THE BOUNDED SPHERICAL FUNCTIONS ON THE CARTAN MOTION GROUP AND GENERALIZATIONS FOR THE EIGENSPACES OF THE LAPLACIAN ON $\mathbb{R}^n$

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Abstract. The bounded spherical functions are determined for a real Cartan Motion group which is a generalization for the case when the Cartan Motion group is complex written by Sigurdur Helgason [1]. Also, I will do a further step of the Laplacian on $\mathbb{R}^n$. I consider the case when $K$ is transitive on the spheres about 0 in $\mathbb{R}^n, n > 1$.

1. Introduction

Consider a symmetric space $X = G/K$ of noncompact type, $G$ being a connected noncompact semisimple lie group with finite center and $K$ a maximal compact subgroup. Let $g = k + p$ be the corresponding Cartan decomposition, $p$ being the orthocomplement of $k$ relative to the killing form of $g$. Let $a \subset p$ be a maximal abelian subspace. Let $G_0$ be the Cartan Motion group. This group is defined as the semidirect product of $K$ and $p$ with respect to the adjoint action of $K$ on $p$. The $X_0 = G_0/K$ is naturally identified with the Euclidean space $p$. The element $g_0 = (k, Y)$ actions on $p$ by

$$g_0(Y') = Ad(k)Y' + Y \quad k \in K, Y, Y' \in p.$$ 

So the algebra $\mathbb{D}(X_0)$ of $G_0$-invariant differential operators on $X_0$ is identified with the algebra of $Ad(K)$-invariant constant coefficient differential operators on $p$. The corresponding spherical functions on $X_0$ are given by

$$\psi_\lambda(Y) = \int_k e^{i\lambda(Ad(k)Y)}dk \quad \lambda \in a_c^*$$

and $\psi_\lambda = \psi_\mu$ if and only if $\lambda$ and $\mu$ are $W$-conjugate. See e.g. [2],IV§4. Again, the maximal ideal space of $L^2(G_0)$ is up to $W$-invariance identified with the set of $\lambda$ in $a_c^*$ for which $\psi_\lambda$ is bounded. Since $\rho$ is relative to the curvature of $G/K$ it is natural to expect the bounded $\psi_\lambda$ to come from replacing $c(\rho)$ by the origin, where $c(\rho)$ is for the semisimple case also proved by Sigurdur Helgason [3]. In the words, $\psi_\lambda$ is would be expected to be bounded if and only if $\lambda$ is real, that is $\lambda \in a^*$. In [1], Sigurdur Helgason proved when $G$ is complex, the spherical function $\psi_\lambda$ on $G_0$ is bounded if and only if $\lambda$ is real, i.e. $\lambda \in a^*$ mainly by using two results proved by Harish-Chandra [4] and [5]. In this paper, I use a different way to prove

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when $G$ is real, the spherical function $\psi_\lambda$ on $G_0$ is bounded also if and only if $\lambda$ is real. In this way, we generalize Sigurdur Helgason’s results.

In Sigurdur Helgason’s another paper [6], he considered Eigenspaces of the Laplacian on $\mathbb{R}^n$. Let $L$ denote the usual Laplacian on $\mathbb{R}^n$ and for each $\lambda \in \mathbb{C}$ let $\mathcal{E}_\lambda(\mathbb{R}^n)$ denote the eigenspace $\mathcal{E}_\lambda(\mathbb{R}^n) = \{f \in \mathcal{E}(\mathbb{R}^n) \mid Lf = -\lambda^2 f\}$ with the topology induced by that of $\mathcal{E}(\mathbb{R}^n)$. Let $G$ denote the group of all isometries of $\mathbb{R}^n$, and $K$ the group of rotations $O(n)$. Sigurdur Helgason mainly proved the natural action of $G$ on $\mathcal{E}_\lambda(\mathbb{R}^n)$ is irreducible if and only if $\lambda \neq 0$. I will prove when $K \subset O(n)$ is transitive on the spheres about 0 in $\mathbb{R}^n$, $n > 1$ instead of $O(n)$ and $G = K \times \mathbb{R}^n$, the same results holds. In this way, we do a further step of this kind of problem. Meanwhile, I will specific all the groups $K$ which is transitive on the spheres about 0 in $\mathbb{R}^n$.

Finally, according to [7], we know when $K \subset O(n)$ is transitivity on the spheres about 0, the specific form of the spherical functions on $K \times \mathbb{R}^n/K \cong \mathbb{R}^n$. Then I will give a estimation for it when $r \to \infty$.

2. The main theorem

The notion of induced spherical function mirrors the notion of induced representation. Let $Q \subset G$ be a closed subgroup such that $K$ is transitive on $G/Q$, i.e. $G = KQ$, i.e. $G = QK$, i.e. $G$ is transitive on $G/K$. Let $\zeta: Q \to \mathbb{C}$ be spherical for $(Q, Q \cap K)$. The induced spherical function is $\text{Ind}_G^Q(\zeta)(g) = \int_K \tilde{\zeta}(kg) d\mu_K(k)$ where $\tilde{\zeta}(kq) = \zeta(q) \Delta_{G/Q}(q)^{-\frac{1}{2}}$

Here $\Delta_{G/Q}: Q \to \mathbb{R}^n$ is the quotient of modular functions, $\Delta_{G/Q}(q) = \Delta_G(q)/\Delta_Q(q) = \Delta_Q(q)^{-1}$

**Theorem 2.1.** Let $\lambda \in a_+^*$. Then $\psi_\lambda(Y)$ is the induced spherical function $\text{Ind}_G^Q(\varphi_\lambda)$, where $\varphi_\lambda(Y) = e^{i\lambda(Y)}$ for every $Y \in a \subset p$.

**Proof.** Apply above formula to $\varphi_\lambda$ with $Q = P$. Since $G = K \times p$ and $p$ are unimodular, it says that the induced spherical function is given by $\text{Ind}_G^Q(\varphi_\lambda)(k, Y) = \int_K \varphi_\lambda(Ad(k)Y) dk = \int_K e^{i\lambda(Ad(k)Y)} dk = \int_K e^{i\lambda(k,Y)} dk = \psi_\lambda(k, Y)$ for $Y \in p, k \in K$. Note that $(k_0,0)(k,Y) = (k_0 k, Ad(k_0) Y)$. \hfill $\square$

We apply the Mackey little group method to $G$ relative to its normal subgroup $p$. If $\psi$ is an irreducible unitary representation of $G$, then it can be constructed (up to unitary equivalence) as follows: If $\varphi_\lambda(Y) = e^{i\lambda(Y)}$, where $\lambda \in p^*, Y \in p$, let $K_{\varphi_\lambda} = \{k \in K \mid \varphi_\lambda(Ad(k)Y) = \varphi_\lambda(Y) \forall Y \in p\}$. $K_{\varphi_\lambda}$ is a closed subgroup of $K$. Let $G_{\varphi_\lambda} = K_{\varphi_\lambda} \times p$. Write $\tilde{\varphi}$ for the extension of $\varphi$ to $G_{\varphi_\lambda}$ given by $\tilde{\varphi}((k, Y)) = \varphi_\lambda(Y)$. If $\gamma$ is an irreducible unitary representation of $K_{\varphi_\lambda}$, let $\tilde{\gamma}$ denote its extension of $G_{\varphi_\lambda}$ given by $\tilde{\gamma}((k, Y)) = \gamma(k)$. Denote $\psi_{\varphi_\lambda, \gamma} = \text{Ind}_G^Q(\varphi_\lambda)(\tilde{\varphi} \otimes \tilde{\gamma})$, then there exist choices of $\varphi_\lambda$ and $\gamma$ such that $\psi = \psi_{\varphi_\lambda, \gamma}$.

**Theorem 2.2.** In the notation above, $\psi_{\varphi_\lambda, \gamma}$ has a $K$-fixed vector is given (up to scalar multiple by $u((k, Y)) = e^{-i\lambda(Ad(k^{-1}) Y)}$, if $\varphi_\lambda = e^{i\lambda(Y)}$. 

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Proof. The representation space \( H_{\psi} \) of \( \psi = \psi_{\varphi,\gamma} \) consists of all \( L^2 \) functions \( f : G \to H_{\psi} \) such that \( f(g'(k',x')) = \gamma(k')^{-1}\varphi_{\lambda}(x')^{-1}f(g') \) for \( g' \in G, \ x' \in p, k' \in K_{\varphi_{\lambda}} \), and \( \psi \) acts by \((\psi(g)f)(g') = f(g^{-1}g')\).

Now suppose that \( 0 \neq f \notin H_{\psi} \) is fixed under \( \psi(K) \). If \( k' \in K_{\varphi_{\lambda}} \), then \( \gamma(k')f(1) = f(1) \). If \( f(1) = 0 \), then \( f(G_{\varphi_{\lambda}}) = 0 \) and \( K \)-invariance says \( f = 0 \), contrary to the assumption. Thus \( f(1) \neq 0 \) and irreducibility of \( \gamma \) forces \( \gamma \) to be trivial.

Conversely, if \( \gamma \) is trivial, then \( u((k,Y)) = e^{-i\lambda(Ad(k^{-1})Y)} \) is a nonzero \( K \)-fixed vector in \( H_{\psi} \). And it is the only one, up to scalar multiple, because any two \( K \)-fixed vectors must be proportional. \( \square \)

**Lemma 2.3.** In the notation above, \( \text{Ind}_{G_{\varphi_{\lambda}}}^{G}(\varphi_{\lambda}) \) is unitary equivalent to the subrepresentation of \( \text{Ind}_{p}^{G}(\varphi_{\lambda}) \) generated by the \( K \)-fixed unit vector \( u((k,Y)) = e^{-i\lambda(Ad(k^{-1})Y)} \), if \( \varphi_{\lambda} = e^{i\lambda(Y)} \).

**Theorem 2.4.** Let \( \varphi \) be a \((G,K)\)-spherical function. Then \( \varphi \) is positive definite if and only if it is of the form \( u_{\lambda} \) for some \( \lambda \in a^{*} \). Further, if \( \lambda, \lambda' \in a^{*} \), then \( \psi_{\lambda} = \psi_{\lambda'} \) if and only if \( \lambda' \in Ad(k)\lambda \).

**Proof.** Let \( \lambda \in a^{*} \). \( \sum_{i,j} e^{i\lambda(Ad(k)(-Y_{j}+Y_{i}))}c_{i}c_{j} = \sum_{i,j} e^{-i\lambda(Ad(k)Y_{j})}c_{i}e^{i\lambda(Ad(k)Y_{i})}c_{j} = (\sum_{i} e^{i\lambda(Ad(k)Y_{i})}c_{i})(\sum_{j} e^{-i\lambda(Ad(k)Y_{j})}c_{j}) \geq 0. \)

Since \( \psi_{\lambda} \) is a limit of non-negative linear combinations of positive definite functions on \( \mathbb{R}^{n} \), so it is positive definite.

Now let \( \varphi \) be a positive definite \((G,K)\)-spherical function. Let \( \Pi_{\varphi} \) be the associated irreducible unitary representation, and \( H_{\varphi} \) the representation space, such that there is a \( K \)-fixed unit vector \( u_{\varphi} \in H_{\varphi} \) and let \( \varphi(g) = \langle u_{\varphi}, \Pi_{\varphi}(g)u_{\varphi} \rangle \) for all \( g \in G \). Following the discussion of the Mackey little group method, and Theorem 2.2, we have \( \varphi_{\lambda}(Y) \) for some \( \lambda \in a^{*}, Y \in p \), s.t. \( \Pi_{\varphi} \) is unitarily equivalent to \( \text{Ind}_{G_{\varphi_{\lambda}}}^{G}(\varphi_{\lambda}) \). Making the identification one, \( K \)-fixed unit vector in \( H_{\varphi} \) is given by \( u((k,Y)) = e^{-i\lambda(Ad(k^{-1})Y)} \).

We have \( \lambda \in a^{*} \) s.t. \( \varphi_{\lambda} = e^{i\lambda(Y)} \) and from above several Theorems and Lemma, we compute:

\[
\varphi(Y) = \langle u, \Pi_{\varphi}(Y)u \rangle = \varphi(Y) = \langle u, \text{Ind}_{G_{\varphi_{\lambda}}}^{G}(\varphi_{\lambda})(Y)u \rangle = \varphi(Y) = \langle u, \text{Ind}_{G_{p}}^{G}(\varphi_{\lambda})(Y)u \rangle = \text{Ind}_{G_{p}}^{G}(\varphi_{\lambda})(Y) = \psi_{\lambda}(Y) = \int_{K} e^{i\lambda(Ad(k)Y)}dk
\]

For the second, if \( \lambda' = Ad(k_{0})\lambda \) for some \( k_{0} \in K \), we have:

\[
\psi_{\lambda'}(Y) = \int_{K} e^{i\lambda'(Ad(k)Y)}dk = \int_{K} e^{i\lambda(Ad(k_{0})Ad(k)Y)}dk = \int_{K} e^{i\lambda(Ad(k_{0}^{-1})Ad(k)Y)}dk = \int_{K} e^{i\lambda(Ad(k)Y)}dk = \psi_{\lambda}(Y).
\]

Conversely, suppose that \( \lambda', \lambda \in a^{*} \) with \( \psi_{\lambda'} = \psi_{\lambda} \). Then (up to unitary equivalence) \( \text{Ind}_{G_{\varphi_{\lambda'}}}^{G}(\varphi_{\lambda'}) = \text{Ind}_{G_{\varphi_{\lambda}}}^{G}(\varphi_{\lambda}) \). That gives us direct integral decompositions.
\[
\int_{K}^{\oplus} \psi_{\text{Ad}(k)\lambda} dk = \text{Ind}_{G_{\phi\lambda}}^{G_{\phi\lambda'}} (\widetilde{\phi}_{\lambda}) \mid p = \text{Ind}_{G_{\phi\lambda'}}^{G_{\phi\lambda}} (\widetilde{\phi}_{\lambda'}) \mid p = \int_{K}^{\oplus} \psi_{\text{Ad}(k)\lambda'} dk \quad \Box
\]

**Theorem 2.5.** If \( N \) is an \( n \)-step group with \( n \geq 3 \), then there are no Gelfand pairs \((K, N)\), where \( K \in \text{Aut}(N) \).

**Theorem 2.6.** We first consider \( K \)-spherical functions associated to a Gelfand pair \((K, N)\).

Suppose \( \phi \) is a bounded \( K \)-spherical function on \( N \). Then there is a \( \pi \in \hat{N} \) and a unit vector \( \xi \in H_{\pi} \) such that

\[
(2.7) \quad \phi(x) = \int_{K} < \pi(k.x)\xi, \xi > dk
\]

for each \( x \in N \)

**Proof.** Let \( \lambda_{\phi} : L_{K}^{1}(N) \to \mathbb{C} \) be given by integration against \( \phi \).

Since \( L_{K}^{1}(N) \) is a symmetric Banach *-algebra,[8], there is a representation \( \pi_{\phi} \) of \( L_{K}^{1}(N) \) and a one-dimensional subspace \( H_{\phi} \) of \( H_{\pi} \) such that \((\pi_{\phi} |_{L_{K}^{1}(N)}, H_{\phi})\) is equivalent to \((\lambda_{\phi}, \mathbb{C})\). As \( \lambda_{\phi} \) is irreducible, the extension \( \pi_{\phi} \) is also irreducible(cf.[9]). Using approximate identities at each point of \( N \), one can show that \( \pi_{\phi} \) is the integrated version of some \( \pi \in \hat{N} \), with \( H_{\pi} = H_{\phi} \).

Choose \( \xi \in H_{\phi} \) with \( \|\xi\| = 1 \). Then for each \( f \in L_{K}^{1}(N) \), \( \pi(f)\xi = \lambda_{\phi}(f)\xi \), so that

\[
(2.8) \quad < \phi, f >= \lambda_{\phi}(f) = < \pi(f)\xi, \xi > = \int_{N} f(x) < \pi(x)\xi, \xi > dx = \int_{K} \int_{N} f(k^{-1}.x) < \pi(x)\xi, \xi > dx dk
\]
since \( f \) is \( K \)-invariant

\[
(2.9) \quad = \int_{K} \int_{N} f(k.x) < \pi(x)\xi, \xi > dx dk
\]

Since \( \phi \) is \( K \)-invariant, we change the order of integration and obtain

\[
(2.10) \quad \phi(x) = \int_{K} < \pi(x)\xi, \xi > dk
\]

\( \Box \)

A complex-valued continuous function \( \phi \) on a locally compact group \( G \) is called positive definite if \( \sum_{i,j=1}^{n} \phi(x^{-1}x)_{ij} \alpha_{i}\bar{\alpha}_{j} \geq 0 \) for all finite sets \( x_{1}, \ldots, x_{n} \) of elements in \( G \) and any complex numbers \( \alpha_{1}, \ldots, \alpha_{n} \).

**Theorem 2.11.** For Gelfand pair \((K, N)\), where \( N \) is at most 2-step nilpotent Lie group, if \( \phi \) is a bounded \( K \)-spherical function on \( N \) is and only if \( \phi \) is positive definite.
Proof. If ϕ is a bounded K-spherical function on N, for all finite sets \( x_1, \ldots, x_n \) of elements in G and any complex numbers \( a_1, \ldots, a_n \), we have:
\[
\sum_{i,j=1}^{n} \phi(x_i^{-1}x_j) = \int_K \pi(k(x_i^{-1}x_j))dk = \int_K \pi(k(x_i^{-1}k(x_j)))dk.
\]
Therefore, \( \sum_{i,j=1}^{n} \phi(x_i^{-1}x_j) a_i a_j = \sum_{i,j=1}^{n} \phi(x_i^{-1}x_j) a_i a_j \int_K \pi(k(x_i^{-1}k(x_j)))dk = \int_K \phi(x_i^{-1}x_j)dk = \int_K \phi(x_i^{-1}k(x_j))dk > 0.
\]

Conversely, if \( \phi \) is a bounded K-spherical function on N then \( \phi \) is positive definite. Let \( \varphi \) be a positive definite \((G, K)\)-spherical function. Let \( \Pi_{\varphi} \) be the associated irreducible unitary representation, and \( H_{\varphi} \) the representation space, such that there is a \( K \)-fixed unit vector \( u_{\varphi} \in H_{\varphi} \) and let \( \varphi(g) = \langle u_{\varphi}, \Pi_{\varphi}(g)u_{\varphi} \rangle \) for all \( g \in G \). Then we have \( |\varphi(g)| = |\langle u_{\varphi}, \Pi_{\varphi}(g)u_{\varphi} \rangle| \leq \langle u_{\varphi}, u_{\varphi} \rangle > \frac{1}{2} \times \langle \Pi_{\varphi}(g)u_{\varphi}, \Pi_{\varphi}(g)u_{\varphi} \rangle \leq |\langle u_{\varphi}, u_{\varphi} \rangle| = 1
\]
Therefore, \( \varphi \) is bounded.

\[\square\]

**Theorem 2.12.** In the notation just above, assume the group \( G \) real. The spherical function \( \psi_\lambda \) on \( G_0 \) is bounded if and only if \( \lambda \) is real, i.e. \( \lambda \in a^* \).

**Proof.** According to Theorem 2.4, we obtain The spherical function \( \psi_\lambda \) on \( G_0 \) is positive definite if and only if \( \lambda \) is real, i.e. \( \lambda \in a^* \). According to Theorem 2.11, since \( p \) is abelian, we know that \( \psi_\lambda \) is a bounded \( K \)-spherical function on \( p \) if and only if \( \psi_\lambda \) is positive definite. Therefore, \( \psi_\lambda \) on \( G_0 \) is bounded if and only if \( \lambda \) is real, i.e. \( \lambda \in a^* \).

\[\square\]

3. **Generalizations for the Eigenspaces of the Laplacian on \( \mathbb{R}^n \)**

**Lemma 3.1.** [7] Let \( K \) be any closed subgroup of \( O(n) \), if \( K \) is transitive on the spheres about 0 in \( \mathbb{R}^n \), then \( \mathcal{D}(G/K) = \mathbb{C}[\Delta] \), algebra of polynomials in the Laplace-Beltrami operator \( \Delta = -\sum \partial^2/\partial x_i^2 \).

**Proof.** It is clear that \( \mathbb{C}[\Delta] \subset \mathcal{D}(G/K) \). Now let \( D \in \mathcal{D}(G/K) \) be of order \( m \). Then the \( m \)th order symbol of \( D \) is a polynomial of degree \( m \) constant on spheres about 0 in \( \mathbb{R}^n \), in other words a multiple \( cr^m \) with \( m \) even and \( r^2 = \sum x_i^2 \). Now \( D - c(\Delta)^m/2 \in \mathcal{D}(G/K) \) and \( D - c(\Delta)^m/2 \) has order < \( m \). By induction on the order, \( D - c(\Delta)^m/2 \in \mathcal{D}(G/K) \), so we have \( D \in \mathcal{D}(G/K) \).

Let \( L \) denote the usual Laplacian on \( \mathbb{R}^n \) and for each \( \lambda \in \mathbb{C} \) let \( \mathcal{E}_\lambda(\mathbb{R}^n) \) denote the eigenspace

\[(3.2) \quad \mathcal{E}_\lambda(\mathbb{R}^n) = \{ f \in \mathcal{E}(\mathbb{R}^n) \mid Lf = -\lambda^2 f \}\]

with the topology induced by that of \( \mathcal{E}(\mathbb{R}^n) \). Let \( G = K \times \mathbb{R}^n \) and \( K \) is the closed subgroup of \( O(n) \) as well as acting transitive on the spheres about 0 in \( \mathbb{R}^n \).

**Theorem 3.3.** The natural action of \( G \) on \( \mathcal{E}_\lambda(\mathbb{R}^n) \) is irreducible if and only if \( \lambda \neq 0 \).
Proof. It is clear that each function

\[ f(x) = \int_{S^{n-1}} e^{i\lambda(x,w)}F(w)dw, \quad F \in L^2(S^{n-1}), \]

lies in \( E_\lambda(\mathbb{R}^n) \); here \((,\)\) denotes the usual inner product on \( \mathbb{R}^n \) and \( dw \) the normalized volume element. \( \square \)

**Lemma 3.5.** Let \( \lambda \neq 0 \). Then the mapping \( F \to f \) defined by (3.4) is one-to-one.

**Proof.** Let \( p(\zeta) = p(\zeta_1, \cdots, \zeta_n) \) be a polynomial and \( D \) the corresponding constant coefficient differential operator on \( \mathbb{R}^n \) such that

\[ \int_{\mathbb{R}^n} e^{i(x,\zeta)}D_x(e^{-\frac{1}{2}|x|^2}) = p(\zeta)e^{-\frac{1}{2}(\zeta_1^2 + \cdots + \zeta_n^2)} \]

for \( \zeta \in \mathbb{C}^n \). If \( f \equiv 0 \) in (1) we deduce from (3.6) that

\[ \int_{S^{n-1}} p(\lambda w_1, \cdots, \lambda w_n)F(w)dw = 0 \]

Since \( \lambda \neq 0 \), this implies \( F \equiv 0 \). \( \square \)

**Lemma 3.8.** Let \( \lambda \neq 0 \). The \( K \)-finite solutions \( f \) of the equation \( Lf = -\lambda^2f \) are precisely

\[ f(x) = \int_{S^{n-1}} e^{i\lambda(x,w)}F(w)dw \]

where \( F \) is a \( K \)-finite function on \( S^{n-1} \).

**Proof.** Let \( \delta \) be an irreducible representation of \( K \) and if \( \Sigma \) is any sphere in \( \mathbb{R}^n \) with center at 0 let \( E_\delta(\Sigma) \) denote the space of \( K \)-finite functions in \( E(\Sigma) \) of type \( \delta \). We know from Lemma 1.5 p.134 in [10] that if \( \Sigma \) is suitably chosen each function \( f \mid \Sigma \) to \( \Sigma \). With \( F \) and \( f \) as in (3.4) it follows that the maps

\[ F \to f \mid \Sigma, \quad F \to f \quad F \in L^2(S^{n-1}) \]

are one-to-one and commute with the action of \( K \). For reasons of dimensionality, the first must therefore map \( E_\delta(S^{n-1}) \) onto \( E_\delta(\Sigma) \). The lemma now follows. \( \square \)

For \( \lambda \neq 0 \) let \( \mathcal{H}_\lambda \) denote the space of functions \( f \) as defined in (3.4); \( \mathcal{H}_\lambda \) is a Hilbert space if the norm of \( f \) is the \( L^2 \) norm of \( F \) on \( S^{n-1} \).

**Lemma 3.10.** Let \( \lambda \neq 0 \). Then the space \( \mathcal{H}_\lambda \) is dense in \( E_\lambda(\mathbb{R}^n) \).

**Proof.** Each eigenfunction of \( L \) can be expanded in a convergent series of \( K \)-finite eigenfunctions (cf. Sect. 5 [6]) so the lemma follows from Lemma 3.8. \( \square \)
We can now prove Theorem 3.3. We first prove that $G$ acts irreducibly on $\mathcal{H}_\lambda$. Let $V \neq 0$ be a closed invariant subspace of $\mathcal{H}_\lambda$. Then there exists an $h \in V$ such that $h(0) = 1$. We write
\[
(3.11) \quad h(x) = \int_{S^{n-1}} e^{i\lambda(x,w)} H(w)dw
\]
and the average $h^2(x) = \int_K h(k.x)dk$ is then
\[
(3.12) \quad h^2(x) = \varphi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda(x,w)}dw.
\]
If $f$ in (3.4) lies in the annihilator $V^0$ of $V$ the functions $F$ and $H$ are orthogonal on $S^{n-1}$. Since $V^0$ is $K$-invariant this remains true for $H$ replaced by its integral over $K$, in other words $\varphi_\lambda$ belongs to the double annihilator $(V^0)^0 = V$. Now, since $V$ is invariant under translations it follows that for each $t \in \mathbb{R}$ the function
\[
(3.13) \quad x \rightarrow \int_{S^{n-1}} e^{i\lambda(x,w)} e^{i\lambda(t,w)}dw
\]
belongs to $V$. But then Lemma 3.5 shows that the annihilator of $V$ in $\mathcal{H}_\lambda$ is 0, whence the irreducibility of $G$ on $\mathcal{H}_\lambda$.

Passing now to $\mathcal{E}_\lambda$ let $V \subset \mathcal{E}_\lambda$ be a closed invariant subspace. Then $V \cap \mathcal{H}_\lambda$ is an invariant subspace of $\mathcal{H}_\lambda$; Schwartz’ inequality shows easily that it is closed. Thus, by the above, $V \subset \mathcal{E}_\lambda$ is $\{0\}$ or $\mathcal{H}_\lambda$. In the second case $V = \mathcal{E}_\lambda$ by Lemma 3.10. In the first case consider for each $f \in V$ the convergent expansion
\[
(3.14) \quad f = \sum_{\delta \in K} \alpha_\delta \ast f
\]
where $\alpha_\delta = d(\delta) \chi_\delta^*$ and
\[
(3.15) \quad (\alpha_\delta \ast f)(x) = \int_K \alpha_\delta(k)f(k^{-1}.x)dk
\]
$\chi_\delta$ being the character of $\delta$. Then $\alpha_\delta \ast f \in \mathcal{H}_\lambda$ by Lemma 3.8. Let $V^0 \subset \mathcal{E}'(\mathbb{R}^n)$ be the annihilator of $V$. Then $V^0$ is $G$-invariant and if $T \in V^0$,
\[
(3.16) \quad \int_{\mathbb{R}^n} (\alpha_\delta \ast f)(x)dT(x) = \int_K \alpha_\delta(k) \int_{\mathbb{R}^n} f(x)dT(k.x)dk
\]
so $\alpha_\delta \ast f$ belongs to the double annihilator $(V^0)^0 = V$. Thus $\alpha_\delta \ast f \in V \cap \mathcal{H}_\lambda = \{0\}$ so, by (3.6), $f = 0$. Thus $V = \{0\}$ so the proof is finished.

4. A ESTIMATION FOR SOME SPHERICAL FUNCTIONS AND THE GROUPS $K$

From [7], we know if $K$ is transitive on the spheres about 0 in $\mathbb{R}^n$, then the spherical function on $K \times \mathbb{R}^n \simeq \mathbb{R}^n$ is of the form: $\varphi_s(r) = \varphi(r,s) = \int_{S^{n-1}} e^{is(\xi,x)}d\sigma(\xi) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n!}} \int_0^\pi e^{s\sqrt{\sin \theta}} \sin^{n-2} \theta d\theta$.

Where $s$ is a complex number, and $r = \|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$, $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and $\sigma$ the normalized surface measure on $S^{n-1}$. 

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If $\text{Res} = 0$ then it follows from above equation that $|\varphi_s(x)| \leq 1$ for all $x \in G$.

Clearly, $\varphi_s = \varphi_{-s}$. We just need to consider the case $\text{Res} \geq 0$.

**Theorem 4.1.** If $\text{Res} > 0$, we have $\varphi(r, s) \sim \frac{\Gamma\left(\frac{2}{s}ight) \frac{2}{s}}{\sqrt{2\pi}} \frac{e^{sr}}{(sr)^{n+2}}$.

**Proof.** From above equation, we know that, by an elementary substitution,

$$\varphi_s(r) = \varphi(r, s) = \frac{\Gamma\left(\frac{2}{s}\right)}{\sqrt{2\pi} \frac{2}{s}} \int_{-1}^{1} e^{sr} (1 - t^2)^{-\frac{n}{2}} dt$$

and, setting $t = 1 - \frac{r}{s}$, we obtain

$$\varphi(r, s) = \frac{\Gamma\left(\frac{2}{s}\right)}{\sqrt{2\pi} \frac{2}{s}} \frac{e^{sr}}{r^2} \int_{0}^{2r} e^{-su} u^{\frac{n-3}{2}} (2 - u) \frac{n+3}{2} du$$

For $\text{Res} > 0$, we get, using Lebesgue’s dominated convergence theorem,

$$\lim_{r \to \infty} \int_{0}^{2r} e^{-su} u^{\frac{n-3}{2}} (2 - u) \frac{n+3}{2} du = \frac{\Gamma\left(\frac{2}{s}\right) \frac{2}{s}}{\sqrt{2\pi}}$$

\[ \square \]

Finally, I will give all the possible $K$, which is transitive on the spheres about 0 on $\mathbb{R}^n$, $n > 1$. [11]

When $K$ is transitive on the spheres about 0 in $\mathbb{R}^n$, $n > 1$, its identity component $K^0$ is also transitive, and $K = K^0F$ where $F$ is a finite subgroup of the normalizer $N_{O(n)}(K^0)$. The possibilities for $K^0$ are as follows:

1. $n > 1$ and $K^0 = SO(n)$,
2. $n = 2m$ and (i) $K^0 = SU(m)$ or (ii) $U(m)$,
3. $n = 4m$ and (i) $K^0 = Sp(m)$ or (ii) $Sp(m).U(1)$ or $Sp(m)Sp(1)$,
4. $n = 7$ and $K^0$ is the exceptional group $G_2$,
5. $n = 8$ and $K^0 = Spin(7)$, and
6. $n = 16$ and $K^0 = Spin(9)$.

In case (1), $N_{O(n)}(K^0) = O(n)$, so the relevant choices for $F$ are $\{I\}$ and $\{I, -I\}$, so $K$ is either $SO(n)$ or $O(n)$.

In case (2)(i), $N_{O(n)}(K^0) = U(m) \cup \alpha U(m)$ where $\alpha$ is a complex conjugation of $\mathbb{C}^n$ over $\mathbb{R}^m$. The relevant choices for $F$ are the finite subgroups of $U(1) \cup \alpha U(1)$ where $U(1)$ consists of the unitary scalar matrices $e^{ix}I$, $x$ real. Those are the cyclic groups $\mathbb{Z}_l = \{e^{2\pi ik/l}I\}$ of order $l \geq 1$ and the dihedral groups $D_l = \mathbb{Z}_l \cup \alpha \mathbb{Z}_l$, so $K$ is a group $SU(m)\mathbb{Z}_l$ or $SU(m)D_l$. In the case (2)(ii) the relevant possibilities for $F$ are $\{I\}$ and $\{\alpha, I\}$, so $K$ is either $U(m)$ or $U(m) \cup \alpha U(m)$.

In case (3)(i),(3)(iii),(4),(5),(6), $K^0$ has no outer automorphism, so we may take $F$ in the centralizer $Z_{O(n)}(K^0)$. Thus in the case (3)(i), $F$ can be any subgroup of $Sp(1)$, in the other words, a cyclic group $\mathbb{Z}_l$ of order $l$, a binary dihedral group $D^*_l$ of order $4l$, a binary tetrahedral group $T^*$ of order 24, a binary octahedral group $O^*$ of order 48, or a binary icosahedral group $I^*$ of order 60. Thus $K$ is a group $Sp(m)\mathbb{Z}_l, Sp(m)D^*_l, Sp(m)T^*, Sp(m)O^*$ or $Sp(m)I^*$. In case (3)(ii) the relevant possibilities for $F$ are $\{I\}$ and $\{\beta, I\}$, where the $U(1)$ factor of $K^0$ consists of all quaternion scalar multiplications by complex numbers $e^{ix}$, $x$ is real, as in the case (2), and
$ß$ is quaternion scalar multiplication by $j$. Thus $K$ is either $Sp(m)U(1)$ or $(Sp(m)U(1)) \cup (Sp(m)U(1))ß$. In case (3)(iii), $K^0$ is its own $O(n)$-centralizer so $F = \{I\}$ and $K = Sp(m)Sp(1)$.

In case (4),(5),(6), $K^0$ is absolutely irreducible on $\mathbb{R}^n$, so relevant $F$ would have to consist of real scalars. As $G_2$ does not contain $-I$ we see that the relevant $F$ for case (4) are $\{I\}$ and $\{I, -I\}$, resulting in $K = G_2$ and $K = G_2 \cup (-I)G_2$. Both $Spin(7)$ and $Spin(9)$ do contain $-I$, so $F$ is trivial in case (5),(6). That gives $K = Spin(7)$ in case (5) and $K = Spin(9)$ for case (6).

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