A three-state cyclic voter model extended with Potts energy

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The cyclically dominated voter model on a square is extended by taking into consideration the variation of Potts energy during the nearest neighbor invasions. We have investigated the effect of surface tension on the self-organizing patterns maintained by the cyclic invasions. A geometrical analysis is also developed to study the three-color patterns. These investigations indicate clearly that in the “voter model” limit the pattern evolution is governed by the loop creation due to the overhanging during the interfacial roughening. Conversely, in the presence of surface tension the evolution is governed by spiral formation whose geometrical parameters depend on the strength of cyclic dominance.

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I. INTRODUCTION

In systems with several species (particles, opinions, etc.) the cyclic invasion processes can maintain a self-organizing domain structure. Although this phenomenon is investigated extensively in different areas, such as the chemical reactions on crystal surfaces [1,2], biological (Lotka-Volterra) systems [3–6], Rock-Scissors-Paper (RSP) games in evolutionary game theories [7], cyclically dominated voter models [8–11], the mechanism sustaining the polydomain patterns is not well understood. At the same time, this type of spatial self-organizations is believed to play crucial role in the biological evolution and antivortices (rotating clockwise and anti-clockwise) transform into $\frac{A}{2}$ or to $\frac{B}{2}$. For $P = 0$ this system is equivalent to the voter model [12,13] exhibiting growing domains whose correlation length is proportional to $\sqrt{t}$ if the system is started from a random initial state. Contrary, for $P = 1/2$ $A$ beats $B$ beats $C$ beats $A$ as it happens in the RSP game. In this former case the above rules sustain a self-organizing, three-color polydomain structure in which the linear domain size can be characterized by a correlation length of $\xi \simeq 2.5$ (measured in lattice unite). Tainaka and Itoh have shown that the typical domain size diverges when $P \to 0$ [8]. Their numerical analysis is focused on the density of vortices which are defined by those points of a three-color map where the three states (domain walls) meet. In fact, one can distinguish vortices and antivortices (rotating clockwise and anti-clockwise) and they are created and annihilated in pairs during the evolution of domain structure. According to the early Monte Carlo (MC) simulations the vortex density can be approximated by a power law behavior in the limit $P \to 0$ [8].

In the above system the domain walls separating two homogeneous domains are very irregular. This irregularity prevents the observation of expected spirals to be formed by the rotating vortex arms for smooth interfaces [8,9]. In the absence of cyclic invasion (and related vortex rotation) some features of the rough interfaces are already studied by several authors considering the two-state voter model [10,11]. In the presence of cyclic invasion, however, the topological and geometrical features of domain structure are not yet investigated rigorously even for smooth interfaces. Very recently some geometrical features and drift of a single spiral wave are studied by using a continuous reaction-diffusion model [21,22]. Now we report a model which exhibits a transition between different self-organizing patterns involving those where smooth, rotating spiral arms can be observed.

The present work is devoted to study the effect of surface tension on the self-organizing domain structures maintained by cyclic invasions on a square lattice. For this purpose we have introduced a model where the nearest neighbor invasion is affected by an interfacial (Potts) energy [8,13] whose strength is controlled by a parameter $K$. Using MC simulations we have investigated the vortex density in the stationary state for different values of $P$ and $K$. In order to have a deeper and more quantitative insight into the domain structure we have developed a method to study some geometrical features of the interfaces. This analysis confirms the necessity of these types of sophisticated descriptions.

II. THE MODEL

We consider a square lattice where at each site $x = (i,j)$ ($i$ and $j$ are integers) there is a state variable with...
three possible states, namely $s_x = 0, 1, 2$. The Potts energy for a configuration $s = \{s_x\}$ is defined as

$$H = -\sum_{<x,y>} [\delta(s_x, s_y) - 1]$$

where the summation runs over the nearest neighbor sites and $\delta(s, s')$ indicates the Kronecker’s delta $[23,24]$. Notice that the coupling constant is chosen to be energy unit. In the present form the Potts energy measures the length of equivalent interfaces (in lattice unit $a = 1$) separating the three types of domains and its inverse estimates the average domain radius $[23]$. Evidently, in the threefold degenerated (homogeneous) ground state $H = 0$.

The configuration evolves in time according to elementary invasions between two nearest neighbor sites (x and y) chosen randomly. More precisely, a pair of neighboring state variables $(s_x, s_y)$ (assuming $s_x \neq s_y$) transforms into $(s_y, s_y)$ with a probability

$$\Gamma[(s_x, s_y) \rightarrow (s_y, s_y)] = \frac{1}{1 + \exp(K\Delta H + PD)}$$

where

$$\Delta H = H_f - H_i$$

is the energy difference between the final and initial states, and $K$, as an inverse temperature, controls the effect of Potts energy on this single site flip. The second term in the argument of exponential function describes the cyclic dominance with a strength $P$, where

$$D = \begin{cases} +1 & \text{if } s_x = (s_y + 1) \mod 3, \\ -1 & \text{if } s_x = (s_y + 2) \mod 3. \end{cases}$$

In the case $K = 0$ the present model is equivalent to those introduced by Tainaka and Itoh $[3]$. Evidently, the three-state voter model is reproduced if $P = 0$ and $K = 0$. For $K > 0$, however, the interfaces become more smooth because the present dynamics suppresses those elementary processes which increase the interfacial (Potts) energy.

We have to emphasize that for $P = 0$ this system exhibits domain growth independently of the value of $K$. Apparently, for $P = 0$ Eq. (3) satisfies the condition of detailed balance at a temperature $1/K$ as defined for the kinetic Potts model (on the analogy of kinetic Ising model) that undergoes a symmetry breaking phase transition when increasing $K$. There exists, however, a relevant difference due to the constraint of invasion dynamics. Namely, in the present model the new state at a given site should be equivalent to one of the neighboring one. This means that the changes are localized along the boundaries separating the homogeneous domains.

Notice that the above rules conserve the cyclic symmetry among the three states. As a result in a sufficiently large system the three states are present with the same probability (1/3). For small system, however, one of the species can extinct due to the effect of fluctuations and finally the system evolves toward one of the three (homogeneous) absorbing states. Henceforth our analyses will be restricted to the large system limit that is provided by choosing the system size to be much larger than any length characteristic to the corresponding pattern.

The above model is investigated by MC simulations under periodic boundary conditions on a square lattice consisting of $N = L \times L$ sites. The system is started from a random initial state where the three states are present with equal probabilities. During the simulations we have recorded the number of vortices and antivortices defined above. For this purpose we have counted those $2 \times 2$ block configurations containing all the three possible states $[3-14]$. After a suitable transition time we have determined the average vortex density as well as its fluctuation defined in $[14]$. The system size is varied from $L = 400$ to 2000 to have sufficiently large number of vortices in the stationary states. The numerical results of vortex densities are summarized in a log-log plot as demonstrated in Fig. 1.

![Vortex densities as a function of $P$ for $K = 1$ (closed triangles), $1/4$ (open triangles), $1/16$ (closed squares), $1/64$ (open squares), and $0$ (pluses). The solid line shows the predicted power law behavior if $K = 0$. The dashed line (with a slope of 2.05) indicates the best power law fit for $K = 1$.](image)

FIG. 1. Vortex densities as a function of $P$ for $K = 1$ (closed triangles), $1/4$ (open triangles), $1/16$ (closed squares), $1/64$ (open squares), and $0$ (pluses). The solid line shows the predicted power law behavior if $K = 0$. The dashed line (with a slope of 2.05) indicates the best power law fit for $K = 1$.

If $P >> \max(K, 1)$ then the dynamics is governed by the deterministic RSP rule that maintain a self-organizing state with small domain sizes ($\xi \approx 2.5$) as mentioned above. Consequently the vortex density becomes independent of $K$ for sufficiently large values of $P$ as demonstrated in Fig. 1.

In the case $K = 0$ the $P$-dependence of vortex density can be well described by a power law, namely, $\rho_v \approx P^\beta$ within the range $0.003 < P < 0.3$. The best fit is found for $\beta = 0.29(1)$ confirming our previous result $[14]$. At the same time Fig. 1 demonstrates clearly that the vortex density is dramatically reduced when the interfacial energy is switched on. For $K = 1$ the MC data can be well approximated by another power law with an exponent $\beta = 2.05(9)$. It should be emphasized that within the statistical error our data is consistent with a quadratic
behavior. Similar behavior can be conjectured from the
trends represented by MC data for lower \( K \) in Fig. 1. Unfortunately, we couldn’t confirm this expectation by de-
termining the leading term in the \( P \)-dependence of vortex
density for lower \( K \) values because this analysis requires
extremely long run time and large systems. Just to in-
dicate the difficulties, the determination of a data point
at low vortex densities has required more than four-week
run time on a fast PC. We think that further numerical
analyses are necessary to justify (or modify) the above
conjecture.

Due to the long run times we could derive the vor-
tex density fluctuations (\( \chi \)) with an adequate accuracy.
As demonstrated in Fig. 2 the numerical data indicate
the divergency of the vortex density fluctuation in the
absence of interfacial energy (\( K = 0 \)). Within the in-
vestigated region, this fluctuation can be approximated
as \( \chi \sim P^{-\gamma} \) with \( \gamma = 0.3(1) \) in good agreement with a
previous result [10].

![Plot showing vortex density fluctuations versus \( P \)](image)

FIG. 2. Fluctuation of vortex density versus \( P \) for different
\( K \) values denoted by the same symbols as in Fig. 1. The solid
line indicates the fitted power law divergency in the absence
of interfacial energy.

In the presence of interfacial energy (\( K > 0 \)) the vortex
density fluctuation vanishes with \( P \) as indicated in Fig. 2.
The vortex density fluctuation seems to be proportional
to the vortex density for sufficiently low \( \rho_v \). Similar features
have been found for some systems of particles and antiparticles
performing branching annihilating random walks [1,10]. This is the reason why we have reinvesti-
gated a parallel drawn between the vortex dynamics in present model and a system of particles and antiparticles
as suggested in [10]. According to a simple idea the ro-
tating vortices form spirals whose long and narrow arms
enhance the probability of the creation of a new vortex-
and antivortex pair. The movement of vortices can be well
approximated by a random walk on a lattice. Further-
more, a vortex and antivortex annihilate each other when
meeting at the same site during their random walks. The
balance between the annihilation process and pair cre-
ation yields an average concentration in the stationary
state. Within this framework a simply mean-field anal-
ysis (details are given in [10]) predicts that a quadratic
behavior (\( \rho_v \sim P^2 \)) can be reproduced if the pair creation
is proportional to \( P^3/2 \) or \( P^2 \rho_v \). From the view-
point of vortex dynamics both possibilities demand a better un-
derstanding about the relationship between the creation
of vortex-antivortex pairs and the geometry of interfaces.

As mentioned above the whole length of interfaces is
equivalent to the Potts energy defined by Eq. (1). During
the simulations we have determined the expected value of
Potts energy per sites,

\[
E = \frac{1}{N}\langle H \rangle
\]

(5)

where \( \langle \cdots \rangle \) indicates the average over the sampling time.
The results of our simulations are summarized in a log-
log plot (see Fig. 3).

![Plot showing Potts energy per sites as a function of \( P \)](image)

FIG. 3. Average Potts energy per sites as a function of
\( P \) for the same values of \( K \) plotted in Fig. 1. The solid
and dashed lines indicate the slopes of 0.08 (\( K = 0 \)) and
1.3 (\( K = 1 \)).

At the first glance the \( P \)- and \( K \)-dependences of the
vortex density and Potts energy seem to be very simi-
lar. However, the detailed numerical analysis gives dif-
frent values for the exponents when fitting a power law
(\( E = aP^\alpha \)) for small \( P \) values. Namely, we have obtained
\( \alpha = 0.08(2) \) and 1.3(1) for \( K = 0 \) and 1 respectively.
The reader can easily check that the traditional scaling
argument (predicting \( \beta = 2\alpha \)) is not valid in the present
cases.

The self-organizing domain structure shows striking
differences depending on wether the interfacial energy is
switched on or not. For the sake of illustration two typical
patterns are shown in Figs. 4 and 5. In the absence of
interfacial energy (\( K = 0 \)) the nearest neighbor invasions
yield irregular boundaries whose overhanging results in
small islands (loops). Their random motion, extension,
shrinking, splitting, and fision seem to play crucial role in
the pattern evolution as well as for the three-state voter
model. In this former case (\( P = 0 \)), however, these el-
ementary events are not able to prevent the growth of
domains whose characteristic linear size increases with
time as \( \sqrt{t} \) [19].
FIG. 4. Typical part (50 × 50) of snapshot in a larger system for \( K = 0 \) and \( P = 0.01 \).

Choosing a particular initial state it is already demonstrated that the cyclic dominance drives the vortex rotation (for \( P > 0 \)) which is accompanied with spiral formation [9,10]. In Figure 4 the rotating spirals are not recognizable due to the irregular interfaces. However, the spiral formation becomes visible when the interfacial roughness is reduced by the surface tension as demonstrated in Fig. 5.

FIG. 5. Spiral formation sustained by the rotating vortices and antivortices is recognizable on a 200 × 200 part of a larger system for \( K = 1/2 \) and \( P = 0.3 \).

In Figure 5 one can easily identify the vortices and antivortices rotating clockwise and anticlockwise respectively. This rotation creates spirals because the average invasion velocity is constant. We have to emphasize that this pattern can not be characterized by a single length unit (e.g. correlation length) because the main features of spirals (armlength, average curvature, average distance, etc.) depend on the model parameters. This is the reason why we have developed a method to study some geometrical features of three-color maps on a square lattice.

III. GEOMETRICAL ANALYSES

On a three-color, continuous, planar map the domains are separated by three types of smooth boundaries. Dedicated points are the vertices where three or more boundaries meet. If such a map evolves smoothly then the appearance of vertices with more than three edges becomes negligible. Thus our analysis can be restricted to those maps which contain only three-edge vortices and antivortices. However, as we show later, the qualitative feature of the system remains unaffected if four-leg vertices are not ignored. One can easily check that these vortices and antivortices are positioned alternately along domain boundaries [10]. Our geometrical analysis will be focused on determining the average value of arclength, rotation of tangent vector, and curvature for those boundaries connecting a vortex and an antivortex. Henceforth the rotating vertex is called vortex.

On a square lattice the boundaries are polygons consisting of unit length parts whose tangential rotation may be \( \Delta \phi = \pm \pi/2 \) and 0. For a given vortex edge the tangential rotation is determined by summarizing these quantities step by step along the edge from a vortex to the connected antivortex. At the same time arclength is also obtained as the number of steps. The elementary step is based on the identification of the \( 2 \times 2 \) block configurations. This algorithm assumes that first we have determined the vortex positions. To reduce the statistical error this procedure was repeated many times during the simulations.

The above algorithm is well defined if the three-color pattern is free of four-edge vertices. Unfortunately, the investigated self-organizing patterns contain undesired four-edge vertices (see Figs. 4 and 5). Some of them (involving all the three states) can be considered as a vortex-antivortex pair just before their annihilation or after they creation [10]. The others involve only two states and make the paths (from vortex to antivortex) indefinite. Both types can be removed by executing an invasion through one of the randomly chosen four edges. Before the geometrical analysis all the investigated distributions are slightly adjusted by repeating the random invasions at the four-edge vertices until they vanish. Evidently, the effect of these modifications on the energy or vortex density is negligible if the typical domains are sufficiently large. The most relevant effect appears at \( K = 0 \) when the density of vortex-antivortex pairs is approximately \( \rho_v/6 \) in the whole region of \( P \) where we studied the system. Consequently, the \( P \)-dependence of vortex density remains power law after the pattern adjustment.

After having removed the four-edge vertices the pattern becomes topologically equivalent to the continuous,
three-color map mentioned above. In this case we can
distinguish two types of boundaries, namely, loops (sur-
rounding an isolated domain) and vortex edges (starting
at a vortex and ending at one of the connected antivor-
tices). Using the mentioned algorithm we have deter-
mined the average length $l_{av}$ and tangential rotation $\phi_{av}$
of vortex edges.

In Figure 6 the log-log plot of the average length of
vortex edges shows that $l_{av}$ increases slowly when $P$
is decreased for $K = 0$. Significantly faster increase can be
observed in the presence of surface tension. The arrange-
ment of MC data for $K = 1$ has inspired us to fit a power
law, $l_{av} = aP^{-\lambda}$ as we had done for the vortex density
and the Potts energy per sites. Within the same region
of $P$ the best fit is found for $\lambda = 1.75(5)$.

The above behavior is not surprising because the av-
erage vortex distance exhibits qualitatively similar $P$-
dependence. For the quantitative analysis an average vortex distance can be deduced from the density of vor-
tices as $d_{av} = 1/\sqrt{\rho_v}$. The striking difference caused by the introduction surface tension and cyclic invasion
becomes visible when the pairs of data $d_{av}$ and $l_{av}$ are plotted on a log-log plot (see Fig. 7). In Figure 7 the
straight line ($l_{av} = 1.05d_{av}$) demonstrates those set of
domain structures that can be well characterized by a single length scale. For example, such a situation can be observed when considering the domain growth in the
three-state Potts model below the critical temperature.

In the absence of interfacial energy the MC data indi-
cates significantly different relation which may be ap-
proximated as $l_{av} \approx d_{av}^{0.8}$ within the given region. The
slower increase of the average length of vortex edges can be
explained by the increasing number of those vortex-
antivortex pairs which have two common (short) edges.
Such a pair is frequently created when the moving is-
lands meet the third type of domain during their random
movements.

An opposite tendency can be recognized for those cases
where the spiral formation becomes relevant because the arclength of spiral arms always exceed the distance be-
tween the corresponding vortex and antivortex. At the
same time our data reflects that the average tangential rotation of a vortex edge increases with $d_{av}$. This state-
ment is supported by those data in Fig. 8 we obtained for
$K > 0$. Particularly, for $K = 1$ one can observe that both $\phi_{av}$ (see Fig. 8) and $l_{av}$ (Fig. 6) increases monotonously
when $P$ goes to zero. Unfortunately, we were not able to study what happens when the average tangential rotation
becomes larger than $2\pi$.

In Figure 8 the angle of tangential rotation is measured
in unit $\pi/2$ which is a natural choice on a square lattice.
One can observe that $\phi_{av}$ becomes practically zero in the
$P \to 0$ limit in the absence of interfacial energy. In the
light of this result one can think that the spiral formation
does not play a dominant role in the pattern formation
for $K = 0$. At the same time, we should keep in mind
that the cyclic invasion ($P > 0$) is required to sustain the self-organizing domain structures, otherwise the domains would grow unlimited. Unfortunately, we cannot explain quantitatively the microscopic mechanism yielding this behavior. Now we can only give some additional arguments supporting the crucial role of islands as mentioned above.

The total interfacial (Potts) energy can be separated into two parts. The first contribution comes from the island boundaries and the second part from the vortex edges. Thus the energy per sites can be written in the following form:

$$E = E_i + 3\rho_v l_{av}$$

(6)

where $E_i$ denotes the contributions of islands to the total Potts energy $E$ defined by Eq. (5). The second term indicates that the contribution of vortex edges can be expressed as a product of the density of (three-edge) vortices ($\rho_v$) and the average length of vortex edges ($l_{av}$). Using this expression we can determine the values of $E_i$ from those data plotted in Figs. 1, 3, and 6. The results of this calculation are illustrated in Fig. 9.

![Fig. 9. The interfacial energies of islands vs. $P$ for different $K$ values. Symbols as in Fig. 1.](image)

For $K = 1$ the interfacial energy contribution of islands vanishes at sufficiently low values of $P$. This tendency can be observed in Fig. 1 for other $K > 0$ values. The lower the value of $K$, the lower the value of $P$ is where $E_i$ becomes negligible. This means that in the stationary state the number of islands is reduced by the interfacial energy. This tendency can be visually checked if the reader compares the two snapshots shown in Figs. 4 and 5.

Notice, furthermore, that for $K = 1$ the energy contribution of islands is negligible for weak cyclic dominance where the $P$-dependence of $E_i$, $\rho_v$, and $l_{av}$ can be well approximated by power laws as mentioned above. The substitution of the corresponding expressions into (6) yields a relation between the exponents, namely $\alpha = \beta - \lambda$. Our numerical data support this result.

The investigation of $E_i$ for $K = 0$ shows surprising results. In the case of strong cyclic dominance the dominant part of interfacial energy comes from the vortex edges. According to our simulations the contribution of $E_i$ to the total interfacial energy increases meanwhile $E$ decrease (see Fig. 6) when decreasing $P$ in the investigated region. Since $E > E_i$ therefore these opposite tendencies imply the possibility of a break point for $P < 0.002$.

### IV. Conclusions

We have studied numerically the effect of surface tension on the self-organizing patterns maintained by cyclic invasions among three species on a square lattice. For this purpose the cyclic voter model introduced by Tainaka and Itoh is extended in a way conserving the cyclic symmetries. In the original model the invasion between two (randomly chosen) nearest neighbors is not affected by the neighborhood. In the extended model the nearest neighbor invasion rate is influenced by the neighborhood via taking the variation of Potts energy into account. Our analyses are restricted to those situations ($K \geq 0$) where this modification favorizes those invasions which reduce the length of interfaces separating the domains.

Our simulations have justified that the introduction of interfacial energy causes relevant changes in the observed patterns. To have a more quantitative and sophisticated picture we have determined the average value of some geometrical features of the interfaces (e.g. arc-length and tangential rotation of vortex edges). This method is based on the analogy to the continuous limit of a three-color map. By this way we could study the contributions of vortex edges and islands separately. In the light of this analysis we can distinguish three types of typical domain structures.

In the deterministic limit ($P \gg \max(K,1)$) the pattern consists of small compact domains and contains many vortices and antivortices. In the absence of interfacial energy ($K = 0$) the typical domain size as well as the contribution of island interfacial energy increases when the the cyclic dominance ($P$) is decreased. Here the island creation via the interfacial roughening seems to be a relevant phenomenon. Conversely, in the presence of interfacial energy the islands vanish when $P$ is decreased and vortex (spiral) rotations dominate the pattern evolution. The transitions among these typical behaviors are smooth.

Our numerical results are obtained in a limited region of the parameter $P$ due to the technical difficulties appearing for large typical domain sizes. For some cases ($K = 0$ and 1) our data can be approximated by power laws in a region of $P$. We are, however, not convinced that these (expected universal) behaviors remain valid for lower $P$ values. For example, we don’t know what happens when the average tangential rotation of vortex edges becomes significantly larger than $2\pi$. Deviations can also appear for $K = 0$ at lower $P$ values where $E_i$ is expected to decrease monotonously with $P$.
The suggested geometrical analyses confirm that the self-organizing patterns cannot be characterized by a single length unit as it happens for many other systems. In these cases two patterns can not be transformed into each other by choosing a suitable length scale. In the presence of interfacial energy this feature is strongly related to the appearance of spiral vortex edges whose average tangential rotation remains unchanged during such a geometrical magnification. We think that this type of geometrical analysis joints different approaches and models, furthermore, it motivates a theoretical effort to find general relations among these quantities.

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[1] R. J. Field and M. Burger, *Oscillations and Traveling Waves in Chemical Systems*, (Wiley Interscience, New York, 1985).
[2] M. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
[3] J. Maynard Smith, *Evolution and Theory of Games* (Cambridge University press, Cambridge, 1982).
[4] A. J. Lotka, Proc. Natl. Acad. Sci. U.S.A. 6, 410 (1920); V. Volterra, *Leçons sur la Théorie Mathématique de la Lutte pour le Vie* (Gauthier-Villars, Paris, 1931).
[5] L. Frachebourg, P. L. Krapivsky, and E. Ben-Naim, Phys. Rev. E 54, 6186 (1996).
[6] E. Ben-Naim, L. Frachebourg, and P. L. Krapivsky, Phys. Rev. E 53, 3078 (1996).
[7] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, England, 1998).
[8] K. Taimaka and Y. Itoh, Europhys. Lett. 15, 399 (1991).
[9] K. Taimaka, Phys. Rev. E 50, 3401 (1994).
[10] G. Szabó, M. A. Santos, and J. F. F. Mendes, Phys. Rev. E 60, 3776 (1999).
[11] J. Maynard Smith, Nature 280, 445 (1979).
[12] M. C. Boerlist and P. Hogeweg, Physica D 48, 17 (1991).
[13] G. Szabó and T. Czárán, Phys. Rev. E 63, 061904 (2001).
[14] G. Szabó and M. A. Santos, Phys. Rev. E 59, R2509 (1999).
[15] P. Clifford and A. Sudbury, Biometrika 60, 581 (1973).
[16] R. Holley and T. M. Liggett, Ann. Probab. 3, 643 (1975).
[17] J. T. Cox and D. Griffeath, Ann. Probab. 14, 347 (1986).
[18] M. J. de Oliveira, J. F. F. Mendes, and M. A. Santos, J. Phys. A 26, 2317 (1993).
[19] J.-M. Drouffe and C. Godrèche, J. Phys. A: Math. Gen. 32, 249 (1999).
[20] I. Dornic, H. Chaté, J. Chave, and H. Hinrichsen, Phys. Rev. Lett. 87, 045701 (2001).
[21] B. Sandstede and A. Scheel, Phys. Rev. Lett. 86, 171 (2001).
[22] O-U. Kheowan, C-K. Chan, V. S. Zykov, O. Rangsiman, and S. Müller, Phys. Rev. E 64, 035201 (2001).
[23] R. B. Potts, Cambridge Philos. Soc. 49, 106 (1952).
[24] F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
[25] O. G. Mouritsen, *Kinetics of Ordering and Growth at Surfaces* ed. M. G. Lagally (Plenum Press, NY, 1990).