Generalized geometry and the Hodge decomposition

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Abstract

In this lecture, delivered at the string theory and geometry workshop in Oberwolfach, we review some of the concepts of generalized geometry, as introduced by Hitchin and developed in the speaker’s thesis. We also prove a Hodge decomposition for the twisted cohomology of a compact generalized Kähler manifold, as well as a generalization of the $\dd\bar{\partial}$-lemma of Kähler geometry.

1 Geometry of $T \oplus T^*$

The sum $T \oplus T^*$ of the tangent and cotangent bundles of an $n$-dimensional manifold has a natural $O(n, n)$ structure given by the inner product

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\xi(Y) + \eta(X)),$$

and we may describe the Lie algebra in the usual way:

$$\mathfrak{so}(n, n) = \wedge^2 T \oplus \text{End}(T) \oplus \wedge^2 T^*.$$

Hence we may view 2-forms $B$ and bivectors $\beta$ as infinitesimal symmetries of $T \oplus T^*$. We may also form the Clifford algebra $\text{CL}(T \oplus T^*)$, which has a spin representation on the Clifford module $\wedge \cdot T^*$ as described by Cartan:

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho,$$

for $X + \xi \in T \oplus T^*$ and $\rho \in \wedge \cdot T^*$. This means that we may view differential forms as spinors$^1$ for $T \oplus T^*$.

From the general theory of spinors, this implies that there is a $\text{Spin}_o(n, n)$-invariant bilinear form $\langle \cdot, \cdot \rangle : \wedge \cdot T^* \times \wedge \cdot T^* \longrightarrow \text{det} T^*$, given by $\langle \alpha, \beta \rangle = [\alpha \wedge \sigma(\beta)]_n$, where $\sigma$ is the anti-automorphism which reverses the order of wedge product.

Another structure emerging from the interpretation of forms as spinors is the Courant bracket $[,]_H$ on sections of $T \oplus T^*$, obtained as the derived bracket (see [7]) of the natural differential operator $d + H \wedge \cdot$ acting on differential forms, where $d$ is the exterior derivative and $H \in \Omega^3_{cl}(M)$. When $H = 0$, we have the following:

**Proposition 1.** The group of orthogonal automorphisms of the Courant bracket for $H = 0$ is a semidirect product of $\text{Diff}(M)$ and $\Omega^2_{cl}(M)$, where $B \in \Omega^2_{cl}(M)$ acts as the shear $\exp(B)$ on $T \oplus T^*$.

In this way we see that the natural appearance of $H \in \Omega^3_{cl}(M)$ and the action of $B \in \Omega^2_{cl}(M)$ coincide precisely with the physicists’ description of the Neveu–Schwarz 3-form flux and the action of the B-field, respectively.

$^1$Actually, the bundle of spinors differs from $\wedge \cdot T^*$ by tensoring with the line bundle $\text{det} T^{1/2}$; we assume a trivialization has been chosen – this is related to the physicists’ dilaton field.
2 Generalized complex geometry

A generalized complex structure is an integrable reduction of the structure group of $T \oplus T^*$ from $O(2n, 2n)$ to $U(n, n)$ (only possible when $\dim_{\mathbb{R}} M = 2n$). This is equivalent to the choice of an orthogonal complex structure

$$\mathcal{J} \in O(T \oplus T^*), \quad \mathcal{J}^2 = -1.$$ 

The integrability condition is that the $+i$-eigenbundle of $\mathcal{J}$,

$$E < (T \oplus T^*) \otimes \mathbb{C},$$ 

must be closed under the Courant bracket. If $H$ is nonzero we call this a twisted generalized complex structure. The Courant bracket is a Lie bracket when restricted to $E$ and therefore we may form the associated differential graded algebra:

$$\mathcal{E} = (\wedge^* E^*, d_E).$$

Because $E^*$ is identified with $\overline{E}$ by the metric on $T \oplus T^*$, we see that it also acquires a Lie bracket. By a general result of Lu, Weinstein, and Xu, this Lie bracket makes $\mathcal{E}$ into a differential Gerstenhaber algebra.

**Theorem 2.1.** The differential Gerstenhaber algebra $\mathcal{E}$ is elliptic and it gives rise to a Kuranishi deformation theory for any generalized complex structure. The tangent space to the deformation space, in the unobstructed case, is $H^2(\mathcal{E})$, and obstructions lie in $H^3(\mathcal{E})$.

For example, let $J \in \text{End}(T)$ be a usual complex structure, and form the generalized complex structure

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

Then, $E = T_{0,1} \oplus T^*_{1,0}$, so that $\mathcal{E}$ is simply the Dolbeault complex of the holomorphic multivectors. Consequently,

$$H^2(\mathcal{E}) = H^0(\wedge^2 T) \oplus H^1(T) \oplus H^2(O).$$

For a complex surface, a holomorphic bivector $\beta$ always integrates to an actual deformation, and so for $\mathbb{C}P^2$, for example, we obtain a new generalized complex structure which is complex along an anticanonical divisor (the vanishing locus of $\beta$) and the B-field transform of a symplectic structure away from the cubic.

This provides an alternative interpretation of the extended deformation parameter $\beta$, which is normally viewed as a noncommutative deformation of the algebra defining $\mathbb{C}P^2$. Note that the usual translation parameter along the commutative elliptic curve can be obtained by differentiating $\beta$ along its vanishing set.

The previous example indicates that the algebraic type of a generalized complex structure may jump along loci in the manifold. Indeed, a generalized complex structure on a $2n$-manifold may have types $0, \cdots, n$, with $0$ denoting the (generic) symplectic type and $n$ denoting the complex type. Type may jump up, but only by an even number.

**Theorem 2.2 (Generalized Darboux theorem \textsuperscript{3}).** Away from type jumping loci, a generalized complex manifold of type $k$ is locally isomorphic, via a diffeomorphism and a $B$-field transform, to $\mathbb{C}^k \times \mathbb{R}^{2n-2k}$, where $\omega_0$ is the usual Darboux symplectic form.

Generalized complex manifolds also have natural sub-objects, called generalized complex submanifolds \textsuperscript{3}. These sub-objects correspond exactly with the physicists’ notion of topological D-branes; in particular, one recovers, in the symplectic case, the co-isotropic A-branes of Kapustin and Orlov \textsuperscript{6}. There is also a natural notion of generalized holomorphic bundle supported on a generalized complex submanifold, a concept which seems to correspond to D-branes of higher rank. One can even see how such a brane could deform into several branes of lower rank.
3 Generalized Riemannian geometry and the Born-Infeld metric

A generalized Riemannian metric is a reduction of the structure group of $T \oplus T^*$ from $O(n,n)$ to $O(n) \times O(n)$. This is equivalent to specifying a maximal positive-definite subbundle, $C_+ < T \oplus T^*$, which can be described as the graph of $b + g$, where $g$ is a usual Riemannian metric and $b$ is a 2-form, each viewed as defining a map $T \rightarrow T^*$ via interior product. The graph of $b - g$ is denoted by $C_-$, the orthogonal complement to $C_+$. These data determine a positive-definite metric on $T \oplus T^*$ by simply taking $\langle \cdot, \cdot \rangle|_{C_+} - \langle \cdot, \cdot \rangle|_{C_-}$. This metric, evaluated on $A, B \in T \oplus T^*$, can be written as $\langle GA, B \rangle$, where $G$ is the obvious involution, expressible in terms of the data as follows:

$$G = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 1 \\ 1 & -g^{-1} \end{pmatrix} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}.$$  

The restriction of this metric to the subbundle $T$ is the Riemannian metric $g - bg^{-1}b$. Note that the volume form induced by this last metric is

$$\text{vol}_G = \det(g - bg^{-1}b)^{1/2} = (\det(g) \det(1 - g^{-1}bg^{-1}))^{1/2} = \frac{\det(g + b)}{\det g^{1/2}}.$$

Let $* = a_1 \cdots a_n$ be the product in $\text{CL}(C_+) < \text{CL}(T \oplus T^*)$ of an oriented orthonormal basis for $C_+$. This volume element acts on the differential forms via the spin representation, and is related to the Hodge star operator $*$ in the following way: if $b = 0$ then

$$*\rho = \sigma(\cdot) \cdot \rho.$$

Since $*^2 = (-1)^{n(n-1)/2}$ and $\langle \alpha, \beta \rangle = (-1)^{n(n-1)/2} \langle \beta, \alpha \rangle$, we see that the volume form,

$$\langle \alpha, \sigma(\cdot) \beta \rangle,$$

is symmetric in $\alpha, \beta$. For $b = 0$, it is nothing but the Hodge volume:

$$\langle \alpha, \sigma(\cdot) \beta \rangle = \alpha \wedge *\beta = g(\alpha, \beta)\text{vol}_g,$$

where $g(\alpha, \beta)$ is the positive-definite Hodge metric on differential forms. In the general case, we obtain a different symmetric positive-definite volume form,

$$\langle \alpha, \sigma(\cdot) \beta \rangle = G(\alpha, \beta)\langle 1, \sigma(\cdot) \rangle = G(\alpha, \beta)\frac{\det(g + b)}{\det g^{1/2}} = g(\alpha, \beta)\text{vol}_G,$$

where $G(\alpha, \beta)$ is a positive-definite metric on forms satisfying $\langle 1, 1 \rangle = 1$. We call this expression the Born-Infeld volume, to coincide with physics terminology. Therefore, for any generalized Riemannian structure, we may define the following positive-definite Hermitian inner product on differential forms:

$$h(\alpha, \beta) = \int_M \langle \alpha, \sigma(\cdot) \beta \rangle,$$

which we call the Born-Infeld inner product. It is a direct generalization of the Hodge inner product of Riemannian geometry.

To develop Hodge theory for generalized Riemannian manifolds, we calculate the adjoint of the twisted exterior derivative $d_H$. Note first that the exterior derivative is such that

$$\langle d_H \alpha, \beta \rangle - (-1)^{\dim M} \langle \alpha, d_H \beta \rangle$$

is exact,

so that for a compact even-dimensional manifold,

$$\int_M \langle d_H \alpha, \beta \rangle \stackrel{\text{exact}}{=} \int_M \langle \alpha, d_H \beta \rangle.$$  

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With this in mind, we may determine the Born-Infeld adjoint of $d_H$:

$$\begin{align*}
h(d_H \alpha, \beta) &= \int_M \langle d_H \alpha, \sigma(*) \bar{\beta} \rangle \\
&= \int_M \langle \alpha, d_H \sigma(*) \bar{\beta} \rangle \\
&= \int_M \langle \alpha, \sigma(*) \ast d_H \sigma(*) \bar{\beta} \rangle \\
&= h(\alpha, \ast d_H \sigma(*) \bar{\beta}),
\end{align*}$$

proving that, for an even-dimensional manifold,

$$d_H^* = \ast \cdot d_H \cdot \sigma(*) = \ast \cdot d_H \cdot \ast^{-1}.$$ 

As in the Riemannian case, the operator $D_+ = d_H + d_{H}^*$ is elliptic (as an operator $\wedge^{ev/od} T^* \rightarrow \wedge^{od/ev} T^*$) and so, therefore, is the Laplacian

$$\Delta_{d_H} = D_+^2 = d_H d_H^* + d_H^* d_H.$$

Proceeding in the usual way, we may conclude that on a compact generalized Riemannian manifold, every $H$-twisted de Rham cohomology class has a unique $\Delta_{d_H}$-harmonic representative. There is a gauge freedom here, in the sense that, given any 2-form $b'$, the automorphism $e^{b'}$ takes harmonic representatives for $(g, b, H)$ to those for $(g, b + b', H - db')$.

## 4 Generalized Kähler geometry and the Hodge decomposition

A generalized Kähler structure is a further integrable reduction of the structure group of $T \oplus T^*$ to $U(n) \times U(n)$. As defined in [3,4], it consists of two commuting generalized complex structures $(J_1, J_2)$ such that the involution $-J_1 J_2 = G$ determines a generalized Riemannian metric on $T \oplus T^*$. The standard example of such a pair is obtained from a usual Kähler structure $(g, J, \omega)$, i.e. a Riemannian metric $g$, a complex structure $J$, and a symplectic structure $\omega$, such that the following diagram commutes:

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    T
    |  \downarrow g
    |   \downarrow \omega
    |    \downarrow J
    \downarrow T 
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By taking

$$J_1 = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix},$$

we see that these generalized complex structures commute and

$$-J_1 J_2 = \begin{pmatrix} g & g^{-1} \\ g^{-1} & g \end{pmatrix},$$

defining a generalized Riemannian metric with $b = 0$.

In the preceding example, the types of the generalized complex structures $(J_1, J_2)$ are $(n, 0)$, since one is complex and the other is symplectic. In general, though type jumping may occur, we have the following constraints on the pair of types:

type$(J_1) + \text{type}(J_2) \equiv n \pmod{2}$, and

type$(J_1) + \text{type}(J_2) \leq n$.  

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In [3], it is proven that generalized Kähler geometry is equivalent to a bi-Hermitian geometry with torsion, first described by Gates, Hull, and Roček [2] in their study of $N = (2, 2)$ supersymmetric sigma models. This equivalence indicates how it is possible to ‘topologically twist’ these models in general. In what follows, we are interested in what implications generalized Kähler geometry has for differential forms, and in particular whether there is a generalization of the Hodge decomposition.

First observe that both $J_1, J_2$ are in $\mathfrak{so}(T \oplus T^*)$, and via the spin representation they act on differential forms. $J_1$ induces a decomposition of forms into its eigenspaces:

$$\wedge^n T^* \otimes \mathbb{C} = U_{-n} \oplus \cdots \oplus U_0 \oplus \cdots \oplus U_n,$$

where $U_k$ is the $ik$-eigenspace of $J_1$. Furthermore, the exterior derivative $d_H$, acting on sections of $U_k$, decomposes as the sum of the two projections $\partial_1, \partial_2$, to $U_{k+1}, U_{k-1}$, respectively. That is,

$$C^\infty(U_k) \xrightarrow{\partial_1} C^\infty(U_{k+1}) \xleftarrow{\partial_2}.$$

The commuting endomorphism $J_2$ engenders a further decomposition of the $U_k$:

$$U_k = U_{k,|k|} \oplus U_{k,|k|-2} \oplus \cdots \oplus U_{k,n},$$

where $U_{p,q}$ is the intersection of the $ip$-eigenspace of $J_1$ and the $iq$-eigenspace of $J_2$. In this way we obtain a $(p, q)$ decomposition of the differential forms into the following diamond:

$$\begin{array}{ccccccc}
\cdots & U_{0,n} & \cdots & \cdots & U_{n-1,1} & U_{n,0} & \cdots \\
U_{-n+1,1} & \cdots & \cdots & U_{n-1,1} & U_{n,0} & \cdots \\
U_{-n+1,-1} & \cdots & \cdots & \cdots & U_{n-1,-1} & \cdots \\
\cdots & U_{0,-n} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}$$

This decomposition is orthogonal with respect to the Born-Infeld metric, and gives rise to the following decomposition of the exterior derivative:

$$d_H = \delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-,$$

where the differential operators act as follows:

$$\begin{array}{ccccccc}
U_{p-1,q+1} & \xrightarrow{\delta_-} & U_{p,q} & \xleftarrow{\partial_1} & U_{p+1,q+1} \\
U_{p-1,q-1} & \xleftarrow{\delta_+} & U_{p,q} & \xrightarrow{\partial_2} & U_{p+1,q-1} \\
\end{array}$$

and where we have, for definiteness, $\partial_1 = \bar{\delta}_+ + \bar{\delta}_-$ and $\partial_2 = \bar{\delta}_+ + \bar{\delta}_-.$

The following proposition gives the crucial relationship between these operators, and is a generalization of the usual Kähler identities:

**Proposition 2 (generalized Kähler identities [4]).** For a generalized Kähler structure, we have the identities

$$\bar{\delta}_+ = -\delta_+ \quad \text{and} \quad \bar{\delta}_- = \delta_-.$$
These simple identities imply the equality of all available Laplacians:
\[ \Delta_d H = 2\Delta \sigma_{1/2} = 2\Delta \sigma_{1/2} = 4\Delta \tau = 4\Delta \delta, \]
and so, finally, we obtain a \((p, q)\) decomposition for the twisted cohomology of any compact generalized Kähler manifold.

**Theorem 4.1 (Hodge decomposition [4]).** The twisted cohomology of a compact twisted \(2n\)-dimensional generalized Kähler manifold carries a Hodge decomposition:
\[ H^*_H(M, \mathbb{C}) = \bigoplus_{|p+q| \leq n \atop p+q \equiv n (\text{mod } 2)} H^{p,q}, \]
where \(H^{p,q}\) are \(\Delta_{d_H}\)-harmonic forms in \(U_{p,q}\).

Note that in the usual Kähler case, this \((p, q)\) decomposition is not the Dolbeault decomposition: it was called the Clifford decomposition by Michelsohn [9], and there is an orthogonal transformation, called the Hodge automorphism, taking it to the usual Dolbeault decomposition. A striking feature of the Clifford decomposition is that a form of type \((p, q)\) is closed if and only if it is co-closed and hence harmonic.

A consequence of this result is a generalization of the well-known \(\partial\bar{\partial}\)-lemma of Kähler geometry. Any (twisted) generalized complex structure gives rise to a real differential operator \(d^J_H = [d_H, J]\) on forms. In the complex case \(d^J = d^c = i(\bar{\partial} - \partial)\), whereas in the symplectic case, \(d^J = \delta\), the Koszul symplectic adjoint of \(d\). The \(dd^J\)-property, studied in more detail by Cavalcanti [1], is then defined as follows.

**Definition (dd\(^J\) property):** A generalized complex manifold \((M, J)\) satisfies the \(dd^J\) property iff the following are equivalent:
- \(\rho\) is \(d\)-closed and \(d^J\)-exact,
- \(\rho\) is \(d^J\)-closed and \(d\)-exact,
- \(\rho = dd^J \tau\) for some \(\tau\).

We have given the property for \(H = 0\); in general simply replace \(d\) by \(d_H\). Now we can state the first corollary of the previous theorem:

**Corollary 4.2 (dd\(^J\) lemma [4]).** A compact twisted generalized Kähler manifold satisfies the \(d_J^H\) property with respect to both \(J_1\) and \(J_2\).

In the usual Kähler case, this implies that both the \(dd^c\) and \(d\delta\)-lemmas are satisfied. As shown by Merkulov (see [1]), the \(d\delta\)-lemma is equivalent to the strong Lefschetz property, which we know is satisfied by any compact Kähler manifold.

A second corollary of the Hodge decomposition concerns a generalization of the fact that the odd Betti numbers of a compact Kähler manifold must be even. Observing that \(\overline{H^{p,q}} = H^{-p,-q}\), we obtain a constraint on the parity of the even or odd twisted Betti numbers \(b_{H}^{ev/od}\).

**Corollary 4.3.** Let \(M\) be a compact twisted generalized Kähler manifold. If \(\dim M = 4k + 2\), then both \(b^{ev}_{H}\) and \(b^{od}_{H}\) must be even. If \(\dim M = 4k\), then the generalized Kähler pair may have types of parity either \((od, od)\) or \((ev, ev)\). In the former case, \(b^{ev}_{H}\) must be even, whereas in the latter case, \(b^{od}_{H}\) must be even.

By applying this corollary, we see at once that the 4-manifold \(\mathbb{C}P^2\) does not admit a generalized Kähler structure with types \((1, 1)\).
References

[1] Cavalcanti, G. *New aspects of the dd^c-lemma*, Oxford D.Phil. thesis (in preparation).

[2] Gates Jr., S., Hull, C., Roček, M., *Twisted multiplets and new supersymmetric nonlinear σ-models*, Nuclear Phys. B 248 (1984), 157–186.

[3] Gualtieri, M. *Generalized complex geometry*, Oxford D.Phil. thesis. (2004) math.DG/0401221.

[4] Gualtieri, M. *Hodge decomposition for generalized Kähler manifolds*, in preparation.

[5] Hitchin, N. *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford, 54 (2003), 281–308.

[6] Kapustin, A., Orlov, D., Remarks on A-branes, Mirror Symmetry, and the Fukaya category, (2001) hep-th/0109098.

[7] Kosmann-Schwarzbach, Y. *Derived brackets*, (2003) math.DG/0312524.

[8] Lu, Z.-J., Weinstein, A., Xu, P. *Manin triples for Lie bialgebroids*, J. Diff. Geom. 45 (1997), 547–574.

[9] Michelsohn, M.-L., *Clifford and Spinor Cohomology of Kähler Manifolds*, Amer. J. of Math., 102 (1980), 1083–1146.