General Bayesian Theories and the Emergence of the Exclusivity Principle

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We construct a general Bayesian framework that can be used to organize beliefs and to update them when new information becomes available. The framework includes classical, quantum, and other alternative models of Bayesian reasoning that may arise in future physical theories. It is only based on the rule of conditional probabilities and the requirement that the agent’s beliefs are consistent across time. From these requirements, we show that ideal experiments within every Bayesian theory must satisfy the exclusivity principle, which is a key to explain quantum correlations.

Introduction.—Quantum theory portrays a world where the outcomes of individual measurements cannot be predicted with certainty. And yet, the quantum predictions are strikingly accurate and successfully explain an astonishingly broad range of phenomena. The reasons for this broad applicability still remain controversial. Does the quantum framework capture a bundle of primitive facts about the world? Or is it just a general-purpose tool for guessing the outcomes of experiments?

Albeit with a variety of nuances, different interpretations of quantum theory tend to favor either one or the other view. For example, Everett’s interpretation [1] holds that the quantum framework refers to a multitude of universes. On the other hand, Bohr’s interpretation [2] holds that the quantum framework concerns what experimenters can say about nature rather than nature in itself. In a similar way, QBism [3] views the quantum framework as a set of rules that constrain how agents make bets about the outcomes of their experiments.

Different positions about the interpretation are reflected into different attitudes towards the countervalue correlations arising in quantum theory. Since Bell [4], we have known that quantum correlations are incompatible with the intuitive worldview known as local realism. But intriguingly, the quantum violations of Bell’s inequalities are not maximal: more general theories compatible with relativistic causality could in principle lead to larger violations [5–7]. Following up on this observation, various physical principles have been proposed to explain the quantum bounds on correlations [8–12]. Behind this approach lies the idea that the quantum bounds are a fact of the world, and, as such, they should be explained in terms of laws the world is subjected to.

However, this is not the only option. Instead of searching for principles constraining how nature behaves, one could search for consistency conditions that an agent should satisfy when assigning probabilities to the outcomes of experiments.

In this paper we examine the consequences of adopting a Bayesian point of view [3], referring to an agent who holds prior beliefs about the object of its bets, and who updates those beliefs when a new piece of information becomes available. The beliefs held by the agent are used to make predictions about the outcomes of future experiments. The key assumptions in this framework are that beliefs undergo updates according to the rule of conditional probabilities, and that beliefs at different times are consistent with each other.

Strikingly, we find that such minimalistic assumptions guarantee the existence of a class of ideal experiments that obey the exclusivity principle [13–17], which is a key to unlock many of the puzzling bounds on quantum correlations. Under mild assumptions, the exclusivity principle implies the Tsirelson bound [18] and, more generally, the quantum bounds for any Bell or Kochen-Specker scenario [19]. Therefore, our result suggests that all such bounds could be explained as a consequence of basic Bayesian reasoning. Whether or not all of quantum theory can be derived from Bayesian consistency conditions remains an open problem, but our result provides evidence that an important part of quantum theory can be reconstructed from elementary rules on how Bayesian agents should place their bets.

Beliefs and probabilities.—Consider the situation of an agent who makes bets about a given physical system. The content of the bets is specified by a sample space X.
equipped with a \( \sigma \)-algebra of events \( \Sigma \), namely a collection of subsets of \( X \) satisfying the properties (i) \( X \in \Sigma \), (ii) \( E \in \Sigma \) implies \( (X \setminus E) \in \Sigma \), and (iii) \( \{E_i\} \subset \Sigma \) implies \( \bigcup E_i \in \Sigma \). In typical cases, \( X \) is a finite set and \( \Sigma \) is the power set of \( X \). The simplest bet is to bet on an event \( E \). Operationally, this means that there exists an experiment with two outcomes, corresponding to \( E \) and its complement \( E = X \setminus E \), and that the agent bets that the experiment will return the outcome \( E \).

More generally, a bet can involve an experiment with multiple outcomes, corresponding to a partition \( \mathcal{E} = \langle E_i \rangle \) of the sample space \( X \) into disjoint events. For brevity, we will identify the experiment and the corresponding partition. These basic experiments will be called principal experiments, in order to distinguish them from more elaborate experiments where the agent performs sequences of operations.

In making a bet, the agent will rely on its beliefs, including beliefs on the laws of physics, or beliefs on the prior history of the physical system involved in the bet. We denote by \( \mathcal{B} \) the set of all possible beliefs. For a given belief \( \beta \) and for a given experiment \( \mathcal{E} = \langle E_i \rangle \), the agent assigns a probability distribution reflecting its expectation that the event \( E_i \) will occur in the experiment \( \mathcal{E} \). In the following, we will make the standard assumption that the probability assignment depends only on the event \( E_i \), and not on the specific experiment \( \mathcal{E} \). In other words, we will assume that every belief \( \beta \) induces a probability distribution \( p : E \mapsto p(E|\beta) \) satisfying the usual conditions (i) \( p(E|\beta) \geq 0 \) for all events \( E \), (ii) \( p(\bigcup E_i|\beta) = \sum p(E_i|\beta) \) whenever all \( E_i \) are mutually disjoint, and (iii) \( p(X|\beta) = 1 \). These conditions are the standard Kolmogorov axioms of probability and are common to most Bayesian approaches to probability [20, 21]. In a Dutch book approach, they amount to the idea that a bookmaker assigns odds to individual events.

It is important to stress that the belief \( \beta \) determines the probability assignment \( p(E|\beta) \), but not vice versa: in general, a belief contains much more than just the outcome probabilities of principal experiments. For example, in quantum theory a belief is described by a density matrix, while a principal experiment corresponds to a projective measurement, with the projectors associated to a given basis. The probability assignment depends only on the diagonal entries of the density matrix in the given basis, and therefore it is not sufficient to identify the density matrix.

Bayesian updates.—The central question in the Bayesian approach is how the agent should update its belief when new information becomes available. Suppose that the agent receives a guarantee that \( E \) is the case. The question is how the agent should update its beliefs in order to incorporate this new piece of information. Intuitively, the new belief \( \beta' \) should be the old belief \( \beta \), adjusted to take into account that \( E \) is the case. This yields an update map, which we assume to depend only on the event \( E \) and on the initial belief \( \beta \). We will denote the new belief as \( \beta' = E\beta \), indicating that the information about the occurrence of \( E \) has been incorporated into \( \beta \).

In the following, whenever the notation \( E\beta \) is used it is assumed that \( p(E|\beta) \) is non-zero. Otherwise, the belief \( \beta \) would be incompatible with event \( E \), meaning that the agent would have no way to combine the prior belief \( \beta \) with the occurrence of \( E \).

The point of updating a belief is to compute conditional probability distributions. We demand that the probability assignment for the updated belief \( E\beta \) is given by the rule of conditional probabilities:

**Axiom 1** (Rule of conditional probabilities). If \( p(E|\beta) \neq 0 \), then the updated belief \( E\beta \) satisfies the rule of conditional probabilities

\[
p(F|E\beta) = \frac{p(F \cap E|\beta)}{p(E|\beta)}.
\]

The rule of conditional probabilities implies many properties of the update map \( \beta \mapsto E\beta \). For example, it implies that, once the agent updates its belief based on the event \( E \), the agent becomes certain of the event \( E \). Indeed, one has \( p(E|E\beta) = 1 \), which follows from letting \( E = F \) in Eq. (1).

We stress that the update map does not represent a physical process on the observed system, but rather an operation internal to the agent. In the following, we formulate two conditions that the update should satisfy in order for the agent to be consistent with its beliefs at all moments of time.

**Forward consistency.—**Suppose that the agent believes that the occurrence of the event \( E \) is certain, \( p(E|\beta) = 1 \). In this case, the occurrence of \( E \) does not add any new information. As a consequence, the update should be trivial, namely \( E\beta = \beta \).

**Axiom 2** (Forward consistency). If the agent is certain that the event \( E \) will occur, then the occurrence of \( E \) does not change the agent’s belief. Mathematically: for every \( \beta \in \mathcal{B} \) and every \( E \in \Sigma \), \( p(E|\beta) = 1 \) implies \( E\beta = \beta \).

Since forward consistency is an axiom about the belief of the agent, it is not a law of physics, but just a consistency requirement. It constrains the agent in how it updates the belief forward in time, coherently with the belief held at the present moment. A simple consequence of forward consistency is that the total event \( E = X \) does not lead to any update, namely \( X\beta = \beta \), \( \forall \beta \in \mathcal{B} \).

**Actions.** So far, we considered the situation where an agent bets directly on the occurrence of a certain event. More generally, the conditions under which a bet is made can be altered by some action before an experiment is performed. We use the notion “action” broadly, including situations in which the system evolves under its natural dynamics. In this sense, actions need not be intentional: what matters is only that the agent associates a change of its belief to what occurred to the physical system under consideration.

We denote the set of all actions as \( \text{Act} \). It is natural to assume that actions can be arbitrarily composed one
We show that quantum theory satisfies EF. The composition is associative, meaning that only the temporal sequence of actions matters, and not how the actions are grouped together. Among the possible actions, we include the trivial action $I$, which preserves the current state of affairs. Mathematically, this makes $\text{Act}$ a monoid [22] (see also [23, 24]).

When an action $A$ is performed, the agent will generally change its belief. We will denote the new belief as $\beta' = A\beta$, meaning that the initial state of affairs described by $\beta$ has undergone the change induced by the action $A$. An action $A$ is reversible if there exists another action $A'$ that acts as its inverse, namely $A'A\beta = AA'\beta = \beta$, $\forall \beta \in B$.

**Backward consistency.**—When actions are included in the picture, additional consistency conditions arise. Suppose that the agent holds the belief $\beta$ and suppose that, in the context of that belief, the proposition “event $E$ is the case after action $A'$” implies the proposition “event $E$ is the case before action $A$”.

Now, consider the following scenarios: **Scenario (1)** The agent is promised that $F$ will be the case, should the action $A$ be performed. **Scenario (2)** The agent performs the action $A$ and, only afterwards, it is promised that $F$ is the case. In **Scenario (1)**, if the agent decides to perform the action $A$, its belief can already be updated to $E\beta'$, even before the action $A$ is performed. After the action $A$, the belief will be updated to $A\beta E\beta$, and finally to $F\beta A\beta$, taking into account that $F$ is promised to be the case.

**Scenario (2)**, the agent will update its belief to $E\beta$, after performing the action $A$, and then to $F\beta A\beta$, upon receiving the guarantee that $F$ is the case. Consistency between the two scenarios requires that the final belief of the agent should be the same, namely, $F\beta A\beta = F\beta E\beta$.

It remains to specify what it means for proposition “$F$ is the case after $A'$” to imply proposition “$E$ is the case before $A$” in the context of the belief $\beta$. To state a relation between these two propositions, the agent can assign a joint probability distribution to the events $(E, F)$, $(E, T)$, $(E, F)$, and $(T, F)$. Let us denote the joint probability distribution by $j_{\beta, A}(Y, Z)$, with $Y \in \{E, T\}$ and $Z \in \{F, T\}$. Consistency with the agent’s beliefs requires $j_{\beta, A}(Y, Z) = p(Z | \text{A} Y \beta)p(Y | \beta)$ when $p(Y | \beta) \neq 0$, and $j_{\beta, A}(Y, Z) = 0$ otherwise. Inferences from the event $Z$ to the event $Y$ can be made in terms of the conditional probability distribution

$$j_{\beta, A}(Y | Z) = \frac{j_{\beta, A}(Y, Z)}{\sum_{Y' \in \{E, T\}} j_{\beta, A}(Y', Z)}. \quad (2)$$

In particular, we say that the proposition “$F$ is the case after $A'$” implies the proposition “$E$ is the case before $A$” in the context of the belief $\beta$ if $j_{\beta, A}(E | F) = 1$.

**Axiom 3** (Backward consistency). If the proposition “event $E$ is the case after action $A'$” implies the proposition “event $E$ is the case before action $A$” in the context of belief $\beta$, then the occurrence of $F$ after $A$ leads to the same updated belief as the occurrence of $F$ after $A$ and after the occurrence of $E$. Mathematically: If $j_{\beta, A}(E | F) = 1$, then $F\beta A\beta = F\beta A\beta$.

In Appendix A we show that quantum theory satisfies forward and backward consistency if the events are represented by orthogonal projectors and the updates follow Lüders’ rule.

**General Bayesian theories.**—A general Bayesian theory is a tuple $(B, \text{Act}, \Sigma, U, p)$, consisting of a set of beliefs $B$, a monoid of actions $\text{Act}$, acting on the beliefs, a $\sigma$-algebra of events $\Sigma$, acting on the beliefs through an update map $U : (E, \beta) \mapsto E\beta$, and a probability assignment $p(E | \beta)$. The update map is required to satisfy Bayes’ rule and the properties of forward and backward consistency.

**Sequential experiments and ideal experiments.**—A sequential experiment consists of a sequence of actions interspersed by principal experiments. For example, $(A, E, B, F)$ represents a sequence consisting of an action $A$, followed by a principal experiment $E$, followed by another action $B$, and by another principal experiment $F$. The joint probability distribution of the outcomes are computed via the rule of conditional probabilities. For example, the probability distribution of the sequential experiment $(A, E, B, F)$ is $p(F | BEA\beta) p(E | A\beta)$, with $E \in F$ and $F \in F$.

We now focus our attention on a special class of experiments that leave the agent with the option of gathering more refined pieces of information in the future. We say that an experiment $(B, F)$ with partition $F = (F_i)_{i \in I}$ is a refinement of another experiment $(A, E)$ with partition $E = (E_i)_i$ if

$$\sum_i p(F_i | B \beta) = p(E_i | A \beta) \quad \forall i, \forall \beta \in B. \quad (3)$$

Intuitively, if an experiment does not disturb any of its refinements then this experiment is disturbing the belief as little as possible and hence captures a central property of sharp measurements in quantum theory [25–27]. Here we say that the experiment $(A, E)$ is sequentially refinable if there exists an action $A'$ such that, for every refinement $(B, F)$ and for every initial belief $\beta$, the probability of the event $F_{i,t}$ in the experiment $(B, F)$ is equal to the joint probability of the events $(E_i, F_{i,t})$ in the sequential experiment $(A, E, A', B, F)$. In formula,

$$p(F_{i,t} | B \beta) = p(F_{i,t} | B A' E_i A \beta) p(E_i | A \beta) \quad (4)$$

for every refinement $(B, F)$, for every event $F_{i,t} \in F$, and for every belief $\beta \in B$ with $p(E_i | A \beta) \neq 0$. This means that the coarse-grained experiment $(A, E)$ does not alter the probability assignment for the fine-grained experiment $(B, F)$, provided that the agent performs the reversing action $A'$ between them.

So far we considered individual experiments. Let us consider now the whole family of experiments of the form $(A, E)$ where the action $A$ is fixed and the partition $E$ is
variable. If there exists a common reversing action \( A' \) such that condition (4) holds for all partitions \( \mathcal{E} \), then we call the action \( A \) ideal. In Appendix B we prove the following

**Theorem 1.** In any general Bayesian theory, every reversible action \( A \) is ideal.

If \( A \) is an ideal action and \( \mathcal{E} \) is a partition, we say that \( (A, \mathcal{E}) \) is an ideal experiment. For every event \( E \in \mathcal{E} \), we call the pair \( (A, E) \) is an ideal result.

**Recovering the exclusivity principle.**—We now show that such ideal results satisfy the exclusivity principle \([13-19, 25, 28]\), also named global exclusivity \([14]\) or consistent exclusivity \([16, 17, 25]\). In the language of general Bayesian theories, the exclusivity principle refers to any set of ideal results \( \{ (A_n, E_n) \} \), which are pairwise mutually exclusive. Two results \( (A_i, E_i) \) and \( (A_j, E_j) \) are mutually exclusive, if there exists an ideal action \( B \) and a partition \( \mathcal{F} \) with events \( F, F' \in \mathcal{F} \), such that \( p(E_i | A_i \beta) = p(F | B \beta) \) and \( p(E_j | A_j \beta) = p(F' | B \beta) \) for all \( \beta \in B \). The exclusivity principle states that, if every pair in a set of ideal results is mutually exclusive, then the corresponding probabilities satisfy the bound

\[
\sum p(E_n | A_n \beta) \leq 1
\]

for every belief \( \beta \in B \). Crucially, the above bound holds in every general Bayesian theory:

**Theorem 2.** The ideal results in every general Bayesian theory satisfy the exclusivity principle.

The proof is provided in Appendix C.

**Conclusions.**—We have built a framework for general Bayesian theories, which can be applied to the classical and quantum domain, or to more general alternatives. In this framework, the agent holds a belief in order to place bets about the outcomes of experiments. The belief is updated when an action is performed and when a new piece of information becomes available. The constraints that the update has to satisfy are the rule of conditional probabilities and the conditions of forward consistency and backward consistency, which express the consistency of beliefs at different moments of time. These three requirements allow us to define a privileged class of ideal experiments. We showed that the results of the class of ideal experiments satisfy the exclusivity principle which, under mild assumptions, implies tight bounds on the set of quantum correlations for any Bell or Kochen-Specker scenario \([19]\). Therefore, our result shows that an important part of quantum theory can be reconstructed from elementary rules on how a Bayesian agent should bet about the outcomes of future experiments.

Whether or not all of quantum theory can be derived from Bayesian consistency conditions remains an open problem. If such a derivation turns out to be possible, it would support the view that quantum theory is a consequence of how agents subjectively organize perceptions \([29]\). If such a derivation is not possible, the attempt to find it might point out to the crucial physical ingredient that identifies quantum theory among other physical theories \([30]\). In contrast to earlier reconstructions of quantum theory \([31-37]\) that have been mostly non-committal about interpretations, such a derivation, even if only partially successful, would have the potential to shed new light on the interpretations of quantum theory.

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**Appendix A: Forward and backward consistency in quantum theory.**

Here we show that forward and backward consistency hold for projective measurements in quantum theory.

We start with forward consistency. Let \( \rho \) be a density matrix, \( E \) be a projector, and \( \overline{E} = I - E \) be its orthogonal complement. \( I \) being the identity matrix. The condition that the event described by \( E \) happens with certainty on \( \rho \) is \( \text{tr}(E \rho) = 1 \). Forward consistency is the statement that, under this condition, the updated state \( E \rho \overline{E} / \text{tr}(E \rho) \) is equal to the original state \( \rho \). The proof is simple. The certainty condition \( \text{tr}(E \rho) = 1 \) is equivalent to \( \text{tr}(\overline{E} \rho \overline{E}) = 0 \), which in turn is equivalent to \( \overline{E} \rho \overline{E} = 0 \). Since this equation is of the form \( AA^\dagger = 0 \) with \( A = \overline{E} \sqrt{\rho} \), we have \( A = 0 \) and thus \( E \sqrt{\rho} \overline{E} = 0 \) and \( \sqrt{\rho} \overline{E} = 0 \). This leads to

\[
\rho = (E + \overline{E}) \rho (E + \overline{E}) = E \rho E
\]

(A1)
proving that quantum theory satisfies forward consistency.

We now show that quantum theory satisfies backward consistency, too. Let \( A \) be a completely positive and trace preserving map with Kraus representation \( A: \rho \mapsto \sum_i A_i \rho A_i^\dagger, \rho \) be a density matrix, \( (E_i) \), and \( F \) be projectors with \( \sum_i E_i = I \). Clearly, the condition that the occurrence of \( F \) implies the occurrence of an event \( E_k \) is equivalent to the condition that the occurrence of \( F \) excludes the occurrence of every event \( E_i \neq E_k \). Hence, we have

\[
\sum_{i \neq k} \text{tr}[FA(E_i \rho E_i)] = 0. \tag{A2}
\]

In turn, this implies \( FA_j(E_i \rho E_i) A_j^\dagger F = 0 \) for all \( j \) and all \( i \neq k \) and by the same argument as above, \( FA_j E_i \sqrt{\eta} = 0 = \sqrt{\eta} E_i A_j^\dagger F \). This way we obtain

\[
FA(\rho) F = \sum_{j,i} FA_j E_i \rho E_i F A_j^\dagger F
= \sum_j FA_j E_k \rho E_k A_j^\dagger F
= FA(\rho E_k) F,
\tag{A3}
\]

proving that quantum theory satisfies backward consistency.

**Appendix B: Proof of Theorem 1**

The proof uses three lemmas, provided in the following:

**Lemma 3.** Let \( (C, F) \) be a refinement of \((I, \mathcal{E})\), with \( \mathcal{E} = \bigcup_i (F_i)_{i \in I} \) and \( \mathcal{E} = (E_i)_{i \in I} \), and let \( F_i \) be the event defined by \( F_i := \bigcup_j F_{i,j} \). Then,

\[
p(F_i | \mathcal{C} \beta) = p(E_i | \beta) \quad \forall i, \forall \beta \in B \tag{B1}
\]

and the equality

\[
F_i \mathcal{C} \beta = CE_i \beta \tag{B2}
\]

holds for every outcome \( i \) and for every \( \beta \in B \) such that \( p(E_i | \beta) \neq 0 \).

**Proof.** Since \((C, F)\) is a refinement of \((I, \mathcal{E})\), Eq. \( \text{(B1)} \) is due to

\[
p(F_i | \mathcal{C} \beta) = \sum_I p(F_{i,j} | \mathcal{C} \beta) = p(E_i | \beta) . \tag{B3}
\]

The first equality follows from the additivity of the probabilities of disjoint events.

Applying Eq. \( \text{(B1)} \) to the initial belief \( E_i \beta \), we obtain the condition

\[
p(F_i | CE_i \beta) = p(E_i | E_i \beta) = \frac{p(E_i \cap E_i | \beta)}{p(E_i | \beta)} = 1, \tag{B4}
\]

the second equality following from Eq. (1) in the main text. Since the agent is sure that the event \( F_i \) will occur, forward consistency implies that no update should take place on the belief \( CE_i \beta \). Hence, we obtain the relation

\[
F_i CE_i \beta = CE_i \beta . \tag{B5}
\]

Now, consider the joint probability distribution \( p(F_i | CE_i \beta) p(E_i | \beta) \). Using Eq. \( \text{(B4)} \), we obtain \( \beta, \forall i, CE_i \beta | F_i = 1 \). Then, backward consistency implies

\[
F_i CE_i \beta = F_i C \beta . \tag{B6}
\]

Combining Eqs. \( \text{(B5)} \) and \( \text{(B6)} \), we obtain Eq. \( \text{(B2)} \).

**Lemma 4.** The trivial action \( I \) is ideal.

**Proof.** For an arbitrary partition \( \mathcal{E} = (E_i) \), we consider an arbitrary refinement \((C,F)\) of \((I, \mathcal{E})\) with \( \mathcal{E} = \bigcup_i (F_i)_{i \in I} \). Let \( F_i \) be the event defined by \( F_i := \bigcup_j F_{i,j} \).

Using the rule of conditional probabilities, we obtain for \( p(F_i | \mathcal{C} \beta) \neq 0 \),

\[
p(F_i | \mathcal{C} \beta) = p(F_i \cap \mathcal{C} \beta) = p(F_i \cap \mathcal{C} \beta | \mathcal{C} \beta) p(F_i | \mathcal{C} \beta) = p(F_i | \mathcal{C} \beta | \mathcal{E} \beta) \tag{B7}
\]

the last equality following from Eqs. \( \text{(B2)} \) and \( \text{(B1)} \). If \( p(F_i | \mathcal{C} \beta) = 0 \), then Eq. \( \text{(B1)} \) implies \( p(E_i | \beta) = 0 \). Eq. \( \text{(B7)} \) is exactly the condition for sequential refinability, Eq. (4) from the main text with \( \mathcal{A} = \mathcal{A}' = I \). Since these considerations hold for any partition \( \mathcal{E} \) and any refinement of \((I, \mathcal{E})\), the action \( I \) is ideal.

We now prove the following reformulation of Theorem 1 in the main text:

**Lemma 5.** If the action \( \mathcal{A} \) is reversible, then \( \mathcal{A} \) is ideal.

**Proof.** We need to show that for every partition \( \mathcal{E} = (E_i)_{i \in I} \), for every refinement \((B,F)\) of \((\mathcal{A}, \mathcal{E})\), and for every belief \( \beta \), the condition of sequential refinability, Eq. (4) from the main text, is satisfied.

Since \( \mathcal{A} \) is reversible, all elements of \( B \) can be obtained as \( \mathcal{A} \beta \). Setting \( \beta' := \mathcal{A} \beta \) and \( \mathcal{C} := \mathcal{B} \mathcal{A} \), Eq. (4) from the main text becomes

\[
p(F_{i,j} | \mathcal{C} \beta') = p(F_{i,j} | \mathcal{C} \beta') p(E_i | \beta') \tag{B8}
\]

for all \( \beta' \in B \) with \( p(E_i | \beta') \neq 0 \). This is exactly the condition of sequential refinability for the trivial action \( I \) (with reversing action \( I' = I \)) applied to the experiment \((C,F)\). Our goal is to show that this condition holds.

With our definitions, the condition that \((B,F)\) is a refinement of \((\mathcal{A},E)\) reads

\[
\sum_{i} p(F_{i,j} | \mathcal{C} \beta') = p(E_i | \beta') \quad \forall \beta' \in B, \tag{B9}
\]

which in turn is equivalent to the fact that \((C,F)\) is a refinement of \((I,E)\). Due to Lemma 4, this implies that for
Let $(I, E)$, the condition of sequential refinability is satisfied for $(C, F)$, that is, Eq. (88) holds. Hence the condition for sequential refinability is satisfied for $(A, E)$ applied to the experiment $(B, F)$. Since the above argument holds for every partition $E$ and the reversing action is always $A'$, the action $A$ is ideal.

Appendix C: Proof of Theorem 2

The proof of Theorem 2 follows the line of argument in Ref. [25].

Lemma 6. Let $(A_1, E_1)$ and $(A_2, E_2)$ be two mutually exclusive ideal outcomes and let $A'_1$ be the reversing action in condition (4) of the main text. Then the equality

$$p(E_2|A_2, A'_1|F_1|A_1) p(F_1|A_1) = p(E_2|A_2, A_1)$$

(C1)

holds for all beliefs $\beta$ with $p(F_1|A_1) \neq 0$, otherwise $p(E_2|A_2, A_1) = 0$.

Proof. Since $(A_1, E_1)$ and $(A_2, E_2)$ are mutually exclusive, there exists an experiment $(B, F)$, with partition $F = (F_1)$, satisfying $p(F_1|B|\beta) = p(E_1|A_1, \beta)$ and $p(F_2|B|\beta) = p(E_2|A_2, \beta)$ for all beliefs $\beta$. The experiment $(B, F)$ is a refinement of the experiment $(A_1, E_1)$ with $E_1 = (E_1, E_1)$. This holds since for any belief $\beta$ we have

$$p(F_1|B|\beta) = p(E_1|A_1, \beta)$$

(C2)

and

$$\sum_{i \neq 1} p(F_i|B|\beta) = 1 - p(F_1|B|\beta)$$

(C3)

Now, the sequential refinability of the experiment $(A_1, E_1)$ implies for every belief $\beta$ with $p(F_1|A_1, \beta) \neq 0$ the equality

$$p(F_2|B|\beta) = p(F_2|B|A'_1|F_1|A_1) p(F_1|A_1, \beta)$$

$$= p(E_2|A_2, A'_1|F_1|A_1) p(F_1|A_1) \cdot$$

(C4)

Since $p(F_2|B|\beta) = p(E_2|A_2, A_1)$ this yields Eq. (C1). Finally, if $p(F_1|A_1, \beta) = 0$, then Eq. (C3) yields immediately $p(E_2|A_2, A_1) = p(F_2|B|\beta) = 0$. \qed

We are now ready to prove the following formulation of Theorem 2:

Lemma 7. Let $\{ (A_n, E_n) \mid n = 1, \ldots, k \}$ be a set of ideal results which are pairwise mutually exclusive. Then the condition

$$\sum_{n=1}^{k} p(E_n|A_n, \beta) \leq 1$$

(C5)

is satisfied for every belief $\beta$.

Proof. We define $E_i = (E_i, \overline{E_i})$, let $A'_i$ denote the ideal reverse of $A_i$. Consider the sequential experiment defined by the following procedure:

(I) Move to step $(R_i)$.

(R_i) Perform the action $(A_i, E_i, A'_i)$. If the outcome was $E_i$, then move to step $(T_i)$. If the outcome was $\overline{E_i}$ and $i + 1 = k$, then move to step $(T_{k+1})$. In all other cases move to step $(R_{i+1})$.

(T_s) Report outcome $s$ and terminate.

The above procedure defines a sequential experiment with outcomes $s = 1, 2, \ldots, k + 1$. If $s \leq k$, the outcome $s$ is the outcome corresponding to the event $E_s$. We denote by $q_\beta(s)$ the probability assigned to the outcome $s$ when the initial belief is $\beta$.

We now show that $q_\beta(s) = p(E_s|A_s, \beta)$ for every $s \leq k$. We start by defining the sequence of beliefs $\beta_0, \beta_1, \ldots, \beta_k$ via $\beta_0 := \beta$ and

$$\beta_j := A'_j \overline{E_j} A_j \beta_{j-1}.$$

(C6)

Then, the belief $\beta_{j-1}$ is the belief of the agent at the beginning of step $(R_i)$. We first treat the case $s = 1$ and obtain

$$q_\beta(1) = p(E_1|A_1, \beta_0) = p(E_1|A_1, \beta).$$

(C7)

For the case $s \geq 2$, we use $(s - 1)$ times Lemma 6, namely for the two mutually exclusive ideal results $(A_s, E_s)$ and $(A_s, \overline{E_s})$ for $j = s - 1, s - 2, \ldots, 1$. We first put aside the cases where $p(\overline{E_j}|A_j, \beta_{j-1}) = 0$ for some $j < s$. Then we obtain

$$q_\beta(s) = p(E_s|A_s, \beta_{s-1}) \prod_{j=1}^{s-1} (1 - q_\beta(j))$$

$$= p(E_s|A_s, A'_{s-1}|\overline{E_{s-1}} A_{s-1} \beta_{s-2})$$

$$\times p(\overline{E_{s-1}}|A_{s-1} \beta_{s-2}) \prod_{j=1}^{s-2} (1 - q_\beta(j))$$

(C8)

$$= p(E_s|A_s, \beta_{s-2}) \prod_{j=1}^{s-2} (1 - q_\beta(j))$$

$$\vdots$$

$$= p(E_s|A_s, \beta_0) = p(E_s|A_s, \beta).$$

Since the sum of the probabilities to obtain one of the outcomes $s = 1, \ldots, k + 1$ must be one, we obtain

$$\sum_{s=1}^{k} p(E_s|A_s, \beta) = \sum_{s=1}^{k} q_\beta(s) \leq 1.$$  

(C9)

For the corner cases, let $t$ be the smallest index such that $p(\overline{E_t}|A_t, \beta_{t-1}) = 0$ holds. According to Lemma 6
this implies $p(E_s|A_s\beta_{s-1}) = 0$ for all $s > t$. We obtain for $s > t$,  
\[
0 = p(E_s|A_s\beta_{s-1}) \prod_{j=1}^{t-1} (1 - q_{s}(j)) \tag{C10}
\]
\[
= p(E_s|A_s\beta_0) = p(E_s|A_s\beta) .
\]

Therefore we have
\[
\sum_s p(E_s|A_s\beta) = \sum_s p(E_s|A_s\beta) = \sum_s q_{s}(s) \leq 1 . \tag{C11}
\]

Where, in fact, equality holds, since $q_{s}(s) = 0$ for all $s > t$. 

\[\square\]

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