ON DISCS IN BIDISCS

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Abstract

Let $\Delta$ be the open unit disc in $\mathbb{C}$. We show that there is no continuous map $F: \Delta \to \Delta^2$, holomorphic on $\Delta$ and such that $F(b\Delta) = b(\Delta^2)$.

Denote by $\Delta$ the open unit disc in $\mathbb{C}$. Given a bounded convex domain $D \subset \mathbb{C}^2$ we consider holomorphic maps $F: \Delta \to D$ which extend continuously to $\overline{\Delta}$ and which are proper, that is, satisfy $F(b\Delta) \subset bD$. We ask how large $F(\overline{b\Delta})$ can be. In particular, can we have $F(\overline{b\Delta}) = bD$? It is known that the answer is positive in the case of the ball $D = B = \{(z, w): |z|^2 + |w|^2 < 1\}$.

**Proposition 1** [G, Cor. 2] There is a continuous map $F: \Delta \to \overline{bB}$, holomorphic on $\Delta$, and such that $F(b\Delta) = b\overline{B}$.

In the present note we show that in general the answer to the preceding question is no. In particular, it is negative for $D = \Delta^2$:

**Proposition 2** There is no continuous map $F: \Delta \to \Delta^2$, holomorphic on $\Delta$ and such that $F(b\Delta) = b(\Delta^2)$.

**Definition** A set of the form $\{\zeta\} \times \Delta$ or $\Delta \times \{\zeta\}$ where $\zeta \in b\Delta$ is called an open face of the bidisc $\Delta^2$.

**Proof of Proposition 2.** Let $F = (f, g): \Delta \to \Delta^2$ be a continuous map, holomorphic on $\Delta$ and such that $F(b\Delta) \subset \overline{b\Delta^2}$.

If one of the components, say $f$, is a constant $\alpha$ then $F(b\Delta) \subset \{\alpha\} \times \overline{\Delta}$ so $F(b\Delta)$ cannot equal $b(\Delta^2)$. So assume that both $f, g$ are nonconstant. We shall show that for any open face $\Phi$ of $\Delta^2$ the set $F(b\Delta) \cap \Phi$ has no cluster point in $\Phi$. In fact, we show that given $\xi \in b\Delta$ there is no injective sequence $\zeta_n \in b\Delta$ such that $f(\zeta_n) = \xi$ for all $n$ and such that the sequence $g(\zeta_n)$ has a cluster point in $\Delta$.}

Assume the contrary, so that $\zeta_n \in b\Delta$ is an injective sequence such that $f(\zeta_n) = \xi$ for all $n$ and such that $g(\zeta_n)$ has a cluster point in $\Delta$. Passing to a subsequence we may assume that $\zeta_n$ converges to $\zeta_0 \in b\Delta$ and that $\lim g(\zeta_n) = g(\zeta_0) \in \Delta$. Passing to a subsequence we may assume that all $\zeta_n$ are contained in a small open arc $J \subset b\Delta$ centered at $\zeta_0$ such that $|g(\zeta)| < 1$ ($\zeta \in J$). Since $F(b\Delta) \subset \overline{b(\Delta^2)}$ it follows that $|f(\zeta)| = 1$ ($\zeta \in J$). Denote by $*$ the reflection across $b\Delta$ so for $z \in \Delta \setminus \{0\}$ write $z^* = 1/\overline{z}$. By the reflection principle there is a narrow open neighbourhood $U$ of $J$ in $\mathbb{C}$ such that $U^* = U$ and such that by defining $f$ on $U \cap (\mathbb{C} \setminus \overline{\Delta})$ by $f(w) = [f(w^*)]^*$ ($w \in U \cap (\mathbb{C} \setminus \overline{\Delta})$.
the function $f$ is holomorphic on $\Delta \cup U$ and satisfies

$$|f(\zeta)| < 1 \ (\zeta \in U \cap \Delta) \ \text{and} \ |f(\zeta)| > 1 \ (\zeta \in U \cap [\mathbb{C} \setminus \Delta]).$$

We show that the derivative $f'$ has no zero on $J$. Indeed, if $f'(z) = 0$ for some $z \in J$ then since $f$ is not a constant, $z$ is an isolated zero of $f'$ and the function $w \mapsto f(w) - f(z)$ has a zero of order at least two at $z$. This means that for every sufficiently small circle $\gamma$ centered at $z$ and contained in $U$ the winding number of $f \circ \gamma$ around $f(z)$ is at least two which is impossible since by (2) $f$ maps $\gamma \cap \Delta$ to $\Delta$ and $\gamma \cap [\mathbb{C} \setminus \Delta]$ to $\mathbb{C} \setminus \Delta$. This implies that $f$ is locally one to one on $J$. In particular, $f$ is one to one in a neighbourhood of $\zeta_0$ in $J$ which contradicts the fact that $f(\zeta_n) = \xi$ for all $n$ and that the injective sequence $\zeta_n$ converges to $\zeta_0$. This proves (1) and the same holds with the roles of $f$ and $g$ interchanged. This completes the proof of Proposition 2.

**REMARK** If one drops the requirement about boundary continuity one can do much more. For instance, one can show that given any bounded convex domain $D$ there is a proper holomorphic embedding $F: \Delta \to D$ such that $\overline{F(\Delta)} \setminus F(\Delta) = bD$ [FGS].

**REFERENCES**

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