A MODEL STRUCTURE FOR LOCALLY CONSTANT FACTORIZATION ALGEBRAS

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Abstract

Several model structures related to the homotopy theory of locally constant factorization algebras are constructed. This answers a question raised by D. Calaque in his habilitation thesis. Our methods also solve a problem related to cosheafification and factorization algebras identified by O. Gwilliam - K. Rejzner in the locally constant case.

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1 Introduction

One of the most ubiquitous family of operads is that of $E_n$. The first incarnation of them is as configurations of n-dimensional disks into a fixed bigger one [31] so it is not surprising

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that they play a relevant role in the study of manifolds. As envisioned in [1] one can in fact
define the homology of a manifold with coefficients in an $E_n$-algebra. This is a smooth, not just
homotopical manifold invariant known as factorization homology [2] or chiral homology [28].
This homology admits a wider class of coefficient systems known as factorization algebras
[4, 5] which are the topological analogues of Beilinson-Drinfeld chiral algebras. Algebras over
the $E_n$ operad are examples of these, which are local in a sense, since n-disks can be interpreted
as local models for n-manifolds [16, 20]. Other kinds of factorization algebras take into account
the global geometry of the manifold where they are defined.

It is often difficult to recognize $E_n$-algebras in nature. For instance, it took several years to
solve the Deligne conjecture, which asserts that the complex computing Hochschild cohomol-
ogy admits an $E_2$-action. Factorization algebras enter the picture to overcome this issue, they
are much easier to recognize, see [21]. This simple fact has proved useful in applications, with
striking consequences such as Costello-Gwilliam’s work on perturbative quantum field theo-
ries [16], Calaque-Scheimbauer’s construction of a fully extended field theory without using
the cobordism hypothesis [11], or Lurie’s non-Abelian Poincaré Duality [28].

Being more precise, factorization algebras are certain algebraic structures which satisfy
(homotopical) codescent conditions. The underlying algebraic structure is simple to describe
and receives the name of prefactorization algebra.

**Definition 1.1.** A prefactorization algebra $A$ over a space $X$ valued in chain complexes is given
by the following data:

- a chain complex $A(U)$ for each open $U$ in $X$,
- a chain map $A(U) \to A(V)$ for any inclusion of open subsets $U \hookrightarrow V$ in $X$,
- a product map $A(U_1) \otimes \cdots \otimes A(U_m) \to A(W)$ for each disjoint inclusion of open subsets
  $U_1 \sqcup \cdots \sqcup U_m \hookrightarrow W$ in $X$,

subject to associativity, unitality and equivariant conditions.

The aforementioned data can be seen as a precosheaf over $X$ with additional multiplication
maps. Prefactorization algebras admit an operadic description in terms of a discrete operad
$D_X$ (see Definition 2.1). The codescent conditions on a factorization algebra (Definition 2.2)
are designed to provide a local to global principle which accounts for multilocal data, i.e.
simultaneous information around finite families of points. Manifold calculus [8] showed how
important is this principle, since apparently global objects like embedding spaces are multilocal
in this sense.

Observe that, since prefactorization algebras are operadic algebras, its homotopy theory
is well understood, as they form a model category in the sense of Quillen (e.g. [19, 39]).
On the other hand, asking this additional local to global property on prefactorization algebras
obstructs the construction of a model category presenting the homotopy theory of factorization
algebras. In fact, it was a question raised by Calaque in his habilitation thesis [10] if such
a model category exists and to our knowledge no satisfactory answer has been provided so
far. Factorization algebras do form a relative category, and this is relevant from a theoretical
viewpoint, but the lack of a model category presentation seriously complicates computations.

We should comment that there are proposals to this respect. For instance, in [33, Sub-
section 9.7] and [38, Example 5.14], it is claimed without proof that a natural left Bousfield
localization of the projective model on prefactorization algebras should work. We doubt that
a left Bousfield localization could serve to force codescent conditions; indeed, in the present
work we encode codescent in terms of a right Bousfield localization. Our approach coincides with the expected behaviour at \( \infty \)-categorical level and the other proposals do not: cosheaves should constitute a coreflective subcategory of precosheaves and not a reflective subcategory, for instance (homotopy) colimits of cosheaves are computed as (homotopy) colimits of the underlying precosheaves and this is not the case for (homotopy) limits. Moreover, note that our approach also produces weak monadicity for factorization algebras (see Definition 2.2) with a right Bousfield localization, while the comment just below [33, Proposition 9.6.1] claims that this property may be found by a suitable left Bousfield localization, which was not provided.

By dual analogy with sheaves, one should recognize two main sources of problems to achieve this goal. On the one hand, cosheafification, as a formal machine to force precosheaves to satisfy a local to global principle, is much more elusive than sheafification. Delving into [34], one can find that usual categories such as sets or Grothendieck abelian categories do permit cosheafification, but we only know its existence by means of abstract adjoint functor theorems, and hence there is no manageable expression for it. On the other hand, one must ensure that, when existing, cosheafification respects the additional algebraic structure that a prefactorization algebra has, and this is quite unlikely. This problem was actually identified by Gwilliam-Rejzner in [22, Remark 2.33].

Assuming that factorization algebras are blind to the size of discs (Definition 2.2), we find a model category presenting the homotopy theory of factorization algebras, answering in this way Calaque’s question. In fact, as a byproduct of the construction, we solve Gwilliam-Rejzner problem for locally constant factorization algebras. The most remarkable achievement of this paper, Theorem 6.14, subsumes answers for both problems.

**Theorem A.** The category of prefactorization algebras over a smooth manifold valued on a suitable symmetric monoidal model category, for instance simplicial sets or chain complexes over a field of characteristic zero, admits a model structure such that the bifibrant objects are the (projectively bifibrant) locally constant factorization algebras and the equivalences between them are just the objectwise equivalences.

It is worth noting that we also explore what happens with less hypotheses on the base symmetric monoidal model category. In such cases we obtain left semimodel categories instead of complete model categories. This subtlety comes from the usage of left Bousfield localizations at a set of maps in the absence of left properness (see [14, 38]).

Along the way, we have found a variety of different Quillen equivalent model structures. The more remarkable one presents the homotopy theory of what we have called enriched factorization algebras (Definition 2.13 and Theorem 5.5). They should be seen as factorization algebras whose algebraic structure is sensible to the topology of embedding spaces, somehow connecting with the ideas of [16, Section 6.3]. Their introduction and study is the core of this work. For them, we construct an explicit functorial cosheafification which forces (homotopical) codescent with respect to Weiss covers preserving all the algebraic structure. This construction is important by itself, but also because it is applied to construct cosheafifications for usual prefactorization algebras in Proposition 6.13.

Our method adheres to the following principle: factorization homology produces fully extended Topological Quantum Field Theories, [3, 11]. Indeed, we construct enriched factorization algebras via a variant of factorization homology, see Section 3 and more concretely Proposition 3.3. For this purpose, we need a technical lemma about factorization homology which appears without proof in [4], see Remark 3.1. Our approach replaces most of their conditions on the operads by a Weiss codescent property for embedding spaces from a finite disjoint union
of discs (Lemma 2.16). All the technicalities arise when reducing every computation to what happens on discs by means of the mentioned Weiss codescent.

Mostly for simplicity in the exposition, we focus on smooth manifolds without boundary, although our methods do apply to much more general settings. Remarkably, they work for smooth manifolds with boundary, conically smooth stratified manifolds, treated in [5], or topological manifolds.

Outline: Let us summarize the content of this paper.

- In Section 2, there is a reminder of notions related to factorization algebras as well as fundamental lemmas about codescent properties of embedding spaces. We include terminology for continuous or enriched variants of factorization algebras and for a weaker notion, that of Weiss algebras.

- Factorization homology (with context) is presented within Section 3, which culminates with Theorem 3.8. This result gives a proof of [4, Theorem 2.15] in the setting of smooth manifolds only relying on the fundamental codescent property stated in Lemma 2.16 and the existence of good Weiss covers (Lemma 2.15).

- Section 4 presents a summary of a general model categorical construction developed in [13] based on a generalization of Bousfield-Friedlander’s Theorem [9].

- Enriched factorization (Weiss) algebras find on Section 5 model categories presenting their homotopy theory, see Theorem 5.5 (resp. Theorem 5.2). These models are constructed via the material in Section 4.

- The analogous model categories for locally constant factorization (Weiss) algebras, Theorem 6.14 (resp. Theorem 6.10 plus Proposition 6.13), are the main goal of Section 6. They are combinations of a left Bousfield localization (forcing local constancy) and a right Bousfield localization (forcing codescent conditions).

- Section 7 is devoted to the discussion of different settings where our results hold. The exposition highlights what ingredients are required, separating them from general abstract arguments. Remarkably, our methods are suitable for topological manifolds or conically smooth stratified manifolds (developed within [4, 5]).

- Finally, Appendix A contains several technical lemmas which are employed in Section 3 and Appendix B contains a required generalization of [4, Lemma 2.16] to the recognition of bifibrant objects in the factorization model (see Lemma 6.11).

Conventions:

- By homotopy cosmos we mean a closed symmetric monoidal model category $\mathcal{V}$ equipped with a lax symmetric monoidal left Quillen functor $\text{Set}_\Delta \to \mathcal{V}$ where $\text{Set}_\Delta$ is considered with the Kan-Quillen model. We also ask that the unit in $\mathcal{V}$ is cofibrant (slight assumption by [32]).

- We fix a homotopy cosmos $\mathcal{V}$ and consider that everything is $\mathcal{V}$-enriched. Otherwise, it will be explicitly specified.

- For two maps $f$ and $g$ in $\mathcal{V}$ sharing their source, we denote by $f \Box g$ its pushout-product. Hence, $f^{\square n}$ denotes the iterated pushout-product of $f$. 


• Fixed a set $O$, recall from [39, Section 3] the definition of the category of colored corollas $\Sigma_O \times O$. For objects in $\Sigma_O$ we will employ the notation $a = [a_1, \ldots, a_m]$ and $a \boxplus b$ for the obvious concatenation of objects. For colored corollas, we use the notation $[a]_R$.

• The category of $O$-symmetric sequences is $[(\Sigma_O \times O)^{op}, \mathcal{V}]$ and a $O$-colored operad is a $o$-monoid in $O$-symmetric sequences, see [33] or [39]. For an operad $O$, we refer to its $\mathcal{V}$-object of operations with inputs $a$ and output $b$ by $O[a]_R$.

• $\text{Alg}_O(\mathcal{V})$ (or simply $\text{Alg}_O$) denotes the category of $O$-algebras in $\mathcal{V}$. When required, $\text{Alg}_O(\mathcal{V})$ carries the projective model structure [39, Section 6], e.g. proj-cofibrant algebras refer to cofibrant objects in this model.

• Let $O$ be an operad. We will denote by $\overline{O}$ the underlying category of $O$. In other words, $\overline{O} \hookrightarrow O$ is the suboperad on unary operations. The functor induced between algebras $\text{Alg}_O \to \text{Alg}_{\overline{O}}$ is denoted by $A \mapsto \overline{A}$.

• $\emptyset$ and $1$ will refer to the initial and the terminal objects, always interpreted in the corresponding context.

• We say that a functor, between homotopical categories, is homotopical if it preserves equivalences.

• Given a cospan of categories $B \xrightarrow{f} C \leftarrow A$, we denote $f \downarrow g$ the ordinary slice category associated to it.

• All manifolds are assumed to be smooth except in Section 7.

2 Preliminaries

Let us collect necessary notions and fix notation. We fix once and for all a number $n \in \mathbb{N}$.

Factorization algebras We recall the definition of factorization algebras on $X$, for $X$ a smooth $n$-manifold. The material presented here about factorization algebras can be found in [16].

Definition 2.1. The operad in sets of disjoint open subsets in $X$, $M_X$, is the operad with colors the open subsets of $X$ and operations

$$M_X \left[ \{U_i\}_{i} \right] \mathcal{V} = \begin{cases} 1 & \text{if } \bigsqcup U_i \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

The full suboperad $D_X$ of $M_X$ is the one spanned by open subsets diffeomorphic to finite disjoint unions of $n$-discs. $E_X$ denotes the full suboperad of $M_X$ spanned by open subsets diffeomorphic to $\mathbb{D}^n$. 

Definition 2.2. Let $Y$ be a space and $\mathcal{A}$ an $M_X$-algebra.

• A Weiss cover of $Y$ is a family of open subsets $(U_i)_{i \in I}$ of $Y$ such that any non-empty finite subset $S \subset Y$ is contained in one of them, $S \subset U_i$. 

A is a *Weiss algebra* if it satisfies homotopical codescent with respect to Weiss covers, i.e. for any Weiss cover \((U_i)_{i \in I}\) of \(U \subseteq X\) open, the map

\[
\underset{S \subseteq I}{\hocolim} A(U_S) \to A(U)
\]

is an equivalence, where the homotopy colimit runs over finite subsets \(S\) of \(I\) and we adopt the standard notation \(U_S = \bigcap_{i \in S} U_i\).

A is a *factorization algebra* on \(X\) if it is a Weiss algebra and satisfies weak monadicity, i.e. for any finite collection of disjoint open subsets \(\{V_j\}_j\) of \(X\), the structure map

\[
\bigotimes_j A(V_j) \to A\left( \bigsqcup_j V_j \right)
\]

is an equivalence.

The full subcategory of \(M_X\)-algebras spanned by factorization algebras on \(X\) is denoted \(\text{Fac}_X(V)\). It is equipped with the class of colorwise equivalences between factorization algebras.

A is *locally constant* if for any inclusion \(U \subset V\) of open subsets diffeomorphic to \(D^n\), the associated map \(A(U) \to A(V)\) is an equivalence. In other words, \(A\) sends unary operations in \(E_X\) to equivalences. We denote by \(\text{Fac}_X^{lc}(V)\) the category of locally constant factorization algebras on \(X\).

**Remark 2.3.** Čech type presentations of the homotopical codescent condition were given in [16]. Moreover, in [20, Section 4], Ginot presents a Čech type diagram which combines weak monadicity and homotopical codescent.

**Remark 2.4.** The definition of (locally constant) factorization algebra can be replicated for full suboperads of \(M_X\) whose colors are a family of open subsets \(\mathcal{O}\) in \(X\). One just has to ensure that \(\mathcal{O}\) is closed under taking finite disjoint unions (and contains inclusions of one disc into another for the locally constancy condition). For example, \(\mathcal{O}\) could be the family of subsets homeomorphic to finite disjoint unions of discs. We employ the notation \(\text{Fac}_\mathcal{O}(V)\) for the resulting category, e.g \(\text{Fac}_{D^*}(V)\). See [10, Section 2.1.2].

We will also employ the notation \(\text{Fac}_{E_X}(V)\). To make it meaningful note that an \(E_X\)-algebra \(A\) can be extended to an \(D_X\)-algebra \(A^\mathcal{O}\) by setting

\[
A^\mathcal{O}(U \sqcup V) = A(U) \otimes A(V).
\]

Then, \(\text{Fac}_{E_X}(V)\) refers to the full subcategory of \(E_X\)-algebras which belong to \(\text{Fac}_{D_X}(V)\) when seen as \(D_X\)-algebras.

(Locally constant) factorization algebras have the same structure as \(M_X\)-algebras, but they enjoy more properties. In fact, these properties permit the following simplifications.

**Proposition 2.5.** The inclusions \(E_X \hookrightarrow D_X \hookrightarrow M_X\) of operads induce equivalences of homotopy categories

\[
\text{Ho Fac}_X(V) \sim \text{Ho Fac}_{D_X}(V) \sim \text{Ho Fac}_{E_X}(V).
\]

The equivalences also hold for locally constant factorization algebras.

**Proof.** Using the homotopical codescent condition together with the fact that every smooth \(n\)-manifold admits a Weiss cover by discs ([8, Proposition 2.10]) it is routine to prove the first equivalence. See [10, Theorem 2.1.9] for the precise construction. Due to weak monadicity, the second functor is an equivalence.
**Enriched factorization algebras** A little detour is in place to give the notion of enriched factorization algebras. Essentially, we need the continuous counterpart of the inclusion of operads

\[ E_X \hookrightarrow D_X \hookrightarrow M_X. \]

We will use a convenient category of topological spaces \( \text{Top} \) in what follows. To be concrete, \( \text{Top} \) is the category of compactly generated, weak Hausdorff topological spaces with continuous maps between them.

**Definition 2.6.** Let \( \text{Mfld} \) denote the symmetric monoidal \( \text{Top} \)-category on smooth \( n \)-manifolds and smooth embeddings with disjoint union as monoidal structure (see [25, Sections 6-7]). In order to avoid set theoretic difficulties later on, we consider every manifold embedded in \( \mathbb{R}^\infty \).

We denote by \( \text{Discs} \) the full symmetric monoidal \( \text{Top} \)-subcategory of \( \text{Mfld} \) spanned by finite disjoint unions of \( n \)-discs, which inherits the monoidal structure.

We denote by \( E \) the full \( \text{Top} \)-suboperad of \( \text{Discs} \) spanned by the color \( \{ D^n \} \).

**Notation 2.7.** We follow the convention of using decorated capital letters to refer to suitably enriched notions and plain capital letters for the discrete analogues, e.g. \( \text{Mfld} \) versus \( \text{Mfld} \) or \( \text{Discs} \) versus \( \text{Discs} \). This can be seen as the higher categorical version versus its underlying ordinary categorical version.

**Remark 2.8.** We are interested in the algebraic structures that the above operads (and variants of them) encode. For example, one should notice that \( E \)-algebras are just \( E_n \)-algebras with additional orientation data. More precisely, they present the framed little \( n \)-disc algebras appearing in [35].

Fix a \( n \)-manifold \( X \). We want to consider manifolds embedded into \( X \).

**Definition 2.9.** The \( \text{Top} \)-category of manifolds with context \( X \), \( \text{Mfld}_{/X} \), has as objects pairs \( \nu_N: N \hookrightarrow X \) where \( N \) is an object of \( \text{Mfld} \), \( \nu_N \in \text{Mfld}(N, X) \) is an embedding, and mapping spaces fitting into homotopy pullback squares

\[
\begin{array}{ccc}
\text{Mfld}_{/X}(\nu_T; \nu_N) & \longrightarrow & \text{Mfld}(T, N) \\
\downarrow & & \downarrow_{(\nu_N)_*} \\
\mathbb{1} & \longrightarrow & \text{Mfld}(T, X).
\end{array}
\]

Composition and units are induced from those of \( \text{Mfld} \). We often abuse notation and refer to an object in \( \text{Mfld}_{/X} \) by its underlying manifold.

**Remark 2.10.** The category \( \text{Mfld}_{/X} \) has to be seen as a homotopical version of the ordinary slice construction. One has to be careful specifying what it means by homotopy pullback above. Providing a strict composition law and strict units for the mapping spaces appearing in a homotopy pullback is a technical question that can be solved as in [25, Section 5].

The main drawback of \( \text{Mfld}_{/X} \) is that it does not inherit the symmetric monoidal structure of \( \text{Mfld} \). However, it can be enhanced to an operad.

**Definition 2.11.** The \( \text{Top} \)-operad \( M_X \) has the same objects as \( \text{Mfld}_{/X} \) and multimapping spaces fitting into homotopy pullback squares

\[
\begin{array}{ccc}
M_X \left[ \begin{array}{c} T_i \\ N \end{array} \right] & \longrightarrow & \prod_i \text{Mfld}_{/X} \left[ \begin{array}{c} T_i \\ N \end{array} \right] \\
\downarrow & & \downarrow_{\text{forget}} \\
\text{Mfld} \left[ \begin{array}{c} \bigcup_i T_i \\ N \end{array} \right] & \longrightarrow & \prod_i \text{Mfld} \left[ \begin{array}{c} T_i \\ N \end{array} \right].
\end{array}
\]

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Composition and units are induced by those of \( \mathcal{M}_{\text{fld}} \) and \( \mathcal{M}_{\text{fld}}/X \).

Analogously, one can define the corresponding notions:

| Without context | \( \mathcal{M}_{\text{fld}} \) sm category | \( \mathcal{D}_{\text{discs}} \) sm category | \( \mathcal{E} \) operad |
|-----------------|---------------------------------|---------------------------------|-----------------|
| With context    | \( \mathcal{M}_X \) operad      | \( \mathcal{D}_X \) operad      | \( \mathcal{E}_X \) operad |

**Remark 2.12.** The continuous and the discrete variants of the operads presented until now are related by the obvious diagram of Top-operads

\[
\begin{align*}
\mathcal{E}_X & \hookrightarrow \mathcal{D}_X \hookrightarrow \mathcal{M}_X \\
\downarrow & \quad \downarrow \quad \downarrow \\
\mathcal{E}_X & \hookrightarrow \mathcal{D}_X \hookrightarrow \mathcal{M}_X .
\end{align*}
\]

Let us turn to the enriched analogue of factorization algebras.

**Definition 2.13.** An *enriched Weiss algebra* on \( X \) is an \( \mathcal{M}_X \)-algebra satisfying homotopical codescent with respect to Weiss covers (see Definition 2.2). An *enriched factorization algebra* is an enriched Weiss algebra which satisfies weak monadicity.

**Remark 2.14.** We prefer to split the axioms for enriched factorization algebras since the weak monadicity axiom for \( \mathcal{M}_X \)-algebras is subtle and it will enjoy a distinguished treatment in the main results. It can be stated as follows: For any pair of embeddings \( \nu_Z : Z \hookrightarrow X \) and \( \nu_Y : Y \hookrightarrow X \) with disjoint images in \( X \), \( (\nu_Z, \nu_Y) : \bigcup Z \cap Y \hookrightarrow X \) is again an embedding. Thus, \( \bigcup Z \cap Y \) is a color of \( \mathcal{M}_X \) which comes with a preferred operation \( m_{Z,Y} \in \mathcal{M}_X \left[ \begin{array}{c} Z,Y \\ \cup Z \cap Y \end{array} \right] \). An \( \mathcal{M}_X \)-algebra \( \mathcal{A} \) satisfies *weak monadicity* if for any pair of embeddings \( \nu_Z, \nu_Y \) into \( X \) with disjoint images, the associated morphism \( \mathcal{A}(m_{Z,Y}) \) is an equivalence.

The fundamental idea is that, in close analogy with Proposition 2.5, one can exploit the chain of inclusions of Top-operads

\[
\mathcal{E}_X \hookrightarrow \mathcal{D}_X \hookrightarrow \mathcal{M}_X
\]

to construct model categories presenting enriched factorization algebras on \( X \). Our approach fundamentally lies on two important results which permit reducing some assertions to their verification for discs embedded in \( X \). We mention them for future reference.

**Lemma 2.15.** Every (smooth) manifold admits a Weiss cover whose elements are finite disjoint unions of discs and their finite intersections remain so.

**Proof.** See [8, Proposition 2.10].

**Lemma 2.16.** Let \( T \) be a finite disjoint union of discs in \( \mathcal{M}_{\text{fld}}/X \), i.e. a color of \( \mathcal{D}_X \). Then, the Top-functor of embeddings \( \mathcal{M}_{\text{fld}}/X(T;\ast) \) is a Weiss cosheaf. That is, for any Weiss cover \( (U_i)_{i \in I} \) of \( Z \in \mathcal{M}_{\text{fld}}/X \), the canonical map

\[
\hocolim_{S \subseteq I} \mathcal{M}_{\text{fld}}/X(T;U_S) \rightarrow \mathcal{M}_{\text{fld}}/X(T;Z)
\]

is an equivalence.
Proof. The result holds if we drop the context $X$ by [8, Equation (10)], i.e.

$$
\operatorname{hocolim}_{S \subseteq I} \mathcal{Mfld}(T; U_S) \to \mathcal{Mfld}(T; Z)
$$

is an equivalence. The claim for embeddings with context $X$ follows by the homotopy pullback definition of $\mathcal{Mfld}_X(T; Z)$.

\[\square\]

Remark 2.17. Lemma 2.16 can be also be deduced from Lurie’s version of the Seifert-van Kampen theorem [28, Proposition A.3.1], which is a source of broad generalizations for this result, such as the following lemma.

Lemma 2.18. Let $T$ be a color in $\mathcal{D}_X$. Then, for any $Z \in \mathcal{Mfld}_X$, there is a canonical equivalence

$$
\operatorname{hocolim}_{V \in \mathcal{D}_Z} \mathcal{Mfld}_X(T; V) \to \mathcal{Mfld}_X(T; Z).
$$

Proof. Note that $\mathcal{Mfld}(T, Y)$ is homotopy equivalent to the total space of a principal bundle over $\operatorname{Conf}_t(Y)$, the configuration space of $t$-points in $Y$ with $t$ the number of discs in $T$. Hence, [28, Proposition A.3.1] applied to the subdiagram of open subsets of $Z$ which are diffeomorphic to finite disjoint unions of discs, i.e. $\mathcal{D}_Z$, gives rise to an equivalence of spaces

$$
\operatorname{hocolim}_{V \in \mathcal{D}_Z} \operatorname{Conf}_t(V) \to \operatorname{Conf}_t(Z),
$$

that, in turn, yields the corresponding equivalence for $\mathcal{Mfld}(T, Z)$. Passing to $\mathcal{Mfld}_X$ only requires a homotopy pullback, which respects homotopy colimits, like in Lemma 2.16.

\[\square\]

3 Factorization homology

In this section, we discuss a variant of factorization homology (see [1, 2, 25]) which introduces a manifold as context. Let us explain this assertion. Factorization homology should be seen as a homology theory for manifolds which uses $\mathcal{D}iscs$-algebras as coefficient systems. More concretely, given a $\mathcal{D}iscs$-algebra $A$ and a manifold $N$, factorization homology is defined by the derived tensor product

$$
\int_N A = \mathcal{Mfld}(i(\ast), N) \otimes_{\mathcal{D}iscs} A,
$$

where $i : \mathcal{D}iscs \hookrightarrow \mathcal{Mfld}$, which is the left derived functor of

$$
A \mapsto \mathcal{Mfld}(i(\ast), N) \otimes_{\mathcal{D}iscs} A = \int_{Y \in \mathcal{D}iscs} \mathcal{Mfld}(i(Y), N) \otimes A(Y).
$$

It is well known that factorization homology computes the derived left adjoint of the forgetful functor $i^* : \mathcal{A}lg_{\mathcal{Mfld}} \to \mathcal{A}lg_{\mathcal{D}iscs}$. See [4, Lemma 2.15].

Introducing a manifold $X$ as context means that we replace $\mathcal{Mfld}$ with $\mathcal{Mfld}_X$. More concretely, factorization homology with context $X$ is defined analogously for a $\mathcal{D}iscs$-algebra $A$ and $Z$ an object in $\mathcal{Mfld}_X$ as the derived tensor product

$$
\mathcal{Mfld}_X(i(\ast), Z) \otimes_{\mathcal{D}iscs_X} A.
$$
Remark 3.1. It is claimed in [4, Lemma 2.16], using a different language, that factorization homology with context $X$ computes the derived left adjoint of $i^*: \Alg_{M_X} \to \Alg_{D_X}$ by an argument similar to that given in the proof of [4, Lemma 2.15]. We feel that such argument is not fully complete as it stands. In fact, we doubt that as stated this result holds due to several reasons. For instance, the finality assumption [4, Lemma 2.16 (5)]: given objects $Z$ and $Z'$ in $M/X$, one may construct $Z \otimes Z'$ in $M$, but one cannot combine their structural maps into $X$ in a canonical way to obtain an object in $M/X$. Hence, the tensor product functor appearing in (5) exists, but $Z \otimes Z'$ is no longer in $M/X$ and hence $(M/X)/Z \otimes Z' \simeq M/Z \otimes Z'$ does not make sense. We are currently discussing these points with the authors.

Nevertheless, one could interpret their claim in two different, although equivalent, ways: the first one says that one can produce an algebra structure on the ordinary left Kan extension making it universal; the second one says that there should be an equivalence between the ordinary left Kan extension and the operadic left Kan extension. We show that this second perspective holds in the case of interest. Our strategy consists on an alternative computation showing that factorization homology with context $X$ computes the derived left adjoint of the forgetful functor $i^*: \Alg_{M_X} \to \Alg_{D_X}$ which essentially uses Lemma 2.16 and some usual filtrations. The main technical result is Proposition 3.3 whose proof is split in several lemmas.

We need to compute the adjoint pair $i^*: \Alg_{D_X} \rightleftarrows \Alg_{M_X}: i^*$. Our first simplification just notes that it can be factored into two adjoint pairs. Since the map $i: D_X \hookrightarrow M_X$ is fully-faithful and injective on colors, the factorization of $i^* \dashv i^*$ corresponds to the factorization of $i$ through the operad with colors $\text{col}(M_X)$, also denoted $D_X$, with multimapping spaces

$$D_X \left[\frac{T}{N}\right] = \begin{cases} M_X \left[\frac{T}{N}\right] & \text{when } T \in \text{col}(D_X)^{=1}, N \in \text{col}(D_X), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $i_2 \dashv i^*$ involves an extension/restriction of colors and an extension/restriction of operations. We will focus on the non-trivial one: the extension/restriction of operations.

In order to distinguish the composite pair and the extension/restriction of operations, we find useful to employ the $\circ$-product [33, Definition 3.3] to express the extension/restriction of operations:

$$M_X \circ_{D_X} \star: \Alg_{D_X} \rightleftarrows \Alg_{M_X}: U.$$ Analogously, one can produce the extension/restriction of unary operations and both constructions sit into a commutative square of adjoint pairs between categories of algebras

$$\begin{array}{cc}
\Alg_{D_X} & \Alg_{M_X} \\
\downarrow \tau & \downarrow \tau \\
\Alg_{D_X} & \Alg_{M_X}
\end{array}$$

Our computation of $L_i$ is based on an analysis of the natural transformation

$$\begin{array}{c}
M_X \circ_{D_X} \star \to M_X \circ_{D_X} \star \to M_X \circ_{D_X} \star \to M_X \circ_{D_X} \star A,
\end{array}$$

induced by the above commutative square, where $A$ is an $D_X$-algebra.
Remark 3.2. By inspection of the colimit defining the ◦-product,
\[
\left( \overline{M}_X \circ_{\mathcal{D}_X} \overline{A} \right)[Z] \cong \int_{Y \in \text{Discs}_X / X} \text{Mfld}_X(i(Y), Z) \otimes \Lambda(Y) \cong \text{Mfld}_X(i(\ast), Z) \circ_{\mathcal{D}_X / X} \Lambda.
\]
Hence, we are simply computing factorization homology with context $X$.

**Proposition 3.3.** The natural transformation
\[
\overline{M}_X \circ_{\mathcal{D}_X} \overline{A} \to \overline{M}_X \circ_{\mathcal{D}_X} \overline{A}
\]
is an equivalence for any proj-cofibrant $\mathcal{D}_X$-algebra $A$.

As mentioned before, we split the proof of Proposition 3.3 into several lemmas which correspond to the analysis of each step in the construction of a cellular projective $\mathcal{D}_X$-algebra, i.e. a cellular object in the projective model structure of $\mathcal{D}_X$-algebras.

**Lemma 3.4 (Free case).** The natural transformation in Proposition 3.3 evaluated on free $\mathcal{D}_X$-algebras is an equivalence.

**Proof.** Let $V$ be a cofibrant object in $\mathcal{V}$ and $Y$ be a color of $\mathcal{D}_X$. With these data, we can form the collection $V(Y)$ in $[\text{col}(M_X), \mathcal{V}]$, which is concentrated on the color $Y$. The free $\mathcal{D}_X$-algebra spanned by $V(Y)$ is
\[
\mathcal{D}_X \circ V(Y) = \bigsqcup_{r \geq 0} \mathcal{D}_X \left[ Y^{\otimes r} \right] \otimes V^{\otimes r}.
\]
Substituting $A$ by this free algebra, $\overline{M}_X \circ_{\mathcal{D}_X} \overline{A} \to \overline{M}_X \circ_{\mathcal{D}_X} \overline{A}$ becomes
\[
\bigsqcup_{r \geq 0} \overline{M}_X \circ_{\mathcal{D}_X} \mathcal{D}_X \left[ Y^{\otimes r} \right] \otimes V^{\otimes r} \to \bigsqcup_{r \geq 0} M_X \left[ Y^{\otimes r} \right] \otimes V^{\otimes r},
\]
where we have applied the commutation of the coproduct with the relative ◦-product on the left (which is possible since the operads involved have unary morphisms only) and the natural isomorphism $M_X \circ_{\mathcal{D}_X} \mathcal{D}_X \circ \ast \cong M_X \circ \ast$. In order to show that this map yields an equivalence, we analyze instead for each $r \geq 0$
\[
\overline{M}_X \circ_{\mathcal{D}_X} \mathcal{D}_X \left[ Y^{\otimes r} \right] \to M_X \left[ Y^{\otimes r} \right],
\]
which just boils down to be induced by operadic composition on $M_X$. This morphism evaluated on colors of $\mathcal{D}_X$ is an isomorphism, since $Y \in \text{col}(\mathcal{D}_X)$. In order to prove that it is an equivalence when evaluated on a general manifold $Z \in \text{col}(M_X)$, we pick a Weiss cover $(U_i)_{i \in I}$ on $Z$ as in Lemma 2.15. Then, we obtain a commutative square
\[
\begin{array}{ccc}
\text{hocolim}_{S \subseteq I} \left( \overline{M}_X \circ_{\mathcal{D}_X} \mathcal{D}_X \left[ Y^{\otimes r} \right] \right)[U_S] & \longrightarrow & \left( \overline{M}_X \circ_{\mathcal{D}_X} \mathcal{D}_X \left[ Y^{\otimes r} \right] \right)[Z] \\
\downarrow & & \\
\text{hocolim}_{S \subseteq I} M_X \left[ Y^{\otimes r} \right]_{U_S} & \longrightarrow & M_X \left[ Y^{\otimes r} \right]_Z
\end{array}
\]
\footnote{The symbol $\boxplus$ was introduced previously on conventions.}
We are interested on showing that the right vertical map is an equivalence, so, using 2 out of 3, it is enough to check that the other maps are equivalences. The left vertical map is an equivalence from evaluation on \( \text{col}(D_X) \). The lower horizontal arrow is an equivalence by Lemma 2.16. Checking our claim for the upper horizontal map requires delving into more details.

We use the coequalizer presentation of the \( \circ \)-product appearing in [33, Definition 3.3] to describe \( \overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m] \) as the coequalizer

\[
\begin{array}{c}
\bigcup_{T,N} \overline{M}_X [T] \otimes \overline{D}_X [N] \otimes D_X [\gamma^m] \\
\downarrow \\
\bigcup_T \overline{M}_X [T] \otimes D_X [\gamma^m] \\
\downarrow \text{coeq} \\
\overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m],
\end{array}
\]

where the actions of \( \overline{D}_X \) on \( \overline{M}_X \) and \( D_X \) are forced to coincide. Again Lemma 2.16 shows that replacing the coequalizer by its derived version, one has an equivalence between diagrams

\[
\begin{array}{c}
\text{hocolim}_{S \subseteq I} \bigcup_{T,N} \overline{M}_X [T] \otimes \overline{D}_X [N] \otimes D_X [\gamma^m] \\
\downarrow \\
\text{hocolim}_{S \subseteq I} \bigcup_T \overline{M}_X [T] \otimes D_X [\gamma^m] \\
\downarrow \text{hocoeq} \\
\text{hocolim}_{S \subseteq I} \left( \overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m] \right) [U_S] \\
\downarrow \text{hocoeq} \\
\left( \overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m] \right) [Z]
\end{array}
\]

Consequently, the right vertical map on the following commutative square is an equivalence since the rest of the maps are so:

\[
\begin{array}{c}
\text{hocolim}_{S \subseteq I} \left( \overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m] \right) [U_S] \\
\downarrow \\
\text{hocolim}_{S \subseteq I} M_X [\gamma^m] \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
\left( \overline{M}_X \circ_{\overline{D}_X} D_X [\gamma^m] \right) [Z] \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
M_X [\gamma^m] \\
\downarrow
\end{array}
\]

The free-algebra case is deduced from equivalence (1) just shown by taking \( \Sigma_r \)-coinvariants (which are homotopical in this situation) and coproducts. \( \Box \)
Lemma 3.5 (Pushout case). Consider that

\[ \mathcal{D}_X \circ W \longrightarrow A \]

\[ \mathcal{D}_X \circ V \longrightarrow B \]

is a pushout square in \( \text{Alg}_{\mathcal{D}_X} \), where \( A \) is proj-cofibrant and the left vertical arrow is of the form \( \mathcal{D}_X \circ j \) for a cofibration \( j : V \rightarrow W \). If the natural transformation in Proposition 3.3 evaluated on \( A \) is an equivalence, then it is an equivalence evaluated on \( B \).

Proof. The natural transformation evaluated on the pushout in the statement yields a commutative cube

\[ \overline{M}_X \circ W \longrightarrow \overline{M}_X \circ A \]

\[ \overline{M}_X \circ V \longrightarrow \overline{M}_X \circ B \]

We have checked in Lemma 3.4 that the diagonal arrows on the left are equivalences over free algebras. Assuming that the diagonal arrow for \( A \) is an equivalence, we now prove that the diagonal arrow for \( B \) too. Our strategy consists on providing a filtration for the right square which allows us to conclude this claim. The filtration before deriving is the content of Lemma A.2. The derived statement, assuming that \( A \) is proj-cofibrant and that we derive the extension of unary operations, just converts every strict notion into a homotopical one, e.g. the pushouts defining the filtrations become homotopy pushouts. Making use of such filtration, we must check that provided the equivalence at \( n-1 \), so is the map at \( n \). This verification is argued over the analogous cube of the one that appears in the proof of Lemma A.2 which corresponds to this particular case. Inspecting it, one deduces that the claim will follow if

\[ \overline{M}_X \circ (\mathcal{D}_X)_B \left[ \mathcal{Y}^{\oplus n} \right] \longrightarrow (\overline{M}_X)_{\mathcal{D}_X} \circ B \left[ \mathcal{Y}^{\oplus n} \right] \]

is an equivalence, by the same kind of argument given in Lemma 3.4.

To check that the above map is an equivalence, we use an inductive argument which lays on the filtration studied in Lemmas A.3 and A.4. Indexing the filtration by \( t \in \mathbb{N} \), the successive
step fits into a cubical diagram

\[
\begin{array}{ccc}
\mathcal{L}_{n,t-1} \xrightarrow{\sim} \mathcal{R}_{n,t-1} \\
\mathcal{L}_{n,t} \\
\mathcal{R}_{n,t}
\end{array}
\]

where the front square represents the square \( \gamma_{t-1} \Rightarrow \gamma_t \) in Lemma A.4 (with a suitable rotation) and the lateral squares are homotopy pushouts defining successive steps. By induction on \( t \in \mathbb{N} \), the claim is translated into showing if

\[
\overline{M}_X \overset{D_{X,A}}{\longrightarrow} \overline{M}_{X,M_{X} \circ D_X A} \overset{V^{\otimes t}}{\longrightarrow} \overline{M}_{X,M_{X} \circ D_X A} \overset{V^{\otimes t}}{\longrightarrow} \overline{M}_{X,M_{X} \circ D_X A} \overset{V^{\otimes t}}{\longrightarrow}
\]

is an equivalence on cofibrant objects with an \( \Sigma_t \)-action (not necessarily \( \Sigma_t \)-cofibrant). Since \( A \) is proj-cofibrant, we can also arrange a filtration of \( D_{X,A} \) (and of \( M_{X,M_{X} \circ D_X A} \)) by the methods described in [39, Section 5] which reduces this question to checking that the canonical map

\[
\overline{M}_X \overset{D_{X,A}}{\longrightarrow} \overline{M}_X \overset{V^{\otimes t}}{\longrightarrow} \overline{M}_X \overset{V^{\otimes t}}{\longrightarrow} \overline{M}_X \overset{V^{\otimes t}}{\longrightarrow}
\]

is an equivalence for any \( Y \in \text{col}(D_X)^{\times m} \). This was the strategy followed in Lemma 3.4 and in this case works by the same reasons.

\[ \square \]

\textbf{Remark 3.6.} A more detailed discussion on comparisons of enveloping operad constructions as the one applied in Lemma 3.5 will appear in [12].

\textbf{Lemma 3.7 (Transfinite case).} Let

\[ A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_{\alpha} \rightarrow \cdots \rightarrow \text{colim}_{\alpha \prec \lambda} A_{\alpha} = A_{\lambda} \]

be a \( \lambda \)-indexed diagram of proj-cofibrations in \( \text{Alg}_{D_X} \) which is colimit preserving and such that \( A_0 \) is proj-cofibrant. Then, if the natural transformation in Proposition 3.3 evaluated on \( A_\alpha \) is an equivalence for any \( \alpha < \lambda \), then it is an equivalence evaluated on \( A_\lambda \).

\textbf{Proof.} The \( \lambda \)-indexed diagram gives rise to a commutative diagram

\[
\begin{array}{cccccccc}
\overline{M}_X \overset{D_{X,A_0}}{\longrightarrow} \overline{M}_X \overset{D_{X,A_1}}{\longrightarrow} \overline{M}_X \overset{D_{X,A_2}}{\longrightarrow} \cdots & \overline{M}_X \overset{D_{X,A_\lambda}}{\longrightarrow} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\overline{M}_X \overset{D_{X,A_0}}{\longrightarrow} \overline{M}_X \overset{D_{X,A_1}}{\longrightarrow} \overline{M}_X \overset{D_{X,A_2}}{\longrightarrow} \cdots & \overline{M}_X \overset{D_{X,A_\lambda}}{\longrightarrow}
\end{array}
\]
and we have to prove that the right vertical map is an equivalence provided all vertical maps except it are equivalences. The horizontal chains can be taken to be towers of cofibrations whose first object is cofibrant. Therefore, the horizontal colimits are homotopical and the induced arrow on colimits an equivalence. One concludes the claim whenever the induced arrow on colimits coincides with the natural transformation

\[
\overline{M_X \circ D_x A} \longrightarrow M_X \circ D_x A.
\]

This fact is just a commutation of sequential colimits with relative ◦-products and inspection of the natural transformation.

Proof of Proposition 3.3. Since equivalences are closed under retracts, we have to check the claim just for \(A\) a cellular proj-cofibrant \(D_x\)-algebra. This claim is consequently reduced to show that it holds in each step in the construction of a cellular algebra. The free case is treated in Lemma 3.4, the pushout case in Lemma 3.5 and the transfinite one in Lemma 3.7.

**Theorem 3.8.** Factorization homology with context computes the left derived adjoint to the forgetful functor \(\text{Alg}_{M_X} \rightarrow \text{Alg}_{D_X}\). In other words, there is a natural equivalence of \(M_X\)-algebras

\[
\overline{M_X \circ D_x A} \sim \overline{M_X \circ D_x A}.
\]

**Proof.** It is a direct consequence of Proposition 3.3 since \(M_X \circ D_x A\) is left Quillen between the projective model structures. □

**Remarks 3.9.**

1. This computation subsumes the well-established result which says that factorization homology is equivalent to operadic left Kan extension along \(\text{Discs} \hookrightarrow \text{Mfld}\), see [2].

2. In [4, Lemma 2.17] it is claimed that this conclusion holds in a broader setting, under several assumptions on the operads. It can be proven from the coequalizer description of relative ◦-products that

\[
\overline{\mathcal{O} \circ P A} \rightarrow \overline{\mathcal{O} \circ P A}
\]

is an isomorphism when \(\mathcal{O}\) and \(\mathcal{P}\) are (partial) symmetric monoidal categories and \(A\) is any \(\mathcal{P}\)-algebra. In Lemma B.2 (and Proposition B.6 its derived version), we discuss a different formulation more related to [4, Lemma 2.16], but the method is not clearly adaptable to broader settings such as the one covered in Theorem 3.8.

3. In [25, Proposition 3.9], it is computed in general the derived operadic left Kan extension using enveloping symmetric monoidal categories. Our result simplifies this presentation since we use the underlying category instead of those envelopes.

## 4 Extension model structure

Local-to-global conditions for sheaves are treated by means of left Bousfield localizations in homotopy theory [18, 27]. By dual analogy, colocality principles for cosheaves may be treated with right Bousfield localizations, although it is not clear what right Bousfield localization could correspond to those colocality principles. Furthermore, classical methods to find right
Bousfield localizations are rarely applicable. In [13], we give a general procedure to construct certain right Bousfield localizations for model categories of operadic algebras which certainly capture colocality principles. Because of their applicability in this work, we include a brief discussion of the results obtained in the cited reference.

Let $\iota: B \to N$ be a map of $\mathcal{V}$-operads. This simple data already determines a local-to-global principle: $N$-algebras that are completely characterized by their restriction to $B$. A canonical choice of reconstruction process from the restricted $B$-algebra to an $N$-algebra is via operadic left Kan extension. More concretely, $\iota$ induces an adjoint pair $\iota_! : \mathcal{Alg}_B \rightleftarrows \mathcal{Alg}_N : \iota^*$. An $N$-algebra $A$ satisfies the above local-to-global principle if $\iota_! \iota^* A \to A$ is an isomorphism, because $\iota^* A$ is the restricted $B$-algebra and $\iota_!$ the operadic left Kan extension.

In order to introduce homotopy theory into the picture, the simplest approach is to enhance the adjunction $\iota_! \dashv \iota^*$ into a Quillen pair. Fixing the projective model for operadic algebras on both sides, one achieves such enhancement. Then, the derived functor $L \iota_!$ gives a homotopy meaningful reconstruction from $B$-algebras to $N$-algebras. To refer to $N$-algebras which satisfy the corresponding homotopical local-to-global principle, we introduce the following notion.

**Definition 4.1.** An $N$-algebra $A$ such that $L \iota_! \iota^* A \to A$ is an equivalence is said to be $B$-colocal.

**Remark 4.2.** The above discussion is quite trivial in nature and that is why is so surprising that it has produced so deep mathematics. The formal idea is the germ of chiral homology [28] and recent local-to-global constructions in algebraic quantum field theories [7]. Replacing operadic left Kan extension by right Kan extension for categories, it appears in manifold calculus [8].

In [13, Section 4], it is discussed in detail how one can produce a right Bousfield localization of the projective model on $\mathcal{Alg}_N$ whose colocal objects are $B$-colocal algebras, i.e. algebras that can be reconstructed from its restriction to $B$. However, this program requires some assumptions to work. We collect them together with their raison d’être.

**Hypothesis 4.3.** We assume the following conditions.

- $\mathcal{V}$ is cofibrantly generated and $B$, $N$ are admissible operads. Necessary to $\iota_! \dashv \iota^*$ being a Quillen pair.
- $L \iota_! : Ho \mathcal{Alg}_B \to Ho \mathcal{Alg}_N$ is fully-faithful. Essential to ensure that restricting $L \iota_! \iota^* A$ and $A$ back to $B$ we get the same result.

Under Hypothesis 4.3, we obtain the desired extension model.

**Theorem 4.4.** The projective model on $\mathcal{Alg}_N$ admits a right Bousfield localization, called the extension model structure, which enjoys the following properties:

- The class of weak equivalences is that of maps of $N$-algebras which are equivalences when restricted to $B$.
- A cofibrant object is a proj-cofibrant algebra which is also $B$-colocal.
- The extension model is cofibrantly generated.

Furthermore, the Quillen pair $\iota_! \dashv \iota^*$ descends to a Quillen equivalence between the projective model on $\mathcal{Alg}_B$ and the extension model on $\mathcal{Alg}_N$.

**Proof.** [13, Theorem 4.9 and Proposition 4.10].
Under restrictive hypothesis on $\mathcal{V}$, one can prove that the extension model on $\mathcal{N}$-algebras is left-proper. Since such discussion would take some effort and space, we leave it to [12]. Note that this fact is useful for the construction of left Bousfield localizations of the extension model. An important example is given by the following result.

**Proposition 4.5.** The projective and the extension model on $\text{Alg}_N$ are left proper when $\mathcal{V}$ is the projective model structure on chain complexes over a field of characteristic 0. If $\mathcal{V}$ is the Kan-Quillen model structure on simplicial sets and $\mathcal{N}$ is $\Sigma$-cofibrant, the same conclusion holds.

**Proof.** See [13, Proposition 4.13] for the first statement. The second one follows from an analogous analysis, since every object in the Kan-Quillen model structure is cofibrant. □

In order to accommodate different examples coming from [12], instead of restricting ourselves to $\text{Ch}_k$ or $\text{Set}_\Delta$, we will ask this left-properness assumption when needed.

## 5 Enriched factorization algebras

Our goal in this section is constructing model structures on $\text{Alg}_{\mathcal{M}_X}$ which present the homotopy theory of enriched Weiss (resp. factorization) algebras.

**Homotopical codescent and enriched Weiss algebras** Let us fix the inclusion of operads $i: \mathcal{D}_X \hookrightarrow \mathcal{M}_X$ along this subsection.

The main idea here is to describe a relation between the homotopical codescent condition for Weiss covers with $\mathcal{D}_X$-colocality introduced in Section 4. Our main tool will be the computation of derived operadic left Kan extension $\mathbb{L}i^\#: \text{Alg}_{\mathcal{D}_X} \to \text{Alg}_{\mathcal{M}_X}$ given in Theorem 3.8. The machinery of Section 4 will produce the claimed model for enriched Weiss algebras.

The main deduction from Proposition 3.3 in this subsection is the recognition principle described in the following result.

**Proposition 5.1.** Let $F$ be a proj-cofibrant $\mathcal{D}_X$-algebra. Then, $i^*_F$ is an enriched Weiss algebra, i.e. satisfies homotopical codescent with respect to Weiss covers.

Moreover, the following conditions are equivalent for an $\mathcal{M}_X$-algebra $A$:

- $A$ is an enriched Weiss algebra (see Definition 2.13).
- $A$ is $\mathcal{D}_X$-colocal (see Definition 4.1).

**Proof.** Let $F$ be a proj-cofibrant $\mathcal{D}_X$-algebra and $(U_t)_{t \in I}$ a Weiss cover of $U \subseteq X$. We claim that the canonical map

$$\text{hocolim}_{S\subseteq I} i^*_t F(U_s) \to i^*_F F(U)$$

is an equivalence.

Note that $i^*_F$ decomposes as an extension of operations and an extension of colors. More concretely, $i^*_F = \mathcal{M}_X \circ \hat{F}$, where $\hat{F}$ is the extension of colors from $\text{col}(\mathcal{D}_X)$ to $\text{col}(\mathcal{M}_X)$, i.e.

$$\hat{F}(Z) = \begin{cases} F(Z) & \text{if } Z \in \text{col}(\mathcal{D}_X), \\ 0 & \text{otherwise}. \end{cases}$$
Using this fact, our claim comes from the equivalence in Lemma 2.16, due to 2 out of 3 applied to the commutative diagram

\[
\begin{array}{ccc}
\text{hocolim} \underset{S \subseteq I}{\leftarrow} i_F(U_S) & \xrightarrow{\sim} & \text{hocolim} \left( \mathcal{M}_X \circ \hat{F} \right)(U_S) \\
\downarrow & & \downarrow \\
\text{hocolim} \left( \mathcal{M}_X \circ \hat{F} \right)(U) & \xrightarrow{\sim} & \text{hocolim} \left( \mathcal{M}_X \left[ \mathcal{U}^{X} \right] \left( \mathcal{U}^{X} \right) \right)
\end{array}
\]

which is a consequence of Proposition 3.3 and consequent remarks.

Now, in order to prove the equivalent statements, fix an \( \mathcal{M}_X \)-algebra \( A \). Observe that \( \mathcal{D}_X \)-colocality means that \( \Omega A \to A \) is an equivalence. Moreover, \( \Omega A = i^*_F \Omega i^* A \) satisfies homotopical codescent with respect to Weiss covers by the former argument and hence \( A \) as well.

Conversely, we deduce \( \mathcal{D}_X \)-colocality of \( A \) by evaluation on a manifold \( Z \) in \( \text{col}(\mathcal{M}_X) \), i.e. showing that \( \Omega A(Z) \to A(Z) \) is an equivalence. Lemma 2.15 provides a particular Weiss cover \( (Z_i)_{i \in I} \) of \( Z \) which we employ to obtain the above equivalence using the commutative square

\[
\begin{array}{ccc}
\text{hocolim} \Omega A(Z_S) & \xrightarrow{\sim} & \text{hocolim} A(Z_S) \\
\downarrow & & \downarrow \\
\Omega A(Z) & \to & A(Z)
\end{array}
\]

since both \( \mathcal{M}_X \)-algebras coincide on colors in \( \text{col}(\mathcal{D}_X) \) and satisfy homotopical codescent with respect to Weiss covers.

The previous characterization of enriched Weiss algebras is the keystone to the construction of the enriched Weiss model.

**Theorem 5.2.** \( \text{Alg}_{\mathcal{M}_X} \) supports the enriched Weiss model characterized by:

- The class of weak equivalences consists on those maps that are equivalences when evaluated on colors of \( \mathcal{D}_X \).
- The class of fibrations is that of proj-fibrations.
- Cofibrant objects are proj-cofibrant enriched Weiss algebras.

**Proof.** It is an immediate application of Theorem 4.4 together with the recognition principle given in Proposition 5.1. \( \square \)

**Weak monadicity and enriched factorization algebras** Now we work instead with the operad inclusion \( j : \mathcal{E}_X \hookrightarrow \mathcal{M}_X \).

As we did with enriched Weiss algebras, we want to further relate weak monadicity with \( \mathcal{E}_X \)-colocality.

**Proposition 5.3.** Let \( K \) be a proj-cofibrant \( \mathcal{E}_X \)-algebra. Then, \( j^*_K \) is an enriched factorization algebra, i.e. satisfies homotopical codescent with respect to Weiss covers and weak monadicity.

Moreover, a colorwise flat \( \mathcal{M}_X \)-algebra \( A \) is an enriched factorization algebra if and only if it is \( \mathcal{E}_X \)-colocal.

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Proof. Let $K$ be a proj-cofibrant $E_X$-algebra. Noting that $j$ can be factored through the morphism $i : D_X \hookrightarrow M_X$, it is clear that $j_* K$ satisfies homotopical codescent with respect to Weiss covers by Proposition 5.1. It remains to check weak monadicity, that is $j_* K(m_{Z,Y})$ is an equivalence for all $Z$ and $Y$ in $\text{col}(M_X)$ disjoint. Unwrapping the definitions, one sees that this operation is induced by disjoint union of embeddings

$$
\bigcup_{T \in T \sqcup T'} M_X \left[ \frac{T_Z}{Z} \right] \otimes M_X \left[ \frac{T_Y}{Y} \right] \twoheadrightarrow M_X \left[ \frac{T}{Z \sqcup Y} \right]
$$

(2)

where $T_i \in \text{col}(E_X)$, i.e. each one is a disc embedded in $X$ ($Z$ and $Y$ being disjoint in $X$ ensure that this map exists). Since the elements that conform the collection $\mathcal{I}$ are all connected, the definition of the multimapping-spaces of $M_X$ implies that (2) is an equivalence. We should pursue the constructions involved to get $j_* K(m_{Z,Y})$ from (2) in order to deduce that weak monadicity holds. A somewhat lengthy but straightforward analysis of the coequalizer that defines $j_* K$ shows that these constructions are just: taking homotopy co limits over finite grupoids, tensor products with cofibrant objects and homotopy coequalizers. Hence, $j_* K(m_{Z,Y})$ is an equivalence.

Now, assume that $A$ is an $M_X$-algebra. If $A$ is $E_X$-colocal, i.e. $\hat{Q} A \xrightarrow{\epsilon} A$ is an equivalence, we use the commutative square

$$
\begin{array}{ccc}
\hat{Q} A(Z) \otimes \hat{Q} A(Y) & \rightarrow & \hat{Q} A(Z \sqcup Y) \\
\epsilon \otimes \epsilon & & \downarrow \epsilon \\
A(Z) \otimes A(Y) & \rightarrow & A(Z \sqcup Y)
\end{array}
$$

(2)

to see that $A$ satisfies weak monadicity. The top arrow is an equivalence by the above argument, $\epsilon$ is an equivalence by definition of $E_X$-colocality and $\epsilon \otimes \epsilon$ is an equivalence since $A$ is colorwise flat and $\hat{Q} A$ is proj-cofibrant.

Conversely, we are going to prove that $\hat{Q} A \xrightarrow{\epsilon} A$ is an equivalence by evaluation on each $Z \in \text{col}(M_X)$. The argument is an adaptation of the one given in Proposition 5.1. By definition, if $Z$ belongs to $\text{col}(E_X)$, i.e. it is an embedded disc into $X$, the claim holds. In general, consider a Weiss cover $(Z_i)_{i \in I}$ of $Z$ by finite disjoint unions of discs whose intersections remain so, Lemma 2.15. Combining homotopical codescent with weak monadicity one can further decompose each finite intersection $Z_S$ of the Weiss cover into its constituent discs $Z_S = \bigsqcup_{j \in J_S} Z_j$ and provide an equivalence

$$
\text{hocolim}_{\bigsqcup_{j \in J_S}} A(Z_j) \xrightarrow{\sim} A(Z)
$$

This is also valid for $\hat{Q} A$ as well. Therefore, the commutative square

$$
\begin{array}{ccc}
\text{hocolim}_{\bigsqcup_{j \in J_S}} \hat{Q} A(Z_j) & \rightarrow & \text{hocolim}_{\bigsqcup_{j \in J_S}} A(Z_j) \\
\sim & & \sim \\
\hat{Q} A(Z) & \rightarrow & A(Z)
\end{array}
$$

allows us to conclude the claim, where the top map is an equivalence due to $A$ being colorwise flat and $\hat{Q} A$ proj-cofibrant. $\Box$
Remark 5.4. The equivalence

\[
\text{hocolim}_{S \subseteq I} \bigotimes_{j \in I_S} A(Z_j) \xrightarrow{\sim} A(Z)
\]

used above appears in the literature as descent for factorizing covers. A clever organization of the diagram indexing the homotopy colimit is given in [20, Section 4] together with the obvious notion of factorizing cover.

Analogously to Theorem 5.2, the former characterization permits the construction of the so-called enriched factorization model structure.

**Theorem 5.5.** \(\text{Alg}_{M_X}\) supports the enriched factorization model characterized by:

- The class of weak equivalences consists on those maps that are equivalences when evaluated on colors of \(E_X\).
- The class of fibrations is that of proj-fibrations.
- Cofibrant objects are proj-cofibrant enriched factorization algebras.

**Proof.** Immediate application of Theorem 4.4 together with Proposition 5.3.

\[
\square
\]

## 6 Locally constant factorization algebras

Next section is devoted to the discretization of the enriched factorization model constructed in Theorem 5.5. This goal will be achieved after two steps. First, using the equivalence [2, Subsection 2.4], we construct a left Bousfield localization which encodes local constancy; and then, a right Bousfield localization will incorporate the local to global properties.

In this section, we perform left Bousfield localizations, see [6, 14, 38], and so our fixed homotopy cosmos should be nice enough to ensure this process. Since left Bousfield localization at a set of maps produces a model structure under left properness and only a left semimodel structure\(^\text{ii}\) without such property, we will distinguish two cases in the sequel.

The best possible scenario will be encoded by the following:

**Hypothesis 6.1.** \(V\) is a left proper combinatorial (resp. cellular) model structure such that the projective/extension model structure on \(\text{Alg}_N\) is left proper for any \(\Sigma\)-cofibrant operad \(N\).

Both simplicial sets with the Kan-Quillen model structure and the projective model structure on chain complexes over a field of characteristic zero satisfy 6.1 by Proposition 4.5.

For more general homotopy cosmoi, in order to ensure that left Bousfield localization at a set of maps exists, we will have to consider:

**Hypothesis 6.2.** \(V\) is a tractable\(^\text{iii}\) combinatorial (resp. cellular) model structure.

\(^\text{ii}\)In this work, by left semimodel structure we will always mean Spitzweck left semimodel structure as in [14].

\(^\text{iii}\)In both cases, tractability means that the generating sets of (trivial) cofibrations have cofibrant domains.
Local constancy on $D_X$-algebras  First, we will explain that a convenient choice of unary operations in $D_X$ (or $E_X$) yields a left Bousfield localization of the projective model on algebras which is Quillen equivalent to that of $D_X$-algebras (resp. $E_X$). The Quillen equivalence is induced by

$$D_X 	o D_X \quad \text{(resp. } E_X 	o E_X).$$

We will focus on $D_X$, because the other case is similar.

Recall from [2, Proposition 2.19] that $\psi: D_X \to D_X$ exhibits $D_X$ as the $\infty$-localization of $D_X$ at the set of isotopy equivalences $J_X$ in $D_X$.

Definition 6.3.  A $D_X$-algebra $F$ is locally constant if it sends $J_X$ to equivalences.

The idea is to represent the set of unary operations $J_X$ as a set of morphisms $S_X$ in $\text{Alg}_{D_X}$ to perform a left Bousfield localization whose local objects are locally constant $D_X$-algebras. Such representation is explained in [15] and it requires the following result.

Proposition 6.4.  If $V$ is combinatorial (resp. left proper or tractably generated) model category, there is a set of objects $G$ in $V$ which jointly detect equivalences, i.e. a map $f$ in $V$ is an equivalence if $\text{Map}_V(x, f)$ is an equivalence $\forall x \in G$, where $\text{Map}_V$ denotes the homotopy mapping space in $V$.

Proof. In the combinatorial case, it is well known that it suffices to take the collection of cofibrant $\kappa$-small objects for a sufficiently large regular cardinal $\kappa$. In the cofibrantly generated cases, the idea is to take (replacements) of domains and codomains of the generating cofibrations (see [17, Proposition A.5]).

With Proposition 6.4, one can make the following definition.

Definition 6.5. The set of maps $S_X$ in $\text{Alg}_{D_X}$ that represents $J_X$ is

$$\left\{ \hat{j}_*(x \otimes f_*) : \hat{j}_*(x \otimes u^*) \to \hat{j}_*(x \otimes v^*) \text{ for } x \in G \text{ and } f \in J_X \right\},$$

where $\otimes$ means $V$-tensoring in $D_X$-algebras, $\hat{j}_*$ is the left adjoint of the restriction functor $j^* : \text{Alg}_{D_X} \to \text{Alg}_{D_X}$ and $u \mapsto u^*$ denotes the covariant enriched Yoneda embedding for $D_X$.

Theorem 6.6. Assume that Hypothesis 6.1 holds. Then, the projective model on $\text{Alg}_{D_X}$ admits a left Bousfield localization at $S_X$ among model categories whose local objects are the locally constant $D_X$-algebras. Moreover,

$$\psi_* : \text{Alg}_{D_X} \rightleftarrows \text{Alg}_{D_X} : \psi^*$$

establishes a Quillen equivalence between the localized model structure on $\text{Alg}_{D_X}$ and the projective model structure on $\text{Alg}_{D_X}$.

If instead Hypothesis 6.2 holds, the conclusions hold but among left semimodel categories.

Proof. The existence of the left Bousfield localization at a set of maps follows from [6, Theorem 4.7] and [24, Theorem 4.1.1] under Hypothesis 6.1 and by [38, Theorem A] and [14, Theorem B] under Hypothesis 6.2. In the last case, we have applied that the projective model structure on $\text{Alg}_{D_X}$ is tractable (because $V$ is tractable and the projective model is obtained by transfer) and additionally in the cellular case that cofibrations with cofibrant domains are effective monomorphisms (since those cofibrations are colorwise cofibrations). The characterization of the local objects is a consequence of the definition of the set $S_X$. The Quillen adjunction is a Quillen equivalence because $D_X \to D_X$ exhibits $D_X$ as the $\infty$-localization of $D_X$ at the set of isotopy equivalences $J_X$ due to [2, Proposition 2.19].

21
Remark 6.7. For more details about this localization process see \cite{15}.

Proposition 2.19 in \cite{2} also proves that $\mathcal{E}_X$ is the $\infty$-localization of $E_X$ at all unary operations. Therefore, the analogous left Bousfield localization for algebras over $E_X$ models local constancy in Definition 2.2.

**Proposition 6.8.** Assume that Hypothesis 6.1 holds. Then, the projective model on $\text{Alg}_{E_X}$ admits a left Bousfield localization at $S_X$ among model categories whose local objects are the locally constant $E_X$-algebras. Moreover,

$$\psi_\#: \text{Alg}_{E_X} \leftrightarrow \text{Alg}_{E_X} : \psi^*$$

establishes a Quillen equivalence between the localized model structure on $\text{Alg}_{E_X}$ and the projective model structure on $\text{Alg}_{E_X}$.

If instead Hypothesis 6.2 holds, the conclusions hold but among left semimodel categories.

**Local constancy on $M_X$-algebras** The content of this subsection subsumes a combination of two Bousfield localizations which finally produce the factorization model for $M_X$-algebras. The idea is to replicate model categorically the equivalence that appears in Proposition 2.5.

Since we need to make references to different model structures and Quillen pairs between them, we adopt the following:

**Notation 6.9.** A subscript decorating a category refers to its model structure, e.g. $(\text{Alg}_{M_X})_{e\text{Weiss}}$ represents the enriched Weiss model of Theorem 5.2. Arrows in diagrams of model categories represent left Quillen functors. The symbol $\sim_Q$ will denote a Quillen equivalence within these diagrams. LBL and RBL are acronyms for left and right Bousfield localization respectively.

Let us denote

$$M_X \leftarrow^t D_X \quad \psi \downarrow \quad \psi$$

$$M_X \leftarrow t D_X$$

the involved operads and maps between them on the subsequent construction.

First, we perform a right Bousfield localization using Theorem 4.4 for the inclusion of operads $M_X \leftarrow t D_X$. This way we get the extension model, $(\text{Alg}_{M_X})_{\text{ext}}$.

Define the set of maps $\tilde{S}_X$ in $\text{Alg}_{M_X}$ as the image of $S_X$ along $t_\#$. Using the existence theorems for left Bousfield localizations as how they were employed in Theorem 6.6, we further produce the Weiss model on $\text{Alg}_{M_X}$.

**Theorem 6.10.** Assume that Hypothesis 6.1 holds. Then, $\text{Alg}_{M_X}$ supports the Weiss model structure, which is the left Bousfield localization at $\tilde{S}_X$ of the extension model constructed in Theorem 4.4. Moreover, the Weiss model sits into a commutative diagram of model categories.

\[
\begin{array}{ccc}
(\text{Alg}_{M_X})_{\text{proj}} & \xrightarrow{\text{RBL}} & (\text{Alg}_{M_X})_{\text{ext}} \\
\Psi_\# & \downarrow & \sim_Q \\
(\text{Alg}_{M_X})_{\text{Weiss}} & \xrightarrow{\sim_Q} & (\text{Alg}_{D_X})_{\text{proj}} \\
\end{array}
\]
If instead Hypothesis 6.2 holds, the left Bousfield localization \((\text{Alg}_{M_X})_{\text{Weiss}}\) exists as a left semimodel category. Moreover, the diagram above is also available but its middle horizontal line consists of left semimodel categories.

**Proof.** The existence of the Weiss model is justified by the sketch right above.

We also need to explain the Quillen equivalences that appear in the diagram above. By Theorem 4.4, both upper and lower horizontal arrows are Quillen equivalences. The middle horizontal arrow is a Quillen equivalence since it comes from left Bousfield localization of equivalent model structures at essentially the same set of maps. Theorem 6.6 says that the right vertical map is an equivalence. The remaining Quillen equivalence is so by 2 out of 3.

An important consequence of the Quillen equivalence between the enriched Weiss model and the Weiss model is a recognition of bifibrant objects in the Weiss model, Proposition 6.13. Before stating it, we present a pair of lemmas which are essential in our proof. Recall from Section 4 the notion of \(D_X\)-colocality.

**Lemma 6.11.** A \(M_X\)-algebra \(A\) is \(D_X\)-colocal if and only if for any \(Z \in \text{col}(M_X)\), the canonical map \(\text{hocolim}(D_X)^{\downarrow} Z \to A(Z)\) is an equivalence.

**Proof.** Note that the endofunctor

\[Q = \mathbb{L} t_! t^* = t_! Q t^* : \text{Alg}_{M_X} \to \text{Alg}_{D_X} \to \text{Alg}_{D_X} \to \text{Alg}_{M_X}\]

can be described alternatively as:

\[Q A = \mathbb{L} t_! t^* A \simeq \mathbb{L} t t^* A = \text{hocolim}(D_X)^{\downarrow} Z \to A,\]

where the equivalence in the middle is the content of Proposition B.6.

With such characterization and adapting [30, Subsections 2.3-2.4] to Weiss covers instead of factorizing covers, one can show the following result.

**Lemma 6.12.** Any locally constant Weiss algebra over \(X\) is \(D_X\)-colocal.

**Proof.** Let \(B\) be a locally constant Weiss algebra over \(X\). By Lemma 6.11, we must show that

\[\text{hocolim}(D_X)^{\downarrow} Z \to B(Z)\]

is an equivalence for any \(Z \in \text{col}(M_X)\) (since \((D_X)^{\downarrow} Z \simeq D_Z\)). A variation of [30, Theorem 2.11] shows this claim using local constancy and codescent for Weiss covers.

Due to Lemmas 6.11 and 6.12, it is possible to produce a cosheafification machine for the Weiss topology that preserves the appropriate algebraic structure, i.e. a functorial construction that turns precosheaves into cosheaves, hence we answer Gwilliam-Rejzner problem [22, Remark 2.33]. Such cosheafification articulates the recognition of bifibrant objects for the Weiss model and its existence is important in its own. One reason is that in usual categories as sets or abelian groups, the formal existence of cosheafification is only known by means of general adjoint functor theorems (see [34]).

**Proposition 6.13.** The bifibrant objects in the Weiss (left semi)model structure are the proj-bifibrant locally constant Weiss algebras.
Proof. Recall that the Weiss model is constructed by two Bousfield localizations,

$$(\text{Alg}_{M_X})_{\text{proj}} \xleftarrow{\text{RBL}} (\text{Alg}_{M_X})_{\text{ext}} \xrightarrow{\text{LBL}} (\text{Alg}_{M_X})_{\text{Weiss}}.$$ 

Right (resp. left) Bousfield localization does not affect fibrant (resp. cofibrant) objects. Hence, the Weiss-cofibrant objects are $D_X$-colocal proj-cofibrant $M_X$-algebras while the Weiss-fibrant objects are locally constant proj-fibrant $M_X$-algebras by Theorem 6.6. In the left semimodel case, we do not know a characterization of Weiss-fibrant objects, but at least we know that the Weiss-bifibrant objects are the locally constant ext-bifibrant $M_X$-algebras, and that suffices to our purposes.

Choose a Weiss-bifibrant object $A$. We will show that $A$ is a locally constant Weiss algebra by constructing one, $\tilde{A}$, which is colorwise equivalent to $A$. The main idea is to make use of the Quillen equivalence

$$\Psi_z : (\text{Alg}_{M_X})_{\text{Weiss}} \rightleftarrows (\text{Alg}_{M_X})_{e\text{Weiss}} : \Psi^*$$

appearing in Theorem 6.10.

Since $A$ is Weiss cofibrant, $\Psi_z A$ is cofibrant in the enriched Weiss model and therefore it is enriched Weiss algebra by Proposition 5.1. In particular, its restriction $\Psi^* \Psi_z A$ is a locally constant Weiss algebra. Moreover, the unit of the Quillen equivalence evaluated at $A$ yields a Weiss equivalence $A \to \Psi^* \Psi_z A$, which is not, a priori, a colorwise equivalence. However, the class of Weiss equivalences between Weiss bifibrant objects is the class of colorwise equivalences, since (co)local equivalences between (co)local objects are ordinary equivalences in a (right) left Bousfield localization.

Thus, it will suffice to show that $\Psi^* \Psi_z A$ is Weiss bifibrant. This is not always the case, but it will be after slightly modifying this algebra.

Recall that Weiss bifibrant means proj-bifibrant locally constant $D_X$-colocal algebra as explained at the beginning of this proof. Let us focus on $D_X$-colocality. By Lemma 6.11, this property is equivalent to the following codescent condition: for any $Z \in \text{col}(M_X)$, the canonical map

$$\text{hocolim}_{(D_X)_{\text{col}}} \Psi^* \Psi_z A (\ast) \to \Psi^* \Psi_z A (Z)$$

is an equivalence. Taking into account that $(D_X)_{\text{col}} \simeq D_Z$, this codescent condition is deduced from $D_X$-colocality in exactly the same way as Weiss codescent in Proposition 5.1 replacing Lemma 2.16 by Lemma 2.18.

It remains to add the projective conditions on $\Psi^* \Psi_z A$. Denoting $R$ (resp. $Q$) a proj-fibrant (resp. proj-cofibrant) replacement in $\text{Alg}_{M_X}$, we get a diagram

$$\begin{array}{ccc}
Q R \Psi^* \Psi_z A & \longrightarrow & \tilde{A} \\
\downarrow & & \\
A & \longrightarrow & R \Psi^* \Psi_z A,
\end{array}$$

where the lift exists by lifting properties in the projective model and it is a Weiss equivalence by 2 out of 3. We conclude the proof since $\tilde{A}$ is a Weiss bifibrant algebra which by construction is colorwise equivalent to $A$.

For the converse, pick a proj-bifibrant locally constant Weiss-algebra $B$. By the discussion at the beginning of the proof, we are reduced to check that $B$ is $D_X$-colocal and this was proven in Lemma 6.12.  \(\square\)
A trivial adaptation of the above arguments replacing $D_X$ by $E_X$, leads to:

**Theorem 6.14.** Assume that Hypothesis 6.1 holds. Then, $\text{Alg}_{M_X}$ supports the factorization model structure, which is the left Bousfield localization at all unary operations in $E_X$ of the extension model constructed in Theorem 4.4. The factorization model sits into a commutative square of model categories and Quillen equivalences

$$
\begin{array}{ccc}
\text{(Alg}_{M_X})_{\text{fact}} & \sim_Q \leftrightarrow & \text{(Alg}_{E_X})_{\text{loc}} \\
\sim_Q \downarrow & & \sim_Q \downarrow \\
\text{(Alg}_{M_X})_{\text{efact}} & \sim_Q \leftrightarrow & \text{(Alg}_{E_X})_{\text{proj}}
\end{array}
$$

Moreover, the bifibrant objects in the factorization model are the proj-bifibrant locally constant factorization algebras over $X$.

If instead Hypothesis 6.2 holds, the left Bousfield localization $(\text{Alg}_{M_X})_{\text{fact}}$ exists as a left semimodel category. Moreover, the square above is also available but its upper horizontal line consists of left semimodel categories. The characterization of bifibrant objects also holds.

7 Variations and generalizations

We have developed model structures that present different kinds of factorization algebras for smooth manifolds, but our methods can be applied in different situations. The choice of ordinary smooth manifolds is due to its simplicity.

Let us summarize our results and their requirements in each setting.

- **Enriched Weiss and factorization model structures:** they are applications of the extension model machine (Theorem 4.4). We perform the pertinent analysis of $\Omega$, which requires a computation of factorization homology with context, Proposition 3.3, that follows from the existence of good Weiss covers (Lemma 2.15) and a cosheaf condition on embedding spaces (Lemma 2.16). The description of $\Omega$ and the codescent conditions of Weiss and factorization algebras are combined to recognize cofibrant objects in Propositions 5.1 and 5.3.

- **Weiss and factorization (left semi)model structures:** they come from a combination of the extension model machine together with a left Bousfield localization ([6, Theorem 4.7] or [24, Theorem 4.1.1]). Left Bousfield localization produces model structures under left properness and left semimodel structures without such hypothesis, so we should distinguish between two classes of homotopy cosmoi depending on this difference. In addition, we need to identify certain $\infty$-localization ([2, Proposition 2.19]) which allows us to define the set of maps at which localize.

On the other hand, one has to recognize the colocality condition for the extension model in this setting. This condition boils down to a codescent property (Lemma 6.11). Since embedding spaces satisfy that property (Lemma 2.18), enriched Weiss (resp. factorization) algebras satisfy this variant of codescent. Proposition 6.13 implies that this suffices to identify bifibrant objects as locally constant Weiss (resp. factorization) algebras.
Next, we collect some settings where one or both of these schemes are valid. They consist on varying the Top-operad diagram

\[
\begin{array}{c}
M_x & \rightarrow & D_x & \rightarrow & E_x \\
\downarrow & & \downarrow & & \downarrow \\
M'_x & \rightarrow & D'_x & \rightarrow & E'_x
\end{array}
\]

(i) *Factorization algebras without context ([16, Definition 3.0.2]):* the chain of fully-faithful inclusions of operads \( \text{Mfld} \leftarrow \mathcal{D}_{\text{Discs}} \leftarrow \mathcal{E} \) may be chosen to mimic the constructions in Section 5. The needed ingredients are justified by the same reasons than the ones that appear in this work (see the mentioned section). Note that Section 6 could not be adapted without context.

(ii) *Factorization algebras with tangent structures:* Introducing tangent structures (with or without context) as in [2, Definition 2.7] everything can be done in exactly the same manner. Codescent conditions on embedding spaces are deduced from [8, Equation (10)] and the proof of Lemma 2.18, since the \( \infty \)-category of spaces is an \( \infty \)-topos. As discussed in [2, Section 2.4], adding tangent structures does not affect the localization result [2, Proposition 2.19].

(iii) *Factorization algebras on topological manifolds:* In [2], the authors prove that Lemma 2.15 holds if we replace smooth by topological and cover by hypercover. Hence, our models in this case present a variant of factorization algebras that satisfy homotopical codescent with respect to Weiss hypercovers. Using [28, Proposition 5.4.1.8] plus [28, Proposition A.3.1], one shows that embedding spaces of topological manifolds satisfy codescent properties of Lemma 2.16 (with respect to hypercovers) and Lemma 2.18. The localization part is again [2, Proposition 2.19].

(iv) *Factorization algebras on manifolds with boundary:* One can provide a Weiss cover by discs and half discs using [8, Proposition 9.1]. Codescent is reduced to [8, Proposition 9.2] if we fix the boundary, or to a combination of [25, Proposition 7.5] with [28, Proposition A.3.1] otherwise. The localization statement with boundaries allowed is also provided by [2].

(v) *Factorization algebras on conically smooth stratified manifolds:* By considering finite disjoint unions of basics ([5, Definition 2.2.1]) one can construct Weiss hypercovers due to [5, Proposition 3.2.23]. Codescent with respect to those hypercovers can be deduced from the proof of [5, Lemma 6.1.1] together with a configuration space argument, simply by evaluation on finite disjoint unions of basics. The localization theorem in this setting is discussed at [4, Proposition 2.22].

A  Appendix: Pushouts of algebras along free maps

This appendix is devoted to a slight improvement of well known filtrations of pushouts of operadic algebras along free maps ([39, Propositions 4.3.17 and 5.3.2]). We use them in the proof of Proposition 3.3; more concretely in Lemma 3.5.
Filtration for algebras

**Lemma A.1.** Let $\mathcal{B}$ be a colored operad and let

$$
\begin{align*}
\mathcal{B} \circ W & \longrightarrow A \\
\text{id} \circ & \quad \gamma \\
\mathcal{B} \circ V & \longrightarrow \mathcal{B}
\end{align*}
$$

be a pushout in $\text{Alg}_{\mathcal{B}}(\mathcal{V})$ where $j$ is concentrated in one color $b \in \text{col}(\mathcal{B})$. Then, the underlying arrow associated to $g$ in $\text{Alg}_{\mathcal{B}}(\mathcal{V})$ is the transfinite composition of a sequence $(g_n)_{n \in \mathbb{N}}$ defined by pushouts in $\text{Alg}_{\mathcal{B}}(\mathcal{V})$:

$$
\begin{align*}
\mathcal{B}_A \left[ b^i \star \right] \otimes s(j^\square n) & \longrightarrow A_{n-1} \\
\text{id} \otimes j^\square n & \quad \gamma \\
\mathcal{B}_A \left[ b^i \star \right] \otimes V^\otimes n & \longrightarrow A_n,
\end{align*}
$$

where $\mathcal{B}_A$ denotes the enveloping operad of the $\mathcal{B}$-algebra $A$.

**Proof.** The result holds in $[\text{col}(\mathcal{B}), \mathcal{V}]$, i.e. without functoriality in $\mathcal{B}$, by [39, Proposition 4.3.17]. Our contribution is just recognizing this functoriality. Due to the pointwise-computation of colimits in functor categories, the proof consists on showing that the above pushouts squares in $[\text{col}(\mathcal{B}), \mathcal{V}]$ also live in $\text{Alg}_{\mathcal{B}}(\mathcal{V})$. We notice that the vertices are $\overline{\mathcal{B}}$-functors by induction and since $\mathcal{B}_A$ is a $\overline{\mathcal{B}}$-module through the canonical map of operads $\mathcal{B} \rightarrow \mathcal{B}_A$. So, it remains to show that the edges of the squares live in $\text{Alg}_{\mathcal{B}}(\mathcal{V})$, i.e.

$$
\begin{align*}
\mathcal{B}_A \left[ b^i \star \right] \otimes s(j^\square n) & \longrightarrow A_{n-1} \quad \text{and} \quad \mathcal{B}_A \left[ b^i \star \right] \otimes V^\otimes n
\end{align*}
$$

are compatible with left $\overline{\mathcal{B}}$-actions.

- $\text{id} \otimes j^\square n$ is compatible with $\overline{\mathcal{B}}$-action because the structure comes from the first tensor factor and the map on this factor is the identity.

- attaching$_n$ is compatible with $\overline{\mathcal{B}}$-action: the explicit description of the attaching map given in [23, Proposition 7.12] can be generalized to the colored-case yielding a decomposition of attaching$_n$. First note that $s(j^\square n)$ is the colimit of a punctured $n$-cube diagram constructed out of $j$, whose vertices are tensor products of $p$ copies of $W$ and $q$ copies of $V$ such that $p + q = n$ and $p > 0$. With this in mind, it is easy to see that attaching$_n$...
factors as

\[
\mathcal{B}_A \left[ b^{p+q} \right] \otimes \Sigma_p \times \Sigma_q \xrightarrow{\text{adjoint map to } \mathcal{B} \circ \mathcal{W} \to \mathcal{A}} \mathcal{B}_A \left[ b^{p+q} \right] \otimes A(b)^{\otimes p} \otimes V^{\otimes q}
\]

\[
\mathcal{B}_A \left[ b^{n+q} \right] \otimes \mathcal{B}_A \left[ b^q \right] \otimes V^{\otimes q}
\]

operadic composition

\[
\mathcal{B}_A \left[ b^{n+q} \right] \otimes V^{\otimes q}
\]

inductively defined map for \( q \)

\[
A_q
\]

\[
A_{n-1}
\]

\[
A_{n-1} \cdots A_{q+1}
\]

\[
A_{n-1}
\]

It suffices to check that each morphism in the composite is compatible with the \( \mathcal{B} \)-action. The first ones does not involve the tensor factor where \( \mathcal{B} \)-acts; the following one is given by right multiplication in \( \mathcal{B} \) and hence, by associativity, commutes with left multiplication (which is the one that defines the \( \mathcal{B} \)-action); compatibility holds for the remaining maps by induction.

\[\square\]

**Lemma A.2.** Let \( \mathcal{B} \to \mathcal{N} \) be a morphism of operads which is the identity on colors. Let \( g: \mathcal{A} \to \mathcal{B} \) be the map of \( \mathcal{B} \)-algebras in Lemma A.1. Then, the natural map \( \mathcal{N} \circ \mathcal{B} \to \mathcal{N} \circ \mathcal{B} \) is the transfinite colimit of

\[
\frac{\mathcal{N} \circ \mathcal{A}}{\mathcal{N} \circ \mathcal{B}} \longrightarrow \cdots \longrightarrow \frac{\mathcal{N} \circ \mathcal{A}_{n-1}}{\mathcal{N} \circ \mathcal{B}} \xrightarrow{\text{id} \circ \gamma_{n-1}} \frac{\mathcal{N} \circ \mathcal{A}_n}{\mathcal{N} \circ \mathcal{B}} \longrightarrow \cdots
\]

where \( \gamma_n \) is defined inductively using the natural transformation \( \mathcal{N} \circ \mathcal{A} \to \mathcal{N} \circ \mathcal{A} \).

**Proof.** It suffices to define \( \gamma_n \) since then, the fact that the functor \( \mathcal{A} \mapsto \mathcal{N} \circ \mathcal{A} \) preserves sequential colimits and the use of Lemma A.1 implies the claim.
To define $\gamma_n$, one observes that it fits (dashed arrow) into a cube whose left and right faces are pushout squares. Hence, the commutativity of the back square implies the existence of $\gamma_n$. This commutativity follows easily from the definition of $\psi$ and $\phi$; both are induced from the map of $\mathcal{B}$-algebras:

$$\mathcal{B}_A \left[ \mathcal{B}_A \left[ \mathcal{B} \mathcal{F} \left( \mathcal{V} \right) \right] \right] \rightarrow \mathcal{N}_n \circ \mathcal{B}_A \left[ \mathcal{B}_A \left[ \mathcal{B} \mathcal{F} \left( \mathcal{V} \right) \right] \right].$$

\[\square\]

Filtration for enveloping operads

**Lemma A.3.** Let $\mathcal{B}$ be a colored operad and let

$$\mathcal{B} \circ \mathcal{W} \rightarrow \mathcal{A}$$

be a pushout in $\text{Alg}_\mathcal{B}(\mathcal{V})$ where $j$ is concentrated in one color $b \in \text{col}(\mathcal{B})$. Then, there is an associated pushout square in $\text{Operad}(\mathcal{V})$

$$\mathcal{F}(\mathcal{W}) \rightarrow \mathcal{B}_A$$

where $\mathcal{V} \mapsto \mathcal{F}(\mathcal{V})$ denotes the free operad functor and $\mathcal{B}_A$ the enveloping operad of the $\mathcal{B}$-algebra $\mathcal{A}$. Evaluating the first variable on $b \in \Sigma_{\text{col}(\mathcal{B})}$, the underlying map of $\mathcal{g}_*$ in $\text{Alg}_\mathcal{B}(\mathcal{V})$ is the transfinite composition of a sequence $(\mathcal{g}_{n+1})_{n \in \mathbb{N}}$ defined by pushouts in $\text{Alg}_\mathcal{B}(\mathcal{V})$:

$$\mathcal{B}_A \left[ \mathcal{B}_A \left[ \mathcal{B} \mathcal{F} \left( \mathcal{V} \right) \right] \right] \rightarrow \mathcal{B}_A \left[ \mathcal{B}_A \left[ \mathcal{B} \mathcal{F} \left( \mathcal{V} \right) \right] \right].$$
Proof. The argument given in Lemma A.1 can be adapted to this situation using [39, Proposition 5.3.2] instead of [39, Proposition 4.3.17]. □

Lemma A.4. Let \( \mathcal{B} \to \mathcal{N} \) be a morphism of operads which is the identity on colors. Let \( g : \mathcal{A} \to \mathcal{B} \) be the map of \( \mathcal{B} \)-algebras in Lemma A.1. Then, the natural map

\[
\mathcal{N} \circ \mathcal{B} \left[ b_{\star}^{[]} \right] \to \mathcal{N} \circ \mathcal{B} \left[ b_{\star}^{[]} \right]
\]

is the transfinite colimit of

\[
\mathcal{N} \circ \mathcal{B} \left[ b_{\star}^{[]} \right] \to \cdots \to \mathcal{N} \circ \mathcal{B} \left[ b_{\star}^{[]} \right] \to \cdots
\]

where \( \gamma_1 \) is defined inductively using the filtration in Lemma A.3.

Proof. Adaptation of the proof of Lemma A.2 for filtrations arising from Lemma A.3. □

B Appendix: Comparison of left Kan extensions

In this appendix, we prove a generalization of [4, Lemma 2.16] which is an essential tool for making computations with factorization homology. Our methods are different from those applied in loc.cit. and our main result should be understood as follows: We want to compare (derived) ordinary Kan extension and (derived) operadic Kan extension along a strong symmetric monoidal functor \( \iota : \mathcal{P} \to \mathcal{O} \) between partial symmetric monoidal categories (see [26, 36] for definitions of partial algebraic structures).

We are going to exploit the following result.

Lemma B.1. Let \( \mathcal{O} \) be a partial symmetric monoidal category seen as an operad. Then, the category of \( \mathcal{O} \)-algebras is equivalent to that of \( \mathcal{O} \)-functors equipped with a (partial) lax monoidal structure.

Proof. Unwrapping definitions, one finds that the claim reduces to the recognition of nonunary operations in a (partial) symmetric monoidal category;

\[
\mathcal{O} \left[ [o_r]_r \right] \cong \begin{cases} \mathcal{O} \left[ [o_r]_r \right] & \text{if } [o_r]_r \in \text{col}(\mathcal{O}) \times m \text{ can be tensored} \\ \emptyset & \text{otherwise.} \end{cases}
\]

In other words, nonunary operations can be recovered using the universal operations coming from the partial monoidal structure \( \text{id}_\mathcal{O} \in \mathcal{O} \left[ [o_r]_r \right] \).

With this in mind, the non-homotopical result is easy.
Lemma B.2. Let \( \iota : \mathcal{P} \to \mathcal{O} \) be a strong symmetric monoidal functor between partial symmetric monoidal categories. Then, the following square of decorated functors commutes:

\[
\begin{array}{ccc}
\text{Alg}_{\mathcal{P}} & \xrightarrow{\iota} & \text{Alg}_{\mathcal{O}} \\
\iota! & & \iota! \\
\iota! & & \iota! \\
\iota & & \iota \\
\text{Alg}_{\mathcal{P}} & \xleftarrow{\iota} & \text{Alg}_{\mathcal{O}}
\end{array}
\]

In other words, there is a natural isomorphism \( \iota! \mathcal{A} \cong \iota! \mathcal{A} \).

Proof. By the universal property satisfied by \( \iota! \mathcal{A} \), one could verify the claim by equipping \( \iota! \mathcal{A} \) with a lax monoidal structure by Lemma B.1 and showing that it is universal. The unit of such monoidal structure comes from the composite

\[
\mathbb{1}_\mathbb{V} \xrightarrow{A \text{-lax}} A(\mathbb{1}_\mathbb{P}) \xrightarrow{\text{unit}} \iota! A(\iota(\mathbb{1}_\mathbb{P})) \xrightarrow{\iota \text{-colax}} \iota! \mathcal{A}(\mathbb{1}_\mathbb{O})
\]

and the multiplication corresponds to the composite

\[
\iota! \mathcal{A}(o_1) \otimes \iota! \mathcal{A}(o_2)
\]

\[
( \int^{b_1} \mathcal{O} \left[ \begin{array}{c} b_1 \\ o_1 \end{array} \right] \otimes A(b_1)) \otimes ( \int^{b_2} \mathcal{O} \left[ \begin{array}{c} b_2 \\ o_2 \end{array} \right] \otimes A(b_2))
\]

\[
\cong \text{Fubini, } \otimes \text{-cocontinuous, symmetry}
\]

\[
\int^{b_1, b_2} \mathcal{O} \left[ \begin{array}{c} b_1 \\ o_1 \end{array} \right] \otimes \mathcal{O} \left[ \begin{array}{c} b_2 \\ o_2 \end{array} \right] \otimes A(b_1) \otimes A(b_2)
\]

\[
\cong \text{tensor } \otimes \text{lax monoidal}
\]

\[
\int^{b_1, b_2} \mathcal{O} \left[ \begin{array}{c} b_1 \boxtimes b_2 \\ o_1 \otimes o_2 \end{array} \right] \otimes A(b_1 \boxtimes b_2)
\]

\[
\cong \text{colax structure } \otimes \text{id}
\]

\[
\int^{b_1, b_2} \mathcal{O} \left[ \begin{array}{c} (b_1 \boxtimes b_2) \\ o_1 \otimes o_2 \end{array} \right] \otimes A(b_1 \boxtimes b_2)
\]

\[
\cong \text{coYoneda lemma}
\]

\[
\int^{b_1, b_2} \mathcal{O} \left[ \begin{array}{c} (b_1 \boxtimes b_2) \\ o_1 \otimes o_2 \end{array} \right] \otimes \left( \int^{b} \mathcal{P} \left[ \begin{array}{c} b \\ b_1 \otimes b_2 \end{array} \right] \otimes A(b) \right)
\]

\[
\cong \text{Fubini, } \otimes \text{-cocontinuous}
\]

\[
\int^{b} \left( \int^{b_1, b_2} \mathcal{O} \left[ \begin{array}{c} (b_1 \boxtimes b_2) \\ o_1 \otimes o_2 \end{array} \right] \otimes \mathcal{P} \left[ \begin{array}{c} b \\ b_1 \otimes b_2 \end{array} \right] \right) \otimes A(b)
\]

\[
\cong \text{composition } \otimes \text{id}
\]

\[
\int^{b} \mathcal{O} \left[ \begin{array}{c} b \\ o_1 \otimes o_2 \end{array} \right] \otimes A(b)
\]

\[
\iota! \mathcal{A}(o_1 \boxtimes o_2).
\]

Checking that these choices fulfil the requirements is a lengthy but easy computation. \(\square\)
Our goal becomes verifying that the natural isomorphism \( \iota! A \cong \iota\# A \) holds, possibly as an equivalence, when replacing left adjoints by their homotopical analogues. Equivalently, comparing derived Kan extensions (Proposition B.6). We achieve this objective showing that in some situations \( \text{Alg}_\Sigma \to \text{Alg}_\Sigma \) preserves cofibrancy (see Lemma B.5), but such result requires a little detour into symmetric monoidal envelopes of operads.

Recall that the forgetful functor from symmetric monoidal categories to operads admits a left adjoint called symmetric monoidal envelope, which is denoted \( \mathcal{B} \mapsto \text{Env}(\mathcal{B}) \). See [25, Definition 1.7] for a description of \( \text{Env}(\mathcal{B}) \) or [28] for a thorough treatment in the higher categorical context. This adjunction can be enhanced to a categorical level.

**Proposition B.3.** The canonical map of operads \( \eta : \mathcal{B} \to \text{Env}(\mathcal{B}) \) induces an equivalence of categories between that of \( \mathcal{B} \)-algebras with the full subcategory of \( \text{Env}(\mathcal{B}) \)-algebras spanned by strong monoidal functors \( \text{Env}(\mathcal{B}) \to \mathcal{V} \).

**Proof.** By Lemma B.1, we already know that \( \text{Alg}_{\text{Env}(\mathcal{B})} \) is equivalent to the category of lax monoidal functors \( \text{Env}(\mathcal{B}) \to \mathcal{V} \) (with monoidal natural transformations). Thus, it suffices to satisfy that the forgetful functor \( \text{Alg}_{\text{Env}(\mathcal{B})} \to \text{Alg}_\mathcal{B} \) admits a retraction into the full subcategory of strong monoidal functors. Such retraction is given by the functorial construction which associates to an \( \mathcal{B} \)-algebra \( A \), the strong monoidal functor \( A \otimes : \mathcal{B} \to \mathcal{V} \).

Using symmetric monoidal envelopes, we will deduce Lemma B.5 from the particular case of symmetric monoidal categories.

**Lemma B.4.** Let \( \mathcal{E} \) be a symmetric monoidal category which is \( \Sigma \)-cofibrant as an operad. Then, the forgetful functor \( \text{Alg}_\mathcal{E} \to \text{Alg}_\Sigma \) preserves cofibrations and cofibrant objects.

**Proof.** The equivalence of Lemma B.1 and the Day convolution product allow us to look at \( \text{Alg}_\mathcal{E} \to \text{Alg}_\Sigma \) as the functor which sends a commutative monoid to its underlying object in the symmetric monoidal category \( \text{Alg}_\Sigma \) [29, Proposition 22.1]. By [37, Theorem 4.6] and its following discussion, \( \text{Alg}_\mathcal{E} \) carries the projective model and the conclusion of the lemma holds if we check the strong commutative monoid axiom ([37, Definition 3.4]) and if the unit in \( \text{Alg}_\Sigma \) is cofibrant. Recall that we assume that the monoidal unit in \( \mathcal{V} \) is cofibrant, and that implies that the unit in \( \text{Alg}_\Sigma \) is cofibrant as well. Hence, we need a explicit description of the pushout product for generating (trivial) cofibrations in this case. A generating set of (trivial) cofibrations for the projective model in \( \text{Alg}_\Sigma \) is

\[
\left\{ \mathcal{E} \left[ \begin{array}{c} a \\
* \end{array} \right] \otimes j \text{ where } a \in \text{ob} \mathcal{E} \text{ and } j \text{ is a gen. (triv.) cof. in } \mathcal{V} \right\}.
\]

Thus, since the Day convolution makes the Yoneda embedding a strong monoidal functor (see the proof of [29, Lemma 3.7]), we have

\[
\left( \mathcal{E} \left[ \begin{array}{c} a \\
* \end{array} \right] \otimes i \right) \Box \left( \mathcal{E} \left[ \begin{array}{c} b \\
* \end{array} \right] \otimes j \right) \cong \mathcal{E} \left[ \begin{array}{c} a \Box b \\
* \end{array} \right] \otimes (i \Box j).
\]

Therefore, the pushout product axiom for \( \mathcal{V} \) combined with the fact that \( \mathcal{E} \) is assumed to be \( \Sigma \)-cofibrant implies that

\[
\left( \mathcal{E} \left[ \begin{array}{c} a \\
* \end{array} \right] \otimes i \right) \Box_n \cong \mathcal{E} \left[ \begin{array}{c} a \Box_n \\
* \end{array} \right] \otimes i \Box_n
\]

is a (trivial) cofibration when \( i \) is so. \( \square \)
Lemma B.5. Let $\mathcal{O}$ be a (partial) symmetric monoidal category which is $\Sigma$-cofibrant as an operad. Then, the forgetful functor $\text{Alg}_{\mathcal{O}} \to \text{Alg}_{\mathcal{O}}^{\Sigma}$ preserves cofibrations and cofibrant objects.

Proof. Let $\eta: \mathcal{O} \to \text{Env}(\mathcal{O})$ be the symmetric monoidal envelope of $\mathcal{O}$ seen as an operad. The strategy of the proof consists on looking at the forgetful functor $\text{Alg}_{\mathcal{O}} \to \text{Alg}_{\mathcal{O}}^{\Sigma}$ as the following composite

$$\text{Alg}_{\mathcal{O}} \xrightarrow{\eta^\#} \text{Alg}_{\text{Env}(\mathcal{O})} \xrightarrow{\text{forget}} \text{Alg}_{\text{Env}(\mathcal{O})}^{\Sigma} \xrightarrow{\pi} \text{Alg}_{\mathcal{O}}^{\Sigma}$$

and analyze each component separately.

To see that the composite coincides with the forgetful functor, note that $\eta^\#$ coincides with the construction $A \mapsto A^\otimes$ in the proof of Lemma B.3. Observe as well that $\eta^\#$ is left Quillen for the projective model structures, so it preserves cofibrations and cofibrant objects.

The preservation of cofibrancy for the middle forgetful functor is Lemma B.4.

Finally, $\pi^\#$ preserves colimits, since they are computed pointwise, and hence it suffices to check that it sends generating (trivial) cofibrations to (trivial) cofibrations. A generating set of (trivial) cofibrations for the projective model in $\text{Alg}_{\text{Env}(\mathcal{O})}^{\Sigma}$ was given in the proof of Lemma B.4. Therefore, the conclusion follows from the canonical identification

$$\text{Env}(\mathcal{O})\left[\delta_r\right] \cong \mathcal{O} \left[\delta_r\right] \cong \begin{cases} \mathcal{O} \left[\delta_r\right] & \text{if } \delta_r \in \text{col}(\mathcal{O})^{\times \mathbb{N}} \text{ can be tensored} \\ \emptyset & \text{otherwise,} \end{cases}$$

that holds because the operad structure on $\mathcal{O}$ comes from a partial symmetric monoidal structure.

Proposition B.6. Let $\iota: \mathcal{P} \to \mathcal{O}$ be a strong symmetric monoidal functor between (partial) symmetric monoidal categories which are $\Sigma$-cofibrant as operads. Then, the following square of decorated functors commutes

$$\begin{array}{ccc} \text{Ho Alg}_{\mathcal{P}} & \xleftarrow{L_\iota} & \text{Ho Alg}_{\mathcal{O}} \\ \pi \uparrow & & \pi \uparrow \\ \text{Ho Alg}_{\mathcal{P}} & \xleftarrow{L_\iota} & \text{Ho Alg}_{\mathcal{O}}. \end{array}$$

In other words, there is a natural equivalence $\mathbb{L}_\iota A \simeq \mathbb{L}_{\iota^\#} A$.

Proof. Immediate combination of Lemmas B.2 and B.5.

Remark B.7. The conclusion of Lemma B.5 holds if instead of the $\Sigma$-cofibrancy assumption, we consider that $\mathcal{E}$ is locally cofibrant and that $\mathcal{V}$ satisfies ($\clubsuit$) in [39, Definition 6.2.1]. Thus, under these hypothesis, Proposition B.6 is also true.

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