Fermionic Linear Optics Revisited

David P. DiVincenzo and Barbara M. Terhal

October 28, 2018

Abstract

We provide an alternative view of the efficient classical simulatibility of fermionic linear optics in terms of Slater determinants. We investigate the generic effects of two-mode measurements on the Slater number of fermionic states. We argue that most such measurements are not capable (in conjunction with fermion linear optics) of an efficient exact implementation of universal quantum computation. Our arguments do not apply to the two-mode parity measurement, for which exact quantum computation becomes possible, see [1].

Dedication to Asher Peres

It is a pleasure to contribute to the Festschrift for Asher Peres’ 70th birthday. To characterize Asher’s achievements in physics and quantum information theory, we would like to quote from a novel by the Austrian writer Robert Musil (1880-1942) The man without qualities written in 1930 [2]. The protagonist of the novel, Ulrich, is full of praise about science:

But one thing, on the other hand, could safely be said about Ulrich: he loved mathematics because of the kind of people who could not endure it. He was in love with science not so much on scientific as on human grounds. He saw that in all the problems that come within its orbit, science thinks differently from the laity. If we translate “scientific outlook” into “view of life,” “hypothesis” into “attempt,” and “truth” into “action,” then there would be no notable scientist or mathematician whose life’s work, in courage and revolutionary impact, did not far outmatch the greatest deeds of history. The man has not yet been born who could say to his followers: ”You may steal, kill, fornicate - our teaching is so strong that it will transform the cesspool of your sins into clear, sparkling mountain streams.” But in science it happens every few years

*IBM Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, USA.
that something till then held to be in error suddenly revolutionizes the field, or that some dim
and disdained idea becomes the ruler of a new realm of thought. Such events are not merely
upheavals but lead us upward like a Jacob’s ladder. The life of science is as strong and carefree
and glorious as a fairy tale. And Ulrich felt: People simply don’t realize it, they have no idea
how much thinking can be done already; if they could be taught to think a new way, they would
change their lives.

Asher Peres is one of those scientists whose work exemplifies the force of logical thinking and of in-
dependent and unorthodox investigation into the nature of physics, as demonstrated by his wide ranging
publications from relativity theory to quantum mechanics. We are grateful for his “revolutionary and coura-
geous acts” in quantum information theory which have been an inspiration for our own work. We also hope
that Asher will appreciate this small paper whose subject is at the boundary of physics and information, a
boundary that Asher has enjoyed crossing during his productive scientific career.

1 Fermionic linear optics and single-mode measurements

A short introduction to second quantization ¹ will serve to set our notation. Suppose that the Hilbert space
of a single electron has dimension $D$, and is spanned by a standard basis $|i\rangle$, $1 \leq i \leq D$. The label $i$ may
indicate both spin and space degrees of freedom. In the language of second quantization these same basis
vectors are indicated as $a_i^\dagger |0\rangle$. $|0\rangle$ is the basis vector of the Hilbert space corresponding to no electrons
(the vacuum state). $a_i^\dagger$ is the creation operator for an electron in mode (or orbital) $i$. Without the dagger it
is a destruction operator. Creation operators are taken to anticommute ² which enforces the Pauli principle
since $(a_i^\dagger)^2 = 0$, i.e., two electrons cannot be put in the same state.

We are interested in fermionic states that contain not just one electron but $N \leq D$ electrons. In second
quantized language, an example of such a state is

$$\Pi_{i=1}^{N} a_i^\dagger |0\rangle. \quad (1)$$

This state is special not because it places electrons in the standard-basis orbitals, but because it can be

¹ Apparently, the second-quantized analysis of fermions developed over the course of a series of papers, principally by P. Jordan;
the most notable in this series is P. Jordan and E. P. Wigner, “About the Pauli exclusion principle”, Z. Physik 47, 631 (1928). It had
assumed essentially its modern form by the time of V. A. Fock, “Configuration space and second quantization”, Z. Physik 75, 622
(1932). It is reviewed in innumerable modern texts, e.g., A. L. Fetter and J. D. Walecka, Quantum theory of many-particle systems,
(McGraw-Hill, New York, 1971), Chap. 1.

² $\{a_i^\dagger, a_j^\dagger\} = \{a_i, a_j\} = 0$ and $\{a_i, a_j^\dagger\} = \delta_{ij}$. 
written as a single term. In the old language of electron physics, Eq. (1) is an example of a single Slater determinant. A general $N$-electron state is a superposition of such Slater determinants, in which the product over orbitals can run over any set of $N$ orthogonal single-particle basis states.

Recent work on quantum information processing with fermions has focussed on this Slater-determinant characterization of electron states. For example, a proposed measure of entanglement of fermi systems \[3, 4\] is the “Slater number”, the number of terms in the expression for the wave function involving the least number of Slater determinants.

It has been known since the very earliest work in computational physics that the simulation of physical properties of electron systems becomes tractable if the states can be approximated by single Slater determinants (this is an essential feature of the Hartree-Fock approximation). This approximation is exact if the Coulomb interaction between electrons can be ignored. In fact this is a rather drastic approximation, and much of the art of atomic (and solid state) modelling has consisted of finding well chosen “mean fields”, atomic potentials that mimic as well as possible the average effect of the interaction of many electrons in the atom. This endeavor has been rather successful, and has provided a basis for the approximate computation of many quantum properties in atomic, molecular, and solid-state physics, as well as in chemistry.

Recent work by the authors \[5\] (see also Ref. \[6\]) has shown that the restricted quantum computational process of fermionic linear optics can be simulated efficiently on a classical computer. Fermionic linear optics on a set of non-interacting electrons are operations such as beam splitters, phase-shifters (delay lines), von Neumann measurements of the electron state, with the choice of quantum operations potentially based on prior measurement results.

It was not explicitly shown in \[5\] that the Slater number remains one under these operations, and we will show it here and argue that, perhaps not surprisingly, it provides the basis for the classical simulatibility. First, the allowed Hamiltonian evolutions in this model are in the class of “one-body interactions”; that is, they arise from forces between the electrons and the controlling apparatus, and not between different electrons. Such Hamiltonians $H_1$ have non-zero matrix elements only between $N$-particle Slater determinant states $\Phi_1$ and $\Phi_2$ with the same $N$, i.e., the Hamiltonian is number conserving (although an extension to fermion-parity conserving Hamiltonians is possible, and has been worked out in \[5, 6\]); $N - 1$ of the orbitals in $\Phi_1$ and $\Phi_2$ should be identical, and just one may be different in the two states. For example, generally

$$\langle \Phi_1 | H_1 | \Phi_2 \rangle \neq 0 \text{ if } | \Phi_1 \rangle = \Pi_{i=1}^{N} a_i^\dagger | 0 \rangle, \quad | \Phi_2 \rangle = (\Pi_{i=1}^{N-1} a_i^\dagger) a_{N+1}^\dagger | 0 \rangle$$

(2)

but

$$\langle \Phi_1 | H_1 | \Phi_3 \rangle = 0 \text{ if } | \Phi_1 \rangle = \Pi_{i=1}^{N} a_i^\dagger | 0 \rangle, \quad | \Phi_3 \rangle = (\Pi_{i=1}^{N-2} a_i^\dagger) a_{N+1}^\dagger a_{N+2}^\dagger | 0 \rangle$$

(3)
In this last example, the matrix element would be nonzero if two body terms in the Hamiltonian (electron-electron interactions) were included. In general, such a non-interacting Hamiltonian can be written as

\[ H(t) = \sum_{i,j=1}^{D} b_{ij}(t) a_i^\dagger a_j. \]  

Equation (4) introduces the Hermitian time-dependent matrix \( b(t) \). We will use the standard result, reviewed in [5], that the action of the time evolution operator,

\[ U(\tau) = T \exp(-i \int_0^\tau dt H(t)) \]  

(\( T \) is the time-ordering operator), on a creation operator is

\[ U(\tau) a_i^\dagger U(\tau)^\dagger = \sum_m V_{im}(\tau) a_m^\dagger = a_i^\dagger(\tau). \]  

Here the unitary matrix \( V \) is given by

\[ V(\tau) = T \exp(-i \int_0^\tau dt b(t)). \]  

The notation introduced in the last part of Eq. (6) (the \( \tau \) dependence) indicates that the resulting operator is just the creation operator for an electron in the time-evolved orbital

\[ |i(\tau)\rangle = \sum_m V_{im}(\tau) |m\rangle. \]  

Under \( U(\tau) \), then, the initial state Eq. (1) evolves to (using \( U|0\rangle = |0\rangle \)),

\[ \Pi_{i=1}^N a_i^\dagger(\tau) |0\rangle, \]  

i.e., still a single Slater determinant in a rotated basis.

We now turn to the other computational step considered by [5], projective measurement of the occupation of a single orbital (call it \( |\kappa\rangle \)). The projector corresponding to the state being occupied is

\[ P_1 = a_\kappa^\dagger a_\kappa, \]  

and for the unoccupied outcome, the projector is

\[ P_0 = 1 - a_\kappa^\dagger a_\kappa = a_\kappa a_\kappa^\dagger. \]  

What is noteworthy is that both projectors consist of a single product of annihilation and creation operators, which would not be the case for bosons.
Let us now show that the state after measurement, under all circumstances, continues to be a single Slater determinant. That is, we show that

\[ P_0(\prod_{i=1}^N a_i^\dagger(\tau)) | 0 \rangle \]  \hspace{1cm} (12)

and

\[ P_1(\prod_{i=1}^N a_i^\dagger(\tau)) | 0 \rangle \]  \hspace{1cm} (13)

are single Slater determinants, of a very simple form. The technical steps are described in an Appendix of [4], we give a simple version of them here for completeness. We first write the orbital \( | \kappa \rangle \) as a linear combination of an orbital in the space spanned by the set \( \{ | i(\tau) \rangle \} \), and an orbital not in that span:

\[ | \kappa \rangle = \alpha | \text{in} \rangle + \beta | \text{out} \rangle \]  \hspace{1cm} (14)

where the two new normalized vectors are defined by

\[ | \text{in} \rangle \in \text{Span}(\{ | i(\tau) \rangle \}), \]  \hspace{1cm} (15)

\[ | \text{out} \rangle \in \text{Ker}(\{ | i(\tau) \rangle \}). \]  \hspace{1cm} (16)

The phase of these orbitals can be chosen so that the coefficients \( \alpha \) and \( \beta \) are real and nonnegative, and \( \alpha^2 + \beta^2 = 1 \). We also rewrite the Slater determinant in terms of a new orthogonal basis of orbitals:

\[ \prod_{i=1}^N a_i^\dagger(\tau) | 0 \rangle = a_{\text{in}}^\dagger \prod_{\mu} a_{\mu}^\dagger | 0 \rangle, \]  \hspace{1cm} (17)

Here we introduce new orbitals \( | m \mu \rangle, \mu = 2, 3, \ldots, N \), such that \( \text{Span}(\{ | i(\tau) \rangle \}) = \text{Span}(\{ | \text{in} \rangle \}, \{ | \mu \rangle \}) \), that is, the space of filled orbitals remains the same. Eq. (17) can be shown by the following steps. Let \( U \) be a fermionic linear-optics transformation such that for \( i = 1, \ldots, N \)

\[ U a_i^\dagger U^\dagger = \sum_{m=1}^D V_{im} a_m^\dagger, \]  \hspace{1cm} (18)

where the \( D \times D \) unitary matrix \( V = \mathcal{V} \oplus \mathcal{W} \) is a block-diagonal matrix with \( \mathcal{V} \in \text{SU}(N) \). Furthermore \( \mathcal{V} \) is such that \( | 1(\tau) \rangle \) is rotated to \( | \text{in} \rangle \) and \( | i(\tau) \rangle, i > 1 \), is rotated to \( | \mu \rangle, \mu = 2, 3, \ldots, N \). Thus, as required, the rotation \( \mathcal{V} \) does not change the space of filled orbitals. By inserting \( U^\dagger U = \mathbf{I} \) and using \( U | 0 \rangle = | 0 \rangle \) we can see that

\[ U \prod_{i=1}^N a_i^\dagger(\tau) | 0 \rangle = a_{\text{in}}^\dagger \prod_{\mu} a_{\mu}^\dagger | 0 \rangle, \]  \hspace{1cm} (19)
while we can also write
\[ U \sum_{i=1}^{N} a_i^\dagger = \sum_{i=1}^{N} V_{1i} a_i^\dagger \sum_{j=1}^{N} V_{2j} a_j^\dagger \sum_{k=1}^{N} V_{3k} a_k^\dagger \ldots \] (20)

There are \( N \) modes, and \( N \) sums in this expression. In order for a term to be nonzero, each mode must appear once and only once (otherwise a mode would be repeated, and \((a^\dagger)^2 = 0\)). Thus we can write using the anti-commutation relations that
\[ U \Pi_i^{N} a_i^\dagger = \prod_{\pi} \text{sign}(\pi) V_{1,\pi(1)} V_{2,\pi(2)} V_{3,\pi(3)} \Pi_i^{N} a_i^\dagger = \text{det} (\mathcal{V}) \Pi_i^{N} a_i^\dagger = \Pi_i^{N} a_i^\dagger. \] (21)

Thus, Eq.(17) is established.

Now we can calculate:
\[ P_1(\Pi_i^{N} a_i^\dagger(\tau)|0) = a_{i\kappa}(\Pi_i^{N} a_i^\dagger(\tau)|0) = a_{i\kappa}(\alpha a_{i\text{in}} + \beta a_{i\text{out}})a_{i\text{in}}^\dagger \Pi_{\mu} a_{\mu}^\dagger|0 = \alpha a_{i\kappa} \Pi_{\mu} a_{\mu}^\dagger|0. \] (22)

Here we have used the fact that \( a_{\text{out}}|0) = 0 \) and \( a_{\text{in}} a_{\text{in}}^\dagger|0 = |0). \) We note that the final state is indeed just another Slater determinant; it is unnormalized, but the coefficient \( \alpha \) just reflects the fact that the probability of this outcome is \( \alpha^2 \). Note that \( \alpha \) is easy to calculate, as it is just the magnitude of the projection of a vector in the single-particle space of dimension \( D \). The outcome of the other projector just takes a little bit more to evaluate:
\[ P_0(\Pi_i^{N} (a_i^\dagger))^\dagger|0) = a_{i\kappa}(\Pi_i^{N} (a_i^\dagger)|0) = a_{i\kappa}(\alpha a_{i\text{in}}^\dagger + \beta a_{i\text{out}}^\dagger)\Pi_{\mu} a_{\mu}^\dagger|0 = \beta a_{i\kappa} a_{i\text{in}}^\dagger \Pi_{\mu} a_{\mu}^\dagger|0. \] (23)

Here we have used \((a_{i\text{in}}^\dagger)^2 = 0\). To go further, we have to introduce another orbital \(|\kappa^{\perp}\rangle\rangle \) perpendicular to \(|\kappa\rangle\rangle \) in the space spanned by \(|\text{in}\rangle\rangle \) and \(|\text{out}\rangle\rangle \); specifically,
\[ |\kappa^{\perp}\rangle = \beta |\text{in}\rangle - \alpha |\text{out}\rangle. \] (24)

Then using the relation
\[ a_{\text{out}}^\dagger a_{\text{in}}^\dagger = a_{\kappa} a_{\kappa^{\perp}}^\dagger, \] (25)
we finish the derivation:
\[ P_0(\Pi_i^{N} a_i^\dagger(\tau)|0) = \beta a_{i\kappa} a_{i\text{in}}^\dagger \Pi_{\mu} a_{\mu}^\dagger|0 = \beta a_{i\kappa} a_{i\kappa^{\perp}}^\dagger \Pi_{\mu} a_{\mu}^\dagger|0 = \beta a_{i\kappa} a_{i\kappa^{\perp}}^\dagger \Pi_{\mu} a_{\mu}^\dagger|0. \] (26)

So all operations keep the state vector in the Slater-determinant form. Indeed, we note that except for the change of normalization, the action of \( P_{0,1} \) on the state is identical to that of some single-particle
Hamiltonian. This gives a new view on the classical simulatability of this suite of operations. Appendix A gives some correspondence between these calculations and ones that appear in the theory of electron energy bands in crystals.

2 Two-mode measurements

We now consider a different scenario for quantum computation, one in which one can perform a two-mode measurement. This can for example be a “charge measurement” that determines how many electrons are in a particular spatial orbital irrespective of their spin state. This means that we imagine a two-mode measurement, in which the two modes are identical spatially and differ only in their spin quantum number (so we might indicate these two orbitals $|m \uparrow\rangle$ and $|m \downarrow\rangle$, for some spatial orbital $m$). In the present analysis we will not wish to make the distinction between spin and orbital labels, so that we will just consider a number measurement in two orthogonal modes labelled $|\kappa\rangle$ and $|\lambda\rangle$. (This generalization means that our analysis is applicable to problems involving spin-orbit interaction, where the distinction between spin and space is not applicable.)

We will assume that this measurement is “nondestructive”, a feature of recent charge measurements in quantum dot structures [7] (but, see the remarks in the Discussion section). Then, similar to a single-mode measurement, we must write down the ‘operation elements’ (see [8]) of the measurement which we take to be projectors (see the Discussion for potential modifications).

For the “0” outcome (both modes unoccupied), this is simple, since it is just given by the product of the two “0” projectors for the two modes separately:

$$P_0 = P_{0\kappa}P_{0\lambda} = a^\dagger_\kappa a_\kappa a^\dagger_\lambda a_\lambda.$$  \hspace{1cm} (27)

The “2” outcome projector (both orbitals occupied) is also simply the product of the two one-orbital projectors:

$$P_2 = P_{1\kappa}P_{1\lambda} = a^\dagger_\kappa a_\kappa a^\dagger_\lambda a_\lambda.$$  \hspace{1cm} (28)

The “1” projector can be written as a sum of two projector products:

$$P_1 = a^\dagger_\kappa a_\kappa a^\dagger_\lambda a_\lambda + a^\dagger_\kappa a_\kappa a_\lambda a^\dagger_\lambda.$$  \hspace{1cm} (29)

The important point for the upcoming analysis is

**Lemma 1** $P_1$ cannot be expressed as a single term; that is, it is not possible to write $P_1$ as $P_1 = f_1 f_2 f_3 ... f_M$, where $M$ is an arbitrary integer, and each $f_i$ is either a creation or an annihilation operator for some arbitrary mode.
**Proof:** We have to study two cases:

1) $f_M$ is a creation operator. We consider a one-particle basis consisting of orbitals $|\kappa\rangle$ and $|\lambda\rangle$, and $D - 2$ orbitals $|\nu\rangle$ orthogonal to $|\kappa\rangle$ and $|\lambda\rangle$. Then the orbital $\phi_M$ that $f_M$ creates can be written

$$|\phi_M\rangle = \alpha |\kappa\rangle + \beta |\lambda\rangle + \sum_{\nu} c_{\nu} |\nu\rangle.$$  

(30)

Consider the (unnormalized) state

$$|\Phi\rangle = (\alpha a_{\kappa}^\dagger + \beta a_{\lambda}^\dagger)\Pi_{\nu} a_{\nu}^\dagger |0\rangle.$$  

(31)

This state has one electron in the space $\text{Span}(|\kappa\rangle, |\lambda\rangle)$, so it is an eigenstate of $P_1$ with eigenvalue 1:

$$P_1 |\Phi\rangle = |\Phi\rangle.$$  

(32)

But $f_M$ annihilates $|\Phi\rangle$ (recall that $(a_x^\dagger)^2 = 0$ for any $x$):

$$f_M |\Phi\rangle = (\alpha a_{\kappa}^\dagger + \beta a_{\lambda}^\dagger + \sum_{\nu} c_{\nu} a_{\nu}^\dagger)(\alpha a_{\kappa}^\dagger + \beta a_{\lambda}^\dagger)\Pi_{\nu} a_{\nu}^\dagger |0\rangle = 0,$$

so the single-term expression $f_1 f_2 f_3 \ldots f_M$ cannot equal $P_1$ in this case.

2) $f_M$ is an annihilation operator. We consider the same orbital expansion as in Eq. (30), and we apply $f_M$ to the state

$$|\Psi\rangle = (\alpha^* a_{\kappa} + \beta^* a_{\lambda})a_{\kappa}^\dagger a_{\lambda}^\dagger |0\rangle.$$  

(34)

This is again an eigenstate of $P_1$, $P_1 |\Psi\rangle = |\Psi\rangle$. However,

$$f_M |\Psi\rangle = (\alpha^* a_{\kappa} + \beta^* a_{\lambda} + \sum_{\nu} c_{\nu}^* a_{\nu}) (\alpha^* a_{\kappa} + \beta^* a_{\lambda}) a_{\kappa}^\dagger a_{\lambda}^\dagger |0\rangle = 0,$$

(35)

(since $(a_x)^2 = 0$ and $a_x |0\rangle = 0$). So, in this case too the single-term expression cannot match $P_1$. □

So, the fact that $P_1$ cannot be written as a single term opens the possibility that this two-mode measurement can lead to more complex quantum time evolution, and thus has the possibility of implementing quantum computation. In particular, when the “1” outcome is obtained, the fact that the minimal expression for $P_1$ contains two terms means that the Slater number (recall above, see [4]) could double after every such measurement; so, if there are $M$ such “1” outcomes, then the state may have an exponentially large $(2^M)$ Slater number, a state for which expectation values are likely to be very hard to calculate classically.
We do not know how to prove that the Slater number will in fact be as large as $2^M$ although we will now prove that generically, when $P_1$ is applied to a single Slater determinant, the result has Slater number two. Nevertheless, the expectation that the Slater number becomes high, and the evolution becomes difficult to simulate, is vindicated by the discovery of Beenakker et al.\[1\] that quantum computation is implementable by linear fermion optics and the two-mode measurement! We will return to more discussion of this measurement after Sec. 3.

### 3 Slater number generically goes from one to two under $P_1$

We now show that if we apply the two-mode projector $P_1$ to a single Slater determinant for $N \geq 2$ electrons, \[|\Psi\rangle = P_1 \Pi_{i=1}^{N} a_i^\dagger |0\rangle = (a_\kappa a_\kappa^\dagger a_\lambda^\dagger a_\lambda + a_\kappa^\dagger a_\kappa a_\lambda a_\lambda^\dagger) \Pi_{i=1}^{N} a_i^\dagger |0\rangle, \] (36) then the resulting state $|\Psi\rangle$ generically has Slater number two. Note that we can always choose a basis such that the initial state has the standard form shown. Furthermore, without loss of generality, we can parameterize the orthogonal orbitals $|\kappa\rangle$ and $|\lambda\rangle$ as \[|\kappa\rangle = \cos \theta |1\rangle + \sin \theta |N + 1\rangle, \] (37) \[|\lambda\rangle = \cos \phi (\sin \theta |1\rangle + \cos \theta |N + 1\rangle) + \sin \phi (|2\rangle + \sin \xi |N + 2\rangle). \] (38)

We can simplify the problem considerably by using a theorem of K. Eckert et al.\[4\], that the Slater number cannot be increased by applying an annihilation operator to a state. Since $|\kappa\rangle$ and $|\lambda\rangle$ do not involve orbitals $|3\rangle$, $|4\rangle$, $|N\rangle$, we can annihilate all of these and be left with a two-electron state: \[|\Psi'\rangle = \Pi_{i=3}^{N} a_i |\Psi\rangle = (a_\kappa a_\kappa^\dagger a_\lambda^\dagger a_\lambda + a_\kappa^\dagger a_\kappa a_\lambda a_\lambda^\dagger) a_1^\dagger a_2^\dagger |0\rangle. \] (39) Using the methods of Sec. 4 we can convert each term of this expression into one involving just two creation operators. After a lengthy calculation (using Mathematica) we find \[|\Psi'\rangle = \sum_{i,j=1,2,N+1,N+2} w_{ij} a_i^\dagger a_j^\dagger |0\rangle. \] (40) Where the $4 \times 4$ antisymmetric matrix $w$ is

\[
\begin{pmatrix}
0 & f_S \cos \theta - f_C \sin \theta & -\frac{\cos \xi \sin 2\phi}{2f_S} & -\frac{\cos \theta \sin^2 \phi \sin 2\xi}{2f_S} & \frac{\sin \theta \sin^2 \phi \sin 2\xi}{2f_C} \\
\cdot & 0 & -f_C \cos \theta - f_S \sin \theta & -\frac{\sin 2\phi \sin \xi}{2f_C} & \frac{\cos \theta \sin^2 \phi \sin 2\xi}{2f_S} \\
\cdot & \cdot & 0 & \frac{\cos \theta \sin^2 \phi \sin 2\xi}{2f_C} & \frac{\sin \theta \sin^2 \phi \sin 2\xi}{2f_S} \\
\cdot & \cdot & \cdot & 0 & 0
\end{pmatrix}
\] (41)
(we don’t show the lower triangle of this antisymmetric matrix), with
\[ f_C = \sqrt{\cos^2 \phi + \cos^2 \xi \sin^2 \phi}, \]
\[ f_S = \sqrt{\cos^2 \phi + \sin^2 \xi \sin^2 \phi}. \] (42)

As discussed by [4], a basis transformation can be made to bring an antisymmetric matrix to a canonical form, consisting of a direct sum of 2 × 2 antisymmetric blocks. The number of nonzero blocks is the Slater number. Obviously, for a 4 × 4 matrix the Slater number is two iff both blocks are nonzero, and iff the determinant of the matrix is nonzero. For an antisymmetric matrix it is more convenient to evaluate the Pfaffian, which is the square root of the determinant. For \( w \) we find that the Pfaffian is
\[ \text{Pf}(w) = \frac{\sin^2 \phi \sin 2\xi}{2f_S f_C}. \] (43)

So, we see that generically, \( \mathbf{P}_1 \) does indeed increase the Slater number from one to two.

4 A no-go theorem

We now explore further the power of nondestructive two-mode measurements. As noted above, Beenakker et al. [1] have shown that the two-mode electric charge measurement above, in conjunction with linear fermion optics, permits the efficient implementation of quantum computation. It was noted, however, that like most of the linear photon optics schemes proposed to date (cf. [9]), this implementation using the three-outcome charge measurement is non-deterministic, i.e., there is some finite chance that the computation fails (in the case of an unlucky combination of measurement outcomes), although the overall probability of failure can be made acceptably low by a suitable implementation strategy. We now argue that this small probability of failure is intrinsic to this implementation:

**Theorem 1** If there exists an efficient implementation of the unitary evolution of a quantum circuit using linear fermion optics (including single-mode measurements) and the three-outcome, two-mode measurement of Sec. 2 that is exact, i.e., has zero probability of failure, then this unitary evolution has an efficient classical simulation.

**Proof:** Suppose the exact implementation exists. The efficient classical simulation of this unitary evolution proceeds as follows: We begin with the standard, single Slater determinant state of Eq. (1). We compute the effect of the first stage of fermionic linear optics on this state as in Eq. (9). Then, we consider the first 0/1/2 charge measurement. We can calculate whether the probability for outcome “1” is 100% or not
by computing the action of $P_1$ on the state (a simple calculation for a single Slater determinant). If the probability of “1” is 100%, if the state is an eigenstate of $P_1$ with eigenvalue one, then the measurement has no effect on the state, and we proceed on with the next stage of computation. If the probability of “1” is not 100%, then at least one of the outcomes “0” or “2” has nonzero probability. Since the implementation of the quantum gates is supposed to be exact (i.e. works for every measurement outcome) we are free to choose any outcome that occurs with non-zero probability. So we choose 0 or 2 (making sure the choice has nonzero probability) and then note that the state after application of $P_0$ or $P_2$ is still a single Slater determinant. By proceeding thus, the classical simulation at all stages need only keep track of a single Slater determinant, which is efficiently doable.

\[ \square \]

Remark: In this proof we have assumed exact classical real-number computation. The proof can be relaxed to treat the case of finite precision classical computations; in that case the quantum computation is simulated approximately, but always with high precision.

Thus if we believe (which we do) that there does not exist an efficient classical simulation of the unitary evolution of polynomially-sized quantum circuits\(^3\), it follows by this Theorem that there will be no exact implementation of quantum circuits using fermionic linear optics and the two-mode three-outcome measurement.

We can modify the two-mode measurement such that some of the outcomes are not distinguished and see what happens. For example, we can consider a two-outcome measurement with projectors $P_{0,1} = P_0 + P_1$ and $P_2$, which only distinguishes whether the two modes are completely filled or not. All such grouped measurements can function in the nondeterministic implementation of quantum computation of \[1\]. But

**Corollary 1** Theorem [1] still holds if the three-outcome measurement is replaced by the two-outcome measurement $P_{0,1}/P_2$, or $P_{1,2}/P_0$.

**Proof:** For both measurements there is an outcome ($P_2$ in the first case, $P_0$ in the second) for which the simulated state remains a single Slater determinant. The rest of the proof then applies.

\[ \square \]

However, for one measurement (the “parity” measurement), this argument does not apply:

*The no-go theorem does not apply to the parity measurement $P_{0,2}/P_1$.*

It would apply if one of the projectors could be written as a single term. We have already demonstrated that $P_1$ cannot be written as a single term. This is easy to show for the projector $P_{0,2} = P_0 + P_2$ as well, by similar arguments: If $f_M$ (see Lemma 1) is a creation operator, it annihilates the $D$-electron Slater determinant.

\(^3\)Note this does not include the final single qubit measurements.
determinant, which is not annihilated by $P_{0,2}$; if $f_M$ is a destruction operator, it annihilates the vacuum $|0\rangle$, which is not annihilated by $P_{0,2}$.

It was this observation that led Beenakker et al. to investigate alternative implementations of quantum circuits using linear fermion optics using the two-mode parity measurement; and, indeed, an exact simulation, which is in some sense much more efficient than any of the known non-deterministic simulations, exists!

5 Discussion

We have presented an alternative description of the fermionic linear optics computation. It is likely that the extension to “fermion-parity preserving” quadratic Hamiltonians which was treated in Ref. [5], can be analyzed similarly using Slater determinants.

We want to close with a few words of caution about the applicability of our results. We have indicated that the two-mode measurement that enables quantum computation is “nondestructive” and uses projective measurement ‘elements’. What happens if we relax these conditions?

If the measurement is destructive, it means that the modes $|\kappa\rangle$ and $|\lambda\rangle$ are no longer available for further processing. The “tracing out” of these two modes that this throwing away implies is implemented in second quantization in the following way: the density matrix of the system, after the application of the two-mode projectors discussed above, is changed by the application of two trace-preserving completely positive maps, $T_\kappa$ and $T_\lambda$. The trace-over-$\zeta$ map $T_\zeta$ is given by

$$\rho' = T_\zeta(\rho) = \sum_{i=1}^{2} A_i \rho A_i^\dagger, \quad A_1 = a_\zeta a_\zeta^\dagger, \quad A_2 = a_\zeta^\dagger a_\zeta.$$  (44)

This map leaves the one-mode measurements unchanged; but the two-mode measurements are changed in a very important way. In particular, for any of the two-mode measurements discussed above, the tracing out leaves the system in a density matrix that is a mixture of single Slater determinants. The evolution of such states is simple (that is, efficiently simulatable on a classical computer), so destructive measurements give none of the quantum computational power of nondestructive ones. This has been anticipated in earlier studies of quantum measurements for quantum computation [10, 11].

Another important modification of the measurement is the following. Instead of the two-mode parity measurement with measurement elements $P_{0,2}$ and $P_1$, suppose we have a two-mode parity measurement with measurement elements $P_{0,2}$ and $P_1$, suppose we have a two-mode parity measurement.

---

4 Actually, the simpler choice of Kraus operators $A_1 = a_\zeta^\dagger, A_2 = a_\zeta$ has the equivalent effect. This corresponds to going to a hole representation for the mode $\zeta$. 

with measurement elements $U_{int}P_{0,2}$, (i.e. not a projector), and $P_1$, where $U_{int}$ is a charge-preserving unitary interaction. The probabilities of outcome of these two measurements are the same but the state after the outcome $0/2$ has occurred will have undergone an additional unitary transformation $U_{int}$ in the second type of measurement.

The status of the no-go theorems is the same for these two measurements, but the construction in Ref. [1] only applies to the first.

This could be important, as there may be situations where $U_{int}$ is nontrivial. In particular, the Beenakker construction [1] is isomorphic to one in which the qubit is coded by one electron in a double quantum dot, with occupation of the orbital in the left dot representing $|0\rangle$ and right-dot orbital occupied being $|1\rangle$ [12]. Then, as is suggested in [1], the required charge parity measurement might be accomplished by placing a single-electron transistor between the right dot of one qubit and the left dot of the adjacent qubit, so that it is sensitive to the charge on both (and can be tuned so that it reads one value of current for both dots empty or occupied, and another level otherwise). However, an analysis of this setup [13] might reveal that $U_{int}$ is nontrivial in this case due to an effective interaction which the measurement sets up between the electrons in the two qubits. More analysis would be worthwhile in this case; and if, in this or in similar situations, $U_{int}$ turns out to be different from the identity, it would be worthwhile to work out an implementation of quantum gates for this case.

**Acknowledgments**

We are grateful for the support of the National Security Agency and the Advanced Research and Development Activity through contract DAAD19-01-C-0056.

**Appendix A: Application to band theory**

The analysis of Sec. [1] can be applied to simple model problems in electron band theory. Not surprisingly, given the long history of band theory, see e.g. [14], some of the objects obtained above have special names, and special significance, in this setting.

Suppose we consider a non-interacting Hamiltonian for electrons on a one-dimensional lattice. If the Hamiltonian only contains nearest-neighbor hopping terms, $t a_i^\dagger a_{i+1}$, and $t < 0$, then the ground state of the system is the Slater determinant

$$\Pi_{|k| \leq k_F} N a_k^\dagger |0\rangle. \tag{45}$$
The orbital $a_k^\dagger |0\rangle = |k\rangle$ is the plane wave

$$|k\rangle = \frac{1}{\sqrt{D}} \sum_x e^{ikx} |x\rangle. \quad (46)$$

Here $D$ is the number of lattice sites (we assume periodic boundary conditions), $a_k^\dagger |x\rangle = |x\rangle$ is the orbital centered at site $x$, $(x = 0, 1, 2, ..., D − 1)$, and $k$ assumes the values $k = 2\pi n / D$ ($k$ lives in the reciprocal space of the crystal), with integer $-(D − 1)/2 \leq n \leq (D − 1)/2$ (The electrons are, in this example, spinless). The filled states have $|k| \leq 2\pi(N − 1)/(2D) = k_F$ where the Fermi-wavenumber $k_F$ is $2\pi(N − 1)/(2D) \simeq \pi \nu$ for $N >> 1$. We assume that $N$ is odd. Here $\nu = ND$ is the filling of the band (the number of electrons per orbital $|x\rangle$). Note that the empty states are those with $k_F < |k| < \pi$, and $k = \pi$, but not $k = −\pi$, corresponding to the rule that $k$s differing by a reciprocal lattice vector, in particular those lying on the boundary of the first Brillouin zone, should not be counted twice.

Suppose that a measurement is done that reveals that an electron is present at the origin. What is the new Slater determinant describing the state? That is, we are specializing the development in the text to the case

$$|\kappa\rangle = |x = 0\rangle = \frac{1}{\sqrt{D}} \sum_k |k\rangle, \quad (47)$$

$$|\text{in}\rangle = \frac{1}{\sqrt{N}} \sum_{|k|\leq k_F} |k\rangle = |W_0\rangle, \quad (48)$$

$$|\text{out}\rangle = \frac{1}{\sqrt{D-N}} \sum_{|k|>k_F} |k\rangle, \quad (49)$$

$$|\kappa\rangle = \sqrt{\nu} |\text{in}\rangle + \sqrt{1-\nu} |\text{out}\rangle, \quad (50)$$

$$|W_s\rangle = \frac{1}{\sqrt{N}} \sum_{|k|\leq k_F} e^{iks}|k\rangle, \quad s = 0, 1, ..., N − 1. \quad (51)$$

Here we have introduced the orbitals $|W_s\rangle$, which are obtained by a Fourier transform over the filled states $|k\rangle$. They are somewhat localized on the lattice, but not perfectly, since they only include the plane waves up to a certain wavelength. The wave function of $W_0$ is

$$W_0(x) = \langle x |W_0\rangle = \frac{\sin(\pi \nu x)}{\pi \nu x}. \quad (52)$$

The other orbitals $|W_s\rangle$, $s \neq 0$, are basically displaced versions of $W_0$:

$$W_s(x) = W_0(x − s/\nu). \quad (53)$$

Note, however, that an analytic continuation of $x$ to the reals is understood here, since for general $\nu$ the $W_s$ wave functions are generally not centered on lattice sites.
The orbitals $|W_s\rangle$ rather resemble the Wannier functions of band theory, in that they are approximately localized states built out of band orbitals. They are different, though, in that Wannier functions are generally defined for full bands only, i.e., only for $\nu = 1$.

So, again, how is the fermi sea modified if the electron number is measured at the origin? With probability $\nu$ the answer will be “1”, and then the new Fermi sea has the $W_0$ orbital replaced by the completely localized orbital $|\kappa\rangle = |0\rangle$, and all the rest unperturbed:

$$a^\dagger_{\kappa} \Pi_{s \neq 0} a^\dagger_{W_s} |0\rangle. \quad (54)$$

One can say that one orbitals’ worth of electrons has been pulled out from $O(1/\nu)$ lattice sites around the origin and concentrated at $x = 0$. The hole that is left in the fermi sea by this process is what is known as the exchange hole in electron physics [15].

With probability $1 - \nu$ the measurement gives answer “0”; then the $W_0$ orbital replaced by $|\kappa^\perp\rangle$:

$$a^\dagger_{\kappa^\perp} \Pi_{s \neq 0} a^\dagger_{W_s} |0\rangle. \quad (55)$$

This modified orbital can be rewritten in an informative way:

$$|\kappa^\perp\rangle = \sqrt{1 - \nu} |\text{in}\rangle - \sqrt{\nu} |\text{out}\rangle = -\sqrt{\nu} \frac{1}{1 - \nu} |0\rangle + \sqrt{\frac{1}{1 - \nu}} |W_0\rangle. \quad (56)$$

We find that the wave function for this orbital $\langle x |\kappa^\perp\rangle$ is zero for $x = 0$, as expected, and also has an exchange hole, but with a reversed sign compared with the other measurement outcome, and with a magnitude that depends on $\nu$. If $\nu$ is near one, the perturbation of the fermi sea is very strong, but also this outcome “0” occurs with vanishingly small probability.

References

[1] C.W.J. Beenakker, D.P. DiVincenzo, C. Emary, and M. Kindermann. Charge detection enables free-electron quantum computation. [http://arxiv.org/abs/quant-ph/0401066](http://arxiv.org/abs/quant-ph/0401066).

[2] Robert Musil. *The man without qualities, Vol.2*. Vintage, 1996. Translation by Sophie Wilkins and Burton Pike.

[3] J. Schliemann, J.I. Cirac, M. Kuś, M. Lewenstein, and D. Loss. Quantum correlations in two-fermion systems. *Phys. Rev. A*, 64:022303, 2001, [http://arxiv.org/abs/quant-ph/0012094](http://arxiv.org/abs/quant-ph/0012094).
[4] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein. Quantum correlations in systems of indistinguishable particles. *Annals of Physics*, 299:88–127, 2002, http://arxiv.org/abs/quant-ph/0203060

[5] B.M. Terhal and D.P. DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. *Phys. Rev. A*, 65:032325/1–10, 2002, http://arxiv.org/abs/quant-ph/0108010

[6] E. Knill. Fermionic linear optics and matchgates. 2001, http://arxiv.org/abs/quant-ph/0108033

[7] R. Hanson, B. Witkamp, L.M.K. Vandersypen, L.H. Willems van Beveren, J.M. Elzerman, and L.P. Kouwenhoven. Zeeman energy and spin relaxation in a one-electron quantum dot. *Phys. Rev. Lett.*, 91:196802, 2002, http://arxiv.org/abs/cond-mat/0303139

[8] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, U.K., 2000.

[9] E. Knill, R. Laflamme, and G. Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409:46–52, 2001, http://arxiv.org/abs/quant-ph/0006088

[10] M.A. Nielsen. Quantum computation by measurement and quantum memory. *Physics Letters A*, 308:96–100, 2003, http://arxiv.org/abs/quant-ph/0108020

[11] D. Leung. Two-qubit projective measurements are universal for quantum computation. Technical report, NSF-ITP-01-174, 2001, http://arxiv.org/abs/quant-ph/0111122

[12] A. Barenco, D. Deutsch, A.K. Ekert, and R. Jozsa. Conditional quantum dynamics and logic gates. *Phys. Rev. Lett.*, 74:4083–4086, 1995, http://arxiv.org/abs/quant-ph/9503017

[13] S.A. Gurvitz. Measurements with a noninvasive detector and dephasing mechanism. 56:15215, 1997, http://arxiv.org/abs/cond-mat/9706074

[14] N. W. Ashcroft and N. D. Mermin. *Solid State Physics*. Saunders College, Philadelphia, 1976.

[15] D. Pines. *Elementary excitations in solids*. Benjamin, Reading, 1963.