Soft-collinear gravity

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Abstract

We study collinear and soft singularities in perturbative quantum gravity by constructing an effective field theory similar to soft-collinear effective theory for QCD (SCET). We find that the soft sector exhibits factorization properties similar to those of SCET. The collinear sector is, however, quite different. While the leading-power collinear effective Lagrangian is trivial, the presence of the metric field $h_{++}$ with negative scaling dimension allows for collinear divergences in loop diagrams with couplings to non-collinear sources. We provide a compact proof of the well-known fact that there are no collinear singularities in perturbative quantum gravity by demonstrating the decoupling of $h_{++}$ from the sources. We briefly discuss the connection of our approach to recent work by Akhoury et al. (Phys. Rev. D84 (2011) 104040) as well as to the Weinberg’s original paper (Phys. Rev. 140 (1965) B516), where the cancellation of the collinear singularities was demonstrated for the first time in the eikonal approximation.
1 Introduction

The story of collinear divergences in general relativity is rather short. In Ref. [1], using the eikonal approximation, Weinberg shows that no additional divergences in gravitational radiation appear in the limit of massless colliding particles. Utilizing the gravitational Ward identity, Akhoury et al. [2] demonstrate without reference to the eikonal approximation that the collinear singularities cancel to any perturbative order when all relevant diagrams are summed over.

There is a simple qualitative explanation of the absence of singular collinear graviton radiation. Consider an energetic particle with virtuality much less than its three-momentum squared \( p^2 \), emitting a nearly collinear graviton with momentum \( k \) such that the angle \( \theta \) between \( p \) and \( k \) is small. The near mass-shell singularity of the emitting particle propagator yields a factor \( \theta^{-2} \) for the splitting amplitude. The on-shell graviton is produced in a state with the definite helicity \( \pm 2 \). Due to helicity and angular momentum conservation, the splitting amplitude should be proportional to the component of the graviton wave function with vanishing projection of angular momentum along the momentum of the initial particle. Quantizing the radiation field in the spherical basis with single-particle states \( |kjm; \lambda \rangle \), where \( \lambda \) denotes helicity and \( jm \) the angular momentum quantum numbers with respect to the quantization axis \( p \), this implies that the emitted graviton must be in a state \( |kj0; \pm 2 \rangle \). The angular dependence of this state is given by a spin-weighted spherical harmonic or Wigner function \( D_{\pm 2,0}^j(k) \propto \sin^2 \theta \), which tends to zero as \( \theta^2 \) in the \( \theta \to 0 \) limit. Thus, the splitting amplitude has no singularity in the collinear limit. In contrast to the graviton, a massless vector boson contributes as \( D_{\pm 1,0}^j(k) \sim \theta \), which leads to a \( \theta^{-1} \) singularity in the amplitude and the well-known logarithmic divergence \( d\theta/\theta \) in the differential cross section.

The above argument refers to physical polarization states of the graviton and thus does not cover the properties of individual Feynman amplitudes in general, in particular covariant gauges, which do have collinear divergences. In order to demonstrate the absence of collinear singularities in a physical process, one needs a factorization theorem that controls the collinear interactions of the unphysical graviton modes and their coupling to a non-collinear environment (“source”). In Ref. [2] the gravitational Ward identity is employed to provide a diagrammatic proof of the factorization and cancellation of collinear divergences, extending Weinberg’s analysis beyond the eikonal approximation.

In order to single out the singular diagrams, the authors of Ref. [2] use power-counting rules which, as mentioned in this paper, are very similar to those used to construct the soft-collinear effective theory (SCET) for QCD [3–6]. In addition to power counting, SCET often simplifies the algebra of factorization proofs, since it displays the relevant properties in the Lagrangian, and avoids reference to individual diagrams. This motivates us to reconsider the problem of soft and collinear graviton physics by constructing “soft-collinear gravity”. That is, following the lines of SCET, we analyze the coupling of soft and collinear field degrees of freedom at the level of an effective Lagrangian instead of classifying all relevant Feynman graphs. We also find it interesting to compare soft-collinear gravity to the SCET for gauge fields, which reveals similarities and differences. As will be shown
below, one of the differences is the presence of a metric field component with negative scaling dimension, which complicates the correspondence between the power expansion of the Lagrangian and the scaling of diagrams. The negative-scaling component $h_{++}$ also plays the crucial role in the interaction of collinear modes with the non-collinear environment; controlling the interactions of $h_{++}$ is the essence of factorization. The field $h_{++}$ could be eliminated by choosing a non-covariant physical gauge, as is expected from the arguments above. However, our aim is to demonstrate the factorization in a covariant gauge, which will be accomplished by a universal field redefinition. For the sake of completeness, we also consider the interactions of soft gravitons, which share many similarities with soft gauge fields.

2 Power-counting rules

The version of SCET we use as a template is based on the position-space representation \[5, 6\]. To refrain from repetition, we review here only the key ideas of the effective theory construction, and refer the reader to Ref. \[5\] for further information. The theory employs separate fields for the collinear and the soft modes with small virtuality compared to the large scale of the process. Each collinear region is characterized by a certain light-like four-vector $n^\mu_\perp$. The complement light-like vector is denoted $n^\mu_\parallel$, such that $n^\mu_\parallel \cdot n^\mu_\perp = 2$, and it is convenient to introduce the notation

$$p_- = n_- \cdot p, \quad p_+ = n_+ \cdot p$$

for the light-cone components. A collinear four-vector $p$ is assumed to have the following scaling of its components:

$$p_+ \sim Q, \quad p_- \sim \lambda^2 Q, \quad p_\perp^\mu = p^\mu - \frac{p_+ n_-^\mu + p_- n_+^\mu}{2} \sim \lambda Q,$$

where $Q$ is the hard scale in a process and $\lambda$ is a small dimensionless parameter. A soft four-vector $q$ has scaling behavior

$$q^\mu \sim \lambda^2 Q$$

for any of its components. The effective fields are defined to create or destroy particles with a certain momentum scaling. Scaling rules for field components can be extracted from the field two-point correlators. For example, a collinear fermion field $\psi_c$ is decomposed into two components with different scaling (see Ref. \[5\]):

$$\psi_c = \xi + \zeta, \quad \xi = \frac{\eta_- \eta_+}{4} \psi_c \sim \lambda, \quad \zeta = \frac{\eta_+ \eta_-}{4} \psi_c \sim \lambda^2,$$

while the components of the collinear gauge field $A_\mu^c$ scale like a collinear momentum. The small field component $\zeta$ is integrated out and not part of the soft-collinear Lagrangian.

The same procedure can be applied to the gravitational field $h_{\mu\nu}$, defined as the metric deviation from flat space,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$
The corresponding expansion of the Einstein-Hilbert action reads as follows:

\[
S = \frac{1}{16\pi G_N} \int \! d^4x \sqrt{-g} \, R
= \frac{1}{2\kappa^2} \int \! d^4x \left[ \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} - \partial_\alpha h \partial^\alpha h - 2 \partial_\mu h^{\mu\nu} (\partial_\alpha h^\nu_\alpha - \partial_\nu h_\alpha) + O(h^3) \right],
\]

(6)

where \( h = h^\alpha_\alpha \), \( \kappa = \sqrt{32\pi G_N} \), and \( G_N \) is the Newton constant. After expansion in \( h_{\mu\nu} \) indices are raised and lowered with the flat-space metric. The gravity coupling \( \kappa \) is dimensional; the dimensionless parameter that controls gravitational perturbation theory is \( \kappa Q \).

Just like the form of soft-collinear QCD does not require the QCD coupling \( g_s \) to be small, since the power counting for soft and collinear fields is not related to the size of \( g_s \), the construction of soft-collinear gravity applies in principle to Planckian and trans-Planckian scattering energies. We shall see below that higher-order terms in the weak-coupling expansion, as well as higher-derivative interactions that must be added to the Einstein-Hilbert action (6) to make it perturbatively renormalizable, are all suppressed for collinear and soft gravitons, provided that \( \lambda \kappa Q \ll 1 \). That is, we must require only that the transverse momentum scale \( \lambda Q \) is sufficiently below the Planck scale, but not the scattering energies themselves.

The gauge is fixed by adding the term

\[
\frac{b}{\kappa^2} \int \! d^4x \left( \partial_\alpha h^\alpha_\mu - \frac{1}{2} \partial_\mu h \right) \left( \partial_\beta h^{\beta\mu} - \frac{1}{2} \partial^\mu h \right)
\]

(7)
to the action (6), where \( b \) is an arbitrary, real, dimensionless parameter. This corresponds to the covariant generalization of de Donder gauge, which is obtained for \( b = 1 \). The graviton propagator thus takes the form:

\[
D_{\mu\nu,\alpha\beta} = \langle 0 | T h_{\mu\nu}(x) h_{\alpha\beta}(y) | 0 \rangle = i\kappa^2 \int \! \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0} \left( P_{\mu\nu,\alpha\beta} + \frac{1-b}{b} S_{\mu\nu,\alpha\beta} \right),
\]

(8)

where

\[
P_{\mu\nu,\alpha\beta} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta}),
\]

\[
S_{\mu\nu,\alpha\beta} = \frac{1}{2p^2} (\eta_{\mu\alpha} p_\nu p_\beta + \eta_{\mu\beta} p_\nu p_\alpha + p_\mu p_\alpha \eta_{\nu\beta} + p_\mu p_\beta \eta_{\nu\alpha}).
\]

(9)

Some components of the propagator (8) vanish identically,

\[
D_{+++,+++} = D_{++,++} = D_{++,++} = D_{--,---} = D_{--,---} = D_{--,---} = 0,
\]

(10)

while the other independent components scale as follows:

\[
D_{+++,++} \sim D_{++,++} \sim \lambda^0,
D_{++,---} \sim D_{++,---} \sim \lambda^1,
D_{++,++} \sim D_{++,++} \sim D_{++,++} \sim \lambda^2,
D_{++,---} \sim D_{++,---} \sim D_{++,---} \sim \lambda^3,
D_{++,---} \sim D_{++,---} \sim D_{++,---} \sim \lambda^4.
\]

(11)
Here the $\perp$ index denotes a contraction with the transverse metric $\eta_{\mu\nu} - (n_+^\mu n_-^\nu + n_-^\mu n_+^\nu)/2$.

The scaling \( (11) \) is only consistent with the following counting rules for the components of the field $h_{\mu\nu}$:

\[
\begin{align*}
  h_{++} &\sim \lambda^{-1}, \\
  h_{+\perp} &\sim 1, \\
  h_{-+} &\sim \lambda, \\
  h_{--} &\sim \lambda^3, \\
  h_{-\perp} &\sim \lambda^2, \\
  h_{\perp\perp} &\sim \lambda.
\end{align*}
\]

(12)

Two points are to be made here. First, it is easy to see that any combination $ah_{\mu\nu} + bh_{\eta_{\mu\nu}}$ with $a \sim b \sim 1$ scales as $h_{\mu\nu}$, since $h = h_\alpha^\alpha \sim \lambda$ and only $\eta_{++}$, $\eta_{-+}$, and $\eta_{\perp\perp}$ are non-zero. This makes our consideration reparametrization invariant; the gravitational field can be defined as the linearized deviation of the contravariant metric density $\sqrt{-g}g^{\mu\nu}$ or the vierbein field $e^{(a)}_{\mu}$ and so on. Second, the power counting presented above depends only on the number of “$-$” components $N_-$ and “$\perp$” components $N_\perp$. The scaling of the propagator components $D_{\mu\nu,\alpha\beta}$ takes the form $D_{\mu\nu,\alpha\beta} \sim \lambda^2 N_--N_\perp-2$, and

\[
h_{\mu\nu} \sim \lambda^{2N_-+N_\perp-1}
\]

(13)

for the gravitational field. Hence the “$\perp$” index contributes as $\lambda$, the “$-$” index as $\lambda^2$, the “$+$” index as $\lambda^0$, and there is one additional factor $\lambda^{-1}$ for every $h_{\mu\nu}$. An immediate and somewhat unusual consequence of this is that the collinear metric component $h_{++}$ is enhanced in the small power-counting parameter $\lambda$. This is the first important difference between collinear gravitational and gauge fields.

It is also easily checked that the contraction of $h_{\mu\nu}$ with any four-vector $V_{\nu}$, collinear to the same $n_-$, that is, with scaling \( (2) \), yields an additional power suppression:

\[
h_{\mu\nu}V_\nu = \frac{1}{2} (h_+^\mu V_- + h_-^\mu V_+) + h_\perp^\mu V_\perp \sim \lambda V^\mu,
\]

(14)

which holds for every component $\mu$ separately. This provides the second main difference between the collinear metric field $h_{\mu\nu}$ and the collinear gauge field $A_\mu$. The coupling to matter is given through an action $S_m$ by

\[
S_{\text{int}} = \int d^4x \left( A^\mu \frac{\delta S_m}{\delta A^\mu} + h_{\mu\nu} \frac{\delta S_m}{\delta h_{\mu\nu}} \right).
\]

(15)

For example, for the coupling to a fermion, $\delta S_m/\delta A^\mu \propto j_\mu = \bar{\psi}\gamma_\mu\psi$, and $\delta S_m/\delta h_{\mu\nu} \propto T^{\mu\nu} = \bar{\psi}\gamma^\mu i\partial^\nu\psi$. In comparison to $\delta S_m/\delta A^\mu$, the variation $\delta S_m/\delta h_{\mu\nu}$ has an additional Lorentz index. If the matter is also a collinear field then the additional Lorentz index implies an additional contraction of the type \( (14) \). This yields power suppression in $\lambda$ relative to the coupling to the collinear gauge field, exactly in line with our qualitative discussion. A consequence of this is that the $\lambda$-expansion of the collinear matter-coupling to gravitation almost coincides with the weak-field expansion. This will be illustrated in more detail for the expansion of the fermion Lagrangian in the next section.

Soft modes of the gravitational field can be also estimated in the same manner. The metric is decomposed into collinear $h_{\mu\nu}$ and soft $s_{\mu\nu}$ fields according to

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + s_{\mu\nu}.
\]

(16)
Derivatives acting on the soft field counted as the corresponding soft momentum (3). The components (10) vanish also for the soft field, while all other projections scale as $\lambda^4$. Therefore, any component of $s_{\mu\nu}$ scales as $\lambda^2$.

3 Soft-collinear Lagrangian

As an example, we consider the massless spinor field coupled to the gravitational field through the action

$$S_m = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \bar{\psi} E_{(a)}^\mu \gamma^a \left( i \vec{D}_\mu \psi \right) - \left( \bar{\psi} i \vec{D}_\mu \right) E_{(a)}^\mu \gamma^a \psi \right].$$

(17)

The covariant derivatives act on fermions as

$$\left( \vec{D}_\mu \psi \right) = \partial_\mu \psi - \frac{i}{2} \Sigma^{ab} \gamma_{abc} e_{(c)} \psi, \quad \left( \bar{\psi} \vec{D}_\mu \right) = \partial_\mu \bar{\psi} + \frac{i}{2} \bar{\psi} \Sigma^{ab} \gamma_{abc} e_{(c)},$$

(18)

where $\Sigma^{ab} = \frac{i}{4} \left[ \gamma^a, \gamma^b \right]$. The vierbein field $e_{(a)}^\mu$, its inverse $E_{(a)}^\mu$, and the spin connection $\gamma_{abc}$ are defined as

$$\eta_{ab} e_{(a)}^\mu e_{(b)}^\nu = g_{\mu\nu}, \quad E_{(a)}^\mu e_{(b)}^\nu = \delta^\mu_\nu, \quad \gamma_{abc} = E_{(b)}^\mu E_{(c)}^\nu D_\nu e_{(a)}^\mu,$$

(19)

respectively. The weak field expansion of the action (17) results in the Lagrangian

$$\mathcal{L}_m = (\partial^\mu - H^\mu_\nu) \bar{\psi} \gamma^\nu \frac{\partial}{\partial x^\mu} \psi, \quad \frac{\partial}{\partial x^\mu} = \frac{1}{2} \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial x^\mu} \right),$$

(20)

where

$$H^\mu_\nu = \frac{1}{2} \left( h^\mu_\nu - h \delta^\mu_\nu \right).$$

(21)

Integrating out the $\zeta$ component of the spinor field (see the definition (4)) and using the power-counting rules (4) and (12), we expand the matter Lagrangian in powers of $\lambda$, $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \ldots$, where the superscript means that $\mathcal{L}^{(n)} \sim \lambda^n \mathcal{L}^{(0)}$. The leading term and the first power correction in the purely collinear Lagrangian are found to be

$$\mathcal{L}^{(0)} = \bar{\xi} \frac{\partial^\mu}{2} \left( i \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial x^\mu} \right) \frac{1}{2} \frac{\partial}{\partial x^\mu} i \frac{\partial}{\partial \tau} \xi,$$

(22)

$$\mathcal{L}^{(1)} = -H^\mu_\nu \bar{\xi} \frac{\partial^\mu}{2} \left( i \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial x^\mu} \right) \xi + i \xi \frac{\partial^\mu}{2} \left[ -\frac{\partial}{\partial \tau} \frac{1}{2} \left( H^\mu_\nu \partial^\mu_{\nu} + \frac{1}{2} H^\mu_{\nu, \mu} \right) \right] \frac{1}{2} \frac{\partial}{\partial x^\mu} \partial_{\tau} \xi,$$

(23)

The inverse derivative $\left( \frac{\partial}{\partial \tau} \right)^{-1}$ should be read as

$$\frac{1}{i \partial_{\tau} + i \delta} \phi(x) = -i \int_{-\infty}^0 du \phi(x + un_{++}),$$

(24)
and an index after the comma denotes an ordinary partial derivative. The leading-power Lagrangian \( L^{(0)} \) has a similar form as in [5] as far as derivatives are concerned, but contrary to the gauge-field case it is non-interacting. All purely collinear interactions are power-suppressed! The fact that the interaction of collinear fields with collinear gravitons is power suppressed is readily apparent from the weak-field expansion [20]. Indeed, the “current” \( \bar{\psi} \gamma^\mu i \partial_\mu \psi \) carries an additional collinear four-vector index, which is contracted with the metric field. As was stressed above (see (14)), this yields power suppression.

A similar reasoning shows that the “decoupling” of collinear gravitons in the leading-power Lagrangian is independent of the type of field, since any collinear Lagrangian contains only collinear four-vectors, which must be contracted at least once with the metric field. This applies, in particular, to the self-interaction of collinear gravitons. Likewise, higher-order corrections to the weak-coupling expansion are further and further suppressed in the purely collinear sector, since additional powers of \( h_{\mu\nu} \) require additional contractions, while \( h = h_{\mu\mu} \) is itself of order \( \lambda \). The same suppression applies to higher-derivative terms that should be added to the Einstein-Hilbert action (6) to make it perturbatively renormalizable. Soft derivatives are power-suppressed by construction, while the collinear derivative terms always come with an additional collinear contraction. This shows that the applicability of the effective theory of soft-collinear quantum gravity is restricted to transverse momenta rather than energies smaller than the Planck scale.

In contrast to the purely collinear interaction, the soft-collinear one is not power suppressed. Writing \( S^\mu_\nu = \frac{1}{2} (s^\mu_\nu - s^\mu \delta^\nu_\nu) \), the leading soft interaction in (20) is contained in

\[
(\delta^\mu_\nu - S^\mu_\nu) \frac{\xi}{2} \left[ \frac{y^+_{\mu}}{n^\mu - x_+} \xi \right] = \frac{j_+ - 1}{2} \frac{1}{s_{-+} - j_+} - \frac{S_{-\perp} j_\perp}{\sim \lambda^5} - \frac{1}{2} \frac{1}{s_{-+} - j_-} \sim \lambda^6,
\]

where

\[
j_{\mu} = \frac{\xi}{2} \frac{y^+_{\mu}}{i \partial_\mu} \xi.
\]

In the expression (25), we indicate the \( \lambda \)-scaling with underbraces, which identifies the second term on the right-hand side as the only leading-power interaction. The structure of this interaction is very similar to the corresponding gauge field interaction \( \xi \frac{\xi}{2} \xi A_\pm \). In further analogy with the soft-collinear gauge interaction, soft fields interacting with collinear fields must be multipole-expanded in their position argument to achieve a homogeneous \( \lambda \)-expansion [5]. If the \( \lambda \)-expansion is limited to the leading order, it is sufficient to replace the argument of the soft field \( x^\mu \) by \( n^\mu_\perp x_+/2 \), where \( n^\mu_\perp \) is the reference direction. Therefore, the leading-order Lagrangian of the collinear fermion field interacting with the soft gravitational field takes the form:

\[
L^{(0)}_{c+s} = \frac{\xi}{2} \left[ \frac{y^+_{\mu}}{i \partial^\perp} - \frac{1}{4} \frac{1}{s_{-+}(x_+)} \frac{1}{i \partial^\perp} \right] \xi.
\]

\[\text{Note the change of notation: the four-vector } n^\mu_\perp x_+/2 \text{ was called } x^\mu_\perp \text{ in Ref. [5], while here we use } x^\mu_+, \text{ or simply } x_+.\]
Figure 1: a) (left panel) Collinear splittings with $h_{++}$-ends generated by three-point vertices. b) (right panel) Source that generates non-collinear lines (double lines) to which the enhanced $h_{++}(x_-)$ field can couple.

We emphasize again that the only component remaining is $s_-$, similar to the leading SCET Lagrangian which contains only the $A_-$ soft component. In both situations this corresponds to the eikonal approximation for soft fields. We also note that on functions $f(x_+) = n_+ x$ only the minus-component $\partial_-$ of $\partial_\mu$ is non-vanishing, hence $\partial_+$ in (27) does not operate on the metric field.

4 Collinear factorization

From the fore-going it might be concluded that collinear factorization is trivial, since collinear interactions are power-suppressed, and that nothing remains to be done except for checking the collinear limit of the leading-power soft (eikonal) interaction as was done in Ref. [1]. This is not quite true. As discussed in Ref. [2], although the purely collinear sector is power suppressed, its interaction with a non-collinear environment is not. In the present framework, this follows from the negative $\lambda$-scaling of the $h_{++}$ metric component, which implies that its coupling to an energetic (or massive) particle not collinear with $h_{\mu\nu}$ should be counted as $\lambda^{-1}$. This negative scaling can compensate the $\lambda$-suppression of the interaction of collinear matter $\xi$ with collinear gravitons, and of collinear graviton self-interactions, thus generating diagrams with a leading-power collinear singularity. The situation is illustrated in Fig. 1. The left panel shows an example of a collinear splitting diagram. The $\lambda$-suppression of the collinear vertices can be compensated only if every line ends with the $h_{++}$ field. In order not to incur further suppression these ends must be tied to non-collinear (with respect to the direction of $h_{++}$) lines, as shown by the double lines in the right panel. The $\xi$ line in this figure stands for the entire collinear sector on the left, which, together with the non-collinear lines, is generated by an unspecified source operator, represented by the shaded blob.

It is instructive to compare this situation to SCET for gauge interactions. The $A_+$ component of the gauge field scales as $\lambda^0$, but it is not enhanced by a negative scaling dimension. For SCET to be a useful effective theory, the interactions of the unsuppressed

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2The right panel in Fig. 1 corresponds to what is referred to as the “rest” of the diagram in Ref. [2].
A\_+ field must be controlled to all orders in the A\_+ field, separately in every order in the \(\lambda\)-expansion. This is accomplished by showing that A\_+ appears in SCET only through collinear Wilson lines \(W_c\) [4,5]. Clearly, in the case of gravity the coupling of the h\_\(++\) field must be also controlled to all orders. In contrast to SCET, the collinear Lagrangian to all orders in the \(\lambda\)-expansion is now potentially relevant, since the \(\lambda\)-suppression of collinear interactions can be compensated by the negative \(\lambda\)-scaling of the h\_\(++\) field coupling to the sources. Hence, we need to factorize h\_\(++\) from the sources without making use of the collinear Lagrangian to any finite order. Below we show how this can be done in complete generality.

The difficulty with negative scaling of h\_\(++\) can be avoided by adopting the ghost-free light-cone gauge [7–9]. The propagator in this gauge reads

\[
\langle 0 | T h_{\mu\nu} (x) h_{\alpha\beta} (y) | 0 \rangle = i\kappa^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0} \left( \rho_{\mu\alpha} \rho_{\nu\beta} + \rho_{\mu\beta} \rho_{\nu\alpha} - \rho_{\alpha\beta} \rho_{\mu\nu} \right),
\]

\[
\rho_{\mu\nu} = \eta_{\mu\nu} - \frac{p_{\mu} n_{\nu} + p_{\nu} n_{\mu}}{p \cdot n}. \tag{28}
\]

The components h\_\(+\mu\) are now non-propagating. Thus we can exclude the unphysical components h\_\(+\mu\) from the path-integral so that only metric components with positive \(\lambda\)-scaling remain. The absence of collinear singularities in gauge-invariant observables is therefore guaranteed. However, to gain a better understanding of factorization, we use here another way to exclude the interaction with h\_\(++\), which does not refer to a non-covariant gauge. Note that in SCET the unsuppressed gauge field component A\_+ can likewise be eliminated in light-cone gauge. But contrary to gravity this does not eliminate collinear divergences, since the collinear Lagrangian contains leading-power interactions with the other gauge field components.

Let us therefore consider the zero-momentum insertion of a generally covariant, local operator

\[
O (\phi_1, ..., \phi_n, \xi, g) = \int d^4 x \sqrt{-g} P (\phi_1, ..., \phi_n, \xi, g), \tag{29}
\]

where \(P\) is the “source” of any number of fields including the collinear one, \(\xi\), that is, a polynomial containing fields \(\phi_i\), \(\xi\), covariant derivatives, vierbeins, and the metric tensor at the same space-time point. The operator (29) corresponds to the vertex depicted as shaded blob in Fig. [1]. Each field \(\phi_i\) in (29) also interacts with h\_\(++\) through some Lagrangian \(L_i (\phi_i, g)\).

An interaction vertex of h\_\(++\) with a non-collinear field \(\phi_i\) should be classified as “hard”, since the momentum transfer is of the order of hard scale \(Q\). Large virtualities \(Q^2\) can be achieved only through the large-momentum component \(p_+\) of the h\_\(++\) line at the vertex. In position space it therefore suffices to consider the interaction with h\_\(++\) (\(x_-\)), where \(x_\perp\) is defined as \(v_\perp^\mu x_- / 2\), thus neglecting the dependence of h\_\(++\) (\(x_+\)) on \(x_+\) and \(x_\perp\). The intermediate, highly virtual non-collinear lines in Fig. [1] should be contracted and represented as an effective vertex in soft-collinear gravity. We shall demonstrate the cancellation of collinear singularities in the sum of diagrams generated by these effective vertices in two
steps: first, we show that through a field redefinition the interactions with \( h_{++} (x_-) \) can be collected into universal factors in \( \mathcal{O} \) and \( \mathcal{L}_1 \) that take the form of a gauge (coordinate) transformation. This should be expected for an unphysical field component, and, as we will see shortly, this transformation is particularly simple when the metric field is \( h_{++} (x_-) \).

This step is very similar to the factorization of collinear modes in QCD and the field redefinition exploited in Refs. [4, 6]. However, in contrast to QCD, in the second step we demonstrate that these factors cancel out due to translation invariance.

Since in the following we are concerned only with the \( h_{++} \) metric component, we may specialize the space-time \([5]\) to

\[
g_{\mu\nu} = \eta_{\mu\nu} + \frac{n_{-\mu} n_{-\nu}}{4} h_{++} (x_-), \tag{30}\]

which can also be denoted as follows:

\[
\hat{g} = \hat{\eta} + \hat{h}, \quad \text{so that} \quad \hat{h}^2 = 0, \quad \text{tr} \hat{h} = 0. \tag{31}\]

The algebra of the matrix \( \hat{h} \) resembles the algebra of a single Grassmann number. Any analytic function of \( \hat{h} \) is a linear function, hence we can find the exact expressions for the contravariant metric tensor and the metric determinant:

\[
\hat{g}^{-1} = \hat{\eta} - \hat{h}, \quad \det \hat{g} = -1. \tag{32}\]

It is straightforward to find the vierbeins

\[
e^{(\alpha)}_{\beta} = \delta^\alpha_\beta + \frac{n_{-\beta} n^\alpha_{-}}{8} h_{++} (x_-), \quad E^{(\beta)}_\alpha = \delta^\beta_\alpha - \frac{n_{-\alpha} n^\beta_{-}}{8} h_{++} (x_-), \tag{33}\]

and the affine connection

\[
\Gamma^\lambda_\mu\nu = \frac{1}{16} n^\lambda_{-\mu} n_{-\nu} \partial_+ h_{++} (x_-). \tag{34}\]

One further verifies that the covariant derivative of the vierbein is zero

\[
D^\nu e^{(\alpha)}_{\mu} = \partial^\nu e^{(\alpha)}_{\mu} - \Gamma^\lambda_\nu e^{(\alpha)}_\lambda = 0, \tag{35}\]

which implies that the spin connection \( \gamma_{abc} \) vanishes. From (34) it also follows that the Riemann tensor \( R^\mu_{\nu\rho\sigma} \) vanishes so the space-time (30) is flat. The transformation to the global inertial frame with coordinates \( y^\mu \) is the local translation

\[
y^\mu = x^\mu + \frac{n^\mu_{-}}{2} w (x_-), \quad w (x_-) = \frac{1}{4} \int_{-\infty}^{x_-} dx' h_{++} (x' -), \tag{36}\]

so that the metric (30) can be obtained via the standard relation

\[
g_{\mu\nu} (x) = \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x'^\nu}. \tag{37}\]
For the first step in the demonstration of the cancellation of collinear singularities, we consider a general covariant Lagrangian $L_i(\phi_i, g, D)$ for some matter field $\phi_i$ in the space-time \((30)\). Due to general covariance, we could go to the “flat” coordinates $y$. Then $g \to \eta$ and $D \to \partial$ in $L_i$, and all interactions with $h_{++}$ disappear, which already proves the decoupling in the Lagrangian.

It is, however, instructive to show the decoupling in the original coordinate system representing all transformations as field redefinitions. Let $\hat{\Gamma}$ be the $4 \times 4$ matrix with entries $\Gamma^\mu_{\nu+}$, and define the parallel transport matrix $\hat{U}$ through

$$U^{\mu}_{\nu}(x-) = \left[ \text{P} \exp \left( \int_{x^-}^{x_+} dx'_- \frac{1}{2} \hat{\Gamma}(x'_-) \right) \right]^{\mu}_{\nu},$$

and the collinear Wilson-line operator as follows:

$$W(x-) = \exp \left[ -\frac{i}{4} \int_{-\infty}^{x_-} dx'_{-} h_{++} (x'_-) \frac{i}{2} \partial_- \right] = \exp \left[ w(x_-) \frac{1}{2} \partial_- \right],$$

$$W^{-1}(x-) = \exp \left[ \frac{i}{4} \int_{-\infty}^{x_-} dx'_{-} h_{++} (x'_-) \frac{i}{2} \partial_- \right].$$

The field is assumed to vanish at $-\infty$, so that the integrals are convergent. Note that $\partial_- \eta$ does not operate on $h_{++}(x_-)$, since it acts only on $x_+ = n_+ x$. Also

$$\frac{1}{2} \partial_- = \frac{n^-}{2} \partial_\mu = \frac{\partial}{\partial (n_- \cdot x)} = \frac{\partial}{\partial x_+}.$$  

Due to the nilpotency of $\hat{\Gamma}$, which follows from \((34)\), the exponential is in fact linear, and the relations

$$U^\alpha_{\mu} = e^{(\alpha)}_{\mu}, \quad (U^{-1})^\mu_{(\alpha)} = E^\mu_{(\alpha)}$$

hold. At the operator level, the Wilson lines acts as

$$W\phi(x) W^{-1} = \phi(x) + w(x_-) \left[ \frac{\partial}{\partial x^+_+}, \phi(x) \right] + \frac{1}{2} w^2(x_-) \left[ \frac{\partial}{\partial x^+_+}, \phi(x) \right] \right] + \ldots$$

The Wilson line \((39)\) has the form of the translation operator to the flat coordinate point,

$$W\phi(x) W^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} w(x_-)^n \frac{\partial^n}{\partial x^+_+} \phi(x) = \phi(x + \frac{n}{2} w(x_-)) = \phi(y).$$

This motivates the following field redefinition. Let $\phi_1^{\nu_1 \nu_2 \ldots \nu_n}$ be an arbitrary field with $n$ generally covariant indices. Then define

$$\phi_1^{\mu_1 \mu_2 \ldots \mu_n} = W^{-1} U^{\mu_1}_{\nu_1} U^{\mu_2}_{\nu_2} \ldots U^{\mu_n}_{\nu_n} \phi_1^{\nu_1 \nu_2 \ldots \nu_n} W.$$  

The interpretation of this expression is clear: due to \((44)\), $U^{\mu_1}_{\nu_1} U^{\mu_2}_{\nu_2} \ldots U^{\mu_n}_{\nu_n} \phi_1^{\nu_1 \nu_2 \ldots \nu_n}$ is the field in the local inertial frame, which in the case at hand is global. The $W$ operators perform
the translation from $x$ to $y$, hence the redefined field should correspond to the decoupled field in $y$ coordinates. Indeed, we now show that in terms of the redefined fields,

$$
\mathcal{L}_i(\phi_i, g, D) = W \mathcal{L}_i(\phi'_i, \eta, \partial) W^{-1},
$$

which implies

$$
\int d^4x \sqrt{-g} \mathcal{L}_i(\phi_i, g, D) = \int d^4x W \mathcal{L}_i(\phi'_i, \eta, \partial) W^{-1} = \int d^4x \mathcal{L}_i(\phi'_i, \eta, \partial),
$$

where, in the last equality, the disappearance of the factors $W$ and $W^{-1}$ is a consequence of translation invariance, i.e., energy-momentum conservation, or simply of dropping total derivative terms. This expresses the decoupling of $h_{++}$ from the non-collinear lines in the original $x$-coordinates, since the metric field no longer appears in the action.

To prove (45), we express the field $\phi^\nu_{1\nu_2...\nu_n}$ in terms of the primed field. We note that $W$ commutes with $U$, since $U$ depends only on $x^-$. It is straightforward to show the identities

$$
\partial_\nu [W \ldots] = U^\nu_\mu W \partial_\mu [\ldots],
$$

$$
D_\nu [(U^{-1})^\mu_1 \ldots (U^{-1})^\mu_n \ldots] = (U^{-1})^\nu_1 \ldots (U^{-1})^\nu_n \partial_\nu [\ldots],
$$

where the second follows from (35) or the vanishing of the spin connection. With the help of these identities we convert all covariant derivatives acting on $\phi^\nu_{1\nu_2...\nu_n}$ into ordinary ones. Furthermore, after applying (47) the $W$ operators and their inverses arising from a product of $\phi$ fields can be cancelled except for one $W$ on the left and one $W^{-1}$ on the right. Since all generally covariant indices in the original Lagrangian must be contracted by $g_{\mu\nu}$ (fields) and $g^{\mu\nu}$ (covariant derivatives), the $U$ and $U^{-1}$ factors must necessarily be multiplied in the form

$$
g_{\mu\nu} (U^{-1})^\mu_\alpha (U^{-1})^\nu_\beta = \eta_{\alpha\beta}, \quad g^{\mu\nu} U^\alpha_\mu U^\beta_\nu = \eta^{\alpha\beta},
$$

where the equalities follow from the identity (41) of $U$ with the vierbein. This removes any appearance of the metric and $U$ from the Lagrangian, thus completing the proof of (45).

At this point after the field redefinition, the interaction with $h_{++}(x^-)$ remains only in the source operator (29). However, nothing above was special to the Lagrangian interactions, and we can apply the same field redefinition to the collinear fields $\xi$ and $h$. Hence, the relations (45), (46) are also valid for the operator (29), that is

$$
P(\phi_1, .., \phi_n, \xi, g, D) = WP(\phi'_1, .., \phi'_n, \xi', \eta, \partial) W^{-1},
$$

and $O(\phi_1, .., \phi_n, \xi, g) = O(\phi'_1, .., \phi'_n, \xi', \eta)$ due to translation invariance. This shows the decoupling of the dangerous $h_{++}$ field with negative scaling dimension from the sources and the non-collinear fields. Due to the power-suppression of collinear self-interactions, this excludes the presence of collinear divergences in physical processes.

We briefly compare the above result to Refs. [1, 2]. As mentioned above, Weinberg [1] works in the eikonal approximation and shows diagrammatically that no additional
collinear singularities arise when any of the non-infrared particles (in our terminology, lines emanating from the source in Fig. 1) becomes massless, provided momentum is conserved at the source vertex, which in our treatment corresponds to the use of translation invariance and a zero-momentum source.

The proof of collinear cancellations in the present paper is not restricted to the eikonal approximation. The general case was also analyzed recently by Akhoury et al. [2] employing diagrammatic factorization methods. Their conclusion is equivalent to ours, but their proof contains the additional result that only collinear three-point but no higher-point vertices in the branchings depicted in the left Figure can lead to diagrams with collinear divergences. No such statement follows from the present treatment. This stronger statement holds only in de Donder gauge ($b = 1$) as was assumed in Ref. [2], and not in the general covariant gauge [8]. In de Donder gauge, any non-zero component of the momentum-space graviton propagator contributes as $1/p^2 \sim \lambda^{-2}$ to the collinear degree of divergence of a given diagram. In the general covariant gauge the propagator [8] contains more singular terms such as $p_+p_+/ (p^2)^2 \sim \lambda^{-4}$, and the power-counting formula must be modified. In de Donder gauge, the structure of the collinear “tree” in Figure 1 is rather special. Since the branches must end in $h_+$, and since in de Donder gauge (contrary to the general covariant gauge, see [11]) the only non-vanishing component of $D_{++\alpha\beta}$ is $D_{++--}$, the relevant component of a triple vertex to which two ending branches attach is $h_--h_--h_++p_+q_+$, where $p, q$ are two collinear momenta at the vertex. Hence, the internal propagator at this vertex has $++$ indices. We can now repeat this argument to conclude that the $++$ index is transported through the tree. It is now clear why four- and higher-point vertices cannot contribute, since this would lead to at least six minus indices at the vertex which cannot all be paired with plus indices. In general covariant gauge this argument fails, since the propagator [11] can link $h_++$ with $h_--$ as well as with $h_+$ or $h_-$. Therefore, the class of vertices involved in collinear “trees” is larger in general. The scaling [12] implies that an $n$-graviton vertex containing two collinear momenta scales as $\lambda^{n-2}$. However, this suppression is compensated by the corresponding growth of the number of $h_+$ lines attached to the non-collinear part of a diagram. Therefore, in the general covariant gauge the “trees” can contain $n$-graviton vertices with $n > 3$.

5 Soft factorization

A similar reasoning shows the factorization of soft graviton interactions, but in this case there is no cancellation. The soft Wilson lines can be defined in a similar manner as [39], but since the leading-power soft-collinear Lagrangian [27] contains only the soft graviton field $s_--(x_+)$, the appropriate expression reads

$$Z_n(x_+) = \exp \left[ -\frac{i}{4} \int_{-\infty}^{x_+} dx'_+ s_--(x'_+) \frac{i}{2} \partial_+ \right] = \exp \left[ z(x_+) \frac{1}{2} \partial_+ \right] ,$$

$$Z_n^{-1}(x_+) = \exp \left[ \frac{i}{4} \int_{-\infty}^{x_+} dx'_+ s_--(x'_+) \frac{i}{2} \partial_+ \right]$$

(51)
with
\[ z(x_+) = \frac{1}{4} \int_{-\infty}^{x_+} dx'_+ s_{--}(x'_+) . \] (52)

Under the coordinate transformation \( x^\mu \to x^\mu + \frac{n_\mu}{2} \epsilon(x_+) \), where \( \epsilon \) is an arbitrary function of \( x_+ \), the soft graviton field and spinor field \( \xi \) have the gauge transformations
\[ s_{--} \to s_{--} - 2 \partial_- \epsilon(x_+), \quad \xi \to \xi - \frac{n_\mu}{2} \epsilon(x_+) \partial_\mu \xi, \] (53)
respectively, and \( Z_n \) transforms as a translation operator:
\[ Z_n(x_+) \to Z_n(x_+) \exp \left[ -\epsilon(x_+) \frac{\partial}{\partial x_-} \right] . \] (54)

To demonstrate the decoupling of the soft graviton from the Lagrangian we need the identity
\[ \partial_- - \frac{1}{4} s_{--}(x_+) \partial_+ = Z_n(x_+) \partial_- Z_n^{-1}(x_+) . \] (55)

We can then express the soft-collinear action (27) as
\[ \int d^4x \mathcal{L}_{c+s}^{(0)} = \int d^4x \hat{\xi} \frac{\hat{y}_+}{2} Z_n(x_+) \left( i \vec{\partial}_- + i \vec{\partial}_\perp \frac{1}{i \vec{\partial}^\perp} \right) Z_n^{-1}(x_+) \xi \]
\[ = \int d^4x \hat{\xi}' \frac{\hat{y}_+}{2} \left( i \vec{\partial}_- + i \vec{\partial}_\perp \frac{1}{i \vec{\partial}^\perp} \right) \xi', \] (56)
where the primed fields, which are invariant under the transformation (53), are defined by
\[ \hat{\xi}' = Z_n^{-1}(x_+) \hat{\xi} Z_n(x_+), \quad \xi' = Z_n^{-1}(x_+) \xi Z_n(x_+). \] (57)

Thus, a field redefinition similar to the collinear redefinition (44) eliminates the soft gravitational field from the Lagrangian. But in this case a scattering amplitude generated by some source operator acquires soft Wilson lines corresponding to different light-like directions \( n_- \), which, contrary to the collinear Wilson lines, do not cancel.

To be specific, consider a process whose initial and final states are clusters of highly energetic collinear particles and soft particles \( X \) including gravitons. The frame is fixed by the time-like 4-vector \( n_0 = (1, 0) \) so that \( P^\mu_{ini} = (P_{ini} \cdot n_0) n_0^\mu \), where \( P_{ini} \) is the total momentum of the initial state. The total momentum of the \( i \)-th cluster \( P_i = (E_i, \mathbf{P}_i) \) is assumed to satisfy \( E_i^2 \gg P_i^2 \). We choose the reference directions as \( n_{\perp i}^\mu = (1, \pm \mathbf{P}_i/|\mathbf{P}_i|) \). If we introduce the hybrid representation \([3,4]\), e.g., \( \xi(x_+, x_-, x_\perp) = \exp \left( -ip_+ x_-/2 \right) \xi_{p_+}(x_+, 0, x_\perp) \), then the field redefinition with the Wilson-line operators (57) results in the following factorization:
\[ \xi_{p_+} = Z(p_+, x_+) \xi'_{p_+}, \quad Z(p_+, x_+) = \exp \left[ -\frac{i}{4} \int_{-\infty}^{x_+} dx'_+ s_{--}(x'_+) \frac{1}{2} p_+ \right] . \] (58)
Therefore, the soft graviton interaction with the $i$-th collinear cluster contributes the factor $Z(n_{i+} \cdot P_i, x_+)$ to the amplitude. Using translation invariance to move the argument $x_+$ of $Z(p_+, x_+)$ to 0, and momentum conservation, the total amplitude $M(P_1, \ldots P_N)$, where $N$ is a number of clusters, thus factorizes into the product $M(P_1, \ldots P_N) = S_N \tilde{M}(P_1, \ldots P_N)$, where $\tilde{M}$ is independent of the soft graviton field and $S_N$ is the soft factor:

$$S_N = \left\langle X \prod_{i=1}^{N} Z(n_{i+} \cdot P_i, 0) \right\rangle.$$  \hspace{1cm} (59)

This is precisely the eikonal form of the amplitude which has been established by Weinberg [1] and which is studied in detail in Refs. [10][11].

6 Conclusion

In a gauge theory with a massless vector boson as the interaction carrier, large logarithmic corrections appear in the collinear limit when particle energies are much larger than their masses and splitting angles tend to zero. The interplay with soft singularities results in double logarithmic corrections and the associated non-trivial dynamics has been intensively studied for collider physics applications. Much recent progress in this area of strong and electroweak physics is based on the construction of an effective theory with explicit separation of soft and collinear modes – SCET [3–6]. Motivated by this and recent work on collinear gravitational interactions [2], we discussed in this paper the effective field theory description of soft and collinear gravitons. It is worth noting that the effective Lagrangians derived here apply to graviton interactions at Planckian and even trans-Planckian energies $E$ provided the transverse momentum scale $p_\perp \sim \lambda E$ is smaller than the Planck scale.

Soft-collinear gravity is quite different from SCET as far as collinear interactions are concerned. In the effective Lagrangian, in leading power in the expansion in $p_\perp/Q$, the collinear sector is trivial and only eikonal-type soft graviton interactions appear. The spin of the graviton prohibits singular collinear splittings like $\text{particle} \rightarrow \text{particle} + \text{graviton}$, thus there is no proliferation of graviton radiation collinear to a very energetic particle, unlike the case of gauge boson radiation.

However, the effective theory contains the collinear metric field $h_{++}$ with an unusual negative scaling power $1/\lambda$, which complicates the discussion. The coupling to a non-collinear environment potentially contributes through real or virtual radiation which interferes with purely collinear splittings. The suppression of collinear self-interactions can be lifted, if the interference occurs through the field with negative scaling power. Nevertheless, contrary to gauge theories with massless vector bosons, collinear divergences, while present in individual diagrams, cancel in physical processes. In the effective field theory treatment this can be shown in various ways. The interference with the non-collinear environment occurs in leading power only through an unphysical metric component, $h_{++}(x_-)$, which can be eliminated in light-cone gauge. In covariant gauge, the coupling of $h_{++}$ can be removed by going to a different coordinate frame, or by the universal field redefinition [14], which in turn is closely related to the coordinate and gauge transformation. Our result
complements the diagrammatic proof \cite{2} and extends it to a general covariant gauge. The final cancellation of collinear singularities then occurs upon using translation invariance, i.e. energy-momentum conservation, as was already observed in Ref. \cite{1} in the soft limit.

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