Symmetry breaking results for problems with exponential growth in the unit disk

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Abstract. We investigate some asymptotic properties of extrema $u_\alpha$ to the two-dimensional variational problem

$$\sup_{u \in H_0^1(B) \atop \|u\| = 1} \int_B \left( e^{\gamma u^2} - 1 \right) |x|^\alpha \, dx$$

as $\alpha \to +\infty$. Here $B$ is the unit disk of $\mathbb{R}^2$ and $0 < \gamma \leq 4 \pi$ is a given parameter. We prove that in a certain range of $\gamma$’s, the maximizers are not radial for $\alpha$ large.

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1. Introduction

Let $0 < \gamma \leq 4 \pi$ be a given number. We consider the maximization problem

$$S(\alpha, \gamma) = \sup_{u \in H_0^1(B) \atop \|u\| = 1} \int_B \left( e^{\gamma u^2} - 1 \right) |x|^\alpha \, dx,$$

where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$, $H_0^1(B)$ is the usual Sobolev space endowed with the Dirichlet norm $\|u\| = \left( \int_B |\nabla u|^2 \, dx \right)^{1/2}$, and $\alpha > 0$. It is readily seen that any maximizer of (1) must satisfy (weakly) the elliptic differential equation

$$-\Delta u = \lambda |x|^\alpha u e^{\gamma u^2}$$

with a Lagrange multiplier $\lambda$ given by

$$\lambda = \frac{1}{\int_B u^2 e^{\gamma u^2} |x|^\alpha \, dx}$$

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Standard regularity theory shows that any weak solution of (2) is classical. Moreover, if \( u \) is a maximizer of problem (1), then so is \(|u|\). Hence we can work with nonnegative functions. We will use freely these facts.

Our problem can be seen as a natural two–dimensional extension of the Hénon–type problem

\[
\sup_{u \in H^1_0(B)} \int_B |u|^p |x| \, dx \tag{4}
\]

in \( \mathbb{R}^n \) with \( n \geq 3 \) and \( 1 < p < 2^* \). Indeed, by the Trudinger–Moser inequality (see [10,12,16]),

\[
\sup_{u \in H^1_0(B)} \int_B \left( e^{\gamma u^2} - 1 \right) \, dx \begin{cases} < \infty & \text{if } \gamma \leq 4\pi \\ = +\infty & \text{otherwise}, \end{cases} \tag{5}
\]

the growth \( \exp(4\pi |\cdot|^2) \) in 2D corresponds (with relevant differences, though) to the critical growth \( |\cdot|^{2^*} \) in dimension \( n \geq 3 \). This can be made precise by introducing the class of Orlicz spaces, but we shall not go into the details.

We refer the interested reader to [1,4].

Recently, Smets et al. ([14]) studied the symmetry of minimizers to the problem

\[
S_{\alpha,p} = \inf_{u \in H^1_0(B) \setminus \{0\}} \frac{\int_B |\nabla u|^2 \, dx}{(\int_B |x|^\alpha |u|^p \, dx)^{2/p}}, \tag{6}
\]

namely problem (4), with

\[
2 < p < +\infty \quad \text{in dimension 2} \\
2 < p < 2^* \quad \text{in higher dimension.}
\]

Since the quotient in (6) is invariant under rotations, it is natural to set up the same minimization problem in the space of radial functions \( H^1_0,\text{rad}(B) \):

\[
S_{\alpha,p}^{\text{rad}} = \inf_{u \in H^1_0,\text{rad}(B) \setminus \{0\}} \frac{\int_B |\nabla u|^2 \, dx}{(\int_B |x|^\alpha |u|^p \, dx)^{2/p}}. \tag{7}
\]

The set \( B \) being bounded, both problems (6) and (7) are compact, and are thus solved by functions \( u_\alpha \) and \( v_\alpha \). A very interesting symmetry–breaking result contained in [14] is the following.

**Theorem 1 (Smets, Su, Willem).** Assume the dimension of the space is greater than or equal to 2. For every \( p \in (2, 2^*) \) (\( p > 2 \) in 2D), there exists \( \alpha^* > 0 \) such that no minimizer of (6) is radial provided that \( \alpha > \alpha^* \). In particular,

\[
S_{\alpha,p} < S_{\alpha,p}^{\text{rad}} \quad \text{for all } \alpha \text{ sufficiently large.}
\]
This result has generated a line of research on Hénon–type equations. For example, it shows in particular that for a certain set of parameter values, the (Hénon) equation associated to (6) admits the coexistence of radial and nonradial positive solutions. Since a radial solution always exists, by a result of Ni, \[11\], for \( \alpha < 2^* + \frac{2\alpha}{n-2} \), a similar phenomenon can be expected also for critical and supercritical growths. Results in this direction have been obtained in \[13\] and \[2\]. See also \[3\] and \[15\] for asymptotic analysis of ground states and other symmetry results.

The symmetry breaking problem for exponential nonlinearities in the unit disk, on the contrary, seems to have been much less studied. Very recently, Calanchi and Terraneo (see \[4\]) proved some results about the existence of non–radial maximizers for the variational problem

\[
T_{\alpha,p,\gamma} = \sup_{\|u\| \leq 1} \int_{B} |x|^\alpha \left( e^{p|u|^\gamma} - 1 - p|u|^\gamma \right) dx
\]

where \( \alpha > 0, \ p > 0 \) and \( 1 < \gamma \leq 2 \) when \( \alpha \to +\infty \). We observe that the functional to be maximized in \( T_{\alpha,p,\gamma} \) contains an extra term with respect to our \( S(\alpha,\gamma) \).

In this paper we present some results about symmetry of solutions to (1), though we are not able to cover the whole range \( (0, 4\pi) \) of the parameter \( \gamma \). The main difficulty is that, unlike (6), our problem (1) is not homogeneous with respect to \( u \). As a consequence, we cannot replace (1) with a more familiar “Rayleigh” quotient.

We consider problem (1) and its radial companion

\[
S^{rad}(\alpha,\gamma) = \sup_{u \in H^1_{0,\text{rad}}(B), \|u\| = 1} \int_{B} \left( e^{\gamma u^2} - 1 \right) |x|^\alpha dx;
\]

since \( H^1_{0,\text{rad}}(B) \subset H^1_0(B) \), it is clear that

\[
S(\alpha,\gamma) \geq S^{rad}(\alpha,\gamma).
\]

Our main concern is to investigate if and when the strict inequality takes place. By standard arguments (see Section 2), both \( S(\alpha,\gamma) \) and \( S^{rad}(\alpha,\gamma) \) are attained for \( \gamma \in (0, 4\pi) \) (this interval is considerably larger for the radial case, see \[4\]).

We first obtain an asymptotic profile type result for the maximizers of (8), as \( \alpha \to \infty \). This result is essential in order to carry out the proof of the main symmetry breaking theorem, and we believe that it is interesting in its own.

In the statements that follow we denote by \( \lambda_1 \) the first eigenvalue of \( -\Delta \) on \( H^1_0(B) \), and by \( \varphi_1 \) the corresponding (positive) eigenfunction, normalized by \( \|\varphi_1\| = 1 \).
**Theorem 2.** Let $\gamma \in (0, 4\pi]$. For every $\alpha > 0$, let $u_\alpha = u_\alpha(|x|)$ be a maximizer for $S^{\text{rad}}(\alpha, \gamma)$. Then
\[
\lim_{\alpha \to +\infty} \sqrt{\frac{2}{\alpha + 2}} u_\alpha(|x|^{\frac{2}{\alpha + 2}}) = \varphi_1 \quad \text{strongly in } H^1_0(B).
\]

We now state the main result of the paper.

**Theorem 3.** There exists $\gamma^* \in [0, 4\pi)$ such that for every $\gamma \in (\gamma^*, 4\pi)$ no maximizer for $S(\alpha, \gamma)$ is radial provided $\alpha$ is large enough. Moreover
\[
\gamma^* \leq \frac{\pi \varphi_1(0)^2}{\lambda_1 \int_B \varphi_1^2 \, dx}.
\] (10)

Of course the upper bound for $\gamma^*$ appearing in the right–hand–side of (10) is strictly smaller than $4\pi$. We do not know if $\gamma^* = 0$; this is one of the interesting open problems connected to $S(\alpha, \gamma)$ and should be the object of further research.

The paper is structured as follows. In Section 2 we obtain the asymptotic description of radial maximizers, while a similar result, perturbative in nature, is given in Section 3 for non radial maximizers. Section 4 is devoted to the proof of the main result, Theorem 3.

## 2. Asymptotic behavior of radial maximizers

In this section we give a precise description of the asymptotic behavior of maximizers of problem (8) as $\alpha \to +\infty$.

To begin with, we fix some notation that we will use throughout the paper. We introduce the variational functional
\[
I(u) = \int_B \left( e^{\gamma u^2} - 1 \right) |x|^\alpha \, dx
\] (11)

which acts formally in the same way both on $H^1_0(B)$ and on $H^1_{0, \text{rad}}(B)$. For the sake of simplicity we suppress the dependence of $I$ on $\alpha$ and $\gamma$. The first (positive) eigenfunction of the Laplace operator $-\Delta$ on $H^1_0(B)$ will be denoted by $\varphi_1$, normalized by $||\varphi_1|| = 1$, and the corresponding eigenvalue by $\lambda_1$.

Throughout the paper, we will make use of polar coordinates in $\mathbb{R}^2$, namely $x = (\rho \cos \theta, \rho \sin \theta)$ with $\rho \geq 0$ and $\theta \in [0, 2\pi)$. With a slight abuse of notation, we will write $u(\rho, \theta) = u(x) = u(\rho \cos \theta, \rho \sin \theta)$ for a given function $u$ on $\mathbb{R}^2$, and, likewise, $u(x) = u(|x|) = u(\rho)$ for a radial function. For further use, we state the variational problem (11) in polar coordinates. Set
\[
\varepsilon = \frac{2}{\alpha + 2}.
\] (12)
For any smooth (or $H^1_0$) function $u$ on $B$, define the new function
\[ v(\rho, \theta) := \frac{1}{\sqrt{\varepsilon}} u(\rho^\varepsilon, \theta) \] (13)
expressed in polar coordinates. Observe that
\[ v_\rho = \sqrt{\varepsilon} \rho^{-1} u_\rho (\rho^\varepsilon, \theta) \]
\[ v_\theta = \frac{1}{\sqrt{\varepsilon}} u_\theta (\rho^\varepsilon, \theta), \]
so that, if $t = \rho^\varepsilon$,
\[ \int_0^1 \int_0^{2\pi} \left( v_t^2 + \frac{\varepsilon^2}{t^2} v_\theta^2 \right) t \, dt \, d\theta = 1 \] (14)
whenever $\int_B |\nabla u|^2 \, dx = 1$. In the variables $(t, \theta)$, the variational functional (11) reads
\[ I(v) = \varepsilon \int_0^1 \int_0^{2\pi} \left( e^{\varepsilon \gamma v^2} - 1 \right) t \, dt \, d\theta. \]
Finally, the original problem (1) can be written by means of (12), (13) as
\[ S(\alpha, \gamma) = \sup \left\{ \varepsilon \int_0^1 \int_0^{2\pi} \left( e^{\varepsilon \gamma v^2} - 1 \right) t \, dt \, d\theta : \right. \]
\[ \left. \int_0^1 \int_0^{2\pi} \left( v_t^2 + \frac{\varepsilon^2}{t^2} v_\theta^2 \right) t \, dt \, d\theta = 1 \right\}. \] (15)

**Remark 1.** We stress that in the new variables the weight $|x|\alpha$ disappears from the functional and the parameter $\varepsilon = \frac{2}{\alpha + 2}$ appears both in the exponent and in front of $|\partial v/\partial \theta|^2$. Notice that if $u$ is radial, then
\[ \int_B |\nabla u|^2 \, dx = 2\pi \int_0^1 v_t^2 \, dt = \int_B |\nabla v|^2 \, dx \] (16)
and
\[ \int_B \left( e^{\gamma u^2} - 1 \right) |x|^\alpha \, dx = 2\pi \varepsilon \int_0^1 \left( e^{\varepsilon \gamma v^2} - 1 \right) t \, dt = \varepsilon \int_B \left( e^{\varepsilon \gamma v^2} - 1 \right) \, dx, \] (17)
so that
\[ S^{\text{rad}}(\alpha, \gamma) = \sup \left\{ 2\pi \varepsilon \int_0^1 \left( e^{\varepsilon \gamma v^2} - 1 \right) t \, dt : 2\pi \int_0^1 v_t^2 \, dt = 1 \right\}. \] (18)

First of all, we deal with the existence of maximizers to (11) and (16).
Proposition 1. There exist a solution $u_\alpha \in H^1_0(B)$ and a solution $u^*_\alpha \in H^1_0,B^0(B)$ (also called $u_\varepsilon$ and $u^*_\varepsilon$ via the change of parameter \(12\)) to problems (1) and (8) respectively, provided $0 < \gamma < 4\pi$.

When $\gamma = 4\pi$, problem (8) has a solution in $H^1_0,B^0(B)$.

Proof. We only give some details, since the argument can be recovered from the existing literature.

First of all we notice that $S(\alpha, \gamma) = \sup_{u \in H^1_0(B)} \int_B (e^{\gamma u^2} - 1) |x|^{\alpha} dx$, and the same for $S^\text{rad}(\alpha, \gamma)$. In the subcritical case $\gamma < 4\pi$, the proof is almost trivial. Indeed, let $\{u_n\}$ be a maximizing sequence for $S(\alpha, \gamma)$ (or for $S^\text{rad}(\alpha, \gamma)$), with $\|u_n\| \leq 1$. By the Sobolev embedding theorem, we can assume without loss of generality that (up to a subsequence) $u_n \rightharpoonup u$, weakly in $H^1_0$ and $u_n \rightarrow u$ a.e. and strongly in $L^q(B)$ for any finite $q \geq 1$.

In particular, $\|u\| \leq 1$. Then, thanks to Lemma 2.1 of \[7\], we have

$$S(\alpha, \gamma) = \lim_{n \to \infty} \int_B (e^{\gamma u^2} - 1) |x|^{\alpha} dx = \int_B (e^{\gamma u^2} - 1) |x|^{\alpha} dx.$$  

This shows that $u \neq 0$, and that $u$ is a maximizer of (1).

The critical case $\gamma = 4\pi$ for $S^\text{rad}(\alpha, \gamma)$ is slightly different. Indeed, equation (17) shows that problem (8) is still “subcritical”, provided that $\varepsilon \gamma < 4\pi$, i.e. $\gamma < 4\pi + 2\pi \alpha$. Therefore, standard arguments prove that $S^\text{rad}(\alpha, 4\pi)$ is actually attained by a radial function. See also the remark at the end of section 3 in \[4\].

Remark 2. It does not seem to be known whether $S(\alpha, 4\pi)$ is attained. For the “unweighted case” $\alpha = 0$ this is a celebrated result due to Carleson and Chang \[5\]. Unfortunately, it does not seem possible to modify their proof so as to take into account the weight $|x|^{\alpha}$. This is an interesting open problem.

We now begin the study of the asymptotic behavior of the radial maximizers.

Take a radial function $v$, compactly supported in $B$, with $\|v\| = 1$. Formally,

$$e^{\varepsilon \gamma v^2} - 1 = \varepsilon \gamma v^2 + \frac{1}{2!} \varepsilon^2 \gamma^2 v^4 + \frac{1}{3!} \varepsilon^3 \gamma^3 v^6 + \ldots$$

$$=: \varepsilon \gamma v^2 + R_\varepsilon(v) \quad \text{(19)}$$

Lemma 1. As $\varepsilon \to 0$,

$$\varepsilon \int_0^1 \left( e^{\varepsilon \gamma v^2} - 1 \right) t \, dt \, d\theta = \varepsilon^2 \gamma \int_0^1 v^2 t \, dt + O(\varepsilon^3), \quad \text{(20)}$$

uniformly for $\|v\| = 1$. 

Proof. Equation (20) is equivalent, via (19), to
\[ \int_0^1 R_\varepsilon(v) t \, dt = O(\varepsilon^2). \] (21)
Now, \( R_\varepsilon(v) = \sum_{k=2}^{\infty} \frac{1}{k!} (\varepsilon \gamma)^k v^{2k} \). Fix any index \( k \geq 2 \); for every \( t \in [0,1] \), we have
\[ v(t) = v(t) - v(1) = \int_1^t v'(s) \, ds \leq \left( \int_1^t |v'(s)|^2 \, ds \right)^{1/2} \left( \int_1^t \frac{1}{s} \, ds \right)^{1/2} \]
\[ \leq \frac{1}{\sqrt{2\pi}} \left( \int_B |\nabla v|^2 \, dx \right)^{1/2} \left( \int_1^t \frac{1}{s} \, ds \right)^{1/2} = \frac{1}{\sqrt{2\pi}} (- \log t)^{1/2}. \]
As a consequence, \( |v(t)|^{2k} \leq \frac{1}{(2\pi)^k} (- \log t)^k \). We multiply by \( t \), integrate this inequality over \([0,1]\) and find
\[ \int_0^1 |v(t)|^{2k} t \, dt \leq \frac{1}{(2\pi)^k} \int_0^1 (- \log t)^k t \, dt. \]
The change of variable \( t = \exp(-x/2) \) yields immediately
\[ \int_0^1 |v(t)|^{2k} t \, dt \leq \frac{1}{(2\pi)^k} \int_0^\infty \frac{1}{2} e^{-x/2} e^{-x/2} \left( \frac{x}{2} \right)^k \, dx \]
\[ = \frac{1}{2^{k+1}(2\pi)^k} \int_0^\infty x^k e^{-x} \, dx \]
\[ = \frac{1}{2^{k+1}(2\pi)^k} \Gamma(k+1) = \frac{1}{2^k(2\pi)^k} \Gamma(k). \] (22)
The Monotone Convergence Theorem implies that we can switch the summation over \( k \) with the integration over \([0,1]\), so that
\[ \int_0^1 R_\varepsilon(v) t \, dt = \sum_{k=2}^{\infty} \int_0^1 \frac{1}{k!} (\varepsilon \gamma)^k |v(t)|^{2k} t \, dt \leq \sum_{k=2}^{\infty} \frac{(\varepsilon \gamma)^k}{k!} \frac{1}{(2\pi)^k} \frac{k!}{2^k(2\pi)^k} \]
\[ = \sum_{k=2}^{\infty} \frac{(\varepsilon \gamma)^k}{2(4\pi)^k} = \frac{1}{2} \sum_{k=2}^{\infty} \frac{(\varepsilon \gamma)^k}{4\pi} = \frac{\gamma \varepsilon^2}{8\pi(4\pi - \gamma \varepsilon)} = O(\varepsilon^2) \]
This completes the proof. ♣

We now establish the asymptotic behavior of a sequence \( u_\alpha \) of maximizers of \( S_{\text{rad}}(\alpha, \gamma) \) as \( \alpha \to +\infty \). For notational convenience, in the statement of the result we denote this sequence by \( u_\varepsilon \), keeping in mind that \( \alpha \) and \( \varepsilon \) are linked by (12).

**Theorem 4.** Let \( \gamma \in (0,4\pi] \) and let \( u_\varepsilon \in H^1_{0,\text{rad}}(B) \) be a maximizer of \( S_{\text{rad}}(\alpha, \gamma) \). Then
\[ \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} u_\varepsilon(|x|^\gamma) = \varphi_1 \quad \text{strongly in } H^1_0(B). \] (23)
Proof. We set \( v_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} u_\varepsilon(t\varepsilon) \). Clearly \( v_\varepsilon \) is a maximizer of problem (18). In particular, \( \|v_\varepsilon\| = 1 \) for all \( \varepsilon \), so that the set \( \{v_\varepsilon\}_\varepsilon \) is bounded in \( H^1_0(B) \); therefore some subsequence, which we still term \( \{v_\varepsilon\}_\varepsilon \), converges weakly to some \( v \) in \( H^1_0(B) \), and strongly in \( L^q(B) \) for all finite \( q \geq 1 \) as \( \varepsilon \to 0 \). Since \( v_\varepsilon \) is a maximizer, for every radial function \( \psi \in H^1_0(B) \) satisfying \( \|\psi\| = 1 \) we have

\[
2\pi \varepsilon \int_0^1 \left( e^{\varepsilon t v_\varepsilon^2} - 1 \right) t \, dt \geq 2\pi \varepsilon \int_0^1 \left( e^{\varepsilon t \psi^2} - 1 \right) t \, dt \geq 2\pi \gamma \varepsilon^2 \int_0^1 \psi^2 \, dt.
\]

Hence by Lemma 1

\[
2\pi \gamma \varepsilon^2 \int_0^1 \psi^2 \, dt \leq 2\pi \gamma \varepsilon^2 \int_0^1 v_\varepsilon^2 \, dt + O(\varepsilon^3).
\]

Dividing by \( \gamma \varepsilon^2 \) and letting \( \varepsilon \to 0 \) we obtain

\[
2\pi \int_0^1 \psi^2 \, dt \leq 2\pi \int_0^1 v^2 \, dt,
\]

namely

\[
\int_B \psi^2 \, dx \leq \int_B v^2 \, dx.
\]

If we now maximize over those \( \psi \in H^1_0(B) \) satisfying \( \|\psi\| = 1 \) we see that

\[
\frac{1}{\lambda_1} \leq \int_B v^2 \, dx.
\]

This shows that \( v \neq 0 \). By a standard semicontinuity argument, we have \( \|v\| \leq 1 \). Therefore

\[
\lambda_1 \leq \frac{\int_B |\nabla v|^2 \, dx}{\int_B v^2 \, dx} \leq \frac{1}{\int_B v^2 \, dx} \leq \lambda_1,
\]

which shows that

\[
\int_B |\nabla v|^2 \, dx = 1 \quad \text{and} \quad \int_B v^2 \, dx = \frac{1}{\lambda_1}.
\]

Since \( \lambda_1 \) is a simple eigenvalue, this means that \( v = \varphi_1 \) and

\[
\int_B v_\varepsilon^2 \, dx \to \int_B \varphi_1^2 \, dx, \quad \int_B |\nabla v_\varepsilon|^2 \, dx \to \int_B |\nabla \varphi_1|^2.
\]

This, together with the weak convergence \( v_\varepsilon \rightharpoonup v = \varphi_1 \), shows that \( v_\varepsilon \to \varphi_1 \) strongly in \( H^1_0(B) \).

\[\Box\]

We complete the description of the asymptotics of problem (8) with the behavior of the levels.
Proposition 2. Let $\gamma \in (0, 4\pi]$. As $\varepsilon \to 0$, i.e. as $\alpha \to +\infty$, we have
\[ S^{\text{rad}}(\alpha, \gamma) = \frac{\gamma}{\lambda_1} \varepsilon^2 + o(\varepsilon^2). \] (28)

Proof. Inserting $v = v_\varepsilon$, as defined in the proof of the previous theorem, in (20), we obtain
\[
2\pi \varepsilon \int_0^1 \left( e^{\gamma \varepsilon^2 v^2} - 1 \right) t \, dt = 2\pi \gamma \varepsilon^2 \int_0^1 v_\varepsilon^2 t \, dt + O(\varepsilon^3)
= \gamma \varepsilon^2 \left( \frac{1}{\lambda_1} + o(1) \right) + O(\varepsilon^3) = \frac{\gamma}{\lambda_1} \varepsilon^2 + o(\varepsilon^2).
\]
♣

3. Asymptotic estimates for non–radial maximizers

In the previous section we have proved that $S^{\text{rad}}(\alpha, \gamma) \approx \frac{\gamma}{\lambda_1} \varepsilon^2$ when $\varepsilon = 2/(\alpha + 2) \to 0$. We now provide a similar estimate for $S(\alpha, \gamma)$, and show that solutions to (1) are never radial, provided $\alpha$ is large and $\gamma \approx 4\pi$.

We begin with a lemma which estimates $S(\alpha, \gamma)$ in terms of the functional without weight.

Lemma 2. Let $\gamma \in (0, 4\pi]$ and $\alpha > 0$. Setting as usual $\varepsilon = 2/(\alpha + 2)$, we have
\[ S(\alpha, \gamma) > \frac{\varepsilon^2}{4} \sup_{||v||=1} \int_B \left( e^{\gamma v^2} - 1 \right) \, dx. \] (29)

Proof. We denote by $p$ be the point $(-\frac{1}{2}, 0) \in B$ and we take a function $\psi \in H^1_0(B_{1/2}(p))$. Notice that in polar coordinates, the function $\psi$ vanishes on $\partial([0, 1] \times [0, 2\pi])$. We extend then $\psi$ by zero outside $[0, 1] \times [0, 2\pi)$ and we still call $\psi$ this extension. Then it makes sense to define, for every $\varepsilon \in (0, 1)$, a function $u = u_\varepsilon : [0, 1] \times [0, 2\pi) \to \mathbb{R}$ by
\[ u(t, \phi) = \psi(t^{1/\varepsilon}, \phi/\varepsilon). \]

Obviously the function $u$ is not radial. Setting $\rho = t^{1/\varepsilon}$ and $\theta = \phi/\varepsilon$ we see that
\[
\int_B |\nabla u|^2 \, dx = \int_0^1 \int_0^{2\pi} \left( u_t^2 + \frac{1}{t^2} u_\theta^2 \right) t \, dt \, d\phi = \int_0^1 \int_0^{2\pi} \left( \frac{t^{2/\varepsilon - 2}}{\varepsilon^2} \psi^2(t^{1/\varepsilon}, \phi/\varepsilon) + \frac{1}{\varepsilon^2 t^2} \psi^2_\theta(t^{1/\varepsilon}, \phi/\varepsilon) \right) t \, dt \, d\phi
= \int_0^1 \int_0^{2\pi} \left( \psi_\rho^2 + \frac{1}{\rho^2} \psi_\theta^2 \right) \rho \, d\rho d\theta = \int_B |\nabla \psi|^2 \, dx = \int_{B_{1/2}(p)} |\nabla \psi|^2 \, dx
\]
and
Therefore we can say that

\[
S(\alpha, \gamma) > \sup \left\{ \varepsilon^2 \int_{B_{1/2}(p)}^{} \left( e^{\gamma \psi^2} - 1 \right) \, dx : \psi \in H^1_0(B_{1/2}(p)), \int_{B_{1/2}(p)} |\nabla \psi|^2 \, dx = 1 \right\}. \tag{30}
\]

Next we define \( v \in H^1_0(B) \) as \( v(x) = \psi(x/2 + p) \). Obviously

\[
\int_B |\nabla v|^2 \, dx = \int_{B_{1/2}(p)} |\nabla \psi|^2 \, dx,
\]

while

\[
\int_B^{} \left( e^{\gamma v^2} - 1 \right) \, dx = 4 \int_{B_{1/2}(p)}^{} \left( e^{\gamma \psi^2} - 1 \right) \, dx.
\]

This means, by (30), that

\[
S(\alpha, \gamma) > \frac{\varepsilon^2}{4} \sup_{||v||=1} \int_B^{} \left( e^{\gamma v^2} - 1 \right) \, dx, \tag{31}
\]

and the proof is complete.

We are now ready to state the main result of this section.

**Theorem 5.** There exists \( \gamma^* \in (0, 4\pi] \) such that, for all \( \gamma \in (\gamma^*, 4\pi) \),

\[
S(\alpha, \gamma) > S^{rad}(\alpha, \gamma), \tag{32}
\]

provided \( \alpha \) is large enough.

**Proof.** By the results of Section 2 we know that

\[
S^{rad}(\alpha, \gamma) = \frac{\gamma}{\lambda_1} \varepsilon^2 + o(\varepsilon^2)
\]
as \( \varepsilon \to 0 \). In view of (31) the proof is done if we show that

\[
\frac{1}{4} \sup_{||v||=1} \int_B^{} \left( e^{\gamma v^2} - 1 \right) \, dx > \frac{\gamma}{\lambda_1}. \tag{33}
\]
The value in the left-hand side of (33), which is attained by the results in [5], is unknown. We are going to estimate it using the same function that appears in [5].

From now on, we assume that \( v \) is a radial function. This is natural, since the supremum in (33) is attained by a radial function. If \( v \) is radial, it is convenient to introduce the function \( w : [0, +\infty) \rightarrow \mathbb{R} \) defined by

\[
w(t) := \sqrt{4\pi} v(e^{-t}).
\]

A straightforward computation shows that

\[
\int_B |\nabla v|^2 \, dx = \int_0^\infty |w'(t)|^2 \, dt
\]

and

\[
\int_B (e^{\gamma v^2} - 1) \, dx = \pi \int_0^\infty (e^{\pi w^2(t)} - 1) \, e^{-t} \, dt.
\]

Since the statement of the Theorem is perturbative in nature with respect to \( \gamma \), and everything depends continuously on \( \gamma \), to complete the proof we can assume \( \gamma = 4\pi \). Explicitly, we focus on the problem

\[
\max_{\int_0^\infty |w'|^2 \, dt = 1} \pi \int_0^\infty e^{w^2} - t \, dt - \pi
\]

Take then \( w: [0, +\infty) \rightarrow \mathbb{R} \) to be

\[
w(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \leq t \leq 2 \\ \sqrt{t - 1} & \text{if } 2 \leq t \leq 1 + e^2 \\ e & \text{if } t \geq 1 + e^2. \end{cases}
\]

This is the function that already appears in [5]. By direct inspection,

\[
\int_0^\infty |w'|^2 \, dt = 1 \quad \text{and} \quad \int_0^\infty e^{w^2} - t \, dt = \frac{2}{e} \int_0^1 e^t \, dt + e.
\]

Hence

\[
\int_0^\infty |w'|^2 \, dt = 1 \quad \text{and} \quad \int_0^\infty e^{w^2} - t \, dt = \frac{2}{e} \int_0^1 e^t \, dt + e.
\]

Hence

\[
\max_{\int_0^\infty |w'|^2 \, dt = 1} \pi \int_0^\infty e^{w^2} - t \, dt - \pi > \pi \left( \frac{2}{e} \int_0^1 e^t \, dt + e \right) - \pi. \tag{35}
\]

If we show that

\[
\frac{1}{4} \left( \frac{2\pi}{e} \int_0^1 e^t \, dt + e\pi - \pi \right) > \frac{4\pi}{\lambda_1}, \tag{36}
\]

then also (33) will be satisfied, by continuity, for \( \gamma \) close enough to \( 4\pi \), and the proof will be finished. We thus check that

\[
\frac{2}{e} \int_0^1 e^t \, dt + e - 1 > \frac{16}{\lambda_1}.
\]
From the characterization of $\lambda_1$ as a zero of the Bessel function $J_0$ (see [6, 17]), we have the approximated value $\lambda_1 \approx 5.783$. If we estimate $\int_0^1 e^{t^2} \, dt$ by expanding the integrand in power series, and taking into account only the first three terms, we get easily that $\int_0^1 e^{t^2} \, dt > 1.453$. Therefore

$$2 e \int_0^1 e^{t^2} \, dt + e - 1 > \frac{2.906}{e} + e - 1 \approx 2.787 > 2.767 \approx \frac{16}{\lambda_1}.$$ 

Although we do not know whether problem (11) admits a solution in the critical case $\gamma = 4\pi$, the previous proof gives the following a priori information.

**Corollary 1.** If $S(\alpha, 4\pi)$ is attained by some function $u$, then $u$ cannot be radial.

4. A nonperturbative estimate for symmetry–breaking

We have seen in Theorem 5 that solutions to (11) are non–radial whenever $\gamma$ is close to $4\pi$ and $\alpha$ is large (depending on $\gamma$). In this final section we present a similar result, whose nature is no longer perturbative with respect to $\gamma$. The technique of the proof is rather different, and resembles that of Theorem 2.1 in [14].

For clarity purposes, we introduce an auxiliary map $N$, defined by

$$N(u) = \frac{u^2}{||u||^2} = \frac{u^2}{\int_B |\nabla u|^2 \, dx} \text{ for all } u \in H^1_0(B) \setminus \{0\}. \quad (37)$$

and a measure $\mu_\alpha$ on Borel subsets $E$ of $\mathbb{R}^2$ by

$$\mu_\alpha(E) = \int_E |x|^\alpha \, dx.$$ 

It follows from straightforward arguments that problem (11) is equivalent to the maximization of the “free” functional

$$F(u) = \int_B \left( e^{\gamma N(u)} - 1 \right) \, d\mu_\alpha$$ 

on $H^1_0(B) \setminus \{0\}$. The use of this functional allows us to embed some homogeneity in the problem, which will be very useful in the computations below.

In the sequel, we denote by $DF(u)$ and $D^2F(u)$ the first and second Fréchet derivatives of $F$ at the point $u \in H^1_0(B)$.

---

1 A good approximation provided by Maple® for the integral is $\int_0^1 \exp(t^2) \, dt \approx 1.462651746$. 

Lemma 3. Assume $u$ is a nonzero critical point of $F$, normalized with $||u|| = 1$. Then, for all $v \in H^1_0(B)$,

$$D^2 F(u)[v, v] =$$

$$\gamma^2 \int_B e^{\gamma u^2} \left( 4u^2 v^2 + 4u^4 \left( \int_B \nabla u \cdot \nabla v \, dx \right)^2 - 8u^3 v \int_B \nabla u \cdot \nabla v \, dx \right) \, d\mu_\alpha + \gamma \int_B e^{\gamma u^2} \left( 2v^2 - 2u^2 \int_B |\nabla v|^2 \, dx \right) \, d\mu_\alpha. \quad (38)$$

Proof. Let $u$ be any nonzero critical point for $F$. Thus,

$$DF(u)[v] = 0 \quad \text{for all } v \in H^1_0(B), \quad (39)$$

where

$$DF(u)[v] = \gamma \int_B e^{\gamma N(u)} DN(u)[v] \, d\mu_\alpha$$

and

$$DN(u)[v] = \frac{2uv \int_B |\nabla u|^2 \, dx - 2u^2 \int_B \nabla u \cdot \nabla v \, dx}{(\int_B |\nabla u|^2 \, dx)^2}.$$ 

For every $v, w \in H^1_0(B)$, the second derivative of $F$ at $u$ is

$$D^2 F(u)[v, w] = \gamma \int_B e^{\gamma N(u)} DN(u)[v] DN(u)[w] \, d\mu_\alpha$$

$$+ \gamma \int_B e^{\gamma N(u)} D^2 N(u)[v, w] \, d\mu_\alpha. \quad (40)$$

We now compute the two integrals. We have

$$D^2 N(u)[v, w] = \left( \int_B |\nabla u|^2 \, dx \right)^{-2} \left( \int_B |\nabla u|^2 \, dx \right)^2 \left( 2vw \int_B |\nabla u|^2 \, dx \right)$$

$$+ 4uv \int_B \nabla u \cdot \nabla w \, dx - 4uv \int_B \nabla u \cdot \nabla v \, dx - 2u^2 \int_B \nabla v \cdot \nabla w \, dx$$

$$- 8 \left( uv \int_B |\nabla u|^2 \, dx - u^2 \int_B \nabla u \cdot \nabla v \, dx \right) \cdot \int_B |\nabla u|^2 \, dx \cdot \int_B \nabla u \cdot \nabla v \, dx$$

$$= \left( \int_B |\nabla u|^2 \, dx \right)^{-2} \left( 2vw \int_B |\nabla u|^2 \, dx + 4uv \int_B \nabla u \cdot \nabla v \, dx 

- 4uv \int_B \nabla u \cdot \nabla v \, dx - 2u^2 \int_B \nabla v \cdot \nabla w \, dx \right)$$

$$- \frac{4 \int_B \nabla u \cdot \nabla w \, dx}{(\int_B |\nabla u|^2 \, dx)^2} \cdot \frac{2uv \int_B |\nabla u|^2 \, dx - 2u^2 \int_B \nabla u \cdot \nabla v \, dx}{(\int_B |\nabla u|^2 \, dx)^2}.$$
If we recall (39), we conclude that
\[
\gamma \int_B e^{\gamma N(u)} D^2 N(u)[v, w] d\mu_\alpha = \frac{\gamma}{(\int_B |\nabla u|^2 dx)^2} \int_B e^{\gamma N(u)} \left( 2vw \int_B |\nabla u|^2 dx + 4uv \int_B \nabla u \cdot \nabla w dx \\
- 4uw \int_B \nabla u \cdot \nabla v dx - 2u^2 \int_B \nabla v \cdot \nabla w dx \right) d\mu_\alpha.
\]
Therefore, choosing \( w = v \), we immediately see that
\[
\gamma \int_B e^{\gamma N(u)} D^2 N(u)[v, v] d\mu_\alpha = \frac{\gamma}{(\int_B |\nabla u|^2 dx)^2} \int_B e^{\gamma N(u)} \left( 2v^2 \int_B |\nabla u|^2 dx - 2u^2 \int_B |\nabla v|^2 dx \right) d\mu_\alpha.
\]
If in addition \( u \) is normalized by \( \int_B |\nabla u|^2 dx = 1 \), then
\[
\gamma \int_B e^{\gamma N(u)} D^2 N(u)[v, v] d\mu_\alpha = \gamma \int_B e^{\gamma u^2} \left( 2v^2 - 2u^2 \int_B |\nabla v|^2 dx \right) d\mu_\alpha. \tag{41}
\]
As far as the first integral in (40) is concerned, by similar but simpler arguments, we obtain, for a normalized critical point,
\[
(DN(u)[v])^2 = 4u^2v^2 + 4u^4 \left( \int_B \nabla u \cdot \nabla v dx \right)^2 - 8u^3v \int_B \nabla u \cdot \nabla v dx. \tag{42}
\]
Finally, equation (38) is an immediate consequence of (41) and (42).

We can now prove the main result of this paper.

Proof (of Theorem 3). Let \( u = u_\epsilon \) be any solution to problem (1), and assume that it is a radial function. Any \( v \in H^1_0(B) \) can be decomposed as \( v = au + w \) with \( a \in \mathbb{R} \) and \( \int_B \nabla u \cdot \nabla w dx = 0 \). It follows from (38) that
\[
D^2 F(u)[au + w, au + w] = 4\gamma^2 \int_B e^{\gamma u^2} u^2 w^2 |x|^\alpha dx + 2\gamma \int_B e^{\gamma u^2} \left( w^2 - u^2 \int_B |\nabla w|^2 \right) |x|^\alpha dx.
\]
Choose now \( w = u\psi f \), where \( \psi \) is a radial function and \( f(\theta) = \sin \theta \) (in polar coordinates). Then, using the fact that
\[
\int_0^{2\pi} f d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} f^2 d\theta = \int_0^{2\pi} f_\theta^2 d\theta,
\]
we see that
\[ D^2 F(u)[au + w\psi f, au + w\psi f] = \]
\[ 4\gamma^2 \int_B e^\gamma u^2 u^4 |x|^\alpha \, dx + 2\gamma \int_B e^\gamma u^2 \psi^2 |x|^\alpha \, dx \]
\[ - 2\gamma \int_B e^\gamma u^2 |x|^\alpha \, dx \left[ \int_B |\nabla (u\psi)|^2 \, dx + \int_B \frac{u^2 \psi^2}{|x|^2} \, dx \right]. \]  
(43)

Since \( u \) is a solution to (1), \( D^2 F(u) \) must be negative semidefinite as a bilinear form on \( H^1_0(B) \). We choose a suitable \( \psi \) and deduce that this can hold (for \( \alpha \) large) only if
\[ \gamma \leq \frac{\pi \varphi_1(0)^2}{\lambda_1 \int_B \varphi_1^2 \, dx}. \] 
(44)

We take \( \psi(r) = r \), and refer to the last remark why we choose this simple candidate.

By direct computation, if \( \psi(r) = r \), then
\[ \int_B |\nabla (u\psi)|^2 \, dx + \int_B \frac{u^2 \psi^2}{|x|^2} \, dx = \int_B |\nabla u|^2 |x|^2 \, dx, \]
and therefore
\[ D^2 F(u)[au + w\psi f, au + w\psi f] = \]
\[ 4\gamma^2 \int_B e^\gamma u^2 u^4 |x|^\alpha \, dx + 2\gamma \int_B e^\gamma u^2 \psi^2 |x|^\alpha \, dx \]
\[ - 2\gamma \int_B e^\gamma u^2 |x|^\alpha \, dx \int_B \frac{u^2 \psi^2}{|x|^2} \, dx \int_B |\nabla u|^2 |x|^2 \, dx. \] 
(45)

Recalling that \( u \) satisfies the Euler–Lagrange equation (2) with
\[ \lambda = \left( \int_B e^\gamma u^2 |x|^\alpha \, dx \right)^{-1}, \]
we can multiply both sides of equation (2) by \( |x|^2 u \) and integrate to obtain
\[ 2\gamma \lambda \int_B e^\gamma u^2 |x|^\alpha \, dx = \int_B |\nabla u|^2 |x|^2 \, dx - 2 \int_B u^2 \, dx. \] 
(46)

We remark that we could have found the last identity by a direct use of condition (39). Inserting (46) into (45) we find
\[ D^2 F(u)[au + w\psi f, au + w\psi f] = \]
\[ 4\gamma^2 \int_B e^\gamma u^2 u^4 |x|^\alpha \, dx - 4\gamma \int_B e^\gamma u^2 |x|^\alpha \, dx \int_B u^2 \, dx. \]
We write $u(|x|) = \sqrt{\varepsilon} v_\varepsilon(|x|^{1/\varepsilon})$, and we recall from Theorem 4 that $v_\varepsilon \to \varphi_1$ strongly in $H_0^1(B)$. Plugging the new variable $v_\varepsilon$ into the previous equation gives

$$D^2F(u)[au + \psi f, au + \psi f] = 4\gamma \varepsilon^3 \left\{ \gamma \int_B e^{\gamma \varepsilon v_\varepsilon^2} v_\varepsilon^4 |x|^{2\varepsilon} \, dx \right.$$ 
$$- \int_B e^{\gamma \varepsilon v_\varepsilon^2} e_{\varepsilon}^2 \, dx \cdot \varepsilon \int_0^{2\pi} \int_0^1 e_{\varepsilon}^2 t^{2\varepsilon-1} \, dt \, d\theta \right\}. \quad (47)$$

By a simple integration by parts, one checks immediately that

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} \int_0^1 e_{\varepsilon}^2 t^{2\varepsilon-1} \, dt \, d\theta = \pi \varphi_1(0)^2.$$ 

We may now conclude that

$$\lim_{\varepsilon \to 0} \frac{1}{4\gamma \varepsilon^3} D^2F(u)[au + \psi f, au + \psi f] =$$

$$\gamma \int_B \varphi_1^4 \, dx - \pi \varphi_1(0)^2 \int_B \varphi_1^2 = \gamma \int_B \varphi_1^4 \, dx - \frac{\pi}{\lambda_1} \varphi_1(0)^2 \leq 0$$

only if condition (44) holds. This completes the proof. ♣

Remark 3. We give a formal motivation why we have chosen $\psi(|x|) = |x|$ in the proof of the theorem. It is clear that equation (43) is homogenious in $\psi$, so that we can assume without loss of generality $\psi(1) = 1$. By inspecting (43), in order to make $D^2F(u)[au + \psi f, au + \psi f]$ negative it seems natural to choose a $\psi$ among radial functions vanishing at zero which keeps the integral

$$\int_B \left( |\nabla \psi|^2 + \frac{\psi^2}{|x|^2} \right) u^2 \, dx$$

as small as possible. The heuristic reason why we have chosen $\psi(r) = r$ is that this is precisely the unique solution to the variational problem

$$\inf \left\{ \int_B \left( |\nabla \psi|^2 + \frac{\psi^2}{|x|^2} \right) \, dx : \psi(1) = 1 \text{ and } \psi(0) = 0 \right\}.$$ 

Remark 4. Our results lead, in a natural way, to the following question: does there exist a “bifurcation point” $\gamma_* < 4\pi$ such that non-radial maximizers of problem (1) exist only when $\gamma > \gamma_*$?

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