Exact Solution for One Type of Lindley’s Equation for Queueing Theory and Network Calculus

Yu Chen, Member, IEEE

Abstract—Lindley’s equation is an important relation in queueing theory and network calculus. In this paper, we develop a new method to solve one type of Lindley’s equation, i.e., the equation \( V(s)T(-s) - 1 = 0 \) only has finite negative real roots. \( V(s) \) and \( T(-s) \) are the Laplace transforms of service time’s probability density function (PDF) and interarrival time’s PDF (evaluated at \(-s\)). For queueing theory, we use this method to derive the exact \( M/M/1 \), \( M/H_2/1 \) and \( M/E_2/1 \) waiting-time distributions, and for the first time find the exact \( D/M/1 \) waiting-time distribution. For network calculus, we use two examples to compare our method with the effective bandwidth model and its dual, the effective capacity model, respectively. We observe that the distribution function of backlog size in the first example can be obtained exactly by our method and partially by the effective bandwidth model; however, such a distribution function in the second example cannot be obtained by our method but can be approximated by the effective capacity model.

Index Terms—Lindley’s equation, queueing theory, network calculus, \( M/M/1 \), \( M/H_2/1 \), \( M/E_2/1 \), \( D/M/1 \), \( M/D/1 \), \( M/I/1 \) waiting-time distribution, CDF of waiting time in queue, effective bandwidth model, effective capacity model, CCDF of backlog size.

I. INTRODUCTION

Queueing theory is the mathematical study of queues with wide applications in real life [1] while network calculus studies the negative effects of buffers in data networks [2]. These two research topics are usually carried out independently although their relations are sometimes discussed [3], [4], [5], [6], [7], [8], [9], [10].

Consider a single-server queueing system with an infinite queue size and first-in first-out (FIFO) queue discipline, shown in Fig. 1. Waiting times in queue \( W_{q}^{(n)} \) and \( W_{q}^{(n+1)} \) of two successive \( n^{th} \) and \( (n+1)^{th} \) customers are related by Lindley’s equation [11]:

\[
W_{q}^{(n+1)} = \max \{0, W_{q}^{(n)} + V^{(n)} - T^{(n)}\}, \quad n \geq 1,
\]

(1)

where \( V^{(n)} \) is the service time of the \( n^{th} \) customer and \( T^{(n)} \) is the interarrival time between the \( n^{th} \) and \( (n+1)^{th} \) customer arrivals. Lindley’s equation is an important general relation in queueing theory because it holds without any requirement on interarrival times or service times, and is commonly used to describe \( G/G/1 \) queueing models. If the service times \( V^{(1)} \), \( V^{(2)} \), \( V^{(3)} \) ... are independent and identically distributed (IID) random variable (RVs) identical to a RV \( V \) and if the interarrival times \( T^{(1)} \), \( T^{(2)} \), \( T^{(3)} \) ... are IID RVs identical to a RV \( T \), then (1) can be expressed as

\[
W_{q}^{(n+1)} = \max \{0, W_{q}^{(n)} + V - T\}, \quad n \geq 1
\]

(2)

Eq. (2) describes a \( GI/GI/1 \) queueing model and is the focus of this paper.

Now let us consider a system model of network calculus, shown in Fig. 2. Assume packet sizes are infinitely divisible. The model consists of a source, a buffer with infinite buffer size and a work-conserving server (a server that is never idle when there is a packet to send). Over a time interval \([0, t]\), denote

1. \( A(t) \) the amount of bits generated by the source,
2. \( S(t) \) the amount of bits that the server is capable of transmitting and
3. \( A'(t) \) the amount of bits left the system.

Let \( A(s, t) = A(t) - A(s) \) and \( S(s, t) = S(t) - S(s) \). The backlog size \( Q(t) \) at time \( t \) and \( A'(t) \) can be expressed as [10]

\[
Q(t) = \max \{A(s, t) - S(s, t)\}, \quad t \geq 0,
\]

(3)

\[
A'(t) = A(t) - Q(t) = \min_{0 \leq s \leq t} \{A(s) + S(s, t)\}, \quad t \geq 0
\]

(4)

The \( \min_{0 \leq s \leq t} \{A(s) - S(s, t)\} \) operation in (4) is called the min-plus convolution developed by [12], [13]. The min-plus convolution together with the max-plus convolution play a central role in network calculus [2], [10]. When the network calculus system model is discrete time (e.g., digital systems), (3) can be derived based on a variant of Lindley’s equation [10]:

\[
Q(t) = \max \{0, Q(t-1) + A(t-1) - S(t-1)\}
\]

(5)

If the arrivals \( A(0, 1), A(1, 2), A(2, 3) \) ... in (5) are IID RVs identical to a RV \( A \) and if the services \( S(0, 1), S(1, 2), S(2, 3) \) ... in (5) are IID RVs identical to a RV \( S \), then (5) can be written as

\[
Q(t) = \max \{0, Q(t-1) + A - S\}, \quad t \geq 1
\]

(6)

The first important observation is that (6) is of the form (2). Lindley in [11] proved that for (2), if interarrival times and service times are IID RVs with finite expectations, and if the mean service time is less than the mean arrival interval, then the distribution functions of \( W_{q}^{(n)} \) and \( W_{q}^{(n+1)} \) tend to a unique limiting distribution function. Let \( W_{q} \) denote the RV of waiting time in queue (in steady state) and \( Q \) denote the RV of backlog size (in steady state). We can claim that

Claim 1: Any method to find the distribution function of \( W_{q} \) in (2) can also find the distribution function of \( Q \) in (6).

In queueing theory, Lindley suggested that (2) can be solved by the Wiener-Hopf method [11]. Based on the Wiener-Hopf method, Smith proposed a method to find an exact solution when the Laplace transforms \( V(s) \) and \( T(-s) \) of service time’s probability density function (PDF) and interarrival time’s PDF

The authors are with the Electronic and Electrical Engineering Department, University College London, London, the U.K.

1 In this paper, we use Kendall’s notation to describe queueing systems. Symbols are listed in Table I.
Based on Lindley's work in [11]. Secondly, we consider basic examples to compare our method with these two models: backlog size in data networks. Because of Claim 1, we use two mathematical models, namely the effective bandwidth model and the effective capacity model. Both models are commonly used to approximate the distribution function of backlog size in data networks. Because of Claim 1, we use two basic examples to compare our method with these two models:

1) arrivals are IID exponential RVs and services are fixed to a constant value;
2) arrivals are fixed to a constant value and services are IID exponential RVs.

We observe that for the first example,

| Characteristic                        | Symbol | Explanation                  |
|---------------------------------------|--------|------------------------------|
| Interarrival-time distribution or service time distribution | $M$    | Exponential                  |
|                                       | $D$    | Deterministic                |
|                                       | $H_k$  | Mixture of $k$ exponentials  |
|                                       | $E_k$  | Erlang type $k$ ($k = 1, 2, ...$) |
|                                       | $G$    | General                      |
|                                       | GI     | General and independent      |

(evaluated at $-s$) are reciprocals of polynomials [14]. Harris continues Smith's work and showed that the Wiener-Hopf method can be greatly simplified if $V(s)$ and $T(-s)$ are rational functions; however, he did not provide an exact solution [15].

In this paper, we first formulate a new boundary condition based on Lindley's work in [11]. Secondly, we consider a special type of Lindley's equation, i.e., the equation $V(s)T(-s) - 1 = 0$ only has finite negative real roots. Thirdly, we integrate Harris's work with the new boundary condition and develop a new method to solve the above type of Lindley's equation. The new method is shown to be simple and can be computerized.

For queueing theory, we use this method to derive the exact waiting-time distributions of the classic $M/M/1$, $M/H_2/1$, and $M/E_2/1$ queueing models, and for the first time find the exact $D/M/1$ waiting-time distribution (the $D/M/1$ mean stationary waiting time in queue was derived in [16]).

For network calculus, we focus on two well-known mathematical models, namely the effective bandwidth model [17], [18], [19] and its dual, the effective capacity model [20]. Both models were developed based on the large deviation theory and are commonly used to approximate the distribution function of backlog size in data networks. Because of Claim 1, we use two basic examples to compare our method with these two models:

1) the distribution function of $Q$ can be found exactly by our method and partially by the effective bandwidth model;
2) the first step that our method and the effective bandwidth model take are identical so the effective bandwidth model can be considered as part of our method; for the second example,

1) neither our method or the effective capacity model is able to find the exact distribution function of $Q$;
2) the effective capacity model is only an approximation model.

The remainder of this paper is organized as follows: our new method is developed in Section II. In Section III, we demonstrate the use of our method by deriving the exact waiting-time distributions of $M/M/1$, $M/H_2/1$, $M/E_2/1$ and $D/M/1$ queueing models. The $M/D/1$ queueing model in this section is a counter example, in which our method is inapplicable. In Section IV, we discuss the relations between our method and two mathematical models in network calculus (the effective bandwidth model and the effective capacity model). Section V summarizes the work and contributions of this paper.

II. METHOD TO SOLVE LINDLEY'S EQUATION

Write $P(\cdot)$ to denote the probability of an event (\cdot). We start with a basic probability relation between $W_q^{(n)}$ and $W_q^{(n+1)}$ developed by Lindley [11]:

$$P\left(W_q^{(n+1)} \leq t\right) = P\left(W_q^{(n)} + V - T \leq t\right).$$

In the steady state, the distribution functions of $W_q^{(n)}$ and $W_q^{(n+1)}$ must be identical to the distribution function of $W_q$:

$$P\left(W_q \leq t\right) = P\left(W_q^{(n)} + V - T \leq t\right).$$

Eqs. (7) and (8) lead to a new probability relation:

$$P\left(W_q \leq t\right) = P\left(W_q + V - T \leq t\right).$$

Let $W_q(t)$ represent the cumulative distribution function (CDF) of $W_q$:

$$W_q(t) = P\left(W_q \leq t\right).$$

It is possible that costumers have zero waiting time in queue, and hence in general $W_q(t)$ will have a discontinuity at the origin and $W_q$ has a discrete probability mass at zero. The PDF
of \( W_q \) can be expressed as a mixture of an auxiliary function and a Dirac delta function at zero (or a degenerate function in the probability theory):  
\[
    f_{w_q}(t) = qf_{w^+}(t) + (1-q)\delta(t), \quad t \geq 0,  
\]
where \( f_{w^+}(t) \) is an auxiliary function with  
\[
    \int_0^\infty f_{w^+}(t) \, dt = 1
\]
and  
\[
    q = P(W_q > 0).
\]

Because of (12), we can consider \( W^+ \) as an auxiliary RV. Eq. (9) can be further expanded based on a standard result from the conditional probability and (13):  
\[
    P(W_q > t) = P(W_q + V - T > t)
\]
\[
= P(W^+ + V - T > t | W_q > 0) + P(V - T > 0 | W_q = 0). \tag{14}
\]
\[
= qP(X^+ + V - T > t) + (1-q)P(V - T > t).
\]

After manipulation of (14), we have  
\[
P(W_q > t) = qP(W^+ + V > T + t) + (1-q)P(V > T + t). \tag{15}
\]

Eq. (15) is the new boundary condition that \( W_q \) shall satisfy. Denote  
\[1) \text{ by } f_q(t) the PDF of service time; \]
\[2) \text{ by } V(s) the two-sided Laplace transforms of the PDF } f_q(t); \]
\[3) \text{ by } T(-s) the two-sided Laplace transforms of the PDF } f_t(t) \text{ evaluated at } -s; \]
\[4) \text{ by } V(s) the two-sided Laplace transform of the PDF } f_t(t); \]
\[5) \text{ the degree of } D_r(s) \text{ is } n, \]
then \( z_1, z_2, ..., z_n \) are \( n \) roots of the equation  
\[
    V(s)T(-s) - 1 = 0
\]
with negative real parts and \( W_q(t) \) is of the form  
\[
    W_q(t) = 1 - \sum_{i=1}^n k_i \exp(z_i t). \tag{19}
\]

In Lemma 2, \( k_1, k_2, ..., k_n \) in (19) are coefficients to be found. If \( N_q(s) \) and \( N_p(s) \) are real constants, Smith’s method can determine \( k_1, k_2, ..., k_n \) [14]. In the next theorem, we will show that if \( z_1, z_2, ..., z_n \) are all negative real roots, then \( k_1, k_2, ..., k_n \) can be determined by adopting the new boundary condition (15):

**Theorem 3**: Suppose that the conditions of Lemma 2 hold and \( z_1, z_2, ..., z_n \) are all negative real numbers. Define  
\[1) \text{ an } n \times 1 \text{ vector } Z \text{ as } \]
\[
    Z = (z_1, z_2, ..., z_n) \tag{19}
\]

where \((Z)\) denotes matrix transpose;  
\[2) \text{ } n \text{ auxiliary RVs: the PDF of the } i^{th} \text{ RV } Z_i^+ \text{ is } \]
\[
    f_{Z_i^+}(x) = -z_i \exp(z_i x); \tag{12}
\]
\[3) \text{ a } 1 \times n \text{ vector C: the } i^{th} \text{ entry } c_i \text{ is } \]
\[
    c_i = P(V > T + \Delta_i); \tag{12}
\]
\[4) \text{ an } n \times n \text{ matrix } P: \text{ the } (i,j)^{th} \text{ entry } p_{i,j} \text{ is } \]
\[
    p_{i,j} = P(Z_i^+ + V > T + \Delta_j); \tag{12}
\]
\[5) \text{ an } n \times n \text{ matrix D: the } (i,j)^{th} \text{ entry } d_{i,j} \text{ is } \]
\[
    d_{i,j} = \exp(z_i \Delta_j) - p_{i,j} + c_i. \tag{12}
\]
\[W_q(t) \text{ is given by } \]
\[
    W_q(t) = 1 - CD^{-1} \exp(Zt). \tag{22}
\]

For a proof of Theorem 3, see Appendix I. For convenience, we let \( \Delta_i = (i-1) \) in this paper.

### III. QUEUEING THEORY: WAITING-TIME DISTRIBUTIONS

By using the new method in Theorem 3, we derive the exact \( W_q(t) \) of the \( M/M/1, M/H_2/1 \) and \( M/E_2/1 \) queueing models in Section III-A, and the \( D/M/1 \) queueing model in Section III-B. In Section III-C, we show that the \( W_q(t) \) of the \( M/D/1 \) queueing model cannot be solved by our method.

#### A. M/M/1, M/H_2/1 and M/E_2/1 Waiting-Time Distributions

**Example 1** \( M/M/1 \) queueing model: Customer arrivals are determined by a Poisson process with a mean arrival rate \( \lambda \) customers/h. Customer service times have an exponential distribution with a mean service rate \( \mu \) customers/h.

The PDFs of service time and interarrival time are  
\[
    f_s(t) = \mu e^{-\mu t}, \quad f_i(t) = \lambda e^{-\lambda t}, \tag{12}
\]

The Laplace transforms \( V(s) \) and \( T(-s) \) are  
\[
    V(s) = \int_0^\infty e^{-st} \mu e^{-\mu t} \, dt = \frac{\mu}{\mu + s}, \tag{12}
\]
\[
    T(-s) = \int_0^\infty e^{st} \lambda e^{-\lambda t} \, dt = \frac{\lambda}{\lambda + s}. \tag{12}
\]

It can be verified that both \( V(s) \) and \( T(-s) \) are rational functions. By letting  
\[
    V(s)T(-s) - 1 = 0, \tag{18}
\]

we have  
\[
    \mu \lambda = (\mu + s)(\lambda - s). \tag{18}
\]

The denominator of \( V(s) \) is a polynomial of degree one. If \( \lambda < \mu \), then (23) has one negative real root  
\[
    z_1 = \lambda - \mu. \tag{18}
\]

In the case of one root, we define  
1) a \( 1 \times 1 \) vector \( Z \); 2) one auxiliary RV \( Z_1^+ \); 3) a \( 1 \times 1 \) vector \( C \); 4) a \( 1 \times 1 \) matrix \( P \); 5) a \( 1 \times 1 \) matrix \( D \).

The entry \( z_1 = \lambda - \mu \) so \( Z \) is
The PDF of the auxiliary RV $Z^*_1$ is
\[ f_{Z^*_1}(t) = (\mu - \lambda) \exp ((\lambda - \mu)t). \]
The entry $c_1$ is
\[ c_1 = P(V > T) = \int_0^\infty P(V > t) \lambda e^{-\lambda t} \, dt = \frac{\lambda}{\lambda + \mu}, \tag{24} \]
so $C$ is
\[ C = \left( \begin{array}{c} \lambda \\ \lambda + \mu \end{array} \right). \tag{25} \]
To compute the entry $p_{1,1}$, we need to know the complementary cumulative distribution function (CCDF) of the sum of $Z^*_1$ and $V$. Since the sum of two exponential RVs with rates $\lambda_1$ and $\lambda_2$ has a hypoexponential distribution, the CCDF of $(Z^*_1 + V)$ is
\[ P(Z^*_1 + V > t) = \frac{\mu - \lambda}{\lambda} \exp (-\mu t) + \frac{\mu}{\lambda} \exp (- (\mu - \lambda) t). \]
The entry $p_{1,1}$ is
\[ p_{1,1} = P(Z^*_1 + V > T) = \int_0^\infty P(Z^*_1 + V > t) \lambda e^{-\lambda t} \, dt = \frac{2\lambda}{\lambda + \mu}. \]
The matrix $D$ is computed from (21) as
\[ D = (1 - p_{1,1} + c_1) = \left( \begin{array}{c} \mu \\ \lambda + \mu \end{array} \right). \tag{26} \]
According to (22), we have
\[ W_q(t) = 1 - CD^{-1} \exp (Zt) = 1 - \frac{\lambda}{\mu} \exp (- (\mu - \lambda) t). \tag{27} \]
Eq. (27) is the exact $W_q(t)$ of the M/M/1 queueing model (p. 65 in [21]).

**Example 2 M/MH/1 queueing model** (p. 288 in [21]): Customer arrivals are determined by a Poisson process with a rate $\lambda = 5$ customers/h ($= 1/12$ customers/min). The service times have two possibilities: exponential with mean 5 minutes (one-third of the time) and exponential with mean 12.5 minutes (two-thirds of the time).

The PDF of $V$ is a hyperexponential distribution (or a mixture of two exponential distributions):
\[ f_V(t) = \left( \frac{2}{25} \right) \exp \left( -\frac{t}{5} \right) + \left( \frac{2}{25} \right) \exp \left( -\frac{2t}{5} \right). \]
The PDF of $T$ is an exponential distribution:
\[ f_T(t) = \frac{1}{12} \exp \left( -\frac{t}{12} \right). \]
The Laplace transforms $V(s)$ and $T(-s)$ are
\[ V(s) = \frac{1}{3 + 15s} + \frac{4}{6 + 75s}, \]
\[ T(-s) = \frac{1}{1 - 12s}. \]
It can be verified that both $V(s)$ and $T(-s)$ are rational functions. By letting
\[ V(s)T(-s) - 1 = 0, \]
we have
\[ \frac{1}{3 + 15s} + \frac{4}{6 + 75s} = 1 - 12s. \tag{28} \]

The denominator of $V(s)$ is a polynomial of degree two and (28) has two negative real roots: $z_1 \approx -0.18$ and $z_2 \approx -0.015$. Therefore, the conditions of Theorem 3 are true. In the case of two roots, we define 1) a $2 \times 1$ vector $Z$; 2) two auxiliary RVs $Z^*_1$ and $Z^*_2$; 3) a $1 \times 2$ vector $C$; 4) a $2 \times 2$ matrix $P$; 5) a $2 \times 2$ matrix $D$.

The entries $z_1$ and $z_2$ are $-0.18$ and $-0.015$ so $Z$ is
\[ Z = (-0.18, -0.015). \]
The PDFs of the first and second RVs $Z^*_1$ and $Z^*_2$ are
\[ f_{Z^*_1}(t) = 0.18 \exp (-0.18t), \]
\[ f_{Z^*_2}(t) = 0.015 \exp (-0.015t). \]
The entries $c_1$ and $c_2$ are
\[ c_1 = P(V > T) = \int_0^\infty P(V > x) \frac{1}{12} \exp \left( -\frac{x}{12} \right) \, dx = 0.4382 \]
and
\[ c_2 = P(V > T + 1) = \int_0^\infty P(V > x + 1) \frac{1}{12} \exp \left( -\frac{x}{12} \right) \, dx = 0.3943 \]
so $C$ is
\[ C = (0.4382, 0.3943). \]
The CCDF of the sum of $Z^*_1$ and $V$ is
\[ P(Z^*_1 + V > t) = 2.8e^{-0.015} - 3e^{-0.2} + 1.2e^{-0.015}, \]
and the CCDF of the sum of $Z^*_2$ and $V$ is
\[ P(Z^*_2 + V > t) = 1.1809e^{-0.015} - 0.027e^{-0.2} - 0.1538e^{-0.015}. \]
The entries $p_{1,1}, p_{1,2}, p_{2,1}$ and $p_{2,2}$ are computed from (20) as
\[ p_{1,1} = \int_0^\infty P(W^*_1 + V > x) \frac{1}{12} e^{-\frac{x}{12}} \, dx = 0.6160, \]
\[ p_{1,2} = \int_0^\infty P(W^*_1 + V > x + 1) \frac{1}{12} e^{-\frac{x}{12}} \, dx = 0.5829, \]
\[ p_{2,1} = \int_0^\infty P(W^*_2 + V > x) \frac{1}{12} e^{-\frac{x}{12}} \, dx = 0.9144, \]
\[ p_{2,2} = \int_0^\infty P(W^*_2 + V > x + 1) \frac{1}{12} e^{-\frac{x}{12}} \, dx = 0.9069. \]
The matrix $D$ is computed from (21) as
\[ D = \begin{pmatrix} 1 - p_{1,1} + c_1 & e^{-0.18} - p_{1,2} + c_2 \\ 1 - p_{2,1} + c_1 & e^{-0.015} - p_{2,2} + c_2 \end{pmatrix} \approx \begin{pmatrix} 0.8222 & 0.6467 \\ 0.5238 & 0.4725 \end{pmatrix}. \tag{29} \]
According to (22), we have
\[ W_q(t) = 1 - CD^{-1} \exp (Zt) = 1 - 0.01 \exp (-0.18t) - 0.82 \exp (-0.015t). \]
Eq. (29) is the exact $W_q(t)$ of the M/H/1 queueing model (p. 290 in [21]).

**Example 3 M/E/1 queueing model** (p. 293 in [21]): Customer arrivals are determined by a Poisson process with an arrival rate $\lambda = 1.2$ customers/h. The service times have an Erlang type-2 distribution with $\mu_1 = \mu_2 = 3$ customers/h.

The PDFs of service time and interarrival time are
\[ f_V(t) = 9t \exp (-3t). \]
The Laplace transforms $V(s)$ and $T(-s)$ are

$$V(s) = \left(\frac{3}{3+s}\right)^2,$$

$$T(-s) = \frac{1.2}{1.2-s}.$$  \hspace{1cm} (30)

The denominator of $V(s)$ is a polynomial of degree two and (30) has two negative real roots: $z_1 \approx -4.39$ and $z_2 \approx -0.41$. Therefore, the conditions of Theorem 3 are true. In the case of two roots, we copy the procedure in Example 2 and have 1) a $2 \times 1$ vector $\mathbf{Z}$ as

$$\mathbf{Z} = (-4.39, -0.41);$$

2) a $1 \times 2$ vector $\mathbf{C}$ as

$$\mathbf{C} = (0.4898, 0.0671).$$

3) a $2 \times 2$ matrix $\mathbf{D}$ as

$$\mathbf{D} = \begin{pmatrix} 0.8905 & -0.0477 \\ 0.6197 & 0.0804 \end{pmatrix}.$$

According to (22), we have

$$W_q(t) = 1 - \mathbf{C} \mathbf{D}^{−1} \exp(\mathbf{Z} t) = 1 + 0.0221 \exp(-4.39 t) - 0.8221 \exp(-0.41 t).$$  \hspace{1cm} (31)

Eq. (31) is the exact $W_q(t)$ of the $M/E_2/1$ queueing model (p. 293 in [21]).

Throughout the above demonstration, our method is shown to be simple and straightforward, and thus this method can be computerized. It worth noting that since the $V(s)$s and $T(-s)$s in Examples 1 and 3 are reciprocals of polynomials, their $W_q(t)$s can also be obtained by Smith’s method. However, the $V(s)$ in Example 2 is not the case so Smith’s method cannot be applied. The conventional method (pp. 288–290 in [21]) to solve Example 2 is shown to be much complicated than our method.

**B. D/M/1 Waiting-Time Distribution**

**Example 4 D/M/1 queueing model:** Customer arrivals are constant with a fixed interarrival time $T = c$ hours. Customer service times have an exponential distribution with a mean service rate $\mu$ customers/h.

Lindley’s equation of Example 4 is

$$W_q(t) = \max\{0, W_q(t) + V - c\}.$$

The PDF of $V$ is an exponential distribution:

$$f_V(t) = \mu \exp(-\mu t).$$

The Laplace transform $V(s)$ is

$$V(s) = \int_0^\infty e^{-st} \mu \exp(-\mu t) dt = \frac{\mu}{\mu + s}.$$  \hspace{1cm} (32)

Here, the constant interarrival time is the primary difficulty because when $T$ equals $c$ with probability “1”, the PDF of $T$ is a Dirac delta function:

$$f_T(t) = \delta(t-c)$$  \hspace{1cm} (33)

so the Laplace transform $T(-s)$ is

$$T(-s) = \int_0^\infty e^{-st} \delta(t-c) dt = e^{-ct}.$$  \hspace{1cm} (34)

Obviously, $V(s)$ is a rational function but $T(-s)$ is not.

To circumvent this problem, one reasonable approach is to use some other functions to approximate the Dirac delta function, such as the Erlang distribution:

**Lemma 4:** If $c$ is greater than $0$, then $\delta(t-c)$ can be approximated by an Erlang type-$k$ distribution with a mean $c/k$ and an infinite $k$:

$$\delta(t-c) = \lim_{k \to \infty} \frac{k^{c/k}}{(k-1)!} \exp\left(-\frac{k}{c} t\right), \quad c > 0.$$  \hspace{1cm} (36)

Eq. (34) is valid because its right-hand side

1) is infinite at $c$:

$$\lim_{k \to \infty} \frac{k^{c/k}}{(k-1)!} \exp\left(-\frac{k}{c} t\right) = \lim_{k \to \infty} \frac{k^k}{(k-1)!} \exp k = \infty,$$

2) satisfies the identity

$$\int_0^\infty \frac{k^{c/k}}{(k-1)!} \exp\left(-\frac{k}{c} t\right) dt = 1, \quad \forall k \geq 1.$$  \hspace{1cm} (35)

Let us choose an arbitrary $k$. The PDF of $T$ is an Erlang type-$k$ distribution:

$$f_T(t) = \frac{k^{c/k}}{(k-1)!} \frac{k}{c} \exp\left(-\frac{k}{c} t\right).$$

The Laplace transform $V(-s)$ is

$$T(-s) = \int_0^\infty e^{-st} \frac{k^{c/k}}{(k-1)!} \frac{k}{c} \exp\left(-\frac{k}{c} t\right) dt = \left(\frac{k}{k-sc}\right)^k.$$  \hspace{1cm} (34)

Now we can verify that both $V(s)$ and $T(-s)$ are rational functions. By letting

$$V(s)T(-s) - 1 = 0,$$

we have

$$\mu + s = \left(1 - \frac{sc}{k}\right)^k.$$  \hspace{1cm} (35)

The denominator of $V(s)$ is a polynomial of degree one, then based on Lemma 2, (35) shall have one root $z_1$ with negative real parts. When $k$ in (35) goes to infinity, the right-hand side of (35) approaches to $\exp(-sc)$ and (35) becomes

$$\lim_{k \to \infty} \left(1 - \frac{sc}{k}\right)^k = \exp(-sc)$$

or equivalently

$$\log\left(\frac{\mu}{\mu + sc}\right) = c.$$  \hspace{1cm} (36)

If the root $z_1$ satisfies (36), it must be the real number. In the case of one root, we copy the procedure in Example 1 and have the $W_q(t)$ as
mean can be simplified to two curves completely overlap with each other.

The PDFs of Eqs. (40) and (41) are the exact steps in [16] to find the

If we replace

Because

from (38):

Clearly, our method is inapplicable in this instance. In fact, the

However, the denominator of V(s) is a polynomial of degree k,
then based on Lemma 2, (44) will have k roots with negative
real parts (e.g., in Example 3, (30) has two roots). If we let k

goes to infinity, then (44) will have infinite number of roots.

IV. NETWORK CALCULUS: THE EFFECTIVE BANDWIDTH
MODEL AND THE EFFECTIVE CAPACITY MODEL

Our method and the effective bandwidth model are compared in Section IV-A, followed by a comparison to the effective capacity model in Section IV-B.

A. Effective Bandwidth Model

Consider a network calculus system model with a constant service c bits/slot. Lindley’s equation (5) is written as

\[ Q(t) = \max \{0, Q(t-1) + A(t-1) - c \}. \]  

Define

1) the asymptotic log-moment generating function \( \Lambda_A(u) \)

of \( A(t) \) as

\[ \Lambda_A(u) = \lim_{t \to \infty} \frac{1}{t} \log E\left[ \exp(uA(t)) \right]. \]  

2) the effective bandwidth \( \alpha_A(u) \) as

\[ \alpha_A(u) = \frac{\Lambda_A(u)}{u}, u \geq 0. \]  

If

1) the arrival process \( A(t-1), t \geq 1 \) is stationary,
2) \( \Lambda_d(u) \) exists and is differentiable for all \( u \in (0, \infty) \), and
3) there is a unique QoS exponent \( u^* > 0 \) that satisfies
\[
\alpha_s(u^*) = c, \tag{50}
\]
then based on the large deviation theory, the backlog size process \( \{Q(t), t \geq 1 \} \) converges in distribution to a RV \( Q \) that satisfies (p. 291 in [23])
\[
\lim_{u \to B} \frac{1}{u} \log P(Q > B) = -u^*, \tag{51}
\]
where \( B \) is a backlog bound. The CCDF of backlog size can be approximated by (51):
\[
P(Q > B) = \exp(-u^* B). \tag{52}
\]
We now use the effective bandwidth model to approximate the CCDF of \( Q \) in the example below:

**Example 6:** The arrivals \( A(0, 1), A(1, 2), A(2, 3) \) ... are IID exponential RVs identical to a single RV \( A \) with a mean arrival rate \( 1/\lambda \) bits/slot. The services \( S(0, 1), S(1, 2), S(2, 3) \) ... are fixed to \( c \) bits/slot.

Lindley’s equation of Example 6 is
\[
Q(t) = \max\{0, Q(t - 1) + A - c\}. \tag{53}
\]
The PDF of arrival \( A \) is
\[
f_a(x) = \lambda e^{-\lambda x}. \tag{54}
\]
The moment generating function (MGF) \( M_A(u) \) of arrival \( A \) is
\[
M_A(u) = \int_0^\infty e^{ux} e^{-\lambda x} \, dx = \frac{\lambda}{\lambda - u}. \tag{55}
\]
Since the arrivals are IID RVs, we have a simplified \( \Lambda_d(u) \) and a simplified \( \alpha_s(u) \):
\[
\Lambda_d(u) = \log E(\exp(uA)) = \log M_A(u), \tag{56}
\]
\[
\alpha_s(u) = \frac{\log M_A(u)}{u}. \tag{57}
\]
If there is a unique QoS exponent \( u^* > 0 \) that satisfies
\[
\log M_A\left(\frac{\lambda}{\lambda - u^*}\right) = \frac{\lambda}{\lambda - u^*} = c, \tag{58}
\]
then the CCDF of \( Q \) is approximated by (52).

From the above example, we observe that (53) is of the form (32) so
1) Example 6 is essentially identical to Example 4 and
2) the CCDF of backlog size in Example 6 can be exactly obtained by our method or taking the same steps in Example 4.

Because the MGF \( M_A(-u) \) is the two-sided Laplace transform of \( f_a(x) \), if \( \lambda \) in (58) equals \( \mu \) in (36) and if \( u^* \) satisfies (58), then \(-u^*\) will automatically satisfy (36). In other words, the effective bandwidth model can also be considered as part of our method.

**B. Effective Capacity Model**

Consider a network calculus system model fed by a constant source data rate \( c \) bits/slot. Lindley’s equation (5) is written as
\[
Q(t) = \max\{0, Q(t - 1) + c - S(t - 1, t)\}. \tag{59}
\]
Define
1) the asymptotic log-moment generating function \( \Lambda_s(-u) \) of \( S(t) \) as
\[
\Lambda_s(-u) = \lim_{t \to \infty} \frac{1}{t} \log E[\exp(-uS(t))], \tag{60}
\]
2) the effective capacity \( \alpha_s(u) \) as
\[
\alpha_s(u) = \frac{\Lambda_s(-u)}{u}, \tag{61}
\]
If
1) the service process \( \{S(t - 1, t), t \geq 1\} \) is stationary,
2) \( \Lambda_s(-u) \) exists and is differentiable for all \( u \in (0, \infty) \) and
3) there is a unique QoS exponent \( u^* > 0 \) that satisfies
\[
\alpha_s(u^*) = c, \tag{62}
\]
then based on the large deviation theory, the backlog process \( \{Q(t), t \geq 1\} \) converges in distribution to a RV \( Q \) that satisfies (20)
\[
\lim_{u \to B} \frac{1}{u} \log P(Q > B) = -u^*. \tag{63}
\]
The CCDF of backlog size can be approximated by (63):
\[
P(Q > B) = \exp(-u^* B). \tag{64}
\]
Let us use the effective capacity model to approximate the CCDF of \( Q \) in the example below:

**Example 7:** The arrivals \( A(0, 1), A(1, 2), A(2, 3) \) ... are fixed to \( c \) bits/slot. The services \( S(0, 1), S(1, 2), S(2, 3) \) ... are IID exponential RVs identical to a single RV \( S \) with a mean service rate \( 1/\mu \) bits/slot.

Lindley’s equation of Example 7 is
\[
Q(t) = \max\{0, Q(t - 1) + c - S\}. \tag{65}
\]
The PDF of service \( S \) is
\[
f_s(x) = \mu e^{-\mu x}. \tag{54}
\]
The MGF \( M_S(-u) \) of service \( S \) is
\[
M_S(-u) = \int_0^\infty e^{-ux} e^{-\mu x} \, dx = \frac{\mu}{\mu + u}. \tag{55}
\]
Since the services are IID RVs, we have a simplified \( \Lambda_s(-u) \) and a simplified \( \alpha_s(u) \):
\[
\Lambda_s(-u) = \log E(\exp(-uS)) = \log M_S(-u), \tag{56}
\]
\[
\alpha_s(u) = \frac{\log M_S(-u)}{u}. \tag{57}
\]
If there is a unique QoS exponent \( u^* > 0 \) that satisfies
\[
\log M_S\left(\frac{\mu}{\mu + u^*}\right) = \frac{\mu}{\mu + u^*} = c, \tag{58}
\]
then the CCDF of \( Q \) is approximated by (64).

From the above example, we have three observations:
1) (65) is of the form (42) so Example 7 is essentially identical to Example 5 and it cannot be solved by our method;
2) the CCDF of backlog in Example 7 can be obtained by slightly modifying (46):
\[
P(Q > B) = 1 - (1 - \mu c) \sum_{k=0}^{m} \frac{(-\mu(B-kc))^k}{k!} \exp(\mu(B-kc)), \tag{67}
\]
\[
m \leq \frac{B}{c} < m + 1;
\]
3) (64) is an approximated CCDF of backlog size so the effective capacity model can be considered as an approximation model.2

Fig. 4 shows actual, approximated and solution CCDFs of backlog size when slot time is 1ms, c = 75 Kbps and 1/μ = 100 Kbps (this parameter setting is used in [20]). The approximated and solution P(Q > B)s in this scenario are calculated from (64) and (67). We can see that the curves of actual and approximated P(Q > B)s completely overlap with each other while the curves of actual and approximated P(Q > B)s have noticeable differences.

V. CONCLUSION

By realizing that Lindley’s equation exists in queuing theory and network calculus, this paper presents a new method to solve one type of Lindley’s equation for these two research branches. Specifically, the equation V(s)T(−s) − 1 = 0 only has finite negative real roots, where V(s) and T(−s) are the Laplace transforms of service time’s PDF and interarrival time’s PDF (evaluated at −s).

For queuing theory, we use this method to derive the exact M/M/1, M/H2/1 and M/E2/1 waiting-time distributions, and for the first time find the exact D/M/1 waiting-time distribution.

For network calculus, we use two examples to compare our method with the effective bandwidth model and the effective capacity model, respectively. We observe that the distribution function of backlog size in the first example can be obtained exactly by our method and partially by the effective bandwidth model; however, such a distribution function in the second example cannot be obtained by our method but can be approximated by the effective capacity model.

Queueing theory and network calculus are mostly studied as two independent research branches. We hope this work would raise a possibility of unifying them.

2 If μ in (66) equals λ in (45) and if u* satisfies (66), then −u* will satisfy (45). However, we have shown that it is impossible to solve Example 4 through (45). The effective capacity model may only be considered as an approximation model.

APPENDIX I: PROOF OF THEOREM 3

If the conditions of Lemma 2 hold, then W_q(t) is of the form

\[ W_q(t) = 1 - \sum_{i=1}^{n} k_i \exp(z_i t) \]  \hspace{1cm} (68)

and the CCDF of W_q is of the form

\[ P(W_q > t) = \sum_{i=1}^{n} k_i \exp(z_i t) \]  \hspace{1cm} (69)

Define an n × 1 vector Z as

\[ Z = (z_1, z_2, \ldots, z_n)^T \]

and a 1 × n vector K as

\[ K = (k_1, k_2, \ldots, k_n). \]

Eq. (68) is written as

\[ W_q(t) = 1 - K \exp(Z t). \]  \hspace{1cm} (72)

If we assume that z_1, z_2, ..., z_n are all negative real numbers (the condition of Theorem 3), then we can use (68), (11) and (13) to express the PDF of W_q

\[ f_{w_i}(t) = q f_{w^*}(t) + (1 - q) \delta(t), \]  \hspace{1cm} (73)

where

\[ f_{w^*}(t) = \frac{1}{q} \sum_{i=1}^{n} k_i z_i \exp(z_i t), \]  \hspace{1cm} (74)

\[ q = \sum_{i=1}^{n} k_i. \]  \hspace{1cm} (75)

In order to determine each entry of K in (72), we have to find N linear equations and it is the time for the boundary condition (15). For convenience, we reproduce this condition here:

\[ P(W_q > t) = q P(W^* + V > T + t) + (1 - q) P(V > T + t). \]  \hspace{1cm} (76)

Define n auxiliary RVs and the PDF of the \( i \)th RV Z_i^+ is

\[ f_{Z_i^+}(t) = -z_i \exp(z_i t). \]  \hspace{1cm} (77)

By using (77), \( f_{w^*}(x) \) in (74) can be expressed as

\[ f_{w^*}(t) = -\frac{1}{q} \sum_{i=1}^{n} k_i f_{Z_i^+}(t). \]  \hspace{1cm} (78)

Then we use (69), (75), (78) and convolution (the symbol “*”) represents a standard convolution operation) to expand (76):

\[
\sum_{i=1}^{N} k_i \exp(z_i x) = \sum_{i=1}^{N} k_i \int_{t=0}^{\infty} \left( f_{Z_i^+} * f_{v_i} \right)(x) f_{v_i}(y) dy dx,
\]

\[ + (1 - q) P(V > T + t) \]

which is equivalent to

\[
\sum_{i=1}^{N} k_i \exp(z_i x) = \sum_{i=1}^{N} k_i \int_{t=0}^{\infty} \left( f_{Z_i^+} * f_{v_i} \right)(x) f_{v_i}(y) dy dx + (1 - q) P(V > T + t). \]  \hspace{1cm} (79)

Eq. (79) can be re-written as

\[
\sum_{i=1}^{N} k_i \left( \exp(z_i x) - P(Z_i^+ + V > T + t) \right) = (1 - q) P(V > T + t). \]  \hspace{1cm} (80)

Replacing t in (80) with \( \Delta_1, \Delta_2, \Delta_3 \ldots, \Delta_n \) generates \( n \) equations:

\[
\sum_{i=1}^{N} k_i \left( \left( 1 - P(Z_i^+ + V > T + \Delta_i) \right) - (1 - q) P(V > T + \Delta_i) \right) \]
\[
\sum_{i=1}^{N} k_i \left( \exp(z_i) - P(Z_i^+ + V > T + \Delta_z) \right) = (1 - q) P(V > T + \Delta_z).
\]

Further define

1. a \(1 \times N\) vector \(C\): the \(i^{th}\) entry \(c_i\) is \(c_i = P(V > T + \Delta_z)\),
2. an \(N \times N\) matrix \(P\): the \((i, j)^{th}\) entry \(p_{i,j}\) is \(p_{i,j} = P(Z_i^+ + V > T + \Delta_z)\),
3. an \(N \times N\) matrix \(D\): the \((i, j)^{th}\) entry \(d_{i,j}\) is \(d_{i,j} = \exp(z_i \Delta_z) - p_{i,j} + c_i\).

The vector \(K\) can be uniquely determined by
\[
K = CD^{-1}.
\]

Substituting \(K\) in (72) with (81) yields
\[
W_q(t) = 1 - CD^{-1} \exp(zt),
\]

which is (22). The essential steps in this appendix to compute \(W_q(t)\) are organized and presented in Theorem 3.

REFERENCES

[1] J. J. Lieberman and F. S. Hillier, Introduction to Operations Research, 9th ed. McGraw-Hill Higher Education, 2009.
[2] J.-Y. le Boudec and P. Thiran, Network Calculus: A Theory of Deterministic Queueing Systems for the Internet. Online Version of the Book Springer Verlag - LNCS 2050, 2012.
[3] C.-S. Chang, “Stability, queue length, and delay of deterministic and stochastic queueing networks,” vol. 39, no. 5, pp. 913–931, 1994.
[4] K. Pandit, J. Schmitt, and R. Steinmetz, “Network calculus meets queuing theory - a simulation based approach to bounded queues,” in Quality of Service, 2004, pp. 114–120.
[5] K. Angrishi and U. Killat, “An approach using node operating point for performance analysis with network calculus,” 2011 Int. Symp. Perform. Eval. Comput. Telecommun. Syst., pp. 30–37, 2011.
[6] F. Ciucu, “Network Calculus Delay Bounds in Queuing Networks with Exact Solutions,” in Proceedings of the 20th International Teletraffic Conference on Managing Traffic Performance in Converged Networks, 2007, pp. 495–506.
[7] F. Ciucu and J. Schmitt, “Perspectives on network calculus: no free lunch, but still good value,” SIGCOMM Comput. Commun. Rev., vol. 42, no. 4, pp. 311–322, 2012.
[8] F. Ciucu, F. Poloczek, and J. Schmitt, “Sharp Bounds in Stochastic Network Calculus,” in Proceedings of the ACM SIGMETRICS/International Conference on Measurement and Modeling of Computer Systems, 2013, pp. 367–368.
[9] Y. Jiang, “Network calculus and queuing theory: two sides of one coin: invited paper,” in Proceedings of the Fourth International ICST Conference on Performance Evaluation Methodologies and Tools, 2009, pp. 37:1–37:12.
[10] Y. Jiang, “Stochastic network calculus for performance analysis of Internet networks—An overview and outlook,” in Computing, Networking and Communications (ICNC), 2012, no. 1, pp. 638–644.
[11] D. V Lindley, “The theory of queues with a single server,” Math. Proc. Cambridge Philos. Soc., vol. 48, pp. 277–289, 1952.
[12] R. Agrawal, R. L. Cruz, C. Okino, and R. Rajan, “Performance bounds for flow control protocols,” IEEE/ACM Trans. Netw., vol. 7, no. 3, 1999.
[13] J.-Y. le Boudec, “Application of network calculus to guaranteed service networks,” Inf. Theory, IEEE Trans., vol. 44, no. 3, pp. 1087–1096, 1998.
[14] W. L. Smith, “On the distribution of queueing times,” Mathematical Proceedings of the Cambridge Philosophical Society, vol. 49, no. 03, p. 449, 1953.
[15] C. M. Harris, “A note on mixed exponential approximations for GI/GI/1 waiting times,” Comput. Oper. Res., vol. 12, no. 3, pp. 285–289, 1985.
[16] B. Jansson, “Effective bands for the multi-type UAS channel,” Queueing Syst. Theory Appl., vol. 9, no. 1–2, pp. 17–28, Oct. 1991.
[17] D. Wu and R. Negi, “Effective capacity: a wireless link model for support of quality of service,” Wirel. Commun. IEEE Trans., vol. 2, no. 4, pp. 630–643, Jul. 2003.
[18] D. Gross, J. F. Shortle, J. M. Thompson, and C. M. Harris, Fundamentals of Queueing Theory, 4th ed. New York, NY, USA: Wiley-Interscience, 2008.
[19] C. D. Crommelin, “Delay Probability Formulas When the Holding Times Are Constant,” P. O. Electr. Engr. J., no. 25, pp. 41–50, 1932.
[20] C.-S. Chang, Performance Guarantees in Communication Networks. Springer; 2000 edition, 2000, p. 7.