Quasiclassical approach to the weak levitation of extended states in the quantum Hall effect

M. M. Fogler
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540
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The two-dimensional motion of a charged particle in a random potential and a transverse magnetic field is believed to be delocalized only at discrete energies $E_N$. In strong fields there is a small positive deviation of $E_N$ from the center of the $N$th Landau level, which is referred to as the “weak levitation” of the extended state. I calculate the size of the weak levitation effect for the case of a smooth random potential re-deriving earlier results of Haldane and Yang [Phys. Rev. Lett. 78, 298 (1997)] and extending their approach to lower magnetic fields. I find that as the magnetic field decreases, this effect remains weak down to the lowest field $B_{\text{min}}$ where such a quasiclassical approach is still justified. Moreover, in the immediate vicinity of $B_{\text{min}}$ the weak levitation becomes additionally suppressed. This indicates that the “strong levitation” expected at yet even lower magnetic fields must be of a completely different origin.

Not long ago, the study of the weak levitation phenomenon has been pioneered by Khmel’nitskii and Laughlin (KL) almost fifteen years ago. Based on scaling arguments (see Ref. 3 for review) KL further suggested that the energies $E_N$ of the extended states can be obtained by solving the equation

$$\sigma_{xy}(E_F = E_N) = (N + 1/2) \frac{e^2}{2\pi\hbar},$$

(1)

where $\sigma_{xy}$ is the unrenormalized (short length scale) Hall conductance, $E_F$ is the Fermi energy, and $N$ runs through the set of integer values $N = 0, 1, \ldots$. Using the Drude-Lorentz formula

$$\sigma_{xy} = \frac{e^2}{m} \frac{\omega_c}{\omega_c^2 + \tau^{-2}} n(E_F),$$

(2)

for the left-hand side, they obtained

$$E_N = (N + 1/2) \hbar \omega_c \left[ 1 + (\omega_c \tau)^{-2} \right],$$

(3)

where $\omega_c$ is the cyclotron frequency, $\tau$ is the zero-field momentum relaxation time, and $n$ is the electron density. Thus, in strong fields where $\omega_c \tau \gg 1$ the energy $E_N$ of the $N$th extended state is close to the center of the $N$th Landau level $E_N^\infty = \hbar \omega_c (N + 1/2)$. As the magnetic field decreases, $E_N$ floats upward with respect to $E_N^\infty$, so that the relative deviation $\delta E_N/E_N^\infty$ increases

$$\delta E_N/E_N^\infty \equiv (E_N - E_N^\infty)/E_N^\infty = (\omega_c \tau)^{-2},$$

(4)

I will call the regimes $\delta E_N/E_N^\infty \ll 1$ and $\delta E_N/E_N^\infty \gg 1$ a “weak” and a “strong” levitation regimes, respectively. The regime of strong levitation is, of course, the most interesting. Unfortunately, this regime is also the hardest one to study. Up to date there is no progress in analytical treatment of this problem; as for the original arguments they suffer from the absence of completely satisfactory derivation of the scaling laws. Since the total energy of the electron is equal to the sum of the energies of the cyclotron and the drift degrees of freedom,
and the cyclotron energy is quantized in $\hbar \omega_c$ quanta, in this approximation $E_N$ is equal to $E_N^\infty$.

The calculation of the the weak levitation correction requires taking into account the Landau level mixing. HY demonstrated that such a mixing simply modifies the form of the potential in which the guiding center drifts. The new potential is not statistically symmetric under sign change. In fact, its percolation level is at higher energy. Correspondingly, $E_N$ is larger than $E_N^\infty$.

Using the quantum-mechanical perturbation theory, HY found that

$$\delta E_N \sim (N + 1/2)(W^2/\hbar \omega_c)(1/d)^4,$$

where $W$ is the rms amplitude of the random potential. I retained only the first term in the perturbation series obtained by HY. The next term is smaller if $d \gg lN^{1/2}$. In addition, HY require that $\hbar \omega_c \gg W$. Denote by $R_c$ the classical cyclotron radius at energy $E_N$. It is easy to see that $R_c = \sqrt{2N + 1}l$, so that Eq. (7) becomes

$$\delta E_N/E_N^\infty \sim (W/E_N)^2(R_c/d)^4,$$

and the condition $d \gg lN^{1/2}$ is simply $R_c/d \ll 1$. It immediately hits the eye that HY’s result is expressed in terms of purely classical quantities, given the particle’s energy is equal to $E_N$. Note also that the size of the effect is different from KL’s formula [6]. For instance, it is much larger provided that $R_c/d \gg W/E_N$. For a weak random potential, $W \ll E_F \approx E_N$, this inequality can be met simultaneously with $R_c/d \ll 1$.

Naively, one might think that the weak levitation mechanism of HY stops functioning in lower magnetic fields where $R_c \gg d$. This turns out not to be true; however, the dependence of the weak levitation effect on magnetic field becomes slower,

$$\delta E_N/E_N^\infty \sim (W/E_N)^2(R_c/d)^4 \sim (\omega_c \tau)^{-1}.$$  

[The final expression follows from $\tau \approx (d/v_F)(E_F/W)^2$.]

Equation (8) is the central result of this paper. It is represented graphically in Fig. 1 together with the previous two. The plot should be understood as the dependence of the quantity $\delta E_N/E_N^\infty$ at the topmost Landau level on the ratio $R_c/d$. In other words, I assumed that the Fermi energy $E_F$ is fixed but the magnetic field is changing. For each value of the magnetic field one has to choose $E_N$ closest to $E_F$. Of course, discreteness of $N$ leads to some fine details on the curve in Fig. 1. Such details are insignificant for large $N$, which is assumed to be the case for the most points on the plot.

As one can see from Eq. (8), $\delta E_N/E_N^\infty$ monotonically grows as the magnetic field decreases. Naturally, one would like to know if it ever becomes of the order of one. The answer is negative: even at the lowest magnetic field $B_{\min}$ where the present approach is justified, the quantity $\delta E_N/E_N^\infty$ is still small. To verify that one needs to know what $B_{\min}$ is. Clearly, $B_{\min}$ is the largest of the two fields, at which the two simplifying considerations mentioned in the beginning of the paper breaks down. One is the field $B_c$ where the separation into the cyclotron and drift motion ceases to be valid, and the other is the field where the quantum uncertainty ($\sim l$) of the guiding center position becomes of the order of $d$. The crossover field $B_c$ was calculated in Ref. [7]. It corresponds to the point $R_c/d \approx (E_F/W)^{2/3}$ where the characteristic frequency of the drift motion becomes of the order of $\omega_c$. Combining this with the other condition, one obtains the largest value of the ratio $R_c/d$ where the calculation is still valid,

$$(R_c/d)_{\max} = \min \{((E_F/W)^{2/3}, k_F d)$$

($k_F$ is the Fermi wavevector in zero field). Substituting this value into Eq. (8) and keeping in mind that $W \ll E_F$, one obtains that $(\delta E_N/E_N^\infty)_{\max} \ll 1$ (see also Fig. 1).

Let us now turn to the derivation of Eqs. (6) and (7). I am going to show that the effect is completely classical and therefore the constraint $W \ll \hbar \omega_c$ imposed by HY is extraneous. The relevant condition is just $R_c/d \ll (R_c/d)_{\max}$. Also, in contrast to the quantum-mechanical treatment of Ref. [5], my instrument will be the classical perturbation theory. For this reason I will drop the subscript “$\infty$” in $E_F$. To ensure continuity with the previous paper [5] I will assume that the magnetic field is in the negative $\hat{z}$-direction, so that the cyclotron gyration is clockwise and the guiding center coordinates are given by

$$\rho_x = x + (v_y/\omega_c), \quad \rho_y = y - (v_x/\omega_c),$$

where $x$ and $y$ are the coordinate of the electron, and $v = -v (\sin \theta, \cos \theta)$ is its velocity. The most convenient form of the equation of motion for $\rho$ is obtained if $U$ is re-expressed in terms $\rho$ and $\theta$ only, i.e., as a new function

$$V(\rho, \theta) = U[\rho_x + (v/\omega_c) \cos \theta, \rho_y - (v/\omega_c) \sin \theta],$$

where $v = \sqrt{2(E - V)/m}$ because of the energy conservation. The equation of motion for $\rho$ is
This equation is of the Hamiltonian form with $\theta/\omega_c$, $\rho_\theta$, $m\omega_c\rho_x$, and $V$ being the time variable, the canonical coordinate, momentum, and the Hamiltonian function, respectively. The return to the original time variable can be accomplished by means of the equation

$$\dot{\theta} = \omega_c + \hat{\theta} / (m\nu^2).$$

It is convenient to expand $V(\rho, \theta)$ in Fourier series,

$$V(\rho, \theta) = \sum_{k=-\infty}^{\infty} V_k(\rho) e^{-ik\theta}. \hspace{1cm} (10)$$

If $|k| \lesssim R_c/d$, the absolute value of $V_k$ is of the order of $W_0 = W(d/R_c)^{1/2}$; otherwise, it is much smaller. Substituting Eq. (10) into Eq. (3), one obtains

$$d\rho/d\theta = \frac{1}{m\omega_c^2} \sum_{k=-\infty}^{\infty} \left[ \hat{\rho} \times \nabla \rho V_k(\rho) \right] e^{-ik\theta}. \hspace{1cm} (11)$$

If one retains only the $k = 0$ term, then the right-hand side does will not depend on $\theta$ and thus the guiding center motion will decouple from the cyclotron one. In this approximation the guiding center performs the drift along the level lines of the potential $V_0$. In the $R_c \ll d$ limit $V_0$ is very close to the original potential $U$ in agreement with the qualitative picture given above. If $R_c$ is larger than $d$, then quite a few $k \neq 0$ terms are of the same magnitude as $k = 0$ one. In this case one can not simply ignore them; however, they can be made smaller by means of series of canonical (or almost canonical) transformations. Each consequent transformation reduces the oscillating terms by a factor of the order of $\gamma \equiv W_0/(m\omega_c^2d^2) \ll 1$. In the end they become suppressed by a factor $\exp(-\text{const}/\gamma)$. This program can be realized only if $\gamma \ll 1$. Equation $\gamma = 1$ thus determines the magnetic field $B_c$ (see above) where the crossover from the adiabatic drift to the random walk of the guiding center occurs.

For calculation of $\delta E_N$ to the first nonvanishing order in $\gamma$ only one such a transformation suffices. Let $p$ and $q$ be the new canonical coordinates after the transformation. Define the “renormalized” guiding center coordinates, $\rho_x^{(1)} = p/(m\omega_c)$ and $\rho_y^{(1)} = q$. It is easy to see that $\rho^{(1)}$ has the following form

$$\rho^{(1)} = \rho + \frac{1}{m\omega_c^2} \sum_{k=0}^{\infty} \frac{1}{ik} \hat{\rho} \times \nabla \rho V_k + O(\gamma^2d). \hspace{1cm} (12)$$

The $\theta$-independent term $V_0$ in the Hamiltonian function is transformed into $V_{\text{eff}}$ given by

$$V_{\text{eff}} = V_0 + \frac{1}{m\omega_c^2} \sum_{k=1}^{\infty} \frac{\hat{\rho}}{ik} [\nabla \rho V_{-k} \times \nabla \rho V_k] + O(\gamma^2W_0).$$

Define

$$U_k(\rho, K) \equiv \int \frac{d\phi}{2\pi} e^{-ik\phi} U[\rho_x + R \cos \phi, \rho_y + R \sin \phi],$$

where $K$ has the meaning of the kinetic energy and $R = \sqrt{2K/m\omega_c^2}$ of the corresponding cyclotron radius. Note a useful formula

$$\tilde{U}_k(q, K) = ik e^{-ik\theta_k} J_0(qR) \tilde{U}(q), \hspace{1cm} (13)$$

where $\tilde{t}$ symbolizes the Fourier transform, $q = q(\cos \theta_k, \sin \theta_k)$, and $J_0$ is the Bessel function.

It is easy to see that

$$V_{\text{eff}}(\rho, K) = U_0 + \frac{1}{m\omega_c^2} \sum_{k=1}^{\infty} (Y_k - Z_k) + O(\gamma^2W_0), \hspace{1cm} (14)$$

$$Y_k = \frac{\hat{\rho}}{ik} [\nabla U_{-k} \times \nabla U_k], \hspace{1cm} Z_k = \frac{1}{R \nabla R}(U_0)^2. \hspace{1cm} (15)$$

The obtained expression agrees with the effective potential of HY in the limit $R_c \ll d$. To see that one has to quantize the kinetic energy $K = E - V_{\text{eff}} = h \omega_c(N + 1/2)$ and keep only $k = 1$ and $k = 2$ terms, which dominate the sum in this limit. The levitation correction can be estimated as

$$\delta E_N \sim (V_{\text{eff}})_{\text{SP}}, \hspace{1cm} (16)$$

where “SP” stands for saddle-points, i.e., the points where

$$(A) \hspace{1cm} U_0^x = U_0^y = 0, \hspace{1cm} (B) \hspace{1cm} U_0^{xx}U_0^{yy} - U_0^{xy}U_0^{yx} < 0 \hspace{1cm} (17)$$

(the superscripts denote the partial derivatives). The rest of the paper is devoted to the derivation of Eq. (8) from Eqs. (3,4,6). I will assume that $U(x, y)$ is an isotropic Gaussian random potential with zero mean.

It turns out that for each $k$, $Y_k$ and $Z_k$ are correlated with at most one of the sets $\{U_0^x, U_0^y\}$ and $\{U_0^{xx}, U_0^{yy}, U_0^{xy}\}$. Therefore, each time one needs to calculate either $(Y_k)_A$ or $(Y_k)_B$, or simply the unrestricted average $(Y_k)$ (and similarly for $Z_k$). The conditions “$A$” and “$B$” are given by Eq. (17). Notice that $Y_k$ and $Z_k$ are bilinear in $U$. This allows us to perform the $(\ldots)_A$ averaging by means of the following general formula. Let $X_1$ and $X_2$ be linear in $U$, then

$$\langle X_1 X_2 \rangle_A = \langle X_1 X_2 \rangle - \langle U_0^x U_0^y \rangle^{-1} \times (\langle X_1 U_0^x \rangle \langle X_2 U_0^x \rangle + \langle X_1 U_0^y \rangle \langle X_2 U_0^y \rangle). \hspace{1cm} (18)$$

Similarly,

$$\langle X_1 X_2 \rangle_B = \langle X_1 X_2 \rangle - \langle U_0^x U_0^y \rangle^{-1} (\langle X_1 U_0^y \rangle \langle X_2 U_0^x \rangle - \frac{1}{2} \langle X_1 U_0^{xx} \rangle \langle X_2 U_0^{yy} \rangle - \frac{1}{2} \langle X_1 U_0^{xy} \rangle \langle X_2 U_0^{yx} \rangle). \hspace{1cm} (19)$$

Equations (18) and (19) can be obtained using general properties of Gaussian potentials and isotropicity of $U$. The $(\ldots)_A$ averaging is needed for $Y_2$ and $Z_1$, the $(\ldots)_B$ averaging is needed for $Y_1$, $Y_3$, and $Z_2$. All other $Y_k$'s
and $Z_k$'s are to be averaged over the entire plane. The calculation is trivial but lengthy, and so I will give only the final result:

$$\langle V_{\text{eff}} \rangle_{\text{SP}} = \frac{1}{m\omega_c^2} \left[ \frac{15A_{13}^2 + A_{23}^2}{36A_{04}} - \frac{A_{22}(2A_{02} + A_{22})}{4A_{02}} \right],$$

where $A_{kn}$ is defined as follows

$$A_{kn} = \int \frac{d^2q}{(2\pi)^2} \tilde{C}(q) J_0(qR) J_k(qR) q^n,$$

with $C$ being the correlator $C(r) = \langle U(0)U(r) \rangle$. If $R_c \gg d$, then

$$\langle V_{\text{eff}} \rangle_{\text{SP}} \simeq \frac{A_{02}}{4m\omega_c^2} \sim \frac{W^2}{m\omega_c^2 R_c^2}.$$  \hspace{1cm} (21)

This concludes the derivation because Eqs. (1) and (21) immediately give Eq. (7).

The correction $(V_{\text{eff}} - V_0)$ has quite peculiar properties: it is typically positive at the points where the gradient squared $(\nabla U_0)^2$ is smaller than its average value, typically negative otherwise, and almost vanishes on average. Indeed, one can show that the unrestricted spatial average of $V_{\text{eff}}$ is $(V_{\text{eff}}) = -A_{01}/(2m\omega_c^2 R) \sim -\langle V_{\text{eff}} \rangle_{\text{SP}} (d/R_c)^2$, which is much smaller than $\langle V_{\text{eff}} \rangle_{\text{SP}}$ by the absolute value.

I speculate that the last property becomes important in the vicinity of $B_c$, where the crossover from the drift to the diffusion occurs. In the diffusive regime the trajectory of the guiding center is no longer bound to the level line $V_{\text{eff}} = \text{const}$ but samples the entire area. Thus, $\langle V_{\text{eff}} \rangle_{\text{SP}}$ should approach $(V_{\text{eff}})$. The latter is indistinguishable from zero within the accuracy of such an unrigorous argument. Thus, I expect the ultimate downfall of the solid curve in Fig. 1 near its termination point.

Concluding this paper, let us emphasize that the discrepancy between KL's formula (11) and Eqs. (13) and (17) does not contradict to Eq. (1), which comes from the scaling arguments. Indeed, for $B < B_c$ the “classical” or the “unrenormalized” Hall conductance is determined not by the average density $n$ but by the density $n_p$ near the percolation contour which is the area responsible for the transport; therefore,

$$\sigma_{xy} \simeq \frac{e^2}{m\omega_c} n_p.$$  \hspace{1cm} (22)

Equation (22) now follows from Eqs. (1), (2), and $n_p = (m/\pi R)(E_F - V_p)$ (“$p$" again means “percolation”). At $B > B_c$ the percolation contour is of no importance so that Eq. (22) becomes valid, and presumably so does Eq. (13) as well.

In conclusion, I showed that the relative size $\delta E_N/E_N^c$ of the weak levitation effect is always much smaller than one. Moreover, it is expected to decrease near its termination point $B = B_c$. This strongly suggests that this effect has nothing to do with the strong levitation predicted by Khmelnitskii and Laughlin. Finally, I should also mention that $\delta E_N$ can be measured experimentally by charting the global phase diagram of the quantum Hall effect. Very interesting and puzzling findings have been reported recently. However, it seems that the electron-electron interaction plays a crucial role in the observed phenomena. This complicated issue is beyond the scope of the present paper.

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