Find Subtrees of Specified Weight and Cycles of Specified Length in Linear Time

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Abstract

We introduce a variant of DFS which finds subtrees of specified weight in linear time, by which, as observed by Mohr, cycles of specified length in planar hamiltonian graphs can be found. We show, for example, that every planar hamiltonian graph $G$ with minimum degree $\delta \geq 4$ has a cycle of length $k$ for every $k \in \{\lfloor \frac{|V(G)|}{2} \rfloor, \ldots, \lceil \frac{|V(G)|}{2} \rceil + 3\}$ with $3 \leq k \leq |V(G)|$, that every planar hamiltonian graph $G$ with $\delta \geq 5$ has a cycle of length $k$ for every $k \in \{\lfloor \frac{1}{4}|V(G)| \rfloor, \ldots, \lceil \frac{3}{4}|V(G)| \rceil + 3\}$ with $3 \leq k \leq |V(G)|$, and that if $|V(G)| \geq 8$ is even, every 3-connected planar hamiltonian graph $G$ with $\delta \geq 4$ has a cycle of length $\frac{|V(G)|}{2} - 1$ or $\frac{|V(G)|}{2} - 2$. Each of these cycles can be found in linear time if a Hamilton cycle of the graph is given. Another interesting consequence follows from our tool is that, given an instance of the number partitioning problem, i.e. a multiset of positive integers $\{a_1, \ldots, a_N\}$, if $\sum_{i=1}^{N} a_i \leq 2N - 2$, then a partition always exists and can be found in linear time.

1 Introduction

The cycle spectrum $CS(G)$ of a graph $G$ is defined to be the set of integers $k$ for which there is a cycle of length $k$ in $G$. $G$ is said to be hamiltonian if $|V(G)| \in CS(G)$ and pancyclic if its cycle spectrum has all possible lengths, i.e. $CS(G) = \{3, \ldots, |V(G)|\}$. Cycle spectra of graphs have been extensively studied in many directions, in this paper we study cycle spectra of planar hamiltonian graphs with minimum degree $\delta \geq 4$.

1.1 Cycle Spectra of Planar Graphs

In 1956, Tutte [24] proved his seminal result that every 4-connected planar graph is hamiltonian. Motivated by Tutte’s theorem together with the metaconjecture proposed by Bondy [2] that almost any non-trivial conditions for hamiltonicity of a graph should also imply pancyclicity, Bondy [2] conjectured in 1973 that every 4-connected planar graph $G$ is almost pancyclic, i.e. $|CS(G)| \geq |V(G)| - 3$, and Malkevitch [16] conjectured in 1988 that every 4-connected planar graph is pancyclic if it contains a cycle of length 4 (see [14, 15] for other variants).

These two conjectures remain open, while 4 is the only known cycle length that can be missing in a cycle spectrum of a 4-connected planar graph. For example, the line graph of a cyclically 4-edge-connected cubic planar graph with girth at least 5 is a 4-regular 4-connected planar graph with no cycle of length 4, see also [14, 23]. If we relax the connectedness, more cycle lengths

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*This work was partially supported by the DFG grant SCHM 3186/1-1 and by DAAD (as part of BMBF, Germany) and the Ministry of Education Science, Research and Sport of the Slovak Republic within the project 57320575.
are known for being absent in some cycle spectra. Choudum [6] showed that for every integer \( k \geq 7 \), there exist 4-regular 3-connected planar hamiltonian graphs of order larger than \( k \) each has cycles of all possible lengths except \( k \), which means every integer \( k \geq 7 \) can be absent in the cycle spectra of some 4-regular 3-connected planar hamiltonian graphs. Another interesting example was constructed by Malkevitch [14], which is, for every \( p \in \mathbb{N} \), a 4-regular planar hamiltonian graph \( G \) of order \( |V(G)| = 6p \) whose cycle spectrum \( CS(G) = \{3, 4, 5, 6\} \cup \{r \in \mathbb{N} : \frac{|V(G)|}{2} \leq r \leq |V(G)|\} \), as shown in Figure 1.

![Figure 1: A 4-regular planar hamiltonian graph \( G \) which has no cycle of length between 7 and \( \frac{|V(G)|}{2} - 1 \).](image)

So far we have seen which cycle lengths can be absent in some cycle spectra, we now ask the opposite question, i.e. which cycle lengths must be present in all cycle spectra. It is known that every planar graph with \( \delta \geq 4 \) must contain cycles of length 3, 5 [26] and 6 [8], which is shown to be best possible by the aforementioned examples. It is also known that every 2-connected planar graph with \( \delta \geq 4 \) must have a cycle of length 4 or 7 [10], a cycle of length 4 or 8 and a cycle of length 4 or 9 [13]. While the presence of a cycle of length 3 follows easily from Euler’s formula, the rest of them were shown by the discharging method.

Another power tool in searching cycles of specified length is the so-called Tutte path method, which was first introduced by Tutte in his proof of hamiltonicity of 4-connected planar graphs. Using this technique, Nelson (see [19, 22]), Thomas and Yu [21] and Sanders [20] showed that every 4-connected planar graph contains cycles of length \( |V(G)| - 1 \), \( |V(G)| - 2 \) and \( |V(G)| - 3 \), respectively. Note that we always assume \( k \geq 3 \) when we say a graph contains a cycle of length \( k \).

Chen et al. [3] noticed that the Tutte path method cannot be generalized for smaller cycle lengths, they hence combined Tutte paths with contractible edges and showed the existence of cycles of length \( |V(G)| - 4 \), \( |V(G)| - 5 \) and \( |V(G)| - 6 \). Following this approach, Cui et al. [7] showed that every 4-connected planar graph has a cycle of length \( |V(G)| - 7 \). To summarize, every 4-connected planar graph contains a cycle of length \( k \) for every \( k \in \{|V(G)|, |V(G)| - 1, \ldots, |V(G)| - 7\} \) with \( k \geq 3 \).

### 1.2 Cycles of Length Close to \( \frac{|V(G)|}{2} \) and Mohr’s Transformation

With the knowledge of these short and long cycles, Mohr [18] asked whether cycles of length close to \( \frac{|V(G)|}{2} \) also exist, and he answered his question by showing that every planar hamiltonian graph \( G \) satisfying \( |E(G)| \geq 2|V(G)| \) has a cycle of length between \( \frac{1}{3}|V(G)| \) and \( \frac{2}{3}|V(G)| \). We present his
simple and elegant argument in the following.

Let \( G^* = (V, E^*) \) be the dual graph of the plane graph \( G \) and \( C \) be a Hamilton cycle of \( G \). Note that \( C \) separates the Euclidean plane into two open regions \( C_{\text{int}} \) and \( C_{\text{ext}} \) containing no vertex. Let \( G_{\text{int}} \) and \( G_{\text{ext}} \) be the graphs obtained from \( G \) by deleting the edges in \( C_{\text{ext}} \) and in \( C_{\text{int}} \), respectively. We always assume that \( |E(G_{\text{int}})| \geq |E(G_{\text{ext}})| \). As \( C \) is a Hamilton cycle, its dual disconnects \( G^* \) into two trees say \( T_{\text{int}} \) lying on \( C_{\text{int}} \) and \( T_{\text{ext}} \) on \( C_{\text{ext}} \) (see Figure 2). By Euler’s formula, we have \( |V(G^*)| = |E(G)| - |V(G)| + 2 \), and hence \( |V(T_{\text{int}})| \geq \frac{1}{2} |V(G^*)| \geq \frac{1}{2} |V(G)| + 1 \). We define vertex weight \( c(v) := d_{G^*}(v) - 2 \geq 1 \) for every vertex \( v \in V(G^*) \cap V(T_{\text{int}}) \), where \( d_{G^*}(v) \) is the degree of \( v \) in \( G^* \), or equivalently, the face length of \( v \) in \( G \) (see Figure 2(a)). It is not hard to see that for every subtree \( S \) of \( T_{\text{int}} \), the set of edges of \( G^* \) having exactly one endvertex in \( S \) is indeed the dual of an edge set of a cycle in \( G \) of length \( c(S) + 2 \), where \( c(S) := \sum_{v \in V(S)} c(v) \) (see Figure 2(b)).

So, the problem of finding a cycle of specified length is transformed to the problem of finding a subtree of specified weight: the existence of a subtree \( S \) of weight \( k \) in \( T_{\text{int}} \) implies the existence of a cycle of length \( k + 2 \) in \( G \). It is left to show that there is a subtree \( S \) in \( T_{\text{int}} \) with \( \frac{1}{3} |V(G)| - 2 \leq c(S) \leq \frac{2}{3} |V(G)| - 2 \). First note that \( c(v) \leq \frac{1}{2} |V(G)| - 2 \) for all \( v \in V(T_{\text{int}}) \); otherwise \( c(T_{\text{int}}) > \frac{1}{2} |V(G)| - 2 + |V(T_{\text{int}})| - 1 \geq |V(G)| - 2 \), which is not possible as \( T_{\text{int}} \) corresponds to the Hamilton cycle of length \( |V(G)| \). If there is a vertex \( v \in V(T_{\text{int}}) \) with \( c(v) \geq \frac{1}{3} |V(G)| - 2 \), then we can simply take \( S \) to be this single vertex \( v \). Suppose \( c(v) < \frac{1}{3} |V(G)| - 2 \) for all \( v \in V(T_{\text{int}}) \). We take \( S \) to be a maximal subtree of \( T_{\text{int}} \) with \( c(S) \leq \frac{2}{3} |V(G)| - 2 \), it is clear that \( c(S) \geq \frac{1}{3} |V(G)| - 2 \). Thus \( G \) has a cycle of length between \( \frac{1}{3} |V(G)| \) and \( \frac{2}{3} |V(G)| \).

Our work is inspired by this transformation, and we will focus on finding subtrees of specified weight. We recapitulate the main content of Mohr’s proof. Given a planar hamiltonian graph \( G \), we can have a tree \( T \) (in the dual graph) of at least \( \frac{1}{2} |E(G)| - \frac{1}{2} |V(G)| + 1 \) vertices with vertex weights \( c: V(T) \to N \) such that \( c(T) = |V(G)| - 2 \) and \( c(v) \leq c(T) - |V(T)| + 1 \leq \frac{2}{3} |V(G)| - \frac{1}{2} |E(G)| - 2 \) for all \( v \in V(T) \). And, if there is a subtree of weight \( k \) in \( T \), then there is a cycle of length \( k + 2 \) in \( G \).

In Section 3 we will develop a tool in order to prove the following lemma which deals with subtrees of specified weight.
Lemma 1. Let \( k, g, N_1, N_2 \in \mathbb{N} \) and \( h \in \mathbb{Z} \) such that \( g + h > 2 \), \( 2N_1 \geq N_2 + h \), \( 1 \leq k \leq N_2 \) and \( 2k - 4g - h + 3 \leq N_2 \leq 2k + g + h - 2 \). Let \( T \) be a tree of \( N_1 \) vertices and let \( c : V(T) \to \mathbb{N} \) be vertex weights such that \( c(T) = N_2 \) and \( c(v) \leq k \) for all \( v \in V(T) \). Then there exists a subtree \( S \) of \( T \) of weight \( k - g + 1 \leq c(S) \leq k \) and \( S \) can be found in linear time.

By Mohr’s transformation, a corollary follows immediately:

Corollary 2. Let \( G \) be a planar hamiltonian graph with \( |E(G)| \geq (2 + \gamma)|V(G)| \) for some real number \(-1 \leq \gamma < 1\). Let \( k, g, N \in \mathbb{N} \) such that \( g + |\gamma|V(G)| + 2 > 0 \), \( 3 \leq k \leq |V(G)| \) and \( \frac{1}{2} \leq \frac{1}{2} - \frac{1}{2} \leq \frac{|V(G)| - 2}{2} \leq \frac{|V(G)|}{2} + \gamma |V(G)| + 2 = c(T) + \gamma |V(G)| + 4 \) implies \( 2N_1 \geq N_2 + h \), and for every \( v \in V(T) \), \( c(v) \leq \frac{2}{3} |V(G)| - \frac{1}{2} |E(G)| - 2 \leq \frac{2}{3} |V(G)| - \frac{2 + \gamma}{2} |V(G)| - 2 = \frac{1 - \gamma}{2} |V(G)| - 2 \) implies \( c(v) \leq \frac{2}{3} |V(G)| - 2 \leq \tilde{k} \). As all conditions are satisfied, by Lemma 1, there exists a subtree \( S \) of weight \( k - g + 1 \leq c(S) \leq \tilde{k} \) which can be found in linear time. Hence \( G \) has a cycle \( K \) of length \( k - g + 1 \leq |V(K)| \) which can be found in linear time provided a Hamilton cycle of \( G \) is given, since every planar graph can be embedded in plane in linear time \([5]\) and the tree \( T_{int} \) can be easily constructed from the planar embedding in linear time.

Proof. Let \( T \) be the tree with vertex weights \( c \) that we mentioned before. We set \( \tilde{k} := k - 2 \geq 1 \) and \( h := |\gamma|V(G)| + 4 \). We check the conditions required for applying Lemma 1 on the parameters \( k, g, h, N_1 \) and \( N_2 \) as follows. First we have that \( g + h > 2 \), \( 1 \leq \tilde{k} \leq N_2 = |V(G)| - 2 \), \( 2k \leq \left[(1 + \gamma)|V(G)| + 4g - 1\right] = N_2 + 4g + h - 3 \), and \( \tilde{k} \geq \frac{|V(G)|}{2} - 2 \geq \frac{|V(G)|}{2} - \frac{3}{2} \geq N_2 - \frac{3}{2} - \frac{h}{2} + 1 \). Note also that \( 2|V(T)| \geq |E(G)| - |V(G)| + 2 \geq |V(G)| + 2 = c(T) + |V(G)| + 4 \) implies \( 2N_1 \geq N_2 + h \), and for every \( v \in V(T) \), \( c(v) \leq \frac{2}{3} |V(G)| - \frac{1}{2} |E(G)| - 2 \leq \frac{2}{3} |V(G)| - \frac{2 + \gamma}{2} |V(G)| - 2 = \frac{1 - \gamma}{2} |V(G)| - 2 \) implies \( c(v) \leq \frac{2}{3} |V(G)| - 2 \leq \tilde{k} \). As all conditions are satisfied, by Lemma 1, there exists a subtree \( S \) of weight \( k - g + 1 \leq c(S) \leq \tilde{k} \) which can be found in linear time. Hence \( G \) has a cycle \( K \) of length \( k - g + 1 \leq |V(K)| \) which can be found in linear time provided a Hamilton cycle of \( G \) is given, since every planar graph can be embedded in plane in linear time \([5]\) and the tree \( T_{int} \) can be easily constructed from the planar embedding in linear time.

In particular, by setting \( g := 1 \) and \( \gamma := 0 \) in Corollary 2, we have that every planar hamiltonian graph \( G \) with \( \delta \geq 4 \) has a cycle of length \( k \) for every \( k \in \left\{ \left\lfloor \frac{|V(G)|}{2} \right\rfloor, \ldots, \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 3 \right\} \) with \( 3 \leq k \leq |V(G)| \), and by setting \( \gamma := \frac{1}{2} \) instead, we have that every planar hamiltonian graph \( G \) with \( \delta \geq 5 \) has a cycle of length \( k \) for every \( k \in \left\{ \left\lfloor \frac{1}{2} |V(G)| \right\rfloor, \ldots, \left\lfloor \frac{3}{2} |V(G)| \right\rfloor + 3 \right\} \) with \( 3 \leq k \leq |V(G)| \). Note that Malkevitch’s example (see Figure 1) illustrates that not every planar hamiltonian graph \( G \) with \( \delta \geq 4 \) can have a cycle of length \( \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1 \). As a further application we will show in Section 4 that every 3-connected planar hamiltonian graph \( G \) with \( \delta \geq 4 \) has a cycle of length \( \frac{|V(G)|}{2} - 1 \) or \( \frac{|V(G)|}{2} - 2 \) if \( |V(G)| \geq 8 \) is even.

We emphasize that our result gives not only the existence of cycles of specified length but also a linear time algorithm for finding them provided a Hamilton cycle is given. We mention that this Hamilton cycle is not needed when the graph is 4-connected and planar, since it is known that in this case a Hamilton cycle can be found in linear time \([4]\).

There are also other results concerning subtrees in weighted tree. We refer the reader to \([25, 11]\) for tree partitioning problems, and to \([9, 1]\) for optimization problems regarding weighted trees or connected graphs. However, the author is not aware of result similar to ours on finding one subtree of specified weight in a tree with vertex weights.

1.3 Relation to the Number Partitioning Problem

We remark that the constraint \( 2N_1 \geq N_2 + h \) in Lemma 1 is a key ingredient for that a subtree of specified weight exists and can be found efficiently. By setting \( k := \frac{N_2}{2} \), \( g := 1 \) and \( h := 2 \) in Lemma 1, we can deduce that for \( N_1 \in \mathbb{N} \) and even \( N_2 \in \mathbb{N} \) with \( 2N_1 \geq N_2 + 2 \), and a tree \( T \) of \( N_1 \) vertices with vertex weights \( c : V(T) \to \mathbb{N} \) satisfying \( c(T) = N_2 \), a subtree of weight \( \frac{N_2}{2} \) is always guaranteed and can be obtained in linear time, as we have \( c(v) \leq N_2 - N_1 + 1 \leq \frac{N_2}{2} \) for
all \( v \in V(T) \). However this problem becomes \textbf{NP}-complete if we drop the contraint \( 2N_1 \geq N_2 + 2 \), since the classic \textbf{NP}-complete problem of number partitioning \cite{12} can simply be reduced to our problem as follows.

Given a multiset of \( N_1 \) positive integers \( \{a_1, \ldots, a_{N_1}\} \) with \( \sum_{i=1}^{N_1} a_i = N_2 \), let \( T \) be the \textit{star} with \( N_1 - 1 \) leaves and \( V(T) := \{v_1, \ldots, v_{N_2}\} \). Define \( c(v_i) := a_i \) for every \( i = 1, \ldots, N_1 \). Then the multiset can be partitioned into two subsets the sum of the elements of each is \( \frac{N_2}{2} \) if and only if there is a subtree of weight \( \frac{N_2}{2} \) in \( T \). This implies also that for \( N_1 \in \mathbb{N} \) and even \( N_2 \in \mathbb{N} \) with \( N_1 \geq \frac{N_2}{2} + 1 \), and a multiset of \( N_1 \) positive integers with total sum \( N_2 \), there always exists a subset the sum of whose elements is \( \frac{N_2}{2} \), which can be found in linear time. The contraint \( N_1 \geq \frac{N_2}{2} + 1 \) is best possible, since for an odd \( N_1 = \frac{N_2}{2} \), we can take a multiset all of whose elements are 2, then there is no subset of it the sum of whose element can equal the odd number \( N_1 \). We remark that our finding matches the fact that the number partitioning problem becomes \textit{easier} when the instances are \textit{smaller}; for more details we refer to \cite{17}.

2 \hspace{1cm} \textbf{Notation}

We use minus sign to denote set subtraction, and parentheses would be omitted for single elements if it causes no ambiguity.

We consider only simple graphs in this paper. Let \( G \) be an undirected graph. We denote by \( V(G) \) and \( E(G) \) the vertex set and the edge set of \( G \) and call \( |V(G)| \) and \( |E(G)| \) order and size of \( G \), respectively. We denote by \( d_G(v) \) the degree of vertex \( v \in V(G) \) in the graph \( G \). The minimum degree of \( G \) is defined as \( \delta(G) := \min_{v \in V(G)} d_G(v) \). Given a vertex set \( W \subseteq V(G) \), \( G[W] \) is defined to be the induced subgraph of \( G \) on \( W \). Let \( c : V(G) \to \mathbb{N} \) be vertex weights. For \( i \in \mathbb{N} \), we denote by \( V_i(G) \subseteq V(G) \) the set of vertices \( v \) in \( G \) with \( c(v) = i \). We write \( V_i := V_i(G) \) if there is no ambiguity. Let \( H \) be a subgraph of \( G \), we define \( c(H) := \sum_{v \in V(H)} c(v) \).

In an undirected graph \( G \) we denote by \( vw \) or \( wv \) the edge with endvertices \( v, w \in V(G) \). We abuse the notation of a sequence of vertices as follows. Let \( t \in \mathbb{N} \). For \( t \) distinct vertices \( v_1, v_2, \ldots, v_t \), we denote by \( v_1v_2 \ldots v_t \) the \textit{path} \( P \) with endvertices \( v_1, v_t \) such that \( V(P) := \{v_1, v_2, \ldots, v_t\} \) and \( E(P) := \{v_1v_2, v_2v_3, \ldots, v_{t-1}v_t\} \). For \( t \geq 3 \) distinct vertices \( v_1, v_2, \ldots, v_t \), we denote by \( v_1v_2 \ldots v_1v_1 \) the \textit{cycle} \( K \) of length \( t \) such that \( V(K) := \{v_1, v_2, \ldots, v_t\} \) and \( E(K) := \{v_1v_2, v_2v_3, \ldots, v_{t-1}v_t, v_tv_1\} \).

In a directed graph \( G \) we denote by \( vw \) the edge directed from \( v \) to \( w \) for \( v, w \in V(G) \). Let \( C \) be a directed cycle. For \( u, v \in V(G) \), we define \( [u, v]_C \) to be the path directed from \( u \) to \( v \) along \( C \). Subscripts can be omitted if it is clear from the context. Let \( vw \) be an edge in \( C \), we define \( v^+ := w \) and \( w^- := v \).

For a plane graph \( G \), we identify the faces of \( G \) not only with the vertices in the dual graph \( G^* \) but also with the cycles in the boundaries of the faces provided that \( G \) is 2-connected and is not a cycle.

Let \( T \) be a tree. For \( vw \in E(T) \), we denote by \( T[vw; v] \) the connected component of \( T - vw \) containing \( v \). Given a vertex \( a \in V(T) \), we specify the tree \( T \) rooted at \( a \) by \( T(a) \). For \( v \in V(T) \), \( T_{v}^{(a)} \) is defined as the subtree of \( T \) containing \( v \) and all of its descendants in \( T(a) \).

A graph \( G \) is said to be \( \kappa \)-connected for some \( \kappa \in \mathbb{N} \) if either \( G \) is the complete graph of order \( \kappa + 1 \) or \( G - U \) is connected for every \( U \subseteq V(G) \) with \( |U| < \kappa \).

3 \hspace{1cm} \textbf{Find Subtrees of Specified Length}

To prove the existence of the subtree \( S \) in Lemma 1 we will assume, towards a contradiction, that there is no subtree of \( T \) has weight between \( k - g + 1 \) and \( k \). In particular, under this assumption
we have $c(v) \leq k - g$ for all $v \in V(T)$, and $k < N_2$. Moreover, we claim that there are at least two distinct vertices in $T$ each of them has weight at least $g + 1$, i.e. $\sum_{i \geq g+1} |V_i| \geq 2$. Suppose there is at most one such vertex, we can take a vertex $v \in V(T)$ such that $c(w) \leq g$ for all $v \neq w \in V(T)$. As $c(v) \leq k$, we can take a maximal subtree $S$ of $T$ satisfying $v \in V(S)$ and $c(S) \leq k$. It is clear that $k - g + 1 \leq c(S) \leq k$; otherwise we can grow $S$ by one more vertex and the new subtree still has weight at most $k$, which contradicts the maximality of $S$.

In Section 3.2 we will give a weaker assumption which nevertheless suffices to prove the existence, and it will give us an algorithm which finds the desired subtree in linear time.

3.1 An Overload-Discharge Approach

We first see how we can count the number of vertices of weight 1. The equation $\sum_{v \in V(T)} c(v) = \sum_{i \geq 1} i|V_i| = N_2$ yields the following:

$$N_2 = \sum_{i \geq 1} i|V_i|$$

$$= 2\sum_{i \geq 1} |V_i| + \sum_{1 \leq i \leq g} (i - 2)|V_i| + \sum_{i \geq g+1} (i - 2)|V_i|$$

$$= 2N_1 - |V_1| + \sum_{2 \leq i \leq g} (i - 2)|V_i| + \sum_{i \geq g+1} (i - g - 1)|V_i| + \sum_{i \geq g+1} (g - 1)|V_i|$$

$$= \sum_{i \geq g+1} (i - g - 1)|V_i| + \sum_{i \geq 2} \min\{i - 2, g - 1\}|V_i| + 2N_1 - |V_1|.$$

As $2N_1 \geq N_2 + h$ and $\sum_{i \geq g+1} |V_i| \geq 2$, and hence we have the following lower bound on $|V_1|$:

$$|V_1| \geq \sum_{i \geq g+1} (i - g - 1)|V_i| + 2g + h - 2. \quad (\diamond)$$

The intuitive idea of our proof of Lemma 1 is that if $|V_1|$ is large enough, i.e. there are many vertices of weight 1, then it should facilitate the search of subtree of the desired weight. Therefore, by assuming that Lemma 1 does not hold, there would be some upper bound on the number of vertices of weight 1 which shows that the inequality $(\diamond)$ must be contradicted. The upper bound is realized by the following observation.

Observation 3. Let $g, k \in \mathbb{N}$. Let $S$ be a subtree of $T$ with $c(S) > k$, $l$ be a leave of $S$ with $c(S - l) < k - g + 1$, $M$ be a subset of $V(S) - l$ and $n$ a vertex in $S - M - l$ such that $S - M$ remains as a tree, $n$ is a leaf of $S - M$, $c(S - M) > k$ and $c(S - M - n) < k - g + 1$. Then we have

$$|M \cap V_1| \leq c(S) - (k + 1) \leq c(l) - g - 1. \quad (*)$$

The vertex set $M$ can be seen as a set of vertices which are collected from a leave-cutting process, i.e. we cut leaves (other than $l$) one by one from $S$ with that the weight of the remainder still larger than $k$, and it becomes less than $k - g + 1$ once we further cut the vertex $n$. Note that $l$ is not cut from the subtree and it always stays as a leaf in the remaining part.

Proof. Since $c(S) - c(M) = c(S - M) > k$ and $c(S) - c(l) = c(S - l) < k - g + 1$, we have

$$|M \cap V_1| \leq c(M) \leq c(S) - (k + 1) \leq c(l) - g - 1.$$
Note that the conditions given in Observation 3 come naturally when we have a tree $T$ which has no subtree of weight between $k-g+1$ and $k$. We carry out an overload-discharge process as follows. We grow a subtree (say a single vertex) which is of weight less than $k-g+1$ until we grow it with a vertex $l$ which makes the weight of the subtree at least $k-g+1$. As we assume that no subtree is of weight between $k-g+1$ and $k$, when we halt the growth, the weight of the subtree is actually not only at least $k-g+1$ but greater than $k$. We then start to cut its leaves (other than $l$) one by one until the weight declines to be less than $k-g+1$ again. The overload and discharge steps can always be achieved provided that $N_2 > k$ and $c(v) \leq k-g$ for all $v \in V(T)$.

In Observation 3, we say that $l$ overloads the overloading subtree $S$ and a discharge $M \cup \{n\}$ containing the last discharge $n$ follows, and that $(S,l,M,n)$ is an overload-discharge quadruple. It is clear that the vertex $l$ which overloads a subtree and the discharge vertex $n$ must have weights larger than $g$ (and hence $\sum_{i \geq g+1} |V_i| \geq 2$).

Let us have a look of a crude argument on how a contradiction would occur. Suppose we have a family of overload-discharge quadruples $(S_f,l_f,M_f,n_f)$ (with some indices $f$) such that the vertices of weight 1 in $T$ is covered by the discharges, i.e. $V_1 \subseteq \bigcup_f M_f$, and each overloading vertex $l_f$ corresponds to only one overload-discharge quadruple, then, by the inequality $(\diamond)$, we can simply deduce the following contradiction to the inequality $(\diamond)$:

$$|V_1| \leq \sum_f |M_f \cap V_1| \leq \sum_f (c(l_f) - g - 1) \leq \sum_{i \geq g+1} (i - g - 1)|V_i|.$$ 

Although it is not always possible to find such a family of quadruples, we are still able to have some sufficiently good family which leads to a contradiction to the inequality $(\diamond)$.

### 3.2 Overloading Subtrees by DFS

To meet the goal mentioned in the previous section, we consider the overload-discharge quadruples collected by DFS. We actually assume a fixed planar embedding of the tree $T$ and we walk around it, i.e. we see edges of $T$ as walls perpendicular to the plane and we walk on the plane along the walls. This walk yields a cycle of size $2(|V(T)| - 1)$.

To make it precise, we define the auxiliary directed cycle graph $C_T$ as follows. For each $v \in V(T)$, we enumerate the edges incident to $v$ in the clockwise order according to the planar embedding and denote them by $e_{v,1}, e_{v,2}, \ldots, e_{v,d_T(v)}$. The vertex set $V(C_T)$ consists of $d_T(v)$ vertices $w_{v,1}, w_{v,2}, \ldots, w_{v,d_T(v)}$ for each $v \in V(T)$. Let $w_{v,d_T(v)+1} := w_{v,1}$. And, for every edge $uv \in E(T)$, say $uv = e_{u,i} = e_{v,j}$ for some $i \in \{1, \ldots, d_T(u)\}$ and $j \in \{1, \ldots, d_T(v)\}$, $E(C_T)$ contains the edges $w_{u,i}w_{u,j+1}$ and $w_{v,j}w_{u,i+1}$. It is clear that $C_T$ is our desired cycle of size $2(|V(T)| - 1)$ (see Figure 3(a)).

Note that a directed path in $C_T$ can be naturally corresponded to a subtree in $T$. Moreover, growing a subtree by this walking-around-walls DFS in $T$ is equivalent to growing a directed path in $C_T$.

We define the mapping $\rho$ (a homomorphism) from $V(C_T)$ to $V(T)$ by $\rho(w_{v,i}) := v$ for $w_{v,i} \in V(C_T)$ with $v \in V(T)$ and $i \in \{1, \ldots, d_T(v)\}$. We also extend this mapping for paths $[u,v]$ directed from $u$ to $v$ in $C_T$ ($u,v \in V(C_T)$) by defining $\rho([u,v]) := T(\{\rho(w) : w \in V([u,v])\})$. We then extend the weight function $c$ to the vertices $w$ and directed paths $[u,v]$ in $C_T$ ($w,u,v \in V(C_T)$) by $c(w) := c(\rho(w))$ and $c([u,v]) := c(\rho([u,v]))$.

We now demonstrate how an overload-discharge quadruple can be formed by considering paths in $C_T$. Let $u,v$ be two distinct vertices of $C_T$. If $c([u,v^-]) < k-g+1$ but $c([u,v]) > k$, then there exists $w \in V([u,v^-])$ such that $c([w,v]) > k$ and $c([w^+,v]) < k-g+1$. It is clear that $\rho(v)$
overloads the subtree $\rho([u, v])$ and we call

$$(\rho([u, v]), \rho(v), V(\rho([u, v])) - V(\rho([w, v])), \rho(w)) =: Q_{u,v}$$

an overload-discharge quadruple associated with $u, v$.

An overload-discharge quadruple $Q_{u,v}$ associated with $u, v \in V(C_T)$ is maximal if $c([u^-, v^-]) > k$ holds (see Figure 3(b)). We let $Q(T; c, k) =: Q$ be the family of all maximal overload-discharge quadruples associated with some $u, v \in V(C_T)$.

Here we state an assumption (towards a contradiction) which we adopt from now on:

(Ω) There is no $x, y \in V(C_T)$ with $k - g + 1 \leq c([x, y]) \leq k$.

In other words there is no subtree of $T$ with weight between $k - g + 1$ and $k$ can be found by DFS. Therefore it is weaker than the assumption that $T$ has no subtree $S$ of weight $k - g + 1 \leq c(S) \leq k$. However it suffices and basically all materials we have discussed so far still hold under our new assumption, for instance, $c(v) \leq k - g$ for all $v \in V(T)$, $N_2 > k$ and $\sum_{i \geq g + 1} |V_i| \geq 2$ hold.

### 3.3 Support Vertices and Support Subtree

In order to see how the overloading subtrees from $Q$ would be packed in the weighted tree $T$, we need to study its structure in more detail. We introduce the notions of support vertices and support subtree of the weighted tree $T$ in this section.

We first fix an arbitrary vertex $a \in V(T)$ and consider the rooted tree $T^{(a)}$. Note that there always exists a vertex $r$ such that $c(T^{(a)}_r) > k$ and $c(T^{(a)}_w) \leq k$ for all children $w$ of $r$ in $T^{(a)}$, as we assume $c(T^{(a)}) = N_2 > k$. We then take one such vertex $r$ and consider the tree $T^{(r)}$ rooted at $r$. Let $r_1, \ldots, r_t (t \in \mathbb{N})$ be the vertices each satisfies that $c(T^{(r)}_{r_i}) > k$ and $c(T^{(r)}_{w}) \leq k$ for all children $w$ of $r_i$ in $T^{(r)}$ ($i = 1, \ldots, t$). We call $r, r_1, \ldots, r_t$ support vertices of $T$, and the minimal subtree $T^*$ containing all support vertices support subtree of $T$.

Note that $T^{(a)}_w$ is exactly the same subtree as $T^{(a)}_w$ for every $w \in V(T^{(a)}) - r$, therefore $c(T^{(r)}_w) = c(T^{(a)}_w) \leq k$ and $r_i \notin V(T^{(a)}_r) - r$ for every $i = 1, \ldots, t$. If there are two distinct support vertices
Lemma 4. Let $vw$ be an edge in $T$. If $c(T[vw; w]) \geq k - g$, then there exist $(S, l, M, n) \in \mathcal{Q}$ with $v \in M \cup \{n\}$.

Proof. Let $i \in \{1, \ldots, d_T(v)\}$ such that the edge $vw$ is $e_{v,i}$. We grow a path in $C_T$ from $u := w_{v,i}$ to obtain an overload-discharge quadruple $Q_u,y$ associated with $u, y$ for some $y \in V(C_T)$. The corresponding situation in $T$ is that a subtree starts growing at $v$, then traverses along the edge $e_{v,i}$ immediately. It will overload, i.e. the weight reaches larger than $k$, without revisiting $v$, since $c(v) + c(T[vw; w]) \geq k - g + 1$.

We can augment the path $[u, y]$ backwards along the cycle $C_T$ to obtain $[x, y]$ such that $c([x, y^-]) < k - g + 1$ but $c([x^-, y^-]) > k$. Then we have $Q_{x,y} \in \mathcal{Q}$. Note that $u$ is the only vertex in $[u, y]$ with $\rho(u) = v$, and $u$ cannot stay in the path after discharge since $c([u, y]) \geq k - g + 1$. Therefore the discharge of $Q_{x,y}$ must contain $v$. \hfill $\square$

We remark that the proof above has used the assumption (Ω) as it says that once the weight the subtree grown by DFS reaches $k - g + 1$, it will be larger than $k$. In the rest of the paper such usage of the assumption (Ω) would occur in an implicit way.

Now we give a necessary condition for a vertex to be an overloading vertex in some quadruple in $\mathcal{Q}$.

Lemma 5. Let $l \in V(T)$ and $Q_{x,y} \in \mathcal{Q}$ be an overload-discharge quadruple associated with some $x, y \in V(C_T)$. We have $c(T[\rho(y)\rho(y^-); \rho(y^-)]) + c(\rho(y)) > k$.

Proof. It is clear that the subtree $\rho([x, y^-])$ is contained in the subtree $T[\rho(y)\rho(y^-); \rho(y^-)]$. Therefore, $c(T[\rho(y)\rho(y^-); \rho(y^-)]) + c(\rho(y)) \geq c([x, y]) > k$ as $\rho(y)$ overloads $\rho([x, y])$. \hfill $\square$

We next show that if more than one overloading subtrees having the same overloading vertex, then the mutual intersection amongst these subtrees can only be the overloading vertex, and such a vertex must be in the support subtree. We also prove upper bounds on the these discharges.
Let $Q_{x_1,y_1}$ and $Q_{x_2,y_2}$ be two distinct overload-discharge quadruples associated with $x_1,y_1$ and $x_2,y_2$, where $x_1,y_1,x_2,y_2 \in V(C_T)$, in $Q$, respectively. If $\rho(y_1) = \rho(y_2) =: l$, we have

$$V(\rho([x_1,y_1])) \cap V(\rho([x_2,y_2])) = \{l\}$$

and $l$ must be a vertex in the support subtree $T^*$.

Proof. As $Q_{x_1,y_1}$ and $Q_{x_2,y_2}$ are distinct overload-discharge quadruples, by the choice of maximality of elements of $Q$, $y_1$ must be different from $y_2$. Hence we have distinct indices $i,j \in \{1,\ldots,d_T(l)\}$ such that $y_1 = w_{i,l+1}$ and $y_2 = w_{i,j+1}$. Note that $y_f$ ($f = 1,2$) is the only vertex in $[x_f,y_f]$ with $\rho(y_f) = l$ since $l$ is the overloading vertex. It means that $l$ is not in the subtree $\rho([x_f,y_f^-])$ ($f = 1,2$). Moreover, the subtree $\rho([x_1,y_1^-])$ is a subtree of $T[e_{l,i};\rho(y_1^-)]$, i.e. the component not containing $l$ when deleting the edge $e_{l,i}$, and similarly, $\rho([x_2,y_2^-])$ is a subtree of the component not containing $l$ when deleting the edge $e_{l,j}$. Therefore $V(\rho([x_1,y_1^-])) \cap V(\rho([x_2,y_2^-])) = \emptyset$ and $V(\rho([x_1,y_1])) \cap V(\rho([x_2,y_2])) = \{l\}$.

Suppose $l \notin V(T^*)$. Let $u \in V(T^*)$ and $w$ be the neighbor of $l$ such that $u \in T[lw;w]$. By the definition of the support subtree, for every neighbor $v$ of $l$ other than $w$, we have $c(T[lv;v]) + c(v) \leq k$. By Lemma 5, there are at most one overloading subtree whose overloading vertex is $l$.

Lemma 7. Let $T_0$ be a subtree of $T$. Let $l$ be an overloading vertex shared by $t \in \mathbb{N}$ quadruples $(S_i,l,M_i,n_i) \in Q$, for $i = 1,\ldots,t$, such that $S_i$ is a subtree of $T_0$ for every $i = 1,\ldots,t$. We have

$$\sum_{i=1}^{t} |M_i \cap V_l| \leq c(l) + (t-2)(c(l) - k - 1) + c(T_0) - 2k - 2 \leq c(T_0) - k - 1.$$

Proof. By Lemma 6, the overloading subtrees share only the overloading vertex $l$, hence

$$c(l) + \sum_i (c(S_i) - c(l)) \leq c(T_0).$$

By Observation 3 and the assumption $c(v) \leq k$ for all $v \in V(T)$, we have

$$\sum_i |M_i \cap V_l| \leq \sum_i (c(S_i) - (k + 1)) = \sum_i (c(S_i) - c(l)) + t(c(l) - (k + 1)) \leq c(T_0) - c(l) + t(c(l) - (k + 1)) = c(T_0) + c(l) - 2(k + 1) + (t-2)(c(l) - k - 1) = c(T_0) - k - 1 + (t-1)(c(l) - k - 1) \leq c(T_0) - k - 1.$$

As the last preparation for the proof of Lemma 1 we show that a reasonable portion of vertices will be covered by the discharges from $Q$.

Lemma 8. If $|V(T^*)| > 1$, then we have

$$\bigcup_{(S,l,M,n) \in Q} (M \cup \{n\}) = V(T).$$
If \(|V(T^*)| = 1\) and \(N_2 \geq 2k - 2g - D\) for some \(D \in \mathbb{N}\), then we have \(|\bigcup_{(S,l,M,n) \in Q}(M \cup \{n\})| \geq |V(T)| - D\). If \(|V(T^*)| = 1\) and \(N_2 \geq 2k - 2g\), then we have
\[
\bigcup_{(S,l,M,n) \in Q}(M \cup \{n\}) \supseteq V(T) - V(T^*).
\]

Proof. If \(|V(T^*)| > 1\), for a vertex \(v\) in \(T\), we can take a support vertex \(u \neq v\) such that \(v \notin T^*[u]\). Let \(w\) be the vertex adjacent to \(v\) such that \(u \in T[vw,w]\). We have \(c(T[vw,w]) \geq c(T^*[u]) \geq k + 1 \geq k - g\), and hence, by Lemma 4, there exists \((S,l,M,n) \in Q\) such that \(v \in M \cup \{n\}\). Thus \(\bigcup_{(S,l,M,n) \in Q}(M \cup \{n\}) = V(T)\).

If \(|V(T^*)| = 1\) and \(N_2 \geq 2k - 2g - D\) for some \(D \in \mathbb{N}\), let \(r\) be the only one support vertex. Consider the tree \(T^{(r)}\) rooted at \(r\). Let \(U\) be the set of vertices \(v\) with \(c(T^{(r)}_v) \geq N_2 - k + g + 1\). For \(v \in V(T) - r - U\), let \(w\) be the parent of \(v\) in \(T^{(r)}\), we have \(c(T[vw,w]) \geq N_2 - (N_2 - k + g) = k - g\) and hence, by Lemma 4, \(v\) is covered by some discharge from \(Q\). We can assume that \(U\) is not empty (otherwise at most one vertex, namely the root \(r\), can be not covered by any discharge from \(Q\)).

We consider the subtree \(T[U]\) of \(T\) induced by \(U\). If \(T[U]\) has two leaves \(v, w\) other than \(r\), then \(|U| \leq 2 + c(T[U] - v - w) \leq 2 + N_2 - c(T^{(r)}_v) - c(T^{(r)}_w) \leq 2 + N_2 - 2(N_2 - k + g + 1) = -N_2 + 2k - 2g \leq D\). Otherwise \(T[U]\) is a path with \(r\) as one of the endvertices, say \(r v_1 v_2 \ldots v_t\) for some integer \(t \geq 0\). If \(t > D\), then \(c(T^{(r)}_{v_t}) \geq \sum_{i=1}^{t-1} c(v_i) + c(T^{(r)}_{v_i}) \geq D + c(T^{(r)}_{v_t}) \geq D + (N_2 - k + g + 1) \geq k - g + 1\) which contradicts the definition of the support subtree \(T^*\) as in this case \(v_t\) should be in \(V(T^*)\). If \(t = D\), similarly as above, we have \(c(T^{(r)}_{v_t}) \geq k - g\) and hence, by Lemma 4, \(r\) is covered by some discharge from \(Q\). In any case, we have that there are at most \(D\) vertices which are not covered by any discharge from \(Q\), i.e. \(|\bigcup_{(S,l,M,n) \in Q}(M \cup \{n\})| \geq |V(T)| - D\).

If \(|V(T^*)| = 1\) and \(N_2 \geq 2k - 2g\), let \(r\) be the only support vertex and \(r \neq v \in V(T)\) be a vertex in \(T\). By the definition of the support subtree, we have that \(c(T^{(r)}_v) \leq k - g\). Let \(w\) be the parent of \(v\) in \(T^{(r)}\). We have \(c(T[vw,w]) = N_2 - c(T^{(r)}_v) \geq (2k - 2g) - (k - g) = k - g\). Therefore

\[
V(T) - r \subseteq \bigcup_{(S,l,M,n) \in Q}(M \cup \{n\}).
\]

\[\square\]

### 3.5 Proof of Lemma 1

In this section we prove Lemma 1. We first consider the case that \(N_2 \geq 2k - 2g\). If \(|V(T^*)| = 1\), let \(r\) be the only support vertex. If \(c(r) < g + 1\), then by Lemmas 8 and 6 and the condition that \(g + h > 2\), we have

\[
|V_i| \leq \sum_{(S,l,M,n) \in Q} |M \cap V_i| + 1
\]
\[
\leq \sum_{(S,l,M,n) \in Q} (c(l) - g - 1) + 1
\]
\[
\leq \sum_{i \geq g+1} (i - g - 1)|V_i| + 1
\]
\[
< \sum_{i \geq g+1} (i - g - 1)|V_i| + 2g + h - 2.
\]
Otherwise, \( c(r) \geq g + 1 \) and \( r \) can be an overloading vertex and we apply Lemmas 7 to bound the corresponding discharges as follows:

\[
|V_1| \leq \sum_{(S, l, M, n) \in Q, l \neq r} |M \cap V_1| + \sum_{(S, l, M, n) \in Q, l = r} |M \cap V_1|
\]

\[
\leq \sum_{(S, l, M, n) \in Q, l \neq r} (c(l) - g - 1) + \max\{0, c(r) - g - 1, c(r) + N_2 - 2k - 2\}
\]

\[
\leq \sum_{(S, l, M, n) \in Q, l \neq r} (c(l) - g - 1) + c(r) + g + h - 4
\]

\[
\leq \sum_{i \geq g + 1} (i - g - 1)|V_i| + 2g + h - 3.
\]

The third inequality follows from the condition that \( N_2 \leq 2k + g + h - 2 \). In any case the inequality \((\diamond)\) is contradicted.

If \( |V(T^*)| > 1 \), then, by Lemma 8, all vertices in \( V_1 \) are covered by some discharge from \( Q \). For a vertex \( u \in V(T^*) \), by Lemma 7 and Observation 3, we have

\[
\sum_{(S, u, M, n) \in Q} |M \cap V_1| \leq \max\{0, c(T^*[u]) - k - 1\} + d_{T^*}(u) \max\{0, c(u) - g - 1\}.
\]

Define \( U_1 \) to be the set of vertices \( u \in V(T^*) \) satisfying \( d_{T^*}(u) = 1 \), \( U_2 \) the set of vertices \( u \in V(T^*) \) satisfying \( d_{T^*}(u) > 1 \) and \( c(T^*[u]) \geq k + 1 \), and \( U_3 \) the set of vertices \( u \in V(T^*) \) satisfying \( c(u) \geq g + 1 \). Recall that \( U_1 \) is exactly the set of support vertices and \( c(T^*[u]) \geq k + 1 \) for all \( u \in U_1 \). As \( U_1 \) is disjoint with \( U_2 \), we have

\[
N_1 \geq \sum_{u \in U_1 \cup U_2} c(T^*[u]) + \sum_{u \in U_3 - (U_1 \cup U_2)} c(u),
\]

\[
\sum_{(S, u, M, n) \in Q, u \in V(T^*)} |M \cap V_1|
\]

\[
\leq \sum_{u \in U_1 \cup U_2} (c(T^*[u]) - k - 1) + \sum_{u \in U_3} d_{T^*}(u) (c(u) - g - 1)
\]

\[
= \sum_{u \in U_1 \cup U_2} (c(T^*[u]) - k - 1) + \sum_{u \in U_3} (c(u) - g - 1) + \sum_{u \in U_3 - U_1} (d_{T^*}(u) - 1) (c(u) - g - 1)
\]

\[
\leq \sum_{u \in U_1 \cup U_2} c(T^*[u]) + \sum_{u \in U_3 - (U_1 \cup U_2)} c(u) + \sum_{u \in U_3} (c(u) - g - 1) + \sum_{u \in U_1} (c(u) - g - 1)
\]

\[
+ \sum_{u \in U_3 - U_1} ((d_{T^*}(u) - 1) (c(u) - g - 1) - k - 1) + \sum_{u \in U_3 - (U_1 \cup U_2)} ((d_{T^*}(u) - 1) (c(u) - g - 1) - c(u))
\]

\[
\leq N_2 + \sum_{u \in U_3} (c(u) - g - 1) + \sum_{u \in U_3 - U_1} (d_{T^*}(u) - 2) (-k - 1) + 2 (-k - 1)
\]

\[
+ \sum_{u \in U_3 - U_1} (d_{T^*}(u) - 2) (c(u) - g - 1)
\]

\[
\leq \sum_{u \in U_3} (c(u) - g - 1) + N_2 - 2k - 2.
\]

In the third inequality we utilize the basic fact about tree that \( \sum_{u \in U_1} 1 = \sum_{u \in U_1} d_{T^*}(u) = \)

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\[ \sum_{u \in V(T^*) - U_1} (d_{T^*}(u) - 2) + 2. \] Thus we have

\[
\begin{align*}
|V_1| &\leq \sum_{(S,l,M,n) \in Q, l \notin V(T^*)} |M \cap V_1| + \sum_{(S,l,M,n) \in Q, l \in V(T^*)} |M \cap V_1| \\
&\leq \sum_{(S,l,M,n) \in Q, l \notin V(T^*)} (c(l) - g - 1) + \sum_{l \in U_3} (c(l) - g - 1) + N_2 - 2k - 2 \\
&\leq \sum_{i \geq g+1} (i - g - 1)|V_1| + g + h - 4,
\end{align*}
\]

which contradicts the inequality (\(\diamondsuit\)).

We now consider the case that \(N_2 < 2k - 2g\). Note that in this case \(|V(T^*)| = 1\) always holds. Let \(r\) be the only support vertex. Set \(D := 2g + h - 3\) in Lemma 8, we have

\[
|V_1| \leq \sum_{(S,l,M,n) \in Q, l \neq r} (c(l) - g - 1) + \max\{0, c(r) - g - 1, c(r) + N_2 - 2k - 2\} + D \\
\leq \sum_{(S,l,M,n) \in Q, l \neq r} (c(l) - g - 1) + \max\{0, c(r) - g - 1, c(r) - 2g - 3\} + 2g + h - 3 \\
\leq \sum_{i \geq g+1} (i - g - 1)|V_1| + 2g + h - 3,
\]

which contradicts the inequality (\(\diamondsuit\)).

Thus it is proved the existence of a subtree \(S\) with weight \(k - g + 1 \leq c(S) \leq k\). As the assumption (\(\Omega\)) cannot hold, it is not hard to see that the subtree \(S\) can be found by the iterative overload-discharge process in linear time, see Algorithm 1. This completes the proof of Lemma 1.

---

**Algorithm 1**

**Input:** A tree \(T\) of \(N_1\) vertices and vertex weights \(c: V(T) \rightarrow \mathbb{N}\) with \(c(T) = N_2\) such that \(g + h > 2, 2N_1 \geq N_2 + h, 1 \leq k \leq N_2\) and \(2k - 4g - h + 3 \leq N_2 \leq 2k + g + h - 2\) \((k, g, N_1, N_2 \in \mathbb{N}, h \in \mathbb{Z})\).

**Output:** A subtree \(S\) of \(T\) with \(k - g + 1 \leq c(S) \leq k\).

1. Construct the directed cycle \(C_T\) and choose an arbitrary vertex \(v\) of \(C_T\). Set \(s := v\) and \(t := v\).
2. **while** \(c([s, t]) < k - g + 1\) **do**
3. \[ \text{Set } t := t^+\]
4. **if** \(c([s, t]) \leq k\) **then**
5. \[ \text{Output } \rho([s, t]).\]
6. **while** \(c([s, t]) > k\) **do**
7. \[ \text{Set } s := s^+.\]
8. **if** \(c([s, t]) \geq k - g + 1\) **then**
9. \[ \text{Output } \rho([s, t]).\]
10. **go to 2.**

---

4 3-Connected Planar Hamiltonian Graphs

In this section we prove that we can have more cycle lengths assured for 3-connected planar hamiltonian graphs with \(\delta \geq 4\).
Theorem 9. Let $G$ be a 3-connected planar hamiltonian graph with minimum degree $\delta(G) \geq 4$. If $|V(G)| \geq 8$ is even, there exists a cycle of length either $\frac{1}{2}|V(G)| - 2$ or $\frac{1}{2}|V(G)| - 1$ in $G$, and it can be found in linear time if a Hamilton cycle is given.

Proof. We adopt the notations defined in Section 1.2. If every face of $G_{\text{int}}$ is of length either $|V(G)|$ or less than $\frac{1}{2}|V(G)|$, i.e. $c(v) \leq \frac{1}{2}|V(G)| - 3$ for every $v \in V(T_{\text{int}})$, by Lemma 1 (set $g := 2$ and $h := 4$), there exists a subtree of weight either $\frac{1}{2}|V(G)| - 4$ or $\frac{1}{2}|V(G)| - 3$ in $T_{\text{int}}$ and hence a cycle of length either $\frac{1}{2}|V(G)| - 2$ or $\frac{1}{2}|V(G)| - 1$ in $G$.

Recall that $|E(G_{\text{int}})| \geq \frac{3}{2}|V(G)|$. If $|E(G_{\text{int}})| > \frac{3}{2}|V(G)|$, then $|V(T_{\text{int}})| \geq \frac{1}{2}|V(G)| + 2 = \frac{1}{2}c(T_{\text{int}}) + 3$ and $c(v) \leq c(T_{\text{int}}) - |V(T_{\text{int}})| + 1 \leq \frac{1}{2}|V(G)| - 3$ for all $v \in V(T_{\text{int}})$. By Lemma 1 (set $g := 1$ and $h := 6$), there exists a subtree of weight $\frac{1}{2}|V(G)| - 3$ in $T_{\text{int}}$ and hence a cycle of length $\frac{1}{2}|V(G)| - 1$ in $G$.

Now we can assume that $|E(G_{\text{int}})| = \frac{3}{2}|V(G)|$ and $G_{\text{int}}$ has a face of length $\frac{1}{2}|V(G)|$. It holds immediately that $|E(G_{\text{ext}})| = \frac{3}{2}|V(G)|$ since $|E(G_{\text{int}})| + |E(G_{\text{ext}})| = |E(G)| + |V(G)| \geq 3|V(G)|$ and $|E(G_{\text{int}})| \geq |E(G_{\text{ext}})|$. And we can also assume that $G_{\text{ext}}$ has a face of length $\frac{1}{2}|V(G)|$. In this case we have $d_G(v) = 4$ and $d_{G_{\text{int}}}(v) + d_{G_{\text{ext}}}(v) = 6$ for every $v \in V(G)$, and that there are exactly one face of length $|V(G)|$, one face of length $\frac{1}{2}|V(G)|$ and $\frac{1}{2}|V(G)|$ faces of length 3 in each of $G_{\text{int}}$ and $G_{\text{ext}}$. We denote by $F_{\text{int}}$ and $F_{\text{ext}}$ be the faces of length $\frac{1}{2}|V(G)|$ in $G_{\text{int}}$ and $G_{\text{ext}}$, respectively.

We claim that $G$ is the square of a cycle of length $|V(G)|$, which is obtained from a cycle of length $|V(G)|$ by adding edges for every pair of vertices having distance 2 (see Figure 4). It is obvious that the square of a cycle is pancyclic. We call a face of length 3 an $i$-triangle ($i = 0, 1, 2$) if it contains exactly $i$ edges of the Hamilton cycle $C$. We assume that the plane graph $G$ has the maximum number of $i$-triangles over all of its planar embeddings. Let the Hamilton cycle $C$ of $G$ be $v_0v_1v_2 \ldots v_{|V(G)| - 1}v_0$ (indices modulo $|V(G)|$).

Figure 4: The square of a cycle of length 16.

Suppose there is a 0-triangle $v_0v_1v_2v_0$ in $G$, say it is also in $G_{\text{int}}$, for some $0 < i - 1 < j - 2 < |V(G)| - 3$. If $i > 2$, then the face in $G_{\text{int}}$ containing the path $v_1v_0v_iv_{i-1}$ is of length larger than 3 and smaller than $|V(G)|$. As there is exactly one such face in $G_{\text{int}}$, namely $F_{\text{int}}$, we can assume that $i = 2$ and $j = 4$. Then $d_{G_{\text{int}}}(v_1) = d_{G_{\text{int}}}(v_3) = 2$ and $d_{G_{\text{ext}}}(v_1) = d_{G_{\text{ext}}}(v_3) = 4$. Let $v_1, v_2$ be the neighbors of $v_1$ other than $v_0, v_2$, and $v_1, v_3, v_4$ be the neighbors of $v_3$ other than $v_2, v_4$, for some $2 < i_1 < i_3 < i_4 < |V(G)| + 2$.

If $v_1$ is adjacent to $v_3$ in $G_{\text{ext}}$, i.e. $i_1 = 3$ and $i_4 = |V(G)| + 1$, and the face in $G_{\text{ext}}$ containing $v_2v_1v_3v_2$ is a face of length 3, i.e. $i_2 = i_3$, then it must be a 0-triangle. We can assume that $i_2 = i_3 = 5$. Clearly, $\{v_0, v_5\}$ is a separator of $G$, which contradicts that $G$ is 3-connected. If $v_1$ is adjacent to $v_3$ in $G_{\text{ext}}$, but the face in $G_{\text{ext}}$ containing $v_2v_1v_3v_2$ is a face of length larger than 3, then the faces in $G_{\text{ext}}$ containing $v_0v_1v_2$ and $v_3v_4$ must be 2-triangles and $\{v_2, v_3\} = \{v_1, v_5\}$ is a separator of $G$, contradiction.
If \( v_1 \) is not adjacent to \( v_3 \), then the face in \( G_{\text{ext}} \) containing \( v_1 v_2 v_3 v_4 \) is of length larger than 3, and hence \( v_1 v_0 v_1 v_{-1} \) and \( v_2 v_1 v_3 v_5 \) must be 2-triangles in \( G_{\text{int}} \). In this case we can swap \( v_0 \) and \( v_1 \) and swap \( v_3 \) and \( v_4 \) to obtain a planar embedding with more 2-triangles (see Figure 5(a)), which contradicts the maximality of the number of 2-triangles. Thus there is no 0-triangle in the plane graph \( G \).

Suppose there is a 1-triangle \( v_0 v_1 v_2 v_0 \) in \( G \), say also in \( G_{\text{int}} \), for some \( 2 < i < |V(G)| - 1 \). It is not hard to see that we can assume that the face in \( G_{\text{int}} \) containing \( v_0 v_1 v_{i+1} \) is \( F_{\text{int}} \). Under this assumption we must have a sequence of \( i-1 \) faces of length 3 such that all faces are 1-triangles except the last one which is a 2-triangle, namely \( v_0 v_1 v_0, v_1 v_{-1} v_1, v_1 v_2 v_1, \ldots, v_{[\frac{i}{2}]-1} v_{[\frac{i}{2}]+1} v_{[\frac{i}{2}]-1} \).

We claim that \( i \leq 4 \). Suppose \( i > 4 \), we prove the claim for odd \( i \), it can be proved for even \( i \) in a similar way. It is clear that \( d_{G_{\text{ext}}} (v_{[\frac{i}{2}]-3}) \leq 3 \), \( d_{G_{\text{ext}}} (v_{[\frac{i}{2}]+1}) = 3 \) and \( d_{G_{\text{ext}}} (v_{[\frac{i}{2}]+2}) = 4 \). Let \( v_{i_1}, v_{i_2} \) be the neighbors of \( v_{[\frac{i}{2}]} \) other than \( v_{[\frac{i}{2}]+1} \), and \( v_{i_3} \) be the neighbor of \( v_{[\frac{i}{2}]-1} \) other than \( v_{[\frac{i}{2}]-2}, v_{[\frac{i}{2}]+1} \), for some \( i < i_1 < i_2 \leq i_3 \leq |V(G)| \). Note that the face in \( G_{\text{ext}} \) containing \( v_1 v_{[\frac{i}{2}]-1} v_{[\frac{i}{2}]+1} \) is of length larger than 3. Therefore the face in \( G_{\text{ext}} \) containing \( v_{i_3} v_{[\frac{i}{2}]-1} v_{[\frac{i}{2}]+1} \) and that containing \( v_{[\frac{i}{2}]-3} v_{[\frac{i}{2}]-2} v_{[\frac{i}{2}]-1} v_{[\frac{i}{2}]+1} \) must be of length 3. It implies that \( v_{[\frac{i}{2}]-3} = v_{i_3} = v_{i_2} \) and \( d_{G_{\text{ext}}} (v_{[\frac{i}{2}]-3}) \geq 4 \), contradiction.

Now we consider the case when \( i = 4 \). It is clear that \( d_{G_{\text{ext}}} (v_2) = 4 \) and \( d_{G_{\text{ext}}} (v_3) = 3 \). Let \( v_1 \) be the neighbor of \( v_3 \) other than \( v_1, v_2, v_4 \), and \( v_{i_2}, v_{i_3} \) be the neighbors of \( v_2 \) other than \( v_1, v_3 \), for some \( 4 < i_1 < i_2 < i_3 \leq |V(G)| \). If the face in \( G_{\text{ext}} \) containing \( v_1 v_2 v_3 \) is of length larger than 3, then \( i_1 = i_2 = |V(G)| - 1 \), \( i_3 = |V(G)| \) and \( \{v_{i-1}, v_4\} \) is separator of \( G \). If the face in \( G_{\text{ext}} \) containing \( v_1 v_4 v_3 \) is of length 3 but that containing \( v_1 v_2 v_4 \) is of length larger than 3, then \( i_1 = 5, i_2 = |V(G)| - 1, i_3 = |V(G)| \) and \( \{v_{i-1}, v_3\} \) is a separator of \( G \). If the faces in \( G_{\text{ext}} \) containing \( v_1 v_3 v_4 \) and \( v_1 v_3 v_2 v_4 \) are of length 3 but that containing \( v_1 v_2 v_3 \) is of length larger than 3, then \( i_1 = i_2 = 5, i_3 = |V(G)| \) and \( \{v_0, v_5\} \) is a separator of \( G \). If the faces in \( G_{\text{ext}} \) containing \( v_1 v_3 v_4 \), \( v_1 v_3 v_2 v_4 \) and \( v_{i_2} v_{i_3} v_4 \) are of length 3, then \( i_1 = i_2 = 5, i_3 = 6 \) and \( \{v_0, v_6\} \) is a separator of \( G \). In any case it contradicts that \( G \) is 3-connected.

Finally, we consider the case when \( i = 3 \). It is clear that \( d_{G_{\text{ext}}} (v_1) = 3 \) and \( d_{G_{\text{ext}}} (v_2) = 4 \). Let \( v_{i_1}, v_{i_2} \) be the neighbors of \( v_2 \) other than \( v_1, v_3 \), and \( v_{i_3} \) be the neighbor of \( v_1 \) other than \( v_0, v_2, v_3 \), for some \( 3 < i_1 < i_2 \leq i_3 \leq |V(G)| \). If the face in \( G_{\text{ext}} \) containing \( v_1 v_2 v_3 \) is of length larger than 3, then \( i_1 = |V(G)| - 2 \) and \( i_2 = i_3 = |V(G)| - 1 \), which has been shown to be not possible. If the face in \( G_{\text{ext}} \) containing \( v_1 v_2 v_3 \) is of length 3 but that containing \( v_1 v_2 v_4 \) is of length larger than 3, then \( i_1 = 4, i_2 = i_3 = |V(G)| - 1 \) and \( d_{G_{\text{int}}} (v_0) = 4 \). Let \( v_{i_4} \) be the neighbor of \( v_0 \) other than \( v_{i-1}, v_1, v_3 \) for some \( 3 < i_4 \leq |V(G)| - 2 \). If the faces in \( G_{\text{int}} \) containing \( v_{i-1} v_0 v_{i_4} \) is of length larger than 3, then \( i_4 = 4 \), which has been shown to be not possible. Hence \( v_{i-1} v_0 v_{i_4} v_{i-1} \) is 2-triangle, \( i_4 = |V(G)| - 2 \) and \( \{v_{i-2}, v_4\} \) is a separator of \( G \), which is not possible. If the faces in \( G_{\text{ext}} \) containing \( v_1 v_2 v_3 \) and \( v_{i_1} v_{i_2} v_{i_3} \) are of length 3, then \( i_1 = 4 \) and \( i_2 = 5 \). Swapping \( v_2 \) and \( v_3 \) yields a planar embedding of more 2-triangles (see Figure 5(b)), which contradicts the maximality of the number of 2-triangles. Hence we can conclude that there is no 1-triangle in the plane graph \( G \).

It is clear that \( G \) is the square of a cycle of length \( |V(G)| \) if it has a planar embedding with neither 0- nor 1-triangle. To find a cycle of the desired length, one can apply Algorithm 1 for \( T_{\text{int}} \) if \( |E(G_{\text{int}})| > \frac{3}{2} |V(G)| \), or if there is no face of length \( \frac{1}{2} |V(G)| \) in \( G_{\text{int}} \) and \( G_{\text{ext}} \), otherwise, do swaps of some vertex pairs at most once for each face to obtain a planar embedding of the square of a cycle of length \( |V(G)| \) with neither 0- nor 1-triangle, then a cycle of length \( \frac{1}{2} |V(G)| \) can be easily found in such planar embedding in linear time.
Figure 5: Swap vertices to obtain planar embedding of more 2-triangles. Edges in $C$, $C_{\text{int}}$ and $C_{\text{ext}}$ are indicated as black, green and red, respectively.

Acknowledgments

The author is very thankful to Samuel Mohr for motivating the problem and sharing his elegant idea; to Tomáš Madaras for a comprehensive survey on the subject; and to Matthias Kriesell and Jens M. Schmidt for helpful discussions.

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