Analyzing Nonblocking Switching Networks using Linear Programming (Duality)

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Abstract

The main task in analyzing a switching network design (including circuit-, multirate-, and photonic-switching) is to determine the minimum number of some switching components so that the design is non-blocking in some sense (e.g., strict- or wide-sense). We show that, in many cases, this task can be accomplished with a simple two-step strategy: (1) formulate a linear program whose optimum value is a bound for the minimum number we are seeking, and (2) specify a solution to the dual program, whose objective value by weak duality immediately yields a sufficient condition for the design to be non-blocking.

We illustrate this technique through a variety of examples, ranging from circuit to multirate to photonic switching, from unicast to \( f \)-cast and multicast, and from strict- to wide-sense non-blocking. The switching architectures in the examples are of Clos-type and Banyan-type, which are the two most popular architectural choices for designing non-blocking switching networks.

To prove the result in the multirate Clos network case, we formulate a new problem called DYNAMIC WEIGHTED EDGE COLORING which generalizes the DYNAMIC BIN PACKING problem. We then design an algorithm with competitive ratio \( 5.6355 \) for the problem. The algorithm is analyzed using the linear programming technique. A new upper-bound for multirate wide-sense non-blocking Clos networks follow, improving upon a decade-old bound on the same problem.

Keywords: Nonblocking, multirate, switching, linear programming, duality, dynamic weighted edge coloring.

1 Introduction

The two most important architectures for designing non-blocking switching networks are Clos-type \(^5\) and Banyan-type \(^12\). The Clos network not only played a central role in classical circuit-switching theory \(^1\)\(^15\), but also was the bedrock of multirate switching \(^4\)\(^11\)\(^19\)\(^22\)\(^25\)\(^22\) (e.g., in time-divisioned switching environments where connections are of varying bandwidth requirements), and photonic-switching \(^13\)\(^24\)\(^27\)\(^28\). The Banyan network is isomorphic to various other “bit-permutation” networks such as Omega, baseline, etc., \(^2\); they are called Banyan-type networks and have been used extensively in designing electronic and optical switches, as well as parallel processor architectures \(^9\).

In particular, the multilog design which involves the vertical stacking of a number of inverse Banyan planes has been used in circuit- and photonic-switching environments because they have small depth (\( \log N \)), self-routing capability, and absolute signal loss uniformity \(^17\)\(^18\)\(^20\)\(^29\)\(^34\).

In analyzing Clos networks, the most basic task is to determine the minimum number of middle-stage crossbars so that the network satisfies a given nonblocking condition. This holds true in space-, multirate-, and photonic-switching, in unicast, \( f \)-cast and multicast, and broadcast traffic patterns, and in all nonblocking types (strict-sense, wide-sense, and rearrangeable). Similarly, analyzing multilog networks often involves determining the minimum

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number of Banyan planes so that the network satisfies some requirements. This paper shows that a simple and effective linear programming (LP) based two-step strategy can be employed in the analysis:

- First, the minimum value we are seeking (e.g., the number of middle-stage crossbars in a Clos network or the minimum number of Banyan planes in a multilog network) is upper-bounded by the optimum value of a linear program (LP) of the form \( \max \{ c^T x \mid Ax \leq b, x \geq 0 \} \). The maximization objective is often required by worst-case analysis, such as the maximum number of middle-stage crossbars in a Clos network which is insufficient to carry a new request. The constraints of the LP are used to express the fact that no input or output can generate or receive connection requests totaling more than its capacity.

- Second, by specifying any feasible solution, say \( y^* \), to the dual program \( \min \{ b^T y \mid A^T y \geq c \} \), and applying weak duality we can use the dual-objective value \( b^T y^* \) as an upper bound for the minimum value being sought.

In some cases, we may not need the second step because the primal LP is small with only a few variables. In most cases, however, the LP and its dual are very general, dependent on various parameters of the switch design. In such cases, it would be difficult to come up with a primal-optimal solution. Fortunately, we can supply a dual-feasible solution to quickly “certify” the bound.

The LP-duality technique was first used in our recent paper [26] to analyze the (unicast) strictly nonblocking multilog architecture in the photonic-switching case, subject to general crosstalk constraints. This paper demonstrates that the technique can be applied to a wider range of switching network analysis problems. Our main contributions are as follows. First, we formulate a new problem called DYNAMIC WEIGHTED EDGE COLORING (DWECC) of graphs, which generalizes the classic DYNAMIC BIN PACKING problem [6] and the routing problem for multirate widesense nonblocking Clos networks. Using the LP-technique, we design an algorithm with competitive ratio 5.6355. A new upper bound for the multirate Clos network problem follows. Since BIN PACKING and its variations have been very useful in both theory and practice, we believe that DWECC and our results on it are of independent interest. Second, we use the LP-technique to prove general sufficient conditions for the multilog network to be \( f \)-cast nonblocking under the so-called window algorithm, both under the link-blocking model and the crosstalk-free model. To the best of our knowledge, these are the first \( f \)-cast results for the multilog design. We show that many known results are immediate corollaries of these general conditions.

The rest of this paper is organized as follows. Section 2 presents notations and terminologies. Section 3 illustrates the strength of the LP-duality technique on analyzing several problems on the Clos networks. The DWECC problem is also defined and analyzed. Section 4 proves non-blocking results for \( f \)-cast multilog architecture. Section 5 does the same with the crosstalk constraint.

## 2 Preliminaries

Throughout this paper, for any positive integers \( k, d \), let \( [k] \) denote the set \( \{1, \ldots, k\} \), \( \mathbb{Z}_d \) denote the set \( \{0, \ldots, d - 1\} \) which can be thought of as \( d \)-ary “symbols,” let \( \mathbb{Z}_d^k \) denote the set of all \( d \)-ary strings of length \( k \), \( |s| \) the length of any \( d \)-ary string \( s \) (e.g., \(|1142| = 4\)), and \( s_{i..j} \) the substring \( s_i \cdots s_j \) of a string \( s = s_1 \cdots s_l \in \mathbb{Z}_d^l \) (if \( i > j \) then \( s_{i..j} \) is the empty string).

### 2.1 Switching environments

Consider an \( N \times N \) switching network, i.e. a switching network with \( N \) inputs and \( N \) outputs. There are three levels of nonblockingness of a switching network. A network is rearrangeably nonblocking (RNB) if it can realize any one-to-one mapping between inputs and outputs simultaneously; it is widesense nonblocking (WSNB) if a new request from a free input to a free output can be realized without disturbing existing connections, as long as all requests are routed according to some algorithm; finally, it is strictly nonblocking (SNB) if a new request from a free input to a free output can always be routed no matter how existing connections were arranged. In the multicast case, RNB, WSNB, and SNB are defined similarly. The reader is referred to [15] for more details on non-blocking concepts.

In circuit switching, a request is a pair \((a, b)\) where \( a \) is an unused input and \( b \) is an unused output. A route \( R(a, b) \) realizes the request if it does not share any internal link with existing routes. In an \( f \)-cast switching network,
each multicast request is of the form \((a, B)\) where \(a\) is some input and \(B\) is a subset of at most \(f\) outputs. The number \(f\) is called the fanout restriction. An \(N \times n\) multicast network without fanout restriction is equivalent to an \(N\)-cast network.

In the multirate case, each link has a capacity (e.g., bandwidth). All inputs and outputs have the same capacity normalized to 1. An input cannot request more than its capacity. Neither can outputs. A request is of the form \((a, b, w)\) where \(a\) is an input, \(b\) is an output, and \(w \leq 1\) is the requested rate. If existing requests have used up to \(x\) and \(y\) units of \(a\’s\) and \(b\’s\) capacity, respectively, then the new requested rate \(w\) can only be at most \(\min\{1 - x, 1 - y\}\). An internal link cannot carry requests with total rate more than 1.

### 2.2 The 3-stage Clos networks

The Clos network \(C(n_1, r_1, m, n_2, r_2)\) is a 3-stage interconnection network, where the first stage consists of \(r_1\) crossbars of size \(n_1 \times m\), the last stage has \(r_2\) crossbars of dimension \(m \times n_2\), and the middle stage has \(m\) crossbars of dimension \(r_1 \times r_2\) (see Figure 1). Each input crossbar \(I_i\) \((i = 1, \ldots, r_1)\) is connected to each middle crossbar \(M_j\) \((j = 1, \ldots, m)\). Similarly, the middle stage and the last stage are fully connected. When \(n_1 = n_2 = n\) and \(r_1 = r_2 = r\), the network is called the symmetric 3-stage Clos network, denoted by \(C(n, m, r)\).

### 2.3 The \(d\)-ary multilog networks

Let \(N = d^n\). We consider the \(\log_d(N, 0, m)\) network, which denotes the stacking of \(m\) copies of the \(d\)-ary inverse Banyan network \(BY^{-1}(n)\) with \(N\) inputs and \(N\) outputs. (See Fig. 2 and 4) Label the inputs and outputs of \(BY^{-1}(n)\) and the \(d \times d\) switching elements (SE) of each stage of \(BY^{-1}(n)\) as illustrated in Fig. 2 We label the inputs and outputs of a \(BY^{-1}(n)\)-plane with \(d\)-ary strings of length \(n\). Specifically, each input \(u \in Z_d^n\) and output \(v \in Z_d^n\) have the form \(u = u_1 \cdots u_n, v = v_1 \cdots v_n\), where \(u_i, v_i \in Z_d, \forall i \in [n]\). Also, label the \(d \times d\) switching elements in each of the \(n\) stages of a \(BY^{-1}(n)\)-plane with \(d\)-ary strings of length \(n - 1\). An input \(x\) (respectively, output \(y\)) is connected to the switching element labeled \(x_{1..n-1}\) in the first stage (respectively, \(y_{1..n-1}\) in the last stage). A switching elements labeled \(z = z_1 \cdots z_{n-1}\) in stage \(i \leq n - 1\) is connected to \(d\) switching elements in stage \(i + 1\) numbered \(z_1 \cdots z_{i-1} * z_{i+1} \cdots z_{n-1}\), where * is any symbol in \(Z_d\).

For the sake of clarity, let us first consider a small example. Consider the unicast request \((x, y) = (01001, 10101)\) when \(d = 2, n = 5\). The input \(x = 01001\) is connected to the switching element labeled 0100 in the first stage, which is connected to two switching elements labeled 0100 and 1100 in the second stage, and so on. The unique path from \(x\) to \(y\) in the \(BY^{-1}(n)\)-plane can be explicitly written out (see Figure 3):

| input \(x\)     | 01001 |
|-----------------|-------|
| stage-1 switching element | 0100  |
| stage-2 switching element | 1100  |
| stage-3 switching element | 1010  |
| stage-4 switching element | 1010  |
| stage-5 switching element | 1010  |
| output \(y\)    | 10101 |
We can see clearly the pattern: the prefixes of $y_{1,n-1}$ are “taking over” the prefixes of $x_{1,n-1}$ on the path from $x$ to $y$. In general, the unique path $R(x, y)$ in a $BY^{-1}(n)$-plane from an arbitrary input $x$ to an arbitrary output $y$ is exactly the following:

| input $x$ | $x_1x_2 \ldots x_{n-1}x_n$ |
|-----------|-------------------------------|
| stage-1 switching element | $x_1x_2 \ldots x_{n-1}$ |
| stage-2 switching element | $y_1x_2 \ldots x_{n-1}$ |
| stage-3 switching element | $y_1y_2 \ldots x_{n-1}$ |
| ... | ... |
| stage-$n$ switching element | $y_1y_2 \ldots y_{n-1}$ |
| output $y$ | $y_1y_2 \ldots y_{n-1}y_n$ |

Now, consider two unicast requests $(a, b)$ and $(x, y)$. From the observation above, on the same $BY^{-1}(n)$-plane the two routes $R(a, b)$ and $R(x, y)$ share a switching element (also called a node) if and only if there is some $j \in [n]$ such that $b_{1..j-1} = y_{1..j-1}$ and $a_{j..n-1} = x_{j..n-1}$. In this case, the two paths intersect at a stage-$j$ switching element. It should be noted that two requests’ paths may intersect at more than one switching element.

For any two $d$-ary strings $u, v \in \mathbb{Z}_d$, let $\text{PRE}(u, v)$ denote the longest common prefix, and $\text{SUF}(u, v)$ denote the longest common suffix of $u$ and $v$, respectively. For example, if $u = 0100110$ and $v = 0101010$, then $\text{PRE}(u, v) =$
010 and $\text{SUF}(u, v) = 10$. The following propositions straightforwardly follow (for more details, see e.g. [35]).

Proposition 2.1. Let $(a, b)$ and $(u, v)$ be two unicast requests. Then their corresponding routes $R(a, b)$ and $R(u, v)$ in a $BY^{-1}(n)$-plane share at least a common SE if and only if

\[ |\text{SUF}(a_{1..n-1}, u_{1..n-1})| + |\text{PRE}(b_{1..n-1}, v_{1..n-1})| \geq n - 1. \]  

(1)

Moreover, the routes $R(a, b)$ and $R(u, v)$ intersect at exactly one SE if and only if

\[ |\text{SUF}(a_{1..n-1}, u_{1..n-1})| + |\text{PRE}(b_{1..n-1}, v_{1..n-1})| = n - 1, \]  

(2)

in which case the common SE is an SE at stage $|\text{PRE}(b_{1..n-1}, v_{1..n-1})| + 1$ of the $BY^{-1}(n)$-plane.

Proposition 2.2. Let $(a, b)$ and $(u, v)$ be two unicast requests. Then their corresponding routes $R(a, b)$ and $R(u, v)$ in a $BY^{-1}(n)$-plane share at least a common link iff

\[ |\text{SUF}(a_{1..n-1}, u_{1..n-1})| + |\text{PRE}(b_{1..n-1}, v_{1..n-1})| \geq n. \]  

(3)

3 Results on the Clos Networks

3.1 Two classic examples in circuit switching

To illustrate the LP-duality technique, we begin with two simple examples which have become classic textbook materials.

Example 3.1 (The SNB Case). Consider the symmetric Clos network $C(n, m, r)$. Consider a new request from an input of input crossbar $I$ to an output of output crossbar $O$. A middle crossbar cannot carry this request if it already carried some request from $I$ or some request to $O$. Let $x$ (resp. $y$) be the number of middle crossbars which already carry some requests from $I$ (resp. to $O$). Since the number of existing requests from $I$ or to $O$ is at most $n - 1$, we have $x \leq n - 1$ and $y \leq n - 1$. The number of unavailable middle crossbars is thus bounded above by the optimal value of the LP

\[ \max \{x + y \mid x \leq n - 1, y \leq n - 1, x, y \geq 0\}. \]

The dual program is

\[ \min \{(n - 1)(\alpha + \beta) \mid \alpha \geq 1, \beta \geq 1, \alpha, \beta \geq 0\}. \]

Setting $\alpha = \beta = 1$ is certainly dual-feasible, and thus its objective value $2n - 2$ is an upper bound on the number of unavailable middle crossbars. We conclude that $m \geq 2n - 1$ is sufficient for $C(n, m, r)$ to be SNB.
Example 3.2 (The WSNB Case). This example is a classic result by Benes [1]. Consider the $C(n, m, 2)$ network. The routing algorithm is simply the following rule: *reuse a busy middle crossbar whenever possible.*

For any $i, j \in \{1, 2\}$, let $M_{ij}$ be the set of middle crossbars carrying an $I_i, O_j$-request. The sets $M_{ij}$ certainly change over time as requests come and go. However, it is easy to show by induction that the routing rule ensures $|M_{11} \cup M_{22}| \leq n$ and $|M_{12} \cup M_{21}| \leq n$ at all times. To see this, without loss of generality consider a new $I_1, O_1$-request. If we can find a crossbar in $M_{22}$ to route the new request, then the union $M_{11} \cup M_{22}$ does not change and thus $|M_{11} \cup M_{22}| \leq n$ by induction hypothesis. If every crossbar in $M_{22}$ is not available for the new request, then it must be the case that $M_{22} \subseteq M_{11}$. There are at most $n - 1$ existing requests out of $I_1$. Thus, $|M_{11}| \leq n - 1$. Hence, before routing the new $I_1, O_1$-request we have $|M_{11} \cup M_{22}| = |M_{11}| \leq n - 1$. Consequently, after realizing the new request, we have $|M_{11} \cup M_{22}| \leq n$.

Next, again without loss of generality, consider a new request from $I_1$ to $O_1$. If $M_{22} \setminus M_{11} \neq \emptyset$, then we have a busy crossbar to reuse. Otherwise, the number of unavailable middle-crossbars for this new request is precisely $|M_{11} \cup M_{12} \cup M_{21}| = |M_{11}| + |M_{12} \cup M_{21}|$. Just before the arrival of this new request, the number of existing requests from $I_1$ to $O_1$ is at most $n - 1$, i.e. $|M_{11} \cup M_{12}| = |M_{11}| + |M_{12}| \leq n - 1$, and $|M_{11} \cup M_{21}| = |M_{11}| + |M_{21}| \leq n - 1$. The number of unavailable middle crossbars is thus bounded by the optimal value of the following LP, where we think of set cardinalities as variables:

$$\begin{align*}
\max & \quad |M_{11}| + |M_{12} \cup M_{21}| \\
\text{s.t.} & \quad |M_{11}| + |M_{12}| \leq n - 1 \\
& \quad |M_{11}| + |M_{21}| \leq n - 1 \\
& \quad |M_{12}| + |M_{21}| \leq n \\
& \quad |M_{12} \cup M_{21}| - |M_{12}| - |M_{21}| \leq 0
\end{align*}$$

The last inequality is the straightforward union bound. Obviously, all cardinalities are non-negative. The dual LP is

$$\begin{align*}
\min & \quad (n - 1)(y_1 + y_2) + ny_3 \\
\text{s.t.} & \quad y_1 + y_2 \geq 1 \\
& \quad y_2 + y_3 - y_4 \geq 0 \\
& \quad y_1 + y_3 - y_4 \geq 0 \\
& \quad y_4 \geq 1, \quad y_1, y_2, y_3 \geq 0
\end{align*}$$

Setting $y_1 = y_2 = y_3 = 1/2$ and $y_4 = 1$ is certainly dual-feasible with objective value $3n/2 - 1$. Hence, by weak duality the number of unavailable middle-crossbars for the new $I_1, O_1$-request is at most $\lfloor 3n/2 \rfloor - 1$, which means $m \geq \lfloor 3n/2 \rfloor$ is sufficient for $C(n, m, 2)$ to be WSNB. It is not hard to show that $m \geq \lfloor 3n/2 \rfloor$ is also necessary [1]. This ($r = 2$) is the only case for which a necessary and sufficient condition is known for the Clos network $C(n, m, r)$ to be WSNB!

### 3.2 Multirate switching and the DWEC problem

It is known that $C(n, m, r)$ is multirate WSNB when $m \geq 5.75n / 3$ [1]. This section uses the LP technique to improve this bound via solving a much more general problem called DYNAMIC WEIGHTED EDGE COLORING (DWEC).

**Definition 3.3 (The DWEC problem).** Let $G = (V, E)$ be a fixed simple graph called the base graph. Let $G_0 = (V, \emptyset)$ be an empty graph with the same vertex set. At time $t$, either an arbitrary edge $e$ is removed from $G_{t-1}$, in which case $G_t = G_{t-1} \setminus \{e\}$, or a copy of some edge $e \in E$ “arrives” along with a weight $w_e \in (0, 1]$, in which case define $G_t = G_{t-1} \cup \{e\}$. Note that $G_t$ can be a multi-graph as many copies of the same edge may arrive over time. The arriving edge is to be colored so that, in $G_t$, the total weight of same-color edges incident to the same vertex is at most 1.

The objective is to design a coloring algorithm so that the number of colors used is minimized, compared to an off-line algorithm which colors edges of $G_t$ subject to the same constraint. Formally, let $\text{OPT}(t)$ denote the number of colors used by an optimal off-line algorithm on $G_t$. Let $\tilde{\text{OPT}}(t) = \max_{i \leq t} \text{OPT}(i)$. For any online coloring algorithm
A, let $\bar{A}(t)$ be the number of colors ever used by $A$ up to time $t$. Algorithm $A$ has competitive ratio $\rho$ if, for any sequence of edge arrivals/departures with arbitrary weights, we always have $\bar{A}(t) \leq \rho \cdot \text{OPT}(t)$, $\forall t$.

The DYNAMIC BIN PACKING problem is exactly the DWEC problem when the base graph $G = K_2$, where each color is a bin. The best competitive ratio for DYNAMIC BIN PACKING is known to be between 2.5 and 2.788 [6]. We will show that the DWEC’s best competitive ratio is somewhere between 4 and 5.6355 for any base graph $G$.

**Theorem 3.4.** There is an algorithm for DWEC with competitive ratio 5.6355.

**Proof.** For the sake of presentation clarity, we will prove a slightly weaker ratio of 5.675, and then indicate how to obtain the better ratio 5.6355. The two proofs are identical, but the one we present is cleaner.

At any time $t$, let $W^u(t)$ denote the total weight of edges incident to $u$ in $G_t$, and let $d^u(t)$ denote the number of edges of weight $> 1/2$ incident to $u$. Let $\bar{W}(t) = \max_{i \leq t} \max_u W^u(i)$ and $\bar{A}(t) = \max_{i \leq t} \max_u d^u(i)$. It is not hard to see that $|W(t)| \leq \text{OPT}(t)$ and $|A(t)| \leq \text{OPT}(t)$.

Refer to an edge a type-0, type-1, type-2, or type-3, if its weight belongs to the interval $(\frac{1}{2}, 1], (\frac{3}{2}, 1], (\frac{7}{2}, \frac{5}{2}]$, or $(0, \frac{1}{2}]$, respectively. Our coloring algorithm is as follows. Maintain 4 disjoint sets of colors $C_i(t), 0 \leq i \leq 3$. Let $x_0, x_1, x_2, x_3$ be constants to be determined. For each $i = 0, 3$, we will maintain the following time-invariant conditions: $|C_i(t)| = |x_i W(t)|$ for $1 \leq i \leq 3$ and $|C_0(t)| = |x_0 A(t)|$.

If $|W(t)|$ or $|A(t)|$ is increased at some time $t$, we are allowed to add new colors to the sets $C_i(t)$ to maintain the invariants. Note that $|W(t)|$ and $|A(t)|$ are non-decreasing in $t$; hence, colors will never be removed from the $C_i(t)$. The colors in $C_0(t)$ are used exclusively for edges of type-0. The coloring for edges of types $i, 1 \leq i \leq 3$ is done as follows. If a type-$i$ edge arrives at time $t$, find a color in $C_i(t)$ to color it. If $C_i(t)$ cannot accommodate this edge, try $C_{i+1}(t)$, and so on until $C_3(t)$. We next show that if the constants $x_i$ are feasible solutions to a certain LP, then it is always possible to color an arriving edge.

Suppose a zero-edge edge $e = (u, v)$ arrives at time $t$. If we cannot find a color in $C_0(t)$ for $e$, then $|C_0(t)| \leq d^u(t - 1) + d^v(t - 1) = (d^u(t) - 1) + (d^v(t) - 1) < 2\bar{A}(t)$. Hence, as long as $x_0 \geq 2$ we can color $e$.

Next, suppose $e = (u, v)$ of type 1 arrives at time $t$ and we cannot find a color in $C_1(t) \cup C_2(t) \cup C_3(t)$ to color $e$.

For a color $c \in C_1(t)$ to be unavailable for $e$, there must be at least two type-1 color-$c$ edges incident to either $u$ or $v$. Thus, the total type-1 weight at $u$ and $v$ is $\geq \frac{4}{5}|C_1(t)|$. Similarly, for each color $c \in C_2(t)$, the total $c$-weight incident to $u$ and $v$ must be $\geq 1/2$, which means this color $c$ “carries” either at least two type-1 edges, or one type-1 edge and one type-2 edge, or at least two type-2 edges. Thus, the total color-$c$ weight incident to $u$ and $v$ must be $\geq \frac{2}{3}|C_2(t)|$. Lastly, for each color $c \in C_3(t)$, the total color-$c$ weight incident to $u$ and $v$ must be $\geq 1/2|C_3(t)|$. Note that the total weight at $u$ and $v$ is $< 2\bar{W}(t)$. Consequently, we will be able to find a color for $e$ if

$$\frac{4}{5}|C_1(t)| + \frac{2}{3}|C_2(t)| + \frac{1}{2}|C_3(t)| \geq 2\bar{W}(t),$$

which would hold if $\frac{4}{5}x_1 + \frac{2}{3}x_2 + \frac{1}{2}x_3 \geq 2$. Similarly, a newly arriving type-2 edge is colorable if $\frac{2}{3}x_2 + \frac{1}{2}x_3 \geq 2$, and a new type-3 edge is colorable if $\frac{2}{3}x_3 \geq 2$. Consequently, our coloring algorithm works if the $x_i$ are feasible for the following LP:

$$\begin{align*}
\min & \quad x_0 + x_1 + x_2 + x_3 \\
n & \quad s.t. \\
& \quad \frac{4}{5}x_1 + \frac{2}{3}x_2 + \frac{1}{2}x_3 \geq 2 \\
& \quad \frac{2}{3}x_2 + \frac{1}{2}x_3 \geq 2 \\
& \quad \frac{2}{3}x_3 \geq 2 \\
& \quad x_0, x_1, x_2, x_3 \geq 0.
\end{align*}$$

The solution $x_0 = 2, x_1 = 3/8, x_2 = 3/10, x_3 = 3$ is certainly feasible. The total number of colors used is

$$[x_0 \bar{A}(t)] + \sum_{i=1}^{3} [x_i W(t)] \leq (x_0 + x_1 + x_2 + x_3) \text{OPT}(t) + \frac{7}{8} + \frac{9}{10} \leq 5.675 \text{OPT}(t) + 1.8.$$ 

As is customary in online/dynamic algorithm analysis, we ignore the constant term of 1.8, as we let $\text{OPT}(t) \to \infty$. To prove the better ratio 5.6355, divide the rates into 5 types belonging to the intervals $(1/2, 1], (2/5, 1/2), (1/3, 2/5], (11/43, 1/3], and (0, 11/43].$
Corollary 3.5. The Clos network \( C(n, m, r) \) is multirate WSNB if \( m \geq 5.6355n + 4 \).

Proof. Consider the multirate WSNB problem on the Clos network \( C(n, m, r) \). We formulate a \( DWEC \) instance generalizing the problem. The base graph is the complete bipartite graph \( G = \mathcal{I} \times \mathcal{O} \), where \( \mathcal{I} \) is the set of input crossbars and \( \mathcal{O} \) is the set of output crossbars. When a new request \((a, b, w)\) arrives at time \( t \), add an edge \( e = (I, O) \) to \( G_{t-1} \) where \( I \) is the input crossbar to which \( a \) belongs and \( O \) is the output crossbar to which \( b \) belongs. Set the edge weight \( w_e = w \). Think of each middle-crossbar as a color. Obviously, the maximum number of colors ever used by an algorithm \( A \) is also a sufficient number of middle crossbars needed for \( C(n, m, r) \) to be non-blocking.

In the above algorithm, \( \Delta(t) \leq n \) because the number of requests with rate \( > 1/2 \) coming out of the same input crossbar or into the same output crossbar is at most \( n \) (one per input/output). Moreover, \( \overline{W}(t) \leq n \) because the total rate of requests from/to an input/output is at most \( n \). Hence, the number of middle-stage crossbars (i.e. colors) needed is at most \( 5.6355n + 4 \).

Remark 3.6. Our strategy can also give a better sufficient condition than the best known in [11] for the case when there’s internal speedup in the Clos network. However, for the ease of exposition, we refrain from stating the most general result we can prove.

4 Analyzing \( f \)-cast wide-sense nonblocking multilog networks

Let \( f, t \) be given integers with \( 0 \leq t \leq n \), and \( 1 \leq f \leq N = d^n \). This section analyzes \( f \)-cast wide-sense nonblocking \( \log_d(N, 0, m) \) networks under the window algorithm with window size \( d^t \). The algorithm was proposed and analyzed for one window size \( d^{|n/2|} \) in [31], and later analyzed more carefully for varying window sizes in [7]. Both papers considered the multicast case with no fanout restriction. We will derive a more general theorem for the \( f \)-cast case.

- The Window Algorithm with window size \( d^t \): Given any integer \( t, 0 \leq t \leq n \), divide the outputs into “windows” of size \( d^t \) each. Each window consists of all outputs sharing a prefix of length \( n - t \), for a total of \( d^n - t \) windows. Denote the windows by \( W_w, 0 \leq w \leq d^n - t - 1 \). Given a new multicast request \((a, B)\), where \( a \) is an input and \( B \) is a subset of outputs, the routing rule is, for every \( 0 \leq w \leq d^n - t - 1 \), the subrequest \((a, B \cap W_w)\) is routed entirely on one single \( \mathcal{B}Y^{-1}(n) \)-plane. (Different sub-requests can be routed through the same or different \( \mathcal{B}Y^{-1}(n) \)-planes.)

Remark 4.1. There is a subtle point about the window algorithm due to which the original authors in [31] thought their multilog network was strictly nonblocking instead of wide-sense nonblocking. Basically, for some specific values of the parameters the algorithm is no algorithm at all. In those cases, any sufficient condition for the network to be nonblocking under the window algorithm is in fact a strictly nonblocking condition, not a wide-sense nonblocking condition.

For example, in the unicast case we have \( f = 1 \), which means the window algorithm does not specify any routing strategy; consequently, any nonblocking condition is actually a strictly non-blocking condition. Another example is when \( t = 0 \). In this case, the routing rule says that each branch of \( a \) (multicast) request should be routed on some plane, independent of other branches. Because there is no restriction on how to route the branches, any nonblocking condition is a strictly non-blocking one.

Yet another example is when \( t = n \). Here, the routing rule is for each request to be routed entirely on some plane. If the \( 1 \times m \)-SE stage of the multilog network has fanout capability, then the rule does restrict how we route requests, and thus we indeed have a wide-sense nonblocking situation. However, if the \( 1 \times m \)-SE stage is implemented with \( 1 \times m \)-unicast crossbars or \( 1 \times m \)-demultiplexers, then we have to route each request entirely on some plane. Thus, any sufficient condition is a strictly nonblocking condition.

4.1 Setting up the linear program and its dual

Let \((a, B)\) be an arbitrary \( f \)-cast request to be routed using the window algorithm with window size \( d^t \). Following the window algorithm, due to symmetry without loss of generality we can assume that \( B = \{b^{(1)}, \ldots, b^{(k)}\} \) where all the outputs \( b^{(l)} \) \((l \in [k])\) belong to the same window \( W_0 \), and \( k \leq \min\{f, d^t\} \). The \( b^{(l)} \) thus share a common prefix
of length $n - t$. (This is because subrequests to the same window are routed through the same plane and different subrequests of the same request are routed independently from each other and they do not block one another.)

For each $i \in \{0, \ldots, n - 1\}$, let $A_i$ be the set of inputs $u$ other than $a$, where $u_{1..n-1}$ shares a suffix of length exactly $i$ with $a_{1..n-1}$. Formally, define

$$A_i := \{ u \in \mathbb{Z}_d^n - \{a\} \mid \text{SUF}(u_{1..n-1}, a_{1..n-1}) = i \}.$$ 

For each $j \in \{0, \ldots, n - 1\}$, let $B_j$ be the set of outputs other than those in $B$ which share a prefix of length exactly $j$ with some member of $B$, namely

$$B_j := \left\{ v \in \mathbb{Z}_d^n - B \mid \exists l \in [k], \text{PRE}(v_{1..n-1}, b^{(l)}_{1..n-1}) = j \right\}.$$ 

Note that

$$|A_i| = d^{n-i} - d^{n-1-i}, 0 \leq i \leq n - 1,$$

$$|B_j| = d^{n-j} - d^{n-1-j}, 0 \leq j \leq n - t - 1.$$

Define $A = \bigcup_{i=0}^{n-1} A_i$. It is easy to see that

$$\bigcup_{j=0}^{n-t-1} B_j = \bigcup_{w=1}^{d^{n-1}} W_w,$$

$$\bigcup_{j=n-t}^{n-1} B_j = W_0 - B.$$ 

Furthermore, for each $j \leq n - t - 1$, $B_j$ is the disjoint union of precisely $d^{n-j-t} - d^{n-1-j-t}$ windows each of size $d^t$.

Note that the sets $B_j$ for $0 \leq j \leq n - t - 1$ are mutually disjoint. On the other hand, the sets $B_j$ for $n - t \leq j \leq n - 1$ are not necessarily disjoint, because for the same output $v \in W_0 - B$ it might be the case that $\text{PRE}(v_{1..n-1}, b^{(l)}_{1..n-1}) = j$ and $\text{PRE}(v_{1..n-1}, b^{(l')}_{1..n-1}) = j'$ for $j \neq j'$, $l \neq l'$. The following simple observation turns out to be an important analytical detail in many of the proofs.

**Proposition 4.2.** Let $q$ be an integer such that $n - t \leq q \leq n - 1$. Then,

$$\left| \bigcup_{j=q}^{n-1} B_j \right| \leq \min\{d^q - k, k(d^{n-q} - 1)\},$$

and

$$\left| \bigcup_{j=n-t}^{n-1} B_j \right| = d^t - k.$$ 

**Proof.** To see the inequality, note that $\left| \bigcup_{j=q}^{n-1} B_j \right|$ counts the number of strings $v$ in $W_0 - B$ for which

$$\text{PRE}(v_{1..n-1}, b^{(l)}_{1..n-1}) \geq q$$

for some $b^{(l)}, l \in [k]$. As $|W_0| = d^t$, the upper-bound $d^t - k$ for the number of such strings is trivial. On the other hand, the number of strings $v$ where $\text{PRE}(v_{1..n-1}, b^{(l)}_{1..n-1}) \geq q$ for a fixed string $b^{(l)}$ is at most $d^{n-q} - 1$ (discounting $b^{(l)}$ itself). Hence, we get the upper-bound $k(d^{n-q} - 1)$ via a simple application of the union bound. The equality trivially holds because $\bigcup_{j=n-t}^{n-1} B_j = W_0 - B$. 

\[ \square \]
For every input $u \in A$, let $i(u)$ denote the index $i$ such that $u \in A_i$. For every $w \in [d^{n-t} - 1] = \{1, \ldots, d^{n-t} - 1\}$, let $j(w)$ be the index $j$ such that $W_w \subseteq B_j$. For every $v \in W_0 - B$, let $j(v)$ denote the largest $j$ for which $v \in B_j$. Note that $j(v) \geq n - t$ for such output $v$ because $W_0 - B = \bigcup_{j=n-t}^{n-1} B_j$.

**Lemma 4.3.** For each input $u \in A$ and each $w \in [d^{n-t} - 1]$ such that $i(u) + j(w) \geq n$, define a variable $x_{u,w}$. Also, for each input $u \in A$ and each output $v \in W_0 - B$ such that $i(u) + j(v) \geq n$, define a variable $x_{u,v}$. Then, the number of Banyan planes blocking the new multicast request $(a, B)$ is upperbounded by the optimal value of the following linear program:

\[
\begin{align*}
\max & \sum_{u,w} x_{u,w} + \sum_{u,v} x_{u,v} \\
\text{s.t.} & \sum_{u} x_{u,w} \leq d^t \quad w \in [d^{n-t} - 1] \\
& x_{u,w} \leq 1 \quad \forall u, w \\
& \sum_{v} x_{u,v} \leq 1 \quad \forall u \in A \\
& \sum_{u} x_{u,v} \leq 1 \quad \forall v \in W_0 - B \\
& \sum_{w} x_{u,w} + \sum_{v} x_{u,v} \geq f \quad \forall u \in A \\
& x_{u,w}, x_{u,v} \geq 0 \quad \forall u, w, v 
\end{align*}
\]

(4)

Obviously, the sums and the constraints only range over values for which the variables are defined.

**Proof.** Suppose the network $\log_d(N, 0, m)$ already had some routes established. Consider a BY$^{-1}(n)$-plane which blocks the new request $(a, B)$. There must be one route $R(u, v)$ on this plane for which $R(u, v)$ and $R(a, b^{(l)})$ share a link, for some $l \in [k]$. Note that the branch $R(u, v)$ could be part of a multicast tree from input $u$, but we only need an arbitrary branching branch $(u, v)$ of this tree. Note also that $u \neq a$ because subrequests from the same input are parts of the same request and thus their routes do not block one another. Let $S$ be the set constructed by arbitrarily taking exactly one blocking branch $(u, v)$ per blocking plane. Then, the number of blocking planes is $|S|$. 

**Fact 1:** if $(u, v)$ and $(u', v')$ are both in $S$ then $v$ and $v'$ must belong to different windows; because, if they belong to the same window, the window algorithm would have routed them through the same plane, and $S$ only contains one branch per blocking plane.

**Fact 2:** each output $v$ can only appear once in $S$, because each output can only be part of at most one existing request.

**Fact 3:** if $(u, v) \in S$, then $(u, v) \in A_i \times B_j$ for some $i + j \geq n$, thanks to Proposition 2.2.

Straightforwardly, we will show that $S$ defines a feasible solution to the linear program with objective value precisely $|S|$. Set $x_{u,w} = 1$ if there is some $(u, v) \in S$ such that $v \in W_w$; and $x_{u,v} = 1$ if there is some $(u, v) \in S$ such that $v \in W_0 - B$. All other variables are set to 0. Due to Fact 3, the procedure does not set value for an undefined variable. Certainly $|S|$ is equal to the objective value of this solution.

We next verify that the solution satisfies all the constraints. The first constraint expresses the fact that each output in a window $W_w$ of size $d^t$ only appears at most once in $S$ (Fact 2). The second and third constraints are a restatement of Fact 1. Note that the sumin the third constraint is only over $v \in W_0 - B$. The fourth constraint says that each output $v \in W_0 - B$ appears at most once in $S$ (Fact 2 again). The fifth constraint says that each input can only be part of at most $f$ members of $S$, due to the $f$-cast nature of the network.

The dual linear program can be written as follows.

\[
\begin{align*}
\min & \sum_{w} d^t \alpha_w + \sum_{u,w} \beta_{u,w} + \sum_{u} \gamma_u + \sum_{v} \delta_v + \sum_{u} f \epsilon_u \\
\text{s.t.} & \alpha_w + \beta_{u,w} + \epsilon_u \geq 1, \quad x_{u,w} \text{ defined (DC-1)} \\
& \gamma_u + \delta_v + \epsilon_u \geq 1, \quad x_{u,v} \text{ defined (DC-2)} \\
& \alpha_w, \beta_{u,w}, \gamma_u, \delta_v, \epsilon_u \geq 0 \quad \forall u, v, w
\end{align*}
\]

(5)

Note that the dual-constraints only exist over all $u, v, w$ for which $x_{u,w}$ and $x_{u,v}$ are defined, in particular they exist for pairs $(u, w)$ such that $i(u) + j(w) \geq n$ and pairs $(u, v)$ such that $i(u) + j(v) \geq n$. 


4.2 Specifying a family of dual-feasible solutions

To illustrate the technique, let us first derive a couple of known results “for free.” The first is Theorem III.2 in [35].

Corollary 4.4 (Theorem III.2 in [35]). Let \( r = \lfloor \log_d f \rfloor \). Suppose the \( 1 \times m \)-SE stage of the \( \log_d(N, 0, m) \) network does not have fanout capability, then when \( f \leq d^{n-2} \) the network is \( f \)-cast strictly non-blocking if

\[
m \geq d^{\frac{n-r}{2}} + f \left( d^{\frac{n-r}{2}} - 1 \right).
\]

When \( f > d^{n-2} \) the network is \( f \)-cast strictly non-blocking if \( m \geq d^{n-1} \).

Proof. Recall Remark 4.1: routing using the window algorithm with window size \( t = n \) is the same as routing arbitrarily in the network when the \( 1 \times m \)-SE stage cannot fanout. Thus any sufficient condition for the window algorithm to work is a strictly nonblocking condition. Note that when \( t = n \) the dual constraints (DC-1) do not exist! We construct a feasible solution to the dual linear program as follows.

When \( f > d^{n-2} \), set \( \gamma_u = 1 \) for all \( u \in \bigcup_{i=1}^{n-1} A_i \), and all other variables to be 0. The dual objective value in this case is

\[
\sum_{u \in \bigcup_{i=1}^{n-1} A_i} \gamma_u = \sum_{i=1}^{n-1} |A_i| = \sum_{i=1}^{n-1} (d^{n-i} - d^{n-i-1}) = d^{n-1} - 1,
\]

and hence one more plane (i.e. \( m \geq d^{n-1} \)) is sufficient. Note that this solution is dual feasible, because for \( u \in A_0 \) there is no \( v \) for which \( i(u) + j(v) \geq n \). In other words, there is no dual constraint for which \( u \in A_0 \).

Next, suppose \( f \leq d^{n-2} \). Define \( q = \lfloor \frac{n-r}{2} \rfloor + 1 \). Note that \( r+1 \leq q \leq n-1 \) in this case; in particular, \( kd^{n-q} < d^{r+1}d^{n-q} \leq d^n \), which implies \( \min\{d^n - k, k(d^{n-q} - 1)\} = kd^{n-q} \). Set \( \gamma_u = 1 \) for all \( u \) with \( i(u) \geq n-q+1 \) and \( \delta_v = 1 \) for all \( v \in \bigcup_{j=q}^{n-1} B_j \). All other dual variables are 0. The solution is dual feasible because, for any pair \((u, v)\) for which \( i(u) + j(v) \geq n \), we must either have \( i(u) \geq n-q+1 \) or \( j(v) \geq q \) (which is the same as saying \( v \in \bigcup_{j=q}^{n-1} B_j \)). Recalling Proposition 4.2, the dual objective value is

\[
\sum_{u : i(u) \geq n-q+1} \gamma_u + \sum_{v \in \bigcup_{j=q}^{n-1} B_j} \delta_v = \sum_{i=n-q+1}^{n-1} |A_i| + \bigg| \bigcup_{j=q}^{n-1} B_j \bigg| \\
\leq d^{n-1} - 1 + \min\{d^n - k, k(d^{n-q} - 1)\} \\
= d^{n-1} - 1 + k(d^{n-q} - 1) \\
\leq d^{n-1} + f(d^{n-q} - 1) - 1,
\]

This is an upper bound on the number of blocking planes. Hence, one more plane is sufficient to route the new (arbitrary) request. \( \square \)

Because unicast is 1-cast, by setting \( r = 0 \) in the previous corollary we obtain the following corollary, whose proof was about 5 pages long in [14]. Recall remark 4.3 which ensures that our result is a strictly nonblocking condition rather than a wide-sense nonblocking one.

Corollary 4.5 (Theorem 1 in [14]). For \( \log_d(N, 0, m) \) to be unicast strictly nonblocking, it is sufficient that \( m \geq d^{(n/2)-1} + d^{(n/2)} - 1 \).

Corollary 4.4 solves the \( t = n \) case. We will consider \( 0 \leq t < n \) henceforth. We next specify a family of dual-feasible solutions to the dual-LP [3]. The main remaining task will be simple calculus as we pick the best dual-feasible solution depending on the parameters \( f, n, d, t \) of the problem.

The family of dual-feasible solution is specified with two integral parameters where \( 0 \leq p \leq n-t-1 \) and \( n-t \leq q \leq n \). The parameter \( p \) is used to set the variables \( \epsilon_u, \alpha_w \) and \( \beta_{u,w} \), and the parameter \( q \) is used to set the variables \( \gamma_u \) and \( \delta_v \). As we set the variables, we will also verify the feasibility of the constraints (DC-1) and (DC-2), and the contributions of those variables to the final objective value.
• **Specifying the \( \epsilon_u \) variables.** Set \( \epsilon_u = 1 \) if \( i(u) \geq n - p \) and 0 otherwise. The contribution of the \( \epsilon_u \) to the objective is

\[
\sum_u f\epsilon_u = \sum_{i=n-p}^{n-1} f \sum_{u:i(u)=i} 1 = \sum_{i=n-p}^{n-1} f|A_i| = f(d^p - 1).
\]

• **Specifying the \( \alpha_w \) and \( \beta_{u,w} \) variables.** Next, we define the \( \alpha_w \) and \( \beta_{u,w} \). The constraints (DC-1) with \( i(u) \geq n - p \) are already satisfied by the \( \epsilon_u \), hence we only need to set the \( \alpha_w \) and \( \beta_{u,w} \) to satisfy (DC-1) when \( j(w) \geq p + 1 \). (If \( j(w) \leq p \), then for the constraint to exist we must have \( i(u) \geq n - j(w) \geq n - p \).) The variables \( \alpha_w \) and \( \beta_{u,w} \) are set differently based on three cases as follows.

**Case 1.** If \( t \geq \lfloor \frac{n}{2} \rfloor \), then set \( \beta_{u,w} = 1 \) whenever \( p + 1 \leq j(w) \leq n - t - 1 \) and \( n - j(w) \leq i(u) \leq n - p - 1 \), and set all other \( \alpha_w \) and \( \beta_{u,w} \) to be 0. It can be verified straightforwardly that all constraints (DC-1) are satisfied. Recall that the number of windows \( W_n \) for which \( j(w) = j \) is precisely \( d^{n-t} - d^{n-j-1} \). Thus, the contributions of the \( \alpha_w \) and \( \beta_{u,w} \) to the dual objective value is

\[
\sum_{u,w} \beta_{u,w} = \sum_{p+1 \leq j(w) \leq n-t-1} \sum_{n-j(w) \leq i(u) \leq n-p-1} |\{w : j(w) = j\}| \sum_{i=n-j}^{n-p-1} |\{u \in A : i(u) = i\}|
\]

\[
= \sum_{j=p+1}^{n-t-1} (d^{n-j-t} - d^{n-j-1}) \sum_{i=n-j}^{n-p-1} |A_i|
\]

\[
= \sum_{j=p+1}^{n-t-1} (d^{n-j-t} - d^{n-j-1})(d^t - d^p)
\]

\[
= (n - t - 1 - p)(d^{n-t} - d^{n-t-1}) - d^{n-t-1} + d^p.
\]

**Case 2.** When \( p + 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1 \), set \( \beta_{u,w} = 1 \) whenever \( p + 1 \leq j(w) \leq t \) and \( n - j(w) \leq i \leq n - p - 1 \), and \( \alpha_w = 1 \) for \( t + 1 \leq j(w) \leq n - t - 1 \), and all other \( \alpha_w \) and \( \beta_{u,w} \) to be 0. All constraints (DC-1) are thus satisfied. The \( \alpha_w \)'s and \( \beta_{u,w} \)'s contributions to the objective is

\[
\sum_{w} d^t \alpha_w + \sum_{u,w} \beta_{u,w} = \sum_{j=t+1}^{n-t-1} d^t \cdot |\{w : j(w) = j\}| +
\]

\[
\sum_{j=p+1}^{t} |\{w : j(w) = j\}| \sum_{i=n-j}^{n-p-1} |\{u \in A : i(u) = i\}|
\]

\[
= \sum_{j=t+1}^{n-t-1} d^t(d^{n-j-t} - d^{n-j-1}) +
\]

\[
\sum_{j=p+1}^{t} (d^j - d^p)(d^{n-j-t} - d^{n-j-1})
\]

\[
= (t - p)(d^{n-t} - d^{n-t-1}) + d^{n+p-2t-1} - d^t.
\]

**Case 3.** When \( t \leq p \) (which is \( \leq n - t - 1 \)), set \( \alpha_w = 1 \) for \( p + 1 \leq j(w) \leq n - t - 1 \) and all the \( \beta_{u,w} \) to be
k is non-decreasing in Lemma 4.7. Let \( m \) be as small as possible. Then, derive an upperbound \( C \) using Theorem 4.6. The idea is, for a given \( k \), it is a very straightforward though somewhat analytically tedious task to derive the best possible sufficient condition.

4.3 Selecting the best dual-feasible solution

The feasibility of all the constraints (DC-2) is easy to verify. The contribution to the objective value is

\[
\sum_{p < j(w) < n-t} d^p \alpha_w = \sum_{j=p+1}^{n-t-1} d^p \cdot |\{w : j(w) = j\}|
\]

\[
= \sum_{j=p+1}^{n-t-1} d^p (d^{n-j-t} - d^{n-j-t-1})
\]

\[
= d^{n-p-1} - d^p.
\]

• Specifying the \( \gamma_u \) and \( \delta_v \) variables. Here, there are two cases

When \( q = n-t \), set \( \delta_v = 1 \) for all \( v \in \bigcup_{j=n-t}^{n-1} B_j \) and all \( \gamma_u = 0 \). The dual-objective contribution in this case is

\[
\sum_{v \in \bigcup_{j=n-t}^{n-1} B_j} \delta_v = \left| \bigcup_{j=n-t}^{n-1} B_j \right| = d^t - k.
\]

When \( n-t+1 \leq q \leq n \), define \( \delta_v = 1 \) for all \( v \in \bigcup_{j=q}^{n-1} B_j \), \( \gamma_u = 1 \) for all \( u \) such that \( n-q+1 \leq i(u) \leq n-p-1 \), and all other \( \delta_v \) and \( \gamma_u \) are set to be zero. From Proposition 4.2, the total contribution of the \( \gamma_u \) and \( \delta_v \) to the dual-objective is at most

\[
\sum_{n-q+1 \leq i(u) \leq n-p-1} \gamma_u + \sum_{v \in \bigcup_{j=q}^{n-1} B_j} \delta_v = \sum_{i=n-q+1}^{n-p-1} |\{u : i(u) = i\}| + \left| \bigcup_{j=q}^{n-1} B_j \right|
\]

\[
\leq \sum_{i=n-q+1}^{n-p-1} |A_i| + \min\{d^t - k, k(d^n - q - 1)\}
\]

\[
= d^{n-1} - d^p + \min\{d^t - k, k(d^n - q - 1)\}.
\]

The feasibility of all the constraints (DC-2) is easy to verify.

Define the “cost” \( c(k, p, q) \) to be the total contribution of all variables to the dual-objective value. We summarize the values of \( c(k, p, q) \) in Figure 5. We just proved the following.

Theorem 4.6. The above family of solutions is feasible for the dual linear program (5) with objective value equal to \( c(k, p, q) \). Consequently, for the network \( \log_d(N, 0, m) \) to be wide-sense nonblocking under the window algorithm with window size \( d^t \), it is sufficient that

\[
m \geq 1 + \max_{1 \leq k \leq \min(f, d^t)} \min_{p, q} c(k, p, q).
\]

(6)

4.3 Selecting the best dual-feasible solution

It is a very straightforward though somewhat analytically tedious task to derive the best possible sufficient condition using Theorem 4.6. The idea is, for a given \( k \leq \min(f, d^t) \), we first choose \( p = p_k, q = q_k \) so that \( c(k, p_k, q_k) \) is as small as possible. Then, derive an upperbound \( C(t, f) \geq \max_k c(k, p_k, q_k) \). The sufficient condition is then \( m \geq C(t, f) + 1 \).

We first need a technical lemma.

Lemma 4.7. Let \( d, n, k \) be positive integers, and \( x = \lfloor \log_d k \rfloor \). Then, the following function

\[
h(k) = d^\left\lfloor \frac{k}{d} \right\rfloor + k \left( d^{n-\left\lfloor \frac{k}{d} \right\rfloor} - 1 \right)
\]

(7)

is non-decreasing in \( k \).
The objective value $c(k, p, q)$

For $t \geq \lfloor \frac{n}{2} \rfloor$ and $q = n - t$,
\[
c(k, p, q) = f(d^p - 1) + (n - t - 1 - p)(d^{n-t} - d^{n-t-1}) - d^{n-t} + d^p + d^t - k.
\]
For $t \geq \lfloor \frac{n}{2} \rfloor$ and $q > n - t$,
\[
c(k, p, q) = f(d^p - 1) + (n - t - 1 - p)(d^{n-t} - d^{n-t-1}) - d^{n-t} + d^p + d^t - k.
\]
For $p + 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1$ and $q = n - t$
\[
c(k, p, q) = f(d^p - 1) + (t - p)(d^{n-t} - d^{n-t-1}) + d^{n-p+2t-1} - k.
\]
For $p + 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1$ and $q > n - t$
\[
c(k, p, q) = f(d^p - 1) + (t - p)(d^{n-t} - d^{n-t-1}) + d^{n-p+2t-1} - d^p - q > n - t\]
\[
- \min(d^t, k(d^{n-q} - 1))
\]
For $t \leq p$ and $q = n - t$,
\[
c(k, p, q) = f(d^p - 1) + d^{n-p-1} - d^t + d^q - d^p + d^t - d^q - k.
\]
For $t \leq p$ and $q > n - t$
\[
c(k, p, q) = f(d^p - 1) + d^{n-p-1} - d^t + d^q - d^p + \min(d^t, k(d^{n-q} - 1)).
\]

Figure 5: The dual objective value of the family of dual-feasible solutions.

The upper-bound $C(t, f)$

To shorten the notations, let $r = \lfloor \log_d f \rfloor$.
\[
C(t, f) = \begin{cases} 
    f \left( \left\lfloor \frac{n+x}{2} \right\rfloor - 1 \right) + d^{\left\lfloor \frac{n+x}{2} \right\rfloor} - 1 & t < \lfloor \frac{n}{2} \rfloor, r \leq n - 2t - 1 \\
    t(d - 1)d^{n-t-1} + d^{n-2t-1} - 1 & t \geq \lfloor \frac{n}{2} \rfloor, r \leq n - 2t \\
    (n - t - 1)(d - 1) - 1 |d^{n-t} - d^{n-t-1} + d^t - (d - 1)d^{2t-n-1} & t \geq \lfloor \frac{n}{2} \rfloor, r \geq n - t \\
    f \left( d^{n-t-r-1} - 1 \right) + r(d - 1)d^{n-t-1} + d^{n-t-r-1} - d^t - (d - 1)d^{2t-n-1} & r \leq \min(2t - n - 2, n - t - 1) \\
    f \left( d^{n-t-r-1} - 1 \right) + r(d - 1)d^{n-t-1} + d\left\lfloor \frac{n+x}{2} \right\rfloor + f \left( d^{\left\lfloor \frac{n+x}{2} \right\rfloor} - 1 \right) & t \geq \lfloor \frac{n}{2} \rfloor, r \geq n - t - 1 \\
\end{cases}
\]

Figure 6: We show in Theorem 4.8 that $C(t, f) \geq \max_k \min_{p, q} c(k, p, q)$

Proof. We induct on $k$. The inequality trivially holds when $k = 1$. Consider $k > 2$. First, suppose $k$ is not an exact power of $d$, i.e. $k > d^x$. In this case, we have
\[
h(k - 1) = d^{\left\lceil \frac{n+x}{2} \right\rceil} + (k - 1) \left( d^{n - \left\lfloor \frac{n+x}{2} \right\rfloor} - 1 \right)
\leq d^{\left\lceil \frac{n+x}{2} \right\rceil} + k \left( d^{n - \left\lfloor \frac{n+x}{2} \right\rfloor} - 1 \right) = h(k).
\]
Second, consider the case when $k = d^x$. It can be verified that, no matter what the parities of $n$ and $x$ are, the multiset $\{ \left\lfloor \frac{n+x}{2} \right\rfloor, \left\lceil \frac{n+x}{2} \right\rceil - 1 \}$ is exactly equal to the multiset $\{ \left\lfloor \frac{n+x}{2} \right\rceil, \left\lfloor \frac{n+x}{2} \right\rfloor - 1 \}$. Thus, noting that $\left\lceil \log_d (k-1) \right\rceil = x - 1,$
we have

\[
\begin{align*}
h(k-1) &= d^\left\lceil\frac{n+x}{2}\right\rceil + (k-1) \left(d^n - \left\lceil\frac{n+x}{2}\right\rceil - 1\right) \\
&= d^\left\lceil\frac{n+x}{2}\right\rceil + (d^x-1) \left(d^\left\lceil\frac{n-x}{2}\right\rceil - 1\right) \\
&= d^\left\lceil\frac{n+x}{2}\right\rceil + d^\left\lceil\frac{n+x}{2}\right\rceil - d^\left\lceil\frac{n-x}{2}\right\rceil - d^x + 1 \\
&\leq d^\left\lceil\frac{n+x}{2}\right\rceil + d^\left\lceil\frac{n+x}{2}\right\rceil - d^x \\
&= h(k).
\end{align*}
\]

\[\Box\]

**Theorem 4.8.** The \(\log_d(N, 0, m)\) network is nonblocking under the window algorithm with window size \(d^t\) if \(m \geq 1 + C(t, f)\) where \(C(t, f)\) is defined in Figure 8.

**Proof.** Consider 5 cases in the definition of \(C(t, f)\). We specify for each \(k\) how to set the values \(p_k\) and \(q_k\). The straightforward task of verifying that \(c(k, p_k, q_k) \leq C(t, f)\) is mostly omitted due to space constraint, except for situations when it is tricky to verify.

- **Case 1:** \(t < \left\lceil\frac{n}{2}\right\rceil\), \(r \geq n - 2t - 1\). For any \(k\), choose \(p_k = \left\lceil\frac{n-x}{2}\right\rceil\) and \(q_k = n-t\).
- **Case 2:** \(t < \left\lceil\frac{n}{2}\right\rceil\), \(r \geq n - 2t\). For any \(k\), set \(p_k = 0\) and \(q_k = n-t\).
- **Case 3:** \(t \geq \left\lceil\frac{n}{2}\right\rceil\), \(r \geq n - t\). This case is a little trickier analytically. Define \(x = \lfloor\log_d k\rfloor\). We set \(p_k\) and \(q_k\) differently depending on how large \(x\) is, so that the inequality \(c(k, p_k, q_k) \leq C(t, f)\) always holds.

If \(0 \leq x \leq 2t - n - 2\), which can only hold when \(t \geq \frac{n+1}{2}\), then set \(q_k = \left\lceil\frac{n+x}{2}\right\rceil + 1\) and \(p_k = 0\). Note that \(q_k > n-t\) and \(x+1+n-q_k < t\). Thus \(kd^{n-q_k} < d^t\). Recall from Lemma 4.7 that function \(h(k)\) defined in (7) is non-increasing, and the fact that in this case \(k \leq d^{t+1} - 1 \leq d^{2t-n-1} - 1\), we have

\[
c(k, p_k, q_k) = \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + d^x - 1 + \min\{d^t - k, (d^n - q_k - 1)\}
= \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + d^x - 1 + k(d^n - q_k - 1)
= \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + h(k)
= \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + h(d^{2t-n-1} - 1)
= \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + d^x - 1 + (d^{2t-n-1} - 1)(d^n - t - 1)
< \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + d^x - 1 + d^{2t-n-1} - 1)(d^n - t - 1)
= \left\lfloor(n-t-1)(d-1) - 1\right\rfloor d^{n-t-1} + d^x - (d-1)d^{2t-n-1}
= C(t, f).
\]

If \(x = 2t - n - 1\) and \(k \leq d^{t+1} - d^x\), then set \(q_k = \left\lfloor\frac{n+x}{2}\right\rfloor + 1 = t\) and \(p_k = 0\). If \(x = 2t - n - 1\) and \(k \geq d^{t+1} - d^x + 1\), then set \(q_k = n-t\) and \(p_k = 0\). Finally, when \(x \geq 2t - n\), we again set \(q_k = n-t\) and \(p_k = 0\).

- **Case 4:** \(t \geq \left\lceil\frac{n}{2}\right\rceil\), \(2t - n - 2 < r \leq n - t - 1\). Note that this case can only happen when \(t \leq 2n/3\). In particular, if \(t > 2n/3\) and \(r \leq n - t - 1\) we would be in case 5. Set \(p_k = n-t-r-1\) and \(q_k = \left\lceil\frac{n+x}{2}\right\rceil + 1\). Proving \(c(k, p_k, q_k) \leq C(t, f)\) is almost identical to Case 3 where we consider different ranges of \(x = \lfloor\log_d k\rfloor\).

- **Case 5:** \(t \geq \left\lceil\frac{n}{2}\right\rceil\), \(r \leq \min(2t - n - 2, n - t - 1)\). Set \(p_k = n-t-r-1\) and \(q_k = \left\lceil\frac{n+x}{2}\right\rceil + 1\). Showing \(c(k, p_k, q_k) \leq C(t, f)\) is similar to Case 3. The only slight variation is, instead of bounding \(k \leq d^{t+1} - 1\) we apply \(k \leq f\) directly. The function \(h(k)\) is then bounded by \(h(f)\). Furthermore, we do not have to consider the cases when \(x \geq 2t - n - 1\) because \(x \leq r \leq 2t - n - 2\).

\[\Box\]
4.4 Some quick consequences of Theorem 4.8

All we have to do is to plug in the parameters $t$ and $f$ and compute $1 + C(t, f)$ to get the following results.

**Corollary 4.9** (Theorem 4 in [10]). Let $r = \lfloor \log_d f \rfloor$. The network $\log_d(N, 0, m)$ is $f$-cast strictly non-blocking if

$$m \geq f \left( d^{\left\lfloor \frac{n-r}{t} \right\rfloor - 1} - 1 \right) + d^n - \left\lfloor \frac{n-r}{t} \right\rfloor - 1.$$

**Proof.** This corresponds to the $t = 0$ case of the window algorithm, which becomes a strictly nonblocking condition as noted earlier.

$$C(0, f) = f \left( d^{\left\lfloor \frac{n-r}{t} \right\rfloor - 1} - 1 \right) + d^n - \left\lfloor \frac{n-r}{t} \right\rfloor - 1.$$

The following result took about 6 pages in [7] to be proved (in two theorems) with combinatorial reasoning. The result is on the general multicast case, without the fanout restriction $f$. In our setting, we can simply set $f = N = d^n$. In fact, even though the corollary states exactly the same results as in [7], the statement is simpler.

**Corollary 4.10** (Theorems 1 and 2 in [7]). The $d$-ary multi-log network $\log_d(N, 0, m)$ is wide-sense nonblocking with respect to the window algorithm with window size $d^t$ if

$$m \geq \begin{cases} d^{n-2t-1} + td^{n-t-1}(d-1) & \text{when } t \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ d^{n-t-1}[(d-1)(n-t-1) - 1] + d^t - d^{t-r-n-1}(d-1) + 1 & \text{when } t \geq \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

5 Analyzing crosstalk-free $f$-cast wide-sense nonblocking multilog networks

When the multi-log architecture is employed to design a photonic switch, each $2 \times 2$ switching element (SE) needs to be replaced by a functionally equivalent optical component. For instance, when $d = 2$ we can use so-called directional couplers as SEs. However, directional couplers and many other optical switching elements suffer from optical crosstalk between interfering channels, which is one of the major obstacles in designing cost-effective switches. To cope with crosstalk, the crosstalk-free constraint is a common requirement, which states that no two routes can share a common SE and each output $v$ in $W_0 - B$ such that $i(u) + j(v) \geq n - 1$, define a variable $x_{u,v}$. Then, the number of Banyan planes blocking $(a, B)$ is upperbounded by the optimal value of the linear program (4), whose dual is (5).

5.1 Setting up the linear program and its dual

We use identical notations as in the previous section. The following lemma is the crosstalk-free analog of Lemma 4.3.

**Lemma 5.1.** For each input $u \in A$ and each $w \in [d^{n-t} - 1]$ such that $i(u) + j(w) \geq n - 1$, define a variable $x_{u,w}$. Also, for each input $u \in A$ and each output $v \in W_0 - B$ such that $i(u) + j(v) \geq n - 1$, define a variable $x_{u,v}$. Then, the number of Banyan planes blocking $(a, B)$ is upperbounded by the optimal value of the linear program (4), whose dual is (5).

We next derive some quick consequences of the formulation.
Corollary 5.2 (Theorem III.1 in [35]). Let \( r = \lfloor \log_{\frac d f} \rfloor \). Suppose the \( 1 \times m \)-SE stage of the \( \log_d(N, 0, m) \) network does not have fanout capability, then when \( f \leq d^{n-2}(d-1) \) the network is crosstalk-free \( f \)-cast strictly non-blocking if

\[
m \geq d\left\lfloor \frac{n-r+1}{2} \right\rfloor + f\left( d\left\lfloor \frac{n-r+1}{2} \right\rfloor - 1 \right).
\]

When \( f > d^{n-2}(d-1) \) the network is \( f \)-cast strictly nonblocking if \( m \geq d^n - d^{n-2}(d-1) \).

Proof. Routing using the window algorithm with window size \( t = n \) is the same as routing arbitrarily in the network when the \( 1 \times m \)-SE stage cannot fanout. Thus any sufficient condition for the window algorithm to work is an strictly nonblocking condition. Note that when \( t = n \) the dual constraints (DC-1) do not exist. Consider a solution to the dual LP as follows.

When \( f > d^{n-2}(d-1) \), consider two cases. If \( k > d^{n-2}(d-1) \), set \( \delta_v = 1 \) for all \( v \in \bigcup_{j=0}^{n-1} B_j \) and all other variables to be 0. Then, the dual objective value is

\[
\sum_{v \in \bigcup_{j=0}^{n-1} B_j} \delta_v = n \left( \bigcup_{j=0}^{n-1} B_j \right) = d^n - k \leq d^n - d^{n-2}(d-1) - 1.
\]

Thus, in this case \( d^n - d^{n-2}(d-1) \) Banyan planes is sufficient. Next, suppose \( k \leq d^{n-2}(d-1) \), in which case \( kd < d^n \). Set \( \gamma_u = 1 \) for all \( u \) with \( i(u) \geq 1 \), \( \delta_v = 1 \) for all \( v \in B_{n-1} \), and all other variables to be 0. The solution is dual-feasible with dual objective value

\[
\sum_{u : i(u) \geq 1} \gamma_u + \sum_{v \in B_{n-1}} \delta_v = \sum_{i=1}^{n-1} |A_i| + |B_{n-1}|
\]

\[
\leq \sum_{i=1}^{n-1} (d^{n-i} - d^{n-i-1}) + \min\{d^n - k, k(d-1)\}
\]

\[
= d^{n-1} - 1 + k(d-1)
\]

\[
\leq d^n - d^{n-2}(d-1)^2 - 1
\]

\[
= d^n - d^{n-2}(d-1) - 1.
\]

and thus again \( d^n - d^{n-2}(d-1) \) Banyan planes is sufficient.

Next, consider the case when \( f \leq d^{n-2}(d-1) \). In this case \( r \leq n - 2 \). Let \( p = \left\lfloor \frac{n-r-1}{2} \right\rfloor \). Then, \( 1 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Furthermore,

\[
kdp < fdp < d^{p+1}d\left\lfloor \frac{n-r+1}{2} \right\rfloor = d\left\lfloor \frac{n+r+1}{2} \right\rfloor \leq d^n.
\]

Set \( \gamma_u = 1 \) for all \( u \) with \( i(u) \geq p \), \( \delta_v = 1 \) for all \( v \in \bigcup_{j=p}^{n-1} B_j \) and all other variables to be 0. The solution is dual feasible because, for any pair \((u, v)\) for which \( i(u) + j(v) \geq n - 1 \), we must either have \( i(u) \geq p \) or \( j(v) \geq n - p \) (which is the same as saying \( v \in \bigcup_{j=n-p}^{n-1} B_j \)). Recalling Proposition 4.2 and the fact that \( kd^p < d^n \) shown above, the dual objective value is

\[
\sum_{u : i(u) \geq p} \gamma_u + \sum_{v \in \bigcup_{j=n-p}^{n-1} B_j} \delta_v = \sum_{i=p}^{n-1} |A_i| + \left| \bigcup_{j=n-p}^{n-1} B_j \right|
\]

\[
\leq \sum_{i=p}^{n-1} (d^{n-i} - d^{n-i-1}) + \min\{d^n - k, k(dp-1)\}
\]

\[
= d^{n-p} - 1 + k(dp-1)
\]

\[
\leq d^{n-p} + f(dp-1) - 1
\]

Hence, in this case \( d^{n-p} + f(dp-1) \) is a sufficient number of Banyan planes. \( \square \)
5.2 Specifying a family of dual-feasible solutions

The family of dual-feasible solution is specified with two integral parameters where \( 0 \leq p \leq n - t - 1 \) and \( n - t \leq q \leq n \). The parameter \( p \) is used to set the variables \( \epsilon_u, \alpha_u \) and \( \beta_{u,w} \), and the parameter \( q \) is used to set the variables \( \gamma_u \) and \( \delta_w \). As we set the variables, we will also verify the feasibility of the constraints (DC-1) and (DC-2), and the contributions of those variables to the final objective value.

- **Specifying the \( \epsilon_u \) variables.** Set \( \epsilon_u = 1 \) if \( i(u) \geq n - p \) and 0 otherwise. The contribution of the \( \epsilon_u \) to the objective is

\[
\sum_{u} f \epsilon_u = \sum_{i=n-p}^{n-1} f \sum_{u : i(u) = i} 1 = \sum_{i=n-p}^{n-1} f |A_i| = f(d^p - 1).
\]

- **Specifying the \( \alpha_w \) and \( \beta_{u,w} \) variables.** Next, we define the \( \alpha_w \) and \( \beta_{u,w} \). The constraints (DC-1) with \( i(u) \geq n - p \) are already satisfied by the \( \epsilon_u \), hence we only need to set the \( \alpha_w \) and \( \beta_{u,w} \) to satisfy (DC-1) when \( j(w) \geq p \). (If \( j(w) \leq p - 1 \), then for the constraint to exist we must have \( i(u) \geq n - 1 - j(w) \geq n - p \).) The variables \( \alpha_w \) and \( \beta_{u,w} \) are set differently based on three cases as follows.

**Case 1.** If \( t \geq \left\lceil \frac{q}{2} \right\rceil \), then set \( \beta_{u,w} = 1 \) whenever \( p \leq j(w) \leq n - t - 1 \) and \( n - 1 - j(w) \leq i(u) \leq n - p - 1 \), and set all other \( \alpha_w \) and \( \beta_{u,w} \) to be 0. It can be verified straightforwardly that all constraints (DC-1) are satisfied. Thus, the contributions of the \( \alpha_w \) and \( \beta_{u,w} \) to the dual objective value is

\[
\sum_{u} \beta_{u,w} \sum_{\substack{p \leq j(w) \leq n-t-1 \\text{ and } n-1-j(w) \leq i(u) \leq n-p-1}} (d^{n-j-t} - d^{n-j-t-1}) \sum_{i=n-j}^{n-p} |A_i| = (n - t - p)(d^{n-t+1} - d^{n-t}) - d^{n-t} + d^t.
\]

The second equality follows from the fact that the number of windows \( W_w \) for which \( j(w) = j \) is precisely \( d^{n-j-t} - d^{n-j-t-1} \).

**Case 2.** When \( p + 1 \leq t \leq \left\lceil \frac{q}{2} \right\rceil - 1 \), set \( \beta_{u,w} = 1 \) whenever \( p \leq j(w) \leq t - 1 \) and \( n - 1 - j(w) \leq i \leq n - p - 1 \), and \( \alpha_w = 1 \) for \( t \leq j(w) \leq n - t - 1 \), and all other \( \alpha_w \) and \( \beta_{u,w} \) to be 0. All constraints (DC-1) are thus satisfied. The \( \alpha_w \)’s and \( \beta_{u,w} \)’s contributions to the objective is

\[
\sum_{w} \alpha_w d^t + \sum_{\substack{t \leq j(w) < n-t \\text{ and } p \leq j(w) \leq t-1 \\text{ and } n-1-j(w) \leq i \leq n-p-1}} \beta_{u,w} = \sum_{j=t}^{n-t-1} d^t (d^{n-j-t} - d^{n-j-t-1}) + \sum_{j=p}^{t-1} (d^{n-j-t} - d^{n-j-t-1})(d^{j+1} - d^t) = (t - p)(d^{n-t+1} - d^{n-t}) + d^{n-2t+p} - d^t.
\]

**Case 3.** When \( t \leq p \) (which is \( \leq n - t - 1 \)), set

\[
\alpha_w = \begin{cases} 1 & p \leq j(w) \leq n - t - 1 \\ 0 & \text{otherwise} \end{cases}
\]
and all the $\beta_{u,w}$ to be zero. Again, the feasibility of the constraints (DC-1) is easy to verify. The contribution to the objective value is
\[
\sum_{p \leq j(w) < n-t} d^p \alpha_w = \sum_{j=p}^{n-1} d^j \cdot |\{w : j(w) = j\}| = \sum_{j=p}^{n-1} d^j (d^{n-j-t} - d^{n-j-t-1}) = d^{n-p} - d^t.
\]

- **Specifying the $\gamma_u$ and $\delta_v$ variables.** When $q = n - t$, set $\delta_v = 1$ for all $v \in \bigcup_{j=n-t}^{n-1} B_j$ and all $\gamma_u = 0$. The dual-objective contribution in this case is
\[
\sum_{v \in \bigcup_{j=n-t}^{n-1} B_j} \delta_v = | \bigcup_{j=n-t}^{n-1} B_j | = d^t - k.
\]

When $n-t+1 \leq q \leq n$, define $\delta_v = 1$ for all $v \in \bigcup_{j=q}^{n-1} B_j$, $\gamma_u = 1$ for all $u$ such that $n-q \leq i(u) \leq n-p-1$, and all other $\delta_v$ and $\gamma_u$ are set to be zero. From Proposition 4.2, the total contribution of the $\gamma_u$ and $\delta_v$ to the dual-objective is
\[
\sum_{u \in \bigcup_{j=q}^{n-1} B_j} \gamma_u + \sum_{v \in \bigcup_{j=q}^{n-1} B_j} \delta_v = \sum_{i=n-q}^{n-p-1} |\{u : i(u) = i\}| + | \bigcup_{j=q}^{n-1} B_j | \leq \sum_{i=n-q}^{n-p-1} |A_i| + \min\{d^t - k, k(d^{n-q} - 1)\} = d^t - d^p + \min\{d^t - k, k(d^{n-q} - 1)\}.
\]

The feasibility of all the constraints (DC-2) is easy to verify.

Define the “cost” $g(k, p, q)$ to be the total contribution of all variables to the dual-objective value. We summarize the values of $g(k, p, q)$ in Figure 7. We just proved the following.

**Theorem 5.3.** The above family of solutions is feasible for the dual LP with objective value equal to $g(k, p, q)$. (Recall that, in this problem we are working on the dual constraints for which $i(u) + j(v) \geq n - 1$ and $i(u) + j(w) \geq n - 1$.) Consequently, for the network $\log_d(N, 0, m)$ to be crosstalk-free $f$-cast wide-sense nonblocking under the window algorithm with window size $d^t$, it is sufficient that
\[
m \geq 1 + \max_{1 \leq k \leq \min(f, d^t)} \min_{p,q} g(k, p, q). \quad (9)
\]

### 5.3 Selecting the best dual-feasible solution

The proof of the following technical lemma is similar to that of Lemma 4.7 and thus we omit the proof.

**Lemma 5.4.** Let $d, n, k$ be positive integers, and $x = \lfloor \log_d k \rfloor$. Then, the following function
\[
\bar{h}(k) = d^\lfloor \frac{x+1}{2} \rfloor + k \left( d^n - \lfloor \frac{x+1}{2} \rfloor - 1 \right)
\]

is non-decreasing in $k$. 

### The objective value $g(k, p, q)$

For $t \geq \left\lceil \frac{n}{2} \right\rceil$ and $q = n - t$,

\[
g(k, p, q) = f(d^p - 1) + (n - t - p)(d^{n-t+1} - d^{n-t}) - d^n + d^t - k.
\]

For $t \geq \left\lceil \frac{n}{2} \right\rceil$ and $q > n - t$,

\[
g(k, p, q) = f(d^p - 1) + (n - t - p)(d^{n-t+1} - d^{n-t}) - d^n + d^t + \min\{d^t - k, k(d^q - q - 1)\}.
\]

For $p + 1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and $q = n - t$

\[
g(k, p, q) = f(d^p - 1) + (t - p)(d^{n-t+1} - d^{n-t}) + d^n + p - 2t - k.
\]

For $p + 1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and $q > n - t$

\[
g(k, p, q) = f(d^p - 1) + (t - p)(d^{n-t+1} - d^{n-t}) + d^n + p - 2t - d^t + \min\{d^t - k, k(d^q - q - 1)\}.
\]

For $t \leq p$ and $q = n - t$,

\[
g(k, p, q) = f(d^p - 1) + d^{n-p} - d^t + d^3 + d^3 + \min\{d^3 - k, k(d^q - q - 1)\}.
\]

For $t \leq p$ and $q > n - t$,

\[
g(k, p, q) = f(d^p - 1) + d^{n-p} - d^t + d^3 + d^3 + \min\{d^3 - k, k(d^q - q - 1)\}.
\]

Figure 7: The dual objective value of the family of dual-feasible solutions.

### The upper-bound $G(t, f)$

To shorten the notations, let $r = \left\lceil \log_d f \right\rceil$.

\[
G(t, f) = \begin{cases} 
\frac{d^n}{n-t}((n-t)(d-1)-1) + d^t - d^{2t-n+1}(d-1) & t > n/2, r = \max\{2t - n - 2, n - t + 1\} \\
f(d^{n-t-r} - 1) + rd^{n-t}(d-1) - d^n + d^t[\frac{r+n+1}{2}] + f(d^n[\frac{r+n+1}{2}] - 1) & t > n/2, r \leq \min\{2t - n - 3, n - t\} \\
d^n - ((n-t)(d-1)-1) + d([\frac{r+n+1}{2}] + f(d^n[\frac{r+n+1}{2}] - 1) & t > n/2, n-t+1 \leq r \leq 2t - n - 3 \\
f(d^{n-t-r} - 1) + d^{n-r} + f(d^{n-r}(d-1) - 1) + d^t - d^{2t-n-2}(d-1) & t > n/2, 2t - n - 2 \leq r \leq n - t \\
d^{n-t}((n-t)(t-1) - 1) + d^t & t = n/2 \\
f(d^n[\frac{r+n+1}{2}] - 1) + d^n[\frac{r+n+1}{2}] - 1 & t < n/2, r \leq n - 2t \text{ and } f \leq d^{n-2t}(d-1) \\
f(d^r - 1) + d^{n-t-1}(d^t - d^t + 1) + 1 & t < n/2, r \leq n - 2t, f > d^{n-2t}(d-1) \\
f(d^{n-t-r} - 1) + (2t - n - r)(d-1)d^{n-t} + d^{2n-3t-r} - 1 & t < n/2, n - 2t + 1 \leq r \leq n - t \\
t(d-1)d^n + d^{n-2t} - 1 & t < n/2, n - t < r
\end{cases}
\]

Figure 8: We show in Theorem 5.5 that $G(t, f) \geq \max_k \min_{p, q} g(k, p, q)$

### Theorem 5.5

The $\log_d(N, 0, m)$ network is crosstalk-free nonblocking under the window algorithm with window size $d^t$ if $m \geq 1 + G(t, f)$ where $G(t, f)$ is defined in Figure 8.

**Proof.** We specify for each $k$ how to set the values $p_k$ and $q_k$. The straightforward task of verifying that $g(k, p_k, q_k) \leq G(t, f)$ is mostly omitted due to space constraint.

Suppose $t > n/2$, i.e. $2t \geq n + 1$. We consider four cases as follows. In all cases, define an integral variable $x = \left\lfloor \log_d k \right\rfloor$.

**Case 1.** $r \geq \max\{2t - n - 2, n - t + 1\}$.

If $k \geq d^{2t-n-2}(d-1) + 1$, then pick $p_k = 0$ and $q_k = n - t$. On the other hand, if $k \leq d^{2t-n-2}(d-1)$ then we pick $p_k = 0$ and $q_k = \left\lfloor \frac{r+n+1}{2} \right\rfloor > n - t$. Note that $kd^{n-q_k} \leq d^t$. Thus, recall from Lemma 5.4 that the
function $\bar{h}(k)$ defined in [10] is non-decreasing in $k$, we have

\[ g(k, p_k, q_k) = (n-t)(d^{n-t+1} - d^{n-t}) - d^{n-t} + d^{\frac{\bar{h}(k)}{2}} + \min \left\{ d^t - k, k \left( d^n - \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \right) \right\} \]

\[ = (n-t)(d^{n-t+1} - d^{n-t}) - d^{n-t} + d^{\frac{\bar{h}(k)}{2}} + k \left( d^n - \left\lfloor \frac{n+1}{2} \right\rfloor \right) \]

\[ = (n-t)(d^{n-t+1} - d^{n-t}) - d^{n-t} + \bar{h}(k) \]

\[ \leq (n-t)(d^{n-t+1} - d^{n-t}) - d^{n-t} + \bar{h}(d^{2t-n-2}(d-1)) \]

\[ = d^{n-t}[\min(2t - n - 3, n - t - 1) - t] + d^t - d^{2t-n+1}(d-1). \]

- **Case 2.** $r \leq \min\{2t - n - 3, n - t\}$, set $p_k = n - t - r \leq t - 1$ and $q_k = \left\lfloor \frac{n+1}{2} \right\rfloor > n - t$.

- **Case 3.** $n - t + 1 \leq r \leq 2t - n - 3$. This is case we set $p_k = 0$ and $q_k = \left\lfloor \frac{n+1}{2} \right\rfloor > n - t$.

- **Case 4.** $2t - n - 2 \leq r \leq n - t$. In this case we set $p_k = n - t - r$ and consider two sub-cases as in case 1: $k \geq d^{2t-n-2}(d-1) + 1$ or $k \leq d^{2t-n-2}(d-1)$.

When $t = n/2$, we simply set $q_k = n - t$ and $p_k = 0$. Then,

\[ g(k, p_k, q_k) = d^{n-t}[(n-t)(d-1) - 1] + d^t. \]

Finally, suppose $t < n/2$. We will always pick $p_k = q - t$ in this situation. Also consider four cases:

- **Case 1.** $r \leq n - 2t$ and $f \leq d^{n-2t}(d-1)$, set $p_k = \left\lfloor \frac{n-r-1}{2} \right\rfloor \geq t$.

- **Case 2.** $r \leq n - 2t$ and $f > d^{n-2t}(d-1)$, set $p_k = t - 1$.

- **Case 3.** $n - 2t + 1 \leq r \leq n - t$, set $p_k = n - t - r$.

- **Case 4.** $n - t < r$, set $p_k = 0$.

\[ \square \]

**Corollary 5.6 (Theorems 1 in [23]).** The $d$-ary multi-log network $\log_d(N, 0, m)$ is crosstalk-free wide-sense non-blocking with respect to the window algorithm with window size $d^t$ if

\[ m \geq \begin{cases} 
  d^{n-2t} + td^{n-t}(d-1) & t < n/2 \\
  d^{n-t}[(n-t)(d-1) - 1] + d^t + 1 & t = n/2 \\
  d^{n-t}[(n-t)(d-1) - 1] + d^t - d^{2t-n-2}(d-1) + 1 & t > n/2.
\end{cases} \]

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