PERVERSE SHEAVES ON THE NILPOTENT CONE AND LUSZTIG’S GENERALIZED SPRINGER CORRESPONDENCE

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Abstract. In this note, we consider perverse sheaves on the nilpotent cone. We prove orthogonality relations for the equivariant category of sheaves on the nilpotent cone in a method similar to Lusztig’s for character sheaves. We also consider cleanness for cuspidal perverse sheaves and the (generalized) Lusztig–Shoji algorithm.

1. Introduction

Let $G$ be a connected, reductive algebraic group, and let $\mathcal{N}$ be its nilpotent cone. We consider $D^b_G(\mathcal{N})$, the $G$-equivariant derived category of constructible sheaves on $\mathcal{N}$. This category encodes the representation theory of the Weyl group $W$ of $G$ via a perverse sheaf called the Springer sheaf $A$ (see Section 2 for a definition) and Springer’s correspondence. Of course, the category $\text{Perv}(\mathcal{N})$ of perverse sheaves on $\mathcal{N}$ contains much more information—there are $G$-equivariant perverse sheaves on $\mathcal{N}$ that are not part of the Springer correspondence. In [L1], Lusztig accounted for the extra information and related it to the representation theory of relative Weyl groups in his generalized Springer correspondence. His classification relies on understanding the cuspidal data associated to $G$ (see Definition 2.3).

Lusztig’s work also reveals that understanding the stalks of these perverse sheaves on $\mathcal{N}$ (given by generalized Green functions) is an important part of the computation of characters of finite groups of Lie type. Using the Lusztig–Shoji algorithm in [L5], these stalks can be computed using only knowledge of Weyl (or relative Weyl) group representations.

Our primary goal is to understand Lusztig’s work on the generalized Springer correspondence using methods inspired by [A] in the Springer setting. One of our main results is Theorem 3.5, an orthogonal decomposition:

$$D^b_G(\mathcal{N}) \cong \bigoplus_{c/\sim} D^b_G(\mathcal{N}, A_c).$$

Here, $c$ denotes a cuspidal datum and $D^b_G(\mathcal{N}, A_c)$ is the triangulated category generated by the simple summands of $A_c$, the perverse sheaf induced from the cuspidal datum $c$. This result relies on a key property enjoyed by cuspidal local systems on the nilpotent cone in good characteristic proven by Lusztig. That is, distinct cuspidals have distinct central characters.

In Section 5, we interpret the definitions of generalized Green functions directly in terms of perverse sheaves on the nilpotent cone avoiding characteristic functions on the corresponding group. We reprove the Lusztig–Shoji algorithm in Theorem 5.5 using the re-envisioned Green functions. Our proofs are heavily influenced by

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those of Lusztig in [L5]. From this point of view, Theorem 3.5 may be viewed as a categorical version of the orthogonality relations among generalized Green functions.

In Section 4, we prove cleanness for cuspidal local systems (see Proposition 4.2). Cleanness for cuspidal local systems was proven by Lusztig in [L5]. Logically, our proof is similar in spirit to Lusztig’s. However, a key role in the argument is played by the orthogonal decomposition in Theorem 3.5 considerably simplifying the exposition. In particular, from this point of view, it is easy to see that cuspidality for local systems is implied by the fact that the triangulated categories generated by the corresponding simple perverse sheaf and its Verdier dual are orthogonal to the ‘rest’ of the category.

Proposition 4.6 provides a computation of the Ext-groups between perverse sheaves on the nilpotent cone in terms of relative Weyl group invariants. This proposition is a first step towards proving formality for non-Springer blocks of sheaves on the nilpotent cone. Formality for the other blocks would complete the description of the equivariant derived category on the nilpotent cone, as was initiated by the first author [Ri] in the Springer case.

Organization of the paper. We review perverse sheaves on the nilpotent cone and the generalized Springer correspondence in Section 2. In Section 3, we prove our orthogonal decomposition (Theorem 3.5). In Section 4, we prove cleanness for cuspidal local systems (4.2), give some Ext computations (Proposition 4.6), and discuss purity for these Ext-groups (Corollary 4.7). In Section 5, we discuss generalized Green functions and the Lusztig–Shoji algorithm. Finally, in the appendix, we review some properties of central characters for equivariant perverse sheaves. In particular, we prove Proposition A.8 that perverse sheaves with distinct central characters have no extensions between them.

2. Preliminaries

We fix an algebraically closed field $k$. All varieties we consider will be defined over $k$ except in Section 5 when we need to employ mixed sheaves as developed in [De] and [BBD] Section 5. For an action of an algebraic group $G$ on a variety $X$, we consider the categories, denoted $\text{Perv}_G(X) \subset D_G(X)$, of $G$-equivariant perverse sheaves on $X$ and the $G$-equivariant (bounded) derived category of sheaves on $X$. See [BL] for background and definitions related to these categories. All of our sheaves will have $\mathbb{Q}_\ell$ coefficients. For $\mathcal{F}, \mathcal{G} \in D_G(X)$, we let $\text{Hom}(\mathcal{F}, \mathcal{G}) := \text{Hom}_{D_G(X)}(\mathcal{F}, \mathcal{G}[i])$. All sheaf functors are understood to be derived. We denote the constant sheaf on $X$ by $\underline{\mathbb{C}}_X$ or just $\underline{\mathbb{C}}$ when there is no ambiguity.

2.1. The Springer correspondence. From now on, we fix a connected, reductive algebraic group $G$, and we let $\mathcal{N}$ denote its nilpotent cone. We also assume that $k$ has good characteristic. We consider $G$-equivariant sheaves on $\mathcal{N}$ with respect to the adjoint $G$-action.

The nilpotent cone $\mathcal{N}$ is a singular variety and has a well-studied desingularization called the Springer resolution, denoted $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. Let $B$ be a Borel subgroup of $G$ with Levi decomposition $B = TU$, where $T$ is a maximal torus in $G$ and $U$ is the unipotent radical of $B$. Then, $\tilde{\mathcal{N}} = G \times^B U$, where $U = \text{Lie}(U)$. We also note that $\tilde{\mathcal{N}}$ can be identified with the cotangent bundle of the flag variety $T^*G/B$. 
Hence, we have maps
\[ \mathcal{N} \xleftarrow{\mu} \widetilde{\mathcal{N}} \xrightarrow{\pi} G/B. \]

**Definition 2.1.** The Springer sheaf, denoted \( \mathcal{A} \), is defined by
\[ \mathcal{A} := \mu_! \pi^* \underline{\mathcal{C}}[2d], \]
where \( \underline{\mathcal{C}} \) is the constant sheaf on \( G/B \) and \( d = \dim u \).

The Springer sheaf \( \mathcal{A} \) is a semisimple perverse sheaf by the Decomposition Theorem (see [BBD]), and it is \( G \)-equivariant. Furthermore, its endomorphism ring is isomorphic as an algebra to the group algebra of the Weyl group \( W \) of \( G \),
\[ \text{End}(\mathcal{A}) \cong \mathbb{C}[W]. \]
See [BM1]. This ring isomorphism allows us to link the simple summands of \( \mathcal{A} \) with irreducible \( W \) representations. This is known as the Springer correspondence.

### 2.2. The generalized Springer correspondence and cuspidal data.

In general, not all simple (\( G \)-equivariant) perverse sheaves on \( \mathcal{N} \) occur as part of the above correspondence. The goal of Lusztig’s generalized Springer correspondence [L1] is to systematically identify the missing pieces and to assign some representation theoretic meaning to them. To this end, we define analogues of the maps and varieties in (2.1).

Let \( P \) be a parabolic subgroup of \( G \) with Levi decomposition \( P = LU \). We denote by \( \mathcal{N}_L \) the nilpotent cone for \( L \) and \( u = \text{Lie}(U) \). We consider the following \( G \)-varieties and \( G \)-maps
\[ \widetilde{\mathcal{N}}^P := G \times^P (u + \mathcal{N}_L) \text{ and } \mathcal{C}_P := G \times^P \mathcal{N}_L, \]
\[ \mathcal{N} \xleftarrow{\mu^P} \widetilde{\mathcal{N}}^P \xrightarrow{\pi^P} \mathcal{C}_P. \]

The variety \( \widetilde{\mathcal{N}}^P \) is called a partial resolution of \( \mathcal{N} \) and is studied in [BM2]. When there is no ambiguity, we omit the subscript \( P \) on the maps for simplicity of notation. Note that \( \mu \) is proper and \( \pi \) is smooth of relative dimension \( d \), so we have \( \mu_! = \mu_\mu \) and \( \pi^! = \pi^! [2d] \), where \( d \) is the dimension of \( u \). We consider the following functors
\[ (2.2) \quad I^G_P = \mu_! \pi^*[d] \cong \mu_\mu \pi^! [-d], \quad R^G_P = \pi_\pi \mu_\mu \pi^! [-d], \quad \text{and } \widehat{R}^G_P = \pi^! \pi_\mu \mu_\mu [d], \]
which we will refer to as induction and restriction functors. We have adjoint pairs \((I^G_P, R^G_P)\) and \((\widehat{R}^G_P, I^G_P)\). In the case when our parabolic is a Borel \( B \), we get the usual Springer resolution diagram from (2.1). By (equivariant) induction equivalence [B], we have an equivalence of categories \( D^b_B(C_P) \cong D^b_N(\mathcal{N}_L) \) that preserves the perverse \( t \)-structure. Thus, often we will think of the induction (respectively, restriction) functor as having domain (respectively, codomain) \( D^b_N(\mathcal{N}_L) \):
\[ I^G_P : D^b_N(\mathcal{N}_L) \to D^b_G(\mathcal{N}) \text{ and } R^G_P, \widehat{R}^G_P : D^b_G(\mathcal{N}) \to D^b_N(\mathcal{N}_L). \]

The following theorem is due to Lusztig; see [L2, Theorem 4.4] for part of the proof in the setting of character sheaves. For a proof in the setting of more general coefficient rings (Noetherian commutative ring of finite global dimension), see [AHR, Proposition 4.7].

**Theorem 2.2.** The functors \( I^G_P, R^G_P, \text{ and } \widehat{R}^G_P \) are exact with respect to the perverse \( t \)-structure.
Proof. The result relies on Lusztig’s generalized Springer correspondence as a classification of the simple perverse sheaves on \( N \) by a direct sum of copies of \( \mathcal{I}_C \).

Proposition 2.6. Let \( c = (L, O_L, L) \) be a cuspidal datum for \( G \). Then \( R_G^0 \mathcal{I}_C^0 \mathcal{I}_C \) is a direct sum of copies of \( \mathcal{I}_C \). Moreover, the same statement holds if \( R_G^0 \) is replaced by \( R_G^\ell \).

Proof. The result relies on Lusztig’s generalized Springer correspondence as a classification of the simple perverse sheaves on \( N \).
Let \( S \) be a simple in \( \mathcal{P}erv(L(N_L)) \) that is a simple summand of \( R^G_P I^G_P IC_c \). We have that \( S \) is a direct summand of a Lusztig sheaf \( I^G_Q IC_{c'} \) on \( N_L \), where \( c' \) is a cuspidal datum for \( L \). Note that we allow the case where \( Q = L \). Hence, \( Hom(S, R^G_P I^G_P IC_c) \neq 0 \) implies

\[
Hom(I^G_Q IC_{c'}, R^G_P I^G_P IC_c) \neq 0.
\]

Let \( \tilde{Q} = QU_P \), where \( U_P \) is the unipotent radical of \( P \). Then \( \tilde{Q} \) is parabolic in \( G \). By adjunction and composition of inductions \( I^G_{\tilde{Q} Q} \cong I^G_P I^L_Q \) [L2 Proposition 4.2],

\[
Hom(I^G_{\tilde{Q} Q} IC_{c'}, I^G_P IC_c) \neq 0.
\]

This implies that \( I^G_{\tilde{Q} Q} IC_{c'} \) and \( I^G_P IC_c \) have a simple summand in common. However, Lusztig’s generalized Springer correspondence partitions the set of simple \( G \)-equivariant perverse sheaves on \( N \) into distinct classes, each labeled by a unique cuspidal datum up to \( G \)-conjugation. Hence, if \( I^G_{\tilde{Q} Q} IC_{c'} \) and \( I^G_P IC_c \) have a simple summand in common, then it must be that

\[
I^G_{\tilde{Q} Q} IC_{c'} \cong I^G_P IC_c,
\]

each labeled by cuspidal data that are \( G \)-conjugate. Since \( Q \subset L \), we have \( L = Q \). Thus, \( S \) is a summand of (and so, must be equal to) \( I^G_Q (IC_{c'}) \cong IC_c \).

The case for \( \tilde{R}^G_P \) follows in a similar manner. \( \square \)

3. Orthogonal decomposition

In this section, we prove our main result: an orthogonal decomposition of the category \( D^b_G(N) \) into blocks, each corresponding to a cuspidal datum for \( G \).

**Definition 3.1.** Let \( \mathcal{T} \) be a triangulated category. We say that \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), two triangulated subcategories of \( \mathcal{T} \), are orthogonal if for all objects \( \mathcal{F} \in \mathcal{T}_1 \) and \( \mathcal{G} \in \mathcal{T}_2 \), we have that

\[
Hom(\mathcal{F}, \mathcal{G}) = Hom(\mathcal{G}, \mathcal{F}) = 0.
\]

If we have a (finite) collection \( \mathcal{T}_i \) of triangulated subcategories of \( \mathcal{T} \) that are pairwise orthogonal and that generate \( \mathcal{T} \) such that each \( \mathcal{T}_i \) cannot be split further, then we call them blocks. The equivalence

\[
\mathcal{T} \cong \bigoplus_i \mathcal{T}_i
\]

is called a block decomposition of \( \mathcal{T} \).

Suppose we have cuspidal perverse sheaves \( IC_c \) on \( N_L \) and \( IC_{c'} \) on \( N_{L'} \), where \( L \) and \( L' \) are the Levi factors of parabolics \( P \) and \( P' \) in \( G \). Let \( E = N_{P \cap L} \times N_{L \cap L'} \). \( N_{P \cap L'} \).

**Proposition 3.2.** If \( P' \cap L \times P \cap L' \) is properly contained within \( L \times L' \), then \( H^i_c(E, IC_c \boxtimes IC_{c'}) = 0 \) for all \( i \).

**Proof.** Consider the following diagram.
Clearly, it is enough to show that
\[
\phi
\]
(JLAURA RIDER AND AMBER RUSSELL), (resp. \(P\)) ∈ \(x\) all \(i\)

We will first show an analogous statement for cohomology with compact \(G\) also denote the

The outline of our proof follows that of a similar result due to Lusztig [L3, Prop. 7.2] in the setting of character sheaves.

By base-change, we have an isomorphism \(\tilde{\omega}i^*\) is easily seen to be the non-equivariant version of restriction \(\tilde{R}\) as in \([22]\). Hence, \(\omega i^*(IC_c \boxtimes IC_{c'}) = 0\) since \(IC_c \boxtimes IC_{c'}\) is cuspidal on \(N_L \times N_{L'}\) and \(P' \cap L \times P \cap L'\) is a proper parabolic in \(L \times L'\). Hence, we have that \(0 = \Delta^* \omega i^*(IC_c \boxtimes IC_{c'}) = \tilde{\omega}i^*\tau^*(IC_c \boxtimes IC_{c'})\). Therefore, \(H^*_c(E, IC_c \boxtimes IC_{c'}) = 0\) for all \(i\).

Let \(Z = \tilde{N}^P < N \tilde{N}^P\) be a generalized Steinberg variety. We use \(IC_c\) and \(IC_{c'}\) to also denote the \(G\)-equivariant perverse sheaves on \(\mathcal{C}_P\) and \(\mathcal{C}_{P'}\) which correspond to the cuspidal perverse sheaves on \(N_L\) and \(N_{L'}\) under induction equivalence. Consider the composition

\[
(\tau : \tilde{N}^P \times_N \tilde{N}^P') \mapsto \tilde{N}^P \times \tilde{N}^P';
\]

Our goal in the following proposition is to prove that the cohomology groups \(H^*_G(Z, \tau^1 IC_c \boxtimes IC_{c'})\) vanish for all \(i\) when \(IC_c\) and \(IC_{c'}\) arise from cuspidals with non-conjugate Levi's. The outline of our proof follows that of a similar result due to Lusztig [L3, Prop. 7.2] in the setting of character sheaves.

**Proposition 3.3.** Suppose \(P\) and \(P'\) are parabolics in \(G\) such that their Levi factors \(L\) and \(L'\) are non-conjugate. Let \(Z\) be as above. Then, \(H^i(Z, \tau^1 IC_c \boxtimes IC_{c'}) = 0\) for all \(i\).

**Proof.** We will first show an analogous statement for cohomology with compact support. Let \(x \in G/P \times G/P'\), and consider the following diagram.

\[
\phi^{-1}(x) \xleftarrow{\iota} Z \xrightarrow{\phi} G/P \times G/P' \xrightarrow{pt}
\]

Clearly, it is enough to show that \(\phi\iota\tau^*(IC_c \boxtimes IC_{c'}) = 0\). This holds if all the stalks of \(\phi\iota\tau^*(IC_c \boxtimes IC_{c'})\) vanish. This is equivalent to \(\phi_x(\iota^*\tau^*(IC_c \boxtimes IC_{c'})) = 0\) for all \(x \in G/P \times G/P'\) by base change. In other words, it is enough to show that \(H^*_c(\phi^{-1}(x), \tau^*(IC_c \boxtimes IC_{c'})) = 0\) for all \(x\). Since \(\phi\iota\tau^*(IC_c \boxtimes IC_{c'})\) is \(G\)-equivariant, it is enough to check

\[
\phi_x(\iota^*\tau^*(IC_c \boxtimes IC_{c'})) = 0
\]
for a single $x$ in each $G$-orbit of $G/P \times G/P'$. Let $n \in G$ be such that $L$ and $nL'n^{-1}$ share a maximal torus and consider $Z(n) = \phi^{-1}((P, nP'))$, a subvariety of $Z$. We must show that $H^i_c(Z(n), \tau^*(IC_c \boxtimes IC_c')) = 0$ for all $i$. Consider the following commutative diagram where $L'' = nL'n^{-1}, P'' = nP'n^{-1}, f : N_{L''} \to N_{L'}$, and $E = N_{P'' \cap L} \times N_{L \cap L''} \cap N_{P \cap L''}$.

$$
\begin{array}{cccc}
Z & \stackrel{\tau}{\longrightarrow} & C_P \times C_{P''} & \stackrel{\sim}{\longrightarrow} & C_P \times C_{P''} \\
\downarrow & & \downarrow & & \downarrow \\
N_L \times N_{L''} & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow \\
N_{P'' \cap L} \times N_{L \cap L''} & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow \\
Z(n) & \sim & N_P \cap N_{P''} & \stackrel{\alpha}{\longrightarrow} & E
\end{array}
$$

Let $IC_{c''} = f^* IC_{c'}$. Hence, $H^i_c(Z(n), \tau^*(IC_c \boxtimes IC_{c'})) \cong H^i_c(Z(n), \alpha^*(IC_c \boxtimes IC_{c''}))$. The map $\alpha$ is smooth of relative dimension $d$ where $d = \dim U_{P \cap P''}$, hence $\alpha^*[2d] \cong \alpha^!$.

Finally, we apply a special case of Braden’s hyperbolic localization: equation (1) in [80 Section 3]. In the diagram below, we let the multiplicative group $\mathbb{G}_m$ act on all varieties with compatible positive weights. Let $e : \{(0, 0)\} \to E$ be the inclusion of the fixed point of this action, and let $\ell : E \to \{(0, 0)\}$ be the map that sends every point to its limit. Note that Proposition 4.2 implies that $\ell_1 IC_c \boxtimes IC_{c''} = 0$.

$$
\begin{array}{ccc}
\{0\} & \xrightarrow{a} & \mathbb{A}^n \\
\uparrow & & \uparrow h \\
b \downarrow & \downarrow e \downarrow & \downarrow \ell \\
N_P \cap N_{P''} & \xrightarrow{\alpha} & E
\end{array}
$$

Then hyperbolic localization implies that we have an isomorphisms $\ell_1 \cong \ell_1$ and $a_1 \cong (b \circ c)_1$. Furthermore, the diagram commutes. Combining these, we see that

$$a_1 \alpha^! c_1 h_1 \alpha^! c_1 h_1 \ell_1 \cong c_1 h_1 \ell_1 \alpha^! c_1 h_1 \ell_1$$

Now, we apply this to the cuspidal perverse sheaf $IC_c \boxtimes IC_{c''}$ to see that

$$a_1 \alpha^! IC_c \boxtimes IC_{c''} \cong c_1 h_1 \ell_1 IC_c \boxtimes IC_{c''} = 0.$$
Suppose Corollary 3.7. □

Proof in [A, Theorem 5.1].

implies the base case. The argument then follows without modification from the computations that Lusztig carried out in [L1] and [L2].

**Theorem 3.5.** We have an orthogonal decomposition indexed by cuspidal data up to equivalence

\[
\mathcal{D}_G^b(\mathcal{N}) \cong \bigoplus_{c/\sim} \mathcal{D}_G^b(\mathcal{N}', \mathbb{A}_c).
\]

**Proof.** Let \( c = (L, \mathcal{O}, \mathcal{L}) \) and \( c' = (L', \mathcal{O}', \mathcal{L}') \) be two cuspidal data such that \( c \not\sim c' \) with Lusztig sheaves \( \mathbb{A}_c \) and \( \mathbb{A}_{c'} \). We want to show that the two triangulated categories \( \mathcal{D}_G^b(\mathcal{N}, \mathbb{A}_c) \) and \( \mathcal{D}_G^b(\mathcal{N}, \mathbb{A}_{c'}) \) are orthogonal in \( \mathcal{D}_G^b(\mathcal{N}) \). It is sufficient to show that \( \text{Hom}^i(S, S') = \text{Hom}^i(S', S) = 0 \) for all \( i \in \mathbb{Z} \) and all simple summands \( S \) of \( \mathbb{A}_c \) and \( S' \) of \( \mathbb{A}_{c'} \).

We have two cases to consider: the Levis \( L \) and \( L' \) are either conjugate or not. In the second case, we apply equation (8.6.4) from [CG, Lemma 8.6.1] to yield

\[
\text{Hom}^i(\mathbb{A}_c, \mathbb{A}_{c'}) \cong H^i_G(Z, \tau^i(\mathbb{D} \mathbb{IC}_c \boxtimes \mathbb{IC}_{c'})),
\]

which vanishes by Proposition 3.3.

In the first case,

\[
\text{Hom}^i(\mathbb{A}_{c'}, \mathbb{A}_c) = \text{Hom}^i(\mathbb{IC}_{c'}, \mathcal{R}_{L'}^{G} \mathcal{I}_{L}^{G} \mathbb{IC}_c).
\]

Here, we apply Proposition 2.6 to see that \( \mathcal{R}_{L'}^{G} \mathcal{I}_{L}^{G} \mathbb{IC}_c \) is a finite direct sum of copies of \( \mathbb{IC}_c \). Hence, it is sufficient to see that \( \text{Hom}^i(\mathbb{IC}_{c'}, \mathbb{IC}_c) = 0 \) for non-isomorphic cuspidals \( \mathbb{IC}_{c'} \) and \( \mathbb{IC}_c \). Lusztig proves in [L1] that non-isomorphic cuspidal perverse sheaves have distinct central characters. Therefore we may apply Proposition A.8 to complete the proof. □

Let \( u : Y \hookrightarrow \mathcal{N} \) be a \( G \)-stable locally closed subvariety. Fix a Lusztig sheaf \( \mathbb{A}_c \). We define \( \mathcal{D}_G^b(Y, \mathbb{A}_c) \) as the triangulated subcategory of \( \mathcal{D}_G^b(Y) \) generated by restrictions \( u^*S \) for all simple summands \( S \) of \( \mathbb{A}_c \). (Note that \( \mathcal{D}_G^b(Y, \mathbb{A}_c) \) may become trivial.)

**Corollary 3.6.** Let \( Y \) be a \( G \)-stable locally closed subvariety of \( \mathcal{N} \). Then restriction to \( Y \) preserves orthogonality. That is, we have an equivalence

\[
\mathcal{D}_G^b(Y) \cong \bigoplus_{c/\sim} \mathcal{D}_G^b(Y, \mathbb{A}_c).
\]

**Proof.** The proof is by induction on the number of \( G \)-orbits in \( \bar{Y}/Y \). Theorem 3.5 implies the base case. The argument then follows without modification from the proof in [A Theorem 5.1]. □

**Corollary 3.7.** Suppose \( \mathcal{L} \) is a local system on the orbit \( \mathcal{O} \) of \( \mathcal{N} \) which appears as a composition factor of \( \mathcal{H}^i(\mathbb{IC}_\chi \! | \! \mathcal{O}) \) for \( \chi \in \text{Irr} \ W(L) \) for some Levi \( L \). Then \( \mathbb{IC}(\mathcal{O}, \mathcal{L}) \cong \mathbb{IC}_\psi \) for some \( \psi \in \text{Irr} \ W(L) \) where \( \mathbb{IC}_\chi \) and \( \mathbb{IC}_\psi \) are in the same block.
4. Applications: cleanness and Ext computations

**Definition 4.1.** For a nilpotent orbit $\mathcal{O}$ of $\mathcal{N}$ consider the inclusion $j : \mathcal{O} \hookrightarrow \mathcal{N}$. A local system $\mathcal{L}$ on $\mathcal{O}$ is called clean if $j_! \mathcal{L} = j_* \mathcal{L} = \text{IC}(\mathcal{O}, \mathcal{L})$.

The following statement is well known in the setting of perverse sheaves on the nilpotent cone and character sheaves. In good characteristic, this follows from work of Lusztig [L1, L2, L3, L4] and in particular, [L5, Theorem 23.1]. His argument uses the fact that all cuspidal local systems in good characteristic must have distinct central characters, and he goes on to show that this implies an orthogonality relationship which gives the result. See [L2, Proposition 7.9] for character sheaves.

**Proposition 4.2.** Cuspidal local systems are clean.

**Proof.** Let $\mathcal{L}$ be a cuspidal local system on the orbit $\mathcal{O}$, and here, let $j : \mathcal{O} \hookrightarrow \mathcal{N}$. By the orthogonal decomposition, $\text{IC}(\mathcal{O}, \mathcal{L})$ is orthogonal to all other simple perverse sheaves on $\mathcal{N}$. Let $\iota : \mathcal{O}' \hookrightarrow \mathcal{N}$ be the inclusion of an orbit $\mathcal{O}' \subset \mathcal{O}$ with $\mathcal{O} \neq \mathcal{O}'$.

Suppose that $\iota^* \text{IC}(\mathcal{O}, \mathcal{L}) \neq 0$. We assume without loss of generality that $\mathcal{O}'$ is minimal among orbits in $\mathcal{O}$ with this property, i.e. we assume that no orbit in $\mathcal{O}'/\mathcal{O}'$ has this property. Then, there exists $i \in \mathbb{Z}$ and a simple perverse sheaf $S$ (a shifted local system) on $\mathcal{O}'$ such that $\text{Hom}^i(\iota^* \text{IC}(\mathcal{O}, \mathcal{L}), S) \neq 0$. By adjunction, we have $\text{Hom}^i(\text{IC}(\mathcal{O}, \mathcal{L}), \iota_* S) = 0$. Consider the distinguished triangle

$$p_{i*} S \rightarrow \iota_* S \rightarrow A \rightarrow$$

where $A \in \mathcal{P} \mathcal{D}^{>0}$. This gives an exact sequence

$$\text{Hom}^i(\text{IC}(\mathcal{O}, \mathcal{L}), \iota^* S) \rightarrow \text{Hom}^i(\text{IC}(\mathcal{O}, \mathcal{L}), \iota_* S) \rightarrow \text{Hom}^i(\text{IC}(\mathcal{O}, \mathcal{L}), A).$$

Note that $\text{Hom}^i(\text{IC}(\mathcal{O}, \mathcal{L}), A) = 0$ since we assumed minimality of $\mathcal{O}'$ and the support of $A$ is contained within $\overline{\mathcal{O}'}/\mathcal{O}'$. In particular, this implies that $\iota^* S$ has $\text{IC}(\mathcal{O}, \mathcal{L})$ as a direct summand since $\text{IC}(\mathcal{O}, \mathcal{L})$ is orthogonal to all other simple perverse sheaves on $\mathcal{N}$. However, the support of $p_{i*} S$ is contained within $\overline{\mathcal{O}'}$, which is a contradiction. Therefore, it must be that the stalk of $\text{IC}(\mathcal{O}, \mathcal{L})$ at any $x \in \mathcal{O}'$ is 0. Thus, $j_! \mathcal{L} = \text{IC}(\mathcal{O}, \mathcal{L})$, where $d = \text{dim} \mathcal{O}$.

Recall that $\mathcal{D} \mathcal{L}$ is also cuspidal on $\mathcal{O}$ (Remark 2.3). The above argument proves that $j_! (\mathcal{D} \mathcal{L}^\vee)[d] = \text{IC}(\mathcal{O}, \mathcal{L}^\vee)$, where $\mathcal{L}^\vee$ is the dual local system to $\mathcal{L}$. Thus, $\mathcal{D} j_* \mathcal{L}[d] = \mathcal{D} \text{IC}(\mathcal{O}, \mathcal{L})$ which implies $j_* \mathcal{L}[d] \cong \text{IC}(\mathcal{O}, \mathcal{L})$. $\square$

4.1. Some Ext computations. For the remainder of this section, let us assume we are in a particular block of the decomposition corresponding to the cuspidal datum $c = (L, \mathcal{O}, \mathcal{L})$. This block will have cuspidal simple perverse sheaf $\text{IC}_c$ on $\mathcal{N}_L$. The simple summand of $\mathcal{A}_c$ corresponding to the representation $V_\psi$ of $W(L)$ will be denoted $\text{IC}_\psi$.

**Lemma 4.3.** We have that

$$\text{Hom}^i_{\mathcal{D} \mathcal{L}(\mathcal{N}_L)}(\text{IC}_c, \text{IC}_c) \cong H_L^i(\mathcal{O}) \cong H_G^i(G \times^P \mathcal{O}).$$
In particular, $\text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\text{IC}_c, \text{IC}_c)$ is pure of weight $i$ where Frobenius acts by $q^{i/2}$ and vanishes for $i$ odd.

Proof. Since cuspidal local systems are clean, we have that

$$\text{Hom}^i(\text{IC}_c, \text{IC}_c) = \text{Hom}^i(j_!L_i, j_!L_i).$$

This reduces to a computation of local systems: $\text{Hom}^i(j_!L_i, j_!L_i) = \text{Hom}^i(L_i, L_i).$ On the other hand, $\text{Hom}^i(L_i, L_i) = H^i(\mathcal{O}, R\text{Hom}(L_i, L_i))$, and for local systems, $R\text{Hom}(L_i, L_i) = \mathbb{Z}[\mathbb{G}_a]$. Thus, $\text{Hom}^i(\text{IC}_c, \text{IC}_c) \cong H^i_L(\mathcal{O})$, as desired.

We assume that all algebraic groups are split over $\mathbb{F}_q$. Let $x$ be a closed point in $\mathcal{O}$ fixed by Frobenius, so $\mathcal{O} \cong L \cdot x$. Let $\text{Stab}_L(x)$ be the stabilizer of $x$ in $L$, and $Z(L)^o$ be the identity component of the center of $L$. By [L6, Section 2.3]), the group $Z(L)^o$ is a (maximal) central torus in $\text{Stab}_L(x)$. It follows that

$$\text{H}^i_L(\mathcal{O}) \cong \text{H}^i_{\text{Stab}_L(x)}(x) \cong \text{H}^i_{Z(L)^o}(x).$$

The second isomorphism follows from [L6, 1.12 (a)]. Now, it is well known that $\text{H}^i(\mathbb{P}^\infty)^r$ where $r = \text{rank} Z(L)^o$ and that Frobenius acts by $q^{id}$ in degree 2.

**Lemma 4.4.** Let $\text{IC}_\psi$ be a simple summand of $\mathcal{A}_c$ as described above. We have an isomorphism $\mathcal{R}_\psi^G(\text{IC}_\psi) \cong \tilde{\mathcal{R}}_\psi^G(\text{IC}_\psi) \cong \text{IC}_c \otimes V_\psi^*.$

Proof. By Proposition [2.3] we know $\mathcal{R}_\psi^G(\text{IC}_\psi)$ is contained in the block $D^b_c(\mathcal{N}_L, \text{IC}_c)$. Since our restriction functor $\mathcal{R}_\psi^G$ is $i$-exact, we know $\mathcal{R}_\psi^G(\text{IC}_\psi)$ must be perverse. As $\text{IC}_c$ is the only simple perverse sheaf in $D^b_c(\mathcal{N}_L, \text{IC}_c)$, we have $\mathcal{R}_\psi^G(\text{IC}_\psi) \cong \text{IC}_c \otimes \text{Hom}(\text{IC}_c, \mathcal{R}_\psi^G(\text{IC}_\psi))$. Furthermore, we have

$$\text{Hom}(\text{IC}_c, \mathcal{R}_\psi^G(\text{IC}_\psi)) \cong \text{Hom}(\mathcal{A}_c, \text{IC}_\psi) \cong \text{Hom}(\text{IC}_\psi \otimes V_\psi, \text{IC}_\psi) \cong V_\psi^*.$$

Now, we need only to show that $\mathcal{R}$ and $\tilde{\mathcal{R}}$ give isomorphic objects when restricted to the category of perverse sheaves. Following the same reasoning as above, we have $\tilde{\mathcal{R}}_\psi^G(\text{IC}_\psi) \cong \text{IC}_c \otimes V_\psi$. Since $W(L)$ is a Weyl group, $V_\psi \cong V_\psi^*$, and the result follows.

**Proposition 4.5.** We have an isomorphism

$$\text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\mathcal{A}_c, \mathcal{A}_c) \cong H^i_L(\mathcal{O}) \otimes \mathbb{C}[W(L)].$$

Proof. First, using a similar argument as in the proof of Lemma [4.4] and Lusztig’s isomorphism [2.3], we have that $\mathcal{R}_\psi^G \mathcal{A}_c \cong \text{IC}_c \otimes \text{Hom}(\text{IC}_c, \mathcal{R}_\psi^G \mathcal{A}_c) \cong \text{IC}_c \otimes \mathbb{C}[W(L)]$.

$$\text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\mathcal{A}_c, \mathcal{A}_c) \cong \text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\text{IC}_c, \mathcal{R}_\psi^G \mathcal{A}_c)$$

$$\cong \text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\text{IC}_c, \text{IC}_c \otimes \mathbb{C}[W(L)])$$

$$\cong \text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\text{IC}_c, \text{IC}_c) \otimes \mathbb{C}[W(L)].$$

Finally, we apply Lemma [4.3] and the result follows.

**Proposition 4.6.** For simple perverse sheaves $\text{IC}_\psi$ and $\text{IC}_\chi$ in the same block corresponding to $\psi, \chi \in \text{Irr} W(L)$, we have an isomorphism

$$\text{Hom}^i_{D^b_c(\mathcal{N}_L)}(\mathcal{R}_\psi^G(\text{IC}_\psi), \mathcal{R}_\psi^G(\text{IC}_\chi)) \cong V_\psi \otimes H^i_L(\mathcal{O}) \otimes V_\chi^*.$$
Moreover, we have that
\[ \text{Hom}_{D^b_c(N)}^i(\text{IC}_\psi, \text{IC}_\chi) \cong \text{Hom}_{D^b_c(N)}^i(\mathcal{R}_P^G(\text{IC}_\psi), \mathcal{R}_P^G(\text{IC}_\chi))^{W(L)} \cong (V_\psi \otimes H^i_L(O) \otimes V_\chi^{\ast})^{W(L)}. \]

**Proof.** The first statement follows quickly from combining Lemmas 4.3 and 4.4. For the second statement, the proof follows in the same vain as in [A, Theorem 4.6]. \( \square \)

**Corollary 4.7.** Assume that \( k \) has positive, good characteristic. Let \( P_1, P_2 \in \mathcal{Perv}_G(N) \). Then \( \text{Hom}_{D^b_c(N)}^i(P_1, P_2) \) is pure of weight \( i \) for all even \( i \in \mathbb{Z} \) and vanishes for all odd \( i \).

**Proof.** First, suppose that \( P_1 \) and \( P_2 \) are simple. If they are not in the same block, then \( \text{Hom}^i(P_1, P_2) \) vanishes by the orthogonal decomposition, Theorem 4.5. If they are in the same block, then the result follows from Proposition 4.6 and the fact that \( H^i_L(O) \) is pure of weight \( i \) for \( i \) even and vanishes for \( i \) odd. For general \((G\text{-equivariant)} \) perverse sheaves on \( N \), we use the facts that \( \mathcal{Perv}_G(N) \) is a semisimple category and that \( \text{Hom}^i(-, -) \) commutes with direct sum. \( \square \)

### 5. Generalized Green functions

In this section, we talk about generalized Green functions. For this, we need to work in the mixed category. For \( X \) an algebraic variety defined over \( \overline{\mathbb{F}}_q \), we consider the category of mixed \( \ell \)-adic complexes \( D^b_m(X) \). There is a natural functor forgetting the Weil structure \( \xi : D^b_m(X) \rightarrow D^b_c(X \times_{\text{Spec}(\overline{\mathbb{F}}_q)} \text{Spec}(\overline{\mathbb{F}}_q)) \). The standard reference for the definition and properties of \( D^b_m(X) \) is [BBD, Section 5].

We define \( K(X) \) as the quotient of the Grothendieck group \( K(D^b_m(X))/\sim \) where we identify isomorphism classes of simple perverse sheaves \([S_1] \sim [S_2]\) if \( S_1 \) and \( S_2 \) have the same weight and \( \xi(S_1) \cong \xi(S_2) \). We fix a square root of the Tate sheaf. Then \( K(X) \) has the structure of a \( \mathbb{Z}[t^{1/2}, t^{-1/2}] \)-module so that the action of \( t \) corresponds to Tate twist: \([F(i/2)] = t^{-i/2}[F]\). For the rest of this section, we assume that \( G \) is a split connected, reductive algebraic group defined over \( \mathbb{F}_q \) and let \( N \) denote its nilpotent cone. We also assume that \( \mathbb{F}_q \) is sufficiently large so that all nilpotent orbits are non-empty. If \( O \) is a nilpotent orbit, then \( K(O) \) is the free \( \mathbb{Z}[t^{1/2}, t^{-1/2}] \)-module generated by classes of (weight 0) irreducible local systems on \( O \).

In what follows, we define four sets of polynomials \( p_{S_1S_2}, \lambda_{S_1S_2}, \tilde{p}_{S_1S_2}, \) and \( \omega_{S_1S_2} \) for a pair of simple perverse sheaves \( S \) and \( S' \). In the case that \( S \) and \( S' \) are in different blocks, we have that \( p_{S_1S_2} = \tilde{p}_{S_1S_2} = \omega_{S_1S_2} = 0 \) by Theorem 5.6 and Corollary 5.7. Thus, for simplicity of notation, we will assume throughout the section that \( S \) and \( S' \) are in the same block. For \( \chi, \psi \in \text{Irr}(W(L)) \), let \( \text{IC}_\chi \) and \( \text{IC}_\psi \) be the simple perverse sheaves that are pure of weight 0 corresponding to \( V_\chi^* \) and \( V_\psi^* \). In this case, we will instead use the notation: \( p_{\chi, \psi}(t), \lambda_{\chi, \psi}(t), \tilde{p}_{\chi, \psi}(t), \) and \( \omega_{\chi, \psi}(t) \). Also, we denote by \( L_\psi \) the local system such that \( \text{IC}_\psi |_{O_\psi} = L_\psi [\dim O_\psi](\frac{1}{2} \dim O_\psi) \), where \( O_\psi \) is the orbit open in the support of \( \text{IC}_\psi \).

We define \( p_{\chi, \psi}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \) as
\[
[\text{IC}_\chi |_{O}] = \sum_{\psi | O_\psi = O} p_{\chi, \psi}(t)[L_\psi].
\]
We also define ‘dual’ polynomials. Let \( j_\mathcal{O} : \mathcal{O} \to \mathcal{N} \) be the inclusion of an orbit. Let \( \tilde{p}_{\chi,\psi}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \) be given by
\[
(5.2) \quad [j_\mathcal{O}^! IC_\chi] = \sum_{\psi : \mathcal{O}_\psi = \mathcal{O}} \tilde{p}_{\chi,\psi}(t)[L_\psi].
\]

**Lemma 5.1.** The polynomials \( p \) and \( \tilde{p} \) satisfy the following relation: \( \tilde{p}_{\chi,\psi}(t^{-1}) = t^{\dim \mathcal{O}} p_{\chi^\vee,\psi^\vee}(t) \).

**Proof.** Let \( IC_\chi = IC(\mathcal{O}_\chi, L_\chi) \). First, we have that \( j_\mathcal{O}^!(IC_\chi) \cong \mathbb{D} j_\mathcal{O}^! \mathbb{D} IC_\chi = \mathbb{D}(IC(\mathcal{O}_\chi, L_\chi^\vee) |_{\mathcal{O}}) \), where \( L_\chi^\vee \) denotes the dual local system to \( L_\chi \). Furthermore, Verdier dual transforms local systems in the following way:
\[
(5.3) \quad \mathbb{D}(L(-i)) = L^\vee[2 \dim \mathcal{O}][\dim \mathcal{O} + i].
\]
Hence, Verdier dual induces a morphism \( \mathbb{D} : K(\mathcal{O}) \to K(\mathcal{O}) \) given by \( t^i[L] \mapsto t^{-\dim \mathcal{O} - i}[L^\vee] \). We apply \( \mathbb{D} \) to equation (5.1) to get
\[
[j_\mathcal{O}^! IC_\chi] = [\mathbb{D} j_\mathcal{O}^! IC(\mathcal{O}_\chi, L_\chi^\vee)]
= \mathbb{D} \sum_{\psi : \mathcal{O}_\psi = \mathcal{O}} p_{\chi^\vee,\psi}(t)[L_\psi]
= \sum_{\psi : \mathcal{O}_\psi = \mathcal{O}} t^{-\dim \mathcal{O}} p_{\chi^\vee,\psi}(t^{-1})[L_\psi]
= \sum_{\psi : \mathcal{O}_\psi = \mathcal{O}} t^{-\dim \mathcal{O}} p_{\chi^\vee,\psi^\vee}(t^{-1})[L_\psi].
\]
Since the irreducible local systems (of weight 0) \([L_\psi]\) are linearly independent in \( K(\mathcal{O}) \), we have that
\[
\tilde{p}_{\chi,\psi}(t) = t^{-\dim \mathcal{O}} p_{\chi^\vee,\psi^\vee}(t^{-1}).
\]
The result follows. \( \square \)

Since we know \( IC_\chi |_{\mathcal{O}_\chi} = L_\chi[\dim \mathcal{O}_\chi](1/2 \dim \mathcal{O}_\chi) \) and \( IC_\chi \) vanishes off \( \overline{\mathcal{O}_\chi} \), we have
\[
(5.4) \quad p_{\chi,\psi}(t) = \begin{cases} 
  t^{-\dim \mathcal{O}_\chi/2} & \text{if } \chi = \psi \\
  0 & \text{if } \mathcal{O}_\psi \not\subset \overline{\mathcal{O}_\chi} \text{ or if } \mathcal{O}_\psi = \mathcal{O}_\chi \text{ with } \chi \neq \psi
\end{cases}
\]
We also define \( \lambda_{\chi,\psi}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \) by
\[
(5.5) \quad [R \Gamma_\chi(\mathcal{O}_\chi, L_\chi \otimes L_\psi)] = \lambda_{\chi,\psi}(t)[\mathbb{F}_{\mathcal{O}_\chi}]
\quad \lambda_{\chi,\psi}(t) = 0 \quad \text{if } \mathcal{O}_\chi \not\subset \mathcal{O}_\psi.
\]
and \( \omega_{\chi,\psi}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \) by
\[
(5.6) \quad [\mathbb{D} R \text{Hom}(IC_\chi, IC_\psi)] = \omega_{\chi,\psi}(t)[\mathbb{F}_{\mathcal{O}_\chi}].
\]
Using Corollary 4.7, we can reformulate the definition of \( \omega \) in the following way:
\[
(5.7) \quad \omega_{\chi,\psi}(t) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(\mathbb{D} R \text{Hom}(IC_\chi, IC_\psi)) t^{i/2}
\quad = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}^{-i}(IC_\chi, IC_\psi) t^{i/2}.
\]
Lemma 5.2. For any simple perverse sheaves $IC_{\chi}$ and $IC_{\psi}$ in $\mathcal{Perv}_G(\mathcal{N})$, we have that $\omega_{\chi,\psi}(t) = \omega_{\psi,\chi}(t)$. Furthermore, if $IC_{\chi}$ and $IC_{\psi}$ are in different blocks, then $\omega_{\chi,\psi}(t) = \lambda_{\chi,\psi}(t) = 0$.

Proof. If $IC_{\chi}$ and $IC_{\psi}$ are in different blocks, then $\omega_{\chi,\psi}(t) = \omega_{\psi,\chi}(t) = 0$ follows from Theorem 3.5 and Corollary 3.7 implies $\lambda_{\chi,\psi}(t) = \lambda_{\psi,\chi}(t) = 0$.

Now assume that $IC_{\chi}$ and $IC_{\psi}$ are in the same block. Proposition 4.4 implies

\[ \dim(\text{Hom}(V_{\psi} \otimes H^*_{\mathcal{L}}(\mathcal{O}) \otimes V_{\psi}^*)^{W(L)}) = \dim(\text{Hom}(V_{\chi} \otimes H^*_{\mathcal{L}}(\mathcal{O}) \otimes V_{\psi})^{W(L)}) \]

Since $W(L)$ is a Weyl group, we have $V \cong V^*$ for any $W(L)$-representation $V$. In particular, $\dim(\text{Hom}(V_{\psi} \otimes H^*_{\mathcal{L}}(\mathcal{O}) \otimes V_{\psi}^*)^{W(L)}) = \dim(\text{Hom}(V_{\chi} \otimes H^*_{\mathcal{L}}(\mathcal{O}) \otimes V_{\psi})^{W(L)})$. Hence, using equation (5.7) we obtain $\omega_{\chi,\psi}(t) = \omega_{\psi,\chi}(t)$. \hfill \Box

The following is a refinement of [A] Lemma 6.6 to include perverse sheaves that are not self dual.

Lemma 5.3.

\[ [\mathcal{D} \text{Hom}(j_{\mathcal{O}}' IC_{\chi}, j_{\mathcal{O}}' IC_{\psi})] = \sum_{\{\phi, \phi' | \mathcal{O}_{\phi} = \mathcal{O}_{\phi'} = \mathcal{O}\}} p_{\chi,\phi}(t)\lambda_{\phi,\phi'}(t)p_{\psi,\phi'}^\vee(t)[\mathcal{O}_\mathcal{A}] \]

Proof. First, we note that $[\mathcal{D}(\text{Hom}(\mathcal{F}(n), \mathcal{S}(m))) = (\mathcal{D}(\text{Hom}(\mathcal{F}, \mathcal{S}))(n-m))$. Thus, $\mathcal{D}(\text{Hom}(\mathcal{F}, \mathcal{S}))_\mathcal{A} \mathcal{F}(n) \times \mathcal{O} \rightarrow \mathcal{O}$ is $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ linear in the first variable and antilinear in the second variable with respect to the involution $t^{1/2} \mapsto t^{-1/2}$. Hence,

\[ [\mathcal{D} \text{Hom}(j_{\mathcal{O}}' IC_{\chi}, j_{\mathcal{O}}' IC_{\psi})] = \sum_{\{\phi, \phi' | \mathcal{O}_{\phi} = \mathcal{O}_{\phi'} = \mathcal{O}\}} p_{\chi,\phi}(t)p_{\psi,\phi'}^\vee(t^{-1}) \mathcal{D}[\text{Hom}(L_{\phi}, L_{\phi'})]. \]

Now, we apply Lemma 5.1 to get

\[ [\mathcal{D} \text{Hom}(j_{\mathcal{O}}' IC_{\chi}, j_{\mathcal{O}}' IC_{\psi})] = t^{\dim \mathcal{O}} \sum_{\{\phi, \phi' | \mathcal{O}_{\phi} = \mathcal{O}_{\phi'} = \mathcal{O}\}} p_{\chi,\phi}(t)p_{\psi,\phi'}^\vee(t)[\mathcal{D} \text{Hom}(L_{\phi}, L_{\phi'})]. \]

To finish the proof, it suffices to show $[\mathcal{D} \text{Hom}(L_{\phi}, L_{\phi'})] = t^{-\dim \mathcal{O}} \lambda_{\phi,\phi'}(t)[\mathcal{O}_\mathcal{A}]$. Let $a : \mathcal{O} \rightarrow \text{pt}$. Then $\text{Hom}(L_{\phi}, L_{\phi'}) \cong \mathcal{R}(\mathcal{O}, L_{\phi}^* \otimes L_{\phi'}) \cong a_* (L_{\phi}^* \otimes L_{\phi'})$. Now, we apply Verdier dual to get $a_! (L_{\phi}^* \otimes L_{\phi'}) [2 \dim \mathcal{O}](\dim \mathcal{O})$. Hence, we have that $\mathcal{D} \text{Hom}(L_{\phi}, L_{\phi'}) = t^{-\dim \mathcal{O}}[\mathcal{R}(\mathcal{O}, L_{\phi} \otimes L_{\phi'})]. \hfill \Box$

We need more notation for the proof of the following result. In particular, $\mathcal{O}_{\phi} \leq \mathcal{O}_{\phi'}$ means $\mathcal{O}_{\phi} \subset \overline{\mathcal{O}_{\phi'}}$ and $\mathcal{O}_{\phi} < \mathcal{O}_{\phi'}$ means $\mathcal{O}_{\phi} \subset \overline{\mathcal{O}_{\phi'}} \setminus \mathcal{O}_{\phi'}$. We also use the fact that $\mathcal{O}_{\phi} = \mathcal{O}_{\phi'}$.

Remark 5.4. The following theorem is proven in [A] Theorem 24.8. We include its proof here for completeness. The inductive steps of the proof illustrate a method for computing the unknown polynomials $p$ and $\lambda$ from $\omega$ which is known as the Lusztig–Shoji algorithm.

Theorem 5.5. (1) For all $\chi, \psi$, $p_{\chi,\psi}(t) = p_{\chi^\vee,\psi^\vee}(t)$ and $\lambda_{\chi,\psi}(t) = \lambda_{\psi,\chi}(t)$.

(2) The polynomials $p$ and $\lambda$ are the unique ones satisfying

\[ \omega_{\chi,\psi}(t) = \sum_{\phi, \phi'} p_{\chi,\phi}(t)\lambda_{\phi,\phi'}(t)p_{\psi,\phi'}^\vee(t) \]

and conditions (5.4) and (5.5).
We will now use the above equation to prove (1), and thus, prove (2).

This can be improved because \( \lambda_{\phi,\phi'} \) is zero when \( O_{\phi} \neq O_{\phi'} \). Thus, we see that

\[
\omega_{\chi,\psi}(t) = \sum_{\phi,\phi'} p_{\chi,\phi}(t) \lambda_{\phi,\phi'}(t) p_{\psi,\phi'}(t).
\]

We will now use the above equation to prove (1), and thus, prove (2).

We prove both statements of (1) simultaneously by induction on \( d = \dim O_{\psi} \). If \( d < 0 \), we know the statement is trivially true. Let us suppose then that \( p_{\chi,\psi}(t) = p_{\chi,\psi}(t) \) and \( \lambda_{\chi,\psi}(t) = \lambda_{\chi,\psi}(t) \) for \( d \leq k - 1 \). First, we will use (5.8) to show \( \lambda_{\chi,\psi}(t) = \lambda_{\chi,\psi}(t) \) for \( d = k \). Since we know \( \lambda_{\chi,\psi}(t) \) vanishes otherwise, we can assume that \( O_{\chi} = O_{\psi} \). Recall that \( p_{\phi,\phi} = t^{-\dim O_{\psi}/2} \) for any \( \phi \). Now, with these conditions we have

\[
\omega_{\chi,\psi}(t) = p_{\chi,\chi}(t) p_{\psi,\psi}(t) \lambda_{\chi,\psi}(t) + \sum_{O_{\phi} < O_{\chi}, O_{\phi'} < O_{\psi}} p_{\chi,\phi}(t) \lambda_{\phi,\phi'}(t) p_{\psi,\phi'}(t)
\]

By the inductive hypothesis, \( p_{\chi,\psi}(t) = p_{\psi,\phi}(t) \) since \( d = \dim O_{\psi} \leq k - 1 \). After rearranging, we get

\[
t^{-\dim O_{\psi}} \lambda_{\chi,\psi}(t) = \omega_{\chi,\psi}(t) - \sum_{O_{\phi} < O_{\chi}, O_{\phi'} < O_{\psi}} p_{\chi,\phi}(t) \lambda_{\phi,\phi'}(t) p_{\psi,\phi'}(t),
\]

Since the right hand side of this equation is symmetric with respect to \( \chi \) and \( \psi \) by the inductive hypothesis and Lemma 5.2, we know the left hand side must be as well. Thus, \( \lambda_{\chi,\psi}(t) = \lambda_{\chi,\psi}(t) \) for \( d = k \).

Now, we need to show \( p_{\chi,\psi}(t) = p_{\chi,\psi}(t) \) when \( d = k \). From (5.8), we may assume \( O_{\psi} \leq O_{\chi} \). Suppose \( \alpha \) is such that \( O_{\alpha} = O_{\psi} \), so that \( \dim O_{\alpha} = k \). Again, we use (5.8).

\[
\omega_{\alpha,\chi}(t) = \sum_{O_{\alpha} = O_{\alpha}} p_{\alpha,\alpha}(t) p_{\chi,\psi}(t) \lambda_{\alpha,\phi}(t) + \sum_{O_{\alpha} < O_{\alpha}, O_{\alpha} = O_{\phi'}} p_{\alpha,\phi}(t) \lambda_{\phi,\phi'}(t) p_{\chi,\phi'}(t)
\]

= \( t^{-\dim O_{\alpha}/2} \sum_{O_{\alpha} = O_{\alpha}} p_{\chi,\psi}(t) \lambda_{\alpha,\phi}(t) + \sum_{O_{\alpha} < O_{\alpha}, O_{\alpha} = O_{\phi'}} p_{\alpha,\phi}(t) \lambda_{\phi,\phi'}(t) p_{\chi,\phi'}(t) \).
Here, the induction hypothesis gives \( p_{\chi^\vee,\phi^\vee}(t) = p_{\chi,\phi}(t) \) in the second sum since \( d = \dim \mathcal{O}_{\phi'} < \dim \mathcal{O}_\alpha = k \). Similarly, we also have

\[
\omega_{\chi,\alpha}(t) = \sum_{\mathcal{O}_a = \mathcal{O}_a} p_{\chi,\phi}(t)p_{\lambda_{\phi,\alpha}}(t) + \sum_{\mathcal{O}_a < \mathcal{O}_a, \mathcal{O}_b = \mathcal{O}_{\phi'}} p_{\chi,\phi}(t)\lambda_{\phi,\alpha}(t)p_{\lambda_{\phi,\alpha}}(t)
= t^{-(\dim \mathcal{O}_\alpha)/2} \sum_{\mathcal{O}_a = \mathcal{O}_a} p_{\chi,\phi}(t)\lambda_{\phi,\alpha}(t) + \sum_{\mathcal{O}_a < \mathcal{O}_a, \mathcal{O}_b = \mathcal{O}_{\phi'}} p_{\chi,\phi}(t)\lambda_{\phi,\alpha}(t)p_{\alpha,\phi}(t).
\]

Keeping in mind the induction hypothesis and Lemma 5.2, we subtract these two formulas to get

\[
(5.10) \sum_{\mathcal{O}_a = \mathcal{O}_a} (p_{\chi^\vee,\phi^\vee}(t) - p_{\chi,\phi}(t))\lambda_{\alpha,\phi}(t) = 0.
\]

The classes of irreducible local systems on \( \mathcal{O} \) form a basis in \( K(\mathcal{O}) \), and therefore, the matrix \( (\lambda_{\phi,\alpha})_{\mathcal{O}_a = \mathcal{O}_{\phi'}} \) must be nonsingular. Thus, (5.10) implies \( p_{\chi^\vee,\phi^\vee}(t) = p_{\chi,\phi}(t) \). This completes the induction argument and the proof of (1). From (5.10) we get a recursive formula which implies the uniqueness of part (2) which completes the proof. \(\square\)

**Appendix A. Central characters**

Throughout this section we will assume that we have a connected algebraic group \( G \) acting on a variety \( X \) with finitely many orbits so that the center \( Z \) acts trivially. The \( G \)-action induces an action of the quotient \( G/Z \) on \( X \), and the action map \( G \times X \to X \) factors as

\[
G \times X \xrightarrow{(q, \text{id}_X)} G/Z \times X \xrightarrow{\alpha} X
\]

where \( q : G \to G/Z \) is the quotient.

**Proposition A.1.** The category of \( G \)-equivariant local systems on \( G/Z \) (for the left translation action of \( G \)) is equivalent to the category of \( Z/Z^0 \)-representations as a tensor category.

**Proof.** By induction equivalence, the category of \( G \) equivariant sheaves on \( G/Z \) is equivalent to the category of \( Z \) equivariant sheaves on a point. Furthermore, the category of \( Z \) equivariant sheaves on a point is equivalent to the category of \( Z/Z^0 \) representations. (The \( Z \)-equivariant fundamental group of a point is \( Z/Z^0 \).) \(\square\)

For a character \( \chi \) of \( Z/Z^0 \), we denote the corresponding local system under this equivalence by \( \mathcal{L}_\chi \).

**Definition A.2.** We say that a \( G \)-equivariant perverse (or constructible) sheaf \( \mathcal{F} \) admits a central character \( \chi \) if there is an isomorphism \( \alpha^* \mathcal{F} \cong \mathcal{L}_\chi \boxtimes \mathcal{F} \).

**Lemma A.3.** A simple \( G \)-equivariant perverse sheaf admits a central character.

**Proof.** Let \( P = \text{IC}(\mathcal{O}, \mathcal{L}) \) be a simple \( G \)-equivariant perverse sheaf with \( \mathcal{O} \) a \( G \)-orbit in \( X \). The \( G \)-equivariant fundamental group of \( \mathcal{O} \) is \( \pi_1^G(\mathcal{O}) = \text{Stab}_G(x)/\text{Stab}_G(z)^0 \), where \( x \in \mathcal{O} \). Since \( Z \) acts trivially on \( X \), we have a map \( Z/Z^0 \to \pi_1^G(\mathcal{O}) \). Thus, if \( \mathcal{L} \) is a local system on \( \mathcal{O} \) corresponding to the \( \pi_1^G(\mathcal{O}) \) representation \( V_\mathcal{L} \), the central character of \( \mathcal{L} \) is the local system \( \mathcal{L}_\chi \) corresponding to the restriction \( V_\mathcal{L}|_{Z/Z^0} \). To see that \( \mathcal{L}_\chi \) is the central character of \( P \), use base change with the diagram.
Under base change, the natural map $\alpha^*j_*\mathcal{L} \to \alpha^*j_*\mathcal{L}$ goes to $j'_!\alpha'^*\mathcal{L} \cong j'_!(\mathcal{L}_\chi \boxtimes \mathcal{L}) \to j'_!\alpha'^*\mathcal{L} \cong j'_!(\mathcal{L}_\chi \boxtimes \mathcal{L})$. Hence, $\alpha^*P \cong \mathcal{L}_\chi \boxtimes P$.

We record some facts about external tensor product for our convenience.

(1) $t$-exact
(2) $\mathcal{D}(\mathcal{F} \boxtimes \mathcal{G}) \cong \mathcal{D}(\mathcal{F} \boxtimes \mathcal{G})$
(3) $(f \times g)^*(\mathcal{F} \boxtimes \mathcal{G}) \cong f^*\mathcal{F} \boxtimes g^*\mathcal{G}$
(4) $(f \times g)_*(\mathcal{F} \boxtimes \mathcal{G}) \cong f_*\mathcal{F} \boxtimes g_*\mathcal{G}$
(5) $(f \times g)_!(\mathcal{F} \boxtimes \mathcal{G}) \cong f_*\mathcal{F} \boxtimes g_*\mathcal{G}$

Lemma A.4. Suppose $\mathcal{F}$ has central character $\chi$ and $\mathcal{G}$ has central character $\psi$. Then $\mathcal{F} \boxtimes^L \mathcal{G}$ has central character $\chi \otimes \psi$.

Proof.

$\alpha^*(\mathcal{F} \boxtimes^L \mathcal{G}) = \alpha^*\mathcal{F} \boxtimes^L \alpha^*\mathcal{G}
= (\mathcal{L}_\chi \boxtimes \mathcal{F}) \boxtimes^L (\mathcal{L}_\psi \boxtimes \mathcal{G})
= p_{\mathcal{L}/Z}^\alpha \mathcal{L}_\chi \boxtimes p_X^*\mathcal{F} \boxtimes p_{\mathcal{L}/Z}^\alpha \mathcal{L}_\psi \boxtimes p_X^*\mathcal{G}
= p_{\mathcal{L}/Z}^\alpha (\mathcal{L}_\chi \boxtimes \mathcal{L}_\psi) \boxtimes p_X^*(\mathcal{F} \boxtimes \mathcal{G})
= (\mathcal{L}_\chi \boxtimes \mathcal{L}_\psi) \boxtimes (\mathcal{F} \boxtimes \mathcal{G})$

Lemma A.5. Suppose $\mathcal{F}$ has central character $\chi$. Then $\mathcal{D}\mathcal{F}$ has central character $\chi^{-1}$.

Proof. Since $\alpha$ is flat with $d = \dim G/Z$ dimensional fibers, we have $\alpha^! \cong \alpha^*[2d]$. Thus, we see that

$\alpha^* \mathcal{D}\mathcal{F} = \alpha^! \mathcal{D}\mathcal{F}[-2d]
= \mathcal{D}\alpha^*\mathcal{F}[-2d]
= \mathcal{D}(\mathcal{L}_\chi \boxtimes \mathcal{F})[-2d]
= \mathcal{L}_\chi \boxtimes \mathcal{D}\mathcal{F}[-2d]
= \mathcal{L}_\chi \boxtimes \mathcal{D}\mathcal{F}[2d - 2d]
= \mathcal{L}_\chi^{-1} \boxtimes \mathcal{D}\mathcal{F}$

Lemma A.6. Suppose that $f : X \to Y$ is a $G$-equivariant map between $G$-varieties on which $Z$ acts trivially. Then $f^!$, $f^*$, $f_*$, and $f_!$ preserve central character.

Proof. We consider the commutative diagram:
First, we will show pull-backs preserve central character. Suppose $F$ is a $G$-equivariant sheaf on $Y$ with central character $\chi$. Then we have
\[
\alpha^* f^* F \cong (\text{id} \times f)^*(\alpha^* F)
\]
\[
\cong (\text{id} \times f)^*(L_\chi \boxtimes F)
\]
\[
\cong L_\chi \boxtimes f^* F
\]

Similarly,
\[
\alpha^* f^! F \cong (\text{id} \times f)^!\alpha^* F
\]
\[
\cong (\text{id} \times f)^!(L_\chi \boxtimes F)
\]
\[
\cong L_\chi \boxtimes f^! F
\]

By base change and flatness of the action maps $\alpha$, we have that
\[
\alpha^* f_* F \cong (\text{id} \times f)_*\alpha^* F
\]
\[
\cong (\text{id} \times f)_*(L_\chi \boxtimes F)
\]
\[
\cong L_\chi \boxtimes f_* F
\]

Similarly,
\[
\alpha^* f^i F \cong (\text{id} \times f)^i\alpha^* F
\]
\[
\cong (\text{id} \times f)^i(L_\chi \boxtimes F)
\]
\[
\cong L_\chi \boxtimes f^i F
\]

Lemma A.7. Suppose $F$ has central character $\chi$ and $G$ has central character $\psi$. Then the sheaf $R\text{Hom}(F, G)$ has central character $\chi^{-1} \otimes \psi$.

Proof. We have an isomorphism
\[
R\text{Hom}(F, G) \cong D(F \otimes D^i G).
\]
Of course, $F \otimes D^i G$ has central character $\chi \otimes \psi^{-1}$ by Lemmas A.5 and A.4, and we apply Lemma A.5 again to get the result.

Proposition A.8. Suppose that two perverse sheaves on $X$ have distinct central characters. Then $R\text{Hom}(F, F') = 0$. In particular, $\text{Hom}^i(F', F) = \text{Hom}^i(F, F') = 0$ for all $i$. 

Proof. First, we note that on $pt$, the only possible central character is trivial. Let $\mathcal{F}$ and $\mathcal{F}'$ be perverse sheaves on $X$ with central characters $\chi$ and $\psi$ (and $\chi \neq \psi$). Let $a : X \rightarrow pt$ be the constant map. Recall that $R\mathcal{H}om(\mathcal{F}, \mathcal{F}') = a_* R\mathcal{H}om(\mathcal{F}, \mathcal{F}')$. If $R\mathcal{H}om(\mathcal{F}, \mathcal{F}') \neq 0$, then Lemma A.7 implies that $R\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ has nontrivial central character equal to $\chi^{-1} \otimes \psi$. Then, $a_* R\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ also has central character $\chi^{-1} \otimes \psi$ by Lemma A.6 which is impossible. $\square$

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