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Constructing Subsets of a Given Packing Index in Abelian Groups

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By definition, the sharp packing index \( \text{ind}_p(A) \) of a subset \( A \) of an Abelian group \( G \) is the smallest cardinal \( \kappa \) such that for any subset \( B \subset G \) of size \( |B| \geq \kappa \) the family \( \{b + A : b \in B\} \) is not disjoint. We prove that an infinite Abelian group \( G \) contains a subset \( A \) with given index \( \text{ind}_p(A) = \kappa \) if and only if one of the following conditions holds: (1) \( 2 \leq \kappa \leq |G|^\ast \) and \( \kappa \notin \{3,4\} \); (2) \( \kappa = 3 \) and \( G \) is not isomorphic to \( \bigoplus_{i=1}^3 \mathbb{Z}_3 \); (3) \( \kappa = 4 \) and \( G \) is not isomorphic to \( \bigoplus_{i=1}^3 \mathbb{Z}_2 \) or to \( \mathbb{Z}_4 \oplus (\bigoplus_{i=1}^2 \mathbb{Z}_2) \).

The famous problem of optimal sphere packing traces its history back to B. Pascal and belongs to the most difficult problems of combinatorial geometry [CS]. In this paper we consider an analogous problem in the algebraic setting. Namely, given a non-empty subset \( A \) of an Abelian group \( G \) we study the cardinal number

\[
\text{ind}_p(A) = \sup \{|B| : B \subset G \text{ and } (B - B) \cap (A - A) = \{0\}\}
\]

called the \textit{packing index} of \( A \) in \( G \). Note that the equality \((B - B) \cap (A - A) = \emptyset\) holds if and only if \((b + A) \cap (b' + A) = \emptyset\) for any distinct points \( b, b' \in B \). Therefore, \( \text{ind}_p(A) \) can be thought as the maximal number of pairwise disjoint shift copies of \( A \) that can be placed in the group \( G \). In this situation it is natural to ask if such a maximal number always exists. In fact, this was a question of D. Dikranjan and I. Protasov who asked in [DP] if for each subset \( A \subset \mathbb{Z} \) with \( \text{ind}_p(A) \geq \aleph_0 \) there exists an infinite family of pairwise disjoint shifts of \( A \). The answer to this problem turned out to be negative, see [BL1], [BL2]. So the supremum in the definition of \( \text{ind}_p(A) \) cannot be replaced by the maximum.

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To catch the difference between sup and max, let us adjust the definition of the packing index \( \text{ind}_p(X) \) and define the cardinal number

\[
\text{ind}_p^\dag(A) = \min\{\kappa : \forall B \subseteq G |B| \geq \kappa \Rightarrow (B - B \cap A - A \neq \{0\})\}
\]
called the *sharp packing index* of \( A \) in \( G \). In terms of the sharp packing index the question of D. Dikranjan and I. Protasov can be reformulated as finding a subset \( A \subseteq \mathbb{Z} \) with \( \text{ind}_p^\dag(A) = \aleph_0 \). According to [BL_2] (and [BL_1]) such a set \( A \) can be found in each infinite (abelian) group \( G \). Having in mind this result, I. Protasov asked in a private conversation if for any non-zero cardinal \( \kappa \leq |G| \) there is a set \( A \subseteq G \) with \( \text{ind}_p(A) = \kappa \). In this paper we answer this question affirmatively (with three exceptions). Firstly, we treat a similar question for the sharp packing index because its value completely determines the value of \( \text{ind}_p(A) \):

\[
\text{ind}_p(A) = \sup\{\kappa : \kappa < \text{ind}_p^\dag(A)\}.
\]

Our principal result is

**Main Theorem.** An infinite Abelian group \( G \) contains a subset \( A \subseteq G \) with sharp packing index \( \text{ind}_p^\dag(A) = \kappa \) if and only if one of the following conditions holds:

1. \( 2 \leq \kappa \leq |G|^+ \) and \( \kappa \notin \{3, 4\} \).
2. \( \kappa = 3 \) and \( G \) is not isomorphic to \( \bigoplus_{i \in I} \mathbb{Z}_3 \).
3. \( \kappa = 4 \) and \( G \) is not isomorphic to \( \bigoplus_{i \in I} \mathbb{Z}_2 \) or to \( \mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2) \).

Using the relation between the packing and sharp packing indices, we can derive from the above theorem an analogous characterization of possible values of the packing index.

**Corollary.** An infinite Abelian group \( G \) contains a subset \( A \subseteq G \) with packing index \( \text{ind}_p(A) = \kappa \) if and only if one of the following conditions holds:

1. \( 1 \leq \kappa \leq |G| \) and \( \kappa \notin \{2, 3\} \).
2. \( \kappa = 2 \) and \( G \) is not isomorphic to \( \bigoplus_{i \in I} \mathbb{Z}_3 \).
3. \( \kappa = 3 \) and \( G \) is not isomorphic to \( \bigoplus_{i \in I} \mathbb{Z}_2 \) or to \( \mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2) \).

**1. Preliminaries**

In the proof of Main Theorem we shall exploit a combinatorial lemma proved in this section. For a set \( A \) by \([A]^2 = \{B \subseteq A : |B| = 2\}\) we denote the family of all two element subsets of \( A \).

We shall say that a map \( f : [A]^2 \mapsto [B]^2 \)

- is *separately injective* if for any \( a \in A \) the map \( f_a : x \mapsto f(\{x, a\}) \) is injective;
- *preserves intersection* if for any \( a_0, a_1, a_2 \in A \) the intersection \( f(\{a_0, a_1\}) \cap f(\{a_0, a_2\}) \) is not empty.
Lemma 1. If $|A| \geq 5$ and a map $f: [A]^2 \mapsto [B]^2$ is separately injective and preserves intersections, then $|A| \leq |B|$.

Proof. Fix any point $a_0 \in A$ and consider the family $\{f(\{a, a_0\}): a \in A \setminus \{a_0\}\}$. Since $f$ preserves intersection we have that $f(\{a, a_0\}) \cap f(\{a', a_0\}) \neq \emptyset$ for any distinct $a, a' \in A$. Using the separate injectivity of $f$ and the inequality $|A| \geq 5$ we can prove that the intersection $\bigcap_{a \in A \setminus \{a_0\}} f(\{a, a_0\})$ is not empty and hence contains some element $b_0$. Thus we obtain that $f: \{a, a_0\} \mapsto \{b, b_0\}$. And since $f$ is separately injective we obtain an injective map from $A \setminus \{a_0\}$ into $B \setminus \{b_0\}$ implying the desired inequality $|A| \leq |B|$.

We shall also need one structure property of Abelian groups. By $\mathbb{Z}$ we denote the additive group of integer numbers and by $\mathbb{Z}(p^\infty) = \{z \in \mathbb{C} : \exists n \in \mathbb{N} \text{ with } z^{p^n} = 1\}$ the quasicyclic $p$-group for a prime number $p$.

Proposition 1. Each infinite Abelian group $G$ contains an infinite subgroup isomorphic to $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ or the direct sum of finite cyclic groups.

Proof. If $G$ contains an element $g$ of infinite order, then it generates a cyclic subgroup isomorphic to $\mathbb{Z}$. Otherwise, $H$ is a torsion group and by Theorem 8.4 [Fu] decomposes into the direct sum $G = \oplus_p A_p$ of $p$-groups $A_p$. If each group $A_p$ is finite, then $G$ contains an infinite direct product of finite cyclic group. If for some prime number $p$ the $p$-group $A_p$ is infinite, then there are two possibilities. Either $A_p$ contains a copy of the quasicyclic $p$-group $\mathbb{Z}(p^\infty)$ or else each element of $A_p$ has finite height. In the latter case, take any infinite countable subgroup $H \subset A_p$ and apply Theorem 17.3 of [Fu] to conclude that $H$ is the direct sum of finite cyclic groups.

2. The proof of the "only if" part of main theorem

The proof of the "only if" part of Main Theorem is divided into two lemmas.

Lemma 2. If a group $G$ contains a subset $A \subset G$ with $\text{ind}_p^*(A) = 3$ (which is equivalent to $\text{ind}_p(A) = 2$), then $G$ is not isomorphic to the direct sum $\oplus_{i \in I} \mathbb{Z}_3$.

Proof. On the contrary suppose that $G$ is isomorphic to the direct sum $\oplus_{i \in I} \mathbb{Z}_3$ and take a subset $A$ of $G$ with $\text{ind}_p^*(A) = 3$. The latter is equivalent to $\text{ind}_p(A) = 2$ which means that there is a subset $B_2 \subset G$ of size 2 such that the family $\{b + A : b \in B_2\}$ is disjoint. Note that for every $b' \in G$ the family $\{b + A : b \in b' + B_2\}$ is disjoint too. So without loss of generality we can assume that $B_2 = \{0, b_1\}$. The family $\{b + A : b \in B_2\}$ is disjoint and hence

$$A \cap (b_1 + A) = \emptyset.$$
Adding to both sides $b_1$ and $2b_1$ we get

$$(b_1 + A) \cap (2b_1 + A) = \emptyset;$$

$$(2b_1 + A) \cap (3b_1 + A) = \emptyset.$$  

Since $G$ is isomorphic to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_3$ we get $3b_1 = 0$. Thus we conclude that $\{b_1 + A : b \in \{0, b_1, 2b_1\}\}$ is disjoint and so $\text{ind}_p(A) > 2$ and $\text{ind}_q(A) > 3$, which contradicts our assumptions. □

**Lemma 3.** If a group $G$ contains a subset $A \subset G$ with $\text{ind}_p(A) = 4$ (which is equivalent to $\text{ind}_q(A) = 3$), then $G$ cannot be isomorphic neither to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_2$ nor to $\mathbb{Z}_4 \oplus \left( \bigoplus_{i \in I} \mathbb{Z}_2 \right)$.

**Proof.** Conversely suppose that $G$ is isomorphic to $\bigoplus_{i \in I} \mathbb{Z}_2$ or to $\mathbb{Z}_4 \oplus \left( \bigoplus_{i \in I} \mathbb{Z}_2 \right)$ and there exists a subset $A$ of $G$ with $\text{ind}_p(A) = 4$. This is equivalent to $\text{ind}_p(A) = 3$ and from the definition we get that there is a three-element subset $B_3 \subset G$ such that the family $\{b + A : b \in B_3\}$ is disjoint. Note that for any $b' \in G$ the family $\{b + A : b \in b' + B_3\}$ is disjoint too. So, without loss of generality we can assume that $B_3 = \{0, b_1, b_2\}$. Since the family $\{b + A : b \in B_3\}$ is disjoint we conclude that

$$(1) \quad A \cap (b_1 + A) = \emptyset;$$

$$(2) \quad A \cap (b_2 + A) = \emptyset;$$

$$(3) \quad (b_1 + A) \cap (b_2 + A) = \emptyset.$$  

We consider three cases.

**Case 1.** Suppose one of the element $b_1$, $b_2$ is of order 2. Let it be $b_1$. Then $2b_1 = 0$ and

$$(2) + b_1 : \quad (b_1 + A) \cap (b_1 + b_2 + A) = \emptyset;$$

$$(3) + b_1 : \quad A \cap (b_1 + b_2 + A) = \emptyset.$$  

Thus we get that the family $\{b + A : b \in \{0, b_1, b_2, b_1 + b_2\}\}$ is disjoint and hence $\text{ind}_p(A) > 3$ and $\text{ind}_q(A) > 4$, which contradicts our assumption. Thus we complete the proof of the Case 1.

Next we consider two cases both $b_1$ and $b_2$ are of order 4. In this case the group $G$ is isomorphic to $\mathbb{Z}_4 \oplus \left( \bigoplus_{i \in I} \mathbb{Z}_2 \right)$. Therefore there are two possibilities: $b_1 = (g, x)$, $b_2 = (g, y)$ or $b_1 = (g, x)$, $b_2 = (-g, y)$ where $x, y \in \bigoplus_{i \in I} \mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is order 4.

**Case 2.** Suppose $b_1 = (g, x), b_2 = (g, y)$ where $x, y \in \bigoplus_{i \in I} \mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is of order 4.

Recall that $B_3 = \{(0, 0), (g, x), (g, y)\}$ and consider the set

$$B_4 = \{(0, 0), (g, x), (g, y), (0, x + y)\}.$$
We claim that the family \( \{b + A : b \in B_3\} \) is disjoint. Indeed, since \( \{b + A : b \in B_3\} \) is disjoint we have:

\[
\begin{align*}
(1) & \quad A \cap ((g, x) + A) = \emptyset; \\
(2) & \quad A \cap ((g, y) + A) = \emptyset; \\
(3) & \quad ((g, x) + A) \cap ((g, y) + A) = \emptyset.
\end{align*}
\]

Then

\[
\begin{align*}
(3) + (3g, y) & \quad : \quad ((0, x + y) + A) \cap A = \emptyset; \\
(2) + (0, x + y) & \quad : \quad ((0, x + y) + A) \cap ((g, x) + A) = \emptyset; \\
(1) + (0, x + y) & \quad : \quad ((0, x + y) + A) \cap ((g, y) + A) = \emptyset;
\end{align*}
\]

Hence, the family \( \{b + A : b \in B_4\} \) is disjoint which implies \( \ind_p(A) > 3 \) and \( \ind_p^+(A) > 4 \), a contradiction with the assumption.

**Case 3.** Suppose \( b_1 = (g, x), b_2 = (-g, y) \) where \( x, y \in \bigoplus_{i=1}^n \mathbb{Z}_2 \) and \( g \in \mathbb{Z}_2 \) and \( g \in \mathbb{Z}_4 \) is of order 4.

In this case \( B_3 = \{(0, 0), (g, x), (-g, y)\} \). Put \( B_4 = \{(0, 0), (g, x), (-g, y), (2g, x + y)\} \). We claim that the family \( \{b + A : b \in B_4\} \) is disjoint. Indeed, since \( \{b + A : b \in B_3\} \) is disjoint we have:

\[
\begin{align*}
(1) & \quad A \cap ((g, x) + A) = \emptyset; \\
(2) & \quad A \cap ((-g, y) + A) = \emptyset; \\
(3) & \quad ((g, x) + A) \cap ((-g, y) + A) = \emptyset.
\end{align*}
\]

Then

\[
\begin{align*}
(3) + (g, y) & \quad : \quad ((2g, x + y) + A) \cap A = \emptyset; \\
(2) + (2g, x + y) & \quad : \quad ((2g, x + y) + A) \cap ((g, x) + A) = \emptyset; \\
(1) + (2g, x + y) & \quad : \quad ((2g, x + y) + A) \cap ((-g, y) + A) = \emptyset.
\end{align*}
\]

Hence the family \( \{b + A : b \in B_4\} \) is disjoint and thus \( \ind_p(A) > 3 \) and \( \ind_p^+(A) > 4 \), which contradicts our assumption.

Thus if \( G \) contains a subset \( A \subset G \) with \( \ind_p^+(A) = \kappa \) then one of the condition (1)–(3) holds.

3. **The proof of the if part “of” main theorem**

To prove the “if” part of the Main Theorem, we should construct a subset \( A \subset G \) with \( \ind_p^+(A) = \kappa \) for any cardinal \( \kappa \) satisfying one of the conditions (1)–(3). First we shall construct a subset \( A_\kappa \) assuming that we have in disposal an auxiliary subset \( B_\kappa \) with some properties. Next, a subset \( B_\kappa \) with the desired properties will be constructed in each group \( G \).

**Proposition 2.** An infinite Abelian group \( G \) contains a subset \( A_\kappa \) with \( \ind_p^+(A_\kappa) = \kappa \) if there exists a subset \( B_\kappa = -B_\kappa \) of \( G \) with the following properties:
(1,) for every cardinal $\alpha < \kappa$ there is a subset $B_{\alpha}$ of size $|B_{\alpha}| = \alpha$ such that $B_{\alpha} - B_{\alpha} \subset B_{\kappa}$;

(2,) $B_{\kappa} - B_{\kappa} \not\subset B_{\kappa}$ for any subset $B_{\kappa} \subset G$ of size $\kappa$;

(3,) $F + B_{\kappa} \neq G$ for any subset $F \subset G$ of size $|F| < |G|$.

By $|A|$ we denote the cardinality of a set $A$.

**Proof.** Let $B_{\kappa} = B_{\kappa} \setminus \{0\}$. We shall construct a subset $A_{\kappa} \subset G$ such that $(B_{\kappa} + A_{\kappa}) \cap A_{\kappa} = \emptyset$. Moreover, the subset $A_{\kappa}$ will be constructed so that $G \setminus B_{\kappa} \subset A_{\kappa} - A_{\kappa}$.

Let $\lambda = |G \setminus B_{\kappa}|$ and $G \setminus B_{\kappa} = \{g_{\alpha} : \alpha < \lambda\}$ be an enumeration of $G \setminus B_{\kappa}$ by ordinals $\alpha < \lambda$.

We put $A_{\kappa} = \bigcup_{\alpha < \lambda} \{a_{\alpha}g_{\alpha} + a_{\alpha}\}$, where a sequence $(a_{\alpha})_{\alpha < \lambda}$ is to be defined later. This clearly forces that $G \setminus B_{\kappa} \subset A_{\kappa} - A_{\kappa}$.

The task is now to find a sequence $(a_{\alpha})_{\alpha < \lambda}$ such that $(B_{\kappa} + A_{\kappa}) \cap A_{\kappa} = \emptyset$. We define this sequence by induction.

We start with $a_0 = 0$. Assuming that for some $\alpha$ the points $a_{\beta}, \beta < \alpha$, have been constructed, put $F_{\alpha} = \{a_{\beta}g_{\beta} + a_{\beta} : \beta < \alpha\}$.

According to the property (3,) of the set $B_{\kappa}$ we can pick a point $a_{\alpha} \in G$ so that $a_{\alpha} \notin (F_{\alpha} + B_{\kappa}) \cup (F_{\alpha} - g_{\alpha} + B_{\kappa})$.

This gives $(B_{\kappa} + A_{\kappa}) \cap A_{\kappa} = \emptyset$.

It remains to show that $A_{\kappa}$ satisfies the conclusion of the theorem.

According to the property (1,) of the set $B_{\kappa}$, for any cardinal $\alpha < \kappa$ there is $B_{\alpha}$ such that $B_{\alpha} - B_{\alpha} \subset B_{\kappa}$. From the fact that $B_{\kappa} + A_{\kappa} \cap A_{\kappa} = \emptyset$ we conclude that $(b - b' + A_{\kappa}) \cap A_{\kappa} = \emptyset$ for all distinct $b, b' \in B_{\alpha}$. Thus for any cardinal $\alpha < \kappa$ there is $B_{\alpha}$ such that the family $\{b + A_{\kappa} : b \in B_{\alpha}\}$ is disjoint and so $\text{ind}_{\kappa}^{d}(A_{\kappa}) = \kappa$.

Let us show that $\text{ind}_{\kappa}^{d}(A_{\kappa}) = \kappa$. According to the property (2,), for any subset $B_{\alpha} \subset G$ of size $\kappa$ there are $b, b' \in B_{\kappa}$ such that $b, b' \notin B_{\kappa}$. Therefore $b - b' \in G \setminus B_{\kappa} \subset A_{\kappa} - A_{\kappa}$. Hence $b + A_{\kappa} \cap b' + A_{\kappa} = \emptyset$, which yields $\text{ind}_{\kappa}^{d}(A_{\kappa}) \leq \kappa$. Combining the two inequalities, we get $\text{ind}_{\kappa}^{d}(A_{\kappa}) = \kappa$.

The proof of the Main Theorem will be completed as soon as we construct a subset $B_{\kappa}$ with properties (1,) — (3,). This will be done in the following five lemmas.

**Lemma 4.** Let $\kappa = 3$ and $G$ be an infinite Abelian group which is not isomorphic to the direct sum $\bigoplus_{i \in \mathbb{Z}_3} \mathbb{Z}_3$. Then $G$ contains a subset $B_3$ with the properties (1,3) — (3,3).

**Proof.** Pick any nonzero point $g \in G$ whose order is not equal to 3 and consider the set $B_3 = B_2 - B_2 = \{0, \pm g\}$ where $B_2 = \{0, g\}$. It is clear that $B_3$ has the properties (1,3), (3,3). So it is enough to show that $B_3$ satisfies the property (2,3). Note that if $2g = 0$ then $B_3 = \{0, g\}$ is a subgroup of $G$ and hence has the property (2,3).
So we assume that $2g \neq 0$ which yields that $B_3 = \{0, g, -g\}$ contains three elements. To prove that $B_3$ has property $(2_3)$ fix some subset $B_3 \subseteq G$ of size $3$ and pick any point $b_0 \in B_3$. If there is $b \in B_3$ with

$$b = b_0 \notin B_3 = \{0, g, -g\}$$

then there is nothing to prove. Otherwise we have that

$$B_3 - b_0 \subseteq B_3.$$

Since $|B_3| = 3$ there are $b, b' \in B_3$ such that $b - b_0 = g; b' - b_0 = -g$. Hence we get $b - b' = 2g$. From the choice of elements $g$ we get that $2g \notin B_3$. Hence $b_2 - b \notin B_3$ and $B$ has the property $(2_3)$ which completes the proof of the lemma. □

**Lemma 5.** Let $\kappa = 4$ and $G$ be an infinite Abelian group which is not isomorphic to $\bigoplus_{i \in \mathbb{Z}/2}\bigoplus_{j \in \mathbb{Z}/2}$ or to $\mathbb{Z}_4 \oplus \bigoplus_{i \in \mathbb{Z}/2}$. Then $G$ contains a subset $B_4$ with properties $(1_4), (3_4)$.

**Proof.** We consider three cases.

**Case 1.** Suppose a group $G$ contains an element $g$ with order $> 5$.

Put $B_4 = B_3 - B_3 = \{0, \pm g, \pm 2g\}$ where $B_3 = \{0, g, -g\}$. It is easily to check that $B_4$ has the properties $(1_4), (3_4)$. We claim that $B_4$ satisfies the property $(2_4)$.

To derive a contradiction, suppose that there is a subset $B_4 \subseteq G$ of size $|B_4| = 4$ such that $B_4 - B_4 \subseteq B_4 = \{0, g, -g, 2g, -2g\}$.

Fix some element $b_0 \in B_4$. Since $B_4 - b_0 \subseteq B_4$ there are $b, b' \in B_4$ such that $b - b_0 = -g; b' - b_0 = 2g$ or $b - b_0 = g; b' - b_0 = -2g$.

Then $b' - b = 3g$ or $b' - b = -3g$.

Note that since the order of $g$ is greater than $5$, neither $3g \in B_4$ nor $-3g \in B_4$. Thus we get $b' - b \notin B_4$, a contradiction with the assumption. Hence $B_4$ satisfies the property $(2_4)$ and we complete the proof of Case 1.

**Case 2.** Assume that $G$ contains no element of order greater that $5$. Then $G$ is the direct sum of cyclic groups according to Theorem 17.2 of [Fu]. More precisely, $G$ is isomorphic either to $\bigoplus_{i \in \mathbb{Z}/2}\bigoplus_{j \in \mathbb{Z}/4}$ or to $\mathbb{Z}_4 \oplus \bigoplus_{i \in \mathbb{Z}/2}$ or to $\mathbb{Z}_6 \oplus \bigoplus_{i \in \mathbb{Z}/5}$. Since $G$ is not isomorphic to $\bigoplus_{i \in \mathbb{Z}/2}$ or $\mathbb{Z}_4 \oplus \bigoplus_{i \in \mathbb{Z}/2}$, we have to consider the following two cases: $G$ contains a subgroup isomorphic to $Z_3$ and $G$ contains a subgroup isomorphic to $Z_i \oplus Z_j$ for some $4 \leq i, j \leq 5$.

**Case 2a.** Suppose that $G$ contains a subgroup $H$ isomorphic to $Z_3$.

In this case we put $B_4 = H$ and see that $B_4$ has the properties $(1_4) - (3_4)$.

**Case 2b.** Suppose $G$ contains a subgroup isomorphic to the direct sum of $Z_i \oplus Z_j$ for some $4 \leq i, j \leq 5$.  

75
We shall identify $\mathbb{Z}_i \oplus \mathbb{Z}_j$ with a subgroup of $G$ and shall find a subset $B_4 \subset \mathbb{Z}_i \oplus \mathbb{Z}_j$ with the properties $(1_4) - (3_4)$. Obviously $B_4$ has the same properties in the whole group $G$.

Put $B_4 = B_3 - B_3$ where $B_3 = \{(0,0), (g_1,0), (0,g_2)\}$. It is clear that $B_4$ has the properties $(1_4), (3_4)$. We claim that $B_4$ has property $(2_4)$. Indeed, assuming the converse, we would find a subset $B_4 \subset G$ of size $|B_4| = 4$ with $B_4 - B_4 \subseteq B_3 - B_3$.

Fix any point $b_0 \in B_4$. Then

$$B_4 - b_0 \subseteq \mathbb{Z}_4 = \{(0,0), (g_1,0), (0,g_2), (-g_1,0), (0,-g_2), (g_1,-g_2), (-g_1, g_2)\}.$$  

Let us show that $(g_1,0) \notin B_4 - b_0$. Since the elements $g_1$ and $g_2$ have order $\geq 4$,

$$\begin{align*}
(g_1,0) - (-g,0) &\notin \mathbb{Z}_4; \\
(g_1,0) - (-g_1, g_2) &\notin \mathbb{Z}_4; \\
(g_1,0) - (0,-g_2) &\notin \mathbb{Z}_4.
\end{align*}$$

Thus if there is $b \in B_4$ with $b - b_0 = (g_1,0)$ then

$$B_4 - b_0 \subseteq \mathbb{Z}_4 = \{(0,0), (g_1,0), (0,g_2), (g_1,0), (g_1,-g_2)\}.$$  

From the above and the fact that $|B_4| = 4$ we get that there are $b_1, b_2 \in B_4$ such that $b_1 - b_0 = (0,g_2)$ and $b_2 - b_0 = (g_1,-g_2)$. Hence $b_2 - b_1 = (g_1, -2g_2) \notin \mathbb{Z}_4$, a contradiction with the assumption that $B_4 - B_4 \subseteq \mathbb{Z}_4$. So, we conclude that $(g_1,0) \notin B_4 - b_0$.

In the same manner we can show that none of the elements $(0,g_2), (-g_1,0), (0,-g_2)$ belong to $B_4 - b_0$, which contradicts the fact that $B_4 - B_4 \subseteq \mathbb{Z}_4$. This completes the proof of Lemma.

**Lemma 6.** If $\kappa > 4$ is a finite cardinal, then each infinite Abelian group $G$ contains a subset $B_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$.

**Proof.** It is easy to check that each subset $B_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$ in a subgroup $H \subset G$ has these properties in the whole group $G$. This observation combined with Proposition 1 reduces the problem to constructing a set $B_\kappa$ in the groups $\mathbb{Z}, \mathbb{Z}(p^{\infty})$ or the direct sum of finite cyclic groups. This will be done separately in the following three cases.

**Case 1.** We construct a subset $B_\kappa$ in the group $\mathbb{Z}$.

In this case put $B_\kappa = B_{\kappa-1} - B_{\kappa-1}$ where $B_{\kappa-1} = \{i: 1 \leq i \leq \kappa - 1\}$. It is easy to check that $B_\kappa$ has properties $(1_\kappa) - (3_\kappa)$ in $\mathbb{Z}$.

**Case 2.** We construct a set $B_\kappa$ in the quasicyclic $p$-group $\mathbb{Z}(p^{\infty})$.

Choose $n$ such that $z^{p^n} \in \{e^{i\phi}: 2\pi \phi < \frac{2\pi}{\kappa - 1}\}$. Then put $B_\kappa = B_{\kappa-1} - B_{\kappa-1}$ where

$$B_{\kappa-1} = \left\{e^{i\phi}: \phi = \frac{2\pi l}{p^n}, 1 \leq l \leq \kappa - 1\right\}.$$
It is easy to check that $B_k$ has the properties $(1_k) - (3_k)$ in $\mathbb{Z}(p^{\infty})$.

**Case 3.** We construct a subset $B_k$ in the direct sum of cyclic groups $\bigoplus_{i \in \mathbb{N}} \langle g_i \rangle$.

Put $B_k = B_{k-1} - B_{k-1} = \{g_i : 1 \leq i \leq k - 1\}$. Obviously $B_k$ has properties $(1_k), (3_k)$. We claim that $B_k$ has property $(2_k)$. To obtain a contradiction assume that there exists a subset $B_k \subseteq G$ with size $|B_k| = k$ such that

$$B_k - B_k \subseteq B_{k-1} - B_{k-1}.$$

Consider the sets $S = \{i : 1 \leq i \leq k - 1\}$ and $F = \{i : 1 \leq i \leq k\}$. We can enumerate the sets $B_{k-1}$ and $B_k$ as $B_{k-1} = \{g_i : i \in S\}$ and $B_k = \{b_i : i \in F\}$.

Since $B_k - B_k \subseteq B_{k-1} - B_{k-1}$ we can define a map $f : [F]^2 \mapsto [S]^2$ assigning to each pair $\{i,j\} \in [F]^2$ a unique pair $\{k,l\} \in [S]^2$ such that $b_i - b_j = \pm (g_k - g_l)$. A desired contradiction will follow from Lemma 1 as soon as we check that $f$ is separately injective and preserves intersections.

**Claim 1.** The map $f$ preserves intersection.

To derive a contradiction, suppose that there are distinct $i, i' \in F$ and $j \in F$ such that $f(\{i,j\}) \cap f(\{i',j\}) = \emptyset$.

Then

$$b_i - b_j = g_k - g_l \quad \text{and} \quad b_{i'} - b_j = g_n - g_m$$

where $k, l, n, m$ are pairwise distinct.

Hence $b_{i'} - b_i = g_n - g_m - g_k + g_l \notin B_{k-1} - B_{k-1}$, which contradicts the assumption that $B_k - B_k \subseteq B_{k-1} - B_{k-1}$.

**Claim 2.** The map $f$ is separately injective.

To derive a contradiction, suppose that there are distinct $i, i' \in F$ and $j \in F$ such that $f(\{i,j\}) = f(\{i',j\}) = \{k,l\}$.

Since $b_i, b_{i'}$ are distinct we get

$$b_i - b_j = g_k - g_l \quad \text{and} \quad b_{i'} - b_j = g_l - g_k$$

and thus $b_f - b_i = 2(g_l - g_k) \neq 0$. Note that $2(g_l - g_k) \in B_{k-1} - B_{k-1}$ iff $2(g_l - g_k) = g_k - g_l$. Thus we get that $3g_k = 0$ and $3g_l = 0$.

Since $|F| = k > 4$ we can chose $r \in F \setminus \{i,i',j\}$. The map $f$ preserves intersection so $f(\{r,j\}) \cap \{k,l\} \neq \emptyset$. Also note that $f(\{r,j\}) \cap \{k,l\} \neq \{k,l\}$ otherwise $b_r = b_i$ or $b_r = b_{i'}$. So without loss of generality we can assume that $f(\{r,j\}) \cap \{k,l\} = \{k\}$.

Hence $b_{r} - b_j = g_s - g_k$ or $b_{r} - b_j = g_k - g_s$ for some $s$.

Consequently, $b_r - b_i = g_s - 2g_k + g_l$ or $b_r - b_{i'} = 2g_k - g_s - g_l$.

Note that $2g_l \neq 0$ since $3g_k = 0$.

So we get $b_r - b_i = g_s - 2g_k + g_l \notin B_{k-1} - B_{k-1}$ or $b_r - b_{i'} = 2g_k - g_s - g_l \notin B_{k-1} - B_{k-1}$.

This contradicts the assumption that $B_k - B_k \subseteq B_{k-1} - B_{k-1}$. \qed
Lemma 7. Let $\kappa$ be an infinite non-limit cardinal with $\kappa \leq |G|$ where $G$ is an infinite Abelian group. Then there exists a subset $B_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$.

Proof. Since $\kappa$ is infinite non-limit cardinal there exists cardinal $\alpha$ such that $\kappa = \alpha^+$. Put $B_\kappa = B_\alpha - B_\alpha$ where $B_\alpha$ is any subset of $G$ with size $|B_\alpha| = \alpha$. Obviously $B_\kappa$ satisfies property $(1_\kappa)$.

Since $|B_\kappa| = \alpha$ and $|B_\kappa - B_\alpha| = \kappa = \alpha^+$ for any subset $B_\kappa \subset G$ of size $\kappa$ we get $B_\kappa - B_\kappa \not\subset B_\kappa$. Therefore $B_\kappa$ has property $(2_\kappa)$.

The last property $(3_\kappa)$ follows from the fact $|F| + |\mathbb{B}_\kappa| \leq |F| \cdot |\mathbb{B}_\kappa| < |G|$. $\square$

Lemma 8. Let $\kappa$ be a limit cardinal and $G$ be an infinite Abelian group with $\kappa \leq |G|$. Then there exists a subset $B_\kappa \subset G$ with the properties $(1_\kappa) - (3_\kappa)$.

Proof. Note that it is enough to show that each group $G$ of size $\kappa$ contains a subset $B_\kappa$ with properties $(1_\kappa) - (3_\kappa)$. When $|G| > \kappa$ then we can take any subgroup $H \subset G$ of size $|H| = \kappa$ and find a subset $B_\kappa$ of with properties $(1_\kappa) - (3_\kappa)$ in $H$. Then the subset $B_\kappa$ will have the properties $(1_\kappa) - (3_\kappa)$ in the whole group.

So it remains to prove that such a set $B_\kappa$ exists in each group $G$ of size $\kappa$.

First we describe a sequence of symmetric subsets $F_\alpha \subset G$ of size $\alpha$ such that $G = \bigcup_{\alpha < \kappa} F_\alpha$ and $F_\alpha \supset \bigcup_{\beta < \alpha} F_\beta$. Enumerate the group $G$ so that $G = \{g_\alpha : \alpha < \kappa\}$ and $g_0 = e$. Then put $F_\alpha = \{g_\beta : g_\beta < \alpha\}$ for all $\alpha < \kappa$.

We put

$$B_\kappa = \bigcup_{\alpha < \kappa} B_\alpha - B_\alpha$$

where a set $B_\alpha = \{b_\beta : \beta < \alpha\} \subset G$ of size $\alpha$ will be chosen later.

To simplify notation we write $\mathbb{B}_{<\alpha}$ instead of $\bigcup_{\beta < \alpha} (B_\beta - B_\beta)$ and $\mathbb{B}_{>\alpha}$ instead of $\bigcup_{\alpha < \beta < \kappa} (B_\beta - B_\beta)$. By $B_{<\beta}^{<\alpha}$ we shall denote the initial interval $\{b_\gamma : \gamma < \beta\}$ of $B_\beta$.

Now we are in a position to define a sequence of sets $B_\alpha$ forcing the set $B_\kappa$ to satisfy the properties $(2_\kappa)$ and $(3_\kappa)$. To ensure property $(3_\kappa)$ we will also construct a transfinite sequence of points $(h_\alpha)_{\alpha < \kappa}$ of $G$ such that $h_\alpha \notin F_\alpha + \mathbb{B}_\kappa$.

We start putting $B_0 = \{e\}$ and taking any non-zero point $h_0 \in G$. Assume that for some ordinal $\alpha < \kappa$ the sets $B_\beta$ and the points $h_\beta$, $\beta < \alpha$, have been constructed. Then pick any point $h_\alpha \in G$ with

$$h_\alpha \notin F_\alpha + \mathbb{B}_{<\alpha}.$$ 

Such a point exists because the size of the set $F_\alpha + \mathbb{B}_{<\alpha}$ is equals $\alpha < \kappa = |G|$. Let

$$H_\alpha = \{h_\beta, -h_\beta : \beta < \alpha\}.$$ 

Next we define inductively elements of $B_\alpha = \{b_\beta : \beta < \alpha\}$.

We pick any $b_\beta$ with $b_\beta \in G \setminus \mathbb{B}_{<\alpha}$. Next, we choose $b_\beta$ with
To ensure properties (a), (b), (c) we have to avoid the sets of size \( \alpha \), which is possible because \( |G| = \kappa \).

Now let us prove that the constructed set \( B_k \) satisfies the properties \((1_\kappa) - (3_\kappa)\). In fact the property \((1_\kappa)\) is evident while \((3_\kappa)\) follows immediately from \((c)\). It remains to prove

**Claim.** The set \( B_k \) has property \((2_\kappa)\).

Let \( B_k \) be a subset of \( G \) of size \( |B_k| = \kappa \). Fix any pairwise distinct points \( c_1, c_2, c \in B_k \).

If \( B - B_k \subseteq B_\gamma \) then \( B_k \subseteq \bigcap_{i=1}^3 c_i + B_\kappa \) and \( \kappa = |B_k| \leq |\bigcap_{i=1}^3 c_i + B_\kappa| \).

So to prove our claim it is enough to show that \( |\bigcap_{i=1}^3 c_i + B_\kappa| < \kappa \). Find an ordinal \( \alpha < \kappa \) such that \( c_p - c_q \in F_\alpha \) for any \( 1 \leq p, q \leq 3 \). Assuming that \( |\bigcap_{i=1}^3 c_i + B_\kappa| = \kappa \) we may find a point \( b \in \bigcap_{i=1}^3 (c_i + B_\kappa) \backslash \{a\} \). A contradiction will be reached in three steps.

**Steps 1.** First show that there is \( \beta > \alpha \) with \( b \in \bigcap_{i=1}^3 (c_i + B_\beta) \).

Otherwise, \( b - c_p \in B_\gamma - B_\gamma \) and \( b - c_q \in B_\beta \) for some \( \gamma > \beta > \alpha \) and some \( p \neq q \). Find \( i, j < \gamma \) with \( b - c_p = b_i^\gamma - b_j^\gamma \). The inequality \( b \neq c_p \) implies \( i \neq j \).

If \( i < j \) then \( b_i^\gamma - b + c_p = b_i^\gamma - b + c_q - c_q + c_p \in b_i^\gamma - B_\beta + B_\beta + + F_\gamma \subseteq B_\gamma^{<j} + B_{<\gamma} + F_\gamma \), which contradicts \((a)\).

If \( i < j \) then \( b_i^\gamma = b_i^\gamma - b - c_p = b_i^\gamma + B_\beta - B_\beta + c_q - c_p \in B_\gamma^{<j} + B_{<\gamma} + + F_\gamma \), which again contradicts \((a)\).

**Step 2.** We claim that if \( b - c_p = b_i^\beta - b_j^\beta \) and \( b - c_q = b_p^\beta - b_\beta^\beta \) then \( \max \{i, j\} = \max \{s, t\} \).

It follows from the hypothesis that \( c_q - c_p = b_j^\beta - b_i^\beta + b_p^\beta - b_\beta^\beta \). To obtain a contradiction assume that \( \max \{i, j\} > \max \{s, t\} \).

If \( j < i \) then \( b_i^\beta = c_q - c_p + b_i^\beta - b_p^\beta + b_\beta^\beta \in F_\beta + B_\beta^{<j} - B_\beta^{<j} + B_\beta^{<i} \), which contradicts \((b)\).

If \( i < j \) then \( b_j^\beta = c_q - c_p + b_j^\beta + b_p^\beta - b_\beta^\beta \in F_\beta + B_\beta^{<j} - B_\beta^{<j} - B_\beta^{<i} \), again a contradiction with \((b)\).

**Step 3.** According to the previous step there exist \( \beta > \alpha \) and \( l \) such that
\[
\begin{align*}
b - c_1 &= b_i^\beta - b_j^\beta \text{ where } \max \{i, j\} \text{ is equal to } l; \\
b - c_2 &= b_s^\beta - b_t^\beta \text{ where } \max \{s, t\} \text{ is equal to } l; \\
b - c_3 &= b_q^\beta - b_r^\beta \text{ where } \max \{q, r\} \text{ is equal to } l.
\end{align*}
\]

In this case we obtain a dichotomy: either among three numbers \( i, s, q \) two are equal to \( l \) or among \( j, t, r \) two are equal to \( l \).

In the first case we lose no generality assuming that \( i = s = l \); in the second, that \( j = t = l \).
In the first case we get $F_a \exists c_2 - c_1 = b'_\beta - b'_\beta$, which contradicts (a).
In the second case we get $F_a \exists c_2 - c_1 = b'_\beta - t'_\beta$, which contradicts (a) again.
Therefore, there is no $b \in \bigcap_{i=1}^{3} (c_i + \mathbb{B}_{>a}) \setminus \{c_i\}$ and hence $|\bigcap_{i=1}^{3} c_i + + \mathbb{B}_{>a}| < \kappa$.

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