A new integrable two parameter model of strongly correlated electrons in one dimension

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Abstract

A new one-dimensional fermion model depending on two independent interaction parameters is formulated and solved exactly by the Bethe ansatz method. The Hamiltonian of the model contains the Hubbard interaction and correlated hopping as well as pair hopping terms. The density-density and pair correlations are calculated which manifest superconducting properties in certain regimes of the phase diagram.

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The study of strongly correlated fermion systems has attracted considerable attention during the last decade. In particular the discovery of high temperature superconductivity [1] has led to the investigation of various electronic mechanisms for an explanation of this phenomenon. In one space dimension several new integrable models have been found exhibiting different physical behaviour [2–9]. Notably the model with correlated hopping [5], the anisotropic $tJ$ model [7], and models with pair hopping processes [8] show dominant superconducting correlations.

Due to integrability many properties of these models can be found nonperturbatively by an exact solution on the basis of a Bethe ansatz [10–13] and conformal field theory [14,15]. On the other hand integrability imposes severe conditions on the interaction parameters of a model. Usually, integrable cases are restricted to discrete points or depend on one parameter only. Therefore, the study of competition of different interaction terms is difficult.

In this letter we report on a new integrable two parameter model of strongly correlated electrons unifying several previously known integrable systems such as the correlated hopping model and the $tJ$ model. The phase diagram is very rich with a crossover from a regime with dominating density-density correlations to a regime with dominating (superconducting) pair correlations.

Our starting point is a one-dimensional Hamiltonian with correlated hopping terms, with the Hubbard on site interaction, and with pair hopping processes

$$H = -\sum_{j,\sigma} (c_{j\sigma}^+ c_{j+1\sigma} + c_{j+1\sigma}^+ c_{j\sigma}) \exp \left[ -\frac{1}{2} (\eta - \sigma \gamma) n_{j,-\sigma} - \frac{1}{2} (\eta + \sigma \gamma) n_{j+1,-\sigma} \right] + \sum_j \left[ Un_{j\uparrow} n_{j\downarrow} + tp \left( c_{j\uparrow}^+ c_{j\downarrow}^+ c_{j+1\downarrow}^+ c_{j+1\uparrow} + h.c. \right) \right],$$

where $j$ denotes the sites and we use standard notation for fermion operators. Tight binding Hamiltonians of this type have been considered since a long time [16,17] and have been studied as effective one-band Hamiltonians for the description of cuprate superconductors, see [18,19] and references therein. In (1) we have included an anisotropy $\gamma$ in the correlated hopping term and also a pair hopping term $tp$. Obviously, the integrable Hubbard chain [11] is a special case of (1) with $\eta = \gamma = tp = 0$. For $\eta = \pm \gamma$ and $tp = U = 0$ we recover the correlated hopping model [5]. The exact solution for the particular case $\eta = \ln 2, \gamma = 0$ was studied in [9,20].

Here we shall establish the integrability of (1) under the condition

$$tp = U/2 = \varepsilon \left[ 2e^{-\eta} (\cosh \eta - \cosh \gamma) \right]^{1/2}, \quad \varepsilon = \pm 1,$$

leaving us with two continuous parameters $\eta$, $\gamma$ ($|\eta| > \gamma \geq 0$ for hermiticity), and the discrete sign $\varepsilon$ where $\varepsilon = \pm 1$ corresponds to repulsive and attractive pair processes, respectively. The exact solution for the eigenstates and eigenvalues of Hamiltonian (1) can be obtained from a Bethe ansatz. The structure of the Bethe ansatz equations follows from the solution of the two particle scattering problem. The nonvanishing elements of the $S$ matrix are

$$S^{11}_{11}(\lambda) = S^{22}_{22}(\lambda) = 1,$$

$$S^{11}_{22}(\lambda) = S^{22}_{11}(\lambda) = e^{\frac{i}{2}(\eta + \gamma)}.$$
where \( \lambda = \lambda_1 - \lambda_2 \), and the \( \lambda_j \) are suitable particle rapidities related to the momenta of the electrons by

\[
k(\lambda) = \begin{cases} \Theta(\lambda, a), & \varepsilon = +1, \\ \pi - \Theta(\lambda, a), & \varepsilon = -1, \end{cases}
\]

with the function \( \Theta \) and the parameter \( a \) defined by

\[
\Theta(\lambda, a) = 2 \arctan \left( \coth a \tan \lambda \right), \
a = \frac{1}{4} \left\{ \ln \left[ \frac{\sinh \frac{1}{2}(\eta + \gamma)}{\sinh \frac{1}{2}(\eta - \gamma)} \right] - \gamma \right\}.
\]

A necessary and sufficient condition for the applicability of the Bethe ansatz are the Yang-Baxter equations [10,21]. These equations are satisfied by (3), i.e. by the \( S \) matrix of (1) under the condition (2).

The Bethe ansatz equations are derived following the standard procedure by imposing periodic boundary conditions. Each state of the Hamiltonian is specified by a set of particle rapidities \( \lambda_j, j = 1, \ldots, N \), and a set of spin rapidities \( \Lambda_\alpha, \alpha = 1, \ldots, N_\downarrow \), where \( N \) is the total number of electrons and \( N_\downarrow \) the number of down spin electrons. All rapidities within a given set have to be different, corresponding to Fermi statistics. They have to satisfy the Bethe ansatz equations

\[
\sin \left( \lambda_j - i\alpha \right) = \prod_{\alpha=1}^{N} \frac{\sin \left( \lambda_j - \Lambda_\alpha + i\gamma/2 \right)}{\sin \left( \lambda_j - \Lambda_\alpha - i\gamma/2 \right)},
\]

\[
\sin \left( \Lambda_\alpha - \lambda_j + i\gamma/2 \right) = -\prod_{\beta=1}^{N} \frac{\sin \left( \Lambda_\alpha - \Lambda_\beta + i\gamma \right)}{\sin \left( \Lambda_\alpha - \Lambda_\beta - i\gamma \right)},
\]

where \( L \) is the number of lattice sites. The total energy and momentum of the system are given in terms of the particle rapidities \( \lambda_j \) as

\[
E = -2 \sum_{j=1}^{N} \cos k_j = 2\varepsilon \sum_{j=1}^{N} \left[ \cosh 2a - \frac{\sinh^2 2a}{\cosh 2a - \cos 2\lambda_j} \right],
\]

\[
P = \sum_{j=1}^{N} k(\lambda_j).
\]

Next we discuss the physical properties of the model. First consider the limit \( \eta \to -\infty, \varepsilon = +1 \). In this limit double occupations on the same site are excluded
and the remaining dynamics is identical to that of the supersymmetric $tJ$ model [2,3,7] as can be seen from the Bethe ansatz equations and also from a canonical transformation of the Hamiltonian

$$H_{\text{eff}} = e^A H e^{-A},$$

(8)

generated by $A = e^{-|\eta|/2}(d - d^+)$ where

$$d = \sum_{t < m, \sigma = \pm 1} (-1)^{t-m} P_{t+1,m} c^+_{t+1 \sigma} c_{t \sigma} (1 - n_{t,-\sigma}).$$

(9)

$P_{a,b}$ denotes the cyclic shift operator acting on the sites $a$, $a + 1$, ..., $b$, moving the state at site $a$ to site $b$. In first order of $A$ the effective Hamiltonian (8) does not involve interactions describing transitions between states with doubly occupied sites and those without doubly occupied sites. In the limit $\eta \to -\infty$ the resulting Hamiltonian is that of the supersymmetric $tJ$ Hamiltonian. A more detailed account will be given in a separate publication.

Secondly, a particle-hole transformation $T$ together with a sublattice rotation $(c^+_{j \sigma} \to (-1)^j c^+_{j \sigma})$ applied to $H(\eta)$ (1,2) yields a resulting Hamiltonian of the same form, namely

$$T H(\eta) T^{-1} = e^\eta H(-\eta) + (L - N)U.$$ 

(10)

Therefore we can restrict the further investigation to one specific sign of $\eta$. We choose $\eta > 0$ corresponding to repulsive (physical) correlated hopping of the electrons and positive values for $a$ (5).

For the groundstate the magnetization is zero, $N_\downarrow = N/2$, and the second set of equations in (2) can be solved for the $\Lambda_\alpha$ in terms of the $\lambda_j$ similar to the treatment in [5] (Appendix of the second paper). Inserting the result into the first set of equations in (2) and introducing the density function $\rho(\lambda)$ for the distribution of $\lambda_j$ in the thermodynamic limit, we obtain the linear integral equation

$$2\pi \rho(\lambda) = \frac{2 \sinh 2a}{\cosh 2a - \cos 2\lambda} + \int_I \varphi(\lambda - \mu) \rho(\mu) d\mu,$$

(11)

where

$$\varphi(\lambda) = 1 + 2 \sum_{n=1}^\infty e^{-n\gamma} \frac{\cos 2n\lambda}{\cosh n\gamma}.$$ 

(12)

In order to minimize the groundstate energy

$$\frac{E_0}{L} = 2\varepsilon \int_I \left[ \cosh 2a - \frac{\sinh^2 2a}{\cosh 2a - \cos 2\lambda} \right] \rho(\lambda) d\lambda,$$

(13)

the integration interval $I$ in (11,13) has to be chosen symmetrically around 0 ($I = [-K,+K]$) or $\pi/2$ ($I = [\pi/2 - K, \pi/2 + K]$), for $\varepsilon = +1$ or $-1$, respectively. The parameter $K$ is determined by the subsidiary condition for the total density $\rho = N/L$ of the electrons

$$\int_I \rho(\lambda) d\lambda = \rho.$$ 

(14)
Results for the groundstate energy per lattice site $E_0$ obtained by numerical solutions of these equations are presented in Figs. 1 and 2 for various values of $\eta$ and $\gamma$. For large values of the correlated hopping parameter $\eta$ the processes described by Hamiltonian (1) simplify drastically. The only single particle process allowed is the hopping of electrons from singly occupied sites to empty sites with hopping rate $t = -1$. In addition, pair hopping processes occur from doubly occupied sites to empty sites with parameters $t_p = U/2 = \varepsilon$. For $\varepsilon = +1$ the groundstate consists of only single electrons for densities $0 < \rho < 1$, with dependence of $E_0$ on $\rho$ as for free spinless fermions ($E_0 = -\frac{2}{\pi} \sin \pi \rho$). Upon further filling doubly occupied sites are created. However, the system freezes in as all hopping processes are suppressed. All states with arbitrary distributions of singly and doubly occupied sites are eigenstates of the Hamiltonian where the energy is just given by the Hubbard interaction. Therefore, $E_0$ depends linearly on $\rho$ for $1 < \rho < 2$, see Fig. 1a). For the case $\varepsilon = -1$ the formation of doubly occupied sites is favoured for all densities. The dynamics of the pairs is described by free spinless fermions ($E_0 = -\frac{2}{\pi} \sin \pi \rho^2 - \rho$) for all densities $\rho$, see Fig. 2a).

We next study the long distance behaviour of correlation functions by means of finite size studies and application of conformal field theory. For details of this method the reader is referred to [14,15,5]. There are two different kinds of excitations in model (1,2). One type of excitations corresponds to gapless particle-hole excitations, i.e. a redistribution of the $\lambda_j$ parameters, the second type consists of spin excitations with gap. The latter excitations become gapless for $\gamma = 0$ which situation will be studied in a separate publication. The central charge of model (1,2) is $c = 1$. The long-distance behaviour of the density-density and pair correlations is given by

$$\langle \rho(r)\rho(0) \rangle \simeq \rho^2 + A_1 r^{-2} + A_2 r^{-\alpha} \cos(2k_F r),$$

and

$$\langle c_{r\uparrow}^+ c_{r\downarrow}^+ c_{0\downarrow} c_{0\uparrow} \rangle \simeq B r^{-\beta},$$

where

$$2k_F = \pi \rho.$$  \hfill (17)

The exponents $\alpha$ and $\beta$ describing the algebraic decay are calculated from the dressed charge function $\xi(\lambda)$ which has to satisfy the integral equation

$$2\pi \xi(\lambda) = 2\pi + \int_I \varphi(\lambda - \mu) \xi(\mu) d\mu.$$  \hfill (18)

We obtain

$$\alpha = 1/\beta = \xi(K)^2/2$$

for $\varepsilon = +1$, and a similar equation with $K$ replaced by $\pi/2 + K$ for $\varepsilon = -1$.

The one-particle Green’s function $\langle c_{r\sigma}^+ c_{0\sigma} \rangle$ shows exponential decay since hole excitations have a mass gap. Consequently the momentum distribution function $\langle n_{k\sigma} \rangle$ is analytic in $k$, in particular there is no singularity at $k_F$.  

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The model does not show finite off-diagonal long-range order. However, we observe a longer range of the pair correlations in comparison to density-density correlations for certain regimes of interactions and particle density. Numerical results for the dependence of the exponent $\beta$ on the density $\rho$ are shown in Figs. 1 and 2. For all values of the interaction parameters $\eta$ and $\gamma$ there is a crossover from a regime with dominant density-density correlations ($1 < \beta < 2$) to a regime with dominant pair correlations ($\beta < 1$) at a “critical density” $\rho_c$. For repulsive pair processes ($\varepsilon = +1$) the size of regime “$\beta < 1$” is shrinking for increasing values of $\eta$ and decreasing values of $\gamma$ ($\rho_c \to 2$ for $\gamma \to 0$), see Figs. 1a) and b). For attractive pair processes ($\varepsilon = -1$) regime “$\beta < 1$” is growing for increasing values of $\eta$ and decreasing values of $\gamma$ ($\rho_c \to 0$ for $\gamma \to 0$ and $\eta \to \infty$), see Figs. 2a) and b).

In summary we have presented a new integrable model for strongly correlated electron systems depending on two free parameters. We have obtained exact results for the groundstate energy of the system as well as the critical exponents for correlation functions showing generically a crossover between regimes with dominant density-density correlations and pair correlations, respectively.

After completion of this work we learnt about the formal proof of integrability of model (1) in the particular case $\gamma = 0$ [22,23].

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**Figure captions**

Figure 1: a) Depiction of the density dependence of the groundstate energy per site $E_0$ and the exponent $\beta$ of the pair correlation function for values $\varepsilon = +1$, $\gamma = 0.5$, and $\eta = 0.6, 1, 2, 4$. b) Similar to a) for $\varepsilon = +1$, $\eta = 1$, and $\gamma = 0.1, 0.3, 0.5, 0.7, 0.9$.

Figure 2: Similar to 1a) and b) but for $\varepsilon = -1$. 
