On volume functions of special flow polytopes associated to the root system of type $A$

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Abstract

In this paper, we consider the volume of a special kind of flow polytope. We show that its volume satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type $A$.

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1 Introduction

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [4]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [9]).

We consider a flow polytope associated to the root system of type $A$. As explained in [2, 3], the cone spanned by the positive roots is divided into several polyhedral cones called chambers, and the combinatorial property of a flow polytope depends on a chamber. Moreover, there is a specific chamber called the nice chamber, which plays a significant role in [11]. In this paper, we call a flow polytope for the nice chamber a special flow polytope. Also in [2, 3], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([3, Lemma 21]). We also refer to [1] for similar formulas for other chambers in more general settings. Moreover, we
mention that a generalization of the Lidskii formula is shown in [3, Theorem 38], there
is a geometric proof of the Lidskii formula in [12], and combinatorial applications of this
formula are given in [5, 7].

The purpose of this paper is to characterize the volume function of a flow polytope for
the nice chamber in terms of a system of differential equations, based on a result in [3]. In
order to state the main results, we give some notation. Let \( e_1, \ldots, e_{r+1} \) be the standard
basis of \( \mathbb{R}^{r+1} \) and let
\[
A_r^{+} = \{ e_i - e_j \mid 1 \leq i < j \leq r + 1 \}
\]
be the positive root system of type \( A \) with rank \( r \). We assign a positive integer \( m_{i,j} \) to
each \( i \) and \( j \) with \( 1 \leq i < j \leq r + 1 \). Let us set \( m = (m_{i,j}) \) and \( M = \sum_{1 \leq i < j \leq r+1} m_{i,j} \).
For \( a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1} \), where \( a_i \in \mathbb{R}_{\geq 0} \) \((i = 1, \ldots, r)\), the
following polytope \( P_{A_r^+, m} (a) \) is called the flow polytope associated to the root system of
type \( A \):
\[
P_{A_r^+, m} (a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \left| \begin{array}{l}
1 \leq i < j \leq r + 1, 1 \leq k \leq m_{i,j}, y_{i,j,k} \geq 0, \\
\sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a
\end{array} \right. \right\}.
\]
Note that the flow polytopes in [3] include the case that some of \( m_{i,j} \)'s are zero, whereas
we exclude such cases in this paper. We denote the volume of \( P_{A_r^+, m} (a) \) by \( v_{A_r^+, m} (a) \).

The open set
\[
\mathcal{C}_{\text{nice}} := \{ a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in \mathbb{R}^{r+1} \mid a_i > 0, i = 1, \ldots, r \}
\]
in \( \mathbb{R}^{r+1} \) is called the nice chamber. We are interested in the volume \( v_{A_r^+, m} (a) \) when \( a \) is in
the closure of the nice chamber, and then it is written by \( v_{A_r^+, m, \mathcal{C}_{\text{nice}}} \). It is a homogeneous
polynomial of degree \( M - r \). The first result of this paper is the following.

**Theorem 1.** Let \( a = \sum_{i=1}^r a_i (e_i - e_{r+1}) \in \mathcal{C}_{\text{nice}} \), and let \( v_{A_r^+, m, \mathcal{C}_{\text{nice}}} (a) \) be the volume of
\( P_{A_r^+, m} (a) \). Then \( v = v_{A_r^+, m, \mathcal{C}_{\text{nice}}} (a) \) satisfies the system of differential equations as follows:
\[
\begin{cases}
\partial_i^{m_{i,r+1}} v = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\
\vdots \\
(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0,
\end{cases}
\]
where \( \partial_i = \frac{\partial}{\partial a_i} \) for \( i = 1, \ldots, r \). Conversely, the polynomial \( v = v(a) \) of degree \( M - r \)
satisfying the above equations is equal to a constant multiple of \( v_{A_r^+, m, \mathcal{C}_{\text{nice}}} (a) \).

We remark that it is known that the volume function \( v_{A_r^+, m} (a) \) of \( P_{A_r^+, m} (a) \), as a
distribution on \( \mathbb{R}^r \), satisfies the differential equation
\[
Lv_{A_r^+, m} (a) = \delta(a)
\]
in general, where \( L = \prod_{1 \leq i < j} (\partial_i - \partial_j)^{m_{i,j}} \) and \( \delta(a) \) is the Dirac delta function on \( \mathbb{R}^r \) ([8, 11]). Note that \( \partial_{r+1} \) in the definition of \( L \) is supposed to be zero. The above theorem

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characterizes the function $v_{A^+_r,m\cdot c_{\text{nice}}} (a)$ on $c_{\text{nice}}$ more explicitly. It might be interesting to see what kind of properties of the volume can be derived from Theorem 1.

In addition, in Theorem 20, we show the volume $v_{A^+_r,m',c_{\text{nice}}} (a')$ is written by a linear combination of $v_{A^+_{r-1},m',c_{\text{nice}}} (a')$ and its partial derivatives, where $m' = (m_{i,j})_{2 \leq i < j \leq r+1}$, $c'_{\text{nice}}$ is the nice chamber of $A^+_{r-1}$, and $a' = \sum_{t=2}^{r+1} a_i (e_i - e_{r+1}) \in c'_{\text{nice}}$. It might be interesting to ask whether there is a relation between this theorem and the inductive formulas of Schmidt–Bincer [13, (4.1), (4.24)].

This paper is organized as follows. In Section 2, we recall the iterated residue, the Jeffrey-Kirwan residue, and the nice chamber based on [2], [3], [6] and [10]. Also, we give some examples of $P_{A^+_r,m}(a)$ and the calculations of the volume $v_{A^+_r,m\cdot c_{\text{nice}}} (a)$. In Section 3, we prove the main theorems.

2 Preliminaries

In this section, we set up the tools to prove the main theorems based on [2], [3], [6] and [10].

2.1 Flow polytopes and its volumes

Let $e_1, \ldots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$, and let

$$V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \left| \sum_{i=1}^{r+1} a_i = 0 \right. \right\}.$$ 

We consider the positive root system of type $A$ with rank $r$ as follows:

$$A^+_r = \{ e_i - e_j \mid 1 \leq i < j \leq r+1 \}.$$ 

Let $C(A^+_r)$ be the convex cone generated by $A^+_r$:

$$C(A^+_r) = \{ a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \mid a_1, \ldots, a_r \in \mathbb{R}_{\geq 0} \}.$$ 

We assign a positive integer $m_{i,j}$ to each $i$ and $j$ with $1 \leq i < j \leq r+1$, and it is called a multiplicity. Let us set $m = (m_{i,j})$ and $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$.

**Definition 2.** Let $a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in C(A^+_r)$. We consider the following polytope:

$$P_{A^+_r,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \left| 1 \leq i < j \leq r+1, 1 \leq k \leq m_{i,j}, y_{i,j,k} \geq 0, \sum_{1 \leq i < j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a \right. \right\},$$

which is called the flow polytope associated to the root system of type $A$.

**Remark 3.** The flow polytopes in [3] include the case that $m_{i,j} = 0$ for some $i$ and $j$. 


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The elements of $A_r^+$ generate a lattice $V_Z$ in $V$. The lattice $V_Z$ determines a measure $da$ on $V$.

Let $du$ be the Lebesgue measure on $\mathbb{R}^M$. Let $[\alpha_1, \ldots, \alpha_M]$ be a sequence of elements of $A_r^+$ with multiplicity $m_{i,j}$, and let $\varphi$ be the surjective linear map from $\mathbb{R}^M$ to $V$ defined by $\varphi(\varepsilon_k) = \alpha_k$. The vector space $\ker(\varphi) = \varphi^{-1}(0)$ is of dimension $d = M - r$ and it is equipped with the quotient Lebesgue measure $du/da$. For $a \in V$, the affine space $\varphi^{-1}(a)$ is parallel to $\ker(\varphi)$, and thus also equipped with the Lebesgue measure $du/da$. Volumes of subsets of $\varphi^{-1}(a)$ are computed for this measure. In particular, we can consider the volume $v_{A_r^+, m}(a)$ of the polytope $P_{A_r^+, m}(a)$.

2.2 Total residue and iterated residue

Let $A_r = A_r^+ \cup (-A_r^+)$, and let $U$ be the dual vector space of $V$. We denote by $R_{A_r}$ the ring of rational functions $f(x_1, \ldots, x_r)$ on the complexification $U_C$ of $U$ with poles on the hyperplanes $x_i - x_j = 0$ ($1 \leq i < j \leq r + 1$) or $x_i = 0$ ($1 \leq i \leq r$). A subset $\sigma$ of $A_r$ is called a basis of $A_r$ if the elements $\alpha \in \sigma$ form a basis of $V$. In this case, we set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}$$

and call such a element a simple fraction. We denote by $S_{A_r}$ the linear subspace of $R_{A_r}$ spanned by simple fractions. The space $U$ acts on $R_{A_r}$ by differentiation: $(\partial(u)f)(x) = (\frac{d}{du})f(x + \varepsilon u)|_{\varepsilon = 0}$. We denote by $\partial(U)R_{A_r}$ the space spanned by derivatives of functions in $R_{A_r}$. It is shown in [6, Proposition 7] that $R_{A_r} = \partial(U)R_{A_r} \oplus S_{A_r}$. The projection map $\text{Tres}_{A_r} : R_{A_r} \to S_{A_r}$ with respect to this decomposition is called the total residue map.

We extend the definition of the total residue to the space $\hat{R}_{A_r}$ consisting of functions $P/Q$ where $Q$ is a finite product of powers of the linear forms $\alpha \in A_r$ and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with $P_k$ of degree $k$. As the total residue vanishes outside the homogeneous component of degree $-r$ of $A_r$, we can define $\text{Tres}_{A_r}(P/Q) = \text{Tres}_{A_r}(P_{q-r}/Q)$, where $q$ is degree of $Q$. For $a \in V$ and multiplicities $m = (m_{i,j}) \in (\mathbb{Z}_{\geq 0})^M$ of elements of $A_r^+$, the function

$$F := \prod_{i,j}^{r} \frac{\epsilon^{a_1x_1 + \cdots + a_rx_r}}{x_i^{m_{i,j} + 1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}}$$

is in $\hat{R}_{A_r}$. We define $J_{A_r^+, m}(a) \in S_{A_r}$ by

$$J_{A_r^+, m}(a) = \text{Tres}_{A_r} F.$$

Next, we describe the iterated residue.

**Definition 4.** For $f \in R_{A_r}$, we define the iterated residue by

$$\text{Ires}_{x \to 0} f = \text{Res}_{x_1 = 0} \text{Res}_{x_2 = 0} \cdots \text{Res}_{x_r = 0} f(x_1, \ldots, x_r).$$

Since the iterated residue $\text{Ires}_{x \to 0} f$ vanishes on the space $\partial(U)R_{A_r}$ as in [3], we have

$$\text{Ires}_{x \to 0} J_{A_r^+, m}(a) = \text{Ires}_{x \to 0} F.$$

(1)
2.3 Chambers and Jeffrey–Kirwan residue

Definition 5. Let $C(\nu)$ be the closed cone generated by $\nu$ for any subset $\nu$ of $A_r^+$ and let $C(A_r^+)^{\text{sing}}$ be the union of the cones $C(\nu)$ where $\nu$ is any subset of $A_r^+$ of cardinal strictly less than $r = \dim V$. By definition, the set $C(A_r^+)^{\text{reg}}$ of $A_r^+$-regular elements is the complement of $C(A_r^+)^{\text{sing}}$. A connected component of $C(A_r^+)^{\text{reg}}$ is called a chamber.

The Jeffrey–Kirwan residue [10] associated to a chamber $c$ of $C(A_r^+)$ is a linear form $f \mapsto \langle \langle c, f \rangle \rangle$ on the vector space $S_{A_r}$ of simple fractions. Any function $f$ in $S_{A_r}$ can be written as a linear combination of functions $f_\sigma$, with a basis $\sigma$ of $A_r$ contained in $A_r^+$. To determine the linear map $f \mapsto \langle \langle c, f \rangle \rangle$, it is enough to determine it on this set of functions $f_\sigma$. So we assume that $\sigma$ is a basis of $A_r$ contained in $A_r^+$.

Definition 6. For a chamber $c$ and $f_\sigma \in S_{A_r}$, we define the Jeffrey–Kirwan residue $\langle \langle c, f_\sigma \rangle \rangle$ associated to a chamber $c$ as follows:

- If $c \subset C(\sigma)$, then $\langle \langle c, f_\sigma \rangle \rangle = 1$.
- If $c \cap C(\sigma) = \emptyset$, then $\langle \langle c, f_\sigma \rangle \rangle = 0$,

where $C(\sigma)$ is the convex cone generated by $\sigma$.

Remark 7. More generally, as in [3, Definition 11], the Jeffrey–Kirwan residue $\langle \langle c, f_\sigma \rangle \rangle$ is defined to be $\frac{1}{\text{vol}(\sigma)}$ if $c \subset C(\sigma)$, where $\text{vol}(\sigma)$ is the volume of the parallelepiped $\bigoplus_{\alpha \in \sigma}[0,1] \alpha$, relative to our Lebesgue measure $da$. In our case, the volume $\text{vol}(\sigma)$ is equal to 1 since $A_r$ is unimodular.

The volume $v_{A_r^+,m}(a)$ of the flow polytope $P_{A_r^+,m}(a)$ is written by the function $J_{A_r^+,m}(a)$ and the Jeffrey–Kirwan residue in the following.

Theorem 8 (Baldoni–Vergne [3]). Let $c$ be a chamber of $C(A_r^+)$. Then, for $a \in \tilde{c}$, the volume $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$ is given by

$$v_{A_r^+,m}(a) = \langle \langle c, J_{A_r^+,m}(a) \rangle \rangle.$$ 

We denote by $v_{A_r^+,m,f}(a)$ the polynomial function of $a$ coinciding with $v_{A_r^+,m}(a)$ when $a \in \tilde{c}$. It is a homogeneous polynomial of degree $M - r$.

2.4 Nice chamber

Definition 9. The open subset $c_{\text{nice}}$ of $C(A_r^+)$ is defined by

$$c_{\text{nice}} = \{a \in C(A_r^+) \mid a_i > 0 \ (i = 1, \ldots, r)\}.$$ 

The set $c_{\text{nice}}$ is in fact a chamber for the root system $A_r^+$ ([3]). The chamber $c_{\text{nice}}$ is called the nice chamber.

Lemma 10 (Baldoni–Vergne [3]). For the nice chamber $c_{\text{nice}}$ of $A_r^+$ and $f \in S_{A_r}$, we have

$$\langle \langle c_{\text{nice}}, f \rangle \rangle = \text{Ires}_{x=0} f.$$
From Theorem 8, Lemma 10 and (1), we have the following corollary.

**Corollary 11** (Lidskii formula [3]). Let \( a \in \mathfrak{c}_{\text{nice}} \). Then the volume function \( v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}(a) \) is given by

\[
v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Res}_{x=0} F.
\]

### 2.5 Examples

In this subsection, we give some examples of the flow polytopes for \( A_1, A_2, \) and \( A_3 \), and calculate their volumes.

**Example 12.** When \( r = 1 \), the nice chamber of \( A^+_1 \) is \( \mathfrak{c}_{\text{nice}} = \{ a = a_1(e_1 - e_2) \mid a_1 > 0 \} \). For \( a = a_1(e_1 - e_2) \in \mathfrak{c}_{\text{nice}} \),

\[
P_{A^+_1,m}(a) = \{(y_{i,j,k}) \in \mathbb{R}^{m_1,2} \mid y_{i,j,k} \geq 0, \ y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,m_1,2} = a_1 \}.
\]

From Corollary 11, we have

\[
v_{A^+_1,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Res}_{x=0} \left( \frac{e^{a_1 x_1}}{x_1^{m_1,2}} \right) = \frac{1}{(m_1,2-1)!} a_1^{m_1,2-1}.
\]

**Example 13.** When \( r = 2 \), there are two chambers \( \mathfrak{c}_1, \mathfrak{c}_2 \) of \( A^+_2 \) as below, and the nice chamber \( \mathfrak{c}_{\text{nice}} \) of \( A^+_2 \) is \( \mathfrak{c}_1 \).

\[
\text{Figure 1 : The chamber of } A^+_2.
\]

For example, we set \( m_{1,2} = n \ (n \in \mathbb{Z}_{>0}), \ m_{1,3} = 1, \) and \( m_{2,3} = 1 \). For \( a = a_1 e_1 + a_2 e_2 - (a_1 + a_2) e_3 \in \mathfrak{c}_{\text{nice}} \),

\[
P_{A^+_2,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{n+2} \mid \begin{array}{l}
y_{i,j,k} \geq 0 \\
y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,n} + y_{1,3,1} = a_1 \\
y_{1,2,1} - y_{1,2,2} - \cdots - y_{1,2,n} + y_{2,3,1} = a_2
\end{array} \right\}.
\]

From Corollary 11, we have

\[
v_{A^+_2,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Res}_{x=0} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) = \text{Res}_{x=0} \text{Res}_{x=0} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) = \frac{1}{n!} a_1^n.
\]
Example 14. When \( r = 3 \), there are seven chambers of \( A_3^+ \) as below ([1]), and the nice chamber \( c_{\text{nice}} \) of \( A_3^+ \) is \( c_1 \).

For example, we set \( m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2, \) and \( m_{3,4} = 2 \). For \( a = \sum_{i=1}^{3} a_i (e_i - e_4) \in c_{\text{nice}} \),

\[
P_{A_3^+,m} (a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^9 \left| \begin{array}{c} y_{i,j,k} \geq 0 \\ y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} = a_1 \\ -y_{1,2,1} + y_{2,3,1} + y_{2,4,1} + y_{2,4,2} = a_2 \\ -y_{1,3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} = a_3 \end{array} \right. \right\}.
\]

From Corollary 11, we have

\[
v_{A_3^+,m,c_{\text{nice}}} (a) = \text{Ires}_{x=0} \left( \frac{e^{a_1 x_1 + a_2 x_2 + a_3 x_3}}{x_1^2 x_2^2 x_3^2 (x_1 - x_2) (x_1 - x_3) (x_2 - x_3)} \right) = \frac{1}{360} a_1^3 (a_1^3 + 6 a_1^2 a_2 + 3 a_1^2 a_3 + 15 a_1 a_2^2 + 15 a_1 a_2 a_3 + 10 a_2^3 + 30 a_2^2 a_3). \]

3 Main theorems

In this section, we prove the main theorems of this paper. Let \( c_{\text{nice}} \) be the nice chamber of \( A_3^+ \) and let \( a = \sum_{i=1}^{3} a_i (e_i - e_4) \in c_{\text{nice}} \).

Theorem 15. For \( a \in c_{\text{nice}} \), let \( P_{A_3^+,m} (a) \) be the flow polytope as in Definition 2 and let \( v_{A_3^+,m,c_{\text{nice}}} (a) \) be the volume of \( P_{A_3^+,m} (a) \). Then \( v = v_{A_3^+,m,c_{\text{nice}}} (a) \) satisfies the system of differential equations as follows:

\[
\begin{align*}
(\partial_{r-1} - \partial_r)^{m_{r-1},r+1} v &= 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1},r} \dot{\partial}_{r-1}^{m_{r-1},r+1} v &= 0 \\
&\vdots \\
(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \dot{\partial}_1^{m_{1,r+1}} v &= 0,
\end{align*}
\]

\[\text{(2)}\]
where \( \partial_i = \frac{\partial}{\partial a_i} \) for \( i = 1, \ldots, r \).

**Proof.** We will prove the first two relations. Let 
\[
F = \prod_{i=1}^{r} \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_i^{m_i} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}}.
\]
It is easy to see that
\[
P(\partial_1, \ldots, \partial_r)(\text{Res}_{x=0} F) = \text{Res}_{x=0}(P(\partial_1, \ldots, \partial_r) F) = \text{Res}_{x=0}(P(x_1, \ldots, x_r) F),
\]
where \( P \) is a polynomial. Since 
\[
e^{a_1 x_1 + \cdots + a_r x_r} \prod_{i=1}^{r} x_i^{m_i} \prod_{1 \leq i < j \leq k} (x_i - x_j)^{m_{i,j}}
\]
is holomorphic at \( x_k = 0 \),
\[
\text{Res}_{x_k=0} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_k^{m_k} \prod_{1 \leq i < j \leq k} (x_i - x_j)^{m_{i,j}}} \right) = 0
\]
for \( k = 1, \ldots, r \). Therefore, from Corollary 11, (3) and (4), we obtain
\[
\partial_r^{m_r+1} v = \partial_r^{m_r+1} \text{Res}_{x=0} F = \text{Res}_{x=0} \partial_r^{m_r+1} F
\]
and
\[
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_r^{m_{r-1,r+1}} v
= \text{Res}_{x=0} (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_r^{m_{r-1,r+1}} F
\]
\[
= \text{Res}_{x=0} (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_r} \prod_{i=1}^{r-2} x_i^{m_{i,r}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)
\]
\[
= \text{Res}_{x=0} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_r+1} \prod_{i=1}^{r-1} x_i^{m_{i,r}} \prod_{1 \leq i < j \leq r-1} (x_i - x_j)^{m_{i,j}}} \right)
\]
\[
= \text{Res}_{x_r=0} \cdots \left( \text{Res}_{x_{r-1}=0} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_r+1} \prod_{i=1}^{r-2} (x_i - x_j)^{m_{i,j}}} \right) \right) = 0.
\]

Similarly, we can verify the remaining expressions. \( \square \)

**Remark 16.** In general, it is known that the volume function \( v_{A_r^+, m} \) of \( P_{A_r^+, m} \), as a distribution on \( V \), satisfies the differential equation
\[
Lv_{A_r^+, m} = \delta(a),
\]
where \( L = \prod_{i<j} (\partial_i - \partial_j)^{m_{i,j}} \) and \( \delta(a) \) is the Dirac delta function on \( V \) ([8, 11]). Note that \( \partial_{r+1} \) in the definition of \( L \) is supposed to be zero. Theorem 15 above, together with Proposition 17 and Theorem 18 as below, characterizes the function \( v_{A_r^+, m, \text{nice}} \) on \( \text{nice} \) more explicitly.
Let $M_\ell = \sum_{i=\ell+1}^{r+1} m_{\ell,i}$ for $\ell = 1, \ldots, r$. Then we have the following proposition.

**Proposition 17.** The coefficient of $a_1^{M_1-1} a_2^{M_2-1} \cdots a_{r-1}^{M_{r-1}-1} a_r^{M_r-1}$ in the volume function $v_{A_r^+, m, c_{\text{nice}}}(a)$ is given by

$$\frac{1}{(M_1-1)! (M_2-1)! \cdots (M_{r-1}-1)! (M_r-1)!}.$$

**Proof.** From the Lidskii formula in Corollary 11, we have

$$v_{A_r^+, m, c_{\text{nice}}}(a) = \sum_{|i|=\ell} \frac{a_i^{i_1} \cdots a_r^{i_r}}{i_1! \cdots i_r!} \text{Res}_{x=0} \left( \prod_{i=1}^{r} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}} \right),$$

where $|i| = i_1 + \cdots + i_r$. When $i_\ell = M_\ell - 1$ for $\ell = 1, \ldots, r$,

$$\text{Res}_{x=0} \left( \frac{x_1^{M_1-1} \cdots x_r^{M_r-1}}{\prod_{i=1}^{r} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right) = \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \text{Res}_{x_r=0} \left( \frac{x_1^{(\sum_{i=2}^{r} m_{i,1})-1} \cdots x_{r-1}^{m_{r-2,r-1}-1}}{x_r \prod_{1 \leq i < j \leq r-1} (x_i - x_j)^{m_{i,j}}} \right) = \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \left( \frac{x_1^{(\sum_{i=2}^{r} m_{i,1})-1} \cdots x_{r-2}^{m_{r-2,r-1}-1}}{x_{r-1} \prod_{1 \leq i < j \leq r-1} (x_i - x_j)^{m_{i,j}}} \right) = \text{Res}_{x_1=0} \frac{1}{x_1} = 1.$$

Thus we obtain the proposition. \qed

**Theorem 18.** Let $\phi_r = \phi(a_1, \ldots, a_r)$ be a homogeneous polynomial of $a_1, \ldots, a_r$ with degree $d$ and let $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$. Suppose $\phi_r$ satisfies the system of differential equations as follows:

$$\begin{cases}
\partial_r^{m_{r,r+1}} \phi_r = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_r^{m_{r-1,r+1}} \phi_r = 0 \\
\vdots \\
(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r = 0.
\end{cases}$$

(i) If $M - r < d$, then $\phi_r = 0$.

(ii) If $0 < d \leq M - r$, then there is a non trivial homogeneous polynomial $\phi_r$ satisfying (4).

(iii) If $d = M - r$ in particular, $\phi_r$ is equal to a constant multiple of $v = v_{A_r^+, m, c_{\text{nice}}}(a)$. 

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Proof. We argue by induction on $r$. In the case that $r = 1$, we write

$$\phi_1 = \phi(a_1) = pa_1^d,$$

where $p$ is a constant. If $m_{1,2} - 1 < d$ and $\phi_1$ satisfies the differential equation $\partial_1^{m_{1,2}} \phi_1 = 0$, then $p = 0$ and hence $\phi_1 = 0$. If $0 \leq d \leq m_{1,2} - 1$, then for any $p \neq 0$, $\partial_1^{m_{1,2}} \phi_1 = 0$.

Also, if $d = m_{1,2} - 1$, in particular, then $\phi_1 = pa_1^{m_{1,2} - 1}$, while $v = \frac{1}{(m_{1,2} - 1)!}a_1^{m_{1,2} - 1}$ as in Example 12. Hence $\phi_1$ is equal to a constant multiple of $v$.

We assume that the statement of this theorem holds for $r - 1$. We write $\phi_r$ as

$$\phi_r = \phi(a_1, \ldots, a_r) = g_d(a_2, \ldots, a_r) + a_1g_{d-1}(a_2, \ldots, a_r) + \cdots + a_1^dg_0(a_2, \ldots, a_r),$$

where $g_k$ is a homogeneous polynomial of $a_2, \ldots, a_r$ with degree $k$ for $k = 0, 1, \ldots, d$. Then for $k = 0, 1, \ldots, d$, $g_k$ satisfies the differential equations as follows:

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_r^{m_{r+1}}g_k = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1}} \partial_r^{m_{r-1}+1}g_k = 0 \\
\vdots \\
(\partial_2 - \partial_3)^{m_{2,4}}(\partial_3 - \partial_4)^{m_{2,4}} \cdots (\partial_2 - \partial_r)^{m_{2,r}} \partial_r^{m_{2,r}+1}g_k = 0.
\end{array} \right.
\end{align*}$$

(6)

We set $h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1)$. From the inductive assumption, if $0 \leq k \leq h$, then $g_k$ is a homogeneous polynomial. On the other hand, if $h + 1 \leq k \leq d$, then $g_k = 0$, namely,

$$g_d(a_2, \ldots, a_r) = g_{d-1}(a_2, \ldots, a_r) = \cdots = g_{h+1}(a_2, \ldots, a_r) = 0.$$  

(i) We consider the case of $M - r < d$. Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. Now we compare the coefficients of $a_1^{d-h-M_1+n}$ in $(\partial_1 - \partial_2)^{m_{1,2}}(\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r$ for $n = 0, \ldots, h$. For $q = 1, \ldots, M_1 - m_{1,r+1}$, we define

$$D_q = \sum_{2 \leq i_1 < \cdots < i_q \leq r} \left( \prod_{1 \leq i < k} \binom{m_{i,i}}{m_{i_1,i}} \right) \partial_{i_1}^{p_{i_1}} \partial_{i_2}^{p_{i_2}} \cdots \partial_{i_q}^{p_{i_q}},$$

Then we have the following equation:

$$\begin{align*}
\frac{(d - h + n)!}{(d - h - M_1 + n)!}g_{h-n}(a_2, \ldots, a_r) &- \frac{(d - h + n - 1)!}{(d - h - M_1 + n)!}D_1g_{h-n+1}(a_2, \ldots, a_r) \\
+ \cdots + (-1)^j \frac{(d - h + n - j)!}{(d - h - M_1 + n)!}D_jg_{h-n+j}(a_2, \ldots, a_r) \\
+ \cdots + (-1)^{M_1-m_{1,r+1}} \frac{(d - h + n - (M_1 - m_{1,r+1})!}{(d - h - M_1 + n)!}D_{M_1-m_{1,r+1}}g_{h-n+M_1,r}(a_2, \ldots, a_r) \\
= 0.
\end{align*}$$

(8)
When \( n = 0 \), from (7) and (8), we have
\[
g_h(a_2, \ldots, a_r) = 0.
\]

When \( n = 1 \), we have
\[
\frac{(d - h + 1)!}{(d - h - M_1 + 1)!} g_{h-1}(a_2, \ldots, a_r) - \frac{(d - h)!}{(d - h - M_1 + 1)!} D_1 g_h(a_2, \ldots, a_r) = 0.
\]

Thus we have
\[
g_{h-1}(a_2, \ldots, a_r) = 0.
\]

Similarly, we have
\[
g_{h-2}(a_2, \ldots, a_r) = g_{h-3}(a_2, \ldots, a_r) = \cdots = g_0(a_2, \ldots, a_r) = 0
\]
and hence \( \phi_r = 0 \).

(ii) We consider the case of \( 0 \leq d \leq M - r \). By the inductive assumption, there is a non trivial homogeneous polynomial \( g_{h-n_1+i} \) satisfying (6) for \( i = 1, \ldots, n_1 \), where \( n_1 = M - r - d + 1 \). We can take
\[
g_{h-n_1+i}(a_2, \ldots, a_r) \neq 0.
\]

When \( n = n_1 \), from (7) and (8),
\[
g_{h-n_1}(a_2, \ldots, a_r) = \frac{(d - h + n_1 - 1)!}{(d - h + n_1)!} D_1 g_{h-n_1+1}(a_2, \ldots, a_r)
\]
\[
- \frac{(d - h + n_1 - 2)!}{(d - h + n_1)!} D_2 g_{h-n_1+2}(a_2, \ldots, a_r)
\]
\[
+ \cdots - (-1)^n_1 - 1 \frac{(d - h)!}{(d - h + n_1)!} D_{n_1} g_h(a_2, \ldots, a_r).
\]

When \( n = n_1 + 1 \),
\[
g_{h-(n_1+1)}(a_2, \ldots, a_r) = \frac{(d - h + n_1)!}{(d - h + n_1 + 1)!} D_1 g_{h-n_1}(a_2, \ldots, a_r)
\]
\[
- \frac{(d - h + n_1 - 1)!}{(d - h + n_1 + 1)!} D_2 g_{h-n_1+1}(a_2, \ldots, a_r)
\]
\[
+ \cdots + (-1)^n_1 \frac{(d - h)!}{(d - h + n_1 + 1)!} D_{n_1+1} g_h(a_2, \ldots, a_r).
\]

Similarly, for \( n = n_1 + 2, \ldots, h \), we can express \( g_{h-j}(a_2, \ldots, a_r) \) \( (j = n_1, n_1 + 1, \ldots, h) \) in terms of \( g_{h-j+i}(a_2, \ldots, a_r) \) \( (i = 1, \ldots, j) \) and their partial derivatives. Namely, we can express \( \phi_r \) in terms of \( g_{h-n_1+i}(a_2, \ldots, a_r) \) and their partial derivatives. It follows that \( \phi_r \neq 0 \) when \( 0 \leq d \leq M - r \).

(iii) If \( d = M - r \) in particular, then \( n_1 = 1 \), and \( g_{h-j} \) \( (j = 1, \ldots, h) \) becomes the linear combination of \( g_h \) and their partial derivatives. Therefore \( \phi_r \) is uniquely determined by
Moreover, from the inductive assumption, \( g_h = C \cdot v_{A_{r-1}^+, m', c_{\text{nice}}} \), where \( C \) is a constant, \( m' = (m_{i,j})_{2 \leq i < j \leq r+1} \), and \( c_{\text{nice}} \) is a nice chamber of \( A_{r-1}^+ \). Hence the solution of (5) is unique up to a constant multiple. On the other hand, by Theorem 15, \( v_{A_{r}^+, m, c_{\text{nice}}} \) satisfies the system of differential equations (5). Hence \( \phi_r \) is equal to a constant multiple of \( v_{A_{r}^+, m, c_{\text{nice}}} \).

Recall that in the proof of Theorem 18, we have defined the operator

\[
D_q = \sum_{2 \leq i_1 \leq \cdots \leq i_r \leq q} \frac{(m_{i_1, q})}{q!} \partial_{i_1}^q + \cdots + \sum_{p_1 + \cdots + p_k = q} \left( \prod_{1 \leq i \leq k} \frac{(m_{i, q})}{p_i!} \right) \partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \cdots \partial_{i_k}^{p_k}
\]

\[+ \cdots + \sum_{2 \leq i_1 < \cdots < i_q \leq r} \left( \prod_{1 \leq i \leq q} \frac{(m_{i, q})}{1!} \right) \partial_{i_1} \partial_{i_2} \cdots \partial_{i_q}
\]

for \( q = 1, \ldots, M_1 - m_{1, r+1} \).

**Remark 19.** Let \( M_1 = \sum_{i=2}^{r+1} m_{1, i} \). When \( d = M - r \), from the proof of Theorem 18 (iii), \( g_{h-j} \ (j = 1, \ldots, h) \) is uniquely determined as follows:

\[
\begin{align*}
g_{h-1} &= \frac{(M_1-1)!}{M_1!} D_1 g_h \\
g_{h-2} &= \frac{(M_1-1)!}{(M_1+1)!} (D_1^2 - D_2) g_h \\
g_{h-3} &= \frac{(M_1-1)!}{(M_1+2)!} (D_1^3 - 2D_1D_2 + D_3) g_h \\
& \quad \vdots \\
g_0 &= \frac{(M_1-1)!}{(M-r)!} (D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_1h) g_h.
\end{align*}
\]

Let \( m' = (m_{i,j})_{2 \leq i < j \leq r+1} \), \( c_{\text{nice}}' \) a nice chamber of \( A_{r-1}^+ \) and \( a' = \sum_{i=2}^r a_i(e_i - e_{r+1}) \in \mathcal{C}_{\text{nice}} \). From Proposition 17 and Remark 19, we obtain the following theorem.

**Theorem 20.** Let \( h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1) \) and let \( D_q \ (q = 1, \ldots, h) \) be as in (9). Then \( v_{A_{r}^+, m, c_{\text{nice}}} (a) \) is written by the linear combination of \( v_{A_{r-1}^+, m', c_{\text{nice}}'} (a') \) and its partial derivatives as follows:

\[
v_{A_{r}^+, m, c_{\text{nice}}} (a) = \left\{ \begin{array}{l}
a_1^{M_1-1} \frac{1}{(M_1-1)!} D_1 + a_1^{M_1} \frac{1}{M_1!} (D_1^2 - D_2) \\
\quad + a_1^{M_1+1} \frac{1}{(M_1+1)!} (D_1^3 - 2D_1D_2 + D_3) + \cdots \\
\quad + a_1^{M_1+2} \frac{1}{(M_1+2)!} (D_1^4 - 2D_1^2D_2 + 2D_1D_3 - D_4) + \cdots \\
\quad + a_1^{M_1+r} \frac{1}{(M-r)!} (D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_1h) \end{array} \right\} v_{A_{r-1}^+, m', c_{\text{nice}}'} (a').
\]

**Example 21.** Let \( r = 3 \), let \( a = \sum_{i=1}^3 a_i(e_i - e_4) \in \mathcal{C}_{\text{nice}} \) and let \( a' = \sum_{i=2}^3 a_i(e_i - e_4) \in \mathcal{C}_{\text{nice}}' \). We set \( m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2 \) and \( m_{3,4} = 2 \) as in Example 14. Then we have

\[
v_{A_3^+, m, c_{\text{nice}}} (a) = \frac{1}{360} v_1^3 \left( 3a_1^2 + 6a_1^2a_2 + 3a_2^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^2 + 30a_3^2a_3 \right).
\]

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We can check that \( v = v_{A^3, m, \epsilon_{nice}}(a) \) satisfies the system of differential equations as follows:

\[
\begin{align*}
(\partial_1 - \partial_2)(\partial_1 - \partial_3)\partial_1^2 v &= 0 \\
(\partial_2 - \partial_3)\partial_2^2 v &= 0 \\
\partial_3^2 v &= 0.
\end{align*}
\]

Also, from Proposition 17, the coefficient of the term \( a_3^2 a_2 a_3 \) is \( \frac{1}{3!2!1!} = \frac{1}{12} \). When \( r = 2 \),

\[
v_{A^3, m^r, \epsilon_{nice}}(a') = \frac{1}{6} a_2^2(a_2 + 3a_3).
\]

Therefore, we have

\[
\left\{ \frac{a_3^3}{6} + \frac{a_4^3}{24} D_1 + \frac{a_5^3}{120} (D_1^2 - D_2) + \frac{a_6^3}{720} (D_1^3 - 2D_1D_2 + D_3) \right\} v_{A^3, m^r, \epsilon_{nice}}(a')
\]

\[
= \frac{a_3^3 a_2^2}{36} + \frac{a_3^3 a_2}{12} + \frac{a_4^3 a_2}{24} + \frac{a_4^4 a_2}{24} + \frac{a_5^2 a_2}{60} + \frac{a_5^3}{120} + \frac{a_6^2}{360} = v_{A^3, m, \epsilon_{nice}}(a)
\]

as in (10).

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