Spinless basis for spin-singlet FQH states

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Abstract

We investigate an alternative description of the SU($M$)-singlet FQH state by using the spinless basis. The SU($M$)-singlet Halperin state is obtained via the $q$-deformation of the Laughlin state and its root of unity limit, by applying the Yangian Gelfand-Zetlin basis for the spin Calogero-Sutherland model. The squeezing rule for the SU($M$) state is also investigated in terms of the spinless basis.

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1 Introduction

Recently it has been shown the Jack polynomial provides a universal description of the fractional quantum Hall states (FQH states) \[1, 2, 3\], satisfying the admissible condition \[4\] which characterizes the generalized statistics of the FQH states. This description is understood from the viewpoint of the conformal field theory (CFT): the Jack polynomial for the generic \((k, r)\)-admissible condition is regarded as the conformal block of the CFT with the extended chiral algebra \(WA_{k-1}(k+1, k+r)\) \[5, 6, 7\]. It is also applied to the FQH states in the presence of the internal (spin) degrees of freedom \[8, 9, 10, 11\] by considering the non-symmetric Jack polynomial \[12, 13, 14, 15\].

The Jack polynomial is the eigenfunction of the Laplace-Beltrami operator, which is regarded as the Hamiltonian of the Calogero-Sutherland model, up to the gauge transformation. If we consider the spin degrees of freedom, we have to deal with the spin Laplace-Beltrami operator. In this case there are degeneracies in its spectrum due to the spin degeneracy. Thus it is difficult to apply it naively to numerical applications \[16, 17\]. On the other hand, it is shown that such a degeneracy is resolved by utilizing the Yangian symmetry of the spin Calogero-Sutherland model \[18, 19\]. Indeed the orthogonal basis for the spin Calogero-Sutherland model is constructed by applying such a basis. The Yangian Gelfand-Zetlin basis for the spin Calogero-Sutherland model is realized as the root of unity limit of the \(q\)-deformed theory \[19\]. This means we have an alternative method to describe
the SU($M$)-singlet FQH state, i.e. the root of unity limit of the Macdonald polynomial, which is called the Uglov polynomial [20], instead of the non-symmetric Jack polynomial.

In this paper we investigate a novel description of the FQH states with the spin degrees of freedom by utilizing the Yangian Gelfand-Zetlin basis, which is well investigated in [19] for the spin Calogero-Sutherland model. First the SU($M$)-singlet condition is discussed by considering the generic property of the Lie algebra. We obtain the Fock condition as well as the standard SU(2)-singlet Halperin state. Starting from the Laughlin state, we construct the spin-singlet Halperin state through the $q$-deformation of the U(1) primary field. This procedure is based on the Yangian Gelfand-Zetlin basis for the spin Laplace-Beltrami operator, which is well studied from the viewpoint of the spin Calogero-Sutherland model. We then show the relation between the SU($M$)-singlet state and a certain spinless state in terms of the occupation number representation. The squeezing rule for the SU($M$) state is consistently translated into the spinless basis. We finally comment on the underlying CFT for the SU($M$)-singlet state, and its relation to the four dimensional gauge theory on the orbifold and the root of unity limit of the $q$-deformed CFT.

2 Spin-singlet FQH states

Before discussing the spinful FQHE states, let us first introduce the most fundamental example of the FQH state, which is called the Laughlin state [21],

$$\Phi_L(\{z_i\}) = \prod_{i<j}^N (z_i - z_j)^r. \quad (2.1)$$

For convenience we now omit the Gaussian factor, $\exp\left[-\sum_i |z_i|^2/(4\ell^2)\right]$, with the magnetic length being $\ell^2 = \hbar/(eB)$. The filling fraction of this state is given by $\nu = 1/r$, and the power $r$ has to be odd integer due to the anti-symmetricity of the wavefunction for a fermionic system. Thus this wavefunction gives rise to the odd denominator series of the FQH states.

Actually this Laughlin state does not include internal degrees of freedom. However we will see the spin-singlet FQH states are built from this fundamental wavefunction.

2.1 SU(2) theory

A natural generalization of this Laughlin state (2.1) is the Halperin state [8], which accompanies SU(2) spin degrees of freedom,

$$\Phi_H(\{z_i, w_i\}) = \prod_{i<j}^{N_1} (z_i - z_j)^r \prod_{i<j}^{N_2} (w_i - w_j)^r \prod_{i,j} (z_i - w_j)^s. \quad (2.2)$$

Here $z_i$ and $w_i$ stand for positions of up- and down-spin particles, respectively. Since we usually investigate the FQH system in the presence of strong magnetic fields, it is natural to consider fully spin-polarized states. This means the spin degrees of freedom are frozen in
such a case. However, when there are pseudo-spin degrees, e.g. valley degeneracy, multi-layer systems and so on, the spinful FQH state would provide a good description.

We consider the spin operators for this wavefunction to discuss the spin-singlet state. When the spin-raising operator \( S^+ \) acts on this wavefunction (2.2), one of the down-spin particles is changed into the up-spin state, i.e. \( w_i \rightarrow z_{N_t+1} \). We often omit the index labeling the particle. If the state is fermionic, it has to be anti-symmetrized with all the up-spin particles,

\[
S^+_t \Phi_H(\{z_i, w_i\}) = \Phi_H(\{z_i, w_i\}) - \sum_{j=1}^{N_t} \Phi_H(z_j \leftrightarrow w_i). \tag{2.3}
\]

Thus the spin-raising operator yields

\[
S^+_t = 1 - \sum_{j=1}^{N_t} e(w_i, z_j) \quad \text{(for fermions)}, \tag{2.4}
\]

where the operator \( e(w_i, z_j) \) exchanges \( w_i \) and \( z_j \). On the other hand, for a bosonic state, the wavefunction has to be symmetrized with up-spin particles. This means the sign factor arising in the spin-raising operator is modified as

\[
S^+_t = 1 + \sum_{j=1}^{N_t} e(w_i, z_j) \quad \text{(for bosons)}. \tag{2.5}
\]

Similarly an up-spin particle is changed into the down-spin state under the spin-lowring operation, \( z_i \rightarrow w_{N_t+1} \). Therefore the spin-lowering operator is written as

\[
S^- = 1 + \sum_{j=1}^{N_t} e(z_i, w_j). \tag{2.6}
\]

The sign factor takes minus for fermionic and plus for bosonic systems. The other spin operator is simply given by \( S_z = (N_t - N_\uparrow)/2 \). Therefore the spin-singlet condition, which is given by

\[
S^+_t \Phi_H(\{z_i, w_i\}) = 0, \quad S^- \Phi_H(\{z_i, w_i\}) = 0, \quad S_z \Phi_H(\{z_i, w_i\}) = 0, \tag{2.7}
\]

yields \( r = s + 1 \) and \( N_{\uparrow} = N_{\downarrow} \) for the Halperin state (2.2). This is just regarded as the Fock condition [22]. This also implies that \( r \) is odd/even for fermions/bosons.

### 2.2 SU(\(M\)) theory

We extend the singlet-condition (2.7) to arbitrary \( M \)-component systems. A simple generalization of the Halperin state (2.2) is given by

\[
\Phi_H^M(\{z_i^{(u)}\}_{u=1,\ldots,M}) = \prod_{u=1}^{M} \prod_{i<j}^{N^{(u)}} (z_i^{(u)} - z_j^{(u)})^r \prod_{u<v}^{M} \prod_{i,j}^{N^{(u,v)}} (z_i^{(u)} - z_j^{(v)})^s. \tag{2.8}
\]
This is an $M$-state wavefunction: $z_i^{(u)}$ stands for a position of an $i$-th $u$-state particle.

To assign a similar manipulation to this wavefunction, we then need the raising and lowering operators for SU($M$) group. According to the general theory of the Lie algebra, the generators can be split into $H_i$, $E_i$ and $F_i$. Here $E_i$ and $F_i$ are just regarded as the raising and lowering operators, while Cartan subalgebra consists of $H_i$. They are simply represented by introducing creation and annihilation operators for the $u$-th state, $a^{(u)\dagger}$ and $a^{(u)}$,

$$H_u = a^{(u)\dagger}a^{(u)} - a^{(u+1)\dagger}a^{(u+1)} \quad (u = 1, \cdots, M-1),$$

$$E_{(u,v)} = a^{(u)\dagger}a^{(v)} \quad (u > v),$$

$$F_{(u,v)} = a^{(u)\dagger}a^{(v)} \quad (u < v).$$

This means $E_{(u,v)}$ and $F_{(u,v)}$ convert $v$-th state into $u$-th state. For $u > v$ and $u < v$, we call it raising and lowering, respectively. Remark the numbers of these generators are $M(M-1)/2$ for $E$ and $F$, $M-1$ for $H_i$. Thus the total dimension of SU($M$) becomes $M^2 - 1$. For example, SU(2) algebras are represented as $S_+ = a_+^\dagger a_+$, $S_- = a_-^\dagger a_-$ and $S_z = (a_+^\dagger a_- - a_-^\dagger a_+)/2$. The total number of the generators is consistent with $\text{dim SU}(2) = 3$.

We then explicitly show the singlet condition for SU($M$) theory. The corresponding condition to (2.7) is given by

$$H_u \Phi_H^M = 0, \quad E_{(u,v)} \Phi_H^M = 0, \quad F_{(u,v)} \Phi_H^M = 0.$$  

(2.12)

The first one is simply satisfied when $N^{(1)} = N^{(2)} = \cdots = N^{(M)}$. As the case of SU(2), the other operators are represented as

$$1 + \sum_{i=1}^{N^{(u)}} e^{i(z_i^{(u)}, z_i^{(v)})} = \begin{cases} E_{(u,v)} & (u > v) \\ F_{(u,v)} & (u < v) \end{cases}.$$  

(2.13)

Again the sign factor takes minus for fermionic and plus for bosonic systems. Thus we have an essentially the same condition for SU($M$) theory as the SU(2) theory. The singlet condition is just given by $r = s + 1$ and $N^{(1)} = N^{(2)} = \cdots = N^{(M)}$.

Remark the filling fraction of the Halperin state (2.8) satisfying this SU($M$)-singlet state is given by

$$\nu = \frac{M}{M(r-1)+1}.$$  

(2.14)

3 From Laughlin to Halperin

We show the spin-singlet FQH state can be obtained from a certain spinless state. Here we concentrate on the Abelian FQH state, i.e. the Laughlin state. Its validity for the generic cases is discussed in section 4.

The method we discuss here is based on the Yangian Gelfand-Zetlin basis for the spin Calogero-Sutherland model [19], which is also applied to the gauge theory partition function.
for the orbifold theory \[23, 24\]. We often use the non-symmetric Jack polynomial to describe the spinful model. However there are degeneracies in its spectrum due to the internal spin degrees of freedom. By resolving the spin degeneracy with the Yangian symmetry, we obtain the alternative basis for the spinful model, which can be written in terms of symmetric polynomials. We apply this method to the FQH states to obtain an alternative way of describing the spinful FQH states.

### 3.1 The $q$-bosonic field

We introduce the $q$-deformed FQH state by considering the $q$-bosonic field. The $q$-boson is quite similar to the standard free boson field, satisfying the slightly modified commutation relations \[25, 26\],

\[
[a_n, a_m] = n \frac{1 - q^{|n|}}{1 - t^{|n|}} \delta_{n+m,0}, \quad [a_n, Q] = \frac{1}{r} \delta_{n,0},
\]

where we parametrize \( t = q^r \). The corresponding bosonic field is given by

\[
\varphi(z) = Q - a_0 \log z - \sum_{n \neq 0} \frac{1}{n} \frac{1 - t^{|n|}}{1 - q^{|n|}} a_n z^{-n}.
\]

As the case of the standard CFT, the operator product expansion (OPE) of this $q$-boson plays an essential role in considering the conformal block, which is interpreted as the FQH wavefunction. The singular part of the OPE is given by

\[
\varphi(z)\varphi(w) \sim -\log \left( \frac{(w/z; q)_\infty}{(tw/z; q)_\infty} z^r \right),
\]

where we use the Pochhammer symbol, \((x; q)_n = \prod_{m=0}^{n-1} (1 - xq^m)\). As a result, the conformal block of the $q$-primary field \( V(z) = : \exp(i\varphi(z)) : \) is calculated by utilizing this OPE relation,

\[
\langle V(z_1) \cdots V(z_N) \rangle = \prod_{i<j} \left( \frac{z_i/z_j; q}_\infty \right) \frac{z_i^r}{z_j^r}.
\]

Remark this is almost the same as the $q$-Vandermonde determinant, which is just the weight function of the Macdonald polynomial \[27\], with the extra contribution of the zero mode. Actually, as well known, the Laughlin state is interpreted as the conformal block of the U(1) primary fields. Thus, we now regard this conformal block (3.4) as the $q$-Laughlin state.

### 3.2 The root of unity limit

We can easily show the $q$-deformed state (3.4) goes back to the standard Laughlin state (2.1) by taking the limit, \( t = q^r \) and then \( q \to 1 \). Note that the Macdonald polynomial is reduced to the Jack polynomial in this limit.

On the other hand, to implement the Yangian Gelfand-Zetlin basis, we take another limit of the $q$-state, i.e. the root of unity limit,

\[
q \to \omega_M q, \quad t \to \omega_M t = \omega_M q^r, \quad q \to 1,
\]

5
where \( \omega_M = \exp(2\pi i/M) \) is the \( M \)-th primitive root of unity. This limit is singular because there are infinite products appearing in the \( q \)-state, thus its convergence radius is \( |q| = 1 \). Therefore we have to deal with it appropriately \([23, 24]\).

Using the formula for the root of unity limit (3.5) \([23, 24]\),

\[
\frac{(z; q)_\infty}{(t; q)_\infty} \rightarrow (1 - z^M)^{(r-1)/M}(1 - z),
\]

the OPE of the \( q \)-state (3.2) in the limit (3.5) turns out to be

\[
\varphi(z) \varphi(w) \sim -\log(z^M - w^M)^{(r-1)/M}(z - w).
\]

To discuss the spin-singlet wavefunction, we now introduce another set of coordinates,

\[
z^M = Z, \quad w^M = W.
\]

This means the original one is described as \( z = \omega_M^{n_z} Z^{1/M}, \ w = \omega_M^{n_w} W^{1/M}, \) with \( n_z, n_w = 0, 1, \ldots, M - 1 \). Here \( n_z \) and \( n_w \) stand for the branches of the \( M \)-th roots of \( Z \) and \( W \), respectively, as shown in Fig. 1. We rewrite the OPE (3.7) with these variables,

\[
\varphi(Z) \varphi(W) \sim -\log(Z - W)^{(r-1)/M}(\omega_M^{n_z} Z^{1/M} - \omega_M^{n_w} W^{1/M}).
\]

We then show the second part of this OPE singularity depends on the choice of the \( M \)-th root branch. In the case of \( n_z = n_w \), its singular behavior is given by

\[
\log(\omega_M^{n_z} Z^{1/M} - \omega_M^{n_w} W^{1/M}) \sim \log(Z - W).
\]

This is because when we write the right hand side as \( (Z - W) = (Z^{1/M} - W^{1/M})(Z^{(M-2)/M} W^{1/M} + \cdots + W^{(M-1)/M}) \), the latter part becomes regular at \( Z \sim W \). Thus the OPE (3.9) should be rewritten as

\[
\varphi(Z) \varphi(W) \sim -\log(Z - W)^{(r-1)/M+1}.
\]

On the other hand, when \( n_z \neq n_w \), \( \log(\omega_M^{n_z} Z^{1/M} - \omega_M^{n_w} W^{1/M}) \) is regular at \( Z \sim W \). In other words, we cannot obtain the form of \( \log(Z - W) \) from \( \log(\omega_M^{n_z} Z^{1/M} - \omega_M^{n_w} W^{1/M}) \), as
discussed in the case of $n_z = n_w$, without modifying its singular behavior at $Z \sim W$ due to the identity

$$\omega_M^{(M-1)n} x^{M-1} + \omega_M^{(M-2)n} y^{M-2} + \cdots + y^{M-1} = 0 \quad \text{at} \quad x = y, \quad n \not\equiv 0 \pmod{M}. \quad (3.12)$$

Therefore the OPE (3.9) behaves as

$$\varphi(Z)\varphi(W) \sim -\log(Z - W)^{(r-1)/M}. \quad (3.13)$$

As a result, the singularity of the OPE (3.9) depends on the choice of the branch for the $M$-th root of the variable. Thus we parametrize positions of particles as $z_i = \omega_M^u(z^{(u)}_I)^{1/M}$, satisfying $z_i^M = z^{(u)}_I$. Then the FQH state, which is represented as the corresponding conformal block of the primary field, is given by

$$\Phi\left(\left\{z_i^{(u)}\right\}\right) = \prod_{u=1}^{M} \prod_{I<J}^{N^{(u)}} (z^{(u)}_I - z^{(u)}_J)^{(r-1)/M + 1} \prod_{u<v}^{M} \prod_{I,J} (z^{(u)}_I - z^{(v)}_J)^{(r-1)/M}. \quad (3.14)$$

This multi-component FQH state is just the SU($M$)-singlet Halperin state (2.8). Here we obtain this spin-singlet state from only one kind of the $q$-boson field by taking the root of unity limit. This is a specific result coming from its SU($M$) symmetry. We have to introduce the corresponding $M$ types of bosons to construct a generic SU($M$) state, i.e. a non-singlet state. Although the singlet-state (3.14) includes the index labeling the particle state, $u = 1, \cdots, M$, its role can be freely changed due to the SU($M$)-singlet property.

The filling fraction of this state is obtained by substituting $r \rightarrow (r - 1)/M + 1$ into the formula (2.14),

$$\nu = \frac{M}{M((\frac{r}{M} - 1) + 1) + 1} = \frac{M}{r}. \quad (3.15)$$

This is almost the same as the fraction of the Laughlin state $\nu = 1/r$, up to the factor corresponding to the number of internal states. This factor is interpreted as a result of changing the variables as $z_i^M = z^{(u)}_I$.

4 Spinful FQH states with admissible condition

We extend the relation between the spinless and spin-singlet FQH wavefunctions to more generalized FQH states. In this section we describe them in terms of the occupation number representation with emphasis on the admissible condition and its connection to the Jack polynomial, which is useful to study the generic FQH states. The scheme discussed in this section can describe not only the Abelian FQH state shown in section 3 but also the non-Abelian state. Thus it is expected that the spinless description of the spin-singlet state is possible even for the non-Abelian state.
4.1 Admissible condition

We introduce the occupation number representation of the FQH states\(^1\). We represent a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)\) with length \(\ell_\lambda \leq N\) as a (bosonic) occupation number configuration \(n(\lambda) = \{n_m(\lambda), m = 0, 1, 2, \ldots\}\). This means each of the lowest Landau level (LLL) orbitals, where \(n_m(\lambda)\) is the multiplicity of \(m\) in \(\lambda\). We can implement the dominance order for partitions \(\lambda > \mu\), with the squeezing rule which connects configurations \(n(\lambda) \rightarrow n(\mu)\). The ground state of the FQH system on the sphere can be uniquely represented with the root partition, which is the most dominant partition with fixing the particle number.

The Jack polynomial is a symmetric polynomial labeled by a partition \(\lambda\), which can be expanded with non-interacting states (monomial polynomials) obtained by squeezing procedure, \(J_\lambda^\alpha(\{z_i\}) = m_\lambda(\{z_i\}) + \sum_{\mu<\lambda} c_{\lambda\mu}(\alpha)m_\mu(\{z_i\})\). This is just an eigenstate of the Laplace-Beltrami operator, \(H_{LB} = \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} z_i - z_j \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \). It has been shown that when the parameter \(\alpha\) is negative as \(\alpha = -(k+1)/(r-1)\), the Jack polynomial obeys the admissible condition \([4]\), \(\lambda_i - \lambda_i+i \geq r\).

The root configuration for the \((k,r)\)-admissible state, whose \(L_z\) component of angular momentum is zero, is given by \(|\Phi(k,r)\rangle = |k0^{r-1}k0^{r-1}k0^{r-1}\ldots\rangle\) and the corresponding magnetic flux for the spherical system yields \(N_\phi = \frac{r}{k}N - r \equiv \frac{1}{\nu}N - r\). Therefore the filling fraction reads \(\nu = k/r\). The Laughlin state \([2.1]\) satisfies the \((k = 1, r)\) admissible condition, \(|\Phi_L\rangle = |10^{r-1}10^{r-1}10^{r-1}\ldots\rangle\) and the corresponding non-Abelian FQH states, e.g. Moore-Read \([28]\) and Read-Rezayi states \([29]\), can be also described in this way: the RR state associated with \(\mathbb{Z}_k\) parafermion obeys the \((k,2)\)-admissible condition \([1, 2, 3]\).

We then consider the mapping from the spinless to the spin-singlet states, discussed in section 3, in terms of these representations. The corresponding partition to the root configuration for the \((k, r, M = 1)\) state \(|k0^{r-1}k0^{r-1}\ldots\rangle\) is given by

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) = (r(N/k - 1), \ldots, r(N/k - 1), 2r, \ldots, 2r, r, \ldots, r, 0, \ldots, 0) \quad (4.5)
\]

\(^1\)See, for example, \([1, 2, 3]\).
where $N$ is the particle number and we can see $\lambda_1 = N_{\phi}$ defined in (4.4). In order to obtain the spin-singlet state as discussed in section 3, we then introduce the following partition from this root configuration (4.5), which corresponds to the modified variables (3.8),

$$\tilde{\lambda} = \left( \left[ \frac{\lambda_1}{M} \right], \ldots, \left[ \frac{\lambda_{N-1}}{M} \right], \left[ \frac{\lambda_N}{M} \right] \right) = \left( \left[ \frac{r}{M} (N/k - 1) \right]/k, \ldots, \left[ \frac{r}{M} (N/k - 1) \right]/k \right), \ldots, \left[ \frac{r}{M} \right], \left[ \frac{0}{M} \right], \ldots, \left[ \frac{0}{M} \right].$$

(4.6)

Here $[x]$ denotes the floor function, providing the largest integer not greater than $x$. We can read the following relation between the parameters of the spinless and the singlet state from the expression shown in (3.14)

$$\tilde{r} = \frac{r - 1}{M} + 1 \quad \iff \quad r = M(\tilde{r} - 1) + 1.$$  

(4.7)

Thus each component is written as

$$\left[ \frac{r}{M} p \right] = \left[ \frac{r}{M} (p - 1) \right] = \begin{cases} \tilde{r} - 1 & \text{for } p \not\equiv 0 \pmod{M} \\ \tilde{r} & \text{for } p \equiv 0 \pmod{M} \end{cases}.$$

(4.9)

In the occupation number basis this configuration is represented as

$$|X0^{\tilde{r}-1}X0^{\tilde{r}-1}\cdots\rangle \quad \text{with} \quad X = k^{(0)} 0^{\tilde{r}-2} k^{(1)} 0^{\tilde{r}-2} \cdots k^{(M-1)}.$$

(4.10)

Superscripts stand for $\lambda_i$ modulo $M$, and correspond to internal degrees of freedom of the SU($M$)-singlet state. This is just the root configuration for the spin-singlet FQH state [14, 15]. Note that the state $u = 0$ is equivalent to $u = M$.

The SU($M$)-singlet $(k, \tilde{r})$-state is obtained from the spin Laplace-Beltrami operator [15]

$$\mathcal{H}_{\text{slLB}} = \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \left( z_i \frac{\partial}{\partial z_i} \right) \left( z_j \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} \right) - \frac{1}{\alpha} \sum_{i\neq j} (1 - K_{ij}) \frac{z_i z_j}{(z_i - z_j)^2},$$

(4.11)

where $K_{ij}$ stands for the exchange operator. In this case the condition (4.3) is modified as

$$\lambda_i - \lambda_{i+k} \geq r - 1.$$  

(4.12)

If $\lambda_i - \lambda_{i+k} = r - 1$, the spin part satisfies

$$\sigma_i > \sigma_{i+k}.$$

(4.13)

Note that the definition we use in this paper is slightly different from the notation used in [15]. Thus we can see (4.10) is just the root configuration for the SU($M$)-singlet $(k, \tilde{r})$-state.
Conversely, starting with an \( M \)-component state labeled by \((\tilde{l}, \sigma)\), where the latter stands for the internal degree with \(0 \leq \sigma \leq M - 1\), the corresponding spinless state can be recovered as

\[
\lambda_i = M\tilde{\lambda}_i + \sigma_i.
\] (4.14)

Essentially this relation has been already shown in [19, 30, 23, 24], and corresponds to the method discussed in section 3. This translation procedure is already explicitly shown in [19] for the positive parameter case \( \alpha > 0 \).

4.2 Squeezing rule

We study the squeezing rule for the spin-singlet state in terms of the spinless basis. From the root configuration we squeeze \( a \)-th and \( b \)-th components in \( i \)-th and \( j \)-th blocks of \( X \) given in (4.10), and exchange their states, as shown in Fig. 2. First we consider the case \( i < j \). This operation is represented in terms of \((\tilde{l}, \sigma)\) as

\[
((i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1), a) \rightarrow ((i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1) + 1, b),
\]

\[
((j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1), b) \rightarrow ((j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1) - 1, a).
\] (4.15)

The corresponding operation in terms of the spinless basis \( \lambda \) is just given by squeezing \( M - a + b \) boxes,

\[
M[(i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1)] + a \rightarrow M[(i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1) + 1] + b,
\]

\[
M[(j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1)] + b \rightarrow M[(j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1) - 1] + a.
\] (4.16)

We can see \( M - a + b > 0 \), since they satisfy \(0 \leq a, b \leq M - 1\).

Next let us study the case \( i = j \). This operation is given by

\[
((i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1), a) \rightarrow ((i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1), b),
\]

\[
((j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1), b) \rightarrow ((j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1), a),
\] (4.17)

and the corresponding one yields

\[
M[(i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1)] + a \rightarrow M[(i - 1)(M(\tilde{r} - 1) + 1) + a(\tilde{r} - 1)] + b,
\]

\[
M[(j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1)] + b \rightarrow M[(j - 1)(M(\tilde{r} - 1) + 1) + b(\tilde{r} - 1)] + a.
\] (4.18)

In this case the number of squeezed boxes is \( b - a \). Therefore we can perform this operation only when \( b > a \). Fig. 2 shows the squeezing rule for the SU(\( M \)) states and the corresponding spinless description.
Figure 2: The squeezing rule for the singlet admissible states in terms of (a) the original spinful configuration and (b) the spinless description. The parameters are related as $r = M(\tilde{r} - 1) + 1$.

As an example we now consider the $(k, \tilde{r}, M) = (1, 2, 2)$ state, which is well investigated in [15]. The corresponding spinless state yields the $(k, r, M) = (1, 3, 1)$ state. The root configuration for $N = 4$ is given by

$$|\uparrow, \downarrow, 0, \uparrow, \downarrow\rangle.$$ (4.19)

This is also represented by the following partitions,

$$(\tilde{\lambda}, \sigma) = (4^{(\uparrow)}, 3^{(\downarrow)}, 1^{(\uparrow)}, 0^{(\downarrow)}) \iff \lambda = (9, 6, 3, 0),$$ (4.20)

where we assign $\uparrow \equiv 1$, $\downarrow \equiv 0$ (mod 2). The descendants of the root configuration (4.19) are given by

$$|\uparrow, \downarrow, 0, \uparrow, \downarrow\rangle = (4^{(\uparrow)}, 3^{(\downarrow)}, 1^{(\uparrow)}, 0^{(\downarrow)}) \iff (9, 6, 2, 1)$$

$$|\downarrow, \uparrow, 0, \uparrow, \downarrow\rangle = (4^{(\downarrow)}, 3^{(\uparrow)}, 1^{(\uparrow)}, 0^{(\downarrow)}) \iff (8, 7, 3, 0)$$

$$|\uparrow, \downarrow, 0, \downarrow, \uparrow\rangle = (4^{(\uparrow)}, 3^{(\downarrow)}, 1^{(\downarrow)}, 0^{(\uparrow)}) \iff (8, 7, 2, 1)$$

$$|\uparrow, \uparrow, 0, \downarrow, \downarrow\rangle = (4^{(\uparrow)}, 3^{(\downarrow)}, 1^{(\downarrow)}, 0^{(\uparrow)}) \iff (8, 6, 3, 1)$$

(4.21)

The last one is equivalent to $|\downarrow, \downarrow, 0, \uparrow, \uparrow\rangle$ due to the spin-singlet condition, which is also represented as

$$|\downarrow, \downarrow, 0, \uparrow, \uparrow\rangle = (4^{(\downarrow)}, 3^{(\uparrow)}, 1^{(\uparrow)}, 0^{(\downarrow)}) \iff (9, 7, 2, 0).$$ (4.22)

Note that $(9, 7, 2, 0)$ cannot be obtained from the root configuration $(9, 6, 3, 0)$ by squeezing it. Therefore, to be consistent with the symmetry, it turns out to be the zero-weight state, as discussed in [13, 15].

5 Relation to conformal field theory

One of the most important properties of the FQH state is its exotic statistics. Such a crucial property can be well described in terms of the two dimensional CFT: the non-Abelian
statistics is interpreted as a consequence of the fusion rule of the corresponding CFT. In this paper we have discussed a novel description of the spin-singlet state by utilizing the $q$-deformed state. Thus, in this section, we comment on CFT relevant to the spin-singlet FQH state with emphasis on its relation to the $q$-deformed CFT.

The typical underlying CFT for the FQH state is the $\mathbb{Z}_k$-parafermion model. It describes the $k$-th Read-Rezayi state $[29]$, which includes the Moore-Read state $[28]$ at $k = 2$. The $\mathbb{Z}_k$-parafermion model can be obtained from the coset construction,

$$\frac{\text{SU}(2)_k}{\text{U}(1)}.$$  \hspace{1cm} (5.1)

Furthermore, it has been shown the generic $(k, r)$-admissible state is coming from the extended chiral algebra, $\mathcal{W}A_{k-1}(k+1, k+r)$ $[5]$ $[6]$ $[7]$. A CFT model for the FQH state has been also investigated for the spin-singlet states. The SU($M$)-singlet state corresponds to the generalized parafermion model,

$$\frac{\text{SU}(M+1)_k}{\text{U}(1)^M}.$$  \hspace{1cm} (5.2)

This is a natural extension of the $k$-th Read-Rezayi state $[10]$ $[11]$.

The interesting property of the parafermion models is the level-rank duality. For example, for the $k$-th Read-Rezayi state, we have

$$\frac{\text{SU}(2)_k}{\text{U}(1)} = \frac{\text{SU}(k)_1 \times \text{SU}(k)_1}{\text{SU}(k)_2}.$$  \hspace{1cm} (5.3)

This means the $\mathbb{Z}_k$-parafermion is also realized as the lowest one of the $\mathcal{W}$-minimal series,

$$\frac{\text{SU}(k)_l \times \text{SU}(k)_1}{\text{SU}(k)_{l+1}},$$  \hspace{1cm} (5.4)

which gives rise to the central charge,

$$c = k - 1 - k(k^2 - 1)\frac{(p - q)^2}{pq}.$$  \hspace{1cm} (5.5)

Here we identify $k+l = p/(q-p)$, $k+l+1 = q/(q-p)$. This duality is also applied to the spin-singlet theory,

$$\frac{\text{SU}(M+1)_k}{\text{U}(1)^M} = \frac{(\text{SU}(k)_1)^{M+1}}{\text{SU}(k)_{M+1}}.$$  \hspace{1cm} (5.6)

Recently the corresponding CFT series, which reproduces the model in the right hand side of (5.6) at the lowest level, has been proposed $[15]$.

$$\frac{\text{SU}(k)_l \times (\text{SU}(k)_1)^M}{\text{SU}(k)^{M+l}}.$$  \hspace{1cm} (5.7)

Substituting $k+l = Mp/(q-p)$, $k+l+M = Mq/(q-p)$, the central charge is given by

$$c = M(k - 1) - \frac{k(k^2 - 1)}{M} \frac{(p - q)^2}{pq}.$$  \hspace{1cm} (5.8)

\footnote{See a review article, e.g. $[31]$.}
This novel CFT series is not well investigated yet, but let us comment on its connection to the four dimensional gauge theory. Recent progress on the supersymmetric gauge theory reveals the remarkable relation between the four dimensional gauge theory and the two dimensional CFT. The instanton partition function of the gauge theory is directly interpreted as the conformal block of the two dimensional CFT. The standard SU(\(N\)) Yang-Mills theory corresponds to the \(A_{N-1}\) Toda CFT. Its central charge is given by

\[ c = N - 1 + N(N^2 - 1)Q^2, \]

(5.9)

where \(Q\) is related to the regularization parameter of the four dimensional theory. This CFT is essentially the same as the model shown in (5.4): they are equivalent under the identification \(Q^2 = -(p - q)^2/(pq)\).

A similar connection is also suggested for the generalized model (5.7). In this case the gauge theory on the type \(A_{M-1}\) ALE space, which is given by resolving the singularity of the orbifold \(\mathbb{C}^2/\mathbb{Z}_M\), gives rise to the corresponding CFT. Its central charge is given by

\[ c = M(N - 1) + \frac{N(N^2 - 1)}{M}Q^2. \]

(5.10)

When we consider a gauge group \(G\), the central charge is written in a generic form,

\[ c = Mr_G + \frac{d_G h_G}{M}Q^2, \]

(5.11)

where \(r_G\), \(d_G\) and \(h_G\) are rank, dimension and the dual Coxeter number of the group \(G\), respectively. This generalized CFT would be realized as the following coset model,

\[ G_l \times (G_1)^M \mathcal{G}_{M+l}. \]

(5.12)

Let us then comment on the \(q\)-deformed CFT. It has been shown that, by taking the limit, \(t = q^r\) and then \(q \to 1\), the central charge of the corresponding CFT is given by \([25, 26]\)

\[ c = 1 - \frac{6(1 - r)^2}{r}. \]

(5.13)

When we start with the \(q\)-deformed \(W\) CFT, it becomes

\[ c = 1 - N(N^2 - 1)\frac{(1 - r)^2}{r}. \]

(5.14)

Indeed they are equivalent to the SU(\(N\)) minimal models via \(r = p/q\). Thus, when we apply the root of unity limit, \(q \to \omega_M q\), \(t \to \omega_M q^r\) and \(q \to 1\), it is natural to obtain the CFT, which describes the SU(\(M\))-singlet FQH state as a result of the Yangian Gelfand-Zetlin basis. Its central charge is expected to be given by

\[ c = M(N - 1) - \frac{N(N^2 - 1) (1 - r)^2}{M}r. \]

(5.15)

This model corresponds to the standard Macdonald polynomial, which is associated with the type \(A\) root system. Thus we now expect that the central charge of the model, which is coming from the \(q\)-deformed theory related to other root systems, e.g. the \(BC\) type theory \([35, 36]\), can be written in a form similar to (5.11).

\(^3\)An attempt to connect the gauge theory partition function with the FQH state is found in \([33]\).
6 Conclusion

We have investigated the SU($M$)-singlet FQH states with the spinless basis. We have shown the raising and lowering operators for SU($M$) states with emphasis on its similarity to the standard SU(2) spinful states. The SU($M$)-singlet condition can be written in a quite similar form to the SU(2)-singlet condition for the Halperin state. We have obtained the SU($M$)-singlet Halperin state from the corresponding Laughlin state by considering the $q$-deformation and its root of unity limit. This is just the prescription to implement the Yangian Gelfand-Zetlin basis in terms of a certain spinless state, which is proposed in [19]. As well known, the FQH trial wavefunction is regarded as the correlation function of the primary fields in the corresponding CFT. Thus, to discuss such a correlation function, we have studied the $q$-boson fields and its OPE in the root of unity limit. We have also shown the relation between the SU($M$)-singlet and the corresponding spinless states for much generalized FQH states, which obey the $(k, r)$-admissible condition. We have discussed them in terms of the occupation number representation of the FQH states, and thus the squeezing rule is naturally assigned for them. We then have commented on the underlying CFT for the FQH states discussed in this paper. There would be the interesting structure in the CFT for the multi-component states, and the root of unity limit of the $q$-deformed CFT.

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