Hochschild cohomology and characteristic classes for star-products
Dedicated to V.I. Arnol’d for his 60th birthday

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Abstract

We show that the Hochschild cohomology of the algebra obtained by formal deformation quantization on a symplectic manifold is isomorphic to the formal series with coefficients in the de Rham cohomology of the manifold. The cohomology class obtained by differentiating the star-product with respect to the deformation parameter is seen to be closely related to the characteristic class of the quantization. A fundamental role in the analysis is played by “quantum Liouville operators,” which rescale the deformation parameter in the same way in which Liouville vector fields scale the Poisson structure (or the units of action). Several examples are given.

1 Introduction

Deformation quantizations of a symplectic manifold $M$ have been completely classified through the combined efforts of many people, beginning with the seminal paper [1] which set out the problem and established the importance of the second de Rham cohomology. The role of this cohomology was further clarified in [11]. The existence of deformation quantizations was established in [8], so

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that the moduli space of deformation quantizations could then be identified as an affine space over the vector space $H^2(M[[\hbar]]$. Identification of an origin in this affine space, i.e. the choice of a distinguished isomorphism class of quantizations, was carried out by several people, so that there are now many versions of the classification \cite{2,3,8,9,14}. The result is that the isomorphism type of a deformation quantization is encoded by a characteristic class in $H^2(M[[\hbar]]$.

In much of the work cited above, as well as the influential paper \cite{17}, an important role was played by local transformations which rescale the deformation parameter $\hbar$, as well as the corresponding infinitesimal transformations, which we will call quantum Liouville operators, since their classical limits are the Liouville vector fields, whose flows rescale the symplectic structure. In particular, De Wilde \cite{6} identifies all but one term of the characteristic class of a quantization as the obstruction to the existence of a quantum Liouville operator when the underlying symplectic structure is exact (i.e. admits a Liouville vector field), while Deligne \cite{5} treats the general case and shows that the characteristic class (again with the exception of a “stray” term) is the obstruction to the existence of an object (“gerbe”) which globalizes the local quantum Liouville operators in a somewhat more complicated way.

The purpose of the present paper is to obtain some of the results described above by considering the derivative of a star-product with respect to the deformation parameter as a Hochschild 2-cocycle $c$ on the noncommutative algebra $C^\infty(M[[\hbar]]$ given by the quantization, and then attaching to this cocycle a de Rham or Čech cocycle which represents (a derivative of) the characteristic class.

Our first step is thus to construct a mapping $\Phi$ (which turns out to be an isomorphism) from the Hochschild cohomology of the deformed algebra with the Čech or de Rham cohomology of $M$ (with a formal variable adjoined). Although results very close to this can be found in the literature (e.g. \cite{3} and \cite{14}), we have not found a complete proof of what we need. In any case, our very simple proof, which is a near-replica of A. Weil’s proof \cite{20} of the isomorphism between Čech and de Rham cohomology, seems to be new.

The second task is to compute the image $\Phi(c)$ in the cohomology of $M$. For this purpose, we use the method of quantum exponential mappings \cite{24} to transfer the problem to the bundle of Weyl algebras on the tangent spaces of $M$ and thereby to relate $\Phi(c)$ to the Weyl curvature of a Fedosov connection and hence to the usual characteristic class of a deformation. The relation matches that already obtained in Proposition 4.4. of \cite{5} by a very different method.

In the last section, we present several examples of symplectic (and Poisson) manifolds and their quantizations to illustrate possible relations between the existence of classical and quantum Liouville operators.

On the day this manuscript was completed, we received the preprint of Kontsevich \cite{12}, in which the classification of deformation quantizations is shown to be completely equivalent to the formal classification of Poisson structures. We hope that our rather direct approach to the characteristic class will be useful in the interpretation of this equivalence.

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\footnote{Throughout this paper, smooth functions and de Rham cohomology will be considered to have complex values.}
2 Deformation quantization

In this section, we will recall some basic ingredients of Fedosov’s construction of star-products on a symplectic manifold, as well as some useful notation. For details, readers should consult [9] [10] [24].

Let \((M,\omega)\) be a symplectic manifold of dimension \(2n\). Each tangent space \(T_xM\) is equipped with a linear symplectic structure, which can be quantized by the standard Moyal-Weyl product to produce an algebra \(W_x\). More precisely,

**Definition 2.1** The formal Weyl algebra \(W_x\) associated to \(T_xM\) is an associative algebra with a unit over \(\mathbb{C}\), whose elements consist of formal power series in the formal parameter \(\hbar\) with coefficients being formal polynomials on \(T_xM\). In other words, each element has the form

\[
a(y,\hbar) = \sum \hbar^k a_{k,\alpha} y^\alpha
\]

where \(y = (y^1, \ldots, y^{2n})\) are linear coordinates on \(T_xM\), \(\alpha = (\alpha_1, \ldots, \alpha_{2n})\) is a multi-index and \(y^\alpha = (y^1)^{\alpha_1} \cdots (y^{2n})^{\alpha_{2n}}\). The product is defined by the Moyal-Weyl rule:

\[
a \ast b = \sum_{k=0}^{\infty} \left( \frac{i\hbar}{2} \right)^k \frac{1}{k!} \pi^{i_1j_1} \cdots \pi^{i_kj_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}.
\]

Let \(W = \cup_{x \in M} W_x\). Then \(W\) is a bundle of algebras over \(M\), called the Weyl bundle over \(M\). Its space of smooth sections \(\Gamma W\) forms an associative algebra with unit under fiberwise multiplication. One may think of the sections of \(W\) as the functions on a “quantized tangent bundle” of \(M\). As a vector space, \(\Gamma W\) may be identified with the infinite jets along the zero section of functions on \(TM\) (with \(\hbar\) adjoined). The algebra structure is a deformation quantization of the Poisson structure given by fiberwise Poisson bracket (using the constant symplectic structure on each tangent space).

The center \(Z(W)\) of \(\Gamma W\) consists of sections not containing \(y\)'s and can be naturally identified with \(C^\infty(M)[[\hbar]]\).

By assigning degrees to \(y\)'s and \(\hbar\) with \(\text{deg}y^i = 1\) and \(\text{deg}\hbar = 2\), we obtain a natural filtration

\[C^\infty(M)[[\hbar]] \subset \Gamma(W_1) \subset \cdots \Gamma(W_i) \subset \Gamma(W_{i+1}) \cdots \subset \Gamma(W)
\]

with respect to the total degree \(2k + l\) of the terms in the series in Equation (1).

A differential form with values in \(W\) is a section of the bundle \(W \otimes \wedge^q T^*M\), which can be expressed locally as

\[
a(x, y, \hbar, dx) = \sum \hbar^k a_{k,i_1 \cdots i_p,j_1 \cdots j_q} y^{i_1} \cdots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q}.
\]

Here the coefficient \(a_{k,i_1 \cdots i_p,j_1 \cdots j_q}\) is symmetric with respect to \(i_1 \cdots i_p\) and antisymmetric in \(j_1 \cdots j_q\). For short, we denote the space of sections of the bundle \(W \otimes \wedge^q T^*M\) by \(\Gamma W \otimes \Lambda^q\).

The fiberwise commutator in \(\Gamma W\) extends to \(\Gamma W \otimes \Lambda^q\) by the standard procedure for differential forms with values in a bundle of Lie algebras. In addition, the usual exterior derivative induces, in
a way special to the present situation, an operator $\delta$ on $W$-valued differential forms:

$$\delta a = dx^i \wedge \frac{\partial a}{\partial y^i}, \quad \forall a \in \Gamma W \otimes \Lambda^*.$$  \hspace{1cm} (4)

Alternatively, one can write:

$$\delta a = -i \hbar \omega_{ij} y^i dx^j, a.$$  \hspace{1cm} (5)

Note that $\delta$ is an “algebraic” operator in that it does not involve derivatives with respect to $x$.

Let $\nabla$ be a torsion-free symplectic connection on $M$ and $\partial : \Gamma W \rightarrow \Gamma W \otimes \Lambda^1$ its induced covariant derivative.

Consider a connection on $W$ of the form:

$$D = -\delta + \partial + \frac{i}{\hbar} [\gamma, \cdot],$$  \hspace{1cm} (6)

with $\gamma \in \Gamma W \otimes \Lambda^1$.

Clearly, $D$ is a derivation with respect to the Moyal-Weyl product, i.e.,

$$D(a \ast b) = a \ast Db + Da \ast b.$$  \hspace{1cm} (7)

A simple calculation yields that

$$D^2a = -i \hbar \Omega, a, \quad \forall a \in \Gamma W,$$  \hspace{1cm} (8)

where

$$\Omega = \omega - R - \delta \gamma - \partial \gamma - \frac{i}{\hbar} \gamma^2.\hspace{1cm} (9)$$

Here $R = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l$ and $R_{ijkl} = \omega_{im} R_{m}^{\ jkl}$ is the curvature tensor of the symplectic connection.

A connection of the form (6) is called Abelian if $\Omega$ is a scalar 2-form, i.e., $\Omega \in \Omega^2(M)[[\hbar]]$. For an Abelian connection, the Bianchi identity implies that $d\Omega = D\Omega = 0$, i.e., $\Omega \in Z^2(M)[[\hbar]]$. In this case, $\Omega$ is called the Weyl curvature.

**Theorem 2.2** \hspace{1cm} (\cite{Fedosov}) Let $\nabla$ be any torsion free symplectic connection, and $\Omega = \omega + \hbar \omega_1 + \cdots \in Z^2(M)[[\hbar]]$ a perturbation of the symplectic form in the space $Z^2(M)[[\hbar]]$. There exists a unique $\gamma \in \Gamma W_3 \otimes \Lambda^1$ such that $D$, given by Equation (6), is an Abelian connection which has Weyl curvature $\Omega$ and satisfies

$$\delta^{-1} \gamma = 0.$$  \hspace{1cm}

Such a connection $D$ is often called a Fedosov connection.

Given a Fedosov connection $D$, the space of all parallel sections $W_D$ automatically becomes an associative algebra. Fedosov proved that $W_D$ can be naturally identified with $C^\infty(M)[[\hbar]]$, and therefore induces a star-product on $C^\infty(M)[[\hbar]]$, which we will call a Fedosov star-product. More precisely, let $\sigma$ denote the projection from $W_D$ to its center $C^\infty(M)[[\hbar]]$ defined as $\sigma(a) = a|_{y=0}$. 


Theorem 2.3 ([7]) For any \( a(x, h) \in C^\infty(M)[[h]] \) there is a unique section \( \tilde{a} \in W_D \) such that \( \sigma(\tilde{a}) = a \). Therefore, \( \sigma \) establishes an isomorphism between \( W_D \) and \( C^\infty(M)[[h]] \) as vector spaces. Moreover, the equation:

\[
a *_h b = \sigma((\sigma^{-1}a) * (\sigma^{-1}b)), \quad \forall a, b \in C^\infty(M)[[h]],
\]

(10)
defines a star-product on \( M \).

If \( \nabla \) is flat and \( \Omega = \omega \), the Fedosov connection is simply given by \( D = -\delta + \partial \). In this case, for any \( a \in C^\infty(M), \tilde{a} = \sigma^{-1}(a) \) can be expressed explicitly as

\[
\tilde{a} = \sum_{k=0}^{\infty} \frac{1}{k!}(\partial_{i_1} \cdots \partial_{i_k} a)y^{i_1} \cdots y^{i_k},
\]

which is just the Taylor expansion of \( \exp_x^* a \) at the origin. So the correspondence \( C^\infty(M)[[h]] \rightarrow W_D \) is indeed the pullback by the \((C^\infty\text{-jet at the origin of the})\) usual exponential map. Thus for a general Fedosov connection, it is sometimes useful to think of \( \sigma^{-1} : C^\infty(M)[[h]] \rightarrow GW \) as a quantum exponential map.

More precisely,

**Definition 2.4** A quantum exponential map is an \( h \)-linear map \( \rho : C^\infty(M)[[h]] \rightarrow GW \) such that

(i). \( \rho(C^\infty(M)[[h]]) \) is a subalgebra of \( GW \);

(ii). \( \rho(a)|_{y=0} = a, \forall a \in C^\infty(M)[[h]] \);

(iii). \( \rho(a) = a + \delta^{-1}da, \forall a \in C^\infty(M), \text{ mod } W_2 \);

(iv). \( \rho(a) \) can be expressed as a formal power series in \( y \) and \( h \), with coefficients being derivatives of \( a \).

Theorem 2.5 ([24]) Quantum exponential maps are equivalent to Fedosov connections. Every star-product, up to an isomorphism, admits a quantum exponential map and is thus isomorphic to a Fedosov star-product.

The final conclusion of this theorem can already be found in [14].

The characteristic class of a star-product algebra \( \mathcal{A} \) (isomorphic as an algebra over \( \mathbb{C}[[h]] \) to \( C^\infty(M)[[h]] \)) is the class

\[
cl(\mathcal{A}) = \frac{1}{h}\Omega
\]

(11)
in \( H^2(M)[h^{-1}, h]] \), where \( \Omega \) is the Weyl curvature of a corresponding Fedosov connection \( D \). (According to Fedosov [3, 10], it does not depend on the choice of connection.)

We end this section by introducing the following notation. By \( W^+ \), we will denote the extended Weyl bundle on \( M \), i.e., the vector bundle obtained by extending the “ground ring” \( \mathbb{C}[[h]] \) to the field \( \mathbb{C}[h^{-1}, h]] \) of formal Laurent series. It is obvious that any Fedosov connection \( D \) naturally extends to \( W^+ \), whose space of parallel sections is \( W_D[h^{-1}, h]] \).
3 Hochschild cohomology

Let $*_h$ be a star-product on a symplectic manifold $M$, and $\mathcal{A} = C^\infty(M)[[h]]$ its star-product algebra. Consider $\tilde{\mathcal{A}} = C^\infty(M)[h^{-1}, h]$, the space of formal Laurent series in $h$ with coefficients in $C^\infty(M)$. Then $\tilde{\mathcal{A}}$ has an induced associative algebra structure. Let $C^k(\tilde{\mathcal{A}}, \tilde{\mathcal{A}})$ be the space of $h$-linear maps from $\tilde{\mathcal{A}} \otimes \cdots \otimes \tilde{\mathcal{A}}$ to $\tilde{\mathcal{A}}$, which are multi-differential operators when being restricted to $C^\infty(M) \otimes \cdots \otimes C^\infty(M)$. Clearly, each element of $C^k(\tilde{\mathcal{A}}, \tilde{\mathcal{A}})$ can be identified with a multi-linear map $c = \sum_{l=-N}^\infty h^l c_l$:

$$C^\infty(M) \otimes \cdots \otimes C^\infty(M) \to C^\infty(M)[h^{-1}, h],$$

where each $c_l$ is a $k$-multi-differential operator on $M$.

Define the coboundary operator

$$b : C^k(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}) \to C^{k+1}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}),$$

as $\frac{i}{h} \tilde{b}$, where $\tilde{b}$ is the usual Hochschild coboundary operator:

$$(\tilde{b}c)(u_0, \cdots, u_k) = u_0 *_h c(u_1, \cdots, u_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(u_0, \cdots, u_i *_h u_{i+1}, \cdots, u_k)$$

$$+ (-1)^{k+1} c(u_0, \cdots, u_{k-1}) *_h u_k,$$

for $u_0, \cdots, u_k \in \tilde{\mathcal{A}}$ and $c \in C^k(\tilde{\mathcal{A}}, \tilde{\mathcal{A}})$.

The Hochschild cohomology $H^*(\mathcal{A}, \mathcal{A})$ of the star-product algebra $\mathcal{A}$ is defined as the cohomology group of this complex.

**Theorem 3.1** If the symplectic manifold $M$ is contractible, then $H^k(\mathcal{A}, \mathcal{A}) = 0$ for $k \geq 1$. In addition, we have $H^k(W, W) = 0$ for $k \geq 1$, where $W$ is the formal Weyl algebra defined on the symplectic vector space $\mathbb{R}^{2n}$.

In fact, if $c = \sum_{l=-N}^\infty h^l c_l$ is a k-cocycle, we can always write $c = bH$ for some cochain $H$ of the form $H = \sum_{l=-N}^\infty h^l H_l$.

**Proof.** We outline a proof here for the case $k = 2$. The general case follows by a longer version of the same argument.

Assume that $c = \sum_{k=0}^\infty h^k c_k$ is a Hochschild 2-cocycle for the star-product. By $b_0$, we denote the Hochschild coboundary operator for the commutative algebra $C^\infty(M)$. It follows from $bc = 0$ that $b_0 c_0 = 0$ ($b_0 c_0$ is the $1/h$ term of $bc$). Therefore, we can write

$$c_0 = b_0 T_0 + c_0'$$

for some differential operator $T_0$ on $M$ and a bivector field $c_0' \in \Gamma(\wedge^2 TM)$.

(a) Assume that $T_0 = 0$.

By inspecting the $h^0$-term of $bc$, we have

$$-\frac{i}{2} \{\{u_0, c_0(u_1, u_2)\} - c_0(\{u_0, u_1\}, u_2)$$

$$+ c_0(u_0, \{u_1, u_2\}) - \{c_0(u_0, u_1), u_2\} + b_0 c_1 = 0$$
Taking the alternating sum over all the permutations of \(u_0, u_1, u_2\), one obtains

\[
[\pi, c_0] = 0,
\]

where \(\pi\) is the Poisson tensor on \(M\). Since \(H^2_\pi(M) \cong H^2(M) = 0\), we may write \(c_0 = [X_0, \pi]\) for some vector field \(X_0 \in \mathfrak{X}(M)\). (In the formal case, we use the formal Poincaré lemma.) On the other hand, it is simple to see that \(bX_0 = \frac{i}{\hbar}[X_0, \pi] + O(\hbar)\). Let \(H_0 = 2X_0\). Then

\[
c = bH_0 + O(\hbar).
\]

(b). In general, we have \(bT_0 = i\hbar b_0T_0 + O(1)\). Thus, \(c - b(-i\hbar T_0) = c_0 - b_0T_0 + O(\hbar) = c'_0 + O(\hbar)\). This would reduce to the situation in Case (a).

Finally, the conclusion follows by using the process above repeatedly.

For a general symplectic manifold \(M\), it turns out that

\[
H^k(A, A) \cong H^k(M)[h^{-1}, h]).
\]  

(13)

In what follows, we will prove this fact for the case of \(k = 2\) by using Fedosov star-products. At the end, we explain how to extend the argument to general \(k\).

Theorem 3.2 Let \(W_D\) be the Fedosov star-product algebra corresponding to a Fedosov connection \(D\). Then

\[
H^2(W_D, W_D) \cong H^2(M)[h^{-1}, h]].
\]

Our method is analogous to the proof of de Rham theorem. The main idea is to consider a double complex whose cohomology of horizontal and vertical complexes are all zero except for the first row and the first column, where the cohomologies are Hochschild cohomology and de Rham cohomology, respectively. Then a standard argument using spectral sequences will lead to the conclusion. However, below we will use a more elementary method similar to Weil’s proof [20] of de Rham’s theorem. This will allow us to obtain an explicit construction of the isomorphism, which will be useful in our computation in next section.

Consider the double complex \(C^{p,q}\), where \(C^{p,q}\) is the tensor product of the \(p\)-multilinear bundle maps from \(W^+\) to itself tensored with \(q\)-forms on \(M\). The first differential \(B : C^{p,q} \rightarrow C^{p+1,q}\) is the fiberwise Hochschild coboundary operator \(b\) (multiplied by the factor \(i/\hbar\) as in Equation (12)), tensored with the identity on differential forms. The second differential \(\tilde{D} : C^{p,q} \rightarrow C^{p,q+1}\) is the Fedosov connection \(D\), extended from the Weyl bundle to its “associated bundle” of multilinear maps in the natural way, i.e.: given any \(\varphi \in C^{p,q}\), \(\tilde{D}\varphi \in C^{p,q+1}\) is defined by,

\[
(\tilde{D}\varphi)(u_1, \cdots, u_p) = D\varphi(u_1, \cdots, u_p) - \sum_{i=1}^p \varphi(u_1, \cdots, Du_i, \cdots, u_p),
\]

for any \(u_1, \cdots, u_p \in W^+\). It is easy to check that \(B^2 = 0\) and \(\tilde{D}^2 = 0\). By \(H^{p,q}_B\) and \(H^{p,q}_\tilde{D}\), we denote the cohomology group of the horizontal complex \(B : C^{p,q} \rightarrow C^{p+1,q}\), and the vertical complex \(\tilde{D} : C^{p,q} \rightarrow C^{p,q+1}\), respectively.
Lemma 3.3  
(i). \( \tilde{D}B = B\tilde{D} \);
(ii). \( H^{p,q}_B = 0 \), if \( p \geq 1 \);
(iii). \( H^{p,q}_D = 0 \), if \( q \geq 1 \).

Proof. (i) can be proved by a straightforward verification, and (ii) follows from Theorem 3.1 directly.
For (iii), assume that \( \varphi \in C^{p,q} \) satisfying \( \tilde{D}\varphi = 0 \). It thus follows that for any \( u_1, u_2, \cdots, u_p \in W_D \), \( D\varphi(u_1, u_2, \cdots, u_p) = 0 \).

Let \( \psi \in \Gamma W^+ \otimes \Omega^{q-1} \) be the solution of the following equation:

\[
\begin{cases}
D\psi &= \varphi(u_1, u_2, \cdots, u_p) \\
\psi|_{y=0} &= 0
\end{cases}
\]  
(14)

According to Theorem 5.2.5 in [10], this equation always has a solution. Moreover, \( \psi \) can be made to depend on \( u_i \) multilinearly. Thus \( \psi \) is a multilinear map from \( W_D \) to \( W^+ \). It is clear that \( \psi \) is a local map. It thus extends to a \( p \)-multilinear bundle map from \( W \) to \( W^+ \) tensored with \( (q-1) \)-forms. In other words, it is an element in \( C^{p,q-1} \), which will be denoted by the same symbol \( \psi \). Then, we have \( \tilde{D}\psi = \varphi \). To see this, we only need to note that both \( \tilde{D}\psi \) and \( \varphi \) are elements in \( C^{p,q} \) and they coincide when being restricted to sections in \( W_D \).

\[\square\]

Denote by \( F^{p,q} \) the subspace of \( C^{p,q} \) consisting of those elements \( \varphi \) satisfying \( \tilde{D}B\varphi = 0 \), and let \( \mathcal{F}^{p,q} \subseteq F^{p,q} \) be the subspace spanned by those elements which satisfy either the equation \( \tilde{D}\varphi = 0 \) or \( B\varphi = 0 \).

Proposition 3.4

\[ F^{1,0} / F^{1,0} \cong F^{0,1} / F^{0,1}. \]  
(15)

Proof. Given any \( \varphi \in F^{1,0} \), consider the following equation for \( \psi \):

\[ B\psi = \tilde{D}\varphi \]  
(16)

By Lemma 3.3 (i) and (ii), this equation always has a solution. Moreover any two solutions differ by an element of \( \mathcal{F}^{0,1} \). Thus we obtain a well defined linear map

\[ \mu : F^{1,0} \to F^{0,1} / \mathcal{F}^{0,1} \]

(a). \( \mu \) is surjective.

For any \( \psi \in F^{0,1} \), it follows from the equation \( \tilde{D}B\psi = 0 \) and Lemma 3.3 (i) and (iii) that there is \( \varphi \in C^{1,0} \) such that \( \tilde{D}\varphi = B\psi \). Since \( \tilde{D}B\varphi = B\tilde{D}\varphi = B^2\psi = 0 \), \( \varphi \in F^{1,0} \).
(b). \( \ker \mu = \mathcal{F}^{1,0} \)

Assume that \( \mu \varphi = [\psi] = 0 \). By definition, \( \psi = X + Y \) with the properties that \( \tilde{D}X = 0 \) and \( BY = 0 \) for some \( X, Y \in \mathcal{F}^{0,1} \). Thus \( B \psi = BX \). Since \( \tilde{D} \varphi = B \psi, \tilde{D} \varphi = BX \).

On the other hand, we can always write \( X = \tilde{D}X \) by Lemma 3.3. Thus \( \tilde{D} \varphi = B \tilde{D}X = \tilde{D}BX \). Hence, \( \tilde{D}(\varphi - BX) = 0 \). Write \( Y_1 = \varphi - BX \). Then \( \varphi = BX_1 + Y_1 \), which is easily seen to be in \( \mathcal{F}^{1,0} \).

Conversely, given any \( \varphi \in \mathcal{F}^{1,0} \), we can write \( \varphi = X + Y \) such that \( \tilde{D}X = 0 \) and \( BY = 0 \). Again by Lemma 3.3, we have \( Y = BY_1 \) for some \( Y_1 \in \mathcal{C}^{0,0} \). It thus follows that \( \tilde{D} \varphi = \tilde{D}Y = \tilde{D}BY_1 = B \tilde{D}Y_1 \). Thus \( \mu \varphi = [\tilde{D}Y_1] = 0 \).

It follows from (a) and (b) that \( \mu \) descends to an isomorphism:

\[
\mu : \mathcal{F}^{1,0} / \mathcal{F}^{1,0} \rightarrow \mathcal{F}^{0,1} / \mathcal{F}^{0,1}.
\]

(17)

\( \Box \)

**Proposition 3.5**

\( \mathcal{F}^{1,0} / \mathcal{F}^{1,0} \cong H^2(W_D, W_D) \).

**Proof.** Let \( \varphi \in \mathcal{F}^{1,0} \), then \( \tilde{D}B \varphi = 0 \). Define \( \tilde{\varphi} \) to be \( B \varphi \) restricted to \( W_D \). It is simple to see that \( D \tilde{\varphi}(u_1, u_2) = 0 \) if \( u_1, u_2 \in W_D \), so \( \tilde{\varphi} \in \mathcal{C}^2(W_D, W_D) \). Then \( b \tilde{\varphi} = B^2 \varphi|_{W_D} = 0 \), and so we obtain a well-defined map:

\[
\tau : \mathcal{F}^{1,0} \rightarrow H^2(W_D, W_D)
\]

\[
\varphi \rightarrow [\tilde{\varphi}]
\]

(a) \( \tau \) is surjective.

Given any 2-cocycle \( \tilde{\varphi} \in \mathcal{C}^2(W_D, W_D) \). It naturally extends to a bundle map \( W \otimes W \rightarrow W^+ \), which will be denoted by the same symbol. Then \( \tilde{\varphi} \) is a fiberwise two-cocycle. According to Lemma 3.3, there is \( \varphi \in \mathcal{C}^{1,0} \) such that \( \tilde{\varphi} = B \varphi \). On the other hand, for any \( u_1, u_2 \in W_D \),

\[
(\tilde{D} \tilde{\varphi})(u_1, u_2) = \tilde{D} \tilde{\varphi}(u_1, u_2) - \tilde{\varphi}(Du_1, u_2) - \tilde{\varphi}(u_1, Du_2)
= \tilde{D} \tilde{\varphi}(u_1, u_2)
= 0,
\]

since \( \tilde{\varphi}(u_1, u_2) \in W_D \). This implies that \( \tilde{D} \tilde{\varphi} \) is identically zero since \( W_D \) spans \( W \) at each fiber. Therefore, \( \tilde{D}B \varphi = \tilde{D} \tilde{\varphi} = 0 \). Hence \( \varphi \in \mathcal{F}^{1,0} \) and \( \tau(\varphi) = [\tilde{\varphi}] \).

(b) \( \ker \tau = \mathcal{F}^{1,0} \).

Assume that \( \varphi \in \mathcal{C}^{1,0} \) such that \( B \varphi = 0 \). It follows from its definition, that \( \tilde{\varphi} = 0 \). On the other hand, if \( \tilde{D} \varphi = 0 \), then \( \varphi(W_D) \subset W_D^+ \). This shows that \( \varphi \in \mathcal{C}^1(W_D, W_D) \). Thus, \( \tilde{\varphi} \in B^2(W_D, W_D) \) by definition. Therefore \( \tau(\varphi) = [\tilde{\varphi}] = 0 \). This shows that \( \mathcal{F}^{1,0} \subseteq \ker \tau \).
Conversely, assume that $\tau(\varphi) = 0$. Then $\bar{\varphi} \in C^2(W_D, W_D)$ is a 2-coboundary. Write $\bar{\varphi} = b\varphi_1$, with $\varphi_1 \in C^1(W_D, W_D)$. Extend $\varphi_1$ to a bundle map $W \to W^+$, which will be denoted by the same symbol. Then $B\varphi = B\varphi_1$ when being restricted to $W_D$. It thus follows that $B(\varphi - \varphi_1) = 0$ identically. Write $X = \varphi - \varphi_1$. Then $\varphi = X + \varphi_1$, where $BX = 0$ and $D\varphi_1 = 0$. The latter follows from the fact that $\varphi_1(W_D) \subseteq W^+_D$. Hence ker $\tau \subseteq F^{1,0}$.

It follows from (a) and (b) that $\tau$ descends to an isomorphism:

$$\tau : F^{1,0}/F^{0,1} \cong H^2(W_D, W_D).$$

(18)

Proposition 3.6

$$F^{0,1}/F^{0,1} \cong H^2(M)[h^{-1}, h]].$$

Proof. For any $\varphi \in F^{0,1}$, $\bar{D}\varphi = D\varphi$ by definition. Hence $BD\varphi = B\bar{D}\varphi = 0$. This means that $D\varphi$ is in the center of $\Gamma(W^+) \otimes \Omega^2(M)$, so it is a scalar 2-form. Moreover, $d(D\varphi) = D(D\varphi) = D^2\varphi = 0$. Thus $D\varphi$ is a closed 2-form in $Z^2(M)[h^{-1}, h]]$.

Let $\lambda : F^{0,1} \to H^2(M)[h^{-1}, h]]$ be the map defined by $\lambda(\varphi) = [D\varphi]$.

(a). $\lambda$ is surjective.

Given any closed 2-form $\theta \in Z^2(M)[h^{-1}, h]]$. Let $\varphi \in \Gamma(W^+) \otimes \Omega^1(M)$ be any solution of the equation: $D\varphi = \theta$. Thus, $B\bar{D}\varphi = BD\varphi = B\theta = 0$ since $\theta$ is a scalar 2-form. This means that $\varphi \in F^{0,1}$. It is clear that $\lambda\varphi = [D\varphi] = [\theta]$.

(b). ker $\lambda \subseteq F^{0,1}$.

Assume that $\varphi \in C^{0,1}$ satisfying $\bar{D}\varphi = 0$. Then $D\varphi = 0$, and therefore $\lambda\varphi = [D\varphi] = 0$. On the other hand, if $B\varphi = 0$, $\varphi$ is a scalar 1-form. Then $D\varphi = d\varphi$. Hence $\lambda\varphi = [d\varphi] = 0$. This shows that $F^{0,1} \subseteq$ ker $\lambda$.

Conversely, assume that $\lambda\varphi = 0$. It follows, from definition, that $D\varphi = d\theta$ for some $\theta \in \Omega^1(M)[h^{-1}, h]]$. Write $\varphi_1 = \varphi - \theta$. Then $D\varphi_1 = D\varphi - d\theta = 0$. It is clear that $B\theta = 0$. Hence $\varphi \in F^{0,1}$. Therefore ker $\lambda \subseteq F^{0,1}$.

A combination of (a) and (b) implies that $\lambda$ descends to an isomorphism:

$$\lambda : F^{0,1}/F^{0,1} \to H^2(M)[h^{-1}, h]].$$

(19)

This concludes the proof.

\[\square\]

Proof of Theorem 3.2. This is a direct consequence of Propositions 3.4, 3.5 and 3.6. The desired isomorphism is established by

$$\Phi = \lambda_0 \mu_0 \tau^{-1} : H^2(W_D, W_D) \to H^2(M)[h^{-1}, h]].$$

(20)
Remark The same argument in Proposition 3.4 can be used to prove that for $p \geq 1$ and $q \geq 0$,
\[
F^{p,q}/F^{p,q} = F^{p-1,q+1}/F^{p-1,q+1}.
\]
Similarly one can prove that for any $k \geq 0$,
\[
F^{k,0}/F^{k,0} \simeq H^{k+1}(W_D, W_D)
\]
and
\[
F^{0,k}/F^{0,k} \simeq H^{k+1}(M)[h^{-1}, h]
\]
using a similar argument as in Proposition 3.5 and 3.6. As a consequence one obtains the isomorphism: $H^k(W_D, W_D) \simeq H^k(M)[h^{-1}, h]$. We are informed by Tsygan that he also has obtained a proof of this result [19].

The next result shows how the isomorphism $\Phi$ given by Equation (20) depends on the Fedosov connection.

**Theorem 3.7** Assume that $W_{D_1}$ and $W_{D_2}$ are isomorphic Fedosov algebras coming from Fedosov connections $D_1$ and $D_2$, respectively. Let $T : W_{D_1} \rightarrow W_{D_2}$ be an isomorphism of star-products, and $T_* : H^2(W_{D_1}, W_{D_1}) \rightarrow H^2(W_{D_2}, W_{D_2})$ its induced isomorphism. Then the following diagram:

\[
\begin{array}{ccc}
H^2(W_{D_1}, W_{D_1}) & \xrightarrow{T_*} & H^2(W_{D_2}, W_{D_2}) \\
\Phi_1 \downarrow & & \downarrow \Phi_2 \\
H^2(M)[h^{-1}, h] & \xrightarrow{id} & H^2(M)[h^{-1}, h]
\end{array}
\]

commutes.

**Proof.** Since $T$ is local, $T$ extends to a bundle map $T : W \rightarrow W$, which is easily seen to be a fiberwise isomorphism. Hence,
\[
T(1) = 1.
\]

The following lemma is essential to the proof of this theorem.

**Lemma 3.8** Under the same assumption as in Theorem 3.7, we have
\[
D_1 = T^{-1}D_2T. \tag{22}
\]

**Proof.** It is easy to check that $T^{-1}D_2T$ is also a flat connection on the Weyl bundle since $T$ is a bundle map. Consider $D = D_1 - T^{-1}D_2T$. Then $Df u = fDu$ for any $f \in C^\infty(M)[[h]]$ and $u \in \Gamma(W)$. On the other hand, $Du = 0$ for any $u \in W_{D_1}$. Thus, $D$ is zero identically. This concludes the proof.
Now $T$ induces an isomorphism $T_* : C^{p,q} \rightarrow C^{p,q}$ by

$$(T_* \varphi)(u_1, \cdots, u_p) = T \varphi(T^{-1}u_1, \cdots, T^{-1}u_p), \quad \forall u_1, \cdots, u_p \in W.$$  

It follows from Lemma 3.8 that

$$\tilde{D}_2 T_* = T_* \tilde{D}_1.$$  

On the other hand, it is clear that $T_*$ commutes with $B$:

$$B \circ T_* = T_* \circ B.$$  

Therefore we have the following commuting diagrams:

\begin{equation}
\begin{array}{ccc}
F^1_{1,0} / F^0_{1,0} & \xrightarrow{T_*} & F^1_{2,0} / F^0_{2,0} \\
\downarrow \mu_1 & & \downarrow \mu_2 \\
F^0_{1,1} / F^0_{1,1} & \xrightarrow{T_*} & F^0_{2,1} / F^0_{2,1},
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
F^1_{1,0} / F^1_{1,0} & \xrightarrow{T_*} & F^1_{2,0} / F^1_{2,0} \\
\downarrow \tau_1 & & \downarrow \tau_2 \\
H^2(W_{D_1}, W_{D_1}) & \xrightarrow{T_*} & H^2(W_{D_2}, W_{D_2}),
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{ccc}
F^0_{1,1} / F^0_{1,1} & \xrightarrow{T_*} & F^0_{2,1} / F^0_{2,1} \\
\downarrow \lambda_1 & & \downarrow \lambda_2 \\
H^2(M)[h^{-1}, h]] & \xrightarrow{id} & H^2(M)[h^{-1}, h]]
\end{array}
\end{equation}

The conclusion thus follows immediately.
Given any star-product on $M$, the star-product algebra $A$ is isomorphic to some Fedosov algebra $W_D$. So one obtains an isomorphism $H^k(A,A) \cong H^k(M)[h^{-1}, h]$. Theorem 3.7 implies that this isomorphism is independent of the choice of the Fedosov algebra, and therefore is canonical.

It turns out that the isomorphism between Hochschild and de Rham cohomology can also be constructed without the explicit choice of a Fedosov connection. To do this, we consider the bundle $\mathcal{J}M$ given by infinite jets of complex-valued functions, with $h$ adjoined as usual. This bundle carries a natural flat connection $D_{\mathcal{J}}$, and the choice of a star-product on $M$ endows it with a fiberwise star-product. Its fiberwise Hochschild coboundary operator extends to differential forms and commutes with the natural extension of $D_{\mathcal{J}}$. So we have a natural double complex, and we can use it as above to get the Hochschild-de Rham isomorphism.

In fact, according to [14], there is an isomorphism between $\mathcal{J}M$ and the Weyl bundle $W$, under which $D_{\mathcal{J}}$ goes to a Fedosov connection $D$ in $W$. So the double complex arising from the jet bundle becomes essentially the same in the proof of Theorem 3.2.

Finally, we note that it is possible to construct directly an isomorphism between Hochschild and Čech cohomology by using a double complex whose cochains are Čech cochains with values in Hochschild cocycles for the deformed algebra. One chooses a covering by open subsets with all intersections contractible and uses the non-formal version of Theorem 3.1.

4 Derivative of a star-product

Let $\ast_h$ be a star-product on a symplectic manifold $M$. For any $f, g \in C^\infty(M)$, set

$$c(f, g) = \frac{d}{dh}(f \ast_h g).$$

Since $c$ is $1/h$ times the infinitesimal deformation corresponding to the rescaling of $h$, it defines a Hochschild 2-cocycle on $A = C^\infty(M)[[h]]$.

The main theorem is the following:

**Theorem 4.1** Under the isomorphism $\Phi : H^2(A, A) \rightarrow H^2(M)[h^{-1}, h]$ as constructed in Theorem 3.2,

$$\Phi[c] = -i h^2 \frac{d}{dh}(cl(A)),$$

where $cl(A)$ is the characteristic class of the star-product algebra (see Equation (11)).

We shall divide our proof into several steps. First note that $c$ induces a family of 2-cocycles $\tilde{c}$ on the fibres of the Weyl bundle via pull back by the quantum exponential map. A crucial step is to write $\tilde{c} = BH$ for some $h$-linear bundle map $H : W \rightarrow W$. Unfortunately, it is difficult to find an explicit formula for this $H$. On the other hand, there exists another fiberwise 2-cocycle $c_1$ on the Weyl bundle obtained simply by taking the $h$-derivative of the fiberwise star-product. In general, $\tilde{c}$ and $c_1$ are quite different since the quantum exponential map itself involves $h$. Nevertheless we
can obtain useful information about \( \tilde{c} \) by relating it to \( c_1 \). For \( c_1 \), it is simple to write it as a 2-coboundary:

**Proposition 4.2** Let \( E = -\frac{i}{2}(\sum_j y_j \partial_{y_j}) \) be \( \frac{i}{2} \) times the Euler vector field of the fiberwise linear symplectic structure on \( TM \), where \((y_1, \cdots, y_{2n})\) are the linear coordinates in the fibers of \( TM \). Then,

\[ c_1 = BE. \]

Define a \( \mathbb{C} \)-linear (NOT \( \hbar \)-linear) bundle map \( \rho : W \rightarrow W \):

\[ \rho(u) = \frac{h}{i} \tilde{u} + Eu, \quad \forall u \in W. \tag{27} \]

Given any flat section \( \tilde{u} \in W_D \), consider the following equation for \( v \in \Gamma(W) \):

\[
\begin{aligned}
Dv &= D\rho(\tilde{u}) \\
v|_{y=0} &= 0.
\end{aligned}
\tag{28}
\]

Clearly, this equation has a unique solution since the right side of Equation (28) is \( D \)-closed. Second, the solution \( v \) depends on \( \tilde{u} \) \( \hbar \)-linearly. Hence, we obtain a \( \hbar \)-linear map

\[ H : W_D \rightarrow \Gamma(W) \quad \tilde{u} \rightarrow v. \]

It is clear that \( H \) extends to a \( \hbar \)-linear bundle map from the Weyl bundle \( W \) to itself. The following lemma indicates that \( H \) coincides with \( \rho \) on a special class of parallel sections of \( W_D \).

**Lemma 4.3** If \( \tilde{u} \in W_D \) is a flat section such that \( \tilde{u}|_{y=0} \) is \( \hbar \)-independent, then

\[ H(\tilde{u}) = \rho(\tilde{u}) \]

**Proof.** Since \( \rho(\tilde{u}) \) clearly satisfies the first part of Equation (28), it suffices to check that \( \rho(\tilde{u}) \) satisfies the initial condition.

\[
\begin{aligned}
\rho(\tilde{u})|_{y=0} &= (\frac{h}{i} \tilde{u} + E\tilde{u})|_{y=0} \\
&= \frac{h}{i} \tilde{u}|_{y=0} \\
&= \frac{h}{i} \frac{d}{dh}(\tilde{u}|_{y=0}) \\
&= 0.
\end{aligned}
\]

Here the last step follows from the assumption that \( \tilde{u}|_{y=0} \) is \( \hbar \)-independent.
The following result is a direct consequence of the lemma above together with Equation (27).

**Proposition 4.4** For any \( f, g \in C^\infty(M) \),

\[
c(f, g) = (BH)(\tilde{u}, \tilde{v})|_{y=0},
\]

where \( c \) is defined as in Equation (26), \( \tilde{u} \) and \( \tilde{v} \) are parallel sections such that \( \tilde{u}|_{y=0} = f \) and \( \tilde{v}|_{y=0} = g \), respectively.

In other words,

\[
\tilde{c} = BH.
\]

The next step is to write \( \tilde{D}H = BK_0 \) for some \( K_0 \in \Gamma(W) \otimes \Lambda^1 \). For this purpose we need the following

**Proposition 4.5** For any \( \tilde{u} \in W_D \), we have

\[
D\rho(\tilde{u}) = [\frac{i}{\hbar}K_0, \tilde{u}],
\]

where

\[
K_0 = -(Er - i\hbar\dot{r} + i\dot{r} + \frac{i}{2}\omega_{ij}y^idx^j) \in \Gamma(W) \otimes \Lambda^1.
\]

An immediate consequence is

**Corollary 4.6** We have

\[
\tilde{D}H = BK_0.
\]

**Proof.** Equation (29) implies that \((\tilde{D}H)(\tilde{u}) = (BK_0)(\tilde{u})\), if \( \tilde{u} \in W_D \) and \( \tilde{u}|_{y=0} \) is \( \hbar \)-independent according to Lemma 4.3. The conclusion thus follows immediately since \( \tilde{D}H \) and \( BK_0 \) are \( \hbar \)-linear and \( W_D \) spans each fiber of \( W \).

To prove Proposition 4.3, we need a couple of lemmas first.

**Lemma 4.7** (i). \([\partial, E] = 0\);

(ii). \([\delta, E] = -\frac{i}{\hbar}\delta = -\frac{1}{\hbar^2}\omega_{ij}y^idx^j\).  

**Proof.** (1) can be easily verified, and is left to the reader.

As for (2), let \( a \in \Gamma(W) \otimes \Lambda^{|a|} \),

\[
[\delta, E]a = \delta Ea - E\delta a
\]

\[
= \delta\left(-\frac{i}{2}\sum_j y^j\frac{\partial a}{\partial y^j}\right) - E\left(\sum_k dx^k \wedge \frac{\partial a}{\partial y^k}\right)
\]

\[
= -\frac{i}{2}\sum_j dx^j \wedge \frac{\partial a}{\partial y^j}
\]

\[
= -\frac{i}{2}\delta a
\]

\[
\]
Lemma 4.8 For any $a \in \Gamma(W) \otimes \Lambda^{[a]}$,

(i). $[D, \frac{d}{dh}]a = -[\left(\frac{i}{\hbar}r, a\right), a] - \frac{i}{\hbar}(c_1(r, a) - (-1)^{|a|}c_1(a, r))$;

(ii). $[D, E]a = \frac{i}{2}\delta a - \left[\frac{i}{\hbar}Er, a\right] + (c_1(r, a) - (-1)^{|a|}c_1(a, r))$.

Proof.

$[D, \frac{d}{dh}]a$

$= D\dot{a} - \frac{d}{dh}(-\delta a + \partial a + \left[\frac{i}{\hbar}r, a\right])$

$= D\dot{a} - (-\delta \dot{a} + \partial \dot{a} + \left[\frac{i}{\hbar}r\right], a)$

$\quad + \left[\frac{i}{\hbar}r, \dot{a}\right] + c_1\left(\frac{i}{\hbar}r, a\right) - (-1)^{|a|}c_1(a, \frac{i}{\hbar}r)$

$= D\dot{a} - (D\dot{a} + \left[\left(\frac{i}{\hbar}\right)^r, a\right] + \frac{i}{\hbar}c_1(r, a) - (-1)^{|a|}\frac{i}{\hbar}c_1(a, r))$

$= -\left(\left(\frac{i}{\hbar}\right)^r; a\right) - \frac{i}{\hbar}(c_1(r, a) - (-1)^{|a|}c_1(a, r))$.

(2)

$EDa = -E\delta a + E\partial a + E\left[\frac{i}{\hbar}r, a\right]$.

$E\left[\frac{i}{\hbar}r, a\right] = \frac{i}{\hbar}(E(r*a) - (-1)^{|a|}E(a*r))$

$= \frac{i}{\hbar}Er*a + r*\frac{i}{\hbar}Ea - c_1(r, a)$

$\quad + (-1)^{|a|}(-\frac{i}{\hbar}Ea*r - a*\frac{i}{\hbar}Er + c_1(a, r))$

$= \left[\frac{i}{\hbar}Er, a\right] + \left[r, \frac{i}{\hbar}Ea\right] - c_1(r, a) + (-1)^{|a|}c_1(a, r)$

On the other hand,

$DEa = -\delta Ea + \partial Ea + \left[\frac{i}{\hbar}r, Ea\right]$.

Hence,

$[D, E]a = -[\delta, E]a + [\partial, E]a - \left[\frac{i}{\hbar}Er, a\right] + c_1(r, a) - (-1)^{|a|}c_1(a, r)$

$= -[\delta, E]a - \left[\frac{i}{\hbar}Er, a\right] + c_1(r, a) - (-1)^{|a|}c_1(a, r)$

$= \frac{i}{2}\delta a - \left[\frac{i}{\hbar}Er, a\right] + (c_1(r, a) - (-1)^{|a|}c_1(a, r))$. 

16
Proof of Proposition 4.5} It follows from Lemma 4.8 that

\[ D\dot{\tilde{u}} = -[(\frac{i}{\hbar}r), \tilde{u}] - \frac{i}{\hbar}(c_1(r, \tilde{u}) - c_1(\tilde{u}, r)), \]

and

\[ DE\tilde{u} = \frac{i}{2}\delta\tilde{u} - \frac{i}{\hbar}E r, \tilde{u}] + (c_1(r, \tilde{u}) - c_1(\tilde{u}, r)). \]

Thus,

\[ D\rho(\tilde{u}) = \frac{h}{i}D\dot{\tilde{u}} + DE\tilde{u} = -\frac{h}{i}[(\frac{i}{\hbar}r), \tilde{u}] - \frac{i}{\hbar}E r, \tilde{u}] + \frac{i}{2}\delta\tilde{u} = [\frac{i}{h}K_0, \tilde{u}], \]

where \( K_0 = -(E r - i\hbar \dot{r} + i r + \frac{i}{2}\omega_{ij}y^idx^j). \)

Finally we need to compute \( DK_0:\)

**Proposition 4.9**

\[ DK_0 = i(\Omega - \hbar \dot{\Omega}) = -i\hbar^2 \frac{d}{dh}(\frac{1}{h}\Omega), \]

(31)

where \( \Omega \) is the Weyl curvature of the Fedosov connection \( D. \)

**Proof.** Let \( L : \Gamma(W) \otimes \Lambda^k \rightarrow \Gamma(W) \otimes \Lambda^k \) be the operator \( L = E - i\hbar \frac{d}{dh}. \) According to Lemma 4.8, we have,

\[ [D, \frac{d}{dh}]r = -[(\frac{i}{\hbar}r), r] - 2\frac{i}{h}c_1(r, r), \]

and

\[ [D, E]r = \frac{i}{2}\delta r - \frac{i}{\hbar}Er, r] + 2c_1(r, r). \]

Thus,

\[ [D, L]r = [D, E]r - i\hbar [D, \frac{d}{dh}]r \]

\[ = \frac{i}{2}\delta r - \frac{i}{\hbar}Er, r] - [\dot{r}, r] + \frac{2}{h}r^2. \]

On the other hand, we have

\[ \Omega = \omega - R - D r + \frac{i}{h}r^2. \]  

(32)  

17
Thus,
\[ Dr = \omega - \Omega - R + \frac{i}{\hbar} r^2. \]

Hence
\[ EDr = -ER + \frac{i}{\hbar} E(r \cdot r) = iR + \frac{i}{\hbar} E(r \cdot r), \]
and
\[ -i\hbar \frac{d}{dh} Dr = -i\hbar(-\dot{\Omega} + (\frac{i}{\hbar} r^2)) \]
\[ = i\hbar \dot{\Omega} - \frac{1}{\hbar} r^2 + [\dot{r}, r] + c_1(r, r). \]

Hence,
\[ LDr = EDr - i\hbar(Dr) \]
\[ = iR + \frac{i}{\hbar} E(r \cdot r) + i\hbar \dot{\Omega} - \frac{1}{\hbar} r^2 [\dot{r}, r] + c_1(r, r) \]
and
\[ DLr = [D, L]r + LDr \]
\[ = \frac{i}{2} \delta r + \frac{1}{\hbar} r^2 + iR + i\hbar \dot{\Omega}. \]

On the other hand,
\[ D(i\frac{\omega_{ij} y^i dx^j}{2}) \]
\[ = \frac{i}{2} \omega_{ij} dx^i \wedge dx^j + [\frac{i}{\hbar} \omega_{ij} y^j dx^i] \]
\[ = -i\omega + \frac{i}{2} [\frac{i}{\hbar} [\dot{r}, \omega_{ij} y^j dx^i]] \]
\[ = -i\omega + \frac{i}{2} [\frac{i}{\hbar} \omega_{ij} y^i dx^j, r] \]
\[ = -i\omega - \frac{i}{2} \delta r. \]

Therefore
\[ DK_0 = -(DLr + iDr + D(i\frac{\omega_{ij} y^i dx^j}{2})) \]
\[ = i(\Omega - \hbar \dot{\Omega}) \]
\[ = -i\hbar^2 \frac{d}{dh} \left( \frac{1}{\hbar} \Omega \right). \]

This concludes the proof. The cancellation of the expressions involving \( c_1 \) still seems rather mysterious to us.

\[ \square \]
Proof of Theorem 4.1 Proposition 4.4 implies that $\tau^{-1}[\tilde{c}] = [H]$. On the other hand, it follows from Corollary 4.6 that $\mu[H] = [K_0]$. Now $\lambda[K_0] = [DK_0] = -ih^2\frac{d}{dh}(\frac{1}{h}\Omega)$ according to Proposition 4.9. Thus the conclusion follows immediately.

We define a quantum Liouville operator to be an $h$-linear local operator $X$ on $A$ such that $h\frac{d}{dh} + X$ is a derivation (intuitively, it generates a 1-parameter group of algebra automorphisms which rescale $h$). It is easy to see that $X$ is such an operator if and only if its Hochschild coboundary is a constant multiple of the derivative cocycle $c$. The next result then follows immediately from Theorem 4.1.

**Corollary 4.10** A star-product algebra $A$ admits a quantum Liouville operator if and only if $\frac{d}{dh}(cl(A)) = 0$.

**Remark** This consideration of quantum Liouville operators connects our Theorem 4.1 to Theorem 4.4 of Deligne in [5].

5 Some examples

We have just seen that the characteristic class of a deformation quantization is nearly the same as the obstruction to the existence of a global Liouville operator. Thus, it is interesting to have an example where the characteristic class is nonzero, but a Liouville operator exists nevertheless. We will exhibit such an example on the cotangent bundle of a 2-torus.

On $M = T^*\mathbb{T}^2$, with coordinates $(\theta_1, \theta_2, p_1, p_2)$, we consider the $h$-dependent symplectic structure $\omega = \omega_0 + h\omega_1$, where $\omega_0$ is the canonical symplectic structure and $\omega_1$ is the cohomologically nontrivial form $d\theta_1 \wedge d\theta_2$. The corresponding Poisson tensor $\pi$ is the sum of the canonical Poisson structure $\pi_0$ and $h$ times the bivector $\pi_1 = -\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2}$. Denote by $X$ the usual Liouville vector field $p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$.

We obtain a star-product on $M$ by applying the usual exponential formula for the Moyal-Weyl product (see Equation (2)) to the $h$-dependent bivector field $\pi$. This is a Fedosov star-product for which $\Omega = \omega$, so its characteristic class is $\frac{1}{h}[\omega_0] + [\omega_1]$. Since $\omega_0$ is exact but $\omega_1$ is not, the characteristic class is nonzero, but its derivative with respect to $h$ vanishes.

It follows from Corollary 4.10 that this star-product should admit a quantum Liouville operator. In fact, it is easy to check (using the usual rule for the derivative of an exponential and the Lie derivative formulas $L_X\pi_0 = \pi_0$ and $L_X\pi_1 = 2\pi_1$) that the Liouville vector field $X$ functions as a quantum Liouville operator for our star-product (just as it does for the usual Moyal-Weyl product corresponding to $\omega_0$).

We note further that, replacing the term $h\omega_1$ by $h^2\omega_1$ in the definition of our $h$-dependent symplectic structure, we obtain a deformation quantization for the standard symplectic structure which does not admit a quantum Liouville operator.
Since the symplectic structure on a compact manifold cannot be exact, no star-product on a compact symplectic manifold can admit a quantum Liouville operator. The following example shows that we can find such an operator on a compact Poisson manifold.

There do exist compact regular Poisson manifolds admitting Liouville vector fields. For instance (see \[22\] for more details), we can take \(M\) to be the unit tangent bundle of a compact surface \(\Sigma\) of negative curvature, with a Poisson structure for which the symplectic leaves are the unstable manifolds of the geodesic flow. One sees from the structure equations of the group \(SL(2, \mathbb{R})\), of which \(M\) is a quotient, that the generator \(X\) of the geodesic flow is a Liouville vector field. For such a tangential Poisson structure, the obstruction to lifting \(X\) to a quantum Liouville operator lies in the 2nd de Rham cohomology along the symplectic leaves. At our instigation, Ratner \[18\] has provided a proof that this cohomology vanishes as long as \(1/4\) is not an eigenvalue of the laplacian on functions on \(\Sigma\), so that the quantum Liouville field must exist.\(^2\)

There is an alternative way to obtain the quantum Liouville operator for \(M\); it works even if \(\Sigma\) has \(1/4\) as an eigenvalue. The Poisson structure on \(M\) is generated by an action of a 2-dimensional solvable group \(G\) of triangular matrices contained in \(SL(2, \mathbb{R})\), so it can be quantized with the aid of a quantum \(R\)-matrix, as in \[23\]. This \(R\) matrix gives a left invariant quantization of a left invariant Poisson structure on \(G\), which can be given a left-invariant quantization by Fedosov’s method. An equivariant version of our method in this paper shows that the obstruction to finding a left-invariant quantum Liouville operator lies in the cohomology of left-invariant forms on \(G\), which is zero. The resulting operator \(Y\) can then be “pushed forward” by the \(G\) action to give a Liouville operator on \(M\).

Here is an example which falls just short of having a quantum Liouville operator. Let \(A\) be an element of \(SL(3, \mathbb{Z})\) whose eigenvalues \(\lambda_j\) are all real, with \(0 < \lambda_1 < 1 < \lambda_2 \leq \lambda_3\). Let \(\pi\) be a translation invariant Poisson structure on \(T^3\) whose symplectic leaves are planes parallel to the 2-dimensional expanding subspace of \(A\). The linear transformation defined by \(A\) acts on the torus, where it multiplies the Poisson structure by \(\lambda = \lambda_2 \lambda_3\); it has a similar effect on the corresponding Moyal-Weyl product, rescaling the deformation parameter by \(\lambda\). Thus we have a discrete family of \(h\) rescaling transformations, but not a 1-parameter group. In fact, it follows from consideration of its fundamental homology class (see \[22\]) that this Poisson structure does not admit a Liouville vector field (i.e. a vector field satisfying \(L_X \pi = \pi\)).

By multiplying \(T^3\) by \(\mathbb{R}\) and using a suitable symplectic structure, one can obtain a symplectic manifold admitting no Liouville vector field, but admitting a discrete group of maps which scale the symplectic structure nontrivially.

Another interesting phenomenon occurs in the case of \(SU(2)\) as a Poisson Lie group with the Bruhat-Poisson structure \[13\]. This Poisson manifold admits no Liouville vector field (as can be seen from consideration of its modular vector field), but it does admit a 1-parameter group of continuous mappings which scale the Poisson structure. (This makes sense because the mappings are smooth on each symplectic leaf.) This reflects the fact \[16\] that the \(C^*\)-completions of the \(^2\)Ratner’s proof uses the representation theory of \(SL(2, \mathbb{R})\) on the functions on \(M\). A related result for measurable cochains which are smooth only along the leaves, based on the nonexistence of a transverse invariant measure, is given as Theorem 4.27 in \[15\]. Incidentally, vanishing of the (smooth) cohomology implies that any two Poisson structures on \(M\) associated with this foliation are isomorphic (at least up to sign), as are any two tangential star-products quantizing such a Poisson structure.
algebras of functions on the corresponding quantum $SU(2)$'s are isomorphic as the deformation parameter varies through nonzero values.

Finally, we remark that the independence of the isomorphism class of a deformation quantization on the (nonzero) value of the deformation parameter is important for the link between deformation quantization and the “E-theory” of asymptotic morphisms of $C^*$ algebras.

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