APS Instability and the Topology of the Brane-World

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ABSTRACT

As is well known, classical General Relativity does not constrain the topology of the spatial sections of our Universe. However, the Brane-World approach to cosmology might be expected to do so, since in general any modification of the topology of the brane must be reflected in some modification of that of the bulk. Assuming the truth of the Adams-Polchinski-Silverstein conjecture on the instability of non-supersymmetric AdS orbifolds, evidence for which has recently been accumulating, we argue that indeed many possible topologies for accelerating universes can be ruled out because they lead to non-perturbative instabilities. PACS-1996: 98.80.Cq, 11.25.-w Keywords: Branes, Topology, AdS Orbifolds

1. A Theoretical Perspective on Cosmic Topology

It is clear that Einstein’s equation alone does not fix the topology of a cosmological model [1]. Since there are many possible topologies consistent with the familiar FRW geometries, it is natural to ask: what physical principle does fix topology in cosmology? Here we consider this question in the light of the observed acceleration of the Universe [2], which may indicate that the basic geometry [though not necessarily the global topology] of our world is that of de Sitter spacetime.

The widely popular brane-world approach [3] to cosmology allows us to attack this problem. For when de Sitter spacetime is formulated in this way [4][5], as a brane-world in AdS$_5$, the conformal infinity of the brane-world actually resides on the conformal infinity of AdS$_5$. [See [6] for the details.] It follows that, in the brane-world picture, non-trivial spatial topology in cosmology necessarily implies non-trivial topology for the boundary of the local AdS$_5$ in which the brane is embedded. This in turn gives us a possible way of testing the physical acceptability of candidate topologies, since the physics of AdS$_5$ and its orbifolds [6][7][8] has been studied intensively. The boundary of the standard simply-connected version of AdS$_5$ has topology $\mathbb{R} \times S^3$, so a locally de Sitter brane with non-trivial topology will have to be embedded in a version of AdS$_5$ which has a boundary where $S^3$ is replaced by some non-singular quotient. Taking such quotients will certainly affect the AdS/CFT dual field theory, for example in the way recently discussed.
by Dowker [9]. [It would also have major effects in the context of the proposed “dS/CFT correspondence” [10][11][12], which however we shall not be using here.] Again, it turns out that such non-singular boundary quotients give rise to orbifold singularities in the bulk. It is this effect that we shall study here.

In AdS$_5$ there are sources of instability which arise when one considers non-supersymmetric orbifolds. This was pointed out by Adams, Polchinski, and Silverstein [13], who conjectured that the condensation of closed string tachyons coming from the twisted sector would tend to resolve the orbifold singularity and restore supersymmetry. This restoration of the “deficit angle” cannot, however, be confined to the vicinity of the (former) singularity: the jump in the deficit is produced by a dilaton pulse which expands outward at the speed of light, ultimately restoring the geometry to its pre-orbifold state. Strong evidence in favour of this conjecture has recently been obtained by studying both the late-time structure [14][15] and the internal consistency of the proposed mechanism [16][17]. It has been argued by Horowitz and Jacobson [18] that a similar phenomenon can be expected in non-supersymmetric orbifolds of AdS. The AdS/CFT correspondence then predicts a similarly radical instability for the matter fields on the de Sitter brane. The upshot is that the brane-world picture must be considered inconsistent if the brane-world is required to reside in an AdS$_5$ orbifold which is not supersymmetric. Since topologically non-trivial de Sitter branes are associated with AdS$_5$ orbifolds, we clearly have here a potentially powerful criterion for ruling out many candidate topologies: we must check whether the relevant AdS$_5$ orbifold is supersymmetric.

There are infinitely many purely spatial quotients of dS$_4$. For the sake of clarity we shall concentrate on one of these, namely the de Sitter version of the “dodecahedral Universe” proposed in [19]. The general case then follows by similar techniques. [We focus on this particular topology because it illuminates the general case. We stress that we have nothing to say here about the motivation or observational status of the dodecahedral model: for that, see [20].]

We begin with a brief explanation of the structure of the dodecahedral space in the context of de Sitter cosmology. We then examine the corresponding AdS$_5$ orbifold and show explicitly that it has no surviving supersymmetries. In view of the above, we can use this to rule out the dodecahedral topology, and, similarly, many other candidate topologies, assuming the validity of the Adams-Polchinski-Silverstein argument.

### 2. The Dodecahedral Cosmos as a Brane-World

The de Sitter solution of the Einstein equation is valid for any three-manifold having the local geometry of S$^3$. However, even if we confine ourselves to “Copernican” models, that is, those with spatial sections which are homogeneous, then there are still infinitely many locally spherical candidates to be considered. These fall into an ADE classification of the kind familiar to string theorists: there are two infinite families together with a special class consisting of just three (isometry classes of) manifolds. The most complex of these, corresponding to E$_8$ in the ADE classification, is the Poincaré dodecahedral space, also known as the Poincaré homology sphere. It is obtained simply by identifying all of the opposite faces of a dodecahedron, after consistently applying a $\pi/5$ twist. (The other two spaces in the E-series are obtained in an analogous way from the regular tetrahedron and
the regular octahedron.) One can obtain a basic model of an accelerating Universe in this 
way by replacing the $S^3$ spatial sections of de Sitter spacetime with copies of the Poincaré 
dodecahedral space, thereby giving the dodecahedral Universe the basic dynamics of an 
accelerating spacetime.

Topologically, the dodecahedral space has the structure $S^3/\tilde{I}_{120}$, where $\tilde{I}_{120}$ is a finite 
subgroup of SU(2). This group is called the binary icosahedral group; it is a group of 
120 elements, such that $\tilde{I}_{120}/\mathbb{Z}_2 = I_{60}$, the icosahedral group. This is the 60-element 
group of symmetries of a regular dodecahedron or icosahedron, the dual polyhedron of 
the dodecahedron. (Throughout this work, “symmetries” of a polygon or polyhedron 
will always mean “orientation-preserving symmetries in three dimensions”.) Since $I_{60}$ is 
a group of symmetries of a geometric object (it is a subgroup of SO(3)), it is easier to 
visualise than $\tilde{I}_{120}$, and this will be useful to us.

Combining these observations, we can obtain an accelerating Universe with the Poincaré 
dodecahedral space as spatial sections simply by taking de Sitter spacetime $dS_4(S^3)$ and 
factoring $S^3$ by $\tilde{I}_{120}$, to obtain $dS_4(S^3/\tilde{I}_{120})$. If we do this, we obtain a spacetime which 
is locally indistinguishable from de Sitter spacetime, but which has a different global 
structure. In particular, while $dS_4(S^3)$ is spatially homogeneous and globally isotropic, 
d$S_4(S^3/\tilde{I}_{120})$ is homogeneous but not globally isotropic.

Now let us embed this version of de Sitter spacetime in the appropriate version of 
AdS$_5$. Five-dimensional anti-de Sitter spacetime, AdS$_5$, is defined as the locus 

$$-A^2 - B^2 + w^2 + x^2 + y^2 + z^2 = -L^2,$$

in a flat six-dimensional space of signature (2,4). This is a space of constant negative 
curvature $-1/L^2$. It is not hard to see that in AdS$_5$ there is a copy of $dS_4$ at each point 
of the bulk which is sufficiently “near” to the boundary. To be precise, there is such a 
copy corresponding to each value of $B$ such that $|B| > L$. Choosing coordinates on AdS$_5$ 
which cover this region only, one can in fact express the AdS$_5$ metric as 

g(AdS$_5$) = $d\rho^2 + \sinh^2(\rho/L) [-d\tau^2 + L^2 \cosh^2(\tau/L)\{d\chi^2 + \sin^2(\chi) [d\theta^2 + \sin^2(\theta) d\phi^2]\}], \tag{2}$

or 

$$g(AdS_5) = d\rho^2 + \sinh^2(\rho/L) g(dS_4), \tag{3}$$

where $g(dS_4)$ is the usual global metric for de Sitter spacetime. Thus, we can put a de 
Sitter brane at $\rho = c$ for some constant $c$; points in AdS$_5$ corresponding to larger values 
of $\rho$ are cut away, in the usual Randall-Sundrum manner. However, the time coordinate 
on the brane is related to the global radial AdS$_5$ coordinate $r$ by the equation 

$$\sinh(r/L) = \sinh(c/L) \cosh(\tau/L), \tag{4}$$

so we see that the temporal conformal infinity of the brane ($\tau \rightarrow \pm \infty$) actually resides 
on the spatial conformal infinity of the bulk ($r \rightarrow \infty$). Thus the brane still has access to 
the conformal infinity of the bulk, despite the cutting away of the region $\rho > c$. It follows 
that if we factor $S^3$ in the de Sitter brane by a finite group such as $\tilde{I}_{120}$, then we have 
no option but to do the same to the $S^3$ in the boundary of AdS$_5$. That is, we are forced 
to allow $\tilde{I}_{120}$ to act on the coordinates $w, x, y,$ and $z$ in equation (1) and then take the 
quotient. We can do this because $\tilde{I}_{120}$ is contained in the isometry group of AdS$_5$; in fact
it just acts on the angular coordinates in equation (2), preserving the spherical part of the metric.

Recall now that the spatial sections of AdS$_5$ are copies of the hyperbolic space $H^4$. Any finite group of isometries of $H^4$ has a (common) fixed point, and so, unlike dS$_4(S^3/\tilde{I}_{120})$, the quotient AdS$_5/\tilde{I}_{120}$ is singular: it is an orbifold. One might suspect that this orbifold singularity at the centre of AdS$_5$ arises from the special, highly symmetric geometry of AdS$_5$, but this is not correct: no matter how we perturb the geometry of the quotient, it remains singular unless (perhaps) the perturbation is so large that some curvature becomes positive. This follows from a theorem of Cartan ([21], page 111); see [6] for the details. This means that we still expect an AdS$_5$ orbifold to be the correct background here even if the exact geometry near the origin is not identical to that of AdS$_5$.

Thus, if the dodecahedral model is valid, then this tells us that the bulk is an orbifold. The symmetry group of this AdS$_5$ orbifold is given by

$$\text{Isom}(AdS_5/\tilde{I}_{120}) = O(2) \times SO(3);$$

this agrees with the conformal group of the quotient CCM$_4/\tilde{I}_{120}$, where CCM$_4$ is the conformal compactification of Minkowski space; this is of course in accord with AdS/CFT expectations. (Note that when AdS$_5$ is obtained as a string background, orientation-reversing isometries are not matter symmetries, so in this context we should state the symmetry group as SO(2) $\times$ SO(3) rather than O(2) $\times$ SO(3).) We see that factoring by finite groups drastically reduces the size of the spacetime isometry group, from fifteen dimensions to four, from non-compact to compact. This prepares us for the still more drastic reduction of supersymmetry to be discussed below.

3. Stringy Instability of AdS$_5$/\tilde{I}_{120}

Quotients of flat spacetimes by ADE finite groups have been studied extensively; see for example [22]. The survival of supersymmetry in such cases can often be understood in terms of holonomy theory. In particular, taking the quotient of $R^4$ by a finite subgroup of one of the SU(2) factors of SO(4) results in an orbifold with holonomy large enough to break half of the supersymmetries.

The case of orbifolds of AdS$_5$ is quite different. For whereas $R^4$ has trivial holonomy, AdS$_5$ already has the maximal possible holonomy group for a (time and space orientable) Lorentzian five-manifold, namely SO$^+(1,4)$. Since the action of $\tilde{I}_{120}$ on AdS$_5$ preserves time and space orientation (that is, the action does not involve time, and the Poincaré dodecahedral space is orientable in the ordinary sense, since $\tilde{I}_{120}$ is completely contained in SO(4), not just O(4) ), it follows that taking the quotient of AdS$_5$ by $\tilde{I}_{120}$ cannot change the holonomy group in any way: it is already as large as it can be if no orientation is reversed. Hence we cannot extend our intuitions regarding the preservation of supersymmetry from the flat case to the anti-de Sitter case. Fortunately, the question of supersymmetry on finite group quotients of anti-de Sitter space has been studied [23], and the degree to which AdS$_5$/\tilde{I}_{120} is supersymmetric can be settled by means of an explicit calculation.

First, let us simplify the problem as follows. Inspection of the regular dodecahedral reveals that its symmetry group, $I_{60}$, contains the symmetry group of the tetrahedron, $T_{12}$. (There is a standard way to fit a tetrahedron inside a dodecahedron; see
for an excellent picture of this. Ignore the symmetries of order 5 associated with the pentagonal faces. The remaining symmetries are just those which define $T_{12}$. In the same way one sees that $T_{12}$ is a subgroup of the group, $O_{24}$, of symmetries of a regular octahedron. The tetrahedral group has only 12 elements. Inspection of the regular tetrahedron reveals that $T_{12}$ in its turn contains a (normal) subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (Each $\mathbb{Z}_2$ is generated by a symmetry of the tetrahedron which acts by rotation through $\pi$ about an axis joining the midpoints of a chosen pair of opposite edges. There are three such pairs of opposite edges, but a combination of the two rotations corresponding to any two pairs generates the rotation corresponding to the third, so the group consists of two copies of $\mathbb{Z}_2$, not three. The obvious $\mathbb{Z}_3$ symmetry of the tetrahedron permutes the three non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.) Thus $T_{12}$, and therefore $I_{60}$, contain $\mathbb{Z}_2 \times \mathbb{Z}_2$ in a natural way. When we lift $I_{60}$ to $\tilde{I}_{120}$, we must therefore also lift $\mathbb{Z}_2 \times \mathbb{Z}_2$ to a subgroup of SU(2), and it is not hard to show that this subgroup is $Q_8$, the quaternionic group $\{\pm 1, \pm i, \pm j, \pm k\}$, where $i$, $j$, and $k$ are the usual basis quaternions; here we are thinking of SU(2) as the group of all unit quaternions, the symplectic group $Sp(1)$. (One sees that $Q_8$ projects to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by pretending that $i$, $j$, and $k$ commute and square to $+1$ instead of $-1$.) Thus $Q_8$ is contained in $\tilde{T}_{24}$, the binary tetrahedral group; since, as we saw above, $T_{12}$ is a subgroup of both $O_{24}$ and $I_{60}$, it follows that $Q_8$ is also contained in the binary octahedral group $\tilde{O}_{48}$ and also, most importantly, in the binary icosahedral group $\tilde{I}_{120}$.

Now AdS$_5$ can be represented using quaternions by taking the coordinates used in equation (1) and defining

$$D = A + iB$$

$$C = w + ix + jy + kz.$$  \hspace{1cm} (6)

If $\tilde{C}$ represents the quaternion conjugate of $C$, defined by reversing the sign of the vector part of the quaternion but not its scalar part, then the definition of AdS$_5$ may be written as

$$-\tilde{D}D + \tilde{C}C = -L^2.$$ \hspace{1cm} (7)

We see at once from this that $Q_8$ acts on AdS$_5$ by $q : (D, C) \to (D, qC)$ for each $q \in Q_8$, since $\tilde{qC} = \tilde{C}\tilde{q}$ and $\tilde{q}q = 1$. As $Q_8$ is generated by $i$ and $j$, the action of $Q_8$ on AdS$_5$ can be fully understood by studying the effect of these two elements. Since we have

$$i(w + ix + jy + kz) = -x + iw - jz + ky$$

$$j(w + ix + jy + kz) = -y + iz + jw - kx,$$ \hspace{1cm} (8)

the action of $Q_8$ on AdS$_5$ is therefore fully described by the maps

$$i : (A, B, w, x, y, z) \to (A, B, -x, w, -z, y)$$

$$j : (A, B, w, x, y, z) \to (A, B, -y, z, w, -x),$$ \hspace{1cm} (9)

where we denote the map by the corresponding quaternion.
In order to make a comparison with the work of Ghosh and Mukhi [23], let us switch from quaternions to ordinary complex coordinates for the embedding space of AdS$_5$, with $Z_i$, $i = 1,2,3$, defined by

\[
\begin{align*}
Z_1 &= A + iB \\
Z_2 &= w + ix \\
Z_3 &= y + iz,
\end{align*}
\]

so that AdS$_5$ is

\[-Z_1\overline{Z_1} + Z_2\overline{Z_2} + Z_3\overline{Z_3} = -L^2,
\]

where the bar denotes the ordinary complex conjugate. A useful set of coordinates $(\theta_1, \theta_2, \delta, \alpha, \beta)$ is defined [23] by

\[
\begin{align*}
Z_1 &= L\cosh(\frac{\theta_1}{2})e^{i\delta} \\
Z_2 &= L\sinh(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})e^{i\alpha} \\
Z_3 &= L\sinh(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})e^{i\beta},
\end{align*}
\]

and the Killing spinors on AdS$_5$ are given by [23]

\[
\epsilon = e^{\frac{1}{2}\Gamma_3 \theta_1}e^{-\frac{1}{2}\Gamma_4 \theta_2}e^{\frac{1}{2}\Gamma_2 \alpha}e^{\frac{1}{2}\Gamma_4 \delta}e^{\frac{1}{2}\Gamma_3 \beta}e^{\frac{1}{2}\Gamma_1}e_0,
\]

where the $\Gamma_i$ all square to unity except for $\Gamma_3$ (which squares to $-1$) and where $e_0$ is a constant spinor.

Now in terms of the $Z_i$ coordinates, the action of $i$ and $j$ given in equations (9) are expressed as

\[
\begin{align*}
i : (Z_1, Z_2, Z_3) &\rightarrow (Z_1, iZ_2, iZ_3) \\
j : (Z_1, Z_2, Z_3) &\rightarrow (Z_1, -\overline{Z_3}, \overline{Z_2});
\end{align*}
\]

notice that both of these square to the map $(Z_1, Z_2, Z_3) \rightarrow (Z_1, -Z_2, -Z_3)$, and they anti-commute, as they should according to the quaternion multiplication table. In terms of the coordinates given by equations (12), the actions of $i$ and $j$ are given by

\[
\begin{align*}
i : (\theta_1, \theta_2, \delta, \alpha, \beta) &\rightarrow (\theta_1, \theta_2, \delta, \alpha + \frac{\pi}{2}, \beta + \frac{\pi}{2}) \\
j : (\theta_1, \theta_2, \delta, \alpha, \beta) &\rightarrow (\theta_1, \theta_2 + \pi, \delta, -\beta, -\alpha).
\end{align*}
\]

We can now see the effects of $i$ and $j$ on the Killing spinor $\epsilon$ given by equation (13):

\[
\begin{align*}
i &\rightarrow e^{\frac{1}{4}\Gamma_4 \theta_1}e^{-\frac{1}{2}\Gamma_4 \theta_2}e^{\frac{1}{2}\Gamma_2 \alpha}e^{\frac{1}{2}\Gamma_4 \delta}e^{\frac{1}{2}\Gamma_3 \beta}e^{\frac{1}{2}\Gamma_1}e_0 \\
j &\rightarrow e^{\frac{1}{4}\Gamma_4 \theta_1}e^{-\frac{1}{2}\Gamma_4 \theta_2}e^{\frac{1}{2}\Gamma_2 \alpha}e^{\frac{1}{2}\Gamma_4 \delta}e^{\frac{1}{2}\Gamma_3 \beta}e^{\frac{1}{2}\Gamma_1}e_0.
\end{align*}
\]
Now suppose that we construct the quotient $\text{AdS}_5/Q_8$, an orbifold which contains a non-singular brane-world with the local geometry of de Sitter spacetime but with $S^3/Q_8$ as spatial sections. (Note that $S^3/Q_8$ can be visualised by simply taking a cube and identifying all opposite faces after a consistent rotation by $\pi/2$.) Then this quotient will retain some supersymmetry if $\epsilon$ is invariant with respect to both $i$ and $j$. From the first equation in the set (16), we see at once that for $\epsilon$ to be invariant with respect to $i$, the constant spinor $\epsilon_0$ has to satisfy

$$\Gamma_{24} \epsilon_0 = \Gamma_{15} \epsilon_0. \quad (17)$$

Of course, not every $\epsilon_0$ can satisfy this, but some do: in fact [23], there is a two-dimensional space of solutions of (17), and so the quotient $\text{AdS}_5/\mathbb{Z}_4$, where $\mathbb{Z}_4$ is generated by $i$, retains precisely half of the supersymmetries. Similarly, the quotient of $\text{AdS}_5$ by the $\mathbb{Z}_4$ generated by $j$ is also half-supersymmetric. But now suppose that we require $\epsilon$ to be invariant with respect to both $i$ and $j$. Then, noting that neither $i$ nor $j$ affects $\theta_1$, we see that the condition for the invariance of $\epsilon$ under the action of $j$ is

$$e^{-\frac{\pi}{4} \Gamma_{14}} e^{\frac{1}{2} \Gamma_{24} \beta} e^{\frac{1}{2} \Gamma_{36} \delta} e^{-\frac{1}{2} \Gamma_{15} \alpha} \epsilon_0 = e^{-\frac{1}{2} \Gamma_{24} \alpha} e^{\frac{1}{2} \Gamma_{36} \delta} e^{\frac{1}{2} \Gamma_{15} \beta} \epsilon_0. \quad (18)$$

But now, using equation (17) — that is, requiring simultaneous invariance under $i$ and $j$ — we can define a spinor $\eta$ by

$$\eta = e^{\frac{1}{2} \Gamma_{24} \beta} e^{\frac{1}{2} \Gamma_{36} \delta} e^{-\frac{1}{2} \Gamma_{15} \alpha} \epsilon_0 = e^{-\frac{1}{2} \Gamma_{24} \alpha} e^{\frac{1}{2} \Gamma_{36} \delta} e^{\frac{1}{2} \Gamma_{15} \beta} \epsilon_0, \quad (19)$$

and then equation (18) becomes simply

$$e^{-\frac{\pi}{4} \Gamma_{14}} \eta = \eta, \quad (20)$$

but this is not possible except for trivial $\epsilon_0$. Thus some supersymmetry generators can survive factoring by either $i$ or $j$ — but none can survive both.

We conclude that $\text{AdS}_5/Q_8$ is a non-supersymmetric orbifold of $\text{AdS}_5$. (That it is indeed an orbifold and not a manifold can be seen from equations [29]: clearly all those points of the form $(A,B,0,0,0,0)$, with $A^2 + B^2 = L^2$ (see equation [1]) are left unmoved by every element of $Q_8$.) But we saw earlier, using the geometry of the regular polyhedra, that $Q_8$ is a subgroup of all of the binary polyhedral groups. Since no Killing spinor on $\text{AdS}_5$ can survive factoring by $Q_8$, it follows that no Killing spinor is invariant by those groups either, and we see that all of the spaces $\text{AdS}_5/\hat{T}_{24}, \text{AdS}_5/\hat{O}_{48}$, and $\text{AdS}_5/\hat{I}_{120}$ are non-supersymmetric orbifolds.

In fact, of all the homogeneous quotients of $S^3$, the only ones that lead to a supersymmetric quotient of $\text{AdS}_5$ are those in the A-series of the ADE classification mentioned above in section 2. To see this, note that we have already dealt with the three E-groups, $\hat{T}_{24}, \hat{O}_{48}$, and $\hat{I}_{120}$, so we can turn to the D-groups and then the A-groups. The D-groups are the generalized quaternionic groups, $Q_{4n}$, of order $4n$, for all $n \geq 2$. For $n \geq 3$ they are the groups which cover the dihedral groups, $D_{2n}$, the groups of symmetries of the regular $n$-sided polygons; that is, $Q_{4n}/\mathbb{Z}_2 = D_{2n}$. We can regard $Q_{4n}$ as being generated by the quaternion $i$ together with another unit quaternion $q$ of order $2n$. A somewhat more intricate version of the calculation given above shows that, as in the case of $Q_8$, there are Killing spinors which can survive factoring by the cyclic groups generated by either $i$ or $q$, but none can survive factoring by both. (This actually follows from our discussion
above if n is even, for then q can be chosen to be a root of j, but a separate argument is needed when n is odd.) Thus none of the quotients AdS$_5$/Q$_{4n}$ is supersymmetric.

Next, the “A-quotients” are the (homogeneous) lens spaces, generalizing the quotient by either i or j but not both. It is clear that all of these lead to quotients of AdS$_5$ which are supersymmetric: they are half-supersymmetric, since the quotients (by cyclic groups of any order) are like the quotients of AdS$_5$ by the $\mathbb{Z}_4$ generated by i or j, which retain a two-dimensional space of Killing spinors.

Finally we note that there is a huge class of S$^3$ quotients [24] which are not homogeneous; these are usually ignored for “Copernican” reasons, though one can question whether we have the right to assume that we are not at a special place in space, given that we do seem to find ourselves at a special point in time, a time when the dark energy has “recently” begun to dominate [2]. “de Sitter” spacetimes with the simplest inhomogeneous lens spaces as spatial sections are obtained as brane-worlds in an AdS$_5$ orbifold — recall that the action by any finite group on the spatial sections has a fixed point — by factoring AdS$_5$ by the $\mathbb{Z}_m$ generated by the map

$$ (Z_1, Z_2, Z_3) \rightarrow (Z_1, \gamma Z_2, \gamma^b Z_3), \quad (21) $$

where $\gamma$ is a primitive $m$th root of unity and $b$ is an integer, relatively prime to $m$, with $1 < b \leq m/2$. For a Killing spinor to survive this projection, condition (17) above is replaced by

$$ \Gamma_{24} \epsilon_0 = b \Gamma_{15} \epsilon_0. \quad (22) $$

However, the eigenvalues of the matrix $-\Gamma_{15} + b \Gamma_{15}$ can easily be computed [23]: they are

$$ (1 + b), -(1 + b), (1 - b), -(1 - b). \quad (23) $$

In view of the conditions on $b$, none of these is zero, and so (22) cannot be satisfied by any non-trivial $\epsilon_0$. This proves that de Sitter branes with inhomogeneous lens spaces as spatial sections cannot reside in a supersymmetric AdS$_5$ orbifold. Since the other inhomogeneous quotients of S$^3$ are all obtained [24] by factoring by groups which contain subgroups acting, after extension from the brane to AdS$_5$, as in (21), we see that none of the versions of de Sitter spacetime with inhomogeneous spatial sections can occur as brane-worlds in supersymmetric AdS$_5$ orbifolds. All of these results can be verified tediously but explicitly by noting that all elements of SO(4), including those which act on S$^3$ such that the quotient is not homogeneous, can be represented by a pair of unit quaternions $(q_1, q_2)$, modulo $\pm(1, 1)$, acting on a quaternion $C$ by $C \rightarrow q_1 C q_2^{-1}$. If $C$ is the quaternion given in equation (6), then in the coordinates given by (12) we have

$$ C = L \sinh(\frac{\theta_1}{2})[\cos(\frac{\theta_2}{2})\cos(\alpha) + i \cos(\frac{\theta_2}{2})\sin(\alpha) + j \sin(\frac{\theta_2}{2})\cos(\beta) + k \sin(\frac{\theta_2}{2})\sin(\beta)], \quad (24) $$

and it is therefore possible to compute explicitly the action of any element of SO(4) on the Killing spinor in equation (13) by means of quaternion multiplication. The results agree with those obtained above.

We have seen explicitly that AdS$_5$/I$_{120}$ is a non-supersymmetric orbifold. In fact, we have a much stronger statement. Combining all of the results of the present section, we see that among all of the possible actions by finite groups on S$^3$, only a small subset extend
from the brane to AdS$_5$ in such a way that the quotient is supersymmetric. This subset consists of actions by finite cyclic groups such that the quotient $S^3/\mathbb{Z}_n$ is homogeneous: that is, the $S^3$ quotient is a homogeneous lens space. The final conclusion is that among all the versions of de Sitter spacetime with topologically non-trivial spatial sections, the only ones which can be self-consistently interpreted as brane-worlds within string theory are the ones with homogeneous lens spaces as spatial sections. (In addition, there are other ways of modifying the topology of de Sitter spacetime, involving quotients which affect the time axis. Most of these can be ruled out in the same way: see [6].)

4. Conclusion

The idea that the spatial sections of the four-dimensional Universe should take the form $S^3/\text{[non-trivial finite group]}$ is extremely natural from the string point of view. For such constructions have arisen before: the famed Calabi-Yau manifolds used in compactifications of heterotic $E_8 \times E_8$ string theory are precisely of the form [compact Riemannian manifold]/[non-trivial finite group], the non-triviality being necessary for gauge symmetry breaking by “Wilson loops” (see [25] for a recent discussion of this). Among the vast variety of quotients of $S^3$, the dodecahedral space $S^3/\tilde{I}_{120}$ has a strong claim to be the most interesting; among many other remarkable properties, it corresponds to $E_8$ in the ADE classification of the homogeneous quotients of $S^3$. It is remarkable that it cannot arise as a model for the spatial sections of an accelerating brane-world cosmology in string theory. In fact, the only survivors of APS instability are the homogeneous lens spaces, which clearly deserve further study.

References

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