)e^{-\frac{(4n+1)^2\delta}{2t}}.
\]

Obviously, \(p(t) \leq \frac{2\delta}{\sqrt{2\pi t^3}} e^{\frac{t}{2}}\) when \(t\) is small enough. Then it is easy to see for any \(c_3 > 0\), we have \(\mathbb{E}(\exp\left(\frac{c_3}{\tau_{1,\delta}}\right)) < +\infty\). Moreover, according to [7, Page 30], there exists a positive constant \(c_4 > 0\) such that for any \(c_5 \in [0, c_4]\), \(\mathbb{E}(e^{c_5 \tau_{1,\delta}}) < +\infty\). Then we can apply the Cramér theorem [5, Page 27] to i.i.d. random walk \(\{\frac{1}{p_{1,\delta}} + \frac{1}{p_{2,\delta}} + \cdots + \frac{1}{p_{i,\delta}}\}_{i \in \mathbb{N}}\) and \(\{\tau_{i,\delta}\}_{i \in \mathbb{N}}\) respectively. Notice that \(\mathbb{E}(\tau_{1,\delta}), \mathbb{E}(\tau_{2,\delta}) \in (0, +\infty), q'' < q'\) and time \(t\) is not depend on \(n\), so when \(n\) is large enough there exist \(c_1, c_2 > 0\) such that

\[
\mathbb{P}\left(\sum_{k=1}^{[\ln n]} \frac{1}{\rho_{k,\delta}} \geq \frac{\left(\ln n\right)^q}{C_1}\right) \leq \mathbb{P}\left(\frac{\sum_{k=1}^{[\ln n]} \rho_{k,\delta}^{-1}}{\left[\left(\ln n\right)^{q''}\right]} \geq 2\mathbb{E}\left(\frac{1}{\tau_{1,\delta}}\right)\right) \leq e^{-c_1\left[\left(\ln n\right)^{q''}\right]}
\]

and

\[
\mathbb{P}(\tau_{\left\lfloor \ln n \right\rfloor} \leq t) \leq \mathbb{P}\left(\frac{\left(\ln n\right)^{q''}}{\left[\left(\ln n\right)^{q''}\right]} \leq \frac{\mathbb{E}(\tau_{1,\delta})}{2}\right) \leq e^{-c_2\left[\left(\ln n\right)^{q''}\right]}.
\]

Combining with (3.12), we get, as \(n \to +\infty\),

\[
n^p\mathbb{P}(\overline{X}_t \geq (\ln n)^{(1+q)}) \leq n^p(\ln n)^{q''} e^{-c_1\left[\left(\ln n\right)^{q''}\right]} + n^p e^{-c_2\left[\left(\ln n\right)^{q''}\right]} \to 0.
\]

So we obtain (3.1) under situation (3.3).

Next, if (3.3) is not hold, that is to say, \(\min\{a_0 - a, b - b_0\} \leq \max\{a' - a_0, b_0 - b'\}\). Let \([x] := \inf\{j \in \mathbb{N} : j \geq x\}\), define \(m := \frac{3\max\{a' - a_0, b_0 - b'\}}{\min\{a_0 - a, b - b_0\}}\), \(\delta' := \frac{\min\{a_0 - a, b - b_0\}}{2}\), \(a_i := \min\{a_0 + i\delta', a\}\), \(b_i := \max\{b_0 - i\delta', b'\}\), and

\[
\overline{X}_{t,i} := -\ln \inf_{x \in [a_i, b_i]} \mathbb{P}^x(\forall s \in [\frac{a_i}{m}, \frac{b_i}{m}]: B_s - \beta(W_s - W_{\frac{a_i}{m}}) \in [a, b], B_{\frac{(i+1)m}{\beta}} - \beta(W_{\frac{(i+1)m}{\beta}} - W_{\frac{a_i}{m}}) \in [a_{i+1}, b_{i+1}]|W, B_{\frac{a_i}{m}} = x).
\]

By the Markov property we can see that \(\overline{X}_t \leq \sum_{i=0}^{m-1} \overline{X}_{t,i}\).

Notice that \(\overline{X}_{t,i}\) has the same law as \(\overline{X}'_{t,i}\), where

\[
\overline{X}'_{t,i} := -\ln \inf_{x \in [a_i, b_i]} \mathbb{P}^x(\forall s \in [0, \frac{a_i}{m}]: B_s - \beta W_s \in [a, b], B_{\frac{a_i}{m}} - \beta W_{\frac{a_i}{m}} \in [a_{i+1}, b_{i+1}]|W).
\]

As \(q' < q\), we have \((\ln n)^q \geq m(\ln n)^{q'}\) when \(n\) is large enough. Consequently,

\[
n^p\mathbb{P}(X_t \geq (\ln n)^q) \leq n^p \sum_{i=0}^{m-1} \mathbb{P}(\overline{X}_{t,i} \geq (\ln n)^{q'}) = \sum_{i=0}^{m-1} n^p\mathbb{P}(\overline{X}_{t,i} \geq (\ln n)^{q'}) = (3.13)
\]
Note that for any $i \in [0, m-1] \cap \mathbb{N}$, $a, b, a_i, b_i, a_{i+1}, b_{i+1}$ satisfy the relationship (3.3), so we have
\[
\lim_{n \to 0} n^p \mathbb{P}(X^j_{t,i} \geq (\ln n)^q) = 0, \quad i \in [0, m-1] \cap \mathbb{N}.
\]
Combining with (3.13), we complete the proof of (3.1).

Moreover, for large enough $n$, according to (3.1) we have $\mathbb{P}(X^j_t \geq n) \leq e^{-\sqrt{n}}$, which implies that (3.2) holds.

\[
\square
\]

4. Proof of the main result

In this section, we will show how to use the Kingman’s subadditive ergodic theorem [8, Theorem 9.14] to prove Theorem 2.1. To simplify the statement, we first introduce some notations. Let $\gamma$ be a non-negative function of two variables $\gamma : (0, +\infty) \times \mathbb{R} \to [0, +\infty)$, which is essential for Theorem 2.1.

Let $t_2 > t_1 \geq 0$, analogous to the definition of (2.3), we define
\[
I_{a,b,t1}^{W_{a.b},\beta} := [a + \beta W_s - \beta W_{t1}, b + \beta W_s - \beta W_{t1}].
\]

Denote $r_{t_1,t_2}(a, b, a', b', x, \beta) := \mathbb{P}(\forall t_1 \leq t \leq t_2 B_s \in I_{a,b,t1}^{W_{a.b},\beta}, B_{t2} \in I_{a',b',t1}^{W_{a',b}',\beta} | W, B_{t1} = x)$,
\[
p_{t_1,t_2}(a, b, a', b', x, \beta) := \inf_{x \in [a', b']} r_{t_1,t_2}(a, b, a', b', x, \beta),
\]
\[
q_{t_1,t_2}(a, b, a', b', \beta) := -\ln p_{t_1,t_2}(a, b, a', b', \beta).
\]

Without causing confusion, sometimes $p_{t_1,t_2}(a, b, a', b', \beta)$ and $q_{t_1,t_2}(a, b, a', b', \beta)$ are abbreviated as $p_{t_1,t_2}$ and $q_{t_1,t_2}$ respectively in the rest part of the paper. The following lemma is essential for Theorem 2.1.

**Lemma 4.1** Under the situation $a < a' < b$ and the relationship (2.1), there exists a non-negative function of two variables $\gamma : (0, +\infty) \times \mathbb{R} \to [0, +\infty)$ such that
\[
\lim_{n \to +\infty} \frac{q_{0,n}(a, b, a', b', \beta)}{n} = \gamma(b - a, \beta), \quad \text{a.s. and in } L^1.
\]

**Proof of Lemma 4.1.** We divide the proof into two steps.

**Step 1.** Showing that $\frac{q_{0,n}(a, b, a', b', \beta)}{n}$ has an almost surely and $L^1$ degenerate limit.

By the Markov property, we know
\[
p_{0,n} \geq \inf_{x \in [a', b']} \mathbb{P}(\forall m \leq s \leq n B_s \in I_{a', b'}^{W_{a', b'}, \beta}, B_n \in I_{a', b'}^{W_{a', b'}, \beta} | W, B_m = x)
\]
\[
= p_{0,m} p_{m,n}, \quad 0 \leq m < n.
\]
Hence we have $q_{0,n} \leq q_{0,m} + q_{m,n}$. This is the subadditivity condition \([8, (9.9)]\) of the Kingman’s subadditive ergodic theorem.

If we denote $W^i(s) := W_{t+s} - W_t$, $s \in [0, 1]$, it is easy to see the sequence \(\{W^i\}_{i \in \mathbb{N}}\) is i.i.d. and the randomness of $q_{m,n}$ is only depend on \(\{W_{m+s} - W_m, s \in [0, n-m]\}\). From these facts and the stationary independent increments property of Brownian motion, we know that for any fixed $k$, the random sequence $q_{0,k}, q_{k,2k}, \ldots, q_{nk,(n+1)k}, \ldots$ is i.i.d., and for every $l \in \mathbb{N}$, random sequence $q_{l,l+1}, q_{l,l+2}, \ldots, q_{l,l+n}, \ldots$ has the same distribution as $q_{0,1}, q_{0,2}, \ldots, q_{0,n}, \ldots$. These mean that \(\{q_{m,n}\}_{1 \leq m \leq n}\) fulfills the conditions \([8, (9.7)]\) and \([8, (9.8)]\) respectively. According to Theorem 3.1, we know $\mathbb{E}(q_{0,1}) < +\infty$, which is the integrability condition of the Kingman’s subadditive ergodic theorem. And obviously, for each $n$, $\frac{\mathbb{E}(q_{0,1})}{n} \geq 0 > -\infty$. So far we have verified all conditions of the Kingman’s subadditive ergodic theorem.

Besides, for every $k$, the sequence $q_{0,k}, q_{k,2k}, \ldots, q_{nk,(n+1)k}, \ldots$ is ergodic since it is i.i.d., and thus we can conclude that $\frac{q_{0,n}}{n}$ converges to a constant almost surely and in $L_1$. Here we denote the limit by $\gamma(a, b, a', b', \beta)$. Consequently, we have

$$
\lim_{n \to +\infty} \frac{q_{0,n}(a, b, a', b', \beta)}{n} = \gamma(a, b, a', b', \beta), \quad \text{a.s. and in } L^1.
$$

**Step 2.** Let $a < a'' < b'' < b$, showing that $\gamma(a, b, a', b', \beta) = \gamma(a, b, a'', b'', \beta)$.

Without loss of generality, we assume $a' \leq a'' \leq b'' \leq b'$. Obviously,

$$q_{0,n}(a, b, a'', b', \beta) \leq -\ln \inf_{x \in [a', b']} r_{0,n}(a, b, a'', b'', x, \beta)

\leq -\ln \inf_{x \in [a', b']} r_{0,1}(a, b, a'', b'', x, \beta) + q_{1,n}(a, b, a'', b', \beta). \quad (4.2)
$$

By step 1 and the stationary increments property of Brownian motion, we know $\frac{q_{1,n}(a, b, a'', b', \beta)}{n-1} \to \gamma(a, b, a'', b'', \beta)$ in probability. Moreover, applying the Kingman’s subadditive ergodic theorem again, we can see $\frac{q_{1,n}(a, b, a'', b'', \beta)}{n-1}$ converges to a constant almost surely. Hence we have

$$
\frac{q_{1,n}(a, b, a'', b'', \beta)}{n-1} \to \gamma(a, b, a'', b'', \beta), \quad \text{a.s.}
$$

According to Theorem 3.1 and the Borel-Cantelli 0-1 law, it is easy to see that for any function $\alpha : \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$,

$$
\lim_{n \to +\infty} -\ln \inf_{x \in [a', b']} \frac{r_{\alpha(n), \alpha(n)+k}(a, b, a', b', x, \beta)}{n} = 0, \quad \text{a.s..} \quad (4.3)
$$

Hence from (4.2) and (4.3) we can obtain

$$
\lim_{n \to +\infty} -\ln \inf_{x \in [a', b']} \frac{r_{0,n}(a, b, a'', b'', x, \beta)}{n} = \gamma(a, b, a'', b'', \beta), \quad \text{a.s.} \quad (4.4)
$$

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On the other hand, we have

\[ q_{0,n}(a, b, a', b', \beta) \leq -\ln \inf_{x \in [a', b']} r_{0,n}(a, b, a'', b'', x, \beta) \leq q_{0,n-1}(a, b, a', b', \beta) - \ln \inf_{x \in [a', b']} r_{n-1,n}(a, b, a'', b'', x, \beta). \]

Analogous to the above discussion, we can also obtain

\[ \lim_{n \to +\infty} \frac{-\ln \inf_{x \in [a', b']} r_{0,n}(a, b, a'', b'', x, \beta)}{n} = \gamma(a, b, a', b', \beta), \quad \text{a.s..} \]

Combining with (4.4), we have \( \gamma(a, b, a', b', \beta) = \gamma(a, b, a'', b'', \beta) \).

If \( a', b', a'', b'' \) can not satisfy the relationship \( "a' \leq a'' \leq b' \leq b" \), then without loss of generality, we assume \( a < a' \leq a'' \leq b' \leq b'' < b \). From the above conclusion, we have

\[ \gamma(a, b, a', b', \beta) = \gamma(a, b, a'', b', \beta) = \gamma(a, b, a'', b'', \beta). \]

So it is reasonable to write \( \gamma(a, b, a', b', \beta) \) as \( \gamma(a, b, \beta) \). Moreover, by the basic property of Brownian motion, it is easy to see for any \( c \in \mathbb{R} \), we have \( \gamma(a + c, b + c, \beta) = \gamma(a, b, \beta) \). Hence we can further denote \( \gamma(a, b, \beta) \) by \( \gamma(b - a, \beta) \). This is the end of the proof of Lemma 4.1.

\[ \square \]

More information of \( \gamma(c, \beta) \) has been listed in the following proposition, which is also an important preparation for the proof of Theorem 2.1.

**Proposition 4.1** The function \( \gamma(c, \beta) \) has been introduced in Lemma 4.1.

1. For each fixed \( \beta \), the function \( c \mapsto \gamma(c, \beta) \) is convex on \((0, +\infty)\).
2. For each fixed \( c > 0 \), the function \( \beta \mapsto \gamma(c, \beta) \) is even and convex.

**Proof of proposition 4.1.**

In this proof, we set \( a, b, a', b' \) satisfy the relationship 2.1 and \( a < a' \leq 0 \leq b' < \min\{b_1, b_2, b\} \). Denote \( d_{n,T}(x) \) is the joint density function of \((B_{T/n}, B_{2T/n}, \ldots, B_{nT/n})\) from \( \mathbb{R}^n \) to \( \mathbb{R}^+ \). By basic calculation, we know for any \( n, T \), function \( d_{n,T} \) is a log-concave function, that is to say, for any \( \lambda \in [0, 1] \) and \( x, y \in \mathbb{R}^n \), it has the relationship

\[ d_{n,T}(\lambda x + (1 - \lambda)y) \geq d_{n,T}(x)^\lambda d_{n,T}(y)^{1-\lambda}. \]

Let \( g_1(s), g_2(s), h_1(s), h_2(s) \) be real functions defined on \([0, 1]\) such that for every \( s \in [0, 1], g_1(s) \leq h_1(s), g_2(s) \leq h_2(s) \). If we denote the \( k \)-th coordinate of \( x, y \in \mathbb{R}^n \) by \( x_k, y_k \), it is obvious that

\[ 1_{\forall k \leq n, x_k + (1 - \lambda)y_k \in [\lambda g_1(k/n) + (1 - \lambda)g_2(k/n), \lambda h_1(k/n) + (1 - \lambda)h_2(k/n)]} \geq (1_{\forall k \leq n, x_k \in [g_1(k/n), h_1(k/n)]})^\lambda (1_{\forall k \leq n, y_k \in [g_2(k/n), h_2(k/n)]})^{1-\lambda}. \]

Denote \( H_{n,T}^\lambda(x) := d_{n,T}(x)1_{\forall k \leq n, x_k \in [\lambda g_1(k/n) + (1 - \lambda)g_2(k/n), \lambda h_1(k/n) + (1 - \lambda)h_2(k/n)], \) then we have

\[ H_{n,T}^\lambda(\lambda x + (1 - \lambda)y) \geq (H_{n,T}^1(x))^{\lambda}(H_{n,T}^0(y))^{1-\lambda}. \quad (4.5) \]
Moreover, by Theorem 3.1, we know for each \( m \in \mathbb{N} \),
\[
q_{0,m}(a, b, a', b', \beta) < +\infty, \quad \text{a.s.} \quad (4.6)
\]
(4.5) and (4.6) are the two conditions of the Prekopa-Leindler inequality \([6, \text{Theorem 7.1}]\). According to the Prekopa-Leindler inequality, we have
\[
\int_{\mathbb{R}^n} H_{n,m}^\lambda(x)dx \geq \left( \int_{\mathbb{R}^n} H_{n,m}^\alpha(x)dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} H_{n,m}^0(x)dx \right)^\lambda. \quad (4.7)
\]
If we set
\[
g_i(s) := \begin{cases} 
\beta W_{s,m} + a, & s \in [0,1) \quad \text{and} \quad h_i(s) := \begin{cases} 
\beta W_{s,m} + a', & s \in [0,1) \\
\beta W_{s,m} + b', & s = 1,
\end{cases} \quad \text{for } i = 1, 2,
\end{cases}
\]
then (4.7) means that except the zero measure set \( \{ \omega : q_{0,m}(a, b, a', b', \beta) = +\infty \} \), we always have
\[
\mathbb{P} \left( \forall 1 \leq k \leq n, k \in \mathbb{N} B_{km/n} \in I_{a,b_2+(1-\lambda)b_2}, B_m \in I_{a',\hat{b}'}, \mathbb{P}(W, B_0 = 0) \right) 
\geq \left[ \mathbb{P} \left( \forall 1 \leq k \leq n, k \in \mathbb{N} B_{km/n} \in I_{a,b_1}, B_m \in I_{a',\hat{b}'}, \mathbb{P}(W, B_0 = 0) \right) \right]^{\lambda} 
\times \left[ \mathbb{P} \left( \forall 1 \leq k \leq n, k \in \mathbb{N} B_{km/n} \in I_{a,b_2}, B_m \in I_{a',\hat{b}'}, \mathbb{P}(W, B_0 = 0) \right) \right]^{1-\lambda}.
\]
Let \( n \to +\infty \), we deduce that for each \( m \in \mathbb{N} \), almost surely we have
\[
r_{0,m}(a, \lambda b_1 + (1-\lambda)b_2, a', b', 0, \beta) \geq r_{0,m}^\lambda(a, b_1, a', b', 0, \beta) r_{0,m}^{1-\lambda}(a, b_2, a', b', 0, \beta).
\]
For any \( c \in [a', b'] \), by the same way, we can prove
\[
r_{0,m}(a - c, \lambda b_1 + (1-\lambda)b_2 - c, a' - c, b' - c, 0, \beta) \geq r_{0,m}^\lambda(a - c, b_1 - c, a' - c, b' - c, 0, \beta) \times r_{0,m}^{1-\lambda}(a - c, b_2 - c, a' - c, b' - c, 0, \beta),
\]
which means that
\[
\inf_{x \in [a', b']} r_{0,m}(a, \lambda b_1 + (1-\lambda)b_2, a', b', x, \beta) 
\geq \inf_{x \in [a', b']} \left( r_{0,m}^\lambda(a, b_1, a', b', x, \beta) \times r_{0,m}^{1-\lambda}(a, b_2, a', b', x, \beta) \right) 
\geq \inf_{x \in [a', b']} r_{0,m}^\lambda(a, b_1, a', b', x, \beta) \times \inf_{y \in [a', b']} r_{0,m}^{1-\lambda}(a, b_2, a', b', y, \beta), \quad \text{a.s.}
\]
That is to say
\[
q_{0,m}(a, \lambda b_1 + (1-\lambda)b_2, a', b', \beta) 
\leq \lambda q_{0,m}(a, b_1, a', b', \beta) + (1-\lambda)q_{0,m}(a, b_2, a', b', \beta). \quad \text{a.s.} \quad (4.8)
\]
Therefore, by Lemma 4.1 we can see the function \( c \mapsto \gamma(c, \beta) \) is convex on \((0, +\infty)\).
Therefore, we can deduce that if \( a < a \) is even since the standard Brownian motion \( W \) is symmetric. Hence we only need to consider \( \beta \in [0, +\infty) \). For any \( \beta_1, \beta_2 \geq 0 \), if we set

\[
g_i(s) := \begin{cases} 
\beta_i W_{sm} + a, & s \in [0, 1) \\
\beta_i W_{sm} + a', & s = 1,
\end{cases}
\]

and

\[
h_i(s) := \begin{cases} 
\beta_i W_{sm} + b, & s \in [0, 1) \\
\beta_i W_{sm} + b', & s = 1,
\end{cases}
\]

then we can obtain Proposition 4.1 (2) by repeating the step (4.5)-(4.8) similarly.

Now we will prove Theorem 2.1.

**Proof of Theorem 2.1.** In fact, in Lemma 4.1 we have shown that under condition \( a < a_0 \leq a' < b' \leq b_0 < b \) and the relationship (2.1), it has

\[
\lim_{n \to +\infty} -\frac{\ln \inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta)}{n} = \gamma(b - a, \beta), \quad \text{a.s..} \quad (4.9)
\]

Next, we will divide the proof into four steps.

**Step 1. Showing**

\[
\lim_{n \to +\infty} -\frac{\ln \sup_{x \in \mathbb{R}} r_{0,n}(a, b, a, b, x, \beta)}{n} = \gamma(b - a, \beta), \quad \text{a.s..} \quad (4.10)
\]

In the case of \( a < a' \leq a_0 < b_0 \leq b' < b \), we have

\[
\inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta) \geq p_{0,n}(a, b, a_0, b_0, x, \beta)
\]

and

\[
\inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta) \inf_{x \in [a', b']} r_{n,n+1}(a, b, a_0, b_0, x, \beta) \leq p_{0,n+1}(a, b, a_0, b_0, x, \beta).
\]

Therefore, we can deduce that if \( a < a' \leq a_0 < b_0 \leq b' < b \), we also have

\[
\lim_{n \to +\infty} -\frac{\ln \inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta)}{n} = \gamma(b - a, \beta). \quad \text{a.s..} \quad (4.11)
\]

Choosing an \( \varepsilon > 0 \) arbitrarily. Let \( M := \lceil \frac{b-a}{\varepsilon} \rceil \), \( y_i = \min\{a+i\varepsilon, b\}, i = 0, 1, 2, \ldots, M \).

\[
\sup_{x \in \mathbb{R}} r_{0,n}(a, b, a, b, x, \beta) = \sup_{x \in [a, b]} r_{0,n}(a, b, a, b, x, \beta)
\]

\[
\leq \max_{0 \leq i \leq M-1} \inf_{x \in [y_i, y_{i+1}]} r_{0,n}(a - 4\varepsilon, b + 4\varepsilon, a - 2\varepsilon, b + 2\varepsilon, x, \beta)
\]

\[
:= \max_{0 \leq i \leq M-1} \tau_{i,n}. \quad (4.12)
\]

By (4.12), we know that for each positive integer \( i \in [0, M - 1] \), it always has

\[
\lim_{n \to +\infty} -\frac{\ln r_{0,n}}{n} \geq \gamma(b - a + 8\varepsilon, \beta). \quad \text{Besides, for fixed} \ \varepsilon > 0, \ M \ \text{is finite. Thus we have}
\]

\[
\lim_{n \to +\infty} -\frac{\ln \sup_{x \in \mathbb{R}} \mathbb{P}\left(\forall s \in [0, n] \right) \beta W_s + a \leq B_s \leq \beta W_s + b | W \right)}{n} \geq \gamma(b - a + 8\varepsilon, \beta), \quad \text{a.s..}
\]

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Moreover, Lemma 4.1 implies that
\[
\limsup_{n \to +\infty} -\ln \sup_{x \in \mathbb{R}} \mathbb{P}^x(\forall s \in [0,n] \ W_s + a \leq B_s \leq \beta W_s + b|W) \leq \gamma(b - a, \beta), \text{ a.s.}
\]

By Proposition 4.1(1), for each fixed \( \beta \), the function \( c \mapsto \gamma(c, \beta) \) is convex hence it is continue. Let \( \varepsilon \to 0 \), we get (4.10).

**Step 2. Changing time axis from \( n \in \mathbb{N} \) to \( t \in \mathbb{R}^+ \).**

Assuming that \( t \in (n, n + 1) \). Just notice that when \( a' < a'' < b'' < b' \), we have
\[
\frac{1}{t} \ln \sup_{x \in \mathbb{R}} r_{0,t}(a, b, a, b, x, \beta) \leq \frac{1}{n} \ln \sup_{x \in \mathbb{R}} r_{0,n}(a, b, a, b, x, \beta)
\]
and
\[
\frac{\ln \inf_{x \in [a_0, b_0]} r_{0,t}(a, b, a', b', x, \beta)}{t} \geq \frac{\inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a'', b'', x, \beta)}{n + 1} + \frac{\inf_{x \in [a', b'] \setminus [a'' b''] \setminus [a'' b] \setminus [a b']} r_{n,n+1}(a', b', a', b', x, \beta)}{n + 1}.
\]

Utilizing (4.3) (4.9) and (4.10), we complete the step 2. According to the above discussion, we have shown the almost surely convergence in (2.2). (The only difference is the expression of \( \gamma(\beta) \)).

**Step 3. Showing the \( L^p(p \geq 1) \) convergence in (2.2).**

Because we have proved that the convergence in (2.2) is almost surely, step 3 is equivalent to show \( \{ \frac{-\ln \inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta)}{n} \}_{n \in \mathbb{N}} \) is \( L^p \) uniformly integrable when \( a < a_0 \leq a' < b' \leq b < b_0 \). Denote \( r_{i,i+1} := -\ln \inf_{x \in [a_0, b_0]} r_{i,i+1}(a, b, a', b', x, \beta) \).

Note that
\[
0 \leq \frac{-\ln \inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta)}{n} \leq \sum_{i=0}^{n-1} r_{i,i+1}.
\]

Theorem 3.1 shows \( \mathbb{E}(r_{0,1}^p) < +\infty \) for any \( p \geq 1 \). Therefore, by the Birkhoff ergodic theorem, we know
\[
\sum_{i=0}^{n-1} r_{i,i+1} \to \mathbb{E}(r_{0,1}), \text{ a.s. and in } L^p(p \geq 1).
\]

Therefore, \( \{ \frac{\sum_{i=0}^{n-1} r_{i,i+1}}{n} \}_{n \in \mathbb{N}} \) is \( L^p \) uniformly integrable. By (4.13), we know the sequence \( \{ \frac{-\ln \inf_{x \in [a_0, b_0]} r_{0,n}(a, b, a', b', x, \beta)}{n} \}_{n \in \mathbb{N}} \) is also \( L^p \) uniformly integrable. That is to say, we have proved along the discrete time axis \( n \in \mathbb{N} \) the convergence in (2.2) is \( L^p \ (p \geq 1) \). Just note the right-hand side of (4.13) and (4.14) are both converge.
to \( \gamma(b - a, \beta) \) in \( L^p \) (\( p \geq 1 \)) since (4.3) also holds in the sense of \( L^p \) (\( p \geq 1 \)). Hence we can also change time axis from \( n \in \mathbb{N} \) to \( t \in \mathbb{R}^+ \) by the same way of step 2.

**Step 4.** Define \( \gamma(\beta) := \gamma(1, \beta) \) and show \( \gamma(1) \leq 4\pi^2 \), \( \gamma(0) = \frac{\pi^2}{2} \) and \( \gamma(\beta) \geq \frac{\pi^2(1+\beta^2)}{2} \).

Firstly, \( \gamma(1) \leq 4\pi^2 \) can be derived directly from [4, Corollary 4.4].

According to step 1-2 of this proof, we can see that for any \( x \in (a, b) \),

\[
\lim_{t \to +\infty} \frac{\rho_{a,t}(a, b, a, b, x, \beta)}{t} = \gamma(b - a, \beta), \quad \text{a.s.}
\]

Moreover, note that for each \( t, d > 0 \), if \( a < 0 < b \), we have

\[
P^0(\forall s \leq t B_s \in [da + \beta W_s, db + \beta W_s]|W) \xrightarrow{d} P^0(\forall s \leq t B_s/d^2 \in [a + \beta W_s/d^2, b + \beta W_s/d^2]|W),
\]

where \( "X \overset{d}{=} Y" \) means that \( X \) and \( Y \) have the same distribution. That implies \( \gamma(b - a, \beta) = \frac{\gamma(1, \beta)}{d} \). Therefore, it is reasonable to define \( \gamma(\beta) := \gamma(1, \beta) \). So far we have given the whole proof of (2.2).

The only rest thing is to show \( \gamma(\beta) \geq \frac{\pi^2(1+\beta^2)}{2} \). We can use the method which has also been used in the corresponding part in [3]. By the Jensen’s inequality we have

\[
\mathbb{E}(\ln P^0(\forall s \leq t B_s - \beta W_s \in [-1/2, 1/2]|W)) > -\ln \mathbb{E}(P^0(\forall s \leq t B_s - \beta W_s \leq 1/2|W)).
\]

Let \( \tilde{B} \) be a Brownian motion with parameters \( \mathbb{E}(\tilde{B}_t) = 0, \mathbb{E}(\tilde{B}_t^2) = (1 + \beta^2)t, \forall t \geq 0 \). Then the annealed expectation

\[
\mathbb{E}(P^0(\forall s \leq t |B_s - \beta W_s| \leq 1/2|W)) = \mathbb{P}(\forall s \leq t \tilde{B}_s \in [-1/2, 1/2]).
\]

It is well known that

\[
\lim_{t \to +\infty} -\frac{\ln \mathbb{P}(\forall s \leq t |\tilde{B}_s| \leq \frac{1}{2})}{t} = \frac{\pi^2(1 + \beta^2)}{2}, \quad \gamma(0) = \lim_{t \to +\infty} -\frac{\ln \mathbb{P}(\forall s \leq t |B_s| \leq \frac{1}{2})}{t} = \frac{\pi^2}{2}.
\]

Hence we have \( \gamma(\beta) \geq \frac{\pi^2(1+\beta^2)}{2} \). Moreover, combining with Proposition 4.1 (2) which shows that \( \gamma(\beta) \) is even and convex, we know \( \gamma(\beta) \) is strictly increasing to \( +\infty \) on \([0, +\infty)\) and strictly decreasing on \((-\infty, 0]\).

\[\square\]

5. Proof of Corollary 2.1 and 2.2

By scaling property of Brownian motion, it is easy to see that the convergence in (2.4) and (2.5) are in Probability. Thanks to (3.2), we can strengthen it to almost surely.

**Proof of Corollary 2.1.** The proof of the upper bound (2.5) is more easier and similar with the lower bound (2.4), so here we only prove (2.4). We choose an \( A > 0 \) arbitrarily. Denote \( M := \lfloor A^{-1} t^{1-2\alpha} \rfloor, z_i := i A t^{2\alpha}. \) Without loss of generality, we
assume \( a_0 < a' < b' < b_0 \) and choose \( a'', b'' \) such that \( a' < a'' < b'' < b' \). It is not hard to see

\[
\inf_{x \in [a_0 t^\alpha, b_0 t^\alpha]} \ln \Pr_x (\forall s \leq t B_s - \beta W_s \in [at^\alpha, bt^\alpha], B_t - \beta W_t \in [a't^\alpha, b't^\alpha]|W) \geq \frac{1}{A} \frac{1}{A^{-1} t^{1-2\alpha}} \left( \sum_{i=0}^{M-1} V_i(t) + U_M(t) \right),
\]

where

\[
V_i(t) = \inf_{x \in [a_0 t^\alpha, b_0 t^\alpha]} \ln \Pr (\forall z_i \leq s \leq z_{i+1} B_s - \beta (W_s - W_{z_i}) \in [at^\alpha, bt^\alpha], B_{z_{i+1}} - \beta (W_{z_{i+1}} - W_{z_i}) \in [a''t^\alpha, b''t^\alpha]|W, B_{z_i} = x),
\]

\[
U_M(t) = \inf_{x \in [a_0 t^\alpha, b_0 t^\alpha]} \ln \Pr (\forall z_M \leq s \leq z_{M+1} B_s - \beta (W_s - W_{z_M}) \in [a't^\alpha, b't^\alpha]|W, B_{z_M} = x).
\]

Note that for each \( t > 0 \), \( U_M(t) \overset{d}{=} \inf_{x \in [a', b']} \Pr_x (\forall s \leq A B_s - \beta W_s \in [a', b']|W) \). According to Theorem 3.1, we know \( \mathbb{E}(U_M(t)) < +\infty \) for any \( j \in \mathbb{N} \), hence \( \lim_{t \to +\infty} \frac{U_M(t)}{t^{1-2\alpha}} = 0 \).

We should note that for any fixed \( t > 0 \), the sequence \( V_0(t), V_1(t), \ldots, V_{M-1}(t) \) are i.i.d. Moreover, for any fixed \( t > 0 \), \( \forall i \in [0, M-1] \cap \mathbb{N}, V_i(t) \) has the same distribution with

\[
V_0(1) = \inf_{x \in [a_0, b_0]} \Pr_x (\forall s \leq A B_s - \beta W_s \in [a, b], B_A - \beta W_A \in [a'', b'']|W).
\]

Now we use Borel-Cantelli 0-1 law to show that

\[
\lim_{t \to +\infty} \frac{\sum_{i=0}^{M-1} V_i(t)}{A^{-1} t^{1-2\alpha}} = \mathbb{E}(V_0(1)).
\]

Let \( V'_i(t) := V_i(t) - \mathbb{E}(V_i(t)) = V_i(t) - \mathbb{E}(V_0(1)) \). Choosing an even positive integer \( m \) such that \( \frac{(1-2\alpha)m}{2} > 1 \). According to (3.2), for any \( \varepsilon > 0 \), there exists a finite constant \( C \) depend on \( m \) such that

\[
\mathbb{P} \left( \left| \frac{\sum_{i=0}^{M-1} V'_i(t)}{M} \right| \geq \varepsilon \right) \leq \mathbb{E} \left( \left( \frac{\sum_{i=0}^{M-1} V'_i(t)}{M} \right)^m \right) \leq \frac{C C_M^m}{M^{m \varepsilon^m}} + o(M^{m/2}),
\]

where \( C \) is the combinatorial number. By Borel-Cantelli 0-1 law, we can obtain

\[
\lim_{t \to +\infty} \frac{\sum_{i=0}^{M-1} V_i(t)}{A^{-1} t^{1-2\alpha}} = \mathbb{E}(V_0(1)).
\]

Combining with (5.1), it implies that for any \( A > 0 \), we have

\[
\inf_{x \in [a_0 t^\alpha, b_0 t^\alpha]} \ln \Pr_x (\forall s \leq t B_s - \beta W_s \in [at^\alpha, bt^\alpha], B_t - \beta W_t \in [a't^\alpha, b't^\alpha]|W) \geq \frac{1}{A} \mathbb{E} \left( \inf_{x \in [a_0, b_0]} \Pr_x (\forall s \leq A B_s - \beta W_s \in [a, b], B_A - \beta W_A \in [a'', b'']|W) \right), \text{ a.s.}
\]
According to the $L^1$ convergence in Theorem 2.1, we get the lower bound (2.4) by taking $A \to +\infty$.

At last, we give the proof of Corollary 2.2.

**Proof of Corollary 2.2.** Recalling the notation at the beginning of section 4. The key step of this proof is to observe that for any $u, v > 0$, it always has

$$
\lim_{t \to +\infty} \inf_{x \in [a_0, b_0]} \frac{\ln r_{ut,(u+v)t}(a, b, a', b', x, \beta)}{vt} = \frac{-\gamma(\beta)}{(b-a)^2}, \text{ a.s.,} \quad (5.2)
$$

$$
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \frac{\ln r_{ut,(u+v)t}(a, b, a, b, x, \beta)}{vt} = \frac{-\gamma(\beta)}{(b-a)^2}, \text{ a.s.} \quad (5.3)
$$

Now let us first prove (5.2) and (5.3). For any $m \in \mathbb{N}$, denote $K := \lceil \frac{vt}{m} \rceil$, $z_k := ut + km, k \in \mathbb{N}$. Choosing $a'', b''$ such that $a' < a'' < b'' < b'$, by Markov property we have

$$
-\ln \inf_{x \in [a_0, b_0]} r_{ut,(u+v)t}(a, b, a', b', x, \beta)
\leq \frac{\sum_{k=0}^{K-1} G_k(t) + q_{z_K,z_K+1}(a', b', a'', b'', \beta)}{K} \times \frac{K}{vt},
$$

$$
-\ln \sup_{x \in \mathbb{R}} r_{ut,(u+v)t}(a, b, a, b, x, \beta)
\geq \frac{\sum_{k=0}^{K-1} G_k(t)}{K} \times \frac{K}{vt},
$$

Where

$$
G_k(t) := -\ln \inf_{x \in [a_0, b_0]} r_{z_k,z_k+1}(a, b, a'', b'', x, \beta),
\quad G_k(t) := -\ln \sup_{x \in \mathbb{R}} r_{z_k,z_k+1}(a, b, a, b, x, \beta).
$$

Notice that for any fixed $t > 0$, $\{G_k(t)\}_{k \in \mathbb{N}}$ and $\{G_k(t)\}_{k \in \mathbb{N}}$ are both i.i.d. sequence. And for any $t > 0, k \in [0, K - 1] \cap \mathbb{N}$,

$$
\overline{G}_k(t) \overset{\text{d}}{=} \inf_{x \in [a_0, b_0]} r_{0,m}(a, b, a'', b'', x, \beta), \quad G_k(t) \overset{\text{d}}{=} \sup_{x \in \mathbb{R}} r_{0,m}(a, b, a, b, x, \beta).
$$

Using Borel-Cantelli 0-1 law, similar with the corresponding part of the proof of Corollary 2.1, we can get (5.2) and (5.3).

Define

$$
v := \min \left\{ a_0 - f(0), \ g(0) - b_0, \ \frac{\inf_{x \in [0,1]} (g(s) - f(s))}{3} \right\}.
$$

It is obvious that $v > 0$ since $f(s), g(s)$ are both continue functions in closed interval $[0, 1]$ and $f(s) < g(s)$. Moreover, there exists $A_0 > 0$, for each $A \geq A_0, A \in \mathbb{N}$, only if $|s_1 - s_2| \leq \frac{1}{A}$, we have

$$
\max \{|f(s_1) - f(s_2)|, |g(s_1) - g(s_2)|\} < \frac{v}{2}.
$$
Denote
\[ Q_t := \sup_{x \in \mathbb{R}} \mathbb{P}^{x} \left( \forall s \leq t \, \beta W_s + f \left( \frac{s}{t} \right) \leq B_s \leq \beta W_s + g \left( \frac{s}{t} \right) \right), \]
\[ Q_t := \inf_{x \in [a_0, b_0]} \mathbb{P}^{x} \left( \forall s \leq t \, f \left( \frac{s}{t} \right) \leq B_s - \beta W_s \leq g \left( \frac{s}{t} \right), \right. \]
\[ \left. a' \leq B_t - \beta W_t \leq b' \right| W \right). \]

For \( i = 0, 1, \ldots, A - 1 \), define
\[ f_{i,A} := \inf_{s \in [it/A, (i+1)t/A]} f(s), \quad \overline{f}_{i,A} := \sup_{s \in [it/A, (i+1)t/A]} f(s); \]
\[ g_{i,A} := \inf_{s \in [it/A, (i+1)t/A]} g(s), \quad \overline{g}_{i,A} := \sup_{s \in [it/A, (i+1)t/A]} g(s). \]

By Markov property we get
\[ Q_t \leq \prod_{i=0}^{A-1} \sup_{x \in \mathbb{R}} \mathbb{P}^{x} \left( \overline{f}_{i,A}, \overline{g}_{i,A}, f_{i,A}, g_{i,A}, x, \beta \right) \tag{5.4} \]
and
\[ Q_t \geq \prod_{i=0}^{A-2} \mathbb{P}^{x} \left( \overline{f}_{i,A}, \overline{g}_{i,A}, f \left( \frac{i+1}{A} \right) + \nu, g \left( \frac{i+1}{A} \right) - \nu, x, \beta \right) \]
\[ \times \inf_{x \in [f \left( \frac{i}{A} \right) + \nu, g \left( \frac{i}{A} \right) - \nu]} \mathbb{P}^{x} \left( \overline{f}_{i+1,A}, \overline{g}_{i+1,A}, f_{i+1,A}, g_{i+1,A}, a', b', x, \beta \right). \tag{5.5} \]

Notice that
\[ \lim_{A \to +\infty} - \sum_{i=0}^{A-1} \frac{\left( g_{i,A} - \overline{f}_{i,A} \right)^2}{A} = \lim_{A \to +\infty} - \sum_{i=0}^{A-1} \frac{\left( \overline{g}_{i,A} - \overline{f}_{i,A} \right)^2}{A} = C_{f,g}, \]

Applying (5.2) (5.3) to (5.4) (5.5) we complete the proof of Corollary 2.2.

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