Uncertainty Relations and Indistinguishable Particles

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I. INTRODUCTION

Quantum theory imposes fundamental limitations on our ability to simultaneously predict the outputs of measurements of different observables. This observation was formalised by the Uncertainty Principle, which has played a pivotal role in the conceptual clarification of quantum theory. It also plays an important role in quantum information theory and cryptography.

Most of the analysis of uncertainty relations is done within the framework of distinguishable particles, important exceptions being the analysis of Bohr and Rosenfeld and the work of Glauber. These exceptions explore uncertainty relations for field observables such as components of the electric and magnetic field and mode quadratures. The central goal of this paper is to derive and understand uncertainty relations for measurements of particle quantum numbers (q-numbers) on multi-particle states. Naively one may expect uncertainty relations for single particle observables to extrapolate in a simple way. Is this true? Does a conventional understanding of single particle uncertainty relations ever break down for multi-particle states?

II. ENTROPIC UNCERTAINTY

Consider two q-numbers \( A \) and \( B \) for a particle, each with (finite) \( n \) mutually exclusive possible outcomes. Consider the corresponding observables which we represent by operators \( A_1 \) and \( B_1 \) in an \( n \)-dimensional Hilbert space \( \mathcal{H} \). Let \( \{ |i\> \} \) and \( \{ |j\> \} \) be the complete bases of eigenstates of \( A_1 \) and \( B_1 \) respectively. A good measure of the uncertainty of an observable is the Shannon entropy of its probability distribution. For a pure state \( |\psi\> \in \mathcal{H} \), the Shannon entropies for \( A \) and \( B \) are given by

\[
H(A_1; \psi) := - \sum_{i=1}^{n} P_i(A_1; \psi) \ln P_i(A_1; \psi),
\]

\[
H(B_1; \psi) := - \sum_{j=1}^{n} P_j(B_1; \psi) \ln P_j(B_1; \psi)
\]

where \( P_i(A_1; \psi) := |\langle i | \psi \rangle|^2 \), \( P_j(B_1; \psi) := |\langle j | \psi \rangle|^2 \), and the units are in nats (which we use throughout the paper). Note the maximum value of a Shannon entropy is \( \ln n \). The sum of these Shannon entropies is constrained by the uncertainty relation:

\[
H(A_1; \psi) + H(B_1; \psi) \geq -2 \ln c,
\]

where

\[
c := \max_{j, k} |U_{jk}| \quad \text{and} \quad U_{jk} := \langle j | \tilde{k} \rangle.
\]

By definition, \( c \) is bounded: \( 0 \leq c \leq 1 \). As Shannon entropies for discrete probability distributions are positive
definite, for observables with \( c < 1 \), relation (3) implies at least one of the entropies must be non-zero.

We have characterized relation (3) for operators \( A_1 \) and \( B_1 \), which refer to particle q-numbers. However, the result applies to any \( s \)-outcome operators acting on an \( s \)-dimensional Hilbert space \( \mathcal{H} \) where \( s \in \mathbb{N} \). We can therefore apply (3) to multi-particle states. The distinguishability of particle ordering becomes important when one considers more than 1 particle. Does an extension of (3) to multi-particle states depend on whether one considers distinguishable or indistinguishable particles?

III. \( N \)-PARTICLE UNCERTAINTY RELATION FOR DISTINGUISHABLE PARTICLES

Consider \( N \) distinguishable particles. There are \( n \) possible outcomes for each particle, giving us \( n^N \) possible outcomes in total. Let \( A^{(i)} \) be the operator representing the q-number \( A \) for the \( i \)th particle with a basis of eigenstates \( \{ |m_i \rangle \} \) such that

\[
A^{(i)} |m_i \rangle := \alpha_m |m_i \rangle ,
\]

where \( |m_i \rangle \in \mathcal{H}_i \) and \( \mathcal{H}_i \) is the Hilbert space for the \( i \)th particle. The \( N \)-particle state \( |\psi_N \rangle \) is then an element of \( \bigotimes_{i=1}^{N} \mathcal{H}_i \). Define a new observable whose eigenstates correspond to the distinct outcomes of a measurement of \( A \) on the system, where the measurement can distinguish particle ordering:

\[
A_N := \sum_{m_1, \ldots, m_N=1}^{N} e_{m_1 \ldots m_N} \left\{ \bigotimes_{i=1}^{N} |m_i \rangle \right\} \left\{ \bigotimes_{j=1}^{N} |m_j \rangle \right\},
\]

where the eigenvalues \( e_{m_1 \ldots m_N} \) are chosen to be non-degenerate. Then for an \( N \)-particle state \( |\psi_N \rangle \), the Shannon entropy of \( A \) is given by

\[
H(A; |\psi_N \rangle) := - \sum_{i_1, \ldots, i_N=1} \ln P_{i_1 \ldots i_N}(A; |\psi_N \rangle),
\]

where \( P_{i_1 \ldots i_N}(A; |\psi_N \rangle) = |\langle i_1 | \cdots | i_N | \psi_N \rangle|^2 \).

Note that \( H(A; |\psi_N \rangle) \) is the joint Shannon entropy for the outcomes of \( A^{(i)} \) for each particle. Define an analogous observable and entropy for \( B \). Utilizing relation (3) but replacing \( A_1 \) with \( A_N \) and \( B_1 \) with \( B_N \), one finds an entropic uncertainty relation

\[
H(A_N; |\psi_N \rangle) + H(B_N; |\psi_N \rangle) \geq -2 \ln \left( \max_{m_1, \ldots, m_N} \left| \sum_{i=1}^{N} \langle m_i | \bigotimes_{j=1}^{N} |\tilde{r}_{ij} \rangle \right| \right)
\]

where \( c \) is defined as in Eq. (4).

This result is easily understood for cases where \( |\psi_N \rangle \) is unentangled:

\[
|\psi_N \rangle = |\psi^{(1)}\rangle \cdots |\psi^{(N)}\rangle.
\]

The average information gained about \( A_N \) over the ensemble equals the sum of the average information gained about \( A^{(i)} \) for each particle:

\[
H(A_N; |\psi_N \rangle) = \sum_{i=1}^{N} H(\psi^{(i)}; |\psi^{(i)} \rangle),
\]

Then,

\[
H(A_N; |\psi_N \rangle) + H(B_N; |\psi_N \rangle)
\]

\[
\geq -2N \ln c,
\]

which gives the same bound on \( H(A_N; |\psi_N \rangle) + H(B_N; |\psi_N \rangle) \) as relation (8) for the special case where \( |\psi_N \rangle \) is unentangled.

Relation (11) highlights an interesting feature of relation (8); that it is unaffected by possible entanglement of \( |\psi_N \rangle \). It is not immediately obvious to us why this is the case. Our thoughts go as follows:

(a) The state of particle \( i \), \( \rho^{(i)} = \text{tr}_{\bar{i}}[\psi_N \langle \psi_N |] \), can become mixed, i.e. the von Neumann entropy \( S(\rho^{(i)}) = -\text{tr}[\rho^{(i)} \ln \rho^{(i)}] > 0 \). A non-zero \( S(\rho^{(i)}) \) constrains the Shannon entropy of \( A \) and \( B \) for particle \( i \):\n
\[
H(\rho^{(i)}; |\psi_N \rangle) \geq S(\rho^{(i)}).
\]

This suggests it may be difficult for entangled states to saturate the bound (8).

(b) The subadditivity of the Shannon entropy implies

\[
H(A_N; |\psi_N \rangle) < \sum_{i=1}^{N} H(\rho^{(i)}; |\psi_N \rangle),
\]

for correlated observables \( A^{(i)} \) and similarly for \( B \). This disallows a direct use of our previous observation in explaining why entangled states cannot overcome relation (11).

The salient feature of both relations (8) and (11) is the proportionality to \( N \) of the lower bounds. We claim that this feature is indicative of the assumption that it is possible to define probabilities for the outcomes of measurements of q-numbers \( A \) and \( B \) for each particle.

It is well known [11] that, for a given quantum state, relation (8) is not necessarily the strongest possible bound. Accordingly, neither are relations (5) and (11).
IV. COMPARING DISTINGUISHABLE AND INDISTINGUISHABLE PARTICLES

To compare uncertainties of particle q-numbers for distinguishable and indistinguishable particles, we have to be careful about the precise meaning of the probabilities of the various outcomes for multi-particle states.

For distinguishable particles, the meaning is simple. For a 2 particle state we can consider observables for each particle $O^{(1)}$ and $O^{(2)}$ such that
\[ [O^{(1)}, O^{(2)}] = 0. \] (14)
This defines the natural tensor product decomposition $H_1 \otimes H_2$ where $H_i$ corresponds to the $i$-th particle. Suppose we have a state $\rho$ which is pure and unentangled
\[ \rho = |\psi^{(1)}_1\rangle_1 |\psi^{(2)}_2\rangle_2 = \frac{1}{2} (|\psi^{(1)}_1\rangle_1 |\psi^{(2)}_2\rangle_2 + |\psi^{(2)}_2\rangle_2 |\psi^{(1)}_1\rangle_1), \] (15)
where $|\psi^{(1)}_1\rangle_1 \in H_1$ and $|\psi^{(2)}_2\rangle_2 \in H_2$. Let $O^{(1)}$ and $O^{(2)}$ be position projectors for positions $x_1$ and $x_2$ respectively, then
\[ \langle O^{(1)}(x_1)O^{(2)}(x_2) \rangle = |\psi^{(1)}(x_1)|^2 |\psi^{(2)}(x_2)|^2, \] (16)
which is the probability for particle 1 to be measured at $x_1$ and particle 2 to be measured at $x_2$.

Consider indistinguishable particles. Now all observables commute with permutations of the particle states i.e., for a permutation operator $P_{1 \leftrightarrow 2}$ that permutes the states in $H_1$ and $H_2$ and any observable $O$,
\[ [P_{1 \leftrightarrow 2}, O] = 0. \] (17)
This defines a superselection rule to constrain direct measurements to irreducible representations of the permutation group i.e., symmetric and anti-symmetric under $P_{1 \leftrightarrow 2}$.

Importantly for our considerations, the constraint (17) means all observables must act non-locally on the tensor product decomposition $H_1 \otimes H_2$. The observable which then corresponds closest to $O^{(1)}O^{(2)} = |x_1\rangle_1 |x_1\rangle_1 \otimes |x_2\rangle_2 |x_2\rangle_2$ is then,
\[ O^{(12)} := \frac{1}{2} (|x_1\rangle_1 |x_2\rangle_2 \pm |x_2\rangle_2 |x_1\rangle_1) (|x_1\rangle_2 |x_2\rangle_1 \pm |x_2\rangle_1 |x_1\rangle_2), \] (18)
with + or − corresponding to bosons or fermions respectively. The expectation value of this observable gives us the probability that we shall measure a particle at $x_1$ and another at $x_2$.

The probabilities given by $\langle O^{(1)}O^{(2)} \rangle$ and $\langle O^{(12)} \rangle$ have fundamentally different meanings. Predictions for distinguishable particles are counterfactual because they assume the existence of particle labels that cannot be measured.

The probabilities and hence Shannon entropies in relation (3) are given physical meaning by the standard axioms of quantum theory that the outcomes of a measurement (eigenstates of an observable) correspond to outputs of a measuring apparatus with probabilities for occurrence defined in the normal way. What is of physical interest are these outputs. We assume:

(a) An ideal apparatus whose outputs are in one-to-one correspondence with the outcomes.

(b) Meaningful probabilities can be defined for these outcomes.

Then a Shannon entropy can be defined on the probability distribution of the outputs. Assumption (a) ensures direct usefulness to physics. This Shannon entropy of the outputs rather than the outcomes (eigenstates of some observable) can then be used to compare different theories whose probabilities have very different meanings.

Suppose we perform measurements of q-numbers $A$ and $B$ on an infinite, homogeneous ensemble of pure $N$-particle states. Define the Shannon entropy of the outputs of the measurement apparatus as $H_N(A)$ and $H_N(B)$ for measurements of $A$ and $B$, respectively.

Identifying the outputs with outcomes for measurements of $N$ distinguishable particles gives us,
\[ H_N(A) = H(A_N; \psi_N), \] (19)
and similarly for $B$.

For the outputs of measurements of $A$ and $B$ over $N$-particle states we can write
\[ H_N(A) + H_N(B) \geq \chi_{\min}(N; A, B), \] (20)
where $\chi_{\min}(N; A, B)$ is the real minimum over the $N$-particle Hilbert space. In this context, the uncertainty relation for distinguishable particles gives
\[ \chi_{\min}^{\text{dist}}(N; A, B) = -2N \ln c. \] (21)

V. $N$-PARTICLE UNCERTAINTY RELATION FOR INDISTINGUISHABLE PARTICLES

We now define $H_N(A)$ and $H_N(B)$ for $N$-particle states of indistinguishable particles.

In order to define $H_N(A)$, the type of measurement interaction must be considered. Consider particles with two compatible q-numbers $A$ and $C$, with eigenmodes $\alpha_i$ and $c_j$ respectively. Let $m$ be the number of $C$ eigenmodes and consider the case where $m > n$. The two-particle states $|\alpha_1 c_1; \alpha_2 c_2\rangle$ and $|\alpha_1 c_1; \alpha_1 c_2\rangle$ are reliably distinguishable in an experiment measuring $A$ if the interaction also couples to modes of $C$ and the modes of $C$ then act as a reference frame for the effective ordering distinguishability of the particles.
As an example, $\mathcal{C}$ may relate to the position of the particle. Measurements can then distinguish particle ordering if the measurement interaction has position dependence, such as the magnetic field in a Stern-Gerlach experiment being localized in space.

For an $N$-particle state, the $n^N$ possible outputs we have for distinguishable particles correspond to indistinguishable particle states where the modes of $\mathcal{C}$ are different for each particle and also fixed over the set of outputs. These states—where the modes of $\mathcal{C}$ are different for each particle—form only a subspace of the $N$-particle states for indistinguishable particles. It is not guaranteed that the other $N$-particle states satisfy the same bounds for $H_N(A) + H_N(B)$.

We require the states of distinguishable particles to correspond to states of indistinguishable particles with modes of $\mathcal{C}$ fixed for the following reasons. With distinguishable particles, for states that are elements of $\otimes_i H_i$ and measurements of $\mathcal{A}$ or $\mathcal{B}$, it doesn’t matter what the q-number $\mathcal{A}$ is referenced to. With indistinguishable particles where the reference mode $\mathcal{C}$ is modelled, different reference modes are distinguished; i.e., the states $|\alpha_1, \alpha_2, \alpha_3, h\rangle$ and $|\alpha_1, h, \alpha_2, \alpha_3\rangle$ are different but give equivalent probabilities for $\mathcal{A}$ and $\mathcal{B}$. Thus, they may be identified with the same state $|1\rangle_1|2\rangle_2$.

Consider particle creation operators $\hat{a}_{ij}^\dagger$ (fermionic or bosonic) which create a particle in mode ‘$j$’ for $\mathcal{A}$ and mode ‘$l$’ for $\mathcal{C}$ such that the state $|\alpha_{ij}^\dagger |0\rangle$ (where $|0\rangle$ is the Fock vacuum) corresponds to the $A_1$ eigenstate $|j, l\rangle_1$. One can also define ladder operators $\hat{a}_{ij}^\dagger$ such that $\hat{a}_{ij}^\dagger |0\rangle$ corresponds to an eigenstate of $B_1$, $|i, b\rangle_1$. The two sets of operators are related by the Bogoliubov transformation

$$\hat{a}_{ij}^\dagger := \sum_{k=1}^n U_{ik} a_{kl}^\dagger$$

For $N$-particle states of indistinguishable particles, the Shannon entropy of the outcomes of $\mathcal{A}$ is constructed in the following way. Define the observable

$$A_{N|C} := \sum_{i_{11}, \ldots, i_{nm}=0}^{N} f_{i_{11} \ldots , i_{nm}} \delta(\sum_{b,c} i_{bc}) N \times \left\{ \prod_{j=1, l=1}^{n,m} \left( \frac{a_{jl}^\dagger}{\sqrt{i_{jl}}} |0\rangle \right) \right\} \left\{ \prod_{k=1, r=1}^{n,m} \left( \frac{a_{kr}}{\sqrt{i_{kr}}} \right) \right\}$$

where $\delta(\sum_{b,c} i_{bc})$ is a Kronecker delta. The eigenvalues $f_{i_{11} \ldots , i_{nm}}$ are chosen to be non-degenerate and the ordering of the ladder operators arbitrary. (The ordering of ladder operators does not affect any result of this paper. In cases where the ordering isn’t specified, the reader is free to choose it.) Then the corresponding Shannon entropy is

$$H(A_{N|C}; \phi) = \sum_{i_{11}, \ldots, i_{nm}=0}^{n} P_{i_{11} \ldots , i_{nm}} (A_{N|C}; \phi) \ln P_{i_{11} \ldots , i_{nm}} (A_{N|C}; \phi),$$

where

$$P_{i_{11} \ldots , i_{nm}} (A_{N|C}; \phi) = \delta(\sum_{b,c} i_{bc}) N \left| \langle 0 | \prod_{k=1, l=1}^{n,m} \frac{a_{kl}}{\sqrt{i_{kl}}} |\phi\rangle \right|^2.$$  

Identifying outputs to outcomes gives us

$$H_N(A) = H(A_{N|C}; \phi)$$

for a state of $N$ indistinguishable particles $|\phi\rangle$ and an apparatus that couples to $\mathcal{A}$ and $\mathcal{C}$. An analogous entropy for $B$ can also be defined with an observable $B_{N|C}$ which is given by $A_{N|C}$ with the ladder operators $\hat{a}_{ij}^\dagger$ replaced by $\hat{a}_{ji}^\dagger$.

We are now in a position to construct an uncertainty relation for $H_N(A)$ and $H_N(B)$. We choose the Hilbert space to be the $N$-particle subspace of the entire Fock space of these modes. Applying relation (23) to $A_{N|C}$ and $B_{N|C}$ instead of $A_1$ and $B_1$, we obtain

$$H(A_{N|C}; \phi) + H(B_{N|C}; \phi) \geq -2 \ln \left\{ \max_{i_{11}, \ldots , i_{nm}=0}^{n,m} \delta(\sum_{b,c} i_{bc}) N G(i,j) \right\},$$

where

$$G(i,j) := \langle 0 | \prod_{k=1, l=1}^{n,m} \frac{a_{kl}}{\sqrt{i_{kl}}} \prod_{t=1, r=1}^{n,m} \frac{\hat{a}_{tr}}{\sqrt{i_{tr}}} |0\rangle.$$

So how does (27) compare to (21)? Firstly, a sanity check. If we restrict ourselves to an $N$-particle subspace where the modes of $\mathcal{C}$ are different for each particle and also fixed, the outcomes and probabilities of a measurement of $\mathcal{A}$ or $\mathcal{B}$ should be in one-to-one correspondence with the distinguishable particles case. Thus, with this restriction, $H(A_{N|C}; \psi) + H(B_{N|C}; \psi)$ should never be smaller than $-2N \ln c$. We can compute the RHS of (27) with this restriction. The overlap between eigenstates of $A_{N|C}$ and $B_{N|C}$ can be given without loss of generality by

$$G(i,j) = \langle 0 | a_{i_1} \cdots a_{i_N} \hat{a}_{j_1}^\dagger \cdots \hat{a}_{j_N}^\dagger |0\rangle = U_{i_{1j_1}} \cdots U_{i_{Nj_N}}.$$  

Thus with this restricted subspace,

$$H(A_{N|C}; \psi) + H(B_{N|C}; \psi) \geq -2N \ln c.$$  

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3 This correspondence can be given by the isomorphism $f_\chi$ defined in [24].

4 They are equivalent in terms of giving the same probabilities for particle q-numbers. The Fock space however, allows for measurements of q-numbers which are impossible for a quantum theory with distinguishable particles, i.e., superpositions of occupation number eigenstates.
choices of $A$

Firstly, it is proportional to proportionality. In contrast, the equations above for bosons and $\chi$ with each other.

Define

$$V(\tilde{k}, \tilde{l}) = U_{t_1 k_1} \cdots U_{t_n k_N}, \quad W(\tilde{k}, \tilde{l}') = \delta_{\tilde{k}'}_{\tilde{k}} \cdots \delta_{\tilde{l}'}_{\tilde{l}},$$

and

$$M(\tilde{k}, \tilde{l}, \tilde{l}') = \left\{ \sum_{\sigma \in P_N} W(\sigma \tilde{k}, \tilde{k}') W(\sigma \tilde{l}, \tilde{l}') \right\}^{1/2},$$

where $k_i, t_i \leq n$ and $l_i \leq m$. Also define

$$G_{\text{boson}}(\tilde{k}, \tilde{l}, \tilde{l}') = \frac{1}{M} \sum_{\sigma \in P_N} V(\sigma \tilde{k}, \tilde{l}) W(\sigma \tilde{l}, \tilde{l}') ,$$

$$G_{\text{ferm}}(\tilde{k}, \tilde{l}, \tilde{l}') = \sum_{\sigma \in P_N} \text{sgn}(\sigma) V(\sigma \tilde{k}, \tilde{l}) W(\sigma \tilde{l}, \tilde{l}') .$$

Then for bosons and fermions respectively, relation (27) gives us

$$\chi_{\text{boson}}^\text{min}(N; A, B) = -2 \ln \left| \max_{\tilde{k}, \tilde{l}, \tilde{l}'} G_{\text{boson}} \right|, \quad (36)$$

and

$$\chi_{\text{ferm}}^\text{min}(N; A, B) = -2 \ln \left| \max_{\tilde{k}, \tilde{l}, \tilde{l}'} G_{\text{ferm}} \right|. \quad (37)$$

We can immediately compare these uncertainty relations to Eq. (21). Eq. (21) has two nice properties. Firstly, it is proportional to $N$. Secondly, different choices of $A$ and $B$ affect only the slope of this proportionality. In contrast, the equations above for bosons and fermions cannot be expected to have these properties. This is because they involve sums of $V(\tilde{k}, \tilde{l})$ which generally are complex numbers and thus generally interfere with each other.

VI. BOSONS

Let us focus on state that do not correspond to states of distinguishable particles. Any interesting differences between (36) and (21) are likely to show up in this region of the Hilbert space. Consider for instance an $N$-boson eigenstate of $A_{|C}$ where every particle is in the same $C$ eigenmode $'P'$. For example, let $\tilde{A} = \tilde{A}_I', \tilde{B} = \tilde{A}_I''$, 

$$|\omega\rangle := |\omega\rangle / \sqrt{\langle \omega | \omega \rangle}, \quad \langle \omega | := a_{\tilde{A}_I|} \cdots a_{\tilde{A}_{I_N}|} |0\rangle. \quad (38)$$

By construction, $H(A_{|C}; \omega) = 0$. An upper bound can be found for $H(B_{|C}; \omega)$ using the following argument:

$$H(B_{|C}; \omega) \leq \ln(\text{no. possible outcomes}). \quad (39)$$

The Bogoliubov transformation (22) does not change (a) the number of particles, and (b) the $C$ eigenmodes. Each component of $|\omega\rangle$ in the $B$-mode basis must have $N$-particles where each particle is still in $C$ eigenmode $'P'$. How many possible components with these properties are there? The question is isomorphic to the question, “how many ways can I place $N$ indistinguishable balls into $n$ distinguishable boxes?” The answer (21) is $\binom{N+n-1}{N}$. Thus,

$$H(A_{|C}; \omega) + H(B_{|C}; \omega) \leq \ln \left( \frac{N + n - 1}{N} \right). \quad (40)$$

For $N \geq n,$

$$\ln \left( \frac{N + n - 1}{N} \right) = (n-1) \ln N - \sum_{k=1}^{n-1} \ln k + O(1/N), \quad (41)$$

and hence this upper bound is of order $N$, which shows that Eq. (36), unlike Eq. (21), cannot be proportional to $N$.

Relation (10) provides an upper bound on $\chi_{\text{min}}$

$$\chi_{\text{min}}^\text{boson}(N; A, B) \leq \ln \left( \frac{N + n - 1}{N} \right), \quad (42)$$

which is dependent only on $N$ and $n$. This is demonstrated in Fig. 4.

To illustrate relation (42), consider an $N = 2 = n$ example with photons, given by the $A_{|C}$ eigenstate

$$|\gamma\rangle := a_{1|k} a_{2\dagger|k} |0\rangle, \quad (43)$$

where 1 and 2 denote polarization eigenmodes of $A$ and $k$ denotes an eigenmode of $C$. The q-number $C$ could be for instance position, momentum, or a combination of angular momentum q-numbers. Choose eigenmodes of $B$ as rotations of 1 and 2 through an angle $\alpha$,

$$a_{1|k} = \cos \alpha a_{1\alpha|k} + \sin \alpha a_{2\alpha|k}, \quad \text{and} \quad a_{2|k} = -\sin \alpha a_{1\alpha|k} + \cos \alpha a_{2\alpha|k}, \quad (44)$$
such that
\[
|\gamma\gamma\rangle = \sqrt{2} \cos \alpha \sin \alpha \left( \frac{1}{\sqrt{2}} a_{\alpha_2 k}^\dagger a_{\alpha_2 k} - \frac{1}{\sqrt{2}} a_{\alpha_1 k}^\dagger a_{\alpha_1 k} \right) |0\rangle 
+ (\cos^2 \alpha - \sin^2 \alpha) a_{\alpha_1 k}^\dagger a_{\alpha_2 k} |0\rangle.
\]
Suppose \( \mathcal{B} \) is complementary to \( \mathcal{A} \), i.e. \( \alpha = \pi/4 \); then,
\[
H(A_2|C; \gamma\gamma\rangle) + H(B_2|C; \gamma\gamma\rangle) = \ln 2,
\]
which is smaller than the RHS of Eq. (21) and \( \frac{1}{2} \) the lower bound (21) for \( H_N(\mathcal{A}) + H_N(\mathcal{B}) \) with distinguishable particles. Interestingly, the upper bound (12) can be saturated for \( \alpha = \frac{1}{2} \arctan \sqrt{2} \) where \( \mathcal{B} \) is not complementary to \( \mathcal{A} \):
\[
H(B_2|C|\alpha=\frac{1}{2} \arctan \sqrt{2}; \gamma\gamma\rangle) = \ln 3. \tag{47}
\]

VII. FERMIONS

Let’s turn our attention to fermions, whose ladder operators we shall denote by \( b_{ij} \) and \( b_{ij}^\dagger \). For the case \( N = n \), the RHS of Eq. (37) can be computed in the following way. Consider the state
\[
|\lambda_f\rangle := b_{1l}^\dagger \cdots b_{nl}^\dagger |0\rangle \tag{48}
\]
which is an eigenstate of \( A_n|C\rangle \). Rewriting the state in terms of modes of \( \mathcal{B} \) gives
\[
|\lambda_f\rangle = \prod_{i_1, \ldots, i_n = 1}^n U_{i_1 i}^{-1} \cdots U_{i_n i_n}^{-1} b_{i_1 l}^\dagger \cdots b_{i_n l}^\dagger |0\rangle
= \prod_{i_1, \ldots, i_n = 1}^n U_{i_1 i}^{-1} \cdots U_{i_n i_n}^{-1} \epsilon_{i_1 \cdots i_n} b_{i_1 l}^\dagger \cdots b_{i_n l}^\dagger |0\rangle
= \det (U^{-1}) b_{1l}^\dagger \cdots b_{nl}^\dagger |0\rangle, \tag{49}
\]
where we also used the fact that \( |0\rangle \) is invariant under Bogoliubov transformations (22) that do not mix creation and annihilation operators. Thus
\[
\langle 0 | \prod_{k=1}^n b_{kl}^\dagger \prod_{r=1}^n b_{rl}^\dagger |0\rangle = |\det (U^{-1})| = 1 \tag{50}
\]
which is the maximum possible value for the overlap between the bases of eigenstates of \( A_n|C\rangle \) and \( B_n|C\rangle \). Hence \( \chi_{\text{form}} \) is zero for \( N = n \). This is in stark contrast to the distinguishable \( n \)-particle uncertainty relation that constrains \( H_N(\mathcal{A}) + H_N(\mathcal{B}) \geq -2n \ln c \).

By considering the exclusion principle, we can find more states that circumvent Eq. (21): Fermions are constrained to be in orthogonal modes. These are modes of the product of all compatible q-numbers of the particles. The only way to create two or more particles in mode \( 'i' \) for q-number \( \mathcal{A} \) is for the \( \mathcal{C} \) modes of the particles to be orthogonal. Consider again an \( N (\leq n) \)-particle state where every particle is in the same \( \mathcal{C} \) mode,
\[
|\lambda_f'|_{i_1, \ldots, i_N} := \prod_{j=1}^N b_{ij}^\dagger |0\rangle. \tag{51}
\]
The q-number \( \mathcal{C} \) does not act as a reference frame for \( \mathcal{A} \). There are only \( \binom{n}{b} \leq n^b \) allowed choices for the labels \( i_j \) which correspond to different outcomes for a measurement of \( \mathcal{A} \). This also applies to \( \mathcal{B} \). Following the same arguments we made for bosons, a limit on the number of outcomes puts a limit on the Shannon entropy of the probability distribution over the outcomes; for a generic normalized superposition of the states \( |\lambda_f'|_{i_1, \ldots, i_N} \) with no reference frame for \( \mathcal{A} \)
\[
|\lambda_f'|_{i_1, \ldots, i_N} := \sum_{c_{i_1 \cdots i_N}} c_{i_1 \cdots i_N} |\lambda_f'|_{i_1, \ldots, i_N}, \tag{52}
\]
the entropies for the outcomes of \( \mathcal{A} \) and \( \mathcal{B} \) are constrained from above, by the exclusion principle:
\[
H(A_{N|C}; \lambda_f') \leq \ln \binom{n}{N}, \tag{53}
\]
\[
H(B_{N|C}; \lambda_f') \leq \ln \binom{n}{N}. \tag{54}
\]
The extreme case of these constraints is when \( N = n \) such that both entropies must be zero. This is equivalent to the results obtained for the state \( |\lambda_f\rangle \).

The constraints (53) and (54) can be extended to arbitrary \( N = an + b \) where \( b < n \). This is done by combining the states \( |\lambda_f\rangle \) and \( |\lambda_f'| \):
\[
|\lambda_f''\rangle = \sum_{i_1, \ldots, i_b} c_{i_1 \cdots i_b} \prod_{j=1}^b b_{ij}^\dagger k_a \prod_{l=1}^{a-1} \prod_{r=1}^n b_{rl}^\dagger |0\rangle, \tag{55}
\]
where \( k_1, \ldots, k_a \) label orthogonal modes of \( \mathcal{C} \). Then,
\[
H(A_{N|C}; \lambda_f'') \leq \ln \binom{n}{N \mod n}, \tag{56}
\]
\[
H(B_{N|C}; \lambda_f'') \leq \ln \binom{n}{N \mod n}. \tag{57}
\]
If one has \( c_{i_1 \cdots i_b} = 1 \) for some choice of \( i_1, \ldots, i_b \) and the other choices all zero, then \( H(A_{N|C}; \lambda_f'') = 0 \) without changing (57). Thus we arrive at an upper bound on \( \chi_{\text{min}} \):
\[
\chi_{\text{min}}(N; \mathcal{A}, \mathcal{B}) \leq \ln \binom{n}{N \mod n}. \tag{58}
\]
As an example of the extreme case \( n = N \), where \( H_N(\mathcal{A}) = 0 = H_N(\mathcal{B}) \), consider a dineutron state where both neutrons are in an s-wave and their spins are anti-aligned on the z-axis. The q-number \( \mathcal{A} \) will correspond to these spins on the z-axis. Denote \( a_{\alpha q}^\dagger \) and \( a_{\alpha q}^\dagger \) as the creation operators for the up and down s-wave neutrons.
This gives (bosons) for varying particle number $N$ and indistinguishable particles (relation (58) for fermions and relation (42) for bosons) for varying particle number $N$. The $q$-numbers are chosen to be complementary with $n = 6$ distinct outcomes. This gives $c = 1/\sqrt{6}$.

with all other $q$-numbers given by $q$. The state is given by

$$|nn⟩ := b_{1q}^{†} b_{1q}^{†} |0⟩.$$  

(59)

For a single spin half particle in a pure state, there exists only one axis where the particle has definite spin. For $|nn⟩$, both particles have definite spin in every direction: consider $B$ modes as modes of spin rotated from the $z$-axis by $θ$. Let $U_{ij} = (exp \{iθ · σ\})_{ij}$, where $σ = (σ_1, σ_2, σ_3)$ is the Pauli matrix vector such that

$$b_{ij}^{†} = \sum_{j=\uparrow, \downarrow} \left( exp \{iθ · σ\} \right)_{ij} b_{ij}^{†}. \tag{60}$$

Then

$$|nn⟩ = b_{1q}^{†} b_{1q}^{†} |0⟩ \quad ∀ \ θ.$$  

(61)

Another two particle example is a deuteron state. Consider both nucleons to be in an $s$-wave and their spins aligned. The $q$-numbers $A$ and $B$ with $H_N (A) = 0 = H_N (B)$ correspond to choices of isospin basis. Measurements of superpositions of a proton and neutron would have to overcome the charge superselection rule.

VIII. REMARKS

In this paper, we derive three $N$-particle uncertainty relations: Eq. (21) for distinguishable particles and (36) and (42) for indistinguishable bosons and fermions respectively. We develop an understanding of how particle $q$-number ordering information manifests itself with indistinguishable particles. Not all states of indistinguishable particles have effective distinguishability for the $q$-number of interest. This leads us to upper bounds (42) and (58) for $χ_{min}$ (the minimum of $H_N (A) + H_N (B)$), which shows that $χ_{min}$ for bosons or fermions is generally much smaller than $χ_{min}$ for distinguishable particles. These upper bounds are due to the structure of indistinguishable particles and are only dependent on the number of particles $N$ and the number of outcomes $n$ for $A$ and $B$. That is, they are not dependent on the compatibility of $A$ and $B$.

Relation (58) in particular has an infinite number of zeroes, which occur when $N$ is an integer multiple of $n$. This means that fermions are not bound by a minimum total uncertainty for finite $n$.

Different experimental situations have to be considered separately. One situation is when the measurement interaction does not couple to any reference modes. The minimum total uncertainty $χ_{min}$ will generally be smaller than the lower bound given by Eq. (21). This can be considered akin to coarse graining where we are not interested in all the information the state has to offer, such as particle ordering information. For the other situation, where the measurement interaction does couple to reference modes, any loss of effective particle distinguishability occurs at a fundamental level: ordering information cannot be accessed irrespective of the capabilities of the measurement apparatus.

Our analysis shows that in the wider context of multiparticle states, the Heisenberg uncertainty limits must be reunderstood. Apparent fundamental limitations on the precision of measurements of particle $q$-numbers can in principle be overcome.

We emphasize several points:

(a) The no coupling case has a maximum $(N+n−1)$ outputs, which is the number obtained by simply ignoring particle ordering information. We have not explored bounds on $H_N (A) + H_N (B)$ for distinguishable particles with particle ordering ignored. For this situation, there is an upper bound on the minimum of $H_N (A) + H_N (B)$ which is the same as what we derived for bosons. Simply ignoring particle ordering will still give different results compared to bosons. The differences between these two situations are explored in.

(b) The definition of $A_{N|C}$ is symmetric under a swap between $A$ and $C$, i.e. $A_{N|C} = C_{N|A}$. Each state we considered in this paper had definite particle number for each mode of $C$. If one considers states where this is not the case, $H(A_{N|C}; δ) = H(C_{N|A}; δ)$ is a quantification of the uncertainty in both $A$ and $C$.

(c) For ease of exposition, we described states such as $|ω⟩$ and $|i⟩$ in terms of the labels of the ladder operators. For such states, where one loses a reference frame for $A$ and $B$, the identity of the particles for measurements of $A$ and $B$ is obscured and independence of the outcomes for these $q$-numbers becomes ambiguous. For instance, consider again the state $|λ_F⟩$ where $H(A_{N|C}; λ) = 0 = H(B_{N|C}; λ)$. One is
attempted to say that every particle has definite values for $\mathcal{A}$ and $\mathcal{B}$ simultaneously.

The particles in the various modes of $\mathcal{A}$ and $\mathcal{B}$ are ‘elements of physical reality’ according to the EPR criterion \[22, 28\]. However it is meaningless to say whether a particle in a certain mode of $\mathcal{A}$ is simultaneously in a mode of $\mathcal{B}$. The particles have lost every label which might distinguish one from the other. We consider the circumvention of relation \[15\] by states such as $|\omega\rangle$ and $|\lambda_f\rangle$ as the manifestation of this lack of distinguishability.

(d) In many situations, the very concept of a particle becomes difficult to define. The context of our arguments are less restrictive than in relativistic quantum field theory where the Hilbert space is defined by asymptotically non-interacting multiparticle states. We have assumed the Fock space of a non-interacting field theory can define all multiparticle states. We have also implicitly assumed that there exists particle q-number measurements that can be performed on all interacting multiparticle states.

Particles must have a long enough lifetime relative to the interaction time such that the states can be considered approximate energy eigenstates. The dineutron for example is not stable \[27\].

One must also consider that approximate energy eigenstates where particle fields begin to overlap will often not be given as simple products of ladder operators. For instance, multiple electrons bound in an atom interact with each other leading to energy eigenstates that are not simply products of ladder operators that create single electron energy eigenstates.

These are some of the factors that must be considered if one attempts to experimentally reach the new limits found in this paper.

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