Quantum Dynamical Resource Theory under Resource Non-increasing Framework

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Abstract. We define the resource non-increasing (RNI) framework to study the dynamical resource theory. With such a definition, we propose several potential quantification candidates under various free operation sets. For explicit demonstrations, we quantify the quantum dynamical coherence in the scenarios with and without post-selective measurements. Correspondingly, we show that maximally incoherent operations (MIO) and incoherent operations (IO) in the static coherence resource theory are free in the sense of dynamical coherence. We also provide operational meanings for the measures by the quantum discrimination tasks. Moreover, for the dynamical total coherence, we also present convenient measures and give the analytic calculation for the amplitude damping channel.

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1. Introduction

In recent years, quantum features, such as quantum entanglement [1–3] and quantum discord [4–8] are taken regarded as quantum resources and quantitatively investigated under rigorous mathematical methods, i.e., quantum resource theories (QRTs), which has been systematically developed since quantum coherence was formally and quantitatively studied in Ref. [9]. QRTs are powerful approaches to investigate quantum characteristics and have deep effects on our recognition of quantum science. Up to now, QRTs have been widely applied to other quantum features: nonlocality [10], contextuality [11], non-Gaussianity [12] and asymmetry [13] and so on. QRTs are usually defined by two fundamental ingredients: free states and free quantum operations. Free states are those states without any resource and free quantum operations are referred to as those that could not generate quantum resources if operated on free states. In such a framework, QRTs bring new insight in studying quantum features on the static level: measures for quantum features not only evaluate the weight of a system but also have operational meanings corresponding to specific quantum processes. Such a framework for static resources is quite rigorous, and could even develop a connection to different areas. For example, in the QRTs of quantum coherence, researchers proposed multiple measures [14–20] and operational interpretations [21–25]. It has been shown that quantum coherence closely correlated with other quantum features like quantum entanglement [2,3], quantum discord [4–8], quantum asymmetric [26–28] et al.

Quantum states might get evolved under dynamic processes or quantum operations. Typically, quantum states can be viewed as some special quantum channels. Most quantum processes can be characterized by quantum channels which are mathematically the set of completely positive trace preserving (CPTP) maps. Corresponding to static states, quantum channels are dynamical quantum resources in quantum science and carry much more information than static systems. Therefore, one natural idea would be whether to investigate dynamical channels by QRTs. In this sense, one possible way is to follow and upgrade the two elements in static QRTs to dynamical levels, and find out what could be gotten in dynamical resources through the QRTs. With such motivations, researchers are working on this area and have made progresses [29–43]. Some studied the dynamical coherence via different free channels, for example, the Choi isomorphism of classical channels is considered [29]. In Ref. [31], the resource theory proposed free Positive-Operator-Valued Measures (POVMs) and free detection/creation incoherence in the sense of quantum computational setting [32]. In addition, the dynamical entanglement has been considered in Ref. [34], and a quantitative relationship between the dynamical coherence and the dynamical entanglement are introduced in Ref. [35].

In dynamical QRTs, the two ingredients are free operations and free superoperations (completely CP and TP preserving maps). The free operations are those without expected quantum features, and free superoperations are defined as those that could not map a free operation to a resourceful operation. However, one would find that the definitions of free superoperation sets depend on the physical considerations. Even for the same quantum feature, the free sets are not unique. Another tough problem is how to measure dynamical
quantum features. Due to the diversity of channel representations, the analytical solutions rarely exist, thus measures are usually given in numerical results. Moreover, realizations of superoperations are not unique either. To sum up, dynamical QRTs are developing in different directions, which leads to various QRTs.

In this paper, we direct a new path to define the free operation sets for quantum dynamical resources. We call it resource non-increasing (RNI) frameworks, in which the free channel won’t increase the resourcefulness of the input states. It will be shown that our RNI mindset provides a straight comprehension for dynamical free and meanwhile guarantee RNI framework has no conflict with the well-defined RNG framework. We also refer to static resource theories and design appropriate superoperations for dynamical QRTs. The free dynamical sets in the RNI frameworks are fairly pellucid. We present several potential quantifications of dynamical resources under different free operation sets. To investigate the dynamical coherence, we demonstrate that maximally incoherent operations (MIO) and incoherent operations (IO) in the static coherence resource theory are free in the sense of dynamical coherence. In this sense, we give the corresponding measure of the quantum dynamical coherence in the case with and without post-selective measurements. Semidefinite programming (SDP) is also applied to quantify dynamical coherence without post selective measurements. In addition, we also study the quantification of the dynamical total coherence, for which an analytic calculation is given for the amplitude dumping channel. We organize the remaining parts of this paper as follows. In Sec. II, we review the dynamical QRTs, propose our RNI framework and present several alternative quantifications. In Sec. III, we establish dynamical QRTs for quantum coherence in different scenarios. We illustrate the operational meanings of measures in quantum discrimination tasks. In Sec. IV, we investigate the dynamical total coherence and give an analytic calculation as an example. We summarize the paper in Sec. V.

2. Resource theory of quantum channels

RNI framework has a direct definition of the free dynamical set, i.e. the free operations could not increase any quantum static resource for arbitrary static input. Therefore, to demonstrate the RNI frameworks, we need the unambiguous definitions of static resource measures first. As mentioned previously, for static resource theory, the free states are those without any resource, and the set of free states are denoted by $\mathcal{F}$. The free operations with its Kraus operators $\{K_i\}$ are defined by $K_i\delta K_i^\dagger \in \mathcal{F}$ for any free state $\delta$. Thus a valid static resource measure can be given as follows.

**Proposition 1.**- A static resource measure $M$ for some certain quantum resource $\mathcal{R}$ should fulfill (i) Faithfulness: $M \geq 0$ and vanishes for free states; (iia) Strong monotonicity: the average resource under selective free operations can not be increased; (iib) Monotonicity: the resource of the state after free operations can not be increased; (iii) Convexity: mixing states would not increase their resourcefulness. These constraints are widely applied in measuring entanglement, coherence et al [9].

It is known to all that the strong monotonicity combined with the convexity would lead
to the general monotonicity. The measures that fulfill the strong monotonicity imply that such quantum channels are post-selective measurements allowed. These selective measurements exist in a scenario that the results are accessible with post-selective operations. With the static resource measure, we can present our free operations in the RNI framework (RNI-free operations).

**Theorem.** 2- RNI-free operations in the dynamical resource theory are consistent with the free operations in the static resource theory.

**Proof.** Based on the idea of the RNI framework, the RNI-free operations $\mathcal{E}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ with respect to a valid static resource measure $\mathcal{M}$ is defined as $\sum_n \text{Tr}((K_i\rho K_i^\dagger))\mathcal{M}(K_i\rho K_i^\dagger/\text{Tr}((K_i\rho K_i^\dagger))) \leq \mathcal{M}(\rho)$ for any density matrix $\rho$. If $\rho \in \mathbb{F}$, due to the faithfulness of static measures, $\mathcal{M}(\rho) = 0$, so $\mathcal{M}(K_i\rho K_i^\dagger/\text{Tr}((K_i\rho K_i^\dagger))) = 0$, which is consistent with the definition of free operations in the static resource theory. On the contrary, if $\mathcal{E}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ is the static free operations, based on the strong monotonicity, the mentioned definition for the RNI-free operations is also satisfied.

Note that if a quantum channel is considered in a black-box scenario, which means the measurement is non-selective, one can only require monotonicity in a non-increasing framework, i.e. $\mathcal{M}(\mathcal{E}(\rho)) \leq \mathcal{M}(\rho)$. This corresponds to the RNI framework in the sense of monotonicity. Similarly, the free operations can also be naturally defined subject to monotonicity, which will be directly used later and won’t be explicitly elucidated here.

Another ingredient in dynamical QR Ts is the free superoperations. Similar to the static resource theory, the free superoperation has a primal constraint that maps a free operation to free operation. Besides, we stress that the construction of superoperations should not contain certain resourceful ingredients. Now we propose our free superoperations with the following structure.

**Definition.** 3- A superoperation represented by Kraus operators $\{\Phi_n\}$ is free if and only if $\delta_n[\mathcal{N}]$ can be written into the sequence as

$$\delta_n[\mathcal{N}] = \mathcal{E}_{i_1,\Phi_n^{i_1}}, \cdots, \mathcal{E}_{i_{\ell},\Phi_n^{i_{\ell}}}\mathcal{E}_{i_1,\Phi_n^{i_1}},$$  \hspace{1cm} (1)

where $\mathcal{E}_{i_n,\Phi_n^{i_n}}$ denotes the $j_n$th Kraus element of the superoperation $\{\mathcal{E}_{i_n,\Phi_n^{i_n}}\}$ with the corresponding free operation $\{\Phi_n^{i_n}\}$ and

$$\mathcal{E}_{i_1=0,\Phi_n^{i_1}}[\mathcal{N}] = \text{Tr}[\mathcal{N}],$$

$$\mathcal{E}_{i_1=1,\Phi_n^{i_1}}[\mathcal{N}] = \Phi_n^{i_1} \circ \mathcal{N}, \mathcal{E}_{i_1=2,\Phi_n^{i_1}}[\mathcal{N}] = \Phi_n^{i_1} \otimes \mathcal{N},$$

$$\mathcal{E}_{i_1=3,\Phi_n^{i_1}}[\mathcal{N}] = \mathcal{N} \circ \Phi_n^{i_1}, \mathcal{E}_{i_1=4,\Phi_n^{i_1}}[\mathcal{N}] = \mathcal{N} \otimes \Phi_n^{i_1}. \hspace{1cm} (2)$$

The above definition implies the tensor product structure is automatically satisfied if one replaces $\Phi_n^{i_n}$ by identity operator $\mathbb{1}$. Furthermore, it is not difficult to find that our superoperations would be free for every single Kraus operator. Combing the two characteristics one can conclude that our superoperations are separately (every Kraus operator free) and completely (tensor product structure) free. Hence, our free superoperations meet the requirement of our motivation. With the two ingredients defined, we can define the measure of RNI dynamical resource as follows.
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**Definition 4.** A measure $T(\cdot)$ quantifying dynamical resourcefulness of arbitrary quantum channel $\mathcal{N}$ is a qualified measure if the following are satisfied.

1. **Faithfulness:** $T(\mathcal{N}) \geq 0$, where equality holds iff $\mathcal{N}$ is free;
2. **Monotonicity or strong monotonicity:** $T(\mathcal{N}) \geq T(\tilde{\mathcal{N}})$ or $T(\mathcal{N}) \geq \sum_n p_n T(\mathcal{N}_n)$ for free superoperations
   
   $\tilde{\mathcal{N}} = \sum_n p_n \tilde{\mathcal{N}}_n$ with $\mathcal{N}_n = \tilde{\mathcal{N}}_n[\mathcal{N}]$ and $\sum_n p_n = 1$,
3. **Convexity:** $T(\mathcal{N})$ is convex.

The strong monotonicity is an operational constraint where the measures obey the monotonicity under selective measurements. Our free superoperations allow us to read out information of the post states. However, if one could not have enough information about the resourceful superoperation (such as a black box in some practical scenarios mentioned previously), it is enough to consider the monotonicity, even though in a QRT strong monotonicity is required. Our free superoperations can also be explicitly given in the sense of quantum computational settings, which is similar to Ref. [44].

Up to now, we have well established the fundamental requirements for a dynamical resource measure in the RNI framework. Thus in the following, we present two natural quantification approaches, one is based on the distance from the free operation set, the other is directly based on the violation of the definition of free operations.

**Definition 5.** Dynamical resource of the channel $\mathcal{N}(\cdot) = \sum_i K_i \cdot K_i^\dagger$ can be measured by the minimal distance from the free set $S$ with appropriate distance functions $\| \cdot \|_*$ as

$$T(\mathcal{N}) = \min_{\mathcal{F} \in S} \| \mathcal{N} - \mathcal{F} \|_*, \quad (3)$$

or by the magnitude of the violation of the free operation as

$$\tilde{T}(\mathcal{N}) = \max \{ \Delta M_\ast(\mathcal{N}), 0 \}, \quad (4)$$

with $\Delta M_\ast(\mathcal{N}) = \max_\rho \sum_i \text{Tr}[K_i \rho K_i^\dagger] M(K_i \rho K_i^\dagger / \text{Tr}[K_i \rho K_i^\dagger])$ where $\ast$ denotes the proper distance functions in Eq. (3) or static resource measure in Eq. (10) such that both $T(\mathcal{N})$ or $\tilde{T}(\mathcal{N})$ satisfies definition 4.

Finally, we’d like to emphasize that the dynamical resource in the RNI framework makes sense for the quantum channel with given Kraus operators since our definitions are based on the strong monotonicity. Of course, one can consider the channel without post-selection in the sense of monotonicity.

### 3. Dynamical quantum coherence in resource non-increasing framework

Now we will consider the dynamical resource theory of coherence in the RNI framework. Since we have shown that the RNI-free operations are the same as free operations in the static
resource theory, the RNI-incoherent operations in the dynamical resource theory naturally correspond to the incoherent operations in the static coherence quantification. As mentioned in the previous section, the RNI dynamical resource theory can be considered in the sense of both strong monotonicity and monotonicity. One can find that the free operations subject to the dynamical coherence are separately the incoherent operations (IO) and the maximally incoherent operations (MIO). Thus, we will directly employ the proposed approaches to quantify the dynamical coherence based on definition 5.

**Theorem 6.-** Let the free operation set denote by IO and MIO corresponding to the RNI coherence with and without post-selection scenarios, the dynamical coherence for a quantum channel \( \mathcal{N} \) can be measured by

\[
T_{1/\diamond}(\mathcal{N}) = \min_{\mathcal{F} \in \text{IO/MIO}} \| \mathcal{N} - \mathcal{F} \|_{1/\diamond},
\]

where the induced trace norm and the diamond norm are defined as

\[
\| \Phi \|_1 = \max \{ \| \Phi(X) \|_1 : X \in \mathcal{L}(X), \| X \|_1 \leq 1 \},
\]

\[
\| \Phi^{A\rightarrow B} \|_\diamond = \| \Phi^{A\rightarrow B} \otimes I^C \|_1.
\]

**Proof.-** Firstly, \( T_{1/\diamond}(\mathcal{N}) \) vanishes for free operations since it is defined by the minimal distance. For the convexity, let’s consider a channel \( \mathcal{N} \) mixed by a set \( \mathcal{N}_m \) with probabilities \( q_m \) and denote the corresponding optimal free operation in set IO/MIO by \( \mathcal{F}_m \), the average coherence is

\[
\sum m q_m T_{1/\diamond}(\mathcal{N}_m) = \sum m q_m \| \mathcal{N}_m - \mathcal{F}_m \|_{1/\diamond}
\geq \| \sum m q_m \mathcal{N}_m - \sum m q_m \mathcal{F}_m \|_{1/\diamond} \geq \| \mathcal{N} - \mathcal{F} \|_{1/\diamond}
= T_{1/\diamond}(\mathcal{N}).
\]

For the strong monotonicity, let’s consider the free superchannel \( \tilde{\mathcal{N}} = \sum m p_m \tilde{\mathcal{N}}_m \) given in Eq. (1). One can denote \( T_{1/\diamond}(\tilde{\mathcal{N}}_m[\mathcal{N}]) = T_{1/\diamond}(\mathcal{N}_m) \). Since the induced trace norm is sub-multiplicative and sub-multiplicative with respect to tensor product, which indicates that \( T_{1/\diamond}(\mathcal{N}_m) \leq T_{1/\diamond}(\mathcal{N}) \) for every single Kraus operatore in free superchannels. Thus, the strong monotonicity holds by the following inequality

\[
\sum m p_m T_{1/\diamond}(\mathcal{N}_m) \leq \sum m p_m T_{1/\diamond}(\mathcal{N}) = T_{1/\diamond}(\mathcal{N}).
\]

Finally one can directly obtain the monotonicity based on the strong monotonicity and convexity.

Let \( T_{1/\diamond,\text{non}} \) denotes \( T_{1/\diamond} \) without post-selective measurements. It is shown that \( T_{1/\diamond,\text{non}} \) have a direct operational meaning in the quantum channels discrimination task and can be calculated by semidefinite programming (SDP). We will illustrate the details in the appendix.
Besides the distance measures, one can also employ the maximal violation in definition 5 to define the dynamical coherence as follows.

**Theorem 7.** Given a quantum channel \( \mathcal{N}(\cdot) = \sum_n K_n \cdot K_n^\dagger \), the dynamical coherence can be well quantified by

\[
\bar{T}(\mathcal{N}) = \max_{\rho \in \mathcal{D}} \{\Delta \mathcal{M}_{np/\rho}(\mathcal{N}), 0\},
\]

where \( \mathcal{D} \) denotes the set of all density matrices in the space, and

\[
\Delta \mathcal{M}_{np}(\mathcal{N}) = \max_{\rho \in \mathcal{D}} C(\mathcal{N}(\rho)) - C(\rho)
\]

without post-selective measurements,

\[
\Delta \mathcal{M}_{p}(\mathcal{N}) = \max_{\rho \in \mathcal{D}} \sum_n p_n C(\rho_n) - C(\rho).
\]

with post-selective measurements. Here \( p_n = \text{Tr}((K_n \rho K_n^\dagger)) \) and \( \rho_n = K_n \rho K_n^\dagger / p_n \).

**Proof.** At first, the definition of free operations in either scenario directly implies \( \bar{T}(\mathcal{N}) \geq 0 \) which is saturated if and only if \( \mathcal{N} \subset \text{IO/MIO} \).

For the strong monotonicity, one will have to consider definition 3 in detail, which shows free superoperations can be written as \( \tilde{\mathcal{S}}(\mathcal{N}) = \sum_m q_m \tilde{\mathcal{N}}_m(\mathcal{N}) = \sum_m q_m \mathcal{N}_m \). From the following, one can find that every \( \tilde{\mathcal{N}}_m \) implies \( \bar{T}(\mathcal{N}_m) \leq \bar{T}(\mathcal{N}) \), which immediately leads to \( \sum_m p_n \bar{T}(\mathcal{N}_m) \leq \sum_m p_n \bar{T}(\mathcal{N}) = \bar{T}(\mathcal{N}) \). (i) Discarding the system with \( \mathcal{E}_{i_n=0} \) makes a free state; (ii) Attaching ancilla by \( \mathcal{E}_{i_n=2,4} \) means \( \mathcal{N}(\rho) = \text{Tr}_A[(\Theta_A \otimes \mathcal{N})(\sigma_A \otimes \rho)] = \text{Tr}_A[\Theta_A(\sigma_A) \otimes \mathcal{N}(\rho)] = 1 \cdot \mathcal{N}(\rho) \); (iii) Linking a free operation by \( \mathcal{E}_{i_n=1} \) corresponds to \( C(\Theta \circ \mathcal{N}(\rho)) \leq C(\mathcal{N}(\rho)) \) for any state \( \rho \in \mathcal{D} \) and any free channel \( \Theta \) due to the monotonicity of static measures \( C \); (iv) Linking a free operation for \( \mathcal{E}_{i_n=3} \) can be verified as

\[
\bar{T}(\mathcal{N} \circ \Theta) = \max_{\rho} \sum_n p_n C(\frac{K_n \rho \Theta K_n^\dagger}{p_n}) - C(\rho)
\]

\[
\leq \max_{\rho} \sum_n p_n C(\frac{K_n \rho \Theta K_n^\dagger}{p_n}) - C(\rho_{\Theta})
\]

\[
\leq \max_{\rho} \sum_n p_n C(\frac{K_n \rho K_n^\dagger}{p_n}) - C(\rho)
\]

\[
= \bar{T}(\mathcal{N}),
\]

where \( \Theta \) represents any free channel and \( \rho_{\Theta} = \Theta(\rho) \) is the post state operated by the free channel. The first inequality comes from the monotonicity of static measures, i.e., \( C(\rho_{\Theta}) \leq C(\rho) \) and the second holds since the maximum overall the density matrix spaces is definitely not less than the maximum overall the subspace. Eq. (13) mainly focuses on \( \Delta \mathcal{M}_{p}(\mathcal{N}) \). A similar proof can be easily obtained for \( \Delta \mathcal{M}_{np}(\mathcal{N}) \) (not given here). So (i)~(iv) prove the strong monotonicity.

For the convexity, let’s first prove the scenario with post-selective measurements. Consider the dynamical coherence of mixing a set of channels \( \{\mathcal{N}_i\} \) with probabilities \( \{q_i\} \).
Let a state $\rho$ undergo channel $N_i$, with its Kraus operator $K_{in}$, the post-measurement state is denoted by $\sigma_{i,n} = K_{in}\rho K_{in}^\dagger$, then we have

$$\tilde{T}(\sum_i q_i N_i) = \max_{\rho \in D} \sum_{i,n} q_i \text{Tr}(\sigma_{i,n}) C\left(\frac{\sigma_{i,n}}{\text{Tr}(\sigma_{i,n})}\right) - C(\rho)$$

(14)

$$= \sum_{i,n} q_i \text{Tr}(\sigma_{i,n}^0) C\left(\frac{\sigma_{i,n}^0}{\text{Tr}(\sigma_{i,n}^0)}\right) - C(\rho^0)$$

(15)

$$\leq \sum_{i,n} q_i \text{Tr}(\sigma_{i,n}^j) C\left(\frac{\sigma_{i,n}^j}{\text{Tr}(\sigma_{i,n}^j)}\right) - C(\rho^j)$$

(16)

$$= \sum_{i,n} q_i \max_{\rho^j \in D} p_{i,n}^j C(\rho^j) - C(\rho) = \sum_i q_i \tilde{T}(N_i),$$

(17)

where the superscript on $\sigma_{i,n}^j$ denotes the optimal state of the maximum in the sense of Eq. (14) and $\sigma_{i,n}^j$ is the optimal one for $N_i$.

In the case without post-selection, for the mixed channel $\sum_i q_i N_i$, we have

$$\tilde{T}(\sum_i q_i N_i) = \max_{\rho \in D} C\left(\sum_i q_i N_i(\rho)\right) - C(\rho)$$

(18)

$$= C\left(\sum_i q_i N_i(\sigma)\right) - C(\sigma)$$

(19)

$$\leq \sum_i q_i C(N_i(\sigma)) - C(\sigma)$$

(20)

$$\leq \sum_i q_i \max_{\rho^j \in D} C(N_i(\rho)) - C(\rho_i) = \sum_i q_i \tilde{T}(N_i),$$

(21)

where $\sigma$ denotes the optimal state for the mixed channel $\sum_i q_i N_i$, and the inequality Eq. (19) comes from the convexity for static measures.

Up to now, we have proved the convexity, which will directly lead to the monotonicity associated with the strong monotonicity. □

The dynamical coherence in Eq. (11) has a similar form with cohering power in Ref. [45], but our maximum is taken over all the density matrices rather than incoherent states. The MIO set was proposed as the maximal set of free operations in the static QRT, and here we show that the free set of RNI with the non-selective measurements is exactly the MIO. The RNI free set subject to MIO can provide a new operational interpretation to the RNG framework. In this sense, one can find an alternative dynamical coherence measure assisted by the dephasing channel $\Delta(\cdot) = \sum_i \langle i | \cdot | i \rangle |i\rangle\langle i|$, as follows.

**Theorem 8.** Given a quantum channel $N(\cdot) = \sum_n K_n \cdot K_n^\dagger$, the dynamical coherence without post measurements can be quantified by

$$T_{a,\text{non}}(N) = \min_{\mathcal{F} \in \text{MIO}, \delta \in I} \| (N - \mathcal{F}) \delta \|_1$$

(22)

$$= \min_{\mathcal{F} \in \text{MIO}} \| (N - \mathcal{F}) \Delta \|_1.$$

(23)

**Proof.** (1) The distance functions guarantee $T_{a,\text{non}}$ can faithfully detect dynamical coherence. (2) The convexity holds because of the absolute homogeneity and the triangle
inequality. Considering two quantum channels $\mathcal{N}$ and $\mathcal{M}$ such that $T_{a,\non}(\mathcal{N}) = \|\mathcal{N} - \mathcal{F}_i\|_1$ and $T_{a,\non}(\mathcal{M}) = \|\mathcal{M} - \mathcal{F}_2\|_1$, for any $0 \leq t \leq 1$, one can find

$$T_a(t\mathcal{N} + (1-t)\mathcal{M}) = \min_{\mathcal{F} \in \text{MIO}} \|(t\mathcal{N} + (1-t)\mathcal{M} - \mathcal{F})\|_1$$

(24)

$$\leq \|(t\mathcal{N} + (1-t)\mathcal{M})\Delta - (t\mathcal{F}_1 + (1-t)\mathcal{F}_2)\|_1$$

$$= \|t(\mathcal{N}\Delta - \mathcal{F}_1\Delta) + (1-t)(\mathcal{M}\Delta - \mathcal{F}_2\Delta)\|_1$$

$$\leq t\|\mathcal{N} - \mathcal{F}_1\|_1 + (1-t)\|\mathcal{M} - \mathcal{F}_2\|_1$$

$$= tT_a(\mathcal{N}) + (1-t)T_a(\mathcal{M}),$$

which is the convexity. (3) The strong monotonicity $T(\mathcal{N}) \geq \sum_i q_i T(\mathcal{N}_i)$ can be proved by

$$\sum_i q_i T_a(\mathcal{N}_i) = \sum_i q_i \min_{\mathcal{F}_i \in \text{MIO}} \|(\mathcal{N}_i - \mathcal{F}_i)\Delta\|_1$$

(25)

$$= \sum_i q_i \|(\mathcal{N}_i - \mathcal{F}_i^*)\|_1$$

(26)

$$\leq \sum_i q_i \min_{\mathcal{F} \in \text{MIO}} \|(\mathcal{F} - \mathcal{F}_i(X))\|_1$$

(27)

$$\leq \min_{\mathcal{F} \in \text{MIO}} \|\mathcal{N} - \mathcal{F}\|_1 = T_a(\mathcal{N}),$$

(28)

in which $\mathcal{N}_i = \mathcal{F}_i(\mathcal{N})$ and $\mathcal{F}_i$ are Kraus operators of superoperation $\mathcal{F}$, $\mathcal{F}^*$ represents the channel achieving the minimum, inequality (27) holds for that $\mathcal{F}_i(X)$ couldn’t be the optimal, and inequality (28) is valid due to the sub-multiplicativity. (4) The strong monotonicity combined with convexity leads to monotonicity. □

$T_{a,\non}$ is a success probability in channel discrimination tasks if the participant allows the specific free dephasing operation or the incoherent states. This numerical result is very close to the one studied in Ref. [44], where the authors analyzed detection-incoherent settings. Since the MIO is the maximal free set in static QR Ts of coherence, constraints (such as applying a dephasing channel) would definitely shrink the free set (to dephasing incoherent channels).

4. Dynamical total coherence in the RNI framework

Quantum total coherence is one type of basis-independent coherence of which the static QRT was studied in Ref. [46]. Similarly, it can also be investigated in the sense of dynamical resource theory, i.e., the dynamical total coherence of a channel. In Ref. [46], it is explicitly given that the free operations with post-selective measurements are the mixed unitary channels defined as $\mathcal{U}(\cdot) = \sum_x q_x U_x(\cdot)U_x^\dagger$ with unitary $U$ and $\sum_x q_x = 1$, and the free operations without post-selective measurements are the unital channels given by $\mathcal{A}(\cdot) = \sum_x A_x(\cdot)A_x^\dagger$ with $\sum_x A_x(\cdot)A_x^\dagger = I$. Following the above section, for a given quantum channel $\mathcal{N}$ one can straightforwardly get the corresponding measures of the dynamical total coherence by replacing the set 'IO/MIO' in Theorem 6 and replacing the static coherence measure $\mathcal{M}$ by a proper total coherence measure. One can also easily show that these measures satisfy the necessary conditions for a valid dynamic resource theory.
Considering the computability, we’d like to mention that the static total coherence measure based on the $l_2$ norm is a good measure. In this sense, an explicit measure of the dynamical total coherence can be raised similar to theorem 7 as follows.

**Theorem 9.** For a quantum channel $N$ with Kraus operators $\{K_n\}$, the dynamical total coherence in the RNI framework can be quantified as

$$\tilde{T}_{l_2}(\{K_n\}) = \max_\rho \max_n \sum p_n \text{Tr}[\rho_n^2] - \text{Tr}[\rho^2], 0$$

with post-selective measurements, and

$$\tilde{T}_{l_2}(N) = \max_\rho \max_n \text{Tr}[\sum p_n \rho_n] - \text{Tr}[\rho^2], 0$$

without post-selective measurements, where $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ and $p_n = \text{Tr}(K_n \rho K_n^\dagger)$.

**Proof.** Since the static total coherence based on $l_2$ norm is a qualified measure (i.e. satisfy all constraints for measures including the monotonicity and the strong monotonicity) [47], this theorem can entirely be understood as an explicit example of theorem 7. □

As demonstrations, we consider the dynamical coherence of amplitude damping channels which characterize the energy dissipation in the quantum process. With a given dissipation rate $\eta$, the Kraus operators of this channel can be given as [48]

$$K_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta} \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \sqrt{\eta} \\ 0 & 0 \end{bmatrix}. \quad (31)$$

Given a density matrix in Bloch representation

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + z & x - i y \\ x + i y & 1 - z \end{bmatrix}, \quad (32)$$

where the parameters $x, y$ and $z$ are subject to $x^2 + y^2 + z^2 \leq 1$, the dynamical coherence measured by $\tilde{T}_{l_2}(\{K_n\})$ with Eq. (29) reads

$$\tilde{T}_{l_2}(\{K_n\}) = \max_{x,y,z} \max \left\{ \frac{(1+z)^2 + (1-\eta)^2(1-z)^2 + 2(1-\eta)(x^2 + y^2)}{2(2-\eta + \eta z)} \\
+ \frac{\eta(1-z)}{2} - \frac{1 + x^2 + y^2 + z^2}{2}, 0 \right\}$$

To maximize $\tilde{T}_{l_2}(\{K_n\})$, we would like to apply the method of residual Multipliers:

$$\mathcal{L}(x, y, z, \lambda) = f(x, y, z) + \lambda(x^2 + y^2 + z^2 - 1), \quad (34)$$

where $f(x, y, z)$ is the main part (ignore the maximize) of Eq.(33) and the residual part is the qubit constraint. The maximum of $f(x, y, z)$ can be obtained when the partial derivatives for all parameters equal zero and the inequality constraint holds for the equality. Thus, we conclude the following equations:
It should be noted that for any $x^2 + y^2 + z^2 = 1$, $f(x, y, z) = 0$, which implies that the maximum lies within the feasible region. Thus, we have the optimal solution for Eq. (35) as:

$$
\begin{align*}
\lambda &= x = y = 0, \\
z &= \frac{2\eta + \sqrt{9 - 8\eta - 9}}{2\eta}.
\end{align*}
$$

(36)

Thus the analytic total coherence measure for amplitude damping channels is given by

$$
\tilde{T}_2(\{K_n\}) = \frac{9 \left(\sqrt{9 - 8\eta - 3} - 4\eta (2\eta + 2 \sqrt{9 - 8\eta - 9})\right)}{4\eta^2}.
$$

(37)

For the situation without post-selective measurements, the dynamical total coherence coherence Eq. (30) is

$$
\tilde{T}_2(N) = \max \left\{ \sum_{i=1}^{3} \frac{\xi_i^2 a_i^2}{2(1 - \xi_i^2)} + \frac{a_i^2}{2}, 0 \right\},
$$

(38)

where $a_i = \frac{1}{2} \text{Tr}N(\sigma_i)$, $M_{ij} = \frac{1}{2} \text{Tr}\sigma_j N(\sigma_j)$, and $\xi_i$ is the singular value of the matrix $M$, $|\tilde{a}\rangle = U^T |a\rangle$ with $U$ determined by the singular value decomposition $M = U\Lambda V^T$ [49]. In Fig. we compare the dynamical total coherence of amplitude damping channels with or without post-selective measurement. It is clear that the dynamical coherence with post-selective measurements is larger than that post-selective measurements.
5. Conclusion and discussion

Quantum channels are dynamical resources that contain more information than static states. Investigating dynamical resources in QRTs has a pre-request that identifies the free sets in different physical backgrounds. In this paper, we introduce a framework in the sense of resource non-increasing. We introduce the free operation sets with and without post selective measurements. It should be understood that the range of the RNI framework does not exceed the well-defined resource non-generating (RNG) framework, but it gives a new sight to determine the free set under different scenarios. We design free super-operations with fundamental ingredients and give some measures to quantify dynamical resources. As a demonstration, we quantify dynamical (total) coherence in our frameworks. MIO and IO are free (incoherent) operations corresponding to the case without and with post selective measurements, respectively. The operational meanings in quantum tasks are also given for the dynamical coherence. In particular, the analytical calculation is given for the dynamical coherence of the amplitude damping channel, and the semidefinite programming is provided for dynamical coherence without post selective measurements.

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Appendix A. Operational meaning of $T_{\diamond, \text{non}}$

We first demonstrate the operational meaning for $T_{\diamond, \text{non}}$ in channel discrimination tasks \cite{50-53}. In such a task, Bob wants to distinguish two channels with allowed operations. Another participant Alice prepares a probabilistic state in classic register $Z$. The classic register will be state 0 with probability $\lambda$ and state 1 with probability $1 - \lambda$. Alice read out the state 0 or 1 in register $Z$, then sent an initial state (which is prepared in arbitrary register $X$) through quantum channels $N_1$ and $N_2$ respectively, and the final state ($\rho_1$ and $\rho_2$) was sent to Bob in register $Y$. Bob has to determine the final state had experienced which quantum channel. And Bob tries best to improve his successful probability by maximize

$$
\frac{1}{2} \| \lambda \rho_1 - (1 - \lambda) \rho_2 \|.
$$

We set $\lambda = \frac{1}{2}$ for a better illustration. Meanwhile, with practical consideration, Bob is allowed to hold an auxiliary register $R$ and manipulate some operations $\Psi^{AR}$. In a non-selective scenario, Bob can only rely on the information located in register $Y$. Now, Bob, has a successful probability \cite{44, 54} as

$$
\text{Prob}(N_1, N_2) = \frac{1}{2} + \frac{1}{4} \max_{\Psi^{AR}, \sigma^{AR}} \| \Psi^{AR}(N_1 - N_2)^A \otimes 1^R \sigma^{AR} \|_1, \tag{A.1}
$$

where $\sigma^{AR}$ denotes the bipartite final state in registers $Y$ and $R$. And the allowed manipulations $\Psi^{AR}$ can be determined by Bob if are applied or not. If Bob holds the free set MIO and we
minimize the diamond norm between resource channel $N_1$ and $N_2 \in \text{MIO}$, then we will find that

$$T_\diamond(N_1) = \min_{N_2^A \in \text{MIO}} ||N_1^A - N_2^A||_\diamond$$

$$\geq \min_{N_2^A \in \text{MIO}} \max_{F^{AR} \in \text{MIO}} ||F^{AR}(N_1 - N_2)^A \otimes 1^R||_1$$

$$= \min_{N_2^A \in \text{MIO}} \max_{F^{AR} \in \text{MIO}} \max_{P^{AB} \in \text{MIO}} \text{Tr}[P^{AB}(N_1 - N_2)^A \otimes 1^A \sigma^{AR}]$$

$$= \min_{N_2^A \in \text{MIO}} \max_{F^{AR} \in \text{MIO}, \sigma^{AR}} ||F(N_1 - N_2)^A \otimes 1^A \sigma^{AR}||_1$$

(A.2)

(A.3)

(A.4)

where (A.2) originates from the monotonicity of dynamical measures, (A.3) is an alternative definition for the trace norm $||\cdot|| = \max_P \text{Tr}[P(\cdot)]$ where $P$ denotes projective operators. Since any projective operators are in the set of MIO, thus we have (A.4).

Comparing (A.4) with (A.1), one can easily see that in such a scenario the probability is exactly given by

$$\text{Prob}(N_1, N_2) = \frac{1}{2} + \frac{1}{4} T_{\diamond,\text{non}}(N_1),$$

(A.5)

since the maximum can be reached at least when Bob do an identity operation (which belongs to MIO, i.e. a free operation). Hence, the boundary gives the operational meaning to $T_{\diamond,\text{non}}$.

Furthermore, if Bob doesn’t hold the ancilla reference R, one will directly derive an operational meaning for $T_{1,\text{non}}$ based on Eq. (A.5).

**Appendix B. Semidefinite programming for $T_{\diamond,\text{non}}$**

A quantum channel $N$ has its Choi representation (sometimes called Choi-Jamiolkowski isomorphism) [55, 56] as

$$J(N) = 1 \otimes N(\phi_+),$$

(B.1)

where $\phi_+$ is unnormalized maximally entangled state $\phi_+ = \sum_{ij} |i\rangle \langle j| \otimes |i\rangle \langle j|$. Its dynamical coherence can be measured by

$$T_\diamond(N) = \min_{F \in \text{FREE}} ||N - F||_\diamond.$$  

(B.2)

Under the RNI and non-selective background, the free set is MIO. Thus, the dynamical coherence measure

$$T_{\diamond,\text{non}}(N) = \min_{F \in \text{MIO}} ||N - F||_\diamond.$$  

(B.3)
and evaluated by semidefinite programming (with polynomial algorithms \[57\]). According to its definition, diamond norm for operator \( ||N - F||_\diamond \) has its primal problem

\[
\text{Primal} \quad \begin{align*}
\text{minimize} \quad & 2\|\text{Tr}_B(Z)\|_\infty \\
\text{s.t.} \quad & Z \geq J(N - F) \\
& Z \geq 0.
\end{align*}
\]  

Then \( T_{\diamond,\text{non}}(N) \) is the optimal value of

\[
\text{minimize} : \quad 2\|\text{Tr}_B(Z)\|_\infty \quad \text{(B.5)}
\]

\[
\text{s.t.} \quad Z \geq J(N - F) \\
& Z \geq 0 \\
& F \in \text{MIO}.
\]

Applying constraints on \( F \) and Choi representation properties for MIO, the primal problem becomes:

\[
\text{minimize} : \quad 2\|\text{Tr}(Z)\|_\infty \quad \text{(B.6)}
\]

\[
\text{s.t.} \quad Z \geq J(N) - M \\
& Z \geq 0 \\
& M \geq 0 \\
& \text{Tr}_B(M) = 1_A \\
& \text{Tr}_A(M) - \Delta \text{Tr}_A(M) = 0.
\]

which equals to

\[
\text{minimize} : \quad a \quad \text{(B.7)}
\]

\[
\text{s.t.} \quad a \geq 0 \\
1_A \cdot a - 2\text{Tr}_B(Z) \geq 0 \\
Z \geq J(N) - M \\
& Z \geq 0 \\
& M \geq 0 \\
& \text{Tr}_B(M) = 1_A \\
& \text{Tr}_A(M) - \Delta \text{Tr}_A(M) = 0.
\]

The Lagrangian of the primal problem is given by:

\[
L(a, Z, M, \tilde{X}, X, Y_1, Y_2) = a + \text{Tr}[(2\text{Tr}_B(Z) - a1_A)\tilde{X}] + \text{Tr}[(J(N) - M - Z)X] + \text{Tr}[(\text{Tr}_A M - \Delta \text{Tr}_A M)Y_1] + \text{Tr}[(\text{Tr}_B M - 1_A)Y_2] \quad \text{(B.8)}
\]
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and its dual function (see more details for prime and dual problems in Ref. [58]) is

\[ q(\overline{X}, X, Y_1, Y_2) = \inf_{a, Z, M} L(a, Z, M, \overline{X}, X, Y_1, Y_2) \]
\[ = \inf_{a, Z, M} \text{Tr}[J(N)] - \text{Tr}[Y_2] + a[1 - \text{Tr}(\overline{X})] \]
\[ + \text{Tr}[2 \cdot \overline{X} \otimes \mathbb{1}_B - X Z] + \text{Tr}[(\mathbb{1}_A \otimes Y_1 - \mathbb{1}_A \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_B - X) M]. \]

The dual function has value \( \text{Tr}[J(N)X] - \text{Tr}[Y_2] \) if \( \overline{X} \leq 1 \land 2 \cdot \overline{X} \otimes \mathbb{1}_B - X \geq 0 \land \mathbb{1}_A \otimes Y_1 - \mathbb{1} \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_B - X \geq 0 \) and \(-\infty\) in other cases.

Thus, the dual problem is to maximize \( \text{Tr}[J(N)X] - \text{Tr}[Y_2] \) with constraints for \( \overline{X}, X, Y_1, Y_2 \). To make the constraint clear and easy reading, we simplify the four constraints:

1. \( \overline{X} \) is nonnegative and trace less than one;
2. \( X \geq 0 \);
3. \( 2 \cdot \overline{X} \otimes \mathbb{1}_B - X \) to one constraint as \( X \leq 2 \cdot \rho \otimes \mathbb{1}_B \) which \( \rho \) is a density matrix. Then we can construct a \( \tilde{X} := \frac{1}{\text{Tr}X} \overline{X} \) is trace one, positive semidefinite and will keep \( 2 \cdot \overline{X} \otimes \mathbb{1}_B - X \) positive semidefinite for all \( X \) satisfy \( 2\overline{X} \otimes \mathbb{1}_B - X \geq 0 \). Hence, the dual problem is

\[
\text{Dual} \quad \begin{align*}
\text{maximize} & \quad \text{Tr}[J(N)X] - \text{Tr}[Y_2] \\
\text{s.t.} & \quad X \leq 2 \cdot \rho \otimes \mathbb{1}_B : \rho \text{ is density matrix} \\
& \quad \mathbb{1}_A \otimes Y_1 - \mathbb{1} \otimes \Delta Y_1 + Y_2 \otimes \mathbb{1}_B - X \geq 0 \\
& \quad X \geq 0 \\
& \quad Y_1 = Y_1^\dagger \\
& \quad Y_2 = Y_2^\dagger.
\end{align*}
\]

In the end, we have to show that the primal and the dual problem reaches the same optimal value, that is the strong duality holds in this programming. The strong duality obtains if the Slater condition [59] holds: there exists some \( Z^* \) and \( W^* \) satisfy all the equality constraints in primal problem and all the inequality constraints strictly hold. It is easy to find that the Slater condition holds when

\[ Z^* = \mathbb{1}_A \otimes \mathbb{1}_B + J(N) \]
\[ M^* = \frac{1}{|B|} \mathbb{1}_A \otimes \mathbb{1}_B. \]

and the strong duality would promise the optimal value can be reached.

Example. The quantum channel \( \mathcal{K} \) has two Kraus operators:

\[ K_0 = \begin{bmatrix} 0.2096 & -0.3956 \\ -0.2564 & -0.3719 \end{bmatrix}, \]
\[ K_1 = \begin{bmatrix} -0.6197 & 0.6418 \\ -0.7116 & -0.5415 \end{bmatrix}. \]

By solving SDP in software CVX [60], this quantum channel has total coherence as \(+0.186758\).
1 Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81(2) 865–942 URL https://link.aps.org/doi/10.1103/RevModPhys.81.865
2 Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Phys. Rev. Lett. 78(12) 2275–2279 URL https://link.aps.org/doi/10.1103/PhysRevLett.78.2275
3 Yu C S and Song H S 2009 Phys. Rev. A 80(2) 022324 URL https://link.aps.org/doi/10.1103/PhysRevA.80.022324
4 Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88(1) 017901 URL https://link.aps.org/doi/10.1103/PhysRevLett.88.017901
5 Datta A, Shaji A and Caves C M 2008 Phys. Rev. Lett. 100(5) 050502 URL https://link.aps.org/doi/10.1103/PhysRevLett.100.050502
6 Piani M, Cavalcanti D, Aolita L, Boixo S, Modi K and Winter A 2011 APS Meeting Abstracts
7 Luo S 2008 Phys. Rev. A 77(4) 042303 URL https://link.aps.org/doi/10.1103/PhysRevA.77.042303
8 Giorda P and Paris M G A 2010 Phys. Rev. Lett. 105(2) 020503 URL https://link.aps.org/doi/10.1103/PhysRevLett.105.020503
9 Baumgratz T, Cramer M and Plenio M B 2014 Phys. Rev. Lett. 113(14) 140401 URL https://link.aps.org/doi/10.1103/PhysRevLett.113.140401
10 Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 Rev. Mod. Phys. 86(2) 419–478 URL https://link.aps.org/doi/10.1103/RevModPhys.86.419
11 Kirchmair G, Zähringer F, Gerritsma R, Kleinmann M, Gühne O, Cabello A, Blatt R and Roos C F 2009 Nature 460 494–497 URL https://doi.org/10.1038/nature08172
12 Strobel H, Muessel W, Linnemann D, Zibold T, Hume D B, Pezzè L, Smerzi A and Oberthaler M K 2014 Science 345 424–427 ISSN 0036-8075 URL https://science.sciencemag.org/content/345/6195/424
13 Marvian I and Spekkens R 2014 Nat. Commun. 5 3821
14 Yu X D, Zhang D J, Xu G F and Tong D M 2016 Phys. Rev. A 94(6) 060302(R) URL https://link.aps.org/doi/10.1103/PhysRevA.94.060302
15 Rana S, Parashar P and Lewenstein M 2016 Phys. Rev. A 93(1) 012110 URL https://link.aps.org/doi/10.1103/PhysRevA.93.012110
16 Yao Y, Xiao X, Ge L and Sun C P 2015 Phys. Rev. A 92(2) 022112 URL https://link.aps.org/doi/10.1103/PhysRevA.92.022112
17 Zhao H Q and Yu C S 2018 Sci. Rep. 8 299
18 Yu C S 2017 Phys. Rev. A 95(4) 042337 URL https://link.aps.org/doi/10.1103/PhysRevA.95.042337
19 Bu K, Singh U, Fei S M, Pati A K and Wu J 2017 Phys. Rev. Lett. 119(15) 150405 URL https://link.aps.org/doi/10.1103/PhysRevLett.119.150405
20 Wu Z, Zhang L, Fei S M and Li-Jost X 2020 J. Phys. A: Math. Theor. 54 015302 URL https://doi.org/10.1088/1751-8121/abcab7
21 Winter A and Yang D 2016 Phys. Rev. Lett. 116(12) 120404 URL https://link.aps.org/doi/10.1103/PhysRevLett.116.120404
22 Napoli C, Bromley T R, Cianciaruso M, Piani M, Johnston N and Adesso G 2016 Phys. Rev. Lett. 116(15) 150502 URL https://link.aps.org/doi/10.1103/PhysRevLett.116.150502
23 Rana S, Parashar P, Winter A and Lewenstein M 2017 Phys. Rev. A 96(5) 052336 URL https://link.aps.org/doi/10.1103/PhysRevA.96.052336
24 Zhu H, Hayashi M and Chen L 2018 Phys. Rev. A 97(2) 022342 URL https://link.aps.org/doi/10.1103/PhysRevA.97.022342
25 Patel D, Patro S, Vararasa C, Chakraborty I and Pati A K 2021 Phys. Rev. A 103(2) 022422 URL https://link.aps.org/doi/10.1103/PhysRevA.103.022422
26 Marvian I, Spekkens R W and Zanardi P 2016 Phys. Rev. A 93(5) 052331 URL https://link.aps.org/doi/10.1103/PhysRevA.93.052331
27 Piani M, Cianciaruso M, Bromley T R, Napoli C, Johnston N and Adesso G 2016 Phys. Rev. A 93(4) 042107 URL https://link.aps.org/doi/10.1103/PhysRevA.93.042107
28 Ioffe L and Mézard M 2007 Phys. Rev. A 75(3) 032345 URL
Reference Page:

1. [29] Korzekwa K, Czachórski S, Puchała Z and Życzkowski K 2018 New J. Phys. 20 043028 URL https://doi.org/10.1088/1367-2630/aaaff3
2. [30] Wang X, Wilde M M and Su Y 2019 New J. Phys. 21 103002 URL https://doi.org/10.1088/1367-2630/ab451d
3. [31] Bischof F, Kampermann H and Bruß D 2019 Phys. Rev. Lett. 123(11) 110402 URL https://link.aps.org/doi/10.1103/PhysRevLett.123.110402
4. [32] Saxena G, Chitambar E and Gour G 2020 Phys. Rev. Research 2(2) 023298 URL https://link.aps.org/doi/10.1103/PhysRevResearch.2.023298
5. [33] Xu J 2021 Phys. Lett. A 387 127028 ISSN 0375-9601 URL https://www.sciencedirect.com/science/article/pii/S0375960120308951
6. [34] Gour G and Scandolo C M 2020 Phys. Rev. Lett. 125(18) 180505 URL https://link.aps.org/doi/10.1103/PhysRevLett.125.180505
7. [35] Theurer T, Satyajit S and Plenio M B 2020 Phys. Rev. Lett. 125(13) 130401 URL https://link.aps.org/doi/10.1103/PhysRevLett.125.130401
8. [36] Chitambar E and Gour G 2019 Rev. Mod. Phys. 91(2) 025001 URL https://link.aps.org/doi/10.1103/RevModPhys.91.025001
9. [37] Liu Y and Yuan X 2020 Phys. Rev. Research 2(1) 012035(R) URL https://link.aps.org/doi/10.1103/PhysRevResearch.2.012035
10. [38] Kuroiwa K and Yamasaki H 2020 Quantum 4 355 ISSN 2521-327X URL https://doi.org/10.22331/q-2020-11-01-355
11. [39] Díaz M G, Desef B, Rosati M, Egloff D, Calsamiglia J, Smirne A, Skotiniotis M and Huelga S F 2020 Quantum 4 249 ISSN 2521-327X URL https://doi.org/10.22331/q-2020-04-02-249
12. [40] Designolle S, Uola R, Luoma K and Brunner N 2021 Phys. Rev. Lett. 126(22) 220404 URL https://link.aps.org/doi/10.1103/PhysRevLett.126.220404
13. [41] Gour G and Winter A 2019 Phys. Rev. Lett. 123(15) 150401 URL https://link.aps.org/doi/10.1103/PhysRevLett.123.150401
14. [42] Masini M, Theurer T and Plenio M B 2021 Phys. Rev. A 103(4) 042426 URL https://link.aps.org/doi/10.1103/PhysRevA.103.042426
15. [43] Hsieh C Y 2021 PRX Quantum 2(2) 020318 URL https://link.aps.org/doi/10.1103/PRXQuantum.2.020318
16. [44] Theurer T, Egloff D, Zhang L and Plenio M B 2019 Phys. Rev. Lett. 122(19) 190405 URL https://link.aps.org/doi/10.1103/PhysRevLett.122.190405
17. [45] Mani A and Karimipour V 2015 Phys. Rev. A 92(3) 032331 URL https://link.aps.org/doi/10.1103/PhysRevA.92.032331
18. [46] Yang S R and Yu C S 2018 Ann. Phys. 388 305–314 ISSN 0003-4916 URL https://www.sciencedirect.com/science/article/pii/S0003491617303500
19. [47] Yu C S, Yang S R and Guo B Q 2016 Quantum Inf. Process 15 3773–3784
20. [48] Nielsen M A and Chuang I 2000 Quantum computation and quantum information (Cambridge University Press)
21. [49] ren Yang S and shui Yu C 2021 Quantifying dynamical total coherence in a resource non-increasing framework (Preprint 2110.14267)
22. [50] Chiribella G, D’Ariano G M and Perinotti P 2008 Phys. Rev. Lett. 101(18) 180501 URL https://link.aps.org/doi/10.1103/PhysRevLett.101.180501
23. [51] Cooney T, Mosonyi M and Wilde M M 2016 Commun. Math. Phys. 344 797–829 URL https://doi.org/10.1007/s00220-016-2645-4
24. [52] Hayashi M 2009 IEEE Trans. Inf. Theory 55 3807–3820
25. [53] Duan R, Feng Y and Ying M 2009 Phys. Rev. Lett. 103(21) 210501 URL https://link.aps.org/doi/10.1103/PhysRevLett.103.210501
26. [54] Matthews W, Wehner S and Winter A 2009 Communications in Mathematical Physics 291 813–843
27. [55] Choi M D 1975 Linear Algebra Appl. 10 285–290 ISSN 0024-3795 URL https://www.sciencedirect.com/science/article/pii/0024379575900750
[56] Jamiołkowski A 1972 *Rep. Math. Phys.* **3** 275–278 ISSN 0034-4877 URL https://www.sciencedirect.com/science/article/pii/0034487772900110
[57] Khachiyan L 1980 *USSR Comput. Math. Phys.* **20** 53–72 ISSN 0041-5553 URL https://www.sciencedirect.com/science/article/pii/0041555380900610
[58] Watrous J 2018 *The theory of quantum information* (Cambridge university press)
[59] Giorgi G and Kjeldsen T H 2014 *Traces and emergence of nonlinear programming* (Springer)
[60] Grant M and Boyd S Cvx: Matlab software for disciplined convex programming http://cvxr.com