A DIRECT IMAGING METHOD FOR INVERSE SCATTERING BY UNBOUNDED ROUGH SURFACES

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Abstract. This paper is concerned with the inverse scattering problem by an unbounded rough surface. A direct imaging method is proposed to reconstruct the rough surface from the scattered near-field Cauchy data generating by point sources and measured on a horizontal straight line segment at a finite distance above the rough surface. Theoretical analysis of the imaging algorithm is given for the case of a penetrable rough surface, but the imaging algorithm also works for impenetrable surfaces with Dirichlet or impedance boundary conditions. Numerical experiments are presented to show that the direct imaging algorithm is fast, accurate and very robust with respect to noise in the data.

Key words. Inverse scattering, unbounded rough surfaces, Cauchy data, Dirichlet boundary conditions, impedance boundary conditions, transmission conditions

AMS subject classifications. 78A46, 35P25

1. Introduction. The ability to effectively find the geometrical information of unknown rough surfaces from the knowledge of the scattered wave field is of great importance in various applications such as radar and sonar detection, remote sensing, geophysics and nondestructive testing.

The aim of this paper is to study the inverse scattering problem by an unbounded rough surface, and in particular, the imaging of the rough surface from the scattered near-field Cauchy data. The wave propagation is governed by the Helmholtz equation. See Fig. 1.1 for the problem geometry. In this paper, we are restricted to the two-dimensional case for simplicity. However, our imaging method and its analysis can be generalized to the three-dimensional case with appropriate modifications.

![Fig. 1.1. The scattering problems by an impenetrable rough surface (left) or a penetrable rough surface (right).](image-url)
Many optimization or iteration methods have been developed for inverse scattering problems [1,2,3,29]. This kind of methods can recover the unknown surface very accurately, but they are time-consuming and also need to know the physical property of the unknown surface in advance. Recently, many methods have been proposed to avoid the huge computation. For example, the reverse time migration method, the direct sampling method and the orthogonality sampling method are proposed to reconstruct bounded obstacles in [10,21,28], respectively. For the case of unbounded rough surfaces, a fast super-resolution method was proposed in [6,7], based on the transformed field expansion technique, under the assumption that the rough surface is a small and smooth deformation of a plane surface. Further, many related inverse scattering problems by unbounded surfaces have been extensively studied for the case when the unbounded surface is a local perturbation of a plane surface [5,20,32], for the case when the unbounded surface is periodic [2,8] and for the time-dependent case [9,12].

In this paper, we propose a direct imaging method to reconstruct the unbounded rough surface from the scattered near-field Cauchy data generating by point sources and measured on a horizontal straight line segment at a finite distance above the rough surface. A main feature of our imaging method is its capability of depicting the profile of the surface only through computing the inner products of the measured data and the fundamental solution in the homogeneous background medium at each sampling point, leading to very cheap computation cost. Further, our method does not require a prior knowledge of the physical property of the surface, that is, the type of boundary conditions on the rough surface does not need to know in advance. Thus, our imaging method works for both penetrable and impenetrable rough surfaces. Furthermore, numerical experiments show that our imaging method can give an accurate and reliable reconstruction of the unbounded rough surface, even for the case with a fairly large amount of noise in the measured data.

To understand why the direct imaging method works, a theoretical analysis of the imaging method is presented. To do this, we introduce Green’s function \( G(x, y) \) for the impedance half-plane (see [14]) which plays an important role when analyzing the asymptotic behavior of the scattered field. Further, an integral identity concerning the fundamental solution of the Helmholtz equation is established for the unbounded rough surface, which is similar to the Helmholtz-Kirchhoff identity for bounded obstacles [10]. In addition, a reciprocity relation is proved for the total field of the forward scattering problem. Based on these results, the main results of the paper (Theorems 3.7-3.10) are established which lead to the required imaging function of the direct imaging method.

The remaining part of the paper is organized as follows. Section 2 gives a brief description of the scattering problems and introduces some notations and inequalities that will be used in this paper. Moreover, the well-posedness of the forward scattering problems, based on the integral equation method, will be also presented without proof in this section. In Section 3 we first conduct a theoretical analysis of the continuous imaging function and then propose the direct imaging method for the inverse problem. Finally, numerical examples are carried out in Section 4 to illustrate the effectiveness of the imaging method.

We conclude this section by introducing some notations used throughout the paper. For \( h \in \mathbb{R} \), define \( U_h := \{ x \in \mathbb{R}^2 \mid x_2 > h \} \). For \( V \subset \mathbb{R}^n \) (\( n = 1, 2 \)), denote by \( BC(V) \) the set of bounded and continuous functions in \( V \), a Banach space under the norm \( \| \psi \|_{\infty, V} := \sup_{x \in V} |\psi(x)| \). We write \( \| \cdot \|_{\infty, \mathbb{R}^n} \) for \( \| \cdot \|_{\infty, \mathbb{R}^n} \). For \( 0 < \alpha \leq 1 \), denote
by $BC^{0,\alpha}(V)$ the Banach space of functions $\phi \in BC(V)$ which are uniformly Hölder continuous with exponent $\alpha$, equipped with the norm $\|\cdot\|_{0,\alpha,V}$ defined by $\|\phi\|_{0,\alpha,V} := \|\phi\|_{\infty} + \sup_{x \neq y \in V} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}$. Given an open set $V \subset \mathbb{R}^2$ and $\nu \in L^\infty(V)$, denote by $\partial_\nu$ the (distributional) derivative $\partial_\nu v(x)/\partial x_j$, $j = 1, 2$. Define $BC^{1,1}(V) := \{\varphi \in BC(V) \mid \partial_j \varphi \in BC(V), j = 1, 2\}$ with the norm $\|\varphi\|_{1,1,V} := \|\varphi\|_{\infty,V} + \|\partial_1 \varphi\|_{\infty,V} + \|\partial_2 \varphi\|_{\infty,V}$. Finally, we introduce some spaces of smooth functions on the boundary $\Gamma$. Let $BC^{1,\alpha}(\Gamma) := \{\varphi \in BC(\Gamma) \mid \text{Grad } \varphi \in C^{0,\alpha}(\Gamma)\}$ with the norm $\|\varphi\|_{1,\alpha,\Gamma} := \|\varphi\|_{\infty,\Gamma} + \|\text{Grad } \varphi\|_{\infty,\Gamma}$, where Grad denotes the surface gradient. For simplicity, we assume that $f \in B(c_1, c_2) := \{f \in BC^{1,1}(\mathbb{R}) \mid f(s) \geq c_1, s \in \mathbb{R} \text{ and } \|f\|_{1,1,\mathbb{R}} \leq c_2\}$ for some $c_1, c_2 > 0$.

2. Problem Formulation. In this section, we introduce the mathematical model of interest and propose some existed results on the well-posedness for the forward scattering problems. Some useful notations and inequalities used in the paper will also be presented.

2.1. The forward scattering problems. Given $f \in BC^{1,1}(\mathbb{R})$ with $f_- := \inf_{x_1 \in \mathbb{R}} f(x_1) > 0$, define the two-dimensional regions $D_+$ and $D_-$ by

$$
D_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > f(x_1)\},
$$
$$
D_- := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 < f(x_1)\},
$$

so that the unbounded rough surface $\Gamma$ is given by

$$
\Gamma := \{x = (x_1, f(x_1)) \mid x_1 \in \mathbb{R}\}.
$$

Let $u_t(x,y) := \Phi_k(x,y)$ be an incident point source, where $\Phi_k(x,y)$ is the fundamental solution to the Helmholtz equation in two dimensions given by

$$
\Phi_k(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y.
$$

(2.1)

Here, $H_0^{(1)}$ is the Hankel function of the first kind of order zero and $k$ is the wavenumber. The forward scattering problem is to determine the unknown scattered wave $u^s$ in $D_+$ and the unknown transmitted wave $u^t$ in $D_-$ such that the total field

$$
u := \begin{cases} u^t + u^s & \text{in } D_+, \\ u^t & \text{in } D_- \end{cases}$$

satisfies the Helmholtz equation

$$
\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma
$$

with $k = k_+ > 0$ in $D_+$ and $k = k_- > 0$ in $D_-$, respectively, where $k_+ \neq k_-$. We restrict our attention to the following three cases:

1. the case when the total field vanishes on the boundary, so that the scattered field $u^s$, the solution of the Helmholtz equation in $D_+$, satisfies the Dirichlet boundary condition $u^s = -u^t$ on $\Gamma$.

2. the case when the total field satisfies the homogeneous impedance boundary condition $\partial_\nu u - ik_\nu u = 0$ on $\Gamma$, where $\nu(x)$ stands for the unit normal vector at $x \in \Gamma$ pointing out of $D_+$ and $\partial_\nu u$ is the normal derivative of $u$;
the case when the rough surface is penetrable and satisfies the *transmission boundary condition* \( u^s + u^t = u^p, \partial_n u^s + \partial_n u^t = \partial_n u^p \) on \( \Gamma \).

In order for the problem to have a unique solution, we adopt the so-called *upward propagating radiation condition (UPRC)* and *downward propagating radiation condition (DPRC)* [16,18,19]. The scattered field \( u^s \) is required to satisfy UPRC: for some \( h_1 > f_+ := \sup_{x_1 \in \mathbb{R}} f(x_1) \) and \( \phi_1 \in L^\infty(\Gamma_{h_1}) \),

\[
    u^s(x) = 2 \int_{\Gamma_{h_1}} \frac{\partial \Phi_{k_+}(x,y)}{\partial y_2}\phi_1(y)ds(y), \quad x \in U_{h_1}
\]  

and the transmitted field \( u^t \) is required to satisfy DPRC: for some \( h_2 < f_- \) and \( \phi_2 \in L^\infty(\Gamma_{h_2}) \),

\[
    u^t(x) = -2 \int_{\Gamma_{h_2}} \frac{\partial \Phi_{k_-}(x,y)}{\partial y_2}\phi_2(y)ds(y), \quad x \in \mathbb{R}^2 \setminus U_{h_2},
\]

respectively. Here, \( \Phi_{k_\pm} \) is defined as \( \Phi \) with \( k \) replaced by \( k_\pm \).

The above scattering problems can now be formulated as the following boundary value problems for the scattered field \( u^s \) and the transmitted field \( u^t \).

**Dirichlet scattering problem (DSP):** Given \( g \in BC(\Gamma) \), determine \( u^s \in C^2(D_+) \cap C(\overline{D}_+) \) such that

(i) \( u^s \) is a solution of the Helmholtz equation

\[
    \Delta u^s + k_+^2 u^s = 0 \quad \text{in} \quad D_+,
\]  

(ii) \( u^s = g \) on \( \Gamma \),

(iii) For some \( \beta \in \mathbb{R}, \)

\[
    \sup_{x \in D_+} x_2^\beta |u^s(x)| < \infty,
\]

(iv) \( u^s \) satisfies the UPRC [22].

Let \( \mathcal{R}(D_+) \) denote the set of functions \( w \in C^2(D_+) \cap C(\overline{D}_+) \) whose normal derivative defined by \( \partial_n w(x) := \lim_{h \to 0^+} v(x) \cdot \nabla w(x + hv(x)) \) exists uniformly for \( x \) on any compact subset of \( \Gamma \).

**Impedance scattering problem (ISP):** Given \( g \in BC(\Gamma), \rho \in BC(\Gamma) \), determine \( u^s \in \mathcal{R}(D_+) \) such that

(i) \( u^s \) is a solution of the Helmholtz equation [24] in \( D_+ \),

(ii) \( \partial_n u^s - ik_+ \rho u^s = g \) on \( \Gamma \),

(iii) \( u^s \) satisfies [25] for some \( \beta \in \mathbb{R}, \)

(iv) \( u^s \) satisfies the UPRC [22],

(v) for some \( \theta \in (0,1) \) and some constant \( C_\theta > 0, \)

\[
    |\nabla u(x)| \leq C_\theta |x_2 - f(x_1)|^{\theta - 1}
\]

for \( x \in D_+ \setminus U^*_h \) with \( b = f_+ + 1 \).

The scattering problem by a penetrable rough surface can be formulated as follows.

**Transmission scattering problem (TSP):** Let \( \alpha \in (0,1), h_1 > f_+ \) and \( h_2 < f_- \). Given \( g_1 \in BC^{1,\alpha}(\Gamma) \) and \( g_2 \in BC^{0,\alpha}(\Gamma), \) determine a pair of functions \( (u^s, u^t) \) with \( u^s \in C^2(D_+) \cap BC^1(\overline{D}_+ \setminus U_{h_1}) \) and \( u^t \in C^2(D_-) \cap BC^1(\overline{D}_- \setminus U_{h_2}) \) such that
(i) \( u^s \) is a solution of the Helmholtz equation (2.4) in \( D_+ \) and \( u^t \) is a solution of the Helmholtz equation \( \Delta u^t + k^2 u^t = 0 \) in \( D_- \),
(ii) \( u^s - u^t = g_1, \partial_n u^s - \partial_n u^t = g_2 \) on \( \Gamma \),
(iii) \( u^s \) satisfies (2.7) and \( u^t \) satisfies
\[
\sup_{x \in D_-} x_2^\beta |u^t(x)| < \infty
\]
for some \( \beta \in \mathbb{R} \),
(iv) \( u^s \) satisfies the UPRC (2.2) and \( u^t \) satisfies the DPRC (2.3).

2.2. Some useful notations. In this subsection we introduce some basic notations and fundamental functions that will be needed in the subsequent discussions. First, note that \( H_0^{(1)} = -H_1^{(1)} \) and that by [3, equation(9.2.7)],
\[
H_n^{(1,2)}(t) = \sqrt{\frac{2}{\pi t}} e^{\pm i(t-\frac{\pi}{2})} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty, \quad n = 0, 1, \ldots.
\] (2.6)
This, combined with equation (2.1), implies that
\[
|\Phi_k(x, y)| \leq C |x-y|^{-1/2},
\]
\[
\left| \frac{\partial \Phi_k(x, y)}{\partial x_i} \right| \leq C |x_i - y_i| |x-y|^{-3/2}, \quad i = 1, 2, \tag{2.7}
\]
\[
|\Phi_k(x, y) - \Phi_k(x, y')| \leq C (1 + |x_2|)(1 + |y_2|)(|x-y|^{-3/2} + |x-y'|^{-3/2}) \tag{2.8}
\]
for \( |x - y| \geq \delta > 0 \), \( x = (x_1, x_2), \ y = (y_1, y_2), \ y' = (y_1, -y_2) \) with \( C > 0 \) depending only on \( k, \delta \).

In this paper, we also need the following impedance Green’s function for the Helmholtz equation \( (\Delta + k^2)u = 0 \) in the half-planes \( U_0^\pm := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq a\} \). For any \( k > 0 \) define
\[
G_k^\pm(x, y; a) := \Phi_k(x, y) + \Phi_k(x, y'_a) + P_k^\pm(x - y'_a), \quad x, y \in U_0^\pm,
\]
where
\[
P_k^\pm(z) := \frac{|z| e^{ik|z|}}{\pi} \int_0^\infty t^{-1/2} e^{-k|z|t} \left| |z| \pm z_2 (1 + it) \right| dt, \quad z \in \overline{U_0^\pm}
\]
with the square root being taken so that \( -\pi/2 < \arg(\sqrt{-2i}) < 0 \) and \( y'_a = (y_1, 2a - y_2) \). From [17], it is known that \( P_k^\pm \in C(\overline{U_0^\pm}) \cap C^\infty(\overline{U_0^\pm} \setminus \{0\}) \) and \( G_k^\pm(x, y; a) \) is a radiating solution in \( U_0^\pm \) and satisfies the impedance boundary condition \( \partial_n P_k^\pm \pm ikP_k^\pm = 0 \) on \( \Gamma_a := \{x = (x_1, x_2) \mid x_1 \in \mathbb{R}, \ x_2 = a\} \). Further, it is shown in [27, equation(2.14)] that
\[
|G_k^\pm(x, y; a)| \leq C (1 + |x_1 - y_1|)^{-3/2}
\]
\[
|\nabla_x G_k^\pm(x, y; a)| \leq C (1 + |x_1 - y_1|)^{-3/2}
\]
\[
|\nabla_y G_k^\pm(x, y; a)| \leq C (1 + |x_1 - y_1|)^{-3/2}
\]
\[
|\nabla_x \partial_n G_k^\pm(x, y; a)| \leq C (1 + |x_1 - y_1|)^{-3/2}
\]
for \( x \in \Gamma_H, \ y \in \Gamma, \ |x_2 - y_2| \geq \delta > 0 \), with \( C > 0 \) depending only on \( a, k, \delta, \Gamma \) and \( H \).
We end this subsection by introducing certain layer potentials and boundary integral operators. For \( a < f_\cdot \) define the single- and double-layer potentials: for \( x \in U_a^+ \setminus \Gamma \)

\[
(S_{k_+,a}^+ \phi)(x) := \int_\Gamma G_{k_+}^+(x,y;a)\phi(y)ds(y),
\]

\[
(D_{k_+,a}^+ \phi)(x) := \int_\Gamma \frac{\partial}{\partial \nu(y)} G_{k_+}^+(x,y;a)\phi(y)ds(y)
\]

and the boundary integral operators: for \( x \in \Gamma \)

\[
(S_{k_+,a}^- \phi)(x) := \int_\Gamma G_{k_+}^-(x,y;a)\phi(y)ds(y),
\]

\[
(K_{k_+,a}^- \phi)(x) := \int_\Gamma \frac{\partial}{\partial \nu(y)} G_{k_+}^-(x,y;a)\phi(y)ds(y),
\]

\[
(K_{k_+,a}'^- \phi)(x) := \int_\Gamma \frac{\partial}{\partial \nu(x)} G_{k_+}^-(x,y;a)\phi(y)ds(y),
\]

\[
(T_{k_+,a}^- \phi)(x) := \frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial}{\partial \nu(y)} G_{k_+}^-(x,y;a)\phi(y)ds(y).
\]

Further, for \( a > f_\cdot \) the layer-potential operators \( S_{k_-,a}^-, D_{k_-,a}^- \) for \( x \in U_a^- \setminus \Gamma \) and the boundary integral operators \( S_{k_-,a}^-, K_{k_-,a}^-, K_{k_-,a}'^-, T_{k_-,a}^- \) for \( x \in \Gamma \) can be defined similarly.

### 2.3. Well-posedness of the forward scattering problems.

The well-posedness of the forward scattering problems described in Section 2 has been studied by using the variational and integral equation methods (see, e.g. [13,18,19,27,31]). In this subsection, we present these well-posedness results based on the integral equation method, which will be needed in the remaining part of this paper.

**Theorem 2.1.** (see [18]) Assume that \( f \in B(c_1,c_2) \). Then the problem (DSP) has exactly one solution in the form

\[
u^s(x) = (D_{k_+,0}^- \phi)(x), \quad x \in D_+.
\]

(2.14)

Here, the density function \( \phi \in BC(\Gamma) \) is the unique solution to the boundary integral equation

\[
A_D \phi := \left( -\frac{1}{2}I + K_{k_+,0}^+ \right) \phi = g,
\]

(2.15)

where the integral operator \( A_D \) is bijective (and so boundedly invertible) in \( BC(\Gamma) \). Further, for each \( g \in BC(\Gamma) \) we have the estimate

\[
|u^s(x)| \leq C x_2^{1/2} \|g\|_{\infty,\Gamma}
\]

(2.16)

for some constant \( C > 0 \) independent of \( g \).

**Theorem 2.2.** (see [31]) Assume that \( f \in B(c_1,c_2) \) and \( \rho \in B(c_1,c_2) \). Then the problem (ISP) has exactly one solution in the form

\[
u^s(x) = (S_{k_+,0}^+ \phi)(x), \quad x \in D_+.
\]

(2.17)
Here, the density function $\varphi \in BC(\Gamma)$ is the unique solution to the boundary integral equation

$$A_I \varphi := \left( \frac{1}{2} I + K^+_{k,0} - ik_\rho S^+_{k,0} \right) \varphi = g,$$  \hspace{1cm} (2.18)

where the integral operator $A_I$ is bijective (and so boundedly invertible) in $BC(\Gamma)$. Moreover, for each $g \in BC(\Gamma)$ we have the estimate

$$|u^*(x)| \leq Cx_2^{1/2} \|g\|_{\infty, \Gamma}$$  \hspace{1cm} (2.19)

for some constant $C > 0$ independent of $g$.

**Theorem 2.3.** (see [1, 2, 3, 7]) Given $g_1 \in BC^{1,\alpha}(\Gamma)$, $g_2 \in BC^{0,\alpha}(\Gamma)$ and $f \in B(c_1, c_2)$, and for $h \in \mathbb{R}$ with $-h < f_- < f_+ < h$, the problem (TSP) has exactly one solution $(u^*, u^t)$ in the form

$$u^*(x) = (D^+_{k_+,-h}(\varphi_1)(x) + (S^+_{k_+,-h}(\varphi_2))(x), \quad x \in D_+,$$  \hspace{1cm} (2.20)

$$u^t(x) = (D^-_{k_-,h}(\varphi_1)(x) + (S^-_{k_-,h}(\varphi_2))(x), \quad x \in D_-.$$  \hspace{1cm} (2.21)

Here, $\varphi := (\varphi_1, \varphi_2)^T \in BC^{1,\alpha}(\Gamma) \times BC^{0,\alpha}(\Gamma)$ is the unique solution to the boundary integral equation

$$A_T \varphi = G$$  \hspace{1cm} (2.22)

with

$$A_T = \begin{pmatrix} K^+_{k_+,-h} - K^-_{k_-,h} + I & S^+_{k_+,-h} - S^-_{k_-,h} \\ T^+_{k_+,-h} - T^-_{k_-,h} & K^+_{k_+,-h} - K^-_{k_-,h} - I \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$  \hspace{1cm} (2.23)

where the integral operator $A_T$ is bijective (and so boundedly invertible) in $[BC(\Gamma)]^2$. Moreover, $u^*, u^t$ depend continuously on $\|g_1\|_{1,\alpha, \Gamma}$ and $\|g_2\|_{1,\alpha, \Gamma}$, and $\nabla u^*, \nabla u^t$ depend continuously on $\|g_1\|_{1,\alpha, \Gamma}$ and $\|g_2\|_{0,\alpha, \Gamma}$.

**Remark 2.4.** It has been shown in [1] that once the unique solvability of the boundary integral equations (2.15), (2.18) or (2.22) has been established in the space of bounded and continuous functions, the unique solvability in $L^p$ $(1 \leq p \leq \infty)$ can be obtained. This general result has been applied in [1] to the integral equation formulation proposed in [1, 18, 19, 27, 31].

### 3. Direct imaging method for the inverse problem.

This section presents a direct imaging method to solve the inverse problem. To do this, we first establish certain results for the forward scattering problems associated with incident point sources. In the following proofs, the constant $C > 0$ may be different at different places.

**Lemma 3.1.** Assume that $(u^*, u^t)$ is the solution to the problem (TSP) with the boundary data $g = (g_1, g_2)^T$. If $g \in [L^p(\Gamma)]^2$ with $1 \leq p \leq \infty$, then $u^*, \partial_\nu u^t \in L^p(\Gamma_H)$ for any $H > f_+$ and $u^t, \partial_\nu u^t \in L^p(\Gamma_H)$ for any $h < f_-$.  

**Proof.** From Theorem [2.3] and Remark [2.4] the scattered field $u^*$ can be written in the form

$$u^*(x) = \int_\Gamma \frac{\partial G^+_{k_+}(x, \xi)}{\partial \nu(\xi)} \varphi_1(\xi) d\nu(\xi) + \int_\Gamma G^+_{k_+}(x, \xi) \varphi_2(\xi) d\nu(\xi), \quad x \in D_+,$$  \hspace{1cm} (3.1)
where \((\varphi_1, \varphi_2)^T \in [L^p(\Gamma)]^2\) is the unique solution to the boundary integral equation (2.22) with \(g \in [L^p(\Gamma)]^2\). Define \(\phi := (\phi_1, \phi_2)^T \in [L^p(\mathbb{R})]^2\) by
\[
\phi_1(s) := \varphi_1((s, f(s))), \quad \phi_2(s) := \varphi_2((s, f(s))), \quad s \in \mathbb{R}.
\]
It is obvious that \(\|\phi\|_{[L^p(\mathbb{R})]^2} \leq \|\varphi\|_{[L^p(\Gamma)]^2} < \infty\). Using (2.10) and (2.12), we can deduce that for \(x \in \Gamma_H\),
\[
|u^s(x)| \leq \int_{\Gamma} \left| \frac{\partial G^+_{k,x}(x, \xi)}{\partial \nu(\xi)} \right| |\varphi_1(\xi)| ds(\xi) + \int_{\Gamma} \left| G^+_{k,x}(x, \xi) \right| |\varphi_2(\xi)| ds(\xi)
\leq C \int_{\Gamma} (1 + |x_1 - \xi_1|)^{-3/2} (|\varphi_1(\xi)| + |\varphi_2(\xi)|) ds(\xi)
= C \int_{-\infty}^{\infty} (1 + |x_1 - s|)^{-3/2} (|\varphi_1(s)| + |\varphi_2(s)|) \sqrt{1 + f'(s)^2} ds
\]
with the constant \(C > 0\) depending only on \(k, \delta\). Since \((1 + |\cdot|)^{-3/2} \in L^1(\mathbb{R})\), \(\phi_1 + \phi_2 \in L^p(\mathbb{R})\) and \(f \in B(c_1, c_2)\), we have by Young’s inequality that
\[
\|u^s\|_{L^p(\Gamma_H)} = \|u^s(\cdot, H)\|_{L^p(\mathbb{R})} \leq \|1 + |\cdot|\|_{L^1(\mathbb{R})} \|\phi_1 + \phi_2\|_{L^p(\mathbb{R})} < \infty,
\]
that is, \(u^s \in L^p(\Gamma_H)\).

Furthermore, by (3.1) we have that for \(x \in \Gamma_H\),
\[
\partial_\nu u^s(x) = \frac{\partial}{\partial \nu(x)} \left\{ \int_{\Gamma} \frac{\partial G^+_{k,x}(x, \xi)}{\partial \nu(\xi)} \varphi_1(\xi) ds(\xi) + \int_{\Gamma} G^+_{k,x}(x, \xi) \varphi_2(\xi) ds(\xi) \right\}
\leq \int_{\Gamma} \frac{\partial^2 G^+_{k,x}(x, \xi)}{\partial (x, \nu)(x, \nu)} \varphi_1(\xi) ds(\xi) + \int_{\Gamma} \frac{\partial G^+_{k,x}(x, \xi)}{\partial (x, \nu)(x, \nu)} \varphi_2(\xi) ds(\xi).
\]
Using this representation and the inequalities (2.11) and (2.13) and arguing similarly as above, we obtain that \(\partial_\nu u^s \in L^p(\Gamma_H)\).

The results for the transmitted field \(u^t\) can be shown similarly. The lemma is thus proved.

**Corollary 3.2.** For \(y \in D_+\) let \(u^s(\cdot, y)\) and \(u^t(\cdot, y)\) be the scattered and transmitted fields, respectively, by the penetrable rough surface \(\Gamma\) and generated by the incident point source \(u^i(x, y) = \Phi_{k_+}(x, y)\) located at \(y\), then for any \(\varepsilon_1 > 0\) and \(\varepsilon_2 > 1/3\), \(u^t(\cdot, y) \in L^{2+\varepsilon_2}(\Gamma_H)\) and \(\partial_\nu u^s(\cdot, y) \in L^{2/3+\varepsilon_2}(\Gamma_H)\) for any \(H > f_+\) and \(u^t(\cdot, y) \in L^{2+\varepsilon_1}(\Gamma_h)\) and \(\partial_\nu u^s(\cdot, y) \in L^{2/3+\varepsilon_2}(\Gamma_h)\) for any \(h < f_-\).

**Proof.** From (2.27) and (2.28) it follows that for any \(\varepsilon_1 > 0\),
\[
(-\Phi_{k_+}(\cdot, y), -\partial_\nu \Phi_{k_+}(\cdot, y)) \in [L^{2+\varepsilon_1}(\Gamma)]^2,
\]
so, by Lemma 3.1 we have
\[
u^s(\cdot, y) \in L^{2+\varepsilon_1}(\Gamma_H).
\]

Now define
\[
\tilde{u}^i(x, y) := \Phi_{k_+}(x, y) - \Phi_{k_+}(x, y'), \quad \tilde{u}^s(x, y) := u^s(x, y) + \Phi_{k_+}(x, y').
\]
(3.2)
It is easy to see that \((\tilde{u}^s, u^t)\) is the solution to the problem (TSP) corresponding to the incident wave \(\tilde{u}^i\). From (2.29) and (2.30) it follows that for any \(\varepsilon_2 > 0\),
\[
(-\Phi_{k_+}(\cdot, y) + \Phi_{k_+}(\cdot, y'), -\partial_\nu \Phi_{k_+}(\cdot, y) + \partial_\nu \Phi_{k_+}(\cdot, y')) \in [L^{2/3+\varepsilon_2}(\Gamma)]^2.
\]
Again, by Lemma 3.1 it is known that for any \( \varepsilon_2 > 1/3 \),

\[
\tilde{u}^s(\cdot, y) \in L^{2/3 + \varepsilon_2}(\Gamma_H), \quad \partial_x \tilde{u}^s(\cdot, y) \in L^{2/3 + \varepsilon_2}(\Gamma_H).
\]

By this and (3.2), we have

\[
\partial_x u^s(\cdot, y) = \partial_x \tilde{u}^s(\cdot, y) - \partial_x \Phi_k(\cdot, y') \in L^{2/3 + \varepsilon_2}(\Gamma_H).
\]

The results for the transmitted field \( u^t \) can be shown similarly. This completes the proof. \( \square \)

**Remark 3.3.** With the help of Theorems 2.1 and 2.2 and Remark 2.4, Corollary 3.2 can also be extended to the cases of an impenetrable rough surface.

The following lemma is similar to the Helmholtz-Kirchhoff identity which plays an important role in the case of inverse scattering by bounded obstacles [10, Lemma 3.1].

**Lemma 3.4.** For any \( H \in \mathbb{R} \) we have

\[
\int_{\Gamma_H} \left( \frac{\partial \Phi_k(x, y)}{\partial \nu(x)} \Phi_k(x, z) - \Phi_k(x, y) \frac{\partial \Phi_k(x, z)}{\partial \nu(x)} \right) ds(x) = \frac{i}{4\pi} \int_{S^1} e^{ik\hat{x} \cdot (z - y)} ds(\hat{x}), \quad y, z \in \mathbb{R}^2 \setminus U_H,
\]

where \( \nu \) denotes the unit normal vector on \( \Gamma_H \) pointing into \( U_H \).

**Proof.** Let \( \partial B_R^+ \) be the half circle above \( \Gamma_H \) centered at \((0, H)\) and with radius \( R > 0 \) and define \( \Gamma_{H, R} := \{ x \in \Gamma_H \mid |x_1| \leq R \} \). Denote by \( \Omega \) the bounded region enclosed by \( \Gamma_{H, R} \) and \( \partial B_R^+ \). Using Green’s theorem in \( \Omega \), we obtain that

\[
0 = \int_{\Omega} \left( \Delta \Phi_k(x, y) + k^2 \Phi_k(x, y) \right) \Phi_k(x, z) dx = \int_{\Gamma_{H, R}} \left( - \frac{\partial \Phi_k(x, y)}{\partial \nu(x)} \Phi_k(x, z) + \Phi_k(x, y) \frac{\partial \Phi_k(x, z)}{\partial \nu(x)} \right) ds(x) + \int_{\partial B_R^+} \left( \frac{\partial \Phi_k(x, y)}{\partial \nu(x)} \Phi_k(x, z) - \Phi_k(x, y) \frac{\partial \Phi_k(x, z)}{\partial \nu(x)} \right) ds(x). \tag{3.3}
\]

From (2.1) and (2.6) we know that

\[
\Phi_k(x, y) = \frac{e^{ik|x|} e^{i\pi/4}}{\sqrt{|x|} \sqrt{8\pi k}} e^{-ik\hat{x} \cdot y} + O \left( \frac{1}{|x|} \right), \quad |x| \to \infty.
\]

By this and the Sommerfeld radiation condition

\[
\frac{\partial \Phi_k(x, y)}{\partial \nu(x)} - ik \Phi_k(x, y) = O(|x|^{-3/2}), \quad |x| \to \infty,
\]
we obtain on letting $R \to \infty$ in (3.3) that
\[
\int_{\Gamma_R} \left( \frac{\partial \Phi_k(x,y)}{\partial \nu(x)} \Phi_k(x,z) - \Phi_k(x,y) \frac{\partial \Phi_k(x,z)}{\partial \nu(x)} \right) ds(x)
\]
\[
= \lim_{R \to \infty} \int_{\partial B_R^+} \left( \frac{\partial \Phi_k(x,y)}{\partial \nu(x)} \Phi_k(x,z) - \Phi_k(x,y) \frac{\partial \Phi_k(x,z)}{\partial \nu(x)} \right) ds(x)
\]
\[
= \lim_{R \to \infty} 2ik \int_{\partial B_R^+} \Phi_k(x,y) \Phi_k(x,z) ds(x)
\]
\[
= \lim_{R \to \infty} \frac{i}{4\pi} \int_{\partial B_R^+} \frac{1}{|x|} e^{ik\hat{x} \cdot (z-y)} ds(x)
\]
\[
= \frac{i}{4\pi} \int_{\partial B_R^+} e^{ik\hat{x} \cdot (z-y)} ds(\hat{x}).
\]
This completes the proof. \( \square \)

We also need the reciprocity relation $u(x,y) = u(y,x)$ for an unbounded rough surface. The reciprocity relation can be found in [11, Chapter 3.3] for the case of bounded obstacles. For a locally rough surface, since the scattered field $u^s$ satisfies the Sommerfeld radiation condition [5, 30], the reciprocity relation can be proved similarly as for the case of bounded obstacles. For the case of a global unbounded rough surface, the reciprocity relation has been established in [25, Theorem 3.14] by using the assumption that the scattered field generated by a point source satisfies the Sommerfeld radiation condition. However, there is no rigorous proof for this assumption in [15, 25]. Here, we give a proof of the reciprocity relation for the case of a globally rough surface without this assumption.

**Lemma 3.5.** (Reciprocity relation) Let $u$ denote the total field in $\mathbb{R}^2 \setminus \Gamma$ generated by a penetrable rough surface $\Gamma$ corresponding to the incident point source $u^i = \Phi_k(x,y)$, that is,
\[
u = \begin{cases} u^i + u^s & \text{in } D_+, \\ u^i & \text{in } D_- \end{cases}
\]
Then $u(x,y) = u(y,x)$, $x, y \in \mathbb{R}^2 \setminus \Gamma$.

**Proof.** For $z \in \mathbb{R}^2$, let $z = (z_1, z_2)$. For $b, L, \varepsilon > 0$ define $D_{b,L,\varepsilon} := \{ z \in \mathbb{R}^2 \mid |z_2| < b, |z_1| < L, |z - x| > \varepsilon \}$. Choose $\varepsilon$ sufficiently small and $b, L$ large enough such that $\overline{B_\varepsilon(x)} \subset D_{b,L,\varepsilon}, \overline{B_\varepsilon(y)} \subset D_{b,L,\varepsilon}$ and $\overline{B_\varepsilon(x)} \cap \overline{B_\varepsilon(y)} = \emptyset$. Then, since, by Theorem 2.3, $u^i(y,y), u^i(x,x) \in C^2(D_{b,L,\varepsilon}) \cap C(\overline{D_{b,L,\varepsilon}})$, we can apply Green’s second theorem in $D_{b,L,\varepsilon}$ to give that
\[
0 = \int_{\partial D_{b,L,\varepsilon}} \left( \frac{\partial u(z,y)}{\partial \nu(z)} u(z,x) - \frac{\partial u(z,x)}{\partial \nu(z)} u(z,y) \right) ds(z). \tag{3.4}
\]
For any $v(z) \in L^2(\Gamma_0)$ with $b \in \mathbb{R}$, let $\mathcal{F}v$ denote the Fourier transform of $v$ with respect to $z_1$, that is,
\[
(\mathcal{F}v)(\xi, z_2) \bigg|_{z_2 = b} = \int_{-\infty}^{+\infty} e^{-iz_1\xi} v(z_2) dz_2.
\]
Then, by [18] (2.7) UPRC can be rewritten in the angular spectrum representation
\[
u(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(z_2 - b)\sqrt{k_x^2 - \xi^2}} e^{iz_1\xi} \hat{u}_h(\xi) d\xi, \quad z_2 > h, \quad \xi \in \mathbb{R}, \quad h > 0.
\]
where \( \hat{u}_h(\xi) := (\mathcal{F} u)(\xi, h) \). It follows from the representation (5.5) that

\[
(\mathcal{F} u)(\xi, z_2) \big|_{z_2 = b} = e^{i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi), \quad b > h, \tag{3.6}
\]

\[
(\mathcal{F} \partial_v u)(\xi, z_2) \big|_{z_2 = b} = \frac{\partial}{\partial z_2} (\mathcal{F} u)(\xi, z_2) \big|_{z_2 = b} = i \sqrt{k_+^2 - \xi^2} e^{i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi), \quad b > h, \tag{3.7}
\]

where \( \nu \) denotes the unit normal vector on \( \Gamma_h \) pointing into \( U_b \). Thus, by (3.6), (3.7) and Parseval’s formula we have

\[
\int_{\Gamma_h} \left( \frac{\partial u(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z)
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ (\mathcal{F} \partial_v u)(\xi, z_2; y) \big|_{z_2 = b} (\mathcal{F} u)(\xi, b; x)
\right.
\]

\[
- (\mathcal{F} u)(\xi, b; x)(\mathcal{F} \partial_v u)(\xi, z_2; y) \big|_{z_2 = b} \big] d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} i \sqrt{k_+^2 - \xi^2} e^{i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi; y) e^{-i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi; x)
\]

\[
- i \sqrt{k_+^2 - \xi^2} e^{i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi; y) e^{-i(b-h)\sqrt{k_+^2 - \xi^2}} \hat{u}_h(\xi; x) \big] d\xi.
\tag{3.8}
\]

In the above equation \( x \) and \( y \) are used to indicate the dependence on the locations of the incident point sources. Since \( (\mathcal{F} \hat{u})(\xi) = (\mathcal{F} u)(-\xi) \), we get

\[
\int_{-\infty}^{+\infty} i \sqrt{k_+^2 - \xi^2} \left( (\mathcal{F} u)(\xi, h; y)(\mathcal{F} \hat{u})(\xi, h; x) - (\mathcal{F} u)(\xi, h; x)(\mathcal{F} \hat{u})(\xi, h; y) \right) d\xi
\]

\[
= \int_{-\infty}^{+\infty} i \sqrt{k_+^2 - \xi^2} \left( (\mathcal{F} u)(\xi, h; y)(\mathcal{F} u)(-\xi, h; x)
\right.
\]

\[
- (\mathcal{F} u)(\xi, h; x)(\mathcal{F} u)(-\xi, h; y) \big] d\xi. \tag{3.9}
\]

Noting that \( a(\xi) = i \sqrt{k_+^2 - \xi^2} \) is an even function of \( \xi \), we obtain that the right-hand side of (3.9) vanishes. This, together with (3.8), implies that

\[
\int_{\Gamma_h} \left( \frac{\partial u(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z) = 0. \tag{3.10}
\]

Similarly, we can show that

\[
\int_{\Gamma_{-h}} \left( \frac{\partial u(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z) = 0.
\]

Using [15] Theorem 5.1, we derive that

\[
\int_{\Gamma_{\pm L}} \left( \frac{\partial u(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z) = 0, \quad L \to \infty. \tag{3.11}
\]
Applying Green’s second theorem to $u^s(z, y)$ and $u(z, x)$ in $B_ε(y)$ gives
\[ 0 = \int_{\partial B_ε(y)} \left( \frac{\partial u^s(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} u^s(z, y) \right) ds(z). \] (3.12)

Then applying Green’s second theorem to $u^s(z, x)$ and $u(z, y)$ in $B_ε(x)$ gives
\[ 0 = \int_{\partial B_ε(x)} \left( \frac{\partial u(z, y)}{\partial \nu(z)} u^s(z, x) - \frac{\partial u^s(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z). \] (3.13)

Now subtracting (3.10)-(3.13) into (3.4) and using Green’s representation theorem [11, (2.5)] yield
\[ 0 = \int_{\partial B_ε(y)} \left( \frac{\partial \Phi_{k_+}(z, y)}{\partial \nu(z)} u(z, x) - \frac{\partial u(z, x)}{\partial \nu(z)} \Phi_{k_+}(z, y) \right) ds(z) \]
\[ + \int_{\partial B_ε(x)} \left( \frac{\partial u(z, y)}{\partial \nu(z)} \Phi_{k_+}(z, x) - \frac{\partial \Phi_{k_+}(z, x)}{\partial \nu(z)} u(z, y) \right) ds(z) = u(y, x) - u(x, y). \]

The proof is thus complete. □

**Remark 3.6.** The reciprocity relation is also valid for the cases of impenetrable rough surfaces.

### 3.1. The penetrable rough surface.

We now prove the following theorem for the penetrable rough surface which leads to the imaging function required for the imaging method.

**Theorem 3.7.** Assume that $(u^s(x, y), u^t(x, y))$ is the solution to the problem (TSP) with the boundary data $g(x, y) = (-\Phi_{k_+}(x, y), -\partial \Phi_{k_+}(x, y)/\partial \nu(x))^T$, $x \in \Gamma$.

For $z \in U_H^+ \setminus \overline{U}_H^+$ with $H > f_+$ and $h < f_-$ define
\[ U^s(y, z) = \int_{\Gamma_H} \left( \frac{\partial u^s(x, y)}{\partial \nu(x)} \Phi_{k_+}(x, z) - u^s(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x) \]
\[ - \frac{i}{4\pi} \int_{\Sigma^+} e^{ik_+ z \cdot \hat{\nu}} ds(\hat{x}), \quad y \in D_+, \] (3.14)
\[ U^t(y, z) = \int_{\Gamma_H} \left( \frac{\partial u^t(x, y)}{\partial \nu(x)} \Phi_{k_+}(x, z) - u^t(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x), \quad y \in D_- \] (3.15)

Then $(U^s(y, z), U^t(y, z))$ is well-defined and solves the problem (TSP) with the boundary data
\[ G(y, z) = \left( \frac{i}{2} J_0(k_+ |y - z|), -\frac{i}{2} \frac{\partial J_0(k_+ |y - z|)}{\partial \nu(y)} \right)^T, \quad y \in \Gamma. \]

**Proof.** We first prove that $(U^s(y, z), U^t(y, z))$ is well-defined. By Corollary 3.2 it follows that for any $\varepsilon_1 > 0$, $\varepsilon_2 > 1/3$,
\[ u^s(\cdot, y) \in L^{2+\varepsilon_1}(\Gamma_H), \quad \partial_u u^s(\cdot, y) \in L^{2/3+\varepsilon_2}(\Gamma_H). \]

Further, by 2.7 and 2.8 we have
\[ \Phi_{k_+}(\cdot, z) \in L^{2+\varepsilon_1}(\Gamma_H), \quad \partial_u \Phi_{k_+}(\cdot, z) \in L^{2/3+\varepsilon_2}(\Gamma_H). \]
Choose \( \varepsilon_1 > 0, \varepsilon_2 > 1/3 \) such that \( 1/(2+\varepsilon_1) + 1/(\varepsilon_2 + 2/3) = 1 \). Then, by Hölder’s inequality it is obtained that \( \|u^x(y)\partial_x \Phi_{k_+}(\cdot, z)\|_{L^1(\Gamma_H)} < \infty \), \( \|\partial_x u^x(y)\Phi_{k_+}(\cdot, z)\|_{L^1(\Gamma_H)} < \infty \). This implies that \( U^x(y, z) \) is well-defined. It can be shown similarly that \( U^t(y, z) \) is well-defined.

Since \((u^x(y, x), u^t(y, x))\) is a solution to the problem (TSP) with the boundary data \( g(y, x) = (-\Phi_{k_+}(y, x), -\partial_y \Phi_{k_+}(y, x)\partial_\nu(y)) \), \( y \in \Gamma \), Then, by Theorem 2.3, there exists \( \varphi_x(y) := (\varphi_{1, x}(y), \varphi_{2, x}(y)) \in BC^{1, \alpha}(\Gamma) \times BC^{0, \alpha}(\Gamma) \) such that

\[
\begin{align*}
\varphi^x(y, x) &= (D_{k_+, h}^+ \varphi_1, y) + (S_{k_+, h}^+ \varphi_2, y), \quad y \in D_+, \\
\varphi^t(y, x) &= (D_{k_-, h}^- \varphi_1, y) + (S_{k_-, h}^- \varphi_2, y), \quad y \in D_-,
\end{align*}
\]

where \( \varphi_x(y) \) satisfies the integral equation \( A_T \varphi_x = g(\cdot, x) \) with \( A_T \) given in (2.23) and the integral operator \( A_T \) is bijective (and so boundedly invertible) in \([BC(\Gamma)]^2\). Here, we use the subscript \( x \) to indicate the dependence on the point \( x \). Further, since the boundary data \( g(y, x) \) is differentiable for \( x \in \Gamma_H \) and \( y \in \Gamma \), we obtain that \( \varphi_x \) is differentiable with respect to \( x \) with \( \partial \varphi_x / \partial x = A_T^{-1} \partial g(\cdot, x) / \partial x \). Now define \( \varphi_x := \partial \varphi_x / \partial x \), \( g(\cdot, x) := \partial g(\cdot, x) / \partial x \) and

\[
\begin{align*}
\tilde{u}^x(y, x) &= (D_{k_+, h}^+ \tilde{\varphi}_1, x) + (S_{k_+, h}^+ \tilde{\varphi}_2, x), \quad y \in D_+, \\
\tilde{u}^t(y, x) &= (D_{k_-, h}^- \tilde{\varphi}_1, x) + (S_{k_-, h}^- \tilde{\varphi}_2, x), \quad y \in D_-.
\end{align*}
\]

Noting that \( A_T \tilde{\varphi}_x = \tilde{g}(\cdot, x) \), we know that \((\tilde{u}^x(\cdot, x), \tilde{u}^t(\cdot, x))\) is a solution to the problem (TSP) with the boundary data \( \tilde{g}(\cdot, x) \). Since \( D_{k_+, h}^+ \) and \( S_{k_+, h}^+ \) in (3.18) are bounded linear operators (see [13, 27, 31]), it follows that

\[
\frac{\partial u^x(\cdot, x)}{\partial x_2} = \tilde{u}^x(\cdot, x),
\]

Similarly, it can be proved that

\[
\frac{\partial u^t(\cdot, x)}{\partial x_2} = \tilde{u}^t(\cdot, x).
\]

By the above two equations and the reciprocity relation in Lemma 3.5, we know that \( \partial u^x(\cdot, x) / \partial x_2, \partial u^t(\cdot, x) / \partial x_2 \) is the solution to the problem (TSP) with the boundary data \( \partial g(y, x) / \partial x_2 \), \( y \in \Gamma \).

Define

\[
\begin{align*}
\tilde{\varphi}_z(y) &:= \int_{\Gamma_H} \left( \varphi_x(y) \Phi_{k_+}(x, z) - \varphi_x(y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x), \quad y \in \Gamma, \\
\tilde{g}(y, z) &:= \int_{\Gamma_H} \left( g(y) \Phi_{k_+}(x, z) - g(y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x), \quad y \in \Gamma, \\
\tilde{u}^x(y, z) &:= \int_{\Gamma_H} \left( \frac{\partial u^x(x, y)}{\partial x_2} \Phi_{k_+}(x, z) - u^x(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x), \quad y \in D_+, \\
\tilde{u}^t(y, z) &:= \int_{\Gamma_H} \left( \frac{\partial u^t(x, y)}{\partial x_2} \Phi_{k_+}(x, z) - u^t(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x), \quad y \in D_-.
\end{align*}
\]

By (2.7), (2.8) and Remark 2.4 and since

\[
A_T \tilde{\varphi}_z(y) = \tilde{g}(y, z), \quad y \in \Gamma,
\]

Direct imaging of unbounded rough surfaces.

13
we can show that \( \tilde{g}(y, z) \) and \( \tilde{\varphi}_2(\cdot) \) are well-defined. Further, by (3.16) and (3.18) we know that
\[
\tilde{u}^s(y, z) = (D_+^{k, -h} \hat{\varphi}_1)(y) + (S_+^{k, -h} \hat{\varphi}_2)(y).
\] (3.23)

Similarly, we can prove that
\[
\tilde{u}^t(y, z) = (D_-^{k, -h} \hat{\varphi}_1)(y) + (S_-^{k, -h} \hat{\varphi}_2)(y).
\] (3.24)

From (3.22), (3.23) and (3.24), we know that \( (\tilde{u}^s(y, z), \tilde{u}^t(y, z)) \) is the solution to the problem (TSP) with the boundary data \( \tilde{g}(y, z), y \in \Gamma \). Thus, \( (U^s(y, z), U^t(y, z)) \) is a solution to the problem (TSP) with the boundary data \( G(y, z) = (h(y, z), \partial h(y, z)/\partial \nu(y))^T \), where
\[
\begin{align*}
\Phi(y, z) &= -\int_{\Gamma} \left( \frac{d\Phi_{k, x}(x, y)}{d\nu(x)} \Phi_{k, x}(x, z) - \Phi_{k, x}(x, y) \frac{d\Phi_{k, x}(x, z)}{d\nu(x)} \right) ds(x) \\
&\quad - \frac{i}{4\pi} \int_{S^2} e^{ik\hat{x}(y' - z')} ds(\hat{x}), \quad y \in \Gamma.
\end{align*}
\]

By using Lemma 3.4 and the Funk-Hecke formula [28], it follows that
\[
\Phi(y, z) = \frac{i}{4\pi} \int_{S^2} e^{ik\hat{x}(z \cdot y)} ds(\hat{x}) - \frac{i}{4\pi} \int_{S^2} e^{ik\hat{x}(y' - z')} ds(\hat{x}) = \frac{i}{2} \mathcal{J}_0(k |y - z|).
\]

The proof is thus complete. \( \square \)

Remark 3.8. From Theorems 2.3 and 3.4 we know that
\[
\begin{align*}
U^s(y, z) &= (D_+^{k, -h} \psi_1)(y) + (S_+^{k, -h} \psi_2)(y), \quad x \in D_+, \\
U^t(y, z) &= (D_-^{k, -h} \psi_1)(y) + (S_-^{k, -h} \psi_2)(y), \quad x \in D_-,
\end{align*}
\] (3.25) (3.26)

where \( \psi := (\psi_1, \psi_2)^T \) is the unique solution to the integral equation \( A_T \psi = G \) with
\[
G(y, z) = \left( -\frac{i}{2} \mathcal{J}_0(k |y - z|), -\frac{i}{2} \frac{\partial \mathcal{J}_0(k |y - z|)}{\partial \nu(y)} \right)^T =: (g_1(y), g_2(y))^T.
\]

Here, we use the subscript \( z \) to indicate the dependence on the point \( z \). Further, since \( A_T \) is bijective (and so boundedly invertible) in \( [BC(\Gamma)]^2 \), we have
\[
\begin{align*}
C_1 \left( \|g_1\|_{\infty, \Gamma} + \|g_2\|_{\infty, \Gamma} \right) &\leq \|\psi_1\|_{\infty, \Gamma} + \|\psi_2\|_{\infty, \Gamma} \\
&\leq C_2 \left( \|g_1\|_{\infty, \Gamma} + \|g_2\|_{\infty, \Gamma} \right)
\end{align*}
\] (3.27)

for some positive constants \( C_1, C_2 \). Note that the Bessel functions \( \mathcal{J}_0 \) and \( \mathcal{J}_1 \) have the following behavior [11] Section 3.4] (see also Figure 3.4). For \( n = 0, 1, 2, \ldots \)
\[
\begin{align*}
\mathcal{J}_n(t) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n + p)!} \left( \frac{1}{2} \right)^{n+2p}, \quad t \in \mathbb{R} \\
\mathcal{J}_n(t) &= \sqrt{\frac{2}{\pi t}} \cos \left( t - \frac{nt}{2} - \frac{n\pi}{4} \right) \left\{ 1 + O \left( \frac{1}{t} \right) \right\}, \quad t \to \infty
\end{align*}
\]
and $J'_0(t) = -J_1(t)$. Thus

$$
\begin{align*}
g_{1,z}(y) &= \begin{cases} 
-\frac{i}{2} & \text{if } y = z, \\
O(|y - z|^{-1/2}) & \text{if } |y - z| \gg 1, 
\end{cases} \\
g_{2,z}(y) &= \begin{cases} 
0 & \text{if } y = z, \\
O(|y - z|^{-1/2}) & \text{if } |y - z| \gg 1. 
\end{cases}
\end{align*}
$$

By (3.27), we have

$$
\begin{align*}
\|\psi_{1, z}\|_{\infty, \Gamma} + \|\psi_{2, z}\|_{\infty, \Gamma} &\geq C_1 \frac{1}{2} \\
\|\psi_{1, z}\|_{\infty, \Gamma} + \|\psi_{2, z}\|_{\infty, \Gamma} &= O(d(z, \Gamma)^{-1/2})
\end{align*}
$$

From this and (3.25), it is expected that the scattered field $U^s(y, z)$ takes a large value when $z \in \Gamma$ and decays as $z$ moves away from $\Gamma$.

### 3.2. The impenetrable rough surfaces.

In this subsection, we briefly present some results for the cases of an impenetrable rough surface similar to Theorem 3.7 which can be shown similarly.

**Theorem 3.9.** Assume that $u^s(x, y), x, y \in D_+, $ is the solution to the problem (DSP) with the boundary data $g(x, y) = -\Phi_{k_+}(x, y), x \in \Gamma$. For $z \in U^+_h \setminus U^+_H$ with $H > f_+$ and $h < f_-$ define

$$
U^s(y, z) = \int_{\Gamma_H} \left( \frac{\partial u^s(x, y)}{\partial \nu(x)} \Phi_{k_+}(x, z) - u^s(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x) \\
- \frac{i}{4\pi} \int_{S^1} e^{ik_+ \hat{x} \cdot (y' - z')} ds(\hat{x}), \quad y \in D_+
$$

(3.28)

which is independent of $h$. Then $U^s(y, z)$ is well-defined and solves the problem (DSP) with the boundary data

$$
G(y, z) = -(i/2)J_0(k_+|y - z|), \quad y \in \Gamma.
$$

**Theorem 3.10.** Assume that $u^s(x, y), x, y \in D_+, $ is the solution to the problem (ISP) with the boundary data $g(x, y) = -(\partial/\partial \nu(x) - ik_+\rho(x))\Phi_{k_+}(x, y), x \in \Gamma$. For
$z \in U_{H_+}^+ \setminus \overline{U_H^+}$ with $H > f_+$ and $h < f_-$ define

$$U^*(y, z) = \int_{\Gamma_H} \left( \frac{\partial u^*(x, y)}{\partial \nu(x)} \Phi_{k_+}(x, z) - u^*(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x)$$

$$- \frac{i}{4\pi} \int_{S^1_+} e^{ik_+ \hat{x} \cdot (y' - z')} ds(\hat{x}), \quad y \in D_+$$

which is independent of $h$. Then $U^*(y, z)$ is well-defined and solves the problem (ISP) with the boundary data

$$G(y, z) = -\frac{i}{2} \left( \frac{\partial}{\partial \nu(y)} - i k_+ \rho(y) \right) J_0(k_+ |y - z|), \quad y \in \Gamma.$$ 

**Remark 3.11.** From Theorems 3.1, 3.2, 3.9 and 3.10, and by a similar argument as for the case of a penetrable rough surface, it can be obtained that the scattered field $U^*(y, z)$ for the case of an impenetrable rough surface has a similar behavior as for the case of a penetrable rough surface given in Remark 3.8, that is, it is expected that for any $y$ in each compact subset of $D_+$ the scattered field $U^*(y, z)$ for the case of an impenetrable rough surface takes a large value when $z \in \Gamma$ and decays as $z$ moves away from $\Gamma$.

### 3.3. The imaging function

Motivated by the above discussion, we introduce the following imaging function

$$I(z) = \int_{\Gamma_H} \left| \int_{\Gamma_H} \left( \frac{\partial u^*(x, y)}{\partial \nu(x)} \Phi_{k_+}(x, z) - u^*(x, y) \frac{\partial \Phi_{k_+}(x, z)}{\partial \nu(x)} \right) ds(x) \right|^2 \|ds(y)\), \quad y \in \Gamma.$$ 

where $u^*(x, y)$ is the scattered field to one of the three scattering problems mentioned above. Using the asymptotic properties of $U^*(y, z)$ given in Remark 3.10, we can expect that the imaging function $I(z)$ takes a large value when $z \in \Gamma$ and decays as $z$ moves away from the rough surface $\Gamma$.

In numerical computation, the infinite integration interval $\Gamma_H$ in (3.30) is truncated to be $\Gamma_{H, A} := \{x \in \Gamma_H \mid |x| < A\}$ which will be discretized uniformly into $2N$ subintervals so the step size is $h = A/N$. In addition, the lower-half circle $S^1_-$ in the second integral in (3.30) will also be uniformly discretized into $M$ grids with the step size $\Delta \theta = \pi/M$. Then for each sampling point $z$ we get the following discrete form of (3.30)

$$I_A(z) = h \sum_{j=0}^{2N} \left( \sum_{i=0}^{2N} \left( \frac{\partial u^*(x_i, y_j)}{\partial \nu(x)} \Phi_{k_+}(x_i, z) - u^*(x_i, y_j) \frac{\partial \Phi_{k_+}(x_i, z)}{\partial \nu(x)} \right) \right)^2$$

$$- \frac{i \Delta \theta}{4\pi} \sum_{k=0}^{M} e^{ik_+ d_k \cdot (y' - z')} \left| ds(\hat{x}) \right|^2.$$ 

Here, the measurement points are denoted by $x_i = (-A + ih, H)$, $i = 0, 1, ..., 2N$, the incident source positions are $y_j = (-A + jh, H)$, $j = 0, 1, ..., 2N$, and $d_k = (\sin(-\pi + k \Delta \theta), \cos(-\pi + k \Delta \theta))$, $k = 0, 1, ..., M$. 

X. Liu, B. Zhang, and H. Zhang
The direct imaging method based on (3.31) can be described in the following algorithm.

**Algorithm 3.1.** Let $K$ be the sampling region which contains the part of the rough surface that we want to recover.

1. Choose $\mathcal{T}_m$ to be a mesh of $K$ and $\Gamma_{H,A}$ to be a straight line segment above the rough surface.
2. Collect the Cauchy data $(u^s(x_i, y_j), \partial u^s(x_i, y_j)/\partial n(x))$ on the measurement points $x_i$, $i = 0, \ldots, 2N$, on $\Gamma_{H,A}$ corresponding to the incident point sources $u^i(x, y_j) = \Phi_k(x, y_j)$, $j = 0, \ldots, 2N$.
3. For each sampling point $z \in \mathcal{T}_m$, compute the approximate imaging function $I_A(z)$ in (3.31).
4. Locate all those sampling points $z \in \mathcal{T}_m$ such that $I_A(z)$ takes a large value, which represent the part of the rough surface on the sampling region $K$.

4. **Numerical examples.** In this section, we present several numerical experiments to demonstrate the effectiveness of our imaging method and compare the reconstructed results by using different parameters. To generate the synthetic data, we use the Nyström method to solve the forward scattering problems for the case of global rough surfaces [24, 26, 31]. The noisy Cauchy data are generated as follows

$$u^s_k(x) = u^s(x) + \delta(\zeta_1 + i\zeta_2) \max_x |u^s(x)|,$$

$$\partial_\nu u^s_k(x) = \partial_\nu u^s(x) + \delta(\zeta_1 + i\zeta_2) \max_x |\partial_\nu u^s(x)|,$$

where $\delta$ is the noise ratio and $\zeta_1, \zeta_2$ are the standard normal distributions. In all examples, we choose $N = 100$ and $M = 256$.

In each figure, we use a solid line to represent the actual rough surface against the reconstructed rough surface.

**Example 1.** In this example, $\Gamma_1$ and $\Gamma_2$ are two Dirichlet rough surfaces with

$$\Gamma_1 : f_1(x_1) = 0.8 + 0.1 \sin(2\pi x_1) + 0.1 \sin(\pi x_1),$$

$$\Gamma_2 : f_2(x_1) = 0.8 + 0.025 \sin(5\pi(x_1 - 1)) + 0.1 \sin(0.5\pi(x_1 - 1)).$$

The Cauchy data are measured on $\Gamma_{H,A}$ with $H = 1.5, A = 10$. Fig. 4.1 presents the reconstructed surfaces from noisy data with $20\%$ noise for the wave numbers $k_+ = 10, 20, 30$, respectively. From Fig. 4.1 it can be seen that the macro-scale features of the rough surface are captured with a smaller wave number $k_+ = 10$ and the whole rough surface is accurately recovered with a larger wave number $k_+ = 30$.

**Example 2.** We now consider two impedance rough surfaces $\Gamma_3$ and $\Gamma_4$ with

$$\Gamma_3 : f_3(x_1) = 0.8 + 0.16 \sin(\pi x_1),$$

$$\Gamma_4 : f_4(x_1) = 0.8 + 0.1 e^{-25(0.3x_1-0.5)^2} + 0.2 e^{-49(0.3x_1+0.6)^2} - 0.25 e^{-8x_1^2}.$$

The wave number and the impedance function are set to be $k_+ = 15$ and $\rho(x_1, f(x_1)) = 5 + \exp(2\pi x_1)$, respectively. Fig. 4.2 presents the reconstructed surfaces from noisy data with $20\%$ noise for the cases when the Cauchy data are measured on $\Gamma_{H,A}$ with $H = 1.5, A = 4, H = 1.5, A = 10$ and $H = 3, A = 10$, respectively. The reconstruction results show that the reconstruction is getting better if the measurement surface $\Gamma_{H,A}$ is getting closer to the rough surface and is also getting longer.
Fig. 4.1. Reconstruction of Dirichlet rough surfaces at different wave numbers. The top row is the reconstructed result of $\Gamma_1$ and the bottom row is the reconstructed result of $\Gamma_2$.

Example 3. This example considers two penetrable rough surfaces $\Gamma_5$ and $\Gamma_6$:

$\Gamma_5 : f_5(x_1) = 0.8 + 0.3 \sin(0.7\pi x_1)e^{-0.4x_1^2}$,
$\Gamma_6 : f_6(x_1) = 0.8 + 0.1 \sin(0.4\pi x_1)e^{-\sin(1.2x_1^2)}$.

The wave numbers are set to be $k_+ = 20$, $k_- = 8$, and the Cauchy data are measured
Direct imaging of unbounded rough surfaces

on $\Gamma_{H,A}$ with $H = 1.5$, $A = 10$. Fig. 4.3 presents the reconstructed results from data without noise, with 20% noise and 40% noise, respectively.

![Fig. 4.3. Reconstruction of a penetrable rough surface from data at different noise levels. The top row is the reconstructions of $\Gamma_5$ and the bottom row is the reconstructed results of $\Gamma_6$.](image)

The above numerical examples and several other examples carried out but not presented here illustrate that the direct imaging method gives an accurate and stable reconstruction of unbounded rough surfaces. The method is very robust to noise in the measured data and is independent of the physical property of the rough surfaces.

5. Conclusion. We proposed a direct imaging method for inverse scattering problems by an unbounded rough surface. Our imaging method does not need to know the property of the rough surface in advance, so it can be used to reconstruct both penetrable and impenetrable rough surfaces. Numerical experiments have also been carried out to show that the reconstruction is accurate and robust to noise. Further, our imaging method can be extended to many other cases such as inverse elastic scattering problems by unbounded rough surfaces. We will report such results in a forthcoming paper.

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