THE DIRICHLET PROBLEM FOR NONLOCAL OPERATORS
MATTHIEU FELSINGER, MORITZ KASSMANN, AND PAUL VOIGT

ABSTRACT. In this note we set up the elliptic and the parabolic Dirichlet problem for linear nonlocal operators. As opposed to the classical case of second order differential operators, here the “boundary data” are prescribed on the complement of a given bounded set. We formulate the problem in the classical framework of Hilbert spaces and prove unique solvability using standard techniques like the Fredholm alternative.

1. Introduction

Given an open and bounded set $\Omega \subset \mathbb{R}^d$ and functions $f : \Omega \to \mathbb{R}$ and $g : \partial \Omega \to \mathbb{R}$, the classical Dirichlet problem is to find a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega. \quad (1.1a)$$

A more general problem is to solve, instead of equation (1.1a), the equation

$$-\text{div} (A(\cdot)\nabla u) = f \quad \text{in } \Omega, \quad (1.2)$$

where $A(x) = (a_{ij}(x))$ is a $d \times d$-matrix which is uniformly positive definite and uniformly bounded. When working in Hilbert spaces a standard assumption is $f \in H^{-1}(\Omega) = (H^1_0(\Omega))^*$ and $g \in H^1(\Omega)$. $H^1_0(\Omega)$ denotes the Hilbert space of all $L^2(\Omega)$-functions with a distributional derivate in $L^2(\Omega)$ and $H^1(\Omega)$ is the closure of $C^\infty_c(\Omega)$ with respect to the norm of $H^1(\Omega)$. The Riesz representation Theorem resp. the Lax-Milgram Lemma imply that there is a unique function $u \in H^1(\Omega)$ such that $u - g \in H^1_0(\Omega)$ and for every $v \in H^1_0(\Omega)$

$$(A(\cdot)\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v \rangle. \quad (1.3)$$

This is one way to solve the Dirichlet problem. Note that the bilinear form on the left-hand-side of (1.3) is not necessarily symmetric. An important further extension is given when one considers additional terms of lower order in (1.2).

The aim of this article is to provide a similar set-up for the Dirichlet problem associated with nonlocal operators. Our main objects are nonlocal operators $\mathcal{L}$ of the form

$$(\mathcal{L}u)(x) = \lim_{\varepsilon \to 0^+} \int_{y \in \mathbb{R}^d \setminus B_\varepsilon(x)} (u(x) - u(y))k(x, y) \, dy, \quad (1.4)$$

where $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is measurable. Our focus is on kernels $k$ with the following properties: (i) $k$ is not necessarily symmetric, (ii) $k$ might be singular on the diagonal and

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(iii) $k$ is allowed to be discontinuous. These properties imply that $\mathcal{L}u$ cannot be evaluated in the classical sense at a given point $x \in \mathbb{R}^d$ even if $u$ is smooth.

Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and $f \in L^2(\Omega)$. Given a function $g : \mathbb{C}\Omega \to \mathbb{R}$, we want to find a solution $u : \mathbb{R}^d \to \mathbb{R}$ to the problem

\[
\begin{align*}
\mathcal{L}u &= f \quad \text{in } \Omega \\
u &= g \quad \text{on } \mathbb{C}\Omega.
\end{align*}
\] (1.5a)

In order to do so, we need to answer the following questions: Which is a natural space for the data $g$ and for the solution $u$? In which sense is equation (1.5a) to be interpreted? Note that, opposed to the classical case of second order differential operators, here the data $g$ are prescribed on the complement of a given bounded set. The term “complement value problem” would be more appropriate for (1.5) than “boundary value problem”.

**Remark 1.1.** It makes sense to study – instead of (1.4) – nonlocal operators of the form

\[
(\mathcal{L}u)(x) = \lim_{\varepsilon \to 0^+} \int_{y \in \Omega \setminus B_\varepsilon(x)} (u(x) - u(y))k(x, y) \, dy.
\]

In that case it would not make sense to describe data on $\mathbb{C}\Omega$. For a large class of kernels one could study associated boundary value problems with boundary data $g : \partial \Omega \to \mathbb{R}$. Note that such operators appear as generators of so-called censored processes. We do not study these cases in this article.

In this work we establish a Hilbert space approach to the nonlocal Dirichlet problem (1.5). We prove results on existence and uniqueness of solutions under various assumptions on the kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ which we assume to be measurable. The symmetric and anti-symmetric parts of $k$ are defined by

\[
k_s(x, y) = \frac{1}{2} (k(x, y) + k(y, x)) \quad \text{and} \quad k_a(x, y) = \frac{1}{2} (k(x, y) - k(y, x)).
\]

Clearly, $|k_a(x, y)| \leq k_s(x, y)$ for almost all $x, y \in \mathbb{R}^d$. Throughout this article we assume that the symmetric part of the kernel satisfies the following integrability condition:

\[
x \mapsto \int_{\mathbb{R}^d} \left( 1 \wedge |x - y|^2 \right) k_s(x, y) \, dy \in L^1_{\text{loc}}(\mathbb{R}^d).
\] (L)

We call a kernel $k$ **integrable** if, for every $x \in \mathbb{R}^d$ the quantity $\int_{\mathbb{R}^d} k_s(x, y) \, dy$ is finite and the mapping $x \mapsto \int_{\mathbb{R}^d} k_s(x, y) \, dy$ is locally integrable. We call a kernel **non-integrable** if it is not integrable in the sense above. In this work we deal with integrable and non-integrable kernels at the same time. In Section 6 we provide several characteristic examples. A simple integrable example is given by $k(x, y) = \mathbf{1}_{B_1}(x - y)$, a simple non-integrable example by $k(x, y) = |x - y|^{-d - 1}$, both kernels being symmetric.

For a certain class of kernels $k$ the Lax-Milgram Lemma turns out to be a good tool whereas for another class the Fredholm alternative is more appropriate. Rather than going through all assumptions in detail, for this introduction let us restrict ourselves to an example in the simple setting where $\Omega$ equals the unit ball $B_1 \subset \mathbb{R}^d$.

**Example 1.** Assume $0 < \beta < \frac{d}{2} < 1$. Let $I_1, I_2$ be arbitrary nonempty open subsets of $S^{d-1}$ with $I_1 = -I_1$. Set $\mathcal{C}_j = \{ h \in \mathbb{R}^d \left| \frac{h}{|h|} \in I_j \right. \}$ for $j \in \{1, 2\}$ and

\[
k(x, y) = |x - y|^{-d - \alpha} \mathbf{1}_{\mathcal{C}_1}(x - y) + |x - y|^{-d - \beta} \mathbf{1}_{\mathcal{C}_2}(x - y) \mathbf{1}_{B_1}(x - y).
\]
The part involving $|x - y|^{-d-\beta}$ can be seen as a lower order perturbation of the main part of the kernel resp. integro-differential operator produced by $|x - y|^{-d-\alpha}1_{C_1}(x - y)$. We discuss this example in detail in Section 6. Define complement data $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = \begin{cases} \left((|x| - 1)^\gamma \right) & \text{if } 1 \leq |x| \leq 2, \\ 0, & \text{else.} \end{cases}$$

where $\gamma$ is an arbitrary real number satisfying $\gamma > \frac{\alpha - 1}{2}$. Note that $g$ may be unbounded. Let $f \in L^2(\Omega)$ be arbitrary. For such a kernel $k$ and such data $g$ and $f$ our results imply that the Dirichlet problem (1.5) for $\Omega = B_1 \subset \mathbb{R}^d$ has a unique variational solution $u$.

Another interesting example to which our theory applies, is given by $k(x, y) = b(x)|x - y|^{-d-\alpha(x)}$ under certain assumptions on the functions $b : \mathbb{R}^d \to (0, \infty)$ and $\alpha : \mathbb{R}^d \to (0, 2)$, see Example 14 in Section 6.

In Corollary 5.5 we provide an existence result for parabolic equations (see (5.9)) where the operator contains a time-dependent kernel $k_t$ and where the complement data and the initial value are prescribed by a single function $h(t, x)$. This result covers the case $k_t = k$ and $h(t, x) = g(x)$ with $g$ and $k$ as above.

Let us comment on related results in the literature. It is remarkable that, although the questions and the framework of our studies are quite basic, the results have not been established yet. One reason might be that integro-differential operators not satisfying the transmission property have been studied only recently when considered in bounded domains. On the other hand, the proof of the weak maximum principle for nonsymmetric nonlocal operators, Theorem 4.1, turns out to be quite tricky.

To our best knowledge, the first profound study of a nonlocal Dirichlet problem for integro-differential operators violating the transmission property is provided in [6, Ch. VII] using balayage spaces and the method of Perron, Wiener and Brelot. A detailed study of nonlocal Dirichlet problems including a Hilbert space approach can be found in [13]. Section 6 of [13] addresses the questions of this article but assumes the bilinear forms to be symmetric. As the proofs show, symmetry of $k$ simplifies the situation greatly. Symmetry is not assumed in the previous sections of [13] but there, the symbol (i.e. the generator) is used which would lead to strong assumptions on the kernel $k$. Note that in our framework, typically the integro-differential operator cannot be evaluated pointwise, even when applied to smooth functions.

The Dirichlet problem for nonlocal operators is studied for fully nonlinear problems in [4] using viscosity solutions. There, the complement data are chosen independently from the kernels which is very different from our approach where, for every $k$ there is an appropriate function space for the data $g$.

Variational solutions to nonlocal problems have been already considered by groups working in the theory of peridynamics, e.g. in [1, 8, 5, 20, 21]. This theory describes a nonlocal continuum model for problems involving discontinuities or other singularities in the deformation. The corresponding integration kernels are often integrable but non-integrable kernels are considered too. What we call “complement data” in this article is called “volume constraints” in articles on peridynamics. An important tool in the articles above are appropriate Poincaré-Friedrichs inequalities. It occurs that Lemma 2.7, in the case of scalar functions, is more general than corresponding Poincaré-Friedrichs inequalities in the aforementioned articles. A major difference of our approach is that we study general (non-symmetric) kernels and non-zero complement data together with appropriate function spaces.
We do not at all discuss regularity of solutions. Regularity up to the boundary is studied carefully in [22] for the case of the fractional Laplacian.

Results on nonsymmetric nonlocal Dirichlet forms are related to our work, see our discussion of \( (K) \) below. We refer the reader to [14], [17], [11] and [23]. We believe that our examples can be helpful when studying Hunt processes generated by nonsymmetric nonlocal (semi-)Dirichlet forms.

The article is organized as follows. In Section 2 we define the function spaces needed for our approach and explain their basic properties. We also define what a solution of \( (1.5) \) shall be. Section 2.3 is dedicated to nonlocal versions of the Poincaré-Friedrichs inequality. In Section 3 we prove a Gårding inequality, comment on the sector condition and apply the Lax-Milgram Lemma to the nonlocal Dirichlet problem. Section 4 is devoted to the weak maximum principle for integro-differential operators in bounded domains. This tool is applied when using the Fredholm alternative in Section 4.2. Our existence and uniqueness results are extended to the parabolic Dirichlet problem in Section 5. Finally, in Section 6 we provide many detailed examples of kernels \( k \) and discuss their properties. Although, formally, this section could be omitted, it presumably is one of the most important one for every reader.

A short index with all conditions used in this work is given at the end of Section 6 on p. 28.

## 2. Function spaces and variational solutions

In this section we derive a weak resp. variational formulation of \( (1.5) \). We start with the definition of the relevant function spaces.

### 2.1. Function spaces.

**Definition 2.1** (Function spaces). Let \( \Omega \subset \mathbb{R}^d \) be open and assume that the kernel \( k \) satisfies (L). We define the following linear spaces:

1. \( L^p_\Omega(\mathbb{R}^d) = \{ u \in L^p(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{C} \Omega \} \).
2. Define

\[
V(\Omega;k) = \left\{ v : \mathbb{R}^d \rightarrow \mathbb{R} : v|_\Omega \in L^2(\Omega), (v(x) - v(y)) k^{1/2}(x,y) \in L^2(\Omega \times \mathbb{R}^d) \right\},
\]

\[
[u,v]_{V(\Omega;k)} = \int_{\Omega \times \mathbb{R}^d} [u(x) - u(y)][v(x) - v(y)] k(x,y) \, dy \, dx.
\]

A seminorm on \( V(\Omega;k) \) is given by \( [v,v]_{V(\Omega;k)} \). Let us emphasize that \( \Omega = \mathbb{R}^d \) is allowed in this definition. In this case we write \( V(\mathbb{R}^d;k) = H(\mathbb{R}^d;k) \) and a norm – which is obviously induced by a scalar product – on this space is defined by

\[
\|v\|^2_{H(\mathbb{R}^d;k)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k(x,y) \, dy \, dx.
\]

3. \( H_\Omega(\mathbb{R}^d;k) = \{ u \in H(\mathbb{R}^d;k) : u = 0 \text{ a.e. on } \mathbb{C} \Omega \} \) endowed with the norm \( \| \cdot \|_{H(\mathbb{R}^d;k)} \).

**Remark 2.2.**

- a) It is clear from this definition that

\[
(H_\Omega(\mathbb{R}^d;k), \| \cdot \|_{H(\mathbb{R}^d;k)}) \hookrightarrow (H(\mathbb{R}^d;k), \| \cdot \|_{H(\mathbb{R}^d;k)}) \tag{2.1}
\]

and \( H(\mathbb{R}^d;k) \subset V(\Omega;k) \). Moreover, if \( g \in V(\Omega;k) \) and \( g = 0 \text{ a.e. on } \mathbb{C} \Omega \), then \( g \in H_\Omega(\mathbb{R}^d;k) \).
b) In the case \(k(x, y) = \alpha(2 - \alpha)|x - y|^{-d - \alpha}, \alpha \in (0, 2)\), we write

\[
\mathcal{E}^\alpha(u, v) = [u, v]_{H(\mathbb{R}^d; k)}.
\]

In this case we recover the standard fractional Sobolev-Slobodeckij spaces and the norms are equivalent to the standard norms given on these spaces. More precisely, we obtain \(H^{\alpha/2}(\mathbb{R}^d) = H(\mathbb{R}^d; k)\) and if \(\Omega\) is a Lipschitz domain we have (cf. [19, Theorem 3.33])

\[
H_\Omega(\mathbb{R}^d; k) = H_0^{\alpha/2}(\Omega) \quad (= \text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^{\alpha/2}(\mathbb{R}^d)}) .
\]

If additionally \(\alpha \neq 1\) then

\[
\text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^{\alpha/2}(\mathbb{R}^d)} = \text{completion of } C_c^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{H^{\alpha/2}(\Omega)} .
\]

The following example illustrates that finiteness of the seminorm on \(V(\Omega; k)\) requires some regularity of the function across \(\partial\Omega\):

**Example 2.** Let \(\Omega = B_1(0), \alpha \in (0, 2)\) and define \(k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)\) by

\[
k(x, y) = \alpha(2 - \alpha)|x - y|^{-d - \alpha}.
\]

In this case, \(H(\mathbb{R}^d; k)\) coincides with the fractional Sobolev space \(H^{\alpha/2}(\mathbb{R}^d)\) and we have \(\mathcal{E}^k(u, v) = \mathcal{E}^\alpha(u, v)\) for \(u, v \in H(\mathbb{R}^d; k)\).

Define \(g : \mathbb{R}^d \to \mathbb{R}\) by

\[
g(x) = \begin{cases} (|x| - 1)^\beta & \text{if } 1 \leq |x| \leq 2, \\ 0, & \text{else.} \end{cases}
\]

We show that \(g \in V(\Omega; k)\) if and only if \(\beta > \frac{\alpha - 1}{2d}\). Since \(\alpha\) is fixed we omit the factor \(\alpha(2 - \alpha)\). Then

\[
[g, g]_{V(\Omega; k)} = \iint_{B_1 \times \mathbb{R}^d} (g(x) - g(y))^2 |x - y|^{-d - \alpha} \, dx \, dy = \iint_{B_1 \setminus B_2 \setminus B_1} ((|x| - 1)^2)^\beta |x - y|^{-d - \alpha} \, dx \, dy
\]

\[
= \int_{B_1 \setminus B_1} (|x| - 1)^2 \int_{B_1} |x - y|^{-d - \alpha} \, dy \, dx .
\]

For \(1 < |x| < 2\) choose \(\xi = \left(\frac{3 - |x|}{2|x|}\right) x\). Then we may estimate

\[
\int_{B_1} |x - y|^{-d - \alpha} \, dy \geq \int_{B_1 \setminus B(x)} |x - y|^{-d - \alpha} \, dy \geq \int_{B_1 \setminus B(|x| - 1/2)(\xi)} (2(|x| - 1))^d |x - y|^{-d - \alpha} \, dy
\]

\[
\geq |B_1| \left(\frac{|x| - 1}{2}\right)^d (2(|x| - 1))^{-d - \alpha} .
\]

Thus \([g, g]_{V(\Omega; k)} \geq C \int_{B_2 \setminus B_1} ((|x| - 1)^2)^\beta \, dx\) for some constant \(C = C(d) > 0\). This integral is finite if \(2\beta - \alpha > -1\). On the other hand, for \(x \in B_2 \setminus B_1\), we have

\[
\int_{B_1} |x - y|^{-d - \alpha} \, dy \leq \int_{B_2 \setminus B_1} |x - y|^{-d - \alpha} \, dy \leq C' \text{ dist}(x, \partial B_1)^{-\alpha} .
\]
Therefore, \( [g, g]_{V(\Omega; k)} \leq C' \int_{B_2 \setminus B_1} (|x| - 1)^{2\beta - \alpha} \, dx \), which shows that \( g \in V(\Omega; k) \) if and only if \( \beta > \frac{\alpha - 1}{2} \).

In the second order case (\( \alpha = 2 \)) the function \( g \) has a trace on \( \partial \Omega \) if and only if \( \beta > \frac{1}{2} \). We note that \( g \notin H^s(\mathbb{R}^d) \) for \( s > \frac{1}{2} \) because of the discontinuity at \( |x| = 2 \).

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^d \) be an open set. The spaces \( H_\Omega(\mathbb{R}^d; k) \) and \( H(\mathbb{R}^d; k) \) are separable Hilbert spaces.

**Proof.** The proof follows the argumentation in [24, Theorem 3.1].

First we show the completeness of \( H(\mathbb{R}^d; k) \). Let \( (f_n) \) be a Cauchy sequence with respect to the norm \( \| \cdot \|_{H(\mathbb{R}^d; k)} \). Set

\[
v_n(x, y) = (f_n(x) - f_n(y))\sqrt{k_s(x, y)}.
\]

Then, by definition of \( \| \cdot \|_{H(\mathbb{R}^d; k)} \) and the completeness of \( L^2(\mathbb{R}^d) \), \( (f_n) \) converges to some \( f \) in the norm of \( L^2(\mathbb{R}^d) \). We may chose a subsequence \( f_{n_k} \) that converges a.e. to \( f \). Then \( v_{n_k} \) converges a.e. on \( \mathbb{R}^d \times \mathbb{R}^d \) to the function

\[
v(x, y) = (f(x) - f(y))\sqrt{k_s(x, y)}.
\]

By Fatou’s Lemma,

\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x) - f(y)|^2 k_s(x, y) \, dx \, dy \leq \liminf_{k \to \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v_{n_k}(x, y)|^2 \, dx \, dy \leq \sup_{k \in \mathbb{N}} \|v_{n_k}\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.
\]

Since \( (v_n) \) is a Cauchy sequence (and hence bounded) in \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \), this shows that \( [f, f]_{H(\mathbb{R}^d; k)} < \infty \), i.e. \( f \in H(\mathbb{R}^d; k) \). Another application of Fatou’s Lemma shows that

\[
[f_{n_k} - f, f_{n_k} - f]_{H(\mathbb{R}^d; k)} = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v_{n_k}(x, y) - v(x, y)|^2 \, dx \, dy
\]

\[
\leq \liminf_{l \to \infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v_{n_k}(x, y) - v_{n_l}(x, y)|^2 \, dx \, dy \xrightarrow{k \to \infty} 0.
\]

This shows that \( \|f_{n_k} - f\|_{H(\mathbb{R}^d; k)} \to 0 \) for \( k \to \infty \) for the subsequence \( (n_k) \) chosen above and thus \( \|f_n - f\|_{H(\mathbb{R}^d; k)} \to 0 \) as \( n \to \infty \), since \( (f_n) \) was assumed to be a Cauchy sequence. The completeness of \( H(\mathbb{R}^d; k) \) follows immediately.

The mapping \( \mathcal{I} \)

\[
\mathcal{I} : H(\mathbb{R}^d; k) \to L^2(\Omega) \times L^2(\Omega \times \mathbb{R}^d),
\]

\[
\mathcal{I}(f) = \left( f, (f(x) - f(y))\sqrt{k_s(x, y)} \right),\tag{2.2}
\]

is isometric due to the definition of the norm in \( H(\mathbb{R}^d; k) \). Having shown the completeness of \( H(\mathbb{R}^d; k) \) we obtain that \( \mathcal{I}(H(\mathbb{R}^d; k)) \) is a closed subspace of the Cartesian product on the right-hand side of (2.2). This product is separable, which implies (cf. [24, Lemma 3.1]) the separability of \( H(\mathbb{R}^d; k) \).

Hence, \( H(\mathbb{R}^d; k) \) is separable and so is \( H_\Omega(\mathbb{R}^d; k) \) being a subspace of \( H(\mathbb{R}^d; k) \). \qed
2.2. Variational formulation of the Dirichlet problem. Define a bilinear form by

\[ \mathcal{E}^k(u, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))v(x)k(x, y) \, dy \, dx. \] (2.3)

In order to prove well-posedness of this expression and that the bilinear form is associated to \( \mathcal{L} \), we need to impose an condition on how the symmetric part of \( k \) dominates the anti-symmetric part of \( k \). We assume that there exist a symmetric kernel \( k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty] \) with \(|\{y \in \mathbb{R}^d | \tilde{k}(x, y) = 0, k_a(x, y) \neq 0\}| = 0\) for all \( x \), and constants \( A_1 \geq 1, A_2 \geq 1 \) such that

\[ \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 \tilde{k}(x, y) \, dx \, dy \leq A_1 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_s(x, y) \, dx \, dy \] (\( \tilde{K}_1 \))

for all \( u \in H(\mathbb{R}^d; k) \), and at the same time

\[ \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{k_s^2(x, y)}{k(x, y)} \, dy \leq A_2. \] (\( \tilde{K}_2 \))

A natural choice is \( \tilde{k} = k_s \) because in this case \( (\tilde{K}_1) \) trivially holds and \( \tilde{k}(x, y) = 0 \) implies \( k_a(x, y) = 0 \). Assumption \( (\tilde{K}) \) would then reduce to the condition

\[ \sup_{x \in \mathbb{R}^d} \int_{\{k_s(x, y) \neq 0\}} \frac{k_s^2(x, y)}{k_s(x, y)} \, dy \leq A. \] (K)

Condition (K) appears in [23, (1.1)] and is sufficient for that \((\mathcal{E}, C^0_c(\mathbb{R}^d))\) extends to a regular lower bounded semi-Dirichlet form. Note that our assumption \((\tilde{K})\) is weaker and thus we can extend [23, Thm 1.1], see Lemma 2.4. In Section 6 we provide an example illustrating the difference between \((\tilde{K})\) and (K).

Let us show that the bilinear form defined in (2.3) is associated to \( \mathcal{L} \) and that the integrand in \((2.3)\) is – in contrast to the integrand in \((1.4)\) – integrable in the Lebesgue sense.

Lemma 2.4. Let \( \Omega \subset \mathbb{R}^d \) open and assume that \( k \) satisfies \((L)\) and \((\tilde{K})\). Define for \( n \in \mathbb{N} \) the set \( D_n = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d: |x - y| > 1/n\} \) and

\[ \mathcal{L}_n u(x) = \int_{\mathcal{L}B_{1/n}(x)} (u(x) - u(y))k(x, y) \, dy, \]

\[ \mathcal{E}^k_n(u, v) = \iint_{D_n} (u(x) - u(y))v(x)k(x, y) \, dy \, dx. \]

Then we have \((\mathcal{L}_n u, v)_{L^2(\mathbb{R}^d)} = \mathcal{E}^k_n(u, v)\) and \( \lim_{n \to \infty} \mathcal{E}^k_n(u, v) = \mathcal{E}^k(u, v) \) for all \( u, v \in C^\infty_c(\Omega) \). Moreover, \( \mathcal{E}^k : H(\mathbb{R}^d; k) \times H(\mathbb{R}^d; k) \to \mathbb{R} \) is continuous. By (2.1), \( \mathcal{E}^k \) is also continuous on \( H_\Omega(\mathbb{R}^d; k) \).

As mentioned above, our proof is an extension of the proof of [23, Theorem 1.1].

Proof. Assume \( u, v \in C^\infty_c(\mathbb{R}^d) \). Splitting \( k \) in its symmetric and antisymmetric part yields

\[ (\mathcal{L}_n u, v)_{L^2(\mathbb{R}^d)} = \iint_{\mathbb{R}^d \times \mathcal{L}B_{1/n}(x)} (u(x) - u(y))k(x, y) \, dy \, v(x) \, dx \]
The first integral is finite due to (L). In order to show the integrability of the second integrand we use (K) with $A = \max(A_1, A_2)$ and the Cauchy-Schwarz inequality:

$$
\begin{align*}
\int_{D_n} |u(x) - u(y)| |v(x)| |k_a(x, y)| \, dy \, dx \\
&\leq \left( \int_{D_n} |u(x) - u(y)|^2 \tilde{k}(x, y) \, dx \, dy \right)^{1/2} \left( \int_{\mathbb{R}^d} \int_{B_{1/n}(x)} \frac{k_a^2(x, y)}{\tilde{k}(x, y)} \, dy \, dx \right)^{1/2} \\
&\leq A \left( \int_{D_n} |u(x) - u(y)|^2 k_a(x, y) \, dx \, dy \right)^{1/2} \|v\|_{L^2(\mathbb{R}^d)}.
\end{align*}
$$

This shows $(\mathcal{L}_n u, v)_{L^2(\mathbb{R}^d)} = \mathcal{E}_n^k(u, v)$ and that all expressions in this equality are well-defined. In particular, by dominated convergence $\lim_{n \to \infty} \mathcal{E}_n^k(u, v) = \mathcal{E}^k(u, v)$. Moreover, $\mathcal{E}^k(u, v) < \infty$ for $u, v \in H(\mathbb{R}^d; k)$. Now let us prove the continuity of $\mathcal{E}: H(\mathbb{R}^d; k) \times H(\mathbb{R}^d; k) \to \mathbb{R}$. Let $u, v \in H(\mathbb{R}^d; k)$. Again by the symmetry of $k_a$ and by (K) we obtain

$$
\begin{align*}
|\mathcal{E}^k(u, v)| &= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y)) v(x) k_a(x, y) \, dy \, dx \right| \\
&\leq \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y)) v(x) k_a(x, y) \, dy \, dx \right| \\
&\quad + \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y)) \tilde{k}(x, y)^{1/2} |v(x)| k_a(x, y) \tilde{k}^{-1/2} \, dy \, dx \right| \\
&\leq \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)| |v(x) - v(y)| k_a(x, y) \, dy \, dx \right| \\
&\quad + \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)| \tilde{k}(x, y)^{1/2} |v(x)| k_a(x, y) \tilde{k}^{-1/2} \, dy \, dx \right| \\
&\leq \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_a(x, y) \, dy \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (v(x) - v(y))^2 k_a(x, y) \, dy \, dx \right)^{1/2}
\end{align*}
$$
\[ A \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dy \, dx \right)^{1/2} \leq C \| u \|_{H(\mathbb{R}^d; k)} \| v \|_{H(\mathbb{R}^d; k)} \cdot \]

This shows that \( E^k \) is a continuous bilinear form on \( H(\mathbb{R}^d; k) \) and on \( H_\Omega(\mathbb{R}^d; k) \).

Finally, we are able to provide a variational formulation of the Dirichlet problem (1.5) with the help of the bilinear form \( E^k \):

**Definition 2.5.** Assume (L) and (K). Let \( \Omega \) be open and bounded, \( f \in H^*_0(\mathbb{R}^d; k) \).

(i) \( u \in H_\Omega(\mathbb{R}^d; k) \) is called a solution of

\[
\begin{cases}
Lu = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

if

\( E^k(u, \varphi) = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H_\Omega(\mathbb{R}^d; k) \). \tag{2.4}

(ii) Let \( g \in V(\Omega; k) \). A function \( u \in V(\Omega; k) \) is called a solution of

\[
\begin{cases}
Lu = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]

if \( u - g \in H_\Omega(\mathbb{R}^d; k) \) and (2.4) holds.

In the subsequent sections we will show how to solve this problem.

**Remark 2.6.** If \( C^\infty_c(\Omega) \) is dense in \( H_\Omega(\mathbb{R}^d; k) \) then \( H^*_0(\mathbb{R}^d; k) \) is a space of distributions on \( \Omega \). In this case solutions in the sense of Definition 2.5 are weak solutions to (D0) and (D), respectively.

### 2.3. Poincaré-Friedrichs inequality

Let us formulate a nonlocal version of the Poincaré-Friedrichs inequality in our set-up:

There exists a constant \( C_P > 0 \) such that for all \( u \in L^2_\Omega(\mathbb{R}^d) \)

\[
\| u \|^2_{L^2(\Omega)} \leq C_P \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dx \, dy. \tag{P}
\]

This inequality appears as an assumption, explicitly or implicitly, in all of our existence results. Below, we provide sufficient conditions on \( k_s \) for (P) to hold. Note that (P) may hold for integrable kernels as well as for non-integrable kernels \( k \), see the table in Section 6.

The following result generalizes the Poincaré-Friedrichs inequalities from [1] and [2, Prop. 1], respectively, to a larger class of integrable and non-integrable kernels. (In these references, the Poincaré-Friedrichs inequality is stated for functions with values in \( \mathbb{R}^d \).)

**Lemma 2.7.** Let \( \Omega \subset \mathbb{R}^d \) open and bounded and let \( k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) be measurable. Assume that there is a symmetric, a.e. nonnegative function \( L \in L^1(\mathbb{R}^d) \) satisfying the following properties: \(|\{L > 0\}| > 0 \) and there is \( c_0 > 0 \) such that for all \( u \in L^2(\Omega) \)

\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k(x,y) \, dy \, dx \geq c_0 \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 L(x - y) \, dy \, dx. \tag{2.5}
\]
Then the following Poincaré–Friedrichs inequality holds: There is \( C_P = C_P(\Omega, c_0, L) > 0 \) such that for all \( u \in H_\Omega(\mathbb{R}^d; k) \)

\[
\|u\|_{L^2(\mathbb{R}^d)}^2 \leq C_P \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k(x, y) \, dy \, dx.
\] (2.6)

The example in [3, Remark 6.20] shows that Lemma 2.7 fails to hold if one replaces the domain of integration \( \mathbb{R}^d \times \mathbb{R}^d \) by \( \Omega \times \Omega \) in (2.5) and (2.6).

For the proof of the Poincaré–Friedrichs inequality, we need the following technical Lemma taken from [9, Lemma 10].

**Lemma 2.8.** Let \( q \in L^1(\mathbb{R}^d) \) be nonnegative almost everywhere and let \( \text{supp} \, q \subset B_\rho(0) \) for some \( \rho > 0 \). Then the following Poincaré-Friedrichs inequality holds: There is \( 0 < \rho_0 < \rho \)

\[
\|u\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{B_R} \int_{B_R} (u(x) - u(y))^2 (q * q)(x - y) \, dy \, dx \leq 4 \|q\|_{L^1(\mathbb{R}^d)} \int_{B_{R+\rho}} \int_{B_{R+\rho}} (u(x) - u(y))^2 q(x - y) \, dy \, dx.
\]

**Proof of Lemma 2.7.** Let \( L \) satisfy the assumptions of the lemma. Without loss of generality \( \Omega \subset 0 \) (otherwise shift \( \Omega \)). Furthermore, we may assume that there is \( \rho > 0 \) such that \( \text{supp} \, L \subset B_\rho(0) \) (otherwise replace \( L \) by \( L \mathbb{1}_{B_\rho(0)} \)). Fix \( R > 0 \) such that \( \Omega \subset B_R(0) \). For \( \nu \in \mathbb{N} \) define

\[
L_\nu = L \ast L \ast \ldots \ast L,
\]

By the properties of \( L \) we have

\[
(L \ast L)(0) = L_1(0) = \int_{\mathbb{R}^d} L(z) L(-z) \, dz = \int_{\mathbb{R}^d} L^2(z) \, dz > 0
\]

and \( L_1 = L \ast L \in C_0(\mathbb{R}^d) \), which implies that we may find \( \delta > 0 \) (depending on \( L \)) such that \( L_1 > 0 \) on \( B_\delta(0) \). By the property of the convolution there is \( m \in \mathbb{N} \) depending on \( L \) and \( \Omega \) such that \( L_m > 0 \) on \( B_R(0) \). Let \( u \in H_\Omega(\mathbb{R}^d; k) \). Then we may estimate

\[
E_{BR}^{L_m}(u, u) := \int_{B_R} \int_{B_R} (u(x) - u(y))^2 L_m(x - y) \, dy \, dx
\]

\[
\geq \int_\Omega u^2(x) \int_{\Omega \cap B_R} L_m(x - y) \, dy \, dx \geq C(L, \Omega) \|u\|_{L^2(\mathbb{R}^d)}^2.
\] (2.7)

Iterated application of Lemma 2.8 (with \( \rho' = 2^{m-1} \rho \) and \( q = L_j, j = m - 1, \ldots, 0 \)) yields

\[
E_{BR}^{L_m}(u, u) \leq 4 \|L_{m-1}\|_{L^1(\mathbb{R}^d)} E_{BR+\rho'}^{L_{m-1}}(u, u) \leq \ldots \leq 4^m E_{BR+m\rho'}^{L_1}(u, u) \prod_{j=0}^{m-1} \|L_j\|_{L^1(\mathbb{R}^d)}.
\] (2.8)

(2.7), (2.8) and the assumption (2.5) imply

\[
\|u\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{C(L, \Omega)} E_{BR}^{L_m}(u, u) \leq \frac{4^m}{C(L, \Omega)} \prod_{j=0}^{m-1} \|L_j\|_{L^1(\mathbb{R}^d)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 L(x - y) \, dy \, dx
\]

\[
\leq \frac{4^m}{c_0 C(L, \Omega)} \prod_{j=0}^{m-1} \|L_j\|_{L^1(\mathbb{R}^d)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k(x, y) \, dy \, dx.
\]

This finishes the proof of Lemma 2.7.
Theorem 3.5. Given the statement about the independence of \( u \in L^2(\mathbb{R}^d) \) such that \( \alpha \) nonnegative almost everywhere. We assume that for some \( \alpha \in (0,2) \), some \( \lambda > 0 \) and all \( u \in L^2(\mathbb{R}^d) \) the kernel \( k \) satisfies (E\( a \)) (see p. 16). Then there is \( C_P > 0 \) such that for all \( u \in L^2(\mathbb{R}^d) \)

\[
\|u\|_{L^2(\mathbb{R}^d)} \leq C_P \int_{\mathbb{R}^d} (u(x) - u(y))^2 k(x,y) \, dy \, dx.
\] (2.9)

Given \( \alpha_0 \in (0,2) \) and \( \alpha \in [\alpha_0,2) \), the constant \( C_P \) can be chosen independently of \( \alpha \).

The proof of the main assertion is simple. The statement about the independence of \( C_P \) on \( \alpha \) is proved in [18, Theorem 1].

3. Gårding inequality and Lax-Milgram Lemma

In this section we discuss basic properties of the bilinear form \( \mathcal{E}^k \) which can be used in order to prove solvability of the Dirichlet problem. First, we establish a Gårding inequality under the conditions (L) and (\( \tilde{K} \)). Then we comment on the sector condition. In a second subsection we show that if, in addition, the Poincaré-Friedrichs inequality and a certain cancellation property hold, the bilinear form \( \mathcal{E}^k \) is positive definite and coercive. This allows to establish a first existence result with the help of the well-known Lax-Milgram Lemma, Theorem 3.5.

3.1. Gårding inequality.

Lemma 3.1 (Gårding inequality). Let \( k \) satisfy (L) and (\( \tilde{K} \)). Then there is \( \gamma = \gamma(A_1, A_2) > 0 \) such that

\[
\mathcal{E}^k(u,u) \geq \frac{1}{4} \|u\|_{H(\mathbb{R}^d; k)}^2 - \gamma \|u\|_{L^2(\mathbb{R}^d)}^2 \quad \text{for all } u \in H(\mathbb{R}^d; k).
\] (3.1)

Proof. Let \( u \in H(\mathbb{R}^d; k) \) and let \( A = \max\{A_1, A_2\} \). By (\( \tilde{K} \)) we obtain

\[
\mathcal{E}^k(u,u) \geq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dy \, dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(u(x) - u(y))u(x)k_a(x,y)| \, dy \, dx
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dy \, dx
\]

\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (u(x) - u(y))\tilde{k}^{1/2}(x,y)u(x)k_a(x,y)\tilde{k}^{-1/2}(x,y) \right| \, dy \, dx
\]

\[
\geq \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dy \, dx
\]

\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \varepsilon |u(x) - u(y)|^2 \tilde{k}(x,y) + \frac{1}{4\varepsilon} u^2(x)k^2_a(x,y)\tilde{k}^{-1}(x,y) \right] \, dy \, dx
\]

\[
\geq \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_s(x,y) \, dy \, dx - \frac{1}{4\varepsilon} A \|u\|_{L^2(\mathbb{R}^d)}^2
\]

\[
\geq \frac{1}{4} \|u\|_{H(\mathbb{R}^d; k)}^2 - \gamma \|u\|_{L^2(\mathbb{R}^d)}^2,
\]
if we choose \( \varepsilon \) sufficiently small such that \( A\varepsilon < \frac{1}{4} \) and then \( \gamma = \gamma(A) \) sufficiently large. \( \square \)

**Remark 3.2.** As the above proof shows, assumption \((\tilde{K}_2)\) is tailor-made for an estimate which shows that \( \mathcal{E}^{k_{\alpha}} \) is dominated by \( \mathcal{E}^{k_{\mathrm{e}}} \). Thus, another consequence of \((\mathrm{L})\) and \((\tilde{K})\) is the estimate

\[
|\mathcal{E}^{k_{\alpha}}(u, v)| \leq \left( \int (u(x) - u(y))^2 \cdot \tilde{k}(x, y) \, dx \, dy \right)^{1/2} \left( \int v^2(x)k_{\alpha}^2(x, y)\tilde{k}^{-1}(x, y) \, dx \, dy \right)^{1/2}
\]

\[
\leq \sqrt{A_1} \sqrt{\mathcal{E}^{k_{\alpha}}(u, u)} \sqrt{A_2} \|v\|_{L^2}
\]

for functions \( u, v \in H(\mathbb{R}^d; k) \). This observation implies that the bilinear forms \( (\mathcal{E}^k, H(\mathbb{R}^d; k)) \) and \( (\mathcal{E}^k, H_{\Omega}(\mathbb{R}^d; k)) \) satisfy a sector condition under an additional assumption, see Proposition 3.6 below.

3.2. **Application of the Lax-Milgram Lemma.** To verify that the bilinear form \( \mathcal{E}^k \) is positive definite, we assume the following cancellation condition:

\[
\inf_{x \in \mathbb{R}^d} \liminf_{\varepsilon \to 0^+} \int_{B_{\varepsilon}(x)} k_a(x, y) \, dy \geq 0. \tag{C}
\]

**Remark 3.3.** As the proof below shows, assumption \((C)\) can be relaxed. It is sufficient to assume

\[
\inf_{x \in \mathbb{R}^d} \liminf_{\varepsilon \to 0^+} \int_{B_{\varepsilon}(x)} k_a(x, y) \, dy > -\frac{1}{2C_P},
\]

with \( C_P \) as in \((P)\). The bilinear form \( \mathcal{E}^k \) would still be coercive. \( \odot \)

There are many cases for which condition \((C)\) holds. If \( k_a(x, y) \) depends only on \( x - y \), then for every \( x \in \mathbb{R}^d \) and every \( \varepsilon > 0 \) one obtains \( \int_{B_{\varepsilon}(x)} k_a(x, y) \, dy = 0 \) which trivially implies \((C)\).

But there are also many interesting cases for which condition \((C)\) is not satisfied, see Section 6.

**Proposition 3.4.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded. Let \( f \in H_{\Omega}(\mathbb{R}^d; k) \) and let \( k \) satisfy \((\mathrm{L})\), \((\mathrm{P})\), \((\tilde{K})\) and \((C)\). Then there is a unique solution \( u \in H_{\Omega}(\mathbb{R}^d; k) \) to \((\mathrm{D}_0)\).

**Proof.** In Lemma 2.4 it was shown that \( \mathcal{E}^k \) is a continuous bilinear form on \( H_{\Omega}(\mathbb{R}^d; k) \). First, we show that \((C)\) implies that \( \mathcal{E}^k \) is positive definite. Let \( u \in H_{\Omega}(\mathbb{R}^d; k) \). Observe that \( k = k_s + k_a \) and for every \( \varepsilon > 0 \)

\[
\iint_{\{|x-y|>\varepsilon\}} (u(x) - u(y))u(y)k_{\alpha}(x, y) \, dy \, dx = \frac{1}{2} \iint_{\{|x-y|>\varepsilon\}} (u(x) - u(y))(u(x) + u(y))k_{\alpha}(x, y) \, dy \, dx
\]

\[
= \frac{1}{2} \iint_{\{|x-y|>\varepsilon\}} u^2(x)k_{\alpha}(x, y) \, dy \, dx - \iint_{\{|x-y|>\varepsilon\}} u^2(y)k_{\alpha}(x, y) \, dy \, dx
\]

\[
= \int_{\mathbb{R}^d} u^2(x) \int_{B_{\varepsilon}(x)} k_{\alpha}(x, y) \, dy \, dx.
\]
From (C) we obtain
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))u(y)k_a(x, y) \, dy \, dx = \lim_{\varepsilon \to 0} \iint_{\{|x-y| > \varepsilon\}} (u(x) - u(y))u(y)k_a(x, y) \, dy \, dx \geq 0.
\]

Hence,
\[
\mathcal{E}^k(u, u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))u(x)k(x, y) \, dy \, dx \geq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_4(x, y) \, dy \, dx,
\]

i.e. \( \mathcal{E}^k(u, u) \geq 0 \) for all \( u \in H(\mathbb{R}^d; k) \). By (P) and (3.2)
\[
\mathcal{E}^k(u, u) \geq \frac{1}{4C_P} \|u\|_{L^2(\Omega)}^2 + \frac{1}{4} [u, u]_{H(\mathbb{R}^d; k)} \geq \frac{1}{4C_P} \|u\|_{H(\mathbb{R}^d; k)}^2,
\]

which shows that \( \mathcal{E}(u, u) \) is coercive.

By the Lax-Milgram Lemma, there is a unique \( u \) in \( H^k_0(\mathbb{R}^d) \), such that
\[
\mathcal{E}^k(u, \varphi) = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H^k_0(\mathbb{R}^d).
\]

Next, we show that the Dirichlet problem with suitable complement data \( g \) has also a unique solution.

**Theorem 3.5.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded and let \( k \) satisfy (L), (P) and (C). Assume further that there is \( \bar{k} \) such that
\[
\iint_{\Omega \times \mathbb{R}^d} (u(x) - u(y))^2 \bar{k}(x, y) \, dx \, dy \leq A_1 \iint_{\Omega \times \mathbb{R}^d} (u(x) - u(y))^2 k_4(x, y) \, dx \, dy.
\]

for all \( u \in V(\Omega; k) \) and (3.2) holds true. Then (D) has a unique solution \( u \in V(\Omega; k) \). Moreover,
\[
[u, u]_{V(\Omega; k)} \leq C \left( \|f\|_{H^s(\mathbb{R}^d; k)}^2 + [g, g]_{V(\Omega; k)} \right), \quad (3.3)
\]

where \( C = C(C_P, A_1, A_2) \) is a positive constant.

**Proof:** To prove the theorem we show that under the above assumptions on \( g \) the problem (D) can be transformed into a problem of the form (D). If \( \bar{u} \in H(\mathbb{R}^d; k) \) is a solution to
\[
\begin{aligned}
\mathcal{L}\bar{u} &= f - Lg & \text{in } \Omega \\
\bar{u} &= 0 & \text{on } \partial\Omega
\end{aligned}
\]

then \( u = \bar{u} + g \) belongs to \( V(\Omega; k) \) and solves (D). In order to apply Proposition 3.4 to (3.4) it remains to show that \( \mathcal{L}g = \mathcal{E}^k(g, \cdot) \in H^s(\mathbb{R}^d; k) \). We have
\[
|\mathcal{E}^k(g, \varphi)| \leq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (g(x) - g(y))\varphi(x)k_4(x, y) \, dy \, dx \\
\leq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (g(x) - g(y))(\varphi(x) - \varphi(y))k_4(x, y) \, dx \, dy + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (g(x) - g(y))\varphi(x)k_a(x, y) \, dx \, dy \\
=: I_1 + I_2.
\]
Since \( \varphi = 0 \) a.e. on \( \Omega \), an application of the Cauchy-Schwarz inequality yields

\[
I_1 \leq \left( \int_{\mathbb{R}^d} (g(x) - g(y))^2 k_s(x,y) \, dy \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} (\varphi(x) - \varphi(y))^2 k_s(x,y) \, dy \, dx \right)^{1/2} = [g,g]_{V(\Omega;k)}^{1/2} [\varphi, \varphi]_{H(\mathbb{R}^d;k)}^{1/2}.
\]

The term \( I_2 \) can be estimated as follows: By \( (\tilde{K}_1) \) and \( (\tilde{K}_2) \)

\[
I_2 \leq \int_{\mathbb{R}^d} \max\{0, (g(x) - g(y))^2 \} \tilde{k}^{{1}/2}(x,y) \max\{0, |\varphi(x)|, |k_s(x,y)|\} \tilde{k}^{{-1}/2}(x,y) \, dx \, dy
\]

\[
\leq \left( \int_{\Omega} (g(x) - g(y))^2 \tilde{k}(x,y) \, dx \, dy \right)^{1/2} \left( \int_{\Omega} \varphi^2(x) \frac{k^2_a(x,y)}{k(x,y)} \, dx \, dy \right)^{1/2}
\]

\[
\leq A_1^{1/2} A_2^{1/2} [g,g]^{1/2}_{V(\Omega;k)} \| \varphi \|^2_{L^2(\Omega)}.
\]

This shows the continuity of \( E(g,\cdot): H_\Omega(\mathbb{R}^d;k) \to \mathbb{R} \) and hence (3.4) has a unique solution \( \tilde{u} \in H_\Omega(\mathbb{R}^d;k) \).

In order to prove estimate (3.3) we apply \( \tilde{u} \in H_\Omega(\mathbb{R}^d;k) \) as test function and obtain

\[
\langle f, \tilde{u} \rangle_{H^1_\Omega(\mathbb{R}^d;k)} + \mathcal{E}^k(g,\tilde{u}) = \mathcal{E}^k(\tilde{u},\tilde{u}) = \int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y)) \tilde{u}(x) k_s(x,y) \, dx \, dy
\]

\[
= \int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y)) \tilde{u}(x) k_s(x,y) \, dx \, dy + \int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y)) \tilde{u}(x) k_a(x,y) \, dx \, dy,
\]

where the second term on the right-hand side is non-negative due to (C). Hence,

\[
\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y))^2 k_s(x,y) \, dx \, dy \leq \langle f, \tilde{u} \rangle_{H^1_\Omega(\mathbb{R}^d;k)} + \mathcal{E}^k(g,\tilde{u})
\]

\[
\leq \| f \|^2_{H^1_\Omega(\mathbb{R}^d;k)} \| \tilde{u} \|^2_{H^1_\Omega(\mathbb{R}^d;k)} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (g(x) - g(y)) \tilde{u}(x) k_s(x,y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (g(x) - g(y)) \tilde{u}(x) k_a(x,y) \, dx \, dy.
\]

The Young inequality and the fact that \( v = 0 \) a.e. on \( \Omega \) imply for \( \varepsilon > 0 \), to be specified later,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (g(x) - g(y)) \tilde{u}(x) k_s(x,y) \, dx \, dy \leq \int_{\Omega} (g(x) - g(y)) \| \tilde{u}(x) - \tilde{u}(y) \| k_s(x,y) \, dx \, dy
\]

\[
\leq \frac{1}{4\varepsilon} \int_{\Omega} (g(x) - g(y))^2 k_s(x,y) \, dx \, dy + \varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\tilde{u}(x) - \tilde{u}(y)|^2 k_s(x,y) \, dx \, dy.
\]
Similarly, using \((\tilde{K}_1')\) and \((\tilde{K}_2)\) we obtain
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (g(x) - g(y))\tilde{u}(x)\tilde{u}(y) \, dx \, dy \leq \iint_{\mathbb{R}^d} |g(x) - g(y)| \, |\tilde{u}(x)| \, |k_n(x, y)| \, dy \, dx
\]
\[
\leq \frac{A}{4\varepsilon} \iint_{\mathbb{R}^d} |g(x) - g(y)|^2 \tilde{k}(x, y) \, dy \, dx + \frac{\varepsilon}{A} \iint_{\mathbb{R}^d} \tilde{u}^2(x) \frac{k^2_n(x, y)}{\tilde{k}(x, y)} \, dy \, dx
\]
\[
\leq \frac{A^2}{4\varepsilon} \iint_{\mathbb{R}^d} |g(x) - g(y)|^2 k_n(x, y) \, dy \, dx + \varepsilon \|\tilde{u}\|^2_{L^2(\Omega)}.
\]
Altogether we obtain
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y))^2 k_n(x, y) \, dx \, dy \leq 2 \|f\|_{H^1_0(\mathbb{R}^d; k)} \|\tilde{u}\|_{H(\mathbb{R}^d; k)}
\]
\[
+ \frac{1}{2\varepsilon} \iint_{\mathbb{R}^d} |g(x) - g(y)|^2 k_n(x, y) \, dy \, dx + 2\varepsilon \iint_{\mathbb{R}^d} |\tilde{u}(x) - \tilde{u}(y)|^2 k_n(x, y) \, dx \, dy
\]
\[
+ \frac{A^2}{2\varepsilon} \iint_{\mathbb{R}^d} |g(x) - g(y)|^2 k_n(x, y) \, dy \, dx + 2\varepsilon \|\tilde{u}\|^2_{L^2(\Omega)}
\]
\[
\leq \frac{1}{2\varepsilon} \|f\|^2_{H^1_0(\mathbb{R}^d; k)} + \left(\frac{A^2 + 1}{2\varepsilon}\right) [g, g]_{V(\Omega; k)} + 4\varepsilon \|\tilde{u}\|^2_{H(\mathbb{R}^d; k)}.
\]
Applying the Poincaré-Friedrichs inequality (P), and choosing \(\varepsilon = \frac{1}{16C_P}\) we deduce
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y))^2 k_n(x, y) \, dx \, dy \leq c_1 \|f\|^2_{H^1_0(\mathbb{R}^d; k)} + c_2 [g, g]_{V(\Omega; k)},
\]
where \(c_1 \geq 1\) depends on \(C_P\) and \(c_2 \geq 1\) depends on \(A\) and \(C_P\). Since \(H^1_0(\mathbb{R}^d; k) \subset V(\Omega; k)\) and \(u = \tilde{u} + g\) the assertion (3.3) follows. \(\square\)

As mentioned in Remark 3.2, condition (C) ensures the sector condition in the sense of [16]:

**Proposition 3.6.** Let \(k\) satisfy (L), \((\tilde{K})\) and (C). Then the bilinear forms \((\mathcal{E}^k, H(\mathbb{R}^d; k))\) and \((\mathcal{E}^k, H^1_0(\mathbb{R}^d; k))\) satisfy the weak sector condition. If, in addition, \(k\) satisfies (P), then the bilinear form \((\mathcal{E}^k, H^1_0(\mathbb{R}^d; k))\) satisfies the strong sector condition.

4. **Weak maximum principle and Fredholm alternative**

The goal in this section is to prove existence and uniqueness when the bilinear form \(\mathcal{E}^k\) is no longer positive definite. To this end we establish a weak maximum principle implying that the homogeneous equation has only the trivial solution. Then we apply Fredholm’s alternative.

4.1. **Weak maximum principle.** In this subsection we prove a weak maximum principle when the kernels \(k\) exhibit a non-integrable singularity at the diagonal. We need to impose two different assumptions on the class of admissible kernels.
We assume that for some $\alpha \in (0,2)$, some $\lambda > 0$ and all $u \in L^2(\mathbb{R}^d)$ the estimate

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 k_s(x, y) \, dy \, dx \geq \lambda \alpha(2 - \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d + \alpha}} \, dy \, dx \quad (E_{\alpha})$$

holds true. Further, we assume that there is $D > 1$ such that for almost every $x, y \in \mathbb{R}^d$

$$|k(x, y)| \leq D^{-1} k_s(x, y) \quad (4.1)$$

Condition $(E_{\alpha})$ requires some minimal singularity of $k_s$ at the diagonal. It is not restrictive when considering non-integrable kernels. Concerning condition $(4.1)$ note that, by definition, the inequality $|k(x, y)| \leq k_s(x, y)$ holds for almost every $x, y \in \mathbb{R}^d$. Condition $(4.1)$ is satisfied by several examples, e.g. for

$$k(x, y) = |x - y|^{-d - \alpha} + g(x, y) 1_{B_1}(x - y)|x - y|^{-d - \beta},$$

if $0 < \beta < \alpha/2$ and $\|g\|_{\infty} \leq \frac{1}{2}$, cp. Example $(11)$ in Section 6. But there are also examples which violate the condition, e.g. $k(x, y) = |x - y|^{-d - \alpha} 1_{\mathbb{R}^d_+}(x - y)$, cp. Example $(10)$. Under the above conditions we can prove the following weak maximum principle:

**Theorem 4.1.** Let $k$ satisfy $(L)$, $(K)$, $(E_{\alpha})$, $(4.1)$. Let $u \in H_{\Omega}(\mathbb{R}^d; k)$ satisfy

$$\mathcal{E}^k(u, \varphi) \leq 0 \quad \text{for all } \varphi \in H_{\Omega}(\mathbb{R}^d; k). \quad (4.2)$$

Then $\sup_{\Omega} u \leq 0$.

**Remark 4.2.** As the proof reveals, it is possible to weaken assumption $(4.1)$ significantly because the estimate under consideration is needed only in an integrated sense. However, it seems challenging to provide a simple appropriate alternative to $(4.1)$. ♦

**Proof.** We apply a strategy which is often used in the proof of the weak maximum principle for second order differential operators (e.g. [12]). We first show that $u$ attains its supremum on a set of positive measure. In a second step we show that this leads to a contradiction if the supremum is positive.

We choose as test function $v = (u - k)^+$, where $0 \leq k < \sup_{\Omega} u$. Then $v \in H_{\Omega}(\mathbb{R}^d; k)$ and

$$\mathcal{E}^k(u, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))v(x)k_s(x, y) \, dy \, dx \leq 0 \quad (4.3)$$

Since $(u(x) - u(y))v(x) = [(u - k)_+(x) + (u - k)_+(y) + (u - k)_-(y)]v(x)$ we deduce

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_s(x, y) \, dy \, dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - k)^-(u(x) - k)^+ k(x, y) \, dy \, dx \leq - \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))v(x)k_s(x, y) \, dy \, dx,$$

and since the second term on the left hand side is positive

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_s(x, y) \, dy \, dx \leq - \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))v(x)k_s(x, y) \, dy \, dx.$$
Now by Cauchy-Schwarz and the assumptions on $k_u$
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) v(x) k_s(x, y) \, dy \, dx \\
\leq 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v(x) - v(y)| k_s^{1/2}(x, y) |v(x)| k_s^{-1/2}(x, y) |k_u(x, y)| \, dy \, dx \\
\leq 2A^{1/2} \|v\|_{L^2(\mathbb{R}^d)} \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_s(x, y) \, dy \, dx \right)^{1/2},
\]
or equivalently
\[
\left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_s(x, y) \, dy \, dx \right)^{1/2} \leq 2A^{1/2} \|v\|_{L^2(\mathbb{R}^d)}.
\]
By $(E_\alpha)$ and since $v = 0$ on $\mathcal{C} \Omega$, the Sobolev and the Hölder inequality imply that there is a constant $C = C(d) > 0$ such that
\[
\|v\|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \leq C \|v\|_{L^2(\Omega)} \leq C \|v\|_{H^{\alpha/2}\mathcal{C}^{d/(d-\alpha)}(\Omega)}.
\]
(If $d \leq 2$ the critical exponent may be replaced by any number greater than 2.) Thus
\[
|\supp v| \geq C^{-2d/\alpha}.
\]
This inequality is independent of $k$ and therefore it holds for $k \not\supset \supp_\Omega u$. Therefore $u$ must attain its supremum on a set of positive measure. This completes the first step of the proof.

We now derive a contradiction. Without loss of generality we may assume $\supp_\Omega u = 1$. Set $v = u^+$. We define a new function $\overline{v}$ by
\[
\overline{v} = \frac{v}{1 - v} = \frac{1}{1 - v} - 1.
\]
We want to use $\overline{v}$ as a test function in (4.2) but it is not clear whether $\overline{v}$ belongs to $H_\Omega(\mathbb{R}^d; k)$. Thus we look at approximations and define for small $\varepsilon > 0$
\[
v_\varepsilon = (1 - \varepsilon)v \quad \text{and} \quad \overline{v}_\varepsilon = \frac{v_\varepsilon}{1 - v_\varepsilon}.
\]
The function $\overline{v}_\varepsilon$ is an admissible test function. However, in order to simplify the presentation, we use $\overline{v}$ instead of $\overline{v}_\varepsilon$ and postpone this issue until the end of the proof. Plugging $\overline{v}$ into (4.3), we obtain
\[
\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) \left( \frac{1}{1 - v(x)} - \frac{1}{1 - v(y)} \right) k_s(x, y) \, dy \, dx \\
\leq - \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) \frac{v(x)}{1 - v(x)} k_u(x, y) \, dy \, dx \\
= - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) \left( \frac{v(x)}{1 - v(x)} + \frac{v(y)}{1 - v(y)} \right) k_u(x, y) \, dy \, dx.
\]
This is equivalent to
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx \\
\leq - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))}{(1 - v(x))^{1/2}(1 - v(y))^{1/2}} \left( \frac{v(x)(1 - v(y)) + v(y)(1 - v(x))}{(1 - v(x))^{1/2}(1 - v(y))^{1/2}} \right) k_s(x, y) \, dy \, dx .
\]

An application of the Young inequality leads to
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx \leq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx \\
+ \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v(x)(1 - v(y)) + v(y)(1 - v(x))|^2}{k_s(x, y)} k_s(x, y) \, dy \, dx
\]
and hence
\[
\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx \\
\leq \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v(x)(1 - v(y)) + v(y)(1 - v(x))|^2}{k_s(x, y)} k_s(x, y) \, dy \, dx .
\]

Using \( v = 0 \) on \( \mathbb{C} \), \( v \leq 1 \) and that \( \frac{k_s^2(x, y)}{k_s(x, y)} \) is symmetric, the right hand side can be estimated from above as follows:
\[
\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v(x)(1 - v(y)) + v(y)(1 - v(x))|^2}{k_s(x, y)} k_s(x, y) \, dy \, dx \\
\leq \iint_{\Omega} \left( \frac{v^2(x)(1 - v(y))}{1 - v(x)} + v(x)v(y) \right) \frac{k_s^2(x, y)}{k_s(x, y)} \, dy \, dx \\
\leq \theta^2 \iint_{\Omega} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx + \left( \frac{\theta}{\theta - 1} + 1 \right) \iint_{\Omega} \frac{k_s^2(x, y)}{k_s(x, y)} \, dy \, dx \\
\leq \frac{\theta^2}{D} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx + \left( \frac{\theta}{\theta - 1} + 1 \right) A|\Omega|,
\]
where we have applied Lemma 4.3 and (4.1). Now, we choose \( \theta = \sqrt{\frac{D+1}{2}} \) such that \( \frac{\theta^2}{D} < 1 \). Combining the above estimate and (4.4) leads to
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))^2}{(1 - v(x))(1 - v(y))} k_s(x, y) \, dy \, dx \leq c_1 A|\Omega|,
\]
for some positive constant \( c_1 = c_1(D) \). Next, we want to estimate the left-hand side from below. We apply the inequality
\[
\frac{(a - b)}{ab} = (a - b)(b^{-1} - a^{-1}) \geq (\log a - \log b)^2,
\]
which holds for positive reals $a, b$, to $a = 1 - v(y)$ and $b = 1 - v(x)$. Thus we obtain
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left( \log(1 - v(x)) - \log(1 - v(y)) \right)^2 k_s(x, y) \, dy \, dx \leq c_1 A |\Omega|.
\]

Due to condition (E\textsubscript{a}) we can apply the Sobolev inequality and obtain
\[
\|w\|_{L^{2d/(d-\alpha)}} \leq c_2 A |\Omega|
\]
where $c_2 \geq 1$ and $w = \log(1 - v)$. Recall that, in fact, we have proved $\|w_\varepsilon\|_{L^{2d/(d-\alpha)}} \leq c_2 A |\Omega|$ for $w_\varepsilon = \log(1 - v_\varepsilon)$ and every $\varepsilon \in (0, \frac{1}{2})$, where $c_2$ is independent of $\varepsilon$.

By Fatou’s lemma, this contradicts the fact that $v = u^+$ attains is supremum 1 on a set of positive measure. The proof is complete. \hfill \Box

**Lemma 4.3.** Assume $\theta > 1$ and $a, b \in [0, 1)$. Then
\[
\frac{1 - a}{1 - b} \leq \theta^2 \frac{(b - a)^2}{(1 - b)(1 - a)} + \frac{\theta}{\theta - 1} \tag{4.5}
\]
\[
\frac{1 - b}{1 - a} + \frac{1 - a}{1 - b} \leq 2\theta^2 \frac{(b - a)^2}{(1 - b)(1 - a)} + \frac{2\theta}{\theta - 1} \tag{4.6}
\]

**Proof.** It is sufficient to establish assertion (4.5) since it implies (4.6). For the proof of (4.5) it is sufficient to assume $a \leq b$. Assume $\theta > 1$ and $0 \leq a \leq b < 1$. Then, for $t = \frac{b}{a}$
\[
0 \leq a < 1 \leq t < \frac{1}{a}
\]
and inequality (4.5) reads
\[
\frac{1 - a}{1 - ta} \leq \theta^2 \frac{a^2(t - 1)^2}{(1 - ta)(1 - a)} + \frac{\theta}{\theta - 1}. \tag{4.7}
\]

**Case 1:** $\frac{1}{a} - 1 \leq \theta(t - 1)$. In this case
\[
\left(\frac{1}{a} - 1\right)^2 \leq \theta^2(t - 1)^2 \quad \Rightarrow \quad (1 - a) \leq \theta^2 \frac{a^2(t - 1)^2}{(1 - a)} \quad \Rightarrow \quad \frac{1 - a}{1 - ta} \leq \theta^2 \frac{a^2(t - 1)^2}{(1 - ta)(1 - a)},
\]
which proves (4.7).

**Case 2:** $\frac{1}{a} - 1 > \theta(t - 1) \iff (t - 1) < \frac{1}{\theta} \left(\frac{1}{a} - 1\right) \iff t < \frac{1}{\theta} \left(\frac{1}{a} - 1\right) + 1$. Therefore
\[
\frac{1 - a}{1 - ta} = a \frac{\left(\frac{1}{a} - 1\right)}{a \left(\frac{1}{a} - t\right)} \leq \frac{\frac{1}{a} - 1}{\frac{1}{a} - 1 - \frac{1}{\theta} \left(\frac{1}{a} - 1\right)} = \frac{1}{a} \frac{1 - 1}{\frac{1}{a} - 1 - \frac{1}{\theta} \left(\frac{1}{a} - 1\right)} = \frac{\theta}{\theta - 1},
\]
which again proves (4.7). \hfill \Box

4.2. **Fredholm alternative.** The aim of this subsection is to prove existence and uniqueness of solutions to (D) without assuming positive definiteness of the bilinear form $E^h$, i.e. without assuming (C). As for the weak maximum principle we assume the kernel $k$ to satisfy (E\textsubscript{a}) and (4.1). Note that (E\textsubscript{a}) implies (P) by Lemma 2.9.
We prove the following well-posedness result:

**Theorem 4.4.** Let \( \Omega \subseteq \mathbb{R}^d \) be open and bounded. Let \( f \in H^s_\Omega(\mathbb{R}^d; k) \) and let \( k \) satisfy \((L), (K), (4.1)\) and \((E_a)\). Then the Dirichlet problem \((D)\) has a unique solution \( u \in V(\Omega; k) \). Moreover, there is a constant \( C = C(C_F, A, D) > 0 \) such that

\[
[u, u]_{V(\Omega; k)} \leq C \left( \|f\|^2_{H^s_\Omega(\mathbb{R}^d; k)} + \|g\|^2_{L^2(\Omega)} + [g, g]_{V(\Omega; k)} + \|u\|^2_{L^2(\Omega)} \right). \tag{4.8}
\]

**Proof.** We use the Fredholm alternative (see e.g. [10]).

**Step 1:** We will use \((E_a)\) to show that the embedding \( H^s_\Omega(\mathbb{R}^d; k) \hookrightarrow L^2_{\Omega}(\mathbb{R}^d) \) is compact. Since the embedding \( L^2(\Omega) \hookrightarrow H^s_\Omega(\mathbb{R}^d; k) \) is continuous we obtain then the compactness of the embedding \( H^s_\Omega(\mathbb{R}^d; k) \hookrightarrow H^s_\Omega(\mathbb{R}^d; k) \).

Let \( \mathcal{A} \subset H^s_\Omega(\mathbb{R}^d; k) \) with \( \|u\|_{H^s(\mathbb{R}^d; k)} \leq C \) for all \( u \in \mathcal{A} \) and some \( C < \infty \). Let \( B \subset \mathbb{R}^d \) be an open ball with \( \Omega \subset B \). Let us recall that the embedding \( H^{s/2}(B) \hookrightarrow L^2(B) \) is compact. Then, for \( u \in \mathcal{A} \)

\[
\|u\|^2_{H^{s/2}(B)} = \iint_{B \times B} (u(x) - u(y))^2 |x - y|^{-d-\alpha} \, dy \, dx
\]

\[
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 |x - y|^{-d-\alpha} \, dy \, dx
\]

\[
\leq \lambda^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 k_s(x, y) \, dy \, dx \leq \lambda^{-1} C^2,
\]

where we used \((E_a)\). Therefore \( \mathcal{A} \) is bounded in \( H^{s/2}(B) \) and thus pre-compact in \( L^2(B) \).

By the definition of \( H^s_\Omega(\mathbb{R}^d; k) \) we know \( u = 0 \) on \( \overline{\Omega} \) and thus the set \( \mathcal{A} \) is also pre-compact in \( L^2_{\Omega}(\mathbb{R}^d) \) and in \( H^s_\Omega(\mathbb{R}^d; k) \).

**Step 2:** Existence and uniqueness of \((D_0)\). By Lemma 3.1 the bilinear form

\[
(u, v) \mapsto \mathcal{E}^k(u, v) + \gamma(u, v)_{L^2(\Omega)}
\]

is coercive for some \( \gamma = \gamma(A) > 0 \) and therefore there is a unique solution \( u \in H^s_\Omega(\mathbb{R}^d; k) \) to the problem

\[
\begin{cases}
\mathcal{E}^k(u, v) + \gamma(u, v)_{L^2(\Omega)} = \langle f, v \rangle & \text{for all } v \in H^s_\Omega(\mathbb{R}^d; k), \\
u = 0 & \text{on } \overline{\Omega}.
\end{cases} \tag{4.9}
\]

Moreover, due to Lemma 3.1 the solution \( u \) satisfies

\[
\|u\|^2_{H^s_\Omega(\mathbb{R}^d; k)} \leq 4\mathcal{E}^k(u, u) + 4\gamma \|u\|^2_{L^2(\Omega)} = 4 \langle f, v \rangle \leq 4 \|f\|_{H^s_\Omega(\mathbb{R}^d; k)} \|u\|_{H^s_\Omega(\mathbb{R}^d; k)}.
\]

This estimate together with Step 1 shows that the operator \( K : H^s_\Omega(\mathbb{R}^d; k) \to H^s_\Omega(\mathbb{R}^d; k) \), which maps the inhomogeneity \( f \) to the solution \( u \in H^s_\Omega(\mathbb{R}^d; k) \), is a compact operator. Fredholm’s theorem in combination with the weak maximum principle Theorem 4.1 shows that \((D_0)\) has a unique solution \( u \in H^s_\Omega(\mathbb{R}^d; k) \).

**Step 3:** The well-posedness of \((D)\) follows in the same way as in the proof of Theorem 3.5. It remains to prove the estimate \((4.8)\). Let \( u \) be the solution of \((D)\). We apply \( v = u - g \in \mathcal{D}(\Delta) \) and find that

\[
\mathcal{E}^k(u, v) + \gamma(u, v)_{L^2(\Omega)} = \langle f, v \rangle = \langle f, u - g \rangle.
\]

Since \( \mathcal{E}^k(u, u) \) is coercive and \( \gamma(u, v) \) is bounded, we have

\[
\|u\|^2_{H^s_\Omega(\mathbb{R}^d; k)} \leq \mathcal{E}^k(u, u) + \gamma(u, v)_{L^2(\Omega)} = \langle f, u - g \rangle \leq \|f\|_{H^s_\Omega(\mathbb{R}^d; k)} \|u\|_{H^s_\Omega(\mathbb{R}^d; k)}.
\]

This proves the theorem.
$H_\Omega(\mathbb{R}^d, k)$ as test function:

$$
(f, v)_{H_\Omega^2(\mathbb{R}^d, k)} - \mathcal{E}^k(g, v) = \mathcal{E}^k(v, v)
$$

$$
= \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))v(x)k(x, y) \, dx \, dy
$$

$$
= \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))v(x)k_\alpha(x, y) \, dx \, dy + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) v(x)k_\alpha(x, y) \, dx \, dy ,
$$

As in the proof of Theorem 3.5 we may estimate:

$$
\iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_\alpha(x, y) \, dx \, dy \leq \frac{1}{2\varepsilon} \|f\|^2_{H_\Omega^2(\mathbb{R}^d, k)} + 2\varepsilon \|v\|^2_{H_\Omega^2(\mathbb{R}^d, k)}
$$

$$
+ \frac{1}{2\varepsilon} \iint_{\Omega \times \mathbb{R}^d} |g(x) - g(y)|^2 k_\alpha(x, y) \, dx \, dy + 2\varepsilon \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v(x) - v(y)|^2 k_\alpha(x, y) \, dx \, dy
$$

$$
+ \frac{A^2}{2\varepsilon} \iint_{\Omega \times \mathbb{R}^d} |g(x) - g(y)|^2 k_\alpha(x, y) \, dy \, dx + 2\varepsilon \|v\|^2_{L^2(\Omega)}
$$

$$
+ 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v(x) - v(y)| |v(x)| |k_\alpha(x, y)| \, dx \, dy .
$$

Due to $v = 0$ on $\mathbb{C} \Omega$, the last term can be estimated as follows:

$$
\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v(x) - v(y)| |v(x)| |k_\alpha(x, y)| \, dx \, dy
$$

$$
\leq \varepsilon \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y))^2 k_\alpha(x, y) \, dx \, dy + \frac{A}{4\varepsilon} \|v\|^2_{L^2(\Omega)} .
$$

Hence, after choosing $\varepsilon$ appropriately,

$$
\iint_{\Omega \times \mathbb{R}^d} (v(x) - v(y))^2 k_\alpha(x, y) \, dx \, dy \leq \|f\|^2_{H_\Omega^2(\mathbb{R}^d, k)} + c_1 \|g\|_{V(\Omega, k)} + c_2 \|v\|^2_{L^2(\Omega)} ,
$$

where $c_1, c_2 > 0$ depend on $A$. This implies (4.8). \hfill \square

5. PARABOLIC PROBLEM

In this section we prove well-posedness of the initial boundary value problem

$$
\begin{cases}
\partial_t u + Lu = f & \text{in } (0, T) \times \Omega ,
\quad u = 0 & \text{on } [0, T] \times \mathbb{C} \Omega ,
\quad u = u_0 & \text{on } \{0\} \times \Omega ,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $Q_T = (0, T) \times \Omega$ and

$$
f \in L^2(Q_T) , \quad u_0 \in L^2(\Omega) .
$$

To put this problem in a functional analytic framework we define for a Banach space $B$

$$
W(0, T) = W(0, T; B) = \{ u \in L^2(0, T; B) : u' \text{ exists and } u' \in L^2(0, T; B^*) \} .
$$
W(0, T) is a Banach space endowed with the norm
\[ \|u\|_{W(0, T)} = \int_0^T \|u(t)\|_B^2 \, dt + \int_0^T \|u'(t)\|^2 \, dt. \]

We consider the Gelfand triplet
\[ H_\Omega(\mathbb{R}^d; k) \hookrightarrow L^2(\Omega) \hookrightarrow H^*_\Omega(\mathbb{R}^d; k), \]
where the two embeddings are continuous and each space is dense in the following one.

Define for \( u, v \in B \)
\[ \mathcal{E}^k(t; u, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y)) v(x) k_t(x, y) \, dx \, dy, \quad (5.3) \]
where we assume that
\[ k_t(x, y) = a(t, x, y)k(x, y) \quad (5.4) \]
with a measurable function \( a: \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to [\frac{1}{2}, 1] \) which is symmetric with respect to \( x \) and \( y \), and a measurable kernel \( k: \mathbb{R}^d \to \mathbb{R}^d \to [0, \infty] \).

**Definition 5.1** (Parabolic variational formulation). Let \( u_0 \in L^2(\Omega) \) and \( f \in L^2(0, T; H^*_\Omega(\mathbb{R}^d; k)) \). We say that \( u \in W(0, T; H^*_\Omega(\mathbb{R}^d; k)) \) is a solution of (5.1) if for all \( v \in H_\Omega(\mathbb{R}^d; k) \)
\[ \frac{d}{dt}(u(t), v)_{L^2(\Omega)} + \mathcal{E}^k(t; u(t), v) = \langle f, v \rangle \quad \text{for a.e. } t \in (0, T), \quad (5.5a) \]
\[ u(0) = u_0. \quad (5.5b) \]

We refer to **Definition 5.4** for the corresponding problem with nonzero complement data.

**Remark 5.2.**

a) (5.5a) is satisfied if and only if for all \( v \in H^*_\Omega(\mathbb{R}^d; k) \) and for all \( \phi \in C_c^\infty(0, T) \)
\[ -\int_0^T (u(t), v)_{L^2(\Omega)} \phi'(t) \, dt + \int_0^T \mathcal{E}^k(t; u(t), v) \phi(t) \, dt = \int_0^T \langle f, v \rangle \phi(t) \, dt. \]

b) The initial condition (5.5b) is well-defined due to the embedding
\[ W(0, T) \hookrightarrow C([0, T]; L^2(\Omega)), \]
see e.g. [24].

c) **Remark 2.6** holds analogously.

A well-known parabolic analog (e.g. [25, Corollary 23.26]) of the Lax-Milgram Lemma asserts that there is a unique solution to (5.1) provided the bilinear form \( \mathcal{E}(t; \cdot, \cdot): \mathcal{B} \times \mathcal{B} \to \mathbb{R} \) has the following properties:

For all \( u, v \in \mathcal{B} \) the mapping \( t \mapsto \mathcal{E}^k(t; u, v) \) is measurable on \( (0, T) \),
\[ (5.6a) \]
there is \( M > 0 \) such that for all \( t \in (0, T) \) and \( u, v \in \mathcal{B} \) we have
\[ \left| \mathcal{E}^k(t; u, v) \right| \leq M \|u\|_\mathcal{B} \|v\|_\mathcal{B}, \quad (5.6b) \]
there are \( m > 0 \) and \( L_0 \geq 0 \) such that for all \( t \in (0, T) \) and \( u \in \mathcal{B} \) we have
\[ \mathcal{E}^k(t; u, u) \geq m \|u\|^2_\mathcal{B} - L_0 \|u\|^2_{L^2(\Omega)}, \quad (5.6c) \]
The measurability condition (5.6a) follows immediately from the measurability of \( k_t \). Due to the special structure (5.4) of \( k_t \) we may refer to Lemma 2.4 for the proof of the continuity condition (5.6b) and to Lemma 3.1 for the Gårding inequality (5.6c). Therefore we obtain the following result:

**Theorem 5.3** (Well-posedness of the parabolic problem). Assume that \( k_t \) is of the form (5.4), where \( k \) satisfies (L) and \((\tilde{K})\). Then there is a unique solution \( u \in W(0, T; H_{\Omega}(\mathbb{R}^d; k)) \) of (5.1). Moreover, for all \( u_0 \in L^2(\Omega) \) and \( f \in L^2(0, T; H_{\Omega}^*(\mathbb{R}^d; k)) \) there is a constant \( C > 0 \) such that

\[
\|u\|_{W(0, T)} \leq C \left( \|f\|_{L^2(0, T; H_{\Omega}^*(\mathbb{R}^d; k))} + \|u_0\|_{L^2(\Omega)} \right).
\]

The constant \( C \) depends on the constants \( M, m \) and \( L_0 \) in (5.6b), (5.6c).

In order to consider non-homogeneous complement data we assume that these values are prescribed by a function \( g: [0, T] \times \mathbb{R}^d \to \mathbb{R} \). For the functional analytic treatment of this problem we need a modification of the space \( V(\Omega; k) \) that turns this linear space into a Hilbert space. One possibility is to define

\[
H(\Omega; k) = \left\{ g \in L^2(\mathbb{R}^d) : \langle g(x) - g(y) \rangle k_1^{1/2} (x, y) \in L^2(\Omega \times \mathbb{R}^d) \right\}.
\]

From the proof of Lemma 2.3 it can be seen that this space is a separable Hilbert space with the inner product defined by

\[
(g, h)_{H(\Omega; k)} = (g, h)_{L^2(\mathbb{R}^d)} + [g, h]_{V(\Omega; k)}.
\]

**Definition 5.4** (Parabolic variational formulation). Let \( f \in L^2(Q_T) \) and \( g \in W(0, T; H(\Omega; k)) \). We say that \( u \in W(0, T; H(\Omega; k)) \) is a solution of

\[
\begin{align*}
\partial_t u + Lu &= f & \text{in } (0, T) \times \Omega, \\
u &= g & \text{on } [0, T] \times \partial\Omega, \\
u &= g(0, \cdot) & \text{on } \{0\} \times \Omega,
\end{align*}
\]

if (5.5a) holds, if \( u - g \in W(0, T; H_{\Omega}(\mathbb{R}^d; k)) \) and if \( u(0) = g(0) \in H(\Omega; k) \).

**Corollary 5.5.** Assume that \( k_t \) is of the form (5.4), where \( k \) satisfies (L) and \((\tilde{K})\). Then there is a unique solution \( u \in W(0, T; H(\Omega; k)) \) of (5.9).

**Proof.** By Theorem 5.3 there is a unique solution \( u \in W(0, T; H_{\Omega}(\mathbb{R}^d; k)) \) to the problem

\[
\begin{align*}
\partial_t u + Lu &= f - \partial_t g - Lg & \text{in } (0, T) \times \Omega, \\
u &= 0 & \text{on } [0, T] \times \partial\Omega, \\
u &= 0 & \text{on } \{0\} \times \Omega.
\end{align*}
\]

Then \( \tilde{u} = u + g \) satisfies \( \tilde{u} \in W(0, T; H(\Omega; k)) \) and (5.9). \( \square \)

Let us emphasize that the conditions on the complement data \( g \) are far from being optimal. On the one hand, it would be desirable to relax the condition on the spatial decay imposed by \( g(t, \cdot) \in L^2(\Omega) \) as in Section 3. On the other hand, one could try to follow the program as in the case of second order parabolic equations (e.g. [15]) in order to relax the regularity assumptions of \( g \) with respect to \( t \) and \( x \).
6. Examples of kernels

We provide several examples of kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ to which the theory above can be applied. Recall that all kernels studied in this work satisfy assumption (L). Further, we distinguish two cases. We call a kernel $k$ integrable if, for every $x \in \mathbb{R}^d$ the quantity $\int_{\mathbb{R}^d} k_s(x, y) \, dy$ is finite and the mapping $x \mapsto \int_{\mathbb{R}^d} k_s(x, y) \, dy$ is locally integrable. We call a kernel non-integrable if it is not integrable in the sense above. At the end of this section we list all examples together with their corresponding properties.

6.1. Integrable kernels. Let us start with a simple observation. Every kernel with the property that the antisymmetric part is of the form $k_a(x, y) = g(x - y)$ for some function $g$ satisfies the assumption (C). This follows from the fact that for $x \in \mathbb{R}^d$

$$\int_{\mathbb{C}B_s(x)} k_a(x, y) \, dy = \int_{\mathbb{C}B_s(x)} g(x - y) \, dy = \int_{\mathbb{C}B_s(0)} g(z) \, dz = 0.$$  

(1) $k(x, y) := 1_{B_1}(x - y)$. The kernel is obviously symmetric. Thus it satisfies (C). It also satisfies the Poincaré-Friedrichs inequality (P) as shown in Section 2.3.

(2) $k(x, y) := 1_{B_r \setminus B_r}(x - y)$ for some numbers $0 < r < R$. Again, (C) and the Poincaré-Friedrichs inequality (P) hold.

(3) $k(x, y) := 1_{B_1 \cap \mathbb{R}_+^d}(x - y)$. Symmetrization leads to

$$k_s(x, y) = \frac{1}{2} 1_{B_1 \cap \mathbb{R}_+^d}(x - y) + \frac{1}{2} 1_{B_1 \cap \mathbb{R}_+^d}(y - x) = \frac{1}{2} 1_{B_1}(x - y)$$

$$k_a(x, y) = \frac{1}{2} 1_{B_1 \cap \mathbb{R}_+^d}(x - y) - \frac{1}{2} 1_{B_1 \cap \mathbb{R}_+^d}(x - y).$$

Since $k$ depends only on $x - y$, condition (C) holds. Concerning the Poincaré-Friedrichs inequality (P), $k$ is not different from example (1).

(4) This example is more general than Example 3. Set $k(x, y) := 1_{B_1}(x - y) 1_C(x - y)$ where the set $C$ is defined by $C = \{h \in \mathbb{R}^d : \frac{y}{|y|} \in I\}$ and $I$ is an arbitrary nonempty open subset of $S^{d-1}$. If $I$ is of the form $I = B_r(\xi) \cap S^{d-1}$ for some $\xi \in S^{d-1}$ and some $r > 0$, then $C$ is a cone. In any case, we obtain

$$k_s(x, y) = \frac{1}{2} 1_{B_1 \cap (C \cup -C)}(x - y),$$

$$k_a(x, y) = \frac{1}{2} 1_{B_1 \cap C}(x - y) - \frac{1}{2} 1_{B_1 \cap -C}(x - y).$$

In the examples above, $k(x, y)$ depends only on $x - y$. As a result, one can choose $L(z) = k(0, y - x)$ in the condition (2.5). Let us look at examples where this is not possible.

(5) $k(x, y) := g(x, y) 1_{B_1}(x - y)$, where $g$ is any measurable bounded function satisfying $g \geq c$ almost everywhere for some constant $c > 0$. Note that $g$ does not need to be symmetric. Then

$$k_s(x, y) = \frac{1}{2} (g(x, y) + g(y, x)) 1_{B_1}(x - y)$$

$$k_a(x, y) = \frac{1}{2} (g(x, y) - g(y, x)) 1_{B_1}(x - y).$$

Condition (C) does not hold in general but (K) holds because $k_s(x, y) \geq c 1_{B_1}(x - y)$ which allows us to apply the Poincaré-Friedrichs inequality (P) choosing $L(z) = c 1_{B_1}(z)$ in (2.5).

(6) Here, we set $d = 1$ and define a kernel $k : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ as follows. Define

$D = [-1, 0] \times [0, 1] \cup \{(x, y) \in \mathbb{R}^2 : (x \leq y \leq x + 1)\}$. 

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Set \( k(x, y) := 2 \cdot 1_D(x, y) \). Then the antisymmetric part of \( k \) is given by
\[
k_a(x, y) = 1_D(x, y) - 1_{(-D)}(x, y).
\]
Due to the construction of \( D \) we obtain for \( |x| > 1 \) \( \lim_{\varepsilon \to 0^+} \int_{\mathbb{C}B_\varepsilon(x)} k_a(x, y) \, dy = 0 \) whereas for \( x \in (-1, 1) \) we obtain \( \lim_{\varepsilon \to 0^+} \int_{\mathbb{C}B_\varepsilon(x)} k_a(x, y) \, dy = -x \), which implies that \( k \) does not satisfy condition (C). Though, conditions (K) and (2.5) hold true because of \( k_a(x, y) \geq 1_{B_1}(x - y) \).

(7) Again, set \( d = 1 \). We define \( k(x, y) \) by \( k(x, y) = 2 \cdot 1_{(-4, 4)}(x - y) + k_a(x, y) \) where
\[
k_a(x, y) = \begin{cases} g(x, y, y) & \text{if } x < y \\ -g(y, y, y) & \text{else} \end{cases}
\]
and
\[
g(x, y) = \text{sgn}(xy) 1_{(-1,1) \times (-1,1)}(x, y).
\]
By construction \( k_a \) is antisymmetric and satisfies condition (C). Thus \( k \) is not a function of \( x - y \) but still satisfies (C). Conditions (K) and (2.5) hold true, too.

6.2. Non-integrable kernels. Here are several examples of kernels \( k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty] \) with a singularity at the diagonal. See above for our definition of when we call a kernel non-integrable. Recall that we want all examples to satisfy (L). Throughout this section (with one exception) \( \alpha \in (0, 2) \) is an arbitrary fixed number.

(8) \( k(x, y) := |x - y|^{-d-\alpha} \). Obviously, \( k \) is symmetric and satisfies (L). Conditions (C) and (K) hold due to the symmetry. Lemma 2.9 can be directly applied. This kernel \( k \) is very special because the space \( H(\mathbb{R}^d; k) \) is isomorphic to the fractional Sobolev space \( H^{\alpha/2}(\mathbb{R}^d) \) (cf. Remark 2.2b). There is a constant \( C \geq 1 \), independent of \( \alpha \), such that for all \( v \in C_0(\mathbb{R}^d) \)
\[
C^{-1} \|v\|_{H^{\alpha/2}(\mathbb{R}^d)} \leq \alpha(2 - \alpha) \|v\|_{H(\mathbb{R}^d; k)} \leq C \|v\|_{H^{\alpha/2}(\mathbb{R}^d)},
\]
where \( \|v\|_{H^{\alpha/2}(\mathbb{R}^d)} = \int (1 + |\xi|^2)^{\alpha/2} |\hat{v}(\xi)|^2 \, d\xi \). Thus, for fixed \( v \in C_0(\mathbb{R}^d) \),
\[
\begin{align*}
\alpha(2 - \alpha) [v, v]_{H(\mathbb{R}^d; k)} &\to [v, v]_{H^1(\mathbb{R}^d)} \\
\alpha(2 - \alpha) \|v\|_{H(\mathbb{R}^d; k)} &\to \|v\|_{L^2(\mathbb{R}^d)}
\end{align*}
\]
for \( \alpha \to 2^- \) and \( \alpha \to 0^+ \). Similar results hold true for \( \mathbb{R}^d \) replaced by a bounded domain \([7, 18]\).

(9) Let \( I \) be an arbitrary nonempty open subset of \( S^{d-1} \) with the property \( I = -I \). Set \( \mathcal{C} = \{h \in \mathbb{R}^d \mid \frac{h}{|h|} \in I\} \) and \( k(x, y) := |x - y|^{-d-\alpha} 1_{\mathcal{C}}(x - y) \). Again, \( k \) is symmetric and satisfies (L). It turns out that \( k \) is comparable to example (8) in the sense of Lemma 2.9. The only difference is that the constant \( \lambda \) depends on \( I \).

(10) \( k(x, y) := |x - y|^{-d-\alpha} 1_{\mathbb{R}_+^d}(x - y) \). This example is different from example (9) because \( k \) is not symmetric anymore. The symmetric and antisymmetric parts are given by
\[
k_s(x, y) = \frac{1}{2} |x - y|^{-d-\alpha}
\]
\[
k_a(x, y) = |x - y|^{-d-\alpha} \left( \frac{1}{2} 1_{\mathbb{R}_+^d}(x - y) - \frac{1}{2} 1_{\mathbb{R}_+^d}(x - y) \right).
\]
Lemma 2.9 can still be applied but conditions (K) and (\( \widetilde{K} \)) do not hold. Condition (C) does hold, though.
The following example appears in [9, Example 12]. Assume that \(0 < \beta < \frac{d}{2}\) and \(g: \mathbb{R}^d \times \mathbb{R}^d \to [-K, L]\) measurable for some \(K, L > 0\). Define
\[
k(x, y) := |x - y|^{-d-\alpha} + g(x, y) 1_{B_1}(x - y)|x - y|^{-d-\beta}
\]
Additionally, we assume that \(k\) is nonnegative. This property does not follow in general under the assumptions above. However, for every choice of \(K\) there are many admissible cases with \(\inf g = -K\). We obtain \(k_\alpha(x, y) \geq \frac{1}{2}|x - y|^{-d-\alpha}\) for \(|x - y| \leq (2K)^{-\frac{1}{d-\alpha}}\). Since \(k_\alpha\) is nonnegative, we can apply Lemma 2.9 and (P) holds. Further \((\tilde{K})\) is satisfied with \(k(x, y) = |x - y|^{-d-\alpha} 1_{B_1}(x - y)\) and \(A_1 = 1\). Conditions (C) and (K) hold for some but not for all choices of \(g\).

The following example is an extension and, at the same time, a special case of Example (11). Example (12) shows that our condition \((\tilde{K})\) is indeed a relaxation of (K) or [23, (1.1)].

Assume \(0 < \beta < \frac{d}{2}\). Let \(I_1, I_2\) be arbitrary nonempty disjoint open subsets of \(S^{d-1}\) with \(I_1 = -I_1\) and \(|-I_2 \setminus I_2| > 0\). Set \(C_j = \{h \in \mathbb{R}^d | h \notin I_j\} \) for \(j \in \{1, 2\}\). Set
\[
k(x, y) = |x - y|^{-d-\alpha} 1_{C_1}(x - y) + |x - y|^{-d-\beta} 1_{C_2}(x - y) 1_{B_1}(x - y).
\]
The symmetric and antisymmetric parts of \(k\) are given by
\[
k_s(x, y) = |x - y|^{-d-\alpha} 1_{C_1}(x - y) + \frac{1}{2}|x - y|^{-d-\beta} 1_{C_2 \cup (C_2 \setminus C_1)}(x - y) 1_{B_1}(x - y),
k_a(x, y) = \frac{1}{2}|x - y|^{-d-\beta} 1_{C_2 \cap B_1}(x - y) - \frac{1}{2}|x - y|^{-d-\beta} 1_{(C_2 \setminus C_1) \cap B_1}(x - y).
\]
Let us show that condition (K) does not hold, i.e. \((\tilde{K}_2)\) is not satisfied for \(\tilde{k} = k_s\). Let \(h, h_a, h_s, \tilde{h}: \mathbb{R}^d \to [0, \infty]\) be defined by \(h(x - y) = k(x, y)\) and \(h_a, h_s, \tilde{h}\) accordingly. Note that \(|h_a| = h_s\) on \(C_2 \cup \tilde{C}_2\). Then
\[
\sup_{x \in \mathbb{R}^d} \int_{\{k_s(x,y) \neq 0\}} \frac{k_s(x,y)^2}{k_s(x,y)} \, dy = \int_{\mathbb{R}^d} \frac{h_a^2(z)}{h_a(z)} 1_{B_1}(z) \, dz = \int_{\{C_2 \cup (C_2 \setminus C_1)\}} \frac{h_s^2(z)}{h_s(z)} 1_{B_1}(z) \, dz + \infty
\]
Let us explain why \((\tilde{K}_1)\) and \((\tilde{K}_2)\) hold for \(\tilde{k}(x, y) = |x - y|^{-d-\alpha}\). \((\tilde{K}_1)\) follows easily from \(k_s(x, y) \geq |x - y|^{-d-\alpha} 1_{C_1}(x - y)\), the constant \(A_1\) needs to be chosen in dependence of \(C_1\) resp. \(I_1\). Let us check \((\tilde{K}_2)\):
\[
\sup_{x \in \mathbb{R}^d} \int \frac{k_s^2(x,y)}{k_s(x,y)} \, dy = \int_{\mathbb{R}^d} \frac{h_s^2(z)}{h_s(z)} 1_{B_1}(z) \, dz = \int_{\{C_2 \cup (C_2 \setminus C_1)\}} \frac{h_s^2(z)}{h_s(z)} 1_{B_1}(z) \, dz 
\]
where \(A_2\) depends on \(I_2\) and \(\alpha/2 - \beta\). Note: If we modify the example by choosing \(I_1 = S^{d-1}\), i.e. \(C_1 = \mathbb{R}^d\), then condition (K) does hold.

The following example appears in [9, Example 12]. Assume \(0 < b < 1\) and \(0 < \alpha' < 1 + \frac{1}{b}\). Define \(\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 | |x_1| \geq |x_2|^b\text{ or }|x_2| \geq |x_1|^b\}\) and set
\[
k(x, y) = 1_{\Gamma \cap B_1}(x - y)|x - y|^{-d-\alpha'}.
\]
Note that the kernel $k$ depends only on $x - y$ and is symmetric but condition (L) is not obvious. Using integration in polar coordinates one can show that there is $C \geq 1$ such that for $\alpha = \alpha' - (1/b - 1)$, $h(z) = k(x, x + z)$ and every $r(\in (0, 1)$

$$
\frac{r^2}{B_r} \int_{|z|^2} |z|^2 h(z) \, dz + \int_{B_1} h(z) \, dz \leq C r^{-\alpha}.
$$

Thus $\alpha$ is the effective order of differentiability of the corresponding integro-differential operator. In [9] the comparability of the quadratic forms needed for Lemma 2.9 is established. From the point of view of this article the kernel $k$ is very similar to the kernel $|x - y|^{-d-\alpha} \mathbb{1}_{B_r}(x - y)$. Of course, one can now produce related nonsymmetric examples.

(14) The following example is taken from [11], [23]. It provides a nonsymmetric kernel $k$ with a singularity on the diagonal which is non-constant. Assume $0 < \alpha_1 \leq \alpha_2 < 2$ and let $\alpha : \mathbb{R}^d \to [\alpha_1, \alpha_2]$ be a measurable function. We assume that $\alpha$ is continuous and that the modulus of continuity $\omega$ of the function $\alpha$ satisfies

$$
\int_0^1 \frac{(\omega(r) \log r)^2}{r^{1+\alpha_2}} \, dr < \infty.
$$

Note that, as a result, there are $\beta \in (0, 1)$ and $C_H > 0$ such that $[\alpha]_{C^0,\beta(\mathbb{R}^d)} \leq C_H$. Let $b : \mathbb{R}^d \to \mathbb{R}$ be another measurable function which is bounded between two positive constants and satisfies $|b(x) - b(y)| \leq c|\alpha(x) - \alpha(y)|$ as long as $|x - y| \leq 1$ for some constant $c > 0$. Finally, set

$$
k(x, y) = b(x)|x - y|^{-d-\alpha(x)}.
$$

In [23] it is proved, that $k$ satisfies (L) and (K). Since $\alpha$ is bounded from below by $\alpha_1$, the Poincaré-Friedrichs inequality (P) holds. Condition (4.1) does not hold for this example since

$$
\lim_{|x - y| \to \infty} \frac{k_0(x, y)}{k_s(x, y)} = 1.
$$

Let us slightly modify the example and look at $k'(x, y) = \mathbb{1}_{B_R}(x - y)k(x, y)$ for some $R \gg 1$. Then conditions (L), (K) and (P) still hold true for $k'$.

**Lemma 6.1.** The kernel $k'$ satisfies (4.1).

**Proof.** We have to show that $\frac{|k'(x, y)|}{k_s(x, y)} \leq \Theta < 1$ for all $x, y \in \{|x - y| < R\}$. By assumption there are $c_1, c_2 > 0$ such that

$$
c_1 \leq b(x) \leq c_2 \quad \text{for all } x \in \mathbb{R}^d.
$$

We can assume that $\alpha(x) \leq \alpha(y)$ due to the symmetry of $\frac{|k_s'(x, y)|}{k_s(x, y)}$.

**Case 1a:** $|x - y| \leq 1$ and $k_s'(x, y) > 0$. Then

$$
\frac{|k_s'(x, y)|}{k_s(x, y)} = \frac{b(x)|x - y|^{-d-\alpha(x)} - b(y)|x - y|^{-d-\alpha(y)}}{b(x)|x - y|^{-d-\alpha(x)} + b(y)|x - y|^{-d-\alpha(y)}}
$$

$$
= \frac{b(x) - b(y)|x - y|^\alpha(x) - \alpha(y)}{b(x) + b(y)|x - y|^\alpha(x) - \alpha(y)} \leq \frac{1 - \frac{\alpha_2}{c_2}}{1 + \frac{\alpha_2}{c_2}} =: \Theta_1.
$$

**Case 1b:** $|x - y| \leq 1$ and $k_s'(x, y) < 0$. Then

$$
\frac{|k_s'(x, y)|}{k_s(x, y)} = \frac{b(y)|x - y|^{-d-\alpha(y)} - b(x)|x - y|^{-d-\alpha(x)}}{b(x)|x - y|^{-d-\alpha(x)} + b(y)|x - y|^{-d-\alpha(y)}}
$$

$$
= \frac{b(y) - b(x)|x - y|^\alpha(y) - \alpha(x)}{b(y) + b(x)|x - y|^\alpha(y) - \alpha(x)}.
$$
Since $|\alpha(y) - \alpha(x)| \leq C_H |x - y|^\beta$, we obtain

$$|x - y|^{\alpha(y) - \alpha(x)} \geq |x - y|^{C_H |x - y|^\beta} \geq \delta(C_H, \beta) > 0.$$ 

Thus

$$\frac{|k'_a(x, y)|}{k'_a(x, y)} \leq \frac{1 - \frac{c_1}{c_2}}{1 + \frac{c_1}{c_2}} =: \Theta_2$$ 

**Case 2a:** $1 < |x - y| < R$ and $k'_a(x, y) < 0$. Then

$$\frac{|k'_a(x, y)|}{k'_a(x, y)} = \frac{b(y)|x - y|^{-d - \alpha(y)} - b(x)|x - y|^{-d - \alpha(x)}}{b(x)|x - y|^{-d - \alpha(x)} + b(y)|x - y|^{-d - \alpha(y)}}$$

$$= \frac{b(y) - b(x)|x - y|^{\alpha(y) - \alpha(x)}}{b(y) + b(x)|x - y|^{\alpha(y) - \alpha(x)}} \leq \frac{1 - \frac{c_1}{c_2}}{1 + \frac{c_1}{c_2}} = \Theta_1$$

**Case 2b:** $1 < |x - y| < R$ and $k'_a(x, y) > 0$. Then

$$\frac{|k'_a(x, y)|}{k'_a(x, y)} = \frac{b(x)|x - y|^{-d - \alpha(x)} - b(y)|x - y|^{-d - \alpha(y)}}{b(x)|x - y|^{-d - \alpha(x)} + b(y)|x - y|^{-d - \alpha(y)}}$$

$$= \frac{b(x) - b(y)|x - y|^{\alpha(x) - \alpha(y)}}{b(x) + b(y)|x - y|^{\alpha(x) - \alpha(y)}} \leq \frac{1 - \frac{c_1}{c_2} R^{\alpha_1 - \alpha_2}}{1 + \frac{c_1}{c_2} R^{\alpha_1 - \alpha_2}} =: \Theta_3.$$ 

We have shown that $k'$ satisfies all conditions needed in order to apply Theorem 4.4.

Although we have provided several different examples, our class is still rather small. All examples of non-integrable kernels from above relate, in one way or another, to the standard kernel $|x - y|^{-d - \alpha}$ for some $\alpha \in (0, 2)$ and the Sobolev-Slobodeckij space $H^{\alpha/2}(\mathbb{R}^d)$. We could also study examples with kernels which relate to a generic standard kernel $|x - y|^{-d} \phi(|x - y|^2)^{-1}$ where $\phi$ itself can be chosen from a rather general class of functions, e.g. the class of complete Bernstein functions.

Let us summarize the examples from above in a table. Recall that all examples satisfy (L). In the tabular below, the symbol ? indicates that the answer depends on the concrete specification of the example.

| Examples: (P) | (C) | (K) | symmetry |
|---------------|-----|------|----------|
| (1)           | ✓   | ✓    | ✓       |
| (2)           | ✓   | ✓    | ✓       |
| (3)           | ✓   | ✓    | ✓       |
| (4)           | ✓   | ✓    | ✓       |
| (5)           | ✓   | ✓    | ✓       |
| (6)           | ✓   | ✓    | ✓       |
| (7)           | ✓   | ✓    | ✓       |
| (8)           | ✓   | ✓    | ✓       |
| (9)           | ✓   | ✓    | ✓       |
| (10)          | ✓   | ✓    | ✓       |
| (11)          | ✓   | ✓    | ✓       |
| (12)          | ✓   | ✓    | ✓       |
| (13)          | ✓   | ✓    | ✓       |
| (14)          | ✓   | ✓    | ✓       |

**Integrable kernels**

**Non-integrable kernels**

**Index of Conditions.**

Condition (L) can be found on p. 2.

Conditions ($\bar{K}$) and (K) on p. 7.

Condition (P) on p. 9.

Condition (C) on p. 12.

Condition (E) on p. 16.
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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld
E-mail address: m.felsinger@math.uni-bielefeld.de

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld
E-mail address: moritz.kassmann@uni-bielefeld.de

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld
E-mail address: pvoigt@math.uni-bielefeld.de