On effects of non-Euclidean geometry in quantum theory

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Abstract

Theory of scattering of a quantum-mechanical particle on a cosmic string is developed. S-matrix and scattering amplitude are determined as functions of the flux and the tension of the string. We reveal that, in the case of the nonvanishing tension, the high-frequency limit of the differential scattering cross section does not coincide with the differential cross section for scattering of a classical pointlike particle on a string.

Keywords: cosmic string, Bohm-Aharonov effect

(based on talks given at: the International Workshop ”Frontiers of Particle Astrophysics”, June 21-24, 2004, Kyiv, Ukraine; the George Gamow Memorial International Conference ”Astrophysics and Cosmology after Gamow - Theory and Observations”, August 8-14, 2004, Odessa, Ukraine; the IV International Conference ”Non-Euclidean Geometry in Modern Physics and Mathematics”, September 7-11, 2004, Nizhny Novgorod, Russia)

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1 Introduction

Usually, the effects of non-Euclidean geometry are identified with the effects which are due to the curvature of space. It can be immediately shown that this is not the case and there are spaces which are flat but non-Euclidean.

A simplest example is given by a twodimensional space (surface) which is obtained from a plane by cutting a segment of a certain angular size and then sewing together the edges. The resulting surface is the conical surface which is flat but has a singular point corresponding to the apex of the cone. To be more precise, the intrinsic (Gauss) curvature of the conical surface is proportional to the twodimensional delta-function placed at the apex; the coefficient of proportionality is the deficit angle. Usual cones correspond to positive values of the deficit angle, i.e. to the situation when a segment is deleted from the plane. But one can imagine a situation when a segment is added to the plane; then the deficit angle is negative, and the resulting flat surface can be denoted as a saddle-like cone. The deleted segment is bounded by the value of $2\pi$, whereas the added segment is unbounded. Thus, deficit angles for possible conical surfaces range from $-\infty$ to $2\pi$.

It is evident that an apex of the conical surface with the positive deficit angle can play a role of the convex lens, whereas an apex of the conical surface with the negative deficit angle can play a role of the concave lens. Really, two parallel trajectories coming from infinity towards the apex from different sides of it, after bypassing it, converge (and intersect) in the case of the positive deficit angle, and diverge in the case of the negative deficit angle. This demonstrates the non-Euclidean nature of conical surfaces. It is interesting that this item provides a basis for understanding such physical objects as cosmic strings. In the present paper we shall discuss peculiarities of quantum theory and its quasiclassical limit, which are due to non-Euclidean geometry of locally flat space-times.

2 Space-time in the presence of a cosmic string

Cosmic strings are topological defects which are formed as a result of phase transitions with spontaneous breakdown of symmetries at early stages of evolution of the universe, see, e.g., reviews in Refs.[1, 2]. In general, a cosmic string is characterized by two quantities: flux

$$\Phi = \int_{\text{core}} d^2 x \sqrt{g} B^3, \quad (1)$$

and tension

$$\mu = \frac{1}{16\pi G} \int_{\text{core}} d^2 x \sqrt{g} R; \quad (2)$$

here the integration is over the transverse section of the core of the string, $B^3$ is the field strength which is directed along the string axis, $R$ is the scalar curvature, $G$ is the gravitational constant, and units $\hbar = c = 1$ are used. The space-time metric outside the string core is

$$d s^2 = d t^2 - (1 - 4G \mu)^{-1} d \overset{\sim}{r}^2 - (1 - 4G \mu) \overset{\sim}{r}^2 d \varphi^2 - d z^2 = d t^2 - d r^2 - r^2 d \varphi^2 - d z^2, \quad (3)$$

where

$$\overset{\sim}{r} = r \sqrt{1 - 4G \mu}, \quad 0 \leq \varphi < 2\pi, \quad 0 \leq \overset{\sim}{\varphi} < 2\pi (1 - 4G \mu). \quad (4)$$

A surface which is transverse to the axis of the string is isometric to the surface of a cone with a deficit angle equal to $8\pi G \mu$. Such space-times were known a long time ago (M. Fierz, unpublished, see footnote in Ref.[3]) and were studied in detail by Marder [4]. In the present context, as cosmological objects and under the name of cosmic strings, they were introduced in seminal works of Kibble [5] and Vilenkin [6].
A cosmic string resulting from a phase transition at the scale of the grand unification of all interactions is characterized by the values of tension

\[ \mu \sim (10^{-7} \div 10^{-6})G^{-1}. \]  

(5)

The nonvanishing of the string tension leads to various cosmological consequences and, among them, to a very distinctive gravitational lensing effect. A possible observation of such an effect has been reported recently [2], and this has revived an interest towards cosmic strings.

The flux parameter (1) is nonvanishing for the so-called gauge cosmic strings, i.e. strings corresponding to spontaneous breakdown of local symmetries. If tension vanishes (\( \mu = 0 \)), then a gauge cosmic string becomes a magnetic string, i.e. a tube of the magnetic flux lines in Euclidean space. If the tube is impenetrable for quantum-mechanical charged particles, then scattering of the latter on the magnetic string depends on flux \( \Phi \) periodically with period \( 2\pi e^{-1} \) (\( e \) is the coupling constant – charge of the particle). This is known as the Bohm-Aharonov effect [3], which has no analogue in classical physics, since the classical motion of charged particles cannot be affected by the magnetic flux from the impenetrable for the particles region. The natural question is, how the nonvanishing string tension (\( \mu \neq 0 \)) influences scattering of quantum-mechanical particles on the string. Thus, the subject of cosmic strings, in addition to tantalizing phenomenological applications, acquires a certain conceptual importance.

3 Quantum scattering on a cosmic string

Due to nonvanishing flux \( \Phi \) and tension \( \mu \), the quantum scattering of a test particle by a cosmic string is a highly nontrivial problem. It is impossible to choose a plane wave as the incident wave, because of the long-range nature of the interaction inherent in this problem. A general approach to quantum scattering in the case of long-range interactions was elaborated by Hormander [9]. This approach covers the cases of scattering on a Coulomb center and on a magnetic string (\( \mu = 0 \)), but is not applicable to the case of scattering on a cosmic string (\( \mu \neq 0 \)). Therefore the last case needs a special consideration and a thorough substantiation.

When the effects of the core structure of a cosmic string are neglected and the transverse size of the core is negligible, the field strength and the scalar curvature are presented by two-dimensional delta-functions. Scattering of a quantum-mechanical particle on an idealized (without structure) cosmic string was considered in Refs.[10][11][12][13]. A general theory of quantum-mechanical scattering on a cosmic string, permitting to take into account the effects of the core structure, was elaborated in Ref.[14].

According to this theory, the \( S \)-matrix in the momentum representation is

\[
S(k, \varphi; k', \varphi') = \frac{1}{2} \frac{\delta(k-k')}{\sqrt{k'k}} \left\{ \Delta(\varphi - \varphi') + \frac{4G\mu\pi}{1-4G\mu} \exp \left[ -\frac{i\varphi\Phi}{2(1-4G\mu)} \right] \right\} + \\
+ \Delta \left( \varphi - \varphi' - \frac{4G\mu\pi}{1-4G\mu} \right) \exp \left[ \frac{i\varphi\Phi}{2(1-4G\mu)} \right] + \delta(k-k') \sqrt{\frac{i}{2\pi k}} f(k, \varphi - \varphi'),
\]  

(6)

where the initial (\( k \)) and final (\( k' \)) twodimensional momenta of the particle are written in polar variables, \( f(k, \varphi - \varphi') \) is the scattering amplitude, and \( \Delta(\varphi) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{in\varphi} \) is the angular part of the twodimensional delta-function. Note that in the case of short-range interaction one has \( 2\Delta(\varphi - \varphi') \) instead of the figure brackets in Eq.(6). Thus, one can see that, due to the long-range nature of interaction, even the conventional relation between \( S \)-matrix and scattering amplitude is changed, involving now a distorted unity matrix (first term in Eq.(6)) instead of the usual one, \( \delta(k-k')\Delta(\varphi - \varphi')(k'k)^{-1/2} \).

In view of the comparison with the Bohm-Aharonov effect [3], we shall be interested in the situation when the string core is impenetrable for the particle. The scattering amplitude in this case takes form:

\[
f(k, \varphi) = f_0(k, \varphi) - \sqrt{\frac{2}{\pi ik}} \sum_{n=-\infty}^{\infty} \exp[in\varphi - i(\alpha_n - |n|\pi)] J_{\alpha_n}(kr_c) H_{\alpha_n}^{(1)}(kr_c),
\]  

(7)
where $r_c$ is the radius of the string core, $J_{\nu}(u)$ and $H^{(1)}_{\nu}(u)$ are the Bessel and the first-kind Hankel functions of order $\nu$,

$$\alpha_n = \left| n - \frac{e\Phi}{2\pi} \right| (1 - 4G\mu)^{-1}, \quad (8)$$

and

$$f_0(k, \phi) = \frac{1}{\sqrt{2\pi} i k} \left\{ \exp \left[ i \left[ \frac{e\Phi}{2\pi} \right] (\phi + \frac{4G\mu\pi}{1 - 4G\mu}) - \frac{ie\Phi}{2(1 - 4G\mu)} \right] - \exp \left[ i \left[ \frac{e\Phi}{2\pi} \right] (\phi - \frac{4G\mu\pi}{1 - 4G\mu}) + \frac{ie\Phi}{2(1 - 4G\mu)} \right] \right\}$$

$$ \left/ \left( 1 - \exp \left[ -i \left( \phi - \frac{4G\mu\pi}{1 - 4G\mu} \right) \right] \right) \right. \right) \quad (9)$$

is the amplitude of scattering on an idealized (without structure) cosmic string, $[u]$ is the integer part of $u$. Sum over $n$ in Eq.(7) describes the core structure effects. In the low-frequency limit ($k \to 0$) these effects die out, and the differential cross section (i.e. the square of the absolute value of the amplitude) takes form

$$\frac{d\sigma}{d\phi} = \frac{1}{4\pi k} \left\{ \frac{1}{2\sin^2 \left( \frac{1}{2} (\phi + \frac{4G\mu\pi}{1 - 4G\mu}) \right)} + \frac{1}{2\sin^2 \left( \frac{1}{2} (\phi - \frac{4G\mu\pi}{1 - 4G\mu}) \right)} - \frac{\cos \left[ \frac{e\Phi}{4G\mu} - (2[\frac{e\Phi}{2\pi}] + 1) \right]}{\sin \left( \frac{1}{2} (\phi + \frac{4G\mu\pi}{1 - 4G\mu}) \right) \sin \left( \frac{1}{2} (\phi - \frac{4G\mu\pi}{1 - 4G\mu}) \right) } \right\}. \quad (10)$$

### 4 Differential cross section in the limit of high frequency of scattered particle

In the high-frequency limit ($k \to \infty$) the first term in Eq.(7) dies out, and the differential cross section takes form

$$\frac{d\sigma}{d\phi} = \frac{1}{2} r_c (1 - 4G\mu)^2 \left\{ \sum_l \sqrt{\cos \left[ \frac{1}{2} (1 - 4G\mu) (\phi - \pi + 2l\pi) \right]} \times \right.$$  

$$\times \exp \left\{ i e\Phi l - 2ikr_c \cos \left[ \frac{1}{2} (1 - 4G\mu) (\phi - \pi + 2l\pi) \right] \right\} \right\}^2, \quad (11)$$

where the summation is over integer $l$ satisfying condition

$$-\frac{\phi}{2\pi} - \frac{2G\mu}{1 - 4G\mu} < l < -\frac{\phi}{2\pi} + 1 + \frac{2G\mu}{1 - 4G\mu}. \quad (12)$$

Note that results (10) and (11) are periodic in the value of flux $\Phi$ with period equal to $2\pi e^{-1}$. This feature is common with the scattering on a purely magnetic string ($\mu = 0$). The difference is that the Bohm-Aharonov differential cross section in the low frequency limit ($k \to 0$) diverges in the forward direction, $\phi = 0$, while Eq.(10) diverges in two symmetric directions, $\phi = \pm 4G\mu(1 - 4G\mu)^{-1}$. The difference becomes much more crucial in the high-frequency limit ($k \to \infty$). In the $\mu = 0$ case one gets

$$\frac{d\sigma}{d\phi} = \frac{1}{2} r_c \sin \frac{\phi}{2}, \quad (13)$$

which is the cross section for scattering of a classical pointlike particle by an impenetrable cylindrical shell of radius $r_c$; evidently, the dependence on fractional part of $e\Phi(2\pi)^{-1}$ disappears in this limit. In
the \( \mu \neq 0 \) case the dependence survives, see Eq.(11). In particular, if \( 0 < \mu < (8G)^{-1} \), which is most interesting from the phenomenological point of view, then the cross section at \( k \to \infty \) takes the following form in the region of the cosmic string shadow, \( -\frac{4G\mu\pi}{1-4G\mu} < \varphi < \frac{4G\mu\pi}{1-4G\mu} \):

\[
\frac{d\sigma}{d\varphi} = r_c(1-4G\mu)^2 \left( \cos\left[\frac{1}{2}(1-4G\mu)\varphi\right] \sin(2G\mu\pi) + \right.
\]
\[
+ \sqrt{\sin^2(2G\mu\pi) - \sin^2\left[\frac{1}{2}(1-4G\mu)\varphi\right]} \cos\left\{ e\Phi + 4kr_c \sin\left[\frac{1}{2}(1-4G\mu)\varphi\right] \cos(2G\mu\pi) \right\} \right). \tag{14}
\]

Integrating Eq.(14) over the region of the shadow and the appropriate expression (which is independent of \( \Phi \)) over the region out of the shadow, we obtain the total cross section in the \( k \to \infty \) limit:

\[
\sigma_{\text{tot}} = 2r_c(1-4G\mu). \tag{15}
\]

The high-frequency limit is usually identified with the quasiclassical limit. Although this identification is valid for the total cross section, it is found to be invalid for the differential cross section, see Eqs.(11) and (14) revealing the periodic dependence on the flux, which is a purely quantum effect.

These results are generalized to the case of scattering of a particle with spin.

5 Acknowledgements

This work was supported by the State Foundation for Basic Research of Ukraine (project 2.7/00152).

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