Self-Dual Chern-Simons Theories*

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Abstract

In these lectures I review classical aspects of the self-dual Chern-Simons systems which describe charged scalar fields in $2 + 1$ dimensions coupled to a gauge field whose dynamics is provided by a pure Chern-Simons Lagrangian. These self-dual models have one realization with nonrelativistic dynamics for the scalar fields, and another with relativistic dynamics for the scalars. In each model, the energy density may be minimized by a Bogomol'nyi bound which is saturated by solutions to a set of first-order self-duality equations. In the nonrelativistic case the self-dual potential is quartic, the system possesses a dynamical conformal symmetry, and the self-dual solutions are equivalent to the static zero energy solutions of the equations of motion. The nonrelativistic self-duality equations are integrable and all finite charge solutions may be found. In the relativistic case the self-dual potential is sixth order and the self-dual Lagrangian may be embedded in a model with an extended supersymmetry. The self-dual potential has a rich structure of degenerate classical minima, and the vacuum masses generated by the Chern-Simons Higgs mechanism reflect the self-dual nature of the potential.

1 Introduction : Self-Dual Theories

“Self-duality” is a powerful notion in classical mechanics and classical field theory, in quantum mechanics and quantum field theory. It refers to theories in which the interactions have particular forms and special strengths such that the second order equations of motion (in general, a set of coupled nonlinear partial differential equations) reduce to first order equations which are simpler to analyze. The “self-dual point”, at which the interactions and coupling strengths take their special self-dual values, corresponds to the minimization of some functional, often the energy or the action. This gives self-dual theories crucial physical significance. For example, the

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self-dual Yang-Mills equations have minimum action solutions known as instantons, the Bogomol’nyi equations of self-dual Yang-Mills-Higgs theory have minimum energy solutions known as monopoles, and the Abelian Higgs model has minimum energy self-dual solutions known as vortices. In these lectures, I discuss a new class of self-dual theories, self-dual Chern-Simons theories, which involve charged scalar fields minimally coupled to gauge fields whose ‘dynamics’ is provided by a Chern-Simons term in $2 + 1$ dimensions. The physical context in which such self-dual models arise is that of anyonic quantum field theory. An interesting novel feature of these self-dual Chern-Simons theories is that they permit a realization with either relativistic or nonrelativistic dynamics for the scalar fields. In the nonrelativistic case, the self-dual point corresponds to a quartic scalar potential, with overall strength determined by the Chern-Simons coupling strength. The nonrelativistic self-dual Chern-Simons equations may be solved completely for all finite charge solutions, and the solutions exhibit many interesting relations to two dimensional (Euclidean) integrable models. In the relativistic case, while the general exact solutions are not explicitly known, the solutions correspond to topological and nontopological solitons and vortices, many characteristics of which can be deduced from algebraic and asymptotic data. These self-dual Chern-Simons theories also have the property that, at the self-dual point, they may be embedded into a model with an extended supersymmetry, a general feature of self-dual theories.

Before introducing the self-dual Chern-Simons theories, I briefly review some other important self-dual theories, in part as a means of illustrating the general idea of self-duality, but also because various specific properties of these theories appear in our analysis of the self-dual Chern-Simons systems. More details concerning some of these models can be found in the lectures of Professor C. Lee on “Instantons, Monopoles and Vortices” from this symposium.

Perhaps the most familiar, and in a certain sense the most fundamental, self-dual theory is that of four dimensional self-dual Yang-Mills theory. The Yang-Mills action is

$$S_{YM} = \int d^4 x \ tr \ (F_{\mu\nu} F^{\mu\nu})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the gauge field curvature. The Euler-Lagrange equations form a complicated set of coupled nonlinear partial differential equations:

$$D_\mu F^{\mu\nu} = 0$$

where $D_\mu = \partial_\mu + [A_\mu, \ ]$ is the covariant derivative. However, in four dimensional Euclidean space the Yang-Mills action (1) is minimized by solutions of the self-dual (or anti-self-dual) Yang-Mills equations:

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$$

where $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}/2$ is the dual field strength. Note that the self-dual equations (3) are first order equations (in contrast to the second order equations of motion...
(2)), and their “instanton” solutions are known in detail [1]. We shall see that the nonrelativistic self-dual Chern-Simons equations have an interesting connection with these self-dual Yang-Mills equations.

Another important class of self-dual equations are the “Bogomol’nyi equations”

\[ D_i \Phi = -\epsilon_{ijk} F_{jk} \]  

which arise in the theory of magnetic monopoles in 3 + 1 dimensional space-time. These equations arise from a minimization of the static energy functional of a Yang-Mills-Higgs system in a special parametric limit known as the BPS limit [2, 3]. It is interesting to note that these Bogomol’nyi equations can be obtained from the (anti-) self-dual Yang-Mills equations (3) by a ‘dimensional reduction’ in which all fields are taken to be independent of \( x^4 \), and \( A_4 \) is identified with \( \Phi \):

\[ F_{14} = F_{23} \rightarrow D_1 \Phi = -F_{23} \]
\[ F_{24} = -F_{13} \rightarrow D_2 \Phi = F_{13} \]
\[ F_{34} = F_{12} \rightarrow D_3 \Phi = -F_{12} \]  

(5)

We shall see that the nonrelativistic self-dual Chern-Simons equations may also be obtained from the self-dual Yang-Mills equations by a dimensional reduction. Furthermore, the relativistic self-dual Chern-Simons equations involve a special algebraic embedding problem (that of embedding \( SU(2) \) into the gauge algebra) which also plays a crucial role in the analysis of the Bogomol’nyi equations (4).

The abelian Higgs model in 2 + 1 dimensions is a model of a complex scalar field \( \phi \) interacting with a \( U(1) \) gauge field with conventional Maxwell dynamics. For a special quartic potential, with a particular overall strength, the static energy functional is minimized by solutions to the following set of self-duality equations:

\[ D_j \Phi = -i \epsilon_{jk} D_k \Phi \]
\[ F_{12} = 1 - |\phi|^2 \]  

(6)

These self-duality equations have vortex solutions [4, 5, 6] which are important in the phenomenological Landau-Ginzburg theory of superconductors. The self-duality equations we find in the self-dual Chern-Simons systems also arise from minimizing the energy functional in a 2 + 1 dimensional theory, and the resulting Chern-Simons self-duality equations have a similar form to the abelian Higgs model self-duality equations (6).

Yang [6] proposed an approach to the four dimensional self-dual Yang-Mills equations (3) in which they can be viewed as the consistency conditions for a set of first order differential operators. This idea is fundamental to the notion of “integrability” of systems of differential equations, a subject with many connections to self-dual theories [4, 8]. If the self-dual Yang-Mills equations (3) are rewritten in terms of the null coordinates \( u = (x^1 + ix^2)/\sqrt{2} \) and \( v = (x^3 + ix^4)/\sqrt{2} \), they become

\[ F_{uv} = 0 \]
\[ F_{\bar{u}v} = 0 \]
\[ F_{uv} + F_{v\bar{u}} = 0 \] (7)

These express the consistency conditions for the first order equations

\[ (D_u - \zeta D_v) \psi = 0 \]
\[ (D_v + \zeta D_u) \psi = 0 \] (8)

where \( \zeta \) is known as a “spectral parameter”. The first two equations in (7) can be solved locally to give

\[ A_u = H^{-1} \partial_u H \]
\[ A_v = H^{-1} \partial_v H \]
\[ A_{\bar{u}} = K^{-1} \partial_{\bar{u}} K \]
\[ A_{\bar{v}} = K^{-1} \partial_{\bar{v}} K \] (9)

where \( H \) and \( K \) are gauge group elements. Then, defining \( J = HK^{-1} \), the third of the self-duality equations in (7) becomes

\[ \partial_{\bar{u}} \left( J^{-1} \partial_u J \right) + \partial_{\bar{v}} \left( J^{-1} \partial_v J \right) = 0 \] (10)

If we now make a dimensional reduction in which the fields are chosen to be independent of \( x^2 \) and \( x^4 \), this equation becomes the two dimensional equation

\[ \partial_\mu \left( J^{-1} \partial_\mu J \right) = 0 \] (11)

which is known as the chiral model equation. The chiral model equation will play a very important role in our analysis of the nonrelativistic self-dual Chern-Simons equations. Also note that if \( J \in SU(N) \) and \( J \) is further restricted to satisfy the condition \( J^2 = 1 \), then (11) is the equation of motion for the \( CP^{N-1} \) model [1, 9].

The final class of models which we shall recall in this introduction are known as Toda theories. The original Toda system [10] described the displacements of a line of masses joined by springs with an exponential spring tension. The equations of motion for the Toda lattice are

\[ \ddot{y}_i = -C_{ij} e^{y_j} \] (12)

where the matrix \( C_{ij} \) is the tridiagonal discrete approximation to the second derivative, and can be chosen for periodic or open boundary conditions. This system is classically integrable in the limit of an infinite number of masses, in the sense that it possesses an infinite number of conserved quantities in involution. The Toda lattice system also has a deep algebraic structure due to the fact that the matrix \( C_{ij} \) in (12) is the Cartan matrix of the Lie algebra \( SU(N) \) (or its affine extension). Indeed, this relationship allows one to extend the original Toda system to a Toda lattice based on other Lie algebras [11, 12, 13].

The Toda system generalizes still further, to an integrable set of partial differential equations

\[ \nabla^2 y_i = -C_{ij} e^{y_j} \] (13)
which is not only integrable, but also solvable, in the sense that the solution may be written in terms of $2r$ arbitrary functions, where $r$ is the rank of the classical Lie algebra whose Cartan matrix appears in \( (13) \) [11, 12]. For $SU(2)$ the classical Toda system reduces to the nonlinear Liouville equation

$$\nabla^2 y = -2e^y$$

which was solved by Liouville [14]. Both the Liouville and Toda equations, together with their solutions, appear prominently in the analysis of the nonrelativistic self-dual Chern-Simons models. Moreover, the Toda equations also arise from the Bogomol’nyi equations [4] when one looks for spherically symmetric monopole solutions [13]. This reduction involves an algebraic embedding problem very similar to one that appears in the treatment of the relativistic self-dual Chern-Simons models.

The self-dual Chern-Simons theories discussed in these lectures describe charged scalar fields in $2 + 1$ dimensional space-time, minimally coupled to a gauge field whose dynamics is given by a Chern-Simons Lagrangian rather than the conventional Maxwell (or Yang-Mills) Lagrangian. The possibility of describing gauge theories with a Chern-Simons term rather than with a Yang-Mills term is particular to odd-dimensional space-time, and $2+1$ dimensions is special in the sense that the derivative part of the Chern-Simons Lagrangian is quadratic in the gauge fields. To conclude this introduction, I briefly review some of the important properties [16, 17, 18] of the Chern-Simons Lagrange density:

$$L_{CS} = \epsilon^{\mu\nu\rho} \text{tr} \left( \partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right)$$

The gauge field $A_\mu$ takes values in a finite dimensional representation of the gauge Lie algebra $\mathcal{G}$. The totally antisymmetric $\epsilon$-symbol $\epsilon^{\mu\nu\rho}$ is normalized with $\epsilon^{012} = 1$. The Euler-Lagrange equations of motion derived from this Lagrange density are simply

$$F_{\mu\nu} = 0$$

which follows directly from the fact that

$$\frac{\delta L_{CS}}{\delta A_\mu} = \epsilon^{\mu\nu\rho} F_{\nu\rho}$$

The equations of motion (16) are gauge covariant under the gauge transformation

$$A_\mu \rightarrow A_\mu^g \equiv g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

and so the Lagrange density (15) defines a sensible gauge theory even though (15) is not invariant under the gauge transformation (18). Indeed, under a gauge transformation $L_{CS}$ transforms as

$$L_{CS}(A) \rightarrow L_{CS}(A) - \epsilon^{\mu\nu\rho} \partial_\mu \text{tr} \left( \partial_\nu g g^{-1} A_\rho \right) - \frac{1}{3} \epsilon^{\mu\nu\rho} \text{tr} \left( g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g \right)$$
For an abelian Chern-Simons theory, the final term in (19) vanishes and the change in $L_{CS}$ is a total space-time derivative. Hence the action $S = \int d^3x L_{CS}$ is gauge invariant. However, for a nonabelian Chern-Simons theory the final term in (19) is proportional to the winding number of the group element $g$, and the action changes by a constant. To ensure that $\exp(iS)$ remains invariant, the Chern-Simons Lagrange density (15) must be multiplied by a dimensionless coupling parameter $\kappa$ which assumes quantized values $[16, 17]$

$$\kappa = \frac{\text{integer}}{4\pi}$$  

(20)

The Chern-Simons term describes a topological gauge field theory [17] in the sense that there is no explicit dependence on the space-time metric. This follows because the Lagrange density (15) can be written directly as a 3-form $tr(AdA + A^3)$. This fact implies that if the Chern-Simons Lagrange density $L_{CS}$ is coupled to other fields, then it will not contribute to the energy momentum tensor. This may also be understood by noting that $L_{CS}$ is first order in space-time derivatives

$$L_{CS} = \epsilon^{ij} tr \left( A_i \dot{A}_j \right) + tr \left( A_0 F_{12} \right)$$  

(21)

The time derivative part of $L_{CS}$ contributes to the canonical structure of the theory, the $A_0$ part contributes to the Gauss law constraint, and there is no contribution to the Hamiltonian. It is very important that $L_{CS}$ is first order in space-time derivatives, because in the self-dual Chern-Simons theories discussed in these lectures the self-duality equations (which should be first order) involve the Chern-Simons equations of motion directly.

2 Nonrelativistic SDCS Theories

2.1 Nonrelativistic Self-Dual Chern-Simons Equations

The nonrelativistic self-dual Chern-Simons system is a model in 2 + 1 dimensional space-time describing charged scalar fields $\Psi$ with nonrelativistic dynamics, minimally coupled to gauge fields $A_\mu$ with Chern-Simons dynamics [19, 20, 21]. The Lagrange density for such a system is:

$$\mathcal{L} = -\kappa L_{CS} + i tr \left( \Psi^\dagger D_0 \Psi \right) - \frac{1}{2m} tr \left( (D_i \Psi)^\dagger D_i \Psi \right) + \frac{1}{4m\kappa} tr \left( [\Psi, \Psi^\dagger]^2 \right)$$  

(22)

where $L_{CS}$ is the Chern-Simons Lagrange density (15). I have chosen to work with adjoint coupling of the scalar and gauge fields (for other couplings of matter and gauge fields see [18]), with the covariant derivative in (22) being $D_\mu \Psi \equiv \partial_\mu \Psi + [A_\mu, \Psi]$. The scalar fields $\Psi$ and the gauge fields $A_\mu$ take values in the same representation of the
gauge Lie algebra $\mathcal{G}$. In these lectures, $\mathcal{G}$ will usually be taken to be $SU(N)$, but much of the formal structure generalizes straightforwardly to other gauge algebras. The parameter $\kappa$ appearing in (22) is the dimensionless Chern-Simons coupling constant, while $m$ denotes the scalar field mass. Notice that the scalar field potential appearing in (22) has a particular quartic form, with an overall scale depending on both $m$ and $\kappa$. This form of the potential is fixed by the condition of self-duality, as shown below.

The Euler-Lagrange equations of motion that follow from the nonrelativistic self-dual Chern-Simons Lagrange density (22) are:

$$i D_0 \Psi = -\frac{1}{2m} \bar{D}^2 \Psi - \frac{1}{2m\kappa} [ [\Psi, \Psi^\dagger], \Psi]$$  \hspace{1cm} (23)

$$F_{\mu\nu} = -\frac{i}{2\kappa} \epsilon_{\mu\nu\rho} J^\rho$$  \hspace{1cm} (24)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the gauge curvature, and $J^\rho$ is the covariantly conserved ($D^\mu J^\mu = 0$) nonrelativistic matter current

$$J^0 = [\Psi, \Psi^\dagger]$$
$$J^i = -\frac{i}{2m} \left( [\Psi^\dagger, D_i \Psi] - [(D_i \Psi)^\dagger, \Psi] \right)$$  \hspace{1cm} (25)

In addition there is an abelian current $Q^\rho$

$$Q^0 = tr(\Psi \Psi^\dagger)$$
$$Q^i = -\frac{i}{2m} tr \left( \Psi^\dagger D_i \Psi - (D_i \Psi)^\dagger \Psi \right)$$  \hspace{1cm} (26)

which is ordinarily conserved ($\partial_\mu Q^\mu = 0$). The matter field equation of motion (23) is referred to as the gauged planar nonlinear Schrödinger equation [19]. The study of the nonlinear Schrödinger equation in $2+1$-dimensional space-time is partly motivated by the significance of the $1+1$-dimensional nonlinear Schrödinger equation. Here we consider a gauged nonlinear Schrödinger equation in which we have not only the nonlinear potential term for the matter fields, but also we have a coupling of the matter fields to the gauge fields. The gauge equation of motion (24) relates the matter and gauge fields via a Chern-Simons coupling. Notice that even though the Chern-Simons Lagrange density $\mathcal{L}_{CS}$ is not strictly invariant under a gauge transformation, the equations of motion (23, 24) are gauge covariant.

The Hamiltonian density corresponding to the Lagrange density (22) is

$$\mathcal{E} = \frac{1}{2m} tr \left( (D_0 \Psi)^\dagger D_0 \Psi \right) - \frac{1}{4m\kappa} tr \left( [\Psi, \Psi^\dagger]^2 \right)$$  \hspace{1cm} (27)

where we recall that the Chern-Simons term $\mathcal{L}_{CS}$ does not contribute to the energy density since it is first order in space-time derivatives. The energy density (27) is supplemented by the Gauss law constraint

$$J^0 = -2i \kappa F_{12}$$  \hspace{1cm} (28)
which is the zero\textsuperscript{th} component of the gauge equations of motion (24). To obtain a Bogomol’nyi - style lower bound for the energy density we employ the following useful identity:

$$ tr \left( (D_i \Psi)^\dagger D_i \Psi \right) = tr \left( (D_- \Psi)^\dagger D_- \Psi \right) - i tr \left( \Psi^\dagger [F_{12}, \Psi] \right) $$

(29)

where $D_{\pm} \equiv D_1 \pm i D_2$.

Using this identity in (27), together with the Gauss law constraint (28) which relates the “magnetic field” $F_{12}$ to the nonrelativistic matter charge density $[\Psi, \Psi^\dagger]$, we see that the energy density can be written as

$$ E = \frac{1}{2m} tr \left( (D_- \Psi)^\dagger D_- \Psi \right) $$

(30)

This energy density is therefore minimized by solutions of the nonrelativistic self-dual Chern-Simons equations:

$$ D_- \Psi = 0 $$

(31)

$$ \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = \frac{1}{\kappa} [\Psi, \Psi^\dagger] $$

(32)

Notice that these self-duality equations are indeed first-order in derivatives of the fields, in contrast to the gauged nonlinear Schrödinger equation (23) which is second order.

Since the self-dual solutions minimize the Hamiltonian density, they provide static solutions to the Euler-Lagrange equations of motion (23,24). Alternatively, one can see this directly from inspection of the static equations of motion. Note that if $D_- \Psi = 0$, then the currents take the simple form

$$ J^+ \equiv J^1 + i J^2 = - \frac{i}{2m} [\Psi^\dagger, D_+ \Psi] $$

(33)

The gauge equation of motion (24) then implies that $A_0 = \frac{i}{4m\kappa} [\Psi, \Psi^\dagger]$. Together with the identity

$$ \bar{D}^2 \Psi \equiv D_+ D_- \Psi + i [F_{12}, \Psi] = D_+ D_- \Psi - \frac{1}{2\kappa} [\Psi, [\Psi^\dagger, \Psi]] $$

(34)

this reduces the matter equation of motion (23) to

$$ i \partial_0 \Psi = - \frac{1}{2m} D_+ D_- \Psi $$

(35)
the RHS of which which vanishes for self-dual solutions.

In fact, owing to a remarkable dynamical $SO(2,1)$ symmetry of the nonrelativistic self-dual Chern-Simons model \( (22) \), it is possible to show that the self-dual solutions \( (31,32) \) saturate all static solutions of the equations of motion \( (21,22) \). For the Abelian models, this fact has recently been formulated in terms of a Kaluza-Klein reduction of a relativistic symmetry \( (23) \).

An important property of the nonrelativistic self-dual Chern-Simons equations \( (31,32) \) is that they can be obtained by dimensional reduction from the four dimensional self-dual Yang-Mills equations for a nonAbelian gauge theory. The signature \((2,2)\) SDYM equations are

\[
F_{12} = F_{34} \quad F_{13} = F_{24} \quad F_{14} = -F_{23} \quad (36)
\]

Taking all fields to be independent of \( x^3 \) and \( x^4 \), these reduce to

\[
F_{12} = [A_3, A_4] \quad D_1 A_3 = D_2 A_4 \quad D_1 A_4 = -D_2 A_3 \quad (37)
\]

which are just the nonrelativistic self-dual Chern-Simons equations \( (31,32) \) with the identification \( \Psi = \sqrt{\kappa} (A_3 - i A_4) \). These dimensionally reduced self-dual Yang-Mills equations have been studied in the mathematical literature \( (24,25) \).

2.2 **Algebraic Ansätze and Toda Theories**

Before classifying the general solutions to the nonrelativistic self-dual Chern-Simons equations, it is instructive to consider certain special cases in which simplifying algebraic Ansätze for the fields reduce \( (31,32) \) to familiar integrable nonlinear equations. Note that since we are considering static fields, the self-duality equations have the appearance of equations of motion in two dimensional Euclidean space.

First, choose the fields to have the following Lie algebra decomposition

\[
A_i = \sum_{a=1}^r A_i^a H_a \quad \Psi = \sum_{a=1}^r \psi^a E_a \quad (38)
\]

Here, \( H_a \) refers to the Cartan subalgebra generators and \( E_a \) to the simple root step operator generators of the gauge Lie algebra, normalized according to a Chevalley basis (for ease of presentation we consider only simply-laced algebras here) \( (26) \):

\[
[H_a, H_b] = 0 \quad [E_a, E_{-b}] = \delta_{ab} H_a \quad [H_a, E_{\pm b}] = \pm C_{ab} E_{\pm b}
\]

\[
tr (E_a E_{-b}) = \delta_{ab} \quad tr (H_a H_b) = C_{ab} \quad tr (H_a E_{\pm b}) = 0 \quad (39)
\]
The indices \( a \) and \( b \) run over \( 1 \ldots r \), where \( r \) is the rank of the gauge algebra \( G \). The \( r \times r \) matrix \( C_{ab} \) is the Cartan matrix of \( G \), which expresses the inner products of the simple roots \( \alpha^{(a)} \):

\[
C_{ab} = \frac{2\alpha^{(a)} \cdot \alpha^{(b)}}{|\alpha^{(b)}|^2}
\]  

(40)

For \( SU(N) \), the classical Cartan matrix is the \((N-1) \times (N-1)\) symmetric tridiagonal matrix (familiar from the theory of numerical analysis):

\[
C = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & & 0 \\
0 & -1 & 2 & -1 & 0 \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -1 & 2
\end{pmatrix}
\]  

(41)

With the ansatze (38) for the fields, the first of the nonrelativistic self-dual Chern-Simons equations, \( D_-\Psi = 0 \), reduces to the set of equations

\[
\partial_- \log \psi_a = - \sum_{b=1}^{r} C_{ab} A_b
\]  

(42)

When combined with its adjoint, and with the other nonrelativistic self-dual Chern-Simons equation, we find the classical Toda equations

\[
\nabla^2 \log \rho_a = - \frac{1}{\kappa} \sum_{b=1}^{r} C_{ab} \rho_b
\]  

(43)

where \( \rho_a \equiv |\psi^a|^2 \). For \( SU(2) \), \( r = 1 \) and the Cartan matrix is just the single number 2, so the Toda equations (43) reduce to the Liouville equation

\[
\nabla^2 \log \rho = - \frac{2}{\kappa} \rho
\]  

(44)

which Liouville showed to be integrable and indeed "solvable" [14] - in the sense that the general real solution can be expressed in terms of a single holomorphic function \( f = f(x^-) \):

\[
\rho = \kappa \nabla^2 \log \left( 1 + f(x^-) \bar{f}(x^+) \right)
\]  

(45)

Kostant [11], and Leznov and Saveliev [12] have shown that the classical Toda equations (43) are similarly integrable (and indeed solvable), with the general real solutions for \( \rho_a \) being expressible in terms of \( r \) arbitrary holomorphic functions, where \( r \) is the rank of the algebra. For \( SU(N) \) it is possible to adapt the Kostant-Leznov-Saveliev solutions to a simple form reminiscent of the Liouville solution (45):

\[
\rho_a = \kappa \nabla^2 \log \det \left( M^+_a(x^+) M_a(x^-) \right)
\]  

(46)
where $M_a$ is the $N \times a$ rectangular matrix $M_a = (u \partial_- u \partial^2_- u \ldots \partial^{a-1}_- u)$, with $u$ being an $N$-component column vector containing $N - 1$ arbitrary holomorphic functions $f_1(x^-), f_2(x^-), \ldots, f_{N-1}(x^-)$:

$$u = \begin{pmatrix} 1 \\ f_1(x^-) \\ f_2(x^-) \\ \vdots \\ f_{N-1}(x^-) \end{pmatrix}$$  \hspace{1cm} (47)

An alternative, extended, ansatz for the fields involves the matter field choice

$$\Psi = \sum_a \psi^a E_a + \psi^M E_M$$  \hspace{1cm} (48)

where $E_M$ is the step operator corresponding to minus the maximal root. With the gauge field still as in (38), the nonrelativistic self-dual Chern-Simons equations then combine to give the affine Toda equations

$$\nabla^2 \log \rho_a = - \frac{1}{\kappa} \sum_{b=1}^{r+1} \tilde{C}_{ab} \rho_b$$  \hspace{1cm} (49)

where $\tilde{C}$ is the $(r+1) \times (r+1)$ affine Cartan matrix. These affine Toda equations are also known to be integrable [11, 12, 13], although it is not possible to write simple convergent expressions such as (46) for the solutions.

There is another useful way to understand these various algebraic reductions of the nonrelativistic self-dual Chern-Simons equations. In two dimensional space we can express the gauge field as

$$A_- = G^{-1} \partial_- G$$

$$A_+ \equiv -A^\dagger_+$$  \hspace{1cm} (50)

where $G$ is an element of the complexification of the gauge group [8, 27]. $G$ can be decomposed as

$$G = H U$$  \hspace{1cm} (51)

where $H$ is hermitean and $U$ is unitary. Note that only with $H = 1$ does (50) correspond to a pure gauge. Gauge transformations on $A_\pm$ correspond to different choices of the unitary factor $U$. In general, the field strength corresponding to (50) is

$$F_{+-} = -U^\dagger \left( H \partial_+ \left( H^{-2} \partial_- H^2 \right) H^{-1} \right) U$$  \hspace{1cm} (52)

With the gauge field represented as in (50), the solution to the self-duality equation $D_- \Psi = 0$ is trivially:

$$\Psi = G^{-1} \Psi_0(x^+) G$$  \hspace{1cm} (53)
for any $\Psi_0(x^+)$. Inserting this solution in the other self-duality equation yields the gauge invariant equation for $H$:

$$\partial_+ \left( H^{-2} \partial_- H^2 \right) = \Psi_0^\dagger H^{-2} \Psi_0 H^2 - H^{-2} \Psi_0 H^2 \Psi_0^\dagger$$

(54)

Thus far, no special choices have been made and equation (54) is still completely general. Now, if we choose to write $H^2$ as

$$H^2 = e^\Phi$$

(55)

where $\Phi$ is restricted to the Cartan subalgebra, then (54) simplifies to

$$\partial_+ \partial_- \Phi = \Psi_0^\dagger e^{-\Phi} \Psi_0 e^\Phi - e^{-\Phi} \Psi_0 e^\Phi \Psi_0^\dagger$$

(56)

These equations follow as equations of motion from the two-dimensional Euclidean Lagrange density

$$\mathcal{L} = tr \left( \partial_\mu \Phi \partial^\mu \Phi + \Psi_0^\dagger e^{-\Phi} \Psi_0 e^\Phi \right)$$

(57)

If $\Psi_0(x^+)$ is now chosen to be the constant field $\Psi_0 = \sum_a E_a$ then this Lagrangian (57) becomes that of the classical $SU(N)$ Toda theory, while if $\Psi_0(x^+)$ is chosen to be the constant field $\Psi_0 = \sum_a E_a + E_{-M}$ then it becomes that of the affine $SU(N)$ Toda theory. With these choices for $\Psi_0$ the self-duality equation (56) reduces to the classical or affine Toda system, respectively.

2.3 Chiral Model, Unitons and General Solutions

Having considered some special cases in which the nonrelativistic self-dual Chern-Simons equations reduce to well-known integrable equations in two-dimensional Euclidean space, we now consider the question of finding the most general solutions. The key to the possibility of finding all solutions lies in the fact that there exists a special gauge transformation $g$ which converts the two equations (31,32) into a single equation

$$\partial_- \chi = [\chi^\dagger, \chi]$$

(58)

where $\chi$ is the gauge transformed matter field $\chi = \sqrt{\frac{1}{\kappa}}g \Psi g^{-1}$. The existence of such a gauge transformation $g^{-1}$ follows from the following zero-curvature formulation of the self-dual Chern-Simons equations [21, 28]. Define

$$A_+ \equiv A_+ - \sqrt{\frac{1}{\kappa}}\Psi$$

$$A_- \equiv A_- + \sqrt{\frac{1}{\kappa}}\Psi^\dagger$$

(59)

Then the nonrelativistic self-dual Chern-Simons equations (31,32) together imply that the gauge curvature associated with $A_\pm$ vanishes:

$$\mathcal{F}_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$$
Therefore, at least locally, one can write $A_\pm$ as a pure gauge

$$A_\pm = g^{-1} \partial_\pm g$$  \hspace{1cm} (61)

for some $g$ in the gauge group. Gauge transforming the nonrelativistic self-dual Chern-Simons equations \([51,52]\) with this group element $g^{-1}$ leads to the single equation \([58]\).

Equation \([58]\) can be converted into the chiral model equation

$$\partial_+ (h^{-1} \partial_- h) + \partial_- (h^{-1} \partial_+ h) = 0$$  \hspace{1cm} (62)

upon defining

$$\chi \equiv \frac{1}{2} h^{-1} \partial_+ h$$  \hspace{1cm} (63)

for some $h$ in the gauge group (the fact that it is possible to write $\chi$ in this manner is a consequence of \([58]\)). Given any solution $h$ of the chiral model equation \([62]\), or alternatively any solution $\chi$ of \([58]\), we automatically obtain a solution of the original nonrelativistic self-dual Chern-Simons equations:

$$\Psi^{(0)} = \sqrt{\kappa} \chi, \quad A_+^{(0)} = \chi, \quad A_-^{(0)} = -\chi^\dagger$$  \hspace{1cm} (64)

The chiral model equations are also referred to as the “harmonic map equations” because if we regard $J_\pm = h^{-1} \partial_\pm h$ as a connection, then it satisfies both

$$\partial_+ J_- + \partial_- J_+ = 0$$
$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$  \hspace{1cm} (65)

and so has zero divergence and zero curl.

The global condition which permits the classification of solutions to the chiral model equation \([62]\) is the condition of finiteness of the chiral model “action functional” (also referred to in the literature as the “energy functional”)

$$E[h] \equiv -\frac{1}{2} \int d^2 x \, tr(h^{-1} \partial_- h h^{-1} \partial_+ h)$$  \hspace{1cm} (66)

This finiteness condition has direct physical relevance in the related nonrelativistic Chern-Simons system because

$$E[h] = 2 \int d^2 x \, tr(\chi \chi^\dagger)$$
$$E[h] = \frac{2}{\kappa} \int d^2 x \, tr(\Psi \Psi^\dagger)$$
$$E[h] = \frac{2}{\kappa} Q^0$$  \hspace{1cm} (67)
where $Q^0$ is the conserved gauge invariant matter charge in (26). Thus, the finite action solutions of the chiral model equations correspond precisely to the finite charge solutions of the nonrelativistic self-dual Chern-Simons equations.

In addition to being physically significant, this finiteness condition is mathematically crucial because it permits the chiral model solutions on $\mathbb{R}^2$ to be classified by conformal compactification to the sphere $S^2$. Indeed, Uhlenbeck has classified all finite action chiral model solutions for $SU(N)$ in terms of “uniton” factors (which will be discussed below).

Before discussing the general classification of finite charge solutions, we introduce the simplest such solutions, the “single unitons”, upon which the general solutions are constructed. A “single uniton” solution, $h$, of the $SU(N)$ chiral model equation (62) has the form

$$h = 2p - 1$$

(68)

where $p$ is a “holomorphic projector” satisfying:

$$p^\dagger = p$$

(69)

$$p^2 = p$$

(70)

$$(1 - p)\partial_+ p = 0$$

(71)

These single uniton solutions are fundamental to the chiral model system because as a consequence of the conditions (71) we find that

$$h^{-1} \partial_\pm h = \pm 2\partial_\pm p$$

(72)

From this it immediately follows that $h$ satisfies the chiral model equation (62). These single uniton solutions are also solutions of the $CP^{N-1}$ model since $h$ satisfies the additional $CP^{N-1}$ condition, $h^2 = 1$, as a result of $p$ being a projector. In terms of the $\chi$ field defined in (63), the single uniton solutions take the simple form

$$\chi = \partial_+ p$$

(73)

It is straightforward to check that, as a consequence of the conditions (71) satisfied by $p$, $\chi$ satisfies the equation (58), and therefore gives a solution to the nonrelativistic self-dual Chern-Simons equations as in (74).

The general holomorphic projector satisfying the conditions (71) can be expressed as

$$p = M \left(M^\dagger M\right)^{-1} M^\dagger$$

(74)

where $M$ is any (rectangular) matrix such that $M = M(x^-)$ [30]. It is easy to see that such an $M$ is a *hermitean* projector. The third condition (71) is equivalent to $\partial_+ p p = 0$, which follows immediately from the fact that

$$\partial_+ p = M \left(M^\dagger M\right)^{-1} \partial_+ M^\dagger (1 - p)$$

(75)
The next step towards the construction of general solutions involves the process of “composing” uniton solutions, as follows. Suppose $h_1 = 2p_1 - 1$ is a single uniton solution with $p_1$ satisfying the conditions (71) for a holomorphic projector. Further, let $h_2 = 2p_2 - 1$ be such that $p_2 = p_2^\dagger$ and $p_2^2 = p_2$. Then $h = h_1h_2$ is a solution of the chiral model equation (62) provided the following first-order algebro-differential conditions are met:

\begin{align*}
(i) & \quad (1 - p_2) \left( \partial_+ + \frac{1}{2} h_1^{-1} \partial_+ h_1 \right) p_2 = 0 \\
(ii) & \quad (1 - p_2) \left( \frac{1}{2} h_1^{-1} \partial_- h_1 \right) p_2 = 0
\end{align*}

Given these conditions,

$$h^{-1} \partial_\pm h = \pm 2 (\partial_\pm p_1 + \partial_\pm p_2)$$

and so, once again, the chiral model equation (62) is immediately satisfied.

This procedure of composing uniton-type solutions can be continued, but since the $p$ matrices involved are projectors, there is a limit to how many independent projections can be made. For $SU(N)$, at most $N - 1$ such terms can be combined in this manner, as expressed in Uhlenbeck’s theorem:

**THEOREM** (K. Uhlenbeck [30]; see also J. C. Wood [32]): Every finite action solution $h$ of the $SU(N)$ chiral model equation (62) may be uniquely factorized as a product of “uniton” factors

$$h = \pm h_0 \prod_{i=1}^m (2p_i - 1)$$

where:

a) $h_0 \in SU(N)$ is constant;

b) each $p_i$ is a Hermitean projector ($p_i^\dagger = p_i$ and $p_i^2 = p_i$);

c) defining $h_j = h_0 \prod_{i=1}^j (2p_i - 1)$, the following linear relations must hold:

\begin{align*}
(1 - p_i) \left( \partial_+ + \frac{1}{2} h_{i-1}^{-1} \partial_+ h_{i-1} \right) p_i & = 0 \\
(1 - p_i) h_{i-1}^{-1} \partial_- h_{i-1} p_i & = 0
\end{align*}

d) $m \leq N - 1$.

The $\pm$ sign in (78) has been inserted to allow for the fact that Uhlenbeck and Wood considered the gauge group $U(N)$ rather than $SU(N)$.

An important implication of this theorem is that for $SU(2)$ all finite action solutions of the chiral model have the “single uniton” form

$$h = (2p - 1)$$
with \( p \) being a holomorphic projector satisfying the conditions (71). These single uniton solutions are essentially the \( CP^1 \) model solutions [34, 9]. For \( SU(N) \) with \( N \geq 3 \) one must consider composite unitons in addition to the single unitons. It becomes increasingly difficult to give a simple characterization of all possible projectors satisfying the subsidiary linear conditions specified in Uhlenbeck’s construction. However, Wood has presented a systematic parametrization of these higher unitons, for any \( SU(N) \), in terms of a sequence of projectors into Grassmannian factors. A detailed analysis of the \( SU(3) \) and \( SU(4) \) cases is also given in [35].

At this point, it is important to ask how these multi-uniton solutions to the chiral model equations relate to the special explicit Toda-type solutions discussed previously in (43-46). While the algebraic Ansätze (38,48) each leads to a non-Abelian charge density \( \rho = [\Psi^\dagger, \Psi] \) which is diagonal, the chiral model solutions (64) have charge density \( \rho^{(0)} = \kappa [\chi^\dagger, \chi] \) which need not be diagonal. However, \( \rho \) is always hermitean, and so it can be diagonalized by a gauge transformation. It is still a nontrivial algebraic task to implement this diagonalization explicitly, but this can be done for the \( SU(N) \) solutions, revealing an interesting new link between the chiral model and the Toda system [28].

It is instructive to illustrate this procedure with the \( SU(2) \) case first. Here, Uhlenbeck’s theorem implies that the only finite charge solution has the form \( \chi = \partial_+ p \), where \( p \) is a holomorphic projector as in (74). For \( SU(2) \) we can only project onto a line in \( \mathbb{C}^2 \), so we take
\[
M(x^-) = \begin{pmatrix} 1 \\ f(x^-) \end{pmatrix}
\]  
(80)

This then leads to the projection matrix
\[
p = \frac{1}{1 + \bar{f} f} \begin{pmatrix} 1 & \bar{f} \\ f & \bar{f} \end{pmatrix}
\]  
(81)

and the corresponding solution \( \chi \) can be expressed in terms of the single holomorphic function \( f(x^-) \):
\[
\chi = \partial_+ p = \frac{f \partial_+ \bar{f}}{(1 + \bar{f} f)^2} \begin{pmatrix} -1 & 1/f \\ -f & 1 \end{pmatrix}
\]  
(82)

The corresponding matter density is
\[
[\chi^\dagger, \chi] = -\frac{\partial_+ \bar{f} \partial_- f}{(1 + \bar{f} f)^3} \begin{pmatrix} 1 - \bar{f} f & 2\bar{f} \\ 2f & -1 + \bar{f} f \end{pmatrix}
\]  
(83)

which may be diagonalized by the unitary matrix
\[
g = \frac{1}{\sqrt{1 + \bar{f} f}} \begin{pmatrix} -\bar{f} & 1 \\ 1 & f \end{pmatrix}
\]  
(84)

to yield the diagonalized charge density
\[
g^{-1}[\chi^\dagger, \chi] g = \partial_+ \partial_- \log(1 + f(x^-)\bar{f}(x^+)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
In this diagonalized form we recognize Liouville’s solution \((43)\) to the classical \(SU(2)\) Toda equation. It is worth emphasizing that for \(SU(2)\) the nonrelativistic self-dual Chern-Simons equations \((31,32)\) can be converted, by suitable algebraic ansatze as discussed in the previous section, into the classical Toda (i.e. Liouville) equation or the affine Toda (i.e. sinh-Gordon) equation. However, the above analysis shows that only the classical Toda case (i.e. Liouville) corresponds to finite charge.

A similar construction is possible for the \(SU(N)\) case \([28,29]\). Specifically, let

\[
h = (-1)^{\frac{1}{2}N(N+1)} \prod_{a=1}^{N-1} (2p_a - 1)
\]

be a product where each \(p_a\) is a holomorphic projector onto the \(a\)-dimensional subspace spanned by the columns of the \(N \times a\) rectangular matrix \(M_a(x^-)\) in \((10,11)\):

\[
M_a = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
-f_1 & \partial_- f_1 & \ldots & \partial^{(a-1)} f_1 \\
-f_2 & \partial_- f_2 & \ldots & \partial^{(a-1)} f_2 \\
\vdots & \vdots & \ddots & \vdots \\
f_{N-1} & \partial_- f_{N-1} & \ldots & \partial^{(a-1)} f_{N-1}
\end{pmatrix}
\]

Then \(h\) is a finite action solution of the \(SU(N)\) chiral model equation \((62)\) and the corresponding solution of the nonrelativistic self-dual Chern-Simons equations is

\[
\chi = \sum_{a=1}^{N-1} \partial_+ p_a
\]

The charge density \([\chi^\dagger, \chi]\) may be diagonalized by an \(SU(N)\) gauge transformation \(g\) yielding a diagonal form

\[
g^{-1}[\chi^\dagger, \chi]g = \sum_{a=1}^{N-1} \{\partial_+ \partial_- \log \det(M_a^\dagger M_a)\} H_a
\]

where the \(H_a\) are the Cartan subalgebra generators of \(SU(N)\) in the Chevalley basis. This diagonal form of the charge density corresponds precisely to the \(SU(N)\) Toda solution \((46)\).

Another useful result which follows from the relationship between the nonrelativistic self-dual Chern-Simons equations \((31,32)\) and the chiral model equation \((62)\) is that the chiral model energy \((66)\) is quantized in integral multiples of \(8\pi\) \([33]\). This implies that the abelian Chern-Simons charge \(Q^0 \equiv \int tr(\Psi^\dagger \Psi)\) is quantized in integral multiples of \(4\pi\kappa\). A related quantization condition has been noted in \([18]\), where the non-Abelian charges \(Q^0_a \equiv \int \rho_a\) are quantized in integral multiples of \(4\pi\kappa\) for the \(SU(N)\) Toda-type solutions \((46)\). In this case the abelian charge is the sum of the individual nonAbelian charges : \(Q^0 = \sum_a Q^0_a\).
3 Relativistic SDCS Theories

3.1 Relativistic Self-Dual Chern-Simons Equations

In this section we discuss the relativistic generalization of the nonrelativistic self-dual Chern-Simons theories. The existence of vortex solutions in 2 + 1-dimensional relativistic gauge-Higgs models including Chern-Simons terms has been known for some time [36]. The importance of self-duality was first noticed in the context of abelian theories [37, 38], where vortices in the relativistic Chern-Simons-Higgs model were shown to be related to a self-duality condition reminiscent of Bogomol’nyi’s analysis [3] of vortices in the abelian Higgs model. With a particular sixth order scalar potential there is a lower bound on the energy functional which is saturated by topological solitons and nontopological vortices [39]. An extension of these abelian models is possible, to nonabelian relativistic self-dual Chern-Simons theories with a global $U(1)$ symmetry [40], once again with a special sixth order potential. However, while the self-dual structure of the model generalizes in a relatively straightforward manner, the analysis of the nonabelian relativistic self-dual Chern-Simons equations themselves is significantly more complicated, and correspondingly more interesting. The richness of the nonabelian theory is compounded by the many available choices: of gauge group, of representation, of matter coupling, etc... [40, 41, 42, 43]. Matter fields in the defining representation have been studied in [41], while the most interesting case once again seems to be the case of adjoint coupling [40, 42, 44, 45]. The self-dual structure of these relativistic self-dual Chern-Simons systems is related at a fundamental level to extended supersymmetry in 2 + 1 dimensions [46, 47, 48], in the sense that the self-dual Lagrangian is the bosonic portion of a Lagrangian with an extended supersymmetry. This is in accordance with a general relationship between self-duality and extended supersymmetry [49].

Consider the Lagrange density

$$\mathcal{L} = -\kappa \mathcal{L}_{CS} - \text{tr} \left( (D_\mu \phi^\dagger) D^\mu \phi \right) - V(\phi, \phi^\dagger)$$

(90)

where $\mathcal{L}_{CS}$ is the Chern-Simons Lagrange density in (13), and the scalar field potential $V(\phi, \phi^\dagger)$ is

$$V(\phi, \phi^\dagger) = \frac{1}{4\kappa^2} \text{tr} \left( \left( [ [ \phi, \phi^\dagger ], \phi ] - v^2 \phi \right)^\dagger \left( [ [ \phi, \phi^\dagger ], \phi ] - v^2 \phi \right) \right).$$

(91)

The space-time metric is taken to be $g_{\mu\nu} = \text{diag} (-1, 1, 1)$ and, as before, $\text{tr}$ refers to the trace in a finite dimensional representation of the compact simple Lie algebra $\mathcal{G}$ to which the gauge fields $A_\mu$ and the charged matter fields $\phi$ and $\phi^\dagger$ belong. The $v^2$ parameter appearing in the potential (91) will play the role of a mass parameter (see (128)). Under a gauge transformation both the potential $V$ and the scalar field kinetic term $\text{tr} \left( (D_\mu \phi^\dagger) D^\mu \phi \right)$ remain invariant. However, the Chern-Simons Lagrange
density is not invariant and the dimensionless coupling coefficient $\kappa$ must be quantized in order for the corresponding quantum theory to be invariant under large gauge transformations [10]. The particular sixth-order form of the scalar field potential (91), together with its overall strength depending on the Chern-Simons coupling parameter $\kappa$, are fixed by the condition of self-duality, as shown below.

The Euler-Lagrange equations of motion obtained from the Lagrange density (90) are:

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger}$$

$$- \kappa \varepsilon^{\mu\nu\rho} F_{\nu\rho} = i J^\mu$$

In the matter equation of motion (92), $\frac{\partial V}{\partial \phi^\dagger}$ is defined by the change in the potential $V$ under a variation of $\phi^\dagger$:

$$\delta V \equiv tr \left( \delta \phi^\dagger \frac{\partial V}{\partial \phi^\dagger} \right)$$

In the gauge equation of motion (93), $J^\mu$ is the relativistic nonabelian current

$$J^\mu \equiv -i \left( \left[ \phi^\dagger, D^\mu \phi \right] - \left[ (D^\mu \phi)^\dagger, \phi \right] \right)$$

which is covariantly conserved: $D_\mu J^\mu = 0$. This system also has an abelian current, $Q_\mu$,

$$Q_\mu = -i tr \left( \phi^\dagger D_\mu \phi - (D_\mu \phi)^\dagger \phi \right),$$

which is ordinarily conserved: $\partial_\mu Q^\mu = 0$.

The energy density corresponding to the Lagrange density (90) is

$$\mathcal{H} = tr \left( (D_0 \phi)^\dagger D_0 \phi \right) + tr \left( (D_i \phi)^\dagger D_i \phi \right) + V \left( \phi, \phi^\dagger \right),$$

supplemented by the Gauss law constraint

$$[\phi^\dagger, D_0 \phi] - [(D_0 \phi)^\dagger, \phi] = 2\kappa F_{12},$$

which is the zeroth component of the gauge field equations of motion (93). Notice that, as is familiar for Chern-Simons theories, the Chern-Simons term $\mathcal{L}_{CS}$ in the Lagrange density (90) does not contribute to the energy, while it does affect the canonical structure and the constraints [16, 17, 18].

To find self-dual solutions which minimize the energy, we re-express the energy density in a modified form, using an adaptation of the Bogomol’nyi method for vortices in the abelian Higgs model [3]. Using the identity (29) together with the Gauss law constraint (98), we can write

$$tr \left( (D_\phi)^\dagger D_i \phi \right) = tr \left( (D_\phi)^\dagger D_- \phi \right)$$
where we recall that $D_\pm \equiv D_1 \pm i D_2$. The second term on the RHS of (99) may be cancelled in the energy density (97) by a term from $\text{tr} \left( (D_0 \phi)^\dagger D_0 \phi \right)$ if we write

$$\text{tr} \left( (D_0 \phi)^\dagger D_0 \phi \right) = \text{tr} \left( \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] \right)^\dagger \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] \right) \right)$$

$$- \frac{i}{2\kappa} \text{tr} \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] \right)^\dagger \left( D_0 \phi - \left[ \left[ \phi, \phi^\dagger \right], \phi \right] \right) (D_0 \phi)^\dagger$$

$$- \frac{1}{4\kappa^2} \text{tr} \left( \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right) \right)^\dagger \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)$$

(100)

One could then interpret the final term on the RHS of (100) as (minus) a potential, in which case the energy density (97) could be expressed in a manifestly positive form. However, this choice would result in a sixth order scalar field potential without a mass term, and this is unsuitable for a number of reasons discussed below. Rather, one should be more general and explicitly introduce a mass term in this potential by writing (generalizing the decomposition (100))

$$\text{tr} \left( (D_0 \phi)^\dagger D_0 \phi \right) =$$

$$\text{tr} \left( \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right) \right)^\dagger \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)$$

$$- \frac{i}{2\kappa} \text{tr} \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)^\dagger \left( D_0 \phi - \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)$$

$$- \frac{1}{4\kappa^2} \text{tr} \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)^\dagger \left( \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)$$

(101)

The final term in this expression (101) is recognized as (minus) the potential $V(\phi, \phi^\dagger)$ defined in (91), and so the energy density (97) can be expressed as

$$\mathcal{E} = \text{tr} \left( \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right) \right)^\dagger \left( D_0 \phi - \frac{i}{2\kappa} \left[ \left[ \phi, \phi^\dagger \right], \phi \right] - v^2 \phi \right)$$

$$+ \text{tr} \left( (D_- \phi)^\dagger D_- \phi \right) + \frac{i v^2}{2\kappa} \text{tr} \left( \phi^\dagger (D_0 \phi) - (D_0 \phi)^\dagger \phi \right)$$

(102)

The first two terms in (102) are manifestly positive and the third gives a lower bound for the energy density, which may be written in terms of the time component, $Q^0$, of the abelian relativistic current defined in (96):

$$\mathcal{E} \geq \frac{v^2}{2\kappa} Q^0$$

(103)
This lower bound (103) is saturated when the following two conditions (each first order in spacetime derivatives) hold:

\[ D^- \phi = 0 \] (104)

\[ D_0 \phi = \frac{i}{2\kappa} \left( [ [\phi, \phi^\dagger], \phi] - v^2 \phi \right) \] (105)

The consistency condition of these two equations states that

\[ (D_0 D^- - D^- D_0) \phi \equiv [F_{0-}, \phi] \]
\[ = -\frac{i}{2\kappa} [\phi, (D_+ \phi)^\dagger], \phi] \]
\[ = \frac{1}{2\kappa} [J_-, \phi] \] (106)

which expresses the gauge field Euler-Lagrange equation of motion

\[ F_{0-} = \frac{1}{2\kappa} J_- \]
\[ = -\frac{1}{2\kappa} [\phi, (D_+ \phi)^\dagger] \] (107)

for the spatial components of the current. The other gauge field equation, \( F_{+-} = \frac{1}{\kappa} J_0 \), may be re-expressed using equation (105) in a form not involving explicit time derivatives. We thus arrive at the relativistic self-dual Chern-Simons equations:

\[ D^- \phi = 0 \] (108)

\[ F_{+-} = \frac{1}{\kappa^2} [v^2 \phi - [[\phi, \phi^\dagger], \phi]\dagger] \] (109)

At the self-dual point, we can use equation (105) to express the energy density as

\[ \mathcal{E}_{SD} = \frac{v^2}{2\kappa^2} tr \left( \phi^\dagger (v^2 \phi - [[\phi, \phi^\dagger], \phi]) \right) \] (110)

Recall that all solutions to the nonrelativistic self-duality equations (31,32) correspond to the static zero-energy solutions to the Euler-Lagrange equations of motion [15]. Here, in the relativistic theory, the situation is rather different. First, the lower bound (103) on the energy density is not necessarily zero, and the solutions of (105) are time dependent. Furthermore, unlike in the nonrelativistic case, it is possible to have nontrivial solutions for \( \phi \) while having \( F_{+-} = 0 \). These solutions do have zero energy, and are gauge equivalent to solutions of the algebraic equation

\[ [[\phi, \phi^\dagger], \phi] = v^2 \phi. \] (111)
Solutions of this equation also correspond to the minima of the potential (91), and these potential minima are clearly degenerate.

A class of solutions to the self-duality equations (109) is given by the following zero energy solutions of the Euler-Lagrange equations:

\[
\phi = g^{-1}\phi_{(0)} g \\
A_\pm = g^{-1}\partial_\pm g \\
A_0 = g^{-1}\partial_0 g
\]

where \(\phi_{(0)}\) is any solution of (111), and \(g = g(\vec{x}, t)\) takes values in the gauge group. It is clear that these solutions satisfy \(D_0\phi = 0, D_\phi = 0, F_{+-} = 0\), as well as the algebraic equation (111), which implies that they are self-dual, and that they have zero magnetic field and zero charge density. While this class of solutions may look somewhat trivial, it is still important because the solutions, \(\phi_{(0)}\), of the algebraic equation (111) classify the minima of the potential \(V\), and the finite nonzero energy solutions of the self-duality equations must be gauge equivalent to such a solution at infinity:

\[
\phi \to g^{-1}\phi_{(0)} g \quad \text{as} \quad r \to \infty
\]

It is important to check explicitly the consistency of the self-duality equations (108,109) with the Euler-Lagrange equation of motion (92,93). Note that

\[
D_\mu D^\mu \phi = -D_0 D_0 \phi + D_i D_i \phi
\]

For self-dual solutions \(D_\phi = 0\), and using the self-duality equation for \(D_0\phi\) we find that

\[
\iota [F_{12}, \phi] = \frac{1}{2\kappa^2}[[\phi, [\phi^\dagger, [\phi, \phi^\dagger]]], \phi] + \frac{\nu^2}{2\kappa^2}[[\phi, [\phi, \phi^\dagger]], [\phi, \phi^\dagger]]
\]

and

\[
-D_0 D_0 \phi = \frac{\nu^4}{4\kappa^2} \phi + \frac{\nu^2}{2\kappa^2}[[\phi, [\phi, \phi^\dagger]], [\phi, \phi^\dagger]] + \frac{1}{4\kappa^2}[[\phi, [\phi, \phi^\dagger]], [\phi, \phi^\dagger]]
\]

Therefore,

\[
D_\mu D^\mu \phi = \frac{\nu^4}{4\kappa^2} \phi + \frac{\nu^2}{\kappa^2}[[\phi, [\phi, \phi^\dagger]], [\phi, \phi^\dagger]]
\]

\[
+ \frac{1}{4\kappa^2} \left( [[\phi, [\phi, \phi^\dagger]], [\phi, \phi^\dagger]] + 2[[\phi, [\phi^\dagger, [\phi, \phi^\dagger]]], \phi] \right)
\]

It is a straightforward matter to verify that (117) does indeed yield the correct charged scalar field Euler-Lagrange equation of motion (92) with the potential \(V(\phi, \phi^\dagger)\) given by equation (91).
To verify that this model is the natural nonabelian generalization of the abelian relativistic model \([37, 38]\), we take the ‘abelian limit’ by choosing a special algebraic restriction of \(SU(2)\). (Such an abelian limit is familiar from the corresponding nonrelativistic models \([21]\)). Consider the Chevalley basis for the \(SU(2)\) Lie algebra generators:

\[
\begin{align*}
[E_+, E_-] &= H \\
[H, E_\pm] &= \pm 2E_\pm \\
\text{tr} \,(E_+ E_-) &= 1 \\
\text{tr} \,(H^2) &= 2
\end{align*}
\] (118)

where \(H\) is the Cartan subalgebra generator and \(E_\pm\) are the step operators. For example, in the defining representation of \(SU(2)\), this basis may be taken as:

\[
E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (119)

Further, choose the fields to have the following Lie algebraic decomposition (note that this is an \textit{ansatz}, not simply a gauge choice):

\[
\begin{align*}
\phi &= \psi E_+ \\
\phi^\dagger &= \bar{\psi} E_- \\
A_- &= aH \\
A_+ &= -\bar{a}H
\end{align*}
\] (120)

Then \(D_- \phi = (\partial_- \psi + 2a \bar{\psi}) E_+\), and the self-duality equations \([108, 109]\) become

\[
\begin{align*}
a &= -\frac{1}{2} \partial_- \ln |\psi| \\
\partial_+ a + \partial_- \bar{a} &= -\frac{1}{\kappa^2} |\psi|^2 \left(2|\psi|^2 - v^2\right)
\end{align*}
\] (121)

These two equations may be combined to yield the single equation satisfied by the gauge invariant scalar, \(|\psi|^2 \equiv tr \left(\phi \phi^\dagger\right)\):

\[
\partial_+ \partial_- \ln |\psi|^2 = 2 \frac{\kappa^2}{\kappa^2} |\psi|^2 \left(2|\psi|^2 - v^2\right)
\] (122)

This is the same (apart from trivial rescalings resulting from different normalization) as the abelian self-duality condition found in \([37, 38]\) in their analysis of abelian self-dual Chern-Simons vortices. With the Lie algebraic \textit{ansatz} \((120)\) for the fields, the potential \((111)\) reduces to

\[
V = \frac{1}{4\kappa^2} |\psi|^2 \left(2|\psi|^2 - v^2\right)^2
\] (123)
which is the same as the self-dual sixth-order potential found in [37, 38]. Further, with the fields as in (120), the self-dual energy density (111) becomes
\[ \mathcal{E}_{SD} = \frac{-v^2}{2\kappa^2} |\psi|^2 \left( 2|\psi|^2 - v^2 \right) \] (124)

From (121) we recognize this self-dual energy density as being proportional to the ‘abelian’ magnetic field strength, \( f_{+-} \equiv \frac{1}{2} \text{tr} (HF_{+-}) = \partial_+ a + \partial_- \bar{a} \), and so
\[ \mathcal{E}_{SD} = \frac{v^2}{2} f_{+-} \] (125)

which shows that the energy is bounded below by a magnetic flux, as in the abelian model [37, 38].

We note here that the positive sign of the RHS of (122) is significant. For example, in the massless case when \( v^2 = 0 \), the equation
\[ \partial_+ \partial_- \ln|\psi|^2 = \frac{4}{\kappa^2} |\psi|^4 \] (126)
can be solved exactly, but it has no real, regular and integrable solutions for \( |\psi|^2 \). This lack of real, regular solutions when \( v^2 = 0 \) is one reason for introducing the \( v^2 \) mass term in the potential (91).

Another reason for introducing the mass term in the potential is that it permits the taking of the nonrelativistic limit. Without a mass scale for the scalar field \( \phi \) there is no meaning to such a limit. Restoring factors of \( c \) to the potential (91), we find a mass term [42]
\[ \frac{\nu^4}{4\kappa^2 c^4} \text{tr} \left( \phi^\dagger \phi \right) \equiv m^2 \text{c}^2 \text{tr} \left( \phi^\dagger \phi \right) \] (127)
which yields a scalar field mass
\[ m = \frac{v^2}{2\kappa c^3} \] (128)

To maintain a finite mass in the nonrelativistic limit, the \( c \to \infty \) limit must be accompanied by the \( v^2 \to \infty \) limit in such a way that \( \frac{\nu^2}{c^4} \) is kept constant. Separating out the rest-mass energy as
\[ \phi = \frac{1}{\sqrt{2m}} e^{-imct} \Psi, \] (129)
and keeping only the dominant terms in inverse powers of \( c \), the Lagrange density (70) reduces to the nonrelativistic Lagrange density (the Chern-Simons Lagrange density, \( \mathcal{L}_{CS} \), is unchanged) in (22):
\[ \mathcal{L} = -\kappa \mathcal{L}_{CS} + i \text{tr} \left( \Psi^\dagger (\partial_t \Psi + [A_0, \Psi]) \right) - \frac{1}{2m} \text{tr} \left( (D_i \Psi)^\dagger D_i \Psi \right) + \frac{1}{4m\kappa c} \text{tr} \left( \left( [\Psi, \Psi^\dagger] \right)^2 \right) \] (130)
Further, in the nonrelativistic limit, the relativistic self-duality equations (108,109) reduce to
\[
D_- \Psi = 0,
\]
\[
F_{+-} = \frac{1}{\kappa} [\Psi, \Psi^\dagger]
\]
which are the nonrelativistic self-dual Chern-Simons equations (31,32). Finally, at the self-dual point (where \( D_- \Psi = 0 \)), the Schrödinger equation (23) becomes
\[
iD_t \Psi = -\frac{1}{4m\kappa c} [[\Psi, \Psi^\dagger], \Psi]
\]
which is the nonrelativistic limit of the relativistic self-duality equation (105).

3.2 Classification of Minima

The sixth order potential (91) has degenerate minima given by fields \( \phi_{(0)} \) which solve
\[
[[\phi, \phi^\dagger], \phi] = \phi
\]
where a factor of \( v \) has been absorbed into the field \( \phi \). We recognize the condition (133) as the \( SU(2) \) commutation relation. For a general gauge algebra, finding the solutions to (133) is the classic Dynkin problem [50] of embedding \( SU(2) \) into a general Lie algebra. It is interesting to note that this type of embedding problem also plays a significant role in the theory of spherically symmetric magnetic monopoles and the Toda molecule equations [15].

It is clear that in order to satisfy (133) for a general gauge algebra, \( \phi = \phi_{(0)} \) must be a linear combination of the step operators for the positive roots of the algebra. Further, since we have the freedom of global gauge invariance, we can choose representative gauge inequivalent solutions \( \phi_{(0)} \) to be linear combinations of the step operators of the positive simple roots. It is therefore convenient to work in the Chevalley basis (39) for the gauge algebra (for ease of presentation we shall consider \( SU(N) \)). Expand \( \phi_{(0)} \) in terms of the positive simple root step operators as:
\[
\phi_{(0)} = \sum_{a=1}^{N-1} \phi^a_{(0)} E_a
\]
Then \([\phi_{(0)}, \phi^\dagger_{(0)}] \) is diagonal,
\[
[\phi_{(0)}, \phi^\dagger_{(0)}] = \sum_{a=1}^{N-1} |\phi^a_{(0)}|^2 H_a.
\]
The Chvalley basis commutation relations (39) then imply that

\[
[\phi(0), \phi^\dagger(0), \phi(0)] = \sum_{a=1}^{N-1} \sum_{b=1}^{N-1} |\phi^a(0)|^2 \phi_b(0) C_{ba} E_b
\]  

(136)

which, like \(\phi(0)\), is once again a linear combination of just the simple root step operators. Thus, for suitable choices of the coefficients \(\phi^a(0)\), it is possible for the \(SU(N)\) algebra element \(\phi(0)\) to satisfy the \(SU(2)\) commutation relation \([\phi, \phi^\dagger], \phi] = \phi\).

For example, one can always choose \(\phi(0)\) proportional to a single step operator, which by global gauge invariance can always be taken to be \(E_1\):

\[
\phi(0) = \frac{1}{\sqrt{2}} E_1
\]  

(137)

In the other extreme, the \(SU(N)\) “maximal embedding” case, with all \(N - 1\) step operators involved in the expansion (134), the solution for \(\phi(0)\) is:

\[
\phi(0) = \frac{1}{\sqrt{2}} \sum_{a=1}^{N-1} \sqrt{a(N-a)} E_a
\]  

(138)

All other solutions for \(\phi(0)\), intermediate between the two extremes (137) and (138), can be generated by the following systematic procedure. If one of the simple root step operators, say \(E_b\), is omitted from the summation in (134) then this effectively decouples the \(E_{\pm a}\)’s with \(a < b\) from those with \(a > b\). Then the coefficients for the \((b - 1)\) step operators \(E_a\) with \(a < b\) are just those for the maximal embedding (see equation (138)) in \(SU(b)\), and the coefficients for the \((N - b - 1)\) \(E_a\)’s with \(a > b\) are those for the maximal embedding in \(SU(N - b)\):

\[
\phi(0) = \frac{1}{\sqrt{2}} \sum_{a=1}^{b-1} \sqrt{a(b-a)} E_a + \frac{1}{\sqrt{2}} \sum_{a=b+1}^{N-1} \sqrt{a(N-b-a)} E_a
\]  

(139)

Diagrammatically, we can represent the maximal embedding case (138) with the Dynkin diagram of \(SU(N)\):

\[
\underset{N-1}{\text{o o o o ... o o}}
\]

(140)

which shows the \(N - 1\) simple roots of the algebra, each connected to its nearest neighbours by a single line. Omitting the \(b^{th}\) simple root step operator from the sum in (134) can be conveniently represented as breaking the Dynkin diagram in two by deleting the \(b^{th}\) dot:

\[
\underset{b-1}{\text{o o ... o x o ... o}} \quad \underset{N-b-1}{\text{o o ... o}}
\]

(141)

\[1\text{In general, the squares of the coefficients for the maximal embedding case are the coefficients, in the simple root basis, of (one half times) the sum of all positive roots of the algebra.}\]
With this deletion of the $b^{th}$ dot, the $SU(N)$ Dynkin diagram breaks into the Dynkin diagram for $SU(b)$ and that for $SU(N-b)$. Since the remaining simple root step operators decouple into a Chevalley basis for $SU(b)$ and another for $SU(N-b)$, the coefficients required for the summation over the first $b-1$ step operators are just those given in (138) for the maximal embedding in $SU(b)$, while the coefficients for the summation over the last $N-b-1$ step operators are given by the maximal embedding for $SU(N-b)$, as indicated in (139).

It is clear that this process may be repeated with further roots being deleted from the Dynkin diagram, thereby subdividing the original $SU(N)$ Dynkin diagram, with its $N-1$ consecutively linked dots, into subdiagrams of $\leq N-1$ consecutively linked dots. The final diagram, with $M$ deletions made, can be characterized, up to gauge equivalence, by the $M+1$ lengths of the remaining consecutive strings of dots. A simple counting argument shows that the total number of ways of doing this (including the case where all dots are deleted, which corresponds to the trivial solution $\phi(0) = 0$) is given by the number, $p(N)$, of (unrestricted) partitions of $N$.

The $SU(4)$ case is sufficient to illustrate this procedure. There are 5 partitions of 4, and they correspond to the following solutions for $\phi(0)$:

\[
\begin{align*}
o - o - o & \quad \phi(0) = \frac{1}{\sqrt{2}} \left( \sqrt{3} E_1 + 2 E_2 + \sqrt{3} E_3 \right) \\
o - o - x & \quad \phi(0) = E_1 + E_2 \\
o - x - o & \quad \phi(0) = \frac{1}{\sqrt{2}} E_1 + \frac{1}{\sqrt{2}} E_3 \\
o - x - x & \quad \phi(0) = \frac{1}{\sqrt{2}} E_1 \\
x - x - x & \quad \phi(0) = 0
\end{align*}
\]

Thus we have a simple constructive procedure, and a correspondingly simple labelling notation, for finding all $p(N)$ gauge inequivalent solutions $\phi(0)$ to the algebraic embedding condition (133). Recall that each such $\phi(0)$ characterizes a distinct minimum of the potential $V$, as well as a class of zero energy solutions to the selfduality equations (108,109).

Since each vacuum solution $\phi(0)$ corresponds to an embedding of $SU(2)$ into $SU(N)$, an alternative shorthand for labelling the different vacua consists of listing the block diagonal spin content of the $SU(2)$ Cartan subalgebra element $[\phi(0), \phi(0)^\dagger] \sim J_3$. For example, consider the matter fields $\phi$ taking values in the $N \times N$ defining representation. Then, for each vacuum solution, $[\phi(0), \phi(0)^\dagger]$ takes the $N \times N$ diagonal sub-blocked form:

\[\text{27}\]
Each spin $j$ sub-block has dimension $2j + 1$, and so it is therefore natural to associate this particular $\phi(0)$ with the following partition of $N$:

$$N = (2j_1 + 1) + (2j_2 + 1) + \ldots + (2j_M + 1)$$

For example, the $SU(4)$ solutions listed in (142) may be labelled by the partitions 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1, respectively.

**3.3 Vacuum Mass Spectra**

Having classified all possible gauge inequivalent vacua of the potential $V$, we now determine the spectrum of massive excitations in each vacuum. In the abelian model there is only one nontrivial vacuum, and a consequence of the particular sixth order self-dual form of the potential is that in this broken vacuum the massive gauge excitation and the remaining real massive scalar field are degenerate in mass. This degeneracy of the gauge and scalar masses in the broken vacuum is also true of the 2 + 1 dimensional Abelian Higgs model. In the nonabelian models considered here the situation is considerably more complicated, due to the presence of many fields and also due to the many different gauge inequivalent vacua. Nevertheless, we shall see that an analogous mass degeneracy pattern exists, reflecting the self-dual character of the potential (91).

Regarded as a symmetry breaking problem, the relativistic self-dual Chern-Simons system with Lagrange density is rather different from a conventional Higgs system. First, in 3 + 1 dimensional field theory one most commonly considers symmetry breaking potentials of $\phi^4$ form, but here in 2 + 1 dimensions we consider a (renormalizable) sixth order potential. This means that the extraction of the scalar masses in the broken vacua is algebraically more complicated. The second, and more significant, difference is that the Higgs mechanism for generating massive gauge degrees of freedom behaves very differently in a 2 + 1 dimensional theory with a Chern-Simons term present for the gauge field. There are three separate possibilities:
• The gauge masses are produced by the Higgs mechanism alone.
• The gauge masses are produced by a Chern-Simons term alone.
• The gauge masses are produced by both a Higgs potential and a Chern-Simons term.

It is sufficient to illustrate these cases with an abelian theory. The first case corresponds to a Lagrange density

\[\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} -(D_{\mu}\phi)^\dagger D^\mu \phi - V(\phi) \]  

where \( V(\phi) \) has some nontrivial vacuum \( \phi(0) \). Note that \( e^2 \) has dimensions of mass in \( 2 + 1 \) dimensions. In the broken vacuum, after shifting the scalar field \( \phi \) by \( \phi(0) \), we find the quadratic part of the gauge field Lagrange density to be

\[\mathcal{L}_{\text{quad}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - |\phi(0)|^2 A_{\mu} A^{\mu}\]  

from which we deduce, as usual, a gauge mass

\[m_{\text{gauge}} = \sqrt{2e} |\phi(0)|\]  

However, if the Lagrange density includes also a Chern-Simons term, then the quadratic part of the gauge Lagrange density in the broken vacuum is

\[\mathcal{L}_{\text{quad}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2e^2} \epsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} - |\phi(0)|^2 A_{\mu} A^{\mu}\]  

which has two massive degrees of freedom

\[m_{\pm} = \frac{\kappa}{2} \left( \sqrt{1 + \frac{8e^2 |\phi(0)|^2}{\kappa^2}} \pm 1 \right)\]  

Notice that with the Chern-Simons coupling in (148), the Chern-Simons coupling parameter \( \kappa \) has dimensions of mass. These two masses (149) may be deduced from the gauge propagator in a covariant gauge [51], by a self-dual factorization of the Maxwell-Chern-Simons-Proca equation [52], or by a Schrödinger representation analysis of the quadratic Hamiltonian [53].

The third possibility (the one that is realized in the relativistic self-dual Chern-Simons systems [50] considered in this paper) is that the Lagrange density has a Chern-Simons term but no Maxwell term [54]. Then, in the broken vacuum, the quadratic part of the gauge Lagrange density is

\[\mathcal{L}_{\text{quad}} = -\frac{\kappa}{2e^2} \epsilon^{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} - |\phi(0)|^2 A_{\mu} A^{\mu}\]  

(150)
from which we deduce a single massive mode, with mass

\[ m_{CSH} = \frac{2e^2|\phi(0)|^2}{\kappa} \]  

(151)

This pure Chern-Simons Higgs mechanism (150,151) can be considered as the limit of the Maxwell-Chern-Simons Higgs mechanism (148,149) in which \( e^2 \to \infty \) with \( \kappa/e^2 \) fixed.

A simple physical picture of these three different forms of gauge mass generation in 2 + 1 dimensions comes from the analogy of Chern-Simons quantum mechanics [55], in terms of which the conventional Higgs mechanism corresponds to the planar quantum mechanics of a particle in a harmonic well, while the Maxwell-Chern-Simons -Higgs mechanism corresponds to the planar quantum mechanics of a particle in a harmonic well and a perpendicular external magnetic field. In this latter case, there are two characteristic frequencies, and these are precisely the two masses found in (149) [53]. The pure Chern-Simons Higgs mechanism corresponds to the lowest Landau level projection in which the external magnetic field becomes very strong, so that the cyclotron frequency scale is ‘frozen out’ leaving a single frequency scale which matches the mass in (151).

Having discussed the general situation, we now return to the specific case of the relativistic self-dual Chern-Simons system with Lagrange density (90), regarded as a symmetry breaking problem. The scalar masses in the vacuum \( \phi(0) \) are determined by expanding the shifted potential \( V(\phi + \phi(0)) \) to quadratic order in the field \( \phi \):

\[
V(\phi + \phi(0)) = \frac{\nu^4}{4\kappa^2} tr \left( \left[ [\phi(0), \phi^\dagger], \phi(0) \right] + \left[ [\phi, \phi^\dagger(0)], \phi(0) \right] + \left[ [\phi(0), \phi^\dagger(0)], \phi \right] - \phi \right)^2 
\]

(152)

With the fields normalized appropriately, the masses are then given by the square roots of the eigenvalues of the \( 2(N^2 - 1) \times 2(N^2 - 1) \) mass matrix in (152).

In the unbroken vacuum, with \( \phi(0) = 0 \), there are \( N^2 - 1 \) complex scalar fields, each with mass

\[ m = \frac{\nu^2}{2\kappa} \]  

(153)

In one of the broken vacua, where \( \phi(0) \neq 0 \), some of these \( 2(N^2 - 1) \) massive scalar degrees of freedom are converted to massive gauge degrees of freedom. The gauge masses are determined by expanding \( \nu^2 tr \left( \left( D_{\mu} (\phi + \phi(0)) \right)^\dagger \left( D_{\mu} (\phi + \phi(0)) \right) \right) \) and extracting the piece quadratic in the gauge field \( A \):

\[
\nu^2 tr \left( [A_{\mu}, \phi(0)]^\dagger [A_{\mu}, \phi(0)] \right) 
\]

(154)

The gauge masses are determined by finding the eigenvalues (not the square roots of the eigenvalues) of the \( (N^2 - 1) \times (N^2 - 1) \) mass matrix in (154). Allowing for the
nonabelian normalization factors, we see from (151) that the gauge masses are given
by multiples of the same mass scale, $v^2/2\kappa$, as the scalar masses.

This procedure of finding the eigenvalues of the scalar and gauge mass matrices,
must be performed for each of the $p(N)$ gauge inequivalent minima $\phi(0)$ of $V$. The
results for $SU(3)$ and $SU(4)$ are presented here in Tables 1 and 2 (see also Ref. [45]).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\text{vacuum} & \text{gauge masses} & & \\
& $\phi(0)$ & $\phi(0)$ & \\
& & real & complex & \\
& & fields & fields & \\
\hline
$o \times o$ & 2 & 1/2 & 1/2 & 1 \\
$o \times o$ & 6 & 1 & 2 & 5 \\
\hline
\end{tabular}
\caption{$SU(3)$ vacuum mass spectra, in units of the fundamental mass scale $v^2/2\kappa$, for the inequivalent nontrivial minima $\phi(0)$ of the potential $V$. Notice that for each vacuum the total number of massive degrees of freedom is equal to $2(N^2 - 1) = 16$, although the distribution between gauge and scalar fields is vacuum dependent.}
\end{table}

A number of interesting observations can be made at this point, based on the
evaluation of these mass spectra for the various vacua in $SU(N)$ for $N$ up to 10.

(i) All masses, both gauge and scalar, are integer or half-odd-integer multiples
of the fundamental mass scale $m = v^2/2\kappa$. The fact that all the scalar masses are
proportional to $m$ is clear from the form of the potential $V$ in (91). The fact that
the gauge masses are multiples of the same mass scale depends on the fact that the
Chern-Simons coupling parameter $\kappa$ has been included in the overall normalization
of the potential in (91). This is a direct consequence of the self-duality of the model.

(ii) In each vacuum, the masses of the real scalar excitations are equal to the masses
of the real gauge excitations, whereas this is not true of the complex scalar and gauge
fields (by ‘complex’ gauge fields we simply mean those fields which naturally appear
as complex combinations of the nonhermitean step operator generators). Indeed, in
some vacua the number of complex scalar degrees of freedom and complex gauge
degrees of freedom is not even the same. This will be discussed further below.

(iii) In each vacuum, each mass appears at least twice, and always an even number
of times. For the complex fields this is a triviality, but for the real fields this is only
true as a consequence of the feature mentioned in (ii). This pairing of the masses is
a reflection of the $N = 2$ supersymmetry of the relativistic self-dual Chern-Simons
systems [16, 17].
(iv) While the distribution of masses between gauge and scalar modes is different in the different vacua, the total number of degrees of freedom is, in each case, equal to $2(N^2 - 1)$, as in the unbroken phase.

| vacuum $\phi(0)$ | gauge masses | scalar masses |
|------------------|--------------|---------------|
|                  | real fields  | complex fields |
| $o - x - x$      | 2            | 1/2 1/2 1/2 1/2 1 |
| $o - x - o$      | 2 2          | 1 1 1 1 1 2 |
| $o - o - x$      | 2 6          | 1 1 2 2 2 5 |
| $o - o - o$      | 2 6 12       | 1 2 3 5 8 11 |

Table 2: $SU(4)$ vacuum mass spectra, in units of the fundamental mass scale $\frac{v^2}{2\kappa}$, for the inequivalent nontrivial minima $\phi(0)$ of the potential $V$. Notice that for each vacuum the total number of massive degrees of freedom is equal to $2(N^2 - 1) = 30$, although the distribution between gauge and scalar fields is vacuum dependent.

The most complicated, and most interesting, of the nontrivial vacua is the “maximal embedding” case, with $\phi(0)$ given by (138). For this vacuum, the gauge and scalar mass spectra have additional features of note. First, this “maximal embedding” also corresponds to “maximal symmetry breaking”, in the sense that in this vacuum all $N^2 - 1$ gauge degrees of freedom acquire a mass. The original $2(N^2 - 1)$ massive scalar modes divide equally between the scalar and gauge fields. The mass spectrum reveals an intriguing and intricate pattern, as shown in Table 3. It is interesting to note that for the $SU(N)$ maximal symmetry breaking vacuum, the entire scalar mass spectrum is almost degenerate with the gauge mass spectrum: there is just one single complex component for which the masses differ!

3.4 Mass Matrices for Real Fields

The masses of the real fields exhibit further special simple properties, which we discuss in this section. As mentioned above, in each vacuum $\phi(0)$ the number of real scalar modes is equal to the number of real gauge modes. Furthermore, the two mass spectra coincide exactly, and are all integer multiples of the mass scale $m$ in (128).
The real gauge fields come from the diagonal algebraic components $H_a$, while the real scalar fields come from the simple root step operator components $E_a$. Indeed, the real scalar fields correspond to those fields shifted by the symmetry breaking minimum field $\phi_{(0)}$, which is decomposed in terms of the simple root step operators as in (134). This means that the number of real scalars in a given vacuum $\phi_{(0)}$ is given by the number of nonzero coefficients $\phi_{(0)}^a$ in the decomposition (134). This can be seen explicitly for $SU(3)$ and $SU(4)$ in the Tables 1 and 2. This also serves as an easy count of the number of real gauge masses. This also means that to determine the mass matrix for the real gauge fields we can expand $A_\mu$ in terms of the Cartan subalgebra elements $H_a$ (the other, off-diagonal, algebraic components do not mix with these ones at quadratic order). In fact, in order to normalize the gauge fields correctly, it is more convenient to expand the $A_\mu$ in another Cartan subalgebra basis, $h_a$, for which the traces are orthonormal (in contrast to the traces (39) in the Chevalley basis which
Such basis elements, $h_a$, are related to the Chevalley basis elements, $H_a$, by

$$h_a = \sum_{b=1}^{r} \omega_a^{(b)} H_b$$

(156)

where $\omega^{(b)}$ is the $b^{th}$ fundamental weight of the algebra [20], satisfying

$$\sum_{b=1}^{r} \omega_a^{(b)} \alpha_c^{(b)} = \delta_{ac}$$

(157)

where $\alpha^{(b)}$ is the $b^{th}$ simple root. For $SU(N)$ we can be more explicit:

$$h_a = \frac{1}{\sqrt{a(a+1)}} \sum_{b=1}^{a} b H_b$$

(158)

The orthogonality relation (157) means that the correspondence can be inverted to give

$$H_a = \sum_{b=1}^{r} \alpha_b^{(a)} h_b$$

(159)

The fundamental weights $\omega^{(b)}$ and simple roots $\alpha^{(b)}$ are also related by

$$\tilde{\alpha}^{(a)} = \sum_{b=1}^{r} C_{ba} \omega^{(b)}$$

(160)

These new basis elements have the following commutation relations with the simple root step operators:

$$[h_a, E_b] = \alpha_a^{(b)} E_b$$

(161)

Given the traces in (155) and the commutation relations (161), it is now a simple matter to expand the quadratic gauge field term (154) to find the following mass matrix:

$$M_{ab}^{(\text{gauge})} = 2 m \sum_{c=1}^{r} |\phi_c^{(0)}|^2 \alpha_{a}^{(c)} \alpha_{b}^{(c)}$$

(162)

where $m$ is the fundamental mass scale in (128). For the maximal embedding vacuum (138) in $SU(N)$ this leads to a mass matrix

$$M_{ab}^{(\text{gauge})} = m \sum_{c=1}^{N-1} c(N-c) \alpha_{a}^{(c)} \alpha_{b}^{(c)}$$

(163)
This matrix has eigenvalues

$$2, 6, 12, 20, \ldots, N(N - 1)$$  \hspace{1cm} (164)

in multiples of $m$. For any vacuum $\phi_{(0)}$ other than the maximal symmetry breaking one, the mass matrix for the real gauge fields decomposes into smaller matrices of the same form, according to the particular partition of the original $SU(N)$ Dynkin diagram, as described in Section 3.2.

The real scalar field mass matrix can be computed by expanding the $\phi$ field appearing in (152) in terms of the positive root step operators. With such a decomposition for $\phi$, the quadratic term (152) simplifies considerably to give a mass (squared) matrix

$$M_{ab}^{\text{scalar}} = 4 m^2 \phi_{(0)}^a \phi_{(0)}^b \sum_{c=1}^r |\phi_{(0)}^c|^2 C_{ac} C_{bc}$$  \hspace{1cm} (165)

where $C$ is the Cartan matrix (40). For the $SU(N)$ maximal symmetry breaking vacuum (138) this mass matrix is

$$M_{ab}^{\text{scalar}} = m^2 \sqrt{ab} (N - a) (N - b) \sum_{c=1}^{N-1} c (N - c) C_{ac} C_{bc}$$  \hspace{1cm} (166)

which has eigenvalues

$$(2)^2, (6)^2, (12)^2, (20)^2, \ldots, (N(N - 1))^2$$  \hspace{1cm} (167)

in units of $m^2$. It is interesting to note that the eigenvalues in (167) are the squares of the eigenvalues (164) of $M^{\text{gauge}}$, even though $M^{\text{scalar}}$ is not the square of the matrix $M^{\text{gauge}}$ in this basis. Nevertheless, as the real scalar masses are given by the square roots of the eigenvalues in (167), we see that the real scalar masses do indeed coincide with the real gauge masses, a consequence of the $N = 2$ supersymmetry of the theory.

4 Conclusion

In these lectures I have reviewed certain selected aspects of self-dual Chern-Simons theories. The self-dual Chern-Simons theories are 2+1 dimensional models of charged scalar fields interacting with gauge fields whose dynamics is described by a Chern-Simons Lagrangian rather than a Maxwell-Yang-Mills Lagrangian. Both nonrelativistic and relativistic dynamics for the scalar fields may be considered, and in each case there exists a classical notion of self-duality whereby the classical energy functional is minimized by solutions of first-order self-duality equations. In the nonrelativistic case, the self-dual equations are integrable and we have a complete understanding of the static self-dual solutions. In the relativistic case, the abelian self-dual equations
have been shown to fail a Painlevé test for integrability, as does the 2 + 1 dimensional abelian Higgs model \[56\]. Nevertheless, the existence of vortex-like solutions has been established for the self-dual Chern-Simons system \[57\], just as for the abelian Higgs model \[5\]. Even though general exact solutions are not available, many properties of these Chern-Simons solitons may be deduced from asymptotic and/or numerical information \[39, 40\]. The vacuum structure of these self-dual theories exhibits a rich structure, on which perturbative analyses of the quantum theory will be based.

The only known exact classical solutions are gauge transforms of the constant fields which minimize the potential \( (91) \), which saturate the self-dual energy bound. It would be interesting to explore the possibility of finding other, less trivial, solutions - possibly by some restrictive algebraic ansatz and/or by restricting to radially symmetric solutions \[56\]. It is also important to search for time dependent solutions. In the nonrelativistic case one can generate time dependent solutions by transforming static solutions, using the dynamical conformal symmetry of the system \[22\]. However, nothing is known about other truly time dependent finite energy non-self-dual solutions.

At the classical level of the Lagrangian and the equations of motion the self-dual Chern-Simons systems exhibit a rich space-time symmetry structure. The self-dual sixth order potential \( (91) \) in the relativistic self-dual Chern-Simons theory may be fixed by requiring that the Lagrange density \( (90) \) be embedded into a supersymmetric theory with an extended \( N = 2 \) supersymmetry \[10, 17\]. This is consistent with the general relationship between self-duality and extended supersymmetry \[48, 49\]. A similar property holds for the nonrelativistic self-dual Chern-Simons system. There, the fourth order form of the self-dual potential in \( (22) \) may be fixed by requiring that the Lagrange density \( (22) \) be embedded into a theory with an \( N = 2 \) superconformal Galilean symmetry \[58\]. This entire picture may be generalized to include both Chern-Simons and Maxwell dynamics for the gauge field, in which case the gauge field is truly dynamical. Such an extension requires the inclusion of additional scalar fields, in both the relativistic \[59\] and nonrelativistic \[60, 58\] cases. These extra fields may be interpreted as extra superpartners in a model with extended supersymmetry.

The most interesting open questions concern the quantization of the self-dual Chern-Simons theories. For the nonrelativistic self-dual Chern-Simons system, the quantized field theory is a nonrelativistic quantum field theory whose multi-particle sector corresponds to the multi-particle quantum mechanics of anyons, and which provides a field theoretic description of Aharonov-Bohm scattering \[61\]. The relativistic self-dual Chern-Simons system is a quantum field theory of anyons. One can then ask: what is the quantum significance of the classical self-duality symmetry which minimizes the classical energy functional? In the nonrelativistic system perturbative analyses of Aharonov-Bohm scattering indicate that the quartic potential, which corresponds quantum mechanically to a \( \delta \)-function hard-core inter-particle potential, is necessary for renormalization \[62\]; and, moreover, the classical conformal invariance is preserved at the self-dual point \[63\]. It has also been shown that the one-loop con-
tribution to the effective potential vanishes with the self-dual quartic self-interaction \[64\]. Considerably less is known about the quantization of the relativistic self-dual Chern-Simons systems. One would like to understand better the quantum significance of the classical self-dual solitons in, for example, a collective coordinate formulation. Further issues, such as renormalization \[65\], vacuum tunnelling, and the perturbative fate of the self-dual potential remain to be resolved.

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