Stability analysis and investigation of higher order Schrödinger equation for strongly dispersive ion-acoustic wave in plasma

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Abstract. Much interest was shown towards the studies on nonlinear stability in the late sixties. Plasma instabilities play an important role in plasma dynamics. More attention has been given towards stability analysis after recognizing that they are one of the principal obstacles in the way of a successful resolution of the problem of controlled thermonuclear fusion. Nonlinearity and dispersion are the two important characteristics of plasma instabilities. Instabilities and nonlinearity are the two important and interrelated terms. In our present work, the continuity and momentum equations for both ions and electrons together with the Poisson equation are considered as cold plasma model. Then we have adopted the modified reductive perturbation technique (MRPT) from Demiray [1] to derive the higher order equation of the Nonlinear Schrödinger equation (NLSE). In this work, detailed mathematical expressions and calculations are done to investigate the changing character of the modulation of ion acoustic plasma wave through our derived equation. Thus we have extended the application of MRPT to derive the higher order equation. Both progressive wave solutions as well as steady state solutions are derived and they are plotted for different plasma parameters to observe dark/bright solitons. Interesting structures are found to exist for different plasma parameters.

Keywords: NLSE, MRPT, higher order nonlinearity and dispersion, modulational instability, solitons.

1. Introduction

Nonlinear plasma dynamics have attracted a great deal of interesting in the last few decades. The nonlinear phenomena have been taken more than half a century to reproduce the interesting and exciting description on soliton formation and its propagation through the Korteweg-de-Vries (KdV) equation. The nonlinear behaviour of the ion acoustic waves in collisionless plasma has been investigating extensively both experimentally and theoretically during the later part of the last century [3, 4]. In many field of nonlinear dynamics, local nonlinear effects such as the modulational instabilities have played a very important role in optical fibre communication system. The slow modulation of a monochromatic plane wave in plasma can lead to the formation of envelope solitons, which are described by the nonlinear Schrödinger equation (NLSE) [5]. It is a classical field equation with applications to optics [6], water waves and NLSE describes the propagation of the wave through a nonlinear medium. Several theoretical as well as experimental research works on envelop solitary...
wave have appeared in different branches of plasma dynamics. The nonlinear modulation of ion acoustic electrostatic wave packets was indeed studied by Sallahuddin et al [7]. The stability solution is important in many theory and applications of the differential equations. It is valuable to study the stability of solitary wave solutions. Recently Jukui and Rongam [8] studied the modulational instability of the amplitude of the carrier wave oblige to the direction of its propagation. The three dimensional stability solutions of NLSE have been studied by Infeld [9]. Kakutani and Sugimoto have considered both the wave number shifts as well as the long time slow modulation of the amplitude by using an extended form of the Krylov-Bogoliubov-Mitropolsky (KBM) method. The KBM method was developed for the study of the theory of nonlinear oscillation. The most exciting methods of studying nonlinear wave are to use of either potential theory [10, 11] or reductive perturbation technique (RPT) [12]. Both these techniques are of current interest and have been found to be useful in observing the soliton theory in plasmas. The RPT was first established for long wave approximation and then for wave modulation problems. Very recently Demiray [13] has presented a method so called modified RPT (MRPT) to examine the contribution of higher order terms in RPT and applied the method to ion acoustic plasma waves for weakly dispersive case. The basic idea in this method was the inclusion of higher order dispersive effects through the introduction of the scaling parameter \( g \), to balance the higher order nonlinearity and dispersion. The contribution of higher order terms in RPT for strongly dispersive case had been studied by Ichikawa [14] et al for nonlinear ion plasma waves. Later based on MRPT, Demiray [1] extended this works to strongly dispersive case and investigated the modulation of nonlinear waves in ionic cold plasma. In our earlier work [15] we have derived the NLSE for strongly dispersive ion acoustic plasma waves taking into consideration the electron inertia effect through the parameter \( Q (= m_e/m_i) \) to study the nonlinear wave modulation and stability. Also we have derived a progressive wave solution, steady state solution and Duffing like equation from NLSE. In our present work we have extended this work for investigation of higher order equation of the NLSE, which is named as linear Schrödinger equation with nonhomogenous term [1] for strongly dispersive ion acoustic wave in plasma. Moreover we have derived both progressive wave solution as well as steady state solution of higher order equation and plotted for different plasma parameters.

2. Basic Equations

In our model we have considered [15] the continuity and momentum equations for both the species-ions and electrons together with the Poisson equation. So the basic equations are as follows

For ions,

\[
\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0,
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\partial \psi}{\partial x} = 0,
\]

For electrons,

\[
\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_e)}{\partial x} = 0,
\]

\[
\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} - \frac{1}{Q} \left( \frac{\partial \psi}{\partial x} - \frac{1}{n_e} \frac{\partial n_e}{\partial x} \right) = 0
\]

Poisson,

\[
\frac{\partial^2 \psi}{\partial x^2} + n_i - n_e = 0
\]

Where \( u_i, u_e \) are the flow variables of ions and electrons, \( n_i, n_e \) are the ions and electrons density, \( \psi \) is the electric potential. Denoting the fluctuation of the ions and electrons density from its equilibrium value by \( n_i = 1 + n_{i_0}, n_e = 1 + n_{e_0} \).

3. Derivation of the higher order equation of NLSE

Introducing the slow variables \( \xi = \epsilon(x - \lambda t), \tau = \epsilon^2 gt \)
Where $\varepsilon$ are small parameter measuring the weakness of certain physical properties like nonlinearity; $\lambda \geq 0$ and $g$ are some scale parameters. In this work it is assumed that the field variables are functions of slow variables $(\xi, \tau)$ as well as the fast variables $(z, t)$. We further assume that the variables and the scale parameter $g$ may be expressed as asymptotic series of $\varepsilon$ as given below

$$
n_i = \sum_{j=1}^{\infty} \varepsilon^j n_{ij}(z, t, \xi, \tau), \quad n_e = \sum_{j=1}^{\infty} \varepsilon^j n_{ej}(z, t, \xi, \tau), \quad u_i = \sum_{j=1}^{\infty} \varepsilon^j u_{ij}(z, t, \xi, \tau)$$

$$
u_e = \sum_{j=1}^{\infty} \varepsilon^j u_{ej}(z, t, \xi, \tau), \quad g = 1 + \sum_{j=1}^{\infty} \varepsilon^j g_j, \quad \psi = \sum_{j=1}^{\infty} \varepsilon^j \psi_j(z, t, \xi, \tau)$$

where the coefficient functions $n_{ij}, n_{ej}, u_{ij}, u_{ej}, \psi_j$ and the constants are to be determined from the solution of the field equations. After some mathematical calculations we have derived the standard NLS equation [15].

$$i \frac{\partial N}{\partial \tau} + p \frac{\partial^2 N}{\partial \xi^2} + q |N|^2 N = 0, \quad N = n_1$$

Which is the nonlinear Schrödinger equation with $p$ and $q$ as the dispersive and nonlinear coefficients respectively and are defined as

$$p = \left[ \frac{1}{\omega^2} + \frac{Q}{4k^2} + \frac{1}{4k^2(1 - \omega^2)} \right] \left[ \frac{k}{1 - \omega^2} + \frac{1}{\omega k} \right]$$

$$q = \left[ \frac{1}{\omega^2} \left( 1 + \frac{Q}{k^2} \right) - \frac{24Q}{4k^2} \omega^2 + 4(1 - \omega^2) - 6\omega Q + 8\omega^2 Q^2 + \frac{\omega^4}{1 - \omega^2} (8 + Q) \right] + \frac{9}{2} + 2Q - \frac{1}{2\omega^2}$$

$$- 4Q(1 - \omega^2) + \frac{2(1 - \omega^2)^2}{3\omega^2} Q(2 - \frac{\omega}{\omega^2}) + \frac{(1 - \omega^2)^3}{2} \left( \frac{3}{2} + Q \right) - 4Qk^2 \left[ \frac{k}{1 - \omega^2} + \frac{1}{\omega k} \right]$$

In deriving equation (4), we have used the charge neutrality condition, $N_i = N_e = N$. This NLSE is different from the NLSE derived by Kakutani and Sugimoto [5] in respect of the linear term, which they have neglected in their works. So it is clear that the NLSE equation that we have derived is more accurate in the case of studies related to nonlinearity of the highly dispersive wave structures.

And after some mathematical calculations we have derived the higher order equation of NLSE which is of the following form

$$i \frac{\partial N_2}{\partial \tau} + P \frac{\partial^2 N_2}{\partial \xi^2} + ig_1 \frac{\partial N}{\partial \tau} + iE \frac{\partial^3 N}{\partial \xi^3} + a|N|^2 N_2 + bN^2 N_2 + ic \left( \frac{\partial N^2}{\partial \xi} + i dN \frac{\partial \bar{N}}{\partial \xi} \right) = 0$$

Where $g_1$ characterizes the contribution of higher order dispersive effects and $a, b, c, d, E, P$ are the coefficients in terms of $\omega, k, Q$. Thus we find the application of MRPT in deriving higher order equation known as linear Schrödinger equation with non-homogenous term.

**4. Solution sets of the higher order equation of NLSE**

**4.1 Progressive wave solution for NLSE (4)**
Using the transformation, \( N(\xi, \tau) = f(\zeta) \exp[i(K\zeta - \Omega \tau)] \), \( \zeta = \xi - c\tau \)

Where \( K, \Omega, c \) are constants and \( f(\zeta) \) is an unknown real function to be determined through the solution of the NLSE (4) which becomes

\[
f''(\zeta) + \left( \frac{\Omega}{p} - K^2 \right) f(\zeta) + \frac{q}{p} f^3(\zeta) = 0, \quad \text{where} \quad c = 2kp
\]  

(6)

Introducing \( f(\zeta) = A \tanh(\beta \zeta) \), where \( A \) is the amplitude and \( \beta \) is constant in (6), the solution is obtained as \( N(\xi, \tau) = A \tanh\left\{ \frac{q}{2p} A(\zeta - (K^2 p - q A^2) \tau) \right\} \), \( \beta = \frac{q}{\sqrt{2p} A} \)  

(7)

4.2 Progressive wave solution for higher order equation of NLSE

The solution for \( N_2^{(i)}(N_2^{(i)} = n_2 \) to next order equation of \( \zeta \) will be of the form

\[
N_2^{(i)} = h(\zeta) \exp[i(K\zeta - \Omega \tau)]
\]  

(8)

Here \( h(\zeta) \) is an unknown function to be determined through the solution of (5). Introducing (7), (8) into equation (5) we have

\[
h'' + \frac{1}{p^2} [\Omega - PK^2 + (a + b) f^2] + \frac{1}{p} \left[ f(g_1 \Omega + EK^3) - \frac{(c - d)}{p} \right] K A^3 z^3 + i \frac{(c + d)}{p} f^2 f' = 0
\]  

(9)

The form of equation (9) reveals that the function \( h(\zeta) \) is complex and can be expressed as \( h(\zeta) = h_1(\zeta) + ih_2(\zeta) \). The real and complex parts of this function satisfy the following differential equations respectively

\[
h_1'' + h_1' \left[ \Omega - PK^2 + (a + b) f^2 \right] + \frac{1}{p} f (g_1 \Omega + EK^3) - \frac{(c - d)}{p} K A^3 z^3 = 0
\]  

(10)

\[
h_2'' + h_2' \left[ \Omega - PK^2 + (a + b) f^2 \right] + \frac{1}{p} f'' + \frac{(c + d)}{p} f^2 f' = 0
\]  

(11)

For the solution of these equations, it might be convenient to employ the tangent hyperbolic method. For this purpose we introduce the new variable \( z = \tanh(\beta \zeta) \). In terms of this new variable the differential equations (10) and (11) become

\[
\frac{q A^3}{2p} \left[(1 - z^2)^z \zeta \frac{d^2 h_1}{dz^2} + 2(z^2 - 2) \frac{dh_1}{dz} + \frac{1}{p} \left[ K^2 p - q A^2 - PK^2 + (a + b) A^2 z^3 \right] h_1 + \frac{A}{p} (g_1 \Omega + EK^3) z - \frac{(c - d)}{p} K A^3 z^3 \right] = 0
\]  

(12)

\[
\frac{q A^3}{2p} \left[(1 - z^2)^z \zeta \frac{d^2 h_2}{dz^2} + 2(z^2 - z) \frac{dh_2}{dz} + \frac{1}{p} \left[ K^2 p - q A^2 - PK^2 + (a + b) A^2 z^3 \right] h_2 + \frac{2 E}{p} A \left( \frac{q}{2p} \right)^{3/2} A^3 \right] + z^2 \left[ \frac{(c + d)}{2p} A^3 \left( \frac{q}{2p} \right)^{1/2} A \right] - z^4 \left[ \frac{2AE}{p} \left( \frac{q}{2p} \right)^{3/2} A^3 + \frac{(c + d)}{p} A^3 \left( \frac{q}{2p} \right)^{1/2} A \right] = 0
\]  

(13)

For the solution of equation (12) we shall propose the following function \( h_1 = b_1 z \)  

(14)

where \( b_1 \) is a constant. The solution of this type is possible if and only if \( g_1 \) is of the following forms

\[
g_1 = - \frac{E}{\Omega} K^3
\]  

(15)

The method present here makes it possible to determine the scale parameter \( g_1 \) as a part of the solution. As it is clear from equation (15) if \( g_1 \) vanishes, the solution obtained here is not valid anymore. The solution for \( h_2 \) may be expressed as \( h_2 = a_0 + a_2 z^2 \)  

(16)
Where the coefficients \( a_0 \) and \( a_2 \) are given by

\[
a_0 = \frac{1}{[1 - (-4 + \frac{P}{q} \Omega - PK^2)]} \frac{1}{(a-b)A^2} \frac{E}{p} \left[ \frac{q}{2p} A^2 \frac{q}{2p} \right]^{1/2} (-4 + \frac{P}{q} \Omega - PK^2) \]

\[
a_2 = \left[ \frac{1}{(a-b)A^2} \frac{q}{2p} \right]^{1/2} \left[ \frac{E}{p} \left[ \frac{q}{2p} A^2 \frac{q}{2p} \right]^{1/2} (-4 + \frac{P}{q} \Omega - PK^2) \right] \]

Hence the solution for \( N_2^{(1)} \) may be given by

\[
N_2^{(1)} = [b_1 \tanh \beta \xi + i(a_0 + a_2 \tanh^2 \beta \xi)] \exp[i(K \xi - \Omega \tau)]
\]

(17)

4.3 Steady state solution for NLSE

For steady state, the transformation \( N(\xi, \tau) = f(\xi) \exp(-i\alpha \tau) \), (with \( \alpha \) is a constant, \( f(\xi) \) is an unknown function ) equation (4) is modified as \( \alpha f(\xi) + pf^*(\xi) + qf^2(\xi) = 0 \)

(18)

Using \( f(\xi) = A \tanh \beta \xi \), where \( A \) is the amplitude and \( \beta \) is constant, the solution is obtained as

\[
N(\xi, \tau) = A \tanh \left( \frac{\sqrt{q/2p}}{A} \xi \right) \exp[(i\alpha^2) \tau], \quad \beta = \sqrt{q/2p} A
\]

(19)

4.4 Steady state solution for higher order equation of NLSE

For the unknown \( N_2^{(1)} \) in equation (5) we shall seek solutions of the following form

\[
N_2^{(1)} = h(\xi) \exp(-i\alpha \tau)
\]

(20)

Here \( h(\xi) \) is another unknown function to be determined from the solution of (5). Introducing the solution (19) and (20) into equation (5) we have

\[
h^* + h \left[ \frac{\gamma}{P} + \frac{1}{P} (a+b) f^2 \right] + \frac{E}{P} f^* + \frac{1}{P} \gamma g, f + i \frac{\gamma + d}{P} f^2 f' = 0
\]

(21)

The form of equation (21) reveals that the function \( h(\xi) \) is complex and can be expressed as \( h(\xi) = h_1(\xi) + i h_2(\xi) \). The real and complex parts of this function satisfy the following differential equations

\[
h_1' + h_1 \left[ \frac{\gamma}{P} + \frac{1}{P} (a+b) f^2 \right] + \frac{1}{P} \gamma g, f = 0
\]

(22)

\[
h_2' + h_2 \left[ \frac{\gamma}{P} + \frac{1}{P} (a-b) f^2 \right] + \frac{E}{P} f^* + \frac{\gamma + d}{P} f^2 f' = 0
\]

(23)

For the solution of these equations, it might be convenient to employ the tangent hyperbolic method.

For this purpose we introduce the new variable \( z \) as \( z = \tanh \left( \frac{\sqrt{q/2p}}{A} \xi \right) \)

(24)

In terms of this new variable in the differential equation (22) and (23) becomes

\[
(1 - z^2) \frac{d^2 h_1}{dz^2} + 2(z^3 - z) \frac{dh_1}{dz} + 2 \left[ \frac{-P}{P} + \frac{(a+b)}{P} \right] z^2 h_1 - 2 \frac{P}{P} g, A z = 0
\]

(25)
Here we have utilized the differential relation
\[ \frac{d}{dz} = \beta (1 - z^2) \frac{d}{dz} \]

The simplest solution of (25) is \( h_1 = 0 \) and \( g_1 = 0 \) and we adopt it in this case. For the solution of \( h_2 \), we propose
\[ h_2 = a_2 z^2 + a_1 z + a_0 \]  
(27)

Where \( a_0, a_1 \) and \( a_2 \) are some constants to be determined form the solution of (25). Introducing (27) into (26) and equating the coefficients of like powers of \( z \) equal to zero we have
\[
a_0 = \frac{1}{p} A^2 (\frac{q}{2p})^{1/2} \{ E - \frac{1}{E - \frac{1}{E - \frac{1}{E}} \} \frac{E}{q} (\frac{a - b}{q}) \}
\]
\[
a_1 = \{ \frac{1}{E - \frac{1}{E}} \} \frac{E}{q} (\frac{a - b}{q}) \}
\]

Here the value of \( a_2 \) is zero. Thus the solution for \( N_2^{(1)} \) becomes
\[ N_2^{(1)} = i (a_0 + a_1 \tanh^2 \beta \xi) \exp(-i \alpha \tau) \]  
(28)

5. Results and Discussions

We have made an effort to study the solution set of higher order equation of NLSE by applying the MRPT for strongly dispersive cold plasma waves and their modulations in the light of their structures. The plane wave solution of the NLSE is modulationally unstable if \( pq > 0 \) and stable if \( pq < 0 \), depending on the wave number \( k \). A progressive and steady state solutions of higher order equation are derived. The progressive wave solution (7) of NLSE as well as solution (17) of higher order equation (5) of NLSE with respect to \( \xi \) for \( k = 0.2, c = 0.03, \omega = 1.3 \) to study the nonlinear wave modulation which are depicted in figure1 (a) and (b) respectively. Higher order wave modulation can easily be seen by comparing 1(b) with 1(a). By changing \( k \) to the value 0.7 we see the modulation for progressive wave solution of higher order equation of NLSE in figure (2). The value of the parameter \( Q = 0.00054, \omega = 1.3, A \) (constant amplitude) = 0.5, \( b_1 = 5, \tau = 400 \) are fixed for all numerical results throughout this works. In case of higher order solution, the wave modulation is found to be more for \( k < 1 \).
Figure 2 Progressive wave solution of higher order equation of NLSE (17) vs $\xi$ is plotted for $k=0.7$, $b_1=5$, $\omega=1.3$, $c=0.03$, $\tau=400$.

Figure 3 (a), (b) Plot showing the comparison of steady state solution of NLSE (19) and higher order equation of NLSE (28) vs $\xi$ is plotted for $k=0.7$, $\omega=1.3$, $\tau=400$.

Figure 4 Steady state solution of (28) vs $\xi$ is plotted for $\omega=1.3$, $\tau=400$, $k=1.47$. 
In case of higher order solution, the wave modulation is found to be more for $k<1$. Also the higher order solution increases the modulation depth of the wave. Progressive wave solution through figure 2 shows the possibility of soliton formation with oscillating tails for $k<1$. Moreover, figure 3(a), (b) shows the comparisons of the formation of dark soliton for the steady state solution of NLSE (19) and higher order equation of NLSE (28). The remarkable change in case of higher order solution of the NLSE is change of the origin of the dark soliton. Figure 4 shows the possibility of the formation of bright solitons for wave number greater than 1.

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