The Numerical Simulation of the rivalry between aerobic and anaerobic bacteria species in a chemostat model

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Abstract: We investigate a chemostat model that contains interaction between two bacteria species, the aerobic species, and a facultative anaerobic species. The competition is restricted on the dissolved oxygen where the aerobic species consumes the oxygen for growth, on the other hand, the facultative anaerobic species do not need the dissolved oxygen for growth. We found that the aerobic growth rate is more extensive compared to that of its competition, the anaerobic species. During our study of the chemostat system, we found three equilibria solutions. The first one is found at the initial dissolved oxygen concentration with the condition of both species washed out. The second equilibrium point is seen when both the dissolved oxygen and the aerobic species present, and finally, the third equilibrium point is found when the dissolved oxygen and the facultative anaerobic species present. We used MATLAB software to simulate these aforementioned three equilibria.

1. Introduction

The relationship between biology and mathematics is well described in mathematical modeling and so-called the chemostat model is one of the remarkable examples of this relation, see [1-6]. A chemostat is widely used in many different applications such as ecology, chemical engineering, pharmacology, and medicine. In this model, there is a continuous stirred-tank reactor (CSTR) receiving a continuous rate of liquid and dissolved oxygen inflow to a chamber which represents the growth environment. In the growth chamber there exist two kinds of microorganism which are: the aerobic species, and the anaerobic species. Both species compete for the dissolved oxygen as a nutrition source in a well-stirred mixture of fluid and culture which form the medium flow. The outflow is leaving the growth chamber at the same rate of the inflow to keep the mixture steady and uniform (see Fig 1).

In the growth chamber, the high stirring rate would cause damage to both species while the low stirring rate would prevent the reactor from reaching the steady state. The outflow rate should not be very large.
because it could cause a wash out to all the species from the growth chamber. To study the problem described in above, a mathematical model needs to be developed in a differential equation system form. The objective is to study and describe the competition between the aerobic species with the anaerobic species for the dissolved oxygen. The growth rate of aerobic and the anaerobic depend on the oxygen substrate, while in the absence of the oxygen, only the anaerobic can survive in the growth chamber, see [7, 8]. The main focus is to set up a system of equations that appropriately models the scenario. After the system is formed parameters, must be selected in a way that is realistic for the biological system, see [9]. Once this is complete, analysis can be done finding the steady-state solutions and evaluating the stability at each point, given its corresponding eigenvalues using the Jacobian, see [10]. We must also determine reasonable dilution rates in which the equilibrium points are valid. Using MATLAB see [11], we can model our system and compare results.

The work in this paper is classified as follows: in section 2 we describe the mathematical model. In section 3 we analyse the model to find the equilibrium points and the dynamic stability for each point. In section 4 we show some computational results after considering a specific value for the parameters in the model. In section 5 we discuss the results and do the conclusion.

2. Mathematical Model

As time goes on, represent the mixture in the growth chamber by the following variables: $x_1(t)$ denote the aerobic concentration, $x_2(t)$ denote the anaerobic concentration, and the $S(t)$ denote the dissolved oxygen concentration. Hence the model is formulated as system of differential equations with three ordinary differential equations (ODE): The first ODE is used to demonstrate the rate of change of oxygen within the system, dependent on the growth of both species. The second and third ODEs describe the growth rate of the aerobic and anaerobic species, respectively. The growth of the aerobic species is solely dependent on the Michaelis-Menten reaction kinetics see [1], where the facultative anaerobic growth depends on this kind of reaction kinetics as well as a standard inhibition kinetic. Within the governing equations examined within section 2.2, the time-dependent variables within the model are $S(t)$, $x_1(t)$, and $x_2(t)$ with units $[g/L], [g], [g]$ respectively. In the model, $S(t)$

![Figure 1: A chemostat as CSTR system where inflow with oxygen is introduced into the growth chamber in a constant rate, and the outflow is leaving the growth chamber in the same rate of the inflow.](image-url)
represents the concentration of dissolved oxygen. Within the ODE model and the remainder of the paper, \( S(t) \) will be written as \( S \) for simplification. The variables \( x_1(t) \) and \( x_2(t) \) represent the concentration of aerobic and facultative anaerobic species, respectively. These two variables are dependent on time as well, for simplicity, they are written and referenced as \( x_1 \) and \( x_2 \). Below, Table 1 defines the parameters found within the three different equations.

| Parameter | Description | Units of Measure |
|-----------|-------------|------------------|
| Q         | Flow rate   | L                |
| V         | Culture Value | L/hr             |
| D         | Dilution rate | 1/hr             |
| \( S^0 \) | Dissolved Oxygen in flow | g/L |
| \( y_1 \) | Yield Coefficient of aerobic species | - |
| \( y_2 \) | Yield Coefficient of facultative anaerobic species | - |
| \( \mu_1 \) | Maximal growth rate of aerobic species | 1/hr |
| \( \mu_2 \) | Maximal growth rate of facultative anaerobic species (dependent on Oxygen) | 1/hr |
| \( \mu_3 \) | Maximal growth rate of facultative anaerobic species (Not dependent on Oxygen) | 1/hr |
| \( k_1 \) | Half Saturated concentration of Oxygen w.r.t. aerobic species | g/L |
| \( k_2 \) | Half Saturated concentration of Oxygen w.r.t. facultative anaerobic species | g/L |
| \( k_3 \) | Half Saturated concentration of Oxygen w.r.t. facultative anaerobic species | g/L |

Table 1: The constants used within the governing equations, with each defined constant, is the described representation within the model. Within the table, \( D \) and \( S^0 \) are generally defined, where all other parameters are defined to a species. Each description of the parameter is the correlating units of measure. (w.r.t. used within the description represents concerning).

2.1 Assumptions

Within the development of the model, assumptions were made to make the model represent a realistic biological system. The first assumption made was that the species do not die; it is assumed that the species do not have a set lifespan, and instead, they undergo binary fission to produce daughter cells. Within the chemostat, it was assumed that \( D > 0 \), in doing so, it creates a more dynamic system. If \( D = 0 \), there wouldn't be any removal factor of the species, and the dissolved oxygen would not be replaced within the system. It is assumed that \( S^0 > 0 \), this is a necessary condition, if \( S^0 = 0 \), the aerobic species would not have a chance to grow and compete within the chemostat model. In the model \( S, x_1, x_2 \geq 0 \) as \( t \to \infty \), in doing so, it provides all the necessary conditions for competition. Making these assumptions will hopefully result in finding the different equilibrium points and allow for computational results to be found within the \( R^+_3 \) domain.

2.2 Governing Equations

The three ordinary differential equations which represent the flux of the dissolved oxygen and the growth rates of the aerobic and anaerobic species are defined as:

\[
\frac{dS}{dt} = D(S^0 - S) - \frac{1}{y_1} \frac{\mu_1 S}{k_1 + S} x_1 - \frac{1}{y_2} \frac{\mu_2 S}{k_2 + S} x_2 \quad \ldots (1)
\]
\[
\frac{dx_1}{dt} = -Dx_1 + \frac{\mu_1 S}{k_1 + S} x_1
\]  
\[\ldots (2)\]

\[
\frac{dx_2}{dt} = -Dx_2 + \frac{\mu_2 S}{k_2 + S} x_2 + \frac{\mu_3 k_3}{k_3 + S} x_2 + \frac{\mu_3 k_3}{k_3 + S} x_2
\]  
\[\ldots (3)\]

Where \( S(0) > 0, x_1(t) > 0, x_2(t) > 0 \)

Within the model, the dilution rate is assumed to have the form \( D = Q/V \), such that the parameter identifies what is entering and leaving the chemostat based on volume. Each of the three equations can be broken up into different parts and examined independently. Looking at equation (1), the first part of this equation examines the difference in the inflow of the dissolved oxygen entering and exiting the system. The remaining aspect of the equation models the consumption of the dissolved oxygen from the two species through their growth within the system. The monodic kinetic equation found in all three equations is used to describe the growth of the aerobic and anaerobic species within the system.

The structural design within equation (2) and (3) are very similar; both equations have a dilution factor \( Dx \) which accounts for the species that are removed from the system. As well when dissolved oxygen is present, the aerobic and anaerobic have a growth-rate dependent on the monodic kinetic equation. What separates the growth of the two species is the last term in equation (3). This term is known as the standard inhibition kinetic equation and is used to show the growth of the anaerobic species when there isn't any dissolved oxygen present within the system. When the concentration of the dissolved oxygen is low, this term dominates and results in an anaerobic growth.

2.3 Computational Methods

The numerical approach plays a powerful role in solving the differential systems compared with the analytic solution of such systems. Therefore, we choose this approach in solving equations (1 - 3) which represents the first order of differential equations. We use the function ode45 in MATLAB of version R14 as a computational method for solving the chemostat model (equations (1 - 3). The function ode45 is a function that solves the differential system by using the famous method of Rung-Kutta fourth order \( (RK^4) \) [4]. In this function, we define the right-hand side (RHS) of system (4-6) in a function-mat. file; in the script file, the solution is determined by assuming the time span as a vector \([0, T]\), where \( T \) is the terminal time, and assuming an initial point \( x0 = [S, x_1, x_2] \).

3. Analysis

Within this section, the governing equations (1 - 3) undergo rigorous analysis to determine the steady states, the Jacobian, the eigenvalues at each steady-state, and the correlating stability.

3.1 Determining Steady States

To determine the steady states within the system, the equations (1 - 3) must be set to zero, so that the system is investigated at equilibrium. As a result, the equations become:

\[
0 = D(S^0 - S) - \frac{1}{y_1} \frac{\mu_1 S}{k_1 + S} x_1 - \frac{1}{y_2} \frac{\mu_2 S}{k_2 + S} x_2
\]  
\[\ldots (4)\]

\[
0 = -Dx_1 + \frac{\mu_1 S}{k_1 + S} x_1
\]  
\[\ldots (5)\]
\[ 0 = -D x_2 + \frac{\mu_2 S}{k_2 + S} x_2 + \frac{\mu_2 k_3}{k_3 + S} x_2 \quad \ldots (6) \]

When determining the steady states within the system, the trivial steady states can be determined the easiest. If \( x_1^*, x_2^* = 0 \) and \( S^* > 0 \) then the steady state found is

\((S^*, x_1^*, x_2^*) = (S^0, 0, 0)\)

If \( x_1^* = 0 \) and \( S^*, x_2^* > 0 \) then the corresponding steady state found is

\[(S^*, x_1^*, x_2^*) = \left( \frac{Dk_2}{\mu_2}, 0, \frac{Dy_2}{\mu_2} \left( S^0 - \frac{Dk_2}{\mu_2} \right) \right) \]

Finally, if \( x_2^* = 0 \) and \( S^*, x_1^* > 0 \) then the corresponding steady state found is

\[(S^*, x_1^*, x_2^*) = \left( \frac{Dk_2}{\mu_2 - D}, S^0 - \frac{Dk_2}{\mu_2 - D}, 0 \right) \]

It is required to check for non-trivial steady states within the system, these steady states must satisfy the condition of \( S^*, x_1^*, x_2^* > 0 \). From the manipulation of equations (4-6) it was found that

\[ S^* = -\left( \frac{(\mu_2 k_3 + k_2 k_3 - D k_2 - D k_3) \pm \sqrt{(\mu_2 k_3 + k_2 k_3 - D k_2 - D k_3)^2 - 4(k_3 + \mu_2 - D)(-D k_2 k_3)}}{2(k_3 + \mu_2 - D)} \right) \]

As a result, the non-trivial steady state becomes

\[(S^*, x_1^*, x_2^*) = \left( S^*, \frac{Dy_1}{\mu_1} (S^0 - S^*) - \frac{y_1 k_2}{y_2} x_2^*, \frac{(S^0 - S^*) \left( D^2 y_1 - \frac{D y_1 S^*}{k_2} \right)}{D y_1 \mu_2 - \frac{y_1 k_2 S^*}{y_2}} \right) \]

Although it is algebraically possible to determine the non-trivial systems associated with the equations (3 - 6), as stated previously \( S^*, x_1^*, x_2^* > 0 \). With the parameter values defined within Section 4, it was determined that this non-trivial steady state is not within the domain of the system.

### 3.2 System stability

The Jacobian and its corresponding eigenvalues can be used to determine the stability at each equilibrium points. Using MATLAB, the general form of the Jacobian is:

\[
J = \begin{bmatrix}
\mu S x_1^* & \mu x_1^* & \mu S x_2^* \\
\mu x_1^* & \mu S x_2^* & \mu x_1^* \\
\mu x_2^* & \mu S x_2^* & \mu x_2^*
\end{bmatrix}
\]

The Jacobian matrix helps in understanding the system's behavior near the equilibrium points. It allows us to determine whether the system is stable, unstable, or neutrally stable.
After applying the first steady state $E_1 = (S^0, 0, 0)$ to the general Jacobian, it was determined that the corresponding Jacobian and eigenvalues are of the form:

$$J = \begin{bmatrix} -D, & -\mu_1 S^0 & -\mu_2 S^0 \\ 0, & \frac{\mu_1 S^0}{S^0 + k_1}, & 0 \\ 0, & 0, & \frac{\mu_2 S^0}{S^0 + k_2} + \frac{\mu_3 k_3}{S^0 + k_3} - D \end{bmatrix}, \quad \lambda(E_1) = \begin{bmatrix} -D \\ \frac{\mu_1 S^0}{S^0 + k_1} - D \\ \frac{\mu_2 S^0}{S^0 + k_2} + \frac{\mu_3 k_3}{S^0 + k_3} - D \end{bmatrix}.$$

The second steady state $E_2 = \left( \frac{Dk_2}{\mu_1 - D}, S^0 - \frac{Dk_2}{\mu_1 - D}, 0 \right)$ was applied to the general Jacobian in use to determine the stability and is of the form:

$$J(E_2) = \begin{bmatrix} -D - \frac{\mu_1 (S^0 + \frac{Dk_2}{\mu_1 - D})}{y_1 (k_1 - D - \mu_1)}, & 0, & -Dp_i, & -Dp_j, & -\frac{Dy_j}{\mu y_z} \\ \frac{\mu k_1 (D - \mu_1)(Dk_2 - \mu_1 S^0 + DS^0)}{(Dk_2 - Dk_1 + \mu_1 k_1)^2}, & 0, & -Dp_i, & \frac{Dp_i k_2}{Dk_2 - k_1(D - \mu_1) - D}, & 0 \\ 0, & 0, & \frac{\mu k_3}{k_3 - D - \mu_1}, & 0, & \frac{Dp_i k_2}{Dk_2 - Dk_3 + \mu_2 k_3} \end{bmatrix}.$$

The third steady state $E_3 = \left( \frac{Dk_2}{\mu_2 \left(1 - \frac{D}{\mu_2} \right)}, 0, \frac{Dy_j}{\mu_1} \left( S^0 - \frac{Dk_2}{\mu_2 \left(1 - \frac{D}{\mu_2} \right)} \right) \right)$ was applied to the general Jacobian in use to determine the stability and is of the form:

$$J(E_3) = \begin{vmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{vmatrix},$$

where

$$j_{11} = -D (\mu_2^2 k_2 + D S^0 + D y_2^2 + \mu_1^2 S^0 y_2^2 + D^2 k_3 y_3^2 - 2D K_2 S^0 y_2^2 - D y_2 k_3 y_3^2), \quad j_{12} = -\frac{-D \mu_1 k_3 y_1}{Dk_2 - Dk_1 + \mu_2 k_1}, \quad j_{13} = \frac{-D \mu_1 k_3 y_1}{Dk_2 - Dk_1 + \mu_2 k_1}.$$

$$j_{21} = 0, \quad j_{22} = \frac{\frac{Dy_j (D - \mu_2)}{(Dk_2 - Dk_1 + \mu_2 k_2 + \mu_3 k_3)}}{Dk_2 - Dk_1 + \mu_2 k_1}, \quad j_{23} = 0,$n

$$j_{31} = \frac{Dy_j (D - \mu_2)}{\mu_2^2 k_2 (Dk_2 - Dk_3 + \mu_2 k_3)^2}, \quad j_{32} = 0.$$

$$j_{33} = \frac{\frac{\mu_3 k_3}{k_3 + \frac{D}{\mu_1}} - D}{\mu_3 k_3 (D - \mu_2)}.$$

Hence,
\[ \lambda(E_3) = \begin{bmatrix} D^2k_1 - D^2k_2 + D\mu_1k_2 - D\mu_2k_1 \\ Dk_2 - Dk_1 + \mu_1k_1 - \mu_2k_2 \\ a_{\text{term}} \\ b_{\text{term}} \\ a_{\text{term}} \\ b_{\text{term}} \end{bmatrix} \]

Both \( a_{\text{term}} \) and \( b_{\text{term}} \) are too large and complex fractions to be displayed within the paper. When parameter values are implemented, values can be found and allow for the analysis of the third steady state.

3.3 Numerical stability analysis

Using the values from Table 1 allows for a numerical stability analysis of the steady states see [12].

When analyzing the stability of the equilibrium points, one parameter significantly changes the behavior within the system and that is the dilution rate. As a result, the equilibrium points will be analyzed when \( D > \mu_1, \mu_1 > D > 0 \), but \( D \) is still relatively close in value to \( \mu_1 \) and \( \mu_1 > D > 0 \) but \( D \) is relatively close in value to 0.

Condition 1:

Let \( D = 4.00 \) then the correspond Jacobian and eigenvalues for the system are:

\[
J(E_1) = \begin{bmatrix} -4.0000 & -50.8249 & -27.5028 \\ 0 & -0.7980 & 0 \\ 0 & 0 & -2.2574 \end{bmatrix}, \quad \lambda(E_1) = \begin{bmatrix} -4.0000 \\ -0.7980 \\ -2.2574 \end{bmatrix}
\]

\[
J(E_2) = \begin{bmatrix} -558.6854 & -71.7736 & -34.1880 \\ 34.9452 & -0.5217 & 0 \\ 0 & -2.0667 & 0 \end{bmatrix}, \quad \lambda(E_2) = \begin{bmatrix} -554.1637 \\ -4.0000 \\ -2.0667 \end{bmatrix}
\]

\[
J(E_3) = \begin{bmatrix} -676.9796 & -330.1587 & -63.4921 \\ 0 & -16.8000 & 0 \\ 27.0680 & 0 & -1.0227 \end{bmatrix}, \quad \lambda(E_3) = \begin{bmatrix} -674.4275 \\ -3.5748 \\ 16.8000 \end{bmatrix}
\]

Condition 2:

Let \( D = 3.00 \), then the correspond Jacobian and eigenvalues for the system are:

\[
J(E_1) = \begin{bmatrix} -3.0000 & -50.8249 & -27.5028 \\ 0 & 0.2020 & 0 \\ 0 & 0 & -1.2574 \end{bmatrix}, \quad \lambda(E_1) = \begin{bmatrix} -3.0000 \\ 0.2020 \\ -1.2574 \end{bmatrix}
\]

\[
J(E_2) = \begin{bmatrix} -40.3636 & -45.8554 & -25.6410 \\ 2.3539 & -0.1111 & 0 \\ 0 & 0 & -1.3065 \end{bmatrix}, \quad \lambda(E_2) = \begin{bmatrix} -37.4748 \\ -3.0000 \\ -1.3065 \end{bmatrix}
\]

\[
J(E_3) = \begin{bmatrix} -159.7347 & -45.8554 & -47.6190 \\ 0 & -5.6667 & 0 \\ 5.5543 & 0 & -0.6250 \end{bmatrix}, \quad \lambda(E_3) = \begin{bmatrix} -158.0546 \\ -2.3051 \\ 5.6667 \end{bmatrix}
\]
Condition 3:

Let $D = 0.80$, then the correspond Jacobian and eigenvalues for the system are:

$$J(E_1) = \begin{bmatrix} -0.8000 & -50.8249 & -27.5028 \\ 0 & 2.4020 & 0 \\ 0 & 0 & -0.9426 \end{bmatrix}, \quad \lambda(E_1) = \begin{bmatrix} -0.8000 \\ 2.4020 \\ 0.9426 \end{bmatrix}$$

$$J(E_2) = 10^3 \begin{bmatrix} -2.3127 & -0.0092 & -0.0068 \\ 0.1456 & -0.0002 & 0 \\ 0 & 0 & 0.0005 \end{bmatrix}, \quad \lambda(E_2) = 10^3 \begin{bmatrix} -2.3121 \\ -0.0008 \\ 0.0005 \end{bmatrix}$$

$$J(E_3) = \begin{bmatrix} -24.1770 & -18.5482 & -12.6984 \\ 0.4137 & 0.3685 & 0 \\ 0.6090 & 0.3685 & 0 \end{bmatrix}, \quad \lambda(E_3) = \begin{bmatrix} -23.9632 \\ 0.3952 \\ 0.3685 \end{bmatrix}$$

In each case it can be seen that as the dilution rate decreases the behavior at each steady state changes. As the dilution rate decreases it is seen that the behavior at $E_1$ transforms from stable, to stable saddle and then to an unstable saddle. Looking at $E_2$, it is observed that the stability changes from a stable saddle point into a stable saddle point. Finally, it can be observed that the stability of point $E_3$ progresses from a stable saddle point into an unstable saddle point.

3.4 Break-Even Analysis

Within a chemostat model, there is a dilution process in which dissolved oxygen is added to the system and the aerobic and anaerobic species are removed. Due to this parameter, if set too high both species can be removed from the system and will result in the system to converge to the steady state $E_1$. Depending on the value of the adjusted dilution rate the system can approach the steady states $E_2$ or $E_3$ as well. The relationship that is desirable to analyze is the break-even concentration, for more details see [13], which is the necessary dissolved oxygen such that the growth of the species is equal to the dilution rate. Below is the analysis demonstrating the required dissolved oxygen concentration such that the aerobic species and the anaerobic species can survive within the system.

For aerobic species:

$$\frac{dx_1}{dt} = -D + \frac{\mu_1S}{k_1+S} = 0 \quad \Rightarrow \quad \begin{cases} D = \frac{\mu_1S}{k_1+S} \\ S_1 = \frac{k_1}{\mu_1-D} \end{cases} \quad \ldots \ (7)$$

For anaerobic species:

$$\frac{dx_2}{dt} = -D + \frac{\mu_2S}{k_2+S} + \frac{\mu_3k_3}{k_3+S} = 0 \quad \Rightarrow \quad \mu_2S(k_3+S) + \mu_3k_3(k_2+S) = D(k_2+S)(k_3+S)$$

$$S^2(\mu_2-D) + S(\mu_2k_3 + \mu_3k_3 - Dk_2 - Dk_3) + (\mu_3k_3k_2 - Dk_3k_2) = 0$$

$$S_2 = -\frac{(-\mu_2k_3 + \mu_3k_3 - Dk_2 - Dk_3 + (\mu_2k_3 + \mu_3k_3 - Dk_2 - Dk_3)^2 - 4(\mu_2-D)(\mu_3k_3k_2 - Dk_3k_2))}{2(\mu_2-D)} \quad \ldots \ (8)$$
To determine the constraints on $D$ such that $S > 0$, the values from Table 1 were implemented into (7) and (8).

Below is the condition required on the dilution rate such that the aerobic species are able to reach and overcome the dilution rate: $\mu_1 > D$

The condition required on the dilution rate such that the anaerobic species can survive within the chemostat is: $\mu_2 > D > \mu_3$

From Table 1 it is seen that $\mu_1 > \mu_2$, to have a system where both species are able to withstand the break-even point the dilution rate domain must be $\mu_2 > D > \mu_3$

Figure 2: These graphs demonstrate the relation between the dissolved oxygen and the growth rate for each species. In graph (A) it determined that for the aerobic species to overcome the dilution rate it is required that $S \geq 1.29$. In graph (B) it is seen that the anaerobic species require more dissolved oxygen to overcome the dilution rate as these species require condition such that $S \geq 1.68$.

Figure 3: This graph combines both graphs (A) and (B) in Figure 3, illustrating how both species growth rates are affected through the flux of dissolved oxygen within the chemostat and demonstrates the conditions in which both species need to survive within the system.
4. Computational Results

We run a numerical simulation to our model (1 - 3) in two cases. Case 1 assumes that there is a low dilution rate $D$ in the growth chamber while Case 2 assumes that the dilution rate $D$ is high in the growth chamber. The experimental data that is considered in experimenting our two cases are defined as follows:

$$
\begin{array}{cccccccc}
\mu_1 & \mu_2 & \mu_3 & k_1 & k_2 & k_3 & Y_1 & Y_2 & S^0 \\
3.25 & 1.75 & 1.25 & 1.5 & 1.00 & 0.80 & 0.063 & 0.063 & 100 \\
\end{array}
$$

Table 2: The experimental data for the growth functions of both the aerobic $x_1$ and anaerobic $x_2$.

We consider the values of $\mu_1, \mu_2, \mu_3, k_1, k_2$ and $k_3$ from [2] while we consider the values of $Y_1$ and $Y_2$ from [1]. Now we study our chemostat model (equations 1 - 3) in two cases when we have low $D$ in the growth chamber and when we have a higher $D$ in the growth chamber. Specifically, we consider $D \in [0.9, 3.0]$, we restrict $D$ to these this interval because we do not want $D$ to exceed the break-even concentration.

Case 1: This case examines 150 the dilution rate when $D = 0.8$, the time $t \in [0,1]$, and also consider the initial point $(S^0, x_{10}, x_{20}) = (20, 10, 10)$. Using these parameter values the following Figures can be observed:

![Figure 4: The trajectories of the dissolved oxygen $S$, the aerobic species $x_1$, and the anaerobic species $x_2$ when $D$ is low, w.r.t. time and $x_0$.](image)

From figure 4 we noticed that when the dilution rate $D$ is low in the growth chamber, at $t_0$ there is an abundance of dissolved oxygen, and it allows for the condition in which both species compete to consume the gas. As time progresses the level of oxygen within the chamber decreases significantly due to the dilution rate. When the oxygen is completely consumed, the aerobic species start to wash out while the anaerobic species continue to thrive under these conditions. The solution curve for Case 1 is given by:
Figure 5: The solution curve \((S, x_1, x_2)\) for the governing equations (1 - 3) w.r.t. Case 1.

Figure 5 attempts to draw a relationship between the competition within the chemostat chamber and the concentration of the dissolved oxygen. As a result of the consumption of the nutrient, it is seen that the population of the anaerobic species dominates. This is a result of the aerobic species not capable of growing under limited oxygen.

Case 2: This case examines the dilution rate when \(D = 3.0\), the time \(t \in [0, T]\) and also consider the initial point \((S^0, x_{10}, x_{20}) = (20, 10, 10)\). Using these parameter values the following Figures can be observed:

Figure 6: The trajectories of the dissolved oxygen \(S\), the aerobic species \(x_1\), and the anaerobic species \(x_2\) when \(D\) is high, w.r.t. time and \(x_0\).

From figure 6 we noticed that when the dilution rate is high, then the dissolved oxygen is never completely consumed because the species have less chance to compete; therefore, both species survive in slow rate compared with Case 1; however, the aerobic species dominate in the growth chamber because its growth rate is higher than the anaerobic species growth rate. The solution curve for Case 2 is as follows:
Figure 7: The solution curve \( (S, x_1, x_2) \) for the equations (1 - 3) w.r.t. Case 2.

Within Figure 7 it can be observed that the population size of both the aerobic and anaerobic species decreases significantly as time progresses. This is due to the higher dilution rate where \( D > \mu_1 > \mu_2 + \mu_3 \). With the higher dilution rate, the system replenishes the dissolved oxygen and as a result the aerobic species with the larger growth rates population size decreases at a slower rate compared to that of the anaerobic species.

From the last two cases, we assumed that the dilution rate \( D \) is less than the dissolved oxygen inflow \( S^0 \). We are going to show that \( D \) should not be greater than or equal to \( S^0 \) because this will cause a wash out to all the species in the growth chamber. Therefore, we construct a new case in below.

**Case 3**: This scenario examines the conditions when \( D = S^0 = 100 \), the time \( t \in [0, 1] \), and also consider the initial point \( (S_0, x_{1_0}, x_{2_0}) = (20, 10, 10) \). Using these parameter values the following Figures can be observed:

Figure 8: The trajectories of the dissolved oxygen \( S \), the aerobic species \( x_1 \), and the anaerobic species \( x_2 \) when \( D = S^0 \), w.r.t. the time interval \( t \in [0, 1] \) and \( x_2 = (20, 10, 10) \).

Figure 8 shows that when \( D = S^0 \) then both the aerobic \( x_1 \) and the anaerobic \( x_2 \) are washed out from the growth chamber. The dissolved oxygen \( S \) is increased from the initial value \( S = 20 \) to the dissolved oxygen inflow \( S^0 \), \( S = 100 \) because there is no consumption from the species. The solution curve for Case 3 is given by:
Figure 9: The solution curve $(S, x_1, x_2)$ for the model w.r.t. Case 3.

Figure 9 illustrate the effect of an extremely dilution on the two species. Due to the overpowering washout rate, the aerobic and anaerobic species are unable to sustain its population and grow. Regardless of the dissolved oxygen concentration the growth rate is significantly lower than the dilution rate and prevents growth within the chamber.

5. Conclusion

We considered a chemostat in which two species: an aerobic species $x_1$ and a facultative anaerobic species $x_2$ compete for dissolved oxygen $S$. While the aerobic species requires oxygen for growth, the facultative anaerobic species can also grow when the dissolved oxygen is absent. With the defined parameter values in conditions of oxygen abundance the aerobic growth rate $\mu_1$ is larger compared to that of its competition the anaerobic species that grows at a rate of $\mu_2$.

Through analysis of the chemostat system we found three steady state solutions. The first equilibrium point $E_1$ is found at the initial dissolved oxygen concentration with the condition of both species washed out. The second equilibrium point $E_2$ is found when both the dissolved oxygen and the aerobic species present, and lastly, the third equilibrium point $E_3$ is found when the dissolved oxygen and the facultative anaerobic species present.

It was found that when the dilution rate decreases the steady states start to lose stability, whether from changing from a stable point to a stable saddle point or from a stable saddle point to an unstable saddle point. Based on the dilution rates obtained from analysis we found that with a dilution rate well below the oxygen in-flow level, both species will be able to grow resulting in oxygen being depleted which inhibits growth of the aerobic species while the anaerobic species is still able to grow. If the dilution rate is increased while still being below oxygen inflow level there is a point at which oxygen will not be depleted and both species survive and compete for oxygen. Last, if the dilution rate is above the oxygen inflow-rate then both species of bacteria will be washed out of the chemostat with only oxygen remaining.
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