New exact results for the two-phase model with several conserved currents

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Abstract

We consider the macroscopic transport properties of two-dimensional random binary mixtures with identical spatial distributions of the two phases. Previous studies have obtained exact analytical results for the electrical conductivity of a single layer with and without a magnetic field, as well as for the thermoelectric response of a magnetic field-free double-layer. Here, we generalize these exact solutions to the magneto-thermoelectric response of a single layer and to the thermoelectric response of a double-layer. The magneto-thermoelectric transport coefficients of the double-layer are calculated perturbatively for weak magnetic field.

1. Introduction

Random mixtures of several phases are common in nature. They appear, for example, near first order phase transitions, in percolating systems and when competing short and long-range interactions are present\cite{1}. Here, we are concerned with the situation where the inhomogeneity occurs on sufficiently large scales, such that each phase can be characterized by its own bulk kinetic properties, \textit{e.g.}, its electrical and thermal conductivities. Typically, the calculation of the response of the entire system is intractable due to the complicated distribution of currents in the sample. However, an exact analytical solution to this problem exists for two-dimensional systems comprised of two phases with statistically identical distributions, as occurs at the percolation transition point. The solution relies on the existence of a self-duality transformation, which interchanges the roles of the currents and the driving fields. The duality can be realized provided that the fields are potential gradients and the currents are conserved.

The aforementioned approach is originally due to Dykhne, who applied it to the case of a single (electrical) current\cite{2}. Subsequently, it was generalized to include the effects of a magnetic field\cite{3,4}, and to the case of two conserved currents (electrical and heat) but without a magnetic field\cite{5}.

In this paper we present several new exact solutions to the problem. First, we study a time-reversal symmetric system with three conserved currents. Such
a setting applies when measuring drag in a bilayer of electrically isolated films that are strongly coupled thermally. Second, we provide a solution to the case of two conserved currents in the presence of a magnetic field, $H$. Finally, we return to the bilayer problem and solve it perturbatively in $H$.

2. Statement of the Problem

Consider a two-dimensional system composed of two isotropic phases whose random spatial distributions are statistically equivalent. Assume that there is a set of $N$ conserved current densities in the problem $\mathbf{J} = (J_1, \ldots, J_N)^T$, with

$$\nabla \cdot \mathbf{J} = 0,$$

(1)

that can be induced by a set of $N$ external forces, $\mathbf{F} = (F_1, \ldots, F_N)^T$, satisfying

$$\nabla \times \mathbf{F} = 0.$$

(2)

Within linear response the currents and forces are related by

$$\mathbf{J}(\mathbf{r}) = \hat{L}(\mathbf{r})\mathbf{F}(\mathbf{r}),$$

(3)

where $\hat{L}(\mathbf{r})$ is a matrix containing the kinetic coefficients, which takes one of two values $\hat{L}_1$ or $\hat{L}_2$ corresponding to the phase present at point $\mathbf{r}$. We choose the forces such that the local entropy production rate is given by $\mathbf{J}(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r})/T(\mathbf{r})$. Consequently, Onsager’s relations and the isotropy of the constituent phases assure that in the absence of a magnetic field $\hat{L}$ is symmetric.

Our goal is to calculate the macroscopic response of the system, as given by $\hat{L}_{\text{eff}}$ relating the spatial averages of $\mathbf{J}$ and $\mathbf{F}$

$$\langle \mathbf{J} \rangle = \hat{L}_{\text{eff}} \langle \mathbf{F} \rangle.$$  

(4)

To this end, we follow Dykhne [2] and introduce an auxiliary transport problem defined in terms of

$$\mathbf{J}'_\alpha = U_{\alpha\beta} \hat{n} \times \mathbf{F}_\beta, \quad \mathbf{F}'_\alpha = U_{\alpha\beta}^{-1} \hat{n} \times \mathbf{J}_\beta,$$

(5)

where $\hat{U}$ is a symmetric matrix whose components are of the same physical dimensions as the components of $\hat{L}$, $\hat{n}$ is a unit vector perpendicular to the plane and where repeated indices are summed over. One can easily check that the new currents and forces satisfy the requirements $\nabla \cdot \mathbf{J}'_\alpha = 0$ and $\nabla \times \mathbf{F}'_\alpha = 0$. Furthermore, owing to the linear nature of the transformation $U_5$ they are related by $\mathbf{J}'(\mathbf{r}) = \hat{L}'(\mathbf{r})\mathbf{F}(\mathbf{r})$. From Eqs. $U_3$-$U_5$ it follows that $\hat{L}'(\mathbf{r}) = \hat{U} \hat{L}^{-1}(\mathbf{r})\hat{U}$ and

$$\hat{L}'_{\text{eff}} = \hat{U} \hat{L}_{\text{eff}}^{-1} \hat{U}.$$  

(6)

A particularly useful choice for $\hat{U}$ is the one that interchanges the two components, i.e.,

$$\hat{L}_2 = \hat{U} \hat{L}_1^{-1} \hat{U}.$$  

(7)
For such a duality transformation the auxiliary problem corresponds to a system that is obtained from the original one by replacing one phase by the other. Consequently, their statistical equivalence implies that \( \hat{L}'_{\text{eff}} = L_{\text{eff}} \), which together with Eq. (6) leads to \( \hat{L}_{2} = \hat{L}_{\text{eff}}^{-1} \hat{L}_{1} \hat{L}_{\text{eff}} \). (8)

This algebraic equation determines the macroscopic response of the system.

2.1. Examples

In the case of a single conserved (electrical) current the matrices \( \hat{L}_{1,2} \) become the conductivities of the two phases, \( \sigma_{1,2} \), and Eq. (8) reduces to \( \sigma_{2} = \sigma_{\text{eff}}^{2} \sigma_{1}^{-1} \) with the solution \( \sigma_{\text{eff}} = \sqrt{\sigma_{1} \sigma_{2}} \). (9)

For two or more conserved currents the non-commutativity of \( \hat{L}_{1} \) and \( \hat{L}_{2} \) plays an essential role. As an example, consider the thermoelectric response of a two-dimensional film described by

\[
\begin{pmatrix}
    j(r) \\
    q(r)
\end{pmatrix} = \hat{L}(r) \begin{pmatrix}
    -\nabla T(r) \\
    E(r)
\end{pmatrix},
\]

where \( j \) and \( q \) are the electrical and heat current densities, respectively, while \( E \) and \( T \) are the electric field and the temperature. Note that \( q \) is conserved within linear response that neglects second order Joule heating effects. The solution to Eq. (8) in this case is \( \hat{L}_{\text{eff}} = c \left( \hat{L}_{1} \sqrt{d_{1}} + \hat{L}_{2} \sqrt{d_{2}} \right) \), (11)

where \( d_{j} = \text{det} \hat{L}_{j} \) and the constant \( c \) is

\[
c = (d_{1}d_{2})^{1/4} \left[ \text{det} \left( \hat{L}_{1} \sqrt{d_{1}} + \hat{L}_{2} \sqrt{d_{2}} \right) \right]^{-1/2}.
\]

For completeness, we present the derivation of this result in the Appendix.

3. Thermoelectric response of a two-phase double-layer

Next, we consider the case with three conserved currents, motivated by drag experiments in double-layer graphene \([10,13]\). In these experiments the two graphene layers are electrically isolated from each other, but they are in sufficient proximity such that they may be considered as a single layer from a thermal point of view. This thermalization is primarily due to intra and inter-layer electron-electron inelastic scattering, while the electron-phonon coupling is much
weaker and does not lead to a significant violation of the presumed conservation of heat in the system. Under such conditions the linear response is described by

\[
\begin{pmatrix}
  j_u(r) \\
  j_d(r) \\
  q(r)
\end{pmatrix}
= \hat{L}(r)
\begin{pmatrix}
  E_u(r) \\
  E_d(r) \\
  \mathbf{v}(r)
\end{pmatrix},
\]

(13)

where \(j_{u,d}\) are the electrical current densities in the upper and lower layers, respectively, and \(q\) is the total heat current density through the system. Similarly, \(E_{u,d}\) are the electric fields in the two layers, and \(T\) is the temperature field, which is identical in both. Furthermore, we will study the case where a domain of a particular phase in the upper layer appears above a similar domain in the lower layer. As a result, \(\hat{L}(r)\) can take two values

\[
\hat{L}_j = \begin{pmatrix}
  \sigma_j & \eta_j & \alpha_j \\
  \eta_j & \sigma_j & \alpha_j \\
  \alpha_j & \alpha_j & 2\kappa_j
\end{pmatrix}, \quad j = 1, 2
\]

(14)

where \(\sigma\) is the electrical conductivity, \(\alpha/T\sigma\) is the thermopower and \(\kappa/T\) is the thermal conductivity (provided it is much larger than \(\alpha^2/T\sigma\), as in metals [14]). Finally, \(\eta\) is the drag conductivity due to interaction-induced momentum transfer between the layers.

Beyond momentum transfer there is another drag mechanism which originates from inter-layer energy transfer [15–17]. Interfaces between phases act as thermocouples. A current driven through one layer generates local temperature gradients by the Peltier effect. The strong thermal coupling in the system causes these temperature gradients to propagate to the second layer, which is under open circuit conditions, and generate thermopower there by the Seebeck effect.

Equation (6) can be exactly solved for the model defined by Eq. (14), as described in the Appendix. Here, we focus on two relevant special cases and begin by considering a system for which \(\eta_1 = \eta_2 = 0\), where any resulting drag is due to inter-layer energy transfer alone. Solving Eq. (8) we obtain the effective conductivity

\[
\sigma_{\text{eff}} = \frac{1}{2} \sqrt{\sigma_1 \sigma_2} \left( 1 + \frac{\nu^{1/2} \cos \theta_1 + \nu^{-1/2} \cos \theta_2}{\sqrt{\nu + \nu^{-1} + 2 \cos(\theta_1 + \theta_2)}} \right),
\]

(15)

with

\[
\sin \theta_j = \frac{\alpha_j}{\sqrt{\kappa_j \sigma_j}}, \quad \nu = \frac{\sigma_1 \kappa_2}{\sigma_2 \kappa_1},
\]

(16)

where the angles \(\theta_j\) are defined in the range \([-\pi/2, \pi/2]\). The other effective transport coefficients are given in terms of \(\sigma_{\text{eff}}\) according to

\[
\eta_{\text{eff}} = \sigma_{\text{eff}} - \sqrt{\sigma_1 \sigma_2},
\]

(17)

\[
\alpha_{\text{eff}} = \frac{\alpha_1 \sqrt{d_2} + \alpha_2 \sqrt{d_1}}{\sigma_1 \sqrt{d_2} + \sigma_2 \sqrt{d_1}} (2\sigma_{\text{eff}} - \sqrt{\sigma_1 \sigma_2}),
\]

(18)

\[
\kappa_{\text{eff}} = \frac{\kappa_1 \sqrt{d_2} + \kappa_2 \sqrt{d_1}}{\sigma_1 \sqrt{d_2} + \sigma_2 \sqrt{d_1}} (2\sigma_{\text{eff}} - \sqrt{\sigma_1 \sigma_2}),
\]

(19)
where, as before, \( d_{1,2} = \det \hat{L}_{1,2} \).

Next, we treat the case with inter-layer momentum transfer but where the two phases differ only by the sign of their charge carriers, while all other characteristics such as mobility and mass are identical, \( i.e., \)

\[
\hat{L}_{1,2} = \begin{pmatrix}
\sigma & \eta & \pm\alpha \\
\eta & \sigma & \pm\alpha \\
\pm\alpha & \pm\alpha & 2\kappa
\end{pmatrix}.
\] (20)

For this model we find

\[
\sigma_{\text{eff}} = \sigma \cos^2 \left( \frac{\theta}{2} \right) - \eta \sin^2 \left( \frac{\theta}{2} \right),
\] (21)

\[
\eta_{\text{eff}} = -\sigma \sin^2 \left( \frac{\theta}{2} \right) + \eta \cos^2 \left( \frac{\theta}{2} \right),
\] (22)

\[
\alpha_{\text{eff}} = 0,
\] (23)

\[
\kappa_{\text{eff}} = \kappa \cos \theta,
\] (24)

where

\[
\sin \theta = \frac{\alpha}{\sqrt{\kappa(\sigma + \eta)}}.
\] (25)

4. A system in a magnetic field

Our next goal is to extend the discussion to include a magnetic field, \( H \), along the \( \hat{z} \) direction. In the presence of the field the components of \( \hat{L}(r) \) and \( \hat{L}_{\text{eff}} \) in Eqs. (3,4) become \( 2 \times 2 \) tensors, whose off-diagonal terms are antisymmetric owing to the assumption of isotropic phases. For example, the electrical conductivity tensor of the \( j = 1, 2 \) phases is

\[
\hat{\sigma}_j = \begin{pmatrix}
\sigma_j & \sigma_{Hj} \\
-\sigma_{Hj} & \sigma_j
\end{pmatrix},
\] (26)

where \( \sigma_H \) is the Hall conductivity.

Equations (3) and (4) may be condensed by representing two-dimensional vectors as complex numbers, \( e.g., \mathbf{E} = E_x + iE_y \), leading to current densities and forces of the form

\[
\mathbf{J} = (J_{1x} + iJ_{1y}, \cdots, J_{Nx} + iJ_{Ny})^T,
\] (27)

\[
\mathbf{F} = (F_{1x} + iF_{1y}, \cdots, F_{Nx} + iF_{Ny})^T.
\] (28)

In this representation Eq. (3) becomes

\[
\mathbf{J}(\mathbf{r}) = \hat{\mathbf{L}}(\mathbf{r})\mathbf{F}(\mathbf{r}),
\] (29)

where the complex response matrix \( \hat{\mathbf{L}}(\mathbf{r}) \) takes one of two values in the corresponding phases \( j = 1, 2 \),

\[
\hat{L}_j = \hat{L}_j - i\hat{L}_{Hj},
\] (30)
expressed in terms of $N \times N$ symmetric matrices $\hat{L}_j$ and $\hat{L}_{Hj}$ holding the longitudinal and Hall parts, respectively, of the kinetic coefficients.

The magnetic field also requires that the duality transformation, Eq. 6, be generalized to include a component proportional to the original fields 3, 6. In the complex notation it reads

$$\mathcal{J}'(\mathbf{r}) = \hat{A}\mathcal{J}(\mathbf{r}) - i\hat{B}\mathcal{F}(\mathbf{r}), \quad \mathcal{F}'(\mathbf{r}) = \hat{C}\mathcal{F}(\mathbf{r}) - i\hat{D}\mathcal{J}(\mathbf{r}),$$

(31)

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are $N \times N$ matrices that have to be real in order for the transformed fields to still satisfy the conditions $\nabla \cdot \mathcal{J}' = 0$ and $\nabla \times \mathcal{F}' = 0$. Substituting Eqs. (29, 31) into $\mathcal{J}'(\mathbf{r}) = \hat{C}'(\mathbf{r})\mathcal{F}'(\mathbf{r})$ yields the relation

$$\hat{A}\hat{C}(\mathbf{r}) - i\hat{B} = \hat{C}'(\mathbf{r}) \left[ \hat{C} - i\hat{D}\hat{C} \right].$$

(32)

A similar relation

$$\hat{A}\hat{L}_{\text{eff}} - i\hat{B} = \hat{C}' \hat{L}_{\text{eff}} \left( \hat{C} - i\hat{D}\hat{L}_{\text{eff}} \right).$$

(33)

holds for the effective response matrix $\hat{L}_{\text{eff}}$ connecting the spatially averaged fields $\langle \mathcal{J} \rangle = \hat{L}_{\text{eff}} \langle \mathcal{F} \rangle$.

As before, we are interested in the transformation that fulfills Eq. 32 for the cases $\mathcal{L} = \mathcal{L}_1$, $\mathcal{L}' = \mathcal{L}_2$ and $\mathcal{L} = \mathcal{L}_2$, $\mathcal{L}' = \mathcal{L}_1$. Because $\mathcal{L}_{1,2}$ are symmetric matrices one can verify that a solution to the first case is also a solution to the second, provided that we take $\hat{B}$ and $\hat{D}$ to be symmetric and set $\hat{C} = -\hat{A}^T$. Employing this choice and separating the real and imaginary parts of Eq. 32 we arrive at the defining relations for the required duality transformation

$$\hat{A}\hat{L}_1 + \hat{L}_2\hat{A}^T + \hat{L}_2\hat{D}\hat{L}_{H1} + \hat{L}_{H2}\hat{D}\hat{L}_1 = 0,$$

(34)

$$\hat{B} + \hat{A}\hat{L}_{H1} + \hat{L}_{H2}\hat{A}^T - \hat{L}_2\hat{D}\hat{L}_1 + \hat{L}_{H2}\hat{D}\hat{L}_{H1} = 0.$$  

(35)

We note that these equations constitute only $2N^2$ conditions for the $2N^2 + N$ independent entries in $\hat{A}, \hat{B}$ and $\hat{D}$, thereby leaving $N$ of them undetermined. Nevertheless, we find that the resulting freedom in choosing the duality transformation does not manifest itself in the effective response matrix. The latter is obtained, once $\hat{A}, \hat{B}$ and $\hat{D}$ have been established, by solving Eq. 33 with $\hat{L}_{\text{eff}} = \hat{L}_{\text{eff}} = \hat{L}_{\text{eff}} - i\hat{L}_{\text{Heff}}$, which translates to solving

$$\hat{A}\hat{L}_{\text{eff}} + \hat{L}_{\text{eff}}\hat{A}^T + \hat{L}_{\text{Heff}}\hat{D}\hat{L}_{\text{Heff}} + \hat{L}_{\text{Heff}}\hat{D}\hat{L}_{\text{Heff}} = 0,$$

(36)

$$\hat{B} + \hat{A}\hat{L}_{\text{Heff}} + \hat{L}_{\text{Heff}}\hat{A}^T - \hat{L}_{\text{eff}}\hat{D}\hat{L}_{\text{eff}} + \hat{L}_{\text{Heff}}\hat{D}\hat{L}_{\text{Heff}} = 0.$$  

(37)

4.1. The thermoelectric response of a two-phase single-layer in a magnetic field

The above scheme can be used to numerically calculate $\hat{L}_{\text{eff}}$ for a general two-phase layer. We were able to obtain closed analytical results for the thermoelectric response in the simple case where the two phases differ only by the sign of the charge carriers, implying

$$\hat{L}_{1,2} = \left( \begin{array}{cc} \sigma & \pm \alpha \\ \pm \alpha & \kappa \end{array} \right), \quad \hat{L}_{H1,2} = \left( \begin{array}{cc} \pm \sigma_H & \alpha_H \\ \alpha_H & \pm \kappa_H \end{array} \right).$$

(38)
In this case \( \hat{B} \) and \( \hat{D} \) are diagonal matrices, and the latter can be chosen as

\[
\hat{D} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},
\]

with \( a \) and \( b \) arbitrary constants of dimensions \([\sigma^{-1}]\) and \([\kappa^{-1}]\), respectively, reflecting the aforementioned freedom. In terms of them and the Hall angles

\[
\tan \theta_{\sigma} = \frac{\sigma H}{\sigma}, \quad \tan \theta_{\kappa} = \frac{\kappa H}{\kappa}, \quad \tan \theta_{\alpha} = \frac{\alpha H}{\alpha},
\]

defined in the range \([-\pi/2, \pi/2]\], the matrix \( \hat{A} \) takes the form

\[
\hat{A} = \frac{\alpha}{\sin(\theta_{\sigma} + \theta_{\kappa})} \begin{pmatrix} \frac{\kappa \cos \theta_{\alpha} b}{\sigma \cos \theta_{\kappa}} & 0 \\ 0 & \frac{\kappa \cos \theta_{\sigma} \alpha}{\cos \theta_{\kappa}} \end{pmatrix} \begin{pmatrix} \cos(\theta_{\sigma} + \theta_{\kappa} - \theta_{\alpha}) & -\cos(\theta_{\sigma} + \theta_{\kappa} - \theta_{\alpha}) \\ \cos(\theta_{\sigma} + \theta_{\kappa} - \theta_{\alpha}) & \cos(\theta_{\sigma} + \theta_{\kappa} - \theta_{\alpha}) \end{pmatrix},
\]

while \( \hat{B} \) is given by Eq. (35). Substituting the transformation back into Eqs. (36,37) and solving for \( \hat{L}_{\text{eff}} \) yields 16 solutions. All are independent of \( a \) and \( b \) but only one produces real transport coefficients that are also consistent with the physical requirements \( \sigma_{\text{eff}} \geq 0 \) and \( \kappa_{\text{eff}} \geq 0 \). This solution is

\[
\sigma_{\text{eff}} = \frac{\sigma}{\cos \theta_{\sigma}} \sqrt{1 - \frac{\alpha^2 \cos \theta_{\sigma} \cos \theta_{\kappa}}{\sigma \kappa \cos^2 \left( \frac{\theta_{\sigma} + \theta_{\kappa}}{2} \right)}},
\]

\[
\kappa_{\text{eff}} = \frac{\kappa}{\cos \theta_{\kappa}} \sqrt{1 - \frac{\alpha^2 \cos \theta_{\sigma} \cos \theta_{\kappa}}{\sigma \kappa \cos^2 \left( \frac{\theta_{\sigma} + \theta_{\kappa}}{2} \right)}},
\]

\[
\alpha_{\text{Heff}} = \alpha \left[ \tan \theta_{\alpha} - \tan \left( \frac{\theta_{\sigma} + \theta_{\kappa}}{2} \right) \right],
\]

\[
\alpha_{\text{eff}} = \sigma_{\text{Heff}} = \kappa_{\text{Heff}} = 0,
\]

which agrees with Eq. (11) when applied to model (38) in the limit of vanishing Hall angles.

4.2. The thermoelectric response of a two-phase double-layer in a magnetic field

Obtaining a complete analytical solution to Eqs. (36,37) becomes difficult even for the simplest models when the number of conserved currents is increased beyond two. However, one can treat the problem perturbatively in the magnetic field, \( H \), as we next demonstrate for the double-layer model defined in Eq. (20) and augmented by the Hall part of the thermoelectric response of the two phases

\[
\hat{L}_{H1,H2} = \begin{pmatrix} \pm \sigma_H & \pm \eta_H & \alpha_H \\ \pm \eta_H & \pm \sigma_H & \alpha_H \\ \alpha_H & \alpha_H & \pm 2\kappa_H \end{pmatrix}.
\]

To proceed we assume that \( \hat{L}_{1,2} \) are independent of \( H \) and \( \hat{L}_{H1,H2} \) are linear in \( H \). Based on our knowledge of the field-free case we look for a solution where
\( \hat{B}, \hat{D}, \text{ and } \hat{L}_{\text{eff}} \) are even functions of \( H \) while \( \hat{A} \) and \( \hat{L}_{\text{Heff}} \) are odd, implying the expansions \( \hat{A} = \hat{A}^{(1)} + \hat{A}^{(3)} + \ldots \), and \( \hat{B} = \hat{B}^{(0)} + \hat{B}^{(2)} + \ldots \), etc. Plugging these expansions into Eqs. (34, 35) one finds at zeroth order

\[
\hat{B}^{(0)} = \hat{L}_2 \hat{D}^{(0)} \hat{L}_1.
\]  

(47)

This equation leaves 3 parameters undetermined, which we denote by \( a, b, z \) and incorporate into the form of \( \hat{D}^{(0)} \)

\[
\hat{D}^{(0)} = \begin{pmatrix}
  a + b & a - b & z \\
  a - b & a + b & z \\
  z & z & \frac{\sigma + \eta}{\kappa} a
\end{pmatrix}.
\]  

(48)

\( \hat{B}^{(0)} \) is then given by Eq. (47), and the zeroth order contribution to \( \hat{L}_{\text{eff}} \) is determined from the corresponding order of Eq. (37)

\[
\hat{B}^{(0)} = \hat{L}_{\text{eff}}^{(0)} \hat{D}^{(0)} \hat{L}_{\text{Heff}}^{(0)} \hat{L}_1.
\]  

(49)

This equation yields 8 solutions, of which one is physical and properly agrees with Eqs. (21)-(24).

The first order equation, which determines \( \hat{A}^{(1)} \) and derived from Eq. (34)

\[
\hat{A}^{(1)} \hat{L}_1 + \hat{L}_2 \hat{A}^{(1)T} + \hat{L}_2 \hat{D}^{(0)} \hat{L}_{\text{Heff}}^{(0)} \hat{L}_1 + \hat{L}_{\text{Heff}}^{(0)} \hat{D}^{(0)} \hat{L}_1 = 0,
\]  

(50)

has a solution provided we set \( z = 0 \) in \( \hat{D}^{(0)} \) (and consequently in \( \hat{B}^{(0)} \)). The solution includes a new undetermined constant, \( c \), and can be written in the form

\[
\hat{A}^{(1)} = \begin{pmatrix}
  0 & 0 & \frac{\sigma + \eta}{\kappa} \\
  0 & 0 & \frac{\sigma + \eta}{\kappa} \\
  c & c & 0
\end{pmatrix} \begin{pmatrix}
  \left(\frac{\sigma \mu + \eta \mu}{\sigma + \eta} + \frac{\sigma \mu - 2 \alpha \mu}{\kappa} \right) \alpha a - \frac{z}{2} \\
  \left(\frac{\sigma \mu + \eta \mu}{\sigma + \eta} + \frac{\sigma \mu - 2 \alpha \mu}{\kappa} \right) \alpha a - \frac{z}{2} \\
  c & c & 0
\end{pmatrix}.
\]  

(51)

Substituting this result into Eq. (39) and expanding to first order yields an equation that is the same as Eq. (50) upon substituting \( \hat{L}_1 = \hat{L}_2 = \hat{L}_{\text{eff}}^{(0)} \) and \( \hat{L}_{\text{Heff}}^{(0)} = \hat{L}_{\text{Heff}}^{(1)} \). Solving for the latter gives the first order contribution to the effective Hall components

\[
\sigma_{\text{Heff}}^{(1)} = 0, \quad \eta_{\text{Heff}}^{(1)} = 0, \quad \kappa_{\text{Heff}}^{(1)} = 0,
\]  

(52)

\[
\alpha_{\text{Heff}}^{(1)} = \alpha_H - \frac{\alpha}{2} \left(\tan \theta_{\sigma} + \tan \theta_{\kappa}\right),
\]  

(53)

where for brevity we have introduced

\[
\sigma_{\pm} = \sigma \pm \eta, \quad \tan \theta_{\sigma} = \frac{\sigma_H \pm \eta_H}{\sigma \pm \eta}.
\]  

(54)

Finally, the second order equation derived from Eq. (55)

\[
\hat{B}^{(2)} + \hat{A}^{(1)} \hat{L}_{\text{Heff}}^{(1)} + \hat{L}_{\text{Heff}}^{(2)} \hat{A}^{(1)T} + \hat{L}_2 \hat{D}^{(2)} \hat{L}_1 + \hat{L}_{\text{Heff}}^{(0)} \hat{D}^{(0)} \hat{L}_{\text{Heff}}^{(1)} = 0,
\]  

(55)
may be solved for $\hat{B}^{(2)}$ and $\hat{D}^{(2)}$. The solution, which depends on 3 additional free parameters is then substituted into Eq. (37) and results in

$$\hat{B}^{(2)} + A^{(1)} A^{(1)T} = \hat{B}^{(2)} + \Lambda^{(0)} \hat{D}^{(0)}.$$  

This in turn yields the second order corrections contained in $\hat{L}^{(2)}_{\text{eff}}$

$$\sigma^{(2)}_{\text{eff}} = \frac{\sigma_{\pm}}{4} \cos \theta \left[ \tan^2 \theta \sigma_{\pm} + \frac{\tan^2 \theta}{4} \left( \tan \theta \sigma_{\pm} - \tan \theta \kappa \right) \right] + \frac{\sigma_{-}}{4} \tan^2 \theta \sigma_{-},$$  

$$\eta^{(2)}_{\text{eff}} = \sigma^{(2)}_{\text{eff}} - \frac{\sigma_{-}}{2} \tan^2 \theta \sigma_{-},$$  

$$\alpha^{(2)}_{\text{eff}} = 0,$$  

$$\kappa^{(2)}_{\text{eff}} = \kappa \cos \theta \left[ \tan^2 \theta \kappa + \frac{\tan^2 \theta}{4} \left( \tan \theta \sigma_{\pm} - \tan \theta \kappa \right) \right],$$

which are again invariant with respect to the freedom in the duality transformation.

5. Discussion

In this work we have provided new exact solutions to the two-phase model in multi-layer systems with both electrical and heat currents. Our analysis relies on several assumptions. First, only the linear response regime is considered. Second, that the systems is electrically and thermally isolated. Third, we assume strong thermal coupling between layers in a multi-layered system but no electrical current leakage. Obviously, these assumptions are an idealization of reality, and cease to hold true beyond the relaxation lengths set by inter-layer electrical leakage and thermal coupling to the environment. Nevertheless, our results are relevant on scales shorter than these relaxation lengths, provided that they are much larger than the typical inhomogeneity scale.

The ability to obtain an exact solution to the problem crucially depends on the statistical equivalence between the spatial distributions of the two phases. Violating this condition, e.g., by moving away from the percolation critical point, necessitates a perturbative approach or a numerical solution. Our results provide benchmarks for the latter.

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Appendix A.

Derivation of Eqs. (11,12)

Making use of the fact that \( \hat{L}_{1,2} \), and therefore \( \hat{U} = \hat{L}_{\text{eff}} \), are symmetric matrices we expand them as

\[
\hat{U} = u + \hat{u}, \quad \hat{L}_j = l_j + \hat{l}_j, \tag{A.1}
\]

where \( u = u_0 I \) and \( \hat{u} = u_x \hat{\sigma}_x + u_z \hat{\sigma}_z \). Here, \( I \) is the unit matrix, \( \hat{\sigma}_{x,z} \) are the Pauli matrices and \( u_\mu \) are constants. Similar definitions hold for \( l \) and \( \hat{l} \).

Equation (7) then implies

\[
\{l_2 + \hat{l}_2, \frac{1}{\sqrt{d_1 d_2}}(u - \hat{u})\} = \{u + \hat{u}, \frac{1}{d_1}(l_1 - \hat{l}_1)\}, \tag{A.2}
\]

where \( \{\cdot, \cdot\} \) is the anticommutator and where we have defined \( d_j = \det \hat{L}_j \) and used the relation

\[
\det \hat{U} = \sqrt{d_1 d_2}, \tag{A.3}
\]

which follows from Eq. (7). In turn, Eq. (A.2) yields

\[
u \left( \frac{\hat{l}_1}{\sqrt{d_1}} + \frac{\hat{l}_2}{\sqrt{d_2}} \right) = \left( \frac{l_1}{\sqrt{d_1}} + \frac{l_2}{\sqrt{d_2}} \right) \hat{u},
\]

\[
2u \left( \frac{l_1}{\sqrt{d_1}} - \frac{l_2}{\sqrt{d_2}} \right) = \left\{ \frac{l_1}{\sqrt{d_1}} - \frac{l_2}{\sqrt{d_2}}, \hat{u} \right\}, \tag{A.4}
\]

whose solution is readily obtained as Eqs. (11,12).

The solution of the double-layer problem without magnetic field

Here we detail the solution of Eq. (7) for the double-layer model defined by Eq. (14). We begin by representing the matrix \( \hat{U} = \hat{L}_{\text{eff}} \) in the form

\[
\hat{U} = \begin{pmatrix}
  u_1 & u_2 & u_4 \\
  u_2 & u_1 & u_4 \\
  u_4 & u_4 & u_3
\end{pmatrix}.	ag{A.5}
\]

Next, noticing that \( \det^2(\hat{U}) = \det \hat{L}_1 \det \hat{L}_2 \) we rewrite Eq. (7) as

\[
\hat{Q} \equiv \beta \det(\hat{L}_1) \hat{U} \hat{L}_1^{-1} - \det(\hat{U}) \hat{L}_2 \hat{U}^{-1} = 0, \tag{A.6}
\]

where \( \beta = (\det \hat{L}_2 / \det \hat{L}_1)^{1/2} \).

The solution of Eq. (A.6) is now obtained by the following steps. First we solve \( Q_{33} = 0 \) for \( u_3 \),

\[
u_3 \equiv \frac{\beta}{2} \left( (u_1 - u_2)u_4 \alpha_2 + (u_2^2 - u_1^2) \kappa_2 + u_1 \alpha_1 (\eta_1 - \sigma_1) \right). \tag{A.7}
\]
Next we substitute $u_3$ in the equation $Q_{13} = 0$ and solve for $u_4$,

$$u_4 = \frac{(u_1 + u_2)[(u_1 - u_2)\alpha_2 + \alpha_1(\sigma_1 - \eta_1)\beta]}{\beta(\sigma_1^2 - \eta_1^2) + (u_1 - u_2)(\eta_2 + \sigma_2)} \quad \text{(A.8)}$$

We substitute the above expressions for $u_3$ and $u_4$ in the equation $Q_{11} + Q_{12} = 0$ and solved it for $u_2$,

$$u_2 = u_1 - \sqrt{(\sigma_1 - \eta_1)(\sigma_2 - \eta_2)} \quad \text{(A.9)}$$

Finally, substituting the above expressions in the equation $Q_{11} = 0$ turns it into a quadratic equation for $u_1$, whose physical root ($u_1 > 0$) completes the general solution for the model.

References

[1] M. Seul and D. Andelman, *Science* 267, 476 (1995).
[2] A. M. Dykhne, Zh. Eksp. Teor. Fiz. 59, 110 (1970) [JETP 32 63, (1971)].
[3] A. M. Dykhne, Zh. Eksp. Teor. Fiz. 59, 641 (1970) [JETP 32 348, (1971)].
[4] B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 72, 288 (1977) [JETP 45 152, (1977)].
[5] G. W. Milton, Phys. Rev. B 38, 11296 (1988).
[6] A. M. Dykhne and I. M. Ruzin, Phys. Rev. B 50, 2369 (1994).
[7] B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. 108, 2202 (1995) [JETP 81 1200 (1995)].
[8] B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. 81, 665 (1981) [JETP 54 355 (1981)].
[9] A. A. Snarskii and I. V. Bezsudnov, arXiv:1108.4122
[10] S. Kim, I. Jo, J. Nah, Z. Yao, S. K. Banerjee, and E. Tutuc, Phys. Rev. B 83, 161401(R) (2011).
[11] R. V. Gorbachev, A. K. Geim, M. I. Katsnelson, K. S. Novoselov, T. Tudorovskiy, I. V. Grigorieva, A. H. MacDonald, S. V. Morozov, K. Watanabe, T. Taniguchi, and L. A. Ponomarenko, Nat. Phys. 8, 896 (2012).
[12] S. Kim and E. Tutuc, Solid State Commun. 152, 1283 (2012).
[13] J. I. A. Li, T. Taniguchi, K. Watanabe, J. Hone, A. Levchenko, and C. R. Dean, Phys. Rev. Lett. 117, 046802 (2016).
[14] A. A. Abrikosov, *Fundamentals of the Theory of Metals* (North-Holland, New York, 1988).
[15] J. C. W. Song, and L. S. Levitov, Phys. Rev. Lett. 109, 236602 (2012).
[16] J. C. W. Song, and L. S. Levitov, Phys. Rev. Lett. 111, 126601 (2013).
[17] J. C. W. Song, D. A. Abanin, and L. S. Levitov, Nano Lett. 13, 3631 (2013).