FUNDAMENTAL DOMAINS OF CLUSTER CATEGORIES INSIDE MODULE CATEGORIES

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Abstract. Let $H$ be a finite dimensional hereditary algebra over an algebraically closed field, and let $\mathcal{C}_H$ be the corresponding cluster category. We give a description of the (standard) fundamental domain of $\mathcal{C}_H$ in the bounded derived category $D^b(H)$, and of the cluster-tilting objects, in terms of the category $\text{mod}\Gamma$ of finitely generated modules over a suitable tilted algebra $\Gamma$. Furthermore, we apply this description to obtain (the quiver of) an arbitrary cluster-tilted algebra.

1. Introduction

Let $k$ be an algebraically closed field and $Q$ a finite acyclic quiver with $n$ vertices. Let $H = kQ$ be the associated path algebra. The cluster category $\mathcal{C}_H$ was introduced and investigated in [7], motivated by the cluster algebras of Fomin-Zelevinsky [10]. By definition we have $\mathcal{C}_H = D^b(H)/\tau^{-1}[1]$, where $\tau$ denotes the AR-translation. An important class of objects are the cluster tilting objects $T$, which are the objects $T$ with $\text{Ext}^1_{\mathcal{C}_H}(T, T) = 0$, and $T$ maximal with this property. They are shown to be exactly the objects induced by tilting objects over some path algebra $kQ'$ derived equivalent to $kQ$.

A crucial property of the cluster tilting objects $T = T_1 \oplus \cdots \oplus T_j$ where the $T_i$ are indecomposable, and $T_i$ is not isomorphic to $T_{i'}$ for $i \neq i'$, is that $j = n$ and for each $i = 1, \ldots, n$ there is a unique indecomposable object $T^*_i$ not isomorphic to $T_i$ in $\mathcal{C}_H$, such that $(T/T_i) \oplus T^*_i$ is a cluster tilting object. This is a more regular behavior than what we have for tilting modules (of projective dimension at most one) over a finite dimensional algebra $A$. In general there is at most one replacement for each indecomposable summand.

The maps in $\mathcal{C}_H$ are defined as follows, as usual for orbit categories. Choose the fundamental domain $\mathcal{D}$ of $\mathcal{C}_H$ inside $D^b(H)$, whose indecomposable objects are the indecomposable $H$-modules, together with $P_1[1], \ldots, P_n[1]$, where the $P_j$ are the indecomposable projective $H$-modules. Let $X$ and $Y$ be in $\mathcal{D}$. Then $\text{Hom}_{\mathcal{C}_H}(X, Y) = \bigoplus_{i \in \mathcal{Z}} \text{Hom}_{D^b(H)}(X, (\tau^{-1}[1])^i(Y))$.

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In [2] the authors considered the triangular matrix algebra \( \Lambda = \begin{pmatrix} \text{H} & 0 \\ DH & \text{H} \end{pmatrix} \), where \( D = \text{Hom}_k(-,k) \). They chose a fundamental domain for \( \mathcal{C}_H \) inside the category \( \text{mod} \Lambda \) of finite dimensional \( \Lambda \)-modules, by using the \( H \)-modules together with \( \text{ind}\tau^{-1}_\Lambda(DH) \). They established a bijection between cluster tilting objects in \( \mathcal{C}_H \) and a certain class of tilting modules in \( \text{mod} \Lambda \), which was shown in [3] to be all tilting modules (of projective dimension at most 1).

The present paper is inspired by [2]. Instead of using the algebra \( \Lambda \) which normally has global dimension 3, we use a smaller triangular matrix algebra \( \Gamma \) which has global dimension at most 2, and is a tilted algebra. We obtain a similar connection between cluster tilting objects in \( \mathcal{C}_H \) and tilting modules in \( \text{mod} \Gamma \) and give an alternative proof for the special property of complements mentioned above. The projective injective modules play a crucial role here, as in [2].

If \( T \) is a tilting \( H \)-module, a description of the quiver of \( \text{End}_{\mathcal{C}_H}(T) \), on the basis of the quiver of \( \text{End}_H(T) \), is given in [1] (see [9] for finite type). For each relation in a minimal set of relations in \( \text{add} \, T \), an arrow is added in the opposite direction. We obtain a similar description for \( T \) in the fundamental domain, but not necessarily being an \( H \)-module. Again the projective injective modules play an essential role. Now we consider relations where we allow factoring also through the projective injective modules, in addition to \( \text{add} \, T \). Then we obtain the same result about adding arrows in the opposite direction as before. When \( T \) is a tilting \( H \)-module, then no maps in \( \text{add} \, T \) factor through projective injective modules.

We now describe the content section by section. In section 2 we give some preliminary results on describing the indecomposable \( \Lambda \)-modules of projective dimension at most 1. In particular, we show that all predecessors of a module of projective dimension 1 have projective dimension at most 1. In section 3 we introduce the algebra \( \Gamma \) which replaces \( \Lambda \) in our work, starting with motivation on how to choose \( \Gamma \) smallest possible, without losing essential information. We show that the indecomposable \( \Gamma \)-modules of projective dimension at most 1 are exactly the modules in the left part \( \mathcal{L}_\Gamma \) of indecomposable modules where the predecessors have projective dimension at most 1. Further, this class consists of the indecomposable modules in our fundamental domain, together with the indecomposable projective injective \( \Gamma \)-modules. In section 4 we show how to describe the quiver of \( \text{End}_{\mathcal{C}_H}(T) \) for any \( T \) in the fundamental domain.

2. Duplicated algebras

In this section we recall work from [2] and improve the statement of the main theorem in [2]. Throughout the paper we assume that \( H \) is a basic hereditary algebra over an algebraically closed field \( k \) and \( \Lambda \) is the duplicated algebra of \( H \), that is, \( \Lambda = \begin{pmatrix} \text{H} & 0 \\ DH & \text{H} \end{pmatrix} \). We denote by \( \text{mod} \Lambda \) the category of finitely generated left \( \Lambda \)-modules, and we use the usual description of the left \( \Lambda \)-modules as triples \( (X,Y,f) \), with \( X,Y \) in \( \text{mod} \, H \) and \( f \in \text{Hom}_H(DH \otimes_H X,Y) \).
Then the full subcategory of mod $\Lambda$ generated by the modules of the form $(0,Y,0)$ is closed under predecessors and canonically isomorphic to mod $H$. We will use this isomorphism to identify mod $H$ with the corresponding full subcategory of mod $\Lambda$ and give some alternative proofs. The opposite algebra $\Lambda^{op}$ is isomorphic to the triangular matrix algebra $\begin{pmatrix} H^{op} & 0 \\ DH & H^{op} \end{pmatrix}$. Under these identifications, the duality $D : \text{mod} \Lambda \rightarrow \text{mod} \Lambda^{op}$ is given by $D(X,Y,f) = (DY,DX,Df)$, where $Df \in \text{Hom}_{H^{op}}(DY,D(DH \otimes_H X)) \cong \text{Hom}_{H^{op}}(DY,\text{Hom}_H(DH,DX)) \cong \text{Hom}_{H^{op}}(DY \otimes_H DH,DX) \cong \text{Hom}_{H^{op}}(DH \otimes_{H^{op}} DY,DX)$.

We recall (see [12] or [5], III, Proposition 2.5) that the indecomposable projective $\Lambda$-modules are given by triples isomorphic to those of the form $(0,P,0)$ or $(P,DH \otimes_H P,1_{DH \otimes_H P})$, with $P$ indecomposable projective in mod $H$. The former are the projective $H$-modules, and the latter are projective-injective $\Lambda$-modules. The remaining indecomposable injective $\Lambda$-modules are of the form $(I,0,0)$, with $I$ injective in mod $H$.

We denote by $pd_{\Lambda}M$ and $id_{\Lambda}M$ the projective dimension and the injective dimension of the $\Lambda$-module $M$, respectively. When $M$ is in mod $H$ we have $pd_{H}M = pd_{\Lambda}M$, and for that reason we will write just $pdM$. We denote by $\text{rad}X$ and $\text{soc}X$ the radical and the socle of the $\Lambda$-module $X$, respectively.

Let $\text{ind}_{\Lambda}$ denote the full subcategory of mod $\Lambda$ where the objects are a chosen set of nonisomorphic indecomposable $\Lambda$-modules.

We denote by $D^b(H)$ the bounded derived category of $H$, by $C_H$ the cluster category of $H$, and by $\tau$ the Auslander-Reiten translation in mod $\Lambda$ or $D^b(H)$. Note that the injective $H$–modules are not $\Lambda$–injective, so that $\tau^{-1}(I_i)$ is indecomposable for each indecomposable $H$-module $I_i = I_0(S_i)$, where $S_i$ is a simple $H$-module. Then $\{\tau^{-1}(I_i)\}$ in mod $\Lambda$ will play a similar role as $\{P_i[1]\}$ in the derived category $D^b(H)$. In particular, $\text{add(ind}_{H} \cup \{\tau^{-1}(DH)\}) \subseteq \text{mod}_{\Lambda}$ can be considered as a fundamental domain $\mathcal{D}_{\Lambda}$ inside mod $\Lambda$ of the cluster category $C_H$ (see [2]).

We recall that given $X,Y \in \text{ind}_{\Lambda}$, a path from $Y$ to $X$ is a sequence of nonzero morphisms $Y \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \ldots \rightarrow X_t \xrightarrow{f_t} X$, with the $X_i \in \text{ind}_{\Lambda}$. When such a path exists, $Y$ is a predecessor of $X$, and $X$ is a successor of $Y$. The left part $\mathcal{L}_{\Lambda}$ of mod $\Lambda$, defined in [14], is the full subcategory of $\text{ind}_{\Lambda}$ consisting of the modules whose predecessors have projective dimension at most 1. That is, $\mathcal{L}_{\Lambda} = \{X \in \text{ind}_{\Lambda} \mid pdY \leq 1 \text{ for any predecessor } Y \text{ of } X\}$. The main result of [2] is the following.

**Theorem 2.1.** (a) The fundamental domain $\mathcal{D}_{\Lambda}$ of $\mathcal{C}_{H}$ lies in $\text{add}_{\Lambda}$, and the only other indecomposable $\Lambda$-modules in $\mathcal{L}_{\Lambda}$ are projective-injective.

(b) There is induced a bijection between cluster-tilting objects in $\mathcal{C}_{H}$ and tilting modules in mod $\Lambda$ whose indecomposable non projective-injective summands lie in $\mathcal{L}_{\Lambda}$.
Note that in [3] it is shown that the bijection in (b) is with all tilting modules. Using the following results for \( \Lambda \)-modules with projective dimension at most one, we give a different approach to the improved version.

**Proposition 2.2.** Let \( X \in \text{ind} \Lambda \). Then \( \text{pd}_\Lambda X \leq 1 \) if and only if \( \tau_\Lambda X \in \text{mod} H \). In other words, the indecomposable \( \Lambda \)-modules \( X \) such that \( \text{pd}_\Lambda X \leq 1 \) are those in the fundamental domain of \( \mathcal{C}_H \), together with the indecomposable projective-injective \( \Lambda \)-modules.

**Proof.** We have that \( \text{pd}_\Lambda X > 1 \) if and only if \( \text{Hom}_\Lambda(D\Lambda, \tau X) \neq 0 \), and this last condition implies \( \tau X \notin \text{mod} H \), since the injective \( \Lambda \)-modules do not belong to \( \text{mod} H \). Conversely, if \( \tau X \notin \text{mod} H \), there is a projective-injective \( \Lambda \)-module \( E \) such that \( \text{Hom}_\Lambda(E, \tau X) \neq 0 \), and this implies \( \text{pd}_\Lambda X > 1 \).

**Lemma 2.3.** Let \( X,Y \in \text{ind} \Lambda \) be such that \( \text{Hom}_\Lambda(X,Y) \neq 0 \) and \( \text{pd}_\Lambda Y = 1 \). Then \( \text{pd}_\Lambda X \leq 1 \).

**Proof.** Let \( f : X \rightarrow Y \) be a nonzero morphism. We want to show that \( \text{pd}_\Lambda X \leq 1 \). Suppose that this is not the case. Then \( f \) is not an isomorphism, and so it factors through the minimal right almost split morphism \( E \rightarrow Y \). Since \( f \neq 0 \), we can choose an indecomposable direct summand \( E_0 \) of \( E \), and morphisms \( g_0 : E_0 \rightarrow Y \) and \( h_0 : X \rightarrow E_0 \) with \( g_0 h_0 \neq 0 \) and \( g_0 \) irreducible. Then \( E_0 \notin \text{mod} H \), because its predecessor \( X \) is not in \( \text{mod} H \). Now suppose \( E_0 \) is projective. Then it is also injective, since all indecomposable projective \( \Lambda \)-modules which are not in \( \text{mod} H \) are injective. Hence \( \tau Y \cong \text{rad} E_0 \), because \( g_0 : E_0 \rightarrow Y \) is irreducible. Now \( h_0 : X \rightarrow E_0 \) factors through \( \text{rad} E_0 \), so \( \text{rad} E_0 \notin \text{mod} H \), and we conclude that \( \tau Y \cong \text{rad} E_0 \notin \text{mod} H \). By Proposition 2.2 this contradicts our hypothesis \( \text{pd} Y = 1 \). Therefore \( E_0 \) is not projective, and then there is an irreducible morphism \( \tau E_0 \rightarrow \tau Y \). On the other hand, \( \text{pd} Y = 1 \) implies that \( \tau Y \) is in \( \text{ind} \Lambda \). Hence \( \tau E_0 \) is in \( \text{ind} \Lambda \), and thus \( \text{pd} E_0 = 1 \). Therefore our original morphism \( f : X \rightarrow Y \) can be replaced by \( h_0 : X \rightarrow E_0 \), and so we can iterate the process to obtain an arbitrarily long path of irreducible morphisms \( E_m \xrightarrow{g_m} E_{m-1} \xrightarrow{g_{m-1}} \ldots \xrightarrow{g_2} E_1 \xrightarrow{g_1} E_0 \) with \( g_1 g_2 \ldots g_m \neq 0 \). But this is a contradiction, because each \( E_i \in \tau^{-1} \text{ind} H \setminus \text{mod} H \), i.e. it is a direct summand of \( \tau^{-1}DH \), and this implies that all the \( g_i \) are in \( \text{rad} \text{End}(\tau^{-1}DH) \), which is nilpotent.

**Proposition 2.4.** If \( X \in \text{ind} \Lambda \) and \( \text{pd}_\Lambda X = 1 \), then \( X \in \mathcal{L}_\Lambda \).

**Proof.** Suppose that the result does not hold, and let \( Y \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \rightarrow \ldots \rightarrow X_t \xrightarrow{f_t} X \) be a path in \( \text{ind} \Lambda \), with \( \text{pd} X = 1 \) and \( \text{pd} Y > 1 \). Then \( Y \notin \text{mod} H \), because \( H \) is hereditary. Since \( \text{mod} H \) is closed under predecessors in \( \text{mod} \Lambda \), then the \( X_i \) and \( X \) do not belong to \( \text{mod} H \) either. Now, let us choose the path so that it has minimal length among
those with \(pdX = 1\) and \(pdY > 1\). By Lemma 2.3, \(t \geq 1\). By minimality, \(X_i\) is projective (and hence injective) for \(1 \leq i \leq t\). The map \(f_{t-1}\) factors through \(radX_t\), which is not injective, and thus not projective. Then, by minimality we must have \(pd(radX_t) > 1\) and \(t = 1\). Since \(f_1\) factors through \(\frac{X_1}{socX_t}\) - which is not projective - we must have \(pd\left(\frac{X_1}{socX_t}\right) = 1\), by Lemma 2.3. By Proposition 2.2, \(\tau(\frac{X_1}{socX_t}) \in \text{mod } H\). But then \(1 < pd(radX_1) = pd(\tau(\frac{X_1}{socX_t})) \leq 1\). This contradiction ends the proof of the proposition. \(\Box\)

It follows that the only indecomposable \(\Lambda\)-modules \(X\) with \(pd\Lambda X \leq 1\) and \(X\) not in \(\mathcal{L}_\Lambda\), are projective-injective. Note that we do not necessarily have that \(\mathcal{L}_\Lambda\) consists exactly of the indecomposable \(\Lambda\)-modules of projective dimension at most 1. (See Example in [2]). We now have the following improvement of Theorem 2.1.

**Theorem 2.5.** Let \(\Lambda = \begin{pmatrix} H & 0 \\ DH & H \end{pmatrix}\) as before.

(a) If \(X\) is indecomposable and not projective-injective in \(\text{mod } \Lambda\), then \(X\) is in \(\mathcal{L}_\Lambda\) if and only if \(pd\Lambda X \leq 1\).

(b) The fundamental domain \(\mathcal{D}_\Lambda\) of \(\mathcal{C}_H\) inside \(\text{mod } \Lambda\) lies in \(\text{add } \mathcal{L}_\Lambda\), and the remaining modules in \(\mathcal{L}_\Lambda\) are projective-injective.

(c) There is a one to one correspondence between the multiplicity-free cluster tilting objects in \(\mathcal{C}_H\) and the basic tilting \(\Lambda\)-modules.

It was proven in [2] that the global dimension \(\text{gldim } \Lambda\) of \(\Lambda\) is at most 3. We end this section with a more precise description of \(\text{gldim } \Lambda\). We will give a proof of this result using the description of \(\Lambda\)-modules as triples, which allows us to calculate the global dimension of \(\Lambda\) more precisely. This shows that \(\Lambda\) is normally of global dimension 3.

**Proposition 2.6.** \(\text{gldim } \Lambda \leq 3\). Moreover:

(a) \(\text{gldim } \Lambda = 1\) if and only if \(H\) is semisimple.

(b) \(\text{gldim } \Lambda = 2\) if and only if \(\tau_2 H = 0\) and \(H\) is not semisimple.

Proof. We calculate \(\text{gldim } \Lambda = \max\{pd S : S \text{ simple } \Lambda\text{-module}\}\)
\[= \max\{pd (X,0,0), pd (0,X,0) : X \in \text{ind } H\}.\]

For \(M\) in \(\text{mod } H\), \(P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0\) denotes a minimal projective presentation. Let \(X \in \text{ind } H\). Then \(pd (0,X,0) \leq 1\), for \(\text{mod } H\) is closed under predecessors in \(\text{mod } \Lambda\). Suppose \(X\) is projective in \(\text{mod } H\). Then the following is a minimal projective resolution:
\[0 \rightarrow (0, P_1(DH \otimes X), 0) \rightarrow (0, P_0(DH \otimes X), 0) \rightarrow (X, DH \otimes X, 1) \rightarrow (X,0,0) \rightarrow 0.\]

Now, if \(\text{gldim } \Lambda \leq 1\), then \(P_1(DH \otimes X) = 0\) for every projective \(X\). Since \(DH \otimes -\) is the Nakayama equivalence between projective and injective \(H\)-modules, this is to say that...
every injective $H$-module is also projective, i.e. $H$ is semisimple. This establishes (a), since $\Lambda$ is clearly hereditary when $H$ is semisimple.

Assume now that $X$ is not projective. Then we have an exact sequence $0 \rightarrow \tau X \rightarrow D(P_1(X)^*) \rightarrow D(P_0(X)^*)$. Using that for projective $P$ there is a functorial isomorphism $DH \otimes P \simeq DP^*$, we obtain the following minimal projective resolution:

$$0 \rightarrow (0, P_1(\tau X), 0) \rightarrow (P_1(X), DH \otimes P_1(X), 1) \rightarrow (P_0(X), DH \otimes P_0(X), 1) \rightarrow (X, 0, 0) \rightarrow 0.$$ 

Thus $\text{pd } (X, 0, 0) \leq 2$ if and only if $P_1(\tau X) = 0$, if and only if $\tau^2 X = 0$. The proposition now follows right away. $\square$

**Corollary 2.7.** For each algebraically closed field $k$, there are only a finite number of basic indecomposable hereditary $k$-algebras $H$ such that $\text{gldim } \Lambda \leq 2$.

**Proof.** By Proposition 2.6, $\text{gldim } \Lambda \leq 1$ if and only if $\tau^2 \Lambda = 0$, i.e. if each $\Lambda$-module is either projective or injective. Hence $H$ is of finite representation type and so its ordinary quiver $Q$ has no multiple arrows. In addition, $(i \rightarrow j \rightarrow k)$ is not a subquiver of $Q$, because the simple module $S_j$ would be neither projective nor injective in such case. Finally, $(i\niceleft\rightarrow j\niceleft\rightarrow k\niceleft\rightarrow l)$ is not a subquiver of $Q$, because otherwise the module $j_k$ would be neither projective nor injective. Therefore $Q$ must be one of the following four quivers: $A_1, A_2, A_3$ with nonlinear order (\niceleft\rightarrow \niceleft\rightarrow). $\square$

Denote by $\hat{H} = \begin{pmatrix} \cdots & H & D_H & H & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$ the infinite dimensional repetitive algebra associated with the finite dimensional hereditary algebra $H$. As explained in [2], we have the following relationship:

$$\text{mod } H \subset D \subset \mathcal{L}_\Lambda \subset \text{mod } \Lambda \subset \text{mod } \hat{H} \to \text{mod } \hat{H} \overset{\simeq}{\to} \text{D}^b(H) \to \mathcal{C}_H.$$ 

The following more precise relationship will be useful.

**Proposition 2.8.** Let $\alpha : (X, Y, f) \rightarrow (X', Y', f')$ be a nonzero map in $\text{mod } \Lambda$. Then $\alpha$ factors through a projective injective $\Lambda$-module if and only if it factors through a projective module in $\text{mod } \hat{H}$.

**Proof.** The projective injective $\Lambda$-modules are additively generated by $(H, DH, id)$. For $\hat{H}$, the projective modules, which coincide with the injective ones, are additively generated by modules of the form $\begin{pmatrix} 0 & H & 0 \\ 0 & DH & 0 \\ \vdots & \ddots & \ddots \end{pmatrix}$. If $\alpha$ factors through a projective injective $\Lambda$-module, it is clear that it does the same when considered as a map in $\text{mod } \hat{H}$.

Conversely, assume that $\alpha$ factors through a projective $\hat{H}$-module. The possible projective modules must come from one or more of the following pictures:
In case (2) we must have a commutative diagram

\[
\begin{array}{ccc}
DH & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
DH & \overset{\delta}{\longrightarrow} & X'
\end{array}
\]

which is impossible since \(\delta \neq 0\). In case (3) the diagram

\[
\begin{array}{ccc}
DH \otimes Y & \overset{DH \otimes \gamma}{\longrightarrow} & DH \otimes H \\
\downarrow & \simeq & \downarrow \\
0 & \longrightarrow & DH
\end{array}
\]

commutes. Then \(DH \otimes \gamma\) must be zero. But since \(H\) is hereditary and \(\gamma : Y \rightarrow H\) is nonzero, there is an indecomposable summand \(Y_1\) of \(Y\) which is projective, with \(\gamma|_{Y_1} : Y_1 \rightarrow H\) nonzero. Since \(DH \otimes -\) gives an equivalence from the category of projective \(H\)-modules to the category of injective ones, then \(DH \otimes \gamma|_{Y_1}\) and hence \(DH \otimes \gamma\) is nonzero. This gives a contradiction.

Hence we must have case (1), which implies that \(\alpha\) factors through a projective injective \(\Lambda\)-module. \(\square\)

We end this section with some discussion about fundamental domains (see [2]). For the cluster category \(C_H\) we have a natural functor \(\Pi : D^bH \rightarrow C_H\). Let \(D\) be the additive subcategory of \(D^b(H)\) whose indecomposable objects are the indecomposable \(H\)-modules together with the shift \([1]\) of the indecomposable projective \(H\)-modules \([7]\). Then \(D\) is a convex subcategory of \(D^b(H)\), and \(\Pi\) induces a bijection between the indecomposable objects of \(D\) and those of \(C_H\). In order to find other “fundamental domains”, one is looking for similar properties. In particular, it is nice to use appropriate module categories rather than derived categories. A step in this direction was made in [2], by considering the duplicated algebra \(\Lambda = \begin{pmatrix} H & 0 \\ DH & H \end{pmatrix}\) of a hereditary algebra \(H\). Here there is a natural functor from \(\text{mod} \Lambda\) to \(C_H\), as discussed above, and \(\text{mod} \Lambda\) is naturally embedded into \(\text{mod} \Lambda\). In addition, the indecomposable objects \(\tau^{-1}_\Lambda(I)\), for \(I\) indecomposable injective \(H\)-module, are added to \(\text{mod} \Lambda\) to form a fundamental domain \(D_\Lambda\) inside \(\text{mod} \Lambda\), giving a desired bijection with the indecomposables in \(C_H\), from our functor \(\text{mod} \Lambda \rightarrow C_H\).

Here the fundamental domain is not only convex, but is also closed under predecessors in \(\text{mod} \Lambda\) which are not projective-injective. We shall see that we have a similar situation when replacing the duplicated algebra \(\Lambda\) by a smaller algebra \(\Gamma\).
3. The algebra Γ.

In this section we will replace the duplicated algebra Λ by a smaller algebra Γ such that also mod Γ contains the fundamental domain \( D \) of \( C_H \).

We start with a lemma, which will be needed later.

**Lemma 3.1.** Let \( A \) be a basic artin algebra, and let \( S, S' \) be simple projective \( A \)-modules. Then:

(a) If \( I \) is an indecomposable injective module not isomorphic to \( I_0(S) \), then \( \text{Hom}_A(I, I_0(S)) = 0 \).

(b) \( \text{End}_A I_0(S) \cong \text{End}_A S \).

(c) \( \text{End}_A I_0(S \oplus S') \cong \text{End}_A I_0(S) \times \text{End}_A I_0(S') \).

(d) \( S \cong \text{End}_A S \) as an \( \text{End}_A S \)-vector space.

**Proof.** (a) and (b) follow using the Nakayama equivalence \( * \) from the category of injective \( \Lambda \)-modules to the category of projective \( \Lambda \)-modules, and (c) is a direct consequence of (a).

(d) Let \( A = S \oplus Q \). Then \( \text{End}_A S \cong \text{Hom}_A(A, S) = \text{End}_A S \oplus \text{Hom}_A(Q, S) = \text{End}_A S \), since \( \text{Hom}_A(Q, S) = 0 \), because \( A \) is basic. \( \square \)

Let \( P \) be a projective \( \Lambda \)-module. We recall that \( \text{Hom}_\Lambda(P, -) : \text{mod } \Lambda \rightarrow \text{mod}(\text{End}_\Lambda P)^\text{op} \) is an equivalence, where \( \text{mod } P \) is the full subcategory of \( \text{mod } \Gamma \) consisting of the modules with a presentation in \( \text{add } P \). Now we can take \( \Gamma = (\text{End}_\Lambda P)^\text{op} \), with the projective \( P \) such that \( \text{mod } P \subseteq \text{Gen } P \), it is clear that \( \text{add } P \) must contain \( H \oplus P_0(\tau^{-1}_\Lambda DH) \). Next we show that this is enough.

We denote by \( \Delta \) the sum of the nonisomorphic simple projective \( H \)-modules. That is, \( \Delta \) is a basic \( \Lambda \)-module such that \( \text{add } \Delta = \text{add } \text{soc } H = \text{add } \text{soc } \Lambda \). Let \( \Delta P = H \oplus I_0^\Delta(\Delta) \). We will prove in the next proposition that \( P_0(\tau^{-1}_\Lambda DH) = I_0^\Delta(H) \), and that the basic projective module \( P \) has the required properties.

**Proposition 3.2.** Let \( \Delta P = H \oplus I_0^\Delta(\Delta) \). Then:

(1) \( \text{add } \overline{P} \) is closed under predecessors in \( P(\Lambda) = \{ \text{projective } \Lambda\text{-modules} \} \).

(2) \( \text{mod } \overline{P} = \text{Gen } P = \{(X, Y, f) \in \text{mod } \Lambda : X \in \text{add } \Delta \} \).

(3) Let \( P \) be an indecomposable projective \( H \)-module. Then \( \tau^{-1}_\Lambda \text{Hom}_H(P, H) = (\text{soc } P, I_1^H(P), \pi) \), where \( 0 \rightarrow P \rightarrow I_0^H(P) \xrightarrow{\pi} I_1^H(P) \rightarrow 0 \) is a minimal injective resolution in \( \text{mod } H \).

(4) \( I_0^\Delta(H) = P_0(\tau^{-1}_\Lambda DH) \).

(5) \( \text{mod } H \cup \{ \tau^{-1}_\Lambda DH \} \subseteq \text{mod } \overline{P} \).

(6) If \( \overline{Q} \) is a projective \( \Lambda \)-module such that \( \text{mod } H \cup \{ \tau^{-1}_\Lambda DH \} \subseteq \text{mod } \overline{Q} \), then \( \overline{P} \) is a direct summand of \( \overline{Q} \).
Proof. (1) Let \( Q \to P \) be a nonzero morphism between indecomposable projective \( \Lambda \)-modules, with \( P \in \text{add} \mathcal{P} \). We have to prove that \( Q \in \text{add} \mathcal{P} \). We may assume that \( P, Q \) are projective-injective. Hence \( P \) is in \( \text{add} I^\Lambda_0(\Delta) \), and the result follows from Lemma 3.1(a).

(2) The first equality follows from (1). Now, \( \mathcal{P} = (\Delta, H \oplus I^H_0(\Delta), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) ) \), where we identify \( DH \otimes_H \Delta \) with \( I^H_0(\Delta) \). Thus \( (X, Y, f) \in \text{Gen} \mathcal{P} \) if and only if \( X \in \text{Gen} \Delta(= \text{add} \mathcal{P}) \).

(3) and (4). We proceed to calculate \( \tau^{-1}_\Lambda D \text{Hom}_H(P, H) = \text{Tr}_\Lambda \text{Hom}_H(P, H) \). Let \( P^* = \text{Hom}_H(P, H) \). Since \( D \text{Hom}_H(P, H) \cong (0, DP^*, 0) \) in \( \text{mod} \Lambda \), then \( \text{Hom}_H(P, H) \cong (P^*, 0, 0) \), and the following is a minimal projective presentation:

\[
(0, P_0(P^* \otimes_H DH), 0) \to (P^*, P^* \otimes_H DH, 1) \to (P^*, 0, 0) \to 0.
\]

Applying \( \text{Hom}_\Lambda(\cdot, \Lambda) \), we obtain

\[
0 \to (0, P, 0) \to ((P_0(P^* \otimes_H DH))^*, DH \otimes_H (P_0(P^* \otimes_H DH))^*, 1) \to \text{Tr}_\Lambda \text{Hom}_H(P, H) \to 0.
\]

Since \( P^* \otimes_H DH \cong DP \), then \( (P_0(P^* \otimes_H DH))^* \cong (P_0(DP))^* = (D(I_0))^* = I_0(\text{soc} P) = \text{soc} P \). (We used that \( H \) is hereditary in the last step).

Hence \( P_0(\text{Tr}_\Lambda \text{Hom}_H(P, H)) = (\text{soc} P, I_1(\text{soc} P), 1) = I^\Lambda_0(\text{soc} P) \). The last equality follows from the description of injective \( \Lambda \)-modules as triples, and the fact that \( \text{soc} P \) is a projective \( H \)-module. Therefore the above sequence is

\[
0 \to (0, P, 0) \to I^\Lambda_0(\text{soc} P) \to \tau^{-1}_\Lambda D \text{Hom}_H(P, H) \to 0,
\]

so \( \tau^{-1}_\Lambda \text{Hom}_H(P, H) = (\text{soc} P, I_1(P), \pi) \). This establishes (3). Adding all the indecomposable projective \( H \)-modules yields the projective resolution

\[
0 \to H \to I^\Lambda_0(\text{soc} H) \to \tau^{-1}_\Lambda DH \to 0
\]

and proves (4).

(5) We have \( \text{mod} H \subseteq \text{mod} \mathcal{P} \), since \( H \) is a direct summand of \( \mathcal{P} \) and \( \text{mod} H \) is closed under predecessors in \( \text{mod} \Lambda \). Now the projective resolution (*) shows that \( \tau^{-1}_\Lambda DH \subseteq \text{mod} \mathcal{P} \).

(6) follows from (2), (4) and (5). \( \square \)

Now we define \( \Gamma = \text{End}_\Lambda(\mathcal{P})^{\text{op}} \). The next proposition describes \( \Gamma \) as a triangular matrix ring.

**Proposition 3.3.** Let \( \Gamma = \text{End}_\Lambda(\mathcal{P})^{\text{op}} \). Then:

(a) \( \Gamma \) is isomorphic to the triangular matrix ring \( \left( \begin{array}{cc} K & 0 \\ J & H \end{array} \right) \), where \( K = \text{End}_H \Delta^{\text{op}} \) is a basic semisimple algebra, and \( J = I^H_0(\Delta) \). In particular, the \( \Gamma \)-modules can be described in terms of triples \( (KX, HY, f) \), with \( f : J \otimes_K X \to Y \).
(b) For \( X \in \text{add} \Delta \) there is an isomorphism \( J \otimes K \text{Hom}_H(\Delta, X) \xrightarrow{\psi} DH \otimes H X \) of \( H \)-modules which is functorial in \( X \).

(c) \((H X, H Y, f) \mapsto (K \text{Hom}_H(\Delta, X), H Y, f \psi)\) is an equivalence from \( \text{mod} P \) to \( \text{mod} \Gamma \).

**Proof.** (a) Since \( \Lambda P = H \oplus I_0^\Lambda(\Delta) \), then
\[
\Gamma = \text{End}_\Lambda(P)^{op} \simeq \left( \begin{array}{cc} \text{End}_\Lambda H & \text{Hom}_\Lambda(I_0^\Lambda(\Delta), H) \\ \text{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & \text{End}_\Lambda I_0^\Lambda(\Delta) \end{array} \right)^{op}
\]
\[
\simeq \left( \begin{array}{cc} H^{op} & 0 \\ \text{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & \text{End}_\Lambda I_0^\Lambda(\Delta)^{op} \end{array} \right)^{op} \simeq \left( \begin{array}{cc} \text{End}_\Lambda I_0^\Lambda(\Delta)^{op} & 0 \\ \text{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) & H \end{array} \right).
\]

Now, by Lemma 3.1, \( \text{End}_\Lambda I_0^\Lambda(\Delta)^{op} \simeq \text{End}_\Lambda \Delta \simeq \text{End}_H \Delta \simeq K^{op} \) is a basic semisimple algebra.

Finally, since \( I_0^\Lambda(\Delta) = (\Delta, I_0^H(\Delta), 1) \), we have \( \text{Hom}_\Lambda(H, I_0^\Lambda(\Delta)) \simeq \text{Hom}_H(H, I_0^H(\Delta)) \simeq I_0^H(\Delta) = J \) as \( H-K \)-bimodule.

(b) Since the functors \( J \otimes K \text{Hom}_H(\Delta, -) \) and \( DH \otimes H - \) are additive, we can assume that \( X \) is simple projective. By Lemma 3.1(c), we have \( K \simeq \prod_{S \in \text{ind} \Delta} \text{End}_H(S)^{op} \), so that
\[
J \otimes K \text{Hom}_H(\Delta, X) \simeq I_0^H(X)^{op} \otimes \text{End}_H(X)^{op} \text{End}_H(X) \simeq \text{End}_H(X) \otimes \text{End}_H(X) I_0^H(X) \simeq I_0^H(X).
\]

But \( DH \otimes H X \) is also isomorphic to \( I_0^H(X) \) when \( X \) is simple projective.

(c) Let \((H X, H Y, f) \in \text{mod} P\). By Proposition 3.2(2), we have that \( X \) is a semisimple projective \( H \)-module. Now the statement follows easily from (b). \( \Box \)

Note that the equivalence given in Proposition 3.3(c) is just \( \text{Hom}_\Lambda(P, -) : \text{mod} P \rightarrow \text{mod} \Gamma \), stated in terms of triples. We will identify \( \text{mod} \Gamma \) with the full subcategory \( \text{mod} P \) of \( \text{mod} \Lambda \). Under this identification, the fundamental domain \( \mathcal{D}_\Lambda \) of \( C_H \) in \( \text{mod} \Lambda \) is in \( \text{mod} \Gamma \), \( \Lambda \Gamma = \Lambda P \), and \( I_0^\Gamma(H) = I_0^\Lambda(H) = P_0^\Lambda(I^{-1}_\Lambda DH) = P_0^\Gamma(I^{-1}_\Gamma DH) \). From Proposition 3.2(2), it follows easily that a minimal \( \Lambda \)-projective resolution of a \( \Gamma \)-module \( M \) is in \( \text{mod} \Gamma \). Hence also \( pd_\Gamma M = pd_\Lambda M \) for \( M \) in \( \text{mod} \Gamma \).

We illustrate the situation with the following example.

**Example 3.4.** For the hereditary algebra \( H \) given below we indicate the corresponding algebras \( \Lambda \) and \( \Gamma \).

\[
\begin{array}{cc}
H : & 1 \rightarrow 2 \rightarrow 3 \\
\Lambda : & 1 \rightarrow 2 \rightarrow 3 \\
\Gamma : & 1 \rightarrow 2 \rightarrow 3
\end{array}
\]
By using $\Gamma$ instead of $\Lambda$ we get improved versions of Propositions 2.4 and 2.6.

**Proposition 3.5.** (a) $\Gamma$ is a tilted algebra.

(b) The set of indecomposable $\Gamma$−modules with projective dimension $\leq 1$ is closed under predecessors, and consists of the indecomposable objects in the fundamental domain $D_{\Gamma}$ of $C_H$ plus the indecomposable projective-injective $\Gamma$−modules. In particular, $L_{\Gamma}$ consists of the indecomposable $\Gamma$-modules of projective dimension at most one.

**Proof.** (a) Let $U = DH \oplus I^0_0(\Delta)$. By [4], Lemma 2.1, it suffices to show that $U$ is a convex tilting $\Gamma$-module.

We have $\text{pd}_U = \text{pd}_H(DH) \leq 1$ because $I^0_0(\Delta)$ is projective and $H$ is hereditary. Since $\text{mod} H$ is closed under predecessors, then $\text{Ext}_H^1(U,U) = 0$. Finally, $|\text{ind add } U| = \text{rk } K_0(\Gamma)$, where $K_0(\Gamma)$ denotes the Grothendieck group of $\Gamma$. Hence $U$ is tilting.

Now let us see that $U$ is convex:

Let $T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s$ be a path in $\text{ind } \Gamma$ with $T_0, T_s \in \text{add } U$, where we assume that all $f_i$ are non-isomorphisms. If $T_s \in \text{mod } H$, then $T_i$ are $H$−modules, and therefore they are all $H$−injective. On the other hand, if $T_s \notin \text{mod } H$, then $T_s \in \text{add } I_0(\Delta)$. Then $T_s$ is projective and $f_s$ factors through $\text{rad } T_s$. Since $\text{rad } T_s \in \text{mod } H$ and is an injective $H$-module, we are in the previous case, so we are done.

(b) Let $f : X \to Y$ be a non-zero map with $X,Y \in \text{ind } \Gamma$ and $\text{pd} Y \leq 1$. If $\text{pd} Y = 1$ then, by Proposition 3.5(a), $\text{pd} X \leq 1$. Thus we can assume $Y$ is projective and $f$ is not surjective. Hence $f$ factors through $\text{rad } Y$. Since $\text{rad } Y \in \text{mod } H$, the proposition follows. $\Box$

**Corollary 3.6.** Let $F = \{X \in \text{mod } \Gamma : \text{pd} X \leq 1\}$, $T = \text{add}(\text{ind } \Gamma \setminus F) = \text{add}\{X \in \text{ind } \Gamma : \text{pd} X = 2\}$. Then $(T,F)$ is a split torsion pair in $\text{mod } \Gamma$.

Now we will prove that the global dimension of $\text{End}_\Gamma(T)$ remains less than or equal to 2 when $T$ is a tilting $\Gamma$−module. We will use results from [13], which we collect in the following lemma.

**Lemma 3.7.** Let $A$ be an artin algebra with finite global dimension and $\Gamma T$ be a tilting module such that $\text{pd} T = 1$ and $id T = s$. Let $B = (\text{End}_A T)^{op}$. Then:

(a) ([13] Prop. 2.1) We have $s \leq \text{gldim } B \leq s + 1$.

(b) ([13] Thm. 3.2) If $s \geq 1$, then $s = \text{gldim } B$ if and only if $\text{Ext}_A^s(\tau T, T) = 0$.

**Proposition 3.8.** Let $T$ be a tilting $\Gamma$−module. Then $\text{gldim } \text{End}_\Gamma(T) \leq 2$.

**Proof.** By Proposition 3.5(a), $\Gamma$ is a tilted algebra. Thus $\text{gldim } \Gamma \leq 2$. Then we may assume that $\text{pd} T = 1$. Let $s = id T$. If $s \leq 1$, the proposition follows from Lemma 3.7(a),
so assume $s = 2$. By Lemma 3.7(b), it is enough to prove that $\mathrm{Ext}^2_{\Gamma}(\tau T, T) = 0$. We have $\mathrm{Ext}^2_{\Gamma}(\tau T, T) \cong \mathrm{Ext}^1_{\Gamma}(\Omega \tau T, T) \cong \mathrm{DHom}_{\Gamma}(T, \tau \Omega \tau T)$. Therefore it suffices to show that $\tau \Omega \tau T = 0$. By Proposition 3.3(b), $p \tau T \leq 1$. Hence $\tau \Omega \tau T$ is projective and $\tau \Omega \tau T = 0$. □

We have seen that the algebra $\Gamma$ is a tilted algebra and $\text{mod } \Gamma$ contains the fundamental domain $\mathcal{D}_\Gamma$ of the cluster category $\mathcal{C}_H$ as a full subcategory, closed under predecessors in $\text{mod } \Gamma$. An analogous statement to Theorem 2.5(c) also holds for tilting $\Gamma$-modules. We will use a preliminary lemma.

**Lemma 3.9.** There is a bijective correspondence between the set of (isoclasses of) basic tilting $\Gamma$-modules and the set of (isoclasses of) basic tilting $\Lambda$-modules, given by $\Gamma T \rightarrow \Lambda T \oplus I^\Lambda_0(DH)/I^\Lambda_0(\Delta)$.

**Proof.** Let $T$ be a basic tilting $\Gamma$-module and let $\overline{Q} = I^\Lambda_0(DH)/I^\Lambda_0(\Delta)$. Then $\Lambda \Lambda \simeq \overline{P} \oplus \overline{Q}$. Hence the basic projective-injective $\Lambda$-module $\overline{Q}$ is not in $\text{mod } \Gamma$, and $T \oplus \overline{Q}$ is a basic partial tilting $\Lambda$-module. But $|\text{ind}(T \oplus \overline{Q})| = |\text{ind } T| + |\text{ind } \overline{Q}| = |\text{ind } \Gamma| + |\text{ind } \Lambda| - |\text{ind } \overline{P}| = |\text{ind } \Lambda|$. Thus $T \oplus \overline{Q}$ is a basic tilting $\Lambda$-module. Conversely, let $T'$ be a basic tilting $\Lambda$-module. Then the basic projective-injective $\Lambda$-module $\overline{Q}$ is a direct summand of $T'$. Now, since $pd T' \leq 1$, by Proposition 2.2, we have that $\tau \Lambda T'$ is in $\text{mod } H$. Thus $T'$ is in $\text{add}(\tau \Lambda^{-1} \text{mod } H \cup \{\Lambda \Lambda\}) \subseteq \text{add}(\text{mod } \Gamma \cup \{Q\})$, and therefore $T'/\overline{Q}$ is a basic partial tilting $\Gamma$-module, which must be a tilting $\Gamma$-module, by the counting argument used before. □

**Theorem 3.10.** There is a bijective correspondence $\theta$ between the multiplicity-free cluster-tilting objects in the cluster category $\mathcal{C}_H$ of $H$ and the basic tilting $\Gamma$-modules. For a cluster-tilting object represented by $T$ in the fundamental domain $\mathcal{D}_\Gamma$ of $\mathcal{C}_H$, the corresponding tilting $\Gamma$-module is $\theta(T) = T \oplus (\Lambda)^0_0(DH)/\Lambda^0_0(\Delta)$.

**Proof.** Let $T \in \mathcal{D}_\Gamma$ be a multiplicity-free cluster-tilting object in $\mathcal{C}_H$. By [2] Thm. 10, $T \oplus I^\Lambda_0(DH)$ is a basic tilting $\Lambda$-module. Thus, by Lemma 3.9, $\theta(T) = T \oplus I^\Lambda_0(\Delta) = T \oplus I^\Lambda_0(\Delta)$ is a basic tilting $\Gamma$-module. Conversely, if $T'$ is a basic tilting $\Gamma$-module then, by Lemma 3.9, $T' \oplus I^\Lambda_0(DH)/I^\Lambda_0(\Delta)$ is a basic tilting $\Lambda$-module. Hence, by Theorem 2.5(c), $T'/I^\Lambda_0(\Delta) = T/I^\Lambda_0(\Delta)$ is in the fundamental domain $\mathcal{D}_\Lambda$, and represents a multiplicity-free cluster-tilting object in $\mathcal{C}_H$. □

As a consequence of this result we obtain the following result of [7].

**Corollary 3.11.** Let $H$ be a hereditary algebra. Then each almost complete cluster tilting object in $\mathcal{C}_H$ has exactly two complements.

**Proof.** Let $T'$ be an almost complete cluster tilting object in $\mathcal{C}_H$. Then $T' \oplus I^\Lambda_0(\Delta)$ is an almost complete tilting module in $\text{mod } \Gamma$. Since $\text{add } \Delta = \text{add } (\text{soc } \Gamma)$, then $I^\Lambda_0(\Delta)$ is a faithful $\Gamma$-module, since $\Gamma \subseteq I^\Lambda_0(\Delta)$. We know from a result of Happel and Unger that then $T' \oplus I^\Lambda_0(\Delta)$ has exactly two complements, thus so does $T'$ in $\mathcal{C}_H$ (see [15]). □
The following result, building upon Proposition 3.12, will be useful in the next section. Here \( \mathcal{D}_\Lambda \) and \( \mathcal{D}_T \) denote the categories \( \mathcal{D}_\Lambda \) and \( \mathcal{D}_T \) modulo the projective injective \( \Lambda \)-modules, respectively the projective injective \( \Gamma \)-modules.

**Proposition 3.12.** (a) A map \( \alpha : (X,Y,f) \to (X',Y',f') \) in \( \mathcal{D}_\Lambda \subset \mod \Lambda \) factors through a projective injective \( \Lambda \)-module if and only if it factors through \( I^\Lambda_0(\soc H) \).

(b) We have an embedding of \( \mathcal{D}_T \) into \( D^b(H) \) via the composition \( \mathcal{D}_T = \mathcal{D}_\Lambda \subset \mod \Lambda \subset \mod \hat{H} = \mod \hat{H} \to \mod \hat{H} \to D^b(H) \).

**Proof.** The indecomposable objects in \( \mathcal{D}_\Lambda \) are the indecomposable \( H \)-modules together with the \( \tau^{-1}_\Lambda(I) \) for \( I \) an indecomposable injective \( H \)-module. Similarly, the indecomposable objects in \( \mathcal{D}_T \) are the indecomposable \( H \)-modules together with the \( \tau^{-1}_T(I) \) for \( I \) an indecomposable injective \( H \)-module. If \( f : X \to Z \) is an irreducible map between indecomposable modules in \( \mod \Lambda \), with \( Z \) projective injective and \( X \) in \( \mod H \), it is clear that \( Z = \left( \frac{S}{i_0(S)} \right) \), where \( S \) is simple projective, since all proper predecessors of \( Z \) should be \( H \)-modules. Since then only summands of \( I^\Lambda_0(\soc H) \) are amongst the projective injective \( \Lambda \)-modules which are summands of the middle term of an almost split sequence \( 0 \to I \to E \to \tau^{-1}_\Lambda(I) \to 0 \) in \( \mod \Lambda \), we see that \( \tau^{-1}_T(I) = \tau^{-1}_\Lambda(I) \), so that \( \mathcal{D}_T = \mathcal{D}_\Lambda \).

It follows from the above that the only indecomposable projective injective \( \Lambda \)-modules which have a nonzero map to \( \tau^{-1}_\Lambda(I) \) are the summands of \( I^\Lambda_0(\soc H) \). Hence no nonzero map \( g : D \to D' \) in \( \mathcal{D}_\Lambda \) can factor through any other projective injective \( \Lambda \)-modules. (Note, however, that there might be additional projective injective \( \Lambda \)-modules belonging to \( \mathcal{L}_\Lambda \)). Now we conclude that \( \mathcal{D}_T = \mathcal{D}_\Lambda \), and the rest follows.

\[ \Box \]

4. A Description of the Cluster Tilted Algebras.

In this section our aim is to describe the quivers of cluster tilted algebras, that is, of the endomorphism algebras of cluster-tilting objects in \( \mathcal{C}_H \), using the fundamental domain \( \mathcal{D}_T \) for \( \mathcal{C}_H \) inside \( \mod \Gamma \). Note that a cluster tilted algebra is determined by its quiver \[ [6] \]. Let \( \hat{T} \) be a cluster-tilting object in \( \mathcal{C}_H \). We assume that \( \hat{T} \) is represented by \( T \) in \( \mathcal{D}_T \subset \mod \Gamma \). For \( X,Y \) in \( \mathcal{D}_T \), regarded as objects in \( D^b(H) \), we have that \( \Hom_{\mathcal{C}_H}(X,Y) = \Hom_{D^b(H)}(X,Y) \oplus \Hom_{D^b(H)}(F^{-1}X,Y) \), where \( F = \tau^{-1}[1] \) in \( D^b(H) \) (see \[ 6 \]). By Proposition 3.12 we have \( \Hom_{D^b(H)}(X,Y) = \Hom_{\mod \Gamma}(X,Y) \). We want to investigate how to describe \( \Hom_{D^b(H)}(F^{-1}X,Y) \) in terms of \( \mod \Gamma \), for \( X,Y \in \add T \).

We first assume that \( T \) is an \( H \)-module. In this case \( \Hom_{D^b(H)}(F^{-1}T,T) \cong \Ext^2_B(DB,B) \), where \( B = \End_{D^b(H)}(T) \) (\[ 1 \], proof of Theorem 2.3). The top of this \( B-B \)-bimodule is generated as \( k \)-vector space by a minimal set of relations of \( B \) (\[ 1 \], 2.2 and 2.4). These relations correspond to relations between projective \( B \)-modules. Since projective \( B \)-modules are of the form \( \Hom_{D^b(H)}(T,T') = \Hom_H(T,T') \), such relations correspond to relations between indecomposable modules in \( \add T \).
By this we mean the following. Let \( T = T_1 \oplus \cdots \oplus T_n \) with the \( T_i \) indecomposable and pairwise non-isomorphic. Let now \( i \neq j \). We will consider maps \( f : T_i \to T_j \) which are irreducible in \( \text{add}T \), that is, the maps which do not factor through a module in \( \text{add}(T/(T_i \oplus T_j)) \). Let \( A(i,j) \) be the space \( \text{Hom}_H(T_i,T_j) \) modulo the maps which factor through \( \text{add}(T/(T_i \oplus T_j)) \). For each pair \((i,j)\) with \( i \neq j \), choose a set of irreducible maps in \( \text{add}T \) representing a basis of \( A(i,j) \), and let \( B \) be the union of all these bases. To each path of maps in \( B \) we associate the corresponding composition map in \( \text{mod}H \). A linear combination of such paths is a relation for \( \text{add}T \) if the corresponding map is zero in \( \text{mod}H \). A set \( R \) of such relations is a minimal set of relations for \( \text{add}T \) if \( R \) is a minimal set of generators of the ideal of relations for \( \text{add}T \). This means that for any relation \( g : T_r \to T_s \) we have \( g = \sum a_i \gamma_i \rho_i \gamma_i' \), with \( a_i \) in \( k, \rho_i \) in \( R \), and \( \gamma_i, \gamma_i' \) paths in \( \text{add}T \); and no proper subset of \( R \) has this property.

We will prove that a similar statement holds also when the \( \Gamma \)-module \( T \) is not an \( H \)-module. In this case we have to consider a minimal set of relations between indecomposable summands of \( T \) in \( \text{add}(T \oplus I^\Gamma_0(\Delta)) \).

Consider the following example. Let \( H = kQ \), where \( Q \) is the quiver \( 1 \to 2 \). Then \( \Gamma \) is the path algebra of the quiver \( 2' \to 1 \to 2 \). Let \( D \) be the fundamental domain of \( C_H \).

\[
\mathcal{D} : \quad \begin{array}{cccc}
1 & 1 & 2[1] & 2[1] \\
2 & \frac{1}{2} & 2' & 1 & \frac{1}{2} [1]
\end{array}
\]

and let \( T_1 = 2 \), \( T_2 = \frac{1}{2}[1] \), and \( T = T_1 \oplus T_2 \). Then \( T \) defines a cluster tilting object in \( C_H \) and is not an \( H \)-module. The \( \Gamma \)-module corresponding to \( T \) under the identification of \( D \) with \( \text{mod} \Gamma \) is \( 2 \oplus 2' \). Moreover, \( \text{Hom}_{C_H}(T_2,T_1) = \text{Hom}_{D^bH}(\tau T_2[-1],T_1) = \text{Hom}_{D^bH}(\tau \frac{1}{2},T_1) \neq 0 \), but there are no relations in \( \text{add}T \) from \( T_1 \) to \( T_2 \). However, \( 2 \to \frac{1}{2} \to 2' \) is a zero relation from \( T_1 \) to \( T_2 \) in \( \text{add}(T \oplus I) \), where \( I = \frac{1}{2} \frac{1}{2} \) is the injective envelope of the simple 2 in \( \text{mod} \Gamma \).

To study the general case we will define an appropriate hereditary algebra \( \hat{H} \), and use that the above mentioned result of \( \Pi \) holds for tilting \( \hat{H} \)-modules, to prove our desired result.

We start with defining a hereditary algebra \( \hat{H} \) such that there is an exact embedding \( G : \text{mod} \Gamma \to \text{mod}(\hat{H}) \) with the property that tilting \( \Gamma \)-modules map to tilting \( \hat{H} \)-modules.

We recall from Proposition \( 3.3 \) that \( U = DH \oplus \Gamma^\Gamma_I(\Delta) \) is a complete slice in \( \text{mod} \Gamma \). We consider another complete slice, \( \Sigma = \tau_1^{-1}DH \oplus \Gamma^\Gamma_I(\Delta) \). Then \( \hat{H} = (\text{End}_\Gamma(\Sigma))^{op} \) is a hereditary algebra of type \( \Sigma \). Let \( (\mathcal{T}, \mathcal{F}) \) be the split torsion pair in \( \text{mod} \Gamma \) of Corollary \( 3.6 \). Then \( \text{ind} \mathcal{F} \) coincides with the predecessors of \( \Sigma \). Also \( D\Sigma \) is a tilting \( \hat{H} \)-module, \( \Gamma = \text{End}_{\hat{H}}(D\Sigma)^{op} \) and the functors \( L = \text{Hom}_{\hat{H}}(D\Sigma, -) \) and \( \text{Ext}_{\hat{H}}^{1}(D\Sigma, -) : \text{mod} \hat{H} \to \)
mod $\Gamma$ induce equivalences $\mathcal{T}_{D\Sigma} \rightarrow \mathcal{F}$ and $\mathcal{F}_{D\Sigma} \rightarrow \mathcal{T}$, respectively, where $(\mathcal{T}_{D\Sigma}, \mathcal{F}_{D\Sigma})$ is the torsion pair associated to the tilting $\tilde{H}$-module $D\Sigma$.

Let $G = D\Sigma \otimes_{\Gamma} - : \text{mod } \Gamma \rightarrow \text{mod } \tilde{H}$. Then $L$ and $G$ are adjoint functors, and the restrictions $L|_{T_{D\Sigma}} : T_{D\Sigma} \rightarrow \mathcal{F}$ and $G|_{\mathcal{F}} : \mathcal{F} \rightarrow T_{D\Sigma}$ are inverse equivalences of categories. Moreover, $G(\Sigma) = D\tilde{H}$, because $D\tilde{H} \simeq GLD\tilde{H} = G\text{Hom}_{\tilde{H}}(D\Sigma, D\tilde{H}) \simeq G\text{Hom}_{\tilde{H}\text{op}}(\tilde{H}, \Sigma) \simeq G(\Sigma)$.

**Example 4.1.** We illustrate the situation for the hereditary algebra $H$ indicated below.

The Auslander Reiten quiver of $\Gamma$ is:

Here, $\Gamma_{\Sigma}$ is the sum of the five modules in frames, $T$ is given by the two modules inside dotted circles, $\mathcal{F}$ is indicated by the curve, and $\tilde{H}$ is the algebra with quiver:

The Auslander Reiten quiver of $\tilde{H}$ is
In this case \( H D\Sigma = 4 \oplus \frac{4}{3} \oplus \frac{3}{4} \oplus \frac{2}{3} \oplus \frac{3}{4} \oplus \frac{1}{4} \), \( F_{DS} \) consists of the two modules inside dotted circles, and \( T_{DS} \) consists of the 15 modules inside the regions indicated with curves.

We know from the Brenner-Butler theorem that \( F = \ker \text{Tor}^1_{\Gamma}(D\Sigma, -) \), so that \( G = D\Sigma \otimes_{\Gamma} - : \text{mod} \Gamma \to \text{mod} \bar{H} \) restricted to \( F \) is exact. In particular, \( G|_{\text{mod} \bar{H}} \) is exact. Thus \( G \) induces an embedding

\[
\hat{G} : D^b(\text{mod} \bar{H}) \to D^b(\text{mod} \bar{H})
\]

such that \( \hat{G}(M[i]) = (\hat{G}(M))[i] \).

**Proposition 4.2.** Let \( T \) be a tilting \( \Gamma \)-module. Then \( G(T) \) is a tilting \( \bar{H} \)-module.

**Proof.** Since \( \Gamma \) and \( \bar{H} \) have the same number of nonisomorphic simple modules, we only need to prove that \( \text{Ext}^1_{\bar{H}}(G(T), G(T)) = 0 \). Since \( T \in F \) because \( \text{pd} T \leq 1 \), it follows that \( G(T) \in T_{DS} \). Then \( \text{Ext}^1_{\Gamma}(\text{LG}(T), \text{LG}(T)) = \text{Ext}^1_{\bar{H}}(G(T), G(T)) \). Thus \( \text{Ext}^1_{\bar{H}}(G(T), G(T)) \cong \text{Ext}^1_{\Gamma}(T, T) = 0 \), since \( \text{LG}(T) \cong T \) because \( T \in F \).

We observe that in general the modules \( \tau(G(X)) \) and \( G(\tau X) \) are not isomorphic, for \( X \in \text{mod} \Gamma \). We will prove next that they are isomorphic when \( X = \tau^{-1}_\Gamma I \), for any injective \( H \)-module \( I \). We start with three lemmas.

**Lemma 4.3.** Let \( f : X \to Y \) be a morphism in \( \text{mod} \Gamma \), with \( Y \in F \), such that \( G(f) : GX \to GY \) is (minimal) right almost split in \( \text{mod} \bar{H} \). Then \( f \) is (minimal) right almost split in \( \text{mod} \Gamma \).

**Proof.** The key is that \( F \) is closed under predecessors and \( G|_F : F \to T_{DS} \) is an equivalence. From this we obtain right away that \( X \in F \), and \( f \) is not a split epimorphism since \( G(f) \) is not a split epimorphism. Now, let \( Z \) be an indecomposable \( \Gamma \)-module and \( h : Z \to Y \) a morphism which is not a split epimorphism. Again, \( Z \in F \) and \( G(h) \) is not an isomorphism. Hence there is a morphism \( g : G(Z) \to G(X) \) such that \( G(h) = G(f)g \). Using that \( G|_F : F \to T_{DS} \) is an equivalence once more, we deduce there is a \( g' : Z \to X \) such that \( G(g') = g \) and \( h = fg' \). Thus \( f \) is right almost split. The minimality is deduced in the same way. \( \square \)
Let again $\Sigma$ denote the complete slice in $\text{mod} \Gamma$ which consists of $\text{ind}(\tau_{\Gamma}^{-1} DH \oplus I_0^0(\text{soc}H))$.

**Lemma 4.4.** The indecomposable projective-injective modules (i.e. those in $\text{ind}(I_0^0(\text{soc}H))$) are sources of $\Sigma$.

**Proof.** Let $P \in \text{ind}(I_0(\text{soc}H))$ and let $f : X \to P$ be a nonzero non-isomorphism in $\text{ind} \Gamma$. Then $\text{Im} f \subseteq \text{rad} P$, which is an injective $H$-module. Thus $X \in \text{mod} H$, since $\text{mod} H$ is closed under predecessors in $\text{mod} \Gamma$. But then $X \notin \Sigma$. □

**Lemma 4.5.** Let $I$ be an indecomposable injective $\tilde{H}$-module, and let $M \to I$ be a minimal right almost split morphism in $\text{mod} \tilde{H}$. Then $M \in T_{D\Sigma}$.

**Proof.** Let $M'$ be an indecomposable direct summand of $M$, and assume $M' \notin T_{D\Sigma}$. Then $M'$ is not injective and there is an irreducible morphism $I \to \tau^{-1} M'$. Since $\tilde{H}$ is hereditary, then $\tau^{-1} M'$ is injective. Thus $\tau^{-1} M' \in T_{D\Sigma}$, and since $M' \notin T_{D\Sigma}$ we conclude that $\tau^{-1} M'$ is Ext-projective in $T_{D\Sigma}$. Therefore, there exists $N \in \text{ind} \Gamma$ such that $N$ is Ext-projective in $\mathcal{F}$ and $G N = \tau^{-1} M'$. Since $\mathcal{F}$ is closed under predecessors, then $N$ is projective in $\text{mod} \Gamma$. But we also have $N \in \text{add} \Sigma$, because $G$ maps $\Sigma$ to $D\tilde{H}$, as we observed before Example 4.1. Since $\tau^{-1} DH$ contains no nonzero projective direct summands, then $N \in \text{ind}(I_0(\text{soc}H))$, i.e. $N$ is projective-injective. By the preceding lemma, $N$ is a source in $\Sigma$. Thus $\tau^{-1} M' = G N$ is a source in $\text{ind}(D\tilde{H})$, which contradicts the already established existence of an irreducible morphism $I \to \tau^{-1} M'$. □

**Proposition 4.6.** Let $I$ be an indecomposable injective $H$-module. Then $G \tau_{\Gamma}^{-1} I = \tau_{\tilde{H}}^{-1} GI$.

**Proof.** Since $\tau_{\Gamma}^{-1} I \in \Sigma$, then $G \tau_{\Gamma}^{-1} I$ is $\tilde{H}$-injective. Let $f : M \to G \tau_{\Gamma}^{-1} I$ be a minimal right almost split morphism. By Lemma 4.3 we have $M \in T_{D\Sigma}$. Then there is a morphism $g : N \to \tau_{\Gamma}^{-1} I$ in $\text{mod} \Gamma$ with $N \in \mathcal{F}$, such that $f = G g$. By Lemma 4.3, $g$ is minimal right almost split in $\text{mod} \Gamma$. Then the almost split sequence $0 \to I \to N \xrightarrow{g} \tau_{\Gamma}^{-1} I \to 0$ is contained in $\mathcal{F}$, and applying $G$ we obtain an exact sequence $(\ast) \ 0 \to GI \to M \xrightarrow{f} G \tau_{\Gamma}^{-1} I \to 0$.

Since $f$ is minimal right almost split, then the sequence $(\ast)$ is almost split. Hence $GI = \tau_{\tilde{H}} G \tau_{\Gamma}^{-1} I$, and the result follows by applying $\tau_{\tilde{H}}^{-1}$. □

**Proposition 4.7.** Let $\hat{T}$ be a cluster tilting object in $C_H$ represented by $T$ in the fundamental domain $D_T$ of $C_H$, which we consider embedded in $\text{mod} \Gamma$ as before. Let $T_1$, $T_2$ be indecomposable summands of $T$. Then top $\text{Hom}_{D_T}(\hat{T}^{-1} T_1, T_2)$ is a vector space with basis given by a minimal set of relations from $T_2$ to $T_1$ in $\text{add}(T \oplus I_0^0(\Delta))$. 
Proof. As we observed above, the result holds for summands $T_1, T_2$ of $T$ which are $H$-modules. So we only need to consider the case when $T_1 \not\equiv \text{mod } H$, that is, $T_1 = \tau^{-1}_r I$, where $I$ is an indecomposable injective module in $\text{mod } H$. For if $T_1 \equiv \text{mod } H$ and $T_2 \not\equiv \text{mod } H$, we have $\text{Hom}(\tau T_1[-1], T_2) = 0$ since $T_2 = P_i[1]$ for $P_i$ indecomposable projective \cite{2}.

Then $\tau^{-1}_r I = \tau^{-1}_b(H) I = P[1]$ in $D^b(H)$, where $top P = soc I$, via our identification. Then, for $X \in \text{mod } H$ we have

$$\text{Hom}_{D^b(H)}(F^{-1}(\tau^{-1} I), X) = \text{Hom}_{D^b(H)}(I[-1], X) \simeq \text{Hom}_{D^b(H)}(\hat{G}(I[-1]), \hat{G}X) =$$

$$\text{Hom}_{D^b(H)}((GI)[1], GX) = \text{Hom}_{D^b(H)}(F^{-1}\tau^{-1}(GI), GX).$$

From Proposition \ref{4.6}, we know that $G\tau^{-1} I = \tau^{-1}GI$. Thus

$$\text{Hom}_{D^b(H)}(F^{-1}(\tau^{-1} I), X) \simeq \text{Hom}_{D^b(H)}((F^{-1}G(\tau^{-1} I)), GX).$$

Using Proposition \ref{4.6} again we can prove that this isomorphism induces an isomorphism between the corresponding tops.

Recall that $T \oplus I_0^b(\Delta)$ is a tilting $\Gamma$-module (Theorem \ref{3.10}) and therefore $G(T \oplus I_0^b(\Delta))$ is a tilting $\hat{H}$-module (Proposition \ref{3.2}).

Now consider $X = T_2$. Since both $G(\tau^{-1} I) = G(T_1)$ and $G(T_2)$ are modules over the hereditary algebra $\hat{H}$, we can apply \cite{2} to the tilting module $G(T \oplus I_0^b(\Delta))$ and conclude that $top \text{Hom}_{D^b(H)}((F^{-1}G(T_1), GT_2)$ has a basis in correspondence with a minimal set of relations from $G(T_2)$ to $G(T_1)$ in $\text{add}(G(T \oplus I_0^b(\Delta))$. Since $G|_{\text{add}(T \oplus I_0^b(\Delta))}$ is an equivalence of categories, we obtain minimal relations as stated. \hfill \Box

Let $T$ and $\hat{T}$ be as in the previous proposition. We are now in the position to describe the ordinary quiver $Q_C$ of the cluster-tilted algebra $C = \text{End}_{\hat{H}}(\hat{T})$, in terms of $\text{mod } \Gamma$.

**Theorem 4.8.** Let $C = \text{End}_{\hat{H}}(\hat{T})$, where $\hat{T}$ is a basic cluster-tilting object in $C_H$ represented by $T = \bigoplus T_i$ in $\text{mod } \Gamma$, with $T_i$ indecomposable. Let $B = \text{End}_{\Gamma}(T)$, and let $i$ denote the vertex of $Q_C$ associated to $\text{Hom}_\Gamma(T, T_1)$. Then, for vertices $i, j$ of $Q_C$ the number of arrows from $i$ to $j$ is equal to the number of arrows from $i$ to $j$ in $Q_B$ plus the cardinality of a minimal set of relations from $T_i$ to $T_j$ in $\text{add}(T \oplus I_0^b(\Delta)) \subset \text{mod } \Gamma$.

**Proof.** The number of arrows from $i$ to $j$ equals $\dim \text{top } \text{Hom}_{C_H}(T_j, T_i) = \dim \text{top } \text{Hom}_{D^b(H)}(T_j, T_i) \oplus \dim \text{top } \text{Hom}_{D^b(H)}(F^{-1}T_j, T_i)$. Now the result follows from the previous proposition and the fact that $\dim \text{top } \text{Hom}_{D^b(H)}(T_j, T_i)$ is equal to the number of arrows from $i$ to $j$ in $Q_B$, because $\text{Hom}_{D^b(H)}(T_j, T_i) \simeq \text{Hom}_\Gamma(T_j, T_i)$, by Proposition \ref{3.12} (b).

**Remark 4.9.** In the above statement, for each pair of vertices $i$ and $j$, only one of the summands describing the number of arrows from $i$ to $j$ is nonzero.
Example 4.10. For the hereditary algebra $H$ given below we indicate the corresponding algebra $\Gamma$.

\[
H: \quad 1 \rightarrow 2 \rightarrow 3
\]

\[
\Gamma: \quad 1 \rightarrow 3
\]

Then the AR-quiver of $\Gamma$ is:

\[
\begin{array}{c}
1 \\
3 \\
2 \\
1
\end{array}
\]

and the fundamental domain of $\mathcal{C}_H$ corresponds to the region enclosed by the curve. Let $T = \frac{2}{3} \oplus 2 \oplus \frac{3'}{2}$. Then $T \oplus \frac{3'}{3}$ is a tilting $\Gamma$-module, so $T$ defines a cluster tilting object $\mathcal{T}$ in $\mathcal{C}_H$.

We notice that nonzero maps $\frac{2}{3} \rightarrow \frac{3'}{1} \rightarrow \frac{3'}{2}$, or $\frac{2}{3} \rightarrow 2 \rightarrow \frac{3'}{2}$ have always nonzero composition. However, there are nonzero maps $\frac{2}{3} \rightarrow \frac{3'}{1} \oplus 2 \rightarrow \frac{3'}{2}$ with zero composition, and this relation from $\frac{2}{3}$ to $\frac{3'}{2}$ is unique, up to scalar multiples. Therefore $\dim \text{Hom}_{\mathcal{C}_H}(\frac{3'}{2}, \frac{2}{3}) = 1$.

Since $\dim \text{Hom}_{\Gamma}(\frac{2}{3}, 2) = 1$, and $\dim \text{Hom}_{\Gamma}(2, \frac{3'}{2}) = 1$, the ordinary quiver of the cluster tilted algebra $\text{End}_{\mathcal{C}_H}(\mathcal{T})$ is

\[
\begin{array}{c}
1 \\
3 \\
2
\end{array}
\]

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