Analytical renormalization of large-size expansion for polygonal Wilson loops in effective string theory

P.V. Pobylitsa

Petersburg Nuclear Physics Institute
Gatchina, 188300, St. Petersburg, Russia

Abstract
Schwarz-Christoffel (SC) mapping plays a crucial role in the calculation of the large-size expansion for polygonal Wilson loops in confining gauge theories using effective string theory (EST). Recently a new analytical regularization based on SC mapping was suggested and successfully applied to the calculation of the two-loop contribution of EST in the case of triangular Wilson loops. We prove that this analytical renormalization produces finite results for arbitrary polygonal Wilson loops and show that the result of the analytical renormalization for a given polygonal contour is independent of the choice of SC mapping for this polygon.
1 Introduction

1.1 Large-size expansion for Wilson loops

The problem of quark confinement in Quantum Chromodynamics (QCD) still remains a challenge. This theoretical problem simplifies if one turns from the full-fledged QCD with light quarks to the gauge theory containing only heavy quarks. The interaction between a heavy quark-antiquark pair allows for a non-relativistic description in terms of quark-antiquark potential that can be expressed via Wilson loops \[ W(C) = \langle \text{Tr} P \exp \left[ i \oint_C dx^\mu A_\mu (x) \right] \rangle \] (1.1) computed in the pure gauge theory with Euclidean time using the path-ordered exponent taken along a closed contour \(C\).

In a wide class of gauge theories Wilson loops obey area law \[ \ln W(C) \mid_{C \to \infty} \sim -\sigma S(C) \] (1.2) where \(S(C)\) is the area of the minimal surface spanned on (generally non-flat) contour \(C\) and notation \(\mid_{C \to \infty}\) stands for the large-size limit of contour \(C\).

It is a tradition to refer to gauge theories obeying area law (1.2) as confining gauge theories for brevity (implying the confinement of a heavy quark-antiquark pair embedded in the pure gauge theory), although the problem of quark confinement in real QCD is much more complicated.

Area law (1.2) sends us a message that some sort of string theory may stand behind Wilson loops in confining gauge theories. Starting from qualitative and heuristic arguments [2], one tried to justify the stringy approach to Wilson loops using various limits and expansions: large size, large number of colors [3]-[6], large number of space-time dimensions [7]-[10], Regge limit [10]-[12].

If one is interested only in large-distance properties in gauge theories satisfying area law, it is natural to expect that the stringy picture needs for its justification only the large-size limit and nothing else. This point of view is implemented in effective string theory (EST). EST assumes that the asymptotic behavior of Wilson loops in the limit of large size of contour \(C\) can be described by a functional integral over surfaces \(\Sigma\) bounded by contour \(C\):

\[
W(C) \to \text{const} \int_{\partial \Sigma = C} D\Sigma \exp (-S_{\text{EST}}[\Sigma]).
\] (1.3)

The idea that one can go beyond naive string models and can interpret functional integral (1.3) as a theoretical tool for the construction of a systematic large-size expansion of Wilson loops has a long history. An important step in this direction was made by M. Lüsher, G. Münster, K. Symanzik and P. Weisz [13]-[15]. In the computation of the first terms of the large-size expansion, one can approximate \(S_{\text{EST}}[\Sigma]\) by Nambu action

\[
S_{\text{Nambu}}(\Sigma) = \sigma_0 S(\Sigma)
\] (1.4)
where $S(\Sigma)$ is the area of surface $\Sigma$ and $\sigma_0$ is the bare string tension (different from the renormalized physical string tension $\sigma$ appearing in area law (1.2)). If one wants to use EST for the construction of higher terms of the large-size expansion then $S_{\text{EST}}[\Sigma]$ must be understood as an infinite series containing all possible terms compatible with the symmetries of the problem.

EST has developed in various directions including

- derivation of general constraints on terms appearing in EST action $S_{\text{EST}}[\Sigma]$ [10]-[25],
- calculation of EST loop corrections for rectangular Wilson loops [24], [26]-[29] and for other quantities like correlation functions of Polyakov lines and spectra of closed and open strings [16], [18]-[23],
- analysis of string finite-width effects [15], [30], [31].

EST was successfully tested by lattice Monte Carlo simulations (see [24], [32]-[39] and references therein).

EST leads to the following expansion of Wilson loops in the large-size limit:

$$\ln W(\lambda C)^{\lambda \to \infty} f_{-2}(C) \lambda^2 + f_{-1}(C) \lambda + f_{\ln}(C) \ln \lambda + f_0(C) + f_2(C) \lambda^{-2} + f_3(C) \lambda^{-3} + \ldots$$

(1.5)

Here $\lambda C$ stands for contour $C$ uniformly rescaled by factor $\lambda$. In this paper we assume that $C$ is a flat polygon. A detailed discussion of this expansion can be found in ref. [40]. A short summary is:

- Coefficient $f_{-2}(C)$ corresponds to area law [12].
- Coefficient $f_{-1}(C)$ is proportional to the length of contour $C$.
- Coefficients $f_{\ln}(C)$, $f_0(C)$ can be expressed via the functional determinant of the two-dimensional Laplace operator in the region bounded by contour $C$ with Dirichlet boundary conditions. This Laplace determinant was computed in ref. [41] for arbitrary polygons using Schwarz-Christoffel (SC) mapping.
- Coefficient $f_2(C)$ was computed for rectangular contours $C$ in refs. [24], [27] but later Billo et al. [28], [29] detected an arithmetic error in the result of refs. [24], [27] and corrected it.
- In ref. [40] coefficient $f_2(C)$ was computed for triangular diagrams and a certain progress was made towards the calculation of $f_2(C)$ for arbitrary polygonal contours $C$. 

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EST leads to the following general expression for $f_2(C)$ [40]

$$f_2(C) = \frac{1}{\sigma} \frac{D-2}{(8\pi)^2} \left[ I_{1,\text{ren}}^\text{ren}(C) + \frac{2}{9} (D-2) I_{2,\text{ren}}^\text{ren}(C) \right]. \quad (1.6)$$

Here

- $\sigma$ is the string tension appearing in Wilson law (1.2),
- $D$ is the space-time dimension of the confining gauge theory,
- quantities $I_{m,\text{ren}}^\text{ren}(C)$ depend only contour $C$ and on nothing else.

The structure of expression (1.6) is determined by a certain (8-shaped) Feynman diagram of EST [26], [27]. The dynamics of the underlying confining gauge theory enters only via string tension $\sigma$. The dependence on dimension $D$ comes from the tensor algebra of the Feynman diagram of EST for $f_2(C)$. Common factor $D-2$ in (1.6) reflects the triviality of $D=2$ pure gauge theories and appears in EST via the number of transverse degrees of freedom.

Once $\sigma$ and $D$ dependence is factored out (1.6), the nontrivial part of the calculation of $f_2(C)$ is localized in $I_{m,\text{ren}}^\text{ren}(C)$.

The Feynman diagram of EST for $f_2(C)$ has ultraviolet divergences that must be renormalized (in the sense of effective field theories). When choosing an ultraviolet regularization, one has to find a reasonable compromise between a solid theoretical status of the calculation and computational efficiency.

If one is interested in the calculation of $f_2(C)$ for arbitrary polygons $C$ then as shown in ref. [40] a certain progress may be achieved by using

- Schwarz-Christoffel (SC) mapping for polygon $C$,
- analytical regularization formulated in terms of parameters of SC mapping.

### 1.2 From triangles to general polygons

In ref. [40] the problem of the computation of $f_2(C)$ was completely solved for triangular contours $C$ but with only a partial progress for general polygonal contours $C$. In case of arbitrary polygons the results of ref. [40] were limited to

- derivation of naive ultraviolet divergent integral representations for non-renormalized analogs $I_m(C)$ of renormalized functions $I_{m,\text{ren}}^\text{ren}(C)$ appearing in (1.6),
- suggesting a procedure of regularization for these divergent integrals $I_m(C)$ using analytical continuation without proving that this analytical continuation exists and that it gives a finite result for $I_{m,\text{ren}}^\text{ren}(C)$,
- demonstration that in the special case of triangular contours $C$ this analytical continuation really exists and produces a finite result (which was explicitly computed).
In the current work we prove that the analytical continuation suggested in ref. [40] exists and leads to finite values of \( I_{\text{ren}}(C) \) for arbitrary polygons \( C \).

### 1.3 SC mapping

SC transformation \( \zeta(z) \) is a conformal mapping of the upper complex \( z \) semi-plane

\[
\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}
\]  

(1.7)
to a polygon in complex \( \zeta \) plane. The real axis of the \( z \) plane is mapped onto the boundary of the polygon whereas \( n_v \) real points \( z_1, \ldots, z_{n_v} \) are mapped to vertices \( \zeta_1, \ldots, \zeta_{n_v} \) of the polygon.

For a given polygon there exists an infinite set of conformal mappings from \( \mathbb{C}_+ \) to this polygon because linear fractional transformations

\[
z' = \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{R})
\]  

(1.8)
with real coefficients \( a, b, c, d \) map upper semiplane \( \mathbb{C}_+ \) to itself.

Using this freedom of linear fractional transformations, one can always choose SC mapping so that

\[
z_{n_v} = \infty
\]  

(1.9)
i.e. infinity of the \( z \) plane is mapped to vertex \( \zeta_{n_v} \) of the polygon.

This special type of SC mappings simplifies calculations and is always assumed in our analytical renormalization procedure. We will refer to SC mappings with \( z_{n_v} = \infty \) as SC\(_\infty\) mappings. SC\(_\infty\) mapping is described by differential equation

\[
\frac{d\zeta}{dz} = \tilde{A} \prod_{k=1}^{n_v-1} (z - z_k)^{-\beta_k}
\]  

(1.10)
where

\[
\tilde{A}, \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1}
\]  

(1.11)
are SC\(_\infty\) parameters of the polygon.

Although condition (1.9) restricts the freedom of linear fractional transformations (1.8), for any given polygon we still have an infinite amount of SC\(_\infty\) mappings from \( \mathbb{C}_+ \) to this polygon (see section 8.4).

Interior angles \( \theta_k \) of the polygon associated with SC vertices \( z_k \) are given by

\[
\theta_k = \pi (1 - \beta_k) \quad (1 \leq k \leq n_v).
\]  

(1.12)
Parameters \( \beta_k \) appear in SC equation (1.10) only for \( 1 \leq k \leq n_v-1 \) but we define also \( \beta_{n_v} \) by extrapolating relation (1.12) to \( k = n_v \). Then the geometric property

\[
\sum_{k=1}^{n_v} (\pi - \theta_k) = 2\pi
\]  

(1.13)
leads to

\[ \beta_{n_v} = 2 - \sum_{k=1}^{n_v-1} \beta_k. \]  

(1.14)

Interior angles of the polygon belong to the range

\[ 0 < \theta_k < 2\pi, \quad \theta_k \neq \pi. \]

Together with eq. (1.12) this leads to

\[ 0 < \beta_k < 2, \quad \beta_k \neq 1 \quad (1 \leq k \leq n_v). \]

(1.15)

1.4 Functions \( \Pi_P^{(n)} \)

In ref. [40] quantities \( I_{m}^{\text{ren}} \) appearing in eq. (1.12) were expressed via certain functions \( \Pi_P^{(n)} \). The explicit expression for \( I_{m}^{\text{ren}} \) via functions \( \Pi_P^{(n)} \) will be given below in eqs. (1.19), (1.20). But first it makes sense to discuss the status of arguments of these functions \( \Pi_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1}) \):

- \( \alpha, \{\gamma_k\}_{k=1}^{n_v-1} \) are generally complex variables,
- \( \{z_k\}_{k=1}^{n_v-1} \) are real variables,
- \( P(z, z^*) \) is a polynomial of two variables.

We use a widespread but slightly misleading notation: from algebraic point of view \( P(z, z^*) \) is a polynomial of two independent variables \( z, z^* \), e.g. this interpretation can used in eq. (1.16), however, quite often, e.g. in eq. (1.17), \( z \) and \( z^* \) are treated as complex conjugate variables.

We assume that polynomial \( P(z, z^*) \) has property

\[ P(z, z^*) = P(z^*, z). \]

(1.16)

Functions \( \Pi_P^{(n)} \) are formally defined by the integral

\[ \Pi_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1}) = \int_{\mathbb{C}_+} d^2z \left| \text{Im} \ z \right|^{\alpha-1} \prod_{k=1}^{n_v-1} \left| z - z_k \right|^{\gamma_k - \alpha - 1} P(z, z^*) \]

running over upper complex semiplane \( \mathbb{C}_+ \). Our normalization of the integration measure is

\[ d^2z = (d\text{Re} \ z)(d\text{Im} \ z). \]

(1.17)

The integral on the RHS of (1.17) may be divergent. In ref. [40] it was suggested to understand function \( \Pi_P^{(n)} \) as follows:

1) concentrate on the dependence of \( \Pi_P^{(n)} \) on complex variables \( \alpha, \{\gamma_k\}_{k=1}^{n_v} \) but fix real parameters \( \{z_k\}_{k=1}^{n_v} \) and polynomial \( P \),
2) first define function $\Pi_P^{(n)}$ in the subset of complex space $\mathbb{C}^{n+1}$ of variables $\alpha, \{\gamma_k\}_{k=1}^n$ where integral $\text{(1.17)}$ is convergent,

3) then analytically continue $\Pi_P^{(n)}$ in $\alpha, \{\gamma_k\}_{k=1}^n$ as far as possible (expecting to obtain a meromorphic function in $\mathbb{C}^{n+1}$).

These steps are rather nontrivial:

- One must prove that integral $\text{(1.17)}$ is convergent in some non-empty region of variables $\alpha, \{\gamma_k\}_{k=1}^n$.

- One must prove that analytical continuation of integral $\text{(1.17)}$ leads to function $\Pi_P^{(n)} (\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n)$ that is meromorphic in $\alpha, \{\gamma_k\}_{k=1}^n$ (in the sense of the theory of functions of several complex variables).

- One must prove that function $\Pi_P^{(n)} (\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n)$ is regular at those values of $\alpha, \{\gamma_k\}_{k=1}^n$ arguments which are used for the computation of $I_m^{\text{ren}}$ in eqs. $\text{(1.19)}, \text{(1.20)}$.

These properties of functions $\Pi_P^{(n)}$ were announced in ref. [40] for arbitrary polygons but checked only for triangles. The proof of these statements for arbitrary polygons is one of the main subjects of this paper.

1.5 Expressions for $I_m^{\text{ren}}$

In ref. [40] it was suggested

- to compute Feynman diagram of EST contributing to $f_2 (C)$ $\text{(1.6)}$ using $\text{SC}_\infty$ mapping $\text{(1.10)}$ for the polygon bounded by contour $C$,

- to perform renormalization of ultraviolet divergences of this Feynman diagram using analytical renormalization formulated in terms of $\text{SC}_\infty$ parameters $\text{(1.11)}$.

In this approach quantities $I_m^{\text{ren}} (C)$ arise as functions of $\text{SC}_\infty$ parameters $\text{(1.11)}$. In ref. [40] $I_m^{\text{ren}}$ were expressed via functions $\Pi_P^{(n)}$ $\text{(1.17)}$

$$I_1^{\text{ren}} \left( \tilde{A}, \{\tilde{\beta}_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right) = \left| \tilde{A} \right|^{-2} \Pi_1^{(n_v-1)} \left( -3, \{2\beta_k - 2\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right),$$

$$\text{(1.19)}$$

$$I_2^{\text{ren}} \left( \tilde{A}, \{\tilde{\beta}_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right) = \left| \tilde{A} \right|^{-2} \Pi_P^{(n_v-1)} \left( 1, \{2\beta_k - 2\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right).$$

$$\text{(1.20)}$$

Here $\Pi_1^{(n_v-1)}$ stands for $\Pi_P^{(n_v-1)}$ with trivial polynomial $P (z, z^*) = 1$ and in $\Pi_P^{(n_v-1)}$ polynomial $P_2$ is given by

$$P_2 (z, z^*) = T_2 (z) T_2 (z^*)$$

$$\text{(1.21)}$$
where polynomial $T_2$ is defined by

$$T_2(z) = T_2 \left( z; \{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \right) = \left\{ \sum_{k=1}^{n_v} \frac{\beta_k}{(z-z_k)^2} - \frac{1}{2} \left( \sum_{k=1}^{n_v} \frac{\beta_k}{z-z_k} \right)^2 \right\} \prod_{k=1}^{n_v} (z-z_k)^2. \quad (1.22)$$

Although the expression on the RHS contains terms with negative powers of $(z-z_k)$, after the expansion of brackets one arrives at a polynomial in $z$.

### 1.6 Polynomials $T_2$ and $P_2$

This section can be ignored at the first reading. Here we discuss some subtleties concerning polynomials $P_2$ and $T_2$. Index 2 in notation $P_2, T_2$ comes from the fact that these polynomials are associated with quantity $I_2^{en}$ (1.20). Usually we use compact notation $T_2(z)$ for the RHS of (1.22) concentrating on the $z$ dependence at fixed $\{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v}$. But in some cases expanded notation $T_2 \left( z; \{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \right)$ containing the full set of arguments is preferable. The expanded notation may be also useful for $P_2$:

$$P_2 \left( z; z^*; \{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \right) = T_2 \left( z; \{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \right) T_2 \left( z^*; \{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \right). \quad (1.23)$$

Obviously $T_2(z)$ is a holomorphic polynomial of $z$ at any fixed $\{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v}$. If parameters $\{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v}$ are real then

$$T_2(z^*) = [T_2(z)]^* \quad (1.24)$$

We (almost) always keep parameters $\{\beta_k\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v}$ real so that relation (1.24) holds (almost) always. Whenever we mention analytical continuation of functions $\Pi_p^{(n)} \left( \alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n} \right)$, we imply analytical continuation in complex variables $\alpha, \{\gamma_k\}_{k=1}^{n}$, at fixed real $\{z_k\}_{k=1}^{n}$ and at fixed polynomial $P$. In particular, expressions (1.19), (1.20) must be understood as follows:

1) construct function $\Pi_p^{(n)} \left( \alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n} \right)$ starting from its integral representation (1.17) for those complex $\alpha, \{\gamma_k\}_{k=1}^{n}$ and real $\{z_k\}_{k=1}^{n}$ where the integral is convergent,

2) next continue function $\Pi_p^{(n)} \left( \alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n} \right)$ in complex $\alpha, \{\gamma_k\}_{k=1}^{n}$ at fixed real $\{z_k\}_{k=1}^{n}$ and at fixed polynomial $P$;

3) only after that use the resulting (analytically continued) functions $\Pi_p^{(n)} \left( \alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n} \right)$ in relations (1.19), (1.20), substituting real values

$$\{2\beta_k - 2\}_{k=1}^{n_v}, \{z_k\}_{k=1}^{n_v} \quad (1.25)$$

for arguments of $\Pi_p^{(n)}$. 

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We never use analytical continuation in $\beta_k$ and we (almost) always keep $\beta_k, z_k$ real.

For real $\{\beta_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1}$ (which is almost always our case) we have

$$P_2(z, z^*) = P_2(z^*, z), \quad (1.26)$$

i.e. $P_2$ obeys general condition [1.16].

1.7 Outline of the work

1.7.1 Analytical continuation

In ref. [40] some preliminary comments were made about the construction of the rigorous definition of functions $\Pi_P^{(n)}$ and some basic properties of functions $\Pi_P^{(n)}$ were announced without a proof for arbitrary polygons, including

- meromorphy of $\Pi_P^{(n)}$ in $\alpha, \{\gamma_k\}_{k=1}^{n}$,

- regularity of $\Pi_P^{(n)}$ at points appearing in the above expressions for $I_m^{en}$ [1.19], [1.20].

The aim of the current paper is to provide proofs of these properties.

The main steps of our work are as follows:

1. In section 4 we prove that integral on the RHS of eq. (1.17) defining function $\Pi_P^{(n)}$ is convergent in a certain non-empty region of complex parameters $\alpha, \{\gamma_k\}_{k=1}^{n}$ for any fixed real parameters $\{z_k\}_{k=1}^{n_v-1}$ and for a fixed polynomial $P$.

2. In section 5 we prove that function $\Pi_P^{(n)}$ (originally defined for $\alpha, \{\gamma_k\}_{k=1}^{n}$ in the convergence region) allows for an analytical continuation to a meromorphic function in the full $\mathbb{C}^{n+1}$ space of complex variables $\alpha, \{\gamma_k\}_{k=1}^{n}$. In other (slightly oversimplified) words, this analytical continuation of $\Pi_P^{(n)}$ is regular ‘almost for all complex $\alpha, \{\gamma_k\}_{k=1}^{n}$’ except for simple pole singularities.

3. In section 5 we derive a representation which exhibits the pole structure of $\Pi_P^{(n)}$:

$$\Pi_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n}) = \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{1}{2} \left( -2M_P + (n - 1)(\alpha + 1) - \sum_{k=1}^{n} \gamma_k \right) \right) \prod_{j=1}^{n} \Gamma \left( \frac{\gamma_j}{2} \right) \times H_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n}). \quad (1.27)$$

Here

- $H_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n})$ is an entire function in $\mathbb{C}^{n+1}$ space of complex variables $\alpha, \{\gamma_k\}_{k=1}^{n}$ (i.e. regular analytical function in the whole space $\mathbb{C}^{n+1}$).
• $M_P$ is an integer number depending on polynomial $P(z, z^*).$ $M_P$ is defined by eq. (3.33) in section 3.6.

Representation (1.27) separates $\alpha, \{\gamma_k\}_{k=1}^{n}$ singularities of functions $\Pi_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n})$ in terms of Euler $\Gamma$ functions.

In Appendix A we show that the explicit expression for the special case of $\Pi_1^{(2)}$ computed in ref. [40] agrees with general pole structure (1.27).

4. After the derivation of representation (1.27), the rest of the work is rather simple. Analytical continuation of function $\Pi^{(n)}_P$ from the convergence region of integral (1.17) is unambiguous: representation (1.27) shows that function $\Pi^{(n)}_P$ has no branching singularities so that this analytical continuation does not depend on the path in the $C^{n+1}$ space of parameters $\alpha, \{\gamma_k\}_{k=1}^{n}$.

5. The proof of the finiteness of analytical renormalization for $I^{\text{ren}}_m$ reduces to testing that arguments of functions $\Pi^{(n_v-1)}_P$ appearing on the RHS of eqs. (1.19), (1.20) do not overlap with poles of Euler $\Gamma$ functions on the RHS of eq. (1.27). This work is done in section 7.

1.7.2 Invariance with respect to SC reparametrization

As discussed above, quantities $I^{\text{ren}}_m(C)$ appearing in eq. (1.6) must depend only on the geometry of polygonal contour $C.$ But in the computation of the EST Feynman diagram in SC$_\infty$ representation, quantities $I^{\text{ren}}_m$ (1.19), (1.20) arise as functions of SC$_\infty$ parameters (1.11). As mentioned in section 1.3 any polygon allows for many different SC$_\infty$ mappings with different SC$_\infty$ parameters. A priori it is not obvious that starting from different SC$_\infty$ parametrizations $\tilde{A}, \{\beta_k\}_{k=1}^{n_v-1}$ of the same polygon, one arrives at the same results for $I^{\text{ren}}_m\left(\tilde{A}, \{\beta_k\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1}\right)$ (1.19), (1.20).

Fortunately there is no problem: in section 8 we prove that the result for $I^{\text{ren}}_m$ depends only on the geometry of the polygon and not on its SC$_\infty$ parametrization.

1.7.3 Physics and mathematics of functions $\Pi^{(n)}_P$

The central subject of this paper is properties of analytical regularization (and renormalization) for the two-loop term $f_2(C)$ in EST expansion (1.5) and not mathematics of functions $\Pi^{(n)}_P.$ Our approach to the analysis of functions $\Pi^{(n)}_P$ is rather utilitarian and devoid of mathematical elegance and perfectionism. The arguments use standard and rather simple mathematical methods. Most of calculations and proofs are described in detail but experts may easily find their own path to the results of this work after looking through the basic guidelines.

1.7.4 Ideas and technical details

The paper is structured in a way that can help those readers who are interested more in ideas rather than in technical details: we start from the discussion of
main final results and from basic underlying ideas. Then we explain how these
final statements may be derived from auxiliary technical results and in the end
prove these technical results.

2 Conventions and assumptions

2.1 Functions $\Pi^{(n)}_P$

In the sections devoted to properties of functions $\Pi^{(n)}_P(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n)$ we
make the following assumptions (if the opposite is not explicitly claimed):

1) $n$ is an integer number obeying condition

$$n \geq 2.$$  \hspace{1cm} (2.1)

2) Parameters $z_j$ are real

$$\text{Im } z_j = 0.$$ \hspace{1cm} (2.2)

3) All parameters $z_j$ are different

$$j \neq k \implies z_j \neq z_k.$$ \hspace{1cm} (2.3)

4) Usually we assume that parameters $z_j$ are ordered

$$z_1 < z_2 < \ldots < z_n.$$ \hspace{1cm} (2.4)

However, in section [8] we do not impose this constraint.

5) Polynomial $P(z, z^*)$ has symmetry property (1.16).

7) In most statements about convergence and analytical continuation we
assume that $P \neq 0$ and avoid comments about the exceptional but trivial case
$P = 0$.

2.2 Convergence of integrals and analytical continuation

We use notation $\mathbb{C}_+$ \hspace{1cm} (1.7) for the upper complex semiplane.
Integration measure $d^2 z$ in the complex $z$ plane is normalized by condition
(1.18).
Whenever convergence of integrals is discussed, e.g.

$$\int_U d^2 z f(z),$$ \hspace{1cm} (2.5)

we imply absolute convergence:

$$\int_U d^2 z |f(z)| < \infty.$$ \hspace{1cm} (2.6)

When we speak about analytical continuation of functions
$\Pi^{(n)}_P(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n)$ from the convergence region, we always mean
analytical continuation in the $C^{n+1}$ space of complex variables $\alpha, \{\gamma_k\}_{k=1}^n$ at
fixed $\{z_k\}_{k=1}^n$, and at fixed $P$ starting from the region of this $C^{n+1}$ space where
integral (1.17) defining function $\Pi^{(n)}_P$ is absolutely convergent.
3 Regions of integration

3.1 From complex semi-plane to complex plane

Functions $\Pi_p^{(n)}$ are formally defined by integral (1.17) running over the upper semi-plane $\mathbb{C}^+$. Changing the integration variable in (1.17) $z \to z^*$ (3.1)

and using property (1.16), we find a similar representation in terms of the integration over the lower complex semi-plane $\mathbb{C}^-$

$$C_- = \{z \in \mathbb{C} : \text{Im } z < 0 \} ,$$ (3.2)

$$\Pi_p^{(n)}(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n) = \int_{\mathbb{C}^-} d^2z |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k-\alpha-1} P(z, z^*) .$$ (3.3)

Taking the average of (1.17) and (3.3), we arrive at

$$\Pi_p^{(n)}(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n) = \frac{1}{2} \sum_j \int_{D_j} d^2z |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k-\alpha-1} P(z, z^*) .$$ (3.4)

The problems of convergence and analytical continuation can be studied using any of integral representations (1.17), (3.3) or (3.4). The choice of the representation is a matter of convenience. Integral (1.17) over upper semi-plane $\mathbb{C}^+$ is convenient for the determination of convergence region whereas integral (3.4) over complex plane $\mathbb{C}$ is preferable at some stages of the study of the analytical continuation in complex variables $\alpha, \{\gamma_k\}_{k=1}^n$, e.g. in section 6.3.

3.2 Splitting complex plane in regions

In order to proceed with the analysis of convergence region and with analytical continuation, we want to split the original semi-plane or plane integration region into subregions. Let us start from the integral over complex plane $\mathbb{C}$ (3.4) and split $\mathbb{C}$ in

$$\mathbb{C} = \bigcup_j D_j$$ (3.5)

so that

$$j \neq m \quad \Rightarrow \quad \text{measure } (D_j \cap D_m) = 0 .$$ (3.6)

Then

$$\Pi_p^{(n)}(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n)$$

$$= \frac{1}{2} \sum_j \int_{D_j} d^2z |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k-\alpha-1} P(z, z^*) .$$ (3.7)

Our regions $D_k$ will be invariant under complex conjugation $z \to z^*$. 

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3.3 Case of semiplane

Let us define

\[ D_{j,+} = D_j \cap C_+, \quad (3.8) \]

\[ C_+ = \bigcup_j D_{j,+}. \quad (3.9) \]

Starting from (1.17) and using (3.9), we derive an equivalent representation for \( \Pi_P^{(n)} \):

\[ \Pi_P^{(n)}(\alpha, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n) = \sum_j \int_{D_{j,+}} d^2z \, |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^\gamma_k - \alpha - 1 \, P(z, z^*) . \quad (3.10) \]

3.4 Choice of \( D_j \)

Now we want to specify \( n + 2 \) regions \( D_j \) labelled by index \( j \) running in the interval

\[ 0 \leq j \leq n + 1. \quad (3.11) \]

Regions \( D_j \) will be constructed from circles and their complements. They are shown in Fig. Fig. 1. Below a formal description of this region structure follows.
Let us denote the circle with center at \( a \) and with radius \( r \) by
\[
C(a, r) = \{ z : |z - z_0| \leq r \}.
\] (3.12)

We start from the definition of \( D_k \) with \( k \) in the range \( 1 \leq k \leq n \). For each \( z_k \) argument of \( \Pi_P^{(n)}(a, \{\gamma_k\}_{k=1}^n, \{z_k\}_{k=1}^n) \) we define region \( D_k \) as circle \( C(z_k, r_k) \)
\[
D_k = C(z_k, r_k) \quad (1 \leq k \leq n).
\] (3.13)

The radii \( r_k \) are chosen small enough so that each circle \( C(z_k, r_k) \) contains no \( z_j \) with \( j \neq k \):
\[
z_j \in D_k \iff j = k \quad (1 \leq j, k \leq n)
\] (3.14)
and so that the circles do not intercept
\[
1 \leq j, k \leq n; \ j \neq k \implies D_j \cap D_k = \emptyset.
\] (3.15)

Next, we choose \( R \) so that
\[
1 \leq k \leq n \implies |z_k| < R.
\] (3.16)

We also require that all circles \( D_k = C(z_k, r_k) \) labelled by \( 1 \leq k \leq n \) are inside circle \( C(0, R) \):
\[
1 \leq k \leq n \implies C(z_k, r_k) \subset C(0, R).
\] (3.17)

Now we define \( D_0 \) as the complement of \( \bigcup_{k=1}^n D_k \) in circle \( C(0, R) \):
\[
D_0 = C(0, R) \setminus \left( \bigcup_{k=1}^n D_k \right)
= \{ z : (|z| \leq R) \& (\forall k |z - z_k| > r_k) \}.
\] (3.18)

Finally we define \( D_{n+1} \) as the complement to circle \( C(0, R) \):
\[
D_{n+1} = \mathbb{C} \setminus C(0, R) = \{ z : |z| > R \}.
\] (3.19)

With this choice of regions \( D_j \) \((0 \leq j \leq n + 1)\) we obey condition (3.5)
\[
\mathbb{C} = \bigcup_{j=0}^{n+1} D_j
\] (3.20)
and we also satisfy condition (3.6)
\[
0 \leq j, k \leq n + 1, \ j \neq k \implies \text{measure} (D_j \cap D_k) = 0.
\] (3.21)

Now regions \( \{D_k\}_{k=0}^{n+1} \) can be used in eq. (3.7).
3.5 Compact notation for other schemes of splitting in regions

In the previous section basic regions $D_j$ were defined with index $j$ running in the interval

$$0 \leq j \leq n + 1.$$  \hfill (3.22)

In our work we will sometimes need other integration regions which will denoted as $D_A$ with some multi-index $A$, e.g. $D_{0,1}$. This multi-index notation will help us avoid confusion with basic regions $D_j$ discussed in the previous section and labelled by a single index $j$. For both basic and alternative regions $D_A$ (i.e. for simple indices $A$ and for multi-indices $A$) we will use compact notation

$$S_A(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_A} d^2z \left|\text{Im} \ z\right|^{\alpha - 1} \prod_{k=1}^n |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*) ,$$  \hfill (3.23)

$$U_A(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_A} d^2z \left|\text{Im} \ z\right|^{\alpha - 1} \prod_{k=1}^n |z - z_k|^{\Re(\gamma_k - \alpha - 1)} |P(z, z^*)| .$$  \hfill (3.24)

The integrand of (3.24) is equal to the absolute value of the integrand of (3.23). Functions $S_A$ arise in the problem of analytical continuation in $\alpha, \{\gamma_k\}_{k=1}^n$ whereas functions $U_A$ appear in the analysis of convergence of integrals. In our analysis of convergence conditions we always imply absolute convergence in the sense of integrals (3.24).

In both case we are interested in dependence of $S_A$ and $U_A$ on variables $\alpha, \{\gamma_k\}_{k=1}^n$ at fixed $D_A$, $P$ and $\{z_k\}_{k=1}^N$. Therefore for brevity we do not write fixed objects $D_A$, $P$ and $\{z_k\}_{k=1}^N$ explicitly in the list of arguments of $S_A$ and $U_A$.

Note that functions $U_A(\alpha, \{\gamma_k\}_{k=1}^n)$ depend on $\alpha, \{\gamma_k\}_{k=1}^n$ via real parts $\Re \alpha, \{\Re \gamma_k\}_{k=1}^n$. Nevertheless we use a universal notational scheme for functions $S_A$ and $U_A$ because this allows for performing routine calculations in a form common for the problems of convergence and analytical continuation using ‘substitution dictionary’

$$\left|\text{Im} \ z\right|^{\alpha - 1} \rightarrow \left|\text{Im} \ z\right|^{\Re \alpha - 1},$$  \hfill (3.25)

$$\prod_{k=1}^n |z - z_k|^{\gamma_k - \alpha - 1} \rightarrow \prod_{k=1}^n |z - z_k|^{\Re(\gamma_k - \alpha - 1)},$$  \hfill (3.26)

$$P(z, z^*) \rightarrow |P(z, z^*)| ,$$  \hfill (3.27)

$$S_A \rightarrow U_A.$$  \hfill (3.28)
3.6 Degree of polynomials $P$ and parameter $M_P$

The standard definition of the degree of polynomial $P$ is based on assigning

$$\text{degree } (z^k z^*^m) = k + m, \quad (3.29)$$

$$\text{degree } \left( \sum_{k,m} c_{km} z^k z^*^m \right) = \max_{r_{km} \neq 0} \text{degree } (z^k z^*^m), \quad (3.30)$$

For this standard degree of polynomial $P(z, z^*)$ we use notation

$$N_P = \text{degree } (P). \quad (3.31)$$

In addition to $N_P$ we will need a different integer quantity $M_P$ characterizing polynomial $P(z, z^*)$. Let us define $M_P$ as the minimal integer number such that

$$(zz^*)^M P \left( z^{-1}, (z^*)^{-1} \right) = Q_M (z, z^*) \quad (3.32)$$

is a polynomial of $z, z^*$, i.e.

$$M_P = \min_{M \geq 0} M: \quad (zz^*)^M P \left( z^{-1}, (z^*)^{-1} \right) = \text{Polynomial } (z, z^*). \quad (3.33)$$

Instructive examples:

$$P = z^* z \quad \Rightarrow \quad N_P = 2, \quad M_P = 1, \quad (3.34)$$

$$P = z^* + z \quad \Rightarrow \quad N_P = 1, \quad M_P = 1. \quad (3.35)$$

Note that for any $P$

$$N_P \leq 2M_P \leq 2N_P. \quad (3.36)$$

In case of factorizable polynomials

$$P(z, z^*) = T(z) [T(z)]^* \quad (3.37)$$

we have

$$N_P = 2M_P. \quad (3.38)$$

The degree of holomorphic polynomial $T_2$ can be easily derived from its definition [1.22]

$$N_{T_2} = 2(n_v - 2). \quad (3.39)$$

Factorizable polynomial $P_2$ [1.21] defined via holomorphic polygon $T_2$ has degree

$$N_{P_2} = 2N_{T_2} = 4(n_v - 2). \quad (3.40)$$

Now $M_{P_2}$ can be computed using (3.38)

$$M_{P_2} = \frac{1}{2} N_{P_2} = 2(n_v - 2). \quad (3.41)$$

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4 Convergence of integrals

4.1 Results

As discussed above, functions $\Pi_P^{(n)}$ are defined first in the convergence region of integral (1.17). More exactly, we want to start from the region where integral (1.17) is absolutely convergent, i.e.

$$
\int_{C_+} d^2z \left| \text{Im} \ z^{\alpha-1} \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*) \right| = \int_{C_+} d^2z \left| \text{Re}(\alpha-1) \prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} |P(z, z^*)| \right| \text{ is convergent (4.1)}
$$

Note that the convergence of this integral depends only on real parts of generally complex parameters $\alpha, \{\gamma_k\}$.

**Statement 1.**

The set of conditions

$$
\text{Re} \ \alpha > 0, \quad (4.2)
$$

$$
\text{Re} \ \gamma_k > 0, \quad (4.3)
$$

$$
\text{Re} \ \alpha > -1 + \frac{1}{n-1} \left( N_P + \sum_{k=1}^{n} \text{Re} \gamma_k \right) \quad (4.4)
$$

is *sufficient* for the absolute convergence (4.1). Here $N_P$ is degree (3.31) of polynomial $P$.

For a special case of polynomials $P$ one can make a stronger statement:

**Statement 2.**

If polynomial $P$

1) has the form

$$
P(z, z^*) = T(z) \left[ T(z) \right]^* \quad (4.5)
$$

where $T(z)$ is a holomorphic polynomial of $z$,

2) for any $k = 1, \ldots, n$

$$
T(z_k) \neq 0 \quad (4.6)
$$

then the combination of conditions (4.2), (4.3), (4.4) is not only *sufficient* but also *necessary* for the absolute convergence (4.1).

**Remark.** For factorizable polynomials (4.5) according to (3.38) we have $2M_P = N_P$ where $N_P$ is usual degree (3.31) of polynomial $P$.

4.2 Naive derivation of convergence conditions

Conditions (4.2), (4.3), (4.4) can be easily ‘explained’ in terms of naive order counting near potentially singular regions of integral (4.1):
The singularity at $|\text{Im } z| \to 0$ is integrable under condition (4.2).

The convergence of integral at $z \to z_j$ is provided by condition (4.3).

Convergence at $|z| \to \infty$ is provided by condition (4.4).

This order counting is straightforward. In the next section the case $z \to z_j$ is discussed in detail.

### 4.3 Example: convergence at $z \to z_j$

Decomposing $z$ in real and imaginary parts we have at $z \to z_j$:

$$z = x + iy,$$  

(4.7)

$$x \to z_j,$$  

(4.8)

$$y \to 0.$$  

(4.9)

We obtain in this limit

$$|\text{Im } z|^{\text{Re}(\alpha - 1)} = |y|^{\text{Re}(\alpha - 1)}$$  

(4.10)

$$\prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} \sim |z - z_j|^{\text{Re}(\gamma_j - \alpha - 1)}.$$  

(4.11)

Concentrating on the case

$$|x - z_j| \sim y \sim |z - z_j| \to 0,$$  

(4.12)

we find

$$\prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} \sim |z - z_j|^{\text{Re}(\gamma_k - \alpha - 1)}.$$  

(4.13)

so that

$$|\text{Im } z|^{\text{Re}(\alpha - 1)} \prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} \sim |z - z_j|^{\text{Re}(\gamma_j - 2)}.$$  

(4.14)

If

$$P(z_j, z_j^*) \neq 0$$  

(4.15)

then

$$|\text{Im } z|^{\text{Re}(\alpha - 1)} \prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} |P(z, z^*)|^2 \sim |y|^{\text{Re}(\gamma_k - 2)}.$$  

(4.16)

so that convergence of integral (4.1) at $z \to z_k$ requires condition

$$\text{Re } \gamma_j > 0.$$  

(4.17)
in agreement with inequality (4.3) in rigorous Statement 1,
In the special case when
\[ P(z_j, z^*_j) = 0 \] (4.18)
the situation is different because this zero of polynomial \( P \) mitigates the singularity of the integrand at \( z \to z_j \). This explains why in Statement 1 the set of conditions \((4.2) - (4.4)\) is considered as sufficient for the convergence, whereas in Statement 2, this set of conditions is both sufficient and necessary.

4.4 Careful derivation of convergence conditions

Although the simple arguments of the previous sections lead to correct convergence conditions, this naive order counting cannot be considered as a rigorous proof of Statements 1, 2. In order to upgrade this simple order counting argument to a careful proof, we use decomposition (3.10) with regions \( D_{j,+} \) defined in section 3.4 and shown in Fig. 2.

The integral controlling absolute convergence (4.1) can be decomposed as
\[
\int_{C_+} d^2z \left| \text{Im} \ z^{\alpha-1} \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*) \right| = \sum_{j=0}^{n} \int_{D_{n,+}} d^2z \left| \text{Im} \ z^{\alpha-1} \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*) \right|. \] (4.19)

so that the problem reduces to the analysis convergence conditions for each separate integral
\[
U_{n,+}(\alpha, \{\gamma_k\}_{k=1}^{n}) = \int_{D_{n,+}} d^2z \left| \text{Im} \ z^{\text{Re}(\alpha-1)} \prod_{k=1}^{n} |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} |P(z, z^*)| \right|. \] (4.20)
where we use compact notation \((3.24)\).

If one is interested in sufficient convergence conditions for integrals \(U_{n,+}\) over separate regions \(D_{n,+}\) then the results of our analysis have a simple summary:

- convergence in \(D_{0,+}\): condition \((4.2)\),
- convergence in \(D_{k,+}\) \((1 \leq k \leq n)\): combination of conditions \((4.2)\) and \((4.3)\),
- convergence in \(D_{n+1,+}\): combination of conditions \((4.3)\) and \((4.4)\).

The combination of all these sufficient conditions leads to Statement 1.

In case of both sufficient and necessary condition formulated in Statement 2 the situation is more subtle (see sections devoted to the detailed analysis of convergence conditions in each separate region).

4.5 Convergence in region \(D_{0,+}\)

4.5.1 Plan

Region \(D_{0,+}\) is defined by eq. \((3.8)\) with \(D_{0}\) defined by eq. \((3.18)\).

\[
D_{0,+} = \{ z : (|z| \leq R) \& (\forall k |z - z_k| > r_k) \& (0 < \text{Im} \ z) \}. \tag{4.21}
\]

We are interested in the convergence region of integral

\[
U_{0,+}(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{0,+}} d^2z \text{Im} \ z \text{Re}(\alpha-1) \left[ \prod_{k=1}^n |z - z_k|^{\text{Re}(\gamma_k - \alpha - 1)} \right] |P(z, z^*)| . \tag{4.22}
\]

Here we use compact notation \((3.24)\).

Our aim is

1) to prove that condition \((4.23)\)

\[
\text{Re} \ \alpha > 0 \tag{4.23}
\]

is sufficient for the convergence of this integral,

2) to prove that in the case of factorizable polynomials \(P(z, z^*)\) of Statement 2 obeying eqs. \((4.5)\), \((4.6)\) condition \((4.23)\) is not only sufficient for the convergence of \((4.22)\) but also necessary.

4.5.2 Splitting \(D_{0,+}\) in subregions

In order to proceed we must divide region \(D_{0,+}\) \((4.21)\) in subregions. Let us split region \(D_{0,+}\) by cutting \(D_{0,+}\) with line

\[
\text{Im} \ z = \eta. \nonumber
\]

We choose \(\eta\) so that

\[
\eta > 0 \tag{4.24}
\]
and for all $k$ ($1 \leq k \leq n$) we have
\[ \eta < r_k . \] (4.25)

This cut splits $D_{0,+}$ in $n+2$ disconnected components. An example of the new region structure is shown in Fig. 3 for the case $n = 2$. Now we turn to the formal description of arising subregions. First we introduce notation

\[ D_{0,1} = D_{0,+} \cap \{ z : \text{Im} \ z < \eta \} = \{ z : (|z| \leq R) \& (\forall k \ |z - z_k| > r_k) \& (0 < \text{Im} \ z < \eta) \} , \] (4.26)

\[ D_{0,2} = D_{0,+} \cap \{ z : \text{Im} \ z \geq \eta \} = \{ z : (|z| \leq R) \& (\forall k \ |z - z_k| > r_k) \& (\text{Im} \ z \geq \eta) \} \] (4.27)

Obviously
\[ D_{0,+} = D_{0,1} \cup D_{0,2} , \] (4.28)
\[ D_{0,1} \cap D_{0,2} = \emptyset . \] (4.29)

As discussed below (and shown in Fig. 3), region $D_{0,1}$ is disconnected but we still have the decomposition of function $U_{0,+}$ (4.22)

\[ U_{0,+} (\alpha, \{ \gamma_k \}_{k=1}^{n}) = U_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^{n}) + U_{0,2} (\alpha, \{ \gamma_k \}_{k=1}^{n}) \] (4.30)

where $U_{0,1}$ and $U_{0,2}$ are defined by eq. (3.24).

Note that

- the integrand of $U_{0,2}$ has no singularities in the integration region $D_{0,2}$ (and in its small vicinity),

Figure 3: Regions $D_{3,+}$, $D_{0,2}$, and $D_{0,1,j}$ for function $\Pi_{\mu}^{(n)}$ with $n = 2$. 
• region $D_{0,2}$ has a finite size.

Hence the integral defining $U_{0,2} (\alpha, \{\gamma_k\}_{k=1}^n)$ is convergent for any complex
$\alpha, \{\gamma_k\}_{k=1}^n$. Therefore integrals $U_{0,1}$ and $U_{0,2}$ have a
common convergence region in the space of parameters $\alpha, \{\gamma_k\}_{k=1}^n$.

4.5.3 Region $D_{0,1}$

Thus the problem is reduced to the analysis of the convergence region in the
$\alpha, \{\gamma_k\}_{k=1}^n$ space for integral

$$U_{0,1} (\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{0,1}} d^2 z \left| \text{Im} \ z \right|^{\Re (\alpha-1)} \left[ \prod_{k=1}^n \left| z - z_k \right|^{\Re (\gamma_k-\alpha-1)} \right] |P(z, z^*)| .$$

(4.31)

Region $D_{0,1}$ (4.26) consists of $n + 1$ disconnected components which will be
denoted $D_{0,1,j}$ $(0 \leq j \leq n)$. Fig. 3 illustrates the case of $n = 2$.

The formal description of regions $D_{0,1,j}$ follows from the definition of $D_{0,1}$
(4.26) and from constraints (2.4), (3.17), (3.15), (4.25):

$$D_{0,1,j} = \{ z = x + iy : (x_j^- (y) < x < x_j^+ (y)) \& (0 < y < \eta) \} ,$$

(4.32)

$$(0 \leq j \leq n) .$$

(4.33)

Here

$$x_0^- (y) = -\sqrt{r^2 - y^2} ,$$

(4.34)

$$x_0^+ (y) = z_1 - \sqrt{r_1^2 - y^2} ,$$

(4.35)

$$x_n^- (y) = z_n + \sqrt{r_n^2 - y^2} ,$$

(4.36)

$$x_n^+ (y) = \sqrt{R^2 - y^2} ,$$

(4.37)

for $1 \leq j \leq n - 1$:

$$x_j^- (y) = z_j + \sqrt{r_j^2 - y^2} ;$$

(4.38)

$$x_j^+ (y) = z_{j+1} - \sqrt{r_{j+1}^2 - y^2} .$$

(4.39)

All $x_j^\pm (y)$ with $0 \leq j \leq n$ have a universal form

$$x_j^\pm (y) = X_j^\pm + \sigma_j^\pm \sqrt{\rho_j^\pm} - y^2$$

(4.40)
with obvious expressions from real parameters $X_j^\pm$, $\beta_j^\pm$ and for sign factors $\sigma_j^\pm$ taking values $\pm 1$.

Thus we have

$$D_{0,1} = \bigcup_{j=0}^n D_{0,1,j},$$  \hspace{1cm} (4.41)

if $j \neq k$ then $D_{0,1,j} \cap D_{0,1,k} = \emptyset$.  \hspace{1cm} (4.42)

This leads to decomposition

$$U_{0,1} (\alpha, \{\gamma_k\}_{k=1}^n) = \sum_{j=0}^n U_{0,1,j} (\alpha, \{\gamma_k\}_{k=1}^n).$$  \hspace{1cm} (4.43)

As usual, functions $U_{0,1,j}$ are defined by general relation (3.24).

We see that the problem of the convergence region in the $\alpha, \{\gamma_k\}_{k=1}^n$ space for integral $U_{0,1,j}$ reduces to the problem of the convergence regions for all separate integrals $U_{0,1,j}$.

We have according to (3.24)

$$U_{0,1,j} (\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{0,1,j}} d^2 z \left| \Im \ z \right|^{\Re(\alpha-1)} \left| \prod_{k=1}^n |z - z_k|^{\Re(\gamma_k - \alpha - 1)} \right| |P(z, z^*)|$$

$$= \int_0^\eta |y|^{\alpha-1} \int_{x_j^-(y)}^{x_j^+(y)} dx \prod_{k=1}^n \left| (x - z_k)^2 + y^2 \right|^{(\gamma_k - \alpha - 1)/2}$$

$$\times |P(x + iy, x - iy)|.$$  \hspace{1cm} (4.44)

The integrand of

$$G_j(y; \alpha, \{\gamma_k\}_{k=1}^n) = \int_{x_j^-(y)}^{x_j^+(y)} dx \prod_{k=1}^n \left| (x - z_k)^2 + y^2 \right|^{(\gamma_k - \alpha - 1)/2} |P(x + iy, x - iy)|$$

has no singularities in the integration region. The integration limits are also regular functions. The only object that can slightly violate regularity is absolute value $|P(x + iy, x - iy)|$ which has cusps at zeros of polynomial $P(x + iy, x - iy)$. Anyway $G_j(y; \alpha, \{\gamma_k\}_{k=1}^n)$ is a continuous function of all its arguments. Therefore integral representation

$$U_{0,1,j} (\alpha, \{\gamma_k\}_{k=1}^n) = \int_0^\eta |y|^{\alpha-1} G_j(y; \alpha, \{\gamma_k\}_{k=1}^n)$$

guarantees that this integral is convergent if condition (4.23) holds.

### 4.5.4 Sufficient conditions for absolute convergence

Once we have proved that condition $\Re \alpha > 0$ is sufficient for the convergence of all integrals $U_{0,1,j}$, we can trace back our arguments which have reduced the problem of convergence of the original integral (4.22) to the problem of convergence of integrals $U_{0,1,j}$. Thus we have proved that condition (4.23) is sufficient for the convergence of integral $U_{0,+,1}$ (4.22).
4.5.5 Comments on necessary conditions for absolute convergence

Now we turn to the question about a necessary condition for convergence the convergence of $U_{0,1,j}$. If function $G_j(y;\alpha,\{\gamma_k\}_{k=1}^n)$ could vanish at $y = 0$ then integral (4.46) would be convergent at some negative values of Re$\alpha$ so that condition Re$\alpha > 0$ would not be necessary for convergence. But can $G_j(y;\alpha,\{\gamma_k\}_{k=1}^n)$ vanish at $y = 0$? In this case integral representation (4.45) would lead to

$$P(x,x) = 0$$

in a finite interval of real $x$. This vanishing is possible if $P$ has the form

$$P(z,z^*) = (z - z^*) R(z,z^*)$$

(4.48)

where $R(z,z^*)$ is a polynomial of $z, z^*$. Symmetry (1.16) leads to

$$R(z,z^*) = -R(z^*,z)$$

(4.49)

which results in decomposition

$$R(z,z^*) = (z - z^*) Q(z,z^*)$$

(4.50)

where $Q$ is a symmetric polynomial

$$Q(z,z^*) = Q(z^*,z).$$

Thus

$$P(z,z^*) = (z - z^*)^2 Q(z,z^*) = -4(\text{Im}z)^2 Q(z,z^*).$$

(4.51)

Thus the problem with the interpretation of (4.23) as a necessary condition for convergence arises only if polynomial $P(z,z^*)$ has a factor of $(\text{Im}z)^2$. This problem is obvious from the very beginning because extra factors $(\text{Im}z)^2$ coming from $P(z,z^*)$ effectively can be interpreted as a shift of parameter $\alpha$ and a change of polynomial $P$ in integral (1.17).

Anyway this problem does not affect our final results: Statement 1 provides only sufficient conditions for the absolute convergence and Statement 2 deals with factorizable polynomials (4.5) which cannot contain factors of $(z - z^*)^2$. 

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4.6 Convergence in regions $D_{j,+}$ ($1 \leq j \leq n$)

Regions $D_j$ with $1 \leq j \leq n$ are circles (3.13). Region $D_{j,+}$ is the corresponding semicircle in the upper complex semiplane. Therefore

$$\int_{D_{j,+}} d^2z \left| \text{Im} \, z \right|^{\operatorname{Re}(\alpha - 1)} \left[ \prod_{k=1}^{n} \left| z - z_k \right|^{\operatorname{Re}(\gamma_k - \alpha - 1)} \right] |P(z, z^*)|$$

$$= \int_{|z - z_j| < r_j, \text{Im} \, z > 0} d^2z \left| \text{Im} \, z \right|^{\operatorname{Re}(\alpha - 1)} \left[ \prod_{k=1}^{n} \left| z - z_k \right|^{\operatorname{Re}(\gamma_k - \alpha - 1)} \right] |P(z, z^*)|$$

$$= \int_{|z| < r_j, \text{Im} \, z > 0} d^2z \left| \text{Im} \, z \right|^{\operatorname{Re}(\alpha - 1)} \left[ \prod_{k=1}^{n} \left| z + z_j - z_k \right|^{\operatorname{Re}(\gamma_k - \alpha - 1)} \right] \times |P(z + z_j, z^* + z_j)|. \quad (4.52)$$

Next we factor the integrand in singular and regular parts

$$F_{\text{sing}} = |\text{Im} \, z|^{\operatorname{Re}(\alpha - 1)} |z|^{\operatorname{Re}(\gamma_j - \alpha - 1)}, \quad (4.53)$$

$$F_{\text{reg}} = |P(z + z_j, z^* + z_j)| \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \left| z + z_j - z_k \right|^{\operatorname{Re}(\gamma_k - \alpha - 1)} \quad (4.54)$$

so that

$$\int_{D_{j,+}} d^2z \left| \text{Im} \, z \right|^{\operatorname{Re}(\alpha - 1)} \left[ \prod_{k=1}^{n} \left| z - z_k \right|^{\operatorname{Re}(\gamma_k - \alpha - 1)} \right] |P(z, z^*)|$$

$$= \int_{|z| < r_j, \text{Im} \, z > 0} d^2z F_{\text{sing}} F_{\text{reg}}. \quad (4.55)$$

In order to proceed we need

**Statement 3.** Integral

$$\int_{|z| < r_j, \text{Im} \, z > 0} d^2z \left| \text{Im} \, z \right|^{\operatorname{Re}(\alpha - 1)} |z|^{\operatorname{Re}(\gamma_j - \alpha - 1)} \quad (4.56)$$

is convergent if and only if two conditions hold

$$\operatorname{Re} \alpha > 0, \quad (4.57)$$

$$\operatorname{Re} \gamma_j > 0. \quad (4.58)$$

This Statement follows from the results of Appendix [B].

Combining Statement 3 and regularity of $F_{\text{reg}}$ in the integration region, we conclude that conditions (4.57), (4.58) are sufficient the convergence of integral (4.55). This is the contribution of region $D_{j,+}$ to the full set of sufficient convergence conditions of Statement 1.
If
\[ P(z, z^*) \neq 0 \]  \hspace{1cm} (4.59)
then we can choose \( r_j \) so small that in the integration region \(|P(z, z^*)|\) is bounded by two nonzero positive constants
\[ |z| < r_j \implies 0 < C_1 < P(z, z^*) < C_2. \]  \hspace{1cm} (4.60)
Combining this double bound with Statement 3, we see that in case (4.59) conditions (4.57), (4.58) are both necessary and sufficient for the convergence of integral (4.55). This argument provides the contribution of region \( D_{j,+} \) to the proof of Statement 2 of section 4.1.

### 4.7 Convergence in region \( D_{n+1,+} \)

Region \( D_{n+1,+} \) is defined by eqs. (3.19), (3.8)
\[ D_{n+1,+} = \{ z : (|z| > R) \& (\text{Im } z > 0) \}. \]  \hspace{1cm} (4.61)
Using inversion
\[ z' = -\frac{1}{z}, \]  \hspace{1cm} (4.62)
we can map region \( D_{n+1,+} \) to semicircle
\[ \{ z : (|z| < 1/R) \& (\text{Im } z > 0) \}. \]  \hspace{1cm} (4.63)
Now the problem of the convergence in region \( D_{n+1,+} \) reduces to the same type as the convergence in semicircle regions \( D_{j,+} \) \((1 \leq j \leq n)\) which was considered in section 4.6.

\[
\int_{D_{n+1,+}} d^2 z \left| \frac{\text{Im } z}{|z|^4} \right| \left| \prod_{k=1}^n \left| \frac{z-z_k}{1-\frac{1}{z}-z_k} \right| \right| P(z, z^*) \right| \\
= \int_{|z|<1/R, \text{Im } z>0} \frac{d^2 z}{|z|^4} \left| \frac{1}{z} \right| \left| \prod_{k=1}^n \left| -\frac{1}{z-z_k} \right| P\left( -\frac{1}{z}, -\frac{1}{z^*} \right) \right| \\
= \int_{|z|<1/R, \text{Im } z>0} \frac{d^2 z}{|z|^4} \left| \frac{1}{z} \right| \left| \prod_{k=1}^n \left| z_k \right| \right| P\left( -\frac{1}{z}, -\frac{1}{z^*} \right). \]  \hspace{1cm} (4.64)
Then

\[ \int_{D_{n+1,+}} d^2 z \mid \text{Im} \mid z \mid \Re(\alpha - 1) \left[ \prod_{k=1}^{n} \left| z - z_k \right| \Re(\gamma_k - \alpha - 1) \right] |P(z, z^*)| \]

\[ = \left[ \prod_{k=1}^{n} \left| z_k \right| \Re(\gamma_k - \alpha - 1) \right] \int_{|z|<1/R, \text{Im} \ z > 0} d^2 z \mid z \mid^{-4 - 2 \Re(\alpha - 1) - N_P - \sum_{k=1}^{n} \Re(\gamma_k - \alpha - 1)} \]

\[ \times \mid \text{Im} \mid z \mid \Re(\alpha - 1) \left[ \prod_{k=1}^{n} \left| z + z_k - 1 \right| \Re(\gamma_k - \alpha - 1) \right] \mid z \mid^{N_P} \mid P(-1/z, -1/z^*)\mid \]

\[ = \int_{|z|<1/R, \text{Im} \ z > 0} d^2 z \tilde{F}_{\text{sing}} \tilde{F}_{\text{reg}} \]

(4.65)

where

\[ \tilde{F}_{\text{sing}} = \mid z \mid^{(n-2) \Re(\alpha + 1) - N_P - \sum_{k=1}^{n} \Re \gamma_k} \mid \text{Im} \mid z \mid \Re(\alpha - 1), \]

(4.66)

\[ \tilde{F}_{\text{reg}} = \left[ \prod_{k=1}^{n} \left| z_k \right| \Re(\gamma_k - \alpha - 1) \right] \left[ \prod_{k=1}^{n} \left| z + z_k - 1 \right| \Re(\gamma_k - \alpha - 1) \right] \mid z \mid^{N_P} \mid P(-1/z, -1/z^*)\mid . \]

(4.67)

Here we have the same structure of the integral as in eq. (4.55). Note that factor

\[ \mid z \mid^{N_P} \mid P(1/z, 1/z^*)\mid \]

may have a soft singularity (divergent derivatives) at \( z \to 0 \) but still is bounded in the integration limit

\[ |z| < R^{-1} \implies |z|^{N_P} |P(1/z, 1/z^*)| < \text{const.} . \]

(4.68)

Therefore for the derivation of sufficient convergence conditions we still can use methods applied earlier to integral (4.55). Note that parameter \( \gamma_j \) appearing in eq. (4.55) is now replaced by

\[ \gamma_j \to (n-1) \Re(\alpha + 1) - N_P - \sum_{k=1}^{n} \Re \gamma_k . \]

(4.69)

Making this replacement in convergence old sufficient conditions (4.57), (4.58) for integral (4.55), we arrive at sufficient conditions for the new integral (4.64)

\[ \Re \alpha > 0 , \]

(4.70)

\[ (n-1) \Re(\alpha + 1) - N_P - \sum_{k=1}^{n} \Re \gamma_k > 0 . \]

(4.71)

The last condition can be rearranged to the form (4.4).
Thus sufficient convergence conditions in region $D_{n+1,+}$ lead to conditions (4.2) and (4.4) in the full set of sufficient convergence conditions of Statement 1.

Now we turn to the necessary convergence conditions. In case of semicircle regions $D_{j,+}$ ($1 \leq j \leq n$) the derivation of necessary convergence conditions was based on assumption (4.59) and on bound (4.60). In case of region $D_{n+1,+}$ the role of polynomial $P(z,z^*)$ is played by function

$$Q(z,z^*) = |z|^NP\left(-1/z, -1/z^*\right).$$

(4.72)

Generally speaking, this function is not a polynomial and it is not regular at $z \to 0$ (although it is bounded at $z \to 0$) so that we cannot derive the analog of bound (4.60). But in the special case of factorizable polynomials $P$, $Q(z,z^*)$ is a polynomial with property

$$[Q(z,z^*)]_{z=0} \neq 0$$

(4.73)

so that the methods used in regions $D_{j,+}$ ($1 \leq j \leq n$) for the derivation of necessary convergence conditions work also in our case. Thus in the case of factorizable polynomials $P$, conditions (4.70) and (4.71) are both necessary and sufficient for the convergence of integral (4.64).

This completes the analysis of the contribution of region $D_{n+1,+}$ to the full set of necessary and sufficient convergence conditions of Statement 2.

5 Analytical continuation

5.1 Starting analytical continuation from the convergence region

Thus we have proved that integral (1.17) defining function $\Pi_p^{(n)}$ is convergent in the region $U$ of parameters $\alpha, \{\gamma_k\}_{k=1}^n$ specified by conditions (4.2) – (4.4) of Statement 1. This convergence region is non-empty. Obviously function $\Pi_p^{(n)}$ is holomorphic in this $\alpha, \{\gamma_k\}_{k=1}^n$ region of $C^{n+1}$. The next step of the work is to study the analytical continuation of $\Pi_p^{(n)}$ from this region $U \subset C^{n+1}$ to the full space $C^{n+1}$ and to prove announced meromorphic structure (1.27).

Note that region $U$ is connected. This excludes a possible unpleasant situation when analytical continuations starting from different disconnected regions could lead to different analytical continuations.

The above analysis of conditions (4.2) – (4.4) for the absolute convergence of the integral defining function $\Pi_p^{(n)}$ proceeded in terms of integral (1.17) over complex semiplane $C_+$. When it comes to the problem of analytical continuation of $\Pi_p^{(n)}$ in $\alpha, \{\gamma_k\}_{k=1}^n$ from the convergence region $U$ to the full complex space $C^{n+1}$ of parameters $\alpha, \{\gamma_k\}_{k=1}^n$, it is more convenient to work with the equivalent representation for $\Pi_p^{(n)}$ with $z$ integral (3.4) running over full complex plane $C$. The advantage of this representation becomes clear at later stages of the work.
(see section 6.3) but it makes sense to pass from the original semiplane integral representation for $\Pi^{(n)}_P$ to the plane representation (3.4) right now.

Our first step is to split integral (3.4) in the sum of integrals over regions $D_j$ (3.7) and to study the problem of analytical continuation in $\alpha, \{\gamma_k\}_{k=1}^n$ for each separate integral

$$S_j (\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_j} d^2 z \ |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k - \alpha - 1} P(z,z^*) . \quad (5.1)$$

In order to simplify notation, on the LHS we omit the dependence on quantities $z_k, P$ which are kept fixed in our analytical continuation.

Alternatively we can work with regions $D_{j,+}$ defined by (3.8) and shown in Fig. 2:

$$S_{j,+} (\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{j,+}} d^2 z \ |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k - \alpha - 1} P(z,z^*) . \quad (5.2)$$

Obviously

$$S_j (\alpha, \{\gamma_k\}_{k=1}^n) = 2S_{j,+} (\alpha, \{\gamma_k\}_{k=1}^n) . \quad (5.3)$$

- in the convergence region of $\alpha, \{\gamma_k\}_{k=1}^n$ space,
- after the analytical continuation in $\alpha, \{\gamma_k\}_{k=1}^n$.

The choice of $D_j$-decomposition or $D_{j,+}$-decomposition is a matter of convenience.

For each separate region $D_j$ we will prove that function $S_j (\alpha, \{\gamma_k\}_{k=1}^n)$ originally defined in the convergence region can be analytically continued to a meromorphic function with the pole structure

$$S_j (\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma_j (\alpha, \{\gamma_k\}_{k=1}^n) H_j (\alpha, \{\gamma_k\}_{k=1}^n) \quad (5.4)$$

where

- $H_j (\alpha, \{\gamma_k\}_{k=1}^n)$ is an entire function of $\alpha, \{\gamma_k\}_{k=1}^n$ (i.e. holomorphic in $\mathbb{C}^{n+2}$),
- $\Gamma_j (\alpha, \{\gamma_k\}_{k=1}^n)$ is a certain product of Euler $\Gamma$ functions depending on linear combinations of variables $\alpha, \{\gamma_k\}_{k=1}^n$.

After we complete the proof of the pole structure (5.4) of functions $S_j (\alpha, \{\gamma_k\}_{k=1}^n)$, it is straightforward to derive pole structure (1.27) from

$$\Pi^{(n)}_P (\alpha, \{\gamma_k\}_{k=1}^n) = \sum_j S_j (\alpha, \{\gamma_k\}_{k=1}^n) = \sum_j \Gamma_i (\alpha, \{\gamma_k\}_{k=1}^n) H_b (\alpha, \{\gamma_k\}_{k=1}^n) \quad (5.5)$$
5.2 Results for $\Gamma_j$

For functions $\Gamma_j(\alpha, \{\gamma_k\}_{k=1}^n)$ appearing in decomposition (5.4) one can derive the following expressions

$$\Gamma_0(\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma\left(\frac{\alpha}{2}\right),$$  \hspace{1cm} (5.6)

$$\Gamma_j(\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\gamma_j}{2}\right) \quad (0 \leq j \leq n),$$  \hspace{1cm} (5.7)

$$\Gamma_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2} \left(-2M_P - 1 - \alpha - \sum_{k=1}^n (\gamma_k - \alpha - 1)\right)\right) \prod_{j=1}^n \Gamma\left(\frac{\gamma_j}{2}\right).$$  \hspace{1cm} (5.8)

Parameter $M_P$ appearing on the RHS eq. (5.8) related to polynomial $P$ of functions $\Pi_P^n$ is defined by eq. (3.33).

5.3 Rough pole structure

It should be stressed that all factorized decompositions of meromorphic function into regular and pole factors

- intermediate result (5.4),
- final result (1.27),

are somewhat rough in the sense that poles of Gamma functions may be sometimes compensated by zeros of regular factors. In particular, the derivation of (1.27) from (5.4) is straightforward if one combines all singular $\Gamma$ factors appearing on the RHS of eqs. (5.6) – (5.8) in one common product

$$\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1}{2} \left(-2M_P - 1 - \alpha - \sum_{k=1}^n (\gamma_k - \alpha - 1)\right)\right) \prod_{j=1}^n \Gamma\left(\frac{\gamma_j}{2}\right).$$  \hspace{1cm} (5.9)

appearing on the RHS of (1.27). Therefore representation (1.27) creates an illusion that the structure of singularities is more severe than it really is. Although final representation (1.27) exaggerates the pole singularities, this exaggeration does not interfere with our final aim, the proof that our analytical regularization provides a finite result for $f_2(C)$, because ‘exaggerated pole factor’ (5.9) is regular at the final point of the analytical continuation used in eqs. (1.19), (1.20).

6 Calculation of singular factors $\Gamma_j$

6.1 Preliminary remarks

Now we turn to the calculation of expressions for $\Gamma_j$ announced in eqs. (5.6) – (5.8). These expressions hint that Gamma functions appearing in $\Gamma_j$ are
determined by $z$ singularities of the integrand of (5.1) in region $D_j$:

$$\text{Im } z \to 0 \implies \Gamma \left( \frac{\alpha}{2} \right), \quad (6.1)$$

$$z \to z_k \implies \Gamma \left( \frac{\gamma_j}{2} \right), \quad (6.2)$$

$$|z| \to \infty \implies \Gamma \left( \frac{1}{2} \left( -2M_P - 1 - \alpha - \sum_{k=1}^{n} (\gamma_k - \alpha - 1) \right) \right). \quad (6.3)$$

Note that region $D_0$ contains only the singularity at $\text{Im } z \to 0$ so that $\Gamma_0$ has only one Gamma function $\Gamma(\alpha/2)$. Region $D_j$ with $1 \leq j \leq n$ has two singularities ($\text{Im } z \to 0$ and $z \to z_k$) which lead to two associated Gamma functions $\Gamma(\alpha/2)$ and $\Gamma(\gamma_j/2)$ in eq. (5.7).

Region $D_{n+1}$ has also two singularities ($\text{Im } z \to 0$ and $|z| \to \infty$) which lead to the two Gamma function in $\Gamma_{n+1}$ appearing on the RHS of eq. (5.8).

### 6.2 Calculation of $\Gamma_0$

The case of $\Gamma_0$ is the simplest because in $D_0$ we have only one singularity $\text{Im } z \to 0$ and expect only one associated Euler Gamma function $\Gamma(\alpha/2)$ in $\Gamma_0(\alpha, \{\gamma_k\}_{k=1}^n)$. Function $S_0$ is defined by eq. (5.1) with $j = 0$

$$S_0(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_0} d^2 z \left| \text{Im } z \right|^{\alpha-1} \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*). \quad (6.4)$$

but we prefer to work with its $D_{0,+}$ analog (5.2)

$$S_{0,+}(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{0,+}} d^2 z \left| \text{Im } z \right|^{\alpha-1} \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*). \quad (6.5)$$

obeying relation (5.3)

$$S_0(\alpha, \{\gamma_k\}_{k=1}^n) = 2S_{0,+}(\alpha, \{\gamma_k\}_{k=1}^n), \quad (6.6)$$

As discussed in the proof of Statement 1, integral (6.5) is absolutely convergent in the region constrained by condition (4.2)

$$\text{Re } \alpha > 0. \quad (6.7)$$

The same obviously holds for integral (6.4).

Our aim is to prove that function $S_0(\alpha, \{\gamma_k\}_{k=1}^n)$ can be analytically continued to a meromorphic function in $\alpha, \{\gamma_k\}$ with pole structure (5.4), (5.6)

$$S_0(\alpha, \{\gamma_k\}_{k=1}^n) = 2S_{0,+}(\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma \left( \frac{\alpha}{2} \right) H_0(\alpha, \{\gamma_k\}_{k=1}^n) \quad (6.8)$$

where $H_0(\alpha, \{\gamma_k\}_{k=1}^n)$ is an entire function.
In order to derive meromorphic factorization (6.8), we will split \( D_0 \) in sub-regions. This splitting is essentially the same as in our analysis of convergence conditions in section 4.5.

Now we can reuse the work of section 4.5 inverting replacements (3.25) – (3.28). We have

\[
S_{0,+} (\alpha, \{ \gamma_k \}_{k=1}^n) = S_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^n) + S_{0,2} (\alpha, \{ \gamma_k \}_{k=1}^n) .
\] (6.9)

This equation is first derived in absolute convergence region (6.7) of integral (6.5). Note that the integrand (3.23) for \( S_{0,2} \) is regular in associated region \( D_{0,2} \) (and in its small vicinity) so that

\[
S_{0,2} (\alpha, \{ \gamma_k \}_{k=1}^n) = \text{entire function in } \mathbb{C}^{n+1}.
\] (6.10)

Combining this fact with decomposition (6.9), we conclude that the problem of the derivation of representation (6.8) for the analytical continuation of \( S_0 \) reduces to the derivation its analog for \( S_{0,1} \):

\[
S_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^n) = \Gamma \left( \frac{\alpha}{2} \right) H_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^n)
\] (6.11)

where \( H_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^n) \) is an entire function.

Due to constraint (4.25) region \( D_{0,1} \) consists of \( n + 1 \) disconnected components \( D_{0,1,j} \).

Using (3.23) we define functions \( S_{0,1,j} \) associated with regions \( D_{0,1,j} \). Then

\[
S_{0,1} (\alpha, \{ \gamma_k \}_{k=1}^n) = \sum_{j=0}^{n} S_{0,1,j} (\alpha, \{ \gamma_k \}_{k=1}^n) .
\] (6.12)

We have associated functions \( S_{0,1,j} \) (3.23)

\[
S_{0,1,j} (\alpha, \{ \gamma_k \}_{k=1}^n)
= \int_{D_{0,1,j}} d^2z \arg \left[ z^{\alpha-1} \prod_{k=1}^n |z - z_k|^{\gamma_k - \alpha - 1} P(z, z^*) \right]
= \int_0^\eta dy \left| y \right|^{\alpha-1} \int_{x_j(y)}^{x_j^+(y)} dx \prod_{k=1}^n \left| (x - z_k)^2 + y^2 \right|^{(\gamma_k - \alpha - 1)/2} P(x + iy, x - iy)
\] (6.13)

This integral representation is absolute convergent in the region (6.7)

\[
\text{Re } \alpha > 0
\] (6.14)

inherited from the absolute convergence region of integral (6.4).

Next we want to continue functions \( S_{0,1,j} \) analytically to arbitrary complex \( \alpha, \{ \gamma_k \}_{k=1}^n \).
The problem of the derivation of meromorphic decomposition (6.11) reduces to the problem of derivation of meromorphic structure

\[ S_{0,j} (\alpha, \{ \gamma_k \}_{k=1}^n) = \Gamma \left( \frac{\alpha}{2} \right) H_{0,1,j} (\alpha, \{ \gamma_k \}_{k=1}^n) \quad (0 \leq j \leq n) \]  

(6.15)

where \( H_{0,1,j} (\alpha, \{ \gamma_k \}_{k=1}^n) \) are entire functions.

Using symmetry (1.16) of the polynomial \( P(z, z^*) \) we can write

\[ [P(z, z^*)]_{z=x+iy} = Q(x, y^2) \]

(6.16)

where \( Q(x, q) \) is a polynomial of its arguments \( x, q \).

Therefore in absolute convergence region (6.14) we derive from (6.13)

\[ S_{0,j} (\alpha, \{ \gamma_k \}_{k=1}^n) = \int_{\eta_0}^{\eta} dy \frac{1}{y^{\alpha-1}} \int_{x_j^- (y^{1/2})}^{x_j^+ (y^{1/2})} dx \prod_{k=1}^n \left| (x-z_k)^2 + y^2 \right|^{(\gamma_k-\alpha-1)/2} Q(x, y^2). \]  

(6.17)

Changing integration variable

\[ y = \sqrt{q} \]

(6.18)

we find

\[ S_{0,j} (\alpha, \{ \gamma_k \}_{k=1}^n) = \frac{1}{2} \int_0^{\sqrt{\eta}} dq \ q^{(\alpha/2)-1} \int_{x_j^- (q^{1/2})}^{x_j^+ (q^{1/2})} dx \prod_{k=1}^n \left| (x-z_k)^2 + q \right|^{(\gamma_k-\alpha-1)/2} Q(x, q). \]  

(6.19)

Let us define

\[ F_{0,j} (q) = \frac{1}{2} \int_{x_j^- (q^{1/2})}^{x_j^+ (q^{1/2})} dx \prod_{k=1}^n \left| (x-z_k)^2 + q \right|^{(\gamma_k-\alpha-1)/2} Q(x, q). \]  

(6.20)

Then

\[ S_{0,j} (\alpha, \{ \gamma_k \}_{k=1}^n) = \int_0^{\sqrt{\eta}} dq \ q^{(\alpha/2)-1} F_{0,1,j} (q). \]  

(6.21)

Note that integration limits \( x_j^\pm (q^{1/2}) \) on the RHS of (6.20) given by (4.40)

\[ x_j^\pm (q^{1/2}) = X_j^\pm + \sigma_j^\pm \sqrt{(\rho_j^\pm)^2 - q} \]  

(6.22)

are infinitely differentiable functions of \( q \) in the integration region of (6.21)

\[ 0 \leq q \leq \sqrt{\eta} \]  

(6.23)

because we have

\[ \eta < \rho_j^\pm \]  

(6.24)
since \( \eta \) obeys constraint \((4.25)\).

Next, the integrand of \((6.20)\) is also an infinitely differentiable function of \(x, q\) in the integration region

\[
x_j^- \left( q^{1/2} \right) < x < x_j^+ \left( q^{1/2} \right),
\]

\(0 \leq q \leq \sqrt{\eta}\)

because

1) points \(z_k\) are outside of the integration region so that all factors \(\left| (x - z_k)^2 + q \right|^{(\gamma_k - \alpha - 1)/2}\) are regular in this integration region,

2) \(Q(x, q)\) is a polynomial of \(x\) and \(q\).

Thus both integrand and integration limits on the RHS of \((6.20)\) are infinitely differentiable functions. Therefore \(F_{0,1,j}(q)\) is also an infinitely differentiable function of \(q\) in the range \(0 \leq q \leq \sqrt{\eta}\).

Now the problem reduces to the study of analytical continuation of integral \((6.20)\) in \(\alpha\) starting from the convergence region \((6.14)\). This analytical continuation can be done iteratively integrating by parts

\[
S_{0,1,j} (\alpha, \{\gamma_k\}_{k=1}^n) = \int_{0}^{\sqrt{\eta}} dq \; q^{(\alpha)/2 - 1} F_{0,1,j} (q)
\]

\[
= \left( \frac{\alpha}{2} \right)^{-1} q^{\alpha/2} F_{0,1,j} (q) |_{0}^{\sqrt{\eta}} - \left( \frac{\alpha}{2} \right)^{-1} \int_{0}^{\sqrt{\eta}} dq \; q^{\alpha/2} F'_{0,1,j} (q)
\]

\[
= \left( \frac{\alpha}{2} \right)^{-1} q^{\alpha/2} F_{0,1,j} (q) |_{0}^{\sqrt{\eta}} - \left( \frac{\alpha}{2} \right)^{-1} \left( \frac{\alpha}{2} + 1 \right)^{-1} q^{(\alpha/2) + 1} F_{0,1,j} (q) |_{0}^{\sqrt{\eta}}
\]

\[
+ \left( \frac{\alpha}{2} \right)^{-1} \left( \frac{\alpha}{2} + 1 \right)^{-1} \int_{0}^{\sqrt{\eta}} dq \; q^{(\alpha/2) + 1} F''_{0,1,j} (q) = \ldots
\]

\((6.27)\)

The poles appearing in this analytical continuation may lie only at points

\[
\frac{\alpha}{2} = 0, -1, -2, \ldots
\]

\((6.28)\)

This proves pole structure \((6.15)\) of \(S_{0,1,j}\). This also completes the derivation of the chain of related meromorphic representations for \(S_{0,1}\) \((6.11)\) and for \(S_{0}\) \((6.8)\) as well as the derivation of expression \((5.6)\) for \(\Gamma_0\).
6.3 Calculation of $\Gamma_j$ ($1 \leq j \leq n$)

We want to derive meromorphic structure (5.4), (5.7) of functions $S_j$ ($1 \leq j \leq n$). We start from eq. (5.1)

$$S_j(\alpha, \{\gamma_k\}_{k=1}^{n}) = \int_{D_j} d^2z \ |\text{Im} \ z|^{\alpha-1} \left[ \prod_{k=1}^{n} |z - z_k|^{\gamma_k - \alpha - 1} \right] P(z, z^*)$$

(6.29)

Let us introduce notation

$$M(\alpha, \{\gamma_k\}, \{z_k\}; z, z^*) = P(z + z_j, z^* + z_j^*) \prod_{k=1, k \neq j}^{n} |z + z_j - z_k|^{\gamma_k - \alpha - 1}.$$  

(6.30)

Then

$$S_j(\alpha, \{\gamma_k\}_{k=1}^{n}) = \int_{|z| < r_j} d^2z \ |\text{Im} \ z|^{\alpha-1} |z|^{\gamma_j - \alpha - 1} M(\alpha, \{\gamma_k\}, \{z_k\}; z, z^*).$$  

(6.31)

Note that function $M(\alpha, \{\gamma_k\}, \{z_k\}; z, z^*)$

- is infinitely differentiable in Re $z$ and in Im $z$ in a vicinity of the integration region $|z| < r_j$,
- is an entire function of $\alpha, \gamma_j$ at any fixed $z$ in the region $|z| < r_k$.

Remember that in our choice of regions we are free to choose $r_k$ as small as we like. Therefore we can replace $M(\alpha, \{\gamma_k\}, \{z_k\}; z, z^*)$ by its Taylor series in small $z, z^*$ or in small Re $z$, Im $z$ and concentrate on the analytical continuation in variables $\alpha, \gamma_j$ keeping other parameters $\gamma_k$ with $k \neq j$ fixed.

Thus we can turn to a simplified problem of analytical continuation in $\alpha, \gamma$ for function $\tilde{S}(\alpha, \gamma)$ defined by integral

$$\tilde{S}(\alpha, \gamma) = \int_{|z| < \rho} d^2z \ |\text{Im} \ z|^{\alpha-1} |z|^{\gamma_j - \alpha - 1} \tilde{M}(\text{Re} \ z, \text{Im} \ z).$$

(6.32)

with $\tilde{M}$ is represented by a convergent power series

$$\tilde{M}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m y^n.$$  

(6.33)
In this analysis we are free to choose parameter \( \rho \) as small as we like in order to improve the convergence of series whenever this is needed.

Absolute-convergence (sufficient) condition for integral (6.32) is determined by Statement 3 of section 4.6

\[
\Re \alpha > 0, \quad (6.34) \\
\Re \gamma > 0. \quad (6.35)
\]

Working in this convergence region we can use the symmetry of integration region under reflections

\[
x \to -x, \quad (6.36) \\
y \to -y. \quad (6.37)
\]

which leads to

\[
\tilde{S}(\alpha, \gamma) = \int_{|z| < \rho} d^2z \, |\Im z|^{\alpha - 1} \, |z|^{\gamma - \alpha - 1} \, \frac{1}{4} \sum_{\varepsilon_1 = \pm 1} \sum_{\varepsilon_2 = \pm 1} \tilde{M}(\varepsilon_1 \Re z, \varepsilon_2 \Im z). \quad (6.38)
\]

After this symmetrization only the even powers of the Taylor series survive:

\[
\frac{1}{4} \sum_{\varepsilon_1 = \pm 1} \sum_{\varepsilon_2 = \pm 1} \tilde{M}(\varepsilon_1 x, \varepsilon_2 y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{2m, 2n} x^{2m} y^{2n}. 
\]

Thus

\[
\tilde{S}(\alpha, \gamma) = \int_{|z| < \rho} d^2z \, |\Im z|^{\alpha - 1} \, |z|^{\gamma - \alpha - 1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{2m, 2n} (\Re z)^{2m} (\Im z)^{2n}. \quad (6.39)
\]

Next we can express

\[
(\Re z)^2 = |z|^2 - (\Im z)^2 \quad (6.40)
\]

and rearrange the power series in terms of new variables \(|z|^2\) and \((\Im z)^2\)

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{2m, 2n} (\Re z)^{2m} (\Im z)^{2n} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} d_{kl} |\Im z|^{2k} |z|^{2l}. \quad (6.41)
\]

Thus

\[
\tilde{S}(\alpha, \gamma) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} d_{kl} \int_{|z| < \rho} d^2z \, |\Im z|^{\alpha - 1 + 2k} \, |z|^{\gamma - \alpha - 1 + 2l}. \quad (6.42)
\]

This result was derived in convergence region (6.34), (6.35).

The integrals can be computed using (B.1), (B.19)

\[
\int_{|z| < \rho} d^2z \, |\Im z|^{\alpha - 1 + 2k} \, |z|^{\gamma - \alpha - 1 + 2l} = \frac{2\sqrt{\pi}}{\gamma + 2k + 2l} \frac{\Gamma \left( \frac{\alpha}{2} + k \right)}{\Gamma \left( \frac{\alpha + \gamma}{2} + k \right)} \rho^{\gamma + 2(k+l)}. \quad (6.43)
\]
Hence
\[ \tilde{S}(\alpha, \gamma) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} d_{kl} \frac{1}{\left( \frac{\alpha}{2} + (k + l) \right)} \frac{\Gamma\left( \frac{\alpha}{2} + k \right)}{\Gamma\left( \frac{\alpha+1}{2} + k \right)} \rho^{\gamma + 2(k+l)}. \] (6.44)

Now we can start with analytical continuation in \( \alpha \) and \( \gamma \). There are two explicit factors obstructing analytical continuation in \( \alpha \) and \( \gamma \)
\[ \Gamma\left( \frac{\alpha}{2} + k \right) \frac{1}{\frac{\alpha}{2} + (k + l)}. \] (6.45)

These singularities can be absorbed in
\[ \Gamma\left( \frac{\alpha}{2} \right) \Gamma\left( \frac{\gamma}{2} \right). \] (6.46)

In other words, we can rewrite (6.44) in the form
\[ \tilde{S}(\alpha, \gamma) = \Gamma\left( \frac{\alpha}{2} \right) \Gamma\left( \frac{\gamma}{2} \right) \tilde{H}(\alpha, \gamma) \] (6.47)

where
\[ \tilde{H}(\alpha, \gamma) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} d_{kl} \frac{1}{\Gamma\left( \frac{\alpha}{2} + (k + l) \right)} \frac{\Gamma\left( \frac{\alpha}{2} + k \right)}{\Gamma\left( \frac{\alpha+1}{2} + k \right)} \frac{1}{\Gamma\left( \frac{\alpha+1}{2} + k \right)} \rho^{\gamma + 2(k+l)}. \] (6.48)

Here all factors
\[ \frac{1}{\Gamma\left( \frac{\alpha}{2} + (k + l) \right)}; \] (6.49)
\[ \frac{\Gamma\left( \frac{\alpha}{2} + k \right)}{\Gamma\left( \frac{\gamma}{2} \right)} = \text{Polynomial}(\alpha), \] (6.50)
\[ \frac{1}{\Gamma\left( \frac{\alpha+1}{2} + k \right)} \] (6.51)

are entire functions of \( \alpha, \gamma \) so that the full product
\[ \frac{1}{\Gamma\left( \frac{\alpha}{2} + (k + l) \right)} \frac{\Gamma\left( \frac{\alpha}{2} + k \right)}{\Gamma\left( \frac{\alpha+1}{2} + k \right)} \rho^{\gamma + 2(k+l)} \] (6.52)
is also an entire function of \( \alpha, \{\gamma_k\} \). Therefore in eq. (6.48) we have a power series in variable \( \rho \) (which can be chosen arbitrarily small) of entire functions in \( \alpha, \gamma \) so that \( \tilde{H}(\alpha, \gamma) \) is also an entire function.

Thus analytical continuation of function \( \tilde{S}(\alpha, \gamma) \) defined by integral (6.32) in its convergence region leads to a meromorphic function with the pole structure described by representation (6.47).

Returning to our original problem of the analytical continuation of function \( S_j(\alpha, \{\gamma_k\}_{k=1}^{n}) \) in \( \alpha \) and in \( \gamma_j \) we conclude that this function has the pole structure
\[ S_j(\alpha, \{\gamma_k\}_{k=1}^{n}) = \Gamma\left( \frac{\alpha}{2} \right) \Gamma\left( \frac{\gamma_j}{2} \right) H_j(\alpha, \{\gamma_k\}_{k=1}^{n}) \] (6.53)
where \( H_j(\alpha, \{\gamma_k\}_{k=1}^{n}) \) is an entire function. Thus we have derived representation (5.4) for \( S_j(1 \leq j \leq n) \) and corresponding expression (5.7) for \( \Gamma_j \).
6.4 Calculation of $\Gamma_{n+1}$

We want to derive meromorphic structure (6.4), (6.7) of $S_{n+1}$. We have

$$S_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{D_{n+1}} d^2z \, |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^\gamma_k - \alpha - 1 \, P(z, z^*)$$

(6.54)

We proceed similarly to our work in section 4.7 where we studied convergence condition for region $D_{n+1}$. First we perform inversion (4.62) and derive by analogy with (4.64), (4.65)

$$S_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) = \int_{|z| > R} d^2z \, |\text{Im } z|^{\alpha-1} \prod_{k=1}^n |z - z_k|^\gamma_k - \alpha - 1 \, P(z, z^*) .$$

(6.55)

Next we use definition (3.33) of integer parameter $M_P$

$$R(z, z^*) \equiv (zz^*)^{MP} P(-1/z , -1/z^*) = \text{Polynomial}(z, z^*) .$$

(6.56)

Hence

$$S_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) = \left[ \prod_{k=1}^n |z_k|^\gamma_k - \alpha - 1 \right] \int_{|z| < 1/R} d^2z \, |z|^{(n-2)(\alpha+1) - \sum_{k=1}^n \gamma_k} \times |\text{Im } z|^{\alpha-1} \prod_{k=1}^n \left| z + \bar{z_k}^{-1} \right|^{\gamma_k - \alpha - 1} R(z, z^*) .$$

(6.57)

This integral has the same structure as integral (6.31) but with replacement

$$\gamma_j \rightarrow -2M_P + (n-1)(1+\alpha) - \sum_{k=1}^n \gamma_k .$$

(6.58)

Therefore analytical continuation of new integral (6.57) is described by representation (6.55) with replacement (6.58)

$$S_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) = \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{1}{2} \left( -2M_P + (n-1)(1+\alpha) - \sum_{k=1}^n \gamma_k \right) \right) H_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n) .$$

(6.59)

Here $H_{n+1}(\alpha, \{\gamma_k\}_{k=1}^n)$ is an entire function. Thus we have derived representation (5.4) for $S_{n+1}$ with $\Gamma_{n+1}$ given by (5.8).
7 Analytical renormalization produces finite results

7.1 Is analytical continuation to the physical point finite?

Now we want to check that the renormalization procedure suggested in ref. \[40\] for the calculation of two-loop EST correction \(f_2(C)\) really renormalizes ultraviolet divergences and provides a finite expression for \(f_2(C)\). Eq. \[1.6\] expresses \(f_2(C)\) via functions \(I_{1\text{ren}}^{(n)}\) and \(I_{2\text{ren}}^{(n)}\) so that we must test that our final expressions for \(I_{1\text{ren}}^{(n)}\) in eqs. \[1.19\] and \[1.20\] are finite. In other words we must test that functions \(\Pi_{1\text{ren}}^{(n)}\) in eqs. \[1.19\] and \[1.20\] are finite at points appearing in \[1.19\] and \[1.20\].

In other words, we must check that arguments of functions \(\Pi_{1\text{ren}}^{(n)}\) in eqs. \[1.19\] and \[1.20\] do not overlap with poles of \(\Gamma\) functions that describe pole structure \[1.27\] of \(\Pi_{1\text{ren}}^{(n)}\).

7.2 Case of \(I_{1\text{ren}}^{(n)}\)

Function \(\Pi_{1\text{ren}}^{(n_v-1)}\) appearing in expression \[1.19\] for \(I_{1\text{ren}}^{(n)}\) has the following pole representation \[1.27\]

\[
\Pi_{1\text{ren}}^{(n_v-1)}(-3,\{2\beta_k -2\}_{k=1}^{n_v-1},\{z_k\}_{k=1}^{n_v-1})
= \Gamma\left(-\frac{3}{2}\right) \Gamma\left(1\right) \Gamma\left(1\right) \prod_{j=1}^{n_v-1} \Gamma(\beta_k -1)
\times H_{1\text{ren}}^{(n_v-1)}(-3,\{2\beta_k -2\}_{k=1}^{n_v-1},\{z_k\}_{k=1}^{n_v-1}). \tag{7.1}
\]

Here \(M_1\) is \(M_P\) parameter \[3.33\] for the trivial polynomial \(P = 1\) so that

\[
M_1 = 0. \tag{7.2}
\]

Now we can simplify

\[
\Gamma\left(1\right) \Gamma\left(1\right) \prod_{j=1}^{n_v-1} \Gamma(\beta_k -1)
= \Gamma\left(1 - \sum_{k=1}^{n_v-1} \beta_k\right). \tag{7.3}
\]

Thus on the RHS of \[7.1\] we have the product of potentially dangerous \(\Gamma\) functions

\[
\Gamma\left(1 - \sum_{k=1}^{n_v-1} \beta_k\right) \prod_{j=1}^{n_v-1} \Gamma(\beta_k -1). \tag{7.4}
\]

According to \[1.14\]

\[
1 - \sum_{k=1}^{n_v-1} \beta_k = \beta_{n_v} - 1 \tag{7.5}
\]
so that

\[ \Gamma \left( 1 - \sum_{k=1}^{n_v-1} \beta_k \right) \prod_{j=1}^{n_v} \Gamma (\beta_j - 1) = \prod_{j=1}^{n_v} \Gamma (\beta_j - 1). \quad (7.6) \]

Parameters \( \beta_k \) (\( 1 \leq k \leq n_v \)) are related to inside angles of the polygon \( \theta_k \) and belong to the range \( (1.15) \) so that

\[ -1 < \beta_k - 1 < 1, \quad (7.7) \]
\[ \beta_k - 1 \neq 0. \quad (7.8) \]

Therefore for all \( k \) in the range \( 1 \leq k \leq n_v \) we have

\[ \Gamma (\beta_k - 1) = \text{finite}. \quad (7.9) \]

Thus all \( \Gamma \) functions appearing in eq. \( (7.1) \) are regular.

7.3 Case of \( I_{2}^{\text{ren}} \)

Function \( \Pi_{P_2}^{(n_v-1)} \) appearing in expression \( (1.20) \) for \( I_{2}^{\text{ren}} \) has the following pole representation \( (1.27) \)

\[ \Pi_{P_2}^{(n_v-1)} \left( 1, \{2\beta_k - 2\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right) \]
\[ = \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \left( -2M_{P_2} + 2(n_v - 2) - \sum_{k=1}^{n_v-1} (2\beta_k - 2) \right) \right) \prod_{j=1}^{n_v} \Gamma (\beta_j - 1) \]
\[ \times H_{P_2}^{(n_v-1)} \left( 1, \{2\beta_k - 2\}_{k=1}^{n_v-1}, \{z_k\}_{k=1}^{n_v-1} \right). \quad (7.10) \]

Polynomial \( P_2 \) is given by eqs. \( (1.21), (1.22) \).

Using expression \( (3.41) \) for \( M_{P_2} \), we find

\[ \Gamma \left( \frac{1}{2} \left( -2M_{P_2} + 2(n_v - 2) - \sum_{k=1}^{n_v-1} (2\beta_k - 2) \right) \right) = \Gamma \left( 1 - \sum_{k=1}^{n_v-1} \beta_k \right). \quad (7.11) \]

Thus we arrive at the same product of \( \Gamma \) functions as in the case of \( (7.4) \) which is regular.

8 Analytical renormalization and SC reparametrization

8.1 Problem

In section \( (1.7.2) \) we mentioned a problem: it is not obvious that starting from different SC parametrizations of a fixed polygon, one arrives at the same values
for $I_{m}^{\text{ren}}$ after our analytical renormalization formulated in terms of SC parameters. Indeed our renormalization procedure for $I_{m}^{\text{ren}}$ is formulated in terms of SC parameters and not directly in terms of the geometry of the polygon.

Remember that our analytical renormalization is formulated in terms of a special class of $\text{SC}_{\infty}$ mappings defined by eq. (1.10) and assuming that one SC vertex is kept at infinity. In section 1.3 it was suggested to refer to this type of SC transformation as $\text{SC}_{\infty}$ mapping.

The following statement shows that our procedure of analytical renormalization is independent of the choice of the $\text{SC}_{\infty}$ mapping for a given polygon:

**Statement 4.** If two $\text{SC}_{\infty}$ mappings $\zeta^{(a)}(z)$ with $\text{SC}_{\infty}$ parameters

\[
\left\{ \tilde{A}^{(a)}, \left\{ \tilde{\beta}^{(a)}_k \right\}_{k=1}^{n_v-1}, \left\{ \tilde{z}^{(a)}_k \right\}_{k=1}^{n_v-1} \right\} \quad (a = 1, 2) \quad (8.1)
\]

map the upper complex semiplane to the same polygon then for quantities $I_{m}^{\text{ren}}$ given by (1.19), (1.20) we have

\[
I_{m}^{\text{ren}} \left( \tilde{A}^{(1)}, \left\{ \tilde{\beta}^{(1)}_k \right\}_{k=1}^{n_v-1}, \left\{ \tilde{z}^{(1)}_k \right\}_{k=1}^{n_v-1} \right) = I_{m}^{\text{ren}} \left( \tilde{A}^{(2)}, \left\{ \tilde{\beta}^{(2)}_k \right\}_{k=1}^{n_v-1}, \left\{ \tilde{z}^{(2)}_k \right\}_{k=1}^{n_v-1} \right). \quad (8.2)
\]

### 8.2 Permutation symmetry

In our previous work we assumed that real points $\{z_k\}_{k=1}^{n}$ are monotonically ordered (2.4). This assumption simplifies intermediate technical calculations when one is interested in such properties of functions $\Pi^{(n)}_{P}$ like convergence region of their integral representations or in the analytical continuation of $\Pi^{(n)}_{P}$ in $\alpha, \gamma_k$ at fixed $z_k$.

Functions $\Pi^{(n)}_{P}$ have another important property: symmetry under permutations of their arguments including permutations of $z_k$. This symmetry property will simplify our following work. Therefore in part of our work devoted to the derivation of relation (8.2) we do not impose constraint (2.4).

Let $R$ be an arbitrary permutation of indices $k = 1, 2, \ldots, n$. Functions $\Pi^{(n)}_{P}$ obey relation

\[
\Pi^{(n)}_{P} \left( \alpha, \{\gamma_k\}_{k=1}^{n}, \{z_k\}_{k=1}^{n} \right) = \Pi^{(n)}_{P} \left( \alpha, \{\gamma_{R(k)}\}_{k=1}^{n}, \{z_{R(k)}\}_{k=1}^{n} \right). \quad (8.3)
\]

The method of derivation is standard: one first proves this relation in the region of parameters $\alpha, \{\gamma_k\}_{k=1}^{n}$ where the integrals (1.17) representing LHS and RHS are convergent. After that one can continue identity (8.3) analytically in $\alpha, \{\gamma_k\}_{k=1}^{n}$ at fixed $z_k$. This program meets no problems because

- the set of convergence constraints (4.2) - (4.4) on $\alpha, \{\gamma_k\}_{k=1}^{n}$ is symmetric with respect to permutations of $\gamma_k$,
- pole structure of $\Pi^{(n)}_{P}$ (1.27) is also symmetric with respect to permutations of $\gamma_k$. 

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One should distinguish

- properties of functions \( \Pi_P^{(n)} \) with respect to permutations,
- application of these properties of \( \Pi_P^{(n)} \) to the calculation of \( I_m^{\text{ren}} \),
- resulting symmetry properties of \( I_m^{\text{ren}} \left( \tilde{A}, \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1} \right) \).

In other words, we should not mix

- intrinsic symmetry properties of functions \( \Pi_P^{(n)} \),
- intrinsic symmetry properties of problems that are solved using functions \( \Pi_P^{(n)} \).

Our work consists of several stages:

\[
C \rightarrow \zeta(z) \rightarrow \tilde{A}, \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1} \rightarrow I_m^{\text{ren}} \left( \tilde{A}, \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1} \right) \quad \text{(8.4)}
\]

Permutations play an important role at each stage. For example, \( SC_{\infty} \) mapping \( \zeta(z) \) satisfies differential equation (1.10). The RHS of this differential equation is a function of \( \{ \beta_k, z_k \}_{k=1}^{n_v-1} \) which is symmetric under permutations of \( k \).

Next, in relation (8.3) \( P(z, z^*) \) stands for an arbitrary polynomial. We have the same polynomial on the LHS and on the RHS of (8.3). When it comes to the calculation of \( I_2^{\text{ren}} \) based on eq. (1.20) then we use polynomial \( P_2 \) (1.23) which has property

\[
P_2 \left( z, z^*; \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_R(k) \}_{k=1}^{n_v-1} \right) = P_2 \left( z, z^*; \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1} \right) \quad \text{(8.5)}
\]

that follows from eqs. (1.22), (1.23).

If one combines

- permutation symmetry property (8.3) of function \( \Pi_P^{(n)} \),
- other permutation symmetry properties like symmetry of the RHS of differential equation (1.10) and symmetry of polynomial \( P_2 \) (8.5)

then one easily derives from eqs. (1.19), (1.20)

\[
I_m^{\text{ren}} \left( \tilde{A}, \{ \beta_k \}_{k=1}^{n_v-1}, \{ z_k \}_{k=1}^{n_v-1} \right) = I_m^{\text{ren}} \left( \tilde{A}, \{ \beta_R(k) \}_{k=1}^{n_v-1}, \{ z_R(k) \}_{k=1}^{n_v-1} \right). \quad \text{(8.6)}
\]

for any permutation \( R \).
8.3 \textbf{SC}_\infty \text{ reparametrization step by step}

A careful proof of Statement 4 requires some work. For a given polygon there is an infinite set of \text{SC}_\infty mappings of the upper complex semiplane to this polygon. Within this infinite set one defines the concept of \text{SC}_\infty mappings differing by an \textit{elementary change}. When can define these elementary changes so that any two \text{SC}_\infty mappings \(\zeta^{(1)}\) and \(\zeta^{(2)}\) representing the same polygon can be connected by a finite chain of sequential elementary changes:

\[
\zeta^{(1)} \to \zeta^{(I)} \to \zeta^{(II)} \to ... \to \zeta^{(2)}. \tag{8.7}
\]

Therefore it is sufficient to prove identity (8.2) only for the case when mappings \(\zeta^{(1)}\) and \(\zeta^{(2)}\) differ by an elementary change.

A careful definition of \textit{elementary changes} involves several subtleties but the general idea is rather simple: elementary changes include:

1) translation in the \(z' = z + b\),
2) dilation \(z' = az\),
3) inversion \(z' = -1/z\).

8.4 \textbf{Linear fractional transformations}

As already mentioned, linear fractional transformations (1.8)

\[
V(z) = \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{R}) \tag{8.8}
\]

map upper complex semiplane \(\mathbb{C}_+\) to itself. Therefore two \textit{general SC} (not necessarily \text{SC}_\infty) mappings \(\zeta^{(1)}\) and \(\zeta^{(2)}\) map semiplane \(\mathbb{C}_+\) to the same polygon if and only if they are connected by a linear fractional transformation \(V\)

\[
\zeta^{(2)}(z) = \zeta^{(1)}(V(z)) \tag{8.9}
\]

or in short

\[
\zeta^{(2)} = \zeta^{(1)} \circ V. \tag{8.10}
\]

In principle, this statement also holds for \text{SC}_\infty mappings \(\zeta^{(1)}\) and \(\zeta^{(2)}\) but there is one subtlety: if

- \(\zeta^{(1)}\) is \text{SC}_\infty mapping,
- \(V\) is linear fractional mapping (1.8)

then \(\zeta^{(1)} \circ V\) is \text{SC} mapping but not necessarily \text{SC}_\infty mapping.

Therefore the precise statement is

\textbf{Statement 5.}

If two \text{SC}_\infty transformations \(\zeta^{(1)}\) and \(\zeta^{(2)}\) map \(\mathbb{C}_+\) to the same polygon then there exists such linear fractional transformation that relation (8.10) holds.

One can be also interested in a different question: starting from a given \text{SC}_\infty mapping which linear fractional transformations \(V\) can be used for \(\zeta^{(2)}\) to be also...
an SC\(\infty\) mapping. The answer is obvious: SC\(\infty\) mappings have one SC vertex at infinity. Linear fractional transformation \(V\) must respect this property.

Hence we have proved

**Statement 6.**

Let \(\zeta^{(1)}\) be SC\(\infty\) mapping with \(n_v - 1\) finite vertices \(\{z_k^{(1)}\}_{k=1}^{n_v-1}\) and one hidden vertex at infinity. Then \(\zeta^{(1)} \circ V\) will be SC\(\infty\) mapping for those and only for those linear fractional mappings \(V\) which have the property

\[
V(\infty) = \infty \quad (8.11)
\]

or

\[
V(\infty) = z_k \text{ for some } k \text{ in the range } 1 \leq k \leq n_v - 1. \quad (8.12)
\]

According to (8.8)

\[
V(\infty) = \frac{a}{c}. \quad (8.13)
\]

Thus we have proved

**Statement 7.**

Let \(\zeta^{(1)}\) be SC\(\infty\) mapping with \(n_v - 1\) finite vertices \(\{z_k^{(1)}\}_{k=1}^{n_v-1}\). Then \(\zeta^{(1)} \circ V\) will be SC\(\infty\) mapping only for the following set of linear fractional transformations \(V(z)\)

- **case 1**

  \[
  V(z) = az + b \quad (a > 0, \ b \in \mathbb{R}) \quad (8.14)
  \]

- **case 2**

  for some \(k\) in the range \(1 \leq k \leq n_v - 1:\)

  \[
  V(z) = z_k - \frac{a}{z + b} \quad (a > 0, \ b \in \mathbb{R}). \quad (8.15)
  \]

### 8.5 Elementary changes of SC\(\infty\) mappings

In section 8.3 it was suggested to define elementary changes of SC\(\infty\) mappings in such a way that any two SC\(\infty\) mappings \(\zeta^{(1)}\) and \(\zeta^{(2)}\) of \(\mathbb{C}_+\) to the same polygon can be connected by a chain of elementary changes (8.7) so that at each step of this chain we have SC\(\infty\) mappings \(\zeta^{(l)}, \zeta^{(l+1)}, \ldots\) to the same polygon.

Statement 7 suggest the following definition of *elementary changes* from SC\(\infty\) mapping \(\zeta^{(1)}(z)\) to SC\(\infty\) mapping \(\zeta^{(2)}(z)\)

1) **translation**

\[
\zeta^{(2)}(z) = \zeta^{(1)}(z + b) \quad (b \in \mathbb{R}), \quad (8.16)
\]

2) **dilation**

\[
\zeta^{(2)}(z) = \zeta^{(1)}(az) \quad (a > 0), \quad (8.17)
\]
3) inversion
\[
\zeta^{(2)}(z) = \zeta^{(1)}(-1/z). \tag{8.18}
\]

Making chain (8.7) of translation (8.17) of dilation (8.16), one can generate case (8.14) of Statement 7.

Inversion (8.18) corresponds to the special case
\[
a = 1, \tag{8.19}
b = 0, \tag{8.20}
z_k = 0 \text{ in } \zeta^{(1)}(z) \tag{8.21}
\]
of case (8.15) in Statement 7. Making a chain containing inversion, translations and dilations, one can generate the general non-elementary modification of SC$_\infty$ mapping.

Note that in cases of translations (8.17) and dilation (8.16) it is sufficient to assume that only \(\zeta^{(1)}(z)\) is SC$_\infty$ mapping. Mappings \(\zeta^{(2)}\) defined by relations (8.17), (8.16) will be automatically SC$_\infty$ mappings.

In case of inversion (8.18) the situation is different. Staring from SC$_\infty$ mappings \(\zeta^{(1)}(z)\) and applying (8.18), we may arrive at non-SC$_\infty$ mapping \(\zeta^{(2)}(z)\). Only if condition (8.21) holds for mapping \(\zeta^{(1)}\) (i.e. \(\zeta^{(1)}\) is defined by eq. (1.10) with \(z_k = 0\) for some \(k\)) then \(\zeta^{(2)}(z)\) will be SC$_\infty$ mapping.

Anyway the set of eqs. (8.17), (8.16) and (8.18) provides a complete set of elementary changes and the problem of the proof of (8.2) reduces to the case of SC$_\infty$ mappings connected by elementary changes.

Let us formulate precisely what we must prove.

**Statement 8.**

If

1) two SC$_\infty$ mappings \(\zeta^{(a)}(z) (a = 1, 2)\) are generated by eq. (1.10) with parameters
\[
\left\{ A^{(a)}, \left\{ \beta_k^{(a)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(a)} \right\}_{k=1}^{n_v-1} \right\} (a = 1, 2), \tag{8.22}
\]

2) these two SC$_\infty$ mappings \(\zeta^{(a)}(z)\) obey one of relations (8.17), (8.16) and (8.18) then

\[
I_m^{\text{ren}} \left( A^{(1)}, \left\{ \beta_k^{(1)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(1)} \right\}_{k=1}^{n_v-1} \right) = I_m^{\text{ren}} \left( A^{(2)}, \left\{ \beta_k^{(2)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(2)} \right\}_{k=1}^{n_v-1} \right). \tag{8.23}
\]

If we prove Statement 8 then it will be the end of the work.

In order to prove Statement 8, we must consider three different cases: (8.17), (8.16) and (8.18). In the case of translations Statement 8 is trivial. The case of dilation requires some simple work. Only in the case of inversion (8.18) the proof of Statement 8 requires a certain effort. Below we concentrate on this problematic case.
8.6 Invariance of $I_{ren}^m$ with respect to inversion

8.6.1 Main result

Thus we must prove Statement 8 for the case of inversion (8.18), e.g. we must prove

Statement 9.

If

1) two SC$_\infty$ mappings $\zeta^{(a)}(z)$ ($a = 1, 2$) are generated by eq. (1.10) with parameters

$$\left\{ A(a), \left\{ \beta_k^{(a)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(a)} \right\}_{k=1}^{n_v-1} \right\} \quad (a = 1, 2), \quad (8.24)$$

2) these two SC$_\infty$ mappings $\zeta^{(a)}(z)$ obey relation

$$\zeta^{(2)}(z) = \zeta^{(1)}(-1/z) \quad (8.25)$$

then

$$I_{ren}^m \left( A(1), \left\{ \beta_k^{(1)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(1)} \right\}_{k=1}^{n_v-1} \right) = I_{ren}^m \left( A(2), \left\{ \beta_k^{(2)} \right\}_{k=1}^{n_v-1}, \left\{ z_k^{(2)} \right\}_{k=1}^{n_v-1} \right). \quad (8.26)$$

Statement 9 assumes that both $\zeta^{(1)}(z)$ and $\zeta^{(2)}(z)$ are SC$_\infty$ mappings. As already discussed, combining this with (8.25), one arrives at (8.21), i.e.

$$z_{k_1}^{(1)} = 0 \quad \text{for some } k_1 \text{ in the range } 1 \leq k_1 \leq n_v - 1. \quad (8.27)$$

Since relation (8.25) is symmetric under exchange $\zeta^{(1)} \leftrightarrow \zeta^{(2)}$, we also have

$$z_{k_2}^{(2)} = 0 \quad \text{for some } k_2 \text{ in the range } 1 \leq k_2 \leq n_v - 1. \quad (8.28)$$

According to results of section 8.2 we are free to change the numeration of SC$_\infty$ parameters $\left\{ z_k^{(1)}, \beta_k^{(1)} \right\}_{k=1}^{n_v-1}$ as we like so that we can simplify property (8.27) $z_{k_1}^{(1)} = 0$ to $z_1^{(1)} = 0$. We can apply the same argument to the set of parameters and reduce property (8.28) $z_{k_2}^{(2)} = 0$ to $z_1^{(2)} = 0$.

Thus we can proceed assuming that

$$z_1^{(1)} = z_1^{(2)} = 0. \quad (8.29)$$

Remember that $\zeta^{(1)}$ and $\zeta^{(2)}$ map $\mathbb{C}_+$ to the same polygon. We can again use the freedom of numeration of parameters $\left\{ z_k^{(a)}, \beta_k^{(a)} \right\}_{k=1}^{n_v-1}$ so that

$$\zeta^{(1)} \left( z_k^{(1)} \right) = \zeta^{(2)} \left( z_k^{(2)} \right) \quad \text{if } 2 \leq k \leq n_v - 1. \quad (8.30)$$

Combining (8.30) and (8.25), we find

$$z_k^{(2)} = -1/z_k^{(1)}. \quad (8.31)$$

Conclusion. It is sufficient to prove Statement 9 for the case when parameters $z_k^{(a)}$ obey relations (8.29), (8.31).
8.6.2 Straightforward way is not efficient

Our aim is to derive relation (8.26) assuming assumptions of Statement 9. At this step one may think about expressing $I^{\text{ren}}_m$ via functions $\Pi^{(n_v-1)}_P$ according to eqs. (1.19), (1.20) and to reduce the derivation of (8.26) to the derivation of certain relations for functions $\Pi^{(n_v-1)}_P$. In principle, one can prove Statement 9 in this way. However,

- this straightforward method requires a rather boring calculation,
- one can learn too little from this calculation because the final result comes as a sort of miracle.

It is much more instructive to use another way.

8.6.3 Alternative representation for analytical regularization

Let us derive another representation for $I^{\text{ren}}_m$

- which is equivalent to (1.19), (1.20) representation
- but allows for a much better tracing of the covariance with respect to inversion (and with respect to other linear fractional transformations).

Note that we want to use the same analytical regularization as before. But we do not want to work in terms of functions $\Pi^{(n_v-1)}_P$. We prefer to work in terms of functions that carry the same information as $\Pi^{(n_v-1)}_P$ (including analytical properties) but in a better representation.

- explicitly showing invariance of the original nonregularized expressions for $I_m$ with respect to inversion (and other fractional transformations),
- demonstrating how analytical regularization modifies the behavior of $I_m$ under inversion,
- showing how invariance with respect to inversion is restored after the analytical renormalization.

In principle, one can derive this new covariant representation for $I^{\text{ren}}_m$ directly from our expressions $I^{\text{ren}}_m$ from (1.19), (1.20) via functions $\Pi^{(n_v-1)}_P$.

But it is much more instructive an much easier to read this new covariant representation from the results of ref. [40]. Eqs. (4.30) and (4.31) in ref. [40] provide non-regularized and ultraviolet divergent expressions for $I_1$ and $I_2$:

$$I_1 = \int_{\mathbb{C}_+} d^2 z \left| \frac{d\zeta}{dz} \right|^{-2} \left| \text{Im } z \right|^{-4},$$  \hspace{1cm} (8.32)

$$I_2 = \int_{\mathbb{C}_+} d^2 z \left| \frac{d\zeta}{dz} \right|^{-2} \left| \{\zeta, z\} \right|^2.$$  \hspace{1cm} (8.33)
Here $\zeta(z)$ is a conformal mapping from $z$ semiplane $\mathbb{C}_+$ to the polygon placed on the plane of complex variable $\zeta$. On the RHS of (8.33) there appears Schwarz derivative $\{\zeta, z\}$ defined by
\[
\{\zeta, z\} = \frac{\zeta'''}{\zeta'} - \frac{3}{2} \left( \frac{\zeta''}{\zeta'} \right)^2
\]  
(8.34)
where the prime stands for derivative $d/dz$:
\[
\zeta' = \frac{d\zeta}{dz}, \quad \zeta'' = \frac{d^2\zeta}{dz^2}, \quad z''' = \frac{d^3\zeta}{dz^3}.
\]  
(8.35)
Representations (8.32) and (8.33) have one bad and one good feature:

- **Integrals** (8.32) and (8.33) are divergent on the boundary of $\mathbb{C}_+$ (i.e. on the real axis and at and at infinity).

- The *integrands* of integrals (8.32) and (8.33) are invariant with respect to linear fractional transformations (8.8) if one combines the integrand and the Jacobian corresponding to the change of the integration variable:
\[
\left| \frac{d\zeta}{dz} \right|^{-2} |\text{Im } z'|^{-4} d^2 z' = \left| \frac{d\zeta}{dz} \right|^{-2} |\text{Im } z|^{-4} d^2 z, 
\]  
(8.36)
\[
\left| \frac{d\zeta'}{dz} \right|^{-2} |\{\zeta, z'\}|^2 d^2 z' = \left| \frac{d\zeta}{dz} \right|^{-2} |\{\zeta, z\}|^2 d^2 z.
\]  
(8.37)
This invariance appears naturally in the context of ref. [40]. Relation (8.36) is trivial. Relation (8.37) follows from standard properties of Schwarz derivative (8.34) with respect to linear fractional transformations.

Let us introduce compact notation
\[
J_1 \left[ \zeta, z \right] = \left| \frac{d\zeta}{dz} \right|^{-2} |\text{Im } z|^{-4}, 
\]  
(8.38)
\[
J_2 \left[ \zeta, z \right] = \left| \frac{d\zeta}{dz} \right|^{-2} |\{\zeta, z\}|^2.
\]  
(8.39)
Then
\[
I_m = \int_{\mathbb{C}_+} d^2 z J_m \left[ \zeta, z \right].
\]  
(8.40)
Under linear fractional transformations (8.8)
\[
J_m \left[ \zeta', z' \right] d^2 z' = J_m \left[ \zeta, z \right] d^2 z.
\]  
(8.41)
Analytical regularization corresponds to the introduction of the temporary factor
\[
|\text{Im } z|^\lambda \prod_{k=1}^{n_k-1} |z - z_k|^{\mu_k}
\]  
(8.42)
in the integrands of the non-regularized version (8.40)
\[ I_{\text{reg}}^{m} \left( \lambda, \{ \mu_k \}_{k=1}^{n_v-1} ; \{ z_k \}_{k=1}^{n_v-1} | \zeta \right) = \int_{C_+} d^2 z J_m [\zeta, z] \prod_{k=1}^{n_v-1} |z - z_k|^{\mu_k} . \]
(8.43)

Here it is assumed that \( \zeta(z) \) is SC\(_{\infty}\) mapping of the form (1.10) and parameters \( z_k \) are taken from (1.10). This regularized version \( I_{\text{reg}}^{m} \) depends on conformal mapping \( \zeta(z) \) representing the polygon so that we include \( \zeta \) in the list of arguments of \( I_{\text{reg}}^{m} \).

Some comments must be made about the list arguments for \( I_{\text{reg}}^{m} \) which contains real parameters \( \{ z_k \}_{k=1}^{n_v-1} \) and SC\(_{\infty}\) mapping \( \zeta \) (understood as a function and not as a complex variable). Obviously objects \( \{ z_k \}_{k=1}^{n_v-1} \) and \( \zeta \) are not independent: they are connected by equation (1.10). Nevertheless in our work with relations containing different SC\(_{\infty}\) mappings \( \zeta^{(a)} \) it is convenient to include both \( \{ z_k^{(a)} \}_{k=1}^{n_v-1} \) and \( \zeta^{(a)} \) in the argument list of \( I_{\text{reg}}^{m} \) for a careful tracing of \( a \)-dependences.

If
- one computes \( J_m [\zeta, z] \) in terms of elementary functions,
- properly represents parameters \( \lambda, \{ \mu_k \}_{k=1}^{n_v-1} \) as a linear combinations of native parameters \( \alpha, \{ \nu_k \}_{k=1}^{n_v-1} \),

then one will see that (8.43) provides the same analytical regularization as eqs. (1.19), (1.20).

However, there is no need to do this work because
- the above statements are a sort of reverse engineering of the work that has already been done in ref. [40];
- in order to prove Statement 9, we do not need explicit expressions for \( \lambda, \{ \mu_k \}_{k=1}^{n_v-1} \) via \( \alpha, \{ \nu_k \}_{k=1}^{n_v-1} \); it is sufficient to know that functions \( I_{\text{reg}}^{m} \) contain essentially the same pole structure as functions \( \Pi^{(n)} \) so that both \( I_{\text{reg}}^{m} \) and \( \Pi^{(n)} \) are regular at the ‘final physical point’ corresponding to the transition from regularization to renormalization.

The rest of the work is straightforward. One must compute integrals (8.43) in the region of \( \lambda, \{ \mu_k \}_{k=1}^{n_v-1} \) where these integrals are convergent and then one must perform analytical continuation to ‘physical point’ \( \lambda = 0, \mu_k = 0 \):
\[ I_{\text{ren}}^{m} = \left[ I_{\text{reg}}^{m} \left( \lambda, \{ \mu_k \}_{k=1}^{n_v-1} ; \{ z_k \}_{k=1}^{n_v-1} | \zeta \right) \right]_{\text{analyt. cont.}} \lambda \to 0, \mu_k \to 0 . \]
(8.44)

### 8.6.4 Inversion and alternative representation for analytical regularization

Now we return to the proof of Statement 9. We want to prove that SC\(_{\infty}\) mappings \( \zeta^{(1)}(z) \) and \( \zeta^{(2)}(z) \) connected by relation (8.18) lead to the same \( I_{\text{ren}}^{m} \).

We can apply eq. (8.44) to both SC\(_{\infty}\) mappings \( \zeta^{(1)} \) and \( \zeta^{(2)} \):
\[ I_{\text{reg}} \left( \lambda, \{ \mu_k \}_{k=1}^{n_v-1}; \left\{ z_k^{(a)} \right\}_{k=1}^{n_v-1} \right) \mid \zeta^{(a)} \right) 
= \int_{C^+} d^2 z J_m \left[ \zeta^{(a)}, z \right] \text{Im} \left( z^{n_v-1} \prod_{k=1}^{n_v-1} \left| z - z_k^{(a)} \right|^{\mu_k} \right). \] (8.45)

Our aim is to show that
\[ I_{\text{ren}} \left( \zeta^{(1)} \right) = I_{\text{ren}} \left( \zeta^{(2)} \right). \] (8.46)

Combining (8.44), (8.46) and Conclusion in the end of section 8.6.1, we see that the proof of Statement 9 reduces to

Statement 10.

If
1) \( \zeta^{(1)} \) and \( \zeta^{(2)} \) are \( \text{SC}_\infty \) mappings related by inversion:
\[ \zeta^{(2)}(z) = \zeta^{(1)}(-1/z), \] (8.47)
2) \( \left\{ z_k^{(a)} \right\} \) are parameters of \( \text{SC}_\infty \) mapping \( \zeta^{(a)} \) as they appear in eq. (1.10),
3) \( z_k^{(a)} \) obey relations
\[ z_1^{(2)} = z_1^{(1)} = 0, \]
\[ z_k^{(2)} = -1/z_k^{(1)} \quad (2 \leq k \leq n_v - 1). \] (8.49)

then
\[ \left[ I_{\text{reg}} \left( \lambda, \{ \mu_k \}_{k=1}^{n_v-1}; \left\{ z_k^{(1)} \right\}_{k=1}^{n_v-1} \right) \right]_{\text{analyt. cont.}} \lambda \to 0, \mu_k \to 0 
= \left[ I_{\text{reg}} \left( \lambda, \{ \mu_k \}_{k=1}^{n_v-1}; \left\{ z_k^{(2)} \right\}_{k=1}^{n_v-1} \right) \right]_{\text{analyt. cont.}} \lambda \to 0, \mu_k \to 0, \] (8.50)

Statement 10 will be derived from Statement 11 in section 8.6.6.

8.6.5 Transformation of \( I_{\text{reg}} \) under inversion

In order to derive Statement 10 we need

Statement 11.

If one
a) makes assumptions 1, 2, 3 of Statement 10,
\[ \lambda^{(1)}, \left\{ \mu_k^{(1)} \right\}_{k=1}^{n_v-1} \]
as independent complex variables,
\[ \lambda^{(2)} = \lambda^{(1)}, \] (8.51)
\[ \mu_1^{(2)} = -2\lambda^{(1)} - \sum_{k=1}^{n_v-1} \mu_k^{(1)}, \quad (8.52) \]

\[ \mu_k^{(2)} = \mu_k^{(1)} \quad (2 \leq k \leq n_v - 1) \quad (8.53) \]

then the following identity holds

\[ I_{\text{reg}}^m \left( \lambda^{(1)}, \left\{ \mu_k^{(1)} \right\}_{k=1}^{n_v-1} ; \left\{ z_k^{(1)} \right\}_{k=1}^{n_v-1} | \zeta^{(1)} \right) = \left( \prod_{k=2}^{n_v-1} |z_k^{(1)}|^{\mu_k^{(1)}} \right) I_{\text{reg}}^m \left( \lambda^{(2)}, \left\{ \mu_k^{(2)} \right\}_{k=1}^{n_v-1} ; \left\{ z_k^{(2)} \right\}_{k=1}^{n_v-1} | \zeta^{(2)} \right) . \quad (8.54) \]

**Remark.** LHS and RHS of eq. (8.54) are meromorphic functions of independent complex variables \( \lambda^{(1)}, \left\{ \mu_k^{(1)} \right\}_{k=1}^{n_v-1} \) at fixed mappings \( \zeta^{(a)} \) and fixed \( \left\{ z_k^{(a)} \right\}_{k=1}^{n_v-1} \).

**8.6.6 From regularization to renormalization**

Statement 10 is a trivial consequence of Statement 11. Indeed, point

\[ \lambda^{(1)} = 0, \mu_k^{(1)} = 0 \quad (1 \leq k \leq n_v - 1) \quad (8.55) \]

is a regular point of the meromorphic function represented by identity (8.54). According to eqs. (8.51) – (8.53) at this point we have

\[ \lambda^{(2)} = 0, \mu_k^{(2)} = 0 \quad (1 \leq k \leq n_v - 1) . \quad (8.56) \]

This completes the derivation of Statement 10 from Statement 11.

Now only one problem remains: we must prove Statement 11.

**8.6.7 Final step: proof of Statement 11**

Let us set in eq. (8.45)

\[ a = 1 \quad (8.57) \]

and let us add superscript (1) to variables \( \lambda, \left\{ \mu_k \right\}_{k=1}^{n_v-1} : \)

\[ I_{\text{reg}}^m \left( \lambda^{(1)}, \left\{ \mu_k^{(1)} \right\}_{k=1}^{n_v-1} ; \left\{ z_k^{(1)} \right\}_{k=1}^{n_v-1} | \zeta^{(1)} \right) = \int_{\mathbb{C}^+} d^2z J_m \left[ \zeta^{(1)}, z \right] \left| \text{Im} \ z \right|^{\lambda^{(1)}} \prod_{k=1}^{n_v-1} \left| z - z_k^{(1)} \right|^{\mu_k^{(1)}} . \quad (8.58) \]
We know that the convergence region of this integral in the $\lambda^{(1)}, \{\lambda^{(1)}\}_{k=1}^{n_v-1}$ space is non-empty and we work in this convergence region. We change the integration variable $z = -1/z'$ and we use eqs. (8.41), (8.48). We obtain

\[
I_{m}^{\text{reg}} \left( \lambda^{(1)}, \left\{ \mu_{k}^{(1)} \right\}_{k=1}^{n_v-1} ; \left\{ z_{k}^{(1)} \right\}_{k=1}^{n_v-1} | \varphi^{(1)} \right) \\
= \int_{\mathbb{C}_{+}} d^{2}z' J_{m} \left[ \zeta^{(2)}, z' \right] \text{Im} \left[ - (z')^{-1} \right] \prod_{k=1}^{n_v-1} | (z')^{-1} - z_{k}^{(1)} |^{\mu_{k}^{(1)}}. \tag{8.59}
\]

Using (8.48), (8.49), we obtain

\[
\text{Im} \left[ - (z')^{-1} \right] \prod_{k=1}^{n_v-1} | (z')^{-1} - z_{k}^{(1)} |^{\mu_{k}^{(1)}} \\
= |z'|^{-2} \text{Im} \left[ z' \right] |z'|^{-\mu_{1}^{(1)}} \prod_{k=2}^{n_v-1} | (z')^{-1} - z_{k}^{(1)} |^{\mu_{k}^{(1)}} \\
= \text{Im} \left[ z' \right] |z'|^{-\mu_{1}^{(1)}} - 2\lambda^{(1)} \prod_{k=2}^{n_v-1} | (z')^{-1} z_{k}^{(1)} |^{\mu_{k}^{(1)}} \prod_{k=2}^{n_v-1} | z' + (z_{k}^{(1)})^{-1} |^{\mu_{k}^{(1)}} \\
= \left( \prod_{k=2}^{n_v-1} | z_{k}^{(1)} |^{\mu_{k}^{(1)}} \right) \text{Im} \left[ z' \right] |z' - 2\lambda^{(1)} \sum_{k=1}^{n_v-1} \mu_{k}^{(1)} \prod_{k=2}^{n_v-1} | z' - z_{k}^{(2)} |^{\mu_{k}^{(1)}} \\
= \left( \prod_{k=2}^{n_v-1} | z_{k}^{(1)} |^{\mu_{k}^{(1)}} \right) \text{Im} \left[ z' \right] |z' - z_{2}^{(2)} |^{-2\lambda^{(1)} - \sum_{k=1}^{n_v-1} \mu_{k}^{(1)}} \prod_{k=2}^{n_v-1} | z' - z_{k}^{(2)} |^{\mu_{k}^{(1)}}. \tag{8.60}
\]

Hence

\[
I_{1}^{\text{reg}} \left( \lambda^{(1)}, \left\{ \mu_{k}^{(1)} \right\}_{k=1}^{n_v-1} ; \left\{ z_{k}^{(1)} \right\}_{k=1}^{n_v-1} | \varphi^{(1)} \right) \\
= \int_{\mathbb{C}_{+}} d^{2}z' J_{m} \left[ \zeta^{(2)}, z' \right] \\
\times \left( \prod_{k=2}^{n_v-1} | z_{k}^{(1)} |^{\mu_{k}^{(1)}} \right) \text{Im} \left[ z' \right] |z' - z_{2}^{(2)} |^{-2\lambda^{(1)} - \sum_{k=1}^{n_v-1} \mu_{k}^{(1)}} \prod_{k=2}^{n_v-1} | z' - z_{k}^{(2)} |^{\mu_{k}^{(1)}}. \tag{8.61}
\]
Let us change the notation of the integration variable from $z'$ to $z$ and use (8.53)

\[ I_{\text{reg}}^{m}(\lambda(1), \{\mu_k^{(1)}\}_{k=1}^{n_v-1}; \{z_k^{(1)}\}_{k=1}^{n_v-1} | \zeta^{(1)}) \]

\[ = \left( \prod_{k=2}^{n_v-1} |z_k^{(1)}|^{\mu_k^{(1)}} \right) \int_{C} d^2 z J_m \left[ \zeta^{(2)}, z \right] \]

\[ \times |\text{Im } z|^\lambda^{(2)} \prod_{k=1}^{n_v-1} \left| z - z_k^{(2)} \right|^{\mu_k^{(2)}}. \]  

(8.62)

Now let us set $a = 2$ in eq. (8.45)

\[ I_{\text{reg}}^{m}(\lambda^{(2)}, \{\mu_k^{(2)}\}_{k=1}^{n_v-1}; \{z_k^{(2)}\}_{k=1}^{n_v-1} | \zeta^{(2)}) \]

\[ = \int_{C} d^2 z J_m \left[ \zeta^{(2)}, z \right] |\text{Im } z|^\lambda^{(2)} \prod_{k=1}^{n_v-1} \left| z - z_k^{(2)} \right|^{\mu_k^{(2)}}. \]  

(8.63)

Comparing eqs. (8.62) and (8.62), we obtain identity (8.54).

Thus Statement 11 is proved. This also completes the proof the main Statement 4.

9 Conclusions

The results of this work generalize observations made in ref. [40] for triangular diagrams to the case of arbitrary polygons. We have proved that analytical regularization formulated in terms of SC\(_{\infty}\) mapping is internally consistent:

- Integrals representing functions $\Pi_P^{(n)}$ are convergent in a non-empty region of complex parameters $\alpha, \{\gamma_k\}_{k=1}^{n}$.
- After defining $\Pi_P^{(n)}$ in this convergence region one can perform analytical continuation in $\mathbb{C}^{n+1}$ space of parameters $\alpha, \{\gamma_k\}_{k=1}^{n}$.
- Resulting function $\Pi_P^{(n)}$ is meromorphic in $\mathbb{C}^{n+1}$, i.e. it has only poles but no branching singularities.
- Analytical continuation of functions $\Pi_P^{(n)}$ to the physical point (i.e. to the values of $\alpha, \{\gamma_k\}_{k=1}^{n}$ needed for the calculation of $I_{\text{ren}}^{m}$) is regular so that our analytical renormalization provides finite results for renormalized quantities $I_{\text{ren}}^{m}$.
- The absence of branching singularities in $\Pi_P^{(n)}$ means that analytical continuation of $\Pi_P^{(n)}$ from the convergence region to the physical point does not depend on the path of continuation, i.e. our analytical regularization has no ambiguities.
• The final result for $I_{m}^{\text{ren}}$ is independent of the SC$_{\infty}$ parametrization of the polygon.

These results put analytical regularization in terms of SC$_{\infty}$ mapping on solid ground.

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A Pole structure of $\Pi_{1}^{(2)}$

In this appendix we check that function $\Pi_{1}^{(2)}$ has a pole structure compatible with the general pole representation (1.27). In this special case:

$$n = 2,$$

(A.1)

function $\Pi_{1}^{(n)}$ was computed in ref. [40]:

$$P(z, z^*) = 1$$

(A.2)

$\Pi_{1}^{(2)}(\alpha, \{\gamma_{1}, \gamma_{2}\}, \{z_{1}, z_{2}\})$

$$= \left[\frac{\sqrt{\pi}}{2} \left|z_{1} - z_{2}\right|^{-\gamma_{3}} \Gamma\left(\frac{\alpha}{2}\right) \prod_{k=1}^{3} \Gamma\left(\alpha + 1 - \frac{\gamma_{k}}{2}\right)\right]_{\gamma_{3}=(\alpha+1)-(\gamma_{1}+\gamma_{2})}.$$  (A.3)

The product on the RHS can be rearranged:

$$\Pi_{1}^{(2)}(\alpha, \{\gamma_{1}, \gamma_{2}\}, \{z_{1}, z_{2}\})$$

$$= \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\gamma_{1}}{2}\right) \Gamma\left(\frac{\gamma_{2}}{2}\right) \Gamma\left(\alpha + 1 - \frac{\gamma_{1} + \gamma_{2}}{2}\right) H_{1}^{(2)}(\alpha, \{\gamma_{1}, \gamma_{2}\}, \{z_{1}, z_{2}\}).$$  (A.4)

Here

$$H_{1}^{(2)}(\alpha, \{\gamma_{1}, \gamma_{2}\}, \{z_{1}, z_{2}\})$$

$$= \left\{\frac{\sqrt{\pi}}{2} \left|z_{1} - z_{2}\right|^{-\gamma_{3}} \prod_{k=1}^{3} \Gamma\left(\frac{\alpha + 1 - \frac{\gamma_{k}}{2}}{2}\right)\right\}^{-1}$$

(A.5)
is an entire function of \( \alpha, \{ \gamma_1, \gamma_2 \} \) (at any fixed real \( z_1 \neq z_2 \)).

Expression (A.4) agrees with the general representation (1.27). Indeed, in the case of trivial polynomial \( P = 1 \) we have according to (3.33):

\[
M_P = 0
\]  

(A.6)

so that in eq. (1.27) the \( M_P \) dependent \( \Gamma \) function simplifies to

\[
\Gamma \left( \frac{1}{2} \left( -2M_P + (n - 1) (\alpha + 1) - \sum_{k=1}^{n} \gamma_k \right) \right) = \Gamma \left( \frac{\alpha + 1 - \gamma_1 + \gamma_2}{2} \right).
\]  

(A.7)

\section{B Integral \( I (\alpha, \gamma) \)}

In this appendix we determine the convergence region of integral

\[
I (\alpha, \gamma) = \int_{|z| < 1} d^2z \, |\text{Im } z|^{\alpha-1} |z|^{-\alpha-1}
\]  

(B.1)

and compute it in this convergence region.

In terms of real integration variables

\[
x = \text{Re } z ,
\]  

(B.2)

\[
y = \text{Im } z
\]  

(B.3)

we have

\[
I (\alpha, \gamma) = \int_{x^2 + y^2 < 1} dx dy \, y^{\alpha-1} \left( x^2 + y^2 \right)^{\frac{\alpha-1}{2}} \\
= 4 \int_{x>0, y>0, x^2 + y^2 < 1} dx dy \, y^{\alpha-1} \left( x^2 + y^2 \right)^{\frac{\alpha-1}{2}}.
\]  

(B.4)

Next we change integration variables:

\[
p = x^2,
\]  

(B.5)

\[
q = y^2.
\]  

(B.6)

Then

\[
I (\alpha, \gamma) = \int_{0 < p, q \atop p + q < 1} dpdq \, q^{\frac{\alpha}{2}-1} p^{-1/2} (p + q)^{\frac{\alpha-1}{2}}.
\]  

(B.7)

Our next change of integration variables is

\[
q = st,
\]  

(B.8)
\[ p = (1 - s) t. \]  
(B.9)

with Jacobian
\[ \frac{D(p, q)}{D(t, s)} = \det \begin{pmatrix} 1 - s & -t \\ s & t \end{pmatrix} = t. \]  
(B.10)

Then
\[ I(\alpha, \gamma) = \int_0^1 ds \int_0^1 dt \frac{(st)^{\alpha - 1} [(1 - s) t]^{-1/2} t^{\frac{\alpha - 1}{2}}}{s^{\frac{\alpha - 1}{2}} t^{\frac{\gamma - 1}{2}}} = \int_0^1 ds \int_0^1 dt \frac{(1 - s)^{-1/2} (1 - s)^{-1/2}}{s^{\frac{\alpha - 1}{2}} t^{\frac{\gamma - 1}{2}}} \left[ \int_0^1 dt t^{\frac{\gamma - 1}{2}} \right]. \]  
(B.11)

The two integrals on the RHS are (absolutely) convergent if two conditions hold:
\[ \Re\alpha > 0, \]  
(B.12)

\[ \Re\gamma > 0. \]  
(B.13)

The integrals on the RHS of (B.11) can be easily computed (in their convergence regions):
\[ \int_0^1 ds s^{\frac{\alpha - 1}{2}} (1 - s)^{-1/2} = B \left( \frac{\alpha}{2}, \frac{1}{2} \right) = \sqrt{\pi} \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{\alpha + 1}{2} \right)}, \]  
(B.14)

\[ \int_0^1 dt t^{\frac{\gamma - 1}{2}} = \frac{2}{\gamma}. \]  
(B.15)

and we arrive at

**Statement.**

Integral
\[ I(\alpha, \gamma) = \int_{|z|<1} d^2z |\text{Im} \ z|^{\alpha - 1} |z|^{\gamma - \alpha - 1} \]  
(B.16)

is absolutely convergent if and only if
\[ \Re\alpha > 0, \]  
(B.17)

\[ \Re\gamma > 0. \]  
(B.18)

and in this region of \(\alpha, \gamma\)
\[ I(\alpha, \gamma) = \frac{2\sqrt{\pi}}{\gamma} \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{\alpha + 1}{2} \right)}. \]  
(B.19)
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