The disappearing $Q$ operator

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Abstract

In the Schrödinger formulation of non-Hermitian quantum theories a positive-definite metric operator $\eta \equiv e^{-Q}$ must be introduced in order to ensure their probabilistic interpretation. This operator also gives an equivalent Hermitian theory, by means of a similarity transformation. If, however, quantum mechanics is formulated in terms of functional integrals, we show that the $Q$ operator makes only a subliminal appearance and is not needed for the calculation of expectation values. Instead, the relation to the Hermitian theory is encoded via the external source $j(t)$. These points are illustrated and amplified for two non-Hermitian quantum theories: the Swanson model, a non-Hermitian transform of the simple harmonic oscillator, and the wrong-sign quartic oscillator, which has been shown to be equivalent to a conventional asymmetric quartic oscillator.

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I. INTRODUCTION

The recent interest in non-Hermitian quantum mechanics stems from the seminal paper by Bender and Boettcher\textsuperscript{[1]}, who showed numerically that Hamiltonians of the form

\[ H = \frac{1}{2} p^2 - g(ix)^N \]

(1)

possessed a real, positive spectrum for \( N \geq 2 \). This reality was attributed to an unbroken \( PT \) symmetry of \( H \) and was subsequently proved analytically by Dorey et al.\textsuperscript{[2]}, exploiting the connection between ordinary differential equations and integrable models.

The more general framework of pseudo-Hermiticity, encompassing \( PT \) symmetry, was subsequently formulated by Mostafazadeh\textsuperscript{[3]}, whereby

\[ H^\dagger = \eta H \eta^{-1}, \]

(2)

in which \( \eta \) is a positive-definite, Hermitian operator. Essentially the same idea had appeared previously under the name of “quasi-Hermiticity” by Scholtz et al.\textsuperscript{[4]}, and indeed goes back to much earlier work by Pauli\textsuperscript{[5]}.

Reality of the eigenvalues is not sufficient for a viable non-Hermitian quantum theory: for a probabilistic physical interpretation we must also have a positive-definite Hilbert-space metric. Such a metric may be constructed, but it is not given a priori, and depends on the particular Hamiltonian \( H \). In the context of \( PT \) symmetry a new operator, the \( C \) operator, was introduced in Ref. \[ 6 \], with the help of which a positive-definite, \( CPT \) inner product could be constructed. In a subsequent paper\textsuperscript{[7]} a systematic way of constructing \( C \) in perturbation theory was developed, which was greatly facilitated by the introduction of the \( Q \) operator, defined by \( PC = e^{-Q} \). The connection with the operator \( \eta \) of Eq. (2) is

\[ \eta = e^{-Q}. \]

(3)

This appears in the overlap \( \langle \langle \psi, \varphi \rangle \rangle \), defined by

\[ \langle \langle \psi, \varphi \rangle \rangle \equiv \langle \psi, e^{-Q} \varphi \rangle, \]

(4)

and in the expression for the equivalent Hermitian Hamiltonian\textsuperscript{[3]}:

\[ h = e^{-\frac{i}{2}Q} H e^{\frac{i}{2}Q}, \]

(5)

showing that the two operators are related by a similarity transformation (rather than a unitary transformation).

In most cases one must resort to perturbation theory or some other approximation in order to construct \( Q \) and \( h \). However, in the two theories we study below they are known exactly. These are the Swanson\textsuperscript{[8]} Hamiltonian,

\[ H = \omega a^\dagger a + \alpha a^2 + \beta a^\dagger a^2, \]

(6)

where \( a \) and \( a^\dagger \) are simple harmonic oscillator annihilation and creation operators for unit frequency and \( \omega, \alpha \) and \( \beta \) are real, and the “wrong-sign” quartic oscillator,

\[ H = \frac{1}{2} p_z^2 + \frac{1}{2} m^2 z^2 - gz^4, \]

(7)
where we have written $z$, $p_z$ in place of $x$, $p$ in recognition of the fact that for the energy eigenvalue equation to be well defined, it must be formulated on a contour in the lower- half complex $z$ plane.

The $Q$ operator (or operators, because it is not unique) can be found exactly\[10, 11\] for Eq. (6), and the equivalent Hermitian Hamiltonian is simply a harmonic oscillator. In the case of Eq. (7) it can again be found exactly for a particular choice of contour\[9\], leading to an equivalent Hermitian Hamiltonian which is a standard quartic oscillator, together with a linear term, as discovered earlier by Andrianov\[12\] and Buslaev and Grecchi\[13\].

Apart from the standard formulation of quantum mechanics in terms of operators $\hat{x}$ and $\hat{p}$ acting on Schrödinger wave functions we may also approach it from the point of view of functional integrals, which indeed is the natural language if a generalization to higher dimensions is to be attempted. A brief discussion of the Swanson model in this context was given in Ref. [14] and a rederivation of the results of Ref. [9] using functional methods was given in [15], subsequently corrected in [16]. These papers were primarily concerned with the derivation of the equivalent Hermitian Hamiltonian $h$ and did not address the question of overlaps or expectation values. Indeed, the quantity considered was the partition function, i.e. the vacuum-generating functional for $j \equiv 0$, rather than the true functional $Z[j]$, from which Green functions can be obtained by functional derivation.

In the following two sections we will explore the full vacuum generating functional, finding that in this formalism the operator $Q$ appears only fleetingly and is not involved in the calculation of overlaps or expectation values. It is rather the dependence of $Z[j]$ on $j$ that encodes the similarity transformation induced by $Q$. In passing, we also note the role of $Q$ at the classical level as being a generator of canonical transformations. In the final section we discuss the implications of our results.

II. $Z[j]$ FOR THE SWANSON MODEL

In terms of $x$ and $p$ the Hamiltonian of Eq. (6) reads

$$H = ax^2 + bp^2 + c\{x, p\},$$

(8)

where $a = \frac{1}{2}(\omega + \alpha + \beta)$, $b = \frac{1}{2}(\omega - \alpha - \beta)$ and $c = \frac{1}{2}i(\alpha - \beta)$. It is non-Hermitian for $\alpha \neq \beta$.

The Euclidean phase-space functional integral for $Z[j]$ corresponding to Eq. (8) is

$$Z[j] = \int[D\varphi][D\pi] \exp\left\{ -\int dt[i\dot{\varphi}\pi + a\dot{\varphi}^2 + b\pi^2 + 2c\varphi\pi - j\varphi]\right\},$$

(9)

where, to correspond to the notation in field theory, we have written $\varphi$ for $x$ and $\pi$ for $p$.

The different solutions for $Q$ and $h$ are quite simply obtained by completing the square in the integrand in different ways\[1\]

\[1\] Below we have illustrated the two extreme cases where $Q = Q(\varphi)$ and $Q = Q(\pi)$. The more general case $Q = Q(\varphi, \pi)$, in which both $\varphi$ and $\pi$ are shifted, does not add anything essential to the case $Q = Q(\pi)$. 

3
A.  $Q = Q(\varphi)$

This corresponds to completing the square by shifting $\pi$, according to

$$a\varphi^2 + b\pi^2 + 2c\varphi\pi = b(\pi + \frac{c}{b}\varphi)^2 + (a - \frac{c^2}{b})\varphi^2 \equiv b\tilde{\pi}^2 + \tilde{a}\varphi^2,$$

(10)
giving

$$Z[j] = \int [D\varphi][D\tilde{\pi}] \exp \left\{ -\int dt [i\dot{\varphi}(\tilde{\pi} - \frac{c}{b}\varphi) + \tilde{a}\varphi^2 + b\tilde{\pi}^2 - j\varphi] \right\}$$

= \int [D\varphi][D\tilde{\pi}] \exp \left\{ -\int dt [i\dot{\varphi}\tilde{\pi} + \tilde{a}\varphi^2 + b\tilde{\pi}^2 - j\varphi] \right\},$$

(11)
The $Q(x)$ operator has previously been determined in the operator formalism[11] as

$$Q(x) = \frac{\alpha - \beta}{\omega - \alpha - \beta}x^2 = -i\frac{c}{2b}x^2$$

(12)

Its time derivative, written in terms of $\varphi$, appears in the first line of Eq. (11), but then disappears in the following line, precisely because it is a total derivative. The momentum variable can now be integrated out$^2$, to leave

$$Z[j] = \int [D\varphi] \exp \left\{ -\int dt \left[ \frac{\dot{\varphi}^2}{4b} + \tilde{a}\varphi^2 - j\varphi \right] \right\},$$

(13)
the standard form for the vacuum generating functional of the simple harmonic oscillator with frequency $\Omega^2 \equiv 4\tilde{a}b$. Note that by completing the square in this particular way, we have effectively changed $\pi$ to $\tilde{\pi} \equiv \pi + (c/b)\varphi$ but left $\varphi$ unchanged. In the operator formalism this would be the result of the transformations

$$p \to P \equiv e^{\frac{i}{2}Q(x)} p e^{-\frac{i}{2}Q(x)} = p + \frac{1}{2}iQ'(x),$$

$$x \to X \equiv e^{\frac{i}{2}Q(x)} x e^{-\frac{i}{2}Q(x)} = x.$$  

(14)
In terms of classical mechanics this is a canonical transformation, generated by

$$F_2(x, P) = xP - \frac{1}{2}iQ(x)$$

(15)
according to the standard equations

$$X = \frac{\partial F_2}{\partial P}, \quad p = \frac{\partial F_2}{\partial x}.$$  

(16)
It transforms the original non-Hermitian Hamiltonian $H$ of Eq. (8) into the equivalent Hermitian Hamiltonian

$$h = bP^2 + \tilde{a}X^2,$$

(17)

$^2$ Here, and in subsequent integrations, we ignore overall multiplicative factors.
while the corresponding Lagrangians, $\ell$ and $L$, differ precisely by the time derivative of $Q$.

In this way of constructing $h$ the field $\varphi$ remains an observable in the original theory, and Green functions of $\varphi$ are obtained in the standard way by functional differentiation with respect to $j(t)$. Since the single-variable functional integral for $Z[j]$ obtained from Eq. (9) is identical that of Eq. (13), the Green functions of the Swanson Hamiltonian can be obtained in the functional formalism without reference to $Q$, in contrast with the corresponding calculations within the standard Schrödinger framework, where matrix elements have to be calculated with the metric $\eta = e^{-Q}$, as in Eq. (4).

B. $Q = Q(\pi)$

In this case, there will be a shift in $\varphi$ between the original and the Hermitian Hamiltonian. For that reason, and to conform with the following section, we change the notation so that the field appearing in Eq. (9) becomes $\psi$:

$$Z[0] = \int [D\psi][D\pi] \exp \left\{ -\int dt [i\dot{\psi}\pi + a\psi^2 + b\pi^2 + 2c\psi\pi] \right\}, \quad (18)$$

reserving $\varphi$ for the shifted field appearing in the Hermitian version. Note that for the moment we have set $j = 0$ in Eq. (18). This is because we will ultimately wish to calculate Green functions of the observable $\varphi$, and hence will insert $j$ in the latter form and then transform back to determine the $j$ dependence in Eq. (18). Completing the square according to

$$a\psi^2 + b\pi^2 + 2c\psi\pi = a(\psi + \frac{c}{a}\pi)^2 + (b - \frac{c^2}{a})\pi^2$$

$$\equiv a\varphi^2 + \tilde{b}\pi^2,$$  \quad (19)

we obtain

$$Z[0] = \int [D\varphi][D\pi] \exp \left\{ -\int dt [i(\dot{\varphi} - \frac{c}{a}\pi)\pi + a\varphi^2 + \tilde{b}\pi^2] \right\}$$

$$= \int [D\varphi][D\pi] \exp \left\{ -\int dt [i(\dot{\varphi}\pi + a\varphi^2 + \tilde{b}\pi^2] \right\},$$  \quad (20)

after discarding the derivative of $Q(\pi) \equiv -i\pi^2(c/2a)$ appearing in the first line of Eq. (20). Performing the integration over $\pi$ we obtain

$$Z[0] = \int [D\varphi] \exp \left\{ -\int dt \left[\frac{\dot{\varphi}^2}{4b} + a\varphi^2 \right] \right\},$$

which is a rescaled version of Eq. (13) with $j = 0$. The corresponding operator transformations would be

$$p \rightarrow P \equiv e^{\frac{i}{2}Q(\pi)}p e^{-\frac{i}{2}Q(\pi)} = p$$

$$x \rightarrow X \equiv e^{\frac{i}{2}Q(\pi)}x e^{-\frac{i}{2}Q(\pi)} = x - \frac{1}{2}iQ'(p),$$  \quad (22)

while the corresponding transformation in classical mechanics would be generated by

$$F_2(x, P) = xP - \frac{1}{2}iQ(P).$$  \quad (23)
In order to obtain Green functions of $\varphi$ we now restore $j$ to Eq. (21):

$$Z[j] = \int [D\varphi] \exp \left\{ -\int dt \left[ \frac{\dot{\varphi}^2}{4b} + a\varphi^2 - j\varphi \right] \right\}, \quad (24)$$

or equivalently in Eq. (20):

$$Z[j] = \int [D\varphi][D\pi] \exp \left\{ -\int dt \left[ i\dot{\varphi}\pi + a\varphi^2 + \tilde{b}\pi^2 - j\varphi \right] \right\}, \quad (25)$$

and transform back to the original field variable $\psi$ by substituting $\varphi = \psi + (c/a)\pi$. The resulting expression in the exponent:

$$\left[ \right] = i\dot{\psi}\pi + b\pi^2 + a\psi^2 - 2c\psi\pi - j\psi - j(c/a)\pi$$

can be written in the form

$$\left[ \right] = b \left[ \pi + \frac{1}{2b} \left( i\dot{\psi} + 2c\psi - \frac{c}{a}j \right) \right]^2 + \frac{1}{4b} \dot{\psi}^2 + \tilde{a}\psi^2 - \frac{\Omega^2}{4ab} j\psi + \frac{ic}{2ab} j\dot{\psi} - \frac{c^2}{4a^2b} j^2, \quad (26)$$

which gives

$$Z[j] = \int [D\psi] \exp \left\{ -\int dt \left[ \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\Omega^2\psi^2 - \frac{\Omega^2}{2a} \frac{j}{\sqrt{2b}} \psi + \frac{ic}{a} \frac{j}{\sqrt{2b}} \dot{\psi} - \frac{c^2 j^2}{4a^2b} \right] \right\}, \quad (27)$$

after performing the $\pi$ integration and rescaling $\psi$ by $\psi \rightarrow \psi \sqrt{2b}$.

Let us now verify that we obtain the correct expressions for some low-order Green functions in $\varphi$ by functional differentiation of Eq. (27). A single differentiation $\delta/\delta j(t)$ gives

$$\left. \frac{1}{Z} \frac{\delta Z}{\delta j} \right|_{j=0} = \frac{1}{\sqrt{2b}} \left\langle \frac{\Omega^2}{2a} \psi - \frac{ic}{a} \dot{\psi} \right\rangle, \quad (28)$$

which is clearly zero, the correct result for $\langle \varphi \rangle$.

For a somewhat less trivial check we perform a double differentiation $\delta^2/\delta j_1\delta j_2$, using an obvious subscript notation for $j$ at the two times $t_1, t_2$. Thus

$$\left. \frac{1}{Z} \frac{\delta^2 Z}{\delta j_1 \delta j_2} \right|_{j=0} = \frac{1}{2b} \left\langle \left( \frac{\Omega^2}{2a} \psi_1 - \frac{ic}{a} \dot{\psi}_1 \right) \left( \frac{\Omega^2}{2a} \psi_2 - \frac{ic}{a} \dot{\psi}_2 \right) \right\rangle + \frac{c^2}{2a^2b} \delta(t_1 - t_2), \quad (29)$$

This is simply evaluated by noting that for the harmonic oscillator Lagrangian appearing in Eq. (27) the expectation values are

$$\langle \psi_1\psi_2 \rangle = \frac{1}{2\Omega} e^{-\Omega|t_1 - t_2|}$$

$$\langle \dot{\psi}_1\dot{\psi}_2 \rangle = -\langle \psi_1\dot{\psi}_2 \rangle = \frac{1}{2} \varepsilon(t_2 - t_1)e^{-\Omega|t_1 - t_2|}$$

$$\langle \dot{\psi}_1\dot{\psi}_2 \rangle = -\frac{\Omega}{2} e^{-\Omega|t_1 - t_2|} + \delta(t_1 - t_2)$$

The result is

$$\left. \frac{1}{Z} \frac{\delta^2 Z}{\delta j_1 \delta j_2} \right|_{j=0} = \frac{\Omega}{4a} e^{-\Omega|t_1 - t_2|}, \quad (31)$$
after using the relation $c^2 + \Omega^2/4 = ab$. On the other hand, from Eq. (24) we expect
\[
\langle \varphi_1 \varphi_2 \rangle = \frac{\bar{b}}{\Omega} e^{-\Omega |t_1 - t_2|},
\]
which is indeed the same because $\Omega^2 = 4\bar{a}b$.

The lesson we draw from these simple calculations is that the Green functions of the observable field $\varphi$ appearing in the Lagrangian form of the equivalent Hermitian Hamiltonian $h$ can be calculated using the functional integral arising from the non-Hermitian Hamiltonian $H$ without any reference to the $Q$ operator, which is needed to calculate expectation values in the Schrödinger formulation of $H$. Instead, the information about the transformation induced by $Q$ is encoded in the functional dependence of $Z[j]$ on $j$.

III. $Z[j]$ FOR $V(z) = -gz^4$

As already mentioned, the Hamiltonian of Eq. (7) gives rise to a well-defined eigenvalue problem only if it is defined on a suitable contour in the lower-half complex $z$ plane. The essential requirement on this contour is that it must lie asymptotically within the Stokes wedges\[1\], which in this case extend from the real axis down to an angle of $\pi/3$. Along the centre of the wedges the wave-function behaves purely exponentially, changing to purely oscillatory at the edges. The particular contour chosen in Ref. \[9\],
\[
z = -2i \sqrt{(1 + ix)},
\]
happens to go to infinity at an angle of $-\pi/4$, i.e. not down the centre of the wedge, but nonetheless it has some very special properties that enable an exact evaluation of the $Q$ operator and the equivalent Hermitian Hamiltonian. The results are that
\[
Q = -\frac{p^3}{3\alpha} + 2p,
\]
where $\alpha \equiv 16g$, and
\[
h = \frac{(p^2 - 4m^2)^2}{16\alpha} - \frac{1}{4}p + \alpha x^2
\]
Notice that this has the unusual feature that $p$ appears to the fourth power. However, by a Fourier transform $h$ reduces to a standard, asymmetric quartic anharmonic oscillator. Note also that because $Q = Q(p)$ it induces the transformations of Eq. (22), whereby
\[
P = p = -i \frac{d}{dx} = -\frac{2}{z} \frac{d}{dz}.
\]
The result for $h$ has been reproduced in the functional integral formalism\[15, 16\], but again the quantity considered was $Z[0]$, not the full vacuum generating functional $Z[j]$. In the present section we review the functional integral derivation, and include an external source in the Hermitian formulation, tracing it back to the non-Hermitian functional integral.
A.  $Z[0]$

We start with the single functional integral

$$Z = \int_C [D\psi] \exp \left\{ - \int dt \left[ \frac{1}{2} \dot{\psi}^2 + \frac{1}{2} m^2 \psi^2 - g\psi^4 \right] \right\}, \quad (37)$$

defined, as discussed above, on the curve $C$ given by

$$\psi = -2i\sqrt{(1 + i\varphi)}, \quad (38)$$

where $\varphi$ is real. Making the change of variable from $\psi$ to $\varphi$, and including the additional effective potential induced by this non-linear change$[9, 17]$, we obtain

$$Z[0] = \int [D\varphi] \det \sqrt{1 + i\varphi} \exp \left\{ - \int dt \frac{1}{2} (\dot{\varphi} + \frac{1}{4})^2 \right\}. \quad (39)$$

Now we represent the functional determinant by a functional integral over $\pi$:

$$\frac{1}{\det \sqrt{1 + i\varphi}} = \int [D\pi] \exp \left\{ - \int dt \frac{1}{2} (1 + i\varphi) \left( \pi - \frac{i\dot{\varphi} + \frac{1}{4}}{1 + i\varphi} \right)^2 \right\}, \quad (40)$$

thus producing the phase-space functional integral

$$Z[0] = \int [D\varphi][D\pi] \exp \left\{ - \int dt \left[ \frac{1}{2} (1 + i\varphi) \pi^2 - i\dot{\varphi} \pi - \frac{\pi}{4} - 2m^2 (1 + i\varphi) - \alpha(1 + i\varphi)^2 \right] \right\}. \quad (41)$$

Here the expression in square brackets in the exponent can be written as

$$[ \ ] = \alpha \left[ (\varphi - i) + \frac{i}{2\alpha} \left( \frac{1}{2} (\pi^2 - 4m^4) + \dot{\pi} \right) \right]^2 + \frac{\pi^2}{4\alpha} - \frac{\pi}{4} + \frac{(\pi^2 - 4m^2)^2}{16\alpha}. \quad (42)$$

So after rescaling $\varphi \rightarrow \varphi/\sqrt{2\alpha}$, $\pi \rightarrow \pi\sqrt{2\alpha}$ and performing the $\varphi$ integration we obtain

$$Z[0] = \int [D\pi] \exp \left\{ - \int dt \left[ \frac{\dot{\pi}^2}{2} - \sqrt{\frac{1}{8\pi}} \frac{\pi}{4} + \frac{(\pi^2 - 2m^2)^2}{4\alpha} \right] \right\}, \quad (43)$$

in which (with $p \leftrightarrow \pi\sqrt{2\alpha}$) we can recognize the Lagrangian corresponding to the Hermitian Hamiltonian $h$ of Eq. (35). Note again that in the transition to this last equation $Q$ put in a brief appearance in the form of its derivative $\dot{Q}(\pi)$, which was immediately dropped.

B.  $Z[j]$

Now let us couple $\pi$ to an external source $j$ by the addition of a term $-j\pi$ to $[ \ ]$ and work backwards to see how $j$ appears in the the original functional integral. In Eqs. (42) and (41) the term $\pi/4$ is replaced by

$$\frac{\pi}{4} \rightarrow \frac{\pi}{4}(1 + 4\tilde{j}), \quad (44)$$
where $\tilde{j} = j/\sqrt{2\alpha}$. Then the $\pi$-dependent terms in the new version of Eq. (41) can be written as

$$\frac{1}{2}(1 + i\varphi)\pi^2 - i\dot{\varphi}\pi - \frac{\pi}{4}(1 + 4\tilde{j})$$

$$= \frac{1}{2}(1 + i\varphi)\left(\pi - \frac{i\dot{\varphi} + \frac{1}{4}(1 + 4\tilde{j})}{1 + i\varphi}\right)^2 + \frac{1}{2}\frac{\dot{\varphi}^2}{1 + i\varphi} - \frac{1}{32}\frac{1}{1 + i\varphi} - \frac{4i\dot{\varphi}\tilde{j} + \tilde{j} + 2\tilde{j}^2}{4(1 + i\varphi)} \quad (45)$$

So in the \ldots of Eq. (39) we will have the additional terms

$$-\frac{4i\dot{\varphi}\tilde{j} + \tilde{j} + 2\tilde{j}^2}{4(1 + i\varphi)} \quad (46)$$

Writing these in terms of $\psi$ using $4(1 + i\varphi) = -\psi^2$, so that $i\dot{\varphi} = -\frac{1}{2}\psi\dot{\psi}$, we finally get

$$Z[j] = \int_C[D\psi]\exp\left\{ -\int dt \left[ \frac{1}{2}(\dot{\psi}^2 + m^2\psi^2) - g\psi^4 - \tilde{j}\left(2\frac{\dot{\psi}}{\psi} - \frac{1}{\psi^2}\right) + \frac{2\tilde{j}^2}{\psi^2}\right] \right\}. \quad (47)$$

A functional differentiation with respect to $j$ gives

$$\frac{1}{Z} \frac{\delta Z}{\delta j} \bigg|_{j=0} \equiv \langle \pi \rangle = \frac{1}{\sqrt{2\alpha}} \left\langle 2\frac{\dot{\psi}}{\psi} - \frac{1}{\psi^2} \right\rangle. \quad (48)$$

Unfortunately, although we are assured that $H$ and $h$ are equivalent, with identical energy spectra, we are unable to solve either exactly. Likewise we are not able to evaluate $\langle \pi \rangle$ exactly in either theory. However, in the next subsection we will check Eq. (48) up to first order in perturbation theory.

### C. Perturbation theory for $\langle \pi \rangle$

#### 1. Hermitian formulation

First we calculate $\langle \pi \rangle$ using normal Schrödinger perturbation theory for the Hermitian Hamiltonian $h$ of Eq. (35). As a change of notation, let $p = y\sqrt{2\alpha}$, so that $x = p_y/\sqrt{2\alpha}$, and $h$ now reads

$$h = \frac{1}{2}p_y^2 + \frac{1}{4}\alpha \left( y^2 - \frac{2m^2}{\alpha}\right)^2 - y\sqrt{\frac{\alpha}{8}}. \quad (49)$$

In order to perform perturbation theory in the standard way we need to shift $y$ according to $y \rightarrow y + \beta$ so as to eliminate the linear term. This yields the cubic equation

$$\alpha\beta^3 - 2m^2\beta - \sqrt{\frac{\alpha}{8}} = 0. \quad (50)$$

Although in principle it is possible to solve this equation exactly, it is extremely cumbersome to do so: instead we will make successive approximations in powers of $\alpha$. In fact, the lowest-order approximation is that $\beta$ is large, of order $1/\sqrt{\alpha}$. We thus write $\beta = \lambda/\sqrt{\alpha}$, to obtain the equation

$$\lambda^3 - 2m^2\lambda = \frac{\alpha}{2\sqrt{2}}. \quad (51)$$
whose solutions are approximately $\lambda = \pm m\sqrt{2}, -\alpha/(4m^2\sqrt{2})$. The overall minimum is at the first of these, $\lambda = m\sqrt{2}$, and the lowest-order mass term is $\frac{1}{2}M^2y^2$, where $M = 2m$. Thus the zeroth-order result for $\langle \pi \rangle$ is given by the classical position of the minimum, i.e.

$$\langle \pi \rangle = \beta = \frac{2m}{\sqrt{2\alpha}}.$$  \hfill (52)

In order to calculate the first-order result we need to do two things: first calculate the position of the classical minimum to higher precision, and then calculate quantum corrections. Calculating the minimum to next order we find that

$$\beta = \frac{2m}{\sqrt{2\alpha}} \left(1 + \frac{g}{m^3}\right),$$  \hfill (53)

where we recall that $\alpha = 16g$. The shifted Hamiltonian now reads

$$h = \frac{1}{2}p_y^2 + \frac{1}{2}M^2y^2 + m\sqrt{2\alpha} \left(1 + \frac{g}{m^3}\right) y^3 + \frac{1}{4}\alpha y^4,$$  \hfill (54)

which gives a quantum contribution to $\langle y \rangle$ from a tadpole diagram arising from the cubic coupling. Its contribution is

$$\Delta \langle y \rangle = -3m\sqrt{2\alpha} \int dt' e^{-M|t-t'|} \frac{1}{2M} = -3m\sqrt{2\alpha} \frac{1}{2M^3}.$$  \hfill (55)

Combining this with the classical shift of Eq. (53) we finally obtain

$$\langle \pi \rangle = \frac{2m}{\sqrt{2\alpha}} \left(1 - \frac{2g}{m^3}\right).$$  \hfill (56)

### 2. Functional integral formulation

Now we wish to rederive these results from the functional integral of Eq. (47) and the expression for $\langle \pi \rangle$ of Eq. (48). In the latter equation a considerable simplification can be made by noting that $\dot{\psi}/\psi$ is a perfect time derivative, and hence its expectation value vanishes, since there is no preferred origin of time. The remaining term, $1/\psi^2$, might seem singular, but we recall that the contour $C$ stays away from the origin. In fact $C$ is not suitable for performing perturbation theory, because it is right on the edge of the Stokes wedge for the harmonic oscillator. But, and this is the great advantage of the functional integral approach compared with the Schrödinger approach to the non-Hermitian problem, the contour may be analytically continued\(^3\) to a contour compatible with the asymptotic requirements of both the harmonic oscillator and negative quartic oscillator potentials and still avoiding the origin.

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\(^3\) This is not possible in the Schrödinger approach, because the calculation of expectation values also involves the complex conjugate of the wave-function.
The calculation of \( \langle \pi \rangle \) to lowest order can be achieved by the use of functional integration by parts. We start from the identity

\[
0 = \int [d\psi] \frac{\delta}{\delta \psi(t_1)} \left[ \frac{1}{\psi(t_2)} e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} \right] = \int [d\psi] \left[ -\frac{\delta(t_1 - t_2)}{\psi(t_1)^2} - \frac{1}{\psi(t_2)} \int dt' G_0^{-1}(t_1 - t') \psi(t') \right] e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} \]  

(57)

where \( G_0(t-t') = \frac{e^{-m|t-t'|}}{2m} \), this proves that

\[
\langle \pi^2 \rangle = -2m .
\]  

(60)

and hence \( \langle \pi \rangle = \frac{2m}{\sqrt{2\alpha}} \), in accordance with Eq. (52). This last equation, Eq. (60), might seem rather odd, but it must always be remembered that \( \psi \) is complex. In order to obtain the first-order correction, we must keep the interaction term in the exponent. The analogue of Eq. (57) is now

\[
0 = \int [d\psi] \frac{\delta}{\delta \psi(t_1)} \left[ \frac{1}{\psi(t_2)} e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} + g \int dt' \psi(t') \right] = \int [d\psi] \left[ -\frac{\delta(t_1 - t_2)}{\psi(t_1)^2} - \frac{1}{\psi(t_2)} \int dt' G_0^{-1}(t_1 - t') \psi(t') + 4g \psi(t_1)^3 \right] e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} \]  

(61)

Now integrate \( \int dt_1 G_0(t_2 - t_1) \) to obtain

\[
0 = \int [d\psi] \left[ -\frac{G_0(0)}{\psi(t_2)^2} - 1 + \frac{4g}{\psi(t_2)} \int dt_1 G_0(t_2 - t_1) \psi(t_1)^3 \right] e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} \]  

(62)

i.e.

\[
0 = \int [d\psi] \left[ -\frac{1}{\psi(t_2)^2} - 2m + \frac{4g}{\psi(t_2)} \int dt_1 e^{-m|t_2 - t_1|} \psi(t_1)^3 \right] e^{-\frac{i}{2} \int dt dt' \psi(t) G_0^{-1}(t-t') \psi(t')} \]  

(63)

Given the fact that in the Hermitian sector we found that \( 2m \rightarrow 2m - 4g/m^2 \), we need to show that

\[
I \equiv \int dt_1 e^{-m|t_1 - t_2|} \left\langle \frac{1}{\psi(t_2)^2} \psi(t_1)^3 \right\rangle = \frac{1}{m^2} .
\]  

(65)

This is an unfamiliar kind of expectation value, but we can represent it in terms of more familiar quantities by using the integral representation

\[
\frac{i}{\psi(t_2) - i\varepsilon} = \int_0^\infty d\lambda e^{i\lambda \psi(t_2)}
\]  

(66)
In fact, precisely because our contour does not go through the origin, we may drop the $i\varepsilon$ in the calculation of $I$, which now reads

$$I = -i \int dt_1 e^{-m|t_1-t_2|} \int_0^\infty \sum_{n=0}^\infty \frac{(i\lambda)^{2n+1}}{(2n+1)!} \langle \psi_2^{2n+1} \psi_1^3 \rangle .$$

(67)

Thinking graphically in terms of contractions, these can occur in two ways:

(a) $\langle \psi_1 \psi_1 \rangle \langle \psi_1 \psi_2 \rangle \langle \psi_2 \psi_2 \rangle^n$

This occurs with coefficient $3(2n+1)!!$ and value $e^{-m|t_1-t_2|}/(2m)^{n+2}$

(b) $\langle \psi_1 \psi_2 \rangle^4 \langle \psi_1 \psi_2 \rangle^{n-1}$

This occurs with coefficient $2n(2n+1)!!$ and value $e^{-3m|t_1-t_2|}/(2m)^{n+2}$

Thus

$$I_a = \frac{3}{4m^2} \int_0^\infty dt_1 e^{-2m|t_1-t_2|} \int_0^\infty d\lambda \lambda e^{-\lambda^2/(4m)} = \frac{3}{2m^2},$$

(68)

while

$$I_b = -\frac{1}{8m^3} \int_0^\infty dt_1 e^{-4m|t_1-t_2|} \int_0^\infty d\lambda \lambda^3 e^{-\lambda^2/(4m)} = -\frac{1}{2m^2}.$$

(69)

Thus $I \equiv I_a + I_b = 1/m^2$, in agreement with Eq. (65).

An alternative way of proceeding, which of course gives the same result, is to treat $I$ in terms of a vacuum expectation value of a time-ordered product of operators and insert a complete set of states. The states involved are only the first and third excited states, and the required matrix elements of $1/\psi_2$ can be calculated from those of $\psi_2$ by writing $\langle (\psi_2)^{2r} \rangle = \langle (\psi_2)^{2r+1} \times (1/\psi_2) \rangle$ for $r = 0, 1$ and inserting the requisite number of intermediate states.

IV. DISCUSSION

The main thrust of this paper has been to illustrate how the metric, in the guise of the $Q$ operator, makes only a fleeting appearance when the theory is formulated in terms of functional integrals rather than the operator formalism. Green functions are calculated as functional integrals in the normal way. The information about observables, namely the transforms of Hermitian operators in the equivalent Hermitian formulation, is carried in the non-Hermitian version by the functional dependence of $Z[j]$ on the external current coupled to those observables in the latter formulation. In the case of the wrong-sign quartic oscillator, there is in fact a technical advantage in the functional formulation compared with the normal Schrödinger formulation, namely that the original contour can be distorted so as to make perturbative calculations of Green function possible.

However, it must be admitted that the number of non-Hermitian theories, regarded as fundamental theories, that are interesting and tractable, is at the moment rather limited. One other theory that has been the subject of extensive investigation is the $i\epsilon x^3$ potential. In this case one can stay on the real line, but the $Q$ operator, and hence the equivalent Hermitian Hamiltonian $h$, is only known in low-order perturbation theory\[18, 19\]. It has been shown that perturbative calculations of the energy levels are considerably easier in the
context of the non-Hermitian theory, regarded as a field theory in 1 dimension, than in the equivalent Hermitian theory, where the potential in $h$ contains a number of momentum-dependent terms\cite{20}. However, calculations of Green functions are likely to be difficult in both versions\cite{21}, and indeed by the methods of the present paper, because the transformations induced by $Q$ are complicated, with $Q$ being a function of both $x$ and $p$.

An alternative way of viewing non-Hermitian theories is as effective theories, which happen to be useful in deriving tractable results, the prime example being the non-unitary Dyson mapping used by Snyman and Geyer\cite{22} to convert the Hermitian Richardson Hamiltonian into an equivalent non-Hermitian boson-fermion Hamiltonian.

Nevertheless, the $-gz^4$ potential is still of great interest as a fundamental theory because of its possible generalization to a 4-dimensional $-g\phi^4$ theory. The reason is that the indications in perturbation theory are that such a theory could be asymptotically free, as was already stressed in the original paper of Bender and Boettcher\cite{1}, and spelt out in somewhat more detail in Ref. \cite{14}. The first attempts at a generalization to higher dimensions were made in Ref. \cite{15}, but it remains a challenging problem.
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