PRODUCTS OF METRIC SPACES, 
COVERING NUMBERS, PACKING NUMBERS 
AND CHARACTERIZATIONS OF ULTRAMETRIC SPACES

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Abstract. We describe some Cartesian products of metric spaces and find 
conditions under which products of ultrametric spaces are ultrametric.

1. Introduction

Let \((X, d)\) be a metric space. The closed balls with a center \(c \in X\) and radius \(r, \ 0 < r < \infty\), are denoted by 
\[B(c, r) = B_d(c, r) = \{x \in X : d(x, c) \leq r\}.
\]
Let \(W\) be a subset of \(X\) and let \(\varepsilon > 0\). A set \(C \subseteq X\) is an \(\varepsilon\)-net for \(W\) if 
\[W \subseteq \bigcup_{c \in C} B(c, \varepsilon).
\]
A set \(W \subseteq X\) is called totally bounded (or precompact) if for every \(\varepsilon > 0\) there 
is a finite \(\varepsilon\)-net for \(W\). The covering number of a totally bounded set \(W \subseteq X\) is 
the smallest cardinality of subsets of \(W\) which are \(\varepsilon\)-nets for \(W\). A set \(A \subseteq X\) is 
called \(\varepsilon\)-distinguishable if \(d(x, y) > \varepsilon\) for every distinct points \(x, y \in A, [7]\). The packing number of a precompact set \(W \subseteq X\) is the maximal cardinality of the \(\varepsilon\)-distinguishable sets \(A \subseteq W\).

We denote by \(N_\varepsilon(W)\) and by \(M_\varepsilon(W)\) the covering number and, respectively, 
the packing number of a totally bounded set \(W \subseteq X\). These quantities have been 
invented by Kolmogorov [6] in order to classify compact metric sets. Note that 
the function \(\log_2 N_\varepsilon(W)\) is the so-called metric entropy and it has been widely 
applied in approximation theory, geometric functional analysis, probability theory 
and complexity theory, see, for example, [2, 5, 7, 8].

A main general fact about packing and covering numbers is the simple double 
inquality
\[(1.1) \quad M_{2\varepsilon}(W) \leq N_\varepsilon(W) \leq M_\varepsilon(W).
\]
In the second section of this paper we consider some transfinite generalizations of 
covering numbers and packing ones and obtain a more exact version of inequality 
\[(1.1), \quad \text{see Lemma 2.6.} \]
It implies the characterization of ultrametric spaces as spaces 
for which packing numbers equal covering numbers. In the third and fourth sections 
we introduce some “natural” metrics on the products of metric spaces and discuss 
conditions under which the products of ultrametric spaces are ultrametric.

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\(\varepsilon\)-capacity.
2. The equality between covering numbers and packing numbers

Let \((X, d)\) be a metric space. Denote by \(t_0 = t_0(d)\) the supremum of positive numbers \(t\) for which the function \((x, y) \mapsto (d(x, y))^t\) is a metric on \(X\). This quantity has the following characterization, see [3].

**Lemma 2.1.** Let \(x, y\) and \(z\) be points in a metric space \((X, d)\). If the inequality
\[
\max\{d(x, z), d(z, y)\} < d(x, y)
\]
holds, then there exists a unique solution \(s_0 \in [1, \infty]\) of the equation
\[
(d(x, y))^s = (d(x, z))^s + (d(z, y))^s.
\]

For points \(x, y\) and \(z\) in \(X\) write
\[
s(x, y, z) := \begin{cases} s_0 & \text{if } (2.1) \text{ holds} \\ +\infty & \text{otherwise} \end{cases}
\]
where \(s_0\) is the unique root of equation \((2.2)\).

**Proposition 2.2.** The equality
\[
t_0(d) = \inf\{s(x, y, z) : x, y, z \in X\}
\]
holds in every metric space \((X, d)\).

**Remark 2.3.** A point \(z\) in a metric space \((X, d)\) lies between two distinct points \(x\) and \(y\) if \(d(x, z) + d(z, y) = d(x, y)\) and \(x \neq z \neq y\), see [9, p. 55]. Now \(t_0 = t_0(d)\) can be called the *betweenness exponent* of the space \((X, d)\).

Recall that a metric space \((X, d)\) is *ultrametric* if the metric \(d\) satisfies the *ultra-triangle inequality* \(d(x, y) \leq \max\{d(x, z), d(z, y)\}\) for all \(x, y, z \in X\). In this case \(d\) is called an *ultrametric*. Since \((2.1)\) never holds in an ultrametric space \((X, d)\) we have \(t_0(d) = \infty\) in this case. In fact, \((X, d)\) is ultrametric if and only if \(t_0(d) = \infty\).

**Lemma 2.4.** Let \(B(a, r)\) be a closed ball in a metric space \((X, d)\). Then we have the inequality
\[
diam(B(a, r)) \leq 2^{\frac{1}{t_0}} r
\]
where \(t_0 = t_0(d)\) is the betweenness exponent of \((X, d)\).

**Proof.** If \(t_0(d) = \infty\), then \((X, d)\) is ultrametric and \(diam(B(a, r)) \leq r\) for every ball \(B(a, r)\), see, for example, [4, p. 43]. In the case \(t_0(d) < \infty\) the function \((x, y) \mapsto (d(x, y))^{t_0}\) is a metric. Hence, by the triangle inequality, we have
\[
d^{t_0}(x, y) \leq d^{t_0}(x, a) + d^{t_0}(y, a) \leq 2^{r_{t_0}}
\]
for all \(x, y \in B(a, r)\). The last inequality implies \((2.4)\). \(\square\)

There is the possibility of more refined classification of nonprecompact metric spaces by means of an extension of the range of values of the functions \(\mathcal{N}_\varepsilon\) and \(\mathcal{M}_\varepsilon\) to transfinite cardinal numbers.

Let \(W\) and \(A\) be subsets of \(X\). Define
\[
\mathcal{N}_\varepsilon^A(W) := \min\{\text{card}(C) : C \text{ is an } \varepsilon\text{-net for } W \text{ and } C \subseteq A\}.
\]
Moreover for the sake of simplicity, write \(\mathcal{N}_\varepsilon(W) := \mathcal{N}_\varepsilon^W(W)\).

For convenience we introduce an additional definition.

**Definition 2.5.** A set \(A\) is *maximal \(\varepsilon\)-distinguishable* with respect to \(W\) if \(A\) is \(\varepsilon\)-distinguishable, \(A \subseteq W\) and for every \(\varepsilon\)-distinguishable \(B \subseteq W\) the inclusion \(A \subseteq B\) implies the equality \(A = B\).
Write $\tilde{M}_\varepsilon(W)$ for the smallest power of maximal $\varepsilon$-distinguishable sets $A \subseteq W$ and define the quantity $\mathcal{M}_\varepsilon^*(W)$ as the smallest cardinal number which is greater than or equal to $\text{card}(A)$ for every $\varepsilon$-distinguishable $A \subseteq W$. It is clear that

$$\mathcal{M}_\varepsilon^*(W) = M_\varepsilon(W) \quad \text{and} \quad \hat{N}_\varepsilon(W) = N_\varepsilon(W)$$

for every precompact $W$.

**Lemma 2.6.** Let $W$ be a set in a metric space $(X, d)$. Then for every $\varepsilon > 0$ we have the following inequalities

$$\mathcal{M}_\varepsilon^*(W) \leq \hat{N}_\varepsilon X(W) \leq \hat{N}_\varepsilon(W) \leq \check{M}_\varepsilon(W) \leq M_\varepsilon^*(W)$$

where $t_0$ is the betweenness exponent of $X$.

**Proof.** The first inequality from the right is immediate. For the proof of the second one note that every maximal $\varepsilon$-distinguishable set $A \subseteq W$ is an $\varepsilon$-net for $W$. The inequality $\hat{N}_\varepsilon X(W) \leq N_\varepsilon(W)$ is clear from the definitions. To prove the first inequality from the left it suffices to show $\text{card}(A) \leq \hat{N}_\varepsilon X(W)$ for every $2^{\frac{1}{\varepsilon}}$-distinguishable set $A \subseteq W$. Let $\{x_i : i \in I\}$ be an $\varepsilon$-net for $W$ with $\text{card}(I) = \hat{N}_\varepsilon X(W)$. Suppose that there exists a $2^{\frac{1}{\varepsilon}}$-distinguishable set $A_0 \subseteq W$ for which $\text{card}(A_0) > \text{card}(I)$. This inequality and the inclusion

$$A_0 \subseteq \bigcup_{i \in I} B(x_i, \varepsilon)$$

imply that there exists a ball $B(x_i, \varepsilon)$ which contains at least two distinct points $y_i, z_i \in A_0$. (In the opposite case $A_0$ and some subset of $I$ have the same cardinality.) Lemma 2.4 implies that $d(y_i, z_i) \leq 2^{\frac{1}{\varepsilon}}$. This contradicts the assumption that $A_0$ is $2^{\frac{1}{\varepsilon}}$-distinguishable. \qed

**Corollary 2.7.** Let $X$ be a nonprecompact metric space. Then for some $\varepsilon_0 > 0$ there is an $\varepsilon_0$-distinguishable, countable infinite set $A \subseteq X$.

**Example 2.8.** Let $X$ be a set of a power $\alpha > 2$ and let $a$ be an element of $X$. For every two distinct $x, y \in X$ write

$$d(x, y) = \begin{cases} 2^t & \text{if } x \neq a \neq y \\ 1 & \text{otherwise} \end{cases}$$

where $t \in [1, \infty]$ and put $d(x, y) = 0$ if $x = y$. Proposition 2.2 implies that the metric space $(X, d)$ has the betweenness exponent $t_0(d) = t$. If we define a set $W$ as $W = X \setminus \{a\}$, then

$$\hat{N}_\varepsilon(W) = \check{M}_\varepsilon(W) = \mathcal{M}_\varepsilon^*(W) = \text{card}(W)$$

but

$$\hat{N}_\varepsilon X(W) = \mathcal{M}_\varepsilon^*(X) = 1 = \check{M}_\varepsilon(X)$$

for every $\varepsilon \in ]1, 2^t[.$

**Theorem 2.9.** Let $(X, d)$ be a metric space. The following statements are equivalent.

(i) The space $X$ is ultrametric.
(ii) For every $W \subseteq X$ the equalities
\begin{equation}
\mathcal{M}_\varepsilon(W) = \mathcal{N}_\varepsilon^X(W) = \mathcal{N}_\varepsilon(W) = \mathcal{M}_\varepsilon(W)
\end{equation}
hold for all $\varepsilon > 0$.

(iii) For every compact $W \subseteq X$ and every $\varepsilon > 0$ we have the equality
\[\mathcal{N}_\varepsilon(W) = \mathcal{M}_\varepsilon(W)\].

Proof. Since $t_0(d) = \infty$ holds if $d$ is an ultrametric, inequalities (2.10) imply (2.6) for ultrametric spaces. The implication (ii)$\Rightarrow$(iii) is trivial. If $(X, d)$ is not an ultrametric space, then there are points $a, b, c \in X$ such that
\begin{equation}
d(a, b) > \max\{d(a, c), d(b, c)\}.
\end{equation}
Write $\varepsilon := \max\{d(a, c), d(b, c)\}$. It follows from (2.8) that $\mathcal{M}_{\varepsilon}(\{a, b, c\}) \geq 2$. Moreover, since $B(c, \varepsilon) \supseteq \{a, b, c\}$, we see that $\mathcal{N}_{\varepsilon}(\{a, b, c\}) \leq 1$. Hence $\mathcal{N}_{\varepsilon}(\{a, b, c\}) \neq \mathcal{M}_{\varepsilon}(\{a, b, c\})$. □

Consider now equalities (2.7) for non ultrametric spaces.
Recall that a cardinal number $\alpha$ is the density of a metric space $X$ if $\alpha = \min_A(\text{card}(A))$

where the minimum is taken over the family of all dense sets $A \subseteq X$. For the density of $X$ we use the symbol $\text{den}X$. For convenience we repeat some definitions related to the confinality of the cardinals, see, for example [10]. We understand the ordinal numbers as some special well-ordered sets $\alpha, \beta, ...$ for which the statements:
- $\alpha$ is similar to an initial segment of $\beta$ and $\alpha \neq \beta$, $\alpha \prec \beta$;
- $\alpha$ is proper subset of $\beta$, $\alpha \varsubsetneq \beta$;
- $\alpha$ belongs to $\beta$, $\alpha \in \beta$
are equivalent. An ordinal number $\beta$ is an initial ordinal if for all ordinals $\alpha$ we have the implication
\[(\alpha \prec \beta) \Rightarrow (|\alpha| \leq |\beta|)\]

where $|\alpha|$ and $|\beta|$ are corresponding cardinality of $\alpha$ and $\beta$. By cardinal numbers we mean initial ordinals. An ordinal number $\alpha$ is confinal in an ordinal $\beta$ if there is an one-to-one increasing mapping $f : \alpha \rightarrow \beta$ such that for every ordinal $\gamma \in \beta$ there exists an ordinal $\delta \in \alpha$ with
\[\gamma \prec f(\delta) \quad \text{or} \quad \gamma = f(\delta).
\]
The confinality of an ordinal $\beta$ is the least ordinal $\alpha$ with $\alpha$ confinal in $\beta$. We write $\text{cf}(\beta)$ for the confinality of $\beta$. If $\alpha$ is the confinality for some $\beta$, then $\alpha$ is a cardinal, [10, p.91].

**Theorem 2.10.** Let $W$ be a subset of a metric space $X$. Suppose that $\text{den}(W)$ is a cardinal of an uncountable confinality. Then there is $\varepsilon_0 > 0$ such that the equalities
\begin{equation}
\mathcal{N}_{\varepsilon}^X(W) = \mathcal{N}_{\varepsilon}(W) = \mathcal{M}_{\varepsilon}(W) = \mathcal{M}_{\varepsilon}^*(W) = \text{den}(W)
\end{equation}
hold for all $\varepsilon \in \{0, \varepsilon_0\}$.

Write, as usual, $\aleph_0$ for $\text{card}(\mathbb{N})$ and $\varepsilon = 2^{\aleph_0} = \text{card}(\mathbb{R})$.

**Corollary 2.11.** Let $W$ be a subset of a metric space $X$. If $\text{den}(W) = \varepsilon$, then there is $\varepsilon_0 > 0$ such that the equalities
\begin{equation}
\mathcal{N}_{\varepsilon}^X(W) = \mathcal{N}_{\varepsilon}(W) = \mathcal{M}_{\varepsilon}(W) = \mathcal{M}_{\varepsilon}^*(W) = \varepsilon
\end{equation}
hold for all $\varepsilon \in \{0, \varepsilon_0\}$.

Proof. Since for each infinite cardinal $\gamma$ we have $\gamma \prec \text{cf}(2^\gamma)$, see [10, Theorem 44,p.93], $\varepsilon$ has an uncountable confinality. □
Corollary 2.12. Let \((X, \tau)\) be a metrizable topological space, let \(W \subseteq X\) be a set such that \(\text{den}(W)\) is a cardinal of an uncountable confinality and let \(D\) be a finite family of metrics \(d\) each of which induces the topology \(\tau\) on \(X\). Then there is \(\varepsilon_0 > 0\) such that for all \(\varepsilon \in [0, \varepsilon_0]\) the values \(\tilde{\mathcal{N}}_\varepsilon(W), \mathcal{N}^*_\varepsilon(W), \mathcal{M}_\varepsilon(W)\) and \(\mathcal{M}^*_\varepsilon(W)\) do not depend on the choice of \(d \in D\).

Proof of Theorem 2.10. The definitions of cardinal numbers \(\mathcal{N}_\varepsilon(W)\) and \(\text{den}(W)\) imply that the inequality

\[
\mathcal{N}_\varepsilon(W) \leq \mathcal{N}_\varepsilon(W) \leq \text{den}(W)
\]

holds for all \(\varepsilon > 0\). Hence, by (2.6), we have

\[
\mathcal{M}_\varepsilon(W) \leq \mathcal{M}^*_\varepsilon(W) \leq \text{den}(W)
\]

if \(\varepsilon > 0\). Moreover, if there is \(\varepsilon_0 > 0\) such that

\[
\text{den}(W) \leq \mathcal{N}_{\varepsilon_0}(W),
\]

then the last inequality and (2.6) imply

\[
\mathcal{N}_{\varepsilon_0}(W) \geq \mathcal{M}^*_{\varepsilon_0}(W) \geq \mathcal{M}_{\varepsilon_0}(W) \geq \mathcal{N}_{\varepsilon_0}(W) \geq \text{den}(W).
\]

Therefore, it is sufficient to show (2.11) with some \(\varepsilon_0 > 0\).

If \(D\) is a dense subset of \(W\), then for every \(k \in [0,1]\) and all \(\varepsilon > 0\) we have the double inequality

\[
\mathcal{N}_{\varepsilon_0}(W) \geq \mathcal{N}_{\varepsilon_0}(D) \geq \mathcal{N}_{\varepsilon_0}(W).
\]

Indeed, if \(C = \{c_i : i \in I\}\) is a \(k\varepsilon\)-net for \(W\) with \(\text{card}(C) = \mathcal{N}_{k\varepsilon}(W)\), then the density of \(D\) in \(W\) implies that for every \(c_i \in C\) there is \(b_i \in D\) such that \(B(b_i, \varepsilon) \supseteq B(c_i, k\varepsilon)\). Hence we have

\[
D \subseteq W \subseteq \bigcup_{i \in I} B(c_i, k\varepsilon) \subseteq \bigcup_{i \in I} B(b_i, \varepsilon),
\]

i.e., \(\{b_i : i \in I\}\) is an \(\varepsilon\)-net for \(D\), so the first inequality in (2.12) is proved. Similarly, if \(P = \{p_i : i \in I\}\) is an \(\varepsilon\)-net for \(D\) with \(\text{card}(P) = \mathcal{N}_{\varepsilon}(D)\), then for every \(x \in W\) there is \(p_i \in P\) such that

\[
x \in B(p_i, \varepsilon/\kappa).
\]

Hence \(P\) is an \(\varepsilon\)-net for \(W\), that implies the second inequality in (2.12).

Let \(D\) be a dense subset of \(W\) such that

\[
\text{card}(D) = \text{den}(W).
\]

Consider a sequence of positive numbers \(\varepsilon_1, \varepsilon_2, \ldots\) with \(\lim_{i \to \infty} \varepsilon_i = 0\). Suppose that a set \(D_i\) is an \(\varepsilon_i\)-net for \(D\) with \(D_i \subseteq D\) and with

\[
\text{card}(D_i) = \mathcal{N}_{\varepsilon_i}(D)
\]

for every \(i \in \mathbb{N}\). The set

\[
\tilde{D} := \bigcup_{i=1}^\infty D_i
\]

is a dense subset of \(W\) and \(\tilde{D} \subseteq D\). Hence, by (2.13), \(\text{card}(\tilde{D}) = \text{den}(W)\). Suppose also that the inequality

\[
\text{card}(D_i) \leq \text{den}(W)
\]

holds for each \(D_i\). Let \(\gamma\) be an initial ordinal such that \(|\gamma| = \text{card}(\tilde{D})\) and let \(f: \gamma \to \tilde{D}\) be a bijection. Inequality (2.13) implies that for every ordinal \(\alpha_i :=
\]
A study of metric products was originated at the paper of A. Bernig, T. Foertsch and V. Schroeder [1].

Contrary to the supposition of the theorem, thus there is $\varepsilon$ such that the following diagram

$$
\begin{array}{c}
\mathbb{R}^+ \\
\downarrow \\
D_X \times D_Y
\end{array}
\xrightarrow{d_X \otimes d_Y}
\begin{array}{c}
(X \times Y) \times D_Y \\
\downarrow \\
(X \times X) \times (Y \times Y)
\end{array}
\xrightarrow{d}
\begin{array}{c}
(X \times Y) \times (X \times Y) \\
\downarrow \\
\mathbb{R}^+
\end{array}
$$

3. Metrics on products of metric spaces

Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces.

**Definition 3.1.** A metric $d$ on the product $X \times Y$ is said to be *distance-increasing* if

$$
d((x_1, y_1), (x_2, y_2)) \leq d((x_3, y_3), (x_4, y_4))
$$

whenever

$$
d_X(x_1, x_2) \leq d_X(x_3, x_4) \quad \text{and} \quad d_Y(y_1, y_2) \leq d_Y(y_3, y_4);
$$

$d$ is *partial distance-preserving* if we have the equalities

$$
d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) \quad \text{and} \quad d((x_1, y_1), (x_2, y_2)) = d_Y(y_1, y_2)
$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

**Remark 3.2.** When

$$
d_X(x_1, x_2) = d_X(x_3, x_4), \quad d_Y(y_1, y_2) = d_Y(y_3, y_4),
$$

we obtain from (3.1) and (3.2) that

$$
d((x_1, y_1), (x_2, y_2)) = d((x_3, y_3), (x_4, y_4)),
$$

i.e., the distance-function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}^+$ depends only on “partial” distance-functions $d_X$ and $d_Y$. Consequently, there is a mapping $F : D_X \times D_Y \to \mathbb{R}^+$ with

$$
D_X := \{d_X(x, y) : x, y \in X\}, \quad D_Y := \{d_Y(x, y) : x, y \in Y\}
$$

such that the following diagram

$$
\begin{array}{c}
(X \times Y) \times (X \times Y) \\
\downarrow \\
(X \times X) \times (Y \times Y)
\end{array}
\xrightarrow{d_X \otimes d_Y}
\begin{array}{c}
D_X \times D_Y \\
\downarrow \\
\mathbb{R}^+
\end{array}
\xrightarrow{F}
$$

is commutative. Here $\text{Id}$ is an identification mapping

$$
\text{Id}((x_1, y_1), (x_2, y_2)) = ((x_1, x_2), (y_1, y_2))
$$

and $d_X \otimes d_Y$ is the direct product of the partial distance functions $d_X$ and $d_Y$,

$$
d_X \otimes d_Y((x_1, x_2), (y_1, y_2)) = (d_X(x_1, x_2), d_Y(y_1, y_2)).
$$

Diagram (3.7) shows that we can find the metric properties of the product $X \times Y$ using the corresponding ones of the function $F$. This approach to the study of metric products was originated at the paper of A. Bernig, T. Foertsch and V. Schroeder [1].

**Example 3.3.** For every $p \in [1, \infty]$ let $d_p$ be a metric on $X \times Y$ defined as

$$
d_p((x_1, y_1), (x_2, y_2)) = ((d_X(x_1, x_2))^p + (d_Y(y_1, y_2))^p)^{\frac{1}{p}}
$$

if $1 \leq p < \infty$ and

$$
d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$

if $p = \infty$. 

Let $\varepsilon$ be a positive number such that

$$
\alpha_i \leq \beta_i \leq \gamma_i + \varepsilon
$$

for all $i \in \mathbb{N}$. From this and (2.15) it follows that $\varepsilon_0$ is confinal in the ordinal number $\text{den}(W)$, contrary to the supposition of the theorem. Thus there is $\varepsilon_0 > 0$ such that $\text{card}(D_{\varepsilon_0}) = \text{den}(W)$. This equality and (2.14) imply (2.11) with $\varepsilon_0 = k\varepsilon_0$. □
if \( p = \infty \). It is clear that the metrics \( d_p \) are distance-increasing and partial distance-preserving for every \( p \in [1, \infty) \).

**Proposition 3.4.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces and let \( d \) be a distance-increasing, partial distance-preserving metric on the product \( X \times Y \). Then the following double inequality holds for all and \((x_i, y_i) \in X \times Y, i = 1, 2\),

\[
\begin{align*}
(3.10) & \quad d_\infty((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_2, y_2)) \leq d_1((x_1, y_1), (x_2, y_2))

de\text{where metrics } d_\infty \text{ and } d_1 \text{ are defined by (3.9) and (3.8), respectively.}
\end{align*}
\]

**Proof.** To prove the first inequality in (3.10) we may assume that

\[
(3.11) \quad d_\infty((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2).
\]

Since \( d_Y(y_1, y_2) \geq 0 = d_Y(y_1, y_1) \) and \( d \) is distance-increasing,

\[
d((x_1, y_1), (x_2, y_2)) \geq d((x_1, y_1), (x_2, y_1))
\]

This inequality, the first equality in (3.3) and (3.11) imply that

\[
d((x_1, y_1), (x_2, y_2)) \geq d_X(x_1, x_2) = d_\infty((x_1, y_1), (x_2, y_2)),
\]

i.e., the first inequality in (3.10) holds.

To prove the right hand side of (3.10) consider the following triangle inequality for the metric \( d \)

\[
(3.12) \quad d((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_1, y_2)) + d((x_1, y_2), (x_2, y_2)).
\]

From this and (3.3) we obtain

\[
d((x_1, y_2), (x_2, y_2)) \leq d_X(x_1, x_2) + d_Y(y_1, y_2) = d_1((x_1, y_1), (x_2, y_2)),
\]

as required. \( \square \)

Recall that there is a natural topology on the product space, it is the coarsest topology for which the canonical projections to the factors are continuous.

**Corollary 3.5.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. All distance-increasing, partial distance-preserving metrics on the product \( X \times Y \) induce the natural topology on this product.

**Proof.** Let \( d \) be a partial distance-preserving, distance-increasing metric on \( X \times Y \). Inequality (3.10) implies that \( d_\infty \leq d \leq 2d_\infty \). Hence the spaces \((X \times Y, d_\infty)\) and \((X \times Y, d)\) have the coinciding sets of convergent sequences. Consequently these spaces have the same topology. Moreover, it is well-known that \( d_\infty \) induces the natural topology on \( X \times Y \). Therefore the topology of the space \((X \times Y, d)\) also is natural. \( \square \)

Proposition 3.4 admits a partial converse.

**Proposition 3.6.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. If \( d \) is a metric on \( X \times Y \) such that double inequality (3.11) holds, then \( d \) is partial distance-preserving and

\[
(3.13) \quad d((x_1, y_1), (x_2, y_2)) \leq 2d((x_3, y_3), (x_4, y_4))
\]

whenever inequalities (3.2) hold.

**Proof.** The first part of the proposition directly follows from (3.10), because \( d_\infty \) and \( d_1 \) is partial distance-preserving. To prove the second part we may use the following elementary inequality

\[
a + b \leq 2 \max\{a, b\}
\]

which holds for all \( a, b \in \mathbb{R} \). \( \square \)
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
 & \( x_1 \) & \( x_2 \) & \( x_3 \) & \( y_1 \) & \( y_2 \) & \( y_3 \) \\
\hline
\( x_1 \) & 0 & 1 & 2 & 1 & 2 & 2 & 2 \\
\( y_1 \) & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\
\hline
\( x_2 \) & 2 & 1 & 0 & 2 & 2 & 1 & 2 & 2 \\
\( y_2 \) & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 2 \\
\hline
\( x_3 \) & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 2 \\
\( y_3 \) & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 0 \\
\hline
\end{tabular}

Figure 1. The distance-matrix of a space \((X \times Y, d)\) for \(X = \{x_1, x_2, x_3\}\) and \(Y = \{y_1, y_2, y_3\}\).

**Example 3.7.** Let \((X, d_X)\) and \((Y, d_Y)\) be two three-point metric spaces such that

\[d_X(x_i, x_j) = d_Y(y_i, y_j) = |i - j|\]

for all \(x_i, x_j \in X\) and all \(y_i, y_j \in Y\), \(i, j \in \{1, 2, 3\}\). Consider the metric space \((X \times Y, d)\) for which the metric \(d\) is defined by the distance-matrix from Fig. 1. Then double inequality (3.10) holds for all \((x_i, y_i) \in X \times Y\), \(i = 1, 2\), and moreover we have the equalities

\[1 = d((x_1, y_1), (x_2, y_2)) = \frac{1}{2}d((x_2, y_2), (x_3, y_3)).\]

Consequently \(d\) is not distance-increasing and 2 is the best possible constant in inequality (3.13).

The product space \((X \times Y, d)\) inherits many useful properties of the factors if \(d\) is distance-increasing and partial distance-preserving. Recall that a metric space \((X, d)\) is \emph{proper} if each closed and bounded set \(A \subseteq X\) is compact.

**Proposition 3.8.** Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. If \(d\) is a metric on \(X \times Y\) such that (3.10) holds, then the following statements are true.

(i) \((X \times Y, d)\) is bounded if and only if \((X, d_X)\) and \((Y, d_Y)\) are bounded.
(ii) \((X \times Y, d)\) is complete if and only if \((X, d_X)\) and \((Y, d_Y)\) are complete.
(iii) \((X \times Y, d)\) is proper if and only if \((X, d_X)\) and \((Y, d_Y)\) are proper.

\textbf{Proof.} Propositions (i) and (ii) can be obtained by the standard arguments.

For the proof of (iii) observe that a metric space \((Z, \rho)\) is proper if and only if every closed ball

\[B_{\rho}(a, r) := \{x \in Z : \rho(x, a) \leq r\}\]
is compact. Suppose that \((X, d_X)\) and \((Y, d_Y)\) are proper. From the first inequality in (3.10) we obtain
\[ B_d((x_1, y_1), r) \subseteq B_{d_\infty}((x_1, y_1), r) = B_{d_X}(x_1, r) \times B_{d_Y}(y_1, r). \]

The last direct product is compact because the balls \(B_{d_X}(x_1, r)\) and \(B_{d_Y}(y_1, r)\) are compact. Hence \(B_d((x_1, y_1), r)\) is compact as a closed subset of a compact set.

Suppose that \((X \times Y, d)\) is proper. By Proposition 3.9, \(d\) is partial distance-preserving. Hence for every closed ball \(B_d((x_1, y_1), r)\) the sets
\[(3.14) \quad (X \times \{y_1\}) \cap B_d((x_1, y_1), r) \text{ and } (\{x_1\} \times Y) \cap B_d((x_1, y_1), r) \]
are isometric to the balls \(B_{d_X}(x_1, r)\) and \(B_{d_Y}(y_1, r)\), respectively. Since sets \(X \times \{y_1\}\) and \(\{x_1\} \times Y\) are closed, the sets in (3.14), and hence the closed balls \(B_{d_X}(x_1, r)\) and \(B_{d_Y}(y_1, r)\), are compact.

**Theorem 3.9.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \(d\) be a partial distance-preserving metric on \(X \times Y\) such that the inequality
\[(3.15) \quad d_\infty((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_2, y_2))\]
holds for all \((x_i, y_i) \in X \times Y, i = 1, 2\). Then \(d\) is an ultrametric if and only if \(d_X\) and \(d_Y\) are ultrametrics and \(d = d_\infty\).

**Proof.** Suppose that \(d_X\) and \(d_Y\) are ultrametrics. Then for all \((x_i, y_i) \in X \times Y, i = 1, 2, 3\), we obtain
\[
\max\{d_\infty((x_1, y_1), (x_2, y_2)), d_\infty((x_2, y_2), (x_3, y_3))\}
= \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\},
\]
\[
\max\{d_X(x_1, x_2), d_X(x_2, x_3)\},
\]
\[
\max\{d_Y(y_1, y_2), d_Y(y_2, y_3)\}
\]
\[
\geq \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} = d_\infty((x_1, y_1), (x_3, y_3)),
\]
i.e., \((X \times Y, d_\infty)\) is an ultrametric space if \((X, d_X)\) and \((Y, d_Y)\) are ultrametric.

Conversely, let \((X \times Y, d)\) be an ultrametric space. Since \(d\) is partial distance-preserving we have
\[
d_X(x_1, x_3) = d((x_1, y), (x_3, y))
\leq \max\{d((x_1, y), (x_2, y)), d((x_2, y), (x_3, y))\}
= \max\{d_X(x_1, x_2), d_X(x_2, x_3)\}
\]
for every \(y \in Y\) and \(x_1, x_2, x_3 \in X\). Hence \(d_X\) is an ultrametric. A similar argument yields that \(d_Y\) is an ultrametric if \(d\) is an ultrametric. To prove that \(d = d_\infty\) it is sufficient to show that the inequality
\[(3.16) \quad d((x_1, y_1), (x_2, y_2)) \leq d_\infty((x_1, y_1), (x_2, y_2))\]
holds for all \((x_1, y_1), (x_2, y_2) \in X \times Y\). Since \(d\) is a partial distance-preserving ultrametric, we have
\[
d((x_1, y_1), (x_2, y_2))
\leq \max\{d((x_1, y_1), (x_1, y_2)), d((x_1, y_2), (x_2, y_2))\}
= \max\{d_Y(y_1, y_2), d_X(x_1, x_2)\},
\]
i.e., (3.16) holds. □

**Remark 3.10.** Let \((X, d_X), (Y, d_Y)\) and \((X \times Y, d)\) be metric spaces such that \(d_\infty \leq d\). It follows from Proposition 3.6 and inequality (3.12) that a metric \(d\) is partial distance-preserving if and only if \(d \leq d_1\).
Corollary 3.11. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \(d\) be a distance-increasing and partial distance-preserving metric on the product \(X \times Y\). Then \(d\) is an ultrametric if and only if \(d_X\) and \(d_Y\) are ultrametrics and \(d = d_\infty\).

Proof. It follows from Theorem 3.9 and Proposition 3.4. \(\square\)

4. Products of packing numbers and products of ultrametric spaces

In this section we give some conditions under which a product of metric spaces is ultrametric.

Theorem 4.1. Let \((X, d_X)\) and \((Y, d_Y)\) be ultrametric spaces and let \(d\) be a partial distance-preserving metric on \((X \times Y)\) such that the inequality
\[
d_\infty((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_2, y_2))
\]
holds for all \(((x_1, y_1), (x_2, y_2)) \in (X \times Y) \times (X \times Y)\). Then the following statements are equivalent.

(i) \(d\) is an ultrametric on \(X \times Y\).

(ii) The equality
\[
\mathcal{M}_\varepsilon(W \times Z) = \mathcal{M}_\varepsilon(W) \cdot \mathcal{M}_\varepsilon(Z)
\]
holds for all compact sets \(W \subseteq X\) and \(Z \subseteq Y\) and every \(\varepsilon > 0\).

Lemma 4.2. Let \((X, d_X)\), \((Y, d_Y)\) and \((X \times Y, d)\) be ultrametric spaces. Suppose that \(d\) is partial distance-preserving and \((4.1)\) holds for all \(((x_1, y_1), (x_2, y_2)) \in (X \times Y) \times (X \times Y)\). Then the equalities \((4.2)\),
\[
\mathcal{N}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W) \cdot \mathcal{N}_\varepsilon(Z),
\]
and
\[
\mathcal{M}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W \times Z)
\]
hold for all compact sets \(W \subseteq X\), \(Z \subseteq Y\) and every \(\varepsilon > 0\).

Proof. Let \(W\) and \(Z\) be compact sets \(W \subseteq X\), \(Z \subseteq Y\) and let \(\varepsilon > 0\). Theorem 3.9 implies that \(d = d_\infty\) if the conditions of the lemma hold. It follows from the definition of the covering numbers that
\[
\mathcal{N}_\varepsilon(W \times Z) \leq \mathcal{N}_\varepsilon(W) \cdot \mathcal{N}_\varepsilon(Z).
\]
Indeed, if \(C_W\) and \(C_Z\) are finite \(\varepsilon\)-nets for \(W\) and, respectively, for \(Z\), then the direct product \(C_W \times C_Z\) is a finite \(\varepsilon\)-net for \(W \times Z\) in the space \((X \times Y, d_\infty)\). Consequently, we obtain
\[
\mathcal{N}_\varepsilon(W \times Z) \leq \text{card}(C_W) \cdot \text{card}(C_Z).
\]
Using this inequality for \(C_W\) and \(C_Z\) with \(\text{card}(C_W) = \mathcal{N}_\varepsilon(W)\) and \(\text{card}(C_Z) = \mathcal{N}_\varepsilon(Z)\) we obtain \((4.5)\). Similarly, the definition of the packing numbers implies the inequality
\[
\mathcal{M}_\varepsilon(W \times Z) \geq \mathcal{M}_\varepsilon(W) \cdot \mathcal{M}_\varepsilon(Z).
\]
for the subspace \(W \times Z\) of the space \((X \times Y, d_\infty)\).

Statement (iii) of Theorem 2.9 gives
\[
\mathcal{M}_\varepsilon(W) = \mathcal{N}_\varepsilon(W) \quad \text{and} \quad \mathcal{M}_\varepsilon(Z) = \mathcal{N}_\varepsilon(Z)
\]
because \((X, d_X)\) and \((Y, d_Y)\) are ultrametric spaces. The metric \(d = d_\infty\) induces the natural topology on \(X \times Y\). Thus \(W \times Z\) is compact in \((X \times Y, d_\infty)\), so Theorem 2.9 (iii) implies also the equality
\[
\mathcal{M}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W \times Z).
\]
Consequently, from (4.5) and (4.6) we obtain

\[ \mathcal{N}_\varepsilon(W) \mathcal{N}_\varepsilon(Z) = \mathcal{M}_\varepsilon(W) \mathcal{M}_\varepsilon(Z) \leq \mathcal{M}_\varepsilon(W \times Z) \]

\[ = \mathcal{N}_\varepsilon(W \times Z) \leq \mathcal{N}_\varepsilon(W) \mathcal{N}_\varepsilon(Z). \]

Equalities (4.2)–(4.4) are proved. □

**Proof of Theorem 4.1.** It was shown in Lemma 4.2 that (i) ⇒ (ii). To prove (ii) ⇒ (i) suppose that (4.2) holds for every \( \varepsilon > 0 \) and all compacts \( W \subseteq X \), \( Z \subseteq Y \) but \( (X \times Y, d) \) is not ultrametric. Then, by Theorem 3.9, there are points \( (x_i, y_i) \in X \times Y, \ i = 1, 2 \), such that

\[ (4.7) \quad \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} < d((x_1, y_1), (x_2, y_2)). \]

Write

\[ (4.8) \quad W := \{x_1, x_2\}, \quad Z := \{y_1, y_2\} \]
and

\[ (4.9) \quad \varepsilon := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}. \]

Then we evidently have

\[ (4.10) \quad \mathcal{M}_\varepsilon(W) = \mathcal{M}_\varepsilon(Z) = 1. \]

Note also that inequality (4.7) implies that the set \( \{(x_1, y_1), (x_2, y_2)\} \) is an \( \varepsilon \)-distinguishable subset of \( W \times Z \) in the space \( (X \times Y, d) \). Hence, we have the inequality \( \mathcal{M}_\varepsilon(W \times Z) \geq 2 \). This inequality and (4.10) contradict (4.2). Hence, the implication (ii) ⇒ (i) holds. □

If \( d \) is partial distance-preserving and \( d_\infty \leq d \) but only one from the spaces \( (X, d_X) \) and \( (Y, d_Y) \) is ultrametric, then, generally, the metric space \( (X \times Y, d) \) may be nonultrametric even if (4.2) holds for all compact sets \( W \subseteq X \), \( Z \subseteq Y \) and every \( \varepsilon > 0 \).

**Example 4.3.** Let \( X = \{x\} \) be an one-point metric space. Then \( X \) is ultrametric and for every \( (Y, d_Y) \) there is a unique partial distance-preserving metric \( d = d_\infty \) on \( X \times Y \), i.e., the function

\[ (X \times Y, d) \ni (x,y) \mapsto y \in (Y, d_Y) \]

is an isometry if \( d \) is partial distance-preserving. Furthermore, it is clear that every \( W \subseteq X \) is either empty or one-point and

\[ \mathcal{M}_\varepsilon(\emptyset) = \mathcal{M}_\varepsilon(X) - 1 = 0. \]

Hence (4.2) holds for all compact sets \( W \subseteq X \), \( Z \subseteq Y \) and every \( \varepsilon > 0 \) but \( (X \times Y, d) \) is ultrametric if and only if \( (Y, d_Y) \) is ultrametric.

**Proposition 4.4.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces and let \( d \) be a partial distance-preserving metric on \( X \times Y \) such that \( d_\infty \leq d \). Then the space \( (X \times Y, d) \) is ultrametric if and only if the equalities

\[ (4.11) \quad \mathcal{N}_\varepsilon(W \times Z) = \mathcal{M}_\varepsilon(W \times Z) \]

and (4.2) hold for all compact sets \( W \subseteq X \), \( Z \subseteq Y \) and every \( \varepsilon > 0 \).

The following fact is included in the proof of Theorem 3.9.

**Lemma 4.5.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces and let \( d \) be a partial distance-preserving ultrametric on \( X \times Y \). Then \( (X, d_X) \) and \( (Y, d_Y) \) are ultrametric spaces.
Proof of Proposition 4.1. If \((X \times Y, d)\) is ultrametric, then, by Lemma 4.3, \((X, d_X)\) and \((Y, d_Y)\) are ultrametric. Consequently, (4.2) follows from Theorem 4.1. The set \(W \times Z\) is compact if \(W\) and \(Z\) are compact. Hence, (4.11) follows from Theorem 2.9 (iii).

Now suppose that (4.11) and (4.2) hold for all compact sets \(W \subseteq Z, Z \subseteq Y\) and every \(\varepsilon > 0\). To prove that \((X \times Y, d)\) is ultrametric, it is sufficient, by Theorem 4.1, to show that \((X, d_X)\) and \((Y, d_Y)\) are ultrametric spaces. Using (4.11) with an one-point set \(W\) we see that \(N_1(Z) = M_2(Z)\) for every compact set \(Z \subseteq Y\) and every \(\varepsilon > 0\), because \(d\) is partial distance-preserving. Hence, by Theorem 2.9 \(Y\) is an ultrametric space. Similarly \(X\) is an ultrametric space. \(\square\)

The following example shows that in Theorem 4.1 the packing numbers cannot be replaced by covering numbers.

| \(x_1\) | \(1\) | \(y_1\) | \(1\) | \(1\) | \(a\) |
| \(x_2\) | \(y_2\) | \(1\) | \(0\) | \(1\) | \(1\) |
| \(x_2\) | \(y_1\) | \(1\) | \(1\) | \(0\) | \(1\) |
| \(x_2\) | \(y_2\) | \(a\) | \(1\) | \(1\) | \(0\) |

Figure 2. The distance-matrix of a metric space \((X \times Y, d)\) for \(X = \{x_1, x_2\}\) and \(Y = \{y_1, y_2\}\). Here \(a\) is an arbitrary real number from \([1, 2]\).

Example 4.6. Let \(X = \{x_1, x_2\}\) and \(Y = \{y_1, y_2\}\) be two-point metric spaces with metrics \(d_X, d_Y\) such that

\[
d_X(x_1, x_2) = d_Y(y_1, y_2) = 1.
\]

Let \((X \times Y, d)\) be a product of the spaces \((X, d_X)\) and \((Y, d_Y)\) such that \(d\) is generated by the distance-matrix from Fig. 2. Then \(d\) is a partial distance-preserving and \(d_\infty \leq d\). Moreover, a computation shows that (4.3) holds for all \(W \subseteq X, Z \subseteq Y\) and every \(\varepsilon > 0\). Specifically we have

\[
N_1(X \times Y) = N_1(X) \cdot N_1(Y)
\]

because

\[
B_d((x_1, y_2), 1) \supseteq X \times Y.
\]

Note that \((X \times Y, d)\) is not an ultrametric space if \(1 < a \leq 2\).

Proposition 4.7. Let \((X, d_X)\) and \((Y, d_Y)\) be ultrametric spaces and let \(d\) be a partial distance-preserving metric on \(X \times Y\) such that \(d_\infty \leq d\). Suppose that (4.2) holds for all compact sets \(W \subseteq Z, Z \subseteq Y\) and every \(\varepsilon > 0\). Then

\[
(4.12) \quad \min\{d((x_1, y_1), (x_2, y_2)), d((x_2, y_1), (x_1, y_2))\} = d_\infty((x_1, y_1), (x_2, y_2))
\]

holds for all \(\{x_1, x_2\} \subseteq X\) and \(\{y_1, y_2\} \subseteq Y\).

Proof. Suppose that (4.12) does not hold for some \(x_1, x_2 \in X\) and \(y_1, y_2 \in Y\). Then using the inequality \(d_\infty \leq d\) we see that

\[
(4.13) \quad d((x_1, y_1), (x_2, y_2)) > \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}
\]
and
\[(4.14) \quad d((x_2, y_1), (x_1, y_2)) > \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.\]
Write
\[W := \{x_1, x_2\}, \quad \varepsilon := d_\infty((x_1, x_2), (y_1, y_2)).\]
Then it is clear that
\[(4.15) \quad \mathcal{N}_\varepsilon(W) = \mathcal{N}_\varepsilon(Z) = 1.\]
Moreover, inequalities \[(4.13)\] and \[(4.14)\] imply that
\[(W \times Z) \setminus B_d((x, y), \varepsilon) \neq \emptyset\]
for every \((x, y) \in X \times Y\). Consequently, we have \(\mathcal{N}_\varepsilon(W \times Z) > 1\). To complete the proof, it suffices to observe that the last inequality and \[(4.15)\] contradict \[(4.3)\]. \(\square\)

**Corollary 4.8.** Let \((X, d_X)\) and \((Y, d_Y)\) be ultrametric spaces and let \(d\) be a partial distance-preserving metric on \(X \times Y\) such that \(d_\infty \leq d\). Suppose that the equality
\[(4.16) \quad d((x_1, y_1), (x_2, y_2)) = d((x_2, y_1), (x_1, y_2))\]
holds for all \(\{x_1, x_2\} \subseteq X\) and all \(\{y_1, y_2\} \subseteq Y\). Then \((X \times Y, d)\) is ultrametric if and only if \[(4.3)\] holds for all compact sets \(W \subseteq X, \ Z \subseteq Y\) and every \(\varepsilon > 0\).

**Proof.** Suppose that \((X \times Y, d)\) is ultrametric. Then \[(4.3)\] holds, see Lemma 4.2. Conversely, if \[(4.3)\] holds for all compact \(W \subseteq X, \ Z \subseteq Y\) and every \(\varepsilon > 0\), then, by Proposition 4.7, we have \[(4.12)\]. Note that \[(4.12)\] and \[(4.16)\] imply the equality \(d = d_\infty\). Using Theorem 3.9, we see that \((X \times Y, d)\) is an ultrametric space. \(\square\)

**Remark 4.9.** If the distance function \(d : (X \times Y) \times (X \times Y) \to \mathbb{R}^+\) depends only on “partial” distances \(d_X\) and \(d_Y\), see diagram \[(3.7)\], then \[(4.16)\] evidently holds. Note that \[(4.16)\] holds for all points from the space \((X \times Y, d)\) in Example 3.7 but, in this case, there is no function \(F\) for which diagram \[(3.7)\] is commutative.

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