A VERY AMPLENESS RESULT

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Abstract. Let \((M, L)\) be a polarized manifold. The aim of this paper is to establish a connection between the generators of the graded algebra \(\bigoplus_{i \geq 1} H^0(M, iL)\) and the very ampleness of the line bundle \(rL\). Some applications are given.

1. Introduction

Let \(L\) be an ample line bundle on an algebraic manifold \(M\), the problem of finding the least \(n\) such that \(nL\) is very ample is a basic one in the classification theory of polarized varieties. Many attempts were made in order to establish a general formula. A fundamental result due to Matsusaka [9] says that there is a constant \(c\), depending only on the Hilbert polynomial of \((M, L)\), such that \(cL\) is very ample. Moving in another direction Fujita conjectured [3, §2, Conjecture b] that \(K_M + (n + 2)L\) is very ample and \(K_M + (n + 1)L\) is spanned for every polarized manifold \((M, L)\) with \(n = \text{dim } M\). In this paper we consider the graded algebra:

\[
G(M, L) = \bigoplus_{i \geq 1} H^0(M, iL).
\]

Now the generators of this algebra describe the whole structure of \((M, L)\). They also allows us to determine the embedding of \(M\) via the linear system \(|rL|\) for \(r \gg 0\), so there must be a connection between these generators and the very ampleness of \(rL\). The aim of this paper is pointing out this connection. The paper is organized as follows: in section 2 we prove the main theorem asserting the very ampleness of \(rL\), outside the base locus of \(L\), for \(L\) a \(r\)-generated line bundle. This also implies the very ampleness of \(2L\) for any ample \(2\)-generated line bundle \(2,7\). In section 3 some applications of the main theorem \(2.7\) are given, concerning: a) sectionally hyperelliptic polarized varieties of type \((-\))\), b) surfaces of general type, c) del Pezzo surfaces.

2. The main theorem

All notation used in this paper are standard in algebraic geometry. Let \(L\) be a line bundle on a projective manifold \(M\) and consider the associated graded algebra \(G(M, L)\). The following definition generalizes the notion of simply generated line bundle.

Definition 2.1. Let \(r \geq 1\) be an integer. A line bundle \(L\) on \(M\) is \(r\)-generated if \(G(M, L)\) is generated by the sections of \(H^0(M, L), \ldots, H^0(M, rL)\). We also say that the pair \((M, L)\) is \(r\)-generated.
Theorem 2.2. Given a polarized manifold \((M, L)\) with \(L\) effective and \(r\)-generated, then \(\varphi_{|rL|}\) is an embedding of \(M \setminus Bs \mid L \mid\).

Proof. Set \(N = M \setminus Bs \mid L \mid\) and \(\varphi = \varphi_{|rL|}\). First of all we observe that \(\varphi\) is well defined in \(N\), since \(L\) is spanned on \(N\) and hence so are its multiples.

By hypothesis there is an integer \(k \geq 1\) such that \(kL\) is very ample. Take \(p, q \in N\) and by contradiction suppose that no section of \(H^0(iL)\) with \(1 \leq i \leq r\) separates \(p\) and \(q\). Then each section \(s \in H^0(M, iL)\) that vanishing at \(p\) must vanish at \(q\) too.

Consider two sections \(s_1, s_2 \in H^0(iL)\), and define \(\gamma, \lambda\) so that \(s_1(p) = \gamma s_1(q) e s_2(p) = \lambda s_2(q)\). If we take \(s = s_1(p)s_2 - s_2(p)s_1\) we have that \(s(p) = 0\) and necessarily \(0 = s(q) = s_1(q)s_2(q)(\gamma - \lambda)\).

This implies that \(\gamma = \lambda\). We note that if one of the two sections vanishes at \(q\) it must vanish also at \(p\), otherwise it will separate \(p\) and \(q\), but in this situation we may equally take \(\gamma = \lambda\).

The previous argument shows that there exist some constants \(\gamma_i\) associate to each \(H^0(iL)\) for \(1 \leq i \leq r\) such that for each section \(s \in H^0(iL)\) we have \(s(p) = \gamma_i s(q)\).

Now note that since \(G(M, L)\) is an algebra, the constants \(\gamma_i\) have to satisfy the relations:

\[ \gamma_i = \gamma^i \text{ where } \gamma := \gamma_1. \]

For, since \(L\) is effective there exists \(\sigma \in H^0(L)\), then \(\sigma^i(p) = \gamma^i \sigma^i(q)\), but \(\sigma^i \in H^0(iL)\) so we have also \(\sigma^i(p) = \gamma_i \sigma^i(q)\) and this implies the previous equation.

Now let us consider a section \(s \in H^0(kL)\). By the \(r\)-generation hypothesis the sections of \(H^0(M, kL)\) are linear combinations of products of sections in \(H^0(iL)\) with \(1 \leq i \leq r\). Then we can write \(s = \sum \xi_j s_j\) where \(\xi_j \in \mathbb{C}\) and \(s_j = \prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)\) with \(\sigma_{n_i} \in H^0(iL)\), where \(\alpha_i\) is the number of sections of \(H^0(iL)\) that appears in the product. Note that \(\sum_{i=0}^{r} i\alpha_i = k\) because \(s_j \in H^0(kL)\). We have

\[ s(p) = \sum \xi_j s_j(p) = \sum \xi_j (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p)) = \sum \xi_j (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \gamma^i \sigma_{n_i}(q)) = \]

\[ = \sum \xi_j \prod_{i=1}^{r} (\gamma^{i\alpha_i} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q)) = \sum \xi_j (\sum_{i=1}^{r} \gamma^{i\alpha_i} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q)) = \]

\[ = \sum \xi_j \gamma^k (\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(q)) = \gamma^k \sum \xi_j s_j(q) = \gamma^k s(q). \]

We thus obtained that the sections of \(H^0(kL)\) do not separate points, which is absurd. Then there exists a section \(\sigma \in H^0(iL)\) which separates \(p\) and \(q\), i.e. \(\sigma(p) = 0\) and \(\sigma(q) \neq 0\). Since \(L\) is spanned outside \(Bs \mid L \mid\) we may take a section \(\delta \in H^0(L)\) such that \(\delta(q) \neq 0\). Now the section \(\sigma \delta^{r-i} \in H^0(rL)\) separates \(p\) and \(q\).
Corollary 2.3. Theorem 2.2 immediately gives the following.

In order to show that the map \( \varphi \) is an immersion at each point \( p \in N \) we have to show that for each vector \( \tau \in T_pM \), there exists a section \( \sigma \in H^0(iL) \) such that \( ds(\tau) \neq 0 \) and \( \sigma(p) = 0 \). By contradiction suppose that such a section does not exist, then for each \( 1 \leq i \leq r \) there exists a constant \( \eta_i \) such that for each \( s \in H^0(iL), ds(\tau) = \eta_is(p) \). To show this let \( s_1, s_2 \in H^0(iL) \) and consider \( \alpha, \beta \) such that \( ds_1(\tau) = \alpha s_1(p) \) and \( ds_2(\tau) = \beta s_2(p) \). Now consider the section \( s = s_1(p)s_2 - s_2(p)s_1; \) we have \( s(p) = 0 \) and \( 0 = ds(\tau) = s_1(p)s_2(p)(\beta - \alpha) \). This implies \( \alpha = \beta \). Define \( \eta := \eta_1 \). Take \( \sigma^i \in H^0(iL) \) then \( \eta_i\sigma^i(p) = d[\sigma^i](\tau) = id\sigma(\tau)\sigma^{-1}(p) = i\eta\sigma^i(p) \). Hence we have:

\[ \eta_i = i\eta. \]

Now for a section \( s \in H^0(kL) \) we have:

\[
ds(\tau) = \sum \xi_j d\xi_j(\tau) = \sum \xi_j d[\prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}](\tau) = \sum \xi_j \left( \sum_{h=1}^{r} d[\prod_{n_h=1}^{\alpha_h} \sigma_{n_h}](\tau) \prod_{i=1, i \neq h}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) \right) = \sum \xi_j \left( \sum_{h=0}^{r} h\eta_1 \prod_{n_h=1}^{\alpha_h} \sigma_{n_h}(p) \prod_{i=1, i \neq h}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) \right) = \sum \xi_j \left( \sum_{h=1}^{r} h\alpha_h\eta_1 \prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) \right) = \sum \xi_j \left( \sum_{h=1}^{r} h\alpha_h\eta_1 \prod_{i=1}^{r} \prod_{n_i=1}^{\alpha_i} \sigma_{n_i}(p) \right) \]

This implies that the sections of \( H^0(kL) \) do not separate \( p \) from the vector \( \tau \), but this contradicts the very ampleness of \( kL \).

\[ \Box \]

Theorem 2.2 immediately gives the following

**Corollary 2.3.** Let \( (M, L) \) a polarized manifold with \( L \) spanned and \( r \)-generated, then \( rL \) is very ample.

Note that theorem 2.2 cannot be inverted. The following examples show very ample line bundles \( L \) which are not \( 1 \)-generated.

**Example 2.4.** Let \( (S, L) \) be an abelian surface polarized by a very ample line bundle. Let \( L^2 = 2d \), since \( S \) is abelian \( K_S \) is the trivial bundle, so we have \( h^i(L) = h^i(K_S + L) = 0 \) for
It follows that $h^0(L) = \chi(L) = \chi(O_S) + \frac{L^2 - LK_S}{2} = d$. Let $S^2(H^0(L))$ denote the second symmetric power of $H^0(L)$. Then:

$$\dim S^2(H^0(L)) = \frac{d(d+1)}{2}.$$ 

Now if $L$ is simply generated we have $S^2(H^0(L)) \cong H^0(2L)$, but $h^0(2L) = \chi(O_S) + \frac{4L^2 - 2LK_S}{2} = 4d$. So if $d \leq 6$ we have $h^0(2L) = 4d > \frac{d(d+1)}{2} = S^2(H^0(L))$. It is well known that there exist abelian surfaces of degree $2d = 10$ in $\mathbb{P}^3$.

Example 2.5. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, we have Pic($Q$) $\cong \mathbb{Z} \times \mathbb{Z}$. Let $L_1, L_2 \in \text{Pic}(Q)$ be two generators; then $2L_1 + 4L_2$ is very ample [II, ch. 2.18, p. 380]. So there exists a smooth curve $C \subset 2L_1 + 4L_2$. Now consider the polarized curve $(C, L)$, where $L = O_{\mathbb{P}^3}(1)_C$. Since $C$ is smooth we have: $2g(C) - 2 = (2L_1 + 4L_2 + K_2)(2L_1 + 4L_2) = (2L_1 + 4L_2 - 2L_1 - 2L_2)(2L_1 + 4L_2) = 2L_2(2L_1 + 4L_2) = 4$. Hence $g(C) = 3$ and $\deg L = (L_1 + L_2)(2L_1 + 4L_2) = 6 > 2g(C) - 1$. Then $h^1(L) = 0$ and $h^0(L) = 4$. From this it follows that: $\dim S^2(H^0(L)) = h^0(2L) = 10$. Now consider the exact sequence:

$$0 \rightarrow \mathcal{I}_C \rightarrow O_{\mathbb{P}^3} \rightarrow O_C \rightarrow 0$$

Tensoring with $O_{\mathbb{P}^3}(2)$ and taking cohomology, we obtain:

$$0 \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(O_{\mathbb{P}^3}(2)) \rightarrow H^0(2L) \rightarrow \ldots .$$

Since $C$ is contained in a quadric it follows that $H^0(\mathcal{I}_C(2)) \neq 0$, and recalling that $H^0(O_{\mathbb{P}^3}(2)) = S^2(H^0(L))$ we immediately see that the map: $S^2(H^0(L)) \rightarrow H^0(2L)$ cannot be surjective.

Theorem 2.2 can be regarded as a generalization of the following well known fact:

Proposition 2.6. Let $(M, L)$ be a polarized manifold and assume that $L$ is simply generated (i.e. 1-generated), then $L$ is very ample.

We have only to show that in this case $Bs | L | = \emptyset$. By contradiction suppose that there exists $p \in Bs | L |$ and let $tL$ be very ample. since $L$ is 1-generated, each section of $tL$ is a sum of products of sections belonging to $H^0(L)$, then $tL$ would not be spanned at $p$.

Theorem 2.2 also allows us to prove the following:

Theorem 2.7. Let $(M, L)$ be a polarized manifold with effective and 2-generated line bundle; then $2L$ is very ample.

Proof: From theorem 2.2 we have only to show that:

- $2L$ is spanned.
- $2L$ separates $p, q$, with at least one in $Bs | L |$.
- $2L$ separates $p \in Bs | L |$ from each $\tau \in T_pM$. 
Let \( tL, t \geq 2 \), be a very ample line bundle. Each section \( s \in H^0(tL) \) is of the form

\[
s = \sum_{i=0}^{a} (\prod_{i=0}^{a} \alpha_i \prod_{j=0}^{b} \beta_j),
\]

with \( \alpha_i \in H^0(2L) \) and \( \beta_j \in H^0(L) \) so that \( 2a + b = t \). Now \( 2L \) is spanned outside \( Bs \mid L \), since \( Bs \mid 2L \subset Bs \mid L \). Let \( p \in Bs \mid L \), if every \( \alpha \in H^0(2L) \) vanishes at \( p \) then each product in \( L \) must vanish, so \( tL \) is not spanned.

If \( p \in Bs \mid L \) and \( q \notin Bs \mid L \) take \( \beta \in H^0(L) \) such that \( \beta(q) \neq 0 \), \( \beta^2 \in H^0(2L) \) separates \( p \) and \( q \). Let \( p, q \in Bs \mid L \). As in the proof of theorem 2.2 the same shows that there exists a section in \( H^0(L) \) or in \( H^0(2L) \) separating \( p \) from \( q \), then \( 2L \) separates \( p \) and \( q \).

If \( p \in Bs \mid L \) and \( \tau \in T_p M \), then as before we know that exists a section of \( H^0(L) \) or of \( H^0(2L) \) separating \( p \) and \( \tau \). But in this case we may have a \( \beta \in H^0(L) \) with \( d\beta_p(\tau) \neq 0 \). This doesn’t tell anything on \( H^0(2L) \) because \( \beta^2(p) = 0 \) but also \( d\beta^2_p(\tau) = 0 \). Now we prove that, if \( 2L \) does not separate \( p \) and \( \tau \), such a \( \beta \) does not exists. We have two cases:

\( t \) is even. Let \( \eta \) be as in theorem 2.2 such that \( d\alpha_p(\tau) = \eta \alpha(p) \), for each \( \alpha \in H^0(2L) \). Let \( t = 2r \) and consider \( s \in H^0 = (tL) \):

\[
ds_p(\tau) = \sum_{i=1}^{a} (d(\prod_{i=1}^{a} \alpha_i) \prod_{2j=0}^{b} \beta_j)(\tau)) = \sum_{i=1}^{r} (d(\prod_{i=1}^{r} \alpha_i)(\tau)) = \\
\sum_{i=1}^{r} \sum_{j=1}^{r} (\prod_{i=1}^{r} \alpha_i(p)) \eta \alpha_i(p)(\tau) = \sum_{i=1}^{r} \sum_{j=1}^{r} \eta (\prod_{j=1}^{r} \alpha_j(p)) = \\
\sum_{j=1}^{r} r \eta (\prod_{j=1}^{r} \alpha_j(p)) = r \eta s(p)
\]

So \( tL \) does not separate \( p \) and \( \tau \).

\( t \) is odd. Since \( \beta_j(p) = 0 \) for each \( j \), we have:

\[
s(p) = \sum_{i=0}^{a} (\prod_{i=0}^{a} \alpha_i(p) \prod_{2j+1=0}^{b} \beta_j(p)) = 0.
\]

Then \( tL \) would not span in \( p \).

3. Some applications

In this section we discuss some applications of theorem 2.2.

First we consider a polarized manifold \((M, L)\) with \( Bs \mid L \) = \( \{p_1, \ldots, p_n\} \) (a finite set) and we obtain some information on the generator of \( G(M, L) \). To apply the main technique: ”the Apollonius method” see [4], recall that given \((M, L)\) be a polarized manifold, a sequence of irreducible and reduced subvarities \( D_i \) of \( M \) such that \( D_i \in \mid L_{D_{i-1}} \mid \) and \( D_i \in \mid L \mid \) is called a ladder of \((M, L)\) and the \( D_i \) are said the rungs of the ladder. In the second application we study some
connection between the generator of the canonical ring $G(S, K_S)$ of a surface of general type and the very ampleness of $tK_S$. Finally we study the very ampleness of $-tK_S$ for a del Pezzo surface.

3.1. Hyperelliptic polarized manifold of type $(-)$. In this subsection we apply the main theorem to provide an evidence for the first conjecture of Fujita in a special case. First of all we recall Fujita’s conjecture, see [3, §2, Conjecture b]

**Conjecture 3.1.** For every polarized manifold $(M, L)$ with $\dim M = n$, $K_M + (n + 2)L$ is very ample.

We recall that the delta-genus $\Delta(M, L)$ of a polarized manifold $(M, L)$ is defined as $\Delta(M, L) = L^n + n - h^0(M, L)$. Here we deal with polarized manifolds with $\Delta(M, L) = 1$. In this case for $L^n \geq 3$ it is known [3] that $L$ is very ample. It is also easy to prove that for $L^n = 2$ the line bundle $(g + 1)L$ is very ample [8].

The case $L^n = 1$ is very difficult to classify and there are only partial results, see [4]. Here we consider only polarized manifolds sectionally hyperelliptic of type $(-)$. In this case Fujita [4] proved the following facts:

\[(2)\] $H^q(M, tL) = 0$ for $q \geq 1$, $t \in \mathbb{Z}$,

\[(3)\] $K_M = (2g - n - 1)L$ where $g = g(M, L)$ and $n = \dim M$,

\[(4)\] $\text{Pic} M \cong \mathbb{Z}$, generated by $L$

\[(5)\] $(M, L)$ has a ladder.

Properties [2] and [3] are also true for the rungs $(D, L_D)$ of this ladder. In this situation we may apply theorem 2.2 to prove the very ampleness of $K_M + (n + 2)L = (2g + 1)L$.

We start with a lemma on the $r$-generation inspired by [3, 2.3]

**Lemma 3.2.** Let $(M, L)$ be a polarized manifold with $\Delta(M, L) = d(M, L) = 1$, sectionally hyperelliptic of type $(-)$; if the line bundle $L_C$ is $r$-generated, then $L$ is $r$-generated.

**Proof.** Consider a ladder of $(M, L)$: $(M_1, L_1) = (M, L), \ldots, (M_n, L_n) = (C, L_C)$ and proceed by induction on $n$.

For $n = 1$ the assumption is true. Now suppose that it is true for $k + 1$ and consider the exact sequence:

$$0 \rightarrow \mathcal{O}_{M_k}(-L_k) \rightarrow \mathcal{O}_{M_k} \rightarrow \mathcal{O}_{M_{k+1}} \rightarrow 0,$$

tensor with $\mathcal{O}_{M_k}(tL_k)$ and consider the exact cohomology sequence, recalling [3] we get:

\[(6)\] $0 \rightarrow H^0(M_k, (t - 1)L_k) \rightarrow H^0(M_k, tL_k) \rightarrow H^0(M_{k+1}, L_{k+1}) \rightarrow 0$.

So for each $t$ the restriction map $\psi$:

$$H^0(M_k, tL_k) \rightarrow H^0(M_{k+1}, tL_{k+1})$$
is surjective. Let us take a set of generators $\gamma_1, \ldots, \gamma_i$ of $G(M_{k+1}, L_{k+1})$ and their inverse images $\eta_1, \ldots, \eta_i$ in $G(M_k, L_k)$, if $\sigma \in H^0(M_k, L_k)$ is the defining section of $M_{k+1}$ then $G(M_k, L_k)$ is generated by the $\eta_j$ and $\sigma$. To show this, observe that by $\exists H^0(M_k, L_k)$ is generated by $\sigma$ and by the $\eta_j$ that belong to $H^0(M_{k+1}, L_{k+1})$. Proceeding by induction on $t$ it is simple to prove the assertion. Finally observe that $\eta_1, \ldots, \eta_i$ and $\sigma$ belong to $H^0(M_k, tL_k)$ with $t \leq r$. This completes the proof.

Considering the ladder of $(M, L)$, by adjunction from 3 we have:

$$K_{M_i} = (2g - i - 1)L_i.$$  

Then $K_C = (2g - 2)L_C = (2g - 2)p$, with $p = Bs | L$. Hence $g(M, L) = g = g(C)$ the genus of the curve $C$. Then since $C$ is an hyperelliptic curve we have that $L_C$ is exactly $2g + 1$-generated. Then by the lemma $(M, L)$ is $2g + 1$-generated.

The following theorem extends what proved in [7] for $g = 1$.

**Theorem 3.3.** Let $(M, L)$ be a polarized manifold with $\Delta(M, L) = L^n = 1$, sectionally hyperelliptic of type $(-)$, then $(2g + 1)L$ is very ample.

**Proof.** Since $L$ is $(2g + 1)$-generated, by 2 we know that $\varphi_{|(2g+1)L|}$ is an embedding of $M \setminus \{p\}$.

To see that $(2g + 1)L$ is spanned on $p$ we proceed by induction. Clearly $(2g + 1)L_C$ is very ample. Consider a smooth $D \in | L |$ (we know that such a $D$ exists by the ladder property), then consider the exact sequence:

$$0 \rightarrow H^0(2gL) \rightarrow H^0((2g + 1)L) \rightarrow H^0((2g + 1)L_D) \rightarrow 0.$$  

By induction we have that $(2g + 1)L_D$ is spanned on $p$ then there exists $\sigma \in H^0((2g + 1)L_D)$ with $\sigma(p) \neq 0$. By the surjectivity of the restriction map we have a $\gamma \in H^0((2g + 1)L)$ such that $\gamma(p) \neq 0$.

To see that $(2g + 1)L$ defines an embedding in $p$, consider a vector $\tau \in T_pM$. We say that exists a section $\eta \in H^0(M, L)$ such that $d\eta(\tau) \neq 0$. To construct $\eta$ we may take an hypersurface $D \in | L |$ such that $\tau \notin T_pD$ then the section that define $D$ clearly have a differential that does not vanish on $\tau$. Suppose by contradiction that for each $D \in | L |$ $\tau \in T_pD$ then the intersection of $n$ such $D_i$ cannot be transverse, so $L^n > 1$, absurd. Section $\eta^{2g+1}$ defines an embedding in $p$.

Now consider a point $q \in M$ different from $p$, we have to find a section in $H^0((2g + 1)L)$ that separates $p$ and $q$ i.e. that vanishes on $p$ but not on $q$. To see this observe that every section of $H^0(L)$ vanish on $p$, since $p = Bs | L |$, but there is a section $\sigma \in H^0(L)$ that does not vanish on $q$, otherwise $q \in Bs | L |$. Now consider $\sigma^{2g+1} \in H^0((2g + 1)L)$, this section separates $p$ from $q$.

**Theorem 3.3** is as best as possible as proved in the following:

**Proposition 3.4.** Given a polarized manifold $(M, L)$ with $\Delta(M, L) = d(M, L) = 1$, sectionally hyperelliptic of type $(-)$, the least integer $t$ such that $tL$ is very ample is $2g + 1$ i.e. $K_M + rL$ is not very ample for $r \leq n + 1$. 


Proof. By contradiction suppose that $2gL$ is very ample and consider a regular ladder of $(M, L)$. Since the ladder is regular, the very ampleness of $2gL$ implies the very ampleness of the line bundles $L_M$.

In particular we have that $2gp$ is very ample, where $p = Bs | L |$ and $2gp$ is a divisor on $C$, where $C$ is the one-dimensional element of the ladder. Note that $C$ is an hyperelliptic curve and that $g(M, L) = g(C)$.

Now we may compute canonical bundle by adjunction. By 6 we have $K_M = (2g - n - 1)L$, then we obtain: $K_D = (K_M + L)_D = (2g - n)L_D$, and by induction: $K_C = (2g - 2)L_C$. But $L_C = Bs | L | = p$ so we have $K_C = (g - 1)2p$. Since $C$ is hyperelliptic we know that $h^0(2p) = 2$.

Now consider the canonical map $\varphi|_{K_C}$ di $C$ and let $< \sigma, \tau >$ is a basis of $H^0(K_C)$, a basis of $H^0(K_C)$ is: $< \sigma^{g-1}, \sigma^{g-2}\tau, \ldots, \tau^{g-1} >$.

Now observe that $< \sigma^g, \sigma^{g-1}\tau, \ldots, \tau^g >$ is a basis of $H^0(K_C + 2p)$. In fact by Riemann-Roch we have $h^0(K_C + 2p) = g + 1$. This theorem implies that also $\varphi|_{K_C+2p}$ is a double covering of $\mathbb{P}^1$ followed by the Veronese embedding, but this implies that $(2g + 1)L_C = K_C + 2p$ is not very ample.

3.2. Surfaces of general type. Let $S$ be a minimal surface of general type (i.e. of Kodaira dimension 2) and let $K_S$ be its canonical bundle. It is known [1] that $K$ is ample outside the set of $(-2)$-curves, i.e. $KC = 0$ if and only if $C$ is a $(-2)$-curve.

In literature there are results [1] on the $C$-isomorphism property of the maps $\varphi_n = \varphi|_{nL}$, there are also results [2] on the generation of the canonical ring:

$$\bigoplus_{t \geq 1} H^0(S, tK_S).$$

As theorem 2.2 shows there must be a connection between these results. In the following table the results on the generation of the canonical ring, for surfaces with $q = H^1(S, \mathcal{O}_S) = 0$, are compared with the very ampleness of $rK_S$. For each pair of values of $K^2_S$ and $P_g(S) = h^0(S, K_S)$ are compared the $C$-isomorphism property of $\varphi|_{rK_S}$ and the $r$-generation of $K_S$.

| $K_S^2$ | $\varphi rK_S$ is a $C$-isomorphism | $P_g$ | $K_S$ is $r$-generated |
|---------|-------------------------------------|-------|----------------------|
| 1       | 5                                   | 0     | $\leq 6$             |
|         |                                     | 1     | $\leq 4$             |
|         |                                     | 2     | 5                    |
| 2       | 4                                   | 0     | $\leq 6$             |
|         |                                     | 1     | $\leq 4$             |
|         |                                     | 2     | 3(*)                 |
|         |                                     | 3     | 4                    |
| 3       | $3K_S$ is spanned and $\varphi_3$ is a birational morphism | 0     | $\leq 6$             |
|         |                                     | 1     | $\leq 4$             |
|         |                                     | 2     | 3(*)                 |
|         |                                     | 3     | 2(*)                 |
A very ampleness result

Where $(\star)$ means that the result is true if $|K_S|$ contains an irreducible element and $(\bullet)$ means that the result is true if $K_S$ is base point free and $\varphi|_{K_S}(S)$ is not a rational surface of degree $r - 1$ in $\mathbb{P}^r (r = h^0(K_S))$.

We observe that $r$-generation of $K_S$ is not so far from the very ampleness of $rK_S$ and by theorem 2.2 we have also the following propositions:

**Proposition 3.5.** Let $S$ be a surface of general type with $P_g = K^2 = 2$ and $q = 0$. If the generic element of $|K_S|$ is irreducible, then $\varphi|_{3K_S}$ is a $C$-isomorphism outside $B_s |K_S|$.

**Proof.** By theorem 3.7 of [2], $K_S$ is 3-generated so the proof follows from theorem 2.2.

Similarly one can prove:

**Proposition 3.6.** Let $S$ a surface of general type with $q = 0$, $P_g \geq 3$, $K_S^3 \geq 3$ and $K_S$ spanned, then we have two cases:

- $\varphi|_{K_S}(S)$ is a rational surface of degree $r - 1$ in $\mathbb{P}^r (r = h^0(K_S))$
- $2K_S$ is a $C$–isomorphism

3.3. Del Pezzo surfaces. In the following table is resumed the correspondence between very ampleness and $r$-generation for a Del Pezzo Surface. Here $L = -K_S$

| degree | very ampleness | $r$-generation |
|--------|---------------|----------------|
| $L^2 = 3$ | $L$ | 1-generated |
| $L^2 = 2$ | $2L$ | 2-generated |
| $L^2 = 1$ | $3L$ | 3-generated |

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