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Multi-stage Stern–Gerlach experiment modeled

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Abstract
In the classic multi-stage Stern–Gerlach experiment conducted by Frisch and Segrè, the Majorana (Landau–Zener) and Rabi formulae diverge far from the experimental observation while the physical mechanism for electron-spin collapse remains unidentified. Here, introducing the physical co-quantum concept provides a plausible physical mechanism and predicts the experimental observation in absolute units without fitting (i.e. no parameters adjusted), with a p-value less than one per million, which is the probability that the co-quantum theory happens to match the experimental observation purely by chance. Further, the co-quantum concept is corroborated by exactly statistically reproducing the wave function, density operator, and uncertainty relation for electron spin in Stern–Gerlach experiments.

Supplementary material for this article is available online

Keywords: Stern–Gerlach experiment, electron spin, Majorana formula, Landau–Zener formula, co-quantum dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Performed three years before the successful development of quantum mechanics, the 1922 Stern–Gerlach experiment on silver atoms [1] quickly proved fundamental to quantum physics [2, 3]. The benchmark experiment led to the quantization of all angular momenta, discovery of electron spin, study of the measurement problem and superposition, direct investigation of the ground-state properties of atoms without electronic excitation, and selection of fully spin-polarized atoms [2]. Within a few weeks, Einstein and Ehrenfest concluded that spin collapse cannot be interpreted by radiation, which would take 100 years [4]. Recently, Wennerström and Westlund numerically simulated that relaxation of 1 μs qualitatively reproduced the double branched collapse pattern [5], and Norsen interpreted spin collapse using the de Broglie–Bohm pilot-wave theory [6]. The significance of the Stern–Gerlach experiment and relevant works are detailed in a 2016 inspiring review [2], concluding that ‘The physical mechanism responsible for the alignment of the silver atoms remained and remains a mystery’ and quoting Feynman, ‘… instead of trying to give you a theoretical explanation, we will just say that you are stuck with the result of this experiment …’ [7].

Immediately, Heisenberg and Einstein proposed multi-stage Stern–Gerlach experiments to explore deeper mysteries of directional quantization [2]. Ten years later, Phipps and Stern reported the first effort [8], which was unfortunately discontinued owing to Phipps’ involuntary return to the US [2]. A year later, Frisch and Segrè modified the same apparatus by adopting Einstein’s suggestion on the use of...
a single wire instead of three electromagnets to rotate spin; they also improved magnetic shielding, slit filtering, and signal detection [2]. Despite the use of three layers of magnetic shielding for the middle stage (i.e. the inner rotation or IR chamber), the remnant or residual fringe magnetic field was still $0.42 \times 10^{-4}$ T (or 0.42 G). Rather than fight the fringe magnetic field further, they took advantage of it. The magnetic field from the wire in the middle stage cancels the remnant field to produce a magnetic null point, around which the field is approximated as a magnetic quadrupole; consequently, they successfully observed nonadiabatic spin flip [9]. Note that only the magnetic field near a null point is effective for nonadiabatic spin flip; thus, the field far from a null point does not significantly affect transition, and its detailed distribution is of little import. Frisch and Segrè varied the wire current, which is the only independent variable controlled here, over nearly two orders of magnitude approximately uniformly on a logarithmic scale to observe the peak fraction of spin flip and its entire range. They started and ended with sufficiently extreme currents that yielded negligible fractions of spin flip. Having reached a nearly zero fraction of spin flip at the highest current might be the reason that they ceased increasing the current further. Further, the calculation of the fraction automatically obviates the requirement for absolute calibration. This data set suggests they designed and executed the experiment with great care.

Frisch and Segrè found that their observation [9] unexpectedly diverges from the Majorana formula (figure 1) [10, 11], which was stimulated by the experiment of Frisch and Segrè. The Majorana formula is a variant of the Landau–Zener formula, which is better-known despite the concurrent publications of all four related papers in the same year [12–14]. For a historical comparison of the four papers, please refer to [15]. Fermi suggested that interaction among atoms could be responsible for the divergence, but atoms were sufficiently sparse to be treated independently [9]. Rabi acknowledged ‘Professor E. Segrè for discussions on the details of the Frisch and Segrè experiment’, recognized the role of the nuclear magnetic moment, and revised the Majorana formula through hyperfine coupling [16]. Rabi’s revised formula, however, did not overcome the divergence (figure 1).

Multi-stage Stern–Gerlach (Frisch–Segrè) experiments are much more difficult to model than single-stage ones. Multiple stages produce far more nuanced observation because the middle stage can vary the electron spin orientation over a wide range after polarization by the first stage. A correct single-stage theory must pass the more stringent test of the multi-stage experiment. This spin-flip divergence in multi-stage Stern–Gerlach experiments remains unresolved [2]. One may only speculate why the 1933 discrepancy [9] has not been resolved. The seminal paper has not been republished in English, which might have limited its visibility.

Here, a theory, called co-quantum dynamics (CQD) [17], is presented to both provide a collapse mechanism and predict the Frisch–Segrè experimental observation (figure 1) [9]. CQD is theoretically verified by reproducing, for electron spin in Stern–Gerlach experiments, the quantum mechanical wave function, density operator, and uncertainty relation as well as, in a recent publication [17], the Schrödinger–Pauli equation. In section 2, CQD is presented in three subsections, including the equations of motion, branching condition, and pre-collapse state function and prediction expression. In section 3, Stern–Gerlach experiments in both single and multiple stages are modeled. For flow continuity, lengthy interpretations are postponed to section 4, and detailed mathematical derivations are presented in Appendices (supplement material). The CQD derivation of the uncertainty relation is deferred to the last appendix.

The following table (table 1) compares briefly CQD with the representative existing quantum mechanical theories for collapse [18], e.g. the Ghirardi–Rimini–Weber model [19] and continuous spontaneous localization model [20, 21]. CQD, based on the classical Bloch equation (or its Landau–Lifshitz–Gilbert derivative) and the two postulates, provides a physical instead of phenomenological mechanism for electron spin collapse. In the presence of an external magnetic field, the nuclear magnetic moment is responsible for the collapse of electron spin. The absence of fitting with any adjustable parameters and the high coefficient of determination $R^2$ (or high correlation coefficient) led to the small $p$-value ($p < 8 \times 10^{-7}$) [22, 23]. In general, fitting with more and more adjustable parameters, one may improve $R^2$ towards unity. While $R^2$ is not penalized for the number of adjustable parameters used relative to the number of experimental data points available, the $p$-value is. Therefore, one may achieve an arbitrarily high $R^2$ at the expense of the $p$-value. The $p$-value is an objective measure of agreement between a theory and the experiment. As a standard definition, the $p$-value quantifies the probability of observing results at least as extreme as the ones observed given that the null hypothesis is true. For stringent discoveries, high-energy physics,
for example, requires $p < 3 \times 10^{-7}$, which corresponds to $5\sigma$ [24]. The Laser Interferometer Gravitational-wave Observatory (LIGO) observation of gravitational waves applied a similar criterion [25]. The agreement of CQD with the experiment is at a similar level as well. While the LIGO observed a chirp signal, which is common in various forms in nature, the Frisch–Segrè experimental data follow an uncommon shape, which is even more unlikely to be matched by random chance. Therefore, the value of $p < 8 \times 10^{-7}$ claims a statistical significance that cannot be ignored objectively. The probability that CQD happens to match the experimental observation so well purely by chance is less than one in a million. It is even less likely for an incorrect theory to match an incorrect experiment by chance if one doubts the Frisch–Segrè experimental data. Because the Majorana or Rabi formula, if correct, follows a monotonic trend, it would be difficult to fathom that some experimental imperfections caused the fraction of spin flip to increase at low currents and to decrease at high currents. Matching a theory with the experiment so well without using any adjustable parameters inspires conviction. Further, CQD is corroborated by statistically reproducing exactly the wave function, density operator, and uncertainty relation for electron spin. This corroboration may be considered supporting evidence because an incorrect theory would highly unlikely be able to reproduce so many fundamental aspects of quantum mechanics.

### 2. Methods

#### 2.1. CQD equations of motion

In classical electrodynamics, the motion of a magnetic dipole moment, $\hat{\mu}$, is described by the Bloch equation,

$$\frac{d\hat{\mu}}{dt} = \gamma \hat{\mu} \times \vec{B},$$

where caret denotes a unit vector, $t$ time, $\gamma$ the gyromagnetic ratio, and $\vec{B}$ the magnetic flux density. Majorana stated that both the classical and the quantum-mechanical treatments on spin flip require integration of the same differential equations [10, 11]. It is known that the Schrödinger or von Neumann equation for a unitary two-level system can be converted to the Bloch equation or its analog [7, 26, 27].

We now extend the Bloch equation to the Landau–Lifshitz–Gilbert equation [28],

$$\frac{d\hat{\mu}}{dt} = \gamma \hat{\mu} \times \vec{B} - k_i \hat{\mu} \times \frac{d\hat{\mu}}{dt}$$

where the dimensionless $k_i$ is called the induction factor here. Although this equation was originally intended for condensed matter, the underlying physical mechanism for the added term is compatible with CQD (see paragraph 1 in section 4). In fact, the author had developed CQD before realizing its connection with the Landau–Lifshitz–Gilbert equation. If $k_i = 0$, the Bloch equation is recovered.

Henceforth, subscripted $e$ and $n$ denote electron and nucleus, respectively. The default atom, to match the Frisch–Segrè experiment [9], is potassium ($^{39}$K). The scope of the manuscript is limited to potassium in the Stern–Gerlach or Frisch–Segrè experiment.

The torque-averaged magnetic flux densities from $\hat{\mu}_n$ and $\hat{\mu}_e$ applied on each other are respectively (appendix 1)

$$\vec{B}_n = \frac{5\mu_0}{16\pi R^3} \vec{B}_n$$

and

$$\vec{B}_e = \frac{5\mu_0}{16\pi R^3} \vec{B}_e,$$

where $\mu_0$ is the vacuum permeability ($4\pi \times 10^{-7}$ H m$^{-1}$) and $R$ is the van der Waals atomic radius ($2.75 \times 10^{-10}$ m) [29].

Chieflly because the nucleus is more massive, $\mu_e$ ($9.285 \times 10^{-24}$ J T$^{-1}$)$ \gg \mu_n$ ($1.977 \times 10^{-27}$ J T$^{-1}$); thus, $B_e$ ($558.1 \times 10^{-4}$ T)$ \gg B_n$ ($0.119 \times 10^{-4}$ T), where $10^{-4}$ T = 1 Gauss. CQD refers to $\hat{\mu}_e$ as the principal quantum and $\hat{\mu}_n$ in the same atom as the co-quantum. Postulate 1 states that induction between the electron and the nucleus tends to increase $|\theta_e - \theta_n|$, where $\theta$ denotes the polar angle relative to the quantization axis (see paragraph 1 in section 4). We (1) apply the Landau–Lifshitz–Gilbert equation to both $\hat{\mu}_e$ and $\hat{\mu}_n$, (2) express the unit vectors in spherical coordinates, and (3) revise...
the signs of the induction terms to implement the above postulate, leading to the following CQD equations of motion (appendix 2):

\[
\dot{\theta}_e = -\gamma_e \left[ B_e \cos \phi_e + B_n \sin \theta_e \sin (\phi_n - \phi_e) \right] - \text{sgn} (\theta_n - \theta_e) k_i |\dot{\phi}_e| \sin \theta_e,
\]

\[
\dot{\theta}_n = -\gamma_n \left[ B_e \cos \phi_n + B_n \sin \theta_n \sin (\phi_n - \phi_n) \right] - \text{sgn} (\theta_n - \theta_n) k_i |\dot{\phi}_n| \sin \theta_n,
\]

\[
\dot{\phi}_e = -\gamma_e \left[ B_e + B_n \cos \theta_n - \cot \theta_e \left[ B_e \sin \phi_e + B_n \sin \theta_n \cos (\phi_n - \phi_e) \right] \right] - \frac{\text{sgn} \dot{\phi}_e k_i \dot{\theta}_e}{\sin \theta_e},
\]

and

\[
\dot{\phi}_n = -\gamma_n \left[ B_e + B_n \cos \theta_n - \cot \theta_n \left[ B_e \sin \phi_n + B_n \sin \theta_n \cos (\phi_n - \phi_n) \right] \right] - \frac{\text{sgn} \dot{\phi}_n k_i \dot{\theta}_n}{\sin \theta_n}.
\]

Here, \( \phi \) denotes the azimuthal angle; \( B_e \) and \( B_n \) represent, respectively, the \( y \) (axis of the atomic beam) and \( z \) components of the external magnetic flux densities; \( B_0 \) is neglected for brevity; \( \text{sgn} \) denotes the sign function. When \( \theta_e = 0 \) or \( \pi \), equation (7) is replaced with \( \dot{\phi}_e = 0 \); when \( \theta_n = 0 \) or \( \pi \), equation (8) is replaced with \( \dot{\phi}_n = 0 \). Primarily because the nucleus is more massive again, \( \gamma_n (-1.761 \times 10^{11} \text{ rad Hz}^{-1}) \) in absolute value is four orders of magnitude greater than \( \gamma_n (1.250 \times 10^7 \text{ rad Hz}^{-1}) \). If \( B_n = 0 \) and \( k_i = 0 \), equations (5) and (7) reduce to the equations shown by Majorana [10, 11].

### 2.2. CQD branching condition

Postulate 2 states that the polar angle of the co-quantum, \( \theta_n \), varies negligibly \((\ll \pi)\) during flight in typical Stern–Gerlach experiments, where the duration is too short for the co-quantum to collapse (see paragraph 2 in section 4). The external main field, \( B_0 \), along the \( z \) axis is usually much stronger than \( B_e \) and \( B_n \). While the fast motion of \( \dot{\mu}_e \) is precession about the main field, the secondary motion is collapse due to the induction term, which yields the following trend from equation (5):

\[
\tan \frac{\theta_e (t)}{2} = \tan \frac{\theta_e (0)}{2} \exp \left[ -\text{sgn} (\theta_n - \theta_e) k_i |\Delta \phi_e (t)| \right].
\]

Here, \( \Delta \phi_e \) denotes the traversed azimuthal angle (i.e. unwrapped phase). If the Larmor frequency of the electron magnetic moment \( \omega_e \) is constant, we simply have \( \Delta \phi_e = \omega_e t \). As time evolves, \( \theta_n \) approaches either 0 or \( \pi \) according to the following branching condition:

\[
\text{sgn} (\theta_n - \theta_e) = \begin{cases} 
1 & \text{if } \theta_n > \theta_e, \\
0 & \text{if } \theta_n = \theta_e, \\
-1 & \text{else}.
\end{cases}
\]

Therefore, \( \dot{\mu}_e \) collapses to either \( +z \) or \( -z \) while precessing about \( B_0 \), depending on the polar angle of the co-quantum \( \theta_n \) relative to \( \theta_e \) (figure 2).

The number of precession cycles required to vary \( \tan (\theta_e/2) \) by a factor of \( e \) is given by

\[
N_e = \frac{1}{2\pi k_i}
\]

regardless of the strength of the external magnetic field. For a constant Larmor frequency, \( \omega_e \), the collapse time constant is

\[
T_e = N_e \frac{2\pi}{|\omega_e|} = \frac{1}{k_i |\omega_e|}.
\]

### 2.3. CQD pre-collapse state function and CQD prediction expression

The CQD pre-collapse state function is denoted by \( |\dot{\mu}_e \otimes \dot{\mu}_n> \), where the co-quantum, \( \dot{\mu}_n \), is prefixed with \( \otimes \) for clarity. \( |\dot{\mu}_e \otimes \dot{\mu}_n> \) represents \( \dot{\mu}_e \) accompanied with \( \dot{\mu}_n \), both governed by the CQD equations of motion. The CQD prediction expression for Stern–Gerlach experiments is written as

\[
|\dot{\mu}_e \otimes \dot{\mu}_n> = C_+ (\dot{\mu}_e, \dot{\mu}_n) |+z> + C_- (\dot{\mu}_e, \dot{\mu}_n) \exp (i\phi_e) |-z>.
\]

The equal sign functions as a right arrow \((\rightarrow)\) because the right side predicts the measurement outcome. A given \( \dot{\mu}_e \) collapses to either \(+z\) or \(-z\) according to the branching condition (equation (10)). The two real and positive \( C \) coefficients take on mutually exclusive binary values while \( \exp (i\phi_e) \) captures the phase information. If \( \theta_n > \theta_e \), then \( C_+ = 1 \) and \( C_- = 0 \); if \( \theta_n < \theta_e \), then \( C_+ = 0 \) and \( C_- = 1 \). In either case, \( C_+ \cdot C_- = 0 \) and \( C_+ + C_- = 1 \).
Figure 3. Multi-stage Stern–Gerlach (SG) experiment conducted by Frisch and Segrè [9]. The atomic beam from the oven is sent through (1) Stage SG1 to collapse $\hat{\mu}_e$ (principal quantum), (2) the magnetically shielded inner rotation (IR) chamber to rotate $\hat{\mu}_e$, (3) a slit (not shown) to select a branch, and (4) Stage SG2 to measure the fraction of spin flip. The red solid line and filled circle represent the current-carrying wire, and the gray sphere in cutaway view represents magnetic shielding. Inset (a) Angular distributions of $\hat{\mu}_n$ (co-quanta) before and after Stage SG1. Inset (b) Magnetic field lines within the IR chamber; NP: null point, formed by the cancelation of the magnetic field from the wire by the vertical remnant (residual) fringe magnetic field. Here, the vertical distance of the atomic beam from the center of the wire, $z_a = 1.05 \times 10^{-4}$ m; the most likely speed of atoms, $v = 800 \text{ m s}^{-1}$; the uniformly distributed remnant (residual) fringe magnetic flux density, $B_r = 0.42 \times 10^{-4} \text{T}$, which is parallel with the $+z$ axis (up in the figure); and the current carried by the wire, $I$, points along the $-x$ axis (into the screen).

3. Results
3.1. Single-stage Stern–Gerlach experiment

To describe the angular distribution of $\hat{\mu}_e$ or $\hat{\mu}_n$ in an ensemble of atoms, we define the angular probability density function, $p(\theta, \phi)$, as the probability of $\hat{\mu}$ pointing to the vicinity of $(\theta, \phi)$ per unit infinitesimal solid angle, with the following normalization:

$$\int_0^\pi \int_0^{2\pi} p(\theta, \phi) \sin \theta d\phi d\theta = 1.$$  
(14)

If the azimuthal distribution is isotropic, the integral reduces to

$$\int_0^\pi p(\theta, \phi) 2\pi \sin \theta d\theta = 1.$$  

The angular distribution of $\hat{\mu}_n$ for atoms immediately out of the oven is presumed to be isotropic as given by (figure 3, inset a, dashed circle)

$$p_{n0}(\theta_n, \phi_n) = \frac{1}{4\pi}.$$  
(15)

In a single-stage Stern–Gerlach experiment (figure 3, SG1), the probabilities of collapse for a given $\theta_e$ are related to the binary coefficients through ensemble averaging of the pre-averaging density operator defined in appendix 3 (equation (70)) over $p_{n0}$. The outcome is summarized as

$$\langle C^+ \rangle_n^2 = \int_{\theta_e}^\pi p_{n0} 2\pi \sin \theta_n d\theta_n = \cos^2 \frac{\theta_e}{2}$$  
(16)

and

$$\langle C^- \rangle_n^2 = \int_0^{\theta_e} p_{n0} 2\pi \sin \theta_n d\theta_n = \sin^2 \frac{\theta_e}{2}.$$  
(17)

The angle brackets, with the subscripts denoting nuclear, represent ensemble averaging with the integration limits determined by the branching condition (equation (10)). The two probabilities are proportional to the solid angles formed by the down and up sides of the cone shaped by the initial Bloch vector [15] precessing over one cycle. Each solid angle determines the probability of having the co-quantum on the corresponding side of the cone.

From equations (16) and (17), the pre-collapse state function (equation (13)) averages to the following familiar
quantum mechanical wave function for a pure state (appendix 3):

\[ |\hat{\mu}_\epsilon \rangle = \cos \frac{\theta_\epsilon}{2} |+z\rangle + \sin \frac{\theta_\epsilon}{2} \exp(i\phi_\epsilon) |-z\rangle. \quad (18) \]

If \( \hat{\mu}_\epsilon \) is also isotropically distributed as

\[ p_{\epsilon 0}(\theta_\epsilon, \phi_\epsilon) = \frac{1}{4\pi}, \quad (19) \]

the probabilities of collapse are predicted by averaging equations (16) and (17) over \( p_{\epsilon 0} \) (appendix 3):

\[ \langle C_+ \rangle_{n,\epsilon}^3 = \int_0^\pi \cos^2 \frac{\theta_\epsilon}{2} p_{\epsilon 0} 2\pi \sin \theta_\epsilon d\theta_\epsilon = \frac{1}{2}; \quad (20) \]

and

\[ \langle C_- \rangle_{n,\epsilon}^3 = \int_0^\pi \sin^2 \frac{\theta_\epsilon}{2} p_{\epsilon 0} 2\pi \sin \theta_\epsilon d\theta_\epsilon = \frac{1}{2}. \quad (21) \]

The \( \epsilon \) subscripts denote electron. The outcomes agree with the familiar quantum mechanical prediction for a maximally mixed state of atoms immediately out of the oven, represented by a density operator (equation (87), appendix 3).

3.2. Multi-stage Stern–Gerlach experiment

In the multi-stage Stern–Gerlach experiment conducted by Frisch and Segrè (figure 3) [9], Stage SG1 collapses \( \hat{\mu}_\epsilon \) into two branches. The inner rotation (IR) chamber rotates \( \hat{\mu}_\epsilon \) by an angle of \( \alpha_\epsilon \) using the magnetic field shown in inset b. A slit (not shown) selects one branch: the \( +z \) branch is chosen here. Stage SG2 collapses \( \hat{\mu}_\epsilon \) and measures the fraction of spin flip. Therefore, Stage SG1 serves as a polarizer, the IR chamber a rotator, and Stage SG2 an analyzer.

The probability of spin flip has been predicted [10, 11] by quantum mechanics as (see equation (17), set \( \theta_\epsilon = \alpha_\epsilon \))

\[ W_{qm} = (-z\alpha_\epsilon)^2 = \sin^2 \frac{\alpha_\epsilon}{2}, \quad (22) \]

which leads to the following Majorana formula (appendix 4, equation (117)) [10, 11]:

\[ W_m = \exp \left(-\frac{\pi z_a}{4\gamma e B_0} \right). \quad (23) \]

Here, \( z_a \) is the vertical distance of the atomic beam from the center of the wire, and \( v \) is the most likely speed of the atoms. The spin flip is because \( B_0 \) vanishes and reverses its sign near the null point (figure 3, inset b). Because \( B_0 \) is inversely proportional to the current carried by the wire, \( I \) (equations (92) and (94) in appendix 4), the Majorana formula predicts a probability of spin flip approaching 100% with increasing currents (figure 4, Curve m), i.e. as \( B_0 \to 0 \), \( W_m \to 1 \); yet, the experimental outcome decreases to nearly zero after peaking at 31% (figure 4, circles) [9]. Consequently, \( W_m \) yields a negative coefficient of determination (\( R^2 \)). Using the dimensionless adiabaticity parameter \( k_m \) (equation (103) in appendix 4), one can express the above equation concisely as \( W_m = \exp(-\pi k_m/2) \) (equation (116)). Rabi revised the Majorana formula to \( W_m^{1/4} \) [16], which, however, overestimates the starting points, underestimates the peak, and continues to diverge thereafter; as a result, the \( R^2 \) remains negative (figure 1).

In CQD, Stage SG1 varies \( \theta_n \) negligibly according to Postulate 2. However, polarization selection by the slit reshapes the co-quantum angular distribution from the original isotropic \( p_{\epsilon 0} \) (equation (15)) to

\[ p_{\epsilon 1}(\theta_n, \phi_n) = p_{\epsilon 0}(\theta_n, \phi_n) \cdot \frac{\theta_n}{\int_0^{\pi} p_{\epsilon 0} 2\pi \sin \theta_\epsilon d\theta_\epsilon} = \frac{1 - \cos(\theta_n)}{4\pi}. \quad (24) \]

Here, the pre-factor 2 compensates for the overall slit rejection of the opposite polarization (equation (20)), \( p_{\epsilon 0} \) is given by equation (19), and the integration limits are based on the branching condition (equation (10)). Because atoms with smaller \( \theta_n \) are deflected to the blocked \( -z \) branch with greater probabilities, \( p_{\epsilon 1} \) forms a heart shape (figure 3, inset a, solid line; paragraph 3 in section 4).

The heart shape is assumed to be approximately maintained throughout the IR chamber owing to the extension of Postulate
2 (see paragraph 2 in section 4). The co-quantas engender the following four effects on the principal quanta.

First, the probability of spin flip is derived by ensemble averaging over \( p_{\alpha 1} \) instead of \( p_{\alpha 0} \) (equation (89) with \( \theta_c = \alpha_r \) in appendix 3):

\[
W_{\text{eqd}} = (-2i\alpha_r)^2 = \int_{p_{\alpha 1}}^{2\pi} \sin \theta_x d\theta_y = \sin^4 \left( \frac{\alpha_r}{2} \right),
\]

which equals \( W^2_{\text{eqd}} \) (equation (22)). As shown by Curve 1 in figure 4, simply squaring \( W_m \) (equation (23)) already brings the solution much closer to the observation at low currents, but with an overcorrection near \( I = 0.03 \) A. This squaring effect evolves the probability of spin flip from \( W_m \) to

\[
W_1 = W^2_m = \exp \left( -\frac{\pi z_a}{c} |\gamma_e| B_y \right),
\]

where \( B_y \) is computed from the remnant (residual) fringe magnetic flux density, \( B_z \), using equations (92) and (94) in appendix 4. Using the dimensionless adiabaticity parameter \( k_m \) (equation (103)), one can express the above equation concisely as

\[
W_1 = \exp \left( -\pi k_m \right)
\]

(see equation (154) in appendix 5).

Second, the \( z \) component of \( \vec{B}_n \) (equation 3), represented by \( B_n \cos(\theta_n) \), offsets the upward \( B_z \). We substitute \( B_x + B_n \cos(\theta_n) \) (equation (119) in appendix 5) for \( B_z \) to update \( B_x \) to \( B'_x \) (equation (122)). The heart shape (equation (24)) yields \( (\theta'_n) = 5\pi/8 \) (equation (118)). The magnitude of \( B_n \cos(\theta_n) = -0.045 \times 10^{-4} \) T exceeds 10% of \( B_z (0.42 \times 10^{-4} \) T), producing an appreciable remnant-alteration effect. As shown by Curve 2 in figure 4, the corrected curve passes through the first two data circles and grazes the third one. If the co-quantum distribution were isotropic, \( \langle \theta_n \rangle \) would be \( \pi/2 \); \( B_n \cos(\theta_n) \) would vanish, so would the remnant-alteration effect. Effect 2 evolves \( W_1 \) to (see equation (136))

\[
W_2 = \exp \left( -\frac{\pi z_a}{c} |\gamma_e| B'_x \right),
\]

where \( B'_x \), however, is computed using \( B_z + B_n \cos(\theta_n) \) instead of \( B_z \). Using the dimensionless adiabaticity parameters \( k_0 \) (equation (125)), one can express the above equation concisely as

\[
W_2 = \exp \left( -\pi k_0 \right)
\]

(see equation (153)).

Third, the co-quantum saturate the rotation. As shown by equation (27), \( W_2 \) increases with decreasing \( B'_x \). However, the weakness of \( B'_x \) is spoiled by the transverse (xy) component of \( \vec{B}_n \), denoted by \( B_n \sin(\theta_n) \). Substitution of

\[
\sqrt{B'_x^2 + (B_n \sin(\theta_n))^2}
\]

for \( B'_x \) (see equation (141)) evolves \( W_2 \) to

\[
W_3 = \exp \left( -\frac{\pi z_a}{c} |\gamma_e| \sqrt{B'_x^2 + (B_n \sin(\theta_n))^2} \right).
\]

Using the dimensionless adiabaticity parameters \( k_0 \) (equation (125)) and \( k_1 \) (equation (126)), one can express the above equation concisely as

\[
W_3 = \exp \left( -\pi \sqrt{k_0^2 + k_1^2} \right)
\]

(see equation (152)).

As shown by Curve 3 in figure 4, this rotation-saturation effect clamps the overshoot in Curve 2. The clamped curve passes through the first four data circles. The current is divided into two regions at \( 0.067 \) A, where \( B'_1 = B_n \sin(\theta_n) = 0.11 \times 10^{-4} \) T. At low currents before the fourth data point \( I = 0.05 \) A and \( B'_1 = 0.15 \times 10^{-4} \) T, \( B'_1 \) is greater than \( B_n \sin(\theta_n) \); at high currents, conversely, \( B_n \sin(\theta_n) \) becomes dominant and saturates the curve. If the co-quantum distribution were isotropic, then \( (\theta_n) = \pi/2 \); consequently, both the squaring and remnant-alteration effects (effects 1 and 2) would vanish. In this case, the rotation-saturation effect (effect 3) alone could not bring the Majorana solution down sufficiently in the low-current region because as the current decreases \( B'_1 \) increasingly overpowers \( B_n \); thus, the effect of the co-quantum would become negligible.

Combining the three effects, CQD accurately predicts the low-current observation in absolute units without fitting (i.e. no parameters are adjusted). The coefficient of determination \( R^2 \) for the low-current regime reaches 0.9495 as computed using the natural logarithm of the fractions of flip to suppress the exponential dependence (equation (28)). Therefore, effecting the three modifications to the Majorana formula has already shown evidence for the existence of both the co-quantum and the derived heart-shaped distribution.

Fourth, in the high-current regime, the precession of \( \vec{B}_n \) generates substantial nuclear-resonant rotation, due to precession resonance between \( \vec{\mu}_s \) and \( \vec{\mu}_e \) when their Larmor frequencies are matched (see appendix 5 for details). This effect evolves \( W_3 \) to (equation (167) in appendix 5)

\[
W_4 = W_3 \exp \left( -c_1 \vec{F} \right),
\]

where the resonant-rotation coefficient, \( c_{11} \), is given by equation (163). The fraction of spin flip peaks near \( I = 0.1 \) A, where \( B'_1 = 0.074 \times 10^{-4} \) T, comparable to but less than \( B_n \sin(\theta_n) = 0.11 \times 10^{-4} \) T. As shown by Curve 4 in figure 4, this effect increases with the current and bends down Curve 3. At the maximum current \( (I = 0.5 \) A), \( B'_1 = 0.015 \times 10^{-4} \) T, far less than \( B_n \sin(\theta_n) \); the fraction of spin flip decreases to nearly zero. Expression of the above equation based on dimensionless parameters is discussed below equation (150). Using the dimensionless adiabaticity parameters \( k_0 \) (equation (125)) and \( k_1 \) (equation (126)) as well as \( f_{11} \) (equation (146)), one can express the above equation concisely as

\[
W_4 = \exp \left( -\pi \sqrt{k_0^2 + k_1^2} - \frac{1}{2} \pi k_1 \right) \]
distribution. In comparison, without taking the logarithm of the fractions of flip, \( R^2 \) is computed to be 0.9621.

Thus far, we have set the induction factor \( k_i = 0 \) for the flight in the IR chamber, owing to the low field (appendix 5). Including \( k_i \) yields the following combined probability of spin flip (appendix 5, equation (160)):

\[
W_{\text{cqd}} = \exp \left[ -\sqrt{(c_{j0}/I)^2 + c_{r2}^2 - c_{r1}I^2 - c_{c2}I} \right],
\]

where \( c_{j0}, c_{r2}, c_{r1}, \) and \( c_{r2} \) represent null-point rotation, rotation saturation, nuclear-resonant rotation, and induction rotation, respectively. The current, \( I \), controls the external magnetic field in the IR chamber. Taken from Frisch and Segré [9], the only device-specific parameters for the predictions include \( B_f \) (0.42 \( \times \) \( 10^{-4} \) T), \( z_0 \) (1.05 \( \times \) \( 10^{-4} \) m), and \( v \) (800 m s\(^{-1}\)). The theoretical predictions from equations (161)–(163) without adjusting any parameters are \( c_{j0} = 0.054 \), \( c_{r2} = 0.80 \), and \( c_{r1} = 48 \) A\(^{-1}\). Substitution of these coefficients into equation (30) with \( c_{r1} = 0 \) produces Curve 4 in figure 4, where no free parameters are used.

Despite its small contribution in the IR chamber, the induction factor is estimated for its order of magnitude. While holding all three other parameters constant at the predicted values, fitting \( W_{\text{cqd}} \) (equation (30)) for the experimental data in figure 4 yields \( c_{r1} \sim 0.57 \) A. Substitution into equation (164) produces \( k_i \sim 7.4 \times 10^{-6} \). Further substitution into equation (11) concludes that electron-spin collapse takes on the order of \( N_c \sim 220 \) precession cycles.

4. Discussion

CQD postulates that the electron and nuclear magnetic moments in an external field \( B_0 \) along \( z \) repel in the polar direction, which results in a revision to the sign of the induction term in the Landau–Lifshitz–Gilbert equation. Whereas precession is governed by the terms from the Bloch equation, collapse is modeled by the revised induction term. If \( k_i = 0 \), the equation of motion reduces to the Bloch equation, which agrees with the Stern–Gerlach experimental observation. Numerical solutions to the CQD equations of motion, to be reported separately, have illustrated collapse with the induction term and none without. This postulate is consistent with the Pauli exclusion principle for two identical fermions, where the two magnetic moments repel towards anti-alignment. Therefore, one may regard this postulate as an extension to the Pauli exclusion principle. Note that while diamagnetism explains the collapse term, paramagnetism is expected to perturb the precession term slightly, which is neglected here.

CQD also postulates that the polar angle of \( \vec{\mu}_n \) in flight varies negligibly. Because the nuclear Larmor frequency is four orders of magnitude smaller (i.e. \( |\omega_n| \ll |\omega_e| \)), nuclear spin collapses much more slowly than electron spin. Because no data on the collapse rates have been found in the literature, we refer to the \( T_1 \) relaxation times. Typical \( T_1 \) relaxation times in electron paramagnetic resonance are on the \( \mu s \) scale [35], consistent with the previous estimation of the collapse time scale of \( \vec{\mu}_e \) [5]. In contrast, typical \( T_1 \) relaxation times in gas-phase nuclear magnetic resonance are on the ms scale [36], indicating the order-of-magnitude collapse time of \( \vec{\mu}_n \). In a typical Stern–Gerlach experiment [5, 37], the main external field \( B_0 \) along \( z \) is at least 0.3 T (\( B_0 > B_e > B_{\mu} \), the Paschen–Back regime [16]), the length of the main field is \( \sim 35 \) mm, and the most likely atomic speed is \( v \sim 800 \) m s\(^{-1}\). Consequently, the flight time through the main field is only \( \sim 44 \) ms, which is long enough for \( \vec{\mu}_e \) to collapse but too short for \( \vec{\mu}_n \) to collapse. In fact, the fringe field on the source side of the main field collapses \( \vec{\mu}_e \) [2]. Besides the two distinct collapse branches due to the quantization of \( \vec{\mu}_e \), no additional branches due to the quantization of \( \vec{\mu}_n \) have been observed by Frisch and Segré [9] despite the prediction of up to eight branches total [38]. For \( N_c \sim 220 \) (equation (11)) estimated from the Frisch–Segré experimental data shown in figure 4, the collapse time constants (\( T_c \), equation (12) and its nuclear counterpart) at the main field strength are computed to be \( \sim 3 \times 10^{-8} \) and \( \sim 4 \times 10^{-4} \) s for \( \vec{\mu}_e \) and \( \vec{\mu}_n \), respectively, which are consistent with the above-mentioned corresponding \( T_1 \) relaxation times in orders of magnitude [35, 36]. This postulate, extended to the weaker-field IR chamber, is consistent with the selection rule for observing an electron-spin–resonance transition, stating that the magnetic quantum number of the nuclear spin remains constant (i.e. \( \Delta m_I = 0 \) [39].
transmittance, proportional to the solid angle formed by the atoms propagate through the slit further; otherwise, the atoms and Rabi used an isotropic angular distribution instead. In comparison, the Majorana or Landau–Zener formuations neglected the nuclear magnetic moment altogether, off. In comparison, the Majorana or Landau–Zener formuations were tuned to fit the experimental data). If the heart rate while all parameters were given (i.e. no parameters were included, the model becomes more and more accurate.

Polarization. The subsequent effects are illustrated using the cone of quantum (i.e. the nuclear magnetic moment) is changed from an isotropic shape (equation (15)) due to the polarization. The subsequent effects are illustrated using the evolution of the curves in figure 4. As more effects are included, the model becomes more and more accurate while all parameters were given (i.e. no parameters were tuned to fit the experimental data). If the heart shape were incorrect, the agreement would be completely off. In comparison, the Majorana or Landau–Zener formula neglected the nuclear magnetic moment altogether, and Rabi used an isotropic angular distribution instead of the heart shape [16]. Note that as the wire current approaches infinity, Rabi’s formula predicts a maximum of $\frac{1}{\sqrt{2}}$, which is well below the experimental peak of 31% (figure 1); here, $I = \frac{1}{2}$ denotes the nuclear spin number for potassium-39. Further, Rabi’s standard hyperfine coupling does not contain the induction terms in CQD and hence does not model collapse. Also, the torque-averaged fields provide greater agreement than the self-averaged fields (see appendix 1).

Quantum mechanics, celebrated for its countless triumphs, still pose mysteries as discussed insightfully in recent literature [30, 40–43]. The Copenhagen interpretation construes that an electron spin is simultaneously in both eigenstates and collapses statistically upon measurement to either [7]. The collapse is not modeled by the original Schrödinger equation but stated separately as a measurement postulate [30]. Debatable inconsistency has been found in thought experiments, such as ‘Schrödinger’s cat’ [44–46].

CQD potentially offers new insight. If co-qua are isotropically distributed, CQD has been verified with quantum mechanics by exactly reproducing the wave function and the density operator (appendix 3) as well as the uncertainty relation (appendix 6). The probabilities of reaching the two eigenstates split according to $\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}$; the wave function is reproduced in equation (18). However, if the co-qua have, for example, a heart-shaped distribution (equation (24)), the split becomes $1 - \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}$ (equations (88) and (89) in appendix 3); the wave function is revised accordingly (equation (90)). The density operator is found to originate from a pre-averaging counterpart with independent realizations (appendix 3). The measurement uncertainty product, explained by co-qua, depends on the initial phase of the principal quanta and the measurement sequence, as shown by the uncertainty inequality (equation (186)), which leads to the familiar quantum mechanical inequality (equation (187)). CQD has also enabled the derivation from the classical Bloch equation to the quantum Schrödinger–Pauli equation [17], while the latter has thus far been treated as a postulate.

CQD can be further tested with atoms having nuclear spins of 0 ($I = 0$), which may collapse differently in Stern–Gerlach experiments. Examples include $^{38mK}$, $^{50K}$, $^{94Ag}$, and $^{108Ag}$, which are isotopes of the stable $^{38K}$, $^{107Ag}$, and $^{109Ag}$. Unfortunately, these isotopes have short lifetimes ranging from 100s to 10s of ms. Note that free electrons have not been used owing to the Lorentz force and orbital magnetic moment.

Since the submission of this manuscript, our team has produced several new manuscripts to support this work. Titimbo et al numerically modeled the Frisch–Segrè experiment using CQD via the Bloch equation [47], whereas He et al numerically modeled the experiment using CQD via the Schrödinger equation [48]. Both works have numerically confirmed the analytical solution presented here and the equivalence between the Bloch equation and the Schrödinger equation stated by Majorana. Interestingly, Majorana wrote the ‘Bloch’ equation [10, 11] 14 years before Bloch published his eponymous equation [49]. The author recently derived from the Bloch equation, which Bloch...
intended for macroscopic magnetization instead of individual nuclear magnetic moments [49], to the Schrödinger or Schrödinger–Pauli equation [17]. Kahraman et al. demonstrated that the standard existing treatment of hyperfine interaction, consistent with the Breit–Rabi formula [38], cannot model the Frisch–Segrè experiment accurately but can be improved by incorporating CQD features [50]. The treatment also does not agree with the Rabi formula [16].

While no alternative theory, to the best of our knowledge, matches the Frisch–Segrè experiment, a recent multi-stage Stern–Gerlach experiment on superatomic icosahedral cage-clusters Mn@Sn also reveals discrepancy of the Landau–Zener formula from experimental observation [51].

5. Conclusions

CQD, based on the sign-modified Landau–Lifshitz–Gilbert equation, provides a plausible collapse mechanism for electron spin in Stern–Gerlach experiments. CQD models both spin evolution and collapse by the same equations of motion. With an anisotropic angular distribution of co-quanta, CQD revises the wave function and accurately predicts the Frisch–Segrè experimental observation in absolute units without fitting with adjustable parameter, achieving $p < 8 \times 10^{-7}$—an objective statistical indication that reflects both correlation and degrees of freedom. Therefore, it is extremely unlikely that CQD happens to match the experimental observation so well by sheer chance. Further, with an isotropic angular distribution of co-quanta, CQD is theoretically corroborated by quantum mechanics. Both the strong experimental evidence and the exact quantitative agreement with quantum mechanics in diverse forms collectively support CQD. Like statistical mechanics [52], which uses molecular properties to predict macroscopic properties by ensemble averaging, CQD reproduces quantum mechanical properties by ensemble averaging over co-quanta (appendix 3). If orthodox quantum mechanics is incomplete [44], CQD may stimulate development for a complete theory.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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