QUANTIFIER ELIMINATION FOR TAME FIELDS

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Abstract. In this paper, we give appropriate languages in which the theory of tame fields (of any characteristic) admits (relative) quantifier elimination.

1. Introduction

In [3], the first author asked the question: do there exist functions definable in the theory of tame fields such that in the language of valued fields enriched by these functions, every substructure admits a henselization which is a tame field? If this is true, it follows from the results in [3] that the theory of tame fields admits quantifier elimination relative to what is called ‘amc-structures of level 0’ in [3], in this enriched language. In the present paper, we show that such functions do exist, more precisely, that one can enrich the language of valued fields with countably many predicates which do the job. However, the amc-structures are a little complicated to use. So we will simplify the results by passing to residue-valuation structures (RVs), which were studied in [1]. Our main theorems (Theorem 4.5 and Theorem 4.9) are that, in the language of valued fields together with the predicates $A_n$, $E_n$, and $H_n$, for $n \in \mathbb{N}$, the theory of tame fields of equal characteristic admits quantifier elimination relative to the RVs; and in the language of valued fields together with the predicates $A'$ and $E'_n$, for $n \in \mathbb{N}$, the theory of tame fields of any characteristic admits quantifier elimination relative to the RVs.

The paper is organized as follows. In the next section, we list preliminaries about RVs and tame fields required for the paper. The goal of Section 3 is to prove Theorem 3.6. The proof is essentially the same as in [3], but rewritten in the RV language. In Section 4, we introduce the predicates $A_n$, $E_n$, and $H_n$, and $A'$ and $E'_n$, and prove our main results.

2. Preliminaries

Given a valued field $K=(K,v)$, we use the following notation:

- $vK = \{va \mid 0 \neq a \in K\}$: the value group
- $\mathcal{O}_K = \{a \in K \mid va \geq 0\}$: the valuation ring
- $m_K = \{a \in K \mid va > 0\}$: the valuation ideal
- $Kv = \mathcal{O}_K/m_K$: the residue field
- $\pi : \mathcal{O}_K \to K_v$: the residue map
- $\pi(c)$ or $\overline{c}$: the residue of an element $c \in \mathcal{O}_K$.

By $K^\times$, $\mathcal{O}_K^\times$, and $(Kv)^\times$ we denote the set of units of $K$, $\mathcal{O}_K$, and $Kv$ respectively. The group $1 + m_K$ of 1-units is a normal subgroup of $K^\times$ under multiplication. We

The second author is partially supported by Laskowski’s NSF grant DMS-0901336.
denote the factor group as
\[ \text{RV}_K := \frac{K^\times}{(1 + m_K)} , \]
and the natural quotient map as
\[ \text{rv}_K : K^\times \to \text{RV}_K . \]
To extend the map to all of \( K \), we introduce a new symbol “\( \infty \)” (as we do with value groups) and define \( \text{rv}_K(0) = \infty \). Though \( \text{RV}_K \) is defined merely as a group, it inherits much more structure from \( K \).

To start with, since the valuation \( v \) on \( K \) is given by the exact sequence
\[ 1 \longrightarrow O_K^\times \xrightarrow{v} K^\times \longrightarrow vK \longrightarrow 0 \]
and since \( 1 + m_K \leq O_K^\times \), the valuation descends to \( \text{RV}_K \) giving the following exact sequence
\[ 1 \longrightarrow (Kv)^\times \xrightarrow{\text{rv}} \text{RV}_K \xrightarrow{vK} vK \longrightarrow 0 \]
(note that \( O_K^\times/(1 + m_K) \cong (O_K/m_K)^\times = (Kv)^\times \)). \( \text{RV}_K \) also inherits an image of addition from \( K \) via the relation
\[ \oplus(x_{rv}^{(1)}, \ldots, x_{rv}^{(n)}, z_{rv}) = \exists z, x^{(1)}, \ldots, x^{(n)} \in K \left(z_{rv} = \text{rv}_K(z) \land x_{rv}^{(1)} = \text{rv}_K(x^{(1)}) \land \cdots \land x_{rv}^{(n)} = \text{rv}_K(x^{(n)}) \land z = x^{(1)} + \cdots + x^{(n)}\right). \]

For more details on \( \text{RV}_K \), see \[7\] Section 9 and [1]. We construe \( \text{RV}_K \) as a structure in the language of rings \( L_{\text{rv}} := \{ +, -, 0, 1, v_{rv} \} \), the residue field \( Kv \) as a structure in the language of rings \( L_r := \{ +, -, 0, 1 \} \), and the value group \( vK \) as a structure in the language of ordered abelian groups \( L_{OG} := \{ +, -, 0, < \} \). We then have the following important result, see [1] Proposition 3.1.4.

**Proposition 2.1.** \( vK \) and \( Kv \) are interpretable in \( \text{RV}_K \).

This proposition essentially tells us that every formula \( \phi \in L_r \) can be encoded by a formula \( \phi_{rv} \in L_{\text{rv}} \), such that for all \( a_1, \ldots, a_n \in Kv \), we have \( Kv \models \phi(a_1, \ldots, a_n) \) if and only if \( \text{RV}_K \models \phi_{rv}(\iota(a_1), \ldots, \iota(a_n)) \). And similarly every formula \( \varphi \in L_{OG} \) can be encoded by a formula \( \varphi_{rv} \in L_{\text{rv}} \) such that for all \( \gamma_1, \ldots, \gamma_n \in vK \) and all \( a_1, \ldots, a_n \in \text{RV}_K \) with \( v_{rv}(a_i) = \gamma_i \) for \( 1 \leq i \leq n \), we have \( vK \models \varphi(\gamma_1, \ldots, \gamma_n) \) if and only if \( \text{RV}_K \models \varphi_{rv}(a_1, \ldots, a_n) \).

We construe a valued field \( K \) as a structure in the language of fields together with a divisibility predicate \( L_{\text{def}} := \{ +, -, \cdot, -1, 0, 1, D_v \} \), where the divisibility predicate \( D_v \) is defined as \( D_v(x, y) \) if and only if \( v(x) \leq v(y) \). The valuation ring \( O_K \) and the maximal ideal \( m_K \) can be defined in this language as \( O_K = \{ x \in K \mid D_v(1, x) \} \) and \( m_K = \{ x \in K \mid D_v(1, x) \land \neg D_v(x, 1) \} \). Because of this observation and because of the way we defined \( \text{RV}_K \), it can be shown that \( \text{RV}_K \) is interpretable in \( K \). In other words, every formula \( \psi_{rv} \in L_{\text{rv}} \) can be encoded by a formula \( \psi_{\text{def}} \in L_{\text{def}} \), such that for all \( r_1, \ldots, r_n \in \text{RV}_K \) and all \( a_1, \ldots, a_n \in K \) with \( \text{rv}_K(a_i) = r_i \) for \( 1 \leq i \leq n \), we have \( K \models \psi_{\text{def}}(a_1, \ldots, a_n) \) if and only if \( \text{RV}_K \models \psi_{rv}(r_1, \ldots, r_n) \).

An algebraic extension \( (L, v)(K, v) \) of henselian fields is called **tame** if for every finite subextension \( (K', v)(K, v) \), the following holds:

(T1) the residue field extension \( K'v/Kv \) is separable,
(T2) if \( p = \text{char}(Kv) > 0 \), then the ramification index \( (vK' : vK) \) is prime to \( p \),
(T3) the extension is defectless, i.e.,

\[ [K' : K] = [K'v : Kv][vK' : vK] \]

A henselian field is called a defectless field if each of its finite extensions is defectless. An arbitrary valued field is called a defectless field if its henselizations are defectless fields. A valued field is called a tame field if it is henselian and every algebraic extension is a tame extension. A tame field \((K, v)\) is characterized as being a henselian defectless field with a perfect residue field and a \(p\)-divisible value group \((p = \text{char}(Kv) > 0 \text{ or } p = 1 \text{ otherwise})\) [6, Lemma 3.10]. A tame field is also perfect [6, Lemma 3.6]. Note that the properties of being a defectless field or a tame field are first-order [6, Sections 4 and 7]. Examples of tame fields (and hence, defectless fields) include algebraically closed valued fields, henselian fields with residue characteristic 0, and algebraically maximal Kaplansky fields. See [2] for details on tame fields.

A general not necessarily algebraic extension \((L, v)| (K, v)\) will be called pre-tame if the following holds:

(P1) the residue field extension \(Lv|Kv\) is separable,

(P2) if \(p = \text{char}(Kv) > 0\), then the order of every torsion element of \(vL/vK\) is prime to \(p\).

Note that every extension of a tame field is pre-tame, and that every algebraic pre-tame extension of a henselian defectless field is tame.

An extension \((L, v)| (K, v)\) of valued fields is called immediate if \(vL = vK\) and \(Lv = Kv\). A valued field \(K\) is called algebraically maximal if it does not admit proper immediate algebraic extensions. Since henselizations are immediate algebraic extensions, every algebraically maximal field is henselian.

We end this section by mentioning two important properties that the elementary class \(\mathfrak{T}\) of tame fields has. For details, see [2].

(IME) “Immediate extensions are equivalent” : if \(K, L, F \in \mathfrak{T}\) and \(L\) and \(F\) are immediate extensions of \(K\), then \(L \equiv_K F\).

(RAC) “Relative algebraic closures” : if \(L \in \mathfrak{T}\), the quotient \(vL/vK\) is a torsion group and the extension \(Lv|Kv\) is algebraic, then the relative algebraic closure \(L'\) of \(K\) in \(L\) is an element of \(\mathfrak{T}\), and \(L|L'\) is immediate.

3. Embedding Lemma

Our main goal in this section is to show that embeddings at the level of RVs can be extended to embeddings of tame fields under suitable conditions. Before we make this more precise, let us first recall from [3] the structure of a finite tame extension \(L|K\) of henselian fields.

Suppose \(L|K\) is a finite tame extension of henselian fields. Then the residue field extension \(Lv|Kv\) is finite and separable, hence simple. Let \(\bar{c}\) be a generator of it. We choose a monic polynomial \(f(x) \in \mathcal{O}_K[x]\) whose reduction modulo \(v\) is the irreducible polynomial of \(\bar{c}\) over \(Kv\). Since the latter is separable, we may use Hensel’s Lemma to find a root \(c \in L\) of \(f(x)\) with residue \(\bar{c}\). From general valuation theory, it follows that the extension \(K(c)|K\) is of the same degree as \(Kv(\bar{c})|Kv\) and that \(K(c)v = Kv(\bar{c}) = Lv\).

Similarly, since \(L|K\) is a finite extension, the group \(vL/vK\) is a finite torsion group, say

\[ vL/vK = \mathbb{Z} \cdot (\alpha_1 + vK) \times \cdots \times \mathbb{Z} \cdot (\alpha_r + vK) \]
(we set $r = 0$ if $vL = vK$). For any $\alpha \in vL \setminus vK$, take $n \in \mathbb{N}$ to be the order of $\alpha + vK$, and $a \in L$ such that $v(a) = \alpha$. Then $v(a^n) = n\alpha \in vK$, and thus there is some $b \in K$ such that $v(ba^n) = 0$. Then $\pi(ba^n) \in Lv = K(c)v$ is not zero, and there is some $h(x) \in \mathcal{O}_K[x]$ with $\pi(a^{-n}b^{-1}h(c)) = 1$. By minimality of $n$ and condition (T2) for tame extensions, $n$ is prime to $p$ if char$(Kv) = p > 0$. Hence, by Hensel's Lemma, we can find $a_0 \in L$ such that $a_0^n = a^{-n}b^{-1}h(c)$; putting $d = aa_0 \in L$, we get $bd^n = h(c)$. Using this procedure for $1 \leq i \leq r$, we choose elements

- $n_i \in \mathbb{N}$ such that $0 \neq n_i\alpha_i \in vK$,
- $b_i \in K$ with $vb_i = -n_i\alpha_i$,
- $d_i \in L$ with $vd_i = \alpha_i$ and $b_id_i^n = h_i(c)$, where
- $h_i \in \mathcal{O}_K[X]$ with $vh_i(c) = 0$.

We show that $L = K(c,d_1,\ldots,d_r)$. We have that $\alpha_i = vd_i \in vK(c,d_1,\ldots,d_r) \subseteq vL = vK \oplus \mathbb{Z} \cdot \alpha_1 \oplus \cdots \oplus \mathbb{Z} \cdot \alpha_r$ for $1 \leq i \leq r$, so $vK(c,d_1,\ldots,d_r) = vL$. Similarly, $\tilde{c} \in K(c,d_1,\ldots,d_r)v \subseteq Lv = Kv(\tilde{c})$, so $K(c,d_1,\ldots,d_r)v = Lv$. Being an algebraic extension of the tame field $K$, also $(K(c,d_1,\ldots,d_r),v)$ is a tame field (cf. [3]). Hence, the finite extension $(L,v)|(K(c,d_1,\ldots,d_r),v)$ must be defectless. As it is also immediate, it must be trivial, showing that $L = K(c,d_1,\ldots,d_r)$, as asserted.

Moreover, if $K \subseteq F$ and $z,t_1,\ldots,t_r \in F$ are such that $f(z) = 0$ and $b_it_i^n = h_i(z)$, then

$$(c,d_1,\ldots,d_r) \mapsto (z,t_1,\ldots,t_r)$$

induces an embedding of $L$ in $F$ over $K$. Since $K$ is henselian, this is valuation preserving, i.e., an embedding of $L$ in $F$ over $K$.

Using this "normal form" for finite tame extensions, we prove the main embedding lemma for tame algebraic extensions:

**Lemma 3.1.** Let $K$ be an arbitrary valued field, $L$ a tame algebraic extension of some henselization of $K$, and $F$ an arbitrary henselian extension of $K$. If $L$ is embeddable in $F$ over $K$, then $\text{RV}_L$ is embeddable into $\text{RV}_F$ over $\text{RV}_K$. Conversely, every embedding $\tau$ of $\text{RV}_L$ in $\text{RV}_F$ over $\text{RV}_K$ may be lifted to an embedding of $L$ in $F$ over $K$ which induces $\tau$.

**Proof.** Since $\text{RV}_K$ can be interpreted in $K$, the proof of the first statement is obvious.

For the second part, let $\tau$ be an embedding of $\text{RV}_L$ in $\text{RV}_F$ over $\text{RV}_K$. Since both $L$ and $F$ are assumed to be henselian, they both contain henselizations of $K$.

By the uniqueness property of henselizations, these are isomorphic over $K$ and we may identify them. This henselization has the same RV structure as $K$: for every $a$ in a henselization of $K$ there is some $a' \in K$ such that $v(a - a') > 0$. But then by [2] Lemma 9.1, $\text{rv}_K(a) = \text{rv}_K(a')$, i.e., they have the same images under $\text{rv}_K$. Hence, it suffices to prove our lemma under the additional hypothesis that $K$ is henselian.

As in [3] Lemma 3.1], it further suffices to prove our lemma only in the case of $L|K$ a finite extension of henselian fields. By the remark preceding our lemma, it suffices to find an image in $F$ for the tuple $(c,d_1,\ldots,d_r)$ in order to obtain an embedding of $L$ in $F$ over $K$. Recall that this tuple satisfies

$$\tilde{f}(\tilde{c}) = 0 \quad \text{and} \quad \bigwedge_{1 \leq i \leq r} \tilde{h}_i(\tilde{c}) \neq 0 \land v(\tilde{h}_i(\tilde{c})) = \tilde{b}_id_i^n,$$
where \( \tilde{b}_i = rv_L(b_i) \) and \( \tilde{d}_i = rv_L(d_i) \). Since \( \tau \) is an embedding of \( RV_L \) in \( RV_F \) over \( RV_K \), we find \( x = \tau(\tilde{c}) \in Fv \) and \( y_i = \tau(\tilde{d}_i) \in RV_F \) for \( 1 \leq i \leq r \), such that the following is satisfied in \( F \):

\[
\tilde{f}(x) = 0 \quad \text{and} \quad \bigwedge_{1 \leq i \leq r} h_i(x) \neq 0 \land \iota(h_i(x)) = \tilde{b}_i y_i^{n_i}.
\]

Since \( \tilde{f} \) is irreducible and separable over \( Kv \), the zero \( x \) is simple and thus, by Hensel’s Lemma, gives rise to a zero \( z \in F \) of \( f \) with residue \( x \). Now, for each \( 1 \leq i \leq r \), pick \( e_i \in F \) such that \( rv_F(e_i) = y_i \). Since \( h_i(x) \neq 0 \), we have

\[
rv_F(h_i(z)) = \iota(h_i(\bar{z})) = \iota(h_i(x)) = \tilde{b}_i y_i^{n_i} = rv_F(b_i e_i^{n_i}),
\]

that is, \( h_i(z) b_i^{-1} e_i^{-n_i} \equiv 1 \mod m_F \). So the polynomial

\[
P(X) = X^{n_i} - h_i(z) b_i^{-1} e_i^{-n_i} \in O_F[X]
\]

reduces modulo \( v \) to the polynomial \( X^{n_i} - 1 \) which admits \( 1 \) as a simple root since \( n_i \) is not divisible by the characteristic of \( Kv \). By virtue of Hensel’s Lemma, \( P(X) \) admits a root \( e'_i \) in the henselian field \( F \). Putting \( t_i := e'_i e_i \in F \), we obtain \( b_i t_i^{n_i} = h_i(z) \). Consequently, the assignment \( (e, d_1, \ldots, d_r) \mapsto (z, t_1, \ldots, t_r) \) induces an embedding of \( L \) in \( F \) over \( K \).

It remains to show that this embedding is a lifting of \( \tau \). This will follow if we are able to show that the assignment \( (\bar{c}, d_1, \ldots, d_r) \mapsto (x, y_1, \ldots, y_r) \) determines an embedding of \( RV_L \) in \( RV_F \) over \( RV_K \) uniquely. Since \( \bar{c} \) generates \( Lv \) over \( Kv \), it just remains to show that the elements \( \bar{d}_1, \ldots, \bar{d}_r \) generate \( RV_L \) over the group compositum \( RV_K \cdot \iota(Lv)^{\times} \). Given an element \( a \in L \), our choice of the \( d_i \)'s implies that there exist integers \( m_1, \ldots, m_r \), an element \( d' \in K \) and an element \( g(c) \in O_K[c] \) of value \( 0 \) such that the value of \( a^{-1} d_1^{m_1} \cdots d_r^{m_r} \cdot \iota(g(c)) \) is \( 0 \) and its residue is \( 1 \). Hence,

\[
rv_L(a) = \tilde{d}_1^{m_1} \cdots \tilde{d}_r^{m_r} \cdot rv_K(d') \cdot \iota(\bar{g}(\bar{c}))
\]

with \( rv_K(d') \in RV_K \) and \( \iota(\bar{g}(\bar{c})) \in \iota(Lv)^{\times} \). This concludes our proof. \( \square \)

Let us now deal with non-algebraic extensions, and first handle the case where \( K \) is a defectless field, \( F \) is a henselian extension field of \( K \) and \( L[K] \) a pre-tame extension which admits a valuation transcendence basis \( T \), i.e., \( T \) is a transcendence basis of \( L/K \) of the form

(1) \( T = \{ x_i, y_j \mid i \in I, j \in J \} \) such that :

(a) the values \( vx_i, i \in I \), form a maximal system of values in \( vL \)
which are rationally independent over \( vK \), and
(b) the residues \( y_j, j \in J \), form a transcendence basis of \( Lv|Kv \).

Now an embedding \( \rho \) of \( RV_L \) over \( RV_K \) induces an embedding \( \sigma \) of \( Lv \) in \( Fv \), and an embedding \( \sigma \) of \( Lv \) in \( vF \) over \( vK \). Choose a set \( T' = \{ x'_i, y'_j \mid i \in I, j \in J \} \subset F \) such that \( vx'_i = \sigma(vx_i) \) and \( y'_j = \rho(y_j) \). Then \( T' \) is a valuation transcendence basis of the subextension \( (K(T'), v)|K \) of \( F \). As shown in the proof of Lemma 5.6 (Embedding Lemma I) of [5], the assignment \( x_i \mapsto x'_i, i \in I \), \( y_j \mapsto y'_j, j \in J \), induces a valuation preserving isomorphism from \( (K(T), v) \) onto \( (K(T'), v) \) over \( K \) which induces the embedding \( \sigma \) on the value groups and the embedding \( \tau \) on the residue fields. As in the proof of Lemma 3.1, we wish to
show that it determines a unique isomorphism of the residue-valuation structures of $K(\mathcal{T})$ and $K(\mathcal{T}')$.

By Lemma 2.2 of [6], the residues $\bar{y}_j, j \in J$, generate $K(\mathcal{T})v$ over $Kv$. Therefore, it remains to show that the elements $\bar{x}_i = r_{v(\mathcal{T}')}v(x_i)$ generate $RV_{K(\mathcal{T})}v$ over the compositum $RV_K \cdot i(K(\mathcal{T})v)^\times$. Again by the cited lemma, the values $v(x_i), i \in I$, generate $vK(\mathcal{T})$ over $vK$. Hence for every element $a \in K(\mathcal{T})$, there exist $i_1, \ldots, i_r \in I$, integers $m_1, \ldots, m_r$, an element $d' \in K$ and an element $g \in K(\mathcal{T})$ of value 0 such that the value of $a^{-1}x_1^{m_1} \cdots x_r^{m_r}d'g$ is 0 and its residue is 1. Hence,

$$rv(\mathcal{T})v(a) = x_1^{m_1} \cdots x_r^{m_r} \cdot rv_K(d') \cdot i(\bar{g})$$

with $rv_K(d') \in RV_K$ and $i(\bar{g}) \in i(K(\mathcal{T})v)^\times$. Similarly, one shows that $RV_{(\mathcal{T}')}v$ is generated by the elements $\tilde{x}_i' = rv_{(\mathcal{T}'},v)(x_i')$ over $RV_K$ and the image of $K(\mathcal{T})v$, which in turn is generated by the residues $\bar{y}'_j$. Thus, the above isomorphism between $K(\mathcal{T})$ and $K(\mathcal{T}')$ is a lifting of the isomorphism of their respective residue-valuation structures, which in turn is a restriction of $\tau$. Therefore, by identifying $K(\mathcal{T})$ and $K(\mathcal{T}')$ as a common valued subfield of $L$ and $F$, and applying Lemma 3.1, we get:

**Lemma 3.2.** Let $K$ be a common subfield of the henselian fields $L$ and $F$. Assume that $L$ admits a valuation transcendence basis $\mathcal{T}$ such that $L$ itself is a tame extension of some henselization $(K(\mathcal{T}), v)^h$. Then, for every embedding $\tau$ of $RV_L$ in $RV_F$ over $RV_K$, there is an embedding of $L$ in $F$ over $K$ which induces $\tau$.

Let us now look at the case of a general pre-tame extension $L|K$ which admits a valuation transcendence basis $\mathcal{T}$. In this case, we need a certain degree of saturation of $F$ to be able to lift an embedding $\tau$ of $RV_L$ in $RV_F$ over $RV_K$ to an embedding of $L$ in $F$ over $K$. Let $L$ be $[L]^\tau$-saturated. Then, to be able to embed $L$ in $F$ over $K$, it is enough to embed every finitely generated subextension $L'|K$ of $L|K$ and to derive that the so-obtained embedding of $L$ in $F$ is a lifting of the embedding of the respective RV-structures, one has to expand the language to incorporate the embeddings and use saturation on the expanded language, see [6] Lemma 5.7. Now every finitely generated subextension $L|^h_K$ is contained in a finitely generated pre-tame subextension $L_0|K$ which admits a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ as its valuation transcendence basis. Thus, $L_0$ is a finite extension of $(K(\mathcal{T}_0), v)$. By general valuation theory, it follows that $vL_0/vK(\mathcal{T}_0)$ and $L_0v/K(\mathcal{T}_0)v$ are finite. Further, $vL_0/vK$ and $L_0v/Kv$ are finitely generated by [6] Corollary 2.3. Since $L_0|K$ is pre-tame, it follows that $vL_0 = \Gamma \oplus \mathbb{Z} \cdot \gamma_1 \oplus \cdots \oplus \mathbb{Z} \cdot \gamma_r$ for some $\gamma_1, \ldots, \gamma_r \in vL_0$ (where $r = \dim Q \otimes vL_0/vK$), with $\Gamma/vK$ finite and torsion prime to $p$ (which is the character $Kv$). Thus, we may choose elements $x_i$ as part of a new transcendence basis $\mathcal{T}_1$ such that $vx_i = \gamma_i$, $1 \leq i \leq r$, and $p$ does not divide $\gamma_1 \cdots \gamma_r$ in $vL_0$. Since $L_0v/Kv$ is separable by assumption, we may choose elements $y_j$ to form $\mathcal{T}_1$ such that their residues $y_jv$ form a separating transcendence basis of $L_0v/Kv$. Since $K$ is assumed to be a defectless field, the same is true for $(K(\mathcal{T}_1), v)$ by virtue of [2] Theorem 3.1. This shows that for $(K(\mathcal{T}_1), v)^h$, the henselization inside of $L_0^h$, the finite extension $L_0^h|(K(\mathcal{T}_1), v)^h$ is tame. The lifting of an embedding of $\mathcal{T}$-structures to an embedding of $L_0^h$ (and hence of $\mathcal{T}'$) in $F$ over $K$ now follows by Lemma 3.2.

Finally, we have to deal with the case where $L|K$ does not admit a valuation transcendence basis. Take $\mathcal{T}$ as in (1) and form the subfield $L'' := (K(\mathcal{T}), v)$ of $L$. By definition of $\mathcal{T}$, the quotient $vL/vL''$ is a torsion group and $Lv/L''v$ is algebraic. Let $L'$ be the relative algebraic closure of $L''$ in $L$. Then $L'|K$ admits a valuation transcendence basis and $L|L'$ is an immediate extension. If $L$ is a member of some
elementary class $K$ which has the (RAC) property, then $L' \in K$. We have thus proved the following:

**Lemma 3.3.** Let $K$ be a common defectless subfield of the henselian fields $L$ and $F$ such that $L|K$ is a pre-tame extension. Assume that $L$ is a member of an elementary class $K$ of valued fields which has the (RAC) property. Then there exists a subfield $L' \in K$ of $L$ which admits a valuation transcendence basis and such that $L|L'$ is immediate. Moreover, if $F$ is an $|L|^+$-saturated extension of $K$, then for every embedding $\tau$ of $RV_L$ in $RV_F$ over $RV_K$, there is an embedding of $L'$ in $F$ over $K$ which induces $\tau$.

Coming back to model theory, since $RV_K$ can be interpreted in $K$, one derives the following

**Lemma 3.4.** If $K^*$ is an elementary extension of $K$ and a special model of cardinality $\kappa > \text{card } K$, then $RV_{K^*}$ is an elementary extension of $RV_K$ which is also a special model of cardinality $\kappa$.

By virtue of this lemma and a standard back and forth argument, Lemma 3.3 yields

**Lemma 3.5.** Let $K$ be an elementary class of henselian fields which has the (RAC) property, and let $K$ be a common defectless subfield of $L, F \in K$ such that $L|K$ and $F|K$ are pre-tame extensions. If $RV_L \equiv_{RV_K} RV_F$, then there exist elementary extensions $L'$ and $F'$ of $L$ and $F$ which contain relatively algebraically closed subfields $L'$ and $F'$ respectively such that

(a) $L', F' \in K$,
(b) $L'$ and $F'$ are isomorphic over $K$,
(c) $L'|L'$ and $F'|F'$ are immediate extensions,
(d) $RV_{L'} \cong_{RV_{K'}} RV_{F'}$.

As a result, we obtain the following:

**Theorem 3.6.** Let $K$ be an elementary class of henselian fields which has the properties (IME) and (RAC). Further, let $K$ be a common defectless subfield of $L, F \in K$ such that $L|K$ and $F|K$ are pre-tame extensions. Then $L \equiv_K F$ is equivalent to $RV_L \equiv_{RV_K} RV_F$.

Since the elementary class $\mathfrak{T}$ of tame fields enjoys the properties (IME) and (RAC), we get the following as an immediate corollary to this theorem.

**Corollary 3.7.** Assume that $L$ and $F$ are tame fields and that $K$ is a common valued subfield of $L$ and $F$. If any henselization of $K$ is a tame field, then $L \equiv_K F$ is equivalent to $RV_L \equiv_{RV_K} RV_F$.

4. Quantifier Elimination relative to RV

Our goal in this section is to show that the theory of tame fields of equal characteristic admits quantifier elimination relative to the RVs in an enriched language. For this, we will use the substructure completeness test, which says that a theory eliminates quantifiers if and only if every two models of the theory are equivalent over every common substructure. Corollary 3.7 gives a hint. If we can enrich our language by adding definable predicates that guarantee that every substructure of
a tame field admits a henselization which is a tame field, we will be done. This is what we are going to achieve in this section.

Recall that for valued fields of residue characteristic zero, “henselian”, “henselian defectless” and “algebraically maximal” are all equivalent, and also henselian valued fields of residue characteristic zero are tame. In positive residue characteristic, “henselian defectless” implies “algebraically maximal”, but the converse is not true; “algebraically maximal” implies “henselian”, but does not imply “defectless” in general. However, it holds for perfect fields of positive characteristic [6, Part (a) of Corollary 3.12]. We will now give a different characterization of algebraically maximal fields in terms of a notion called “extremality”, which in turn is easier to express by a predicate.

**Definition 4.1.** If $f$ is a polynomial over $K$ in $n$ variables, then $(K, v)$ is said to be **K-extremal with respect to** $f$ if the set

$$\{v(f(a_1, \ldots, a_n)) \mid a_1, \ldots, a_n \in K\} \subseteq vK \cup \{\infty\}$$

has a maximum. A valued field $(K, v)$ is called **K-extremal** if for all $n \in \mathbb{N}$, it is $K$-extremal with respect to every polynomial in $n$ variables with coefficients in $K$.

With this notion of extremity, we have the following theorem, see [5, Theorem 1.5, Theorem 1.8].

**Theorem 4.2.** A henselian valued field $(K, v)$ of characteristic $p > 0$ is algebraically maximal if and only if it is $K$-extremal with respect to every polynomial in one variable over $K$ of the form

$$b_0 + \sum_{i=1}^{n} b_i x^{p^{i-1}} , \quad n \in \mathbb{N}, b_0, b_1, \ldots, b_n \in K.$$

We now give a characterization of tame fields in positive characteristic.

**Theorem 4.3.** Let $K = (K, v)$ be a valued field of char $K = p > 0$. Then $K$ is tame if and only if $K$ is algebraically maximal and for every $c \in K$ with $vc < 0$, there is $d \in \mathcal{O}_K^{\times}$ or $d = 0$ such that $x^p - x - c - d$ has a zero in $K$.

**Proof.** ($\Rightarrow$) The fact that $K$ is algebraically maximal follows from [6, Lemma 3.6].

Now, let $\eta$ be a root of $x^p - x - c$ in some extension field of $K$. If $\eta \in K$ then we set $d = 0$ and are done. If $\eta \notin K$, then $x^p - x - c$ is irreducible over $K$ and $[K(\eta) : K] = p$.

Consider the set

$$v(\eta - K) = \{v(\eta - a) \mid a \in K\}.$$ 

If $v(\eta - K) \subseteq vK^{<0}$, then it has no maximal element; this is seen as follows. Pick any $a \in K$. Then $v(\eta - a) < 0$. Now,

$$\eta - a = \eta^p - \eta - a^p + a = c - a^p + a.$$

Since $v(\eta - a) < 0$, we have $v((\eta - a)^p) = pv(\eta - a) < v(\eta - a)$. Thus, $v(\eta - a) = 1 \frac{1}{p} v(c - a^p + a)$. Set $b = (c - a^p + a)^{1/p}$; note that $b \in K$ since a tame field of positive characteristic is perfect (cf. [6]). Then, $v(b) = v(\eta - a)$. Moreover,

$$\frac{1}{p} v((\eta - a - b)^p) - v((\eta - a - b) = c - a^p + a - b^p + b = b.$$ 

Therefore, $v(\eta - a - b) = \frac{1}{p} v(b) = 0$.
Proof. Let $\mathbf{L}$ and $\mathbf{F}$ be two tame fields of equal characteristic zero with a common valued subfield $\mathbf{K}$, also of equal characteristic zero. Since $\mathbf{L}$ and $\mathbf{F}$ are tame, they are also henselian. Therefore, they contain henselizations of $\mathbf{K}$. By the uniqueness property of henselizations, these are isomorphic over $\mathbf{K}$ and we may identify them.

Now, the henselization of $\mathbf{K}$ is a henselian field of equal characteristic zero, and hence, tame. Thus, by Corollary 2.17, $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ if and only if $\text{RV}_\mathbf{L} \equiv_{\text{RV}_\mathbf{K}} \text{RV}_\mathbf{F}$. \square

For equal characteristic $p > 0$, we expand our language by adding predicates $A$, $E_n$ and $H_n$ for $n \in \mathbb{N}$, defined as follows:

\[
A(x) := \exists d \exists y \left( (d = 0) \lor (v(d) = 0) \right) \land (v^p \geq x - y - d = 0) \\
E_n(x_0, \ldots, x_n) := \exists z \forall y \left( v(x_0 + \sum_{i=1}^{n} x_i z^{p^{i-1}}) \geq v(x_0 + \sum_{i=1}^{n} x_i y^{p^{i-1}}) \right) \\
H_n(x_1, \ldots, x_n, y) := \exists z \left( v(y - z) > 0 \land z^n + \sum_{i=1}^{n} x_i z^{p^{i-1}} = 0 \right)
\]

We define our new language $\mathcal{L}_{\text{pref}} := \mathcal{L}_{v \mathbf{f}} \cup \{A\} \cup \{E_n \mid n \in \mathbb{N}\} \cup \{H_n \mid n \in \mathbb{N}\}$ for valued fields in equal characteristic $p > 0$. We now prove our first relative quantifier elimination result for tame fields in equal characteristic $p$. 

\[\frac{1}{p} v(\eta - a) > v(\eta - a).\] With $b_0 = a + b \in K$, we have $v(\eta - b_0) > v(\eta - a)$, and so $v(\eta - a)$ is not a maximal element of $v(\eta - K)$.

We have now shown that if $v(\eta - K) \subseteq vK^{\geq 0}$, then $v(\eta - K)$ has no maximal element. Note that the extension of the valuation $v$ from $K$ to $K(\eta)$ is unique since the tame field $(K, v)$ is henselian. So we can apply part (1) of [Lemma 2.21] to deduce that $(K(\eta))|K(v)$ is immediate. Since $\mathbf{K}$ is algebraically maximal, it follows that $K(\eta) = K$, which is a contradiction.

Hence, $v(\eta - K) \cap vK^{\geq 0} \neq \emptyset$. Take $a \in K$ such that $v(\eta - a) \geq 0$. Since $v(\eta - a) \geq 0$, we have $v((\eta - a)p) = pv(\eta - a) \geq 0$. In view of [2], we obtain that

\[v(c + a - a^p) \geq \min\{v(\eta - a), v((\eta - a)p)\} \geq 0.\]

If $v(c + a - a^p) > 0$, we set $d = 0$. If $v(c + a - a^p) = 0$, we set $d = -(c + a - a^p)$; in this case, $v(d) = 0$, i.e., $d \in \mathcal{O}_K^\times$. Now the equation $f(x) = x^p - x - (c + a - a^p) - d = 0$ reduces to $\bar{f}(\bar{x}) = \bar{x}^p - \bar{x} = 0$ in the residue field $Kv$. We observe that $\bar{f}(\bar{x}) = 0$ has a simple root in $Kv$, for instance $\bar{x} = \bar{1}$. Thus, by Hensel’s Lemma there is a root $\beta \in K$ of $f(x) = 0$, i.e., $\beta^p - \beta - (c + a - a^p) - d = 0$. But then, $(\beta + a)^p - (\beta + a) - c - d = 0$, which means $x^p - x - c - d = 0$ has a solution in $K$, namely $x = \beta + a$. Thus, $d$ works.

\[\iff\] Conversely, suppose for every $c \in K$ with $vc < 0$, there is $d = 0$ or $d \in \mathcal{O}_K^\times$ such that $x^p - x - c - d$ has a zero in $K$. Then it follows from the proof of [4 Corollary 2.17] that $vK$, $p$-divisible and $Kv$ is perfect. The fact that $(K, v)$ is tame now follows from [5 Lemma 3.10]. \square

We are now ready to define our language. For equal characteristic zero, it is enough to have the language $\mathcal{L}_{v \mathbf{f}}$ of valued fields.

**Theorem 4.4.** The theory of tame fields in equal characteristic zero admits quantifier elimination relative to the RVs in the language $\mathcal{L}_{v \mathbf{f}}$.

**Proof.** Let $\mathbf{L}$ and $\mathbf{F}$ be two tame fields of equal characteristic zero with a common valued subfield $\mathbf{K}$, also of equal characteristic zero. Since $\mathbf{L}$ and $\mathbf{F}$ are tame, they are also henselian. Therefore, they contain henselizations of $\mathbf{K}$. By the uniqueness property of henselizations, these are isomorphic over $\mathbf{K}$ and we may identify them. Now, the henselization of $\mathbf{K}$ is a henselian field of equal characteristic zero, and hence, tame. Thus, by Corollary 3.7, $\mathbf{L} \equiv_{\mathbf{K}} \mathbf{F}$ if and only if $\text{RV}_\mathbf{L} \equiv_{\text{RV}_\mathbf{K}} \text{RV}_\mathbf{F}$. \square

For equal characteristic $p > 0$, we expand our language by adding predicates $A$, $E_n$ and $H_n$ for $n \in \mathbb{N}$, defined as follows:
**Theorem 4.5.** The theory of tame fields in equal characteristic $p > 0$ admits quantifier elimination relative to the RVs in the language $\mathcal{L}_{\text{vrf}}$.

**Proof.** Let $L$ and $F$ be two tame fields of equal characteristic $p > 0$ with a common substructure $K$ in the language $\mathcal{L}_{\text{vrf}}$. Clearly $K$ is a common valued subfield of $L$ and $F$ also of equal characteristic $p$.

Observe that $K$ is henselian: Let $f(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ be a polynomial over $\mathcal{O}_K$, and let $b \in \mathcal{O}_K$ be such that $v_f(b) > 0 = v_f'(b)$. Since $a_1, \ldots, a_n, b$ also belong to $L$, and $L$ is henselian, there is $c \in L$ such that $v(b - c) > 0$ and $f(c) = 0$. Thus, $L \models H_n(a_0, \ldots, a_n, b)$. Since $K$ is a substructure of $L$ in the language $\mathcal{L}_{\text{vrf}}$, we have $K \models H_n(a_0, \ldots, a_n, b)$, i.e., there is $d \in K$ such that $v(b - d) > 0$ and $f(d) = 0$. In particular, $K$ is henselian.

Now, observe that $K$ is algebraically maximal: because of Theorem 4.2, it is enough to show that $K$ is extremal with respect to every one-variable polynomial over $K$ of the form $f(x) = b_0 + \sum_{i=1}^n b_i x^{p^i}$. Since $b_0, \ldots, b_n$ also belong to $L$ and $L$ is algebraically maximal, there is a maximum element in $L$ of the set $\{vf(a) \mid a \in L\}$. Thus, $L \models E_n(b_0, \ldots, b_n)$. Since $K$ is a substructure of $L$ in the language $\mathcal{L}_{\text{vrf}}$, we have $K \models E_n(b_0, \ldots, b_n)$, i.e., the set $\{vf(a) \mid a \in K\}$ has a maximum in $K$. In particular, $K$ is algebraically maximal.

Finally, observe that $K$ is tame: because of Theorem 4.3, it is enough to show that for any $c \in K$ with $v(c) < 0$, there is $d \in \mathcal{O}_K^c$ or $d = 0$ such that $x^p - x - c - d$ has a zero in $K$. Since $c$ also belongs to $L$ and $L$ is tame, there is $e \in \mathcal{O}_L^c$ or $e = 0$ such that $x^p - x - c - e$ has a zero in $L$. Thus, $L \models A(c)$. Since $K$ is a substructure of $L$ in the language $\mathcal{L}_{\text{vrf}}$, we have $K \models A(c)$, i.e., there is $d \in \mathcal{O}_K^c$ or $d = 0$ such that $x^p - x - c - d$ has a zero in $K$. In particular, $K$ is tame.

Thus, by Corollary 3.7, we have $L \equiv_{K}^{\mathcal{L}_{\text{vrf}}} F$ if and only if $\text{RV}_L \equiv_{\mathcal{L}_{\text{vrf}}} \text{RV}_F$. For this last statement, note that since $A(x), E_n(x_0, \ldots, x_n)$ and $H_n(x_1, \ldots, x_n, y)$, for $n \in \mathbb{N}$, are definable predicates in the language $\mathcal{L}_v$ of valued fields, it follows that $L \equiv_{K}^{\mathcal{L}_{\text{vrf}}} F$ if and only if $L \equiv_{K}^{\mathcal{L}_{\text{vrf}}} F$.

This, however, leaves open the mixed characteristic case. Now we will give an alternate language which will handle the mixed characteristic case too. For this, we will use the following alternate characterization of tame fields, see [6] Lemma 3.10.

**Theorem 4.6.** A valued field $(K, v)$ of residue characteristic $p > 0$ is tame if and only if $(K, v)$ is algebraically maximal, $vK$ is $p$-divisible and $Kv$ is perfect.

To use this result to define the right predicates, we also need the following result, see [3] Theorem 1.8.

**Theorem 4.7.** A valued field $(K, v)$ is algebraically maximal if and only if it is $K$-extremal with respect to every polynomial in one variable with coefficients in $K$.

We now convert the restrictions on the value group and the residue field to an elementary condition.

**Theorem 4.8.** Let $(K, v)$ be a valued field of residue characteristic $p > 0$. Then $vK$ is $p$-divisible and $Kv$ is perfect if and only if for every $a \in K$ there is $b \in K$ such that $v(a - b^p) > va$.

**Proof.** [$\Rightarrow$] Pick $a \in K$. 

Since \( vK \) is \( p \)-divisible, there is \( c \in K \) such that \( va = pvc \), i.e., \( va = vc^p \).
Therefore, \( v\left(\frac{a}{c^p}\right) = 0 \).

Since \( Kv \) is perfect, there is \( d \in K \) such that \( \left(\frac{a}{c^p}\right)v = (dv)^p \), i.e., \( \left(\frac{a}{c^p}\right)v = d^pv \).
By [7, Lemma 9.1], it follows that \( v\left(\frac{a}{c^p} - d^p\right) > 0 \).

Hence, \( v(a - b^p) > va \).
Set \( b = cd \).
Thus, \( vK \) is \( p \)-divisible.

\[ \Leftarrow \] Conversely, let \( va \in vK \).
Pick \( b \in K \) such that \( v(a - b^p) > va \).
Then, \( va = vb^p = pb \).
Thus, \( vK \) is \( p \)-divisible.

Similarly, let \( av \in Kv \). Thus, \( a \in K \) with \( va \geq 0 \).
Pick \( b \in K \) such that \( v(a - b^p) > va \).
By [7, Lemma 9.1], it follows that \( av = b^pv = (be)^p \).
Hence, \( Kv \) is perfect. \( \square \)

For residue characteristic \( p > 0 \), we expand our language by adding predicates \( A' \) and \( E'_{n} \), for \( n \in \mathbb{N} \), defined as follows:

\[
A'(x) := \exists b(v(x - b^p) > vx)
\]

\[
E'_n(x_0, \ldots, x_n) := \exists z \forall y \left( v(\sum_{i=0}^{n} x_i z^i) \geq v(\sum_{i=0}^{n} x_i y^i) \right)
\]

We define our new language \( L_{rvf} := L_{raf} \cup \{ A' \} \cup \{ E'_{n} \mid n \in \mathbb{N} \} \) for valued fields in residue characteristic \( p > 0 \). We now prove our second relative quantifier elimination result for tame fields in residual characteristic \( p \).

**Theorem 4.9.** The theory of tame fields in residue characteristic \( p > 0 \) admits quantifier elimination relative to the RVs in the language \( L_{rvf} \).

**Proof.** Let \( L \) and \( F \) be two tame fields of residue characteristic \( p > 0 \) with a common substructure \( K \) in the language \( L_{raf} \). Clearly \( K \) is a common valued subfield of \( L \) and \( F \) also of residue characteristic \( p \).

Observe that \( K \) is algebraically maximal: because of Theorem 4.7, it is enough to show that \( K \) is \( K \)-extremal with respect to every one-variable polynomial over \( K \). Let \( f(x) = \sum_{i=0}^{n} b_i x^i \) be such a polynomial over \( K \). Since \( b_0, \ldots, b_n \) also belong to \( L \) and \( L \) is algebraically maximal, there is a maximum element in \( L \) of the set \( \{ vf(a) \mid a \in L \} \). Thus, \( L \models E'_n(b_0, \ldots, b_n) \). Since \( K \) is a substructure of \( L \) in the language \( L_{raf} \), we have \( K \models E'_n(b_0, \ldots, b_n) \), i.e., the set \( \{ vf(a) \mid a \in K \} \) has a maximum in \( K \). In particular, \( K \) is algebraically maximal.

Also, observe that \( K \) is tame: because of Theorems 4.6 and 4.8, it is enough to show that for any \( a \in K \), there is \( b \in K \) such that \( v(a - b^p) > va \). Since \( a \) also belongs to \( L \) and \( L \) is tame, there is \( e \in L \) such that \( v(a - e^p) > va \). Thus, \( L \models A'(a) \). Since \( K \) is a substructure of \( L \) in the language \( L_{raf} \), we have \( K \models A'(a) \), i.e., there is \( b \in K \) such that \( v(a - b^p) > va \). In particular, \( K \) is tame.

Thus, by Corollary 3.7, we have \( L \equiv_{K}^{L_{raf}} F \) if and only if \( RV_{L} \equiv_{RV_{K}}^{L_{raf}} RV_{F} \).
For this last statement, note that since \( A'(x) \) and \( E'_n(x_0, \ldots, x_n) \), for \( n \in \mathbb{N} \), are
definable predicates in the language $\mathcal{L}_{vf}$ of valued fields, it follows that $L \equiv_{K}^{\mathcal{L}_{vf}} F$ if and only if $L \equiv_{K}^{\mathcal{L}_{rvf}} F$.

References

[1] J. Flenner. The relative structure of Henselian valued fields. Ph.D. thesis, available at [http://www.nd.edu/~jflenner/papers/dissertation.pdf](http://www.nd.edu/~jflenner/papers/dissertation.pdf) 2008.

[2] F.-V. Kuhlmann. Henselian function fields and tame fields. Preprint (extended version of Ph.D. thesis), Heidelberg, available at [http://math.usask.ca/~fvk/hftf.pdf](http://math.usask.ca/~fvk/hftf.pdf) 1990.

[3] F.-V. Kuhlmann. Quantifier elimination for Henselian fields relative to additive and multiplicative congruences. *Israel Journal of Mathematics*, 85:277–306, 1994.

[4] F.-V. Kuhlmann. Value groups, residue fields and bad places of rational function fields. *Transactions of the American Mathematical Society*, 356:4559–4600, 2004.

[5] F.-V. Kuhlmann. A classification of Artin Schreier defect extensions and characterizations of defectless fields. *Illinois Journal of Mathematics*, 54(2):397–448, 2010.

[6] F.-V. Kuhlmann. The algebra and model theory of tame valued fields. Submitted, available at [http://math.usask.ca/~fvk/MTHTVF.pdf](http://math.usask.ca/~fvk/MTHTVF.pdf) 2012.

[7] K. Pal. Multiplicative valued difference fields. *Journal of Symbolic Logic*, 77(2):545–579, 2012.