A MATRIX AND ITS INVERSE: REVISITING MINIMAL RANK COMPLETIONS

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Abstract

We revisit a formula that connects the minimal ranks of triangular parts of a matrix and its inverse and relate the result to structured rank matrices. We also address the generic minimal rank problem.

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1 Introduction

In this paper we revisit the following result from [22]:

Let \([T_{ij}]_{i,j=1}^n = (S_{ij})_{i,j=1}^n\) be block matrices with sizes that are compatible for multiplication. Other than the full matrix (which is of size \(N\), say), none of the blocks need to be square. Then

\[
\min \text{ rank } \begin{pmatrix}
T_{11} & ? & \cdots & ? \\
T_{21} & T_{22} & \cdots & ? \\
\vdots & \ddots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{pmatrix} + \min \text{ rank } \begin{pmatrix}
? & ? & \cdots & ? \\
S_{21} & ? & \cdots & ? \\
\vdots & \ddots & \ddots & \vdots \\
S_{n1} & \cdots & S_{n,n-1} & ?
\end{pmatrix} = N.
\] (1.1)

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With the recent interest in numerical algorithms that make effective use of matrices with certain rank structures (see, e.g., [4], [21], [18], [7], [9], and references therein), it seems appropriate to revisit this formula that captures many of the rank considerations that go into these algorithms. The nullity theorem due to [11] is a particular case. The papers [17] and [19] show the recent interest in the nullity theorem. It is our hope that this general formula (1.1) enhances the insight in rank structured matrices.

In addition, in Section 3 we will address the so-called ”generic minimal rank problem”. This problem was introduced by Professors Gilbert Strang and David Ingerman.

2 Minimal ranks of matrices and their inverses

Let us recall the notion of partial matrices and their minimal rank. Let \( \mathbb{F} \) be a field and let \( n, m, \nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_m \) be nonnegative integers. The pattern of specified entries in a partial matrix will be described by a set \( J \subset \{1, \ldots, n\} \times \{1, \ldots, m\} \). Let now \( A_{ij}, (i, j) \in J \), be given matrices with entries in \( \mathbb{F} \) of size \( \nu_i \times \mu_j \). We will allow \( \nu_i \) and \( \mu_j \) to equal 0. The collection of matrices \( A = \{ A_{ij}; (i, j) \in J \} \) is called a partial block matrix with the pattern \( J \). A block matrix \( B = (B_{ij})_{i,j=1}^{m,n} \) with \( B_{ij} \in \mathbb{F}^{\nu_i \times \mu_j} \) is called a completion of \( A \) if \( B_{ij} = A_{ij}, (i, j) \in J \). The minimal rank of \( A \) (notations: \( \min \text{ rank}(A) \)) is defined by \( \min \text{ rank}(A) = \min \{ \text{rank } B : B \text{ is a completion of } A \} \).

A completion of \( A \) with rank \( \min \text{ rank}(A) \) is called a minimal rank completion of \( A \). When all the blocks are of size 1 \( \times \) 1 (i.e., \( \nu_i = \mu_j = 1 \) for all \( i \) and \( j \)), we will simply talk about a partial matrix. Clearly, any block matrix as above may be viewed as a partial matrix of size \( N \times M \) as well, where \( N = \nu_1 + \ldots + \nu_n, M = \mu_1 + \ldots + \mu_m \). It will be convenient to represent partial block matrices in matrix format. As usual a question mark will represent an unknown block. For instance, \( A = \{ A_{ij} : 1 \leq j \leq i \leq n \} \) will be represented as

\[
A = \begin{pmatrix}
A_{11} & ? & \ldots & ? \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & ? \\
A_{n1} & \ldots & \ldots & A_{nn}
\end{pmatrix}
\]

The formula that connects the minimal ranks of triangular parts of a matrix and its inverse is the following. The result appeared originally in [22] (see also [23] and Chapter 5 of [24]).

**Theorem 2.1** [22] Let \( T = (T_{ij})_{i,j=1}^{n} \) be an invertible block matrix with \( T_{ij} \) of size \( \nu_i \times \mu_j \), where \( \nu_i \geq 0, \mu_j \geq 0 \) and \( N = \nu_1 + \ldots + \nu_n = \mu_1 + \ldots + \mu_n \). Put \( T^{-1} = (S_{ij})_{i,j=1}^{n} \) where \( S_{ij} \) is of size \( \mu_i \times \nu_j \). Then

\[
\min \text{ rank} \begin{pmatrix}
T_{11} & ? & \ldots & ? \\
T_{21} & T_{22} & \ldots & ? \\
\vdots & \ddots & \ddots & \vdots \\
T_{n1} & T_{n2} & \ldots & T_{nn}
\end{pmatrix} + \min \text{ rank} \begin{pmatrix}
? & ? & \ldots & ? \\
S_{21} & ? & \ldots & ? \\
\vdots & \ddots & \ddots & \vdots \\
S_{n1} & \ldots & S_{n,n-1} & ?
\end{pmatrix} = N.
\]
As we will see, one easily deduces from Theorem 2.1 that the inverse of an upper Hessenberg matrix has the lower triangular part of a rank 1 matrix. The strength of Theorem 2.1 lies in that one easily deduces a multitude of such results from it.

From the same paper [22] we would also like to recall the following result.

**Theorem 2.2** [22] The partial matrix $\mathcal{T} = \{T_{ij} : 1 \leq j \leq i \leq n\}$ has minimal rank

$$\min \text{rank } \mathcal{T} = \sum_{i=1}^{n} \text{rank } \begin{pmatrix} T_{i1} & \ldots & T_{ii} \\ \vdots & \ddots & \vdots \\ T_{ni} & \ldots & T_{ni} \end{pmatrix} - \sum_{i=1}^{n-1} \text{rank } \begin{pmatrix} T_{i+1,1} & \ldots & T_{i+1,i} \\ \vdots & \ddots & \vdots \\ T_{n1} & \ldots & T_{ni} \end{pmatrix}.$$

After the $n = 2$ case of Theorem 2.2 is obtained it is straightforward to prove the general case by introduction. For the $2 \times 2$ case of Theorem 2.2 one needs to observe that the minimal rank of

$$\begin{pmatrix} T_{11} & ? \\ T_{21} & T_{22} \end{pmatrix}$$

will at least be the rank of $\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$ plus the minimal number of columns in $T_{22}$ that together with the columns of $T_{21}$ span the column space of $(T_{21} \ T_{22})$. Once such a minimal set of columns in $T_{22}$ has been identified, put any numbers on top of these columns. Now any other columns in $T_{22}$ can be completed to be a linear combination of fully completed columns. Doing this leads to a completion of rank

$$\text{rank } \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} + \text{rank } (T_{21} \ T_{22}) - \text{rank } T_{21},$$

yielding the $n = 2$ case of Theorem 2.2.

The proof of Theorem 2.1, which can be found in [22] is easily derived from Theorem 2.2 and the nullity theorem, which we recall now.

**Theorem 2.3** [11] Consider

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$ 

Then $\dim \ker C = \dim \ker R$.

**Proof.** Since $CP = -DR$, $P[\ker R] \subseteq \ker C$. Likewise, since $RA = -SC$, we get $A[\ker C] \subseteq \ker R$. Consequently,

$$AP[\ker R] \subseteq A[\ker C] \subseteq \ker R.$$ 

Since $AP + BR = I$, $AP[\ker R] = \ker R$, thus

$$A[\ker C] = \ker R.$$
This yields $\dim \ker C \geq \dim \ker R$. By reversing the roles of $C$ and $R$ one obtains also that $\dim \ker R \geq \dim \ker C$. This gives $\dim \ker R = \dim \ker C$, yielding the lemma. □

The nullity theorem is in fact the $n = 2$ case of Theorem 1. Indeed, if

$$T^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

we get from Theorem 1 that

$$\text{rank} \left( \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} \right) + \text{rank} \left( \begin{pmatrix} T_{21} & T_{22} \end{pmatrix} \right) - \text{rank} T_{21} + \text{rank} S_{21} = N. \quad (2.2)$$

As $T$ is invertible we have that $\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$ and $\begin{pmatrix} T_{21} & T_{22} \end{pmatrix}$ are full rank, so (2.2) gives

$$\mu_1 + \nu_2 - \text{rank} T_{21} + \text{rank} S_{21} = \mu_1 + \mu_2 = \nu_1 + \nu_2,$$

and thus

$$\nu_2 - \text{rank} T_{21} = \mu_2 - \text{rank} S_{21},$$

which is exactly Theorem 3.

To make the connection with some of the results in the literature we need the following proposition.

**Proposition 2.4** Let $\mathcal{T} = \{t_{ij} : 1 \leq j \leq i \leq n\}$ be a scalar valued partial matrix. Then $\min \text{rank} (\mathcal{T}) = n$ if and only if $t_{ii} \neq 0$, $i = 1, \ldots, n$, and $t_{ij} = 0$ for $i > j$.

**Proof.** The "if" part is immediate. For the only if part write

$$\min \text{rank} \mathcal{T} = \text{rank} \begin{pmatrix} t_{11} \\ \vdots \\ t_{n1} \end{pmatrix} + \sum_{i=2}^{n} s_i, \quad (2.3)$$

where

$$s_i = \text{rank} \begin{pmatrix} t_{i1} & \cdots & t_{ii} \\ \vdots & \vdots & \vdots \\ t_{n1} & \cdots & t_{ni} \end{pmatrix} - \text{rank} \begin{pmatrix} t_{i1} & \cdots & t_{i,i-1} \\ \vdots & \vdots & \vdots \\ t_{n1} & \cdots & t_{n,i-1} \end{pmatrix}.$$

All the terms in (2.3) are at the most 1, and as there are exactly $n$ terms they need to all be equal to 1 for $\min \text{rank}(\mathcal{T}) = n$ to be satisfied. But then $s_n = 1$ implies $t_{n1} = \cdots = t_{n,n-1} = 0$ and $t_{nn} \neq 0$. Inductively, one can then show that $s_k = 1$ implies $t_{k1} = \cdots = t_{k,k-1} = 0$ and $t_{kk} \neq 0, k = n-1, \ldots, 2$. Finally the first column of $\mathcal{T}$ needs to have rank 1. As $t_{ij} = 0, j = 2, \ldots, n$, was already established we get that $t_{11} \neq 0$. This proves the result. □

We now easily obtain the following corollary, due to Asplund [1].
Corollary 2.5 [1] Let $p \geq 0$ and $A = (a_{ij})_{i,j=1}^{N}$ be an $N \times N$ scalar matrix with inverse $B = (b_{ij})_{i,j=1}^{N}$. Then $a_{ij} = 0$ for all $i$ and $j$ with $j > i + p$, and $a_{ij} \neq 0$, $j = i + p$ if and only if there exist a $N \times p$ matrix $F$ and a $p \times N$ matrix $G$ so that $b_{ij} = (FG)_{ij}$, $i < j + p$.

**Proof.** Let $(S_{ij})_{i,j=N-p+1}^{N} = A$, where $S_{i1}$ is of size $1 \times p$, $i = 1, \ldots, n - p$, $S_{N-p+1,j}$ has size $p \times 1$, $j = 2, \ldots, N - p + 1$, and all the other $S_{ij}$ are $1 \times 1$. Let $B = (T_{ij})_{i,j=N-p+1}^{N}$ be partitioned accordingly. Then, it follows from (1.1) that

$$\min \text{ rank } \begin{pmatrix} \cdot & \cdots & \cdot \\ S_{21} & \cdots & ? \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{n,n-1} \end{pmatrix} = N - p$$

if and only if

$$\min \text{ rank } \begin{pmatrix} T_{11} & \cdots & ? \\ T_{21} & T_{22} & \cdots \\ \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} = p.$$ 

Using Proposition 2.4 the result now follows. □

In a similar way it is easy to deduce results by [3], [16], [14], [15], [13] and [8] from Theorem 2.1. For instance, if $T_{ij}$ and $S_{ij}$ are scalars, and $T_{21}, \ldots, T_{n,n-1} \neq 0$ and $T_{ij} = 0$ for $i > j + 1$, then the left hand term in (1.1) is $n - 1$. Since $N = n$, we get that the lower triangular partial matrix $(S_{ij})_{i \geq j}$ has minimal rank 1. Thus one easily obtains that $S_{ij} = u_{i}v_{j}, i \geq j$, where $u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{n}$ are scalars. Examples like this show that Theorem 2.1 is useful in the contexts of semi-separability and quasi-separability (see, e.g., [20] and [6] for an overview of these notions). We hope that the simplicity of formula (1.1) will help in the further development of these notions.

3 The generic minimal rank completion problem

Recently D. Ingerman and G. Strang posed the following problem. Suppose that a partial matrix (over some field $F$) has the property that all of its fully specified submatrices are of full rank and so that every $k \times k$ partial submatrix has at most $(2k - r)r$ entries specified. Is it true that one can always complete to a matrix of rank $\leq r$? The count of $(2k - r)r$ specified entries comes from the consideration that if $r$ columns and $r$ rows in a $k \times k$ submatrix are specified, one can complete this submatrix to a rank $r$ one (due to the fact that the submatrix in the overlap of the $r$ columns and the $r$ rows has full rank). However, as soon as one adds one specified entry to these $r$ columns and $r$ rows, immediately a $(r + 1) \times (r + 1)$ submatrix is specified, and the minimal rank will be at least $r + 1$. 
Ingeman and Strang showed that the above statement is correct for $r = 1$. However, the following example shows that in general it is not correct for $r \geq 2$.

**Example.** Consider the matrix

$$A := \begin{pmatrix} 6 & 3 & x & 1 \\ 3 & 1 & 1 & y \\ z & 1 & 2 & 3 \\ 1 & w & 1 & 1 \end{pmatrix},$$

where $x, y, z$ and $w$ are the unknowns. Note that this partial matrix satisfies the requirements stated in the first paragraph. Furthermore, suppose that rank $A = 2$. Then we have that

$$\begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z & 1 \\ 1 & w \end{pmatrix} = 0,$$

and since the rank of the first term is 2, the second term must also have rank 2. Thus, we have that $xy \neq 1$ and $zw \neq 1$. Next, we also have that

$$\begin{pmatrix} z & 1 \\ 1 & w \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}^{-1} \begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix} = 0.$$

Multiplying on both sides with $xy - 1$, the off-diagonal entries yield the following equations

$$xy - 6y - 3x + 10 = 0, \quad xy - 6y - 3x + 8 = 0.$$ 

These are not simultaneously solvable (as long as we are in a field where $8 \neq 10$). It should be noted that this is a counterexample for any field in which $6 \neq 9, 6 \neq 1, 3 \neq 1, 9 \neq 1$ (so that we have full rank specified submatrices) and $8 \neq 10$. As an aside, we note that for some of the small fields it may impossible to fulfill the nondegeneracy requirement on the data. E.g., when $\mathbb{F} = \{0, 1\}$ a $2 \times 2$ matrix can only be nonsingular if zeroes are allowed in the matrix.

It should be noted that if one associates the bipartite graph with the partial matrix (see, e.g., [5]) one obtains a minimal eight cycle. Consequently, the bipartite graph is not bipartite chordal as bipartite chordality requires by definition the absence of minimal cycles of length 6 or greater. Notice that in the $r = 1$ case the condition on the density of the specified entries prevents the existence of minimal cycles of length 6 or more. We now arrive at the following conjecture.

**Conjecture 3.1** Consider a partial matrix for which the bipartite graph is bipartite chordal. Suppose furthermore that any fully specified submatrix has full rank and that any $k \times k$ submatrix has at most $(2k - r)r$ entries specified. Then there exists a completion of rank $r$.

We can prove the conjecture for the subclass of banded patterns (cf. [25]).
Theorem 3.2 Consider a partial matrix with a banded pattern (as defined in [25]). Suppose furthermore that any fully specified submatrix has full rank and that any $k \times k$ submatrix has at most $(2k - r)r$ entries specified. Then there exists a completion of rank $r$.

Proof. By Theorem 1.1 in [25] it suffices to show that for every triangular subpattern (for the definition, see [25]) we have that the minimal rank is $\leq r$. But a triangular subpattern can always embedded in a pattern that corresponds to $r$ rows and columns specified (due to the condition that in any $k \times k$ submatrix has at most $(2k - r)r$ entries are specified). But then the result follows. \qed

Observe that the proof shows that if the bipartite chordal minimal rank conjecture in [5] (see also Chapter 5 in [24]) is true, then the above conjecture is true as well. The techniques developed in [2] and/or [12] may be helpful in proving the conjecture above.

References

[1] Asplund, Edgar Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i+p$. Math. Scand. 7 1959 57–60.

[2] Bakonyi, Mihály; Bono, Aaron Several results on chordal bipartite graphs. Czechoslovak Math. J. 47(122) (1997), no. 4, 577–583.

[3] Barrett, Wayne W.; Feinsilver, Philip J. Inverses of banded matrices. Linear Algebra Appl. 41 (1981), 111–130.

[4] Bini, Dario A.; Gemignani, Luca; Pan, Victor Y. Fast and stable QR eigenvalue algorithms for generalized companion matrices and secular equations. Numer. Math. 100 (2005), no. 3, 373–408.

[5] Cohen, Nir; Johnson, Charles R.; Rodman, Leiba; Woerdeman, Hugo J. Ranks of completions of partial matrices. The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988), 165–185, Oper. Theory Adv. Appl., 40, Birkhäuser, Basel, 1989

[6] Eidelman, Y.; Gohberg, I. On generators of quasiseparable finite block matrices. Calcolo 42 (2005), no. 3-4, 187–214.

[7] Eidelman, Y.; Gohberg, I.; Olshevsky, Vadim Eigenstructure of order-one-quasiseparable matrices. Three-term and two-term recurrence relations. Linear Algebra Appl. 405 (2005), 1–40.

[8] Elsner, L. A note on generalized Hessenberg matrices. Linear Algebra Appl. 409 (2005), 147–152.
[9] Fasino, Dario; Gemignani, Luca A Lanczos-type algorithm for the QR factorization of Cauchy-like matrices. Fast algorithms for structured matrices: theory and applications (South Hadley, MA, 2001), 91–104, Contemp. Math., 323, Amer. Math. Soc., Providence, RI, 2003.

[10] Gohberg, I.; Kailath, T.; Koltracht, I. Linear complexity algorithms for semiseparable matrices. Integral Equations Operator Theory 8 (1985), no. 6, 780–804.

[11] William H. Gustafson, *A note on matrix inversion*, Linear Algebra Appl. 57 (1984), 71–73.

[12] Johnson, Charles R.; Miller, Jeremy Rank decomposition under combinatorial constraints. Linear Algebra Appl. 251 (1997), 97–104.

[13] Rózsa, Pál; Romani, Francesco; Bevilacqua, Roberto On generalized band matrices and their inverses. Proceedings of the Cornelius Lanczos International Centenary Conference (Raleigh, NC, 1993), 109–121, SIAM, Philadelphia, PA, 1994

[14] Rózsa, Pál; Bevilacqua, Roberto; Romani, Francesco; Favati, Paola On band matrices and their inverses. Proceedings of the First Conference of the International Linear Algebra Society (Provo, UT, 1989). Linear Algebra Appl. 150 (1991), 287–295.

[15] Rózsa, Pál; Bevilacqua, Roberto; Favati, Paola; Romani, Francesco On the inverse of block tridiagonal matrices with applications to the inverses of band matrices and block band matrices. The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988), 447–469, Oper. Theory Adv. Appl., 40, Birkhuser, Basel, 1989.

[16] Rózsa, Pál Band matrices and semiseparable matrices. Numerical methods (Miskolc, 1986), 229–237, Colloq. Math. Soc. Jnos Bolyai, 50, North-Holland, Amsterdam, 1988

[17] Strang, Gilbert; Nguyen, Tri The interplay of ranks of submatrices. SIAM Rev. 46 (2004), no. 4, 637–646 (electronic).

[18] Tyrtyshnikov, E. Piecewise separable matrices. Calcolo 42 (2005), no. 3-4, 243–248.

[19] Vandebril, Raf; Van Barel, Marc A note on the nullity theorem. J. Comput. Appl. Math. 189 (2006), no. 1-2, 179–190.

[20] Vandebril, R.; Van Barel, M.; Golub, G.; Mastronardi, N. A bibliography on semiseparable matrices. Calcolo 42 (2005), no. 3-4, 249–270.

[21] Vandebril, Raf; Van Barel, Marc; Mastronardi, Nicola An implicit QR algorithm for symmetric semiseparable matrices. Numer. Linear Algebra Appl. 12 (2005), no. 7, 625–658.
[22] Woerdeman, H. J. The lower order of lower triangular operators and minimal rank extensions. Integral Equations Operator Theory 10 (1987), no. 6, 859–879.

[23] Woerdeman, H. J. Minimal rank completions for block matrices. Linear Algebra Appl. 121 (1989), 105–122.

[24] Woerdeman, H. J. Matrix and operator extensions. CWI Tract, 68. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[25] Woerdeman, Hugo J. Minimal rank completions of partial banded matrices. Linear and Multilinear Algebra 36 (1993), no. 1, 59–68.