Speed of Light in Gravitational Fields

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Abstract

A spherically symmetric and static metric that describes physical coordinates is introduced. It is defined to be a metric that gives coordinate independent results for physically observable quantities without a further coordinate transformation. The suggested metric also makes a prediction for the second order gravitational red shift effect that can be utilized for a precision experimental test in the future. A possible new experimental test would be provided by a modern Michelson Morley experiment on the earth with the two arms in vertical and horizontal directions to see the validity of isotropy or anisotropy for the speed of light. The possibility of using pulsars and GPS (Global Positioning System) for a general relativity test is discussed.

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I. INTRODUCTION

The coordinates in the Schwartzschild metric or the Eddington isotropic metric do not correspond to physically observable coordinates in the gravitational field of a mass. One should specify the relationship between these coordinates and physically observable coordinates in order to compare observations with the theoretical predictions of general relativity. In this article, we introduce a physical metric as one in which the coordinates in the metric represent observable coordinates, by showing that a coordinate independent result is reproduced by the physical metric. In particular, the time delay experiment is found to be crucial for the determination of the physical metric, while all other experimental tests of general relativity that have been done in the past are insensitive to the choice of the metric in the first order of gravity correction.

II. THE PHYSICAL METRIC

The physical metric for a spherically symmetric and static point mass $M$,

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (1)

is related to the Schwartzschild metric,

$$ds^2 = (1 - \frac{r_s}{r'}) dt^2 - \frac{1}{(1 - \frac{r_s}{r'})} dr'^2 - r'^2(d\theta^2 + \sin^2 \theta d\phi^2),$$ \hspace{1cm} (2)

by the transformation, $r' = re^{\mu(r)/2}$, where $r_s = 2GM/c^2$ is the Schwartzschild radius. Then one gets

$$e^{\nu(r)} = 1 - (r_s/r)e^{-\mu(r)/2} = 1 + a_1(r_s/r) + a_2(r_s/r)^2 + \cdots,$$ \hspace{1cm} (3)

$$e^{\lambda(r)} = \left(\frac{d}{dr}(re^{\mu(r)/2})\right)^2/(1 - (r_s/r)e^{-\mu(r)/2}) = 1 + b_1(r_s/r) + b_2(r_s/r)^2 + \cdots,$$ \hspace{1cm} (4)

$$e^{\mu(r)} = 1 + c_1(r_s/r) + c_2(r_s/r)^2 + \cdots,$$ \hspace{1cm} (5)
where asymptotic expansion is used and

\[ a_1 = -1, \quad \text{and} \quad a_2 = c_1/2, \] (6)

\[ b_1 = 1, \quad \text{and} \quad b_2 = 1 - c_1/2 + c_1^2/4 - c_2. \] (7)

To first order of \( r_s/r \), the metric is expressed as

\[ e^\nu(r) = 1 - r_s/r + \cdots, \quad e^\lambda(r) = 1 + r_s/r + \cdots, \quad \text{and} \quad e^\mu(r) = 1 + c_1(r_s/r) + \cdots. \] (8)

All the experimental tests of general relativity so far can be expressed in terms of this metric.

### III. THE GEODESIC EQUATIONS

The geodesic equations can be obtained from variations of the line integral over an invariant parameter \( \tau \), \( \int \frac{ds}{d\tau}^2 \, d\tau \), and their integrals are given by

\[ \frac{dt}{d\tau} = e^{-\nu(r)}, \] (9)

\[ \frac{d\phi}{d\tau} = J_\phi e^{-\mu(r)}/(r \sin \theta)^2, \] (10)

\[ \left( \frac{d\theta}{d\tau} \right)^2 = (J_\theta^2 - J_\phi^2 / \sin^2 \theta) e^{-2\mu(r)}/r^4. \] (11)

Restricting the plane of motion to \( \frac{d\theta}{d\tau} = 0, \theta = \pi/2 \), the radial part of the geodesic integral is given by

\[ \left( \frac{dr}{d\tau} \right)^2 = e^{-\lambda(r)}(e^{-\nu(r)} - J^2 e^{-\mu(r)}/r^2 - E) \] (12)

where \( J_\phi, J_\theta \) and \( E \) are constants of integration and

\[ J^2 = J_\phi^2 = J_\theta^2. \] (13)

for the above restriction on the plane of motion. The constant \( E \) is 0 for light propagation.
IV. TIME DELAY AND SPEED OF LIGHT

From Eq. (9) and Eq. (12), it follows that

\[ \frac{dt}{dr} = \pm \frac{e^{-\nu(r)}}{\sqrt{e^{-\nu(r)} - J^2 e^{-\nu(r)}/\lambda(r)}} \]

\[ = \pm \frac{r}{\sqrt{r^2 - r_0^2}} \left( 1 + \frac{(b_1 - a_1) r_s}{2r} + \frac{(c_1 - a_1) r_0 r_s}{2r (r + r_0)} + \cdots \right) \quad (15) \]

for light propagation, where \( r_0 \) is the impact parameter. Integrating from the distance between the planet and the sun, \( r_1 \), to the distance between the earth and the sun, \( r_2 \), one gets the expression for the time delay experiment of Shapiro et al. (for the return trip of the light),

\[ \Delta t = 2 r_s \ln \left( \frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_2 - \sqrt{r_2^2 - r_0^2}} \right) + \frac{c_1 + 1}{2} \left( \frac{r_1 - r_0}{r_1 + r_0} + \frac{r_2 - r_0}{r_2 + r_0} \right). \quad (16) \]

(See the ref. 2 for the calculation for the Schwartzschild metric, \( c_1 = 0 \).)

The second term of Eq. (16) depends on the choice of the value of \( c_1 \) and can be eliminated by a further coordinate transformation,

\[ r = r'' e^{\mu(r'')/2} = 1 + c_1''/2(r_s/r'') + \cdots. \quad (17) \]

Therefore, the coordinate independent prediction of general relativity should be

\[ \Delta t = 2 r_s \ln \left( \frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_2 - \sqrt{r_2^2 - r_0^2}} \right). \quad (18) \]

This is the result also obtained by the PPN (Post Newtonian Method) 3, and agrees with the most recent observational data 4 with high accuracy (1 in 1000 accuracy). By comparing Eqs. (16) and (18), we conclude that the physical metric is determined by the condition,

\[ c_1 = -1. \quad (19) \]

We note that the parameter values

\[ a_1 = -1, \quad and \quad b_1 = 1 \quad (20) \]

are coordinate independent and determined by being the solution of the Einstein equation and the physical boundary condition. Thus we conclude that Eq. (19) is the condition for the physical metric.
On the other hand, the coordinate speed of light in the gravitational field represented by the physical metric is obtained as

\[ c_g = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\phi}{dt}\right)^2} = e^{\nu(r)} \sqrt{e^{-\nu(r)} - J^2 e^{-\mu(r)} + J^2 e^{-2\mu(r)}/r^2}, \]  

(21)

where \( J^2 = r_0^2 e^{\mu(r_0)-\nu(r_0)} \). Using asymptotic expansion, one gets

\[ c_g = 1 - \frac{r_s}{r} (\cos^2 \omega + \frac{c_1 + 1}{2} \sin^2 \omega) + \cdots \]  

(22)

where \( \omega \) is the angle between the direction of the source of gravity and the direction of the propagation of light. Here the bending of light can be neglected as a higher order correction, and \( (\frac{r_0}{r})^2 = \sin^2 \omega \).

With the choice, Eq. (19), one gets

\[ c_g = 1 - \frac{r_s}{r} \cos^2 \omega + \cdots, \]  

(23)

for the coordinate speed of light. This is consistent with the coordinate speed of light obtained by the condition, \( ds^2 = 0 \). If one uses the local time,

\[ d\tau_p = e^{\nu(r)/2} dt, \]  

(25)

the coordinate speed of light becomes

\[ c_g' = \frac{c_g dt}{d\tau_p} = c_g (1 + \frac{r_s}{2r} + \cdots) \]  

(26)

\[ = 1 - \frac{r_s}{2r} (2 \cos^2 \omega - 1) + \cdots = 1 - \frac{r_s}{2r} \cos 2\omega + \cdots \]  

(27)

The question remains what is the observable speed of light in an environment of gravity such as on the earth. If one defines the speed of light by

\[ c_g'' = \sqrt{e^{\lambda(r)} \left(\frac{dr}{d\tau_p}\right)^2 + e^{\mu(r)} \left(r \frac{d\phi}{d\tau_p}\right)^2} = e^{\nu(r)/2} \sqrt{e^{-\nu(r)} - J^2 e^{-\mu(r)} + J^2 e^{-2\mu(r)}/r^2} = 1 \]  

(28)

This implies that in coordinates for which the radial length, \( dr \), is stretched as \( (1 + \frac{r_s}{2r} + \cdots) dr \) and the angular length, \( rd\phi \), is shortened as \( (1 - \frac{r_s}{2r} + \cdots) rd\phi \), the speed of light is equal to 1 (\( = c \)). It is the author’s opinion that this statement does not correspond to the observable
speed of light. This can be seen in the following manner. Suppose one tries to make a modern Michelson Morley experiment [5] by using two laser cavities, one in a horizontal direction and the other in the vertical direction. Prepare two identical cavities lying in the horizontal direction. By bringing one of the cavities to the vertical direction, its length is shortened by the force of gravity. If the material of the cavities has a very high Young’s modulus, the lengths of the two cavities are almost identical. Then, a modern Michelson Morley experiment with this instrument (with an appropriate correction for the gravitational shrinkage effect) should show a difference in the speed of light based on Eq. (24) or Eq. (27), but not based on Eq. (28).

V. THE OTHER EXPERIMENTAL TESTS

The other tests of general relativity are shown to be insensitive to the presence of the $c_1$ term. For the bending of light, one uses the formula,

$$\frac{d\phi}{dr} = \pm e^{-\mu(r)+\lambda(r)/2}/r^2 \sqrt{e^{-\nu(r)}/J^2 - e^{-\mu(r)/r^2}}$$

(29)

$$= \pm \left(1 + \frac{r}{r_0} \left(\frac{b_1}{2} - \frac{a_1 r}{2 r_0 (r + r_0)} + \frac{c_1}{2} \left(\frac{r}{r_0 (r + r_0)} - \frac{1}{r} \right) \right) \right) \cdots .$$

(30)

Integrating this from a large distance, one gets the well known expression for the bending of light,

$$\Delta \phi = (b_1 - a_1) \frac{r_s}{r_0} = \frac{2 r_s}{r_0}.$$  

(31)

The integration of the $c_1$ term in Eq. (30) gives a vanishingly small value and therefore insensitive to the value of $c_1$, as is seen from Eq. (31).

For the advancement of perihelia, one uses the formula

$$\frac{d\phi}{dr} = \pm e^{-\mu(r)+\lambda(r)/2}/r^2 \sqrt{e^{-\nu(r)/J^2} - e^{-\mu(r)/r^2} - E}$$

(32)

$$= \pm \left(1 + \frac{r_s}{2} \left(\frac{b_1}{r} + (-a_1 + \frac{a_2}{a_1} \frac{r_- + r_+}{r_+ r_-}) + c_1 \left(\frac{1}{r_+} + \frac{1}{r_-} - \frac{1}{r} \right) \right) \right) \cdots$$

(33)

where $r_\pm$ are the semi major and minor axis of the elliptical orbit. The appearance of $a_2$ is necessitated by the cancellation of the lowest term for the determination of the constants $J^2$ and $E/J^2$. Integration over the ellipse yields the advancement of perihelion,

$$\Delta \phi = \pi r_s \frac{1}{2} \left(\frac{1}{r_+} + \frac{1}{r_-} \right) (b_1 - 2 a_1 + c_1 + \frac{2 a_2}{a_1}).$$

(34)
Due to the relationship, Eq. (6), \( c_1 + \frac{2a_2}{a_1} = 0 \), one obtains

\[
\Delta \phi = \frac{\pi r_s}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right) (b_1 - 2a_1) = \frac{3\pi r_s}{2} \left( \frac{1}{r_+} + \frac{1}{r_-} \right).
\]

(35)

It is remarkable that the \( c_1 \) term and \( a_2 \) term cancel each other and the final result is again independent of \( c_1 \). In other words, both equations, Eq. (31) and Eq. (35), which have been supported by observational data, are insensitive to the value of \( c_1 \). The reason for these phenomena is that the bending of light and the advancement of perihelia are variations in the angular variables, which are less ambiguous coordinates. On the other hand, the time delay experiment, Eq. (16), and the speed of light, Eq. (23), formally depend on the parameter \( c_1 \).

**VI. SUGGESTED EXPERIMENTAL TESTS**

In the following, the author suggests possible experiments of various types.

(i) Pulsar time delay experiments: In order to improve the statistics of time delay experiments, the author suggests doing a time delay experiment on pulsars with small declination angles in ecliptic coordinates. Such pulsars cross, graze the sun or nearly do so once a year and provide an opportunity to perform the experiment. Candidates for such pulsars are listed with J-names and the J2000 ecliptic RA and DE in parentheses (in degrees) in the order of small DE angle [7]: J1022+10 (153.864, -0.06982), J0540+2329 (86.139, 0.10214), J1744-2334 (266.495, -0.17392), J1817-2312 (273.915, 0.18425), J1730-2304 (263.186, 0.19150), J1800-2343 (270.012, -0.27921), J1801-2451 (270.221, 0.33398), J1801-2316 (270.306, 0.33426), J1822-2256 (275.290, 0.38682), J1733-2228 (263.866, 0.82122), J0614+2229 (93.299, -0.89891), J1757-2421 (269.4470, -0.92775), J0629+2415 (96.629, 0.98950). Since the angular size of the radius of the sun is 0.267 degrees, the first 5 pulsars in the list cross the sun. The time delay experiments by binary pulsars have been performed.

(ii) Speed of light experiments:

A recent series of laser beat experiments, which is called modern Michelson-Morley experiments, tests the isotropy of the speed of light in horizontal directions with good accuracy (on the order of \( \delta c/c \approx 10^{-15} \)). It is desirable to do a modern Michelson-Morley experiment with the two arms in vertical and horizontal directions in order to see the effect of the earth’s gravity on the variation of the speed of light or absence of it, as was suggested earlier in this
article. Since the characteristic parameter at the surface of the earth is $r_s/r = 1.39 \times 10^{-9}$ and a simple minded application of the Schwarzschild metric gives anisotropy for speed of light of this order, Eq. (23) with $c_1 = 0$, it is worthwhile to examine isotropy or anisotropy in this type with the accuracy of $10^{-10}$.

(iii) Use of GPS. It is known that distance (or time) measurement by GPS has errors of the order of a few meters due to the atmospheric index of refraction and other factors. It is therefore essential to reduce the errors in order to perform a general relativity experiment with GPS. One possible suggestion is to use the measurement of a LEO (Low Earth Orbit) satellite to subtract the effect of the free electron density and get a distance measurement between a GPS and a LEO satellite. There is still the remaining effect of the upper atmosphere to be eliminated. An alternative test of general relativity by GPS would be a time dilation test like the Pound-Rebka experiment\[9\]. The time difference between the atomic clocks on a GPS and on the ground (such as NIST) is recorded as a monitoring operation. Subtracting the effects of the Doppler shifts from the GPS motion and the earth’s rotation, one can get the time delay by (general and special) relativity. This is equivalent to the Pound Rebka experiment. The advantage of the GPS experiment is that it can improve statistics by a continuous operation.

(iv) A second order test of the gravitational red shift. Using the physical metric derived in this article, one can derive a formula for gravitational red shift in second order in the gravitational constant. From Eq. (13), Eq. (19) and Eq. (25), it follows that

$$d\tau_p = \sqrt{1 + a_1(r_s/r) + a_2(r_s/r)^2 + \cdots dt}$$

$$= \sqrt{1 - (r_s/r) - (r_s/r)^2/2 + \cdots dt}.$$  

Here, the use of the physical metric enables us to get a coordinate independent prediction based on

$$a_2 = c_1/2 = -1/2.$$  

This prediction can be utilized as a second order test of general relativity when a precise measurement of the gravitational red shift becomes available in the future. One possible direction is the measurement of spectral red shifts from binary white dwarfs.
VII. DISCUSSIONS

Some discussions are due. Eq. (1) can be generalized to

$$ds^2 = e^{\nu(r)}dt^2 + 2e^{\kappa(r)}dtdr - e^{\lambda(r)}dr^2 - e^{\mu(r)}r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

This introduces an extra parameter in the theory,

$$e^{\kappa(r)} = d_1(r_s/r) + \cdots.$$ \hspace{1cm} (40)

It is easy to see that the only change in first order in Eq. (2) through Eq. (12) is in the geodesic integral

$$e^{\nu(r)} \frac{dt}{d\tau} = 1 - e^{\kappa(r)} \frac{dr}{d\tau}.$$ \hspace{1cm} (41)

The change in the geodesic integral for the radial coordinate is in a second order of $r_s/r$, while those for the angular coordinates are invariant. As a result, one obtains the change for Eq. (24),

$$c_g = 1 - \frac{r_s}{r}(\cos^2 \omega + d_1 \cos \omega) + \cdots.$$ \hspace{1cm} (42)

An additional term for time delay experiments cancels for the return trip or gives an unobservable constant shift for a one-way trip in pulsar time delay experiments. The predictions for the bending of light and the advancement of perihelia are not affected by the change in Eq. (41). In other words, the final result for the rest of discussion in this article is unchanged.

Finally, the author emphasizes that Eq. (16) or Eq. (23) provides a challenge for new experimental tests of general relativity.

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