Examples of pathological dynamics of the subgradient method for Lipschitz path-differentiable functions

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Abstract

We show that the vanishing stepsize subgradient method—widely adopted for machine learning applications—can display rather messy behavior even in the presence of favorable assumptions. We establish that convergence of bounded subgradient sequences may fail even with a Whitney stratifiable objective function satisfying the Kurdyka-Lojasiewicz inequality.

Moreover, when the objective function is path-differentiable we show that various properties all may fail to occur: criticality of the limit points, convergence of the sequence, convergence in values, codimension one of the accumulation set, equality of the accumulation and essential accumulation sets, connectedness of the essential accumulation set, spontaneous slowdown, oscillation compensation, and oscillation perpendicularity to the accumulation set.

1 Introduction

In our previous work [5], we investigate the vanishing-step subgradient method applied to a nonsmooth, nonconvex objective function $f$ in the hope of finding

$$\arg\min_{x \in \mathbb{R}^n} f(x).$$

This paper is intended as a companion to [5], as it presents two examples that show that the results obtained there are sharp in several senses. We also aim here to provide insight into the types of dynamics that the subgradient algorithm presents in the asymptotic limit, and we evaluate some of the ideas that are believed to show promise towards a proof of convergence of the algorithm, such as the Kurdyka–Lojasiewicz inequality. We refer the reader to [5] for some discussion of the historical background.

We shall now give some definitions that will allow us to discuss our results.

For a locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$, we denote by $\partial f(x)$ the Clarke subdifferential of $f$ at $x \in \mathbb{R}^n$, that is, the convex envelope of the set of vectors $v \in \mathbb{R}^n$ such that there is a sequence $\{y_i\}_i \subset \mathbb{R}^n$ such that $f$ is differentiable at $y_i$, $y_i \to x$ and $\nabla f(y_i) \to v$.

**Definition 1** (Small-step subgradient method). Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, and $\{\varepsilon_i\}_i$ be a sequence of positive step sizes such that

$$\sum_{i=0}^{\infty} \varepsilon_i = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \varepsilon_i = 0.$$

Given $x_0 \in \mathbb{R}^n$, consider the recursion, for $i \geq 0$,

$$x_{i+1} = x_i - \varepsilon_i v_i, \quad v_i \in \partial f(x_i).$$

Here, $v_i$ is chosen freely among $\partial f(x_i)$. The sequence $\{x_i\}_{i \in \mathbb{N}}$ is called a subgradient sequence.
Since the dynamics of the subgradient method in the case of $f$ locally Lipschitz had been shown\textsuperscript{6} to be too unwieldy, in \textsuperscript{7} we instead discuss the dynamics of the subgradient method for $f$ path-differentiable.

**Definition 2 (Path-differentiable functions).** A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is path-differentiable if for each Lipschitz\textsuperscript{4} curve $\gamma : \mathbb{R} \to \mathbb{R}^n$, for almost every $t \in \mathbb{R}$, the composition $f \circ \gamma$ is differentiable at $t$ and the derivative is given by

$$(f \circ \gamma)'(t) = v \cdot \gamma'(t)$$

for all $v \in \partial f(\gamma(t))$.

**Definition 3 (Weak Sard condition).** We will say that $f$ satisfies the weak Sard condition if it is constant on each connected component of its critical set $\text{crit} f = \{x \in \mathbb{R}^n : 0 \in \partial f(x)\}$.

Recall that the accumulation set $\text{acc}\{x_i\}_i$ of the sequence $\{x_i\}_i$ is the set of points $x \in \mathbb{R}^n$ such that, for every neighborhood $U$ of $x$, the intersection $U \cap \{x_i\}_i$ is an infinite set. Its elements are known as limit points.

**Definition 4 (Essential accumulation set).** Given sequences $\{x_i\}_i \subset \mathbb{R}^n$ and $\{\varepsilon_i\}_i \subset \mathbb{R}_{>0}$, the essential accumulation set $\text{ess acc}\{x_i\}_i$ is the set of points $x \in \mathbb{R}^n$ such that, for every neighborhood $U$ of $x$,

$$\limsup_{N \to +\infty} \sum_{0 \leq i < N} \varepsilon_i \left( \sup_{x_i \in U} \varepsilon_i \right) > 0.$$  \hspace{1cm} (1)

**Definition 5 (Whitney stratifiable functions).** Let $X$ be a nonempty subset of $\mathbb{R}^m$ and $0 < p \leq +\infty$. A $C^p$ stratification $\mathcal{X} = \{X_i\}_{i \in I}$ of $X$ is a locally finite partition of $X = \bigcup_i X_i$ into connected submanifolds $X_i$ of $\mathbb{R}^m$ of class $C^p$ such that for each $i \neq j$

$$\overline{X_i} \cap X_j \neq \emptyset \implies X_j \subset \overline{X_i} \setminus X_i.$$

A $C^p$ stratification $\mathcal{X}$ of $X$ satisfies Whitney’s condition A if, for each $x \in \overline{X_i} \cap X_j$, $i \neq j$, and for each sequence $\{x_k\}_k \subset X_i$ with $x_k \to x$ as $k \to +\infty$, and such that the sequence of tangent spaces $\{T_{x_k}X_i\}_k$ converges (in the usual metric topology of the Grassmanian) to a subspace $V \subset T_x \mathbb{R}^m$, we have that $T_x X_j \subset V$. A $C^p$ stratification is Whitney if it satisfies Whitney’s condition A.

With the same notations as above, a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is Whitney $C^p$-stratifiable if there exists a Whitney $C^p$ stratification of its graph as a subset of $\mathbb{R}^{n+k}$.

**Summary of the results.** Let

- $n > 0$,
- $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz, path-differentiable function,
- the sequence $\{\varepsilon_i\}_i \subset \mathbb{R}_{>0}$ of step sizes satisfy $\lim_{i \to +\infty} \varepsilon_i = 0$, and
- $\{x_i\}_i$ be a bounded subgradient sequence with stepsizes $\{\varepsilon_i\}_i$.

\footnote{In other parts of the literature (see e.g. \textsuperscript{4}), this definition is given with absolutely-continuous curves, and this is equivalent because such curves can be reparameterized (for example, by arclength) to obtain Lipschitz curves, without affecting their role in the definition.}
The main questions we address here are the following:

Q1. Does the sequence \( \{x_i\} \) converge in general?

While it is tempting to hope for the sequence to converge since we have proven \[5, \text{Theorems 6(i),7(i),7(ii)}\] that the sequence slows down indefinitely, in Section 2 we give an example in which the sequence forever accumulates around a circle and never converges. The function we construct satisfies the weak Sard condition, so even with that assumption there is no hope for the convergence of \( \{x_i\} \). The function also satisfies the Kurdyka–Lojasiewicz inequality; see Q9.

In contrast, it can be proven \[3\] that if \( f \) satisfies the weak Sard condition and the Kurdyka–Lojasiewicz inequality, then the flow lines \( x: \mathbb{R} \to \mathbb{R}^n \) of the continuous-time subgradient flow, which satisfy

\[
-\dot{x}(t) \in \partial f(x(t)),
\]

always converge. Thus the example in Section 2 shows that the convergence of the continuous-time process may not guarantee the convergence of the discrete subgradient sequence.

Q2. Do the values \( \{f(x_i)\} \) converge for a general path-differentiable function \( f \)?

Although this convergence can be proved when \( f \) satisfies the weak Sard condition \[5, \text{Theorem 7(v)}\], the example in Section 3 shows that the convergence of the values \( f(x_i) \) fails in general. In fact, in that example we have \( f(\text{acc}\{x_i\}) = [0,1] = f(\text{ess acc}\{x_i\}) \).

Q3. Must \( \text{acc}\{x_i\} \) be a subset of \( \text{crit} f \) in general?

The example in Section 3 shows that in general the set \( \text{acc}\{x_i\} \setminus \text{ess acc}\{x_i\} \) may not intersect \( \text{crit} f \). This contrasts with results that \( \text{ess acc}\{x_i\} \) is always contained in \( \text{crit} f \) \[5, \text{Theorem 6(iii)}\], and that \( \text{acc}\{x_i\} \) is contained in \( \text{crit} f \) if \( f \) satisfies the weak Sard condition \[5, \text{Theorem 7(iv)}\].

Q4. Do we always have \( \text{ess acc}\{x_i\} = \text{acc}\{x_i\} \)?

No, in the example in Section 3 we have a situation in which the set \( \text{ess acc}\{x_i\} \) is strictly smaller than \( \text{acc}\{x_i\} \). We do not know the answer to this question with more stringent assumptions, such as \( f \) satisfying the weak Sard condition.

Q5. Can the essential accumulation set \( \text{ess acc}\{x_i\} \) be disconnected?

Yes. Although for simplicity we do not construct an example here, the reader will surely understand that the example in Section 3 can be easily modified (by taking several copies of \( \Gamma \) and joining them with curves having roles similar to the one played by \( J \)) to produce a situation in which \( \text{ess acc}\{x_i\} \) is disconnected. This contrasts with the fact that \( \text{acc}\{x_i\} \) is always connected because \( \text{dist}(x_i, x_{i+1}) \leq \varepsilon_i \text{Lip}(f) \to 0 \) as \( i \to +\infty \), where \( \text{Lip}(f) \) is the Lipschitz constant for \( f \) in a compact set that contains \( \{x_i\} \).

Q6. A certain spontaneous slowdown phenomenon is proved in \[5, \text{Theorem 6(i)}\] of the fragments of the subgradient sequence as (roughly speaking) it traverses the piece of \( \text{acc}\{x_i\} \) starting at a point \( x \) and ending at another point \( y \), such that \( x, y \in \text{acc}\{x_i\} \) verify \( f(x) \leq f(y) \) (see the precise statement below).

Is there any hope of proving, for general \( f \), that this phenomenon always occurs uniformly throughout the accumulation set, regardless of the restriction \( f(x) \leq f(y) \)?
No, the example in Section 3 shows that the speed of drift of the sequence can remain high forever between points that do not satisfy this inequality.

To be precise, the result in [5, Theorem 6(i)] is this: Let \( x \) and \( y \) be two distinct points in \( \text{acc}\{x_i\} \), satisfy \( f(x) \leq f(y) \), and take subsequences \( \{x_{i_k}\}_k \) and \( \{x'_{i_k}\}_k \) such that \( x_{i_k} \to x \), \( x'_{i_k} \to y \) as \( k \to +\infty \), and \( i'_{k} > i_{k} \) for all \( k \). Then

\[
\lim_{k \to +\infty} \sum_{p=i_k}^{i'_{k}} \varepsilon_p = +\infty.
\]

This is verified independently of the subsequences taken.

On the other hand, the endpoints \( x \) and \( y \) of the curve \( J \) in the example in Section 3 are contained in \( \text{acc}\{x_i\} \), satisfy \( f(x) > f(y) \), and we can take subsequences \( \{x_{i_k}\}_k \) and \( \{x'_{i_k}\}_k \) converging to \( x \) and \( y \), respectively, and with \( i'_{k} > i_{k} \), for which we additionally have

\[
\sup_{k} \sum_{p=i_k}^{i'_{k}} \varepsilon_p < +\infty.
\]

Q7. Does the oscillation compensation phenomenon described in [5, Theorem 6(ii)] occur on the entire accumulation set in general?

While we are able to prove an oscillation compensation result [5, Theorem 7(iii)] that holds throughout \( \text{acc}\{x_i\} \), with the assumption that \( f \) satisfies the weak Sard condition, the example in Section 3 shows that in general, in the absence of the weak Sard condition, there need not be any oscillation compensation on \( \text{acc}\{x_i\} \setminus \text{ess acc}\{x_i\} \), which in the example corresponds to the curve \( J \). For a precise statement, please refer to C7 in Section 3.

Q8. Can the perpendicularity of the oscillations of \( \{x_i\} \), verified around \( \text{ess acc}\{x_i\} \) in [5, Remark 9] be proved on the entire accumulation set?

No, as is shown in the example of Section 3 this may fail on \( \text{acc}\{x_i\} \setminus \text{ess acc}\{x_i\} \) for general \( f \). The perpendicularity can, however, be proved to happen on \( \text{ess acc}\{x_i\} \) or, if \( f \) satisfies the weak Sard condition, on all of \( \text{acc}\{x_i\} \); see [5, Remark 9].

Q9. Would it be possible to prove the convergence of \( \{x_i\} \) if \( f \) is Whitney stratifiable (cf. Definition 3) and satisfies a Kurdyka–Lojasiewicz inequality?

No; more assumptions are necessary. The objective function \( f \) in the example in Section 2 is Whitney \( C^\infty \) stratifiable and satisfies a Kurdyka–Lojasiewicz inequality of the form

\[
\|\nabla f(x)\| \geq \frac{1}{2} \quad \text{for all } x \notin \text{crit } f,
\]

but we also construct a bounded subgradient sequence that fails to converge. However, in the case of \( f \) smooth, the Kurdyka–Lojasiewicz inequality does suffice to prove convergence of the subgradient method [1].

Q10. Recall that the Hausdorff dimension of a set \( X \) is

\[
\dim X = \inf\{d \in \mathbb{R} : \mathcal{H}^d(X) = 0\},
\]

where \( \mathcal{H}^d(X) \) is the \( d \)-dimensional Hausdorff outer measure,

\[
\mathcal{H}^d(X) := \liminf_{r \to 0-} \{\sum_i r_i^d : \text{there is a cover of } X \text{ by balls of radii } 0 < r_i < r\}.
\]
Must the Hausdorff dimension of the accumulation set of \( \{ x_i \}_i \) be \( \dim \text{acc}\{ x_i \}_i \leq n - 1 \)?

No, the example in Section 3 gives a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and a subgradient sequence \( \{ x_i \}_i \) such that the Hausdorff dimension satisfies

\[
1 < \dim \text{acc}\{ x_i \}_i = \dim \text{ess acc}\{ x_i \}_i \leq \frac{\log 4}{\log 3} \approx 1.26,
\]

and actually depends on a parameter \( \alpha \) that can be tweaked to produce any value of the Hausdorff dimension in this range; see Lemma 11. Although the function \( f \) in that example does not satisfy the weak Sard condition, the example can be easily modified (by changing the value of \( f \) on \( \Gamma \cup J \) to a constant) to satisfy also this condition and still have the dimension attain any value in the range (2).

This contrasts with the result [5, Remark 10] that, if \( f \) is Whitney \( C^m \) stratifiable, then

\[
\dim \text{acc}\{ x_i \}_i \leq n - 1.
\]

Q11. Can the set of limit closed measures of the interpolant curve be infinite?

Yes. This is the case in the situation of the example in Section 2 (and also in the example of Section 3 but for simplicity we will not prove it in that case). Please refer to Section 2.2 for the full definitions and an explanation.

Q12. Would the answer to any of the previous questions Q1–Q11 be different if one enforced that the sequence be contained in the (full measure) set of differentiability points of the function \( f \)?

No, all our claims are based on constructive existence proofs of subgradient sequences \( \{ x_i \}_i \) such that each point \( x_i \) is contained in a ball in which the objective function \( f \) is \( C^\infty \).

Notation. Given two sets \( A \) and \( B \), denote by \( B^c \) the complement of \( B \) and by \( A \setminus B = A \cap B^c \). Let \( n \) be a positive integer, and let \( \mathbb{R}^n \) denote \( n \)-dimensional Euclidean space. For two vectors \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) in \( \mathbb{R}^n \), we let \( u \cdot v = \sum_{i=1}^n u_i v_i \) and \( \| u \| = \sqrt{u \cdot u} \). We will denote the gradient of \( f \) at \( x \) by \( \nabla f(x) \). We denote \( \log_b a = \log a / \log b \) the logarithm of \( a \) in base \( b \). We denote the unit circle by \( S^1 \), and the open ball of radius \( r \) centered at \( x \) by \( B_r(x) \). A number with a subindex \( b \) is in base \( b \); for example, \( 0.129 = 1/9 + 2/81 \). For a Lipschitz function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \), we denote by

\[
\text{Lip}(g) = \sup_{x, y \in \mathbb{R}^n} \frac{\| g(x) - g(y) \|}{\| x - y \|}.
\]

2 Example on the circle

We construct a path-differentiable function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and a subgradient sequence \( \{ x_i \}_i \) that does not converge and instead accumulates around a circle. The function \( f \) additionally has the property that it is Whitney \( C^\infty \) stratifiable and satisfies a Kurdyka-Łojasiewicz inequality. The construction is given in Section 2.1 and the main properties are collected in Proposition 6.

In the context of the theory developed in [5, §4.2], it is also interesting that the dynamics in this example induce, through the interpolant curve, infinitely-many limiting closed measures. This is discussed in Section 2.2.
2.1 Construction and main properties

For \(i \geq 2\), let (see Figure 1)

\[ x_i = [1 + (-1)^i] (\cos \vartheta_i, \sin \vartheta_i) \quad \text{with} \quad \vartheta_i = \frac{1}{2} \sum_{k=2}^{i} \frac{1}{k \log k} \]

and

\[ \varepsilon_i = \|x_{i+1} - x_i\|, \quad v_i = -\frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|}, \]

so that \(x_{i+1} = x_i - \varepsilon_i v_i\). Note that \(\varepsilon_i\) satisfies, for large \(i\),

\[ \frac{2}{i+1} < \varepsilon_i < \frac{1}{i} + \frac{1}{i+1} + \frac{1}{i \log i} < \frac{2}{i}, \]

so that \(\varepsilon_i \to 0\), \(\sum_i \varepsilon_i = +\infty\), and \(\sum_i \varepsilon_i^2 < +\infty\).

We want to obtain a function \(f\) that is very close to the function \(\phi\) given by the distance to the circle,

\[ \phi(x) = |1 - \|x\||, \]

yet satisfies

\[ \nabla f(x) = v_i \quad \text{for all} \quad x \in B_{1/2i}(x_i). \quad (3) \]

Let \(\psi: \mathbb{R}^2 \to [0, 1]\) be a \(C^\infty\) function with radial symmetry (i.e. \(\psi(x) = \psi(y)\) for \(\|x\| = \|y\|\)), such that \(\psi(x) = 1\) for \(x \in B_1(0)\), \(\psi(x) = 0\) for \(\|x\| \geq 2\), and decreases monotonically on rays emanating from the origin. Let

\[ \psi_i(x) = \psi(2^i (x - x_i)), \]

so that \(\psi_i\) equals 1 on \(B_{1/2^i}(x_i)\) and vanishes outside \(B_{1/2^{i-1}}(x_i)\). Note that the supports of the functions \(\psi_i\) are pairwise disjoint.

Define

\[ V_i(x) = (x - x_i) \cdot v_i + \frac{1}{i}. \]

**Proposition 6.** Let \(i_0 \geq 2\) and

\[ f(x) = \left(1 - \sum_{i=i_0}^{\infty} \psi_i(x)\right) \phi(x) + \sum_{i=i_0}^{\infty} \psi_i(x) V_i(x). \quad (4) \]

Then we have:
i. The function $f$ is $C^\infty$ on $\mathbb{R}^2 \setminus S^1$.

ii. The function $f$ satisfies (3), so that $\{x_i\}_i$ is a subgradient sequence with stepsizes $\{\varepsilon_i\}_i$.

iii. Let $p$ be a point in the unit circle, then $\partial f(p) = \{ap : a \in [-1, 1]\} = \partial \phi(p)$.

iv. The critical set of $f$ is $\text{crit } f = S^1 \cup \{0\}$.

v. The function $f$ is Lipschitz path-differentiable.

vi. The function $f$ is Whitney $C^\infty$ stratifiable.

vii. If $i_0$ is large enough, $f$ satisfies a Kurdyka-Lojasiewicz inequality of the form

$$\|\nabla f(x)\| > 1/2$$

for $x \in \mathbb{R}^2 \setminus \text{crit } f$.

To prove the proposition we need

**Lemma 7.** For $i$ large enough we have the estimates

$$\left\| v_i - (-1)^i \frac{x_i}{\|x_i\|} \right\| \leq \frac{6}{\log i} \tag{5}$$

and, if $\text{dist}(x_i, y) \leq 2^{1-i}$,

$$\left\| \frac{x_i}{\|x_i\|} - \frac{y}{\|y\|} \right\| \leq 3 \text{dist}(x_i, y). \tag{6}$$

**Proof.** To show (5), first observe that, in the definition of $x_i$, the jump in the direction tangential to the circle has magnitude $\dot{\vartheta}_i - \dot{\vartheta}_{i-1} = 1/i \log i$, while the jump in the direction normal to the circle has magnitude $1/i + 1/i + 1$. It follows that

$$\frac{1}{2i \log i} \leq (x_{i+1} - x_i) \cdot \frac{x_i}{\|x_i\|} \leq \frac{2}{i \log i},$$

$$\frac{2}{i+1} \left( 1 - \frac{1}{\log i} \right) \leq \frac{2}{i+1} \sqrt{1 - \frac{1}{\log^2 i}} \leq (x_{i+1} - x_i) \cdot \frac{x_i}{\|x_i\|} \leq \|x_{i+1} - x_i\|,$$

where $(a, b)^\perp = (-b, a)$ and we have used the Cauchy–Schwarz inequality. Since $1/i \leq \varepsilon_i = \|x_{i+1} - x_i\| \leq 2/i$, together with $v_i = -(x_{i+1} - x_i)/\varepsilon_i$ and the estimates above, we also have

$$\frac{1}{2 \log i} \leq \left| v_i \cdot \frac{x_i}{\|x_i\|} \right| \leq \frac{2}{\log i}, \tag{7}$$

and

$$\frac{i}{i+1} \left( 1 - \frac{2}{\log i} \right) \leq \left| v_i \cdot \frac{x_i}{\|x_i\|} \right| \leq 1. \tag{8}$$

The estimate (5) follows from (7) and (8):

$$\left\| v_i - (-1)^i \frac{x_i}{\|x_i\|} \right\| = \sqrt{\left( v_i \cdot \frac{x_i}{\|x_i\|} - 1 \right)^2 + \left( v_i \cdot \frac{x_i}{\|x_i\|} \right)^2} \leq \sqrt{\left( \frac{i}{i+1} \left( \frac{2}{\log i} + 1 \right) - 1 \right)^2 + \left( \frac{2}{\log i} \right)^2} \leq \frac{4}{\log i} + \frac{2}{i} \leq \frac{6}{\log i}.$$
Estimate (5) can be deduced by letting \( w = y - x_i \), so that \( \|w\| = \text{dist}(x, y) \) and observing that
\[
1 - \frac{2}{i} \leq \|x_i\| \leq 1 + \frac{2}{i} \quad \text{and} \quad \|x_i + w\| = \|y\| \geq 1 - \frac{2}{i},
\]
which means that, for \( i \) large, we have
\[
\left\| \frac{x_i}{\|x_i\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x_i}{\|x_i\|} - \frac{x_i + w}{\|x_i + w\|} \right\|
\leq 2 \|x_i\| \|w\| \|x_i + w\| \leq 2 \frac{1 + \frac{2}{i}}{(1 - \frac{2}{i})^2} \|w\| \leq 3 \|w\|. \hfill \Box
\]

**Proof of Proposition 6.** Item [i] becomes evident once we realize that the sum (4) reduces to \( f(x) = (1 - \psi_i(x))\phi(x) + \psi_i(x)V_i(x) \) for \( x \) in \( B_{1/2^i-1}(x_i) \) and to \( f(x) = \phi(x) \) elsewhere, since \( \psi_i, V_i \) and \( \phi \) are \( C^\infty \) on \( \mathbb{R}^2 \setminus (S^1 \cup \{0\}) \).

To prove item [ii], note that, for \( x \in B_{1/2^i}(x_i) \), we have \( f(x) = V_i(x) \) and \( \nabla f(x) = \nabla V_i(x) = v_i \) so that \( x_{i+1} - x_i = -\epsilon_i v_i = -\epsilon_i \nabla f(x_i) \).

In order to prove item [iii] let \( p \in S^1 \). Let us first show that, as \( y \in \mathbb{R}^2 \) with \( \|y\| < 1 \) tends \( p \), \( \nabla f(y) \to -p \). If \( y \notin \bigcup_i B_{1/2^i-1}(x_i) \) is near \( p \), then
\[
\|\nabla f(y) + p\| = \|\nabla \phi(y) + p\| = \left\| -\frac{y}{\|y\|} + p \right\|,
\]
which clearly tends to 0 as \( y \to p \). If \( y \in B_{1/2^i-1}(x_i) \) (and since \( \|y\| < 1 \) we must have \( i \) odd), then we have, by a Taylor expansion, \( \nabla \phi(x_i) = -x_i/\|x_i\| \), the Cauchy–Schwarz inequality, and (5),
\[
|V_i(y) - \phi(y)| = \left| (y - x_i) \cdot v_i + \frac{1}{2} - \phi(x_i) - \nabla \phi(x_i) \cdot (y - x_i) \right| + 2 \|y - x_i\|^2
\leq 2 \|y - x_i\| \left| v_i + \frac{x_i}{\|x_i\|} \right| + 1 + \frac{1}{i} + 2 \|y - x_i\|^2
\leq 2 \|y - x_i\| \left| v_i + \frac{x_i}{\|x_i\|} \right| + 2 \left( \frac{1}{2^i - 1} \right)^2
\leq 2 \frac{1}{2^i - 1} \frac{6}{\log i} = \frac{12}{2^i - 1} \log i
\]
and, since also \( \nabla \phi(y) = -y/\|y\| \), \( \nabla V_i(y) = v_i \), \( \text{Lip}(\nabla \psi_i) = 2 \text{Lip}(\nabla \psi) \), \( |\psi_i(y)| \leq 1 \), the triangle
inequality, the estimates from Lemma 7 and $y \in B_{1/2^{i-1}}(x_i)$,
\[
\| \nabla f(y) + \frac{y}{\|y\|} \| = \left\| \nabla[(1 - \psi_i(y))\phi(y) + \psi_i(y)V_i(y)] + \frac{y}{\|y\|} \right\|
\]
\[
= \left\| \nabla\psi_i(y)(V_i(y) - \phi(y)) + \psi_i(y) \left( \nabla V_i(y) + \frac{y}{\|y\|} \right) \right\|
\]
\[
\leq \text{Lip}(\nabla \psi_i)|V_i(y) - \phi(y)| + \left\| v_i + \frac{y}{\|y\|} \right\|
\]
\[
\leq 2\text{Lip}(\nabla \psi)|V_i(y) - \phi(y)| + \left\| v_i + \frac{x_i}{\|x_i\|} \right\| + \left\| \frac{x_i}{\|x_i\|} - \frac{y}{\|y\|} \right\|
\]
\[
\leq 2\text{Lip}(\nabla \psi) \frac{12}{2^{i-1}\log i} + \frac{6}{\log i} + \frac{3}{2^{i-1}}
\]
\[
= (12\text{Lip}(\nabla \psi) + 6) \frac{2}{\log i} + \frac{3}{2^{i-1}} \to 0 \quad \text{as} \ i \to +\infty.
\]
It follows from the triangle inequality that
\[
\| \nabla f(y) + p \| \leq \left\| \nabla f(y) + \frac{y}{\|y\|} \right\| + \left\| p - \frac{y}{\|y\|} \right\|
\]
so that, as $y \to p$ with $\|y\| < 1$, we have $\nabla f(y) \to -p$. A similar argument yields that, as $y \to p$ with $\|y\| > 1$, we have $\nabla f(y) \to p$, which proves item (iii).

To prove item (v) note that, by items (i) and (iii), if a Lipschitz curve $\gamma$ satisfies either $\gamma(t) \in S^1$ and $\gamma'(t)$ tangent to $S^1$ or $\gamma(t) \in \mathbb{R}^2 \setminus S^1$, then indeed we have $(f \circ \gamma)'(t) = v \circ \gamma'(t)$ for all $v \in \partial f(\gamma(t))$. On the other hand, the set of points $t$ in the domain of $\gamma$ such that $\gamma(t) \in S^1$ but $\gamma'(t)$ is not tangent to $S^1$ is at most countable (these points $t$ can be covered by disjoint open sets) and hence has measure zero; see also the proof of [8, Theorem 5.3]. It follows that the chain rule condition for path differentiability is satisfied for almost all $t$. Since this is true for all curves $\gamma$, $f$ is path-differentiable.

Item (vi) is clear in view of items (i) and (iii).

If follows from item (iii) that $S^1 \subseteq \text{crit } f$. Recall $f = \phi$ in a neighborhood of $0$ and $0 \in \text{crit } \phi$, so $0 \in \text{crit } f$. If $x \notin \bigcup_i B_{1/2^{i-1}}(x_i)$, then $\|\nabla f(x)\| = \|\nabla \phi(x)\| = 1$ and $\nabla f(x)$ is the only element of $\partial f(x)$, so $x \notin \text{crit } f$. If $x \in B_{1/2^{i-1}}(x_i)$, then, taking $i_0$ large enough, we can ensure that, for $i \geq i_0$, we have, by the triangle inequality and the estimates above,
\[
\|\nabla f(x)\| \geq \left\| \frac{x}{\|x\|} - \nabla f(x) - \frac{x}{\|x\|} \right\| > \frac{1}{2}.
\]
This settles items (vi) and (vii).

\[\Box\]

2.2 Limiting measures

Here we recall some of the theory of [5, Section 4], and we show that in the example constructed in Section 2.1, the set of limiting measures is uncountable. We also compute those measures explicitly.

The interpolating curve and its associated closed measures. Given a measure $\xi$ on $X$ and a measurable map $g: X \to Y$, the pushforward $g_*\xi$ is defined to be the measure on $Y$ such that, for $A \subset Y$ measurable, $g_*\xi(A) = \xi(g^{-1}(A))$.

Recall that the support $\text{supp } \mu$ of a positive Radon measure $\mu$ on $\mathbb{R}^n$ is the set of points $x \in \mathbb{R}^n$ such that $\mu(U) > 0$ for every neighborhood $U$ of $x$. It is a closed set.
Definition 8. A compactly-supported, positive, Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ is closed if, for all functions $f \in C^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \nabla f(x) \cdot v \, d\mu(x, v) = 0.$$ 

Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection $\pi(x, v) = x$. To a measure $\mu$ in $\mathbb{R}^n \times \mathbb{R}^n$ we can associate its projected measure $\pi_* \mu$. We have $\text{supp } \pi_* \mu = \pi(\text{supp } \mu) \subseteq \mathbb{R}^n$.

Let $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be the curve linearly interpolating the sequence $\{x_i\}_i$ with $\gamma(t_i) = x_i$ for $t_i = \sum_{j=0}^{i-1} \epsilon_j$ and $\gamma'(t) = v_i$ for $t < t_i < t_{i+1}$.

For a bounded set $B \subseteq \mathbb{R}_{\geq 0}$, we define a measure on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\mu_{|B} = \frac{1}{|B|} \int \mu_{\pi} = \int_{\mathbb{R}^n} (\gamma, \gamma') \, \text{Leb}_B,$$

where $|B| = \int_{\mu} 1 \, dt$ is the length of $B$, and $\text{Leb}_B$ is the Lebesgue measure on $B$ (so that $\text{Leb}_B(A) = |A|$ for $A \subseteq B$ measurable). If $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is measurable, then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi \, d\mu_{|B} = \frac{1}{|B|} \int_B \varphi(\gamma(t), \gamma'(t)) \, dt.$$ 

Lemma 9 ([5, Lemmas 20 and 21]). In the weak* topology, the set of limit points of the sequence $\{\mu_{|[0,N]}\}_N$ is nonempty, and its elements are closed probability measures. Also,

$$\bigcup_{\mu \in \text{acc } \{\mu_{|[0,N]}\}_N} \pi(\text{supp } \mu) = \text{ess } \text{acc } \{x_i\}_i.$$ 

A measure $\mu$ on $\mathbb{R}^n \times \mathbb{R}^n$ can be fiberwise disintegrated as

$$\mu = \int_{\mathbb{R}^n} \mu_x \, d\pi_* \mu(x),$$

where $\mu_x$ is a probability on $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. We define the centroid field $\bar{v}_x$ of $\mu$ by

$$\bar{v}_x = \int_{\mathbb{R}^n} v \, d\mu_x(v).$$

An important intermediate result of [5] is

Theorem 10 (Subgradient-like closed measures are trivial [5, Theorem 23]). Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a path-differentiable function. Let $\mu$ be a closed measure on $\mathbb{R}^n \times \mathbb{R}^n$, and assume that every $(x, v) \in \text{supp } \mu$ satisfies $-v \in \partial^c f(x)$. Then the centroid field $\bar{v}_x$ of $\mu$ vanishes for $\pi_* \mu$-almost every $x$.

Analysis of the example. Let $\gamma$ be the interpolating curve of the sequence $\{x_i\}_i$, as defined in Section 2.1. In this example, the set of limit points of the sequence $\{\mu_{|[0,N]}\}_N$ consists of all measures on $T\mathbb{R}^2$ given by

$$\mu_{\theta_0} = \int_{\theta_0}^{\theta_0 + 2\pi} \frac{\delta_{(r(\theta), r(\theta))} + \delta_{(r(\theta) + \pi, r(\theta))}}{2} \, \frac{e^{\theta - \theta_0}}{1 - e^{-2\pi}} \, d\theta, \quad \theta_0 \in \mathbb{R},$$

(9)

where $r(\theta) = (\cos \theta, \sin \theta)$ and $\delta_{(u,v)}$ denotes the Dirac delta in $\mathbb{R}^2 \times \mathbb{R}^2$ concentrated at $(u, v)$. This is the measure that captures the dynamics occurring whenever $x_N$ has angle $\vartheta_N$ close to $\theta_0$. Of course,
we have $\mu^{\theta_1} = \mu^{\theta_2}$ if $\theta_2 - \theta_1$ is an integer multiple of $2\pi$, as well as $R^\xi \mu^{\theta_0} = \mu^{\xi + \theta_0}$ for $R^\xi$ the rotation by angle $\xi$.

Before proving (9), we remark that in accordance with Theorem 10 we have, for $x \in S^1$,

$$\mu^{\theta_0}_x = \frac{\delta(x,x) + \delta(x,-x)}{2}$$

and

$$\bar{v}_x = \int_{\mathbb{R}^2} v \, d\mu_x(v) = x - x = 0.$$

Also the conclusion of Lemma 9 is verified: we have

$$\text{ess acc} \{x_i\}_i = \pi(\text{supp } \mu^{\theta_0}) = S^1,$$

and each $\mu^{\theta_0}$ is a closed probability measure.

Let us see how to arrive at (9). From the construction, it is clear that these measures must have the form

$$\int_{\theta_0 - 2\pi}^{\theta_0} \frac{\delta(r(\theta),r(\theta)) + \delta(r(\theta),-r(\theta))}{2} \rho(\theta) \, d\theta$$

for some density $\rho$ on $\mathbb{R}$; the sum of Dirac deltas in (9) can be deduced from the fact that the vectors $v_1$ asymptotically approach $y$ and $-y$ as $x_i \to y \in S^1$ (with a subsequence), together with $\gamma'(t) = v_i$ for $t_i < t < t_{i+1}$.

Let us compute the density $\rho$. Let $I \subset \mathbb{R}$ be an interval of length $0 < \alpha = |I| \leq 2\pi$. Considering $I$ as an arc in the circle, we will write

$$\beta \in I \mod 2\pi$$

if $\beta \in \mathbb{R}$ and there is some $k \in \mathbb{Z}$ such that $\beta + 2\pi k \in I$. Let

$$m_0 = \min \{i : \vartheta_i \in I \mod 2\pi\}.$$

Writing $P \approx Q$ if $P/Q \to 1$ as $N \to +\infty$, if $m < n$ are two integers such that $\alpha = n \pi - m \pi = \sum_{k=m}^{n-1} \frac{1}{k \log k}$, then

$$\alpha \approx \int_m^n \frac{dx}{x} \log x = \log \log n - \log \log m;$$

thus $n \approx m^\alpha$. In other words, the intervals $J \subset \mathbb{N}$ such that $\vartheta_i + 2\pi k \in I \mod 2\pi$ if $i \in J$ are approximately

$$[m_0, m_0^{2k}], [m_0^{2k}, m_0^{2k+2\pi}], [m_0^{4k}, m_0^{4k+4\pi}], \ldots, [m_0^{2k}, m_0^{2k+2\pi}], \ldots$$
Letting $k_N \in \mathbb{N}$ be such that $N = m_0^{\alpha + 2k_N}$, we compute
\[
\sum_{\vartheta \in \mathcal{I}} \vartheta_i \leq N \varepsilon_i \approx \sum_{\vartheta \in \mathcal{I}} \frac{2}{i} \approx \sum_{i=2}^{N} \frac{2}{i} \approx \frac{1}{\log N} \sum_{k=0}^{k_N} \int_{m_0^{2k}}^{
 m_0^{\alpha + 2k\pi}} \frac{dx}{x} \approx \frac{1}{\log N} \sum_{k=0}^{k_N} (e^\alpha - 1)e^{2k\pi} \log m_0 \approx \frac{(e^\alpha - 1) \log m_0 e^{2\pi(k_N + 1)} - 1}{e^{2\pi} - 1} \rightarrow 1 - \frac{1}{1 - e^{-2\pi}} =: p(\alpha)
\]
as $N \to +\infty$. To compute $\varrho$, we apply that to an interval $I$ of the form $[\theta, \theta_0] = [\theta_0 - \alpha, \theta_0]$ and we take the derivative
\[
\varrho(\theta) = \frac{dp(\theta_0 - \theta)}{d\theta} = \frac{d}{d\theta} \frac{1 - e^{-(\theta_0 - \theta)} - e^{\theta_0 - 2\pi}}{1 - e^{-2\pi}}, \quad \theta \in [\theta_0 - 2\pi, \theta_0).
\]

3 Example on a fractal set

In the spirit of Whitney’s counterexample [11] to the Morse–Sard theorem, we construct a function $f: \mathbb{R}^2 \to \mathbb{R}$ and a bounded subgradient sequence $\{x_i\}_i$ satisfying:

C1. $f$ is path-differentiable,

C2. $f(\text{crit } f) \supset f(\text{ess acc } \{x_i\}_i) = f(\text{acc } \{x_i\}_i) = [0, 1],$

C3. The accumulation set $\text{acc } \{x_i\}_i$ is not contained in $\text{crit } f$, and

\[
\text{ess acc } \{x_i\}_i \neq \text{acc } \{x_i\}_i.
\]

C4. $\{x_i\}_i$ and $\{f(x_i)\}_i$ do not converge.

C5. The Hausdorff dimensions of $\text{ess acc } \{x_i\}_i$ and $\text{acc } \{x_i\}_i$ are greater than 1 and satisfy (2).

C6. There are points $x$ and $y$ in $\text{ess acc } \{x_i\}_i \setminus \text{acc } \{x_i\}_i$ such that we can take subsequences $\{x_{i_k}\}_k$ and $\{x_{i_k}'\}_k$ converging to $x$ and $y$, respectively, with $i_k < i_k' < i_{k+1}$ for all $k$ and

\[
\sup_k \sum_{p=i_k}^{i_k'} \varepsilon_p < +\infty.
\]

C7. There is no oscillation compensation on $\text{acc } \{x_i\}_i \setminus \text{ess acc } \{x_i\}_i$. This means, precisely, that there is a continuous function $Q: \mathbb{R}^n \to [0, 1]$ such that

\[
\lim_{N \to +\infty} \left| \frac{\sum_{i=0}^{N} \varepsilon_i Q(x_i)}{\sum_{i=0}^{N} \varepsilon_i Q(x_i)} \right| > 0.
\]
Figure 2: The first three steps $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ in the construction of the set $\Gamma$.

Crucially, since we strive to show that the dynamics on $\text{acc}\{x_i\} \setminus \text{ess acc}\{x_i\}$ may be very different to the one displayed on $\text{ess acc}\{x_i\}$, we are not requiring the condition from [5, Theorem 6(ii)], namely, the existence of a sequence $\{N_i\}$ with

$$\liminf_{j \to +\infty} \frac{\sum_{i=1}^{N_j} \varepsilon_i Q(x_i)}{\sum_{i=0}^{N_j} \varepsilon_i} > 0,$$

which would force the focus to be on the dynamics around $\text{ess acc}\{x_i\}$.

C8. The oscillations near $\text{acc}\{x_i\} \setminus \text{ess acc}\{x_i\}$ are not asymptotically perpendicular to $\text{acc}\{x_i\}$.

Outline. To construct the function $f$, we will first define a fractal curve $\Gamma$ and $f$ on it, aiming to have $\Gamma \subset \text{crit} f$ and $f(\Gamma) = [0,1]$. We will also define a curve $J$ such that $\Gamma \cup J$ is a closed loop and $J$ only intersects $\text{crit} J$ at its endpoints. We will construct an auxiliary path-differentiable function $h$ coinciding with $f$ on the curve $\Gamma$, and in Lemma 12 we will prove some properties of $h$. We will next construct a series of loops $T_0, T_1, T_2, \ldots$ that will help us define the sequence $\{x_i\}$, which we carefully specify so that it is almost a subgradient sequence of $h$. The dynamics of $\{x_i\}$ around $\Gamma$ will mimic that of the sequence in the example of Section 2, and near $J$ it will instead move relatively fast. To obtain $f$, we modify $h$ slightly in a way that ensures that $\{x_i\}$ is a subgradient sequence. In Proposition 13 we show that $f$ has certain properties, which we will finally link, in our concluding remarks, to claims C1–C8 above.

The reader will find this example easier to follow after having looked at the construction of Section 2.1. The role of the function $\phi$ in that construction is taken by the function $h$ in the one presented below.

A fractal curve. Pick $\frac{1}{4} < \alpha \leq \frac{1}{3}$. We begin by constructing a set $\Gamma \subset \mathbb{R}^2$ recursively as illustrated in Figure 2. For the first step, we pick four disjoint squares of side $\alpha$ inside the unit square, and we let $\Gamma_1$ be the closed set consisting of the five disjoint paths joining the left and bottom sides of the unit square with those four squares successively, as in the figure. In each of the following inductive steps, we rescale the set $\Gamma_i$ we had for the previous step and we place new copies inside each of the four squares, perhaps rotated by an angle $\pi/2$, so that the paths making up $\Gamma_1$ connect with those of each rescaled copy of $\Gamma_i$. The set $\Gamma_{i+1}$ is then the union of $\Gamma_1$ with the four rescaled and appropriately rotated copies of $\Gamma_i$. This defines an increasing sequence of sets $(\Gamma_i)_{i \in \mathbb{N}}$ and $\Gamma = \bigcup_i \Gamma_i$. 
We proceed to parameterize $\Gamma$ with a continuous curve $\phi: [0, 1] \to \mathbb{R}^2$. To do this, we will imitate the procedure in the construction of the Cantor staircase. Thus, we first divide $[0, 1]$ into nine contiguous intervals of equal length, namely, the nine intervals (we write in base 9)

$$[0_9, 0.1_9), [0.1_9, 0.2_9), \ldots, [0.7_9, 0.8_9), [0.8_9, 1_9].$$

We define the map $\phi$ on each of the five odd-numbered intervals

$$[0_9, 0.1_9), [0.2_9, 0.3_9), [0.4_9, 0.5_9), [0.6_9, 0.7_9), [0.8_9, 1_9]$$

to map the corresponding interval to one of the intervals making up $\Gamma_1$ (in Figure 2 these are the five blue curves in the left-hand diagram). Then iteratively, at step $i$, we divide each of the remaining intervals into nine equal subintervals, and we map the odd-numbered subintervals into the pieces of $\Gamma_i \backslash \Gamma_{i-1}$. Thus for example the interval $[0.1_9, 0.2_9)$ gets divided into

$$[0.1_9, 0.11_9), [0.11_9, 0.12_9), \ldots, [0.17_9, 0.18_9), [0.18_9, 0.2_9),$$

and the images of the intervals $[0.1_9, 0.11_9)$ and $[0.18_9, 0.2_9)$ will touch the images of the intervals $[0.0_9, 0.1_9)$ and $[0.2_9, 0.3_9)$, but the intervals

$$[0.12_9, 0.13_9), [0.14_9, 0.15_9), [0.16_9, 0.17_9)$$

will not touch the image of the curve defined in the previous step; refer to the middle diagram in Figure 2. The map $\phi$ is the unique continuous extension of the thus-defined function.

The resulting curve $\phi$ has infinite arc length. Indeed at each construction step of the $\Gamma_i$, the paths in $\Gamma_i \backslash \Gamma_{i-1}$ are contained in $4i^{-1}$ squares, each of them contributing in an increase of at least $2\alpha^2$ in the total length. This results in a global increase of at least $(4\alpha)^i/2 > 1/2$ in the $i$-th step.

Let $p: [0, 1] \to \mathbb{N} \cup \{+\infty\}$ be the function that assigns to a number $t$ the first appearance of an even digit after the decimal point in its base 9 expansion, so that for example $p(0_9) = 1$ and $p(0.757823_9) = 4$. Thus if $t \in [0, 1]$ and $p(t) < +\infty$, then $\phi(t) \in \Gamma_{p(t)} \backslash \Gamma_{p(t)-1}$, and if $p(t) = +\infty$ then $\phi(t)$ is a point in the Cantor set $\Gamma \backslash \bigcup \Gamma_i$ at the intersection of all the squares used in the construction.

**Lemma 11.** The Hausdorff dimension of $\Gamma$ is $\log_9 \frac{1}{4}$.

**Proof.** The definition of Hausdorff dimension was recalled in [Q10] in Section [I].

Let $r > 0$. As explained above, the length of $\Gamma_i \backslash \Gamma_{i-1}$ is at least $(4\alpha)^i/2$. Thus, a lower bound on the number of balls of radius $r$ necessary to cover $\Gamma_i \backslash \Gamma_{i-1}$ is $(4\alpha)^i/2r - 1$ balls, for $i$ such that $\alpha^i > r$, i.e., $i < \log_9 r$. We have, for $d > 0$,

$$\mathcal{H}^d(\Gamma) \geq \mathcal{H}^d(\bigcup_i \Gamma_i)$$

$$\geq \liminf_{r \to 0} \sum_{i=1}^{\log_9 r - 1} r^d \left(\frac{(4\alpha)^i}{2r} - 1\right)$$

$$= \liminf_{r \to 0} \frac{1}{2} \sum_{i=1}^{\log_9 r - 1} \frac{(4\alpha)^i}{2r} - 1 - (\log_9 r - 1)r^d$$

$$= \liminf_{r \to 0} \frac{1}{2} \frac{r^{d-1} (4\alpha)^{\log_9 r}}{4\alpha - 1}$$

$$= \liminf_{r \to 0} \frac{1}{2} \frac{1}{4\alpha - 1} \exp((d - 1 + \log_9 (4\alpha)) \log r)$$

$$= \liminf_{r \to 0} \frac{1}{2} \frac{1}{4\alpha - 1} \exp((d - \log_9 \frac{4}{3}) \log r).$$
Hence in order to have $H^d(\Gamma) = 0$ it is necessary that $d > \log_4 \frac{1}{4}$ because this $\lim$ must vanish and $\log r \to -\infty$. This translates to $\dim \Gamma \leq \log_4 \frac{1}{4}$.

Let us prove the opposite inequality. For $r > 0$ we cover $\Gamma_i \setminus \Gamma_{i-1}$ with $A(4\alpha)^i/r$ balls of radius $r$ for $i$ such that $\alpha^i \geq r$; here $A > 0$ is taken so that $4A\alpha^i$ is an upper bound for the contribution of the paths in each of the $4^{i-1}$ squares added. Since $\Gamma \setminus \bigcup \Gamma_i$ is also the intersection of the squares in the construction above, we know that it can be covered by $4^k$ balls of radius $2\alpha^k$, and these balls will cover the remaining part of $\bigcup \Gamma_i$. Hence we have, with $r = 2\alpha^k$ and its consequence $\log_4 r = k + \log_4 2$,

$$H^d(\Gamma) \leq H^d(\Gamma \setminus \bigcup \Gamma_i) + H^d(\bigcup \Gamma_i)$$

$$\leq \liminf_{k \to +\infty} 4^k r^d + \liminf_{k \to +\infty} \sum_{i=1}^{\log_4 r} i^d \frac{A(4\alpha)^i}{r}$$

$$= \liminf_{k \to +\infty} 4^k (2\alpha^k)^d + \liminf_{k \to +\infty} \sum_{i=1}^{k + \log_4 2} (2\alpha^k)^d \frac{A(4\alpha)^i}{2\alpha^k}$$

$$\leq \liminf_{k \to +\infty} 4^k (4\alpha)^{k+d\log_4 2} + \liminf_{k \to +\infty} A(2\alpha^k)^d 2^{\alpha^k} 2^{\frac{1}{\alpha^k} - 1}$$

which vanishes unless $\log 4 + d \log \alpha > 0$, that is, unless $d < \log_4 \frac{1}{4}$. This gives $\dim \Gamma \geq \log_4 \frac{1}{4}$.

**Defining $f$ on $\Gamma \cup J$.** We define $f$ on $\Gamma$ imitating the construction of the Cantor staircase as follows. For a point $q \in \Gamma_i$, we let $t = \phi^{-1}(q)$, and we express $t$ in base 9, so that the first $k = p(t) - 1$ numbers $a_1, a_2, \ldots, a_k$ in the base 9 expansion $x = (a_1 a_2 a_3 \ldots)_9$ are odd. We then let, for $1 \leq i \leq k$, $b_i = (a_i - 1)/2$, and $f(q) = (0.b_1 b_2 \ldots b_k)_4$ in base 4. The values so-assigned for $f$ are illustrated in Figure 3. The reader will convince herself that with this definition, $f$ is constant on each path-connected component of $\bigcup \Gamma_i$ and can be uniquely extended to a continuous function on all of $\Gamma$.

We remark that the function $f \circ \phi: [0,1] \to [0,1]$, just like the Cantor staircase, is continuous but not absolutely continuous; indeed, since it is constant on the intervals where $p$ is constant, its derivative $(f \circ \phi)'$ vanishes almost everywhere on $[0,1]$, yet $f \circ \phi$ is not constant, contradicting the fundamental theorem of calculus, which is valid for absolutely continuous functions.

Let $J \subset \mathbb{R}^2 \setminus ((0,1) \times (0,1))$ be a smooth, non-self-intersecting curve joining the two intersections of $\Gamma$ with the boundary of the unit square. We define $f$ on $J$ to smoothly and strictly monotonously take the values between 0 and 1, keeping $f$ continuous.

**Lipschitz continuity of $f$ on $\Gamma \cup J$.** Let $j: (0,1) \times (0,1) \to \mathbb{N} \cup \{+\infty\}$ be, for each pair of points $s$ and $t$ in $(0,1)$, the position of the first digit of the base-9 expansion $s$ and $t$ that differs; thus for example $j(0.1129, 0.1239) = 2$. Since we have $|s - t| > 9^{-j(s,t)-1}$, as $|s - t| \to 0$ we necessarily have $j(s,t) \to +\infty$.

Note also that if $j(s,t) > 0$, then $\phi(s)$ and $\phi(t)$ must be contained in the same square of side $\alpha^{j(s,t)-1}$. Thus, for some $A, B > 0$, $|\phi(s) - \phi(t)| \geq A\alpha^{j(s,t)}$ and $|f \circ \phi(s) - f \circ \phi(t)| \leq B4^{-j(s,t)}$.

Thus if $x, y \in \Gamma$, letting $s$ and $t$ be such that $\phi(s) = x$ and $\phi(t) = y$, we have $\|x-y\| = |\phi(s) - \phi(t)| \geq \ldots$
Figure 3: Values of $f$ on the set $\Gamma_3$. All numbers are in base 4.
$A \alpha^{j(s,t)}$ or, equivalently, $-j(s,t) \leq -\log_\alpha \|x-y\|$. Also,

$$\ |f(x) - f(y)| = |f \circ \phi(s) - f \circ \phi(t)| \leq B 4^{j(s,t)} \leq B 4^{\log_\alpha \|x-y\|} = \frac{B}{A} \|x-y\|^{\log_\alpha \frac{1}{4}} \leq \frac{B}{A} \|x-y\|.$$ 

because $\alpha > 1/4$ so $\log_\alpha 1/4 > 1$. This, together with the smoothness of $f$ on $J$ implies that $f$ on $\Gamma \cup J$ is Lipschitz. Let $\text{Lip}(f|\Gamma \cup J)$ be the Lipschitz constant of $f$ on $\Gamma \cup J$.

**The auxiliary function $h$.** Let $C$ be a connected component of $J \cup \Gamma_i$ for some $i$, without its endpoints. As such, $C$ is a smooth, non-self-intersecting curve, diffeomorphic to an open interval. As is well known (see for example [10, p. 109]) there exists a tubular neighborhood $W_C$ around $C$, by which we mean specifically:

- there is an open set $W_C \subset \mathbb{R}^2$ that contains $C$,
- there is an open set $U \subset \mathbb{R}^2$ of the form $(a,b) \times (-c,c)$ for some $a,b,c \in \mathbb{R}$, $c > 0$, and
- there is a smooth, bijective function $\varphi_C: U \rightarrow W_C$ such that
  - the map $x \mapsto \varphi_C(x,0)$ is a parameterization of $C$ by arclength,
  - the map $y \mapsto \varphi_C(x,y)$ is a parameterization, by arclength, of the segment perpendicular to $C$ and passing through $x$.

We will refer to $\varphi_C$ as the *chart* of $W_C$, and to the number $c > 0$ as the *thickness* of $W_C$.

The statement of existence of the tubular neighborhoods $W_C$ is obvious if we choose all $\Gamma_i$ and $J$ to be composed of straight line segments and circle arcs, so readers unfamiliar with the general case may assume that this is the case.

**Lemma 12.** There is a function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

i. $h$ is locally Lipschitz and path-differentiable.

ii. $h$ coincides with $f$ on $\Gamma \cup J$.

iii. $h$ is $C^1$ on $\mathbb{R}^2 \setminus (\Gamma \cup J)$.

iv. On a tubular neighborhood $W_C$ of each connected component $C$ of $J \cup \Gamma_i$, $h$ is defined by

$$h(\varphi_C(x,y)) = f(\varphi_C(x,0)) + 2L|y|,$$

where $\varphi_C$ is the chart of $W_C$. Hence $h$ is piecewise $C^\infty$ in $W_C$, with the singular locus of $h$ within $W_C$ coinciding exactly with $C$.

v. Let $L = \text{Lip}(f|\Gamma \cup J)$. If $p \in \Gamma_i$ for some $i$, and if $\mathbf{n}$ is a unit vector normal to $\Gamma_i$ at $p$, then

$$\partial^\mathbf{n} h(p) = \{\lambda \mathbf{n} : -2L \leq \lambda \leq 2L\}, \quad p \in \bigcup_i \Gamma_i.$$
More precisely, the gradients of \( h \) on each side of \( \Gamma_i \) at \( p \) are asymptotically equal to \( 2L\mathbf{n} \) and \( -2L\mathbf{n} \), respectively, pointing away from \( \Gamma_i \).

Similarly, if now \( p \in J \), \( \mathbf{n} \) is a unit vector normal to \( J \) at \( p \), and \( \mathbf{t} \) is the unit vector tangent to \( J \) at \( p \) that points in the clockwise direction (for the loop \( \Gamma \cup J \)) and if \( a > 0 \) is the magnitude of the derivative of \( f|J \) at \( p \), then

\[
\partial^c h(p) = \{-at + \lambda \mathbf{n} : -2L \leq \lambda \leq 2L\}, \quad p \in J.
\]

More precisely, the gradients of \( h \) on each side of \( J \) at \( p \) are asymptotically equal to \( -at + 2L\mathbf{n} \) and \( -at - 2L\mathbf{n} \), respectively, pointing away from \( J \).

vi. The norm of the Hessian of \( h \) is bounded on each connected component of \( W_C \setminus C \), for \( C \) and \( W_C \) as in item [iv].

vii. \( \Gamma \subseteq \text{crit} h \).

This lemma will be proved in Appendix A.

**A skeleton curve for the sequence.** We shall now define a sequence of smooth loops \( T_1, T_2, \ldots \) that will guide the trajectory of the sequence \( \{x_i\}_i \). Figure 4 illustrates the shapes of the first elements of the sequence of closed curves that we now proceed to construct.

The first one, \( T_0 \), will simply be a small loop around the origin, containing \( J = T_0 \setminus ((0,1) \times (0,1)) \) and closing it up with a circular arc contained in \([0,1] \times [0,1]\).

For \( i > 0 \), the path \( T_i \) will be equal to \( \Gamma_i \cup J \) together with some small circular arcs glued to close up the loose ends in such a way that we obtain a smooth loop that does not touch the \( 4i+1 \) smaller squares of side \( \alpha^{i+1} \) involved in the construction of \( \Gamma_{i+1} \).

**Specification of the sequence \( \{x_i\}_i \).** Unlike what we did for the example described in Section 2, we will not try here to define \( \{x_i\}_i \) explicitly; instead, we will take the lesson from that example as to what this sequence should look like. We pick \( \{x_i\}_i \) to be a sequence of distinct points with \( \|x_{i+1} - x_i\| \to 0 \) successively bouncing around each path \( T_1, T_2, \ldots \). Thus, the sequence will start near \( J \), it will go around \( T_0 \) a few times, and while it is at \( J \), it will start going around \( T_1 \), which it will do a few times, and then \( T_2 \), and so on.
Let $L = \text{Lip}(f|_{\Gamma \cup J})$ and let $I_0, I_1, \cdots \subset \mathbb{N}$ be the intervals during which each of the paths $T_j$, respectively. We will choose an initial value $i_0 > 0$ such that the sequence $\{x_i\}_{i=i_0}^{\infty}$ will satisfy:

S1. Not self-accumulating. We require the sequence $\{x_i\}_i$ to be such that, for each $i$, there is some $r > 0$ such that $B_r(x_i) \cap \{x_j\}_{j \neq i} = \emptyset$.

S2. If $i \in I_j$ then

$$\frac{2}{iL} \leq \text{thickness}(W_C)$$

for all connected components $C$ of $\Gamma_j \cup J$.

S3. Bouncing. If $j > 0$ and $i, i + 1 \in I_j$, the points $x_i$ and $x_{i+1}$ are on opposite sides of $T_j$.

S4. Distance to $T_j$. For $i \in I_j$, we require the points $x_i$ to remain at a distance

$$\left| \text{dist}(x_i, T_j) - \frac{1}{iL} \right| \leq \frac{1}{i^2}.$$  

S5. Around $\Gamma_j \cup J$. Recall from Lemma 12 that $h$ is piecewise smooth near $\Gamma_j \cup J$. If $j > 0$, $i \in I_j$ and the closest point of $T_j$ to $x_i$ is $y \in \Gamma_j \cup J$, and if $t$ is the unit vector tangent to $\Gamma_j \cup J$ pointing in the clockwise direction, then we require $h$ to be differentiable at $x_i$ and

$$\left\| [x_{i+1} - x_i + \frac{1}{iL} \nabla h(x_i)] \cdot t - \frac{1}{i \log i} \right\| \leq \frac{1}{i^2}.$$  

S6. Around the circle arcs $T_j \setminus (\Gamma_j \cup J)$. If $j > 0$, $i \in I_j$, and the point $y$ of $T_j$ closest to $x_i$ is in $T_j \setminus (\Gamma_j \cup J)$, and if $t$ is a unit vector tangent to $T_j$ at $y$ pointing in the clockwise direction, we require

$$\frac{3}{iL} \leq (x_{i+1} - x_i) \cdot t \leq \frac{4}{iL}.$$  

S7. Small jumps. For all $i$ in the situation of S5

$$\|x_{i+1} - x_i + \frac{1}{iL} \nabla h(x_i)\| \leq \frac{3}{iL}.$$  

For all $i$ in the situation of S6

$$\|x_{i+1} - x_i\| \leq \frac{6}{iL}.$$  

Let us explain how such a sequence can be constructed. First, we choose $i_0 > 0$ large enough that if $C_1$ is the connected component of $J \cup \Gamma_1$ containing $J$, then $2/i_0 L \leq \text{thickness}(W_{C_1})$. We then choose $x_{i_0}$ in $W_{C_1}$ such that the point of $C_1$ closest to $x_{i_0}$ is in $J$, and such that S4 is satisfied with $j = 0$.

By induction, assuming that for some $i \geq i_0$ we have chosen $x_i$ satisfying S1, S7, we let $x_{i+1}$ be a point in the component on the opposite side of $T_j$ (thus complying with S3) of the nonempty set $X_i$ determined by S4 and S7 together with either S5 or S6, depending on the location of $x_i$. The set $X_i$ is indeed nonempty because the inequality in S4 determines two stripes going parallel to $T_j$, while S5 and S6 determine stripes perpendicular to $T_j$. So they intersect (with at least one connected component of the intersection on each side of $T_j$) as long as the step size is small enough with respect to the curvature of $T_j$; this can be ensured in the case of $j = 0$ by increasing $i_0$, and in the case of $j > 0$ by increasing the amount of times the sequence goes around $T_{j-1}$ before moving on to $T_j$. 

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Although the intersection of the condition in S4 and those of either S5 or S6 may also include points located far from \( x_i \), S7 forces the choose a connected component that is directly ahead along \( T_j \), and it is impossible that the sequence will jump very far. Thus S3–S7 can be complied with.

To see that S1 can be complied with as well, note that, by S4, together with S5 and S6, a ball of radius \( r = 1/2^i \) works automatically once the other conditions have been satisfied. To ensure S2 is true, we let the sequence go around each \( T_j \) a few times until \( i \) grows enough that the inequality in S2 becomes true.

We remark that the precise form of S6 will not be used explicitly, and its only purpose is to keep the sequence moving around the circular arcs \( T_j \setminus (\Gamma_j \cup J) \) at a moderate rate.

**Construction of \( f \).** Choose real numbers \( \{r_i\}_i \subset \mathbb{R} \) such that \( 0 < r_i < 1/i^2 \) and such that the disk \( B_{3r_i}(x_i) \) of radius \( 3r_i \) centered at \( x_i \) does not intersect \( \Gamma \) and all the disks \( B_{3r_i}(x_i) \) are disjoint. This is possible because of our specification S1.

Let \( \psi: \mathbb{R}^2 \to [0, 1] \) be a \( C^\infty \) function with radial symmetry, \( \psi(x) = \psi(y) \) for \( \|x\| = \|y\| \), such that \( \psi(x) = 1 \) for \( x \in B_1(0), \psi(x) = 0 \) for \( \|x\| \geq 2 \), and decreases monotonically on rays emanating from the origin. Let

\[
\psi_i(x) = \psi\left(\frac{x - x_i}{r_i}\right),
\]

so that \( \psi_i \) equals 1 on \( B_{r_i}(x_i) \) and vanishes outside \( B_{2r_i}(x_i) \). Denote by Lip(\( \psi \)) > 0 the Lipschitz constant of \( \psi \), and by Lip(\( \nabla \psi \)) > 0 the Lipschitz constant of its gradient. Note that the supports of the functions \( \psi_i \) are pairwise disjoint and Lip(\( \nabla \psi_i \)) = \( \frac{1}{r_i} \) Lip(\( \nabla \psi \)).

Let

\[
v_i = iL(x_i - x_{i+1}),
\]

and define, for \( h \) as in Lemma 12

\[
V_i(x) = (x - x_i) \cdot v_i + h(x_i).
\]

**Proposition 13.** Let

\[
f(x) = \left(1 - \sum_{i=0}^\infty \psi_i(x)\right)h(x) + \sum_{i=0}^\infty \psi_i(x)V_i(x).
\]

Then we have

i. \( f \) is piecewise \( C^\infty \) in a tubular neighborhood \( W_C \) of each connected component \( C \) of \( \Gamma \cup J \), \( i > 0 \).

ii. \( \{x_i\}_i \) is a subgradient sequence for \( f \) with stepsizes

\[
\varepsilon_i = \frac{1}{iL}.
\]

In particular, \( \sum \varepsilon_i = +\infty \) and \( \sum \varepsilon_i^2 < +\infty \).

iii. \( acc\{x_i\}_i = \Gamma \cup J \).

iv. Let \( p \) be a point in \( \Gamma \cup J \) for some \( i > 0 \). Then

\[
\partial^c f(p) = \partial^c h(p).
\]

v. The critical set of \( f \) contains \( \Gamma \), but \( J \cap \text{crit } f \) consists only of the two endpoints of \( J \).

vi. \( f \) is locally Lipschitz and path-differentiable.
Note that it follows from S3, S4, and S5 and item (v) of Lemma 12 that, if \( y \) as

Thus \( \partial f(x_i) = \nabla V(x_i) = v_i \).

Thus \( \partial f(x_i) = \{ v_i \} \) and

\[
x_i - \varepsilon_i v_i = x_i - \frac{1}{iL} iL(x_{i+1} - x_i) = x_{i+1},
\]

which is the statement of item [ii].

Item (iii) is then clear from the construction of \( \Gamma \) and the loops \( J_j \supset J \), together with the specification S4 that forces the sequence \( \{ x_i \}_{i} \) to get ever closer to \( \Gamma \).

Let us prove item (iv). Fix \( j > 0 \) and \( p \in \Gamma_j \cup J \), and denote by \( C \) the connected component of \( \Gamma_j \cup J \) that contains \( p \). Consider a point \( y \) near \( p \). In particular, we may assume that \( y \) is not in the situation described in S6. If \( y \notin \bigcup_j B_{2r_i}(x_i) \), then \( f = h \) on a neighborhood of \( y \) and we have nothing to prove. Otherwise, we have \( y \in B_{2r_i}(x_i) \) for some \( i > 0 \), and by S2 we may assume \( B_{2r_i}(x_i) \) is contained in the neighborhood \( W_C \) of item (iv) in Lemma 12. Item (v) in Lemma 12 means that the derivative of \( \nabla h \) (the Hessian of \( h \)) is bounded on \( W_C \), which means in particular that \( \nabla h \) is Lipschitz in \( W_C \); in other words, there is some \( K > 0 \), depending only on \( C \) such that, for all \( z \in B_{2r_i}(x_i) \),

\[
\| \nabla h(z) - \nabla h(x_i) \| \leq K \| z - x_i \|.
\]

Note that it follows from S3, S4, and S5 and item (v) of Lemma 12 that, if \( i \) is large enough,

\[
\varepsilon_i \left\| v_i - \nabla h(x_i) - \frac{L}{\log i} t \right\| \leq \frac{2}{i^2}. \tag{13}
\]

By \( (12) \), the Lipschitzity of \( \nabla \psi \), the fact that \( 0 \leq \psi_i \leq 1 \), a Taylor expansion with \( w \) a point in the segment joining \( x_i \) and \( y \), the definition of \( K \), the Cauchy-Schwarz and triangle inequalities, and (13),

\[
\| \nabla f(y) - \nabla h(y) \| = \| \nabla [(1 - \psi_i(y))h(y) + \psi_i(y)V_i(y)] - \nabla h(y) \|
\leq \text{Lip}(\nabla \psi_i)(\|V_i(y) - h(y)\| + \|v_i - \nabla h(y)\|)
\leq \frac{1}{r_i} \text{Lip}(\nabla \psi)(\|h(x_i) + v_i \cdot (y - x_i) - h(x_i) - \nabla h(w) \cdot (y - x_i)\| + \|v_i - \nabla h(y)\|)
\leq \frac{1}{r_i} \text{Lip}(\nabla \psi)(\|v_i - \nabla h(w)\| \|y - x_i\| + \|v_i - \nabla h(y)\|)
\leq \frac{1}{r_i} \text{Lip}(\nabla \psi)(2r_i)(\|v_i - \nabla h(x_i)\| + \|\nabla h(x_i) - \nabla h(w)\|)
\leq 2 \text{Lip}(\nabla \psi)(\|v_i - \nabla h(x_i)\| + 2Kr_i) + \|v_i - \nabla h(x_i)\| + 2Kr_i
\leq (2 \text{Lip}(\nabla \psi) + 1) \left( \frac{4L}{\log i} + 2Kr_i \right) \to 0,
\]

as \( y \to p \) because, in that case, \( i \to +\infty \). So item (iv) follows.
Item (v) follows from item [iv] together with the same being true for \( h \); see items (v) and (vii) in Lemma 12.

Item (vi) follows from item (i) in Lemma 12 the form of (12) on \( \mathbb{R}^2 \setminus (\Gamma \cup J) \), which ensures that the path differentiability of \( h \) is inherited by \( f \) on that region, and from item (iv) above, which ensures the modification (12) of \( h \) does not change the path differentiability property on \( \Gamma \cup J \).

\[ \square \]

Lemma 14. \( \text{ess acc}\{x_i^1\} = \Gamma \).

Proof. We will first show that \( \bigcup_i \Gamma_i \subseteq \text{ess acc}\{x_i^1\} \), and from the fact that \( \text{ess acc}\{x_i^1\} \) is closed it will follow that \( \Gamma \) is contained in it. We use the notation \( P \approx Q \) to mean that \( P/Q \to 1 \).

Let \( j > 0, p \in \Gamma_j \) that is not an endpoint of the connected component \( C \) of \( \Gamma \) containing \( p \), and \( \{N_i\}_i \subset \mathbb{N} \) be a subsequence such that \( \lim_i x_{N_i} = p \). Let \( \alpha > 0 \) be smaller than the distance between \( p \) and the closest of the two endpoints of \( C \). Let also \( \{M_i\}_i \subset \mathbb{N} \) be a subsequence such that \( q = \lim_i x_{M_i} \) is a point on \( C \) at arclength \( \alpha \) from \( p \), \( M_i < N_i \) for all \( i \), and \( \text{dist}(x_k, C) < 2/kL \) for all \( M_i \leq k \leq N_i \).

In view of (S4) for each \( i \) the sequence \( x_{M_i}, x_{M_{i+1}}, \ldots, x_{N_i} \) is bouncing around the segment of \( C \) of length \( \alpha \) that starts at \( q \) and ends at \( p \). By item (v) of Lemma 12 we know that \( \partial^e h \) on the points of \( C \) contains only vectors that are normal to \( C \). Thus, using the modification (12) that \( \text{ess acc}\{x_i^1\} \) is at least 1, \( e^{-\alpha/2} \log N_i \). This proves that \( p \in \text{ess acc}\{x_i^1\} \), and thus also that \( \Gamma \subseteq \text{ess acc}\{x_i^1\} \).

In view of item (iii) of Proposition 13 and the fact that \( \text{ess acc}\{x_i^1\} \subseteq \text{acc}\{x_i^1\} = \Gamma \cup J \), we now need to show that if \( p' \in J \setminus \Gamma \) then \( p' \notin \text{ess acc}\{x_i^1\} \). For such \( p' \) we pick an open ball \( U \) containing \( p' \) such that \( U \cap \Gamma = 0 \) and

\[ \kappa_U := \inf_{x \in U} \text{dist}(0, \partial^e f(x)) > 0, \]

as is possible because of item (iv) of Proposition 13, together with the fact that \( f \) is strictly monotonous on \( J \). Let \( a > 0 \) be the arclength of \( J \cap U \). Then from (S5) it follows that if \( i_1 < i_2 \) are such that for all \( i_1 \leq k \leq i_2 \) we have \( x_k \in U \), while \( x_{i_1-1}, x_{i_2+1} \notin U \), then

\[ 2a \geq \sum_{k=i_1}^{i_2} \varepsilon_k \| v_k \| \geq \sum_{k=i_1}^{i_2} \varepsilon_k \kappa_U \]

For \( \ell > 0 \), let \( p_\ell \) denote the number of times the sequence goes around \( T_\ell \). If \( N > 0 \) is in \( I_j \), so that the sequence is bouncing around \( T_j \), then

\[ \sum_{x_k \in U \cap N} \varepsilon_k \leq \frac{2a}{\kappa_U} \sum_{\ell=0}^{j} p_\ell. \]
On the other hand, to estimate \( j \) as a function of \( N \) we compute a lower bound of the length of the path traversed by \( x_1, \ldots, x_N \),

\[
\sum_{i=1}^{j-1} p_i \text{ arc length } \Gamma_i \geq \sum_{i=1}^{j-1} p_i \left( \frac{(4\alpha)^k}{2} \right) = \sum_{i=1}^{j-1} \frac{(4\alpha)^{i+1} - 1}{2(4\alpha - 1)} \geq \frac{1}{2(1 - 4\alpha)^2}(4\alpha)^{j+1} + O(j).
\]

To turn this lower bound on the length of the path into an lower bound of the number \( N \) of steps we use S5 and the fact that \( \nabla h \) is normal to \( \Gamma_k \), so that we have

\[
\frac{1}{2(1 - 4\alpha)^2}(4\alpha)^{j+1} + O(j) \leq \sum_{k=2}^{N} \frac{1}{k \log k} \approx \log \log N.
\]

Whence

\[
j \leq A \log \log N
\]

for some \( A > 0 \), and (1) can be bounded by

\[
\sum_{k \leq N}^{\mathbb{N}} \frac{\varepsilon_k}{L} \leq \frac{2a/\kappa_U}{(\log N)/L} \sum_{\ell=0}^{j} p_{\ell} = O \left( \frac{\sum_{\ell=0}^{j} p_{\ell}}{e^{\ell j}} \right).
\]

Because of the fractal form of the construction of \( \Gamma \), we see that the thickness of the tubular neighborhoods around the connected components of \( \Gamma_i \cup J \) and those around the connected components around \( \Gamma_{i+1} \cup J \) are related by a factor \( \alpha \). From our calculation above we conclude that, the number of steps it takes to traverse each \( T_j \) increases rapidly, so that in view of S2, we see that \( p_{\ell} \) can be uniformly bounded. This means that \( \sum_{\ell=0}^{j} p_{\ell} \leq C j \) for some \( C > 0 \), and hence, as \( j \to +\infty \),

\[
\sum_{\ell=0}^{j} p_{\ell} e^{\ell j} \leq C j e^{\ell j} \to 0.
\]

This proves that \( J \setminus \Gamma \) is not in ess acc \( \{ x_i \} \), and concludes the proof of the lemma.

**Conclusion.** Claim C4 was proved as item (vi) of Proposition 13.

It follows from item (v) in Proposition 13 and Lemma 14 that \( \Gamma = \text{ess acc} \{ x_i \} \subset \text{crit } f \), and since \( f(\Gamma) = [0,1] \), \( f \) satisfies claim C2.

Claim C3 is true by item (iii) of Proposition 13 and Lemma 14.

Since \( f(\Gamma) = [0,1] \) and \( \{ x_i \} \) bounces endlessly around \( \Gamma \cup J \) by item (iii) in Proposition 13, the sequence \( \{ f(x_i) \} \) also does not converge, which is claim C4.

Claim C5 follows from Lemmas 11 and 14 and item (iii) of Proposition 13.
Claim \( \text{C6} \) requires some analysis. Let \( x \) and \( y \) be distinct points in \( J \) with \( f(x) > f(y) \). Let \( \{x_{ik}\}_k \) and \( \{x_{i'k}\}_k \) be subsequences that converge to \( x \) and \( y \), respectively, and such that \( i_k < i'_k \) for all \( k \). Let \( u : [0, T] \rightarrow J \) be a parameterization of \( J \) such that \( \|u'(t)\| = -(f \circ u)'(t) \) (this determines \( T > 0 \)), so that \( u \) is a gradient curve, that is, \( -u'(t) \in \partial f(u(t)) \). Then it follows from item \( \text{(v)} \) in Lemma 12 item \( \text{(iv)} \) in Proposition 13, and \( \text{S5} \) that the subgradient sequence \( \{x_i\}_i \) goes along \( J \) at about the same speed as the neighboring curve \( u \), so a very rough estimate of the amount of time it takes for it to go between \( x \) and \( y \) is \( \sup_k \sum_{p=i_k}^{i'_k} \varepsilon_p \leq 2T \), which is claim \( \text{C6} \).

Claim \( \text{C7} \) is true because, if we choose the function \( Q \) so that its support intersects \( J \) but not \( \Gamma \), then it follows from item \( \text{(v)} \) in Lemma 12 item \( \text{(iv)} \) in Proposition 13 and \( \text{S5} \) that the averages in the lim inf in (10) asymptotically approach

\[
\left\| \frac{\int_0^T Q(u(t))u'(t)dt}{\int_0^T Q(u(t))dt} \right\| \neq 0,
\]

with \( u \) as in our discussion of claim \( \text{C6} \) above, which immediately implies inequality (10).

Claim \( \text{C8} \) follows immediately from item \( \text{(v)} \) in Lemma 12 and assumptions \( \text{S4} \) and \( \text{S5} \).

### A Proof of Lemma 12

For \( i > 0 \) and a connected component \( C \) of \( J \cup \Gamma_i \), and let \( W_C \) be its tubular neighborhood with chart \( \varphi_C \).

On the tubular neighborhood \( W_C \), we define \( h \) by \( \text{(11)} \). Observe that with this definition, \( h \) is \( C^\infty \) on each of the two connected components of \( W_C \setminus C \), settling items \( \text{(iv)} \) and \( \text{(vi)} \).

Since the coordinates given by the charts \( \varphi_C \) are compatible for the different connected components \( C \) of the sets \( J \cup \Gamma_i \), \( i \in \mathbb{N} \), this defines \( h \) on the closure of the union \( R = \bigcup_C W_C \). In particular \( \Gamma \subset R \). From \( \text{(11)} \), we see that \( h \) is smooth on each connected component of \( R \setminus \Gamma \).

In the following, we will extend \( h \) continuously, so the fact that \( h \) coincides with \( f \) on \( \Gamma \cup J \), item \( \text{(ii)} \) will follow from the observation that, as we see from \( \text{(11)} \), it is true on each of connected component of \( J \cup \bigcup_i \Gamma_i \).

Item \( \text{(v)} \) also follows directly from \( \text{(11)} \) because the components of elements of the Clarke subdifferential in the \( t \) and \( n \) directions coincide with the derivatives in the \( x \) and \( y \) variables, respectively, since

\[
\left\| \frac{\partial \varphi_C}{\partial x}(x,0) \right\| = 1 \quad \text{and} \quad \left\| \frac{\partial \varphi_C}{\partial y}(x,y) \right\| = 1.
\]

It follows from item \( \text{(v)} \) that \( C \cap \Gamma \subset \text{crit } h \) for each connected component \( C \) of \( J \cup \bigcup_i \Gamma_i \), so in order to conclude that \( \Gamma \subset \text{crit } h \), item \( \text{(vii)} \) we observe that \( \Gamma = \bigcup_C (\Gamma \cap C) \) and recall that the graph of the Clarke subdifferential is closed.

Recall

**Lemma 15** (Whitney partition of unity \[ \text{[2 Lemma 2.5]} \). Let \( K \) be a compact subset of \( \mathbb{R}^n \). There exists a countable family of functions \( \phi_i \in C^\infty(\mathbb{R}^n \setminus K) \), \( i \in \mathbb{N} \), such that

1. for each \( x \in \mathbb{R}^n \setminus K \) there are at most \( 3^n \) numbers \( i \in \mathbb{N} \) such that \( x \in \text{supp } \phi_i \),
2. \( \phi_i \geq 0 \) for all \( i \in \mathbb{N} \), and \( \sum_i \phi_i(x) = 1 \) for all \( x \in \mathbb{R}^n \setminus K \),
3. \( 2 \text{dist}(\text{supp } \phi_i, K) \geq \text{diam}(\text{supp } \phi_i) \) for all \( i \in \mathbb{N} \),

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4. there exist constants $C_k > 0$, depending only on $k$ and $n$, such that, if $x \in \mathbb{R}^n \setminus K$, then

$$\|D^k \phi_i(x)\| \leq C_k \left(1 + \frac{1}{\text{dist}(x, K)^{|k|}} \right)$$

where $D^k g$ denotes the $k^{th}$ derivative of $g$.

Let $\{\phi_i\}_i$ be a Whitney partition of unity of $\mathbb{R}^2 \setminus R$, as in Lemma 15 with $K = R$. For each $i$, choose a point $p_i$ in $R$ minimizing the distance to supp $\phi_i$. Although the definition (11) does not give $h$ on a neighborhood of each $p_i$ not on $\Gamma$, we may assume that $\nabla h(p_i)$ is well defined, perhaps after shrinking $R$ slightly. Let $g_i : \mathbb{R}^2 \to \mathbb{R}$ be the affine function given by

$$g_i(x) = \begin{cases} h(p_i) + \nabla h(p_i) \cdot (x - p_i), & p_i \notin \Gamma, \\ h(p_i), & p_i \in \Gamma. \end{cases}$$

Similar to the proof [2] to Whitney’s extension theorem, for $x \in \mathbb{R}^2 \setminus R$ we define

$$h(x) = \sum_i g_i(x) \phi_i(x).$$

On $\mathbb{R}^2 \setminus R$, it is clear that $h$ is smooth, because locally it is a finite sum of smooth functions.

On the other hand, on the set $(\partial R) \setminus \Gamma$ that is the boundary of $R$ with $\Gamma$ removed, by construction $h$ is differentiable and its gradient is continuous. To see why, one can use the same technique as in the well-known proof of Whitney’s extension theorem [2, Theorem 2.3]; we sketch the main ideas. To show that $h$ is differentiable at $r \in (\partial R) \setminus \Gamma$, it is enough to show that

$$|h(x) - h(r) - \nabla h(r) \cdot (x - r)| = O(|x - r|^2)$$

for $x \in \mathbb{R}^2 \setminus R$ such that the point $r$ minimizes the distance from $x$ to $R$. For this, one uses the fact that $h$ is smooth on each connected component of $W_C \setminus (\Gamma \cup J)$, so that for $p_i$ near $r$ we have the Taylor estimates

$$|h(p_i) + \nabla h(p_i) \cdot (r - p_i) - h(r)| = O(|r - p_i|^2)$$

and

$$\|\nabla h(r) - \nabla h(p_i)\| = O(|r - p_i|).$$

These give

$$|h(x) - h(r) - \nabla h(r) \cdot (x - r)|$$

$$= \left| \sum_i g_i(x) \phi_i(x) - h(r) - \nabla h(r) \cdot (x - r) \right|$$

$$\leq \sum_i \phi_i(x) |g_i(x) - h(r) - \nabla h(r) \cdot (x - r)|$$

$$= \sum_i \phi_i(x) |h(p_i) + \nabla h(p_i) (x - p_i) - h(r) - \nabla h(r) \cdot (x - r)|$$

$$\leq \sum_i \phi_i(x) (|h(p_i) + \nabla h(p_i) \cdot (r - p_i) - h(r)| + |\nabla h(p_i) \cdot (x - r) - \nabla h(r) \cdot (x - r)|)$$

$$= \sum_i \phi_i(x) O(|r - p_i|^2) + O(|r - p_i| |x - r|)$$

$$= \sum_i \phi_i(x) O(|x - r|^2) = O(|x - r|^2),$$

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since, by item 3 in Lemma 15
\[ \|x - p_i\| \leq \text{diam}(\text{supp } \phi_i) + \text{dist}(\text{supp } \phi_i, R) \leq \text{diam}(\text{supp } \phi_i) + \text{dist}(x, R) \leq 2 \text{dist}(\text{supp } \phi_i, R) + \text{dist}(x, R) \leq 3 \text{dist}(x, R) = 3\|x - r\| \]
and
\[ \|r - p_i\| \leq \|r - x\| + \|x - p_i\| \leq 4\|x - r\|. \]
Similarly, since \( \sum_i \nabla \phi_i(x) = 0 \) because \( \sum_i \phi_i(x) = 1 \), so that also \( \sum_i \nabla \phi_i(x)(\nabla h(r) \cdot (x - r)) = 0 \), and using the triangle inequality and Lemma 15
\[
\|\nabla h(r) - \nabla h(x)\| = \left\|\nabla h(r) - \nabla \sum_i g_i(x) \phi_i(x)\right\|
= \left\|\nabla h(r) - \nabla \sum_i \phi_i(x) (h(p_i) + \nabla h(p_i) \cdot (x - p_i))\right\|
\leq \left\|\sum_i \phi_i(x) (\nabla h(r) - \nabla h(p_i))\right\| + \left\|\sum_i \nabla \phi_i(x) (h(p_i) + \nabla h(p_i) \cdot (x - p_i))\right\|
= \left\|\sum_i \phi_i(x) (\nabla h(r) - \nabla h(p_i))\right\| + \left\|\sum_i \nabla \phi_i(x) [(h(p_i) + \nabla h(p_i) \cdot (r - p_i))
+ (\nabla h(p_i) \cdot (x - r) + \nabla h(r) \cdot (x - r)))]\right\|
\leq \sum_i \phi_i(x) O(\|r - p_i\|) + \sum_i \|\nabla \phi_i(x)\| (O(\|r - p_i\|^2)
+ O(\|p_i - r\|) O(\|x - r\|))
\leq O(\|r - p_i\|) + 3^2 C_1 \left(1 + \frac{1}{\|x - r\|}\right) (O(\|r - p_i\|^2)
+ O(\|p_i - r\|) O(\|x - r\|))
= O(\|x - r\|)
\]

Thus \( h \) is \( C^1 \) on \( \mathbb{R}^2 \setminus (\Gamma \cup J) \), which settles item (iii).

It also follows from that, together with the fact that from (11) we know that \( h \) is Lipschitz with constant \( \leq 3L \) on \( R \), that \( h \) is locally Lipschitz.

To prove that \( h \) is path-differentiable, let \( \gamma: \mathbb{R} \to \mathbb{R}^2 \) be a Lipschitz curve. By Lemma 16 \( \gamma^{-1}(\mathbb{R} \setminus \bigcup \Gamma_i) \) is a set of measure zero. The set of points \( t \in \mathbb{R} \) such that \( \gamma(t) \in \bigcup \Gamma_i \) with \( \gamma(t) \) not tangent to \( \bigcup \Gamma_i \) is countable as each such \( t \) is isolated. If \( \gamma(t) \) is in \( J \cup \Gamma_i \) for some \( i \), and \( \gamma(t) \) is tangent to \( J \cup \Gamma_i \), then it follows from item (v) that the chain rule condition for path-differentiability holds at \( t \), and this condition also holds on \( \mathbb{R}^2 \setminus (\Gamma \cup J) \) because of item (iii). This proves item (i).

**Lemma 16.** If \( \gamma: \mathbb{R} \to \mathbb{R}^2 \) is Lipschitz with \( \gamma'(t) \neq 0 \) for almost every \( t \), then \( \gamma^{-1}(\mathbb{R} \setminus \bigcup \Gamma_i) \) has measure zero.

*Proof.* Write \( \gamma = (\gamma_1, \gamma_2) \) for the two coordinate components of \( \gamma \). Let \( P_i \) be the projection of \( \Gamma \setminus \bigcup \Gamma_i \) into the \( \ell \)-th coordinate axis, \( \ell = 1, 2 \). Note that since \( \alpha \leq \frac{1}{3} \), \( P_i \) is a Cantor set of measure zero, \( |P_i| = 0 \).
Because of Rademacher’s theorem and the fact that $\gamma'(t) \neq 0$ for almost every $t \in \mathbb{R}$, the sets $A_1$ and $A_2$ where the derivatives $\gamma'_1(t)$ and $\gamma'_2(t)$, respectively, are well-defined and nonzero, satisfy that $A_1 \cup A_2$ is a set of full measure. If $B$ is the null set of real numbers $t \in \mathbb{R}$ such that $\gamma'(t)$ is either not defined or equal to zero, then $A_1 \cup A_2 \cup B = \mathbb{R}$.

For $i \in \{1, 2\}$ and $p \in P$, $\gamma_i^{-1}(p) \cap A_i$ is countable because the isolated points in this set are only countably-many, and the non-isolated points $t \in \gamma_i^{-1}(p) \cap A_i$ either satisfy $\gamma'_i(t) = 0$ (as can be seen by taking the limit in the definition of the derivative restricting to points in $\gamma_i^{-1}(p) \cap A_i$) or $\gamma_i(t)$ is not defined; in other words, if $t \in \gamma_i^{-1}(p) \cap A_i$ is not isolated, then $t \in B$.

Thus $\gamma_i^{-1}(P) \cap A_i$ can be written as a countable, disjoint union $\bigcup_{j=1}^{\infty} Q_j^i$ of measurable sets $Q_j^i \subset A_i$ such that $\gamma_i(Q_j^i \cup B) = P_i$ and $\gamma_i$ is injective on $Q_j^i$. Since, by the change of variable formula $[9, \text{p. 99}]$, 
\[ 0 \leq \int_{Q_j^i} \|\gamma'_i(t)\| \, dt = |\gamma_i(Q_j^i)| \leq |P_i| = 0, \]
and since this is only possible if $|Q_j^i| = 0$ because the integrand is strictly positive throughout $Q_j^i$, all the sets $Q_j^i$ must be Lebesgue null. As a consequence, $\gamma_i^{-1}(P) \cap A_i$ is a countable union of null sets, and it is hence null.

Now, 
\[ \gamma^{-1}(T \setminus \bigcup_i \Gamma_i) = \gamma_1^{-1}(P_1) \cap \gamma_2^{-1}(P_2) \subseteq (\gamma_1^{-1}(P_1) \cap A_1) \cup (\gamma_2^{-1}(P_2) \cap A_2) \cup B, \]
and the three sets on the right-hand side are null, so this proves the lemma. \qed

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References

[1] Hedy Attouch, Jérôme Bolte, and Benar Fux Svaiter. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward–backward splitting, and regularized Gauss–Seidel methods. Mathematical Programming, 137(1-2):91–129, 2013.

[2] Edward Bierstone. Differentiable functions. Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society, 11(2):139–189, 1980.

[3] Jérôme Bolte, Aris Daniilidis, Olivier Ley, and Laurent Mazet. Characterizations of lojasiewicz inequalities: subgradient flows, talweg, convexity. Transactions of the American Mathematical Society, 362(6):3319–3363, 2010.

[4] Jérôme Bolte and Edouard Pauwels. Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning. Mathematical Programming, 2020.

[5] Jérôme Bolte, Edouard Pauwels, and Rodolfo Rios-Zertuche. Long term dynamics of the subgradient method for Lipschitz path differentiable functions. Preprint. arXiv:2006.00098 [math.OC].

[6] Jonathan Borwein, Warren Moors, and Xianfu Wang. Generalized subdifferentials: a Baire categorical approach. Transactions of the American Mathematical Society, 353(10):3875–3893, 2001.

[7] Aris Daniilidis and Dmitriy Drusvyatskiy. Pathological subgradient dynamics. SIAM Journal on Optimization, 30(2):1327–1338, 2020.
[8] Damek Davis, Dmitriy Drusvyatskiy, Sham Kakade, and Jason D. Lee. Stochastic subgradient method converges on tame functions. *Foundations of Computational Mathematics*, 01 2019.

[9] Lawrence Craig Evans and Ronald F Gariepy. *Measure theory and fine properties of functions*. CRC Press, 2015.

[10] Serge Lang. *Differential and Riemannian manifolds*, volume 160 of *Graduate Texts in Mathematics*. Springer, 2012.

[11] Hassler Whitney. A function not constant on a connected set of critical points. *Duke Mathematical Journal*, 1(4):514–517, 1935.