LOCAL CONVEXITY ESTIMATES FOR MEAN CURVATURE FLOW

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ABSTRACT. We develop a local version of Huisken–Stampacchia iteration, using it to obtain local versions of a host of important sharp curvature pinching estimates for mean curvature flow. The local estimates we obtain do not depend on the quality of noncollapsing of the solution and the method adopted applies in a host of other settings.

Developing a quantitative structure theory for singularities in geometric flows is a fundamental problem, since their emergence prevents the flow from reaching an equilibrium state, and hence obstructs many desired applications. One powerful tool for analyzing singularity formation in extrinsic geometric flows is Huisken–Stampacchia iteration, which is the main tool in the proof of various important curvature pinching estimates that improve at the onset of singularities. For mean curvature flow, these a priori estimates imply that a compact, strictly $(m+1)$-convex hypersurface evolving by mean curvature becomes weakly convex and either becomes strictly $m$-convex or forms a high quality $m$-neck in regions of sufficiently high curvature \cite{5,8,11} (see \cite{4} for a different approach for embedded flows). This provides a powerful description of singularity formation; however, there remains the major drawback that the estimates depend on global data from the initial (compact) hypersurface, whereas singularity formation is a local phenomenon. This prevents application of the estimates to noncompact solutions. It also prevents iteration of the estimates in a neighbourhood of a singularity, which seems to be a basic requirement for extending the Huisken–Sinestrari surgery algorithm for 2-convex hypersurfaces \cite{8} to weaker intermediate convexity conditions.

We will localize these pinching estimates by introducing suitable cutoff functions and developing a local version of the Huisken–Stampacchia method. Fixing data $n \in \mathbb{N} \setminus \{1\}, m \in \{0, \ldots, n-1\}, \alpha > 0, \Theta < \infty$.
$V < \infty$, $q > 0$, $\lambda > 0$ and $\delta \in (0, \lambda)$ such that $\sqrt{q} \Theta \geq (\lambda - \delta)^{-1}$ and $\Theta V \geq 1$, and a scale parameter $R > 0$, our main result may be stated as follows.

**Theorem 1** (Local convexity and cylindrical estimates). If a mean curvature flow is properly defined in $B_{\lambda R} \times \left(0, \frac{1}{2n} R^2\right) \subset \mathbb{R}^{n+1} \times \mathbb{R}$ and satisfies

\[
\begin{align*}
-\inf_{B_{\lambda R} \times \left\{0\right\}} & \frac{\kappa_1 + \cdots + \kappa_{m+1}}{|A|} \geq \alpha > 0, \\
-\sup_{B_{\lambda R} \times \left\{0\right\} \cup B_{\lambda (\delta) R} \times \left(0, \frac{1}{2n} R^2\right)} \frac{1}{n} H \leq \Theta R^{-1}, \text{ and} \\
-\frac{R^2}{2n} \int_{B_{\lambda R}} H^q \, d\mu_0 + \delta^{-2} \int_0^1 \int_{B_{\lambda R} \setminus B_{\lambda (\delta) R}} H^q \, d\mu \, dt \leq \left(\Theta R^{-1}\right)^q (VR)^{n+2},
\end{align*}
\]

then, given any $\varepsilon > 0$ and $\vartheta \in (0, 1)$, it satisfies the estimates

\[
\kappa_1 \geq -\varepsilon H - C_\varepsilon R^{-1}
\]

and

\[
\sum_{j=m+1}^n (\kappa_n - \kappa_j) \leq \sum_{j=1}^m \kappa_j + \varepsilon H + C_\varepsilon R^{-1}
\]

in $B_{\lambda (\delta) \vartheta R} \times \left(0, \frac{1}{2n} R^2\right)$, where $C_\varepsilon = c(n, \alpha, q, \varepsilon) \Theta \left[\frac{\Theta V}{1-\vartheta}\right]^\frac{2}{q}$.

**Remark 1.**

- The hypotheses of the theorem are very close in nature to those of the aforementioned global estimates; however, since the estimates are local, appropriate conditions must also be assumed at the spatial boundary. Cf. [13].
- Taking $\lambda$ sufficiently large, we see that the (global) pinching estimates for mean curvature flow are an immediate consequence of Theorem 1.
- Local estimates of a similar nature have been obtained in [4] (cf. [13]) assuming a noncollapsing condition (in the sense of [11]; however, the mean curvature flow in Theorem 1 need not be noncollapsing, nor even embedded; it need only be proper, which we take to mean that the pair $(X, t) : M \times (t_i, t_f) \to \mathbb{R}^{n+1} \times \mathbb{R}$ forms a proper map with respect to the subset $B_{\lambda R} \times \left(-\frac{1}{2n} R^2, 0\right) \subset \mathbb{R}^{n+1} \times \mathbb{R}$, where $X : M \times (t_i, t_f) \to \mathbb{R}^{n+1}$ is a parametrization for the flow and $t : M \times (t_i, t_f) \to \mathbb{R}$ is the projection onto the second (time) factor. Moreover, the method adopted here can be applied in other contexts, such as flows by nonlinear speeds, high codimension mean curvature flow, or free-boundary mean curvature flow.
– The estimates can be localized in more general open subsets of \( \mathbb{R}^{n+1} \), or parabolically open subsets of \( \mathbb{R}^{n+1} \times \mathbb{R} \), by exploiting different cut-off functions.
– Due to the pointwise curvature bound of the second hypothesis in Theorem 1, the integral curvature bound of the third hypothesis may be replaced by an area bound of the form

\[
\frac{R^2}{2n} \int_{B_{\lambda R}} d\mu_0 + \delta^{-2} \int_0^t \int_{B_{\lambda R} \setminus B_{(\lambda - \delta) R}} d\mu_t \, dt \leq (VR)^{n+2}.
\]

In fact, as can be easily deduced from the proof, each of these hypotheses need only be made in regions where \( \varepsilon \)-pinching fails.
– Even in the global setting, Theorem 1 provides a more precise accounting of the dependence of the constant \( C_\varepsilon \) on the boundary data.
– By applying maximum principle type arguments similar to those of [8, Theorems 6.1 and 6.3], Theorem 1 yields corresponding local derivative estimates for the second fundamental form \( A \), so long as \( m < \frac{2(n-1)}{3} \), and therefore also a corresponding local neck detection lemma (cf. [8, Lemma 7.4]). This makes a local surgery algorithm possible when \( m = 1 \) (and \( n \geq 3 \)).

**Proof of Theorem 1** Let \( G \) be given either by

\[
G \doteq -\kappa_1 \quad \text{or by} \quad G \doteq \kappa_n - \frac{1}{n-m} H
\]

and set, for any \( \varepsilon > 0 \) and \( \sigma \in (0,1) \),

\[
G_\varepsilon \doteq G - \varepsilon (H - \frac{\sigma}{2} |A|), \quad G_{\varepsilon, \sigma} \doteq G_\varepsilon H^{\sigma-1} \quad \text{and} \quad G_{\varepsilon, \sigma+, \doteq} \max \{G_{\varepsilon, \sigma}, 0\}.
\]

Well known calculations then show that

\[
(3) \quad \frac{(\partial_t - \Delta)G_{\varepsilon, \sigma}}{G_{\varepsilon, \sigma}} \leq \sigma |A|^2 - \gamma \frac{\nabla |A|^2}{H^2} + \gamma^{-1} \frac{\nabla G_{\varepsilon, \sigma}}{G_{\varepsilon, \sigma}^2}
\]

in \( B_\lambda \times (0, \frac{1}{2n}) \cap \text{spt } G_{\varepsilon, \sigma} \) in the distributional sense\(^2\), where \( \gamma = \gamma(n, \alpha, \varepsilon) > 0 \) (see, for example, [2, Proposition 12.9] and [11, Section 3]).

Given any \( \zeta \in C^\infty_0(B_\lambda) \), set \( \psi \doteq \zeta \circ X \). Setting \( v \doteq G_{\varepsilon, \sigma+, \psi} \) we then obtain

\[
(\partial_t - \Delta)\frac{\psi^2 v^2}{\psi^2 v^2} = 2 \frac{(\partial_t - \Delta)\psi}{\psi} - 2 \frac{\nabla \psi |A|^2}{\psi^2} + \frac{p(\partial_t - \Delta)G_{\varepsilon, \sigma}}{G_{\varepsilon, \sigma}^2} - 4 \frac{p-1}{p} \frac{\nabla v |A|^2}{v^2} - 8 \left\langle \frac{\nabla \psi}{\psi}, \frac{\nabla v}{v} \right\rangle
\]

\(^2\)In what follows, all differential inequalities are intended in the distributional sense.
wherever $\psi v > 0$. Applying Young’s inequality to the final term and recalling (3) we obtain, for $p \geq 6(1 + \gamma^{-1})$,

$$
\frac{(\partial_t - \Delta)\psi^2 v^2}{\psi^2 v^2} \leq 2 \frac{(\partial_t - \Delta)\psi}{\psi} + 6 \frac{\nabla\psi^2}{\psi^2} - \left(2 - \frac{4}{p}\right) \frac{\nabla v^2}{v^2}
$$

$$
+ p \left(\sigma |A|^2 - \gamma \frac{|\nabla A|^2}{H^2} + \frac{4}{\gamma p^2} \frac{|\nabla v|^2}{v^2}\right).
$$

Fix $r$ and $R$ so that $0 < r < R \leq \lambda$. If we choose the function $\zeta : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ so that

1. $\zeta(X) = 0$ when $X \notin B_R$ and $\zeta(X) = 1$ when $X \in B_r$,
2. $|D_i \zeta|^2 \leq 10(R - r)^{-2} \zeta$ for each $i$, and
3. $|D_i D_j \zeta| \leq 10(R - r)^{-2}$ for each $i$ and $j$,

then

$$
(\partial_t - \Delta)\psi \leq 10n(R - r)^{-2} \chi_{B_R \setminus B_r} \quad \text{and} \quad \frac{\nabla\psi^2}{\psi} \leq 10n(R - r)^{-2} \chi_{B_R \setminus B_r},
$$

where $\chi_{B_R \setminus B_r}$ denotes the characteristic function of the set $B_R \setminus B_r$ (pulled back by the flow parametrization $X$).

We thus obtain

$$
\frac{(\partial_t - \Delta)\psi^2 v^2}{\psi^2 v^2} \leq \frac{100n \chi_{B_R \setminus B_r}}{(R - r)^2} - 2 \left(1 - \frac{2 + 2\gamma^{-1}}{p}\right) \frac{\nabla v^2}{v^2}
$$

$$
- \gamma p \frac{|\nabla A|^2}{H^2} + \sigma p |A|^2
$$

$$
\leq \frac{100n \chi_{B_R \setminus B_r}}{(R - r)^2} - \frac{4}{3} \frac{|\nabla v|^2}{v^2} - \gamma p \frac{|\nabla A|^2}{H^2} + \sigma p |A|^2
$$

(4)

wherever $\psi v > 0$.

The $L^2$-estimate. Applying (4) and the gradient flow property of mean curvature flow, we obtain

$$
\frac{d}{dt} \int \psi^2 v^2 d\mu + \int \psi^2 v^2 H^2 d\mu = \int \partial_t (\psi^2 v^2) d\mu
$$

$$
\leq \int \psi^2 v^2 \left(\sigma p |A|^2 - \frac{4}{3} \frac{|\nabla v|^2}{v^2} - \gamma p \frac{|\nabla A|^2}{H^2}\right) d\mu
$$

$$
+ \frac{100n}{(R - r)^2} \int_{B_R \setminus B_r} \psi v^2 d\mu.
$$

(5)
We now apply the Poincaré-type inequality \( [11, \text{Proposition 2.7}] \) to \( u = \psi v \). This yields, for any \( \beta > 0 \),
\[
\int \psi^2 v^2 |A|^2 d\mu \leq \beta \int \psi^2 v^2 \left| \frac{\nabla \psi}{\psi} + \frac{\nabla v}{v} \right|^2 d\mu + P(1 + \beta^{-1}) \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2} d\mu \\
\leq 2\beta \int (\psi^2 |\nabla v|^2 + v^2 |\nabla \psi|^2) d\mu \\
+ P(1 + \beta^{-1}) \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2} d\mu,
\]
where \( P = P(n, \alpha, \varepsilon) \). Setting \( \beta = p^{-\frac{1}{2}} \) and recalling (5), we find that
\[
\frac{d}{dt} \int \psi^2 v^2 d\mu \leq \frac{(100 + 20\sigma p^{-\frac{1}{2}})n}{(R - r)^2} \int_{B_R \setminus B_r} \psi v^2 d\mu \\
+ \left( 2\sigma p^{-\frac{1}{2}} - \frac{4}{3} \right) \int \psi^2 |\nabla v|^2 d\mu \\
+ \left( \sigma P(1 + p^{-\frac{1}{2}}) - \gamma \right) p \int \psi^2 v^2 \frac{|\nabla A|^2}{H^2} d\mu.
\]
Choosing \( p \geq L \) and \( \sigma \leq \ell p^{-\frac{1}{2}} \), where \( \ell = \ell(n, \alpha, \varepsilon) \leq 4 \) and \( L = L(n, \alpha, \varepsilon) \), yields
\[
\frac{d}{dt} \int \psi^2 v^2 d\mu \leq \frac{200n}{(R - r)^2} \int_{B_R \setminus B_r} \psi v^2 d\mu.
\]
If we choose \( R = \lambda \) and \( r = \lambda - \delta \), then, integrating in time, we find that
\[
\sup_{t \in (0, \frac{1}{2n} R^2)} \int_{B_{\lambda - \delta}} v^2 d\mu \leq \int_{B_{\lambda}} v^2 d\mu_0 + 200n\delta^{-2} \int_{B_{\lambda} \setminus B_{\lambda - \delta}} \int v^2 d\mu_i dt,
\]
and hence
\[
\int_{B_{\lambda - \delta}} v^2 d\mu dt \leq \frac{1}{2n} \int_{B_{\lambda}} v^2 d\mu_0 + 100\delta^{-2} \int_{B_{\lambda} \setminus B_{\lambda - \delta}} \int v^2 d\mu_i dt,
\]
so long as \( p \geq L \) and \( \sigma \leq \ell p^{-\frac{1}{2}} \).

**From \( L^2 \) to \( L^\infty \).** Stampacchia iteration will now allow us to pass from \( L^2 \) to \( L^\infty \). We assume the reader is familiar with [13, Section 5] or [10, Chapter II: Appendices B and C].

Given \( k \geq k_0 \equiv \Theta^\sigma = \sup_{B_{\lambda} \setminus B_{\lambda - \delta}} G_{\varepsilon, \sigma} \) and \( R \leq \lambda - \delta \), consider
\[
v_k^2 \equiv (G_{\varepsilon, \sigma} - k)^{p_+} \quad \text{and} \quad A_{k,R} \equiv \{(x,t) \in X^{-1}(B_R) : v_k(x,t) > 0\}
\]
and set
\[ u(k, R) \doteq \int \int_{A_{k, R}} v_k^2 \, d\mu_t \, dt \quad \text{and} \quad a(k, R) \doteq \int \int_{A_{k, R}} d\mu_t \, dt. \]

Note that, for any \( h \geq k > 0 \) and any \( 0 < r \leq R \leq \lambda - \delta \),
\[(h - k)^p a(h, r) \leq u(k, r). \tag{8}\]

We need an estimate for \( u(k, r) \). First observe that, computing as in (5), we can estimate
\[
\frac{d}{dt} \int \psi^2 v_k^2 \, d\mu + \int_{A_{k, r}} |\nabla v_k|^2 \, d\mu + \int_{A_{k, r}} v_k^2 H^2 \, d\mu \leq \frac{100n}{(R - r)^2} \int_{A_{k, R}} v_k^2 \, d\mu 
+ \sigma p \int_{A_{k, R}} G_{\varepsilon, \sigma}^p |A|^2 \, d\mu
\]
for any \( k > 0 \) and \( r < R \leq \lambda - \delta \), where \( \psi \) is a cut-off function satisfying \( \psi \equiv 1 \) on \( B_r \) and \( \psi \equiv 0 \) outside of \( B_R \). On the other hand, the Sobolev inequality of Michael and Simon [12, Theorem 2.1] yields
\[
\left( \int_{A_{k, r}} v_k^{2^*} \, d\mu \right)^{\frac{2}{2^*}} \leq c_S \int_{A_{k, r}} \left( |\nabla v_k|^2 + H^2 v_k^2 \right) \, d\mu,
\]
where for \( n \geq 3 \), \( \frac{1}{2^*} = \frac{1}{2} - \frac{1}{n} \) and \( c_S \) depends only on \( n \), so that
\[
\frac{d}{dt} \int \psi^2 v_k^2 \, d\mu + \frac{1}{c_S} \left( \int_{A_{k, r}} v_k^{2^*} \, d\mu \right)^{\frac{2}{2^*}} \leq \frac{100n}{(R - r)^2} \int_{A_{k, R}} v_k^2 \, d\mu 
+ \sigma p \int_{A_{k, R}} G_{\varepsilon, \sigma}^p |A|^2 \, d\mu.
\]
Integrating with respect to \( t \) then yields
\[
\sup_{t \in (0, \frac{1}{2n} R^2)} \int_{A_{k, r}} v_k^2 \, d\mu + \int \left( \int_{A_{k, r}} v_k^{2^*} \, d\mu \right)^{\frac{2}{2^*}} \, dt 
\leq \frac{100nc_S}{(R - r)^2} \int_{A_{k, R}} v_k^2 \, d\mu \, dt + c_S \sigma p \int \int_{A_{k, R}} G_{\varepsilon, \sigma}^p |A|^2 \, d\mu \, dt.
\]
By the interpolation inequality,
\[
\int_{A_{k, r}} v_k^{\frac{2(n+2)}{n}} \, d\mu \leq \left( \int_{A_{k, r}} v_k^2 \, d\mu \right)^{\frac{2}{n}} \left( \int_{A_{k, r}} v_k^{2^*} \, d\mu \right)^{\frac{2}{2^*}}.
\]

\(^3\)We can interpret the left hand side as the square of the \( L^\infty \)-norm when \( n = 2 \) with \( 2^* \) any fixed number bigger than one and the constant \( c_S \) depending additionally on \( 2^* \) and the measure of the support of \( v_k \). Cf. [5].
and hence, by Young’s inequality,

\[
\left(\int\int_{A_{k,r}} v_k^\frac{2(n+2)}{n} \, d\mu \, dt\right)^\frac{n}{n+2} \leq \left(\sup_{t \in (0, \frac{1}{2} R^2)} \int_{A_{k,r}} v_k^2 \, d\mu \right)^\frac{2}{n+2} \left(\int\int_{A_{k,r}} v_k^\frac{2\sigma}{n} \, d\mu \, dt\right)^\frac{\frac{n}{n+2}}{2}.
\]

\[
\leq \frac{2}{n+2} \sup_{t \in (0, \frac{1}{2} R^2)} \int_{A_{k,r}} v_k^2 \, d\mu + \frac{n}{n+2} \int\int_{A_{k,r}} v_k^\frac{2\sigma}{n} \, d\mu \, dt.
\]

Thus,

\[
\left(\int\int_{A_{k,r}} v_k^\frac{2(n+2)}{n} \, d\mu \, dt\right)^\frac{n}{n+2} \leq \frac{100 n c_S}{(R-r)^2} \int\int_{A_{k,R}} v_k^2 \, d\mu \, dt
\]

\[
+ c_S \sigma p \int\int_{A_{k,R}} G^p_{\varepsilon,\sigma} |A|^2 \, d\mu \, dt.
\]

(9)

Applying Hölder’s inequality and (choosing \(\ell\) slightly smaller\(^4\)), we estimate, for \(\sigma' \equiv \sigma + \frac{2}{p}\) and some soon-to-be-determined \(\rho \geq 1\),

\[
\int\int_{A_{k,R}} H^2 G^p_{\varepsilon,\sigma'} \, d\mu \, dt = \int\int_{A_{k,R}} G^p_{\varepsilon,\sigma'} \, d\mu \, dt
\]

\[
\leq a(k, R)^{1-\frac{1}{p}} \left(\int\int_{A_{k,R}} G^p_{\varepsilon,\sigma'} \, d\mu \, dt\right)^\frac{1}{p}
\]

\[
\leq a(k, R)^{1-\frac{1}{p}} \left(\int\int_{B_{\lambda-\delta}} G^p_{\varepsilon,\sigma'} \, d\mu \, dt\right)^\frac{1}{p}.
\]

(10)

Similarly, we may estimate, for any \(k > k_0\) and \(R < \lambda - \delta\),

\[
u(k, R) \leq a(k, R)^{1-\frac{1}{p}} \left(\int\int_{B_{\lambda-\delta}} G^p_{\varepsilon,\sigma'} \, d\mu \, dt\right)^\frac{1}{p}.
\]

(11)

Finally, we estimate

\[
u(k, r) \leq a(k, r)^{\frac{2}{n+2}} \left(\int\int_{A_{k,r}} v_k^\frac{2(n+2)}{n} \, d\mu \, dt\right)^\frac{n}{n+2}.
\]

(12)

\(^4\)Depending now also on \(\rho\), which will be fixed momentarily.
Since $|A| \leq \alpha^{-1}H$ and $G \leq H$, combining (8)–(12) and the $L^2$-estimate (7) yields

\[(h - k)^p (R - r)^2 a(h, r) \leq c a(k, R)^\gamma (1 + \sigma p \Theta^2 (R - r)^2).\]

\[
\left( \frac{1}{2n} \int_{B_\lambda} G_{\varepsilon, \sigma_+}^{pp} d\mu_0 + \frac{100}{\delta^2} \int \int_{B_\lambda \setminus B_{\lambda - \delta}} G_{\varepsilon, \sigma_+}^{pp} d\mu dt \right)^{\frac{1}{p}}
\leq c a(k, R)^\gamma (1 + \sigma p \Theta^2 (R - r)^2).
\]

\[
\left( \frac{1}{2n} \int_{B_\lambda} H_{\sigma p}^{\sigma p} d\mu_0 + \frac{100}{\delta^2} \int \int_{B_\lambda \setminus B_{\lambda - \delta}} H_{\sigma p}^{\sigma p} d\mu dt \right)^{\frac{1}{p}}
\leq c a(k, R)^\gamma (1 + \sigma p \Theta^2 (\lambda - \delta)^2) \Theta^{\sigma p (1 - \frac{1}{p})} \Lambda^{\frac{\sigma p}{p}}.
\]

so long as $p \geq L(n, \alpha, \varepsilon, \rho)$ and $\sigma \leq \ell(n, \alpha, \varepsilon, \rho)^{-\frac{1}{2}}$, where $c = c(n, \alpha, \rho)$, $\gamma = 1 + \frac{2}{n+2} - \frac{1}{\rho}$, and

\[
\Lambda \triangleq \left( \frac{1}{2n} \int_{B_\lambda} H_{\sigma p}^{\sigma p} d\mu_0 + \frac{100}{\delta^2} \int \int_{B_\lambda \setminus B_{\lambda - \delta}} H_{\sigma p}^{\sigma p} d\mu dt \right)^{\frac{1}{\sigma p}}.
\]

At this point, we fix some $\rho = \rho(n) > 1 + \frac{2}{\sigma}$ (so that $\gamma = \gamma(n) > 1$) and choose $p = p(n, \alpha, \varepsilon, q) \geq L$ and $\sigma = \sigma(n, \alpha, \varepsilon, q) \leq \ell p^{-\frac{1}{2}}$ such that $\sigma p = q$. Stampacchia’s Lemma [15, Lemma 5.1] then yields

\[a(k_0 + d, \vartheta R_0) = 0,\]

where $R_0 \triangleq \lambda - \delta$ and

\[d^p \triangleq \frac{2^{(p+2)\gamma}}{\gamma} c (1 + \sigma p \Theta^2 (\lambda - \delta)^2) \Theta^{\sigma p (1 - \frac{1}{p})} \Lambda^{\frac{\sigma p}{p}} a(k_0, R_0)^{-\gamma}.\]

We may estimate, using the $L^2$-estimate (7),

\[a(k_0, R_0) \leq k^{-p} \int \int_{U_{R_0}} G_{\varepsilon, \sigma_+}^{pp} d\mu dt \leq k^{-p} \left( \frac{1}{2n} \int_{B_\lambda} G_{\varepsilon, \sigma_+}^{pp} d\mu_0 + \frac{100}{\delta^2} \int \int_{B_\lambda \setminus B_{\lambda - \delta}} G_{\varepsilon, \sigma_+}^{pp} d\mu dt \right) \leq k^{-p} \Lambda^{\sigma p}.\]
Since we chose $k_0 = \Theta^\sigma$ and $\sigma p = q$, we thus obtain

\[
G_{\varepsilon,\sigma} \leq k_0 + d
\]

\[
= k_0 \left( 1 + 2^{(p+2)/2} \left[ \frac{c(1 + \sigma p R_0^2 \Theta^2)}{(1 - \vartheta)^2 R_0^2} \right]^{\frac{1}{p}} \frac{\Theta^\sigma (1 - \frac{1}{2}) \Lambda^{\frac{1}{p+2} - \frac{1}{2}}}{k_0^{1 - \frac{1}{p}} \frac{1}{k_0^{n+2}}} \right)
\]

\[
\leq \Theta^\sigma \left( 1 + c(n, \alpha, q, \varepsilon) \left[ \frac{(\lambda - \delta)^{-2} + q \Theta^2}{(1 - \vartheta)^2 V^2} \right]^{\frac{1}{p}} \right)
\]

in $B_{\vartheta(\lambda - \delta)}$. Young’s inequality then yields

\[
G \leq 2\varepsilon H + c(n, \alpha, q, \varepsilon) \Theta \left( \Theta V \left( \frac{1}{1 - \vartheta} \right) \right)^{\frac{1}{p}}
\]

in $B_{\vartheta(\lambda - \delta)}$. This completes the proof of Theorem 1. \qed

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