Parity Effects and Higher Order Tunneling in Superconducting SET Transistors

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Abstract

Single electron tunneling into small superconducting islands is sensitive to the gap energy of the excitations created in the process and, hence, depends on the parity of the electron number in the island. We study these effects by analyzing the kinetics of the system. Since the interplay of Coulomb blockade and parity effects leads to a blocking of single electron tunneling, higher order tunneling processes become important. This is well-established for two-electron tunneling. Here we study processes of third and fourth order. We estimate the order of magnitude for these processes and discuss their relevance for recent experiments.

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I. INTRODUCTION

Experiments with systems including small superconducting islands showed effects which depend on the electron number parity in the island [1–4]. The effect has been interpreted as the even-odd asymmetry predicted by Averin and Nazarov [5]. It arises since single electron tunneling from the ground state, where all electrons near the Fermi surface of the superconducting island are paired, leads to a state with one extra electron - the "odd" one - in an excited state. In a small island, where charging effects prevent further tunneling, the odd electron does not find another excitation for recombination. Hence the energy of this state stays (at least metastable) above that of the equivalent normal system by the gap energy. Only at larger gate voltages another electron can enter the island, and the system can relax to the ground state.

At higher temperatures, above a crossover value \(T_{cr}\), only the \(e\)-periodic behavior typical for normal metal systems has been observed. This crossover has been explained by free energy arguments in Ref. [1]. We had developed an alternative approach by studying the kinetics of the system [6]. Thus, we can describe nonequilibrium situations and derive the \(I-V\) characteristics of driven systems such as the SET transistors. We analyze the rate of tunneling of electrons between the lead and the island (reviewed in Section II) paying particular attention to the tunneling of the “odd” electron (or the tunneling of an electron into the partner state of the odd electron, resulting in an instantaneous condensation into a Cooper pair). The well studied single electron tunneling with rate \(\Gamma\) between a superconducting island and lead electrodes depends on the difference in the energy before and after the process. This includes the charging energy, but also the excitation energy \(\Delta_i\) in the island. At low temperature processes which cost energy are suppressed. This renders the tunneling of the excited “odd” electron important. The tunneling rate \(\gamma\) of this single electron is smaller by a factor \(1/N_{eff}\) than the rate \(\Gamma\). Here \(N_{eff}\) is the effective number of quasiparticle states available for an excitation at low temperature; in mesoscopic islands it is typically of the order of \(10^4\). On the other hand, in an important range of parameters the rate \(\gamma\) is not exponentially suppressed, since the excitation energy in the island is regained in this tunneling process. Hence \(\gamma \approx \Gamma e^{\Delta_i/k_B T}/N_{eff}\). Parity effects are observable as long as this single electron tunneling rate is relevant \(\gamma > \sim \Gamma\), from which we obtain directly the cross-over temperature \(k_B T_{cr} \approx \Delta_i/\ln N_{eff}\).

Our analysis of the tunneling rates can be applied to derive the \(I-V\) characteristics of normal-superconducting-normal (NSN) transistors (Section III). In NSN transistors we combine single electron tunneling and 2\(e\)-tunneling via Andreev reflection to obtain a richly structured \(I-V\) characteristic, which reproduces quantitatively many features observed in the experiments of Refs. [3,4,7]. Our \(I-V\) characteristic shows wide regions where both single and two-electron tunneling are blocked due to a combination of the superconducting gap and the Coulomb blockade, even at comparatively high transport voltages. The earlier experiments [4] showed more structure, which could be identified as arising from the influence of the electromagnetic environment and could be removed by improved shielding [7]. Even the latest experiments, however, show more structure than the theoretical results which account for single and two-electron tunneling. In particular the experiments show a ridge-like structure with a period \(e\) in the gate charge \(Q_g\) at voltages \(eU_{tr} \approx \Delta_i\). Cotunneling
processes are expected to produce an unstructured background rather than an effect which depends in a pronounced way on the gate voltage. Hence, the question arises whether higher than second order processes noticeably contribute to the characteristic. Here we consider coherent three-electron tunneling (Section IV). The process which turns out to be important is a coherent combination of a single electron transition and an Andreev reflection. The threshold voltage for this process and the order of magnitude of the rate match well with experimental results.

Another interesting process is cotunneling of two electrons. As this process is of fourth order in the junction conductance its contribution is usually small compared to lower order processes. However, in the limit of small \( U_{tr} \) both single electron tunneling and Andreev reflection (as well as their coherent combination) are suppressed due to the superconducting gap and Coulomb blockade. In this case cotunneling processes play the leading role. One of these processes - single electron cotunneling - has been investigated in Ref. [5]. The process of two-electron cotunneling, i.e. the process of two coherent Andreev reflection events will be discussed in Section V. We will show that in the limit of low \( T \) and \( U_{tr} \) either one- or two-electron cotunneling processes dominate depending on the system parameters. A discussion of our results is presented in Section VI.

II. SINGLE ELECTRON TUNNELING RATES

In order to review some basic results we consider an electron box [2] assuming \( \Delta_i < E_C \). The results are easily extended to cover transport in SET transistors and systems with \( \Delta_i > E_C \). The electron box is a closed circuit consisting of a small superconducting island with total capacitance \( C = C_J + C_g \) connected by a tunnel junction with capacitance \( C_J \) to a lead electrode and by a capacitance \( C_g \) to a voltage source \( U_g \). The charging energy of the system depends on \( Q_g = C_g U_g \) and the number \( n \) of charges on the island

\[
E_{ch}(n, Q_g) = \frac{(ne - Q_g)^2}{2C}.
\]  

The normal state conductance of the junction can be expressed by the tunneling matrix element and the normal density of states \( N_{i/l}(0) \) and volume \( V_{i/l} \) of the island and lead:

\[
1/R_t = 4\pi e^2 N_{i/l}(0)V_{i/l}(0)V_l|T|^2/\hbar
\]

The transition rate \( \Gamma^+ \) for a single electron tunneling (SET) process from the lead to the island, where the number of excess charges on the island changes from \( n \) to \( n + 1 \), is

\[
\Gamma^+(Q_g) = \frac{1}{e^2 R_t} \int_{-\infty}^{\infty} d\epsilon \int_{-\infty}^{\infty} d\xi N_i(\epsilon) f_l(\xi)[1 - f_i(\epsilon)] \delta(\xi - \epsilon - \delta E_{ch}(Q_g)) .
\]  

The \( \delta \)-function expressing energy conservation depends on the change of the charging energy before and after the process

\[
\delta E_{ch}(Q_g) = E_{ch}(n + 1, Q_g) - E_{ch}(n, Q_g).
\]

If the distribution functions of lead and island are equilibrium Fermi functions the expression for the rate reduces to
\[
\Gamma^+(Q_g) = \frac{1}{e} I_t(\delta E_{ch}(Q_g)) \frac{1}{\exp[\delta E_{ch}(Q_g)/k_B T] - 1}.
\] (3)

The SET tunneling rate depends on the difference in the charging energy. In the superconducting state it depends further on the energy gap, which enters via the BCS densities of states \( N_i(\epsilon) = \Theta(|\epsilon| - \Delta_i)|\epsilon|/\sqrt{\epsilon^2 - \Delta_i^2} \) of the island into the well known quasiparticle tunneling characteristic \( I_t(eV) \) [10].

As long as the distributions are equilibrium Fermi functions the rate for the reverse process \( \Gamma^- \) is given by the same expression as (3), however the sign of \( \delta E_{ch} \) is reversed. Both satisfy the condition of detailed balance \( \Gamma^-(Q_g) = \Gamma^+(Q_g)e^{\delta E_{ch}/k_B T} \). At low temperature in the superconducting state the rates \( \Gamma^\pm \) are large only if the gain in charging energy exceeds the sum of the energies of the excitation created in the island \( \pm \delta E_{ch} + \Delta_i < 0 \). They are exponentially suppressed otherwise.

The assumption of equilibrium Fermi distributions is sufficient as long as we start from the even state. For definiteness let us assume that we started from \( n = 0 \) and that the gate voltage is chosen such that \( |Q_g| \leq e \). Hence, the rate of tunneling from an even to an odd state is

\[
\Gamma^{oe}(Q_g) = \Gamma^+(Q_g).
\] (4)

However, in a superconductor with an “odd” unpaired electron, occupying a quasiparticle state above the gap, the distribution differs from an equilibrium one. Also the odd electron can tunnel back to the lead, enhancing the tunneling from odd to even states. Its initial energy in the island is at least \( \Delta_i \). This makes its tunneling rate large in a range of gate voltages where the competing processes, the tunneling of all the other electrons described by \( \Gamma^- \), are still exponentially suppressed. At finite temperature it is reasonable to assume that the odd state of the island is described by a thermal Fermi distribution but with a shifted chemical potential \( f_{\delta \mu}(\epsilon_i) = [e^{(\epsilon - \delta \mu)/T} + 1]^{-1} \). The shift in chemical potential is fixed by the constraint

\[
1 = N_i(0)V_i \int_{-\infty}^{\infty} d\epsilon N_i(\epsilon)[f_{\delta \mu}(\epsilon) - f_0(\epsilon)].
\] (5)

This reduces at low temperatures to

\[
\delta \mu = \Delta_i - T \ln N_{eff},
\] (6)

where

\[
N_{eff}(T) = N_i(0)V_i \sqrt{2\pi \Delta_i T}
\] (7)

is the number of states available for quasiparticles near the gap [1]. The tunneling rate from the odd state to the even state \( \Gamma^{oe} \) is given by the expression (5), however the island distribution function is replaced by \( f_{\delta \mu}(\epsilon) \). For the following discussion it is useful to decompose this rate as

\[
\Gamma^{oe}(Q_g) = \Gamma^-(Q_g) + \gamma(Q_g),
\] (8)
where $\Gamma^-$ has been defined above, and

$$\gamma(Q_g) = \frac{1}{e^2 R_i} \int_{-\infty}^{\infty} d\epsilon_i \int_{-\infty}^{\infty} d\xi_i \mathcal{N}_i(\epsilon_i)[f_{\delta\mu}(\epsilon_i) - f_0(\epsilon_i)][1 - f_0(\xi_i)]\delta(\xi_i - \epsilon_i - \delta E_{ch}) .$$

(9)

In the important range of parameters the rate $\gamma$ reduces to

$$\gamma(Q_g) = \frac{1}{2e^2 R_i N(0) V_i} \begin{cases} 1 & \text{for } \Delta_i + \delta E_{ch} > T \\ e^{\frac{\Delta_i + \delta E_{ch}}{kT}} & \text{for } -\delta E_{ch} - \Delta_i > T \end{cases}$$

(10)

In comparison to $\Gamma^-$ the odd electron tunneling rate $\gamma$ contains a small prefactor $1/N_{eff}(T)$. On the other hand, there exists a range of gate voltages at low temperature where $\gamma$ is larger than $\Gamma^-$, since it is not exponentially suppressed. The reason is that the gap energy of the island is regained in the process described by $\gamma$.

For $\exp(-\Delta_i/T) \ll 1$ the ratio of the rates for two transitions is

$$\Gamma^{oe}/\Gamma^{eo} = \exp[(\delta E_{ch} + \delta \mu)/k_BT] .$$

(11)

The rates $\Gamma^{eo}$ and $\Gamma^{oe}$ obey a detailed balance relation, however, depending on the free energy difference, which in addition to charging energy involves the shift of the chemical potential $\delta \mu$ as well. This free energy difference coincides with that introduced in Ref. [1].

Above we described the range $0 \leq Q_g \leq e$ where tunneling occurs between the island states $n = 0$ and $n = 1$. The range $e \leq Q_g \leq 2e$ can be treated analogously. The tunneling now connects the states $n = 1$ and $n = 2$. In this case, except for the single electron tunneling $\Gamma^\pm$, which creates further excitations, one electron can tunnel into one specific state $(-k, -\sigma)$, the partner state of the excitation $(k, \sigma)$ which is already present. Both condense immediately; the state with two excitations only exists virtually. The latter process is described again by $\gamma(Q_g)$. The symmetry implies $\Gamma^{eo/oe}(Q_g) = \Gamma^{eo/oe}(2e - Q_g)$. Since the properties of the system are $2e$-periodic in $Q_g$ we have provided the description for all values of the gate voltage.

Given the rates $\Gamma^{eo}$ and $\Gamma^{oe}$ we can analyze where the transition between the even and the odd state occurs. We can describe the incoherent sequential tunneling of charges between the island and the lead by a master equation for the occupation probabilities of the even and odd states $W_e(Q_g)$ and $W_o(Q_g)$. It is

$$\frac{dW_e(Q_g)}{dt} = -\Gamma^{eo}(Q_g)W_e(Q_g) + \Gamma^{oe}(Q_g)W_o(Q_g)$$

(12)

with $W_e(Q_g) + W_o(Q_g) = 1$. The equilibrium solutions are

$$W_{e(o)}(Q_g) = \Gamma^{oe eo}(Q_g)/\Gamma_{\Sigma}(Q_g) ,$$

(13)

where $\Gamma_{\Sigma}(Q_g) = \Gamma^{oe}(Q_g) + \Gamma^{eo}(Q_g)$. For $\Gamma^{oe} \gg \Gamma^{eo}$ we have $W_e(Q_g) \approx 1$, i.e. the system occupies the even state, while for $\Gamma^{eo} \gg \Gamma^{oe}$ the island is in the odd state. Finally, the crossover temperature $T_{cr}$ is determined by $\gamma(e/2) \approx \Gamma[\delta E^+(e/2)]$. The result

$$T_{cr} = \Delta_i/\ln N_{eff}(T_{cr}) ,$$

(14)

coincides with that found in Refs. [98]. $T_{cr}$ can be interpreted as the temperature, at which the average number of excitations in the island is at least one. This result is valid also for SET transistors.
III. I – V CHARACTERISTICS OF SET TRANSISTORS

The analysis presented above can be extended in a straightforward way to describe even-odd effects in SET transistors, which consist of an island coupled to two tunnel junctions (left and right) and a gate capacitance. In this system the charging energy depends on the gate voltage $U_g$ and the transport voltage. The total capacitance of the island $C = C_g + C_l + C_r$ defines the energy scale $E_C = e^2/2C$. The energy differences for tunneling processes onto the island in the left and right junctions are

$$\delta E_{ch,l/r} = E_C - \frac{e(Q_g \pm Q_{tr}/2)}{C}. \quad (15)$$

For clarity we assume a symmetric bias $U_l = -U_r = U_{tr}$ and $C_l = C_r$. We further define $Q_{tr} = CU_{tr}$.

We focus our attention on NSN transistors with an energy gap which is larger than the charging energy $\Delta_i > E_C$. In these systems the odd states have a large energy. Hence a mechanism which transfers 2 electrons between the normal metal and the superconductor becomes important. The Andreev reflection provides such a mechanism [11]. The master equation description can be generalized to include also this process. In the limit considered the rate for Andreev reflection is given by the same expression as that for single charge tunneling [2] with the following modifications [12]: (i) the charge transferred in an Andreev reflection is $2e$, and the charging energy changes accordingly, (ii) the expression for $I_t(V)$ is linear as for normal state tunneling, but (iii) the effective conductance is of second order $G_A = R_K/(4N_{ch}R_t^2)$. Here $R_K = h/e^2$ is the resistance quantum, and the number of channels $N_{ch} \approx 10^3$ depends on the correlations of the electron propagation in the lead and the island [13,14]. An important conclusion is that Andreev reflection is also subject to Coulomb blockade [12]. Because of the similarity of the rate to that of single electron tunneling it is clear that the shape of the $I – V$ characteristic due to Andreev reflection also takes a similar form as in normal transistors. At low temperatures a set of parabolic current peaks is found centered around the degeneracy points $Q_g = \pm e, \pm 3e, \ldots$ [11]

$$I_A^A(\delta Q_g; U_{tr}) = G_A^A(U_{tr} - 4(\delta Q_g)^2/U_{tr}C^2). \quad (16)$$

Here $\delta Q_g = Q_g - e$ for $Q_g$ close to $e$ and similar at the other degeneracy points.

Depending on the gate and transport voltage we find different mechanisms to dominate the $I – V$ characteristic. Upon increasing $U_{tr}$ we enter a regime where single electron tunneling dominates. This “poisons” or blocks the condition for Andreev scattering [11]. Hence after an initial increase with increasing $U_{tr}$ the current suddenly drops at the threshold voltage

$$eU_{tr} = 2 \left( E_C - \frac{eQ_g}{C} + \Delta_i \right) \quad (17)$$

to the value limited by the odd electron tunneling, i.e. an “escape” current of the order

$$I_{esc} \sim \frac{1}{2eR_tN_i(0)V_i} \quad (18)$$
For a detailed comparison of our calculations with available experimental data from Refs. [7] we solved the master equation using the parameters of the experiments.

Fig. 1 shows the $I-V$ characteristic as a function of both gate and transport voltage. At small transport voltage we find $2e$-periodic peaks due to Andreev reflection; the peaks at larger transport voltages arise due to subsequent incoherent steps of single electron tunneling and Andreev reflection processes (which we will refer to as AQP cycles [7]). As can be seen in Fig. 1 there are regions of $Q_g$ where even for transport voltages $U_{tr} \gtrsim \Delta_i/e$ both single electron and two-electron tunneling is blocked. In the experiment, however, a current is found in these regions. Especially, one finds ridge-like structures at $Q_g = 0, \pm 1, \ldots$ which start at a transport voltage of $U_{tr} \sim 250 \mu eV$. We argue that these structures cannot be explained by the presence of several unpaired quasiparticles or inelastic cotunneling. Consider the system at $Q_g = 0$. There is no process of first or second order to alter the number of electrons in the island, hence, it cannot be explained why unpaired quasiparticles should appear on the island at the considered transport voltages. Moreover, increasing the gate voltage one approaches the AQP peaks and the probability to have unpaired quasiparticles increases, so one expects that a correspondingly large background has to be added to the current due to AQP processes. The experiment, however, indicates that instead of such a background a well-defined ($e$-periodic) structure is added to the features in Fig. 1. This motivated us to investigate higher order processes.

IV. COHERENT THREE-ELECTRON TUNNELING

An argument in selecting the important third-order contributions is that the processes may neither produce a large number of excitations nor change the number of electrons on the island considerably. This favors processes which are coherent combinations of two-electron tunneling in one junction and quasiparticle tunneling in the other (see the discussion below).

The calculation of the rate is analogous to the calculation of the two-electron tunneling rate [11]. The rate for a process with two-electron tunneling in the left junction and one quasiparticle tunneling in the right can be expressed as

$$\Gamma^{(3)} = \frac{2\pi}{\hbar} \sum_{ll'kr} |M_{ll'kr}|^2 f(\xi_l)f(\xi_{l'})(1 - f(\epsilon_k))(1 - f(\xi_r)) \times$$

$$\times \delta(\xi_r + \epsilon_k - \xi_l - \xi_{l'} - \frac{3}{2}eU_{tr} + \delta E_{ch}) .$$  \hspace{1cm} (19)

It contains the Fermi functions for electrons with energies $\xi_l$, $\xi_{l'}$ in the left lead, $\xi_r$ in the right lead and for excitations with energy $\epsilon_k$ on the island. We note the important fact that the process has a threshold voltage

$$eU_{thr} = \frac{2}{3}(\Delta_i + \delta E_{ch}) .$$  \hspace{1cm} (20)

Below this threshold the rate is exponentially suppressed. The matrix element $M_{ll'kr}$ has the form
\[ M_{l'k'} = \sum_{k'} \frac{T_{l-k'}^{(l)} T_{l_k}^{(l)} T_{r_k}^{(r)} v_{l'} u_{k'} v_k}{(\epsilon_k + \epsilon_{k'} + \xi_r - \xi_l - eU_{tr})(\epsilon_k + \xi_r + \delta E_{ch}(-e) - eU_{tr}/2)} + 5 \text{ other terms} \]  

where \( u_k, v_k \) denote the coherence factors in the island and \( T^{(l/r)} \) the tunneling matrix elements of the left and right junctions. The combination of energy denominators written explicitly corresponds to quasiparticle tunneling in the first and Andreev tunneling in the second and third coherent step. We assume \( U_{tr} \approx U_{thr} \) and therefore set \( \xi_l, \xi_{l'}, \xi_r \approx 0 \) and \( \epsilon_k \approx \Delta_i \). Since we expect the rate to be important at integer values of \( Q_g/e \), we replace \( \delta E_{ch} \) by \( E_C \). Carrying out the integrations and estimating the expressions for the parameters of the experiment we find

\[ M_{l'k'} \approx 4 \left\langle T_{l-k'}^{(l)} T_{l_k}^{(l)} T_{r_k}^{(r)} v_k N_i(0) V_i \right\rangle E_C, \]  

where the bracket \( \langle \ldots \rangle_p \) denotes averaging over the directions of \( p \).

In the next step, we calculate the voltage dependence of the rate, assuming that \( T \) is much less than any other energy difference. The result is

\[ \Gamma^{(3)} = \eta \cdot \left\{ 1 \left[ \left( \frac{3}{2} eU_{tr} - \delta E_{ch} \right) \frac{\Delta_i}{\Delta_i} \right]^2 + 2 \right\} \sqrt{\left( \frac{3}{2} eU_{tr} - \delta E_{ch} \right) \frac{\Delta_i}{\Delta_i}} - 1 \]

\[ - \left( \frac{3}{2} eU_{tr} - \delta E_{ch} \right) \arccosh \left( \frac{3}{2} eU_{tr} - \delta E_{ch} \right) \} . \]  

where

\[ \eta = \frac{4\pi}{\hbar} \langle |M_{l'k'}|^2 \rangle W_{l'k'} N_i(0)^2 V_i^2 N_r(0) V_i N_i(0) V_i \Delta_i^3 \]  

The momentum average \( \langle M_{l'k'} \rangle_{W_{l'k'}} \) is proportional to the product of the quasiparticle conductance of the right junction and the Andreev conductance of the left junction. Using the experimental parameters, without further assumptions concerning the averages of the tunneling matrix elements, we estimate \( \eta \sim 10^6 \text{s}^{-1} \). This compares well with the experiments.

The master equation is completed with the rates \( \Gamma^{(3)} \) for the various processes which change the island charge by \( \pm e \). The result is shown in Fig. 2. As we have expected there are no longer regions with blockade of the current at higher transport voltages. We also note that the ratio of the heights of Andreev and AQP peaks is changed. The simulation, however, does not reproduce the peaked structure of the experimental results, especially the ridges at \( Q_g = 0, \pm 1, \ldots \) are missing.

Finally, we return to the discussion of the criteria to select the important three-electron contribution. Other possible realizations are combinations of single electron tunneling and Andreev reflection in the same junction, coherent tunneling of three electrons with the creation of three quasiparticles in the island (tunnel events at the same or at different junctions) and combinations of elastic cotunneling and single electron tunneling. The first and the second of these realizations can be excluded, since they yield much higher threshold
voltages than given in (20). Furthermore, the experiment shows that elastic cotunneling plays no role for the $I-V$ characteristic at transport voltages $eU_{tr} \sim \Delta_i$. This is confirmed by an estimate using the theory of Ref. [4]. Hence, higher order processes including elastic cotunneling can be neglected.

V. TWO-ELECTRON COTUNNELING

As we have discussed above, the 2-electron tunneling process through NS interfaces via Andreev reflection is to a large extent similar to the process of single electron tunneling between two normal metals. We can proceed further with this analogy and study the process of two-electron cotunneling in NS transistors (similarly to the process of single electron cotunneling in normal tunnel junctions [15-18]). Let us consider two successive Andreev reflection events at two NS boundaries of the NSN transistor. One of these events corresponds to tunneling of two electrons from a normal metal into a superconducting island through one of the NS boundaries (with conversion of two normal electrons into a Cooper pair). For small values of the external voltage and at low $T$ this process is energetically suppressed because it increases the charging energy of the superconducting island by $\sim 4E_C$. The island, however, can be immediately discharged due to another coherent tunneling event of a Cooper pair from the island into the normal metal through the second NS boundary. Thus Coulomb interaction induces time correlation between two acts of Andreev reflection at two NS interfaces and the Coulomb blockade is lifted for such a process similarly to the case of single electron cotunneling. For a nonzero transport voltage $U_{tr}$ the above process provides a finite contribution to the current through the system. This contribution is proportional to $R_t^{-4}$ but nevertheless it can become significant in the limit of small transport and gate voltages, i.e. in the case of Coulomb blockade of Andreev reflection at each of the two NS interfaces.

In order to evaluate the two-electron cotunneling current through NSN transistors we shall make use of the method developed in Ref. [17] for the case of single electron cotunneling in chains of normal tunnel junctions. It enables us to evaluate the contribution from the cotunneling process to the free energy $F_{2e}^{\text{cot}}$ and then to calculate the current by means of the formula $I_{2e}^{\text{cot}} = 4e\text{Im}F_{2e}^{\text{cot}}$. As the two electron cotunneling mechanism is important for small transport and gate voltages we restrict ourselves to the case $Q_g = 0$ and $eU_{tr} \ll E_C$. Proceeding perturbatively in $\alpha_{1,2}^A = R_q G_{1,2}^A$ ($G_{1,2}^A$ are the Andreev conductances of two NS junctions) and assuming that $G_{1,2}^A$ do not depend on the external voltage in the limit of small $U_{tr}$, one can easily show that $I_{2e}^{\text{cot}} \propto U_{tr}^3$ in complete analogy with single electron cotunneling through normal metallic grains [15-18]. A rigorous calculation yields [14]

$$I_{2e}^{\text{cot}} = \frac{(4e)^4 \alpha_{1,2}^A U_{tr}^3}{3\pi^2 E_C^2} \left( \frac{\Delta_i}{\Delta_i^2 - E_C^2} \right)^4 \left\{ \frac{\Delta_i - E_C}{\Delta_i + E_C} \arctan \sqrt{\frac{\Delta_i - E_C}{\Delta_i + E_C}} \right\}^4$$

This result is valid provided the Andreev conductance of each NS interface is Ohmic. As it was demonstrated in Refs. [13,14] under certain physical conditions due to the proximity effect the Andreev conductance of NS interfaces may become non-Ohmic $G_{1,2}^A \propto U_{tr}^{-1}$ in the limit of small voltages. In this case we find $I_{2e}^{\text{cot}} = G_{2e}^{\text{cot}} U_{tr}$, where [14]
\[ G_{\text{cot}}^A = \frac{1}{2\pi^3 e^6 R_{t1}^2 R_{t2}^2 N_i(0)^2 A_1 A_2 d_{N1} d_{N2} E_C^2} \times \]
\[ \times \frac{\Delta_i^4}{(\Delta_i^2 - E_C^2)^2} \left\{ \arctan \sqrt{\frac{\Delta_i - E_C}{\Delta_i + E_C}} \right\}^4, \] (26)

i.e. the two-electron cotunneling conductance \( G_{\text{cot}}^A \) turns out to be Ohmic. For reasonable experimental parameters we can estimate \( G_{\text{cot}}^A \sim 10^{-10} \, \Omega^{-1} \). Thus the effect is in the measurable range and can be important in the limit of small external voltages.

The effect of environment on two-electron cotunneling can be treated in complete analogy with that for the case of single electron cotunneling [17,18]. In the case of Ohmic external impedance \( R_x \) the above results for the cotunneling conductance should be multiplied by a factor \( \sim (2eU_{tr} R_x C)^4 e^2 R_x / \pi \).

Finally let us point out that the effect of two-electron cotunneling also occurs in SNS structures. The corresponding cotunneling conductance of such structures is identical to that for NSN systems with equivalent parameters.

**VI. DISCUSSION**

In conclusion, we have developed a theory of parity effects in small superconducting islands by analyzing the rates of various tunneling processes. The tunneling of the single odd excitation created in earlier tunneling processes is crucial at low temperatures. By comparing its rate \( \gamma \) to the tunneling rate \( \Gamma^\pm \) of all the other electrons we explain the transitions and the crossover conditions in an electron box. Our approach can easily be generalized to derive the \( I-V \) characteristics of more complicated systems, such as SET transistors, which allows us to explain numerous details observed in recent experiments. The agreement with experiment can be improved by including coherent third-order processes.

We considered a superconducting island which always remains coupled to the leads. In this respect our model differs from that considered in Refs. [19–21]. These authors study a system with fixed electron number parity, either in the even or in the odd state, while single electron tunneling processes which allow for transition between the two states are forbidden. This restricts the excitation spectrum, leading to unusual thermodynamic properties. If the island remains coupled to the lead the excited states alternate between even and odd parity, resulting in rather different thermodynamic properties.

Our analysis yields a richly structured \( I-V \) curves for NSN transistors and allows us to distinguish different charge transfer mechanisms important for different values of the external voltage. In particular, we found that in the low temperature limit the following processes are important for relatively large gate and/or transport voltages \( Q_g \) and \( U_{tr} \):

- a) Single electron tunneling (influenced by parity effects) [8]
- b) Two-electron tunneling (Andreev reflection) [11] and
- c) Coherent three-electron tunneling.

We have shown that the correspondence between the simulations and the experimental results is improved if the theoretical model takes into account three-electron processes. This model, however, does not reproduce the ridges in the \( I-V \) characteristic in [7]. On the other hand, we mention that the results of Ref. [8] also show no signs of these ridges.
For small values of the external voltage and low $T$ the processes $a) - c)$ are suppressed due to the superconducting gap and Coulomb blockade. Under these conditions the system conductance is determined by one of the cotunneling processes:

$d)$ Single electron cotunneling \[5\] or

$e)$ Two-electron cotunneling \[14\].

As the corresponding contributions to the system conductance depend on different parameters (in the case $d$), the conductance is inversely proportional to $R^2 N_i(0) \Delta_i$ \[5\], whereas the conductance $e)$ is defined by (25), (26)) each of them can dominate depending on particular experimental situation.

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REFERENCES

[1] M.T. Tuominen, J.M. Hergenrother, T.S. Tighe, and M. Tinkham, Phys. Rev. Lett. 69, 1997 (1992); Phys. Rev. B 47, 11599 (1993).
[2] P. Lafarge, P. Joyez, D. Esteve, C. Urbina, and M.H. Devoret, Phys. Rev. Lett. 70, 994 (1993).
[3] T.M. Eiles, J.M. Martinis, and M.H. Devoret, Phys. Rev. Lett. 70, 1862 (1993).
[4] J.M. Hergenrother, M.T. Tuominen, and M. Tinkham, Phys. Rev. Lett. 72, 1742 (1994).
[5] D.V. Averin and Yu.V. Nazarov, Phys. Rev. Lett. 69, 1993 (1992).
[6] G. Schönh and A.D. Zaikin, Europhys. Lett. 26, 695 (1994); G. Schönh, J. Siewert and A. D. Zaikin, Physica B 203, 340 (1994).
[7] J. M. Hergenrother, M.T. Tuominen, J.G. Lu, D.C. Ralph, and M. Tinkham, Physica B 203, 327 (1994).
[8] D.V. Averin and K.K. Likharev, J. Low Temp. Phys. 62, 345 (1986).
[9] G. Schönh and A.D. Zaikin, Phys. Rept. 198, 237 (1990).
[10] see e.g. M. Tinkham, Introduction to Superconductivity, McGraw Hill (1975).
[11] F.W.J. Hekking, L.I. Glazman, K.A. Matveev, and R.I. Shekhter, Phys. Rev. Lett. 70, 4138 (1993).
[12] F. Guinea and G. Schönh, Physica B 152, 165 (1988).
[13] F.W.J. Hekking and Yu.V. Nazarov, Phys. Rev. Lett. 71, 1625 (1993); Phys. Rev. B 49, 6847 (1994).
[14] A.D. Zaikin, Physica B 203, 417 (1994).
[15] D.V. Averin and A.A. Odintsov, Phys. Lett. A 140, 251 (1989).
[16] D.V. Averin and Yu.V. Nazarov, in Single Charge Tunneling, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992).
[17] A.D. Zaikin and D.S. Golubev, Phys. Lett. A 169, 475 (1992).
[18] A.A. Odintsov, V.Bubanjga and G.Schön, Phys. Rev. B 46, 6875 (1992).
[19] B. Jankó, A. Smith and V. Ambegaokar, Phys. Rev. B 50, 1152 (1994).
[20] D.S. Golubev and A.D. Zaikin, Phys. Lett. 195, 380 (1994); this volume.
[21] D.V. Averin and Yu.V. Nazarov, Physica B 203, 310 (1994).
[22] A.A. Odintsov, G. Falci, and G. Schönh, Phys. Rev. B 44, 13089 (1991); M.H. Devoret et al., Phys. Rev. Lett. 64, 1824 (1990).

Figure Captions:

Fig. 1: The current $I(Q_g, U_{tr})$ through a NSN transistor. Only single and two-electron tunneling has been taken into account. The parameters are chosen to coincide with those of Ref. [7], Fig. 3a ($E_C = 99\mu eV, \Delta_i = 245\mu eV$).

Fig. 2: The current $I(Q_g, U_{tr})$ through a NSN transistor including three-electron tunneling. The parameters are chosen to coincide with those of Ref. [7], Fig. 3a ($E_C = 99\mu eV, \Delta_i = 245\mu eV$); for the prefactor of the three-electron rate we set $\eta = 2 \cdot 10^6 \text{s}^{-1}$. 