Abstract

We show how to extend the Covering Spectrum (CS) of Sormani-Wei to two spectra, called the Extended Covering Spectrum (ECS) and Entourage Spectrum (ES) that are new for Riemannian manifolds but defined with useful properties on any metric on a Peano continuum. We do so by measuring in two different ways the “size” of a topological generalization of the δ-covers of Sormani-Wei called “entourage covers”. For Riemannian manifolds \( M \) of dimension at least 3, we characterize entourage covers as those covers corresponding to the normal closures of finite subsets of \( \pi_1(M) \). We show that \( \text{CS} \subset \text{ES} \subset \text{MLS} \) and that for Riemannian manifolds these inclusions may be strict, where MLS is the set of lengths of curves that are shortest in their free homotopy classes. We give equivalent definitions for all of these spectra that do not actually involve lengths of curves. Of particular interest are resistance metrics on fractals for which there are no non-constant rectifiable curves, but where there is a reasonable notion of Laplace Spectrum (LaS). The paper opens new fronts for questions about the relationship between LaS and subsets of the length spectrum for a range of spaces from Riemannian manifolds to resistance metric spaces.

Keywords: length spectrum, covering spectrum, Laplace spectrum, homotopy critical spectrum, resistance metrics on fractals

1 Introduction

The Length Spectrum (LS) of a Riemannian manifold is the set of lengths of closed geodesics, with various notions of multiplicity. The notion goes back at least to Huber’s papers in the late 1950’s ([16], [17]) in which the notion of LS is defined for what seems to be the first time. He showed that for compact Riemann surfaces, LS and the Laplace Spectrum (LaS) determine one another. Put another way, two Riemann surfaces have the same LS if and only if they are isospectral. These results have been followed by a decades-long investigation...
into the relationship between LS and LaS. We will not give a detailed history here, but will mention the fundamental open question of whether isospectral compact Riemannian manifolds have the same Weak Length Spectrum (defined as LS ignoring multiplicities, also called the Absolute Length Spectrum). For Riemann surfaces it is also known that LS and LaS are each completely determined by a finite subset, the size of which is bounded in the first case by the injectivity radius (Theorem 10.1.4, [7]), and in the second case by the injectivity radius and the genus (Theorem 14.10.1, [7]). This result suggests that there may be interesting relationships among geometrically and topologically significant subsets of LS, LaS, and other spectra from geometric analysis.

One of the most important subsets of LS is what Carolyn Gordon called the [L]-spectrum ([12]) and Christina Sormani and Guofang Wei called the Minimum Length Spectrum (MLS) in [28]. MLS is the set of lengths of curves that are shortest in their free homotopy classes. As is well-known (more on this later), in a compact Riemannian manifold there is always a shortest curve in every free homotopy class and that curve must be a closed geodesic, hence MLS ⊂ LS. In [12], Gordon showed that there are isospectral manifolds with distinct MLS, considering multiplicity as the number of distinct free homotopy classes.

In [28], Sormani-Wei introduced a subset of $\frac{1}{2}$MLS called the Covering Spectrum (CS). The existence of isospectral compact Riemannian manifolds with different CS was established in 2010 by Bart de Smit, Ruth Gornet and Craig Sutton ([10], dimensions ≥ 3 and [11], surfaces). Although CS and MLS are not “spectral invariants”, these spectra have mathematical applications, some of which we will mention below, and they may have nicer properties than LS. For example MLS is discrete and CS is finite for any compact Riemannian manifold, but LS may not be discrete ([25]).

In this paper we show how to extend CS to two spectra, called the Extended Covering Spectrum (ECS) and the Entourage Spectrum (ES), that are new even for compact Riemannian manifolds. ECS is discrete for arbitrary metrics on Peano Continua (compact, connected, locally path connected spaces), but may contain arbitrarily small values and so generally is not contained in $\frac{1}{2}$LS. ES contains 2CS and is contained in MLS, and therefore is discrete for Riemannian manifolds, but we do not know whether it is discrete for arbitrary compact geodesic spaces (metric spaces in which every x, y are joined by a curve, called a geodesic, of length equal to $d(x, y)$). We show by example that for compact Riemannian manifolds, CS may be properly contained in ECS and ES, and ES may be properly contained in MLS; that is, in general all of these spectra are distinct. Counting multiplicity, the cardinality of ES, like MLS but unlike CS, is a topological invariant, independent of any metric.

We also show how to extend the notion of MLS to compact geodesic spaces in general, where, contrary to statements in [13] and [14], there may be no shortest curve in a free homotopy class, and when there is one it may not be a closed geodesic ([3]). For all of these length-type spectra except LS itself, we give equivalent alternative definitions for geodesic spaces that don’t actually involve lengths of curves. That is, we define “length spectra” when there may be no length. Of particular interest are resistance metrics on self-similar fractals.
such as the Sierpinski Gasket and Carpet (which are Peano continua). These metric spaces have a meaningful notion of Laplacian (see \cite{18, 34} for general references), but they are generally far from being geodesic spaces. In fact there may be no non-constant rectifiable curves at all, and hence empty LS. On the other hand, we show in \cite{22} that all of the generalized spectra discussed in the present paper have infinitely many values for these resistance metric spaces, hence they provide good proxies for questions about the relationship between subsets of LS and LaS.

There are at least five characterizations of CS for compact geodesic spaces, most of which we will use at some point in this paper, and we will add two more. The original definition of CS due to Sormani-Wei (\cite{28}) begins with a general construction of Spanier (\cite{33}) that produces a regular covering map determined by an open cover $\mathcal{U}$ of a connected, locally path connected space $X$. In $\pi_1(X)$, let $\Gamma(\mathcal{U})$ be the (normal) subgroup generated by the set of homotopy classes of loops of the form $\overline{\alpha} \ast \lambda \ast \alpha$ where $\lambda$ lies entirely in one of the open sets in $\mathcal{U}$ and $\alpha$ starts at the basepoint. Here $\ast$ denotes concatenation and $\overline{\alpha}$ is the reverse parameterization of $\alpha$. According to Spanier there is a regular covering map of $X$ such that $\Gamma(\mathcal{U})$ is the image of $\pi_1(X)$ via the homomorphism induced by the covering map. The deck group $\pi_1(X)/\Gamma(\mathcal{U})$ of the covering map is in a sense a “fundamental group at the scale of $\mathcal{U}$” because it “ignores small holes” contained in elements of $\mathcal{U}$ when modding out by $\Gamma(\mathcal{U})$. Sormani-Wei took for $\mathcal{U}$ the open cover of $X$ by (open) $\delta$-balls $B(x, \delta)$, and called the resulting covering map the $\delta$-cover of $X$. In the case when $X$ is a compact geodesic space, Sormani-Wei showed that the equivalence type of $\delta$-covers changes at certain values, which they called the Covering Spectrum, that are discrete in $(0, \infty)$. Viewing $\delta$ as a parameter, as $\delta$ shrinks from the diameter of $X$ to 0, $X$ “unrolls” more and more at the discrete values in CS. If $X$ has a universal cover $\tilde{X}$, then $\tilde{X}$ is the $\delta$-cover for all sufficiently small $\delta > 0$ and CS is finite. By “universal cover” we mean in a categorical sense, which for compact geodesic spaces is equivalent to finiteness of CS (see Theorem 3.4 in \cite{28} and \cite{35} for related equivalent conditions).

Another way to characterize CS uses the discrete homotopy methods of Berestovskii-Plaut, developed in 2001 for topological groups (\cite{4}) then uniform spaces in 2007 (\cite{5}). In 2010, Plant-Wilkins (\cite{20}) focused on the special case of metric spaces, where discrete homotopy theory means replacing continuous curves and homotopies by discrete sequences and homotopies called $\varepsilon$-chains and $\varepsilon$-homotopies, respectively. An $\varepsilon$-chain is a finite sequence of points $\{x_0, ..., x_n\}$ in a metric space such that for all $i$, $d(x_i, x_{i+1}) < \varepsilon$. Discrete homotopies consist of finitely many steps adding or removing a single point in an $\varepsilon$-chain (fixing the endpoints) so that the sequence remains an $\varepsilon$-chain at each step. Discrete homotopies “ignore small holes” simply by skipping over them. As in \cite{4} and \cite{5}, one can imitate the classical construction of the universal covering space, substituting $\varepsilon$-chains for curves and $\varepsilon$-homotopies for homotopies. This produces what Plant-Wilkins called $\varepsilon$-covers $\phi_\varepsilon : X_\varepsilon \to X$. The fact that both ways of “ignoring small holes” are essentially the same (despite the very different constructions) was shown by Plant-Wilkins in \cite{21}: for compact geodesic spaces, the Sormani-Wei $\delta$-covers are equivalent to the Plaut-Wilkins $\varepsilon$-covers when
\[ \varepsilon = \frac{3}{4} \delta. \] Plaut-Wilkins also defined the Homotopy Critical Spectrum (HCS) in [20] to be the set of all \( \varepsilon \) such that there is an \( \varepsilon \)-loop that is not \( \varepsilon \)-null (homotopic) but is \( \delta \)-null when considered as a \( \delta \)-chain for any \( \delta > \varepsilon \). They also showed in [21] that \( \text{CS} = \frac{3}{4} \text{HCS} \) for compact geodesic spaces. In this paper, since we will refer frequently to [20] and [21], we will generally use the notation of \( \varepsilon \)-covers. Plaut-Wilkins also defined special closed geodesics called “essential circles” whose lengths are precisely three times the values of HCS, discussed in more detail later in this paper.

ECS, ES, and our generalized definition of MLS involve expanding the class of \( \varepsilon \)-covers to a larger class of covering spaces called entourage covers. Entourage covers are defined using the original construction of Berestovskii-Plaut for uniform spaces ([5]). The present paper is written so that no special knowledge of uniform spaces is required, and but the language and framework of uniform spaces are useful. If \( E \) is an “entourage” in a uniform space \( X \), which is a special symmetric set containing an open subset of the diagonal in \( X \times X \), one may define “\( E \)-chains” to be sequences \( \{x_0, \ldots, x_n\} \) such that for all \( i \), \((x_i, x_{i+1}) \in E\). Then \( E \)-homotopies and the corresponding covering map \( \phi_E : X_E \to X \) may be defined analogous to \( \varepsilon \)-homotopies and \( \varepsilon \)-covers. We will provide more details in the next section. Compact topological spaces have a unique uniform structure in which entourages are just any symmetric subsets of \( X \times X \) containing an open set containing the diagonal, and therefore the maps \( \phi_E \) are determined only by the topology (whereas \( \varepsilon \)-covers are determined by the metric).

In metric spaces there are metric entourages \( E_\varepsilon := \{(x, y) : d(x, y) < \varepsilon\} \) for \( \varepsilon > 0 \). \( E_\varepsilon \)-chains and \( E_\varepsilon \)-homotopies are precisely the \( \varepsilon \)-chains and \( \varepsilon \)-homotopies previously described. Metric entourages form a basis for a uniform structure uniquely determined by the metric (although there may be other uniform structures compatible with the topology). That is, entourages in a metric space are simply symmetric subsets of \( X \times X \) that contain some \( E_\varepsilon \). We also use an analogous notation for \( E \)-balls: \( B(x, E) := \{y \in X : (x, y) \in E\} \). If an \( E \)-loop is \( E \)-homotopic to the trivial chain, we say it is \( E \)-null.

In general the covering maps \( \phi_E \) can be problematic, especially if the balls \( B(x, E) \) are not connected. For example, if \( X \) is a compact metric space that is not geodesic, the \( \varepsilon \)-covers many have infinitely many components and HCS may not only not be discrete, it may even be dense in \([0, 1]\) ([8], [36]). Disconnected metric balls also may occur in resistance metrics on finite graphs ([2], Remark 7.19) and have been numerically verified by Cucuringu-Strichartz for certain resistance metrics on the Sierpinski Gasket ([9], Section 4). For non-geodesic spaces, it is not clear whether focusing on metric entourages is likely to be a successful strategy. In some sense, these problems occur because there is a disconnect between the metric and the underlying topology, which is improperly “viewed” by metric balls.

We address this problem by restricting attention to what we call “chained entourages”: an entourage \( E \) is chained if it is contained in the closure of its interior, which is assumed compact if the space is not, and whenever \((x, y) \in E\), \( x, y \) may be joined by an \( F \)-chain that lies entirely in \( B(x, E) \cap B(y, E) \), for any entourage \( F \). That is, \( x \) and \( y \) may be joined in \( B(x, E) \cap B(y, E) \) by “arbitrarily
fine” chains. Entourage covers are by definition those covers $\phi_E$ such that $E$ is a chained entourage; we will sometimes call it the $E$-cover of $X$. In a geodesic space, metric entourages are always chained. This is true because by the triangle inequality, any geodesic joining $x, y$ must stay inside $B(x, \varepsilon) \cap B(y, \varepsilon)$. Then one may simply take arbitrarily fine chains along the geodesic.

One advantage of discrete methods is that they are amenable to counting arguments. For example, while Sormani-Wei showed using convergence methods that the size of CS is bounded in any Gromov-Hausdorff precompact class, in [20] we actually give an explicit bound. Moreover, since for compact Riemannian manifolds $\pi_1(M) = \pi_\varepsilon(E)$ for all sufficiently small $\varepsilon > 0$, by building a simplicial model of the space, Plaut-Wilkins were able to give an explicit fundamental group finiteness theorem generalizing those of Anderson (11) and Shen-Wei (26). Similarly, we are able to prove the following explicit finiteness theorem, where $C(X, \varepsilon)$ denotes the number of $\varepsilon$-balls needed to cover $X$ and $\sigma(E) := \sup\{\varepsilon : E_\varepsilon \subset E\}$ (which is a measure of “size” of $E$).

**Theorem 1** Let $X$ be a compact geodesic space and $\varepsilon > 0$. Then the number $NC(\varepsilon)$ of equivalence classes of $E$-covers $\phi_E : X_E \rightarrow X$ such that $\sigma(E) \geq \varepsilon$ is at most

$$2^{C(X, \varepsilon)^{NC(X, \varepsilon)}}.$$ 

In order to apply the above theorem in more generality, recall that the Bing-Moise Theorem (6, 19) says, in modern terminology, that every Peano continuum has a compatible geodesic metric. “Compatible” means precisely that every metric entourage in the original metric contains a metric entourage in the geodesic metric, and vice versa. We immediately obtain:

**Corollary 2** If $X$ is a Peano Continuum with a given (possibly non-geodesic) metric and $\varepsilon > 0$, then $NC(\varepsilon) < \infty$.

We are now in a position to modify the original Sormani-Wei definition of CS to apply to entourage covers. The only complication is that $E$-covers, unlike $\varepsilon$-covers, are not totally ordered by the relation of one space covering another. But by Corollary 2 there are certain discrete values of $\varepsilon$ such that $NC(\varepsilon)$ strictly increases, and we define ECS to be those values. We may also define the *multiplicity* of a value $\varepsilon$ in ECS to be $NC(\varepsilon) - NC(\delta)$ for $\delta < \varepsilon$ sufficiently close to $\varepsilon$. With a little more effort we show:

**Theorem 3** If $X$ is a compact geodesic space then $CS \subset ECS$.

An immediate consequence of Corollary 2 is that ECS is discrete in $(0, \infty)$ for any metric on a Peano continuum.

**Remark 4** By Gromov’s Precompactness Criterion (13, 14), a corollary of Theorem 7 is that $NC(\varepsilon)$, and hence the number of elements of ECS greater than $\varepsilon$, is uniformly bounded below for any fixed $\varepsilon$ in any Gromov-Hausdorff precompact class of compact geodesic spaces. This extends the corresponding statement about CS.
We already know that any $\varepsilon$-cover for any geodesic metric is an entourage cover, and a natural question is: in general, which regular covers are entourage covers? We have the following necessary algebraic condition. The meaning of “covering map corresponding to $N$” is standard from algebraic topology and will be reviewed as part of the proof.

**Theorem 5** Let $X$ be a semi-locally simply connected Peano continuum and $N \subset \pi_1(X)$ be a normal subgroup. If the covering map corresponding to $N$ is an entourage cover, then $N$ is the normal closure of a finite set.

**Remark 6** The normal closure of a subset of a group is by definition the smallest normal subgroup containing it. Being the normal closure of a finite set is intimately connected with the study of finitely presented groups, since a quotient of a finitely presented group is finitely presented if and only if the kernel is the normal closure of a finite set. We are unable to find a reference for this equivalence, although the proof of one implication may be found in [15], Lemma 3, which cites Siebenmann's dissertation for the statement. The other implication is an exercise in basic algebra using the formal definition of “finitely presented”.

**Remark 7** Suppose that $G$ is a finitely presented group with a quotient that is not finitely presented (such $G$ are well-known to exist). Then as is also classically known, one may construct a compact 4-manifold $M$ with $G$ as its fundamental group. Therefore $M$ must have a regular cover that is not an entourage cover.

For manifolds of dimension at least 3, the condition in Theorem 5 is also sufficient:

**Theorem 8** Let $M$ be any compact smooth manifold of dimension at least 3. If $G \subset \pi_1(M)$ is the normal closure of a finite set then $M$ has a Riemannian metric for which the cover corresponding to $G$ is an $\varepsilon$-cover for the metric.

**Corollary 9** If $M$ is a smooth manifold of dimension at least 3, a normal subgroup $G$ of $\pi_1(X)$ corresponds to an $E$-cover if and only if $G$ is the normal closure of a finite set. Moreover, we may always take $E$ to be an open entourage.

For closed manifolds of dimension 1 there is a simple statement: the only entourage covers of the circle are the trivial and universal covers (which are always entourage covers for manifolds), see Example 68. We do not know much about the situation for closed surfaces, including, for example, exactly which covers of the 2-torus are entourage covers. See also Example 69 concerning the Möbius Band.

We will define later in this paper (Definition 45) a notion of $E$-homotopy (or free $E$-homotopy) of curves, which essentially means the two curves may be “$E$-subdivided” into $E$-chains that are $E$-homotopic (or freely $E$-homotopic). Equivalently, the curve lifts as a loop to $X_E$ (Lemma 46). We say a chained entourage is “essential” if there is an $E$-loop that is not $E$-null. The next proposition (correctly) generalizes the classical statement mentioned above about shortest curves in free homotopy classes in Riemannian manifolds. We will use the
term “ε-geodesic” to describe a curve that is arclength parameterized and minimizing on all intervals of length ε, i.e. the distance between the endpoints of such segments is ε. This is similar to the concept of 1/ε-geodesic of Sormani ([27]) but we obtain more precise results by allowing arbitrary values of ε. A closed curve c that is an ε-geodesic for some ε > 0 for any reparameterization involving a parameter shift is called a “closed ε-geodesic” or simply a “closed geodesic” if a particular ε isn’t specified. Briefly, we express this by saying c is minimal on all segments of length ε, understanding that when c is closed this includes segments that have the common start/end point in the interior.

**Proposition 10** Let X be a compact geodesic space, E be a chained entourage in X, and c be a closed curve in X. Then c has a shortest curve c in its free E-homotopy class, and for any such c,

1. c is non-constant if and only if E is essential, and
2. if c is non-constant then c is a closed 2ε-geodesic whenever Eε ⊂ E.

We now have the following characterizations of MLS and CS for Riemannian manifolds. Each pair consists of a statement involving lengths of curves (which in a general metric space might be infinite), and another that makes sense and is always finite in any metric space. By the length \( L(\alpha) \) of a finite chain \( \alpha = \{x_0, \ldots, x_n\} \) in a metric space we mean \( \sum_{i=1}^{n} d(x_{i-1}, x_i) \).

**Theorem 11** Suppose that M is a compact Riemannian manifold. Then over the set of all essential entourages E,

1. MLS is the set of
   (a) lengths of non-constant closed curves that are shortest in their free E-homotopy class.
   (b) lengths of non-trivial E-loops that are shortest in their free E-homotopy class.
2. CS is the set of
   (a) half the shortest lengths of closed curves that are not freely E-null.
   (b) half the shortest lengths of E-loops that are not freely E-null.

We may now simply use Theorem 1.1a as the definition of MLS for compact geodesic spaces, and with this definition Theorem 1.1 is true for any compact geodesic space (Theorem 67). We may use Theorem 1.1b and Theorem 1.2b as the definitions for arbitrary metric spaces, although if the underlying space is not a Peano continuum then there may not be many (or any!) essential entourages.
Remark 12 We do not know whether, for arbitrary metric spaces, the definition using Theorem 11.1b is equivalent to the definition of CS for metric spaces given in [10], namely that CS is the collection of all $\varepsilon > 0$ such that some covering map is maximally evenly covered on all $\varepsilon$-balls.

While CS = $\frac{3}{2}$HCS for compact geodesic spaces, the approaches of Sormani-Wei and Plaut-Wilkins diverge when more general $E$-covers are added to the mix. The Sormani-Wei playbook leads to ECS as we have already described. To apply the Plaut-Wilkins approach, for an open, chained entourage $E$ in a metric space $X$ we define an $E$-loop $\lambda$ (resp. curve loop $c$) to be $E$-critical if $\lambda$ (resp. $c$) is not $E$-null but is $\overline{E}$-null, where $\overline{E}$ is the closure of $E$. If $E$ has a critical $E$-loop then we will say that $E$ is critical, and we let $\psi(E) := \inf \{ L(\lambda) : \lambda$ is an $E$-critical $E$-loop $\}$. We define the Entourage Spectrum ES to be the set of $\psi(E)$ for all critical entourages $E$. Note that this definition does not involve lengths of curves. We show:

Theorem 13 Let $X$ be a compact geodesic space. Then

1. For every open, chained entourage $E$ there is a critical $E$-loop $\lambda$ if and only if there is a critical loop $c$.
2. If $E$ is a critical entourage then there is a critical $E$-loop (resp. critical loop $c$) such that $L(c) = L(\lambda) = \psi(E)$.
3. $2CS = 3HCS \subset ES \subset MLS$.

Moreover, there are compact Riemannian manifolds for which both of the above inclusions are proper. There also are pairs of diffeomorphic Riemannian manifolds that have the same CS but different ES, and pairs that have the same ES but different MLS.

Remark 14 The proof of Theorem 13 uses in essential ways the fact that the metric is geodesic—in particular by lifting the metric to a geodesic metric on $X_E$. However, the “lifted metric” defined later in this paper is defined for any metric, and in particular some of these methods may be modified for arbitrary metrics on Peano continua [22].

Remark 15 It seems there are interesting questions involving these metric invariants akin to classical results in the metric geometry of Riemannian manifolds. For example, suppose that $M$ is a compact Riemannian manifold. Are there metrics having a particular fixed spectrum (pick one of CS, ECS, ES, MLS) that are optimal with respect to some other geometric parameters, for example having minimal volume with fixed bounds on sectional curvature? And if so what is the regularity of those optimal metrics? In effect this question fixes the size of certain significant “holes” in the space and asks how the space can minimally be stretched to maintain those sizes while constraining curvature.
Remark 16 Sormani-Wei have explored ways in which to extend ideas related to CS to non-compact spaces ([30], [31]). Certainly CS has meaning for non-compact geodesic spaces, for example, but it will “miss” any loops that are homotopic to arbitrarily small loops, for example in the surface obtained by revolving the graph of $y = e^x$ around the $x$-axis. Along these lines, we note that some of the basic and technical results in this paper only require a kind of uniform local compactness (Remark 37)–which is why we state in the definition of “chained” that the closure of the entourage is compact when the space is not. An alternative approach to understand such spaces might be to consider all possible uniform structures that are compatible with the underlying topology of a given metric on the space, rather than just the uniform structure induced by the particular metric. For example, for the surface mentioned above the uniform structure compatible with the metric does not “see” the fundamental group but the uniform structure of the same space metrized as a flat cylinder has the universal cover as an $\varepsilon$-cover.

2 Basic Constructions

This section has a mixture of background from [5], extensions of some results in [20], and a completely new basic result called the Ball Continuity Lemma. The length of a curve is defined in the standard way for metric spaces and it is a classical result that curves having finite length (i.e. rectifiable curves) in metric spaces always have monotone reparameterizations proportional to arclength. We will always assume rectifiable curves are parameterized this way. See [23] for a review, with references, of many basic concepts from metric geometry.

We now recall a bit of basic terminology for uniform spaces. One should keep in mind the two fundamental examples mentioned in the Introduction: metric spaces and compact topological spaces. We have already defined the metric entourage $E_\varepsilon$ in a metric space $X$. In general, a uniform structure on an (always Hausdorff) topological space $X$ is a collection of symmetric subsets of $X \times X$ that contain an open set containing the diagonal, which are called entourages. Moreover, entourages have the following properties: (UA) Their intersection is the diagonal (equivalent to Hausdorff), and (UB) for every entourage $E$ there exists an entourage $F$ such that $F^2 \subset E$. We may also iteratively define, for any entourage $F$, $F^n$. Equivalently, in the terminology from the Introduction, $F^n$ consists of all $(x, y) \in X \times X$ such that there is an $F$-chain $\{x = x_0, ..., x_n = y\}$.

As mentioned in the Introduction, for an entourage $E$ in a uniform space $X$, an $E$-chain consists of a finite sequence $\alpha = \{x_0, ..., x_n\}$ in $X$ such that for all $i$, $(x_i, x_{i+1}) \in E$. We define $\nu(\alpha) = n$. The concatenation of two chains $\alpha = \{x_0, ..., x_n\}$ and $\beta = \{y_0 = x_n, y_1, ..., y_m\}$ is the chain $\alpha * \beta = \{x_0, ..., x_n = y_0, y_1, ..., y_m\}$ and the reversal of $\alpha$ is the chain $\overline{\alpha} = \{x_n, ..., x_0\}$. An $E$-homotopy
between $E$-chains $\alpha$ and $\beta$ consists of a finite sequence $\{\alpha = \eta_0, ..., \eta_n = \beta\}$ of $E$-chains $\eta_i$ all having the same endpoints such that for all $i$, $\eta_i$ differs from $\eta_{i+1}$ by one of the following two basic moves:

1. Insert a point $x$ between $x_i$ and $x_{i+1}$, which we will denote by

$$\{x_0, ..., x_i, x, x_{i+1}, ..., x_n\}$$

and which is “legal” provided $(x, x) \in E$ and $(x, x_{i+1}) \in E$.

2. Remove a point $x_i$ (but never an endpoint!), which we will denote by

$$\{x_0, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n\}$$

and which is legal provided $(x_{i-1}, x_{i+1}) \in E$.

The $E$-homotopy equivalence class of an $E$-chain $\alpha$ is denoted by $[\alpha]_E$. We will sometimes abuse notation by dropping brackets, for example writing $[x_0, ..., x_n]_E$ rather than $\{(x_0, ..., x_n)\}_E$. We note that if $E \subset F$ then $\alpha$ may also be considered as an $F$-chain, and $[\alpha]_F$ also makes sense. Fixing a basepoint $*$, the collection of all $[\alpha]_E$ such that the first point of $\alpha$ is $*$ is called $X_E$. For any entourage $F \subset E$ in $X$, we define $F^* \subset X_E \times X_E$ to be the set of all ordered pairs $([\alpha]_E, [\beta]_E)$ such that $[\alpha \ast \beta]_E = [x_n, y_m]_E$, where $x_n, y_m$ are the endpoints of $\alpha, \beta$, respectively, and $(x_n, y_m) \in F$. The sets $F^*$ form (a basis of) a uniform structure on $X_E$, which we will call the “lifted uniform structure”.

**Remark 17** It is an easy exercise, worthwhile for the unfamiliar reader, to check that this definition of $F^*$ is equivalent to the more cumbersome but sometimes useful original from [5], namely that up to $E$-homotopy we may write $\alpha = \{\ast = x_1, ..., x_{n-1}, x_n\}$ and $\beta = \{\ast = x_1, ..., x_{n-1}, y_n\}$ with $(x_n, y_n) \in F$.

Since $E$-homotopies don’t change endpoints, the endpoint map $\phi_E : X_E \to X$, $\phi_E([x_0, ..., x_n]_E) := x_n$ is well-defined, and its restriction to any $E^*$-ball $B([\alpha]_E, E^*)$ is a bijection onto its image $B(x, E)$ under $\phi_E$, where $x$ is the endpoint of $\alpha$. Since $\phi_E$ is a local bijection, $X_E$ has a unique topology such that $\phi_E$ is a local homeomorphism, with a basis given by all $F^*$-balls with $F \subset E$. This topology is compatible with the lifted uniform structure.

Concatenation is compatible with $E$-homotopies—i.e., if $\alpha_1, \beta_1$ are $E$-homotopic to $\alpha_2, \beta_2$, respectively, then $[\alpha_2 \ast \beta_2]_E = [\alpha_1 \ast \beta_1]_E$. Concatenation induces a group structure on the set of all $E$-homotopy classes $\pi_E(X)$ of $E$-loops starting and ending at the basepoint $*$. That is, $[\lambda_1]_E [\lambda_2]_E = [\lambda_1 \ast \lambda_2]_E$ and $[\lambda]^{-1}_E = [\lambda]_E$, with $[*]_E$ as identity. The group $\pi_E(X)$ acts on $X_E$ induced by pre-concatenation of any loop to an $E$-chain starting at $*$, and the resulting maps are uniform homeomorphisms (i.e. preserve the uniform structure). With this topology, $\phi_E$ is a regular covering map with deck group $\pi_E(X)$ such that the $E$-balls are evenly covered by disjoint unions of $E^*$-balls. Moreover, $X$ is identified with the quotient space $X_E/\pi_E(X)$. Two $E$-loops $\lambda_1$ and $\lambda_2$ are said to be freely $E$-homotopic if there exist $E$-chains $\alpha$ and $\beta$ starting at a common point $x_0$, to the initial points of $\lambda_1$ and $\lambda_2$, respectively, such that

$$\overline{\alpha \ast \lambda_1} \ast \alpha$$

is $E$-homotopic to $\overline{\beta \ast \lambda_2} \ast \beta$. (1)
It is easy to check, and we will use without reference, the following facts: If we can satisfy Formula 1 for some \( x_0 \) then we can do it for any other point, including the basepoint \( * \). Likewise \( \lambda_1 \) and \( \lambda_2 \) are freely \( E \)-homotopic if and only if given an \( E \)-chain \( \alpha \) from \( x_0 \) to the initial point of \( \lambda_1 \) then we can always find a \( \beta \) so that Formula 1 is satisfied.

**Remark 18** One can equivalently define free \( E \)-homotopies of \( E \)-loops by adding a special “homotopy step” to enable one to move the common start/end point. That is, “double it” by adding a repeat of the start/end point, then add a new point (which becomes the new start/end point) between the doubled points. However, this does not seem simpler in all of its details, or more useful than the definition we have given.

Whenever \( E \subset F \) there is a natural covering map \( \phi_{EF} : X_E \to X_F \) that simply treats an \( E \)-chain as an \( F \)-chain. That is, \( \phi_{EF}(\alpha|_E) = [\alpha]|_F \). Given \( D \subset E \subset F \), by definition \( \phi_{FD} = \phi_{FE} \circ \phi_{ED} \). The restriction of \( \phi_{EF} \) to \( \pi_F(X) \) is a homomorphism denoted by \( \theta_{EF} : \pi_F(X) \to \pi_E(X) \). This homomorphism is injective (resp. surjective) if and only if \( \phi_{EF} \) is injective (resp. surjective), and plays a critical role in this paper. Note that the mapping \( \phi_{EF} \) may be identified with the quotient mapping from \( X_F \) to \( X_F / \ker \theta_{EF} = X_E \). In the special case of metric entourages, we denote \( \phi_{E \epsilon} \) by \( \phi_{\epsilon \delta} \) as in [20].

We will need the following general lemma, which partly justifies why we require that the balls in a chained entourage be contained in the closure of their interior.

**Lemma 19** (Ball Continuity) Suppose \( E \) is an entourage in a metric space \( X \) and \( x_i \to x \) in \( X \). Then

1. If \( y_i \to y \) in \( X \) and \( (x_i, y_i) \in E \) for all \( i \), then \( (x, y) \in \overline{E} \).
2. If \( E \) has compact closure then \( B(x, E) \) is Hausdorff convergent to a subset of \( B(x, \overline{E}) \).
3. If \( E \) is open then \( B(x, E) \) is the union of the sets \( B(x, E) \cap B(x_i, E) \).
4. If \( E \) is contained in the compact closure of its interior then \( B(x_i, E) \to \overline{B(x, E)} \) (meaning convergence in the Hausdorff metric).

**Proof.** The first part is true in any uniform space: just note that if \( U \) and \( V \) are open sets containing \( x \) and \( y \), respectively, then \( U \times V \) eventually contains some \( (x_i, y_i) \in E \).

For the second part, let \( A \) be the (compact) set of all limits of convergent sequences \( (z_i) \) such that \( z_i \in B(x_i, E) \). By the first part, \( A \subset B(x, \overline{E}) \). For any \( \varepsilon > 0 \), we may cover \( A \) by finitely many balls \( B(p_1, \frac{\varepsilon}{2}), ... , B(p_m, \frac{\varepsilon}{2}) \). Then for sufficiently large \( k \) there are points \( p'_j \in B(x_k, E) \) such that \( d(p_j, p'_j) < \frac{\varepsilon}{2} \), and from the triangle inequality it follows that the \( \varepsilon \)-neighborhood of \( B(x_k, E) \) contains \( A \). Now suppose that for all \( k \), the \( \varepsilon \)-neighborhood of \( A \) does not contain
B(x_k, E). That is, for all k there exist w_k ∈ B(x_k, E) such that d(w_k, z) ≥ ε for all z ∈ A. Since (w_k, x_k) ∈ E and E has compact closure, by taking a subsequence if necessary we may assume that (w_k, x_k) is convergent to (w, x) ∈ E. By definition, w ∈ A, so d(w_k, w) ≥ ε for all k, a contradiction to w_k → w.

For the third part, let y ∈ B(x, E). Since E is open there exist open U, V in X such that (x, y) ∈ U × V ⊂ E. Then for large enough i, x_i ∈ U and therefore (x_i, y) ∈ U × V ⊂ E, i.e. y ∈ B(x_i, E).

Finally, in the special case when E is open with compact closure, the fourth part follows from the second and third parts. The proof in general is now finished by observing that in general if A_i → A then any sequence of dense subsets of A_i is Hausdorff convergent to any dense subset of A. □

We will now extend the idea of the “lifted metric” from [20] to this more general situation. The following definition is the same as the corresponding part of Definition 12, [20], with “E” replacing “ε”:

Definition 20 Let X be a metric space and [α]_E, [β]_E ∈ X_E. We define

$$|[α]_E| := \inf \{ L(κ) : [α]_E = [κ]_E \}$$

and

$$d([α]_E, [β]_E) = |[α]_E * [β]_E|.$$

The proof that d is a metric on X_E is essentially identical to the proof of this statement (Proposition 13) in [20]. Likewise, one may prove that the deck group π_E(X) acts as isometries. However in [20], Proposition 14, we also proved that ϕ_E preserves all distances less than ε. We will verify here an important analog of this statement, namely:

Lemma 21 If X is a metric space, E is an entourage, and ([α]_E, [β]_E) ∈ E^* then d([α]_E, [β]_E) = d(ϕ_E([α]_E), ϕ_E([β]_E)). In particular, if F^2 ⊂ E then the restriction of ϕ_E to any F^*-ball is an isometry onto an F-ball.

Proof. If ([α]_E, [β]_E) ∈ E^* then by definition of E^*, [α]_E * [β]_E = [α, β]_E, where x_n, y_m are the endpoints of α, β, respectively. Since [x_n, y_m]_E is the shortest possible E-chain joining x_n, y_m, d([α]_E, [β]_E) = d(x_n, y_m) = d(ϕ_E([α]_E), ϕ_E([β]_E)).

To prove the second part, suppose that [α]_E, [β]_E ∈ B([γ]_E, F^*). By definition, ([α]_E, [γ]_E) ∈ F^* and ([β]_E, [γ]_E) ∈ F^*. This in turn means that if the endpoints of α, β, γ are x_n, y_m, z_k, respectively, then [α]_E * [γ]_E = [x_n, z_k]_E and [β]_E * [γ]_E = [z_k, y_m]_E with (x_n, z_k), (z_k, y_m) ∈ F. This implies that (x_n, y_m) ∈ F^2 ⊂ E. Next note that

$$[α]_E * [β]_E = [α]_E * [γ]_E * [β]_E = [x_n, z_k, z_k, y_m]_E = [x_n, z_k, y_m]_E.$$

Since (x_n, y_m) ∈ E, removing z_k is a legal move for an E-homotopy, proving that ([α]_E, [β]_E) ∈ E^*. By the first part of this lemma, d([α]_E, [β]_E) is preserved by ϕ_E. We know from [20] (and it is easy to verify) that the restriction of ϕ_E to any F^*-ball is a bijection onto an F-ball, finishing the proof. □
Since $\phi_E$ is a local isometry, it preserves the lengths of curves. If $X$ is a
godesic space then the lifted metric is a godesic metric and $\phi_E$ is distance
non-increasing. In fact, one can check that in this case the metric we have
defined is the unique metric with these properties (cf. [20], Proposition 23).

3 Refinement and Approximation

The underlying assumption for the main results in [20] and [21] is that the space
in question is a godesic space. Two fundamental issues appear when attempting
to extend results from metric entourages in godesic spaces to entourages in
general. First, there is the issue of refinement. In a godesic space, when
$0 < \delta < \varepsilon$, any $\varepsilon$-chain $\alpha = \{x_0, ..., x_n\}$ can always be “refined” into a $\delta$-chain
that is in the same $\varepsilon$-homotopy class as $\alpha$. This is accomplished simply by
subdividing a godesic joining $x_i$ and $x_{i+1}$. Lack of some method of refinement
in some sense “causes” the problems observed in [8] and [36] concerning the HCS
in non-godesic metric spaces. As we will see, restricting to chained entourages
solves this problem.

The second issue in this generality is that metric entourages are totally or-
dered by inclusion, but entourages in general are not. This issue is unavoidable
and has many implications. For example, as soon as there are two entourage
covers of a space, neither of which covers the other, then it is impossible that
those two entourage covers may be simultaneously an $\varepsilon_1$-cover and an $\varepsilon_2$-cover
for a single godesic metric. Hence it is generally impossible to realize all en-
tourage covers as $\varepsilon$-covers of a single godesic metric. However, recall that
Theorem [8] shows that for smooth manifolds of dimension $\geq 3$, every entourage
cover is realized as an $\varepsilon$-cover for some Riemannian metric.

Definition 22 Let $X$ be a uniform space, $E \subset F$ be entourages, and $\alpha = \{x_0, ..., x_n\}$ be an $F$-chain. An $E$-refinement of $\alpha$ is an $E$-chain of the form

$$\{x_0 = m_0, ..., m_0k_0 = x_1, ..., x_r = m_r, ..., m_rk_r = x_{r+1}, ..., x_n\}$$

such that each $E$-chain $\{x_j = m_{j0}, ..., m_{jk_j} = x_{j+1}\}$ lies entirely in $B(x_j, F) \cap B(x_{j+1}, F)$.

Remark 23 As mentioned above, in [20] we defined “$\varepsilon$-refinement” of a $\delta$-chain
$\alpha$ in a godesic space using subdivisions of geodesics joining successive pairs of
points in $\alpha$. An $\varepsilon$-refinement as defined in that paper is an $E_\varepsilon$-refinement in
the sense of the present paper, but due to the special method of construction,
not every $E_\varepsilon$-refinement in the present sense (even in a godesic space) is an $\varepsilon$-
refinement in the sense of [20]. For this reason, we will maintain a distinction
between $\varepsilon$-refinements and $E_\varepsilon$-refinements. Note also that we cannot expect
existence of $\varepsilon$-refinements even in a godesic space if the refinement involves a
non-metric entourage $E$. This is because it is possible that no godesic joining
a pair $(x, y) \in E$ stays inside $B(x, E) \cap B(y, E)$.
Remark 24 There is some potential for confusion because any $F$-chain $\alpha$ may be considered as a $D$-chain when $F \subset D$, and the notion of $E$-refinement depends on whether we consider $\alpha$ as an $F$-chain or a $D$-chain. That is, an $E$-refinement when $\alpha$ is considered as an $F$-chain is always an $E$-refinement when $\alpha$ is considered as a $D$-chain, but not always conversely.

Remark 25 Note that it is immediate from the definition in the Introduction that if $E$ is a chained entourage and $F \subset E$ then every $E$-chain has an $E$-refinement. We will see later (Lemmas 31 and 32) that $F$-refinements do not change the $E$-homotopy class and have a size that can be uniformly controlled in any Gromov-Hausdorff precompact class.

Recall that a subset $A$ of a uniform space $X$ is called chain connected if for every pair of points $x, y \in A$ and entourage $E$ there is an $E$-chain in $A$ joining $x$ and $y$ (see [5] and note that this definition is equivalent to what is sometimes known as uniformly connected in the literature). It is easy to check that connected implies chain connected but the converse is not true (e.g., the rational numbers). For compact subsets of uniform spaces, connected and chain connected are equivalent. It is also easy to check that if $X$ has a basis consisting of entourages with open, connected balls then $X$ is connected if and only if $X$ is chain connected. In particular, any chain connected geodesic space is connected.

Remark 26 We are using chain connectedness rather than connectedness not simply to gain a little extra generality. Chain connectedness is a far more natural condition in the context of these discrete methods; many arguments are simplified using it even when the sets in question are ultimately known to be connected; see for example Lemmas 30 and 31. Two easy-to-check and useful properties of chain connected sets that are not true for connected sets are: the closure $\overline{A}$ of a set $A$ is chain connected if and only if $A$ is chain connected; and the Hausdorff limit of a sequence of chain connected subsets of a metric space is chain connected.

The next lemma complements the definition in the Introduction, and we will use it without reference.

Lemma 27 An entourage $E$ in a uniform space is chained if and only if $E$ is contained in the closure of its interior and for every $(x, y) \in E$ there is a chain connected set $C \subset B(x, E) \cap B(y, E)$ that contains $x$ and $y$.

Proof. Sufficiency is obvious. Suppose that for every $(x, y) \in E$ and entourage $F$ there is an $F$-chain joining $x$ and $y$ in $B(x, E) \cap B(y, E)$. Let $C$ denote the set of all points $z$ in $B(x, E) \cap B(y, E)$ such that for every entourage $F$, $z$ is joined to $x$ and to $y$ by an $F$-chain in $B(x, E) \cap B(y, E)$. If $v, w \in C$ then for any $F$, we may join $v$ to $x$ with an $F$-chain, then $x$ to $y$ with an $F$-chain, then $y$ to $w$ with an $F$-chain, all of them lying in $B(x, E) \cap B(y, E)$. In other words, $C$ is a chain connected subset of $B(x, E) \cap B(y, E)$, and since it also contains $x, y$, the proof is complete. ■
Remark 28 Note that the above lemma implies that every \( B(x, E) \) is chain connected. This in turn has another important consequence: since \( E \)-balls are chain connected, \( X_E \) is chain connected and for any \( F \subset E, \phi_{EF} : X_F \to X_E \) is surjective (Proposition 71 in \([5]\)).

We will use the following lemma only in the case of a union of two entourages but this statement isn’t much harder to prove:

**Lemma 29** If \( \{E_\alpha\} \) is a collection of entourages in a uniform space \( X \) then \( E := \bigcup E_\alpha \) is an entourage and for any \( x \in X \), \( B(x, E) = \bigcup E(x, E_\alpha) \). In addition, if each \( E_\alpha \) is chained then \( E \) is chained.

**Proof.** Clearly \( E \) is an entourage. Now \( y \in B(x, E) \) if and only if \( (x, y) \in E_\alpha \) for some \( \alpha \), which is equivalent to \( y \in B(x, E_\alpha) \) for some \( \alpha \). This proves \( B(x, E) = \bigcup B(x, E_\alpha) \). Next observe that \( B(x, E) \cap B(y, E) \) contains \( B(x, E_\beta) \cap B(y, E_\beta) \) for any \( \beta \), which itself contains a chain connected set containing \( x \) and \( y \) because \( E_\beta \) is chain connected. It is an exercise in elementary point set topology to show that \( E \) is contained in the closure of its interior. \( \blacksquare \)

In a geodesic space, not only does every \( \delta \)-chain \( \alpha \) have an \( \varepsilon \)-refinement when \( \delta < \varepsilon \), we can control its size in terms of \( \nu(\alpha) \). This is because we may choose points of distance arbitrarily close to \( \delta \) along a geodesic joining \( x_i, x_{i+1} \), which has length less than \( \varepsilon \). So we may always refine by adding at most \( \frac{2}{\varepsilon} \) points between any point and its successor. For more general chained entourages, we are only able to control this number in compact spaces (but uniformly in any Gromov-Hausdorff precompact class).

**Lemma 30** Let \( X \) be a compact metric space and \( \varepsilon > 0 \). Then any two points in a chain connected subset \( A \) of \( X \) may be joined by an \( \varepsilon \)-chain in \( A \) having at most \( 2C(X, \frac{\varepsilon}{\varepsilon}) \) points.

**Proof.** By definition, for any \( x, y \in A \) there is some \( \varepsilon \)-chain \( \{x = x_0, ..., x_n = y\} \) with a minimal number of points in \( A \) joining \( x, y \). Let \( Z \) be an \( \frac{\varepsilon}{\varepsilon} \)-dense in \( X \) having \( C(X, \frac{\varepsilon}{\varepsilon}) \) points. For each \( x_i \), choose some \( z_i \) such that \( d(x_i, z_i) < \frac{\varepsilon}{\varepsilon} \). We claim that each element of \( Z \) can be paired in this way at most twice, which completes the proof. To prove the claim, suppose that \( z_i = z_j \) for some \( i < j \). Then by the triangle inequality \( d(x_i, x_j) < \varepsilon \). If \( j > i + 1 \) then we could reduce the size of the chain by eliminating the points \( x_{i+1}, ..., x_{j-1} \), contradicting minimality. In other words, if a point of \( Z \) is used more than once, it must be used for precisely two adjacent points in the chain. \( \blacksquare \)

**Lemma 31** Suppose that \( E \) is a chained entourage in a compact metric space \( X \), and \( E_\varepsilon \subset F \subset E \) for some entourage \( F \) and \( \varepsilon > 0 \). Then every \( E \)-chain \( \alpha \) has an \( F \)-refinement \( \alpha' \) such that \( \nu(\alpha') \leq 2\nu(\alpha) \cdot C(X, \frac{\varepsilon}{\varepsilon}) \).

**Proof.** It suffices to show that an \( E \)-chain \( \{x_0, x_1\} \) has an \( E_\varepsilon \)-refinement of at most \( 2C(X, \frac{\varepsilon}{\varepsilon}) \) points; such refinements are also \( F \)-refinements and may then
Lemma 32 Let \( X \) be a uniform space and \( F \subset E \) be entourages. If \( \alpha = \{x_0, \ldots, x_n\} \) is an \( F \)-chain in \( B(x_0, E) \) then \( \alpha \) is \( E \)-homotopic to \( \{x_0, x_n\} \). In particular, every \( F \)-refinement of an \( E \)-chain \( \beta \) is \( E \)-homotopic to \( \beta \) and any two \( F \)-refinements of \( \beta \) are \( E \)-homotopic.

Proof. Since \( x_2 \in B(x_0, E) \), \( \{x_0, \widehat{x_2}, x_3, \ldots x_n\} \) is a legal move. Likewise \( \{x_0, \widehat{\widehat{x_2}}, x_3, \ldots x_n\} \) is a legal move and the proof of the first statement is finished after finitely many steps. The last statements are immediate consequences of the first and the definitions.

The next lemma formalizes a process that is a discrete version of the Arzela-Ascoli Theorem, used several times in this paper and originating in [20].

Lemma 33 (Chain Normalizing) Let \( X \) be a compact metric space, \( E \) be an entourage in \( X \), and \( \alpha_i = \{x_{i0}, x_{i1}, \ldots, x_{in_i}\} \) be a sequence of \( E \)-chains of bounded length. Then:

1. Up to \( E \)-homotopy we may assume that for some \( n \), \( n_i = n \) for all \( i \).

2. By choosing a subsequence if necessary we may assume that for all \( 0 \leq j \leq n \), \( x_{ij} \to x_j \) for some \( x_j \in X \).

Proof. Let \( L \) be an upper bound for \( \{L(\alpha_i)\} \) and suppose that \( E_\varepsilon \subset E \). For any \( i \), choose some representative of \( [\alpha_i]_E \) of length \( L_i \leq L \). We may assume that the maximum number of values \( d(x_{ij}, x_{i(j+1)}) \) that are smaller than \( \frac{\varepsilon}{2} \) is at most \( \frac{4L}{\varepsilon} \). Otherwise there would have to be two consecutive distances smaller than \( \frac{\varepsilon}{2} \) and we could remove one point while staying in the same \( E \)-homotopy class and not increasing length. In other words, \( L \geq L_i \geq \frac{4L}{\varepsilon} \cdot \frac{\varepsilon}{2} \) and we conclude that \( n_i \leq \frac{4L}{\varepsilon} \). Now by adding repeated points, if needed, we may ensure that \( \nu(\alpha_i) \) is precisely \( n := \left\lfloor \frac{4L}{\varepsilon} \right\rfloor + 1 \). The second part is an immediate consequence of compactness.

For many problems it is important to have some version of “close \( E \)-chains are \( E \)-homotopic”. Proposition 34 is analogous to Proposition 15 in [20], using \( E = E_\varepsilon \) and \( F \subset E_2 \). The proof here uses the same homotopy steps as the one in [20], but due to the differences in assumptions we write out a full proof here.

Proposition 34 Let \( X \) be a uniform space, \( E \) be an entourage, and \( F \) be an entourage such that \( F^2 \subset E \). If \( \alpha = \{x_0, \ldots, x_n\} \) is an \( F \)-chain and \( \beta = \{x_0 = y_0, \ldots, y_n = x_n\} \) is a chain such for all \( i \), then \( \beta \) is an \( E \)-chain that is \( E \)-homotopic to \( \alpha \).
**Proof.** We will construct an $E$-homotopy $\eta$ from $\alpha$ to $\beta$, using the fact that $F^2 \subset E$ to see that in each step that the resulting chain is an $E$-chain, i.e. the step is legal. For example, the first step in the second line below is justified by the fact that $(x_0, x_1) \in F$ and $(x_1, y_1) \in F$, so $(x_0, y_1) \in F^2 \subset E$. The remaining justifications are similar.

\[
\alpha = \{x_0, x_1, \ldots, x_n\} \rightarrow \{x_0, \widehat{x_1}, x_1, \ldots, x_n\} \rightarrow \{x_0, x_1, \widehat{y_1}, x_1, \ldots, x_n\}
\]

\[
\rightarrow \{x_0, x_1, y_1, x_1, \ldots, x_n\} \rightarrow \{x_0, y_1, \widehat{x_2}, x_2, \ldots, x_n\}
\]

\[
\rightarrow \{x_0, y_1, x_2, \widehat{y_2}, x_2, \ldots, x_n\} \rightarrow \{x_0, y_1, x_2, y_2, x_2, \ldots, x_n\}
\]

\[
\rightarrow \{x_0, y_1, x_2, y_2, x_2, \ldots, x_n\} \rightarrow \{x_0, y_1, y_2, \widehat{x_3}, x_3, \ldots, x_n\} \rightarrow \cdots \rightarrow \beta
\]

\[\blacksquare\]

**Proposition 35** Let $X$ be a metric space and $E$ be a chained entourage. Suppose that $\alpha_i := \{x_0 = x_0, \ldots, x_m = x_n\}$ is a sequence of $E$-chains converging to $\alpha = \{x_0, \ldots, x_n\}$. Then $\alpha$ is an $E$-chain such that for all sufficiently large $i$, $\alpha_i$ is $E$-homotopic to some (hence any) $E$-refinement of $\alpha$.

**Proof.** The Ball Continuity Lemma (Lemma 19) implies that $\alpha$ is an $E$-chain. Note that in any metric space and for any $0 < \varepsilon < \delta$, by continuity of the distance function, $E_{\varepsilon} \subset \{(x, y) : d(x, y) \leq \varepsilon\} \subset E_{\delta}$.

In particular we may find some closed entourage $F$ such that $F^2 \subset E$. Let $\alpha'_i$ be an $F$-refinement of $\alpha$ for every $i$. Since $\nu(\alpha_i) = n$ for all $i$, $L(\alpha_i) \rightarrow L(\alpha)$ and hence $\{L(\alpha_i)\}$ is bounded. By the Chain Normalizing Lemma ( Lemma 33), we may suppose that for some fixed $k$, and for all $i$,

\[
\alpha'_i = \{x_0 = m^i_{00}, \ldots, m^i_{0k} = x_1, \ldots, x_r = m^i_{r0}, \ldots, m^i_{rk} = x_r+1, \ldots, x_n\}
\]

and that for all $j, m, m^i_{mj} \rightarrow m^i_{mj}$, for some $m_j \in X$. In addition,

\[
\alpha' := \{x_0 = m_0, \ldots, m_k = x_1, \ldots, x_r = m_r, \ldots, m_{rk} = x_{r+1}, \ldots, x_n\}
\]

is an $F$-chain since $F$ is closed. By the Ball Continuity Lemma (Lemma 19), each $m_{jk}$ lies in $B(x_j, E) \cap B(x_{j+1}, E)$ and therefore $\alpha'$ is an $F$-refinement, hence an $E$-refinement, of the $E$-chain $\alpha$. For all large enough $i$ and all $j, m$, $(m^i_{mj}, m^i_{mj}) \in F$, and Proposition 34 tells us that $\alpha'_i$ is $E$-homotopic to $\alpha'$. Since each $\alpha_i$ is $E$-homotopic to $\alpha'_i$ by Lemma 32, the proof is complete. \[\blacksquare\]

**Corollary 36** Let $X$ be a metric space and $E$ be a chained entourage. Suppose that $\alpha_i := \{x_0, \ldots, x_m = x_0\}$ is a sequence of $E$-loops converging to an $E$-loop $\alpha = \{x_0, \ldots, x_n = x_0\}$. Then for all sufficiently large $i$, $\alpha_i$ is freely $E$-homotopic to some (hence any) $E$-refinement of $\alpha$. 

17
Proof. Let $\beta = \{* = y_0, \ldots, y_m = x_0\}$ be any $E$-chain. Then for large enough $i$, $(x_0, x_{i0}) \in E$ and therefore
\[
\alpha'_i := \{* = y_0, \ldots, x_0, x_{i0}\} \ast \alpha_i \ast \{x_{i0}, x_0, \ldots, y_0 = *\}
\]
is an $E$-loop at $*$. Then $\alpha'_i \rightarrow \alpha' := \overline{\beta} \ast \alpha \ast \beta$ and by Proposition \([35]\) for all large $i$, $\alpha'_i$ is $E$-homotopic to any $E$-refinement of the $E$-loop $\alpha'$. By definition of freely $E$-homotopic, the proof is finished. \(\blacksquare\)

**Remark 37** While we will not need it here, in analogy with the notion of “proper metric space” (meaning all closed metric balls are compact), one could define a “proper uniform space” to be a uniform space $Y$ such that $Y \times Y$ has a basis of, and is the countable union of compact entourages. Of course covering spaces of a compact space $X$ may not be compact, but one can show that $X_E$ is still a proper uniform space for any chained entourage $E$. Proposition \([35]\) and various other results in this paper may be applied to proper uniform spaces.

### 4 Curves and the Fundamental Group

Many of the next few results extend basic theorems concerning covering space theory both discrete and classical, with $\phi_E$ playing the role of the universal covering map. The standard proofs of some of these results in the classical theory use the homotopy lifting property, which one could try to emulate here: i.e. lifting $F$-homotopies to $F^*$-homotopies in $X_E$. However, it is not clear whether this approach is worth the effort. For the results we need, we are able to get by only with chain lifting. See also \([35]\) for some lifting results concerning $\varepsilon$-chains in geodesic spaces.

**Lemma 38 (Chain Lifting)** Let $X$ be a uniform space and $E$ be an entourage. Suppose that $\beta := \{x_0, \ldots, x_n\}$ is an $E$-chain and $[\alpha]_E$ is such that $\phi_E([\alpha]_E) = x_0$. Let $y_i := [\alpha \ast \{x_0, \ldots, x_i\}]_E$. Then $\tilde{\beta} := \{y_0 = [\alpha]_E, y_1, \ldots, y_n = [\alpha \ast \beta]_E\}$ is the unique “lift” of $\beta$ to $[\alpha]_E$. That is, $\tilde{\beta}$ is the unique $E^*$-chain in $X_E$ starting at $[\alpha]_E$ such that $\phi_E(\tilde{\beta}) = \beta$.

Proof. Since the endpoint of $\alpha \ast \{x_0, \ldots, x_i\}$ is $x_i$, $\phi_E(y_i) = x_i$; i.e., $\phi_E(\tilde{\beta}) = \beta$. Also, by definition of $E^*$, $\tilde{\beta}$ is an $E^*$-chain (in fact this is easiest to see using the original definition from \([5]\), see Remark \([16]\)). So we need only show uniqueness, which we will prove by induction on $n$. For $n = 0$, the starting point $[\alpha]_E$ is determined by assumption, so $\{[\alpha]_E\}$ is the unique lift. Suppose we have proved the statement for $n - 1$. Then given $\beta = \{x_0, \ldots, x_n\}$ and using the above notation we know that $[y_0, \ldots, y_{n-1}]$ is the unique lift of $\{x_0, \ldots, x_{n-1}\}$. But we already know that $\phi_E$ is injective from $B(y_{n-1}, E^*)$ onto $B(x_{n-1}, E)$ and therefore $x_n$ has a unique preimage in $B(y_{n-1}, E^*)$. Therefore $y_n$, which lies in $B(y_{n-1}, E^*)$ and satisfies $\phi_E(y_n) = x_n$, is the only possibility. \(\blacksquare\)
Lemma 39 Let $X$ be a uniform space and $E$ be an entourage. Let $[\lambda]_E \in \pi_E(X)$ and $[\alpha]_E \in X_E$. Then considering $[\lambda]_E$ as a deck transformation of $X_E$, $[\lambda]_E([\alpha]_E)$ is the endpoint of the lift of $\alpha$ starting at $[\lambda]_E$ in $X_E$. In particular, the unique lift of $\alpha$ starting at the basepoint ends at $[\alpha]_E$.

Proof. By definition, $[\lambda]_E([\alpha]_E) = [\lambda \ast \alpha]_E$, which by the Chain Lifting Lemma (Lemma 38) is the endpoint of the lift of $\lambda \ast \alpha$ to $[s]_E$. But by uniqueness this is the endpoint of the lift of $\alpha$ starting at the endpoint $z$ of the lift $\hat{\lambda}$ of $\lambda$ starting at $[s]_E$. By the Chain Lifting Lemma, $z = [\lambda]_E$, completing the proof.

We also need to extend the definition of “stringing” from [20], which among other things allows us to understand the relationship between the fundamental group and the groups $\pi_E(X)$. Note that the definition below is slightly stronger than the one in [20], but the new one is more natural and necessary for the current paper; replacing the definition in [20] by the new one would have essentially no impact on [20].

Definition 40 Let $\alpha := \{x_0, \ldots, x_n\}$ be an $E$-chain in a metric space $X$, where $E$ is an entourage in a uniform space. A stringing of $\alpha$ consists of a path $\hat{\alpha}$ formed by concatenating paths $\gamma_i$ from $x_i$ to $x_{i+1}$ where each path $\gamma_i$ lies entirely in $B(x_i, E) \cap B(x_{i+1}, E)$. Conversely, let $c : [0, 1] \to X$ be a curve. An $E$-subdivision of $c$ is an $E$-chain $\beta = \{c(t_0), \ldots, c(t_n)\}$, where $0 = t_0 \leq \cdots \leq t_n = 1$ are such that $c$ is a stringing of $\beta$.

Remark 41 Note that, unlike the typical definition of partitions, our definition of $E$-subdivision allows “repeated points” (as do chains in general) due to the inequality $t_i \leq t_{i+1}$. This simplifies matters, for example when considering limits of chains and curves.

Lemma 42 If $X$ is a metric space, $c : [0, 1] \to X$ is a curve and $E$ is an entourage in $X$, then $c$ has an $E$-subdivision. Moreover, any two $E$-subdivisions of $c$ are $E$-homotopic and curves $c_1$ and $c_2$ with the same endpoints are $E$-homotopic if and only if any $E$-subdivision of $c_1$ is $E$-homotopic to any $E$-subdivision of $c_2$.

Proof. For existence, it suffices to find an $E_\varepsilon$-subdivision for some $E_\varepsilon \subset E$. Since $c$ is uniformly continuous there exists some $\delta > 0$ such that if $|s - t| < \delta$ then $d(c(s), c(t)) < \varepsilon$. Now subdivide $[0, 1]$ into intervals of length less than $\delta$ with endpoints $t_0 = 0 < t_1 < \cdots < t_n = 1$. Then the image of the restriction of $c$ to any interval $[t_i, t_{i+1}]$ lies entirely in $B(c(t_i), \varepsilon) \cap B(c(t_{i+1}), \varepsilon)$ and hence the $\varepsilon$-chain $\{c(t_0), \ldots, c(t_n)\}$ is an $E_\varepsilon$-subdivision.

For the second statement, suppose that $\alpha_1, \alpha_2$ are $E$-subdivisions of $c$, corresponding to partitions $\tau_1, \tau_2$ of $[0, 1]$. We claim that $s = \tau_1 \cup \tau_2$ has the property that if $s = \{s_0 \leq \cdots \leq s_m\}$ then $\alpha := \{c(s_0), \ldots, c(s_m)\}$ is an $E$-refinement of both $\alpha_1$ and $\alpha_2$. This will complete the proof by Lemma 32. Moreover, by symmetry it suffices to show that $\alpha$ is an $E$-refinement of $\alpha_1$. Consider some $t_i \leq t_{i+1} \in \tau_1$, and suppose that for some $j \leq k$, $t_i \leq s_j \leq \cdots \leq s_k < t_{i+1}$ with $s_m \in \tau_2$. Since $\alpha_1$ is an $E$-subdivision of $c$, the restriction of $c$ to $[t_i, t_{i+1}]$ lies
Lemma 46  Let \( E \) freely \((\text{resp. freely } E)\) \( \alpha \) is an \( X \) also that the proof of the lemma only requires that \( c \) starting and ending at the same point. Then \( E \) same point, and \( \alpha \) is an \( \alpha_1 \), as required. ■

Lemma 43  Suppose that \( X \) is a Peano continuum, \( E \) is a chained entourage and \( \alpha \) is an \( E \)-chain. Then for some chained entourage \( F \subset E \), \( \alpha \) has an \( F \)-refinement with a stringing. In particular, \([\alpha]_E\) contains a representative with a stringing.

Proof. Let \( X \) have a geodesic metric (by the Bing-Moise Theorem). Since \( E \) is a chained entourage, there is some \( E \)-refinement \( \beta = \{x_0, \ldots, x_n\} \) of \( \alpha \) when \( E_\varepsilon \subset E \). Now \( x_i \) and \( x_{i+1} \) may be joined by a geodesic, which remains inside \( B(x_i, E_\varepsilon) \cap B(x_{i+1}, E_\varepsilon) \subset B(x_i, E) \cap B(x_{i+1}, E) \). The concatenation of these geodesics is an \( F \)-stringing, hence an \( E \)-stringing of \( \beta \). ■

Remark 44  Note that the stringing of \( \beta \) in the above lemma may not be a stringing of \( \alpha \) since there is no reason why the concatenated curve must lie inside \( B(y_i, E) \cap B(y_{i+1}, E) \), where \( y_i \) and \( y_{i+1} \) are consecutive points in \( \alpha \). Note also that the proof of the lemma only requires that \( X \) be a uniform space with a uniformly equivalent geodesic metric—for example any smooth, possibly non-compact, manifold with the uniform structure given by a particular Riemannian metric.

Definition 45  If \( c_1, c_2 \) are curves in a uniform space starting and ending at the same point, and \( E \) is an entourage, we say that \( c_1 \) and \( c_2 \) are \( E \)-homotopic (resp. freely \( E \)-homotopic) if there exist \( E \)-subdivisions \( \kappa_i \) of \( c_i \), that are \( E \)-homotopic (resp. freely \( E \)-homotopic).

Lemma 46  Let \( E \) be an entourage in a uniform space \( X \) and \( c_1, c_2 \) be curves starting and ending at the same point. Then \( c_1 \) and \( c_2 \) are \( E \)-homotopic if and only if one lift (equivalently any lifts) of \( c_1 \) and \( c_2 \) to \( X_E \) starting at the same point end at the same point. In particular, a curve \( c \) lifts as a loop to \( X_E \) if and only if \( c \) is freely \( E \)-null.

Proof. By arguments analogous to standard ones for covering space theory involving concatenations and uniqueness of lifts, the entire statement reduces to showing the following: If \( c \) is a loop at the basepoint then \( c \) lifts as a loop at the basepoint if and only if it is \( E \)-null. Let \( \lambda := \{* = c(t_0), \ldots, c(t_n) = *\} \) be a \( E \)-subdivision of \( c \). By the Chain Lifting Lemma its unique lift \( \tilde{\lambda} \) starting at the basepoint ends at \([\lambda]_E\), i.e. \( \tilde{\lambda} \) is a loop if and only if \( \lambda \), hence \( c \), is \( E \)-null. ■

The next two statements are extensions of [20], Proposition 20 and Corollary 21, replacing \( \varepsilon \) by \( E \) or \( E^* \) as appropriate. Note only that \( E \) is not assumed to be chained, so that \( X_E \) may not be connected (and we are not guaranteed that stringings exist). But this does not affect the statement or arguments.

Proposition 47  Let \( E \) be an entourage in a uniform space \( X \) and \( \alpha \) be an \( E \)-chain starting at the basepoint. Then the unique lift of any stringing \( \hat{\alpha} \) starting at the basepoint \([*]_E \) in \( X_E \) has \([\alpha]_E \) as its endpoint.

20
Proof. By uniqueness in the Chain Lifting Lemma and the second statement in Lemma \[\text{Lemma 39}\] it suffices to show that if \(c : [0, 1] \to X\) is a stringing of \(\alpha := \{x_0, ..., x_n\}\) with \(x_i = c(t_i)\), and \(\tilde{c}\) is the unique lift of \(c\) to the basepoint, then \(\{\tilde{c}(t_0), ..., \tilde{c}(t_n)\}\) is an \(E^*\)-chain. But by definition, for any \(i\), the segment of \(c\) from \(x_i\) to \(x_{i+1}\) lies entirely in \(B(x_i, E)\), so lifts into \(B(\tilde{c}(t_i), E^*)\), proving that \((\tilde{c}(t_i), \tilde{c}(t_{i+1})) \in E^*\). ■

The next corollary now follows from the homotopy lifting property for covering spaces, verifying that homotopies of curves are “stronger than” \(E\)-homotopies. Recall that even with a chained entourage in a geodesic space we are not guaranteed that stringings exist, but Lemma \[\text{Lemma 13}\] tells us we can always find a stringing of an \(E\)-homotopic \(E\)-chain.

Corollary 48 Let \(E\) be an entourage in a metric space \(X\) and \(\alpha, \beta\) be \(E\)-chains. If there exist stringings \(\hat{\alpha}\) and \(\hat{\beta}\) that are path homotopic then \(\alpha\) and \(\beta\) are \(E\)-homotopic.

Corollary 49 Let \(X\) be a Peano continuum, \(\lambda = \{x_0, ..., x_n\}\) be an \(E\)-loop, and \(c_i\) be a path from \(x_i\) to \(x_{i+1}\), denoting by \(c\) the concatenation of those paths. If the lift of \(c\) to any point \([\alpha]_E\) in \(X_E\) ends at \([\alpha + \lambda]_E\), then \(c\) is \(E\)-homotopic to some, hence any stringing of \(\lambda\).

Proof. The statement and the proof use the Chain Lifting Lemma (Lemma \[\text{Lemma 38}\]), and we will use it without further explicit reference. First, in order for the hypothesis to make sense, \(\alpha\) must be an \(E\)-chain from the basepoint to \(x_0\). By Lemma \[\text{Lemma 43}\] up to \(E\)-homotopy we may let \(\hat{\alpha}\) be a stringing of \(\alpha\). Then \(\hat{\lambda} * \hat{\alpha}\) is a stringing of \(\lambda * \alpha\). According to Proposition \[\text{Proposition 47}\] the endpoint of the unique lift \(\tilde{\lambda} * \tilde{\alpha}\) of \(\hat{\lambda} * \hat{\alpha}\) is \([\lambda * \alpha]_E\). By assumption, the lift \(\tilde{c}\) of \(c\) to \([\alpha]_E\) also ends in \([\lambda * \alpha]_E\). By Lemma \[\text{Lemma 46}\] \(\hat{\lambda} * \hat{\alpha}\) is \(E\)-homotopic to \(c * \tilde{\alpha}\) and therefore \(c\) is \(E\)-homotopic to \(\lambda\) as required. ■

The above Corollary is useful because it “frees us” from having to rely on stringings in some situations; for example the next proposition is useful in the proof of Theorem \[\text{Theorem 67}\].

Proposition 50 Let \(X\) be a geodesic space and \(E\) be a chained entourage. Then

1. If \(\lambda\) is an \(E\)-loop then there is a curve loop \(c\) such that some (hence any) \(E\)-subdivision of \(c\) is \(E\)-homotopic to \(\lambda\) and \(L(c) = L(\lambda)\).

2. Conversely, if \(c\) is a curve loop then there is an \(E\)-loop \(\lambda\) such that \(L(c) = L(\lambda)\) and \(\lambda\) is \(E\)-homotopic to some (hence any) \(E\)-subdivision of \(c\).

Proof. For the first part, let \(X_E\) have the lifted geodesic metric (see the end of the second section). Let \(\tilde{\lambda}\) be the unique lift of \(\lambda\) at a point \([\alpha]_E \in X_E\), which is an \(E^*\)-chain. By Lemma \[\text{Lemma 21}\] \(L(\tilde{\lambda}) = L(\lambda)\). Since \(X_E\) is a geodesic space we may join each point in \(\tilde{\lambda}\) to its successor by a geodesic, concatenating to produce a curve \(\tilde{c}\) such that \(L(\tilde{c}) = L(\tilde{\lambda}) = L(\lambda)\). Since \(\phi_E\) preserves the lengths of curves,
\[ c := \phi_E(\tilde{c}) \] has the same length as \( \lambda \). Moreover, the endpoint of \( \tilde{c} \), which by uniqueness is the lift of \( c \) at \([\alpha]_E\), is by construction the same as the endpoint of \( \tilde{\lambda} \). That endpoint is equal to \([\alpha \ast \lambda]_E\) by Proposition 17.

Now let \( \lambda' \) be any \( E \)-subdivision of \( c \); that is, \( c \) is a stringing of \( \lambda' \). Again by uniqueness of lifts, the lift \( \tilde{\lambda}' \) of \( \lambda' \) at \([\alpha]_E\) lies on \( c \) and also ends at \([\alpha \ast \lambda]_E\). By Proposition 17 \([\alpha \ast \lambda']_E = [\alpha \ast \lambda]_E\), and hence \([\lambda']_E = [\lambda]_E\), completing the proof of the first part.

For the second part, let \( F \) be an open entourage contained in \( E \) begin with any \( F \)-subdivision \( \omega \) of \( c \). By definition of the length of a curve \( L(\omega) \leq L(c) \).

If the lengths are equal, we are finished. Otherwise, suppose that \( \delta := L(c) - L(\omega) > 0 \) and choose any point \( x_i \) and its successor \( x_{i+1} \) on \( \omega \). Since \( F \) is open we may choose \( n \) large enough that, setting \( \varepsilon := \frac{\delta}{2n} \), \( B(x_i, F) \cap B(x_{i+1}, F) \) contains \( B(x_i, 2\varepsilon) \). Because the space is geodesic, we may pick a point \( z \in B(x_i, 2\varepsilon) \) so that \( d(x_i, z) = \frac{\delta}{2n} \) (for example just apply the Intermediate Value Theorem to any geodesic from \( x_i \) to \( x_{i+1} \)). Now we make the following legal moves for an \( E \)-homotopy:

\[
\{x_i, \overset{z}{\to}, x_{i+1}\} \to \{x_i, z, x_i, x_{i+1}\} \to \{x_i, z, x_i, z, x_{i+1}\} \to \cdots
\]

Each pair of moves, adding \( z \) then \( x_i \), increases the length of \( \lambda \) by precisely \( 2\delta \). Therefore after \( n \) such moves we have attained precisely \( L(c) \). □

Note that the next statement is not true when replacing “\( E \)-homotopic” with “homotopic”, for example if the space is not semi-locally simply connected.

**Proposition 51** Let \( c_i : [0, 1] \to X \) be a sequence of curves in a compact metric space \( X \) of uniformly bounded length, uniformly convergent to \( c : [0, 1] \to X \), and let \( E \) be any chained entourage.

1. If the \( c_i \) have the same start and endpoints then \( c_i \) is \( E \)-homotopic to \( c \) for all large \( i \).

2. If the \( c_i \) are closed then \( c_i \) is freely \( E \)-homotopic to \( c \) for all large \( i \).

**Proof.** If the statement were not true then we could, by taking a subsequence, assume that for all \( i \), \( c_i \) is never \( E \)-homotopic to \( c \). Let \( \varepsilon > 0 \) be such that \( \overline{E}_\varepsilon \subset E \). For every \( i \), let \( \lambda_i = \{x_{0i} = c_i(0), x_{1i} = c_i(t_i), \ldots, x_{ni} = c_i(1)\} \) be an \( E_i \)-subdivision of \( c_i \), where \( 0 \leq t_i \leq \cdots \leq t_{(n_i-1)i} \leq 1 \). By Lemma 31 we may assume that these chains have bounded length, and hence we may apply the Chain Normalizing Lemma (Lemma 33). That is, we may assume that \( ni = n \) for some fixed \( n \) and all \( i \), and \( x_{ji} \to x_j \) for some \( x_j \), for all \( j \). Since \( c_i \) converges uniformly to \( c \), \( x_j = c(t) \) for some \( t_j \in [0, 1] \) with \( 0 \leq t_1 \leq \cdots \leq t_n = 1 \). We will next show that \( \lambda = \{x_0, \ldots, x_n\} \) is an \( \overline{E}_\varepsilon \)-subdivision, hence an \( E \)-subdivision, of \( c \). Let \( c_{ji} \) denote the restriction of \( c_i \) to \([t_{ji}, t_{(j+1)i}]\); by definition of subdivision, the image of \( c_{ji} \) lies in \( B(x_{(j+1)i}, E) \cap B(x_{(j+1)i}, E) \). Since \( \{c_i\} \) converges uniformly, the Ball Continuity Lemma (Lemma 19) implies that if \( c_j \) denotes the restriction
of \(c\) to \([t_j, t_{j+1}]\) then \(c_j\) lies in \(B(x_j, E_j) \cap B(x_{j+1}, E_j)\). This shows that \(\lambda\) is an \(E^\epsilon\)-subdivision of \(c\). It now follows from Proposition 35 that \(\lambda\) is an \(E^\epsilon\)-loop, hence an \(E\)-loop that is \(\epsilon\)-homotopic (hence \(E\)-homotopic) to \(\lambda_i\) for all large \(i\). That is, \(c_i\) is \(E\)-homotopic to \(c\), completing the proof of the first part.

The proof of the second part is the same as the proof of the first, using Corollary 36 rather than Proposition 35.

Remark 52 We do not need it for this paper, but the above statement may be proved with a little more work without the condition that the curves have uniformly bounded length. The idea is that even if the curves do not have uniformly bounded length, the fact that the convergence is uniform allows one to divide any \(c_i\) into a concatenation of segments, the number of which is uniformly bounded and each of which lies in an \(\frac{\epsilon}{2}\)-ball. Each of those segments then has an \(E^\epsilon\)-subdivision consisting only of two points, and those together provide an \(E^\epsilon\)-subdivision of \(c_i\) with length bounded independent of \(i\).

The next definition extends a notion from [20]:

Definition 53 If \(X\) is a uniform space and \(E\) is an entourage, an \(E\)-loop of the form \(\lambda = \alpha \ast \tau \ast \overline{\alpha}\), where \(\nu(\tau) = 3\), will be called \(E\)-small. Note that this notation includes the case when \(\alpha\) consists of a single point—i.e. \(\lambda = \tau\). In this case, we will call \(\lambda\) an \(E\)-triad.

Note that any \(E\)-small loop is \(E\)-null since two of the points in \(\tau\) may be removed one by one, followed by the points in \(\alpha\) and \(\overline{\alpha}\).

The next essential proposition is an extension of Proposition 29 in [20], with a similar proof (essentially replacing \(\epsilon\) by \(E\) and \(\delta\) by \(D\)). However, we write out the complete proof here so it is clear how our new definition of refinement is used (including the obvious fact that the reversal of a refinement is a refinement).

Proposition 54 Let \(X\) be a uniform space, \(D\) be a chained entourage in \(X\) and \(E \subset D\) be an entourage. Suppose \(\alpha, \beta\) are \(E\)-chains and \(\langle \gamma_0, ..., \gamma_n \rangle\) is a \(D\)-homotopy such that \(\gamma_0 = \alpha\) and \(\gamma_n = \beta\). Then \([\beta]_E = [\lambda_1 \ast \cdots \ast \lambda_r \ast \alpha \ast \lambda_{r+1} \ast \cdots \ast \lambda_n]_E\), where each \(\lambda_i\) is an \(E\)-refinement of a \(D\)-small loop.

Proof. We will prove by induction that for every \(k \leq n\), an \(E\)-refinement \(\gamma'_k\) of \(\gamma_k\) is \(E\)-homotopic to \(\lambda_1 \ast \cdots \ast \alpha \ast \cdots \ast \lambda_k\), where each \(\lambda_i\) is an \(E\)-refinement of a \(D\)-small loop. Since any two \(D\)-refinements of an \(E\)-chain are \(E\)-homotopic (Lemma 52), this will complete the proof.

The case \(k = 0\) is trivial. Suppose the statement is true for some \(0 \leq k < n\). The points required to \(E\)-refine \(\gamma_k\) to \(\gamma'_k\) will be denoted by \(m_i\). Suppose that \(\gamma_{k+1}\) is obtained from \(\gamma_k\) by adding a point \(x\) between \(x_i\) and \(x_{i+1}\). Let \(\{x_i, a_1, ..., a_k, x\}\) be an \(E\)-refinement of \(\{x_i, x\}\) and \(\{x, b_1, ..., b_m, x_{i+1}\}\) be an \(E\)-refinement of \(\{x, x_{i+1}\}\), so

\[\gamma'_{k+1} := \{x_0, m_0, ..., x_i, a_1, ..., a_k, x, b_1, ..., b_m, x_{i+1}, m_r, ..., x_j\}\]
is an $E$-refinement of $\gamma_{k+1}$. Defining $\mu_{k+1} := \{x_0, m_0, ..., x_i\}$ and
$$\kappa_{k+1} = \{x_i, a_1, ..., a_k, x, b_1, ..., b_m, x_{i+1}, m_r, ..., x_i\}$$
we have
$$[\gamma'_{k+1}]_E = [\mu_{k+1} \ast \kappa_{k+1} \ast \mu_{k+1} \ast \gamma'_{k}]_E$$
and since the homotopy is a $D$-homotopy, $\lambda_{k+1} := \mu_{k+1} \ast \kappa_{k+1} \ast \mu_{k+1}$ is an $E$-refinement of a $D$-small loop. The case when a point is removed from $\gamma_k$ is similar, except that the refined $D$-small loop is multiplied on the right.

**Corollary 55** Let $X$ be a uniform space, $D$ be a chained entourage in $X$ and $E \subset D$ be an entourage. Then $\ker \theta_{DE}$ is equal to the subgroup of $\pi_E(X)$ generated by all $E$-homotopy classes of $E$-refinements of $D$-small loops.

**Proof.** It follows from Proposition 54 that every element of $\ker \theta_{ED}$ (i.e. an $E$-homotopy class of an $E$-loop that is $D$-null) is a product of $E$-homotopy classes of $E$-refinements of $D$-small loops. On the other hand, any concatenation of $E$-refinements of $D$-small loops is $D$-null and hence its $E$-homotopy class is in $\ker \theta_{DE}$. ■

Further extending the results of [20] we define for any entourage $E$ in a metric space $X$, a map from fixed-endpoint homotopy classes of continuous paths to $E$-homotopy classes of $E$-chains as follows. Suppose $c: [0, 1] \to X$ is continuous. We set $\Lambda_E([c]) := [\alpha]_E$, where $\alpha$ is any $E$-subdivision of $c$. By Lemma 42 $\Lambda_E$ is well-defined. Note that if $E$ is a chained entourage then by Lemma 43 every $E$-chain $\alpha$ may be assumed, up to $E$-homotopy, to have a stringing $\pi$. By definition $\Lambda_E(\pi) = [\alpha]_E$; that is, $\Lambda_E$ is surjective. Restricting $\Lambda_E$ to the fundamental group at any base point yields a homomorphism $\pi_1(X) \to \pi_E(X)$ that we will also refer to as $\Lambda_E$.

Continuing to assume that $E$ is a chained entourage, fix a basepoint and suppose that $[c] \in \ker \Lambda_E$. In other words, any $E$-subdivision $\{c(t_0), ..., c(t_n) = c(t_0)\} := \lambda$ is $E$-null. Taking $D = E$ in Proposition 54 we see that $\lambda$ is $E$-homotopic to a product of $E$-small $E$-loops. Since $\Lambda_E$ is a homomorphism, $c$ is homotopic to the concatenation of stringings of $E$-small loops. We have shown:

**Theorem 56** Let $X$ be a Peano continuum that has a (compatible) geodesic metric and $E$ be a chained entourage. Then for any basepoint, $\Lambda_E : \pi_1(X) \to \pi_E(X)$ is a surjective homomorphism and $\ker \Lambda_E$ is the subgroup of $\pi_1(X)$ generated by homotopy classes of stringings of $E$-small loops.

**Remark 57** The above theorem actually only requires that $X$ have a geodesic metric compatible with the uniform structure, see Remark 44.

**Remark 58** Note that one may also restrict $\Lambda_E$ to the set of equivalence classes of curves starting a fixed basepoint, which is by definition the universal covering space of $X$ when $X$ is semi-locally simply connected. According to Theorem 26.2 in [20], if $X$ is a compact, semi-locally simply connected geodesic space, then $\Lambda_{E_{\varepsilon}} : \pi_1(X) \to \pi_{E_{\varepsilon}}(X)$ is length preserving when $\varepsilon$ is a lower bound for HCS.
That is, \( \Lambda_{E_\varepsilon} \) restricts to a bijection from the universal cover \( \tilde{X} \) of \( X \) to \( X_{\varepsilon} \). That is, we may identify the universal cover \( \tilde{X} \) of \( X \) with \( X_{\varepsilon} \) and \( \pi_1(X) \) with \( \pi_{\varepsilon}(X) \).

Remark 59 In the above theorem we see a hint of the relationship between our construction and the construction of Spanier used by Sorman-Wei, referred to in the Introduction. For a metric entourage \( E_\varepsilon \) in a geodesic space, we may take stringings using geodesics, and such geodesics always remain in \( B(x_i, \frac{3\varepsilon}{2}) \). That is, \( \ker \Lambda_{E_\varepsilon} \) is precisely the Spanier subgroup used by Sormani-Wei, and this is how the equivalence of \( \varepsilon \)-covers and \( \delta \)-covers was proved in [21]. One could take the Sormani-Wei use of Spanier a bit farther by applying it to the covering of a space by entourage balls for a fixed entourage, but it seems unlikely that the resulting covering maps would not be equivalent to entourage covers as we have defined them.

Remark 60 As for basepoints, as was observed in [5], as long as \( X \) is chain connected, the various groups and homomorphisms defined above are independent of the basepoint, up to natural isomorphisms induced by basepoint change. This is why we have not included basepoints in our notation. When necessary we can always assume that all mappings are basepoint-preserving (take basepoints to basepoints).

5 Properties of Entourage Covers

Two chained entourages \( E_1, E_2 \) in a uniform space will be called equivalent if \( \phi_{E_1} \) and \( \phi_{E_2} \) are equivalent as covering maps.

Proposition 61 Let \( X \) be a uniform space, \( D \subset E \) be chained entourages, \( G \) be a normal subgroup of \( \pi_D(X) \), and \( \pi : X_D \to X_D/G = Y \) be the quotient covering map. Then there is a covering map \( h : X_E \to Y \) such that \( h \circ \phi_{ED} = \pi \) if and only if \( \ker \theta_{ED} \subset G \). In particular, the covers \( \phi_E : X_E \to X \) and the induced cover \( \phi : Y \to X \) are equivalent if and only if \( G = \ker \theta_{ED} \).

Proof. If \( \ker \theta_{ED} \subset G \) then as mentioned in the background section we have \( X_E = X_D/\ker \theta_{ED} \) and so \( G/\ker \theta_{ED} \) acts properly discontinuously on \( X_E \) with quotient space naturally identified with \( X_D/G \) (cf. Theorem 1.6.11 in [33]). That is, the quotient map

\[
h : X_E = X_D/\ker \theta_{ED} \to X_E/(G/\ker \theta_{ED}) = X_D/G = Y
\]

is a (regular) covering map that by definition satisfies \( h \circ \phi_{ED} = \pi \).

Conversely, suppose that there is a covering map \( h : X_E \to Y \) such that \( h \circ \phi_{ED} = \pi \). By composing with a covering equivalence we may suppose that \( h \) is basepoint preserving. Now suppose that \( \theta_{ED}([\lambda]_D) = [\ast]_E \). Then since \( h \) is basepoint preserving, \( h \circ \theta_{ED}([\lambda]_D) = \pi([\ast]_D) \). That is, \( [\lambda]_D \in \pi^{-1}(\pi([\ast]_E) = G \).
Corollary 62 Let $D$, $E$, $F$ be chained entourages in a uniform space $X$ with $D \subseteq E \cap F$. Then

1. $E$ and $F$ are equivalent if and only if $\ker \theta_{ED} = \ker \theta_{FD}$.

2. There is a non-trivial covering map $h : X_E \to X_F$ if and only if there is some $F$-triad with a $D$-refinement that is not $E$-null.

Proof. The first statement is an obvious consequence of Proposition 61 which also says that there is a non-trivial covering map $h : X_E \to X_F$ if and only if $\ker \theta_{ED}$ is a proper subset of $\ker \theta_{FD}$. Equivalently, there is a $D$-loop $\lambda$ that is $F$-null but not $E$-null. Equivalently, by Proposition 54

$$[\lambda]_D = [\lambda_1 \cdots \lambda_n]_D$$

where each $\lambda_i$ is a $D$-refinement of an $F$-small loop, at least one of which is not $E$-null. The proof is now finished by the definition of $F$-small.

Remark 63 Corollary 62 can be considered as an extension of Corollary 31 in [3] to include the situation when neither $E$ nor $F$ may be contained in the other, but there is a covering equivalence between $\phi_E$ and $\phi_F$.

Theorem 64 If $X$ is a geodesic space and $\varepsilon > 0$ then there is a set $S \subseteq \pi_2(X)$ with $|S| \leq C(X, \hat{\delta})^{\text{ACC}(X, \hat{\varepsilon})}$ such that if $E$ is a chained entourage with $E_{\varepsilon} \subseteq E$ then $\ker \theta_{EE_{\varepsilon}}$ is the normal closure of some subset of $\Gamma$.

Proof. Let $S$ be the set of all $[\beta]_\varepsilon \subseteq \pi_2(X)$ such that $L(\beta) \leq 10\varepsilon C(X, \hat{\delta})$. By Theorem 3.2 of [20] $|S| \leq C(X, \hat{\delta})^{\text{ACC}(X, \hat{\varepsilon})}$. Therefore we need only prove that for any such $E$, $\ker \theta_{EE_{\varepsilon}}$ is the normal closure of a set of elements of length at most $10\varepsilon C(X, \hat{\delta})$. By Corollary 55 $\ker \theta_{EE_{\varepsilon}}$ is equal to the subgroup of $\pi_2(X)$ generated by the collection of all $[\alpha \ast \tau \ast \pi]_\varepsilon$ where $\tau$ is an $E_{\varepsilon}$-refinement of an $E$-triad. But it is an easy algebraic argument that this means that $\ker \theta_{EE_{\varepsilon}}$ is the normal closure of elements of the form $[\alpha_{\varepsilon} \ast \tau \ast \alpha_{\varepsilon}]_\varepsilon$ where (1) $\tau$ is an element of a set $\Gamma$ of $E_{\varepsilon}$-refinements of $E$-triads that contains exactly one representative of each free $E_{\varepsilon}$-homotopy class of $E_{\varepsilon}$-refinements of $E$-triads, and (2) $\alpha_{\varepsilon}$ is any $\varepsilon$-chain from the basepoint to the start/end point of $\tau$. In fact, any generator of $\ker \theta_{EE_{\varepsilon}}$ is conjugate to some such $[\alpha_{\varepsilon} \ast \tau \ast \alpha_{\varepsilon}]_\varepsilon$.

According to Lemma 59 the chains $\alpha_{\varepsilon}$ from the above paragraph may be chosen to have at most $2C(X, \hat{\delta})$ points. Likewise, we may produce an $E_{\varepsilon}$-subdivision of $\tau$ having at most $6C(X, \hat{\delta})$, and therefore there are $E_{\varepsilon}$-refinements of the $E$-small loops $\alpha_{\varepsilon} \ast \tau \ast \alpha_{\varepsilon}$ having at most $10C(X, \hat{\delta})$ points and hence length at most $10\varepsilon C(X, \hat{\delta})$.

Theorem 4 now follows from Theorem 64 and Proposition 62.

To prove Theorem 3 suppose that $\sigma$ is any value of CS. By definition the corresponding $\sigma$-cover is not equivalent to any $\delta$-cover for $\delta > \sigma$. Now suppose that $E$ is an entourage that contains $E_\delta$ for some $\delta > \sigma$. Then $\phi_{EE_{\varepsilon}} = \phi_{E_\delta E} \circ \phi_{E_{\varepsilon} E_{\varepsilon}}$ and since $\phi_{E_\delta E}$ is not injective (and all maps are surjective), $\phi_{EE_{\varepsilon}}$ is
not either. That is, there is at least one chained entourage (namely $E_\sigma$) that contains $E_\sigma$ and is not equivalent to any $E$ with $\sigma(E) > \sigma$, hence $\sigma \in \text{ECS}$.

To prove Theorem 5 we may assume that $X$ is a geodesic space by the Bing-Moise Theorem. Since $X$ is semilocally simply connected, Corollary 43 of [20] implies that for all sufficiently small $\varepsilon$, $\phi_\varepsilon : X_\varepsilon \to X$ is the universal covering map of $X$ and $\pi_\varepsilon(X) = \pi_1(X)$. By definition, $\phi_E$ is the covering map corresponding to $\ker E_\varepsilon$, and the proof of Theorem 5 is finished by Theorem 64.

Remark 65 In [20], Theorem 37, Berestovskii-Plaut showed that for any entourage $E$ in a compact uniform space, if $\phi_E$ is chain connected (which is true when $E$ is chained according to our new definition) then $\pi_E(X)$ is finitely generated. Essentially the same argument as the proof of Theorem 5 together with Remark 6 shows that if $X$ is a Peano continuum and $E$ is a chained entourage then $\pi_E(X)$ is in fact finitely presented.

For the proof of Theorem 8 we need to some facts from [20] and [21] concerning essential circles, along with a new result (Proposition 66). Essential circles are defined (Definition 5, [20]) to be continuous paths of length $L$ that are not $\varepsilon$-null for $\varepsilon = \frac{L}{3}$ (we may also refer to it as an essential $\varepsilon$-circle). According to Lemma 33, [20] this means that essential circles are characterized as curves of positive length that are shortest in their $\varepsilon$-homotopy class for some $\varepsilon$. Essential circles are special closed geodesics that are minimal on half their length ("2-geodesics" in the parlance of Sormani-Wei, $\frac{L}{2}$-geodesics in our terminology) whose lengths are three times the lengths of the values in HCS, or equivalently $\frac{2}{3}$ the values of CS ([20], Theorem 6). If $C$ is an essential $\varepsilon$-circle then any triple of points $T = \{x_0, x_1, x_2\}$ on $C$ such that $d(x_i, x_j) = \varepsilon$ when $i \neq j$ is an essential $\varepsilon$-triad (or just essential triad when $\varepsilon$ is unspecified), meaning that no $\varepsilon$-subdivision of $T$ is $\varepsilon$-null. Essential triads are characterized by the fact that if the points on them are joined by geodesics the resulting curve is an essential circle. To summarize, essential triads are precisely the discrete analogs of essential circles: adding "edges" to an essential triad creates an essential circle, and any triad of equally spaced points on an essential circle is an essential triad.

Two essential triads $\tau_1, \tau_2$ are said to be equivalent if they are both essential $\varepsilon$-triads for some $\varepsilon > 0$, and some, hence any $\varepsilon$-subdivision of $\tau_1$ is $\varepsilon$-homotopic to an $\varepsilon$-subdivision of $\tau_2$ or $\overline{\tau_2}$. Two essential circles are said to be equivalent if the corresponding essential triads are equivalent (see [20] for more details). As we will now show, the smallest essential circles are generally easiest to find.

Proposition 66 Let $X$ be a compact geodesic space that is semilocally simply connected. If $c$ is a path loop that is not null-homotopic and whose length is equal to the 1-systole $\sigma_1$ of $X$ (i.e. the length of the shortest curve that is not null-homotopic) then $c$ is a shortest essential circle. Moreover, any two shortest essential circles are equivalent if and only if one is freely homotopic to the other or its reversal.

Proof. The fact $\sigma_1$ is positive and is the shortest possible length of any essential circle is part of Corollary 43 in [20]. It follows from Theorem 6 in [20] that
there is at least one essential circle of length $\sigma_1$. Now suppose that $c$ is not null-homotopic of length $\sigma_1$. Taking $\varepsilon := \frac{\sigma_1}{N}$, to show $c$ is an essential circle we need only show that it is not $\varepsilon$-null. Suppose it were; that is, some $\varepsilon$-subdivision $\lambda$ of $c$ must be $\varepsilon$-null. Taking $\delta = \varepsilon$ in Proposition 30 of [20] (which is analogous to Proposition 24 in the present paper), for some choice of $\alpha$, $\tau \ast \lambda \ast \alpha$ is $\varepsilon$-homotopic to a product of $\varepsilon$-small $\varepsilon$-loops, one of which must not be $\varepsilon$-null. But recall that an $\varepsilon$-small loop is of the form $\overline{\beta} \ast \tau \ast \overline{\beta}$, where $\tau = \{x_0, x_1, x_2, x_0\}$ satisfies $d(x_i, x_j) < \varepsilon$ for all $i, j$. But such a $\tau$ is necessarily $\varepsilon$-null, a contradiction.

Next, suppose that $c_1, c_2$ are equivalent shortest essential circles. Reversing $c_2$, if necessary, this means that for $\varepsilon = \frac{\sigma_1}{N}$, some essential triad $\tau_i$ on $c_1$ has an $\varepsilon$-subdivision $\tau_i'$ that is $\varepsilon$-homotopic to $\tau_j'$ for $j \neq i$. That is, $c_1$ is $\varepsilon$-homotopic to $c_2$. By Theorem 26 of [20], $X_\varepsilon$ is the universal covering space of $X$, meaning that two loops lift to a loop in $X_\varepsilon$ if and only if they are homotopic. But the same statement is true for $\varepsilon$-homotopic curves by Proposition 47 completing the proof. □

In [21], Theorem 27, when $0 < \delta < \varepsilon$, we showed that $\phi_{\varepsilon, \delta} : X_\delta \to X_\varepsilon$ is characterized as the quotient map of $X_\delta$ via the (normal) subgroup $K_\varepsilon(\mathcal{T})$ of $\pi_1(X)$ generated by all $\varepsilon$-loops of the form $\tau \ast \alpha \ast \tau$, where is in a set $\mathcal{T} = \{\tau_1, ..., \tau_k\}$ containing exactly one essential triad $\tau_i$ representing each equivalence class of essential $\mu$-triads with $\varepsilon \leq \mu < \delta$. Via the argument in the proof of Theorem 6.4 $K_\varepsilon(\mathcal{T})$ is in fact the normal closure of the finite set of all $\alpha \ast \tau \ast \overline{\tau}$ for any specific choice of $\varepsilon$-chain $\alpha$.

**Proof of Theorem 8.** In [10], DeSmit, Gornet, and Sutton introduced (Definition 2.3) the notion of a length map on a group $H$ with identity 1, namely a function $m : H \to \mathbb{R}^+$ such that (a) $m$ is positive except $m(1) = 0$, (b) $m(gh^{-1}) = m(g)$ for all $g, h \in H$, and (c) $m(g^k) \leq |k| m(g)$ for all $g \in H$ and $k \in \mathbb{Z}$. A particular example of a length map is what Sormani-Wei ([28]) called the Minimum Marked Length Map $m_g$ in on a manifold with Riemannian metric $g$: $m_g$ assigns to each element of the fundamental group the length of the shortest curve in its free homotopy class. So the values of $m_g$ are precisely MLS.

We will need Theorem 2.9 of [10], which we will describe in a weaker simplified form using Example 5.5 [10], and we will shift the subscripts to improve the exposition for our purposes. Let $M$ be a Riemannian manifold of dimension at least 3. Let $\mathcal{F}(M)$ denote the collection of un-oriented free homotopy classes of loops in $M$. Suppose that $\{c_0, ..., c_k\}$ is a set of distinct elements of $\mathcal{F}(M)$ with $c_0$ trivial, and $l_0 = 0 < l_1 \leq \cdot \cdot \cdot \leq l_k$ is a sequence of real numbers such that $2l i \geq l_k$. Then by Theorem 2.9 and Example 5.5 of [10] there is a Riemannian metric $g$ on $M$ such that $m_g(c_i) = l_i$ for all $i$ and $m_g(c) \geq l_k$ otherwise. In other words, one may prescribe the length of the shortest geodesics in the free homotopy class of all $c_i$, and force all other values of MLS to be at least $l_k$.

Suppose that $G$ is the normal closure of a finite set $\{g_1, ..., g_{k-1}\}$, all distinct and none of which is trivial. If $G$ is equal to $\pi_1(M)$ then the corresponding covering map is trivial, hence an $\varepsilon$-cover for any Riemannian metric. Assume there is some nontrivial $g_k \notin G$. Now let $c_0$ be the trivial free homotopy class and for each $1 \leq i \leq k$ let $c_i$ be the free homotopy class of some, hence any loop in $g_i$. Define $l_0 := 0$, $l_k = 1.5$ and $l_i = 1$ for $i = 1, ..., k - 1$. By Example 5.5
in [10] (noting that our indexing begins with 0 rather than 1) and Theorem 2.9 of [10] there is Riemannian metric $g$ on $M$ such that $m_g(c_i) = 1$ for all $i \neq 0, k$ and $m_g(c) \geq 1.5$ for every free homotopy class $c \neq c_i$ for any $i < k$. Let $\kappa_i$ be shortest curves representing each $c_i$ with $1 \leq i \leq k - 1$. By Proposition 68, each $\kappa_i$ is a shortest essential circle, and two $\kappa_i$ are non-equivalent as essential circles if and only if they lie in different $c_i$. Moreover, any other essential circle, being shortest in its homotopy class, must have length at least 1.5.

As discussed in Remark 58, we may suppose that $\varepsilon > 0$ is small enough that we may identify, via the function $\Lambda_{E_\varepsilon}$, $M_\varepsilon$ with the universal cover $\hat{M}$ of $M$ and identify $G$ with a normal subgroup of $\pi_1(M) = \pi_1(\hat{M})$. Under this correspondence, $G$ is the normal closure of $\{[\lambda_i]_\varepsilon\}$, where $\lambda_i$ is any $E_\varepsilon$-subdivision of a loop of the form $f_i * c_i * f_i$, where $f_i$ is any curve from the basepoint to the start point of the essential circle $\kappa_i$. That is, $G$ is precisely equal to $K(\varepsilon)(\varepsilon)$ (see discussion prior to this proof), where $T := \{\tau_i\}_{i=1}^{k-1}$ is a collection of essential triads representing all equivalence classes of smallest essential circles, with exactly one representative $\tau_i$ for each free homotopy class. On the other hand, letting $\delta := \frac{\varepsilon^2}{3} = \frac{\varepsilon}{3}$, Theorem 27 of [21] states that since $T$ contains a representative for each essential $\sigma$-triad with $\frac{\varepsilon}{3} < \sigma < \varepsilon$, $K(\varepsilon) = \ker \theta_{E_\varepsilon}$. That the covering space $\hat{M}/G$ is equivalent to $M_\delta$ now follows from Proposition 61. ■

**Proof of Proposition 10.** The statements about inessential $E$ are obvious. The proof of the rest is very similar to the proof of the statement for Riemannian manifolds involving free homotopy classes, replacing the fact that small loops are null-homotopic by the fact that small loops are $E$-null. Suppose that $E$ is essential, $c$ is not freely $E$-null and let $c_i : [0, 1] \to X$ be $E$-loops that are freely $E$-homotopic to $c$, parameterized proportional to arclength, with lengths converging to

$$L := \inf \{L(f) : f \text{ is a curve loop that is } E\text{-homotopic to } c\}.$$  

Since the lengths of the $c_i$ are bounded, a standard application of the Ascoli-Arzelà Theorem shows we may assume, taking a subsequence if needed, that $c_i$ converges uniformly to some $\tau : [0, 1] \to X$ with $L(\tau) \leq L$. By Proposition 51 for all large $i$, $\tau$ is freely $E$-homotopic to $c_i$, hence to $c$. So $\tau$ is the desired shortest curve. To see that any such $\tau$ is a closed $\frac{3\varepsilon}{2}$-geodesic, suppose that a segment $S$ of length $\leq \frac{3\varepsilon}{2}$ is not minimal. By definition we may join its endpoints by a new curve $S'$ of length less than $\frac{3\varepsilon}{2}$. Then $S$ and $S'$ together form a loop of length less than $3\varepsilon$, which is $E$-null (Lemma 33, [20]), hence $E$-null. Therefore the curve obtained by replacing $S$ by $S'$ has length shorter than $L$ and is $E$-homotopic to $c$, a contradiction. ■

**Proof of Theorem 11.** In any geodesic space, if $c$ is shortest in its $E$-homotopy class, it is shortest in its homotopy class by Corollary 48 showing that the set described in Part 1a is contained in MLS. On the other hand, suppose that $c$ is shortest in its homotopy class. As in the proof of Proposition 66, let $\varepsilon > 0$ be small enough that $X_\varepsilon$ is the universal covering space of $X$, so that curves are freely homotopic if and only if they are freely $\varepsilon$-homotopic. That is, $c$ is shortest in its $\varepsilon$-homotopy class, i.e. its $E$-homotopy class for $E = E_\varepsilon$. This
proves Part 1a. The remaining parts are true for any compact geodesic space, as will be shown next. ■

**Theorem 67** Theorem 11 holds when $M$ is simply assumed to be a compact geodesic space, using Theorem 11.1a as the definition of MLS.

**Proof.** If $c$ is non-constant and shortest in its free $E$-homotopy class then according to Proposition 50.1 there is some $E$-loop $\lambda$ having the same length as $c$ that is $E$-homotopic to any $E$-subdivision of $c$. But Proposition 50.2 now shows that $\lambda$ must also be shortest in its $E$-homotopy class. An analogous argument shows that if $\lambda'$ is shortest in its $E$-homotopy class then there must be a curve of the same length as $\lambda'$ that is shortest in its $E$-homotopy class. That is, the quantities described by Parts 1a and 1b in Theorem 67 are the same.

For the second part, note that in [10], Section 3, de Smit, Gornet, and Sutton gave the following equivalent definition of CS in any compact geodesic space $X$: CS consists of half the lengths of loops that lift as a non-loop to any covering space of $X$. For the covering spaces $X_E$, Lemma 46 implies that this is precisely the length of a shortest loop that is not $E$-null, so any value in Part 2a is contained in CS. On the other hand, CS = $\frac{3}{2}HCS$ consists of the lengths of all essential circles. Since every essential $\varepsilon$-circle has length $3\varepsilon$ and curves of length less than $3\varepsilon$ are $\varepsilon$-null (Lemma 33, [20]), then essential circles are the shortest possible loops that are not $E$-null for $E = E_\varepsilon$. This completes the proof that CS is consists of the values in Part 2a.

Note that if $c$ is a shortest loop that is not $E$-null then $c$ must be shortest in its $E$-homotopy class. Proposition 50.1 tells us that there is an $E$-loop of the same length as $c$ that is $E$-homotopic to any $E$-subdivision of $c$. By Proposition 50.2, there cannot be a shorter such loop. This justifies the term “shortest” in 2b. Moreover, as in the proof of Part 1, Proposition 50 tells us that the values in 2a and 2b are the same. ■

**Proof of Theorem 13** For the first part, note that if $c$ is $E$-critical then any $E$-subdivision $\lambda$ of $c$, being also an $E$-subdivision of $c$, is also $E$-critical. Likewise, any stringing $\hat{\lambda}$ of an $E$-critical $E$-loop $\lambda$ is also a stringing of $\lambda$ when considered as an $E$-chain. It is immediate that $\lambda$ is an $E$-critical loop.

The proof of the second part is similar to the proof of Proposition 10 and we will only state the essential steps for the argument involving $c$. Let $c_i : [0, 1] \to X$ be $E$-loops that are $E$-critical, parameterized proportional to arclength, with lengths converging to $\psi(E)$. We may assume that $c_i$ converges uniformly to some $c : [0, 1] \to X$ with $L(c) \leq \psi(E)$. For all large $i$, $c$ is freely $E$-homotopic and $E$-homotopic to $c_i$. That is, $c$ is also $E$-critical and has shortest length. Proposition 13 now finishes the proof.

For the third part, recall that HCS consists of $\frac{1}{3}$ the lengths of essential triads. But any essential $\varepsilon$-triad $T$, having length $3\varepsilon$, is $E_\varepsilon$-null, but by definition has no $\varepsilon$-null $\varepsilon$-subdivision. That is, any $\varepsilon$-subdivision of $T$ (which also has length $3\varepsilon$) is $E_\varepsilon$-critical. By definition, CS $\subset$ ES. On the other hand, if $c$ is a shortest $E$-critical curve then $c$ must be shortest in its free $E$-homotopy class. If
not, it would be $E$-homotopic, hence $\overline{E}$-homotopic to a shorter curve—but that curve would still be $E$-critical, a contradiction. The final two statements are shown in Example 71.}

6 Examples

Example 68 Let $X$ be a circle with the unique geodesic metric of circumference 1 and let $E$ be a chained entourage. According to Theorem 57, $\ker \Lambda_E$ is generated by homotopy classes of stringings of $E$-small loops. We will show that every $E$-triad has a stringing that represents either the trivial homotopy class or the class of a generator of $\pi_1(X) = \mathbb{Z}$. From this it follows that the only two possible $E$-covers of the circle are the trivial cover and the universal cover.

If an $E$-triad $\tau = \{x_0, x_1, x_2\}$ has no $E$-null stringing then in particular all three points must be distinct. We assume that the points are ordered in the clockwise direction and let $A_i$ denote the arc in the clockwise direction from $x_i$ to $x_{i+1}$. Since $E$ is chained, each $B(x_i, E) \cap B(x_{i+1})$ must contain $A_i$ or $A_i \cup A_k$ with $j,k \neq i$. If $B(x_i, E) \cap B(x_{i+1})$ only contains $A_i$ for all $i$, then the only possible stringings of $\tau$ are $E$-homotopic to the circle itself, meaning that the $\phi_E$ is trivial.

Otherwise, without loss of generality we may suppose that $B(x_0, E) \cap B(x_1, E)$ contains $A_1 \cup A_2$. Case 1: Suppose that $A_2 \subset B(x_2, E) \cap B(x_0, E)$. There are two subcases. 1a: $A_1 \subset B(x_1, E) \cap B(x_2, E)$. In this case (slightly abusing notation by considering each $A_i$ as a path), $A_2 \ast A_1 \ast A_1 \ast A_2$ is a stringing of $\tau$ that is clearly $E$-null. 1b: $A_0 \cup A_2 \subset B(x_1, E) \cap B(x_2, E)$. In this case, $A_2 \ast A_1 \ast A_0 \ast A_2 \ast A_2$ is a stringing of $\tau$ that represents a generator of $\pi_1(X)$.

Now observe that we have considered the three essential cases (up to re-ordering): Each $B(x_i, E) \cap B(x_{i+1})$ contains only $A_i$, exactly two of the $B(x_i, E) \cap B(x_{i+1})$ contain only $A_i$, or exactly one of the $B(x_i, E) \cap B(x_{i+1})$ contains only $A_i$, so the proof is complete.

The next examples are related to the question of identifying entourage covers for 2-dimensional manifolds (with or without boundary).

Example 69 Let $M$ be the Moebius Band. First note that $M$ is homeomorphic to $\mathbb{RP}^2$ with a small disk removed. Therefore we may Gromov-Hausdorff approximate $\mathbb{RP}^2$ by Mobius bands, taking the standard metric on $\mathbb{RP}^2$ and induced geodesic metric on $M$ with smaller and smaller disks removed. Since the double cover of $\mathbb{RP}^2$ is its universal cover, hence an $\varepsilon$-cover for all sufficiently small $\varepsilon$, the double cover of $M$ is also an $\varepsilon$-cover for small enough $\varepsilon$ with respect to the induced metrics with small enough disks removed. This follows from the convergence result Proposition 37, since small enough $\varepsilon > 0$ is not a homotopy critical value of $\mathbb{RP}^2$. Note that $M$ deformation retracts onto the circle and yet the double cover of the circle is not an entourage cover by Example 68. This example shows one way that convergence can be used to identify entourage covers but also shows the limitations of “enlarging” spaces to try to find entourage covers.
Remark 70 Let $X$ be a compact geodesic space such that $HCS$ has $n$ elements, counting multiplicity as defined in [20]. In other words, there are a total of $n$ equivalence classes of essential circles. As pointed out in the proof of Theorem 13, every essential $\varepsilon$-circle is $E_{\varepsilon}$-critical, and as defined in [20], the multiplicity is the number of distinct $E_{\varepsilon}$-homotopy classes of $\varepsilon$-circles. Since being $E$-critical is a topological property these essential circles will still be critical loops in any other metric on $X$, and by definition their lengths will be elements of $ES$. That is, $ES$ in any metric will always have at least $n$ elements, but the lengths of curves, and hence the values and multiplicities of $ES$ will be different. In particular, as soon as there are two geodesic metrics with $CS$ having different sizes, then the space with the smaller $CS$ must have $CS$ strictly contained in $ES$, counting multiplicity. In the next example we essentially carry this out in a carefully controlled setting, allowing us to insure that the multiplicities are 1 and hence control the absolute size of the spectra.

Example 71 Let $M$ be a compact smooth manifold of dimension 3 or higher with fundamental group $Z_4$, which we will denote by $\{[c_0], [c_1], [c_2], [c_3]\}$. (There are many other possibilities for $\pi_1(M)$, but we are using $Z_4$ for simplicity.) Denote the double cover of $M$ (corresponding to the subgroup generated by $[c_2]$) by $M'$ and the universal cover of $M$ by $M''$. By standard covering space theory, $c_2$ is the only loop, up to free homotopy, that lifts as a loop to $M'$. We now make the following assignments: $[c_1] \to 1.2$, $[c_2] \to 1.1$, $[c_3] \to 1.2$ and apply Theorem 2.9 of [10]. In the resulting metric, the shortest loop lifting to a non-loop in $M'$ must be freely homotopic to either $c_1$ or $c_3$ and hence has length 1.2. According to Proposition 6B, $M''$ is equivalent to $M_{1,1}$. In particular, $HCS = \{\frac{1}{1.1}, \frac{1.2}{3}\}$ and $CS = \{\frac{1}{1.1}, \frac{1.2}{3}\}$. Moreover, as pointed out in the proof of Theorem 13, if $c$ is an essential $\frac{1}{3}$-circle then $c$ is $E$-critical, where $E := E_{\frac{1}{3}}$. Note that since $c$ lifts as a non-loop to $M_E$, $c$ is freely homotopic to $c_2$.

Now use the following assignments: $[c_1] \to 1.1$, $[c_2] \to 1.2$, $[c_3] \to 1.3$ and apply Theorem 2.9 of [10]. With this metric, any shortest loop lifting as a non-loop to either $M'$ or $M''$ has length 1.1. That is, $HCS = \{\frac{1}{1.1}\}$ and $CS = \{\frac{1}{1.1}\}$. Since being $E$-critical is a topological property, the loop $c$ is still $E$-critical and by definition the length of the shortest curve $c'$ in its free $E$-homotopy class is an element of $ES$. But any curve in the free $E$-homotopy class of $c$ must lift to a loop in $M' = M_E$. Since $c_2$ is the only such curve, up to free homotopy, $c'$ must be freely homotopic to $c_2$ and therefore has length 1.2. That is, $ES$ contains $\{1.1, 1.2\}$ and in particular $ES$ strictly contains $3HCS = 2CS$. Moreover, since we may change the value assigned to $[c_3]$ by any small amount, it is possible for different Riemannian metrics on $M$ to have the same $CS$ but different $ES$. Finally, since there are only two non-trivial entourage covers of $M$, hence at most two elements in $ES$, $ES = \{1.1, 1.2\}$. But 1.3 is an element of $MLS$ by definition, so $ES$ is strictly contained in $MLS$. Again, the value assigned to $[c_3]$ may be changed by a small amount without impacting $ES$, and therefore one obtains Riemannian metrics on $M$ that have the same $ES$ but different $MLS$. This example verifies Theorem 13.2.d-e.
Example 72 In [3], Berestovskii-Plaut-Stallman gave two examples of compact geodesic spaces with free homotopy classes having no closed geodesics in them. The first is 1-dimensional, consisting of a circle in the plane with smaller and smaller straight segments added to join points around the perimeter circle, with the induced geodesic metric. The circle itself is not a closed geodesic because any segment may be “bypassed” by a shorter straight segment, but it is, up to monotone reparametization, the only curve, hence the shortest curve, in its homotopy class. Now for any entourage $E$, the circle is $E$-homotopic to a shorter curve that is a piecewise segment and is shortest in its $E$-homotopy class. Note that although this segment visually has “corners” in the construction, it is still a closed geodesic with the induced geodesic metric.

The other example is the infinite torus, the countable product $T^\infty$ of circles with the Tychonoff topology. This space can be metrized by making the sizes of the circles square summable and taking the geometric product metric (see [23]). This space is of course a compact topological group and this metric is bi-invariant. For this example, Stallman proved in his dissertation that (1) there is a unique 1-parameter subgroup (i.e. a homomorphism $\theta : \mathbb{R} \to T^\infty$) in each free homotopy class of a curve, (2) there are free homotopy classes containing no rectifiable curves (3) if there are rectifiable curves then the shortest one is the unique 1-parameter subgroup in it, but (4) the 1-parameter subgroup may not be a closed geodesic. It would be interesting to see whether the shortest curves in free $E$-homotopy classes given by Proposition 10 are also 1-parameter subgroups.

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