Two-Photon Exchange Contribution to Proton Form Factors in Time-Like region

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Abstract

We estimate two-photon exchange contribution to the process \(e^+ + e^- \rightarrow p + \bar{p}\). The two-photon exchange corrections to double spin polarization observables and form factors in the time-like region are calculated. The corrections are found to be small in magnitude, but with a strong angular dependence at fixed momentum transfer. These two features are the same as those in the space-like region. In the view of experiment, the double spin polarization observable \(P_z\) deserves to be considered.

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1 Introduction

The electromagnetic form factors in both space-like \((Q^2 > 0)\) and time-like \((Q^2 < 0)\) regions are essential to understand the intrinsic structure of hadrons. The experimental data of elastic form factors over several decades, including recent high precision measurement at Jefferson Lab [1, 2] and elsewhere [3], have provided considerable insight into the detail structure of the nucleon. Generally, in Born amplitude for one photon exchange, the proton current operator is parameterized in terms of Dirac \((F_1)\) and Pauli \((F_2)\) form factors,

\[
\Gamma_\mu = F_1(q^2)\gamma_\mu + i\frac{F_2(q^2)}{2m_N}\sigma_{\mu\nu}q^\nu,
\]

where \(q\) is the momentum transfer to the nucleon and \(m_N\) is the nucleon mass. The resulting differential cross section depends on two kinematic variables, conventionally taken to be \(Q^2 \equiv -q^2\) (or \(\tau\), in order to consistent with the case in the time-like region, we take \(\tau \equiv q^2/4m_N\) other than \(\tau \equiv Q^2/4m_N\)) and the scattering angle \(\theta_e\) (or virtual photon polarization \(\varepsilon \equiv [1 + 2(1 - \tau)\tan^2(\theta_e/2)]^{-1}\)). The reduced Born cross section, in terms of the Sachs electric and magnetic form factors, is

\[
\frac{d\sigma}{d\Omega} = C(Q^2, \varepsilon) \left[ G_M^2(Q^2) - \frac{\varepsilon}{\tau} G_E^2(Q^2) \right].
\]
Lab [7], with very small systematic errors were achieved by detecting the recoiling proton rather than the electron, is also consistent with the earlier LT results. It should be mentioned that polarized lepton beams give another way to access the form factors [8]. In the Born approximation, the polarization of the recoiling proton along its motion ($p_t$) is proportional to $G_M^2(Q^2)$ while the component perpendicular to the motion ($p_n$) is proportional to $G_E(Q^2)G_M(Q^2)$. Then the form factor ratio $R$ can be determined through a measurement of $p_t/p_n$, with

$$\frac{p_t}{p_n} = -\sqrt{-\frac{2\varepsilon}{\tau(1+\varepsilon)}} \frac{G_E(Q^2)}{G_M(Q^2)}.$$  

This method has been applied only recently in Jefferson Lab [1], since it needs high-intensity polarized beams, large solid-angle spectrometers, and advanced techniques of polarimetry in GeV range. The measurement about the electron-to-proton polarized transfer in $e^- + p \rightarrow e^- + \bar{p}$ shows that the ratio of Sachs form factors [9,10] is monotonically decreasing with increasing of $Q^2$, which strongly contradicts to the scaling ratio determined by the traditional Rosenbluth separation method [11]. In order to explain the discrepancy, radiative corrections, especially the two-photon contribution, have been involved [12–18]. In Ref. [14], only the intermediate proton state considered, it is found that the two-photon corrections have the proper sign and magnitude to resolve a larger part of the discrepancy between the two experimental techniques. Furthermore, Ref. [15] considered the intermediate $\Delta^+$ state as well as the proton. In Ref. [16] a partonic calculation of the two-photon exchange contribution to the form factors is given. It is concluded that for $Q^2$ in the range of $2 \sim 3 GeV^2$, the ratio extracted using LT method including the two-photon corrections agrees well with the polarization transfer results. Consequently, it shows that the two-photon exchange corrections can, at least, partly explain the discrepancy of the two methods of the separation.

For a stable hadron, in the space-like region the form factors are real, while its time-like form factors have a phase structure reflecting the final-state interactions of the outgoing hadrons, therefore, form factors are complex. So far, there are not many precise experimental data in this region as in the space-like one. In the theoretical point of view, it seems unavoidable to check the two-photon exchange contribution to the nucleon form factors in the time-like region. Actually, some works have been done. Refs. [19–21] employed the general arguments based on crossing symmetry for the processes of $e^- + h \rightarrow e^- + h$ and $e^+ + e^- \rightarrow h + \bar{h}$, and showed the general expressions for the polarization observables of the reaction $\bar{p} + p \rightarrow e^+ + e^-$ in terms of three independent complex amplitudes and in presence of two-photon exchange. Ref. [21] also tried to search some experimental evidences for the two-photon exchange from the experimental data of $e^+ + e^- \rightarrow p + \bar{p} + \gamma$. However, a negative conclusion is obtained due to the level of the present precision. A total contribution of the radiative corrections to the angular asymmetry is under 2%, while the asymmetry getting from the experimental data is always compatible with zero and the typical error is about 5%. In this reference, the polarization observables are not discussed.

Difference with the above work, we calculate the two-photon exchange correction to the unpolarized differential cross section as well as the double spin polarization observables. Some qualitative properties based on the crossing symmetry and C-invariance are discussed in section 2. Moreover, the analytical forms of the unpolarized differential cross section and polarization observables are presented in section 3. In section 4, we will directly calculate the two-photon exchange contribution to the differential cross section and polarization observables. In section 5, some numerical results and discussions are given.

2 Crossing Symmetry and C-invariance

In quantum field theory, crossing symmetry is a symmetry that relates to the S-matrix elements. In general, the S-matrix for any process involving a particle with momentum $p$ in the initial state is equal to the S-matrix for an otherwise identical process but with an anti-particle of momentum $k = -p$ in the
approximation as shown in Fig. (1), the crossing symmetry can be expressed by the following relation

\[ M(\phi(p) + \cdots + \phi) = M(\cdots + \phi(k)), \]

where \( \phi \) stands for anti-particle and \( k = -p \). We notice that there is no any realistic value of \( p \) for which \( p \) and \( k \) are both physically allowed. So technically we should say that either amplitude can be obtained from the other by analytic continuation. The crossing symmetry provides a relation between the scattering channel \( e^- + p \rightarrow e^- + p \) and the annihilating channel \( e^+ + e^- \rightarrow p + \bar{p} \). In the one-photon approximation as shown in Fig. (1), the crossing symmetry can be expressed by the following relation

\[ |M(e^- p \rightarrow e^- p)|^2 = f(s, t) = |M(e^+ e^- \rightarrow p\bar{p})|^2. \]

The line over \( M \) denotes the sum over the polarization of all particles in the initial and final states. The Mandelstan variables \( s \) and \( t \) are defined as follows:

\[ s = (k_1 + p_1)^2 = m_N^2 + 2E_1m_N \geq m_N^2, \]
\[ t = (k_1 - k_2)^2 = \bar{e}^2 < 0, \]

for the scattering channel (with \( E_1 \) being the energy of the incoming electron in the Lab frame), and

\[ s = (k_1 - p_1)^2 = m_N^2 - 2e^2 + 2\sqrt{e^2 - m_N^2} \cos \theta \leq 0, \]
\[ t = (k_1 + k_2)^2 = 4e^2 > 4m_N^2, \]

for the annihilating channel with \( \bar{e} \) being the energy of the initial electron (or final proton) and \( \theta \) being the hadron production angle.

Considering Lorentz, parity, time-reversal, and helicity conservation in the limit of \( m_e \rightarrow 0 \), the \( T \)–matrix for the elastic scattering of two Dirac particles can be expanded in terms of three independent Lorentz structures. Then, the proton current operator through the Lorentz structure \([8]\) is

\[ \Gamma_\mu = \bar{F}_1(s, t)\gamma_\mu + i\bar{F}_2(s, t)\sigma_{\mu\nu}q^\nu + \bar{F}_3(s, t)\gamma \cdot KP_\mu \]

with

\[ P = \frac{1}{2}(p_2 + p_1), \quad K = \frac{1}{2}(k_1 + k_2), \]

in the scattering channel, and

\[ P = \frac{1}{2}(p_2 - p_1), \quad K = \frac{1}{2}(k_1 - k_2), \]

in the annihilating channel. Similar to the Sachs form factor, we can recombine the form factors \( \bar{F}_{1,2} \) as

\[ \bar{G}_E(q^2, \cos \theta) = \bar{F}_1(q^2, \cos \theta) + \tau \bar{F}_2(q^2, \cos \theta), \]
\[ \bar{G}_M(q^2, \cos \theta) = \bar{F}_1(q^2, \cos \theta) + \bar{F}_2(q^2, \cos \theta). \]

Taking the proton current operator defined in Eq. (8) which includes the multi-photon exchange, we can express \( f(s, t) \) in Eq. (5) in the form:

\[ f(s, t) = \frac{8e^4}{(4m_N^2 - t)} \left\{ \text{Re} \left[ 8\bar{G}_E \right]^2 m_N^2 \left[ m_N^4 - 2sm_N^2 + s(s + t) \right] - |\bar{G}_M|^2 t \left[ 2m_N^4 - 4m_N(s + t) + 2s^2 + t^2 + 2st \right] \right. \]
\[ -m_N^{-2} \left[ 2m_N^6 - m_N^4(6s + t) + 2m_N^2s(3s + 2t) - s(2s^2 + 3ts + t^2) \right] \}
\[ \left. \text{Re} \left[ (4m_N^2 \bar{G}_E - t\bar{G}_M)^* F_3 \right] \right\}. \]
In the one-photon mechanism for $e^+ + e^- \rightarrow p + \bar{p}$, the conservation of the total angular momentum $J$ allows only one value of $J = 1$. This is due to the quantum numbers of the photon: $J^p = 1^-, C(1\gamma) = -1$. The selection rule combined with $C$ and $P$ invariances allows two states for $e^+ e^-$ (and $p \bar{p}$):

$$S = 1, \quad \ell = 0 \quad \text{and} \quad S = 1, \quad \ell = 2 \quad \text{with} \quad J^p = 1^-,$$  

where $S$ is the total spin and $\ell$ is the orbital angular momentum of the $e^+ e^-$ (or $p \bar{p}$) system. As a result the $\theta$ dependence of the differential cross section for $e^+ + e^- \rightarrow p + \bar{p}$, in the one-photon exchange mechanism, has the following general form

$$\frac{d\sigma^{1\gamma}}{d\Omega} = a(t) + b(t) \cos^2 \theta. \quad (14)$$

Similar analysis can be done for the $\cos \theta$ dependence of the $1\gamma \otimes 2\gamma$ interference contribution to the differential cross section of this process. In general, the spin and parity of the $2\gamma$ states are not fixed, but only a positive $C-$ parity, $C(2\gamma) = +$, is allowed, then the $\cos \theta$ dependence of the $1\gamma \otimes 2\gamma$ interference contribution to the differential cross section can be predicted on the basis of its $C-$ odd nature as:

$$\frac{d\sigma^{int}}{d\Omega} = \cos \theta [c_0(t) + c_1(t) \cos^2 \theta + c_2(t) \cos^4 \theta + ...]. \quad (15)$$

In the one-photon exchange mechanism, the differential cross section is angular symmetric. However, after considering the two-photon exchange, this symmetry is broken. Define the asymmetry of the total differential cross section as

$$A_{2\gamma}(q^2, \theta) = \frac{\frac{d\sigma^{1\gamma}}{d\Omega}(q^2, \theta) - \frac{d\sigma^{1\gamma}}{d\Omega}(q^2, \pi - \theta)}{\frac{d\sigma^{1\gamma}}{d\Omega}(q^2, \theta) + \frac{d\sigma^{1\gamma}}{d\Omega}(q^2, \pi - \theta)}, \quad (16)$$

after some algebraic simplification, we have

$$A_{2\gamma}(q^2, \theta) = \frac{d\sigma^{int}}{d\Omega}(q^2, \theta) / \frac{d\sigma^{1\gamma}}{d\Omega}(q^2, \theta). \quad (17)$$

Then based on the general forms of $d\sigma^{1\gamma}/d\Omega$ and $d\sigma^{int}/d\Omega$ as shown in Eq. (14) and Eq. (15), One can easily conclude that the angular asymmetry of the total differential cross section is also an odd function of $\cos \theta$.

## 3 Differential Cross Section and Polarization Observables

In order to represent the polarization vector of outgoing anti-proton in a straight way for the process of $e^+ + e^- \rightarrow p + \bar{p}$, we define a coordinate frame in center of mass system (CMS) of the reaction in such a way that the $z$ axis directs along the three-momentum of the anti-proton and the angle between the incoming electron and outgoing anti-proton is defined as $\theta$. In such a frame, according to the approaches used in Refs. [22–24], one has

$$\mathcal{M} = \frac{e^2}{q^2} j_\mu J^\mu \quad (18)$$

with leptonic current

$$j_\mu = \bar{u}(-k_2)\gamma_\mu u(k_1)$$

and hadronic current

$$J_\mu = \bar{u}(p_2) \left[ \tilde{F}_1(s,t) \gamma_\mu + i \frac{\tilde{F}_2(s,t)}{2m_N} \sigma_{\mu\nu} q^\nu + \tilde{F}_3(s,t) \frac{\gamma \cdot K P_\mu}{m_N^2} \right] u(-p_1) \quad (19)$$
Then the differential cross section of the reaction in the CMS is

$$\frac{d\sigma}{d\Omega} = \alpha^2 \beta \frac{q^6}{q^2} L_{\mu\nu} H_{\mu\nu}, \quad L_{\mu\nu} = j_\mu j_\nu, \quad H_{\mu\nu} = J_\mu J_\nu,$$

\(\alpha = e^2/4\pi\) is the fine structure constant and \(\beta = \sqrt{1 - 4M^2/q^2}\) is the nucleon velocity in the CMS. In this work we consider the unpolarized incoming positron and longitudinally polarized incoming electron with the polarization four-vector \(s\), and in the final state, the anti-proton is polarized with polarization four-vector \(s_1\), then the leptonic and hadronic vectors can be divided into unpolarized and polarized parts

$$L_{\mu\nu} = L_{\mu\nu}(0) + L_{\mu\nu}(s), \quad H_{\mu\nu} = H_{\mu\nu}(0) + H_{\mu\nu}(s_1).$$

In the current operator shown in Eq. (8), the Lorentz structure functions are not only the function of \(q^2\) but also depend on hadron production angle \(\theta\), and they can relate to the Dirac and Pauli form factors

$$\tilde{F}_{1,2}(q^2, \cos \theta) = F_{1,2}(q^2) + \Delta F_{1,2}(q^2, \cos \theta)$$

and \(\tilde{G}_{E,M}(q^2, \cos \theta)\) related to the Sachs form factors

$$\tilde{G}_{E,M}(q^2, \cos \theta) = G_{E,M}(q^2) + \Delta G_{E,M}(q^2, \cos \theta).$$

The unpolarized differential cross section of the process \(e^+ + e^- \rightarrow p + \bar{p}\) is in the form

$$\frac{d\sigma_{un}}{d\Omega} = \frac{\alpha^2 \beta}{4q^2} L_{\mu\nu}(0) H_{\mu\nu}(0) = \frac{\alpha^2 \beta}{4q^2} D,$$

with the current operator in Eq. (8) and the definition in Eq. (11), \(D\) can be expressed as:

$$D = |\tilde{G}_M|^2 (1 + \cos^2 \theta) + \frac{1}{\tau} |\tilde{G}_E|^2 \sin^2 \theta - 2 \sqrt{\tau(\tau - 1)} Re[(\tilde{G}_M - \frac{1}{\tau}\tilde{G}_E)\tilde{F}_{3}^*] \sin \theta \cos \theta. \tag{25}$$

Notice that in Eq. (25), \(\Delta G_{E,M}\) and \(\tilde{F}_3\) caused by the two-photon exchange is in the order of \(\alpha \simeq 1/137\), so that the terms \(\Delta G_{E,M} \Delta G_{E,M}\) and \(\Delta G_{E,M} \tilde{F}_3\) are negligible, then,

$$D = |G_M|^2 (1 + \cos^2 \theta) + \frac{1}{\tau} |G_E|^2 \sin^2 \theta + 2 Re[G_M \Delta G_M^*](1 + \cos^2 \theta) + \frac{2}{\tau} Re[G_E \Delta G_E^*] \sin^2 \theta - 2 \sqrt{\tau(\tau - 1)} Re[(G_M - \frac{1}{\tau}G_E)\tilde{F}_{3}^*] \sin \theta \cos \theta. \tag{26}$$

From \(C\)-invariance and the above expression of \(D\), we have the general properties of the form factors,

$$\Delta G_{E,M}(q^2, + \cos \theta) = -\Delta G_{E,M}(q^2, - \cos \theta),$$

$$\tilde{F}_3(q^2, + \cos \theta) = \tilde{F}_3(q^2, - \cos \theta). \tag{27}$$

which is equivalent to the symmetry relations of the scattering channel [22].

Generally, the polarization four-vector \(S_\mu\) of a relativistic particle with three-momentum \(p\) and mass \(m\), is connected with the polarization vector, \(\xi\), by a Lorentz boost:

$$\vec{S} = \vec{\xi} + \frac{\vec{p} \cdot \vec{\xi}}{m(E + p)} \cdot \hat{p} = \frac{\vec{p} \cdot \vec{S}}{m}. \tag{28}$$

Where \(E = \sqrt{m^2 + p^2}\) is the energy of the particle. In the CMS defined above, we have the polarization vectors of the anti-proton

$$\vec{\xi}_x = (1, 0, 0), \quad s_{1x} = (0, 1, 0),$$

$$\vec{\xi}_y = (0, 1, 0), \quad s_{1y} = (0, 0, 1),$$

$$\vec{\xi}_z = (0, 0, 1), \quad s_{1z} = (\sqrt{\tau - 1}, 0, 0, \sqrt{\tau}). \tag{29}$$
$P_y$ is a single-spin polarization observable, which relates to one polarized particle along the $y-$ axis. Since the time-like form factors are complex, then it appears in the Born approximation in the process $e^+ + e^- 	o p + \bar{p}$. In this work we consider the outgoing anti-proton polarized. The general expression for $P_y$ is

$$P_y = \frac{1}{Dq^4} L_{\mu \nu} H_{\mu \nu}(s_{1y}) = \frac{1}{Dq^4} \left[ L_{\mu \nu}(0) H_{\mu \nu}(s_{1y}) + L_{\mu \nu}(s) H_{\mu \nu}(s_{1y}) \right].$$

(30)

After some algebraic calculation [25], we can find $P_y$ does not depend on the polarization of incoming electron, that means the second term in Eq. (30) has no contribution to $P_y$. With the proton current operator in Eq. (8) we have,

$$P_y = \frac{2 \sin \theta}{D \sqrt{\tau}} \left[ \text{Re}\{G_M G_E^* \} \cos \theta - \sqrt{\tau(\tau - 1)} \left( \text{Im}\{G_E F_3^*\} \sin^2 \theta + \text{Im}\{G_M F_3^*\} \cos^2 \theta \right) \right].$$

$$= \frac{2 \sin \theta}{D \sqrt{\tau}} \left[ \text{Re}\{G_M G_E^* + G_M \Delta G_E^* + \Delta G_M G_E^* \} \cos \theta - \sqrt{\tau(\tau - 1)} \left( \text{Im}\{G_E F_3^*\} \sin^2 \theta + \text{Im}\{G_M F_3^*\} \cos^2 \theta \right) \right].$$

(31)

Similar definitions are employed for the double spin polarization observables $P_x$ and $P_z$. For $P_x$ and $P_z$, the polarization of the incoming electron is necessary, and the unpolarized incoming electron has no contribution, that is $L_{\mu \nu}(0) H_{\mu \nu}(s_{1x,z}) = 0$. Since $L_{\mu \nu}(0) H_{\mu \nu}(s_{1x,z}) \propto a_{\mu \nu} c_{\rho \lambda} d_{\lambda \mu \rho \lambda} = \epsilon^{\mu \nu \rho \lambda}$ and $a$, $b$, $c$, $d$ are out of $s_{1x,z}$, $k_1$, $k_2$, $p_1$, $p_2$, and all of those four-vectors have zero $y-$ component, then the contribution of the unpolarized electron vanishes. The double spin polarization observables with proton current operator in Eq. (8) are

$$P_x = \frac{2 \sin \theta}{D \sqrt{\tau}} \left\{ \text{Re}\{G_M G_E^* \} + \text{Re}\{G_M F_3^*\} \sqrt{\tau(\tau - 1)} \cos \theta \right\}$$

$$= \frac{2 \sin \theta}{D \sqrt{\tau}} \left\{ \text{Re}\{G_M G_E^* + G_M \Delta G_E^* + \Delta G_M G_E^* \} + \text{Re}\{G_M F_3^*\} \sqrt{\tau(\tau - 1)} \cos \theta \right\},$$

$$P_z = \frac{2}{D} \left\{ |G_M|^2 \cos \theta - \text{Re}\{G_M F_3^*\} \sqrt{\tau(\tau - 1)} \sin^2 \theta \right\}$$

$$= \frac{2}{D} \left\{ |G_M|^2 \cos \theta + 2 \text{Re}\{G_M \Delta G_M\} \cos \theta - \text{Re}\{G_M F_3^*\} \sqrt{\tau(\tau - 1)} \sin^2 \theta \right\}. $$

(32)

In Eqs. (31,32), if we set $\Delta G_{E,M} = 0$ and $F_3 = 0$, the polarization observables reduces to the results in the one-photon approximation. Considering the two-photon exchange contribution to the double spin polarization observables, we define $\delta(P_{x,z})$ as the ratio between the contributions of $1\gamma \otimes 2\gamma$ interference terms and the results in the one-photon mechanism, that is

$$\delta(P_{x,z}) = \frac{P_{int}}{P_{1\gamma}}.$$

with Eq. (32) we have,

$$\delta(P_x) = \frac{\text{Re}\{G_M \Delta G_E^* + G_E \Delta G_M^*\}}{\text{Re}\{G_M G_E^*\}} + \sqrt{\tau(\tau - 1)} \frac{\text{Re}\{G_M F_3^*\}}{\text{Re}\{G_M G_E^*\}} \cos \theta,$$

$$\delta(P_z) = \frac{2 \text{Re}\{G_M \Delta G_M\}}{|G_M|^2} - \sqrt{\tau(\tau - 1)} \frac{\text{Re}\{G_M F_3^*\}}{|G_M|^2} \sin \theta \tan \theta. $$

(33)

One can see both $\delta(P_x)$ and $\delta(P_z)$ are the odd functions of $\cos \theta$.

4 Two-Photon Exchange Contribution

This section is devoted to a directly numerical calculation for the two-photon exchange. We know that much work has been done in the space-like region. Naturally, it is expected that the same calculation
In the soft approximation should be performed in the time-like region. After considering the two-photon exchange, the amplitude $\mathcal{M}$ will be essentially modified, that is,

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{2\gamma},$$  \hspace{1cm} (34)

where $\mathcal{M}_0$ is the contribution of the one-photon exchange and $\mathcal{M}_{2\gamma}$ denotes the two-photon exchange. Therefore, to the first order of $\alpha$ ($\alpha = e^2/4\pi$), we have,

$$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2 = |\mathcal{M}_0|^2 (1 + \delta_{2\gamma})$$

with

$$\delta_{2\gamma} = 2 \frac{Re\{\mathcal{M}_{2\gamma}\mathcal{M}_0^*\}}{|\mathcal{M}_0|^2}. \hspace{1cm} (35)$$

From the analysis in section 2, one can see that $A_{2\gamma}(q^2, \theta)$ and $\delta_{2\gamma}$ are identical.

To proceed a direct calculation, the amplitude of the two-photon exchange from the direct box (Fig. 2a) and crossed box diagram (Fig. 2b) has the form

$$\mathcal{M}_{2\gamma} = e^4 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{N_a(k)}{D_a(k)} + \frac{N_b(k)}{D_b(k)} \right], \hspace{1cm} (36)$$

where the numerators are the matrix elements. For the direct box diagram,

$$N_a(k) = j(a)_{\mu\nu} J^\mu_{(a)}^\nu,$$

with

$$j^\mu_{(a)}^\nu = \bar{u}(-k_2)\gamma^\mu(\hat{k} - \hat{k})\gamma^\nu u(k_1),$$

$$J^\mu_{a,b} = \bar{u}(p_2)\Gamma^\mu(k_1 + k_2 - \hat{k}) - \hat{p}_1 - m_N)\Gamma^\nu(k)u(-p_1), \hspace{1cm} (37)$$

with $\hat{k} \equiv \gamma \cdot k$ and $\Gamma_{\mu}(k)$ defined in Eq. (1). The denominators in Eq. (36) are the products of the scalar propagators,

$$D_a(k) = [k^2 - \lambda^2][(k_1 + k_2 - k)^2 - \lambda^2][(k_1 - k)^2 - m_N^2][(k - p_1)^2 - m_N^2], \hspace{1cm} (38)$$

where an infinitesimal photon mass, $\lambda$, has been introduced in the photon propagator to regulate the IR divergence. Similarly we can write down the expressions of $N_b(k)$ and $D_b(k)$ for Fig. 2b.

For the $1\gamma \otimes 2\gamma$ interference term, we define the leptonic and hadronic tensors as,

$$L^{(a,b)}_{\mu\nu\rho} = j^{(a,b)}_{\mu\nu} j^\rho_{(a)}, \quad H^{(a,b)}_{\mu\nu\rho} = J^{(a,b)}_{\mu\nu} J^\rho_{(a)}. \hspace{1cm} (39)$$

Here the current operator in the hadronic current $J_{\rho}$ is the same as the one in $J_{\mu\nu}$, then,

$$\frac{d\sigma^{int}}{d\Omega} \propto \mathcal{M}_{2\gamma} \mathcal{M}_0 = e^6 \frac{q^2}{\lambda^2} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{L^{(a)}_{\mu\nu\rho} H^{(a)}_{\mu\nu\rho}}{D_a(k)} + \frac{L^{(b)}_{\mu\nu\rho} H^{(b)}_{\mu\nu\rho}}{D_b(k)} \right]. \hspace{1cm} (40)$$

For the unpolarized differential cross section only $L^{(a,b)}_{\mu\nu\rho}(0)H^{(a,b)\mu\nu\rho}(0)$ survives. From the crossing symmetry, we conclude that the expressions of $\delta_{2\gamma}$ are identical with Mandelstan variables for both the scattering channel and the annihilating channel, that is,

$$\delta_{2\gamma}(s,t)_{e-\gamma-p-e-\gamma-p} = g(s,t) = \delta_{2\gamma}(s,t)_{e-\gamma-\bar{p}+e-p+\bar{p}}. \hspace{1cm} (41)$$

In the soft approximation $g(s,t)$ can be expressed as

$$g(s,t)_{soft} = -2\frac{\alpha}{\pi} \ln \left| \frac{s - m_e^2 - m_N^2}{s + t - m_e^2 - m_N^2} \right| \ln \left| \frac{t}{\lambda^2} \right|. \hspace{1cm} (42)$$
For the double spin polarization observables $P_1$ and $P_2$ the $1\gamma \otimes 2\gamma$ interference contribution is

$$\begin{align*}
\frac{p_{\text{int}}}{s,z} &= \frac{e^2}{q^2 D} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{L_{\mu\nu\rho}(s_{1z},z)}{D_a(k)} + \frac{L_{\mu\nu\rho}(s_{1z},z)}{D_b(k)} \right] \\
&= \frac{e^2}{q^2 D} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{L_{\mu\nu\rho}(S)(s_{1z},z)}{D_a(k)} + \frac{L_{\mu\nu\rho}(S)(s_{1z},z)}{D_b(k)} \right].
\end{align*}$$

As in the one-photon exchange approximation, the unpolarized leptonic vector has no contribution to the double spin polarization observables. For the term of $L_{\mu\nu\rho}(0)H^{\mu\nu\rho}(s_{1z},z)$, after some algebraic calculations, we find that the non-vanishing contributions are in the forms of $\varepsilon^{a b c k}$, $a'\cdot k\varepsilon^{a b c k}$, $a'\cdot k' b'\cdot k\varepsilon^{a b c k}$, $k^2 a'$, $k\varepsilon^{a b c k}$, and $\{a', b', a, b, c\} \in \{s_1, k_1, k_2, p_1, p_2\}$. Since $a'$, $a$, $b$, $b'$ and $c$ have zero-$\gamma$ component, then the non-vanishing terms are the odd functions of $k_y$. Namely $L_{\mu\nu\rho}(0)H^{\mu\nu\rho}(s_1) = f(s, t, k_0, k_x, k_y, k^2)k_y$. Since the denominators in Eq. (43) are the even functions of $k_y$, the contribution of $L_{\mu\nu\rho}(0)H^{\mu\nu\rho}(s_1)$, therefore, vanishes.

5 Numerical Results and Discussion

In this work, we calculate the contributions of direct box diagram (Fig. 2a) and crossed box diagram (Fig. 2b) to the unpolarized differential cross section and the double spin polarization observables. In this calculation a simple monopole form of the form factors is employed. This phenomenological form factor is $G_E(q^2) = G_M(q^2)/\mu_p = G(q^2) = -\Lambda^2/(q^2 - \Lambda^2)$, with $\Lambda = 0.84$ GeV, which is consistent with the size of the nucleon. Practically, for the interaction of the outgoing hadrons, the time-like form factors have a phase structure, which means the form factors are complex in the time-like region. In this work what we concern is the ratio $\delta_{2\gamma}$ and double spin polarization observables $P_1$ and $P_2$. Moreover, the phenomenological form factors appear in both denominator and numerator of these physical observables. In such cases, the form of form factors varies the ratio and polarization observables in a very limited extension. The same conclusion can be drawn from the results of two-photon exchange corrections to space-like form factors in Ref. [17].

In our calculation, the loop integrals of the two-photon exchange contribution, firstly, were evaluated analytically in terms of the four-point Passarino-Veltman functions [27] using package FeynCalc [28]. Then, the Passarino-Veltman functions were evaluated numerically with LoopTools [29]. The IR divergence in the $1\gamma \otimes 2\gamma$ is proportional to $\ln \lambda$. This conclusion can be drawn by analyzing the integral in Eq. (35) as well as by crossing symmetry and previous results in the scattering channel. Furthermore, the previous calculations in the scattering channel have shown that the IR divergence in the two-photon exchange contribution is exactly canceled by the corresponding terms in the bremsstrahlung cross section involving the interference between the real photons emitted from the electron and from the proton. With crossing symmetry, the IR divergence in the annihilating channel caused by the two-photon exchange can also be ignored.

From our previous analysis, the two-photon contribution to unpolarized differential cross section $\delta_{2\gamma}$ is identical to the angular asymmetry $A_{2\gamma}$, which means $\delta_{2\gamma}$ is also the odd function of $\cos\theta$. The numerical results of the two-photon contribution to unpolarized differential cross section $\delta_{2\gamma}$ are presented in Fig. 8, where we show a comparison of $\delta_{2\gamma}$ (defined as in Eq. (35)) between the results of the full calculation and the soft approximation. The full circles in the figure are the full calculation, the dotted curves are the results with soft approximation and the full curves are the polynomial fits to the full calculation. We find a polynomial in the form of $\cos\theta[a_0(t) + a_1(t) \cos^2\theta + a_2(t) \cos^4\theta + ...]$ can give a good fit with a power series of $\cos\theta$ (no more than $\cos^5\theta$). The left panel of Fig. 8 shows the results with momentum transfer $q^2 = 4$ GeV$^2$, which is near the threshold of the reaction $e^+ + e^- \rightarrow p + \bar{p}$. We see that the two-photon exchange contribution to the unpolarized differential cross section is rather small, only about $\pm 0.6\%$ at $\theta = \pi(0)$. In addition, with the coefficients $a_0 = -9.6 \times 10^{-3}$, $a_1 = 4.9 \times 10^{-3}$, $a_2 = -1.5 \times 10^{-3}$ we see that the polynomial gives a good fit of the full calculation. The right panel of Fig. 8 is the
results at $q^2 = 16 \text{ GeV}^2$, the contribution of the two-photon exchange is relatively large, nearly 4%, the parameters of the fit for the full calculation are $a_0 = 2.9 \times 10^{-3}, a_1 = 5.7 \times 10^{-2}, a_2 = -1.9 \times 10^{-2}$. We conclude that at a fixed momentum transfer, the contribution of the two-photon exchange is strongly dependent on $\cos \theta$. In magnitude, the contribution is rather limited in small momentum transfer region, with $q^2$ increasing, the contribution becomes larger. This conclusion is consistent with the results in the space-like region.

In Fig. (4) we show the $\cos \theta$ dependence of the real part of corrections to the proton time-like form factors caused by the two-photon exchange at $q^2 = 4 \text{ GeV}^2$. For $\Delta G_E/G$ and $\Delta G_M/G$, significant $\cos \theta$ dependences are observed, while $\tilde{F}_3/G$ weakly depends on $\cos \theta$. For the parity, $\Delta G_E/G$ and $\Delta G_M/G$ are odd, and $\tilde{F}_3/G$ is even. These features are consistent with our general analysis. The electric form factor is relatively more sensitive to the two-photon exchange corrections, is about 2.5% at $\theta = 0(\cos \theta = 1)$ and $-2.5\%$ at $\theta = \pi(\cos \theta = -1)$. The correction to magnetic form factor, $\Delta G_M/G$ varies from 1% to $-1\%$ with $\theta$ from zero to $\pi$, while $\tilde{F}_3/G_E$ is about 1% in the whole $\theta$ region.

From our previous analysis in Sec. 3, one can see the two-photon contributions to double spin polarization observables $P_x$ and $P_z$ are even functions of $\cos \theta$. In our previous numerical results in Fig. (4) we find $\tilde{F}_3$ is not zero at $\theta = \pi/2$, then $\delta(P_z)$ will be proportional to $\tan \theta$ at the limit of $\theta \to \pi/2$ and will be infinity at $\theta = \pi/2$. Our numerical results of the $\cos \theta$ dependence of $\delta(P_{z,z})$ at $q^2 = 4 \text{ GeV}^2$ are displayed in Fig. (5). We can see that the two-photon exchange contribution to the double spin polarization is strongly $\theta$–dependence, and is the odd function of $\cos \theta$, which is consistent with our general analysis. For $\tilde{F}_3$, the variation caused by the two-photon exchange reaches maximum at $\theta = \pi(0)$ (about 4%). It seems that one can more easily find the signal of the two-photon exchange at the backward ($\theta = \pi$) and forward ($\theta = 0$) angle. However, notice Eq. (32), we know that $P_{z,\gamma}$ is proportional to $\sin \theta$. It means when $\theta$ is very small (close to 0) or very large (close to $\pi$), $P_{z,\gamma}$ will be compatible to 0, and therefore, the absolute variation caused by the two-photon exchange will be very limited. Thus, it will still be difficult to find any signal of the two-photon exchange in the observable $P_z$. For $P_x$, the contribution of the two-photon exchange reaches maximum when $\theta = \pi/2$. In the one photon mechanism $P_{z,\gamma}$ is proportional to $\cos \theta$, which suggests that no matter what kinds of form factors we employed, $P_{z,\gamma}$ vanishes at $\theta = \pi/2$. While taking the two-photon exchange contribution into consideration, as in Eq. (33), $P_x(\pi/2)$ is not equal to zero any more. From the experimental point of view, the nonzero $P_z$ at $\theta = \pi/2$ might be a strong evidence of the two-photon exchange in the process of $e^+ + e^- \rightarrow p + \bar{p}$.

According to our numerical results, one can see the two-photon exchange contribution to the unpolarized differential cross section $\delta_{2\gamma}$, which is identical to angular asymmetry $A_{2\gamma}$, is rather small at small momentum transfer. With present experimental precision, it is rather difficult to find any evidence of the two-photon exchange from the unpolarized observable in the process $e^+ + e^- \rightarrow p + \bar{p}$, especially at low momentum transfer region. With $q^2$ increasing, the contribution of the two-photon exchange becomes important. It can be several percent at $q^2$ about 16 GeV$^2$. Furthermore, for the double spin polarization observables, $P_z$ deserves to be considered in further experiment. In conclusion, the precise measurements of the unpolarized differential cross section at high momentum transfer and the double spin polarization observable $P_z$ especially at $\theta = \pi/2$ are expected to show some evidences of the two-photon exchange in this process.

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Fig. 1: One-photon approximation for the crossed channels. The left one represents the annihilating channel of $e^+ + e^- \rightarrow h + \bar{h}$, and the right one shows the scattering channel of $e^- + h \rightarrow e^- + h$.

Fig. 2: Two-photon exchange box and crossed box diagrams in annihilating channel.
Fig. 3: $\cos \theta$ dependences of the finite $2\gamma$ contribution to the unpolarized differential cross section. The full circles are the results of full calculation, the dotted curves are those with soft approximation, and the solid lines are the polynomial fit for the full calculation. The left panel is the result at $q^2 = 4 \, \text{GeV}^2$ and right one is at $q^2 = 16 \, \text{GeV}^2$.

Fig. 4: $\cos \theta$ dependence of the two-photon contribution to the proton form factors in the time-like region at $q^2 = 4 \, \text{GeV}^2$. 
Fig. 5: $\cos \theta$ dependences of the two-photon contribution to the polarization observables at $q^2 = 4\text{GeV}^2$. The solid curve stands for the results of $\delta(P_x)$, and the dotted curve represents the results of $\delta(P_z)$. 