ON KREBES’ TANGLE

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Abstract. A genus-1 tangle \( G \) is an arc properly embedded in a standardly embedded solid torus \( S \) in the 3-sphere. We say that a genus-1 tangle embeds in a knot \( K \subseteq S^3 \) if the tangle can be completed by adding an arc exterior to the solid torus to form the knot \( K \). We call \( K \) a closure of \( G \). An obstruction to embedding a genus-1 tangle \( G \) in a knot is given by torsion in the homology of branched covers of \( S \) branched over \( G \). We examine a particular example \( A \) of a genus-1 tangle, given by Krebes, and consider its two double-branched covers. Using this homological obstruction, we show that any closure of \( A \) obtained via an arc which passes through the hole of \( S \) an odd number of times must have determinant divisible by three. A resulting corollary is that if \( A \) embeds in the unknot, then the arc which completes \( A \) to the unknot must pass through the hole of \( S \) an even number of times.

1. Introduction

We choose a standardly embedded solid torus \( S^1 \times D^2 \subset S^3 \), denoted by \( S \). Then a genus-1 tangle is a properly embedded arc in \( S \). Just as we may discuss embedding ordinary tangles in \( B^3 \) into knots and links (see [1], [2], and [3]), we may consider embedding genus-1 tangles in knots. We say that a genus-1 tangle \( G \) embeds in a knot \( K \) if \( G \) can be completed by an arc exterior to \( S \) to form the knot \( K \); that is, there exists some arc in \( S^3 - \text{Int}(S) \) such that upon gluing this arc to \( G \) along their boundary points, we have a knot in \( S^3 \) which is isotopic to \( K \). We say that \( K \) is a closure of \( G \).

Let \( l \) denote a longitude for \( S \) which is contained in \( \partial S \) and avoids the genus-1 tangle. A closure \( K \) of \( G \) is called odd (respectively, even) with respect to \( l \) if \( lk(K,l) \) is odd (respectively, even). If \( l \) is chosen to be the longitude which circles the central hole of \( S \) as in Fig. [1] and we span the longitude \( l \) by a disk \( \Delta \) filling the hole, then \( lk(K,l) \) is the number of transverse intersections counted with sign of the arc which completes \( G \) to \( K \) with \( \Delta \). Thus, in this case we can say more colloquially that \( K \) is an odd (respectively, even) closure with respect

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to \( l \) if the arc which completes \( G \) to \( K \) passes through the hole of \( S \) an odd (respectively, even) number of times.

In \([1]\), Krebes asks whether the genus-1 tangle given in Fig. 1 embeds in the unknot. We denote this tangle by \( A \), and when discussing this example, we always use the longitude \( l \) drawn in Fig. 1. Using the following results from \([4]\), we are able to partially answer the question posed by Krebes.

**Theorem 1.1** (Ruberman). Suppose \( M \) is an orientable 3-manifold with connected boundary, and \( i : M \hookrightarrow N \) where \( N \) is an orientable 3-manifold with \( H_1(N) \) torsion. Then the inclusion map \( i_* \) induces an injection of the torsion subgroup \( T_1(M) \) of \( H_1(M) \) into \( H_1(N) \).

This theorem has a useful corollary which can easily be proved directly using a Meyer-Vietoris sequence.

**Corollary 1.2** (Ruberman). Let \( M \) and \( N \) be as in Theorem 1.1 but suppose \( H_1(N) = 0 \). Then \( H_1(M) \) is torsion-free.

One obtains an obstruction to embedding genus-1 tangles in knots from Theorem 1.1 by applying the result to branched covers of \( S \) branched over genus-1 tangles.

Recall, for a given \( n \), each \( n \)-fold cover of \( S \) branched over a genus-1 tangle \( G \) is associated to a homomorphism \( \varphi : H_1(S - G) \to \mathbb{Z}_n \) which maps the meridian \( m \) of \( G \) to one. The remaining generator \( l \) of \( H_1(S - G) \) may be sent to any element of \( \mathbb{Z}_n \); we use \( \varphi(l) \) to index the \( n \)-fold branched covers. So, \( Y_{G,i} \) denotes the \( n \)-fold cover of \( S \) branched over \( G \) associated to the homomorphism \( \varphi \) which maps \( l \) to \( i \).

If a genus-1 tangle \( G \) embeds in a knot \( K \), then the \( n \)-fold cover \( X_K \) of \( S^3 \) branched over \( K \) restricts to some \( n \)-fold cover \( Y_{G,i} \) of \( S \) branched over \( G \). In this case, we say that the closure \( K \) induces the cover \( Y_{G,i} \). Then according to Theorem 1.1, the torsion subgroup \( T_1(Y_{G,i}) \) of \( H_1(Y_{G,i}) \) injects into \( H_1(X_K) \).

Note that if \( K \) is the unknot, then \( X_K \) is \( S^3 \) and according to Corollary 1.2, the torsion subgroup \( T_1(Y_{G,i}) \) is trivial. Thus, if there is any
torsion in the homology of $Y_{G,i}$, then any closure of $G$ which induces the cover $Y_{G,i}$ is not the unknot.

After applying this obstruction to the double-branched covers of $S$ branched over $A$, we prove the following results:

**Theorem 1.3.** If a knot $K$ in $S^3$ is an odd closure of $A$, then $\det(K)$ is divisible by 3.

**Corollary 1.4.** If $A$ embeds in the unknot, then the unknot is an even closure of $A$.

Before further discussion, we need to make a remark about the definition of genus-1 tangles.

**Remark 1.5.** Note that when defining genus-1 tangles, we fix a standardly embedded solid torus $S$ in the 3-sphere. The reason that we restrict to a fixed embedding is that there are many ways to re-embed a solid torus inside $S^3$.

For instance, if we perform a meridional twist on $S$ along the disk indicated in Fig. 2, the image of $A$ under this twist can be easily seen to embed in an unknot via the exterior arc pictured in Fig. 2. Thus it is necessary to specify the embedding of $S^1 \times D^2$ in the case of genus-1 tangles, and we restrict to a fixed standardly embedded solid torus in our definition.

2. **Surgery descriptions for double-branched covers**

For the purposes of this paper, we restrict our attention to double-branched covers of $S$ branched over $A$. Since a homomorphism $\varphi : H_1(S - A) \to \mathbb{Z}_2$ must map the specified longitude $l$ to either zero or one, there are two double-branched covers, $Y_{A,0}$ and $Y_{A,1}$. We call $Y_{A,0}$ the even double-branched cover because it is induced by all even
closures of $A$ (with respect to $l$). Similarly, since $Y_{A,1}$ is induced by all odd closures of $A$, we call it the odd double-branched cover.

In this section, we adapt Rolfsen’s technique to find surgery descriptions for these double-branched branched covers.

Following [3], we perform surgery near a carefully selected crossing (see Fig. 3) in such a way that after surgery we may essentially unwind $A$ (via sliding its endpoints around the boundary in the complement of $l$) so that it looks trivial. This process, illustrated in Fig. 4, results in a nice surgery description of $A$ inside $S$. Note that in the last drawing of Fig. 4, we choose to draw this surgery description in a particular way because it makes constructing branched covers easier.
Figure 5. Constructing the odd double-branched cover $Y_{A,1}$ of $S$ branched over $A$.

Now we construct the odd cover, $Y_{A,1}$. Construction is dictated by the homomorphism $\varphi : H_1(S - A) \to \mathbb{Z}_2$ corresponding to the cover. If $\varphi$ maps a generator of $H_1(S - A)$ to a non-zero element, then we cut the solid torus along a disk transverse to that generator. Thus, we have two cuts to make in the case of the odd cover.

First, we cut $S$ along a disk which is transverse to the meridian $m$ of $A$ and whose boundary is made up of the unwound genus-1 tangle $A$ together with an arc in $\partial S$. Then, because $\varphi$ sends $l$ to one, we cut $S$ along a disk which is transverse to $l$ and whose boundary is contained in $\partial S$. We then take two copies of the resulting manifold and glue them together carefully to obtain a surgery description for $Y_{A,1}$. This process is illustrated in Fig. 5.

Although it is not needed in the proof of Theorem 1.3, we also give a surgery description of the even double-branched cover $Y_{A,0}$ in Fig. 6.

3. Homology of the covers

Now we compute the homology of the odd double-branched cover. From Fig. 5 we see that the surgery description for $Y_{A,1}$ is given by
Figure 6. Obtaining a surgery description of $Y_{A,0}$.

Figure 7. A surgery description of $Y_{A,1}$

a 2-component surgery link inside a genus-2 handlebody. We denote the components of the surgery link by $\sigma$ and $\tau$, and let $H$ denote the genus-2 handlebody. The complement of $H$ in $S^3$ is a neighborhood of the handcuff graph $G$, pictured in Fig. 7 which is composed of loops $\alpha_1$ and $\alpha_2$ joined together by an arc. Then the complement of $\sigma \cup \tau$ in $H$ can be viewed as the complement of $\sigma \cup \tau \cup G$ in $S^3$. One can see that $H_1(S^3 - (\sigma \cup \tau \cup G))$ is isomorphic to $H_1(S^3 - (\sigma \cup \tau \cup \alpha_1 \cup \alpha_2))$ which is free on four generators: the meridians of $\sigma$, $\tau$, $\alpha_1$, and $\alpha_2$.

Completing the surgery by gluing in two solid tori according to $\sigma$ and $\tau$ introduces two relations on these four generators, which are given by the linking numbers of $\sigma$ and $\tau$ with each of $\sigma$, $\tau$, $\alpha_1$, and $\alpha_2$. Then $H_1(Y_{A,1})$ is isomorphic to $H_1(S^3 - (\sigma \cup \tau \cup \alpha_1 \cup \alpha_2))$ modulo these two relations, and we can get a presentation for $H_1(Y_{A,1})$ using linking
numbers. Thus, we have the following presentation matrix for $H_1(Y_{A,1})$:

$$\begin{bmatrix}
\sigma & \tau & \alpha_1 & \alpha_2 \\
\sigma & 1 & 2 & 0 & 0 \\
\tau & 2 & 1 & 0 & 0 \\
\end{bmatrix}.$$

Using row and columns operations we obtain a simpler presentation matrix:

$$\begin{bmatrix}
\sigma & \tau & \alpha_1 & \alpha_2 \\
\sigma & 1 & 0 & 0 & 0 \\
\tau & 0 & 3 & 0 & 0 \\
\end{bmatrix}.$$

Therefore, $H_1(Y_{A,1}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$ and we are now able to prove the main theorem. Corollary 1.4 follows immediately.

**Proof of Theorem 1.3** Let $K$ be an odd closure of $A$, and let $X_K$ denote the double cover of $S^3$ branched over $K$. Since $K$ is an odd closure of $A$, it induces a restriction from $X_K$ to $Y_{A,1}$. Then according to Theorem 1.1, we have that $T_1(Y_{A,1}) = \mathbb{Z}_3 \hookrightarrow H_1(X_K)$. Thus $|T_1(Y_{A,1})| = 3$ divides $|H_1(X_K)| = \det(K)$. □

We are unable to use this method to restrict all closures of $A$ because $Y_{A,0}$ has a torsion-free first homology group. Indeed, the statement in Remark 1.5 allows us to see that the even cover does embed in $S^3$ and so must have torsion-free first homology. Of course, this can be verified by deriving a presentation for the homology of $Y_{A,0}$ using the procedure above.

**References**

[1] D. A. Krebes, An obstruction to embedding 4-tangles in links, *J. Knot Theory and its Ramifications* 8 (1999) 321-352.

[2] J.H. Przytycki, D.S. Silver and S.G. Williams, 3-manifolds, tangles, and persistent invariants, *Math. Proc. Camb. Phil. Soc.* 139 (2005) 291–306.

[3] D. Rolfsen, *Knot and Links* (Publish or Perish, Berkeley, CA, 1976).

[4] D. Ruberman, Embedding tangles in links, *J. Knot Theory and its Ramifications* 9 (2000) 523-530.

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