A \( C^1 \)-CONFORMING PETROV-GALERKIN METHOD FOR CONVECTION-DIFFUSION EQUATIONS AND SUPERCONVERGENCE ANALYSIS OVER RECTANGULAR MESHES

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Abstract. In this paper, a new \( C^1 \)-conforming Petrov-Galerkin method for convection-diffusion equations is designed and analyzed. The trial space of the proposed method is a \( C^1 \)-conforming \( Q_k \) (i.e., tensor product of polynomials of degree at most \( k \)) finite element space while the test space is taken as the \( L^2 \) (discontinuous) piecewise \( Q_{k-2} \) polynomial space. Existence and uniqueness of the numerical solution is proved and optimal error estimates in all \( L^2, H^1, H^2 \)-norms are established. In addition, superconvergence properties of the new method are investigated and superconvergence points/lines are identified at mesh nodes (with order \( 2k-2 \) for both function value and derivatives), at roots of a special Jacobi polynomial, and at the Lobatto lines and Gauss lines with rigorous theoretical analysis. In order to reduce the global regularity requirement, interior a priori error estimates in the \( L^2, H^1, H^2 \)-norms are derived. Numerical experiments are presented to confirm theoretical findings.

1. Introduction

The Petrov-Galerkin method, using different trail and test spaces, has been widely used in solving various partial differential equations such as second-order wave equations \[31\], electromagnetic problems \[3, 5\], fluid mechanic equations \[8, 25\], and so on. Classified by the continuity of the approximation space, the existing Petrov-Galerkin method can be roughly divided into three categories, i.e., the family of \( C^0 \) elements that require the continuity of the numerical solution, the \( L^2 \) elements (was also called discontinuous Petrov-Galerkin method) whose trail space is not necessary to be continuous, and the \( C^1 \) elements that require the continuity of the trial space and its first-order derivatives. Comparing with the \( C^0 \) and \( L^2 \) elements Petrov-Galerkin method (see e.g., \[11, 21, 22\]) or the counterpart \( C^1 \) finite element method (see e.g., \[28, 32, 24\]), the \( C^1 \) Petrov-Galerkin method is still far from fully developed.

The Petrov-Galerkin method we study in this paper is \( C^1 \)-conforming, where we use \( C^1 \)-conforming piecewise \( Q_k \) polynomials (the tensor product space) as
trial functions and the $L^2$ discontinuous piecewise $Q_{k-2}$ polynomials as test functions. Comparing with the continuous Galerkin (i.e., $C^0$ element) and discontinuous Galerkin (i.e., $L^2$ element) methods, the most attractive feature of the proposed $C^1$-conforming method is the continuity of the derivative approximation across the element interface. Note that the total degrees of freedom of the $C^1$-conforming method is the same or less than the counterpart $C^0$ and/or $L^2$ element methods over rectangular meshes with the same accuracy. In other words, the $C^1$-conforming method provides a better approximation for derivatives (including the second-order derivatives) without increasing the computational cost. Furthermore, the discontinuous test space is used so that the test functions can be locally computed on each element, which makes the assembly of global matrices simpler than the counterpart $C^1$-conforming finite element method, where some test functions across several elements.

The objective of the present study is to develop a $C^1$-$L^2$ pair of Petrov-Galerkin method (i.e., the trial space is $C^1$ while the test space is chosen as $L^2$), using the two-dimensional convection-diffusion equations as model problems. We provide a unified mathematical approach to establish convergence theory for the proposed method including the optimal error estimates in all $H^1$, $L^2$, $H^2$-norms and superconvergence results at some special points and lines. Note that superconvergence behavior has been investigated for many years. For an incomplete list of references, we refer to [4, 9, 17, 23, 26, 33, 34, 40] for $C^0$ finite element methods, and [10, 14, 16, 20, 36] for $C^0$ finite volume methods, [4] for discontinuous Galerkin methods, and [55, 59] for spectral Galerkin methods. Regardless of rich literatures on the superconvergence study, the relevant work for $C^1$ element methods is far from satisfied. Only very special and simple cases have been discussed (see, e.g., [34, 7, 6]). To the best of our knowledge, no superconvergence analysis of the $C^1$ Petrov-Galerkin method has been published yet until our recent work on $C^1$ Petrov-Galerkin and Gauss collocation methods for 1D two-point boundary value problems in [11].

The main superconvergence results established in this paper include: 1) $h^{2k-2}$ superconvergence rate for approximations of both function value and the first-order derivatives at mesh nodes; 2) $h^{k+2}$ superconvergence rate for the function value approximation at roots of a special Jacobi polynomial; 3) $h^{k+1}$ superconvergence rate for the first-order and $h^k$ superconvergence rate for the second-order derivative approximations at Lobatto lines and Gauss lines, respectively; 4) as a by-product, we also prove that the Petrov-Galerkin solution is superconvergent towards a particular Jacobi projection of the exact solution in $H^2$, $H^1$, and $L^2$-norms. By doing so, we present a full picture of superconvergence theory for the $C^1$ Petrov-Galerkin method, which gives us some insights into the difference among the $C^0$, $C^1$, $L^2$ element methods. We have found that the superconvergence points of the solution and its first-order derivative for the $C^1$ Petrov-Galerkin method are different from those for the existing $C^0$ Galerkin methods (e.g., FEM, FVM) and $L^2$ discontinuous Galerkin methods. The superconvergence of the second-order derivative approximation for the $C^1$ Petrov-Galerkin method is also novel. Comparing with the $Q_k$ $C^0$ Galerkin method (see, e.g., FEM in [17], FVM in [16]) for the Poisson equation over rectangular meshes, which converges with rate $h^{2k}$ at nodal points, the convergence rate at mesh nodes for the $C^1$ Petrov-Galerkin method drops to $h^{2k-2}$, while
the convergence rate of the first-order derivative at mesh nodes improves from $h^k$ to $h^{2k-2}$, which almost doubles the optimal convergence rate $h^k$.

The rest of the paper is organized as follows. In Section 2, we present a $C^1$-$L^2$ Petrov-Galerkin method for two-dimensional convection-diffusion equations over rectangular meshes. In Section 3, we prove the existence and uniqueness of the numerical scheme. In Section 4, we construct a $C^1$-conforming Jacobi projection of the exact solution and study the approximation and superconvergence properties of the special Jacobi projection. Section 5 is the main and most technical part, where optimal error estimates in $L^2, H^1, H^2$-norms and superconvergence behavior at the mesh points (for solution and its first-order derivative approximations), at interior roots of Jacobi polynomials (solution approximation), at Lobatto lines (the first-order derivative approximation) and Gauss lines (the second-order derivative approximation) are investigated. In Section 6, we establish some interior a priori error estimates in $H^2, H^1, L^2$-norms. Numerical experiments supporting our theory are presented in Section 7. Some concluding remarks are provided in Section 8.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\| \cdot \|_{m,p,D}$ and semi-norm $| \cdot |_{m,p,D}$. When $D = \Omega$, we omit the index $D$; and if $p = 2$, we set $W^{m,p}(D) = H^{m}(D)$, $\| \cdot \|_{m,p,D} = \| \cdot \|_{m,D}$, and $| \cdot |_{m,p,D} = | \cdot |_{m,D}$. Notation $A \lesssim B$ implies that $A$ can be bounded by $B$ multiplied by a constant independent of the mesh size $h$. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$.

2. A $C^1$ Petrov-Galerkin method

We consider the following convection-diffusion problem

$$- \nabla \cdot (\alpha \nabla u) + \beta \cdot \nabla u + \gamma u = f, \quad \text{in } \Omega = (a, b) \times (c, d),$$

\[ u = 0, \quad \text{on } \partial \Omega, \]

where $\alpha \geq \alpha_0 > 0, \gamma - \frac{\nabla \cdot \beta}{2} \geq 0, \gamma \geq 0, \alpha, \beta = (\beta_1, \beta_2), \gamma \in L^\infty(\Omega)$, and $f$ is real-valued function defined on $\Omega$. Without loss of generality, we assume that $\alpha, \beta, \gamma$ are all constants. The assumption is not essential since the analysis can be applied to that for variable coefficients as long as the above conditions are satisfied.

Let $a = x_0 < x_1 < \cdots < x_M = b$ and $c = y_0 < y_1 < \cdots < y_N = d$. For any positive integer $r$, we define $\mathbb{Z}_r = \{1, 2, \ldots, r\}$, and denote by $T_h$ the rectangular partition of $\Omega$. That is,

$$T_h = \{ \mathcal{T}_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] : (i, j) \in \mathbb{Z}_M \times \mathbb{Z}_N \}. $$

For any $\mathcal{T} \in T_h$, we denote by $h_x, h_y$ the lengths of $x$- and $y$-directional edges of $\mathcal{T}$, respectively. $h$ is the maximal length of all edges, and $h_{\min} = \min_{\mathcal{T}} (h_x, h_y)$. We assume that the mesh $T_h$ is quasi-uniform in the sense that there exists a constant $c$ such that $h \leq c h_{\min}$.

We define the $C^1$ finite element space as follows:

$$V_h := \{ v \in C^1(\Omega) : v|_{\mathcal{T}} \in \mathbb{Q}_k(x, y) = \mathbb{P}_k(x) \times \mathbb{P}_k(y), \mathcal{T} \in T_h \},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree not more than $k$. Let

$$V_h^0 := \{ v \in V_h : v|_{\partial \Omega} = 0 \}.$$
To design the Petrov-Galerkin method, we define the test space $W_h$ as follows:

\[(2.2) \quad W_h := \{ v \in L^2(\Omega) : v|_\tau \in Q_{k-2}(x, y) = P_{k-2}(x) \times P_{k-2}(y), \tau \in \mathcal{T}_h \}. \]

Then the $C^1$ Petrov-Galerkin method for solving (2.1) is: Find a $u_h \in V_h^0$ such that

\[(2.3) \quad a(u_h, v_h) := -\nabla \cdot (a \nabla u_h) + \beta \cdot \nabla u_h + \gamma u_h, v_h = (f, v_h), \quad \forall v_h \in W_h. \]

Here $(u, v) = \sum_{\tau \in \mathcal{T}_h} \int_{\tau} (uv)(x, y)dxdy$.

We would like to point out that the method we proposed here is only one of the several ways to define a $C^1$ Petrov-Galerkin method. Actually, different choices of the test space $W_h$ may lead to different numerical schemes. For example, other than the $L^2$ test space, we can also choose the $C^0$ space as our test space, i.e., $W_h \subset C^0(\Omega)$ is some subspace of the continuous finite element space. Throughout this paper, we focus our analysis on the $L^2$ test space, i.e., $W_h \subset L^2(\Omega)$ is defined by (2.2).

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of the $C^1$ Petrov-Galerkin method (2.3). We begin with some estimates for the bilinear form $a(\cdot, \cdot)$ of the Petrov-Galerkin method, which plays an important role in our later analysis.

**Lemma 3.1.** Given any $v \in V_h^0$, suppose that $\varphi \in H^2(\Omega)$ is the solution the following dual problem:

\[(3.1) \quad -\nabla \cdot (a \nabla \varphi) - \beta \cdot \nabla \varphi + \gamma \varphi = v \quad \text{in} \quad \Omega, \quad \text{and} \quad \varphi = 0, \quad \text{on} \quad \partial \Omega.
\]

Denote by $I_h \varphi \in Q_1 \subseteq W_h$ the bi-linear interpolation function of $\varphi$. Then

\[(3.2) \quad ||v||_0^2 \lesssim h^4(||v_{xx}||_0^2 + ||v_{xy}||_0^2) + |a(v, I_h \varphi)|,
\]

\[(3.3) \quad ||\Delta v||_0^2 + ||v_{xx}||_0^2 + ||v_{xy}||_0^2 \lesssim |a(v, v_{xyy})| + ||v||_0^2. \]

**Proof.** First, from the dual problem (3.1) and the integration by parts, we have

\[
\begin{align*}
||v||_0^2 & = (v, -(\nabla \cdot (a \nabla \varphi) - \beta \cdot \nabla \varphi + \gamma \varphi) = (v, -\nabla \cdot (a \nabla v) + \beta \cdot \nabla v + \gamma v, \varphi) \\
& = (v, -\nabla \cdot (a \nabla v) + \beta \cdot \nabla v + \gamma v, \varphi - I_h \varphi + I_h \varphi) \\
& \lesssim h^2(\|v_{x}\|_0 + ||v||_1)||\varphi||_2 + |a(v, I_h \varphi)| \lesssim h^2(||\Delta v||_0 + h^{-1}||v||_0)||v||_0 + |a(v, I_h \varphi)|.
\end{align*}
\]

Here in the last step, we have used the $H^2$ regularity $||\varphi||_2 \lesssim ||v||_0$ and the inverse inequality

$$||v||_1 \lesssim h^{-1}||v||_0, \quad \forall v \in V_h.$$

Consequently, if $h$ is sufficiently small, then

\[(3.4) \quad ||v||_0^2 \lesssim h^4||\Delta v||_0^2 + |a(v, I_h \varphi)|. \]

On the other hand, noticing that for any function $v \in V_h^0$, $\partial^i_x v, i \geq 1$ is continuous about $y$ satisfying $\partial^i_x v(x, c) = \partial^i_x v(x, d) = 0$. Similarly, $\partial^i_y v, i \geq 1$ is a continuous function about $x$ satisfying $\partial^i_y v(a, y) = \partial^i_y v(b, y)$. Then

$$v_{xx}(x, y) = \int_c^y v_{xx}(x, y)dy, \quad v_{yy}(x, y) = \int_a^x v_{yy}(x, y)dx.$$ 

By Poincaré inequality,

$$||v_{xx}||_0 + ||v_{yy}||_0 \lesssim ||v_{xx}||_0 + ||v_{yy}||_0.$$
Therefore,
\[ \|\Delta v\|_0 \leq \|v_{xx}\|_0 + \|v_{yy}\|_0 \lesssim \|v_{xxy}\|_0 + \|v_{yyx}\|_0, \]
which yields (together with (3.4)) the desired result (3.2).

We next consider (3.3). By integration by parts, the inverse inequality and (3.2), there holds for any positive constant \( \epsilon \),
\[ (v_{xy}, v_{xy}) = -(v_x, v_{xxy}) \leq \frac{1}{4\epsilon} \|v\|_0^2 + \epsilon \|v_{xxy}\|_0^2 \]
\[ \leq C\|v\|_0^2 + \epsilon (\|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2 + \|\Delta v\|_0^2). \]
Consequently,
\[ |(\beta \cdot \nabla v, v_{xxy})| = |(v_{xy}, \beta_1 v_{xxy} + \beta_2 v_{yyx})| \leq \frac{c_0}{\alpha} \|v_{xxy}\|_0^2 + \frac{\alpha}{4} (\|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2) \]
\[ \leq \left( \frac{c_0 \epsilon}{\alpha} + \frac{\alpha}{4} \right) (\|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2) + C\|v\|_0^2 + \frac{c_0 \epsilon}{\alpha} \|\Delta v\|_0^2, \]
where \( c_0 = \max(\beta_1^2, \beta_2^2) \). Recalling the definition of \( a(\cdot, \cdot) \) and using the integration by parts again, we derive that
\[ a(v, v_{xxy}) = \alpha (\|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2) + \gamma \|v_{xy}\|_0^2 + (\beta \cdot \nabla v, v_{xxy}) \]
\[ \geq \frac{3\alpha}{4} - \frac{c_0 \epsilon}{\alpha} (\|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2 - \epsilon \|\Delta v\|_0^2 + \gamma \|v_{xy}\|_0^2 - C\|v\|_0^2) \]
\[ \geq \frac{3\alpha}{4} - C_0 \|v_{xxy}\|_0^2 + \|v_{yyx}\|_0^2) + \gamma \|v_{xy}\|_0^2 - C\|v\|_0^2. \]

By choosing a small \( \epsilon \), we obtain \( (3.3) \) directly. This finishes our proof. \( \square \)

Now we are ready to prove the existence and uniqueness results for the \( C^1 \) Petrov-Galerkin method.

**Theorem 3.2.** The \( C^1 \) Petrov-Galerkin method (2.3) has one and only one solution, provided that the mesh size is sufficiently small.

**Proof.** We shall prove that the homogeneous problem has a unique zero solution. To this end, we assume that \( f = 0 \) and prove the numerical scheme (2.3) admits a solution \( u_h = 0 \).

Noticing that \( \partial_{xxy}^2 u_h \in W_h, I_h \varphi \in W_h \) for any function \( \varphi \), we have
\[ a(u_h, \partial_{xxy}^2 u_h) = 0, \quad a(u_h, I_h \varphi) = 0. \]
Then from (3.2) and (3.3), we have \( \|u_h\|_0 = 0 \) and thus
\[ u_h \equiv 0. \]
This finishes the proof. \( \square \)

4. A specially constructed Jacobi projection

In this section, we define a \( C^1 \) Jacobi projection of the exact solution and study its approximation property, which is essential for the establishment of the superconvergence results for the numerical solution \( u_h \), especially the discovery of superconvergence points.
We begin with some preliminaries. First, we introduce the four Hermite interpolant basis functions on the interval $[-1, 1]$, which are given by

\[
\psi_{-1}(s) = \frac{1}{4}(s + 2)(1 - s)^2, \quad \psi_1(s) = \frac{1}{4}(2 - s)(1 + s)^2,
\]
\[
\chi_{-1}(s) = \frac{1}{4}(s + 1)(1 - s)^2, \quad \chi_1(s) = \frac{1}{4}(s - 1)(1 + s)^2.
\]

Second, we denote by $J_{n}^{r,l}(s)$, $r, l > -1$, the standard Jacobi polynomials of degree $n$ over $(-1, 1)$, which are orthogonal with respect to the Jacobi weight function $\omega_{r,l}(s) := (1 - s)^r(1 + s)^l$. That is,

\[
\int_{-1}^{1} J_{n}^{r,l}(s) J_{m}^{r,l}(s) \omega_{r,l}(s) ds = \kappa_{n,m}^{r,l} \delta_{mn},
\]

where $\delta$ denotes the Kronecker symbol and $\kappa_{n,m}^{r,l} = \|J_{n}^{r,l}\|_{\omega_{r,l}}^2$. Note that when $r = l = 0$, the Jacobi polynomial $J_{n}^{r,l}$ is reduced to the standard Legendre polynomial. That is $J_{n}^{0,0}(s) = L_n(s)$ with $L_n(s)$ being the Legendre polynomial of degree $n$ over $[-1, 1]$. We extend the definition of the classical Jacobi polynomials to the case where both parameters $r, l \leq -1$

\[
J_{n}^{r,l}(s) := (1 - s)^{-r}(1 + s)^{-l} J_{n+r+l}^{r,l}(s), \quad r, l \leq -1.
\]

By taking $r = l = -2$ in (4.1), we get a sequence of Jacobi polynomials $\{J_{n}^{2,-2}\}_{n=4}$ with

\[
J_{n}^{2,-2}(s) := (1 - s)^2(1 + s)^2 J_{n-4}^{2,2}(s), \quad \forall n \geq 4.
\]

Apparently, there holds

\[
J_{n}^{2,-2}(\pm 1) = 0, \quad \partial_s J_{n}^{2,-2}(\pm 1) = 0.
\]

Denoting

\[
J_{0}^{2,-2}(s) = \psi_{-1}(s), \quad J_{1}^{2,-2}(s) = \psi_1(s), \quad J_{2}^{2,-2}(s) = \chi_{-1}(s), \quad J_{3}^{2,-2}(s) = \chi_1(s),
\]

then $\{J_{n}^{2,-2}\}_{n=0}^{\infty}$ constitutes the basis function of $C^1$ over $[-1, 1]$. We also refer to [30] for more detailed information and discussions about the Jacobi polynomials.

Third, we denote by $\phi_{n+1}$ for $n \geq 1$ the Lobatto polynomial of degree $n + 1$ over $[-1, 1]$, which is defined by

\[
\phi_{n+1}(s) = \int_{-1}^{s} L_n(s) ds = \frac{1}{2n+1}(L_{n+1} - L_{n-1}) = \frac{1}{n(n+1)}(s^2 - 1) L'_n(s).
\]

The above Jacobi and Lobatto polynomials will be frequently used in our later superconvergence analysis.

Now we are ready to present the truncated Jacobi projection. Given any function $u \in C^1(\Omega)$, suppose $u(x, y)$ has the following Jacobi expansion in each element $\tau_{ij}$, $(i, j) \in Z_M \times Z_N$

\[
u(x, y) |_{\tau_{ij}} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} u_{pq} J_{ij}^{2,-2}(x) J_{j,q}^{2,-2}(y),
\]
where \( J_p^{\alpha\beta}(x) = J_p^{\alpha\beta}(\frac{2x-x_1-x_1}{h_1}) = J_p^{\alpha\beta}(s), \quad s \in [-1,1], \) is the Jacobi polynomial of degree \( p \) over \((x_{i-1}, x_i), \) and \( u_{pq} \) are some coefficients to be determined. Then the truncated Jacobi projection \( u_I \in V_h \) of \( u \) is defined as follows:

\[
u_I(x,y)|_{\tau_{ij}} := \sum_{p=0}^{k} \sum_{q=0}^{k} u_{pq} J_{i,p}^{\alpha\beta}(x) J_{j,q}^{\alpha\beta}(y).
\]

Note that when \( k = 3 \), the truncated Jacobi projection \( u_I \) is reduced to the Hermite interpolation of \( u \).

For all \( \tau = \tau_{ij} \), a direct calculation yields

\[
u - u_I|_{\tau} = \sum_{p=k+1}^{\infty} \sum_{q=k+1}^{\infty} u_{pq} J_{i,p}^{\alpha\beta}(x) J_{j,q}^{\alpha\beta}(y) = (E^x u + E^y u - E^x E^y u),
\]

where

\[
u^x u(x,y)|_{\tau_{ij}} = \sum_{p=k+1}^{\infty} \sum_{q=0}^{\infty} u_{pq} J_{i,p}^{\alpha\beta}(x) J_{j,q}^{\alpha\beta}(y),
\]

\[
u^y u(x,y)|_{\tau_{ij}} = \sum_{p=0}^{\infty} \sum_{q=k+1}^{\infty} u_{pq} J_{i,p}^{\alpha\beta}(x) J_{j,q}^{\alpha\beta}(y),
\]

\[
u^x E^y u(x,y)|_{\tau_{ij}} = \sum_{p=k+1}^{\infty} \sum_{q=k+1}^{\infty} u_{pq} J_{i,p}^{\alpha\beta}(x) J_{j,q}^{\alpha\beta}(y).
\]

Note that \( E^x u \) is actually the one dimensional residual functions along the \( x \)-direction while the other variable \( y \) is fixed. Similar for \( E^y u, E^x E^y u \).

We have the following properties for the residual functions \( E^x u, E^y u \) and \( E^x E^y u \).

**Lemma 4.1.** Assume that \( u \in W^{l,\infty}(\Omega) \cap C^1(\Omega) \) with \( 0 < l \leq k + 1 \), and \( u_I \) is the truncated Jacobi projection of \( u \) defined by (4.6). Let \( u - u_I = E^x u + E^y u - E^x E^y u \) with \( E^x u, E^y u, E^x E^y u \) given by (4.8)-(4.11). There holds for \( m = \infty, 2 \) and \( p = 0, 1 \) the following results:

1. The function \( E^x u(x,\cdot) \in C^1(\cdot, y) \) is continuous about \( y \) and

\[
\partial_y E^x u(x_1, y) = 0, \quad \partial_x E^x u(x, y) = E^x (\partial^1_x u), \quad \forall n,
\]

\[
\|E^x u\|_{0,m} + h \|\partial_x E^x u\|_{0,m} + h^2 \|\partial^2_x E^x u\|_{0,m} \lesssim h\|u\|_{l,m}.
\]

2. The function \( E^y u(x,\cdot) \in C^1(x, \cdot) \) is continuous about \( x \) and

\[
\partial_y E^y u(x, y_2) = 0, \quad \partial_y E^y u(x, y_2) = E^y (\partial^1_y u), \quad \forall n,
\]

\[
\|E^y u\|_{0,m} + h \|\partial_y E^y u\|_{0,m} + h^2 \|\partial^2_y E^y u\|_{0,m} \lesssim h\|u\|_{l,m}.
\]

3. The function \( E^x E^y u \) is of high order, i.e., there holds for \( r, l \leq k + 1 \),

\[
\|E^x E^y u\|_{0,m} + h \|E^x E^y u\|_{1,m} + h^2 \|E^x E^y u\|_{2,m} \lesssim h^{r+l}\|u\|_{r+l,m}.
\]

Here we omit the proof and refer to [11] for more detailed discussions about the one dimensional truncated Jacobi projection.

We next study the approximation and superconvergence properties of \( u_I \). To this end, we first introduce some special points and lines on the whole domain and then prove that \( u_I \) is superconvergent at this special points and lines.
Let $R_p, p \in \mathbb{Z}_{k-3}$ be the $k-3$ zeros of $J_{k+1}^{2,2}(s)$ except the point $s = -1, 1,$ and $l_p, p \in \mathbb{Z}_k$ the $k$ Gauss-Lobatto points, i.e., $l_p, p \in \mathbb{Z}_k$ are zeros of $\partial_s J_{k+1}^{2,2}(s),$ and $G_p, p \in \mathbb{Z}_{k-1}$ the Gauss points of degree $k-1$ in $[-1,1]$ (i.e., the zeros of $L_{k-1}$), respectively. Then for all $\tau = \tau_{i,j} \in T_h, (i,j) \in \mathbb{Z}_M \times \mathbb{Z}_N,$ denote $h_i^x = x_i - x_{i-1}, h_j^y = y_j - y_{j-1},$ and

$$\mathcal{R}_\tau = \{ P : P = (R^\tau_{p,x}, R^\tau_{q,y}), p,q \in \mathbb{Z}_{k-3}\}, \quad \mathcal{R} = \bigcup_{\tau \in T_h} \mathcal{R}_\tau,$$

where

$$R^\tau_{p,x} = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}} + h_i^x l_p), \quad R^\tau_{p,y} = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}} + h_j^y l_p).$$

Denote by

$$\mathcal{E}^l_i = \{(x,y) : x = l^x_{\tau,p}, y \in [c,d], p \in \mathbb{Z}_k, \tau \in T_h\},$$

$$\mathcal{E}^l_j = \{(x,y) : y = l^y_{\tau,p}, x \in [a,b], p \in \mathbb{Z}_k, \tau \in T_h\}$$

the set of vertical and horizontal edges of all interior Lobatto points along the $x$-direction and the $y$-direction, respectively. Here

$$l^x_{\tau,p} = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}} + h_i^x l_p), \quad l^y_{\tau,p} = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}} + h_j^y l_p), \quad \forall \tau = \tau_{i,j}.$$

Similarly, the interior Gauss lines along the $x,y$-direction are defined as

$$\mathcal{E}^g_x = \{(x,y) : x = g^x_{\tau,p}, y \in [c,d], p \in \mathbb{Z}_{k-1}, \tau \in T_h\},$$

$$\mathcal{E}^g_y = \{(x,y) : y = g^y_{\tau,p}, x \in [a,b], p \in \mathbb{Z}_{k-1}, \tau \in T_h\},$$

where $(g^x_{\tau,p}, g^y_{\tau,p}), (p,q) \in (k-1) \times (k-1)$ denotes the $(k-1)^2$ Gauss points in $\tau.$

We have the following approximation properties for the Jacobi projection $u_I.$

**Proposition 1.** Assume that $u \in W^{l,\infty}(\Omega), l \geq 2$ is the solution of (4.1), and $u_I$ is the truncated Jacobi projection of $u$ defined by (4.6). The following orthogonality and approximation properties hold true.

1. **Optimal error estimates:**

   $$\|u - u_I\|_{0,m} \lesssim h^r\|u\|_{r,m}, \quad r \leq \min(k + 1, l), \quad m = 2, \infty.$$  

2. **Exactly the same at the mesh nodes for both the function value and the first-order derivative:**

   $$(u - u_I)(x_i, y_j) = 0, \quad \nabla(u - u_I)(x_i, y_j) = 0.$$  

3. **Superconvergence of function value approximation on roots of $J_{i,k+1}^{2,2}(x)J_{j,k+1}^{2,2}(y):**

   $$|(u - u_I)(P)| \lesssim h^{r-1}\|u\|_{r,\infty}, \quad \forall P \in \mathcal{R}, \quad r \leq \min(k + 2, l).$$  

4. **Superconvergence of the first-order derivative on Gauss-Lobatto lines, i.e., for $r \leq \min(k + 2, l),$**

   $$|\partial_x(u - u_I)(P_1)| + |\partial_y(u - u_I)(Q_1)| \lesssim h^{r-1}\|u\|_{r,\infty},$$

   where $P_1 \in \mathcal{E}^l_i, Q_1 \in \mathcal{E}^l_j$ denotes the Lobatto lines along the $x$ and $y$ directions, respectively.
5. Superconvergence of the second-order derivative on Gauss lines and Lobatto point, i.e., there holds for \( r \leq \min(k+2, l) \),

\[
|\partial_{xx}^2(u - u_I)(P_2)| + |\partial_{yy}^2(u - u_I)(Q_2)| + |\partial_{xy}^2(u - u_I)(P_3)| \lesssim h^{r-2}|u|_{r,\infty},
\]

where \( P_2 \in \mathcal{E}_p^2, Q_2 \in \mathcal{E}_q^2 \) denotes the Gauss lines along the \( x \) and \( y \) direction, respectively, and \( P_3 \in \mathcal{L} \) with \( \mathcal{L} \) the set of Lobatto points on the whole domain, i.e., \( \mathcal{L} = \{ (l_p^r, l_q^r) : r \in T_h, (p,q) \in k \times k \} \).

**Proof.** For any fixed \( y \), there holds for any \( r \leq \min(k+2, l) \) (see \([11]\))

\[
|E^x u(R_m^{r,x}, y)| + h|\partial_x E^x u(R_m^{r,x}, y)| + h^2|\partial_{xx} E^x u(g_p^{r,x}, y)| \lesssim h^r||u||_{r,\infty}.
\]

Following the same arguments, we have

\[
|E^y u(x, R_m^{r,y})| + h|\partial_y E^y u(x, R_m^{r,y})| + h^2|\partial_{yy} E^y u(x, g_p^{r,y})| \lesssim h^r||u||_{r,\infty}.
\]

Then the desired results follow from the error equation \([17]\) and the estimates of \( E^x u, E^y u, E^z E^y u \) in Lemma \([11]\). \(\square\)

As we may observe, if we choose \( l = k + 1 \), the point-wise error estimates in the above Proposition indicate a superconvergent phenomenon of \( u_I \) at mesh nodes, at roots of the Jacobi polynomial, and at the Lobatto lines and Gauss lines.

### 5. Error estimates and superconvergence analysis

In this section, we present error estimates and study superconvergence properties of the \( C^1 \) Petrov-Galerkin method for \([21]\). In the rest of this paper, we use the following notations:

\[
e = u - u_h, \quad \xi = u_I - u_h, \quad \eta = u - u_I.
\]

#### 5.1. Optimal error estimates.

**Theorem 5.1.** Assume that \( u \in H^1(\Omega) \) is the solution of \([2.1]\), and \( u_h \) is the solution of \([2.3]\). Then

\[
||u - u_h||_0 + h||u - u_h||_1 + h^2||u - u_h||_2 \lesssim h^r||u||_{r+1}, \quad r \leq \min(l - 1, k + 1).
\]

**Proof.** In light of \([3.3]\) and the orthogonality \( a(e, w) = a(\xi + \eta, w) = 0 \) for all \( w \in W_h \), we have

\[
||\Delta \xi||_0^2 + ||\xi_{xyy}||_0^2 + ||\xi_{xyy}||_0^2 \lesssim |a(\xi, \xi_{xyy})| + ||\xi||_0^2 \lesssim |a(\eta, \xi_{xyy})| + |a(\eta, \xi_{xyy})|,
\]

where \( \varphi \) is the solution of the problem \([3.1]\) with \( v = \xi \). By \([17]\), we have

\[
a(\eta, w) = a(E^x u, w) + a(E^y u, w) - a(E^x E^y u, w), \quad \forall w \in W_h.
\]

Recalling the definition of the bilinear form and using and the integration by parts and the properties of \( E^x u \) in Lemma \([4.1]\) we derive for all \( \mu \leq \min(k+1, l - 2) \)

\[
a(E^x u, \xi_{xyy}) = |(-\alpha E^x u_{yy} + \beta_1 E^x u_{y} + \beta_2 E^x u_x + \gamma E^x u, \xi_{xyy})| = ||(\partial_{yy} (\alpha E^x u_{yy} - \beta_2 \partial_{yy} E^x u - \gamma E^x u, \xi_{xyy})| + |\beta_1 (\partial_{y} E^x u_{y} + \gamma E^x u, \xi_{xyy})| \lesssim h^{\mu-1}||u||_{\mu+2}||\xi_{xyy}||_0 + ||\xi_{xyy}||_0.
\]

Consequently,

\[
|a(E^x E^y u, \xi_{xyy})| \lesssim h^{\mu-1}||E^y u||_{\mu+2}||\xi_{xyy}||_0 + ||\xi_{xyy}||_0 \lesssim h^{\mu-1}||u||_{\mu+2}||\xi_{xyy}||_0 + ||\xi_{xyy}||_0.
\]
By the same arguments, there holds
\[ |a(E_h u, \xi_{xxyy})| \lesssim h^{\mu-1} \|u\|_{\mu+2}(\|\xi_{xxyy}\|_0 + \|\xi_{xyy}\|_0).\]
Then
\[ (5.3) \quad |a(\eta, \xi_{xxyy})| \lesssim h^{\mu-1} \|u\|_{\mu+2}(\|\xi_{xyy}\|_0 + \|\xi_{xxyy}\|_0), \quad \mu \leq \min(k+1, l-2). \]

Now we consider the term \(a(\eta, I_h \varphi)\). Noticing that \(\varphi = 0\) on \(\partial \Omega\), we have from the integration by parts
\[ |a(\eta, I_h \varphi)| = |a(\eta, I_h \varphi - \varphi)| + |a(\eta, \varphi)| \]
\[ = |a(\eta, I_h \varphi - \varphi)| + |(\eta - \alpha \Delta \varphi - \beta \cdot \nabla \varphi + \gamma \varphi)| \]
\[ \lesssim (h^2 \|\eta\|_2 + \|\eta\|_0) \|\varphi\|_2 \lesssim h^{\mu'} \|u\|_{\mu'} \|\xi\|_0, \]
where \(\mu' \leq \min(l, k + 1)\), and in the last step, we have used the \(H^2\) regularity \(\|\varphi\|_2 \lesssim \|\xi\|_0\). Then we choose \(v \in \xi\) in (5.2) to obtain
\[ \|\xi\|_0^2 \lesssim h^{2\mu'} \|u\|_{\mu'}^2 + h^4(\|\xi_{xxyy}\|_0^2 + \|\xi_{xyy}\|_0^2), \]
\[ \|\Delta \xi\|_0^2 + \|\xi_{xyy}\|_0^2 + \|\xi_{xxyy}\|_0^2 \lesssim h^{2(\mu' - 1)} \|u\|_{\mu'+2}^2 + \|\xi\|_0^2. \]

Consequently, there holds for \(\mu \leq \min(k + 1, l - 2)\), \(\mu' \leq \min(l - 1, k + 1)\),
\[ (5.4) \quad \|\Delta \xi\|_0 + \|\xi_{xyy}\|_0 + \|\xi_{xxyy}\|_0 \lesssim h^{-1} \|u\|_{\mu+2}, \quad \|\xi\|_0 \lesssim h^{\mu'} \|u\|_{\mu'+1}. \]

As for the \(H^1\)-norm error estimate, a direct calculation from the integration by parts yields
\[ (\xi_x, \xi_x) + (\xi_y, \xi_y) = -(\xi, \xi_{xx}) - (\xi, \xi_{yy}) \lesssim \|\xi\|_0 \|\xi\|_2 \lesssim h^{2(\mu' - 1)} \|u\|_{\mu'+2}^2. \]

Then the desired result (5.2) follows from the triangle inequality and approximation properties of \(u_I\). The proof is complete. \(\square\)

5.2. Superconvergence analysis. In this subsection, we study superconvergence properties of the \(C^1\) Petrov-Galerkin methods. As the superconvergence analysis would require more strong regularity assumption on the smoothness of \(u\) than one would need to obtain the counterpart optimal convergence rate, we suppose the exact solution \(u\) is smooth enough in our superconvergence analysis. In our later section, we discuss the interior estimates, i.e., the error in an interior domain \(\Omega\), with less requirements on the smoothness of \(u\) on the whole domain \(\Omega\).

**Theorem 5.2.** Assume that \(u \in H^{k+3}(\Omega)\) is the solution of (2.1), and \(u_h\) is the solution of (2.3). The following superconvergence properties hold true.

1. **Superconvergence result between \(u_h\) and \(u_I\) in all \(H^2, H^1, L^2\)-norms:**
\[ (5.5) \quad \|u_h - u_I\|_1 + h \|u_h - u_I\|_2 \lesssim h^{k+1} \|u\|_{k+3}, \quad \|u_h - u_I\|_0 \lesssim h^{\min(k+2, 2k-2)} \|u\|_{k+3}. \]

2. **Superconvergence of the function value on roots of \(J_{k+1}^{-2, -2}(x)J_{k+1}^{-2, -2}(y)\) in average sense for \(k \geq 4\), i.e.,**
\[ (5.6) \quad e_{u, J} := \left( \frac{1}{NM} \sum_{P \in \mathcal{R}} (u - u_h)^2(P) \right)^{\frac{1}{2}} \lesssim h^{k+2} \|u\|_{k+3}. \]
3. Superconvergence of the first-order derivative on Lobatto lines in average sense, i.e.,

\[(5.7)\]
\[e_{\nabla u,l} := \left( \frac{1}{N_x} \sum_{P_i \in \mathcal{E}_l^x} \partial_x (u - u_h)^2 (P_i) + \frac{1}{N_y} \sum_{Q_j \in \mathcal{E}_l^y} \partial_y (u - u_h)^2 (Q_j) \right)^{\frac{1}{2}} \lesssim h^{k+1} \|u\|_{k+3}.
\]

4. Superconvergence of the second-order derivative on Gauss line in average sense. That is,

\[(5.8)\]
\[e_{\Delta u,g} := \left( \frac{1}{N_x} \sum_{P_i \in \mathcal{E}_l^x} \partial_{xx}^2 (u - u_h)^2 (P_i) + \frac{1}{N_y} \sum_{Q_j \in \mathcal{E}_l^y} \partial_{yy}^2 (u - u_h)^2 (Q_j) \right)^{\frac{1}{2}} \lesssim h^k \|u\|_{k+3}.
\]

Here \( N_x, N_y, M_x, M_y \) denote the cardinalities of \( \mathcal{E}_l^x, \mathcal{E}_l^y, \mathcal{E}_g^x, \mathcal{E}_g^y \), respectively.

**Proof.** First, by choosing \( \mu = k + 1 \) in (5.4), we get

\[
\|\nabla \xi\|_0 + \|\xi_{xyy}\|_0 + \|\xi_{xxy}\|_0 \lesssim h^k \|u\|_{k+3}.
\]

By using (3.2) and the orthogonality \( a(\xi + \eta, \psi) = 0 \) for all \( v \in W_h \), we have

\[
\|\xi\|^2 \lesssim h^{2k+4} \|u\|^2_{k+3} + |a(\eta, I_h \varphi)| = h^{2k+4} \|u\|^2_{k+3} + |a(E^x u + E^y u - E^x E^y u, I_h \varphi)|.
\]

Here \( \varphi \) is the solution of (3.1) with \( \alpha_E \parallel \nabla \cdot \psi \parallel_{E} \). Noticing that \( E^x u \parallel \mathbb{P}_0(x), \partial_x E^x u \parallel \mathbb{P}_1(x) \) for \( k \geq 4 \), then

\[
|a(E^x u, I_h \varphi)| = |(-\alpha E^x u_{yy} + \beta_2 E^x u_y + \gamma E^x u, I_h \varphi - \bar{\varphi})| \lesssim h^{k+2} \|u\|_{k+3} \|\varphi\|_1,
\]

where \( \bar{\varphi} \) denotes the cell average of \( \varphi \). As for \( k = 3 \), we use the integration by parts to obtain

\[
|a(E^x u, I_h \varphi)| \lesssim h^{k+1} \|u\|_{k+3} \|\varphi\|_1.
\]

Consequently,

\[
|a(E^x u, I_h \varphi)| \lesssim h^{\min(k+2,2k-2)} \|u\|_{k+3} \|\varphi\|_1.
\]

Similarly, there holds

\[
|a(E^y u, I_h \varphi)| + |a(E^x E^y u, I_h \varphi)| \lesssim h^{\min(k+2,2k-2)} \|u\|_{k+3} \|\varphi\|_1,
\]

and thus

\[
(5.9) \quad |a(\eta, I_h \varphi)| \lesssim h^{\min(k+2,2k-2)} \|u\|_{k+3} \|\varphi\|_1,
\]

which yields, together with the \( H^2 \) regularity \( \|\varphi\|_2 \lesssim \|\xi\|_0 \),

\[
\|\xi\|_0 \lesssim h^{\min(k+2,2k-2)} \|u\|_{k+3}.
\]

We next estimate \( \|\nabla \xi\|_0 \). Given any \( \zeta \in [C^1(\Omega)]^2 \), let \( \psi \) be the solution of the following dual problem

\[-\nabla \cdot (a \nabla \psi) - \beta \cdot \nabla \psi + \gamma \psi = -\nabla \cdot \zeta \quad \text{in} \quad \Omega, \quad \text{and} \quad \psi = 0, \quad \text{on} \quad \partial \Omega.\]

By using the integration by parts,

\[
(\nabla \xi, \zeta) = - (\xi, \nabla \cdot \zeta) = (\xi, -(\nabla \cdot (a \nabla \psi) - \beta \cdot \nabla \psi + \gamma \psi)
\]

\[
= -(\nabla \cdot (a \nabla \xi) + \beta \cdot \nabla \xi + \gamma \xi, \psi - I_h \psi + I_h \psi)
\]

\[
\lesssim h \|\xi\|_2 \|\psi\|_1 + |a(\xi, I_h \psi)| = h \|\xi\|_2 \|\psi\|_1 + |a(\eta, I_h \psi)|.
\]
Consequently, by the regularity result $\|\psi\|_1 \lesssim \|\nabla \cdot \xi\|_{-1} \lesssim \|\xi\|_0$, and the estimate of $\|\xi\|_2$, we derive

$$|(\nabla \xi, \xi)| \lesssim h^{k+1} \|u\|_{k+3} \|\xi\|_0.$$  

Since the set of all such $\xi$ is dense in $L^2(\Omega)$, the above inequality indicates that (5.10)

$$\|\nabla \xi\|_0 \lesssim h^{k+1} \|u\|_{k+3}.$$  

This finishes the proof of (5.5).

In light of the superconvergence properties of $u_I$ in Proposition 4, we have

$$e_{u,I} \lesssim \left( \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \|\xi\|_{0,\infty}^2 \right) \frac{1}{2} + h^{k+2} \|u\|_{k+3}.$$

Then (5.18) follows. Similarly, there hold

$$e_{\nabla u,I} \lesssim \|\nabla \xi\|_0 + h^{k+1} \|u\|_{k+3}, \quad e_{\Delta u,I} \lesssim \|\nabla \xi\|_0 + h^{k} \|u\|_{k+3}.$$  

Then (5.7) and (5.8) follow from the estimates of $\|\nabla \xi\|_0$ and $\|\xi\|_2$ directly. This finishes our proof. \(\square\)

In the following, we study the highest superconvergence result of the $C^1$ Petrov-Galerkin approximation at the mesh nodes. We use the idea of correction function to achieve our superconvergence goal. The basic idea of the correction function is the construction of a specially designed function $w_h \in V_h^0$ such that $\tilde{u}_I = u_I - w_h$ is superconvergent towards the numerical solution $u_h$ in some norms, e.g., $H^2$ or $L^2$-norm, with higher order of accuracy.

Denote

$$\tilde{\xi} = \tilde{u}_I - w_h = u_I - w_h - u_h.$$  

In light of (3.2)-(3.3), the errors $\|\xi\|_0$ and $\|\xi\|_2$ are dependent on two terms: $a(\tilde{\xi}, \tilde{\xi}_{xyy})$ and $a(\tilde{\xi}, \tilde{T}_h \varphi)$. By the orthogonality, we have

$$a(\tilde{\xi}, \theta) = -a(\eta + w_h, \theta), \quad \forall \theta \in W_h.$$  

In other words, to achieve our superconvergence goal, the function $w_h \in V_h$ should be specially construct such that

$$a(\eta, \theta) + a(w_h, \theta) = a(E^x E^x u + E^y E^y u, \theta) + a(w_h, \theta), \quad \forall \theta \in W_h$$

is of high order. Note that if we choose $w_h = 0$, then we get the superconvergence results presented in Theorem 2.2, which is one order higher than the counterpart optimal convergence rate.

The next Proposition shows the existence of the correction function $w_h$, which satisfies our superconvergence goal.

**Proposition 2.** Let $u \in W^{2k+1,\infty}(\Omega)$. There exists a $w_h \in V^0_h$ such that

(5.13) \hspace{1cm} $\|w_h\|_{0,\infty} \lesssim h^{\min (k+2,2k-2)} \|u\|_{2k+1,\infty}, \quad \|w_h\|_{1,\infty} + h \|w_h\|_{2,\infty} \lesssim h^{k+1} \|u\|_{2k+1,\infty},$

(5.14) \hspace{1cm} $|w_h(x_i, y_j)| + |\nabla w_h(x_i, y_j)| \lesssim h^{2k-2} \|u\|_{2k+1,\infty}.$

Furthermore, there holds for any $\theta \in W_h$

(5.15) \hspace{1cm} $|a(u - u_I + w_h, \theta)| \lesssim h^{2k-2} \|u\|_{2k+1,\infty} \|\theta\|_0.$
Theorem 5.3. Assume that \( u \in W^{2k+1,\infty}(\Omega) \) is the solution of (2.1), and \( u_h \) is the solution of (5.3). Then
\[
e_{u,n} \lesssim h^{2k-2} \|u\|_{2k+1,\infty}, \quad e_{\varphi,u,n} \lesssim h^{2k-2} \|u\|_{2k+1,\infty},
\]
where
\[
e_{v,n} = \left( \frac{1}{MN} \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (v - v_h)^2 (x_i, y_j) \right)^{\frac{1}{2}}, \quad v = u, \nabla u.
\]

Proof. By (3.2)-(3.3), (5.11) and (5.15), we have
\[
\|\Delta \tilde{\xi}\|_0^2 + \|\tilde{\xi}_{xy}\|_0^2 + \|\tilde{\xi}_{yy}\|_0^2 \lesssim |a(\eta + w_h, \tilde{\xi}_{xyy})| + |a(\eta + w_h, \mathcal{I}_h \varphi)| \lesssim h^{2k-3} \|u\|_{2k+1,\infty} \|\tilde{\xi}_{xyy}\|_0 + h^{2k-2} \|u\|_{2k+1,\infty} \|\mathcal{I}_h \varphi\|_0,
\]
where \( \varphi \) is the solution of (5.11) with \( v = \tilde{\xi} \), and in the last step, we have used the inverse inequality \( \|\tilde{\xi}_{xyy}\|_0 \lesssim h^{-1} \|\tilde{\xi}_{yy}\|_0 \). Using the inequality \( \|\mathcal{I}_h \varphi\|_0 \lesssim \|\varphi\|_2 \lesssim \|\tilde{\xi}\|_0 \lesssim \|\Delta \tilde{\xi}\|_0 \), we immediately get
\[
\|\Delta \tilde{\xi}\|_0^2 + \|\tilde{\xi}_{xy}\|_0^2 + \|\tilde{\xi}_{yy}\|_0 \lesssim h^{2k-3} \|u\|_{2k+1,\infty}.
\]
Then we follow the same argument as what we did in Theorem 5.2 to obtain
\[
\|\tilde{\xi}\|_0 \lesssim h^{2k-2} \|u\|_{2k+1,\infty}, \quad \|\tilde{\xi}\|_0 \lesssim \|\tilde{\xi}\|_1 \lesssim h^{2k-2} \|u\|_{2k+1,\infty}.
\]
By the property of \( u_f \) and (5.11), we get
\[
|\{(u - u_h)(x, y)\}| = |\{(u_f + w_h - u_h)(x, y) - w_h(x, y)\}| \leq \|\tilde{\xi}\|_{0,\tau_{i,j}} + h^{2k-2} \|u\|_{2k+1,\infty},
\]
and thus,
\[
e_{u,n} \lesssim \|\tilde{\xi}\|_0 + h^{2k-2} \|u\|_{2k+1,\infty} \lesssim h^{2k-2} \|u\|_{2k+1,\infty}.
\]
Following the same argument, we have
\[
e_{\varphi,u,n} \lesssim \|\nabla \tilde{\xi}\|_0 + h^{2k-2} \|u\|_{2k+1,\infty} \lesssim h^{2k-2} \|u\|_{2k+1,\infty}.
\]
Then (5.16) follows. This finishes our proof. \( \square \)

With the help of the correction function \( w_h \), we can also improve our superconvergence results from the average sense to the point-wise sense for \( k \geq 4 \).

Theorem 5.4. Suppose all the conditions of Theorem 5.3 hold true. Then
\[
|\{(u - u_h)(x, y)\}| \lesssim h^{2k-2} |\ln h|^\frac{1}{2} \|u\|_{2k+1,\infty}, \quad |(u - u_h)(P)| \lesssim h^{k+2} \max(1, h^{k-4} \ln h^2) \|u\|_{2k+1,\infty},
\]
\[
|\partial_x (u - u_h)(P_1)| + |\partial_y (u - u_h)(Q_1)| \lesssim h^{k+1} \|u\|_{2k+1,\infty},
\]
\[
|\partial^2_{xx} (u - u_h)(P_2)| + |\partial^2_{yy} (u - u_h)(Q_2)| + |\partial^2_{xy} (u - u_h)(P_3)| \lesssim h^k \|u\|_{2k+1,\infty},
\]
where \( P \in \mathcal{R}, P_1 \in \mathcal{E}_x, Q_1 \in \mathcal{E}_y, P_2 \in \mathcal{E}_x, Q_2 \in \mathcal{E}_y \) and \( P_3 \in \mathcal{L} \).
\textbf{Proof.} We first define the $C^0$-conforming finite element space $S_h$ as follows:
\[ S_h = \{ v \in C^0(\Omega) : v|_{\partial \Omega} = 0, v|_{\tau} \in Q_{k}(x,y) = P_k(x) \times P_k(y), \tau \in T_h \}. \]
We denote by $a_e(\cdot, \cdot)$ the bilinear form of the finite element method, that is,
\[ a_e(u,v) = (\alpha \nabla u \cdot \nabla v) + (\beta \nabla u, v) + (\gamma u, v). \]
Note that $a_e(u,v)$ is coercive and continuous in the $H_0^1$ space. By Lax-Milgram Lemma, there exists a $g_h \in S_h$ such that
\begin{equation}
(5.21) \quad a_e(v_h, g_h) = v_h(x,y), \quad \forall v_h \in S_h.
\end{equation}
Especially, we choose $v_h = g_h$ to obtain
\begin{equation}
(5.22) \quad \|g_h\|_1^2 \leq |a_e(g_h, g_h)| = |g_h(x,y)| \leq \|g_h\|_{0, \infty}.
\end{equation}
Since (cf.,[31], p.84, Theorem 2.8) we have
\[ \|g_h\|_1 \lesssim \ln h^{1/2}, \quad \forall v_h \in S_h, \]
we have
\[ \|g_h\|_1 \lesssim \ln h^{1/2}. \]
By choosing $v_h = \tilde{\xi}$ in (5.21) and use the integration by parts and (5.13),
\[ \|\tilde{\xi}\|_{0, \infty} \leq |a_e(\tilde{\xi}, g_h)| = |a(\tilde{\xi}, g_h - R_h g_h) - a(\eta + w_h, R_h g_h)| \lesssim h \|\tilde{\xi}\|_2 \|g_h\|_1 + h^{2k-2} \|g_h\|_{0, 2k+1, \infty} \lesssim h^{2k-2} \ln h^{1/2} \|\eta\|_{2k+1, \infty}. \]
Here $R_h$ denotes the $L^2$ projection of $S_h$ onto $W_h$. Then the desired results (5.13) follow from the approximation properties of $u_f$ and the estimates of $w_h$ in Proposition 2. The proof is complete. \qed

\section{Interior estimates for the $C^1$ Petrov-Galerkin method}

In this section, we study interior a priori error estimates in $H^2, H^1, L^2$-norms, which can be estimated with an error in a strong norm on a smaller domain plus an error in a weaker norm over a slightly larger domain. We begin with some preliminaries.

Let $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_m \subset \Omega$ be separated by $d \geq c_0 h$, with $\Omega_i, i \leq m$ the rectangular domain. For any domain $D$, we define
\[ W_h^0(D) := \{ v \in W_h : v|_{\partial D} = 0, \supp v \subset \overline{D} \}, \]
\[ V_h^0(D) := \{ v \in V_h : v|_{\partial D} = 0, \supp v \subset \overline{D} \}. \]
Define
\begin{equation}
(6.1) \quad \|v\|_D^2 := \|\nabla v\|_{0,D}^2 + \|\partial_x \partial_y v\|_{0,D}^2 + \|\partial_y \partial_{xx} v\|_{0,D}^2.
\end{equation}
Denote by $B(u,v)$ the bilinear form which is defined as
\[ B(u,v) := a(u,v_{xyy}). \]
\textbf{Lemma 6.1.} Let $\Omega_0 \subset \subset \Omega'$ and $p \geq 0$ be a fixed but arbitrary integer. Suppose $e \in V_h^0(\Omega')$ is the solution of the problems
\[ a(e, \zeta) = 0, \quad \forall \zeta \in W_h(\Omega') \quad \text{or} \quad B(e, \partial _{xyy}) = 0, \quad \forall \theta \in V_h^0(\Omega'). \]
Then
\[(6.2) \quad \|\tilde{e}\|_{1, \Omega_0} \lesssim h\|\Delta \tilde{e}\|_{0, \Omega} + \|\tilde{e}\|_{-p, \Omega}, \quad \|\tilde{e}\|_{0, \Omega_0} \lesssim h^2\|\Delta \tilde{e}\|_{0, \Omega} + \|\tilde{e}\|_{-p, \Omega}, \]
\[(6.3) \quad \|\tilde{e}\|_{2, \Omega_0} \lesssim h^2\|\tilde{e}\|_{0, \Omega} + \|\tilde{e}\|_{-p, \Omega}.\]

**Proof.** We only consider the case that \(\tilde{e}\) satisfies \(a(\tilde{e}, \zeta) = 0\) since the same argument can be applied to the case in which \(B(\tilde{e}, \theta_{xyy}) = 0\).

For any \(s \leq 1, v \in H^{-s}(\Omega_1)\), denote by \(\varphi \in H^{-s+2}(\Omega_1)\) the solution of (5.31).

Let \(w = 1\) on \(\Omega_0\) and \(w \in C^\infty(\Omega_1)\) with \(\Omega_1 \subset \subset \Omega'\). Then
\[
\|w\tilde{e}\iota, \Omega_1 \lesssim \sup_{\varphi \in H^{-s+2}_0(\Omega_1)} \frac{|(w\tilde{e}, v)|}{\|v\|_{-s, \Omega_1}} = \sup_{\varphi \in H^{-s+2}_0(\Omega_1)} \frac{|a(w\tilde{e}, \varphi)|}{\|\varphi\|_{-s, \Omega_1}} = \sup_{\varphi \in H^{-s+2}_0(\Omega_1)} \frac{|a(\tilde{e}, w\varphi) + I|}{\|\varphi\|_{-s, \Omega_1}},
\]
where \(I = \{(-\alpha w\Delta \tilde{e} - 2\alpha \nabla w \nabla \tilde{e}, \varphi)\} = \{(2\alpha \nabla \cdot (\nabla w \varphi) - \alpha \Delta w, \tilde{e}\}| \lesssim \|\tilde{e}\|_{s-1, \Omega_1} \|\varphi\|_{2-s, \Omega_1}.

Consequently,
\[
\|w\tilde{e}\|_{s, \Omega_0} \lesssim \|w\tilde{e}\|_{s, \Omega_1} \lesssim \sup_{\varphi \in H^{-s+2}_0(\Omega_1)} \frac{|a(\tilde{e}, \varphi)|}{\|\varphi\|_{-s, \Omega_1}} \lesssim h^{\min(2-s, 2)}(\|\Delta \tilde{e}\|_{0, \Omega_0} + \|\tilde{e}\|_{1, \Omega_0}) + \|\tilde{e}\|_{s-1, \Omega_1}.
\]

Especially, by choosing \(s = 0\) and iterating the above inequality \(p\) times, we get
\[
\|\tilde{e}\|_{0, \Omega_0} \lesssim h^2(\|\Delta \tilde{e}\|_{0, \Omega_0} + \|\tilde{e}\|_{1, \Omega_0}) \lesssim h^2(\|\Delta \tilde{e}\|_{0, \Omega_0} + \|\tilde{e}\|_{1, \Omega_0}) \lesssim h^2(\|\Delta \tilde{e}\|_{0, \Omega_p} + \|\tilde{e}\|_{1, \Omega_p}) \lesssim h^2(\|\Delta \tilde{e}\|_{0, \Omega_p} + \|\tilde{e}\|_{-p, \Omega_p}).
\]

Similarly, we choose \(s = 1\) to obtain
\[
\|\tilde{e}\|_{1, \Omega_0} \lesssim h(\|\Delta \tilde{e}\|_{0, \Omega_{p+1}} + \|\tilde{e}\|_{1, \Omega_{p+1}}) \lesssim h(\|\Delta \tilde{e}\|_{0, \Omega_{p+1}} + \|\tilde{e}\|_{-p, \Omega_{p+1}}).
\]

Let \(\Omega_{p+1} \subset \subset \Omega_{p+2} \subset \subset \cdots \subset \Omega_{2p} = \Omega'\) and iterate the above inequality \(p\) times, we obtain
\[(6.4) \quad \|\tilde{e}\|_{1, \Omega_0} \lesssim h(\|\Delta \tilde{e}\|_{0, \Omega_{2p}} + h^{p+1}\|\tilde{e}\|_{1, \Omega_{2p}} + \|\tilde{e}\|_{-p, \Omega_{2p}}.
\]

Then (6.3) follows by using the inverse inequality.

We next estimate \(\|\Delta \tilde{e}\|_{0, \Omega_0}\). Recalling the definition of the bilinear form of \(a(\cdot, \cdot)\), we have
\[
\|\Delta (w\tilde{e})\|_{0, \Omega_1}^2 \lesssim |a(w\tilde{e}, \Delta (w\tilde{e}))| + \|w\tilde{e}\|_{1, \Omega_1}^2 = |a(\tilde{e}, w\Delta (w\tilde{e})) + (-\alpha \tilde{e}\Delta w - 2\alpha \nabla w \nabla \tilde{e}, \Delta (w\tilde{e}))| + \|w\tilde{e}\|_{1, \Omega_1}^2 \lesssim |a(\tilde{e}, w\Delta (w\tilde{e}) - \theta)| + \|w\tilde{e}\|_{1, \Omega_1}^2 \lesssim (\|\Delta \tilde{e}\|_{0, \Omega_0} + \|\tilde{e}\|_{1, \Omega_0})\|w\Delta (w\tilde{e}) - \theta\|_{0, \Omega_1} + \|\tilde{e}\|_{1, \Omega_1}^2, \quad \forall \theta \in W_h(\Omega_1).
\]

By the standard approximation theory, there holds
\[
\|w\Delta (w\tilde{e}) - \theta\|_{0, \Omega_1} \lesssim h^{k-1}(\|\Delta \tilde{e}\|_{k-1} + \|\nabla \tilde{e}\|_{k-1} + \|\tilde{e}\|_{k-1}) \lesssim h^{k-1}(\|\partial_{x}^{k-1} \partial_{xy} \tilde{e}\|_{0, \Omega_0} + \|\partial_{xx}^{k-1} \partial_{xy} \tilde{e}\|_{0, \Omega_0} + \|\Delta \tilde{e}\|_{k-2, \Omega_0}) \lesssim h(\|\tilde{e}\|_{1, \Omega_0} + \|\tilde{e}\|_{1, \Omega_0}).
\]
Here in the last step, we have used the inverse inequality. Consequently,
\[ \| \tilde{e} \|_{2, \Omega_0}^2 \leq \| w \|_{2, \Omega_1}^2 \lesssim \| \nabla (w \tilde{e}) \|_{2, \Omega_1}^2 \lesssim h \| \tilde{e} \|_{1, \Omega_1}^2 + \| \tilde{e} \|_{1, \Omega_1}^2, \]
which yields (together with (6.2)) the desired result (6.3). The proof is complete.

Given any \( v \in C^1(\Omega_1) \cap H^4_0(\Omega_1) \), let \( Pv \in V_h^0(\Omega_1) \) and \( P^*v \in V_h^0(\Omega_1) \) be defined as the solutions of the equations
\[ B(v - Pv, \varphi) = 0, \quad B(\varphi, v - P^*v) = 0, \quad \forall \varphi \in V_h^0(\Omega_1). \]
By the same argument as what we did in Theorem 3.2, we can prove that \( Pv \) and \( P^*v \) are uniquely defined.

In light of the conclusions in Lemma 3.1, we easily obtain, by using (6.5) and the Cauchy-Schwarz inequality, the integration by parts and the homogenous boundary condition \( v|_{\partial \Omega_1} = 0 \),
\[ \| \|Pv\|_{1, \Omega_1}^2 \lesssim B(Pv, Pv) + a(Pv, I_h \varphi) = B(v, Pv) + a(v, I_h \varphi) \lesssim \|v\|_{1, \Omega_1} \|Pv\|_{0, \Omega_1} + \|v\|_{2, \Omega_1} \|\varphi\|_{0, \Omega_1}, \]
where \( \varphi \) is the solution of (6.1) with \( v \) replaced by \( Pv \), and in the second step, we have used the identity
\[ a(v, I_h \varphi) = B(v, \varphi_1) = B(Pv, \varphi_1) = a(Pv, I_h \varphi) \]
with \( \varphi_1 \in V_h^0 \) satisfying \( \partial_x \partial_{yy} \varphi_1 = I_h \varphi \). Using the \( H^2 \) regularity assumption \( \|\varphi\|_{0, \Omega_1} \lesssim \|Pv\|_{0, \Omega_1} \lesssim \|\Delta Pv\|_{0, \Omega_1} \), we get
\[ \| \|Pv\|_{1, \Omega_1}^2 \lesssim \|v\|_{1, \Omega_1} \|v\|_{0, \Omega_1} + \|v\|_{2, \Omega_1} \|\varphi\|_{0, \Omega_1} \]
Similarly, we can prove that the same result holds true for \( P^*v \).

Let \( w = 1 \) on \( \Omega_0 \) and \( w \in C_0^\infty(\bar{\Omega}') \) with \( \Omega_0 \subset \subset \Omega' \). Set \( \tilde{u} = wu \) and denote \( \tilde{e} = \tilde{u} - \tilde{P}\tilde{u} \). By using (6.2), (6.3), (6.6) and the integration by parts, we get
\[ \| \tilde{e} \|_{2, \Omega'} \lesssim |a(\tilde{e}, \tilde{e}) + a(\tilde{e}, I_h \varphi)| = |B(\tilde{e}, \tilde{e})| = |B(\tilde{e}, \tilde{e} - P^*\tilde{e})| \lesssim \|\tilde{u} - \tilde{u}_I\|_{\Omega'} \|\tilde{e} - P^*\tilde{e}\|_{\Omega'} \lesssim h^{\mu-1} \|\tilde{u}\|_{\mu+2, \Omega'}, \quad \mu \leq k. \]
Consequently,
\[ \|u - \tilde{P}u\|_{\Omega_0} \lesssim \|\tilde{e}\|_{\Omega'} \lesssim h^{\mu-1} \|\tilde{u}\|_{\mu+2, \Omega'}. \]
We next estimate \( \|P\tilde{u} - u_h\|_{\Omega_0} \).

**Lemma 6.2.** Assume that \( \Omega_0 \subset \subset \Omega' \) and \( p \geq 0 \) is a fixed but arbitrary integer. Let \( u_h \) be the solution of (6.2), \( \tilde{u} = wu \) with \( w = 1 \) on \( \Omega_0 \) and \( w \in C_0^\infty(\bar{\Omega}') \), and \( \tilde{P}\tilde{u} \) be defined by (6.5). Then for sufficiently small \( h \),
\[ \|P\tilde{u} - u_h\|_{\Omega_0} \lesssim \|P\tilde{u} - u_h\|_{-p, \Omega'}. \]

**Proof.** First, we note that
\[ B(u - u_h, v) = 0, \quad \forall v \in V_h^0(\Omega_0), \]
which yields (together with (6.5))
\[ B(u_h - \tilde{P}\tilde{u}, v) = a(u_h - \tilde{P}\tilde{u}, v_{xyy}) = 0, \quad \forall v \in V_h^0(\Omega_0). \]
Let $\tilde{e} = u_h - P\tilde{u}$. Then

$$
\|\tilde{e}\|_{\Omega_0} \leq \|w\tilde{e}\|_{\Omega_1} \leq \|(w\tilde{e}) - P(w\tilde{e})\|_{\Omega_1} + \|P(w\tilde{e})\|_{\Omega_1}.
$$

As for $\|(w\tilde{e}) - P(w\tilde{e})\|_{\Omega_1}$, we have from (5.22)-(5.33), (6.10) and the integration by parts that

$$
\|\|(w\tilde{e}) - P(w\tilde{e})\|_{\Omega_1}^2 \leq B((w\tilde{e}) - P(w\tilde{e}), (w\tilde{e}) - P(w\tilde{e})) = B((w\tilde{e}) - P(w\tilde{e}), (w\tilde{e})_I)
\leq \|\|(w\tilde{e}) - P(w\tilde{e})\|_{\Omega_1} \|(w\tilde{e}) - (w\tilde{e})_I\|_{\Omega_1}.
$$

Here $(w\tilde{e})_I$ denotes the truncated Jacobi projection of $w\tilde{e}$. From the property of $u_I$ in Lemma 4.4, we derive

$$
\|u - u_I\|_{\Omega_1} \leq \|(u - u_I)_{xyy}\|_{0, \Omega_1} + \|(u - u_I)_{xxy}\|_{0, \Omega_1} \lesssim h^{k-1}\|u\|_{k+2, \Omega_1}.
$$

Consequently,

$$
\|\|(w\tilde{e}) - P(w\tilde{e})\|_{\Omega_1} \leq \|\|(w\tilde{e}) - (w\tilde{e})_I\|_{\Omega_1} \lesssim h^{k-1}\|w\tilde{e}\|_{k+2, \Omega_1}
\leq h^{k-1}\|\tilde{e}\|_{k, \Omega_1} \lesssim h\|\tilde{e}\|_{2, \Omega_1}.
$$

Here in the last step, we have used the inverse inequality $\|\tilde{e}\|_{k, \Omega_1} \lesssim h^{-k}\|\tilde{e}\|_{2, \Omega_1}$.

Let $\varphi$ be the solution of (3.1) with $v = P(w\tilde{e})$. Following the same argument as what we did in (6.6), we derive

$$
\|P(w\tilde{e})\|_{\Omega_1} \lesssim |B(w\tilde{e}, P(w\tilde{e})) + a(w\tilde{e}, I_h\varphi)|
= |B(\tilde{e}, wP(w\tilde{e})) + a(\tilde{e}, wI_h\varphi) + I|
= |B(\tilde{e}, wP(w\tilde{e})) - (wP(w\tilde{e}))_I + a(\tilde{e}, wI_h\varphi - I_h(wI_h\varphi)) + I|
\lesssim h\|\tilde{e}\|_{0, \Omega_1} (wP(w\tilde{e}))_{2, \Omega_1} + \|wI_h\varphi\|_{1, \Omega_1} + |I|,
$$

where in the third step and last step, we have used (6.10) and the integration by parts, respectively, and

$$
I = \int_{\Omega_1} (-\alpha\triangle w\tilde{e} - 2\alpha\nabla w\nabla \tilde{e}) ((P(w\tilde{e}))_{xyy} + I_h\varphi) \, dx \, dy.
$$

Again we use the integration by parts and the Cauchy-Schwarz inequality to obtain

$$
|I| \lesssim \|w\tilde{e}\|_2 (\|P(w\tilde{e})\|_{\Omega_1} + \|\varphi\|_{0, \Omega_1}),
$$

which yields, together with the $H^2$ regularity $\|\varphi\|_{2, \Omega_1} \lesssim \|P(w\tilde{e})\|_{0, \Omega_1} \lesssim \|P(w\tilde{e})\|_{\Omega_1},$

$$
\|P(w\tilde{e})\|_{\Omega_1} \lesssim h\|\tilde{e}\|_{0, \Omega_1} + \|\tilde{e}\|_{2, \Omega_1}.
$$

Substituting (6.12)-(6.13) into (6.11) and using (6.8) with $\Omega_1, \Omega'$ replaced by $\Omega_1, \Omega_2$, we get

$$
\|\tilde{e}\|_{\Omega_0} \lesssim h^2\|\tilde{e}\|_{\Omega_2} + \|\tilde{e}\|_{-p, \Omega_2} \lesssim h\|\tilde{e}\|_{\Omega_3} + \|\tilde{e}\|_{-p, \Omega_3}.
$$

By integrating the above inequality $p + 2$ times and using the inverse inequality again, we obtain

$$
\|\tilde{e}\|_{\Omega_0} \lesssim h^{p+3}\|\tilde{e}\|_{0, \Omega_{p+5}} + \|\tilde{e}\|_{-p, \Omega_{p+5}} \lesssim \|\tilde{e}\|_{-p, \Omega'}.
$$

This finishes our proof. □

Now we are ready to present our interior estimates in all $H^2, H^1, L^2$-norms.
Theorem 6.3. Let \( \Omega_0 \subset\subset \Omega_1 \subset\subset \Omega \), \( u \in H^l(\Omega_1) \) and \( u_h \) be the solutions of (2.1) and (2.3), respectively. Suppose that \( p \geq 0 \) is a fixed but arbitrary integer. Then for \( \mu \leq \min(k+1, l-2) \),

\begin{align*}
(6.14) \quad && \|u - u_h\|_{\Omega_0} &\lesssim h^{\mu - 1}\|u\|_{\mu+2, \Omega_1} + \|u - u_h\|_{-p, \Omega_1}, \\
(6.15) \quad && \|u - u_h\|_{1, \Omega_0} &\lesssim h^{\mu}(\|u\|_{\mu+2, \Omega_1} + \|u\|_{1, \Omega}) + \|u - u_h\|_{-p, \Omega_1}, \\
(6.16) \quad && \|u - u_h\|_{0, \Omega_0} &\lesssim h^{\mu+1}(\|u\|_{\mu+2, \Omega_1} + \|u\|_{1, \Omega}) + \|u - u_h\|_{-p, \Omega_1}.
\end{align*}

Furthermore, if \( u \in H^k(\Omega) \cap H^{k+2}(\Omega_1) \), there hold the following optimal interior estimates:

\begin{align*}
(6.17) \quad && \|u - u_h\|_{0, \Omega_0} + h\|u - u_h\|_{1, \Omega_0} + h^2||u - u_h||_{\Omega_0} &\lesssim h^{k+1}(\|u\|_{k+2, \Omega_1} + \|u\|_{3, \Omega}).
\end{align*}

Proof. Let \( \Omega' \subset\subset \Omega_1 \). As a direct consequence of (6.8)-(6.9),

\begin{align*}
\|u - u_h\|_{\Omega_0} &\lesssim h^{\mu - 1}\|u\|_{\mu+2, \Omega_1} + \|P\tilde{u} - u_h\|_{-p, \Omega'} \\
&\lesssim h^{\mu - 1}\|u\|_{\mu+2, \Omega_1} + \|P\tilde{u} - u_h\|_{-p, \Omega'} + \|u - u_h\|_{-p, \Omega_1} \\
&\lesssim h^{\mu - 1}\|u\|_{\mu+2, \Omega_1} + \|P\tilde{u} - u_h\|_{1, \Omega'} + \|u - u_h\|_{-p, \Omega_1}.
\end{align*}

Replacing \( \Omega_0, \Omega_1 \) by \( \Omega', \Omega_1 \) in (6.8) yields

\[ \|P\tilde{u} - u_h\|_{1, \Omega'} \lesssim \|P\tilde{u} - u_h\|_{\Omega'} \lesssim h^{\mu-1}\|u\|_{\mu+2, \Omega_1}. \]

Then the desired result (6.14) follows.

Note that \( a(\cdot, \cdot) = 0 \) for all \( \zeta \in W_h \). Following the same argument as what we did in (6.4), we have

\[ \|\varepsilon\|_{0, \Omega_0} \lesssim h\|\Delta\varepsilon\|_{0, \Omega'} + h^{p+1}\|\varepsilon\|_{1, \Omega'} + \|\varepsilon\|_{-p, \Omega'} \]

\[ \|\varepsilon\|_{0, \Omega_0} \lesssim h^2\|\Delta\varepsilon\|_{0, \Omega'} + h^{p+1}\|\varepsilon\|_{1, \Omega'} + \|\varepsilon\|_{-p, \Omega'} \]

Then (6.15) and (6.16) follows from (6.2).

As for the term \( \|\varepsilon\|_{-p, \Omega_1} \), we first suppose \( \varphi \) is the solution of the problem (3.1) and \( \|\varphi\|_{p+2, \Omega} \lesssim \|v\|_{p, \Omega} \). Then from the integration by parts,

\[ \|\varepsilon\|_{-p, \Omega} = \sup_{v \in C^0_0(\Omega)} \frac{|(\varepsilon, v)|}{\|v\|_{p, \Omega}} = \sup_{\varphi \in C^0_0(\Omega)} \frac{|a(\varepsilon, \varphi)|}{\|\varphi\|_{p+2, \Omega}} \leq h^{\min(k-1, p)} \|\varepsilon\|_{2, \Omega}. \]

Substituting the estimate of \( \|\varepsilon\|_{-p, \Omega} \) and (5.2) into (6.14)-(6.16), we obtain (6.17) directly. The proof is complete. \( \Box \)

Remark 6.4. The interior error estimates in (6.14)-(6.16) indicates that errors in the \( H^2, H^1, L^2 \)-norms over any compact subdomain \( \Omega_1 \) of \( \Omega_1 \) may be estimated with an almost optimal order of accuracy that is possible locally for the subspace \( V_h \) plus an error in a much weak norm \( H^{-p}(\Omega) \). Just as pointed out in [27], the significance of the negative norm is that, under some very important circumstances, one can prove high order convergence rate in negative norms with relatively less requirements on the global smoothness of \( u \).

Following the same arguments, we can also obtain interior estimates for the error \( u_h - u_1 \) in \( H^2, L^2, H^1 \)-norms. For simplicity, we discuss only the interior error \( \|u_1 - u_h\|_{\Omega_0} \). Similar argument can be applied to estimating other norms by some tedious calculations.
Note that
\begin{equation}
B(u_I - u_h, v) = a(u_I - u, v_{xyy}), \quad \forall v \in V_h^0(\Omega_0).
\end{equation}
As we may observe, the only difference between the above equation and \ref{eq:diff} lies
in the right hand side. Following the same argument as what we did in Lemma \ref{lem:interior}
and choosing \( \mu = k + 1 \) in \ref{eq:interior}, we get
\[
\| u_I - u_h \|_{\Omega_0} \lesssim \| u_I - u_h \|_{p, \Omega_1} + h^k \| u \|_{k + 3, \Omega_1} \\
\lesssim \| u_I - u \|_{p, \Omega_1} + h^k \| u \|_{k + 3, \Omega_1} + \| e \|_{p, \Omega_1}.
\]
Using the error decomposition of \( u - u_I \) and the properties of \( u_I \) in Lemma \ref{lem:error}
we have
\[
\| \eta \|_{-p, \Omega} = \sup_{v \in C^p(\Omega)} \frac{\| (\eta, v) \|}{\| v \|_{p, \Omega}} \leq \sup_{v \in C^p(\Omega)} \| a(\eta, \varphi) \| \lesssim h^{\min(k - 1, p)} \| \eta \|_{2, \Omega}.
\]
Therefore, by choosing \( p = k - 1 \) and using the error estimates \( \| \eta \|_2 + \| e \|_2 \lesssim h \| u \|_{1, \Omega}, \)
\[
\| u_I - u_h \|_{\Omega_h} \lesssim h^k (\| u \|_{k + 3, \Omega_1} + \| u \|_{3, \Omega}),
\]
which indicates a superconvergence result for the interior error \( \| u_I - u_h \|_{\Omega_0} \).

7. Numerical experiments

In this section, we present some numerical examples to verify our theoretical
findings in previous sections.

In our experiments, we adopt the \( C^1 \) Petrov-Galerkin method \ref{eq:petrov}
for the convection-diffusion equation \ref{eq:diff} with \( k = 3, 4, 5 \), respectively. We test various
errors for \( u - u_h \), including \( e_{u,n} \) and \( e_{\nabla u,n} \) defined in Theorem \ref{thm:convergence}
the maximum error on roots of \( J_{k+1}^{-2,-2}(x), J_{k+1}^{-2,-2}(y) \), the derivative error on the Lobatto lines,
and the second order derivative error on the Gauss lines and Lobatto points, which are
defined as:
\[
e_{u} = \max_{P \in R} | (u - u_h)(P) |, \\
e_{\nabla u} = \max_{P_1, Q_1 \in \mathcal{E}_n^0} | \partial_x (u - u_h)(P_1) | + \max_{Q_1 \in \mathcal{E}_n^0} | \partial_x (u - u_h)(Q_1) |, \\
e_{\Delta u} = \max_{P_2, Q_2 \in \mathcal{E}_n^0} | \partial_{xx} (u - u_h)(P_2) | + \max_{Q_2 \in \mathcal{E}_n^0} | \partial_{yy} (u - u_h)(Q_2) | + \max_{P_3 \in \mathcal{E}_n^0} | \partial_{xy} (u - u_h)(P_3) |.
\]
We obtain our meshes by dividing the domain into \( M \times N \) rectangles, which is
generated by randomly and independently perturbing each node in the \( x \) and \( y \)
axes of a uniform mesh as
\[
x_i = \frac{i}{M} + \varepsilon \frac{1}{M} \sin\left( \frac{i \pi}{M} \right) \text{randn}, \quad 0 \leq i \leq M, \\
y_j = \frac{j}{N} + \varepsilon \frac{1}{N} \sin\left( \frac{j \pi}{N} \right) \text{randn}, \quad 0 \leq j \leq N,
\]
where \text{randn()} returns a uniformly distributed random number in \((0, 1)\). If not
otherwise stated, we choose \( M = N \) and \( \varepsilon = 0.001 \).

Example 1: We consider the problem \ref{eq:diff} and take the constant coefficients as
\[
\alpha = \gamma = 1, \quad \beta = (1, 1).
\]
The right-hand side function \( f(x, y) \) is chosen such that the exact solution is
\[
u(x, y) = \sin(\pi x) \sin(\pi y).
\]
In Figure 1, we show error curves of various approximation errors calculated from the $C^1$ Petrov-Galerkin method for $k = 3, 4, 5$, respectively. We observe that both convergence rates for the function value error (i.e., $e_{u,n}$) and the first-order derivative error (i.e., $e_{\nabla u,n}$) at mesh nodes can reach as high as $h^{2k-2}$. As for the errors $e_u$ (i.e., the function value error at roots of the Jacobi polynomial $J_{k+1}^{-2,-1}(x)J_{k+1}^{-2,-1}(y)$), $e_{\nabla u}$ (i.e., the first-order derivative error on the Lobatto lines), $e_{\Delta u}$ (i.e., the second-order derivative error on the Gauss lines and Lobatto points), convergence rates are $h^{k+2}$, $h^{k+1}$, $h^k$, respectively. They are all consistent with error bounds established in Theorems 5.3-5.4. We also test the supercloseness between the $C^1$ Petrov-Galerkin solution $u_h$ and the Jacobi projection $u_I$. As expected, the convergence rates for errors $\|u_h - u_I\|_0$, $\|u_h - u_I\|_1$, $\|u_h - u_I\|_2$ are $h^{\min(k+2,2k-2)}$, $h^{k+1}$, $h^k$, respectively. These results verify our theoretical findings (5.5) in Theorem 5.2.

![Figure 1. Error curves for Example 1 with $\alpha = 1$, $\beta = (1,1)$, and $\gamma = 1$. (Left: $k = 3$, Middle: $k = 4$, Right: $k = 5$)](image)

To show the effect of the coefficients on the convergence rate, we further test different choice of coefficients. Presented in Figures 2-4 are error curves of $\|u_h - u_I\|_m$, $0 \leq m \leq 2$ in three cases: $\alpha = 1, \beta = (0,0), \gamma = 1$, $\alpha = 1, \beta = (1,1), \gamma = 0$, and $\alpha = 1, \beta = (0,0), \gamma = 0$. We observe that the convergence rate for the case $\alpha = 1, \beta = (1,1), \gamma = 0$ is the same at that for the counterpart $\alpha = 1, \beta = (1,1), \gamma = 1$ in Figure 1. However, in the case $\beta = (0,0)$, it seems that the convergence rate improves for $k = 4, 5$. To be more precise, we observe a convergence rate $h^{k+2}$ for $\|u_h - u_I\|_1$ and $h^{k+1}$ for $\|u_h - u_I\|_2$ when $k = 4, 5$, one order higher than the case $\beta = (1,1)$. In other words, it seems that the convection coefficient has effect on the superconvergence rate.

![Figure 2. Error curves for Example 1 with $\alpha = 1$, $\beta = (0,0)$, and $\gamma = 1$. (Left: $k = 3$, Middle: $k = 4$, Right: $k = 5$)](image)
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Figure 3. Error curves for Example 1 with $\alpha = 1$, $\beta = (1, 1)$, and $\gamma = 0$. (Left: $k = 3$, Middle: $k = 4$, Right: $k = 5$)

We also present in Figure 4 the error curves for $e_{u,n}$ and $e_{\nabla u,n}$ for the Poisson equation, i.e., $\alpha = 1$, $\beta = (0, 0)$, $\gamma = 0$. We observe a convergence rate of $h^{2k-2}$ for both $e_{u,n}$, $e_{\nabla u,n}$. Note that this superconvergence phenomenon for two-dimensional case is different from that for the one-dimensional case, where both errors $e_{u,n}$, $e_{\nabla u,n}$ equal to zero (see, [11]).

Figure 4. Error curves for Example 1 with $\alpha = 1$, $\beta = (0, 0)$, and $\gamma = 0$. (Left: $k = 3$, Middle: $k = 4$, Right: $k = 5$)

Example 2: We consider the problem (2.1) with the following variable coefficients:

$$\alpha(x, y) = e^{xy}, \quad \beta(x, y) = (x^2y, xy^2), \quad \gamma(x, y) = 2xy.$$ 

The right-hand side function $f(x, y)$ is chosen such that the exact solution is

$$u(x, y) = xy(1 - e^{x-1})(1 - e^{y-1}).$$

The corresponding error curves for $k = 3, 4, 5$ are presented in Figure 5. We see that, both convergence rates for $e_{u,n}$ and $e_{\nabla u,n}$ are $h^{2k-2}$, and convergence rates for $e_{u}$, $e_{\nabla u}$, $e_{\Delta u}$ are $h^{k+2}$, $h^{k+1}$, $h^{k}$, respectively. All these results again verify our theoretical findings in Theorems 5.3-5.4. Just the same as the constant coefficient case in Example 1, we observe a convergence rate $h^{\min(k+2, 2k-2)}$ for $\|u_h - u_f\|_0$, $h^{k+1}$ for $\|u_h - u_f\|_1$, and $h^{k}$ for $\|u_h - u_f\|_2$. Again, all these results are consistent with the error bounds established in Theorems 5.2-5.3.
8. Conclusion

In this work, we have proposed a new $C^1$-$L^2$ Petrov-Galerkin method for convection-diffusion equations over rectangular meshes. The numerical scheme is designed to use the $C^1$-conforming $Q_k$ element as our trial space and $L^2$ piecewise $Q_{k-2}$ polynomials as our test space. We prove that the designed numerical method is convergent with optimal rates in the $H^1, L^2, H^2$-norms, respectively. Furthermore, we have presented a unified approach to study the superconvergence property of the Petrov-Galerkin method and establish the superconvergence results including: 1) the function value and the first-order derivative are superconvergent with a rate of $h^{2k-2}$ at all mesh nodes; 2) the function value approximation is superconvergent with rate $h^{k+2}$ at roots of $J_{k+1}^{-2,-2}(x) \otimes J_{k+1}^{-2,-2}(y)$; 3) the first-order and second-order derivatives are superconvergent with rates of $h^{k+1}$ and $h^k$ along the Lobatto lines and Gauss lines, respectively; 4) the numerical solution $u_h$ is superconvergent towards the special Jacobi projection $u_I$ of the exact solution in all $L^2, H^1, H^2$-norms. Numerical experiments demonstrate that all the established error bounds are optimal.

We would like to point out that in principle it is straightforward to generalize the methodology we adopt in this paper to convection-diffusion equations with variable coefficients. However, it requires very tedious and lengthy arguments to carry on the argument, in a mathematically rigorous way. Our numerical results demonstrate that the same convergence and superconvergence results still hold true for convection-diffusion equations with variable coefficients.

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9. Appendix

This section is dedicated to the construction of the correction function \( w_h \) satisfying the conditions of Proposition 2.

In light of (5.2) and the estimates of \( E^x E^y u \) in Lemma 4.1, we have

\[
a(u - u_L + w_h, \theta) = a(E^x u + E^y u + w_h, \theta) + O(h^{2k-2})\|\theta\|_0.
\]

In other words, to achieve our superconvergence goal, we need to construct correction functions \( w_h^x, w_h^y \in V_h \) to separately correct the two low-order errors \( a(E^x u, \theta) \) and \( a(E^y u, \theta) \) such that

\[
a(E^x u + w_h^x, \theta) = O(h^{k-1+l})\|\theta\|_0, \quad a(E^y u + w_h^y, \theta) = O(h^{k-1+l})\|\theta\|_0
\]

for some positive \( l \).

9.1. Correction function for the error \( a(E^x u, \theta) \). We begin with the introduction of some operators \( Q_h^x, Q_h^y \). For any function \( v(x, y) \), we define a special operator \( Q_h^x \) along the \( y \)-direction as follows: \( Q_h^x v \big|_{\tau_j} \in \mathbb{P}_k(\tau_j^y) \) satisfies

\[
(Q_h^x v - v)(\cdot, y_j) = \partial_y (Q_h^x v - v)(\cdot, y_j) = 0, \quad j \in \mathbb{Z}_N,
\]

\[
\int_{y_j-1}^{y_j} (Q_h^x v - v)(\cdot, y)\theta dy = 0, \quad \forall \theta \in \mathbb{P}_{k-4}(\tau_j^y), \quad k \geq 4.
\]

Note that the operator \( Q_h^x \) is actually the one dimensional truncated Jacobi expansion along the \( y \)-direction while the other variable is fixed.

\[
\|v - Q_h^x v\|_{p, \infty, \tau_j^y} = \|E^y v\|_{p, \infty, \tau_j^y} \lesssim h^{k+1-p}\|v\|_{p, \infty, \tau_j^y}, \quad \forall p \leq k + 1.
\]

Similarly we can define the operator \( Q_h^y \) along the \( x \)-direction.

In light of (4.3), we have

\[
E^x u \big|_{\tau_{ij}} = \sum_{p=k+1}^{\infty} c_{i,p}(y)J_{j,p}^{-2}(x).
\]
Define
\[ u_0(x, y)|_{\tau} := c_{i+k+1}(y)J_{i,k+1}^{-2,-2}(x) + c_{i+k+2}(y)J_{i,k+2}^{-2,-2}(x), \]
By the orthogonality of Jacobi polynomials, i.e., \( J_{n+1}^{-2,-2} \perp \mathbb{P}_{n-4}, \partial_y J_{n+1}^{-2,-2} \perp \mathbb{P}_{n-3} \) (see [11]), we have
\[ a(E^x u, \theta) = (-\alpha E^x u_{yy} + \beta_1 \partial_x E^x u + \beta_2 E^x u_y + \gamma E^x u, \theta) \]
\[ = (\beta_1 \partial_x u_0, \theta) + ((-\alpha \partial^2_{yy} + \beta_2 \partial_y + \gamma)u_0, \theta). \]
Let
\[ \lambda_1(y) = Q_h^y c_{i,k+1}(y), \quad \lambda_2(y) = Q_h^y c_{i,k+2}(y), \]
and define
\[ w_0(x, y)|_{\tau} = \lambda_1(y)J_{i,k+1}^{-2,-2}(x) + \lambda_2(y)J_{i,k+2}^{-2,-2}(x), \]
\( w_{-1}(x, y) = 0. \)
For all \( l \in \mathbb{Z}_{k-2}, \) we define a series of functions \( w_l \in V_h \) as follows:
\[ \alpha(\partial_{xx} w_l, \theta) = (\beta_1 \partial_x w_{l-1} + (\beta_2 \partial_y - \alpha \partial^2_{yy} + \gamma)w_{l-2}, \theta), \quad \forall \theta \in S^x, \]
\[ \partial_x w_l(x, y) = 0, \quad w_l(a, y) = 0, \quad w_l(x, c) = w_l(x, d) = 0, \quad \forall (x, y) \in \Omega. \]
where
\[ S^x := \{ \theta(x, y) : \theta|_{\tau} \in (\mathbb{P}_{k-2}(x) \setminus \mathbb{P}_0(x)) \times \mathbb{P}_{k-2}(y), \quad \forall \tau \in T_h \}. \]

**Lemma 9.1.** Define
\[ W_h^y := \{ v(y) \in C^1([c, d]) : v|_{\tau} \in \mathbb{P}_k, \quad v(c) = v(d) = 0, \quad j \in \mathbb{Z}_N \}. \]
Given any smooth function \( g, \) assume that \( v(y) \in W_h^y \) is the solution of the following problem:
\[ \int_c^d v(y) \theta(y) dy = \int_c^d g(y) \theta(y) dy, \quad \forall \theta \in \mathbb{P}_{k-2}(\tau^y). \]
Then \( v(y) \) is well defined. Moreover, there holds
\[ \| \partial^n_y v \|_{0, \infty, [c, d]} \lesssim \| \partial^n_y g \|_{0, \infty, [c, d]}, \quad \forall n \leq k. \]

**Proof.** To prove the uniqueness of \( v, \) we only need to show that the right hand side function \( g = 0 \) can yield a unique zero solution, i.e., \( v = 0. \) To this end, we choose \( \theta = \partial_{yy} v \) in (9.10) and use the integration by parts to obtain
\[ \int_c^d (\partial_y v)^2 dy = 0, \]
which yields, together with the homogenous boundary condition,
\[ \partial_y v = 0, \quad v = 0. \]
Consequently, \( v(y) \) is well defined.

To estimate (9.11), we define a special function \( R_h g \in W_h^y \) of \( g \) as follows:
\[ R_h g|_{\tau} := Q_h^y g, \quad j \neq 1, N, \quad R_h g(y_{j,m}) = g(y_{j,m}), \quad j = 1, N, m \in \mathbb{Z}_{k-3}, \]
where \( y_{j,m} \) can be chosen as any interpolation points. By the approximation theory, we have for all \( n \leq k, \)
\[ \| g - R_h g \|_{0, \infty, \tau^y} \lesssim \begin{cases} h^{n+1} \| g \|_{n+1, p, \tau^y}, & j \neq 1, N, \\ h^n \| g \|_{n, p, \tau^y}, & j = 1, N. \end{cases} \]
Then we choose $\theta = \partial_{yy}(v - R_h g)$ in (9.10) to derive
\[
\|\partial_y(v - R_h g)\|_1^2 = \|(v - R_h g, \partial_{yy}(v - R_h g))\| = \|(g - R_h g, \partial_{yy}(v - R_h g))\| \\
= \sum_{j=0}^{N-1} |(\partial_y g - R_h g, \partial_y(v - R_h g))| + h^{n+1} \|g\|_{n,\infty, \tau_0} \|v - R_h g\|_{2,\infty, \tau_0} \\
\lesssim h^{n-\frac{1}{2}} \|g\|_{n,\infty} \|v - R_h g\|_1,
\]
where $\tau_0 = \tau^y_1 \cup \tau^y_N$, and in the last step, we have used the inverse inequality $\|v\|_{2,\infty} \lesssim h^{-\frac{1}{2}} \|v\|_1$ for any finite element function $v$. Again, by the inverse inequality, we have
\[
\|v - R_h g\|_n \lesssim h^{1-n} \|v - R_h g\|_1 \lesssim h^{\frac{1}{2}} \|g\|_{n,\infty}, \quad \forall n \leq k,
\]
and thus
\[
\|v - R_h g\|_n \lesssim h^{-\frac{1}{2}} \|v - R_h g\|_n \lesssim \|g\|_{n,\infty}.
\]
Consequently,
\[
\|v\|_{n,\infty} \lesssim \|v - R_h g\|_{n,\infty} + \|R_h g\|_{n,\infty} \lesssim \|g\|_{n,\infty}.
\]
This finishes the proof of (9.11). □

Lemma 9.2. Assume that $u$ has the Jacobi expansion (4.5) in each $\tau_{ij}$, and $\lambda_1, \lambda_2$ are defined in (4.5). Then the correction functions $w_l, 1 \leq l \leq 2$ in (9.7)-(9.8) are well defined. Furthermore, there exist functions $\mu_{l,p}(y) \in W^l_h$ such that
\[
(9.12) \quad \partial_{x} w_l|_{\tau_i^j} = h^{l-1} \sum_{p=k-l}^{k-1} \mu_{l,p}(y) \phi_{i,p}(x), \quad \|\partial^l_y \mu_{l,p}\|_{0,\infty} \lesssim \|\lambda_1\|_{n+l,\infty} + \|\lambda_2\|_{n+l,\infty}.
\]
Consequently, if $u \in W^{k+1+l+n,\infty}$ with $n$ being some positive integer, then
\[
(9.13) \quad \|\partial^l_y \partial_{x} w_l\|_{0,\infty} \lesssim h^{k+l} \|u\|_{k+l+n+1,\infty}, \quad \|\partial^l_y \partial_{y} w_l\|_{0,\infty} \lesssim h^{\min(k+l+1,2k-2)} \|u\|_{k+l+n+1,\infty}.
\]
Here $\phi_{i,n}(x)$ denotes the Lobatto polynomial of degree $n$ on $\tau_i^j$, i.e.,
\[
\phi_{i,n}(x) = \phi_n\left(\frac{2x - x_i - x_{i-1}}{h}\right) = \phi_n(s), \quad s \in [-1,1], \quad x \in (x_{i-1}, x_i).
\]

Proof. Note that for any $w_l \in V_h$, we have $\partial^2_{xx} w_l \in P_{h-2}(\tau_i^j)$. Suppose
\[
\partial_{x} w_l|_{\tau_i^j} = \sum_{p=0}^{k-2} c_{l,p}(y) L_{i,p}(x).
\]
Recalling the definition of $w_l$ in (9.7), we easily obtain that $c_{l,p}(y)$ is the solution of (9.11) with the right hand side function
\[
g(y) = \frac{2p+1}{h^2 \alpha} \int_{\tau_i^j} (\beta_1 \partial_{x} w_{l-1} + (\beta_2 \partial_{y} - \alpha \partial^2_{yy} + \gamma) w_{l-2})(x,y)y L_{i,p}(x) dx.
\]
In other words, $c_{l,p}(y)$ is uniquely determined, and thus $\partial_{xx} w_l$ is well-defined. Then the homogenous boundary condition in (9.8) indicates a unique function $w_l$ from $\partial_{xx} w_l$. Consequently $w_l$ is uniquely defined.
We prove (9.12) by the method of mathematical induction. We first show (9.12) is valid for \( l = 1 \). Note that \( w_0(x, y) = 0, y \in [c, d], i \in \mathbb{Z}_M \) and \( \partial_r \theta|_{r^\tau} \in \mathbb{P}_{k-3}(x) \) for any \( \theta \in W_h \). By (9.7) and the integration by parts, we have

\[
\alpha(\partial_x w_1, \partial_x \theta) = \beta_1(w_0, \partial_x \theta) = \beta_1 \sum_{i=1}^M \int_c^d \lambda_1(y)dy \int_{x_i}^{x_i+1} J_{i,k+1}^{-2, -2}(x) \partial_x \theta(x, y)dx
\]

\[
= -\frac{4\beta_1(k-1)(k-2)}{2k-1} \sum_{i=1}^M \int_c^d \lambda_1(y)dy \int_{x_i}^{x_i+1} \phi_{i,k-1}(x, y) \partial_x \theta(x, y)dx.
\]

Here in the last step, we have used \( J_{n+1}^{-2, -2}(s) = \frac{4(n-1)(n-2)}{2n-1} (\phi_{n+1} - \phi_{n-1}) (s) \) and the fact \( \phi_{k+1} \perp \mathbb{P}_{k-2} \). Since \( \lambda_1(c) = \lambda_1(d) = 0 \) and the above equation holds for any \( \theta \), then

\[
\partial_x w_1|_{r^\tau} = -\frac{4\beta_1(k-1)(k-2)}{2k-1} \lambda_1(\phi_{i,k-1}(x)),
\]

and thus (9.12) holds true for \( l = 1 \) with

\[
\mu_{1,k-1}(y) = -\frac{4\beta_1(k-1)(k-2)}{2k-1} \lambda_1(y), \quad \|\partial_y^p \mu_{1,k-1}\|_{0, \infty} \leq \|\partial_y^p \lambda_1\|_{0, \infty} \leq \|\lambda_1\|_{n, \infty}.
\]

Now we suppose (9.12) is valid for all \( l \) and prove it holds true for \( l = 1 \) with \( l \leq k-3, k \geq 4 \). By the induction assumption and the orthogonality of the Lobatto polynomials, we get

\[
\partial_x w_1|_{r^\tau} = h^{l-1} \sum_{p=k-l}^{k-1} \mu_{i,p}(y) \phi_{i,p}(x) \perp \mathbb{P}_0(r^\tau), \quad \forall l \leq k-3,
\]

and thus

\[
w_1(x, y) = \int_a^x \partial_x w_1(x, y)dx = h^{l-1} \sum_{p=k-l}^{k-1} \mu_{i,p} \int_{x_i}^{x} \phi_{i,p}(x)dx
\]

\[
= h^{l} \sum_{p=k-l}^{k-1} \frac{\mu_{i,p}}{2p-1} (\phi_{i,p+1} - \phi_{i,p-1})(x) = h^{l} \sum_{p=k-l-1}^{k} \frac{\mu_{i,p-1}}{2p-3} - \frac{\mu_{i,p+1}}{2p+1} \phi_{i,p}(x),
\]

where we use the notation \( \mu_{i,p} = 0 \) for all \( p = k - l - 1, k - l - 2, k + 1 \), and in the forth step, we have used (4.4). Similarly, there holds

\[
w_{l-1}(x, y)|_{r^\tau} = h^{l-1} \sum_{p=k-l}^{k} \frac{\mu_{i-1,p-1}}{2p-3} - \frac{\mu_{i-1,p+1}}{2p+1} \phi_{i-1,p}(x) \perp \mathbb{P}_0(r^\tau), \quad \forall l \leq k-3.
\]

By defining

\[
\partial_x^{-1} v(x, \cdot) = \int_a^x v(x, \cdot)dx,
\]

and using the fact that \( w_{l-1} \perp \mathbb{P}_0(r^\tau) \) and (4.4), we get

\[
\partial_x^{-1} w_{l-1}(x, y) = h^{l-1} \sum_{p=k-l}^{k} \frac{\mu_{i-1,p-1}}{2p-3} - \frac{\mu_{i-1,p+1}}{2p+1} \int_{x_i}^{x} \phi_{i,p}(x)dx
\]

\[
= h^{l} \sum_{p=k-l-1}^{k} \mu_{i-1,p} \phi_{i,p}(x),
\]
In light of the conclusion in Lemma 9.1, we have

\[ \bar{\mu}_{l-1} = \left( \frac{\mu_{l-1,p-2}}{2p-5} - \frac{\mu_{l-1,p}}{2p-3} \right) \frac{1}{2p-3} - \left( \frac{\mu_{l-1,p-1}}{2p-1} - \frac{\mu_{l-1,p+2}}{2p+3} \right) \frac{1}{2p+1}, \]

with \( \mu_{l-1,p} = 0, \forall p \leq k - l, \) or \( p \geq k + 1. \) Note that

\[ w_l(x, \cdot) = \partial_x^{-1} w_{l-1}(x, \cdot) = 0, \ \forall i \in \mathbb{Z}_M, l \leq k - 3. \]

By (9.13) and the integration by parts,

\[ \alpha(xw_{l+1}, \partial_x \theta) = (\beta_1 w_l + (\beta_2 \partial_y - \alpha \partial^2_{yy} + \gamma) \partial_x^{-1} w_{l-1}, \partial_x \theta) = \]

\[ = h^l \sum_{\tau_{ij} \in T, p=k-l-1} (\bar{e}_{i,p} \phi_{i,p}, \partial_x \theta)_{\tau_{ij}}. \]

Here \((u, v)_x = \int \int (uv)(x, y)dxdy, \) and

\[ \bar{e}_{i,p} = \beta_1 \left( \frac{\mu_{l-1,p-1}}{2p-3} - \frac{\mu_{l-1,p+1}}{2p+1} \right) + (\beta_2 \partial_y - \alpha \partial_{yy} + \gamma) \bar{\mu}_{l-1,p}. \]

Consequently,

\[ \partial_x w_{l+1}|_{\tau_{ij}} = h^l \sum_{p=k-l-1}^{k-1} \mu_{l+1,p}(y) \phi_{i,p}(x) \]

with \( \mu_{l+1,p}(y) \in W^W_h \) is the solution of the following equation:

\[ \int_c^d \mu_{l+1,p}(y) v(y) dy = \int_c^d \bar{e}_{i,p} v(y) dy, \ \forall v \in \mathbb{P}_{k-2}(y). \]

In light of the conclusion in Lemma 9.1, we have

\[ \| \partial_y^n \mu_{l+1,p} \|_{0, \infty} \lesssim \| \partial_y^n \mu_{l,p-1} \|_{0, \infty} + \| \partial_y^n \mu_{l+1,p} \|_{0, \infty} + \| \partial_y^n \bar{\mu}_{l-1,p} \|_{0, \infty} \]

\[ \lesssim \| \lambda_1 \|_{n+l+1, \infty} + \| \lambda_2 \|_{n+l+1, \infty}. \]

In other words, (9.12) is also valid for \( l + 1 \) and thus holds true for all \( l \leq k - 2. \)

We next prove (9.13). By (9.12), we easily get

\[ \| \partial_y^n \partial_x w_l \|_{0, \infty} \lesssim h^{l-1} \sum_{p=k-l-1}^k \| \partial_y^n \mu_{l,p} \|_{0, \infty} \lesssim h^l (\| \lambda_1 \|_{n+l+1, \infty} + \| \lambda_2 \|_{n+l+1, \infty}). \]

Recalling the definition of \( \lambda_i, i = 1, 2 \) and using the estimates for \( E^x u, \) we have

\[ \| \partial_y^{n+l} \lambda_1 \|_{0, \infty} + \| \partial_y^{n+l} \lambda_2 \|_{0, \infty} \lesssim \| \partial_y^{n+l} E^x u \|_{0, \infty} \lesssim h^{k+l} \| u \|_{k+l+n+1, \infty}. \]

Then

\[ \| \partial_y^n \partial_x w_l \|_{0, \infty} \lesssim h^{k+l} \| u \|_{k+l+n+1, \infty}, \ \forall l \in \mathbb{Z}_{k-2}. \]

This finishes the proof of the first inequality of (9.13). Similarly, by using (9.14) and the estimates of \( \lambda_1, \lambda_2, \) we have for all \( l \leq k - 3, \)

\[ \| \partial_y^n w_l \|_{0, \infty} \lesssim h^{l} \sum_{p=k-l-1}^k \| \partial_y^n \mu_{l,p} \|_{0, \infty} \lesssim h^{k+l+1} \| u \|_{k+l+n+1, \infty}. \]

As for \( l = k - 2, \) we have, from the Poincaré inequality,

\[ \| \partial_y^k w_{k-2} \|_{0, \infty} \lesssim \| \partial_y^k \partial_x w_{k-2} \|_{0, \infty} \lesssim h^{2k-2} \| u \|_{2k-1+n, \infty}. \]

Then the second inequality of (9.13) follows. This finishes our proof. \( \square \)
Now we are ready to construct the correction function $w^x_h$. Define

$$w^x_h(x, y) = \sum_{l=1}^{k-2} w_l(x, y)$$

with $w_l$ defined by (9.10)-(9.8). We have the following property for the correction function $w^x_h$.

**Theorem 9.3.** Let $w^x_h(x, y) \in V_h$ be defined by (9.16). Then

$$w^x_h(a, y) = w^x_h(x, c) = w^x_h(x, d) = 0, \quad w^x_h(b, y) = w_{k-2}(b, y).$$

Furthermore, if $u \in W^{2k+1, \infty}(\Omega)$, then

$$|a(E^x u + w^x_h, \theta)| \lesssim h^{2k-2} \|u\|_{2k+1, \infty}\|\theta\|_0, \quad \forall \theta \in W_h.$$  

**Proof.** First, (9.11) follows directly from the conclusions in Lemma 9.2, 9.3 and (9.15). Note that any $\theta \in W_h$ can be decomposed into

$$\theta = \theta_0 + \theta_1, \quad \theta_1|_{\tau} \in (\mathbb{P}_{k-2}(x) \setminus \mathbb{P}_0(x)) \times \mathbb{P}_{k-2}(y), \quad \theta_0|_{\tau} \in \mathbb{P}_0(x) \times \mathbb{P}_{k-2}(y).$$

Since $\theta_1 \in S^x_h$ with $S^x_h$ defined by (9.9), we have, from (9.6)-(9.7),

$$a(w^x_h, \theta_1) = \sum_{l=1}^{k-2} (-\alpha \Delta w_l + \beta \cdot \nabla w_l + \gamma w_l, \theta_1)$$

$$= ((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)(w_{k-2} - w_0 + w_{k-3}), \theta_1) + \beta_1(\partial_x w_{k-2} - \partial_x w_0, \theta_1)$$

$$= I_{\theta_1} - ((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)w_0, \theta_1) - \beta_1(\partial_x w_0, \theta_1),$$

where $w_0$ is defined in (9.10), and

$$I_{\theta} = ((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)(w_{k-2} - w_{k-3}), \theta) + \beta_1(\partial_x w_{k-2}, \theta).$$

As for the term $a(w^x_h, \theta_0)$, we use the properties of $w_l$ in (9.12) to obtain that

$$\partial_x w_n \perp \mathbb{P}_0(x), \quad n \in \mathbb{Z}_{k-2}, \quad \partial_x w_m \perp \mathbb{P}_0(x), \quad m \in \mathbb{Z}_{k-3}, \quad w_l \perp \mathbb{P}_0(x), \quad l \in \mathbb{Z}_{k-4}, \text{ if } k \geq 4,$$

and thus,

$$a(w^x_h, \theta_0) = ((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)(w_{k-2} - w_{k-3}), \theta_0) + \beta_1(\partial_x w_{k-2}, \theta_0) = I_{\theta_0}.$$

Consequently,

$$a(w^x_h, \theta) = a(w^x_h, \theta_0 + \theta_1) = I_{\theta} - ((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)w_0, \theta_1) - \beta_1(\partial_x w_0, \theta_1)$$

$$= I_{\theta} - a(w_0, \theta_1).$$

On the other hand, we have from (9.2),

$$a(E^x u, \theta) = ((\beta_1 \partial_x + \beta_2 \partial_y - \alpha \partial_{yy} + \gamma)u_0, \theta_1 + \theta_0) = a(u_0, \theta_1) + a(u_0, \theta_0),$$

and thus

$$a(E^x u + w^x_h, \theta) = I_{\theta} + a(u_0, \theta_0) + a(u_0 - w_0, \theta_1).$$

By (9.13) and Cauchy-Schwarz inequality, we have

$$|I_{\theta}| \lesssim \left( \|\partial_x w_{k-2}\|_0 + \sum_{n=0}^{2} \sum_{l=1}^{k-3} \|\partial_y^n w_l\|_0 \right) \|\theta\|_0 \lesssim h^{2k-2} \|u\|_{2k+1, \infty}\|\theta\|_0.$$  

As for the term $a(u_0, \theta_0)$, noticing that $\theta_0 \in \mathbb{P}_0(x)$, we have from (9.3)

$$|a(u_0, \theta_0)| = |((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)u_0, \theta_0)| \lesssim h^{m}\|u\|_{m+2, \infty}\|\theta_0\|_0, \quad m \leq k + 1$$
for $k = 3$. While for $k \geq 4$, we have $u_0 \perp \mathbb{P}_{k-4}$, which yields

$$a(u_0, \theta_0) = 0.$$  

Consequently,

$$a(u_0, \theta_0) \lesssim h^{k+1}||u||_{k+3, \infty}||\theta||_0 \lesssim h^{2k-2}||u||_{2k, \infty}||\theta||_0, \quad \forall k \geq 3.$$  

To estimate the error $u_0 - w_0$, we recall the definition of $u_0, w_0$ in (9.3) and (9.6) and then use the estimate of $E^x u$ to obtain

$$||\partial_n^p (u_0 - w_0)||_{0, \infty, \tau, \gamma} \lesssim ||\partial_n^p (Q^y_h E^x u - E^x u)||_{0, \infty} \lesssim h^{2k+1-n}||u||_{2k+1, \infty}.$$  

Similarly, we get

$$||\partial_x (u_0 - w_0)||_{0, \infty, \tau, \gamma} \lesssim h^{-1}||Q^y_h E^x u - E^x u||_{0, \infty} \lesssim h^{2k}||u||_{2k+1, \infty}.$$  

Consequently,

$$||((\beta_2 \partial_y - \alpha \partial_{yy} + \gamma)(u_0 - w_0) \theta_1) + \beta_1 (\partial_x (u_0 - w_0) \theta_1)|| \lesssim h^{2k-2}||u||_{2k+1, \infty}||\theta||_0.$$  

Substituting (9.20) and (9.21) into (9.19) yields

$$|a(E^y u + w^y_h, \theta)| \lesssim h^{2k-2}||u||_{2k+1, \infty}(||\theta||_0 + ||\theta_1||_0) \lesssim h^{2k-2}||u||_{2k+1, \infty}||\theta||_0.$$  

This finishes the proof of (9.18). \q\quad \Box

9.2. Correction function for the error $a(E^y u, \theta)$. The construction of the correction function $w^y_h$ for $a(E^y u, \theta)$ is similar to that of $w^x_h$. To be more precise, we suppose in each element $\tau_{ij}$,

$$E^y u|_{\tau_{ij}} = \sum_{q=k+1}^{\infty} s_{j,q}(x) J_{j,q}^{-2}u^{-2}(y).$$

Let

$$\bar{w}_0(x, y)|_{\tau_{ij}} = Q^y_h s_{j,k+1}(x) J_{j,k+1}^{-2}u^{-2}(y) + Q^y_h s_{j,k+2}(x) J_{j,k+2}^{-2}u^{-2}(y), \quad \bar{w}_{-1}(x, y) = 0.$$  

For all $l \in \mathbb{Z}_{k-2}$, we define a series of functions $\bar{w}_l \in V_h$ as follows:

$$\alpha(\partial_y \partial_y \bar{w}_l, \theta) = (\beta_2 \partial_y \bar{w}_l - 1 + (\beta_1 \partial_x - \alpha \partial_{xx} + \gamma) \bar{w}_{l-2}, \theta), \quad \forall \theta \in S^y_h,$$

$$\partial_y \bar{w}_l(x, y) = 0, \quad \bar{w}_l(x, c) = 0, \quad \bar{w}_l(a, y) = \bar{w}_l(b, y) = 0, \quad \forall (x, y) \in \Omega,$$

where

$$S^y_h := \{\theta(x, y) : \theta \in \mathbb{P}_k(x) \times (\mathbb{P}_{k-2}(y) \setminus \mathbb{P}_0(y))\}.$$  

Following the same argument as that in Lemma 9.2, we get $\bar{w}_l(x, y_j) = 0$ for all $l \leq k - 3$ and

$$||\partial_n^p \partial_y \bar{w}_l||_{0, \infty} \lesssim h^{k+l}||u||_{k+l+n+1, \infty}, \quad ||\partial_n^p \bar{w}_l||_{0, \infty} \lesssim h^{\min(k+l+1, 2k-2)}||u||_{k+l+n+1, \infty}.$$  

Define

$$w^y_h(x, y) = \sum_{l=1}^{k-2} \bar{w}_l(x, y),$$

and follow what we have done in Theorem 9.3, we get

$$w^y_h(x, c) = w^y_h(a, y) = w^y_h(b, y) = 0, \quad w^y_h(x, d) = \bar{w}_{k-2}(x, d),$$

and

$$|a(E^y u + w^y_h, \theta)| \lesssim h^{2k-2}||u||_{2k+1, \infty}||\theta||_0, \quad \forall \theta \in W_h.$$
9.3. Proof of Proposition 2 Define the correction function by
\[ w_h(x, y) = (w_h^w + w_h^h)(x, y) - \frac{x - a}{b - a} w_{k-2}(b, y) - \frac{y - c}{d - c} w_{k-2}(x, d), \]
where \( w_h^w, w_h^h \) are given by (9.16), (9.7)-(9.8) and (9.23). As a direct consequence of (9.17) and (9.24), we have
\[ w_h(a, y) = w_h(b, y) = w_h(x, c) = w_h(x, d) = 0. \]
In other words, \( w_h \in V_h^0 \).

Now we are ready to prove the conclusion of Proposition 2.

**Proof.** By using the properties and estimates of \( E^z E^y u \) in Lemma 4.1, we have
\[ |a(E^z E^y u, \theta)| \lesssim h^{2k-2} \|u\|_{2k-1} \|\theta\|_0, \quad \forall \theta \in W_h. \]
Let
\[ \tilde{w}(x, y) = \frac{x - a}{b - a} w_{k-2}(b, y) - \frac{y - c}{d - c} w_{k-2}(x, d). \]
By (9.13) and (9.22), we have
\[ a(\tilde{w}, \theta) \lesssim \sum_{n=0}^2 \|\tilde{w}\|_{0, \infty} \|\tilde{w}\|_{0, \infty} \|\tilde{w}\|_{2k-2, \infty} \|\tilde{w}\|_{2k-2, \infty} \|\theta\|_0 \lesssim h^{2k-2} \|u\|_{2k+1, \infty} \|\theta\|_0, \]
which yields, together with (9.18) and (9.25),
\[ |a(\eta + w_h, \theta)| \lesssim h^{2k-2} \|u\|_{2k+1, \infty} \|\theta\|_0. \]
Then (5.15) follows.

We next prove (5.13)-(5.14). By (9.13) and (9.22), we have
\[ \|w_h\|_{0, \infty} \lesssim \sum_{l=1}^{k-2} (\|w_l\|_{0, \infty} + \|\tilde{w}_l\|_{0, \infty}) \lesssim h^{(k+2)(2k-1)} \|u\|_{2k+1, \infty}, \]
\[ \|w_h\|_{m, \infty} \lesssim \sum_{l=1}^{k-2} (\|w_l\|_{m, \infty} + \|\tilde{w}_l\|_{m, \infty}) \lesssim h^{k+2-m} \|u\|_{2k+1, \infty}, \quad m = 1, 2. \]
Then (5.13) follows. By (9.12) and (9.15), we have \( \partial_x w_h^y(x, y_j) = 0 \) and
\[ |\partial_y^n w_h^y(x, y_j)| = |\partial_y^n w_{k-2}(x, y_j)| \leq \|\partial_y^n w_{k-2}\|_{0, \infty} \lesssim h^{2k-2} \|u\|_{2k+1, \infty}, \quad n = 0, 1. \]
Similarly, there holds
\[ |\partial_x^n w_h^y(x, y_j)| \leq \|\partial_x^n \tilde{w}_{k-2}(x, y_j)| \lesssim h^{2k-2} \|u\|_{2k+1, \infty}, \quad n = 0, 1, \quad \partial_y w_h^y(x, y_j) = 0. \]
Consequently,
\[ |w_h(x, y)| + |\nabla w_h(x, y_j)| \lesssim h^{2k-2} \|u\|_{2k+1, \infty}. \]
This finishes the proof of (5.14). The proof is complete. \( \square \)

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