Compactification of the Bruhat-Tits building of PGL by seminorms

Annette Werner
Mathematisches Institut, Universität Münster, Einsteinstr. 62, D - 48149 Münster
e-mail: werner@math.uni-muenster.de

Abstract
We construct a compactification $\overline{X}$ of the Bruhat-Tits building $X$ associated to the group $PGL(V)$ which can be identified with the space of homothety classes of seminorms on $V$ endowed with the topology of pointwise convergence. Then we define a continuous map from the projective space to $\overline{X}$ which extends the reduction map from Drinfeld’s $p$-adic symmetric domain to the building $X$.

2000 MSC: 20E42, 20G25

Introduction

It is well-known that the Bruhat-Tits building $X$ of $PGL(V)$ can be identified with the set of homothety classes of norms on $V$. It can also be described with lattices of full rank in $V$. In [We] we constructed a compactification of $X$ which takes into account lattices of arbitrary rank in $V$.

The goal of the present paper is to describe a related compactification $\overline{X}$, which is more adapted to the description of the building in terms of norms. It is in some sense dual to the construction given in [We]. Whereas there the boundary of $X$ consists of all buildings corresponding to subspaces of $V$, here we attach the buildings corresponding to quotient spaces of $V$. We prove that our construction leads to a compact, contractible space $\overline{X}$, and that $PGL(V)$ acts continuously on $\overline{X}$.

A nice feature of this new construction is that it has a very natural description: Namely, $\overline{X}$ can be identified with the set of homothety classes of seminorms, endowed with the topology of pointwise convergence.

In the final section, we use the compactification $\overline{X}$ in order to show that the reduction map $r : \Omega^m \to X$ from the Berkovich analytic space corresponding to Drinfeld’s $p$-adic symmetric domain to the building $X$ has a natural extension to a continuous,
$PGL(V)$-equivariant map

$$r : \mathbb{P}(V)^{an} \rightarrow \overline{X},$$

where $\mathbb{P}(V)^{an}$ is the Berkovich analytic space induced by the projective space. Besides we show that $r$ has a continuous section $j : \overline{X} \rightarrow \mathbb{P}(V)^{an}$, which induces a homeomorphism between $\overline{X}$ and a closed subset of $\mathbb{P}(V)^{an}$.

We hope that the compactification $\overline{X}$ can be used to construct a kind of Satake compactification for Bruhat-Tits buildings associated to arbitrary reductive groups.

**Acknowledgements:** I am much indebted to Matthias Strauch for some very fruitful discussions. In fact, the idea to modify the construction in [We] in order to extend the reduction map from Drinfeld’s symmetric domain to the building was born during one of them. Besides, it is a pleasure to thank Vladimir Berkovich for some useful conversations about his analytic spaces.

## 1 Notation and conventions

Throughout this paper we denote by $K$ a non-archimedean local field, by $R$ its valuation ring and by $k$ the residue class field. Besides, $v$ is the valuation map, normalized so that it maps a prime element to 1, $q$ is the number of elements in the residue field and $|x| = q^{-v(x)}$ the absolute value on $K$.

We adopt the convention that “$\subset$” always means strict subset, whereas we write “$\subseteq$”, if equality is permitted.

Let $V$ be an $n$-dimensional vector space over $K$, and let $G$ be the algebraic group $PGL(V)$.

## 2 Seminorms

We call a map $\gamma : V \rightarrow \mathbb{R}_{\geq 0}$ a seminorm, if $\gamma$ is not identically zero, and satisfies

i) $\gamma(\lambda v) = |\lambda|\gamma(v)$ for all $\lambda \in K$ and $v \in V$, and

ii) $\gamma(v + w) \leq \sup\{\gamma(v), \gamma(w)\}$ for all $v, w$ in $V$.

We call $\gamma$ canonical with respect to a basis $v_1, \ldots, v_n$ of $V$, if

$$\gamma(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \sup\{|\lambda_1|\gamma(v_1), \ldots, |\lambda_n|\gamma(v_n)\}$$

for all $\lambda_1, \ldots, \lambda_n$ in $K$. 

2
A seminorm $\gamma$ satisfying

iii) $\gamma(x) > 0$ for all $x \neq 0$,

is a norm on $V$.

A seminorm $\gamma$ on $V$ induces in a natural way a norm on the quotient space $V/\ker\gamma$. By [Go-I], Proposition 3.1, for any norm $\gamma$ there exists a basis with respect to which $\gamma$ is canonical. Looking at the norms induced on a quotient, we find that the same holds for seminorms.

Two norms or seminorms $\gamma_1$ and $\gamma_2$ on $V$ are called equivalent, iff there is a positive real constant $c$ such that $\gamma_1 = c\gamma_2$.

3 Compactification of one apartment

We fix a basis $v_1, \ldots, v_n$ of $V$, which defines a maximal $K$-split torus $T$ in $G$, induced by the torus $T^\sim$ of diagonal matrices in $GL(V)$ with respect to $v_1, \ldots, v_n$. We denote by $\chi_i$ the character of $T^\sim$ mapping a diagonal matrix to its $i$-th entry. Then all $\chi_i/\chi_j$ define a character $a_{ij}$ of $T$.

Let $N$ be the normalizer of $T$ in $G$. We denote by $T = T(K), N = N(K)$ and $G = G(K)$ the groups of rational points. Then $W = N/T$ is the Weyl group of the root system $\Phi = \{a_{ij} : i \neq j\}$ corresponding to $T$. We can identify $W$ in a natural way with the group of permutations of $\{1, \ldots, n\}$. By embedding $W$ as the group of permutation matrices in $N$, we find that $N$ is the semidirect product of $T$ and $W$.

By $X_*(T)$ respectively $X^*(T)$ we denote the cocharacter respectively the character group of $T$. Let $A$ be the $\mathbb{R}$-vector space $A = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We take $A$ as our fundamental apartment in the definition of the Bruhat-Tits building associated to $G$.

Let $\eta_i : \mathbb{G}_m \to T$ be the cocharacter induced by mapping $x$ to the diagonal matrix with diagonal entries $d_1, \ldots, d_n$ such that $d_k = 1$ for $k \neq i$ and $d_i = x$. Then $\eta_1 + \ldots + \eta_n = 0$, and $\eta_1, \ldots, \eta_{n-1}$ is an $\mathbb{R}$-basis of $A$. If $t \in T$ is induced by the diagonal matrix with entries $d_1, \ldots, d_n$, we define a point $\nu(t) \in A$ by $\nu(t) = -v(d_1)\eta_1 - \ldots - v(d_n)\eta_n$.

Every $t \in T$ acts on $A$ as translation by $\nu(t)$, Besides, $W$ acts as a group of reflections on $A$. Since $N$ is the semidirect product of $T$ and $W$, we can define an action of $N$ on $A$ by affine bijections.

It is well-known that $A$ can be identified with the set $N_c$ of equivalence classes of those norms on $V$ which are canonical with respect to the basis $v_1, \ldots, v_n$, cf. [Br-T]. To be more explicit, define a map

$$\varphi : A \longrightarrow N_c$$
by mapping \( x = x_1\eta_1 + \ldots + x_n\eta_n \) in \( A \) to the class of norms represented by

\[
\gamma(\lambda_1v_1 + \ldots + \lambda_nv_n) = \sup\{|\lambda_1|q^{-x_1}, \ldots, |\lambda_n|q^{-x_n}|.
\]

It is easily seen that \( \varphi \) is well-defined, bijective and \( N \)-equivariant, if we let \( N \) act on the class of equivalence classes of norms by \( \gamma \mapsto \gamma \circ n^{-1} \).

Let us denote by \( S'_c \) the set of all seminorms on \( V \) which are canonical with respect to \( v_1, \ldots, v_n \), and by \( S_c \) the quotient of \( S'_c \) by the equivalence relation on seminorms defined above. We will now define a compactification \( \overline{A} \) of \( A \) so that \( \varphi \) can be extended to a homeomorphism from \( \overline{A} \) to \( S_c \).

We write \( n \) for the set \( \{1, \ldots, n\} \). Let \( J \) be a non-empty subset of \( n \), and let \( V_J \) be the subspace of \( V \) generated by the \( v_i \) for \( i \in J \). We write \( G^V_J \) for the subgroup of \( G = PGL(V) \) consisting of the elements leaving the subspace \( V_J \) invariant, and \( G_J \) for the algebraic group \( PGL(V_J) \). Then we have a natural restriction map \( \rho_J : G^V_J \to G_J \).

The torus \( T \) is contained in \( G^V_J \), and its image under \( \rho_J \) is a maximal \( K \)-split torus \( T_J \) in \( G_J \), namely the torus induced by the diagonal matrices with respect to the base \( \{v_i : i \in J\} \) of \( V_J \).

Besides, we have the quotient map \( q_J : V \to V/V_J \). There is a natural homomorphism

\[
\sigma_{\varnothing \setminus J} : G^V_J \to PGL(V/V_J).
\]

Then for all subsets \( I \subset n \) the homomorphism \( \sigma_I \) maps \( T \) to a split torus \( T_I \) in \( PGL(V/V_{\varnothing \setminus I}) \). Put \( T_I = T_I(K) \). Besides, we put

\[
A_I = X_\ast(T_I) \otimes \mathbb{R}.
\]

Then \( \sigma_I \) induces a surjective homomorphism

\[
s_I : A \to A_I,
\]

mapping the cocharacter \( \eta_i \) to zero, if \( i \) is not contained in \( I \), and to the induced cocharacter \( \overline{\eta}_i \) of \( T_I \), if \( i \) is contained in \( I \). Then \( A_I \) is generated by the cocharacters \( \overline{\eta}_i \) for \( i \in I \), subject to the relation \( \sum_{i \in I} \overline{\eta}_i = 0 \).

Besides, we define a homomorphism \( \nu_I : T_I \to A_I \) by \( \nu_I(t) = \sum_{i \in I} -v(d_i)\overline{\eta}_i \), if \( t \) is induced by the matrix with diagonal entries \( d_i \) for all \( i \in I \). Obviously, \( \nu_I \circ \sigma_I = s_I \circ \nu \) on \( T \).

Now put

\[
\overline{A} = A \cup \bigcup_{\emptyset \neq I \subseteq n} A_I = \bigcup_{\emptyset \neq I \subseteq n} A_I.
\]

Here of course \( A_{\emptyset} = A \) and \( s_\emptyset \) is the identity.
Let us denote by $S_c'(I)$ the set of all seminorms on $V$ which are canonical with respect to $v_1, \ldots, v_n$, and whose kernel is equal to $V_{n,I}$, and by $S_c(I)$ the corresponding quotient space with respect to equivalence of norms. Then we define a map

$$\varphi : A_I \rightarrow S_c(I)$$

by associating to the point $x = \sum_{i \in I} x_i \eta_i$ in $A_I$ the seminorm represented by

$$\gamma(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \sup\{ |\lambda_i| q^{-x_i} : i \in I \}$$

Obviously, $\varphi : A_I \rightarrow S_c(I)$ is bijective.

Combining all these maps yields a bijection $\varphi : \overline{A} \rightarrow S_c$.

Now we want to define an action of $N$ on $\overline{A}$. First of all, we use the homomorphism $\nu_I \circ \sigma_I = s_I \circ \nu : T \rightarrow A_I$ to define an action of $T$ on $A_I$ by affine transformations. For $w \in W$ we denote the induced permutation of the set $n$ also by $w$, i.e. we abuse notation so that $w(v_i) = v_{w(i)}$. Now define an action of $w$ on $\overline{A}$ by putting together the maps

$$w : A_I \rightarrow A_{w(I)},$$

sending $\eta_i^I$ to $\eta_{w(I)}^{w(I)}$. Note that it is compatible with the action of $w$ on $A$, i.e. we have

$$w \circ s_I = s_{w(I)} \circ w$$

on $A$. These two actions give rise to an action of $N = T \times W$ on $\overline{A}$, which we denote by $\nu$. If $N$ acts in the usual way on $S_c$, i.e. by $\gamma \mapsto \gamma \circ n^{-1}$, it can easily be checked that the bijection $\varphi$ is $N$-equivariant.

Let us now define a topology on $\overline{A}$. For all $I \subset n$ we put

$$\Delta_I = \sum_{i \notin I} \mathbb{R}_{\geq 0} \eta_i \subset A.$$ 

For all open and bounded subsets $U \subset A$ we define

$$\Gamma_U^I = (U + \Delta_I) \cup \bigcup_{I \subset J \subset n} s_J(U + \Delta_I).$$

We take as a base of our topology on $\overline{A}$ the open subsets of $A$ together with these sets $\Gamma_U^I$ for all non-empty $I \subset n$ and all open bounded subsets $U$ of $A$.

Note that every point $x \in \overline{A}$ has a countable fundamental system of neighbourhoods. This is clear for $x \in A$. If $x$ is in $A_I$ for some $I \subset n$, then choose some $z \in A$
with $s_I z = x$, and choose a countable decreasing fundamental system of bounded open neighbourhoods $(V_k)_{k \geq 1}$ of $z$ in $A$. Put $U_k = V_k + \sum_{i \in I} k \eta_i$. This is an open neighbourhood of $z + k \sum_{i \in I} \eta_i$. Then $(V_{U_k})_{k \geq 1}$ is a fundamental system of open neighbourhoods of $x$.

Our next goal is to compare the previous construction to the compactification of the apartment $A$ which was used in [We]. Denote by $V^*$ the dual vector space corresponding to $V$. The map $g \mapsto g^{-1}$ induces an isomorphism $\alpha : PGL(V^*) \to PGL(V)$, which maps the torus $T'$, given by the diagonal matrices with respect to the dual basis $v_1^*, \ldots, v_n^*$, to the torus $T$. This defines an isomorphism of the cocharacter groups

$$\alpha_* : X_*(T') \to X_*(T).$$

Note that $\alpha_*$ introduces a sign, i.e., if $\eta'_i$ is given by the map sending $x$ to the diagonal matrix with $i$-th entry $x_i$ and entries 1 at the other places, then $\alpha_*$ maps $\eta'_i$ to the cocharacter $-\eta_i$. If we denote by $\Lambda'$ the apartment in the building of $PGL(V^*)$ given by the torus $T'$, the map $\alpha_*$ induces an isomorphism of $\mathbb{R}$-vector spaces $\beta : \Lambda' \to A$.

Let $I$ be a non-empty subset of $\underline{n}$, and let $(V^*)_I$ be the subspace of $V^*$ generated by all $v_i^*$ for $i \in I$. Then $(V^*)_I \simeq (V/V_{\underline{n}\setminus I})^*$. Hence $\alpha$ induces an isomorphism $\alpha_I : PGL((V^*)_I) \to PGL(V/V_{\underline{n}\setminus I})$ making the following diagram commutative

$$\begin{array}{ccc}
PGL(V^*)(V^*)_I & \longrightarrow & PGL(V)^{V_{\underline{n}\setminus I}} \\
\rho_I \downarrow & & \downarrow \sigma_I \\
PGL((V^*)_I) & \longrightarrow & PGL(V/V_{\underline{n}\setminus I})
\end{array}$$

Restricting $\alpha_I$ to the torus $T'_I$ induced by the diagonal matrices with respect to $v_i^*$ for $i \in I$, we get an isomorphism $T'_I \to T_I$. This induces an isomorphism

$$(\alpha_I)_* : X_*(T'_I) \to X_*(T_I).$$

If $\eta_i'$ denotes the cocharacter induced by mapping $x$ to the diagonal matrix with entry $x$ at the place $i$, and with entry 1 at the places $j \neq i$ in $I$, we have $\alpha_*(\eta_i') = -\eta_i'$. Again we put $\Lambda'_I = X_*(T'_I) \otimes \mathbb{R}$. Then $(\alpha_I)_*$ induces an $\mathbb{R}$-linear isomorphism $\beta : \Lambda'_I \to A_I$.

Now we put as in [We], section 3, $\overline{\Lambda} = \Lambda' \cup \bigcup_{\emptyset \neq I \subset \underline{n}} \Lambda'_I$. Then we have defined a bijection

$$\beta : \overline{\Lambda} \to \overline{\Lambda}.$$ 

Note that by definition $\beta$ is a homeomorphism, if $\overline{\Lambda}$ is endowed with the topology of $[\overline{\Lambda}$, section 3. Besides, in [We], section 3, we defined an action $\nu'$ of $N'$ on $\overline{\Lambda}$, where $N'$ is the normalizer of $T'$. Via the isomorphism $\alpha : N \to N'$, this is compatible with the $N$-action on $\overline{\Lambda}$ defined above.

Hence we can deduce
Theorem 3.1  i) The topological space $\overline{A}$ is compact and contractible, and $A$ is an open, dense subset of $\overline{A}$.

ii) The action $\nu : N \times \overline{A} \rightarrow \overline{A}$ is continuous.

Proof: This follows immediately from [We], Theorem 3.4 and Lemma 3.5.

We can endow the space $S_c'$ of canonical seminorms with the topology of pointwise convergence, i.e. with the coarsest topology such that for all $v \in V$ the map $\gamma \mapsto \gamma(v)$ from $S_c'$ to $\mathbb{R}_{\geq 0}$ is continuous. On $S_c$ we have the quotient topology. Both $S_c'$ and $S_c$ are Hausdorff. We will now show that the bijection $\varphi$ defined above is in fact a homeomorphism.

Proposition 3.2 The $N$-equivariant bijection $\varphi : \overline{A} \rightarrow S_c$ is a homeomorphism.

Proof: Note that it is enough to show that $\varphi$ is continuous, since a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.

Let $x = \sum_{i \in I} x_i \eta_i$ be a point in $A_I$, and assume that $x_{i_0} = 0$ for some $i_0 \in I$. We want to show that $\varphi$ is continuous in $x$. If $\gamma$ is the seminorm $\gamma(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \sup\{|\lambda_i| q^{-x_i+\lambda_i} : i \in I\}$, then $\varphi(x)$ is represented by $\gamma$.

If $U$ is an open neighbourhood of $\varphi(x) = \{\gamma\}$, then it contains a set of the form

$$\{\{\beta\} \in S_c : |\beta(v_i) - \gamma(v_i)| < \varepsilon \text{ for all } i = 1, \ldots, n\}.$$

We find an open and bounded subset $V$ of $A$ so that all $\sum_{i=1}^n y_i \eta_i \in V$ satisfy

$$|q^{-y_i+y_{i_0}} - q^{-x_i}| < \varepsilon \quad \text{for all } i \in I \text{ and}$$

$$|q^{-y_i+y_{i_0}}| < \varepsilon \quad \text{for all } i \notin I.$$

Then $\Gamma_V = (V + \Delta_I) \cup \bigcup_{I \subseteq J \subseteq N} s_J(V + \Delta_I)$ is an open neighbourhood of $x$ in $\overline{A}$. Let us show that $\varphi(\Gamma_V)$ is contained in $U$. If $y$ is a point in $\Gamma_V$, say $y = s_J(z)$ for some $I \subseteq J$ and some $z = \sum_{i=1}^n z_i \eta_i \in V + \Delta_I$, then $\varphi(y)$ is represented by the seminorm $\beta$ with $\beta(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \sup\{|\lambda_i| q^{-z_i+\lambda_i}+z_{i_0} : i \in J\}$. Hence $|\beta(v_i) - \gamma(v_i)| < \varepsilon$ for all $i = 1, \ldots, n$, so that $\varphi(y) \in U$. Therefore $\varphi$ is indeed continuous.

4 Compactification of the whole building

Let us first recall the construction of the Bruhat-Tits building $X$ corresponding to $G$. For every root $a = a_{ij}$ we denote by $U_a$ the root group in $G$, i.e. the group of matrices
$U = (u_{kl})_{k,l}$ such that the diagonal elements $u_{kk}$ are equal to one, and all the other entries apart from $u_{ij}$ are zero. Then we have a homomorphism

$$\psi_a : U_a \rightarrow \mathbb{Z} \cup \{\infty\}$$

by mapping the matrix $U = (u_{kl})_{k,l}$ to $v(u_{ij})$. Put for all $l \in \mathbb{Z}$

$$U_{a,l} = \{u \in U_a : \psi_a(u) \geq l\}.$$ 

We also define $U_{a,\infty} = \{1\}$, and $U_{a,-\infty} = U_a$. For all $x \in A$ let now $U_x$ be the group generated by $U_{a,-a(x)} = \{u \in U_a : \psi_a(u) \geq -a(x)\}$ for all $a \in \Phi$. Besides, put $N_x = \{n \in N : \nu(n)x = x\}$, and

$$P_x = U_xN_x = N_xU_x.$$ 

Then the building $X = X(\text{PGL}(V))$ is given as

$$X = G \times A/\sim,$$

where the equivalence relation $\sim$ is defined as follows:

$$(g, x) \sim (h, y), \quad \text{iff there exists an element } n \in N \text{ such that } \nu(n)x = y \text{ and } g^{-1}hn \in P_x.$$ 

There is a continuous action of $G$ on $X$ via left multiplication on the first factor, which extends the $N$-action on $A$. For all $x \in A$ the group $P_x$ is in fact the stabilizer of $x$.

Now we define for all non-empty subsets $\Sigma$ of $\mathbb{A}$ and all roots $a \in \Phi$

$$f_{\Sigma}(a) = \inf\{t : \Sigma \subseteq \{z \in A : a(z) \geq -t\}\}$$

$$= -\sup\{t : \Sigma \subseteq \{z \in \Lambda : a(z) \geq t\}\}$$

Here we put $\inf\emptyset = \sup\mathbb{R} = \infty$ and $\inf\mathbb{R} = \sup\emptyset = -\infty$. Obviously, $f_x(a) = -a(x)$, if $x$ is contained in $A$.

We define a subgroup

$$U_{a,\Sigma} = U_{a,f_{\Sigma}(a)} = \{u \in U_a : \psi_a(u) \geq f_{\Sigma}(a)\}$$

of $U_a$, where $U_{a,\infty} = 1$ and $U_{a,-\infty} = U_a$. By $U_\Sigma$ we denote the subgroup of $G$ generated by all the $U_{a,\Sigma}$ for roots $a \in \Phi$.

Recall from the previous section that the isomorphism $\alpha : \text{PGL}(V^*) \rightarrow \text{PGL}(V)$ induces a homeomorphism $\beta : \overline{\Lambda} \rightarrow \overline{\Lambda}$. Besides, $\alpha$ induces a bijection between the root systems $\alpha^* : \Phi = \Phi(T) \rightarrow \Phi(T')$. Let us define as in [We], section 4, for all non-empty subsets $\Omega$ of $\overline{\Lambda}$ and all roots $a \in \Phi(T')$ the subgroup $U_{a,\Omega}$ of the group $\text{PGL}(V^*)$ as $\{u \in U_a : \psi_a(u) \geq f_{\Omega}(a)\}$ for $f_{\Omega}(a) = \inf\{t : \Omega \subseteq \{z \in \Lambda' : a(z) \geq -t\}\}$. Besides,
let $U_\Omega$ be the subgroup of $PGL(V^*)$ generated by all $U_{a,\Omega}$, and put $P_\Omega = N_\Omega U_\Omega$. Then $f_\Omega(\alpha^*(a)) = f_\beta(\Omega)(a)$, and hence $\alpha(U_{\alpha^*(a),\Omega}) = U_{a,\beta(\Omega)}$ for all roots $a \in \Phi$, which implies

$$\alpha(U_\Omega) = U_{\beta(\Omega)}.$$  

Besides, we define for all non-empty $\Sigma \subseteq \overline{A}$ the group $N_\Sigma = \{ n \in N : \nu(n)x = x \text{ for all } x \in \Sigma \}$, and

$$P_\Sigma = U_\Sigma N_\Sigma = N_\Sigma U_\Sigma.$$  

Since $N_\Sigma$ normalizes $U_\Sigma$, which can be shown as in $[\text{We}], 4.4$, this set is indeed a group. Besides, we have

$$\alpha(P_\Omega) = P_{\beta(\Omega)}.$$  

Now denote by $Z \subset T$ the kernel of the map $\nu : T \to A$. We fix the point $0 \in A$. The group $U_0^\wedge = U_0Z$ is compact and open in $G$, see $[\text{La}], 12.12$. We define the compactification $\overline{X}$ of the building $X$ as

$$\overline{X} = U_0^\wedge \times \overline{A} / \sim,$$  

where the equivalence relation $\sim$ is defined as follows:

$$(g, x) \sim (h, y), \text{ iff there exists an element } n \in N \text{ such that } \nu(n)x = y \text{ and } g^{-1}hn \in P_x.$$  

Since $[\text{We}], 4.4$ implies that $nP_xn^{-1} = P_{\nu(n)x}$ for all $n \in N$ and $x \in \overline{A}$, it is easy to check that $\sim$ is indeed an equivalence relation. We equip $\overline{X}$ with the quotient topology. Hence $X$ is open and dense in $\overline{X}$. In $[\text{We}],$ section 4, we gave a similar definition of a compactification $\overline{X}'$ of the building $X'$ corresponding to the group $PGL(V^*)$, using the groups $P_x$ for $x \in \overline{A}$ recalled above, and the group $(U_0^\wedge)' = U_0Z'$ for $Z' = \ker(\nu' : T' \to \Lambda')$ and $0 \in \Lambda'$. Since $\alpha(P_x) = P_{\beta(x)}$, the homeomorphism

$$(\alpha, \beta) : (U_0^\wedge)' \times \overline{\Lambda'} \longrightarrow U_0^\wedge \times \overline{A}$$  

induces a homeomorphism

$$\overline{X}' \longrightarrow \overline{X}.$$  

Therefore we can use the results of $[\text{We}]$ to deduce

**Theorem 4.1** $\overline{X}$ is a compact and contractible topological space, containing the building $X$ as an open, dense subset.
Claim: Assume for the moment that we have proven the following claim:

Since $U$ converges in $\mathbb{A}$ to some subsequence of it) converges to $\mu$ in $\mathbb{A}$. Hence it suffices to show that every $g \in G$ acts continuously on $\mathbb{A}$. Let us denote the quotient map $\pi : \mathbb{A} \to \mathbb{A}$, defined by $\mu_g(u,x) = \pi(v, \nu(n)x)$, if $gu = vnh$ is a decomposition according to $G = U_0^\wedge NP_x$, is continuous.

Assume that $u_k$ is a sequence in $U_0^\wedge$ converging to $u \in U_0^\wedge$, and that $x_k$ is a sequence in $\mathbb{A}$ converging to $x \in \mathbb{A}$. Then we have to show that $\mu_g(u_k, x_k)$ (or at least a subsequence of it) converges to $\mu_g(u, x)$. Write $gu_k = v_k \gamma_k$ for $v_k \in U_0^\wedge$ and $\gamma_k \in NP_x$.

Since $U_0^\wedge$ is compact, we can pass to a subsequence of the $(u_k, x_k)$ and assume that $v_k$ converges in $U_0^\wedge$ to some element $v \in U_0^\wedge$. Then the sequence $\gamma_k$ converges in $G$.

Assume for the moment that we have proven the following claim:

Claim: If $x_k$ is a sequence in $\mathbb{A}$ converging to $x \in \mathbb{A}$, and $\gamma_k$ is a sequence in $NP_x$, converging to some $\gamma \in G$, then (after possibly passing to a subsequence of $x_k$) we can
write $\gamma_k = n_k h_k$ with $n_k \in N$ and $h_k \in P_{x_k}$ such that $n_k$ converges to some $n \in N$, and $h_k$ converges to some $h \in P_x$. In particular, $\gamma$ lies in $NP_x$.

Believing this result for a moment we can write $\gamma_k = n_k h_k$ with $n_k \to n \in N$ and $h_k \to h \in P_x$. As $g u_k = v_k n_k h_k \in U_0^\wedge NP_{x_k}$ converges to $gu = vh \in U_0^\wedge NP_x$, the continuity of the $N$-action on $\bar{A}$ implies that $\mu_y(u_k, x_k) = \pi(v_k, \nu(n) x_k)$ converges to $\pi(v, \nu(n) x) = \mu_y(u, x)$.

Hence it remains to prove the claim. As a first step, we will prove it under the condition that the sequence $x_k$ is contained in $A$. The limit point $x$ lies in some component $A_I$ for $I \subseteq n$. We fix an index $i_0 \in I$, and write $x_k = \sum_i x_{k,i} \eta_i$ with $x_{k,i_0} = 0$, and $x = \sum_{i \in I} x_i \eta_i$ with $x_{i_0} = 0$. The convergence $x_k \to x$ translates into

$$
x_{k,i} \to x_i, \quad \text{if } i \in I \quad \text{and} \quad x_{k,i} \to \infty, \quad \text{if } i \notin I.
$$

By the pigeon-hole principle there must be one ordering $i_1 \prec i_2 \prec \ldots \prec i_r$ of the set $n \setminus I = \{i_1, \ldots, i_r\}$ such that infinitely many members of the sequence $x_k$ satisfy

$$
x_{k,i_1} \geq x_{k,i_2} \geq \ldots \geq x_{k,i_r}.
$$

We replace $x_k$ by the subsequence of all $x_k$ satisfying these inequalities. Now we choose any ordering $\prec$ of the set $I$ and define a linear ordering $\prec$ on the whole of $n$ by combining the orderings on $I$ and $n \setminus I$ subject to the condition $i \prec j$, if $i \notin I$ and $j \in I$. Then $\Phi^+ = \{a_{ij} : i \prec j\}$ is the set of positive roots with respect to a suitable chamber corresponding to $\Phi$ (cf. [Bou], chapter V and VI).

If $a_{ij}$ is contained in the set $\Phi^-$ of negative roots, i.e. we have $j \prec i$, then

$$
f_{x_k}(a_{ij}) = -a_{ij}(x_k) = x_{k,j} - x_{k,i} \begin{cases} \geq 0, & \text{if } i, j \notin I \\ \to \infty, & \text{if } i \in I, j \notin I \\ \to x_j - x_i, & \text{if } i, j \in I.
\end{cases}
$$

Hence there is a real constant $C$ such that $f_{x_k}(a) \geq C$ for all $x_k$ and all $a \in \Phi^-$.

Now let us denote by $U_{\Phi^+}$, respectively $U_{\Phi^-}$ the corresponding subgroup of $G$ (see [Bo] 21.9), and by $U_{\Phi^+}$, respectively $U_{\Phi^-}$, their $K$-rational points. For all $y \in A$ we put

$$
U^+_y = U_{\Phi^+} \cap U_y \quad \text{and} \quad U^-_y = U_{\Phi^-} \cap U_y.
$$

It follows from [We], Theorem 4.7, that the multiplication map induces a bijection $\prod_{a \in \Phi^\pm} U_{a,y} \to U^\pm_y$, where the product on the left hand side may be taken in arbitrary order. Hence the fact that for all $a \in \Phi^-$ the numbers $f_{x_k}(a)$ are bounded from below implies that all $U^-_{x_k}$ are in fact contained in a compact subset of $U^-_{\Phi^-}$.
It follows from [We], Corollary 4.8, that the group $P_y$ can be written as

$$P_y = U_y^- U_y^+ N_y = N_y U_y^+ U_y^-.$$

Therefore the element $\gamma_k \in NP_{x_k}$ has a product decomposition as $\gamma_k = n_k u_k^+ u_k^-$ with $n_k \in N$, $u_k^+ \in U_{x_k}^+$, and $u_k^- \in U_{x_k}^-$. Since all $u_k^-$ are contained in a compact subset of $U_{\Phi^-}$, we can pass to a subsequence and assume that $u_k^-$ converges to some element $u^- \in U_{\Phi^-}$. By [We], Lemma 4.3 and Theorem 4.7, we find that $u^-$ lies in $U_x^-$. Besides, after passing to a suitable subsequence of $\gamma_k$, all $n_k \in N$ lie in the same coset modulo $T$, i.e. $n_k = mt_k$ for some $m \in N$. Then $t_k u_k^+$ is a converging sequence in the Borel group $TU_{\Phi^+}$. Hence $t_k$ converges to some $t \in T$, and that $u_k^+$ converges to some $u^+ \in U_{\Phi^+}$. Again we deduce from [We], 4.3 and 4.7, that $u^+$ lies in fact in $U_x^+$. Hence $\gamma_k = mt_k u_k^+ u_k^- = n_k h_k$ for $n_k = mt_k \in N$ and $h_k = u_k^+ u_k^- \in P_{x_k}$, and we have shown that $n_k$ converges to $mt \in N$ and that $h_k$ converges to $u^+ u^- \in P_x$. Therefore our claim holds if the $x_k$ are contained in $A$.

In the general case we denote again by $A_I$ the piece of $\overline{A}$ containing $x$. After passing to a subsequence of $x_k$ we can assume that all $x_k$ lie in the same piece $A_J$. Then $I$ must be contained in $J$. Besides, after passing to a subsequence of $\gamma_k$ we find some $m_0 \in N$ such that $m_0 \gamma_k$ is contained in $TP_{x_k}$ for all $k$.

Now recall the commutative diagram

$$\begin{array}{ccc}
PGL(V^*)^{(V^*)_J} & \overset{\alpha}{\longrightarrow} & PGL(V)^{V_{\overline{\Delta}}^J} \\
\rho_J \downarrow & & \sigma_J \downarrow \\
PGL((V^*)_J) & \overset{\alpha_J}{\longrightarrow} & PGL(V/V_{\overline{\Delta}}^J)
\end{array}$$

from section 3. From the proof of [We], Theorem 5.7, we know that for all $y' \in A'_J$ the map $\rho_J$ induces a surjection $\rho_J : P_{y'} \to P_{y_J'}$, where $P_{y_J'}$ is defined in the same way as $P_y$, replacing $PGL(V^*)$ by $PGL((V^*)_J)$ and the appartment $A'$ by $A'_J$. Besides, $P_{y'}$ is the full preimage of $P_{y_J'}$ under $\rho_J$. We know that for $y = \beta(y')$ the equality $\alpha(P_{y'}) = P_y$ holds, and it is easily checked that also $\alpha_J(P_{y_J'}) = \overline{\mathcal{P}}_{y_J'}$, where for every point $y \in A_J$ the group $\overline{\mathcal{P}}_{y_J'}$ is the subgroup of $PGL(V/V_{\overline{\Delta}}^J)$ defined in the same way as $P_y$, replacing the group $PGL(V)$ by $PGL(V/V_{\overline{\Delta}}^J)$, and the appartment $A$ by $A_J$. Hence it follows that $\sigma_J$ induces a surjection $\sigma_J : P_y \to \overline{\mathcal{P}}_{y_J}$, and that $P_y$ is the full preimage of $\overline{\mathcal{P}}_{y_J}$.

Since $m_0 \gamma_k$ lies in $PGL(V)^{V_{\overline{\Delta}}^J}$, we can apply $\sigma_J$ to this sequence and get a converging sequence in $PGL(V/V_{\overline{\Delta}}^J)$. Since $x_k$ is a sequence in $A_J$, we can apply the case of our claim already proven, this time working in the building corresponding to $PGL(V/V_{\overline{\Delta}}^J)$.

It follows that $\sigma_J(m_0 \gamma_k)$ can be written as $\sigma_J(m_0 \gamma_k) = \tilde{m}_k \tilde{h}_k$ with $\tilde{m}_k \in N(\overline{T}_J)$ converging to some $\tilde{m}$ in $N(\overline{T}_J)$ and $\tilde{h}_k \in \overline{T}_{x_J}$, converging to some $\tilde{h}$ in $\overline{T}_x$. 

12
After passing to a subsequence, we find elements $m_k$ in $N$ projecting to $\tilde{m}_k$ under $\sigma_J$ such that $m_k$ converges to some $m \in N$ with $\sigma_J(m) = \tilde{m}$. Then $\sigma_J$ maps $m_k^{-1}m_0\gamma_k$ to the element $\tilde{h}_k \in \mathcal{P}_x$. Hence $m_k^{-1}m_0\gamma_k$ lies in $P_{x_k}$. Besides, $m_k^{-1}m_0\gamma_k$ converges to $m^{-1}m_0\gamma$, and this projects via $\sigma_J$ to $\tilde{h}$. Now $\tilde{h}$ lies in $\mathcal{P}_x$ for the limit point $x \in A_I$, and it is easily checked that $\sigma_J^{-1}(\mathcal{P}_x)$ is contained in $\sigma_I^{-1}(\mathcal{P}_x)$, which is equal to $P_x$. Hence $m^{-1}m_0\gamma \in P_x$. Therefore we can put $n_k = m_k^{-1}m_k$ and $h_k = m_k^{-1}m_0\gamma_k$ to obtain the desired decomposition of $\gamma_k$. The sequence $n_k$ converges to $n = m_0m^{-1}$ in $N$, and the sequence $h_k$ converges to $m^{-1}m_0\gamma$ in $P_x$.

This finishes the proof of the theorem.

5 $\mathcal{X}$ as the space of seminorms on $V$

We want to extend the homeomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{S}_c$, where $\mathcal{S}_c$ is the space of equivalence classes of canonical seminorms with respect to $v_1, \ldots, v_n$, to the whole compactified building $\mathcal{X}$. Let $\mathcal{S}'$ denote the set of all seminorms on $V$, and denote by $\mathcal{S}$ the quotient of $\mathcal{S}'$ with respect to the equivalence relation on seminorms. On $\mathcal{S}'$ we have the topology of pointwise convergence, i.e. the coarsest topology such that for all $v \in V$ the map $\gamma \mapsto \gamma(v)$ is continuous. The quotient space $\mathcal{S}$ is equipped with the quotient topology. Both $\mathcal{S}'$ and $\mathcal{S}$ are Hausdorff. Note that $\mathcal{S}'$ carries a natural $GL(V)$-action given by $\gamma \mapsto g(\gamma) = \gamma \circ g^{-1}$. This induces a $G = PGL(V)$-action on the quotient space $\mathcal{S}$.

Theorem 5.1 The map $G \times \mathcal{A} \rightarrow \mathcal{S}$, which associates to $(g, x) \in G \times \mathcal{A}$ the point $g(\varphi(x)) \in \mathcal{S}$, induces a $G$-equivariant homeomorphism

$$\varphi : \mathcal{X} \rightarrow \mathcal{S}.$$  

Proof: In order to show that $\varphi$ is well-defined we have to check that for all $x \in \mathcal{A}$ the group $P_x$ stabilizes the seminorm class $\varphi(x)$. We have $P_x = U_xN_x$. Since $\varphi$ is $N$-equivariant on $\mathcal{A}$, the group $N_x$ stabilizes $\varphi(x)$. It remains to show that for all $a \in \Phi$ the group $U_{a,x}$ fixes $\varphi(x)$. Let $A_I$ be the piece of $\mathcal{A}$ containing $x$. Let us first assume that $i \notin I$. Every $u \in U_{a_{ij}}$ leaves the vectors $v_l$ for $l \neq j$ invariant and maps $v_j$ to $v_j + \omega v_i$ for some $\omega \in K$. If $\gamma$ is a seminorm representing $\varphi(x)$, then $\gamma \circ u^{-1}(v_j) = \gamma(v_j - \omega v_i) = \gamma(v_j)$, as $\gamma(v_i) = 0$. Hence every $u \in U_{a_{ij}}$ stabilizes $\gamma$. If $i \in I$ and $j \notin I$, then $U_{a_{ij},x} = 1$, and there is nothing to prove. It remains to treat the case that both $i$ and $j$ are contained in $I$. If $x = \sum_{i \in I} x_i \mathcal{P}_x$, then every $u \in U_{a_{ij},x}$ maps $v_j$ to $v_j + \omega v_i$ for some $\omega \in K$ satisfying $\nu(\omega) > \nu((a_{ij}) = x_j - x_i$. For the usual seminorm $\gamma$ representing $\varphi(x)$ we can calculate

$$\gamma(u^{-1}(v_j)) = \gamma(v_j - \omega v_i) = \sup \{q^{-x_j}, |\omega|q^{-x_j}\} = q^{-x_j} = \gamma(v_j).$$
As $u$ leaves the other $v_i$ invariant, the group $U_{\alpha_{ij},x}$ fixes $\varphi(x)$.

The map $\varphi$ is obviously $G$-equivariant. Since every seminorm on $V$ is canonical with respect to a suitable basis, $\varphi$ is surjective.

Let us now show that $\varphi$ is injective. If $g(\varphi(x))$ coincides with $h(\varphi(y))$, it is easy to see that there exists some $n \in N$ satisfying $n(\varphi(x)) = \varphi(y)$. Since $\varphi$ is $N$-equivariant and injective on $A$, this implies $\nu(n)x = y$. Now $g^{-1}hn$ is contained in the stabilizer of $\varphi(x)$. Hence it remains to show that this stabilizer is equal to $P_x$. If $x$ is contained in $A$, we have the Bruhat decomposition $G = P_xNP_x$. Hence every $g$ stabilizing $\varphi(x)$ can be written as $g = pmq$ with $p$ and $q$ in $P_x$. We already know that $P_x$ is contained in the stabilizer of $\varphi(x)$, hence it follows that $n$ stabilizes $\varphi(x)$. Since $\varphi : A \to S_c$ is $N$-equivariant and injective, this implies that $n$, and hence also $g$, is contained in $P_x$.

If $x$ is a boundary point, i.e. $x \in A_I$ for some $I \subset n_\circ$, then $\varphi(x)$ induces an equivalence class of norms on the quotient space $V/V_{n_\circ I}$. Every element $g$ stabilizing $\varphi(x)$ leaves $V_{n_\circ I}$ invariant, so that we can apply $\sigma_I : PGL(V)^{V_{n_\circ I}} \to PGL(V/V_{n_\circ I})$ to $g$. By the first case, applied to the apartment $A_I$ in the building associated to $PGL(V/V_{n_\circ I})$, we find that $\sigma_I(g)$ lies in the group $\overline{P}_x$, i.e. the subgroup of $PGL(V/V_{n_\circ I})$ defined in the same way as $P_x$, replacing $PGL(V)$ by $PGL(V/V_{n_\circ I})$ and $A$ by $A_I$. We have seen in the proof of 4.2 that $\sigma_I^{-1}(\overline{P}_x) = P_x$, so that $g$ lies indeed in $P_x$.

Let us now show that $\varphi$ is continuous. Since $\varphi$ is induced by the composition

$$G \times \overline{A} \xrightarrow{\text{id} \times \varphi} G \times S_c \to S,$$

we have to show that the second map is continuous. Hence we have to check that the action $GL(V) \times S'_c \to S'$ is continuous, which amounts to checking that for all $v \in V$ the map $\psi_v : GL(V) \times S'_c \to \mathbb{R}$, given by $(g, \gamma) \mapsto \gamma(g^{-1}v)$ is continuous.

We claim that for all $v \in V$, $\gamma_0 \in S'_c$ and $\varepsilon > 0$ there exists an open neighbourhood $W$ of $v$ in $V$ and an open neighbourhood $\Gamma$ of $\gamma_0$ in $S'_c$ such that for all $w \in W$ and all $\gamma \in \Gamma$ we have $|\gamma(w) - \gamma_0(w)| < \varepsilon$.

Let us believe the claim for a second. Then we find for $v \in V$ and $\gamma_0$ in $S'_c$ and for every $\varepsilon > 0$ open neighbourhoods $H$ of $1$ in $GL(V)$ and $\Gamma$ of $\gamma_0$ in $S'_c$ such that for all $h \in H$ and all $\gamma \in \Gamma$ the estimate $|\gamma(h^{-1}v) - \gamma_0(h^{-1}v)| < \varepsilon$ holds. Besides, we can make $H$ so small that all $h \in H$ satisfy $|\gamma_0(h^{-1}v) - \gamma_0(v)| < \varepsilon$. Hence

$$|\gamma(h^{-1}v) - \gamma_0(v)| \leq |\gamma(h^{-1}v) - \gamma_0(h^{-1}v)| + |\gamma_0(h^{-1}v) - \gamma_0(v)| < 2\varepsilon.$$ 

This shows that $\psi_v$ is continuous in $(1, \gamma_0)$, hence everywhere.

It remains to show the claim. Let $v = \sum_{i=1}^n \mu_i v_i$. Let us first assume that $\gamma_0(v) \neq 0$. Choose some $0 < \delta < \min\{1, |\mu_i| : \mu_i \neq 0\}$ such that $\delta \gamma_0(v_i) \leq \gamma_0(v)$ for all $i = 1, \ldots, n$.

Then we put $W = \{w = \sum_i \lambda_i v_i : |\lambda_i - \mu_i| < \delta\}$. For all $w = \sum_i \lambda_i v_i \in W$ the
conditions on \( \delta \) imply that \( |\lambda_i| = |\mu_i| \) for all \( i \) such that \( \mu_i \neq 0 \). If on the other hand \( \mu_i = 0 \), then \( |\lambda_i| < \delta \), which implies \( |\lambda_i| \gamma_0(v_i) \leq \gamma_0(v) \). Hence for all \( w \in W \)

\[
\gamma_0(w) = \sup \{|\lambda_1| \gamma_0(v_1), \ldots, |\lambda_n| \gamma_0(v_n)\} = \gamma_0(v)
\]

and

\[
\gamma(w) = \sup \{|\lambda| \gamma(v) : \mu_i = 0\}.
\]

Let us denote by \( \Gamma \) the open neighbourhood of \( \gamma_0 \) in \( S_\epsilon' \) consisting of all \( \gamma \) satisfying \( |\gamma(v) - \gamma_0(v)| < \epsilon \) and \( |\gamma(v_i) - \gamma_0(v_i)| < \epsilon \) for \( i = 1, \ldots, n \).

If \( \gamma \in \Gamma \) satisfies \( \gamma(w) = |\lambda_i| \gamma(v_i)| \) for some \( i \) with \( \mu_i = 0 \), then

\[
\gamma(v) \leq |\lambda_i| \gamma(v_i) \leq |\lambda_i| (\gamma_0(v_i) + \epsilon) \leq \gamma_0(v) + \epsilon
\]

\[
< \gamma(v) + 2\epsilon,
\]

which implies

\[
|\gamma(w) - \gamma_0(w)| = |\lambda_i| \gamma(v_i) - \gamma_0(v)|
\]

\[
\leq |\lambda_i| |\gamma(v_i) - \gamma(v)| + |\gamma(v) - \gamma_0(v)|
\]

\[
< 3\epsilon.
\]

If \( \gamma(w) = \gamma(v) \), this estimate holds trivially, so that our claim follows.

Now we treat the case that \( \gamma_0(v) = 0 \), i.e. we have \( \gamma_0(v_i) = 0 \) for all \( i \) such that \( \mu_i \neq 0 \). Choose some \( 0 < \delta < \min\{1, |\mu_i| : \mu_i \neq 0\} \), put \( W = \{w = \sum \lambda_i v_i : |\lambda_i - \mu_i| < \delta\} \), and let \( \Gamma \) be the open neighbourhood of \( \gamma_0 \) consisting of all \( \gamma \) satisfying \( |\gamma(v)| < \epsilon \) and \( |\gamma(v_i) - \gamma_0(v_i)| < \epsilon \) for all \( i = 1, \ldots, n \). As above, we have for all \( w \in W \)

\[
\gamma(w) = \sup \{|\lambda| \gamma(v) : \mu_i = 0\}.
\]

If \( \gamma \in \Gamma \) satisfies \( \gamma(w) = \gamma(v) \), then for all \( i \) such that \( \mu_i = 0 \) the estimate \( |\lambda_i| \gamma(v_i) \leq \gamma(v) < \epsilon \) holds. Since \( \gamma_0(w) = |\lambda_i| \gamma_0(v_i) \) for some \( i \) satisfying \( \mu_i = 0 \), this implies that

\[
|\gamma(w) - \gamma_0(w)| = |\gamma(v) - |\lambda| \gamma_0(v)|
\]

\[
\leq |\gamma(v) - |\lambda| \gamma(v_i)| + |\lambda| |\gamma(v_i) - \gamma_0(v_i)| < 2\epsilon.
\]

If \( \gamma \in \Gamma \) satisfies \( \gamma(w) = |\lambda_i| \gamma(v_i) \) for some \( i \) with \( \mu_i = 0 \), then we have for all \( j \) such that \( \mu_j = 0 \):

\[
|\lambda_j| \gamma_0(v_j) \leq |\lambda_j| \gamma(v_j) + \epsilon
\]

\[
\leq |\lambda_i| \gamma(v_i) + \epsilon \leq |\lambda_i| \gamma_0(v_i) + 2\epsilon.
\]
If $\gamma_0(w) = |\lambda_j|\gamma_0(v_j)$, we also have $|\lambda_i|\gamma_0(v_i) \leq |\lambda_j|\gamma_0(v_j)$, whence

$$|\gamma(w) - \gamma_0(w)| = ||\lambda_i|\gamma(v_i) - |\lambda_j|\gamma_0(v_j)|| \leq \varepsilon.$$ 

Hence the claim is proven.

Therefore $\varphi$ is a continuous bijection from a compact space to a Hausdorff space, hence it is a homeomorphism.

6 The reduction map from Drinfeld’s symmetric domain

Let $\mathbb{P}(V) = \text{Proj} \text{Sym} V$ be the projective space corresponding to $V$, i.e. points in $\mathbb{P}(V)$ correspond to lines in the dual space of $V$. Drinfeld’s $p$-adic symmetric domain $\Omega$ is the complement in $\mathbb{P}(V)$ of the union of all $K$-rational hyperplanes. $\Omega$ carries the structure of a rigid analytic variety. There is a reduction map $r : \Omega \rightarrow X$ from $\Omega$ onto the building $X$, see [Dr], §6, which is defined as follows: Every $\overline{K}$-rational point $x$ in $\Omega$ induces a line in the dual space of $(V \otimes \overline{K})$. For every element $z \neq 0$ on this line, the map $v \mapsto |z(v)|_{\overline{K}}$ defines a norm on $V$. Then $r(x)$ is the point in $X$ associated to this norm via the bijection $\varphi$ discussed in the previous section.

We want to extend this reduction map to a map from the whole projective space to our compactification $\overline{X}$ of the building. In the following, we identify $\overline{X}$ with the set of equivalence classes of seminorms on $V$ without specifying the homeomorphism $\varphi$ any longer.

Also we consider Berkovich spaces instead of rigid analytic varieties. Namely, let $\mathbb{P}(V)^{an}$ and $\Omega^{an}$ be the analytic spaces in the sense of [Be1] corresponding to the projective space and the $p$-adic symmetric domain. Then $\mathbb{P}(V)^{an}$ can be identified with the set of equivalence classes of multiplicative seminorms on the polynomial ring $\text{Sym} V$ extending the absolute value on $K$, which do not vanish identically on $V$. Here two such seminorms $\alpha$ and $\beta$ are equivalent, iff there exists a constant $c > 0$ such that for all homogeneous polynomials $f$ of degree $d$ we have $\alpha(f) = c^d \beta(f)$, see [Be2].

In [Be2], Berkovich defines a continuous, $\text{PGL}(V)$-equivariant reduction map $r : \Omega^{an} \rightarrow X$, and a right-inverse $j : X \rightarrow \Omega^{an}$, which identifies $X$ homeomorphically with a closed subset of $\Omega^{an}$.

We can now generalize these results to the compactifications $\mathbb{P}(V)^{an}$ and $\overline{X}$ by using almost verbatim the same constructions for $r$ and $j$ as in [Be2].

For every point in $\mathbb{P}(V)^{an}$ represented by the seminorm $\alpha$ on the polynomial ring $\text{Sym} V$, the restriction of $\alpha$ to $V$ induces a seminorm on $V$, hence a point in $\overline{X}$. This
induces a $PGL(V)$-equivariant map

$$r: \mathbb{P}(V)^{an} \rightarrow \overline{X}.$$ 

On the other hand, let $x$ be a point in $\overline{X}$, corresponding to the class of seminorms on $V$ represented by $\gamma$. Then $\gamma$ is canonical with respect to a basis $w_1, \ldots, w_n$ of $V$. If $f = \sum_{\nu=(\nu_1, \ldots, \nu_n)} a_\nu w_1^{\nu_1} \ldots w_n^{\nu_n}$ is a polynomial in $\text{Sym} V$, we put

$$\alpha(f) = \sup \left\{ |a_\nu| \gamma(w_1)^{\nu_1} \ldots \gamma(w_n)^{\nu_n} \right\}.$$ 

Then $\alpha$ is a multiplicative seminorm on $\text{Sym} V$ extending the absolute value of $K$, hence it induces a point in $\mathbb{P}(V)^{an}$.

This defines a $PGL(V)$-equivariant map

$$j: \overline{X} \rightarrow \mathbb{P}(V)^{an}.$$ 

**Proposition 6.1** The maps $r: \mathbb{P}(V)^{an} \rightarrow \overline{X}$ and $j: \overline{X} \rightarrow \mathbb{P}(V)^{an}$ are continuous and satisfy $r \circ j = \text{id}_{\overline{X}}$. Besides, $j$ is a homeomorphism from $\overline{X}$ to its image $j(\overline{X})$, which is a closed subset of $\mathbb{P}(V)^{an}$.

**Proof:** The continuity of $r$ follows immediately from the definitions. Since $j$ is obviously continuous on $\overline{A}$ and $PGL(V)$-equivariant, it is continuous on the whole of $\overline{X}$. By construction, we have $r \circ j = \text{id}_{\overline{X}}$, so that $j$ is a homeomorphism onto its image, which is a closed subset of $\mathbb{P}(V)^{an}$, since $\overline{X}$ is compact and $\mathbb{P}(V)^{an}$ is Hausdorff.

**References**

[Be1] V. G. Berkovich: Spectral theory and analytic geometry over non-archimedean fields. Math Surveys Monographs 33. American Mathematical Society 1990.

[Be2] V. G. Berkovich: The automorphism group of the Drinfeld upper half-plane. C.R. Acad. Sci. Paris 321, Série I (1995) 1127-1132.

[Bo] A. Borel: Linear Algebraic Groups. Second edition. Springer 1991.

[Bou] N. Bourbaki: Groupes et algèbres de Lie. Chapitres 4, 5 et 6. Herrmann 1968.

[Br-Ti] F. Bruhat, J. Tits: Schémas en groupes et immeubles des groupes classiques sur un corps local. Bull. Soc. math. France 112 (1984) 259-301.

[Dr] V. G. Drinfeld: Elliptic modules. Math USSR Sbornik 23 (1974) 561-592.
[Go-I] O. Goldman, N. Iwahori: *The space of $p$-adic norms*. Acta math. **109** (1963) 137-177.

[La] E. Landvogt: A compactification of the Bruhat-Tits building. Lecture Notes in Mathematics **1619**. Springer 1996.

[We] A. Werner: *Compactification of the Bruhat-Tits building of PGL by lattices of smaller rank*. Documenta Math. **6** (2001) 315-342.