Quasi-local energy and the choice of reference

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Abstract
A quasi-local energy for Einstein’s general relativity is defined by the value of the preferred boundary term in the covariant Hamiltonian formalism. The boundary term depends upon a choice of reference and a time-like displacement vector field (which can be associated with an observer) on the boundary of the region. Here, we analyze the spherical symmetric cases. For the obvious analytic choice of reference based on the metric components, we find that this technique gives the same quasi-local energy values using several standard coordinate systems and yet can give different values in some other coordinate systems. For the homogeneous-isotropic cosmologies, the energy can be non-positive, and one case which is actually flat space has a negative energy. As an alternative, we introduce a way to determine the choice of both the reference and displacement by extremizing the energy. This procedure gives the same value for the energy in different coordinate systems for the Schwarzschild space, and a non-negative value for the cosmological models, with zero energy for the dynamic cosmology which is actually Minkowski space. The time-like displacement vector comes out to be the dual mean curvature vector of the two-boundary.

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1. Introduction
The identification of a good expression which well describes the energy for gravitating systems (and really this means for all physical systems) still remains an outstanding issue. One consequence of the equivalence principle is that there is no well-defined (i.e. covariant) local description of the energy momentum for gravitating systems (see, e.g., the discussion in section 20.4 of [1]). The modern idea is that properly energy is quasi-local (i.e. is associated with a closed two-surface, for a comprehensive review see [2]). One approach is to regard energy as the value of the generator of dynamical changes with time, the Hamiltonian. Here, we consider in particular the covariant Hamiltonian formalism [3–6]. Within that approach
a certain preferred Hamiltonian boundary term was identified [6]. The quasi-local energy
is given by the value of this boundary term with a suitable choice of time evolution vector
field on the closed two-surface. In addition to the spacetime displacement vector field and
the dynamical variables, this boundary term also depends in general on a choice of certain
reference values for the dynamical variables (which represent the ground state with vanishing
quasi-local quantities). Unlike the case for other fields, the reference values for gravity theories
based on dynamic geometry (the metric and connection) cannot be taken to have trivial values.
Essentially this is because the natural ground state for dynamic geometry, Minkowski space,
has a non-vanishing metric. Hence, the choice of reference for such theories in general, and
thus in particular for general relativity (GR), necessarily requires some suitable way to select
an appropriate Minkowski geometry at the points of the closed two-surface. Currently, this is a
quite active research topic [7, 8, 9]. Here, following [10], we consider this problem for the most
important special case: spherically symmetric solutions (more specifically, Schwarzschild and
the homogeneous isotropic cosmologies) to Einstein’s gravity theory, GR. We consider two
approaches. The first we name analytic—essentially one takes the limit of the physical
metric components in some coordinate system when the physical parameters take on trivial
values. This type of approach goes back to [4, 11], and shows that the quasi-local energy
is reference and displacement vector dependent. This naturally raises a question: Is there a
minimum (maximum) value among all these available choices? which leads to the second
approach: find the optimal reference via a variational principle extremizing the energy. Here,
we show that the analytic approach can give the same standard quasi-local energy value for
several choices of the spatial coordinate system and yet will lead to different energy values
for different time coordinates. The value of the obtained quasi-local energy is not necessarily
non-negative. On the other hand, the energy extremization always gives a coordinate-frame-
independent quasi-local energy value which is, moreover, non-negative and vanishing only for
Minkowski space.

The outline of this work is as follows. In section 2, we briefly introduce the covariant
Hamiltonian approach, which leads to the Hamiltonian boundary term that gives the quasi-
local energy. Section 3 concerns the analytic approach to choosing the reference: section
3.1 includes the analysis of the Schwarzschild geometry in three different spatial coordinate
systems and the Eddington–Finkelstein (EF) and Painlevé–Gullstrand (PG) time slicings; the
homogeneous-isotropic cosmology metrics are considered in section 3.2. In section 4, we use
the method of extremizing the quasi-local energy to determine the choice of reference and also
the displacement vector. Section 5 is the conclusion.

2. The covariant Hamiltonian approach

Our approach to quasi-local energy is via the covariant Hamiltonian formalism, which has
been described in detail in a series of works [3–6]. The construction of the energy expression
is briefly outlined here. It begins from a first-order Lagrangian 4-form for some k-form field:

\[ L = d\varphi \wedge p - \Lambda(\varphi, p). \]  

The variation has the form

\[ \delta L = d(\delta \varphi \wedge p) + \delta \varphi \wedge \frac{\delta L}{\delta \varphi} + \frac{\delta L}{\delta p} \wedge \delta p. \]  

Hamilton’s principle applied to the action obtained by integrating the first-order Lagrangian
over a region leads to the pair of first-order field equations:

\[ 0 = \frac{\delta L}{\delta \varphi} := -(-1)^k dp - \partial_\psi \Lambda, \quad 0 = \frac{\delta L}{\delta p} := d\varphi - \partial_\psi \Lambda. \]
From infinitesimal diffeomorphism invariance (generated by a vector field $N$) for $\mathcal{L}$ it follows that (2) must become an identity under the replacement $\delta \to \xi_N$ (where $\xi_N = d\Gamma_N + i_N d$ on the components of form fields):

$$d\xi_N \mathcal{L} \equiv \xi_N \mathcal{L} \equiv d(\xi_N \varphi \wedge p) + \xi_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \xi_N p. \quad (4)$$

From this identity it follows that the 3-form

$$\mathcal{H}(N) := \xi_N \varphi \wedge p - i_N \mathcal{L} \quad (5)$$

satisfies the (a particular case of Noether’s first theorem) identity

$$-d\mathcal{H}(N) \equiv \xi_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \xi_N p. \quad (6)$$

Hence (when the field equations are satisfied), the 3-form $\mathcal{H}(N)$ is a conserved current. From an expansion of its definition (5) it can be seen to have the form $\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + dB(N)$. Inserting this expansion into the identity (6) gives an identity with terms proportional to $dN^\mu$.

The boundary term $B(N)$ arising from an expansion of its definition (5) it can be seen to have the form $B(N) = \int_{\partial \Sigma} \mathcal{B}(N)$ for a time-like vector field $\mathcal{B}$ yielding the Hamilton equations $\xi_N \varphi = \delta \mathcal{H}(N)/\delta p$, $\xi_N p = -\delta \mathcal{H}(N)/\delta \varphi$. The key point is that requiring functional differentiability of the Hamiltonian (i.e. the vanishing of the boundary term in the variation of the Hamiltonian) determines the boundary conditions built into the Hamiltonian. Hence, one should thus choose the particular form of the Hamiltonian boundary term $\mathcal{B}$ that gives the desired type of boundary condition for the dynamical variables (e.g. Dirichlet or Neumann) which is suitable for the physical problem.

The boundary term $\mathcal{C}(N)$ in the variation of the Hamiltonian (8) will not have vanishing limiting value at infinity with the usual field falloffs [12, 13]—unless one adjusts by hand the total differential (i.e. the boundary term) in the Hamiltonian.

Investigations led to several explicit alternative boundary term expressions. In general, these expressions require, along with the dynamical variables on the boundary, certain non-dynamical reference values that represent the ground state of the physical system. Whereas,
it is generally possible, and indeed appropriate, to choose trivial (i.e. vanishing) magnitudes for these reference values for all the other physical fields that cannot be done for dynamic geometry gravity theories, simply because the ground state of the metric is not a vanishing value but rather the non-vanishing Minkowski metric. Moreover, in a general coordinate system, the Minkowski connection also has non-vanishing components.

Among the possible boundary terms corresponding to various boundary conditions, a certain preferred boundary term for the covariant Hamiltonian for Einstein’s GR was identified which should be suitable for most applications. It corresponds to holding the metric fixed on the boundary. This choice has the virtue of directly giving not only the ADM quantities at spatial infinity but also the Bondi energy and flux at null infinity, and moreover under certain conditions it will give a positive quasi-local energy. Our preferred boundary term for GR is given by

\[ B(N) = \frac{1}{2\kappa} (\Delta \Gamma^\mu_\nu \wedge \iota_N \eta^\nu_\mu + \tilde{D}^\mu_\nu \tilde{N}^\nu \Delta \eta_{\mu\nu}), \] (9)

where \( \tilde{D}_\nu \tilde{N}_\mu = \bar{g}^{\nu\lambda} \tilde{D}_\nu \tilde{N}_\mu = \bar{g}^{\nu\lambda} (\partial_\nu \tilde{N}_\mu + \Gamma^\gamma_\mu_\lambda \tilde{N}^\gamma), \) \( \Delta \Gamma^\mu_\nu = \Gamma^\mu_\nu - \bar{\Gamma}^\mu_\nu \) is the difference of the connection 1-form between the dynamic space and the reference space, with a bar referring to the reference objects, \( \eta_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \partial^\alpha \wedge \partial^\beta \), and \( \partial^\nu \) is the orthonormal coframe.

For a given dynamical region and given dynamical fields, the value of this boundary expression with suitably selected vector fields can be used to determine values for the quasi-local energy momentum. However, to find the specific values for these quasi-local quantities, one still needs to explicitly select the reference configuration and the appropriate displacement vector field. A natural choice for the reference is one with a Minkowski metric. But one must determine exactly which Minkowski space should be used. In the following sections we explore two approaches to achieving this.

3. Choice of reference: the analytic approach

The boundary expression (9) includes the reference values for the dynamical fields, but gives no restriction as to what the reference should be. In general, the reference space could be any four-dimensional manifold endowed with a Lorentz signature metric tensor and a connection. Here, we take the reference space to have a Minkowski metric. The quasi-local energy (7) is the value determined by the boundary integral with a certain time displacement. The boundary term (9) includes the reference variables for (certain projected components of) the four-dimensional metric and connection on the two-boundary. A reasonable requirement for choosing the reference is isometric embedding of the two-boundary into the chosen reference space. Without additional conditions, however, the embedding is not unique.

Here, we consider just spherical symmetric spacetimes. This is an important yet relatively simple special case for finding an isometric embedding easily, and also it simplifies the Hamiltonian boundary expression (9), so that only the first term \( \Delta \Gamma^\mu_\nu \wedge \iota_N \eta^\nu_\mu \) contributes.

3.1. Schwarzschild geometry

For the static, spherically symmetric Schwarzschild metric, we will consider five different representations related to different time and spatial coordinates: the standard \( \{t, r\} \) where \( r \) is the area coordinate, the isotropic spherical \( \{t, R\} \) and isotropic Cartesian \( \{t, x, y, z\} \) coordinates, and the EF \( \{\tilde{t}, r\} \) and PG \( \{\tau, r\} \) coordinates. In each case, the choice of reference is obtained analytically by taking \( m = 0 \) in the metric and connection coefficients of the dynamic spacetime in the particular coordinates chosen. In general, it can be expected that the resulting quasi-local energy value will depend on the choice of coordinates.
Let us first illustrate the procedure in detail using the standard area coordinate.

### 3.1.1. Standard Schwarzschild

The standard form of the Schwarzschild metric is

\[
\begin{align*}
\text{d}s^2 &= -\left(1 - \frac{2m}{r}\right) \text{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \text{d}r^2 + r^2 \text{d}\theta^2 + r^2 \sin^2 \theta \text{d}\phi^2.
\end{align*}
\]

(10)

The radial coordinate is determined geometrically in terms of the area of a 2-sphere:

\[A = 4\pi r^2.\]

Take the orthonormal coframe to be

\[
\begin{align*}
\vartheta^0 &= \sqrt{1 - \frac{2m}{r}} \text{d}t, \\
\vartheta^1 &= \frac{1}{\sqrt{1 - \frac{2m}{r}}} \text{d}r, \\
\vartheta^2 &= r \text{d}\theta, \\
\vartheta^3 &= r \sin \theta \text{d}\phi.
\end{align*}
\]

(11)

The Lévi-Civitá connection 1-form coefficients are obtained using the torsion-free condition,

\[
\text{d}\vartheta^\mu + \Gamma^1_{\mu\nu} \wedge \vartheta^\nu = 0.
\]

Due to the metric compatibility condition, the orthonormal frame connection coefficients are anti-symmetric. The independent connection coefficients are

\[
\begin{align*}
\Gamma^1_2 &= -\sqrt{1 - \frac{2m}{r}} \text{d}\theta, \\
\Gamma^1_3 &= -\frac{r}{\sqrt{1 - \frac{2m}{r}}} \sin \theta \text{d}\phi, \\
\Gamma^2_3 &= -\cos \theta \text{d}\phi, \\
\Gamma^0_1 &= \frac{m}{r^2} \text{d}t, \\
\Gamma^0_2 &= 0, \\
\Gamma^0_3 &= 0.
\end{align*}
\]

(12)

Take \(m = 0\) in (11) and (12) to obtain the reference geometry components. Then, the non-vanishing differences of the connection components in (9) become

\[
\begin{align*}
\Delta \Gamma^0_1 &= \frac{m}{r^2} \text{d}t, \\
\Delta \Gamma^1_2 &= \left(1 - \sqrt{1 - \frac{2m}{r}}\right) \text{d}\theta, \\
\Delta \Gamma^1_3 &= \left(1 - \sqrt{1 - \frac{2m}{r}}\right) \sin \theta \text{d}\phi.
\end{align*}
\]

(13)

Note that the term \(\Delta \eta_{\mu\nu}\) vanishes in the boundary integral for this reference choice, so the boundary expression (9) reduces to just the first term.

Another important role in the boundary expression is played by the displacement vector. We assume that it is a future time-like vector field. Suppose \(N\) is normal to the two-surface (which we choose here to be the constant \(t, r\) sphere, with \(e_2, e_3\) being the two-surface tangent vectors); then, the factor \(i_{N} \eta_{\mu\nu}\) is obtained using \(N = N^0 e_0 + N^1 e_1\):

\[
\begin{align*}
i_{N} \eta^0_1 &= 0, \\
i_{N} \eta^0_2 &= -r \sin \theta \, N^1 \text{d}\phi, \\
i_{N} \eta^0_3 &= r N^1 \text{d}\theta, \\
i_{N} \eta^1_2 &= r \sin \theta N^0 \text{d}\phi, \\
i_{N} \eta^1_3 &= -r N^0 \text{d}\theta, \\
i_{N} \eta^2_3 &= N^0 \sqrt{1 - \frac{2m}{r}} \text{d}r - \sqrt{1 - \frac{2m}{r}} N^1 \text{d}t.
\end{align*}
\]

Only the purely angular components of the quasi-local boundary term will contribute to the integral over the 2-sphere \(S\):

\[
\begin{align*}
2\kappa B(N) &= \Delta \Gamma^\mu_\nu \wedge i_{N} \eta^\nu_\mu \\
&= 2[\Delta \Gamma^1_2 \wedge i_{N} \eta^1_2 + \Delta \Gamma^1_3 \wedge i_{N} \eta^1_3] \\
&= 4(1 - \sqrt{1 - \frac{2m}{r}}) r N^0 \sin \theta \text{d}\phi \wedge \text{d}\phi.
\end{align*}
\]

(14)

The quasi-local energy obtained from the integral over the 2-sphere boundary at constant \(t, r\) then comes out to be

\[
E_S(N) = r (1 - \sqrt{1 - \frac{2m}{r}}) N^0.
\]

(15)
One possible choice of the displacement is the time-like Killing vector of the reference \( \mathbf{N} = \partial_t = \sqrt{1 - 2m/r} \mathbf{e}_0 \), yielding

\[
E_S(\partial_t) = r \sqrt{1 - 2m/r} (1 - \sqrt{1 - 2m/r}),
\]

(16)

with the horizon and asymptotic limits

\[
E_S(\partial_t)_{r \to \infty} \approx m \left( 1 - \frac{m}{2r} \right) \to m, \quad E_S(\partial_t)_{r = 2m} = 0.
\]

(17)

On the other hand, for the choice of \( \mathbf{N} = \mathbf{e}_0 \) (this is the unit dual mean curvature vector of the two-surface), the quasi-local energy value comes out to be

\[
E_S(\mathbf{e}_0) = r (1 - \sqrt{1 - 2m/r}),
\]

(18)

with the horizon and asymptotic limits

\[
E_S(\mathbf{e}_0)_{r \to \infty} \approx m \left( 1 - \frac{m}{2r} \right) \to m, \quad E_S(\mathbf{e}_0)_{r = 2m} = 2m.
\]

(19)

The value (18) is the famous result first found by Brown and York [14]. It turns out that several approaches yield this same value, so we will refer to it as the standard value. From an examination of these two results we have noticed a curious fact:

\[
E_S \left( \frac{1}{2} (\partial_t + \mathbf{e}_0) \right) = m.
\]

(20)

We do not know whether this has any significance.

The calculations for several other metric expressions of interest to us here follow a similar common procedure, to avoid unnecessary repetition we have done the general calculation in the appendix; we briefly report in the following the specific results.

3.1.2. Isotropic Schwarzschild. The spherical isotropic Schwarzschild metric can be obtained from (10) using the coordinate transformation \( r = R \left( 1 + \frac{m}{2R} \right)^2 \):

\[
ds^2 = - \left( \frac{1 - \frac{m}{2R}}{1 + \frac{m}{2R}} \right)^2 \ dt^2 + \left( 1 + \frac{m}{2R} \right)^2 \ (dR^2 + R^2 \ d\theta^2 + R^2 \sin^2 \theta \ d\phi^2).
\]

(21)

We chose the coframe

\[
\vartheta^0 = \frac{1 - m/2R}{1 + m/2R} \ dt, \quad \vartheta^1 = (1 + m/2R)^2 \ dR,
\]

\[
\vartheta^2 = (1 + m/2R)^2 R \ d\theta, \quad \vartheta^3 = (1 + m/2R)^2 R \sin \theta \ d\phi.
\]

(22)

Then, we worked out the corresponding Lévi-Civita connection \( \Gamma^\mu_{\nu\rho} \) and took \( m = 0 \) to get the reference values (\( \Gamma^0_{\nu\rho} \) and \( \bar{\vartheta}^\mu \)). The displacement vector in the normal space of the two-surface has the general form \( \mathbf{N} = N^0 \mathbf{e}_0 + N^1 \mathbf{e}_1 \). Only the angular components of the quasi-local boundary term contribute.

This procedure leads to the quasi-local energy for the spherical isotropic metric. The value can be calculated from expression (A.20) in the appendix, which gives

\[
E_{SI}(\mathbf{N}) = N^0 E_{SI}(\mathbf{e}_0) = m (1 + m/2R) N^0.
\]

(23)

If we choose \( \mathbf{N} = \partial_t \) (the time-like Killing field of the reference), which means \( N^0 = \frac{1 - m/2R}{1 + m/2R} \), then

\[
E_{SI}(\partial_t) = m (1 - m/2R), \quad E_{SI}(\partial_t)_{R \to m/2} = 0, \quad E_{SI}(\partial_t)_{R \to \infty} = m.
\]

(24)

Another choice for the evolution vector is \( \mathbf{N} = \mathbf{e}_0 \), which gives

\[
E_{SI}(\mathbf{e}_0) = m (1 + m/2R),
\]

(25)
with the limits

\[ E_{SI}(e_0)_{R=0} = 2m, \quad E_{SI}(e_0)_{R \to \infty} = m. \]  

The value (25) is just the standard result (18), after taking into account the transformation 
\[ r = R(1 + m/2R)^2. \]

Furthermore, we considered another metric form which is also isotropic, but in the Cartesian coordinate system \( x^\mu = \{t, x, y, z\} \). This is an important check on our techniques, since for this representation the reference connection vanishes when we take \( m = 0 \) in the dynamic connection. With 
\[ x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta, \quad R^2 = x^2 + y^2 + z^2, \]

The metric then has the form

\[ ds^2 = -N^2 dt^2 + \Phi^2 (dx^2 + dy^2 + dz^2), \]

where \( N = \frac{1-m/2R}{1+R/2m}, \quad \Phi = (1+m/2R)^2 \). We choose the obvious coframe:
\[ \vartheta^0 = N dt, \quad \vartheta^i = \Phi dx^i, \quad x^i = \{x, y, z\}. \] (28)

Suppose that \( N^0 \) depends on \( (t, R) \) only. Then, from the calculation given in more detail in the appendix, we find for the Cartesian coordinate isotropic Schwarzschild metric exactly the same quasi-local energy result (25) as was found using spherical coordinates for the isotropic Schwarzschild metric.

Here, we have calculated the quasi-local energy for the Schwarzschild metric using analytic matching in three different coordinate representations and obtained in each case the standard result. This may give one some confidence in these techniques as well as in the standard value. On the other hand, as we see in the following, there are other coordinate systems in which this analytic technique for determining the reference will lead to other values.

3.1.3. Eddington–Finkelstein. The EF form of the Schwarzschild metric,

\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 - 2\zeta \frac{2m}{r} dr \, dt + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\Omega^2, \]

(29)

(where \( \zeta = -1 \) is for incoming and \( \zeta = +1 \) is for outgoing and \( (t, r, \theta, \phi) \) is the standard coordinate system of the Schwarzschild metric) follows from the time coordinate transformation \( \tilde{t} = t - \zeta 2m \ln \left(\frac{r}{\Delta m} - 1\right) \), which makes the outgoing(incoming) radial null geodesics into straight lines of slope \( \pm 1 \) in the \( \tilde{t} - r \) plane. A principal virtue of this form of the metric is that it is regular at the horizon, \( r = 2m \). Rewriting this metric in the ADM form,

\[ ds^2 = -\left(1 + \frac{2m}{r}\right)^{-1} dt^2 + \left(1 + \frac{2m}{r}\right) \left(dr - \zeta 2m/r \frac{dr}{1+2m/r} dt\right)^2 + r^2 d\Omega^2, \]

(30)

leads to the coframe
\[ \vartheta^0 = \frac{1}{\sqrt{1+2m/r}} \, dt, \quad \vartheta^1 = \sqrt{1+2m/r} \left(dr - \zeta 2m/r \frac{dr}{1+2m/r} dt\right), \]
\[ \vartheta^2 = r \, d\theta, \quad \vartheta^3 = r \sin \theta \, d\phi. \]

(31)

The quasi-local energy is obtained by straightforwardly calculating the corresponding Lévi-Civitá connection and taking \( m = 0 \) as the reference. For \( N = N^0 e_0 + N^1 e_1 \) it works out to be

\[ E_{EF}(N) = r \left(1 - \frac{1}{\sqrt{1+2m/r}}\right) N^0 - \zeta \frac{2m}{\sqrt{1+2m/r}} N^1, \]

(32)
where $e_0 = \sqrt{1 + 2m/r} \partial_t + \zeta \sqrt{1 + 2m/r} \partial_r$, and $e_1 = \frac{1}{\sqrt{1 + 2m/r}} \partial_r$. For the choice of the reference time-like Killing field, $N = \partial_t$, this expression yields the value

$$E_{\text{EF}}(\partial_t) = 2m - r \left(1 - \frac{1}{1 + 2m/r} \right),$$

(33)

with the asymptotic and horizon limits

$$E_{\text{EF}}(\partial_t)_{r \to \infty} = m, \quad E_{\text{EF}}(\partial_t)_{r = 2m} = \sqrt{2} m.$$  

(34)

For the alternative choice of $N = e_0$, the quasi-local energy obtained from (32) is

$$E_{\text{EF}}(e_0) = r \left(1 - \frac{1}{1 + 2m/r} \right) = \frac{2m}{\sqrt{1 + 2m/r} (1 + \sqrt{1 + 2m/r})},$$

(35)

with the asymptotic and horizon limits

$$E_{\text{EF}}(e_0)_{r \to \infty} = m, \quad E_{\text{EF}}(e_0)_{r = 2m} = 2 - 2\sqrt{2}.$$  

(36)

From these two time choices we find the curious fact again:

$$E_{\text{EF}}\left(\frac{1}{2} (\partial_t + e_0)\right) \equiv m.$$  

(37)

Another choice is the unit dual mean curvature vector (outside the horizon)

$$\hat{N}^\perp = \frac{1}{\sqrt{1 - 4m^2/r^2}} \left( e_0 - \zeta e_1 \right);$$

the associated quasi-local energy is

$$E_{\text{EF}}(\hat{N}^\perp) = r \left( \frac{1}{\sqrt{1 - 4m^2/r^2}} - \sqrt{1 - \frac{2m}{r}} \right),$$

(38)

$$E_{\text{EF}}(\hat{N}^\perp)_{r \to \infty} = m, \quad E_{\text{EF}}(\hat{N}^\perp)_{r = 2m} \to \infty.$$  

(39)

3.1.4. Painlevé–Gullstrand. Another form of the Schwarzschild metric which is regular at the horizon is the PG form:

$$ds^2 = -(1 - 2m/r) dr^2 - 2\zeta \sqrt{2m/r} \, d\tau \, dr + dr^2 + r^2 \, d\Omega^2$$

$$= -dt^2 + (dr - \zeta \sqrt{2m/r} \, d\tau)^2 + r^2 \, d\Omega^2$$

(40)

(where $\zeta = -1$ means incoming and $\zeta = +1$ means outgoing). The PG time coordinate is given by the relation $dr = dt - \zeta \sqrt{2m/r} \, d\tau$. The most noteworthy feature of this form of the Schwarzschild metric is that the geometry of the spatial $\tau = \text{constant}$ surfaces is flat. We choose the coframe

$$\theta^0 = dt, \quad \theta^1 = dr - \zeta \sqrt{2m/r} \, d\tau, \quad \theta^2 = r \, d\theta, \quad \theta^3 = r \sin \theta \, d\phi.$$  

(41)

Some of the connection 1-form components, namely $\Gamma^2_1$, $\Gamma^3_1$, $\Gamma^3_2$, have the same values as in Minkowski space (so the corresponding $\Delta \Gamma$ vanish); whereas, the others are

$$\Gamma^1_0 = -\frac{\zeta}{2} \sqrt{2m/r} \, \theta^1, \quad \Gamma^2_0 = \zeta \sqrt{2m/r} \, \theta^2, \quad \Gamma^3_0 = \zeta \sqrt{2m/r} \, \theta^3$$

(42)

(with the corresponding $\Delta \Gamma$ having the same values).

Using the same procedure as above, the quasi-local energy now works out to be

$$E_{\text{PG}}(N) = -\zeta \sqrt{2m r} \, N^1.$$  

(43)

For the reference time-like Killing choice, $N = \partial_t = e_0 - \zeta \sqrt{2m/r} e_1$, this expression yields the value

$$E_{\text{PG}}(\partial_t) = 2m \quad \text{everywhere.}$$  

(44)
Whereas, it is obvious that if $N = e_0$, then
\[ E_{\text{PG}}(e_0) = 0 \quad \text{everywhere.} \]  
(45)

This latter quasi-local value is consistent with the well-known fact that the ADM energy vanishes for the PG metric (since the spatial metric of the constant $\tau$ surfaces is just that of the flat Euclidean space). Once again we find the curious result:
\[ E_{\text{PG}}\left(\frac{1}{2}(\partial_\tau + e_0)\right) = m. \]  
(46)

On the other hand, for the unit dual mean curvature vector outside the horizon,
\[ \hat{N}^\perp = \frac{1}{\sqrt{1 - 2m/r}}(e_0 - \zeta \sqrt{2m/r} e_1), \]  
(47)

the PG quasi-local energy has the value
\[ E_{\text{PG}}(\hat{N}^\perp) = \frac{2m}{\sqrt{1 - 2m/r}}, \]  
(48)

with the asymptotic and horizon limits
\[ E_{\text{PG}}(\hat{N}^\perp)_{r \to \infty} = 2m, \quad E_{\text{PG}}(\hat{N}^\perp)_{r = 2m} \to \infty. \]  
(49)

Note that this value does not approach the ADM energy at spatial infinity. It is well known that that desirable property can only be expected to hold for metrics which fall off faster than $O(r^{-1/2})$, see, e.g., [15].

### 3.2. FLRW cosmology

Now let us consider dynamic spherically symmetric metrics. The homogeneous-isotropic FLRW cosmological metric has several equivalent manifestly-isotropic-about-one-point forms

\[ ds^2 = -dt^2 + a(t)^2 dl^2, \]  

where

\[ dl^2 = d\chi^2 + \Sigma^2(\chi) d\Omega^2 \]  

\[ = (1 - kr^2)^{-1} dr^2 + r^2 d\Omega^2 \]  

\[ = [1 + kR^2/4]^{-2}(dR^2 + R^2 d\Omega^2) \]  

\[ = [1 + k(x^2 + y^2 + z^2)/4]^{-2}(dx^2 + dy^2 + dz^2). \]  

The first uses the proper radial coordinate $\rho = \chi$, with $\Sigma(\chi) = \{\sinh \chi, \chi, \sin \chi\}$ respectively corresponding to the spatial curvature signature $k = \{-1, 0, +1\}$.

Here, we will take the reference metric and connection components to be obtained analytically from the respective dynamical ones by taking $\bar{a}(t) = 1$, $\bar{k} = 0$ ($\bar{\Sigma} = \chi$), and will use the general quasi-local energy expression derived in the appendix.

For the first metric form (50), for the quasi-local energy of a sphere at constant $t, \rho$ from (A.15) with $A = a(t), B = a(t)\Sigma(\chi)$, we find
\[ E_{\text{FLRW}} = -a\Sigma \Delta \Sigma', \]  
(54)

which is, respectively,
\[ E_{\text{FLRW}} = a[\sinh \chi (1 - \cosh \chi), 0, \sin \chi (1 - \cos \chi)]. \]  
(55)

For the area coordinate $\rho = r$, from the metric form (51), $A = a(1 - kr^2)^{-1/2}, B = ar$. Then, from (A.15) for the quasi-local energy of the sphere at constant $t, r$ we find
\[ E_{\text{FLRW}} = -ar\Delta \sqrt{1 - kr^2} = ar(1 - \sqrt{1 - kr^2}). \]  
(56)
For isotropic spherical coordinates take $\rho = R$, and from the metric form (52) $A = a/[1 + (k/4)R^2]$, $B = AR$. From (A.15) for the quasi-local energy of a sphere at constant $t, R$ we find

$$E_{\text{FLRW}} = \frac{akR^3}{2[1 + (k/4)R^2]^2}. \quad (57)$$

We note that the isotropic Cartesian formula (A.17) with $\Phi = a[1 + (k/4)R^2]^{-1}$ obtained from the metric form (53) gives exactly the same value.

Although the above results may appear to be different they are in fact identical, as can readily be verified using $\Sigma(\chi) = r = R/(1 + kR^2/4)$ with due consideration to the respective ranges of the radial coordinates used in these various representations of the FLRW metric. In summary, for FLRW we have the respective equivalent quasi-local energy values:

$$E_{k=-1} = a \sinh(1 - \cosh \chi) = ar[1 - \sqrt{1 + r^2}] = \frac{-aR^3}{2(1 - R^2/4)^2}, \quad (58)$$

$$E_{k=0} = 0, \quad (59)$$

$$E_{k=+1} = a \sin(1 - \cos \chi) = ar[1 - \sqrt{1 - r^2}] = \frac{aR^3}{2(1 + R^2/4)^2}. \quad (60)$$

It is noteworthy that, according to this measure, the sign of the quasi-local energy is proportional to $k$, being negative for the open universe, vanishing for the flat case and positive for the closed case—but (just as it should be) vanishing when the whole universe is considered.

These results (which were first reported in [16]) may be compared with those obtained using the same quasi-local Hamiltonian boundary expression applied to homogenous but generally non-isotropic Bianchi cosmological models, using a homogeneous choice of reference [17]. That analysis found a vanishing quasi-local value for all Bianchi class A models (which includes as special cases the isoropic FLRW $k = 0$ and $k = +1$ models) and a negative quasi-local energy for all class B models (including as a special case the isotropic FLRW $k = -1$ model).

It is also noteworthy that in the FLRW $k = -1$ model with vanishing matter, one finds $a(t) = t$. It can be directly verified that the geometry is then really flat Minkowski space; yet, our quasi-local Hamiltonian boundary term expression gives a non-vanishing energy, which, moreover is negative. The fact that a negative quasi-local energy value for certain cosmological models can be physically appropriate has been discussed in the work cited in the previous paragraph. In the present case, the negative quasi-local energy is related to the choice of dynamically expanding comoving observers and their associated choice of reference. The next section describes an alternative technique for choosing the reference that will yield a different value for the FLRW quasi-local energy in general and for this curious special case in particular.

**Remark.** The analytic choice of reference coframe and connection was obtained by taking trivial values for some specific constant parameters of the dynamic spacetime, for example for the Schwarzschild case taking $m = 0$. For this kind of reference choice the quasi-local energy may depend on the coordinate systems along the same displacement vector (e.g. $E_S(\mathbf{N}) \neq E_{\text{EF}}(\mathbf{N})$). This can happen because this kind of reference choice may lead to different reference connections. For example, let the standard Schwarzschild reference connection be denoted by $\Gamma_S$ and the EF one by $\Gamma_{\text{EF}}$. It is clear that when taking $m = 0$ for the reference connection components, both of them remain anti-symmetric. If the dynamic orthonormal coframes are related by $\theta_{\text{EF}} = \Lambda \theta_S$, then for the reference connection coefficients expressed in the dynamical frames $\Gamma_{\text{EF}} \neq -d\Lambda \Lambda^{-1} + \Lambda \Gamma_S \Lambda^{-1}$. 


4. Choice of reference: extremization of energy

An alternative strategy for obtaining the reference and displacement vector is via extremization of the quasi-local energy. We note that Wang–Yau have used this technique to select a reference for their quasi-local energy expression \[7, 8\]. This is reasonable in light of the usual desiderata that quasi-local energy should be non-negative and should vanish iff the dynamical space is actually flat Minkowski space. For if one supposes that the quasi-local energy expression for any reasonable choice of reference indeed were non-negative, and vanished only if the dynamical variables were actually those of Minkowski space, then the quasi-local energy could be expected to have a unique minimum for some reference. Extremizing the energy w.r.t. the reference can be viewed as selecting a Minkowski reference that is ‘closest’ to the dynamical space (where the energy value is used to measure how close). Here, we assume that the reference space has a Minkowski metric and a connection (which, however, need not necessarily turn out to be flat; technically we will simply keep the anti-symmetric shape of the reference connection when it is expressed in the dynamic orthonormal coframe).

Since we consider here only spherical symmetric physical spacetimes, the natural choice of the quasi-local two-surface is a constant \(t, r\) 2-sphere; the tangent space of this surface is expressed by the spherical orthonormal frame basis \(e_2, e_3\). This simplifies the boundary expression and also makes it easier to determine an isometric embedding of the two-boundary into the reference space. The extremization comes from extremizing the value of the quasi-local energy over the reference gauge choices, i.e. the extremal value of energy over the reference coordinate transformations. Through the extremizing process, the choice of reference variables and the displacement vector are tied together with the dynamic connection. The displacement vector comes out to be the dual mean curvature vector of the two-boundary. Using this approach, we are able to obtain a quasi-local energy value for the Schwarzschild metric which is independent of the choice of the \(t, r\) coordinates and which, moreover, gives zero energy for the \(a = t, k = -1\) FLRW cosmology, i.e. for the dynamic representation of Minkowski space.

Let us now introduce this process. Suppose the reference metric has the form

\[
\text{d}\tilde{s}^2 = -d\tilde{T}^2 + d\tilde{R}^2 + R^2(d\Theta^2 + \sin^2\Theta \text{ d}\Phi^2). \tag{61}
\]

To determine the quasi-local energy, we have to obtain the reference connection which is pulled back from the reference space to the dynamic space via a coordinate transformation. This means finding \(\{T, R, \Theta, \Phi\}\), which are in general functions of \(\{t, r, \Theta, \Phi\}\). Because of the special simplicity of the spherically symmetric metrics, we can assume the coordinate transformation to have the restricted form

\[
T = T(t, r), \quad R = R(t, r), \quad \Theta = \Theta, \quad \Phi = \Phi. \tag{62}
\]

Then, (61) becomes

\[
\text{d}\tilde{s}^2 = -(\dot{T}^2 - \dot{R}^2) \text{ d}t^2 + 2(R\dot{R}' - \dot{T}T') \text{ d}t \text{ d}r + (R^2 - T^2) \text{ d}r^2 + R^2 \text{ d}\Omega^2
\]

\[
= \tilde{g}_{00} \text{ d}t^2 + 2\tilde{g}_{01} \text{ d}t \text{ d}r + \tilde{g}_{11} \text{ d}r^2 + \tilde{g}_{22} \text{ d}\Theta^2 + \tilde{g}_{33} \text{ d}\Phi^2, \tag{63}
\]

where \(\dot{R} = \partial R/\partial t, T' = \partial T/\partial r, \dot{R} = \partial R/\partial t, R' = \partial R/\partial r\). We can rewrite (63) in the ADM form:

\[
\text{d}\tilde{s}^2 = \frac{\tilde{g}_{01}^2 - \tilde{g}_{00} \tilde{g}_{11}}{\tilde{g}_{11}} \text{ d}t^2 + \left(\sqrt{\tilde{g}_{11}} \text{ d}r + \frac{\tilde{g}_{01}}{\sqrt{\tilde{g}_{11}}} \text{ d}r\right)^2 + \tilde{g}_{22} \text{ d}\Theta^2 + \tilde{g}_{33} \text{ d}\Phi^2. \tag{64}
\]

Choose the coframe from (64):

\[
\tilde{\gamma}^0 = \sqrt{\frac{\tilde{g}_{01}^2 - \tilde{g}_{00} \tilde{g}_{11}}{\tilde{g}_{11}}} \text{ d}t, \quad \tilde{\gamma}^1 = \sqrt{\tilde{g}_{11}} \text{ d}r + \frac{\tilde{g}_{01}}{\sqrt{\tilde{g}_{11}}} \text{ d}r.
\]
where the explicit form of the component functions is

\[ \delta^2 = \sqrt{S_{22}} \, d\theta = R \, d\theta, \quad \delta^3 = \sqrt{S_{33}} \, d\phi = R \sin \theta \, d\phi, \]  

and the corresponding orthonormal frame is denoted by \( \hat{e}_\mu \). Next define the connection of the reference:

\[ \Gamma^0_1 = \Gamma^1_0 = \cdots \quad \text{not needed below}, \]

\[ \Gamma^0_2 = \Gamma^2_0 = \bar{P} \, d\theta, \quad \Gamma^0_3 = \Gamma^3_0 = \bar{P} \sin \theta \, d\phi, \]

\[ \Gamma^1_2 = -\Gamma^2_1 = \bar{Q} \, d\theta, \quad \Gamma^1_3 = -\Gamma^3_1 = \bar{Q} \sin \theta \, d\phi, \]

where the explicit form of the component functions is

\[ \bar{P} = -\frac{\bar{g}_{01} R' - \bar{g}_{11} \bar{R}}{\sqrt{\bar{g}_{11} \bar{g}_{00} - \bar{g}_{01} \bar{g}_{11}}} = \pm \frac{T'}{\sqrt{R^2 - T^2}}, \]  

\[ \bar{Q} = -\frac{R'}{\sqrt{\bar{g}_{11}}} = -\frac{R'}{\sqrt{R^2 - T^2}}. \]  

The dynamic connection coefficients have a similar form. The ones that we will explicitly need can likewise be parameterized by two functions:

\[ \Gamma^0_2 = \Gamma^2_0 = P \, d\theta, \quad \Gamma^0_3 = \Gamma^3_0 = P \sin \theta \, d\phi, \]

\[ \Gamma^1_2 = -\Gamma^2_1 = Q \, d\theta, \quad \Gamma^1_3 = -\Gamma^3_1 = Q \sin \theta \, d\phi. \]  

We assume that the displacement vector \( N = N^0 e_0 + N^1 e_1 \) is in the normal plane of the constant \( t, r \) surface, where \( N^0, N^1 \) are functions of \( (t, r) \) only, independent of \( T' \) and \( R' \). The second term of (9) is not vanishing in general. Considering the spherical symmetric case, the boundary integral over the constant \( (t, r) \) surface \( S \) involves the \( \Delta \eta_{01} \) term:

\[ I = \frac{1}{\kappa} \int (\bar{D}^1 \bar{N}^0 - \bar{D}^0 \bar{N}^1)(g_{22} - \bar{g}_{22}) \sin \theta \, d\theta \wedge d\phi \]

\[ = \frac{1}{2} (\bar{D}^1 \bar{N}^0 - \bar{D}^0 \bar{N}^1)(g_{22} - \bar{g}_{22})|_\pi, \]  

(70)

where \( \bar{N}^\mu \) is the component expressed in the holonomic basis of reference: \( N = N^0 e_0 + N^1 e_1 = N^0 \partial_x + N^1 \partial_y \).

The quasi-local energy works out to be

\[ E = \frac{1}{16\pi} \oint \mathcal{B} = \frac{1}{16\pi} \oint \Delta \Gamma^\nu_\beta \wedge \iota_N \eta^\nu_\beta + I \]

\[ = \frac{1}{4\pi} \oint \left( \left( Q + \frac{R'}{\sqrt{R^2 - T^2}} \right) \frac{N^0}{\sqrt{g_{22}}} - \left( P \pm \frac{T'}{\sqrt{R^2 - T^2}} \right) \frac{N^1}{\sqrt{g_{22}}} \right) \delta^2 \wedge \delta^3 + I \]

\[ = \sqrt{g_{22}} \left( Q + \frac{R'}{\sqrt{R^2 - T^2}} \right) N^0 - \sqrt{g_{22}} \left( P \mp \frac{T'}{\sqrt{R^2 - T^2}} \right) N^1 + I. \]  

(71)

In the previous section, the choices of reference variables for Schwarzschild were obtained by taking \( m = 0 \). The term \( I \) in (71) vanishes because \( (g_{22} - \bar{g}_{22})|_\pi = 0 \). In this section, we will let the functions \( T(t, r) \) and \( R(t, r) \) be undetermined, and through the extremization of energy find out what these functions should be. However, we will require \( R(t, r)|_\pi = \sqrt{g_{22}} \). This makes the two-surfaces in the dynamic spacetime and the reference space isometric. Since the Hamiltonian boundary term is a quantity dependent on the quasi-local two-surface, the isometric requirement of the two-surface is reasonable, and furthermore, it simplifies the boundary expression by making \( I = 0 \).
For any given fixed $N$, extremize the energy by requiring the vanishing of the partial derivative with respect to $T'$ (it is easy to check that taking the partial derivative of (71) w.r.t. $R'$ gives the same condition):

$$\frac{\partial E}{\partial T'} = \sqrt{g_{22}} \frac{R'(T'N^0 \pm R'N^1)}{(R^2 - T'^2)^{3/2}} = 0, \quad T' = \mp \frac{N^1}{N^0} R'.$$

(72)

Then, substitute into (71)

$$E = \sqrt{g_{22}}[(Q + N^0)N^0 - (P + N^1)N^1].$$

(73)

Suppose we choose the normalized time-like displacement $N$ which means $-(N^0)^2 + (N^1)^2 = -1$; then, (73) becomes

$$E = \sqrt{g_{22}}(1 + Q N^0 - P N^1).$$

(74)

This result implies that the quasi-local energy depends on the free choice of $N$. We further look at the extremal value w.r.t. all the displacements. Let $N^0 = \cosh \alpha$, $N^1 = \sinh \alpha$; take the extremization of the quasi-local energy value (74):

$$\frac{\partial E}{\partial \alpha} = \sqrt{g_{22}}(Q \sinh \alpha - P \cosh \alpha) = 0, \quad \Rightarrow \quad \frac{\sinh \alpha}{\cosh \alpha} = \frac{N^1}{N^0} = \frac{P}{Q},$$

(75)

then we have the relation

$$T' = \mp \frac{P}{Q} R',$$

(76)

and

$$N^0 = \frac{1}{\sqrt{1 - P^2/Q^2}}, \quad N^1 = \frac{P}{Q \sqrt{1 - P^2/Q^2}}.$$  

(77)

Consequently, the quasi-local energy (71) has the extreme value

$$E_{\text{ex}} = \sqrt{g_{22}}(1 + Q \sqrt{1 - P^2/Q^2}).$$

(78)

We can see that $P, Q$ are determined purely by the metric of the dynamic spacetime. There is no longer any information of the reference frame or the displacement vector in this energy expression. With the vectors $\{e_2, e_1\}$ tangent to the two-surface, the dual mean curvature vector of the two-surface in the dynamic space is $N^\perp = -k e_0 + p e_1$, where $k = 2Q/\sqrt{g_{22}}$ is the extrinsic curvature w.r.t. the space-like normal $e_1$ and $p = -2P/\sqrt{g_{22}}$ is the extrinsic curvature w.r.t. the time-like normal $e_0$. Then, the vector $\hat{N}^\perp := N^\perp/|N^\perp|$ has the components (77), where $|N^\perp| := \sqrt{-1}/(|N^\perp|, N^\perp)$.

Rewrite (78) by replacing $Q$ and $P$ with the extrinsic curvature $k$ and $p$:

$$E_{\text{ex}} = \frac{g_{22}}{2}(2/\sqrt{g_{22}} + (-k)\sqrt{1 - p^2/k^2}).$$

(79)

Here, we use the definition of extrinsic curvature which is $k_{ab} := -\langle \nabla_a e_1, e_b \rangle$, and $p_{ab} := -\langle \nabla_a e_0, e_b \rangle$, $a, b = 2, 3$; the trace is $k = g^{ab}k_{ab}$ and $p = g^{ab}p_{ab}$. By this convention $k$ is negative and $N^\perp$ is time-like for the dual mean curvature vector, so that $(-k)\sqrt{1 - p^2/k^2} = \sqrt{k^2 - p^2} = |N^\perp|$. Equation (69) implies that the trace of the reference extrinsic curvature $k_0$ is given by taking $Q = -1$, so that $k_0 = -2/\sqrt{g_{22}}$. Consequently,

$$E_{\text{ex}} = \frac{g_{22}}{2}(|N^\perp| - k_0),$$

(80)

which is the same as the Liu–Yau result [9]. Now let us check the following cases.
4.1. Standard Schwarzschild

The functions necessary here are found from (10) and (12) to be
\[ \sqrt{g^{22}} = r, \quad P = 0, \quad Q = -\sqrt{1 - 2m/r}. \] (81)

With these expressions, the extreme energy (78) works out to have the standard Brown–York value:
\[ E_{\text{exS}} = r(1 - \sqrt{1 - 2m/r}). \] (82)

In this case, as \( P = 0 \) we have \( T' = 0 \) and the displacement vector \( N \) is equal to \( e_0 \). The reference here could be found from \( T = T(t), R = R(t, r), \Theta = \theta, \Phi = \varphi \), with the restriction \( R(t, r)|_\sigma = r \).

4.2. Eddington–Finkelstein

The necessary functions obtained from (29) now have the values
\[ \sqrt{g^{22}} = r, \quad P = \zeta \frac{2m}{r}, \quad Q = -\frac{1}{\sqrt{1 + 2m/r}}. \] (83)

With these expressions the extreme energy (78) again comes out to be
\[ E_{\text{exEF}} = r(1 - \sqrt{1 - 2m/r}), \] (84)

which is again the standard value. The condition which restricts the choice of reference is
\[ T' = -\frac{P}{Q} R' = \zeta \frac{2m}{r} R'. \] (85)

Then, one can set any function \( R(t, r) \) with the restriction \( R|_\sigma = r \), and solve for \( T(t, r) \). The displacement vector can also be determined from (77) to be
\[ N = \left( e_0 - \zeta \frac{2m}{r} e_1 \right) / \sqrt{1 - 4m^2/r^2}, \] (86)

which is the dual mean curvature vector.

4.3. Painlevé–Gullstrand

The necessary functions found from (40) are now
\[ \sqrt{g^{22}} = r, \quad P = \zeta \sqrt{2m/r}, \quad Q = -1. \] (87)

Using these expressions the extreme energy (78) works out to be
\[ E_{\text{exPG}} = r(1 - \sqrt{1 - 2m/r}). \] (88)

Thus, once again we found it to have the standard value. The condition which restricts the choice of reference is
\[ T' = -\frac{P}{Q} R' = \zeta \frac{2m}{r} R'. \] (89)

The displacement vector is \( N = (e_0 - \zeta \sqrt{2m/r} e_1) / \sqrt{1 - 2m/r} \).

In a similar fashion, if one considers the functions \( \sqrt{g^{22}}, P \) and \( Q \) associated with the spherical isotropic Schwarzschild coframe (22), one will once again obtain from (78) the standard quasi-local energy value.
4.4. FLRW cosmology

For the dynamic FLRW cosmological models, from the extreme energy expressions (78) it is sufficient to calculate the quasi-local energy using the metric functions \( \sqrt{g} = a(t)r \), \( P = \dot{a}r \), \( Q = -\sqrt{1 - kr^2} \) obtained from (50)—the other forms of the FLRW metric would lead to the same answer. The result is

\[
E_{\text{exFLRW}} = ar\left(1 - \sqrt{1 - kr^2 - \dot{a}^2r^2}\right) = \frac{ar^3[k + \dot{a}^2]}{1 + \sqrt{1 - kr^2 - \dot{a}^2r^2}}.
\] (90)

In contrast to the analytic result (56), which can be negative, with the aid of the Friedmann cosmological equation,

\[
\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3}\rho,
\] (91)

it can be seen that this value is non-negative. There is no contradiction: the present quasi-local energy value (90) corresponds to non-expanding observers using a different reference.

Consider in particular the special test case \( k = -1, a(t) = t \). We find the energy value \( E = 0 \). We can find a simple reference choice \( R = tr \) (actually, this needs to be satisfied only on the two-boundary, not necessary in the whole space); then, from (76) we obtain \( T = t\sqrt{1 + r^2} \). It is well known that the 4-geometry

\[
-dr^2 + \frac{t^2}{1 + r^2} dr^2 + t^2 r^2 d\Omega^2
\] (92)

is actually Minkowski space, and zero energy is just the value we expect.

4.5. Discussion

In contrast to the analytic approach, here we used energy extremization to select the reference and displacement vector field. We tested the resulting quasi-local energy expression on several forms of the Schwarzschild metric, obtaining in each case the standard quasi-local energy value. We also tested the expression on the FLRW cosmological metric, obtaining a new result for the FLRW quasi-local energy. In both cases, the time displacement vector field turns out to be the dual mean curvature vector, and the quasi-local energy value has the desirable property of being non-negative, vanishing iff the dynamic geometry is flat Minkowski space.

5. Conclusion

The covariant Hamiltonian formalism has been incomplete in one aspect. The Hamiltonian boundary term, whose value determined the quasi-local quantities, in addition to depending on the dynamical fields, also necessarily depends on the reference values for these dynamical fields (which specify the ground state with vanishing quasi-local quantities) along with a spacetime displacement vector field. However, no specific proposal had been made as to how to choose these unspecified quantities. Here, for Einstein’s GR, for certain spherically metrics (the static Schwarzschild metric and the dynamical homogeneous isotropic cosmologies), following [10] we have considered two techniques for choosing the reference and vector field.

The first (which goes back to [4, 11]) depends on an analytic choice of reference fields obtained by taking trivial values for certain parameters in the metric and connection coefficients. For the usual time slicing for several spatial metrics this leads to the standard Brown–York quasi-local energy for the Schwarzschild metric, but to different energy expressions for the alternative time slicings of the Eddington–Finkelstein and Painlevé–Gulstrand metrics. For the FLRW cosmological metrics it leads to a quasi-local energy
value which is proportional to the sign of the spatial curvature (and thus a negative energy for a certain dynamical slicing of Minkowski space).

The other approach uses extremization of the quasi-local energy to select an optimal reference and time-like vector. The resulting quasi-local energy for these spherically symmetric metrics is independent of the coordinates and is non-negative (for both the Schwarzschild and FLRW metrics) and vanishes only for Minkowski space. For the Schwarzschild metric it gives the standard quasi-local value. Going beyond the present work, there have been further developments in the energy optimization approach; a brief letter describing this has already appeared [18].

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Appendix. General formulas for spherical analytical reference choice

Here, we briefly present the quasi-local energy calculations using an analytic reference choice for general spherically symmetric metrics in several different coordinates and coframes.

**Spherical frames**

Consider orthonormal co-frames using spherical coordinates of the form

\[
\theta^\tau := N \, d\tau, \quad \theta^\rho := A \, d\rho, \quad \theta^\theta := B \sin \theta \, d\phi,
\]

where \( N, A, B \) are functions only of the general time and radial coordinates \( \tau, \rho \). The (metric-compatible hence anti-symmetric) connection 1-form coefficients are readily obtained from the differentials

\[
d\theta^\tau = (N'/NA)\theta^\rho \wedge \theta^\tau, \quad d\theta^\rho = (\dot{A}/AN)\theta^\tau \wedge \theta^\rho,
\]

\[
d\theta^\theta = (\dot{B}/BN)\theta^\tau \wedge \theta^\theta + (B'/AB)\theta^\rho \wedge \theta^\theta,
\]

\[
d\theta^\phi = (\dot{B}/BN)\theta^\tau \wedge \theta^\phi + (B'/AB)\theta^\rho \wedge \theta^\phi + (1/B) \cot \theta \, \theta^\theta \wedge \theta^\phi,
\]

where the dot and prime represent, respectively, the \( \tau \) and \( \rho \) partial derivatives. The ones of particular interest to us are

\[
\Gamma^\theta_{\rho \rho} = \frac{B'}{AB} \theta^\rho = \frac{B'}{A} d\theta, \quad \Gamma^\rho_{\rho \rho} = \frac{B'}{AB} \theta^\rho = \frac{B'}{A} \sin \theta d\phi.
\]

We will also need the associated reference values

\[
\hat{\Gamma}^\theta_{\rho \rho} = (\dot{B}'/A) \, d\theta, \quad \hat{\Gamma}^\rho_{\rho \rho} = (\dot{B}'/A) \sin \theta d\phi,
\]

which we here have assumed to be given analytically (by taking limits like \( m \to 0 \)) and thereby affecting the transformations \( A \to \hat{A}, B \to \hat{B} \). In our calculation we will need

\[
\Delta \Gamma^\theta_{\rho \rho} = \Delta (B'/A) \, d\theta, \quad \Delta \Gamma^\rho_{\rho \rho} = \Delta (B'/A) \sin \theta d\phi.
\]
Cartesian frame and coordinates

Spherically symmetric metrics may be also be described by Cartesian coframes using Cartesian spatial coordinates (labeled by Latin indices with range 1–3) in the form

\[
\vartheta^\tau = N d\tau, \quad \vartheta^k = \Phi dx^k,
\]

where \(N, \Phi\) are functions of \(\tau, R\) with \(R^2 = x^k x_k\). We find

\[
d\vartheta^\tau = \left(\frac{N'}{N} \Phi^2 R\right) x_k \vartheta^k \wedge \vartheta^\tau, \quad (A.9)
\]

\[
d\vartheta^k = \left(\frac{\Phi'}{N} \Phi^2 R\right) \vartheta^\tau \wedge \vartheta^k + \left(\frac{\Phi'}{\Phi^2 R}\right) x_m \vartheta^m \wedge \vartheta^k. \quad (A.10)
\]

The connection coefficients of particular interest are found to be

\[
\Gamma^{ij} = \left(\frac{\Phi'}{\Phi^2 R}\right) (x^j \vartheta^i - x^i \vartheta^j). \quad (A.12)
\]

We here assume that the associated reference is given by the Minkowski space obtained by analytically restricting these formulas to \(N = 1, \Phi = 1\); thus, the reference connection is such that its values vanish. This is just what we expect for the Minkowski frame determined by the coordinates \(x^\mu\). Since the reference connection vanishes the Cartesian case provides a good check for our calculations.

Our choice of particular non-vanishing reference values for the various spherical representations can be understood as just what is needed, as we see, to arrange to give the same results in all of these frames.

Energy expression

We are interested in the particular quasi-local energy given by our preferred Hamiltonian boundary term 2-form expression (9) with vanishing second term:

\[
2\kappa B(N) := \Delta \Gamma^a_{\rho \beta} \wedge \eta_a^\beta. \quad (A.13)
\]

Here, we will take \(N\) to be the unit time-like displacement on the boundary, which is at constant \(\rho, \tau\). The other choices considered in the text are proportional to this choice. For spherical frames with the displacement choice \(N = e_\perp\) (i.e. one unit of proper time orthogonal to the constant ‘time’ hypersurface), our Hamiltonian boundary term quasi-local energy 2-form expression reduces to

\[
2\kappa B(e_\perp) = \Delta \Gamma^{ab} \wedge \eta_{\perp ab} = 2\Delta \Gamma^{\rho \beta} \wedge \eta_{\perp \rho \beta} + 2\Delta \Gamma^{\rho \phi} \wedge \eta_{\perp \rho \phi} = 4\Delta \Gamma^{\rho \beta} \wedge \eta_{\perp \rho \beta} = 4\Delta \Gamma^{\rho \phi} \wedge \vartheta^{\rho} \wedge \vartheta^{\phi} = -4B \Delta (B'/A) d\Omega. \quad (A.14)
\]

The associated quasi-local energy, obtained by integrating over a 2-sphere at constant \(\tau, \rho\) (with \(\kappa = 8\pi\)), has the value

\[
E_S(e_\perp) = -B \Delta (B'/A). \quad (A.15)
\]

It is notable that the result is not explicitly dependent on the lapse \(N\).
On the other hand, for the Cartesian frame we find

\[
\begin{align*}
2\kappa B(e_\perp) &= \Delta^{ij}_{\perp} \wedge \eta_{\perp ij} = 2(\Phi' / \Phi^2) R x^i \theta^j \wedge \eta_{\perp ij} \\
&= -4(\Phi' / \Phi^2) R x^k \eta_{\perp k} = -4 \Phi' R^2 d\Omega.
\end{align*}
\tag{A.16}
\]

Hence, in this case, the quasi-local energy obtained from integration over the 2-sphere at constant \(\tau, R\) is given simply by

\[
E_C(e_\perp) = -R^2 \Phi'.
\tag{A.17}
\]

**Schwarzschild applications**

In particular, we have for the area coordinate from (11) \(A = (1 - 2m/r)^{-1/2}, B = r\). Using these in the general spherical energy expression (A.15) gives

\[
E_S(e_\perp) = r[1 - (1 - 2m/r)^{1/2}],
\tag{A.18}
\]

which is the standard energy value (18). An equivalent expression,

\[
E_S(e_\perp) = \frac{2m}{1 + \sqrt{1 - 2m/r}},
\tag{A.19}
\]

more clearly reveals the horizon and asymptotic limits. For isotropic spherical coordinates, from (22), \(A = (1 + m/2R)^2, B = R(1 + m/2R)^2\). Using these in the general spherical expression (A.15) yields the quasi-local energy

\[
E_S(e_\perp) = m(1 + m/2R).
\tag{A.20}
\]

Recalling that \(r = R(1 + m/2R)^2\), we find that this is actually the same as the standard value (A.18). On the other hand, for the isotropic Cartesian frame, using from (28), \(\Phi = (1 + m/2R)^2\) in the Cartesian expression (A.17), turns out to give the same value (A.20), equivalent to the standard value (A.18). As this case has a vanishing reference connection, it provides an important confirmation, not only for the standard energy value, but also especially for our analytic technique of choosing the reference.

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