Spectral Portraits in the Semi-Classical Approximation of the Sturm-Liouville Problem with a Complex Potential

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Abstract. We study the problem on the limit behavior of the discrete spectrum of the Sturm-Liouville problem
\[-\varepsilon y''(x) + P(x, \lambda)y(x) = 0, \quad y(a) = y(b) = 0,\]
provided that the physical parameter \(\varepsilon\) tends to zero. It is assumed that \(\lambda\) is the spectral parameter, the function \(P\) is polynomial on \(x\) with analytic coefficients on \(\lambda\) varying on a domain \(G\) in the complex \(\lambda\)-plane \(\mathbb{C}\). The case \(P(x, \lambda) = p(x) - \lambda\) with complex valued function \(p\) corresponds to the usual linear spectral problem. Boundary conditions are formed by arbitrary complex numbers \(a, b\) or \(1\). We shall show that in this case the eigenvalues are concentrated along the so-called limit spectral graph as \(\varepsilon \to 0\). We define three type of curves, forming this graph and find the asymptotic formulae for the eigenvalue distribution along the curves of various types. For the case \(P(x, \lambda) = p(x)\) with real polynomial \(p\) these formulae coincide with well known Bohr-Sommerfeld quantization formulae. Non-self-adjoint Sturm-Liouville problems are often found in mathematical physics. We demonstrate this considering the well-known in hydromechanics Orr-Sommerfeld spectral problem.

1. Introduction and historical remarks
The interest of the authors to the topic of this paper was originated while considering the spectral portraits of the Orr-Sommerfeld equation with large Reynolds number. The well-known in hydrodynamics Orr-Sommerfeld equation is obtained by linearization of the Navier-Stokes equation in the infinite three-dimension spatial layer \((x, \xi, \eta) \in \mathbb{R}^3\), where \(|x| \leq 1\) and \((\xi, \eta) \in \mathbb{R}^2\), assuming that a stationary unperturbed solution for the velocity profile is of the form \((q(x), 0, 0)\). This equation with respect to a function \(y(x)\) has the form (see details in the book of Drazin and Reid [1], for example)
\[(D^2 - \alpha^2)y - i\alpha R\left[q(x)(D^2 - \alpha^2) - q''(x)\right]y = -i\alpha R\lambda(D^2 - \alpha^2)y.\]  
Here \(i\) is the imaginary unit, \(D = d/dx\), \(\alpha\) is the wave number appearing after the separation of the variables \((\xi, \eta) \in \mathbb{R}^2\), \(R\) is the Reynolds number and \(\lambda\) is the spectral parameter. The boundary conditions have to be associated with the spectral problem (1). Often they are assumed of the form
\[y(\pm1) = y'(\pm1) = 0.\]
A challenging problem is to describe the limit behavior of the spectrum of the problem (1), (2) as $R \to \infty$. The Reynolds number $R$ is proportional to the reciprocal of the viscosity $\nu$ of the liquid, therefore, we meet a problem of the spectrum description of the Orr-Sommerfeld equation for the liquid which is almost ideal. For a long time it has been supposed that the spectrum of the Reyleigh problem

\begin{equation}
q(x)(D^2 - \alpha^2)y - q''(x)y = \lambda(D^2 - \alpha^2)y, \\
y(-1) = y(1) = 0
\end{equation}

plays an important role for explaining the limit spectrum behavior of the Orr-Sommerfeld problem. The Reyleigh problem is obtained (after deviding equation (1) by $-i\alpha R$) by formal passing to the limit as $R \to \infty$ and by eliminating the "superfluous" boundary conditions. A vast amount of literature is devoted to the study of the spectrum of problem (3) (references can be found in the cited book of Drazin and Reid [1] and in the papers of Lin [2]). However, it turns out that the spectrum of the Reyleigh problem plays no essential role in the description of the Orr-Sommerfeld problem with large Reynolds numbers. This will be seen in the sequel.

It is known [1] that the spectrum of problem (3) consists of the segment $[m, M]$, where $m$ and $M$ are the minimum and maximum of the function $q$ (it is assumed that the function $q$ is continuous), and possibly of some isolated eigenvalues of finite multiplicity lying outside this segment. It was Heisenberg who noticed that for Couette profile $q(x) = x$ there is no continuity between the spectrum of the Orr-Sommerfeld problem with large $R$ and the spectrum of the Reyleigh problem. Given arbitrary small $\delta > 0$ there is an open domain containing an interval $(m + \delta, M - \delta)$ such that this domain contains no points of the spectrum of the Orr-Sommerfeld problem for sufficiently large $R$. This phenomenon was named "Heisenberg tongue".

In 1924 Heisenberg proved the existence of a system of fundamental solutions for equation (1), having special representation (see [1] and [3]). This result is very essential for the explanation of this phenomenon. However, we are not aware of Heisenberg’s papers containing ideas on the explanation of "Heisenberg tongue" phenomenon. It was Morawetz [3] who proved rigorously that in the case of the Couette profile $q(x) = x$ the spectrum of the problem (1), (2) is concentrated in a $\delta$-neighborhood of the ray $[-i/\sqrt{3}, -i\infty)$, the two segments $[\pm 1, -i/\sqrt{3}]$, and the isolated eigenvalues $\{\mu_k\}$ of the Reyleigh problem for $q(x) = x$. Actually, the Reyleigh problem for $q(x) = x$ has no isolated eigenvalues (see [1]), therefore, the last reservation in the Morawetz’ theorem can be dropped. Moreover, Morawetz made an attempt to prove a similar results for more general functions $q$. She assumed (this assumption is quite essential for her method) that $q$ is an entire function with real values on the real axis and maps bijectively the whole complex plane $\mathbb{C}$ onto itself. Probably she did not noticed that all the functions possessing these properties had representation $q(x) = \Theta x + \theta$ with $\Theta, \theta \in \mathbb{R}, \Theta \neq 0$.

There is another important problem which is left out of the paper [3]. What is the set of accumulation points of the eigenvalues as $R \to \infty$? Morawetz emphasized in [3] that her method did not allow to get the information if the eigenvalues do exist as $R \to \infty$ near each point of the segments $[\pm 1, -i/\sqrt{3}]$.

In the 90s there appeared papers (see [4], [5], for example; more details can be found in [6]), where a simpler problem of the form

\begin{equation}
-i\varepsilon z'' + q(x)z = \lambda z, \\
z(-1) = z(1) = 0
\end{equation}

was associated with (1), (2). Here $\varepsilon$ and $\lambda$ are small and spectral parameter respectively. One can consider this problem as a simplified model for the Orr-Sommerfeld problem (1), (2). The following arguments can confirm this point.

Let us make the substitution $z = (D^2 - \alpha^2)y$ in equation (1). From this equation, taking into
account the conditions \( y(0) = y'(0) = 0 \), we find
\[
y(x) = \frac{1}{\alpha^2} \int_{-1}^{x} z(\xi) \sinh \alpha (x - \xi) \, d\xi.
\]  
(5)

Then equation (1) takes the form
\[
-\iota \varepsilon (D^2 - \alpha^2) z + Fz = \lambda z,
\]  
(6)

where
\[
Fz = \int_{-1}^{x} z(\xi) \sinh (2(x - \xi)) \, d\xi, \quad \varepsilon = \frac{1}{\alpha R}.
\]  
(7)

Taking into account (5), we rewrite boundary conditions (2) in the form
\[
\int_{-1}^{1} z(\xi) \sinh (1 - \xi) \, d\xi = 0, \quad \int_{-1}^{1} z(\xi) \cosh (1 - \xi) \, d\xi = 0,
\]  
(8)

Thus, problem (1), (2) is equivalent to problem (6), (8). Similar arguments had been carried out yet in the paper of Orr in 1915. Now, if we neglect the influence of the integral (compact) operator in (6) (note that \( F = 0 \) in the case \( q(x) = x \)) and suppose that boundary conditions do not play a substantial role in the spectrum behavior as \( \varepsilon \to 0 \) (this is by no means obvious), we arrive at model problem (4) up to the shift of the spectral parameter by \( i \alpha^2 \).

Of course, these arguments are heuristic. However, the similarity of the spectral portraits of the model and Orr–Sommerfeld problems as \( \varepsilon \to 0 \) can be proved rigorously. This was done in the case of Couette (\( q(x) = x \)) and Poiseuille (\( q(x) = 1 - x^2 \)) profiles in papers [7], [8] and [9].

If we put \( \varepsilon > 0 \) instead of \( i \varepsilon \) in equation (4), we get a self-adjoint problem with small parameter. Such a problem was well investigated long ago (see [10], for example). Its spectrum is real, it condenses as \( \varepsilon \to 0 \), and the formulae for the eigenvalue localization can be written down explicitly. They are known as the Bohr–Sommerfeld quantization formulae.

The replacement of \( "\iota \varepsilon" \) by \( "i \varepsilon" \) changes the problem dramatically (we assume here that a function \( q \) in (4) is real). Numerical calculations show that for analytic functions \( q \) the spectrum of problem (4) is concentrated along some curves in the complex \( \lambda \)-plane as \( \varepsilon \to 0 \) (see [4], [5], [6] and the figures below). This raises the following natural questions:

How to define these curves? Are there formulae which give explicit description of these limit spectral curves? How to find the density of concentration of the eigenvalues along these limit spectral curves, provided that these curves are already found?

Certainly, the attempt to answer these questions must first be launched to a simpler problem (4). Heuristic arguments provide a hope that similar results can be obtained for the Orr–Sommerfeld problem as well.

Probably the first rigorous description of the limit spectral curves for problem (4) in the simplest case \( q(x) = x \) was given in the paper [11]. In the same paper an important remark was made: it was claimed that the form of the so-called "Heisenberg tongue" (the domain adjacent to the segment \([m, M] = [q(-1), q(1)]\) which is free of the eigenvalues, provided that \( \varepsilon \) is sufficiently small) is determined by the Stokes lines of the action integrals defined by function (12) (see below for explanations). It turns out that the boundary of this domain form the curves which later we named critical. Then the authors [12], [13] gave a complete description of the spectral portraits as \( \varepsilon \to 0 \) for the problem (4) with the function \( q(x) = x^2 \) and \( q(x) = ax^2 + b + c \) where \( a, b, c \in \mathbb{R} \). Simultaneously the quantization formulae of the eigenvalue distribution as \( \varepsilon \to 0 \) were written down in these papers for linear and quadratic profiles. As we have already mentioned, soon there appear papers [7], [8], [9] in which the correspondence between the model problem (4) and the Orr–Sommerfeld problem (1), (2) with linear and quadratic (Couette and
Poiseuille) profiles was approved. We note that afterwards there appear a great number of works with analytic and numerical studies of different type of the Orr-Sommerfeld spectral problem. In particular, recently we had a chance to be acquainted with a paper [14].

An important observation was made in papers [15] and [16]. It was observed that there are only three types of curves which form the limit spectral portrait (the set of the accumulation points of the eigenvalues as \( \varepsilon \to 0 \)) of problem (4), provided that the analytic function \( q \) is monotone or has the only extremum on the segment \([-1, 1]\). These lines were named singular, critical and main curves (in this work the main curves we call balanced ones). In these papers the quantization formulae along these curves were found as well. It is worth mentioning that the balanced curves there appear in the study of quasi-classical approximations of non-self-adjoint Sturm-Liouville problem as well as in the case of self-adjoint one. This phenomena was observed for the Sturm-Liouville equation on the whole line long ago in [17]. In particular, the quantization formulae along balanced curves have the same representation as in the self-adjoint case.

The cited works dealt only with the Sturm-Liouville equation on a finite interval, mainly with Dirichlet boundary conditions. The case of periodic boundary conditions was studied in papers [18], [19]. There appear some new phenomena in this case.

There are a number of interesting papers investigating the Schrödinger operator with complex polynomial or periodic (in particular the so-called PT-symmetric) potentials on the whole line. In this connection we refer the readers to the works [20]{[24] (see also the references therein).

2. Main results
The objective of the present paper is to investigate a non-self-adjoint Sturm-Liouville problem in the quasi-classical approximation (as the physical parameter at the second derivative tends to zero) in a more general setting. Namely, we deal with the problem

\[
-\varepsilon y''(z) + P(z, \lambda) y(z) = 0, \tag{9}
\]
\[
y(a) = y(b) = 0. \tag{10}
\]

Here \( \lambda \) is the generalized spectral parameter which varies in the domain \( G \) of the complex \( \lambda \)-plane and \( P \) is a polynomial of degree \( n \geq 1 \) in \( z \) with coefficients analytically depending on \( \lambda \in G \). Actually, we may assume that \( P \) is an entire function in \( z \), however, in this case the subsequent results can be formulated without changes only in the case when \( P \) has a finite number of zeros. The points \( a \) and \( b \) at which the boundary conditions are defined we assume to be arbitrary complex numbers; one of them or both may be infinite. With the infinite point \( a \) we associate the ray \( \Upsilon_\varphi = \{re^{i\varphi}, \ r > 0 \} \) and write \( a = e^{i\varphi}_{\infty} \), while the boundary condition \( y(a) = 0 \) in this case we understand in the following sense: \( y = y(re^{i\varphi}) \to 0 \) as \( r \to \infty \) along the ray \( \Upsilon_\varphi \). The set of values \( \lambda \in G \) for which the equation (9) has a non-trivial solution satisfying the boundary conditions (10) we call the spectrum of the problem under consideration. In the usual classical setting the points \( a \) and \( b \) are assumed to be real or equal to \( \pm \infty \) and in this case the problem is considered on the finite real segment \([a, b]\), on the semi-axis \([a, \infty)\) or on the whole axis \((-\infty, \infty)\).

The main results of this section were earlier obtained by the authors in the paper submitted in arXiv [29].

The problem (4) which was discussed in the previous section is a very particular case of (9), (10). First, it is well known, that there is a difference in the study of Sturm-Liouville problems on finite and infinite intervals. Therefore, we come to a more general setting if we understand how to treat the cases of finite and infinite intervals simultaneously. Then, multiplying equation in (4) by \(-i\) we arrive to equation (9) with the function \( P \) of a particular form \( P(z, \lambda) = \lambda - iq(x) \) which is linear in \( \lambda \) and contain a real function \( q(x) \) instead of an arbitrary complex function.
A more general setting, which we present here, by no means complicates the results obtained below, and even makes them simpler formulated.

It should be noted that when talking about the semi-classical approximation one typically uses a small parameter $\varepsilon$ or $h$ at the second derivative. However, we will use in the sequel the WKB asymptotics, and for our purposes it will be more convenient to rewrite equation (9) in the form

$$-y''(z) + k^2 P(z, \lambda) y(z) = 0,$$  \hspace{1cm} (11)

assuming further in the sequel that the parameter $k = 1/\sqrt{\varepsilon} \to \infty$.

In what follows, we exclude the degenerate cases from consideration and require additionally for the coefficient $a_n(\lambda)$ not to vanish in $G$, and for the zeros $z_j, j = 1, \ldots, n$, of the polynomial $P$ to be different functions of $\lambda$, or different germs of a function analytic in $G$ everywhere except some algebraic branching points.

Further we assume that the reader is familiar with the method of phase integrals (WKB method). For details we refer the readers to the papers and books [25, 26, 27, 28]. Here we recall the basic definitions and notations.

Let us consider the function

$$S(z_0, z; \lambda) = \int_{z_0}^{z} \sqrt{P(\zeta, \lambda)} d\zeta.$$  \hspace{1cm} (12)

Here the branch of a root is fixed, and the integration is carried out along a path from $z_0$ to $z$, not crossing the turning points $z_j = z_j(\lambda)$, that is, the zeros of $P(z)$. The system of lines in $\mathbb{C}$, determined for each turning point (zero of $P$) $z_j$:

$$\Gamma_j = \left\{ z \in \mathbb{C} \mid \text{Re} S(z_j, z; \lambda) = 0 \right\},$$

is called the Stokes complex associated with the turning point $z_j$. The maximal connected components of $\Gamma_j$ starting from $z_j$ and containing no other turning points are called Stokes lines. Each Stokes line is either infinite and contains a turning point at the origin or is finite and connects two different turning points. In the second case only the initial turning point belongs to the Stokes line. Thus, the Stokes lines are always understood as oriented curves — either infinite or finite curvilinear semi-intervals — starting at a turning point. The assembly $\mathcal{G}(\lambda)$ of all the Stokes complexes corresponding to all turning points is called the Stokes Graph. The Stokes Graph splits the complex plane into simply connected infinite domains called basic domains. Among them there are two types of domains. The first type relates to domains that are of the half-plane type (the function $S$ maps them conformally to a half-plane $\text{Re} S > \alpha$ or $\text{Re} S < \alpha$, $\alpha \in \mathbb{R}$); the second type relates to the so-called strip type domains (the function $S$ maps them to vertical strips in $\mathbb{C}$).

A domain $D$ of the complex plane is called canonical if the function $S$ maps it one-to-one to the entire complex plane with a finite number of vertical cuts. A canonical domain contains two half-plane type domains and some (possibly empty) set of strip type domains (the common boundaries are also included).

The interior of angles in $\mathbb{C}$ formed by the rays of the form

$$l_j = \left\{ re^{i\varphi_j} - \frac{a_{n-1}}{na_n}, \ r > 0 \right\},$$

$$\varphi_j = \frac{\pi - \arg a_n}{n+2} + \frac{2\pi j}{n+2}, \ j = 0, \ldots, n + 1,$$  \hspace{1cm} (13)

are called the Stokes sectors. These and only these rays are the asymptotes for infinite Stokes lines.
Definition. We say that a point \( \lambda \in G \) is exceptional for the boundary value problem (9), (10), if at least one of the boundary conditions is specified in an infinite point, say, \( a = e^{i\varphi} \infty \), and \( \varphi \) coincides with one of the values \( \varphi_j \) in (13), i.e. the direction of one of the infinite points coincides with the direction of one of the asymptotes to the Stokes lines.

Further in the case \( a_n(\lambda) = a_n = \text{const} \) and \( a = e^{i\varphi} \infty \) we always assume that \( \varphi \neq \varphi_j \) where \( \varphi_j \) are defined by (13). The same we assume for the value \( \psi \) if the second boundary point \( b = e^{i\varphi} \infty \) is also infinite.

**Theorem 1.** Let \( g \) be a compact set in the domain \( G \) such that all points \( \lambda \in g \) are not exceptional for the boundary value problem (9), (10). Then the spectrum of this problem is discrete in a small neighborhood of \( g \).

If \( a_n(\lambda) \neq \text{const} \) and \( a_n(\lambda) \neq 0 \) then \( u(\lambda) := \arg a_n(\lambda) \) is a nonconstant harmonic function on \( G \). Therefore, the conditions \( \arg a_n(\lambda) = \pi - \varphi_j(n + 2) \) determine smooth curves in the domain \( G \). In what follows, we eliminate the points of these curves from consideration passing, if necessary, to a narrower domain \( G' \subset G \) containing no exceptional points. Throughout the rest of the paper we assume that any compact set in \( G \) satisfies the condition of Theorem 1.

According to our assumption, the roots \( z_j(\lambda) \) of the polynomial \( P(z, \lambda) \) are different algebraic functions or germs of an analytic function with algebraic branching points, which can accumulate only to the boundary of \( G \). Further we wish to avoid the branching points and pass to a narrower domain \( G' \subset G \) which does not contain points of this type. Certainly, it may happen that the obtained domain \( G' \) is multi-connected. As before we will write \( G \) instead of \( G' \).

Our nearest goal is to define the curves in \( G \) that play an important role in the description of the spectrum of the problem (9), (10) as \( k \to \infty \).

Recall that a Stokes complex is called simple if it is generated by a simple turning point \( z_j \) and all the three Stokes lines that come out from \( z_j \) are infinite (i.e., this complex contains no other turning points). All the other complexes are called compound. If in addition to \( z_j \) a complex contains \( z_{j_2}, \ldots, z_{j_s} \), turning points (with multiplicity taken into account), then it is called the \( s \)-point complex. We say that a point \( \lambda \in G \) is regular if the Stokes graph \( \Theta(\lambda) \) consists of only simple complexes. All the other points we call singular. If \( \lambda \) is a singular point, then there exists a Stokes complex which includes at least two different turning points, say, \( z_j(\lambda) \) and \( z_l(\lambda) \). Let us consider the integral

\[
S_{j,l}(\lambda) = \int_{z_j(\lambda)}^{z_l(\lambda)} \sqrt{P(\zeta, \lambda)} \, d\zeta, \tag{14}
\]

where the integration is carried out along the Stokes line connecting \( z_j \) and \( z_l \). It can be shown that \( S_{j,l}(\lambda) \) admits the analytic continuation to the domain \( G \) (this continuation is multi-valued and depends on the path in \( G \)). Consequently, the function \( u(\lambda) := \text{Re} S_{j,l}(\lambda) \) is harmonic in any simply connected domain belonging to \( G \), and the set

\[
\gamma_{j,l} = \left\{ \lambda \in G \mid \text{Re} S_{j,l}(\lambda) = 0 \right\} \tag{15}
\]

determines smooth curves in \( G \), possibly with bifurcation points, when \( S_{j,l}'(\lambda) = 0 \).

It follows from the representations (14) and (15) that for all points \( \lambda \in \gamma_{j,l} \) the Stokes graph \( \Gamma_j = \Gamma(z_j, \lambda) \) contains a compound complex connecting the turning points \( z_j \) and \( z_l \). We shall call the curves \( \gamma_{j,l} \) which are defined by (15) singular curves. If different curves \( \gamma_{j,l} \) and \( \gamma_{j,s} \) intersect at a point \( \lambda_0 \), then the Stokes graph \( \Theta(\lambda_0) \) includes at least three-point complex. Intersection of three singular curves at one point generates a four-point complex, etc. Of course, for some indices \( j, l \) the set \( \gamma_{j,l} \) may be empty in \( G \). Thus, a set of singular points in \( G \) consists
of the curves $\gamma_{j,l}$ and in a generic position the number of their intersection points is finite in any compact subset $g \subset G$. We do not exclude the situation when the intersection of different curves $\gamma_{j,l}$ and $\gamma_{p,e}$ forms a continuum.

Suppose that the point $a$ which defines the boundary condition is finite. We say that a point $\lambda \in G$ is critical with respect to $a$ if there is a Stokes complex $\Gamma_j$ with turning point $z_j(\lambda) \neq a$ such that one of its Stokes lines intersects the point $a$. Let us consider the following integral

$$C_j(a, \lambda) = \int_{z_j(\lambda)}^a \sqrt{P(\zeta, \lambda)} \, d\zeta,$$

where the integration is carried out along the Stokes line connecting $z_j$ and $a$. It can be shown that $C_j(a, \lambda)$ admits the analytic continuation in $\lambda$ along any path in $G$ which does not intersects the zeros of $P$. Therefore, $u(\lambda) := \operatorname{Re} C_j(a, \lambda)$ is a locally harmonic function and the set

$$\gamma_j(a) = \left\{ \lambda \in G \mid \operatorname{Re} C_j(a, \lambda) = 0 \right\}$$

determines the curves in $G$ which we shall call critical curves. If the point $b$ is also finite, we define similarly the function $C_j(b, \lambda)$ and the critical curves $\gamma_j(b)$ (for some indices the sets $\gamma_j(a)$ and $\gamma_j(b)$ may be empty or coincide).

Suppose now that both $a$ and $b$ boundary points are finite. A point $\lambda \in G$ will be called balanced point with respect to the boundary points $a$ and $b$ if both these points lie in a canonical domain determined by the Stokes graph $\mathcal{G}(\lambda)$ and

$$\operatorname{Re} B(a, b, \lambda) = 0, \text{ where } B(a, b, \lambda) = \int_a^b \sqrt{P(\zeta, \lambda)} \, d\zeta. \quad (16)$$

Here the integration is carried out along the path which entirety lies in the canonical domain. Evidently, the set of balanced points with respect to $a$ and $b$ consists of smooth curves in $G$ which we shall call balanced curves. Let us denote this set by $\gamma(a, b)$. We remark that the curves of this set take the origin from the intersection points of the critical curves $\gamma_j(a)$ and $\gamma_j(b)$ (if there are such points in $G$), or appear at the boundary of $G$ and finish at similar points of intersection of critical curves, or go to the boundary of $G$ (in particular, go to $\infty$ if $G = \mathbb{C}$).

**Definition.** Given $\lambda \in G$ let $\mathcal{G}(\lambda)$ be the Stokes graph corresponding to this point, and $\Gamma_j$ be a certain Stokes complex in this graph. The domains which the complex $\Gamma_j$ splits the complex plane $\mathbb{C}$ into, we shall call basic domains for this complex. A domain $D \subset \mathbb{C}$ is said to be admissible with respect to the complex $\Gamma_j$ if $D$ intersects no more than two basic domains of this complex. In particular, maximal admissible domains (domains that have no nontrivial admissible extensions) consist of two neighboring basic domains of the complex $\Gamma_j$ and the Stokes line (excluding its starting point) which is their common boundary.

We say that the boundary points $a$ and $b$ are linked with respect to the complex $\Gamma_j$ if they both lie in some admissible domain for $\Gamma_j$. We imply that an infinite point $a = e^{i\varphi}\infty$ belongs to a domain if the ray $\arg z = \varphi$ is asymptotically contained in this domain.

The usefulness of the definition of linkedness with respect to complexes is clarified by the following statement.

**Lemma 1.** Given $\lambda \in G$ assume that the boundary points $a$ and $b$ do not coincide with the turning points of the Stokes graph $\mathcal{G}(\lambda)$. Then the points $a$ and $b$ are linked with respect to all the Stokes complexes $\Gamma_j \subset \mathcal{G}(\lambda)$ if and only if both these points belong to a canonical domain determined by $\mathcal{G}(\lambda)$. 


Balanced curves are determined by the position of the boundary points \( a \) and \( b \) with respect to the Stokes graphs \( \mathfrak{G}(\lambda) \), critical curves depend only on one of the boundary points and singular curves do not depend on \( a \) and \( b \). Next, we shall select some parts of critical and singular curves which play a crucial role in the description of the spectrum of the problem in question as the parameter \( k \to \infty \).

In the case of a finite point \( a \) we shall denote by \( \gamma_j^a(a) \) the set of points \( \lambda \in \gamma_j(a) \) for which the \( a \) and \( b \) are not linked with respect to the Stokes complex \( \Gamma_j \subset \mathfrak{G}(\lambda) \). Similarly, in the case of a finite point \( b \) we shall denote by \( \gamma_j^b(b) \) the set of points \( \lambda \in \gamma_j(b) \) for which the points \( b \) and \( a \) are not linked with respect to the Stokes complex \( \Gamma_j \subset \mathfrak{G}(\lambda) \). Finally, by \( \gamma_j^{a,b} \) we shall denote the set of points \( \lambda \in \gamma_j^{a,b} \) for which the \( a \) and \( b \) are not linked with respect to the Stokes complex \( \Gamma_j = \Gamma_1 \subset \mathfrak{G}(\lambda) \). The selected parts of critical and singular curves will be called essential critical and singular ones.

**Definition.** A set \( T \subset G \) is called limit spectral set or limit spectral graph of the problem (9), (10) if all the points \( \lambda \in T \) and only they are the interior points of \( G \) which are the accumulation points of the spectrum of the problem as \( k \to \infty \).

**Theorem 2.** If both boundary points \( a \) and \( b \) are finite, then the limit spectral set of the problem (9), (10) coincides with the set of balanced, essential critical and essential singular curves, namely,

\[
T = \bigcup_j \gamma_j^b(a) \bigcup_j \gamma_j^a(b) \bigcup_j \gamma_j^{a,b} \bigcup \gamma(a,b).
\]

If the point \( a \) is finite but the point \( b \) is infinite, then

\[
T = \bigcup_j \gamma_j^b(a) \bigcup \gamma_j^{a,b}.
\]

If both points \( a \) and \( b \) are infinite, then:

\[
T = \bigcup_{j,l} \gamma_{j,l}^{a,b}.
\]

Here we assume that the set \( T \) also includes the endpoints of the curves which form \( T \).

The following theorem gives us the formulae for the eigenvalue distribution of the problem along the curves which constitute the limit spectral graph \( T \).

**Theorem 3.** Let \( a \) and \( b \) be finite points, and the set \( \gamma(a,b) \) comprises a curvilinear interval \( \gamma \subset \gamma(a,b) \) such that the closure of \( \gamma \) has no intersections with \( \gamma_j^a(a) \), \( \gamma_j^b(b) \) and \( \gamma_j^{a,b} \). Then in a neighborhood of \( \gamma \) the following quantization formula is valid

\[
k \int_a^b \sqrt{P(\zeta, \lambda_m)} d\zeta \sim m \pi i, \quad k \to \infty, \ m \in \mathbb{Z}.
\]

This means that in a small neighborhood of the curve \( \gamma \) the eigenvalues \( \lambda_m \) of the problem “almost” coincide with the solutions \( \lambda_m^0 \in \gamma \) of the equation

\[
k \int_a^b \sqrt{P(\zeta, \lambda_m^0)} d\zeta = m \pi i, \quad k \to \infty, \ m \in \mathbb{Z}.
\]
that is, \(|\lambda_m - \lambda_k^0| = O(k^{-2})\) as \(k \to \infty\).

If the point \(a\) is finite, \(b\) is arbitrary (finite or infinite) and there exists a curvilinear interval \(\gamma \subset \gamma_j^0(a)\) such that the closure of \(\gamma\) does not have intersections with other curves of the graph \(T\), then in a neighborhood of \(\gamma\) the following quantization formula is valid up to the accuracy \(O(k^{-2})\):

\[
k \int_{z_j(\lambda_m)}^a \sqrt{P(\zeta, \lambda_m)} \, d\zeta \sim \left(-\frac{1}{4} + m\right) \pi i, \quad k \to \infty, \; m \in \mathbb{Z}.
\]

In the case of arbitrary boundary points \(a\) and \(b\) (finite or infinite), the existence of an

curvilinear interval \(\gamma \subset \gamma_j^{a,b}\) whose closure does not intersect the other curves of the graph \(T\), implies that in a small neighborhood of \(\gamma\) the following quantization formula is valid up to the accuracy \(O(k^{-2})\):

\[
k \int_{z_j(\lambda_m)}^\infty \sqrt{P(\zeta, \lambda_m)} \, d\zeta \sim \left(-\frac{1}{2} + m\right) \pi i, \quad k \to \infty, \; m \in \mathbb{Z}.
\]

As was already noted, some of the above–defined curves which form the limit spectral graph \(T\) may also have continual intersections. Depending on the topology of the Stokes graphs \(\Theta(\lambda)\) when \(\lambda\) varies along the intervals of such intersections we either obtain other quantization formulas (for the cases of three–points and more intricate compound Stokes complexes) or the eigenvalues can be divided into series according the number of complexes with respect to which \(a\) and \(b\) are not linked, wherein the quantization formulae for each series are determined by the corresponding Stokes complex. We do not present here exact formulae for such cases as we are planning to do this in a more advanced version of the work. Also, note that the mentioned in the theorem \(O(k^{-2})\) accuracy for quantization formulæ may decrease in small neighborhoods of some critical points (here we also omit details).

At the end, we present the main ideas of the proof for the case when the both boundary points \(a\) and \(b\) are finite. First, note that all points \(\lambda \in G \setminus T\) cannot be the limit points of the spectrum when \(k \to \infty\). Indeed, if \(\lambda_0 \notin T\), then it follows from the definitions that both points \(a\) and \(b\) lie in a canonical domain \(D = D(\lambda_0)\) of the complex \(z\)-plane (this domain is determined by the Stokes graph \(\Theta(\lambda_0)\)). In the domain \(D\) there exist solutions \(v_+\) and \(v_-\) for the equation (9) with the following asymptotic presentations:

\[
v_{\pm}(z, \lambda, k) = \frac{1}{P^{1/4}(z, \lambda)} e^{\pm kS(a, z; \lambda)} (1 + O(k^{-1})), \quad k \to \infty,
\]

uniformly with respect to \(z \in K\) for any compact \(K \subset D\).

Let us take a domain \(d\) which is compactly embeded in \(D\) and contains both points \(a\) and \(b\). There exists a neighborhood \(U(\lambda_0)\) of the point \(\lambda_0\) such that \(\forall \lambda \in U(\lambda_0)\) the lines of the Stokes graphs \(\Theta(\lambda)\) do not cross the domain \(d\) and the asympotics (18) are valid uniformly for all \((\lambda, z) \in U(\lambda_0) \times d\).

The characteristic determinant of the problem (9), (10) has the following form:

\[
\Delta(\lambda, k) = \begin{vmatrix} v_+(a) & v_+(b) \\ v_-(a) & v_-(b) \end{vmatrix} = P^{-1/4}(a, \lambda) P^{-1/4}(b, \lambda) \times
\]

\[
\times \left(e^{kB(a, b, \lambda)} (1 + O(k^{-1})) - e^{-kB(a, b, \lambda)} (1 + O(k^{-1}))\right),
\]

where the function \(B\) is defined in (16). If \(\lambda_0 \notin \gamma(a, b)\), then \(|\Delta(\lambda_0, k)|\) grows exponentially as \(k \to \infty\); this assertion remains true within a small neighborhood \(U(\lambda_0)\), that is, \(\lambda_0\) is not
a limit point of the spectrum. For \( \lambda_0 \in \gamma(a,b) \), if \( \lambda_0 \) does not belong to essential critical and essential singular curves, by using the Rouche’s theorem in a neighborhood of \( \lambda_0 \) we can get the quantization formula (17).

In order to get quantization formulas along essential critical and singular curves one should use the transition formulas for asymptotic solutions from one canonical domain to another one (these formulas are implemented by special transition matrices). For example, if \( \lambda_0 \in \gamma_{a,b}^{ab} \), then the points \( a \) and \( b \) are located in different canonical domains which location is determined by two-point Stokes complexes \( \Gamma_j = \Gamma_l \). In this case the points \( a \) and \( b \) are located in different basic domains of this complex, and these domains have no common boundary. Linking solutions in these domains should be carried out by three main transition matrices:

\[
\begin{pmatrix}
u_+(z) \\ v_-(z)
\end{pmatrix} = \left( \begin{array}{cc} 0 & 1 \\ 1 + O(k^{-1}) & i + O(k^{-1}) \end{array} \right) \left( \begin{array}{cc} 0 & \exp(kS_{j,l}(\lambda)) \\ \exp(-kS_{j,l}(\lambda)) & 0 \end{array} \right) \times
\left( \begin{array}{cc} 0 & 1 \\ 1 + O(k^{-1}) & i + O(k^{-1}) \end{array} \right) \left( \begin{array}{c} u_+(z) \\ u_-(z) \end{array} \right),
\]

where the functions \( S_{j,l} \) defined in (14). Here \( u_+(z) \) and \( u_-(z) \) form a fundamental system of solutions in the basic domain containing point \( b \), while \( v_+(z) \) and \( v_-(z) \) form a fundamental system of solutions with asymptotics (18) in the basic domain containing the point \( a \). Using this relation for calculating the characteristic determinant we get (3). Similarly (or even more simpler), the formulas (18) can be obtained.

3. Examples

To illustrate the results obtained above let us consider two examples.

**Example 1.** (cf. [11]). *The model problem for plane Couette flow.*

\[-y'' + k^2 i(z - \lambda)y = 0, \quad y(-1) = y(1) = 0. \tag{19}\]

In this case \( P(z, \lambda) = i(z - \lambda) \). For each \( \lambda \in \mathbb{C} \) there is the only single turning point \( z_1 = \lambda \). One can explicitly calculate

\[S(z_1, z; \lambda) = \int_{\lambda}^{z} \frac{1}{i(\zeta - \lambda)} d\zeta = \frac{2}{3} e^{i\pi/4}(z - \lambda)^{3/2}.\]

For each \( \lambda \) the Stokes Graph consists of a single complex which Stokes lines which are straight-line rays starting from \( \lambda \):

\[\Gamma(\lambda) = \left\{ \zeta = \lambda + re^{i\psi_k} \mid r \geq 0, \quad \psi_k = \frac{\pi}{6} + \frac{2\pi k}{3}, \quad k = 0, 1, 2 \right\}.\]

From this, in particular, we get the absence of essential singular curves in the limit spectral set. The location of critical and essential critical curves is illustrated on fig.1. Here \( B = -i/\sqrt{3} \) is the intersection point of the critical curves.

Basing on the nature of balanced curves we get that \pm 1 should not only belong to common canonical domain, but furthermore to one basic domain which in our case is valid only in domain \( D \), which lies below all critical curves:

\[D = \left\{ \lambda \in \mathbb{C} \mid \lambda = -i/\sqrt{3} + \xi, \quad \arg \xi \in \left(-\frac{5\pi}{6}, \frac{\pi}{6}\right) \right\}.\]
and the equation for balanced curves has the form: \( \text{Re}\, S(\lambda, 1; \lambda) = \text{Re}\, S(\lambda, -1; \lambda), \lambda \in D. \) Its solution is the infinite interval \( \gamma_\infty = (-i/\sqrt{3}; -i\infty). \)

Eigenvalues of (19) for some sufficiently large \( k \) and the limit spectral set are shown on fig.2.

**Example 2.** *The model problem for plane Couette–Poiseuille flow.*

Consider the problem:

\[
-y'' + k^2 i(z^2 - \frac{1}{2}z - \lambda)y = 0, \quad y(-1) = y(1) = 0.
\]

Now \( P(z, \lambda) = i(z^2 - z/2 - \lambda). \) Both turning points \( z_{1,2} = 1/4 \pm \sqrt{1/4 + 4\lambda} \) are simple when \( \lambda \neq -1/16. \)
Figure 3. Example 2. Limit spectral set (red); examples of singular and critical curves which are not essential singular and not essential critical (black).

Figure 4. Example 2. Location of ±1 with respect to Stokes Graph.
The equation for singular curves is as follows:

\[ \text{Re} \int_{z_1}^{z_2} (i(z_1 - z_2) (\zeta_z - z_2))^{1/2} d\zeta = 0. \]

It is equivalent to the equation \( \text{Im} e^{\pi i/4} (z_2 - z_1)^2 = 0 \), i.e. \( \text{Im} e^{\pi i/4} (1/4 + 4\lambda) = 0 \). The singular curve is a straight line in \( \mathbb{C} \): \( \text{Im} \lambda + \text{Re} \lambda + 1/16 = 0 \).

Let us multiply the basic equation (20) by \( \overline{y} \) and integrate the obtained equation from -1 to 1. Then, separating the real and imaginary parts, we obtain:

\[ \text{Im} \lambda = -\frac{1}{k^2} \langle y', y' \rangle \subset (\infty, 0), \quad \text{Re} \lambda = \frac{\langle (x^2 - x/2)y, y \rangle}{\langle y, y \rangle} \subset (-1/16, 3/2). \]

This analysis allows us to narrow the domain \( G \) where we study the spectrum. In our case it is reduced from the whole plane to the half-strip:

\[ G = \left\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \in (-1/16, 3/2), \; \text{Im} \lambda < 0 \right\}. \]

Figure 3 illustrates the limit spectral set (in red): essential singular curve (\( a \)), essential critical curves: (\( b \), \( c \), \( d \)) and the balanced curve (\( e \)). We also put there an example of singular, but not essential singular curve (\( a' \)) and critical, but not essential critical curve (\( b' \)): along these curves \( \pm 1 \) are linked with respect to all Stokes complexes, for this reason they are not contained in the limit spectral set.

Figure 4 illustrates the location of \( \pm 1 \) with respect to Stokes Graph. Evidently \( \pm 1 \) are not linked in the case (\( a \)). In the case (\( b \)) \( \pm 1 \) are linked with respect to the left complex, but not linked with respect to the right one. On the contrary in the case (\( c \)) \( \pm 1 \) are not linked with respect to the left complex, but linked with respect to the right one. In the case (\( d \)) \( \pm 1 \) are not linked with respect to the upper complex, but linked with respect to the lower one.

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