Matched disturbance rejection for energy-shaping controlled underactuated mechanical systems

Joel Ferguson\textsuperscript{1}, Alejandro Donaire\textsuperscript{2}, Romeo Ortega\textsuperscript{3} and Richard H. Middleton\textsuperscript{1}

\textbf{Abstract}—In this paper, we present a method of applying integral action to enhance the robustness of energy shaping controllers for underactuated mechanical systems with matched disturbances. Previous works on this problem have required a number of technical assumptions to be satisfied, restricting the class of systems for which the proposed solution applies. The design proposed in this paper relaxes some of these technical assumptions.

I. INTRODUCTION

Interconnection and damping assignment passivity-based control (IDA-PBC) is a nonlinear control method whereby the closed-loop system is a passive port-Hamiltonian (pH) system with desired characteristics to comply with the control objectives [1]. Many systematic solutions have been proposed for the stabilization of nonlinear systems using IDA-PBC, but the general procedure is still limited by the designers ability to solve the so called matching equations. Although the matching equation are difficult to solve in some cases, IDA-PBC has been successful applied to a variety of nonlinear systems such as electrical machines [2], [3], power converters [4], [5] and underactuated mechanical systems [6]-[8]. In general, the equilibrium of a mechanical system stabilised with IDA-PBC will be shifted when an external disturbance acts on the system. In this paper we are interested in robustifying IDA-PBC vis-à-vis constant external disturbances.

A general design for the addition of integral action to pH systems with the objective of rejecting disturbances was first presented in [9] and further discussed in [10]. The approach relies on a (possibly implicit) change of coordinates to satisfy the matching equations. The integral action scheme was tailored to fully actuated mechanical systems in [11] and underactuated mechanical systems in [12]. While in both cases the required change of coordinates to satisfy the matching equations were given explicitly, a number of technical assumption were imposed to do so. In both cases, the proposed integral action controllers were shown to preserve the desired equilibrium of the system, rejecting the effects of an unknown matched disturbance.

More recently, an alternative method for the addition of integral action to pH systems was presented in [13], [14]. In these works, the controller is constructed from the open-loop dynamics of the plant. The energy function of the controller is chosen such that it couples the plant and controller states, which allows the matching equations to be satisfied by construction. In addition, the control system studied in [14] has a physical interpretation and is shown to be equivalent to a control by interconnection (CbI) scheme, another PBC technique [15]. The method in [13] was shown to be applicable to mechanical systems with constant mass matrix.

In this paper, we extend the integral action design proposed in [13] to underactuated mechanical systems subject to matched disturbances. The assumption of a constant mass matrix is relaxed, and general mechanical systems are considered. The method proposed in this paper is constructed to directly satisfy the matching equations without the need of the technical assumptions previously used in [12]. Specifically, the presented scheme allows the open-loop mass matrix, shaped mass matrix and input mapping matrix to be state dependant.

\textbf{Notation}. In this paper we use the following notation: Let \( x \in \mathbb{R}^n \), \( x_1 \in \mathbb{R}^{m_1} \), \( x_2 \in \mathbb{R}^r \). For real valued function \( H(x), \nabla H \triangleq \left( \frac{\partial H}{\partial x} \right)^\top \). For functions \( G(x_1, x_2) \in \mathbb{R}, \nabla_x G \triangleq \left( \frac{\partial G}{\partial x_i} \right)^\top \) where \( i \in \{1, 2\} \). For fixed elements \( x^* \in \mathbb{R}^n \), we denote \( \nabla H^* \triangleq \nabla H(x^*) \). For vector valued functions \( C(x) \in \mathbb{R}^m \), \( \nabla_x C \) denotes the transposed Jacobian matrix \( \nabla_x C \).

II. PROBLEM FORMULATION

In this paper, we consider mechanical systems that have been stabilised using IDA-PBC. This class of systems can be expressed as

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0_{n \times n} & M^{-1}(q)M_d(q) \\
-M_d(q)^{-1} & J_2(q, p) - R_d(q)
\end{bmatrix}
\begin{bmatrix}
\nabla_q H_d \\
\nabla_p H_d
\end{bmatrix}
\]

\[
F_m(q, p) + \begin{bmatrix}
0_{m \times n} \\
G_m(q)
\end{bmatrix}^\top (u - d)
\]

\[
y = G^\top(q) \nabla_p H_d,
\]

\textsuperscript{1}See [12] for the detailed explanation and motivation of the problem formulation.
with Hamiltonian

$$H_d(q, p) = \frac{1}{2} p^\top M_d^{-1}(q)p + V_d(q),$$  \hspace{1cm} (2)$$

where \(q, p \in \mathbb{R}^n\) are the generalised configuration and momentum vectors respectively, \(n\) is the number of degrees of freedom of the system, \(u \in \mathbb{R}^m\) is the input, \(y \in \mathbb{R}^m\) is the output, \(d \in \mathbb{R}^m\) is a constant disturbance, \(M(q) > 0\) and \(M_d(q) > 0\) are the open-loop and shaped mass matrices of the system respectively, \(V_d(q)\) is the shaped potential energy, \(G(q)\) is the full-rank input matrix, \(R_d(q) = G(q)K_p(q)G^\top(q)\) for some \(K_p(q) \geq 0\) is the damping matrix and \(J_2(q, p) = -J_2(q, p)\) is a skew-symmetric matrix. We assume that \(\mathbb{2}\) has a strict minimum at the desired operating point \((q, p) = (q^*, 0_{n \times 1})\). For the remainder of the paper, the explicit state dependency of terms and various mapping are assumed and omitted.

The control objective is to develop a dynamic controller \(u = \beta(q, p, \zeta)\), where \(\zeta \in \mathbb{R}^m\) is the state of the controller, that ensures asymptotic stability of the desired equilibrium \((q, p, \zeta) = (q^*, 0, \zeta^*)\), for some \(\zeta^* \in \mathbb{R}^m\), even under the action of constant disturbances \(d\).

III. PREVIOUS WORK

A nonlinear PID controller was proposed in [12] as a solution to the matched disturbance rejection problem. Under the assumptions:

P.1. \(G\) and \(M_d\) are constant
P.2. \(G^{-\frac{1}{2}} \nabla_d(q^\top M^{-1}p) = 0_{(n-m) \times 1}\),

the control law was proposed to be

$$u = -\begin{bmatrix} K_p G^\top M_d^{-1} G K_1 G^\top M^{-1} + K_1 G^\top M^{-1} + K_2 K_1 \\ (K_2^\top + K_3 G^\top M_d^{-1} G K_1) G^\top M^{-1} \end{bmatrix} \nabla V_d$$

$$-\begin{bmatrix} K_1 G^\top M^{-1} \nabla_2 V_d M^{-1} + (G^\top G)^{-1} G^\top J_2 M_d^{-1} \\ K_2 K_1 G^\top M_d^{-1} \end{bmatrix} p$$

$$-\begin{bmatrix} K_2 \end{bmatrix} \nabla V_d$$

The resulting closed-loop can be expressed as

$$\begin{bmatrix} \dot{q} \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 & M^{-1} M_d^{-1} & -\Gamma_2 \\ -M_d M^{-1} & -K_3 G^\top G & -K_3 \end{bmatrix} H_z$$

where

$$H_z = \frac{1}{2} z_2^\top M_d^{-1} z_2 + V_d(q) + \frac{1}{2} (\zeta - \alpha)^\top K_1 (\zeta - \alpha)$$

and

$$z_2 = p + G K_1 G^\top M^{-1} \nabla V_d + G K_2 (\zeta - \alpha)$$

$$\Gamma_1 = M^{-1} G K_1 G^\top M^{-1}$$

$$\Gamma_2 = M^{-1} G K_2$$

$$\alpha = K_1^{-1} (K_p + K_3)^{-1} d.$$  \hspace{1cm} (5)$$

The closed-loop system \(5\) was shown to have a stable equilibrium at \((q, p, \zeta) = (q^*, 0_{n \times 1}, \alpha)\). Furthermore, if the output signal

$$y_{d3} = \begin{bmatrix} G^\top M^{-1} \nabla V_d \\ G^\top M_d^{-1} z_2 \\ K_1 (\zeta - \alpha) \end{bmatrix}$$  \hspace{1cm} (7)$$

is detectable, then the equilibrium is asymptotically stable.

The assumptions P.1 and P.2 are necessary to ensure that the dynamics of \(z_2\) in \(5\) match the dynamics of \(p\) in \(1\), using the transformation \(\mathbb{6}\).

IV. INTEGRAL ACTION FOR UNDERACTUATED MECHANICAL SYSTEMS

In this section we propose an alternative method to add integral action to mechanical systems. This is achieved by first performing a momentum transformation such that the disturbance is pre-multiplied by the identity, rather than \(G\). The integral action control law is then defined in the transformed coordinates. The resulting closed-loop is shown to be unique and preserves the desired operating point \(q^*\) of the original system.

A. Momentum transformation

To solve the integral action problem, we transform the dynamics \(1\) such that the disturbance is pre-multiplied by the identity, rather than \(G\). Such a transformation is always possible utilising the following matrix:

$$T(q) = \begin{bmatrix} (G^\top G)^{-1} G^\top \\ G^\top \end{bmatrix},$$

where \(G^\top \in \mathbb{R}^{m \times n}\) is a full-rank, left annihilator of \(G\).

Lemma 1: Consider the system \(1\) under the change of momentum coordinates \(p = T \dot{p}\). The dynamics can be equivalently expressed as

$$\begin{bmatrix} \dot{q} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ -S_1^\top & S_3 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_m H_d \end{bmatrix}$$

$$+ \begin{bmatrix} 0_{m \times n} \\ I_{m \times m} \\ 0_{m \times s} \end{bmatrix}^\top (u - d)$$

$$y = \nabla_{p_1} H_d$$

$$H_d = \frac{1}{2} p^\top M_d^{-1}(q)p + V_d(q),$$

where \(p = \text{col}(p_1, p_2), p_1 \in \mathbb{R}^m, p_2 \in \mathbb{R}^s, s = n - m, \)

$$M_d = T M_d T^\top$$

$$S_1 = M^{-1} M_d G (G^\top G)^{-1}$$

$$S_2 = M^{-1} M_d G^\top T^\top$$

$$S_31 = (G^\top G)^{-1} G^\top J_p G (G^\top G)^{-1}$$

$$S_32 = (G^\top G)^{-1} G^\top J_p G^\top T^\top$$

$$S_34 = G^\top J_p G^\top T^\top$$

and \(J_p\) is defined by

$$J_p = M_d M^{-1} \nabla_q (T^{-1} p) - \nabla_q (T^{-1} p) M^{-1} M_d$$

$$+ J_2(q, T^{-1} p).$$

As \(J_p = -J_p^\top\), both \(S_{31}\) and \(S_{34}\) are skew-symmetric.
Proof: The proof of this lemma follows along the lines of the proof of [16, Lemma 2], [17, Proposition 1] and [18, Theorem 1], therefore the full proof is omitted. An outline of the proof, however, can be found in the Appendix.

Importantly, the output of the system under the change of momentum, \( y \), remains unchanged. Indeed, 
\[
y = G^T \nabla_p H_d = G^T T^T \nabla_p H_d = G^T [G(G^T G)^{-1} (G^\top)^\top] \nabla_p H_d = [I_m \ 0_{m \times s}] \nabla_p H_d = y.
\]

B. Integral action control law

The integral action control law now proposed for the underactuated mechanical system described in \((q, p)\) coordinates by (9).

Proposition 1: Consider the system (9) in closed-loop with the controller 
\[
u = (-S_{31} + K_p + J_{c1} - R_{c1} - R_{c2}) \nabla_p H_d + (J_{c1} - R_{c1}) \nabla_p H_c,
\]
\[\zeta = -R_{c2} \nabla_p H_d - S_1^\top \nabla_q H_d + S_{32} \nabla_p H_d,
\]
where 
\[H_c = \frac{1}{2}(p_1 - \zeta)^\top K_I (p_1 - \zeta),\]
\(\zeta \in \mathbb{R}^m, K_I > 0\) and \(J_{c1} = -J_c^\top, R_{c1} > 0 R_{c2} > 0\) are constant matrices free to be chosen. Then, the closed-loop dynamics can be written in the \(\dot{p} = F(x)\) form,
\[
\begin{bmatrix}
\dot{q} \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{\zeta}
\end{bmatrix}
= F(x)
\begin{bmatrix}
\nabla_q H_d \\
\nabla_{p1} H_d \\
\nabla_{p2} H_d \\
\nabla_{\zeta} H_c
\end{bmatrix}
- \begin{bmatrix}
0_{n \times 1} \\
d \\
0_{n \times 1} \\
\frac{K_I (p_1 - \zeta)}{2}
\end{bmatrix},
\]
(15)

where 
\[
F(x) = \begin{bmatrix}
0_{n \times n} \\
-S_1^\top \\
-S_2^\top \\
-S_3^\top
\end{bmatrix}
\begin{bmatrix}
S_1 \\
J_{c1} - R_{c1} - R_{c2} \\
-S_{32}^\top \\
R_{c1} - R_{c1}
\end{bmatrix}
\begin{bmatrix}
S_2 \\
-S_{32} \\
-S_{34} \\
-R_{c2}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_{32} \\
S_{34} \\
R_{c2}
\end{bmatrix}
- \begin{bmatrix}
S_1 \\
S_{32} \\
S_{34} \\
R_{c2}
\end{bmatrix}
\]
and \(H_c : \mathbb{R}^{2n+m} \to \mathbb{R}\) is the closed-loop Hamiltonian defined as
\[H_c(q, p, p_2, \zeta) = H_d(q, p_1, p_2) + H_c(p_1, \zeta),\]
(17)

Proof: First notice that \(\nabla_{p1} H_c = -\nabla_\zeta H_c\). Due to this relationship, the dynamics of \(q\) and \(p_2\) in (9) are equivalent to the dynamics of \(q\) and \(p_2\) in (13).

Considering the dynamics of \(\zeta\) in (9) and using \(\nabla_{p1} H_c = -\nabla_{\zeta} H_c\) yields
\[
\dot{\zeta} = -R_{c2} \nabla_p H_d - S_1^\top \nabla_q H_d + S_{32} \nabla_p H_d
\]
\[= -R_{c2} (\nabla_p H_d + \nabla_{p1} H_c - \nabla_{p2} H_c) - S_1^\top \nabla_q H_d
\]
\[= -R_{c2} \nabla_{p1} H_d - R_{c2} \nabla_\zeta H_c - S_1^\top \nabla_q H_c + S_{32} \nabla_p H_d
\]
(18)

which matches the dynamics of \(\zeta\) in (15).

Finally, considering the dynamics of \(p_1\) in (9),
\[
\dot{p}_1 = -S_1^\top \nabla_q H_d + (S_{31} - K_p) \nabla_{p1} H_d + S_{32} \nabla_{p2} H_d + u - d
\]
\[= -S_1^\top \nabla_q H_d + (J_{c1} - R_{c1} - R_{c2}) \nabla_{p1} H_d + S_{32} \nabla_{p2} H_d + (J_{c1} - R_{c1}) \nabla_{p1} H_c - d
\]
\[= -S_1^\top \nabla_q H_d + (J_{c1} - R_{c1} - R_{c2}) \nabla_{p1} H_d + S_{32} \nabla_{p2} H_d
\]
\[+ (J_{c1} - R_{c1}) \nabla_{p1} H_c + R_{c2} \nabla_{p1} H_c - d
\]
\[= -S_1^\top \nabla_q H_d + (J_{c1} - R_{c1} - R_{c2}) \nabla_{p1} H_c + S_{32} \nabla_{p2} H_c
\]
\[= -S_1^\top \nabla_q H_d + (J_{c1} - R_{c1} - R_{c2}) \nabla_{p1} H_c + S_{32} \nabla_{p2} H_c
\]
\[+ R_{c2} \nabla_{p1} H_c - R_{c2} \nabla_\zeta H_c - d,
\]
(19)

which is equivalent to the dynamics of \(p_1\) in (13).

Remark 1: In the case that \(S_{31}\) and \(K_v\) are constant, The choice \(J_{c1} = S_{31}, R_{c1} = K_v\) can be made to simplify the control law (13).

C. Stability

For the remainder of this section, the stability properties of the closed-loop system (15) are considered. It is shown that the integral action control (12) preserves the desired operating point \(q^*\) of the open-loop system. Further, if the original system is detectable, then the closed-loop system is asymptotically stable.

Lemma 2: The closed-loop system (15) has an isolated equilibrium point
\[(q, p, \zeta) = (q^*, 0_{n \times 1}, -K_I^{-1}(J_{c1} - R_{c1})^{-1} d),\]
(20)

Proof: The dynamics of \(q\) in (15) can be simplified to
\[
\dot{q} = M^{-1} M_d T^\top \nabla_p H_d
\]
\[= M^{-1} M_d T^\top M_d^\top p.
\]
(21)

As \(M, M_d, M_d, T\) are full-rank, \(p = 0_{n \times 1}\) and \(\nabla_p H_d = 0_{n \times 1}\) at any equilibrium. As \(\nabla_{p1} H_c = \nabla_{p2} H_d,\)
\[
\nabla_{p1} H_c = 0_{n \times 1}.
\]
(22)

The difference between the dynamics of \(p_1\) and \(\zeta\) are given by \(\dot{p}_1 - \dot{\zeta} = (J_{c1} - R_{c1}) \nabla_{p1} H_d - d\). As \(\nabla_{p1} H_d = 0_{n \times 1}\),
\[
\nabla_{p1} H_c = -\nabla_\zeta H_c = (J_{c1} - R_{c1})^{-1} d.
\]
(23)

Recalling that \(-\nabla_\zeta H_c = -\nabla_\zeta H_c = K_I(p_1 - \zeta)\) and \(p_1 = 0, (23)\) can be rearranged to find \(\zeta = K_I^{-1}(J_{c1} - R_{c1})^{-1} d\).

Substituting the equilibrium gradients (22) and (23) into (15) and considering the dynamics of \(p\), it results in
\[
\dot{p} = -[S_1 \ 0_{n \times n} \ S_2 \ 0_{n \times n} \ S_1]
\]
\[\begin{bmatrix}
\nabla_q H_d \\
\nabla_{p1} H_d \\
\nabla_{p2} H_d \\
\nabla_\zeta H_c
\end{bmatrix},
\]
(24)

which implies that \(\nabla_\zeta H_d = \nabla_q H_d = 0_{n \times 1}\) at any equilibrium as \([S_1 \ S_2]\) is full-rank. The equilibrium gradient \(\nabla_q H_d = 0_{n \times 1}\) is satisfied by \(q = q^*\).

Proposition 2: Consider system (9) subject to unknown matched disturbance in closed-loop with the controller (13).

The following properties hold:
(i) The equilibrium (20) of the closed-loop system is stable.
(ii) If the output
\[ y_{p1} = \begin{bmatrix} \nabla_p \mathcal{H}_d - (J_{c1} - R_{c1})^{-1}d \end{bmatrix} \]  

is detectable, the equilibrium is asymptotically stable.

(iii) If the shaped potential energy \( V_d \) is radially unbounded, then the stability properties are global.

**Proof:** To verify (i), consider the function \( \mathcal{W} = \mathcal{H}_d(q, p_1, p_2) + \frac{1}{2}(z - z^*)^T K_I (z - z^*) \), where \( z = p_1 - \zeta \) and \( z^* = \hat{p}_1^* - \zeta^* = K_I^{-1}(J_{c1} - R_{c1})^{-1}d \), as a Lyapunov candidate for the system. \( \mathcal{W} \) has a strict minimum at \( \mathcal{V} \) as \( \mathcal{H}_d \) is strictly minimised by \( (q, p) = (q^*, 0_{n \times 1}) \) and \( K_I > 0 \).

Defining \( w = \text{col}(q, p_1, p_2, \zeta) \), the closed-loop dynamics \( \dot{w} = F(x) \) can be equivalently expressed as
\[ \dot{w} = F(x) \begin{bmatrix} \nabla_p \mathcal{H}_d - (J_{c1} - R_{c1})^{-1}d \\ \nabla_{p1} \mathcal{H}_{cl} - (J_{c1} - R_{c1})^{-1}d \\ \nabla_{p2} \mathcal{H}_{cl} + (J_{c1} - R_{c1})^{-1}d \\ \nabla_{\zeta} \mathcal{H}_{cl} + (J_{c1} - R_{c1})^{-1}d \end{bmatrix}. \]

The equilibrium is stable since \( F + F^T \leq 0 \), which implies that \( \dot{\mathcal{W}} \leq 0 \) along the trajectories of the closed-loop system. The claim (ii) follows by considering the structure of \( F \) and invoking LaSalle’s invariance principle.

Finally, to verify (iii), first note that the component of \( W \) associated with the controller and disturbance, \( \frac{1}{2}(z - z^*)^T K_I (z - z^*) \), is radially unbounded in \( z \). Then, recalling that \( \mathcal{H}_d \) is of the form \( \mathcal{F}_d \) and \( M_d^{-1} > 0 \), it is clear that \( \mathcal{H}_d \) is radially unbounded in \( p \). Finally, if \( V_d \) is radially unbounded in \( q \), then \( \mathcal{W} \) is radially unbounded. This implies that the closed-loop system is globally stable.

**Corollary:** The output of the system (1) is detectable when \( d = 0_{n \times 1} \) and \( u = 0_{n \times 1} \), then the closed-loop system (15) is asymptotically stable.

**Proof:** By Proposition 2 the equilibrium of the closed-loop (15) is asymptotically stable if \( y_{p1} \) is detectable. The control action (13a), evaluated at \( y_{p1} = 0_{2n \times 1} \) is \( u = d \). Further, using (12), the output of (1) resolves to be \( y = y = \nabla_p \mathcal{H}_d = 0_{n \times 1} \). Substituting \( u = d \) and \( y = 0_{n \times 1} \) into (1) recovers the zero dynamics of the original, undisturbed system. Thus, if (1) is detectable when \( d = 0_{n \times 1} \) and \( u = 0_{n \times 1} \), then the closed-loop system (15) is asymptotically stable.

V. CART PENDULUM EXAMPLE

In this section, we apply the presented integral action scheme to the cart pendulum system. For the existing IDA-PBC laws, the shaped mass matrix \( M_d \) is not constant so the integral action scheme of (12) cannot be used.

Stabilisation control of the cart pendulum using IDA-PBC was solved in [6]. After partial feedback linearisation, the cart pendulum can be modelled as a pH system of the form
\[ \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix} \nabla \mathcal{H} \\ 0_{2 \times 1} \end{bmatrix} + \begin{bmatrix} 0_{2 \times 1} \end{bmatrix} \begin{bmatrix} 1 \\ m_c + m_p \sin^2 \theta \end{bmatrix} \]
\[ \mathcal{H} = \frac{1}{2} p^T M^{-1} p + v, \]

where \( q = [q_1, q_2]^T \) is the configuration vector containing the angle of the pendulum from vertical and the horizontal position of the cart respectively, \( p = [p_1, p_2]^T \) is the generalised momenta.

\[ M = I_{2 \times 2} \]
\[ G = \begin{bmatrix} -b \cos(q_1) \\ 1 \end{bmatrix} \]
\[ v = a \cos(q_1), \]

\( m_c \) and \( m_p \) are the masses of the cart and pendulum respectively, \( a = \frac{4}{3}, b = \frac{1}{2}, g \) is the acceleration due to gravity and \( l \) is the length of the pendulum. The disturbance \( d \) is an unknown constant force collocated with the input \( u \).

Note that the system (25) is not in the form (1) as the disturbance is not constant. In the remainder of this section, the undisturbed system will be stabilised using IDA-PBC and the resulting closed-loop will be converted into the form (1) by defining a new input mapping matrix and input.

A. Energy shaping

In the case that \( d = 0_{m \times 1} \), the cart pendulum can be stabilised around a desired equilibrium \( (q_1, q_2, p) = (0, q_2, 0_{2 \times 1}) \) using the IDA-PBC law
\[ u = \begin{bmatrix} G^T G \end{bmatrix}^{-1} G^T \begin{bmatrix} \nabla_q \mathcal{H} - M_d M_d^{-1} \nabla_q \mathcal{H}_d + J_2 M_d^{-1} p \end{bmatrix} - \frac{1}{(m_c + m_p \sin^2 \theta)^2} \begin{bmatrix} K_p G^T M_d^{-1} p + u' \end{bmatrix} \]
\[ (30) \]

where
\[ M_d = \begin{bmatrix} \frac{kb^2 \cos^3 q_1}{k^2 b^2 \cos^3 q_1} & \frac{kb^2 \cos^2 q_1}{k b \cos q_1 + m_0^0} \\ -\frac{3 b}{k b^2 \cos^2 q_1} & k \cos q_1 + m_0^0 \end{bmatrix} \]
\[ V_d = \begin{bmatrix} \frac{3 b}{k b^2 \cos^2 q_1} + \frac{P}{2} \begin{bmatrix} q_2 - q_2^* + \frac{3}{b} \log (\sec q_1 + \tan q_1) \\ \frac{6 m_0^0}{kb} \tan^2 q_1 \end{bmatrix} \end{bmatrix} \]
\[ J_2 = \begin{bmatrix} p^T M_d^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix} \]
\[ \alpha = \frac{k \gamma_1}{2} \sin q_1 \begin{bmatrix} -b \cos q_1 \\ 1 \end{bmatrix} \]
\[ \gamma_1 = \frac{kb^2}{6} \cos^3 q_1, \]
\[ (31) \]

\( P > 0, k > 0, m_0^0 > 0 \) are tuning parameters, \( K_p > 0 \) is a constant used for damping injection and \( u' \) is an additional input for further control design.
The cart pendulum (28), together with the control law (30), results in the closed-loop
\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0_{2 \times 2} & M^{-1}M_d \\
-M_dM^{-1} & J_2 - G_kG^T
\end{bmatrix}
\begin{bmatrix}
\nabla H_d \\
G^T(\ddot{u} - d)
\end{bmatrix} \\
+ [0_{1 \times 2}]^T (\ddot{u} - d)
\]
(32)
\[
H_d = \frac{1}{2} \dot{p}^T M_d^{-1}(q) \dot{p} + V_d(q),
\]
where \( \ddot{u} = (M + m \sin^2 \theta)p' \), (32) and \( G = \frac{1}{m + m_p \sin^2 \theta}G \).
Clearly, the closed-loop system is of the form (1).

B. Integral action

Before the integral action control law can be applied, the momentum must be transformed as per Section [V-A]
Taking \( G^\bot = (m_c + m_p \sin^2 \theta) \left[ \begin{array}{c}
b & -b \cos q_1 \\
b \cos^2 q_1 + 1 & 1
\end{array} \right] \), the necessary momentum transformation is \( p = T \dot{p} \) with
\[
T(q) = (m_c + m_p \sin^2 \theta) \left[ \begin{array}{c}
b \cos q_1 \\
\frac{1}{b} \cos^2 q_1 + 1
\end{array} \right],
\]
(33)
and results in the transformed Hamiltonian
\[
\mathcal{H}_d = \frac{1}{2} \dot{p}^T T^{-T} M_d^{-1}(q) T^{-1} p + V_d(q).
\]
(34)
In the new momentum coordinates, the \( S \) matrices can be resolved as per (10):
\[
S_1 = \frac{m_c + m_p \sin^2 \theta}{b^2 \cos^2 q_1 + 1} \left[ \begin{array}{c}
-k b^3 \cos^4 q_1 - k \cos^2 q_1 \\
k b^2 \cos^3 q_1 + k \cos q_1 + n^0_{22}
\end{array} \right]
\]
\[
S_{31} = 0
\]
\[
S_{32} = \frac{1}{b^2 \cos^2 q_1 + 1} \left[ \begin{array}{cc}
-k \cos(q_1) \\
1
\end{array} \right],
\]
(35)
\[
\left[ M_d M^{-1} \nabla_q T^{-1}(q) \dot{p} - \nabla_q (T^{-1}(q) p) M^{-1} M_d \\
+ J_2(q, p) \right]
\left[ \begin{array}{c}
1 \\
1
\end{array} \right].
\]
As \( S_{31} = 0 \) and \( K_p \), the integral control law (13) is simplified by making the selection \( R_{c1} = K_p, J_{c1} = 0 \) which results in
\[
\ddot{u} = -R_{c2} \nabla_p \mathcal{H}_d - K_p \nabla_p \mathcal{H}_c
\]
(37a)
and
\[
\dot{\zeta} = -R_{c2} \nabla_p \mathcal{H}_d - S_{31} \nabla_q \mathcal{H}_d + S_{32} \nabla_p \mathcal{H}_d.
\]
(37b)
As discussed in (6), \( V_d \) is radially unbounded on the domain \( Q = \{ -\frac{\pi}{2}, \frac{\pi}{2} \} \times \mathbb{R} \) and the system (32) with \( \ddot{u} = d = 0 \) is detectable. Thus, by Proposition 2 and Corollary 1 the closed-loop system is asymptotically stable with region of attraction given by the set \( \{ Q \times \mathbb{R}^2 \times \mathbb{R} \} \).

C. Numerical simulation

The cart pendulum was simulated using the following plant parameters: \( g = 9.8, M = I_2 \times 2, l = 1, m_c = 1, m_p = 1 \). The desired cart position was selected to be \( q_2^* = 0 \) and the energy shaping control law (30) was implemented with the controller parameters \( k = 1, m_{\text{p2}} = 1, P = 1, K_p = 10 \).
To reject the effects the disturbance \( d \), the control law (37) was applied with the controller storage function \( \mathcal{H}_c(p_1, \zeta) = \frac{1}{2} K_1 (p_1 - \zeta)^2 \) and \( K_1 = 0.05 \).
The system was simulated for 60 seconds with state of the plant initialised at \((q_1(0), q_2(0), p(0)) = (0, 1, 0_{2 \times 1}) \) and the controller initialised at \( \zeta(0) = 0 \). For the time interval \( t \in [0, 30] \) the disturbance was set to \( d = 0 \). At \( t = 30s \), a disturbance of \( d = 2 \) was applied for the remainder of the simulation.

Figure 1 shows that the cart pendulum, together with the integral action control law, tends towards the desired equilibrium on the time interval \( t \in [0, 30] \). At \( t = 30 \), the disturbance \( d = 2 \) is applied and the states move away from the desired equilibrium. On the time interval \( t \in [30, 60] \), the integral control compensates for the disturbance and the system again approaches the desired equilibrium.

VI. CONCLUSIONS

In this paper, a method to robustify IDA-PBC via the addition of integral action to underactuated mechanical systems was presented. The method relaxes technical assumptions required by previous solutions. The control scheme preserves the desired equilibrium of the open-loop system, rejecting the effects of an unknown matched disturbance. Further, the closed-loop system was shown to be asymptotically stable provided that the passive output of the open-loop system is detectable.
REFERENCES

[1] R. Ortega and E. Garcia-Canseco, “Interconnection and damping assignment passivity-based control: A survey,” European Journal of control, vol. 10, no. 5, pp. 432–450, 2004.
[2] V. Petrović, R. Ortega, and A. M. Stanković, “Interconnection and damping assignment approach to control of PM synchronous motors,” IEEE Transactions on Control Systems Technology, vol. 9, no. 6, pp. 811–820, 2001.
[3] H. González, M. A. Duarte-Mermoud, I. Pelisier, J. C. Travieso-Torres, and R. Ortega, “A novel induction motor control scheme using IDA-PBC,” Journal of Control Theory and Applications, vol. 6, no. 1, pp. 59–68, 2008.
[4] H. Rodríguez, R. Ortega, and G. Escobar, “A robustly stable output feedback saturator controller for the Boost DC-to-DC converter,” Systems & Control Letters, vol. 40, no. 1, pp. 1–8, 2000.
[5] ——, “New Family of Energy-Based Non-Linear Controllers for Switched Power Converters,” in IEEE International Symposium on Industrial Electronics, Pusan, 2001, pp. 723–727.
[6] J. Acosta, R. Ortega, and A. Astolfi, “Interconnection and damping assignment passivity-based control of mechanical systems with underactuation degree one,” IEEE Transactions on Automatic Control, vol. 50, no. 12, pp. 1936–1956, 2005.
[7] J. Sandoval, R. Ortega, and R. Kelly, “Interconnection and Damping Assignment Passivity-Based of the Pendubot,” in IFAC World Congress, Seoul, Korea, 2008, pp. 723–727.
[8] J. G. Romero, A. Donaire, and R. Ortega, “Robust energy shaping control of mechanical systems,” Systems and Control Letters, vol. 62, no. 9, pp. 770–780, 2013.
[9] A. Donaire, J. G. Romero, R. Ortega, B. Siciliano, and M. Crespo, “Robust IDA-PBC for Underactuated Mechanical Systems Subject to Matched Disturbances,” International Journal of Robust and Nonlinear Control, vol. 27, no. 8, pp. 1000–1016, 2016.
[10] J. G. Romero, A. Donaire, and R. Ortega, “Robust energy shaping control of mechanical systems,” Systems and Control Letters, vol. 62, no. 9, pp. 770–780, 2013.
[11] A. Donaire, J. G. Romero, R. Ortega, B. Siciliano, and M. Crespo, “Robust IDA-PBC for Underactuated Mechanical Systems Subject to Matched Disturbances,” International Journal of Robust and Nonlinear Control, vol. 27, no. 8, pp. 1000–1016, 2016.
[12] J. Sandoval, R. Ortega, and R. Kelly, “Interconnection and Damping Assignment Passivity-Based of the Pendubot,” in IFAC World Congress, Seoul, Korea, 2008, pp. 723–727.

APPENDIX

Proof of Lemma Let $x_m = \text{col}(q, p)$, $x_m = \text{col}(q, p)$ and $x_m = g_t(x_m) = (q, Tp)$. The transformed Hamiltonian is defined as

$$\mathcal{H}_d(q, p) = \mathcal{H}_d(q, T^{-1}(q)p) = \frac{1}{2}m^T \{\begin{bmatrix} M_d^{-1}(q)T^{-1}(q)p + V_d(q) \\ M_d^{-1}(q) \end{bmatrix} \}_{|x_m=g_t^{-1}(q,p)}.$$

Utilising the differential of $g_t$ (see [19]) can be equivalently expressed in $x_m$ as

$$\dot{x}_m = \{ \nabla_{\dot{x}_m} g_t F_m \nabla x_m g_t \}_{|x_m=g_t^{-1}(q,p)} \nabla x_m \mathcal{H}_d + \{ \nabla_{\dot{x}_m} g_t G_m \}_{|x_m=g_t^{-1}(q,p)} (u - d_m)$$

$$= \{ \begin{bmatrix} I_{l \times l} & 0_{l \times l} \\ \nabla q (Tp) & T \end{bmatrix} \}_{|x_m=g_t^{-1}(q,p)} \begin{bmatrix} \nabla q H_d \\ \nabla p H_d \end{bmatrix} + \{ \begin{bmatrix} I_{l \times l} & 0_{l \times l} \\ \nabla q (Tp) & T \end{bmatrix} \}_{|x_m=g_t^{-1}(q,p)} \begin{bmatrix} M_d^{-1} \nabla q H_d \\ \nabla p H_d \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ 0_{n \times m} \end{bmatrix} T(J_p - R_d)T^\top (u - d),$$

where $J_p$ is defined in [11]. Recalling that $R_d(q) = G(q)K_p(q)G^\top(q)$, the term $TR_dT^\top$ can be simplified to

$$TR_dT^\top = \begin{bmatrix} (G^\top G)^{-1}G^\top \\ G^\perp \end{bmatrix} GK_p G^\top \begin{bmatrix} (G^\top G)^{-1}G^\top \\ G^\perp \end{bmatrix}^\top \begin{bmatrix} K_p & 0_{m \times s} \\ 0_{s \times m} & 0_{s \times s} \end{bmatrix}.$$