Efficient quantum algorithms for solving quantum linear system problem

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Abstract

We transform the problem of solving linear system of equations $A\mathbf{x} = \mathbf{b}$ to a problem of finding the right singular vector with singular value zero of an augmented matrix $C$, and present two quantum algorithms for solving this problem. The first algorithm solves the problem directly by applying the quantum eigenstate filtering algorithm with query complexity of $O(s\kappa \log(1/\epsilon))$ for a $s$-sparse matrix $C$, where $\kappa$ is the condition number of the matrix $A$, and $\epsilon$ is the desired precision. The second algorithm uses the quantum resonant transition approach, the query complexity scales as $O [s\kappa + \log(1/\epsilon) / \log \log(1/\epsilon)]$. Both algorithms meet the optimal query complexity in $\kappa$, and are simpler than previous algorithms.
I. INTRODUCTION

Solving linear system of equations (LSE) is one of the fundamental problems in scientific computation. Given an $N \times N$ matrix $A$ and a vector $b$, the task is to find a vector $x$ such that $Ax = b$. Solving high-dimensional LSE is expensive on a classical computer. Classical linear solvers can be categorized into the direct methods and the iterative methods [1, 2]. The direct methods such as Gaussian elimination solve LSE with runtime scales as $O(N^3)$. There exists more efficient classical linear system solver that scales as $O(N^\nu)$ where $\nu \leq 2.373$ [3, 4], but it is difficult to utilize in practice due to numerical instability. The iterative methods show great advantages when they converge quickly, the iteration number is an indicator for the efficiency of these methods. E.g., for a symmetric positive-definite problem, the steepest descent method needs $O(\kappa \log(1/\epsilon))$ iterations and the conjugate gradient method needs $O(\sqrt{\kappa} \log(1/\epsilon))$, where $\kappa$ is the condition number of the matrix $A$ defined as the ratio between the largest and the smallest singular values of $A$, or $\|A\|\|A^{-1}\|$, where $\|\cdot\|$ denotes vector or matrix 2-norm, and $\epsilon$ is the desired precision of the solution.

Quantum computation provides an efficient way of solving quantum linear system problem (QLSP), which aims to prepare a quantum state that is proportional to the solution vector of a given LSE. Quantum algorithms for solving the QLSP either apply the matrix inversion operator $A^{-1}$ directly on the state $|b\rangle$ to obtain $|x\rangle$, e.g. the HHL algorithm [5, 6], or transform the LSE to an eigenvalue problem where the ground state of the problem Hamiltonian is the state $|x\rangle$ [7, 8]. Recently, a quantum eigenstate filtering (QEF) algorithm [8] was proposed for solving the QLS by combining with adiabatic quantum computing (AQC) or quantum Zeno effect (QZE), and achieves near optimal query complexity of $O(s\kappa \log(1/\epsilon))$, where $s$ is the sparsity of the matrix $A$. The QEF algorithm approximates a spectral projection operator by using the quantum signal processing (QSP) [9] method and project out the quantum state $|x\rangle$ from an initial state prepared through an AQC procedure. Such a procedure can also be realized through QZE by applying a sequence of QEF procedures following the same path as that of the AQC method.

In this work, we solve the QLSP in a simpler way. We first prove that the solution vector to the LSE is contained in a vector proportional to the right singular vector of the augmented matrix $C = \begin{pmatrix} A & \beta^{-1}b \end{pmatrix}$ with corresponding singular value 0, where $\beta$ is a parameter. Then
we construct a Hermitian matrix

\[
B = \begin{pmatrix}
0 & C \\
C^\dagger & 0
\end{pmatrix},
\]

and transform the QLSP to a problem of finding the eigenstate of the matrix \( B \) with eigenvalue 0. In our previous work, we proposed a quantum algorithm for finding an eigenstate with known eigenvalue of a Hermitian matrix through quantum resonant transition (QRT) \([10, 11]\). Here, we set a simple initial state whose overlap with the desired eigenstate of the matrix \( B \) is \( O(1) \), then both the QEF and the QRT algorithms can be applied for solving the QLSP. The QEF algorithm has query complexity of \( O(s\kappa \log (1/\epsilon)) \), and the query complexity of the QRT algorithm scales as \( O[s\kappa + \log (1/\epsilon) / \log \log (1/\epsilon)] \) for solving the problem. Both algorithms achieve the optimal query complexity in \( \kappa \). In previous quantum algorithms, the complexity for preparing the state \( |b\rangle \) and querying the matrix \( A \) are considered separately, while they are combined together in our work. The algorithms do not need to use complex procedures such as phase estimation, amplitude amplification, AQC or QZE, thus they are easier to implement experimentally than previous algorithms.

II. LSE AND SINGULAR VALUE DECOMPOSITION

The LSE \( Ax = b \) can be written in form of

\[
C \begin{pmatrix}
x \\
-\beta
\end{pmatrix} = 0.
\]

By performing singular value decomposition (SVD), we have \( C = SDV^\dagger \), where \( S \) and \( V \) are unitary matrices of dimension \( N \) and \( (N + 1) \), respectively, and 

\[
D = \begin{pmatrix}
D' & 0
\end{pmatrix},
\]

\( D' = \text{diag} (\sigma_1, \ldots, \sigma_N) \), and \( 0 \) is the zero column vector of dimension \( N \). \( \sigma_1 \geq \cdots \geq \sigma_N \) are the singular values of the matrix \( C \). Correspondingly, \( s_1, \ldots, s_N \) are column vectors of \( S \), and \( v_1, \ldots, v_{N+1} \) are column vectors of \( V \).

**Theorem 1.**—Suppose \( x \) is the solution to the equation \( Ax = b \), where \( A \) is an \( N \times N \) nonsingular matrix, \( v_{N+1} \) is the right singular vector corresponding to \( Cv_{N+1} = 0 \), where

\[
C = \begin{pmatrix}
A & \beta^{-1}b
\end{pmatrix},
\]

then the vector \( \begin{pmatrix} x^T, -\beta \end{pmatrix}^T \) that satisfies Eq. (1) is proportional to the singular vector \( v_{N+1} \) of the matrix \( C \).

**Proof.**—Since the rank of the matrix \( A \) is \( N \), let \( \sigma_{N+1} = 0 \), the singular values of the matrix \( C \) satisfy \( \sigma_1 \geq \cdots \geq \sigma_N > \sigma_{N+1} = 0 \). The matrix \( C \) can be written as 

\[
C = \sum_{i=1}^{N} \sigma_i s_i v_i^\dagger.
\]
column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{N+1}$ of the unitary matrix $V$ are orthogonal to each other, therefore to satisfy Eq. (1), the vector $\left( \mathbf{x}^T, -\beta \right)^T$ must be orthogonal to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$, thus it is proportional to the vector $\mathbf{v}_{N+1}$. □

Let $\|A\| \leq 1$, we perform SVD for the matrix $A$, $\bar{S}^T \bar{A} \bar{V} = \text{diag}(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_N)$. The singular values of the matrices $A$ and $C$ satisfy the following relation [1, Corollary 8.6.3, p487]:

$$\sigma_1 \geq \bar{\sigma}_1 \geq \sigma_2 \geq \bar{\sigma}_2 \geq \ldots \geq \bar{\sigma}_N \geq \sigma_{N+1} = 0. \quad (3)$$

Then the energy gap between the singular states $\mathbf{v}_{N+1}$ and $\mathbf{v}_N$ is

$$\Delta = \sigma_N - \sigma_{N+1} \geq \Delta^* = \bar{\sigma}_N = \frac{1}{\kappa}. \quad (4)$$

Suppose $\|\mathbf{b}\| = 1$, then $\|\mathbf{x}\| = \|A^{-1} \|b\|$ is in the range of $[1, \kappa]$. The vector $\left( \mathbf{x}^T, -\beta \right)^T$ can be normalized as $\mathbf{v}_{N+1} = d_0 \left( \mathbf{x}^T/\|\mathbf{x}\|, 0 \right)^T + d_1 \left( 0^T, 1 \right)^T$, where $d_0/d_1 = \|\mathbf{x}\|/\beta$.

In the algorithm, the solution state $\mathbf{x}/\|\mathbf{x}\|$ is obtained in two steps, we first obtain the state $\mathbf{v}_{N+1}$, then extract the solution state $\mathbf{x}/\|\mathbf{x}\|$ from $\mathbf{v}_{N+1}$. The choice of the parameter $\beta$ has dual influence on both the overlap of $\mathbf{v}_{N+1}$ with the initial state and the post-selection of solution state. We first set the parameter $\beta$ in the order of $O(\kappa)$, thus the component $d_1$ is guaranteed to be in the order of $O(1)$, and run the algorithm to obtain the state $\mathbf{v}_{N+1}$. We run the algorithm and perform measurement on the state $\mathbf{v}_{N+1}$ for a number of times to estimate the value of $d_1$. This can be achieved by measuring the $(N + 1)$-th component of the vector $\mathbf{v}_{N+1}$ to obtain the probability of its outcome being 1, and obtain $d_0 = \sqrt{1 - d_1^2}$. After that we reset the parameter $\beta$ such that $\beta$ is in the order of $\|\mathbf{x}\|$ and run the algorithm again. Then both the components $d_0$ and $d_1$ are in the order of $O(1)$. The state $\mathbf{x}^T/\|\mathbf{x}\|$ can be extracted efficiently from the state $\mathbf{v}_{N+1}$.

In the following we describe the algorithms for obtaining the state $\mathbf{v}_{N+1}$. The matrix $B$ has eigenvalues and eigenstates as follows: $B|\varphi_j \rangle = E_j|\varphi_j \rangle$, $j = 1, \ldots, 2N + 1$, and

$E_1 = -\sigma_1, \ldots, E_N = -\sigma_N, E_{N+1} = 0, E_{N+2} = \sigma_N, \ldots, E_{2N+1} = \sigma_1$, and $|\varphi_1 \rangle = (\mathbf{s}_1, -\mathbf{v}_1), \ldots, |\varphi_N \rangle = (\mathbf{s}_N, -\mathbf{v}_N), |\varphi_{N+1} \rangle = |\mathbf{v}_{N+1} \rangle, |\varphi_{N+2} \rangle = (\mathbf{s}_N, \mathbf{v}_N), \ldots, |\varphi_{2N+1} \rangle = (\mathbf{s}_1, \mathbf{v}_1)$. The problem of solving the LSE is transformed to finding the eigenstate $|\mathbf{v}_{N+1} \rangle = (\mathbf{0}, \mathbf{v}_{N+1})$ of the matrix $B$ with eigenvalue 0. The gap between the eigenstate $|\mathbf{v}_{N+1} \rangle$ and its nearest neighboring eigenstate $|\varphi_{N+2} \rangle$ is in the order of $O(\kappa^{-1})$ in the worst case. We set the initial state of the quantum circuit as $|1 \rangle = (\mathbf{0}, \mathbf{0}, 1)$, whose overlap with the desired eigenstate
|v_{N+1}\rangle$ of the matrix $B$ is $d_1 = \langle 1 | v_{N+1} \rangle$ which scales as $O(1)$ in the algorithm. We then apply the QEF and the QRT algorithms for obtaining the state $|v_{N+1}\rangle$ from the initial state $|1\rangle$, and extract the solution state $x/\|x\|$ from the state $v_{N+1}$.

III. SOLVING QLSP THROUGH QEF

The QEF algorithm projects out the desired eigenstate of a Hermitian matrix from an initial state by implementing an eigenstate-filtering function using the QSP method, which is a powerful algorithm for implementing a polynomial function of matrices on a quantum computer with minimal number of ancilla qubits. The matrix $B$ is encoded in a unitary matrix by using the block-encoding technique, then the QSP method is applied to implement the QEF function to project out the eigenstate $|v_{N+1}\rangle$ from the initial state.

The matrix $B$ is represented on an $n$-qubit quantum register, an $(m+n)$-qubit unitary operator $U_B$ is called a $(\alpha, m, \epsilon)$-block-encoding of the matrix $B$, if

$$\| B - \alpha (\langle 0^m | \otimes I_n) U_B (|0^m\rangle \otimes I_n) \| \leq \epsilon,$$

where $I_n$ is an $n$-qubit identity matrix. Block-encoding of $B$ can also be written in form of

$$U_B = \begin{pmatrix} B/\alpha & \cdot \\ \cdot & \cdot \end{pmatrix},$$

where $\alpha$ is a parameter such that $\|B/\alpha\| \leq 1$. It is determined by the largest eigenvalue of the matrix $B$, or the largest singular value $\sigma_1$ of the matrix $C$. We estimate the upper
bound of $\sigma_1$ as follows:

$$\sigma_{\text{max}}^2(C) = \max_{y \neq 0} \frac{\|Cy\|^2}{\|y\|^2}$$

$$= \max_{y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0} \frac{\left\| \begin{pmatrix} A & \beta^{-1}b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|^2}{\|y_1\|^2 + \|y_2\|^2}$$

$$= \max_{y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0} \frac{\|Ay_1 + \beta^{-1}by_2\|^2}{\|y_1\|^2 + \|y_2\|^2}.$$

Then we have:

$$\sigma_{\text{max}}(C) \leq \max_{y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0} \frac{\|Ay_1 + \beta^{-1}by_2\|}{\sqrt{\|y_1\|^2 + \|y_2\|^2}}$$

$$\leq \max_{y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0} \frac{\|Ay_1\|}{\sqrt{\|y_1\|^2 + \|y_2\|^2}} + \max_{y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0} \frac{\beta^{-1} \|b\| \|y_2\|}{\sqrt{\|y_1\|^2 + \|y_2\|^2}}$$

$$\leq \max_{y_1 \neq 0} \frac{\|Ay_1\|}{\|y_1\|} + \max_{y_2 \neq 0} \frac{\beta^{-1} \|b\| \|y_2\|}{\|y_2\|}$$

$$= \sigma_{\text{max}}(A) + \beta^{-1}$$

$$= \bar{\sigma}_1 + \beta^{-1}.$$  \hspace{1cm} (8)

The parameter $\beta$ is in the range of $[1, \kappa]$, thus the maximum singular value of the matrix $C$ is no more than 2.

The polynomial eigenvalue transformation of $B/\alpha$ can be implemented based on the following theorem [8, 14]:

**Theorem 2** (Polynomial eigenvalue transformation with definite parity via quantum signal processing): Let $U_B$ be an $(\alpha, m, 0)$-block-encoding of the Hermitian matrix $B/\alpha$ and
\( P \in \mathbb{R}[w] \) be a degree-\( l \) even or odd real polynomial and \( |P(w)| \leq 1 \) for any \( w \in [-1, 1] \). Then there exists a \((1, m + 1, 0)\)-block-encoding \( U_{\varphi} \) of \( P(B/\alpha) \) using \( l \) queries of \( U_B, U_B^\dagger \), and \( O((m + 1)l) \) other primitive quantum gates.

Let \( B/\alpha = \sum \lambda |\lambda\rangle \langle \lambda| \), and define \( \Pi := |0\rangle \langle 0| \) acting on an auxiliary qubit, and \( \Pi_\phi := e^{i\phi(2\Pi - I)} \). For \( l \) is even, \( U_{\varphi} \) is in form of

\[
U_{\varphi} = \prod_{k=1}^{l/2} \left( \Pi_{\varphi_{2k-1}} U_B^\dagger \Pi_{\varphi_{2k}} U_B \right) = \left( \begin{array}{c} P(B/\alpha) \cdot \\ . \end{array} \right),
\]

(9)

where \( P(B/\alpha) = \sum \lambda P(\lambda) |\lambda\rangle \langle \lambda| \) is a polynomial transform of the eigenvalues of \( B/\alpha \). The phase factors \((\varphi_1, \ldots, \varphi_l)\) can be calculated efficiently \([16, 20]\). The eigenvalue transform can be used to project out the desired eigenstate with known eigenvalue and filter out other unrelated states. For a Hermitian matrix with eigenvalue \( \lambda \) that is known to be separated from other eigenvalues by a gap \( \Delta > 0 \), it has been shown that the following degree-\( (l = 2k) \) polynomial

\[
R_k(w; \Delta) = \frac{T_k \left(-1 + 2\frac{w^2 - \Delta^2}{1 - \Delta^2}\right)}{T_k \left(-1 + 2\frac{\Delta^2}{1 - \Delta^2}\right)}
\]

(10)

is an optimal polynomial for filtering out the unwanted eigenstates \([8]\), where \( T_k(w) \) is the \( k \)th Chebyshev polynomial of the first kind. By using this polynomial in eigenvalue transform, the system can be projected onto the eigenspace corresponding to the eigenvalue \( \lambda \).

The eigenstate \( |v_{N+1}\rangle \) of the matrix \( B \) is separated from the nearest eigenstate of \( B \) by a minimum gap of \( \Delta^* = \kappa^{-1} \). We set the initial state of the quantum circuit as \( |1\rangle \), and apply the QEF algorithm to project out the eigenstate \( |v_{N+1}\rangle \), while filtering out all other eigenstates. The overlap between the states \( |1\rangle \) and \( |v_{N+1}\rangle \) is \( d_1 = O(1) \). Therefore the eigenstate \( |v_{N+1}\rangle \) can be obtained with success probability \( O(1) \). The number of qubits required is \((n + m + 1)\).

The implementation of the algorithm requires querying the matrix \( B \) that contains the matrix \( C \) which is composed of the matrix \( A \) and the column vector \( |b\rangle \). The matrix \( A \) can be accessed by oracles \( O_{A,1} \) and \( O_{A,2} \) as \( O_{A,1}|j,l\rangle = |j, \nu(j, l)\rangle, \ O_{A,2}|j, k, z\rangle = |j, k, A_{jk} \oplus z\rangle, \langle j, k, l, z \in [N]| \), where the oracle \( O_{A,1} \) accepts a row \( j \) index and calculates the column index \( \nu(j, l) \) of the \( l \)th nonzero element in the \( j \)th row of the matrix \( A \), and the oracle \( O_{A,2} \) accepts a row \( j \) and a column \( k \) index and returns the matrix element \( A_{jk} \) \([21]\). The vector
b is prepared by using an oracle \( O_b \) as \( O_b |0\rangle = |b\rangle \), and its elements can be accessed by an oracle \( O_{b,1} \) as \( O_{b,1} |k, z\rangle = |k, b_k \oplus z\rangle \), \( (k, z \in \mathbb{N}) \), where the oracle \( O_{b,1} \) accepts the input \( k \) and returns the \( k \)th element \( b_k \) of the vector \( |b\rangle \). The matrix \( C \) contains the matrix \( A \) and the vector \( |b\rangle \) as the \((N + 1)\)-th column vector. If the matrix \( C \) is \( s \)-sparse, then it can be accessed by using oracles \( O_{C,1} \) and \( O_{C,2} \) similar to the oracles for accessing the matrix \( A \) above as \( O_{C,1} |j, l\rangle = |j, f(j, l)\rangle \), \( O_{C,2} |j, k, z\rangle = |j, k, C_{jk} \oplus z\rangle \), \( (k, l, z \in \mathbb{N}, j \in \mathbb{N} + 1) \), where the oracle \( O_{C,1} \) calculates \( f(j, l) \) which is the column index of the \( l \)th nonzero element in the \( j \)th row of the matrix \( C \), and the oracle \( O_{C,2} \) accepts the input \((j, k)\) and returns the matrix element \( C_{jk} \) of the matrix \( C \). In this case, a \((s, n + 2, 0)\)-block-encoding of \( C \) can be constructed by using the \( O_{C,1} \) and \( O_{C,2} \) [6, 22]. The complexity of the QEF algorithm scales as \( O((\alpha/\Delta^*) \log (1/\epsilon)) \) [8], therefore the complexity of using the QEF algorithm for obtaining the state \( |v_{N+1}\rangle \) scales as \( O(sk \log (1/\epsilon)) \) by querying the oracles \( O_{C,1} \) and \( O_{C,2} \).

IV. SOLVING QLSP VIA QRT

In Ref. [10, 11], we proposed a quantum algorithm for finding an eigenstate with known corresponding eigenvalue of a Hamiltonian based on quantum resonant transition. By coupling a probe qubit to a system, a resonant transition occurs when the transition frequency of the probe qubit matches a transition in the system, and the system is evolved to the eigenstate with known eigenvalue. This algorithm can be applied for solving the QLSP by obtaining the eigenstate \( |v_{N+1}\rangle \) of the matrix \( B \). The algorithm requires \( n + 1 \) qubits with one probe qubit and an \( n \)-qubit register \( R \) representing the matrix \( B \). The Hamiltonian of the algorithm is constructed as

\[
H = -\frac{1}{2} \omega \sigma_z \otimes I_n + H_R + c \sigma_x \otimes I_n,
\]

where

\[
H_R = \varepsilon_0 |1\rangle \langle 1| \otimes \mathbf{1} + |0\rangle \langle 0| \otimes B,
\]

and \( \sigma_x \) and \( \sigma_z \) are the Pauli matrices. The first term in Eq. (11) is the Hamiltonian of the probe qubit, the second term contains the Hamiltonian of the register \( R \) and describes the interaction between the probe qubit and \( R \), and the third term is a perturbation. The parameter \( \varepsilon_0 \) is used as a reference point to the eigenstate \( |v_{N+1}\rangle \) of the matrix \( B \) with
eigenvalue 0, and $c \ll 1$. The condition for resonant transition between states $|1\rangle|1\rangle$ and $|0\rangle|v_{N+1}\rangle$ is satisfied as $E_{N+1} - \varepsilon_0 = \omega$, that is, $\omega = -\varepsilon_0$.

The algorithm is run as follows:

i) Set the initial state of the $n + 1$ qubits as $|1\rangle|1\rangle$, which is an eigenstate of $H_R$ with eigenvalue $\varepsilon_0$.

ii) Implement the unitary operator $U(t) = \exp(-iHt)$ by setting $\omega = 1$ and $\varepsilon_0 = -1$.

iii) Read out the state of the probe qubit.

As the resonant transition occurs, the system is approximately in state
\[
\sqrt{1-p}|1\rangle|1\rangle + \sqrt{p}|0\rangle|v_{N+1}\rangle,
\]
where $p = \sin^2(ctd_1)$ is the decay probability of the probe qubit, and $c < \Delta^*$ and $d_1 = \langle 1|v_{N+1}\rangle$. By performing a measurement on the probe qubit, if the probe decays to its ground state $|0\rangle$, it indicates that a resonant transition occurs and the system evolves to the state $|0\rangle|v_{N+1}\rangle$; otherwise if the probe qubit stays in state $|1\rangle$, it means that the register $R$ remains in state $|1\rangle$, then we repeat steps ii)-iii) until the probe qubit decays to its ground state $|0\rangle$. The number of times the procedures need to be repeated is proportional to $1/p$.

Errors are introduced by excitations from the initial state $|1\rangle|1\rangle$ to the states $|0\rangle|\varphi_j\rangle$ ($j = N + 2, \cdots, 2N + 1$). Similar to the method in Ref. [23], we estimate errors introduced when running the algorithm with the evolution time $t = \pi/(2ctd_1)$ by setting $\varepsilon_0 = -1$ and $\omega = 1$ as follows. Let
\[
H_0 = -\frac{1}{2}\sigma_z \otimes I_n - |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes B,
\]
then the algorithm Hamiltonian $H$ can be written as $H = H_0 + c\sigma_x \otimes I_n$. The Hamiltonian $H_0$ is the unperturbed term and has eigenstates
\[
H_0|1\rangle|1\rangle = -\frac{1}{2}|1\rangle|1\rangle,
\]
and
\[
H_0|0\rangle|\varphi_j\rangle = \left(-\frac{1}{2} + E_j\right)|0\rangle|\varphi_j\rangle,
\]
where $j = 1, \ldots, 2N + 1$. The system is initialized in state $|1\rangle|1\rangle$.

For a probe qubit coupled to a two-level system described by the Hamiltonian in Eq. (11), the maximum transition probability from the ground state to the excited state of the two-level system becomes higher as the transition frequency between the two-level system gets closer to the frequency of the probe qubit. Therefore the upper bound of the transition
probability from the initial state to the states other than $|\varphi_{N+1}\rangle$ can be estimated by assuming all the other states are degenerate at the eigenstate $|\varphi_{N+2}\rangle$ and without considering competition of the transition from the initial state to the target state $|0\rangle|\varphi_{N+1}\rangle$. Then the upper bound of the error of the algorithm is the transition probability from the initial state to the state $|0\rangle|\varphi_{N+2}\rangle$ described by the following Hamiltonian in basis of \{\{|1\rangle|1\rangle, |0\rangle|\varphi_{N+2}\rangle\}\}:

$$H_{\text{err}} = \begin{pmatrix} -\frac{1}{2} & c\sqrt{1 - d_1^2} \\ c\sqrt{1 - d_1^2} & -\frac{1}{2} + E_{N+2} \end{pmatrix}. \quad (16)$$

When the coefficient $c$ is much less than the energy difference between the ground state and the first excited state of $H_0$, the excitation to the excited states can be described by the Rabi’s formula [12, p414]. The transition probability from the initial state $|1\rangle|1\rangle$ to the state $|0\rangle|\varphi_{N+2}\rangle$ is

$$\frac{4c^2 (1 - d_1^2)}{4c^2 (1 - d_1^2) + E_{N+2}^2} \sin^2 \left[ \frac{t}{2} \sqrt{\frac{4c^2 (1 - d_1^2) + E_{N+2}^2}{E_{N+2}^2}} \right]. \quad (17)$$

The upper bound of the transition probability from the initial state to the states $|0\rangle|\varphi_j\rangle$ ($j = N + 2, \ldots, 2N + 1$) can be estimated as

$$\sum_{j=N+2}^{2N+1} p_j \leq \frac{4c^2 (1 - d_1^2)}{4c^2 (1 - d_1^2) + E_{N+2}^2} < \frac{4c^2 (1 - d_1^2)}{E_{N+2}^2} = \frac{4c^2 (1 - d_1^2)}{\sigma_N^2}, \quad (18)$$

which can be controlled as a small number by setting the coefficient $c$ to be small, since $c < \sigma_N$.

The evolution time $t$ of the QRT algorithm is in the order of $1/cd_1$ such that the success probability of the algorithm is $O(1)$. It scales as $O(\kappa)$ since $d_1$ is in $O(1)$, and $\Delta^*$ thus $c$ is in the order of $O(\kappa^{-1})$ in the worst case. The complexity of the QRT algorithm is determined by Hamiltonian simulation of the algorithm $U(t) = \exp(-iHt)$. By applying a $(\alpha, m, 0)$-block-encoding of $H$ in a unitary matrix, the Hamiltonian simulation of $H$ is $U(t) = e^{-i(H/\alpha)t}$, and $\alpha$ scales as $O(1)$. The optimal approach for Hamiltonian simulation is by applying the QSP algorithm, for which the query complexity scales as $\Theta \left( \alpha t + \frac{\log(1/\epsilon)}{\log(e + \log(1/\epsilon) \alpha t)} \right)$ by accessing the Hamiltonian $H$ [15]. The number of times the Hamiltonian $H$ is queried scales as $O[\kappa + \log(1/\epsilon) / \log \log(1/\epsilon)]$ since $t$ scales as $O(\kappa)$ and $\alpha$ scales as $O(1)$. The query complexity of the algorithm by querying the oracles that access the Hamiltonian matrix $H$ scales as $O[s\kappa + \log(1/\epsilon) / \log \log(1/\epsilon)]$ [9] for an $s$-sparse matrix $C$. The total number of qubits required in this algorithm is $(n + m + 2)$. 


V. DISCUSSION

In this work, the problem of solving the linear system of equations is transformed to finding the eigenstate of a Hermitian matrix with eigenvalue 0. The QEF and QRT algorithms are used for obtaining this state by setting an initial state whose overlap with the desired eigenstate is $O(1)$. Because of this property, our algorithm have a better complexity scaling compare with other algorithms where an initial state needs to be prepared through an AQC procedure. The QEF algorithm is applied directly on the initial state to project out the solution state without using the AQC or QZE, thus achieves a better query complexity of $O[sk \log (1/\epsilon)]$ than that of the algorithm in [8]. The QRT algorithm evolves the initial state to the solution state of the QLSP with query complexity of $O[sk + \log (1/\epsilon)/\log \log (1/\epsilon)]$. The complexity of both algorithms have linear scaling in $\kappa$, which is optimal since the dependence on $\kappa$ cannot be made sublinear [5]. The complex procedures such as phase estimation, amplitude amplification, AQC or QZE used in previous algorithms are not needed, only time-independent Hamiltonian simulation is used, for which the QSP algorithm provides an optimal solution, thus our algorithms are easier to implement experimentally. In the algorithms, we obtain the state $|v_{N+1}\rangle$ that contains the solution vector of LSE as $v_{N+1} = d_0 \left( x^T / \|x\| , 0 \right)^T + d_1 \left( 0^T, 1 \right)^T$. The contribution of the state $\left( 0^T, 1 \right)^T$ in applications, such as calculating expectation value of some operators, can be removed through some adjustment.

The linear solver is a basic engine in engineering and scientific computing, and has wide applications in many areas. It paves a way for quantum machine learning, and acts as an important ingredient in linear regression, Bayesian inference, least-squares fitting, least squares support vector machine. The numerical solvers for partial differential equations and ordinary differential equations are also built on it. After numerical discretization, such as the finite element method, finite difference method, or finite volume method, one usually needs to solve a sparse linear system. The quantum algorithms we presented here can be used as a subroutine in solving these problems.
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