A method to derive explicit formulas for an elliptic generalization of the Jack polynomials

Edwin Langmann

Abstract. We review a method providing explicit formulas for the Jack polynomials. Our method is based on the relation of the Jack polynomials to the eigenfunctions of a well-known exactly solvable quantum many-body system of Calogero-Sutherland type. We also sketch a generalization of our method allowing to find the exact solution of the elliptic generalization of the Calogero-Sutherland model. We present the resulting explicit formulas for certain symmetric functions generalizing the Jack polynomials to the elliptic case.

1. Introduction

In this paper we explain a method which yields explicit formulas for the Jack polynomials; see \cite{St89, M79}. We found this method by studying integrable quantum many-body systems of Calogero-Sutherland type \cite{C71, Su71}, and our discussion will be therefore from the quantum integrable systems' point of view. However, we made an effort to make this paper also useful to readers interested in symmetric polynomials and not so much in the physics interpretation of our results. To explain our notation (which is admittedly less elegant than the one used in \cite{M79, St89} but closer to the one used in physics) we first give a definition of the Jack polynomials. (The equivalence of our definition and the one in \cite{St89} follows from Theorem 3.1 in \cite{St89}.) We then recall the relation of the Jack polynomials to the eigenfunctions of the so-called (quantum) Sutherland model \cite{Su71, Su72} and explain some physics terminology which we use. We conclude this introduction with the definition of the elliptic generalization of the Sutherland model \cite{C75, OP77} and an outline of the rest of this paper.

A definition of the Jack polynomials. The Jack polynomials $J_n(z; 1/\lambda)$ are symmetric polynomials of $N > 1$ variables $z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$ labeled by
partitions \( n \) (i.e. \( n \in \mathbb{Z}^N \) with \( n_1 \geq n_2 \geq \ldots \geq n_N \geq 0 \)), which are of the form

\[
J_n(z) = \sum_{m \leq n} v_{n,m} M_m(z), \quad M_m(z) = \sum_{P \in S_N} \prod_{j=1}^N z_j^{m_P_j},
\]

with real coefficients \( v_{n,m} \) fixed up to some non-zero normalization constant \( v_{n,n} \) (which we ignore for simplicity), and which can be defined by the property that they are eigenfunctions of the differential operator

\[
D = \frac{1}{2\lambda} \sum_{j=1}^N z_j^2 \frac{\partial^2}{\partial z_j^2} + \sum_{j,k=1}^N \frac{z_j^2}{z_j - z_k} \frac{\partial}{\partial z_j}
\]

for some parameter \( \lambda > 0 \); the ordering of partitions which we use is defined as follows, \( m \leq n \) if \( n_1 + n_2 + \ldots + n_N = m_1 + m_2 + \ldots m_N \) and \( m_1 + m_2 + \ldots m_j \leq n_1 + n_2 + \ldots n_j \) for all \( j \), and \( S_N \) is the permutation group. \(^{1}\) Thus our \( \lambda, N \), and \( M_m(z) \) correspond to \( 1/\alpha, n \) and \( m_\lambda(x) \) in \([St89]\), and we write vectors with \( N \) components in bold face.] To see that this defines unique symmetric polynomials (up to normalization) one can check that

\[
DM_n = \sum_{m \leq n} b_{n,m} M_m
\]

for certain real coefficients \( b_{n,m} \) which one can compute, and this implies that one can determine the coefficients \( v_{n,m} \), \( m < n \), recursively from \( v_{n,n} \) and the condition \( DJ_n = b_{n,n}J_n \); see \([St89]\), Theorem 3.1 and the discussion thereafter. In principle one can compute straightforwardly the Jack polynomials from this definition \([St89]\), however, the recursion relations one thus obtains for the coefficients \( v_{n,m} \) are complicated, and it is therefore difficult to solve them explicitly and obtain closed formulas.

**The Sutherland model and its relation to the Jack polynomials.** The Sutherland model is defined by the \( N \)-body Schrödinger operator

\[
H = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \gamma \sum_{1 \leq j < k \leq N} V(x_j - x_k),
\]

where

\[
V(r) = \frac{1}{4 \sin^2(r/2)}.
\]

\( x_j \in [-\pi, \pi], \) and

\[
\gamma = 2\lambda(\lambda - 1), \quad \lambda > 0.
\]

This differential operator has a natural physical interpretation as Hamiltonian defining a quantum mechanical model of \( N \) identical particles moving on a circle of length \( 2\pi \) and interacting via the two body potentials \( V \). (To be precise, this Hamiltonian is the self-adjoint operator on \( L^2([-\pi, \pi]^N) \) defined by the Friedrichs extension of the differential operator \( H \) above, and one is only interested in particular eigenfunctions of \( H \) specified below. We set the length of the circle to \( 2\pi \) only to ease the

\(^{1}\) Usually these monomials \( M_m \) are defined with a slightly different normalization obtained sum summing only over the distinct permutation \( P \) \([St89]\); the choice of normalization is irrelevant in our discussion.
notation, and an arbitrary length \( L > 0 \) of space as in \([Su72]\) could be easily introduced by rescaling \( x_j \to (2\pi/L)x_j, \) \( H \to H(2\pi/L)^2, \) etc.) Following pioneering work by Calogero solving\(^2\) a similar model \([C71]\), Sutherland found an algorithm to compute the eigenvalues and eigenfunctions of \( H \) \([Su71, Su72]\), and these exact solutions have made these models famous among theoretical physicists (since usually such quantum models describing interacting particles can only be studied using approximation methods; moreover, there are many interesting physics applications and variants of these models; see \([OP83]\) for review).

Sutherland’s solution method is closely related to our definition of the Jack polynomials above. To be more specific: The well-known ground state \( \Psi_0(x) \) of the Sutherland model (= eigenfunction of \( H \) with the smallest possible eigenvalue) is

\[
\Psi_0(x) = \prod_{1 \leq j < k \leq N} \theta(x_j - x_k)^\lambda
\]

where

\[
\theta(r) = \sin(r/2),
\]

and to obtain the other eigenfunctions of \( H \), Sutherland made the ansatz

\[
\Psi(x) = \Phi(x)\Psi_0(x)
\]

transforming the eigenvalue equation \( H\Psi = \mathcal{E}\Psi \) to \( H'\Phi = \mathcal{E}'\Phi \) where \( H' = [1/\Psi_0(x)]H\Psi_0(x) \) equals

\[
H' = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} - i\lambda \sum_{j<k} \left( e^{ix_j} + e^{ix_k} \right) \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right),
\]

and \( \mathcal{E}' = \mathcal{E} - E_0 \) with \( E_0 = \lambda^2N(N^2 - 1)/12 \) the ground state energy defined through \( H\Psi_0 = E_0\Psi_0 \). Sutherland’s method to compute the eigenfunctions \( \Phi \) of \( H' \) is equivalent to diagonalizing the operator \( D \) in \((1.2)\) by making the ansatz in \((1.1)\) etc., as discussed above. To see this, we change variables to \( z_j = e^{ix_j} \) and obtain

\[
H'(x) = 2\lambda D(z) + [1 - \lambda(N - 1)]P(z), \quad P = \sum_{j=1}^N z_j \frac{\partial}{\partial z_j}.
\]

The operators \( P \) and \( D \) commute, and it is not difficult to see that the eigenfunctions \( \Phi_n(x) = J_n(z; 1/\lambda) \) obtained by Sutherland’s method are equal to the Jack polynomials. We thus can conclude that the eigenfunctions of the Sutherland model are given by

\[
\Psi(x) = e^{ip\sum_{j=1}^N x_j}J_n(z; 1/\lambda)\Psi_0(x), \quad z_j = e^{ix_j},
\]

with arbitrary \( p \in \mathbb{R} \) (we used the fact that, if \( \Psi(x) \) is an eigenfunction of \( H \), then \( \exp(ip\sum_{j=1}^N x_j)\Psi(x) \) is an eigenfunction of \( H \) as well). The corresponding eigenvalues are found to have the following remarkably simple formula, \( \mathcal{E}_0(n) = \sum_{j=1}^N |n_j + p + \lambda(N + 1 - 2j)/2|^2 \). It is interesting to note that the exponential factor on the r.h.s. in \((1.12)\) describes the center-of-mass motion of the system, which is not very interesting and thus often ignored. However, it should be included if one is interested in all eigenfunctions of the model. Moreover, since transformations

\(^2\)By solving a quantum many-body model we mean to determine the eigenvalues and eigenfunctions of the Hamiltonian defining this model.
\( p \to p + k \) and \( n_j \to n_j - k \) for arbitrary integers \( k \leq n_N \) leave the eigenfunction invariant, one should restrict \( p \) to integers and set \( n_N = 0 \) if one wants to avoid over-counting. We also note in passing that the ansatz in (1.1) has a natural physical interpretation: The monomials \( M_n(z) = \sum_{P \in S_N} \exp(i \sum_{j} n_j P j x_j) \) are plane waves providing a complete set of eigenfunctions of the non-interacting Hamiltonian \( H \) with \( \gamma = 0 \), and the Jack polynomials thus correspond to a linear superposition of plane waves.

### The elliptic Calogero-Sutherland model.

In this paper we explain a method to solve the Sutherland model which, different from Sutherland’s method, gives fully explicit formulas for the eigenfunctions \([L01, L04d]\). We will mainly concentrate on the Sutherland model, but a main motivation for writing this paper is that our method can be generalized to the well-known elliptic generalization of the Sutherland model where the interaction potential in (1.5) is replaced by

\[
V(r) = \sum_{m \in \mathbb{Z}} \frac{1}{4 \sin^2([r + im\beta]/2)} \quad \beta > 0
\]

which is essentially the Weierstrass elliptic \( \wp \)-function with periods \( 2\pi \) and \( i\beta \).\(^3\) This so-called elliptic Calogero-Sutherland (eCS) model is known to be integrable [C75, OP77], which suggests that it should be possible to solve it. For \( N = 2 \) the eigenvalue equation of the eCS model is essentially equivalent to the Lamé equation which was studied extensively at the end of the 19th century; see [WW62] for an extensive discussion of the classical results. The problem of finding the general solution of the eCS model (without any restrictions on parameters) seems to be regarded as open, even though there exist various interesting results in this direction \([DI93, EK94, EFK95, FV95a, FV95b, S95, T00, KT02, FNP03]\). As we will discuss, our method also provides an explicit solution of the eCS model. In the elliptic case we will also obtain solutions as in (1.12) and (1.7) above, only the function \( \theta(r) \) is replaced by

\[
\theta(r) = \sin(r/2) \prod_{m=1}^{\infty} (1 - 2q^{2m} \cos(r) + q^{4m}), \quad q = e^{-\beta/2},
\]

which is essentially the Jacobi theta function \( \vartheta_1 \),\(^4\) and the \( J_n(z) \) are symmetric functions which no longer are polynomials but reduce to the Jack polynomials in the limit \( q \downarrow 0 \). We will sketch how to obtain explicit formulas for these elliptic generalization of the Jack polynomials by infinite series.\(^5\)

### Plan of the rest of this paper.

In the next section we explain our explicit solution of the Sutherland model and present our explicit formulas for the Jack polynomials. Our arguments are such that they generalize with minor changes to the elliptic case, as discussed in Section 3. This section also contains a description of our results for the elliptic generalizations of the Jack polynomials. We end with a few remarks in Section 4.

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\(^3\)To be precise: \( V(r) = \wp(r) + c_0 \) where \( c_0 = (1/12) - (1/2) \sum_{m \in \mathbb{Z}} 1/\sinh^2(\beta m/2) \) [WW62].

\(^4\)To be precise: \( \theta(r) = \vartheta_1(r/2)/\left[2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m}) \right] \) [WW62].

\(^5\)These \( q \)-deformed Jack polynomials are different from the Macdonald polynomials [M79].
2. Solution of the Sutherland model

We start by summarizing our solution in a theorem. The proof of this theorem has the character of a derivation which not only proves but, as we hope, also clarifies and explains this result.

**Theorem 2.1.** For \( m \in \mathbb{Z}^N \), let

\[
(f_m(z) = \prod_{j=1}^{N} \left[ \oint_{C_j} \frac{d\xi_j}{2\pi i \xi_j} \xi_j^{m_j} \right] \prod_{1 \leq j < k \leq N} \Theta(\xi_j / \xi_k)^\lambda / \prod_{j,k=1}^{N} \Theta(z_j / \xi_k)^\lambda
\]

where

\[
(2.2) \quad \Theta(\xi) = (1 - \xi),
\]

and the integration contours \( C_j \) are nested circles in the complex plane enclosing the unit circle,

\[
(2.3) \quad C_j : \xi_j = e^{\varepsilon_j e^{iy_j}}, \quad -\pi \leq y_j \leq \pi
\]

for some \( \varepsilon > 0 \). Moreover, for partitions \( n \), let

\[
(2.4) \quad P_n(z) = \sum_{m \in \mathbb{Z}^N} \alpha_n(m) f_m(z)
\]

with

\[
\alpha_n(m) = \delta(n, m) + \sum_{s=1}^{\infty} \gamma^s \sum_{j_1 < k_1} \sum_{s_1} \cdots \sum_{j_s < k_s} \sum_{s_s} \nu_s \delta(m, n + \sum_{r=1}^{s} \nu_r E_{j_r k_r})
\]

\[
(2.5) \quad \frac{1}{\prod_{r=1}^{s} [E_0(n + \sum_{\ell=1}^{r} \nu_{\ell} E_{j_{\ell} k_{\ell}}) - E_0(n)]}
\]

for all \( m \in \mathbb{Z}^N \), \( E_{jk} \) is the vector in \( \mathbb{Z}^N \) with the following components,

\[
(2.6) \quad E_0(m) = \sum_{j=1}^{N} \left[ m_j + \frac{1}{2} \lambda(N + 1 - 2j) \right]^2
\]

for all \( m \in \mathbb{Z}^N \), \( E_{jk} \) is the vector in \( \mathbb{Z}^N \) with the following components,

\[
(2.7) \quad (E_{jk})_{\ell} = \delta_{j, \ell} - \delta_{k, \ell}
\]

for \( j, k, \ell = 1, 2, \ldots, N \), and \( \delta(n, m) := \prod_{j=1}^{N} \delta_n_j, m_j \). Then

\[
(2.8) \quad \Psi_n(x) = P_n(z) \Psi_0(x), \quad z_j = e^{ix_j},
\]

with \( \Psi_0(x) \) defined in (1.4) and (1.5), is an eigenfunction of the Sutherland Hamiltonian \( H \) in (1.4) and (1.5),

\[
(2.9) \quad H \Psi_n(x) = E_n \Psi_n(x),
\]

and the corresponding eigenvalue is

\[
(2.10) \quad E_n = E_0(n).
\]

It is important to note that the sums in (2.4) and (2.5) are finite (i.e. all but a finite number of terms in these sums are zero), and the \( f_m \) are symmetric polynomials. From this we can conclude:

**Lemma 2.2.** The \( P_n(z) \) given in Theorem 2.1 are symmetric polynomials.
From this one can conclude that the $P_n(z)$ are (essentially; see below) equal to the Jack polynomials unless there is a degeneracy, i.e., unless there exists a $m \neq n$ with $\sum_j (m_j - n_j) = 0$ such that $E_0(m) = E_0(n) \text{[L01]}$. To resolve this uncertainty due to possible degeneracies we have recently checked that the $P_n(z)$ also are eigenfunctions of the well-known third order differential operator commuting with Sutherland Hamiltonian \textit{OP77}.\textsuperscript{6} and we thus have convinced ourselves that always,

$$ (z_1 z_2 \cdots z_N)^k P_{n-k \text{e}}(z) = c_{n,k} J_n(z), \quad \text{e} = (1, 1, \ldots, 1) $$

with some normalization constant $c_{n,k}$. We thus get an infinite number of explicit formulas for each Jack polynomial. We checked in the simplest case $N = 2$ that the fact that the l.h.s. of (2.11) is independent of $k$ is non-trivial. Thus (2.11) implies an infinite number of non-trivial identities which are essentially the contents of Theorem 5.1 in \textit{St89}.

It is interesting to note that the functions $f_m$ given above can be non-zero even for certain non-partition $m \in \mathbb{Z}^N$; and in our proof of Theorem 2.1 it does not seem essential to restrict to $n$’s which are partitions. We checked for $N = 2$ that the functions $P_n$ all vanish unless $n$ is a partition, and this is true due to highly non-trivial cancellations.\textsuperscript{7} We do not know how to prove this analytically and for all $N$ using only our approach, and the same is true for the question of completeness, i.e., whether our construction gives all eigenfunctions or not. However, these facts can be proven by comparing our solution with Sutherland’s \textit{Su71, Su72}.

An important aspect of our method is that we expand our eigenfunctions in a set of functions $f_m$ which are much more complicated than the monomials $M_m$. However, as we will discuss, there exist fully explicit formulas for the $f_m$. Moreover, the $f_m$ are much closer to the exact eigenfunctions than the monomials $M_m$ in the sense that the coefficients $a_n(m)$ are much simpler than the coefficients $v_{n,m}$ discussed in the introduction, and we therefore can compute their explicit series representation in (2.5). This later formula is our main result in this paper in addition to our previous results in Reference \textit{L01}.

**PROOF OF THEOREM 2.1.** We derive the result stated in Theorem 2.1 in three steps.

**Step 1: A remarkable identity.** The starting point of our solution method is a particular functional identity.

**Lemma 2.3.** Let

$$ F(x, y) = \frac{\prod_{1 \leq j < k \leq N} \theta(x_j - x_k) \theta(y_k - y_j)}{\prod_{j,k=1}^N \theta(x_j - y_k)} $$

with $\theta(r)$ the function defined in (1.8) and $x, y \in \mathbb{C}^N$. Then

$$ H(x) F(x; y) = H(y) F(x; y), $$

where $H$ is the differential operator defined in (1.4) and (1.5) acting on different arguments $x$ and $y$, as indicated.

\textsuperscript{6}E.L., unpublished.

\textsuperscript{7}These computations were done with the help of MATHEMATICA.
This can be proven by a straightforward but somewhat tedious computation using the well-known identity
\[(2.14) \quad \cot(x) \cot(y) + \cot(x) \cot(z) + \cot(y) \cot(z) = 1 \quad \text{if} \quad z + y + z = 0;
\]
see Appendix A in \cite{L01}. While this provides an elementary proof, it does not explain why this identity is true. We thus mention that we found this identity by studying a particular quantum field theory model, and this provides a natural physics interpretation of this result; see \cite{L04e} for review.

**Remark 2.4.** It is interesting to note that the identity in \[(2.13)\] can also be obtained as a corollary of Proposition 2.1 in Reference \cite{St89}. Moreover, it seems that our method is closely related to the solution of the Sutherland model by separation of variables \cite{KMS03} since \[(2.13)\] seems to give an alternative proof of the crucial Theorem 4.1 in \cite{KMS03}.

**Step 2: Fourier-type transformation of the remarkable identity.** It is useful to note that the identity in \[(2.13)\] remains true if we replace \(F(x; y)\) by
\[(2.15) \quad F'(x; y) = c e^{jP \sum_{j=1}^{N}(y_j - x_k)} F(x; y)
\]
for arbitrary constants \(P, c\) (this is not difficult to verify \cite{L01}). Below we will find it convenient to choose
\[(2.16) \quad P = -\lambda N/2, \quad c = (2/i)^{\lambda[N(N-1)/2-N^2]}.
\]

The idea now is to trade the variables \(y\) for suitable quantum numbers \(n\) by taking the Fourier transform of \[(2.13)\] (with \(F\) replaced by \(F'\)) with respect to the variables \(y\), i.e., apply \((2\pi)^{-N} \int d^N y \exp(i\hat{n} \cdot y)\) with suitable Fourier variables \(\hat{n}\). We need to do this with care since, firstly, the function \(F'(x; y)\) and the Hamiltonian \(H(y)\) have singularities and branch cuts, and secondly, the function \(F'(x; y)\) is not periodic in the variables \(y_j\) but changes by phase factors under \(y_j \rightarrow y_j + 2\pi\). As we will show below in more detail, the second problem can be accounted for by choosing the Fourier modes as
\[(2.17) \quad \hat{n}_j = n_j + \frac{1}{2} \lambda(N + 1 - 2j), \quad n_j \in \mathbb{Z},
\]
while the first problem is solved by shifting the \(y_j\)-integrations in the complex plan as follows, \(y_j = \varphi_j + j\varepsilon\) with real \(\varphi_j \in [-\pi, \pi]\) and some \(\varepsilon > 0\) (one can take the limit \(\varepsilon \downarrow 0\), but this turns out to be unnecessary). Using the fact that
\[1/[4 \sin^2(y/2)] = -\sum_{\nu=1}^{\infty} \nu e^{iuy} \quad \text{for} \quad \Im(y) > 0,
\]
straightforward computations leads to the following result.

**Lemma 2.5.** For all \(n \in \mathbb{Z}^N\), the function
\[(2.18) \quad \hat{F}(x; n) = \left[ \prod_{j=1}^{N} \int_{\pi + i\varepsilon}^{\pi - i\varepsilon} \frac{dy_j}{2\pi} e^{i\hat{n}_j y_j} \right] F'(x; y), \quad \varepsilon > 0,
\]
\(\hat{n}\) in \[(2.17)\], and \(x \in [-\pi, \pi]^N\), are well-defined and obey the identities
\[(2.19) \quad H(x)\hat{F}(x; n) = \mathcal{E}_0(n)\hat{F}(x; n) - \sum_{j<k} \sum_{\nu \in \mathbb{Z}} S_{\nu} \hat{F}(x; n + \nu E_{jk})
\]
with \(\mathcal{E}_0(n)\) and \(E_{jk}\) defined in Theorem \[(2.21)\] and
\[(2.20) \quad S_0 = 0, \quad S_{\nu} = \nu \quad \text{and} \quad S_{-\nu} = 0 \quad \forall \nu > 0.
\]
Moreover,
\begin{equation}
\hat{F}(x; n) = f_n(z)\Psi_0(x)
\end{equation}
with the functions $f_n$ and $\Psi_0$ defined in Theorem 2.1.

Note that the integral in (2.18) is independent of $\varepsilon > 0$ (due to Cauchy’s theorem).

It is not difficult to understand how (2.19) results from (2.13): the l.h.s. is obvious since $H(x)$ commutes with the Fourier transformation, while the r.h.s. arises from a simple computation using

\begin{equation}
H(y) = -\sum_{j=1}^N \frac{\partial^2}{\partial y_j^2} - \gamma \sum_{j<k} S_\nu e^{i\nu(y_j-y_k)} \quad \text{for } \Im(y_j - y_k) > 0,
\end{equation}

with the first term in the r.h.s. of (2.19) coming from the derivative term and partial integrations using $\tilde{n}^2 = E_0(n)$, and the second term comes from the interaction.

To show that $\hat{F}(x; n)$ is well-defined we use $\sin(y/2) = \frac{1}{2}ie^{-i\frac{y}{2}}(1-e^{iy})$ to write

\begin{equation}
e^{i\tilde{n} \cdot y} F'(x; y) = \prod_{j=1}^N \xi^n_j \prod_{1 \leq j < k \leq N} \Theta(\xi_j/\xi_k)^\lambda \prod_{j,k=1}^N \Theta(e^{i\nu_j/\nu_k})^{\lambda} \Psi_0(x)
\end{equation}

with $\Theta(\xi) = (1 - \xi)$ and $\xi_j = e^{i\nu_j}$, where the shifts $\lambda(N + 1 - 2j)/2$ in the pseudo-momenta exactly cancel the terms which would make the integrand non-analytic in the $\xi_j$. This implies (2.21).

**Step 3: Ansatz for eigenfunctions and solution of recursion relation.**

Equation (2.19) suggests the following ansatz for the eigenfunctions of the Sutherland Hamiltonian $H$,

\begin{equation}
\Psi_n(x) = \sum_{m \in \mathbb{Z}^N} \alpha_n(m) \hat{F}(x; m)
\end{equation}

with the normalization condition

\begin{equation}
\alpha_n(m) = \delta(m, n) + O(\gamma).
\end{equation}

This, (2.19), and the eigenvalue equation (2.9) lead to the following relations for the coefficients $\alpha_n(m)$,

\begin{equation}
[H_0(m) - E_n] \alpha_n(m) = \gamma \sum_{j<k} \sum_{\nu \in \mathbb{Z}} S_\nu \alpha_n(m - \nu E_{jk}) := \gamma(S\alpha_n)(m)
\end{equation}

(the last equality defines a convenient shorthand notation), i.e., the latter relations imply (2.9). We now observe that (2.26) and (2.27) yield

\begin{equation}
\alpha_n(m) = \delta(m, n) + \gamma(R_n \alpha_n)(m)
\end{equation}

with the linear operator $R_n$ defined as follows,

\begin{equation}
(R_n \alpha)(m) := \frac{1}{[\mathcal{E}_0(m) - E_n]_n}(S\alpha)(m);
\end{equation}

here and in the following we use the following convenient notation

\begin{equation}
\frac{1}{[\mathcal{E}_0(m) - E_n]_n} := [1 - \delta(m, n)]^{\lambda} \frac{1}{[\mathcal{E}_0(m) - E_n]}.
\end{equation}
Equation (2.27) can be easily solved by iteration setting $\alpha^{(0)}_n(m) = \delta(m, n)$, \footnote{We realized this only when rereading our paper [LO1] during this spring.}

$$\alpha_n(m) = \sum_{s=0}^{\infty} \gamma^s (R_n^s \alpha^{(0)}_n)(m) = \delta(m, n) + \sum_{s=1}^{\infty} \gamma^s \sum_{j_1 < k_1, \nu_1 \in \mathbb{Z}} S_{\nu_1} \cdots \times \sum_{j_s < k_s, \nu_s \in \mathbb{Z}} S_{\nu_s} \frac{\delta(m, n + \sum_{r=1}^{s} \nu_r E_{j_r, k_r})}{\prod_{r=1}^{s}[E_0(n + \sum_{r=1}^{s} \nu_r E_{j_r, k_r}) - E_n]}.$$ \hspace{1cm} (2.30)

Setting $m = n$ we obtain $\alpha_n(n) = 1$ (since $(R_n \alpha_n)(n) = 0$), and this and Equations (2.25) and (2.26) for $m = n$ imply the following condition determining the eigenvalue $E_n$,

$$E_n - E_0(n) = -\gamma(S\alpha_n)(n) = -\sum_{s=1}^{\infty} \gamma^{s+1} \sum_{j_1 < k_1, \nu_1 \in \mathbb{Z}} S_{\nu_1} \cdots \times \sum_{j_{s+1} < k_{s+1}, \nu_{s+1} \in \mathbb{Z}} S_{\nu_{s+1}} \frac{\delta(0, \sum_{r=1}^{s+1} \nu_r E_{j_r, k_r})}{\prod_{r=1}^{s}[E_0(n + \sum_{r=1}^{s} \nu_r E_{j_r, k_r}) - E_n]}.$$ \hspace{1cm} (2.31)

We now recall that $S_{\nu} = 0$ for $\nu \leq 0$, and this simplifies the previous formula considerably: it implies that all terms in the sum on the r.h.s. vanish (since the Kronecker deltas always give zero), and we thus get the result in (2.10). Moreover, for the same reason we can replace all double brackets $[\cdots]$ in (2.29) by normal bracket $\langle \cdots \rangle$ (since we only have the expressions in (2.29) with $m \neq n$). Thus the series for $\alpha_n(m)$ simplifies to the expression given in (2.30). \hfill \Box

A simpler argument to derive (2.10) is as follows: since $S_{\nu \leq 0} = 0$, (2.26) has an obvious triangular structure [LO1], and inserting $m = n$ thus immediately implies (2.10). However, this argument does not generalize to the elliptic case, whereas ours does.

\textbf{Proof of Lemma 2.2} The functions $f_n$ in (2.1) and (2.2) can be computed by Taylor expanding the integrand using the binomial series and computing the $\xi_j$-integrals which project out the $\xi_j$-independent terms of the integrand. One thus obtains (for details see [LO1], Appendix B.3),

$$f_n(z) = \sum_m p_{n,m} M_m(z)$$ \hspace{1cm} (2.32)

with the $M_m(z)$ in (1.1), and the coefficients are

$$p_{n,m} = \sum_{1 \leq j' < k' \leq N} \prod_{1 \leq j' < k' \leq N} \left( \frac{\lambda}{\mu_{j'k'}} \right)^N \prod_{j,k=1}^{N} \left( -\frac{\lambda}{\nu_{jk}} \right)^{-\mu_{j'k'} + \nu_{jk}}$$ \hspace{1cm} (2.33)

where the sum $\sum$ here is over all non-negative integers $\mu_{j'k'}, \nu_{jk}$ restricted by the following $2N$ equations,

$$n_j = \sum_{\ell=1}^{N} \nu_{j\ell} + \sum_{\ell=1}^{j-1} \mu_{\ell j} - \sum_{\ell=j+1}^{N} \mu_{j\ell}, \quad m_j = \sum_{\ell=1}^{N} \nu_{j\ell}$$ \hspace{1cm} (2.34)

and $m_1 \geq m_2 \geq \ldots \geq m_N \geq 0$. This shows that the $f_n$ are symmetric polynomials which are non-zero only if

$$n_j + n_{j+1} + \ldots + n_N \geq 0 \quad \forall j = 1, 2, \ldots, N,$$ \hspace{1cm} (2.35)
and the sum in (2.34) only contains terms with
\begin{equation}
\sum_{j=1}^{N} m_j = \sum_{j=1}^{N} n_j.
\end{equation}
The latter implies that all series in (2.32) and (2.5) are finite (i.e., they truncate after a finite number of terms). We finally need to show that the denominators in (2.5) are always non-zero. For that we write
\begin{equation}
m = n + \hat{\mu} \text{ with } \hat{\mu} = \sum_{j<k} \mu_{jk} E_{jk},
\end{equation}
and with that and (2.40) we compute
\begin{equation}
\mathcal{E}_0(m) - \mathcal{E}_0(n) = \sum_{j=1}^{N} \left( 2 \sum_{k=j+1}^{N} \mu_{jk} [n_j - n_k + (k-j)\lambda] + \left[ \sum_{k<j} \mu_{kj} - \sum_{k>j} \mu_{jk} \right]^2 \right)
\end{equation}
which is manifestly positive if \( n \) is a partition and all \( \mu_{jk} \geq 0 \) with \( \hat{\mu} \neq 0 \). Since obviously only such terms appear in the denominators in (2.30) we conclude that all \( a_n(m) \) in (2.5) are well-defined finite series. □

As already mentioned, in our proof it does not seem essential to restrict to \( n \)'s which are partitions. In fact, we only used this property to have all energy differences \( \mathcal{E}_0(m) - \mathcal{E}_0(n) \) in (2.38) manifestly positive, but we only would have needed that these energy differences are always non-zero which is true also for certain non-partition \( n \)'s, e.g., for \( N = 2 \) and \( \lambda \) non-integer. We feel that this point would deserve a better understanding.

3. Solution of the elliptic Calogero-Sutherland model: Outline

We formulated Theorem 2.1 and its proof so that it straightforwardly extends to the elliptic case: one only needs to replace \( V, \theta, \Theta \) and \( S_\nu \) by their elliptic generalizations. To be more specific we describe the generalizations of steps 1–3 in our proof of Theorem 2.1 in more detail.

Step 1: Lemma 2.3 holds true as it stands with \( H \) the eCS Hamiltonian defined in (1.4) and (1.13) and the function \( \theta \) defined in (1.14). This can be proved by a brute-force computation using the following well-known identify for the Weierstrass elliptic functions \( \xi \) and \( \wp \) [WW62].
\begin{equation}
[\xi(x) + \xi(y) + \xi(z)]^2 = \wp(x) + \wp(y) + \wp(z) \quad \text{if } x + y + z = 0;
\end{equation}
see Appendix A in [L04b]. A quantum field theory proof of this results can be found in [L04a].

Step 2: Lemma 2.5 holds true as it stands with \( S_\nu \) in (2.20) replaced by
\begin{equation}
S_0 = 0, \quad S_\nu = \nu \frac{1}{1 - q^{2 \nu}} \quad \text{and } S_{-\nu} = \nu \frac{q^{2 \nu}}{1 - q^{2 \nu}} \quad \forall \nu > 0
\end{equation}
and the functions \( f_n \) and \( \Psi_0 \) in (2.1) and (1.7) with
\begin{equation}
\Theta(\xi) = (1 - \xi) \prod_{m=1}^{\infty} (1 - q^{2m} \xi)(1 - q^{2m}/\xi)
\end{equation}
and θ in (1.14).

The proof is essentially unchanged, only now the functions Θ and the coefficients \( S_\nu \) in the identities

\[
\theta(y) = \frac{1}{2} \text{ie}^{-iy/2} \Theta(e^{iy})
\]

and

\[
V(y) = -\sum_{\nu=1}^{\infty} S_\nu e^{iy} \quad \text{for } \Im(y) > 0
\]

need to be changed to what is given above, and we also need to assume \( \varepsilon < \beta/N \); see Appendix B in [L01] for a detailed proof.

**Step 3:** The argument given remains unchanged until and including (2.31). The crucial difference now is that \( S_\nu \) no longer vanishes for negative values of \( \nu \), and thus the above-mentioned triangular structure is lost. Due to this (2.31) does not simplify but remains as an implicit equation determining the eigenvalue \( E_n \), and thus the eigenvalues for \( q > 0 \) are much more complicated.

We can summarize the solution of the eCS model thus obtained as follows.

**Theorem 3.1.** The eigenvalues \( E_n \) of the eCS model are determined by (2.31) with \( S_\nu \) in (3.2) and \( E_0(m) \) in (2.6). The corresponding eigenfunctions are given in (2.24), (2.21), (2.1), (3.3), (1.7), and (1.14), and the coefficients \( \alpha_n(m) \) in (2.30).

It is interesting to note that (2.31) can be turned into a fully explicit formula for the eigenvalues as follows: Defining the function

\[
\Phi_n(\xi) := -\sum_{s=0}^{\infty} \gamma^{s+1} \sum_{j_1<k_1, \nu_1 \in \mathbb{Z}} \ldots \sum_{j_s<k_s+1, \nu_s+1 \in \mathbb{Z}} S_{\nu_1} \ldots S_{\nu_s+1} \\
\times \delta(0, \sum_{r=1}^{s+1} \nu_r E_{j_r,k_r}) \prod_{r=1}^{s} \left[ E_0(n + \sum_{\ell=1}^{\nu_s} E_{j_\ell,k_\ell}) - \xi \right]_n
\]

of one complex variable \( \xi \), we can write (2.31) as

\[
E_n = E_0(n) + \Phi_n(\xi_n).
\]

Using Lagrange’s theorem as stated in [WW62], Paragraph 7.32, the latter equation can be solved by the following infinite series,

\[
E_n = E_0(n) + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^{n-1}}{\partial \xi^{n-1}} \Phi_n(\xi)^n \bigg|_{\xi = E_0(n)+a}
\]

where \( a \) is a real parameter which formally can be set to any value such that \( \Phi_n(\xi) \) is non-singular in \( \xi = E_0(n)+a \). From the explicit formula for \( \Phi_n(\xi) \) given above it is straightforward to compute the coefficients in this series explicitly [L04d]. In a similar manner one can find fully explicit formulas for all coefficients \( \alpha_n(m) \).

The arguments sketched above are enough to obtain an explicit solution in the sense of formal power series in \( q \). However, it is obviously important to also check if the analyticity properties of the function \( \Phi_n(\xi) \) are such that Lagrange’s theorem
gives convergent series. For that the parameter $a$ in (3.8) is important: One can prove that, if $a$ is such that
\begin{equation}
\exists \Delta > 0 : \forall m \neq n \text{ with } \sum_{j=1}^{N} (m_j - n_j) = 0 : \quad |\mathcal{E}_0(m) - \mathcal{E}_0(n) - a| > \Delta,
\end{equation}
then the function $\Phi_n(\xi)$ is analytic in a disc around $\xi = \mathcal{E}_0(n) + a$ which, in a finite $q$-interval, is large enough for the series in (3.8) to converge. A similar results holds true for all the coefficients $\alpha_n(m)$. We are confident now to be able to prove that the eigenfunctions $\Psi_n$ are square integrable in a finite $q$-interval as well.

The parameter $\Delta$ is obviously important since it restricts the range of convergence of our series solution. It is not easy to give general lower bounds for $\Delta$, and we only mention one simple case: if $\lambda$ is integer then $\Delta \geq 1/2$ for $a = 1/2$. It is possible to increase $\Delta$ by finding ‘better’ values of $a$, and we believe that this is useful for computing $E_n$ numerically using our results.

We plan to give a more detailed discussion of this solution of the eCS model together with detailed proofs in a future revision of Reference [L04b].

4. Final remarks

1. It would be desirable to compare our perturbative solution of the eCS model with brute-force numeric solutions and thus check if our results are also numerically useful.\[9\]

2. As discussed, the eigenvalues of the eCS model are determined by the implicit equation in (3.7). While the explicit series solution in (3.8) can be regarded as perturbative solutions which continuously deforms the solution for $q = 0$, we do not see any reason to rule out the possibility that there are additional non-perturbative solutions for larger values of $q$. The spectrum of the eCS Hamiltonian as a function of $q$ might therefore be quite complicated with qualitative changes at certain critical values of $q$ (this should be closely related to the intriguing analytical structure of the functions $\Phi_n(\xi)$ defined above). If so the eCS model would challenge notions of quantum integrability based on the simplicity of the spectrum.

3. Lemma 2.2 has an interesting generalization.

**Lemma 4.1.** For non-negative integers $N, M$, let
\begin{equation}
F_{N,M}(x, y) = \frac{\prod_{1 \leq j < k \leq N} \theta(x_j - x_k)^{\lambda} \prod_{1 \leq j < k \leq M} \theta(y_k - y_j)^{\lambda}}{\prod_{j=1}^{N} \prod_{k=1}^{M} \theta(x_j - y_k)^{\lambda}}
\end{equation}
with $\theta(r)$ in (1.8) and $x \in \mathbb{C}^N$, $y \in \mathbb{C}^M$. Then
\begin{equation}
[H_{\lambda,N}(x) - H_{\lambda,M}(y) - c_{N,M}] F_{N,M}(x; y) = 0
\end{equation}
where $H = H_{\lambda,N}(x)$ is the differential operator defined in (1.4)–(1.6) and $c_{N,M} = \lambda^2(N - M)(N - M)^2 - 1)/12$. Similarly, the function
\begin{equation}
\tilde{F}_{N,M}(x, y) = \prod_{1 \leq j < k \leq N} \theta(x_j - x_k)^{1/\lambda} \prod_{1 \leq j < k \leq M} \theta(y_k - y_j)^{1/\lambda} \prod_{j=1}^{N} \prod_{k=1}^{M} \theta(x_j - y_k)
\end{equation}

\[9\] I thank David Gomez-Ullate for stressing this point.
obeys the identity

\[
[H_{\lambda,N}(x) + \lambda H_{1/\lambda,M}(y) - \tilde{c}_{N,M}] \tilde{F}_{N,M}(x;y) = 0
\]

with \( \tilde{c}_{N,M} = [\lambda^2 N(N^2 - 1) + M(M^2 - 1)/\lambda + 3MN(\lambda N + M)]/12. \)

We thus can have different particle numbers for the two sets of variables \( x \) and \( y \), and in addition to the case where the coupling \( \lambda \) is the same there is also a dual relation with reciprocal couplings for the \( x \) and \( y \) variables. Note that, for \( M = 0 \), (4.2) and (4.4) both reduce to the eigenvalue equation \( H\Psi_0 = E_0\Psi_0 \) for the groundstate in (1.7).

We found these identities using a quantum field theory construction, but once known they can also be proven by brute-force computations [L04c]. Taking the Fourier transform of (4.2) and (4.4) etc. (as in Section 2), one can obtain additional explicit formulas, and thus identities, for the Jack polynomials. We suspect that (4.4) would lead to an alternative proof of Theorem 3.3 (duality relation of the Jack polynomials) in [St89].

It is important to note that the generalizations of (4.2) and (4.4) to the elliptic case exist but involve, in general, a term with a \( \beta \)-derivative [L04c], and this term is only absent in the special case stated in Lemma 2.3. We only know how to construct eigenfunctions when this \( \beta \)-derivative term is absent, which is the reason why we only discussed this case in detail.

4. The method explained in this paper can be adapted to give explicit solutions of the original Calogero model [C71] and the \( BC_N \)-variants of the Calogero-Sutherland models [OP83]. It will be interesting to see if this method can also provide an alternative solution of the Ruijsenaars models [R87] and thus can give interesting explicit formulas for Macdonald’s generalization of the Jack polynomials [M79]; see e.g. [AOS96] for interesting results in this direction.

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