We have computed the properties of compact objects like neutron stars based on equation of state (EOS) deduced from a core-envelope model of superdense stars. Such superdense stars have been studied by solving the Einstein’s equation based on pseudo-spheroidal and spherically symmetric space-time geometry. The computed star properties are compared with those obtained based on nuclear matter equations of state. From the mass-radius (\( M - R \)) relationship obtained here, we are able to classify compact stars in three categories: (i) highly compact self-bound stars that represents exotic matter compositions with radius lying below 9 km (ii) normal neutron stars with radius between 9 to 12 km and (iii) soft matter neutron stars having radius lying between 12 to 20 km. Other properties such as Keplerian frequency, surface gravity and surface gravitational redshift are also computed for all the three types. The present work would be useful for the study of highly compact neutron like stars having exotic matter compositions.

**Keywords**: neutron star; core-envelope model; dense matter equation of state.

**PACS numbers**:

1. Introduction

Neutron stars are one of the densest objects in the observable universe. It represents state of matter with highest densities. As such, they are valuable laboratories for the study of dense matter. Such studies include interplay between various disciplines like general relativity, high-energy astrophysics, nuclear and particle physics etc.
Neutron stars have masses of about \((1\sim 3 \, M_\odot)\). These stars with masses about \(1.2 \, M_\odot\) have central densities more than normal nuclear matter density and radius of the order of 10 km. The average mass density \(\rho\) of the neutron star is approximately \(10^{15} \, \text{g cm}^{-3}\), which is about 3 times the nuclear saturation density \(\rho_n = 2.7 \times 10^{14} \, \text{g cm}^{-3}\) and at the core \(\rho > \rho_n\). The magnetic field of such a compact stars lies between \(10^8 \sim 10^{15} \, \text{G}\) and possess gravitational field \(2 \times 10^{11} \, \text{cm s}^{-2}\) times stronger than that of earth’s gravitational fields. The structure of these stars can be considered having an outer and an inner crust. The envelope (outer crust) matter consists of atomic nuclei (ions) and electrons. The thickness of envelope is few hundred meters. The inner crust occurs at a density of \(4 \times 10^{11} \, \text{g cm}^{-3}\) which consists of electrons, free neutrons and neutron-rich atomic nuclei. The thickness of this crust is typically about few kilometers. The outer crust envelopes the inner crust, which expands from the neutron drip density to a transition density \(\rho_{tr} \sim 1.0 \times 10^{14} \, \text{g cm}^{-3}\). And beyond the transition density one enters the core, where all atomic nuclei have been melt down into their components, neutrons and protons. Caused by the high density and Fermi pressure, the core might also contain more massive baryon resonances or possibly a gas of free up, down and strange quarks. Ultimately, \(\pi\) and \(K\) mesons condensates may be found there too. All these dissimilar internal structure lead to different physical equation of state and hence contrasting mass-radius (M-R) relations.

In view of our inadequate knowledge of the equation of state of matter at extremely high densities, when matter density of ultra dense spherical objects is much higher than nuclear saturation density (\(\rho > \rho_n\)), it is difficult to have proper elucidation of matter in the form of an equation of state and quantitative calculations for the structure of neutron stars become obscure. A methodical valuation on the structure and properties of neutron stars can be found in references therein. Many theorists have developed theoretical models for the structure of neutron stars which may be made up of various layers including core (inner and outer), crust (inner and outer) in which atomic nuclei are arranged into a crystal and the liquid ocean composed of the coulomb fluid. The central region, i.e., core contains hyperons or quark matter. A detailed analysis of quarks core models are discussed by Bordbar, Bigdell and Yazdizadeh. Alternative method to study compact high-density astrophysical objects is through the space-time metric of the general theory of relativity and solving the relevant Einstein’s equations. Such attempts particularly for compact object, like the neutron star exist. Thus, for the present study we make use of the core-envelope model for neutron stars studied based on the geometric approach making use of the relativistic model for these regions. The core-envelope model of a neutron star has different physical properties in envelope and core regions. From this we have considered two different EOS, based on anistropic pressure in core or envelope region. The core-envelope models studied by Thomas, Ratanpal and Vinodkumar have considered anisotropic pressure in the envelope region and isotropic pressure in the core region. While in
another case studied by S. Gedela, N. Pant, J. Upreti and R. Pant (SNJR model) have taken both the core and envelope region as anisotropic. In both the cases valid solutions of the Einstein’s equations were studied in appropriate metrics. The EOSs deduced from these models are then used to compute the neutron star properties. Brief descriptions of these two models are given in the following section.

2. Relativistic core-envelope framework

Our primary focus in this paper is based on the models belonging to the core-envelope family as discussed by Thomas et al. (TRV) and Gedela et al. (SNJR). We summarize below only the relevant part of the formalism adopted for the study of compact objects with appropriate geometric consideration. More details can be found in the earlier works.

A nonrotating spherical metric in a most general form can be expressed as

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$  \hspace{1cm} (2.1)

where, \(r\) is the radial coordinate, \(\theta\) is the polar angle and \(\phi\) is the azimuthal angle.

The right boundary condition for the metric is to match (2.1) with the Schwarchild exterior metric at the surface of the star. It is implemented as

$$\nu(r = a) = \ln \left(1 - \frac{2GM}{ac^2}\right)$$  \hspace{1cm} (2.2)

$$\lambda(r = a) = -\ln \left(1 - \frac{2GM}{ac^2}\right)$$  \hspace{1cm} (2.3)

Here, \(a\) and \(M\) is the radius and mass of the star.

The Einstein field equation is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$$  \hspace{1cm} (2.4)

has been solved for the metric given by Eqn (2.1) for an energy momentum tensor relevant for perfect fluid

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - Pg_{\mu\nu} + \pi_{\mu\nu}$$  \hspace{1cm} (2.5)

where \(\pi_{\mu\nu}\) denotes anistropic stress tensor give by

$$\pi_{\mu\nu} = \sqrt{3}S \left[ C_{\mu}C_{\nu} - \frac{1}{3}(u_{\mu}u_{\nu} - g_{\mu\nu}) \right]$$  \hspace{1cm} (2.6)

where \(S = S(r)\) is the magnitude of anisotropy stress tensor and \(C^\mu = (0, -e^{-\lambda}, 0, 0)\), which is a radial vector.

Thomas et al. have discussed core-envelope model on pseudo-spheroidal space-time with core consisting of isotropic distribution of matter and envelope with
anisotropic distribution of matter. While anisotropic core-envelope models by assuming linear equation of state in the core and quadratic equation of state in the envelope have been studied by Gedela et. al.\cite{Gedela} In the following sub-sections we derive important aspects of these two models.

2.1. The TRV core-envelope model

It has been shown that core and envelope regions consist of different physical features. They have chosen ansatz for a pseudo-spheroidal geometry of spacetime to solved the Einstein’s equations. According to their metric, potential for pseudo-spheroidal geometry is expressed as

$$e^{\lambda(r)} = \frac{1 + K \frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}$$  \hspace{1cm} (2.7)

where $K$ and $R$ are geometric variables.

The energy momentum tensor components (2.5) with anisotropic stress tensor $\pi_{\mu \nu}$ has non-vanishing components

$$T_{00} = \rho, \quad T_{11} = -\left( p + \frac{2S}{\sqrt{3}} \right), \quad T_2^2 = T_3^3 = -\left( p - \frac{p}{\sqrt{3}} \right).$$  \hspace{1cm} (2.8)

The magnitude of anisotropic stress is given by\cite{Khubchandani}

$$S = \frac{p_r - p_\perp}{\sqrt{3}}$$  \hspace{1cm} (2.9)

The boundary conditions for the core and envelope regions are

$$S(r) = 0 \text{ for } 0 \leq r \leq R_C \quad \text{and} \quad S(r) \neq 0 \text{ for } R_C \leq r \leq R_E$$  \hspace{1cm} (2.10)

where $R_C$ refers to the core boundary radius and $R_E$ corresponds to the envelope boundary radius which is the same as the radius of the star ($a$) under consideration. Making use of these conditions with Eqn. (2.1), (2.4), (2.7) and (2.8), the Einstein field equations give the equations for density and pressure

Accordingly, the density distribution (core and envelope region) is expressed as

$$\rho = \frac{1}{8\pi R^2} \left[ 3 + 2 \frac{r^2}{R^2} \right]^{-2} \left[ 1 + 2 \frac{r^2}{R^2} \right]^{-2}.$$  \hspace{1cm} (2.11)

where $R$ is a geometrical parameter. Equation (2.11) provides the density distribution in core and envelope region by using boundary condition for $0 \leq r \leq R_C$ for core and $R_C \leq r \leq R_E$ for envelope region.

The radial and transverse pressure in the envelope region is given by
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\[ 8\pi p_c = \frac{C}{R^2} \sqrt{1 + \frac{r^2}{R^2}} \left( 3 + 4 \frac{r^2}{R^2} \right) + D \frac{R^2 (1 + 2 \frac{r^2}{R^2})^2}{\left( C \sqrt{1 + \frac{r^2}{R^2}} + D \right)^2}, \]  

(2.12)

\[ 8\pi p_{c\perp} = 8\pi P_c - \frac{\frac{r^2}{R^2} (2 - \frac{r^2}{R^2})}{R^2 (1 + 2 \frac{r^2}{R^2})^3}. \]  

(2.13)

and anisotropy \( S \) has expression

\[ 8\pi \sqrt{3} S = \frac{\frac{r^2}{R^2} (2 - \frac{r^2}{R^2})}{R^2 (1 + 2 \frac{r^2}{R^2})^3}. \]  

(2.14)

The constants \( C \) and \( D \) are given by

\[ C = -\frac{1}{2} \left( 1 + 2 \frac{a^2}{R^2} \right)^{-\frac{3}{2}}, \]  

(2.15)

\[ D = \frac{1}{2} \sqrt{1 + \frac{a^2}{R^2}} \left( 3 + 4 \frac{a^2}{R^2} \right) \left( 1 + 2 \frac{a^2}{R^2} \right)^{-\frac{5}{2}}. \]  

(2.16)

The radial pressure in the core region is given by

\[ 8\pi p_c = \frac{A}{R^2} \sqrt{1 + \frac{r^2}{R^2}} + B \frac{1}{R^2} \sqrt{1 + \frac{r^2}{R^2}} L(r) + \frac{1}{\sqrt{2}} \sqrt{1 + 2 \frac{r^2}{R^2}} \frac{\left[ A + \sqrt{1 + 2 \frac{r^2}{R^2}} L(r) + \frac{1}{\sqrt{2}} \sqrt{1 + 2 \frac{r^2}{R^2}} \right]}{A + \sqrt{1 + 2 \frac{r^2}{R^2}}} \]  

(2.17)

where

\[ L(r) = \ln \left( \sqrt{2} \sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + 2 \frac{r^2}{R^2}} \right). \]

(2.18)

\[ A = \left[ 5\sqrt{5} - 3\sqrt{2} (\sqrt{3}L(R_c) - \sqrt{2.5}) \right] C + \frac{1}{\sqrt{3}} \left[ 5\sqrt{5} + 2\sqrt{2} (\sqrt{3}L(R_c) - \sqrt{2.5}) \right] D, \]  

(2.19)

\[ B = \frac{1}{\sqrt{3}} \left( 3\sqrt{3}C - 2D \right). \]  

Equation (2.11) implies that the matter density at the center is explicitly related with geometrical variable \( R \) as
\[ R = \sqrt{\frac{3\lambda}{8\pi\rho(a)}} \quad \lambda = \frac{\rho(a)}{\rho(0)} = \frac{1 + \frac{2a^2}{\lambda R}}{(1 + 2\frac{a^2}{\lambda R})^2} \quad (2.20) \]

We have plotted the graph of pressure against density in the TRV model and displayed by solid curve in Fig. 1 for density variation parameter \( \lambda = 0.01 \). The best fit for the pressure-density curve is found to be in the quadratic form

\[ p = \rho_0 + \alpha\rho + \beta\rho^2 \quad (2.21) \]

where \( \rho_0 = -9.30 \times 10^{-4} \), \( \alpha = 406 \) and \( \beta = 1.69 \). It has been shown as a dotted curve in Figure 1. It can be shown that the model reveals quadratic equation of state for different choices of the density variation parameter \( \lambda \).

Fig. 1. (Color online) The radial pressure and density are given by Thomas et al. \cite{16} (given in units of km\(^{-2}\)), is plotted with solid curve. The dashed curve corresponds to the fitted curve with \( \alpha = 406 \), \( \beta = 1.69 \) and \( \rho_0 = -9.30 \times 10^{-4} \). For a density variation (\( \lambda = 0.01 \)).
2.2. The SNJR core-envelope model

In the second case of core-envelope anisotropic model of Gedela et al.\textsuperscript{17} two distinct EOSs for core and envelope region are proposed. For the core region \((0 \leq r \leq R_c)\), here a linear EOS as given below is used\textsuperscript{17}

\[
p_c = (0.170)\rho - (7.833 \times 10^{-5})
\]  

(2.22)

The numerical values appeared in equation (2.22) are the same as given in\textsuperscript{17} The expressions of density and pressure for core region are given by

\[
\rho_c = \frac{c(b^2 - 3)}{8\pi(b^2 + 1)^3}
\]

(2.23)

\[
p_c = \frac{c\alpha(b^2 - 3)}{8\pi(b^2 + 1)^3} - \beta
\]

(2.24)

where \(c, b, \alpha\) and \(\beta\) are constants whose numerical values are \(-0.00735\) km\(^{-2}\), \(0.0038\) km\(^{-2}\), \(0.1707\) km\(^{-2}\) and \(0.7833 \times 10^{-5}\) km\(^{-2}\), respectively. For envelope region \((R_c \leq r \leq R_E)\), they have considered quadratic EOS in the form

\[
p_e = \kappa\rho^2 - \gamma
\]

(2.25)

where \(\kappa\) and \(\gamma\) are constants whose numerical values are \(108\) km\(^{-2}\) and \(1.088 \times 10^{-5}\) km\(^{-2}\), respectively\textsuperscript{17} Further, the density and pressure profile in the envelope region are given by\textsuperscript{17}

\[
\rho_e = \frac{a(b^2 - 3)}{8\pi(b^2 - 1)^3}
\]

(2.26)

\[
p_e = \frac{a^2\kappa(b^2 - 3)^3}{64\pi^2(b^2 + 1)^6} - \gamma
\]

(2.27)
An important feature of both of these core-envelope models (TRV and SNJR) is that they have the stable equilibrium under hydrostatic configuration. Theoretical study of the relativistic core-envelope model using paraboloidal spacetime by Ratanpal and Sharma\textsuperscript{23} have shown that paraboloidal geometry also admit quadratic equation of state. Other EOSs that we have considered in the present work for comparison include those considering different physical compositions of nuclear matter reported by\textsuperscript{24–30} The different models used in this study are listed in Table 1. A very crucial feature of the equation of state is the causal limit (a sound signal cannot propagate faster than the speed of light, $\nu_s^2 = dp/d\rho \leq c^2$). In both cases based on the geometrical models (TRV and SNJR) the causality condition is satisfied. In particular, Thomas et al.\textsuperscript{16} have studied the causality limit for different density variables. The computed speed of sound ($\nu_s$) versus radius as shown in Figure 3. Both the cases clearly indicate the validity of causality condition.
3. Compact Star Structure: Static Equilibrium configurations

It is vital to explore static and spherical symmetrical gravity sources in general relativity, especially when it comes to internal structure of compact objects. For simplicity, we consider only nonrotating, spherically symmetric stars. The geometry inside the star is described by the familiar Tolman–Oppenheimer–Volkoff (TOV) equation, which is valid for a perfect fluid. The equation of state is all that is required to solve the TOV equations. For static, spherically symmetric stars in hydrostatic equilibrium, the TOV equations may be written as a pair of first-order differential equations. The calculation of neutron star structure is obtained by numerically integrating the Tolman-Oppenheimer-Volkoff equation

\[
\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \left(1 + \frac{P(r)}{\rho(r)c^2}\right) \frac{\left[1 + \frac{4\pi r^2 P(r)}{m(r)c^2}\right]}{1 - 2\frac{Gm(r)}{rc^2}},
\]

(3.1)

\[
\frac{dm(r)}{dr} = 4\pi r^2 \rho(r).
\]

(3.2)
Here $P$ is the radial pressure, $\rho$ is the mass density, $r$ is the radial distance measured from the center, and $m(r)$ is the enclosed mass from the center $r = 0$ where $P = P_c$ and $\rho = \rho_c$ to a radial distance $r$. In the present work we have fixed the central density for both geometrical models at $\rho_c = 1.34 \times 10^{15} \text{g cm}^{-3}$. The seven nuclear EOSs with a fixed central density at $\rho_c = 1.0 \times 10^{15} \text{g cm}^{-3}$, equations (3.1) and (3.2) are integrated numerically to determine the global structure (e.g. radius and mass) of a neutron star. To begin with, the density close to the center of the compact star is assumed to be homogeneous, with the density $\rho = \rho_c$, the radius $r = 0.1 \text{ cm}$ and $m(0.1 \text{ cm}) = \frac{4}{3} \pi \rho_c r^3$. Equations (3.1) and (3.2) are integrated numerically from $r = 0.1 \text{ cm}$ to the boundary of the star, where the pressure falls to zero ($P(a) = 0$). The total mass of the star is then given by $M = m(a)$.

Using the data files provided by Özel et al.\textsuperscript{31} we have re-plotted pressure-density profile corresponds to all the model EOS’s listed in Table 1 along with geometric EOS’s of TRV and SNJR. It can be seen that these EOS’s distinctly differ from each other.

Fig. 4. (Color online) Geometrical EOS TRV (wine short dash line) and SNJR (olive solid line) compared to the selected nuclear EOS’s (ALF1 (black solid line), APR (red dash line), BKS19 (green dot line), ENG (blue dash dot line), SLy (magenta dash dot dot line), WWF1 (navy short dash line) and SQM1 (purple dot line). Details of these EOSs are listed in Table I.
3.1. Mass-Radius Relation

In this work, we have considered two general relativity inspired equations of state and compared with seven different nuclear equations of states as listed in Table 1.

Table 1. Nuclear and Geometrical equations of state used for the construction of models of general relativistic static neutron stars

| Label | EOS   | Composition and model                                      | Reference                   |
|-------|-------|------------------------------------------------------------|-----------------------------|
| 1     | ALF1  | nuclear plus quark matter (MIT Bag Model)                  | Alford et al. (2005)        |
| 2     | APR   | $\tau\rho\mu$, variational theory, Nijmegen NN plus Urbana NNN potential | Akmal et al. (1998)         |
| 3     | BKS19 | cold catalyzed nuclear matter analytical unified EOSs       | Potekhin et al. (2013)      |
| 4     | ENG   | Dirac-Brueckner HF asymmetric nuclear matter               | Engvik et al. (1996)        |
| 5     | SLy   | potential method, $n p e \mu$ effective nucleon energy functional | Douchin and Haensel et al. (2001) |
| 6     | WWF1  | variational method dense nucleon matter                    | Wiringa et al. (1988)       |
| 7     | SQM1  | MIT Bag Model (Strange quark matter)                        | Zdunik (2000)               |
| 8     | TRV   | core : isotropic fluid distribution envelope : anisotropic fluid distribution | Thomas et al. (2005)        |
| 9     | SNJR  | core : linear equation of state envelope : quadratic equation of state | Gedela et al. (2019)       |

The composition and model used for all these equation of state and their respective bibliographic references are also listed in Table 1. Making use of these equations of state, we obtained the mass-radius relationship for a compact star.

The mass-radius relations obtained with the help of nuclear equations of state of different compositions are compared with the geometrical equations of state and are plotted in Fig. 4. These plots reiterate the fact that nuclear and geometrical equations of state manifest three distinct types of compact stars. The first one corresponds to the two cases represented by the models 7 and 8 of Table 1, the second one corresponds to the models (1 to 6) largely represented by the nuclear matter EOSs and third type corresponds to the model (9) represented by the geometric model (SNJR). In all the three cases the maximum masses correspond to stable structure varies from 1.4 to 2.3 $M_\odot$, while the radius at their maximum masses lie 8 - 9 kms in the case of the first category, 9 - 12 kms in the cases of (second category) and beyond 12 km in the case of the third category. The central density...
at maximum mass obtained here for the stable configurations are listed in Table 2.

The M-R diagram obtained from the two geometrically deduced models behave differently. We found that TRV equation of state resulted into the mass-radius curve similar to the one obtained for strange quark matter stars (SQM1, label-7).\(^3\) The monotonically increasing mass with radius \((M \propto a^3)\) is expected for the class of ultra compact objects which are self-bound.\(^3\) The surface density of strange star is roughly fourteen orders of magnitude larger than the surface density of normal neutron stars.\(^4\) The TRV model gives a stable configuration in the same orders of magnitude, with the surface density, \(\rho_s \approx 2 \times 10^{14} \text{ g cm}^{-3}\). Thus, it is an appropriate geometrical model for the study of ultra compact stars having exotic matter composition.

The isotropic fluid distribution in the core part of the TRV model\(^1\) is justified if the core matter distribution is of quarks or strange matter, governed by MIT bag model. Further, the envelope with anisotropic fluid distribution can be viewed as due to hadronization to baryonic matter. Thus the TRV model prediction fit well with that
of the strange quark matter stars with its maximum mass, \(M_{\text{max}} = 1.69M_\odot\) and radius, 8.76 km. The SNJR model that predicts the third category in which EOS has the linear behaviour inside the core and quadratic behaviour at the envelope has resulted into the M-R diagram different from all other cases. Its M-R curve is broader as compared to all other cases studied here. And its density is much lower than that of normal neutron like stars. Recent observations of binary neutron-star mergers (GW170817) have reported an estimation for the radius of the neutron star in the range 10.6 to 11.5 kilometers.\(^{42}\)

### 3.2. Keplerian frequency (rotation frequency of neutron star)

The Kepler frequency expresses the balance of centrifugal and gravitational force on a particle on equatorial plane at the surface of a star. It is expressed as

\[
\Omega_c = \sqrt{\frac{M}{a^3}},
\]

where the subscript \(c\) denotes classical symmetry of the centrifugal and gravitational forces, which is the Newtonian expression for the Kepler angular velocity. This equation do to not hold in General Relativity, but as it turn out, it holds to very good accuracy if the right side is multiplied by a prefactor(\(C\)).\(^{44}\) It has been shown by J. M. Lattimer, et al.,\(^{43}\) Haensel et al.\(^{44}\) and B. Haskell et al.\(^{45}\) that the numerical value of the Keplerian frequency, namely the maximum rotational frequency of a neutron star accounting for the effects of general relativity, deformation, and independent on the EOS, can be well fitted from the simple formula

\[
\Omega_K \approx C \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{10 \text{ km}}{a}\right)^{3/2} \text{Hz},
\]

providing the neutron star mass is not very close to the maximum stable value, \(M\) and \(a\) are the mass and the radius of the nonrotating star respectively. The constant \(C\) of Eq.3.4 are given by B. Haskell et al.\(^{45}\) For the self bound compact stars it is given as 1.15 KHz and for other gravitationally bound neutron stars it is given as 1.08 KHz.

The deduction of \(\Omega_K\) generally requires the calculation of rotating general relativistic configurations. Nevertheless, Haensel et al. (2009) have shown to a good degree of accuracy that the mass-shedding frequency \(\Omega_{K,\text{max}}\) can be determined by the EOS-independent empirical formula as given in Eq.(3.4). On the other hand, it allows to determine \(\Omega_K\) using the mass and radius of the nonrotating star.

The calculated Keplerian frequency based on the mass-radius relations obtained using all the nine equations of state are shown in Fig. 6. Here we found that Keplerian frequency corresponds to TRV and SQM1 are similar with higher values of \(\Omega_K\) (14-18 KHz). While other cases \(\Omega_K\) varies from 2 KHz to 18 KHz. The results of
Keplerian frequency for the maximum mass of stable stars are shown in Table 2 for all the nine models.

3.3. Surface Gravity

The surface gravity of neutron stars denoted by $g_s$ (i.e., the acceleration due to gravity as measured on the surface), is an important parameter for the study of neutron star atmospheres. The upper bound of the surface gravity for neutron stars with various baryonic EOSs is studied by Bejger et al. (2004). The surface gravity of neutron star is many orders of magnitude larger than that of other stars; it is $\sim 10^{12}$ times stronger than gravity at the Earth surface, and $10^9$ times larger than that for the white dwarfs.

The expression for $g_s$ is given by:

$$g_s = \frac{GM}{a^2 \sqrt{1 - x_{GR}}}$$

(3.5)
Here, $x_{GR} = 2GM/ac^2 = r_g/a$, where $r_g$ is the Schwarzschild radius. The importance of relativistic effects for a neutron star mass $M$ and radius $a$ is characterized by the compactness parameter $r_g/a$. Usually for a neutron star with $M = 1.4 \, M_\odot$ and radius is about 10 km, surface gravity becomes $(g_s) = 2.43 \times 10^{14} \, \text{cm s}^{-2}$. In consequence it is suitable to measure $g_s$ in units of $10^{14} \, \text{cm s}^{-2}$ and is represented as $g_{s,14} \equiv g_s/(10^{14} \, \text{cm s}^{-2})$. The computed surface gravity, $g_{s,14}$ for all the cases studied here are shown in Fig. 8 against mass expressed in $M_\odot$. The numerically values of $g_{s,14}$ correspond to maximum stable mass of the star are also listed in Table 2.

It is found that for $M = 1.4 \, M_\odot$, $g_{s,14}$ ranges from 1.43 to 2.8 and for $M \approx 2.0 \, M_\odot$ the surface gravity lies between 1.88 to 4.38. The nuclear EOSs (labeled : 1 and 6) with an exotic quarks phase have relatively low $g_{s,max}$. A similar situation occurs for the SNJR EOS that gives lowest value of surface gravity. The only reason SNJR EOS have low surface gravity is that they have a greater radius compared to other EOSs. The TRV EOS (labeled 8) yields $g_{s,max}$ similar to BKS19 and SLy EOSs.
Their values of $g_{s,14(\text{max})}$ range from 4.10 to 4.60.

### 3.4. Gravitational Redshift of Neutron Star

In general relativity the ratio of the emitted wavelength $\lambda_e$ at the surface of a nonrotating neutron star to the observed wavelength $\lambda_0$ received at radial coordinate $r$, is given by $\lambda_e/\lambda_0 = [g_{tt}(a)/g_{tt}(r)]^{1/2}$. From this definition of gravitational redshift, $z \equiv (\lambda_0 - \lambda_e)/\lambda_e$ from the surface of the neutron star as measured by a distant observer ($g_{tt}(r) \to -1$) is given by

$$z = | -g_{tt}(a) |^{-1/2} - 1 = \left( 1 - \frac{2GM}{ac^2} \right)^{-1/2} - 1$$

We compute the limit of the redshift from the surface of a neutron star using Eqn (3.8) where $g_{tt} = -e^{\lambda(r)} = -(1 - 2GM/c^2a)$ is the metric components.\[22\] For a given EOSs the maximum value $z_{\text{surf}}$ increase with increase of $M_{\text{max}}$. Neutron stars of $M \geq M_\odot$ are expected to have sizable $z_{\text{surf}}$. The computed values of $z_{\text{surf}}$ for all the cases studied here are listed in Table 2. The computed values of $z_{\text{surf}}$ are found to lie between 0.2 to 0.3.

![Gravitational redshift at the neutron star surface as a function of the stellar gravitational mass for the nine considered EOS models.](image)

From the listed values of $x_{GR}$ in Table 2, we found that all the models studied here satisfy the Buchdahl inequality,\[23\] $a \geq (9/8)r_g = (9/4)GM/c^2$ which is stricter than
the Schwarzschild bound. A consequence of this is that the gravitational redshift should satisfy \( z \leq 2 \). The precise upper bound on the surface redshift for neutron star is \( z_{\text{surf}} = 0.851 \) for subluminal EOSs. In the present study, we found that \( z_{\text{surf}} \) for all the nine cases computed here lie much below the upper bound for the gravitational redshift.

### 4. Results and Discussion

We have computed several properties of a compact star like neutron star, using nuclear and geometrically deduced equations of state. We have used geometrical equations of state from the core envelope model, that describes different properties of the physics in the core and envelope region. Many similarities and dissimilarities are observed from the properties computed based on the geometrical EOSs and the nuclear EOSs.

| Label | \( M_{\text{max}} \) (\( M_\odot \)) | \( a_{\text{max}} \) (km) | \( \rho_c \) (10^{15} \text{ gm cm}^{-3}) | \( \Omega_k \) (10^4 \text{ s}^{-2}) | \( z_{\text{surf}} \) | \( g_s, 14 \) (cm s^{-2}) | \( x_{GR} \) |
|-------|-----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| 1     | 1.47            | 9.21            | 3.34           | 1.49           | 0.21           | 3.17           | 0.47           |
| 2     | 1.65            | 8.37            | 4.51           | 1.85           | 0.28           | 4.89           | 0.58           |
| 3     | 1.86            | 9.25            | 2.69           | 1.66           | 0.29           | 4.59           | 0.59           |
| 4     | 2.22            | 10.76           | 2.91           | 1.44           | 0.30           | 4.07           | 0.60           |
| 5     | 2.00            | 10.08           | 2.85           | 1.53           | 0.29           | 4.14           | 0.58           |
| 6     | 1.73            | 9.17            | 2.93           | 1.67           | 0.26           | 5.36           | 0.65           |
| 7     | 1.54            | 8.48            | 3.27           | 1.82           | 0.25           | 4.17           | 0.53           |
| 8     | 1.68            | 8.76            | 3.58           | 1.82           | 0.24           | 4.26           | 0.57           |
| 9     | 2.06            | 11.58           | 0.61           | 1.24           | 0.23           | 1.98           | 0.56           |

In Table 2 we have listed computed properties of such a compact star with all the different types of EOSs. Like, maximum mass (\( M_{\text{max}} \)), stellar radius (\( a_{\text{max}} \)) correspond to the maximum mass, central density (\( \rho_c \)), Keplerian frequency (\( \Omega_k \)) correspond to the maximum mass for the stable structure of the star, gravitational redshift (\( z_{\text{surf}} \)), surface gravity (\( g_s \)) and compactness parameter (\( x_{GR} \)). We compared all these properties with the properties obtained from geometrically deduced equations states. The properties obtained from TRV equation of state are in good accordance with the properties obtained from other nuclear matter based models. While the parameters obtained using the SNJR model are quite different from others except for \( z_{\text{surf}} \) and (\( x_{GR} \)). The central density that yields the maximum mass of \( \approx 2 \ M_\odot \) in the case of SNJR is very low and the radius is about 12 km. It is also reflected in the low values of the surface gravity. It is noticed that mass-radius configuration as shown in Fig. 5 obtained from geometrical models will be pertinent for divergent class of compact stars. Particularly, the pseudo-spheroidal spacetime of TRV model seemed to describe the ultra dense compact stars like the strange
self-bound stars. The spacetime geometry adopted for the SNJR model represents low density neutron like star, where radius lie between $12 \leq R \leq 20$ kms. The mass-radius diagram in Fig. clearly classify the nature of compact stars in three categories: (i) highly compact self-bound stars represented by the TRV Model and SQM1 model with exotic matter compositions (ii) the normal neutron stars with nuclear matter EOS and (iii) the ultra soft compact stars represented by the SNJR geometrical model. At the end, we are able to identify a correspondence between the geometric description with the structure of the matter distribution in compact objects like a strange star. To summarize, we have been able to classify neutron like compact stars in three distinct types each one having different internal structures. We hope that TRV model for compact neutron like stars will be useful for the study of superdense self-bound stars having exotic matter compositions.

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References

1. Lattimer James M and Prakash Maddappa, Science. 304 (2004) 536
2. A. Y. Potekhin Physics-Uspekhi 53 (2010) 1235
3. Norman K. Glendenning, Compact stars: Nuclear physics, particle physics and general relativity. (Springer Science & Business Media, 2012)
4. B. Datta, FCPb. 12 (1988) 151-239
5. H. Heiselberg, Int. J. Mod. Phys. B 15 (2001) 1519-1534
6. J. M. Lattimer, and M. Prakash, Physics Reports 621 (2016) 127-164
7. J. M. Lattimer, and M. Prakash, The Astrophysical Journal 550 (2001), 426
8. I. Bednarek and R. Manka, Int. J. Mod. Phys. D 10, 05 (2001) 607-624
9. N. Chamel and P. Haensel, Living Rev. Relativity, 11, 2008, 10
10. P. Haensel, A. Y. Potekhin, D. G. Yakovlev, , Astrophys.Space Sci.Libr, 326 200) pp.1-619
11. S. Balberg and I. Lichtenstadt and G. B. Cook, The Astrophysical Journal Supplement Series. 121 (1999) 515.
12. S. Weissenborn, D Chatterjee and J Schaffner-Bielich, Nuclear Physics A 881 (2012), 62-77
13. E. Witten, Phys. Rev. D 30 (1984), 272.
14. G. H. Bordbar, and M. Bigdeli and T. Yazdizadeh, International Journal of Modern Physics A 21 (2006) 5991-6001.
15. P. C. Vaidya and R. Tikekar, J. Astrophys. Astr.3 (1982) 325-334
16. V. O. Thomas, B. S. Ratanpal, and P. C. Vinodkumar, Int. J. Mod. Phys. D 14, 85 (2005)
17. S. Gedela, N. Pant, J. Upreti, R. Pant, The European Physical Journal C. 79 (2019) 566
18. P. Mafa Takisa, S.D. Maharaj,Astrophys. Sp. Sci. 361, (2016) 262
19. Charles W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation(San Francisco: W. H. Freeman, ISBN 978-0-7167-0344-0 ) 1973
Distinct Classes of Compact Stars Based On Geometrically Deduced Equations of State

20. Max Camenzind, *Compact Objects In Astrophysics*, (Astronomy and Astrophysics Library, Springer Berlin Heidelberg New York, 2007).
21. Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, New York, 1972).
22. J. R. Oppenheimer, G. M. Volkoff, *Physical Review D*, **55**, (1939) 374
23. Ranjan Sharma and B.S. Ratanpal *Int. J. Mod. Phys. D* **22** (2013)
24. M. Alford, M. Braby, M. Paris, and S. Reddy , *Astrophys. J.* **629** (2005), 969
25. A. Akmal, V. R. Pandharipande, *Phys. Rev. C* **58** (1998), 1804
26. A. Potekhin, A. Fantina, N. Chamel, J. Pearson, and S. Goriely *A & A, 560* (2013), 48
27. Engvik, L., Osnes, E., Hjorth-Jensen, M., Bao, G. and Ostgaard, E. *Astrophys. J.* **469** (1996), 794
28. F. Douchin and P. Haensel, *Astron. Astrophys*, **380** (2001), 151
29. R. B. Wiringa, V. Fiks, and A. Fabrocini,*Phys. Rev. C* **38**, (1988), 1010
30. J.L. Zdunik *Astron.Astrophys*, **359**, (2000) 311
31. Feryal ¨Ozel and Paulo Freire, *Annu. Rev. Astron. Astrophys.* **54** (2016), 401-440
32. W. Fridolin, O. Milva, R. Hilario and Y, Shu-Hua, *Proceedings of the International Astronomical Union 8* (2012) 61-66
33. J. L. Friedman, J. R. Ipser and L. Parker, *Phys. Rev. Lett.* **62** (1989), 3015
34. P. Haensel and J. L. Zdunik , *Nature* **340** (1989), 617
35. N. K. Glendenning and F. Weber, *Phys. Rev. D* **50** (1994), 3836
36. E. H. Gudmundsson, C. J. Pethick and R. I. Epstein, *Astrophys. J.* **272** (1983), 286-300
37. M. Bejger and P. Haensel, *Astron. Astrophys*. **420** (2004), 987-991
38. D. Gondek, J. L. Zdunik, *Acta Astron.* **45** (1995), 319-325
39. Ang Li, Ren-Xin Xu and Ju-Fu Lu, *Mon. Not. R. Astron. Soc.* **402** (2010) 2715-2719
40. F. Weber, M. Orsaria, H. Rodrigues and S.-H Yang, *Proceedings of the International Astronomical Union*, **291** (2013) 61-66
41. V. O. Thomas and D. M. Pandya ,*Eur. Phys. J. A* **53** (2017) 120
42. Collin D, T. Ingo, Stephanie M, M. Ben, S. Kumar, S. Brown, Duncan A, K. Badri and R. Sanjay, *Nature Astronomy* (2020) 1-8
43. J. M. Lattimer and M. Prakash, *Science*, **304** (2004) 536
44. P. Haensel, J. L. Zdunik, M. Bejger, and J. M. Lattimer, *Astron. Astrophys*, **520** (2009), 605-610
45. B. Haskell, J. L. Zdunik, M. Fortin, M. Bejger, R. Wijnands, and A. Patruno, *Astron. Astrophys*, **A69** (2018), 620
46. H. A. Buchdahl, *Phys. Rev.* **116**, (1959)1027