STABILITY OF THE POSITIVE MASS THEOREM IN DIMENSION THREE

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Abstract. In this paper, we show that for a sequence of orientable complete pointed uniformly asymptotically Euclidean 3-manifolds \((M_i, g_i, p_i)\) with non-negative integrable scalar curvature \(R_{g_i} \geq 0\), if their mass \(m(g_i) \to 0\), then by subtracting some subsets \(Z_i\) whose boundary area \(\|\partial Z_i\| \leq C m(g_i)^{1/2}\), up to diffeomorphisms, \((M_i \setminus Z_i, g_i, p_i)\) converge to the Euclidean space \(\mathbb{R}^3\) in the pointed flat metric topology. This confirms Huisken-Ilmanen’s conjecture in terms of the flat metric topology. Moreover, if we assume the Ricci curvature bounded from below uniformly by \(\text{Ric}_{g_i} \geq -2\Lambda\), then \((M_i, g_i, p_i)\) converge to \(\mathbb{R}^3\) in the pointed Gromov-Hausdorff topology.

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1. INTRODUCTION

A pointed smooth orientable connected complete Riemannian 3-manifold \((M^3, g, p)\) is called uniformly \((A, B, \sigma)\)-asymptotically Euclidean (or AE), for some given \(A, B > 0, \sigma > \frac{1}{2}\), if there exists a compact subset \(K \subset M\) and a \(C^\infty\)-diffeomorphism \(\Phi : M \setminus K \to \mathbb{R}^3 \setminus B(0, A)\) such that under this identification,

\[ |\partial^k (g_{uv} - \delta_{uv})(x)| \leq B|x|^{-\sigma-|k|}, \]

for all multi-indices \(|k| = 0, 1, 2\), and \(p = \Phi^{-1}(p^*)\) for some fixed \(p^* \in \mathbb{R}^3 \setminus B(0, A)\). Furthermore, we always assume the scalar curvature \(R_g\) is integrable.

Notice that there is only one end in this definition of uniformly AE 3-manifold. In general, a 3-manifold \((M^3, g)\) is called an AE 3-manifold if there exists a compact set \(K \subset M\) such that \(M \setminus K = \bigcup_{k=1}^{N} M^k_{\text{end}}\), where the ends \(M^k_{\text{end}}\) are pairwise disjoint and for some \(A_k, B_k > 0, \sigma > \frac{1}{2}\), each \(M^k_{\text{end}}\) is \((A_k, B_k, \sigma)\)-AE defined as above.

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In the following, we will say \((M^3, g, p)\) is pointed \((A, B, \sigma)\)-AE if it’s an AE 3-manifold with one or more ends, and one of these ends contains the base point \(p\) and is \((A, B, \sigma)\)-AE in the sense of above definition.

The ADM mass from general relativity [ADM61] of an AE 3-manifold \((M, g)\) is defined as
\[
m(g) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \sum_{j,k=1}^3 (g_{jk,j} - g_{jj,k}) v^k dA,
\]
where \(v\) is the unit normal vector to the standard sphere \(S_r \subset \mathbb{R}^3\). It follows from [Bar86] that the mass is finite and independent of AE coordinates.

The positive mass theorem in dimension three is the following.

**Theorem 1.1.** Let \((M^3, g)\) be an AE 3-manifold with nonnegative integrable scalar curvature. Then \(m(g) \geq 0\) with equality if and only if \((M, g) = (\mathbb{R}^3, g_E)\).

This theorem was first proved by Schoen-Yau [SY79a] in 1979 by constructing stable minimal surface. Later Schoen-Yau extended the theorem to the case when the dimension was less than eight [Sch89, SY79b]. In 1981, Witten [Wit81] later proved the positive mass theorem for spin manifolds of any dimension. In 2001, as a byproduct of the proof of the Penrose inequality, Huisken-Ilmanen [HI01] proved the three dimensional positive mass theorem by using the inverse mean curvature flow. Recently, for three dimensional positive mass theorem, Li [Li18] gave a proof using Ricci flow, Bray-Kazaras-Khuri-Stern [BKKS22] gave a proof using a mass inequality.

The positive mass theorem gives a rigidity statement. A natural question to then ask is the stability problem.

**Question 1.2.** Would the smallness of the mass imply that the manifold is close to the Euclidean space in some topology?

It’s not known which topology should be a good choice and suitable for nonnegative scalar curvature. One conjecture about this question in terms of Gromov-Hausdorff topology was stated in 2001 by Huisken-Ilmanen in [HI01].

**Conjecture 1.3.** Suppose \(M_i\) is a sequence of asymptotically Euclidean 3-manifolds with nonnegative scalar curvature and ADM mass tending to zero. Then there is a set \(Z_i \subset M_i\) such that \(|\partial Z_i| \to 0\) and \(M_i \setminus Z_i\) converges to \(\mathbb{R}^3\) in the Gromov-Hausdorff topology.

**Remark 1.1.** Huisken-Ilmanen’s motivation of this conjecture comes from the Riemannian Penrose inequality (see Section 2.1 for more details).

In 2014 Lee-Sormani [LS14] gave a conjecture in terms of intrinsic flat metric topology (see also [S⁺21], Conjecture 10.1). Denote the exterior region by \(M_i^{\text{ext}}\) (see Section 2.1 for more details).

**Conjecture 1.4.** Suppose \(M_i\) is a sequence of asymptotically Euclidean 3-manifolds with nonnegative scalar curvature and ADM mass tending to zero. Suppose \(\Omega_i \subset M_i\) are regions inside \(M_i^{\text{ext}}\) containing \(\partial M_i^{\text{ext}}\), and \(\Sigma_i\) is the outer boundary of \(\Omega_i\) such that \(\text{diam}(\Omega_i) \leq D\) and \(|\Sigma_i| = A_0\), then \(\Omega_i \to \Omega_0\) in the intrinsic flat metric topology, where \(\Omega_0\) is a ball in the Euclidean space such that \(|\partial \Omega_0| = A_0\).

Recently Lee-Naber-Neumayer [LNN20] defined a \(d_p\)-convergence, and one can also formulate a conjecture with respect to this topology.
Assuming additional conditions, there have been several progress toward above question and conjectures, see among others \cite{BF99, FK02, Cor05, Lee09, LS14, HL15, KKL21, ABK22}.

In this paper, motivated by Conjectures 1.3, 1.4 and the recent work of Kazaras-Khuri-Lee \cite{KKL21}, we study the stability of the positive mass theorem without imposing additional conditions.

Inspired by Cheeger-Gromov’s smooth convergence, we modify the flat metric topology in geometric measure theory and give the following definition.

**Definition 1.1.** A sequence of pointed smooth manifolds \((M_i, g_i, p_i)\) with or without boundary is said to converge, up to diffeomorphisms, to \((M, g, p)\) in the pointed flat metric topology, if we can find diffeomorphisms \(\varphi_i : M_i \to \varphi_i(M_i) \subset M\) with \(\varphi_i(p_i) = p \) such that \(\varphi_i#M_i \to \varphi_i(M_i) \in M\) in the pointed flat metric topology as currents on \(M\) (see Definition 31.1 in \cite{Sim83}). In other words, for any fixed \(r > 0\) and any coordinate domain \(x : U \to \mathbb{R}^n\) with \(U \subset B(p, r) \subset M\), we have

\[
\lim_{i \to \infty} \int_U \left| \chi_{\varphi_i(M_i)} \sqrt{\det g_i} - \sqrt{\det g} \right| dx = 0.
\]

Our main result gives one answer to Question 1.2, which also confirms Huisken-Ilmanen’s Conjecture 1.3 in terms of this pointed flat metric topology.

To make the statement precisely, we give some notations. For any \(r > 0\) and a uniformly AE 3-manifold with diffeomorphism \(\Phi\) on the uniform end, define \(M_r\) to be the interior component of \(M \setminus \Phi^{-1}(S_r)\), where \(S_r\) is the standard sphere with radius \(r\) in \(\mathbb{R}^3\). Similarly define \(M_{ext,r}\) to be the bounded region in \(M_{ext} \setminus \Phi^{-1}(S_r)\). If \(r > 0\) is uniform, then for a uniformly pointed AE 3-manifold \((M, g, p)\), without loss of generality, we can assume \(p \in M \setminus M_r\) by a uniform perturbation.

**Theorem 1.5.** Fix \(A, B > 0\), \(\sigma > \frac{1}{2}\). Let \((M_i, g_i, p_i)\) be a sequence of orientable complete pointed \((A, B, \sigma)\)-AE 3-manifolds with nonnegative integrable scalar curvature \(R_{g_i} \geq 0\). If the ADM masses \(m(g_i)\) of \((M_i, g_i)\) converge to zero, then up to a subsequence, there exist \(Z_i \subset M_{i,r_0}\) for some uniform \(r_0 > 0\) with boundary area \(|\partial Z_i| \leq Cm(g_i)^{\frac{1}{4}}\) for a uniform constant \(C > 0\), such that \((M_i \setminus Z_i, g_i, p_i)\) converge, up to diffeomorphisms, to \(\mathbb{R}^3\) in the pointed flat metric topology.

**Remark 1.2.** Ilmanen conjectured that for an asymptotically flat complete Riemannian 3-manifold \((M, g)\) with nonnegative scalar curvature, there exists a surface \(\Sigma\) bounding a region \(\Omega\) such that area \(|\Sigma| \leq 16\pi m(g)^2\) and \(M \setminus \Omega\) is ”flat-like”. Our result implies that we can find such surface with area smaller that \(Cm(g)^{\frac{1}{4}}\) such that \(M \setminus \Omega\) is ”flat-like” in the sense of pointed flat metric topology. This conjecture was told to the author by Prof. Marcus Khuri, and it comes from a private conversation between Prof. Hubert Bray and Prof. Tom Ilmanen.

**Remark 1.3.** If we weaken the smallness on area of boundary, then for any uniform small \(0 < \varepsilon \ll 1\), there exists \(Z_i \subset Z, \subset M_{i,r_0}\) such that \(|\partial Z_i| \leq Cm(g_i)^{\frac{1}{4}} - \varepsilon\) and \((M_i \setminus Z_i, g_i, p_i) \to \mathbb{R}^3\) in the pointed flat metric topology. Also, for any \(\delta > 0\), \((M_i \setminus B_{\delta}(Z_i), g_i, p_i)\) converges to a subset in \(\mathbb{R}^3\) locally in the \(C^0\)-topology, where \(B_{\delta}(Z_i)\) is the \(\delta\)-neighborhood of \(Z_i\).

If we assume stronger conditions on curvature, we can improve the pointed flat metric convergence to pointed Gromov-Hausdorff convergence as following.
Theorem 1.6. Under the same conditions as in Theorem 1.5, for a fixed $\Lambda > 0$, if the Ricci curvature are bounded uniformly from below by $\text{Ric}_g \geq -2\Lambda$, then up to a subsequence, $(M_i, g_i, p_i)$ converge to $\mathbb{R}^3$ in the pointed Gromov-Hausdorff topology.

Now we explain the ideas of the proof. Let’s firstly recall Kazaras-Khuri-Lee’s result in [KKL21], where they proved the Gromov-Hausdorff convergence under additional conditions that $H_2(M, \mathbb{Z}) = 0$ and Ricci curvature bounded uniformly from below by $\text{Ric}_g \geq -2\Lambda$. Their basic ingredient is the mass inequality proved by Bray-Kazaras-Khuri-Stern in [BKKS22], which is the following (see Section 2.1 for more details)

$$m(g) \geq \frac{1}{16\pi} \int_{M_{ext}} \left( \frac{\lvert \nabla^2 u \rvert^2}{\lvert \nabla u \rvert} + R_g \lvert \nabla u \rvert \right) dV,$$

where $\{u_j\}, j \in \{1, 2, 3\}$, are harmonic functions defined on $M_{ext}$ and asymptotic to each AE coordinate function. When $H_2(M, \mathbb{Z}) = 0$, this mass inequality holds on $M$. Then under the Ricci lower bound condition, they used Cheng-Yau’s gradient estimate, Cheeger-Colding’s techniques, like segment inequality, to derive almost pointwise estimate for $u_j$ from the integral inequality, and take such $u_j$ as splitting functions to prove the limit space is $\mathbb{R}^3$ when mass tends to zero.

There are at least two difficulties when generalizing their arguments. If we drop the condition that $H_2(M, \mathbb{Z}) = 0$, then $u_j$ is only defined on $M_{ext}$. Even we only restrict on $M_{ext}$, we have no control on $\lvert \nabla u \rvert$ around $\partial M_{ext}$. So we can not directly take $\{u_j\}$ as splitting functions. If we drop the lower bound on Ricci curvature, then almost nothing could be controlled, and we can not expect the Gromov-Hausdorff convergence. The metric could either concentrate or degenerate inside $M_{ext}$, so a different argument should be involved.

Our idea is inspired by Huisken-Ilmanen’s conjecture. We want to remove a bad subset and consider the convergence on the remaining set, which we call it the regular subregion. The regular subregion $E^\tau$ is defined by the connected component of

$$\{x \in M_{ext} : \sum_{j,k=1}^3 \lvert \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} \rvert^2 \leq \tau\}$$

containing outer region. In Proposition 4.1, we show that such regular subregion is well defined and has smooth boundary with small area. We need uniform effective estimates on $u_j$, which are proved in Section 3, to ensure $E^\tau$ is non-empty and contains the outer region. To show $\partial E^\tau$ has small area, we need to use the co-area formula, the mass inequality and choose some suitable $\tau$ depending on the mass $m(g)$.

Then by uniform estimates on outer region and some topological arguments, we can show that the harmonic map

$$U = (u^1, u^2, u^3) : E^\tau \to \mathbb{R}^3$$

is a diffeomorphism onto its image, under which the metric $g$ is $C^0$-close to the Euclidean metric. Applying integral by parts to the integral of $\lvert \nabla u \rvert^2$, together with the fact that $\partial E^\tau$ has small area, we can show that $E^\tau$ has almost Euclidean volume. Finally, in Section 4, we can prove our main result that up to diffeomorphism $U$, $E^\tau$ converge to $\mathbb{R}^3$ in the pointed flat metric topology.
In the case when Ricci curvature has a uniform lower bound, we always have a subsequence converge to some limit space in the pointed Gromov-Hausdorff topology, and let’s say \((M_i, g_i, p_i) \to (X, d_X, p_X)\). Since \(\partial E^{\tau_i}\) has small area, as an application of Bishop-Gromov volume comparison theorem, we can perturb the geodesic segment between points in \(E^{\tau_i}\) to avoid \(\partial E^{\tau_i}\). This can be used to show that \((E^{\tau_i}, d_i, p_i)\) equipped with the restricted metric converge to \((X, d_X, p_X)\). Also from effective estimate on outer region, we know the outer region in \(X\) is isometric to the outer region in \(\mathbb{R}^3\). Then we can fill the interior region by geodesic segments between points in the outer region. To make sure all points in the interior region of \(X\) can be obtained in this way, we need to use some structure theory on three dimensional Ricci limit space, particularly that \(X\) is a topological manifold and geodesics are non-branching. This filling process gives an isometric embedding

\[ \Phi : X \to \mathbb{R}^3. \]

Intuitively, one may think this map \(\Phi\) as the limit of harmonic maps \(U_i : E^{\tau_i} \to \mathbb{R}^3\). But since \(E^{\tau_i}\) may not be a length space and it’s not clear whether \(U_i\) is Lipschitz on \(E^{\tau_i}\), we can not directly take the limit of \(U_i\). So we need above perturbation and filling process to overcome this difficulty. All these arguments are included in Section 5.

**Notations.** We use \(C\) to denote a uniform constant which may be different from line to line; \(\Psi(\varepsilon)\) to denote a small uniform number satisfying \(\lim_{\varepsilon \to 0} \Psi(\varepsilon) = 0\), and \(\Psi(\varepsilon|a)\) satisfying \(\lim_{\varepsilon \to 0} \Psi(\varepsilon|a) = 0\) for each fixed \(a\). We denote the Euclidean metric and the induced distance by \(g_E, d_E\) respectively; and the geodesic segment and distance between \(x, y \in \mathbb{R}^3\) with respect to \(g_E\) by \([xy], |xy|\). The geodesic ball around \(p \in (M, g)\) is denoted by \(B(p, r)\), so \(B(0, r)\) is a geodesic ball in \(\mathbb{R}^3\).

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### 2. Preliminaries

#### 2.1. Asymptotically Euclidean 3-manifolds

In this section, we recall some basic facts and estimates of asymptotically Euclidean 3-manifold. Let \((M^3, g)\) be a complete AE 3-manifold. By Lemma 4.1 in [HI01], we know inside \(M\) there is a trapped compact region \(T\) whose topological boundary consists of smooth embedded minimal 2-spheres. The exterior region \(M_{ext}\) is defined as the metric completion of a connected component of \(M \setminus T\) associated to one end. Then \(M_{ext}\) is connected and AE, has a compact minimal boundary, and contains no other compact minimal surfaces (even immersed).

We have the following Riemannian Penrose inequality proved in [HI01, Bra01].

**Proposition 2.1.** Let \((M^3, g)\) be a complete AE 3-manifold with nonnegative scalar curvature, with total mass \(m(g)\). For an exterior region \(M_{ext}\) associated to one end, the total area of the boundary is bounded by

\[ |\partial M_{ext}| \leq 16\pi m(g)^2. \]
Given \((M_{ext}, g)\) associated to one end, let \(x^j, j = 1, 2, 3\) denote the given asymptotically Euclidean coordinate system in the end. Then there exist functions \(u^j \in C^\infty(M_{ext})\) satisfying

\[
\Delta_g u^j = 0 \text{ on } M_{ext}, \quad \frac{\partial u^j}{\partial v} = 0 \text{ on } \partial M_{ext}, \quad |u^j - x^j| = o(|x|^{1-\sigma}) \text{ as } |x| \to \infty,
\]

where the Neumann boundary condition can be taken over components in \(\partial M_{ext}\) which do not bound a compact region. We call \(u^j\) to be harmonic function asymptotic to an AE coordinate function \(x^j\). See [BKKS22, KKL21] for more details.

The following mass inequality was proved in [BKKS22].

**Proposition 2.2** (Theorem 1.2 in [BKKS22]). Let \((M_{ext}, g)\) be an exterior region of a complete AE Riemannian 3-manifold \((M, g)\) with mass \(m(g)\). Let \(u\) be a harmonic function on \((M_{ext}, g)\) asymptotic to one of the AE-coordinate functions of the associated end. Then

\[
m(g) \geq \frac{1}{16\pi} \int_{M_{ext}} \left( \frac{|\nabla^2 u|^2}{|\nabla u|} + R_g |\nabla u| \right) dV.
\]

By Proposition 3.1 and Lemma 3.2 in [KKL21], we have the following estimates for harmonic functions.

**Proposition 2.3.** Let \((M, g)\) be an orientable complete \((A, B, \sigma)\)-AE 3-manifold with \(R_g \geq 0\) and mass \(m(g) \leq m_0\). Assume \(u\) is a harmonic function defined on \(M_{ext}\) asymptotic to a AE coordinate function \(x^j\) and satisfies the Neumann boundary condition on \(\partial M_{ext}\). Then there exists a uniform big \(r_0 = r_0(A, B, \sigma) > 0\) such that for any \(r > r_0\) and small \(\delta > 0\), after the normalization such that the average of \(u\) over \(M_r \setminus M_{r_0}\) is zero, we have

\[
\sup_{M_{ext, r}} |u| \leq C(r),
\]

where \(C(r)\) is uniform and depends on \(A, B, \sigma, m_0\). And on \(M_{ext} \setminus M_{ext, r_0}\),

\[
|\nabla u - \partial_{x^j}|(x) \leq C(r_0)|x|^{-\sigma}.
\]

2.2. **Space with Ricci curvature bounded below.** For two pointed metric spaces \((X, x), (Y, y)\), define the pointed Gromov-Hausdorff distance by

\[
d_{GH}((X, x), (Y, y)) = \inf \{d_H(X, Y) + |xy| \},
\]

where the infimum is taken over all admissible distance on \(X \sqcup Y\).

Alternatively, we can use Gromov-Hausdorff approximation (GHA) to define the GH-distance. A map \(f : X \to Y\) is called an \(\varepsilon\)-GHA if it satisfies the following conditions:

1. \(Y \subset B_\varepsilon(f(X))\);
2. \(|d(x_1, x_2) - d(f(x_1), f(x_2))| < \varepsilon\).

Define

\[
\tilde{d}_{GH}((X, x), (Y, y)) = \inf \{ \varepsilon : \exists \varepsilon\text{-GHA } f : X \to Y, f(x) = y \}.
\]

Then \(\frac{1}{2}d_{GH} \leq \tilde{d}_{GH} \leq 2d_{GH}\) (see for example [Ron06] for a proof).

For a compact metric space \(X\), define the capacity and covering functions by

\[
\text{Cap}_X(\varepsilon) = \text{maximum number of disjoint } \frac{\varepsilon}{2}\text{-balls in } X,
\]
\[
\text{Cov}_X(\varepsilon) = \text{minimum number of } \varepsilon\text{-balls it takes to cover } X.
\]
Proposition 2.4 (Gromov’s compactness). For a class $C$ of compact metric spaces with uniformly bounded diameter, $C$ is precompact if and only if $\text{Cap}_X(\varepsilon) \leq N_1(\varepsilon)$ for all $X \in C$, which is also equivalent to $\text{Cov}_X(\varepsilon) \leq N_2(\varepsilon)$ for all $X \in C$.

See [Pet06] for a proof and more details. As corollaries, we have the following lemmas.

Lemma 2.5. If $(X_i, d_i)$ are metric spaces with diameter bounded uniformly by $D < \infty$ and $(X_i, d_i) \to (X, d)$ in the Gromov-Hausdorff topology, for any compact subset $\Omega_i \subset X_i$ with restricted metric, up to a subsequence, $(\Omega_i, d_i) \to (\Omega, d) \subset (X, d)$.

Lemma 2.6. The collection of complete pointed Riemannian $n$-manifolds with Ricci curvature uniformly bounded below by $\text{Ric} \geq -(n - 1)\Lambda$ is precompact in the pointed Gromov-Hausdorff topology.

For manifolds with Ricci curvature bounded from below, we have the Bishop-Gromov’s volume comparison (see [Pet06] for a proof).

Proposition 2.7. Let $(M, g, p)$ be a complete Riemannian $n$-manifold with $\text{Ric} \geq -(n - 1)\Lambda g$. Then for any $0 < r_1 < r_2$, we have

$$\frac{\text{Vol}(B(p, r_1))}{|B_{-\Lambda}(r_1)|} \geq \frac{\text{Vol}(B(p, r_2))}{|B_{-\Lambda}(r_2)|},$$

where $|B_{-\Lambda}(r)|$ is the volume of a geodesic ball with radius $r$ in the space form $S^n_{-\Lambda}$ with constant curvature $-\Lambda$. Moreover, for $v \in T_p M$ a unit tangent vector and $\gamma_v(0), t \in [0, l]$ a minimal geodesic segment with $\gamma_v(0) = p, \gamma_v(0) = v$, for any $0 < t_1 < t_2 < l$, we have

$$\frac{A(t_1 v)}{A(t_1)} \geq \frac{A(t_2 v)}{A(t_2)},$$

where $A(tv)$ is the area element at $\gamma_v(t)$ and $A(t)$ is the corresponding area element in $S^n_{-\Lambda}$.

As an application, we have the following perturbation proposition. Our argument follows from Lemma 3.7 in [Hei10]. We give full details for reader’s convenience.

Lemma 2.8. Assume $(M^n, g)$ is a complete Riemannian manifold with Ricci curvature bounded from below by $\text{Ric} \geq -(n - 1)\Lambda$, and $\Sigma^{-1} \subset M$ is a smooth hypersurface with the area $|\Sigma^{2D}| \leq \varepsilon$, where $\Sigma^D := \Sigma \cap B(p, D)$. Assume $\delta_1, \delta_0 > 0$, and $\text{Vol}(B(p, 1)) \geq \varepsilon_0$.

Then there exists $\delta_0 = \Psi(\varepsilon, n, \delta_0, \Lambda, D, \delta_1)$ such that for any $x, y \in B(p, D)$ and $d(x, \Sigma) > 2\delta_1$, there exists $z \in B(y, \delta_0)$ and a geodesic segment $\gamma_{xz}$ connecting $x, z$ such that $\gamma_{xz} \subset M \setminus \Sigma$.

Proof. Otherwise, for any $z \in B(y, \delta_0)$ and any segment $\gamma_{xz}$, there is a $y \in \Sigma \setminus \gamma_{xz}$. Note $\gamma_{xz} \subset B(p, 2D)$, so $y \in \Sigma^{2D}$.

Denote $B = B(y, \delta_0)$ and $B' \subset B$ the complement of cut locus of $x$, so $B'$ has the same volume as $B$. Let $\Sigma^1$ be the set of all $y \in \Sigma$ which occur as the first intersection with $\Sigma$ of $\gamma_{xz}$ for $z \in B'$. Define $d_1, d_2 : \Sigma^1 \to \mathbb{R}^+$ by $d_1(y) = d(x, y)$ and $d_2(y) = \sup\{t > 0 : \gamma_{xy}(d_1(y) + t) \in B'\}$, $U := \{(y, t) \in \Sigma^1 \times \mathbb{R} : d_1(y) < t \leq d_1(y) + d_2(y)\}$, and $\Phi : U \to M$ by $\Phi(y, t) = \gamma_{xy}(t)$. Then $\delta_1 \leq d_1(y) \leq d_1(y) + d_2(y) \leq 2D$ and

$$\Phi^* (d\text{Vol}_g) |_{(y, t)} = \frac{A(tv)}{A(d_1(y)v_y)} \cos \alpha_y d\text{Vol}_\Sigma \wedge dt,$$
where $v_y = \dot{\gamma}_{xy}(0)$, $\alpha_y$ is the angle between $\dot{\gamma}_{xy}(d_1(y))$ and exterior normal to $\Sigma$ at $y$. Then by Bishop-Gromov’s comparison, integrating over $U$ gives
\[
\text{Vol}(B') \leq \int_\Sigma \int_0^{d_1(y)+d_2(y)} \frac{A(tv_y)}{A(d_1(y)v_y)} dt dA(y) 
\leq \int_\Sigma \int_0^{d_1(y)+d_2(y)} \frac{A(t)}{A(d_1(y))} dt dA(y) 
\leq C(D, \delta_1, \Lambda, n)|\Sigma|^{2D}.
\]

By Bishop-Gromov’s volume comparison again,
\[
\text{Vol}(B, \delta_0) \geq \frac{\text{Vol}_B(A(\delta_0))}{\text{Vol}_B(A(3D))} \cdot \text{Vol}(p, D) \geq \frac{\text{Vol}_B(A(\delta_0))}{\text{Vol}_B(A(3D))} \cdot v_0.
\]

Choosing $\delta_0$ such that $\frac{\text{Vol}_B(A(\delta_0))}{\text{Vol}_B(A(3D))} \cdot v_0 = 2C(D, \delta_1, \Lambda, n)v$ gives a contradiction. \hfill \Box

Remark 2.1. In this proof, we actually proved that if for a set $B \subset B(p, D)$ and for all points $y \in B$, there is a geodesic segment between $x, y$ intersecting $\Sigma$, then $\text{Vol}(B) \leq C(D, \delta_1, \Lambda, n)|\Sigma|$. 

Remark 2.2. If we denote the subset in $B(y, \delta_0)$ by $B$ consisting of points which could be connected to $x$ by segments in $M \setminus \Sigma$, then by the same argument, we can prove that $\text{Vol}(B) \geq (1 - \sqrt{\varepsilon})\text{Vol}(B(y, \delta_0))$. So for any uniform positive number $N$ and points $x_1, \cdots, x_N$ with $d(x_j, \Sigma) > 2\delta_1$, if $\varepsilon$ is small enough depending on $N$, there exists $z \in B(y, \delta_0)$ such that each geodesic segment connecting $z, x_j$ lies in $M \setminus \Sigma$.

We also need some structure theory on Ricci limit space. The first is the non-branching property. The following proposition is a special case of Theorem 1.3 in [Den20].

**Proposition 2.9.** If $(M_i, g_i, p_i)$ is a sequence of Riemannian manifolds with Ricci curvature bounded uniformly from below, and $(M_i, g_i, p_i)$ converge to $(X, d, p)$ in the pointed Gromov-Hausdorff topology, then $(X, d)$ is non-branching. In other words, if there are two geodesics $\gamma_1$ and $\gamma_2$ in $X$, and for some $t > 0$, $\gamma_1(s) = \gamma_2(s)$ for $s \in [0, t]$, then $\gamma_1 = \gamma_2$.

In dimension three, we know Ricci limit space is a manifold. The following proposition is Corollary 1.5 in [ST21].

**Proposition 2.10.** If $(M_i^3, g_i, p_i)$ is a sequence of Riemannian 3-manifolds with Ricci curvature bounded uniformly from below and noncollapsing in the sense that $\text{Vol}(B(p_i, 1)) \geq v_0 > 0$, and $(M_i, g_i, p_i)$ converge to $(X, d, p)$ in the pointed Gromov-Hausdorff topology, then there exists a topological 3-manifold $M$ such that $M = X$ and the topology generated by $d$ is the same as $M$. Moreover, the charts of $M$ can be taken to be bi-Hölder with respect to $d$.

### 3. Effective estimates on outer region

In this section, we always assume $(M, g, p)$ is an orientable complete pointed $(A, B, \sigma)$-AE 3-manifold with nonnegative integrable scalar curvature $R_g \geq 0$, and the mass $m(g) < 1$. Let $\Phi : M \setminus K \to \mathbb{R}^3 \setminus B(0, A)$ be the corresponding $C^\infty$-diffeomorphism. For this end defined by $\Phi$, let $M_{\text{ext}}$ be the associated exterior
region, and \( \{u^j\}, j = 1, 2, 3 \) be harmonic functions asymptotic to AE-coordinate functions \( x^j \circ \Phi \), which are defined over \( M_{ext} \) with Neumann boundary condition on \( \partial M_{ext} \) (see Section 2.1 for more details on rigorous definitions and properties).

Recall that \( M_r \) is defined to be the interior component of \( M \setminus \Phi^{-1}(S_r) \), where \( S_r \subset \mathbb{R}^3 \) is the standard sphere with radius \( r \).

**Lemma 3.1.** For a sufficiently large uniform \( r_0 > 0 \), and any \( x \in M \setminus M_{r_0} \),
\[
|\nabla^2 u^j|(x) \leq \Phi m(g)^{2/3} = \Psi(m(g)).
\]

**Proof.** Fix a \( j \in \{1, 2, 3\} \) and denote \( u^j \) by \( u \) for simplicity. For large \( r_0 \) and \( x \in M \setminus M_{r_0} \), we know
\[
B(x, 2) \subset \Phi^{-1}(\mathbb{R}^3 \setminus B(0, A)),
\]
where the metric \( g \) is \( C^2 \)-close to the Euclidean metric, i.e. \( \forall y \in B(x, 2) \), and for all multi-index \( |l| = 0, 1, 2 \),
\[
|\partial^l(g_{jk} - \delta_{jk})|(y) \leq B|y|^{-|l|} \leq 2B^{1-\sigma} r_0^{-\sigma} \ll 1.
\]

By Proposition 2.3, we have estimate
\[
\sup_{B(x, 1)} |\partial u| \leq C.
\]
From the harmonic equation, \( (|\partial^2 u| + |\partial^3 u|)(y) \leq C \) over \( B(x, 1) \), and thus
\[
\sum_{k=1}^{3} |\nabla^k u|(y) \leq C, \quad \forall y \in B(x, 1).
\]

From the Bochner formula, we have
\[
\frac{1}{2} \Delta |\nabla^2 u|^2 = |\nabla^3 u|^2 + \nabla_j \nabla_i u \cdot (\nabla_k (R(\partial_x^j, \partial_x^i) (\nabla_i u)) + \nabla_j (R(\partial_x^k, \partial_x^i) (\nabla_k u)) + R_{ikij} \cdot \nabla_j \nabla_i u \cdot \nabla_k \nabla_l u.
\]
Now we can apply the Moser iteration to control \( |\nabla^2 u| \) by its \( L^2 \)-integral as following. For any cut-off function \( \varphi \) with support in \( B(x, 1) \), integral by parts gives
\[
-\frac{1}{2} \int \nabla \varphi \cdot \nabla |\nabla^2 u|^2 \geq - \int (\varphi |\nabla^2 u|^2 + \nabla \varphi \cdot |\nabla^2 u| |\nabla u| + \varphi |\nabla^2 u| |\nabla u|^2) ,
\]
where we have used the fact that \( \|Rm\| \ll 1 \). Take a cutoff function \( \eta \) with support in \( B(x, 1) \) and a constant \( \beta \geq 2 \) to be determined below. Choosing \( \varphi = \eta^2 |\nabla^2 u|^{2(\beta-1)} \), we have
\[
(\beta - 1) \int \eta^2 |\nabla^2 u|^{2(\beta-2)} |\nabla |\nabla^2 u|^2|^2 + \int 2\eta |\nabla^2 u|^{2(\beta-1)} \nabla \eta \cdot |\nabla^2 u|^2 + \\
\leq \int \eta^2 |\nabla^2 u|^{2(\beta-1)} |\nabla u|^2 + \int \eta |\nabla \eta||\nabla^2 u|^{2(\beta-1)} \cdot |\nabla^2 u||\nabla u| + \\
+ \int (\beta - 1) \eta^2 |\nabla^2 u|^{2(\beta-2)} |\nabla |\nabla^2 u|^2|^2 + |\nabla^2 u| |\nabla u| + \int \eta^2 |\nabla^2 u|^{2\beta}.
\]
By Hölder inequality and \( |\nabla u| \leq C \), we have
\[
(\beta - 1) \int \eta^2 |\nabla^2 u|^{2(\beta-2)} |\nabla |\nabla^2 u|^2|^2 \leq \frac{C}{\beta - 1} \int |\nabla \eta||\nabla^2 u|^{2\beta} + \int \eta^2 |\nabla^2 u|^{2\beta} + \\
+ C \int (\beta \eta^2 |\nabla^2 u|^{2(\beta-1)} + |\nabla \eta||\nabla^2 u|^{2(\beta-1)}).
Set $v = |\nabla^2 u|^2$. Then
\[
\int (\nabla (\eta v^\beta))^2 \leq C \int |\nabla \eta|^2 v^\beta + C\beta \int (\eta v^\beta)^2 v^\beta + C\beta^2 \int \eta^2 v^\beta_{-1} + C\beta \int |\nabla \eta| v^\beta_{-\frac{1}{2}}
\]
\[
\leq C\beta^2 \int (\eta^2 + \eta |\nabla \eta| + |\nabla \eta|^2) v^\beta_{-1},
\]
where we have used $v \leq C$. By Sobolev inequality,
\[
\|\eta v^\beta\|_{L^2} \leq C \cdot \|\eta v^\beta\|_{W^{1,2}} \leq C\beta^2 \int (\eta^2 + \eta |\nabla \eta| + |\nabla \eta|^2) v^\beta_{-1}.
\]

Let $\eta \geq 0$ be a cutoff function satisfying $\eta(s) = 0$, $s \leq 0$, $\eta(s) = 1$, $s \geq 1$, and $|\eta'| \leq 2$. Set $r_k = \frac{1}{3} + \frac{1}{2^k}$, $\Omega_k = B(x, r_k)$, and
\[
\eta_k(y) = \eta \left( \frac{d(x, y) - r_k}{r_{k-1} - r_k} \right).
\]
Then
\[
\|v\|_{L^{3^k}(\Omega_k)} \leq \|\eta_k v^\beta\|_{L^2(\Omega_{k-1})} \leq \left( C\beta^2 4^k \right)^{\frac{1}{2}} \left( \int_{\Omega_{k-1}} v^\beta_{-1} \right)^{\frac{1}{2}}
\]
\[
\leq (C |B(x, 1)|^{\frac{1}{2}} \beta 4^k)^{\frac{1}{2}} \|v\|_{L^3(\Omega_k)}
\]
\[
\leq (C\beta^2 4^k)^{\frac{1}{2}} \|v\|_{L^3(\Omega_{k-1})}.
\]

Taking $\beta = 2 \cdot 3^{k-1}$ gives
\[
\|v\|_{L^{3^k}(\Omega_k)} \leq C \frac{2}{3^k} \|v\|_{L^{3^{k-1}}(\Omega_{k-1})}
\]
\[
\leq C \frac{4}{3^k} \frac{4}{3^k} \left( 1 - \frac{1}{2^k} \right) \|v\|_{L^3(\Omega_{k-1})}
\]
\[
\leq C \sum_{j=1}^k \frac{4}{3^j} \|v\|_{L^3(B(x, 1))}.
\]

Notice that
\[
\sum_{j=1}^\infty \frac{2}{3^j} \leq \sum_{j=1}^\infty \frac{2}{3^j} < 1,
\]
\[
\prod_{j=0}^{\infty} \left( 1 - \frac{1}{2 \cdot 3^j} \right) > \frac{1}{2} \cdot \frac{5}{6} \prod_{j=2}^{\infty} \left( 1 - \frac{1}{2 \cdot 3^j} \right) > \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{1}{2},
\]
and
\[
\int_{B(x, 1)} |\nabla^2 u|^4 \leq C \int_{B(x, 1)} |\nabla^2 u|
\]
\[
\leq C \left( \int_{B(x, 1)} ( \frac{|\nabla^2 u|^2}{|\nabla u|} ) \right)^{\frac{1}{2}} \left( \int_{B(x, 1)} |\nabla u| \right)^{\frac{1}{2}}
\]
\[
\leq C \cdot m(g)^{\frac{1}{2}}.
\]

So
\[
\|v\|_{L^\infty(B(x, 1))} \leq C \|v\|_{L^3(B(x, 1))} = C m(g)^{\frac{1}{2}} = \Psi(m(g)).
\]
\[\square\]
Lemma 3.2. For a sufficiently large uniform \( r_0 > 0 \), and any \( x \in M \setminus M_{r_0} \),
\[
| \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} |(x) \leq Cm(g)\frac{1}{r_0} = \Psi(m(g)).
\]

Proof. By Proposition 2.3, we know on \( M \setminus M_{r_0} \),
\[
|\nabla u - \partial x^j|(x) \leq C|x|^{-\sigma},
\]
so
\[
| \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} | \leq | \langle \nabla u^j - \partial x^j, \nabla u^k \rangle + \langle \partial x^j, \nabla u^k - \partial x^k \rangle + \langle \partial x^j, \partial x^k \rangle - \delta_{jk} |
\leq C|x|^{-\sigma}.
\]

Choose \( L > 0 \) big enough, depending on \( m(g) \), to be determined below. If \( x \in M \setminus M_L \), then
\[
| \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} |(x) \leq CL^{-\sigma}.
\]
If \( x \in M_L \setminus M_{r_0} \), choose \( y \in \partial M_L \) such that \( |xy| = L - r_0 \) and \( |xy| \subset M_L \setminus M_{r_0} \). Then
\[
| \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} |^2(x) \leq |(\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk})^2(x) - | \langle \nabla u^j, \nabla u^k \rangle - \delta_{jk} |^2(y)|
+ (\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk})^2(y)
\leq L \cdot Cm(g)\frac{1}{r_0} + CL^{-2\sigma}.
\]

Taking \( L = m(g)\frac{1}{r_0} \), then \( \frac{1}{L} = \Psi(m(g)) \) and \( Lm(g)\frac{1}{r_0} = m(g)\frac{1}{r_0} = \Psi(m(g)) \) finishes the proof. \( \square \)

Set
\[
\mathcal{U} := (u^1, u^2, u^3) : M_{ext} \to \mathbb{R}^3.
\]
For \( C(r_0) \) in Proposition 2.3, we can choose \( r_1 \) such that \( C(r_0)r_1^{-\sigma} < \frac{1}{2} \). For \( j \in \{1, 2, 3\} \), we know \( \sup_{M_{r_1}} |u^j| \leq C(r_1) \). So if we choose \( r_2 = 2C(r_1) \), then
\[
\mathcal{U}^{-1}(\mathbb{R}^3 \setminus B(0, r_2)) \subset M \setminus M_{r_1}.
\]

We prove that around outer region, \( \mathcal{U} \) is one to one.

Lemma 3.3. For all \( L \gg r_2 \) big enough, and any \( w \in B(0, L) \setminus B(0, L - 1) \),
\( \mathcal{U}^{-1}\{w\} \) consists of at most one point.

Proof. Assume that \( \mathcal{U}(y) = \mathcal{U}(z) = w \). Then \( y, z \in M \setminus M_{r_1} \). Under the diffeomorphism \( \Phi : M \setminus M_{r_1} \to \mathbb{R}^3 \), we can assume \( y, z \in \mathbb{R}^3 \setminus B(0, r_1) \), where the metric \( g \) is \( C^2 \)-close to the Euclidean metric. Then \( \mathcal{U} : (\mathbb{R}^3 \setminus B(0, r_1), g) \to \mathbb{R}^3 \). Consider the straight line segment \([yz]\) between \( y, z \) and assume \( y', z' \in \partial B(0, r_1) \) are points on \([yz]\cap \partial B(0, r_1) \) closest to \( y, z \) respectively. Denote the unit direction vector of \([yz]\) by \( a \in S^2 \subset \mathbb{R}^3 \). Set \( u^a = \sum_{j=1}^{3} a_j u^j \), and \( l = |yz| \). Then there are two cases.

Case 1): if no such points \( y', z' \) exist, then \( |yz| \subset \mathbb{R}^3 \setminus B(0, r_1) \). So
\[
u^a(z) - u^a(y) = \int_0^l (u^a(y + at))' dt
\]
\[
= \int_0^l \sum_{j,k=1}^{3} \langle a_j \nabla u^j, a_k \partial y^k \rangle
\geq (1 - C(r_0)r_1^{-\sigma})l
\geq \frac{1}{2} l,
\]
which implies that $l = 0$.

Case 2): if such $y’, z’$ exist, then by choosing $L \gg r_2$, we can assume $l \gg r_1$, since otherwise no such $y’, z’$ exist. We can construct a piecewise straight line segment $\gamma’$ between $y’, z’$ satisfying $\gamma’ \subset \mathbb{R}^3 \setminus B(0, r_1)$ and $|\gamma’| \leq 10r_1$. Consider the new piecewise straight line segment $\gamma := yy’ \cup \gamma’ \cup [z’z]$, then $\gamma \subset \mathbb{R}^3 \setminus B(0, r_1)$. Similarly, we can construct a piecewise straight line segment $\gamma$ between $y’, z’$ satisfying $\gamma \subset \mathbb{R}^3 \setminus B(0, r_1)$ and $|\gamma| \leq 10r_1$.

\[ u^a(z) - u^a(z’) \geq \frac{1}{2}||zz’||, \quad u^a(y’) - u^a(y) \geq \frac{1}{2}||yy’||, \]

and
\[ |u^a(z’) - u^a(y’)| \leq C(r_1). \]

So
\[ u^a(z) - u^a(y) \geq \frac{1}{2}(|zz’| + |yy’|) - C(r_1) \geq \frac{1}{2}l - C(r_1) > 0, \]

a contradiction. \hfill \Box

**Lemma 3.4.** For any $L_0 > 0$, there exists $L_1$ big enough such that for all $L > L_1$, $U^{-1}(\mathbb{R}^3 \setminus B(0, L)) \subset M \setminus M_{L_0}$, and $U(M \setminus M_L) \subset \mathbb{R}^3 \setminus B(0, L_0)$.

**Proof.** By Proposition 2.3, we know $\sup_{M_{L_0}} |u| \leq C(L_0)$, so for any $L > C(L_0)$, $U^{-1}(\mathbb{R}^3 \setminus B(0, L)) \subset M \setminus M_{L_0}$. For any $y \in M \setminus M_L$, choose a straight line segment $[xy]$ with $x \in \partial M_{L_0}$ being the closest point to $y$. Denote the unit direction vector of $xy$ by $a$, then similar to above calculation, we know
\[ u^a(y) \geq u^a(x) + \frac{1}{2}|xy| \geq \frac{1}{2}(L - L_0) - C(L_0), \]

which is bigger than $L_0$ if $L$ is chosen big enough. \hfill \Box

In the following, for simplicity, we will use $L_0$ to denote a uniform positive number such that conclusions in Lemma 3.3 and Lemma 3.4 hold.

## 4. Regular Subregion and the Flat Convergence

In this section, we define the regular subregion, where the geometry is locally uniformly controlled. Then we will use this regular subregion to prove our convergence theorem.

For any small $\tau > 0$, we can find $\tau_1 < \tau_2 < \tau$ such that
\[ \{x \in M_{ext} : |\nabla u^j|^2 = 1 + \tau_1^2 \} \]

is a smooth surface.

Define
\[ E^T_1 := M_{ext} \cap \cap_{j=1}^3 \{ |\nabla u^j|^2 \leq 1 + \tau_1^2 \}. \]
By the co-area formula, we have
\[
\int_0^\infty \mathcal{H}_g^2(M_{r_0} \cap E_t^1 \cap \{ \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 = t \}) dt
\]
\[
= \int_{M_{r_0} \cap E_t^1} \left| \nabla \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 \right|
\]
\[
\leq 2((1 + \tau)^2 + 1) \int_{M_{r_0} \cap E_t^1} \sum_{j,k=1}^3 (|\nabla^2 u^j||\nabla u^k| + |\nabla^2 u^k||\nabla u^j|)
\]
\[
\leq C \sum_{j,k=1}^3 \left( \int_{M_{r_0} \cap E_t^1} \frac{|\nabla^2 u^j|^2}{|\nabla u^j|} \right)^\frac{1}{2} \left( \int_{M_{r_0} \cap E_t^1} |\nabla u^j|^3 \right)^\frac{1}{3} \left( \int_{M_{r_0} \cap E_t^1} |\nabla u^k|^3 \right)^\frac{1}{3}
\]
\[
\leq C(r_0)m(g)^\frac{1}{2},
\]
where in the last inequality, we used the mass inequality Proposition 2.2 and the fact that
\[
\int_{M_{r_0} \cap E_t^1} |\nabla u^j|^3 \leq (1 + \tau) \int_{M_{r_0}} |\nabla u|^2 \leq C(r_0),
\]
where the last inequality can be proved by integration by parts (see also the proof of Lemma 4.3).

So for \( \tau_1 := \min\{\tau_1^*\} \), there exists \( \frac{\tau_2}{2} < \tau_2 < \tau_1 \) such that
\[
\{ x \in \text{M}_{\text{ext}} : \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 = \tau_2 \}
\]
is a smooth surface and
\[
\mathcal{H}_g^2(M_{r_0} \cap E_t^1 \cap \{ \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 = \tau_2 \}) \leq \frac{C(r_0)m(g)^\frac{1}{2}}{\tau}.
\]
Define
\[
E_2^\tau := \text{M}_{\text{ext}} \cap \{ \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 \leq \tau_2 \}.
\]
Then \( E_2^\tau \subset E_1^\tau \), and \( \partial\text{M}_{\text{ext}} \cap E_2^\tau = \emptyset \). This is because on \( \partial\text{M}_{\text{ext}} \), \( \langle \nabla u^j, n \rangle = 0 \) for the normal vector \( n \) of \( \partial\text{M}_{\text{ext}} \) and all \( j \in \{1, 2, 3\} \), but \( \{\nabla u\}^3_{j=1} \) is an almost orthonormal basis at any point in \( E_2^\tau \). So
\[
\partial E_2^\tau = \{ \sum_{j,k=1}^3 |\langle \nabla u^j, \nabla u^k \rangle - \delta_{jk}|^2 = \tau_2 \},
\]
which is a smooth surface.

**Proposition 4.1.** For any \( 0 < \varepsilon \ll 1 \), there exists \( \tau = \Psi(m(g)) \) such that
\[
M \setminus M_{r_0} \subset E_2^\tau,
\]
and
\[
\mathcal{H}_g^2(\partial E_2^\tau) \leq Cm(g)^{\frac{1}{2} - \varepsilon} = \Psi(m(g)).
\]
For any fixed small \( \varepsilon > 0 \), we define the regular subregion \( E^\tau \) to be the connected component of such \( E_2^\tau \) containing \( M \setminus M_{r_0} \).
Proof. We can take small $\tau = m(g)^2$ with $0 < \varepsilon < \frac{1}{3m^2}$, which is very big compared with $\Psi(m(g))$ in Lemma 3.2, satisfies $\tau \to 0$ as $m(g) \to 0$. Then
\[ M \setminus M_{r_0} \subset E^r_2, \quad \partial E^r_2 \subset M_{r_0}. \]

By above arguments,
\[ \mathcal{H}^2_\gamma(\partial E^r_2) = \mathcal{H}^2_\gamma(M_{r_0} \cap \partial E^r_2) \leq C(r_0)m(g)^{1 - \varepsilon} = \Psi(m(g)). \]

Now we consider the harmonic map $U$ restricted to $E^r$,
\[ U = (u^1, u^2, u^3) : E^r \to Y^r \subset \mathbb{R}^3, \]
where $Y^r := U(E^r)$. Since $E^r \cap B(p, 3r_0) \neq \emptyset$, we can perturb the base point and without loss of generality we assume that $p \in E^r$, and $U(p) = p^r \in \mathbb{R}^3 \setminus B(0, L_0)$.

By the definition of $E^r$, $U$ is nondegenerate, thus a local diffeomorphism. We will show that $U$ is injective and almost onto in the sense of volume.

Lemma 4.2. $U$ is a diffeomorphism from $E^r$ onto its image $Y^r$, and $\mathbb{R}^3 \setminus B(0, L_0) \subset Y^r$.

Proof. To prove $U$ is a diffeomorphism, it’s enough to show that $U$ is injective. Assume there exist $y, z \in E^r$ such that $\tilde{U}(y) = \tilde{U}(z) = w$. By Lemma 3.4, there exists $v = U(x) \in B(0, 2L_0) \setminus B(0, L_0)$. By connectedness we can take a curve $\gamma \subset Y^r$ connecting $w, v$. Since $U$ is a local diffeomorphism, we can lift $\gamma$ to two curves $\tilde{\gamma}_y$ and $\tilde{\gamma}_z$ with $\tilde{\gamma}_y(0) = y, \tilde{\gamma}_z(0) = z$ respectively. By Lemma 3.3, $U^{-1}\{v\} = \{x\}$. So $\tilde{\gamma}_y(1) = \tilde{\gamma}_z(1) = x$. This implies that $\tilde{\gamma}_y = \tilde{\gamma}_z$, thus $y = z$.

From Lemma 3.4, we know $U$ maps the infinity of $E^r$ to infinity of $\mathbb{R}^3$. Also $U(\partial E^r) \subset B(0, L_0)$. So from the degree theory, we know $\mathbb{R}^3 \setminus B(0, L_0) \subset Y^r$. \( \square \)

For computation simplicity, we will usually take integration over coordinate cylinder. For any fixed unit vector $(a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3$, let $u := \sum_{j=1}^3 a_j u^j$. Define the coordinate cylinder $C_L := D^+_L \cup T_L$, where
\[ D^+_L := \{ x \in M_{ext} : u(x) = \pm L, |U(x)|^2 - u(x)^2 \leq L^2 \}, \]
\[ T_L := \{ x \in M_{ext} : |u(x)| \leq L, |U(x)|^2 - u(x)^2 = L^2 \}. \]

Set $\Omega_L \subset M_{ext}$ be the closure of the bounded component of $M_{ext} \setminus C_L$. In the following, we will also use $\Omega^L_2$ to emphasize that $\Omega_L$ is associated to the harmonic function along $a = (a_1, a_2, a_3)$ direction.

Lemma 4.3. For a fixed $j \in \{1, 2, 3\}$ and $u = u^j$,
\[ \text{Vol}_g(\Omega_L \cap E^r) \geq (1 - 4\tau)(1 - \Psi(m(g)))2\pi L^3 - \Psi(m(g)). \]

Proof. Since $\Delta u = 0$, integration by parts gives
\[ \int_{\Omega_L} |\nabla u|^2 = \int_{T_L \cup D^+_L} u \langle \nabla u, n \rangle + \int_{\partial M_{ext}} u \langle \nabla u, n \rangle \]
\[ = (1 + \Psi(m(g)))2\pi L^3, \]
where we used Lemma 3.2 in the last equality.
Note that $M_{\text{ext}} \setminus E^\tau$ is a bounded domain inside $M_0$ with boundary $\partial M_{\text{ext}} \cup \partial E^\tau$. Integration by parts gives

$$
\int_{M_{\text{ext}} \setminus E^\tau} |\nabla u|^2 = \int_{\partial E^\tau} u \langle \nabla u, n \rangle + \int_{\partial M_{\text{ext}}} u \langle \nabla u, n \rangle
\leq C(r_0)(1 + \tau) \mathcal{H}_2^2(\partial E^\tau)
\leq \Psi(m(g)).
$$

So

$$
(1 + 2\tau)\text{Vol}(\Omega_L \cap E^\tau) \geq \int_{\Omega_L \cap E^\tau} |\nabla u|^2 \geq (1 + \Psi(m(g))) 2\pi L^3 - \Psi(m(g)).
$$

\[\Box\]

**Lemma 4.4.** For any $D > L_0$,

$$|B(p^*, D) \setminus Y^\tau| \leq \Psi(m(g)|D).$$

**Proof.** Otherwise, assume $|B(p^*, D) \setminus Y^\tau| > c > 0$. Choose $L > 2D$, then we have

$$|\Omega_L \cap Y^\tau| \leq |\Omega_L \setminus B(p^*, D)| + |B(p^*, D) \cap Y^\tau|
\leq |\Omega_L \setminus B(p^*, D)| + |B(p^*, D)| - c
\leq |\Omega_L| - c
= 2\pi L^3 - c.
$$

On the other hand, from above lemma, we know

$$|\Omega_L \cap Y^\tau| \geq (1 - 4\tau)\text{Vol}_g(\Omega_L \cap Y^\tau) \geq (1 - 4\tau)^2(1 - \Psi(m(g))) 2\pi L^3 - \Psi(m(g)).$$

Letting $m(g) \to 0$ gives a contradiction. \[\Box\]

**Remark 4.1.** Alternatively, one can prove this lemma by using the remark below Lemma 2.8 and the fact that $\mathcal{H}_2^2(\partial Y^\tau) \leq \Psi(m(g))$. Such argument will also be used later.

Now we prove the convergence theorem in the pointed flat metric topology.

**Proof of Theorem 1.5.** Assume that $(M_i, g_i, p_i)$ is a sequence of orientable complete $(A, B, \sigma)$-AE 3-manifolds with nonnegative integrable scalar curvature $R_{g_i} \geq 0$, and mass $m(g_i) \to 0$.

By Proposition 4.1, there exists $\tau_i \to 0$ such that the regular subregion $E^\tau_i$ is well defined. And we have diffeomorphisms $U_i : E_{\tau_i} \to Y_{\tau_i} \subset \mathbb{R}^3$, under which $g_i^{jk}(y) = \langle \nabla u_i^j, \nabla u_i^k \rangle$ and for any $y \in Y_{\tau_i}$,

$$|g_i^{jk} - \delta_{jk}|(y) \leq \tau_i.
$$

So for any $\delta > 0$, we know $(E_{\tau_i} \setminus B_\delta(\partial E_{\tau_i}), g_i, p_i)$ converges to the Euclidean metric locally in $C^0$-topology.

The associated current of $Y_{\tau_i}$ is

$$T_i(f) := \int_{Y_{\tau_i}} f(y)d\text{Vol}_{g_i}(y) = \int_{\mathbb{R}^3} f(y)\chi_{Y_{\tau_i}}(y)\sqrt{\det g_i(y)}dy, \forall f \in C_c^\infty(\mathbb{R}^3).$$
We claim that $T_i \to \mathbb{R}^3$ in the pointed flat metric topology. By definition, we need to show that for any $D \gg 1$,

$$\int_{B(p^*, D)} |\chi_{Y_i}(y)\sqrt{\det g_i(y)} - 1|dy \to 0.$$ 

By Lemma 4.4, $|B(p^*, D) \setminus Y_{\tau_i}| \leq \Psi(m(g_i)|D) \to 0$. So

$$\int_{B(p^*, D)} |\chi_{Y_i}(y)\sqrt{\det g_i(y)} - 1|dy = \int_{B(p^*, D) \cap Y_{\tau_i}} |\sqrt{\det g_i(y)} - 1| + |B(p^*, D) \setminus Y_{\tau_i}| \leq \Psi(m(g_i)|D) \to 0,$$

as $i \to \infty$.

So we proved that for $\hat{Z}_i := M_i \setminus E_{\tau_i}$, $(M_i \setminus \hat{Z}_i, g_i, p_i) \to \mathbb{R}^3$ in the pointed flat metric topology, $|\partial \hat{Z}_i| \leq C \cdot m(g_i)^{3/2}$, and $(M_i \setminus B_i(\hat{Z}_i), g_i, p_i)$ converge to a subset in $\mathbb{R}^3$ locally in $C^0$-topology.

To get better control on the area of boundary, we can choose a fixed small $\tau_0 \ll 1$ and define $E_i^{\tau_0}$ to be the connected component of

$$\{x \in M_i, \text{ext} : \sum_{j,k=1}^3 |\langle \nabla u_i^j, u_i^k \rangle - \delta_{jk}|^2 \leq \tau_0\}$$

containing $E_{\tau_i}$. The only difference between $E_i^{\tau_0}$ and $E_{\tau_i}$ is that we take $\tau_0$ to be a fixed small number such that over $E_i^{\tau_0}$, $U_i$ is also nondegenerate. So previous arguments also work for $E_i^{\tau_0}$ and in particular, $|\partial E_i^{\tau_0}| \leq C(\tau_0, \tau_0) m(g_i)^{3/2}$, and $U_i$ is a diffeomorphism onto its image when restricted on $E_i^{\tau_0}$. The only missing property is that we may not have locally $C^0$-convergence of $g_i$ away from the boundary $\partial E_i^{\tau_0}$.

If we take $\hat{Y}_i = U_i(E_i^{\tau_0})$, then since on $\hat{Y}_i$, $|\sqrt{\det g_i} - 1| \leq 4\tau_0$, we know that for any $D \gg 1$,

$$\int_{B(p^*, D)} |\chi_{\hat{Y}_i}(y)\sqrt{\det g_i(y)} - 1|dy = \int_{B(p^*, D) \cap Y_{\tau_i}} |\sqrt{\det g_i(y)} - 1|$$

$$+ \int_{B(p^*, D) \cap \partial Y_{\tau_i}} |\sqrt{\det g_i(y)} - 1| + |B(p^*, D) \setminus \hat{Y}_i|$$

$$\leq \Psi(m(g_i)|D) + 4\tau_0 |B(p^*, D) \setminus Y_{\tau_i}|$$

$$\leq \Psi(m(g_i)|D) \to 0$$

as $i \to \infty$.

Define $Z_i := M_i \setminus E_i^{\tau_0}$. So $(M_i \setminus Z_i, g_i, p_i) \to \mathbb{R}^3$ in the pointed flat metric topology, and $|\partial Z_i| \leq C \cdot m(g_i)^{3/2}$. \hfill \Box

5. Removable singularity under Ricci lower bound

In this section, we prove the Gromov-Hausdorff convergence when Ricci curvature is uniformly bounded below. Let’s assume $(M_i, g_i, p_i)$ is a sequence of orientable complete $(A, B, \sigma)$-$\text{AE}$ 3-manifolds with nonnegative integrable scalar curvature $R_{g_i} \geq 0$, Ricci curvature bounded from below by $\text{Ric}_{g_i} \geq -2\Lambda$ and the mass $m(g_i) \to 0$.

By Gromov’s compactness theorem, up to a subsequence,

$$(M_i, g_i, p_i) \to (X, d_X, p_X)$$
in the pointed Gromov-Hausdorff topology, where \((X,d_X)\) is a complete length space.

From Proposition 4.1, there exist regular subregions \(E^\tau \subset M_{i,ext}\), and without loss of generality, assume \(p_i \in E^\tau\). Consider the restricted metric space \((E^\tau, d_i, p_i)\), where \(d_i\) is the restriction of the distance induced by \(g_i\). Up to a subsequence, assume
\[
(E^\tau, d_i, p_i) \to (X', d_X, p_X) \subset (X, d_X).
\]
Similarly, consider the restricted metric and assume \((\partial E^\tau, d_i) \to (\Sigma, d_X) \subset (X', d_X)\).

By Proposition 4.1, \(X' \setminus \Sigma \neq \emptyset\).

In general, \((X', d_X)\) is not a length space and is different from \((X, d_X)\). In our case, by using the fact that \(|\partial E^\tau| \to 0\) and the condition on lower bound of Ricci curvature, we can prove the following.

**Proposition 5.1.** \(X' = X\).

**Proof.** We argue by contradiction. Otherwise, take \(y_0 \in X \setminus X'\). Choose \(\delta_0 > 0\) small enough such that \(d(y_0, X') > 2\delta_0 > 0\). Choose \(x_0 \in X' \setminus \Sigma\) and by choosing \(\delta_0\) smaller, assume \(d(x_0, \Sigma) > 2\delta_0\). Choose \(D > 0\) such that \(x_0, y_0 \in B(p_X, D)\).

Assume \(d(x_0, y_0) = L > 2\delta_0\) by choosing \(\delta_0\) smaller. Take \(y_i, z_i \in B(p_i, D)\) such that \(x_i \to x_0, y_i \to y_0\), and \(x_i \in E^\tau, d_i(x_i, \partial E^\tau) > 2\delta_0\).

By Proposition 4.1 and Lemma 2.8, there is a \(z_i \in B(y_i, \delta_0)\) and a segment \(\gamma_{x_i, z_i} \subset (M_i \setminus \partial E^\tau, g_{i,ext})\). Note that \(M_{i,ext}\) is a connected component of \(M_i \setminus \partial M_{i,ext}\) and \(\gamma_{x_i, z_i} \subset M_i \setminus \partial M_{i,ext}\), so \(\gamma_{x_i, z_i} \subset M_{i,ext}\) and \(z_i \in M_{i,ext}\).

Also \(\sum_{j,k=1}^3 |\langle \nabla u^j_i, \nabla u^k_i \rangle - \delta_{jk}|^2(z_i) < \tau_i\), otherwise since \(\sum_{j,k=1}^3 |\langle \nabla u^j_i, \nabla u^k_i \rangle - \delta_{jk}|^2(x_i) < \tau_i\), there will be an intersection point of \(\gamma_{x_i, z_i}\) and \(\partial E^\tau\). So \(z_i \in E^\tau\).

Taking \(i \to \infty\), \(z_i \to z_0 \in B(y_0, \delta_0) \cap X'\), a contradiction.

Notice that by Lemma 3.4, there exists uniform \(L_0 > 0\) such that
\[
\mathbb{R}^3 \setminus B(0, L_0) \subset \mathcal{U}_i(M_i \setminus \partial M_{i,ext}) \subset Y^\tau,
\]
and
\[
(\mathbb{R}^3 \setminus B(0, L_0), g_i) \to (\mathbb{R}^3 \setminus B(0, L_0), g_E)
\]
locally in \(C^0\) topology. We can assume \(\mathcal{U}_i(p_i) = p^* \in \mathbb{R}^3 \setminus B(0, L_0)\) as before. Moreover, we claim that this convergence also holds in the pointed Gromov-Hausdorff topology, i.e.
\[
(\mathbb{R}^3 \setminus B(0, L_0), d_i) \to (\mathbb{R}^3 \setminus B(0, L_0), d_E) \subset (X', d_X).
\]
This claim follows from the following lemma.

**Lemma 5.2.** For any \(D > 0, \delta > 0\), there exists \(\delta_0 = \Psi(m(g_i)|D, \delta)\) such that for any \(y, z \in E^\tau \cap B(p_i, D)\) with \(d_i(y, \partial E^\tau), d_i(z, \partial E^\tau) \geq \delta > 0\),
\[
|\mathcal{U}_i(y) - \mathcal{U}_i(z)| - d_i(y, z) \leq \Psi(m(g_i)|D, \delta).
\]

**Proof.** Take \(\delta_0 = \Psi(m(g_i)|D, \delta)\) as in Lemma 2.8. For all \(i\) large enough, \(\delta_0 < \frac{1}{4}\delta_0\), and by Proposition 4.1 and Lemma 2.8, there exist \(z'\) and a geodesic segment \(\gamma_{yz'}\) connecting \(y, z'\) such that \(z' \in B(z, \delta_0)\) and \(\gamma_{yz'} \cap \partial E^\tau = \emptyset\). So \(\gamma_{yz'} \subset E^\tau\).

Set \(l = d_i(y, z')\). Note that
\[
|\mathcal{U}_i(y) - \mathcal{U}_i(z')|^2 \leq \sum_{j=1}^3 (u^j_i(\gamma_{yz'}(l)) - u^j_i(\gamma_{yz'}(0)))^2 = \sum_{j=1}^3 \left(\int_0^l \langle \nabla u^j_i, \gamma'_{yz'} \rangle \right)^2,
\]
and

\[ 1 - \frac{3}{j=1} \left( \nabla u_i^j, \gamma_{y}^j \right)^2 \leq 4 \tau_i. \]

So

\[ \| U_i(y) - U_i(z) \| \leq \left( \frac{3}{j=1} \int_0^1 \left( \nabla u_i^j, \gamma_{y}^j \right)^2 \right)^{\frac{1}{2}} \leq (1 + 4 \tau_i)l. \]

Similarly, since \( d_i(z, z') < \frac{1}{2} \delta \) and \( d_i(z, \partial E^{\tau_i}) > \delta \), there exists a geodesic segment connecting \( z, z' \) inside \( E^{\tau_i} \), so we have

\[ \| U_i(z) - U_i(z') \| \leq (1 + 4 \tau_i) d_i(z, z'). \]

Thus

\[ |U_i(y) - U_i(z)| \leq (1 + 4 \tau_i)(d_i(y, y') + d_i(z, z')) \leq (1 + 4 \tau_i)d_i(y, z) + 4 \delta_0, \]
i.e.

\[ |U_i(y) - U_i(z)| - d_i(y, z) \leq \Psi(m(g_i)|D, \delta). \]

On the other hand, denote \( y_i = U_i(y), z_i = U_i(z) \). Then

\[ d_E(y_i, \partial Y^{\tau_i}), d_E(z_i, \partial Y^{\tau_i}) > \frac{1}{2} \delta. \]

Also, from above arguments,

\[ |y_i z_i| \leq d_i(y, z) + \Psi(m(g_i)|D, \delta) \leq 2D + \Psi(m(g_i)|D, \delta). \]

By the area formula,

\[ \mathcal{H}^2_{d_E}(\partial Y^{\tau_i}) = \int_{\partial E^{\tau_i}} |JU_i|dH^2 \leq (1 + 2 \tau_i)\mathcal{H}^2_{d_E}(\partial E^{\tau_i}) \leq \Psi(m(g_i)). \]

Similar arguments as above imply that there exist \( z_i' \in B(z_i, \delta_0) \) and a straight line segment \( [y_i z_i'] \subset Y^{\tau_i} \). Denote \( z' = U_i^{-1}(z_i') \in E^{\tau_i} \). Then \( \gamma_1 := U_i^{-1}([y_i z_i']) \subset E^{\tau_i} \) is a smooth curve between \( y, z' \), \( \gamma_2 := U_i^{-1}([z_i z_i']) \subset E^{\tau_i} \) is a smooth curve between \( z \), \( z' \). So

\[ d_i(y, z) \leq d_i(y, z') + d_i(z, z') \leq L_{g_i}(\gamma_1) + L_{g_i}(\gamma_2) \leq (1 + \Psi(m(g_i)))(|y_i z_i| + \delta_0), \]
i.e.

\[ d_i(y, z) - |U_i(y) - U_i(z)| \leq \Psi(m(g_i)|D, \delta). \]

□

From this lemma, we know

\[ (\mathbb{R}^3 \setminus B(0, L_0), d_X, p_X) = (\mathbb{R}^3 \setminus B(0, L_0), d_E, p^* \subset (X, d_X). \]

Let \( \{ y^1, y^2, y^3 \} \) be the coordinate function of \( \mathbb{R}^3 \) and \( e_j \) the unit vector along \( y^j \)-axis. Let \( q^j_{\pm} = \pm L e_j \) for \( L > L_0 \). By elementary geometry, we know that any \( y \in B(0, L) \subset \mathbb{R}^3 \) is uniquely determined by \( \{ |y q^j_{\pm}| \}_{j=1}^3 \), and particularly there exists a continuous function \( \xi \) with \( \xi(0) = 0 \) such that for any \( y, z \in B(0, L) \),

\[ |y z| = \xi(|y q^j_{\pm}| - |z q^j_{\pm}|). \]

We also choose an \( \frac{1}{N} \)-net \( \{ q^j_{a_k} \}_{k=1}^N \) of \( A(0, L_0, L) \), where \( A(0, L_0, L) = B(0, L) \setminus B(0, L_0) \subset \mathbb{R}^3 \) is the standard annulus. We always assume \( \{ q^j_{a_k} \}_{k=1}^N \subset \{ q^j_{a_k} \}_{k=1}^N. \)
Lemma 5.3. For any fixed \( N > 10 \), there exists \( \delta_0 = \Psi(m(g_i)|N,L) \) such that for any \( y \in Y^\tau \cap B(p^*, L) \), there exists \( y' \in Y^\tau \) with \( d_i(y, y') < \delta_0 \) and

\[
|d_i(y', q^\delta_k) - |y| q^\delta_k|| \leq \Psi(m(g_i)), \forall 1 \leq k \leq N.
\]

Proof. By the remark below Lemma 2.8, there exists \( \hat{B} \subset B_i(y, \delta_0) \subset M_i \) with \( Vol_i(\hat{B}) \geq (1 - \Psi(m(g_i)|N,L))Vol_i(B_i(y, \delta_0)) \) such that for any \( y' \in \hat{B} \), the geodesic segment \( \gamma_{q^\delta_k y'} \) lies in \( Y^\tau \) for all \( 1 \leq k \leq N \). By volume comparison theorem, we know

\[
\frac{\text{Vol}(B_i(y, \delta_0))}{|B_i(\Lambda(\delta_0))|} \geq \frac{\text{Vol}(B_i(y, 2L))}{|B_i(\Lambda(2L))|} \geq \frac{\text{Vol}(B_i(p_i, L))}{|B_i(\Lambda(2L))|} \geq \frac{|B(1)|}{|B_i(\Lambda(2L))|}.
\]

So

\[
\frac{\text{Vol}(B_i(y, \delta_0))}{|B(\delta_0)|} \geq \frac{|B(1)|}{|B_i(\Lambda(2L))|}, \quad \frac{|B_i(\Lambda(\delta_0))|}{|B_i(\Lambda(2L))|} \geq c(L) > 0.
\]

Note that \( \hat{B} \subset Y^\tau \) and \( |\hat{B}| \geq (1 - \Psi)Vol_i(B) \).

We claim that there exists \( y' \in \hat{B} \) such that each straight line segment \( [q^\delta_k y'] \) lies in \( Y^\tau \). Otherwise, by the same argument as the proof of Lemma 2.8, we have

\[
|\hat{B}| \leq C(N, L) \cdot \mathcal{H}^d_{dE}(\partial Y^\tau),
\]

which is a contradiction if we choose \( \delta_0 = \Psi(m(g_i)|N,L) \) bigger.

So for such \( y' \in \hat{B} \), these geodesic segments \( \gamma_{q^\delta_k y'} \) and \( [q^\delta_k y'] \) all lie in \( Y^\tau \). Similar to the proof of Lemma 5.2, by taking integral along these segments, we can get the conclusion.

Now we can define a map from \( X' \) to \( \mathbb{R}^3 \). For any \( L > 2L_0 \) and \( x \in X' \cap B(p_X, \frac{L}{2}) \), assume \( x_i \in E^\tau \cap B(p_i, \frac{L}{2}) \) and \( x_i \to x \) in the GH-topology. Denote \( y_i = U(x_i) \). Then by Lemma 5.2, \( y_i \in Y^\tau \cap B(p^*, L) \). For any fixed \( N > 10 \), by Lemma 5.3, there exists \( y_{N,i}^\prime \in Y^\tau \) with \( d_i(y_i, y_{N,i}^\prime) < \Psi(m(g_i)) \) and

\[
|d_i(y_{N,i}^\prime, q^\delta_k) - |y_{N,i}^\prime| q^\delta_k|| \leq \Psi(m(g_i)), \forall 1 \leq k \leq N.
\]

Up to a subsequence, we can assume \( y_{N,i}^\prime \to y_N^\prime \in B(p^*, L) \subset \mathbb{R}^3 \) in the \( d_E \)-topology. By taking a further subsequence, assume \( y_{N,i}^\prime \to y' \) in the \( d_E \)-topology. Define

\[
\Phi_L : X' \cap B(p_X, \frac{L}{2}) \to B(p^*, L) \subset \mathbb{R}^3, \quad \Phi(y) = y'.
\]

Lemma 5.4. \( \Phi_L \) is well defined.

Proof. It’s enough to show that for \( y \in X' \) with \( y_i^\prime, y_i^\prime \in Y^\tau \cap B(p^*, L) \) satisfying \( y_i'' \to y \) in the GH-topology and

\[
|d_i(y_i^\prime, q_{i^\pm})| = |y_i^\prime q_{i^\pm}| \leq \Psi(m(g_i)), \quad |d_i(y_i'', q_{i^\pm})| = |y_i'' q_{i^\pm}| \leq \Psi(m(g_i)), \forall j \in \{1, 2, 3\},
\]

if \( y_i'' \to y' \) and \( y_i'' \to y'' \) in the \( d_E \)-topology, then \( y' = y'' \). To see this, by the assumptions, we know

\[
d_X(y, q_{i^\pm}) = |y q_{i^\pm}| = |y'' q_{i^\pm}|, \forall j \in \{1, 2, 3\}.
\]

So \( |y''| = \xi(|y q_{i^\pm}| - |y'' q_{i^\pm}|) = \xi(0) = 0 \).

\[ \Box \]

Proposition 5.5. \( \Phi_L : (B(p_X, \frac{L}{2}), d_X) \to (\mathbb{R}^3, d_E) \) is an isometry onto \( B(p^*, \frac{L}{2}) \).
Proof. For any \( y, z \in X' \cap B(p_X, \frac{L}{2}) \) and assume \( y_i \to y, z_i \to z \) in the GH-topology with \( y_i, z_i \in Y^\gamma \cap B(p^*, L) \). Take \( y_{N,i}^{-}, z_{N,i}^{-} \) as above and assume \( y_{N,i}^{-} \to y', z_{N,i}^{-} \to z' \) in the \( d_E \)-topology. So \( \Phi(y) = y' \) and \( \Phi(z) = z' \).

Note that \( d_X(y, z) = \lim_{i \to \infty} d_i(y_i, z_i) \) and

\[
|d_i(y_{N,i}^{-}, z_{N,i}^{-}) - d_i(y_i, z_i)| \leq d_i(y_i, y_{N,i}^{-}) + d_i(z_i, z_{N,i}^{-}) \leq \Psi(m_g(i)),
\]

so \( d_X(y, z) = \lim_{i \to \infty} d_i(y_{N,i}^{-}, z_{N,i}^{-}) \).

If we take \( z = q^a, \forall 1 \leq k < \infty \), then

\[
d_X(y, z^a) = |y'q^a|.
\]

Since \( \{q^a\}_{k=1}^\infty \) is dense in \( A(0, L_0, 0) \), for any \( q^a \in A(0, L_0, 0) \), we know

\[
d_X(y, q^a) = |y'q^a|.
\]

In particular, \( d_X(p_X, y) = |p^*y'| \). Also \( \Phi_L \) is the identity map when restricted on \( A(0, L_0, 0) \), and \( \Phi_L^{-1}(A(0, L_0, 0)) = A(0, L_0, 0) \).

For any \( y, z \in X' \cap B(p_X, L) \), since \( y', z' \in B(p^*, L) \), we can extend \([y'z']\) to a straight line segment in \( B(p^*, L) \subset \mathbb{R}^3 \) such that \( q^a \) is one end point for some \( q^a \in A(0, L_0, 0) \), say \( z' \in [q^a y'] \). Then

\[
|y'z'| = |q^a y' - q^a z'| = d_X(q^a, y) - d_X(q^a, z) \leq d_X(y, z).
\]

So \( \Phi_L \) is an 1-Lipschitz map, and particularly continuous.

We claim that \( \Phi_L \) is injective on \( B(p_X, \frac{L}{2}) \). Otherwise, assume \( \Phi_L(y_1) = \Phi_L(y_2) = y' \) with \( y_1, y_2 \in B(p_X, \frac{L}{2}) \). Choose geodesic segments \( \gamma_1, \gamma_2 \) between \( y_1, y_2 \) and \( q^a \) respectively for some \( q^a \in A(0, L_0, 2L_0) \). Then \( \gamma_1, \gamma_2 \subset B(p_X, L) \).

Let \( \sigma_j = \Phi_L(\gamma_j) \) for \( j = 1, 2 \). Since \( \Phi_L \) is 1-Lipschitz, the length of \( \sigma_j \) in \( \mathbb{R}^3 \) is smaller than the length of \( \gamma_j \) in \( X \), i.e.

\[
|y'q^a| \leq |\sigma| \leq d_X(y_j, q^a) = |y'q^a|.
\]

So \( \sigma_1 = \sigma_2 = [y'q^a] \) is the unique line segment between \( y', q^a \). Since \( \Phi_L \) is one to one around \( q^a \), we know \( \gamma_1 \) coincides with \( \gamma_2 \) around \( q^a \). If \( y_1 \neq y_2 \), then we have branching geodesic segments in \( X \). This is a contradiction with Proposition 2.9.

By Proposition 2.10 and Lemma 5.1, \( X' = B(p_X, \frac{L}{2}) \) is a topological manifold with boundary \( \{x \in X : d(p_X, x) = \frac{L}{2}\} \). Since \( d_X(p_X, x) = |p^*\Phi_L(x)| \), we know \( \Phi_L \) maps the boundary of \( B(p_X, \frac{L}{2}) \) to boundary of \( B(p^*, \frac{L}{2}) \subset \mathbb{R}^3 \). We have proved that \( \Phi_L \) is one to one on \( A(0, L_0, 0) \), so for the induced map \( \Phi_L : (B(p_X, \frac{L}{2}), \partial B(p_X, \frac{L}{2})) \to (B(p^*, \frac{L}{2}), \partial B(p^*, \frac{L}{2})) \), the mod 2 degree is 1. From the degree theory,

\[
\Phi_L(B(p_X, \frac{L}{2})) = B(p^*, \frac{L}{2}).
\]

So \( \Phi_L : B(p_X, \frac{L}{2}) \to B(p^*, \frac{L}{2}) \) is a homeomorphism. To show it’s also isometric, for any \( y, z \in B(p_X, \frac{L}{2}) \), assume \( y' = \Phi_L(y), z' = \Phi_L(z) \). Choose a line segment \([q^a - q^a']\) with \([q^a - q^a'] \in A(0, L_0, 0) \), length \( l \) and \([y'z'] \in [q^a - q^a] \subset B(p^*, \frac{L}{2}) \).

Let \( \gamma := \Phi_L^{-1}([q^a - q^a']) \). Then \( d_X(\gamma(t), s) = |t - s| \) since \( \Phi_L \) is 1-Lipschitz, and

\[
d_X(\gamma(0), \gamma(l)) = d_X(q^a, q^a') = |q^a - q^a'| = l.
\]

So it must hold that

\[
d_X(\gamma(t), s) = |t - s|.
\]

In particular, \( d_X(y, z) = |y'z'| \), i.e. \( \Phi_L \) is an isometry.
Since \( L \) could be any large positive number, \((X, d_X) = \mathbb{R}^3\). This finishes the proof of Theorem 1.6.

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