Two-Parameter Dynamics and Geometry

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ABSTRACT: In this paper we present the two-parameter dynamics which is implied by the law of inertia in flat spacetime. A remarkable perception is that (A)dS_4 geometry may emerge from the two-parameter dynamics, which exhibits some phenomenon of dynamics/geometry correspondence. We also discuss the Unruh effects within the context of two-parameter dynamics. In the last section we construct various invariant actions with respect to the broken symmetry groups.
1 \((A)dS_4\) Geometry from Two-Parameter Dynamics

We start with a triple \((\mathbb{R}^4, \eta, B)\) for the Minkowski metric \(\eta = \text{diag}(-1, 1, 1, 1)\) and a non-degenerate symmetric bilinear form \(B\). We take the following ansatz in terms of the Lorentz coordinates \(\{x^0 = ct, x^1, x^2, x^3\}\)

\[
B_{\mu\nu} = A_0(x) \eta_{\mu\nu} + A_1(x) \frac{\eta_{\alpha\beta}(x^\alpha x^\beta) d_1}{l_1^2} + A_2(x) \frac{\eta_{\alpha\beta}(x^\alpha x^\beta) d_2}{l_2^2} + \cdots + A_n(x) \frac{\eta_{\alpha\beta}(x^\alpha x^\beta) d_n}{l_n^2},
\]

(1.1)

thus there are \(n + 1\) universal constants: \(c, l_1, \cdots, l_n\), which will be all set up to 1 for convenience. Let us consider the following action\[1\]

\[
S = \int \sqrt{B_{\mu\nu} dx^\mu dx^\nu} = \int dt \sqrt{B_{00} + 2B_{0i} v^i + B_{ij} v^i v^j},
\]

(1.2)

where \(v^i = \frac{dx^i}{dt}, i = 1, 2, 3\). Thereby the Lagrangian is given by

\[
L(t, x^i, v^i) = \sqrt{-A_0 + \sum_i A_i t^{2d_i} - 2 \sum_i A_i v^i (tx^i)^{d_i} + A_0 v \cdot v + \sum_i A_i (x^i x^j)^{d_i} v^i v^j},
\]

(1.3)

and the Euler-Lagrange equation reads

\[
\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial v^i} = \frac{\partial}{\partial t} \frac{\partial L}{\partial v^i} + v^j \frac{\partial}{\partial x^j} \frac{\partial L}{\partial v^i} + \frac{dv^i}{dt} \frac{\partial^2 L}{\partial v^j \partial v^i}.
\]

(1.4)

If a free particle of mass \(m = 1\) is assumed to be subject to this action, namely the Euler-Lagrange equation implies the acceleration has to vanish for the free particle (i.e. the law of inertia), we should have

\[
\begin{align*}
\text{det}\left(\frac{\partial^2 L}{\partial v^j \partial v^i}\right) &\neq 0, \\
\frac{\partial L}{\partial x^i} = \frac{\partial}{\partial t} \frac{\partial L}{\partial v^i} + v^j \frac{\partial}{\partial x^j} \frac{\partial L}{\partial v^i},
\end{align*}
\]

(1.5)
We observe that the unique promise of (1.5) being satisfied may arrive when taking $n = 1, d_1 = 1$, then the first equation of (1.5) reduces to

\[ - \partial_t A_0 + \partial_t A_1 t^2 - 2 \partial_t A_1 t x \cdot v + \partial_t A_0 v \cdot v + \partial_t A_1 (x \cdot v)^2 \]

\[ = - 2 \partial_t A_1 t x^i - 2 A_1 x^i + 2 \partial_t A_0 v^i + 2 \partial_t A_1 x^i \cdot v - 2 v \cdot \nabla A_1 t x^i + 2 v \cdot \nabla A_0 v^i + 2 v \cdot \nabla A_1 x^i \cdot v + 2 A_1 x^i v \cdot v \]

\[
\begin{align*}
&= \frac{(- \partial_t A_0 + 2 A_1 t + \partial_t A_1 t^2 - 2 A_1 t x \cdot v - 2 \partial_t A_1 t x \cdot v + \partial_t A_0 v \cdot v + \partial_t A_1 (x \cdot v)^2)(-2 A_1 t x^i + 2 A_0 v^i + 2 A_1 x^i v \cdot v)}{2(-A_0 + A_1 t^2 - 2 A_1 t x \cdot v + A_0 v \cdot v + A_1 (x \cdot v)^2)} \]
\end{align*}
\]

where the following notations are employed

\[ x = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}, \]

\[ v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}, \]

\[ \nabla A_{0/1} = \partial_t A_{0/1} \frac{\partial}{\partial x^1} + \partial_2 A_{0/1} \frac{\partial}{\partial x^2} + \partial_3 A_{0/1} \frac{\partial}{\partial x^3}, \]

and $x \cdot v = \sum_i x^i v^i$, $v \cdot v = \sum_i v^i v^i$, $v \cdot \nabla A_{0/1} = \sum_i x^i \partial_i A_{0/1}$.

Comparing the monomials of the both sides of the equality with the same type, we derive the following 1-order partial differential equations

\[
\begin{align*}
\partial_t A_0 &= 2 A_1 x^i, \\
\partial_t A_0 &= -2 A_1 t, \\
\partial_t A_1 A_0 &= 4 A_1^2 x^i, \\
\partial_t A_1 A_0 &= -4 A_1^2 t,
\end{align*}
\]

thus

\[
\begin{align*}
\partial_\mu A_0 &= 2 A_1 \eta_{\mu \nu} x^\nu, \\
\partial_\mu A_1 A_0 &= 4 A_1^2 \eta_{\mu \nu} x^\nu.
\end{align*}
\]

Obviously, if $A_0$ or $A_1$ is constant, then $A_1$ must vanish and $A_0$ is constant, so everything essentially goes back to the classical theory when $B = \eta$ up to a positive constant. The non-trivial solutions are given by

\[
A_0 = \frac{a}{b + \eta_{\mu \nu} x^\mu x^\nu}; A_1 = -\frac{a}{(b + \eta_{\mu \nu} x^\mu x^\nu)^2},
\]

with two dimensionless constants $a, b$. To check the second condition in (1.5), we only need to show that under the limit $l_1 \to \infty$, which is straightforward calculated

\[
\lim_{l_1 \to \infty} \det\left( \frac{\partial^2 L}{\partial v^i \partial v^j} \right) = -\frac{1}{b^2 (\mathbf{v} \cdot \mathbf{v} - 1)^2} \neq 0.
\]
Moreover if we assume that $B$ has the same signature as $\eta$, then we has to require

\[
B_{00} = -\frac{a(b + x \cdot x)}{(b - t^2 + x \cdot x)^2} < 0,
\]

\[
\tilde{B}_{11} = \frac{a(b + (x^2)^2 + (x^3)^2)}{(b - t^2 + x \cdot x)(b + x \cdot x)} > 0,
\]

\[
\det \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{12} & B_{22} \end{pmatrix} = \frac{a^2(b + (x^3)^2)}{(b - t^2 + x \cdot x)^2(b + x \cdot x)} > 0,
\]

\[
\det \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} \\ \tilde{B}_{12} & B_{22} & B_{23} \\ \tilde{B}_{13} & B_{23} & B_{33} \end{pmatrix} = \frac{a^3b}{(b - t^2 + x \cdot x)^3(b + x \cdot x)} > 0,
\]

where $\tilde{B}_{ij} = B_{ij} - \frac{B_{0i}B_{0j}}{B_{00}}$, namely the following conditions should be imposed

\[
a > 0, b > 0, \quad \text{and} \quad b - t^2 + x \cdot x > 0. \quad (1.9)
\]

or

\[
a < 0, b < 0, b + x \cdot x < 0. \quad (1.10)
\]

For example, let $a = b = 1$, and let the short distance approximation $\frac{x^i}{t^i} \ll 1$ for $i = 1, 2, 3$ be adopted, then the action (1.2) for a free particle is simplified to be integrated out

\[
S = \int \sqrt{|B_{00} + (v^1)^2B_{11} + (v^2)^2B_{22} + (v^3)^2B_{33}|}dt
\]

\[
= \int \sqrt{\frac{1}{1-t^2} - \frac{v \cdot v}{1-t^2}}dt
\]

\[
= \frac{1}{2} \ln \left[\frac{1-v \cdot v(1-t^2) + t}{1-v \cdot v(1-t^2) - t} - \sqrt{v \cdot v} \ln[\sqrt{v \cdot v}t + \sqrt{1-v \cdot v(1-t^2)}]\right]. \quad (1.11)
\]

A remarkable perception is that (A)dS$_4$ geometry may emerge from the two-parameter dynamics of free particle in flat spacetime. This picture exhibits some phenomenon of dynamics/geometry correspondence. (A)dS$_4$ is defined by a hypersurface in 5-dimensional space $\mathbb{R}^5$ with the Minkowski metric $\eta^{(5)} = \text{diag}(-1, 1, 1, 1, 1)$ (or the metric $\tilde{\eta}^{(5)} = \text{diag}(-1, -1, 1, 1, 1)$ with signature 1) via the following equation\cite{2}

\[-T^2 + X^2 + Y^2 + Z^2 + bW^2 = 1(b > 0),
\]

or

\[-T^2 - bW^2 + X^2 + Y^2 + Z^2 = -1(b > 0, T < 1).
\]

Define the following coordinates which cover the half domain $\{W > 0\}$ or $\{W < 0\}$ in (A)dS$_4$

\[
x^0 = \frac{T}{W}, \quad x^1 = \frac{X}{W}, \quad x^2 = \frac{Y}{W}, \quad x^3 = \frac{Z}{W}, \quad (1.12)
\]
Then the induced metric on this hypersurface is given by in terms of the coordinate system \( \{x^0, x^1, x^2, x^3\} \)

\[
g_{\mu\nu} = \frac{b \eta_{\mu\nu}}{b + \eta_{\alpha\beta} x^\alpha x^\beta} - \frac{b \eta_{\mu\alpha} \eta_{\nu\beta} x^\alpha x^\beta}{(b + \eta_{\alpha\beta} x^\alpha x^\beta)^2}, \tag{1.13}
\]

or

\[
g_{\mu\nu} = \frac{b \eta_{\mu\nu}}{b - \eta_{\alpha\beta} x^\alpha x^\beta} + \frac{b \eta_{\mu\alpha} \eta_{\nu\beta} x^\alpha x^\beta}{(b - \eta_{\alpha\beta} x^\alpha x^\beta)^2}, \tag{1.14}
\]

which exactly coincides with our quadratic form \( B \) with condition (1.9) or (1.10) up to an insignificant constant conformal factor.

From this viewpoint, we immediately conclude that the coordinate transformations preserve \( B \) form the group \( O(1,4) \) or \( O(2,3) \) which contains Lorentz group \( O(1,3) \) as a subgroup preserving the pair \( (\eta, B) \), thus preserving the inertial motion. By decomposing a matrix belongs to the group \( O(1,4) \) or \( O(2,3) \) as

\[
\lambda \left( \pm \frac{N}{\sqrt{1 + \eta_{\mu\nu} P^\mu P^\nu}} \right) \frac{P}{\pm 1 + \eta_{\mu\nu} P^\mu P^\nu},
\]

with matrices \( N = (N_{\mu\nu}) \) and \( P = (P^0, P^1, P^2, P^3)^T \) satisfying the relation

\[
N^T \eta N = \eta + \frac{N^T \eta P P^T \eta N}{\mp 1 + \eta_{\mu\nu} P^\mu P^\nu}, \tag{1.15}
\]

where \( \mp \) correspond \( O(1,4) \) and \( O(2,3) \) respectively, and \( \lambda \) is fixed to 1 or \(-1\), we can explicitly determine these coordinate transformations as fractional linear transformations\([3, 4]\)

\[
x^\mu \mapsto \frac{N_{\mu\nu} x^\nu + \sqrt{b} P^\mu}{\mp \eta_{\alpha\beta} N_{\nu\gamma} P^\nu x^\gamma \sqrt{1 + \eta_{\alpha\beta} P^\alpha P^\beta}} + \sqrt{b} \frac{\mp \eta_{\mu\nu} P^\mu P^\nu}{\sqrt{1 + \eta_{\mu\nu} P^\mu P^\nu}}, \tag{1.16}
\]

which come back to Poincaré transformations when the parameter \( l_1 \) tends to infinity. Since the action (2) is invariant under these transformations, there are corresponding conserved charges for a free particle, which can be given via Norther method. Non-trivial charges would reflect the dynamical (not geometrical) symmetries.

## 2 Unruh Effects Within the Context of Two-Parameter Dynamics

A Klein-Gordon-type equation that governs scalar field \( \Phi \) with mass \( m \) under the context of two-parameters dynamics is presented as

\[
\frac{1}{\sqrt{|\det(B_{\mu\nu})|}} \partial_{\mu}\sqrt{|\det(B_{\mu\nu})|} B^{\mu\nu} \partial_{\nu} \Phi - (m^2 + \frac{\xi}{l_1^2}) \Phi = 0 \tag{2.1}
\]

for a dimensionless constant \( \xi \), which is invariant under the transformations (1.16). Here the quadratic form \( B \) is taking the form in (1.13) with \( b = 1 \), and \( B^{\mu\nu} = \left( \frac{\eta^{\mu\nu} - x^\mu x^\nu}{1 - t^2 + x x} \right) \) is
the inverse of $B$. Assume that the scalar field is distributed on a small spatial domain, thus the short distance approximation is valid. Changing variables as
\[
\begin{align*}
\tilde{x}_0 &= \frac{1}{2} \ln \frac{l+1}{l}, \\
\tilde{x}_i &= x_i, i = 1, 2, 3, (2.2)
\end{align*}
\]
the equation (2.1) reduces to
\[
(\eta^{\mu\nu} \tilde{\partial}_\mu \tilde{\partial}_\nu + \tilde{\partial}_0 \ln \sigma(\tilde{x}_0) \tilde{\partial}_0 - \frac{m^2 + \xi}{\sigma(\tilde{x}_0)}) \Phi = 0, (2.3)
\]
where
\[
\sigma(\tilde{x}_0) = 1 - \left( \frac{e^{2\tilde{x}_0} - 1}{e^{2\tilde{x}_0} + 1} \right)^2. (2.4)
\]
Taking ansatz $\Phi_k = \chi_k(\tilde{x}_0)e^{ik \cdot x}$, we obtain an ordinary differential equation satisfied by the coefficient $\chi_k$
\[
\ddot{\chi}_k + F(\tilde{x}_0)\dot{\chi}_k + G(\tilde{x}_0)\chi_k = 0, (2.5)
\]
where
\[
\begin{align*}
F(\tilde{x}_0) &= 2 \sqrt{1 - \sigma(\tilde{x}_0)} = 2 \frac{e^{2\tilde{x}_0} - 1}{e^{2\tilde{x}_0} + 1}, \\
G(\tilde{x}_0) &= \frac{m^2 + \xi}{\sigma(\tilde{x}_0)} + k \cdot k.
\end{align*}
\]
We may introduce a family of solutions $\chi_k = e^{is(\tilde{x}_0)}$ with a parameter $\lambda$, and expand the function $s$ in terms of $\lambda$ as $s = s_0 + i\lambda s_1 + (i\lambda)^2 s_2 + \cdots$. By considering the power of $\lambda$, we have
\[
\begin{align*}
\left( \frac{ds_0}{d\tilde{x}_0} \right)^2 &= w^2(\tilde{x}_0), \\
\frac{d^2 s_0}{d(\tilde{x}_0)^2} - 2 \frac{ds_0 ds_1}{d\tilde{x}_0 d\tilde{x}_0} + F(\tilde{x}_0) \frac{ds_0}{d\tilde{x}_0} &= 0, \\
\frac{d^2 s_1}{d(\tilde{x}_0)^2} - \left( \frac{ds_1}{d\tilde{x}_0} \right)^2 - 2 \frac{ds_0 ds_2}{d\tilde{x}_0 d\tilde{x}_0} + F(\tilde{x}_0) \frac{ds_1}{d\tilde{x}_0} &= 0, \\
&\vdots
\end{align*}
\]
From these recurrence relations we get
\[
\begin{align*}
s_1 &= \ln \sqrt{w} + \int \frac{F(\tilde{x}_0)}{2} d\tilde{x}_0, \\
s_2 &= \frac{1}{4w^2} \frac{dw}{d\tilde{x}_0} + \frac{1}{8} \int d\tilde{x}_0 \left[ \frac{1}{w^3} \left( \frac{dw}{d\tilde{x}_0} \right)^2 + \frac{F^2}{w} + \frac{2}{w} \frac{dF}{d\tilde{x}_0} \right], \\
&\vdots
\end{align*}
\]
Substituting the first order solution
\[
\begin{align*}
\chi_k &= \frac{1}{\sqrt{w}} e^{- \int \frac{F(\tilde{x}_0)}{2} d\tilde{x}_0} e^{\pm i \int w(\tilde{x}_0) d\tilde{x}_0} \\
&= \frac{e^{\tilde{x}_0^0}}{1 + e^{2\tilde{x}_0^0}} \frac{1}{\sqrt{w}} e^{\pm i \int w(\tilde{x}_0) d\tilde{x}_0} \quad (2.6)
\end{align*}
\]
into (2.5) gives rise to the equation controls $w$

$$w^2 = G(x^0) - \frac{1}{2} \left\{ \int w dx^0; x^0 \right\}_S - \frac{1}{2} \frac{dF}{dx^0} - \frac{F^2}{4},$$

where $\left\{ \cdot \right\}_S$ denotes the Schwartz derivative defined by

$$\left\{ \int w dx^0; x^0 \right\}_S = \frac{1}{w} \frac{d^2 w}{d(x^0)^2} - 3 \frac{d w}{d(x^0)}.$$ 

The lowest order approximation is given by

$$w(0)^{(0)} = \pm \sqrt{(m^2 + \xi)(e^{2x^0} + 1)^2 - k \cdot k + 1},$$

(2.7)

and then by iteration we can get other higher order approximate solutions. In particular, when $m = \xi = 0$ we have the exact solution

$$\chi_k = \frac{e^{x^0}}{1 + e^{2x^0}} (k \cdot k - 1)^{-\frac{1}{4}} e^{\pm i k \cdot x - x^0}. 

(2.8)$$

Let us consider a detector that is in the ground state with energy $E_0$ at initial time. If it detects a particle, it will transit to an excited state with energy $E > E_0$ and meanwhile the field $\Phi$ will transit from the vacuum $|0\rangle$ to a certain excited state $|T\rangle$. Roughly speaking, the transition amplitude $\Lambda$ in this process can be calculated as[5]

$$\Lambda \sim \int_{-\infty}^{+\infty} \langle T|\Phi(x)|0\rangle e^{i S[x](E - E_0)} dS

(2.9)$$

where $x$ denotes the worldline of the detector and $S[x]$ stands for the action of the detector. The field $\Phi$ can be expanded in terms of excited modes as

$$\Phi = \int d^3k a_k \psi_k + \int d^3k a_k^\dagger \psi_k^* = \int d^3k a_k e^{ik \cdot x} \chi_k + \int d^3k a_k^\dagger e^{-ik \cdot x} \chi_k^*, 

(2.10)$$

where $a_k$ and $a_k^\dagger$ are viewed respectively as annihilation and creation operators after quantization. Therefore we have

$$\Lambda \sim \int_{-\infty}^{+\infty} e^{-ik \cdot x} \chi_k^* e^{i S(E - E_0)} dS,

(2.11)$$

where $k$ is the 3-momentum of the final state $|T\rangle$. Suppose the detector uniformly moves in a constant speed $v$, namely its action $S$ is given by (1.2). In particular, for a static detector we have $S \sim \tilde{x}^0$ (see (1.11)), hence

$$\Lambda \sim (k \cdot k - 1)^{-\frac{1}{4}} \int_{-\infty}^{+\infty} \frac{e^{x^0}}{1 + e^{2x^0}} e^{i 2x^0 (E - E_0 \pm \sqrt{k \cdot k - 1})} dx^0,

(2.12)$$

thus the Fourier transformation of the function $\frac{e^{x^0}}{1 + e^{2x^0}}$. Consequently, we arrive at

$$\Lambda \sim \frac{\pi (k \cdot k - 1)^{\frac{1}{4}}}{e^{\frac{\pi}{2} (E - E_0 \pm \sqrt{k \cdot k - 1})} + e^{-\frac{\pi}{2} (E - E_0 \pm \sqrt{k \cdot k - 1})}, 

(2.13)$$

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However, we cannot take the minus sign in (2.13) because the probability amplitude of detecting a particle with energy $E_p = \sqrt{(E - E_0)^2 + 1}$ is given by $\mathcal{A} = \int_{-\infty}^{+\infty} \frac{e^\alpha}{1 + e^{2\alpha}} \, d\alpha = \pi$, which is not acceptable physically since it tends to infinity when the parameter $l_1$ tends to infinity. For the plus sign it is easy to see $\lim_{l_1 \to +\infty} \mathcal{A} \to 0$ due to the energy of excited state being higher than that of ground state.

3 Symmetry Breaking

The Lie bracket among the basis $\{M_{AB} = -M_{BA}, A, B = 0, \cdots, 4\}$ of Lie algebra $\mathfrak{o}(1,4)$ is given by

$$[M_{AB}, M_{CD}] = \eta^{(5)}_{AB} M_{BC} + \eta^{(5)}_{BC} M_{AD} - \eta^{(5)}_{AC} M_{BD} - \eta^{(5)}_{BD} M_{AC}.$$ 

Let $J_\mu = \frac{M_{\mu \delta}}{l_1^4}, \mu = 0, \cdots, 3$, then

$$[J_\mu, J_\nu] = -\frac{M_{\mu \nu}}{l_1^4},$$

$$[J_\mu, M_{\alpha \beta}] = \eta_{\mu \alpha} J_\beta - \eta_{\mu \beta} J_\alpha,$$

$$[M_{\mu \nu}, M_{\alpha \beta}] = \eta_{\mu \beta} M_{\nu \alpha} + \eta_{\nu \alpha} M_{\mu \beta} - \eta_{\mu \alpha} M_{\nu \beta} - \eta_{\nu \beta} M_{\mu \alpha}.$$ 

These relations can be realized via the following differential operators

$$J_\mu = \partial_\mu + \frac{\eta_{\mu \alpha} x^\alpha \partial_\nu}{l_1^4},$$

$$M_{\mu \nu} = \eta_{\mu \alpha} x^\alpha \partial_\nu - \eta_{\nu \alpha} x^\alpha \partial_\mu = \eta_{\mu \alpha} x^\alpha J_\nu - \eta_{\nu \alpha} x^\alpha J_\mu.$$ 

Let us introduce the following symbols

$$K_i^\pm = \frac{1}{\sqrt{2}} (M_{0i} \pm M_{1i}), i = 2, 3$$

$$F_i^\pm = \frac{1}{\sqrt{2}} (M_{0i} \pm J_i), i = 1, 2, 3,$$

$$L_i = \frac{1}{2} \epsilon_{ijk} M_{jk}, i, j, k = 1, 2, 3,$$

$$P_i^\pm = \frac{1}{\sqrt{2}} (J_0 \pm J_1), R = M_{01}, T = M_{23}.$$ 

The maximal Lie subalgebras of rank 7 in $\mathfrak{o}(1,4)$ are exhibited in the following list

| Generators | Algebraic Relations |
|------------|---------------------|
| **Type I** $\{K_1^\pm, K_2^\pm, J_2, J_3, P^\pm, R, T\}$ | $[K_i^\pm, K_i^\pm] = 0, [J_i, J_j] = \epsilon_{ij} P^\pm$, $[K_i^\pm, J_i] = \delta_i^j P^\pm$, $[K_i^\pm, P^\pm] = 0, [K_i^\pm, R] = -K_i^\pm, [J_i, T] = \epsilon_{ij} J_j$, $[J_i, P^\pm] = \frac{K_i^\pm}{l_1^4}, [J_i, R] = 0, [J_i, T] = \epsilon_{ij} J_j$, $[P^\pm, R] = \mp P^\pm, [P^\pm, T] = 0, [R, T] = 0$ |
| **Type II** $\{F_1^\pm, F_2^\pm, F_3^\pm, L_1, L_2, L_3, J_0\}$ | $[F_i^\pm, F_i^\pm] = 0, [L_i, J_j] = -\epsilon_{ijk} L_k, [F_i^\pm, L_j] = -\epsilon_{ijk} F_k^\pm$, $[F_i^\pm, J_0] = \pm \frac{F_i^\pm}{l_1^4}, [L_i, J_0] = 0$ |
The little groups in $O(1, 4)$ corresponding to these two types of Lie subalgebras are denoted by $G_1$ and $G_2$ respectively, which exactly coincide with the groups $ISIM(2)^1$ and $O(3) \ltimes T$ ($T$ denoting the 4-dimensional translation group) respectively when the parameter $l_1$ tends to infinity. Some subgroups of $G_1$ and $G_2$ are listed as follows:

| Subgroups | Generators of Lie Algebras |
|-----------|---------------------------|
| $H_1$     | $\{K_1^\pm, K_2^\pm, P\}$ |
| $H_2$     | $\{K_1^\pm, K_2^\pm, P, R\}$ |
| $H_3$     | $\{K_1^\pm, K_2^\pm, P, T\}$ |
| $H_4$     | $\{K_1^\pm, K_2^\pm, P, R, T\}$ |
| $H_5$     | $\{P, R, T\}$ |
| $H_6$     | $\{J_2, J_3, T\}$ |
| $H_7$     | $\{J_2, J_3, R, T\}$ |
| $H_8$     | $\{K_1^\pm, K_2^\pm, P, J_2, J_3, T\}$ |

| Subgroups | Generators of Lie Algebras |
|-----------|---------------------------|
| $K_1$     | $\{F_1^\pm, F_2^\pm, F_3^\pm\}$ |
| $K_2$     | $\{L_1, L_2, L_3\}$ |
| $K_3$     | $\{F_1^\pm, F_2^\pm, F_3^\pm, J_0\}$ |
| $K_4$     | $\{L_1, L_2, L_3, J_0\}$ |
| $K_5$     | $\{F_1^\pm, F_2^\pm, F_3^\pm, L_1, L_2, L_3\}$ |

To construct an action whose symmetry group breaks into the little group $G_1$ or $G_2$ or one of their subgroups, we need find some invariant tensors under these groups.

**Example** 1. For the subgroup $H_1$ or $H_3$, by taking the following matrix representations of generators

$$K_2^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \hspace{1cm} K_3^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} ,$$

$$P^+ = \frac{1}{l_1} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} , \hspace{1cm} T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

we find a second-order non-degenerate symmetric invariant tensor with respect to $H_1$ or $H_3$

$$C = \begin{pmatrix} a & b & 0 & 0 & 0 \\ b & 2b - a & 0 & 0 & 0 \\ 0 & 0 & b - a & 0 & 0 \\ 0 & 0 & 0 & b - a & 0 \\ 0 & 0 & 0 & 0 & b - a \end{pmatrix} . \hspace{1cm} (3.1)$$

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^1In some literatures[6], $ISIM(2)$ means an 8-dimensional maximal subgroup of the Poincaré group generated by $\{K_1^\pm, K_2^\pm, J_1, J_2, P^+, P^-, R, T\}$. 

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with two constants \( a \neq b \), thus a quadratic form

\[
C = a dT^2 + 2 bdTdX + (2b - a)dX^2 + (b - a)dY^2 + (b - a)dZ^2 + (b - a)dW^2. \tag{3.2}
\]

Then the coordinate transformations (1.12) give rises to an induced quadratic form

\[
C = C_{\mu\nu} dx^\mu dx^\nu
\]

\[
= \frac{1}{(1 + \eta_{\mu\nu}x^\mu x^\nu)^2} \left[ b \left( 1 + \mathbf{x} \cdot \mathbf{x} + x^0 x^1 \right)^2 - (b - a)(1 + \mathbf{x} \cdot \mathbf{x}) \right] (dx^0)^2
+ 2b \frac{(1 + \mathbf{x} \cdot \mathbf{x} + x^0 x^1)(1 + \eta_{\mu\nu}x^\mu x^\nu - (x^1)^2 - x^0 x^1)}{1 + \eta_{\mu\nu}x^\mu x^\nu} + (b - a)x^0 x^1 dx^0 dx^1
+ 2[-b \frac{(1 + \mathbf{x} \cdot \mathbf{x} + x^0 x^1)(x^0 x^2 + x^1 x^2)}{1 + \eta_{\mu\nu}x^\mu x^\nu} + (b - a)x^0 x^2 dx^0 dx^2
+ 2[-b \frac{(1 + \mathbf{x} \cdot \mathbf{x} + x^0 x^1)(x^0 x^3 + x^1 x^3)}{1 + \eta_{\mu\nu}x^\mu x^\nu} + (b - a)x^0 x^3 dx^0 dx^3
+ [b \frac{(x^0 x^2 + x^1 x^2)^2}{1 + \eta_{\mu\nu}x^\mu x^\nu} + (b - a)(1 + \eta_{\mu\nu}x^\mu x^\nu - (x^2)^2)](dx^2)^2
+ 2[b \frac{x^2 x^3 (x^0 + x^1)^2}{1 + \eta_{\mu\nu}x^\mu x^\nu} - (b - a)x^2 x^3 dx^2 dx^3
+ [b \frac{(x^0 x^3 + x^1 x^3)^2}{1 + \eta_{\mu\nu}x^\mu x^\nu} + (b - a)(1 + \eta_{\mu\nu}x^\mu x^\nu - (x^3)^2)](dx^3)^2]. \tag{3.3}
\]

Hence the invariant action can be chosen as

\[
S = \int \sqrt{C_{\mu\nu}} dx^\mu dx^\nu. \tag{3.4}
\]

2. Similarly for the subgroup \( H_1 \), we also have a second-order anti-symmetric tensor

\[
D = \begin{pmatrix}
0 & 0 & a & b & c \\
0 & 0 & a & b & c \\
-a & -a & 0 & 0 & 0 \\
-b & -b & 0 & 0 & 0 \\
-c & -c & 0 & 0 & 0
\end{pmatrix}
\tag{3.5}
\]

with three constants \( a, b, c \), thus a 2-form

\[
D = a(dT + dX) \wedge dY + b(dT + dX) \wedge dZ + c(dT + dX) \wedge dW. \tag{3.6}
\]

Therefore the following Yang-Mills-type action is \( H_1 \)-invariant

\[
S = \int d^4x D_{\mu\nu} D_{\alpha\beta} B^{\mu\alpha} B^{\nu\beta}, \tag{3.7}
\]
where the induced 2-form $D$ is given by
\[
D = \frac{1}{2} D_{\mu \nu} dx^\mu \wedge dx^\nu
\]
\[
= \frac{1}{(1 + \eta_{\mu \nu} \omega^\mu \omega^\nu)^2} \left[ -(x^0 + x^1) (ax^2 + bx^3 + c) dx^0 \wedge dx^1
+ (a(1 + (x^1)^2 + (x^3)^2 + x^0 x^1) - bx^2 x^3 - cx^2) dx^0 \wedge dx^2
+ (b(1 + (x^1)^2 + (x^3)^2 + x^0 x^1) - ax^2 x^3 - cx^2) dx^0 \wedge dx^3
+ (a(1 - (x^0)^2 + (x^3)^2 - x^0 x^1) - bx^2 x^3 - cx^2) dx^1 \wedge dx^2
+ (b(1 - (x^0)^2 + (x^3)^2 - x^0 x^1) - ax^2 x^3 - cx^2) dx^1 \wedge dx^3
+ (x^0 + x^1)(ax^3 - bx^2) dx^2 \wedge dx^3 \right],
\]
(3.8)
which can be viewed as the strength of the field
\[
U = U_\mu dx^\mu = \frac{x^0 + x^1}{1 + \eta_{\mu \nu} \omega^\mu \omega^\nu} [x^0 (ax^2 + bx^3 + c) dx^0 - x^1 (ax^2 + bx^3 + c) dx^1
+ (a(1 + \eta_{\mu \nu} \omega^\mu \omega^\nu) - x^2 (ax^2 + bx^3 + c)) dx^2
+ (a(1 + \eta_{\mu \nu} \omega^\mu \omega^\nu) - x^3 (ax^2 + bx^3 + c)) dx^3].
\]
(3.9)
Alternativ choice is taking a Born-Infeld-type action\[7\]
\[
S = \int d^4 x \sqrt{|\det(B - DB^{-1} D)|}.
\]
(3.10)
3. For the subgroup $K_4$ whose generators are explicitly expressed as
\[
F^+_1 = \frac{1}{\sqrt{2l_1}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},
F^+_2 = \frac{1}{\sqrt{2l_1}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},
F^+_3 = \frac{1}{\sqrt{2l_1}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\]
there is an invariant vector
\[
V = (a, 0, 0, 0, -a)^T
\]
(3.11)
with a constant $a$. Therefore we can consider a Finsler-type action\[6, 8\]
\[
S = \int \left( B_{\mu \nu} dx^\mu dx^\nu \right)^{\frac{1-\delta}{2}} (V_\mu dx^\mu)^\delta
\]
(3.12)
with a constant $\delta \neq 0, 1$, thus the corresponding Lagrangian is given by
\[
L(t, x^i, v^i) = (B_{00} + 2B_{0i} v^i + B_{ij} v^i v^j)^{\frac{1-\delta}{2}} (V_0 + V_i v^i)^\delta,
\]
(3.13)
where
\[
\begin{align*}
V_0 &= a(1 + \eta_{\mu \nu} \omega^\mu \omega^\nu)^{-\frac{3}{2}} (1 + x \cdot x - x^0), \\
V_i &= a(1 + \eta_{\mu \nu} \omega^\mu \omega^\nu)^{-\frac{3}{2}} x^i (1 - x^0), i = 1, 2, 3.
\end{align*}
\]
(3.14)
The pair \((B, V)\) is preserved by \(K_1\).

4. Since there are no new invariant tensors for the group \(G_1\) or \(G_2\), any local term appears in the Lagrangian enjoys the full symmetry group \(O(1, 4)\), and hence the symmetry breaking effects are necessarily nonlocal\([9]\). For the vectors \(W = (a, a, 0, 0, 0)^T\) and \(V\), we have

\[ K_1^+ W = J_i W = TW = PW = 0, RW = W, \]
\[ F_1^+ V = J_i V = 0, J_0 V = -V, \]

namely the group \(G_1\) or \(G_2\) preserves the direction of the vector \(W\) or \(V\). Then we can write the following equation contains a nonlocal term for the scalar field \(\Phi\) with mass \(m\)

\[
\frac{1}{\sqrt{|\det B|}} \partial_\mu (\sqrt{|\det B|} B^{\mu \nu} (\partial_\nu + m^2 \frac{W_\nu}{B^{\alpha \beta} W_\alpha \partial_\beta}) \Phi) = 0, \tag{3.15}
\]

where

\[
W = W_\mu dx^\mu = a(1 + \eta_{\mu \nu} x^\mu x^\nu)^{-\frac{3}{2}} \left[(1 + \eta_{\mu \nu} x^\mu x^\nu) d(x^0 + x^1) + (x^0 + x^1)(x^0 dx^0 - x^1 dx^1 - x^2 dx^2 - x^3 dx^3)\right]. \tag{3.16}
\]

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