pp-wave limits and orientifolds

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Abstract

We study the pp-wave limits of various elliptic models with orientifold planes and D7-branes, as well as the pp-wave limit of an orientifold of \( adS_5 \times T^{11} \). Many of the limits contain both open and closed strings. We also present pp-wave limits of theories which give rise to a compact null direction and contain open strings. Maps between the string theory states and gauge theory operators are proposed.

\textsuperscript{1}Research supported in part by the DOE under grant DE-FG02–92ER40706.
\textsuperscript{2}Research supported by the DOE under grant DE-FG02–92ER40706.
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1 Introduction

Recently a particular limiting version of the adS/CFT correspondence [1] between string theory on $adS_5 \times S^5$ and the $\mathcal{N} = 4$ supersymmetric gauge theory was discovered [2] (for earlier work see refs. [3, 4, 5]). On the string theory side the limiting metric is the metric experienced by an observer moving very fast along a particular null geodesic of $adS_5 \times S^5$, viz., a parallel plane (pp) wave metric [6]. The limiting background also contains a non-zero RR five-form. We will refer to this limit interchangeably as a Penrose or pp-wave limit. Type IIB string theory in the pp-wave background is exactly solvable [5, 7]. Moreover, the spectrum of string theory states in this background corresponds to the subset of the single trace operators in the gauge theory with large conformal dimension $\Delta$ and R-charge $J$, but with $H \equiv \Delta - J$ finite. The operators in the superconformal $\mathcal{N} = 4$ SU($N$) gauge theory that correspond to the string theory states in the Penrose limit of $adS_5 \times S^5$ were identified in ref. [2].

In this paper we study examples of the limiting procedure introduced in ref. [2] for a class of models containing orientifold planes (most of the models also contain D7-branes) and therefore potentially having an open string sector.

Sections 2 and 3 contain background material relevant for our later discussions, with section 3 exhibiting the two different classes of Penrose limits encountered in this paper. In section 4 we consider the pp-wave limit of the simplest orientifolded model, $adS_5 \times \mathbb{R} \mathbb{P}^5$ and its dual $\mathcal{N} = 4$ SO/Sp gauge theory. In section 5 we discuss some aspects of the simplest orientifold of $adS_5 \times S^5$ containing D7-branes, whose dual is the $\mathcal{N} = 2$ Sp($2N$) gauge theory.
with \( \oplus 4 \) matter hypermultiplets. The pp-wave limit containing an open string sector was studied in ref. [8]; we focus on another Penrose limit of the same theory, which does not contain open strings. (Other aspects of open strings and D-branes in the pp-wave background have been discussed in refs. [9].)

In sections 6 and 8 we discuss the pp-wave limits of theories whose IIB descriptions contain (before taking the limit) an O7-plane, four D7-branes (plus mirror branes), and a \( \mathbb{Z}_2 \) orbifold; specifically, \( \mathcal{N} = 2 \) SU(\( N \)) with \( 2 \oplus 4 \) matter hypermultiplets, and \( \mathcal{N} = 2 \) Sp(2\( N \))×Sp(2\( N \)) with \( (\bullet, \bullet) \oplus 2(1, \bullet) \oplus 2(\bullet, 1) \) matter hypermultiplets. The adS/CFT correspondence for these elliptic models was discussed in detail in refs. [10, 11]. Both theories possess two different Penrose limits, one of which allows for an open string sector. Moreover, in these models there are several distinct species of open strings, due to the presence of the orbifold projection.

In sections 7 and 9 we discuss two infinite classes of elliptic models [12] for which the models considered in sections 6 and 8 are the first members. The type IIB descriptions of these two classes of models contain an O7-plane, four D7-branes (plus mirror branes), and a \( \mathbb{Z}_2 N_2 \) orbifold, and the dual gauge theories have gauge groups SU(\( N_2 \)) and Sp×SU(\( N_2 - 1 \))×Sp, respectively. We show that these theories possess a scaling limit, similar to the one recently considered in refs. [13, 14], giving rise to a compact null direction. The theories we consider contain both open and closed string sectors.

In section 10 we discuss the pp-wave limit for the correspondence between type IIB string theory on \( adS_5 \times T^{11}/\mathbb{Z}_2^{ori} \) and the \( \mathcal{N} = 1 \) Sp(2\( N \))×Sp(2\( N \)) theory with \( 2(\bullet, \bullet) \oplus 4(1, \bullet) \oplus 4(\bullet, 1) \) matter chiral multiplets. By studying this example we are able to test the ideas of ref. [8] in a more complicated setting. Section 11 summarizes the results of this paper.

2 Gauge theories on branes in orientifold backgrounds

The four-dimensional \( \mathcal{N} = 4, \mathcal{N} = 2, \) and \( \mathcal{N} = 1 \) supersymmetric gauge theories considered in this paper arise as the worldvolume theories on D3-branes in various orientifolded type IIB backgrounds. The \( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) theories arise from D3-branes in a flat background modded out by an orientifold group \( G \). We parameterize the flat six-dimensional space transverse to the D3-brane worldvolume by

\[
\begin{align*}
  z_1 &= x_6 + ix_7 = r \cos \theta_1 \cos \theta_2 e^{i\varphi_1}, & \quad \theta_{1,2} \in [0, \frac{1}{2}\pi], \\
  z_2 &= x_8 + ix_9 = r \cos \theta_1 \sin \theta_2 e^{i\varphi_2}, & \quad \varphi_{1,2,3} \in [0, 2\pi], \\
  z_3 &= x_4 + ix_5 = r \sin \theta_1 e^{i\varphi_3}.
\end{align*}
\]

The coordinates \( z_i \) are subject to various identifications that depend on the form of \( G \).

In the large-\( N \) limit (\( \sim \) large number of D3-branes), the D3-branes significantly modify the background. The near-horizon limit of the resulting background is the space \( adS_5 \times S^5 \), whose metric is given by

\[
\begin{align*}
  ds^2 &= R^2 \left( ds^2_{adS_5} + d\tilde{\Omega}_5^2 \right), \\
  ds^2_{adS_5} &= d\rho^2 - \cos^2 \rho \, dt^2 + \sinh^2 \rho \, d\Omega_3^2, \\
  d\tilde{\Omega}_5^2 &= d\theta_1^2 + \cos^2 \theta_1 \left( d\theta_2^2 + \cos^2 \theta_2 \, d\varphi_1^2 + \sin^2 \theta_2 \, d\varphi_2^2 \right) + \sin^2 \theta_1 \, d\varphi_3^2.
\end{align*}
\]
Modding out by the orientifold group $G$ results in identifications among the coordinates, specified in each case below.

The isometry group of $S^5$ is $\text{SO}(6)$, which is to be identified with the SU(4) $R$-symmetry group of the $\mathcal{N} = 4$ theories. For the $\mathcal{N} = 2$ theories considered in this paper, $G$ reduces the isometry group to $[\text{SO}(4) \times \text{SO}(2)]/G$, where $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$ is the rotation group on $\mathbb{C}^2 \sim \{z_1, z_2\}$, and $\text{SO}(2) \simeq \text{U}(1)_R$ is the rotation group on $\mathbb{C} \sim \{z_3\}$. More precisely, $\text{SU}(2)_L$ and $\text{SU}(2)_R$ act on the left and right of the matrix

$$
\begin{pmatrix}
  z_1 & -\bar{z}_2 \\
  z_2 & \bar{z}_1
\end{pmatrix}.
$$

The diagonal generators $J^3_{L,R}$ of SU(2)$_L,R$ and the generator $J^3_R$ of U(1)$_R$, act as

$$J^3_R + J^3_L = -i\partial\varphi_1, \quad J^3_R - J^3_L = -i\partial\varphi_2, \quad J^3_R = -i\partial\varphi_3.$$  

SU(2)$_R \times \text{U}(1)_R/G$ is the $R$-symmetry group of the $\mathcal{N} = 2$ field theories, whereas SU(2)$_L$ is a global symmetry group. For some of the models we consider SU(2)$_L$ is further reduced to U(1)$_L$. For these models $J^3_L$ is identified with the U(1)$_L$ generator. To facilitate comparison to the results in ref. [11], we note that $J^3_R = \frac{1}{2} q_R$.

In this paper, we also consider the $\mathcal{N} = 1 \text{Sp}(2N) \times \text{Sp}(2N)$ field theory on a stack of D3-branes in the background of an orientifold of the conifold. The orientifold group and the gauge theory on the D3-branes are discussed in more detail in section 10.

3 Penrose limits of orientifolds of $adS_5 \times S^5$

All geodesics of $S^5$ are equivalent, so there is only one Penrose limit of $adS_5 \times S^5$, when the geodesic lies partially in the $S^5$ directions. (If the geodesic lies completely inside $adS_5$, the Penrose limit gives flat space [13].) For the orientifolded theories considered in this paper, however, Penrose limits along different geodesics can give rise to different string theories. In this section, we identify two particular geodesics and the associated Penrose limits.

Penrose limit A

Consider the geodesic parameterized by $\varphi_3$ and contained in the subspace $\rho = 0$, $\theta_1 = \frac{1}{2} \pi$. Define

$$x^+ = \frac{1}{2}(t + \varphi_3), \quad x^- = \frac{1}{2} R^2(t - \varphi_3), \quad r = R\rho, \quad y = R(\frac{1}{2} \pi - \theta_1).$$

(3.6)

In the limit $R \to \infty$ (keeping $x^\pm$, $r$, and $y$ finite), the metric goes to

$$ds^2 = -4dx^+dx^- - (r^2 + y^2)(dx^+)^2 + dr^2 + r^2d\Omega^2_3 + dy^2 + y^2d\tilde{\Omega}^2_3,$$

(3.7)

where

$$d\tilde{\Omega}^2_3 = d\theta_2^2 + \cos^2 \theta_2 d\varphi_1^2 + \sin^2 \theta_2 d\varphi_2^2.$$  

(3.8)

The space transverse to the null geodesic is $\mathbb{R}^4 \times \mathbb{R}^4$. Parameterizing $\mathbb{R}^4 = \mathbb{C}^2$ by

$$w_1 = y \cos \theta_2 e^{i\varphi_1}, \quad w_2 = y \sin \theta_2 e^{i\varphi_2},$$

(3.9)
we can rewrite the metric as
\[ ds^2 = -4dx^+dx^- - (r^2 + |w_1|^2 + |w_2|^2)(dx^+)^2 + dr^2 + r^2d\Omega_3^2 + |dw_1|^2 + |dw_2|^2. \quad (3.10) \]

The \( p^- \) lightcone momentum is given by \( 2p^- = H_A = i\partial_{x^+} = i(\partial_t + \partial_{\varphi_1}) \), which is identified in the dual gauge theory with \( H_A = \Delta - J_A \), where \( \Delta \) is the conformal dimension and \( J_A \) is a global symmetry generator. From (2.3) we see that \( J_A = J_R \). Hence, in Penrose limit A, \( H_A = \Delta - J_R \).

**Penrose limit B**

Consider the geodesic parameterized by \( \varphi_1 \) lying along \( \rho = 0 \) and \( \theta_1 = \theta_2 = 0 \). Define
\[ x^+ = \frac{1}{2}(t + \varphi_1), \quad x^- = \frac{1}{2}R^2(t - \varphi_1), \quad r = R\rho, \quad y_2 = R\theta_2, \quad y_3 = R\theta_1. \quad (3.11) \]

In the limit \( R \to \infty \) (keeping \( x^\pm, r, y_2, \) and \( y_3 \) finite), the metric goes to
\[ ds^2 = -4dx^+dx^- - (r^2 + y_2^2 + y_3^2)(dx^+)^2 + dr^2 + r^2d\Omega_3^2 + dy_2^2 + y_2^2d\varphi_2^2 + dy_3^2 + y_3^2d\varphi_3^2. \quad (3.12) \]

The space transverse to the null geodesic is \( \mathbb{R}^4 \times \mathbb{R}^4 \). Parameterizing \( \mathbb{R}^4 = \mathbb{C}^2 \) by
\[ w_2 = y_2e^{i\varphi_2}, \quad w_3 = y_3e^{i\varphi_3}, \quad (3.13) \]
the metric becomes
\[ ds^2 = -4dx^+dx^- - (r^2 + |w_2|^2 + |w_3|^2)(dx^+)^2 + dr^2 + r^2d\Omega_3^2 + |dw_2|^2 + |dw_3|^2. \quad (3.14) \]

From \( 2p^- = H_B = i(\partial_t + \partial_{\varphi_2}) \) we see that \( J_B = -i\partial_{\varphi_2} \), which using (2.3), implies that \( J_B = J^3_R + J^3_L \). Hence, in Penrose limit B, \( H_B = \Delta - J^3_R - J^3_L \).

The Penrose limits discussed here are limits of the full supergravity/string theory background (as discussed in ref. [3]). The non-trivial RR five-form field strength is also non-zero in the limit (and is responsible for giving masses to the lightcone fermions). As observed in ref. [2], string theory in the pp-wave background is dual to a certain subset of gauge theory operators.

The metrics in the two Penrose limits (3.10) and (3.14) are identical. However, the orientifold group \( G \) imposes different identifications on the coordinates in the two cases, and thus can lead to distinct string theories. These distinct Penrose limits therefore correspond to different subsets of operators in the gauge theory on the D3 branes in the orientifolded background.

Finally, we note that the limit along the geodesic parameterized by \( \varphi_2 \) is equivalent to the limit involving \( \varphi_1 \) (Penrose limit B). On the gauge theory side, the only difference is a trivial sign redefinition of \( J^3_L \).

In section [10] we discuss the Penrose limit of an orientifold of \( adS_5 \times T^{11} \). More details about the orientifold group and the Penrose limit are given in that section.
4 $\mathcal{N} = 4$ Sp($2N$) and SO($N$) gauge theories

Let us start by considering the $\mathcal{N} = 4$ SO($N$) and Sp($2N$) theories. In $\mathcal{N} = 1$ language, these theories contain a vector multiplet, and three chiral multiplets $\phi_{\alpha b}$. The scalars in the chiral multiplets, organized into $SU(2)_R$ multiplets, and their quantum numbers are shown in the table below.

| CFT field | Sp($2N$) or SO($N$) | $J_R$ | $J_R^2$ | $J_L^2$ | $\Delta$ | $H_A$ | $H_B$ |
|-----------|---------------------|-------|---------|---------|-------|-------|-------|
| $\phi_3$  | adjoint             | 1     | 0       | 0       | 1     | 0     | 1     |
| $\phi_3^\dagger$ | adjoint          | -1    | 0       | 0       | 1     | 2     | 1     |
| ($\phi_1, \phi_2$) | adjoint      | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $\frac{1}{2}$ | 1     | (1,1) | (0,1) |
| ($\phi_2, \phi_1$) | adjoint | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $\frac{1}{2}$ | 1     | (1,1) | (1,2) |

Table 1: SU($2)_R$ multiplets of scalar fields in $\mathcal{N} = 4$ Sp($2N$) or SO($N$).

The $\mathcal{N} = 4$ SO($N$) and Sp($2N$) gauge theories arise as the worldvolume theories on a stack of D3-branes parallel to an O3-plane in a flat type IIB background. The generator $\Omega(-1)^F_L R_{456789}$ of the orientifold group $\mathbb{Z}_2^{\text{ori}}$ acts on the $\mathbb{C}^3$ transverse to the D3-branes as

$$z^1 \to -z^1, \quad z^2 \to -z^2, \quad z^3 \to -z^3,$$

where $R_{456789}$ denotes reflection in the directions 4, $\cdots$, 9, $\Omega$ is the worldsheet parity operation, and $(-1)^F_L$ changes the sign of the left-movers in the Ramond sector.

In the near-horizon limit, the geometry becomes $adS_5 \times S^5 / \mathbb{Z}_2^{\text{ori}}$ where the geometric part of $\mathbb{Z}_2^{\text{ori}}$ acts as

$$\varphi_1 \to \varphi_1 + \pi, \quad \varphi_2 \to \varphi_2 + \pi, \quad \varphi_3 \to \varphi_3 + \pi,$$

on the coordinates of $S^5$ (2.3), which yields $adS_5 \times \mathbb{RP}^5$. The adS/CFT correspondence between string theory on this background and the above SO/Sp gauge theories was developed in ref. [16]. The Penrose limit of these theories was briefly studied in ref. [17]; below we give some further details.

Penrose limit A

The action (1.2) of $\mathbb{Z}_2^{\text{ori}}$ on the coordinates in Penrose limit A is an inversion of $\tilde{R}^4$: $w_1 \to -w_1$, $w_2 \to -w_2$, accompanied by a shift $\varphi_3 \to \varphi_3 + \pi$ halfway around the geodesic (see section 3 for details about the notation). This $\mathbb{Z}_2^{\text{ori}}$ action has no fixed points on $S^5$, and therefore no fixed points in the Penrose limit (as we will see in later sections, this fact is related to the absence of an open string sector). We now discuss the operators in the gauge theory that correspond to the states in the closed string theory.

The only candidate for the operator corresponding to the vacuum state of the closed string theory in the pp-wave background (3.10) is

$$\text{tr} \left[ \phi_3^A \right],$$

(4.3)

1Throughout this paper we will not keep track of the overall normalizations of the operators.
which has $\Delta = J_A$ and therefore $H_A = 0$. The effect of modding out by $\mathbb{Z}_2^\text{ori}$ is to halve the periodicity of the geodesic circle, and thus to allow only even integer values of $J_A$ for the vacuum string states. Thus from the string theory perspective one expects the vacuum to correspond to

$$\text{tr} \left[ \phi_3^{2n+1} \right].$$  \hspace{1cm} (4.4)

From the gauge theory perspective, the exponent must be even because the adjoint fields satisfy $\phi_i^T = J \phi_i J$ ($\phi_i^T = -\phi_i$) in the Sp theory (SO theory), so that $\text{tr}(\phi_3^{2n+1})$ vanishes identically.

Consider string states in which a zero-momentum oscillator $a_0^\dagger \mu$ in one of the directions of $\mathbb{R}^4$ acts on the light-cone vacuum. These states correspond to the insertion of $(D_\mu \phi_3) \phi_3$ into (4.4), yielding four operators

$$\text{tr} \left[ (D_\mu \phi_3) \phi_3^{2n+1} \right],$$  \hspace{1cm} (4.5)

which have $\Delta = 2n + 3$ and $J_A = 2n + 2$, and therefore $H_A = 1$.

Next, consider string states in which a zero-momentum oscillator $a_0^\dagger i$, $\bar{a}_0^\dagger i$ in one of the directions $w_i$, $\bar{w}_i$ ($i = 1, 2$) of $\mathbb{R}^4$ acts on the light-cone vacuum. The inversion $(w_1, w_2) \rightarrow (-w_1, -w_2)$ causes $a_0^\dagger i \rightarrow -a_0^\dagger i$, $\bar{a}_0^\dagger i \rightarrow -\bar{a}_0^\dagger i$, so for the string state to be invariant under $\mathbb{Z}_2^\text{ori}$, the value of $J_A$ must be odd. These string states correspond to insertions of $\phi_1 \phi_3$, $\phi_1^\dagger \phi_3$, $\phi_2 \phi_3$, and $\phi_2^\dagger \phi_3$ into (4.4), yielding four additional $H_A = 1$ operators

$$\text{tr} \left[ \phi_1 \phi_3^{2n+1} \right], $$  \hspace{0.5cm} \text{tr} \left[ \phi_1^\dagger \phi_3^{2n+1} \right],$$  \hspace{0.5cm} \text{tr} \left[ \phi_2 \phi_3^{2n+1} \right],$$  \hspace{0.5cm} \text{tr} \left[ \phi_2^\dagger \phi_3^{2n+1} \right].$$  \hspace{1cm} (4.6)

Again, from the gauge theory perspective, the exponents follow from the Sp (SO) constraints on the $\phi_i$'s.

The eight operators (4.5) and (4.6) therefore represent the eight bosonic zero-momentum states in the string theory. The correspondence for the fermionic oscillators as well as for the non-zero-momentum oscillators can be constructed in complete analogy with the original construction [2]; we will not give the details here.

For the case of $adS_5 \times \mathbb{R}^5$, the distinction between the Sp and SO theories was explained in ref. [10]. An interesting question is to understand how they are distinguished in the pp-wave limit.

**Penrose limit B**

The action (4.2) of $\mathbb{Z}_2^\text{ori}$ on the coordinates in Penrose limit B involves an inversion of the transverse $\mathbb{C}^2$: $w_2 \rightarrow -w_2$, $w_3 \rightarrow -w_3$, accompanied by a shift $\varphi_1 \rightarrow \varphi_1 + \pi$ halfway around the geodesic. This action is identical to the one in Penrose limit A after a trivial relabelling of the coordinates. The correspondence of string states to the gauge theory operators is the same as in Penrose limit A except with $\phi_1$ and $\phi_3$ interchanged.

5 \hspace{1cm} $\mathcal{N} = 2 \text{ Sp}(2N)$ with $\square \oplus 4\square$

The $\mathcal{N} = 2 \text{ Sp}(2N)$ theory with one antisymmetric and four fundamental hypermultiplets contains an $\mathcal{N} = 2$ vector multiplet in the adjoint representation, which in $\mathcal{N} = 1$ language
consists of a vector multiplet and a chiral multiplet, $\phi^{a\dot{b}}$, in the adjoint representation. The theory also contains an antisymmetric $\mathcal{N}=2$ hypermultiplet, which in $\mathcal{N}=1$ language consists of a chiral multiplet in the $\mathbf{2}$ representation of $\text{Sp}(2N)$, $A_{ab} = -A_{ba}$, and a chiral multiplet in the $\mathbf{1}$ representation $\tilde{A}^{ab}$ = $\tilde{A}^{ba}$. The indices on the latter can be lowered using the symplectic unit $J^{ab}$, leaving us with two chiral multiplets, $A_{1ab}$ and $A_{2ab}$ = $J^{ac}J^{bd}\tilde{A}_{cd}$, in the representation, transforming as a doublet of $\text{SU}(2)_L$. Finally, the theory contains four fundamental $\mathcal{N}=2$ hypermultiplets, which in $\mathcal{N}=1$ language consists of eight chiral multiplets $Q_{Ia}$ in the representation (after lowering the indices of the four chiral multiplets in the $\mathbf{1}$ representation). The scalars in the chiral multiplets and their quantum numbers are shown in the table below.

| CFT field | Sp(2N) | $J_R$ | $J_R^3$ | $J_L^3$ | $\Delta$ | $H_A$ | $H_B$ |
|-----------|--------|-------|---------|---------|--------|-------|-------|
| $\phi$    | adjoint| 1     | 0       | 0       | 1      | 0     | 1     |
| $\phi^\dagger$ | adjoint| $-1$  | 0       | 0       | 2      | 1     | 1     |
| $(A_1, (J A_2 J)^\dagger)$ | $\mathbf{2}$ | 0     | $(1/2, -1/2)$ | $1/2$  | 1      | (1,1) | (0,1) |
| $(A_2, (J A_1 J)^\dagger)$ | $\mathbf{2}$ | 0     | $(1/2, -1/2)$ | $-1/2$ | 1      | (1,1) | (1,2) |
| $(Q^I, Q^{I\dagger})$ | $\mathbf{2}$ | 0     | $(1/2, -1/2)$ | $0$    | 1      | 1     | $(-1/2, 3/2)$ |

Table 2: SU(2)$_R$ multiplets of scalar fields in $\mathcal{N}=2$ Sp(2N) with $\mathbf{1} \oplus 4 \mathbf{2}$.

The above $\mathcal{N}=2$ Sp(2N) gauge theory arises as the theory on a stack of D3-branes parallel to an O7-plane and four physical D7-branes in a flat background. The orientifold group $\mathbb{Z}_2^{ori}$ = $\{1, \Omega R_{45}(-1)^F\}$, where $R_{45}$ reflects the 45 directions, acts on the space transverse to the D3-branes as $z^3 \to -z^3$. This action fixes the hyperplane $z_3 = 0$, which corresponds to the position of the O7-plane and the D7-branes.

In the near-horizon limit the background becomes $adS_5 \times S^5/\mathbb{Z}_2^{ori}$, where $\mathbb{Z}_2^{ori}$ acts as $\varphi_3 \to \varphi_3 + \pi$ on the coordinates of $S^5$. The fixed point set of this action is an $S^3$ located at $\theta_1 = 0$. The adS/CFT correspondence for this model was discussed in ref. [19].

Penrose limit A

In Penrose limit A, the generator of $\mathbb{Z}_2^{ori}$ simply produces a translation along the geodesic $\varphi_3 \to \varphi_3 + \pi$ (together with the action of $\Omega(-1)^F$) with no action on the transverse coordinates $w_1$ and $w_2$ [3,4]. This limit therefore yields the maximally supersymmetric pp-wave background. However, $\mathbb{Z}_2^{ori}$ projects out half of the discrete values of $J_A$ that would be allowed in the unprojected case.

Closed string sector

The $H_A = 0$ operators

$$\text{tr} \left[ \phi^{2n} \right], \quad (5.1)$$

correspond to the closed string vacuum states, where the effect of modding out by $\varphi_3 \to \varphi_3 + \pi$ is to allow only even integer values of $J_A$. From the gauge theory perspective, the exponent
must be even because the adjoint field satisfies $\phi^T = J\phi J$, implying that $\text{tr} [\phi^{2n+1}]$ vanishes identically.

String states in which a zero-momentum oscillator $a^0_\mu$ in one of the directions of $\mathbb{R}^4$ acts on the light-cone vacuum correspond to the $H_A = 1$ gauge theory operators

$$\text{tr} \left[(D_\mu \phi)\phi^{2n+1}\right].$$

(5.2)

String states in which a zero-momentum oscillator $a^0_i$, $\bar{a}^0_i$ in one of the directions of $\tilde{\mathbb{R}}^4$ acts on the light-cone vacuum correspond to the four $H_A = 1$ operators

$$\text{tr} \left[(A_i J)\phi^{2n}\right], \quad \text{tr} \left[(J A_i^\dagger)\phi^{2n}\right], \quad (i = 1, 2),$$

(5.3)

where we write $A_i J$ for $A_{iac} J^c_b$ (and similarly for $J A_i^\dagger$). Using $(AJ)^T = -J(AJ)J$, it follows that the operator would vanish if the exponent of $\phi$ were odd.

Non-zero-momentum states can be constructed in complete analogy with the original construction.

The fixed point set ($O7$-plane) of $Z_{\text{ori}}^2$ goes away in Penrose limit A, and so do the D7-branes, so there is no open string sector. The absence of an open string sector corresponds to the fact that the operator

$$Q^I J\phi^n Q^K$$

vanishes by using the F-term equations of the gauge theory. This implies that it is not a chiral primary operator, and therefore can not correspond to a vacuum state.

**Penrose limit B**

In Penrose limit B, the geometric action of $Z_{\text{ori}}^2$, $\varphi_3 \rightarrow \varphi_3 + \pi$ produces a reflection of the transverse coordinate $w_3 \rightarrow -w_3$. The geodesic lies in the orientifold plane $w_3 = 0$, and the coincident D7-branes gives rise to an open string sector in addition to the closed string sector.

This limit of the theory was considered in detail by Berenstein et. al. [8], to which we refer the reader for a complete description of the closed and open string sectors and the corresponding gauge theory operators.

**6 $\mathcal{N} = 2$ SU($N$) with $2\mathbb{E} \oplus 4\square$**

The $\mathcal{N} = 2$ SU($N$) theory with two antisymmetric and four fundamental hypermultiplets contains an $\mathcal{N} = 2$ vector multiplet in the adjoint representation, which in $\mathcal{N} = 1$ language comprise a vector multiplet and a chiral multiplet, $\phi_a^b$, in the adjoint representation. The theory also contains two antisymmetric $\mathcal{N} = 2$ hypermultiplets, which in $\mathcal{N} = 1$ language comprise a pair of chiral multiplets $A_{iab}$ ($i = 1, 2$) in the $\mathbb{E}$ representation and a pair of chiral multiplets $\bar{A}^{ab}_i$ ($i = 1, 2$) in the $\tilde{\mathbb{E}}$ representation. Both $(A_1, A_2)$ and $(\bar{A}_1, \bar{A}_2)$ form SU(2)$_L$ doublets. The conjugate fields $(A_1^\dagger, A_2^\dagger)$ transform in the conjugate SU(2)$_L$ representation, but by lowering the indices with the $\epsilon$ tensor, so that $A_1^\dagger = A_2^\dagger$ and $A_2^\dagger = -A_1^\dagger$ we can form
an SU(2)$_L$ doublet ($A^1_1, A^1_2$) from the conjugate fields (and similarly for ($\tilde{A}^1_1, \tilde{A}^1_2$)). Finally, the theory contains four fundamental $\mathcal{N} = 2$ hypermultiplets, which in $\mathcal{N} = 1$ language comprise four chiral multiplets $Q^I_i$ in the $\square$ representation and four chiral multiplets $\tilde{Q}^I_i$ in the $\square$ representation. The quantum numbers of the scalars in the above supermultiplets are shown in the table below.

| CFT field | SU(N) | $J_R$ | $J^R_R$ | $J^L_L$ | $\Delta$ | $H_A$ | $H_B$ |
|-----------|-------|-------|---------|---------|---------|-------|-------|
| $\phi$ | adjoint | 1     | 0       | 0       | 1       | 0     | 1     |
| $\phi^\dagger$ | adjoint | -1    | 0       | 0       | 1       | 2     | 1     |
| $(A_1, \tilde{A}^1_1)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $\frac{1}{2}$ | 1       | (1,1) | (0,1) |
| $(A_2, \tilde{A}^1_2)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $-\frac{1}{2}$ | 1       | (1,1) | (1,2) |
| $(\tilde{A}_1, A^1_1)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $\frac{1}{2}$ | 1       | (1,1) | (0,1) |
| $(\tilde{A}_2, A^1_2)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | $-\frac{1}{2}$ | 1       | (1,1) | (1,2) |
| $(Q^I, \tilde{Q}^I_i)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | 0       | 1       | (1,1) | $(\frac{1}{2}, \frac{3}{2})$ |
| $(\tilde{Q}^I, Q^I_i)$ | $\square$ | 0     | $(\frac{1}{2}, -\frac{1}{2})$ | 0       | 1       | (1,1) | $(\frac{1}{2}, \frac{3}{2})$ |

Table 3: SU(2)$_R$ multiplets of scalar fields in $\mathcal{N} = 2$ SU(N) with $2 \square \oplus 4 \square$.

The F-term equations for the model are

$$A_1 \tilde{A}_2 - A_2 \tilde{A}_1 + Q^I \tilde{Q}_I = 0, \quad \phi A_i + A_i \phi^T = 0, \quad \tilde{A}_i \phi + \phi^T \tilde{A}_i = 0, \quad \phi Q^I = 0 = \tilde{Q}_I \phi. \quad (6.1)$$

The $\mathcal{N} = 2$ SU(N) gauge theory just described arises as the worldvolume theory on a stack of coincident D3-branes in a flat type IIB background modded out by the orientifold group

$$G = \mathbb{Z}_2^{\text{orb}} \times \mathbb{Z}_2^{\text{ori}}, \quad (6.2)$$

where $\mathbb{Z}_2^{\text{orb}} = \{1, R_{6789}\}$ and $\mathbb{Z}_2^{\text{ori}} = \{1, R_{45}, (1, F_{L})\}$ act on the $\mathbb{C}^3$ transverse to the D3-branes as

$$z_3 \xrightarrow{\mathbb{Z}_2^{\text{ori}}} -z_3, \quad (z_1, z_2) \xrightarrow{\mathbb{Z}_2^{\text{orb}}} (-z_1, -z_2). \quad (6.3)$$

The fixed point set of $\mathbb{Z}_2^{\text{ori}}$ is the hyperplane $z_3 = 0$, which corresponds to the position of an O7-plane and four physical D7-branes, while the fixed point set of $\mathbb{Z}_2^{\text{orb}}$ is the six-dimensional hyperplane $z_1 = z_2 = 0$.

The fields of the SU(N) theory discussed above can be obtained via a projection from those of the $\mathcal{N} = 4$ SU(2N) theory, whose three adjoint $\mathcal{N} = 1$ chiral superfields are denoted $\Phi_i$. The independent generators $\gamma_\theta$ and $\gamma_\Omega^\nu$ of the orientifold group are realized as the $2N \times 2N$ matrices \cite{12}

$$\gamma_\theta = i \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}, \quad \gamma_\Omega^\nu = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}, \quad (6.4)$$

\cite{2}The projection really gives a $\text{U}(N)$ theory; however, since we are interested in the large-$N$ limit the $\text{U}(1)$ factor is suppressed and can be ignored.
which induce the following projections on the fields of the gauge theory:

\[
\Phi_{1,2} = -\gamma_\theta \Phi_{1,2} \gamma_\theta^{-1}, \\
\Phi_3 = \gamma_\theta \Phi_3 \gamma_\theta^{-1}, \\
\Phi_{1,2} = \gamma_{\Omega'} \Phi_{1,2} \gamma_{\Omega'}^{-1}, \\
\Phi_3 = -\gamma_{\Omega'} \Phi_3 \gamma_{\Omega'}^{-1},
\]

resulting in \(\Phi_i\)'s of the form

\[
\Phi_1 = \begin{pmatrix} 0 & A_1 \\
-A_1 & 0 \end{pmatrix}, \\
\Phi_2 = \begin{pmatrix} 0 & A_2 \\
A_2 & 0 \end{pmatrix}, \\
\Phi_3 = \begin{pmatrix} \phi & 0 \\
0 & -\phi^T \end{pmatrix}.
\]

In the near-horizon (large-\(N\)) limit of the above orientifolded background one obtains \(\text{adS}_5 \times S^5 / [Z_2^{orb} \times Z_2^{ori}]\), where \(Z_2^{ori}\) acts as \(\varphi_3 \rightarrow \varphi_3 + \pi\), \(\varphi_2 \rightarrow \varphi_2 + \pi\) and \(Z_2^{orb}\) acts as \(\varphi_3 \rightarrow \varphi_3 + \pi\) on the coordinates of \(S^5\). String theory in this background is dual to the \(SU(N)\) gauge theory discussed above (see ref. \[11\] for further details). Below we discuss the Penrose limits for this correspondence.

**Penrose limit A**

In the first Penrose limit, the generator of \(Z_2^{ori}\) simply produces a translation along the geodesic: \(\varphi_3 \rightarrow \varphi_3 + \pi\) (together with the action of \(\Omega(-1)^F\)), with no action on the transverse coordinates \(w_1\) and \(w_2\) \((3.9)\). There is no orientifold fixed plane and therefore no open string sector in this limit, but \(Z_2^{ori}\) projects out half of the discrete values of \(J_A\) that would be allowed in the unprojected case. The generator of \(Z_2^{orb}\) acts solely on the transverse coordinates as \(w_1 \rightarrow -w_1, w_2 \rightarrow -w_2\). The geodesic lies in the orbifold fixed plane, resulting in a twisted sector of the string theory in this pp-wave background. First we describe the operators corresponding to the untwisted sector.

**Untwisted sector**

The identification of the gauge theory operators corresponding to the states of the dual string theory in the Penrose limit is facilitated by going over to the cover space, as in refs. \[20, 13\] (see also \[21\]). The ground state in the untwisted sector corresponds to \(\text{tr}[\Phi_3^{J_A}]\). As a result of the orientifold projection \((6.6)\), this is only nonvanishing for even \(J_A\), yielding the \(H_A = 0\) operator

\[
\text{tr} \left[ \phi^{2n} \right].
\]

The \(H_A = 1\) gauge theory operators \(\text{tr}[(D_\mu \Phi_3) \Phi_3^{2n+1}]\), or equivalently using \((6.6)\),

\[
\text{tr} \left[ (D_\mu \phi) \phi^{2n+1} \right],
\]

correspond to string states in which a single zero-momentum oscillator \(a_0^{I,\mu}\) in one of the directions of \(\mathbb{R}^4\) acts on the light-cone vacuum.

Since the zero-momentum oscillators \(a_0^{I,\mu}\) in the \(\mathbb{R}^4\) directions \((3.9)\) are odd under \(Z_2^{orb}\), an even number of \(a_0^{I,\mu}\)'s must act on the light-cone vacuum. The \(H_A = 2\) gauge theory operators corresponding to the states in which two such oscillators act on the vacuum involve
the insertion of two $H_A = 1$ fields $\Upsilon_I = (\Phi_1, \Phi_1^\dagger, \Phi_2, \Phi_2^\dagger)$ into the ground state operator (the insertion of a single $H_A = 1$ operator into the ground state operator gives zero using (6.6)): 

$$
\sum_{m=0}^{J_A} \text{tr} \left[ \Upsilon_I \Phi_3^m \Upsilon_J \Phi_3^{J_A-m} \right],
$$

(6.9)

where we average over the relative position of the $\Upsilon$'s [2, 22, 23]. The sum over $m$ vanishes unless $J_A = 2n$. Using (6.6), together with $\Phi_1^\dagger = \begin{pmatrix} 0 & \tilde{A}_1^\dagger \\ A_1^\dagger & 0 \end{pmatrix}$, $\Phi_2^\dagger = \begin{pmatrix} 0 & \tilde{A}_2^\dagger \\ A_2^\dagger & 0 \end{pmatrix}$, (6.10)

we obtain the set of ten operators

$$
\sum_{m} (-)^m \text{tr} \left[ \tilde{A}_i \phi^m A_j (\phi^T)^{2n-m} \right], \quad \sum_{m} (-)^m \text{tr} \left[ A_i^\dagger \phi^m A_j^\dagger (\phi^T)^{2n-m} \right], \quad \sum_{m} (-)^m \text{tr} \left[ A_i^\dagger \phi^m A_j (\phi^T)^{2n-m} + \tilde{A}_j \phi^m \tilde{A}_i (\phi^T)^{2n-m} \right].
$$

(6.11)

These correspond to the ten string theory states

$$
a_0^{i_1} a_0^{j_1} |0, p^+\rangle, \quad a_0^{i_1} a_0^{j_1} |0, p^+\rangle, \quad \tilde{a}_0^{i_1} \tilde{a}_0^{j_1} |0, p^+\rangle,
$$

(6.12)

where $a_0^{i_1}$ ($\tilde{a}_0^{i_1}$) is the zero-momentum operator in the direction $w_i$ ($\tilde{w}_i$).

In limit A there are no open strings, since the D7-branes disappear. The gauge theory explanation of this fact is that the candidate open string vacuum $\bar{Q}_I \phi^2 Q^K$ is ruled out since it is not a chiral primary operator, as can be seen from the fact that it vanishes by the F-term equation, $\phi Q^K = 0$.

**Twisted sector**

The ground state in the twisted sector of the closed string theory corresponds to the operator $\text{tr}[\gamma_0 \Phi_3^{J_A}]$. The presence of the operator $\gamma_0$ in the trace requires $J_A$ to be odd, giving

$$
\text{tr} \left[ \phi^{2n+1} \right].
$$

(6.13)

The $H_A = 1$ operators corresponding to the string theory oscillators in the $\mathbb{R}^4$ directions of the pp-wave metric are identified, in the twisted sector, with $\text{tr}[\gamma_0 (D_\mu \Phi_3) \Phi_3^{2n+2}]$, or

$$
\text{tr} \left[ (D_\mu \phi) \phi^{2n+2} \right].
$$

(6.14)

The oscillators in the twisted sector are half-integer moded [24, 25, 20, 26], so there are no zero-momentum oscillators in this sector. Hence the operators that would have corresponded to such oscillators acting on the twisted-sector vacuum

$$
\sum_{m} \text{tr} \left[ \gamma_0 \Upsilon_I \Phi_3^m \Upsilon_J \Phi_3^{2n-m} \right],
$$

(6.15)
must not be chiral primary operators in the gauge theory. Consider for example

\[ \sum (-)^m \mathrm{tr} \left[ \tilde{A}_i \phi^m A_{ij} (\phi^T)^{2n-m} \right]. \]  

(6.16)

By using the F-term equation \( \tilde{A}_i \phi + \phi^T \tilde{A}_i = 0 \), \( \tilde{A}_i \) can be shifted through the \( \phi^T \)'s. Then \( A_{ij} \tilde{A}_{ij} \) can be replaced by \( Q \)'s, using \( A_1 \tilde{A}_2 - A_2 \tilde{A}_1 + Q^I \tilde{Q}_I = 0 \). Finally, using \( \phi \tilde{Q}_I = 0 \), the operator vanishes and is therefore not a chiral primary operator. The other operators in (6.15) are similarly expected to be ruled out.

Although there are no zero-momentum oscillators in the twisted sector, non-zero modes are present. The construction of the corresponding gauge theory operators proceeds in analogy with the discussions in refs. [24, 25, 20, 21].

**Penrose limit B**

In limit B, the geometric action of \( \mathbb{Z}_2^{\text{ori}} \), \( \varphi_3 \to \varphi_3 + \pi \), produces a reflection of the coordinate transverse to the O7-plane: \( w_3 \to -w_3 \), cf. (3.13). Thus the geodesic lies in the orientifold (O7) plane \( w_3 = 0 \), and the coincident D7-branes give rise to an open-string sector. The generator of \( \mathbb{Z}_2^{\text{orb}} \) produces a shift \( \varphi_1 \to \varphi_1 + \pi \) halfway around the geodesic, accompanied by the reflection \( w_2 \to -w_2 \). Since the \( \mathbb{Z}_2^{\text{orb}} \) action has no fixed points, there is no twisted sector in this Penrose limit, however \( \mathbb{Z}_2^{\text{orb}} \) projects out half of the discrete values of \( J_B \) that would be allowed in the unprojected case. The pp-wave background in this limit is the same as the one considered in ref. [8] except for the additional action of \( \mathbb{Z}_2^{\text{orb}} \), which leads to different species of open string states, not encountered in ref. [8].

**Closed string sector**

The vacuum state of the closed string sector corresponds to the \( H_B = 0 \) operator \( \mathrm{tr}[\Phi_1^{2n}] \), which, due to the \( \mathbb{Z}_2^{\text{orb}} \) projection, is only nonvanishing when \( J_B = 2n \),

\[ \mathrm{tr}[\Phi_1^{2n}] = \mathrm{tr} \left[ (A_1 \tilde{A}_1)^n \right]. \]  

(6.17)

(On the gauge theory side the constraint on \( J_B \) is required because we need an equal number of even and odd fields to form a gauge invariant operator.) The absence of the twisted sector vacuum can been seen from the fact that \( \mathrm{tr}[\gamma_0 \Phi_1^n] = 0 \) for all values of \( m \).

String states in which a zero-momentum oscillator \( a_0^\mu \) in one of the directions of \( \mathbb{R}^4 \) acts on the light-cone vacuum correspond to the \( H_B = 1 \) gauge theory operator

\[ \mathrm{tr}[(D_\mu \Phi_1)\Phi_1^{2n+1}] = \mathrm{tr} \left[ D_\mu (A_1 \tilde{A}_1)(A_1 \tilde{A}_1)^n \right]. \]  

(6.18)

String states in which a zero-momentum oscillator \( a_0^\dagger \) in the \( w_2 \) (\( \tilde{w}_2 \)) direction of \( \mathbb{R}^4 \) acts on the light-cone vacuum corresponds to insertions of the \( H_B = 1 \) field \( \Phi_2 \Phi_1 \) into the vacuum state operator:

\[ \mathrm{tr} [\Phi_2 \Phi_1^{2n+1}] = \mathrm{tr} \left[ (A_2 \tilde{A}_1 + A_1 \tilde{A}_2)(A_1 \tilde{A}_1)^n \right], \]

\[ \mathrm{tr} [\Phi_2 \Phi_1^{2n+1}] = \mathrm{tr} \left[ (\tilde{A}_1 \tilde{A}_1 + A_1 \tilde{A}_1^\dagger)(A_1 \tilde{A}_1)^n \right]. \]  

(6.19)
Note that an insertion of \((A_1\tilde{A}_2 - A_2\tilde{A}_1)\) can be replaced by \(Q^I\tilde{Q}_I\) via the F-term equations, and thus turns the would-be \(H_B = 1\) closed string state into the (subleading) open string vacuum (6.23) (this appears to be related to the closed/open string interaction). Since the operator vanishes modulo subleading terms by using the F-term equations it is not protected and is thus ruled out. There is no similar gauge theory argument which rules out the insertion of the \(H_B = 1\) operator \(A_1A_{\dagger 1} - \tilde{A}_{\dagger 1}\tilde{A}_1\). It would be interesting to better understand why this operator is ruled out.

The zero-momentum oscillators in the \(w_3, \bar{w}_3\) directions of \(\mathbb{R}^4\) correspond to insertions of \(\phi\) and \(\phi^\dagger\) into the vacuum state (6.17). These oscillators are odd under \(\mathbb{Z}_2\) and so there should be no operators corresponding to single oscillators acting on the vacuum. This is automatic in the cover space language since the gauge theory operators \(\text{tr}[\Phi_3\Phi_1^{2n}]\) and \(\text{tr}[\Phi_3^\dagger\Phi_1^{2n}]\) vanish identically. An independent argument can also be constructed. The insertion of a single \(\phi\) is not possible since \(\text{tr}[\phi(A_1\tilde{A}_1)^n] = \frac{1}{2}\text{tr}[(\phi A_1 + A_1\phi^T)\tilde{A}_1(A_1\tilde{A}_1)^{(n-1)}]\), which vanishes by the F-term equations. Thus the operator is not a chiral primary (nor is it a “near chiral primary”) and is consequently ruled out. We expect that the insertion of a single \(\phi^\dagger\) is also ruled out as implied by the cover space result.

States with two oscillators in the \(w_3\) direction acting on the ground state are not projected out and correspond to the insertion of a pair of \(\Phi_3\)’s

\[
\sum_{m=0}^{2m} \text{tr} \left[ \Phi_3\Phi_1^m\Phi_3\Phi_1^{2n-m} \right],
\]

which gives

\[
\sum_{m \text{ even}} \text{tr} \left[ \phi \left( A_1\tilde{A}_1 \right)^{\frac{m}{2}} \phi \left( A_1\tilde{A}_1 \right)^{\frac{n-m}{2}} \right] - \sum_{m \text{ odd}} \text{tr} \left[ \phi \left( A_1\tilde{A}_1 \right)^{\frac{m-1}{2}} \tilde{A}_1\phi^T \tilde{A}_1 \left( A_1\tilde{A}_1 \right)^{\frac{n-m+1}{2}} \right].
\]

States where one (or both) of the zero-momentum oscillators are in the \(\bar{w}_3\) direction are obtained from the above expressions by replacing one (or both) of the \(\Phi_3\)’s by \(\Phi_3^\dagger\).

Open string sector

Since the geodesic lies in the orientifold (O7) fixed-plane \(w_3 = 0\) which is also the location of the D7-branes, there is an open-string sector. The gauge symmetry of the D7-branes/O7-plane, \(\text{SO}(8)_F\), is broken to \(\text{SU}(4)_F \times \text{U}(1)\) by the \(\mathbb{Z}_2^{\text{orb}}\) projection. The ground state of the open string sector, which transforms in the adjoint 28 of \(\text{SO}(8)_F\), decomposes into

\[
28 \rightarrow 15_0 \oplus 1_0 \oplus 6_2 \oplus \bar{6}_{-2}.
\]

The \(\mathbb{Z}_2^{\text{orb}}\) projection correlates the states in the \(15_0 \oplus 1_0\) representation with operators in odd-dimensional representations of \(\text{SU}(2)_L\) \([\|]\). These open string vacuum states should therefore correspond to gauge theory operators where the total number of \(A\)’s and \(\tilde{A}\)’s is even. There is a natural candidate

\[
\tilde{Q}_I(A_1\tilde{A}_1)^nQ^K,
\]

\[\text{A similar comment applies to the } H_B = 1 \text{ insertion, } (D_\mu A_1)\tilde{A}_1 - A_1(D_\mu \tilde{A}_1).\]
which indeed belongs to the correct \((15_0 \oplus 1_0)\) SU(4)_F × U(1) representation. Furthermore, the operator (6.23) has \(H_B = 1\), reflecting the non-vanishing zero point energy for the open string sector [8].

The \(\mathbb{Z}^2_{orb}\) projection correlates the states in the \(6_2 \oplus \bar{6}_{-2}\) representation with operators in even-dimensional representations of SU(2)_L [11]. These open string vacuum states should therefore correspond to gauge theory operators where the total number of \(A\)'s and \(\bar{A}\)'s is odd. The natural candidates are

\[
Q[I(A_1 \bar{A}_1)^n Q^K] , \quad \bar{Q}[I(A_1 \bar{A}_1)^n A_1 \bar{Q}_K].
\]

In these expressions, the flavor indices on the \(Q\)'s are antisymmetrized because the operator sandwiched in between them is antisymmetric in gauge indices, which shows that the operators (6.24) belong to the correct SU(4) representations. Furthermore, they have the correct U(1) (quark number) charges.

Notice that in contrast to the model discussed in [8] there are two distinct types of open strings in this model. The first, corresponding to (6.23) are “meson” open strings with a quark at one end and an anti-quark at the other, whereas the other, corresponding to (6.24), have quark number \(\pm 2\), with quarks (or anti-quarks) at both ends.

The open string state in which a zero-momentum oscillator in one of the \(\mathbb{R}^4\) directions act on the open string vacua corresponds to the insertion of the operator \(D_\mu(A_1 \bar{A}_1)\) into the operators in (6.23) and (6.24).

Zero-momentum oscillators \(a_0^\dagger, \bar{a}_0^\dagger\) in the \(w_2, \bar{w}_2\) directions of \(\tilde{\mathbb{R}}^4\) acting on the meson string vacuum (6.23) correspond to the two \(H_B = 2\) operators

\[
\begin{align*}
\sum_m \bar{Q}[I(A_1 \bar{A}_1)^m (A_1 \bar{A}_2 + A_2 \bar{A}_1)(A_1 \bar{A}_1)^{n-m} Q^K], \\
\sum_m \bar{Q}[I(A_1 \bar{A}_1)^m (\bar{A}_1^\dagger \bar{A}_1 + A_1 A_1^\dagger)(A_1 \bar{A}_1)^{n-m} Q^K],
\end{align*}
\]

and analogous insertions into the states (6.24). The insertion of the other linear combination of \(H_B = 1\) fields \((A_1 \bar{A}_2 - A_2 \bar{A}_1)\) results in splitting the operator \(\bar{Q} \cdots Q\) into \(\bar{Q} \cdots Q \bar{Q} \cdots Q\) by using the F-term equations. The latter operator is a “double-trace” operator which is subleading in the 1/\(N\)-expansion. Since the operator vanishes (modulo subleading terms) by using the F-term equations it is not a chiral primary operator (nor is it a “near chiral primary”) and is therefore ruled out.

Zero-momentum oscillators in the \(w_3, \bar{w}_3\) directions of \(\tilde{\mathbb{R}}^4\) are absent from the open string spectrum [8], so the corresponding operators on the gauge theory side should be absent as well. This follows from the fact that the insertion of a \(\phi\) (which has \(H_B = 1\)) can always be commuted to one of the endpoints by using the F-term equations, where it vanishes using the \(0 = \bar{Q}_F \phi\) F-term equation. Hence, such an operator is not chiral primary, and is therefore not protected. Similarly, insertions of \(\phi^\dagger\) should not be allowed, but this is more difficult to show.

The number of bosonic zero modes obtained above is in agreement with the string theory result [8], where it was shown that on the string theory side there are six bosonic and four fermionic zero modes.
7 The $\mathcal{N} = 2$ SU×SU×⋯×SU orientifold

The SU($N$) + 2□ + 4□ theory studied in section 6 is the first in an infinite series of similar theories [12]. The orientifold group for these theories has the form $G = \mathbb{Z}_{2N_2}^{\text{orb}} \times \mathbb{Z}_2^{\text{ori}}$, where the pure orientifold part has the universal form $\mathbb{Z}_2^{\text{ori}} = \{1, \Omega(-1)^{F_L} R_{45}\}$, and the orbifold part is $\mathbb{Z}_{2N_2}^{\text{orb}} = \{1, \theta, \ldots, \theta^{2N_2-1}\}$, where $\theta$ acts on the 6789 directions as $(z_1, z_2) \rightarrow (e^{\pi i/N_2} z_1, e^{-\pi i/N_2} z_2)$. For a particular choice of the orientifold projections acting on the D3-branes, the gauge group becomes [12]

$$SU(v_1) \times \cdots \times SU(v_{N_2}), \quad (7.1)$$

with the matter content (in $\mathcal{N} = 2$ language)

$$\mathbb{E}_1 + \bigoplus_{j=2}^{N_2} (\mathbb{A}_j - 1, \mathbb{A}_j) + \mathbb{E}_{N_2} + \bigoplus_{j=1}^{N_2} w_j \mathbb{A}_j. \quad (7.2)$$

The $w_j$'s are non-negative integers constrained by $\sum_{j=1}^{N_2} w_j = 4$. The $v_j$'s obey constraints arising from the vanishing of the beta-function(s) of the field theory [12]. The adS/CFT correspondence involves the large $N_1$ limit, where $N_1 \equiv v_1$ denotes the rank of the first factor of the gauge group (denoted by $N$ in the previous section). It can be shown [27] that, to leading order in $N_2/N_1$, the $v_j$'s corresponding to the different gauge group factors are equal: $v_j = N_1 [1 + \mathcal{O}(N_2/N_1)]$. Thus for $N_2 \ll N_1$ the gauge group is essentially SU($N_1$)$^{N_2}$. In the following we will not assume that $N_2 \ll N_1$, although there are some motivations for making this assumption as will be discussed later in this section. The quantum numbers of the scalar fields of the SU×⋯×SU gauge theory are given in the table below.

| CFT field | Representation | $J_R$ | $J_R^3$ | $J_L^3$ | $\Delta$ | $H_B$ |
|-----------|----------------|-------|---------|---------|---------|-------|
| $\phi_r$  | adjoint         | 1     | 0       | 0       | 1       | 1     |
| $\phi_r^*$| adjoint         | -1    | 0       | 0       | 1       | 1     |
| $(A_1, \tilde{A}^1)\dagger$ | □_1 | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (0,1) |
| $(\tilde{A}^1, (A_1)\dagger)$ | □_1 | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (1,2) |
| $(A_2, \tilde{A}^2)\dagger$ | □_{N_2} | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (1,2) |
| $(\tilde{A}^2, (A_2)\dagger)$ | □_{N_2} | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (0,1) |
| $(B_p, C_p^\dagger)$ | □_{p-1}, □_p | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (0,1) |
| $(C_p, B_p^\dagger)$ | □_{p-1}, □_p | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | (1,2) |
| $(Q_r, \tilde{Q}^r)$ | $\frac{1}{2}w_r$ □_r | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | 0 | 1 | $\frac{3}{2}$ |
| $(\bar{Q}_r, Q^r)$ | $\frac{1}{2}w_r$ □_r | 0 | $\frac{1}{2}$, $\frac{1}{2}$ | 0 | 1 | $\frac{3}{2}$ |

Table 4: SU(2)$_R$ multiplets of scalar fields in the $\mathcal{N} = 2$ SU×⋯×SU orientifold theory.

In the table, $r$ takes the values $r = 1, \ldots, N_2$, whereas $p$ ranges from 2 to $N_2$. The quantum number $J_L^3$ in the table above denotes a U(1)$_L$ quantum number. Note that there is generically no SU(2)$_L$ symmetry (only when $N_2 = 1$ is there such a symmetry). The correspondence between our notation and that of ref. [13] is as follows: $J_R^3 \rightarrow J'$, $J_L^3 \rightarrow N_2 J$. 

16
We will now show that there is a limit of the SU×⋯×SU theory discussed above in which the null direction \(x^-\) becomes compact, analogous to the limit discovered in refs. [13, 14]. Although the gauge group of the model is very similar to that of the unorientifolded model considered in refs. [13, 14], the matter content differs; in particular, there are no bifundamental fields connecting the last and first factors of the gauge group.

We consider only Penrose limit B, since it is in this limit that the compact null direction appears. Using the results of section 3, we find that the orbifold identifies \((\varphi_1, w_2)\) with \((\varphi_1 + (\pi/N_2), e^{-i\pi/N_2}w_2)\). This identification along the geodesic translates into \(x^+ \sim x^+ + (\pi/2N_2)\) and \(x^- \sim x^- + (\pi R^2/2N_2)\). As observed in [13], if one take a scaling limit in which \(2\pi R_+ \equiv \pi R^2/2N_2 \sim (g_sN_1/N_2)^{1/2}\alpha'\) stays finite, the \(x^-\) direction becomes compact with radius \(R_+\). Since we obtain a background with a compact null direction the string theory can be quantized using the discrete-light-cone-quantization (DLCQ) scheme, familiar from matrix theory [28].

**Closed string sector**

Since there is no representation connecting the last and first factor of the product gauge group, a gauge invariant operator cannot be constructed from a “loop” of \(H_B = 0\) bifundamental fields \(B_p\), as in ref. [13]. However, we may introduce the two \(H_B = 0\) operators

\[
A_{\alpha\bar{b}} = (A_1B_2\cdots B_{N_2})_{\alpha\bar{b}}, \quad \hat{A}^{\alpha\bar{b}} = (B_2\cdots B_{N_2}\hat{A}^2)_{\alpha\bar{b}} \quad \left\{ \begin{array}{l} a = 1, \ldots, v_1 \\ \bar{b} = 1, \ldots, v_N \end{array} \right. \quad (7.3)
\]

From these, a gauge invariant operator

\[
\text{tr}[(A\hat{A}^T)^k] \quad (7.4)
\]

can be constructed which may be identified with vacuum of the closed string sector. By the reasoning in ref. [13], \(k\) is the momentum labeling the different DLCQ sectors.

In ref. [13], the ground state corresponded to a “loop” of bifundamental fields, wrapping \(k\) times around the quiver/moose diagram. It was suggested that this was related to the T-dual type IIA picture of the gauge theory. In a similar way, the two operators in (7.3) can be identified with two half-circles of a quiver/moose diagram which are identified by the orientifold. This is reminiscent of the type IIA picture of the orientifold model, which contains two O6− planes located at diametrically opposite points of the elliptic circle, which reflect the circle onto itself [12]. It should be possible to connect these two pictures by applying a T-duality as in ref. [13]. (T-dualities of the pp-wave background have also been discussed in ref. [24]).

The operators corresponding to the string theory states may also be obtained using a cover space approach, as in ref. [13] and in sec. 6. Beginning with matrices \(\Phi_i\) in the adjoint representation of an SU(2v) theory, where \(v = \sum_r v_r\), and projecting to the SU(\(v_1\))×⋯×SU(\(v_N\)) theory using a generalization [12] of eqs. (6.4), we obtain

\[
\Phi_1 = \begin{pmatrix} B^T & A_1 \\ -\hat{A}^2 & B \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} C & A_2 \\ \hat{A}^1 & C^T \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^T \end{pmatrix}, \quad (7.5)
\]

where \(\phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_{N_2})\) and

\[
A_1 = \text{diag}(A_1, 0, \ldots, 0), \quad \hat{A}^1 = \text{diag}(\hat{A}^1, 0, \ldots, 0), \\
A_2 = \text{diag}(0, \ldots, 0, A_2), \quad \hat{A}^2 = \text{diag}(0, \ldots, 0, \hat{A}^2),
\]
reduce to those in the previous section when \( N \) different DLCQ vacua. There are two kinds of open strings, as in section 6. (The operators \( A_{\alpha}B_{\beta} \cdot \cdot \cdot B_{\gamma} \), from (6.23), (6.24) from these, one can construct operators corresponding to the open string vacua, in analogy with (7.8).

Since the geodesic in Penrose limit B lies in the orientifold \((O7)\) fixed-plane \( w_3 = 0 \), which is coincident with the D7-branes, there is an open-string sector, in contrast to the model studied in ref. [13]. The model (7.1), (7.2) actually comprises several different models, due to the fact that the fundamentals belong to different factors of the gauge group. To be able to treat all the different cases on the same footing we use the following notation. Assuming that \( Q^I \) (or \( \hat{Q}_I \)) for a particular \( I \) belongs to the \( r \)th factor of the gauge group, we can define

\[
Q^I_{\alpha} = \left( Q^I B_{r+1} \cdots B_{N_2} \right)_{\alpha}, \quad \hat{Q}^I_{\alpha} = \left( B_2 \cdots B_r \hat{Q}_I \right)_{\alpha}.
\]

From these, one can construct operators corresponding to the open string vacua, in analogy with (7.23), (7.24)

\[
Q^I \hat{A}^T \left( \mathcal{A} \hat{A}^T \right)^{(k-1)} A_1 \hat{Q}_K, \quad Q^I \hat{A}^2 \left( \mathcal{A}^T \mathcal{A} \right)^k (Q^K)^T, \quad (\hat{Q}_I)^T \left( \mathcal{A}^T \mathcal{A} \right)^k A_1 \hat{Q}_K.
\]

As above, \( k \) is the non-negative integer lightcone momentum quantum number labeling the different DLCQ vacua. There are two kinds of open strings, as in section 5. (The operators (7.9) reduce to those in the previous section when \( N_2 = 1 \).

Excited states in the open string sector are constructed by inserting \( H_B = 1 \) operators, as in section 5. Since there is no level matching condition in the open string sector, a single non-zero-momentum oscillator acting on the vacuum state is allowed.

\[
B = \begin{pmatrix}
0 & B_2 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & B_{N_2} \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
C_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & C_{N_2} & 0
\end{pmatrix}.
\]
Finally, we will discuss the implications of making the additional assumption $N_2 \ll N_1$. For the original model [2] the 't Hooft coupling is $\lambda' = g_{\text{YM}}^2/N/J^2$. It was recently argued [23] that the effective genus counting parameter in the Yang-Mills theory is $g_2^2 = J^4/N^2$, and the effective coupling between a wide class of excited states is $g_2 \sqrt{\lambda'}$ (see ref. [23] for further details; see also [30] for some refinements). Translating these expressions to our case one finds that both the expansion parameters are small provided $N_2 \ll N_1$, thus from this perspective it is natural to make the additional assumption $N_2 \ll N_1$. Note that this assumption corresponds to a large radius approximation for the compact circle.

8 $\mathcal{N} = 2 \text{Sp}(2N) \times \text{Sp}(2N)$ with $(\square, \square) \oplus 2(1, \square) \oplus 2(\square, 1)$

The $\mathcal{N} = 2 \text{Sp}(2N) \times \text{Sp}(2N)$ theory with an $\mathcal{N} = 2$ hypermultiplet in the bifundamental representation $(\square, \square)$ and two $\mathcal{N} = 2$ hypermultiplets in the fundamental representation of each factor of the gauge group contains (in $\mathcal{N} = 1$ language) two chiral multiplets $\phi_{1a}^b$ and $\phi_{2a}^b$ in the adjoint representation, one for each $\text{Sp}(2N)$ factor. In matrix notation, these fields satisfy

$$\phi_1 = J_1 \phi_1^T J_1, \quad \phi_2 = J_2 \phi_2^T J_2,$$

where $J^{ab} = J_1$ and $J_{ab} = J_1^{-1} = -J_1$ is the symplectic unit of the first $\text{Sp}(2N)$ factor, used to raise and lower indices (and similarly for $J^{\bar{a} \bar{b}} = J_2$). The theory also contains two chiral multiplets in the $(\square, \square)$ representation; for convenience, in writing operators in matrix notation, we represent these with two chiral multiplets $A_{i a}^b$ ($i = 1, 2$) in the $(\square, \overline{\square})$ representation of $\text{Sp}(2N) \times \text{Sp}(2N)$ and two chiral multiplets $B_{i \bar{a}}^b$ ($i = 1, 2$) in the $(\overline{\square}, \square)$ representation, with the constraint

$$B_i = -J_2 A_i^T J_1.$$

In this paper, both $(A_1, A_2)$ and $(B_1, B_2)$ are in the $\square$ of $\text{SU}(2)_L$. (The indices may be raised with the $\epsilon$ tensor, so that $(B^1, B^2) \equiv (B_2, -B_1)$ is in the $\overline{\square}$ of $\text{SU}(2)_L$. In ref. [31] and many other papers, $(B^1, B^2)$ is written as $(B_1, B_2)$.)

Finally, the theory also contains four chiral multiplets in the $(\square, 1)$ and $(1, \square)$ representations, denoted by $Q_{1a}^I$ and $Q_{2a}^I$ ($I = 1, \ldots, 4$), respectively. For convenience, we also introduce fields $\tilde{Q}_{1I}^a$ and $\tilde{Q}_{2I}^a$ in the $(\overline{\square}, 1)$ and $(1, \overline{\square})$ representations, related to the above fields by [31]

$$\tilde{Q}_{1I}^a = -g_{IJ} J_1 Q_1^I, \quad \tilde{Q}_{2I}^a = -g_{IJ} J_2 Q_2^I,$$

where $g_{IJ}$ is the metric for either of the two factors of the $\text{SO}(4) \times \text{SO}(4)$ flavor symmetry group. The quantum numbers of the scalar fields are displayed in the table below.
The superpotential for this theory is
\[ W_{\mathcal{N}=2} = \text{tr} \left[ \phi_1 (A_1 B_1 + A_2 B_2) - \phi_2 (B_1^2 A_1 + B_2^2 A_2) \right] + \tilde{Q}_{1I} \phi_1 Q_{1I} + \tilde{Q}_{2I} \phi_2 Q_{2I}, \] (8.4)
giving rise to the F-term equations
\[
\begin{align*}
A_1 B_2 - A_2 B_1 + Q_{1I}^I \tilde{Q}_{1I} &= 0, \\
A_i \phi_2 - \phi_1 A_i &= 0, \\
\phi_1 Q_{1I}^I &= 0, \\
B_1 A_2 - B_2 A_1 + Q_{2I}^I \tilde{Q}_{2I} &= 0, \\
B_i \phi_1 - \phi_2 B_i &= 0, \\
\phi_2 Q_{2I}^I &= 0. 
\end{align*}
\] (8.5)

The \( \mathcal{N} = 2 \) Sp(2N)\times Sp(2N) gauge theory just discussed arises as the theory on a stack of D3-branes in the same IIB background that gave rise to the SU(N) with \( 2 \square \oplus 4 \square \) theory discussed in section 6, namely, flat space modded out by
\[ G = \mathbb{Z}_2^{\text{orb}} \times \mathbb{Z}_2^{\text{ori}}. \] (8.6)
As in that case, the fixed point set of \( \mathbb{Z}_2^{\text{ori}} \) is the hyperplane \( z_3 = 0 \), which corresponds to the position of the O7-plane and the D7-branes, while the fixed point set of \( \mathbb{Z}_2^{\text{orb}} \) is the six-dimensional hyperplane \( z_1 = z_2 = 0 \).

The fields of the \( \mathcal{N} = 2 \) Sp(2N)\times Sp(2N) theory can be obtained via a projection (different from the one in section 6) from the \( \mathcal{N} = 4 \) SU(4N) gauge theory. The independent generators of the orientifold group are realized as the 4N×4N matrices \( \text{I2N} \)
\[ \gamma_\theta = \left( \begin{array}{cc} \text{I2N} & 0 \\ 0 & -\text{I2N} \end{array} \right), \quad \gamma_{\Omega} = \left( \begin{array}{cc} J & 0 \\ 0 & J \end{array} \right). \] (8.7)

The orbifold projection
\[ \Phi_{1,2} = -\gamma_\theta \Phi_{1,2} \gamma_\theta^{-1}, \quad \Phi_3 = \gamma_\theta \Phi_3 \gamma_\theta^{-1}, \] (8.8)
restricts the adjoint fields, $\Phi_i$, of the $\mathcal{N} = 4$ theory to be of the form

$$\Phi_1 = \begin{pmatrix} 0 & A_1 \\ -B^2 & 0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 0 & A_2 \\ B^1 & 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}. \quad (8.9)$$

The orientifold projection

$$\Phi_{1,2} = \gamma_{\Omega} \Phi_{1,2}^T \gamma_{\Omega}^{-1}, \quad \Phi_3 = -\gamma_{\Omega} \Phi_3^T \gamma_{\Omega}^{-1}, \quad (8.10)$$

is equivalent to the constraints (8.1) and (8.2).

The action of $G$ on the coordinates of $adS_5 \times S^5$ is as follows. $Z_2^{\text{orb}}$ acts as $\varphi_1 \rightarrow \varphi_1 + \pi$, $\varphi_2 \rightarrow \varphi_2 + \pi$ and $Z_2^{\text{ori}}$ acts as $\varphi_3 \rightarrow \varphi_3 + \pi$. String theory on this background is dual to the above Sp$\times$Sp gauge theory [10, 11]. Below we discuss the pp-wave limits of this correspondence.

**Penrose limit A**

In the first Penrose limit, the generator of $Z_2^{\text{ori}}$ simply produces a translation along the geodesic: $\varphi_3 \rightarrow \varphi_3 + \pi$ (together with the action of $\Omega(-1)^F \Omega$) with no action on the transverse coordinates $w_1$ and $w_2$ (3.9). There is no orientifold fixed plane and therefore no open string sector in this Penrose limit, but $Z_2^{\text{ori}}$ projects out half of the discrete values of $J_A$ that would be allowed in the unprojected case. The generator of $Z_2^{\text{orb}}$ acts solely on the transverse coordinates as $w_1 \rightarrow -w_1$, $w_2 \rightarrow -w_2$. The geodesic lies in the orbifold fixed plane, resulting in a twisted sector of the string theory in this pp-wave background.

**Untwisted and twisted sectors**

As in the case of the SU($N$) + $2\square \oplus 4\square$ theory in section 3, the operators in the untwisted and twisted sectors are easily obtained through projections of the cover space fields.

The ground state in the untwisted sector corresponds to the operator $\text{tr}[\Phi_3^J]$ whereas the twisted sector ground state corresponds to the operator $\text{tr}[\gamma_0 \Phi_3^{J_A}]$. The $Z_2^{\text{ori}}$ projection (8.10) implies $\text{tr}[\gamma_0 \Phi_3^{J_A}] = (-)^{J_A} \text{tr}[\gamma_0 (\Phi_3^J)^{J_A}] = (-)^{J_A} \text{tr}[\gamma_0 \Phi_3^{J_A}]$, which vanishes unless $J_A = 2n$ for both the untwisted and twisted sectors. Thus, the $H_A = 0$ operators

$$\text{tr} \left[ \phi_{1}^{2n} \pm \phi_{2}^{2n} \right], \quad (8.11)$$

correspond to the vacuum states in the untwisted (+) and twisted (−) sectors of the string theory.

The $H_A = 1$ gauge theory operators $\text{tr}[D_{\mu} \Phi_3 \Phi_3^{2n+1}]$ and $\text{tr}[\gamma_0 (D_{\mu} \Phi_3) \Phi_3^{2n+1}]$ or, explicitly,

$$\text{tr} \left[ (D_{\mu} \phi_1) \phi_1^{2n+1} \pm (D_{\mu} \phi_2) \phi_2^{2n+1} \right], \quad (8.12)$$

correspond to string states in which a single zero-momentum oscillator $a_0^\dagger \mu$ in one of the directions of $\mathbb{R}^4$ acts on the light-cone vacuum in the untwisted (+) and twisted (−) sectors respectively.

Since the zero-momentum oscillators $a_0^\dagger I$ in the transverse $\bar{\mathbb{R}}^4$ directions (3.3) are odd under $Z_2^{\text{orb}}$, only pairs of such oscillators can act on the light-cone vacuum. The $H_A = 2$ gauge theory operators corresponding to the states involving two oscillators in the untwisted
sector involve the insertion of a pair of $H_A = 1$ fields $\Upsilon_I = (\Phi_1, \Phi_1^\dagger, \Phi_2, \Phi_2^\dagger)$ into the ground state operator (the insertion of a single $J_A = 1$ operator into the ground state operator gives zero using (8.9)):

$$
\sum_{m=0}^{J_A} \text{tr} \left[ \Upsilon_I \Phi_3^m \Upsilon_K \Phi_3^{A-m} \right],
$$

(8.13)

where we average over the relative position of the $\Upsilon$'s [22, 23]. Eq. (8.13) vanishes unless $J_A$ is even. Using (8.9), together with

$$
\Phi_1^\dagger = \left( \begin{array}{cc} 0 & B_1^{i1} \\ A_1^{i1} & 0 \end{array} \right), \quad \Phi_2^\dagger = \left( \begin{array}{cc} 0 & B_1^{i2} \\ A_2^{i2} & 0 \end{array} \right),
$$

(8.14)

we obtain the set of ten $H_A = 2$ operators

$$
\sum_m \text{tr} \left[ A_i \phi_2^m B_j \phi_1^{2n-m} \right], \quad \sum_m \text{tr} \left[ B_i \phi_2^m A_j \phi_1^{2n-m} \right], \quad \sum_m \text{tr} \left[ A_i \phi_2^m A_j \phi_1^{2n-m} + B_i \phi_2^m B_j \phi_1^{2n-m} \right],
$$

(8.15)

in agreement with string theory expectations, cf. (6.12).

The oscillators in the directions $w_1$ and $w_2$ are half-integer moded in the twisted sector [24, 25, 24, 20] so there exist no zero-momentum oscillators in this sector. The operators that would have corresponded to such oscillators acting on the twisted-sector vacuum

$$
\sum_m \text{tr} \left[ \gamma_\theta \Upsilon_I \Phi_3^m \Upsilon_K \Phi_3^{A-m} \right],
$$

(8.16)

must therefore not be chiral primary operators in the gauge theory. Consider for example

$$
\sum_m \text{tr} \left[ A_i \phi_2^m B_j \phi_1^{J_A-m} \right],
$$

(8.17)

Using the F-term equation $A_i \phi_2 = \phi_1 A_i$, $A_i$ can be shifted through $(\phi_2)^m$. Then $A_i B_j$ can be replaced by $Q_1$'s, using $A_1 B_2 - A_2 B_1 + Q_1^I Q_1^{LI} = 0$. Finally, using $\phi_1 Q_1^I = 0$, the operator vanishes and is therefore not a chiral primary. The other operators in (8.16) are similarly expected to be ruled out.

Operators corresponding to non-zero-momentum oscillators can be constructed in analogy with the constructions in refs. [25, 20, 20].

**Penrose limit B**

In limit B, the geometric action of $\mathbb{Z}_2^{\text{ori}}$, $\varphi_3 \to \varphi_3 + \pi$, produces a reflection of the transverse coordinate $w_3 \to -w_3$. The geodesic lies in the orientifold (O7) plane $w_3 = 0$, and the coincident D7-branes give rise to an open-string sector. The generator of $\mathbb{Z}_2^{\text{orb}}$ produces a shift $\varphi_1 \to \varphi_1 + \pi$ halfway around the geodesic, accompanied by a reflection of the coordinates transverse to the O7-plane: $w_2 \to -w_2$. Since the $\mathbb{Z}_2^{\text{orb}}$ action has no fixed points, there is no twisted sector in this Penrose limit, however $\mathbb{Z}_2^{\text{orb}}$ projects out half of the discrete values of $J_B$ that would be allowed in the unprojected case.
**Closed string sector**

The vacuum state of the closed string sector corresponds to the $H_B = 0$ operator $\text{tr}[\Phi_1^{\dagger \mu}]$. Due to the $\mathbb{Z}_2^{\text{orb}}$ projection on $\Phi_1$, this is only nonvanishing when $J_B = 2n$,

$$\text{tr}[\Phi_1^{2n}] = \text{tr}[(A_1B_1)^n]. \quad (8.18)$$

String states in which a zero-momentum oscillator $a_0^{\dagger \mu}$ in one of the directions of $\mathbb{R}^4$ acts on the light-cone vacuum correspond to the $H_B = 1$ gauge theory operators

$$\text{tr}[(D_\mu \Phi_1)\Phi_1^{2n+1}] = \text{tr}[D_\mu(A_1B_1)(A_1B_1)^n]. \quad (8.19)$$

String states in which a zero-momentum oscillator $a_0^{\dagger}, a_0^{\dagger}$ in the $w_2, \bar{w}_2$ directions of $\mathbb{R}^4$ acts on the light-cone vacuum corresponds to insertions of the $H_B = 1$ fields $\Phi_2\Phi_1$, $\Phi_2^{\dagger}\Phi_1$ into the vacuum state operator:

$$\text{tr}[\Phi_2\Phi_1^{2n+1}] = \text{tr}[(A_2B_1 + A_1B_2)(A_1B_1)^n],$$

$$\text{tr}[\Phi_2^{\dagger}\Phi_1^{2n+1}] = \text{tr}[(B_1^{\dagger}B_1 + A_1A_1^{\dagger})(A_1B_1)^n]. \quad (8.20)$$

The linear combination $A_2B_1 - A_1B_2$ is ruled out since it is not a chiral primary; see section 3 for further details.

The zero-momentum oscillators in the $w_3, (\bar{w}_3)$ directions of $\mathbb{R}^4$ are odd under $\mathbb{Z}_2^{\text{orb}}$, so there are no states corresponding to single oscillators acting on the vacuum. Correspondingly, the gauge theory operator $\text{tr}[\Phi_3\Phi_1^{\dagger \mu}]$, resp. $\text{tr}[\Phi_3^{\dagger}\Phi_1^{\dagger \mu}]$, vanishes, as can be seen by taking the transpose of $[\phi_1(A_1B_1)^n]$, resp. $[\phi_1^{\dagger}(A_1B_1)^n]$, and using (8.18) and (8.21).

**Open string sector**

Since the geodesic lies in the orientifold (O7) fixed-plane $w_3 = 0$ which is also the location of the D7-branes, there is an open-string sector. The gauge symmetry of the D7-branes in the presence of the O7-plane, SO(8)$_F$, is broken to SO(4)$\times$SO(4) by the $\mathbb{Z}_2^{\text{orb}}$ projection. The ground state of the open string sector, which transforms in the adjoint 28 of SO(8)$_F$, decomposes into

$$28 \rightarrow (6, 1) \oplus (1, 6) \oplus (4, 4). \quad (8.21)$$

The $\mathbb{Z}_2^{\text{orb}}$ projection correlates the states in the $(6, 1) \oplus (1, 6)$ representation with operators in odd-dimensional representations of SU(2)$_L$ [11]. These open string vacuum states therefore correspond to gauge theory operators with an even number of $A$’s and $B$’s. The natural candidates are

$$Q_1^{[I}J_1(A_1B_1)^nQ_1^{K]}, \quad Q_2^{[I}J_2(B_1A_1)^nQ_2^{K]}. \quad (8.22)$$

These operators have $H_B = 1$, reflecting the non-vanishing zero point energy for the open string sector [3]. The antisymmetry of the flavor indices $I, K$ is enforced by the antisymmetry of the matrices $J_1(A_1B_1)^n$ and $J_2(B_1A_1)^n$ (which follows from eq. (8.21)) and guarantees that the operators (8.22) belong to the correct representation of the flavor group.

The $\mathbb{Z}_2^{\text{orb}}$ projection correlates the states in the $(4, 4)$ representation with operators in even-dimensional representations of SU(2)$_L$ [11]. These open string vacuum states therefore
correspond to the gauge theory operators with an odd numbers of $A$’s and $B$’s. The natural candidate is

$$Q_I^1 J_1(A_1 B_1)^n A_1 Q_K^2,$$  \hspace{1cm} (8.23)

which has $H_B = 1$ and belongs to the $(4, 4)$ representation of the flavor group. (The operator $Q_2^K J_2 B_1(A_1 B_1)^n Q_1^1$ is equivalent to (8.23) using the orientifold projections (8.2).)

Zero-momentum oscillators in the $\bar{w}_2$, $\bar{w}_2$ directions of $\tilde{I} \mathbb{R}^4$ acting on the open string vacuum correspond to the insertion of the $H_B = 1$ fields $A_2 B_1 + A_1 B_2$ and $B_1^\dagger B_1 + A_1 A_1^\dagger$ into the operators (8.22) and (8.23).

Zero-momentum oscillators in the $\bar{w}_3$, $\bar{w}_3$ directions of $\tilde{I} \mathbb{R}^4$ (which are the directions transverse to the D7-branes) are absent from the open string spectrum [8], so the corresponding operators on the gauge theory side should be absent as well. This follows from the fact that, as in section 6, the insertion of $\phi_{1,2}$ (which has $H_B = 1$) can be shown to give rise to an operator that vanishes by the F-term equations. It is therefore not a chiral primary operator and is consequently ruled out, in agreement with the string theory result [8].

9 The $\mathcal{N} = 2$ $Sp \times SU \times \cdots \times SU \times Sp$ orientifold

The $Sp(2N) \times Sp(2N)$ theory discussed in section 8 is the first in an infinite series of conformal models with gauge groups $Sp \times SU \times \cdots \times SU \times Sp$ [12]. The orientifold group for these theories has the form $G = \mathbb{Z}_2^{\text{ori}} \times \mathbb{Z}_2^{\text{orb}}$. As before, $\mathbb{Z}_2^{\text{ori}} = \{1, \Omega(-1)^{F_L R_{45}}\}$ and $\mathbb{Z}_2^{\text{orb}} = \{1, \theta, \ldots, \theta^{2N_2 - 1}\}$. The action of the orbifold on the coordinates transverse to the D3-branes is the same as in section 6. For a particular choice of the orientifold projection acting on the D3-branes (different from the one in section 4) the gauge group becomes [12]

$$Sp(v_0) \times SU(v_1) \times \cdots \times SU(v_{N_2 - 1}) \times Sp(v_{N_2}),$$  \hspace{1cm} (9.1)

and the matter content is (in $\mathcal{N} = 2$ language)

$$\bigoplus_{j=1}^{N_2} (\Box_{j-1}, \Box_j) + \frac{1}{2} w_0 \Box_0 + \bigoplus_{j=1}^{N_2-1} w_j \Box_j + \frac{1}{2} w_{N_2} \Box_{N_2}. $$  \hspace{1cm} (9.2)

The $w_j$’s are non-negative integers constrained by the equation $\frac{1}{2}(w_0 + w_{N_2}) + \sum_{j=1}^{N_2-1} w_j = 4$. The $v_j$’s obey constraints arising from the vanishing of the beta-function(s) of the field theory [12]. To leading order in $N_2/N_1$, where $N_1 (\equiv v_0/2)$ equals the rank of the first factor of the product gauge group, the $v_j$’s corresponding to each of the group factors are equal and take the value $2N_1$ [27], and the gauge group is essentially $Sp(2N_1) \times SU(2N_1)^{N_2-1} \times Sp(2N_1)$. We will not make any assumptions about the magnitude of the ratio $N_2/N_1$; see, however, the discussion in section 6. The quantum numbers of the scalar fields of the $Sp \times SU \times \cdots \times SU \times Sp$ gauge theory (9.1) are given in the table below.
In the table, \( r \) takes the values \( r = 0, \ldots, N_2 \), whereas \( p \) ranges from 1 to \( N_2 \). The quantum number \( J^3_L \) in the table above denotes a \( U(1)_L \) quantum number; generically there is no \( SU(2)_L \) symmetry (except when \( N_2 = 1 \)).

Scaling \( N_1 \) and \( N_2 \) to infinity together in Penrose limit \( B \), as in sec. 7, we obtain a background with a compact null direction, in which the string theory can be quantized using the DLCQ scheme, as in ref. [13].

### Closed string sector

Since there is no bifundamental representation connecting the last and first factors of the gauge group, one cannot construct a gauge invariant operator as a “loop” of \( H_B = 0 \) bifundamental fields. However, we may define

\[
A_a^b = (A_1 A_2 \cdots A_{N_2})_a^b, \quad a = 1, \ldots, v_0, \quad b = 1, \ldots, v_{N_2},
\]

and then use the symplectic units \( J_1 \) and \( J_2 \), of \( Sp(v_0) \) and \( Sp(v_{N_2}) \) respectively, to define \( B_a^b \) by \( B = J_2 A^T J_1 \). In terms of these two operators, a gauge invariant \( H_B = 0 \) operator may be constructed

\[
\text{tr}[(AB)^k],
\]

which may be identified with the closed string ground state. Here \( k \) is an arbitrary non-negative integer which labels the DLCQ momentum states as in [13].

The pictorial interpretation of this operator is as a “loop” around a quiver/moose diagram, with two half-circles related via the orientifold projection (again reminiscent of the type IIA picture of the gauge theory).

As in section 7, one can write the gauge invariant operators corresponding to the vacuum states (9.4) as well as the excited states in terms of cover space fields.

### Open string sector

The open string vacuum is constructed out of a string of \( H_B = 0 \) operators with two \( Q \)'s at the ends, exactly as in section 7. Assuming that \( Q^I \) (or \( \tilde{Q}^I \)) for a particular \( I \) belongs to the \( r \)th factor of the gauge group, we can define

\[
Q^I_a = (A_1 \cdots A_r Q^I)_a, \\
\tilde{Q}^I_a = (\tilde{Q}^I A_{r+1} \cdots A_{N_2})^a.
\]
From these, one can construct operators corresponding to the open string vacua, in analogy with (8.22), (8.23)

\[(Q^I)^T J_1 (AB)^k Q^K, \quad \bar{Q}_I (BA)^k J_2 (\bar{Q}_K)^T, \quad \bar{Q}_I B (AB)^k Q^K. \quad (9.6)\]

Further details can be found in section 4.

In addition to the series of models studied in this section and in section 7, there is another infinite series of models containing D7-branes, with gauge group $\text{Sp} \times \text{SU} \times \cdots \times \text{SU} \times \text{SU}$. These models can also be treated using the methods of this paper, and possess both a closed and open string sector in the Penrose limit.

In ref. [12] four additional infinite series of models were constructed that do not contain D7-branes. It should also be possible to analyze the pp-wave limits of these models.

10 \( \mathcal{N} = 1 \text{ Sp}(2N) \times \text{Sp}(2N) \) with \( 2(\square, \square) \oplus 4(1, \square) \oplus 4(\square, 1) \)

In the previous sections, we have discussed various \( \mathcal{N} = 2 \) theories and their associated Penrose limits. It is of interest to extend these results to \( \mathcal{N} = 1 \) theories. The pp-wave limit associated with the duality between the \( \mathcal{N} = 1 \) SU(\(N\)) × SU(\(N\)) theory and string theory on \( \text{adS}_5 \times T^{11} \) was treated in refs. [24, 33, 34]. Orbifolds of this theory and their Penrose limits were discussed in ref. [35]. Various \( \mathcal{N} = 1 \) orbifolds of \( \text{adS}_5 \times S^5 \) have been discussed in ref. [36, 17, 35]; recently non-supersymmetric models have also been discussed [36].

In this section we discuss a particular orientifolded version of the \( \text{adS}_5 \times T^{11} \) model and its Penrose limit. By studying this model we are able to test the ideas of ref. [8] in a more complicated example. The orientifold we study was previously discussed in ref. [27, 31, 37]. The field theory dual is an \( \mathcal{N} = 1 \) Sp(\(2N\)) × Sp(\(2N\)) field theory with matter content (in \( \mathcal{N} = 1 \) language) \( 2(\square, \square) \oplus 4(1, \square) \oplus 4(\square, 1) \). As in ref. [31] and in sec. 8, for notational clarity the bifundamental chiral multiplets are denoted using the doubled set of fields \( A_{ia}^b \) (\( i = 1, 2 \)) and \( B_{ib}^a \) (\( i = 1, 2 \)), with the constraint

\[ B^1 = -J_2 A_T^2 J_1, \quad B^2 = J_2 A_T^1 J_1. \quad (10.1) \]

The chiral multiplets in the \( (\square, 1) \) and \( (1, \square) \) representations will be denoted by \( Q_{1a}^I \) and \( Q_{2a}^I \) (\( I = 1, \ldots, 4 \)), respectively. For convenience, we also introduce fields \( \tilde{Q}_{1I}^a \) and \( \tilde{Q}_{2I}^a \), related to \( Q_{1a}^I \) and \( Q_{2a}^I \) by [31]

\[ \tilde{Q}_{1I}^a = -g_{II} J_1 Q_{1I}^I, \quad \tilde{Q}_{2I}^a = -g_{II} J_2 Q_{2I}^I. \quad (10.2) \]

where \( g_{II} \) is the metric for either of the two factors of the SO(4) × SO(4) flavor symmetry group. The quantum numbers of the scalar fields and their complex conjugates are displayed in the table below.

\(^{4}\)In ref. [31], \( (B^1, B^2) \) was written as \( (B_1, B_2) \).
The superpotential for the $\mathcal{N} = 1 \text{Sp}(2N) \times \text{Sp}(2N)$ theory is given by
\begin{equation}
W_{\mathcal{N}=1} = -\left[ \text{tr}(A_1 B_1^2 A_2 B^2 - B_1^2 A_1 B^2 A_2) + \frac{1}{2} Q_1 i Q_1^I \tilde{Q}_{1I} i Q^I - \frac{1}{2} Q_{2I} Q_{2I}^I \tilde{Q}_{2I} Q^I \right. \\
+ \tilde{Q}_{1I} (A_1 B_1^2 + A_2 B^2) Q^I - \tilde{Q}_{2I} (B_1^2 A_1 + B^2 A_2) Q^I \Bigg] . \tag{10.3}
\end{equation}
from which follow the independent F-term equations
\begin{align}
B_1^2 A_2 B^2 - B_1^2 A_2 B^2 + B_1^2 Q_1^I \tilde{Q}_{1I} - Q_2^I \tilde{Q}_{2I} B_1^2 &= 0, \\
B_2 A_1 B^2 - B_2 A_1 B^2 + B_2 Q_1^I \tilde{Q}_{1I} - Q_2^I \tilde{Q}_{2I} B_2^2 &= 0, \\
(A_1 B_1^2 + A_2 B^2 + Q_1^I \tilde{Q}_{1I}) Q_1^I &= 0, \\
(B_1^2 A_1 + B^2 A_2 + Q_2^I \tilde{Q}_{2I}) Q_2^I &= 0. \tag{10.4}
\end{align}
Additional relations can be obtained by taking the transposes of eqs. (10.4).

The $\mathcal{N} = 1 \text{Sp}(2N) \times \text{Sp}(2N)$ gauge theory just discussed arises as the theory on a stack of D3-branes at a conifold singularity modded out by a $\mathbb{Z}_2^\text{ori}$ orientifold projection. In the near-horizon limit, the geometry becomes $\text{adS}_5 \times T^{11}/\mathbb{Z}_2^\text{ori}$. The metric on $\text{adS}_5$ is given by eq. (22); the metric on $T^{11}$ is
\begin{equation}
ds^2_{T^{11}} = \frac{1}{9}(d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2). \tag{10.5}
\end{equation}
The orientifold acts by interchanging the two spheres [27, 31]: $(\varphi_1, \theta_1) \leftrightarrow (\varphi_2, \theta_2)$. The fixed point set $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$ represents the position of the O7-plane and D7-branes.

**The Penrose limit**

Parameterizing the angular variables in eq. (10.5) as [24, 33, 34]
\begin{equation}
\varphi_i = \xi - \chi_i, \quad \psi = \xi + \chi_1 + \chi_2, \quad \theta_i = \frac{\sqrt{6}}{R} y_i, \tag{10.6}
\end{equation}
introducing
\begin{equation}
x^+ = \frac{1}{2}(t + \xi), \quad x^- = \frac{1}{2} R^2(t - \xi), \quad r = R\rho, \tag{10.7}
\end{equation}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
Field & Representation & $J_R$ & $J_R^\pm$ & $\Delta$ & $H$ & Field & Representation & $J_R$ & $J_R^\pm$ & $\Delta$ & $H$
\hline
$A_1$ & $(\Box, \Box)$ & $\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$ & $(A_1)^\dagger$ & $(\Box, \Box)$ & $-\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$
\hline
$A_2$ & $(\Box, \Box)$ & $\frac{1}{2}$ & $-\frac{1}{2}$ & $\frac{3}{2}$ & $1$ & $(A_2)^\dagger$ & $(\Box, \Box)$ & $-\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $1$
\hline
$B^1$ & $(\Box, \Box)$ & $\frac{1}{2}$ & $-\frac{1}{2}$ & $\frac{3}{2}$ & $1$ & $(B^1)^\dagger$ & $(\Box, \Box)$ & $-\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $1$
\hline
$B^2$ & $(\Box, \Box)$ & $\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$ & $(B^2)^\dagger$ & $(\Box, \Box)$ & $-\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$
\hline
$Q^I_1$ & $(\Box, 1)$ & $0$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$ & $(Q^I_1)^\dagger$ & $(\Box, 1)$ & $0$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$
\hline
$Q^I_2$ & $(\Box, 1)$ & $\frac{1}{2}$ & $0$ & $\frac{3}{2}$ & $0$ & $(Q^I_2)^\dagger$ & $(\Box, 1)$ & $\frac{1}{2}$ & $0$ & $\frac{3}{2}$ & $0$
\hline
$\tilde{Q}^I_1$ & $(1, \Box)$ & $\frac{1}{2}$ & $0$ & $\frac{3}{2}$ & $0$ & $(\tilde{Q}^I_1)^\dagger$ & $(1, \Box)$ & $\frac{1}{2}$ & $0$ & $\frac{3}{2}$ & $0$
\hline
$\tilde{Q}^I_2$ & $(1, \Box)$ & $0$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$ & $(\tilde{Q}^I_2)^\dagger$ & $(1, \Box)$ & $0$ & $\frac{1}{2}$ & $\frac{3}{2}$ & $0$
\hline
\end{tabular}
\caption{Scalar fields in the $\mathcal{N} = 1 \text{Sp}(2N) \times \text{Sp}(2N)$ theory with $2(\Box, \Box) \oplus 4(1, \Box) \oplus 4(\Box, 1)$.}
\end{table}
and then taking the $R \to \infty$ limit gives the standard pp-wave metric:

$$\begin{align}
\text{ds}^2 &= R^2 \left( ds_{adS_5}^2 + ds_{T^{11}}^2 \right), \\
&\to -4 dx^+ dx^- - (r^2 + y_1^2 + y_2^2)(dx^+)^2 + dr^2 + r^2 d\Omega_5^2 + dy_1^2 + y_1^2 d\chi_1^2 + dy_2^2 + y_2^2 d\chi_2^2.
\end{align}$$ (10.8)

The orientifold survives the limit and acts by interchanging $(y_1, \chi_1)$ and $(y_2, \chi_2)$. The D7-branes also survive the limit and are located at the fixed point of the orientifold, i.e. at $y_1 = y_2, \chi_1 = \chi_2$. This gives rise to an open-string sector of the string theory in the Penrose limit. (Contrary to the situation for the $N = 2$ models, there does not appear to exist a limit in which the D7-branes disappear.)

By virtue of eqs. (10.1). It follows that $(N_{11} + N_{22})J_2$ is symmetric whereas $N_{12}J_2, N_{21}J_2$ and $(N_{11} - N_{22})J_2$ are antisymmetric. This implies that the number of insertions of $N_{11} + N_{22}$ into the vacuum state $\text{tr}(N_{12})^n$ must be even. This result corresponds to the fact that the coordinate on the string theory side corresponding to $(N_{11} + N_{22})$ changes sign under the...
orientifold \[31\]. Similarly, the other operator which changes sign under the orientifold is \((A_2)^\dagger A_1 + B^2(B^1)^\dagger\); only an even number of insertions of this operator is possible. This follows from the fact that \([(A_2)^\dagger A_1 + B^2(B^1)^\dagger]J_2\) is symmetric due to eq. \((10.1)\). We have thus shown that the field theory reproduces the string theory result.

**Open string sector**

There are three possible open string vacua formed from an \(H = 0\) operator with two “quarks” at the ends:

\[
Q_1^i J_1(A_1 B^2)^n Q_1^K, \quad Q_2^j J_2(B^2 A_1)^n Q_2^K, \quad Q_1^i J_1(A_1 B^2)^n A_1 Q_2^K.
\]  

(10.12)

Because of the antisymmetry of \(J_2(B^2 A_1)^n\) and \(J_1(A_1 B^2)^n\), the first two operators in \((10.12)\) are antisymmetric in the interchange of \(I\) and \(K\), and therefore belong to the adjoint \(6\) of SO(4). This is consistent with the fact that the flavor symmetry group for the D7’s is SO(4)×SO(4), with the first two operators in \((10.12)\) corresponding to the two independent SO(4) factors (two stacks of D7-branes). The last operator in \((10.12)\) belongs to the \((4, 4)\) representation of SO(4)×SO(4) and represents the vacuum states for strings connecting the two stacks of D7-branes.

As in the closed string sector, excitations of the open string correspond to insertions of \(H = 1\) operators into the open string vacua \((10.12)\). Operators corresponding to insertions of zero-momentum oscillators in the directions transverse to the D7-branes should be absent however \[8\]. As an example of this, consider the insertion of \(N_{11} + N_{22}\) into \(Q_2^i J_2 N_{12}^{n_2} Q_2^K\). We will now show that such an operator vanishes by using the F-term equations. From the F-term equations \((10.4)\), it follows that \(N_{11} + N_{22} + Q_2^i Q_2^i\) commutes with all the \(N_{ij}\)’s. Therefore \(N_{11} + N_{22}\) can be commuted past \(N_{12}\) modulo a \(Q^2\) term. This \(Q^2\) term splits the operator into a “multiple trace” operator of the form \([Q_2 \cdots Q_2][Q_2 \cdots Q_2]\), which is subleading in the \(1/N\) expansion. (It is natural to interpret this as a splitting of the open string into two.) Thus, \(N_{11} + N_{22}\) can be moved to the end of the operator, where, again by the F-term equations \((10.4)\), it gives zero when acting on the \(Q_2\) (again modulo a \(Q^2\) term, which gives a subleading contribution). Since the operator vanishes (modulo subleading terms) when using the F-term equations it is not a protected operator, consequently the corresponding zero-momentum oscillator is absent. Similarly, we expect that an insertion of \((A_2)^\dagger A_1 + B^2(B^1)^\dagger\) should give rise to an unprotected operator, reflecting the absence of the corresponding string zero-momentum state. Thus the field theory reproduces the expected string theory result \[8\].

**11 Summary**

In this paper we have studied the pp-wave limits of a variety of elliptic models with O7-planes and D7-branes, as well as the pp-wave limit of a particular orientifold of \(adS_5 \times T^{11}\). The theories with D7-branes have a Penrose limit with both open and closed string sectors.

For most cases, two different Penrose limits were discussed: limit A, which gives an orbifolded pp-wave and which thus contains both an untwisted and a twisted sector; and limit B, which gives an orientifold of the pp-wave with D7-branes, and which thus contains both closed and open string sectors. In limit A (B) a remnant of the orientifold (orbifold) part
of the full orientifold group is manifested in the classification of the chiral and “near-chiral”
primaries of the theories.

The two infinite series of elliptic models discussed in sections 7 and 8 allow pp-wave
scaling limits analogous to the one discussed in refs. [13, 14], in which there is a compact
light-cone null direction. The models that we discuss contain both open and closed strings.

In section 10 by studying a particular orientifold of $AdS_5 \times T^{11}$ we were able to test the
ideas of ref. [8] in a more complicated example and agreement was found between the string
and field theory results.

We have not touched upon the important topic of interactions, but it would clearly be of
interest to gain a better understanding of the interactions between closed and open strings
and to elucidate the splitting of open strings. For some recent work on interactions in the
pp-wave background and in the dual gauge theory, see e.g. refs. [22, 23, 30, 39].

Acknowledgement

HJS would like to thank the string theory group and Physics Department of Harvard Uni-
versity for their hospitality extended over a long period of time.

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