The Anharmonic Correction in the Soliton Model of the Hyperons

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Abstract

We derive the anharmonic correction to the hyperon energy in the bound state version of the topological soliton model for the hyperons, and show that it represents a negative correction of at most 10\% to the energy of the bound heavy flavour two-meson system in the case of cascade hyperons. The main anharmonic correction arises from the mass term in the Lagrangian density. For large meson masses the consistency of the model requires that the anharmonic correction decreases as the inverse square root of the mass of the heavy flavour meson.
1. Introduction

The bound state version of the topological soliton model for the hyperons originally proposed by Callan and Klebanov [1] has been shown to provide a remarkably good description of the static properties of the hyperons in all heavy flavour generations [2-4]. The version of this model in which the soliton is stabilized by explicit vector meson fields [3] has been shown to provide a smooth interpolating model, which possesses chiral symmetry in the light flavour sectors and heavy quark symmetry [6] in the limit when the heavy meson mass grows beyond limit [7,8]. As the model thus has all the symmetries that are believed to be important in the low energy sector, it should be viewed as realistic enough to have predictive power - as e.g. in its prediction of the existence of nonstrange pentaquark states [9].

The original version of the bound state soliton model [1] is based on the direct extension of Skyrme’s Lagrangian density [10] to $SU(3)$, and an expansion to second order of the isodoublet kaon field which is treated as a quantum fluctuation. The resulting harmonic approximation leads to a wave equation for the kaon field, which has two bound states, the lower one of which describes the $\Lambda - \Sigma$ system ($P$-state) and the excited one the $\Lambda(1405)$ ($S$-state) [1,2,11]. The cascade and $\Omega^-$ particles are then constructed as multimeson states, which in the harmonic approximation do not interact. In this paper we investigate the validity of this harmonic approximation by carrying out the expansion in the meson field to quartic order. Because of the complicated form of the stabilizing term in the Lagrangian density of the Skyrme model [10] we employ the simpler alternate form proposed by Pari [12]. This alternative stabilizing term, which is quartic in the derivatives, is equivalent to Skyrme’s quartic stabilizing term at the level of $SU(2)$, but not in the case of $SU(3)$ fields. In the case of the kaons it has been shown that linear combinations of the two alternate quartic stabilizing terms lead to good predictions for the strange hyperon energies, although if the alternate (Pari) term is employed alone bound $\eta N$ states would appear, in contradiction with empirical findings [13]. As the kaon bound state energy is fairly insensitive to the choice of quartic stabilizing term we shall however here be content to consider only the simpler alternate quartic stabilizing term, as this should suffice for the purpose of estimating the anharmonic correction.
The anharmonic term of lowest order takes the form of a meson-meson interaction, which we treat in lowest order perturbation theory. The matrix elements of this interaction for two-meson hyperon states - i.e. the cascade particles and their heavy flavour analogs - represent the perturbative estimate of the anharmonic correction to the hyperon mass values, the quality of which should be comparable to the standard perturbation theory estimate of the electron-electron interaction energy in the case of the $He$-atom. We find that in the case of the cascade type hyperons the anharmonic correction amounts to a negative correction of at most 10% to the energy of the bound two meson system. In the case of the strange cascade hyperons the correction is -26 MeV, in the case of the $\Xi_c$ -233 MeV and in the case of the predicted $\Xi_b$ hyperons -639 MeV when standard values for the parameters of the model are used. We show that in the case of very large meson mass values the consistency of the model requires that the anharmonic correction is roughly proportional to the inverse square root of the heavy meson mass. The relative smallness of the anharmonic correction indicates that the harmonic approximation is reliable and that the satisfactory predictions that it gives for the hyperon spectra and magnetic moments are fairly robust.

This paper is organized into sections. In section 2 we derive the general form for the lowest anharmonic term in the Lagrangian density of the meson-soliton system. In section 3 we show that the matrix element of this term in the case of a state with two mesons in the ground state can be reduced to a fairly simple form. In section 4 we derive an approximate estimate for the dependence on the heavy flavour meson mass of the anharmonic correction. In section 5 we present the numerical results for the anharmonic correction to the cascade-type hyperons. Section 6 contains a summarizing discussion.

2. The anharmonic Hamiltonian density

2.1 The harmonic Lagrangian

In its simplest version the bound state model for the hyperons is based on a Lagrangian density for an $SU(3)$-valued field $U$ that has three components:
\[ \mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_S + \mathcal{L}_{CSB}. \] (2.1)

The first of these is the Lagrangian density of the non-linear \( \sigma \)-model:

\[ \mathcal{L}_\sigma = -\frac{1}{4} f_\pi^2 \text{Tr} L_\mu L^\mu, \] (2.2)

where the ”left current” \( L^\mu \) is defined as

\[ L^\mu = U^\dagger \partial^\mu U. \] (2.3)

The second term \( \mathcal{L}_S \) is a stabilizing term for which a variety of forms has been considered [10,12,14]. We shall here employ the form

\[ \mathcal{L}_S = \frac{1}{16e^2} \left\{ (\text{Tr} L_\mu L_\nu)^2 - (\text{Tr} L_\mu L^\mu)^2 \right\}, \] (2.4)

that was suggested by Pari [12] as a simpler, although \( SU(2) \) equivalent, alternative to Skyrme’s original quartic stabilizing term [10]. The third term \( \mathcal{L}_{CSB} \) takes into account the chiral symmetry breaking meson mass term and the \( SU(3) \) symmetry breaking implied by the different values for the decay constants of the mesons in the different flavour generations. For this term we employ the form [15]

\[ \mathcal{L}_{CSB} = \frac{f_\pi^2 m_\pi^2 + 2f^2 m^2}{12} \text{Tr} \{ U + U^\dagger - 2 \} \\
+ \frac{f_\pi^2 m_\pi^2 - f^2 m^2}{6} \text{Tr} \{ \sqrt{3} \lambda_8 (U + U^\dagger) \} \\
- \frac{f_\pi^2 - f^2}{12} \text{Tr} \{ (1 - \sqrt{3} \lambda_8)(U \partial_\mu U^\dagger \partial^\mu U + U^\dagger \partial_\mu U \partial^\mu U^\dagger) \}, \] (2.5)

where \( m \) represents the mass of the heavy flavour meson (\( K, D \) or \( B \)) and \( f \) the corresponding decay constant.

This Lagrangian density has to be augmented by the Wess-Zumino action

\[ S_{WZ} = -\frac{iN_C}{240\pi^2} \int d^5x \epsilon^{\mu\alpha\beta\gamma} \text{Tr} \{ L_\mu L_\nu L_\alpha L_\beta L_\gamma \}, \] (2.6)

which contributes to the energy when the field manifold is larger than \( SU(2) \), although it cannot be reduced to the form of a Lagrangian density [1,2].
$N_C$ is the colour number.

For the $SU(3)$ field $U$ we adopt the form [1]

$$U = \sqrt{U_\pi U_m} \sqrt{U_\pi},$$

(2.7)

where $U_\pi$ is the soliton field

$$U_\pi = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$ 

(2.8)

In this expression $u$ is the usual $SU(2)$ hedgehog field:

$$u = e^{i\vec{r} \cdot \hat{r} \theta(r)},$$

(2.9)

where $\theta$ is the chiral angle.

The heavy meson field $U_m$ has the form

$$U_m = \exp\{i\sqrt{2} f \left( \begin{array}{cc} 0 & M \\ M & 0 \end{array} \right) \}.$$ 

(2.10)

Here $M$ represents one of the meson isodoublets $(K^+, K^0)^T$, $(\bar{D}^0, D^-)^T$ or $(B^+, B^0)^T$ [4].

The bound state model for the hyperons is obtained by expanding the Lagrangian density (2.1) to second order in the meson field $M$. This harmonic approximation leads to a linear differential equation of second order for the meson field $M$, in the potential field of the $SU(2)$-soliton. The bound state solutions to this wave equation represent stable hyperon states.

In the harmonic approximation the interactions between the mesons in the hyperons, that are formed as multimeson states as the cascade and $\Omega^-$ particles, are neglected. We here carry out the expansion in $M$ to quartic order in order to obtain the lowest anharmonic correction to the Hamiltonian density for the meson field $M$. This takes the form of an effective meson-meson interaction, which we treat as a first order perturbation. The employment of the alternate quartic stabilizing term (2.2) is motivated by
the fact that it leads to a much simpler form for the anharmonic correction than Skyrme’s original stabilizing term. We show that the alternate term leads to binding energies for the heavy isodoublet pseudoscalar mesons, which are very close to those obtained with the Skyrme term, and hence it should suffice for the purpose of estimating the magnitude of the anharmonic correction. If a realistic description of the interaction between the soliton and the complete pseudoscalar nonet is required one has in general to consider a linear combination of the two quartic stabilizing terms [13].

In the expansion of the Lagrangian density (2.1) in powers of the meson field $M$ the lowest (zero) order term is the usual Lagrangian density of the $SU(2)$ soliton [16]. Because of the stability of the soliton only terms of even order in $M$ appear. To second order one obtains in the case of an unrotated soliton field [1,7] the following expressions:

\begin{equation}
\mathcal{L}^{(2)}_{\sigma} = \frac{1}{\chi^2} \partial_\mu M^\dagger \partial^\mu M + \frac{1}{\chi^2} M^\dagger \left\{ \left( \frac{2 \sin^2 \theta / 2}{r^2} \cos \theta + \frac{\theta^2}{4} \right) - \frac{\sin^2 \theta / 2}{r^2} \vec{\tau} \cdot \vec{E} \right\} M, \tag{2.11a}
\end{equation}

\begin{equation}
\mathcal{L}^{(2)}_S = \frac{1}{e^2 f^2} \left\{ (\theta^2 + \frac{\sin^2 \theta}{r^2}) \partial_\mu M^\dagger \partial^\mu M \right.
\left. + (\theta^2 - \frac{\sin^2 \theta}{r^2}) \vec{\tau} \cdot \nabla M^\dagger \vec{\tau} \cdot \nabla M \right.
\left. + M^\dagger \left[ 2 \cos \theta \frac{\sin^2 \theta / 2}{r^2} (\theta^2 + \frac{\sin^2 \theta}{r^2}) + \frac{\theta^2}{2} \sin^2 \theta \right.ight.
\left. - 2 \frac{\sin^2 \theta / 2}{r^2} (\theta^2 + \frac{\sin^2 \theta}{r^2}) \vec{\tau} \cdot \vec{E} \right\} M \right\}, \tag{2.11b}
\end{equation}

\begin{equation}
\mathcal{L}^{(2)}_{CSB} = -m^2 M^\dagger M + (\chi^2 - 1) \mathcal{L}^{(2)}_{\sigma}. \tag{2.11c}
\end{equation}

Here the factor $\chi$ is the ratio $f/f_\pi$. Finally the Wess-Zumino term (2.6) gives rise to the following contribution to the meson Lagrangian in the harmonic approximation [2]:

\begin{equation}
\mathcal{L}_{WZ} = i \frac{N_c}{4f^2} B^\mu (M^\dagger D_\mu M - (D_\mu M)^\dagger M). \tag{2.12}
\end{equation}
Here $B^\mu$ is the anomalous baryon current of the soliton [16], the time component of which has the explicit form

$$B^0 = -\frac{1}{2\pi^2 r^2} \sin^2 \theta \theta', \quad (2.13)$$

and $D_\mu$ is the "covariant" derivative [1]

$$D_\mu \equiv \partial_\mu + \frac{1}{2}(\sqrt{U^\dagger_\pi} \partial_\mu U_\pi + \sqrt{U_\pi} \partial_\mu \sqrt{U^\dagger_\pi}). \quad (2.14)$$

The second order Lagrangian densities (2.11), (2.12) do, upon a mode decomposition of the kaon field, lead to the following wave equation for the bound meson:

$$a(r) \nabla^2 k + b(r) \hat{r} \cdot \nabla k - c(r) \vec{L}^2 k - [v_0(r) + v_{IL}(r) \vec{I} \cdot \vec{L}] k - m^2 k + 2\omega \lambda(r) k + d(r)\omega^2 k = 0. \quad (2.15)$$

Here $\omega$ is the energy of the bound meson ($\omega < m$) and $k(\vec{r})$ is the meson wave function. The effective (iso)spin operator $\vec{I}$ for the system is defined as $\vec{I} = \vec{r} / 2$. The radial functions $a, ... \lambda$ in eqn. (2.15) are defined as

$$a(r) = 1 + \frac{2}{e^2 f^2} \frac{\sin^2 \theta}{r^2}, \quad (2.16a)$$

$$b(r) = -\frac{4}{e^2 f^2} \frac{\sin \theta}{r^2} \left(\frac{\sin \theta}{r} - \cos \theta \theta'\right), \quad (2.16b)$$

$$c(r) = \frac{1}{e^2 f^2 r^2} (\theta'^2 - \frac{\sin^2 \theta}{r^2}), \quad (2.16c)$$

$$d(r) = 1 + \frac{1}{e^2 f^2} (\theta'^2 + 2\frac{\sin^2 \theta}{r^2}), \quad (2.16d)$$

$$v_0(r) = -2\cos \theta \frac{\sin^2 \theta / 2}{r^2} - \frac{\theta'^2}{4} - \frac{1}{2e^2 f^2} \left(\theta'^2 \frac{\sin^2 \theta}{r^2} \right) + 4\frac{\sin^2 \theta / 2 \cos \theta}{r^2} \left(\theta'^2 + \frac{\sin^2 \theta}{r^2}\right), \quad (2.16e)$$

$$v_{IL}(r) = \frac{4\sin^2 \theta / 2}{r^2} + \frac{4}{e^2 f^2} \frac{\sin^2 \theta / 2}{r^2} (\theta'^2 + \frac{\sin^2 \theta}{r^2}), \quad (2.16f)$$
\[ \lambda(r) = -\frac{N_c \sin^2 \theta \theta'}{8\pi^2 f^2 r^2}. \]  

(2.16g)

The ground state solution to the meson wave equation (2.15) is a \( P \)-state. In the case of the kaon we obtain the ground state energy to be 195 MeV when \( f_K = 1.23f_\pi \), and the parameter values \( f_\pi \) and \( e \) are chosen as in ref. [16] so that the nucleon and \( \Delta_{33} \) resonance have their empirical mass values \( (f_\pi = 64.5 \text{ MeV}, e=5.45) \). This value for the energy of the bound \( K \) meson leads to satisfactory predictions for the masses of the stable strange hyperons [4].

In the case of the heavier flavour hyperons the alternate quartic stabilizing term (2.11b) does not lead to as satisfactory predictions for the energies of the charm and bottom hyperons if the decay constant ratios \( f_D/f_\pi \) and \( f_B/f_\pi \) are varied within the presently accepted uncertainty range. Unless these decay constant ratios are given fairly large values the \( D \) and \( B \) mesons will be predicted to be overbound by a significant amount. We shall here use the value \( f_D/f_\pi = 2.4 \), which gives the value \( \omega_D = 1226 \text{ MeV} \), which is about 100 MeV too small. If the value for \( f_D/f_\pi \) is reduced to the value 1.8 at the upper end of the empirical uncertainty range the predicted value for \( \omega_D \) drops to only 996 MeV, which is unrealistically low. In the case of the \( B \) meson we obtain \( \omega_B=3528 \text{ MeV} \) with the value \( f_B/f_\pi=2.8 \), which is somewhat above the presently accepted uncertainty range. This value for \( \omega_B \) is about 1 GeV below the more realistic value that is obtained with the original Skyrme model stabilizing term [4], which reinforces the conclusion that the alternate quartic stabilizing term should not be used for phenomenological predictions except in combination with Skyrme’s stabilizing term [13]. We nevertheless assume that it is sufficiently realistic to permit an estimate of the relative importance of the anharmonic corrections to the energy.

2.2 The anharmonic Hamiltonian density

We now turn to the derivation of the lowest order anharmonic correction to the Lagrangian density (2.11), (2.12) for the meson field by expanding the full Lagrangian density (2.1) to quartic order in the kaon fields. The quartic anharmonic correction that arises from the non-linear \( \sigma \)-model term (2.2) has the following form:
\[ \mathcal{L}_s^{(4)} = \frac{1}{6\chi^2 f^2} \left\{ \partial_\mu M^\dagger M \partial^\mu M + M^\dagger \partial_\mu M M^\dagger \partial^\mu M \\
- M^\dagger M \partial_\mu M^\dagger \partial^\mu M - M^\dagger \partial_\mu M \partial^\mu M^\dagger M \right\} \]

\[ - \frac{1}{12\chi^2 f^2} \left\{ 6 M^\dagger \vec{l} M^\dagger \vec{r} M + M^\dagger (\vec{l} \cdot \vec{r} + \vec{r} \cdot \vec{l}) M \right\} \]

\[ + M^\dagger M [M^\dagger (\vec{l} + \vec{r}) \cdot \vec{\nabla} M - \vec{\nabla} M^\dagger \cdot (\vec{l} + \vec{r}) M] \]

\[ + 3 M^\dagger (\vec{l} + \vec{r}) M \cdot [M^\dagger \vec{\nabla} M - \vec{\nabla} M^\dagger M]. \]

(2.17)

Here \( \vec{l} \) and \( \vec{r} \) are defined as the left and right currents for the square root of the soliton field:

\[ \vec{l} = \sqrt{U_\perp} \vec{\nabla} \sqrt{U_\parallel}, \]

(2.18a)

\[ \vec{r} = \sqrt{U_\parallel} \vec{\nabla} \sqrt{U_\perp}. \]

(2.18b)

The explicit forms of these are

\[ \vec{l} = i \left\{ \frac{\sin \theta}{2r} \vec{r} + \frac{\sin^2 \theta / 2}{r} \vec{r} \times \hat{r} + \left( \frac{\theta'}{2} - \frac{\sin \theta}{2r} \right) \vec{r} \cdot \hat{r} \right\}, \]

(2.19a)

\[ \vec{r} = -i \left\{ \frac{\sin \theta}{2r} \vec{r} - \frac{\sin^2 \theta / 2}{r} \vec{r} \times \hat{r} + \left( \frac{\theta'}{2} - \frac{\sin \theta}{2r} \right) \vec{r} \cdot \hat{r} \right\}. \]

(2.19b)

The quartic anharmonic correction that arises from the stabilizing term \( \mathcal{L}_S \) (2.4) can be expressed in the following form:

\[ \mathcal{L}_S^{(4)} = \frac{1}{4e^2 f_4} \left\{ (\theta'^2 + \frac{\sin^2 \theta}{r^2}) A_{\mu\nu} - (\theta'^2 - \frac{\sin^2 \theta}{r^2}) \hat{r}_m \hat{r}_n A_{mn} \right\} \]

\[ + \frac{1}{16e^2 f_4} \{ B_{\mu\nu} B^{\mu\nu} - (B_{\mu\mu})^2 \}. \]

(2.20)

Here we have introduced two tensors \( A_{\mu\nu} \) and \( B_{\mu\nu} \) which are defined as follows:

\[ A_{\mu\nu} = \frac{1}{3} \left\{ M^\dagger M (\partial_\mu M^\dagger \partial_\nu M + \partial_\nu M^\dagger \partial_\mu M) \right\} \]

\[ + \partial_\nu M^\dagger M M^\dagger \partial_\mu M + \partial_\mu M^\dagger M M^\dagger \partial_\nu M \]
\[-M^\dagger \partial_\nu MM^\dagger \partial_\mu M - M^\dagger \partial_\mu MM^\dagger \partial_\nu M\]
\[-\partial_\mu M^\dagger M \partial_\nu M^\dagger M - \partial_\nu M^\dagger M \partial_\mu M^\dagger M\]
\[-2M^\dagger \left(\frac{\cos \theta \sin^2 \theta / 2}{r^2} \delta_{mn} + \left(\frac{\theta'}{4} - \frac{\cos \theta \sin^2 \theta / 2}{r^2}\right) \hat{r}_m \hat{r}_n\right) MM^\dagger M\]
\[-6M^\dagger \left[\frac{\sin \theta}{2r^2} \tau_m + \left(\frac{\theta'}{2} - \frac{\sin \theta}{2r}\right) \vec{\tau} \cdot \hat{r} \hat{r}_m\right] MM^\dagger M\]
\[+6M^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_m MM^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_n M\]
\[-iM^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_m \nabla_n MM^\dagger M\]
\[+i\nabla_n M^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_m MM^\dagger M\]
\[+i\nabla_m M^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_n MM^\dagger M\]
\[+3iM^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_m M(\nabla_n M^\dagger M - M^\dagger \nabla_n M)\]
\[+3M^\dagger \kappa(r)(\vec{\tau} \times \hat{r})_n M(\nabla_m M^\dagger M - M^\dagger \nabla_m M)\],

and

\[B_{\mu\nu} = -2(\partial_\mu M^\dagger \partial_\nu M + \partial_\nu M^\dagger \partial_\mu M)\]
\[+M^\dagger (l_m r_n + r_m l_n + l_n r_m + r_m l_n) M\]
\[+2i\kappa(r)[M^\dagger (\vec{\tau} \times \hat{r})_m \partial_\nu M + M^\dagger (\vec{\tau} \times \hat{r})_n \partial_\mu M\]
\[-\partial_\mu M^\dagger (\vec{\tau} \times \hat{r})_n M - \partial_\nu M^\dagger (\vec{\tau} \times \hat{r})_m M]\]

The indices \(m\) and \(n\) in the expression for \(B_{\mu\nu}\) indicate the spatial components of \(B\). Above we have used the notation

\[\kappa(r) = \frac{\sin^2 \theta / 2}{r}.\]

Finally the charge symmetry breaking Lagrangian (2.5) gives rise to the following quartic anharmonic term:

\[\mathcal{L}^{(4)}_{CSB} = \frac{2}{3} \frac{m^2}{f^2} M^\dagger MM^\dagger M\]

\[-\frac{1}{2f^2} (1 - \frac{1}{\chi^2}) \left\{ \frac{4}{3} K^\dagger K \partial_\mu K^\dagger \partial^\mu K + \frac{4}{3} \partial_\mu K^\dagger KK^\dagger \partial^\mu K\right\}\]
The quartic anharmonic Hamiltonian density is obtained by introduction of conjugate momentum fields as

\[ \Pi = \frac{\partial L}{\partial \dot{M}}, \quad \Pi^\dagger = \frac{\partial L}{\partial \dot{M}^\dagger}, \]

and carrying out the Legendre transformation

\[ \mathcal{H} = \Pi^\dagger \dot{M} + \dot{M}^\dagger \Pi - L. \]

The resulting expression for the anharmonic term in \( \mathcal{H} \) is identical to that of the quartic Lagrangian densities (2.17), (2.20) and (2.24), with exception for an overall sign change of those terms that do not contain time derivatives of the meson field or its Hermitean conjugate.

The quartic Hamiltonian density has the form of meson-meson interaction. In general the interacting mesons can be in different states, although we shall here consider the case of both mesons in the ground state. As the ground state is a P-state it is described by a wave function of the form

\[ M = \frac{k(r)}{\sqrt{4\pi}} \vec{\tau} \cdot \hat{r}, \]

where \( k(r) \) denotes the radial wave function and \( A \) is the rotation operator, which transforms the meson isospin operator into an effective spin operator. As the soliton field is rotated by the same \( SU(2) \) rotation operator, i.e. \( u \to AuA^\dagger \), the isospin operator \( \vec{\tau} \) in the Lagrangian densities is converted into the spin operator of the meson as well. Note that the normalization condition for the meson wave function \( k(r) \) is

\[ 2 \int_0^\infty dr r^2 k^2(r)[\omega d(r) + \lambda(r)] = 1, \]

where \( \omega \) and \( \lambda \) are the meson mass and potential, respectively.
a result that is implied by the form of the harmonic Lagrangian for the meson.

The anharmonic correction to the energy of a two-meson state, which corresponds to a hyperon of the cascade type, is obtained as

\[ \Delta E^{(4)} = \int d^3r \mathcal{H}^4, \]  

(2.29)

where \( \mathcal{H}^4 \) is the Hamiltonian density of quartic order in \( M \). The expression for this Hamiltonian density has the general form

\[ \mathcal{H}^{(4)} = M^\dagger O_1 M M^\dagger O_2 M, \]  

(2.30)

where \( O_j \) are operators that depend on the chiral angle \( \theta \) of the soliton and the spin and radial coordinates of the mesons. In the matrix element (2.29) the meson fields \( M \) are replaced by the corresponding wave functions, which in the case of both mesons being in the ground state have the form (2.27). The general expression for the anharmonic correction (2.29) will then be

\[ \Delta E^{(4)} = \frac{2}{(4\pi)^2} \int d^3r r^2 k^4(r) \{ \omega^2 + \frac{1}{r^2} g(r) - \frac{\theta^2}{8} \}. \]  

(3.1)

Note that in the case when both mesons are in the same orbital symmetry they form a spin triplet state because they satisfy Bose statistics.

3. The anharmonic correction for cascade hyperons

In the case of two-meson states, when both mesons have the same flavour and are in the ground state (2.27) the expression for the matrix element of the anharmonic Hamiltonian derived above can be reduced to a very compact form, as in this case \( \vec{r} \cdot \vec{L} = -2 \) and \( \vec{\sigma}^1 \cdot \vec{\sigma}^2 = 1 \) as the mesons form a triplet state. The anharmonic correction that arises from the quadratic term in the Hamiltonian, which is obtained by converting \( \mathcal{L}^{(4)}_\sigma \) (2.17) into a corresponding Hamiltonian density is

\[ \Delta E^{(4)} = -\frac{1}{3\pi\chi^2 f^2} \int dr r^2 k^4(r) \{ \omega^2 + \frac{1}{r^2} g(r) - \frac{\theta^2}{8} \}, \]  

(3.1)
where the function \( g(r) \) is defined as

\[
g(r) = \cos \theta \cos^2 \frac{\theta}{2}.
\]  

(3.2)

The correction to the energy that arises from the quartic stabilizing term \( L_s^{(4)} \) takes the following form:

\[
\Delta E_S^{(4)} = \frac{1}{3 \pi e^2 f^4} \int dr k^2(r) \left\{ k^2(r) \left[ \frac{g(r)}{r^2} (3g(r) - \sin^2 \theta) + \frac{\theta'^2}{4} (2g(r) + \sin^2 \theta) \right] - 6k^2(r)g(r) \right\} \\
- \frac{\omega^2}{2} \left[ k^2(r) (3 \sin^2 \theta + \frac{7}{2} \theta^2 - 12g(r)) - 6r^2 k^2(r) \right].
\]  

(3.3)

Finally the chiral symmetry breaking term \( L_{CSB}^{(4)} \) yields the correction

\[
\Delta E_{CSB}^{(4)} = -\frac{1}{3 \pi f^2} \int dr r^2 k^2(r) \left\{ m^2 k^2(r) \\
- (1 - \frac{1}{\chi^2}) \left[ k^2(r) \left[ -\frac{5}{2} \omega^2 + \frac{3}{8} \theta'^2 - \frac{3}{r^2} g(r) \right] \\
- \frac{3}{2} k^2 \right] \right\}.
\]  

(3.4)

Note that all the coefficients of the wave functions in these expressions are nonsingular at the origin.

Numerically we find the anharmonic quartic correction to the energy of a two-kaon state to be -26 MeV (Table I). This represents only a 7% increase of the energy 390 MeV that is obtained in the harmonic approximation for the two meson state. The relative smallness of the anharmonic correction indicates that the harmonic approximation is sufficiently reliable to use in the study of hyperon structure and that thus the fairly satisfactory prediction for the spectra [4] and magnetic moments [18-20] of the strange hyperons have uncertainties of no more than 10% due to the anharmonic terms. Note
that the 10% correction in the meson energy corresponds to only a 2% correction to the predicted energies of the cascade particles, and therefore may be viewed as insignificant. The contributions $\Delta E_\sigma^{(4)}$, $\Delta E_S^{(4)}$ and $\Delta E_{CSB}^{(4)}$ are listed separately in Table I. Of the value -32 MeV for $\Delta E_{CSB}^{(4)}$ the mass term alone contributes -29 MeV, and it thus represents the dominant term.

The anharmonic correction to the energy of a two $D$-meson state is -233 MeV (Table I). As the energy of the bound two $D$-meson state is 2452 MeV this represents a correction of only 10 percent. In the case of the $B$-meson we find with $\chi = 2.8$ the bound state energy 7056 MeV. In this case the anharmonic correction to be -639 MeV and transfer less significant than in the case of the charmed cascade hyperon. The separate contributions to the total anharmonic correction are given in Table I.

The fact that the relative magnitude of the anharmonic correction does not increase with increasing meson mass may be understood with the help of the following argument. At large distances the meson wave function behaves as

$$k(r) = Ne^{-\sqrt{m^2-\omega^2}r},$$

where $N$ is a constant. In the case of the ground state (2.27) $k(r)$ is finite and nonzero at the origin. The following approximate form for the meson wave function

$$k(r) \simeq Ne^{-\sqrt{m^2-\omega^2}r},$$

has the correct behaviour at the origin and exponential tail at large distances. The normalization constant $N$ is determined by the normalization condition [1]

$$2 \int dr r^2 k^2(r)(\omega d(r) + \lambda(r)) = 1.$$  

For the wave function model (3.6) this implies that

$$N \sim \frac{(m^2 - \omega^2)^{3/4}}{\sqrt{\omega}}$$

(3.7)
when the meson mass and energy is large ($\omega >> \lambda(0)$). For large energies the main anharmonic contribution is obtained from those terms in (3.1), (3.3) and (3.4) which are proportional to $m^2$ (the mass term) and $\omega^2$. If the soliton functions these terms in the integrands are replaced by constants they yield the estimate

$$\Delta E^{(4)} \sim \frac{N^4 \omega^2}{f^4 (m^2 - \omega^2)^{3/2}}.$$  

(3.8)

Here we have used the fact that $\omega \sim m$ for large meson mass values. By (3.7) it then follows that

$$\Delta E^{(4)} \sim \frac{(m^2 - \omega^2)^{3/2}}{f^4}.$$  

(3.9)

In ref. [14] it has been shown that the consistency of the soliton model requires that

$$f \sim \sqrt{m},$$  

(3.10)

and in ref. [21] that to a good approximation for large meson mass values the meson energy is given as

$$\omega = \frac{m}{\sqrt{1 + \frac{3}{\chi^2}}}.$$  

(3.11)

Combination of these results yield the following scaling law for the anharmonic correction:

$$\Delta E \sim \frac{1}{\sqrt{m}},$$  

(3.12)

which shows that the anharmonic correction falls with meson mass and would become insignificant in the case of very large meson mass values.

4. Discussion

The results presented above for the anharmonic correction to the energy of a bound two-meson state show that it reduces the predicted energy in the
harmonic approximation for a two meson state by 10%. This indicates that the harmonic approximation is reliable for the strange, charm and bottom hyperon generations, and that the satisfactory results it leads to for the static properties of the heavy flavour hyperons are fairly stable. As the topological model implies a large colour number approximation its inherent accuracy should in any case not be expected to be better than about 10%. It should also be noted that small adjustments of the values of the decay constants that appear in the Lagrangian density can lead to shifts in the predicted meson energies that are much larger than 10% [3].

The numerical results in this work apply only to the model with the alternate quartic stabilizing term (2.4). But the qualitative conclusion that the anharmonic corrections to the hyperon energies are small is not restricted to this particular version of the soliton model. It has been shown [14] that models with alternate stabilizing terms give qualitatively similar results in the harmonic approximation. The consistency argument for the smallness of the anharmonic correction given in section 3 is also general.

The anharmonic terms do also in principle contribute to the energy of a single meson state in the form of self energy corrections. These corrections, which have not been considered here, require summation over all meson states and hence a method of regularizing the high energy terms in the sum. As the Lagrangian model (2.1) is constructed as an effective low energy interaction the evaluation of these self energy corrections lead outside the range of its validity and would require a systematic approach along the lines of chiral perturbation theory.
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Table I

Numerical values (in MeV) for the different anharmonic contributions to the energy of a 2-meson state with the mesons in the ground state.

|                   | K  | D   | B   |
|-------------------|----|-----|-----|
| $\Delta E^{(4)}_a$ | 3  | -6  | -21 |
| $\Delta E^{(4)}_S$ | 3  | -44 | 144 |
| $\Delta E^{(4)}_{CSB}$ | -32 | -183 | -762 |
| $\Delta E^{(4)}_{TOT}$ | -26 | -233 | -639 |