ONE-SIDED AND INTERNAL CONTROLLABILITY
OF SEMILINEAR WAVE EQUATIONS WITH
INFINITELY ITERATED LOGARITHMS

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Abstract. In two previous works we improved some earlier results of Imanuvilov, Li
and Zhang, and of Zuazua on the boundary exact controllability of one-dimensional
semilinear wave equations by weakening the growth assumptions on the nonlinearity.
Our growth assumption is in a sense optimal. Here we adapt our method for the case
of one-sided control actions. This also enables us to obtain rather general internal
controllability results.

1. Introduction and formulation of the main results. Fix a bounded open
interval \((a, b)\) and a positive number \(T\). Given a function \(f : \mathbb{R} \to \mathbb{R}\) of class \(C^1\),
consider the problem

\[
\begin{cases}
    u_{tt} - u_{xx} - f(u) = 0 \quad \text{in} \quad (a, b) \times (0, T), \\
    u(a, t) = h_a(t) \quad \text{and} \quad u(b, t) = h_b(t) \quad \text{for} \quad t \in (0, T), \\
    u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in} \quad (a, b).
\end{cases}
\]

(1.1)

We will obtain a boundary exact controllability result under suitable, rather
weak growth assumptions on the nonlinearity \(f\). In order to state our result, let us
introduce the \(\text{iterated logarithm}\) functions \(\log_j\) defined by the formulas

\[
\log_0 s := s \quad \text{and} \quad \log_j s := \log(\log_{j-1} s), \quad j = 1, 2, \ldots,
\]

and define the numbers \(e_j\) by the equations \(\log_j e_j = 1:\)

\[
e_0 = 1, \quad e_1 = e, \quad e_2 = e^e, \quad e_3 = e^{e^e}, \ldots
\]

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Modifying a result in [2], we shall prove in section 2 that the formula
\[ L(x) := (1 + x^2)^{1/2} \prod_{k=1}^{\infty} \log(e_k + x^2) = (1 + x^2)^{1/2} \log(e + x^2) \log_2(e^2 + x^2) \ldots \]
defines an everywhere finite, even function of class \( C^\infty \) with \( L(0) = 1 \). Furthermore, \( L(x) \) is increasing for \( x \geq 0 \), and \( L(x) \to \infty \) relatively slowly as \( x \to \infty \), so that
\[ \int_0^\infty \frac{dx}{L(x)} = \infty. \]

Let us also introduce the primitive \( F \) of \( f \) defined by
\[ F(x) = \int_0^x f(s) \, ds, \quad x \in \mathbb{R}. \]

We shall prove the

**Theorem 1.1.** Assume that there exists a positive number \( \beta \) such that
\[ |F(x)| \leq \beta L(x)^2 \quad \text{for all } x. \] (1.2)
If \( T > b - a \), then for any given
\((u_0, u_1), (v_0, v_1) \in H^1(a, b) \times L^2(a, b)\)
there exist control functions
\( h_a, h_b \in H^1(0, T) \)
such that (1.1) has a global solution
\[ u \in C([0, T]; H^1(a, b)) \cap C^1([0, T]; L^2(a, b)) \]
satisfying the final conditions
\[ u(T) = v_0 \quad \text{and} \quad u_t(T) = v_1 \quad \text{in} \quad (a, b). \]

**Remark.** This theorem improves theorem 1.3 in [1] by weakening the growth assumptions on \( f \). Theorem 1.3 in [1] shows that the assumption (1.2) above is essentially optimal: under slightly weaker growth assumptions the system can even become ill posed.

In [1] and [2] we adapted a method of Imanuvilov for the construction of control functions. We are going to modify this approach for the study of controllability by acting at only one end-point. Consider the following problem:

\[
\begin{aligned}
&u_{tt} - u_{xx} - f(u) = 0 \quad \text{in} \quad (a, b) \times (0, T),
&u(a, t) = 0 \quad \text{and} \quad u(b, t) = h_b(t) \quad \text{for} \quad t \in (0, T),
&u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in} \quad (a, b).
\end{aligned}
\] (1.3)

**Theorem 1.2.** Let \( f \) be as in theorem 1.1 and fix \( T > 2(b - a) \). Given
\((u_0, u_1), (v_0, v_1) \in H^1(a, b) \times L^2(a, b)\)
such that
\[ u_0(a) = 0 = v_0(a), \]
there exists a control function \( h_b \in H^1(0, T) \) such that the problem (1.3) has a global solution
\[
u \in C([0, T]; H^1(a, b)) \cap C^1([0, T]; L^2(a, b))
\]
satisfying the final condition
\[
u(T) = v_0 \quad \text{and} \quad u_t(T) = v_1 \quad \text{in} \quad (a, b).
\]

Remark. Of course, an analogous result holds by symmetry if we act at the left endpoint \( a \) instead of \( b \).

For the proof we will require some new technical results.

We shall then study the \( \textit{internal} \) controllability problem
\[
\begin{cases}
u_{tt} - \nu_{xx} - f(u) = h & \text{in} \quad (a, b) \times (0, T), \\
u(a, t) = u(b, t) = 0 & \text{for} \quad t \in (0, T), \\
u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 & \text{in} \quad (a, b).
\end{cases}
\] (1.4)

Applying theorems 1.1 and 1.2 we shall prove the

\textbf{Theorem 1.3.} Let \( f \) be as in theorem 1.1. Fix two points \( a' \), \( b' \) such that
\[
a < a' < b' < b
\]
and set
\[
T > 2 \max\{a' - a, b - b'\}.
\]

Given
\[
(u_0, u_1), \quad (v_0, v_1) \in H^1_0(a, b) \times L^2(a, b)
\]

arbitrarily, there exists a control function
\[
h \in C([0, T]; L^2(a, b))
\]
vanishing outside of \([a', b'] \times [0, T]\) and such that (1.4) has a global solution
\[
u \in C([0, T]; H^1_0(a, b)) \cap C^1([0, T]; L^2(a, b))
\]
satisfying
\[
u(T) = v_0 \quad \text{and} \quad u_t(T) = v_1 \quad \text{in} \quad (a, b).
\]

Theorem 1.3 generalizes substantially theorem 1.3 in [2], by considering controls of arbitrarily small support.

2. \textbf{Some preliminary results.} First we establish some properties of the function \( L \) mentioned in the introduction. Let us observe that \( (e_j) \) is a strictly increasing sequence of positive numbers, rapidly tending to infinity. Note that
\[
e_0 = 1 \quad \text{and} \quad \log_j e_l = e_{l-j} \quad \text{for all} \quad l \geq j \geq 0.
\] (2.1)

We have the following variant of a proposition in [2].

\textbf{Proposition 2.1.} The formula (1.2) defines an even, everywhere finite, \( C^\infty \) function \( L(x) \) which is increasing for \( x \geq 0 \). We have \( L(x) \geq L(0) = 1 \) for all \( x \) and
\[
\int_0^\infty \frac{dx}{L(x)} = \infty.
\] (2.2)
Furthermore, for every $\alpha > 0$ and $\delta > 0$ there exists a constant $c(\alpha, \delta) > 0$ such that
\[
L(x)^2 \leq \delta |x|^{2+2\alpha} + c(\alpha, \delta) \quad \text{for all } x.
\] (2.3)

Finally, the function $\ell(x) := (1 + x^2)^{-1/2}L(x)$ satisfies the inequality
\[
\ell(ab) \leq C\ell(a) + C\ell(b)
\]
with a suitable constant $C > 0$, for all real numbers $a$ and $b$.

**Proof.** The same properties were proved in [2] for a slightly different function $L$. Most of the proofs remain valid with minor changes. (In lemma 2.2 we need $0 \leq x \leq \sqrt{e^2 - e}$ instead of $0 \leq x \leq e^2 - e$.) Since the infinite product defining $L$ converges locally uniformly and since its terms are analytic, $L$ is a $C^\infty$ function. The only part which needs a little more modification is the divergence of the integral
\[
\int_0^\infty \frac{dx}{L(x)}.
\]
Assume on the contrary that the integral converges. Then
\[
\int_0^\infty \frac{dx}{L(x)} \to 0 \quad \text{as } n \to \infty.
\] (2.4)

Performing the change of variable $x = e^t$ we obtain the equalities
\[
\int_{e_n}^{\infty} \frac{dx}{L(x)} = \int_{e_n}^{\infty} \frac{e^t \, dt}{L(e^t)} = \int_{e_n}^{\infty} \frac{e^t \, dt}{(1 + e^{2t})^{1/2} \prod_{k=1}^\infty \log_k(e^k + e^{2t})}.
\]

Observe that
\[
\frac{e^{2t}}{1 + e^{2t}} \geq \frac{e^{2e_n}}{1 + e^{2e_n}} = \frac{e^{2}^{n+1}}{1 + e^{2}^{n+1}} \geq 1 - e^{-2} \geq (1 - e_{n+1})^2
\]
and
\[
\log_k(e^k + e^{2t}) \leq \log_k(e^k e^{2t}) = \log_{k-1}(e^{k-1} + 2t) \leq \log_{k-1}(e^{k-1} + t^2)
\]
for all $k \geq 1$ and $t \geq e_n$ (because $e_k e^{2t} \geq 2$ and $t^2 \geq 2t$). Therefore we deduce from the above equalities the inequalities
\[
\int_{e_n}^{\infty} \frac{dx}{L(x)} \geq (1 - e_{n+1}) \int_{e_n}^{\infty} \frac{dt}{L(t)}.
\]
It follows by induction that
\[
\int_{e_n}^{\infty} \frac{dx}{L(x)} \geq \left( \prod_{j=2}^n (1 - e_j^{-1}) \right) \int_{e_1}^{\infty} \frac{dx}{L(x)}
\]
for $n = 2, 3, \ldots$.

Since the series $\sum e_j^{-2}$ clearly converges (because $e_j \to \infty$ very quickly) and since every $e_j$ is greater than 1, we have
\[
A := \prod_{j=2}^\infty (1 - e_j^{-1}) > 0
\]
and therefore
\[
\int_{e_n}^{\infty} \frac{dx}{L(x)} \geq A \int_{e}^{\infty} \frac{dx}{L(x)} > 0.
\]
follows that

Using also the above Sobolev imbedding, we conclude that

\[ \|L(u)\| \leq \varepsilon \|\nabla u\| + c(\varepsilon, \Omega) L(\|u\|) \] (2.5)

for all \( u \in H^1(\Omega) \), where \( \| \cdot \| \) denotes the usual norm of \( L^2(\Omega) \).

**Proof.** Fix \( N^* > 2 \) such that \( H^1(\Omega) \subset L^{N^*}(\Omega) \) and set \( \gamma = 4(N^* - 1)/N^* \). Then \( \gamma > 2 \) and

\[ \frac{1}{\gamma} = \frac{\gamma - 2}{2} + \frac{1}{\gamma N^*}, \]

so that applying the interpolational inequality we have

\[ \int_\Omega |u|^\gamma \ dx = \|u\|_\gamma^\gamma \leq \|u\|_{\gamma^2}^2. \]

Using also the above Sobolev imbedding, we conclude that

\[ \int_\Omega |u|^\gamma \ dx \leq S(\Omega)\|u\|_{\gamma^2}^2 \|\nabla u\|^2 + S(\Omega)\|u\|^\gamma \] (2.6)

with a constant \( S(\Omega) \) depending only on \( \Omega \).

Given \( \delta > 0 \) arbitrarily, by proposition 2.1 there exists a constant \( c(\delta) > 0 \) such that

\[ L(x)^2 \leq \delta |x|^\gamma + c(\delta) \]

for all real \( x \). Using the above estimate and denoting by \( |\Omega| \) the volume of \( \Omega \), it follows that

\[ \|L(u)\|^2 \leq \int_\Omega \delta |u|^\gamma + c(\delta) \ dx \leq \delta S(\Omega)\|u\|_{\gamma^2}^2 \|\nabla u\|^2 + \delta S(\Omega)\|u\|^\gamma + c(\delta) |\Omega|. \]

Since \( L \geq 1 \) everywhere, in case \( \|u\| \leq 1 \) hence we deduce the estimate

\[ \|L(u)\|^2 \leq \delta S(\Omega)\|\nabla u\|^2 + (\delta S(\Omega) + c(\delta) |\Omega|) L(\|u\|)^2. \] (2.7)

This proves (2.5) in the special case \( \|u\| \leq 1 \). (Choose \( \delta = \varepsilon^2/S(\Omega) \).

Henceforth assume that \( \|u\| > 1 \). Setting \( v := \frac{u}{\|u\|} \) and applying the inequality \( \ell(ab) \leq C\ell(a) + C\ell(b) \),

from proposition 2.1, we have

\[
\int_\Omega L(u)^2 \ dx = \int_{\|u\| \leq 1} L(u)^2 \ dx + \int_{\|u\| > 1} L(u)^2 \ dx \\
\leq |\Omega| L(\|u\|)^2 + \int_{\|u\| > 1} (1 + u^2) \ell(u)^2 \ dx \\
\leq |\Omega| L(\|u\|)^2 + 2C^2 \int_{\|u\| \leq 1} (1 + u^2) \ell(u)^2 \ dx + 2C^2 \int_{\|u\| > 1} (1 + u^2) \ell(\|u\|)^2 \ dx \\
=: |\Omega| L(\|u\|)^2 + 2C^2 I_1 + 2C^2 I_2. \tag{2.8}
\]
Since \( \|u\| > 1 \) implies that
\[
1 + u^2 \leq \|u\|^2(1 + v^2),
\]
we have
\[
I_1 \leq \|u\|^2 \int_{\|u\| > \|u\|} L(v)^2 \, dx \leq \|u\|^2 \|L(v)\|^2.
\]
Furthermore, since \( |u| > \|u\| > 1 \) implies that
\[
1 + u^2 \leq 2u^2,
\]
we have
\[
I_2 \leq 2L(\|u\|)^2.
\]
Substituting them into (2.8) we find that
\[
\|L(u)\|^2 \leq (|\Omega| + 4C^2) L(\|u\|)^2 + 2C^2 \|u\|^2 \|L(v)\|^2.
\]
Applying (2.7) for \( v \) and using the inequality \( L(x) \geq |x| \) hence we obtain that
\[
\|L(u)\|^2 \leq (|\Omega| + 4C^2)L(\|u\|)^2 + 2C^2 \|u\|^2 (\delta S(\Omega) \|v\|^2 + (\delta S(\Omega) + c(\delta)|\Omega|) L(1)^2)
\leq 2C^2 \delta S(\Omega) \|v\|^2 + \{\|\Omega| + 4C^2 + 2C^2 (\delta S(\Omega) + c(\delta)|\Omega|) L(1)^2\} L(\|u\|)^2.
\]
Choosing \( \delta = c^2/(2C^2 S(\Omega)) \) the lemma follows. \( \square \)

**Remark.** In the one-dimensional case \( \Omega = (a, b) \) we may choose \( N^* = \infty, \gamma = 4 \) and
\[
S(\Omega) = \max\{b - a, 1/(b - a)\}
\]
in (2.6). It follows in particular that if the length of \( \Omega \) remains between two fixed positive constants, then \( S(\Omega) \) may be chosen uniformly with respect to \( \Omega \). Indeed, it suffices to establish the estimate
\[
\|u\|_\infty \leq (b - a)^{-1/2} \|u\|_2 + (b - a)^{1/2} \|u'\|_2.
\]
(2.9)
Now, for \( x, y \in \Omega \) given arbitrarily, we have
\[
|u(x)| \leq |u(y)| + \left| \int_x^y u' \, dx \right| \leq |u(y)| + \|u'\|_1.
\]
Integrating by \( y \) in \( \Omega \) and then applying the Hölder inequality we obtain that
\[
(b - a)|u(x)| \leq \|u'\|_1 + (b - a)\|u'\|_1 \leq (b - a)^{1/2}\|u'\|_2 + (b - a)^{3/2}\|u'\|_2.
\]
Since the right-hand side does not depend on \( x \), hence (2.9) follows.

Thanks to proposition 2.1 we have the following special case of a Gronwall type lemma (theorem 2.1) proved in [1]:

**Lemma 2.3.** Let \( \varphi : [0, T] \rightarrow \mathbb{R} \) be a continuous, nonnegative function, satisfying for some numbers \( A, B, C > 0 \) the inequalities
\[
\varphi(t) \leq A + B \int_0^t (C + \varphi(s)) \, ds \quad \text{for all} \quad t \in [0, T].
\]
Then \( \varphi \) is bounded on \( [0, T] \) by a constant depending only on \( A, B, C \) and \( T \).
Now we are ready to generalize a theorem concerning the well-posedness of the Cauchy problem
\[
\begin{aligned}
    u_{tt} - u_{xx} - f(u) &= 0 \quad \text{in} \quad Q, \\
    u(0, \cdot) &= u_0 \quad \text{and} \quad u_t(0, \cdot) = u_1 \quad \text{in} \quad (-d, d),
\end{aligned}
\] (2.10)
where \(Q\) is an isosceles triangle of the form
\[
Q = \{(x, t) \in \mathbb{R}^2 : 0 < t < d, \quad |x| < d - t\}
\]
for some given \(d > 0\). Denoting by
\[
S = \{(x, t) \in \mathbb{R}^2 : 0 < t < d, \quad x = \pm(d - t)\}
\]
the union of its equal sides, we have the

**Proposition 2.4.** Assume that \(f\) satisfies the growth assumption (1.2). Then for every
\[
u_0 \in H^1(-d, d), \quad u_1 \in L^2(-d, d)
\]
the problem (2.10) has a unique solution \(u \in H^1(Q)\) whose traces \(u_t(\cdot, t), u_x(\cdot, t)\) are well defined in \(L^2(t - d, d - t)\) for every \(0 \leq t \leq d\), and the function
\[
t \mapsto \int_{d-t}^{d} u_t^2 + u_x^2 \, dx
\]
is continuous (hence bounded) on \([0, d]\). In particular, \(u \in L^\infty(Q)\).

Finally, the trace of the solution on \(S\) belongs to \(H^1(S)\).

**Proof.** Thanks to lemma 2.3 above, we may repeat the proof of theorem 1.1 and of proposition 5.1 in [1] under the present weaker growth assumption (1.2). Let us correct here a small error in [1]: in the proof of proposition 5.1 the formula (5.3) is incorrect; the correct form is the following:
\[
\sqrt{2} \int_S u_t^2 \, dS = E(0) + \sqrt{2} \int_S F(u) \, dS - 2 \int_{-d}^d F(u) \, dx.
\]
However, the rest of the proof of proposition 5.1 remains valid. \(\square\)

The function \(L\) is also useful in the study of the Cauchy–Goursat problem
\[
\begin{aligned}
    u_{tt} - u_{xx} - f(u) &= 0 \quad \text{in} \quad Q, \\
    u &= \psi \quad \text{on} \quad S, \\
    u_t &= \eta \quad \text{on} \quad S',
\end{aligned}
\] (2.11)
where \(Q\) is an isosceles trapezoid of the form
\[
Q = \{(x, t) \in \mathbb{R}^2 : 0 < t < d', \quad |x| < d - t\}
\]
for some given \(0 < d' < d\),
\[
S' = \{(x, d') \in \mathbb{R}^2 : |x| < d - d'\}
\]
denotes its smaller base, and
\[
S = S' \cup \{(x, t) \in \mathbb{R}^2 : 0 < t < d', \quad x = \pm(d - t)\}
\]
denotes its boundary without the larger base.
Proposition 2.5. Let $f$ satisfy (1.2). Then for every 
\[ \psi \in H^1(S), \quad \eta \in L^2(S'), \]
the problem (2.11) has a unique solution $u \in H^1(Q)$ whose traces $u_t(\cdot,t)$, $u_x(\cdot,t)$ are well defined in $L^2(t-d,d-t)$ for every $0 \leq t \leq d'$, and the function 
\[ t \mapsto \int_{t-d}^{d-t} u_t^2 + u_x^2 \, dx \]
is continuous (hence bounded) on $[0,d']$. In particular, $u \in L^\infty(Q)$.

Remark. In [1] we claimed that the proposition holds true for $d' = d$ when $Q$ becomes a triangle. There was an error in the proof when we stated that $\|u\|_{L^\infty(Q)}$ is bounded by a constant independent of the Lipschitz constant of $f$ if $f$ is globally Lipschitz continuous. Indeed, given a real number $c$, the solution of 
\[ \begin{cases} u_{tt} - u_{xx} + 4cu = 0 & \text{in } Q, \\ u = 1 & \text{on } S \end{cases} \]
is given by the series 
\[ u(t,x) = \sum_{k=0}^{\infty} c^k \frac{(d-t-x)^k}{k!} \frac{(d+t-x)^k}{k!}, \]
so that 
\[ u(0,0) > cd^2 \to \infty \text{ as } c \to \infty. \]

However, all theorems of [1] remain valid (and the proofs become even simpler) by applying the above proposition instead of proposition 5.2 there.

Proof of the proposition. According to the preceding remark, it suffices to prove that if $f$ is globally Lipschitz continuous, then $\|u\|_{L^\infty(Q)}$ is bounded by a constant independent of the Lipschitz constant of $f$. Although this is not true in the triangular case, it is true in the trapezoidal case. In what follows $c$ will denote diverse constants depending on $\|\psi\|_{H^1(S)}$ and $\|\eta\|_{L^2(S')}$ but not on $t$ and on the Lipschitz constant of $f$.

Put 
\[ E(t) := \int_{t-d}^{d-t} \frac{u_t^2 + u_x^2}{2} - F(u) \, dx, \quad (0 \leq t \leq d') \]
for brevity.

Multiplying the differential equation in (2.11) by $u_t$ and integrating by parts we obtain the equality 
\[ E(t) = E(d') + \sqrt{2} \int_{S_t} u_x^2 \, d\sigma \quad (0 \leq t \leq d') \]
where 
\[ S_t = \{(x,s) \in \mathbb{R}^2 : 0 < s < d', \quad x = \pm(d-s)\}. \]

Hence 
\[ \int_{t-d}^{d-t} u_t^2 + u_x^2 \, dx \leq c + c \int_{t-d}^{d-t} L^2(u) \, dx. \]
Applying lemma 2.3 we conclude that
\[
\int_{t-d}^{d-t} u_t^2 + u_x^2 \ dx \leq c + cL^2(\|u(t)\|) \ dx \quad (0 \leq t \leq d'); \tag{2.12}
\]
here and in the sequel we write \( \| \cdot \| \) instead of \( \| \cdot \|_{L^2(t-d,d-t)} \) for brevity.

First of all, (2.12) yields
\[
\|u_t(t)\| \leq c + cL(\|u(t)\|) \ dx \quad (0 \leq t \leq d'). \tag{2.13}
\]
On the other hand, integrating over \([s, d']\) the equality
\[
\frac{d}{dt} \int_{t-d}^{d-t} u^2(t,x) \ dx = -u^2(t, d-t) - u^2(t, t-d) + 2 \int_{t-d}^{d-t} u(t,x)u_t(t,x) \ dx
\]
and then using the Cauchy–Schwarz inequality, we obtain
\[
\|u(s)\|^2 = c + \int_s^{d'} \|u(t)\| \|u_t(t)\| \ dt
\]
for all \(d'' \leq s \leq d'\). Now using (2.13) we have
\[
\|u(s)\|^2 = c + \int_s^{d'} \|u(t)\| \left(c + cL(\|u(t)\|)\right) \ dt \\
\leq c + c \int_s^{d'} \|u(t)\| L(\|u(t)\|) \ dt \leq c + c \int_s^{d'} L(\|u(t)\|^2) \ dt.
\]
Applying lemma 2.3 it follows that \(\|u(t)\|\) is bounded. Then using, (2.1) again, we conclude as in [1] that the function
\[
[0,d'] \ni t \mapsto \int_{t-d}^{d-t} \frac{u_t^2 + u_x^2}{2} \ dx
\]
is bounded, too. \[\square\]

3. **Proof of theorem 1.2.** Let us decompose \(Q := (a,b) \times (0,T)\) into three sub-regions defined by
\[
\begin{align*}
K_1 & := \{(x,t) \in Q : t < b-x\}, \\
K_2 & := \{(x,t) \in Q : T-t < b-x\}, \\
K_3 & := \{(x,t) \in Q : b-x < t < T -(b-x)\}.
\end{align*}
\]
Applying proposition 2.4, the Cauchy problem
\[
\begin{cases}
u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_1, \\
u(a,t) = 0 & \text{for } t \in (0,b-a), \\
u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{in } (a,b)
\end{cases}
\]
has a solution \(u^1\), whose trace on the line segment
\[
S_1 := \{(x,t) \in Q : x + t = b, \ a < x < b\}
\]
belongs to \(H^1(S_1)\).
Similarly, the Cauchy problem
\[
\begin{cases}
  u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_2, \\
  u(a,t) = 0 & \text{for } t \in (T - (b - a), T), \\
  u(T) = v_0 \text{ and } u_t(T) = v_1 & \text{in } (a, b)
\end{cases}
\]
has a solution \( u^2 \), whose trace on the line segment
\[ S_2 := \{(x, t) \in Q : t - x = T - b, a < x < b\} \]
belongs to \( H^1(S_2) \).

Now applying proposition 2.5, the Cauchy–Goursat problem
\[
\begin{cases}
  u_{tt} - u_{xx} - f(u) = 0 & \text{in } K_3, \\
  u(a,t) = u_x(a,t) = 0 & \text{for } t \in (b - a, T - (b - a)), \\
  u = u^1 \text{ on } S_1 \text{ and } u = u^2 \text{ on } S_2
\end{cases}
\]
has a solution \( u^3 \). As in [1], one can readily verify that the formula
\[ u := u^i \text{ in } K_i \text{ for } i = 1, 2, 3 \]
defines a solution of (1.3) with
\[ h_b(t) = u^3(b,t), \quad t \in (0,T). \]
The proof is complete.

4. **Proof of theorem 1.3.** Fix points \( x_i \in (a, b), i = 0, \ldots, n + 1 \), so that
\[ a =: x_0 < a' =: x_1 < x_2 < \cdots < x_n := b' < x_{n+1} := b \]
and
\[ x_{i+1} - x_i < T \quad \text{for } i = 1, \ldots, n - 1. \]

Then, fix a positive number \( \varepsilon \) such that
\[ 2(x_1 - x_0 + \varepsilon) < T, \quad 2(x_{n+1} - x_n + \varepsilon) < T \]
and
\[ x_{i+1} - x_i + \varepsilon < T \quad \text{for } i = 1, \ldots, n - 1. \]

Let us introduce sets \( A_0, \ldots, A_n \) by the formulas
\[
A_0 = [x_0, x_1 + \varepsilon),
A_1 = (x_1, x_2),
A_i = (x_i - \varepsilon, x_{i+1}), \quad i = 2, \ldots, n - 1,
A_n = (x_n - \varepsilon, x_{n+1}].
\]

Since \( A_0, \ldots, A_n \) cover the interval \([a, b]\), there exists a partition of unity
\[ \theta_0, \ldots, \theta_n \in C^\infty([a, b]) \]
satisfying
\[ \theta_i \geq 0 \text{ and } \text{supp } \theta_i \subset A_i \quad \text{for all } i = 0, \ldots, n \]
and
\[ \theta_0 + \cdots + \theta_n = 1. \]
Observe that, in particular,
\[ \theta_0(x) = 1 \quad \text{for all} \quad x \in [x_0, x_1], \]
\[ \theta_n(x) = 1 \quad \text{for all} \quad x \in [x_n, x_{n+1}]. \]

Applying theorem 1.2 on the intervals \( A_0 \) and \( A_n \), there exist control functions \( h_0^0, h_n^0 \in H^1(0, T) \) such that the problem
\[
\begin{aligned}
&u_{tt} - u_{xx} - f(u) = 0 \quad \text{in} \quad A_0 \times (0, T), \\
&u(a, t) = 0 \quad \text{and} \quad u(x_1 + \varepsilon, t) = h_0^0(t) \quad \text{for} \quad t \in (0, T), \\
&u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in} \quad A_0
\end{aligned}
\]
has a solution \( u^0 \) satisfying
\[ u^0(T) = v_0 \quad \text{and} \quad u^0_t(T) = v_1 \quad \text{in} \quad A_0, \]
and the problem
\[
\begin{aligned}
&u_{tt} - u_{xx} - f(u) = 0 \quad \text{in} \quad A_n \times (0, T), \\
&u(x_n - \varepsilon, t) = h_n^0(t) \quad \text{and} \quad u(b, t) = 0 \quad \text{for} \quad t \in (0, T), \\
&u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in} \quad A_n
\end{aligned}
\]
has a solution \( u^n \) satisfying
\[ u^n(T) = v_0 \quad \text{and} \quad u^n_t(T) = v_1 \quad \text{in} \quad A_n. \]

Next, applying theorem 1.1 on each interval \( A_i =: (a_i, b_i), i = 1, \ldots, n - 1 \), there exist control functions \( h_i^1, h_i^2 \in H^1(0, T) \) such that the problem
\[
\begin{aligned}
&u_{tt} - u_{xx} - f(u) = 0 \quad \text{in} \quad A_i \times (0, T), \\
&u(a_i, t) = h_i^1(t) \quad \text{and} \quad u(b_i, t) = h_i^2(t) \quad \text{for} \quad t \in (0, T), \\
&u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in} \quad A_i
\end{aligned}
\]
has a solution \( u^i \) satisfying
\[ u^i(T) = v_0 \quad \text{and} \quad u^i_t(T) = v_1 \quad \text{in} \quad A_i. \]

Finally, define
\[ u(x, t) := \sum_{i=0}^{n} \theta_i(x) u^i(x, t). \]

Let us check that \( u \) has the required properties. First we notice that \( u \) is well defined and has the required regularity in \( x \) and \( t \) by the properties of the functions \( \theta_i \). Moreover,
\[ u(a, t) = 0 = u(b, t) \]
for all \( t \) by construction. Also,
\[
\begin{aligned}
u(0) &= \sum_{i=0}^{n} \theta_i u_0 = u_0, \\
u(T) &= \sum_{i=0}^{n} \theta_i v_0 = v_0, \\
u_t(0) &= \sum_{i=0}^{n} \theta_i u_1 = u_1, \\
u_t(T) &= \sum_{i=0}^{n} \theta_i v_1 = v_1.
\end{aligned}
\]
Let us compute the Dalembertian of $u$:

$$u_{tt} - u_{xx} = \sum_{i=0}^{n} \theta_i u_{tt}^i - \sum_{i=0}^{n} (\theta_i u_{tt}^i)_{xx} = \sum_{i=0}^{n} \theta_i (u_{tt}^i - u_{xx}^i) - \sum_{i=0}^{n} \theta_i'' u_{x}^i - 2 \sum_{i=0}^{n} \theta_i' u_{xx}^i.$$

Therefore

$$u_{tt} - u_{xx} - f(u) = \left[ \sum_{i=0}^{n} \theta_i f(u^i) - f(u) \right] - \sum_{i=0}^{n} [\theta_i'' u_{x}^i + 2 \theta_i' u_{xx}^i] =: h_1 + h_2.$$

Notice that $h_2 = 0$ outside $[a', b'] \times [0, T]$ as $\theta_0$, $\theta_n$ are constant on this set and all other cutoff functions vanish. Moreover, for all $x \in [a, a']$ we have

$$h_1(x, t) = \theta_1 f(u^1) - f(\theta_1 u^1) = f(\theta_1 u^1) - f(\theta_1 u^1) = 0.$$

Similarly, for all $x \in [b', b]$ we have

$$h_1(x, t) = \theta_n f(u^n) - f(\theta_n u^n) = 0.$$

We have thus proved that $h := h_1 + h_2$ has its support contained in $[a', b']$ and the proof is complete.

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