New quantum (anti)de Sitter algebras and discrete symmetries

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Abstract

Two new quantum anti-de Sitter $so(4,2)$ and de Sitter $so(5,1)$ algebras are presented. These deformations are called either ‘time-type’ or ‘space-type’ according to the dimensional properties of the deformation parameter. Their Hopf structure, universal $R$ matrix and differential-difference realization are obtained in a unified setting by considering a contraction parameter related to the speed of light, which ensures a well defined non-relativistic limit. Such quantum algebras are shown to be symmetry algebras of either time or space discretizations of wave/Laplace equations on uniform lattices. These results lead to a proposal for time and space discrete Maxwell equations with quantum algebra symmetry.
1 Introduction

Quantum deformations of the Poincaré algebra have been studied during last years in a search of more general symmetries that could lead to new relativistic theories at the Planck scale. Amongst them, we remark the well known $\kappa$-Poincaré algebra [1, 2, 3] (of Drinfeld–Jimbo type) and the so called null-plane quantum Poincaré algebra [4, 5] (of non-standard or triangular type). The former is currently under a great research activity that explores different traits as, for instance, wave functions and free fields on the associated $\kappa$-Minkowskian space [6], boost transformations [7, 8], $\kappa$-deformed electrodynamics [9] and Hawking radiation [10].

Nevertheless, any quantum Poincaré symmetry should be taken as an intermediate stage in the construction of more general structures such as quantum deformations of the conformal or anti-de Sitter algebra $so(4,2)$. Natural properties expected for such possible quantum conformal algebras should include, at least, a well defined non-relativistic limit to a quantum conformal Galilean algebra, the existence of some kind of Poincaré Hopf subalgebra as well as a clear dimensional interpretation of the deformation parameter. The aim of this paper is to present two new non-standard quantum deformations of $so(4,2)$ fulfilling the above requirements. Furthermore these structures are more manageable than other known non-standard quantum conformal algebras [11, 12] and, by construction, properties known for lower dimensional cases (such as $so(2,2)$ [13]) can be extended to the present dimension.

In the next section, we give a unified description of the three Lie algebras we shall deal with: $so(4,2)$, $so(5,1)$ and their limit to the conformal Galilean algebra. The Hopf structure of the first type of deformations is introduced in section 3; this is called ‘time-type’ as the deformation parameter has dimensions of time. Their role as discrete symmetries on a uniform time lattice is explicitly shown in section 4 through a differential-difference realization. In this way, we obtain a time discretization of conformal invariant equations such as wave or massless Klein–Gordon, Laplace and Maxwell equations. A parallel procedure is performed for ‘space-type’ deformations in the last section.

2 Conformal Lie algebras

The Lie algebras of the groups of conformal transformations of the $(3+1)$D Minkowskian and Galilean spacetimes as well as of the 4D Euclidean space can be studied simultaneously by means of a single real contraction parameter $\omega$; they are denoted collectively $so_\omega(4,2)$. These are spanned by generators of rotations $J_i$, time $P_0$ and space $P_\mu$ translations, boosts $K_i$, special conformal transformations $C_\mu$ and dilations $D$. We will assume sum over repeated indices, latin indices $i, j, k = 1, 2, 3$, greek indices $\mu, \nu = 0, 1, 2, 3$, and three components of a generator will be denoted $X = (X_1, X_2, X_3)$. The non-vanishing commutation relations of $so_\omega(4,2)$ are given by

\begin{align}
[J_i, J_j] &= \varepsilon_{ijk}J_k, \\
[J_i, C_j] &= \varepsilon_{ijk}C_k, \\
[K_i, P_0] &= P_i, \\
[P_0, C_0] &= -2D, \\
[P_i, C_j] &= 2\omega(\delta_{ij}D - \varepsilon_{ijk}J_k), \\
[J_i, K_j] &= \varepsilon_{ijk}K_k, \\
[K_i, K_j] &= -\omega\varepsilon_{ijk}J_k, \\
[K_i, C_0] &= C_i, \\
[C_0, P_\mu] &= 2K_\mu, \\
[D, P_\mu] &= P_\mu, \\
[D, C_\mu] &= -C_\mu.
\end{align}
Each specific Lie algebra is recovered from $so_\omega(4,2)$ once the contraction parameter $\omega$ is particularized to a real value as follows:

- $so(4,2) \cong \mathcal{CM}^{3+1}$ for $\omega > 0$, is the conformal algebra of the $(3 + 1)$D Minkowskian spacetime, or $(4 + 1)$D anti-de Sitter algebra. The contraction parameter is related to the speed of light $c$ through $\omega = 1/c^2$.

- $so(5,1) \cong \mathcal{CE}^4$ for $\omega < 0$, is the conformal algebra of the 4D Euclidean space, or $(4 + 1)$D de Sitter algebra; $P_0$ should be considered as another generator of space translations and $K$ as generators of rotations ($\{J, K\}$ span an $so(4)$ subalgebra).

- $t_9(so(3) \oplus so(2, 1)) \cong \mathcal{CG}^{3+1}$ for $\omega = 0$, where $so(3) = \{J\}$, $so(2,1) = \{P_0, C_0, D\}$ and $t_9 = \{K, P, C\}$. This case corresponds to the conformal algebra of the $(3 + 1)$D Galilean spacetime [14] obtained from $\mathcal{CM}^{3+1}$ through the non-relativistic limit $c \to \infty$.

As is well known $so_\omega(4,2)$ has two remarkable Lie subalgebras: $\{J, K, P, P_0\}$ that generate the (kinematical) algebra of isometries of the space, and $\{J, K, P, P_0, D\}$ that span the Weyl (or similitude) subalgebra $W_\omega$.

Let us consider the spacetime coordinates $x \equiv (x^0, x) \equiv (x^0, x^1, x^2, x^3)$ with metric $(g_{\mu\nu}) = \text{diag}(+1, -\omega, -\omega, -\omega)$. A vector field representation of $so_\omega(4,2)$ reads

\[
\begin{align*}
J_i &= \varepsilon_{ijk} x^k \partial_j, & K_i &= -\omega x^i \partial_0 - x^0 \partial_i, & P_\mu &= \partial_\mu, \\
D &= -x^\mu \partial_\mu - 1, & C_0 &= (\omega x^2 - (x^0)^2) \partial_0 + 2x^0 (x^\mu \partial_\mu + 1), \\
C_i &= (\omega x^2 - (x^0)^2) \partial_i - 2\omega x^i (x^\mu \partial_\mu + 1),
\end{align*}
\]

where $\partial_\mu \equiv \partial/\partial x^\mu$ and $x^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. Under this realization, the Casimir of the kinematical subalgebra $E = P^2 - \omega P_0^2$ gives rise to the following differential equation

\[
(\partial_i^2 + \partial_0^2 + \partial^2 - \omega \partial_0^2)\Phi(x) = 0.
\]

Since $E$ commutes with $\{J, K, P, P_0\}$ and the remaining generators (2.2) verify

\[
[E, D] = -2E, \quad [E, C_0] = 4x^0 E, \quad [E, C_i] = -4\omega x^i E,
\]

we find that all of them are symmetry operators of (2.3), so that $so_\omega(4,2)$ is the symmetry algebra of such an equation. Hence we recover the $(3+1)$D wave or massless Klein–Gordon equation when $\omega > 0$ [13] and the usual 4D Laplace–Beltrami equation when $\omega < 0$ ($x^0$ should be seen as another space coordinate instead of time). The contraction $\omega = 0$ gives rise to a 3D Laplace equation in the Galilean spacetime; the absence of the time coordinate $x^0$ is in full agreement with the known non-relativistic electromagnetic theories that only allow static electric and magnetic limits from the Maxwell equations [16, 17].

## 3 Time-type quantum conformal algebras

Let us consider the non-standard classical $r$ matrix [8, 13] of $sl(2, \mathbb{R}) \simeq so(2,1)$ written in a conformal basis $\{J_3, J_+, J_-\} \equiv \{D, P_0, C_0\}$:

\[
 r = -\tau D \wedge P_0,
\]

where $\tau$ is the deformation parameter. We now follow the same procedure applied to the $so(2,2)$ case [13], that is, we take (3.1) as the $r$ matrix for the whole $so_\omega(4,2)$ algebra. The resulting Hopf structure of the quantum algebra $U_\tau(so_\omega(4,2))$ is as follows.
• Coproduct:

\[
\begin{align*}
\Delta(P_0) &= 1 \otimes P_0 + P_0 \otimes 1, \\
\Delta(J_i) &= 1 \otimes J_i + J_i \otimes 1, \\
\Delta(D) &= 1 \otimes D + D \otimes e^{-\tau P_0}, \\
\Delta(K_i) &= 1 \otimes K_i + K_i \otimes 1 - \tau D \otimes e^{-\tau P_0} P_i, \\
\Delta(C_i) &= 1 \otimes C_i + C_i \otimes e^{-\tau P_0} + 2\tau D \otimes e^{-\tau P_0} K_i - \tau^2(D^2 + D) \otimes e^{-2\tau P_0} P_i.
\end{align*}
\]

(3.2)

• Non-vanishing commutation rules that close the Weyl Hopf subalgebra \(U_\tau(\mathcal{W}_\omega)\):

\[
\begin{align*}
[J_i, J_j] &= \epsilon_{ijk} J_k, \\
[K_i, K_j] &= -\omega \epsilon_{ijk} J_k, \\
[K_i, P_j] &= \omega \frac{e^{\tau P_0} - 1}{\tau}, \\
[D, P_0] &= \frac{1 - e^{-\tau P_0}}{\tau}.
\end{align*}
\]

(3.3)

• Non-vanishing commutation rules that involve special conformal transformations:

\[
\begin{align*}
[J_i, C_j] &= \epsilon_{ijk} C_k, \\
[K_i, C_0] &= C_i, \\
[P_0, C_0] &= -2D, \\
[K_i, C_i] &= C_i, \\
[P_0, C_i] &= -\tau(DC_0 + C_i D), \\
[P_i, C_0] &= K_i + C_i e^{-\tau P_0}, \\
[D, C_i] &= -C_i, \\
[D, C_0] &= -C_0 + \tau D^2.
\end{align*}
\]

(3.4)

The three quantum algebras included within \(U_\tau(so_\omega(4,2))\) are called ‘time-type’ ones since the deformation parameter \(\tau\) has, generically, dimension of a time, with the clear exception of \(U_\tau(so(5,1)) \cong U_\tau(CE^4)\), for which \(\tau\) has dimension of a length (\(\tau\) has the inverse dimension to the translation generator \(P_0\)). In this sense, although of a non-standard nature, this kind of deformation is close to the \(\kappa\)-Poincaré algebra.

Some relevant subalgebras of \(so_\omega(4,2)\) become into Hopf subalgebras of \(U_\tau(so_\omega(4,2))\) after deformation. In particular, besides the Weyl subalgebra \(U_\tau(\mathcal{W}_\omega) \subset U_\tau(so_\omega(4,2))\), we find the following embedding:

\[
U_\tau(so(2,1)) \subset U_\tau(so_\omega(2,2)) \subset U_\tau(so_\omega(3,2)) \subset U_\tau(so_\omega(4,2))
\]

(3.5)

with generators \(\{D, P_0, C_0\} \subset \{D, P_0, P_1, C_0, C_1, K_1\} \subset \ldots\) All of these Hopf subalgebras share the same classical \(\tau\) matrix \(\tau\). Notice that the kinematical subalgebras do not become into Hopf subalgebras, but the presence of the dilation is essential. Other kinds of Hopf subalgebra embeddings similar to \((3.3)\) have been obtained in \(20\). We also stress that the initial Hopf structure in the embedding, \(U_\tau(so(2,1))\), underlies the approach to physics at the Planck scale introduced in \(21\) and \(22\).

According to \(\omega >, =, < 0\), the embedding \((3.3)\) splits into three chains that clearly show the contractions \(\omega = 0\) (vertical arrows):

\[
\begin{align*}
U_\tau(so(2,1)) &\subset U_\tau(CM^{1+1}) \subset U_\tau(CM^{2+1}) \subset U_\tau(CM^{3+1}) \\
U_\tau(so(2,1)) &\subset U_\tau(CG^{1+1}) \subset U_\tau(CG^{2+1}) \subset U_\tau(CG^{3+1})
\end{align*}
\]

(3.6)
The chain \((3.3)\) has an important consequence: properties previously known for a low dimensional case can directly be extended to higher dimensions. A first application is provided by the universal quantum \(R\) matrix of \(U_\tau(sl(2,\mathbb{R})) \simeq U_\tau(so(2,1))\):

\[
R = \exp\{\tau P_0 \otimes D\} \exp\{-\tau D \otimes P_0\}. \tag{3.7}
\]

By construction, this element also gives the universal \(R\) matrix for all the quantum algebras arising in the sequence \((3.3)\). This result may further be used in the construction of quantum anti-de Sitter and de Sitter spaces as well as in the computation of differential calculi in such spaces by means of a matrix realization of \(U_\tau(so_\omega(4,2))\); for a quantum anti-de Sitter space of Drinfeld–Jimbo type see \([23]\).

4 Discrete time symmetries

In this section we extend the time discretization of the \((1 + 1)D\) wave equation associated to \(U_\tau(so(2,2)) \tag{3.3}\) to \((3 + 1)D\). Commutation rules \((3.3)\) and \((3.4)\) naturally include discrete derivatives through terms as \((e^{\pm \tau P_0} - 1)/\tau\). Thus if we take the usual realization of the translation generators \(P_\mu\) as the derivatives \(\partial_\mu\), we obtain a differential-difference realization of \(U_\tau(so_\omega(4,2))\):

\[
\begin{align*}
J_i &= \varepsilon_{ijk} x^k \partial_j, \quad K_i &= -\omega x^i \Delta_0 - x^0 T_0^{-1} \partial_i, \quad D = -x^0 T_0^{-1} \Delta_0 - x^j \partial_j - 1, \\
P_\mu &= \partial_\mu, \quad C_0 = (\omega x^2 + (x^0)^2 T_0^{-1}) \Delta_0 + 2x^0 (x^j \partial_j + 1) + \tau (x^j \partial_j + 1)^2, \\
C_i &= (\omega x^2 - (x^0)^2 T_0^{-2}) \partial_i - 2\omega x^i (x^0 T_0^{-1} \Delta_0 + x^j \partial_j + 1) + \tau x^0 T_0^{-2} \partial_i,
\end{align*}
\] \tag{4.1}

where we have introduced the time shift operator \(T_0 = e^{\tau \partial_0}\) and the discrete derivative in the time direction \(\Delta_0 = (T_0 - 1)/\tau\); these operators act on a function \(\Phi(x) \equiv \Phi(x^0, x)\) as

\[
\begin{align*}
T_0 \Phi(x^0, x) &= \Phi(x^0 + \tau, x), \quad \Delta_0 \Phi(x^0, x) = \frac{\Phi(x^0 + \tau, x) - \Phi(x^0, x)}{\tau}.
\end{align*}
\] \tag{4.2}

Therefor the deformation parameter \(\tau\) is the time lattice constant on the uniform lattice discretized along \(x^0\), while the space coordinates \(x\) remain as continuous variables.

The Casimir of the isometries sector \(\{J, K, P_0\}\) turns out to be

\[E_\tau = P_1^2 + P_2^2 + P_3^2 - \omega \left(\frac{e^{\tau P_0} - 1}{\tau}\right)^2. \tag{4.3}\]

By introducing \([4.1]\) we find a differential-difference equation given by

\[
(\partial^2_t + \partial^2_1 + \partial^2_2 - \omega \Delta^2_0) \Phi(x) = 0. \tag{4.4}
\]

As the operators \((4.1)\) out of the isometries sector verify

\[
[E_\tau, D] = -2E_\tau, \quad [E_\tau, C_0] = 4(x^0 + \tau x^i \partial_i + 2\tau) E_\tau, \quad [E_\tau, C_i] = -4\omega x^i E_\tau, \tag{4.5}
\]

we conclude that \(U_\tau(so_\omega(4,2))\) is the symmetry algebra of the equation \((4.4)\). Each specific quantum algebra, Weyl subalgebra and associated equation that arise for a particular value of \(\omega\) are displayed in table \([4]\); the arrows indicate the contraction \(\omega = 0\) (or \(c \rightarrow \infty\)) for each (sub)algebra/equation. The Lie algebra and continuous picture is recovered when
Table 1: Time-type quantum conformal algebras, $U_\tau(so_\omega(4,2))$, quantum Weyl subalgebras $U_\tau(W_\omega)$ and differential-difference equations according to $\omega = \{-1, 0, +1\}$.

| $\omega$ | Quantum conformal algebra | Quantum Weyl subalgebra | Differential-difference equation on a uniform time lattice |
|----------|---------------------------|-------------------------|----------------------------------------------------------|
| $+1$     | Quantum conf. Minkowskian | $U_\tau(CM^{3+1}) \equiv U_\tau(so(4,2))$ | Discrete $(3+1)$ wave equation $(\partial_t^2 + \partial_x^2 + \partial_y^2 - \Delta_3^2)\Phi = 0$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $0$      | Quantum conf. Galilean    | $U_\tau(CG^{3+1})$     | Continuous 3D Laplace equation $(\partial_t^2 + \partial_x^2 + \partial_y^2)\Phi = 0$ |
| $\uparrow$ | $\uparrow$ | $\uparrow$ |
| $-1$     | Quantum conf. Euclidean   | $U_\tau(CE^4) \equiv U_\tau(so(5,1))$ | Discrete 4D Laplace equation $(\partial_t^2 + \partial_x^2 + \partial_y^2 + \Delta_3^2)\Phi = 0$ |

$\tau \to 0$. Note that $\omega = 0$ leads to a differential equation with continuous space variables associated to $U_\tau(CG^{3+1})$, nevertheless the realization (4.1) is still a differential-difference one with an intrinsic time discretization.

The Hopf structure of $U_\tau(so_\omega(4,2))$ can be transformed into another one with classical commutation rules. Explicitly, if we consider the non-linear map defined by [13, 23]:

$$P_0 = \frac{e^{\tau P_0} - 1}{\tau}, \quad C_0 = C_0 - \tau D^2,$$  

(4.6)

with the remaining generators unchanged, we find that the commutators (3.3) and (3.4) are just the non-deformed ones (2.1), while the coproduct (3.2) is transformed into:

$$\Delta(P_\mu) = 1 \otimes P_\mu + P_\mu \otimes 1 + \tau P_\mu \otimes P_0, \quad \Delta(J_i) = 1 \otimes J_i + J_i \otimes 1,$$

$$\Delta(K_i) = 1 \otimes K_i + K_i \otimes 1 - D \otimes \frac{\tau P_i}{1 + \tau P_0}, \quad \Delta(D) = 1 \otimes D + D \otimes \frac{1}{1 + \tau P_0},$$

$$\Delta(C_0) = 1 \otimes C_0 + C_0 \otimes \frac{1}{1 + \tau P_0} - D \otimes \frac{2\tau}{1 + \tau P_0} D + (D^2 + D) \otimes \frac{\tau^2 P_0}{(1 + \tau P_0)^2},$$

$$\Delta(C_i) = 1 \otimes C_i + C_i \otimes \frac{1}{1 + \tau P_0} + D \otimes \frac{2\tau}{1 + \tau P_0} K_i - (D^2 + D) \otimes \frac{\tau^2 P_i}{(1 + \tau P_0)^2}.$$

(4.7)

At the level of the differential-difference realization (4.1), the non-linear map gives rise to

$$P_0 = \Delta_0, \quad C_0 = (\omega x^2 + (x^0)^2 T_0^{-2}) \Delta_0 + 2x^0 T_0^{-1} (x^i \partial_i + 1) - \tau x^0 T_0^{-2} \Delta_0,$$  

(4.8)

with the remaining operators and equation (4.4) unchanged (the latter now comes from the non-deformed Casimir $E = P^2 - \omega_P^2$). This final form for the realization of $U_\tau(so_\omega(4,2))$ allows us to define ‘time-type’ momenta $\hat{p}_\mu(\partial, x)$ and position operators $\hat{x}_\mu(\partial, x)$ as

$$\hat{p}_0 = \Delta_0, \quad \hat{p}_i = \partial_i, \quad \hat{x}_0 = x^0 T_0^{-1}, \quad \hat{x}_i = x_i,$$  

(4.9)

that fulfil

$$[\hat{p}_\mu, \hat{x}_\nu] = \delta_{\mu\nu}, \quad [\hat{p}_\mu, \hat{x}_\nu] = g_{\mu\nu}, \quad [\hat{x}_\mu, \hat{x}_\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0,$$  

(4.10)

provided that $[\Delta_0, x^0] = T_0$. If we now apply the maps $\partial_\mu \mapsto \hat{p}_\mu, \ x^\mu \mapsto \hat{x}_\mu$ to the differential realization (2.2) and equation (2.3) of $so_\omega(4,2)$, we recover the differential-difference realization (4.8) (with (4.1) for the remaining operators) and equation (4.4) of
The non-standard classical \( r \) matrix of \( sl(2, \mathbb{R}) \) can alternatively be written in the conformal basis \( \{ J_3, J_+, J_- \} \equiv \{ D, P_1, C_1 \} \) as \( r = -\sigma D \wedge P_1 \), where \( \sigma \) is now the deformation parameter. Hence we interchange the role of the generators \( P_0 \leftrightarrow P_1 \), so that \( \sigma \) has dimensions of a length. The resulting Hopf structure of the ‘space-type’ quantum algebras \( U_\sigma(so_\omega(4, 2)) \) is characterized by:

- **Coproduct:**
  \[
  \begin{align*}
  \Delta(P_\mu) &= 1 \otimes P_\mu + P_\mu \otimes e^{\sigma P_1} - \delta_{1\mu} P_\mu \otimes (e^{\sigma P_1} - 1), \\
  \Delta(K_j) &= 1 \otimes K_j + K_j \otimes 1 - \omega \sigma \delta_{1j} D \otimes e^{-\sigma P_1} P_0, \\
  \Delta(J_j) &= 1 \otimes J_j + J_j \otimes 1 + \sigma \varepsilon_{1jk} D \otimes e^{-\sigma P_1} P_k, \\
  \Delta(D) &= 1 \otimes D + D \otimes e^{-\sigma P_1}, \\
  \Delta(C_0) &= 1 \otimes C_0 + C_0 \otimes e^{-\sigma P_1} - 2\sigma D \otimes e^{-\sigma P_1} K_1 + \omega \sigma^2 (D^2 + D) \otimes e^{-2\sigma P_1} P_0, \\
  \Delta(C_j) &= 1 \otimes C_j + C_j \otimes e^{-\sigma P_1} - 2\sigma \varepsilon_{1jk} D \otimes e^{-\sigma P_1} J_k + \omega \sigma^2 (\delta_{2j} + \delta_{3j}) (D^2 + D) \otimes e^{-2\sigma P_1} P_j.
  \end{align*}
  \]

- **Non-vanishing commutation rules closing a Weyl Hopf subalgebra** \( U_\sigma(W_\omega) \):
  \[
  \begin{align*}
  [J_i, J_j] &= \varepsilon_{ijk} J_k, \\
  [J_i, K_j] &= \varepsilon_{ijk} K_k, \\
  [K_i, J_j] &= -\omega \varepsilon_{ijk} J_k, \\
  [J_i, P_j] &= \varepsilon_{ijk} \left( \delta_{1i} P_k + \delta_{1j} e^{-\sigma P_1} P_k + \delta_{1k} \frac{e^{\sigma P_1} - 1}{\sigma} \right), \\
  [K_i, P_j] &= \omega P_0 (1 + \delta_{1i} (e^{-\sigma P_1} - 1)), \\
  [K_i, P_0] &= P_i + \delta_{1i} \left( \frac{e^{\sigma P_1} - 1}{\sigma} - P_1 \right), \\
  [D, P_\mu] &= P_\mu + \delta_{1\mu} \left( \frac{1 - e^{-\sigma P_1}}{\sigma} - P_1 \right).
  \end{align*}
  \]
Non-vanishing commutation rules involving special conformal transformations:

\[ [J_i, C_j] = \epsilon_{ijk}(C_k + \omega_\sigma \delta_{1k} D^2), \quad [C_1, C_\nu] = \omega_\sigma (D C_\nu + C_\nu D), \quad \nu \neq 1, \]
\[ [K_i, C_0] = C_i + \omega_\sigma \delta_{1i} D^2, \quad [K_i, C_i] = \omega_\sigma C_0, \]
\[ [P_i, C_j] = -\omega \epsilon_{ijk} (\delta_{1i} (e^{-\sigma P_1} J_k + J_k e^{-\sigma P_1}) + \delta_{1j} (2 J_k - \sigma \epsilon_{ijk} (DP_i + P_i D))) - 2 \omega \epsilon_{ijk} J_i + 2 \omega \delta_{ij} D, \]
\[ [C_0, P_i] = 2 K_i + \delta_{li} (e^{-\sigma P_1} K_l + K_l e^{-\sigma P_1} - 2 K_l), \quad [P_0, C_0] = -2 D, \]
\[ [P_0, C_i] = 2 K_i + \omega_\sigma \delta_{li} (DP_0 + P_0 D), \quad [D, C_\mu] = -C_\mu - \omega_\sigma \delta_{1\mu} D^2. \]

Chains of Hopf subalgebras similar to the time-type ones (3.5) and (3.6) arise within \( U_\sigma(so_\omega(4, 2)) \). All of them share the same universal \( R \) matrix given by

\[ R = \exp\{\sigma P_1 \otimes D\} \exp\{-\sigma D \otimes P_1\}. \]

In this case, the non-linear map defined by

\[ P_1 = \frac{e^{\sigma P_1} - 1}{\sigma}, \quad C_1 = C_1 + \omega_\sigma D^2, \]

keeping the remaining generators unchanged, allows us to rewrite the Hopf structure of \( U_\sigma(so_\omega(4, 2)) \) with non-deformed commutation rules (2.1) and coproduct given by

\[ \Delta(J_j) = 1 \otimes J_j + J_j \otimes 1 + \sigma \epsilon_{1jk} D \otimes \frac{P_k}{1 + \sigma P_1}, \]
\[ \Delta(K_j) = 1 \otimes K_j + K_j \otimes 1 - \omega_\sigma \delta_{1j} D \otimes \frac{P_0}{1 + \sigma P_1}, \]
\[ \Delta(P_\mu) = 1 \otimes P_\mu + P_\mu \otimes 1 + \sigma P_\mu \otimes P_1, \quad \Delta(D) = 1 \otimes D + D \otimes \frac{1}{1 + \sigma P_1}, \]
\[ \Delta(C_0) = 1 \otimes C_0 + C_0 \otimes \frac{1}{1 + \sigma P_1} - D \otimes \frac{2 \sigma}{1 + \sigma P_1} K_1 + (D^2 + D) \otimes \frac{\omega_\sigma^2 P_0}{(1 + \sigma P_1)^2}, \]
\[ \Delta(C_1) = 1 \otimes C_1 + C_1 \otimes \frac{1}{1 + \sigma P_1} + D \otimes \frac{2 \omega_\sigma}{1 + \sigma P_1} D - (D^2 + D) \otimes \frac{\omega_\sigma^2 P_1}{(1 + \sigma P_1)^2}, \]
\[ \Delta(C_l) = 1 \otimes C_l + C_l \otimes \frac{1}{1 + \sigma P_1} - D \otimes \frac{2 \omega_\sigma \epsilon_{1lk}}{1 + \sigma P_1} J_k + (D^2 + D) \otimes \frac{\omega_\sigma^2 P_l}{(1 + \sigma P_1)^2}. \]

where the index \( l = 2, 3 \). Once \( U_\tau(so_\omega(4, 2)) \) is expressed in this last form, we can apply a space-type discretization rule. Let us consider the space shift operator \( T_1 = e^\sigma \partial_1 \) and the discrete derivative in the \( x^1 \)-space direction \( \Delta_1 = (T_1 - 1)/\sigma \). Similarly to the previous section, we define space-type momenta and position operators, fulfilling (4.10), as

\[ \hat{p}_0 = \partial_0, \quad \hat{p}_1 = \Delta_1, \quad \hat{p}_l = \partial_l, \quad \hat{\Phi}^0 = x^0, \quad \hat{\Phi}^1 = x^1 T_1^{-1}, \quad \hat{\Phi}^l = x^l. \]

By subsituting \( \partial_\mu \mapsto \hat{p}_\mu, \quad x^\mu \mapsto \hat{\Phi}^\mu \) into the differential realization (2.2) of \( so_\omega(4, 2) \), we obtain a differential-difference realization of \( U_\tau(so_\omega(4, 2)) \). The associated differential-difference equation turns out to be

\[ (\Delta_1^2 + \partial_0^2 + \partial_3^2 - \omega \partial_0^2) \Phi(x) = 0. \]

Therefore the deformation parameter \( \sigma \) is the space lattice constant on the uniform lattice discretized along \( x^1 \). These results can be displayed for each particular \( \omega \) as in table I.
Finally, if we consider a differential-difference operator \( \nabla_{\sigma} = (\hat{p}_1, \hat{p}_2, \hat{p}_3) = (\Delta_1, \partial_2, \partial_3) \), we could make the following ansatz for a space discretization of the Maxwell equations with \( U_r(so(4,2)) \)-symmetry:

\[
\nabla_{\sigma} \cdot \mathbf{E} = 0, \quad \nabla_{\sigma} \cdot \mathbf{B} = 0, \quad \nabla_{\sigma} \times \mathbf{E} = -\partial_0 \mathbf{B}, \quad \nabla_{\sigma} \times \mathbf{B} = \frac{1}{c^2} \partial_0 \mathbf{E},
\]

which give rise to a \( \sigma \)-d’Alembertian \( \Box_{\sigma} = \nabla_{\sigma}^2 - \frac{1}{c^2} \partial_0^2 \), such that \( \Box_{\sigma} \mathbf{E} = \Box_{\sigma} \mathbf{B} = 0 \).

Acknowledgments

This work was partially supported by the Ministerio de Ciencia y Tecnología, Spain (Project BFM2000-1055). The author thanks Naru Aizawa for helpful discussions and Susana García-Castrillo for hospitality.

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