FRENKEL-GROSS’ IRREGULAR CONNECTION AND HEINLOTH-NGÔ-YUN’S ARE THE SAME

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We show that the irregular connection on $\mathbb{G}_m$ constructed by Frenkel-Gross (\cite{FG}) and the one constructed by Heinloth-Ngô-Yun (\cite{HNY}) are the same, which confirms the Conjecture 2.14 of \cite{HNY}.

The proof is simple, modulo the big machinery of quantization of Hitchin’s integrable systems as developed by Beilinson-Drinfeld (\cite{BD}). The idea is as follows. Let $\mathcal{E}$ be the irregular connection on $\mathbb{G}_m$ as constructed by Frenkel-Gross. It admits a natural oper form. We apply the machinery of Beilinson-Drinfeld to produce an automorphic D-module on the corresponding moduli space of $G$-bundles, with Hecke eigenvalue $\mathcal{E}$. We show that this automorphic D-module is equivariant with respect to the unipotent group $I(1)/I(2)$ (see \cite{HNY} for the notation) against the non-degenerate additive character $\Psi$. By the uniqueness of such D-modules on the moduli space, one knows that the automorphic D-module constructed using the Beilinson-Drinfeld machinery is the same as the automorphic D-module explicitly constructed by Heinloth-Ngô-Yun. Since the irregular connection on $\mathbb{G}_m$ constructed in \cite{HNY} is by definition the Hecke-eigenvalue of this automorphic D-module, it is the same as $\mathcal{E}$.

1. RECOLLECTION OF [BD]

We begin with the review of the main results of Beilinson-Drinfeld (\cite{BD}). We take the opportunity to describe a slightly generalized (and therefore weaker) version of \cite{BD} in order to deal with the level structures.

Let $G$ be a simple, simply-connected complex Lie group, with Lie algebra $\mathfrak{g}$ and the Langlands dual Lie algebra $\check{\mathfrak{g}}$. Let $X$ be a smooth projective algebraic curve over $\mathbb{C}$. For every closed point $x \in X$, let $\mathcal{O}_x$ be the completed local ring of $X$ at $x$ and let $F_x$ be its fractional field. Let $D_x = \text{Spec} \mathcal{O}_x$ and $D^\times_x = \text{Spec} F_x$. In what follows, for an affine (ind-)scheme $T$, we denote by $\text{Fun}_T$ the (pro)-algebra of regular functions on $T$.

Let $\mathcal{G}$ be an integral model of $G$ over $X$, i.e. $\mathcal{G}$ is a (fiberwise) connected smooth affine group scheme over $X$ such that $\mathcal{G}_{\mathbb{C}(X)} = G_{\mathbb{C}(X)}$, where $C(X)$ is the function field of $X$. Let $\text{Bun}_G$ be the moduli stack of $G$-torsors on $X$. The canonical sheaf $\omega_{\text{Bun}_G}$ is a line bundle on $\text{Bun}_G$. As $G$ is assumed to be simply-connected, we have

Lemma 1. There is a unique line bundle $\omega_{\text{Bun}_G}^{1/2}$ over $\text{Bun}_G$, such that $(\omega_{\text{Bun}_G}^{1/2})^\otimes 2 \simeq \omega_{\text{Bun}_G}$.

Now we assume that $\text{Bun}_G$ is “good” in the sense of Beilinson-Drinfeld, i.e.

$$\dim T^* \text{Bun}_G = 2 \dim \text{Bun}_G.$$ 

In this case one can construct the D-module of the sheaf of critically twisted (a.k.a. $\omega_{\text{Bun}_G}^{1/2}$ twist) differential operators on the smooth site $(\text{Bun}_G)_{\text{sm}}$ of $\text{Bun}_G$, denoted by $\mathcal{D}'$. Let $\mathcal{D}' = (\text{End}(\mathcal{D}'))^{\text{op}}$ be the sheaf of endomorphisms of $\mathcal{D}'$ as a twisted D-module.

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Then $D'$ is a sheaf of associative algebra on $(\text{Bun}_G)_{sm}$ and $D' \simeq (D')^{op}$. For more details, we refer to [BD] §1.

Recall the definition of opers on a curve (cf. [BD] §3]). Let $\text{Op}_{L_\mathfrak{g}}(D^\times_x)$ be the ind-scheme of $L_\mathfrak{g}$-opers on the punctured disc $D^\times_x$. Then there is a natural ring homomorphism

\begin{equation}
 h_x : \text{Fun Op}_{L_\mathfrak{g}}(D^\times_x) \to \Gamma(\text{Bun}_G, D').
\end{equation}

Let us briefly recall its definition. Let $Gr_{G,x}$ be the affine Grassmannian, which is an ind-scheme classifying pairs $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on $X$ and $\beta$ is a trivialization of $\mathcal{F}$ away from $x$. Then we have $Gr_{G,x} \simeq G(F_x)/K_x$, where $K_x = \mathcal{G}(O_x)$. Let $\mathcal{L}_{\text{crit}}$ be the pullback of the line bundle $\omega^1_{\text{Bun}_G}$ on $\text{Bun}_G$ to $Gr_{G,x}$, and let $\delta_e$ be the delta $D$-module on $Gr_{G,x}$ twisted by $\mathcal{L}_{\text{crit}}$. Let

$$\text{Vac}_x := \Gamma(Gr_{G,x}, \delta_e)$$

be the vacuum $\mathfrak{g}_{\text{crit},x}$-module at the critical level.

Remark 1.1. The module $\text{Vac}_x$ is not always isomorphic to $\text{Ind}_{\text{Lie} K_x + C_1}(\text{triv})$, due to the twist by $\mathcal{L}_{\text{crit}}$. For example, if $K_x$ is an Iwahori subgroup,

$$\text{Vac}_x = \text{Ind}_{\text{Lie} K_x + C_1}(\mathbb{C}_{-\rho}),$$

is the Verma module of highest weight $-\rho$ ($-\rho$ is anti-dominant w.r.t. the chosen $K_x$).

Let $\text{Bun}_{G,x}$ be the scheme classifying pairs $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on $X$ and $\beta$ is a trivialization of $\mathcal{F}$ on $D_x = \text{Spec} \mathcal{O}_x$. It admits a $(\mathfrak{g}_{\text{crit},x}, K_x)$ action, and $\text{Bun}_{G,x}/K_x \simeq \text{Bun}_G$. Now applying the standard localization construction to the Harish-Chandra module $\text{Vac}_x$ (cf. [BD] §1) gives rise to

$$\text{Loc}(\text{Vac}_x) \simeq D'$$

as critically twisted $D$-modules on $\text{Bun}_G$. Recall that the center $Z_x$ of the category of smooth $\mathfrak{g}_{\text{crit},x}$-modules is isomorphic to $\text{Fun Op}_{L_\mathfrak{g}}(D^\times_x)$ by the Feigin-Frenkel isomorphism ([BD], §3.2], [F]). The mapping $h_x$ then is the composition

$$\text{Fun Op}_{L_\mathfrak{g}}(D^\times_x) \simeq Z_x \to \text{End}(\text{Vac}_x) \to \text{End}(\text{Loc}(\text{Vac}_x)) \simeq \Gamma(\text{Bun}_G, D').$$

If $\mathcal{G}$ is unramified at $x$, then $h_x$ factors as

$$h_x : \text{Fun Op}_{L_\mathfrak{g}}(D^\times_x) \to \text{Fun Op}_{L_\mathfrak{g}}(D_x) \simeq \text{End}(\text{Vac}_x) \to \Gamma(\text{Bun}_G, D'),$$

where $\text{Op}_{L_\mathfrak{g}}(D_x)$ is the scheme (of infinite type) of $L_\mathfrak{g}$-opers on $D_x$.

The mappings $h_x$ can be organized into a horizontal morphism $h$ of $\mathcal{D}_X$-algebras over $X$ (we refer to [BD] §2.6) for the generalities of $\mathcal{D}_X$-algebras). Let us recall the construction. By varying $x$ on $X$, the affine Grassmannian $Gr_{G,x}$ form an ind-scheme $Gr_\mathcal{G}$ formally smooth over $X$. Let $\pi : Gr_G \to X$ be the projection and $e : X \to Gr_G$ be the unital section given by the trivial $\mathcal{G}$-torsor. Let $\delta_e$ be the delta $D$-module along the section $e$ twisted by $\mathcal{L}_{\text{crit}}$. Then we have a chiral algebra

$$\text{Vac}_X := \pi_!(\delta_e).$$

over $X$ whose fiber over $x$ is $\text{Vac}_x$.

Lemma 2. The sheaf $\text{Vac}_X$ is flat as an $\mathcal{O}_X$-module.
For any chiral algebra \( \mathcal{A} \) over a curve, one can associate the algebra of its endomorphisms, denoted by \( \mathcal{E}nd(\mathcal{A}) \). As sheaves on \( X \),
\[
\mathcal{E}nd(\mathcal{A}) = \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}),
\]
where \( \text{Hom} \) is taken in the category of chiral \( \mathcal{A} \)-modules. Obviously, \( \mathcal{E}nd(\mathcal{A}) \) is an algebra by composition. Less obviously, there is a natural chiral algebra structure on \( \mathcal{E}nd(\mathcal{A}) \otimes \mathcal{O}_X \) which is compatible with the algebra structure. Therefore, \( \mathcal{E}nd(\mathcal{A}) \) is a commutative \( \mathcal{D}_X \)-algebra. If \( \mathcal{A} \) is \( \mathcal{O}_X \)-flat, there is a natural injective mapping \( \mathcal{E}nd(\mathcal{A}) \rightarrow \text{End}(\mathcal{A}) \) which is not necessarily an isomorphism in general, where \( \text{End}(\mathcal{A}_x) \) is the endomorphism algebra \( \mathcal{A}_x \) as a chiral \( \mathcal{A} \)-module. However, this is an isomorphism if there is some open neighborhood \( U \) containing \( x \) such that \( \mathcal{A}|_U \) is constructed from a vertex algebra. We refer to [R] for details of the above discussion.

Let \( U \subset X \) be an open subscheme such that \( \mathcal{G}|_U \simeq G \times U \), then by the above generality, the Feigin-Frenkel isomorphism gives rise to
\[
\text{Spec} \mathcal{E}nd(\mathcal{V}ac_U) \simeq \text{Op}_{Lg}|_U,
\]
where \( \text{Op}_{Lg} \) is the \( \mathcal{D}_X \)-scheme over \( X \), whose fiber over \( x \in X \) is the scheme of \( \mathfrak{g} \)-opers on \( D_x \). Recall that for a commutative \( \mathcal{D}_U \)-algebra \( B \), we can take the algebra of its horizontal sections \( H_{\mathcal{V}}(U, B) \) (or so-called conformal blocks) [BD] \( \S 2.6 \), which is usually a topological commutative algebra. For example,
\[
\text{Spec} H_{\mathcal{V}}(U, \text{Op}_{Lg}) = \text{Op}_{Lg}(U)
\]
is the ind-scheme of \( \mathfrak{g} \)-opers on \( U \) (BD] \( \S 3.3 \)). As \( H_{\mathcal{V}}(U, \mathcal{E}nd(\mathcal{V}ac_U)) \rightarrow H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X)) \) is surjective, we have a closed embedding
\[
\text{Spec} H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X)) \rightarrow \text{Op}_{Lg}(U).
\]
Let \( \text{Op}_{Lg}(X)_{\mathcal{G}} \) denote the image of this closed embedding. This is a subscheme (rather than an ind-scheme) of \( \text{Op}_{Lg}(U) \).

On the other hand, as argued in [BD] \( \S 2.8 \), the mapping \( h_x \) is of crystalline nature so that it induces a mapping of \( \mathcal{D}_X \)-algebras
\[
h : \mathcal{E}nd(\mathcal{V}ac_X) \rightarrow \Gamma(\text{Bun}_{\mathcal{G}}, D') \otimes \mathcal{O}_X,
\]
which induces a mapping of horizontal sections
\[
h_{\mathcal{V}} : H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X)) \rightarrow \Gamma(\text{Bun}_{\mathcal{G}}, D').
\]
Therefore, (1.3) can be rewrite as a mapping
\[
h_{\mathcal{V}} : \text{Fun} \text{Op}_{Lg}(X)_{\mathcal{G}} \rightarrow \Gamma(\text{Bun}_{\mathcal{G}}, D').
\]

We recall the characterization \( \text{Op}_{Lg}(X)_{\mathcal{G}} \).

**Lemma 3.** Let \( X \setminus U = \{x_1, \ldots, x_n\} \). Assume that the support of \( \mathcal{V}ac_{x_i} \) (as an \( \mathfrak{g} \)-module) is \( Z_{x_i} \subset \text{Op}_{Lg}(D_{x_i}) \) (i.e. \( \text{Fun}(\mathcal{Z}_{x_i}) = \text{Im}(\text{Op}_{Lg}(D_{x_i}) \rightarrow \text{End}(\mathcal{V}ac_x)) \)). Then
\[
\text{Op}_{Lg}(X)_{\mathcal{G}} \simeq \text{Op}_{Lg}(U) \times \prod_i \text{Op}_{Lg}(D_{x_i}) \prod Z_{x_i}.
\]
The mapping (1.4) is a quantization of a classical Hitchin system. Namely, there is a natural filtration ([BD] \( \S 3.1 \)) on the algebra \( \text{Fun} \text{Op}_{Lg}(U) \) whose associated graded is the algebra of functions on the classical Hitchin space
\[
\text{Hitch}(U) = \bigoplus_i \Gamma(U, \Omega^{d_i+1})
\]
where \( d_i \)s are the exponent of \( \mathfrak{g} \) and \( \Omega \) is the canonical sheaf of \( X \). On the other hand, there is a natural filtration on \( \Gamma(\text{Bun}_{\mathcal{G}}, D') \) coming from the order of the
Let \( E \) be a \( \mathbb{C} \)-algebra. Then \( \text{Aut}_E \) is a \( \text{Hecke-eigensheaf} \) on \( \text{Bun}_G \) with respect to \( E \) (regarded as a \( L \)-\( G \)-local system).

Remark 1.3. The statement of the above theorem is weaker than the main theorem in [BD] in two aspects: (i) if \( G \) is the constant group scheme (the unramified case), then \( \text{Op}_\ell(\mathfrak{g}) = \text{Op}_\ell(X) \) is the space of \( L \)-\( \mathfrak{g} \)-opers on \( X \). In this case, Beilinson and Drinfeld proved that

\[
\text{Fun Op}_\ell(X) \simeq \Gamma(\text{Bun}_G, D')
\]

and therefore \( \text{Aut}_E \) is always non-zero in this case; (ii) in the unramified case, the automorphic D-module \( \text{Aut}_E \) is holonomic.

The proofs of both assertions are based on the fact that the classical Hitchin map is a complete integrable system. If the level structure of \( G \) is not deeper than the Iwahori level structure (or even the pro-unipotent radical of the Iwahori group), then by the same arguments, the above two assertions still hold. However, it is not obvious from the construction that \( \text{Aut}_E \) is non-zero for the general deeper level structure, although we do conjecture that this is always the case. In addition, for arbitrary \( G \), the automorphic D-modules constructed as above will in general not be holonomic. This is the reason that we need to use a group scheme different from \( \text{Hitch}(-) \) in what follows.

2.

Now we specialize the group scheme \( \mathcal{G} \). Let \( G \) be a simple, simply-connected complex Lie group, of rank \( \ell \). Let us fix \( B \subset G \) a Borel subgroup and \( B^- \) an opposite Borel subgroup. The unipotent radical of \( B \) (resp. \( B^- \)) is denoted by \( U \) (resp. \( U^- \)). Following [HNY], we denote by \( \mathcal{G}(0, 1) \) the group scheme on \( \mathbb{P}^1 \) obtained from the dilatation of \( G \times \mathbb{P}^1 \) along \( B^- \times \{0\} \subset G \times \{0\} \) and \( U \times \{\infty\} \subset G \times \{\infty\} \). Following loc. cit., we denote \( I(1) = \mathcal{G}(0, 1)(\mathcal{O}_\infty) \).

Let \( \mathcal{G}(0, 2) \rightarrow \mathcal{G}(0, 1) \) be the group scheme over \( \mathbb{P}^1 \) so that they are isomorphic away from \( \infty \) and \( \mathcal{G}(0, 2)(\mathcal{O}_\infty) = I(2) := [I(1), I(1)] \). Then \( I(1)/I(2) \simeq \prod_{i=0}^{\ell} U_{\alpha_i} \), where \( \alpha_i \) are simple affine roots, and \( U_{\alpha_i} \) are the corresponding root groups. Let us choose for each \( \alpha_i \) an isomorphism \( \Psi_i : U_{\alpha_i} \simeq \mathbb{G}_a \). Then we obtain a well-defined morphism

\[
\Psi : I(1) \rightarrow I(1)/I(2) \simeq \prod_{i=0}^{\ell} U_{\alpha_i} \simeq \prod_{i=0}^{\ell} \mathbb{G}_a \rightarrow \mathbb{G}_a.
\]

Let \( I_\mathfrak{g} := \ker \Psi \subset I(1) \).

As explained in loc. cit., there is an open substack of \( \text{Bun}_{\mathcal{G}(0,2)} \), which is isomorphic to \( \mathbb{G}_a^{\ell+1} \). For the application of Beilinson-Drinfeld’s construction, it is
convenient to consider \( \text{Bun}_G(0, \Psi) \), where \( \mathcal{G}(0, \Psi) \rightarrow \mathcal{G}(0, 1) \) is an isomorphism away from \( \infty \) and \( \mathcal{G}(0, \Psi)(\mathcal{O}_\infty) = I_{\Psi} \subset I(1) = \mathcal{G}(0, 1) \). Then \( \text{Bun}_G(0, 2) \) is a torsor over \( \text{Bun}_G(0, \Psi) \) under the group \( I_{\Psi}/I(2) \cong \mathcal{G}_d^* \) and there is an open substack of \( \text{Bun}_G(0, \Psi) \) isomorphic to \( \mathcal{G}_a \).

**Lemma 5.** The stack \( \text{Bun}_G(0, \Psi) \) is good in the sense of [BD] §1.1.1.

*Proof.* Since \( \text{Bun}_G(0, \Psi) \) is a principal bundle over \( \text{Bun}_G(0, 1) \) under the group \( I(1)/I_{\Psi} \cong \mathcal{G}_a \), it is enough to show that \( \text{Bun}_G(0, 1) \) is good. It is well-known in this case \( \text{Bun}_G(0, 1) \) has a stratification by elements in the affine Weyl group of \( G \) and the stratum corresponding to \( w \) has codimension \( \ell(w) \) and the stabilizer group has dimension \( \ell(w) \).

Therefore \( \text{Bun}_G(0, 1) \) is good.

Let \( S_w \) denote the preimage in \( \text{Bun}_G(0, \Psi) \) of the stratum in \( \text{Bun}_G(0, 1) \) corresponding to \( w \). Then \( S_1 \cong \mathbb{A}^1 \), and for a simple reflection \( s \), \( S_1 \cup S_s \cong \mathbb{P}^1 \). In particular, any regular function on \( \text{Bun}_G(0, \Psi) \) is constant.

Let us describe \( \text{Op}_{Lg}(X)_{G(0, \Psi)}^{\infty} \) in this case.

At \( 0 \in \mathbb{P}^1 \), \( K_0 = \mathcal{G}(0, \Psi)(\mathcal{O}_0) \) is the the Iwahori subgroup \( I_{\text{op}}^G \) of \( G(F_0) \), which is \( \text{ev}^{-1}(B^-) \) under the evaluation map \( \text{ev} : G(\mathcal{O}) \rightarrow G \), and

\[
\text{Vac}_0 = \text{Ind}_{L_{\text{Lie}}^I}^{\text{Lie}_{G} \times \mathbb{C}^1}(C_{-\rho}).
\]

is just the Verma module \( \mathcal{M}_{-\rho} \) of highest weight \( -\rho \) (\( -\rho \) is anti-dominant w.r.t. \( B^- \)), and it is known ([F, Chap. 9]) that \( \text{Fun} \text{Op}_{Lg}(D_{0}^{\infty}) \rightarrow \text{End}(\mathcal{M}_{-\rho}) \) induces an isomorphism

\[
\text{Fun} \text{Op}_{Lg}(D_{0}^{\infty})_{\text{ev}(0)} \cong \text{End}(\mathcal{M}_{-\rho}),
\]

where \( \text{Op}_{Lg}(D_{0}^{\infty})_{\text{ev}(0)} \) is the scheme of \( L^G \) opers on \( D_{0} \) with regular singularities and zero residue. Let us describe this space in concrete terms.

Let \( f = \sum_i X_{-\alpha_i} \) be the sum of root vectors \( X_{-\alpha_i} \) corresponding negative simple roots \( -\alpha_i \) of \( L^G \).

After choosing a uniformizer \( z \) of the disc \( D_{0} \), \( \text{Op}_{Lg}(D_{0}^{\infty})_{\text{ev}(0)} \) is the space of operators of the form

\[
\partial_z + \frac{f}{z} + L^G b([z])
\]

up to \( L^G(\mathcal{O}) \)-gauge equivalence.

At \( \infty \in \mathbb{P}^1 \), \( K_{\infty} = \mathcal{G}(0, \Psi)(\mathcal{O}_\infty) = I_{\Psi} \).

Denote

\[
\mathbb{W}_{\text{univ}} = \text{Vac}_\infty = \text{Ind}_{L_{\text{Lie}}^I}^{\text{Lie}_{G} \times \mathbb{C}^1}(\text{triv}).
\]

It is known ([FF, Lemma 5]) that \( \text{Fun} \text{Op}_{Lg}(D_{\infty}^{\infty}) \rightarrow \text{End}(\mathbb{W}_{\text{univ}}) \) factors as

\[
\text{Fun} \text{Op}_{Lg}(D_{\infty}^{\infty}) \rightarrow \text{Fun} \text{Op}_{Lg}(D_{\infty}^{\infty})_{1/h} \hookrightarrow \text{End}(\mathbb{W}_{\text{univ}}),
\]

where \( \text{Op}_{Lg}(D_{\infty}^{\infty})_{1/h} \) is the scheme of opers with slopes \( \leq 1/h \) (as \( L^G \)-local systems) and \( h \) is the Coxeter number of \( L^G \).

To give a concrete description of this space, let us complete \( f \) to an \( \mathfrak{s}_2 \)-triple \( \{e, \gamma, f\} \) with \( e \in \mathfrak{l}B \). Let \( L^G e \) be the centralizer of \( e \) in \( L^G \), and decompose \( L^G e = \bigoplus_{i=1}^{\ell} L^G g_i \) according to the principal grading by \( \gamma \). Let \( d_i = \deg(L^G g_i) \). Then after choosing a uniformizer \( z \) on the disc \( D_{\infty} \), \( \text{Op}_{Lg}(D_{\infty}^{\infty})_{1/h} \) is the space of operators of the form

\[
\partial_z + f + \sum_{i=1}^{\ell-1} z^{-d_i-1}(L^G g_i^{\infty})([z]) + z^{-d_{\ell}-2}(L^G g_{\ell}^{\infty})([z])
\]

up to \( L^G(\mathcal{O}) \)-gauge equivalence.
Therefore, $\text{Op}_{\ell_\varphi}(X)_{G(0,\psi)}$ is isomorphic to
\[
\text{Op}_{\ell_\varphi}(X)_{(0,\text{RS}), (\infty, 1/h)} := \text{Op}_{\ell_\varphi}(D_\infty)_{1/h} \times \text{Op}_{\ell_\varphi}(D_\infty^\times) \times \text{Op}_{\ell_\varphi}(G_m) \times \text{Op}_{\ell_\varphi}(D_0^\times) \times \text{Op}_{\ell_\varphi}(D_0) \times (0).
\]
As observed in [FG], $\text{Op}_{\ell_\varphi}(X)_{(0,\text{RS}), (\infty, 1/h)} \simeq \mathbb{A}^1$. Indeed, let $z$ be the global coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. Then the space of such opers are of the form
\[
\nabla = \frac{\partial}{\partial z} + \frac{f}{z} + \lambda e_\theta,
\]
where $f$ is the sum of root vectors corresponding to negative simple roots and $e_\theta$ is a root vector corresponding to the highest root $\theta$.

According to [HNY] there is a ring homomorphism
\[
h_\nabla : \mathbb{C}[\lambda] \to \Gamma(\text{Bun}_{\mathcal{G}(0,\psi)}, D').
\]
Let us describe this mapping more explicitly. Recall that there is an action of $I(1)/I_\psi \simeq \mathbb{G}_a$ on $\text{Bun}_{\mathcal{G}(0,\psi)}$, and therefore the action of $\mathbb{G}_a$ induces an algebra homomorphism
\[
a : U(\text{Lie}I(1)/I_\psi) \to \Gamma(\text{Bun}_{\mathcal{G}(0,\psi)}, D').
\]

**Lemma 6.** We have $h_\nabla(\lambda) = a(\xi)$ for some non-zero element $\xi \in \text{Lie}I(1)/I_\psi \simeq \mathbb{C}$.

**Proof.** Consider the associated graded $h^{cl} : \text{gr} \mathbb{C}[\lambda] \to \Gamma(T^*\text{Bun}_{\mathcal{G}(0,\psi)}, \mathcal{O})$, which is the classical Hitchin map. Recall that the filtration on $\mathbb{C}[\lambda]$ comes from the existence of $h$-opers, and therefore the symbol of $\lambda$ is identified with a coordinate function on
\[
\text{Hitch}(X)_{\mathcal{G}(0,\psi)}
\]
\[
\simeq \bigoplus_{i=1}^{\ell - 1} \Gamma(\mathbb{P}^1, \Omega_i^{d_i + 1}((d_i) \cdot 0 + (d_i + 1) \cdot \infty)) \bigoplus \Gamma(\mathbb{P}^1, \Omega_{d_\ell + 1}((d_\ell) \cdot 0 + (d_\ell + 2) \cdot \infty))
\]
\[
\simeq \mathbb{A}^1.
\]

On the other hand, it is easy to identify the Hitchin map with the moment map associated to the action of $I(1)/I_\psi$ on $\text{Bun}_{\mathcal{G}(0,\psi)}$. Therefore, $h_\nabla(\lambda) = a(\xi) - c$ for some constant $c$. Up to normalization, we can assume that $d_\nabla(\xi) = 1$. We show that $c = 0$. Indeed, consider the automorphic $D$-module $\text{Aut} = \mathcal{D}'/\mathcal{D}'\lambda$ on $\text{Bun}_{\mathcal{G}(0,\psi)}$. It is $I(1)/I_\psi$-equivariant against $c\Psi$, with eigenvalue the local system on $\mathbb{G}_m$ represented by the connection $\partial_z + \frac{\xi}{z}$ by Theorem [HNY] which is regular singular. However, if $c \neq 0$, by [HNY] Theorem 4(1)], the eigenvalue for this $\text{Aut}$ should be irregular at $\infty$. Contradiction. \hfill \Box

Finally, for any $\chi \in \text{Op}_{\ell_\varphi}(X)_{(0,\text{RS}), (\infty, 1/h)}$ given by $\lambda = c$, $\text{Aut}_\varphi = \mathcal{D}'/\mathcal{D}'(\lambda - c)$ is a $D$-module on $\text{Bun}_{\mathcal{G}(0,\psi)}$, equivariant against $(I(1)/I_\psi, c\Psi)$. By the uniqueness of such $D$-modules on $\text{Bun}_{\mathcal{G}(0,\psi)}$ (same argument as in [HNY] Lemma 2.3), this must be the same as the automorphic $D$-module as constructed in [HNY]. We are done.

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