Spiked oscillators: exact solution

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Abstract
A procedure to obtain the eigenenergies and eigenfunctions of a quantum spiked oscillator is presented. The originality of the method lies in an adequate use of asymptotic expansions of Wronskians of algebraic solutions of the Schrödinger equation. The procedure is applied to three familiar examples of spiked oscillators.

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1. Introduction
Spiked harmonic oscillators, i.e. harmonic oscillators to which a singular repulsion at the origin
\[
\frac{\lambda}{r^\alpha}, \quad \lambda > 0, \quad \alpha > 0,
\]
has been added, have deserved a considerable interest since the publication of the pioneering paper of Klauder [1], three decades ago. A very extensive list of articles dealing with the topic can be found in more recent papers by Saad et al [2] and by Liverts and Mandelzweig [3].

Singular (at the origin) potentials were firstly considered in the context of collision of particles [4], the main interest being the adequate definition of the S matrix. Here we are concerned with the determination of the energy levels and the corresponding wavefunctions of a particle bound in a confining potential behaving at infinity like \(r^n\), \(n \geq 2\), and presenting a singularity of the type \(r^{-m}\), \(m > 2\), at the origin.

Besides numerical integration of the Schrödinger equation by procedures adapted to the singular potential [5–7] and iterative methods of the Lanczos type [8], different approximate methods have been implemented to solve spiked oscillator problems. The usual WKB method was discarded from the very beginning in view of the results of Detwiler and Klauder [9] which also showed that conventional perturbative methods could not be used in the case of supersingular potentials. Harrell, in a very lucid paper [10], suggested a modified perturbation theory to a finite order applicable in that case. A perturbative study of nonsingular (\(\alpha < 5/2\)) spiked harmonic oscillators in the two regimes of weak coupling (\(\lambda \ll 1\)) and strong coupling
(λ ≫ 1) has been carried out by Aguilera–Navarro et al [11]. They and other authors [12] have discussed the connection between the expansions obtained in the two regimes. For the case of critical (α = 5/2) singularity, a large coupling perturbative expansion has been obtained [13].

One or more supersingular (α > 5/2) terms added to the harmonic oscillator potential have also been treated perturbatively by Guardiola and Ros [14] and by Hall and co-workers [15, 16]. Nonperturbative procedures have also been used: let us mention different implementations of variational methods [2, 13, 16–19], matricial [20] and Hill-determinant [21] techniques, a conveniently modified WKB method [22], smooth transformations of solvable potentials [23], the pseudoperturbative shifted-l expansion technique [24] and the more recently proposed quasilinearization method [3].

All these procedures provide with reasonably accurate values of the energies, at least for the ground state, but are not able to give the wavefunction in the vicinity of the origin. This fact, in addition to being unsatisfactory from an aesthetical point of view, may lead to incorrect expected values of certain operators.

The first attempt to obtain the correct wavefunctions of spiked harmonic oscillators was made, to our knowledge, by Znojil [25, 26], who looked for solutions of the Schrödinger equation in the form of Laurent series multiplied by a noninteger power of the variable. Substitution of such series in the differential equation leads to a recurrence relation satisfied by the coefficients; convergence of the series occurs only for a specific value of the noninteger exponent. For any given energy, two independent solutions of that kind are obtained. They constitute a basis in the space of solutions of the differential equation for that value of the energy. The wavefunction, that can be written as a linear combination (with only one effective degree of freedom) of these two basic solutions, must be well behaved at the origin and at infinity. This double requirement can be satisfied, by adjusting the degree of freedom of the mentioned linear combination, only when the considered value of the energy is one of its eigenvalues. This is, schematically, the procedure followed by Znojil to find the energies and the wavefunctions of certain spiked harmonic oscillators. The procedure, rigorous in principle, presents the drawback that there is no possibility of evaluating the basic solutions neither at the origin nor at infinity. For this reason, Znojil obtained approximate values of the eigenenergies by requiring the vanishing of the wavefunction at a pair of points \( r_0 \ll 1 \) and \( r_\infty \gg 1 \).

The procedure that we present in this paper is much in the spirit of Znojil’s method, but differs from it in the manner in which the regularity of the wavefunction is imposed: our \( r_0 \) and \( r_\infty \) are actually the origin and the infinity. Although the wavefunction cannot be calculated at these points, we are able to know its asymptotic behaviour in their vicinity and we may require that it be the correct one. The eigenenergies determined in this way are exact, except for errors inherent to the computation process. But these errors can be reduced by increasing the number of digits carried along the calculations.

In the next section we consider basic solutions of the Schrödinger equation for a very general three-dimensional spiked oscillator represented by the potential

\[
V(r) = \sum_q A(q) r^q,
\]

where the index \( q \) runs along a finite set of negative and positive integers and/or rational numbers, whose extremes are \( q_{\text{min}} < 0 \) and \( q_{\text{max}} > 0 \). Moreover, in order to have a true spiked oscillator, we assume that both \( A(q_{\text{min}}) \) and \( A(q_{\text{max}}) \) are positive. Section 3 explains how to determine the behaviour at the origin and at infinity of a general solution. The requirement, in section 4, of a regular behaviour at both singular points provides a quantization condition whose fulfilment determines the energy levels and the wavefunction. The procedure is applied
to three popular cases of spiked oscillators in section 5. Appendices A and B give details of the nontrivial steps of the method.

2. Solutions of the Schrödinger equation

The radial Schrödinger equation for the wave of angular momentum $L$

$$
-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + V(r) R(r) = E R(r),
$$

(2)

with a potential given by (1), can be easily written in the form

$$
-z^2 \frac{d^2 w}{dz^2} + g(z) w = 0,
$$

(3)

with a 'potential' (including centrifugal and energy terms) that, multiplied by $z^2$, becomes a combination of positive and negative integer powers of $z$,

$$
g(z) = \sum_{s=-2}^{2N} g_s z^s, \quad M, N > 0, \quad g_{-2M} > 0, \quad g_{2N} > 0.
$$

(4)

Obviously, $z$ represents a power of the radial variable $r$, with exponent conveniently chosen, and $w(z)$ is the reduced radial wavefunction $R(r)$ multiplied by an adequate function of $r$.

The origin and the infinity are the only singularities, of ranks $M$ and $N$, of the differential equation (3). We are interested in considering the following three pairs of independent solutions.

- **Two Floquet or multiplicative solutions** [28, 29], $w_1$ and $w_2$, that, except for particular sets of values of the parameters $g_s$ in (4), have the form

$$
w_j = z^{\nu_j} \sum_{n=-\infty}^{\infty} c_{n,j} z^n, \quad \text{being} \quad \sum_{n=-\infty}^{\infty} |c_{n,j}|^2 < \infty, \quad j = 1, 2.
$$

(5)

The indices $\nu_j$ are not uniquely defined. They admit addition of any integer (with an adequate relabelling of the coefficients). In the general case, the indices $\nu_j$ and the coefficients $c_{n,j}$ may be complex.

- **Two Thomé formal solutions**, $w_3$ and $w_4$, that have the nature of asymptotic expansions for $z \to \infty$,

$$
w_k(z) \sim \exp \left( \sum_{p=1}^{N} \frac{\alpha_{p,k}}{p} z^p \right) z^{\mu_k} \sum_{m=0}^{\infty} a_{m,k} z^{-m}, \quad a_{0,k} \neq 0, \quad k = 3, 4.
$$

(6)

It is usual to say that these two expansions are associated with each other.

- **Two Thomé formal solutions**, $w_5$ and $w_6$, asymptotic expansions for $z \to 0$, of the form

$$
w_l(z) \sim \exp \left( \sum_{q=1}^{M} \frac{\beta_{q,l}}{q} z^{-q} \right) z^{\rho_l} \sum_{m=0}^{\infty} b_{m,l} z^m, \quad b_{0,l} \neq 0, \quad l = 5, 6.
$$

(7)

Also these expansions are associated.

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1 We adopt the definition of ranks used in [27], mainly because our work is based on Naundorf’s treatment of the homogeneous linear second-order differential equation. According to the definitions used in [28], the singularities should be of ranks $M+1$ and $N+1$.

2 Although the credit of these solutions is attributed to Thomé, they were used before by Fabry and by Poincaré. We are grateful to Professor Alberto Grünbaum for illustrating this fact to us.
The determination of the indices \( \nu_j \) and the coefficients \( c_{n,j} \) of the multiplicative solutions is rather laborious. By substitution of (5) into (3) one obtains the infinite set of homogeneous equations for the coefficients

\[
(n + \nu_j)(n + 1 + \nu_j) c_{n,j} - \sum_{i=2}^{2N} g_i c_{n-i,j} = 0, \quad n = \cdots, -1, 0, 1, \ldots.
\]

that can be interpreted as a nonlinear eigenvalue problem, where the eigenvalue \( \nu \) must be such that

\[
\sum_{n=-\infty}^{\infty} |c_{n,j}|^2 < \infty.
\]

In appendix A we recall the Newton iterative method to solve that problem. In general, two indices, \( \nu_1 \) and \( \nu_2 \), and two corresponding sets of coefficients, \( \{c_{n,1}\} \) and \( \{c_{n,2}\} \), are obtained, but for certain sets of values of the parameters \( g_i \) only one multiplicative solution appears. Any other independent solution must include powers of the variable multiplied by its logarithm. Such logarithmic solutions cannot correspond, usually, to the practical system that one tries to describe and are, therefore, to be discarded. We will assume, from now on, that the parameters \( g_i \) are such that (3) admits two independent multiplicative solutions.

In what concerns the formal solutions \( w_3 \) and \( w_4 \), the exponents \( \alpha \) and \( \mu \) and the coefficients \( a_m \) must be such that the expansions on the right-hand side of (6) satisfy the differential equation. In this way one obtains

\[
\left[ \left( \sum_{p=1}^{N} \alpha_p z^p \right)^2 + \sum_{p=2}^{N} (p-1) \alpha_p z^p - g(z) \right] \sum_{m=0}^{\infty} a_m z^{-m} + 2 \left( \sum_{p=1}^{N} \alpha_p z^p \right) \times \sum_{m=0}^{\infty} (-m + \mu) a_m z^{-m} + \sum_{m=0}^{\infty} (-m + \mu)(-m - 1 + \mu) a_m z^{-m} = 0.
\]

Cancellation of the powers \( z^{2N}, z^{2N-1}, \ldots, z^{N+1} \) inside the bracket of the first term in (10) produces a system of equations

\[
(a_N)^2 - g_{2N} = 0,
\]

\[
2 a_N a_{N-1} - g_{2N-1} = 0,
\]

\[
2 a_N a_{N-2} + (a_{N-2})^2 - g_{2N-2} = 0,
\]

\[
\vdots
\]

\[
2 a_N a_1 + \cdots - g_{N+1} = 0,
\]

which can be solved successively. There are two sets of solutions, \( \{\alpha_{p,3}\} \) and \( \{\alpha_{p,4}\} \), that obviously verify

\[
\alpha_{p,3} = -\alpha_{p,4}, \quad p = 1, \ldots, N.
\]

Let us assign the labels 3 and 4 in such a way that

\[
\alpha_{N,3} = -\sqrt{g_{2N}}, \quad \alpha_{N,4} = +\sqrt{g_{2N}}.
\]

Accordingly, the formal solution \( w_3 \) presents, from the physical point of view, the adequate behaviour at infinity, on the positive real semiaxis, whereas \( w_4 \) should be rejected. Cancellation of the coefficient of \( z^N \) on the left-hand side of (10) gives two exponents \( \mu_k \) in terms of the
previously obtained $\alpha_{p,k}$. Note that
\[
\mu_3 + \mu_4 = -N + 1.
\]
Finally, cancellation of the coefficient of $z^n$, $n = N - 1, N - 2, \ldots, -\infty$, implies a recurrence relation for the coefficients $a_{m,k}$ that allows one to obtain all of them starting with an arbitrarily chosen $a_{0,k} \neq 0$.

Analogously, by requiring the expansions on the right-hand side of (7) to satisfy the differential equation, one obtains for the formal solutions $w_5$ and $w_6$

\[
\left[ \left( \sum_{q=1}^{M} \beta_q z^{-q} \right)^2 + \sum_{q=1}^{M} (q + 1) \beta_q z^{-q} - g(z) \right] \sum_{m=0}^{\infty} b_m z^m = -2 \left( \sum_{q=1}^{M} \beta_q z^{-q} \right) \sum_{m=0}^{\infty} (m + \rho) b_m z^m
\]
\[
+ \sum_{m=0}^{\infty} (m + \rho)(m - 1 + \rho) b_m z^m = 0.
\]

Now, cancellation of the powers $z^{-2M}, z^{-2M+1}, \ldots, z^{-M-1}$ inside the bracket of the first term in (14) gives the system of equations
\[
(\beta M)^2 - g - 2M = 0, \\
2 \beta M \beta_{M-1} - g - 2M+1 = 0, \\
2 \beta M \beta_{M-2} + (\beta M - 1)^2 - g - 2M+2 = 0, \\
\vdots \\
2 \beta M \beta_1 + \cdots - g - M-1 = 0.
\]

The two sets of solutions, $\{\beta_{q,5}\}$ and $\{\beta_{q,6}\}$, verify
\[
\beta_{q,5} = -\beta_{q,6}, \quad q = 1, \ldots, M.
\]

If the labels 5 and 6 are assigned in such a way that
\[
\beta_{M,5} = -\sqrt{g - 2M}, \quad \beta_{M,6} = +\sqrt{g - 2M},
\]
the formal solution $w_5$ vanishes at the origin, whereas $w_6$ diverges. Cancellation of the coefficient of $z^{-M}$ on the left-hand side of (14) allows one to obtain the two exponents $\rho_l$ in terms of the previously calculated $\beta_{q,l}$. They obey the relation
\[
\rho_5 + \rho_6 = M + 1.
\]

The coefficients $b_{m,l}$ can be obtained, starting with an arbitrarily chosen $b_{0,l}$, by making use of the recurrence relation stemming from the cancellation of the coefficient of $z^n$, $n = -M + 1, -M + 2, \ldots, +\infty$, on the left-hand side of (14).

3. The connection factors

Any solution $w$ of the differential equation (3) can be written as a linear combination of the two multiplicative solutions,
\[
w = \zeta_1 w_1 + \zeta_2 w_2.
\]

Its behaviour in the neighbourhood of the singular points can be immediately written if, besides the coefficients $\zeta_1$ and $\zeta_2$, one knows the behaviour of the multiplicative solutions, that is, if one knows the connection factors $T$ of their asymptotic expansions,
\[
w_j \sim T_{j,3} w_3 + T_{j,4} w_4, \quad \text{for } z \to \infty, \quad j = 1, 2.
\]
These connection factors are, obviously, numerical constants, but their values depend on the sector of the complex plane where \( z \) lies. This fact, known as ‘Stokes phenomenon’ [30], introduces a slight complication. As is well known, the connection factor multiplying any one of the asymptotic expansions on the right-hand sides of (19) and (20) takes different values in the sectors of the complex \( z \)-plane separated by a Stokes ray of the associated expansion. On the ray, the value of the connection factor is the average of those two different ones. We are interested in the behaviour of \( w(z) \) in the vicinity of the origin and at infinity on the ray \( \arg z = 0 \). Since this is a Stokes ray for the expansions \( w_4 \) and \( w_6 \), the values of \( T_{j,3} \) and \( T_{j,5} \) for \( \arg z = 0 \) are the respective averages of their values for \( z \) just above and below the positive real semiaxis.

The problem of finding the connection factors \( T \) has been considered by several authors. Most of them [31] refer to a differential equation for which the origin is an ordinary or a regular singular point and the infinity is an irregular singular one. As far as we know, only the procedure developed by Naundorf [27] also becomes applicable to the present case of both the origin and the infinity being irregular singular points. In a former paper [32] we have suggested a modification of Naundorf’s procedure, improving notably its performance, and applied it to find the bound states of anharmonic oscillators, whose Schrödinger equations present an irregular singular point at infinity, the origin being an ordinary or a regular singular one. Some mathematical questions left aside in that paper have been tackled in a posterior one [33]. As we are going to show, our improvement of Naundorf’s method is easily applicable in the case of the origin also being an irregular singular point, as it happens in the Schrödinger equations of spiked oscillators.

Our procedure to calculate the connection factors makes use of the fact that they can be written as quotients of Wronskians of the solutions presented in section 2. We adopt the usual definition of Wronskian of two functions \( f(z) \) and \( g(z) \), namely

\[
\mathcal{W}[f, g](z) = f(z) \frac{dg(z)}{dz} - \frac{df(z)}{dz} g(z).
\]

We benefit from the fact that equation (3) is a second-order linear differential one where the first derivative term is absent and, therefore, the Wronskian of any two of their solutions is a constant. Then, it is immediate to obtain

\[
T_{j,3} = \frac{\mathcal{W}[w_j, w_4]}{\mathcal{W}[w_3, w_4]}, \quad T_{j,4} = \frac{\mathcal{W}[w_j, w_3]}{\mathcal{W}[w_4, w_3]}, \quad j = 1, 2, \tag{21}
\]

\[
T_{j,5} = \frac{\mathcal{W}[w_j, w_5]}{\mathcal{W}[w_5, w_6]}, \quad T_{j,6} = \frac{\mathcal{W}[w_j, w_5]}{\mathcal{W}[w_6, w_5]}, \quad j = 1, 2. \tag{22}
\]

The denominators of the quotients on the right-hand sides of these equations can be computed trivially. One obtains

\[
\mathcal{W}[w_3, w_4] = -\mathcal{W}[w_4, w_3] = -2\alpha_{N,3} a_{0,4}, \tag{23}
\]

\[
\mathcal{W}[w_5, w_6] = -\mathcal{W}[w_6, w_5] = 2\beta_{M,5} b_{0,6}. \tag{24}
\]

The calculation of the numerators is not so easy. Direct computation of any one of them gives a certain power of \( z \) times, an infinite series of negative and positive powers of \( z \). This does not seem to be an adequate expression to determine the (independent of \( z \)) value of the Wronskian. Instead, we have devised a trick that allows one to obtain each one of those numerators. Appendix B contains a detailed description of the procedure.
4. The quantization condition

We have already mentioned that any solution of (3) can be written as a linear combination of the two multiplicative solutions, like in (18). According to (20), its behaviour in the neighbourhood of the origin is given by

\[ w(z) \sim (\zeta_1 T_{1.5} + \zeta_2 T_{2.5}) w_5(z) + (\zeta_1 T_{1.6} + \zeta_2 T_{2.6}) w_6(z) \quad \text{for } z \to 0. \]  

(25)

Since \( w_6 \) diverges as \( z \) goes to zero, the coefficients \( \zeta_1 \) and \( \zeta_2 \) must be such that

\[ T_{1.6} \zeta_1 + T_{2.6} \zeta_2 = 0. \]  

(26)

On the other hand, at infinity,

\[ w(z) \sim (\zeta_1 T_{1.3} + \zeta_2 T_{2.3}) w_3(z) + (\zeta_1 T_{1.4} + \zeta_2 T_{2.4}) w_4(z) \quad \text{for } z \to \infty. \]  

(27)

The correct asymptotic behaviour occurs when

\[ T_{1.4} \zeta_1 + T_{2.4} \zeta_2 = 0. \]  

(28)

The fulfilment of both requirements (26) and (28) implies the quantization condition

\[ T_{1.6} T_{2.4} - T_{1.4} T_{2.6} = 0. \]  

(29)

For given potential (1) and angular momentum, the connection factors are functions only of the energy, through one of the parameters \( g_s \) in (4). Therefore, the left-hand side of (29) is but a function of the energy whose zeros are the eigenenergies of the spiked oscillator.

The problem of finding the wavefunctions is now immediately solved. Any nontrivial solution \( \{ \hat{\zeta}_1, \hat{\zeta}_2 \} \) of the system of equations (26) and (28) gives the (unnormalized) physical solution

\[ w_{\text{phys}}(z) = \hat{\zeta}_1 w_1(z) + \hat{\zeta}_2 w_2(z). \]  

(30)

For the computation of \( w_{\text{phys}} \) in the neighbourhood of the origin or for large values of \( z \), the asymptotic expansions

\[ w_{\text{phys}}(z) \sim (\hat{\zeta}_1 T_{1.5} + \hat{\zeta}_2 T_{2.5}) w_5(z), \quad z \to 0, \]  

(31)

\[ w_{\text{phys}}(z) \sim (\hat{\zeta}_1 T_{1.3} + \hat{\zeta}_2 T_{2.3}) w_3(z), \quad z \to \infty \]  

(32)

should be used instead of (30), which suffers from strong cancellations on the right-hand side for \( z \) in those regions. So, there is no difficulty to normalize \( w_{\text{phys}} \) properly.

5. Some examples

In order to illustrate how our procedure may serve to treat spiked oscillators, we are going to apply it to a few cases that have already been considered, by having recourse to different approximations, by other authors. The numerical results given below have been obtained by using double precision FORTRAN codes. In our opinion, the digits shown in the quoted values of the energies are correct. Our procedure could provide additional correct digits but it would require a higher precision arithmetic. This need is more evident in the case of high values of the potential parameters, angular momentum different from zero, or excited states.

Following a common practice and to facilitate comparison of our results with those of other authors, we assume that \( r \) is a dimensionless variable that represents a distance in a given scale. Analogously, we use dimensionless symbols, \( E \) and \( A_p \), to represent the energy and the intensities of the different terms of the potential that are assumed, expressed in adequate units.
5.1. Potential \( V(r) = A_2 r^2 + A_{-4} r^{-4} \)

The first example to be considered is a three-dimensional spiked harmonic oscillator of potential

\[
V(r) = A_2 r^2 + A_{-4} r^{-4},
\]

which has been most discussed by other authors. We will assume, without loss of generality, that

\[ A_2 = 1. \]

The radial Schrödinger equation (2), written in terms of the variable

\[ z \equiv r, \]

turns into equation (3) for the function

\[ w(z) = R(r), \]

with

\[ g(z) = A_{-4} z^{-2} + \mathcal{L}(\mathcal{L} + 1) - E z^2 + z^4, \]

where, obviously, \( \mathcal{L} \) represents the angular momentum quantum number. So, the ranks of the singularities at the origin and at infinity are, respectively,

\[ M = 1, \quad N = 2. \]

The coefficients \( c_{n,j} \) of the Floquet solutions obey the recurrence relation (omitting the second subindex)

\[
A_{-4} c_{n+6} + [\mathcal{L}(\mathcal{L} + 1) - (n + 4 + \nu)(n + 3 + \nu)] c_{n+4} - E c_{n+2} + c_n = 0.
\]

The Thomé solutions at infinity have exponents

\[ \alpha_{2,3} = -1, \quad \alpha_{1,3} = -1, \quad \alpha_{1,4} = 0, \quad \mu_3 = \frac{-1 + E}{2}, \quad \mu_4 = \frac{-1 - E}{2}, \]

and coefficients \( a_{m,k} \) \( (k = 3, 4) \) obeying the recurrence relation (omitting the second subindex)

\[
2 \alpha_2 m a_m = [(m - 2 - \mu)(m - 1 - \mu) - \mathcal{L}(\mathcal{L} + 1)] a_{m-2} - A_{-4} a_{m-4}.
\]

For the Thomé solutions at the origin the exponents are

\[ \beta_{1,5} = -1, \quad \beta_{1,6} = -A_{-4}^{1/2}, \quad \rho_5 = \rho_6 = 1, \]

and the coefficients \( b_{m,k} \) \( (k = 5, 6) \) satisfy the recurrence relation (omitting the second subindex)

\[
2 \beta_1 m b_m = [m(m - 1) - \mathcal{L}(\mathcal{L} + 1)] b_{m-1} + E b_{m-3} - b_{m-5}.
\]

The definitions of Floquet and Thomé solutions given above fix them up to arbitrary multiplicative constants. To avoid ambiguities in the definition of the connection factors, we assume from now on that those arbitrary constants have been chosen in such a way that

\[ c_{0,1} = c_{0,2} = 1, \quad a_{0,3} = a_{0,4} = 1, \quad b_{0,5} = b_{0,6} = 1. \]

Then, according to equations (23) and (24), we have for the denominators in equations (21) and (22)

\[
\mathcal{W}[w_3, w_4] = -\mathcal{W}[w_4, w_3] = 2, \quad \mathcal{W}[w_5, w_6] = -\mathcal{W}[w_6, w_5] = -2A_{-4}^{1/2}.
\]
Our procedure gives for the numerators in the same equations (for \( j = 1, 2 \))

\[
\mathcal{W}[w_j, w_3] = 2^{n+\delta_1^{(j,3)}} \Gamma(n + 1 + \delta_1^{(j,3)}) \gamma_2^{(j,3)}, \tag{36}
\]

\[
\mathcal{W}[w_j, w_4] = (-1)^n \cos(\pi \delta_1^{(j,4)}) 2^{n+\delta_1^{(j,4)}} \Gamma(n + 1 + \delta_1^{(j,4)}) \gamma_2^{(j,4)}, \tag{37}
\]

\[
\mathcal{W}[w_j, w_5] = A_{-4}^{(n+\nu_j+1)/2} \Gamma(n - \nu_j) \gamma_{-n}^{(j,5)}, \tag{38}
\]

\[
\mathcal{W}[w_j, w_6] = (-1)^{n-1} \cos(\pi \nu_j) A_{-4}^{(n+\nu_j+1)/2} \Gamma(n - \nu_j) \gamma_{-n}^{(j,6)}, \tag{39}
\]

where we have abbreviated

\[
\delta_1^{(j,k)} = (\nu_j + \mu_k + 1)/2, \quad k = 3, 4, \tag{40}
\]

\[
\gamma_m^{(j,k)} = \sum_{s=0}^{\infty} a_{s,k} (\alpha_{2,k} c_{m-s-1,j} - \gamma_j + \mu_k) c_{m+s+1,j}, \quad k = 3, 4, \tag{41}
\]

\[
\gamma_m^{(j,l)} = \sum_{s=0}^{\infty} b_{s,l} (-\beta_{1,l} c_{m-s+2,j} + (2s - m - \nu_j) c_{m-s+1,j}), \quad l = 5, 6. \tag{42}
\]

We report in table 1 our ground-state energies of the spiked oscillator of potential (33), for several values of the parameter \( A_{-4} \). The quoted energies should be compared with those in table 3 of [18] that collects results obtained by several authors with different methods. Obviously, our procedure is superior to variational ones, as it provides with four more correct digits in the energies for the values of \( A_{-4} \) considered. In fact, our double precision FORTRAN calculations give results with an accuracy comparable to the excellent one reached by Buendía et al. [7] thanks to an analytic continuation method or by Roy [6] by using a generalized pseudospectral method. (Note that, due to a different definition of the Schrödinger operator, our energies should be divided by 2 if they are to be compared with those of table 4 of [6].) As mentioned above, larger values of \( A_{-4} \) require a more precise arithmetic to the same accuracy in the results. For completeness, in the same table we also give the indices \( \nu_j \) of the Floquet solutions. We have already mentioned the existing ambiguity in the values of the indices. To

| \( A_{-4} \) | \( E \) | \( \nu_1 \) |
|---|---|---|
| 0.0001 | 3.0222745087 | 0.201485000573E - 03 |
| 0.001 | 3.0687631709 | 0.204586237797E - 02 |
| 0.005 | 3.1483523083 | 0.104967473643E - 01 |
| 0.01 | 3.2050674951 | 0.213850813448E - 01 |
| 0.1 | 3.575519912 | 0.270240464647E + 00 |
| 0.4 | 4.0319714400 | 0.5 + i 0.606 083 134 346 |
| 1 | 4.4941779834 | 0.5 + i 0.203 793 867 918 |
| 10 | 6.6066225120 | 0.5 + i 0.203 793 867 918E + 01 |
| 100 | 11.265080432 | 0.5 + i 0.412 681 646 514 |

We report in table 1 our ground-state energies of a spiked oscillator of potential (33), for several values of the parameter \( A_{-4} \). The index \( \nu_1 \) of one of the Floquet solutions, \( w_1 \), is also given. The index \( \nu_2 \) of the other Floquet solution, \( w_2 \), is immediately obtained from the relation \( \nu_2 = 1 - \nu_1 \).
avoids it, we assume that the indices vary continuously with the parameter $A_{-4}$ and fix their integer part in such a way that $v_1 = 0$ (and, consequently, $v_2 = 1$) for $A_{-4} = 0$, as it should be for a particle of zero angular momentum. As $A_{-4}$ increases, the index $v_1$ also increases until it reaches the value $0.5$ for $A_{-4} = 0.1305 \ldots$ and an eigenenergy $E = 3.6454 \ldots$. Note that, for those critical values of the parameters, $v_1 = v_2$ and only one Floquet solution of the form (5) is obtained. Any other independent solution of the differential equation must contain logarithmic terms. Our procedure, in its present form, is not applicable in this case. A different implementation of the basic idea would be necessary. If $A_{-4}$ continues increasing above the critical value, the real part of $v_1$ remains equal to $0.5$, whereas its imaginary part increases. Obviously, the two Floquet solutions are then complex conjugate to each other. This fact allows one to simplify the calculations in our procedure.

In order to illustrate the trend of the coefficients of the Floquet and Thomé solutions, we show, in tables 2–7, the most relevant of them for two particular cases. Tables 2–4 correspond to the ground state of the example discussed by Znojil [25], namely that of $A_{-4} = 0.4$. In this case, the energy and the indices of the Floquet solutions are

$$E = 4.031\,971\,4400, \quad v_1 = v_2 = 0.5 + i 0.606\,083\,134\,346,$$

and the connection factors

$$T_{1,3} = T_{2,3} = 0.363\,722\,440 \, 835 + i 1.104\,390 \, 656 \, 062,$$
$$T_{1,4} = T_{2,4} = -0.378\,572\,197 \, 756 + i 0.137\,728 \, 550 \, 255,$$

Note that, in tables 2–7, the most relevant of them for two particular cases. Tables 2–4 correspond to the ground state of the example discussed by Znojil [25], namely that of $A_{-4} = 0.4$. In this case, the energy and the indices of the Floquet solutions are

$$E = 4.031\,971\,4400, \quad v_1 = v_2 = 0.5 + i 0.606\,083\,134\,346,$$

and the connection factors

$$T_{1,3} = T_{2,3} = 0.363\,722\,440 \, 835 + i 1.104\,390 \, 656 \, 062,$$
$$T_{1,4} = T_{2,4} = -0.378\,572\,197 \, 756 + i 0.137\,728 \, 550 \, 255,$$

| $n$ | $\Re c_{2n,1}$ | $\Im c_{2n,1}$ |
|-----|-----------------|-----------------|
| $-10$ | -0.142\,317\,651\,396\,E - 23 | 0.690\,261\,395\,778\,E - 23 |
| $-9$  | -0.100\,263\,927\,421\,E - 20 | 0.697\,827\,393\,080\,E - 20 |
| $-8$  | -0.429\,957\,466\,419\,E - 18 | 0.569\,639\,303\,294\,E - 17 |
| $-7$  | 0.167\,970\,940\,662\,E - 17 | 0.365\,781\,757\,971\,E - 14 |
| $-6$  | 0.156\,002\,041\,860\,E - 12 | 0.178\,667\,148\,961\,E - 11 |
| $-5$  | 0.120\,892\,275\,974\,E - 09 | 0.634\,808\,135\,173\,E - 09 |
| $-4$  | 0.492\,753\,885\,758\,E - 07 | 0.154\,076\,769\,690\,E - 06 |
| $-3$  | 0.115\,445\,648\,553\,E - 04 | 0.232\,262\,850\,896\,E - 04 |
| $-2$  | 0.144\,400\,180\,831\,E - 02 | 0.184\,616\,207\,302\,E - 02 |
| $-1$  | 0.780\,262\,882\,557\,E - 01 | 0.537\,272\,549\,009\,E - 01 |
| $0$   | 1               |  |
| $1$   | -0.760\,452\,558\,754\,E + 00 | 0.536\,951\,488\,093\,E + 00 |
| $2$   | 0.199\,603\,204\,168\,E + 00 | -0.202\,477\,697\,179\,E + 00 |
| $3$   | -0.348\,689\,735\,973\,E - 01 | 0.453\,398\,033\,971\,E - 01 |
| $4$   | 0.433\,386\,333\,626\,E - 02 | -0.673\,686\,840\,525\,E - 02 |
| $5$   | -0.431\,181\,023\,059\,E - 03 | 0.781\,829\,457\,162\,E - 03 |
| $6$   | 0.349\,885\,697\,033\,E - 04 | -0.725\,041\,819\,983\,E - 04 |
| $7$   | -0.243\,278\,698\,241\,E - 05 | 0.570\,826\,186\,141\,E - 05 |
| $8$   | 0.146\,155\,963\,833\,E - 06 | -0.385\,089\,511\,701\,E - 06 |
| $9$   | -0.779\,445\,127\,491\,E - 08 | 0.229\,774\,575\,047\,E - 07 |
| $10$  | 0.370\,654\,869\,957\,E - 09 | -0.121\,862\,184\,787\,E - 08 |
Table 3. The first coefficients of the two Thomé solutions at infinity for the particular case of
$A_{-4} = 0.4$, $L = 0$ and energy $E = 4.031 971 440.0$. We have skipped over the coefficients with
the odd first label, since they are equal to zero.

| $m$ | $a_{2m,3}$ | $a_{2m,4}$ |
|-----|------------|------------|
| 0   | 1         | 1         |
| 1   | −0.195 556 745 811 $E + 00$ | 0.221 154 246 581 $E + 01$ |
| 2   | 0.675 581 627 248 $E − 01$ | 0.683 622 152 000 $E + 01$ |
| 3   | −0.552 411 742 467 $E − 01$ | 0.278 260 729 988 $E + 02$ |
| 4   | 0.865 891 090 477 $E − 01$ | 0.140 764 673 339 $E + 03$ |
| 5   | −0.212 297 950 675 $E + 00$ | 0.851 787 239 516 $E + 03$ |
| 6   | 0.709 506 391 974 $E + 00$ | 0.600 154 315 448 $E + 04$ |
| 7   | −0.305 385 835 605 $E + 01$ | 0.482 637 790 136 $E + 05$ |
| 8   | 0.160 735 596 654 $E + 02$ | 0.436 250 311 536 $E + 06$ |
| 9   | −0.100 168 048 970 $E + 03$ | 0.437 841 970 329 $E + 07$ |
| 10  | 0.721 888 409 401 $E + 03$ | 0.483 138 693 754 $E + 08$ |

Table 4. The first coefficients of the two Thomé solutions at the origin for the particular case of
$A_{-4} = 0.4$, $L = 0$ and energy $E = 4.031 971 440.0$.

| $m$ | $b_{m,5}$ | $b_{m,6}$ |
|-----|-----------|-----------|
| 0   | 1         | 1         |
| 1   | 0         | 0         |
| 2   | 0         | 0         |
| 3   | −0.106 251 776 760 $E + 01$ | 0.106 251 776 760 $E + 01$ |
| 4   | 0.251 998 215 001 $E + 01$ | 0.251 998 215 001 $E + 01$ |
| 5   | −0.781 076 937 396 $E + 01$ | 0.781 076 937 396 $E + 01$ |
| 6   | 0.314 392 488 782 $E + 02$ | 0.314 392 488 782 $E + 02$ |
| 7   | −0.150 276 962 631 $E + 03$ | 0.150 276 962 631 $E + 03$ |
| 8   | 0.834 637 749 300 $E + 03$ | 0.834 637 749 300 $E + 03$ |
| 9   | −0.528 962 618 019 $E + 04$ | 0.528 962 618 019 $E + 04$ |
| 10  | 0.376 836 341 629 $E + 05$ | 0.376 836 341 629 $E + 05$ |

$T_{1,5} = T_{2,5} = 0.520 935 174 155 + i 1.431 885 933 657$, $T_{1,6} = T_{2,6} = 0.436 272 922 113 − i 0.158 720 681 109$.

The coefficients of the Floquet solutions can be seen in table 2, and those of the Thomé solutions at infinity and at the origin in tables 3 and 4, respectively. The second particular case, illustrated in tables 5–7, is that of $A_{-4} = 1$ and angular momentum $L = 2$, considered by Aguilera–Navarro and Ley Koo [19, table II]. For the energy and the indices of the Floquet solutions we have obtained

$E = 7.224 287 1639$, $\nu_1 = −2.083 592 228 877$, $\nu_2 = 3.083 592 228 877$, and for the connection factors

$T_{1,3} = −0.704 193 8314$, $T_{2,3} = 0.928 083 1701$, $T_{1,4} = −2.461 700 3408$, $T_{2,4} = −0.046 233 0978$, $T_{1,5} = −7.718 522 8435$, $T_{2,5} = 0.144 961 2757$, $T_{1,6} = 7.992 547 3468$, $T_{2,6} = 0.150 107 7191$. 
Table 5. Coefficients of the Floquet solutions $w_1$ and $w_2$, normalized in such a way that $c_{0,1} = c_{0,2} = 1$, for the particular case of $A_{-4} = 1$, $L = 2$ and energy $E = 7.2242871639$. The coefficients with odd first label are equal to zero.

| $n$ | $c_{2n,1}$       | $c_{2n,2}$     |
|-----|------------------|----------------|
| −10 | 0.418079159329E−21 | 0.508287996884E−16 |
| −9  | 0.210620217409E−18 | 0.151012739304E−13 |
| −8  | 0.879231161250E−16 | 0.349503786349E−11 |
| −7  | 0.298162410599E−13 | 0.607371865088E−09 |
| −6  | 0.801421283847E−11 | 0.753905871024E−07 |
| −5  | 0.165459858432E−08 | 0.621798117715E−05 |
| −4  | 0.251717006169E−06 | 0.303689920293E−03 |
| −3  | 0.266341461331E−04 | 0.705627796961E−02 |
| −2  | 0.179770550637E−02 | 0.404457498845E−01 |
| −1  | 0.668756857204E−01 | −0.120970341064E+00 |
| 0   | 1                 | 1               |
| 1   | 0.906280257537E+00 | −0.489421426933E+00 |
| 2   | 0.180182069867E+01 | 0.121731165609E+00 |
| 3   | −0.209931807620E+01| −0.202632214005E−01 |
| 4   | 0.728418847644E+00 | 0.253270584477E−02 |
| 5   | −0.150546747680E+00| −0.253385266560E−03 |
| 6   | 0.220037264283E−01 | 0.211291907657E−04 |
| 7   | −0.249248011094E−02| −0.151036531706E−05 |
| 8   | 0.230170752863E−03 | 0.944714256302E−07 |
| 9   | −0.179507697993E−04| −0.525254239416E−08 |
| 10  | 0.121105001156E−05 | 0.262829669113E−09 |

Table 6. The first coefficients of the two Thomé solutions at infinity for the particular case of $A_{-4} = 1$, $L = 2$ and energy $E = 7.2242871639$. We have skipped over the coefficients with the odd first label, since they are equal to zero.

| $m$ | $a_{2m,3}$       | $a_{2m,4}$     |
|-----|------------------|----------------|
| 0   | 1                 | 1               |
| 1   | −0.143323523200E+00 | 0.375546710356E+01 |
| 2   | 0.197417671158E−01 | 0.174648768948E+02 |
| 3   | −0.483024944799E−02 | 0.985368170788E+02 |
| 4   | 0.281201497378E−02  | 0.653978669776E+03 |
| 5   | −0.344425647424E−02 | 0.499199894361E+04 |
| 6   | 0.705309939217E−02  | 0.430838076974E+05 |
| 7   | −0.207488754900E−01 | 0.414831591842E+06 |
| 8   | 0.802543608266E−01  | 0.440835793663E+07 |
| 9   | −0.386209034478E+00 | 0.512489320365E+08 |
| 10  | 0.222788471661E+01  | 0.646983198568E+09 |

The ambiguity in the integer part of the indices of the Floquet solutions has been resolved in this second particular case bearing in mind that, since $L = 2$, it is desirable that $\nu_1 \rightarrow −2$ and $\nu_2 \rightarrow 3$ for $A_{-4} \rightarrow 0$.  

Table 7. The first coefficients of the two Thomé solutions at the origin for the particular case of
$A_{-4} = 1$, $L = 2$ and energy $E = 7.224 \, 287 \, 163 \, 9$.

| $m$ | $b_{m,5}$  | $b_{m,6}$  |
|-----|-----------|-----------|
| 0   | 1         | 1         |
| 1   | 3         | −3        |
| 2   | 3         | 3         |
| 3   | −0.120 404 786 066 $E + 01$ | 0.120 404 786 066 $E + 01$ |
| 4   | −0.180 607 179 098 $E + 01$ | −0.180 607 179 098 $E + 01$ |
| 5   | 0.461 214 358 197 $E + 00$  | −0.461 214 358 197 $E + 00$  |
| 6   | 0.524 369 089 808 $E + 01$  | 0.524 369 089 808 $E + 01$  |
| 7   | 0.101 141 803 810 $E + 01$  | −0.101 141 803 810 $E + 01$  |
| 8   | −0.344 418 092 084 $E + 01$ | −0.344 418 092 084 $E + 01$  |
| 9   | 0.125 072 806 620 $E + 02$  | −0.125 072 806 620 $E + 02$  |
| 10  | −0.528 728 562 620 $E + 02$ | −0.528 728 562 620 $E + 02$  |

Once the connection factors are known, the wavefunction is easily obtained following the
procedure indicated in section 4.

5.2. Potential $V(r) = A_2 r^2 + A_{-4} r^{-4} + A_{-6} r^{-6}$

Our second example is a three-dimensional spiked harmonic oscillator which has also received
considerable attention and has become quasi-exactly solvable for certain sets of parameters.
The potential is

$$V(r) = A_2 r^2 + A_{-4} r^{-4} + A_{-6} r^{-6}. \tag{43}$$

Following a common practice, we will assume

$$A_2 = 1.$$  

We can reduce the rank of the singularities of the radial Schrödinger equation by using the
variable

$$z \equiv r^2.$$  

Equation (2) turns in this way into equation (3) for the function

$$w(z) = r^{1/2} R(r),$$

with

$$g(z) = \frac{1}{2} (A_{-6} z^{-2} + A_{-4} z^{-4} + L(L + 1) - 3/4 - E z + z^2).$$

Now the ranks of the singularities at the origin and at infinity are, respectively,

$$M = 1, \quad N = 1.$$  

This fact reveals that we are dealing with a double confluent Heun equation [34]. In a recent
paper [35], we have detailed the algorithm resulting from the application of our method to an
equation of this type. For the coefficients $c_{n,j}$ of the Floquet solutions we have the recurrence
relation (omitting the second subindex $j$)

$$A_{-6} c_{n+4} + A_{-4} c_{n+3} + [L(L + 1) - 3/4 - 4(n + 2 + v)(n + 1 + v)] c_{n+2} - E c_{n+1} + c_n = 0.$$  

The Thomé solutions at infinity have exponents

$$\alpha_{1,3} = -\alpha_{1,4} = -1/2, \quad \mu_3 = E/4, \quad \mu_4 = -E/4,$$
and the recurrence relation for the coefficients $a_{m,k}$ ($k = 3, 4$) is (omitting the second subindex $j$

\[ 8\alpha(m\alpha m [4(m - \mu)(m - 1 - \mu) - L(L + 1) + 3/4] a_{m-1} - A_{-4}a_{m-2} - A_{-6}a_{m-3}. \]

For the Thomé solutions at the origin, the exponents are in this second example

\[ \beta_{1,5} = -\beta_{1,6} = -A_{-6}^{1/2}/2, \quad \rho_5 = 1 + A_{-4}/4A_{-6}, \quad \rho_6 = 1 - A_{-4}/4A_{-6}, \]

and the coefficients $b_{m,l}$ ($l = 5, 6$) obey the recurrence relation (omitting the second subindex $j$)

\[ 8\beta_{1} b_{m,j} = [4(m - 1 + \rho)(m - 2 + \rho) - L(L + 1) + 3/4] b_{m-1} + E b_{m-2} - b_{m-3}. \]

By choosing, as in the first example,

\[ c_{0,1} = c_{0,2} = 1, \quad a_{0,3} = a_{0,4} = 1, \quad b_{0,5} = b_{0,6} = 1, \]

we have, from equations (23) and (24),

\[ \mathbb{V}[w_3, w_4] = -\mathbb{V}[w_4, w_3] = 1, \quad (44) \]

\[ \mathbb{V}[w_5, w_6] = -\mathbb{V}[w_6, w_5] = -A_{-6}^{1/2}. \quad (45) \]

For the numerators in equations (21) and (22) our procedure in this second example (for $j = 1, 2$) gives

\[ \mathbb{V}[w_j, w_3] = 2^{n+\delta(j,3)} \Gamma(n + 1 + \delta(j,3)) \gamma_n^{(j,3)}, \quad (46) \]

\[ \mathbb{V}[w_j, w_4] = (-1)^n \cos \left( \pi \delta(j,4) \right) 2^{n+\delta(j,4)} \Gamma(n + 1 + \delta(j,4)) \gamma_n^{(j,4)}, \quad (47) \]

\[ \mathbb{V}[w_j, w_5] = \left( A_{-6}^{1/2}/2 \right)^{n-\delta(j,5)} \Gamma(n + 1 + \delta(j,5)) \gamma_n^{(j,5)}, \quad (48) \]

\[ \mathbb{V}[w_j, w_6] = (-1)^n \cos \left( \pi \delta(j,6) \right) \left( A_{-6}^{1/2}/2 \right)^{n-\delta(j,6)} \Gamma(n + 1 + \delta(j,6)) \gamma_n^{(j,6)}, \quad (49) \]

with the abbreviations

\[ \delta(j,k) = v_j + \mu_k, \quad k = 3, 4, \quad \delta(j,l) = -v_j - \rho_l, \quad l = 5, 6, \quad (50) \]

\[ \gamma_{m}^{(j,k)} = \sum_{s=0}^{\infty} a_{s,k}(\alpha_{1,k} e_{m+s, j} - (m + 2s + 1 + v_j - \mu_k) e_{m+s+1,j}), \quad k = 3, 4, \quad (51) \]

\[ \gamma_{m}^{(j,l)} = \sum_{s=0}^{\infty} b_{s,l}(-\beta_{1,l} e_{m+s+2, j} + (2s - m - 1 - v_j + \rho_l) e_{m+s+1,j}), \quad l = 5, 6. \quad (52) \]

The connection factors are then obtained immediately from equations (21) and (22).

The ground-state energies of the spiked oscillator of potential (43) obtained with our method, for several values of $A_{-6}$ and $A_{-4}$, are shown in table 8. The chosen values of the parameters allow us to compare our results with the extremely precise ones of Buendía et al [7] and of Roy [6]. The concordance is remarkable. We have also considered the values of $A_{-6}$ and $A_{-4}$, both different from zero, allowing comparison with the variational results of Saad et al [2]. In the same table we also give the indices $v_1$ and $v_2$ of the Floquet solutions. As in the preceding example, the physical solution $w_{phys}$ can be obtained easily following the steps indicated in section 4. Going back to the original variable $r$ and reduced radial wavefunction $R(r)$ is trivial.
Table 8. Ground-state energy of a spiked oscillator of potential (43) for several values of the intensities \( A_{-6} \) and \( A_{-4} \). The column headed by \( E \) contains the results obtained with our method. For comparison, we show, in the column headed by \( E_{\text{lit}} \), three set of values taken from the literature, namely from [7], [6] and [2], respectively. (Note that, due to a different definition of the Hamiltonian, factor 2 has been applied to the results of [6] and, consequently, the last digit may oscillate by one unit.) The last column shows the indices \( v_1 \) and \( v_2 \) of the Floquet solutions.

| \( A_{-6} \) | \( A_{-4} \) | \( E \) | \( E_{\text{lit}} \) | \( v_1 = -v_2 \) |
|---|---|---|---|---|
| 0.001 | 0 | 3.279 855 825 92 | 3.279 855 825 921 856 | 0.249 216 175 554 |
| 0.0025 | 0 | 3.353 919 317 11 | 3.353 919 317 108 725 | 0.247 958 538 878 |
| 0.01 | 0 | 3.505 452 276 00 | 3.505 452 275 995 097 | 0.241 137 578 178 |
| 1 | 0 | 4.659 939 969 57 | 4.659 939 969 573 538 | i 0.466 911 061 788 |
| 10 | 0 | 6.003 209 028 90 | 6.003 209 028 895 745 | i 0.895 534 935 089 |
| 0.005 | 0 | 3.422 884 184 26 | 3.422 884 184 26 | 0.245 761 020 193 |
| 0.05 | 0 | 3.765 540 206 04 | 3.765 540 206 04 | 0.198 535 942 381 |
| 0.5 | 0 | 4.387 909 060 26 | 4.387 909 060 26 | i 0.337 261 268 644 |
| 5 | 0 | 5.513 159 014 18 | 5.513 159 014 18 | i 0.768 433 078 693 |
| 1 | 10 | 6.679 053 664 45 | 6.679 054 | 0.5 − i 1.005 393 093 01 |
| 10 | 1 | 6.140 122 871 78 | 6.140 123 | 0.896 525 791 611 |
| 10 | 10 | 7.138 260 939 98 | 7.138 261 | 0.5 − i 0.320 864 634 688 |

5.3. Potential \( V(r) = r^2 + \lambda r^{-5/2} \)

As a third example, we have chosen the spiked oscillator whose potential

\[
V(r) = r^2 + \lambda r^{-5/2}
\]

presents a critical singularity. The Schrödinger equation (2) adopts the form (3) for the variables

\[
z \equiv r^{1/4} \quad \text{and} \quad w(z) = r^{-3/8} R(r),
\]

with \( g(z) \) now being

\[
g(z) = 16 \left( \lambda z^{-2} + \mathcal{L}(\mathcal{L} + 1) + 15/64 - Ez^8 + z^{16} \right).
\]

The singularities at the origin and at infinity have ranks

\[
M = 1, \quad N = 8.
\]

The recurrence relation for the coefficients \( c_{n,j} \) of the Floquet solutions (5) is (omitting the subindex \( j = 1, 2 \))

\[
\lambda c_{n+18} + [\mathcal{L}(\mathcal{L} + 1) + 15/64 - (n + 16 + \nu)(n + 15 + \nu)/16] c_{n+16} = Ec_{n+8} + c_n = 0.
\]

For the exponents of the Thomé solutions at infinity we have

\[
\alpha_{8,3} = -\alpha_{8,4} = -4, \quad \alpha_{7,j} = \alpha_{6,j} = \cdots = \alpha_{1,j} = 0, \quad \mu_3 = -7/2 + 2E, \quad \mu_4 = -7/2 - 2E,
\]

and for their coefficients \( a_{m,k} \) (omitting the subindex \( k = 3, 4 \))

\[
2\alpha_{8,4} a_{m,k} = [(m - 8 - \mu)(m - 7 - \mu) - 16\mathcal{L}(\mathcal{L} + 1) - 15/4] a_{m-8} - 16\lambda a_{m-10}.
\]

The Thomé solutions at the origin have exponents

\[
\beta_{1,5} = -\beta_{1,6} = -4\lambda^{1/2}, \quad \rho_5 = \rho_6 = 1,
\]

\[
15
\]
and coefficients $b_{m,l}$ that satisfy the recurrence relation (omitting the second subindex $l = 5, 6$)

$$2\beta_1 b_m = [m(m-1) - 16L(L+1) - 15/4] b_{m-1} + 16Eb_{m-9} - 16b_{m-17}.$$  

If we choose, as in the preceding examples,

$$c_{0,1} = c_{0,2} = 1, \quad a_{0,3} = a_{0,4} = 1, \quad b_{0,5} = b_{0,6} = 1,$$

we have for the denominators in equations (21) and (22)

$$W[w_3, w_4] = W[w_4, w_3] = 8, \quad (54)$$

$$W[w_5, w_6] = W[w_6, w_5] = -8 \lambda^{1/2}. \quad (55)$$

The numerators in the same equations (for $j = 1, 2$) are given by

$$W[w_j, w_3] = \sum_{L=0}^7 2^{n+\delta^{(j,3)}} \Gamma(n+1+\delta^{(j,3)}) \gamma^{(j,3)}_{\text{h}+L}, \quad (56)$$

$$W[w_j, w_4] = (-1)^n \sum_{L=0}^7 \cos(\pi \delta^{(j,4)}) 2^{n+\delta^{(j,4)}} \Gamma(n+1+\delta^{(j,4)}) \gamma^{(j,4)}_{\text{h}+L}, \quad (57)$$

$$W[w_j, w_5] = (4\lambda^{1/2})^{-n+\nu_j+1} \Gamma(n-\nu_j) \gamma^{(j,5)}_{\text{h}+L}, \quad (58)$$

$$W[w_j, w_6] = (-1)^n \cos(\pi \nu_j) (4\lambda^{1/2})^{-n+\nu_j+1} \Gamma(n-\nu_j) \gamma^{(j,6)}_{\text{h}+L}, \quad (59)$$

where we have abbreviated

$$\delta^{(j,k)}_L = (v_j + \mu_k + L)/8, \quad k = 3, 4, \quad (60)$$

$$\gamma^{(j,k)}_m = \sum_{s=0}^\infty a_{s,k}(\alpha_{s,k} c_{m+s-7,j} - (m + 2s + 1 + v_j - \mu_k) c_{m+s+1,j}), \quad k = 3, 4, \quad (61)$$

$$\gamma^{(j,l)}_m = \sum_{s=0}^\infty b_{s,l}(-\beta_{1,l} c_{m+s-2,j} + (2s - m - v_j) c_{m+s+1,j}), \quad l = 5, 6. \quad (62)$$

The ground-state energies obtained by using our procedure are reported in table 9. We have taken for $\lambda$ several values already considered by other authors, whose results are also shown in table 9 in order to facilitate comparison. As can be seen, our results are considerably more accurate than those obtained by numerical integration of the Schrödinger equation [13], although we do not reach, with our double precision FORTRAN codes, the impressive accuracy of the results of Buendía et al [7] obtained with the analytic continuation method. Our procedure, however, can provide results with as many correct digits as desired if a sufficiently precise arithmetic is used. And, very importantly, the wavefunction is obtained in a very convenient form (asymptotic expansions and Laurent series) for algebraic manipulations like normalization or computation of expected values.

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Table 9. Ground-state energy of the critical spiked oscillator of potential (53) for several values of the intensity $\lambda$. The column headed by $E$ shows our results. For comparison, we show, in the column headed by $E_{\text{lit}}$, values taken from the literature, namely from [7] (superscript a) and [13] (superscript b), obtained by the analytic continuation method and by numerical integration, respectively. The last column shows the index $v_1$ of one of the Floquet solutions. For the index of the other one we have taken $v_2 = 1 - v_1$.

| $\lambda$ | $E$      | $E_{\text{lit}}$ | $v_1$          |
|-----------|----------|------------------|----------------|
| 0.001     | 3.004 011 251 01 | 3.004 011 251 013 044$^a$ | 0.5 + i 0.244 567 376 746E-04 |
| 0.005     | 3.019 140 107 28 | 3.019 140 107 276 879$^a$ | 0.5 + i 0.612 956 032 070E-03 |
| 0.01      | 3.036 729 472 63 | 3.036 729$^b$       | 0.5 + i 0.245 895 012 676E-02 |
| 0.05      | 3.152 429 441 40 | 3.152 429$^b$       | 0.5 + i 0.625 334 357 268E-01 |
| 0.1       | 3.266 873 026 11 | 3.266 873 026 113 018$^a$ | 0.5 + i 0.248 671 350 579E+00 |
| 0.5       | 3.848 553 172 29 | 3.848 553$^b$       | 0.5 + i 0.196 189 243 685E+01 |
| 1         | 4.317 311 689 25 | 4.317 311 689 247 366$^a$ | 0.5 + i 0.288 463 702 918E+01 |
| 2         | 4.986 135 736 09 | 4.986 135 839 484 474E+01 |
| 5         | 6.296 472 638 90 | 6.296 472$^b$       | 0.5 + i 0.599 616 514 206E+01 |
| 10        | 7.735 111 103 49 | 7.735 111 103 489 141$^a$ | 0.5 + i 0.813 625 698 416E+01 |
| 20        | 9.709 404 096 21 | 9.709 404 096 213 018$^a$ | 0.5 + i 0.110 399 234 356E+02 |

Appendix A. Indices and coefficients of the Floquet solutions

In section 2 we have referred to the Floquet or multiplicative solutions (5) of equation (3) and mentioned that their indices $v_j$ and coefficients $c_{n,j}$ must obey the infinite set of homogeneous equations (8) and condition (9). (Along this appendix we will omit, for brevity, the subindex $j$ in $v_j$ and $c_{n,j}$.) As has been already said, we face a nonlinear eigenvalue problem. Algorithms to solve finite-order problems of this kind have been discussed by Ruhe [36]. Obviously, condition (9) requires

$$\lim_{n \to \pm \infty} |c_n| = 0.$$  \hfill (A.1)

This allows us to truncate our infinite problem (8) by restricting the label $n$ to the interval $-M \leq n \leq N$, both $M$ and $N$ being positive integers large enough to guarantee that the solution of the truncated problem does not deviate significantly from that of the original infinite one. Then, the Newton iteration method can be applied. It consists in moving from an approximate solution, \{\nu^{(i)}, c^{(i)}_n\}, to another one, \{\nu^{(i+1)}, c^{(i+1)}_n\}, by solving the system of equations

$$\begin{align*}
(2n - 1 + 2v^{(i)})c^{(i)}_n (v^{(i+1)} - v^{(i)}) + (n + v^{(i)})(n - 1 + v^{(i)}) c^{(i+1)}_n \\
- \sum_{s=-2M}^{-2N} g_s c^{(i+1)}_{n-s} &= 0, \quad n = -M, \ldots, -1, 0, 1, \ldots, N, \quad (A.2) \\
\sum_{n=-M}^{N} c^{(i+1)*}_n c^{(i+1)}_n &= 1, \quad (A.3)
\end{align*}$$

that results, by linearization [37], from (8) and from the truncated normalization condition

$$\sum_{n=-M}^{N} |c_n|^2 = 1.$$
Obviously, the values of $c_m^{(l)}$ with $m < -M$ or $m > N$ entering in some of equations (A.2) should be taken equal to zero, in accordance with the truncation done. The iteration process is stopped when the difference between consecutive solutions is satisfactory. The outcome may serve as starting point for a new iteration process with larger values of $M$ and $N$, to check the stability of the solution.

The iteration process just described needs initial values \{\nu(0), c_n^{(0)}\} not far from the true solution. The two different values of \(\nu\) can be obtained from the two eigenvalues 

$$\lambda_j = \exp(2i\pi \nu_j), \quad j = 1, 2,$$

of the circuit matrix \(C\) [38] for the singular point at \(z = 0\). The entries of that matrix can be computed by numerically integrating equation (3) on the unit circle, from \(z = \exp(0)\) to \(z = \exp(2i\pi)\), for two independent sets of initial values. If we consider two solutions, \(w_a(z)\) and \(w_b(z)\), obeying, for instance, the conditions

\[
\begin{align*}
w_a(e^0) &= 1, & w_a'(e^0) &= 0, \\
w_b(e^0) &= 0, & w_b'(e^0) &= 1,
\end{align*}
\]

then

\[
\begin{align*}
C_{11} &= w_a(e^{2i\pi}), & C_{12} &= w_b(e^{2i\pi}), \\
C_{21} &= w_a'(e^{2i\pi}), & C_{22} &= w_b'(e^{2i\pi}),
\end{align*}
\]

and

\[

\nu = \frac{1}{2i\pi} \ln \left[ \frac{1}{2} (C_{11} + C_{22} \pm \sqrt{(C_{11} - C_{22})^2 + 4C_{12}C_{21}}) \right].
\]

The two signs in front of the square root produce two different values for \(\nu\), unless the parameters \(g_s\) in equation (3) be such that \((C_{11} - C_{22})^2 + 4C_{12}C_{21} = 0\), in which case only one multiplicative solution appears, any other independent solution containing logarithmic terms. The ambiguity in the real part of \(\nu\) due to the multivaluedness of the logarithm on the right-hand side of (A.5) reflects the fact already mentioned that the indices \(\nu\) are not uniquely defined. Note that

\[
\lambda_1 \lambda_2 = \det C = \mathcal{W}[w_a, w_b] = 1
\]

and, therefore,

\[
\nu_1 + \nu_2 = 0 \pmod{1}.
\]

This may serve as a test for the integration of equation (3) on the unit circle.

Although equation (A.5) is exact, \(C_{mn}\) are obtained numerically and the resulting values of \(\nu\) may only be considered as starting values, \(\nu^{(0)}\), for the Newton iteration process. As starting coefficients \(c_n^{(0)}\) one may use the solutions of the homogeneous system

\[
(n + \nu^{(0)})(n - 1 + \nu^{(0)}) c_n^{(0)} - \sum_{s=-2M}^{2N} g_s c_{n-s}^{(0)} = 0, \quad n = -M, \ldots, -1, 0, 1, \ldots, N,
\]

(A.6)

with the already mentioned truncated normalization condition

\[
\sum_{n=-M}^{N} |c_n^{(0)}|^2 = 1.
\]

(A.7)
Appendix B. Wronskians of Floquet and Thomé solutions

The connection factors of the Floquet solutions with the Thomé ones are given, by equations (21) and (22), as quotients of two Wronskians. Those in the denominators can be obtained immediately and were given in equations (23) and (24). Direct computation of the Wronskians in the numerators must be discarded for the reasons pointed out at the end of section 3. The purpose of this appendix is to give a procedure to compute them. The idea is to find, for each of the needed Wronskians, two functions, one proportional to the other, the proportionality constant being that Wronskian. A comparison of analogous terms in the asymptotic expansions of those functions allows one to obtain the required Wronskian. This idea has already been exploited in the solution of the Schrödinger equation with a polynomial potential [32, 33].

Let us consider the Wronskian of one of the Floquet solutions $w_j$ ($j = 1, 2$), given in (5), and one of the Thomé solutions at infinity $u_k$ ($k = 3, 4$), given in (6). We find convenient to introduce auxiliary functions $u_{j,k} = \exp\left(-\frac{\alpha_{N,k} z^N}{2N} \right) w_j$, $u_k = \exp\left(-\frac{\alpha_{N,k} z^N}{2N} \right) w_k$, $j = 1, 2$, $k = 3, 4$. (B.1)

Obviously, $\mathcal{W}[u_{j,k}, u_k] = \exp\left(-\frac{\alpha_{N,k} z^N}{N} \right) \mathcal{W}[w_j, w_k]$. (B.2)

An asymptotic expansion on the left-hand side of this equation can be calculated by using definitions (B.1) and expansions (5) and (6). It becomes

$$\mathcal{W}[u_{j,k}, u_k] \sim \left( \alpha_{N,k} \frac{z^{-1}}{2N} + 2 \sum_{p=1}^{N-1} \alpha_{p,k} z^{p-1} \right) v_{j,k} - \frac{dv_{j,k}}{dz} S_k + v_{j,k} \frac{dS_k}{dz},$$

where we have denoted

$v_{j,k} = \exp\left(\sum_{p=1}^{N-1} \frac{\alpha_{p,k}}{p} z^p \right) w_j$, $j = 1, 2$, $k = 3, 4$. (B.4)

$S_k = \sum_{m=0}^{\infty} \alpha_{m,k} z^{-m+\mu_k}$, $k = 3, 4$. (B.5)

For the newly introduced function $v_{j,k}$, a convergent Laurent expansion

$v_{j,k} = \sum_{n=-\infty}^{\infty} c_n z^{n+\nu_j}$ (B.6)

can be obtained as the Floquet solution of the differential equation

$$-z^2 \frac{d^2 v_{j,k}}{dz^2} + 2z \left( \sum_{p=1}^{N-1} \alpha_{p,k} z^p \right) \frac{dv_{j,k}}{dz}$$

$$+ \left( \sum_{s=-2M}^{2N} g_s z^s + \sum_{p=2}^{N-1} (p-1) \alpha_{p,k} z^p - \left( \sum_{p=1}^{N-1} \alpha_{p,k} z^p \right)^2 \right) v_{j,k} = 0.$$ (B.7)

Then, by using equations (B.5) and (B.6) in (B.3), we obtain

$$\mathcal{W}[u_{j,k}, u_k] \sim \sum_{n=-\infty}^{\infty} v_n^{(j,k)} z^{n+\nu_j+\nu_k},$$ (B.8)
where
\[ γ_{n}^{(j,k)} = \sum_{m=0}^{\infty} d_{m,k} \left( \alpha_{N,k} \hat{c}_{n+m+1-N,j,k} + 2 \sum_{p=1}^{N-1} \alpha_{p,k} \hat{c}_{n+m+1-p,j,k} \right) - (n + 2m + 1 + \nu_{j} - \mu_{k}) \hat{c}_{n+m+1,j,k} \). \tag{B.9}

The value of \( W[w_{j}, w_{k}] \) can be immediately obtained if we are able to write an asymptotic expansion of \( \exp \left( -\alpha_{N,k} z^{N}/N \right) \), on the right-hand side of (B.2), with the same powers of \( z \) as expansion (B.8) on the left-hand side. With this purpose, we construct \( N \) formal expansions
\[ E_{L}^{(j,k)}(z) = \sum_{n=-\infty}^{\infty} \left( -\alpha_{N,k} z^{N}/N \right)^{n+\delta_{L}^{(j,k)}} \frac{n+\delta_{L}^{(j,k)}}{\Gamma(n+1+\delta_{L}^{(j,k)})}, \tag{B.10} \]
of \( \exp \left( -\alpha_{N,k} z^{N}/N \right) \). Such expansions are but particular forms of the so-called Heaviside’s exponential series
\[ \exp(t) \sim \sum_{n=-\infty}^{\infty} \frac{t^{n+\delta}}{\Gamma(n+1+\delta)}, \tag{B.11} \]
introduced by Heaviside in the second volume of his *Electromagnetic theory* (London, 1899) and which, as proved by Barnes [39], is an asymptotic expansion for arbitrary \( \delta \) and \( |\arg(t)| < \pi \). Expansions of this kind have been already used by Naundorf [27] in his treatment of the connection problem, from which our method is a convenient modification. It becomes evident that, for any set of constants \( \{\kappa_{L}^{(j,k)}\} (L = 0, 1, \ldots, N-1) \) satisfying the restriction
\[ \sum_{L=0}^{N-1} \kappa_{L}^{(j,k)} = W[w_{j}, w_{k}], \tag{B.12} \]
one has from (B.2)
\[ W[u_{j,k}, u_{k}] \sim \sum_{L=0}^{N-1} \kappa_{L}^{(j,k)} E_{L}^{(j,k)}(z). \tag{B.13} \]

By choosing for \( \delta_{L}^{(j,k)} \) in the expansions \( E_{L}^{(j,k)} \) the values
\[ \delta_{L}^{(j,k)} = (\nu_{j} + \mu_{k} + L)/N, \tag{B.14} \]
a comparison, term by term, of the resulting expansion in (B.13) with that in (B.8) can be done. One obtains in this way
\[ \kappa_{L}^{(j,k)} \left( -\alpha_{N,k}/N \right)^{n+\delta_{L}^{(j,k)}} \frac{n+\delta_{L}^{(j,k)}}{\Gamma(n+1+\delta_{L}^{(j,k)})} = \gamma_{nN+L}, \tag{B.15} \]
for any positive integer \( n \) large enough to satisfy
\[ |(n + \nu_{j})(n - 1 + \nu_{j})| > \sum_{s=-2M}^{2N} |g_{s}|. \]

By substituting in (B.12) the values of \( \kappa_{L}^{(j,k)} \) obtained from (B.15) one finally has
\[ W[w_{j}, w_{k}] = \sum_{L=0}^{N-1} \Gamma(n+1+\delta_{L}^{(j,k)}) \left( -\alpha_{N,k}/N \right)^{n+\delta_{L}^{(j,k)}} \gamma_{nN+L}, \tag{B.16} \]
where the minus sign in front of $\alpha_{N,k}$ is to be interpreted as an odd power of $e^{i\pi}$ or $e^{-i\pi}$ so as to have $|\arg(-\alpha_{N,k}z^N)| < \pi$. In the physical problems, one is interested in the connection factors on the positive real semiaxis, that is, on the ray $\arg(z) = 0$. In this case, $\arg(-\alpha_{N,3}z^N) = 0$; then, equation (B.16) gives for one of the numerators in (21)

$$W[w_j, w_3] = \sum_{L=0}^{N-1} \frac{\Gamma(n + 1 + \delta_{L,3})}{(\alpha_{N,3}/N)^{\delta_{L,3}}} \gamma_{n+N+L,}^{(j,3)} \gamma_{n+N+L,}^{(j,3)}, \quad j = 1, 2,$$

and the connection factor $T_{j,4}$ can be obtained immediately from (B.17) and (23). For the computation of the other connection factor, instead, one has to bear in mind the fact that the semiaxis $\arg(z) = 0$ is a Stokes ray for $T_{j,3}$. Actually, for $z$ on this ray, $|\arg(-\alpha_{N,4}z^N)| = \pi$ and expansions (B.10) would not correspond to $\exp(-\alpha_{N,k}z^N/N)$. Following the common practice, we define $T_{j,3}$ on the Stokes ray by the average

$$T_{j,3} = \frac{1}{2}(T_{j,3}^+ + T_{j,3}^-)$$

of its values in the regions separated by that ray. This corresponds to define

$$W[w_j, w_4] = \frac{1}{2}(W[w_j, w_4]^+ + W[w_j, w_4]^-)$$

an average of the Wronskians for $z$ slightly above and below the positive real semiaxis. One obtains in this way

$$W[w_j, w_4] = (-1)^n \sum_{L=0}^{N-1} \cos(\delta_{L,4} \pi) \frac{\Gamma(n + 1 + \delta_{L,4})}{(\alpha_{N,4}/N)^{\delta_{L,4}}} \gamma_{n+N+L,}^{(j,4)} \gamma_{n+N+L,}^{(j,4)}, \quad j = 1, 2,$$

and the expression of $T_{j,3}$ follows then from (B.20) and (23).

The procedure of obtaining the Wronskian of one of the Floquet solutions, $w_j$ ($j = 1, 2$), and one of the Thomé solutions at the origin, $u_l$ ($l = 5, 6$) given in (7), is analogous to that just described. The auxiliary functions are now

$$u_{j,l} = \exp\left(-\frac{\beta_{M,l}}{2M} z^{-M-1}\right) w_j, \quad u_l = \exp\left(-\frac{\beta_{M,l}}{2M} z^{-M}\right) w_j, \quad j = 1, 2, \quad l = 5, 6.$$  

(B.21)

Then,

$$W[u_{j,l}, u_l] = \exp\left(-\frac{\beta_{M,l}}{M} z^{-M}\right) W[w_j, w_l].$$  

(B.22)

By using definitions (B.21) and expansions (5) and (7) one obtains an asymptotic expansion on the left-hand side, namely

$$W[u_{j,l}, u_l] \sim \left(-\beta_{M,l} z^{-M-1} - 2 \sum_{q=1}^{M-1} \beta_{q,l} z^{-q-1} \right) u_{j,l} - \frac{d u_{j,l}}{dz} S_l + v_{j,l} \frac{d S_l}{dz},$$  

(B.23)

where we have denoted

$$v_{j,l} = \exp\left(\sum_{q=1}^{M-1} \frac{\beta_{q,l}}{q} z^{-q}\right) w_j, \quad j = 1, 2, \quad l = 5, 6,$$

$$S_l = \sum_{m=0}^{\infty} b_{m,l} z^{m+\rho_l}, \quad l = 5, 6.$$  

(B.24)

(B.25)
A convergent Laurent expansion

\[ v_{j,l} = \sum_{n=-\infty}^{\infty} \hat{c}_{n,l} z^{n+j} \] (B.26)

for \( v_{j,l} \) can be obtained as the Floquet solution of the differential equation

\[
-\varepsilon^2 \frac{d^2 v_{j,l}}{dz^2} + 2\varepsilon \left( \sum_{q=1}^{M-1} \beta_{q,l} z^{-q} \right) \frac{dv_{j,l}}{dz} \\
+ \left( \sum_{q=1}^{2N} g_q z^q + \sum_{q=1}^{M-1} (q+1) \beta_{q,l} z^{-q} - \left( \sum_{q=1}^{M-1} \beta_{q,l} z^{-q} \right) \right)^2 v_{j,l} = 0. \] (B.27)

From (B.23), by using equations (B.25) and (B.26), we obtain

\[
\mathcal{W}[u_{j,l}, u_l] \sim \sum_{n=-\infty}^{\infty} \gamma^{(j,l)}_n z^{n+j+\rho_l}, \] (B.28)

where

\[
\gamma^{(j,l)}_n = \sum_{m=0}^{\infty} h_m \left( \sum_{q=1}^{M-1} \beta_{M,l} \hat{c}_{n-m+1+q,j,l} - \sum_{q=1}^{M-1} \beta_{q,l} \hat{c}_{n-m+1+q,j,l} \right) \\
+ (-n + 2m - 1 - v_j + \rho_l) \hat{c}_{n-m+1,j,l}. \] (B.29)

In order to obtain an asymptotic expansion of \( \exp(-\beta_{M,l} z^{-M}/M) \), on the right-hand side of (B.22), with the same powers of \( z \) as expansion (B.28) on the left-hand side, we construct \( M \) formal expansions analogous to those in (B.10):

\[
\mathcal{E}^{(j,l)}_L(z) = \sum_{n=-\infty}^{\infty} \frac{(-\beta_{M,l} z^{-M}/M)^n y^{(j,l)}_n}{\Gamma(n + 1 + \delta^{(j,l)}_L)}, \quad L = 0, 1, \ldots, M - 1. \] (B.30)

Let us now consider \( M \) constants \( \{\kappa^{(j,l)}_L\} (L = 0, 1, \ldots, M - 1) \) such that

\[
\sum_{L=0}^{M-1} \kappa^{(j,l)}_L = \mathcal{W}[w_j, w_l]. \] (B.31)

Then,

\[
\mathcal{W}[u_{j,l}, u_l] \sim \sum_{L=0}^{M-1} \kappa^{(j,l)}_L \mathcal{E}^{(j,l)}_L(z). \] (B.32)

If we choose for \( \delta^{(j,l)}_L \) in (B.30) the values

\[
\delta^{(j,l)}_L = (-v_j - \rho_l + L)/M, \] (B.33)

a comparison of the resulting expansion in (B.32) with that in (B.28) allows one to write

\[
\kappa^{(j,l)}_L \frac{(-\beta_{M,l}/M)^n y^{(j,l)}_n}{\Gamma(n + 1 + \delta^{(j,l)}_L)} = y^{(j,l)}_{nM-L}, \] (B.34)

which, substituted in (B.31), gives

\[
\mathcal{W}[w_j, w_l] = \sum_{L=0}^{M-1} \frac{\Gamma(n + 1 + \delta^{(j,l)}_L)}{(-\beta_{M,l}/M)^n y^{(j,l)}_{nM-L}}, \] (B.35)
where the minus sign in front of $\beta_{M,l}$ should be replaced by an odd power of $e^{i\pi}$ or $e^{-i\pi}$ such that $|\arg(-\beta_{M,l}z^{-M})|<\pi$. For $z$ on the positive real semiaxis, one has

$$\mathcal{V}[w_j, w_5] = \sum_{L=0}^{M-1} \frac{\Gamma(n+1+\delta^{(j,6)}_L)}{(\beta_{M,5}/M)^{\theta^{(j,6)}_L}} y_{-\theta^{(j,5)}_L} \gamma_{-nM-L}, \quad j = 1, 2, \quad (B.36)$$

and, analogous to equation (B.20),

$$\mathcal{V}[w_j, w_6] = (-1)^n \sum_{L=0}^{M-1} \cos\left(\frac{\theta^{(j,5)}_L}{\pi}\right) \frac{\Gamma(n+1+\delta^{(j,6)}_L)}{(\beta_{M,6}/M)^{\theta^{(j,6)}_L}} y_{-\theta^{(j,6)}_L} \gamma_{-nM-L}, \quad j = 1, 2. \quad (B.37)$$

The connection factors $T_{j,5}$ and $T_{j,6}$ are then immediately obtained from equations (22) by using (24), (B.36) and (B.37).

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