Development of a Digital Model of a Beta Function Based on Mellin's Integral Transforms

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Abstract. The implementation of a digital model of the beta function for use in computer algorithms is a time-consuming task. This is due to the complexity of the high-precision representation of its integrand functions, which require a large number of intermediate operations, which entails a large load on the computational power. Purpose: Development of the basic theoretical provisions of the integral Mellin transform in relation to the theory of signal processing against the background of noise and research of their discrete representation. Results: It is shown in the paper that the beta function can be considered as a special case of Mellin's integral transforms. Based on this statement, a mathematical model of the beta function was developed. Using the properties of parametrically periodic oscillations belonging to the class of trigonometric-logarithmic functions, it was possible to create a digital model for representing the beta function. Practical relevance: Based on the established digital model can be realized a high-speed algorithm for calculating the beta function with a given accuracy. Such algorithms can serve as a basis for creating signal processing programs in order to detect wideband phase-shift keyed signals against a background of noise with an unknown phase sequence. An example of using such algorithms is the search for Wi-Fi bugs.

1. Introduction

These The integral transforms of Fourier, Hilbert, Mellin played an important role in the development of the theory and applications of signal processing against the background of noise. Their introduction into the theory has enriched the functionality of the signal theory, expanding the area of their application. In modern times, various methods of spectral engineering have become the working tools of engineers and scientists. The Hilbert transform made it possible to solve the problem of finding the complex envelope of a wide class of broadband signals. And this, in turn, made it possible to use methods of digital representation of signals and methods of digital processing, which led to the creation of a new class of signal processing devices. The use of the integral Mellin transforms (MT) for signal processing provides the scale invariance of the decision rules to changes in the duration of the input signal.

In a number of works, the mathematical foundations of the MT are considered, which make it possible to ensure its application for signal processing. However, the results obtained by mathematicians did not become the property of a wide range of development engineers due to the complexity of the mathematical apparatus, and were not adapted to the theory of signal processing.
Therefore, it seems relevant to solve the problem of creating a digital model of the beta function for its implementation in computer systems. The complexity of the digital implementation of the model lies in its high-precision representation of the integrand functions that require so many operations that it becomes a problem even for a modern computer base. All issues of creating such digital models were considered in detail in the work [8].

In this work, the authors adapt the theory they obtained in relation to the digital model of the beta function.

Let’s briefly examine the theoretical basis of the integral Mellin transforms. A pair of integral transforms is given as a direct transform:

\[ M(s) = \int_{x} f(x)\theta(s,x)dx, x \in X, \] (1)

and reverse transformation

\[ f(x) = \int_{s} M(s)\theta^{-1}(x,s)ds, s \in S, \] (2)

where

\[ \theta(s,x) \] is the kernel of the direct transformation,
\[ \theta^{-1}(x,s) \] is the kernel of the inverse transformation,
\[ x \in X \] is the domain of the variable x, where the integral (1) exists.

That is, the pair of integral transformations (1) and (2) is the operator of the mapping of the variable x, when it is defined in the domain of existence of the integral (1) X, to the domain of the variable s, existing in the domain of integral S (2) [9,10,11].

As shown in [8], for further analysis, equation (1) can be written in the form:

\[ M(u) = \int_{0}^{\infty} f(t)\cos(u\ln(t))dt + j\int_{0}^{\infty} f(t)\sin(u\ln(t))dt \]

2. Basis function generator modeling

The studies carried out by the authors of the work showed that the main difficulty of the model is the creation of a basis functions of the form below:

\[ \varphi(t) = \frac{\sin(u\ln(t))}{\sqrt{t}}. \]

Let’s define two limits:

\[ \lim_{t \to 1} \frac{\sin(u\ln(t))}{\sqrt{t}} = 0, \]
\[ \lim_{t \to \infty} \frac{\sin(u\ln(t))}{\sqrt{t}} = 0. \]

Let’s investigate the zeros of the function \( \varphi(t) \):

\[ \sin(u\ln(t)) = 0 \]
\[ u\ln(t) = n\pi, \forall n \in \mathbb{Z} \]
\[ t_{0} = e^\frac{mn}{u}, \text{if} \ t > 1 \]
\[ t_{0} = e^\frac{-mn}{u}, \text{if} \ t < 1. \]
Consider a figure depicting one period $\phi(t)$ for $t<1$

Theorem 1. For the function $\frac{\sin(u \pi t)}{\sqrt{u}}$, for $t>0$, and $u \in (u_{\text{max}}, u_{\text{min}})$, for any finite $u$, all half-periods of the function will be shifted by a constant value $e^{m \pi / u}$, $m = 1, 2, 3 \ldots M$.

Evidence.

Figure 1 shows that the full period will be equal to $T_{01} = T_1 + T_2$

On the other hand, for the previous half-period, we write $T_1 = e^{u} - e^{-u}$, and for the next half-period $T_2 = e^{-u} - e^{-u}$, then the ratio $T_2/T_1$ we get:

$$\frac{T_2}{T_1} = \frac{e^{(n-1)\pi u}}{e^{n\pi u}} - \frac{e^{(n-2)\pi u}}{e^{n\pi u}} = e^{\pi u}.$$ 

Or $T_2 = T_1 e^{\pi u}$, then for $T_{01}$ we have:

$$T_{01} = T_1 + T_1 e^{\pi u} = T_1 (1 + e^{\pi u}).$$

Similarly, for the ratio $T_3/T_2$, we obtain

$$\frac{T_3}{T_2} = e^{\pi u},$$

$$T_{02} = T_3 + T_4 = T_1 e^{2\pi u} + T_1 e^{3\pi u} = T_1 e^{\pi u} (1 + e^{\pi u}) = T_{01} e^{\pi u}$$

$$T_{0m} = T_{01} e^{m\pi u} \text{ Q.E.D.}$$

Express $e^{m \pi u}$ in terms of recursion:

$$U[m] = \rho U[m - 1],$$

$$m = 1 e^{\pi u} = e^{\pi u} e^0 = \rho,$$

$$m = 2 e^{2\pi u} = e^{\pi u} e^{\pi u} = \rho \rho = \rho^2,$$

For

$$m = 3 e^{3\pi u} = e^{\pi u} e^{2\pi u} = \rho \rho^2 = \rho^3,$$

$$\ldots \ldots \ldots \ldots \ldots \ldots$$

$$m = \rho l = 1, 2, 3 \ldots$$

**Figure 1.** Period of the function $\phi(t)$. 
Thus, for the generator of basis functions, it is necessary to calculate one value \( \rho = e^{\pi i} \) for a given \( u \), and then, by successive multiplication, calculate all the coordinates of the zeros of the \( u \)-th harmonic.

### 3. Development of a digital model of the beta function

The Euler integral of the first kind, hereinafter the beta function, is written in two equivalent ways:

\[
B(s, z) = \int_0^1 x^{s-1}(1 - x)^{z-1} dx \quad (3)
\]

\[
x \geq 0; s = \sigma + j\mu; z = \delta + j\nu; \Re\{\sigma\}, \Re\{\delta\} > 0;
\]

\[
u, \mu \in (-\infty, \infty), \quad B(s, z) = \int_0^\infty \frac{y^{\delta-1}}{(1+y)^{\sigma+z}} dy, \ y > 0 \quad (4)
\]

For all \( s \) from all area \( \Re\{s\} > 0 \) and any \( z \) from the area \( \Re\{z\} > 0 \), the equality holds:

\[
(s, z) = \frac{\Gamma(s)\Gamma(z)}{\Gamma(s+z)}, \quad (5)
\]

where

\[
\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx, \quad (6)
\]

\( p = \alpha + j\beta; \ \alpha_1 < \alpha < \alpha_2; \ \beta \in (-\infty, \infty) \)

\( \Gamma(p) \) - is called the gamma function or Euler integral of the second kind.

It is known that the gamma function is a special case of the Mellin transform (MT) [16,17]. Accordingly, the beta function is also the MT of the function \((1 - x)^{z-1}\). Indeed, the direct and inverse MT can be written in the form (7) and (8).

\[
M(S) = \int_0^\infty f(t) t^{s-1} dt, s = \sigma + j\mu; \ \sigma_1 < \sigma < \sigma_2; \quad (7)
\]

\[
u, \mu \in (-\infty, \infty); t > 0.
\]

\[
f(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} M(S) t^{-S} dS. \quad (8)
\]

Substituting \( f(t) = (1 - t)^{z-1} \) from (1), we obtain the beta function

\[
B(s, z) = \int_0^1 (1 - t)^{z-1} t^{s-1} dt. \quad (9)
\]

When developing a digital model (3), it is necessary to conduct additional studies of the function \((1 - x)^{z-1}\). Since for the integral core of the MT \( x^{s-1} \) was created a theory of its digital representation, then combining the theory of digital representation \((1 - x)^{z-1}\) and the theory of \( x^{s-1} \), we obtain a digital model (3).

Let’s transform the function \((1 - t)^{\delta-1+j\nu}\) into a form, convenient for further analysis.

\[
B(1 - t)^{\delta-1+j\nu} = (1 - t)^{\delta-1}(\cos(\nu \ln(1 - t)) + j \sin(\nu \ln(1 - t))). \quad (10)
\]

At \( t = 0 \) function (10) is equal to one,

For \( t = 1 \), function (10) is not defined

This case for \( t \to 1 \) is our subject to additional research. As shown in [8], in this case the reference points are the zeros of the functions:

\[
\sin(\nu \ln(1 - t)), \quad (11)
\]

\[
\cos(\nu \ln(1 - t)). \quad (12)
\]

For definiteness, consider a sine trigonometric logarithmic function.

Obviously, the equality \( \nu \ln(1 - t) = n\pi, \ \forall n = 0, 1, 2, .. \)
Then \( t \) for which the fulfillment of this equality is equal to:

\[
t_0 = 1 - e^{-\pi \nu}. \tag{13}
\]

For the considered limit of integration (3) in the interval 0,1 for (13) we have:

\[
t_0 = 1 - e^{-n \pi \nu}. \tag{14}
\]

Let \( \nu=1 \) in (14), then for the first zero in (14) we obtain:

\[
t_{01} = 1 - e^{-n} = 1 - e^0 = 1 \quad (n = 0). \tag{15}
\]

Second zero of the period:

\[
t_{02} = 1 - e^{-n} \approx 0.956786. \tag{16}
\]

Third zero of the period:

\[
t_{03} = 1 - e^{-2\pi} \approx 0.998133. \tag{17}
\]

For \( n + 1 \) zero:

\[
t_{0n+1} = 1 - e^{\pi n} \quad \forall n = 0,1,2,.. \tag{18}
\]

For the first period, we write:

\[
T_1 = t_{02} - t_{01} = 1 - e^{-2\pi} \tag{19}
\]

For the second period, we get:

\[
T_2 = t_{04} - t_{03} = 1 - e^{-4\pi} - 1 + e^{-2\pi} = e^{-2\pi}(1 - e^{-2\pi}) = T_1 e^{-2\pi} \tag{20}
\]

Generalizing for the \( n \)-th period:

\[
T_n = T_{n-1} e^{-2\pi(n-1)} \quad \forall n = 1,2,3,4,.. \tag{21}
\]

or

\[
T_n = T_1 e^{-2\pi(n-1)} \quad n = 1,2,3,.. \tag{22}
\]

It follows from (19), (20), (21) that the function under consideration has the character of a parametrically periodic trigonometric logarithmic function. Thus, for all periods of the MT harmonic equal to \( \nu=1 \), it is necessary to calculate the value of the first period \( T_1 = 1 - e^{-2\pi} \), and then, calculating \( e^{-2\pi(n-1)} \) for \( n=1,2,3,.. \) we can get the periods of the function \( (1 - x)^{2-1} \) for \( x \in [0,1) \), ie

\[
T_1 = 1 - e^{-2\pi},
T_2 = T_1 e^{-2\pi},
T_3 = T_1 e^{-4\pi},
T_4 = T_1 e^{-6\pi},
T_n = T_1 e^{-2\pi(n-1)}; \text{ for } n = 1,2,3,.. \tag{23}
\]

The next step is to determine the size of the sampling step \( \Delta x \) for the first period of the first harmonic.

It can be determined based on the conditions for applying the Kotelnikov theorem[18]:

\[
\Delta x = \frac{1}{2F_b}, \tag{24}
\]

where \( F_b \) is the upper frequency value of the spectrum of the signal representation.

Since in the analyzed signal \( F_b = \infty \), it is naturally necessary to choose the value of computer infinity[19,20]. This can be done from a given error in representing the area of the period:

\[
\delta = \frac{s_{u}-s_{ol}}{s_u} = 1 - \frac{s_{sa}}{s_u}, \tag{25}
\]
where $S_u$ - the true value; 
$S_{\text{ou}}$ - estimated value.

If $\delta_{\rho\Omega n}$ is given, then it will be equal to

$$\delta_{\rho\Omega n} = 1 - \frac{S_{\text{ou}}}{S_u},$$

then

$$S_{\text{ou}} = \left(1 - \delta_{\rho\Omega n}\right)S_u. \quad (25)$$

And $S_{\text{ou}}$ is a function $\Delta x$, determining it we get the number of samples for the first period:

$$N = \frac{T_1}{\Delta x}.$$ 

After that, for the second period, in order to leave the number of samples equal to $N$, it is necessary to multiply $\Delta x$ by the factor of increasing $T_2$.

$$\Delta x_2 = \Delta x e^{-2\pi},$$

Similarly for the $n$-th period:

$$\Delta x_n = \Delta x e^{-2\pi(n-1)}, \quad (26)$$

Now it is necessary to extend the obtained algorithm to harmonics $v=2,3, \ldots, M$. Extending (15), (16), (17), (18) to $m$ harmonics: $m = 1,2,3, \ldots, M$, where $M$ is the number of analyzed harmonics, we get:

$$T_{1m} = 1 - e^{-\frac{2\pi}{m}}, \quad (27)$$

For example:

$$m = 1, T_{11} = 1 - e^{-\frac{2\pi}{1}} = 1 - e^{-2\pi};$$

$$m = 2, T_{12} = 1 - e^{-\frac{2\pi}{2}} = 1 - e^{-\pi};$$

$$m = 3, T_{13} = 1 - e^{-\frac{2\pi}{3}}.$$

Then (11) can be written for the $m$-th harmonic:

$$T_{nm} = T_{1m}e^{-\frac{2\pi(n-1)}{m}}, \quad n = 1,2,3, \ldots N; \quad m = 1,2,3, \ldots M. \quad (28)$$

From (28) it follows that for each $m$-th harmonic it is necessary to perform the first period and write down the subsequent periods of the $m$-th harmonic relative to it. Figure 2 shows a block diagram of a MT harmonic generator.

\[
\begin{array}{c}
\text{m=1} \\
\downarrow \\
\begin{array}{c}
\text{T_{11}} \\
\text{T_{21}} \\
\text{T_{31}} \\
\vdots \\
\text{T_{n1}}
\end{array} \\
\text{Out 1}
\end{array}
\begin{array}{c}
\text{m=2} \\
\downarrow \\
\begin{array}{c}
\text{T_{12}} \\
\text{T_{22}} \\
\text{T_{32}} \\
\vdots \\
\text{T_{n2}}
\end{array} \\
\text{Out 2}
\end{array}
\begin{array}{c}
\text{m=M} \\
\downarrow \\
\begin{array}{c}
\text{T_{1M}} \\
\text{T_{2M}} \\
\text{T_{3M}} \\
\vdots \\
\text{T_{NM}}
\end{array} \\
\text{Out M}
\end{array}
\]

\textbf{Figure 2.} Block diagram of the harmonic generator of the IMT core.

The model presented in the diagram above can be used as the basis for the creation of digital systems for detecting phase-shift keyed signals in conditions of interference. Such systems can be used as detectors under conditions of a priori uncertainty in the form of the interference correlation function. This is important for the information security of informatization objects.
4. Conclusion
Let’s consider a mathematical model that we will use to create a digital model.
We represent the core of the beta function as:
\[ \alpha^{5-1} = \alpha^{5-1}(\cos(\ln(1-x)) + j \sin(\ln(x)), \] (29)
For the second integrand (1) we get:
\[ (1-x)^{2^{-1}} = (1-x)^{\delta^{-1}}(\cos v \ln(1-x)) + j \sin(v \ln(1-x)), \] (30)
The product (29) and (30) has the following expression:
\[ x^{\delta^{-1}}(1-x)^{\delta^{-1}}(\cos(\ln x + v \ln(1-x)) + j \sin(\ln \alpha + v \ln(1-x))), \] (31)
Performing transformations (20) we finally get:
\[ x^{\delta^{-1}}(1-x)^{\delta^{-1}} = x^{\delta^{-1}}(1-x)^{\delta^{-1}}[\cos(\ln x + v \ln(1-x)) + j \sin(\ln x + v \ln(1-x))] \]
Substituting the resulting expression in (3) we have:
\[ B(\sigma, u; \delta, v) = \int_0^1 x^{\delta^{-1}}(1-x)^{\delta^{-1}} \cos(u \ln x + v \ln(1-x)) \, dx + \int_0^1 x^{\delta^{-1}}(1-x)^{\delta^{-1}} \sin(u \ln x + v \ln(1-x)) \, dx'. \] (32)
Or in common case:
\[ B(\sigma, u; \delta, v) = R_e\{B(\cdot)\} + i m\{B(\cdot)\}, \] (33)
Then for the module we get:
\[ |B(\sigma, u; \delta, v)| = \sqrt{R_e^2\{B(\cdot)\} + i m^2\{B(\cdot)\}}, \] (34)
And for the argument:
\[ \phi(\sigma, u; \delta, v) = -\arctg \frac{i m\{B(\cdot)\}}{R_e\{B(\cdot)\}}. \] (35)
We’ll write the modulus of the imaginary part in discrete form for the first harmonic:
\[ i m\{B(\cdot)\} = \sum_{i=1}^N (i \Delta x_{i1})^{\delta^{-1}}(1-i \Delta x_{i1})^{\delta^{-1}} \sin(\ln i \Delta x_{i1}) + \ln(1-i \Delta x_{i1})) \Delta x_{i1} + \]
\[ + \sum_{i=1}^N (i \Delta x_{i2})^{\delta^{-1}}(1-i \Delta x_{i2})^{\delta^{-1}} \sin(\ln i \Delta x_{i2}) + \ln(1-i \Delta x_{i2})) \Delta x_{i2} + ... \]
\[ + \sum_{i=1}^N (i \Delta x_{iN})^{\delta^{-1}}(1-i \Delta x_{iN})^{\delta^{-1}} \sin(\ln i \Delta x_{iN}) + \ln(1-i \Delta x_{iN})) \Delta x_{iN}. \]
As a result of the theory of beta function research carried out, it managed to create, using the properties of parametrically periodic oscillations belonging to the class of trigonometric-logarithmic functions, a digital model of the beta function representation and a high-speed algorithm for its implementation with a given accuracy.
The developed digital model for implementing the calculation of the beta function may also be of interest for research and development in the field of string theory. The digital model makes it possible to simplify the calculations of the beta function to solve the problem of fulfilling the conditions for maintaining the scale invariance of a two-dimensional quantum field [7]. Therefore, the obtained results require further research in this direction.

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