Berry-Esseen bounds for functionals of independent random variables

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Abstract

We derive Berry-Esseen approximation bounds for general functionals of independent random variables, based on chaos expansions methods. Our results apply to $U$-statistics satisfying the weak assumption of decomposability in the Hoeffding sense, and yield Kolmogorov distance bounds instead of the Wasserstein bounds previously derived in the special case of degenerate $U$-statistics. Linear and quadratic functionals of arbitrary sequences of independent random variables are included as particular cases, with new fourth moment bounds, and applications are given to Hoeffding decompositions, weighted $U$-statistics, quadratic forms, and random subgraph weighing. In the case of quadratic forms, our results recover and improve the bounds available in the literature, and apply to matrices with non-empty diagonals.

Keywords: Stein-Chen method; Berry-Esseen bounds; Kolmogorov distance; $U$-statistics; quadratic forms; Malliavin calculus.

Mathematics Subject Classification: 60F05; 60G57; 60H07.

1 Introduction

Significant progress in probability approximation has been achieved in recent years by combining the Chen-Stein method with the Malliavin calculus. See for example Nourdin and Peccati (2009), Peccati et al. (2010), Peccati and Thäle (2013), for the derivation of distance bounds on the Wiener and Poisson spaces, and also Nourdin et al. (2010a) and Krokowski et al. (2016) in the case of Rademacher sequences. Those results rely on covariance representations based on the inverse of the Ornstein-Uhlenbeck operator $L$ acting on multiple Wiener-Poisson stochastic integrals. While the inverse operator $L^{-1}$ is well adapted to certain random functionals such as multiple stochastic integrals, it can prove more difficult to use in applications.
to other, more specific functionals. Other covariance representations based on the Clark-Ocone representation formula and not relying on $L^{-1}$ have been used in Privault and Torrisi (2013) on the Wiener and Poisson spaces, and in Privault and Torrisi (2015) for Rademacher sequences.

In Last et al. (2016), second order Poincaré inequalities in the Kolmogorov and Wasserstein distances have been obtained for functionals of a Poisson point process by using the iterated Malliavin gradient instead of $L^{-1}$. This approach relies on probabilistic representations for the inverse operator $L^{-1}$ using Mehler’s formula on the Poisson space, see e.g. Lemma 6.8.1 in Privault (2009). Second order Poincaré inequalities for functionals of Rademacher sequences have also been obtained in Krokowski et al. (2017a), with application to renormalized triangle counting using the Kolmogorov distance in the Erdős-Rényi random graph, see also Privault and Serafin (2020a) and references therein for the treatment of arbitrary subgraph counting.

In Privault and Serafin (2018), a general framework for the derivation of Wasserstein distance bounds for functionals of independent random sequence has been developed in the integration by parts setting of Privault (1997), using an analog of the operator $L^{-1}$ on discrete chaos expansions based on discrete multiple stochastic integrals. Bounds in total variance distance have also been obtained therein using Clark-Ocone covariance representation formulas under stronger smoothness conditions. Related results have been obtained in Decreusefond and Halconruy (2019), see Theorem 5.9 therein for a normal Stein approximation bound for functionals of independent random variables, see also Nguyen (2020), and Bobkov et al. (2019) for concentration inequalities. Applications to normal approximation in the Wasserstein distance have been obtained in Privault and Serafin (2020b) for the weights of subgraphs in the Erdős-Rényi random graph.

Our first goal in this paper is to extend existing Stein normal approximation bounds proved in the Kolmogorov distance for Rademacher sequences, see e.g. Krokowski et al. (2017a), Döbler and Krokowski (2019), to general sequences of independent random variables. This is achieved in the general framework of Privault and Serafin (2018), by replacing the Wasserstein distance with the Kolmogorov distance for which obtaining rates is known to be more difficult and requires new ideas. In Theorem 4.1 we derive a general Berry-Esseen bound which is then specialized to sums of multiple stochastic integrals in Proposition 5.1 and then to multiple stochastic integrals in Proposition 5.2. Note that multiple stochastic
integrals of order $d$ coincide with degenerate (generalized) $U$-statistics of order $d$, and can then be used to represent Hoeffding decompositions as a chaos summations, see the examples given below.

Our second goal is to show that the obtained bounds remain sharp despite the very general framework of the paper, as demonstrated in the following examples. Consider a sequence $(X_1, \ldots, X_n)$ of (not necessarily identically distributed) independent random variables, and the $d$-homogeneous random multilinear forms $W_{n,d}$ written in the Hoeffding form as

$$W_{n,d} = \sum_{J \subset \{1, \ldots, n\}, |J|=d} W_J,$$

where, for each $J \subset \{1, \ldots, n\}$, $W_J$ is a random variable with variance $\sigma_J^2$, measurable with respect to the $\sigma$-algebra $\mathcal{F}_J := \sigma(X_j : j \in J)$, and such that $E[W_J | \mathcal{F}_K] = 0$, $J \not\subset K \subset [n]$. In de Jong (1990), a central limit theorem has been proved for the sequence $(W_{n,d})_{n \geq 1}$ under the conditions

$$\lim_{n \to \infty} \max_{1 \leq i \leq n} \sum_{J \ni i} \sigma_J^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} E[W_{n,d}^4] = 3,$$

generalizing earlier results by de Jong (1987) for quadratic random functionals. The results of de Jong (1987; 1990) have been refined by the derivation of bounds in the Wasserstein distance in Theorem 1.3 in Döbler and Peccati (2017) in the case of degenerate $U$-statistics, for which $|J|$ is constrained to a fixed value $|J| = d$ for some $d \in \{1, \ldots, n\}$ in the sum (6.1).

Applications of Proposition 5.1 are given to Kolmogorov distance bounds in Theorem 6.2 for general $U$-statistics, and in Theorems 6.3 and 6.4 for degenerate $U$-statistics. This extends the bounds of Döbler and Peccati (2017) by using the Kolmogorov distance instead of the Wasserstein distance, and by applying to Hoeffding decompositions in full generality and not only to degenerate $U$-statistics. This also extends the bounds in the Kolmogorov distance derived in Döbler and Krokowski (2019) for $U$-statistics in the particular case of Rademacher chaoses, where $(X_1, \ldots, X_n)$ is a sequence of independent Bernoulli random variables.

More specifically, given an i.i.d. sequence $(X_k)_{k \geq 1}$ of centered random variables with unit variance, and the sum

$$Z_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k, \quad n \geq 1,$$

convergence bounds to the standard normal distribution $\mathcal{N}$ of the form

$$d_W(Z_n, \mathcal{N}) \leq \frac{E[|X_1|^3]}{\sqrt{n}}.$$
have been obtained in e.g. Theorem 1.1 in Goldstein (2010) in the Wasserstein distance

\[ d_W(X, N) := \sup_{h \in \text{Lip}(1)} |E[h(X)] - E[h(N)]|. \]

See also Corollary 2.11 of Döbler (2015) for related bounds in the Kolmogorov distance

\[ d_K(X, N) := \sup_{x \in \mathbb{R}} |P(X \leq x) - P(N \leq x)|, \]

including the case of random sums. In the case of quadratic functionals of the form

\[ Q_n := \sum_{1 \leq k,l \leq n} a_{kl} X_k X_l, \tag{1.1} \]

where \( A = (a_{ij})_{1 \leq i,j \leq n} \) is a symmetric matrix, the bound

\[ d_K(Q_n, N) \leq C \left( E \left[ X_1^3 \right] \right)^2 |\lambda_1|, \tag{1.2} \]

where \( \lambda_1 \) denotes the largest absolute eigenvalue of \( A \) and \( C > 0 \) is an absolute constant, has been obtained in Götze and Tikhomirov (1999) when the diagonal of \( A \) vanishes, see e.g. Theorem 1 therein, and also Theorem 3.1 of Shao and Zhang (2019).

In this vanishing diagonals setting, Theorem 6.4 is applied to derive Corollary 7.1 which recovers Theorem 3.1 in Shao and Zhang (2019), and improves on the above bound (1.2) of Theorem 1 in Götze and Tikhomirov (1999). In addition, Corollary 7.1 extends the Kolmogorov bounds of Theorem 1.1 in Döbler and Krokowski (2019), restricted to the quadratic case, from Rademacher sequences to general sequences of random variables by using fourth moment differences as in e.g. Theorem 1.3 of Döbler and Peccati (2017).

In case the diagonal of \( A = (a_{ij})_{1 \leq i,j \leq n} \) may not vanish, the bound

\[ d_K \left( \frac{Q_n}{\sigma_n}, N \right) \leq C(\gamma) \left( \frac{E[|X|^3]}{\gamma E[X^6]} + \gamma \frac{E[X^6]}{\sum_{1 \leq i,j \leq n} a_{ij}^2} \right)^2 |\lambda_1|, \tag{1.3} \]

has been obtained in Theorem 1.1 of Götze and Tikhomirov (2002) for some \( \gamma > 0 \) depending on \( A \). See also Proposition 3.1 in Chatterjee (2008) for a result in the Wasserstein distance using Rademacher sequences, and Theorem 2.2 in Chatterjee (2009) for related normal approximation bounds in total variation distance for a smooth function of finite-dimensional random vectors via second order Poincaré inequalities.

In comparison with Theorem 1.1 of Götze and Tikhomirov (2002), the bound (7.7) in Theorem 7.2 gives better rates under weaker assumptions according to the inequality (7.5).
Theorem 7.2 also provides an additional bound (7.6) which is valid for any i.i.d. sequence \((X_n)_{n \geq 1}\) and holds in the Kolmogorov distance, instead of the Wasserstein distance used in Dörner and Peccati (2017). This bound is related to the so-called fourth moment phenomenon (Nualart and Peccati (2004)), which has been the object of intense research work, see e.g. Nourdin and Peccati (2012) and references therein.

We proceed as follows. In Section 2 we recall the framework of Privault (1997) for the treatment of functionals of independent random sequences, including the construction of discrete multiple stochastic integrals and the associated finite difference gradient operator and integration by parts formula, which are used to derive a fourth moment bound in Section 3. Section 4 contains our main result Theorem 4.1 which states a general Berry-Esseen bounds for general functionals of independent random sequences. In Section 5 those results are applied to the derivation of Kolmogorov bounds for discrete multiple integrals and for sums of discrete multiple integrals. Applications to Hoeffding decompositions, weighted \(U\)-statistics and subgraph counting in the Erdős and Rényi (1959) random graph are given in Section 6. Section 7 focuses on quadratic forms.

2 Preliminaries and notation

Consider an i.i.d. sequence \((U_k)_{k \geq 1}\) of uniformly distributed random variables on the interval \([-1, 1]\), on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Given \(F(U_1(\omega), U_2(\omega), \ldots)\) a functional of the sequence \((U_1(\omega), U_2(\omega), \ldots)\), we consider the shifted sequence \(\Phi_t(\omega)\) defined as

\[
\Phi_t(\omega) := (U_1(\omega), \ldots, U_{\lfloor t/2 \rfloor}(\omega), t - 1 - 2[t/2], U_{\lfloor t/2 \rfloor+2}(\omega), \ldots), \quad t \in \mathbb{R}_+,
\]

and define the finite difference gradient operator \(\nabla\) on random functionals as

\[
\nabla_t F := F \circ \Phi_t - \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} F \circ \Phi_s ds, \quad t \in \mathbb{R}_+, \quad (2.1)
\]

provided that \((F \circ \Phi_s)_{s \in \mathbb{R}_+}\) is integrable on \(\mathbb{R}_+, \mathbb{P}\)-a.s., see Definition 5 and Proposition 10 in Privault (1997).

For any \(X \in L^1(\Omega)\) and \(k \in \mathbb{N}\) we also note the identity

\[
\mathbb{E}[X] = \frac{1}{2} \mathbb{E} \left[ \int_{2k}^{2k+2} X \circ \Phi_u du \right]. \quad (2.2)
\]

Consider now the adjoint \(\nabla^*\) of \(\nabla\), defined by the duality relation

\[
\mathbb{E} \left[ \langle \nabla X, u \rangle_{L^2(\mathbb{R}_+, dx/2)} \right] = \mathbb{E}[X \nabla^*(u)], \quad (2.3)
\]
which shows that $\nabla^*$ and $\nabla$ are closable with domains $\text{Dom}(\nabla^*) \subset L^2(\Omega)$ and $\text{Dom}(\nabla) = \{ X \in L^2(\Omega) : E[\|\nabla X\|_{L^2(\mathbb{R}^+)}^2] < \infty \} \subset L^2(\Omega \times \mathbb{R}^+_+)$, see Proposition 8 in Privault (1997). The operators $(\nabla, \nabla^*)$ are linked by the Skorohod isometry
\[
\mathbb{E}[\nabla^* u \nabla^* v] = \mathbb{E} \left[ \int_0^\infty u_t v_t dt \right] + \mathbb{E} \left[ \int_0^\infty \int_0^\infty \nabla_s u_t \nabla_t v_s ds dt \right],
\]
see Proposition 9 in Privault (1997), which yields the Poincaré inequality
\[
\mathbb{E}[|\nabla^* u|^2] \leq \mathbb{E} \left[ \int_0^\infty |u_t|^2 dt \right] + \mathbb{E} \left[ \int_0^\infty \int_0^\infty |\nabla_s u_t|^2 ds dt \right].
\]

**Definition 2.1** Given $f_n$ in the space $\hat{L}^2(\mathbb{R}^+_n)$ of square integrable symmetric functions on $\mathbb{R}^+_n$ that vanish outside of
\[
\Delta_n := \bigcup_{k_1 \neq k_j \geq 1} [2k_1 - 2, 2k_1] \times \cdots \times [2k_n - 2, 2k_n],
\]
we define the multiple stochastic integral
\[
I_n(f_n) = n! \int_0^\infty \cdots \int_0^\infty f_n(t_1, \ldots, t_n) d(Y_{t_1} - t_1/2) \cdots d(Y_{t_n} - t_n/2),
\]
with respect to the jump process $Y_t := \sum_{k=1}^\infty 1_{[2k-1+U_k,\infty)}(t), \ t \in \mathbb{R}_+$, which satisfies
\[
I_n(f_n) := \sum_{r=0}^n \left( -\frac{1}{2} \right)^{n-r} \binom{n}{r} \prod_{k=1}^n \int_0^\infty \cdots \int_0^\infty f_n(2k_1 - 1 + U_{k_1}, \ldots, 2k_r - 1 + U_{k_r}, y_1, \ldots, y_{n-r}) dy_1 \cdots dy_{n-r}.
\]

The multiple stochastic integral $I_n(f_n)$ satisfies the bound
\[
\mathbb{E} \left[ (I_n(f_n))^2 \right] \leq n! \|f_n\|^2_{L^2(\mathbb{R}^+_n, dx/2)}, \quad n \geq 1,
\]
which allows us to extend the definition of $I_n(f_n)$ to all $f_n \in \hat{L}^2(\mathbb{R}^+_n)$, see Propositions 4 and 6 in Privault (1997). Under the additional condition
\[
\int_{2k-2}^{2k} f_n(t, \ast) dt = 0, \quad k \geq 1,
\]
i.e. $f_n$ is canonical in the sense of Surgailis (2003), the multiple stochastic integral $I_n(f_n)$ can be written as the $U$-statistics of order $n$

$$I_n(f_n) = \sum_{k_1 \neq \cdots \neq k_n \geq 1} f_n(2k_1 - 1 + U_1, \ldots, 2k_n - 1 + U_n),$$

with the isometry and orthogonality relation

$$\mathbb{E}[I_n(f_n)I_m(f_m)] = \mathbf{1}_{\{n=m\}} n! \langle f_n, f_m \rangle_{L^2(\mathbb{R}^n_+, dx/2)^n}, \quad f_n \in \hat{L}^2(\mathbb{R}_+^n), \quad f_m \in \hat{L}^2(\mathbb{R}_+^m), \quad (2.7)$$

see Privault (1997), page 589, which shows that the sequence $(I_n(f_n))_{n \geq 1}$ forms a family of mutually orthogonal centered random variables. Finally, every $X \in L^2(\Omega)$ admits the chaos decomposition

$$X = E[X] + \sum_{n=1}^{\infty} I_n(f_n), \quad (2.8)$$

for some sequence of functions $f_n$ in $\hat{L}^2(\mathbb{R}_+^n)$, $n \geq 1$, cf. Proposition 7 in Privault (1997). Moreover, under the condition (2.6) the sequence $(f_n)_{n \geq 1}$ is unique in $\hat{L}^2(\mathbb{R}_+^n)$ due to the isometry relation (2.7), and in this case we have

$$\mathbb{E}[X^2] = (\mathbb{E}[F])^2 + \sum_{n=1}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}^n_+, (dx/2)^n)}. \quad (2.9)$$

Under the condition (2.6) we also have the relations

$$\nabla^* (I_n(g_{n+1})) := I_{n+1}(1_{\Delta_{n+1}} \tilde{g}_{n+1}) \quad \text{and} \quad \nabla_t I_n(f_n) = nI_{n-1}(f_n(t, *)) \quad t \in \mathbb{R}_+, \quad (2.10)$$

see Proposition 10 in Privault (1997), where $\tilde{g}_{n+1}$ is the symmetrization of $g_{n+1} \in \hat{L}^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+)$ in $n + 1$ variables. The operator $L$ defined on linear combinations of multiple stochastic integrals as

$$LI_n(f_n) := -\nabla^* \nabla_t I_n(f_n) = -nI_n(f_n), \quad f_n \in \hat{L}^2(\mathbb{R}_+^n),$$

is called the Ornstein-Uhlenbeck operator. By (2.8) the operator is invertible for centered $X \in L^2(\Omega)$, and its inverse operator $L^{-1}$ is given by

$$L^{-1}I_n(f_n) = -\frac{1}{n} I_n(f_n), \quad n \geq 1. \quad (2.11)$$

In fact, we can easily derive the form of any real power of $-L$, i.e. it holds

$$(-L)^\alpha I_n(f_n) = n^\alpha I_n(f_n), \quad n \geq 1, \quad \alpha \in \mathbb{R}. \quad (2.12)$$
We also recall that, by Proposition 5.3 in Privault and Serafin (2020b), for every $f_n \in \widehat{L}^2(\mathbb{R}^n_+)$ there exists $\bar{f}_n \in \widehat{L}^2(\mathbb{R}^n_+)$ given by

$$\bar{f}_n(t_1, \ldots, t_n) = \Psi_{t_1} \cdots \Psi_{t_n} f_n(t_1, \ldots, t_i),$$

satisfying (2.6) and such that $I_n(f_n) = I_n(\bar{f}_n)$, where

$$\Psi_{t_i} f(t_1, \ldots, t_n) := f(t_1, \ldots, t_n) - \frac{1}{2} \int_{2[t_i/2]}^{2[t_i/2]+2} f(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n) ds,$$

$i = 1, \ldots, n, t_1, \ldots, t_n \in \mathbb{R}_+$. We end this section with the following multiplication formula for multiple stochastic integrals, see Proposition 5.1 in Privault and Serafin (2018). Letting $n \wedge m := \min(n, m)$, for $0 \leq l \leq k \leq n \wedge m$ we define the contraction $f_n \star_k^l g_m$ of $f_n \in \widehat{L}^2(\mathbb{R}^n_+)$ and $g_m \in \widehat{L}^2(\mathbb{R}^m_+)$ as

$$f_n \star_k^l g_m(y_1, \ldots, y_{n-l}, z_1, \ldots, z_{m-k})$$

$$:= \frac{1}{2l} \int_{\mathbb{R}^l_+} f_n(x_1, \ldots, x_l, y_1, \ldots, y_{n-l}) g_m(x_1, \ldots, x_l, y_{l+1}, \ldots, y_{n-l}, z_1, \ldots, z_{m-k}) dx_1 \cdots dx_l,$$

and we let $f_n \hat{\star}_k^l g_m$ denote the symmetrization

$$f_n \hat{\star}_k^l g_m(x_1, \ldots, x_{n+m-k-l})$$

$$:= \frac{1}{2l} \Delta_{m+n-k-l}(x_1, \ldots, x_{n+m-k-l}) \frac{(m+n-k-l)!}{(m-n-k-l)!} \sum_{\sigma \in \Sigma_{m+n-k-l}} f_n \star_k^l g_m(x_{\sigma(1)}, \ldots, x_{\sigma(m+n-k-l)}).$$

Then, for $f_n \in \widehat{L}^2(\mathbb{R}^n_+)$ and $g_m \in \widehat{L}^2(\mathbb{R}^m_+)$ satisfying (2.6), the following multiplication formula holds:

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} I_{m+n-k-i}(f_n \hat{\star}_k^i g_m),$$

whenever $f_n \star_k^i g_m \in L^2(\mathbb{R}^{m+n-k-i}_+)$ for every $0 \leq i \leq k \leq m \wedge n$.

\section{Fourth moment bound}

The next covariance relation can be obtained as in Proposition 2.1 in Houdré and Privault (2002).

**Proposition 3.1** Let $\alpha \in \mathbb{R}$ and $X, Y \in L^2(\Omega)$ such that $L^{\alpha-1}X \in \text{Dom}(\nabla)$ and $L^{-\alpha}Y \in \text{Dom}(\nabla)$. Then we have the covariance relation

$$\text{Cov}(X, Y) = \frac{1}{2} \mathbb{E} \left[ \int_0^\infty (\nabla_t(-L)^{\alpha-1}X)(\nabla_t(-L)^{-\alpha}Y) \frac{dt}{2} \right],$$

(3.1)
Proof. We have

\[
\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]
\]

\[
= -\mathbb{E}[t(-L)^{\alpha-1}(X - \mathbb{E}[X])(-L)^{-\alpha}(Y - \mathbb{E}[Y])]
\]

\[
= \mathbb{E}[\nabla^*\nabla(-L)^{\alpha-1}(X - \mathbb{E}[X])(-L)^{-\alpha}(Y - \mathbb{E}[Y])]
\]

\[
= \frac{1}{2}\mathbb{E}\left[\int_0^\infty (\nabla(-L)^{\alpha-1}X)(\nabla(-L)^{-\alpha}Y)dt\right].
\]

\[
\square
\]

Although \(\nabla_t\) does not satisfy the chain rule of derivation, we have the following lemma.

**Lemma 3.2** The finite difference operator \(\nabla\) satisfies the relation

\[
\nabla_t(FG) = (F \circ \Phi_t)\nabla_tG + (G \circ \Phi_t)\nabla_tF - \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} (\nabla_tF\nabla_tG + \nabla_uF\nabla_uG)du,
\]

(3.2)

\(t \in \mathbb{R}_+\), provided that \((F \circ \Phi_s)_{s \in \mathbb{R}_+}\), \((G \circ \Phi_s)_{s \in \mathbb{R}_+}\) and \((F^2 \circ \Phi_s)_{s \in \mathbb{R}_+}\), \((G^2 \circ \Phi_s)_{s \in \mathbb{R}_+}\) are integrable on \([2n - 2, 2n]\), \(n \geq 1\), \(\mathbb{P}\)-a.s.

**Proof.** By (2.1), we have

\[
\nabla_t(FG) = \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} ((FG) \circ \Phi_t - (FG) \circ \Phi_u)du
\]

\[
= \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} (F \circ \Phi_u)(G \circ \Phi_t - G \circ \Phi_u)du + \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} (G \circ \Phi_t)(F \circ \Phi_t - F \circ \Phi_u)du
\]

\[
= \frac{1}{2}(F \circ \Phi_t)\int_{2[2/t/2]}^{2[t/2]+2} (G \circ \Phi_t - G \circ \Phi_u)du + \frac{1}{2}(G \circ \Phi_t)\int_{2[2/t/2]}^{2[t/2]+2} (F \circ \Phi_t - F \circ \Phi_u)du
\]

\[
- \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} (F \circ \Phi_t - F \circ \Phi_u)(G \circ \Phi_t - G \circ \Phi_u)du
\]

\[
= (F \circ \Phi_t)\nabla_tG + (G \circ \Phi_t)\nabla_tF - \frac{1}{2}\int_{2[2/t/2]}^{2[t/2]+2} (F \circ \Phi_t - F \circ \Phi_u)(G \circ \Phi_t - G \circ \Phi_u)du.
\]

Furthermore, we have

\[
\int_{2[2/t/2]}^{2[t/2]+2} (F \circ \Phi_t - F \circ \Phi_u)(G \circ \Phi_t - G \circ \Phi_u)du = \int_{2[2/t/2]}^{2[t/2]+2} (\nabla_tF - \nabla_uF)(\nabla_tG - \nabla_uG)du
\]

\[
= \int_{2[2/t/2]}^{2[t/2]+2} (\nabla_tF\nabla_tG + \nabla_uF\nabla_uG)du,
\]

from the equality \(\int_{2[2/t/2]}^{2[t/2]+2} \nabla_uFdu = 0.\)

\[
\square
\]
The next result is a fourth order moment bound stated in terms of the gradient operator $\nabla$.

**Proposition 3.3** For any $X \in L^4(\Omega)$ we have

$$
\mathbb{E} [X^4] \leq 36 \mathbb{E} [\| \nabla X \|_{L^2(\mathbb{R}^+)}^4] + 15 \mathbb{E} [\| \nabla X \|_{L^4(\mathbb{R}^+)}^4] + 2 (\mathbb{E} [X^2])^2.
$$

(3.3)

**Proof.** By the covariance relation (3.1), we have

$$
\mathbb{E} [X^4] = \text{Var} [X^2] + (\mathbb{E} [X^2])^2
$$

$$
= \frac{1}{2} \mathbb{E} \left[ \int_{0}^{\infty} \nabla_t (X^2 - \mathbb{E} [X^2]) \nabla_t L^{-1} (X^2 - \mathbb{E} [X^2]) \, dt \right] + (\mathbb{E} [X^2])^2
$$

$$
\leq \frac{1}{2} \sqrt{\mathbb{E} \left[ \int_{0}^{\infty} \| \nabla_t (X^2) \|^2 dt \right] \mathbb{E} \left[ \int_{0}^{\infty} \| \nabla_t L^{-1} (X^2 - \mathbb{E} [X^2]) \|^2 dt \right]} + (\mathbb{E} [X^2])^2
$$

$$
\leq \frac{1}{2} \mathbb{E} \left[ \int_{0}^{\infty} \| \nabla_t (X^2) \|^2 dt \right] + (\mathbb{E} [X^2])^2,
$$

where we applied (2.10), (2.9) and (2.11). Since

$$(X \circ \Phi_t) \nabla_t X = X \nabla_t X + (X \circ \Phi_t - X) \nabla_t X$$

$$= X \nabla_t X + \left( \nabla_t X - \left( X - \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} X \circ \Phi_u du \right) \right) \nabla_t X,$$

by the relations (3.2) and (2.2) and the bound $(a + b + c)^2 \leq 3 (a^2 + b^2 + c^2)$, we have

$$
\mathbb{E} \left[ \int_{0}^{\infty} \| \nabla_t (X^2) \|^2 dt \right] = \mathbb{E} \left[ \int_{0}^{\infty} \left( 2 (X \circ \Phi_t) \nabla_t X - \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} (\| \nabla_t X \|^2 + \| \nabla_u X \|^2) du \right) \, dt \right]
$$

$$\leq 3 \mathbb{E} \left[ 4 \int_{0}^{\infty} (X \nabla_t X)^2 dt + 4 \int_{0}^{\infty} \left( \left( \nabla_t X - \left( X - \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} X \circ \Phi_u du \right) \right) \nabla_t X \right)^2 dt
$$

$$+ \frac{1}{4} \int_{0}^{\infty} \left( \int_{2[t/2]}^{2[t/2]+2} (\| \nabla_t X \|^2 + \| \nabla_u X \|^2) du \right) \, dt \right]
$$

$$= 12 \mathbb{E} \left[ \int_{0}^{\infty} (X \nabla_t X)^2 dt \right]
$$

$$+ 12 \mathbb{E} \left[ \int_{0}^{\infty} \int_{2[t/2]}^{2[t/2]+2} \left( \left( \nabla_t X - \left( X - \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} X \circ \Phi_u du \right) \right) \nabla_t X \right)^2 dv \, dt \right]
$$

$$+ \frac{3}{4} \mathbb{E} \left[ \int_{0}^{\infty} \left( \int_{2[t/2]}^{2[t/2]+2} (\| \nabla_t X \|^2 + \| \nabla_u X \|^2) du \right) \, dt \right]
$$

$$= 12 \mathbb{E} \left[ X^2 \int_{0}^{\infty} (\nabla_t X)^2 dt \right] + 12 \mathbb{E} \left[ \int_{2[t/2]}^{2[t/2]+2} (\nabla_t X - \nabla_v X \nabla_t X)^2 dv \, dt \right],
$$
\[ + \frac{3}{4} \mathbb{E} \left[ \int_0^\infty \left( \int_{2[t/2]}^{2[t/2]+2} (|\nabla_t X|^2 + |\nabla_u X|^2) du \right)^2 dt \right] \]

\[ \leq 12 \mathbb{E}[X^4] \mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|^2 dt \right)^2 \right] + 12 \mathbb{E} \left[ \int_0^\infty |\nabla_t X|^2 \int_{2[t/2]}^{2[t/2]+2} (|\nabla_t X|^2 + |\nabla_u X|^2) dv \frac{dv}{2} dt \right] \]

\[ + 3 \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] \]

\[ \leq 12 \mathbb{E}[X^4] \mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|^2 dt \right)^2 \right] + 15 \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right]. \]

Thus, we get

\[ \mathbb{E} [X^4] \leq 6 \sqrt{\mathbb{E}[X^4] \mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|^2 dt \right)^2 \right]} + \frac{15}{2} \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] + (\mathbb{E} [X^2])^2. \]

Denoting

\[ a = \sqrt{\mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|^2 dt \right)^2 \right]}, \quad b = \frac{15}{2} \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] + (\mathbb{E} [X^2])^2 \]

and \( x = \sqrt{\mathbb{E} [X^4]} \), we rewrite the last inequality as \( x^2 \leq 6ax + b \), which gives \( x \leq 3a + \sqrt{9a^2 + b} \) and consequently \( x^2 \leq 2(9a^2 + b) + 18a^2 = 36a^2 + 2b \), which yields (3.3). \( \blacksquare \)

## 4 Berry-Esseen bound

Our main result is a Berry-Esseen bound on the Kolmogorov distance \( d_K(X, \mathcal{N}) \) between the standard normal distribution \( \mathcal{N} \) on \( \mathbb{R} \) and a general functional \( X \) of the uniform i.i.d. sequence \( (U_k)_{k \in \mathbb{N}} \) on \([-1, 1] \), using the operators \( \nabla \) and \( L \). This result extends Proposition 4.1 in Krokowski et al. (2017b), see also Theorem 3.1 in Krokowski et al. (2016) and Proposition 2.1 in Privault and Serafin (2020a), from functionals of Bernoulli sequences to more general functionals of independent random variables.

**Theorem 4.1** Let \( X \in \text{Dom}(\nabla) \) be such that \( \mathbb{E}[X] = 0 \). We have

\[ d_K(X, \mathcal{N}) \leq |1 - \mathbb{E}[X^2]| + \sqrt{\text{Var} \left[ \int_0^\infty \nabla_t X \nabla_t L^{-1} X dt \right]} \]

\[ + \frac{3}{2} \sqrt{\mathbb{E} \int_0^\infty (\nabla_t X)^4 dt} \left( \mathbb{E} [X^4] \mathbb{E} \left[ \left( \int_0^\infty |\nabla L^{-1} X|^2 dt \right)^2 \right] \right)^{1/4} + \frac{\sqrt{\pi}}{2} \sqrt{\mathbb{E} \left[ ((-L)^{-1/2} X)^2 \right]} \]

(4.1)
+ 4 \left( \mathbb{E} \left[ \int_0^\infty \left( (I + 2(-L)^{1/2}) \left( |\nabla_t X|^2 \right) \right)^2 dt \right] \right)^{1/4}.

\textbf{Proof.} For any \( x \in \mathbb{R} \), let \( f_x \) denote the unique bounded solution of the Stein equation

\[ f'_x(z) - zf_x(z) = 1_{\{z \leq x\}} - \mathbb{P}(N \leq x), \tag{4.2} \]

which is continuous, infinitely differentiable on \( \mathbb{R} \setminus \{x\} \), and satisfies \( 0 < f_x(y) < \sqrt{\pi/8} \) and \( |f'_x(y)| \leq 1 \), \( y \in \mathbb{R} \), see Lemmas 2.2 and 2.3 in Chen et al. (2011). From the Stein equation (4.2) we have the bound

\[ d_K(X, N) \leq \sup_{x \in \mathbb{R}} \mathbb{E}[f'_x(X) - X f_x(X)]. \]

For every \( f \in C^1(\mathbb{R}) \), the finite difference operator \( \nabla \) satisfies

\[
\nabla_t f(X) = \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} (f(X \circ \Phi_t) - f(X \circ \Phi_s)) ds \\
= \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} \int_{X \circ \Phi_t - X}^{X \circ \Phi_s - X} f'(X + u) dus \\
= \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} \left( \int_{X \circ \Phi_t - X}^{X \circ \Phi_s - X} f'(X + u) - f'(X) du \right) ds \\
= f'(X) \nabla_t X + \frac{1}{2} \int_{2[t/2]}^{2[t/2]+2} \int_{X \circ \Phi_t - X}^{X \circ \Phi_s - X} (f'(X + u) - f'(X)) dus ds, \quad t \in \mathbb{R}_+. 
\]

hence by the duality relation (2.3), we have

\[
\mathbb{E}[f'(X) - X f(X)] = \mathbb{E}[f'(X) - f(X)(-\nabla \nabla^{-1} L X)] \\
= \mathbb{E} \left[ f'(X) - \frac{1}{2} \int_0^\infty \nabla_t f(X)(-\nabla_t L^{-1} X) dt \right] \\
= \mathbb{E} \left[ f'(X) \left( 1 - \frac{1}{2} \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) dt \right) \right] \\
+ \frac{1}{4} \mathbb{E} \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} \int_{X \circ \Phi_t - X}^{X \circ \Phi_s - X} (f'(X + u) - f'(X)) dus \nabla_t L^{-1} X dt \right].
\]

By the covariance relation (3.1) applied with \( \alpha = 0 \) and the fact that \( \mathbb{E}[X] = 0 \), we have

\[
\mathbb{E} [X^2] = \mathbb{E} \left[ \int_0^\infty (\nabla_t X)(-\nabla_t L^{-1} X) dt \right],
\]

hence from the bound \( \|f'_x\|_\infty \leq 1 \) and Jensen’s inequality we obtain

\[
\left| \mathbb{E} \left[ f'(X) \left( 1 - \frac{1}{2} \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) dt \right) \right] \right|
\]
\[
\begin{align*}
&\leq \mathbb{E}\left[ 1 - \frac{1}{2} \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) dt \right] \\
&\leq |1 - \mathbb{E}[X^2]| + \mathbb{E}\left[ \frac{1}{2} \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) dt - \mathbb{E} \left[ \int_0^\infty (\nabla_t X)(-\nabla_t L^{-1} X) \frac{dt}{2} \right] \right] \\
&\leq |1 - \mathbb{E}[X^2]| + \text{Var} \left[ \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) \frac{dt}{2} \right].
\end{align*}
\]

Next, from the Stein equation (4.2) we have
\[
\int_{X \circ \Phi_t - X}^{X \circ \Phi_t - X} (f'_x(X + u) - f'_x(X)) du = A_{s,t}(x, X) + B_{s,t}(x, X), \quad x \in \mathbb{R},
\]
where
\[
A_{s,t}(x, X) := \int_{X \circ \Phi_t - X}^{X \circ \Phi_t - X} ((X + u)f_x(X + u) - Xf_x(X)) du
\]
and
\[
B_{s,t}(x, X) := \int_{X \circ \Phi_t - X}^{X \circ \Phi_t - X} (1_{\{X + u \leq x\}} - 1_{\{X \leq x\}}) du.
\]
Thus, we get
\[
|\mathbb{E}[f'(X) - Xf(X)]| \leq |1 - \mathbb{E}[X^2]| + \text{Var} \left[ \int_0^\infty \nabla_t X(-\nabla_t L^{-1} X) \frac{dt}{2} \right], \quad (4.3)
\]
\[
+ \frac{1}{4} \left| \mathbb{E} \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} A_{s,t}(x, X) ds \nabla_t L^{-1} X dt \right] \right| \\
+ \frac{1}{4} \left| \mathbb{E} \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} B_{s,t}(x, X) ds \nabla_t L^{-1} X dt \right] \right|.
\]

Using the inequality
\[
|(u + w)f_x(u + w) - w f_x(w)| \leq \left( |w| + \frac{\sqrt{2\pi}}{4} \right)|u|, \quad u, w \in \mathbb{R},
\]
see Lemma 2.3 in Chen et al. (2011), we estimate
\[
|A_{s,t}(x, X)| \leq \int_{\min(X \circ \Phi_s - X, X \circ \Phi_t - X)}^{\max(X \circ \Phi_s - X, X \circ \Phi_t - X)} \left( |X| + \frac{\sqrt{2\pi}}{4} \right)|u| du
\]
\[
\leq \left( \frac{\sqrt{2\pi}}{4} + |X| \right) \int_{|X \circ \Phi_t - X|}^{|X \circ \Phi_t - X|} |u| du
\]
\[
= \frac{1}{2} \left( \frac{\sqrt{2\pi}}{4} + |X| \right) (|X \circ \Phi_s - X|^2 + |X \circ \Phi_t - X|^2).
\]

Then, by the Cauchy-Schwarz inequality we have
\[
\mathbb{E} \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} A_{s,t}(x, X) ds \nabla_t L^{-1} X dt \right]
\]
\[
\begin{align*}
&\leq \frac{1}{2} \mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \left( \frac{\sqrt{2\pi}}{4} + |X| \right) \left( |X \circ \Phi_t - X|^2 + |X \circ \Phi_s - X|^2 |\nabla_t L^{-1}X| ds \right) dt \right] \\
&\leq \frac{1}{2} \sqrt{\mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \left( |X \circ \Phi_t - X|^2 + |X \circ \Phi_s - X|^2 \right)^2 ds \right] dt} \\
&\quad \times \left( \sqrt{2\mathbb{E} \left[ \int_0^\infty (X \nabla_t L^{-1}X)^2 dt \right]} + \frac{\sqrt{\pi}}{2} \sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t L^{-1}X)^2 dt \right]} \right). 
\end{align*}
\]

Next, by the inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), formula (3.1) with \(\alpha = 0\) and the relation (2.2), we get
\[
\begin{align*}
&\mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \left( |X \circ \Phi_t - X|^2 + |X \circ \Phi_s - X|^2 \right)^2 ds \ dt \right] \\
&\quad = \frac{1}{2} \mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \int_2^{2[t/2]+2} \left( |X \circ \Phi_t - X \circ \Phi_v|^2 + |X \circ \Phi_s - X \circ \Phi_v|^2 \right)^2 dv \ dt \right] \\
&\quad = \frac{1}{2} \mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \int_2^{2[t/2]+2} \left( |\nabla_t X - \nabla_v X|^2 + |\nabla_s X - \nabla_v X|^2 \right)^2 dv \ dt \right] \\
&\quad \leq \mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \int_2^{2[t/2]+2} \left( |\nabla_t X|^2 + |\nabla_s X|^2 + 2|\nabla_v X|^2 \right)^2 dv \ dt \right] \\
&\quad \leq 3 \mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} \int_2^{2[t/2]+2} \left( \nabla_t X \right)^4 + \left( \nabla_s X \right)^4 + 4(\nabla_v X)^4 dv \ dt \right] \\
&\quad = 72 \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right].
\end{align*}
\]

Furthermore, by Proposition 3.1 applied to \(X\) and \((-L)^{-1}X\) with \(\alpha = 1/2\), we have
\[
\mathbb{E} \left[ \int_0^\infty (\nabla_t L^{-1}X)^2 dt \right] = 2 \mathbb{E} [((-L)^{-1/2}X)^2]
\]
and
\[
\mathbb{E} \left[ \int_0^\infty (X \nabla_t L^{-1}X)^2 dt \right] \leq \sqrt{\mathbb{E} [X^4] \mathbb{E} \left[ \left( \int_0^\infty (\nabla_t L^{-1}X)^2 dt \right)^2 \right]}.
\]

Applying the last three inequalities to (4.4), we finally obtain
\[
\begin{align*}
&\mathbb{E} \left[ \int_0^\infty \int_2^{2[t/2]+2} A_{s,t}(x, X) ds \nabla_t L^{-1}X dt \right] \\
&\quad \leq 6 \sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] \left( \mathbb{E} [X^4] \mathbb{E} \left[ \left( \int_0^\infty (\nabla_t L^{-1}X)^2 dt \right)^2 \right] \right)^{1/4} + \frac{\sqrt{\pi}}{2} \sqrt{\mathbb{E} [((-L)^{-1/2}X)^2]}).
\end{align*}
\]
Regarding the last term in (4.3), we use (2.2) and the equivalence \( \nabla_t L^{-1} X \circ \Phi_v = (\nabla_t L^{-1} X) \), which is valid for \( 2[t/2] \leq v < 2[t/2] + 2 \), and get

\[
\left| E \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} B_{s,t}(x, X) ds \nabla_t L^{-1} X \, dt \right] \right|
\]

\[
= \left| E \left[ \int_0^\infty \int_{2[t/2]}^{2[t/2]+2} \left( \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{x \leq s\}}) du \right) ds \nabla_t L^{-1} X \, dt \right] \right|
\]

\[
= \frac{1}{2} \left| \int_0^\infty \left( \int_{2[t/2]}^{2[t/2]+2} \int_{2[t/2]}^{2[t/2]+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{x \leq s\}}) du \, dv \, ds \right) \nabla_t L^{-1} X \, dt \right| \]

\[
= \frac{1}{2} \left| \sum_{m=0}^{2m+2} K_m(t, X) \nabla_t L^{-1} X \, dt \right|
\]

(4.5)

where

\[
K_m(t, x, X) := \int_{2m}^{2m+2} \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{x \leq s\}}) du \, ds \, dv, \quad 2m \leq t < 2m + 2.
\]

Next, we rewrite \( K_m(t, X) \) as follows

\[
K_m(t, x, X) = \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{X \circ \Phi_t \leq x\}} - 1_{\{X \circ \Phi_s \leq x\}}) \, dv \, du \, ds
\]

\[
+ \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{X \circ \Phi_t \leq x\}}) \, dv \, du \, ds
\]

\[= 4 \nabla_t X \nabla_t 1_{\{X \leq x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{X \circ \Phi_t \leq x\}}) \, dv \, du \]  

\[= -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} (1_{\{u \leq x\}} - 1_{\{X \circ \Phi_t \leq x\}}) \, dv \, du, \quad (4.6)
\]

where we used the equality \( \nabla_t 1_{\{X \leq x\}} = -\nabla_t 1_{\{X > x\}} \). Next, we consider two cases.

(i) If \( X \circ \Phi_t > x \), we have

\[
K_m(t, x, X) = -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t}^{X \circ \Phi_s} 1_{\{u \leq x\}} \, dv \, du \, ds
\]

\[= -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t} X \circ \Phi_s (x - X \circ \Phi_t) \, dv \, ds. \quad (4.7)
\]

Note that the last expression depends only on \( m := \lfloor t/2 \rfloor \) and may be bounded for \( x < \int_{2m}^{2m+2} X \circ \Phi_t \, du/2 \) as follows

\[
0 \leq \int_{2m}^{2m+2} 1_{\{X \circ \Phi_s \leq x\}} (x - X \circ \Phi_t) \, ds
\]

\[= x \int_{2m}^{2m+2} 1_{\{X \circ \Phi_s \leq x\}} \, ds - \int_{2m}^{2m+2} X \circ \Phi_t \, du + \int_{2m}^{2m+2} 1_{\{X \circ \Phi_s > x\}} X \circ \Phi_t \, du
\]
which shows that the inequality 

$$X > x$$

Consequently, for 

$$x < \int_{2m}^{2m+2} X \circ \Phi_u \ du / 2$$

we get

$$\int_{2m}^{2m+2} 1_{\{X \circ \Phi_t > x\}} K_m(t, X) \nabla_t L^{-1} X dt$$

$$\leq 4 \int_{2m}^{2m+2} \left| \nabla_t X \nabla_t 1_{\{X > x\}} \nabla_t L^{-1} X \right| dt$$

$$+ 2 \left| \int_{2m}^{2m+2} \nabla_u X \nabla_u 1_{\{X > x\}} du \right| \left| \int_{2m}^{2m+2} \nabla_t 1_{\{X > x\}} \nabla_t L^{-1} X dt \right|,$$

where we also changed \(1_{\{X \circ \Phi_t > x\}}\) into \(\nabla_t 1_{\{X > x\}}\) in the last integral, which is justified by 

$$\int_{2m}^{2m+2} \nabla_t L^{-1} X dt = 0.$$ 

In order to obtain the same bound in the case 

$$x \geq \int_{2m}^{2m+2} X \circ \Phi_u \ du / 2,$$

we rewrite (4.7) as

$$K_m(t, x, X) = -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} (x - X \circ \Phi_t + \nabla_t X - \nabla_s X) ds$$

$$= 2 \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} (x - X \circ \Phi_t - \nabla_s X) ds$$

$$= 2 \int_{2m}^{2m+2} \nabla_s 1_{\{X > x\}} \nabla_s X ds - 2 \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} (X \circ \Phi_t - x) ds,$$

and we estimate the last integral by

$$0 \leq \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} (X \circ \Phi_t - x) ds$$

$$\leq \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} ds \left( X \circ \Phi_t - \frac{1}{2} \int_{2m}^{2m+2} X \circ \Phi_u du \right)$$

$$= -\nabla_t X \nabla_t 1_{\{X \leq x\}} = \nabla_t X \nabla_t 1_{\{X > x\}},$$

which shows that the inequality (4.8) is valid for all \(x \in \mathbb{R}\) under the condition 

$$x < X \circ \Phi_t.$$ 

Thus, applying the Cauchy-Schwarz inequality several times and using the bound
\[ |\nabla t \mathbf{1}_{\{X \leq x\}}| \leq 1, \text{ we obtain} \]
\[
\left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} \mathbf{1}_{\{X \leq x\}} K_m(t, x, X) \nabla_t L^{-1} X \, dt \right] \right|
\leq 4 \sqrt{\mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} \mathbf{1}_{\{X \leq x\}} |\nabla u|^2 \, du \right] \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} \mathbf{1}_{\{X \leq x\}} (\nabla u L^{-1} X)^2 \, du \right]}
\]
\[ + 4 \sqrt{\mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} (\nabla u \mathbf{1}_{\{X \leq x\}})^2 |\nabla u X|^2 \, du \right] \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} (\nabla u \mathbf{1}_{\{X \leq x\}})^2 (\nabla u L^{-1} X)^2 \, du \right]}
\]
\[ \leq 8 \sqrt{\mathbb{E} \left[ \int_0^\infty |\nabla u \mathbf{1}_{\{X > x\}}| |\nabla u X|^2 \, du \right] \mathbb{E} \left[ \int_0^\infty |\nabla u \mathbf{1}_{\{X > x\}}| (\nabla u L^{-1} X)^2 \, du \right]}.
\]

By the duality relation (2.3), Hölder’s inequality and the formula (2.4), we get
\[
\mathbb{E} \left[ \int_0^\infty |\nabla u \mathbf{1}_{\{X > x\}}| |\nabla u X|^2 \, du \right]
\leq 2 \mathbb{E} \left[ |\nabla u \mathbf{1}_{\{X > x\}}|^2 \sum_{m=0}^{\infty} \mathbb{E} \left[ \int_0^\infty \nabla^* (\text{sgn}(\nabla u \mathbf{1}_{\{X > x\}})) |\nabla u X|^2 \, ds \right] \right]
\leq 2 \sqrt{\mathbb{E} \left[ \left( \nabla^* (\text{sgn}(\nabla u \mathbf{1}_{\{X > x\}})) |\nabla u X|^2 \right)^2 \right]}
\leq 2 \mathbb{E} \left[ \int_0^\infty (\nabla u X)^4 \, dt \right] + \mathbb{E} \left[ \int_0^\infty \left( \nabla^* (\text{sgn}(\nabla u \mathbf{1}_{\{X > x\}})) |\nabla u X|^2 \right)^2 \, ds \right]. \tag{4.9}
\]

Next, we observe that by the covariance relation (3.1) with \( \alpha = \frac{1}{2} \), we have
\[
\mathbb{E} \left[ \int_0^\infty \int_0^\infty \left( \nabla^* \left( \text{sgn}(\nabla u \mathbf{1}_{\{X > x\}}) |\nabla u X|^2 \right) \right)^2 \, ds \, du \right]
\leq \mathbb{E} \left[ \int_0^\infty \int_0^\infty \left( \nabla^* \left( |\nabla u X|^2 \right) \right)^2 \, ds \, du \right]
\leq 2 \mathbb{E} \left[ \int_0^\infty \left( (I + 2(-L)^{1/2}) (|\nabla u X|^2) \right)^2 \, du \right].
\]

Applying this to (4.9), we get
\[
\mathbb{E} \left[ \int_0^\infty |\nabla u \mathbf{1}_{\{X > x\}}| |\nabla u X|^2 \, du \right] \leq 2 \sqrt{\mathbb{E} \left[ \int_0^\infty ((I + 2(-L)^{1/2}) (|\nabla u X|^2))^2 \, du \right]},
\]
and analogously we obtain
\[
\mathbb{E} \left[ \int_0^\infty |\nabla u \mathbf{1}_{\{X > x\}}| (\nabla u L^{-1} X)^2 \, du \right] \leq 2 \sqrt{\mathbb{E} \left[ \int_0^\infty ((I + 2(-L)^{1/2}) ((\nabla u L^{-1} X)^2))^2 \, du \right]}.
\]
which eventually gives us
\[
\left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t > x\}} K_m(t, x, X) \nabla_t L^{-1} X dt \right] \right| \leq 16 \left( \mathbb{E} \left[ \int_0^\infty \left( \left( I + 2(-L)^{1/2} \right) (|\nabla_u X|^2) \right)^2 du \right] \right)^{1/4}.
\]

(ii) In case \( X \circ \Phi_t \leq x \) we observe that, denoting
\[
\tilde{K}_m(t, x, X) := -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t} (1_{\{u \leq x\}} - 1_{\{X \circ \Phi_t \leq x\}}) \, du \, ds,
\]
which comes from (4.6) by changing weak inequalities into strict ones and conversely, and repeating all the above argument, we arrive at
\[
\left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \geq x\}} \tilde{K}_m(t, x, X) \nabla_t L^{-1} X dt \right] \right| \leq 16 \left( \mathbb{E} \left[ \int_0^\infty \left( \left( I + 2(-L)^{1/2} \right) (|\nabla_u L^{-1} X|^2) \right)^2 du \right] \right)^{1/4}.
\]

Next, by (4.6) we have, for \( m = \lfloor t/2 \rfloor \) and \( X \circ \Phi_t \leq x \),
\[
K_m(t, x, X) = -4 \nabla_t X \nabla_t 1_{\{X > x\}} + 2 \int_{2m}^{2m+2} \int_{X \circ \Phi_t} (1_{\{u \leq x\}} - 1_{\{X \circ \Phi_t \leq x\}}) \, du \, ds
\]
\[
= 4 \nabla_t X \nabla_t 1_{\{X \leq x\}} - \int_{2m}^{2m+2} \int_{X \circ \Phi_t} (1_{\{u \geq x\}} - 1_{\{X \circ \Phi_t \geq x\}}) \, du \, ds
\]
\[
= 4 \nabla_t X \nabla_t 1_{\{-X \geq -x\}} - \int_{2m}^{2m+2} \int_{X \circ \Phi_t} (1_{\{-u \leq -x\}} - 1_{\{-X \circ \Phi_t \leq -x\}}) \, du \, ds
\]
\[
= -4 \nabla_t (-X) \nabla_t 1_{\{-X \geq -x\}} + \int_{2m}^{2m+2} \int_{-X \circ \Phi_t} (1_{\{u \geq -x\}} - 1_{\{-X \circ \Phi_t \leq -x\}}) \, du \, ds
\]
\[
= \tilde{K}_m(t, -x, -X).
\]

Thus, using (4.12) with \(-x\) and \(-X\) instead of \(x\) and \(X\) respectively, we get
\[
\left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} 1_{\{X \circ \Phi_t \leq x\}} K_m(t, x, X) \nabla_t L^{-1} X dt \right] \right|
\]
\[
= \left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \int_{2m}^{2m+2} 1_{\{-X \circ \Phi_t \geq -x\}} \tilde{K}_m(t, -x, -X) \nabla_t L^{-1} (-X) dt \right] \right|.
\]

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Combining (4.10) and (4.13) with (4.5), we finally obtain

\[
\frac{1}{4} \left| \mathbb{E} \left[ \int_0^\infty \int_{2^{[t/2]}}^{2^{[t/2]+2}} B_{s,t}(x,X) ds \nabla_t L^{-1} X dt \right] \right| \\
= \frac{1}{8} \left| \mathbb{E} \left[ \sum_{m=0}^{\infty} \sum_{l=2m}^{2m+2} (1_{X \circ \Phi_t > x} + 1_{X \circ \Phi_t \leq x}) K_m(t,X) \nabla_t L^{-1} X dt \right] \right| \\
\leq 4 \left( \mathbb{E} \left[ \int_0^\infty (1 + 2(-L)^{1/2}) (|\nabla_t X|^2)^2 \right] \mathbb{E} \left[ \int_0^\infty (1 + 2(-L)^{1/2}) (\nabla_t L^{-1} X)^2 \right] \right)^{1/4},
\]

which ends the proof. \(\square\)

5 Sums of multiple stochastic integrals

The next proposition applies Theorem 4.1 to sums of multiple stochastic integrals. It extends Theorem 3.1 of Privault and Serafin (2020a) from functionals of Bernoulli sequences to functionals of independent random variables, see also earlier results such as Proposition 3.7 in Nourdin and Peccati (2009) in the case of multiple Wiener integrals.

Proposition 5.1 For any \(X \in L^2(\Omega)\) written as a sum \(X = \sum_{k=1}^d I_k(f_k)\) of multiple stochastic integrals where \(f_k \in \hat{L}^2(\mathbb{R}_+^k)\) satisfies (2.6), \(k = 1, \ldots, d\), we have

\[
d_K(X, \mathcal{N}) \leq \mathbb{E} \left[ |1 - \mathbb{E}[X^2]| \right] \\
+ C_d \sqrt{\sum_{0 \leq i < i \leq d} \|f_i \ast_i f_i\|_{L^2(\mathbb{R}_+^{2(i-1)})}^2 + \sum_{1 \leq i \leq d} \left( \|f_i \ast_i f_i\|_{L^2(\mathbb{R}_+^{2(i-1)})}^2 + \|f_i \ast f_i\|_{L^2(\mathbb{R}_+^{2(i-1)})}^2 \right)},
\]

for some \(C_d > 0\).

Proof. Since \(|\nabla_t X|^2\) and \((\nabla_t L^{-1} X)^2\) are sums of multiple integrals of orders \(2d - 2\) and below, the relation (2.9) shows the bound

\[
\mathbb{E} \left[ (1 + 2(-L)^{1/2}) (|\nabla_t X|^2)^2 \right] \leq 2d \mathbb{E} \left[ (\nabla_t X)^4 \right],
\]

and

\[
\mathbb{E} \left[ (1 + 2(-L)^{1/2}) ((\nabla_t L^{-1} X)^2)^2 \right] \leq 2d \mathbb{E} \left[ (\nabla_t L^{-1} X)^4 \right].
\]
Additionally, by (2.9) we also have
\[
\mathbb{E} \left[ ((-L)^{-1/2}X)^2 \right] \leq \mathbb{E} \left[ X^2 \right] \leq \sqrt{\mathbb{E}[X^4]}.
\]

Applying these inequalities to (4.1) in Theorem 4.1, we get
\[
d_K(X,N) \leq |1 - \mathbb{E}[X^2]| + \sqrt{\text{Var} \left[ \int_0^\infty \nabla_t X \nabla_t L^{-1} X \, dt \right]}
\]
\[
\quad + \frac{3}{2} \left( \mathbb{E} [X^4] \right)^{1/4} \sqrt{\mathbb{E} \int_0^\infty (\nabla_t X)^4 \, dt \left( 1 + \left( \mathbb{E} \left[ (\int_0^\infty (\nabla_t L^{-1} X)^2 \, dt) \right]^2 \right) \right)^{1/4}}
\]
\[
\quad + 6d \sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 \, dt \right] \mathbb{E} \left[ \int_0^\infty (\nabla_t L^{-1} X)^4 \, dt \right]}.
\]

Denoting
\[
R_X := \sum_{1 \leq i \leq j \leq d} \sum_{k=1}^i \sum_{l=0}^k 1_{\{i=j=k=l\}} \| f_i \ast_k^l f_j \|_{L^2(\mathbb{R}_+^{i+j-k-l})}^2,
\]

it follows from the proof of Corollary 3.2 in Privault and Serafin (2020b) that
\[
R_X \leq c_d \left( \sum_{0 \leq l < i \leq d} \| f_i \ast_l^i f_i \|_{L^2(\mathbb{R}_+^{i+l})}^2 + \sum_{1 \leq i \leq j \leq d} \left( \| f_i \ast_l^i f_i \|_{L^2(\mathbb{R}_+^{i+l})}^2 + \| f_i \ast_k^l f_i \|_{L^2(\mathbb{R}_+^{i+k})}^2 \right) \right),
\]
\[
(5.1)
\]

and
\[
\text{Var} \left[ \int_0^\infty \nabla_t X \nabla_t L^{-1} X \, dt \right] \leq c_d R_X, \quad \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 \, dt \right] \leq c_d R_X,
\]
\[
(5.2)
\]

for some $c_d \geq 0$. Taking $L^{-1}X$ as $X$ in the last inequality, we also have
\[
\mathbb{E} \left[ \int_0^\infty (\nabla_t L^{-1} X)^4 \, dt \right] \leq c'_d R_X,
\]

for some $c'_d \geq 0$. Furthermore, since
\[
\nabla_t L^{-1} X = \sum_{k=0}^{d-1} I_k \left( f_{k+1}(t, \cdot) \right)
\]

and the functions $f_k$ satisfy (2.6), the multiplication formula (2.12) gives
\[
\int_0^\infty (\nabla_t L^{-1} X)^2 \, dt = \int_0^\infty \sum_{0 \leq i \leq j < d-1} \sum_{k=0}^i \sum_{l=0}^k c_{i,j,l,k} \int_{j+1-k-l}^\infty \left( \int_0^\infty f_{i+1}(t, \cdot) \nabla_{j+1}^l f_{j+1}(t, \cdot) \, dt \right) \, dt
\]
\[
= \sum_{0 \leq i \leq j < d-1} \sum_{k=0}^i \sum_{l=0}^k c_{i,j,l,k} \left( \int_0^\infty f_{i+1}(t, \cdot) \nabla_{j+1}^l f_{j+1}(t, \cdot) \, dt \right).
\]
for some $c_{i,j,l,k} \geq 0$, and consequently
\[
\mathbb{E} \left[ \left( \int_0^\infty (\nabla_t L^{-1} X)^2 dt \right)^2 \right] \leq c_d \sum_{0 \leq i \leq j < d} \sum_{k=0}^i \sum_{l=0}^k \left\| \left( \int_0^\infty f_{i+1}(t, \cdot) \ast_k^l \ast_{j+1} f_{j+1}(t, \cdot) dt \right) \right\|_{L^2(\mathbb{R}^{i+j-k-l})}^2
\]
\[
= c_d \sum_{0 \leq i \leq j < d} \sum_{k=0}^i \sum_{l=0}^k \left\| (f_{i+1} \ast_{k+1}^l f_{j+1}) \right\|_{L^2(\mathbb{R}^{i+j-k-l})}^2
\]
\[
= c_d \left( \sum_{1 \leq i \leq j < d} \sum_{k=1}^i \sum_{l=1}^k 1_{\{i=j=k=1\}} \left\| (f_i \ast_k^l f_j) \right\|_{L^2(\mathbb{R}^{i+j-k-l})}^2 + \sum_{i=1}^d (f_i \ast_i f_i)^2 \right)
\]
\[
\leq c_d \left( R_X + (\mathbb{E}[X^2])^2 \right).
\]

Similarly, we get for some $C_{i,j,k,l} \geq 0$
\[
\mathbb{E} [X^4] \leq c_d \mathbb{E} \left[ \left( \sum_{0 \leq i \leq j < d} \sum_{k=0}^i \sum_{l=0}^k C_{i,j,l,k} I_{i+j-k-l} (f_i \ast_k^l f_j) \right)^2 \right]
\]
\[
\leq c_d \sum_{0 \leq i \leq j < d} \sum_{k=0}^i \sum_{l=0}^k \left\| f_i \ast_k^l f_j \right\|_{L^2(\mathbb{R}^{i+j-k-l})}^2
\]
\[
= c_d \left( R_X + \sum_{i=1}^d (f_i \ast_i f_i)^2 + \sum_{1 \leq i \leq j \leq d} \left\| f_i \ast_0^j f_j \right\|_{L^2(\mathbb{R}^{i+j})}^2 \right)
\]
\[
= c_d \left( R_X + \sum_{i=1}^d \left\| f_i \right\|_{L^2(\mathbb{R}^i)}^4 + \sum_{1 \leq i \leq j \leq d} \left\| f_i \left\|_{L^2(\mathbb{R}^i)} \right\|_{L^2(\mathbb{R}^j)}^2 \right)^2
\]
\[
\leq c_d \left( R_X + (\mathbb{E}[X^2])^2 \right).
\]

This finally gives us
\[
d_K(X, \mathcal{N}) \leq |1 - \mathbb{E}[X^2]| + c_d \sqrt{R_X} \left( 1 + \left( (R_X + \mathbb{E}[X^2])^{1/4} + 1 \right)^2 \right).
\]

Since $d_K(X, \mathcal{N}) \leq 1$, we may assume that $\mathbb{E}[X^2]$ and $R_X$ are bounded, which implies
\[
d_K(X, \mathcal{N}) \leq |1 - \mathbb{E}[X^2]| + c_d \sqrt{R_X},
\]
and the assertion of the corollary follows from (5.1). \qed

Next, due to the identity $\nabla_t L^{-1} I_d(f_d) = I_{d-1}(f_d(t, *))$, $d \geq 1$, the bound in Theorem 4.1 can be significantly simplified in the case of multiple stochastic integrals $I_d(f_d)$.

**Proposition 5.2** For $X = I_d(f_d)$ a multiple stochastic integral of order $d \geq 1$, we have
\[
d_K(X, \mathcal{N})
\]

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\[
\leq |1 - \mathbb{E}[X^2]| + \frac{1}{d} \sqrt{\text{Var} \left[ \int_0^\infty (\nabla_t X)_t^2 dt \right]} + \frac{12 + 5 \sqrt{\mathbb{E}[X^4]}}{\sqrt{d}} \sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right]}
\]

Proof. In view of the relation

\[
(-L)^{1/2} I_d(f_d) = \frac{1}{\sqrt{d}} I_d(f_d)
\]

and the covariance identity (3.1) applied with \( \alpha = 0 \), Theorem 4.1 gives

\[
d_K(X, \mathcal{N}) \leq |1 - \mathbb{E}[X^2]| + \frac{1}{d} \sqrt{\text{Var} \left[ \int_0^\infty |\nabla_t X|_t^2 dt \right]}
\]

\[
+ \frac{3}{2\sqrt{d}} \sqrt{\mathbb{E} \int_0^\infty (\nabla_t X)^4 dt \left( \left( \mathbb{E}[X^4] \left( \frac{1}{d^2} \text{Var} \left[ \int_0^\infty |\nabla_t X|_t^2 dt \right] + 4(\mathbb{E}[X^2])^2 \right) \right)^{1/4} + \frac{\sqrt{\pi}}{2} \sqrt{\mathbb{E}[X^2]} \right)}
\]

\[
+ \frac{4}{d} \left( \mathbb{E} \left[ \int_0^\infty \left( (1 + 2(-L)^{1/2}) (|\nabla_t X|^2)^2 dt \right) \right] \right)^{1/2}.
\]

Since \( d_K(X, \mathcal{N}) \leq 1 \) by definition, we may assume that \( \sqrt{\text{Var} \left[ \int_0^\infty |\nabla_t X|_t^2 dt \right]} \leq d \) and \( \mathbb{E}[X^2] \leq 2 \). Hence we get

\[
\left( \mathbb{E}[X^4] \left( \frac{1}{d^2} \text{Var} \left[ \int_0^\infty |\nabla_t X|_t^2 dt \right] + 4(\mathbb{E}[X^2])^2 \right) \right)^{1/4} + \frac{\sqrt{\pi}}{2} \sqrt{\mathbb{E}[X^2]}
\]

\[
\leq \sqrt{\mathbb{E}[X^4]} \left( \sqrt{18} + \frac{\sqrt{\pi}}{2} \right) \leq \frac{10}{3} \sqrt{\mathbb{E}[X^4]}.
\]

Furthermore, since \(|\nabla_u X|^2|\) is a sum of multiple integrals of orders \( 2d - 2 \) and below, we have by (2.9)

\[
\mathbb{E}[\left( (2(-L)^{1/2}) + 1 \right) (|\nabla_t X|^2)^2] \leq \left( 2\sqrt{2d - 2} + 1 \right)^2 \mathbb{E}[(\nabla_t X)^4] \leq 9d \mathbb{E}[(\nabla_t X)^4].
\]

Combining all together we obtain the first inequality from the assertion. Next, applying Proposition 3.3, we get

\[
d_K(X, \mathcal{N}) \leq |1 - \mathbb{E}[X^2]| + \sqrt{\text{Var} \left[ \int_0^\infty |\nabla_t X|_t^2 dt \right]} + \sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right]}
\]

\[
\times \left( 12 + 5 \left( 36 \mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|_t^2 dt \right)^2 \right] + 15 \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] + 8 \right)^{1/4} \right).
\]

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Using once again the inequality $d_K(X, N) \leq 1$, we may assume $\sqrt{\text{Var} \left[ \int_0^\infty |\nabla_t X|^2 dt/2 \right]} \leq 1$ and $\sqrt{\mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right]} \leq \frac{1}{17}$, which lets us bound the expression in the last parenthesis as follows

$$12 + 5 \left( 36 \mathbb{E} \left[ \left( \int_0^\infty |\nabla_t X|^2 dt \right)^2 \right] + 15 \mathbb{E} \left[ \int_0^\infty (\nabla_t X)^4 dt \right] + 8 \right)^{1/4} \leq 12 + 5 (81)^{1/4} = 27,$$

which ends the proof. □

6 Applications to $U$-statistics

6.1 Hoeffding decompositions

Recall that given $(X_1, \ldots, X_n)$ a family of independent random variables and $[n] := \{1, \ldots, n\}$, $n \geq 1$, the family $(\mathcal{F}_J)_{J \subset [n]}$ of $\sigma$-algebras is defined as

$$\mathcal{F}_J := \sigma(X_j : j \in J), \quad J \subset [n].$$

**Definition 6.1** A centered $\mathcal{F}_n$-measurable random variable $W_n$ admits a Hoeffding decomposition if it can be written as

$$W_n = \sum_{J \subset [n]} W_J,$$

(6.1)

where $(W_J)_{J \subset [n]}$ is a family of random variables such that $W_J$ is $\mathcal{F}_J$-measurable, $J \subset [n]$, and

$$\mathbb{E} [W_J | \mathcal{F}_K] = 0, \quad J \not\subset K \subset [n].$$

For $J = \{k_1, \ldots, k_{|J|}\}$ with $k_1 < k_2 < \cdots < k_{|J|}$, any $W_J$ in Definition 6.1 can be written as a function $W_J = g_J(X_{k_1}, \ldots, X_{k_{|J|}})$ of $(X_{k_1}, \ldots, X_{k_{|J|}})$, with in particular

$$\mathbb{E} [g_J(X_j : j \in J) | J \setminus \{k\}] = 0, \quad k \in J,$$

(6.2)

and

$$W_n = \sum_{J \subset [n]} g_J(X_{k_1}, \ldots, X_{k_{|J|}}).$$

(6.3)

Note that if $X_i = U_i, i \in [n]$, then the chaos decomposition (2.8) coincides with the Hoeffding decomposition (6.1), by taking

$$W_J := \frac{1}{|J|!} f_{|J|}(2k_1 + 1 + U_1, \ldots, 2k_{|J|} - 1 + U_{|J| - 1}, 2k_{|J|} + 1 + U_{|J|}), \quad J \subset [n],$$

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and Condition (6.2) is equivalent to (2.6).

The next Theorem 6.2 is a consequence of Proposition 5.1, using the fact that any random variable can be represented in distribution as a function of a uniformly distributed random variable, and makes more precise the central limit theorem of de Jong (1987; 1990). In comparison with Theorem 1.3 in Döbler and Peccati (2017), Theorem 6.2 is stated for the Kolmogorov distance instead of the Wasserstein distance, it applies to Hoeffding decompositions in full generality and not only to degenerate $U$-statistics for which $|J|$ is constrained to a fixed value $|J| = d$ for some $d \in \{1, \ldots, n\}$ in the sum (6.1).

**Theorem 6.2** Let $1 \leq d \leq n$. For any $W_n \in L^4(\Omega)$ admitting the Hoeffding decomposition (6.1) with $|J| \leq d$, and such that $\mathbb{E}[W_n^2] = 1$, we have

$$d_K(W_n, \mathcal{N}) \leq C_d \left( \sum_{0 \leq l < i \leq d} \sum_{|J|=i-l} \mathbb{E}\left[ \sum_{|K|=l, K \cap J = \phi} (W_{J \cup K})^2 | \mathcal{F}_J \right] \right)^{1/2},$$

(6.4)

where $C_d > 0$ depends only on $d$.

**Proof.** By representing $X_i$ as $X_i \overset{d}{=} F_i^{-1}((U_i + 1)/2)$ where $F_i^{-1}$ is the generalized inverse of the cumulative distribution function $F_i$ of $X_i$, $i = 1, \ldots, n$, we rewrite (6.3) as the sum of multiple stochastic integrals

$$W_n \overset{d}{=} \sum_{k=1}^{d} I_k(f_k),$$

where

$$f_k(x_1, \ldots, x_k) := \frac{1}{k!} \sum_{J = \{j_1, \ldots, j_k\} \subset [n]} g_J \left( F_{i_1}^{-1} \left( \frac{x_1}{2} - \left\lfloor \frac{x_1}{2} \right\rfloor \right), \ldots, F_{i_k}^{-1} \left( \frac{x_k}{2} - \left\lfloor \frac{x_k}{2} \right\rfloor \right) \right) 1_{[2i_1-2,2i_1) \times \cdots \times [2i_k-2,2i_k)}(x_1, \ldots, x_k),$$

$x_1, \ldots, x_k \in \mathbb{R}^k_+$. Next, denoting

$$\hat{\mathcal{N}}^m := \{(k_1, \ldots, k_m) : k_1, \ldots, k_m \geq 1, k_i \neq k_j \text{ if } i \neq j, 1 \leq i, j \leq m\},$$

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we have
\[
\|f_i \ast f_i \|_{L^2(\mathbb{R}^{d-i})}^2 \leq (i-l)! (l)!^2 \sum_{|J|=i-l} \int [2j_1,2j_1+2] \times \cdots \times [2j_{l-1},2j_{l-1}+2) \left( \sum_{K=\{k_1,\ldots,k_l\}} \int [2k_1,2k_1+2] \times \cdots \times [2k_{l-1},2k_{l-1}+2) (f_i(x_1,\ldots,x_i))^2 dx_1 \cdots dx_l \right)^2 dx_{l+1} \cdots dx_i
\]
for some \( C = C(d) \). Similarly, we get
\[
\|f_i \ast f_i \|_{L^2(\mathbb{R}^{2d-i})}^2 \leq C \sum_{|J|=i-l} \sum_{|K|=l, K \cap J = \emptyset} \mathbb{E} \left[ \left( \sum_{|J|=i-l, K \cap J = \emptyset} \mathbb{E} \left[ (W_{J\cup K})^2 \mid F_J \right] \right)^2 \right]
\]
and
\[
\|f_i \ast f_i \|_{L^2(\mathbb{R}^{d-i})}^2 \leq C \sum_{|J|=i-l} \mathbb{E} \left[ \left( \sum_{|K|=l, K \cap J = \emptyset} \mathbb{E} [W_K W_{J\cup K} \mid F_J] \right)^2 \right], \quad 1 \leq l < i \leq d.
\]
We conclude by applying the above to Proposition 5.1, which yields the required bound.

\[\square\]

### 6.2 Degenerate \( U \)-statistics

In this section we narrow our attention to the degenerate \( U \)-statistics of a given order \( d \geq 1 \), which are random variables \( W_{n,d} \) admitting the Hoeffding decomposition (6.1) with \(|J| = d\).

**Theorem 6.3** For any degenerate \( U \)-statistics \( W_{n,d} \in L^4(\Omega) \) of order \( d \geq 1 \), and such that \( \mathbb{E} [W_{n,d}^2] = 1 \), we have
\[
d_K(W_{n,d}, \mathcal{N}) \leq \sqrt{\text{Var} \left[ \sum_{k=1}^{\infty} \mathbb{E} \left[ \left( W_{n,d} - \mathbb{E} [W_{n,d}|\{X_k\}_k] \right)^2 \right] \mid \{X_k\}_k \right]}
\]
\[+ 24 \sqrt{2 \mathbb{E}\sum_{k=1}^{\infty} \mathbb{E} \left[ (W_{n,d} - \mathbb{E}[W_{n,d}|\{X_k\}^c])^4 \right]}\]

\[\leq C_d \left( \sum_{0 \leq l < d} \sum_{|J|=d-l} \mathbb{E} \left[ \left( \sum_{|K|=l, K \cap J = \emptyset} \mathbb{E} \left[ (W_{J \cup K})^2 | \mathcal{F}_J \right] \right)^2 \right] \right)^{1/2} + \sum_{1 \leq l < d} \sum_{|J|=d-l} \mathbb{E} \left[ \left( \sum_{|K|=l, K \cap (J \cup J_2) = \emptyset} \mathbb{E} \left[ W_{J \cup K} W_{J_2 \cup K} | \mathcal{F}_{J_1 \cup J_2} \right] \right)^2 \right]^{1/2},\]

where \(\{X_k\}^c = \{X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n\}\) and \(C_d > 0\) depends only on \(d\).

**Proof.** The first bound is just the latter bound from Proposition 5.2 rewritten in a different form. Namely, it is enough to take \(f_d\) as in (6.5) and then we have for \(t \in [2k, 2k+2)\)

\[\nabla_t W_{n,d} = \mathbb{E} [W_{n,d} | \{X_k\}^c, X_k = t] - \mathbb{E} [W_{n,d} | \{X_k\}^c].\]

The other bound in the assertion follows from Proposition 5.2 in view of (5.2), (5.1) – where the last sum is vanishing – and the proof of Theorem 6.2. \(\square\)

**Weighted U-statistics**

As an example, we consider classical degenerate weighted U-statistics. Precisely, given \((X_1, \ldots, X_n)\) an i.i.d. sequence of random variables with distribution \(\nu\), we define

\[U_{n,d} = \binom{n}{d}^{-1} \sum_{1 \leq k_1 < \cdots < k_d \leq n} w(k_1, \ldots, k_d) g(X_{k_1}, \ldots, X_{k_d}), \quad 1 \leq d \leq n,\]

(6.6)

where \(w(k_1, \ldots, k_d) \in \mathbb{R}\) is symmetric and vanishes on diagonals, and \(g(X_{k_1}, \ldots, X_{k_d}) \in L^2(\Omega), 1 \leq k_1 < \cdots < k_d \leq n\), satisfies

\[\mathbb{E} [g(X_{x_1}, x_2, \ldots, x_d)] = 0, \quad (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}.\]

(6.7)

The variance \(\sigma^2\) of \(U_{n,d}\) is given by

\[\sigma^2 := \text{Var}[U_{n,d}] = \binom{n}{d}^{-2} \|g\|_{L^2(\mathbb{R}^d, \nu^\otimes d)}^2 \sum_{1 \leq k_1 < \cdots < k_d \leq n} w^2(k_1, \ldots, k_d).\]

The assumption (6.7) plays a technical role, which helps in simplifying the derivations. Nevertheless, it covers important examples of U-statistics such as quadratic forms and their multidimensional generalizations. Sharp bounds have been provided in Chen and Shao (2007).
in case (6.7) is not satisfied, but only in the case of classical (i.e. non-weighted) \( U \)-statistics. See also Krokowski et al. (2016) for weighted first order \( U \)-statistics based on symmetric Rademacher sequences, and Nourdin et al. (2016) for a fourth moment type central limit theorem in case \( g(x_1, \ldots, x_n) = x_1 \cdots x_n \) and \( X_1 \) has a vanishing third moment.

**Theorem 6.4** Let \( U_{n,d} \) be a degenerate weighted \( U \)-statistics of the form (6.6). We have

\[
d_K \left( \frac{U_{n,d}}{\sigma}, N \right) \leq C_d \frac{\|g\|_{L^4(R^d, \nu \otimes d)}^2}{\|g\|_{L^2(R^d, \nu \otimes d)}^2} \sup_{1 \leq l \leq d-1} \sqrt{\sum_{k, r \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w(k, m)w(r, m) \right)^2} \frac{\sum_{m \in \mathbb{N}^l} w^2(m)}{\sum_{m \in \mathbb{N}^d} w^2(m)}
\]

for some \( C_d > 0 \) depending only on \( d = 1, \ldots, n \), where \( \nu \) denotes the distribution of \( X_1 \).

**Proof.** By Theorem 6.3, we have

\[
d_K \left( \frac{U_{n,d}}{\sigma}, N \right) \leq C_d \frac{n}{\sigma^2} \left( \sum_{0 \leq l \leq d-1} \int_{R^l} g^2(x, y)\nu^{\otimes l}(dx) \right)^2 \nu^{\otimes (d-l)}(dy) \sum_{k \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w^2(k, m) \right)^2
\]

\[
+ \sum_{1 \leq l \leq d-1} \left( \int_{R^l} g(x, y)g(x, z)\nu^{\otimes l}(dx) \right)^2 \nu^{\otimes (d-l)}(dy)\nu^{\otimes (d-l)}(dz)
\]

\[
\times \sum_{k, r \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w(k, m)w(r, m) \right)^2
\]

Since \( \nu \) is a probability measure, we have

\[
\int_{R^l} \left( \int_{R^l} g^2(x, y)\nu^{\otimes l}(dx) \right)^2 \nu^{\otimes (d-l)}(dy) \leq \int_{R^d} \int_{R^l} g^4(x, y)\nu^{\otimes (d-l)}(dx)\nu^{\otimes (d-l)}(dy)
\]

\[
= \|g\|_{L^4(R^d, \nu \otimes d)}^4,
\]

as well as

\[
\int_{R^d} \int_{R^d} \left( \int_{R^l} g(x, y)g(x, z)\nu^{\otimes l}(dx) \right)^2 \nu^{\otimes (d-l)}(dy)\nu^{\otimes (d-l)}(dz)
\]

\[
\leq \int_{R^d} \int_{R^d} \int_{R^l} g^2(x, y)\nu^{\otimes l}(dx) \int_{R^l} g^2(x, z)\nu^{\otimes l}(dx)\nu^{\otimes (d-l)}(dy)\nu^{\otimes (d-l)}(dz)
\]

\[
= \|g\|_{L^2(R^d, \nu \otimes d)}^4 \leq \|g\|_{L^4(R^d, \nu \otimes d)}^4.
\]

Using also the inequality

\[
\sum_{k \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w^2(k, m) \right)^2 \leq \sum_{k, r \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w(k, m)w(r, m) \right)^2,
\]

\[27\]
we arrive at
\[
d_K \left( \frac{U_{n,d}}{\sigma}, \mathcal{N} \right) \leq C_d \left( \frac{\|g\|_{L^2(\mathbb{R}^d, \mu \otimes d)}}{\sigma^2(n_d)^{2d}} \right)^{1/2} \left( \sum_{k, r \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w(k, m)w(r, m) \right)^2 \right)^{1/2},
\]
which is the bound in the assertion. \[\Box\]

6.3 Random graphs

Consider the Erdős and Rényi (1959) random graph \( G_n(p) \) constructed by independently retaining any edge in the complete graph \( K_n \) on \( n \) vertices with probability \( p \in (0, 1) \). Here, we assign an independent sample of a random weight \( X \) to every edge in \( G_n(p_n) \), and we consider the renormalized random weight
\[
\tilde{W}_n^G := \frac{W_n^G - \mathbb{E}[W_n^G]}{\sqrt{\text{Var}[W_n^G]}},
\]
where \( W_n^G \) denotes the combined weight of graphs in \( G_n(p_n) \) that are isomorphic to a fixed graph \( G \). By writing the combined weight \( W_n^G \) of graphs in \( G_n(p_n) \) that are isomorphic to a fixed graph \( G \) as a sum of multiple stochastic integrals (which is equivalent to finding its Hoeffding decomposition) we obtain the following result as in Privault and Serafin (2020b), by replacing the use of Theorem 5.1 therein with Theorem 5.1 above.

**Theorem 6.5** Let \( G \) be a graph without isolated vertices. The renormalized weight \( \tilde{W}_n^G \) of graphs in \( G_n(p_n) \) that are isomorphic to \( G \) satisfies
\[
d_K(\tilde{W}_n^G, \mathcal{N}) \leq C \left( \mathbb{E} \left[ (X - \mathbb{E}[X])^4 \right] + (1-p)(\mathbb{E}[X])^2 \right)^{1/2} \left( \sum_{k, r \in \mathbb{N}^{d-l}} \left( \sum_{m \in \mathbb{N}^l} w(k, m)w(r, m) \right)^2 \right)^{1/2},
\]
for some constant \( C = C(e_G) > 0 \).

This extends other Kolmogorov distance bounds previously obtained for triangle counting in Ross (2011), and in Krokowski et al. (2017b) using the Malliavin approach to the Stein method, see also Röllin (2017) for triangle counting and Privault and Serafin (2020a) for arbitrary subgraph counting, and Krokowski et al. (2016) for weighted first order Rademacher
$U$-statistics in the symmetric case $p = 1/2$. As a consequence, if $p_n$ satisfies $p_n < c < 1$, $n \geq 1$, we have

$$d_K(\hat{W}_n^G, \mathcal{N}) \leq C \sqrt{\frac{\mathbb{E}[X^4]}{\mathbb{E}[X^2]}} \left(1 - p_n \right) \min \limits_{H \subseteq G \subseteq \mathbb{H}} n_{\epsilon_H}^{p_n^\epsilon_H} \left(1 - p_n \right) \inf_{H \subseteq G \subseteq \mathbb{H}} n_{\epsilon_H}^{p_n^\epsilon_H}$$

and for $p_n > c > 0$, $n \geq 1$, it holds

$$d_K(\hat{W}_n^G, \mathcal{N}) \leq C \frac{\sqrt{\mathbb{E}[X^4]}}{n \sqrt{1 - p_n \text{Var}[X]}}.$$ 

Applications to cycle graphs, complete graphs trees can be treated as in Privault and Serafin (2020b) by replacing the Kolmogorov distance with the Wasserstein distance.

### 7 Quadratic forms

We consider the quadratic form $Q_n$ defined as

$$Q_n = \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} a_{ij} X_i X_j + \sum_{k=1}^n a_{kk} \left(X_k^2 - \mathbb{E}[X_k^2]\right),$$

where $A_n = (a_{ij})_{1 \leq i, j \leq n}$ is a symmetric matrix, $n \geq 1$, and $(X_k)_{k \geq 1}$ denotes i.i.d. copies of a given random variable $X$ satisfying $\mathbb{E}[X] = 0$. In the sequel, we let $\mu_k := \mathbb{E}[X^k]$, $\bar{\mu}_k := \mathbb{E}\left[(X^2 - \mathbb{E}[X^2])^{k/2}\right]$, $k \geq 2$, and

$$\sigma_n^2 := \text{Var}[Q_n] = \mathbb{E}[Q_n^2] = 2\mu_2^2 \sum_{1 \leq i, j \leq n \\ i \neq j} a_{ij}^2 + \bar{\mu}_4 \sum_{i=1}^n a_{ii}^2.$$

Many papers in the literature are devoted to asymptotical normality of quadratic forms. The best known convergence rates in the general case where the diagonal of $A$ may not vanish are given in Götze and Tikhomirov (2002), as

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C(\gamma) \left(\frac{\mathbb{E}[|X|^3]}{\mathbb{E}[X^6]} \right)^2 + \gamma \mathbb{E}[X^6] \right) \left(|\lambda_1|, \right)$$

see Theorem 1.1 therein, where $\lambda_1$ denotes the largest absolute eigenvalue of $A_n$, $\gamma = \sum_{i=1}^n a_{ii}^2 / \sum_{1 \leq i, j \leq n} a_{ij}^2$, and the constant $C(\gamma)$ blows up when $\gamma$ tends to one, i.e. when the linear part is dominating.
Vanishing diagonals

More is known if we assume the diagonal of $A_n$ to be empty, in which case de Jong (1987) proved the asymptotic normality of $Q_n/\sigma_n$ under the conditions

$$\mathbb{E}[(Q_n/\sigma_n)^4] \to 3 \quad \text{and} \quad \frac{1}{\sigma_n^2} \max_{1 \leq i \leq n} \sum_{j=1}^n a_{ij}^2 \to 0. \quad (7.2)$$

In addition, for $(X_k)_{k \geq 1}$ a Rademacher sequence, Theorem 1.1 in Döbler and Krokowski (2019) restricted to double integrals gives the corresponding bound

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C \left( \sqrt{\mathbb{E}[(Q_n/\sigma_n)^4]} - 3 \right) + \frac{1}{\sigma_n} \sqrt{\max_{1 \leq i \leq n} \sum_{j=1}^n a_{ij}^2}. \quad (7.3)$$

The same bound may be concluded from Döbler and Peccati (2017) for $(X_k)_{k \geq 1}$ being any i.i.d. sequence, but only in Wasserstein distance. Note that the quantity $\max_{1 \leq i \leq n} \sum_{j=1}^n a_{ij}^2$ corresponds to “maximal influence”, see Mossel et al. (2010), Nourdin et al. (2010b).

The bound

$$d_W\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C \frac{\mu_4}{\sigma_n^2} \sqrt{\sum_{i=1}^n \left( \sum_{k=1}^n a_{ik} a_{kj} \right)^2}.$$ \quad (7.4)

has been provided for Rademacher sequences using the Wasserstein distance in Proposition 3.1 of Chatterjee (2008), and has been recently extended to arbitrary i.i.d. sequences using the Kolmogorov distance in Shao and Zhang (2019), Theorem 3.1.

Corollary 7.1 recovers this bound as an immediate consequence of Theorem 6.4 by taking $d = 2$, $w(k_1, k_2) = a_{k_1 k_2}$, $1 \leq k_1, k_2 \leq n$, $k_1 \neq k_2$, and $g(y_1, y_2) = y_1 y_2$. Note however that only the second term is significant in the right-hand side of (7.4), making the conjecture at the end of Section 3.1 in Shao and Zhang (2019) pointless.

**Corollary 7.1** Assume $a_{ii} = 0$, $i = 1, \ldots, n$. Then, there exists a constant $C > 0$ such that

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C \frac{\mu_4}{\sigma_n^2} \sqrt{\sum_{i,j=1}^n \left( \sum_{k=1}^n a_{ik} a_{kj} \right)^2} = C \frac{\mu_4}{\sigma_n^2} \sqrt{\text{Tr}(A_n^4)}, \quad n \geq 1.$$

Corollary 7.1 also improves (7.1) for matrices $A_n$ with empty diagonal, since

$$\sqrt{\text{Tr}(A_n^4)} = \sqrt{\sum_{k=1}^n \lambda_k^4} \leq |\lambda_1| \sqrt{\sum_{k=1}^n \lambda_k^2} \leq |\lambda_1| \sqrt{\sum_{i,j=1}^n a_{ij}^2} \leq \sigma_n |\lambda_1|. \quad (7.5)$$
Non-empty diagonals

Theorem 7.2 below generalizes and improves all the aforementioned results. First, in comparison with the above bound (7.1) of Götze and Tikhomirov (1999; 2002), it gives better rates under weaker assumptions, as noted in (7.5). Furthermore, it extends every other result by applying as well to non-vanishing diagonals. In addition, it completes Corollary 7.1 with an additional bound related to so called fourth moment phenomenon (Nualart and Peccati (2004)), and it also extends (7.3) from the Rademacher case to any distribution. Finally, it deals with the Kolmogorov distance instead of the Wasserstein distance considered in Döbler and Peccati (2017). See also Theorem 3.11 in Bally and Caramellino (2019) for some bounds in total variation and Kolmogorov distances, which however provide worse rates and require slightly stronger assumptions.

**Theorem 7.2** There exist absolute constants $C_1, C_2 > 0$ such that

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C_1 \left(\sqrt{\mathbb{E}[(Q_n/\sigma_n)^4]} - 3\right) + \frac{\alpha_n}{\sigma_n} \sqrt{\max_{1 \leq i \leq n} \sum_{1 \leq j \leq 1} a_{ij}^2},$$  

(7.6)

and

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq C_2 \frac{\beta_n}{\sigma_n^2} \sqrt{\text{Tr}(A_n^4)},$$  

(7.7)

where

$$\alpha_n := \mu_2 + \frac{\mu_4}{\mu_2} \mathbb{I}\{a_{11}^2 + \cdots + a_{nn}^2 > 0\}, \quad \text{and} \quad \beta_n = \mu_4 + \sqrt{\mu_8} \mathbb{I}\{a_{11}^2 + \cdots + a_{nn}^2 > 0\}.$$

**Proof.** The quadratic form $Q_n$ admits the Hoeffding decomposition

$$Q_n = \sum_{1 \leq i, j \leq n} W_{\{i,j\}} + \sum_{k=1}^{n} W_{\{k\}},$$

where

$$W_{\{i,j\}} = 2a_{ij}X_iX_j, \quad W_{\{k\}} = a_{kk} \left(X_k^2 - \mathbb{E}[X_k^2]\right).$$

Thus, Theorem 6.2 gives

$$d_K\left(\frac{Q_n}{\sigma_n}, \mathcal{N}\right) \leq \frac{C}{\sigma_n^2} \left(\hat{\mu}_8 \sum_{i=1}^{n} a_{ii}^4 + 2\mu_4 \sum_{1 \leq i, j \leq n} a_{ij}^4 + 2\mu_2^2 \sum_{1 \leq i, j, k \leq n} a_{ik}^2 a_{ij}^2 \right)$$

$$\quad + \mu_2^4 \sum_{1 \leq i, j \leq n} \left(\sum_{1 \leq k \leq n} a_{ik} a_{kj}\right)^2 + \mu_3^2 \mu_2 \sum_{i=1}^{n} \left(\sum_{1 \leq j \leq n} a_{ij} a_{i,j}\right)^2 \right)^{1/2}.$$  

(7.8)
Next, we estimate this bound by means of $\mathbb{E} [Q^4_n]$ and $\max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2$. A direct calculation shows that

$$
\mathbb{E} [Q^4_n] = S_1 + 3S_2 + 4S_3,
$$

where

$$
S_1 := \tilde{\mu}_8 \sum_{i=1}^n a_{ii}^4 + 16\mu_4^4 \sum_{1 \leq i < j \leq n} a_{ij}^4 + 48\mu_4^2 \mu_4 \sum_{1 \leq i,j,k \leq n} a_{ij}^2 a_{ik}^2
$$

$$
+ 48\mu_4^2 \sum_{1 \leq i,j,k \leq n} a_{iiz}a_{iiz}a_{iiz}a_{iiz} + 48\mu_2^2 \sum_{1 \leq i,j,k \leq n} a_{iia}_{jja_{ik}a_{kj}}
$$

$$
+ 48\mu_3^2 \sum_{1 \leq i,j,k \leq n} a_{kj}a_{ik}a_{ij},
$$

and

$$
S_2 := \tilde{\mu}_1^2 \sum_{i \neq j} a_{ii}^2 a_{jj}^2 + 4\mu_4 \mu_2 \sum_{1 \leq i,j,k \leq n} a_{ii}^2 a_{jk}^2 + 4\mu_4 \sum_{1 \leq i,j,k \leq n} a_{iiz}^2 a_{iiz}^2
$$

and

$$
S_3 := 3\tilde{\mu}_3^2 \sum_{1 \leq i,j \leq n} a_{ii}a_{jj}a_{ij}^2 + 8\mu_3 (\mu_5 - \mu_3\mu_2) \sum_{i \neq j} a_{ii}a_{jj}a_{ij}^2 + 6\mu_2 (\tilde{\mu}_6 + \tilde{\mu}_4 \mu_2) \sum_{1 \leq i,j,k \leq n} a_{ii}a_{jj}a_{ij}^2
$$

$$
+ 12\mu_3^2 \mu_2 \sum_{1 \leq i,j,k \leq n} a_{ii}a_{jj}a_{ij}^2 + 24\mu_2^2 \sum_{1 \leq i,j,k \leq n} a_{ii}a_{ij}a_{ik}a_{kj}.
$$

The sum $S_1$ is supposed to dominate the right-hand side of (7.8), $S_2$ is approximating $\sigma^2$, and $S_3$ contains some remainders that are problematic due to their unknown sign and vanishes if the diagonal of $A$ is empty. First, by

$$
\sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 = \sum_{1 \leq i,j,k \leq n} a_{iiz}a_{iiz}a_{iiz}a_{iiz} + \sum_{1 \leq i,j,k \leq n} a_{ii}a_{jj}a_{ij}^2
$$

and

$$
\sum_{i=1}^n \left( \sum_{1 \leq j \leq n} a_{ij}a_{ij} \right)^2 = \sum_{1 \leq i,j,k \leq n} a_{ii}a_{jj}a_{kj} + \sum_{1 \leq i,j \leq n} a_{ii}a_{ij}^2
$$

we get

$$
S_1 := \tilde{\mu}_8 \sum_{i=1}^n a_{ii}^4 + 16\mu_4^4 \sum_{1 \leq i < j \leq n} a_{ij}^4 + 48\mu_2^2 \mu_4 \sum_{1 \leq i,j,k \leq n} a_{ij}^2 a_{ik}^2
$$
The first two lines dominate the right-hand side of (7.8) with substantial surplus, which will be used to deal with the last term of $S_1$ and some terms of $S_3$. Indeed, by $\mu_3^2 \mu_2 \leq \mu_4^2 \mu_2$ and the inequality of arithmetic and geometric means, we have

$$48\mu_2^2 \sum_{1 \leq i, j, k \leq n} a^2_{kj} a_{ik} a_{ij}$$

$$\leq 46\mu_4^2 \mu_2 \sum_{1 \leq i, j, k \leq n} \frac{1}{2} \left( (a_{kj} a_{ik})^2 + (a_{kj} a_{ij})^2 \right) + \sum_{1 \leq i, j \leq n} \left( \mu_4^2 a^2_{kj} + \left( \mu_2^2 \sum_{1 \leq i \leq n} a_{ik} a_{ij} \right)^2 \right)$$

$$= 46\mu_4^2 \mu_2 \sum_{1 \leq i, j, k \leq n} a^2_{ij} a^2_{ik} + \mu_4^2 \sum_{1 \leq i, j \leq n} a^2_{ij} + \mu_2^2 \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq i \leq n} a_{ik} a_{ij} \right)^2.$$  (7.9)

Since, additionally

$$\mu_2^4 \sum_{1 \leq i, j, k \leq n} a^2_{ik} a^2_{kj} + \mu_4^2 \mu_2 \sum_{1 \leq i, j \leq n} a^2_{ij}$$

$$\leq \sigma_n^2 \mu_2^2 \max_{1 \leq i \leq n} \sum_{j=1}^n a^2_{ij} + \sigma_n^2 \mu_4^2 \frac{1}{\mu_2^2} \left\{ a_{i1}^2 + \ldots + a_{in}^2 > 0 \right\} \max_{1 \leq i \leq n} \sum_{j=1}^n a^2_{ij} \leq \sigma_n^2 \alpha_n^2 \max_{1 \leq i \leq n} \sum_{j=1}^n a^2_{ij},$$  (7.10)

we arrive at

$$d_K \left( \frac{Q_n}{\sigma_n^2}, N \right) \leq \frac{C}{\sigma_n^2} \left( S_1 + 48\sigma_n^2 \alpha_n^2 \max_{1 \leq i \leq n} \sum_{j=1}^n a^2_{ij}$$

$$+ 46\mu_2^2 \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n \atop k \neq i, j} a_{ik} a_{kj} \right)^2 + 47\mu_2^2 \mu_2 \sum_{1 \leq i \leq n} \left( \sum_{1 \leq j \leq n \atop j \neq i} a_{ij} a_{ij} \right)^2 \right)^{1/2}$$

$$\leq \frac{C}{\sigma_n^2} \left( \mathbb{E} \left[ Q_n^4 \right] - 3\sigma_n^4 + 48\sigma_n^2 \alpha_n^2 \max_{1 \leq i \leq n} \sum_{j=1}^n a^2_{ij} + 3(\sigma_n^4 - S_2)$$

$$- 4S_3 - 24\mu_4^2 \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n \atop k \neq i, j} a_{ik} a_{kj} \right)^2 - 24\mu_3^2 \mu_2 \sum_{1 \leq i \leq n} a^2_{ij} \right)^{1/2}. \quad (7.11)$$
Next, in order to bound \(3(\sigma^4_n - S_2)\), we calculate

\[
\sigma^4_n = \left(2\mu_2^2 \sum_{1 \leq i, j \leq n} a_{ij}^2 + \tilde{\mu}_4 \sum_{i=1}^n a_{ii}^2\right)^2
\]

\[
= 4\mu_2^4 \left(\sum_{1 \leq i, j \leq n} a_{ij}^2\right)^2 + \tilde{\mu}_4^2 \left(\sum_{i=1}^n a_{ii}^2\right)^2 + 4\mu_2^2 \tilde{\mu}_4 \left(\sum_{1 \leq i, j \leq n} a_{ij}^2\right) \left(\sum_{i=1}^n a_{ii}^2\right)
\]

\[
= 4\mu_2^4 \left(2 \sum_{1 \leq i, j \leq n} a_{ij}^4 + \sum_{1 \leq i, j \leq n, i \neq i} a_{ij}^2 a_{ij}^2 + 2 \sum_{1 \leq i, j \leq n, j \neq i} a_{ij}^2 a_{jk}^2\right)
\]

\[
+ \tilde{\mu}_4 \left(\sum_{i=1}^n a_{ii}^4 + \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2\right) + 4\mu_2^2 \tilde{\mu}_4 \left(2 \sum_{1 \leq i, j \leq n} a_{ii}^2 a_{ij}^2 + \sum_{1 \leq i, j, k \leq n, j \neq k, j, k \neq i} a_{ij}^2 a_{jk}^2\right),
\]

hence

\[
3 |\sigma^4_n - S_2| = 24\mu_2^4 \left(\sum_{1 \leq i, j \leq n} a_{ij}^4 + \sum_{1 \leq i, j \leq n, j \neq i} a_{ij}^2 a_{ij}^2\right) + 3\tilde{\mu}_4^2 \sum_{i=1}^n a_{ii}^4 + 24\mu_2 \tilde{\mu}_4 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2
\]

\[
\leq \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \left(4\mu_2^4 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 + 27\mu_4 \tilde{\mu}_4 \{a_{i1}^2 + \ldots + a_{nn}^2 > 0\} \sum_{i=1}^n a_{ii}^2\right)
\]

\[
\leq \sigma^2_n \left(4\mu_2^4 + 27\mu_4 \tilde{\mu}_4 \{a_{i1}^2 + \ldots + a_{nn}^2 > 0\}\right) \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2.
\]

(7.12)

Regarding \(S_3\), we have

\[
\left|\tilde{\mu}_4 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij} a_{ij}^2\right| \leq \tilde{\mu}_4 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2 \leq \sigma_n^2 \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2,
\]

and

\[
8\mu_3 (\mu_5 - \mu_3 \mu_2) \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^3 a_{ij}^3 + 6\mu_2 (\tilde{\mu}_6 + \tilde{\mu}_4 \mu_2) \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2
\]

\[
= 2\mathbb{E} \left[4 \sum_{1 \leq i, j \leq n, i \neq j} a_{ii}^3 (X_i^3 - \mathbb{E}[X_i^3]) X_i^3 X_j^3 + 3 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2 (X_i^2 - \mathbb{E}[X_i^2])^2 X_i^2 X_j^2\right]
\]

\[
= 6\mathbb{E} \left[\sum_{1 \leq i, j \leq n, i \neq j} \left(a_{ii}^3 (X_i^3 - \mathbb{E}[X_i^3]) + \frac{2}{3} a_{ij} a_{ij}^2 \right) X_i^3 X_j^3 + 3 \sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^2 a_{ij}^2 (X_i^2 - \mathbb{E}[X_i^2])^2 X_i^2 X_j^2\right] - \frac{8}{3} \mathbb{E} \left[\sum_{1 \leq i, j \leq n, i \neq j} a_{ij}^4 X_i^4 X_j^4\right]
\]

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\[-\frac{8}{3} \mu_4 \sum_{1 \leq i,j \leq n} a_{ij}^4 \geq -\frac{4}{3} \sigma_n^2 \left( \frac{\mu_4}{\mu_2} \right)^2 \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2.\]

Furthermore, using the correction terms from (7.11), we get

\[
12 \mu_3^2 \mu_2 \sum_{1 \leq i,j,k \leq n \atop i \neq j, j \neq k} a_{ii} a_{ij} a_{jk} + 6 \mu_3^2 \mu_2 \sum_{i=1}^n \left( \sum_{1 \leq j \leq n \atop i \neq j} a_{jj} a_{ij} \right)^2
\]

\[
= 6 \mu_3^2 \mu_2 \sum_{i=1}^n \left( \sum_{1 \leq k \leq n \atop k \neq i} a_{ik}^2 + \sum_{1 \leq j \leq n \atop j \neq i} a_{jj} a_{ij} \right)^2 - 6 \mu_3^2 \mu_2 \sum_{i=1}^n \left( \sum_{1 \leq k \leq n \atop k \neq i} a_{ik}^2 \right)^2
\]

\[
\geq -6 \mu_3^2 \mu_2 \left( \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \right) \sum_{i=1}^n \sum_{1 \leq k \leq n \atop k \neq i} a_{ik}^2 \geq -3 \sigma_n^2 \left( \frac{\mu_4}{\mu_2} \right)^2 \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2,
\]

as well as

\[
24 \mu_4 \sum_{1 \leq i,j,k \leq n \atop i \neq j, j \neq k, i \neq k} a_{ii} a_{ij} a_{ik} a_{kj} + 6 \mu_2^4 \sum_{1 \leq i,j \leq n \atop i \neq j} \left( \sum_{1 \leq k \leq n \atop k \neq i,j} a_{ik} a_{kj} \right)^2
\]

\[
= 6 \sum_{1 \leq i,j \leq n \atop i \neq j} \left( 2 \mu_4 a_{ii} a_{ij} + \mu_2^2 \sum_{1 \leq k \leq n \atop k \neq i,j} a_{ik} a_{kj} \right)^2 - 24 \mu_4 \sum_{1 \leq i,j \leq n \atop i \neq j} a_{ij}^2 a_{ij} a_{ij}
\]

\[
\geq -24 \mu_4 \left( \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \right) \sum_{1 \leq i,j \leq n \atop i \neq j} a_{ij}^2 \geq -24 \sigma_n^2 \mu_4 \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2.
\]

Hence, we arrive at

\[
S_3 + 6 \mu_3^2 \mu_2 \sum_{1 \leq i \leq n} \left( \sum_{1 \leq j \leq n \atop i \neq j} a_{jj} a_{ij} \right)^2 + 6 \mu_2^4 \sum_{1 \leq i,j \leq n \atop i \neq j} \left( \sum_{1 \leq k \leq n \atop k \neq i,j} a_{ik} a_{kj} \right)^2
\]

\[
\geq -C \sigma_n^2 \left( \frac{\mu_4}{\mu_2} \right)^2 \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \left\{ a_{i1}^2 + \cdots + a_{in}^2 > 0 \right\} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2
\]

for some $C > 0$, since $S_3$ vanishes if $a_{11} = \cdots = a_{nn} = 0$. Applying this and (7.12) to (7.11), we obtain the first inequality from the assertion. To prove the other one, we use (7.8) and write

\[
d_K \left( \frac{Q_n}{\sigma_n} \right) \leq C \sigma_n^2 \left( \mu_4 \sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq k \leq n \atop k \neq i,j} a_{ik} a_{kj} \right)^2 + \mu_8 \sum_{i=1}^n a_{ii}^4 + \mu_8 \sum_{i=1}^n \left( \sum_{1 \leq j \leq n \atop i \neq j} a_{jj} a_{ij} \right)^2 \right)^{1/2}.
\]
Next, we bound
\[
\sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n, k \neq i,j} a_{ik}a_{kj} \right)^2 = \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} - a_{ii}a_{jj} - a_{ij}a_{ji} \right)^2 
\]
\[
\leq \sum_{1 \leq i, j \leq n} \left[ 2 \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 + 4a_{ii}^2a_{jj} \right] \leq 2\text{Tr}(A_n^4) + 4 \sum_{1 \leq i \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}^2 \right)^2 \leq 6\text{Tr}(A_n^4),
\]
(7.13)
and, by the inequality $ab \leq (a^2 + b^2)/2$,
\[
\sum_{i=1}^n a_{ii}^4 + \sum_{i=1}^n \left( \sum_{1 \leq j \leq n, i \neq j} a_{ij}a_{ij} \right)^2 = \sum_{i=1}^n a_{ii}^4 + \sum_{i=1}^n \sum_{1 \leq j, k \leq n, i \neq j} (a_{ij}a_{kk})(a_{ik}a_{jj}) 
\]
\[
\leq \sum_{i=1}^n a_{ii}^4 + \sum_{1 \leq i, j, k \leq n, i \neq j} (a_{ij}a_{kk})^2 \leq 2 \sum_{1 \leq i \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}^2 \right)^2 \leq 2\text{Tr}(A_n^4),
\]
(7.14)
This ends the proof. □

Contrary to what is stated on page 1590 of Chatterjee (2008), the conditions $\sigma_n^{-2}\sqrt{\text{Tr}(A_n^4)} \to 0$ and $\mathbb{E}[(Q_n/\sigma_n)^4] \to 3$ are not equivalent as $n$ tends to infinity, and therefore fourth moment convergence is not sufficient for the central limit theorem to hold for quadratic functionals.

The next proposition clarifies this fact via inequalities between the quantities appearing in Theorem 7.2. In the sequel, we let $a \wedge b := \min(a, b)$, $a, b \in \mathbb{R}$.

**Proposition 7.3** There exist absolute constants $C_1, C_2, C_3 > 0$ such that
\[
C_1 \frac{\mu_4 \wedge \tilde{\mu}_8}{\sigma_n^4} \text{Tr}(A_n^4) \leq |\mathbb{E}[(Q_n/\sigma_n)^4] - 3| + \frac{\alpha_n^2}{\sigma_n^2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 
\]
\[
\leq C_2 \left( \frac{\beta_n^2}{\sigma_n^4} \text{Tr}(A_n^4) + \frac{\alpha_n^2}{\sigma_n^2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \right) \leq C_3 \frac{\beta_n^2}{\mu_2^2 \sigma_n^2} \sqrt{\text{Tr}(A_n^4)},
\]
where $\alpha_n, \beta_n$ are as in Theorem 7.2

**Proof.** The proof of Theorem 7.2 shows that the right hand side of (7.6) is larger than the right hand side of (7.8) up to an absolute multiplicative constant, hence we have
\[
|\mathbb{E}[(Q_n/\sigma_n)^4] - 3| + \frac{\alpha_n^2 \sigma_n^2}{\mu_2^2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \geq C \frac{\mu_4 \wedge \tilde{\mu}_8}{\sigma_n^4} \left( \sum_{i=1}^n a_{ii}^4 + \sum_{1 \leq i, j \leq n} \left( \sum_{1 \leq k \leq n, k \neq i, j} a_{ik}a_{kj} \right)^2 \right).
\]

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Employing the inequalities \((a + b)^2 \leq 2a^2 + 2b^2\) and \(ab \leq (a^2 + b^2)/2\), we get

\[
\sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 \leq \sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} + a_{ii}a_{jj} \right)^2
\]

\[
\leq \sum_{1 \leq i,j \leq n} \left[ \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 + 8a_{ii}a_{jj} \right]
\]

\[
= \sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq i,j \leq n} a_{ik}a_{kj} \right)^2 + 8 \sum_{1 \leq i \leq n} a_{ii}^2 + 8 \sum_{1 \leq i \leq n} a_{jj}^2
\]

\[
\leq 5 \sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq i,j \leq n} a_{ik}a_{kj} \right)^2 + 12 \sum_{1 \leq i \leq n} a_{ii}^4,
\]

which gives the first inequality in the assertion. In order to justify the latter one, we will show

\[
\left| \mathbb{E} \left[ (Q_n/\sigma_n)^4 \right] - 3 \right| \leq C \left( \frac{\beta^2}{\sigma_n^2} \text{Tr}(A_n^4) + \frac{\alpha^2}{\sigma_n^2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2 \right), \tag{7.15}
\]

for some \(C > 0\). Following notation from the proof of Theorem 7.2, we have \(|\mathbb{E}[(Q_n/\sigma_n)^4] - 3| \leq (|S_1| + 3|S_2 - \sigma_n^4| + 4|S_3|)/\sigma_n^4\). By (7.9), (7.10), (7.12) and bounding terms from the first three sums in \(S_3\) by \(a_{ii}^2 + a_{ij}^2\) and the last two sums from \(S_3\) by

\[
\sum_{i=1}^n \left[ \left( \sum_{1 \leq k \leq n} a_{ik}^2 \right)^2 + \left( \sum_{1 \leq j \leq n} a_{jj}a_{ij} \right)^2 \right],
\]

and

\[
\sum_{1 \leq i,j \leq n} \left[ a_{ii}^2a_{ij}^2 + \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 \right],
\]

respectively, we arrive at

\[
\left| \mathbb{E} \left[ (Q_n/\sigma_n)^4 \right] - 3 \right| \leq C \frac{\beta^2}{\sigma_n^2} \sum_{i=1}^n a_{ii}^4 + \sum_{1 \leq i,j \leq n} \left( \sum_{1 \leq k \leq n} a_{ik}a_{kj} \right)^2 + \sum_{i=1}^n \left( \sum_{1 \leq j \leq n} a_{jj}a_{ij} \right)^2 + \frac{\alpha^2}{\sigma_n^2} \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a_{ij}^2,
\]
and (7.15) follows from (7.13) and (7.14). Finally, the last bound in the assertion is a consequence of

$$\max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}^2 \leq \sqrt{\sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right)^2} \leq \sqrt{\text{Tr}(A_n^4)},$$

and

$$\text{Tr}(A_n^4) \leq \sum_{i,j=1}^{n} \left( \sum_{k=1}^{n} a_{ik}^2 \right) \left( \sum_{k=1}^{n} a_{kj}^2 \right) \leq \frac{\sigma_n^4}{\mu_2^2}.$$

□

Theorem 7.2 and Lemma 7.3 immediately imply

**Corollary 7.4** Assume \((X_i)_{i \in \mathbb{N}}\) is a fixed i.i.d. sequence with zero means and finite 8th moments. The following two conditions are equivalent:

a) \(\mathbb{E} [(Q_n/\sigma_n)^4] \rightarrow 3\) and \(\sigma_n^{-2} \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}^2 \rightarrow 0,\)

b) \(\sigma_n^{-4} \text{Tr}(A_n^4) \rightarrow 0,\)

and they imply \(Q_n/\sigma_n \xrightarrow{L} \mathcal{N}\) with the Kolmogorov rates (7.6) and (7.7).

This extends (7.2) for any matrix \(A_n\) and completes it with the equivalent condition in terms of the trace of \(A_n\).

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