Quantum $\kappa$-deformations of D=4 relativistic supersymmetries

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Abstract. We describe the quantum $\kappa$-deformation of super-Poincaré algebra, with fundamental mass-like deformation parameter $\kappa$. We shall describe the result in graded bicrossproduct basis, with classical Lorentz superalgebra sector which includes half of the supercharges.

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Introduction

In order to define the quantum deformations one should consider the deformed Lie algebra $A = U_\zeta(\mathfrak{g})$ as an algebraic sector of the Hopf algebra $H = (A, \Delta, S, \varepsilon)$, where $A$ denotes associative algebra, the coproduct $\Delta (\Delta : A \to A \otimes A)$ describes the coalgebra structure, $S$ defines the coinverse (antipode) $(S : A \to A)$ and the counit $\varepsilon$ is a complex functional on $A$ ($\varepsilon : A \to C$) (see e.g. [1]). In classical Hopf-Lie algebra with Lie algebra basis $I_i \in \mathfrak{g}$ where $[I_i, I_j] = c_{ij}^k I_k$ the generators $I_i$ are endowed with the primitive (Abelian) coproducts $\Delta^{(0)}(I_i)$

$$I_i \rightarrow \Delta^{(0)}(I_i) = I_i \otimes 1 + 1 \otimes I_i$$

(1)

Following Drinfeld [2] the quantum symmetries in deformed Lie-algebraic framework are described by noncocommutative Hopf algebras, with nonsymmetric coproducts

$$\tau(\Delta(I_i)) \neq \Delta(I_i)$$

(2)

where $\tau$ denotes the flip operator ($\tau(a \otimes b) = b \otimes a$). If the coproduct is symmetric i.e. $\tau(\Delta(I_i)) = \Delta(I_i)$ we deal with the "fake" quantum deformations, which can be obtained by nonlinear transformation of the classical Lie algebra generators: $I_i \rightarrow J_i = J_i(\overrightarrow{T}) (\overrightarrow{T} = (I_1 \ldots I_N; N = \dim \mathfrak{g})$. The coproduct for such nonlinearly realized classical symmetry in arbitrary basis remains symmetric.

In our note we shall discuss the quantum relativistic supersymmetries characterized by mass-like deformation parameter (so-called $\kappa$-deformations). The general quantum deformations of supersymmetries are described by the noncocommutative $Z_2$-graded Hopf superalgebras (see e.g. [3, 4]. We recall that the postulates for the Hopf superalgebra

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\( \tilde{H} = (\tilde{A}, \tilde{\Delta}, \tilde{S}, \tilde{\varepsilon}) \) are obtained from the ones for Hopf algebra \( H = (A, \Delta, S, \varepsilon) \) by introducing the sign factors in accordance with the rules of supermathematics (see e.g. [1]). In particular one introduces \( Z_2 \)-graded super-Lie bracket \( [a, b] = ab = (-1)_{\text{grad}a} grad^b ba \) (grad \( a = 0 \) for bosonic (even) \( a \), grad \( a = 1 \) for fermionic (odd) \( a \)) and the multiplication rule of \( Z_2 \)-graded tensor products \( (a \otimes b)(c \otimes d) = (-1)_{\text{grad}b} grad^c(ac \otimes bd) \). An important class of quantum supersymmetries is generated from classical Hopf-Lie superextension of standardHopf-Lie superalgebra \( (\tilde{H}(0) = (\tilde{A}(0), \tilde{\Delta}(0), \tilde{S}(0), \tilde{\varepsilon}(0)) \) by Drinfeld twist \( \tilde{F} \) which is an even element of superalgebra \( \tilde{A}(0) \otimes \tilde{A}(0) \) \( (\tilde{F} = \Sigma \tilde{F}(1) \otimes \tilde{F}(2)) \), where \( \text{grad} \tilde{F}(1) + \text{grad} \tilde{F}(2) = 0 \) or \( 2 \). In twisted Hopf-Lie superalgebras we modify only the classical formulae for coproducts and antipodes.

Below we shall present the \( \kappa \)-deformation of superPoincaré algebra which is the superextension of standard \( \kappa \)-deformation of Poincaré algebra \([5, 6]\). We shall describe this deformation in bicrossproduct basis with classical Lorentz algebra sector \([6]\), what is suitable for the supersymmetrization of recently studied Doubly Special Relativity framework (see e.g. \([7]\)).

The example of quantum-deformed SUSY: superextension of standard \( \kappa \)-deformation of Poincaré algebra

Let us recall the \( \kappa \)-deformed Poincaré algebra \([5]\) in bicrossproduct basis \([6]\):

\[
\mathcal{P}_4 = O(3, 1) \times T_4 \xrightarrow{\kappa_{\times}} \mathcal{P}_4^{\kappa} = O(3, 1) \triangleright T_4^{\kappa}
\]

where \( T_4^{\kappa} \) is a Hopf algebra of \( \kappa \)-deformed Abelian translation generators \(([P_\mu, P_\nu] = 0)\)

\[
\Delta P_i = P_i \otimes 1 + 1 \otimes P_i \xrightarrow{\kappa_{\times}} P_i \otimes e^{-\frac{P_i}{\kappa}} + 1 \otimes P_i
\]

with the classical coproduct \( \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \) unchanged. The only cross-product relations which are modified look as follows \((M_{\mu \nu} = (M_i = \frac{1}{2} \varepsilon_{ijk} M_{jk}, N_i = M_{i0}))\)

\[
[N_i, P_j] = i \delta_{ij} P_0 \xrightarrow{\kappa_{\times}} [N_i, P_j] = i \delta_{ij} \frac{\kappa}{2} (1 - e^{-\frac{P_i}{\kappa}}) + \frac{1}{2 \kappa} \vec{P}^2 + \frac{1}{\kappa} P_i P_j
\]

We introduce the following modified coproducts defining \( \kappa \)-deformed cross-coproduct

\[
\Delta (N_i) = N_i \otimes 1 + e^{-\frac{P_i}{\kappa}} \otimes N_i + \frac{1}{\kappa} \varepsilon_{ijk} P_j \otimes M_k.
\]

The relations \((4, 6)\) are the only deformed relations of \( \kappa \)-deformed D=4 Poincaré Hopf-Lie algebra.

The D=4 superPoincaré algebra is obtained by adding two complex Weyl supercharges \( Q_\alpha \) (chiral) and its Hermitean conjugates \( \overline{Q}_\alpha = Q_\alpha \) (antichiral), satisfying the relations

\[
\{Q_\alpha, \overline{Q}_\beta\} = 2 (\sigma^\mu P_\mu)_{\alpha \beta}, \quad \{Q_\alpha, \overline{Q}_\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0
\]
\[
\begin{align*}
\{M_{\alpha\beta}, Q_\gamma\} &= \varepsilon_{\alpha\gamma} Q_\beta - \varepsilon_{\beta\gamma} Q_\alpha \quad \{M_{\alpha\beta}, \overline{Q}_\gamma\} = 0 \quad (8) \\
\{M_{\alpha\beta}, \overline{Q}_\gamma\} &= \varepsilon_{\alpha\gamma} Q_\beta - \varepsilon_{\beta\gamma} Q_\alpha \quad \{M_{\alpha\beta}, Q_\gamma\} = 0, \quad (9) \\
[P_\mu, Q_\alpha] &= [P_\mu, \overline{Q}_\alpha] = 0 \quad (10)
\end{align*}
\]

The D=4 Lorentz algebra can be described as \(O(3,1) = SL(2;c) \oplus SL(2;c)\) with the generators \(M_{\alpha\beta} \sim M_i + iN_i \in SL(2;c), \overline{M}_{\alpha\beta} = M_i - iN_i \in SL(2;c)\).

Let us introduce the following graded bicrossproduct description of superPoincaré algebra
\[
\mathcal{P}_{4;1} = (SL(2;c) \oplus SL(2;c) \oplus T_{0;2}) \rtimes T_{4;2} \quad (11)
\]
where \(T_{0;2} = \overline{Q}_\alpha\) and \(T_{4;2} = (P_\mu, Q_\alpha)\) are graded Abelian sub-superalgebras. The \(\kappa\)-deformation of the relation (11) leads to \(\kappa\)-deformed bicrossproduct structure [8]
\[
\mathcal{P}_{4;1}^\kappa = (SL(2;c) \oplus SL(2;c) \oplus T_{0;2} \ltimes T_{4;2}) \quad (12)
\]
described as follows:

i) We introduce the deformed coproducts for supercharges
\[
\Delta^{(0)} Q_\alpha = Q_\alpha \otimes 1 + 1 \otimes Q_\alpha \quad \overline{\Delta} Q_\alpha = Q_\alpha \otimes e^{-\frac{\kappa}{2\kappa}} + 1 \otimes Q_\alpha. \quad (13)
\]
The relation (13) together with the formulae (4) and (10) define the \(\kappa\)-deformed graded-Abelian chiral Hopf superalgebra \(T_{4;2}\).

ii) Besides the commutator (5) the following cross-product relations of \(\mathcal{P}_{4;1}\) in (11) are \(\kappa\)-deformed
\[
\{Q_\alpha, Q_\beta\} = 2(\sigma^\mu P_\mu)_{\alpha\beta} \quad \overline{\{Q_\alpha, Q_\beta\}} = 4\kappa \delta_{\alpha\beta} \sinh \frac{P_0}{2\kappa} - 2e^{\frac{\kappa}{2\kappa}}(p_i \sigma_i)_{\alpha\beta} \quad (14)
\]
\[
[N_i, Q_\alpha] = i \frac{1}{2} (\sigma_i)_{\alpha\beta} Q_\beta \quad \overline{\{N_i, Q_\alpha\}} = i \frac{1}{2} e^{-\frac{\kappa}{2\kappa}} (\sigma_i)_{\alpha\beta} Q_\alpha + \frac{1}{2\kappa} \epsilon_{ijk} P_j (\sigma_k)_{\alpha\beta} Q_\beta \quad (15)
\]

iii) The relations (6), defining coalgebraic part of the bicrossproduct are extended supersymmetrically as follows:
\[
\Delta N_i \sim \overline{\Delta} N_i = \Delta N_i - \frac{i}{4\kappa} (\sigma_i)_{\alpha\beta} Q_\alpha \otimes e^{\frac{\kappa}{2\kappa}} Q_\beta. \quad (16)
\]
The formulae (6) and (14, 15) describe the deformed part of the \(\mathcal{P}_{4;1}^\kappa\) algebra, and the deformed coalgebraic part is described by the relations (4), (13) and (16).

Firstly standard \(\kappa\)-deformation of Poincaré algebra has been obtained by so-called quantum AdS contraction of Drinfeld-Jimbo (DJ) \(q\)-deformation \(U_q(O(3,2))\) of D=4 AdS Lie-Hopf algebra [5]. We assume that in quantum AdS contraction procedure \(q\) depends on \(R\) in accordance with the asymptotic formula \(q(R) = 1 + (\kappa R)^{-1} + \mathcal{O}((\kappa R)^{-2})\),
and \( \lim_{R \to \infty} U_{q(R)}(O(3,2)) = U_\kappa(P_4) \) provides the \( \kappa \)-deformed Poincaré algebra. Similarly if we consider \( q \)-deformed DJ superalgebra \( U_q(OSp(1|4)) \) one obtains \( [9] \)

\[
\lim_{R \to \infty} U_{q(R)}(OSp(1|4)) = U_\kappa(P_{4;1}). \tag{17}
\]

In order to describe the deformation \( [17] \) in the bicrossproduct basis one has to introduce the nonlinear change of basis.

It should be pointed out that recently there was obtained a two-parameter Jordanian twist quantization \( U_{\xi_1,\xi_2}(OSp(1|4)) \) of D=4 AdS superalgebra \( [10] \). It was also shown \( [10] \) that by putting \( \xi_1 = \xi_2 = \frac{1}{\kappa R} \) and performing the quantum contraction procedure

\[
\lim_{R \to \infty} U_{\xi_1,\xi_2}(OSp(1|4)) \bigg|_{\xi_1=\frac{1}{\kappa R}, \xi_2=\frac{1}{\kappa R}} = U_{\kappa}^{LC}(P_{4;1}) \tag{18}
\]

one gets the supersymmetric extension of so-called light-cone \( \kappa \)-deformation of Poincaré algebra (see e.g. \([11, 12]\))

**Final remarks**

It is known that new unified models of fundamental interactions are supersymmetric (e.g. supergravities, superstrings, \( M \)-theory). If the quantization of such theories generate the noncommutative structure of underlying spaces and superspaces, they also require quantum supersymmetries. In this note we presented more explicitly only one example of quantum supersymmetry, leading to Lie-algebraic structure of underlying noncommutative Minkowski spaces and superspaces which can be introduced as quantum representation spaces \( [13] \). Finally we add here that exists also simpler way of deforming D=4 supersymmetries - by the twist function with a carrier spanned by the fourmomenta and supercharges (see e.g. \([14]\)) - but it is outside of the scope of this short presentation.

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