Sharp convergence for sequences of nonelliptic Schrödinger means

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Abstract

We consider pointwise convergence of nonelliptic Schrödinger means $e^{it_n\Box}f(x)$ for $f \in H^s(\mathbb{R}^2)$ and decreasing sequences $\{t_n\}_{n=1}^{\infty}$ converging to zero, where

$$e^{it_n\Box}f(x) := \int_{\mathbb{R}^2} e^{i(x \cdot \xi + t_n \xi_1 \xi_2)} \hat{f}(\xi) d\xi.$$

We prove that when $0 < s < \frac{1}{2}$,

$$\lim_{n \to \infty} e^{it_n\Box}f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^2$$

holds for all $f \in H^s(\mathbb{R}^2)$ if and only if $\{t_n\}_{n=1}^{\infty} \in \ell^{r(s),\infty}(\mathbb{N})$, $r(s) = \frac{s}{1-s}$. Moreover, our result remains valid in general dimensions.

1 Introduction

Consider the generalized Schrödinger equation

$$\begin{cases}
\partial_t u(x,t) - iP(D)u(x,t) = 0 & x \in \mathbb{R}^N, t \in \mathbb{R}^+,
\end{cases}$$

(1.1)

where $D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N})$, $P(\xi)$ is a real continuous function defined on $\mathbb{R}^N$, $P(D)$ is defined via its real symbol

$$P(D)f(x) = \int_{\mathbb{R}^N} e^{ix \cdot \xi} P(\xi) \hat{f}(\xi) d\xi.$$

The solution of (1.1) can be formally written as

$$e^{itP(D)}f(x) := \int_{\mathbb{R}^N} e^{ix \cdot \xi + itP(\xi)} \hat{f}(\xi) d\xi,$$

(1.2)

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where $\hat{f}(\xi)$ denotes the Fourier transform of $f$. The related pointwise convergence problem is
to determine the optimal $s$ for which
\[
\lim_{t \to 0^+} e^{itP(D)} f(x) = f(x)
\]
almost everywhere whenever $f \in H^s(\mathbb{R}^N)$.

In the elliptic case: $P(\xi) = |\xi|^2$, the pointwise convergence problem was first considered by
Carleson [4] and he showed the convergence for $s \geq 1/4$ when $N = 1$. Dahlberg-Kenig [9] showed
that the convergence does not hold for $s < 1/4$ in any dimension, which implies sharpness for
the condition given by Carleson in one-dimensional case. In higher dimension $N \geq 2$, Sjölin [22]
and Vega [26] independently obtained the convergence for $s > 1/2$. In 2016, Bourgain [3] gave a
counterexample showing that it is false if $s < \frac{N}{2(N+1)}$. Recently, Du-Guth-Li [10] for $N = 2$
and Du-Zhang [12] for higher dimensions $N \geq 3$ obtained the sharp result for convergence up
to the endpoint. Moreover, recent progress for the fractional Schrödinger operators when $P(\xi) = |\xi|^\alpha$, 
$\alpha > 1$ can be found in [6].

Another interesting case is the nonelliptic case: $P(\xi) = \xi_1^\alpha - \xi_2^\alpha \pm \xi_3^\alpha \pm \cdots \pm \xi_N^\alpha$. For physical
application of the nonelliptic Schrödinger equation, see for example [25]. Rogers-Vargas-Vega [20]
showed that the pointwise convergence of the solution to the nonelliptic Schrödinger equation,
$i\partial_t u + (\partial_x^2 - \partial_y^2) u = 0$, was proved when $f \in H^s(\mathbb{R}^2)$, if and only if $s \geq 1/2$. Thus the pointwise
behavior is worse than that in the elliptic case. In higher dimensions, they also established
similar results except the endpoint.

One of the natural generalizations of the pointwise convergence problem is to ask a.e. con-
vergence of the Schrödinger means where the limit is taken over decreasing sequences $\{t_n\}_{n=1}^\infty$
converging to zero. That is to investigate relationship between optimal $s$ and properties of
$\{t_n\}_{n=1}^\infty$ such that for each function $f \in H^s(\mathbb{R}^N)$,
\[
\lim_{n \to \infty} e^{it_nP(D)} f(x) = f(x) \ a.e. \ x \in \mathbb{R}^N.
\]
This problem was first considered by Sjölin [23] in general dimensions and later improved
by Sjölin-Strömberg [24] for $P(\xi) = |\xi|^\alpha$, $\alpha > 1$. Dimou-Seeger [8] obtained a sharp characterization
of this problem in the one-dimensional case for $P(\xi) = |\xi|^\alpha$, $\alpha > 0$. More recently, Li-Wang-Yan
[16] improved the previous results of Sjölin [23] and Sjölin-Strömberg [24] for $P(\xi) = |\xi|^2$ in $\mathbb{R}^2$
through the bilinear method.

In this paper, we concentrate ourselves on the nonelliptic case and seek what happens if
$0 < s < \frac{1}{2}$. More concretely, we obtain a sufficient and necessary condition for $\{t_n\}_{n=1}^\infty$ to ensure
a.e. convergence of nonelliptic Schrödinger means. For convenience, we first set $N = 2$. By
changing of variables, the nonelliptic Schrödinger operator can be written as
\[
e^{it\xi_1} f(x) := \int_{\mathbb{R}^2} e^{ix \cdot \xi + it\xi_1 \xi_2} \hat{f}(\xi) d\xi.
\]
In what follows, we always assume that the decreasing sequence $\{t_n\}_{n=1}^\infty$ converges to zero and
$\{t_n\}_{n=1}^\infty \subset (0, 1)$. In order to characterize the convergence of $\{t_n\}_{n=1}^\infty$, we introduce the Lorentz
space $\ell^{r,\infty}(\mathbb{N})$, $r > 0$. The sequence $\{t_n\}_{n=1}^\infty \in \ell^{r,\infty}(\mathbb{N})$ if and only if
\[
\sup_{b > 0} b^{r+1} \left\{ n : t_n > b \right\} < \infty.
\]
Our main results are as follows.
THEOREM 1.1. Let $0 < s < \frac{1}{2}$, $t_n - t_{n+1}$ be decreasing. Then
\[
\lim_{n \to \infty} e^{it_n \Delta} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^2
\]
holds for all $f \in H^s(\mathbb{R}^2)$ if and only if \( \{t_n\}_{n=1}^\infty \in \ell^s(\mathbb{N}) \), \( r(s) = \frac{s}{1-s} \).

Theorem 1.1 provides a sharp condition of \( \{t_n\}_{n=1}^\infty \) for convergence of nonelliptic Schrödinger means to hold. The sufficient and necessary conditions in Theorem 1.1 will be proved in Section 2 and Section 3, respectively. The proof of the sufficient condition depends heavily on the following Theorem 1.2.

THEOREM 1.2. If \( \text{supp} \hat{f} \subset \{\xi : |\xi| \sim \lambda\}, \lambda \geq 1 \), then for any small interval \( I \) with
\[ \lambda^{-2} \leq |I| \leq \lambda^{-1}, \]
we have
\[
\left\| \sup_{t \in I} e^{it \Delta} f(x) \right\|_{L^2(B(0,1))} \leq C\lambda |I|^{\frac{1}{2}} \|f\|_{L^2}, \tag{1.7}
\]
where the constant \( C \) does not depend on \( f \).

The proof of Theorem 1.2 is not hard. Indeed, it follows from Sobolev’s embedding and Plancherel theorem that
\[
\left\| \sup_{t \in I} e^{it \Delta} f \right\|_{L^2(B(0,1))} \leq \|f\|_{L^2} + \left\| \int_{\mathbb{R}^2} e^{i(x\xi + t\xi_1 \xi_2)} \hat{f}(\xi) \, d\xi \right\|_{L^2(B(0,1) \times I)}^{1/2} \\
\times \left\| \int_{\mathbb{R}^2} e^{i(x\xi + t\xi_1 \xi_2)} \xi_1 \xi_2 \hat{f}(\xi) \, d\xi \right\|_{L^2(B(0,1) \times I)}^{1/2} \\
\leq \|f\|_{L^2} + |I|^{\frac{1}{2}} \|\hat{f}\|_{L^2}^{1/2} \|\xi_1 \xi_2 \hat{f}(\xi)\|_{L^2}^{1/2} \\
\leq \|f\|_{L^2} + \lambda |I|^{\frac{1}{2}} \|f\|_{L^2} \\
\leq \lambda |I|^{\frac{1}{2}} \|f\|_{L^2}.
\]

Then we arrive at inequality (1.7).

Rogers-Vargas-Vega [20] applied the stationary phase method to show the sharp estimate
\[
\left\| \sup_{t \in (0,1)} e^{it \Delta} f(x) \right\|_{L^2(B(0,1))} \leq C\lambda \|f\|_{L^2(\mathbb{R}^2)}, \text{ supp} \hat{f} \subset \{\xi : |\xi| \sim \lambda\}. \tag{1.8}
\]

This implies that if \( |I| = \lambda^{-1} \), then
\[
\left\| \sup_{t \in I} e^{it \Delta} f(x) \right\|_{L^2(B(0,1))} \leq C\lambda \|f\|_{L^2(\mathbb{R}^2)}, \tag{1.9}
\]
which coincides with Theorem 1.2 when \( |I| = \lambda^{-1} \). Due to the localizing lemma in Remark 3.1 of Lee-Rogers [15], inequality (1.9) yields inequality (1.8). Therefore, inequality (1.9) is also sharp.

Moreover, by the same method as we applied to prove Theorem 1.1, we can get the corresponding result in general dimensions \( N \geq 2 \).
Theorem 1.3. Let \(0 < s < \frac{1}{2}\), \(t_n - t_{n+1}\) be decreasing, \(P(\xi) = \xi_1^2 - \xi_2^2 \pm \xi_3^2 \pm \cdots \pm \xi_N^2\). Then

\[
\lim_{n \to \infty} e^{it_n P(D)} f(x) = f(x) \text{ a.e. } x \in \mathbb{R}^N
\]

holds for all \(f \in H^s(\mathbb{R}^N)\) if and only if \(\{t_n\}_{n=1}^{\infty} \in \ell^r(\mathbb{N})\), \(r(s) = \frac{s}{1-s}\).

We prefer to omit the proof of Theorem 1.3 since it is very similar with that of Theorem 1.1. But a simple explanation for the proof of the necessary condition in Theorem 1.3 will be given in Section 3.

Conventions: Throughout this article, we shall use the well known notation \(A \gg B\), which means if there is a sufficiently large constant \(G\), which does not depend on the relevant parameters arising in the context in which the quantities \(A\) and \(B\) appear, such that \(A \geq GB\). We write \(A \sim B\), and mean that \(A\) and \(B\) are comparable. By \(A \lesssim B\) we mean that \(A \leq CB\) for some constant \(C\) independent of the parameters related to \(A\) and \(B\). \(B(0,1)\) denotes the unit ball centered at the origin in \(\mathbb{R}^2\).

2 Sufficient condition

By standard arguments, in order to obtain the convergence result, it is sufficient to show the maximal function estimate in \(\mathbb{R}^2\). In order to involve the endpoint \(r(s) = \frac{s}{1-s}\), we adopt the similar decomposition as Proposition 2.3 in [8] to prove Theorem 2.1.

Theorem 2.1. If \(\{t_n\}_{n=1}^{\infty} \in \ell^r(\mathbb{N})\), \(r(s) = \frac{s}{1-s}\). Then for any \(0 < s < \frac{1}{2}\), we have

\[
\left\| \sup_{n \in \mathbb{N}} |e^{it_n \Delta} f| \right\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^2)},
\]

whenever \(f \in H^s(\mathbb{R}^2)\), where the constant \(C\) does not depend on \(f\).

Proof. Set

\[
s = \frac{r}{r+1}, \quad r \in (0,1).
\]

We decompose \(f\) as

\[
f = \sum_{k=0}^{\infty} f_k,
\]

where \(\text{supp} f_0 \subset B(0,1)\), \(\text{supp} f_k \subset \{\xi : |\xi| \sim 2^k\}, k \geq 1\).

We decompose \(\{t_n\}_{n=1}^{\infty}\) as

\[
A_l := \left\{ t_n : 2^{-(l+1)\frac{2}{1-r}} < t_n \leq 2^{-l\frac{2}{1-r}} \right\}, \quad l \in \mathbb{N}.
\]

Since \(\{t_n\}_{n=1}^{\infty} \in \ell^r(\mathbb{N})\) and \(r \in (0,1)\), we have

\[
\|A_l\| \leq C2^{\frac{2l}{1-r}},
\]
Then we have

\[
\sup_{n \in \mathbb{N}} \left| e^{it_n \Box} f \right| = \sup_{n \in \mathbb{N}} \sup_{t_n \in A_i} \left| \sum_{k=0}^{\infty} e^{it_n \Box} f_k \right|
= \sup_{n \in \mathbb{N}} \sup_{t_n \in A_i} \left| \sum_{k \geq 1} e^{it_n \Box} f_k \right| + \sup_{n \in \mathbb{N}} \sup_{t_n \in A_i} \left| \sum_{1 \leq k < l} e^{it_n \Box} f_k \right|
\]

\[
+ \sup_{n \in \mathbb{N}} \sup_{t_n \in A_i} \left| \sum_{k \leq \frac{l}{1 + r}} e^{it_n \Box} f_k \right|
\]

\[
:= I + II + III. \tag{2.3}
\]

Next, we will estimate \( I, II, III \) respectively.

We firstly estimate \( I \). We make the change of variable \( k = l + m \) in \( I \). Inequality (2.2) and Plancherel theorem imply that

\[
\left\| I \right\|_{L^2(B(0,1))} \leq \sum_{m \geq 0} \left( \sum_{l \in \mathbb{N}} \sum_{n \in \mathbb{N}} \left\| e^{it_n \Box} f_{l+m} \right\|_{L^2(B(0,1))}^2 \right)^{1/2}
\]

\[
\leq \sum_{m \geq 0} \left( \sum_{l \in \mathbb{N}} 2^{\frac{m}{1+r}} \left\| f_{l+m} \right\|_{L^2}^2 \right)^{1/2}
\]

\[
= \sum_{m \geq 0} 2^{-\frac{m}{1+r}} \left( \sum_{l \in \mathbb{N}} 2^{\frac{2(l+m)}{1+r}} \left\| f_{l+m} \right\|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left\| f \right\|_{H^s(\mathbb{R}^2)}. \tag{2.4}
\]

For \( II \), we make the change of variable \( k = l - j \). Then we have

\[
II = \sup_{n \in \mathbb{N}} \sup_{t_n \in A_i} \left| \sum_{0 < j \leq \frac{l}{1 + r}} e^{it_n \Box} f_{l-j} \right|
\]

\[
\leq \sum_{j \in \mathbb{N}} \sup_{n \in \mathbb{N}} \sup_{l \in \mathbb{N}} \left| e^{it_n \Box} f_{l-j} \right|.
\]

Therefore, we have

\[
\left\| II \right\|_{L^2(B(0,1))} \leq \sum_{j \geq 0} \left( \sum_{l \in \mathbb{N}, l \geq \frac{j}{1+r}} \left\| \sup_{n \in \mathbb{N}} \left| e^{it_n \Box} f_{l-j} \right| \right\|_{L^2(B(0,1))}^2 \right)^{1/2}
\]

\[
\leq \sum_{j \geq 0} \left( \sum_{l \in \mathbb{N}, l \geq \frac{j}{1+r}} 2^{2(l-j)} 2^{-\frac{m}{1+r}} \left\| f_{l-j} \right\|_{L^2}^2 \right)^{1/2}
\]

\[
= \sum_{j \geq 0} 2^{-\frac{j}{1+r}} \left( \sum_{l \in \mathbb{N}, l \geq \frac{j}{1+r}} 2^{2(l-j)} \left\| f_{l-j} \right\|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left\| f \right\|_{H^s(\mathbb{R}^2)}. \tag{2.5}
\]
where we used Theorem 1.2 to obtain
\[
\left\| \sup_{n;t_n \in A_l} |e^{it_n \Box} f_{l-j}| \right\|_{L^2(B(0,1))} \leq 2^{l-j} 2^{-\frac{r+1}{r}} \| f_{l-j} \|_{L^2},
\]
(2.6)
since
\[
2^{-2(l-j)} \leq 2^{-\frac{2l}{1+r}} \leq 2^{-(l-j)}.
\]

Finally we estimate \(III\). We have
\[
III = \sup_{l \in \mathbb{N}} \sup_{n;t_n \in A_l} \left| \sum_{k<1/r} e^{it_n \Box} f_k \right| \\
\leq \sum_{k \in \mathbb{N}} \sup_{l \in \mathbb{N}} \sup_{(r+1)k \leq n \leq A_l} \left| e^{it_n \Box} f_k \right|.
\]

Notice that when \(l \in \mathbb{N}, A_l \subset (0, 2^{-2k})\), then we have
\[
\|III\|_{L^2(B(0,1))} \leq \sum_{k \geq 0} \left\| \sup_{l \in \mathbb{N}} \sup_{(r+1)k \leq n \leq A_l} |e^{it_n \Box} f_k| \right\|_{L^2(B(0,1))} \\
\leq \sum_{k \geq 0} \left\| \sup_{l \in (0, 2^{-2k})} |e^{it \Box} f_k| \right\|_{L^2(B(0,1))} \\
= \sum_{k \geq 0} \|f_k\|_{L^2} \\
\leq \|f\|_{H^s(\mathbb{R}^2)}.
\]
(2.7)

Combining (2.4), (2.5) and (2.7), inequality (2.1) holds true for all \(f \in H^s\).

\[\square\]

3 Necessary Condition

By Nikishin’s theorem, the weak type estimate (3.1) can be established from the pointwise convergence result. One can see the appendix in [8] for more details. Then the necessary condition in Theorem 1.1 can be obtained from the following theorem.

THEOREM 3.1. Let \(0 < s < \frac{1}{2}, t_n - t_{n+1}\) be decreasing. If the weak type estimate
\[
\left| \left\{ x \in B(0,1) : \sup_{n \in \mathbb{N}} |e^{it_n \Box} f| > \frac{1}{2} \right\} \right| \leq C\|f\|_{H^s}^2
\]
(3.1)
holds for all \(f \in H^s(\mathbb{R}^2)\), then \(\{t_n\}_{n=1}^\infty \in \ell^r(s,\infty)(\mathbb{N}), r(s) = \frac{1}{1-s}\).

PROOF. Since
\[
1 - r(s) = \frac{1 - 2s}{1 - s} > 0,
\]
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according to Lemma 3.2 in [8], if \( \{ t_n \}_{n=1}^{\infty} \notin \ell^{(s),\infty}(\mathbb{N}) \), then we can choose \( \{ b_j \}_{j=1}^{\infty} \) and \( \{ M_j \}_{j=1}^{\infty} \) satisfying

\[
\lim_{j \to \infty} b_j = 0, \quad \lim_{j \to \infty} M_j = \infty,
\]

and

\[
M_j b_j^{1-r(s)} \leq 1, \tag{3.2}
\]

such that

\[
\sharp \left\{ n : b_j < t_n \leq 2b_j \right\} \geq M_j b_j^{-r(s)}. \tag{3.3}
\]

By the similar argument as in [8], Proposition 3.3, when \( t_n \leq b_j \), we have

\[
t_n - t_{n+1} \leq 2M_j^{-1}b_j^{r(s)+1}. \tag{3.4}
\]

For fixed \( j \), choose

\[
\lambda_j = \frac{1}{1000} M_j^{1/2} b_j^{-r(s)+1/2},
\]

\[
\tilde{f}_j(\xi_1, \xi_2) = \frac{1}{\lambda_j} \chi_{[0,\lambda_j] \times [-\lambda_j,-\lambda_j]}(\xi_1, \xi_2).
\]

Therefore,

\[
\|f_j\|_{H^s} = \lambda_j^{s-\frac{1}{2}}. \tag{3.5}
\]

Set

\[
U_j = (0, \frac{\lambda_j b_j}{2}) \times \left( -\frac{1}{1000}, \frac{1}{1000} \right).
\]

Notice that \( U_j \subset B(0,1) \) due to inequality (3.2). We will show that for each \( x \in U_j \),

\[
\sup_{n \in \mathbb{N}} |e^{it_n \square} f_j| > \frac{1}{2}. \tag{3.6}
\]

Indeed, changing of variables shows that for each \( n \in \mathbb{N} \),

\[
|e^{it_n \square} f_j(x)| = \left| \frac{1}{\lambda_j} \int_{-\lambda_j}^{\lambda_j} \int_{-\lambda_j}^{\lambda_j} e^{ix_1 \xi_1 + ix_2 \xi_2 + it_n \xi_1 \xi_2} d\xi_1 d\xi_2 \right|
\]

\[
= \left| \int_{-1}^{1} \int_{0}^{1} e^{i\lambda_j (x_1 - \lambda_j t_n) \eta_1 + ix_2 \eta_2 + it_n \lambda_j \eta_1 \eta_2} d\eta_1 d\eta_2 \right|. \tag{3.7}
\]

For each \( x \in U_j \), there exists a unique \( n(x,j) \) such that

\[
x_1 \in (\lambda_j t_n(x,j)+1, \lambda_j t_n(x,j)].
\]

It is obvious that \( t_n(x,j) \leq b_j \) due to \( t_n(x,j)+1 \leq b_j^2 \), inequality (3.3) and the assumption that \( t_n - t_{n+1} \) is decreasing. Then it follows from inequality (3.4) that

\[
|\lambda_j (x_1 - \lambda_j t_n(x,j)) \eta_1 | \leq 2\lambda_j^2 M_j^{-1}b_j^{r(s)+1} \leq \frac{1}{1000}. \tag{3.8}
\]

Also,

\[
|x_2 \eta_2| \leq \frac{1}{1000}. \tag{3.9}
\]
and by inequality (3.2), we have
\[
|\lambda_j t_{n(x,j)} \eta_1 \eta_2| \leq \lambda_j b_j \leq \frac{1}{1000}.
\] (3.10)

Therefore, if we take \( n = n(x,j) \) in inequality (3.7), then the phase function will be sufficiently small such that
\[
|e^{itn(x,j)\square f_j(x)}| > \frac{1}{2}
\]
for each \( x \in U_j \), which implies inequality (3.6).

By the weak type estimate we have
\[
\left\{ x \in U_j : \sup_{n \in \mathbb{N}} |e^{itn \square f_j}| > \frac{1}{2} \right\} \leq C \|f_j\|_{H^s}^2.
\]

Then it follows from inequality (3.5) and inequality (3.6) that
\[
1 \leq CM_j^{s-1}.
\]

This is not true when \( j \) is sufficiently large. \( \square \)

**Remark 3.2.** The original idea we adopted to construct the counterexample in the proof of Theorem 3.1 comes from [20]. The same idea remains valid in general dimensions. For example, in \( \mathbb{R}^3 \), by changing variables, we can write
\[
e^{itP(D)} f(x) := \int_{\mathbb{R}^3} e^{ix \cdot \xi + it(\xi_1 \xi_2 + \xi_3)} \hat{f}(\xi) d\xi.
\]

In order to prove the necessary condition, we only need to take
\[
U_j = (0, \frac{\lambda_j b_j}{2}) \times (-\frac{1}{1000}, \frac{1}{1000}) \times (-\frac{1}{1000}, \frac{1}{1000})
\]
and
\[
\hat{f}_j(\xi_1, \xi_2, \xi_3) = \frac{1}{\lambda_j} \chi_{[0, \lambda_j] \times [-\lambda_j, -1-\lambda_j] \times (0,1)}(\xi_1, \xi_2, \xi_3).
\]

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