GEOMETRICALLY DISTINCT SOLUTIONS GIVEN BY SYMMETRIES OF VARIATIONAL PROBLEMS WITH THE $O(N)$-SYMMETRY

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Abstract. For variational problems with $O(N)$-symmetry the existence of several geometrically distinct solutions has been shown by use of group theoretic approach in previous articles. It was done by a crafty choice of a family $H_i \subset O(N)$ subgroups such that the fixed point subspaces $E^{H_i} \subset E$ of the action in a corresponding functional space are linearly independent, next restricting the problem to each $E^{H_i}$ and using the Palais symmetry principle. In this work we give a thorough explanation of this approach showing a correspondence between the equivalence classes of such subgroups, partial orthogonal flags in $\mathbb{R}^N$, and unordered partitions of the number $N$. By showing that spaces of functions invariant with respect to different classes of groups are linearly independent we prove that the amount of series of geometrically distinct solutions obtained in this way grows exponentially in $N$, in contrast to logarithmic, and linear growths of earlier papers.

1. Introduction

The purpose of this paper is to describe a group theoretical scheme which arose in works on $O(N)$-invariant variational problems as a method to show the existence of several geometrically different series of solutions distinguished by their symmetry properties. This approach originated from the studies of problem how to find sign-changing solutions of some nonlinear elliptic equations, that is of importance in the PDEs theory. We would like to point out only these works in which the group and representation theory approach have been introduced and developed. This means that we do not mention many important papers and results based on a use of this kind of a symmetry approach. Especially we do not expose works which use the same approach group theoretical approach, but are distinguished by their analytical form.

Up to our knowledge the pioneer of this approach was the paper [5] of Bartsch and Willem, where they used it in very particular form. They studied a semilinear elliptic problem

(1) $-\Delta u + b(|x|) u = f(|x|, u), \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N).$

The weak solutions of (1) correspond to the critical points of functional

$$\Phi(u) := \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla(u)|^2 + \frac{1}{2} b(|x|) u^2 - F(|x|, u) \right) dx,$$

with $F(r, u) = \int_0^u f(r, v)dv$ being the primitive of $f$ (cf. [5], also monographs [11], [20]). Observe that the nonlinear functional $\Phi$ is $O(N)$-invariant with respect to the action of $O(N)$ on $\mathbb{R}^N$. A similar holds for the autonomous nonlinear elliptic problem

(2) $-\Delta u = f(u), \quad x \in \Omega \subset \mathbb{R}^N, \quad u \in H^1(\Omega)$

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where $\Omega$ is either bounded region with smooth boundary or $\Omega = \mathbb{R}^N$ invariant with respect to the action of $O(N)$ on $\mathbb{R}^N$. In this case the corresponding nonlinear functional is equal to

$$\Phi(u) := \int_{\Omega} \left( \frac{1}{2} |\nabla (u)|^2 - F(u) \right) dx$$

with $F(u) = \int_0^u f(v) dv$ being the primitive of $f$ (cf. [5], also monographs [11], [20]). $\Phi$ is $O(N)$-invariant with respect to the action of $O(N)$ on $\mathbb{R}^N$.

The existence of solutions of (1) and (2) is obtained by standard and well-known variational methods (cf. [5], [11], [20]). In particular, analytical assumptions on $f$, a boundary condition, and $\mathbb{Z}_2$-symmetry allow to apply the Ambrosetti-Rabinowitz symmetry mountain pass theorem of [11] or the fountain theorem of [2] (cf. [11], [20] for general references) which gives infinitely many solutions of discussed problem (2). Note that any function space $E$ related to it has a natural $O(N)$ linear action given by

$$g u(x) := u(g^{-1}x).$$

Consequently, with every closed subgroup $G \subset O(N)$ we can associate the corresponding linear subspace $E^G$ of fixed points of $G$, which is infinite-dimensional for the discussed function spaces.

Moreover, by posing the problem (2) in $E^G$, i.e restricting $\Phi$ to $E^G$, finding its critical points and finally using the Palais principle of symmetry (cf. [16]) we get solutions, or respectively infinitely many solutions of studied problem which possess the given symmetry. In particular taking $G = SO(N)$ we get the radial solutions (see [5] for references up to 1993).

In order to get non-radial solutions of (1) Bartsch and Willem took the subgroup $H \subset O(N)$ defined as follows:

$$\forall ~ 2 \leq m \leq N/2, \quad \text{and} \quad 2m \neq N-1 \quad \text{let} \quad H := O(m) \times O(m) \times O(N-2m).$$

Next, they extended $H$ to $G = \{H \cup \{\tau\}\}$, where $\tau \in O(N)$ is the linear map transposing the two first coordinates $(x, y, z) \mapsto (y, x, z)$. Observe that $\tau \in N(H)$, the normalizer of $H$ in $O(N)$. This allows to define a representation $\rho$ of $G$ in a function space $E$ putting

$$(gf)(x) := \rho(\tau h)f(h^{-1}\tau^{-1}x) = -f(h^{-1}\tau^{-1}x), \quad \text{for} \quad g = \tau h \in G,$$

where $\rho : G \to O(1) = \{-1, 1\}$ is a representation of $G$ defined by $\rho(\tau) = 1$, $\rho(h) = 1$ for every $h \in H$. Such functions, thus solutions, are called $\rho$-intertwining (cf. [7]).

It is easy to verify that functions belonging to $E^G$ are sign changing with the zero set containing a hyperplane thus must not be radial. Consequently, restricting $\Phi$ to $E^G$ they have got infinitely many non radial sign-changing solutions $\{u_n\}_{n=1}^\infty$ of (1). We must add that they have to use the space $E^G \subset E^H$ also for an analytical reason. Indeed, since the embedding of $E^H \subset L^4(\mathbb{R}^N)$ is compact due the Lions theorem, the restriction $\Phi_{|E^G}$ of functional $\Phi$ satisfies the Palais condition. Note that the above trick and assumption of the Lions theorem require $m \geq 2$, and $N - 2m \geq 2$ which can be satisfied only if $N = 4$, or $N \geq 6$. Taking any $l \in I_N := \{i \in \mathbb{N} : 2 \leq i \leq N/2, 2i \neq N-1\}$ as $m$ they get an infinite sequences $\{u_n^k\}$ of solutions of (1) and showed that these sequences of solutions are geometrically different with respect to the symmetry, i.e. none of $u_n^m$ is in the $O(N)$-orbit of any $u_n^k$ if $m \neq k$. In other words two solutions $u, v$ are geometrically distinguished with respect to the action of group $O(N)$ if

$$\exists g \in O(N) \text{ such that } v(x) = u(g^{-1}x).$$

Consequently, the number of those sequences of solutions containing elements in different $O(N)$-orbits is at least $\left\lfloor \log_2 \frac{N+2}{3} \right\rfloor$, as shows a careful inspection of [5] Proposition 4.1, p. 457, (here $[r]$ denote the integer part of $r$).

The use of various function spaces distinguished by their symmetries property has been displayed in [3] where the authors made a general remark that the amount of geometrically distinct series of
solutions obtained in this way is related to the number of partitions of $N$. They studied another nonlinear problem which could be handled by this approach:

$$
\begin{cases}
  (-\Delta)^m u = |u|^{q-2} u, & \text{in } \mathbb{R}^N, \\
  u \in D^{m,2}(\mathbb{R}^N)
\end{cases}
$$

where $D^{m,2}(\mathbb{R}^N)$ is a completion of $C^0_0(\mathbb{R}^N)$ in a norm (cf. [3]), $N > 2m$, and $q = \frac{2N}{N-2m}$.

The next step in a developing group theoretical scheme for finding changing-sign solutions of nonlinear elliptic problems has been done in [10]. For two groups of the form $H = O(N_1) \times O(N_2) \times \cdots \times O(N_r) \subset O(N)$ and $K = O(M_1) \times O(M_2) \times \cdots \times O(M_s) \subset O(N)$, $\sum_i N_i = \sum_j M_j = N$, there was given a sufficient condition on partitions $N_1, N_2, \ldots, N_r$ and $M_1, M_2, \ldots, M_s$ which guarantees that the group $G = \langle H, K \rangle$ generated by $H$ and $K$ acts transitively on the sphere $S(\mathbb{R}^N)$. It states that there is not $\tilde{r} < r$ and $\tilde{s} < s$ such that

$$
\sum_{i=1}^{\tilde{r}} N_i = \sum_{j=1}^{\tilde{s}} M_j = N' < N.
$$

To assign to each such subgroup $H$ as above a subspace consisted only of functions changing signs (if are nonzero!) we have to assume that there exists an element $\tau \in N(H) \setminus H$, where $N(H)$ the normalizer of $H$ in $O(N)$, of order two. The latter is satisfied if there exist $i \neq j$ such that $N_i = N_j$. Consequently, every such subgroup $H$ as above defines an infinite series of changing-sign solutions in $E^G$ with $G = \langle H, \tau \rangle$. Next, the condition (5) implies that for two subgroups $H, K$ the corresponding subspaces $E^G, E^{G'}$, with $G' = \langle K, \tau' \rangle$, are linearly independent, because $E^G \cap E^{G'}$ consists of radial functions only, thus the zero here. In [10] we showed that there exists at least $s_N = \text{card } I_N = \lfloor \frac{N-3}{2} \rfloor + (-1)^N$ different pairs of subgroups satisfying (5) by an effective construction of a special partition of $N$. The number $s_N$ does not depend only on the space dimension $N$ but on the amount of constructed partitions. Note also that $s_N \sim N/2$ as $N \to \infty$, but the sequence $\{s_N\}_{N \geq 4}$ is not increasing. This estimate is affected by the fact that to apply the Lions theorem on the compact embedding [12] we had to assume that for all $i, j N_i, M_j \geq 2$ in a constructed partition associated with $H$, respectively $K$. This way we proved the existence of $s_N$ sequences of non-radial, sign-changing weak solutions such that elements in different sequences are mutually distinguished by their symmetry properties. In [10] we considered particular problem

$$
\begin{cases}
  -\Delta_p u + |u|^{p-2} u = K(x)f(u), & x \in \mathbb{R}^N, \\
  u \in W^{1,p}(\mathbb{R}^N),
\end{cases}
$$

when $p > N$, the space dimension $N$ is large enough, and $f$ has an oscillatory behavior at the origin. Here, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the usual $p$-Laplacian of $u$, $K : \mathbb{R}^N \to \mathbb{R}$ is a measurable function, and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

This group theoretical scheme can be applied to study similar problems with different analytical part which have been successfully used by A. Kristaly and co-authors in a couple of papers (see [11] for references).

However, there are still remained open questions and objects for studies:
Q. Questions

(1) Analyze whether it is possible to improve the construction given in [10] to enlarge the number of subgroups for which the spaces $E^H$ of fixed points in $E$ can be used for the construction of subspaces described above. In particular get rid of a fixed order of partitions $(N_1, N_2, \ldots , N_r)$ and $(M_1, M_2, \ldots , M_s)$ in condition (5).

(2) Examine a dependence of a choice of such a subgroup on the orthogonal splitting into subspaces of $\mathbb{R}^N$. In particular what happens if one of the groups $H$, $K$ as above is constructed with respect to a partition of $N$ which is taken the canonical orthonormal basis but the second with respect to a partition corresponding to another orthonormal basis.

(3) Find a largest number $s_N$ of subgroups $H \subset O(N)$ with the normalizers $N(H)$ containing involutions in $N(H) \setminus H$ such that for $H \neq K$ and $G = \langle H, \tau \rangle$ and $G' = \langle K, \tau' \rangle$ we have $E^G \cap E^{G'} = \{0\}$, and for the orbit of $0 \neq v \in E^G$ we have $O(N)v \cap E^{G'} = \emptyset$.

(4) Get an information about the nodal set of every function $u \in E^G$.

In this work we give answers to all the questions formulated in Q. More precisely, the answer to Question (1) is positive and is contained in considerations of Sections 2, 3, and 4 (e.g. Theorem 3.5 Proposition 4.4). An answer to Question (2) states that a choice of orthogonal basis determining each of these two groups does not have an affect on this property. Only a relation between the partitions $(N_1, N_2, \ldots , N_r)$ and $(M_1, M_2, \ldots , M_s)$ is essential (cf. Theorem 4.1). Next, by reducing the problem of estimating the amount $s_N$ of pairs of subgroups $(H, K)$ with this property to an estimate of number of not equivalent special partitions of $N$ we show that the rate of growth of $s_N$ is exponential (cf. Theorem 4.8) which is an answer to Question (3). Finally, Corollary 4.12 gives an answer to Question (4).

Remark 1.1. Since we studying $O(N)$ invariant problems if a function $u$ is a solution then $(gu)(x) = u(g^{-1}x)$ is also a solution. Consequently, we are interested in solutions (series of solutions) which are geometrically distinct. The latter means that none of the solutions from one series is in the $O(N)$ orbit of a solution from another series.

The paper is organized as follows. After the introduction and short opening on the fixed point spaces of a representation, in Section 2 we provide an information on the groups acting transitively on spheres called the Borel groups. Next we introduce the notion of orthogonal Borel subgroups, or correspondingly the maximal orthogonal subgroups (cf. Definitions 2.6 2.6 2.14) and relate them to partial orthogonal flags in $\mathbb{R}^N$ in Proposition 2.10. In Section 3 we discuss the action of $O(N)$ on the set of all partial orthogonal flags and the set of all maximal orthogonal Borel subgroups. We show that these actions coincide which implies the correspondence of the orbits (Proposition 3.2). This gives a combinatorial description of these equivalence classes as the partition of the number $N$ (Theorems 3.5 3.8). Next we derive the Weyl group of a partial flag (Proposition 3.10), thus the Weyl group of a Borel subgroup (Corollary 3.11). We end this section with a survey of information on the amount of partitions of given number $N$, next its partitions without a repetition, and finally the amount of partitions with every summand $\geq 2$. We conclude with Proposition 3.19 in which we effectively construct $s_N$ partitions with nontrivial Weyl groups and each summand $\geq 2$ in the amount $s_N$ growing exponentially in $N$. In the last section we prove our main Theorem 4.8 which proof is based on Proposition 4.5. The latter states that each class of $s_N$ constructed partitions of $N$ determines a subspace of the functional space $E$ in such a way that it is the fixed points space of a subgroup $\tilde{H} \subset O(N)$, and subspaces corresponding to different partitions are linearly independent, i.e. their intersection is equal to $\{0\}$. Restricting the functional to each of these subspaces we get $s_N$ infinite series of geometrically distinct solutions. The thesis of Theorem 4.8...
automatically applies to the results of \cite{3, 10, 11}, and all similar quoted in \cite{12}, giving a much larger number of series of geometrically distinct solutions than of those papers. Finally, in Section \ref{sec:5} we provide additional information about the types of partitions we use and show describe subgroups generated by two maximal orthogonal Borel subgroups in $\mathbb{R}^N$ associated with two partitions of $N$ (Proposition \ref{prop:5.3}, Theorem \ref{thm:5.4}).

2. Subspaces of fixed points of subgroups and Borel subgroups

By $O(N)$ denote the group of all linear orthogonal maps of the Euclidean space of dimension $N$ and by $SO(N) \subset O(N)$ its connected component of $\{e\}$ consisting of maps preserving orientation.

Let $\Omega \subset \mathbb{R}^N$ be an open region in $\mathbb{R}^N$ which, is $O(N)$ invariant, i.e. $\Omega = D^N_r$ if it is bounded, or $\Omega = \mathbb{R}^N$ if it is unbounded. From now on, by a functional space $E$ with a domain $\Omega$ we understand a completion of $C^\infty(\Omega)$ or $C^\infty_0(\Omega)$ in any linear topology, e.g. any topology induced by a norm, such that the natural linear action

$$O(N) \times E \to E, \quad (g, u) \mapsto u(g^{-1}x)$$

is continuous and preserving norm if it is the case.

Let $H, K \subset O(N)$ be two closed subgroups and $E^H$, respectively $E^K$ the fixed point subspaces.

By a general theory of transformations of groups we have

$$u \in E^H \iff gu(x) = u(g^{-1}x) \in E^{gHg^{-1}}.$$  

In particular, if $K = gHg^{-1}$ then every element $u \in E^H$ has the same orbit as $v = gu \in E^K$.

Let $\mathcal{G}(N)$ be the set of all closed subgroups of $O(N)$ with the action of $O(N)$ by conjugation, and next $\mathcal{S}(N)$ be the set of all conjugacy classes of (closed!) subgroups of $O(N)$ and $\{H_s\}_{s \in \mathcal{S}}$ a complete set of representatives of $\mathcal{S}$.

Since we are interested in finding geometrically distinct solutions there is reasonable to take only one representative $H_s \in \mathcal{G}(N)$ of the class $[H_s] \in \mathcal{S}(N)$.

Remark 2.1. \textit{We must emphasize that $[H_s] \neq [H_{s'}] \Rightarrow E^{H_s} \cap E^{H_{s'}} = \{0\}$ in general.}

Next, we will define a class of subgroups of $O(N)$, respectively $SO(N)$, called the Borel subgroups. Beforehand, we present a short opening to justify the name this notion.

Definition 2.2. \textit{We call a closed subgroup $G \subset O(N)$ an \textbf{orthogonal Borel group}, if $G$ acts transitively on the unit sphere $S(\mathbb{R}^N) = S^{N-1}$. If $G$ is connected, i.e. $G \subset SO(N)$ then we call $G$ \textbf{connected orthogonal Borel group}.}

Note that $G$ is an orthogonal Borel group iff it acts transitively on the sphere of any radius $r > 0$. Also we have to point out that the above notion of the orthogonal Borel group is different that a notion of the Borel group used in the algebraic geometry. An expiation of the origin of this notion and an information on the corresponding Borel theorem is included to the Section \ref{sec:5}.

Remark 2.3. \textit{If a group $G$ acts transitively on a $G$-space $X$ then $X$ is ($G$-equivariantlty) homeomorphic to the orbit $Gx$ of any point $x \in X$, i.e. it is homeomorphic to the homogenous space $G/H$, with $H = G_x$. If $G$ acts smoothly, or by isometries respectively then $X$ is diffeomorphic, or correspondingly isometric to $G/G_x$.}

Remark 2.4. \textit{Our assumption of the Definition of orthogonal Borel group, that $G \subset O(N)$, automatically guarantees that it acts effectively.}

Remark 2.5. \textit{Note that if $G_0 \subset G$ acts transitively on $S(\mathbb{R}^N)$ then $G$ does too. For $N \geq 2$, if $G$ acts transitively on $S(\mathbb{R}^N)$ then $G_0$ does too. Indeed, otherwise $S(\mathbb{R}^N)$ would be a finite ($|G/G_0|$) union of closed submanifolds which is impossible.
This means that that for \( N \geq 2 \) a group \( G \subset O(N) \) is a Borel group if and only if its connected component \( G_0 \subset SO(N) \) is a connected Borel group.

Let \( H \subset O(N) \) be a closed subgroup. Then the linear space \( \mathbb{R}^N \) is the space of orthogonal representation of \( H \) given by the restriction to \( H \) the canonical representation of \( O(N) \) in \( \mathbb{R}^N \).

We denote by \((V, \rho_H)\), or shortly \( V \), the space \( \mathbb{R}^N \) with the defined above representation structure \( \rho_H : H \to O(N) \).

**Definition 2.6.** We call a closed subgroup \( H \subset O(N) \) an orthogonal Borel subgroup if:

- \( A_1 \) The restricted representation \((V, H)\) of \( H \) in \( \mathbb{R}^N \) decomposes into a direct sum \( V = \bigoplus_{j=1}^r V_j \), \( r \geq 2 \), of pairwise non-isomorphic irreducible representations \( V_j \) of \( H \).
- \( A_2 \) For every \( 1 \leq j \leq r \) the group \( H \) acts transitively on \( S(V_j) \).

**Definition 2.7.** We call a closed connected subgroup \( H \subset O(N) \) a connected orthogonal Borel subgroup if it is an orthogonal Borel subgroup and is connected.

Note that for any connected Borel subgroup we have \( H \subset SO(N) \).

**Remark 2.8.** Since \( O(1) = \mathbb{Z}_2 = \{-1, 1\} \), but its connected component \( SO(1) \) does not act transitively on \( S(\mathbb{R}) = \{-1, 1\} \subset \mathbb{R} \), for all irreducible representations \( V_j \) of Definition 2.7 we have to assume that

\[
N_j = \dim_{\mathbb{R}} V_j \geq 2.
\]

Let \( H \) be an orthogonal Borel subgroup as in Definition 2.6 and \( N_j = \dim V_j \) the dimension of \( V_j \). Note that \( \sum_{j=1}^r N_j = N \). Furthermore, for every \( 1 \leq j \leq r \) we have canonical projection \( p_j : V \to V_j \) given by

\[
p_j(v) := \int_H \rho_H(h) \chi_j^*(h) d\mu(h),
\]

where \( \mu \) is the Haar measure on \( H \), \( \chi_j : H \to \mathbb{R} \) is the character of irreducible representation \( V_j \), and \( \chi^* \) denote the character of representation conjugated to \( \chi \). Moreover \( V_j = p_j(V) \), and we have a canonical embedding \( \epsilon_j : V_j \hookrightarrow V \), \( v \mapsto (0, 0, \cdots, v, 0, \cdots, 0) \).

Since \( H \subset O(N) \), every \( p_j \) is an orthogonal projection, \( \epsilon_j \) is an orthogonal embedding, and they are \( H \)-equivariant.

Consequently for any \( h \in H \) the formula \( h \mapsto \rho_j(h) \) defines a homomorphism, denoted also \( p_j \), from \( H \) into \( O(N_j) \) which can be defined as

\[
p_j(h)(v) := p_j(\rho_H(h)(v)) \quad \text{where} \quad v \in V_j \cong \mathbb{R}^{N_j}.
\]

Denote by \( H_j = p_j(H) \) the image of \( H_j \) in \( O(N_j) \).

**Lemma 2.9.** For every \( 1 \leq j \leq r \) the group \( H_j = p_j(H) \) is an orthogonal Borel group of \( O(N_j) \).

**Proof.** By definition \( H_j \subset O(N_j) \). On the other hand, by assumption \( A_2 \) \( H \) acts transitively on \( S(V_j) \), but the action of \( H \) on \( S(V_j) \) factorizes through \( H_j \), i.e. for every \( v_j = (0, 0, \cdots, v, \cdots, 0) \in S(V_j) \) we have \( p_j(h)(v_j) = p_j(\rho_j(h)(v_j)) \). Indeed \( \rho_H(h) = p_j(\rho_H(h)(v_j)) \), since \( \rho_H \) preserves each \( V_j \), and \( p_j(\rho_H(h)(v_j)) = p_j\rho_H(h)(\epsilon(v)) = p_j(h)(v) \).

This leads to the following characterization of the orthogonal Borel subgroups of \( O(N) \).

**Proposition 2.10.** Every orthogonal Borel subgroup \( H \) of \( O(N) \) is isomorphic to a product \( H \cong H_1 \times H_2 \times \cdots \times H_r \), where \( H_j \) is an orthogonal Borel group of \( O(N_j) \) and \( \sum_{j=1}^r N_j = N \).
More precisely, if the restricted representation \((V, \rho_H)\), \(\dim V = N\), of \(H\) decomposes into \(r\) irreducible summands \(V = \bigoplus_{j=1}^{r} V_j\), with \(\dim V_j = N_j\), then \(H_j = p_j(H)\) (see (7)).

\textbf{Proof.} The product map \(p := p_1 \times \cdots \times p_r : H \rightarrow H_1 \times \cdots \times H_r\) is a homomorphism as the product of homomorphisms, and is onto by the definition of \(H_j\).

It is enough to show that it is injective. Indeed, let \(p(h) = e = (e, \cdots, e) \in H_1 \times \cdots \times H_r\), i.e. \((p_1(h), \cdots, p_r(h))(v_1, \ldots, v_r) = (v_1, \ldots, v_r)\) for every \(v = (v_1, \ldots, v_r) \in \bigoplus_{j=1}^{r} V_j\). But \(V = \bigoplus_{j=1}^{r} V_j\), and \(\rho_H(h)(v) = \sum_{j=1}^{r} \rho_j(p_j(h))(v_j)\) as we have shown in the proof of Lemma 2.9. This shows that \(\rho_H(h) = \text{id}\) and consequently \(h = e\), since \(\rho_H\) is given by the embedding of the subgroup \(H\) in \(O(N)\). \(\square\)

\textbf{Remark 2.11.} Observe that the assumption that \(\rho_H\) is a direct sum of pairwise non-isomorphic irreducible representations is necessary for the transitivity of action of \(H\) on the spheres \(S(V_j)\) of the summands of this decomposition. Indeed, the canonical projections \(p_j\) (thus canonical decomposition of \(V\)) are projections onto the isogenic subrepresentations of \(V\), i.e. onto subrepresentations which are multiplicities of a given irreducible representation \(\rho_j : H \rightarrow O(N_j)\). If \((V, \rho_H)\) contains a representation \(kV_j, V_j\) irreducible, \(k > 1\), then \(H\) does not act transitively on \(S(kV_j)\), because it acts diagonally on \(kV_j = \bigoplus_{j=1}^{k} V_j\).

For a given orthogonal Borel subgroup \(H \subset O(N)\) and the restricted representation \((V, \rho_H)\), \(V = \bigoplus_{j=1}^{r} V_j\) we put

\[V^j := \bigoplus_{l=1}^{j} V_l.\]

Note that \(V_1 = V^1, V^r = V, V^i \subset V^j\) if \(i < j\), i.e. the family \(\{V^j\}_{j=1}^{r}\) forms a filtration of \(V\) by linear subspaces. Put \(n_j = \sum_{l=1}^{j} N_l\) equals to the dimension of \(V^j\). Additionally put \(V^0 = V_0 = \{0\}\).

Note that a filtration of \(V = \mathbb{R}^N\) by

\[\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \cdots \subsetneq V^r = V = \mathbb{R}^N\]

is called a \textit{partial flag} of \(V\). Furthermore, if \(H \subset O(N)\) preserves this filtration \(\{V^i\}\) (the flag) it preserves also the subspaces of orthogonal complements \(V_i = V^i_{\perp}\) in \(V^i\). This leads to an equivalent definition of an orthogonal Borel subgroup of \(O(N)\).

\textbf{Proposition 2.12.} Let \(H \subset O(N)\) be a subgroup. Then \(H\) is an orthogonal Borel subgroup if and only if there exists a partial flag \(V^1 \subsetneq V^2 \cdots \subsetneq V^r\) of \(\mathbb{R}^N\) which is preserved by \(H\) and for every \(1 \leq i \leq r\) the induced action of \(H\) on the projective space \(\mathbb{P}(V^i/V^{i-1})\) of quotient space \(V^i/V^{i-1}\) is transitive. \(\square\)

The above justifies a use of the following notion. For every partial flag

\[\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \cdots \subsetneq V^r = V\text{ in }V = \mathbb{R}^N\]

we call the corresponding orthogonal decomposition

\[V = \bigoplus_{j=1}^{r} V_j, \text{ where } V_j \supset V^j = V^j_{\perp}\]

the orthogonal partial flag.
Remark 2.13. Note that the isomorphism of $H \simeq H_1 \times H_2 \times \cdots \times H_r$ of Proposition 2.12 depends on the flag $\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \cdots \subsetneq V^r = V = \mathbb{R}^N$.

For our analytical considerations we will use only a special class of orthogonal Borel subgroups, each of them is canonically associated with a partial flag.

Definition 2.14. We will call $O(V_1) \times O(V_2) \times \cdots \times O(V_r) \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r)$ the maximal orthogonal Borel subgroup, and respectively $SO(V_1) \times SO(V_2) \times \cdots \times SO(V_r)$ the maximal connected orthogonal Borel subgroup associated with given partial orthogonal flag $\{V_1, V_2, \ldots, V_r\}$.

By $\mathcal{G}_B$ we denote the set of all maximal orthogonal Borel subgroups, and respectively by $\mathcal{G}_B^0$ the set of all maximal connected orthogonal Borel subgroups.

3. Equivalent maximal orthogonal Borel subgroups

Now we would like to describe a number of non-equivalent orthogonal Borel subgroups that correspond to an action of the group $O(N)$, or $SO(N)$ on the set of orthogonal partial flags. To shorten notation, in this section by an orthogonal, correspondingly connected orthogonal, Borel group we mean the maximal orthogonal Borel, respectively maximal connected orthogonal Borel subgroups.

Let us denote by $\mathfrak{F}_r(N)$ the set of flags of length $1 \leq r \leq N$ in $\mathbb{R}^N$.

3.1. The action of $O(N)$ on the set of partial flags. Observe that if $g \in O(N)$, in particular if $g \in SO(N)$, and $V = \bigoplus_{j=1}^r V_j$ is an orthogonal partial flag then the family of spaces $\{gV_j\}_{j=1}^r$ forms also an orthogonal partial flag of the same length.

Definition 3.1. The mapping $(g, V = \bigoplus_{j=1}^r V_j) \mapsto \{gV_j\}_{j=1}^r$ defines an action of $O(N)$ or $SO(N)$ on $\mathfrak{F}_r(N)$.

The isotropy group of any such an orthogonal partial flag is equal to

$$O(V_1) \times O(V_2) \times \cdots \times O(V_r) \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r),$$

or respectively

$$SO(V_1) \times SO(V_2) \times \cdots \times SO(V_r) \equiv SO(N_1) \times SO(N_2) \times \cdots \times SO(N_r)$$

if we consider the action of $SO(N)$.

Consequently, the orbit of any orthogonal partial flag is equal to

$$O(N)/(O(N_1) \times O(N_2) \times \cdots \times O(N_r))$$

or $SO(N)/(SO(N_1) \times SO(N_2) \times \cdots \times SO(N_r))$

or respectively.

Note that by Definition 2.6 an orthogonal Borel subgroup $H$ associated with given partial orthogonal flag $V = \bigoplus_{j=1}^r V_j$ is contained in the isotropy group of $\{V_j\}$, i.e. in $O(V_1) \times O(V_2) \times \cdots \times O(V_r) \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r)$. By the definition, if it is a connected orthogonal Borel subgroup it is a subgroup of $SO(N)$ thus $\text{a subgroup of the isotropy group } SO(V_1) \times SO(V_2) \times \cdots \times SO(V_r) \equiv SO(N_1) \times SO(N_2) \times \cdots \times SO(N_r)$.

On the other hand we have the orthogonal Borel subgroup, or correspondingly the connected orthogonal Borel subgroup, of the partial orthogonal flag $\{gV_j\}_{j=1}^r$ is equal to $O(gV_1) \times O(gV_2) \times \cdots \times O(gV_r) = gO(V_1)g^{-1} \times gO(V_2)g^{-1} \times \cdots \times gO(V_r)g^{-1} \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r)$ or respectively $SO(gV_1) \times SO(gV_2) \times \cdots \times SO(gV_r) = gSO(V_1)g^{-1} \times gSO(V_2)g^{-1} \times \cdots \times gSO(V_r)g^{-1} \equiv SO(N_1) \times SO(N_2) \times \cdots \times SO(N_r)$.

Summing up, we have the following proposition.
Proposition 3.2. The natural action of the group $O(N)$, correspondingly $SO(N)$, on the set $\mathcal{F}_r(N)$ of all orthogonal partial flags of length $r$ given by $(g, \{V_j\}_{j=1}^r) \mapsto \{gV_j\}_{j=1}^r$ corresponds to the action by conjugation $(g, H) \mapsto gHg^{-1}$ on the set $\mathcal{G}_B(N)$, correspondingly $\mathcal{G}_B^0(N)$ of orthogonal Borel subgroups, respectively the connected orthogonal Borel subgroups, associated with them.

Definition 3.3. We say that two orthogonal partial flags $\{V_j\}_{j=1}^r$ and $\{W_i\}_{i=1}^s$ are equivalent if they are in one orbit of the above action, i.e. if there exists $g \in O(N)$, respectively $g \in SO(N)$, such that

$$g \{V_j\}_{j=1}^r = \{W_i\}_{i=1}^s.$$

Now we show a condition which is necessary and sufficient for the equivalence of orthogonal partial flags. To do it we need new notation. We say that a sequence $N_1, N_2, \ldots, N_r$ of natural numbers such that $N_1 + N_2 + \ldots + N_r = N$ is a partition of $N$ and $r$ is the length of this partition.

Definition 3.4. We say the two partitions $N_1, N_2, \ldots, N_r$ and $M_1, M_2, \ldots, M_s$ of $N$ are equivalent if $r = s$ and there exists a permutation $\sigma$ of the set $\{1, 2, \ldots, r\}$ such that $N_j = M_{\sigma(j)}$.

Note that any partition $N_1, N_2, \ldots, N_r$ of $N$ is equivalent to a partition $\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_r$ such that $\bar{N}_1 \leq \bar{N}_2 \leq \ldots \leq \bar{N}_r$, in particular if $N_j \neq N_i$ for $i \neq j$ then the partition $N_1, N_2, \ldots, N_r$ is equivalent to a partition such that $\bar{N}_1 < \bar{N}_2 < \ldots < \bar{N}_r$.

We are in position to formulate the main result of this section.

Theorem 3.5. Two orthogonal partial flags $\{V_j\}_{j=1}^r$ and $\{W_i\}_{i=1}^s$ are equivalent if and only if $r = s$ and the corresponding to them partitions of $N$: $(N_1, N_2, \ldots, N_r)$ and $(M_1, M_2, \ldots, M_s)$, with $N_j = \dim V_j$, respectively $M_i = \dim W_i$, are equivalent.

Proof. Since an orthogonal linear map $g$ preserves the dimension of spaces and their orthogonality the partitions of $N$ associated with $\{V_j\}_{j=1}^r$ and with $\{gV_j\}_{j=1}^r$ are the same which proves the necessity of condition.

Now suppose that $r = s$ and there exists a permutation $\sigma$ of the set $\{1, 2, \ldots, r\}$ such that $N_j = M_{\sigma(j)}$. Changing indices at the partition $(M_1, M_2, \ldots, M_s)$ and corresponding orthogonal partial flag $\{W_i\}_{i=1}^s$ we can assume that $N_j = M_j$ for $1 \leq j \leq r$. For $1 \leq j \leq r$ let $\{\vec{e}_i^j\}$, $i = 1, \ldots, N_j$, be an orthonormal basis of $V_j$, and $\{\vec{f}_i^j\}$, $i = 1, \ldots, N_j$, the analogous basis of $W_j$. From linear algebra, it follows that there exists an orthogonal map $g \in O(N)$ such that

$$g(\vec{e}_i^j) = \vec{f}_i^j.$$

Obviously, $g(V_j) \subset W_j$, thus $gV_j = W_j$ for every $1 \leq j \leq r$. Moreover, fixing orientations in each $V_j$ and $W_j$ we can choose each basis $\{\vec{e}_i^j\}$ and $\{\vec{f}_i^j\}$ consistent with their orientation. Then, $g \in SO(N)$ and $g_{ij}: V_j \to W_j$ is also a preserving orientation linear orthogonal map. This shows that this condition is sufficient for the equivalence of orthogonal partial flags.

As we stated in Section 2 we are interested in a choice of one orthogonal Borel subgroup from each equivalence (conjugacy) class. By Proposition 3.2 and Theorem 3.5 we get the following statement.

Corollary 3.6. Two orthogonal Borel subgroups $H = O(V_1) \times O(V_2) \times \ldots \times O(V_r) \equiv O(N_1) \times O(N_2) \times \ldots \times O(N_r)$ and $H' = O(V_1') \times O(V_2') \times \ldots \times O(V_s') \equiv O(N_1') \times O(N_2') \times \ldots \times O(N_s')$ are equivalent if and only if $r = s$ and the partitions of $N: N_1, N_2, \ldots, N_r$, and $N_1', N_2', \ldots, N_s'$ are equivalent.

In other words, to chose one representative of each equivalence class of the orthogonal Borel subgroups is enough to fix an orthogonal basis $\mathcal{E} = \{e_1, e_2, \ldots, e_N\}$, next fix a representative $N_1, N_2, \ldots, N_r$ of each equivalence class of partitions, and finally construct a partial orthogonal
flag \{0\} ⊂ V^1 ⊂ V^2 ⊂ ⋯ ⊂ V^r = \mathbb{R}^N, V^i = \text{span}\{e_1, e_2, \ldots, e_{N_j}\}, \text{with } N_j = \sum^r_i N_j, \text{defining the orthogonal subgroup}

\[ H = O(V_1) \times O(V_2) \times \cdots \times O(V_r) = O(N_1) \times O(N_2) \times \cdots \times O(N_r). \]

3.2. A combinatorial description of the action. Now we would like to give a combinatorial condition for two partitions of \(N\) to be equivalent in the sense of Definition 3.4.

Let \(\pi_r(N) = \{N_1, N_2, \ldots, N_r\}\) be a partition of \(N\), denoted shortly by \(\pi\). The set of all partitions of \(N\) of length \(r\) we denote by \(\Pi_r(N)\).

The permutation group of \(r\)-symbols \(S(r)\) acts on \(\Pi_r(N)\) by permuting the indices. Observe that two partitions \(\pi, \pi' \in \Pi_r(N)\) are in one orbit of the action of \(S(r)\) iff they are equivalent in the sense of Definition 3.4. On the other hand they are in the same orbit of the action of \(S(r)\) on \(\Pi(r)\) iff the isotropy groups \(\mathcal{G}_\pi\) and \(\mathcal{G}_{\pi'}\) are conjugated in \(S(r)\) as follows from general theory of actions of groups.

Note that for a partition \(\pi = \{N_1, \ldots, N_r\}\) with \(N_i \neq N_j\) for \(i \neq j\) we have \(\mathcal{G}_\pi = e\) the identity permutation.

Our task is to describe the isotropy group \(\mathcal{G}_\pi\) of a partition \(\pi \in \Pi_r(N)\). To do it we define a function

\[ \phi_\pi : \{1, 2, \ldots, N\} \rightarrow 2^{\{0,1,\ldots,r\}} \text{ defined as } \phi_\pi(n) = \{1 \leq j \leq r : N_j = n\} \]

Note that \(\phi(k) \cap \phi(n) = \emptyset\) if \(k \neq n\).

Now we define next function

\[ \psi_\pi : \{1, 2, \ldots, N\} \rightarrow \{0, 1, \ldots, r\} \text{ defined as } \psi_\pi(n) = |\phi_\pi(n)| \]

where \(|A|\) denotes the cardinality of a finite set \(A\) with a convention that \(|\emptyset| = 0\).

**Lemma 3.7.** Let \(\pi = \{N_1, \ldots, N_r\}\) be a partition of \(N\) such that there is \(q\) different values \(n_1, \ldots, n_q\) in the sequence \(N_1, N_2, \ldots, N_r\). Then the isotropy group \(\mathcal{G}_\pi \in \mathcal{G}(r)\) of the action of \(\mathcal{G}(r)\) on \(\Pi_r(N)\) is equal to \(\mathcal{G}(\phi_\pi(n_1)) \times \mathcal{G}(\phi_\pi(n_2)) \times \cdots \times \mathcal{G}(\phi_\pi(n_q))\).

Note that since the supports of permutations in \(\mathcal{G}(\phi(n_i))\), with \(\phi_\pi(n_i) \neq 0\), are disjoint for different \(i\), all they commute. Consequently,

\[ \mathcal{G}_\pi = \mathcal{G}(\phi_\pi(n_1)) \times \mathcal{G}(\phi_\pi(n_2)) \times \cdots \times \mathcal{G}(\phi_\pi(n_q)) \text{ is a group of order } \psi_\pi(1)! \psi_\pi(2)! \cdots \psi_\pi(N)! \]

with the usual convention that \(0! = 1\).

Note the length \(r\) of any abstract partition should be \(\leq N\), and under our assumption that \(\dim V_j = N_j \geq 2\) we have \(r \leq \frac{N}{2}\).

Now we are in position to formulate a theorem which characterizes combinatorially equivalent partitions of \(N\) of the same length \(r\).

**Theorem 3.8.** Two partitions \(\pi = \{N_1, \ldots, N_r\}\) and \(\pi' = \{N'_1, \ldots, N'_r\}\), \(\sum^r_i N_j = \sum^r_i N'_j = N\) are equivalent if and only if the associated with them functions \(\psi_\pi : \{1, 2, \ldots, N\} \rightarrow \{0, 1, \ldots, r\}\), and respectively \(\psi_{\pi'} : \{1, 2, \ldots, N\} \rightarrow \{0, 1, \ldots, r\}\) are equal. Consequently their (common) orbit is of length

\[ |\mathcal{G}(r)/\mathcal{G}_\pi| = \frac{r!}{\psi_\pi(1)! \psi_\pi(2)! \cdots \psi_\pi(N)!} \]
3.3. The normalizer and Weyl group of a partial flag.

Definition 3.9. Let \( \{ V_j \}_{j=1}^{q} \) be an orthogonal partial flag. By the normalizer of the flag in \( O(N) \), respectively \( SO(N) \), denoted \( \mathcal{N}(\{ V_j \}_{j=1}^{q}) \), we mean the set of all elements \( g \) of \( O(N) \), respectively \( SO(N) \), which map \( \{ V_j \}_{j=1}^{q} \) into itself.

Note that \( \mathcal{N}(\{ V_j \}_{j=1}^{q}) \) is a subgroup of \( O(N) \), respectively \( SO(N) \), containing the orthogonal Borel subgroup, respectively connected orthogonal Borel subgroup of \( \{ V_j \}_{j=1}^{q} \). Moreover, it is the normalizer of the latter in \( O(N) \), respectively \( SO(N) \).

In this subsection we describe the normalizer in \( SO(N) \) of an orthogonal partial flag \( \{ V_j \}_{j=1}^{q} \), or equivalently of an orthogonal Borel subgroup \( H \equiv SO(N_1) \times \cdots \times SO(N_r) \), describe its normalizer in \( SO(N) \). This let us describe the set of all orthogonal partial flags, or equivalently all orthogonal Borel subgroups \( H \), for which the Weyl group \( \mathcal{W}(H) = \mathcal{N}(H)/H \) is nontrivial.

As a consequence, we characterize all orthogonal Borel subgroups which have a nontrivial Weyl group.

Let \( \{ V_j \}_{j=1}^{q} \) be an orthogonal partial flag in \( \mathbb{R}^N \) with \( \dim V_j = N_j \) giving a partition of \( N \). Choosing an orthogonal basis in each \( V_j \) we get a coordinate system in \( \mathbb{R}^N \) in which

\[
\{ V_j \}_{j=1}^{q} = \{ \mathbb{R}^{N_1}, \mathbb{R}^{N_2}, \ldots, \mathbb{R}^{N_r} \}.
\]

Proposition 3.10. Let \( \{ V_j \}_{j=1}^{q} = \{ \mathbb{R}^{N_1}, \mathbb{R}^{N_2}, \ldots, \mathbb{R}^{N_r} \} \) be an orthogonal partial flag in \( \mathbb{R}^N \) with \( \dim V_j = N_j \) defining a partition \( \pi \) of \( N = \sum_{j=1}^{r} N_j \). Let \( \pi = (N_1, \ldots, N_r) \) be a such that there is \( q \) different values \( n_1, \ldots, n_q \) in the sequence \( N_1, N_2, \ldots, N_r \). Then the normalizer \( \mathcal{N}(\{ V_j \}_{j=1}^{q}) \) in \( O(N) \) contains the orthogonal Borel subgroup \( O(N_1) \times O(N_2) \times \cdots \times O(N_r) \) and all permutations \( \sigma_{\pi} \in S_{\pi} = S(\phi_\pi(n_1)) \times \cdots \times S(\phi_\pi(n_q)) \). Moreover, the normalizer is generated by these maps, and consequently the Weyl group of \( \{ V_j \}_{j=1}^{q} \) is a product of \( q \) permutation groups \( \mathcal{W}(\{ V_j \}_{j=1}^{q}) = \cdots \times S(\phi_\pi(n_1)) \times \cdots \times S(\phi_\pi(n_q)) \). Consequently, the Weyl group \( \mathcal{W}(\{ V_j \}_{j=1}^{q}) \) is of order \( \psi_\pi(1)! \psi_\pi(2)! \cdots \psi_\pi(N)! \), and it contains an element of order 2 provided it is not the trivial group.

Proof. It is clear that if an element \( g \in O(N) \) maps \( \{ V_j \}_{j=1}^{q} \) into itself then for every \( 1 \leq j \leq q \) have \( gV_j = V_{\sigma(j)} \) for some permutation \( \sigma : \{1, \ldots, r\} \rightarrow \{1, \ldots, r\} \). Identifying every \( V_j \) with \( \mathbb{R}^{N_j} \) by a choice of an orthonormal basis \( \{ e_i \}_{1 \leq i \leq N_j} \), for any permutation \( \sigma \in S_{\pi} \) we can define a permutation map \( g_{\sigma} \in O(N) \) by

\[
g_{\sigma}(e_i^j) = e_{\sigma(i)}^j
\]

provided only \( N_{\sigma(i)} = N_j \).

Composing \( g \) with \( g_{\sigma}^{-1} \) we get an orthogonal linear map \( g' = g_{\sigma}^{-1}g : \mathbb{R}^N \rightarrow \mathbb{R}^N \) such that \( g'(V_j) = V_{\sigma(j)} \) for every \( 1 \leq j \leq r \). Consequently \( g' \in O(V_1) \times O(V_2) \times \cdots \times O(V_r) \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r) \). This shows that the normalizer \( \mathcal{N}(\{ V_j \}_{j=1}^{q}) \) is generated by the permutations \( \sigma_{\pi} \in S_{\pi} \) and elements of its orthogonal Borel subgroup.

\[\square\]

Corollary 3.11. Let \( H \equiv O(N_1) \times O(N_2) \times \cdots \times O(N_r) \) be a orthogonal Borel subgroup of \( O(N) \). Then the Weyl group of \( H \) is equal to \( \mathcal{W}(H) = S_{\pi} = S(\phi_\pi(n_1)) \times \cdots \times S(\phi_\pi(n_q)) \) with a convention that \( S(\emptyset) = e \).

Remark 3.12. Remind that with respect to our condition on the connected orthogonal Borel subgroups (cf. Definition 2.6 and Remark 2.8) in study of them we had to put the assumption \( N_j \geq 2 \). This complicates a combinatorics of possible partitions.

Remark 3.13.

It is worth of pointing out that if we take the partial orthogonal flag \( \{ V_j \}_{j=1}^{q} \) in \( \mathbb{R}^N \), \( N = 2K \), which is equivalent (i.e. is in the same \( O(N) \)-orbit) to \( \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 \) then its orthogonal Borel
subgroup is conjugated to \(SO(2) \times SO(2) \times \cdots \times SO(2) = \mathbb{T}^K\). In other words it is the maximal torus \(\mathbb{T}^K\) of \(SO(2K)\).

On the other hand, by our considerations this conjugacy class of Borel orthogonal subgroups corresponds to the partition \(2 + 2 + \cdots + 2\) of \(N = 2K\). Finally, from Corollary 3.11 it follows that for the torus \(\mathbb{T}^K \subset SO(2K)\) we have \(\mathcal{W}(\mathbb{T}^K) = \mathcal{G}(K)\), which is a classical fact.

3.4. Partitions of \(N\). In the combinatorial number theory the number the number of writing the integer as a sum of positive integers, where the order of addends is not considered significant is denoted by \(P(N)\), and is sometimes called the number of unrestricted partitions.

Also, the number of ways of writing the integer as a sum of positive integers without regard to order with the constraint that all integers in a given partition are distinct is denoted by \(Q(N)\).

**Definition 3.14** (Definition of \(P(N)\) and \(Q(N)\)). In our terms the function \(P(N)\) is equal to the sum over \(1 \leq r \leq N\) of numbers of equivalence classes of partitions \(\pi \in \Pi(r)\).

Respectively, the function \(Q(N)\) corresponds to the sum over \(1 \leq r \leq N\) of numbers of all equivalence classes of partitions \(\pi \in \Pi(r)\) such that all \(N_{j_1}, N_{j_2}, \ldots, N_{j_r}\) are different.

More information about the functions \(P(N)\) and \(Q(N)\) we present in Section 5. At now let us only mention the asymptotic behavior of them (Hardy-Littlewood 1921):

\[
(11) \quad P(N) \sim \frac{1}{4N\sqrt{3}} e^{\pi \sqrt{2N/3}}, \quad Q(N) \sim \frac{e^{\pi \sqrt{N/3}}}{4N^{3/4} 3^{1/4}}
\]

Let us remind that due to applications we are more interested in a description of the number of conjugacy classes of the orthogonal (connected) Borel subgroups of \(O(N)\) (\(SO(N)\)) with a non-trivial Weyl group.

We denote by

\[
(12) \quad R(N) := P(N) - Q(N)
\]

the number of partitions of \(N\) where the order of addends is not considered significant, but which contain at least two equal summands.

As a consequence of Proposition 3.10 and Corollary 3.11 we have the following.

**Theorem 3.15.** Consider the set \(\mathcal{F} = \bigcup_1^N \mathcal{F}_r\) of partial orthogonal flags in \(\mathbb{R}^N\) of all lengths \(1 \leq r \leq N\) with the action of \(O(N)\), or \(SO(N)\).

The number of classes of equivalence, i.e. the orbits of the action, is equal to \(P(N)\). Consequently the number of conjugacy classes of all orthogonal Borel subgroups of \(O(N)\), or respectively \(SO(N)\) is equal to \(P(N)\). Furthermore, the number of all equivalence classes of partial orthogonal flags, respectively orthogonal Borel subgroups, with the trivial Weyl group is equal to \(Q(N)\). Consequently the number of all equivalence classes of partial orthogonal flags, respectively orthogonal Borel subgroups, with a non-trivial Weyl group is equal to \(R(N) = P(N) - Q(N)\).

**Corollary 3.16.** The rate of growth of number \(R(N)\) of the equivalence classes of orthogonal partial flags of all lengths in \(\mathbb{R}^N\) with a non-trivial Weyl group is exponential:

\[
R(N) \sim P(N) - Q(N) \sim \frac{1}{4N\sqrt{3}} e^{\pi \sqrt{2N/3}} - \frac{e^{\pi \sqrt{N/3}}}{4N^{3/4} 3^{1/4}} = \frac{e^{\pi \sqrt{N/3}}}{4N^{3/4} 3^{1/4}} \left( \frac{e^{\pi \sqrt{2N/3}} - e^{\pi \sqrt{N/3}}}{N^{1/4} \sqrt{3}} - \frac{1}{3^{1/4}} \right)
\]

Due the analytical assumption our task is to study equivalence classes of partial orthogonal flags, respectively orthogonal Borel subgroups, such that each subspace is of dimension \(\geq 2\). To do this we need a new notation.
Definition 3.17. For a given \( N \in \mathbb{N} \) let \( P(N;1) \) denote the number of ways of writing the integer \( N \) as a sum of positive integers, where the order of addends is not considered significant and each of them is greater then 1.

For a given \( N \in \mathbb{N} \) let \( Q(N;1) \) denote the number of ways of writing the integer \( N \) as a sum of positive integers without regard to order with the constraint that all integers in a given partition are distinct and greater then 1.

Finally we define \( R(N;1) := P(N;1) - Q(N;1) \).

We have the following correspondent of Theorem 3.15 describing the number of classes the connected Borel subgroups.

Theorem 3.18. Consider the set \( \mathfrak{F} = \bigcup_{1}^{N} \mathfrak{F}_{r} \) of partial orthogonal flags in \( \mathbb{R}^{N} \) of all lengths \( 1 \leq r \leq N \) with the action of \( SO(N) \) and such that \( N_{j} = \dim V_{j} \geq 2 \) for every summand of such a flag.

The number of classes of equivalence, i.e. the orbits of the action, is equal to \( P(N;1) \). Consequently the number of conjugacy classes of all connected orthogonal Borel subgroups of is equal to \( P(N;1) \). Furthermore, the number of all \( O(N) \) equivalence classes of partial orthogonal flags, respectively connected orthogonal Borel subgroups, with the trivial Weyl group is equal to \( Q(N;1) \).

Consequently the number of all equivalence classes of partial orthogonal flags, respectively connected orthogonal Borel subgroups, with a non-trivial Weyl group is equal to \( R(N;1) \).

Proof. This is a direct consequence of Proposition 3.10. Since in any such partition \( N = N_{1} + N_{2} + \cdots + N_{r} \) we have \( N_{j} \geq 2 \), we have take only such partitions for which each summand is \( > 1 \). The same with partitions consisting of all distinct summands that correspond to flag, or connected orthogonal Borel subgroups, with the trivial Weyl group.

Our next task is to give more effective formulas for the the functions \( P(N;1), Q(N;1) \) and \( R(N;1) \).

At this point we are not able to describe completely the asymptotic behavior of the sequence \( R(N;1) \). Regardless, we show that there is a family of partitions of \( N \) with each of them with nontrivial Weyl group and such the rate of growth of their amount is exponential.

We define these partitions by a formula which has four different forms depending on the class of \( N \) modulo 4.

Suppose first that \( N \equiv 0 \mod 4 \), i.e. it is of the form \( N = 4M, M \geq 1 \). Let \( M_{1} \leq M_{2} \leq \ldots \leq M_{r} \) be a partition of \( M \) of length \( r \).

We define a partition of \( N = 4M \) of length \( r \) by the formula

\[
\pi_{r}(N) = \{2M_{1}, 2M_{1}, 2M_{2}, 2M_{2}, \ldots, 2M_{r}, 2M_{r}\}
\]

For every \( 1 \leq j \leq r \) we have \( N_{2j} = N_{2j+1} = 2M_{j} \geq 2 \) and the Weyl group \( W(\pi_{r}(N)) \) is nontrivial. Indeed the Weyl group of contains the Weyl group of the partition \( M_{1} \leq M_{2} \leq \ldots \leq M_{r} \) of \( M \) and transpositions of the equal summands \( 2M_{i} \mapsto 2M_{i} \).

Suppose now that \( N \equiv 2 \mod 4, N \geq 6 \), i.e. \( N \) is of the form \( 4M + 2, M \geq 1 \). Let \( M_{1} \leq M_{2} \leq \ldots \leq M_{r} \) be a partition of \( M \) of length \( r \). We define a partition of \( N = 2M + 2 \) of length \( r \) by the formula

\[
\pi_{r}(N) = \{2, 2M_{1}, 2M_{1}, 2M_{2}, 2M_{2}, \ldots, 2M_{r}, 2M_{r}\}
\]

For every \( 1 \leq j \leq r \) we have \( N_{j} \geq 2 \) and the Weyl group \( W(\pi_{r}(N)) \) is nontrivial. Indeed the Weyl group of it contains the Weyl group of the partition \( \{M_{1} \leq M_{2} \leq \ldots \leq M_{r}\} \) of \( M \) and transpositions of the equal summands \( 2M_{i} \mapsto 2M_{i} \) for \( i \geq 2 \).
Thirdly, suppose that now \( N \equiv 3 \mod 4, \ N \geq 7, \) i.e. \( N \) is of the form \( N = 4M + 3, \ M \geq 1. \)
Let \( M_1 \leq M_2 \leq \ldots \leq M_r \) be a partition of \( M \) of length \( r. \) We define a partition of \( N = 4M + 3 \) of length \( r \) by the formula
\[
\pi_r(N) = \begin{cases}
\{2M_1, 2M_2, 2M_3, \ldots, 2M_i, 2M_{i+1}, 2M_{i+2}, \ldots, 2M_r\} & \text{if } M_1, M_2, \ldots, M_i \leq 1,
\{3, 2M_1, 2M_2, \ldots, 2M_i\} & \text{if } 2 \leq M_1 \leq M_2 \leq \cdots \leq M_r,
\end{cases}
\]
For every \( 1 \leq j \leq r \) we have \( N_j \geq 2 \) and the Weyl group \( W(\pi_r(N)) \) is nontrivial by the same argument as above.

Finally, suppose that \( N \equiv 3 \mod 4, \ N \geq 9, \) i.e. \( N = 4M' + 1, \ M' \geq 2, \) or equivalently \( N = 4M + 5, \) where \( M = M' - 1. \) Let \( M_1 \leq M_2 \leq \ldots \leq M_r \) be a partition of \( M \) of length \( r. \) We define a partition of \( N = 4M + 5 \) of length \( r \) by the formula
\[
\pi_r(N) = \begin{cases}
\{2M_1, 2M_2, 2M_3, \ldots, 2M_i, 2M_{i+1}, 2M_{i+2}, \ldots, 2M_r\} & \text{if } M_1, M_2, \ldots, M_i \leq 2,
\{5, 2M_1, 2M_2, \ldots, 2M_i\} & \text{if } 3 \leq M_1 \leq M_2 \leq \cdots \leq M_r,
\end{cases}
\]
For every \( 1 \leq j \leq r \) we have \( N_j \geq 2 \) and the Weyl group \( W(\pi_r(N)) \) is nontrivial by the same argument as above.

As a consequence we get the following.

**Proposition 3.19.** For every \( N \geq 4 \) of the form \( N = 4M, \ N = 4M + 2, \ N = 4M + 3, \) or finally \( N = 4M + 5 \) with \( M \geq 1 \) there exists at least \( s_N = P(N) \), \( s_N = P(\frac{N-2}{4}) \), \( s_N = P(\frac{N-3}{4}) \), or respectively \( s_N = P(\frac{N-5}{4}) \) equivalence classes of partitions of \( N \) such that every summand \( N_j \geq 2 \) and for every such partition \( \pi(N) \) the Weyl group is nontrivial.

4. Applications to analytical problems

For analytical applications we have to show that for any two not equivalent orthogonal Borel subgroups \( H, K \subset O(N) \) the subgroup \( \langle H, K \rangle \) generated by them acts transitively on a sphere \( S(V), \) where \( V \subset \mathbb{R}^N \) is a subspace invariant with respect to \( H \) and \( K \) simultaneously. The resulting answer says that it does not depend on a choice of a representatives of the conjugacy classes of \( H, \) and \( K \) in \( O(N). \)

More precisely, by \( G = \langle H, K \rangle \) we denote a subgroup of \( O(N) \) generated topologically by \( H \) and \( K, \) i.e. the closure of the set of products \( \{h_{a_1}^1k_{b_1} \ldots h_{a_m}^mk_{b_m}\}, \) where \( h_i \in H, \ k_i \in K, \ a_i \in \mathbb{Z}, \ b_i \in \mathbb{Z}. \)

**Theorem 4.1.** Let \( H, K \subset O(N) \) be two Borel subgroups

If \( \langle H, K \rangle \) acts transitively on \( S(\mathbb{R}^N) \) then for every \( g \in O(N) \) the group \( \langle H, gKg^{-1} \rangle, \) and dually \( \langle gHg^{-1}, K \rangle \) acts transitively on \( S(\mathbb{R}^N) \).

Moreover, suppose that there exists \( N' < N \) such that \( H(\mathbb{R}^{N'}) \subset \mathbb{R}^{N'}, \ K(\mathbb{R}^{N'}) \subset \mathbb{R}^{N'} \) then \( \langle H, K \rangle(\mathbb{R}^{N'}) \subset \mathbb{R}^{N'} \) and \( \langle H, K \rangle \) acts transitively on the sphere \( S(\mathbb{R}^{N'}). \)

Then statement holds for every \( g' \in O(N') \subset O(N), \) i.e. the group \( \langle H, g'Kg'^{-1} \rangle, \) and dually \( \langle g'Hg'^{-1}, K \rangle \) acts transitively on \( S(\mathbb{R}^{N'}). \)

**Corollary 4.2.** \( \langle H, K \rangle \) acts transitively on \( S(\mathbb{R}^N) \) if and only if for every \( g, \) \( \bar{g} \in O(N) \) the subgroup \( \langle gHg^{-1}, \bar{g}K\bar{g}^{-1} \rangle \) acts transitively on \( S(\mathbb{R}^N) \).

**Proof of Theorem 4.1.** We begin with proving the first part of Proposition 4.1, i.e. that the transitive action of \( \langle H, K \rangle \) on \( S(\mathbb{R}^N) \) implies the transitive action of \( \langle H, gKg^{-1} \rangle \) on \( S(\mathbb{R}^N). \)
First of all note due to Remark 2.5 for \( N \geq 2 \) \( \langle H, K \rangle \) acts transitively on \( S(\mathbb{R}^N) \) if and only if \( \langle H, K \rangle \) acts transitively on \( S(\mathbb{R}^N) \) if and only if \( \langle H_0, K_0 \rangle \) does. From it we can assume that \( H \) and \( K \) are connected.

Next, we assume that \( g \in SO(N) \subset O(N) \) leaving the remaining case to the end of proof.

Now we show that the statement holds if \( g \) is in a small neighborhood of \( e \in SO(N) \). Denote \( \langle H, K \rangle \) by \( G \subset SO(N) \) and \( S(\mathbb{R}^N) \) by \( M \). The transitivity of action of \( G \subset SO(N) \) means that the map \( \phi_m : SO(N) \times \{ m \} \to M \) restricted to \( G \times \{ m \} \) is a surjection. It is enough to show that \( \phi_m \) is a basis of the Grassmanian \( G(l, d) \), where \( l = \dim SO(N) \) and \( d = \dim G = \dim G' \). Equivalently the distance:

\[
\max \rho(x, y) = \{ x \in T_eG \cap S(\mathbb{R}^N), y \in T_eG' \cap S(\mathbb{R}^N) \}
\]

is small. If we change \( H \) to \( K \), and conversely, then the proof is the same.

Now the statement follows from the continuity of \( D\phi(e, m) \). Indeed, for a basis \( \{ x_1, \ldots, x_l \} \), \( x_i \in T_eG' \cap S(\mathbb{R}^N) \) be a basis of \( T_eG \) such that \( \text{rank}\{ D\phi(e, m)(x_i) \} = N - 1 \). By the continuity of \( D\phi(e, m) \) if \( \{ x'_1, \ldots, x'_l \} \) is a basis of \( T_eG' \) such that \( \max \rho(x_i, x'_i) \) is small, then

\[
\text{rank}\{ D\phi(e, m)(x'_i) \} = \text{rank}\{ D\phi(e, m)(x_i) \} = N - 1
\]

which shows that \( D\phi(e, m) : T_eG' \to TS^N \) is a surjection if \( g \) is close to \( e \).

Now let \( \tilde{g} = g_0g \) be an arbitrary element of \( O(N) \) with \( g \in SO(N) \) and \( g_0 \notin SO(N) \). We can assume that \( g_0 \) is an involution \( g_0^2 = \text{id} \) with \( \det(g_0) = -1 \). It is clear that \( \langle H, g_0(gKg^{-1})g_0^{-1} \rangle \) acts transitively on the sphere \( S(V) \) if and only if \( \langle H, gKg^{-1} \rangle \) does so, which reduces this case to the previous.

Now let us take \( H = gHg^{-1} \) and \( K = gKg^{-1}g_0^{-1} = g^2K(g^2)^{-1} \).

By the above \( \langle H, K \rangle, \langle H, gKg^{-1} \rangle, \langle gHg^{-1}, K \rangle, \langle gHg^{-1}, gKg^{-1} \rangle \) act transitively on \( S(\mathbb{R}^N) \). The latter is obvious, because \( \langle gHg^{-1}, gKg^{-1} \rangle = g\langle H, K \rangle g^{-1} \). This implies that \( \langle gHg^{-1}, g^2K(g^2)^{-1} \rangle \) acts transitively on \( S(\mathbb{R}^N) \). Now applying the first part of this proof to \( H \), with \( g = g^2 \) being small, we see that \( \langle H, g^2K(g^2)^{-1} \rangle \) acts transitively on \( S(\mathbb{R}^N) \).

Continuing this argument, we see that for every \( n \in \mathbb{N} \) \( \langle H, g^nK(g^n)^{-1} \rangle \), and respectively \( \langle g^nH(g^n)^{-1}, K \rangle \) act transitively on \( S(\mathbb{R}^N) \) if \( g \in U \) belongs to some small symmetric (i.e. \( U = U^{-1} \)) neighborhood of \( e \). But it is known that any such neighborhood of \( e \) generates every connected compact Lie group, which completes the proof of first part of statement.

A proof of the second part with a \( V' \subset V \) invariant for \( H \) and \( K \) is analogous. \( \Box \)

The above theorem let state that the conjugacy classes of the Borel subgroups are determined by partitions of the number \( N = \sum j N_j \) and isomorphism classes of Borel groups of dimensions \( N_j \). In particular, the conjugacy classes of the orthogonal Borel subgroups are determined by the equivalence classes of partitions of the number \( N = \sum j N_j \).

**Lemma 4.3.** Let \( H = H_1 \times H_2 \times \cdots \times H_r \) where \( H_j = \pi_j(H) \), \( H_j \subset O(N_j) \), and \( \sum_j N_j = N \) be an orthogonal Borel subgroup corresponding to a partial flag \( \{ 0 \} = V^0 \subset V^1 \subset V^2 \cdots \subset V^r = V \) in \( V = \mathbb{R}^N \) with its orthogonal decomposition \( V = \bigoplus_{j=1}^r V_j \), where \( V_j = V_j^1 \supset V_j^0 \) spanned by an orthonormal basis \( \{ e_j^i \} \), \( 1 \leq j \leq r, 1 \leq i \leq N_j \) of \( \mathbb{R}^N \). Let next \( K = H_1' \times H_2' \times \cdots \times H_r' \) be
another orthogonal subgroup, where \( H'_j = \pi_j(K), \) \( H'_j \subset O(N'_j), \) and \( \sum_{j=1}^{\delta} N'_j = N \) be an orthogonal Borel subgroup corresponding to a partial flag \( \{0\} = V'^0 \subsetneq V'^1 \subsetneq V'^2 \cdots \subsetneq V'^s = V \) in \( V = \mathbb{R}^N \) with its orthogonal decomposition \( V = \bigoplus_{j=1}^{\delta} V'_{j}, \) where \( V'_{j} = V'^{j-1} \supset V'^{j} \) of \( \mathbb{R}^N \) spanned by an orthonormal basis \( \{e^i_j\}, 1 \leq j \leq s, 1 \leq i \leq N'_j. \)

Then there exists a representative \( \tilde{K} = gKg^{-1} \) of the conjugacy class of \( K \) such that the corresponding partial flag \( \{\tilde{V}^j\}, \tilde{V}^j = \bigoplus_{j=1}^{a} \tilde{V}^j_j, 1 \leq j \leq s \) is spanned by the basis \( \{e^i_j\} \) which spans the partial flag \( \{V^j\} \) corresponding to the subgroup \( H. \) Moreover, if \( \{e^i_j\}, \) and \( \{e'^i_j\} \) are of the same orientation, in particular if \( H \) and \( K \) are connected orthogonal Borel subgroups, then \( g \in SO(N). \)

Furthermore \( \tilde{N}_j = N'_j \) for every \( 1 \leq j \leq s. \)

**Proof.** By the linear algebra we know that ordering \( e^i_j \mapsto e'^i_j, 1 \leq i \leq N \) extends to an orthogonal map \( g \in O(N). \) If \( \{e^i_j\} \) and \( \{e'^i_j\} \) are in the same orientation class then \( g \in SO(N). \) Then \( \tilde{V}_j = g(V'_j), \) and respectively \( \tilde{V}^j = g(V'^j) \) are the required partial flags of the statement.

Since \( g \) is an isomorphism \( \tilde{N}_j = \dim \tilde{V}_j = \dim V'_j = N'_j \) which completes the proof. \( \square \)

As we already noted, in the basis \( \{e^i_j\} \) the orthogonal Borel subgroup \( H, \) and correspondingly the subgroup \( \tilde{K}, \) defines a partition \( N_j, 1 \leq j \leq r \) of \( N, \) respectively another partition \( \tilde{N}_j = N'_j, 1 \leq j \leq s \) of \( N. \) Consequently, in view of Theorem 4.1 and Lemma 4.3 to study of the problem of action of group \( \langle H, K \rangle \) on the sphere we can take the representatives of \( H \) and \( K \) in the same basis.

This let us adapt and apply the geometrical result of [10, Lemma 4.1].

Let \( N_1^H \leq N_2^H \leq N_3^H \leq \cdots \leq N_r^H, N_1^K \leq N_2^K \leq \cdots \leq N_s^K, \sum_{i=1}^{r} N_i^H = \sum_{i=1}^{s} N_i^K = N \) be two different not decreasing partitions of \( N. \) Let \( u = a, r, s \), \( N_a \leq N, \) be the smallest numbers such that for all \( 1 \leq j \leq a, 1 \leq i \leq a \) we have \( N_j^H = N_i^K, \) and

\[
N_a = \sum_{i=1}^{a} N_i^H = \sum_{i=1}^{a} N_i^K
\]

with the convention that a sum over empty set is zero, i.e. in the case when there is not such a \( a < r, a < s. \) In other words, up to the index \( a \) these partitions are equal and define a partition of \( N_a \leq N. \) If \( a = r = s \) then these partitions of \( N \) are equal.

Let next \( N_a \leq N, N_b > N_a \) be a smallest number such that there exist \( r \geq j_b > a \) and \( s \geq i_b > i_a \) such that

\[
N_a + \sum_{j=a+1}^{j_b} N_j^H = N_a + \sum_{i=a+1}^{i_b} N_i^K = N_b
\]

Of course, it can happen that \( j_b = r. \) Then \( i_b = s, \) and consequently \( N_b = N. \)

Put \( N_{b-a} := N_b - N_a. \) By the above \( N_j^H_{j_a+1}, N_j^H_{j_a+2}, \ldots, N_j^H_{j_b} \) and \( N^K_{a+1}, N^K_{a+2}, \ldots, N^K_i_b \) are partitions of \( N_{b-a} \) and they do not contain a common sub-partition of any \( N' < N_{b-a}. \) As a consequence of [10, Lemma 4.1] we have the following proposition

**Proposition 4.4.** Let \( N_1^H \leq N_2^H \leq N_3^H \leq \cdots \leq N_r^H, N_1^K \leq N_2^K \leq \cdots \leq N_s^K, \sum_{i=1}^{r} N_i^H = \sum_{i=1}^{s} N_i^K = N \) be two different not decreasing partitions of \( N. \) Let \( a, b, \) correspondingly \( N_a < N_b, \) and \( N_{b-a} \leq N \) be as defined above. Let next \( H = O(N_1^H) \times O(N_2^H) \times \cdots \times O(N_r^H), \) correspondingly \( K = O(N_1^K) \times O(N_2^K) \times \cdots \times O(N_s^K), \) be the maximal orthogonal Borel subgroups associated with these partitions.
Then the subspace \( V' = \{0\} \times \{0\} \times \cdots \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_3} \times \{0\} \times \cdots \{0\} \) of dimension \( N_{b-a} \) is preserved by \( H \) and \( K \) and the group \( \langle H, K \rangle \) acts transitively on the sphere \( S(V') \).

Our main analytical observation is the fact that to apply the scheme of [10] that is based on [5] we do not need pairs \( H, K \) of subgroups, with nontrivial the Weyl group, which generate an entire Borel group \( G = \langle H, K \rangle \subset O(N) \). It is enough if they generate a group which acts transitively on \( S(V') \) for a subspace \( V' \subset \mathbb{R}^N \) preserved by both \( H \) and \( K \).

Let \( E \) be a functional space on which \( H, K \subset O(N) \) act by the action induced by the action of \( O(N) \) on the space of variables in \( \mathbb{R}^N \). Suppose the \( \tau \in \mathcal{N}(H) \setminus H \), and correspondingly, \( \tau' \in \mathcal{N}(K) \setminus K \) are elements of order two in \( \mathcal{N}(H) \), and \( \mathcal{N}(K) \) respectively. Suppose next that \( \Upsilon = \{\tau_\alpha\} \), and \( \Upsilon' = \{\tau'_\beta\} \) are finite sets of such elements in \( \mathcal{N}(H) \), and \( \mathcal{N}(K) \) respectively.

Consider \( \tilde{H} = \langle H, \Upsilon \rangle \), and correspondingly \( \tilde{K} = \langle K, \Upsilon' \rangle \), the extended subgroups of \( \mathcal{N}(H) \) generated by \( H \) and \( \Upsilon \), and \( K \) and \( \Upsilon' \) respectively.

Now define an action of the group \( \tilde{H} \subset \mathcal{N}(H) \) on \( E \), and correspondingly an action of \( \tilde{K} \subset \mathcal{N}(K) \), as a representation \( \rho : \tilde{H} \rightarrow GL(E) \), respectively \( \rho : \tilde{K} \rightarrow GL(E) \). For an element of the form \( g = \tau h, \tau \in \Upsilon, h \in H \), or respectively \( g' = \tau' k, \tau' \in \Upsilon', k \in K \), it is given by the formula:

\[
\rho(h)f(x) := f(hx) \quad \rho(\tau)f(x) = -1f(\tau x),
\]

and next extended to a homomorphism \( \tilde{H} \rightarrow GL(E) \). Obviously, it extends to a homomorphism from \( \tilde{H} \) to \( GL(E) \) and does not depend on the representative of \( g \) as \( \tau h \), since \( \tau \in \mathcal{N}(H) \) and \( \tau \) commutes with every linear map of \( E \). Similarly we define \( \rho : \tilde{K} \rightarrow GL(E) \).

We will consider the subspaces \( E^H \), respectively \( E^K \), of the fixed points sets of the above defined action.

The following observation is fundamental

**Proposition 4.5.** Let \( E \) be a function space with an \( O(N) \)-invariant domain, \( N \geq 4 \). Let next \( H, K \) be two subgroups of \( O(N) \) such that:

- the Weyl groups \( \mathcal{W}(H), \mathcal{W}(K) \) are nontrivial and contain elements of order two \( \tau, \tau' \) respectively.
- there exists \( 2 \leq N' \leq N \) such that the group \( G = \langle H, K \rangle \) acts transitively on \( S(V') \) for subspace \( V' \simeq \mathbb{R}^{N'} \) preserved by \( H \), and \( K \).
- there exist \( \tau \in \Upsilon \) or \( \tau' \in \Upsilon' \) which acts trivially on the orthogonal complement \( V'_\perp \) and acts nontrivially on \( V' \).

Then for the groups \( \tilde{H}, \tilde{K} \) defined above we have

\[
E^H \cap E^K = \{0\}.
\]

**Proof.** First note that if \( H(V') \subset V' \) and \( K(V') \subset V' \) then also \( H(V'_\perp) \subset V'_\perp \) and respectively \( K(V'_\perp) \subset V'_\perp \). Let \( (x, y) \) be the coordinates in \( \mathbb{R}^N \) corresponding to the orthogonal decomposition \( \mathbb{R}^N = V' \oplus V'_\perp \).

Assume that there exists \( \tau \in \Upsilon \) such that \( \tau \) acts nontrivially on \( V' \) and trivially on \( V'_\perp \).

Let \( f \in E^H \cap E^K \subset E^H \cap E^K \). Take \( \tilde{H} = \langle H, \tau \rangle \subset \tilde{H} \). So that we have \( f \in E^H \cap E^K \subset E^H \cap E^K \subset E^H \cap E^K \).

Fix \( y \in V'_\perp \). Since \( G = \langle H, K \rangle \) acts transitively on \( S(V') \), the function \( f \) is radial in \( x \), i.e. \( f(x, y) \) depends only on \( |x| \) as \( gf(x, y) = f(gx, gy) \) for every \( g \in \langle H, K \rangle \) and \( \langle H, K \rangle \) acts transitively on \( S(V') \).

Moreover \( f(\tau(x, y)) = f(\tau x, y) \), because \( \tau \) preserves \( V' \), and acts trivially on \( V'_\perp \). This gives

\[
-f(x, y) = f(\tau(x, y)) = f(\tau x, y) = f(x, y),
\]
Theorem 4.8. Let $x \in V'$, since $|\tau x| = |x|$. 

This shows that for every fixed $y$ the section function $f_y(x) = f(x, y)$ is identically equal to 0. Consequently $f(x, y)$ is equal identically zero which proves the statement. □

As we announced we are not going into analytical complexity of particular variational problems that need a special study by fine analytical tools. Instead we assume what usually is an output of this study as our supposition (see [5, Th. 3.2]).

Assumption 4.6 (On the invariant functional $\Phi$). Let $E$ be an infinite Hilbert space with an orthogonal linear action $\rho(g) : E \to E$ of a compact Lie group $G$, $\Phi : E \to \mathbb{R}$ is a $C^1$ functional and the following hold:

i) $\Phi$ is $G$-invariant.

ii) $\dim E^G = \infty$.

iii) $\Phi|_{E^G} : E^G \to \mathbb{R}$ has infinitely many critical points $\{u_k\}$ such that $\|u_k\| \geq c_k$, $c_k \to \infty$.

Remark 4.7. In all the quoted papers [3], [3], [11], [9] and others, e.g. also quoted in [11], the Assumption 4.4 is verified for studied there problems. It is based on a scheme which is called the fountain theorem (see [2] for an excellent exposition), and which began with the famous Ambrosetti Rabinowitz theorem of [1]. The latter is applicable, because in all the cited above works the studied problems lead to a functional $\Phi$ which is even, i.e. $\Phi(-u) = \Phi(u)$.

Notify, since the functional space $E$ consists of functions with domain equal to $\mathbb{R}^N$, the Palais-Smale condition is not satisfied for a functional $\Phi : E \to \mathbb{R}$ in general. But her $\Phi$ is $O(N)$-invariant with the action of $O(N)$ on the functional space $E$ induced by the action of $O(N)$ on the domain of functions equal to $\mathbb{R}^N$, and $G = (H, \tau)$, $H = H_1 \times H_2 \cdots H_r$, $H_j \subset O(N_j)$, $\tau \in \mathcal{W}(H) \setminus H$, $\sum N_j = N$. Consequently, the Palais-Smale condition follow form the Lions theorem [12] which says that for the discussed functional space the embedding

$$E^G \hookrightarrow L^s(\mathbb{R}^N)$$

is compact for $s$ of an interval $[a, b]$ depending on the problem (on $N$) provided $N_j \geq 2$ for every $1 \leq j \leq r$.

This additionally to our Remark 2.5 justifies the assumption $N_j \geq 2$ put on an orthogonal Borel subgroup.

In the terms used here, the problem of multiplicity of infinite series of non-radial sign-changing solutions was reduced to a problem of finding a set $\{H_i\}_{i=1}^{s_N}$, with some $s_N$ depending on $N$, of subgroups of $O(N)$ such that:

- for every $i \neq j$ $H_i$ and $H_j$ are not conjugated in $O(N)$,
- for every $i \neq j$ the group $(H_i, H_j)$ acts transitively on $S(V')$, where $V' \simeq \mathbb{R}^N$ and is invariant for $H$ and $K$,
- for each $1 \leq i \leq s_N$ there exists an element $e \neq \tau \in \mathcal{W}(H_i)$ of order 2, or such that $\tau(V') \subset V'$, and $\tau|_{V'_i} = \text{id}|_{V'_i}$, or an element $e \neq \tau' \in \mathcal{W}(H_i)$ of order 2 of the same property.

With the notation of previous section and Theorem 3.13 we have the following theorem.

Theorem 4.8. Let $N \geq 4$ and $O(N)$ orthogonal group of $\mathbb{R}^N$ and $\Omega \subset \mathbb{R}^N$ be equal to $\mathbb{R}^N$, $D_r^N$ or $S^{N-1}$. Suppose that we have a variational problem which weak solutions correspond to the critical points of a functional $\Phi : E \to \mathbb{R}$ defined on a functional space $E$ on $\Omega$, where $E$ posses the natural linear action induced by the action of $O(N)$ on $\mathbb{R}^N$. Assume also that $\Phi$ is $O(N)$-invariant and satisfies Assumption 4.6 with respect to every closed subgroup $G \subset O(N)$. 

Then there exist \( s_N \) geometrically distinct, with respect to the symmetry, infinite series of solutions \( S_i = \{ u_k^i \} \), \( 1 \leq i \leq s_N \), \( 1 \leq k < \infty \) of this problem, where

\[
s_N = \begin{cases} 
    P\left(\frac{N}{4}\right) & \text{if } N \geq 4 \text{ and } N = 4M, \\
    P\left(\frac{N-2}{4}\right) & \text{if } N \geq 9 \text{ and } N = 4M + 5 = 4M' + 1, \\
    P\left(\frac{N-1}{4}\right) & \text{if } N \geq 6 \text{ and } N = 4M + 2, \\
    P\left(\frac{N}{4}\right) & \text{if } N \geq 7 \text{ and } N = 4M + 3.
\end{cases}
\]

with \( P(N) \) defined in (3.14) with an exponential asymptotic behavior given in (17). Moreover, in each series \( S_i \) the solutions \( u_k^i \) are geometrically distinct for different \( k \).

Note that Theorem 4.8 applies to the problems studied in [5], [3], [10], and [11]. Another trivial observation: \( N = 5 \) is not covered by the cases listed above. Any other number \( N \geq 4 \) appears there.

**Proof.** We will prove the theorem showing that the assumptions of Proposition 4.5 are satisfied for the groups determined by the partitions given in the statement. To do it we first have to show that the assumption of Proposition 4.3 is satisfied.

First of all let us take the unique representative \( M_1 \leq M_2 \leq M_3 \leq \cdots \leq M_r \) of each equivalence class of \( P(M) \) partitions of \( M = \sum M_j \). Then the formula defined in (13), (14), (15), or respectively (16) gives a partition of \( N \) in each of listed above cases modulo 4. Obviously for every summand we have \( N_j \geq 2 \).

Assume for simplicity that \( N = 4M \), i.e we have the first of four cases. Let \( \pi(M) \) and \( \pi'(M) \) be two different partitions of \( M \). Observe that if \( 0 \leq M_a < M_b \leq M \) is the interval on which \( \pi(M) \) and \( \pi'(M) \) do not coincide, with the minimal \( M_a \) with this property, then an interval \( 0 \leq N_a = 4M_a < N_b = 4M_b \leq 4M \) has the corresponding property with respect to the partitions of \( N = 4M \):

\[
\{2M_1, 2M_1, 2M_2, 2M_2, \ldots, 2M_j, 2M_j, 2M_{j_a}, 2M_{j_a+1}, 2M_{j_a+1}, \ldots, 2M_r, 2M_r\} \\
\{2M_1, 2M_1, 2M_2, 2M_2, \ldots, 2M_{i_a}, 2M_{i_a}, 2M'_{i_a+1}, 2M'_{i_a+1}, \ldots, 2M'_r, 2M'_r\}.
\]

(17)

Indeed \( M_j = M'_i \) for \( i = j \leq j_a = i_a \), so \( 2M_j = 2M'_i \) for \( i = j \leq j_a = i_a \), \( 2M_{j_a+1} \neq 2M'_{i_a+1} \) as \( M_{j_a+1} \neq M'_{i_a+1} \) thus \( 2j_a = 2i_a \) is the maximal index up to which partitions (17) of \( N = 4N \) coincide.

We have \( N_a \leq 4M_b \) but it could happen that \( N_b < 4M_b \) in general. Nevertheless, the interval \( N_a, N_b \) on which the partitions do not coincide contains at least one pair \( M_{j_a+1}, M_{j_a+1} \), or dually \( M'_{i_a+1}, M_{i_a+1} \), because it can happen that \( 2M_{j_a+1} + 2M_{j_a+1} = 2M'_{i_a+1} \) or dually \( 2M'_{i_a+1} + 2M_{i_a+1} = 2M_{j_a+1} \).

Now let us take as \( N_b \) the largest number such that the partitions \( \pi(N) \) and \( \pi'(N) \) do not coincide in any subinterval, and \( \tau, \tau' \), the transposition of first pair of coordinates corresponding to \( 2M_{j_a+1}, 2M_{j_a+1} \), ro respectively \( 2M'_{i_a+1}, 2M'_{i_a+1} \) depending which of the above cases happens. (Note that the both cases could happen). Let \( j_b \), or dually \( i_b \) be the index corresponding to \( N_b \), i.e. the index for which

\[
\sum_{j_a+1}^j b \quad \sum_{i_a+1}^{i_b} N'_i
\]

It is clear that \( \tau \), correspondingly \( \tau' \), is an element of \( \mathcal{N}(\pi(N)) \), respectively of \( \mathcal{N}(\pi'(N)) \) of order 2 to (a reflection). Moreover \( \tau \), correspondingly \( \tau' \), does not permute the summands with indices \( \leq 2j_a = \tilde{j}_a = \tilde{i}_a = 2i_a \), and greater then \( j_b \), or \( i_b \) respectively.

On the other hand, by Proposition 3.19 we have \( s_N \) not-equivalent classes of such partitions.
Now, let us fix a frame, i.e., an orthogonal basis $e_1, e_2, \ldots, e_N$ of $V = \mathbb{R}^N$. Let next $0 \nsubseteq V^1 \nsubseteq V^2 \nsubseteq \cdots \nsubseteq V^r = V$, $0 \nsubseteq V^1 \nsubseteq V^2 \nsubseteq \cdots \nsubseteq V^s = V$, $\dim V_j = N_j$, $\dim V'_i = N'_i$ be the partial flags corresponding to $\pi(N)$ and $\pi'(N)$. Finally let

$$H = O(N_1) \times O(N_2) \times \cdots \times O(N_r), \quad H' = O(N'_1) \times O(N'_2) \times \cdots \times O(N'_s)$$

be the maximal orthogonal Borel subgroups corresponding to these partial flags, i.e., corresponding to these two partitions. Taking elements $\tau \in \mathcal{N}(H) \subset O(N)$, correspondingly $\tau' \in \mathcal{N}(H') \subset O(N')$, of their normalizers as described above we get elements which acts nontrivially only on the partial flags $V^{j_1+1} \nsubseteq V^{j_2} \nsubseteq \cdots \nsubseteq V^{j_k}$ and $V'^{i_1+1} \nsubseteq V'^{i_2} \nsubseteq \cdots \nsubseteq V'^{i_k}$.

We put

$$V' := \{0\} \times \cdots \times \{0\} \times \mathbb{R}^{N_{j_1}+1} \times \mathbb{R}^{N_{j_2}+2} \times \cdots \times \mathbb{R}^{N_{j_k}} \times \{0\} \times \cdots \times \{0\}$$

$$\equiv \{0\} \times \cdots \times \{0\} \times \mathbb{R}^{N'_{i_1}+1} \times \mathbb{R}^{N'_{i_2}+2} \times \cdots \times \mathbb{R}^{N'_{i_k}} \times \{0\} \times \cdots \times \{0\} \simeq \mathbb{R}^{N_b-N_0}$$

Moreover we can take $\tau$ in $\mathcal{N}'(H)$, or correspondingly $\tau'$ in $\mathcal{N}'(H')$, being of order two as described above. It is clear that $\tau \in \mathcal{Y}(H)$, or $\tau'$, preserves $V'$ and $\tau \in \mathcal{Y}(H)$, or $\tau' \in \mathcal{Y}'(H')$.

Now, let us consider the groups $\tilde{H} = \langle H, \mathcal{Y}(H) \rangle \subset O(N)$ and $\tilde{H}' = \langle H', \mathcal{Y}'(H') \rangle \subset O(N)$.

Finally, take the corresponding fixed points subspaces $E^{\tilde{H}}$, $E^{\tilde{H}'}$, and the subspace

$$V' := V_{j_1+1} \oplus V_{j_2+2} \oplus \cdots \oplus V_{i_b} = V'_{i_1+1} \oplus V'_{i_2+2} \oplus \cdots \oplus V_{i_b} \simeq \mathbb{R}^{N_{j_1}+1} \times \mathbb{R}^{N_{j_2}+2} \times \cdots \times \mathbb{R}^{N_{j_k}} \simeq \mathbb{R}^{N_b-N_0}.$$

By its construction $V'$ is preserved by $H$ and $H'$ and $\tau$, or $\tau'$ if it is the case. Moreover $\tau$, or $\tau'$ acts trivially on $V'_\perp$.

From the above and Proposition 4.3 it follows that the group $\langle H, H' \rangle$ acts transitively on $S(V')$ and we can apply Proposition 4.3 Consequently $E^{\tilde{H}} \cap E^{\tilde{H}'} = \{0\}$ if $H$, and $H'$ are defined by two different partitions.

Let $\Phi$ be the functional corresponding to the studied variational problem with symmetry. Now we can restrict $\Phi$ to every subspace $E^{\tilde{H}}$ with $H$ the maximal orthogonal Borel subgroup as above. By our assumption 4.6 $\Phi |_{E^{\tilde{H}}}$ are weak solutions of the original variational problem by the Palais symmetry principle.

If $u \in E^{\tilde{H}}$ and $u' \in E^{\tilde{H}'}$ then $u \neq u'$ as they are in the linearly independent subspaces. We let to show that they are geometrically independent, namely that there is not $g \in O(N)$ such that $u'(x) = u(gx)$ for all $\in \Omega$. Indeed, if $u(x) \in E^{\tilde{H}} \subset E^{H}$ and $u' = u(gx)$, $g \neq e$, then $u'(x) = u(gx) \in E^{\tilde{H}g^{-1}}$. But $gHg^{-1}$ is another maximal orthogonal Borel subgroup in the same equivalence class as $H$. This implies that the partition corresponding to $gHg^{-1}$ is the same as that of $H$ which is impossible because we took only one representative $H$ of each conjugacy class of the maximal Borel subgroups.

Finally, if $u_k^i$, $u_k^k$, $1 \leq i \leq s_N$, $k' > k$, are two different solutions in one infinite series $\mathcal{S}_i$ then $\|u_k^i\| > \|u_k^k\|$ by our assumption 4.6. Since we supposed that the action of $O(N)$ on $E$ induced by the action of $O(N)$ on $\Omega$ preserves the norm, it is impossible to have $u_k^k(x) = u_k^k(gx)$. This shows that these two solutions are geometrically distinct.

\[\Box\]

**Remark 4.9.** Note that Assumption 4.6 iii) implies that in each series $\mathcal{S}_i$, $1 \leq i \leq s_N$ of Theorem 4.8 consists of infinitely many geometrically distinct functions $\{u_k^i\}$. If we drop out this assumption then any two different series $\mathcal{S}_i$ of solutions still consist geometrically distinct functions, by the same argument as in the proof of Theorem 4.8 Moreover, since they are given by the variational
principle, the cardinality of each series \( S_i \) is not smaller than the number of critical values of \( \Phi|_{E^G} \). Indeed two solutions being in two distinct critical levels must not be in the same \( O(N) \)-orbit as \( \Phi \) is \( O(N) \)-invariant.

It is clear that series of solutions found in this way are not radial and are sign-changing. Indeed \( u(\tau(x, y)) = u(\tau x, y) = -1 u(x, y) \) implies that the nodal set of \( u \in E^H \) contains the fixed point set of reflection \( \tau \), i.e. a hyperplane \( \{(x, y) : \tau x = x \} \).

In analysis, the questions of finding sign-changing, or correspondingly radial solutions for problems posed in \( D^N \), or \( \mathbb{R}^N \) is of importance. Obviously a radial solution is \( SO(N) \)-invariant and conversely, so the second question can be posed as the existence of \( SO(N) \)-invariant solutions. Of course, this method gives only some families of the radial and sign-changing solutions but indicates that this question can be successfully studied by variational methods as only we are able to defined a sub-representation of the functional space \( E \), which is linearly independent of \( E^{SO(N)} \) is of the form \( E^{G, \rho} \) for a subgroup \( G \subset SO(N) \) and its representation structure \( \rho \). In general, to get not \( SO(N) \)-invariant, or sign-changing solutions one can impose some analytical conditions and use subtle analytical arguments. There several important works, also very recent, studying these problems, so we refer only most close to the analytical problems we have already described \([8], [13], [14]\). In the first of quoted papers, the author showed that for an elliptic problem in \( D^N \) and any group \( G \subset SO(N) \) such that \( G \) does not act transitively on \( S(\mathbb{R}^N) \) there exists infinitely many solutions which are \( G \)-invariant but not \( SO(N) \)-invariant (not radial). He did it by proving that the distribution of critical values corresponding to the functional restricted to \( E^{SO(N)} \) is smaller than the distribution of critical values which correspond to the restriction of functional to \( E^G \). In the second paper the authors studied a similar problem as that of \([5]\) for \( N = 5 \) which dimension is not covered by the approach used in \([5]\) and studied here for the obvious combinatorial reason. But they impose a condition of the form nonlinear term which let them to reduce the studied problem to a problem in \( \mathbb{R}^4 \). Finally, in \([14]\) for a problem like in \([5]\) and every \( N \geq 2 \) but with some additional assumption on the nonlinear part it is shown that for any \( k \geq 7 \) there exist infinitely many \( D_k \times O(N - 2) \)-invariant solutions, where \( D_k \subset O(2) \) is the dihedral group of \( 2k \) elements. Moreover these solutions are not \( O(2) \times O(N - 2) \)-invariant, e.g. for \( N = 4 \) give a nice example of not \( O(N) \)-invariant (radial) solutions which are different than those obtained by the method of \([5]\), i.e. by the scheme studied here.

4.1. Spaces of \( \rho \)-interwinding functions. It is possible to define these functional subspaces associated with a partition \( \pi(N) \) in terms of representations of the Weyl group \( \mathcal{W}(H) \), where \( H \) is the unique maximal orthogonal Borel subgroup defined by \( \pi(N) \). Then it would be seen that the nodal set contains union of fixed points of all elements of order two in \( \mathcal{W}(H) = \mathcal{W}(\pi(N)) \), i.e. the union of fixed points of all transposition, or equivalently reflections, in \( \mathcal{W}(\pi(N)) = \mathcal{W}(H) \). But we have consider these reflections, and their fixed points, only in subspaces on which they are not trivial.

Let \( \{ V_j \}_j = \{ \mathbb{R}^{N_1}, \mathbb{R}^{N_2}, \ldots, \mathbb{R}^{N_r} \} \) be an orthogonal partial flag in \( \mathbb{R}^N \) with \( \dim V_j = N_j \), or equivalently the corresponding maximal orthogonal Borel subgroup uniquely determined the partition \( \pi \) of \( N = \sum_{j=1}^r N_j \).

By Proposition \([5,10]\) and Corollary \([3,11]\) the normalizer \( \mathcal{N}(\{ V_j \}_j) \) in \( O(N) \), or equivalently the normalizer \( H \) in \( O(N) \) consists of the maximal orthogonal Borel subgroup \( H = O(N_1) \times O(N_2) \times \cdots \times O(N_r) \) and all permutations

\[ \sigma_\pi \in \mathcal{G}_\pi = \mathcal{G}(\phi_\pi(n_1)) \times \mathcal{G}(\phi_\pi(n_2)) \times \cdots \times \mathcal{G}(\phi_\pi(n_q)). \]

Moreover the Weyl group of \( H \) is a product of \( q \) permutation groups

\[ \mathcal{W}(H) = \mathcal{G}_\pi = \mathcal{G}(\phi_\pi(n_1)) \times \mathcal{G}(\phi_\pi(n_2)) \times \cdots \times \mathcal{G}(\phi_\pi(n_q)) \]
Let \( \rho^1 : \mathfrak{S}(n) \to \{-1, 1\} = O(1) \) be the unique non-trivial one-dimensional representation of the permutation group \( \mathfrak{S}(n) \) given by \( \sigma \mapsto \text{sign} \sigma \). Correspondingly, let \( \rho^0 : \mathfrak{S}(n) \to \{-1, 1\} = O(1) \) be the trivial one-dimensional representation of the permutation group \( \mathfrak{S}(n) \) given by \( \sigma \mapsto 1 \) for all \( \sigma \in \mathfrak{S}(n) \).

Moreover, since \( W(H) = \mathfrak{S}(\phi_\tau(n_1)) \times \mathfrak{S}(\phi_\tau(n_2)) \times \cdots \times \mathfrak{S}(\phi_\tau(n_q)) \) is the product of group, every one-dimensional orthogonal representation of \( W(H) \), i.e. every homomorphism \( \rho \) from \( W(H) \) to \( \mathbb{Z}_2 = O(1) \) is of the form

\[
\rho(\sigma_1, \sigma_2, \ldots, \sigma_p) = \rho^{\delta_1}_1(\sigma_1) \cdot \rho^{\delta_2}_2(\sigma_2) \cdots \rho^{\delta_p}_p(\sigma_p)
\]

where \( \delta_i \in \{0, 1\} \), and \( \rho^{\delta_i}_i : \mathfrak{S}(\phi_\tau(n_i)) \to \{-1, 1\} \) is either the the unique nontrivial homomorphism, or correspondingly the trivial homomorphism from \( \mathfrak{S}(\phi_\tau(n_i)) \) to \( O(1) \) depending whether \( \delta_i \) is equal to 1, or to 0 respectively. In other words every such representation is determined by the sequence \( (\delta_1, \delta_2, \ldots, \delta_p) \), e.g. \( \rho \) is trivial if and only if this sequence consists of zeros only.

Furthermore, every representation \( \rho \) of \( W(H) \) composed with the natural projection \( \mathcal{N}(H) \to \mathcal{N}(H)/H = W(H) \) defines a representation of \( \mathcal{N}(H) \). Observe that \( \rho|_{\mathcal{N}(H)} \equiv 1 \) for every such \( \rho \).

**Definition 4.10.** Let \( H = O(N_1) \times O(N_2) \times \cdots \times O(N_r) \) be a maximal orthogonal Borel subgroup with nontrivial the Weyl group \( W(H) = \mathfrak{S}(\phi_\tau(n_1)) \times \mathfrak{S}(\phi_\tau(n_2)) \times \cdots \times \mathfrak{S}(\phi_\tau(n_q)) \). Let \( \rho = \rho^{\delta_1}_1(\phi_\tau(n_1)) \cdot \rho^{\delta_2}_2(\phi_\tau(n_2)) \cdots \rho^{\delta_q}_q(\phi_\tau(n_q)) \) be a nontrivial one-dimensional representation of \( \mathcal{N}(H) \). Then in every functional space \( E \) with domain \( \Omega = \mathbb{R}^N \), or \( D^N \) we can define a linear action of \( \mathcal{N}(H) \) (a representation structure) by the formula

\[
gu(x) := \rho(g)u(gx), \quad \text{or equivalently for} \quad g = \sigma h, \quad \sigma \in W(H) \quad \text{and} \quad h \in H \quad \text{by} \quad (\sigma, h)u(x) := \rho(\sigma)u(\sigma hx),
\]

where \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p) \), and the formula does not depend on a representation of \( g \) as a pair \( (\sigma, h) \) since \( \rho|_{\mathcal{N}(H)} \equiv 1 \).

Finally to a pair \( (H, \rho) \) as above we assign the fixed point space of this action denoted by \( E^{(H, \rho)} \) and called (as in [7]) the space of \( \rho \)-interwinding functions.

**Remark 4.11.** Note that the spaces \( E^H \) discussed previously are special cases of spaces defined in Definition 4.10, i.e. every space \( E^H \) is of this form.

Our task was to find a family \( (H_i, \rho^i), 1 \leq i \leq s_N \) as above such that \( E^{(H, \rho)} \cap E^{(H_j, \rho^j)} = \{0\} \) if \( i \neq j \) and with possibly large \( s_N \).

Observe that each \( \mathfrak{S}(\phi_\tau(n_i)) \) is generated by transpositions \( \tau \), geometrically reflections, so \( W(H) = \mathfrak{S}(\phi_\tau(n_1)) \times \mathfrak{S}(\phi_\tau(n_2)) \times \cdots \times \mathfrak{S}(\phi_\tau(n_q)) \) is generated by the compositions of transpositions. As \( u(\tau x) = -1u(x) \) provides \( \tau \in W(H) \) and \( \rho(\tau) = -1 \). The latter implies that \( u(x) = 0 \) if \( \tau x = x \) is a fixed point of \( \tau \). As a consequence we get the following.

**Corollary 4.12.** The zero set (the nodal set if \( u \) is a solution) of every \( u \in E^{(H, \rho)} \) with \( \rho = \rho^{\delta_1}_1 \rho^{\delta_2}_2 \cdots \rho^{\delta_q}_q \) contains

\[
\bigcup_{\delta_i=1} \mathcal{H}_i
\]

where \( \mathcal{H}_i \) is a union of hyperplanes being fixed points of transposition \( \tau \in \mathfrak{S}(\phi_\tau(n_i)) \) interpreted as reflections.

In particular, if \( H = O(2) \times O(2) \times \cdots O(2) \subset O(2N) \), then \( W(H) = \mathfrak{S}(N) \) is equal to \( W(T^N) \subset SO(2N) \). Consequently for every \( u \in E^{(H, \rho)} \) its zero set contains the union of walls of the Weyl chambers of canonical representation of \( O(N) \) in \( \mathbb{R}^N \).
Remark 4.13. It is worth of pointing out that the functional subspaces of \( \rho \)-interwining functions as defined in Definition 4.10 can be define in a context of action of any Coxeter group, but we do not know any application, especially that the part which shows that some of them are orthogonal has not a direct analog.

5. Supplementary information

5.1. Borel groups. In 1950 A. Borel [10] gave a complete classification of all groups acting transitively and effectively on the sphere completing earlier results of D. Montgomery and H. Samelson [15].

Theorem 5.1 (A. Borel). If a connected compact group \( G \) of linear transformations acts effectively transitively on the sphere \( S(\mathbb{R}^N) \) then \( S(\mathbb{R}^N) \) is \( G \)-homeomorphic to the homogenous space \( G/H \) where the pair \((G, H)\) is one of the listed below.

i) If \( N \) is odd, or equivalently \( N - 1 \) is even, then \((G, H) \simeq (SO(N), SO(N - 1))\), or \((G_2, SU(3))\) in the case when \( N = 7 \); 
ii) If \( N = 2s \), or equivalently \( N - 1 = 2s - 1 \) and \( s \) is odd, then \((G, H) \simeq (SO(N), SO(N - 1))\) or \((SU(s), SU(s - 1))\); 
iii) If \( N = 2s \), or equivalently \( N - 1 = 2s - 1 \) and \( s \) is even, then \((G, H) \simeq (SO(N), SO(N - 1)), (SU(s), SU(s - 1)), (Sp(s/2), (Sp(s/2 - 1)), 
iv) \((Spin(9), Spin(7))\) in case \( n = 15 \), or \((Spin(7), G_2)\) in case \( n = 7 \).

Here \( G_2 \subseteq GL(7, \mathbb{R}) \) denotes the automorphism group of the octonion algebra, i.e. the subgroup of \( GL(7, \mathbb{R}) \) of that preserves the non-degenerate 3-form

\[
dx^{123} + dx^{235} + dx^{346} + dx^{450} + dx^{561} + dx^{602} + dx^{013}
\]

(invariant under the cyclic permutation \((012356)\)) with \( dx^{ijk} \) denoting \( dx^i \wedge dx^j \wedge dx^k \) in variables \( x^i, 0 \leq i \leq 6 \) of \( \mathbb{R}^7 \).

Remark 5.2. It is worth of pointing out that original formulation of the Borel theorem is stronger, i.e. it has a weaker supposition that \( G \) is a compact connected Lie group of transformations of a homotopy sphere acting effectively and transitively.

The Borel theorem says, roughly speaking, that only \( SO(N) \) or in few cases its classical linear subgroups are only connected groups that act transitively on \( S(\mathbb{R}^N) \). The latter happens if \( \mathbb{R}^N \) has an extra structure: complex, quaternionic, spinor, or octonion.

We end this subsection with a statement which is generalization of Proposition 4.1 and a positive answer to a question posed in [10] Remark 4.1 in a stronger form.

Once more, let \( \pi_H = \{N^H_1 \leq N^H_2 \leq \cdots \leq N^H_r\} \), \( \pi_K = \{N^K_1 \leq N^K_2 \leq \cdots \leq N^K_s\} \), \( \sum_j N^H_j = \sum_k N^K_k = N \) be two different not decreasing partitions of \( N \) corresponding to the maximal orthogonal Borel subgroups \( H = O(N^H_1) \times O(N^H_2) \times \cdots \times O(N^H_r) \), \( K = O(N^K_1) \times O(N^K_2) \times \cdots \times O(N^K_s) \), or maximal orthogonal connected Borel subgroups \( H = SO(N^H_1) \times SO(N^H_2) \times \cdots \times SO(N^H_r) \), \( K = SO(N^K_1) \times SO(N^K_2) \times \cdots \times SO(N^K_s) \) respectively. Remind, that we say that \( \pi_H \) and \( \pi_K \) contain common partition of \( N' < N \) if there exist 1 \( \leq a \leq r \), 1 \( \leq b \leq s \) such that \( \sum_{j=1}^a N^H_j = \sum_{j=1}^b N^K_j = N' \). We begin with a proposition which is a positive answer to the question posed in [10].

Proposition 5.3. Let \( K, H \subseteq O(N) \) be two maximal orthogonal Borel subgroups as above, respectively \( K, H \subseteq O(N) \) be two maximal orthogonal connected Borel subgroups, corresponding to two different partitions \( \pi_H \) and \( \pi_K \), of \( N \) such that \( N^H_i \geq 2 \) and \( N^K_j \geq 2 \) for all 1 \( \leq i \leq r \), 1 \( \leq j \leq s \).
Then $\pi_H$ and $\pi_K$ do not contain a common partition of $N' < N$ if and only if $\langle H, K \rangle = O(N)$, or respectively $\langle H, K \rangle = SO(N)$ when $H, K$ are connected.

**Proof.** We will show the connected case only.

The part "if" is a direct consequence of [10] Lemma 4.1 shown in [10] Remark 4.1. To prove the part "only if" observe that $\langle H, K \rangle$ acts transitively on $S'(\mathbb{R}^N)$ as follows from [10] Lemma 4.1 (cf. Proposition 4.4). Consequently it is one of the groups listed in the statement of Borel theorem 5.1.

By the dimension assumption $N_1^H \geq 2, N_1^K \geq 2$. Next, since $\pi_H$ and $\pi_K$ are different and do not contain a common partition of $N' < N$, either $N_1^H \geq 3$ or $N_1^K \geq 3$. Consequently the group $SO(3)$ is contained at least in one of the subgroups $H$ or $K$. Suppose that $N_1^H \geq 3$. Is enough to find an orthogonal map $A \in SO(3) \subset SO(N_1^H)$ and its extension $\tilde{A}$ to $H$ such that $A \notin G$ for all $G \subset SO(N)$ listed in Theorem 5.1. Consider $A$ given by the matrix

$$\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}$$

and extended to a linear map of $\mathbb{R}^N$ by the identity on remaining coordinates.

Since $A \in SO(\mathbb{R}^N)$, the element $\tilde{A} = A \times \{e\} \times \{e\} \times \cdots \times \{e\}$ belongs to $H$, thus belongs to $\langle H, K \rangle$. On the other hand $\tilde{A}$ does not belong to any proper subgroup $G$ of $SO(N)$ listed in the Borel theorem 5.1. This shows that $\langle H, K \rangle = SO(N)$ which completes the proof. $\square$

Finally, we are able to describe a subgroup generated by two maximal orthogonal subgroups $H, K \subset O(N)$ associated with two partitions $\pi_H, \pi_K$ of $N$. Reasoning as in before Proposition 4.4 we split the sequences of (non decreasing) partitions $\{N_1^H, N_2^H, \ldots, N_r^H\}, \{K_1^K, N_2^K, \ldots, N_s^K\}$ subsequence following intervals of indices for which these partitions are equal or not comparable. More precisely, let $1 \leq j_a < j_b < j_2 < j_3 \cdots < j_a$ and $1 \leq i_1 < i_2 < i_3 \cdots < i_k$ be indices such that $j_a = i_1$ and $N_j^H = N_i^K$ for all $j, i \leq j_a$, in the next interval of indices $j_a < j_a + 1 < \cdots j_b, i_a < i_{a+1} < \cdots i_b$ be such that $\pi_H$ and $\pi_K$ are not commensurable, i.e. do not exist $j_a < j_c < j_b, i_a < i_c < i_b$ such that $\sum_{j=j_a}^{j_b} N_j^H = \sum_{i=i_a}^{i_b} N_i^K$ but

$$\sum_{j=j_a}^{j_b} N_j^H = \sum_{i=i_a}^{i_b} N_i^K = N_{b_1} - N_{a_1},$$

where $N_{a_1} = \sum_{j=1}^{j_a} N_j^H, N_{b_1} = \sum_{j=1}^{j_b} N_j^H = \sum_{i=1}^{i_b} N_{b_i}$. In the next intervals $j_b = j_a, i_1 \leq i_a$ the partitions are equal and so on alternately. Put $N_{b_1, a_1} := N_{b_1} - N_{a_1} = N_{b_1} - N_{a_1}$.

**Theorem 5.4.** Let $\pi_H = \{N_1^H, N_2^H, \ldots, N_r^H\}$, and $\pi_K = \{K_1^K, K_2^K, \ldots, K_s^K\}$ be two partitions of $N$. Let next $H = O(N_1^H) \times O(N_2^H) \times \cdots \times O(N_r^H), K = O(N_1^K) \times O(N_2^K) \times \cdots \times O(N_s^K)$ be associated with them maximal orthogonal Borel subgroups of $O(N)$, correspondingly $H = SO(N_1^H) \times SO(N_2^H) \times \cdots \times SO(N_r^H), K = SO(N_1^K) \times SO(N_2^K) \times \cdots \times SO(N_s^K)$ maximal connected orthogonal Borel subgroups.

With the above notation, the group $\langle H, K \rangle$ is equal to $O(N_1^H) \times O(N_2^H) \times \cdots \times O(N_{j_{a_1}}) \times O(N_{b_1, a_1}) \times O(N_{j_{a_1}+1}) \times O(N_{j_{a_2}}) \times \cdots \times O(N_{b_k, a_m})$ with the convention that if $j_{b_k} = j_{a_{q+1}}$, or correspondingly $b_k = a_m$ then the corresponding factor $O(N_{j_{b_k}+1}) \times O(N_{j_{a_{q+1}}})$, or $O(N_{b_k, a_m})$ respectively, is equal to $\{e\}$. Moreover the thesis holds for the connected case with a corresponding formulation.
Proof. The statement is a direct consequence of Proposition 5.3 applied consecutively to the partitions $N_{j_{aq}}^H, N_{j_{aq}+1}^H, \ldots, N_{j_{aq}}^H$ and $N_{i_{aq}}^H, N_{i_{aq}+1}^H, \ldots, N_{i_{aq}}^H$. In the intervals in which the partitions are equal the factors of $(H, K)$ are the same as in the original groups $H$ and $K$. \hfill \Box

5.2. Partitions and their properties. The functions $P(N)$ and $Q(N)$ have been studied by several mathematicians for hundreds years, so let us give only references to the survey articles from MathWorld–A Wolfram Web Resource ([18, 19]) where one can find as well basic facts as an expanded bibliography.

Remark 5.5. It is worth of pointing out that the function $R(N)$ is strictly monotonic for $N \geq 4$, and the first its values, for $N = 1, \ldots, 10$ are equal to

$$0, 1, 1, 3, 4, 7, 10, 16, 22, 32$$

Remark 5.6. Note that the functions $P(N; 1), Q(N; 1), R(N; 1)$ are not expressed directly by the classical number theory functions $P(N, k), Q(N, k)$ of the number of partitions of $N$ into summands each of them is smaller or equal to $k$.

Lemma 5.7. For the functions $P(N; 1), Q(N; 1)$ of Definition 7.14 we have

For $N \geq 2$ \quad $P(N; 1) = P(N) − P(N − 1)$

For $N \geq 2$ \quad $Q(N; 1) = Q(N) − Q(N − 1; 1)$

therefore

$$Q(N; 1) = Q(N) − Q(N − 1) + Q(N − 2) + \cdots + (-1)^{N−2}Q(2).$$

Consequently

$$R(N; 1) = P(N) − P(N − 1) − \left(\sum_{i=0}^{N−2} (-1)^i Q(N − i)\right).$$

Proof. Since $P(N)$ measures the number of partitions of $N$ where the order of addends is not considered significant, we can assume that in every such partition $N_1, N_2, \ldots, N_r$ we have $N_1 \leq N_2 \leq \ldots \leq N_{r−1} \leq N_r$. If $N_1 = 1$ then $N_2, N_3, \ldots, N_r$ gives a partition of $N − 1$ of length $r − 1$. Conversely, if $N_2, N_3, \ldots, N_r$ is a partition of $N − 1$ of length $r − 1$ then $1 \leq N_2 \leq \ldots \leq N_{r−1} \leq N_r$ is a partition of $N$, because $N_2 \geq 1$. Consequently the only partitions of $N$ for which $N_1 \geq 2$ are exactly these which are not constructed by the above procedure, which shows that $P(N; 1) = P(N) − P(N − 1)$.

Next, let $N_1 < N_2 < \ldots < N_{r−1} < N_r$ is a partition of $N$ consisting of distinct summands. If $N_1 = 1$ then $N_2, N_3, \ldots, N_r$ is a partition of $N$ of length $N − 1$ consisting of distinct summands and not containing 1. Conversely, if $2 \leq N_2 < \ldots < N_{r−1} < N_r$ is a partition of $N − 1$ of length $r − 1$ consisting of distinct summands and not containing 1, then adding 1 as first summand we get a partition $1 < N_2 < N_3 < \ldots < N_r$ of $N$ of length $r$. This shows that $Q(N; 1) = Q(N) − Q(N−1; 1)$. The last equality follows from the latter by applying it $N−1$ times. \hfill \Box

For a convenience of the reader we present the values of $P(N), Q(N), P(N; 1), Q(N; 1)$, and $R(N; 1)$ for first 10 values of $N$.
The values of $P(N)$ for first 49 natural numbers:

$N = 1, P(N) = 1$;  $N = 2, P(N) = 2$;  $N = 3, P(N) = 2$;  $N = 4, P(N) = 5$;  $N = 5, P(N) = 7$;  
$N = 6, P(N) = 11$;  $N = 7, P(N) = 15$;  $N = 8, P(N) = 22$;  $N = 9, P(N) = 30$;  $N = 10, P(N) = 42$;  
$N = 11, P(N) = 56$;  $N = 12, P(N) = 77$;  $N = 13, P(N) = 101$;  $N = 14, P(N) = 135$;  $N = 15, P(N) = 176$;  
$N = 16, P(N) = 231$;  $N = 17, P(N) = 297$;  $N = 18, P(N) = 385$;  $N = 19, P(N) = 490$;  
$N = 20, P(N) = 627$;  $N = 21, P(N) = 792$;  $N = 22, P(N) = 1002$;  $N = 23, P(N) = 1255$;  $N = 24, P(N) = 1575$;  
$N = 25, P(N) = 1958$;  $N = 26, P(N) = 2436$;  $N = 27, P(N) = 3010$;  $N = 28, P(N) = 3718$;  
$N = 29, P(N) = 4565$;  $N = 30, P(N) = 5604$;  $N = 31, P(N) = 6842$;  $N = 32, P(N) = 8349$;  $N = 33, P(N) = 10143$;  
$N = 34, P(N) = 12310$;  $N = 35, P(N) = 14883$;  $N = 36, P(N) = 17977$;  $N = 37, P(N) = 21637$;  
$N = 38, P(N) = 26015$;  $N = 39, P(N) = 31185$;  $N = 40, P(N) = 37338$;  $N = 41, P(N) = 44583$;  
$N = 42, P(N) = 53174$;  $N = 43, P(N) = 63261$;  $N = 44, P(N) = 75175$;  $N = 45, P(N) = 89134$;  
$N = 46, P(N) = 105558$;  $N = 47, P(N) = 124754$;  $N = 48, P(N) = 147273$;  $N = 49, P(N) = 173525$.

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