Calculation of Stress Intensity Factor on Typical MSD Structure of Aircraft Wing

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Abstract. The research object of this paper is Multiple site damage (MSD) structures which are common in the service of aging aircraft. A complex function modified method is proposed to calculate the stress intensity factor of MSD cracks. On the basis of Muskhelishvili complex function method and its approximate superposition method, the concept of equivalent crack is introduced to solve the accurate stress function of MSD cracks. The stress intensity factor of MSD hole edge crack of infinite plate under far end uniform load is calculated and compared with the finite element method. The method is applied to the typical MSD structure of aircraft wing, and the stress intensity factors of cracks in three modes are compared and analyzed. This method can be used to discuss the damage tolerance of MSD structure, it has certain application value in the reliability evaluation of aircraft.

1. Introduction
The cracks on the edge of the hole are mostly found in the lap rivets of the outer skin of the wing of the aging aircraft. Multiple site damage (MSD) will significantly reduce the residual strength of the structure and bring great hidden danger to flight safety [1]. In order to evaluate the damage tolerance of MSD structure, the stress intensity factor (SIF) of crack should be calculated first, and then the crack growth and residual strength should be studied.

The stress intensity factors of non hole edge MSD cracks have been studied by many scholars [2-6], but the research on hole edge MSD cracks which are common in engineering practice is rare. In this paper, a complex function modification method is used to solve the stress intensity factor of infinite plate containing collinear hole edge MSD cracks under far end uniform load. It has higher accuracy than the traditional complex function method. When applied to the calculation of stress intensity factors of typical MSD structures of wings, a series of valuable conclusions can be obtained.

2. Basic Principle
2.1. Muskhelishvili complex function method
In the case of infinite multi connected bodies, the stress function can be expressed as
\[
\begin{align*}
\Phi(z) &= -\frac{1 + \mu}{8\pi} (X + iY) \ln z + Bz + \phi_0(z) \\
\Psi(z) &= -\frac{3 - \mu}{8\pi} (X - iY) \ln z + (B' + iC')z + \Psi_0(z)
\end{align*}
\]

(1)

Where \(X\) and \(Y\) are the sum of surface forces along the \(x\) and \(y\) directions on the finite boundary respectively, \(B, B'\) and \(C'\) are determined by the stress conditions at infinity.

In order to discuss the boundary conditions conveniently, the auxiliary plane is introduced. There is a transformation relationship between the physical plane \(z\) and the auxiliary plane \(\zeta\). After conformal mapping, the stress function in the plane can be expressed as

\[
\begin{align*}
\varphi(\zeta) &= -\frac{1 + \mu}{8\pi} (X + iY) \ln \zeta + B\omega(\zeta) + \varphi_0(\zeta) \\
\psi(\zeta) &= -\frac{3 - \mu}{8\pi} (X - iY) \ln \zeta + (B' + iC')\omega(\zeta) + \psi_0(\zeta)
\end{align*}
\]

(2)

The stress intensity factor of two-dimensional composite crack is [7]

\[
K = K_1 - iK_2 = 2\sqrt{2\pi} \lim_{\zeta \to \infty} \left\{ \left[ \omega(\zeta) - \omega(\zeta_0) \right] \frac{1}{2} \frac{\varphi' (\zeta)}{\omega' (\zeta)} \right\} = 2\sqrt{\pi} \varphi'(\zeta_0) / \sqrt{\omega''(\zeta_0)}
\]

(3)

### 2.2. Conformal mapping

Suppose an infinite plane with an asymmetric crack at the edge of a single hole, the radius of the circular hole is \(r\), and the crack length is \(b\) and \(c\), as shown in Figure 1. Taking the center of the circular hole as the origin and the straight line where the crack is located as the \(x\)-axis, the rectangular coordinate system is established, and the conformal mapping is performed [8]

\[
z = \omega(\zeta) = \frac{r}{4\zeta} \left\{ f(1 + \zeta^2) + g(1 - \zeta^2) + [(r^2 - 1)(1 + \zeta^2)^2 + 2(fg + 1)(1 - \zeta^2)^2 + (g^2 - 1)(1 - \zeta^2)^4]^{1/2} \right\}
\]

(4)

Where \(g = (r^2 + b^2)/2rb, h = (r^2 + c^2)/2rc\).

This mapping maps the conformal of an infinite plate with an asymmetric crack on the edge of a single hole on the physical plane to the outside of the unit circle on the mathematical plane. \(\omega^{-1}(b) \to 1, \omega^{-1}(-c) \to -1, \omega^{-1}(ri) \to B, \omega^{-1}(-ri) \to B'\). At the same time, the upper and lower shore of \(r\) are mapped to \(A\) and \(A_1\) respectively, and the upper and lower shore of \(-r\) are mapped to \(C\) and \(C_1\) respectively. Obviously, \(\omega(\zeta)\) has only one pole \(\zeta = \infty\).

\[
\text{Res}(\omega(\zeta), \infty) = -r(g + h)/2
\]

(5)

It is assumed that the infinite plate is uniformly loaded \(q\) in the \(y\)-axis direction and the orifice is not stressed, thus \(\bar{X} = Y = X = 0\), \(B = q'A, B' + iC' = q/2\). On the boundary of unit circle, \(\varphi(\zeta) = \frac{q}{4} \omega(\zeta) + \varphi_0(\zeta), \psi(\zeta) = \frac{q}{2} \omega(\zeta) + \psi_0(\zeta), f_s = \frac{q}{2} \omega(\sigma) - \frac{q}{2} \omega(\sigma)\). From Cauchy integral formula at infinity

\[
\varphi_0(\sigma) = \frac{1}{2\pi i} \int_{C} \frac{q}{2} \frac{\omega(\sigma) - \varphi_0(\sigma)}{\sigma - \zeta} d\sigma
\]

(6)

According to residue theorem...
\[
\varphi_0(\sigma) = -q \varphi(\zeta) + \frac{q}{\zeta} \text{Res}(\varphi(\zeta), \infty) = -q \varphi(\zeta) - q \frac{r(\sigma + h)}{2\zeta}
\]

\[
\varphi(\zeta) = -\frac{3q}{4} \varphi(\zeta) - \frac{q(r(\sigma + h))}{2\zeta}
\]

2.3. \textit{Approximate superposition of complex functions}

For MSD structure of infinite plate with arbitrary distribution shown in Figure 2, assuming the total number of cracks at the hole edge is \( M \), the stress function of crack A is obtained by using the approximate superposition method of complex function [9]

\[
\varphi(\zeta) = \sum_{j=1}^{M} \varphi_j(\zeta_j) \quad j = 1 \sim M
\]

Where \( \varphi_1 \) is the complex stress function of hole 1, \( \zeta_1 \) is the coordinate of crack A in the local coordinate of hole 1, and the stress function \( \varphi_j \) of other items represents the influence term of stress field intensity of corresponding hole crack A. Obviously, the stress function \( \varphi_j \) should satisfy the stress boundary conditions of each hole and the stress conditions of infinite distance. The complex stress function of multiple hole-edge crack problem is

\[
K = K_{1i} - K_{ii} = 2\sqrt{\pi} \sum_j \varphi'_j(\zeta_j) / \sqrt{\varphi''_j(\zeta_j)} \quad j = 1 \sim M
\]

3. \textit{Modified Method of the Complex Function}

The conformal mapping function used in the complex function method is a transcendental function that cannot be expanded into a finite term polynomial. It can be treated as a form that can participate in the calculation of the stress function by using the residue theorem, but this method will introduce some errors, resulting in the calculation result of the initial crack growth is smaller than the actual one. The key to eliminate the error is to obtain the exact mapping function of the crack to be solved. Therefore, the equivalent crack is introduced and the complex function method is proposed.

3.1. \textit{Equivalent crack}

By introducing the equivalent crack [10], the hole edge crack to be solved is equivalent to a central crack, and a new MSD structure shown in Figure 3 is constructed. The principle of equivalent crack length is that the stress intensity factors of the two cracks are equal. The crack length before and after equivalence is \( 2a \) and \( 2a' \). Then the stress intensity factor of crack a is

\[
K = q\sqrt{\mu} = f_q\sqrt{\mu}
\]

\[\text{Figure 1. Map asymmetrical crack emanating from the hole into the exterior of unit circle.} \quad \text{Figure 2. MSD structure on infinite plate with arbitrarily distributed holes.}\]
where \( f_h \) is the shape coefficient of crack initiation hole.

In order to keep the interference effect between cracks unchanged, the principle of equivalent crack location is that the distance between the crack tip \( A \) and the nearest hole is the same as the original distance.

The shape coefficient of crack initiation hole depends on the crack shape at the hole edge [11], as shown in Figure 4. In the case of a single crack at the edge of the hole

\[
f_h = \sqrt{1-r/a} \times \left[ 1 + \frac{1}{2x^2 + 1.93x + 0.539} \times \frac{1}{2(x+1)} \right] \times \frac{2+x}{2+2x} \times \left[ 1 + \frac{0.2x}{(1+x)^3} \right]
\]

In the case of symmetrical cracks at the hole edge

\[
f_h = \sqrt{1-r/a} \times \left[ 1 + \frac{1}{2x^2 + 1.93x + 0.539} \times \frac{1}{2(x+1)} \right]
\]

Where \( x = (a-r)/r \).

3.2. Modified complex function and its approximate superposition method

The equivalent crack is regarded as an ellipse with the long axis equal to the crack length \( 2a' \) and the short axis approximately equal to 0. Taking the center of ellipse as the origin and the straight line where the crack is located as the \( x \)-axis, the rectangular coordinate system is established, and the conformal mapping is performed.

\[
z_{eq} = \omega_{eq}(\zeta) = \frac{1}{2} a' (\zeta + \zeta^{-1})
\]

Using the complex function method, the exact stress function of the equivalent crack is obtained.

\[
\phi_{eq}(\zeta) = \frac{q}{8} a' (\zeta - 3\zeta^{-1})
\]

Because the effect of other holes on the equivalent crack does not affect the solution accuracy, that is, the superposition term of stress function does not need to change. Then the complex stress function of the multiple hole-edge crack shown in Figure 3 is

\[
K = K_I - iK_\omega = 2\sqrt{\pi} |\phi_{eq}(\zeta)| + \sum_j \phi_j(\zeta) \left[ \int \phi_{eq}^*(\zeta) \right] j = 2 \sim M
\]
4. Analysis of Typical MSD Structure of Wing

MSD cracks usually occur at the edge of the top row of rivet holes on the outer skin of the wing. As the rivets unload the load to the next row of rivets, the situation of the top row of rivet holes is similar to the situation of collinear MSD hole edge cracks of infinite plates under the far end uniform load [12], and the stress intensity factor analysis model under the typical MSD mode of the wing is established, as shown in Figure 5.

Select the crack spacing $2d=12r$ for calculation, and get the normalized SIF calculation results of each crack in MSD structure, as shown in Figure 6-8. In Figure 6, the calculation results of each crack in MSD-1 mode are compared with the finite element results.

![Figure 5. Three typical MSD modes.](image)

![Figure 6. SIF of cracks in Mode MSD-1.](image)

![Figure 7. SIF of cracks in Mode MSD-2.](image)

![Figure 8. SIF of cracks in Mode MSD-3.](image)

5. Conclusion

In this paper, a modified method of complex function is proposed to solve the stress intensity factor of infinite plate MSD structure. Compared with the finite element results, the method is simple and reliable. It can be used in the calculation of stress intensity factors of typical MSD structure of wing, which is helpful to understand the law of crack growth and judge the dangerous parts of fatigue structure, and is worth popularizing in engineering practice.
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References
[1] C.S. Wang, J.Y. Zhang, R. Bao, et al. Reliability analysis on structure with multiple site damage, Journal of Beijing University of Aeronautics and Astronautics. 32 (2006) 899-902.
[2] Y.Z. Chen, S.C. Chang. Analysis of two dimensional fracture problem with multiple cracks undermixed boundary conditions, Engineering Fracture Mechanics. 34 (1989) 921-934.
[3] J.H. Kuang, C.K. Chen. Equivalence for two interacting parallel cracks, ASME Journal of Pressure Vessel Technology. 120 (1998) 424-430.
[4] G.T. Sha. Stiffness Derivative finite element technique to determine nodal weight functions with singularity elements, Engineering Fracture Mechanics. 19 (1984) 685-699.
[5] T. Fett. Weight functions for plates with periodical edge cracks, International Journal of Fracture. 78 (1996) 45-52.
[6] X. Wang, Q. Glinka. Approximate weight functions for embedded elliptical cracks, Engineering Fracture Mechanics. 36 (1990) 267-285.
[7] N.I. Muskhelishvili. Some basic problems of mathematical theory of elasticity, Noordhoff International Publishing, Gorkingen, 1953.
[8] J.H. Guo, G.T. Liu. Stress Analysis of the Problem about a Circular Hole with Asymmetry Collinear Cracks, Journal of Inner Mongolia Normal University (Natural Science Edition). 36 (2007) 418-422.
[9] Y.Y. Zhang, H.D. Cai. Calculation of stress intensity factor for cracked plates with multiple hole edges, Journal of Shanghai Jiaotong University. 20 (1986) 77-86.
[10] D.P. Rooke. Simple methods of determining stress intensity factors, Engineering Fracture Mechanics. 14 (1981) 397.
[11] China Academy of Aeronautics, Manual of stress intensity factors. Science Press, Beijing, 1981.
[12] D.Z. Yu, Y.L. Chen, C.M. Duan. The evaluation of corrosion tolerant on the former girder of one aircraft, Journal of Naval Aeronautical Engineering Institute. 18 (2003) 411-413.