Foundations for abstract forcing

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Abstract: The foundations of forcing theory are reworked to streamline the presentation and to show how the most basic results are applicable in very general contexts.

INTRODUCTION

After Cohen [Co] invented the method of forcing to settle long-open independence results in set theory, techniques evolved to become simpler as well as more powerful. Our focus is on simplifying foundational aspects of forcing, in its most elementary form, that seem to have been relatively neglected. As will be seen, forcing becomes particularly easy to understand when separated into distinct but interacting techniques, involving posets, generic filters, Boolean algebras, and (for set theory) a weak form of the membership relation. A new general-purpose forcing method, with a presumably much wider range of applications, arises when aspects related to the construction of models for set theory are stripped away and the role of names reexamined. Well-known complications involving the set membership relation $\in$ do not arise if methods requiring well-foundedness are abandoned and simpler ones found. Although forcing models are studied via objects called names, constructed in a standard way (with minor variations) from the poset, details about names will turn out to be irrelevant in the earlier foundational stages studied below. Could new ways of encoding names, with new relations and interpretations of them, yield innovative applications of forcing in areas distant from set theory?

An overview of the most relevant aspects seems appropriate. Forcing has been regarded mainly as a method to construct models of set theory tailored to have a great variety of properties, resolving purely set-theoretic questions such as those concerning possible values of powers of cardinals, or more ‘applied’ ones in areas that include infinite combinatorics, topology, measure theory, and proof theory. Very briefly, one approach starts with a fixed countable transitive model $(M,\in)$ of ZFC which gives rise, nonconstructively, to other more interesting models. It is convenient to work with such an $M$ but its existence is an unnecessarily strong assumption, and several alternate approaches are possible (e.g., see [Ku]). The most crucial aspect in applications is to define a poset, often with great ingenuity, so that it ‘forces’ the new models obtained using it to have the properties aimed at. These properties are discussed in a forcing language, using constants intended to represent (nonfaithfully) names of the intended new sets. Statements are interpreted in a way that depends on the choice, outside $M$, of a ‘generic’ filter in the poset and on the form in which names are represented concretely as certain sets in $M$. This approach already appears in mature form in Shoenfield [Sh], whose presentation incorporated important simplifications. For historical details, see [Mo]. Since the basic ideas of forcing lead to spectacular applications, some easily accessible, they deserve to be made as simple and transparent as possible.
Order-theoretic preliminaries

Order-theoretic concepts usually refer to the fixed poset \((\mathcal{P}, \leq)\), and variables such as \(p, q, r\) range over the elements of \(\mathcal{P}\). Notation \(X^\uparrow = \{p : (\exists x \in X)(x \leq p)\}\) and \(X^\downarrow\) is used for up- and down-closures of subsets \(X\) of \(\mathcal{P}\). Optionally, one can use topological concepts, relative to the topology on \(\mathcal{P}\) formed by the down-closed sets. For example, a subset \(D\) is dense if \(D^\uparrow = \mathcal{P}\).

Instead of working with arbitrary posets, one can use Boolean algebras. This permits techniques of a more algebraic nature and more concise language. In practice, to construct a model tailored to have certain properties, the preferred method is search for an appropriate poset \((\mathcal{P}, \leq)\), which is then embedded into a complete Boolean algebra. One way is to start with the unary operation which takes each subset \(X\) of \(\mathcal{P}\) to a down-closed set \(X' = \mathcal{P}\setminus X^\uparrow\). This yields a closure operator \(X \mapsto X''\) on \(\mathcal{P}\), whose closed elements \((X = X'')\) form a family, here called \(\mathcal{B}(\mathcal{P})\). As usual, the family is closed under arbitrary intersections, making it a complete lattice. The lattice is complemented and is in fact a Boolean algebra, by a result of Byrne [By], as it satisfies Byrne’s axioms: \(\land\) is a semilattice operation, \(0 \neq 0'\), and, most notably, \(X \land Y' = 0\) iff \(X \land Y = X\). Alternatively, \(\mathcal{B}(\mathcal{P})\) consists of the regular open sets of the above topology on \(\mathcal{P}\).

In this paragraph, we make the mild assumption (separativity) that the sets \(p^\downarrow\) (\(p \in \mathcal{P}\)) lie in \(\mathcal{B}(\mathcal{P})\). Then \(\mathcal{P}\) has a certain kind of universal embedding in a complete Boolean algebra. One good source for relevant details is [Kub]. We remark that \(\mathcal{P} \to \mathcal{B}(\mathcal{P})\), \(p \mapsto p^\downarrow\) is such an embedding. As this will not be needed, the details are omitted.

In conventional applications of forcing, the poset \((\mathcal{P}, \leq)\) lies within some model \((M, \in)\) for set theory, and those elements of \(\mathcal{B}(\mathcal{P})\) that lie in \(M\) clearly form a subalgebra \(\mathcal{B}_M(\mathcal{P})\), with sup and inf restricted to subfamilies definable in \(M\). This relativization of \(\mathcal{B}(\mathcal{P})\), viewed within the model, is a complete Boolean algebra. Moreover, the forcing map (see below) which takes a statement \(\varphi\) to \(\{p \in \mathcal{P} : p \models \varphi\}\) will be seen to map into \(\mathcal{B}_M(\mathcal{P})\).

A filter \(G\) is a non-empty up-closed set for which each pair of elements of \(G\) has a lower bound in \(G\). The dense subsets of \(\mathcal{P}\) are clearly those of the form \(X \cup X'\), and no filter can contain points of both \(X\) and \(X'\). Relative to any collection \(\mathcal{D}\) of dense subsets, there is a notion of \(\mathcal{D}\)-generic filters \(G\), by which it is meant that \(G\) has non-empty intersection with each \(D \in \mathcal{D}\). A well-known idea (see [Sh] or [Ku, VII, 2.3]) produces an abundance of \(\mathcal{D}\)-generic filters, enough so that each point \(p\) lies in such filters, whenever \(\mathcal{D}\) is countable.

Abstract forcing

A forcing language will be taken here to be just a first-order language, say with relation symbols but not terms, used to make statements involving constants called ‘names’, which are used in place of free variables. The choice of basic symbols for logical conectives and quantifiers (say \(\neg, \land, \exists\)) matters little in our classical approach. If desired, one could rephrase results in terms of a version of the Lindenbaum algebra, regarded as the Boolean algebra (with some extra structure) naturally induced on the statements of the forcing language modulo logical equivalence.

Variables such as \(\tau, \sigma, \pi\) always range over names, and may be quantified. The abstract approach to names requires nothing more than a function from atomic statements \(\varphi\) to values \([\varphi]\) in the Boolean algebra \(\mathcal{B}(\mathcal{P})\). In set-theoretical applications, the class of names
and the relation \( p \in [[\sigma \in \tau]] \) should be definable by predicates in the base model \((M, \in)\). Extensionality of the new models is obtained by imposing a condition \((E)\), treated below.

Any evaluation of atomic statements extends, in the obvious way, to one for all statements in the language. In particular, \([[\exists \tau \varphi(\tau)]]\) is the sup in \( \mathbf{E}(\mathcal{P}) \) of the elements \([[\varphi(\tau)]]\) as \( \tau \) ranges over all names. Without assuming more about the structure of names, this sup is not usually the union, which could be thought of as a defect when working with individual points in \( \mathcal{P} \). Generic filters provide the remedy.

The countable collection \( \mathcal{D} \) used to define ‘generic’ matters little, so long as it contains all \( X \cup X' \) for which \( X \) is either of the form \([[\varphi]]\) or (letting one of the names used in \( \varphi \) vary freely) of the form \( \cup_\tau [[\varphi(\tau)]] \). This restriction on \( \mathcal{D} \) is essential for the next results, but some class \( \mathcal{D} \) more easily definable than the minimal one is customarily used. To ensure that \( \mathcal{D} \) is countable, the approach for set-theoretic applications makes the strong assumption that \( M \) (viewed from outside) is countable, so that only a countable number of dense subsets of \( \mathcal{P} \) are definable in \( M \). In more general contexts, it would suffice to assume at most enumerably many names and relation symbols (hence also statements).

For each generic filter \( G \), consider the set \( \mathcal{T}_G \) of statements \( \varphi \) for which \([[\varphi]]\) intersects \( G \) nontrivially; i.e., \((\exists \exists p \in G)(p \in [[\varphi]])\). These \( \varphi \) are, in some sense ‘true relative to \( G \)’, and we write \( G \models \varphi \), while statements not in \( \mathcal{T}_G \) are ‘false for \( G \)’. This assignment of truth values respects operations such as \( \wedge, \neg \). To obtain models for \( \mathcal{T}_G \), quantifiers must also be respected. Handling \( \exists \) yields at once the perhaps more surprising result for \( \forall \).

**Lemma:** For statements \( \varphi(\tau) \), with \( \tau \) varying over names, and \( G \) a generic filter,
\[(a) \ G \models \exists \tau \varphi(\tau) \text{ iff } \text{ for some } \tau, G \models \varphi(\tau).\]
\[(b) \ G \models \forall \tau \varphi(\tau) \text{ iff } \text{ for all } \tau, G \models \varphi(\tau).\]

Proof: For this, it suffices to show that, given any family of statements of the form \( \varphi(\tau) \) with \( \tau \) ranging over names, and letting \( X \) be the union of the corresponding subsets \([[\varphi(\tau)]]\), then \([[\exists \exists \tau \varphi(\tau)]]\) is disjoint from \( G \) if \( X \) is. This holds because here \( G \) must contain some point of \( X' \), hence none in \( X'' \). This last set is just \([[\exists \exists \tau \varphi(\tau)]]\).

It should not fail to be mentioned that, for names of the form used in set theory, part (a) of the lemma can be strengthened to say that some suitable name \( \tau \) depends only on \( \varphi \), not on \( G \). See for example [Kub, Th. 9.2]. (The treatment in [Ku, VII, 8.2] is incomplete.) Although statements \( \varphi \) are evaluated in the base model by using the Boolean algebra, the usual aim is to use a generic filter \( G \) to construct a model \( M[G] \) with \( M[G] \models \varphi \) iff \( G \models \varphi \). Note that the previous lemma implies at once that the forcing language has such a model: for now, \( M[G] \) will be taken to be the set \( \mathcal{N} \) of names, endowed via the atomic statements in \( \mathcal{T}_G \) with the relations corresponding to the relation symbols. Later, one may prefer to factor out \( \mathcal{N} \) by some equivalence relation, or use an appropriate image of \( \mathcal{N} \), in order to interpret more naturally symbols such as \( = \) (which should be equality) and \( \in \) (membership).

Each statement \( \varphi \) in the forcing language yields a forcing relation, say as a unary relation on \( \mathcal{P} \), as follows. For points \( p \) of \( \mathcal{P} \), we say that \( p \) forces \( \varphi \), and write \( p \models \varphi \), when \( \varphi \) is true in every model \( M[G] \) with \( G \) generic (relative to \( \mathcal{D} \)) and \( G \models \varphi \).

There are two especially fundamental results concerning forcing. First, it is definable in a way that does not mention generic filters: for each statement \( \varphi \), some predicate in
the base model determines which \( p \in P \) force \( \varphi \). The second, the Truth Lemma, is that a statement \( \varphi \) is valid in a model \( M[G] \) exactly when some \( p \in G \) forces \( \varphi \). Another perhaps noteworthy observation is that the forcing relation does not depend on the choice (restricted as above) of the dense sets \( D \) used to define generic filters. All these results follow immediately from:

**Lemma:** \( p \Vdash \varphi \) iff \( p \in [\varphi] \).

**Proof:** If \( p \in [\varphi] \) then, trivially, \( p \Vdash \varphi \). Conversely, when \( p \notin [\varphi] \) some \( q \leq p \) lies in \([\varphi] \)' and some generic filter \( G \) contains \( q \). Now \( M[G] \not\models \varphi \) and \( p \in G \), giving \( p \not\Vdash \varphi \).

**INTERLUDE ON NON-EXTENSIONAL AND NON-WELL-FOUNDED THEORIES**

In theories and models, the symbol = is almost universally taken to mean equality. Following a principle held by Leibniz, objects which cannot be distinguished by any relevant property should be regarded as equal. The only reason not to identify would be to leave open the possibility of later more refined properties. In models of set theory, or within the language itself, = can be treated as being defined from the basic relation \( \in \), via an even stronger principle, extensionality. This idea, that sets with the same members are equal, seems so central to the concept of what sets are (platonically) that it has rarely been examined critically. When \( \in \) is the only basic relation present, Leibniz extensionality, or quasi-extensionality (with an equivalence relation in place of =) is:

\[
(E) \quad (\forall \pi)(\pi \in \sigma_1 \leftrightarrow \pi \in \sigma_2) \rightarrow (\forall \tau)(\sigma_1 \in \tau \leftrightarrow \sigma_2 \in \tau).
\]

However, set-theoretic forcing tends to produce even laxer models that need further adjustments in order to satisfy (E). Another principle, that sets should be well-founded, is usually imposed here, but this a rare example where such ideas only serve to complicate matters. Now that there is sufficient motivation to examine non-extensionality, in conjunction with non-well-foundedness, we take a broad view of related issues.

The area of mathematical foundations is rich with history and interesting issues, enough to merit continuing scholarly attention. Attitudes to certain topics, for example foundations for set theory, may change for reasons worth elucidating, detailing how groups with different agenda, using different language, influence each other or fail to do so. For example, it is extremely convenient to work with well-founded sets, using transfinite induction on rank. This may explain in part why non-well-founded set theories remained a little-studied curiosity in the shade of ZFC until there was a clear need for them, as tools useful for studying several problems in computer science. For similar reasons, non-extensionality (and intensionality) have in recent years become common topics in computer science, whose influence may in time spread wider.

Despite earlier work in the field, notably by Boffa (in many articles), the use of non-well-founded sets became commonplace thanks to the timely and influential work by Aczel [A]. Not surprisingly, the sort of non-well-founded theories considered are overwhelmingly those based on Aczel’s anti-foundation axiom AFA, which is too restrictive for our taste. Our view is that if objects and a binary relation called \( \in \) are somehow given, and objects with the same ‘members’ are equal, then this is a system of sets, as long as we do not need
to examine objects internally (using more than $\in$) to see what they are ‘really’ made of. Does anyone ask what the empty set is made of? As in other areas of mathematics, only abstract structural properties matter. Following Boffa rather than Aczel (and others), we regard it as desirable to permit sets with different elements to have isomorphic internal $\in$-structures, thereby for example allowing more than one set of the form $x = \{x\}$.

Many equivalent ways have been used to axiomatize the system known as ZF. The exact form of the axioms becomes important when studying systems which omit or modify axioms, and our attention is on Extensionality together with Foundation. One standard source treating axioms for ZF and other set theories is [F], but it makes no claim to completeness and mentions surprisingly little about the particular topics we wish to focus on. For non-extensional theories, the following studies, whose reviews we consulted, seem to be among the most mathematically relevant. Hinnion [Hi] introduces methods which, among other things, simplify earlier work by Gandy [Ga], Scott [Sc], and others. Briefly, the effects of omitting Extensionality are as follows. Scott showed that the resulting system ZF$\neq$ is distinctly weaker than ZF, while Gandy and H. Friedman [Fr] showed that their systems suffer no significant change. The reason is that Scott and Gandy used the usual form of Replacement, involving a functional relation of the form $(\forall x)(\exists!y)\varphi(x,y)$, but Gandy compensated for this weakness by introducing a set formation operator $\lambda x A(x)$ (i.e., $\{x \mid A(x)\}$). The solution we prefer is Friedman’s, who used the Collection axiom (not even mentioned in [F]), which strengthens Replacement when Foundation is not assumed. Collection is: $[(\forall x)(\exists y)\varphi(x,y)] \rightarrow [(\forall X)(\exists Y)(\forall x \in X)(\exists y \in Y)\varphi(x,y)]$. A good exposition of Friedman’s later improvements on [Fr] appears in [Kr, §5]. The conclusion is that, even without Foundation, extensional systems can easily be recovered from non-extensional ones of the right form.

Friedman’s solution is internal to the set theory ($\in$ can be defined from a weaker non-extensional $\varepsilon$, assuming some set-theoretic axioms, but not Foundation), whereas our solution assumes no axioms, but is carried out within the Boolean algebra rather than within the forcing language. It can also be carried out internally in the presence of axioms sufficient to support a definition by transfinite induction. While Friedman recovers extensionality by identifying objects as much as possible, using an approach based on ideas of Aczel, where the preferred bisimulations are maximal, we identify objects only when absolutely necessary to obtain extensionality. The method is so simple that it may have been rediscovered several times, but it does not seem to be widely known, as intensive searches failed to locate a reference in literature or reviews accessible to us. Aczel [A] takes a different approach when studying the least bisimulation, a topic treated earlier by Hinnion [Hi], presumably in a roughly similar way. However, the method we prefer can be found in two recent preprints – [BMW, Prop. 6] and [Fi, §6]. In the second, the idea is used in the process of making an interesting connection between forcing and modal logic.

The solution now given produces the smallest quasi-extensional relation $\in$ (with an equivalence relation in place of equality) generated from an arbitrary binary relation $\varepsilon$ on a set. The relation $\in$ is well-founded precisely when $\varepsilon$ is. The language will temporarily have many binary relations: $\in$, $\varepsilon$, $\sim$ and a family $\sim_\alpha$, where $\alpha$ always ranges over ordinals. For abstract forcing, the universe is the class of names, and only ordinals $\alpha < |P|$ are needed. For more general use, name-free notation will be used.
To start, $\sim_0$ is equality, and $y_1 \sim_1 y_2$ means $(\forall x_1)(x_1 \in y_1 \rightarrow (\exists x_2)(x_1 \sim_0 x_2 \land x_2 \in y_2)) \land (\forall x_2)(x_2 \in y_2 \rightarrow (\exists x_1)(x_1 \sim x_2 \land x_1 \in y_1))$. For limit ordinals $\lambda$, $\sim_\lambda$ is the limit (union) of the earlier relations. It is not hard to see that the $(\sim_\lambda)$ form an increasing family of equivalence relations between names, and $\sim$ is defined to be its limit. Finally, $x \in y$ is defined to mean $\exists x'(x \sim x' \land x' \in y)$. It may be clearer to imagine the transition from $\varepsilon$ to $\in$ in stages, defining $x \varepsilon_\alpha y$ iff there exist $x' \sim_{\alpha} x$ and $y' \sim_{\alpha'} y$ with $x' \varepsilon y'$. The earlier extensionality condition ($E$) clearly holds. The equivalence relation $\sim$ (which is factored out to form models with $=\varepsilon$) and the relation $\varepsilon$, are definable in terms of each other, in the presence of $\varepsilon$. Thus $x \varepsilon y \rightarrow x \in y$ and $\varepsilon$ is a simulation for $\sim$, by which we mean that $x \varepsilon y \land y' \sim y \rightarrow (\exists x') (x' \varepsilon y')$. This gives a clear idea how to construct all possible $\varepsilon$ from a quasi-extensional relation $\varepsilon$ (with an equivalence relation $\equiv$ in place of $=$), not guaranteeing here that $\equiv$ will be the smallest relation $\sim$ compatible with $\varepsilon$.

**Forcing with names and Boolean-valued models**

Returning to forcing, with the enriched language now using names, formulas will be assigned values in a complete Boolean algebra $\mathcal{B}$, starting with atomic formulas and extending in the obvious way. Everything is done within a model $(M, \varepsilon)$ for set theory, in ways definable by predicates in the forcing language, using a given valuation of $\varepsilon$ on the class of names, with values in the Boolean algebra $\mathcal{B} = \mathcal{B}_M(\mathcal{P})$ constructed from a fixed poset $\mathcal{P}$ in the model. If $M$ is countable, we can also concentrate on a model of the forcing language obtained from a generic filter $G$. This initially gives a non-extensional binary relation $\varepsilon_G$ on the names, which generates a quasi-extensional relation $\equiv_G$ in the way described in the previous section. As well-foundedness is not assumed, it no longer seems compelling to try to obtain quotient models $(M[G], \varepsilon)$ that are standard.

The construction using Boolean valuations starts by defining $[[\tau_1 \sim_0 \tau_2]]$ to be $1$ (the set $\mathcal{P}$) if the names $\tau_1$ and $\tau_2$ are identical, and $0$ ($\emptyset$) otherwise. Whatever $G$ is, this just gives the equality relation on names. Other definitions are as expected. Thus, for limit ordinals $\lambda$, $[[\tau \sim_\lambda \tau']] = \bigvee_{\alpha<\lambda}[[\tau \sim_\alpha \tau']]$. Care is required here, as it is conceivable that a generic filter $G$ could be disjoint from the sets $[[\tau \sim_\alpha \tau']] (\alpha < \lambda)$ but not from $[[\tau \sim_\lambda \tau']]$. In set-theoretic models, assuming enough axioms to allow quantification over ordinals, an earlier lemma shows that this problem does not arise. In general, a suitable relation $\sim_\lambda$, no longer claimed to be the smallest one, will be produced. This follows from $[[\tau \sim_\lambda \tau']] = \bigvee_{\alpha<\lambda}[[\tau \sim_\alpha \tau']] \leq [[\sigma \varepsilon \tau \rightarrow (\exists \sigma' \varepsilon \tau') (\sigma \sim_\lambda \sigma')]]$.

Set-theoretic axioms are usually stated in terms of $\varepsilon$, but stronger forms using $\varepsilon$ may be more convenient. The axioms of ZFC are especially well-suited for constructing models that satisfy enough of the axioms (a finite number) to be useful. Names and Boolean valuations can be built in transfinite stages from initial data, in ways ensuring that these axioms then hold in the models $(M[G], \varepsilon_G)$. Now that problems concerning non-extensionality and non-well-foundedness have been adequately resolved, remaining details can be inferred from many sources. Only a little will be sketched here.

To construct names of one of the conventional forms, one can start with $\mathcal{N}_0 = \emptyset$ (say), define $\mathcal{N}_{\alpha+1}$ to be the union of $\mathcal{N}_\alpha$ with the set of functions $\mathcal{N}_\alpha \rightarrow \mathcal{B}$ that lie in $M$, and let $\mathcal{N}_\lambda (\lambda$ a limit ordinal) be the obvious union. Thus each name is ‘created’ at a successor
ordinal in $M$. For names $\sigma, \tau$, the value $[[\sigma \in \tau]]$ is defined to be $\tau(\sigma)$ if $\sigma$ is created before $\tau$, and is 0 otherwise.

As an illustration, the power set axiom will be examined. Write $\sigma' \leq \sigma$ if, for all names $\pi$, $[[\pi \in \sigma']] \leq [[\pi \in \sigma]]$. This provides a sufficiently large supply of names for subsets. Given $\sigma \in \mathcal{N}_\alpha$ one name $\tau : \mathcal{N}_\alpha \to \mathbb{B}$ with $[[\forall \sigma'(\sigma' \in \tau \iff \sigma' \subset \sigma)]] = 1$ is as follows: for all $\sigma' \in \mathcal{N}_\alpha$, $\tau(\sigma') = 1 \in \mathbb{B}$ when $\sigma' \leq \sigma$, while $\tau(\sigma') = 0$ otherwise.

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