A COMPLETE WEAK INVARIANCE FOR KOLMOGOROV STATES ON $B = \otimes_{k \in \mathbb{Z}} M_{d}^{(k)}(\mathbb{C})$

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Abstract
Translation dynamics on $C^*$-algebra $B = \otimes_{k \in \mathbb{Z}} M_{d}^{(k)}$, where $M_{d}(\mathbb{C})$ are $d$ dimensional matrices over the field $\mathbb{C}$ of complex numbers, with two invariant states with Kolmogorov property are weakly isomorphic.

1. Introduction

Let $B = \otimes_{\mathbb{Z}} M_{d}(\mathbb{C})$ be the uniformly hyper-finite $C^*$-algebra over the lattice $\mathbb{Z}$, where $M_{d}(\mathbb{C})$ be the $d \times d$-matrices over complex field $\mathbb{C}$. A state $\omega$ on $B$ is called translation invariant if $\omega(x) = \omega(\theta(x))$ where $\theta$ is the translation induced by $z \to z + 1$ for all $z \in \mathbb{Z}$. A state $\omega$ on $B$ is called a factor state if $\pi_\omega(B)''$ is a factor i.e. it's center is trivial, where $(\mathcal{H}, \pi, \Omega)$ is the GNS space associated with $\omega$ on $B$ [BR1]. It is well known since late 60's [Po] that a translation invariant state is a factor state if and only if $\omega(x\theta^m(y)) \to \omega(x)\omega(y)$ for all $x, y \in \mathcal{M}$ as $|m| \to \infty$. Further this criteria is equivalent to ergodic property of the semi-group of endomorphisms $(B_R, \theta_R, \omega_R)$ where $\theta_R$ is the restriction of $\theta$ on $B_R$. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS space of $(B_R, \omega_R)$. We set support projection $P = [\pi(B_R)'\Omega]$ for the state $\omega_R$ in von-Neumann algebra $\pi(B_R)''$. Then by invariance property of the state $\omega_R(P\theta_R(I - P)P) = 0$ which says that $\omega_R(P) \geq P$. We set unital completely positive map $\tau : \mathcal{M} \to \mathcal{M}$ defined by

(1) $\tau(x) = P\theta_R(PxP)P$

for $x \in \mathcal{M} = P\pi_\omega(B_R)'P$. It is well known [BJKW] that $\omega$ is a factor state if and only if $(\mathcal{M}, \tau, \phi)$ is ergodic. A non trivial consequence of this criteria clubbed with asymptotic abelianness of the dynamics $(\mathcal{B}, \theta_n : n \in \mathbb{Z})$ says that two such factor states on $B$ are either orthogonal to each other or equal (Chapter 4.3.2 in [BR1], Theorem 4.3.19). In spite of these spectacular success a classification upto isomorphism remains an open problem. R. T. Power also shown [Po] that automorphism of $B$ acts transitively on the set of pure states. We take a clue from here and focus on classification problem for the class of translation invariant pure state. Mathematical difficulties or challenge come from the fact that Powers transitive action may not commute with transition action. In this paper we give more details.

In a recent paper [Mo4] an elegant asymptotic criteria on $(\mathcal{M}, \tau, \phi)$ is given for purity of $\omega$. It is also known [BJKW, Ma] that for a type-I factor state $\omega_R$, $\omega$ is pure. This class of translation invariant pure states $\omega$ with type-I factor state $\omega_R$ includes finitely correlated states [FNW1, FNW2] which often appear as ground

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state of integrable models for quantum spin chain. There are translation invariant pure \( \omega \) where \( \omega_R \) is a type-III factor state [AMa]. Theorem 1.1 in [Mo4] says that \( \omega_R \) is either a type-I or type-III factor state when \( \omega \) is a translation invariant pure state on \( B \). In case \( \omega_R \) is type-I, \( \omega \) admits Kolmogorov’s property introduced in [Mo1]. In the next para we briefly recall Kolmogorov’s property [Mo1].

In this paper we consider translation invariant state \( \omega \) on \( B \). We consider the GNS space \((\mathcal{H}, \pi, \Omega) \) of \((B, \omega)\). We set family of projections \( E_n = |\pi(\Theta_n(B_R))/\Omega\rangle \), \( n \in \mathbb{Z} \). \( E_n \) is the support projection of \( \omega \) in \( \pi(\Theta_n(B_R))'' \). Thus we have \( E_n \leq E_{n+1} \) and \( E_n \uparrow \mathbb{I}_{\mathcal{H}} \) as \( n \uparrow \infty \). However in general \( E_n \downarrow E_{-\infty} \) as \( n \downarrow -\infty \) with \( E_{-\infty} \geq |\Omega \rangle < |\Omega |\rangle \) where \( E_{-\infty} \) may not be equal to \( |\Omega \rangle < |\Omega |\rangle \) even when \( \omega \) is pure. We say \( \omega \) admits Kolmogorov property if \( E_{-\infty} = |\Omega \rangle < |\Omega |\rangle \) and call \( \omega \) a Kolmogorov state. A Kolmogorov state is a pure state [Mo1]. For a pure \( \omega, \omega_R \) is either type-I or type-III factor state [Mo4]. In case \( \omega_R \) is type-I, \( \omega \) is a Kolmogorov state [Mo4]. No example of a Kolmogorov state \( \omega \) is known yet for which \( \omega_R \) is a type-III factor state.

Let \((B_k, \theta_k, \omega_k)\) with \( k = 1, 2 \) be two \( C^* \)-dynamical systems where \( B_k \) are \( C^* \)-algebras and \( \theta_k : B_k \to B_k \) are automorphisms preserving state \( \omega_k \) respectively. Let \( \Theta_k : \pi_k(B_k)'' \to \pi_k(B_k)'' \) be the associated automorphisms where \((\mathcal{H}_k, \pi_k, \Omega_k)\) are GNS spaces for \((B_k, \theta_k, \omega_k)\), \( k = 1, 2 \). \((B_1, \theta_1, \omega_1)\) and \((B_2, \theta_2, \omega_2)\) are said to be unitary equivalent or weakly isomorphic if there exists an unitary operator \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) so that \( U\Theta_1(X)U^* = \Theta_2(UXU^*) \) for all \( X \in \pi(B_k)'' \). We say there are isomorphic if there exists \( C^* \) automorphism \( \beta : B_1 \to B_2 \) such that \( \Theta_2 \circ \beta = \beta \circ \Theta_1 \) on \( B_1 \) and \( \omega_2 \beta = \omega_1 \). In this paper we are interested with a single \( C^* \) algebra namely \( B \) and \( \theta_k = \theta^k \) where \( \theta \) is right translation.

Given a translation invariant state \( \omega \) on \( B \), one has two natural numbers:

(a) Mean entropy \( s(\omega) = \lim_{\Lambda_n \uparrow \mathbb{Z}} \frac{1}{|\Lambda_n|} \sum_{\lambda \in \Lambda_n} S_{\omega, \lambda_n} \) where \( S_{\omega, \lambda_n} \) is the von-Neumann entropy of the state \( \omega_{\lambda_n} \) i.e. restricted state to local \( C^* \) algebra \( B_{\lambda_n} \) and \( \Lambda_n = \{-n \leq k \leq n \} \) or more generally a sequence of finite subsets of \( \mathbb{Z} \) such that \( \Lambda_n \uparrow \mathbb{Z} \) in the sense of Van Hove [Section 6.2.4 in BR2]. It is not known yet whether \( s(\omega) \) is an invariance for the translation dynamics \((B, \theta, \omega)\) i.e. whether \( s(\omega) \) can be realized intrinsically as a dynamical entropy of \((B, \theta, \omega)\) for all translation invariant state \( \omega \).

(b) Connes-Stormer dynamical entropy: \( h_{CS}(\omega) \) is a close candidate for such an invariance for the translation dynamics \((B, \theta, \omega)\). It is known that \( 0 \leq h_{CS}(\omega) \leq s(\omega) \). In case \( \omega \) is a product state then it is known that \( h_{CS}(\omega) = s(\omega) \). It is also known that \( h_{CS}(\omega) = 0 \) if \( \omega \) is pure. No example is known yet for which \( h_{CS}(\omega) < s(\omega) \).

The product Bernoulli state \( \mu_\infty = \otimes_{\mathbb{Z}^d} \mu \) on \( C(D^2) \) has several Hann-Banach translation invariant extension to \( B \) where \( C(D^2) \) is viewed as a \( C^* \)-sub-algebra consists of product of diagonal matrices in \( B \) with respect to an orthonormal basis \((e_i)\) for \( \mathbb{C}^d \) (section 3 gives details). Let \( \rho \) be a state on \( M_d \) so that \( \rho(|e_i < e_j|) = \mu_j \). One trivial possibility is to take \( \rho(|e_i > e_j|) = \delta^i_j \mu_j \). In such a case \( \rho \) is a pure state unless \( \mu_j = 1 \) for some \( j \in D \) and associated product state on \( B \) includes in particular canonical tracial state on \( B \) for a suitable choice namely \( \mu_j = \frac{1}{s_j} \). Otherwise we will have type-III for \( \lambda \in (0, 1) \) [Po]. In general we can choose a Hann-Banach extension of the Bernoulli measure to a product state on \( B \) so that it’s mean entropy takes values a prescribed number between 0 and \( \sum_j -\mu_j \ln(\mu_j) \). One can choose a unit vector \( \lambda = (\lambda_j) \) in \( \mathbb{C}^n \) and set pure state.
Here we make use of Mackey’s system of imprimitivity that arises from the symmetry of the state $\omega$ and simplicity of the $C^*$ involved. Main result gives a complete classification upto unitary equivalence of translation dynamics on $B = \otimes_2 M_2(C)$ with Kolmogorov states. Main result says that translation dynamics on $B$ with two invariant states with Kolmogorov’s property are unitary equivalent. However dynamics need not be isomorphic as $C^*$ dynamical system. This shows isomorphism as $C^*$-dynamics is a stronger condition where additional invariance would play a role to classify them. Connes-Størmer dynamical entropy being zero for such class of dynamics, gives room for other possible dynamical invariance. In other words Mackey’s invariance complements a possible dynamical invariance. In the proof simplicity of $B_0$ plays an important role and one can generalize replacing $B_0$ by a simple infinite dimensional separable $C^*$-algebra. Interesting point here to be noted unlike classical Markov chain, any translation invariant state is a quantum Markov state in the sense of Luigi Accardi [Ac1,Ac2] once we take into account it’s generalization to finitely or infinitely correlated states [FNW1],[BJ],[BJKW],[Mo6] for general situation. Thus this simple result gives a hope for reformulation of classical Ornstein problem [Or] in this more general quantum mechanical framework of $C^*$-algebra $B$ where isomorphism need not preserve local sub-algebra $B_{t_0\infty} = \bigcup_{\Lambda\subset Z} B_{\Lambda}$ where $\Lambda$ are taken as all possible finite subsets and $B_{\Lambda}$ are $C^*$ sub-algebras generated by elements supported in $\Lambda$.

2. Inductive limit Kolmogorov state and weak isomorphism:

Let $T$ be either $Z$ or $R$ and $T_+ = \{ t \in T : t \geq 0 \}$. Let $B_0$ be a $C^*$ algebra, $(\lambda_t : t \geq 0)$ be a semi-group of injective endomorphisms and $\psi$ be an invariant state for $(\lambda_t : t \geq 0)$. We extend $(\lambda_t)$ to an automorphism on the $C^*$ algebra $B_{t_{\infty}}$ of the inductive limit

$$B_0 \to^{\lambda_t} B_0 \to^{\lambda_s} B_0$$

and extend also the state $\psi$ to $B_{t_{\infty}}$ by requiring $(\lambda_t)$ invariance. Thus there exists a directed set ( i.e. indexed by $T$ , by inclusion $B_{t-s} \subseteq B_{t-t}$ if and only if $t \geq s$ ) of $C^*$-subalgebras $B_0$ of $B_{t_{\infty}}$ so that the uniform closure of $\bigcup_{s\in T} B_{[s]}$ is $B_{t_{\infty}}$. Moreover there exists an isomorphism

$$i_0 : B_0 \to B_0$$

( we refer [Sa] for general facts on inductive limit of $C^*$-algebras ). It is simple to note that $i_t = \lambda_t \circ i_0$ is an isomorphism of $B_0$ onto $B_{[t]}$ and

$$\psi_{t_{\infty}} \lambda_t = \psi$$

on $B_0$. Let $(H_\pi, \pi, \Omega)$ be the GNS space associated with $(B_{t_{\infty}}, \psi_{t_{\infty}})$ and $(\lambda_t)$ be the unique normal extension to $\pi(B_{t_{\infty}})'$. Thus the vector state $\psi_{\Omega}(X) = <\Omega, X\Omega>$ is an invariant state for automorphism $(\lambda_t)$. As $\lambda_t(B_0) \subseteq B_0$ for all $t \geq 0$, $(\pi(B_0)', \lambda_t, t \geq 0, \psi_{\Omega})$ is a quantum dynamics of endomorphisms. Let $E_t$,
be the support projection of the normal vector state $\Omega$ in the von-Neumann sub-algebra $\pi(B_0)''$. $E_t \in \pi(B_t)'' \subseteq \pi(B_{-\infty})''$ is a monotonically decreasing sequence of projections as $t \to -\infty$. Let the projection $F_{-\infty}$ be the limit. Thus $E_{-\infty} \geq \pi(B_{-\infty})' \Omega \geq \Omega$ ensures that $\psi_{-\infty}$ on $B_{-\infty}$ is in particular pure. We say $\psi_{-\infty}$ is Kolmogorov if $E_{-\infty} = \Omega$. In case we are dealing with $\mathbb{T} = \mathbb{R}$, we assume that the induced semi-group of endomorphisms $\lambda_t : \pi(B_0)'' \to \pi(B_0)''$ is continuous in weak$^*$ topology. So we get a family of strongly continuous projections $F_t = E_t - \Omega$ and a unitary representation $U_t$ of $\mathbb{T}$ by restricting $S_t$ to $I_{H^t} = \Omega$. So we get $U_t F_s U_s^* = F_{s+t}$ for all $s, t \in \mathbb{T}$. Since $F_t \uparrow I_{H^t} = \Omega$ as $t \uparrow \infty$ and $F_t \downarrow 0$ as $t \downarrow -\infty$, we get a non-generate regular Mackey’s system of imprimitivity [Mac]. As a consequence of Mackey’s theorem [Mac] and simplicity of $C^*$ algebra $B_0$ we arrive at the following theorem.

**Theorem 2.1.** Let $(B_0, \lambda_t : t \geq 0)$ be a unital semi-group of injective endomorphisms and $\psi_0, \psi_{-\infty}$ be two $(\lambda_t)$ states with inductive limit states $\psi_{-\infty}$ and $\psi_{-\infty}'$ on $(B_{-\infty}, \lambda_t : t \in \mathbb{T})$ with Kolmogorov’s property. If $B_0$ is simple separable $C^*$-algebra then $(B_{-\infty}, \alpha_t : t \in \mathbb{T}, \psi_{-\infty})$ and $(B_{-\infty}, \alpha_t : t \in \mathbb{T}, \psi_{-\infty})'$ are weakly isomorphic.

**Proof.** In the next section we will give a proof for $\mathbb{T} = \mathbb{Z}$ with $B_0 = \otimes_k M^{(k)}(\mathbb{C})$ which needs little modification to include a proof for the general situation $\mathbb{T} = \mathbb{R}$ and $B_0$.

3. **A complete weak invariance for Kolmogorov states of $B = \otimes_k M^{(k)}(\mathbb{C})$:**

Let $\mathcal{D} = \{1, 2, \ldots, d\}$ be a finite set with $d \geq 1$ and $p : \mathcal{D} \times \mathcal{D} \to [0, 1]$ be a transition probability matrix which admits a stationary probability measure $\mu : \mathcal{D} \to [0, 1]$ i.e. $p_j \geq 0$, $\sum_j p_j = 1$ and $\sum_j \mu_j = 1$, $\sum_j \mu_j p_j^i = \mu_j$. Let $\mathcal{Z}$ be the set of integers and $(X_n : n \in \mathbb{Z})$ be the two sided stationary Markov process or chain defined on a probability space $(\mathcal{D}^\mathbb{Z}, \mathcal{F}, \mathcal{P}_\mu)$ where $\mathcal{D}^\mathbb{Z} = \times_{n \in \mathbb{Z}} \mathcal{D}_n$ is the product set with $\mathcal{D}_n = \mathcal{D}$ and $\mathcal{F}$ be the $\sigma-$fields generated by the cylinder subsets of $\mathcal{D}^\mathbb{Z}$ and $X_n(\zeta) = \zeta_n$ with $\mathcal{P}(\zeta, X_{n+1}(\cdot) = j | X_n(\cdot) = i) = p_{ij}$. Let $\mathcal{F}_n$ be the $\sigma-$field generated by the Markov process upto time $n$ i.e. $\mathcal{F}_n = \sigma\{X_k : -\infty < k \leq n \text{ are measurable}\}$. Such a Markov process is said to admit Kolmogorov property if $\bigcap_{n \in \mathbb{Z}} \mathcal{F}_n = \{\mathcal{D}^\mathbb{Z}, 0\}$. Such a Markov process is also called Bernoulli shift if $p_{ij} = \mu_j$ for all $i, j \in \mathcal{D}$.

Let $\mathcal{H} = L^2(\mathcal{D}^\mathbb{Z}, \mathcal{F}, \mathcal{P}_\mu)$. Let $F_n$ be the orthogonal projection on $L^2(\mathcal{D}^\mathbb{Z}, \mathcal{F}_n, \mathcal{P}_\mu)$ for each $n \in \mathbb{Z}$ and $S_n : n \in \mathbb{Z}$ be the right shift unitary operator on $\mathcal{H}$ defined by $S_n f(\zeta) = f(\theta_n(\zeta))$, $f \in \mathcal{H}$ where $\theta_n(\zeta)(k) = \zeta(k-n)$ is the $n$-shift which preserves the probability measure $\mathcal{P}_\mu$. Thus $X_n$ can be viewed as an element in $L^\infty(\mathcal{D}^\mathbb{Z}, \mathcal{F}_n, \mathcal{P}_\mu)$ and $\alpha : L^\infty(\mathcal{D}^\mathbb{Z}, \mathcal{F}, \mathcal{P}_\mu) \to L^\infty(\mathcal{D}^\mathbb{Z}, \mathcal{F}, \mathcal{P}_\mu)$ is an automorphism defined by $\alpha(f)(\zeta) = f(\theta^{-1}(\zeta))$ for $f \in L^\infty(\mathcal{D}^\mathbb{Z}, \mathcal{F}, \mathcal{P}_\mu)$ and $\Phi_n(f) = f f dP_\mu$ is the translation invariant state on the function space on the compact Hausdorff space $\mathcal{D}^\mathbb{Z} = \otimes_n \mathcal{D}$ where $\mathcal{D}$ is equipped with product topology and so $\mathcal{D}^\mathbb{Z}$ is compact. Thus an invariant state $\mu$ and associated translation invariant state $\Phi_\mu$ exists. We assume further that $\mu$ is also faithful i.e. $\mu_j > 0$ for all $j \in \mathcal{D}$ to avoid trivial modification in our argument. Kolmogorov property is equivalent to $F_n \downarrow P_\Theta$ in strong operator topology as $n \to \infty$, where $P_\Theta$ is the orthogonal projection on constant functions on $\mathcal{D}^\mathbb{Z}$. In such a case the projection $F_n - P_\Theta$ together with shifts $S_n$ restricted to $I - P_\Theta$ give rise to a system of imprimitivity for the group $\mathbb{Z}$ in the sense of G. W. Mackey. So by Mackey theorem $(S_n(I - P_\Theta), F_n - P_\Theta)$ is unitary equivalent to direct sum of copies of standard shifts on $L^2(\mathbb{Z})$. Thus the
number of copies that appears in the direct sum is also an invariance for associated imprimitivity system. A natural question: how this new invariance which intertwines filtration is related to Kolmogorov-Sinai’s invariance of dynamical entropy which may not preserve filtration? We will prove that these two invariance rather then competing with each other they are complementing each other as invariance for the shift.

It is well known [Pa] that a Bernoulli shift admits Kolmogorov’s property and a deeper result in ergodic theory says that the converse is also true i.e. a Markov chain with Kolmogorov’s property is isomorphic with a Bernoulli shift. A celebrated result in ergodic theory [Or] says much more giving a complete invariance for such a class of Markov chain in terms of their dynamical entropy \( h_{\mu}(\theta) \) introduced by Kolmogorov-Sinai [Pa] of the shift. However the isomorphism that intertwines the Kolmogorov’s shifts with equal positive dynamical entropy, need not intertwine the filtration generated by the processes. However one gets finite correlation which we will explain in the following text.

Let \( \omega \) be as before a translation invariant state on \( \mathcal{B} \) and \( E \) be the support projection of \( \omega \) in \( \pi(\mathcal{B}_R)^\prime \prime \) in the GNS space \( (\mathcal{H}_\pi, \pi, \Omega) \) associated with \( (\mathcal{B}, \omega) \) and set \( E_{n_0} = \theta^n( E ) \) for all \( n \in \mathbb{Z} \). So \( \theta^n( E ) \) is the support projection of \( \omega \) in \( \pi(\mathcal{B}_R)^\prime \prime \) and thus \( E_{n_0} \leq E_{n_0+1} \). We say \( \omega \) admits Kolmogorov’s property if \( E_{n_0} \downarrow |\Omega > < \Omega| \) in strong operator topology as \( n \downarrow -\infty \) [Mo2]. In such a case restriction of \( (S_n, E_{n_0}) \) on \( |\Omega > < \Omega| \uparrow = I - |\Omega > < \Omega| \) give rises to a Mackey system of imprimitivity for the group \( \mathbb{Z} \) and \( \theta_H \circ H \mid |\Omega > < \Omega| \) can decomposed into copies of \( l^2(\mathbb{Z}) \) and restriction of \( S_n \) is isomorphic to the standard shift. We say number of copies that appear in the decomposition as Mackey’s index for \( \omega \) which are assumed to admit Kolmogorov’s property. A simple consequence of the main result in section 3 shows that Bernoulli state admits Kolmogorov’s property. Any translation invariant state on \( \mathcal{B} \) with Kolmogorov’s property is pure. However a pure may not in general admit Kolmogorov’s property as we pointed out indication an example in section 4. A translation invariant pure states with \( \pi(\mathcal{B}_L)^\prime \prime \) as type-I factor admits Kolmogorov’s property by Theorem 2.7 in [Mo4]. A valid question: When does a translation invariant state on \( \mathcal{B} \) unitary equivalent to a Bernoulli state on \( \mathcal{B} \) intertwining the shifts? In the following text we formulate the problem precisely and state our main results of this section and compute it’s Connes-Størmer dynamical entropy [NeS] indirectly.

**Proposition 3.1.** Let \( \omega \) be a translation invariant state on \( \mathcal{B} = \otimes \mathcal{M}_d(\mathbb{C}) \) with Kolmogorov’s property. Then it’s Mackey index of shift with respect to filtration generated by \( E = [\pi(\mathcal{B}_R)]^\prime \prime \) is equal to \( \mathcal{S}_0 \). Same holds filtration generated by the projection \( F = [\pi(\mathcal{B}_L)]^\prime \prime \).

**Proof.** Let \( E \) be the support projection of \( \omega \) in \( \pi(\mathcal{B}_R)^\prime \prime \) where \( (\mathcal{H}, \pi, \Omega) \) is the GNS space associated with the state \( \omega \) on \( \mathcal{B} \) i.e. \( E = [\pi(\mathcal{B}_R)]^\prime \prime \). By Haag duality [Mo4] we also have \( E = [\pi_\omega(\mathcal{B}_R)]^\prime \prime \Omega = [\pi_\omega(\mathcal{B}_L)]^\prime \prime \Omega \) and so \( \theta(E) = [\pi_\omega(\theta(\mathcal{B}_L))]^\prime \prime \Omega \). We claim that \( \theta(E) - E \) is infinite dimensional. That \( \theta(E) - E \neq 0 \) follows as \( \theta^n(E) \uparrow I \) as \( n \uparrow \infty \) and \( \theta^n(E) \downarrow |\Omega > < \Omega| \) as \( n \downarrow -\infty \). We choose a unit vector \( f \in \theta(E) - E \) and thus in particular \( f \perp \Omega \) since \( \theta(E)\Omega = \Omega \). We set temporary notation \( F = [\pi_\omega(\mathcal{B}_R)]f \) and note that \( F \subseteq \theta(E) \) as \( B_L \subseteq \theta(B_L) \) and \( F \perp E \) as \( f \perp \Omega \). Thus \( F \subseteq \theta(E) - E \). We claim that \( F \) is an infinite dimensional subspace as otherwise we find a finite dimensional representation of \( \mathcal{B}_R \) on \( F \) by restricting the representation \( \pi \) of \( \mathcal{B}_L \) to \( F \). That contradicts simplicity of \( \mathcal{B}_L \) as a \( C^* \)-algebra.
The proof uses Haag duality property but one may as well get a proof of Proposition 3.1 without using it as follows. Since proof given for Haag duality in [Mo6] is not yet published, in the following we use filtration generated by $F$ to give an alternative proof.

Let $E$ be the support projection of $\omega$ as before in $\pi_\omega(B_R)''$ in the GNS space i.e. $E = [\pi_\omega(B_R)'\Omega]$ and $F = [\pi_\omega(B_L)'\Omega]$. $F \leq E$ as $F \ominus |\Omega \rangle\langle \Omega|$ as well give rises to a system of imprimitivity with action of translation i.e. $\theta(F) \geq F$ and $\theta^n(F) \uparrow I$ by cyclic property of the GNS representation and $0 \leq \theta^n(F) \leq \theta^n(E) \downarrow |\Omega \rangle\langle \Omega|$ as $n \downarrow -\infty$. Thus the Mackey systems generated by the projection $F - |\Omega \rangle\langle \Omega|$ by shift is of index $\aleph_0$.

**Theorem 3.2.** Two translation invariant states on $B$ with Kolmogorov’s property gives unitary equivalent shift on $B$.

**Proof.** Let $\omega, \omega'$ be two such states of $B$. By Mackey’s theorem [Mac], as their Mackey indices are equal to $\aleph_0$ by Proposition 4.1, we get an unitary operator $U : H_{\pi_\omega} \to H_{\pi_{\omega'}}$ such that

$$USU^* = S', \quad UFU^* = F'$$

(2) In particular we get $U\theta(X)U^* = \theta(UXU^*)$ for all $X \in \pi_\omega(B)''$.

**Theorem 3.3.** Let $\omega$ be a translation invariant state on $B$ with Kolmogorov’s property. Then $(B, \theta, \omega)$ is unitary equivalent to a Bernoulli dynamics $(B, \theta, \omega_\lambda)$ where $\lambda \in \mathbb{C}^n$ is a unit vector.

**Proof.** A Bernoulli state on $B$ admits Kolmogorov’s property and thus it is a simple case of Theorem 3.2.

Thus in contrast to classical situation and Connes-Størmer’s theory of Bernoulli shifts we have the following.

**Theorem 3.4.** Let $(B, \theta, \omega)$ be as in Theorem 3.3. Then $(B, \theta, \omega)$ and $(B, \theta^k, \omega)$ are unitary equivalent dynamics with Connes-Størmer dynamical entropy equal to zero but not isomorphic as $C^*$ dynamics for $k > 1$.

**Proof.** By Theorem 3.2 $(B, \theta, \omega)$ and $(B, \theta^k, \omega)$ are isomorphic dynamics as both admits Kolmogorov’s property and their Mackey index are equal to $\aleph_0$. That Connes-Størmer dynamical entropy is zero follows once we use the isomorphism to translation dynamics with a Bernoulli state on $B$. That $\theta$ and $\theta^k, k > 1$ are not isomorphic is well known [CS] since Connes-Størmer dynamical entropy with respective to the unique trace are not equal.

It is already known [NeS] that Connes-Størmer dynamical entropy is zero for a pure translation invariant state. In general for a translation invariant state on $B$, $h_{CS}(\omega) \leq s(\omega)$, where $s(\omega)$ is the mean entropy. One of the standing conjecture in the subject about equality. In particular it is not known yet whether $s(\omega)$ is zero for a translation invariant pure state. As a first step towards the conjecture it seems worth to investigate this question for such a state with Kolmogorov’s property. Note that mean entropy is yet to be interpreted as an invariance for the shift! However it is not hard to verify mean entropy of translation invariant states $\omega_1, \omega_2$ remain same if $(B, \theta, \omega_1)$ and $(B, \theta, \omega_2)$ are isomorphic provided the isomorphism preserves the local algebras $B_{loc} = \bigcup_{|\Lambda| < \infty} B_{\Lambda}$.
Now we conclude this section comparing isomorphic class in quantum situation with that of classical situation studied by [Or]. Any isomorphism between the space of continuous functions on two compact Hausdorff Bernoulli spaces \( D_\beta^c = \otimes_Z D_\beta \) and \( D_\beta^c = \otimes_Z D_\beta \) with product topology is local i.e. any set \( E \in D_\beta^c \) with finite support i.e. cylinder set gets mapped by the isomorphism to a set with finite support in \( D_\beta^c \), i.e. a cylinder set since a compact set goes to a compact set which can be covered by finitely many open sets of finite support since union over sets with finite support is a cover for any set. Besides the isomorphism local, we can use covariance relation with respect to translation action to conclude that isomorphism takes \( F_n \) to \( F_{n+k} \) for all \( n \in \mathbb{Z} \) for some fixed \( k \in \mathbb{Z} \). Similarly it takes backward filtration \( F'_n \) to \( F'_{n+k} \) for all \( n \in \mathbb{Z} \) for some \( k' \in \mathbb{Z} \). On the other hand an isomorphism on \( \mathcal{B} = \otimes M_d(\mathbb{C}) \) need not be local.

In contrast we note that the inter-twiner \( U : \mathcal{H} \rightarrow \mathcal{H} \) between the two dynamics \((\mathcal{B}, \theta, \omega)\) and \((\mathcal{B}, \theta^2, \omega)\) does not preserves local algebra \( \mathcal{B}_{loc} \) where \((\mathcal{H}, \pi, \Omega)\) is the GNS space. Since otherwise, \( \mathcal{B} \) being a simple \( C^* \) algebra, we would have \( U\pi_\omega(x)U^* = \pi_\omega(\beta(x)) \) by faithful property of the representation for some \( C^* \)-isomorphism \( \beta : \mathcal{B} \rightarrow \mathcal{B} \) and further inter-twiner relation \( USU^* = SU \), would have given \( \theta \beta = \beta \theta^2 \) on \( \mathcal{B} \). Such a relation is impossible. This feature in a sense is a deviation from an isomorphism map between two compact Hausdorff spaces representing Bernoulli or Markov shifts [Or]. Thus this feature is a purely quantum phenomenon and a strong deviation of our intuition of classical spin chain. A family of classical Bernoulli states with entropy between 0 and \( \log d \) are embedded continuously into a Bernoulli state \( \omega_\lambda \) on \( \mathcal{B} = \otimes M_d(\mathbb{C}) \) and Kolmogorov-Sinai dynamical entropy is a complete invariance to determine a translation invariant maximal abelian von-Neumann sub-algebra upto isomorphism. An unitary conjugation that makes two Bernoulli dynamics \((\mathcal{B}, \theta, \omega_\lambda)\) and \((\mathcal{B}, \theta^2, \omega_\lambda)\) unitary equivalent does not make them \( C^* \)-isomorphic. We have explored this contrast and it’s property further with a new notion of quantum dynamical entropy [Mo] which is not same as Connes-Størmer [NeS] dynamical entropy.

One natural question that arises here: when two Kolmogorov dynamical systems \((\mathcal{B}, \theta, \omega)\) and \((\mathcal{B}, \theta', \omega')\) are isomorphic? It seems that this problem is far from being understood even in its primitive form. In the following we make a simple observation.

**Proposition 3.5.** Two Kolmogorov dynamical systems \((\mathcal{B}, \theta, \omega)\) and \((\mathcal{B}, \theta', \omega')\) are isomorphic if and only if there exists a unitary operator \( U : \mathcal{H}_{\omega} \rightarrow \mathcal{H}_{\omega'} \) such that (a) \( U\Omega_\omega = \Omega_{\omega'} \), \( US_\omega = S_{\omega'} U \) and (b) \( U^*\pi_\omega(x)U = \pi_{\omega'}(\beta(x)) \) for all \( x \in \mathcal{B} \).

Since \( \mathcal{B} \) is an UHF \( C^* \)-algebra and \( \omega, \omega' \) are pure states on \( \mathcal{B} \), we can apply Power’s theorem (Theorem 3.7 and Corollary 3.8 in [Po]) to find an automorphism \( \beta : \mathcal{B} \rightarrow \mathcal{B} \) such that

\[
\omega(x) = \omega'(\beta(x))
\]

for all \( x \in \mathcal{B} \). It is for states of question of interest, \( \omega \neq \omega' \) and thus \( \beta \) is not equal to any order of translation. So there exists an unitary operator \( U : \mathcal{H}_{\omega} \rightarrow \mathcal{H}_{\omega'} \) so that

\[
U\pi_\omega(x)U^* = \pi_{\omega'}(\beta(x))
\]

On the other hand by Theorem 3.2, there exists \( U : \mathcal{H}_{\omega} \rightarrow \mathcal{H}_{\omega'} \) such that \( US_\omega = S_{\omega'} U \). Proposition 3.5 says that we need both relations to hold simultaneously for some unitary \( U \).
Proof. Assume (a) and (b). Then by (a) we get \( U^*\pi_{\omega}(\theta(x))U = \pi_{\omega}(\theta(\beta(x))) = \pi_{\omega}(\beta(\theta(x))) \). \( B \) being a simple \( C^* \) algebra, any non degenerate representation is faithful. Thus \( \theta\beta = \beta\theta \) and \( \omega'\beta = \omega \). Converse follows along the same line once we trace back the argument. \( \blacksquare \)

4. Appendix

Here we give a sketch leaving details as it requires quite a different involved framework to prove our claim that the unique ground state \( \omega_{XY} \) of \( H_{XY} \) \([\text{AMa}] \) model when restricted to one side of the chain \( B_R \) ( or \( B_L \) ) is faithful. We refer to Chapter 6 of the monograph \([EvKav]\) for details on the mathematical set up for the Fermion algebra and what follows now is based on \([Ma]\). Let \( \mathcal{H} \) be a complex separable infinite dimensional Hilbert space \((\mathcal{H} = L^2(\mathbb{R}))\) and consider the universal simple unital \( C^* \)-algebra \( \mathcal{A}_F \) generated by \( \{a(f) : f \in \mathcal{H}\} \) over \( \mathcal{H} \) where \( a : \mathcal{H} \to \mathcal{A}_F \) is a conjugate linear map satisfying

\[
 a(f)a(g) + a(g)a(f) = 0
\]

\[
 a(f)a(g)^* + a(g)^*a(f) = \langle g, f \rangle > 0
\]

For an orthonormal basis \( \{e_j : j \in \mathbb{Z}\} \) for \( \mathcal{H} \) we fix an unitary operator \( U : e_j \to e_j \) if \( j > 1 \) otherwise \( -e_j \). The universal property of CAR algebra ensures an automorphism \( \alpha_- \) on \( \mathcal{A}_F \) via the second quantization \( a(f) \to a(Uf) \) i.e. \( a(e_j) \to a(e_j) \) if \( j \geq 1 \) otherwise \( a(e_j) \to -a(e_j) \) for \( j \leq 1 \), where \( e_j(i) = \delta_j^i \) is the Dirac functions on \( j \). Let \( \hat{\mathcal{A}}_F = \mathcal{A}_F \times [\mathbb{Z}_2] \) the cross-product \( C^* \)-algebra.

Now we consider Pauli’s \( C^* \)-algebra \( \mathcal{A}_P = \otimes_2 M_2(\mathbb{C}) \) with grading \( \alpha_-\sigma^k_x = \sigma^k_x, \alpha_-\sigma^k_y = -\sigma^k_y \) and \( \alpha_-\sigma^k_z = -\sigma^k_z \) where \( \sigma_x, \sigma_y, \sigma_z \) are Pauli spin matrices if \( k < 1 \) and \( \alpha_-\sigma^k_w = \sigma^k_w \) for all \( w = x, y, z \) for \( k \geq 1 \). We consider the cross-product \( C^* \)-algebra \( \hat{\mathcal{A}}_P = \mathcal{A}_P \times [\mathbb{Z}_2] \). Jordan-Wigner transformation map \( J \) which takes

\[
 \sigma^1_x \to 2(a(e_j)^*a(e_j) - 1), \quad \sigma^2_x \to TS_j(a(e_j) + a(e_j)^*), \quad \sigma^3_y \to TS_j(a(e_j) - a(e_j)^*)
\]

where \( T = \otimes_{1 \leq k \leq 0}\sigma^k_z \) and \( S_j = 1 \) if \( j = 1 \), \( S_j = \otimes_{1 \leq k \leq j - 1}\sigma^k_z \) if \( j \geq 1 \) identifies \( \hat{\mathcal{A}}_P \) with \( \mathcal{A}_F \) as the Jordan map is co-variant with grading automorphism \( \alpha_- \) and thus via this map we have also identified \( \mathcal{A}_F^+ \) with \( \mathcal{A}_F^+ \). There is a one to one affine correspondence between the set of even states of \( \mathcal{A}_P \) and \( \mathcal{A}_F \).

\( XY \) model \( H_{XY} = -\sum_k \sigma^k_x\sigma^{k+1}_x + \sigma^k_y\sigma^{k+1}_y \) is an even Hamiltonian and the KMS state \( \omega_{XY} \) at inverse temperature being unique \( \omega_{XY} \) is also an even state. Thus the low temperature limiting state is also even and the ground state being unique it is also pure and translation invariant. The unique ground state \( \omega_{XY} \) of \( XY \) model \([\text{AMa}] \) once restricted to \( \mathcal{A}_P^+ \) can be identified as restriction of a quasi-free state on \( \mathcal{A}_F \) to \( \mathcal{A}_P^+ \).

Now by a lemma of Antony Wassermann \([\text{Wa}, \text{p}496]\) we have the following: Since the closed real subspace \( \mathcal{K} = \{f \in \mathcal{H} : f(x) = \overline{f(x)}\} \) which we identify with the closed subspace generated by \( \{e_j : j \geq 1\} \) satisfies the condition \( \mathcal{K} + i\mathcal{K} \) dense in \( \mathcal{H} \) (we have equality) and \( \mathcal{K} \cap i\mathcal{K} = \{0\} \), we get \( \Omega \) is also cyclic and separating for \( \pi(\mathcal{A}_F^+)' \) i.e. \( \pi(\mathcal{A}_F^+)'\Omega = [\pi(\mathcal{A}_F^+)'\Omega] \) where \( (\mathcal{H}, \pi, \Omega) \) is the GNS space of \( (\mathcal{A}_F, \omega_{XY}) \), where \( \omega_{XY} \) is unique quasi-free state associated with unique ground state via Jordan map. Now going back to \( \mathcal{A}_P^+ \), we find \( \omega_{XY} \) on \( \mathcal{A}_P^+ \) is faithful.

Since the solution is known explicitly and thus it is not unreasonable to make it possible to compute mean entropy. Since \( \omega_{XY} \) is strongly correlated i.e. two point
correlation does not decay exponentially, it won’t be surprising if \( s(\omega_{XY}) > 0 \). Here we make a conjecture for a translation invariant pure state on \( B \), \( s(\omega) > 0 \) if and only if \( \omega_R \) is a type-III factor state.

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