Chebyshev-type Quadratures for Doubling Weights

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Abstract

A Chebyshev-type quadrature for a given weight function is a quadrature formula with equal weights. In this work we show that a method presented by Kane may be used to determine the order of magnitude of the minimal number of nodes required in Chebyshev-type quadratures for doubling weight functions. This extends a long line of research on Chebyshev-type quadratures starting with the 1937 work of Bernstein.

1 Introduction

Let \( w \) be a non-negative, integrable function on the interval \([-1, 1]\), such that \( \int_{-1}^{1} w(t) dt > 0 \). Such a \( w \) will be called a weight function. A quadrature (formula) of degree \( n \) is a sequence of points \(-1 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1\), called nodes, and a sequence of weights \( w_1, w_2, \ldots, w_N \) such that

\[
\int_{-1}^{1} p(t)w(t)dt = \sum_{i=1}^{N} w_ip(t_i)
\]

for every polynomial \( p \) of degree at most \( n \). If \( w_1 = w_2 = \cdots = w_N \) then the quadrature is called a Chebyshev-type quadrature, or an equal-weight quadrature.

Given \( w \) and the location of the nodes, the equalities (1) reduce to a set of \( n \) linear equations in \( N \) unknowns and it is simple to see from this that a quadrature formula exists whenever \( N \geq n \). The situation becomes more complicated if we require the quadrature formula to have non-negative weights. However, the celebrated Gaussian quadrature formula (see, e.g., [11, Chapter IV, Section 8]) satisfies this restriction and achieves the optimal degree \( n = 2N - 1 \).

Surprisingly, the situation changes dramatically if we require the quadrature to be of Chebyshev-type, i.e., to have equal weights. This was established by Bernstein [2, 3] in 1937 who proved that for

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a constant weight function the minimal possible number of nodes in a Chebyshev-type quadrature of degree $n$ is of order $n^2$.

Bernstein’s result naturally raises the question of understanding the minimal number of required nodes in Chebyshev-type quadratures for other weight functions. For a weight function $w$ and a positive integer $n$, we denote by $N_w(n)$ the minimal number of nodes in a Chebyshev-type quadrature of degree $n$ for $w$. The most well-studied case is that of the Jacobi weight function,

$$w_{\alpha,\beta}(x) := (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1. \tag{2}$$

Kuijlaars [13] (see also [12]) generalized Bernstein’s result by proving that $N_{w_{\alpha,\beta}}(n)$ is of order $n^{2+2\max\{\alpha,\beta\}}$ when $\alpha, \beta \geq 0$. Kane [10] recently extended this result to $\alpha, \beta \geq -1/2$ (the case $\alpha = \beta = -1/2$ is classical as the Gaussian quadrature itself already has equal weights). In the regime $-1 < \alpha, \beta \leq -\frac{1}{2}$, Kuijlaars [15] proved that there exists a $\lambda > 0$ such that if $\alpha = \beta = -\frac{1}{2} - \lambda$ or $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{2} - \lambda$ for some $\lambda < \lambda_0$ then $N_{w_{\alpha,\beta}}(n)$ has order $n$ (and, moreover, $N_{w_{\alpha,\beta}}(n) \leq n+2$).

Regarding results on general classes of weight functions, Kuijlaars proved a universality result in [14], showing that for every weight function $w$ of the form $h(x)(1 - x^2)^{-1/2}$, where $h$ is positive on $[-1,1]$ and analytic in a neighbourhood of $[-1,1]$, $N_w(n)$ is of order $n$. Geronimus [9], continuing ideas of Bernstein and Akhiezer, gave lower bounds for $N_w(n)$ for certain general classes of weight functions. Wagner [20] gave upper bounds on $N_w(n)$ for weight functions which are bounded below by a constant multiple of the weight $w_{\alpha,\alpha}$, $\alpha \geq 0$, and bounded above by a constant. The results in [20] are rather general in that no further assumptions are placed on the weight function (and, additionally, more general types of quadrature formulas are considered). They are, however, somewhat difficult to apply and may overestimate the order of magnitude of $N_w(n)$ (as $n$ tends to infinity). An upper bound with similar merits and disadvantages was given by Rabau and Bajnok [18] (inspired by [19]). In addition, the work [17] gives a simple upper bound on $N_w(n)$ for every weight function (and more generally, every measure). The bound of [17] is, however, far from sharp in many interesting cases, being, for instance, exponential in $n$ when $w$ is the constant weight function.

In this work we determine the order of magnitude of $N_w(n)$ (as $n$ tends to infinity) in great generality, extending most of the previous results on this question. Our results are obtained by combining topological tools developed by Kane [10] with analytic properties of doubling weight functions given by Mastroianni and Totik [16].

A weight function $w$ on $[-1,1]$ is called doubling if there is an $L > 0$ (necessarily $L \geq 2$), called the doubling constant, such that for every $\delta > 0$ and $-1 \leq a \leq 1$,

$$\int_{a-2\delta}^{a+2\delta} w(t) dt \leq L \int_{a-\delta}^{a+\delta} w(t) dt,$$

where $w(t)$ is interpreted as 0 for $t \notin [-1,1]$.

For a weight function $w$ and a positive integer $n$, in addition to the definition of $N_w(n)$ given above, we denote by $\overline{N}_w(n)$ the minimal number such that for every $N \geq \overline{N}_w(n)$ there is a Chebyshev-type
quadrature of degree \( n \) for \( w \) with exactly \( N \) nodes. We trivially have \( N_w(n) \leq \overline{N}_w(n) \) but it may happen that the two quantities have radically different orders of magnitudes; see Förster \([6]\) for an example where \( N_w(n) \) is linear in \( n \) while \( \overline{N}_w(n) \) is exponential in \( n \).

**Theorem 1.1.** Suppose \( w \) is a doubling weight function on \([-1, 1]\) with doubling constant \( L \). Let

\[
R_w(n) := \frac{\int_{-1}^{1} w(t)dt}{\inf_{-1 \leq x \leq 1} \int_{x-\Delta_n(x)}^{x+\Delta_n(x)} w(t)dt},
\]

where \( \Delta_n(x) := \frac{1}{n} \left( \sqrt{1-x^2} + \frac{1}{n} \right) \) and \( w(t) \) is again interpreted as 0 for \( t \notin [-1, 1] \). Then

\[
c(L)R_w(n) \leq N_w(n) \leq \overline{N}_w(n) \leq C(L)R_w(n), \quad n \geq 1,
\]

where \( c(L) \) and \( C(L) \) are positive constants depending only on \( L \).

Many of the weight functions that appear in analysis are doubling. These include the generalized Jacobi weight functions which are the weight functions having the form

\[
h(x)(1-x)^\alpha(1+x)^\beta|x-s_1|^{\gamma_1} \ldots |x-s_\ell|^{\gamma_\ell},
\]

with \( h \) a positive measurable function bounded away from zero and infinity, \( \ell \geq 0, -1 < s_1 < \ldots < s_\ell < 1 \) and \( \alpha, \beta, \gamma_1, \ldots, \gamma_\ell > -1 \). The Jacobi weight function \( w_{\alpha, \beta} \) given by (2) is obtained by setting \( h \equiv 1 \) and \( \ell = 0 \). Theorem 1.1 shows that weight functions \( w \) of this form satisfy that both \( N_w(n) \) and \( \overline{N}_w(n) \) are of order \( n^{\max\{1, 2(\alpha+1), 2(\beta+1), \gamma_1+1, \ldots, \gamma_\ell+1\}} \).

This extends the known results even in the Jacobi weight case, where the order of magnitude of \( N_{w_{\alpha, \beta}}(n) \) was not previously known for all values of \(-1 < \alpha, \beta \leq -\frac{1}{2}\). Moreover, it shows that the special form of the Jacobi weight function plays no role in this problem beyond the type of its zeros and singularities.

Theorem 1.1 is a direct consequence of an analogous theorem for weight functions on the unit circle. This result, which we shall now describe, seems more natural in that the corresponding estimate takes a simpler form. A weight function on the unit circle, or \( 2\pi \)-periodic weight function, is a non-negative, \( 2\pi \)-periodic measurable function \( W \) on the real line, such that \( 0 < \int_{-\pi}^{\pi} W(\theta)d\theta < \infty \). A trigonometric quadrature (formula) of degree \( n \) for such a \( W \) is a sequence of real numbers \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N \), called nodes, and a sequence of weights \( w_1, w_2, \ldots, w_N \) such that

\[
\int_{-\pi}^{\pi} p(\theta)W(\theta)d\theta = \sum_{i=1}^{N} w_ip(\theta_i)
\]

for every trigonometric polynomial \( p \) of degree at most \( n \). If \( w_1 = w_2 = \cdots = w_N \) then the quadrature is called a Chebyshev-type trigonometric quadrature. A \( 2\pi \)-periodic weight function \( W \) is
called *doubling* if there is an $L > 0$ (necessarily $L \geq 2$), called the *doubling constant*, such that for every $\delta > 0$ and real $a$,

$$\int_{a-2\delta}^{a+2\delta} W(\theta) d\theta \leq L \int_{a-\delta}^{a+\delta} W(\theta) d\theta.$$  

For a $2\pi$-periodic weight function $W$ and a positive integer $n$, we denote by $N_{W}^{\text{trig}}(n)$ the minimal number of nodes in a Chebyshev-type trigonometric quadrature of degree $n$ for $W$. We also denote by $\overline{N}_{W}^{\text{trig}}(n)$ the minimal number such that for every $N \geq \overline{N}_{W}^{\text{trig}}(n)$ there is a Chebyshev-type trigonometric quadrature of degree $n$ for $W$ with exactly $N$ nodes.

**Theorem 1.2.** Suppose $W$ is a doubling $2\pi$-periodic weight function with doubling constant $L$. Let

$$R_{W}^{\text{trig}}(n) := \frac{\int_{-\pi}^{\pi} W(\theta) d\theta}{\inf_{x \in \mathbb{R}} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} W(\theta) d\theta}.$$  

(5)

Then

$$c(L)R_{W}^{\text{trig}}(n) \leq N_{W}^{\text{trig}}(n) \leq \overline{N}_{W}^{\text{trig}}(n) \leq C(L)R_{W}^{\text{trig}}(n), \quad n \geq 1,$$

where $c(L)$ and $C(L)$ are positive constants depending only on $L$.

Theorem 1.1 is deduced from Theorem 1.2 via the standard device of defining a $2\pi$-periodic weight function $W$ from a weight function $w$ on $[-1, 1]$ by $W(\theta) := w(\cos \theta) |\sin \theta|$. Theorem 1.2 admits the following heuristic interpretation: A trigonometric polynomial of degree $n$ ‘sees’ a doubling weight $W$ at a resolution of $\frac{1}{n}$, as made precise by Mastroianni and Totik in [16, Theorem 3.1] (see also Theorem 2.2 below). Thus one may hope to obtain a trigonometric quadrature of degree $n$ for $W$ by placing $n$ nodes at $n$ roughly equi-distanced locations along the unit circle, in which case each node should, in a sense, replace $W$ in a neighborhood of size $\frac{1}{n}$ around it. In particular, the weight of each node should roughly equal the integral of $W$ in that neighborhood, so that the minimal weight would be of order $\int_{-\pi}^{\pi} W(\theta) d\theta / R_{W}^{\text{trig}}(n)$. Such a trigonometric quadrature may then be converted into a Chebyshev-type trigonometric quadrature with $N \approx R_{W}^{\text{trig}}(n)$ nodes by replacing each of the $n$ nodes by a cluster of nodes of weight $\int_{-\pi}^{\pi} W(\theta) d\theta / N$, roughly preserving the weight at each location.

The paper is organized as follows. In Section 2 we list the ingredients we will use in our proofs: a proposition of Kane and some facts about doubling weight functions. Theorem 1.1 and Theorem 1.2 are proved in Section 3. In Section 4 we remark upon extensions to non-doubling weight functions, providing a simple upper bound on the minimal number of nodes required in a Chebyshev-type quadrature and discussing some explicit examples. In Section 5 we discuss some open problems for future research.

Throughout the paper we adopt the following policy regarding constants. The constants $C(L)$ and $c(L)$ will always denote positive constants whose value depends only on $L$. The values of these constants will be allowed to change from line to line, even within the same calculation, with the value of $C(L)$ increasing and the value of $c(L)$ decreasing. We similarly use $C$ and $c$ to denote positive absolute constants whose value may change from line to line.
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2 Preliminaries

In this section we gather several existing results which we rely upon in our proofs.

For a non-negative integer \( n \), let \( T_n \) denote the set of real trigonometric polynomials of degree at most \( n \), i.e., the linear space spanned by the \( 2n+1 \) functions \( \{1, \sin \theta, \cos \theta, \sin 2\theta, \cos 2\theta, \ldots, \sin n\theta, \cos n\theta\} \). Let \( T_n^+ \) be the subset of \( T_n \) of non-negative trigonometric polynomials which are not identically zero.

Kane’s bound on the size of Chebyshev-type quadratures

The following proposition is a restriction of Proposition 20 in [10], adapted to our setup. For completeness, a proof is provided in Appendix A.

**Proposition 2.1.** ([10, Proposition 20]) Let \( W \) be a \( 2\pi \)-periodic weight function and let \( n \) be a positive integer. Then for every integer \( N \) satisfying that

\[
N > \frac{\int_{-\pi}^{\pi} W(\theta) d\theta}{2} \sup_{p \in T_n^+} \frac{\int_{-\pi}^{\pi} |p'(\theta)| d\theta}{\int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta}
\]

there is a Chebyshev-type trigonometric quadrature of degree \( n \) for \( W \) with \( N \) nodes. Equivalently,

\[
N_{\text{trig}}^{W}(n) \leq \frac{\int_{-\pi}^{\pi} W(\theta) d\theta}{2} \sup_{p \in T_n^+} \frac{\int_{-\pi}^{\pi} |p'(\theta)| d\theta}{\int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta} + 1.
\]

We use this bound together with the classical \( L^1 \) version of Bernstein’s inequality (see, e.g., [21, Vol. II, Theorem 3.16, p.11]),

\[
\int_{-\pi}^{\pi} |p'(\theta)| d\theta \leq n \int_{-\pi}^{\pi} |p(\theta)| d\theta, \quad p \in T_n.
\]

(6)

Properties of doubling weight functions

Given a \( 2\pi \)-periodic weight function \( W \) and a positive integer \( n \), define

\[
W_n(x) := n \int_{x - \frac{1}{n}}^{x + \frac{1}{n}} W(\theta) d\theta, \quad x \in \mathbb{R}.
\]

(7)

Mastroianni and Totik [16] showed that the result of integrating powers of trigonometric polynomials of degree \( n \) against a doubling weight \( W \) remains unchanged, up to constants, if \( W \) is replaced by \( W_n \). In our work we use only a one-sided bound of this form and only in the special case when the power is 1.
Theorem 2.2. [11] Special case of Theorem 3.1] Let $W$ be a doubling $2\pi$-periodic weight function with doubling constant $L$. Then
\[
\int_{-\pi}^{\pi} |p(\theta)| W_n(\theta) d\theta \leq C(L) \int_{-\pi}^{\pi} |p(\theta)| W(\theta) d\theta, \quad p \in \mathcal{T}_n.
\]

The following simple lemma allows to transfer results from the trigonometric to the algebraic setting.

Lemma 2.3. Let $w$ be a weight function on $[-1,1]$. Define a $2\pi$-periodic weight function $W$ by
\[
W(\theta) := w(\cos \theta) |\sin \theta|, \quad \theta \in \mathbb{R}.
\]
If $w$ is doubling with a doubling constant $L$ then $W$ is also doubling with a doubling constant $C(L)$.

Proof. For every real $u$ and positive $\delta$, let
\[
a(u, \delta) := \min\{\cos(u + t) \mid -\delta \leq t \leq \delta\},
b(u, \delta) := \max\{\cos(u + t) \mid -\delta \leq t \leq \delta\},
\]
and note that, for every real $t$,
\[
\cos(u + 4t) - \cos u = 2 \cos t (\cos(u + 3t) - \cos(u + t)).
\]
Therefore $\cos u - a(u, 4\delta) \leq 2(b(u, 3\delta) - a(u, 3\delta))$ and $b(u, 4\delta) = \cos u \leq 2(b(u, 3\delta) - a(u, 3\delta))$, whence $b(u, 4\delta) - a(u, 4\delta) \leq 4(b(u, 3\delta) - a(u, 3\delta))$. Applying this bound iteratively three times yields that
\[
b(u, 2\delta) - a(u, 2\delta) \leq 4^3 \left( b(u, (3/4)^3 2\delta) - a(u, (3/4)^3 2\delta) \right) \leq 4^3 (b(u, \delta) - a(u, \delta)),
\]
from which it follows that
\[
[a(u, 2\delta), b(u, 2\delta)] \subseteq \left[ \frac{a(u, \delta) + b(u, \delta)}{2}, 2^7 \frac{b(u, \delta) - a(u, \delta)}{2}, \frac{a(u, \delta) + b(u, \delta)}{2} + 2^7 \frac{b(u, \delta) - a(u, \delta)}{2} \right].
\]
For $\delta < \pi$ we get, using the doubling property of $w$ (and continuing to interpret $w$ as 0 outside $[-1,1]$),
\[
\int_{u-2\delta}^{u+2\delta} W(\theta) d\theta = \int_{u-2\delta}^{u+2\delta} w(\cos \theta) |\sin \theta| d\theta \leq 4 \int_{a(u, \delta)}^{b(u, \delta)} w(t) dt \leq 4 \int_{a(u, \delta) + b(u, \delta)}^{b(u, \delta) + b(u, \delta)} w(t) dt \leq 4L^7 \int_{a(u, \delta) + b(u, \delta)}^{b(u, \delta) + b(u, \delta)} w(t) dt = 4L^7 \int_{a(u, \delta)}^{b(u, \delta)} w(t) dt \leq 4L^7 \int_{u-\delta}^{u+\delta} w(\cos \theta) |\sin \theta| d\theta = 4L^7 \int_{u-\delta}^{u+\delta} W(\theta) d\theta.
\]
Furthermore, it is straightforward that for $\delta \geq \pi$ we have
\[
\int_{u-\delta}^{u+\delta} W(\theta) d\theta \leq 3 \int_{u-\delta}^{u+\delta} W(\theta) d\theta,
\]
completing the proof of the lemma. \qed
Lastly, we require the following immediate property of doubling weight functions.

**Lemma 2.4.** If $W$ is a doubling $2\pi$-periodic weight function with doubling constant $L$, then for every real $x, y$ and $\delta > 0$,

$$\int_{y - \delta}^{y + \delta} W(\theta) d\theta \leq L \left(1 + \frac{|x - y|}{\delta}\right)^{\log_2 L} \int_{x - \delta}^{x + \delta} W(\theta) d\theta.$$

**Proof.** Let $r := \left\lceil \log_2 \left(1 + \frac{|x - y|}{\delta}\right) \right\rceil$. Then $[y - \delta, y + \delta] \subseteq [x - 2^r \delta, x + 2^r \delta]$ and hence, by the doubling property,

$$\int_{y - \delta}^{y + \delta} W(\theta) d\theta \leq \int_{x - 2^r \delta}^{x + 2^r \delta} W(\theta) d\theta \leq L^r \int_{x - \delta}^{x + \delta} W(\theta) d\theta \leq L \left(1 + \frac{|x - y|}{\delta}\right)^{\log_2 L} \int_{x - \delta}^{x + \delta} W(\theta) d\theta.$$

3 Proofs

For every integer $m \geq 0$, Let $F_m$ be the following variant of the Fejér kernel,

$$F_m(\theta) := \left(\frac{\sin \left(\frac{2m+1}{2} \theta \right)}{\sin \frac{\theta}{2}}\right)^2,$$  \hspace{1cm} (9)

which is a non-negative trigonometric polynomial of degree $2m$. We frequently use that

$$F_m(\theta) \leq \min \left\{1, \left(\frac{\pi}{(2m+1)\theta}\right)^2\right\} \text{ for every } -\pi \leq \theta \leq \pi$$ \hspace{1cm} (10)

$$F_m \text{ is even, and decreasing on } \left[0, \frac{2\pi}{2m+1}\right],$$ \hspace{1cm} (11)

$$\left(\frac{2}{\pi}\right)^2 \leq F_m \left(\frac{\pi}{2m+1}\right) \leq F_1 \left(\frac{\pi}{3}\right) = \left(\frac{2}{3}\right)^2 \text{ for every } m \geq 1,$$ \hspace{1cm} (12)

where the inequalities in (12) follow by noting that $F_m \left(\frac{\pi}{2m+1}\right) = \left(\frac{2m+1}{\sin \frac{\pi}{2m+1}}\right)^{-2}$ decreases with $m$. The following lemma shows that when integrating powers of $F_m$ against doubling weight functions the main contribution comes from a small neighborhood of the origin.

**Lemma 3.1.** Let $W$ be a doubling $2\pi$-periodic weight function with doubling constant $L$. For every integers $k, \ell, m$ such that $0 < |k| \leq m$, $\ell \geq 5 \log_2 L$,

$$\int_{\frac{\pi}{2m+1}}^{\frac{2k+1}{2m+1}\pi} F_m(\theta)^\ell W(\theta) d\theta \leq \left(\frac{2}{3|k|}\right)^\ell \int_{\frac{\pi}{2m+1}}^{\frac{\pi}{2m+1}} W(\theta) d\theta.$$

**Proof.** With no loss of generality assume $k$ is positive. We have

$$F_m(\theta) \leq \left(\frac{2}{3k}\right)^2, \hspace{1cm} \frac{2k - 1}{2m+1} \pi \leq \theta \leq \frac{2k + 1}{2m+1} \pi.$$
When \( \theta \geq \frac{2\pi}{2m+1} \) this follows from (10) whereas when \( \frac{\pi}{2m+1} \leq \theta \leq \frac{2\pi}{2m+1} \) this follows from (11) and (12). Therefore, by Lemma 2.4

\[
\int_{\frac{2k-1}{2m+1}\pi}^{\frac{2k+1}{2m+1}\pi} F_m(\theta)^{\ell} W(\theta) d\theta \leq \left(\frac{2}{3k}\right)^{2\ell} \int_{\frac{2k-1}{2m+1}\pi}^{\frac{2k+1}{2m+1}\pi} W(\theta) d\theta \leq \left(\frac{2}{3k}\right)^{2\ell} L (2k + 1)^{\log_2 L} \int_{-\frac{\pi}{2m+1}}^{\frac{\pi}{2m+1}} W(\theta) d\theta,
\]

and the lemma follows since for every \( k \geq 1 \) and \( \ell \geq 5 \log_2 L \),

\[
\left(\frac{2}{3k}\right)^{\ell} L (2k + 1)^{\log_2 L} \leq \left(\frac{2}{3k}\right)^{5 \log_2 L} L (3k)^{\log_2 L} = \left(\frac{8}{9} \frac{1}{k^4}\right)^{\log_2 L} \leq 1.
\]

We proceed to prove our main theorems.

**Proof of Theorem 1.2** Denote \( I := \int_{-\pi}^{\pi} W(\theta) d\theta \). Recall the definition of \( W_n \) from (7). Note that \( R_{W}^{\text{trig}}(n) W_n(\theta) \geq I \cdot n \) for every real \( \theta \). For every \( p \in T^+ \), by (6) and Theorem 2.2,

\[
I \int_{-\pi}^{\pi} |p'(\theta)| \, d\theta \leq I \cdot n \int_{-\pi}^{\pi} p(\theta) d\theta \leq R_{W}^{\text{trig}}(n) \int_{-\pi}^{\pi} p(\theta) W_n(\theta) d\theta \leq C(L) R_{W}^{\text{trig}}(n) \int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta
\]

and the upper bound of the theorem follows by Proposition 2.1.

To get the lower bound we need to show that if \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N \) are the nodes of a Chebyshev-type trigonometric quadrature of degree \( n \) for \( W \), then for every real \( x \), \( \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} W(x) d\theta \geq c(L) I / N \).

With no loss of generality we may assume that all nodes are in \([-\pi, \pi] \) and that \( x = 0 \).

Let \( \ell \) be a positive integer and let \( m := \lfloor n / (2(\ell + 1)) \rfloor \). We first show that if \( \ell \geq 5 \log_2 (\pi^2 L) \) then the interval \( \left[ -\frac{\pi}{2m+1}, \frac{\pi}{2m+1} \right] \) contains at least one node of the Chebyshev-type trigonometric quadrature. This is obvious if \( m = 0 \) so we assume \( m \geq 1 \) in the following calculation. Let \( p \) be the polynomial \( p(\theta) := F_m(\theta)^{\ell} \left( F_m(\theta) - F_m(\frac{\pi}{2m+1}) \right) \). We note that

\[
p(\theta) \leq 0, \quad \theta \in [-\pi, \pi] \setminus \left[ -\frac{\pi}{2m+1}, \frac{\pi}{2m+1} \right]
\]

as in this range,

\[
F_m(\theta) = \left( \frac{\sin \frac{(2m+1)\theta}{2}}{(2m+1) \sin \frac{\theta}{2}} \right)^2 \leq \left( \frac{1}{(2m+1) \sin \frac{\pi}{2(2m+1)}} \right)^2 = F_m \left( \frac{\pi}{2m+1} \right).
\]

Therefore, if there is no node of the Chebyshev-type trigonometric quadrature in the interval \( \left[ -\frac{\pi}{2m+1}, \frac{\pi}{2m+1} \right] \), then, using that \( \deg p = 2m(\ell + 1) \leq n \),

\[
\int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta = \frac{I}{N} \sum_{j=1}^{N} p(\theta_j) \leq 0.
\]

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However, using (11) and the doubling property,
\[
\int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta \geq \int_{-2\pi(2m+1)}^{2\pi(2m+1)} p(\theta) W(\theta) d\theta \geq
\]
\[
\geq \left( F_m \left( \frac{\pi}{2(2m+1)} \right) - F_m \left( \frac{\pi}{2(m+1)} \right) \right) F_m \left( \frac{\pi}{2(2m+1)} \right)^\ell \int_{-2\pi}^{2\pi(2m+1)} W(\theta) d\theta \geq
\]
\[
\geq \frac{1}{3} \left( \frac{8}{\pi^2} \right)^\ell \frac{1}{L} \int_{-2\pi}^{2\pi(2m+1)} W(\theta) d\theta.
\]
Moreover, by (13), (10) and using Lemma 3.1, if \( \ell \geq 5 \log_2 L \) then for every integer \( k \) such that 0 < \(|k| \leq m\),
\[
- \int_{-2\pi}^{2\pi} p(\theta) W(\theta) d\theta \leq F_m \left( \frac{\pi}{2m+1} \right) \int_{-2\pi}^{2\pi} F_m(\theta)^k W(\theta) d\theta \leq \left( \frac{2}{3} \right)^\ell \int_{-2\pi}^{2\pi} W(\theta) d\theta,
\]
whence, for \( \ell \geq 5 \log_2 (\pi^2 L) \) there is at least one node of the Chebyshev-type trigonometric quadrature in the interval \([-\pi, \pi] \). Let \( \theta_{j_0} \) be one such node.

Therefore, if \( \ell \geq 5 \log_2 (\pi^2 L) \) then there is at least one node of the Chebyshev-type trigonometric quadrature in the interval \([-\pi, \pi] \). Let \( \theta_{j_0} \) be one such node.

Now, as \( \deg F_m^\ell = 2m\ell \leq n \), it follows, using (11), (12) and Lemma 3.1 that for \( \ell \geq 5 \log_2 (\pi^2 L) \),
\[
\frac{I (2)}{N} \left( \frac{2}{\pi} \right)^{2\ell} \leq \frac{I}{N} F_m \left( \frac{\pi}{2m+1} \right)^\ell \leq \frac{I}{N} \sum_{j=1}^N F_m(\theta_j)^\ell = \int_{-\pi}^{\pi} F_m(\theta)^\ell W(\theta) d\theta =
\]
\[
= \sum_{k=-m}^m \int_{2k+1}^{\pi} F_m(\theta)^\ell W(\theta) d\theta \leq \left( 1 + 2 \left( \frac{2}{3} \right)^\ell \sum_{k=1}^m \frac{1}{k^2} \right) \int_{-\pi}^{\pi} W(\theta) d\theta \leq 3 \int_{-\pi}^{\pi} W(\theta) d\theta.
\]
Taking \( \ell \geq \lceil 5 \log_2 (\pi^2 L) \rceil \), the last inequality and the doubling property imply that
\[
\int_{-\pi}^{\pi} W(\theta) d\theta \geq c(L) I/N, \text{ as required.}
\]

\textbf{Proof of Theorem 7.1} Let \( W \) be the 2\( \pi \)-periodic weight function defined by \( W(\theta) := w(\cos \theta) |\sin \theta| \).

It is easy to verify that if \( (\theta_j)_{j=1}^N \) are the nodes of a Chebyshev-type trigonometric quadrature of degree \( n \) for \( W \), then \( (\cos \theta_j)_{j=1}^N \) are the nodes of a Chebyshev-type quadrature of degree \( n \) for \( w \), and conversely, if \( (t_j)_{j=1}^N \) are the nodes of a Chebyshev-type quadrature of degree \( n \) for \( W \), then \( (\arccos t_j)_{j=1}^N \cup (- \arccos t_j)_{j=1}^N \) are the nodes of a Chebyshev-type trigonometric quadrature of degree \( n \) for \( W \) (where the union is interpreted as a multiset union). Therefore
\[
\frac{1}{2} N_{W}^{\text{trig}}(n) \leq N_{w}(n) \leq N_{w}(n) \leq N_{W}^{\text{trig}}(n), \quad (14)
\]

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By Lemma 2.3, \( W \) is doubling with doubling constant \( C(L) \). Therefore by Theorem 1.2,
\[
c(L) R_W^{\text{trig}} (n) \leq N_W^{\text{trig}} (n) \leq \overline{N}_W^{\text{trig}} (n) \leq C(L) R_W^{\text{trig}} (n).
\] (15)

For every \( u \) let
\[
a(u) := \min \{ \cos(u + \delta) \mid -1/n \leq \delta \leq 1/n \}, \quad b(u) := \max \{ \cos(u + \delta) \mid -1/n \leq \delta \leq 1/n \},
\]
then
\[
\int_{a(u)}^{b(u)} w(t) dt \leq \int_{u - \frac{1}{n}}^{u + \frac{1}{n}} w(\cos \theta) | \sin \theta | d\theta \leq 2 \int_{a(u)}^{b(u)} w(t) dt.
\] (16)

Recall the definition of \( \Delta_n \) from the statement of the theorem. For every real \( u \) and \(-1/n \leq \delta \leq 1/n\),
\[
| \cos u - \cos(u + \delta) | = | (1 - \cos \delta) \cos u + \sin \delta \sin u | \leq (1 - \cos \delta) + | \sin \delta | \cdot | \sin u | \leq \frac{\delta^2}{2} + | \delta | \cdot | \sin u | \leq \frac{1}{n} \left( \frac{1}{n} + | \sin u | \right) = \Delta_n (\cos u).
\]

Therefore,
\[
[a(u), b(u)] \subseteq [\cos u - \Delta_n (\cos u), \cos u + \Delta_n (\cos u)],
\]
whence, using (16),
\[
\int_{u - \frac{1}{n}}^{u + \frac{1}{n}} W(\theta) d\theta = \int_{u - \frac{1}{n}}^{u + \frac{1}{n}} w(\cos \theta) | \sin \theta | d\theta \leq 2 \int_{a(u)}^{b(u)} w(t) dt \leq 2 \int_{\cos u - \Delta_n (\cos u)}^{\cos u + \Delta_n (\cos u)} w(t) dt,
\]
where here and below we again interpret \( w \) as 0 outside \([-1, 1]\). Therefore, since \( \int_{-\pi}^{\pi} W(\theta) d\theta = \int_{-\pi}^{\pi} w(\cos \theta) | \sin \theta | d\theta = 2 \int_{-1}^{1} w(t) dt \),
\[
R_W^{\text{trig}} (n) \geq R_w(n).
\] (17)

For every real \( u \),
\[
\Delta_n (\cos u) = \frac{1}{n} \left( \frac{1}{n} + | \sin u | \right) \leq \frac{1}{n} \left( \frac{1}{n} | \cos u | + \left( 1 + \frac{1}{n} \right) | \sin u | \right) \leq
\]
\[
\leq \frac{\pi^2}{2} \left( 1 - \cos \left( \frac{1}{n} \right) \right) | \cos u | + \pi \sin \left( \frac{1}{n} \right) | \sin u | \leq
\]
\[
\leq \frac{15}{2} \left[ \left( 1 - \cos \left( \frac{1}{n} \right) \right) | \cos u | + \sin \left( \frac{1}{n} \right) | \sin u | \right] =
\]
\[
= \frac{15}{2} \max \left\{ \left| \cos u - \cos \left( u - \frac{1}{n} \right) \right| , \left| \cos u - \cos \left( u + \frac{1}{n} \right) \right| \right\} \leq \frac{15}{2} (b(u) - a(u)).
\]

Therefore,
\[
[\cos u - \Delta_n (\cos u), \cos u + \Delta_n (\cos u)] \subseteq
\]
\[
\subseteq \left[ \frac{a(u) + b(u)}{2} - 16 \frac{b(u) - a(u)}{2} , \frac{a(u) + b(u)}{2} + 16 \frac{b(u) - a(u)}{2} \right],
\]

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whence, using the doubling property of \( w \) and (16),
\[
\int_{\cos u - \Delta_n (\cos u)}^{\cos u + \Delta_n (\cos u)} w(t) dt \leq \int_{a(u) + b(u)}^{a(u) + b(u) + 16 \Delta_n (\cos u)} \frac{w(t) dt}{\Delta_n (\cos u)} \leq L^4 \int_{a(u)}^{b(u)} w(t) dt \leq L^4 \int_{u - \frac{1}{n}}^{u + \frac{1}{n}} w(\cos \theta) |\sin \theta| d\theta = L^4 \int_{u - \frac{1}{n}}^{u + \frac{1}{n}} W(\theta) d\theta.
\]
Thus, using again that \( \int_{-\pi}^{\pi} W(\theta) d\theta = 2 \int_{-1}^{1} w(t) dt \),
\[R^\text{trig}_W(n) \leq 2L^4 R_w(n).\]  

The theorem follows by combining (14), (15), (17) and (18).

4 Additional results for non-doubling weight functions

In this section we briefly remark on Chebyshev-type quadratures for non-doubling weight functions. We shall write \(|A|\) to denote the Lebesgue measure of a Lebesgue measurable set \( A \) in the real line.

4.1 A general upper bound

Here we provide a simple upper bound on the minimal number of nodes required in a Chebyshev-type quadrature which is applicable for any weight function. The advantage of this bound is its generality and the fact that it is sharp in some cases, e.g., for generalized Jacobi weight functions on \([-1, 1]\) (see (3)) as it has the same order of magnitude as the bound given in Theorem 1.1. Its disadvantage is that it is not sharp in general, e.g., for weight functions vanishing on an interval (see also [6, 17]) or for the example given in Section 4.2 below.

**Theorem 4.1.** Let \( n \) be a positive integer and let \( W \) be a \( 2\pi \)-periodic weight function. Then, for every \( 0 < \eta < 1 \), and every measurable \( D \subset [-\pi, \pi] \) such that \(|D| \leq (1 - \eta)^2 \frac{2\pi}{2n + 1}\),
\[N^\text{trig}_W(n) \leq \frac{1}{\eta} \cdot \frac{\int_{-\pi}^{\pi} W(\theta) d\theta}{\operatorname{ess inf}_{\theta \in [-\pi, \pi] \setminus D} W(\theta)} n.
\]

We also provide an analogue of the theorem for weight functions on \([-1, 1]\).

**Corollary 4.2.** Let \( n \) be a positive integer and let \( w \) be a weight function on \([-1, 1]\). Then, for every \( 0 < \eta < 1 \), and every measurable \( D \subset [-1, 1] \) such that \( \int_D \frac{dt}{\sqrt{1-t^2}} \leq (1 - \eta)^2 \frac{\pi}{2n + 1}\),
\[N_w(n) \leq \frac{2}{\eta} \cdot \frac{\int_{-1}^{1} w(t) dt}{\operatorname{ess inf}_{t \in [-1, 1] \setminus D} \sqrt{1 - t^2} w(t)} n.
\]

Theorem 4.1 follows by combining Proposition 2.1, (6) and the following lemma. The lemma and its proof are inspired by [10, Lemma 23].
Lemma 4.3. Let \( n \) be a positive integer and let \( W \) be a \( 2\pi \)-periodic weight function. Suppose \( D \subset [-\pi, \pi] \) is a measurable set such that \( |D| \leq (1 - \eta)^2 \frac{2\pi}{2n+1} \) for some \( 0 < \eta < 1 \). Then

\[
\int_{-\pi}^{\pi} p(\theta)W(\theta)d\theta \geq \eta \ inf_{\theta \in [-\pi, \pi] \setminus D} W(\theta) \cdot \int_{-\pi}^{\pi} p(\theta)d\theta, \quad p \in T_n^+.
\]

Proof. It is easy to verify that

\[
\int_{-\pi}^{\pi} q(\theta)d\theta = \frac{2\pi}{2n+1} \sum_{j=0}^{2n} q \left( \frac{2j}{2n+1} \pi \right), \quad q \in T_{2n}.
\]

Fix \( p \in T_{n}^+ \). As \( \deg p \leq n \) and \( p \) is non-negative,

\[
\int_{-\pi}^{\pi} p(\theta)^2d\theta = \frac{2\pi}{2n+1} \sum_{j=0}^{2n} p \left( \frac{2j}{2n+1} \pi \right)^2 \leq \frac{2\pi}{2n+1} \left( \sum_{j=0}^{2n} p \left( \frac{2j}{2n+1} \pi \right) \right)^2 = \frac{2n+1}{2\pi} \left( \int_{-\pi}^{\pi} p(\theta)d\theta \right)^2.
\]

Therefore, by the Cauchy-Schwartz inequality,

\[
\int_{D} p(\theta)d\theta \leq \sqrt{\int_{D} d\theta} \cdot \sqrt{\int_{D} p(\theta)^2d\theta} \leq \sqrt{|D|} \cdot \sqrt{\int_{-\pi}^{\pi} p(\theta)^2d\theta} \leq \sqrt{(1 - \eta)^2 \frac{2\pi}{2n+1}} \cdot \sqrt{\frac{2n+1}{2\pi}} \int_{-\pi}^{\pi} p(\theta)d\theta = (1 - \eta) \int_{-\pi}^{\pi} p(\theta)d\theta.
\]

Hence

\[
\int_{-\pi}^{\pi} p(\theta)W(\theta)d\theta \geq \int_{[-\pi, \pi] \setminus D} p(\theta)W(\theta)d\theta \geq \inf_{\theta \in [-\pi, \pi] \setminus D} W(\theta) \cdot \int_{[-\pi, \pi] \setminus D} p(\theta)d\theta \geq \eta \ inf_{\theta \in [-\pi, \pi] \setminus D} W(\theta) \cdot \int_{-\pi}^{\pi} p(\theta)d\theta. \quad \Box
\]

Proof of Corollary 4.2. Let \( W \) be the \( 2\pi \)-periodic weight function defined by \( W(\theta) := w(\cos \theta) |\sin \theta| \) and set \( \tilde{D} \) to be the subset of \( [-\pi, \pi] \) satisfying \( -\tilde{D} = \tilde{D} \) and \( \cos(\tilde{D}) = D \). The corollary follows from Theorem 4.1 by observing that \( \tilde{N}_w(n) \leq \tilde{N}_{w^{\text{trig}}}(n) \) by (14), that \( |\tilde{D}| = 2 \int_{D} \frac{dt}{\sqrt{1-t^2}} \), that \( \inf_{\theta \in [-\pi, \pi] \setminus D} W(\theta) = \inf_{t \in [-1, 1]} \sqrt{1-t^2} W(t) \) and that \( \int_{-\pi}^{\pi} W(\theta)d\theta = 2 \int_{-1}^{1} w(t)dt \). \quad \Box

4.2 Exponentially vanishing weights

In this section we study a family of non-doubling weight functions which vanish as a stretched exponential at a point. For \( \alpha > 0 \), let \( W_\alpha \) be the \( 2\pi \)-periodic weight function defined for \( -\pi \leq \theta < \pi \) by

\[
W_\alpha(\theta) := \begin{cases} 
  e^{-|\theta|^{-\alpha}} & \theta \neq 0 \\
  0 & \theta = 0.
\end{cases}
\]
Theorem 1.2 does not apply to this weight function as it is not doubling. Theorem 4.4 applies, and yields that $N^{\text{trig}}_{W_\alpha}(n) \leq \exp(Cn^\alpha)$. As it turns out, however, $N^{\text{trig}}_{W_\alpha}(n)$ is considerably smaller.

Theorem 4.4. Let $W_\alpha$ be the $2\pi$-periodic weight function defined for $-\pi \leq \theta < \pi$ by (19). Then

$$\exp\left(c(\alpha)n^{\alpha/(\log n)^{1/\alpha}}\right) \leq N^{\text{trig}}_{W_\alpha}(n) \leq \overline{N}^{\text{trig}}_{W_\alpha}(n) \leq \exp\left(C(\alpha)n^{\alpha/(1+\alpha)}\right), \quad n \geq 1,$$

where $C(\alpha), c(\alpha)$ are positive constants depending only on $\alpha$.

We mention that the technique used for the proof of the theorem uses similar ideas to the proof of Theorem 1.2 and may be applicable for certain other weight functions. For instance, for the $2\pi$-periodic weight functions $\overline{W}_\alpha$, $\alpha > 0$, defined for $0 < |\theta| \leq \pi$ by $\overline{W}_\alpha(\theta) := \exp(-\exp(|\theta|^{-\alpha}))$ we may show that

$$\exp\left(c(\alpha)n^{\alpha/(\log n)^{1/\alpha}}\right) \leq N^{\text{trig}}_{\overline{W}_\alpha}(n) \leq \overline{N}^{\text{trig}}_{\overline{W}_\alpha}(n) \leq \exp\left(C(\alpha)n^{\alpha/(1+\alpha)}\right), \quad n \geq 1,$$

where $C(\alpha), c(\alpha)$ are positive constants depending only on $\alpha$. It is also worth mentioning in this regard that it is known [17, Theorem 1.4] that for weight functions $w$ on $[-1, 1]$ which are bounded we have $N_w(n) \leq \exp(C_wn)$ for a constant $C(w)$ depending only on $w$.

As a second remark we point out that the fact that the bounds provided by Theorem 4.4 for $N^{\text{trig}}_{W_\alpha}(n)$ and $\overline{N}^{\text{trig}}_{W_\alpha}(n)$ are not as close to each other as in Theorem 1.2 may be essential. Indeed, as mentioned before, there is an example [6] of a weight function $w$ vanishing on an interval for which $N_w(n)$ is linear in $n$ while $N_w(n)$ is exponential in $n$.

The proof of Theorem 4.4 is obtained by combining Proposition 2.1 with the following Remez-type inequality for trigonometric polynomials.

Theorem 4.5. [4] Theorem 5.1.2] Let $p \in T_n$ and denote $M := \max_{\theta \in \mathbb{R}}|p(\theta)|$. Then

$$\{|-\pi \leq \theta < \pi : |p(\theta)| \geq Me^{-4ns}\}| \geq s, \quad 0 < s \leq \pi/2.$$

Proof of Theorem 4.4. We first prove the upper bound. We may assume $3n^{-\frac{1}{\alpha+1}} \leq \pi/2$. Fix $p \in T_n^+$ and denote $M := \max_{\theta \in \mathbb{R}}|p(\theta)|$. Applying Theorem 4.5 with $s = 3n^{-\frac{1}{\alpha+1}}$ yields that

$$\left|\{ -\pi \leq \theta < \pi : |p(\theta)| \geq M \exp\left(-12n^{\alpha/(\alpha+1)}\right)\}\right| \geq 3n^{-\frac{1}{\alpha+1}}.$$

Let

$$A := \left\{ -\pi \leq \theta < \pi : |p(\theta)| \geq M \exp\left(-12n^{\alpha/(\alpha+1)}\right)\right\} \setminus \left(-n^{-\frac{1}{\alpha+1}}, n^{-\frac{1}{\alpha+1}}\right).$$

It follows that $|A| \geq n^{-\frac{1}{\alpha+1}}$ and $W_\alpha(\theta) \geq \exp\left(-n^{\alpha/(\alpha+1)}\right)$ for every $\theta \in A$, whence

$$\int_{-\pi}^{\pi} p(\theta)W_\alpha(\theta)d\theta \geq \int_{A} p(\theta)W_\alpha(\theta)d\theta \geq n^{-\frac{1}{\alpha+1}}M \exp\left(-12n^{\alpha/(\alpha+1)}\right) \exp\left(-n^{\alpha/(\alpha+1)}\right) \geq$$

$$\geq n^{-\frac{1}{\alpha+1}} \exp\left(-13n^{\alpha/(\alpha+1)}\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\theta)d\theta.$$
By Proposition 2.1 and (6) we get that
\[ N_{W_\alpha}^{\text{trig}}(n) \leq C n^{1 + \frac{1}{\alpha + 1}} \exp \left( 13n^\frac{\alpha}{\alpha + 1} \right) \]
and the upper bound follows.

To get the lower bound we need to show that if \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N \) are the nodes of a Chebyshev-type trigonometric quadrature of degree \( n \) for \( W_\alpha \), then \( \log N \geq c(\alpha) n^{\frac{\alpha}{\alpha + 1}} \). Fix such a quadrature. With no loss of generality we may assume that all nodes are in \([-\pi, \pi]\). Let \( m := \left[ \frac{1}{12} 6^\frac{\alpha}{\alpha + 1} n^{\frac{1}{\alpha + 1}} \right], \ell := \left[ \frac{1}{6^{\frac{1}{\alpha + 1}} n^{\frac{1}{\alpha + 1}}} - 1 \right], r := \min\{6^{\frac{1}{\alpha + 1}}, \alpha / 6^{\frac{1}{\alpha + 1}} n^{\frac{1}{\alpha + 1}}\} \). We may assume that \( n \), and hence also \( m, \ell \) and \( r \) are large enough, as functions of \( \alpha \), for the following calculation.

As in the proof of Theorem 1.2 we first show that the interval \([ - \frac{\pi}{2m + 1}, \frac{\pi}{2m + 1} ]\) contains at least one node of the Chebyshev-type trigonometric quadrature. Let \( p \) be the polynomial \( p(\theta) := F_m(\theta)^\ell \left( F_m(\theta) - F_m(\frac{\pi}{2m + 1}) \right) \). Note that \( \deg p = 2m(\ell + 1) \leq n \). If there is no node of the Chebyshev-type trigonometric quadrature in the interval \([ - \frac{\pi}{2m + 1}, \frac{\pi}{2m + 1} ]\) then, as in the proof of Theorem 1.2, \( \int_{-\pi}^{\pi} p(\theta)W_\alpha(\theta)d\theta \leq 0 \). However, using (11),
\[
\int_{-\pi}^{\frac{\pi}{2m + 1}} p(\theta)W_\alpha(\theta)d\theta \geq \int_{-\frac{\pi}{2m + 1}}^{\frac{\pi}{2m + 1}} p(\theta)W_\alpha(\theta)d\theta \geq \frac{\pi}{3(2m + 1)} \left( F_m\left( \frac{2\pi}{3(2m + 1)} \right) - F_m\left( \frac{\pi}{2m + 1} \right) \right) F_m\left( \frac{2\pi}{3(2m + 1)} \right) \ell W_\alpha\left( \frac{\pi}{3(2m + 1)} \right) \geq \frac{c}{2m + 1} \left( \frac{3\sqrt{3}}{2\pi} \right)^{2\ell} \exp\left( -\left( \frac{3(2m + 1)}{\pi} \right)^\alpha \right) \geq \left( \frac{3}{4} \right)^{2\ell}.
\]
In the last inequality we have used that
\[
\lim_{n \to \infty} \frac{1}{2\ell} \left( \frac{3(2m + 1)}{\pi} \right)^\alpha = \frac{1}{12} \left( \frac{3}{\pi} \right)^\alpha < \frac{1}{12}
\]
together with the fact that \( \frac{3\sqrt{3}}{2\pi} e^{-1/12} > \frac{3}{4} \). We also have, using (10), (11) and (12), that
\[
-\int_{\frac{\pi}{2m + 1}}^{\pi} p(\theta)W_\alpha(\theta)d\theta \leq \int_{-\frac{\pi}{2m + 1}}^{\frac{\pi}{2m + 1}} F_m(\theta)^\ell d\theta \leq \frac{\pi}{2m + 1} F_m\left( \frac{\pi}{2m + 1} \right)^\ell \int_{\frac{2\pi}{2m + 1}}^{\pi} \left( \frac{\pi}{2m + 1} \theta \right)^{2\ell} d\theta \leq \frac{\pi}{2m + 1} \left( \frac{2}{3} \right)^{2\ell} + \frac{(2m - 1)\pi}{2m + 1} \left( \frac{1}{2} \right)^{2\ell} \leq \pi \left( \frac{2}{3} \right)^{2\ell},
\]
whence
\[
\int_{-\pi}^{\pi} p(\theta)W_\alpha(\theta)d\theta = 2 \int_{0}^{\pi} p(\theta)W_\alpha(\theta)d\theta > 0.
\]
Therefore there is at least one node of the Chebyshev-type trigonometric quadrature in the interval \([ - \frac{\pi}{2m + 1}, \frac{\pi}{2m + 1} ]\). Let \( \theta_0 \) be one such node. Let \( I := \int_{-\pi}^{\pi} W_\alpha(\theta)d\theta \). Now, since \( \deg ((F_m)^r) = 2rm \leq
n, it follows, using (11), (12) and (10), that

\[
\frac{I}{N} \left( \frac{2}{\pi} \right)^{2r} \leq \frac{I}{N} F_m \left( \frac{\pi}{2m+1} \right)^r = \frac{I}{N} \sum_{j=1}^{N} F_m(\theta_j)^r = \int_{-\pi}^{\pi} F_m(\theta)^r W_\alpha(\theta) d\theta \leq \int_{-\pi}^{\pi} \left( \frac{\pi}{(2m+1)\theta} \right)^{2r} e^{-\frac{1}{\theta^m}} d\theta \leq 2\pi \left( \frac{\pi}{2m+1} \right)^{2r} \left( \frac{2r}{e\alpha} \right)^{1/\alpha} \leq 2\pi \left( \frac{\pi}{6} \right)^{2r},
\]

where in the penultimate inequality we used that \( \left( \frac{2r}{e\alpha} \right)^{2r/\alpha} \) is the maximum of \( \theta^{-2r} \exp(-|\theta|^{-\alpha}) \) and in the last inequality we used that

\[
\frac{\pi}{2m+1} \left( \frac{2r}{e\alpha} \right)^{1/\alpha} \leq \pi \left( \frac{2 \cdot \frac{\alpha}{\theta + \cdot \cdot \cdot n \cdot \cdot \cdot + 1}}{e\alpha} \right)^{1/\alpha} \rightarrow \frac{\pi}{6} \left( \frac{\theta}{6} \right)^{1/\alpha} < \frac{\pi}{6}.
\]

Therefore \( N > \frac{I}{2\pi} \left( \frac{12}{\pi^2} \right)^{2r} \) and the lower bound follows.

\( \square \)

5 Discussion and open questions

In our work we find the order of magnitude of the minimal number of nodes required in Chebyshev-type quadratures for doubling weight functions, and also briefly discuss more general weight functions. It is natural to wonder whether any of the qualitative phenomena observed for doubling weight functions are in fact true in greater generality. In this section we briefly discuss some questions of this type. The questions are formulated in the trigonometric case but are natural also on the interval.

Multiplication by a function

Let \( h \) be a \( 2\pi \)-periodic measurable function satisfying that there exist \( M, m > 0 \) such that

\[
m < h(x) < M
\]

almost everywhere. Let \( W \) be a \( 2\pi \)-periodic weight function. Is it true that the minimal number of nodes required in Chebyshev-type quadratures for \( W \) and for \( hW \) is of the same order of magnitude? Precisely, that there exist \( C(M, m), c(M, m) > 0 \), depending only on \( M \) and \( m \) (in fact, only on \( M/m \) by homogeneity), such that

\[
c(M, m)N_{W}^{\text{trig}}(n) \leq N_{hW}^{\text{trig}}(n) \leq C(M, m)N_{W}^{\text{trig}}(n), \quad n \geq 1,
\]

\[
c(M, m)N_{W}^{\text{trig}}(n) \leq N_{hW}^{\text{trig}}(n) \leq C(M, m)N_{W}^{\text{trig}}(n), \quad n \geq 1.
\]

Our results show that this phenomenon holds for doubling weight functions.

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Sharpness of Kane’s bound

Our upper bounds on $N^\text{trig}_W(n)$ are based on Kane’s general upper bound given in Proposition 2.1. In fact, we do not know any example of a $2\pi$-periodic weight function which is non-zero almost everywhere for which this bound is not sharp up to constants. Does it give the correct order of magnitude of $N^\text{trig}_W(n)$ (as $n$ tends to infinity) for all such weight functions? A positive answer will also verify the relation (21) discussed in the previous section.

It is necessary to make some assumption on the support of the weight function. For instance, as pointed out to us by Kane, if $W$ is the $2\pi$-periodic weight function defined for $-\pi \leq \theta < \pi$ by

$$W(\theta) := \begin{cases} 1 & |\theta| \leq \frac{\pi}{6} \\ 0 & |\theta| > \frac{\pi}{6} \end{cases}.$$  

For even $n$, the bound on $N^\text{trig}_W(n)$ given by Proposition 2.1 is at least $2n$, by considering the polynomial $p(\theta) = (\sin \theta)^n \in T^+$. However, it is fact true that $cn^2 \leq N^\text{trig}_W(n) \leq N^\text{trig}_W(n) \leq Cn^2$. To see this, observe that $W(\theta) = w(\cos \theta)|\sin \theta|$ for the weight function $w$ on $[-1,1]$ defined by

$$w(x) := \begin{cases} \frac{1}{\sqrt{1-x^2}} & x \geq \frac{\sqrt{3}}{2} \\ 0 & x < \frac{\sqrt{3}}{2} \end{cases}.$$  

A Chebyshev-type trigonometric quadrature for $W$ may be obtained from a Chebyshev-type quadrature for $w$ (as in the proof of Theorem 1.1) by mapping the nodes $(t_j)_{j=1}^N$ to the nodes $(\arccos t_j)_{j=1}^N \cup (-\arccos t_j)_{j=1}^N$ (where the union is interpreted as a multiset union). Now, $cn^2 \leq N_w(n) \leq N_w(n) \leq Cn^2$ as follows from Theorem 1.1 applied to $w$ composed with an affine mapping.

Locality

Theorem 1.2 shows that the order of magnitude of $N^\text{trig}_W(n)$ and $N^\text{trig}_W(n)$ is given by $R^\text{trig}_W(n)$ for doubling $2\pi$-periodic weight functions $W$. The order of magnitude of $R^\text{trig}_W(n)$, as a function of $n$, is determined by the expression $\inf_{x \in \mathbb{R}} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} W(\theta) d\theta$. This gives a sense to the idea that the orders of magnitude of $N^\text{trig}_W(n)$ and $N^\text{trig}_W(n)$ depend only on local features of $W$. Is this property shared also by non-doubling weight functions?

To make this question precise, suppose $W_1, W_2, \ldots, W_m$ and $W$ are $2\pi$-periodic weight functions such that for every real $x$, $W$ coincides with some $W_i$ in a neighbourhood of $x$. Is it true that there exists a constant $M$, depending only on $W_1, W_2, \ldots, W_m$ and $W$, for which

$$N^\text{trig}_W(n) \leq M \max_{1 \leq i \leq m} N^\text{trig}_{W_i}(n),$$

$$N^\text{trig}_W(n) \leq M \max_{1 \leq i \leq m} N^\text{trig}_{W_i}(n).$$
for every $n \geq 1$? One cannot expect similar inequalities in the opposite direction to hold, even in the doubling case, as no control is provided on $W_i$ away from the neighbourhoods where it coincides with $W$ and Theorem 1.2 shows that changing $W_i$ in these regions can increase $N_{W_i}^{\text{trig}}(n)$ and $N_{W_i}^{\text{trig}}(n)$ significantly.

**Numerical algorithms**

Our results provide bounds on the minimal number of nodes in Chebyshev-type quadratures for doubling weight functions. From a numerical analysis point of view it is of interest to complement these with efficient algorithms for finding such quadrature formulas. It appears that this problem has not received much attention in the literature and we are not aware of efficient algorithms for it. Research has been focused on identifying in the set of all Chebyshev-type quadratures with a given number of nodes and a given degree of accuracy, such formulas which are optimal in a specified sense. The interested reader is pointed to the recent book of Brass and Petras [5, Chapter 9], to the older survey of Gautschi [8], to the works of Anderson and Gautschi [1] and of Förster and Ostermeyer [7] and to the references there, for discussion of such notions and their properties. The work of Anderson and Gautschi suggests also an algorithm for finding a quadrature formula with certain optimality properties.

**Extensions**

Both the notions of quadrature formula and the notion of doubling are, in fact, defined for the measure given by $w(t)dt$ (or $W(\theta)d\theta$ in the trigonometric case). Thus these notions may be extended naturally to general finite, positive measures. With these extended notions, Theorem 1.1, Theorem 1.2 and their proofs continue to apply, mutatis mutandis, for the class of doubling measures.

Some authors extend the notion of quadrature formula further, allowing the nodes to be outside of $[-1, 1]$ for weight functions on $[-1, 1]$, or allowing the nodes to take complex values for $2\pi$-periodic weight functions. We do not consider these extended notions here.

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[17] R. Peled, Simple universal bounds for Chebyshev-type quadratures, J. Approx. Theory 162 (2010), no. 12, 2317–2348.
A Proof of Proposition 2.1

In presenting the proof of Proposition 2.1 essentially following the proof in [10], we use the following three lemmas.

**Lemma A.1.** Let \( W \) be a 2\( \pi \)-periodic weight function and let \( n \) and \( N \) be positive integers, such that

\[
2N > \int_{-\pi}^{\pi} W(\theta) d\theta \sup_{p \in T_n^+} \int_{-\pi}^{\pi} |p'(\theta)| d\theta.
\]  

(22)

Then there is a finite set \( S \subset [-\pi, \pi) \) such that every non-zero \( q \in T_n \) for which \( \int_{-\pi}^{\pi} q(\theta)W(\theta)d\theta = 0 \) satisfies

\[
2N \max_{x \in S} q(x) > \int_{-\pi}^{\pi} |q'(\theta)| d\theta.
\]  

(23)

**Proof.** We consider \( T_n \) as a subset of the space of continuous 2\( \pi \)-periodic functions endowed with the supremum norm. Let \( V := \{ r \in T_n \mid \int_{-\pi}^{\pi} r(\theta)W(\theta)d\theta = 0 \} \) and observe that

a non-zero \( r \in V \) satisfies \( \max_x r(x) > 0 \) \hspace{1cm} (24)

as non-zero trigonometric polynomials cannot be zero on a set of positive measure. Let \( K := \{ r \in V \mid \max_x r(x) = 1 \} \). We first prove that \( K \) is a compact set, for which it suffices to show that

\[
\sup_{r \in K} \max_x |r(x)| < \infty.
\]  

(25)

Let \( B := \{ q \in V \mid \max_x |q(x)| = 1 \} \). The set \( B \) is compact as a closed and bounded set in the finite-dimensional space \( T_n \). Thus the continuous functional \( f \mapsto \max_x f(x) \) attains a minimum on \( B \), which must be positive due to (24). This implies (25).

Take \( \eta > 1 \) such that \( 2N \) is still bigger than \( \eta \) times the right-hand side of (22). For every \( -\pi \leq x < \pi \) let \( U_x := \{ r \in T_n \mid r(x) > 1/\eta \} \). Noting that \( \{ U_x \}_{-\pi \leq x < \pi} \) is an open cover of the compact set \( K \) we conclude that there is a finite set \( S \subset [-\pi, \pi) \) such that \( K \subseteq \cup_{x \in S} U_x \). Consequently, using (24),

\[
\max_x q(x) \leq \eta \max_{x \in S} q(x), \quad q \in V.
\]  

(26)
Finally, let \( q \in V \) be a non-zero trigonometric polynomial in \( V \). Define \( p_0 := \eta(\max_{x \in S} q(x)) - q \) and note that \( p_0 \in \mathcal{T}_n^+ \) by (20) and \( \max_{x \in S} q(x) > 0 \) by (24) and (26). Thus,

\[
2N > \eta \int_{-\pi}^{\pi} W(\theta) d\theta \sup_{p \in \mathcal{T}_n^+} \frac{\int_{-\pi}^{\pi} |p'(\theta)| d\theta}{\int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta} \geq \eta \int_{-\pi}^{\pi} W(\theta) d\theta \frac{\int_{-\pi}^{\pi} |p'_0(\theta)| d\theta}{\int_{-\pi}^{\pi} p_0(\theta) W(\theta) d\theta} = \frac{\int_{-\pi}^{\pi} |q'(\theta)| d\theta}{\max_{x \in S} q(x)}. \]

**Lemma A.2.** Let \( v_1, v_2, \ldots, v_R \) be points in \( \mathbb{R}^n \), no \( n + 1 \) of which are on the same affine hyperplane, and let \( Q \subset \mathbb{R}^n \) be their convex hull. Let \( -\pi < x_1 < x_2 < \ldots < x_R < \pi \) and let \( N \) be a positive integer. Then there are continuous functions \( f_1, f_2, \ldots, f_N \) from \( Q \) to \([-\pi, \pi]\) such that if \( v \) belongs to a facet of \( Q \) whose vertices are \( v_{i_1}, v_{i_2}, \ldots, v_{i_n} \), \( 1 \leq i_1 < i_2 < \ldots < i_n \leq R \), then \( \{ f_1(v), f_2(v), \ldots, f_N(v) \} \subset [x_{i_1}, x_{i_n}] \) and in each of the intervals \((x_{i_1}, x_{i_2}), \ldots, (x_{i_{n-1}}, x_{i_n})\) there is at most one member of the set \( \{ f_1(v), f_2(v), \ldots, f_N(v) \} \).

**Proof.** Take some triangulation of \( Q \). Let \( v \in Q \). If \( v \) is in the simplex of the chosen triangulation whose vertices are \( v_{l_0}, v_{i_1}, \ldots, v_{i_n} \), then \( v \) can be uniquely presented as a convex combination \( v = \sum_{j=0}^{n} \alpha_j v_{i_j} \). For every \( -\pi \leq x < \pi \) let

\[
\rho_v(x) := \frac{\pi + x}{2\pi} + N \sum_{0 \leq j \leq n, x_{i_j} \leq x} \alpha_j
\]

and observe that this definition is independent of the choice of the simplex of the triangulation which contains \( v \) (if more than one exists). Define \( f_i(v) := \min\{-\pi \leq x < \pi \mid \rho_v(x) \geq i\} \). It is easy to verify that each \( f_i \) is continuous. Let \( v \) be a point on a facet of \( Q \) whose vertices are \( v_{i_1}, v_{i_2}, \ldots, v_{i_n}, 1 \leq i_1 < i_2 < \ldots < i_n \leq R \). Then \( \rho_v(x) < 1 \) for every \(-\pi \leq x < x_{i_1} \) and \( \rho_v(x) > N \) for every \( x_{i_n} < x < \pi \), whence \( \{ f_1(v), f_2(v), \ldots, f_N(v) \} \subset [x_{i_1}, x_{i_n}] \). Also \( \rho_v(y) - \rho_v(x) < 1 \) for every \( x_{i_{k-1}} < x < y < x_{i_k}, 1 < k \leq n \), whence in each of the intervals \((x_{i_1}, x_{i_2}), \ldots, (x_{i_{n-1}}, x_{i_n})\) there is at most one member of the set \( \{ f_1(v), f_2(v), \ldots, f_N(v) \} \).

**Lemma A.3.** [10, Proposition 7] Let \( Q \subset \mathbb{R}^n \) be a convex polytope with 0 in its interior. Let \( F : Q \to \mathbb{R}^n \) be a continuous function such that if \( \{ v \in \mathbb{R}^n \mid u \cdot v = 1 \} \) is an affine hyperplane containing a facet \( T \) of \( Q \) then \( u \cdot F(v) > 0 \) for every \( v \in T \). Then there is a \( v \in Q \) such that \( F(v) = 0 \).

**Proof.** By way of contradiction, assume that \( F(v) \neq 0 \) for every \( v \in Q \). Let \( B^n := \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \) be the closed unit ball of \( \mathbb{R}^n \). For every \( u \in \mathbb{R}^n \) let \( \| u \|_Q := \min\{ \lambda > 0 \mid \frac{1}{\lambda} u \in Q \} \) and define \( h : B^n \to B^n \) by

\[
h(x) := \begin{cases} 
- \frac{F(\frac{|x|}{\|x\|_Q})}{F(0)} x & \text{if } x \neq 0 \\
- \frac{F(0)}{|F(0)|} x & \text{if } x = 0
\end{cases}
\]
The function $h$ is continuous, so by Brouwer’s fixed point theorem, there exists an $x_0 \in B^n$ such that $h(x_0) = x_0$. Let $y_0 := \frac{x_0}{\|x_0\|}$. Since $|h(x)| = 1$ for every $x \in B^n$ we get that $|x_0| = |h(x_0)| = 1$, and therefore $y_0 = \frac{x_0}{\|x_0\|} = \frac{x_0}{\|x_0\|}$ is on the boundary of $Q$. Therefore, $y_0$ belongs to at least one facet $T$ of $Q$. Suppose $T$ is contained in the affine hyperplane $\{x \in \mathbb{R}^n \mid u \cdot x = 1\}$. Then $u \cdot F(y_0) > 0$ and we get a contradiction since

$$1 = u \cdot y_0 = u \cdot \frac{x_0}{\|x_0\|} = \frac{1}{\|x_0\|} u \cdot x_0 = \frac{1}{\|x_0\|} u \cdot h(x_0) =$$

$$= \frac{1}{\|x_0\|} u \cdot \left( -\frac{F(y_0)}{|F(y_0)|} \right) = -\frac{1}{\|x_0\| |F(y_0)|} u \cdot F(y_0) < 0. \quad \square$$

**Proof of Proposition 2.1.** Suppose $N$ is a positive integer such that

$$2N > \int_{-\pi}^{\pi} W(\theta) d\theta \sup_{p \in \mathbb{T}_n} \int_{-\pi}^{\pi} p(\theta) W(\theta) d\theta$$

and let $S$ be the finite set provided by Lemma A.1. Let $\varphi_0 \equiv 1, \varphi_1, \ldots, \varphi_{2n}$ be a basis of $\mathbb{T}_n$. For every $-\pi \leq x < \pi$ let $E(x)$ be the vector $\left( \left( \int_{-\pi}^{\pi} W(\theta) d\theta \right) \varphi_i(x) - \int_{-\pi}^{\pi} \varphi_i(\theta) W(\theta) d\theta \right)_{i=1}^{2n}$ in $\mathbb{R}^{2n}$. It is straightforward to check that no $2n + 1$ elements of $\{E(x) \mid -\pi \leq x < \pi\}$ are on the same affine hyperplane. Define $Q$ to be the convex hull of $\{E(x) \mid x \in S\}$. We claim that 0 is in the interior of $Q$. Indeed, otherwise there exists a non-zero $u \in \mathbb{R}^{2n}$ for which $u \cdot E(x) \leq 0$ for all $x \in S$. However, the non-zero trigonometric polynomial in $T_n$ defined by $p_u(x) := u \cdot E(x)$ satisfies $\int_{-\pi}^{\pi} p_u(\theta) W(\theta) d\theta = 0$ and thus, by (2.3), we have that $\max_{x \in S} p_u(x) > 0$, a contradiction.

Suppose $S = \{x_1, x_2, \ldots, x_R\}$ where $-\pi \leq x_1 < x_2 < \ldots < x_R < \pi$. Let $f_1, f_2, \ldots, f_N$ be the continuous functions from $Q$ to $[-\pi, \pi]$ guaranteed by Lemma A.2. Let $v$ be a point on a facet of $Q$ whose vertices are $E(x_{i_1}), E(x_{i_2}), \ldots, E(x_{i_{2n}})$, where $1 \leq i_1 < i_2 < \ldots < i_{2n} \leq R$. Let $\{w \in \mathbb{R}^n \mid u \cdot w = 1\}$ be the affine hyperplane containing the facet (using that 0 is in the interior of $Q$). Let $q$ be the non-zero trigonometric polynomial in $T_n$ defined by $q(x) := u \cdot E(x)$, so that $\int_{-\pi}^{\pi} q(\theta) W(\theta) d\theta = 0$. Note that $q(x) \leq 1$ for every $x \in S$ as 0 is in the interior of $Q$, whence $2N > \int_{-\pi}^{\pi} |q'(\theta)| d\theta$ by Lemma A.1. Let $J := \{1 \leq j \leq N \mid f_j(v) \notin \{x_{i_1}, x_{i_2}, \ldots, x_{i_{2n}}\}\}$. Note that Lemma A.2 implies that for every $j \in J$ there is a $2 \leq k_j \leq 2n$ such that $f_j(v) \in (x_{i_{k_j-1}}, x_{i_{k_j}})$ and these $k_j$ are distinct. Then, since $q(x_{i_k}) = 1$ for every $1 \leq k \leq 2n$,

$$N > \frac{1}{2} \int_{-\pi}^{\pi} |q'(\theta)| d\theta \geq \frac{1}{2} \sum_{j \in J} \left( \int_{x_{i_{k_j-1}}}^{x_{i_{k_j}}} q'(t) dt \right) =$$

$$= \sum_{j \in J} \frac{|q(f_j(v)) - q(x_{i_{k_j-1}})| + |q(x_{i_{k_j}}) - q(f_j(v))|}{2} =$$

$$= \sum_{j \in J} |1 - q(f_j(v))| = \sum_{j=1}^{N} |1 - q(f_j(v))| \geq N - \sum_{j=1}^{N} q(f_j(v)), \quad \square$$
whence $u \cdot \sum_{j=1}^{N} E(f_j(v)) = \sum_{j=1}^{N} q(f_j(v)) > 0$. Thus, if we define a function $F : Q \to \mathbb{R}^{2n}$ by $F(v) := \sum_{j=1}^{N} E(f_j(v))$ then $F$ satisfies the assumptions of Lemma A.3. Consequently, there is some $v \in Q$ such that $\sum_{j=1}^{N} E(f_j(v)) = F(v) = 0$, i.e., $\{f_j(v)\}_{j=1}^{N}$ are the nodes of a Chebyshev-type trigonometric quadrature of degree $n$ for $W$. \hfill \Box