Multiple Hamiltonian Structures for Toda-type systems

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ABSTRACT

Results on the finite nonperiodic Toda lattice are extended to some generalizations of the system: The relativistic Toda lattice, the generalized Toda lattice associated with simple Lie groups and the full Kostant-Toda lattice. The areas investigated, include master symmetries, recursion operators, higher Poisson brackets, invariants, and group symmetries for the systems. A survey of previous work on the classical Toda lattice is also included.

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I INTRODUCTION

This paper can be divided into two parts. First, we survey some previous work on the Toda lattice dealing with the Poisson Geometry and algebraic structure of the system (section III). The second part contains new results on some systems generalizing the finite nonperiodic Toda lattice (sections IV, V, VI). This system has been studied extensively and therefore we will examine only some aspects of the Toda lattice which are shared by most integrable systems. The list of topics includes bi-Hamiltonian structure, recursion operators, symmetries, master symmetries, Lax formulations, Poisson and symplectic Geometry. This area of integrable systems has been studied extensively for infinite dimensional systems such as the KdV, Burgers, Kadomtsev-Petviashvili, Benjamin-Ono equations and many more. The Toda lattice is the only finite dimensional system where these ideas have been worked out in detail. In this paper we will extend results about the finite nonperiodic Toda lattice to some generalizations of the system: The relativistic Toda lattice, the generalized Toda lattice associated with simple Lie groups and the full Kostant-Toda lattice.

The natural setting for Hamiltonian systems is on symplectic manifolds. These are Poisson manifolds whose Poisson structure is locally isomorphic to the standard one on \( \mathbb{R}^{2N} \). A Poisson structure on a manifold \( M \) may be defined as contravariant tensor field (bivector) \( \pi \) for which the Poisson bracket on \( C^\infty(M) \), \( \{f,g\} = \langle \pi, df \wedge dg \rangle \) satisfies the Jacobi identity. When \( \pi \) has full rank, the dimension of \( M \) is even and the Poisson structure is symplectic. Darboux's theorem provides coordinates which make the structure locally isomorphic to the standard symplectic bracket on \( \mathbb{R}^{2N} \).

To define a Hamiltonian system we consider \( \mathbb{R}^{2N} \) with coordinates \((q_1, \ldots, q_N, p_1, \ldots, p_N)\), and the standard symplectic bracket

\[
\{f, g\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).
\]
Let \( H : \mathbb{R}^{2N} \to \mathbb{R} \) be a smooth function. Hamilton’s equations are the differential equations

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.
\]  

(2)

Using the symplectic bracket (1), Hamilton’s equations take the simple form

\[
\dot{F} = \{F, H\}.
\]

(3)

The condition \( \dot{F} = 0 \) is equivalent to the condition \( \{F, H\} = 0 \). Such a function is called a constant of motion (or first integral).

Smooth functions \( f_1, \ldots, f_r \) on a manifold \( M \) are called independent, if \( df_1 \wedge \ldots \wedge df_r \neq 0 \), except possibly on a submanifold of smaller dimension. The functions \( f_1, \ldots, f_r \) are said to be mutually in involution if \( \{f_i, f_j\} = 0, \quad \forall i, j \). A Hamiltonian system is called integrable if there exists a family \( f_1, \ldots, f_r \) of independent \( C^\infty \) functions mutually in involution. The Hamiltonian is one of these functions (or a function of these functions). The systems we consider will be integrable.

The equations for the Toda systems in consideration will be written in the form

\[
\dot{L}(t) = [B(t), L(t)].
\]

(4)

The pair of matrices \( L, B \) is known as a Lax pair. In the case of the finite nonperiodic Toda lattice \( L \) is a symmetric tridiagonal matrix and \( B \) is the projection onto the skew-symmetric part in the decomposition of \( L \) into skew-symmetric plus lower triangular. In the case of the other generalized Toda systems the matrix \( L \) will lie in some Lie algebra and \( B \) will again be obtained from \( L \) by some projection associated with a decomposition of the Lie algebra. The decomposition plays an important role in the solution of the equations by factorization.

A Lax equation implies that the traces of powers of \( L \) are constants of motion. \( \dot{L} = [B, L] \) implies \( (L^2) = [B, L^2] \), and similarly \( (L^k) = [B, L^k] \). Therefore,

\[
(\text{trace } L^k) = (\text{trace } L^k) = \text{trace } [B, L^k] = 0.
\]

(5)
This calculation shows that the eigenvalues of $L$ do not evolve with time. Therefore, such systems will be integrable.

In the case of Toda lattice the Lax equation is obtained by the use of a transformation due to H. Flaschka [1] which changes the original $(p, q)$ variables to new reduced variables $(a, b)$. The symplectic bracket in the variables $(p, q)$ transforms to a degenerate Poisson bracket in the variables $(a, b)$. This linear bracket is an example of a Lie-Poisson bracket. The functions $H_n = \frac{1}{n} \text{tr} L^n$ are in involution. A Lie algebraic interpretation of this bracket can be found in [2]. We denote this bracket by $\pi_1$. A quadratic Toda bracket, which we call $\pi_2$ appeared in a paper of Adler [3]. It is a Poisson bracket in which the Hamiltonian vector field generated by $H_1$ is the same as the Hamiltonian vector field generated by $H_2$ with respect to the $\pi_1$ bracket. This is an example of a bi-Hamiltonian system, an idea introduced by Magri [4]. A cubic bracket was found by Kuipershmidt [5] via the infinite Toda lattice. We found the explicit formulas for both the quadratic and cubic brackets in some lecture notes by H. Flaschka. The Lenard relations (73) are also in these notes.

In [6] we used master symmetries to generate nonlinear Poisson brackets for the Toda lattice. In essence, we have an example of a system which is not only bi-Hamiltonian but it can actually be given $N$ different Hamiltonian formulations with $N$ as large as we please. The first three Poisson brackets are precisely the linear, quadratic and cubic brackets we mentioned above, but one can use the master symmetries to produce an infinite hierarchy of brackets. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables $(a, b)$) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [7] where a Ninjenhuis tensor for the infinite Toda lattice is calculated. Recursion operators were introduced by Olver [8]. Master symmetries were first introduced by Fokas and Fuchssteiner in [9] in connection with the Benjamin-Ono Equation. Then W. Oevel and B. Fuchssteiner [10] found master symmetries for the Kadomtsev-
Petviashvili equation. Master symmetries for equations in 1 + 1, like the KdV, are discussed in Chen, Lee and Lin [11] and in Fokas [12]. The general theory of master symmetries is discussed in Fuchssteiner [13]. In the case of Toda equations the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. Another approach, which explains these relations is adopted in Das and Okubo [14], and Fernandes [15]. In principle, their method is general and may work for other finite dimensional systems as well. The procedure is the following: One defines a second Poisson bracket in the space of canonical variables \((q_1, \ldots, q_N, p_1, \ldots, p_N)\). This gives rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [16] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables \((a, b)\) to produce a sequence of master symmetries in the reduced space. This approach is discussed at the end of section III.

The Toda lattice has been generalized in several directions: Kostant [2] and Bogoyavlensky [17] generalized the system to the tridiagonal coadjoint orbit of the Borel subgroup of an arbitrary simple Lie group. Therefore, for each simple Lie group there is a corresponding mechanical system of Toda type.

Another generalization is due to Deift, Li, Nanda and Tomei [18] who showed that the system remains integrable when \(L\) is replaced by a full (generic) symmetric \(n \times n\) matrix.

Another variation is the full Kostant-Toda lattice which was studied by S. Singer, N. Ercolani and H. Flaschka [19], [20], [21]. In this case, the matrix \(L\) in the Lax equation is the sum of a lower triangular plus a regular nilpotent matrix.

Finally, there is a relativistic Toda lattice which was introduced by Ruijsenaars [22]. The non-relativistic Toda lattice can be thought as a limiting case of this system.

In section II we present the necessary background on Poisson manifolds, bi-Hamiltonian systems and master symmetries. In the spirit of this special issue we define Cohomology of Lie algebras and
a Cohomology on Poisson manifolds due to Lichnerowicz [23]. Most of the results in this paper can be considered as statements about the Cohomology of Poisson Lie algebras.

Section III is a review of the classical finite nonperiodic Toda lattice. This system was investigated in [1], [24], [25], [26], [27]. We define the quadratic and cubic Toda brackets and show that they satisfy certain Lenard-type relations. We briefly describe the construction of master symmetries and the new Poisson brackets as in [6]. Propositions 2-5 have been known for some time [28], but had not been published before. Theorem 7 which shows the connection with group symmetries appeared in [29].

In section IV, some results on the non-relativistic Toda lattice are extended to the case of the relativistic Toda Systems. The main new result is the hierarchy of higher Poisson brackets which also exists in the finite nonperiodic Toda lattice. We were recently informed by Walter Oevel about reference [30], where the first three local Hamiltonian structures are found using different methods.

In section V we define some integrable systems associated with simple Lie groups. They have been considered by Kostant [2], Bogoyavlensky [17] and Olshanetsky and Perelomov [31]. We present in detail the systems of type $B_n$. We show that they are bi-Hamiltonian and we also construct a recursion operator. These results are new. Similar results hold for other semisimple Lie algebras. We checked small dimensions, but we do not have complete general results yet.

In section VI we define Poisson structures which lead to a tri-Hamiltonian formulation for the full Kostant-Toda lattice. In addition, master symmetries are constructed and they are used to generate the nonlinear Poisson brackets and other invariants. Various deformation relations are investigated. The results have been announced in [32].
II BACKGROUND

The Schouten bracket

We list some properties of the Schouten bracket following Lichnerowicz [23]. Let $M$ be a $C^\infty$ manifold, $N = C^\infty(M)$ the algebra of $C^\infty$ real valued functions on $M$. A contravariant, antisymmetric tensor of order $p$ will be called a $p$-tensor for short. These tensors form a superspace endowed with a Lie-superalgebra structure via the Schouten bracket.

The Schouten bracket assigns to each $p$-tensor $A$, and $q$-tensor $B$, a $(p+q-1)$-tensor, denoted by $[A,B]$. For $p = 1$ we have $[A,B] = L_A B$ where $L_A$ is the Lie-derivative in the direction of the vector field $A$. The bracket satisfies:

$$[A,B] = (-1)^{pq}[B,A]$$  \hspace{1cm} (6)

$$[A,B] = (-1)^{pq}[B,A]$$  \hspace{1cm} (7)

$$(-1)^{pq}[[B,C],A] + (-1)^{qr}[[C,A],B] + (-1)^{pr}[[A,B],C] = 0$$  \hspace{1cm} (8)

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Poisson Manifolds

A Poisson structure on $M$ is a bilinear form, called the Poisson bracket $\{ , \} : N \times N \to N$ such that

$$\{f, g\} = -\{g, f\}$$  \hspace{1cm} (9)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$  \hspace{1cm} (10)

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$  \hspace{1cm} (11)
Properties $i)$ and $ii)$ define a Lie algebra structure on $N$. $ii)$ is called the Jacobi identity and $iii)$ is the analogue of Leibniz rule from calculus. A Poisson manifold is a manifold $M$ together with a Poisson bracket $\{ , \}$.

To a Poisson bracket one can associate a 2-tensor $\pi$ such that
\[
\{f, g\} = \langle \pi, df \wedge dg \rangle .
\] (12)
Jacobi’s identity is equivalent to the condition $[\pi, \pi] = 0$ where $[ , ]$ is the Schouten bracket. Therefore, one could define a Poisson manifold by specifying a pair $(M, \pi)$ where $M$ is a manifold and $\pi$ a 2-tensor satisfying $[\pi, \pi] = 0$. In local coordinates $(x_1, x_2, \ldots, x_n)$, $\pi$ is given by
\[
\pi = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}
\] (13)
and
\[
\{f, g\} = \langle \pi, df \wedge dg \rangle = \sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \wedge \frac{\partial g}{\partial x_j} .
\] (14)
In particular $\{x_i, x_j\} = \pi_{ij}(x)$. Knowledge of the Poisson matrix $(\pi_{ij})$ is sufficient to define the bracket of arbitrary functions. The rank of the matrix $(\pi_{ij})$ at a point $x \in M$ is called the rank of the Poisson structure at $x$.

A function $F : M_1 \to M_2$ between two Poisson manifolds is called a Poisson mapping if
\[
\{f \circ F, g \circ F\}_1 = \{f, g\}_2 \circ F
\] (15)
for all $f, g \in C^\infty(M_2)$. In terms of tensors, $F_\ast \pi_1 = \pi_2$. Two Poisson manifolds are called isomorphic, if there exists a diffeomorphism between them which is a Poisson mapping.

The Poisson bracket allows one to associate a vector field to each element $f \in N$. The vector field $\chi_f$ is defined by the formula
\[
\chi_f(g) = \{f, g\}.
\] (16)
It is called the Hamiltonian vector field generated by $f$. In terms of the Schouten bracket
\[
\chi_f = [\pi, f].
\] (17)
Hamiltonian vector fields are *infinitesimal automorphisms* of the Poisson structure. These are vector fields $X$ satisfying $L_X \pi = 0$. In the case of Hamiltonian vector fields we have

$$L_{\chi_f} \pi = [\pi, \chi_f] = [\pi, [\pi, f]] = -2[[\pi, \pi], f] = 0.$$  

(18)

The Hamiltonian vector fields form a Lie algebra and in fact

$$[[\chi_f, \chi_g]] = \chi_{\{f, g\}}.$$  

(19)

So, the map $f \rightarrow \chi_f$ is a Lie algebra homomorphism.

The Poisson structure defines a bundle map $\pi^* : T^*M \rightarrow TM$ such that

$$\pi^*(df) = \chi_f.$$  

(20)

The rank of the Poisson structure at a point $x \in M$ is the rank of $\pi^*_x : T^*_x M \rightarrow T_x M$.

The functions in the center of $N$ are called *Casimirs*. It is the set of functions $f$ so that $\{f, g\} = 0$ for all $g \in N$. These are functions which are constant along the orbits of Hamiltonian vector fields. The differentials of these functions are in the kernel of $\pi^*$. In terms of the Schouten bracket a Casimir satisfies $[\pi, f] = 0$.

Given a function $f$, there is a reasonable algorithm for constructing a Poisson bracket in which $f$ is a Casimir. One finds two vector fields $X_1$ and $X_2$ such that $L_{X_1} f = L_{X_2} f = 0$. If in addition $X_1$, $X_2$ and $[X_1, X_2]$ are linearly dependent, then $X_1 \wedge X_2$ is a Poisson tensor and $f$ is a Casimir in this bracket. In fact

$$[f, X_1 \wedge X_2] = [f, X_1] \wedge X_2 - X_1 \wedge [f, X_2] = 0.$$  

(21)

More generally, there is a formula due to H. Flaschka and T. Ratiu which gives locally a Poisson bracket when the number of Casimirs is 2 less than the dimension of the space. Let $f_1, f_2, \ldots, f_r$ be functions on $\mathbb{R}^{r+2}$. Then the formula

$$\omega\{g, h\} = df_1 \wedge \ldots \wedge df_r \wedge dg \wedge dh$$  

(22)

where $\omega$ is a non-vanishing $r+2$ form, defines a Poisson bracket on $\mathbb{R}^{r+2}$ and the functions $f_1, \ldots, f_r$ are Casimirs.
Multiplication of a Poisson bracket by a Casimir gives another Poisson bracket. Suppose \([\pi, \pi] = 0\) and \([\pi, f] = 0\). Then
\[
[f \pi, f \pi] = f \wedge [f \pi] \wedge \pi + f \wedge \pi \wedge [\pi, f] + f^2 [\pi, \pi] = 0.
\] (23)

**Examples**

The most basic examples of Poisson brackets are the symplectic and Lie-Poisson brackets.

i) **Symplectic manifolds:** A *symplectic manifold* is a pair \((M^{2n}, \omega)\) where \(M^{2n}\) is an even dimensional manifold and \(\omega\) is a closed, non-degenerate two-form. The associated isomorphism
\[
\mu : TM \rightarrow T^* M
\] (24)
extends naturally to a tensor bundle isomorphism still denoted by \(\mu\). Let \(\lambda = \mu^{-1}, f \in N\) and let \(\chi_f = \lambda(df)\) be the corresponding Hamiltonian vector field. The symplectic bracket is given by
\[
\{f, g\} = \omega(\chi_f, \chi_g).
\] (25)

In the case of \(\mathbb{R}^{2n}\), there are coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\), so that
\[
\omega = \sum_{i=1}^{n} dx_i \wedge dy_i
\] (26)
and the Poisson bracket is the standard one (1).

ii) **Lie Poisson** : Let \(M = G^*\) where \(G\) is a Lie algebra. For \(a \in G\) define the function \(\Phi_a\) on \(G^*\) by
\[
\Phi_a(\mu) = \langle a, \mu \rangle
\] (27)
where \(\mu \in G^*\) and \(\langle , \rangle\) is the pairing between \(G\) and \(G^*\). Define a bracket on \(G^*\) by
\[
\{\Phi_a, \Phi_b\} = \Phi_{[a,b]}.
\] (28)
This bracket is easily extended to arbitrary \(C^\infty\) functions on \(G^*\). The bracket of linear functions is linear and every linear bracket is of this form, i.e., it is associated with a Lie algebra.
Local theory

In his paper [33] A. Weinstein proves the so-called “splitting theorem”, which describes the local behavior of Poisson manifolds.

**Theorem 1** Let $x_0$ be a point in a Poisson manifold $M$. Then near $x_0$, $M$ is isomorphic to a product $S \times N$ where $S$ is symplectic, $N$ is a Poisson manifold, and the rank of $N$ at $x_0$ is zero.

$S$ is called the symplectic leaf through $x_0$ and $N$ is called the transverse Poisson structure at $x_0$. $N$ is unique up to isomorphism.

So, through each point $x_0$ passes a symplectic leaf $S_{x_0}$ whose dimension equals the rank of the Poisson structure on $M$ at $x_0$. The bracket on the transverse manifold $N_{x_0}$ can be calculated using Dirac’s constraint bracket formula. For more details, see Oh [34].

**Theorem 2** Let $x_0$ be a point in a Poisson manifold $M$ and let $U$ be a neighborhood of $x_0$ which is isomorphic to a product $S \times N$ as in Weinstein’s splitting theorem. Let $p_i$, $i = 1, \ldots, 2n$ be functions on $U$ such that

$$N = \{ x \in U \mid p_i(x) = \text{constant} \} . \quad (29)$$

Denote by $P = P_{ij} = \{p_i, p_j\}$ and by $P^{ij}$ the inverse matrix of $P$. Then the bracket formula for the transverse Poisson structure on $N$ is given as follows:

$$\{F, G\}_N(x) = \{\hat{F}, \hat{G}\}_M(x) + \sum_{i,j} \{\hat{F}, p_i\}_M(x)P^{ij}(x)\{\hat{G}, p_j\}_M(x)$$

for all $x \in N$, where $F, G$ are functions on $N$ and $\hat{F}, \hat{G}$ are extensions of $F$ and $G$ to a neighborhood of $M$. Dirac’s formula depends only on $F, G$, but not on the extensions $\hat{F}, \hat{G}$.

Cohomology

Cohomology of Lie algebras was introduced by Chevalley and Eilenberg in [35]. Let $\mathcal{G}$ be a Lie algebra and let $\rho$ be a representation of
$G$ with representation space $V$. A $q$-linear skew-symmetric mapping of $G$ into $V$ will be called a $q$-dimensional $V$-cochain. The $q$-cochains form a space $C^q(G, V)$. By definition, $C^0(G, V) = V$.

We define a coboundary operator $\delta = \delta_q : C^q(G, V) \rightarrow C^{q+1}(G, V)$ by the formula

\[
(\delta f)(x_0, \ldots, x_q) = \sum_{i=0}^q (-1)^q \rho(x_i) f(x_0, \ldots, \hat{x}_i, \ldots, x_q) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q)
\]

(31)

where $f \in C^q(G, V)$ and $x_0, \ldots, x_q \in G$. As can be easily checked $\delta_{q+1} \circ \delta_q = 0$ so that $\{C^q(G, V), \delta_q\}$ is an algebraic complex. Define $Z^q(G, V)$ the space of $q$-cocycles as the kernel of $\delta : C^q \rightarrow C^{q+1}$ and the space $B^q(G, V)$ of $q$-coboundaries as the image $\delta C^{q-1}$. Since $\delta \delta = 0$ we can define

\[
H^q(G, V) = \frac{Z^q(G, V)}{B^q(G, V)}.
\]

(32)

Lichnerowicz [23] considers the following cohomology defined on the tensors of a Poisson manifold. Let $(M, \pi)$ be a Poisson manifold. If we set $B = C = \pi$ in (7) we get

\[
[\pi, [\pi, A]] = 0
\]

(33)

for every tensor $A$. Define a coboundary operator $\partial_\pi$ which assigns to each $p$-tensor $A$, a $(p+1)$-tensor $\partial_\pi A$ given by

\[
\partial_\pi A = -[\pi, A].
\]

(34)

We have $\partial^2_\pi A = [\pi, [\pi, A]] = 0$ and therefore $\partial_\pi$ defines a cohomology. An element $A$ is a $p$-cocycle if $[\pi, A] = 0$. An element $B$ is a $p$-coboundary if $B = [\pi, C]$, for some $(p-1)$-tensor $C$. Let

\[
Z^n(M, \pi) = \{ A \in T_n \mid [\pi, A] = 0 \}
\]

(35)

and

\[
B^n(M, \pi) = \{ B \mid B = [\pi, C] \quad C \in T_{n-1} \}.
\]

(36)

The quotient

\[
H^n(M, \pi) = \frac{Z^n(M, \pi)}{B^n(M, \pi)}
\]

(37)

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is the $n$th cohomology group.

**Proposition 1** Let $(M, \pi_1), (M, \pi_2)$ two Poisson structures on $M$. The following are equivalent:

i) $\pi_1 + \pi_2$ is Poisson.

ii) $[\pi_1, \pi_2] = 0$.

iii) $\partial_{\pi_1} \partial_{\pi_2} = -\partial_{\pi_2} \partial_{\pi_1}$.

iv) $\pi_1 \in Z^2(M, \pi_2)$, $\pi_2 \in Z^2(M, \pi_1)$.

Two tensors which satisfy the equivalent conditions are said to form a **Poisson pair** on $M$. The corresponding Poisson brackets are called **compatible**.

**Lemma 1** Suppose $\pi_1$ is Poisson and $\pi_2 = L_X \pi_1 = -\partial_{\pi_1} X$ for some vector field $X$. Then $\pi_1$ is compatible with $\pi_2$.

**Proof:**

$$[\pi_1, \pi_2] = [\pi_1, -[\pi_1, X]] = -\partial_{\pi_1} \partial_{\pi_1} X = 0 \quad \square . \quad (38)$$

If $\pi_1$ is symplectic, we call the Poisson pair non-degenerate. If we assume a non-degenerate pair we make the following definition: The **recursion operator** associated with a non-degenerate pair is the $(1, 1)$-tensor $R$ defined by

$$R = \pi_2 \pi_1^{-1}. \quad (39)$$

A **bi-Hamiltonian system** is defined by specifying two Hamiltonian functions $H_1, H_2$ satisfying:

$$X = \pi_1 \nabla H_2 = \pi_2 \nabla H_1 . \quad (40)$$

We have the following result due to Magri [4]
Theorem 3 Suppose we have a bi-Hamiltonian system on a manifold $M$, whose first cohomology group is trivial. Then there exists a hierarchy of mutually commuting functions $H_1, H_2, \ldots$, all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows $\chi_i, i = 1, 2, \ldots$, satisfying the Lenard recursion relations
\[ \chi_{i+j} = \pi_i \nabla H_j, \]
where $\pi_{i+1} = R^i \pi_1$ are the higher order Poisson tensors.

Let $\mathcal{G}$ be a Lie algebra and consider the Lie-Poisson manifold $\mathcal{G}^*$. Define a representation $\rho$ of $\mathcal{G}$ with values in $C^\infty(\mathcal{G}^*)$ by
\[ \rho(x_i) f = \sum_{j,k} c_{ij}^k \frac{\partial f}{\partial x_j}, \]
where $x_i$ denotes coordinates on $\mathcal{G}^*$ and at the same time elements of a basis for $\mathcal{G}$. In other words, $\rho(x_i) f = \{x_i, f\}$, where the bracket is the Lie-Poisson bracket on $\mathcal{G}^*$. We denote the $n$th cohomology group of $\mathcal{G}$ with respect to this representation by
\[ H^n(\mathcal{G}, C^\infty(\mathcal{G}^*)). \]

We have the following result:

Theorem 4
\[ H^n(\mathcal{G}^*, \pi) \cong H^n(\mathcal{G}, C^\infty(\mathcal{G}^*)). \]

The proof can be found in [36] or [28].

Master Symmetries

We recall the definition and basic properties of master symmetries following Fuchssteiner [13].

Consider a differential equation on a manifold $M$, defined by a vector field $\chi$. We are mostly interested in the case where $\chi$ is a Hamiltonian vector field. A vector field $Z$ is a symmetry of the equation if
\[ [Z, \chi] = 0. \]
If $Z$ is time dependent, then a more general condition is
\[
\frac{\partial Z}{\partial t} + [Z, \chi] = 0 .
\] (46)

A vector field $Z$ will be called a master symmetry if
\[
[[Z, \chi], \chi] = 0 ,
\] (47)
but
\[
[Z, \chi] \neq 0 .
\] (48)

Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors $J_0, J_1$ and the Hamiltonians $h_0, h_1$. Assume that $J_0$ is symplectic. We define the recursion operator $\mathcal{R} = J_1 J_0^{-1}$, the higher flows
\[
\chi_i = \mathcal{R}^{i-1} \chi_1 ,
\] (49)
and the higher order Poisson tensors
\[
J_i = \mathcal{R}^i J_0 .
\] (50)

Master symmetries preserve constants of motion, Hamiltonian vector fields and generate hierarchies of Poisson structures. For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to W. Oevel [16].

**Theorem 5** Suppose that $Z_0$ is a conformal symmetry for both $J_0$, $J_1$ and $h_0$, i.e., for some scalars $\lambda, \mu, \nu$ we have
\[
L_{Z_0} J_0 = \lambda J_0, \quad L_{Z_0} J_1 = \mu J_1, \quad L_{Z_0} h_0 = \nu h_0 .
\] (51)

Then the vector fields
\[
Z_i = \mathcal{R}^i Z_0
\] (52)
are master symmetries and we have
(a) \[ [Z_i, \chi_j] = (\mu + \nu + (j - 1)(\mu - \lambda)) \chi_{i+j} \] (53)
(b) \[ [Z_i, Z_j] = (\mu - \lambda)(j - i) Z_{i+j} \] (54)
(c) \[ L_{Z_i} J_j = (\mu + (j - i - 1)(\mu - \lambda)) J_{i+j} \] (55)
III THE TODA LATTICE

Definition of the System

The Toda lattice is a Hamiltonian system with Hamiltonian function

\[
H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}. \tag{56}
\]

Hamilton’s equations become

\[
\dot{q}_j = p_j, \quad \dot{p}_j = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}. \tag{57}
\]

This system is integrable. One can find a set of independent functions \( \{H_1, \ldots, H_N\} \) which are constants of motion for Hamilton’s equations. To determine the constants of motion, one uses Flaschka’s transformation:

\[
a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad b_i = -\frac{1}{2} p_i. \tag{58}
\]

Then

\[
\dot{a}_i = a_i (b_{i+1} - b_i), \quad \dot{b}_i = 2 (a_i^2 - a_{i-1}^2). \tag{59}
\]

These equations can be written as a Lax pair \( \dot{L} = [B, L] \), where \( L \) is the Jacobi matrix

\[
L = \begin{pmatrix}
    b_1 & a_1 & 0 & \cdots & \cdots & 0 \\
    a_1 & b_2 & a_2 & \cdots & & \\
    0 & a_2 & b_3 & \cdots & & \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \\
    \vdots & & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & \cdots & \cdots & a_{N-1} & b_N
\end{pmatrix}, \tag{60}
\]

and
It follows that the functions $H_i = \frac{1}{i} \text{tr} L^i$ are constants of motion.

Remark: The Lax equation

$$\dot{L}(t) = [B(t), L(t)], \quad L(0) = L_0$$

(62)

can be solved by factorization. First we perform a Gram-Schmidt factorization $e^{tL_0} = k(t)b(t)$, where $k(t)$ is orthogonal and $b(t)$ upper triangular. The solution is given by

$$L(t) = k(t)^{-1} L_0 k(t).$$

(63)

The form of the solution shows again that the functions trace $L^k$ are independent of $t$.

**Quadratic Toda bracket**

Consider $\mathbb{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

(64)

and the mapping $F : \mathbb{R}^{2N} \to \mathbb{R}^{2N-1}$ defined by

$$F : (q_1,\ldots, q_N, p_1,\ldots, p_N) \to (a_1,\ldots, a_{N-1}, b_1,\ldots, b_N).$$

(65)

Define a bracket on $\mathbb{R}^{2N-1}$ by

$$\{f, g\} = \{f \circ F, g \circ F\}_s.$$
We obtain a bracket defined by
\[
\{a_i, b_i\} = -a_i \\
\{a_i, b_{i+1}\} = a_i.
\] (67)
All other brackets are zero. \(H_1 = b_1 + b_2 + \ldots + b_N\) is the only Casimir. The Hamiltonian in this bracket is \(H_2 = \frac{1}{2} \text{tr } L^2\). We later prove that \(\{H_i, H_j\} = 0\). We denote this bracket by \(\pi_1\).

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let \(\lambda\) be an eigenvalue of \(L\) with normalized eigenvector \(v\). Standard perturbation theory shows that
\[
\nabla \lambda = (2v_1v_2, \ldots, v_{N-1}v_N, v_1^2, \ldots, v_N^2)^T := U^\lambda, 
\] (68)
where \(\nabla \lambda\) denotes \((\frac{\partial \lambda}{\partial a_1}, \ldots, \frac{\partial \lambda}{\partial a_N})\). Some manipulations show that \(U^\lambda\) satisfies
\[
MU^\lambda = \lambda NU^\lambda, 
\] (69)
where \(M\) and \(N\) are skew-symmetric matrices. It turns out that \(N\) is the matrix of coefficients of the Poisson tensor (67), and \(M\), whose coefficients are quadratic functions of the \(a\)'s and \(b\)'s, can be used to define a new Poisson tensor. We denote this quadratic bracket by \(\pi_2\). The defining relations are:
\[
\{a_i, a_{i+1}\} = \frac{1}{2}a_i a_{i+1} \\
\{a_i, b_i\} = -a_i b_i \\
\{a_i, b_{i+1}\} = a_i b_{i+1} \\
\{b_i, b_{i+1}\} = 2 a_i^2 ;
\] (70)
all other brackets are zero. This bracket has \(\det L\) as Casimir and \(H_1 = \text{tr } L\) is the Hamiltonian. The eigenvalues of \(L\) are still in involution. Furthermore, \(\pi_2\) is compatible with \(\pi_1\). We also have
\[
M \nabla \lambda_j = \lambda_j N \nabla \lambda_j \quad \forall j.
\] (71)
So,
\[
M \nabla \frac{1}{l} \sum \lambda_j^l = \sum \lambda_j^{l-1} M \nabla \lambda_j
= \sum \lambda_j^l N \nabla \lambda_j
= N \nabla \frac{1}{l+1} \sum \lambda_j^{l+1}.
\]
(72)

Therefore,
\[
M \nabla H_l = N \nabla H_{l+1}.
\]
(73)

These relations are similar to the Lenard relations for the KdV equation. We will generalize them later.

Finally, we remark that further manipulations with the Lenard relations for the infinite Toda lattice, followed by setting all but finitely many \(a_i, b_i\) equal to zero, yield another Poisson bracket, \(\pi_3\), which is cubic in the coordinates. The defining relations for \(\pi_3\) are:

\[
\begin{align*}
\{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1} \\
\{a_i, b_i\} &= -a_i b_i^2 - a_i^3 \\
\{a_i, b_{i+1}\} &= a_i b_i^2 + a_i^3 \\
\{a_i, b_{i+2}\} &= a_i a_{i+1}^2 \\
\{a_{i+1}, b_i\} &= -a_i^2 a_{i+1} \\
\{b_i, b_{i+1}\} &= 2 a_i^2 (b_i + b_{i+1});
\end{align*}
\]
(74)

all other brackets are zero. The bracket \(\pi_3\) is compatible with both \(\pi_1\) and \(\pi_2\) and the eigenvalues of \(L\) are still in involution.

**Construction of master symmetries**

In [6] a sequence of master symmetries \(X_n\), for \(n \geq -1\) was constructed. These vector fields generate an infinite sequence of contravariant 2-tensors \(\pi_n\), for \(n \geq 1\). Before stating the theorem, we define an equivalence relation on the space of Poisson tensors. A bracket is *trivial* if \(H_i\) is a Casimir \(\forall i\). Two brackets are considered equivalent if their difference is a trivial bracket. An example of a trivial bracket is \(\chi_i \wedge \chi_j\). There are examples of trivial brackets that are not of this form. We summarize the properties of \(X_n\) and \(\pi_n\) in the following:
Theorem 6

i) $\pi_n$ are all Poisson.

ii) The functions $H_n = \frac{1}{n} \text{Tr } L^n$ are in involution with respect to all of the $\pi_n$.

iii) $X_n(H_m) = (n + m)H_{n+m}$.

iv) $L_{X_n} \pi_m = (m - n - 2)\pi_{n+m}$, up to equivalence.

v) $[X_n, \chi_l] = (l - 1)\chi_{l+n}$, where $\chi_l$ is the Hamiltonian vector field generated by $H_l$ with respect to $\pi_1$.

vi) $\pi_n \nabla H_l = \pi_{n-1} \nabla H_{l+1}$, where $\pi_n$ denotes the Poisson matrix of the tensor $\pi_n$.

We give an outline of the construction of the vector fields $X_n$. We define $X_{-1}$ to be

$$\text{grad } H_1 = \text{grad } \text{Tr } L = \sum_{i=1}^{N} \frac{\partial}{\partial b_i}$$

and $X_0$ to be the Euler vector field

$$\sum_{i=1}^{N-1} a_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial b_i}.$$  \hspace{1cm} (76)

We want $X_1$ to satisfy

$$X_1(\text{Tr } L^n) = n\text{Tr } L^{n+1}.$$ \hspace{1cm} (77)

One way to find such a vector field is by considering the equation

$$\dot{L} = [B, L] + L^2.$$ \hspace{1cm} (78)

Note that the left hand side of this equation is a tridiagonal matrix while the right hand side is pentadiagonal. We look for $B$ as a tridiagonal matrix

$$B = \begin{pmatrix}
\gamma_1 & \beta_1 & 0 & \cdots & \cdots \\
\alpha_1 & \gamma_2 & \beta_2 & \cdots & \cdots \\
0 & \alpha_2 & \gamma_3 & \beta_3 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}.$$ \hspace{1cm} (79)
We want to choose the $\alpha_i$, $\beta_i$ and $\gamma_i$ so that the right hand side of equation (78) becomes tridiagonal. One simple solution is $\alpha_n = -(n+1)a_n$, $\beta_n = (n+1)a_n$, $\gamma_n = 0$. The vector field $X_1$ is defined by the right hand side of (78) and:

$$X_1 = \sum_{n=1}^{N-1} \dot{a}_n \frac{\partial}{\partial a_n} + \sum_{n=1}^{N} \dot{b}_n \frac{\partial}{\partial b_n},$$

where

$$\dot{a}_n = -na_n b_n + (n+2)a_n b_{n+1}$$

$$\dot{b}_n = (2n+3)a_n^2 + (1-2n)a_{n-1}^2 + b_n^2.$$

The construction of the vector field $X_2$ is similar. We consider the equation

$$\dot{L} = [B, L] + L^3.$$

The calculations are similar to those for $X_1$. The matrix $B$ is now pentadiagonal and the system of equations slightly more complicated.

We continue the sequence of master symmetries for $n \geq 3$ by

$$[X_1, X_{n-1}] = (n-2)X_n.$$  

Properties of $X_n$ and $\pi_n$.

It is well known that $\pi_1$, $\pi_2$, $\pi_3$ satisfy Lenard relations

$$\pi_n \nabla H_l = \pi_{n-1} \nabla H_{l+1}, \quad n = 2, 3 \quad \forall l.$$

We want to show that these relations hold for all values of $n$. We denote the Hamiltonian vector field of $H_l$ with respect to the $n$th bracket by $\chi_l^n$. In other words,

$$\chi_l^n = [\pi_n, H_l].$$

We prove the Lenard relations in an equivalent form
Proposition 2 $\chi^{n+1}_l = \chi^n_{l+1}$.

Proof: To prove this we need the identity

$$[X_1, \chi^n_l] = (n - 3)\chi^{n+1}_l + (l + 1)\chi^n_{l+1}$$  \hspace{1cm} (87)

which follows easily from $X_1(H_l) = (l + 1)H_{l+1}$ and (86). Therefore,

$$(n - 3)\chi^n_{l+1} = [X_1, \chi^n_l] - (l + 1)\chi^n_{l+1} = (n - 4)\chi^n_{l+1} + (l + 2)\chi^{n-1}_{l+2} - (l + 1)\chi^n_{l+1} \hspace{1cm} (88)$$

Using the Lenard relations we can show that the functions $H_n$ are in involution with respect to all of the brackets $\pi_n$.

Proposition 3 $\{H_i, H_j\}_n = 0$, where $\{ \ , \ \}_n$ is the bracket corresponding to $\pi_n$.

Proof: First we consider the Lie-Poisson Toda bracket. We have

$$\{H_1, H_j\} = 0 \quad \forall j \hspace{1cm} (89)$$

since $H_1$ is a Casimir for $\pi_1$. Suppose that $\{H_{i-1}, H_j\} = 0 \forall j$.

$$i\{H_i, H_j\} = \{X_1(H_{i-1}), H_j\} = -[\chi^j_l, [X_1, H_{i-1}]] = [X_1, \{H_{i-1}, H_j\}] + [H_{i-1}, (j + 1)\chi^{n+1}_j] = (j + 1)\{H_{i-1}, H_{j+1}\} = 0 \hspace{1cm} (90)$$

Now we use induction on $n$. Suppose

$$\{H_i, H_j\}_n = 0 \quad \forall i, j \hspace{1cm} (91)$$

$$\{H_i, H_j\}_{n+1} = \chi^{n+1}_{i+1}(H_j) = \chi^n_i(H_j) = \{H_{i+1}, H_j\}_n = 0 \hspace{1cm} (92)$$
It is straightforward to verify that the mapping
\[ f(a_1, \ldots, a_{N-1}, b_1, \ldots, b_N) = (a_1, \ldots, a_{N-1}, 1 + b_1, \ldots, 1 + b_N) \] (93)
is a Poisson map between \( \pi_2 \) and \( \pi_1 + \pi_2 \). Since \( f \) is a diffeomorphism, we have the isomorphism
\[ \pi_2 \cong \pi_1 + \pi_2 . \] (94)

In other words, the tensor \( \pi_2 \) encodes sufficient information for both the linear and quadratic Toda brackets. An easy induction generalizes this result: i.e.,

**Proposition 4**

\[ \pi_n \cong \sum_{j=0}^{n-1} \binom{n-1}{j} \pi_{n-j} . \] (95)

The function \( \text{tr} \ L^{2-n} \), which is well-defined on the open set \( \det L \neq 0 \), is a Casimir for \( \pi_n \), for \( n \geq 3 \). The proof uses the Lenard type relation
\[ \pi_n \nabla \lambda = \lambda \pi_{n-1} \nabla \lambda \] (96)
satisfied by the eigenvalues of \( L \). To prove the last equation, one uses the relation
\[ \pi_n \sum \lambda_k^{l-1} \nabla \lambda_k = \pi_{n-1} \sum \lambda_k^l \nabla \lambda_k . \] (97)

But
\[ \sum \lambda_k^{l-1}(\pi_k \nabla \lambda_k - \lambda_k \pi_{n-1} \nabla \lambda_k) = 0 , \] (98)
for \( l = 1, 2, \ldots, N + 1 \), has only the trivial solution because the (Vandermonde) coefficient determinant is nonzero.

**Proposition 5** For \( n > 2 \), \( \text{tr} \ L^{2-n} \) is a Casimir for \( \pi_n \) on the open dense set \( \det L \neq 0 \).

**Proof:** For \( n = 3 \),
\[
\pi_3 \nabla \text{tr} \ L^{-1} = \pi_3 \sum_k \frac{1}{\lambda_k} \nabla \lambda_k
= \sum_k \frac{1}{\lambda_k} \lambda_k \pi_2 \nabla \lambda_k
= -\sum_k \pi_1 \nabla \lambda_k
= -\pi_1 \nabla \text{tr} \ L = \chi_1 = 0 .
\] (99)
For $n > 3$ the induction step is as follows:

$$
\pi_n \nabla \text{tr} L^{2-n} = \pi_n \nabla \sum_k \lambda_k^{1/2} = \\
= \sum_k (2-n)1\lambda_k^{n-1} \pi_n \nabla \lambda_k = \\
= \frac{n-2}{n-3} \pi_{n-1} \nabla \text{tr} L^{3-n} = 0 .\tag{100}
$$

### Symmetries of Toda equations

In this subsection we find an infinite sequence of evolution vector fields that are symmetries of Toda equations (59). A symmetry group of a system of differential equations is a Lie group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. The symmetry approach to solving differential equations can be found, for example, in the books of Olver [37], Bluman and Cole [38], Bluman and Kumei [39], and Ovsiannikov [40].

We begin by writing equations (59) in the form

$$
\Gamma_j = \dot{a}_j - a_j b_{j+1} + a_j b_j = 0 \\
\Delta_j = \dot{b}_j - 2a_j^2 + 2a_{j-1}^2 = 0 .\tag{101}
$$

We look for symmetries of Toda equations. i.e., vector fields of the form

$$
v = \tau \frac{\partial}{\partial t} + \sum_{j=1}^{N-1} \phi_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} \psi_j \frac{\partial}{\partial b_j} ,\tag{102}
$$

that generate the symmetry group of the Toda System. The first prolongation of $v$ is

$$
\text{pr}^{(1)} v = v + \sum_{j=1}^{N-1} f_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} g_j \frac{\partial}{\partial b_j} ,\tag{103}
$$
where
\[ f_j = \dot{\phi}_j - \dot{\tau} \dot{a}_j \]
\[ g_j = \dot{\psi}_j - \dot{\tau} \dot{b}_j. \]  

The infinitesimal condition for a group to be a symmetry of the system is
\[ \text{pr}^{(1)}(\Gamma_j) = 0 \]
\[ \text{pr}^{(1)}(\Delta_j) = 0. \]  

Therefore we obtain the equations
\[ \dot{\phi}_j - \dot{\tau} a_j (b_{j+1} - b_j) + \phi_j (b_j - b_{j+1}) + a_j \psi_j - a_j \psi_{j+1} = 0 \]  
\[ \dot{\psi}_j - 2 \dot{\tau} (a_j^2 - a_{j-1}^2) - 4 a_j \phi_j + 4 a_{j-1} \phi_{j-1} = 0. \]  

We first give some obvious solutions:

i) \( \tau = 0, \phi_j = 0, \psi_j = 1. \) This is the vector field \( X_{-1}. \)

ii) \( \tau = -1, \phi_j = 0, \psi_j = 0. \) The resulting vector field is the time translation \(-\frac{\partial}{\partial t}\) whose evolutionary representative is
\[ \sum_{j=1}^{N-1} \dot{a}_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} \dot{b}_j \frac{\partial}{\partial b_j}. \]  

This is the Hamiltonian vector field \( \chi_{H_2}. \) It generates a Hamiltonian symmetry group.

iii) \( \tau = -t, \phi_j = a_j, \psi_j = b_j. \) Then
\[ \mathbf{v} = -t \frac{\partial}{\partial t} + \sum_{j=1}^{N-1} a_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial b_j} = -t \frac{\partial}{\partial t} + X_0. \]  

This vector field generates the same symmetry as the evolutionary vector field
\[ X_0 + t \chi_{H_2}. \]
We next look for some non-obvious solutions. The vector field $X_1$ is not a symmetry, so we add a term which depends on time. We try 

$$
\begin{align*}
\phi_j &= -ja_j b_j + (j + 2)a_j a_{j+1} + t(a_j^2 a_{j+2} - a_j b_j) \\
\psi_j &= (2j + 3)a_j^2 + (1 - 2j)a_j b_j + b_j + \\
&\quad + t(2a_j a_{j+1} + 2a_j^2 - 2a_{j-1} a_j - 2a_{j-1} b_j),
\end{align*}
$$

(111)

with $\tau = 0$.

A tedious but straightforward calculation shows that $\phi_j, \psi_j$ satisfy (106) and (107). It is also straightforward to check that the vector field $\sum \phi_j \frac{\partial}{\partial a_j} + \sum \psi_j \frac{\partial}{\partial b_j}$ is precisely equal to $X_1 + t\chi H_3$. The pattern suggests that $X_n + t\chi H_{n+2}$ is a symmetry of Toda equations.

**Theorem 7** The vector fields $X_n + t\chi_{n+2}$ are symmetries of Toda equations for $n \geq -1$.

**Proof**: Note that $\chi H_1 = 0$ because $H_1$ is a Casimir for the Lie-Poisson bracket. We use the formula

$$
[X_n, \chi_l] = (l - 1)\chi_{n+l}.
$$

(112)

In particular, for $l = 2$, we have $[X_n, \chi_2] = \chi_{n+2}$.

Since the Toda flow is Hamiltonian, generated by $\chi_2$, to show that $Y_n = X_n + t\chi_{n+2}$ are symmetries of Toda equations we must verify the equation

$$
\frac{\partial Y_n}{\partial t} + [\chi_2, Y_n] = 0.
$$

(113)

But

$$
\begin{align*}
\frac{\partial Y_n}{\partial t} + [\chi_2, Y_n] &= \frac{\partial Y_n}{\partial t} + [\chi_2, X_n + t\chi_{n+2}] \\
&= \chi_{n+2} - [X_n, \chi_2] \\
&= \chi_{n+2} - \chi_{n+2} = 0.
\end{align*}
$$

(114)

$\Box$

26
A recursion operator for the Toda lattice

One way of finding master symmetries of finite dimensional systems is the method used by some authors in the case of the finite nonperiodic Toda lattice. We briefly describe the procedure.

The first step is to define a second Poisson bracket on the space of canonical variables \((q_1, \ldots, q_N, p_1, \ldots, p_N)\). This bracket appears in Das and Okubo [14] and Fernandes [15]. We follow the notation from [15]. Let \(J_0\) be the symplectic bracket and define \(J_1\) as follows:

\[
\{q_i, q_j\} = 1, \\
\{p_i, q_j\} = p_i, \\
\{p_i, p_{i+1}\} = e^{q_i - q_{i+1}}.
\] (115)

Also define

\[
h_0 = \sum_{i=1}^N p_i, \quad h_1 = \sum_{i=1}^N \frac{p_i^2}{2} + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.
\] (116)

The recursion operator is defined by \(R = J_1 J_0^{-1}\). It follows easily that the vector field

\[
Z_0 = \sum_{i=1}^N \frac{N+1-2i}{2} \frac{\partial}{\partial q_i} + \sum_{i=1}^N p_i \frac{\partial}{\partial p_i}
\] (117)

is a conformal symmetry for \(J_0\), \(J_1\) and \(h_0\) and therefore, theorem 5 applies. The constants in Theorem 5 turn out to be \(\lambda = -1\), \(\mu = 0\) and \(\nu = 1\). We end up with the following deformation relations:

\[
L_{Z_i} J_j = (j - i - 1) J_{i+j} \\
[Z_i, Z_j] = (j - i) Z_{i+j} \\
[Z_i, \chi_j] = j \chi_{i+j}
\] (118)

Recall the Flaschka transformation (58) \(F : \mathbb{R}^{2N} \to \mathbb{R}^{2N-1}\) defined by

\[
F : (q_1, \ldots, q_N, p_1, \ldots, p_N) \to (a_1, \ldots, a_N, b_1, \ldots, b_N).
\] (119)
The Poisson tensors $J_0$ and $J_1$ reduce to $\mathbb{R}^{2N-1}$. They reduce precisely to the tensors $\pi_1$ and $\pi_2$ of section III. The mapping $F$ is a Poisson mapping between $J_0$ and $\pi_1$. It is also a Poisson mapping between $J_1$ and $\pi_2$. The Hamiltonians $h_0$ and $h_1$ correspond to the reduced Hamiltonians $H_1$ and $H_2$ respectively. The recursion operator $R$ cannot be reduced. Actually, it is easy to see that there exists no recursion operator in the reduced space. The kernels of the two Poisson structures $\pi_1$ and $\pi_2$ are different and, therefore, it is impossible to find an operator that maps one to the other.

The deformation relations (118) become precisely the deformation relations of Theorem 6. Of course, one has to replace $j$ by $j - 1$ in the formulas involving $J_j$ because of the difference in notation between [6] and [15]. We should note that the vector field $Z_1$ corresponds to the vector $X_1$ up to addition of a Hamiltonian vector field. For this reason the reduced $J_i$ correspond to $\pi_i$. However $Z_2$ does not correspond to $X_2$. This implies two things:

i) The master symmetry $X_2$ does not come from the given recursion operator. A question we could ask, is whether every master symmetry in the reduced space comes from a recursion operator.

ii) If one replaces $X_2$ by the reduced $Z_2$ then the relations in Theorem 6 become exact and we do not need an equivalence relation.

IV RELATIVISTIC TODA SYSTEMS

In this section some results on the non-relativistic Toda lattice are extended to the case of the relativistic Toda Systems. The main new result is the hierarchy of higher Poisson brackets which also exists in the finite nonperiodic Toda lattice.

The relativistic Toda lattice was introduced and studied by Ruijsenaars in [22]; see also [30], [41], [42]. In terms of canonical coordinates the relativistic Toda lattice is defined by the Hamiltonian

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{j=1}^{N} e^{p_j} f(q_{j-1} - q_j) f(q_j - q_{j+1}), \quad (120)$$
where \( f(x) = \sqrt{1 + g^2 e^x} \) and, by convention, \( q_0 = -\infty \), \( q_{N+1} = \infty \). The number \( g \) is a coupling constant. To see the connection with the non-relativistic Toda lattice one writes the equation in Newtonian form

\[
\ddot{q}_j = g^2 \dot{q}_j (\dot{q}_{j-1} - q_j) \cdot \exp(q_{j-1} - q_j) / (1 + g^2 \exp(q_{j-1} - q_j)) - \dot{q}_{j+1} \cdot \exp(q_j - q_{j+1}) / (1 + g^2 \exp(q_j - q_{j+1})) ,
\]

\( j = 1, 2, \ldots, N \). Setting \( \dot{q}_j = \dot{Q}_j + c \) and letting \( c \to \infty \) and \( g \to 0 \) in such a way that \( gc = 1 \), one obtains the equations of motion for the classical Toda lattice

\[
\dot{Q}_j = e^{Q_{j-1} - Q_j} - e^{Q_j - Q_{j+1}} .
\]

In the classical case one uses a change of variables to prove integrability. We follow the same technique. Combining the changes of variables from [22], [41] and [42], we set

\[
a_j = g^2 e^{q_j - q_{j+1} + p_j} f(q_{j-1} - q_j) / f(q_j - q_{j+1})
\]

\[
b_j = \dot{q}_j - a_j \]

In these variables the Hamiltonian is homogeneous quadratic and the equations of motion become:

\[
\dot{a}_j = a_j (b_j - b_{j+1} + a_{j-1} - a_{j+1})
\]

\[
\dot{b}_j = b_j (a_{j-1} - a_j) .
\]

These equations can be written as a Lax pair \( \dot{L} = [L, B] \), where \( L \) is the matrix

\[
L = \begin{pmatrix}
a_1 + b_1 & a_1 & 0 & \cdots & \cdots & 0 \\
 a_2 + b_2 & a_2 + b_2 & a_2 & 0 & \cdots & 0 \\
 a_3 + b_3 & a_3 + b_3 & a_3 + b_3 & a_3 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 a_{N-1} + b_{N-1} & \cdots & \cdots & \cdots & a_{N-1} + b_{N-1} & a_{N-1} \\
 b_N & b_N & \cdots & \cdots & \cdots & b_N
\end{pmatrix},
\]

and

29
\[
B = \begin{pmatrix}
0 & a_1 & 0 & \cdots & \cdots & 0 \\
0 & -a_1 & a_2 & \cdots & \cdots & 0 \\
0 & 0 & -a_2 & a_3 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & -a_{N-1}
\end{pmatrix}.
\] (126)

This shows that the functions \( H_n = \frac{1}{n} \text{Tr} \ L^n \) are constants of motion and therefore the system is integrable.

In the new coordinates \( a_j, b_j \) the symplectic bracket is transformed into a new quadratic Poisson bracket defined as follows:

\[
\{a_i, a_{i+1}\} = a_i a_{i+1} \\
\{a_i, b_i\} = -a_i b_i \\
\{a_i, b_{i+1}\} = a_i b_{i+1}.
\] (127)

All other brackets are zero. This bracket has \( \det L = \prod_{i=1}^{N} b_i \) as Casimir, and the eigenvalues of \( L \) are in involution. Taking \( H_1 = \text{Tr} \ L \) as Hamiltonian we obtain equations (124). We denote this bracket by \( \pi_2 \). Note that in the case of non-relativistic Toda lattice the symplectic bracket is transformed into a linear Poisson bracket.

We next define a linear bracket \( \pi_1 \) as follows:

\[
\{a_i, b_i\} = -a_i \\
\{a_i, b_{i+1}\} = a_i \\
\{b_i, b_{i+1}\} = -a_i.
\] (128)

All other brackets are zero. In this bracket \( \text{Tr} \ L \) is the Casimir and \( H_2 = \frac{1}{2} \text{Tr} \ L^2 \) is the Hamiltonian. Therefore we have a bi-Hamiltonian system, a situation similar to the classical case.

To find higher Poisson brackets we work as in section III. We construct a vector field \( X_1 \) which satisfies \( X_1(H_m) = (m + 1)H_{m+1} \). One possibility is

\[
X_1 = \sum_{i=1}^{N-1} r_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{N} s_i \frac{\partial}{\partial b_i},
\] (129)

where
\[ r_i = a_i^2 + (i + 2)a_i b_{i+1} + (1 - i)a_i b_i + (i + 2)a_i a_{i+1} + (1 - i)a_{i-1} a_i \]
\[ s_i = b_i^2 + (i + 1)a_i b_i + (1 - i)a_{i-1} b_i. \]

(130)

Taking the Lie derivative of \( \pi_2 \) in the direction of \( X_1 \), we obtain a cubic Poisson bracket \( \pi_3 \):

\[
\begin{align*}
\{a_i, a_{i+1}\} &= a_i^2 a_{i+1} + a_i a_{i+1}^2 + 2a_i a_{i+1} b_{i+1} \\
\{a_i, a_{i+2}\} &= a_i a_{i+1} a_{i+2} \\
\{a_i, b_i\} &= -a_i b_i (a_i + b_i) \\
\{a_i, b_{i+1}\} &= a_i b_{i+1} (a_i + b_i+1) \\
\{a_i, b_{i+2}\} &= a_i a_{i+1} b_{i+2} \\
\{a_{i+1}, b_i\} &= -a_i a_{i+1} b_i \\
\{b_i, b_{i+1}\} &= a_i b_i b_{i+1}
\end{align*}
\]

(131)

This bracket is compatible with both \( \pi_1 \) and \( \pi_2 \) and the eigenvalues of \( L \) are still in involution. The Casimir in this bracket is \( \text{tr} \, L^{-1} \).

Another iteration of the procedure gives nothing new since \( L_{X_1} \pi_3 = 0 \).

In a similar way we construct a vector field \( X_2 \) which satisfies \( X_2(H_m) = (m + 2)H_{m+2} \). For \( N = 3 \) we can take

\[
X_2 = \sum_{i=1}^{2} r_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{3} s_i \frac{\partial}{\partial b_i}
\]

(132)

where

\[
\begin{align*}
r_1 &= a_1 (a_1^2 + 5a_1 b_1 - a_1^2 + 2a_2 b_1 - 2a_2 b_2 - a_2 b_3 + 4b_1^2 + 2b_1 b_2 - b_2^2) \\
r_2 &= a_2 (3a_1^2 + 4a_1 a_2 + 3a_1 b_1 + 6a_1 b_2 + 2a_1 b_3 + a_1^2 + 4a_2 b_2 + a_2 b_3 - 2b_1 b_2 + 2b_1 b_3 + 3b_2^2 + 2b_2 b_3) \\
s_1 &= b_1 (-2a_1^2 - 2a_1 a_2 - a_1 b_1 - 2a_1 b_2 + b_1^2) \\
s_2 &= b_2 (3a_1^2 + 2a_1 a_2 + 3a_1 b_1 + 4a_1 b_2 - a_1^2 + 2a_2 b_1 - a_2 b_3 + b_2^2) \\
s_3 &= b_3 (2a_2^2 + 2a_1 a_2 - 2a_2 b_1 + 2a_2 b_2 + 3a_2 b_3 + b_3^2).
\end{align*}
\]

(133)

We define \( \pi_4 = L_{X_2} \pi_2 \). This bracket is Poisson and we still have involution of constants of motion. Another iteration, \( L_{X_2} \pi_4 \), gives a trivial bracket. The construction of \( X_2 \) allows one to construct \( X_3 = [X_1, X_2] \) and, inductively, a sequence \( X_1, X_2, \ldots \) which satisfies

\[
[X_n, X_m] = (m - n)X_{n+m}.
\]

(134)
Remark: One should try to find master symmetries of this system using the method of [15]. In the case of the relativistic Toda equations we were unable to construct a second bracket which projects onto either the linear \((\pi_1)\) or cubic \((\pi_3)\) bracket and therefore we cannot implement the same procedure.

V GENERALIZED TODA SYSTEMS ASSOCIATED WITH SIMPLE LIE GROUPS

Definition of the systems

In this section we consider mechanical systems which generalize the finite, nonperiodic Toda lattice. These systems correspond to Dynkin diagrams. It is well known that irreducible root systems classify simple Lie groups. So, in this generalization for each simple Lie algebra there exists a mechanical system of Toda type.

The generalization is obtained from the following simple observation: In terms of the natural basis \(q_i\) of weights, the simple roots of \(A_{n-1}\) are
\[
q_1 - q_2, q_2 - q_3, \ldots, q_{n-1} - q_n.
\]
(135)

On the other hand, the potential for the Toda lattice is of the form
\[
e^{q_1 - q_2} + e^{q_2 - q_3} + \ldots + e^{q_{n-1} - q_n}.
\]
(136)

We note that the angle between \(q_{i-1} - q_i\) and \(q_i - q_{i+1}\) is \(\frac{2\pi}{3}\) and the lengths of \(q_i - q_{i+1}\) are all equal. The Toda lattice corresponds to a Dynkin diagram of type \(A_{n-1}\).

More generally, we consider potentials of the form
\[
U = c_1 e^{f_1(q)} + \ldots + c_l e^{f_l(q)}
\]
(137)

where \(c_1, \ldots, c_l\) are constants, \(f_i(q)\) is linear and \(l\) is the rank of the simple Lie algebra. For each Dynkin diagram we construct a Hamiltonian system of Toda type. These systems are interesting
not only because they are integrable, but also for their fundamental
importance in the theory of semisimple Lie groups. For example
Kostant in [2] shows that the integration of these systems and the
theory of the finite dimensional representations of semisimple Lie
groups are equivalent.

For reference, we give a complete list of the Hamiltonians for each
simple Lie algebra.

\[ A_{n-1} \]

\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} \]

\[ B_n \]

\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n} \]

\[ C_n \]

\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{2q_n} \]

\[ D_n \]

\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_{n-1}+q_n} \]

\[ G_2 \]

\[ H = \frac{1}{2} \sum_{j=1}^{3} p_j^2 + e^{q_1-q_2} + e^{-2q_1+q_2+q_3} \]

\[ F_4 \]

\[ H = \frac{1}{2} \sum_{j=1}^{4} p_j^2 + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3} + e^{-\frac{1}{2}(q_4-q_1-q_2-q_3)} \]

\[ E_6 \]
\[
H = \frac{1}{2} \sum_{j} p_j^2 + \sum_{j} e^{q_j - q_{j+1}} + e^{-(q_1 + q_2)} + e^{\frac{1}{2}(-q_1 + q_2 + \ldots + q_7 - q_8)}
\]

**E₇**

\[
H = \frac{1}{2} \sum_{j} p_j^2 + \sum_{j} e^{q_j - q_{j+1}} + e^{-(q_1 + q_2)} + e^{\frac{1}{2}(-q_1 + q_2 + \ldots + q_7 - q_8)}
\]

**E₈**

\[
H = \frac{1}{2} \sum_{j} p_j^2 + \sum_{j} e^{q_j - q_{j+1}} + e^{-(q_1 + q_2)} + e^{\frac{1}{2}(-q_1 + q_2 + \ldots + q_7 - q_8)}
\]

We should note that the Hamiltonians in the list are not unique. For example, the $A_2$ Hamiltonian is

\[
H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{2} p_3^2 + e^{q_1 - q_2} + e^{q_2 - q_3} .
\]  

(138)

An equivalent system is

\[
H(Q_i, P_i) = \frac{1}{2} P_1^2 + \frac{1}{2} P_2^2 + e^{\sqrt{\frac{3}{2}} (q_1 + q_2)} + e^{-2\sqrt{\frac{3}{2}} q_2} .
\]  

(139)

The second Hamiltonian is obtained from the first by using the canonical transformation

\[
Q_1 = \frac{\sqrt{2}}{4} (q_1 + q_2 - 2q_3) 
\]  

(140)

\[
Q_2 = \frac{\sqrt{6}}{4} (q_2 - q_1) 
\]  

(141)

\[
P_1 = \frac{2}{\sqrt{2}} (p_1 + p_2) 
\]  

(142)

\[
P_2 = \frac{2}{\sqrt{6}} (p_2 - p_1) .
\]  

(143)
Another example is the following two systems, both corresponding to a Lie algebra of type $D_4$:

$$
\sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1} + e^{q_2} + e^{q_3} + e^{\frac{1}{4}(q_4 - q_1 - q_2 - q_3)} \quad (144)
$$

$$
\sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_4} + e^{q_4 + q_4} \quad (145)
$$

A rational bracket for $B_n$-Toda

Another way to describe these generalized Toda systems, is to give a Lax pair representation in each case. It can be shown that the equation $\dot{L} = [B, L]$ is equivalent to the equations of motion generated by the Hamiltonian $H_2 = \frac{1}{2} \text{tr} L^2$ on the orbit through $L$ of the coadjoint action of $B_-$ (lower triangular group) on the dual of its Lie algebra, $B_-^*$. The space $B_-^*$ can be identified with the set of symmetric matrices. This situation, which corresponds to $sl(n, \mathbb{C}) = A_{n-1}$ can be generalized to other semisimple Lie algebras. We use notation and definitions from Humphreys [43].

Let $\mathcal{G}$ be a semisimple Lie algebra, $\Phi$ a root system for $\mathcal{G}$, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ the simple roots, $H$ a Cartan subalgebra and $\mathcal{G}_\alpha$ the root space of $\alpha$. We denote by $x_\alpha$ a generator of $\mathcal{G}_\alpha$. Define

$$
B_- = H \oplus \sum_{\alpha < 0} \mathcal{G}_\alpha .
$$

(146)

There is an automorphism $\sigma$ of $\mathcal{G}$, of order 2, satisfying $\sigma(x_\alpha) = x_{-\alpha}$ and $\sigma(x_{-\alpha}) = x_\alpha$. Let $\mathcal{K} = \{x \in \mathcal{G} | \sigma(x) = -x\}$. Then we have a direct sum decomposition $\mathcal{G} = B_- \oplus \mathcal{K}$. The Toda flow is coadjoint flow on $B_-^*$ and the coadjoint invariant functions on $\mathcal{G}^*$, when restricted to $B_-^*$ are still in involution. The Jacobi elements are of the form

$$
L = \sum_{i=1}^{l} b_i h_i + \sum_{i=1}^{l} a_i (x_{\alpha_i} + x_{-\alpha_i}) .
$$

(147)
We define
\[ B = \sum_{i=1}^{l} a_i (x_{\alpha_i} - x_{-\alpha_i}) \] . \hspace{1cm} (148)

The generalized Toda flow takes the Lax pair form:
\[ \dot{L} = [B, L] . \hspace{1cm} (149) \]

The \( B_n \) Toda systems were shown to be Bi-Hamiltonian. The second bracket can be found in [28]. It turned out to be a rational bracket and it was obtained by using Dirac’s constrained bracket formula (30). The idea is to use the inclusion of \( B_n \) into \( A_{2n} \) and to restrict the hierarchy of brackets from \( A_{2n} \) to \( B_n \) via Dirac’s bracket. Straightforward restriction does not work. We briefly describe the procedure in the case of \( B_2 \).

The Jacobi matrices for \( A_4 \) and \( B_2 \) are given by:
\[ L_{A_4} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & a_3 & 0 \\ 0 & 0 & a_3 & b_4 & a_4 \\ 0 & 0 & 0 & a_4 & b_5 \end{pmatrix} , \hspace{1cm} (150) \]
and
\[ L_{B_2} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & -a_2 & 0 \\ 0 & 0 & -a_2 & 2b_3 - b_2 & -a_1 \\ 0 & 0 & 0 & -a_1 & 2b_3 - b_1 \end{pmatrix} . \hspace{1cm} (151) \]

Note that \( L_{A_4} \) lies in \( \text{gl}(4, \mathbb{C}) \) instead of \( \text{sl}(4, \mathbb{C}) \). Therefore we have added an additional variable in \( L_{B_2} \). We define
\[ p_1 = a_1 + a_4 \]
\[ p_2 = a_2 + a_3 \]
\[ p_3 = b_1 + b_5 - 2b_3 \]
\[ p_4 = b_2 + b_4 - 2b_3 . \hspace{1cm} (152) \]
It is clear that we obtain $B_2$ from $A_4$ by setting $p_i = 0$ for $i = 1, 2, 3, 4$. We calculate the matrix $P = \{p_i, p_j\}$. The bracket used is the quadratic Toda (70) on $A_4$.

$$
\begin{align*}
\{p_1, p_2\} &= \frac{1}{2}(a_1a_2 - a_3a_4) \\
\{p_1, p_3\} &= a_4b_5 - a_1b_1 \\
\{p_1, p_4\} &= a_1b_2 - a_4b_4 \\
\{p_2, p_3\} &= 2(a_3b_3 - 2a_2b_3) \\
\{p_2, p_4\} &= a_3b_4 + 2a_3b_3 - 2a_2b_3 - a_2b_2 \\
\{p_3, p_4\} &= -2a_1^2 - 4a_2^2 + 4a_3^2 + 2a_1^2.
\end{align*}
$$

(153)

If we evaluate at a point in $B_2$ we get

$$
\begin{align*}
\{p_1, p_2\} &= 0 \\
\{p_1, p_3\} &= -2a_1b_3 \\
\{p_1, p_4\} &= 2a_1b_3 \\
\{p_2, p_3\} &= -4a_2b_3 \\
\{p_2, p_4\} &= -6a_2b_3 \\
\{p_3, p_4\} &= 0.
\end{align*}
$$

(154)

Therefore the matrix $P$ is given by

$$
P = \begin{pmatrix}
0 & 0 & -2a_1b_3 & 2a_1b_3 \\
0 & 0 & -4a_2b_3 & -6a_2b_3 \\
2a_1b_3 & 4a_2b_3 & 0 & 0 \\
-2a_1b_3 & 6a_2b_3 & 0 & 0
\end{pmatrix}
$$

(155)

and $P^{-1}$ is the matrix

$$
P^{-1} = \begin{pmatrix}
0 & 0 & \frac{3}{10a_1b_5} & -\frac{1}{5a_1b_5} \\
0 & 0 & \frac{1}{10a_2b_5} & -\frac{1}{10a_2b_5} \\
-\frac{3}{10a_1b_5} & \frac{1}{10a_2b_5} & 0 & 0 \\
-\frac{1}{10a_1b_5} & \frac{1}{10a_2b_5} & 0 & 0
\end{pmatrix}
$$

(156)

Using Dirac’s formula we obtain a homogeneous quadratic bracket on $B_2$ given by:
\{a_1, a_2\} = \frac{a_1a_2(3b_3 - b_2 - 2b_1)}{10b_3}
\{a_1, b_1\} = \frac{-a_1(10b_1b_3 - 2b_1b_2 - 3b_1^2 - a_1^2)}{10b_3}
\{a_1, b_2\} = \frac{a_1(10b_2b_3 - 3b_2^2 - 2b_1b_2 - 4a_2^2 - a_1^2)}{10b_3}
\{a_1, b_3\} = \frac{a_1(b_2 - b_1)}{5}
\{a_2, b_1\} = \frac{a_2(2b_1b_3 - 2b_1b_2 + a_1^2)}{10b_3}
\{a_2, b_2\} = \frac{-a_2(8b_2b_3 - 3b_2^2 - 6a_2^2 - 4a_1^2)}{10b_3}
\{a_2, b_3\} = \frac{a_2(b_3 - b_2)}{5}
\{b_1, b_2\} = \frac{10a_1^2b_3 - 3a_1^2b_2 - 2a_2^2b_1 - 3a_1^2b_1}{5b_3}
\{b_1, b_3\} = \frac{2a_1^2}{5}
\{b_2, b_3\} = \frac{2}{5}(a_2^2 - a_1^2).

(157)

The bracket satisfies the following properties which are analogous to the quadratic $A_n$ Toda (70).

i) It is a homogeneous quadratic Poisson bracket.

ii) It is compatible with the $B_2$ Lie-Poisson bracket.

iii) The functions $H_n = \frac{1}{n} \text{tr} L^n$ are in involution in this bracket.

iv) We have Lenard type relations $\pi_2 \nabla H_i = \pi_1 \nabla H_{i+1}$ where $\pi_1$, $\pi_2$ are the component matrices of the linear and quadratic $B_2$ Toda brackets respectively.

v) The function $\det L$ is the Casimir.

We conjecture that the only bracket satisfying all five properties is the one just obtained. In [28] a quadratic bracket is defined which satisfies properties i)- iv) but not v).
A recursion operator for $B_n$ Toda systems

In this section, we use a different approach to show that polynomial brackets exist in the case of $B_n$ Toda systems. We will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution.

The Hamiltonian for $B_n$ is

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_j - q_2} + \ldots + e^{q_{n-1} - q_n} + e^{q_n}.$$  \hspace{1cm} (158)

We make a Flaschka-type transformation

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad a_n = \frac{1}{2} e^{\frac{1}{2}q_n},$$

$$b_i = -\frac{1}{2} p_i.$$ \hspace{1cm} (159)

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where $L$ is the matrix

$$
\begin{pmatrix}
  b_1 & a_1 & & & & \\
  a_1 & \ddots & \ddots & & & \\
  & \ddots & \ddots & \ddots & & \\
  & & a_{n-1} & b_n & a_n & \\
  & & a_n & 0 & -a_n & \\
  & & & -a_n & -b_n & \ddots \\
  & & & & \ddots & \ddots & -a_1 \\
  & & & & & -a_1 & -b_1
\end{pmatrix} \hspace{1cm} (161)
$$

In the new variables $a_i, b_i$ the symplectic bracket $\pi_1$ is given by

$$\{a_i, b_i\} = -a_i,$$

$$\{a_i, b_{i+1}\} = a_i.$$ \hspace{1cm} (162)

The invariant polynomials for $B_n$, which we denote by

$$H_2, H_4, \ldots H_{2n}$$ \hspace{1cm} (163)
are defined by $H_{2i} = \frac{1}{2i} \text{Tr} \ L^{2i}$.

We look for a bracket $\pi_3$ which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4 .$$ \hspace{1cm} (164)

Using trial and error, we end up with the following homogeneous cubic bracket $\pi_3$.

$$\begin{align*}
\{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1} \\
\{a_i, b_i\} &= -a_i b_i^2 - a_i^3 \hspace{1cm} i = 1, 2, \ldots, n - 1 \\
\{a_n, b_n\} &= -a_n b_n^2 - 2a_n^3 \\
\{a_i, b_{i+2}\} &= a_i a_{i+1}^2 \\
\{a_i, b_{i+1}\} &= a_i b_{i+1}^2 + a_i^3 \\
\{a_i, b_{i-1}\} &= -a_{i-1}^2 a_i \\
\{b_i, b_{i+1}\} &= 2a_i^2 (b_i + b_{i+1}) .
\end{align*}$$ \hspace{1cm} (165)

We summarize the properties of this new bracket in the following:

**Theorem 8** The bracket $\pi_3$ satisfies:

1. $\pi_3$ is Poisson
2. $\pi_3$ is compatible with $\pi_1$.
3. $H_{2i}$ are in involution.
4. $\pi_{j+2} \text{ grad } H_{2i} = \pi_j \text{ grad } H_{2i+2} \forall i, j$.

Define $N = \pi_3 \pi_1^{-1}$. Then $N$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

The proofs of 1. and 2. are straightforward verification of the Jacobi identity. Since (164) holds, 4. follows from properties of the recursion operator. 3. is a consequence of 4.
VI A TRI-HAMILTONIAN FORMULATION OF THE FULL KOSTANT-TODA LATTICE

The full Kostant-Toda lattice is another variation of the Toda lattice. We briefly describe the system: In [2] Kostant conjugates the matrix $L$ in (60) by a diagonal matrix to obtain a matrix of the form

$$X = \begin{pmatrix}
  b_1 & 0 & \cdots & 0 \\
  a_1 & b_2 & \cdots & \vdots \\
  0 & a_2 & b_3 & \vdots \\
  \vdots & \vdots & \ddots & 0 \\
  0 & \cdots & \cdots & a_{n-1} & b_n
\end{pmatrix}. \quad (166)$$

The equations take the form

$$\dot{X}(t) = [X(t), PX(t)] \quad (167)$$

where $P$ is the projection onto the strictly lower triangular part of $X(t)$. This form is convenient in applying Lie theoretic techniques to describe the system.

To obtain the full Kostant-Toda lattice we fill the lower triangular part of $X$ in (166) with additional variables. ($P$ is again the projection onto the strictly lower part of $X(t)$). So, using the notation from [19], [20], [21].

$$\dot{X}(t) = [X(t), PX(t)], \quad (168)$$

where $X$ is in $\epsilon + B_-$ and $PX$ is in $N_-$. $B_-$ is the Lie algebra of lower triangular matrices and $N_-$ is the Lie algebra of strictly lower triangular matrices. In the case of $sl(4, \mathbb{C})$ the matrix $X$ has the form

$$X = \begin{pmatrix}
  f_1 & 0 & 0 & 0 \\
  g_1 & f_2 & 1 & 0 \\
  h_1 & g_2 & f_3 & 1 \\
  k_1 & h_2 & g_3 & f_4
\end{pmatrix}. \quad (169)$$

with $\sum_i f_i = 0$.

We now apply the method of section III to generate nonlinear Poisson brackets for the full Kostant-Toda lattice. The Poisson
brackets are deformations of the Lie Poisson bracket on $B^*_+$ and they are obtained by using master symmetries. The main difference in this version of Toda lattice is that the sequence of Poisson brackets is not infinite. The first three tensors are Poisson, but the remaining ones fail to satisfy the Jacobi identity. We are therefore in the situation investigated in [44] and [45]. In fact, some of the results (and proofs) of Li and Parmentier in [44], on the full symmetric Toda carry over to this system almost without change. The connection with R-matrices and the full symmetric Toda lattice is worth further investigation.

The vector fields in the construction are unique up to addition of a Hamiltonian vector field. Similarly, the Poisson brackets are unique up to addition of a trivial bracket. By generating the second Hamiltonian structure, which turns out to be quadratic, we obtain a bi-Hamiltonian system. One can use this fact to prove involutivity of integrals as in Ratiu [46]. Furthermore, a third Poisson structure is found, which leads to a tri-Hamiltonian formulation of the equations. In this system, all constants of motion, polynomial and rational, are in involution with respect to all three of the Poisson brackets.

Let $G = sl(n)$, the Lie algebra of $n \times n$ matrices of trace zero. Using the decomposition $G = B_+ \oplus N_-$ we can identify $B^*_+$ with the annihilator of $N_-$ with respect to the trace form. This annihilator is $B_-$. Thus we can identify $B^*_+$ with $B_-$ and therefore with $\epsilon + B_-$ as well. The Lie Poisson bracket in the case of $sl(4)$ is given by the following defining relations:

\[
\begin{align*}
\{g_i, g_{i+1}\} &= h_i, \\
\{g_i, f_i\} &= -g_i, \\
\{g_i, f_{i+1}\} &= g_i, \\
\{h_i, f_i\} &= -h_i, \\
\{h_i, f_{i+2}\} &= h_i, \\
\{g_1, h_2\} &= k_1, \\
\{g_3, h_1\} &= -k_1, \\
\{k_1, f_1\} &= -k_1, \\
\{k_1, f_3\} &= k_1.
\end{align*}
\]

All other brackets are zero. Actually, we calculated the brackets on $gl(4, \mathbb{C})$; the trace of $X$ now becomes a Casimir. The Hamiltonian in this bracket is $H_2 = \frac{1}{2} \text{Tr} \; X^2$.  

\[42\]
Remark: If we use a more conventional notation for the matrix $X$, i.e., $x_{ij}$ for $i \geq j$, $x_{ii+1} = 1$, and all other entries zero, then the bracket is simply $\{x_{ij}, x_{kl}\} = \delta_{ij}x_{kj} - \delta_{jk}x_{il}$.

The functions $H_i = \frac{1}{i} \text{Tr} X^i$ are still in involution but they are not enough to ensure integrability. There are, however, additional integrals and the interesting feature of this system is that the additional integrals turn out to be rational functions of the entries of $X$. We describe the constants of motion following references [19], [20], [21].

For $k = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, denote by $(X - \lambda \text{Id})_{(k)}$ the result of removing the first $k$ rows and last $k$ columns from $X - \lambda \text{Id}$, and let

$$\det (X - \lambda \text{Id})_{(k)} = E_{0k} \lambda^{n-2k} + \ldots + E_{n-2k,k} . \quad (170)$$

Set

$$\frac{\det (X - \lambda \text{Id})_{(k)}}{E_{0k}} = \lambda^{n-2k} + I_{1k} \lambda^{n-2k-1} + \ldots + I_{n-2k,k} . \quad (171)$$

The functions $I_{rk}$, $r = 1, \ldots, n - 2k$, are constants of motion for (168).

So, in the case of $gl(4, C)$ the additional integral is

$$I_{21} = \frac{g_1g_2g_3 - g_1f_3h_2 - f_2g_3h_1 + h_1h_2}{k_1} + f_2f_3 - g_2 , \quad (172)$$

and

$$I_{11} = \frac{g_1h_2 + g_3h_1}{k_1} - f_2 - f_3 \quad (173)$$

is a Casimir.

We want to define a second bracket $\pi_2$ so that $H_1$ is the Hamiltonian and

$$\pi_2 \text{ grad } H_1 = \pi_1 \text{ grad } H_2 . \quad (174)$$

i.e., we want to construct a bi-Hamiltonian pair. We will achieve this by finding a master symmetry.

To construct $X_1$, we consider the equation
\[ \dot{X} = [Y, X] + X^2. \]  

(175)

\( Y \) is chosen in such a way that the equation is consistent. One solution is

\[ Y = \sum_{i=1}^{n} \alpha_i E_{ii} + \sum_{i=1}^{n-1} \beta_i E_{i,i+1}, \]

(176)

where

\[ \beta_i = i, \]
\[ \alpha_i = if_i + \sum_{k=1}^{i-1} f_k. \]

The vector field \( X_1 \) is defined by the right hand side of (175). For example, in \( \mathfrak{gl}(4, \mathbb{C}) \) the components of \( X_1 \) are:

\[ X_1(f_1) = 2g_1 + f_1^2, \]
\[ X_1(f_2) = 3g_2 + f_2^2, \]
\[ X_1(f_3) = -g_2 + 4g_3 + f_3^2, \]
\[ X_1(f_4) = -2g_3 + f_4^2, \]
\[ X_1(g_1) = 3h_1 + g_1f_1 + 3g_1f_2, \]
\[ X_1(g_2) = 4h_2 + 4g_2f_3, \]
\[ X_1(g_3) = -h_2 - g_3f_3 + 5g_3f_4, \]
\[ X_1(h_1) = g_1g_2 + 4k_1 + h_1f_1 + h_1f_2 + 4h_1f_3, \]
\[ X_1(h_2) = g_2g_3 + h_2f_3 + 5h_2f_4, \]
\[ X_1(k_1) = g_3h_1 + g_1f_2 + k_1f_1 + k_1f_2 + k_1f_3 + 5k_1f_4. \]

The second bracket, \( \pi_2 \), is defined by taking the Lie derivative of \( \pi_1 \) in the direction of \( X_1 \).

Similarly, we define \( \pi_3 = L_{X_1} \pi_2 \). Another iteration of the procedure gives nothing new since \( L_{X_1} \pi_3 = 0 \).

We define \( X_0 \) to be the Euler vector field and \( X_{-1} = \text{grad} \, H_1 \).

To construct the vector field \( X_2 \) we consider the equation

\[ \dot{X} = [Y, X] + X^3 \]

(177)

We take

\[ Y = \sum_{i=1}^{n} \alpha_i E_{ii} + \sum_{i=1}^{n-1} \beta_i E_{i,i+1} + \sum_{i=1}^{n-2} \gamma_i E_{i,i+2} \]

(178)

with

\[ \alpha_i = if_i^2 + \sum_{k=1}^{i-1} f_k^2 + f_i \sum_{k=1}^{i-1} f_k + i[g_i + g_{i-1}] + 2 \sum_{k=1}^{i-2} g_k, \]
\[ \beta_i = i[f_{i+1} + f_i] + \sum_{k=1}^{i-1} f_k, \]

\[ \gamma_i = i. \]

In the formulas we take \( g_i = 0 \) for \( i = 0, n \).

We complete the sequence of the master symmetries \( X_i \) for \( i \geq 3 \) by taking Lie brackets. We summarize the results in a Theorem. The proofs are almost identical with similar ones for the classical Toda lattice.

**Theorem 9** There exists a sequence of vector fields \( X_i \), for \( i \geq -1 \), and a sequence of contravariant 2-tensors \( \pi_i \), \( i \geq 1 \), satisfying:

i) \( \pi_i \) are Poisson for \( i = 1, 2, 3 \).

ii) The functions \( H_i = \frac{1}{i} \text{Tr} X^i \) are in involution with respect to all of the \( \pi_i \). Actually all invariants, including the rational ones, are in involution.

iii) \( X_i(H_j) = (i + j)H_{i+j} \).

iv) \( L_{X_i} \pi_j = (j - i - 2)\pi_{i+j} \), up to equivalence.

v) \( [X_i, \chi_l] = (l - 1)\chi_{t+i} \).

vi) \( \pi_i \text{ grad } H_l = \pi_{i-1} \text{ grad } H_{t+1}, \ i = 2, 3 \).

vii) A polynomial Casimir for \( \pi_2 \) is \( \det X \) and \( \text{Tr} X^{-1} \) is a Casimir for \( \pi_3 \).

**Remark 1** The master symmetries \( X_i \) preserve constants of motion. It is interesting to see where the rational invariants are mapped in a low dimensional example. We consider the case of \( gl(5, \mathbb{C}) \). We have four rational invariants denoted by \( K_1 = I_{11}, \ K_2 = I_{21}, \ K_3 = I_{31} \) and \( K_4 = I_{12} \). The master symmetry \( X_1 \) behaves in the following way:

i) \( X_1(K_1) = 2K_2 + K_1^2 \)

ii) \( X_1(K_2) = 3K_3 + K_1K_2 \)

iii) \( X_1(K_3) = K_1K_3 \)

iv) \( X_1(K_4) = K_4^2 \).

The master symmetry \( X_2 \) acts in a more complicated way. For example,
\[ X_2(K_3) = \frac{1}{120} H_1^5 - \frac{1}{6} H_1^3 H_2 + H_1 K_1 K_3 - \frac{1}{2} H_1^2 K_3 + \frac{1}{2} H_1 H_3 + \frac{1}{2} H_1^2 H_3 - H_1 H_4 + K_2 K_3 - H_2 H_3 + K_3 H_2 + H_5 \]

**Remark 2)** We also have Lenard-type relations for the rational invariants. For example in the case of \( gl(5, \mathbb{C}) \)

i) \( \pi_1 \text{grad} K_2 = \pi_2 \text{grad} K_1 \)

ii) \( \pi_1 \text{grad} K_3 = \pi_2 \text{grad} K_2 \)

iii) \( \pi_1 \text{grad} K_4 = \pi_2 \text{grad} K_3 \)

Using the master symmetry \( X_1 \) and properties of the Schouten bracket one can derive similar relations between \( \pi_2 \) and \( \pi_3 \). For example \( \pi_2 \text{grad} M_1 = \pi_3 \text{grad} K_1 \), where \( M_1 = K_2 + \frac{1}{2} K_3^2 = \frac{1}{2} X_1(K_1) \). These Lenard-type relations can be used to prove involution of integrals. In the course of the proof one uses the fact that \( K_3 \) is a Casimir for \( \pi_2 \) and \( K_4 \) is a Casimir for both \( \pi_2 \) and \( \pi_3 \).

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