Correlators in 2D string theory with vortex condensation

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Abstract

We calculate one- and two-point correlators of winding operators in the matrix model of 2D string theory compactified on a circle, recently proposed for the description of string dynamics on the 2D black hole background.

1 Introduction

The matrix quantum mechanics (MQM) approach to the 2D string theory has been proven very effective, in particular, in the case of the compactification on a circle of a radius $R$ \textsuperscript{1,2}. Recently, it was shown \textsuperscript{3} that this approach is capable to capture such important phenomena as the emergence of a black hole background and condensation of winding modes (vortices on the world sheet).

From the statistical-mechanical point of view such a model describes a system of planar rotators (XY-model) on a two-dimensional surface with fluctuating metric. In this sense it is a natural generalization of its analogue on the flat 2D space studied long ago by Berezinski, Kosterlitz and Thouless (BKT). The role of the fluctuating surfaces is played by the planar graphs of the MQM and the vortices appear due to the compactification of the time direction.

As it is usual for the matrix model approach, it enables us to apply the powerful framework of classical integrability to the computations of physical quantities, such as partition functions of various genera or correlators of physical operators.

The appropriate MQM model for the compactified 2D string is the inverted matrix oscillator with twisted boundary conditions \textsuperscript{2}. The partition function of such a system as a function of couplings of the potential for the twist angles appears to be a $\tau$-function of the Toda lattice

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hierarchy \[1, 3\]. As such, it satisfies an infinite set of partial difference-differential Hirota equations. The couplings play the role of “times” of commuting flows in the Toda hierarchy. As was shown in \[3\] they are at the same time the couplings of vortex operators of various vorticity charges. The tree approximation for the string theory corresponds to the dispersionless limit of Toda equations.

In this paper we will use Hirota equations to calculate all one-point and two-point correlators of vortex operators with arbitrary vorticity charges for the compactified 2D string in the tree-like approximation in the regime of the radius of compactification \( \frac{1}{2}R_{KT} < R < R_{KT} \) (\( R_{KT} \) is the BKT radius), when condensation of vortices of the lowest charge \( n = \pm 1 \) takes place. Unlike the standard BKT model in flat 2D space, where the Debye screening in the vortex plasma destroys the long range correlations, the BKT vortices in the fluctuating 2D metric can be described as a conformal matter with central charge \( c = 1 \) coupled to the metric.

An important particular case of this model, for \( R = \frac{3}{4}R_{KT} \), claimed in \[3\] to describe the 2D string in the dilatonic black hole background, due to the conjecture of \[5\] based on the results of \[7, 6\]. This conjecture establishes a kind of a weak-strong duality between two CFT’s: the dilatonic black hole and the so called Sine-Liouville theory. We will try to compare our results with a few calculations of correlators in the continuum CFT performed in \[7, 6, 5\] and discuss the difficulties of the direct comparison.

In the next section we briefly describe the basic ideas of the MQM approach to the 2D string theory with vortex excitations; the Toda integrable structure of the model allows to formulate the Hirota equations describing the dynamics of vortices. In section 3, using Hirota equations the two point correlators of arbitrary vorticities are expressed through the one point correlators. In section 4 the one point correlators are explicitly calculated. The explicit expressions for the two-point correlators can be found in section 5. Section 6 is devoted to the comparison with the results of continuous (Liouville) approach and to the conclusions.

2 Toda hierarchy for the compactified 2D string

2.1 Matrix model of 2D string and coupling to the windings

In this section we recall the main results of the paper \[3\] leading to the Toda hierarchy description for the compactified 2D string theory.

This theory is represented by the Polyakov string action:

\[
S(x, g) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\det g} [g^{ab} \partial_a x \partial_b x + \mu'],
\]

(1)

where \( g_{ab} \) is a two dimensional fluctuating world sheet metric and the only bosonic field \( x(\sigma) \) is compactified on a circle of a radius \( R \):

\[
x(\sigma) \sim x(\sigma) + 2\pi R.
\]

(2)

Such a field admits in general Berezinski-Kosterlitz-Thouless vortices (windings) in its configurations. If we denote by \( x^{n_{\pm 1}, n_{\pm 2} \cdots}(\sigma) \) a configuration of the bosonic field containing \( n_{\pm 1} \) vortices
of vorticity charge \( \pm 1 \), \( n_{\pm 2} \) vortices of vorticity charge \( \pm 2 \), etc, the partition function of the theory in this sector will be

\[
Z_{n_{\pm 1}, n_{\pm 2}, \ldots} = \int Dg(\sigma) \int Dx^{n_{\pm 1}, n_{\pm 2}, \ldots} (\sigma) e^{-S(x, g)}.
\]  

(3)

Obviously, only the total vorticity \( \sum_{k=-\infty}^{\infty} kn_k \) is conserved by topological reasons, but we will distinguish configurations with different individual vortices as point-like objects with vorticities \( k \), to give a statistical-mechanical meaning to the system.

Instead of fixing the numbers of vortices \( n_k \) we can also introduce the partition function with fixed fugacities \( t_{\pm 1}, t_{\pm 2}, \ldots \) of the vortices with given charges:

\[
Z_{t_{\pm 1}, t_{\pm 2}, \ldots} = \sum_{n_{\pm 1}, n_{\pm 2}, \ldots} t_{\pm 1}^{n_{\pm 1}} t_{\pm 2}^{n_{\pm 2}} \cdots Z_{n_{\pm 1}, n_{\pm 2}, \ldots}.
\]  

(4)

It is well known that in the case of flat 2D space \( (g_{ab} = \delta_{ab}) \) such a system can be described in a dual, coulomb gas picture, in terms of a 2D field theory of one massless scalar field \( \tilde{x}(\sigma) \) (not necessarily compactified) perturbed by vortex operators built out of this field. In the case of fluctuating metric this scalar field gets coupled to the Liouville field. The action (at least in the small coupling regime \( \phi \to \infty \)) looks as

\[
S = \frac{1}{4\pi} \int d^2z \left[ (\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu' \phi e^{-2\phi} + \sum_{n \neq 0} \lambda'_n e^{(n|R-2)\phi} e^{in\tilde{x}} \right],
\]  

(5)

where \( x = x_L + x_R \), \( \tilde{x} = x_L - x_R \) (\( x_L \) and \( x_R \) are left and right movers). The total curvature is normalized as \( \int d^2z \sqrt{\hat{R}} = 2\pi(2 - 2h) \), where \( h \) is the genus of the surface. Note that we chose the central charge \( c = 26 \) and the Liouville “charges” of vortex operators to make them marginal. The compactification radius \( R \) remains a free parameter of the theory. If we choose the couplings as

\[
\lambda'_n = \frac{\lambda'}{2}(\delta_{n,1} + \delta_{n, -1}),
\]  

(6)

the model will turn into the Sine-Gordon theory coupled to 2D gravity:

\[
S_{SGG} = \frac{1}{4\pi} \int d^2z \left[ (\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu' \phi e^{-2\phi} + \lambda' e^{(R-2)\phi} \cos R\tilde{x} \right].
\]  

(7)

As its counterparts on the flat space, this theory is supposed to be in the same universality class as the compactified 2D string theory \([1]\).

In particular case of vanishing cosmological constant \( \mu' \) and \( R = 3/2 \) (here \( R_{KT} = 2 \)) the CFT \([2]\) was conjectured \([5]\) to be dual to the coset CFT describing the string theory on the dilatonic black hole background \([8, 9]\) (see \([3]\) for the details).

The matrix model version of this theory is represented by the partition function of the MQM on a “time” circle of the radius \( R \)

\[
Z_N[\lambda, g] = \int [d\Omega]_{SU(N)} \exp \left( \sum_{n \in \mathbb{Z}} \lambda_n \text{tr} \Omega^n \right) Z_N(\Omega, g),
\]  

(8)
where
\[ Z_N(\Omega, g) = \int_{M(2\pi R) = \Omega^\dagger M(0) \Omega} D M(x) e^{-tr \int_0^{2\pi R} dx \left[ \frac{1}{2} (\partial_x M)^2 + V(M) \right]}. \]  

(9)

We take the twisted boundary conditions for the hermitian matrix field \( M(x) \), where the twist matrix \( \Omega \) can be chosen without lack of generality as a Cartan element of the \( SU(N) \) group \( \Omega = diag(z_1, z_2, \ldots, z_N) \). The matrix potential is, for example, \( V(M) = \frac{1}{2} M^2 - \frac{g}{3\sqrt{N}} M^3 \). The couplings \( \lambda_n \) will play the same role here as \( \lambda'_n \) in (5) although the exact relation between them depends on the regularization procedure (the shape of the potential \( V(M) \)).

Following the usual logic of the “old” matrix models, the Feynman expansion of this model can be shown (see [3] for the details) to describe the compactified lattice scalar field \( x_i \) living in the vertices \( i = 1, 2, 3, \ldots \) of \( \phi^3 \) type planar graphs. This field can have vortex configurations: the vortices of charges \( n = \pm 1, \pm 2, \ldots \) occur on the faces of planar graphs and are weighted with the factors \( \lambda_n \). So the model (8) represents a lattice analogue of the continuous partition function for the models (1), (4) or (5). The sum over planar graphs represents the functional integral over two dimensional metrics of the world sheet and due to the standard ‘t Hooft argument \( 1/N \) expansion goes over the powers \( 1/N^2 - 2h \) where \( h \) is a genus of the world sheet.

This model has a long history. In [10] some estimates of the matrix approach were compared to the predictions of (4) and appeared to be in full agreement. In [2] the model (1) was formulated in terms of a twisted inverted matrix oscillator (9). The Toda hierarchy description of the model (for the usual stable oscillatorial potential) was proposed in [4]. In the paper [11] the dual version of the matrix model of [1], [10], the Sine-Gordon model on random graphs, was used to find (or rather correctly conjecture) the free energy of (5) as a function of the cosmological constant \( \mu' \), Sine-Gordon coupling \( \lambda' \) and radius \( R \).

The twisted inverted matrix oscillator of [2] appeared to be very useful for the identification of the free energy of the whole theory (4) and of the corresponding matrix model (8) with the \( \tau \)-function of the Toda chain hierarchy of integrable PDF’s [3]. To see how this description emerges we pass to the grand canonical partition function:

\[ Z_{\mu}[\lambda, g] = \sum_{N=0}^{\infty} e^{-2\pi R_{\mu N}} Z_N[\lambda, g]. \]

(10)

The chemical potential \( \mu \) happens to be the cosmological constant similar to \( \tilde{\mu} \) appearing in (7) (see [12] and [2] for the details). Then following the double scaling prescription for the \( c = 1 \) matrix model [13] corresponding to the continuous limit for the lattice world sheets we should send in (8) \( N \to \infty \) and \( g \to g_{\text{crit}} \) in such a way that at the saddle point of the sum over \( N \) in (10)

\[ \mu = \frac{1}{2\pi R} \frac{\partial}{\partial N} \log Z_N[\lambda, g] \]

remains fixed. This amounts to shifting \( M \to M + \sqrt{N} \frac{\lambda}{g} \) and neglecting the cubic term in the potential \( V(M) \), or rather treating it as a cut-off wall at \( M \to \infty \), which gives \( V(M) \simeq -\frac{1}{2} M^2 \). Then the integral [4] can be calculated exactly, giving:

\[ Z_N[\lambda] = \frac{1}{N!} \int \prod_{k=1}^{N} \frac{dz_k}{2\pi i z_k} \frac{e^{2u(z_k)}}{(q^{1/2} - q^{-1/2})} \prod_{j \neq j'} \frac{z_j - z_{j'}}{q^{1/2} z_j - q^{-1/2} z_{j'}}. \]

(11)
where \( u(z) = \frac{1}{2} \sum_n \lambda_n z^n \), \( q = e^{2\pi i R} \).

For the identification with \( \tau \)-function it is convenient to redefine the vortex couplings as

\[
t_n = \frac{\lambda_n}{q^{n/2} - q^{-n/2}}.
\]

Then plugging (11) into (10) we recognize in it the \( \tau \)-function of Toda lattice hierarchy [14, 15]

\[
\tau_l[t] = e^{-\sum_{n} n t_n t^{-n} \sum_{N=0}^{\infty} (q^l e^{-2\pi R \mu})^N Z_N[t]} = e^{-\sum_{n} n t_n t^{-n} Z_{\mu-l}[t]},
\]

where the charge \( l \) giving an extra “lattice” dimension to the Toda equations appears to be an imaginary integer shift of the cosmological constant \( \mu \). The couplings \( t_n \) of vortices, together with \( t_0 = \mu \) turn out to be the “times” of commuting flows of the hierarchy. Due to this the whole sector of the theory describing the dynamics of winding modes of the model (1) is completely solvable and the calculations of particular physical quantities, such as the correlators of vortex operators, are greatly simplified. In particular, the \( \tau \) function (13) represents the generating function for such correlators in the theory (7):

\[
K_{i_1 \cdots i_n} = \left. \frac{\partial^n}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} \log \tau_0 \right|_{\lambda_{\pm 2} = \lambda_{\pm 3} = \cdots = 0}.
\]

with \( \mu \) and \( \lambda_{\pm 1} \) fixed.

Using the Hirota equations of Toda hierarchy we will calculate in this paper the one-point and two-point functions of vortex operators of arbitrary vorticities in the spherical approximation.

### 2.2 Hirota equations

One can show [16] that the ensemble of the \( \tau \)-functions of the Toda hierarchy with different charges satisfies a set of bilinear partial differential equations known as Hirota equations which are written as follows (the derivatives are taken with respect to \( t_n \) rather than \( \lambda_n \))

\[
\sum_{j=0}^{\infty} p_{j+i}(-2y_+) p_j(\tilde{D}_+) \exp \left( \sum_{k \neq 0} y_k D_k \right) \tau_{i+t+1}[t] \cdot \tau_i[t] = \\
\sum_{j=0}^{\infty} p_{j-i}(-2y_-) p_j(\tilde{D}_-) \exp \left( \sum_{k \neq 0} y_k D_k \right) \tau_{i+t}[t] \cdot \tau_{i+1}[t],
\]

where

\[
y_{\pm} = (y_{\pm 1}, y_{\pm 2}, y_{\pm 3}, \ldots),
\]

\[
\tilde{D}_{\pm} = (D_{\pm 1}, D_{\pm 2}/2, D_{\pm 3}/3, \ldots)
\]

represent the Hirota’s bilinear operators

\[
D_n f[t] \cdot g[t] = \left. \frac{\partial}{\partial x} f(t_n + x) g(t_n - x) \right|_{x=0},
\]

with \( u(z) = \frac{1}{2} \sum_n \lambda_n z^n \), \( q = e^{2\pi i R} \).
and $p_j$ are Schur polynomials defined by

$$\sum_{k=0}^{\infty} p_k[t]x^k = \exp \left( \sum_{k=1}^{\infty} t_n x^n \right).$$

(19)

Since the $\tau$-function of the Toda hierarchy is related to the free energy by

$$\tau_s[\mu, t] = \exp \left( F(\mu - is) \right),$$

(20)

the Hirota equations lead to a triangular system of nonlinear difference-differential equations for the free energy of the model (10)-(11). Due to (14) it can be actually thought of as a system of equations for the correlators which we are interested in.

### 2.3 Scaling of winding operators

It turns out that there is a scale in the model which corresponds to the scale given by the string coupling constant in the string theory. Since $g_s \sim e^{\phi}$ it can be associated with the Liouville field $\phi$ or the cosmological constant $\mu'$ coupled with it. All couplings $\lambda'_n$ have definite dimensions with respect to this scale. In the Toda hierarchy the counterparts of $\mu'$ and $\lambda'_n$ are $\mu$ and $t_n$ correspondingly. As one can immediately see from (5) by shifting the zero mode of the Liouville field, the corresponding dimensions of the couplings $t_n$ with respect to the rescaling of the cosmological constant $\mu$ are

$$\Delta[t_n] = 1 - \frac{R|n|}{2}.\quad(21)$$

This scaling leads to an expansion for the free energy which can be interpreted as its genus expansion

$$F = \sum_{h=0}^{\infty} F_h,$$

(22)

$$F_0 = \xi^{-2} f_0(\omega; s) + \frac{R}{2} \mu^2 \log \xi,$$

(23)

$$F_h = \xi^{2h-2} f_h(\omega; s), \quad h > 0,$$

(24)

where the quantities $f_h$ are the functions of the dimensionless parameters

$$s = (s_2, s_3, \ldots), \quad s_n = i \left( -\frac{t_n}{t_1} \right)^{n/2} \xi^{\Delta[t_n]} t_n,$$

(25)

$$\omega = \mu \xi, \quad \text{with} \quad \xi = (\lambda \sqrt{R-1})^{-\frac{2}{\pi}} = (t_1 t_{-1} (R-1))^{-\frac{1}{\pi}}.\quad(26)$$

Remarkably, the Toda equations (15) are compatible with this scaling.

The spherical limit of the string partition function is described by the dispersionless Toda hierarchy which can be obtained taking $\xi \longrightarrow 0$. In this limit the explicit expression for the partition function with vanishing $s_n$ has been conjectured in [11] and proven in [3]. The result formulated in terms of $X_0(\omega) = \partial^2_\omega f_0(\omega)$ reads

$$w = e^{-\frac{1}{\pi} X_0} - e^{-\frac{R-1}{\pi} X_0}.$$

(27)
or, integrating $X_0$ twice over $\omega$

$$F_0 = \frac{1}{2} \mu^2 (R \log \xi + X_0) + \xi^{-2} \left( \frac{3}{4} \frac{R}{R-1} e^{-\frac{R}{2} \xi} x_0 + \frac{3}{4} Re^{-\frac{\mu}{\xi}} - \frac{R^2 - R + 1}{R - 1} e^{-x_0} \right). \quad (28)$$

This result will serve as an input for the following calculation of the correlators in the spherical limit.

### 3 Two-point correlators from the Toda hierarchy

#### 3.1 Equation for the generating function of two-point correlators

To find equations for the two-point correlators in the spherical limit let us take in (15) $i = 0$ and the coefficient in front of $y_n y_m$, $n, m > 0$. Then we obtain the following equation

$$\left[ 4 p_{n+m}(\tilde{D}_+) - 2 p_n(\tilde{D}_+)D_m - 2 p_m(\tilde{D}_+)D_n + D_n D_m \right] \tau_{s+1} \cdot \tau_s = D_n D_m \tau_s \cdot \tau_{s+1},$$

$$\left[ 2 p_{n+m}(\tilde{D}_+) - p_n(\tilde{D}_+)D_m - p_m(\tilde{D}_+)D_n \right] \tau_{s+1} \cdot \tau_s = 0. \quad (29)$$

This equation is valid also at $m = 0$ or $n = 0$ if we set $D_0 \equiv 1$. Such an equation can be obtained from the coefficient in front of $y_n$ in (15).

If we multiply the equation (29) by $x^n y^m$ and sum over all $n$ and $m$ we obtain

$$\left[ 2 \sum_{n,m=0}^{\infty} x^n y^m p_{n+m}(\tilde{D}_+) - \sum_{m=0}^{\infty} y^m D_m \exp \left\{ \sum_{n=1}^{\infty} x^n \tilde{D}_n \right\} \right. - \sum_{n=0}^{\infty} x^n D_n \exp \left\{ \sum_{m=1}^{\infty} y^m \tilde{D}_m \right\} \left] \tau_{s+1} \cdot \tau_s = 0. \quad (30)$$

Furthermore, it is easy to see that the first term can be rewritten as follows:

$$\frac{2}{x - y} \left[ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} x^k \tilde{D}_k - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} y^k \tilde{D}_k \right] \tau_{s+1} \cdot \tau_s. \quad (31)$$

It is easy to check the following remarkable formula

$$f(D)e^F \cdot e^G = e^F e^G f(D + (\partial F \cdot 1 - 1 \cdot \partial G)), \quad (32)$$

which allows to express $\tau$-functions in the equation (31), through the derivatives of the free energy:

$$\frac{2}{x - y} \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} x^k (\tilde{D}_k + \tilde{X}_k) - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} y^k (\tilde{D}_k + \tilde{X}_k) \right) = \left( 1 + \sum_{m=1}^{\infty} y^m (D_m + X_m) \right) \sum_{k=1}^{\infty} x^k (\tilde{D}_k + \tilde{X}_k) + 1 + \sum_{n=1}^{\infty} x^n (D_n + X_n) \sum_{k=1}^{\infty} y^k (\tilde{D}_k + \tilde{X}_k), \quad (33)$$
where we have introduced the notation $\tilde{X}_k = \frac{1}{|k|} X_k$ and 
\[ X_k = \partial_k F_{ \frac{1}{2} } \cdot 1 - 1 \cdot \partial_k F_{ -\frac{1}{2} }, \quad k \neq 0. \] (34)
The 1’s in the right hand side of (33) arise from the definition of $D_0$.

We show in Appendix A that in the dispersionless limit only the second derivatives of the free energy survive. This means that we can put the second and higher derivatives of $X_k$ to zero: $D^n X_k = 0$, $n > 1$. Besides, using (20) we obtain in this limit
\[ \tilde{X}_n = -i \frac{1}{|n|} \frac{\partial^2}{\partial_t n \partial \mu} F_0 := \tilde{X}_{0,n}, \] (35)
\[ \tilde{D}_n \tilde{X}_m = 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial t n \partial t m} F_0 := \tilde{X}_{n,m}. \] (36)
Thus the fields $\tilde{X}_{n,m}$ correspond (up to a numerical factor) to the two-point correlators, whereas $\tilde{X}_{0,n}$ give the one-point ones. We also define the generating function of all $\tilde{X}$’s
\[ F(x, y) = \sum_{n,m=0}^{\infty} x^n y^m \tilde{X}_{n,m}, \] (37)
where we have taken $\tilde{X}_{0,0} = 0$.

With these definitions, trivial manipulations which can be found in Appendix B, lead to the following equation for the generating function
\[ \frac{x + y}{x - y} \left( e^{\frac{1}{2} F(x,x)} - e^{\frac{1}{2} F(y,y)} \right) = x \partial_x F(x, y) e^{\frac{1}{2} F(y,y)} + y \partial_y F(x, y) e^{\frac{1}{2} F(x,x)}. \] (38)

### 3.2 Solution in terms of one-point correlators

It turns out that the differential equation (38) is solvable. First of all, change the variables to
\[ d\eta = e^{\frac{1}{2} F(x,x)} \frac{dx}{x} + e^{\frac{1}{2} F(y,y)} \frac{dy}{y}, \]
\[ d\zeta = e^{\frac{1}{2} F(x,x)} \frac{dx}{x} - e^{\frac{1}{2} F(y,y)} \frac{dy}{y}. \] (39)
It is easy to see that
\[ \zeta = \int_{y}^{x} \frac{dz}{z} e^{\frac{1}{2} F(z,z)} = \log \frac{x}{y} + h(x) - h(y), \] (40)
where
\[ h(x) = \int_{0}^{x} \frac{dz}{z} \left( e^{\frac{1}{2} F(z,z)} - 1 \right) \] (41)
\footnote{We can shift the index $s$ of the free energy $F_s = \log \tau_s$ since due to (20) the resulting equation depends only on the difference of the indices of two $\tau$-functions.}
is a regular function at $x = 0$.

In terms of the new variables the equation reads

$$
\partial_\eta F(x, y) = \frac{1}{2} \frac{x + y}{x - y} \left( e^{-\frac{1}{2}F(y,y)} - e^{-\frac{1}{2}F(x,x)} \right) = \partial_\eta \log \frac{xy}{(x - y)^2}.
$$

Thus

$$
F(x, y) = \log \frac{xy}{(x - y)^2} + g(\zeta),
$$

where the function $g(\zeta)$ is fixed by the requirement for $F(x, y)$ to be analitycal in both arguments.

The analitycity condition together with (40) leads to the following requiremnts on $g(\zeta)$:

i) $g(\zeta) \sim \zeta \to \pm \infty$; ii) $g(\zeta) \sim \zeta \to 0 \log \zeta^2$. There is only one function satisfying both of them, which is given by

$$
g(\zeta) = \log \left( e^\zeta + e^{-\zeta} - 2 \right) = \log \left( 4 \text{sh}^2 \left( \frac{\zeta}{2} \right) \right).
$$

Taking together (43), (44) and (40) we obtain the solution for the generating function of the two-point correlators in terms of a still unknown regular function $h(x)$

$$
F(x, y) = \log \left[ \frac{4xy}{(x - y)^2} \text{sh}^2 \left( \frac{1}{2} (h(x) - h(y) + \log \frac{x}{y}) \right) \right].
$$

From (45) it is easy to see that $h(x)$ is nothing else than the generating function for the one-point correlators since $h(x) = F(x, 0)$.

### 3.3 Two-point correlators with vorticities of opposite signs

Up to now we considered the two-point correlators for vorticities of positive sign only. Now we generalize the results for other cases.

First of all, the case of both negative signs can be directly obtained from the previous one. The corresponding equation which is derived from the coefficient in front of $y - n y - m$ in the Hirota equation looks as (29)

$$
\left[ 2p_{n+m}(\tilde{D}_-) - p_n(\tilde{D}_-)D_{-m} - p_m(\tilde{D}_-)D_{-n} \right] \tau_s \cdot \tau_{s+1} = 0.
$$

The only difference is that $\tau_s$ and $\tau_{s+1}$ are exchanged. Due to this there is an additional minus in the definition of $\tilde{X}_{0,-n}$. So we should distinguish two types of correlators

$$
\tilde{X}_{0,m}^\pm := \mp i \frac{1}{|m|} \frac{\partial^2}{\partial \eta_n \partial \eta_n} F_0, \quad n \neq 0,
$$

$$
\tilde{X}_{n,m}^\pm := 2 \frac{1}{|n||m|} \frac{\partial^2}{\partial \eta_n \partial \eta_m} F_0, \quad n, m \neq 0.
$$

With this definition, the generating functions of the correlators for positive and negative vorticities are

$$
F^\pm(x, y) = \sum_{n, m=0}^{\infty} x^n y^m \tilde{X}_{n,\pm m}^\pm.
$$
Despite of this minus all equations for the case of negative vorticities being written in terms of \( \tilde{X}_{n,m} \) are the same as for the case of positive ones. As a result \( F^-(x,y) \) is given by the same solution (43) as \( F^+(x,y) \). The only difference between them is the generating function of the one-point correlators \( h \) which should be replaced in (43) by \( h^\pm \) correspondingly.

To find the correlators with vorticities of opposite signs, take the coefficient in the Hirota equation in front of \( y_n y_m \) and set \( i = 1 \). Then we obtain the following equation

\[
-2p_n(\tilde{D}_+ D_m \tau_{s+1} \cdot \tau_s) = [-2p_m(\tilde{D}_-) D_n + \tilde{D}_- D_m D_n] \tau_{s+1} \cdot \tau_{s+1},
\]

\[
p_n(\tilde{D}_+ ) D_m \tau_{s+2} \cdot \tau_s = p_m(\tilde{D}_-) D_n \tau_{s+1} \cdot \tau_{s+1}.
\]

The last term in the first line vanishes since it always gives rise to \( D_k (F_{s+1} - 1 \cdot F_{s+1}) = 0 \). In general, the one-point correlators appearing from \( \tau_{s+k} \cdot \tau_s \) are always supplied with the coefficient \( k \) what follows from (20). Due to this in the right hand side there are no one-point correlators whereas in the left hand side they enter with the coefficient 2. This equation is valid also for \( m = 0 \) if we take \( D_0 \equiv -2 \) instead of the previous choice. However, for \( n = 0 \) it is never fulfilled.

The arguments similar to those of Section 1 lead to the following equation for the generating function

\[
A \left[ y \partial_y (G(x,y) - 2h^-(y)) - 2 \right] e^{\frac{1}{2} F^+(x,x) + h^+(x)} = \frac{1}{y} \partial_x G(x,y) e^{\frac{1}{2} F^-(y,y) - h^-(y)},
\]

where we have introduced

\[
A = \exp \left( -\partial_\mu^2 F_0 \right) = \xi^{-R} e^{-\partial_\xi^2 f_0},
\]

\[
G(x,y) = \sum_{n,m=1}^\infty x^n y^m \tilde{X}_{n,m}.
\]

Using (11) we can rewrite (51) as follows

\[
\left( A \partial_x (xe^{h^+(x)}) \right)^{-1} \partial_x + \left( \partial_y (\frac{1}{y} e^{-h^-(y)}) \right)^{-1} \partial_y \right) G(x,y) = -2ye^{-h^-(y)}.
\]

It is clear that the solution of this equation looks as

\[
G(x,y) = 2 \log \left( ye^{-h^-(y)} \right) + f \left( \frac{1}{y} e^{-h^-(y)} - Ax e^{h^+(x)} \right).
\]

The unknown function \( f \) is determined by the analitycity of \( G(x,y) \). Requiring cancelation of the logarithmic singularity at \( y = 0 \) we obtain that \( f(z) = 2 \log z \). As a result we find the generating function

\[
G(x,y) = 2 \log \left( 1 - Ax ye^{h^+(x) + h^-(y)} \right).
\]

Let us note that the equations (43) and (56) resemble the equations for two-point correlators of the two-matrix model found in [17].
4 One-point correlators

4.1 Equation for the generating function

Now we should work out an equation for the one-point correlators. We can always put $t_1 = -t_{-1}$ (corresponding to $\lambda_1 = \lambda_{-1}$, see (12)). From the equations written below for positive vorticities it is clear that in this case $h^\pm(x)$ coincide with each other. For this reason we omit in the following the inessential sign label in $h(x)$. The case of general $t$'s can be obtained from the previous one by the substitution $h^\pm(x) = h\left((-t_{-1}/t_1)^{\pm1/2}\right)$. It can be easily seen from (25) and is due to the total vorticity conservation. Accordingly,

$$F^\pm(x, y) = F\left((-t_{-1}/t_1)^{\pm1/2}x, (-t_{-1}/t_1)^{\pm1/2}y\right),$$

$$G(x, y) = G_{t_1=-t_{-1}}\left((-t_{-1}/t_1)^{1/2}x, (-t_{-1}/t_1)^{-1/2}y\right).$$

The equation we are looking for arises from two facts: i) we know the dependence of the free energy on $t_{\pm1}$ and $\mu$ (27), (28); ii) the system of equations for the correlators is triangular. Due to this we can express $\tilde{X}_{\pm1,n}$ through $\tilde{X}_{0,n}$ by a linear integral-differential operator. Namely, in terms of the generating functions we can write

$$\partial_y F^+(x, 0) = \hat{K}^+(h(x)) + \tilde{X}_{1,0},$$

$$\partial_y G(x, 0) = \hat{K}^-(h(x)),$$

where the operators $\hat{K}^\pm$ are to be found. The last term in the first line is explicitly added since in our notations $F^+(0, 0) = X_{0,0}^+ = 0$. Moreover, we can rewrite these equations in terms of the generating function for the one-point correlators only. Indeed, from (17) and (56) we obtain

$$\hat{K}^+(h(x)) = \frac{2}{x}(1 - e^{-h(x)}) - 2\tilde{X}_{0,1}^+,$$

$$\hat{K}^-(h(x)) = -2Ax e^{h(x)}.$$

To find the operators $\hat{K}^\pm$ we compare the derivatives of the free energy with respect to $\mu$ and $t_{\pm1}$. From (23), (25) and (28) we obtain for $|n| > 1$

$$\frac{\partial}{\partial \mu} \left. \left( \frac{\partial}{\partial t_n} F_0 \right) \right|_{s_k=0} = i \xi^{-|n|R/2} \frac{\partial}{\partial \omega} \left( \frac{\partial}{\partial s_n} f_0(\omega; s) \right) \bigg|_{s_k=0},$$

$$\frac{\partial}{\partial t_{\pm1}} \left. \left( \frac{\partial}{\partial t_n} F_0 \right) \right|_{s_k=0} = \frac{2}{2-R} \left( \omega \frac{\partial}{\partial \omega} - \left( 1 + \frac{R|n|}{2} \pm \frac{(R-2)|n|}{2} \right) \left( \frac{\partial}{\partial s_n} f_0(y; s) \right) \right) \bigg|_{s_k=0}.$$  

These relations together with (33) and (34) imply that the operators are given by

$$\hat{K}^+ = -a \left[ \omega - (1 + (R-1)x) \frac{\partial}{\partial x} \int d\omega \right],$$

$$\hat{K}^- = a \left[ \omega - (1 + x) \frac{\partial}{\partial x} \int d\omega \right],$$

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where \( a = 2 \sqrt{\frac{R}{R-1}} \xi^{-\frac{R}{2}} \). This expression is also valid for the case \( n = \pm 1 \) where the derivatives of the free energy contain additional terms appearing from the logarithmic term in (23).

Inserting (65) and (66) into (61) and (62) and taking the derivative with respect to \( \omega \) we obtain two differential equations for the generating function of the one-point correlators

\[
\left[ -a \left( (R-1)x \frac{\partial}{\partial x} - \omega \frac{\partial}{\partial \omega} \right) + \frac{2}{x} e^{-h(x;\omega)} \frac{\partial}{\partial \omega} \right] h(x;\omega) = 2 \frac{\partial}{\partial \omega} \tilde{X}^+_{0,1} \tag{67}
\]

\[
\left[ a \left( x \frac{\partial}{\partial x} - \omega \frac{\partial}{\partial \omega} \right) - 2Ax e^{h(x;\omega)} \frac{\partial}{\partial \omega} \right] h(x;\omega) = 2xe^{h(x;\omega)} \frac{\partial}{\partial \omega} A. \tag{68}
\]

### 4.2 Solution

Let us take advantage of having two equations for one quantity. Using (67) and (68) we can exclude the derivative with respect to \( x \). The resulting equation is

\[
\left( a(2-R)\omega + 2e^{-h} - 2(R-1)Ax e^h \right) \frac{\partial}{\partial \omega} h = 2\frac{\partial}{\partial \omega} \tilde{X}^+_{0,1} + 2(R-1)\frac{\partial}{\partial \omega} Ax e^h. \tag{69}
\]

From (47), (23) and (28) one can obtain

\[
\tilde{X}^+_{0,1} = -i \frac{\partial^2}{\partial \mu \partial t} F_0 \bigg|_{s_k=0} = \frac{\sqrt{R-1}}{2-R} \xi^{-R/2} ((\omega \frac{\partial}{\partial \omega} - 1) \frac{\partial}{\partial \omega} f_0(\omega) + R\omega) = \frac{R}{\sqrt{R-1}} \xi^{-R/2} e^{-\frac{R-1}{R} X_0}. \tag{70}
\]

Noting that the equation (69) is homogeneous in the derivative \( \frac{\partial}{\partial \omega} \) we can change it by \( \frac{\partial}{\partial X_0} \). Then after the substitution of the definitions of all entries the common multiplier

\[
2\sqrt{R-1} \xi^{-R/2} \left( e^{-\frac{R-1}{R} X_0} + \sqrt{R-1} \xi^{-R/2} xe^{-X_0} \right)
\]

can be canceled. As a result we obtain the following simple equation

\[
\left( 1 - \frac{\xi^{R/2}}{R-1} xe^{-\frac{R-1}{R} X_0} e^{-h} \right) \frac{\partial}{\partial X_0} h = 1. \tag{71}
\]

This equation can be trivially integrated giving after the proper choice of the integration constant

\[
e^{\frac{1}{R} h} - ze^h = 1, \tag{72}
\]

where \( z = \frac{\xi^{-R/2}}{\sqrt{R-1}} xe^{-\frac{R-1}{R} X_0} \). As it is easy to check this solution satisfies both equations (67) and (68).

### 4.3 One-point correlators of fixed vorticities

Explicit expressions for the one-point correlators are given by the coefficients of the expansion in \( x \) by the generating function \( h(x) \) satisfying equation (72). To find them we can use equation
5.2.13.30 in [18]

\[ \frac{1}{b} s^b = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(n(a - 1) + b + 1)} \Gamma(na + b). \]

(73)

\[ s = 1 + zs^a. \]

(74)

Taking \( s = e^{\frac{1}{2}h} \), \( a = R \) we obtain for \( b = kR \)

\[ e^{kh} = kR \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma((n + k)R - n + 1)}. \]

(75)

Also the limit \( b \to 0 \) gives

\[ h = R \sum_{n=1}^{\infty} \frac{z^n}{n! \Gamma(n(R - 1) + 1)}. \]

(76)

This means that the derivative with respect to \( \mu \) of the one-point correlator is

\[ \frac{\partial^2}{\partial \mu \partial t_n} F = \frac{i \Gamma(nR + 1)}{n! \Gamma(n(R - 1) + 1)} \frac{\xi^{-nR/2}}{(R - 1)^{n/2}} e^{-\frac{nR}{R} X_0}. \]

(77)

After integration over \( \mu \) and taking into account the relation (12) we obtain the following one-point correlators of operators of vorticity \( n \) in the spherical limit

\[ \mathcal{K}_n = \frac{1}{2 \sin \pi n R} \frac{\Gamma(nR + 1)}{n! \Gamma(n(R - 1) + 1)} \frac{\xi^{-nR/2}}{(R - 1)^{n/2}} \left( e^{-\frac{n(R - 1) + 1}{R} X_0} - e^{-\frac{n(R - 1) + 1}{R} X_0} \right). \]

(78)

(Here the lower limit of the integration is chosen to be \( X_0 = +\infty \). Only this choice reproduces \( \partial \lambda F_0 \) from (28).) In particular, in the limit \( \mu \to 0 \) (\( X_0 \to 0 \)) corresponding to string theory on the black hole background (in fact, only the point \( R = 3/2 \) is supposed to describe the conventional black hole) we find

\[ \mathcal{K}_n = \frac{1}{2 \sin \pi n R} \frac{\xi^{-nR/2}}{(R - 1)^{n/2}} \left( 2 - R \right) n \Gamma(nR + 1) \frac{(2 - R)n \Gamma(nR + 1)}{(n + 1)! \Gamma(n(R - 1) + 2)}. \]

(79)

5 Results for two-point correlators

5.1 Correlators with vorticities of opposite signs

These correlators are given by the expansion coefficients of the generating function (56). This function can be easily expanded in \( e^h \) for which the series (75) can be used. This leads to the following expansion

\[ D(x, y) = -2 \sum_{k=1}^{\infty} \frac{B^k}{k} z^k(x) z^k(y) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C^k_n C^k_m z^n(x) z^m(y) \]

\[ = -2 \sum_{n,m=1}^{\infty} z^n(x) z^m(y) \sum_{k=1}^{\min(n,m)} \frac{B^k}{k} C^k_{n-k} C^k_{m-k}, \]

(80)
where

\[ B = A(R - 1)\xi^R e^{2\frac{R}{\pi^2} x_0} = (R - 1)e^{\frac{R+2}{\pi} x_0}, \]  

\[ C^k_n = \frac{kR}{n!} \frac{\Gamma((n+k)R)}{\Gamma(n(R-1) + kR+1)}. \]  

The two-point correlators appear to be

\[
K_{n,m} = - \frac{\Gamma(nR + 1)\Gamma(mR + 1)}{2 \sin \pi n R} \frac{\xi^{-(n+m)R/2}}{2 \sin \pi m R} \frac{2}{(R - 1)(n+m)/2} e^{-(n+m)\frac{R+1}{\pi} x_0} \times
\]
\[
\sum_{k=1}^{\min(n,m)} k(R - 1)^k e^{k\frac{R+1}{\pi} X_0} \times
\]
\[
\frac{(n-k)!(m-k)!\Gamma(n(R-1) + k + 1)\Gamma(m(R-1) + k + 1)}{\Gamma(n(R-1) + k + 1)\Gamma(m(R-1) + k + 1)}. \]  

To get the result in the black hole limit it is enough to set \( X_0 = 0 \) in the above expression.

It is worth to give also the result for the ratios of correlators which do not depend on possible leg-factors — wave function renormalisations of the operators (see for ex. [12]):

\[
\frac{K_{n,m}}{K_{n,n-m}} = - \frac{\xi^2}{(2 - R)^2 n m} \sum_{k=1}^{\min(n,m)} k(R - 1)^k(n + 1)!(m + 1)\Gamma(n(R-1) + 2)\Gamma(m(R-1) + 2)\Gamma(n(R-1) + k + 1)\Gamma(m(R-1) + k + 1). \]  

Since they can in principle be compared with CFT calculations in the black hole limit we restrict ourselves only to this particular case.

### 5.2 Correlators with vorticities of same signs

To investigate this case we should expand (12) in \( x \) and \( y \). The expansion in the first argument gives for \( n > 0 \)

\[
\partial_x^n F(0, y) = \frac{2}{y^n} \left( \frac{1}{n} - \sum_{k=1}^{n} \frac{1}{k} \sum_{m=0}^{\infty} z^{n-k}(y) e^{-kh(y)} C^k_{n-k} \right) - \tilde{X}_{0,n}
\]
\[
= -2\frac{\xi^{nR/2}}{(R - 1)^n/2} e^{-n\frac{R+1}{\pi} X_0} \sum_{m=0}^{\infty} z^m(y) \sum_{k=1}^{n} \frac{1}{k} C^k_{n-k} C^{-k}_{n+m} - \tilde{X}_{0,n}
\]
\[
+ \frac{2}{y^n} \left( \frac{1}{n} - \sum_{m=1}^{n} z^{-m}(y) \sum_{k=m}^{n} \frac{1}{k} C^k_{n-k} C^{-k}_{n-m} \right).
\]  

The last term in the right hand side is singular and we know from the analycity of \( F(x, y) \) that it should vanish. The remaining part gives the values of the two-point correlators

\[
K_{n,m} = - \frac{\Gamma(nR + 1)\Gamma(mR + 1)}{2 \sin \pi n R} \frac{\xi^{-(n+m)R/2}}{2 \sin \pi m R} \frac{2}{(R - 1)(n+m)/2} e^{-(n+m)\frac{R+1}{\pi} x_0} \times
\]
\[
\sum_{k=1}^{n} \frac{k}{(n-k)!(m+k)!\Gamma(n(R-1) + k + 1)\Gamma(m(R-1) - k + 1)}. \]  

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As in the previous case the black hole limit of (86) is obtained vanishing \(X_0\). The normalized correlators are

\[
\frac{\mathcal{K}_{n,m}}{\mathcal{K}_n \mathcal{K}_m} = -\frac{\xi^2}{(2-R)^2 nm} \sum_{k=1}^{n} \frac{k(n+1)!m!(n-1)\Gamma(n(R-1)+2)\Gamma(m(R-1)+2)}{(n-k)!(m+k)!\Gamma(n(R-1)+k+1)\Gamma(m(R-1)-k+1)}.
\] (87)

It is clear from the definition that (86) as well as (87) should be symmetric in \(m\) and \(n\). However, our results do not possess this symmetry explicitly. The reason is that the expansion of (45) has been done in a nonsymmetric way. It may be possible to symmetrize them by means of identities between \(\Gamma\)-functions like

\[
\sum_{k=m}^{n} \frac{1}{k} C_{n-k}^k C_{k-m}^{-k} = 0, \quad 1 \leq m < n.
\] (88)

This identity follows from vanishing of the last term in (85).

### 6 Discussion

The main result of this paper is the calculation of one- and two-point vorticity (winding) correlators given by the generating functions (45), (56) and (72). What kind of physics do they describe?

We can look at our model from two points of view: as a statistical-mechanical system describing a gas of vortices and as a string theory.

**Statistical mechanical picture**

In the statistical mechanical picture we interpret the planar graphs of the MQM (9) as random dynamical lattices populated by Berezinski-Kosterlitz-Thouless (BKT) vortices.

Our one point correlator \(\mathcal{K}_n\) (78) should be in principle proportional to the probability to find in the system a vortex of a given vorticity \(n\). The probability should be a positive quantity. However, we see that the explicit expressions (78)-(79) contain the sign-changing factors \(\frac{1}{\sin \pi Rn}\). Their origin is not completely clear to us but we see two possible explanations for them:

i) They appeared as a result of wave function renormalisation in the continuous (double scaling) limit and should be absorbed into the “leg-factors”. The rest of the one-point correlator is strictly positive in the interval \(1 < R < 2\) which we pretend to describe. This point of view is supported by the fact that these factors appear as the same wave-function renormalisations also in the two point correlators (83),(86). Since their origin is completely due to the change of variables (12) these factors appear to be a general feature for all multipoint correlators.

ii) The sign-changing might be a consequence of the fact that our approach based on the Hirota equations (describing directly the double scaling limit) does not keep track of non-universal terms due to the ultraviolet (lattice) cut-off in the system. Such UV divergences are well known in the Coulomb gas description of the BKT system: they usually correspond to the proper energy of a vortex. In that case the correlators we calculated above are only the universal parts of the full correlators containing also big positive non-universal contributions.

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These universal parts cannot be interpreted as full probabilities and have no reason to be positively defined.

To prove or reject these explanations we have to see what happens with the individual vortices in the process of taking the double scaling limit, from planar graphs picture to the inverted oscillator description. It is not an easy, although not hopeless, task which we leave for the future.

**String theory picture**

The MQM describes also the two-dimensional bosonic string theory in compact imaginary time (i.e. at finite temperature). Due to the FZZ conjecture, at the vanishing cosmological constant it can be also interpreted as a string theory in the Euclidean 2D black hole background (at least at $R = 3/2$). Our one-point correlators should contain information about the amplitudes of emission of winding modes by the black hole, whereas the two-point correlators describe the S-matrix of scattering of the winding states from the tip of the cigar (or from the Sine-Liouville wall). However, the exact correspondence is lacking due to the absence of the proper normalization of the operators. In [5] and [6] some two- and three-point correlators were computed in the CFT approach to this theory. It would be interesting to compare their results with the correlators calculated in this paper from the MQM approach. However there are immediate obstacles to this comparison.

First of all, these authors do not give any results for the one point functions of vortices. In the conformal theory such functions are normally zero since the vortex operators have the dimension one. But in the string theory we calculate the averages of a type $\langle \int d^2z \hat{V}_n(z) \rangle$ integrated over the parameterization space. They are already quantities of zero dimension, and the formal integration leads to the ambiguity $0 \ast \infty$ which should in general give a finite result. Another possible reason for vanishing of the one-point correlator could be the additional infinite W-symmetry found in [7] for the CFT (at $R = 3/2$, $\mu' = 0$). The generators of this symmetry do not commute with the vortex operators $\hat{V}_n$, $n \neq \pm 1$. Hence its vacuum average should be zero, unless there is a singlet component under this symmetry in it. Our results suggest the presence of such a singlet component.

Note that the one-point functions were calculated [3] in a non-conformal field theory with the Lagrangian:

$$L = \frac{1}{4\pi} \left[ (\partial x)^2 + (\partial \phi)^2 - 4\tilde{R}\phi + m \left( e^{-\frac{1}{2}\phi} + e^{\frac{1}{2}\phi} \right) \cos\left(\frac{3}{2}x\right) \right].$$

(89)

This theory coincides with the CFT (7) (again at $R = 3/2$ and $\mu' = 0$) in the limit $m \to 0$, $\phi_0 \to -\infty$, with $\lambda = me^{-\frac{1}{2}\phi_0}$ fixed, where $\phi_0$ is a shift of the zero mode of the Liouville field $\phi(z)$. So we can try to perform this limit in the calculated one-point functions directly. As a result we obtain $\mathcal{K}_n^{(m)} \sim m^3 \lambda^{3n-4}$. The coefficient we omitted is given by a complicated integral which we cannot perform explicitly. It is important that it does not depend on the couplings and is purely numerical. We see that, remarkably, the vanishing mass parameter enters in a constant power which is tempting to associate with the measure $d^2z$ of integration. Moreover,

\(^2\text{We thank A.Zamolodchikov for this comment}\)
the scaling in $\lambda$ is the same as in (79) (at $R = 3/2$) up to the $n$-independent factor $\lambda^{-8}$. All these $n$-independent factors disappear if we consider the correlators normalized with respect to $K_{1}^{(m)}$ which is definitely nonzero. They behave like $\sim \lambda^{3(n-1)}$ what coincides with the MQM result.

The two point correlators for the CFT (7) with $\mu' = 0$ are calculated in [6] only in the case of vortex operators of opposite and equal by modulo vorticities. They should be in principle compared to our $K_{n-\mu}$ correlators from (83). However, we should compare only properly normalized quantities, knowing that the matrix model correlators generally differ from the CFT correlators by the leg-factors. To fix the normalization we have to compare the quantities similar to (84) and (87) which are not available in the CFT approach.

A lot of interesting physical information is contained in multipoint correlators (3-point and more) which we did not consider in this paper and some of which are calculated in the CFT approach of [7, 19]. The direct calculations from the Hirota equations look very tedious. Hopefully the collective field theory approach of [20] can help on this way.

A lot is to be done to learn how to extract information about the black hole physics from the correlators. First of all, we have only calculated some vortex (winding modes) correlators in our MQM approach. The vertex (momentum modes, or tachyon) correlators are unavailable in the Toda hierarchy approach used here and do not seem to fit any known integrable structure in the MQM, although they are in principle calculable directly from the MQM in the large $N$ limit. On the other hand, they bear even a more important information about the system than the winding correlators: they describe the $S$-matrix of scattering of tachyons off the black hole and can be used to “see” the black hole background explicitly. The one-point tachyon correlators may give information about the black hole radiation. In general, the MQM approach seems to be a good chance to work out a truly microscopic picture for the 2D black hole physics.

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Appendix A

Let us investigate which terms will survive in the dispersionless limit in the equation (29). This limit corresponds to $\xi \rightarrow 0$. It is clear that all terms have the form

$$\left( \prod_{j=1}^{l} D_{k_{j}} \right) \tau_{s+1} \cdot \tau_{s}$$

(90)
with \( n = \sum_{j=1}^{l} k_j \) fixed. When we rewrite them in terms of the free energy \( F_s = \log \tau_s \) we obtain a set of terms of a kind

\[
e^{F_s + F_{s+1}} \prod_{j=1}^{p} \left( \prod_{i=1}^{q_j} \frac{\partial}{\partial t_{k_{ij}}} \right) (F_{s+1} + (-1)^{q_j} F_s),
\]

(91)

where \( \sum_{j=1}^{p} q_j = l \). In the dispersionless limit (22), (26) and (20) imply

\[
F_{s+1} + F_s \sim 2F_0,
\]

(92)

\[
F_{s+1} - F_s \sim -i\xi \frac{\partial}{\partial \omega} F_0,
\]

(93)

The main contribution in the dispersionless limit comes from the terms with the minimal degree of \( \xi \). The total degree of \( \xi \) for a given term is

\[
\sum_{j=1}^{l} \Delta[t_{k_j}] - 2p + r = l - 2p + r - \frac{nR}{2},
\]

(94)

where \( r \) is the number of odd \( q_j \). It is easy to see that the minimum is achieved when all \( q_j = 1 \) or 2 and it equals \(-\frac{nR}{2}\) what is independent on all particular parameters and hence is similar for all terms in the equation. It means that only two- and one-point correlators survive.

**Appendix B**

Combine together all terms with \( \tilde{D}_k \) and separately with \( \tilde{X}_k \) in the exponents of the eq. (33). Then due to vanishing of \( \tilde{D}^2 \tilde{X}_k \) we can apply the formula \( e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \) which is valid for operators having c-number commutator. This gives

\[
\frac{2}{x-y} \left( \sum_{k=1}^{\infty} x^k \tilde{X}_k e^{\frac{1}{2} \sum_{k,l=1}^{\infty} x^k \tilde{X}_{k,l} - y \sum_{k=1}^{\infty} \frac{1}{2} \sum_{l=1}^{\infty} y^l \tilde{X}_{k,l}} \right) =
\]

\[
\left( 1 + \sum_{m=1}^{\infty} y^m (D_m + X_m) \right) \sum_{k=1}^{\infty} x^k \tilde{X}_k e^{\frac{1}{2} \sum_{k,l=1}^{\infty} x^k \tilde{X}_{k,l}} + \left( 1 + \sum_{n=1}^{\infty} x^n (D_n + X_n) \right) \sum_{k=1}^{\infty} y^k \tilde{X}_k e^{\frac{1}{2} \sum_{k,l=1}^{\infty} y^l \tilde{X}_{k,l}}
\]

(95)

The right hand side of this equation can be rewritten as follows

\[
\left( 1 + \sum_{m=1}^{\infty} m y^m \left( \tilde{X}_m + \sum_{n=1}^{\infty} x^n \tilde{X}_{n,m} \right) \right) \sum_{n=1}^{\infty} x^n \tilde{X}_n e^{\frac{1}{2} \sum_{k,l=1}^{\infty} x^k \tilde{X}_{k,l}} +
\]

\[
\left( 1 + \sum_{n=1}^{\infty} n x^n \left( \tilde{X}_n + \sum_{m=1}^{\infty} y^m \tilde{X}_{n,m} \right) \right) e^{\frac{1}{2} \sum_{k,l=1}^{\infty} y^k \tilde{X}_{k,l}}
\]

(96)

18
Due to (35) we obtain

\[
\frac{x+y}{x-y} \left( e^{\frac{1}{2} \sum_{k,l=0}^{\infty} x^{k+l} \tilde{X}_{k,l}} - e^{\frac{1}{2} \sum_{k,l=0}^{\infty} y^{k+l} \tilde{X}_{k,l}} \right) = \\
\left( \sum_{n,m=0}^{\infty} m x^n y^m \tilde{X}_{n,m} \right) e^{\frac{1}{2} \sum_{k,l=0}^{\infty} x^{k+l} \tilde{X}_{k,l}} + \left( \sum_{n,m=0}^{\infty} n x^n y^m \tilde{X}_{n,m} \right) e^{\frac{1}{2} \sum_{k,l=0}^{\infty} y^{k+l} \tilde{X}_{k,l}} = 0, \quad (97)
\]

what can be written in terms of the generating function (37) as in (38).

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