Abstract. Let \( \Omega \subset \mathbb{R}^n \) be an unbounded open set. We consider the generalized weighted Morrey spaces \( \mathcal{M}^{p(x)}(\Omega, \varphi) \) and the vanishing generalized weighted Morrey spaces \( V \mathcal{M}^{p(x)}(\Omega, \varphi) \) with variable exponent \( p(x) \) and a general function \( \varphi(x, r) \) defining the Morrey-type norm. The main result of this paper are the boundedness of Riesz potential and its commutators on the spaces \( \mathcal{M}^{p(x)}(\Omega, \varphi) \) and \( V \mathcal{M}^{p(x)}(\Omega, \varphi) \). This result generalizes several existing results for Riesz potential and its commutators on Morrey type spaces. Especially, it gives a unified result for generalized Morrey spaces and variable Morrey spaces which currently gained a lot of attentions from researchers in theory of function spaces.

Mathematics subject classification (2010): 42B20, 42B25, 42B35.

Keywords and phrases: Riesz potential, commutator, vanishing generalized weighted Morrey space with variable exponent, \textit{BMO} space.

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