Entanglement swapping under quantum information masking

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Abstract

Quantum information contained in single-particle states can be masked by mapping them to entangled states. In this paper, we consider entanglement swapping under the masking of quantum information. Our work can pave the way for developing the applications of quantum information masking schemes in entanglement-swapping-based quantum cryptography.

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1 Introduction

As one of the core means to ensure information security, cryptography includes various technologies such as authentication, signature, and secure multi-party computing [1]. The security of classical cryptography is based on the complexity of some mathematical problems or the computing power of classical computers, which, unfortunately, has been seriously threatened by powerful quantum computers and quantum algorithms [2]. In this context, quantum cryptography, as a combination of quantum mechanics and cryptography, has attracted much attention because of its unconditional security.

Entanglement plays an important role in quantum cryptography. Using the quantum correlation of a bipartite entangled state, classical information can be hidden. In 2018, Modi et al. showed that quantum information can be hidden in the quantum correlations of bipartite composite systems rather than the subsystems [3]. The masking process is completed by unitary operations called maskers which can map single-particle states to entangled states. Afterwards, Li et al. proposed quantum information masking schemes by using multi-particle entangled states [4]. Quantum information masking has potential applications in entanglement-based quantum cryptography by hiding classical information in entangled states [5].

Using a set of orthogonal and complete bases to measure some of the particles in different entangled states, two new entangled states can be created. Such a process is called entanglement swapping, which, proposed by Zukowski et al. in 1993 [5], has important applications in many fields such as quantum information processing [6]. Entanglement swapping has been studied in any number of 2-level systems and d-level systems with discrete variables. Bose et al. considered the entanglement swapping between 2-level multi-particle cat states [7]. Hardy and Song proposed the entanglement swapping chains, which is realized by swapping d-level bipartite pure states [8]. Bouda and Buzek presented the entanglement swapping between any number of d-level multi-particle maximally entangled states [9]. Karimipour et al. studied entanglement swapping between a d-level maximally entangled state and a Bell state and illustrated its application in quantum cryptography by proposing a secret sharing protocol [10], which is a special case of the entanglement swapping schemes proposed by Bouda and Buzek [6].

In this paper, we consider the swapping of entanglement used in the quantum information masking scheme, and derive the entanglement swapping formulas for such a purpose. We will first consider the entanglement swapping in a 2-level quantum systems under quantum information masking, and then consider the one in a d-level systems. Our work can remove obstacles for finding the application of quantum information masking schemes in quantum cryptography which uses entanglement swapping.

2 Preliminaries

In this section, we will first introduce the definition of quantum information masking and the masking schemes. Then the general entanglement swapping schemes are described.
2.1 Quantum information masking schemes

The definition of quantum information masking introduced by Modi et al. [3] is given by

**Definition** If an unitary operation $\mathcal{M}$ can map the single-particle state $|\mathcal{A}_k\rangle_A$ to the two-particle entangled state $|\mathcal{C}_k\rangle_{AB} = H_A \otimes H_B$ such that all the marginal states of $|\mathcal{C}_k\rangle_{AB}$ are identical, that is,

$$\rho_A = Tr_B (|\mathcal{C}_k\rangle_{AB} \langle \mathcal{C}_k|), \quad \rho_B = Tr_A (|\mathcal{C}_k\rangle_{AB} \langle \mathcal{C}_k|),$$

then $\mathcal{M}$ is called a masker, which can mask the information encoded on the states $|\mathcal{A}_k\rangle_A$.

The action of the masker $\mathcal{M}$ can be depicted as

$$\mathcal{M} |\mathcal{A}_k\rangle_A \otimes |\mathcal{B}_k\rangle_B = |\mathcal{C}_k\rangle_{AB},$$

where $|\mathcal{B}_k\rangle_B$ is an ancillary particle. A simple masker introduced by Modi et al. can be described by

$$\mathcal{M} : |0\rangle \rightarrow \frac{1}{\sqrt{2}} \left([00] + (-1)^i [11] \right), \quad i = 0, 1.$$

Another introduced masker can mask the quantum information encoded on a state belonging to the family of states

$$\left\{|\alpha\rangle_A \mid |\alpha\rangle_A = \sum_{l=0}^{d-1} \eta_l e^{i \theta} |l\rangle, \quad 0 \in [-\pi, \pi], \quad \sum_{l=0}^{d-1} |\eta_l|^2 = 1\right\}.$$

Let us mark the masker by $\mathcal{M}$ as well, and the mapping result by $|\mathcal{C}\rangle_{AB}$, then the corresponding physical process can be expressed as

$$|\mathcal{C}\rangle_{AB} = \mathcal{M} |\alpha\rangle_A \otimes |\mathcal{B}\rangle_B = \sum_{l=0}^{d-1} \eta_l e^{i \theta} |l\rangle |\mathcal{B}\rangle_{AB},$$

where $|\mathcal{B}\rangle_B$ is an ancillary particle. One can get

$$\rho_A = \rho_B = \text{diag} \left[|\eta_0|^2, |\eta_1|^2, \ldots, |\eta_{d-1}|^2\right].$$

In particular,

$$\rho_A = \rho_B = \frac{I_d}{d}, \quad \text{if} \quad \eta_l = \frac{1}{\sqrt{d}}, \quad \forall l = 0, 1, \ldots, d-1.$$

Following the work of Modi et al., Li et al. proposed quantum information masking schemes with multi-particle entangled states [4]. It has been proved that quantum information carried on an arbitrary qubit state $|\psi_d\rangle = a_0 |0\rangle + a_1 |1\rangle$, where $\vec{a} = (a_0, a_1)$, can be masked by the following mapping operations:

$$|0\rangle \rightarrow |\Psi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),$$

$$|1\rangle \rightarrow |\Psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \otimes \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle).$$

With the mapping process, the masker can be described as

$$\mathcal{M}_{\vec{a}} : |\psi_d\rangle = a_0 |0\rangle + a_1 |1\rangle \rightarrow |\Psi_d\rangle = a_0 |\Psi_0\rangle + a_1 |\Psi_1\rangle.$$

Another masker introduced by Li et al. is given by

$$\mathcal{M}_{\vec{a}} : |\psi_d\rangle = a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle \rightarrow |\Psi_d\rangle = a_0 |\Psi_0\rangle + a_1 |\Psi_1\rangle + a_2 |\Psi_2\rangle,$$

where $\vec{a} = (a_0, a_1, a_2)$, and the mapping process is defined by

$$|0\rangle \rightarrow |\Psi_0\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle),$$

$$|1\rangle \rightarrow |\Psi_1\rangle = \frac{1}{\sqrt{3}} (|00\rangle + \zeta |11\rangle + \zeta^2 |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + \zeta |11\rangle + \zeta^2 |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + \zeta |11\rangle + \zeta^2 |22\rangle),$$

$$|2\rangle \rightarrow |\Psi_2\rangle = \frac{1}{\sqrt{3}} (|00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle) \otimes \frac{1}{\sqrt{3}} (|00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle).$$
\[ |2\rangle \rightarrow |\Psi_2\rangle = \frac{1}{\sqrt{3}} \left( |00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle \right) \otimes \frac{1}{\sqrt{5}} \left( |00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle \right) \otimes \frac{1}{\sqrt{3}} \left( |00\rangle + \zeta^2 |11\rangle + \zeta |22\rangle \right), \]  

where \( \zeta = e^{2\pi i/3} \). For d-level systems, Li et al. demonstrated that the general state \( |\psi_{\alpha}\rangle = \sum_{k=0}^{d-1} \alpha_k |k\rangle \) can be masked by the masking process

\[ A_{\alpha} : |\psi_{\alpha}\rangle = \sum_{k=0}^{d-1} \alpha_k |k\rangle \rightarrow |\Psi_\alpha\rangle = \sum_{k=0}^{d-1} \alpha_k |\Psi_k\rangle = \sum_{k=0}^{d-1} \alpha_k \sqrt{\frac{d}{k}} |j\rangle, \]  

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}) \), \( \zeta = e^{2\pi i/d} \), and the mapping process is defined by

\[ |k\rangle \rightarrow |\Psi_k\rangle = \sqrt{\frac{d}{k}} |j\rangle. \]

### 2.2 Entanglement swapping

Entanglement swapping was discovered by Žukowski et al. in 1993, who realized entanglement swapping between two Bell states experimentally [5]. The Bell states are usually expressed by

\[ |\phi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle), \quad |\psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle). \]  

Let us denote two Bell states as

\[ |\mathcal{B}(\lambda_1)\rangle = \frac{1}{\sqrt{2}} \left[ |0a_1\rangle + (-1)^{d_1} |1\bar{a}_1\rangle \right]_{1,2}, \quad |\mathcal{B}(\lambda_2)\rangle = \frac{1}{\sqrt{2}} \left[ |0a_2\rangle + (-1)^{d_2} |1\bar{a}_2\rangle \right]_{3,4}, \]  

respectively, where \( \lambda_i, a_i \in \{0, 1\} \) for \( r = 1, 2 \) and the subscripts 1,2,3,4 denote the particles in the two states. We can express the entanglement swapping between two Bell states as follows,

\[ \bigotimes_{r=1}^{2} |\mathcal{B}(\lambda_r)\rangle_{2r-1,2r} \rightarrow \begin{cases} 
1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle + |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle + |\phi^+\rangle |\phi^+\rangle \right)_{1324} \\
1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle + |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle + |\phi^+\rangle |\phi^+\rangle \right)_{1324}, \text{if } a_1a_2 = 00,
\end{cases} \]

\[ \bigotimes_{r=1}^{2} |\mathcal{B}(\lambda_r)\rangle_{2r-1,2r} \rightarrow \begin{cases} 
1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} \\
1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324}, \text{if } a_1a_2 = 01,
\end{cases} \]

\[ \bigotimes_{r=1}^{2} |\mathcal{B}(\lambda_r)\rangle_{2r-1,2r} \rightarrow \begin{cases} 
1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} \\
1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324}, \text{if } a_1a_2 = 10,
\end{cases} \]

\[ \bigotimes_{r=1}^{2} |\mathcal{B}(\lambda_r)\rangle_{2r-1,2r} \rightarrow \begin{cases} 
1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} \\
1 - (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324} + 1 + (-1)^{d_1+d_2} \left( |\phi^+\rangle |\phi^+\rangle - |\phi^+\rangle |\phi^+\rangle \right)_{1324}, \text{if } a_1a_2 = 11,
\end{cases} \]  

where the coefficient \( \frac{1}{\sqrt{2d}} \) in each state is ignored (the same below).

Bose et al. [7] generalized the entanglement swapping of two bi-qubit systems to any number of multi-qubit cases by studying the entanglement swapping between m-particle cat states which has the form

\[ |C^+ (m)\rangle = \frac{1}{\sqrt{2^m}} \left( \bigotimes_{i=1}^{m} |a_i\rangle \pm \bigotimes_{i=1}^{m} |\bar{a}_i\rangle \right), \quad a_i \in \{0, 1\}, m \geq 2. \]

Bose et al. showed their formula without providing a specific derivation, which can be expressed as

\[ \bigotimes_{r=1}^{n} |C (m_r)\rangle \rightarrow \begin{bmatrix} \sum_{r=1}^{n} k_r \end{bmatrix} \otimes \begin{bmatrix} \sum_{r=1}^{n} m_r - \sum_{r=1}^{n} k_r \end{bmatrix}. \]
where the entanglement swapping is realized by performing a joint measurement on the first \( k \) particles in the \( r \)-th cat state that contains \( m_r \) particles. We remark that Bose et al.’s result is ambiguous. The reason is that the result does not indicate whether the state obtained by measurements and the state that the remaining particles collapse into are in \( |C^+(-\cdot)| \) or \( |C^-(\cdot)| \). Unfortunately, we did not notice this problem in our recent work presented in Ref. [6]. Therefore, we would like to provide a detailed derivation for the entanglement swapping between cat states here. Let us denote \( n \) cat states as

\[
|\mathcal{C}(m_r)\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{i=1}^n [a_i^r]_+ + (-1)^{y_k} \bigotimes_{i=1}^n [a_i^r]_-, \quad A_r, A_r^e \in [0, 1], m_r \geq 2, r = 1, 2, \ldots, n. \tag{19}
\]

Let us perform a joint measurement on the first \( k \) particles in the \( r \)-th cat state, such that a \( (\sum_{r=1}^n m_r - \sum_{r=1}^n k_r) \)-particle cat state and a \( (\sum_{r=1}^n m_r - \sum_{r=1}^n k_r) \)-particle cat state can be created. Let us set \( K = \sum_{r=1}^n k_r \) and \( R = \sum_{r=1}^n m_r - \sum_{r=1}^n k_r \), then we can arrive at

\[
\bigotimes_{r=1}^n |\mathcal{C}(m_r)\rangle
\]

\[
= \bigotimes_{r=1}^n \bigotimes_{i=1}^{m_r} [a_i^r]_+ + (-1)^{y_k} \bigotimes_{i=1}^{m_r} [a_i^r]_+ [a_i^r]_+ + (-1)^{y_k-1} \bigotimes_{i=1}^{m_r} [a_i^r]_+ [a_i^r]_+ \bigotimes_{i=1}^{m_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r} [a_i^r]_+ \\
+ \cdots + (-1)^{\sum_{r=1}^n k_r} \bigotimes_{r=1}^n \bigotimes_{i=1}^{m_r} [a_i^r]_+
\]

\[
\rightarrow \bigotimes_{r=1}^n \bigotimes_{i=1}^{k_r} [a_i^r]_+ \bigotimes_{r=1}^{m_r-k_r} [a_i^r]_+ [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \\
+ (-1)^{y_k-1} \bigotimes_{r=1}^{m_r-k_r} \bigotimes_{i=1}^{k_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [a_i^r]_+ \\
+ \cdots + (-1)^{\sum_{r=1}^n k_r} \bigotimes_{r=1}^n \bigotimes_{i=1}^{m_r} [a_i^r]_+
\]

\[
= \sum_{p,q=0}^{n-1} \left( (-1)^{\sum_{r=1}^n k_r} \bigotimes_{r=1}^n \bigotimes_{i=1}^{k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \right) \]

\[
= \sum_{p,q=0}^{n-1} \left( (-1)^{\sum_{r=1}^n k_r} \bigotimes_{r=1}^n \bigotimes_{i=1}^{k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \bigotimes_{i=1}^{m_r-k_r} \bigotimes_{i=1}^{m_r-k_r} [b_i^r]_+ \right) \]

\[
\text{where}
\]

\[
\begin{align*}
|\mathcal{C}^+ (K)\rangle &= \bigotimes_{i=1}^K [b_i]_+ \pm \bigotimes_{i=1}^K [\bar{b}_i]_+ \\
|\mathcal{C}^- (R)\rangle &= \bigotimes_{i=1}^R [b_i]_+ \pm \bigotimes_{i=1}^R [\bar{b}_i]_+.
\end{align*}
\tag{21}
\]

The formulas of the entanglement swapping of any number of Bell states and GHZ states can be derived from that of the entanglement swapping of cat states if the first particle of each cat state is in \( |\phi(r)\rangle = (0 \pm |1\rangle) \), which is discussed in Ref. [6].

For the entanglement swapping between d-level systems, Boua and Buzek studied entanglement swapping for any number of multi-particle systems [8]. Karimipour et al. considered the entanglement swapping between a d-level maximally entangled state (also known as a d-level cat state) and a Bell state [10]. Hardy and Song proposed entanglement swapping chains for two-particle pure states [8]. Here we would like to briefly review Karimipour et al.’s work presented in [11]. Suppose that a joint measurement is performed on the particle with the mark \( u_1^k \) in a d-level \( m \)-particle cat state and the particle with the mark \( u_1^k \) in a d-level Bell state, the the formula derived by Karimipour et al. can be expressed by [6]

\[
|\phi (u_1^k, u_2^k, \ldots, u_m^k)\rangle \otimes \left| \phi (u_1^k, u_2^k) \right\rangle
\]
be masked, that is, In what follows, we consider the entanglement swapping for the entangled states used in the schemes of quantum proposed by Modi et al. For the masker shown in Eq. 3, let us suppose that there are \( n \) orthogonal basis constructed by \( W \). We remark that although Karimipour et al.'s formula is mathematically correct, it cannot clearly show the two single-particle states of the remaining particles can only be known after calculating \( u_1^* \otimes v_2 \) and \( u_1^* \otimes v_1 \). Therefore, the formula provided in Ref. [6] is clear, which is given by

\[
\frac{1}{d} \sum_{j, l = 0}^{d - 1} \zeta^{j(l)} |j, l \oplus u^*_1\rangle \otimes |\phi(u^*_1 \oplus l, u^*_1 \oplus l)\rangle.
\]

We remark that although Karimipour et al.'s formula is mathematically correct, it cannot clearly show the two single-particle states of the remaining particles can only be known after calculating \( u_1^* \otimes v_2 \) and \( u_1^* \otimes v_1 \). Therefore, the formula provided in Ref. [6] is clear, which is given by

\[
\frac{1}{d} \sum_{j, l = 0}^{d - 1} \zeta^{j(l)} |j, l \oplus u^*_1\rangle \otimes |\phi(u^*_1 \oplus l, u^*_1 \oplus l)\rangle.
\]

Note that several typos can be found in the formula corresponding to this one in Ref. [6] through comparison. More details of entanglement swapping schemes in d-level systems can be found in Refs. [6-11].

3 Swapping the entanglement used for masking quantum information

In what follows, we consider the entanglement swapping for the entangled states used in the schemes of quantum information masking. Let us first consider entanglement swapping for the quantum information masking schemes proposed by Modi et al. For the masker shown in Eq. 4 let us suppose that there are \( n \) single-particle states \( |l\rangle \) to be masked, that is,

\[
\mathcal{M}: |\lambda_r\rangle \rightarrow \frac{1}{\sqrt{2}} \left[ |00\rangle + (-1)^{\lambda_r} |11\rangle \right], \quad \lambda_r \in \{0, 1\}, \quad r = 1, 2, \ldots, n.
\]

Let us mark these entangled states by \( \frac{1}{\sqrt{2}} \left[ |00\rangle + (-1)^{\lambda} |11\rangle \right]_{2r-1,2r} \), where the subscripts \( 2r-1,2r \) denote the two particles in each state. By performing measurements on the first particle in each state using the complete and orthogonal basis constructed by \( n \)-particle Greenberger–Horne–Zeilinger (GHZ) states

\[
|G_d^n\rangle = |0\rangle \bigotimes_{i=2}^{n} |a_i\rangle \pm |1\rangle \bigotimes_{i=2}^{n} |\bar{a}_i\rangle, \quad a_i = 0, 1, d = \sum_{i=2}^{n} a_i^{2^{n-i}},
\]

we can get

\[
\begin{align*}
\sum_{r=1}^{n} \left[ |00\rangle + (-1)^{\lambda} |11\rangle \right]_{2r-1,2r} \\
= \sum_{i=1}^{2n} |0\rangle_{2r-1} \bigotimes_{i=1}^{2n-2} |0\rangle_{2i} \bigotimes_{i=2}^{n} |11\rangle_{2n-1,2n} + (-1)^{\lambda} \sum_{i=1}^{2n-4} |0\rangle_{2r-1} \bigotimes_{i=1}^{2n-4} |0\rangle_{2i} \bigotimes_{i=2}^{n-1} |1100\rangle_{2n-3,2n-2,2n-1,2n} \\
+ \cdots + (-1)^{\lambda} \sum_{i=1}^{n} |11\rangle_{2r-1,2i} \\
\rightarrow \sum_{i=1}^{n} |0\rangle_{2r-1} \bigotimes_{i=1}^{n} |0\rangle_{2i} + (-1)^{\lambda} \sum_{i=1}^{n} |0\rangle_{2r-1} \bigotimes_{i=1}^{n} |0\rangle_{2i} \bigotimes_{i=2}^{n} |1\rangle_{2r-1} \bigotimes_{i=2}^{n} |1\rangle_{2i} \\
+ (-1)^{\lambda} \sum_{i=1}^{n} |0\rangle_{2r-1} \bigotimes_{i=1}^{n} |0\rangle_{2i} \bigotimes_{i=2}^{n} |1100\rangle_{2n-3,2n-2,2n-1,2n} + \cdots + (-1)^{\lambda} \sum_{i=1}^{n} |1\rangle_{2r-1} \bigotimes_{i=1}^{n} |1\rangle_{2i} \\
= \sum_{j=1}^{n} \sum_{a_j'=0}^{a_j} \left[ (-1)^{\lambda} |a_j\rangle_{2j-1} \bigotimes_{i=2}^{n} |0\rangle_{2i} \bigotimes_{i=2}^{n} |a_j\rangle_{2i} + (-1)^{\lambda} \sum_{a_j'=0}^{a_j} \sum_{a_j''=0}^{a_j} \sum_{a_j'''=0}^{a_j''} |1\rangle_{2j-1} \bigotimes_{i=2}^{n} |1\rangle_{2i} \bigotimes_{i=2}^{n} |a_j''\rangle_{2i} \right].
\end{align*}
\]
\[
\begin{align*}
\sum_{p=\sum_{k=1}^{\infty} n_k a_k} (-1)^{\sum_{k=1}^{\infty} n_k a_k} \left( |G_p^+\rangle + |G_p^\perp\rangle \right) \left( |G_p^+\rangle + |G_p^\perp\rangle \right) + (-1)^{\sum_{k=1}^{\infty} n_k a_k} \left( |G_p^+\rangle - |G_p^\perp\rangle \right) \left( |G_p^+\rangle - |G_p^\perp\rangle \right)
\end{align*}
\]

In particular, for the entanglement swapping between two states, we have
\[
\begin{align*}
&\frac{1}{\sqrt{2}} |00\rangle + (-1)^{l_1} |11\rangle, \\
\rightarrow &\left[ 1 + (-1)^{l_1} \left( |\phi^+\rangle + |\phi^-\rangle \right) \right]_{1,2,4} + \left[ 1 - (-1)^{l_1} \left( |\phi^+\rangle + |\phi^-\rangle \right) \right]_{1,3,4} \\
&+ \left[ (-1)^{l_2} + (-1)^{l_3} \right] \left( |\psi^+\rangle + |\psi^-\rangle \right) \right]_{1,3,4} + \left[ (-1)^{l_2} - (-1)^{l_3} \right] \left( |\psi^+\rangle + |\psi^-\rangle \right) \right]_{1,3,4}.
\end{align*}
\]

An interesting conclusion that can be drawn from the above entanglement swapping schemes is that if an even number of states in the \(n\) entangled states are in \(\sum (|00\rangle - |11\rangle)\), two identical GHZ states can be generated after entanglement swapping. Furthermore, the conclusion can be applied to quantum secret sharing and quantum private comparison. Details of this conclusion and its application are provided in Ref. [6].

Next, let us derive the formula of the entanglement swapping for the other quantum information masking scheme introduced by Modi et al.; that is, swapping the entanglement used for masking process presented in Eq. [5]. As before, let us suppose that there are \(n\) entangled states used for masking quantum information, and let us denote them as
\[
\sum_{a=0}^{d-1} \eta_a e^{i\theta_a} |a, a_i\rangle_{2r-1, 2r}, \quad r = 1, 2, \ldots, n.
\]

Measuring the first particle in each state with the basis constructed by the \(n\)-particle maximally entangled states shown in Eq. [23] we can get
\[
\begin{align*}
\sum_{r=1}^{n} \sum_{a=0}^{d-1} \eta_a e^{i\theta_a} |a, a_i\rangle_{2r-1, 2r} \\
\rightarrow &\frac{1}{P}\sum_{v, v_2, \ldots, v_n=0}^{d-1} \sum_{\alpha=0}^{d-1} |\phi(v, v_2, \ldots, v_n)\rangle_{1,2,\ldots,2n} |a_1, a_1' \oplus v_2, a_2 \oplus u_3, \ldots, a_i \oplus v_n\rangle_{2,4,\ldots,2n},
\end{align*}
\]

where \(P\) can be derived through the normalization of probability.

Let us finally consider entanglement swapping under Li et al.'s quantum information masking scheme presented in Eq. [12]. Similar to the expression above, suppose that there are \(n\) entangled states denoted as
\[
\frac{1}{d^2} \sum_{\alpha', \alpha''=0}^{d^2} \alpha'_k \xi^{\omega'k} |a_0, a_0', a_1, a_1', \ldots, a_{d-1}', a_{d-1}'\rangle, \quad \omega' = \sum_{i=0}^{d-1} a_i', \quad r = 1, 2, \ldots, n.
\]

Without losing generality, considering measuring the first particle in each state, we can get
\[
\begin{align*}
\sum_{r=1}^{n} \frac{1}{d^2} \sum_{\alpha', \alpha''=0}^{d^2} \alpha'_k \xi^{\omega'k} |a_0, a_0', a_1, a_1', \ldots, a_{d-1}', a_{d-1}'\rangle \\
\rightarrow &\frac{1}{P}\sum_{k, k', \alpha''=0}^{d^2} \sum_{v, v_2, \ldots, v_n=0}^{d^2} \sum_{\alpha, \alpha'=0}^{d^2} \sum_{\alpha', \alpha'=0}^{d^2} \cdots \sum_{\alpha', \alpha'=0}^{d^2} \xi^{\omega'k - \omega'v} \sum_{r=1}^{n} a_v \phi(v, v_2, \ldots, v_n) \\
&|a_0, a_1, \ldots, a_{d-1}, a_{d-1}' \oplus v_2, a_1, a_1', \ldots, a_{d-1}', a_{d-1}' \oplus v_n, a_0, a_0', \ldots, a_{d-1}', a_{d-1}'\rangle.
\end{align*}
\]

4 Conclusion

We have studied entanglement swapping after masking quantum information with composite quantum systems. Our work has enlightening and guiding significance for finding the application of quantum information masking schemes in entanglement-swapping-based quantum cryptography. The entanglement swapping chains under quantum information masking can be considered.
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