On the Diameter of Lattice Polytopes

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Abstract In this paper we show that the diameter of a $d$-dimensional lattice polytope in $[0, k]^n$ is at most $\lfloor (k - \frac{1}{2}) d \rfloor$. This result implies that the diameter of a $d$-dimensional half-integral polytope is at most $\lfloor \frac{3}{2} d \rfloor$. We also show that for half-integral polytopes the latter bound is tight for any $d$.

Keywords Lattice polytope · Diameter · Linear programming

1 Introduction

The 1-skeleton of a polyhedron $P$ is the graph whose nodes are the vertices of $P$, and that has an edge joining two nodes if and only if the corresponding vertices of $P$ are adjacent on $P$. Given vertices $u, v$ of $P$, the distance $\delta^P(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$ on the 1-skeleton of $P$. We may write $\delta(u, v)$ instead of $\delta^P(u, v)$ when the polyhedron we are referring to is clear from the context. The diameter $\delta(P)$ of $P$ is the smallest number that bounds the distance between any pair of vertices of $P$. 

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In this paper, we investigate the diameter of lattice polytopes, i.e. polytopes whose vertices are integral. Lattice polytopes play a crucial role in discrete optimization and integer programming problems, where the variables are constrained to assume integer values. Our goal is to define a bound on the diameter of a lattice polytope $P$, that depends on the dimension of $P$ and on the parameter $k = \max\{\|x - y\|_\infty : x, y \in P\}$, in order to apply such bound to classes of polytopes for which $k$ is known to be small. A similar approach has been followed by Bonifas et al. [4], who showed that the diameter of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded by a polynomial that depends on $n$ and on the parameter $\Delta$, defined as the largest absolute value of any sub-determinant of $A$. Note that, while $\Delta$ is related to the external description of $P$, $k$ is related to its internal description. However, both $\Delta$ and $k$ are in general not polynomial in $n$ and in the number of the facet-defining inequalities of $P$.

For $k \in \mathbb{N}$, a $(0, k)$-polytope $P \subseteq \mathbb{R}^n$ is a lattice polytope contained in $[0, k]^n$. Naddef [10] showed that the diameter of a $d$-dimensional $(0, 1)$-polytope is at most $d$, and this bound is tight for the hypercube $[0, 1]^d$. Kleinschmidt and Onn [9] extended this result by proving that the diameter of a $d$-dimensional $(0, k)$-polytope cannot exceed $kd$. However, their bound is not tight for $k \geq 2$.

Our main contribution is establishing an upper bound for the diameter of a $d$-dimensional $(0, k)$-polytope, which refines the bound by Kleinschmidt and Onn.

**Theorem 1** For $k \geq 2$, the diameter of a $d$-dimensional $(0, k)$-polytope is at most $\lfloor (k - \frac{1}{2})d \rfloor$.

The proof of Theorem 1 is elementary, as it combines an induction argument with basic tools from linear programming and polyhedral theory. Our proof is also constructive, since it shows how to build a path between two given vertices of $P$, whose length does not exceed our bound.

For $(0, 2)$-polytopes, we show that the upper bound given in Theorem 1 is tight for any $d$.

**Corollary 1** The diameter of a $d$-dimensional $(0, 2)$-polytope is at most $\lfloor \frac{3}{2}d \rfloor$. Moreover, for any natural number $d$, there exists a $d$-dimensional $(0, 2)$-polytope attaining this bound.

The lower bound of Corollary 1 follows by an easy construction based on the cartesian product of polytopes of dimension one and two. It is well-known that, given two polytopes $P_1$ and $P_2$, their cartesian product $P_1 \times P_2$ satisfies $\delta(P_1 \times P_2) = \delta(P_1) + \delta(P_2)$. Now, let $H_1 = [0, 2]$ and $H_2 = \text{conv}\{(0, 0), (1, 0), (0, 1), (2, 1), (1, 2), (2, 2)\}$. For even $d$, let $H_d = (H_2)^{d/2}$, and for odd $d$, let $H_d = H_{d-1} \times H_1$. Thus for all $d \in \mathbb{N}$, $H_d$ is a $d$-dimensional $(0, 2)$-polytope, with $\delta(H_d) = \lfloor \frac{3}{2}d \rfloor$.

Corollary 1 has important implications for the diameter of half-integral polytopes. **Half-integral polytopes** are polytopes whose vertices have components in $\{0, \frac{1}{2}, 1\}$, and they are affinely equivalent to $(0, 2)$-polytopes. The class of half-integral polytopes is very rich, as many half-integral polytopes appear in the literature as relaxations of $(0, 1)$-polytopes arising from combinatorial optimization problems. In some cases, while the $(0, 1)$-polytope defined as the convex hull of the feasible solutions to the combinatorial problem has exponentially many facets, there is a linear relaxation, defined by a polynomial number of constraints, that yields a half-integral polytope.

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There are several classes of polytopes that are known to be half-integral, such as the fractional matching polytope and the fractional stable set polytope [2], the linear relaxation of the boolean quadric polytope and the rooted semimetric polytope [12] (see also [7,14]). An interesting class of half-integral polytopes arises from totally dual half-integral systems, such as the fractional stable matching polytope [1,6], and the fractional matroid matching polytope [8,13].

The rest of the paper is devoted to proving Theorem 1.

2 Proof of Main Result

In order to bound the diameter of a non full-dimensional \((0,k)\)-polytope \(P \subseteq \mathbb{R}^n\), we define the projection of \(P\) onto the \(i\)-coordinate hyperplane as the polytope

\[
\{ \bar{x} \in \mathbb{R}^{n-1} : \exists x \in P \text{ with } x_j = \bar{x}_j \text{ for } j = 1, \ldots, i-1, \\
x_j = \bar{x}_{j-1} \text{ for } j = i+1, \ldots, n \}.
\]

That is, we simply drop the \(i\)-th coordinate from all vectors in \(P\). Since integral vectors are mapped into integral vectors, the next lemma follows from [11, Thm. 3.3].

**Lemma 1** Let \(P\) be a \(d\)-dimensional \((0,k)\)-polytope in \(\mathbb{R}^n\) with \(d \geq 1\). Then there exists a full-dimensional \((0,k)\)-polytope in \(\mathbb{R}^d\) with the same 1-skeleton as \(P\).

For \(d, k \in \mathbb{N}\), we define \(\delta^d_k\) to be the maximum possible diameter of a \((0,k)\)-polytope of dimension at most \(d\), i.e.

\[
\delta^d_k = \max \{ \delta(P) : P \text{ is a } (0,k)\text{-polytope of dimension at most } d \}.
\]

Note that the maximum in the definition of \(\delta^d_k\) always exists. In fact, it follows from Lemma 1 that the number of vertices of a \(d\)-dimensional \((0,k)\)-polytope is at most \((k+1)^d\), thus also its diameter is upper bounded by \((k+1)^d\), which is a number independent on the dimension of the ambient space of \(P\). Moreover, for fixed \(k\), the value \(\delta^d_k\) is clearly non-decreasing in \(d\).

We now present some lemmas that will be used to prove Theorem 1. These results follow by applying the ideas introduced by Kleinschmidt and Onn in [9]. The next lemma shows how to bound the distance \(\delta(u,F)\) between a vertex \(u\) of a lattice polytope \(P\) and a face \(F\) of \(P\), that is defined as \(\delta(u,F) = \min \{ \delta(u,v) : v \text{ is a vertex of } F \}\).

We say that two vertices \(u,v\) of a polytope are neighbors if \(\delta(u,v) = 1\). We denote by \(e^i\), for \(i = 1, \ldots, n\), the \(i\)-th vector of the standard basis of \(\mathbb{R}^n\).

**Lemma 2** Let \(P\) be a lattice polytope, and let \(u\) be a vertex of \(P\). Let \(c\) be an integral vector, \(\gamma = \min \{ cx : x \in P \}\), and \(F = \{ x \in P : cx = \gamma \}\). Then \(\delta(u,F) \leq cu - \gamma\).

**Proof** We show that there exists a vertex \(v\) of \(F\) such that \(\delta(u,v) \leq cu - \gamma\). We prove this statement by induction on the integer value \(cu - \gamma \geq 0\). The statement is trivial for \(cu - \gamma = 0\), as we can set \(v = u\). Assume \(cu - \gamma \geq 1\). Since \(F\) is nonempty, there exists a neighbor \(u'\) of \(u\) with \(cu' < cu\) (see, e.g., [5]). The integrality of \(c\), \(u'\) and \(u\), implies \(cu' \leq cu - 1\). As \(cu' - \gamma \leq cu - \gamma - 1\), by the induction
hypothesis there exists a vertex \( v \) of \( F \) such that \( \delta(u', v) \leq cu' - \gamma \). Therefore \( \delta(u, v) \leq \delta(u, u') + \delta(u', v) \leq 1 + cu' - \gamma \leq cu - \gamma \). \( \square \)

Given two vertices \( u \) and \( v \) and a face \( F \) of a lattice polytope \( P \), we have \( \delta(u, v) \leq \delta(u, F) + \delta(v, F) + \delta(F) \). By applying Lemma 2 to both \( u \) and \( v \), we obtain an upper bound on \( \delta(u, v) \) that depends on \( F \):

**Lemma 3** Let \( P \) be a lattice polytope, and let \( u, v \) be vertices of \( P \). Let \( c \) be an integral vector, \( \gamma = \min\{cx : x \in P\} \), and \( F = \{x \in P : cx = \gamma\} \). Then \( \delta(u, v) \leq \delta(F) + cu + cv - 2\gamma \).

Let \( P \) be a \((0, k)\)-polytope in \( \mathbb{R}^n \) and let \( l = \min\{x_i : x \in P\} \) and \( h = \max\{x_i : x \in P\} \) for some \( i \in \{1, \ldots, n\} \). We can bound the distance between any two vertices \( u \) and \( v \) of \( P \) by bounding their distances from the faces \( L = \{x \in P : x_i = l\} \) and \( H = \{x \in P : x_i = h\} \). If \( u_i + v_i \leq l + h \), Lemma 3 applied with \( F = L \), \( c = e^i \) and \( \gamma = l \) implies \( \delta(u, v) \leq \delta(L) + (h - l) \). If \( u_i + v_i \geq l + h \), Lemma 3 applied with \( F = H \), \( c = -e^i \) and \( \gamma = -h \) implies \( \delta(u, v) \leq \delta(H) + (h - l) \). Since \( L \) and \( H \) are \((0, k)\)-polytopes of dimension at most \( n - 1 \), we have that both \( \delta(L) \) and \( \delta(H) \) are at most \( \delta_k^{n-1} \).

**Lemma 4** Let \( P \) be a \((0, k)\)-polytope in \( \mathbb{R}^n \), and suppose that there exists \( i \in \{1, \ldots, n\} \) such that \( x_i \in [l, h] \) for every \( x \in P \). Then \( \delta(P) \leq \delta_k^{n-1} + (h - l) \).

Given a \( d \)-dimensional \((0, k)\)-polytope \( P \), Kleinschmidt and Onn prove the bound \( \delta(P) \leq kd \) by essentially applying Lemma 1, and then Lemma 4 inductively. Therefore, their bound uses Lemma 2 only with vectors \( c = \pm e^i \). To prove our refined bound, we will use Lemma 2 also with different vectors \( c \). We are now ready to give the proof of Theorem 1.

**Proof of Theorem 1** Let \( P \) be a \( d \)-dimensional \((0, k)\)-polytope, with \( k \geq 2 \). The proof is by induction on \( d \). The base cases are \( d = 0 \) and \( d = 1 \). The diameter of a 0-dimensional polytope is clearly zero, and the diameter of a 1-dimensional polytope is at most one, thus also bounded by \( |k - \frac{k}{2}| = k - 1 \) since \( k \geq 2 \).

We now assume \( d \geq 2 \). Let \( u, v \) be vertices of \( P \). By the induction hypothesis we assume that Theorem 1 is true for \((0, k)\)-polytopes of dimension at most \( d - 1 \). In particular, \( \delta_k^{d-1} \leq \lfloor (k - \frac{1}{2})(d - 1) \rfloor \), and \( \delta_k^{d-2} \leq \lfloor (k - \frac{1}{2})(d - 2) \rfloor \). Thus, in order to prove the inductive step, it is sufficient to show one of the following two inequalities:

\[
\delta(u, v) \leq \delta_k^{d-1} + k - 1, \quad (1)
\]
\[
\delta(u, v) \leq \delta_k^{d-2} + 2k - 1. \quad (2)
\]

\( \square \)

**Claim 1** We can assume that \( P \) is full-dimensional.

**Proof of claim** By Lemma 1, there exists a full-dimensional \((0, k)\)-polytope in \( \mathbb{R}^d \) with the same 1-skeleton as \( P \). \( \square \)
Claim 2 We can assume that $P$ intersects all facets of the hypercube $[0, k]^d$.

**Proof of claim** If there exists a facet $G$ of the hypercube $[0, k]^d$ with $P \cap G = \emptyset$, then let $i \in \{1, \ldots, d\}$ be such that $l \leq x_i \leq h$, with $l \geq 1$ or $h \leq k - 1$. By Lemma 4, $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied.

In the remainder of the paper, we will denote by $k^d$ the $d$-dimensional vector with all entries equal to $k$.

Claim 3 We can assume that $u + v = k^d$.

**Proof of claim** Assume that $u + v \neq k^d$, there exists an index $i \in \{1, \ldots, d\}$ such that $u_i + v_i \leq k - 1$ or $u_i + v_i \geq k + 1$. By Lemma 3 applied with $c = e^t$ or $c = -e^t$, respectively, we obtain $\delta(u, v) \leq \delta(F) + k - 1$, where $F$ is the face of $P$ that minimizes $cx$. As $F$ is a $(0, k)$-polytope of dimension at most $d - 1$, we have $\delta(F) \leq \delta_k^{d-1}$, therefore $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied.

Claim 4 We can assume that $u \in [0, k]^d$.

**Proof of claim** Assume that $u$ has one component $u_i, i \in \{1, \ldots, d\}$, with $1 \leq u_i \leq k - 1$. In this case we show that (2) is satisfied. Since the set $\{x \in P : x_i = 0\}$ is nonempty, there exists a neighbor $s$ of $u$ with $s_i < u_i$ (see, e.g., [5]). By the integrality of $s$ and $u$, this implies $s_i \leq u_i - 1$. Symmetrically, since the set $\{x \in P : x_i = k\}$ is nonempty, $u$ has a neighbor $t$ with $t_i \geq u_i + 1$. If $s_j = t_j = u_j$ for all $j \in \{1, \ldots, d\}$, $j \neq i$, then by setting $\lambda = \frac{t_i - u_i}{s_i - t_i}$ we have $\lambda s + (1 - \lambda)t = u$, contradicting the fact that $u$ is a vertex of $P$. Thus, there exists an index $j \in \{1, \ldots, d\}$ with $j \neq i$ such that either $s_j \neq u_j$ or $t_j \neq u_j$. Therefore there exists a neighbor $w$ of $u$ such that $w_i \neq u_i$ and $w_j \neq u_j$, for distinct indices $i, j \in \{1, \ldots, d\}$ (see Fig. 1i).

We assume without loss of generality that $w_i < u_i$ (if not, we can perform the change of variable $\tilde{x}_i = k - x_i$). Analogously, we assume $w_j < u_j$. As $u + v = k^d$, we have $w_i + w_j + v_i + v_j \leq 2k - 2$. Let $\gamma = \min\{x_i + x_j : x \in P\}$ and $F = \{x \in P : x_i + x_j = \gamma\}$. By Lemma 3 (with $c = e^t + e^j$), $\delta(w, v) \leq \delta(F) + w_i + w_j + v_i + v_j - 2\gamma \leq \delta(F) + 2k - 2 - 2\gamma$ (see Fig. 1ii).

We now show that $\delta(F) \leq \delta_k^{d-2} + \gamma$. Let $\tilde{F}$ be the projection of $F$ onto the $j$-coordinate hyperplane. $\tilde{F}$ is a $(0, k)$-polytope in $\mathbb{R}^{d-1}$ and, by Lemma 1, $\tilde{F}$ has the same 1-skeleton of $F$. Note that, for any $x \in F$, $x_i = \gamma - x_j$ and $x_j \geq 0$ imply $x_i \leq \gamma$. Therefore, $x_i \leq \gamma$ for any $x \in \tilde{F}$. Then, by Lemma 4, $\delta(\tilde{F}) \leq \delta_k^{d-2} + \gamma$, thus $\delta(F) \leq \delta_k^{d-2} + \gamma$.

This implies $\delta(w, v) \leq \delta_k^{d-2} + 2k - 2 - \gamma$ and, since $\gamma \geq 0$ and $\delta(u, w) = 1$, finally $\delta(u, v) \leq \delta(u, w) + \delta(w, v) \leq \delta_k^{d-2} + 2k - 1$, i.e. (2) is satisfied.

By possibly performing the change of variable $\tilde{x}_1 = k - x_1$, we can further assume without loss of generality that $u_1 = k$, and $v_1 = 0$.

Let $F$ be the face of $P$ defined by $F = \{x \in P : x_1 = 0\}$. $F$ is a $(0, k)$-polytope of dimension at most $d - 1$, thus $\delta(F) \leq \delta_k^{d-1}$. By Lemma 2 (with $c = e^1$), there exists a vertex $u'$ of $F$ such that $\delta(u, u') \leq 1$. Observe that both $u'$ and $v$ lie in $F$ and therefore $\delta(u', v) \leq \delta_k^{d-1}$.
Fig. 1 In Claim 4, (i) we construct a neighbor \( w \) of \( u \) with \( w_j < u_j \), and \( w_j < u_j \), (ii) we use Lemma 3 with \( c = e^i + e^j \) to show that \( \delta(w, v) \leq \delta^d - 2 + 2k \).

Fig. 2 To bound the distance between vertices \( u \in \{0, k\}^d \) with \( u_1 = k \) and \( v = k^d - u \), we construct a path from \( u \) to a vertex \( u' \) with \( u'_1 = 0 \). There are two cases: (i) \( u' = (0, u_2, \ldots, u_d) \), thus \( \delta(u, u') = 1 \) and \( \delta(u', v) \leq \delta^d - k + 1 \); (ii) \( u' \neq (0, u_2, \ldots, u_d) \), thus \( \delta(u, u') \leq k \) and \( \delta(u', v) \leq \delta^d - 2 + k - 1 \).

If \( u' = (0, u_2, \ldots, u_d) \), then \( u \) and \( u' \) are adjacent vertices of the hypercube \([0, k]^d\), implying that \( \text{conv}\{u, u'\} \) is an edge of \([0, k]^d\) (see Fig. 2i). As \( P \) is convex and it is contained in \([0, k]^d\), it follows that \( \text{conv}\{u, u'\} \) is also an edge of \( P \). Therefore, \( \delta(u, u') = 1 \) and consequently \( \delta(u, v) \leq \delta^d - 1 + 1 \). As \( k \geq 2 \), it follows \( \delta(u, v) \leq \delta^d - 1 + k - 1 \), i.e. (1) is satisfied.

Thus we now assume \( u' \neq (0, u_2, \ldots, u_d) \) (see Fig. 2ii). Then, there exists an index \( i \in \{2, \ldots, d\} \) such that \( u'_i + v_i \leq k - 1 \) or \( u'_i + v_i \geq k + 1 \). We assume without loss of generality that \( u'_i + v_i \leq k - 1 \) (if not, we can perform the change of variable \( \bar{x}_i = k - x_i \)). Let \( \gamma = \min\{x_i : x \in F\}, F' = \{x \in F : x_i = \gamma\} \). \( F' \) is a \((0, k)\)-polytope, and it has dimension at most \( d - 2 \) because it is contained in the intersection of the two linearly independent hyperplanes \( \{x \in \mathbb{R}^d : x_1 = 0\} \) and \( \{x \in \mathbb{R}^d : x_i = \gamma\} \). It follows that \( \delta(F') \leq \delta^d - 2 \). Then, by applying Lemma 3 to the polytope \( F \) and the vertices \( u' \) and \( v \), we have \( \delta(u', v) \leq \delta(F') + u'_i + v_i \leq \delta^d - 2 + k - 1 \). This implies \( \delta(u, v) \leq \delta(u, u') + \delta(u', v) \leq \delta^d - 2 + 2k - 1 \), i.e. (2) is satisfied. \( \square \)
3 Further Directions

Both our upper bound and the one by Kleinschmidt and Onn are not tight for \( k \geq 3 \). As an example, \( \delta_3^2 = 4 \), as the maximum diameter of a lattice polygon in \([0, 3]^2\) is realized by the octagon. It seems that our approach cannot be easily refined to obtain a tight upper bound for general \( k \).

An interesting direction of research is to study the asymptotic behavior of the function \( \delta_k^d \). It is known that the maximum number of vertices of a 2-dimensional \((0, k)\)-polytope is in \( \Theta(k^{2/3}) \) [3], which implies the asymptotically tight bound \( \delta_k^2 \in \Theta(k^{2/3}) \). Using cartesian products of polytopes, it follows that \( \delta_k^d \in \Omega(k^{2/3}d) \). This provides an asymptotic lower bound on \( \delta_k^d \) that is a fractional power with respect to \( k \) and linear in \( d \). However, the best upper bound on \( \delta_k^d \) is linear both in \( k \) and in \( d \). In other words, there is still a significant gap between the lower and the upper bound.

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