ON CURVES ON SANDWICHED SURFACE SINGULARITIES

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Abstract. Fixed a point $O$ on a non-singular surface $S$ and a complete $m_O$-primary ideal $I$ in $\mathcal{O}_{S,O}$, the curves on the surface $X = \text{Bl}_I(S)$ obtained by blowing-up $I$ are studied in terms of the base points of $I$. Criteria for the principality of these curves are obtained. New formulas for their multiplicity, intersection numbers and order of singularity at the singularities of $X$ are given. The semigroup of branches going through a sandwiched singularity is effectively determined, too.

1. Introduction

Sandwiched singularities are normal surface singularities which birationally dominate a non-singular surface. They are rational surface singularities and among them are included all cyclic quotients and minimal surface singularities. The original interest in sandwiched singularities comes from a question posed by Nash in the early sixties: does a finite succession of Nash transformations or normalized Nash transformations resolve the singularities of a reduced algebraic variety? Hironaka had proved in [14] that after a finite succession of normalized Nash transformations one obtains a surface $X$ having only sandwiched singularities. Some years later, in [23], M. Spivakovsky proves that sandwiched singularities are resolved by normalized Nash transformations, thus giving a positive answer to the original question posed by Nash for the case of surfaces. Since then, a constant interest in sandwiched singularities has been shown, and they have been deeply studied as a nice testing ground for the Nash and the wedge problem by Lejeune-Jalabert and Reguera in [17]. A recent paper by Reguera proves the Nash conjecture for sandwiched singularities [24]. Sandwiched singularities have been also studied from the point of view of deformation theory by de Jong, van Straten in [7] and by Gustavsen in [12].

Any sandwiched surface singularity can be obtained by blowing up a complete $m_O$-primary ideal $I$ in a local ring $\mathcal{O}_{S,O}$ of a non-singular surface $S$. The main purpose of this paper is to study the curves (effective Weil divisors) on the sandwiched surface $X = \text{Bl}_I(S)$ through their relationship to the infinitely near base points of $I$. One motivation for this study comes from [18] and the study of the divisor class group for rational surface singularities in general. Another motivation comes from the fact that the study of the curves on a surface singularity may lead to a deeper understanding of the singularity itself (see [21]). Criteria for the principality of Weil divisors on $X$ are given in this paper. A number of invariants for the curves on $X$ (such as their multiplicity and their order of singularity) are also recovered in terms of the relative position of their projection to $S$ with the base points of $I$. We keep the point of view of [8] and make use of the theory of infinitely near points as revised and developed by Casas-Alvero in [6]. Relevant to our purposes is the fact that any complete ideal has a cluster of infinitely near base points that in turn determines the ideal.

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AMS 2000 subject classification 14H20; 14H50; 13B22; 14B05; 14E05.
The organization of the paper is as follows. Concepts and well-known facts about infinitely near points and sandwiched surface singularities are reviewed from [6] and [8] in section 2. We also give a formula for the multiplicity of curves on $X = Bl_1(S)$ going through a sandwiched singularity in terms of the base points of the ideal $I$ (Corollary 2.5). Section 3 deals with the existence of equations for curves on a sandwiched surface. The main result of the paper is Theorem 3.1, which provides different criteria for their principality. In Section 4 we derive easy consequences and show some examples to illustrate these results. In Section 5, we suggest a procedure to compute effective Cartier divisors containing a given (not necessarily principal) curve $C$. To this aim, we introduce a slight modification of the unloading procedure called partial unloading. It is explained in the Appendix and it may be useful in any other context where precise control of unloading is needed. The procedure of Section 5 gives rise to flags of clusters depending on the ideal $I$ (and hence, on the surface $X$) and the curve $C$. These flags keep deep information about the strict transform of $C$ and they are used in Section 6 to infer formulas for its order of singularity (Proposition 6.1) and to compute effectively the semigroup of a branch on any singularity of $X$ (Proposition 6.5).

2. Preliminaries

Throughout this work the base field is the field $\mathbb{C}$ of complex numbers. A standard reference for most of the material treated here is the book by Casas-Alvero [6]. Let $(R, \mathfrak{m}_R)$ be a regular local two-dimensional $\mathbb{C}$-algebra and $S = \text{Spec}(R)$. A cluster of points of $S$ with origin $O$ is a finite set $K$ of points infinitely near or equal to $O$ such that for any $p \in K$, $K$ contains all points preceding $p$. By assigning integral multiplicities $\nu = \{\nu_p\}$ to the points of $K$, we get a weighted cluster $K = (K, \nu)$, the multiplicities $\nu$ being called the virtual multiplicities of $K$. We write $p \rightarrow q$ if $p$ is proximate to $q$. If

$$\rho_p = \nu_p - \sum_{q \rightarrow p} \nu_q$$

is the excess at $p$ of $K$, consistent clusters are those clusters with no negative excesses. Moreover, $K_+ = \{p \in K | \rho_p > 0\}$ is the set of dicritical points of $K$. Strictly consistent clusters are consistent clusters with no points of virtual multiplicity zero.

If $K$ is a weighted cluster, the equations of all curves going through it define a complete $\mathfrak{m}_R$-primary ideal $H_K$ in $R$ (see [6] 8.3). Two weighted clusters $K$ and $K'$ are equivalent if $H_K = H_{K'}$. Any complete $\mathfrak{m}_R$-primary ideal $J$ in $R$ has a cluster of base points, denoted by $BP(J)$, which consists of the points shared by, and the multiplicities of, the curves defined by a generic element of $J$. Moreover, the maps $J \mapsto BP(J)$ and $K \mapsto H_K$ are reciprocal isomorphisms between the semigroup $\mathcal{I}_R$ of complete $\mathfrak{m}_R$-primary ideals in $R$ and the semigroup of strictly consistent clusters (see [6] 8.4.11 for details). If $p \in N_O$, we denote by $I_p$ the simple ideal generated by the equations of the branches going through $p$ and by $K(p)$ the weighted cluster corresponding to it by the above isomorphism. Clearly, $I_p = H_{K(p)}$. In this language, if $I \in \mathcal{I}_R$ and $K = BP(I)$ is the cluster of base points of $I$, we have that $K = \sum_{i=1}^n \alpha_i K(p_i)$ if and only if $I = \prod_{i=1}^n I_p^{\alpha_i}$ is the Zariski factorization of $I$ into simple ideals. Hence, the exponent $\alpha_p$ of $I_p$ in the factorization of $I$ equals the excess of $K$ at $p$, so we can write

$$I = \prod_{p \in K_+} I_p^{\rho_p}. \tag{2.1}$$

If $K = (K, \nu)$ is consistent, then the virtual codimension of $K$ equals (Proposition 4.7.1 of [6])

$$c(K) = \sum_{p \in K} \frac{\nu_p(\nu_p + 1)}{2} = \dim_c\left(\frac{R}{H_K}\right). \tag{2.2}$$
Consistent clusters are characterized as those clusters whose virtual multiplicities may be realized effectively by some curve on $S$. If $K$ is not consistent, $\overline{K}$ is the cluster given rise from $K$ by unloading. Equivalently, $\overline{K}$ is the unique consistent cluster which is equivalent to $K$ and has the same points (see [6] §4.2 and §4.6 for details).

If $\pi_K : S_K \to S$ is the composition of the blowing-ups of all points in $K$, write $\mathcal{E}_K$ for the exceptional divisor of $\pi_K$ and $\{E_p\}_{p \in K}$ for its irreducible components. Use $|\cdot|$ as meaning intersection number and $[\cdot, \cdot]_P$ as intersection multiplicity at $P$. We denote by $A_K = -P^t_K P_K$ the intersection matrix of $\mathcal{E}_K$, where $P_K$ is the proximity matrix of $K$. If $C$ is a curve on $S$, $v_p(C)$ is the value of $C$ relative to the divisorial valuation associated with $E_p$. Then, $v_p(C) = e_p(C) + \sum_{p \to q} v_q(C)$. We write $v_K = (v^p_K)_{p \in K}$ for the vector of virtual values of $K$. Moreover, if $\widetilde{C}^K$ is the strict transform of $C$ on $S_K$, we have

$$|\widetilde{C}^K \cdot E_p|_{S_K} = e_p(C) - \sum_{q \in K, q \to p} e_q(C), \text{ for all } p \in K. \tag{2.3}$$

and we have the equality (projection formula) for $\pi_K$: $|\pi^*_K(C) \cdot B|_{S_K} = [C, (\pi_K)_* (B)]_O$ for any curve $B$ on $S_K$. We will make use also of the Noether formula for the intersection multiplicity of two curves at $O$ (Theorem 3.3.1 of [6]):

$$[C, C']_O = \sum_p e_p(C)e_p(C'). \tag{2.4}$$

For consistent clusters, write $|[\mathcal{K}, C]|_O = \sum_{p \in K} v_p e_p(C)$ for the intersection multiplicity of $\mathcal{K}$ and a curve $C$ on $S$, and $K^2 = \sum_{p \in K} v_p^2$ for the self-intersection of $\mathcal{K}$.

We will also make the following results for rational surface singularities. Let $(X, Q)$ be a rational surface singularity and let $g : X' \to X$ be any resolution of it. Write $\{E_i\}_{i=1,...,n}$ for the irreducible components of the exceptional locus of $g$. If $A$ is a curve on $X$ (i.e. an effective Weil divisor), it is possible to associate to $A$ a $\mathbb{Q}$-Cartier exceptional divisor $D_A$ on $X'$, being defined by the condition (see II (b) of [20])

$$|D_A \cdot E_i|_{X'} = -|\overline{A} \cdot E_i|_{X'} \forall i \in \{1, \ldots, n\}.$$  

On the other hand, if $D$ is a divisor on $X'$ such that $|D \cdot E_i| = 0$ for each exceptional component, then there exists an element $h \in m_{X', Q}$ such that $D$ is the total transform of the curve defined by $h = 0$ (see the proof of Theorem 4 of [2]). Because of this, we have that

**Lemma 2.1.** $A$ is Cartier if and only if $D_A$ is a divisor on $X'$. 

2.1. **Sandwiched surface singularities.** In this section, we fix notation and recall some facts concerning sandwiched singularities and base points of ideals which will be useful throughout this paper. The main references are [25] and [8]. A curve will be an effective Weil divisor on a surface.

Let $I \in I_R$. The ideal $I$ is fixed throughout the paper, so that no confusion should arise if the notation introduced from now on does not reflect its dependence of $I$. Write $K = (K, \nu)$ for the cluster of base points of $I$ and $\pi_I : X = Bl_I(S) \to S$ for the blowing-up of $I$. The singularities of $X$ are by definition sandwiched singularities and we have a commutative diagram

$$\begin{array}{c}
S_K \xrightarrow{f} X \\
\pi_K \searrow \quad \nearrow \pi_I \\
S
\end{array}$$
where the morphism $f$, given by the universal property of the blowing-up, is the minimal resolution of the singularities of $X$ (Remark 1.4 of [25]).

If $Q$ is in the exceptional locus of $X$, write $m_Q$ for the maximal ideal in $O_{X,Q}$ and $M_Q$ for the ideal sheaf on $X$ associated with $Q$. Then, the ideal $I_Q := \pi_*(M_Q I_{O_X}) \subseteq I$ is complete, $m_{O_Q}$-primary and has codimension one in $I$ and moreover, the map $Q \mapsto I_Q$ defines a bijection between the set of points in the exceptional locus of $X$ and the set of complete $m_Q$-primary of codimension one in $I$ (Theorem 3.5 of [8]).

It can be seen that if $q \in E_K$, and $K_q$ is the weighted cluster obtained from $K$ by adding $q$ as a simple point, then $H_{K_q} \subset I$ has codimension one in $I$, and every complete $m_Q$-primary ideal of codimension one in $I$ has this form (Lemma 3.1 of [9]). Therefore, for every point $Q$ in the exceptional locus of $X$, there exists some $q \in E_K$ such that $I_Q = H_{K_q}$.

Also, $|\pi^{-1}(Q)| = 1$, (see (2.1)) and the set of irreducible components of $\pi^{-1}(Q)$ (Theorem 3.5 of [8]). Given two points $K, K'$ such that $\pi_*(O_{X,K'}) = Y'$, we write $\pi^{-1}(K') = \pi^{-1}(Q)$ for the set of points in the exceptional locus of $X$.

For any sandwiched singularity $Q \in X$, write $T_Q = \{ p \in K \mid f_*(E_p) = Q \}$ so that $\{ E_p \}_{p \in T_Q}$ is the set of the exceptional components of $\tilde{S}_K$ contracting to $Q$. In particular, we have that

$$(2.5) \quad \{ p \in K \mid \pi_p = 0 \} = \bigcup_{Q \in \text{Sing}(X)} T_Q.$$  

Recall that there is a bijection between the set of simple ideals $\{ I_p \}_{p \in \mathcal{K}^+}$ in the (Zariski) factorization of $I$ (see (2.1)) and the set of irreducible components of $\pi^{-1}(O)$ (25 Corollary 1.5; see also [15] Proposition 21.3). We write $\{ L_p \}_{p \in \mathcal{K}^+}$ for the set of these components. If $J \in \mathcal{I}_R$, we write $L_J = \sum_{p \in \mathcal{K}^+} v_p(J)L_p$ and if $C$ is a curve on $S$, $L_C = \sum_{p \in \mathcal{K}^+} v_p(C)L_p$. For $p \in \mathcal{K}^+$, write $L_p = \sum_{q \in \mathcal{K}^q} v_q(I_p)L_q$.

The following proposition is easy, but will be very useful in the future:

**Proposition 2.2** (Projection formula for $\pi_I$). If $C, D$ are curves on $S$, we have that

$$[C,D]_O = [C^* \cdot \tilde{D}]_X.$$  

Also, $|C^* \cdot L_p|_X = 0$, for any $p \in \mathcal{K}^+.$

**Proof.** Let $g: X' \to X$ be a resolution of $X$. To prove the claim it is enough to apply the projection formula for $\pi_K$ and for $g$. \(\square\)

The following technical result is a generalization of Lemma 4.2 of [10]. Given two points $q, p \in K$, the chain $ch(q,p)$ is the subgraph of the dual graph $\Gamma_K$ of $E_K$ of all vertices and edges between the vertices representing $q$ and $p$ in $\Gamma_K$ (see §4.4 of [6]).

**Lemma 2.3.** Let $Q \in X$ be singular and write $K_Q = (K, \nu(Q))$.

(a) By taking the partial order relation of being infinitely near to, there exists a unique minimal point in $T_Q$. We will denote this point by $O_Q$;

(b) $\nu^{(Q)}_Q = \nu_{O_Q} + 1$. If $p \in K$ and $p \neq O_Q$, then $\nu_p - 1 \leq \nu^{(Q)}_p \leq \nu_p$.

**Proof.** (a) Assume that $q_1, q_2 \in T_Q$ are different and minimal among the points of $T_Q$. Let $u_0 \in K$ be such that $q_1$ and $q_2$ are infinitely near to $u_0$ and maximal with this property. Then, $u_0 \in ch(q_1, q_2)$. By the minimality of $q_1$ and $q_2$, $u_0 \notin T_Q$. By Zariski’s Main Theorem (see, for
example, Theorem V 5.2 of [13], the union of the components \( \{ E_p \}_{p \in \tau_Q} \) is connected and by Theorem 1.7 of [1], \( \mathcal{E}_K \) contains no cycles. This leads to contradiction, and (a) follows.

To prove (b), denote by \( S^{(Q)} \) the surface obtained by blowing up the points preceding \( O_Q \), so that \( O_Q \) is a proper point of \( S^{(Q)} \) (\( S^{(Q)} = S \) if \( O_Q = O \)). Denote by \( \varphi_{O_Q} : R \rightarrow O_{S^{(Q)}, O_Q} \) the morphism induced by blowing up. By Corollary 3.6 of [8], we know that \( \nu_p(I_Q) = \nu_p(I) \) and so, \( \nu_p^{(Q)} = \nu_p \) for each \( p \) preceding \( O_Q \). Therefore, the exceptional component of the total transform on \( S^{(Q)} \) of curves going sharply through \( K \) or \( K_Q \) are equal. Write \( z \) for an equation of this exceptional component at \( O_Q \). Then, the ideal \( I_Q \) (resp. \( I \)) generated by \( z^{-1} \varphi_{O_Q}(I_Q) \) (resp. \( z^{-1} \varphi_{O_Q}(I) \)) is complete and \( m_{O_Q} \)-primary in \( O_{S^{(Q)}, O_Q} \). Since \( I_Q \subset I \), we have \( I_Q \subset I \). Moreover, the base points of \( I \) (resp. \( I_Q \)) equal the base points of \( I \) (resp. \( I_Q \)) infinitely near or equal to \( O_Q \). Direct computation using (2.2) shows that \( \dim_C(\frac{I}{I_Q}) = 1 \). Now, the proof of (b) follows as in Lemma 4.2 of [10]. The details are left to the reader. \( \square \)

The above lemma shows that the virtual multiplicities of \( K_Q \) differ as much in one from the original multiplicities of \( K \). The set

\[
B_Q = \{ p \in K \mid \nu_p^{(Q)} = \nu_p - 1 \}
\]

will play an important role in the sequel. Given \( p \in K \), we denote \( \varepsilon_p = -1 \) if \( p \in B_Q \), \( \varepsilon_{O_Q} = 1 \) and \( \varepsilon_p = 0 \), otherwise.

Let us quote some consequences of (2.3). The first one follows immediately from \( \dim_C(\frac{I}{I_Q}) = 1 \) by using (2.2).

**Corollary 2.4.** \( \nu_{O_Q} = \sum_{p \in B_Q} \nu_p \).

We also obtain a new formula for the multiplicity at \( Q \) of curves on \( X \) in terms of \( O_Q \) and the points of \( B_Q \). Namely,

**Corollary 2.5.** \( \nu_Q(C) = e_{O_Q}(C) - \sum_{p \in B_Q} e_p(C) \).

**Proof.** First of all, notice that if \( Q \) is not singular, then \( B_Q \) is empty and \( O_Q = O \), so there is nothing to prove. Hence, we assume that \( Q \) is singular. By the projection formula applied to \( f \), we have that

\[
(2.6) \quad \nu_Q(C) = (\tilde{C}^K \cdot Z_Q)_{S_K} = (\tilde{C}^K \cdot \mathcal{E}_{I_Q} - \mathcal{E}_I)_{S_K}
\]

the last equality by Corollary 3.6 of [8]. Now, since the sheaves \( I_Q O_{S_K} \) and \( IO_{S_K} \) are invertible, we can take two curves \( B \) and \( B' \) going sharply through \( K \) and \( K_Q \), respectively and such that \( \tilde{C}_I \) and \( \tilde{C}_{I_Q} \) share no points with \( \tilde{C}^K \) on \( S_K \). Then, by the projection formula applied to \( \pi_K \),

\[
[C, B]_O = [\tilde{C}^K \cdot (\tilde{C}_I + \mathcal{E}_I)]_{S_K} = [\tilde{C}^K \cdot \mathcal{E}_{I_Q}]_{S_K}
\]

and similarly, \( [C, B']_O = [\tilde{C}^K \cdot \mathcal{E}_{I_Q}]_{S_K} \). Hence, by (2.6) above, \( \nu_Q(C) = [C, B]_O - [C, B']_O \), and by Noether’s formula (see (2.4)), we have \( \nu_Q(C) = \sum_{p \in B} e_p(C)(\nu_p^{(Q)} - \nu_p) \). Finally, by (b) of (2.3) we obtain \( \nu_Q(C) = e_{O_Q}(C) - \sum_{p \in B_Q} e_p(C) \) as claimed. \( \square \)

An easy formula for the multiplicity of a sandwiched singularity \( Q \) is also obtained in terms of the set \( B_Q \). Moreover, it shows that the number of these points is an invariant of the singularity and so, it is independent of the particular ideal blown-up to obtain it. Recall that \( R = O_{S, Q} \) is said
to be maximally regular in \( \mathcal{O}_{X,Q} \) if there are no regular rings \( R_0 \neq R \) such that \( R \subset R_0 \subset \mathcal{O}_{X,Q} \) (see \[19\]).

**Corollary 2.6.** Let \( Q \) be a point in the exceptional locus of \( X \). Then,

(a) \( \text{mult}_Q(X) = 1 + \sharp \mathcal{B}_Q \);

(b) \( \text{mult}_Q(X) \leq 1 + \nu_Q \), and the equality holds if and only if \( R \) is maximally regular in \( \mathcal{O}_{X,Q} \) and all the points in \( \mathcal{B}_Q \) are simple.

**Proof.** It is enough to compute the multiplicity of a transverse hypersurface section of \((X,Q)\), that is, of the strict transform of a curve \( C \) going sharply through \( K_Q \), see Theorem 3.5 of \[8\]. (a) follows from \[2.3\] and \[2.4\] by direct computation. Now, the inequality in (b) is obvious if \( Q \) is regular, so we assume that \( Q \) is singular. Clearly, \( \sharp \mathcal{B}_Q \leq \sum_{p \in \mathcal{B}_Q} \nu_p \). Again by \[2.4\] we have

\[
\text{mult}_Q(X) \leq 1 + \sum_{p \in \mathcal{B}_Q} \nu_p = 1 + \nu_Q \leq 1 + \nu_O,
\]

and the equality holds if and only if the virtual multiplicity of all the points in \( \mathcal{B}_Q \) is one and \( \nu_Q = \nu_O \). Now, if \( O_Q \neq O \), there must be some \( q \) preceding \( O_Q \) with \( \rho_{K_Q} > 0 \) (recall that \( O_Q \) is minimal in \( T_Q \), see \[2.3\]) and so, \( \nu_Q > \nu_Q \). Hence, \( \nu_Q \geq \nu_Q > \nu_Q \) and \( \text{mult}_Q(X) < 1 + \nu_O \).

Now, this is equivalent to the maximally regular condition, since \( \mathcal{O}_{X,Q} \) can always be projected birationally into the surface obtained by blowing up all the points preceding \( O_Q \), i.e. the surface where \( O_Q \) is lying as a proper point, so \( R \subset \mathcal{O}_{O_Q} \subset \mathcal{O}_{X,Q} \). This completes the proof. □

3. Criteria for the principality of curves through a sandwiched singularity

**Theorem 3.1.** Let \( C \) be a curve on \( S \) and write \( \mathcal{L}_C = \sum_{u \in K_+} v_u(C) L_u \). The following conditions are equivalent:

(i) \( \tilde{C} \) is a Cartier divisor on \( X \);

(ii) \( \mathcal{L}_C \in \bigoplus_{u \in K_+} \mathbb{Z} \mathcal{L}_u \);

(iii) there exists a curve \( B \) on \( S \) such that \( \mathcal{L}_B = \mathcal{L}_C \) and \( \tilde{B} \) goes through no singularity of \( X \);

(iv) if \( J_C = \{ g \in R \mid v_p(g) \geq v_p(C), \forall p \in K_+ \} \) and \( \mathcal{Q}_C = BP(J_C) \), then every dicritical point of \( Q_C \) is a dicritical point of \( K \).

Before proving 3.1 we need a technical result.

**Lemma 3.2.** (a) The set \( \{ E_p \}_{p \in K} \) is a basis of the \( \mathbb{Q} \)-vector space generated by \( \{ E_p \}_{p \in K} \). The matrix of the change of basis from \( \{ E_p \}_{p \in K} \) to \( \{ E_p \}_{p \in K} = -A_K \).

(b) The set \( \{ \mathcal{L}_u \}_{u \in K_+} \) is a basis of the \( \mathbb{Q} \)-vector space generated by \( \{ L_u \}_{u \in K_+} \).

**Proof.** We have \( A_K(v_q(I_p)) = -1_p \) where \( 1_p \) is the \( K \)-vector having all its entries equal to 0 but the corresponding to \( p \) which is 1. Thus, \( A_K E_p = -E_p \) and the claim (a) follows. To prove (b), it is enough to show that the \( \{ \mathcal{L}_u \}_{u \in K_+} \) are linearly independent. Assume that there exist rational numbers \( \{ a_u \}_{u \in K_+} \) such that

\[
\sum_{u \in K_+} a_u \mathcal{L}_u = 0.
\]

By multiplying by an integer, we may assume that the \( a_u \in \mathbb{Z} \) for all \( u \in K_+ \). Now, for each \( u \in K_+ \) take \( \gamma_u \) a curve going sharply through \( K(u) \) and missing all points after \( u \) in \( K \), and write

\[
\sum_{u \in K_+} a_u \gamma_u = C_1 - C_2
\]
where \( C_1 = \sum_{a_u > 0} a_u \gamma_u \) and \( C_2 = \sum_{a_u < 0} (-a_u) \gamma_u \). Then, by (3.1) we have that \( \mathcal{L}_{C_1} = \sum_{a_u > 0} a_u \mathcal{L}_u \) and \( \mathcal{L}_{C_2} = \sum_{a_u < 0} (-a_u) \mathcal{L}_u \) are equal. Hence, by taking total transforms on \( S_K \), we see that
\[
\sum_{a_u > 0} a_u \mathcal{E}_{I_u} = \sum_{a_u < 0} (-a_u) \mathcal{E}_{I_u}
\]
against (a). \( \square \)

**Proof of 3.1.** We prove that \((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)\).

\((i) \Rightarrow (ii). \) Assume that \( \tilde{C} \) is Cartier on \( X \) and hence, that \( \mathcal{L}_C \) is so (as \( C^* = \tilde{C} + \mathcal{L}_C \) is always Cartier). Then, by 2.1, \( f^*(C) \) is a divisor on \( S_K \) and so, the coefficients of the components \( \{E_p\}_{p \in T_Q} \) in \( f^*(\mathcal{L}_C) \) are integers, i.e.
\[
(3.2) \quad f^*(\mathcal{L}_C) = \sum_{q \in K} b_q E_q \quad \text{with } b_q \in \mathbb{Z}.
\]
On the other hand, by (b) of (3.2), we can write \( \mathcal{L}_C = \sum_{a_u \in K_+} a_u \mathcal{L}_u \), with \( \{a_u\} \) rational numbers. Now, if \( u \in K_+ \), then \( f^*(\mathcal{L}_u) = \mathcal{E}_{I_u} \) and \( f^*(\mathcal{L}_C) = \sum_{u \in K_+} a_u \mathcal{E}_{I_u} \) is the expression of \( f^*(\mathcal{L}_C) \) in the basis \( \{\mathcal{E}_{I_p}\}_{p \in K} \). Therefore, by (a) of (3.2) and the equality (3.2), \( (a_u)_{u \in K} = -A_K(b_u)_{u \in K} \) and hence all the \( a_u \) are integers.

\((ii) \Rightarrow (iii). \) Assume that \( \mathcal{L}_C = \sum_{u \in K_+} a_u \mathcal{L}_u \), with \( a_u \in \mathbb{Z} \). If \( u \in K_+ \), the projection formula applied to \( \pi \) gives that \( |C^* \cdot \mathcal{L}_u|_X = |(\tilde{C} + \mathcal{L}_C) \cdot \mathcal{L}_u|_X = 0 \). Hence,
\[
|\mathcal{L}_C \cdot \mathcal{L}_u|_X = -|\tilde{C} \cdot \mathcal{L}_u|_X.
\]
In particular, if \( \gamma_p \) is a curve going sharply through \( K(p) \) (\( p \in K_+ \)) and missing all points in \( K \) after \( p \), then
\[
|\mathcal{L}_p \cdot \mathcal{L}_u|_X = -|\gamma_p \cdot \mathcal{L}_u|_X = \begin{cases} -1 & \text{if } p = u \\ 0 & \text{otherwise} \end{cases}
\]
It follows that
\[
|\tilde{C} \cdot \mathcal{L}_u|_X = -|\mathcal{L}_C \cdot \mathcal{L}_u|_X = -\sum_{p \in K_+} a_p |\mathcal{L}_p \cdot \mathcal{L}_u|_X = a_u
\]
and thus, \( a_u \geq 0 \) for all \( u \in K_+ \). Now, let \( B \) be a curve going sharply through \( T = \sum_{u \in K_+} a_u K(u) \) and missing those points of \( K \) not contained in the underlying cluster of \( T \) (Corollary 4.2.8 of [6]). The strict transform \( \tilde{B} \subset X \) cuts transversally each exceptional component \( L_u \) at \( a_u \) difference points and goes through no singularity of \( X \). Moreover, \( \mathcal{L}_B = \sum_{u \in K_+} a_u \mathcal{L}_u = \mathcal{L}_C \).

\((iii) \Rightarrow (iv). \) Since \( \mathcal{L}_B = \mathcal{L}_C \), we have \( v_p(B) = v_p(C) = v_p(J_C) \) \( \forall p \in K_+ \), and hence, \( B \) is defined by an element of \( J_C \). Therefore, \( v_p(B) \geq v_p(J_C) \) \( \forall p \), and so
\[
(3.4) \quad \mathcal{E}_B \geq \mathcal{E}_{J_C} = \sum_{p \in K} v_p(J_C) E_p.
\]
On the other hand, since \( \tilde{B} \) goes through no singularities of \( X \), the total transform of \( \tilde{B} \) by \( f \) has no exceptional part, and \( \pi_K(B) = f^*(\tilde{B} + \mathcal{L}_C) = B^K + f^*(\mathcal{L}_C) \). Thus, \( \mathcal{E}_B = f^*(\mathcal{L}_C) \). Now, if \( C_1 \) goes sharply through \( Q_C, \mathcal{E}_{C_1} = \mathcal{E}_{Q_C} \) and \( \mathcal{L}_{C_1} = \mathcal{L}_C \). Assume that there exists \( q \in K \setminus K_+ \) so that \( Q_C \) has positive excess at it. Then \( \rho_q = 0 \) and by the equality (2.3) in page 4, there is some singularity \( Q \) in \( X \) such that \( q \in T_Q \). Moreover, \( \tilde{C}_1 \) goes through \( Q \) and \( f^*(\tilde{C}_1) = \tilde{C}_1^K + D_{\tilde{C}_1} \) where \( D_{\tilde{C}_1} = \sum_{u \in K \setminus K_+} b_u E_u \) is a non-zero effective divisor. Thus,
\[
\mathcal{E}_{C_1} = D_{C_1} + f^*(\mathcal{L}_{C_1}) = D_{C_1} + f^*(\mathcal{L}_C) = D_{C_1} + \mathcal{E}_B > \mathcal{E}_B
\]
Cartier divisor on $X$.

Corollary 3.3. Let $Q_c$ be a simple ideal (Definition I.3.1 of [25]). In this case, 3.1 has a very easy formulation. 

$$\text{Corollary 3.4.}$$

Let $I_p \subset R$ be a simple ideal and $X = Bl_{I_p}(S)$. If $C \subset S$ is a curve, $\tilde{C}$ is a Cartier divisor on $X$ if and only if $v_p(C)$ is a multiple of $\mathcal{K}(p)^2$. Moreover,

$$m_{\tilde{C}} = \frac{\text{lcm}(v_p(C), \mathcal{K}(p)^2)}{v_p(C)}$$

is the minimal integer $m$ such that $m\tilde{C}$ is a Cartier divisor.

**Proof.** It is clear that $\mathcal{L}_C = v_p(C)L_p$ and $\mathcal{L}_p = v_p(I_p)L_p$. Since $\mathcal{K}(p)^2 = v_p(I_p)$, we infer that $\mathcal{L}_C \in \mathbb{Z}\mathcal{L}_p$ if and only if $v_p(C) \in (\mathcal{K}(p)^2)$. The second claim follows by direct computation. 

Clearly, the question of whether $\tilde{C}$ on $X$ is Cartier is local as it depends only on the existence of an equation for $\tilde{C}$ near the singularities of $X$ (recall that every Weil divisor on a neighbourhood of a regular point is principal, see [13] II.6.11). If $Q$ is in the exceptional locus of $X$, we write $\mathcal{K}_+^Q$ for the set of points $p \in K_+$ such that $Q$ lies on $L_p$. Then, $\{L_p\}_{p \in K_+^Q}$ are the exceptional components going through point $Q$. From 3.1 we get the following local criterion.

**Corollary 3.4.** Let $Q$ be a point in the exceptional locus of $X$. If $C \subset S$, denote by $C_Q$ the curve on $S$ composed of the branches $\gamma$ of $C$ whose strict transform $\tilde{\gamma}$ on $X$ goes through $Q$. Then, $\tilde{C}$ is locally principal in a neighbourhood of $Q$ if and only if

$$\mathcal{L}_{C_Q} \in \bigoplus_{u \in K_+^Q} \mathbb{Z}\mathcal{L}_u.$$ 

**Proof.** From the definition of $C_Q$ the germs of $\tilde{C}$ and $\tilde{C}_Q$ at $Q$ are equal. Moreover, $Q$ is the unique point in the intersection of $\tilde{C}_Q$ with the exceptional locus of $X$. Therefore, $\tilde{C}$ is locally principal near $Q$ if and only if $\tilde{C}_Q$ is Cartier. By 3.1 this is the case if and only if $\mathcal{L}_{C_Q} \in \bigoplus_{p \in K_+^Q} \mathbb{Z}\mathcal{L}_p$. By 3.1 $|\tilde{C}_Q \cdot L_p|_X = |\tilde{C}_Q \cdot L_p|_Q > 0$ if $p \in K_+^Q$ and zero, otherwise. The claim follows. 

From 3.1 and 3.4, an algorithm providing a test to verify if the strict transform on $X$ of a given curve is Cartier or locally principal at some point is deduced. If $C \subset S$ is a curve, then the cluster $\mathcal{Q}_C$ (defined in 3.1) is consistent and equivalent to the (weighted) cluster $\mathcal{T}_C$ defined by the system of virtual values given by

$$v_p = \begin{cases} v_p(C) & \text{if } p \in K_+ \\ 0 & \text{otherwise.} \end{cases}$$

In order to know the dicritical points of $\mathcal{Q}_C$, it is enough to unload $\mathcal{T}_C$. By 3.1, $\tilde{C}$ is Cartier if and only if $(\mathcal{Q}_C)_+ \subset K_+$. To study the local principality of $\tilde{C}$ at some singularity $Q$, proceeding analogously with $\mathcal{Q}_{C_Q}$ is enough. Forthcoming 4.4 and 4.5 provide examples of this.

**Remark 3.5.** A curve on $X$ containing no exceptional component is the strict transform of some curve on $S$, so 3.1 actually gives criteria for the principality of any such curves. The principality of
exceptional curves can be studied componentwise in terms of curves with no exceptional component, taking into account \(2.1\) for and \(p \in \mathcal{K}_+\), write \(\gamma_p\) for a generic curve going sharply through \(\mathcal{K}(p)\) and missing all points in \(\mathcal{K}\) after \(p\). Then, the principality of a curve

\[
\tilde{C} + L = \tilde{C} + \sum_{p \in \mathcal{K}_+} a_p L_p
\]

is equivalent to the principality of the strict transform on \(X\) of \(C + \mathcal{L}\), where

\[
\mathcal{C}_L = \sum_{p \in \mathcal{K}_+} a_p \mathcal{C}^p,
\]

and \(\mathcal{C}^p = \sum_{E_p \cap \mathcal{E}_q \neq \emptyset} \gamma_q\). Indeed, the curves \(\mathcal{C}^p\) are defined in this way so that the equality \(\tilde{D}_C = D_{L_p}\) holds, and we can apply \(2.1\) Now, if \(\tilde{C} + \mathcal{C}_L\) is Cartier or not can be decided by using the equivalent assertions of \(3.1\).

4. Some consequences and examples

Given \(C\) on \(S\), let \(\gamma_1, \ldots, \gamma_s\) be the branches of \(C\) at \(O\) and for each \(i\) denote by \(p_i\) the first non-singular point on \(\gamma_i\) and not in \(\mathcal{K}\). Write \(K'\) for the minimal cluster containing \(K\) and the points \(p_1, \ldots, p_s\). Since the sheaf \(I\mathcal{O}_{S_{K'}}\) is invertible, there is a morphism \(g: S_{K'} \to X\) induced by the universal property of the blowing-up. Moreover, \(g\) factors through \(f: S_K \to X\) because this is the minimal resolution of \(X\). Since \(K'\) contains \(p_1, \ldots, p_s\), the strict transform of \(C\) on \(S_{K'}\) is non-singular, while the total transform has normal crossings only:

\[
\begin{array}{ccc}
S_{K'} & \xrightarrow{g} & X \\
\pi_{K'} & \downarrow & \pi_i \\
S & \xleftarrow{f} & S_K
\end{array}
\]

Remark on the notation. From now on, we are working on the surface \(S_{K'}\) rather than on \(S_K\). For the sake of simplicity in the notation, we will make a slight abuse of language and write \(S'\) for \(S_{K'}, \widetilde{C}'\) and \(D_{\widetilde{C}}\) for the strict transform and the exceptional part of the total transform of \(\widetilde{C}\) on \(S'\), and also \(E_J\) and \(E_C\) for the exceptional parts of \(J \in \mathfrak{I}_R\) and \(C\) on \(S'\), respectively. Recall from \(3.1\) that \(J_C = \{g \in R \mid v_p(g) \geq v_p(C), \forall p \in \mathcal{K}_+\}\) and \(Q_C = BP(J_C)\).

During the proof of \(3.1\) we have proved the following fact relating the coefficients of \(\mathcal{L}_C\) in the base \(\{\mathcal{L}_p\}_{p \in \mathcal{K}_+}\) to the intersection product of \(\widetilde{C}\) with \(\{L_p\}_{p \in \mathcal{K}_+}\).

Theorem 4.1. (corollary of the proof of \(3.1\)) If \(\mathcal{L}_C = \sum_{u \in \mathcal{K}_+} a_u \mathcal{L}_u\), then

\[
a_u = |\widetilde{C} \cdot L_u|_X \geq 0.
\]

Moreover, if \(\widetilde{C}\) is Cartier, then \(J_C = \prod_{u \in \mathcal{K}_+} \mathcal{I}_{u}^{a_u}\) is the (Zariski) factorization of \(J_C\). In particular, \(J_C \mathcal{O}_X = \mathcal{O}_X(-L_C)\).

Corollary 4.2. Let \(I = \prod_{p \in \mathcal{K}_+} \mathcal{I}_p^{\alpha_p}\) be the (Zariski) factorization of \(I\) and let \(Q\) be a point in the exceptional locus of \(X\). If \(C\) goes sharply through \(K_Q\), then \(\widetilde{C}\) is Cartier on \(X\) and \(\widetilde{C} \cdot L_p|_X = \alpha_p\), for all \(p \in \mathcal{K}_+\).
PROOF. We already know that for such a curve \( C \), \( \mathcal{L}_C = \mathcal{L}_I \). Then, it is enough to apply Proposition 3.11.

The following proposition is technical and will be useful later on.

**Proposition 4.3.**

(a) If \( \widetilde{C} \) is Cartier, then \( D_{\widetilde{C}} = \mathcal{E}_C - \mathcal{E}_{Q_C} \).

(b) If \( C_1, C_2 \) are curves on \( S \), then

\[
|\widetilde{C}_1 \cdot \widetilde{C}_2|_X = |C_1, C_2|_O - |Q_{C_1}, C_2|_O.
\]

**Proof.** Since \( \pi^*_I(C) = \widetilde{C} + \mathcal{L}_C \) and \( \pi^*_K(C) = \widetilde{C} + \mathcal{E}_C \), we infer that

\[
\mathcal{E}_C = D_{\widetilde{C}} + g^*(\mathcal{L}_C).
\]

Let \( B \) be a curve going sharply through \( Q_c \) and missing the points in \( K' \setminus K \). Then, \( \mathcal{L}_B = \mathcal{L}_C \) and \( \widetilde{B} \) goes through no singularities of \( X \) and shares no points with \( \widetilde{C} \). Thus, \( \pi^*_K(B) = \widetilde{B} + g^*(\mathcal{L}_C) \) and so, \( \mathcal{E}_{Q_C} = g^*(\mathcal{L}_C) \). (a) follows from Proposition 4.1 above.

Now, by the projection formula applied to \( \pi_K \), we have

\[
|\widetilde{C}_1 \cdot \widetilde{C}_2|_X = |(\widetilde{C}_1 + \mathcal{E}_{C_1}) \cdot \widetilde{C}_2|_{S'} - |\mathcal{E}_{Q_C} \cdot \widetilde{C}_2|_{S'}.
\]

As above, let \( B_1 \) be a curve going sharply through \( Q_{C_1} \) and such that \( \widetilde{B}_1 \) shares no point with \( \widetilde{C}_2 \) on \( S' \). Then,

\[
|Q_{C_1}, C_2|_O = |B_1, C_2|_O = |(\widetilde{B}_1 + \mathcal{E}_{Q_{C_1}}) \cdot \widetilde{C}_2|_{S'} = |\mathcal{E}_{Q_{C_1}} \cdot \widetilde{C}_2|_{S'}
\]

and \( |C_1, C_2|_O = |(\widetilde{C}_1 + \mathcal{E}_{C_1}) \cdot \widetilde{C}_2|_{S'} \). The claim of (b) is derived from (4.2).

To close this section, we illustrate the results in it and in the previous one with some examples.

**Example 4.4.** Take \( I \in I_R \) with base points as on (a) of figure 1. The dicritical points of \( K = BP(I) \) are \( p_2, p_4 \) and \( p_8 \) and so, the surface \( X = Bl_I(S) \) has three exceptional components \( L_{p_2}, L_{p_4} \) and \( L_{p_8} \). \( X \) has two singularities: \( Q_1 \) in the intersection of \( L_{p_2} \) and \( L_{p_4} \), with \( T_{Q_1} = \{p_3\} \) and \( Q_2 \) in the intersection of \( L_{p_2} \) and \( L_{p_8} \), with \( T_{Q_2} = \{p_1, p_5, p_6, p_7\} \). We have \( L_{p_2} = 2L_{p_2} + 2L_{p_4} + 2L_{p_8} \), \( L_{p_4} = 2L_{p_2} + 4L_{p_4} + 2L_{p_8} \) and \( L_{p_8} = 2L_{p_4} + 2L_{p_4} + 9L_{p_8} \). If \( C \) is a curve on \( S \) with singular points as represented on (b) of figure 1 then \( \mathcal{L}_C = 6L_{p_2} + 8L_{p_4} + 11L_{p_8} = \frac{2}{5} L_{p_2} + L_{p_4} + \frac{3}{5} L_{p_8} \). By Proposition 4.1, \( \widetilde{C} \) is not a Cartier divisor on \( X \). However, \( \mathcal{L}_{C_{Q_1}} = 4L_{p_2} + 6L_{p_4} + 4L_{p_8} = L_{p_2} + L_{p_4} \), so \( C \) is principal in a neighborhood of \( Q_1 \), but not near \( Q_2 \) (see Proposition 3.3).
Figure 2. (a) The Enriques diagrams of \( K \); (b) the singular points of the curves \( C \); (c) the singular points of \( D \) (Example 4.5).

Example 4.5. Let \( I \in I_R \) with base points as on (a) of figure 2. The dicritical points of \( K = BP(I) \) are \( p_1, p_4, p_6 \) and \( p_{10} \) and so, the surface \( X = Bl_I(S) \) has exceptional components \( L_{p_1}, L_{p_4}, L_{p_6} \) and \( L_{p_{10}} \). There is only one singularity on \( X \), say \( Q \), and \( L_{p_1} = L_{p_1} + L_{p_4} + 2L_{p_6} + 2L_{p_{10}}, L_{p_6} = L_{p_1} + 4L_{p_4} + 4L_{p_6} + 4L_{p_{10}}, L_{p_6} = 2L_{p_1} + 4L_{p_4} + 12L_{p_6} + 10L_{p_{10}}, L_{p_{10}} = 2L_{p_1} + 4L_{p_4} + 10L_{p_6} + 12L_{p_{10}}. \) Take the curves \( C \) and \( D \) having singularities as (c) and (d) of figure 2. Direct computation shows that \( L_C = 9L_{p_1} + 21L_{p_4} + 42L_{p_6} + 44L_{p_{10}} = L_{p_1} + 2L_{p_4} + L_{p_6} + 2L_{p_{10}}, \) so \( \tilde{C} \) is a Cartier divisor on \( X \) and \( |\tilde{C} \cdot L_{p_1}|_X = 1, |\tilde{C} \cdot L_{p_4}|_X = 2, |\tilde{C} \cdot L_{p_6}|_X = 1 \) and \( |\tilde{C} \cdot L_{p_{10}}|_X = 2. \) The cluster \( Q_c \) is represented on (b) of figure 2. We have that \( |C, D|_O = 88 \) and \( |Q_c, D|_O = 82. \) Thus, in virtue of (b) of [13] \( |C \cdot \bar{D}|_X = 6. \)

5. On Cartier divisors containing a given Weil divisor on \( X \)

In this section, we are interested in studying the (effective) Cartier divisors containing a given curve on \( X \). For the sake of simplicity in the exposition, we restrict our study to curves containing no exceptional components on \( X \) and whose strict transform on \( S' \) is already non-singular. Moreover, we add a minimality condition concerning the values of these curves relative to the divisorial valuations \( \{v_p\}_{p \in T_S^1}. \) To this aim, recall that if \( A \) is a curve on \( X, A \) is a Cartier divisor if and only if \( D_A \) is a divisor on \( S' \) (see [21]). When \( A \) is not Cartier, it is also possible to associate a divisor \( \overline{D}_A \) on \( S' \) to \( A \) such that

\[
|E_p \cdot \overline{D}_A|_{S'} \leq |E_p \cdot D_A|_{S'} \quad \text{for any exceptional component} \ E_p \ \text{for} \ g
\]

and minimal with this property. This divisor \( \overline{D}_A \) can be easily computed by Laufer’s method; moreover, \( A \) is Cartier if and only if \( D_A = \overline{D}_A \) (see [11] §1 for details).

Fix \( C \) be a curve on \( S \). An effective Cartier divisor \( C' \) containing \( \tilde{C} \) will be called \( v \)-minimal (relative to \( C \)) if

(i) \( D_{C'} = \overline{D}_{\tilde{C}}; \)
(ii) the strict transform of \( C' \) on \( S' \) is non-singular and \( g^*(C') \) has normal crossings only;
(iii) \( C' \) contains no exceptional components on \( X. \)

By using partial unloading (see Appendix), we are going to construct flags of clusters that will allow us to give a complete description of the \( v \)-minimal Cartier divisors containing \( \tilde{C}. \) These flags will play a basic role in the forthcoming sections since, as it will be shown there, they carry deep information relative to \( \tilde{C}. \)
Recall that $K'$ is the minimal cluster containing $K$ and the first non-singular points $p_i$ of the branches of $C$ which are not in $K$. Fix non-negative integers $m = \{m_p\}_{p \in K_+}$, and write $K^m$ for the cluster whose underlying set of points is $K'$ and whose excess at $p$ is $m_p$ if $p \in K_+$ and 0, otherwise. Take the sequence of clusters
\begin{equation}
T_0 = K^m < T_1 < \ldots < T_j < \ldots \quad \text{with } T_j = (K', \tau^j)
\end{equation}
defined as follows: for $j \geq 0$, as far as there exists some $p_i$ such that $\tau^j_{p_i} = 0$, $T_{j+1}$ is the cluster obtained from $T_j$ by increasing its multiplicity at $p_i$ by one and performing partial unloading relative to the set $K_+$. Note that as long as there exists such a $p_i$, $H_{T_j} \supset H_{T_{j+1}}$, and moreover
\begin{equation}
dim_c \left( \frac{H_{T_j}}{H_{T_{j+1}}} \right) = 1.
\end{equation}
A flag of clusters as in (5.1) will be called a flag for $(K, C)$ (an $m$-flag if we want to precise the original excesses of $T_0$). For simplicity in the notation and if no confusion may arise, we will write $\{T_j\}$ for the clusters appearing in such a flag.

**Lemma 5.1.** After finitely many steps, this procedure stops giving rise to a cluster $T_n$ such that
\begin{enumerate}
  \item $\tau^n_{p_i} = 1$, for $i = 1, \ldots, s$;
  \item $L_{T_j} = \sum_{p \in K_+} m_p L_p$, for $j = 0, \ldots, n$.
\end{enumerate}
and non-negative integers $\omega = \{\omega_p\}_{p \in K_+}$ such that $\rho^n_{p_i} = m_p - \omega_p$ for $p \in K_+$. Moreover, the number $n$ and the integers $\omega$ depend only on $K$ and the points $p_1, \ldots, p_s$, and not on $m$.

**Proof.** Write $K^m = K^m + \sum_{i=1}^m K(p_i)$. First of all, we show by using induction on $j$ that
\begin{equation}
H_{K^m} \subset H_{T_j}
\end{equation}
for every $T_j$ defined as above. This is clear for $j = 0$. Assume it is also true for some $j \geq 0$. To prove that $H_{K^m} \subset H_{T_{j+1}}$, it is enough to show that the virtual transform relative to the multiplicities of $T_j$ of a generic curve going through $K^m$ goes through the point $p_i$. But this is clear since any curve going sharply through $K^m$ goes effectively through $p_i$. Because of (5.2), this procedure stops after finitely many steps. Hence, we obtain a (not necessarily consistent) cluster $T_n$ such that $\tau^n_{p_i} \geq 1$ for $i = 1, \ldots, s$.

Notice by the way that $n$ is the codimension of $T_n$ in $T_0$.

Now, assume that we have a couple of (not necessarily consistent) clusters $T^{(1)} = (K', \tau^{(1)})$ and $T^{(2)} = (K', \tau^{(2)})$ such that $H_{T^{(1)}} \subset H_{K^m}$ and $\tau^{(1)}_{p_i} \geq 1$ for $i = 1, \ldots, s$. Define a new cluster $T^{(0)} = (K', \tau^{(0)})$ as follows: take $v_0 = (v^0_p)_{p \in K}$ where
\begin{equation}
v^0_p = \min \{v^{(1)}_p, v^{(2)}_p\},
\end{equation}
and define virtual multiplicities for $K'$ by
\begin{equation}
\tau^{(0)}_p = \begin{cases}
1_p \mathbf{P}_K v_0 & \text{if } p \in K \\
0_p & \text{if } p \in K' \setminus K.
\end{cases}
\end{equation}
where $\{\mathbf{P}_K v_0\}_{p \in K'}$ are the virtual multiplicities of $K^m$. Clearly, $H_{T^{(1)}} \subset H_{T^{(0)}}$ and $\tau^{(0)}_p = \tau^0_p$ for $p \in K' \setminus K$. Since $v^{(0)}_p = \min \{v^{(1)}_p, v^{(2)}_p\} \geq v^0_p$ for any $p \in K$, we have $H_{T^{(0)}} \subset H_{K^m}$. By using Artin’s trick as in the proof of (5.3) we see that $\rho^{(0)}_{p_i} \geq 0$ for $p \in K' \setminus K$. This proves the uniqueness of a minimal cluster $T' = (T', \tau')$ with $H_{T'} \subset H_{K^m}$ and $\tau^{(0)}_{p_i} \geq 1$ for $i = 1, \ldots, s$, and shows actually that $\tau'_{p_i} = 1$ for $i = 1, \ldots, s$. From the way the cluster $T_n$ has been constructed, necessarily $T' = T_n$. 

and so $T_n$ does not depend on the choices done when constructing the flag $\mathbb{F}$. In virtue of \([5.3]\), if $j \geq 1$ and $p \in T \setminus K_+$, $\tau_p^{j-1}$ does not depend on the excesses $\{\rho_q^j\}_{q \in K_+}$. By induction, the multiplicities $\{\tau_p^j\}_{p \in K_+}$ do not depend on $m$, and neither the difference $\omega_p = \rho_p^{j-1} - \rho_p^j$. Hence, the numbers $\omega_p = \sum_{j=1}^n \omega_p^j$ $(p \in K_+)$ are independent of $m$ and $\rho_p^j = m_p - \omega_p$ as claimed. \(\Box\)

**Remark 5.2.** Notice that $m \geq \omega$ (this meaning that $m_p \geq \omega_p \forall p \in K_+$) if and only if all the clusters $T_0, T_1, \ldots, T_n$ are consistent. In this case, partial unloading relative to $K_+$ equals usual unloading.

Write $Q_c = (K', e_C)$ for the cluster obtained by taking as virtual multiplicities the effective multiplicities of $C$. If $C$ is already Cartier, the integers $\{\omega_p\}_{p \in K_+}$ given by \([5.1]\) equal the effective values $\{v_p(C)\}_{p \in K_+}$. In fact, in this case we have that $Q_c = T_0\omega$ and $Q_c = T_n\omega$.

We state the following easy lemma for future reference.

**Lemma 5.3.** Let $m \geq \omega$ and $\{T_i\}_{i=0,\ldots,n}$ an $m$-flag for $(K,C)$. For any curve $B$ on $S$, we have that

(a) the value of $[T_j, B]_O - [T_0, B]_O$ does not depend on $m$.

(b) if $B$ shares no points on $X$ with $C$, $[T_j, B]_O = [T_0, B]_O$ for any $j \geq 0$. In particular,

$$\sum_p \tau_p^j e_p(B) = \sum_p \tau_p^0 e_p(B).$$

**Proof.** If $C_j$ is a curve going sharply through $T_j$, we have that $C_j^* = C_j + \mathcal{L}_m$. Then, by the projection formula applied to $\pi$, $[T_j, B]_O = ((C_j + \mathcal{L}_m) \cdot \tilde{B})_X$. The first claim follows immediately. Now, notice that for a generic curve going through $T_0$, $[\tilde{C}_0 \cdot \tilde{B}]_X = 0$. If $\tilde{B}$ shares no points on $X$ with $\tilde{C}$ and $j > 0$, then $[\tilde{C}_j \cdot \tilde{B}]_X = 0$ also. The second assertion is derived now from the first one and (b) of \([4.3]\) \(\Box\)

The following proposition is the main result of this section and gives a complete description of the $v$-minimal Cartier divisors containing $\tilde{C}$.

**Proposition 5.4.** Take $\omega$ as in \([5.1]\)

(a) If $m \geq \omega$, the system of virtual multiplicities $\{\tau_p^n - \epsilon_p(C)\}_{p \in K'}$ is consistent.

(b) Every $\epsilon$-minimal Cartier divisor for $C$ has the form $\tilde{C} + B$, where $B$ is a generic curve going through $\mathcal{C}_m = (K', \epsilon - \epsilon(C))$, for some $m \geq \omega$.

(c) $\mathcal{T}_1 \mathcal{C} = \mathcal{E}_T - \mathcal{E}_0 = \sum_{p \in K'} n_p E_p$, where $n_p$ is the number of unloadings performed on each $p \in K'$.

**Remark 5.5.** After \([5.4]\) one should think of the strict transform of the generic curves going through $\mathcal{C}_m$ (for some $m \geq \omega$) as the curves to be added to $C$ to obtain $(v$-minimal) Cartier divisors on $X$.

**Proof.** Since $m \geq \omega$, the cluster $T_n^m$ is consistent (see \([5.2]\), and from its construction, the points $p_1, \ldots, p_s$ have virtual multiplicity one at it and are maximal in $K'$. It follows that $\rho_p^{m} = 1$ for $i \in \{1, \ldots, s\}$. On the other hand, from its own definition the cluster $Q_c$ has excess equal to one at $\{p_i\}_{i=1,\ldots,s}$, and zero at the remaining points. Since the excesses of $\mathcal{C}_m$ are the difference between the excesses of $T_n^m$ and $Q_c$, we infer that $\mathcal{C}_m$ is consistent.

Let $A$ be an effective Cartier divisor on $X$ containing $\tilde{C}$ and without exceptional components. There exists some curve $B$ on $S$ such that $A$ is the strict transform of $C_B = C + B$. By \([3.1]\), we
have that
\[\mathcal{L}_{C_B} = \sum_{p \in K^+} m_p \mathcal{L}_p \text{ with } \{m_p\} \subset \mathbb{Z}_{\geq 0}.\]
Write \(m = \{m_p\}_{p \in K^+}\). By (5.2) we have that \(m \geq \omega\) and so, the clusters in any \(m\)-flag
\[\mathcal{T}_0^m < \ldots < \mathcal{T}_n^m\]
are consistent. Now, \(C_B\) goes through \(\mathcal{T}_0^m\) and \(e_p(C_B) = 1\) for \(i = 1, \ldots, s\). By the minimality of \(\mathcal{T}_n^m\) (see (5.1)), it follows that \(C_B\) goes through it. Thus, \(C_B\) goes virtually through \(\mathcal{T}_n^m\) and \(\mathcal{Q}_{C_B} = \mathcal{T}_0^m\). Hence, by (a) of (4.3)
\[D_{C_B} = \mathcal{E}_{C_B} - \mathcal{E}_{T_0^m} \geq \mathcal{E}_{T_n^m} - \mathcal{E}_{T_0^m},\]
and the equality holds if and only if \(C_B\) goes through \(\mathcal{T}_n^m\) with effective multiplicities equal to the virtual ones. This is equivalent to say that the curve \(B\) goes through \(C_m\) with effective multiplicities equal to the virtual ones. By definition of the cluster \(K', \tilde{\mathcal{C}}_B\) intersects transversally the exceptional divisor of \(f_C\). Therefore, the strict transform \(\tilde{A}\)' intersects transversally the exceptional divisor of \(f_C\) if and only if for each \(p \in K' \setminus K^+,\) \(C_B\) has \(\rho_{T_p}^m\) branches through \(p\) and misses all points after \(p\) in \(K'\). Therefore, we see that \(A\) is a \(v\)-minimal Cartier divisor containing \(\tilde{C}\) if and only if \(B\) goes through \(C_m\) with effective multiplicities equal to the virtual ones and has \(\rho_{T_p}^m\) branches through \(p\) and misses all points after \(p\) in \(K\). This completes the proof of (b).

Moreover, if \(A\) is \(v\)-minimal, then \(D_A = \mathcal{E}_{T_n^m} - \mathcal{E}_{T_0^m}\) and in particular, \(\mathcal{D}_{\tilde{C}} = \mathcal{E}_{T_n^m} - \mathcal{E}_{T_0^m}\). Now, from the definition of \(\mathcal{T}_n^m\), we know that \(\tau_p^m = e_p(C)\) for every \(p \in K' \setminus K\). Therefore, \(p \notin K\) is a dicritical point of \(\mathcal{T}_n^m\) if and only if \(p = p_i\) for some \(i\). From its own definition, it follows that \(C_m\) has no dicritical points out of \(K\). This gives (c) and completes the proof of the proposition. \(\square\)

We finish with an example

**Example 5.6.** Take \(I \in I_R\) as in Example 4.4. On (a) of figure 3 the clusters \(\mathcal{T}_0\) and \(\mathcal{T}_n\) of a \(m\)-flag for \((K, C)\) are represented. The excesses of \(\mathcal{T}_n\) at \(p_2, p_3\) and \(p_3\) are \(\rho_{p_2}^{T_n} = m_{p_2} - 2, \rho_{p_3}^{T_n} = m_{p_3} - 1, \rho_{p_3}^{T_n} = m_{p_3} - 2\), and so, \(\omega_{p_2} = 2, \omega_{p_3} = 1\) and \(\omega_{p_3} = 2\). By taking \(\mathbf{m} = \omega\), the virtual values of \(\mathcal{T}_n\) and \(\mathcal{T}_0\) are \(v_{\mathcal{T}_n} = \{8, 10, 12, 12, 10, 20, 22, 22\}\) and \(v_{\mathcal{T}_0} = \{7, 10, 11, 12, 9, 18, 20, 22\}\). By (5.4) the \(v\)-minimal Cartier divisors on \(X\) containing \(\tilde{C}\) are the curves \(\tilde{C} + \tilde{B}\), where \(B\) is a generic curve going through \(C_m\). By (b) of (5.3) \(\mathcal{D}_{\tilde{C}} = \mathcal{E}_{T_n} - \mathcal{E}_{T_0} = E_{p_3} + E_{p_1} + E_{p_3} + 2E_{p_0} + E_{p_7}\), so \(\mathcal{D}_{\tilde{C}} = D_{\tilde{C}}^{M_1} + D_{\tilde{C}}^{M_2}\).
6. The order of singularity of curves on $X$

For a given curve $C$ on $S$, write $\delta_X(\tilde{C}) = \sum_{Q} \delta_Q(\tilde{C})$ and call this value the total order of singularity of $\tilde{C}$. Our aim in this section is to get some formulas for the order of singularity of curves on $X$ without exceptional support. Then, we will use them to compute the semigroup of a branch going through a sandwiched singularity.

**Proposition 6.1.** Let $C$ be a curve on $S$ and take a flag $\{T_i\}_{i=0,\ldots,n}$ for $(\mathcal{K}, C)$. Then,

$$\delta_X(\tilde{C}) = [T_n, C]_O - [T_0, C]_O - n.$$

In virtue of [13], the right term of the above equality does not depend on the particular flag $\{T_i\}_{i=0,\ldots,n}$ chosen.

To prove this result we need a technical lemma.

**Lemma 6.2.** Let $T = (T, \tau)$ be a a consistent cluster, and write $\mathcal{K}_T$ for the canonical divisor on the surface $S_T$ obtained by blowing up all the points of $T$. If $A$ is a curve on $S$ and $\mathcal{E}_A$ is the exceptional part of its total transform on $S_T$, then

$$|\mathcal{E}_A \cdot \mathcal{K}_T|_{S_T} = -\sum_{p \in T} e_p(A).$$

**Proof.** The adjunction formula (see [13] V.1.5) says that

$$|E_p \cdot \mathcal{K}_T|_{S_T} = -2 - E_p^2$$

for each $p \in T$.

Write $T'$ for the cluster obtained by taking $\{\tau_p - 1\}_{p \in T}$ as the system of virtual multiplicities. Let $B, B'$ be curves going sharply through $T$ and $T'$, respectively. Direct computation using (2.3) and that $E_p^2 = -\frac{2}{q} \{q \in T \mid q \to p\} - 1$ shows that

$$|E_p \cdot \tilde{B}'_T|_{S_T} = |E_p \cdot \tilde{B}_T|_{S_T} = -2 - E_p^2.$$

Therefore, by the projection formula,

$$|\mathcal{E}_A^T \cdot \mathcal{K}_T|_{S_T} = |\mathcal{E}_A^T \cdot \tilde{B}'_T|_{S_T} - |\mathcal{E}_A^T \cdot \tilde{B}_T|_{S_T} = [A, B']_O - [A, B]_O = -\sum_{p \in T} e_p(A),$$

the last equality by Noether’s formula. □

**Proof of 6.1.** Write $\mathcal{K}$ for the canonical divisor on $S'$. For the sake of simplicity, write $D$ and $\overline{D}$ for $D_{\tilde{C}}$ and $D_{\tilde{C}}$, respectively. Add the subindex $Q$ to denote the exceptional part of $D$ and $\overline{D}$ contracting to some $Q \in X$. By Corollary 2.1.2. of [19], we have the following formula for the order of singularity of $\tilde{C}$ at $Q$:

$$\delta_Q(\tilde{C}) = \frac{1}{2} |D_Q \cdot (\mathcal{K} - D_Q)|_{S'} + \frac{1}{2} |(D_Q - \overline{D}_Q) \cdot (D_Q - \overline{D}_Q - \mathcal{K})|_{S'} =$$

$$= \frac{1}{2} |D_Q \cdot (\overline{D}_Q - 2D_Q + \mathcal{K})|_{S'}.$$

From this and the fact that $D_Q$ and $D_Q'$ are disjoint if $Q \neq Q'$, we have

$$\delta_Q(\tilde{C}) = \frac{1}{2} |\overline{D}_Q \cdot (\overline{D} - 2D + \mathcal{K})|_{S'}.$$

Thus,

$$\sum_{Q} \delta_Q(\tilde{C}) = \frac{1}{2} |\overline{D} \cdot (\overline{D} + \mathcal{K})|_{S'} - |\overline{D} \cdot D|_{S'}.$$
In virtue of (b) of 4.3, $D = \mathcal{E}_T - \mathcal{E}_{T_0}$. Let $B$ be a curve going sharply through $T_n$. Then, $\tilde{B}$ is Cartier on $X$ and by (a) of 4.3, $D_B = \mathcal{T}$. By the projection formula applied to $f_C$, $|D|_{S'} = (E' \cdot E)|_{S'}$ for any exceptional divisor $E$ being contracted by $f_{KC}$. Therefore, we have

\[(6.3) \quad |D \cdot (D + \mathbb{K})|_{S'} = (|E_{T_n} - E_{T_0}| \cdot (-E')|_{S'}).
\]

Now, since a generic curve through $T_n$ (resp. through $T_0$) goes sharply through it and shares no points with $B$ outside $K'$ (see Theorem 4.2.8 of [4]), we have that

\[-(|E_{T_n} - E_{T_0}| \cdot B)|_{S'} = |T_0 \cdot B|_O - |T_n \cdot B|_O.
\]

On the other hand, in virtue of $(6.2)$,

\[|(E_{T_n} - E_{T_0}) \cdot \mathbb{K}|_{S'} = \sum_{p \in K'} (\tau^n_0 - \tau^n_p).
\]

From all of this, $(6.3)$ and the Noether’s formula (see (2.4)), we infer that

\[
|D \cdot (D + \mathbb{K})|_{S'} = |T_0 \cdot B|_O - |T_n \cdot B|_O + \sum_{p \in K'} (\tau^n_0 - \tau^n_p) = \\
\sum_p \tau^n_0 \tau^n_p - \sum_p \tau^n_0 \tau^n_p - 1,
\]

the last equality because $\sum_p \tau^n_0 \tau^n_p = \sum_p (\tau^n_p)^2$ (see (b) of 4.3). Hence,

\[
\frac{1}{2} |D \cdot (D + \mathbb{K})|_{S'} = \frac{1}{2} \sum_p \tau^n_0 (\tau^n_p - 1) - \frac{1}{2} \sum_p \tau^n_0 (\tau^n_p - 1) = \dim \left( \frac{H_{T_n}}{H_{T_0}} \right) = n.
\]

On the other hand, since $D_{\tilde{C}} = \mathcal{T}$, the projection formulas for $f_C$ and $\pi_{KC}$ say that $|D \cdot D|_{S'} = -|\mathcal{T} \cdot \mathcal{T}'|_{S'} = |T_0, C|_O - |T_n, C|_O$. The claim follows from (6.2).

From 6.1 we deduce the following formula for the order of singularity of a Cartier divisor on $X$.

**Corollary 6.3.** If $\tilde{C}$ is a Cartier divisor on $X$ and $d_p = e_p(C) - \tau^n_0$, we have that

\[\delta_X(\tilde{C}) = \delta_O(C) - \delta_O(Q_c) = \sum_{p \in K_C} d_p (d_p - 1).
\]

**Proof.** If $\tilde{C}$ is Cartier, $T^\omega_n = Q'_{C}$ and $T^\omega_0 = Q_C$ (see 5.5). The strict transform of a generic curve $B$ going sharply through $T_0$ shares no points with $\tilde{C}$ on $X$. Therefore, by (b) of 4.3 we have

\[|T_0, C|_O = |B, T_n|_O = |B, T_0|_O = T^2_{\tilde{C}}.
\]

Thus, $|T^\omega_n, C|_O = (Q'_{C})^2$ and also $|T^\omega_0, C|_O = Q^2_{C}$. Using 6.1 and the fact that $\delta_T + c(T) = T^2$, the first equality follows. The second follows by direct computation since by (b) of 4.3 again, $\sum_p d_p \tau^n_0 = 0$.

Let $R_f = \frac{R_f}{(j)}$ be the local ring of $C$ at $O$ and assume that $\tilde{C}$ is Cartier. Write $\{Q_1, \ldots, Q_n\}$ for the points of $\tilde{C}$ in the exceptional locus of $X$ and $J_i$ for the ideal of $\tilde{C}$ in $O_{X,Q_i}$. Clearly, we have a monomorphism

\[R_f \rightarrow R_f^l = \prod_{i=1}^n \frac{O_{X,Q_i}}{J_i}.
\]
induced by the monomorphisms $R \to \mathcal{O}_{X,Q}$. Then, the above corollary says that
\[ \dim C \frac{R^I_f}{R_I} = \delta_O(Q_c). \]

Notice that if $I = m_O$, $R^I_f$ is the ring of $C$ in the first neighbourhood of $O$ introduced by Northcott (see \[22\] \[23\]). Then, \[6.3\] says that $\delta_O(C) - \delta_X(\tilde{C}) = \frac{\alpha(C)(e_Q(C)+1)}{2}$ which is known to be the variation in the order of singularity of $C$ after blowing up the point $O$.

**Example 6.4.** Take the cluster and the curve $C$ of Example 4.4 (see figure \[1\]). We have that $[\mathcal{T}_n^\omega, C]_O = 49$ and $[\mathcal{T}_0^\omega, C]_O = 42$. Direct computation shows that $\dim C(\frac{H_{\mathcal{T}_n^\omega}}{H_{\mathcal{T}_0^\omega}}) = 6$. Therefore, by \[6.1\] we obtain $\delta_X(\tilde{C}) = 1$. Since $Q_1$ is the only singularity of $\tilde{C}$, we derive that $\delta_{Q_1}(\tilde{C}) = 1$.

Let $B$ be a curve going sharply through $\mathcal{C}_\omega$ and let $C_B = C + B$. Then, as noticed above, $\tilde{C}_B$ is Cartier and $Q_{CB} = \mathcal{T}_0^\omega$. By \[6.3\] $\delta_X(\tilde{C}_B) = \delta_O(C_B) - \delta_T = \delta_O(C_B) - \delta_O(\mathcal{T}_0^\omega) = 33 - 28 = 5$. In Example 6.4 we have seen that $\tilde{C}$ is principal in a neighbourhood of $Q_1$, so the germs of $\tilde{C}$ and $\tilde{C}_B$ in a neighbourhood of $Q_1$ are equal, and $\delta_{Q_1}(\tilde{C}_B) = 1$. Therefore, $\delta_{Q_2}(\tilde{C}_B) = 4$.

**The semigroup of a branch on a sandwiched singularity.** Let $C$ be a branch on $S$ and let $Q$ be the point where $\tilde{C}$ intersects the exceptional divisor of $X$. Here we show how the flags for $(\mathcal{K}, C)$ already defined determine the semigroup of $\tilde{C}$ on $(X, Q)$. Recall that since $(\tilde{C}, Q)$ is analytically irreducible, the integral closure $\mathcal{O}_{\tilde{C}, Q} \cong \mathbb{C}[t]$ is a discrete valuation ring and the semigroup $\Sigma_Q(\tilde{C})$ of $\tilde{C}$ at $Q$ is
\[ \Sigma_Q(\tilde{C}) = \{ v_t(g) \mid g \in \mathcal{O}_{\tilde{C}, Q} \} \]
where $v_t$ is the valuation corresponding to $\mathcal{O}_{\tilde{C}, Q}$. It is also known that $\delta_Q(\tilde{C})$ is the number of elements in $\mathbb{N} \cup \{0\} \setminus \Sigma_Q(\tilde{C})$ (see Appendix of \[24\]).

The following proposition gives a description of $\Sigma_Q(\tilde{C})$ in terms of differences of intersection multiplicities of curves at $O$.

**Proposition 6.5.** Take a flag $\mathcal{T}_0 \prec \mathcal{T}_1 \ldots \prec \mathcal{T}_n$ for $(\mathcal{K}, C)$, and for $0 \leq i \leq n$, write
\[ \alpha_C^i = [\mathcal{T}_i, C]_O - [\mathcal{T}_0, C]_O. \]
Then, the integers \( \{ \alpha_C^i \}_{0 \leq i \leq n} \) are the first $n + 1$ elements of the semigroup $\Sigma_Q(\tilde{C})$. Moreover $\alpha_C^i \geq c(Q)$, the conductor of $\Sigma_Q(\tilde{C})$.

**Proof.** For each $i$, let $C_i$ be a curve going sharply through $\mathcal{T}_i$. Then $\mathcal{L}_C_i \in \bigoplus_{u \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_u$ and so, the strict transform $\tilde{C}_i$ is a Cartier divisor on $X$. Let $g_i \in m_{X,Q}$ be a local equation for $\tilde{C}_i$ near $Q$, and $\overline{g}_i$ its class in $\mathcal{O}_{\tilde{C}, Q}$. Then,
\[ [\mathcal{T}_i, C]_O - [\mathcal{T}_0, C]_O = [\tilde{C}_i, \tilde{C}]_Q = v_t(\overline{g}_i) \]
is an element in $\Sigma_Q(\tilde{C})$. On the other hand, $[\mathcal{T}_i, C]_O < [\mathcal{T}_{i+1}, C]_O$ for each $i$. From this it follows that $\alpha_C^i \in \Sigma_Q(\tilde{C})$ and
\[ \# [0, \alpha_C^i] \cap \Sigma_Q(\tilde{C}) \geq n + 1. \]
Since $\delta_Q(\tilde{C})$ is the number of elements in the complement of $\Sigma_Q(\tilde{C})$, \[6.3\] implies that the above inequality is actually an equality.
Now, if \( j \geq \alpha_C^2 \), take a curve \( C' \) going sharply through \( T_n \) and sharing exactly \( j + \lvert T_0, C \rvert_O - \lvert T_n, C \rvert_O \) points with \( C \) outside \( K_C \). Then, \( \mathcal{L}C = \mathcal{L}_T \) and so, \( C' \) is Cartier (see 5.1). From Noether’s formula, we infer that

\[
[C', C]_O = \sum_{p \in K_C} e_p(C')e_p(C) + (j + \lvert T_0, C \rvert_O - \lvert T_n, C \rvert_O) = j + \lvert T_0, C \rvert_O.
\]

Hence, \( j = \lvert C', C \rvert_O - \lvert T_n, C \rvert_O = v_i(g') \) belongs to \( \Sigma_Q(\tilde{C}) \) and so, \( c(\Sigma_Q(\tilde{C})) \leq \alpha_C^2 \).

The above proposition provides an easy way to compute the semigroup of \( \tilde{C} \) once a flag for \((K, C)\) has been computed: the differences \( \alpha_C^i = \lvert T_i, C \rvert_O - \lvert T_0, C \rvert_O \) for \( i = 0, \ldots, N - 1 \) provide the first elements of \( \Sigma_Q(\tilde{C}) \); then add all the integers greater or equal than \( \alpha_C^i \).

**Remark 6.6.** The bound for the conductor of 6.5 is far from being sharp, and in general the semigroup \( \Sigma_Q(\tilde{C}) \) is not symmetric as shown in the following example (so the curves \( \tilde{C} \) needs not to be Gorenstein, see [16] for details).

**Example 6.7.** Take a cluster \( K \) and a curve \( C \) as shown in figure 4. By means of the algorithm explained in section 5, we compute the clusters \( T_n \) and \( T_0 \) for some \( m \)-flag for \((K, C)\). The virtual multiplicities of \( T_n \) and \( T_0 \) are respectively, \( \nu^{T_m} = \{12, 1, 0, 8, 1, 0, 0, 0, 5, 2, 1, 1, 1\} \), \( \nu^{T_0} = \{11, 1, 1, 4, 2, 2, 2, 2\} \). The cluster \( T_n \) is shown on (c) of figure 4. Then, \( \lvert T_n, C \rvert_O = 176 \) and \( \lvert T_0, C \rvert_O = 105 \). Also, we have \( \dim_C(\frac{H_{T_n}}{H_{T_0}}) = 141 - 100 = 41 \). Thus, by 6.1, \( \delta_Q(\tilde{C}) = \lvert T_m, C \rvert_O - \lvert T_0, C \rvert_O - \dim_C(\frac{H_{T_0}}{H_{T_m}}) = 176 - 105 - 41 = 30 \). In virtue of 6.5, we see that

\[
\Sigma_Q(\tilde{C}) = \{0, 7, 12, 14, 19, 21, 24, 26, 28, 31, 33, 35, 36, 38, 40, 41, \ldots, 42, 43, 45, 47, 48, 49, 50, 52, 53, 54, 55, 56, 57, 59, 60, \ldots\}
\]

Therefore, the conductor of the semigroup is 59. Note that the semigroup is not symmetric (see figure 4) so, the curve \( \tilde{C} \) is not Gorenstein (see [11] [3]; see also [3]).

**APPENDIX: PARTIAL UNLOADING**

The goal is to describe a slight modification of the unloading procedure that we call partial unloading. Partial unloading has been used in section 5, but it is explained separately for clarity and because it may be useful in any other context where precise control of unloading is needed.
The main virtue of partial unloading is that, fixed a non-consistent cluster \( \mathcal{K} \) it gives rise to an equivalent cluster, with no variation of the virtual values at some prefixed points and with non-negative excesses at the remaining points.

First of all, we introduce a partial order in the set of weighted (not necessarily consistent) clusters: if \( Q \) and \( Q' \) are weighted clusters, we can assume that they have the same set of points. Then, if \( Q = (Q, \tau) \) and \( Q' = (Q, \tau') \), we write \( Q \leq Q' \) to mean that \( v_p^Q \leq v_p^{Q'} \), for each \( p \in Q \). We denote by \( A_K = -P_K^T P_K \) the intersection matrix of \( E_K \), where \( P_K \) is the proximity matrix of \( K \).

**Lemma 6.8.** Given a weighted cluster \( \mathcal{K} = (K, \nu) \) (consistent or not) and a subset \( K_0 \subset K \), there exists a cluster \( \mathcal{K}' = (K', \nu') \) equivalent to \( \mathcal{K} \) and such that:

(i) \( \rho_p^{K'} \geq 0 \) if \( p \in K \setminus K_0 \);

(ii) \( v_p^{K'} = v_p^K \) if \( p \in K_0 \).

Moreover, with the order relation \( \triangleleft \) there is a unique minimal cluster satisfying (i) and (ii). This cluster will be denoted by \( \mathcal{K}^{K_0} \).

**Proof.** Put \( \mathcal{K}^0 = \mathcal{K} \) and, inductively, as far as \( \mathcal{K}^{i-1} \) has negative excess at some point \( p \in K \setminus K_0 \), define \( \mathcal{K}^i \) from \( \mathcal{K}^{i-1} \) by unloading on \( p \). We claim that there is an \( m \) such that \( \mathcal{K}^m \) has non-negative excess at each point in \( K \setminus K_0 \), and \( v_p^{\mathcal{K}^m} = v_p^K \) if \( p \in K_0 \). To show this, note that the steps of the above procedure are part of an unloading sequence giving rise to a consistent cluster as described in Theorem 4.6.2 of [1]. We reach this cluster after finitely many steps, independently on the order of the unloadings performed. Hence, after finitely many steps we reach a cluster \( \mathcal{K}^m \) satisfying the condition (i). The condition (ii) is clear, because no unloading is performed on any point of \( K_0 \).

Now, assume that \( \mathcal{K}_{(1)} = (K, \nu^{(1)}) \) and \( \mathcal{K}_{(2)} = (K, \nu^{(2)}) \) are equivalent to \( \mathcal{K} \) and verify the conditions (i) and (ii). Then, for \( i = 1, 2 \) and for every \( p \in K \),

\[
(6.4) \quad v_p^K \leq v_p^{\mathcal{K}^{(i)}} \leq v_p^\mathcal{K}'.
\]

Put \( \mathcal{K}^{(0)} = (K, \nu^{(0)}) \) the cluster defined by taking virtual values \( v_p^{\mathcal{K}^{(0)}} = \min\{v_p^{\mathcal{K}^{(1)}}, v_p^{\mathcal{K}^{(2)}} \} \) for every \( p \in K \). Since \( v_p^K \leq v_p^{\mathcal{K}^{(0)}} \leq v_p^\mathcal{K}, \mathcal{K}^{(0)} \) is equivalent to \( \mathcal{K} \). For \( p \in K \), write \( \omega(p) = \sharp\{q \in K \mid q \to p\} + 1 \). Then, if \( p \in K \setminus K_0 \) and say \( v_p^{\mathcal{K}^{(1)}} \leq v_p^{\mathcal{K}^{(2)}} \), we have (Artin’s trick: see Lemma 1.3 [2])

\[
\rho_p^{\mathcal{K}^{(0)}} = -\mathbf{1}_p^T A_K \mathbf{v}^{\mathcal{K}^{(0)}} = \omega(p) v_p^{\mathcal{K}^{(0)}} - \sum_{d(p,q)=1} v_q^{\mathcal{K}^{(0)}} \geq \omega(p) v_p^{\mathcal{K}^{(1)}} - \sum_{d(p,q)=1} v_q^{\mathcal{K}^{(1)}} = -\mathbf{1}_p^T A_K \mathbf{v}^{\mathcal{K}^{(1)}} = \rho_p^{\mathcal{K}^{(1)}} \geq 0
\]

the last inequality by assumption. Moreover, if \( p \in K_0 \), then \( v_p^{\mathcal{K}^{(0)}} = v_p^K \), so \( \mathcal{K}^{(0)} \) verifies the conditions (i) and (ii). From this, the last assertion follows. \( \square \)

In practice, \( \mathcal{K}^{K_0} \) is computed by performing usual unloading on the points not in \( K_0 \) with negative excess, so we say that it is obtained from \( \mathcal{K} \) by partial unloading (relative to \( K_0 \)). Note that if \( K_0 = \emptyset \), partial unloading equals usual unloading.

The following lemma shows that the variation of the excesses from \( \mathcal{K} \) to \( \mathcal{K}^{K_0} \) (also, of the virtual multiplicities and values) is independent of the excesses of \( \mathcal{K} \) in the points of \( K_0 \). Recall that \( \nu_p = \nu_p^K + n \), where \( n \) has non-negative entries and \( n = (0) \) if and only if \( \mathcal{K} \) is consistent.

**Lemma 6.9.** Let \( K \) be a cluster and \( K_0 \subset K \) a proper subset. Let

\[
\mathcal{K}^i = \sum_{p \in K_0} m^i_p \mathcal{K}(p) + \sum_{p \in K \setminus K_0} m_p \mathcal{K}(p), \quad i = 1, 2
\]
be clusters with $m_p \geq 0$ for each $p \in K \setminus K_0$. Write $\widehat{K}_i$ for the cluster obtained from $K_i$ by partial unloading relative to $K_0$. Then, there exists non-negative integers $\omega = \{\omega_p\}_{p \in K}$ such that $\omega_p \leq m_p$ if $p \in K \setminus K_0$ and for $i = 1, 2$

$$\widehat{K}_i = \sum_{p \in K} (m^i_p - \omega_p)K(p) + \sum_{p \in K \setminus K_0} (m_p - \omega_p)K(p).$$

**Proof.** By definition of partial unloading, we have

$$v_{\widehat{K}_i} = v_{K_i} + n^i,$$

where the entries of $n^i = (n^i_p)_{p \in K}$ are non-negative and $n^i_p = 0$ if $p \in K_0$. Take $n^0 = (n^0_p)_{p \in K}$, $n^0_p = \min\{n^1_p, n^2_p\}$, and write $K_0^i$ for the cluster with set of points $K$ and system of values given by $v_{K^i} + n^0$.

I claim that $K^i_0$ satisfies the conditions (i) and (ii) of 6.8. If $p \in K_0$, we have that $n^1_p = n^2_p = 0$. Thus, $v_{K_0^i} = v_{K^i}$ and the condition (ii) follows. For the condition (i), assume that $p \in K \setminus K_0$ and take $j \in \{1, 2\}$ such that $n^j_p = \min\{n^1_p, n^2_p\}$. Then,

$$-1_p^iA_Kn^0 = \omega(p) n^0 - \sum_{d(p,q)=1} n^0_q \geq \omega(p) n^j - \sum_{d(p,q)=1} n^j_q = -1_p^i A_Kn^j.$$

Hence,

$$\rho^i_p = -1_p^i A_Kv_{K^i} - 1_p^i A_Kn^0 \geq -1_p^i A_Kv_{K^i} - 1_p^i A_Kn^j = m_p - 1_p^i A_Kn^j = \rho^i_p \geq 0,$$

the last inequality by definition of $\widehat{K}_i$. By the minimality of $\widehat{K}_i$, we deduce that $n^0 = n^i$ and therefore, $n^1_p = n^2_p$ for each $p \in K$. In particular, for every $p \in K$, $\rho^i_p = -1_p^i A_K(\omega_{K^i} + n^0)$. Now, if we write $\omega_p = 1_p^i A_Kn^0$, we have that for $i = 1, 2$,

$$\rho^i_p = -1_p^i A_Kv_{K^i} - \omega_p = \rho^i_p - \omega_p.$$

Moreover, since $n^0 = 0$, we infer that $\omega_p = 1_p^i A_Kn^0 = \sum_{d(p,q)=1} n^0_q \geq 0$. Now, if $p \in K \setminus K_0$, $\rho^i_p = m_p$ and so, we have $\rho^i_p = m_p$. Since the excess of $\widehat{K}_i$ at $p$ is non-negative, it is clear that $m_p \geq \omega_p$. On the other hand, if $p \in K_0$, we have $\rho^i_p = m^1_p - \omega_p$ and $\rho^i_p = m^2_p - \omega_p$. \Box

In the study of the principal curvatures of curves on a sandwiched surface, partial unloading is specially useful because of [63]. Keeping the notation already used, assume that we want to prove the existence of some Cartier divisors on $X = Bl_I(S)$ satisfying some prefixed properties, or even, we want to construct them. The general sketch of the procedure is the following: write $K = (K, \nu)$ for the cluster $BP(I)$ and take $K^m = \sum_{p \in K^+} m_p K(p)$, where $m_p$ are integers to be determined. Add convenient extra conditions to $K^m$ (i.e. extra points with virtual multiplicity one) in order to force the curves going through the obtained cluster to have the desired properties and perform partial unloading relative to $K^m$. After repeating this procedure a number of times, a new cluster $K^i$ with non-negative excesses at the points of $K \setminus K^m$ is obtained. Moreover, for $p \in K^+$

(i) $\rho^i_p = m_p - \omega_p$;

(ii) $v^i_p = v^m_p$.
Since \( \omega_p \) is independent of \( m \), \( \tilde{K} + 1 \) is consistent if we take the \( m \) big enough. The curves going sharply through it will verify the prefixed conditions. The local principality of the strict transform on \( X \) of generic curves going through this cluster follows from the condition (ii) together with [5.1].

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