NON ABELIAN TODA THEORY:
A COMPLETELY INTEGRABLE MODEL FOR
STRINGS ON A BLACK HOLE BACKGROUND

Adel Bilal

Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, USA

ABSTRACT

The present paper studies a completely integrable conformally invariant model in 1+1 dimensions that corresponds to string propagation on the two-dimensional black hole background (semi-infinite cigar). Besides the two space-time string fields there is a third (internal) field with a very specific Liouville-type interaction leading to the complete integrability. This system is known as non-abelian Toda theory. I give the general explicit classical solution. It realizes a rather involved transformation expressing the interacting string fields in terms of (three) functions $\varphi_j(u)$ and $\bar{\varphi}_j(v)$ of one light-cone variable only. The latter are shown to lead to standard harmonic oscillator (free field) Poisson brackets thus paving the way towards quantization. There are three left-moving and three right-moving conserved quantities. The right (left)-moving conserved quantities form a new closed non-linear, non-local Poisson bracket algebra. This algebra is a Virasoro algebra extended by two conformal dimension-two primaries.
1. Introduction

The study of strings propagating on a curved background is a difficult subject. In general our knowledge is restricted to perturbation in the inverse string tension $\alpha'\ [1]$. Sometimes the $\sigma$-model describing the string can be shown to be equivalent to a known conformal field theory [2]. However, we do not know the general classical solution to the equations of motion of a string on a Schwarzschild background. Even in conformal gauge, one has to solve the coupled non-linear partial differential equations

$$\partial_u \partial_v X^\mu + \Gamma^\mu_{\nu\rho}(X) \partial_u X^\nu \partial_v X^\rho = 0 \quad (1.1)$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol associated with the background metric, and $u = \tau + \sigma, \ v = \tau - \sigma$ are light-cone coordinates on the world-sheet. Of course, particular solutions are easy to find: we can look e.g. for $\sigma$-independent solutions (no oscillator excitations) and find that the string center of mass describes the geodesics, well known for the Schwarzschild metric. Starting from this special solution we can generate others, exploiting the conformal invariance and making a conformal transformation of the world-sheet coordinates, thereby obtaining solutions depending on two arbitrary functions of one variable.

However, if we want to (canonically) quantize the theory, and eventually compute the partition function and the entropy of the string on the black hole background we need the complete general solutions. Obtaining the general solution of eq. (1.1) for the Schwarzschild metric in 4D seems far beyond our present abilities, and one seems forced to consider instead the two-dimensional (euclidean) black hole with metric

$$ds^2 = dr^2 + th^2 r \, dt^2. \quad (1.2)$$

Classical solutions can be obtained by exploiting the (classical) equivalence with the gauged $Sl(2, \mathbb{R})/U(1)$ WZW-model [2]. On the other hand, quantization modifies the metric (1.2) by higher order corrections in the WZW coupling constant $\frac{1}{\tau}$. 
It might be useful to have another model to investigate classical solutions and
quantization. It was noted some time ago [3] that the equations of motion still
are exactly solvable if one adds another ("internal") string field with a particular
Liouville-type exponential interaction. The complete action of this theory is

\[ S = \frac{2}{\gamma^2} \int \sigma \tau \left( \partial_\mu u \partial_\nu u + \partial_\mu \phi \partial_\nu \phi + \text{th}^2 r \partial_\mu t \partial_\nu t + \partial_\mu \phi \partial_\nu \phi + \text{ch} 2 r e^{2\phi} \right) \]  

(1.3)

The first two, \( \phi \)-independent terms, are precisely the 2D black hole sigma-model
action, while the last two, \( \phi \)-dependent terms correspond to an "internal" field
\( \phi \) (or a flat third dimension) and a tachyon potential \( \text{ch} 2 r e^{2\phi} \). The constant \( \gamma^2 \)
plays the role of the Planck constant and will be seen later on to control the central
charge of the conformal algebra. As discussed below, this model is obtained by
gauging a nilpotent subalgebra of the Lie algebra \( B_2 \).

One may view (1.3) as the conformal gauge version of the general sigma-model

\[ S = \frac{1}{\gamma^2} \int d^2z \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu - T(X) + \Phi(X) R^{(2)} \right] \]  

(1.4)

where \( g_{\alpha\beta} \) is the world-sheet metric with curvature \( R^{(2)} \), \( \Phi \) the dilaton, \( G_{\mu\nu} \) the
metric describing the space-time background and \( T \) the so-called tachyon potential.
We have

\[ G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{th}^2 r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = -\text{ch} 2 r e^{2\phi} \]  

(1.5)

In the conformal gauge action (1.3) the dilaton \( \Phi \) has disappeared, but it should be
remembered that the full (improved) stress tensor obtained from (1.4) by varying
the metric \( g_{\alpha\beta} \) equals not only the canonical (Noether) stress tensor \( T_{\alpha\beta} \) obtained
from (1.3) but also includes a contribution from the dilaton \( \sim (\eta_{\alpha\beta} \partial^2 - \partial_\alpha \partial_\beta) \Phi \).

In the present paper, I will be only concerned with the theory defined by the
action (1.3). It will turn out to be quite interesting as a conformal field theory in
its own right. It is known as the non-abelian Toda theory [4] associated with the
Lie algebra $B_2$ [3]. The general solution of the equations of motion is in principle contained in ref. 3 where it is shown how the solutions for an equivalent system of equations can be obtained from the general scheme of ref. 4. However, it is non-trivial to actually spell out the solution and put it in a compact and useful form. This will be done in section 2, after a brief review of the main results of refs. 3, 4. The explicit solutions I obtain for the fields exhibit an amazing factorization into relatively simple factors which makes it possible to compute the three left-moving conserved quantities $T \equiv T_{++}, V^+ \equiv V^+_{++}$ and $V^- \equiv V^-_{++}$ and express them in terms of three functions $f_i(u)$ of one light-cone variable only. The same is true for the three right-moving conserved quantities $\bar{T} \equiv T_{--}, \bar{V}^+ \equiv V^+_{--}$ and $\bar{V}^- \equiv V^-_{--}$ which are expressed in terms of three functions $g_i(v)$ only. Although a priori expected, the way this actually works is highly non-trivial and constitutes a severe consistency check on the solution. Part of the algebraically somewhat involved computations of this section is transferred into the appendix B.

The complete set of conserved charges must form a closed (Poisson bracket) algebra, since otherwise one would generate new conserved quantities. Since the present theory has three fields $r,t,\phi$ one expects that the three (left-moving) conserved quantities $T,V^+$ and $V^-$ form a complete (left-moving) set and hence a closed Poisson bracket algebra. In section 3, I compute this Poisson bracket algebra using the canonical Poisson brackets of the original fields $r,t,\phi$. The $T$ and $\bar{T}$ turn out to generate each a Virasoro algebra with classical central charge $c \sim \frac{1}{\gamma}$. The Poisson brackets of $T$ with $V^+$ and $V^-$ show that the latter have conformal dimension 2 (no anomaly) while the Poisson bracket of $V^\pm$ with $V^\pm$ or $V^\mp$ gives a non-linear, non-local but closed expression of the three generators. The same applies to the $--$ components $\bar{V}^\pm$.

Probably the most important step in solving an integrable model is to obtain the Poisson brackets for the free fields of one variable only, the $f_i(u)$ and the $g_i(v)$, since these or some suitable functions thereof will serve as a basis for quantization. Given the rather involved transformation from the original fields $r,t,\phi$ (and their momenta $\Pi_r,\Pi_t,\Pi_\phi$) to the $f_i(u), g_i(v)$ obtained in section 2 it would be a
formidable task to work out these Poisson brackets directly. There is, however, an
alternative simpler route. The conserved charges $T, V^+, V^-$, resp. $\bar{T}, \bar{V}^+, \bar{V}^-$, are
relatively simple functionals of the $f_i(u)$, resp $g_i(v)$, only. It is not too difficult to
deduce the Poisson brackets of the $f_i$ and of the $g_i$ from those of the $T$’s and $V$’s.
As expected from experience with integrable models, the $f_i$ and $g_i$ Poisson brackets
are simple, but they can be made even simpler: by a further (chiral) transformation
they are turned into harmonic oscillator Poisson brackets, $i\{\varphi_n^j, \varphi_m^k\} \sim n\delta_{n,-m}\delta^{j,k}$.
This completes the classical solution of the theory defined by (1.3), and will be the
starting point for quantization which I intend to discuss in a separate publication.
Here, in section 4, I only make a few remarks about quantization and a couple of
other issues, like zero-modes and periodicity in $\sigma$ of the solution, as well as how
to implement periodicity of the Euclidean (target space) time $t$, or the possible
structure of the quantum algebra of the conserved quantities. Another interesting
topic is the associated hierarchy of integrable partial differential equations leading
probably to some non-abelian version of the KP hierarchy. However, none of these
issues will be fully solved here.

2. The classical solution

2.1. Review of known facts

The equations of motion obtained from the action (1.3) read

$$
\begin{align*}
\partial_u \partial_v r &= \frac{\operatorname{sh} r}{\operatorname{ch}^3 r} \partial_u t \partial_v t + \operatorname{sh} 2r \ e^{2\phi} \\
\partial_u \partial_v t &= -\frac{1}{\operatorname{sh} r \operatorname{ch} r} (\partial_u r \partial_v t + \partial_u t \partial_v r) \\
\partial_u \partial_v \phi &= \operatorname{ch} 2r \ e^{2\phi} .
\end{align*}
$$

(2.1)

Using these equations of motion it is completely straightforward to show that the
following three quantities are conserved [3]:

\[
T \equiv T_{++} = (\partial_u r)^2 + \text{th}^2 r (\partial_u t)^2 + (\partial_u \phi)^2 - \partial_u^2 \phi
\]

\[
V^{\pm} \equiv V^{\pm}_{++} = \frac{1}{\sqrt{2}} (2\partial_u \phi - \partial_u) \left[ e^{\pm i\nu} (\partial_u r \mp i\text{th} r \partial_u t) \right]
\]

\[\text{(2.2)}\]

i.e.

\[
\partial_v T = \partial_v V^{\pm} = 0 .
\]

\[\text{(2.3)}\]

Here \(\nu\) is defined by

\[
\partial_v \nu = \text{ch}^{-2} r \partial_v t , \quad \partial_u \nu = (1 + \text{th}^2 r)\partial_u t
\]

\[\text{(2.4)}\]

where the integrability condition is fulfilled due to the equations of motion (2.1). Similarly one has for the \(-\) components \(\partial_u T = \partial_u V^{\pm} = 0\) where

\[
\bar{T} \equiv T_{--} = (\partial_v r)^2 + \text{th}^2 r (\partial_v t)^2 + (\partial_v \phi)^2 - \partial_v^2 \phi
\]

\[
\bar{V}^{\pm} \equiv V^{\pm}_{--} = \frac{1}{\sqrt{2}} (2\partial_v \phi - \partial_v) \left[ e^{\mp i\mu} (\partial_v r \mp i\text{th} r \partial_v t) \right]
\]

\[\text{(2.5)}\]

and\(^*\)

\[
\partial_u \mu = \text{ch}^{-2} r \partial_u t , \quad \partial_v \mu = (1 + \text{th}^2 r)\partial_v t .
\]

\[\text{(2.6)}\]

The essential point for solving the equations of motion (2.1) is to realize that they can be derived from a Lax pair. Following Gervais and Saveliev [3]\(^\dagger\), one first

\(^*\) The \(\bar{V}^{\pm}\) are not given in ref. 3 but can easily be guessed, or actually deduced from \(V^{\pm}\) using the star operation defined below.

\(^\dagger\) There are some obvious misprints in ref. 3 which are corrected here.
introduces fields $a_1, a_2, a_+$ and $a_-$ subject to the following equations of motion

$$
\begin{align*}
\partial_u \partial_v a_1 &= -2(1 + 2a_+ a_-) e^{-a_1} \\
\partial_u \partial_v (2a_2 - a_1) + 2\partial_u (\partial_v a_+ a_-) &= 0 \\
\partial_u \left( e^{a_1 - 2a_2} \partial_v a_+ \right) &= 2a_+ e^{-2a_2} \\
\partial_u \left[ e^{-a_1 + 2a_2} \left( \partial_v a_- - a_2^2 \partial_v a_+ \right) \right] &= 2a_-(1 + a_+ a_-) e^{-2a_1 + 2a_2}.
\end{align*}
$$

(2.7)

It is then straightforward, although a bit lengthy, to show that we can identify

$$
\phi = -\frac{1}{2} a_1 \\
\text{sh}^2 r = a_+ a_- \\
\partial_u t = \frac{1}{2t} \left[ (1 + 2a_+ a_-) \frac{\partial_u a_-}{a_-} - \frac{\partial_u a_+}{a_+} - 2(1 + a_+ a_-) \partial_u (a_1 - 2a_2) \right] \\
\partial_v t = \frac{i}{2} \left[ (1 + 2a_+ a_-) \frac{\partial_v a_+}{a_+} - \frac{\partial_v a_-}{a_-} \right]
$$

(2.8)

i.e. with these definitions the equations (2.7) and (2.1) are equivalent. Again, the integrability condition for solving for $t$ is given by the equations (2.7).

The advantage of equations (2.7) over equations (2.1) is that they follow, as will be shown next, from the Lie algebraic formulation

$$
\partial_u (g_0^{-1} \partial_v g_0) = [J_-, g_0^{-1} J_+ g_0].
$$

(2.9)

Note, that this is obviously equivalent to the Lax representation

$$
\begin{align*}
[\partial_u - A_u, \partial_v - A_v] &= 0, \\
A_u &= -g_0^{-1} J_+ g_0 \\
A_v &= \partial_v g_0^{-1} g_0 - J_-.
\end{align*}
$$

(2.10)

Here the relevant algebra is $B_2$ with generators $h_1, h_2$ (Cartan subalgebra) and $E_{e_1}, E_{e_2}, E_{e_1-e_2}, E_{e_1+e_2}$ and their conjugates $E^+_\alpha = E_{-\alpha}$; see appendix A for their definitions in terms of fermionic oscillators. Then $H = 2h_1 + h_2, J_+ = E_{e_1}$ and
\( J_- = E_{-e_1} \) span an \( A_1 \) subalgebra. \( H \) induces a gradation on \( B_2 \). The gradation 0 part \( G_0 \) is spanned by \( h_1, h_2, E_{e_2} \) and \( E_{-e_2} = E_{e_2}^+ \). The corresponding group elements \( g_0 \in G_0 \) can be parametrized as

\[
g_0 = \exp(a_+ E_{e_2}) \exp(a_- E_{e_2}^+) \exp(a_1 h_1 + a_2 h_2) .
\]

(2.11)

Using the commutation relations of \( B_2 \), it is then easy to show that with these definitions of \( g_0 \) and \( J_- \), equation (2.9) is equivalent to equations (2.7).

Now, following ref. 3, I will show how to obtain the solution of equation (2.9). What follows applies to any Lie algebra \( G \), not only \( B_2 \). Suppose \( G \) has a grading under which it decomposes as \( G = G_+ \oplus G_0 \oplus G_- \). Then, every element of the corresponding group \( G \) has a unique Gauss decomposition

\[
g = g_0 g_+ .
\]

(2.12)

(Of course, there is also a similar decomposition \( \tilde{g}_+ \tilde{g}_0 \tilde{g}_- \).) Take some fixed elements \( J_\pm \in G_\pm \) and let for every \( g_0 \in G_0 \) the elements \( g_\pm \in G_\pm \) be solutions of the differential equations

\[
\begin{align*}
\partial_u g_+^{-1} &= g_+^{-1}(g_0^{-1} J_+ g_0) \\
\partial_v g_- &= g_-(g_0 J_- g_0^{-1}) .
\end{align*}
\]

(2.13)

Let \( g = g_- g_0 g_+ \). Equations (2.13) define \( g_+ \) and \( g_- \) up to right (resp. left) multiplications by a group element depending only on \( v \) (resp. \( u \)). Thus \( g \) is defined up to \( g \to F(u) g H(v) \). It follows from these differential equations that

\[
\begin{align*}
\partial_u (g^{-1} \partial_v g) &= g_+^{-1} \left\{ \partial_u (g_0^{-1} \partial_v g_0) - [J_-, g_0^{-1} J_+ g_0] \right\} g_+ \\
\partial_v (\partial_u gg^{-1}) &= -g_- g_0 \left\{ \partial_u (g_0^{-1} \partial_v g_0) - [J_-, g_0^{-1} J_+ g_0] \right\} g_0^{-1} g_-^{-1}
\end{align*}
\]

(2.14)

and

\[
\begin{align*}
\partial_u (g^{-1} \partial_v g) &= 0 \quad \text{and} \quad \partial_v (\partial_u gg^{-1}) = 0
\end{align*}
\]

(2.15)

are both equivalent to equation (2.9). The general solution to equations (2.15) is
\( g = g_L(u)g_R(v) \). But each group element \( g_L \) and \( g_R \) has again a Gauss decomposition

\[
\begin{align*}
    g_L(u) &= g_L^-(u)g_{L0}(u)g_{L+}(u), \\
    g_R(v) &= g_R^-(v)g_{R0}(v)g_{R+}(v)
\end{align*}
\]

so that

\[
    g = g_L^-(u)g_{L0}(u)g_{L+}(u)g_R^-(v)g_{R0}(v)g_{R+}(v). \tag{2.17}
\]

On the other hand we also have the decomposition (2.12) and \( g_+ \) and \( g_- \) must obey the differential equations (2.13). The latter translate into

\[
\begin{align*}
    \partial_u g_{L+}(u) &= -\mathcal{F}_L(u)g_{L+}(u), & \mathcal{F}_L(u) &= g_{L0}^{-1}(u)J_+g_{L0}(u), \\
    \partial_v g_{R-}(v) &= g_{R-}(v)\mathcal{F}_R(v), & \mathcal{F}_R(v) &= g_{R0}(v)J_-g_{R0}^{-1}(v). \tag{2.18}
\end{align*}
\]

The strategy then is

1. Pick some arbitrary \( g_{L0}(u), g_{R0}(v) \in G_0 \).

2. Compute the solutions \( g_{L+}(u) \) and \( g_{R-}(v) \) from the first order ordinary differential equations (2.18).

3. Let

\[
    \Gamma = g_{L0}(u)g_{L+}(u)g_{R-}(v)g_{R0}(v) \tag{2.19}
\]

and choose a basis \( |\lambda_\alpha\rangle \) of states annihilated by \( G_+ \). Then using (2.12) and (2.17) we have

\[
    G_{\alpha\beta} \equiv \langle \lambda_\beta|g_0|\lambda_\alpha\rangle = \langle \lambda_\beta|\Gamma|\lambda_\alpha\rangle. \tag{2.20}
\]

This yields all matrix elements of \( g_0 \), solution of equation (2.9), which in turn, as shown above, yields the solution for the \( a_1, a_2, a_+ \) and \( a_- \).
2.2. The explicit solution

I now apply the general strategy to the case of present interest with Lie algebra $B_2$. To carry out point 1, parametrize

$$
\begin{align*}
g_L &= \exp (-f_1(u)h_1 - f_2(u)h_2) \exp (-f_-(u)E_{-e_2}) \exp (-f_+(u)E_{e_2}) \\
g_R &= \exp (-g_+(v)E_{-e_2}) \exp (-g_-(v)E_{e_2}) \exp (-g_1(v)h_1 - g_2(v)h_2)
\end{align*}
$$

(2.21)

so that

$$
g_R = g_L^+ \bigg|_{f \to g}
$$

(2.22)

(where the $f_i, g_i$ are treated as real under hermitian conjugation). More generally, I define a star operation, denoted by $\star$, which essentially interchanges left and right movers. More precisely, it includes hermitian conjugation (treating $f_i$ and $g_i$ as real), the interchange of $f_i$ and $g_i$, and replaces $\int du$ by $-\int dv$ as well as $\partial_u$ by $-\partial_v$ and vice versa:

$$
X \star \equiv \left. X^+ \right|_{f(u) \leftrightarrow g(v), \int du \leftrightarrow -\int dv, \partial_u \leftrightarrow -\partial_v} .
$$

(2.23)

Note that

$$
(X \star) \star = X .
$$

(2.24)

Then $\mathcal{F}_L$ and $\mathcal{F}_R$ are easily obtained from their definition (2.18) with $J_\pm = E_{\pm e_1}$:

$$
\begin{align*}
\mathcal{F}_L(u) &= e^{f_1} \left[ (1 + 2f_+f_-)E_{e_1} - 2f_-E_{e_1-e_2} + 2f_+(1 + f_+f_-)E_{e_1+e_2} \right] \\
\mathcal{F}_R(v) &= \mathcal{F}_L^+ \bigg|_{f(u) \to g(v)} = \mathcal{F}_L^* .
\end{align*}
$$

(2.25)

Note that they do not depend on the functions $f_2$ and $g_2$. The group elements
\(g_{L+}(u)\) and \(g_{R-}(v)\) are given by the solutions of (2.18) as

\[
\begin{align*}
g_{L+}(u) &= 1 - \int u \, dF_L(u_1) + \int u_1 \, dF_L(u_1)F_L(u_2) - \ldots \\
g_{R-}(v) &= 1 + \int v \, dF_R(v_1) + \int v_1 \, dF_R(v_2)F_R(v_1) + \ldots
\end{align*}
\]  

(2.26)

so that

\[
g_{R-}(v) = \left.g_{L+}^+(u)\right|_{f \rightarrow g, \int du \rightarrow \int dv} = g_{L+}^* \Rightarrow g_{R-}g_{R0} = (g_{L0}g_{L+})^*.
\]  

(2.27)

This completes step 2.

All one has to do to implement step 3 is to compute \(|\psi_\alpha(v)\rangle = g_{R-}g_{R0}|\lambda_\alpha\rangle\) for the various \(|\lambda_\alpha\rangle\) annihilated by \(G_+\), i.e. annihilated by \(E_{e_1}, E_{e_1+e_2}\) and \(E_{e_1-e_2}\). Indeed, once the \(|\psi_\alpha(v)\rangle\) are known, one has

\[
G_{\beta\alpha} = \langle \chi_\beta(u)|\psi_\alpha(v)\rangle
\]

(2.28)

where \(\langle \chi_\alpha(u)| = |\psi_\alpha(v)\rangle^*\). As in ref. 3, I choose

\[
|\lambda_1\rangle = b_1^+|0\rangle \quad |\lambda_2\rangle = b_2^+b_1^+|0\rangle \quad |\lambda_3\rangle = b_0^+b_1^+|0\rangle.
\]  

(2.29)

The actual computation of the \(|\psi_\alpha\rangle\) is rather cumbersome, although straightforward. To write the results in a more compact way, introduce the functions of one variable

\[
\begin{align*}
F_1(u) &= -\int u \, e^{f_1}(1 + 2f_+f_-) \\
F_2(u) &= 2\int u \, e^{f_1}f_- \\
F_3(u) &= -2\int u \, e^{f_1}f_+(1 + f_+f_-)
\end{align*}
\]

(2.30)

10
as well as

\[ F_+ = F_1 + f_+ F_2 \quad , \quad F_- = F_3 - f_+ F_1 \]  \hfill (2.31)

and

\[ G_i(v) = F_i(u)^* \quad , \quad G_{\pm}(v) = F_{\pm}(u)^*. \]  \hfill (2.32)

The three vectors \(|\psi_{\alpha}\rangle\) then are

\[
|\psi_1\rangle = g_{R-} g_{R0} b_1^+ |0\rangle
= e^{-g_1} \left\{ b_1^+ + \sqrt{2} G_1 b_0^+ + G_2 b_2^+ + G_3 b_{-2}^+ - (G_1^2 + G_2 G_3) b_{-1}^+ \right\} |0\rangle ,
\]

\[
|\psi_2\rangle = g_{R-} g_{R0} b_2^+ b_1^+ |0\rangle
= e^{-2g_2} \left\{ b_2^+ b_1^+ - \sqrt{2} g_+ b_0^+ b_1^+ - g_1^2 b_{-2}^+ b_1^+
+ \sqrt{2} G_+ b_2^+ b_0^+ - \sqrt{2} G_- b_0^+ b_{-2}^+
- (G_+ - g_+ G_+) b_1^+ b_{-1}^+ + (G_- + g_+ G_-) b_2^+ b_{-2}^+
+ G_+^2 b_{-1}^+ b_2^+ - G_-^2 b_{-1}^+ b_{-2}^+ + \sqrt{2} G_+ G_- b_0^+ b_{-1}^+ \right\} |0\rangle ,
\]

and

\[
|\psi_3\rangle = g_{R-} g_{R0} b_0^+ b_1^+ |0\rangle
= e^{-g_1} \left\{ \begin{array}{l}
- \sqrt{2} g_- b_2^+ b_1^+ + (1 + 2 g_+ g_-) b_0^+ b_1^+ + \sqrt{2} g_+(1 + g_+ g_-) b_{-2}^+ b_1^+
- (2 g_- G_+ + G_2) b_2^+ b_0^+ + [(1 + 2 g_+ g_-) G_- - g_+ G_1] b_0^+ b_{-2}^+
+ \sqrt{2} (1 + g_+ g_-) G_+ - g_+ G_- b_1^+ b_{-1}^+ \\
- \sqrt{2} [g_- G_+ - g_+ g_- G_+ + g_+ G_2] b_1^+ b_{-2}^+
- \sqrt{2} (g_- G_+^2 + G_+ G_2) b_{-1}^+ b_2^+ + \sqrt{2} (g_- G_-^2 - G_- G_1) b_{-1}^+ b_{-2}^+
- (2 g_- G_+ G_- + G_- G_2 - G_+ G_1) b_0^+ b_{-1}^+ \end{array} \right\} |0\rangle .
\]

From these expressions one obtains the matrix elements \(G_{\alpha\beta} = \langle \chi_\alpha(v)|\psi_\beta(u)\rangle\). Obviously, \(G_{12} = G_{13} = G_{21} = G_{31} = 0\). The matrix element \(G_{11}\) is relatively simple, but \(G_{22}, G_{33}\) and \(G_{23}, G_{32}\) look discouragingly complicated at first sight.
(e.g. \( G_{33} \) contains about 150 terms). However, one realizes that the \( G_{\alpha \beta} \) factorize into products of simpler quantities:

\[
\begin{align*}
G_{11} &= e^{-f_1-g_1}Z \\
G_{22} &= e^{-2f_2-2g_2}X^2 \\
G_{23} &= \sqrt{2}e^{-2f_2-g_1}XV \\
G_{32} &= \sqrt{2}e^{-f_1-2g_2}XW \\
G_{33} &= -G_{11} + 2e^{-f_1-g_1}XY .
\end{align*}
\]

(2.36)

The quantities \( X, Y, Z \) and \( V, W \) are

\[
\begin{align*}
X &= 1 + f_+g_+ + F_+G_+ + F_-G_- \\
Y &= (1 + f_+f_-)(1 + g_+g_-) + f_-g_- + (F_1 - f_-F_-)(G_1 - g_-G_-) \\
&\quad + (F_2 + f_-F_+)(G_2 + g_-G_-) \\
Z &= 1 + 2F_1G_1 + F_2G_2 + F_3G_3 + (F_1^2 + F_2F_3)(G_1^2 + G_2G_3) \\
V &= -g_- - f_+ - g_+g_+f_+ - g_-F_+G_+ - g_-F_-G_- + F_-G_1 - F_+G_2 \\
W &= -f_- - g_- - f_-g_+ + f_-F_+G_- + f_-F_-G_+ + F_1G_- + F_2G_+ .
\end{align*}
\]

(2.37)

Note that under the star operation

\[
X^* = X \ , \ Y^* = Y \ , \ Z^* = Z \ , \ V^* = W \ , \ W^* = V .
\]

(2.38)

At this point one has to recall that the \( G_{\alpha \beta} \) are the matrix elements of \( g_Lg_L^* - g_Rg_R^* \) between \( \langle \lambda_\alpha | \) and \( | \lambda_\beta \rangle \). But according to the general discussion of the previous subsection (see point 4 of the strategy) they equal \( \langle \lambda_\alpha | g_0 | \lambda_\beta \rangle \). If one uses the parametrization (2.11) of \( g_0 \) in terms of \( a_1, a_2, a_+ \) and \( a_- \) (each depending on \( u \) and \( v \), in contrast with the \( f_i(u) \) and \( g_i(v) \)) one finds

\[
\begin{align*}
G_{11} &= e^{a_1} \\
G_{22} &= e^{2a_2}(1 + a_+a_-)^2 \\
G_{23} &= \sqrt{2}e^{a_1}a_+(1 + a_+a_-) \\
G_{32} &= \sqrt{2}e^{2a_2}a_-(1 + a_+a_-) \\
G_{33} &= -e^{a_1} + 2e^{a_1}(1 + a_+a_-) .
\end{align*}
\]

(2.39)
Comparing equations (2.39) and (2.36), one obtains the $a_1, a_2, a_+, a_-$ in terms of the $V, W, X, Y, Z$. Note that we have 5 equations for only 4 $a$'s. This implies a relation between the $V, \ldots Z$ as will be shown below. Note also that equations (2.39) have the same factorized form as (2.36) which allows for immediate identifications like $e^{a_1} = e^{-f_1 -g_1} Z$ and $1 + a_+ a_- = \frac{XY}{Z}$. The complete solution is

\begin{align*}
e^{a_1} &= e^{-f_1 -g_1} Z \\
e^{-a_2} &= \pm e^{f_2 +g_2} \frac{Y}{Z} \\
a_+ &= e^{f_1 -2f_2} \frac{V}{Y} \\
a_- &= e^{2f_2 -f_1} \frac{YW}{Z} \quad (2.40)
\end{align*}

while the fifth equation is the relation

\begin{equation}
XY - Z = VW \quad (2.41)
\end{equation}

which is easily verified. Some other relations satisfied by the $V, \ldots Z$ and their derivatives are given in appendix B. It is also noteworthy that under the star operation

\begin{align*}
(e^{a_1})^* &= e^{a_1} , \quad (e^{-a_2})^* = e^{-a_2} \\
(a_+)^* &= e^{2a_2 -a_1} a_- , \quad (a_-)^* = e^{a_1 -2a_2} a_+ \quad (2.42)
\end{align*}

so that $(a_+ a_-)^* = a_+ a_-$. It is then seen from (2.8) that the fields $\phi, r$ and $t$ are invariant under the star operation. Actually, since one only determines $sh^2 r$ (or $ch^2 r$) one has a sign ambiguity for $r$ and one could decide that $r$ changes sign under the star operation. (However from the euclidean black hole point of view, $r$ should be always non-negative, hence is invariant.) Also $t$ is invariant if one assumes $i^* = -i$ (as would be normally implied by hermitian conjugation). If one considers the Minkowskian continuation $\theta = it$ instead, one sees that the star operation corresponds to Minkowskian time reversal.

13
From equations (2.8) one immediately has

\[ \phi = \frac{1}{2} (f_1 + g_1 - \log Z) \]  
\[ \text{sh}^2 r = \frac{VW}{Z^2} = \frac{XY}{Z} - 1 \]  

while some more work is needed to integrate the two equations for \( t \). Using equations (B.2) and (B.3) from appendix B one finds

\[ -2i \partial_t t = 2 \frac{W \partial_t V - X \partial_t Y}{Z} + \partial_v \log \frac{VZ}{W} \]

\[ = -2g - g' + \partial_v \log \frac{V}{W} \]  

and

\[ -2i \partial_u t = 2f - f' + \partial_u \log \frac{V}{W} \]  

which can be integrated to give

\[ t = t_0 + i \int^u f - f'_+ - i \int^v g - g'_+ + \frac{i}{2} \log \frac{V}{W} \]  

(where it is again obvious that \( t^* = t \)). This completes the explicit solution of the field equations of motion.

Before going on, let me make a remark about the reality properties of the solution. For real functions \( f_i, g_i \) the euclidean time \( t - t_0 \) is purely imaginary (provided \( V/W > 0^* \)) so that the Minkowski time \( \theta = it \) is real. Thus real \( f_i, g_i \) are appropriate for a Minkowskian “target space” time. On the other hand, if we use euclidean “world-sheet” coordinates \( u, v \) so that \( u^* = v \) (here * means complex conjugation), we can use complex \( f_i, g_i \) such that \( f_i(u)^* = g_i(v) \), so that the exchange of \( f \) and \( g \) in the star operation is part of the hermitian conjugation. Then \( V^* = W \) simply becomes \( V^* = W \) and \( \log \frac{V}{W} \) is purely imaginary. It follows that in this case \( t - t_0 \) is real. So this is the appropriate setting for euclidian “target space” time \( t \). In both cases, \( r \) and \( \phi \) are real.

* If \( V/W < 0 \) the corresponding \((i/2)i\pi \) can be absorbed into \( t_0 \).
One might want to check directly that the equations of motion are indeed satisfied. This can be done. For example, the $a_1$-equation of motion reduces to equation (B.4). Rather than verifying the $t$-equation of motion directly, one may observe that the integrability of the above equations (2.44) and (2.45) is a severe consistency check on the solution. Moreover, in the next subsection, I express the conserved quantities $T$ and $V^\pm$ in terms of the $f_i(u)$ only. This would almost certainly fail if there were only the slightest error in the solution (2.37), (2.40).

2.3. THE CONSERVED QUANTITIES

Using the equations of motion for $\phi$, $r$ and $t$ it was shown above that the quantities $T \equiv T_{++}$ and $V^\pm \equiv V_{++}^\pm$ are conserved, i.e. can only depend on $u$. This means that they must be expressible entirely in terms of the $f_i(u)$'s. Given the complexity of the solutions (2.43), (2.46) and (2.37) this is highly non-trivial, and, as already mentioned, constitutes a severe consistency check. The same considerations apply to $\bar{T} \equiv T_{--}$ and $\bar{V}^\pm \equiv V_{--}^\pm$.

To begin with, I consider $T_{\pm\pm}$ as given by (2.2) and (2.5). When expressed in terms of $a_1, a_2, a_+$ and $a_-$ they read

$$
\bar{T} \equiv T_{--} = \frac{1}{4}(\partial_v a_1)^2 + \frac{1}{2} \partial_v^2 a_1 + \partial_v a_+(\partial_v a_- - a_-^2 \partial_v a_+) \\
T \equiv T_{++} = \frac{1}{4}(\partial_u a_1)^2 + \frac{1}{2} \partial_u^2 a_1 \\
+ \partial_u \left( e^{2a_2-a_1} a_- \right) \left[ \partial_u \left( e^{a_1-2a_2} a_+ \right) - \left( e^{a_1-2a_2} a_+ \right)^2 \partial_u \left( e^{2a_2-a_1} a_- \right) \right] \\
= T_{--}^* .
$$

(2.47)

It is easier to compute $T_{--}$ first. Using equation (2.40) and performing straightforward algebra one obtains

$$
T_{--} = \frac{1}{4}(g'_1)^2 - \frac{1}{2} g''_1 + \frac{\partial_v^2 Z - g'_1 \partial_v Z}{2Z} - \left( \frac{\partial_v Z}{2Z} \right)^2 + t_{--}
$$

(2.48)
with

\[ t_{--} = \frac{1}{Y^2 Z^2} [Y \partial_v V - V \partial_v Y] \times \]
\[ \times [Y (Z \partial_v W - W \partial_v Z) - W^2 (Y \partial_v V - V \partial_v Y) + WZ \partial_v Y] . \]  

(2.49)

Using the relation (2.41) and its \( \partial_v \) derivative one can simplify \( t_{--} \) to

\[ t_{--} = \frac{1}{Z^2} (Y \partial_v V - V \partial_v Y) (X \partial_v W - W \partial_v X) . \]  

(2.50)

Inserting relations (B.5) and (B.6) of appendix B (\( \alpha \) and \( \beta \) are defined in this appendix) yields

\[ t_{--} = g'_+ (g'_- - g^2_- g'_+) - \frac{g'_+ \beta + (g'_- - g^2_- g'_+)}{Z} + \frac{\alpha \beta}{Z^2} . \]  

(2.51)

Combining this formula with (2.48) one finds for \( T_{--} \) a piece \( \frac{1}{4} (g'_1)^2 - \frac{1}{2} g''_1 + g'_+ (g'_- - g^2_- g'_+) \) that manifestly only depends on \( v \), plus

\[ \Delta = \frac{\partial^2 V - g'_1 g''_1 Z}{2Z} - \left( \frac{\partial V Z}{2Z} \right)^2 - \frac{g'_+ \beta + (g'_- - g^2_- g'_+)}{Z} + \frac{\alpha \beta}{Z^2} . \]  

(2.52)

But this vanishes by relation (B.8), so that finally

\[ T_{--} = \frac{1}{4} (g'_1)^2 - \frac{1}{2} g''_1 + g'_+ (g'_- - g^2_- g'_+) . \]  

(2.53)

This is again a very simple expression, surprisingly similar to (2.47). However, in (2.47) the \( a \)'s are interacting fields depending on \( u \) and \( v \) in a very complicated way, while in (2.53) the \( g_i(v) \) are functions of \( v \) only. The ++ component of \( T \) is obtained by the star operation:

\[ T_{++} = \frac{1}{4} (f'_1)^2 - \frac{1}{2} f''_1 + f'_+ (f'_- - f^2_- f'_+) . \]  

(2.54)

In the next section, I will make a further (although much simpler) field redefinition \( f_i(u) \to \varphi_i(u) \) that brings \( T_{++} \) into the form \( \sim (\partial \varphi)^2 + \partial^2 \varphi \), which is the standard form for a conformal field theory energy momentum tensor.
The next task is to compute the $V^\pm$ and $\bar{V}^\pm$ given by (2.2) and (2.5). First, one has to integrate (2.4) and (2.6) for $\nu$ and $\mu$. Using relations (B.2) and (B.3) this can be done with the result

\[
\nu = 2i \int u^ f - f'_+ + \frac{i}{2} \log\frac{VX}{WY} \\
\mu = -2i \int v^ g - g'_+ - \frac{i}{2} \log\frac{WX}{VY}.
\]  \hspace{1cm} (2.55)

Inserting this and the solutions for $\phi, r$ and $t$ one obtains after some simple algebra, e.g. for $V^+$:

\[
V^+ = \frac{e^{f_1}}{2\sqrt{2}Z} \partial_u \left[ \exp \left( -f_1 - 2 \int f_- f'_+ \right) \frac{1}{X} \left( 2Z\partial_u W - W\partial_u Z - 2WZf_- f'_+ \right) \right].
\]  \hspace{1cm} (2.56)

Then using (B.3) and (2.41) this can be written as

\[
V^+ = \frac{e^{f_1}}{\sqrt{2Z}} \partial_u \left[ \exp \left( -f_1 - 2 \int f_- f'_+ \right) (Y\partial_u W - W\partial_u Y) \right].
\]  \hspace{1cm} (2.57)

Making further use of the relation obtained from (B.6) by the star operation and equation (B.10) one finally arrives at

\[
V^+ = \frac{1}{\sqrt{2}} (f'_1 - \partial_u) \left[ \exp \left( -2 \int f_- f'_+ \right) (f'_- f'_+ - f'_- f'_+) \right].
\]  \hspace{1cm} (2.58)

Again, one should compare this with the starting point (2.2), but now (2.58) is manifestly only a function of $u$. $V^-$ is obtained along similar lines and I only give the result:

\[
V^- = \frac{1}{\sqrt{2}} (f'_1 - \partial_u) \left[ \exp \left( +2 \int f_- f'_+ \right) f'_+ \right].
\]  \hspace{1cm} (2.59)
The $\tilde{V}^\pm$ are obtained from the $V^\pm$ by the star operation:

$$
\tilde{V}^+ = \frac{1}{\sqrt{2}}(g_1' - \partial_v) \left[ \exp \left( -2 \int g_- g'_+ \right) \left( g'_- - g'_+ g'_+ \right) \right] \\
\tilde{V}^- = \frac{1}{\sqrt{2}}(g_1' - \partial_v) \left[ \exp \left( +2 \int g_- g'_+ \right) g'_+ \right].
$$

In conclusion, through the subtle “magic” of integrable models, the conserved quantities manage to be simple expressions of the functions $f_i(u)$, resp. $g_i(v)$ of one variable only. The $V^\pm, \tilde{V}^\pm$ depend in a non-local way on the $f_i, g_i$. In the next section, motivated by the Poisson brackets of the $f_i$’s $(g_i$’s), I make a non-local transformation $f_i(u) \rightarrow \varphi_i(u)$ (and similarly for the $g$’s). The $V^\pm$ then depend locally on the $\varphi_i(u)$.

Many more questions remain to be investigated at the level of the classical solution to the equations of motion, and I will mention some of them in section 4. Now, however, I turn to the discussion of the Poisson brackets derived from the canonical structure.

3. Symplectic structure and constraint algebra

In this section I will determine the Poisson brackets of the various fields and conserved quantities (constraints) encountered in the previous section. The program is the following.

First, in subsection 3.1, using the canonical Poisson brackets of the fields $r, t$ and $\phi$ and their momenta, I compute the Poisson bracket algebra of the conserved

---

* Since $T_{++}$ and $V_{++}$ are conserved under evolution of the light-cone “time” $v$, they can be considered as constraints imposed on the initial data. If one considered a Hamiltonian formalism of the theory one would discover that $T_{++}$ indeed appear as constraints conjugate to Lagrange multipliers that play the role of world-sheet shift and lapse functions. This leaves open the interpretation of the $V^\pm$. The answer is probably closely related to the problem of $W$-gravity, and I expect that the $V^\pm$ are constraints related to symmetries of the target space. In any case I will refer to the $T$ and $V^\pm$ as constraints.
quantities $T$ and $V^\pm$ (constraints) when expressed in terms of $r$, $t$ and $\phi$ and their derivatives. I will find that $T$ behaves as a stress tensor in conformal field theory and obeys the usual Poisson bracket version of the Virasoro algebra with classical central charge. The Poisson bracket of $T$ with $V^\pm$ just shows that $V^\pm$ are conformally primary fields of weight 2. The Poisson bracket of $V^\pm$ with $V^\pm$ or $V^\mp$ is more interesting (and more difficult to obtain). On dimensional grounds one expects that the bracket of $V^\pm$ with $V^\mp$ can contain $T$ and a central term, but also a term $V^\pm V^\mp$. All of them do indeed appear. Due to the $U(1)$ charges $\pm 1$ and 0 we can assign to $V^\pm$ and $T$, the Poisson bracket of $V^\pm$ with $V^\pm$ can only give a $V^\pm V^\pm$ term.

Then, in principle one could deduce the Poisson brackets of the $f_i$ and $g_i$ through the transformation induced by the classical solution (2.43), (2.46). More precisely, one would have to allow formally that the $f_i$ and $g_i$ depend both on $u$ and $v$ since one has to consider the full phase space and not only the manifold of solutions to the equations of motion. Nevertheless, equations (2.43) and (2.46), as well as their time derivatives, constitute a phase space transformation from $r(\tau, \sigma), t(\tau, \sigma), \phi(\tau, \sigma)$ and their momenta $\Pi_r(\tau, \sigma), \Pi_t(\tau, \sigma), \Pi_\phi(\tau, \sigma)$ to new phase space variables $f_i(\tau, \sigma), g_i(\tau, \sigma)$. (Of course, the equations of motion still imply $\partial_v f_i = \partial_u g_i = 0$.) In practice, this would be very complicated to implement. It is much simpler to use the Poisson brackets of the $T$ and $V^\pm$ derived before, and then consider the $T$ and $V^\pm$ (or $\bar{T}$ and $\bar{V}^\pm$) as given in terms of the $f_i$ only (or $g_i$ only). Thus one does the phase space transformation in two steps: $r(\tau, \sigma), t(\tau, \sigma), \phi(\tau, \sigma), \Pi_r(\tau, \sigma), \Pi_t(\tau, \sigma), \Pi_\phi(\tau, \sigma) \rightarrow T_{\pm\pm}(\tau, \sigma), V_{\pm\pm}^+(\tau, \sigma), V_{\pm\pm}^-(\tau, \sigma) \rightarrow f_i(\tau, \sigma), g_i(\tau, \sigma)$ This yields the Poisson brackets of the $f_i$ and of the $g_i$ in a relatively easy way (subsection 3.2). These Poisson bracket are simple, and a final (non-local) transformation (subsection 3.3) turns them into standard harmonic oscillator Poisson brackets.
3.1. The constraint algebra

Recall that the theory under consideration is based on the action (1.3). Writing $u = \tau + \sigma$, $v = \tau - \sigma$, it becomes (as usual, a dot denotes $\partial_\tau$ while a prime denotes $\partial_\sigma$)

$$S = \frac{1}{\gamma^2} \int d\tau d\sigma \left[ \frac{1}{2} (\dot{r}^2 - r'^2) + \frac{1}{2} \theta^2 r (\dot{t}^2 - t'^2) + \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + 2 \text{ch} 2 r e^{2\phi} \right].$$

The constant $\gamma^2$ can be viewed as the Planck constant $2\pi\hbar$, already included in the classical action, or merely as a coupling constant. The canonical momenta $\Pi_r = \gamma^{-2} \dot{r}$, $\Pi_t = \gamma^{-2} \theta^2 r \dot{t}$, $\Pi_\phi = \gamma^{-2} \dot{\phi}$

and the canonical (equal $\tau$) Poisson brackets are

$$\{ r(\tau, \sigma), \Pi_r(\tau, \sigma') \} = \{ t(\tau, \sigma), \Pi_t(\tau, \sigma') \} = \{ \phi(\tau, \sigma), \Pi_\phi(\tau, \sigma') \} = \delta(\sigma - \sigma')$$

$$\{ r(\tau, \sigma), r(\tau, \sigma') \} = \{ t(\tau, \sigma), t(\tau, \sigma') \} = \ldots = 0$$

$$\{ \Pi_i(\tau, \sigma), \Pi_j(\tau, \sigma') \} = 0.$$

It follows that the only non-zero equal $\tau$ Poisson brackets are

$$\{ r(\tau, \sigma), \dot{r}(\tau, \sigma') \} = \gamma^2 \delta(\sigma - \sigma') \quad \{ t(\tau, \sigma), \dot{t}(\tau, \sigma') \} = \frac{\gamma^2}{\theta^2 r} \delta(\sigma - \sigma'),$$

$$\{ \phi(\tau, \sigma), \dot{\phi}(\tau, \sigma') \} = \gamma^2 \delta(\sigma - \sigma') \quad \{ \dot{r}(\tau, \sigma), \dot{t}(\tau, \sigma') \} = \frac{2\gamma^2}{\text{sh} r \text{ch} r} \delta(\sigma - \sigma'),$$

and those derived from them by applying $\partial_\sigma^m \partial_\sigma'^m$.

Before one can compute the Poisson bracket algebra of the $T, V^\pm$ one has to rewrite them in terms of the fields and their momenta. This means in particular that second (and higher) $\tau$-derivatives have to be eliminated first, using the equations of motion. One might object that one is not allowed to use the equations of motion in a canonical formulation. However, the conserved quantities given in...
the previous section are only defined up to terms that vanish on solutions of the equations of motion. So the correct starting point for a canonical formulation are the expression where all higher $\tau$-derivatives are eliminated, while the expressions given in section 2 are merely derived from the canonical ones by use of the equations of motion. One has, for example, using the $\phi$-equation of motion (2.1)

$$\frac{\partial^2 u}{\partial \phi} = \frac{1}{4}(\ddot{\phi} + 2\dot{\phi}' + \phi'') = \frac{1}{2}(\phi'' + \dot{\phi}') + \text{ch}2r e^{2\phi} = \partial_\sigma \partial_u \phi + \text{ch}2r e^{2\phi}$$  \hspace{1cm} (3.5)

where in the canonical formalism

$$\partial_u r = \frac{1}{2}(\gamma^2 \Pi_r + r') \hspace{0.5cm} \partial_u t = \frac{1}{2}(\gamma^2 \text{th}^{-2} \Pi_t + t') \hspace{0.5cm} \partial_u \phi = \frac{1}{2}(\gamma^2 \Pi_\phi + \phi')$$  \hspace{1cm} (3.6)

The canonical expressions for the ++ components of the constraints are

$$T = (\partial_u r)^2 + \text{th}^2 r (\partial_u t)^2 + (\partial_u \phi)^2 - (\partial_u \phi)' - \text{ch}2r e^{2\phi}$$

$$V^\pm = \frac{1}{\sqrt{2}} e^{\pm iv} \left[ 2\partial_u \phi \partial_u r \pm 2i\text{th}r \partial_u \phi \partial_u t + 2\text{th}^3 r(\partial_u t)^2 \mp 2i\text{th}r \partial_u r \partial_u t \right. \
\left. \frac{\text{ch}r}{\text{sh}^2 r} \partial_u r t' + i \frac{\partial_u r t'}{\text{ch}^2 r} - (\partial_u r')' = \text{th}r(\partial_u t)' - \text{sh}2r e^{2\phi} \right]$$  \hspace{1cm} (3.7)

where the substitutions (3.6) are understood.

It is easy to derive from (3.4) the following Poisson brackets needed for computing $\{T, T\}$ (I do not write the $\tau$-argument any longer; all Poisson brackets are at equal $\tau$):

$$\{\partial_u \phi(\sigma), \partial_u \phi(\sigma')\} = \frac{\gamma^2}{2} \delta'(\sigma - \sigma')$$

$$\{\partial_u \phi(\sigma), \partial_u^2 \phi(\sigma')\} = -\frac{\gamma^2}{2} \delta''(\sigma - \sigma') - \gamma^2 \text{ch}2r e^{2\phi} \delta(\sigma - \sigma')$$

$$\{\partial_u^2 \phi(\sigma), \partial_u^2 \phi(\sigma')\} = -\frac{\gamma^2}{2} \delta''(\sigma - \sigma') - \gamma^2 (\partial_\sigma - \partial_\sigma') \left[ \text{ch}2r e^{2\phi} \delta(\sigma - \sigma') \right]$$  \hspace{1cm} (3.8)

$$\{\partial_u r(\sigma), \partial_u r(\sigma')\} = \frac{\gamma^2}{2} \delta'(\sigma - \sigma')$$

$$\{\partial_u t(\sigma), \partial_u t(\sigma')\} = \frac{\gamma^2}{4} (\partial_\sigma - \partial_\sigma') \left[ \text{th}^{-2} r \delta(\sigma - \sigma') \right]$$

$$\{\partial_u r(\sigma), \partial_u t(\sigma')\} = \frac{\gamma^2}{\text{ch}r \text{sh}r} \left( \partial_u t - \frac{t'}{2} \right) \delta(\sigma - \sigma') \hspace{1cm} .$$

21
Then one obtains
\[ \gamma^{-2}\{ T(\sigma), T(\sigma') \} = (\partial_{\sigma} - \partial_{\sigma'}) [ T(\sigma')\delta(\sigma - \sigma') ] - \frac{1}{2}\delta'''(\sigma - \sigma') \]  
(3.9)

and similarly
\[ \gamma^{-2}\{ \bar{T}(\sigma), \bar{T}(\sigma') \} = - (\partial_{\sigma} - \partial_{\sigma'}) [ \bar{T}(\sigma')\delta(\sigma - \sigma') ] + \frac{1}{2}\delta'''(\sigma - \sigma') \]  
(3.10)

while
\[ \{ T(\sigma), \bar{T}(\sigma') \} = 0 . \]  
(3.11)

This are just two copies of the conformal algebra. If \( \sigma \) takes values on the unit circle one can define the modes
\[ L_n = \gamma^{-2} \int_{-\pi}^{\pi} d\sigma \left[ T(\tau, \sigma) + \frac{1}{4} \right] e^{in(\tau+\sigma)} \]  
(3.12)
\[ \bar{L}_n = \gamma^{-2} \int_{-\pi}^{\pi} d\sigma \left[ \bar{T}(\tau, \sigma) + \frac{1}{4} \right] e^{in(\tau-\sigma)} . \]

Then the brackets (3.9) and (3.10) become two Virasoro algebras
\[ i\{ L_n, L_m \} = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \]  
(3.13)
\[ i\{ L_n, \bar{L}_m \} = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \]

while \( \{ L_n, \bar{L}_m \} = 0 \). Here \( c \) is the central charge given by
\[ c = \frac{12\pi}{\gamma^2} . \]  
(3.14)

The occurrence of a central charge already at the classical, Poisson bracket level is due to the \( \partial^2 \phi \)-term in \( T \). This is reminiscent of the well-known Liouville theory. The factor \( i \) on the left hand side of equations (3.13) may seem strange at first sight, but one should remember that quantization replaces \( i \) times the canonical Poisson bracket by the commutator. Hence (3.13) is indeed the Poisson bracket version of the (commutator) Virasoro algebra.
In order to compute Poisson brackets involving \( V^\pm \) one needs the Poisson brackets involving the field \( \nu \). Now, \( \nu \) is only defined through its partial derivatives, and thus only up to a constant. This constant, however, may have a non-trivial Poisson bracket with certain modes of \( \nu \) and/or of the other fields. From (2.4) one has using (3.2)

\[
\nu' = \gamma^2 \Pi_t + t'
\]

whereas one does not need \( \dot{\nu} \) explicitly. Equation (3.15) implies

\[
\partial_\sigma \partial_{\sigma'} \{ \nu(\sigma), \nu(\sigma') \} = \{ \nu'(\sigma), \nu'(\sigma') \} = 2\gamma^2 \delta'(\sigma - \sigma') .
\]

This is integrated to yield \( \{ \nu(\sigma), \nu(\sigma') \} = -\gamma^2 \epsilon(\sigma - \sigma') + h(\sigma) - h(\sigma') \) where I already used the antisymmetry of the Poisson bracket. \( \epsilon(\sigma - \sigma') \) is defined to be +1 if \( \sigma > \sigma' \), −1 if \( \sigma < \sigma' \) and 0 if \( \sigma = \sigma' \). The freedom to choose the function \( h \) corresponds to the above-mentioned freedom to add a constant \( \nu_0 \) to \( \nu \) with \( \{ \nu(\sigma), \nu_0 \} = h(\sigma) \). However, if one imposes invariance under translations \( \sigma \to \sigma + a, \sigma' \to \sigma' + a \) then \( h \) can be only linear and one arrives at

\[
\{ \nu(\sigma), \nu(\sigma') \} = -\gamma^2 \epsilon(\sigma - \sigma') - \gamma^2 \left[ \epsilon(\sigma - \sigma') - \frac{\alpha}{\pi}(\sigma - \sigma') \right] .
\]

There is only one free parameter \( \alpha \) left. Roughly speaking, \( \alpha \) is related to the Poisson bracket of the “center of mass” position and momentum (zero) modes of \( \nu \). Hence the choice of \( \alpha \) depends on the topology of the world-sheet. If \( \sigma \) takes values on the unit circle \( S^1 \) with \( \sigma \in [-\pi, \pi] \) and if \( \nu \) is defined to be periodic on the circle, then \( \alpha = 1 \) so that \( \epsilon_1(-\pi) = \epsilon_1(\pi) \) and \( \epsilon_1 \) can be continued as a periodic function. If \( \sigma \in \mathbb{R} \) it is most appropriate to define \( \nu \) such that \( \alpha = 0 \). For simplicity, I will assume the latter case till further notice. Later on, I will come back to \( \sigma \in S^1 \). The Poisson brackets one needs thus are

\[
\begin{align*}
\{ \nu(\sigma), \nu(\sigma') \} &= -\gamma^2 \epsilon(\sigma - \sigma') \quad , \quad \{ \nu(\sigma), \partial_\sigma t(\sigma') \} = \frac{\gamma^2}{2} (1 + \text{th}^{-2} r) \delta(\sigma - \sigma') , \\
\{ \nu(\sigma), t'(\sigma') \} &= \gamma^2 \delta(\sigma - \sigma') \quad , \quad \{ \partial_\sigma t(\sigma), t'(\sigma') \} = \frac{\gamma^2}{2} \text{th}^{-2} r(\sigma) \delta'(\sigma - \sigma') ,
\end{align*}
\]
together with (3.8), and also \( \{\nu, r\} = \{\nu, \partial_u r\} = \{\nu, \phi\} = \{\nu, \partial_u \phi\} = 0 \).

Before doing the actual calculation it is helpful to show how the result is constrained by dimensional and symmetry considerations. Consider first \( \{V^+(\sigma), V^+(\sigma')\} \). Each \( V^+ \) contains a factor \( e^{i\nu} \). The Poisson bracket of \( e^{i\nu(\sigma)} \) with \( e^{i\nu(\sigma')} \) leads to a term \( \gamma^2 \epsilon(\sigma - \sigma')V^+(\sigma)V^+(\sigma') \). All other terms are local, i.e. involve \( \delta(\sigma - \sigma') \) or derivatives of \( \delta(\sigma - \sigma') \). On dimensional grounds \( \delta(\sigma - \sigma') \) must be multiplied by a dimension 3 object (3 derivatives) and \( \delta'(\sigma - \sigma') \) by a dimension 2 object. Furthermore, these objects must contain an overall factor \( e^{2i\nu} \).

If the \( T, V^+ \) and \( V^- \) form a closed algebra, there are no such objects. The same reasoning applies to \( \{V^-(\sigma), V^-(\sigma')\} \). Thus one expects

\[
\gamma^{-2}\{V^\pm(\sigma), V^\pm(\sigma')\} = \epsilon(\sigma - \sigma')V^\pm(\sigma)V^\pm(\sigma') .
\]

(3.19)

It is a bit tedious, but otherwise straightforward to verify that this is indeed correct. Note that it is enough to do the computation for \( V^+ \) since \( V^- \) is the complex conjugate of \( V^+ \) (treating the fields \( r, t, \phi \) and \( \nu \) and their derivatives as real):

\[
V^- = (V^+)^* .
\]

(3.20)

What can one say about \( \{V^+(\sigma), V^-(\sigma')\} \)? Using (3.20) one sees that \( \{V^+(\sigma), V^-(\sigma')\}^* = -\{V^+(\sigma), V^-(\sigma')\}|_{\sigma \mapsto \sigma'} \). This fact, together with the same type of arguments as used above, imply

\[
\gamma^{-2}\{V^+(\sigma), V^- (\sigma')\} = -\epsilon(\sigma - \sigma')V^+(\sigma)V^- (\sigma') + (\partial_\sigma - \partial_{\sigma'})[a\delta(\sigma - \sigma')] \\
+ ib\delta(\sigma - \sigma') + i(\partial_\sigma^2 + \partial_{\sigma'}^2)[d\delta(\sigma - \sigma')] + \tilde{c}\delta'''(\sigma - \sigma')
\]

(3.21)

where \( a, b, d, \tilde{c} \) are real and have (naive) dimensions 2, 3, 1 and 0. Hence \( \tilde{c} \) is a c-number. Also, \( b, a, d \) cannot contain a factor \( e^{\pm i\nu} \). If one assumes closure of the

\* The naive dimensional counting assigns dimension 1 to each derivative, dimension 0 to all fields \( r, t, \phi, \nu \) and functions thereof, except for functions of \( \phi \). \( e^{2\nu} \) has dimension 2 as seen from the action, while \( \delta(\sigma - \sigma') \) has dimension 1.
algebra one must have \( b = d = 0 \) and \( a \sim T_{++} \). After a really lengthy computation one indeed finds equation (3.21) with \( b = d = 0 \), \( a = T_{++} \) and \( \tilde{c} = -\frac{1}{2} \).

Finally the Poisson bracket of \( T \) with \( V^{\pm} \) simply shows that \( V^{\pm} \) are conformally primary fields of weight (conformal dimension) 2. The complete algebra thus is

\[
\begin{align*}
\gamma^{-2} \{ T(\sigma) , T(\sigma') \} &= (\partial_\sigma - \partial_{\sigma'}) [ T(\sigma') \delta(\sigma - \sigma') ] - \frac{1}{2} \delta'''(\sigma - \sigma') \\
\gamma^{-2} \{ T(\sigma) , V^{\pm}(\sigma') \} &= (\partial_\sigma - \partial_{\sigma'}) [ V^{\pm}(\sigma') \delta(\sigma - \sigma') ] \\
\gamma^{-2} \{ V^{\pm}(\sigma) , V^{\pm}(\sigma') \} &= \epsilon(\sigma - \sigma') V^{\pm}(\sigma) V^{\pm}(\sigma') \\
\gamma^{-2} \{ V^{\pm}(\sigma) , V^{\mp}(\sigma') \} &= -\epsilon(\sigma - \sigma') V^{\pm}(\sigma) V^{\mp}(\sigma') \\
&\quad + (\partial_\sigma - \partial_{\sigma'}) [ T(\sigma') \delta(\sigma - \sigma') ] - \frac{1}{2} \delta'''(\sigma - \sigma') .
\end{align*}
\]

The algebra of the \(-\)\(-\) components \( \bar{T}^{\pm} , \bar{V}^{\pm} \) is obtained by applying the star operation, and looks exactly the same except for the replacements \( \sigma \rightarrow -\sigma \), \( \sigma' \rightarrow -\sigma' \) and hence \( \partial \rightarrow -\partial \), \( \epsilon(\sigma - \sigma') \rightarrow -\epsilon(\sigma - \sigma') \).

The algebra (3.22) is the correct algebra for \( \sigma \in \mathbb{R} \). If \( \sigma \in S^1 \), as outlined above, one must replace \( \epsilon(\sigma - \sigma') \rightarrow \epsilon_1(\sigma - \sigma') \) which is a periodic function. Also \( \delta(\sigma - \sigma') \rightarrow \frac{1}{2\pi} \partial_\sigma \epsilon_1(\sigma - \sigma') = \delta(\sigma - \sigma') - \frac{1}{2\pi} \) while \( \delta'(\sigma - \sigma') \) remains unchanged. But since the right hand sides of (3.22) can be written using only \( \epsilon(\sigma - \sigma') \), \( \delta'(\sigma - \sigma') \) and \( \delta'''(\sigma - \sigma') \), only the replacement \( \epsilon(\sigma - \sigma') \rightarrow \epsilon_1(\sigma - \sigma') \) is relevant. One then defines the modes

\[
\begin{align*}
V_n^{\pm} &= \gamma^{-2} \int_{-\pi}^{\pi} d\sigma V^{\pm}(\tau, \sigma) e^{in(\tau + \sigma)} \\
\bar{V}_n^{\pm} &= \gamma^{-2} \int_{-\pi}^{\pi} d\sigma \bar{V}^{\pm}(\tau, \sigma) e^{in(\tau - \sigma)} .
\end{align*}
\]

(3.23)
The mode algebra is†

\[ i\{L_n, L_m\} = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \]

\[ i\{L_n, V^\pm_m\} = (n - m)V^\pm_{n+m} \]

\[ i\{V^\pm_n, V^\pm_m\} = \frac{12}{c} \sum_{k \neq 0} \frac{1}{k} V^\pm_{n+k} V^\pm_{m-k} \]

\[ i\{V^\pm_n, V^{\mp}_m\} = -\frac{12}{c} \sum_{k \neq 0} \frac{1}{k} V^\pm_{n+k} V^{\mp}_{m-k} + (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \]

(3.24)

This is a non-linear algebra, reminiscent of the \( W \)-algebras. What is new here are the non-local terms involving \( \epsilon(\sigma - \sigma') \), or the \( \frac{1}{k} \) in the mode algebra.

### 3.2. The Poisson brackets of the \( f_i \) and \( g_i \)

In this subsection, I will deduce the Poisson brackets of the \( f_i \) from those of \( T, V^+ \) and \( V^- \). Those of the \( g_i \) follow by the star operation. As in the previous section, the \( T, V^\pm \) as well as the \( f_i \) are considered as functions on phase-space without imposing the equations of motion, so they are supposed to depend on \( \tau \) and \( \sigma \). Again, since all Poisson brackets are at equal \( \tau \), I will not write \( \tau \) explicitly.

Recall

\[ T = \frac{1}{4}(f'_1)^2 - \frac{1}{2} f''_1 + f'_1(f'_- - f^2 f'_+) \]

\[ V^\pm = \frac{1}{\sqrt{2}}(f'_1 - \partial_\sigma)O^\pm \]

(3.25)

where now \( f'_i = \partial_\sigma f_i \), and where I introduced

\[ O^+ = \exp \left( -2 \int B \right) (f'_- - f^2 f'_+) \]

\[ O^- = \exp \left( 2 \int B \right) f'_+ \]

(3.26)

\[ B = f^- f'_+ . \]

† As a further consistency check one can verify that the Jacobi identities are satisfied.
One observes that the part of $T$ involving only $f_1$ already satisfies the correct Virasoro Poisson bracket algebra with central charge $c$ by itself if

$$\{f_1(\sigma), f_1(\sigma')\} = 2\gamma^2 \delta'(\sigma - \sigma').$$  \tag{3.27}$$

It is thus natural to use this and

$$\{f_1(\sigma), f_\pm(\sigma')\} = 0 \tag{3.28}$$

as an Ansatz.

Consider now $\{V^-(\sigma), V^-(\sigma')\} = \gamma^2 \epsilon(\sigma - \sigma')V^-(\sigma)V^-(\sigma')$. By the same type of argument as used in the previous subsection one must have

$$\{O^-(\sigma), O^-(\sigma')\} = [\alpha(\sigma)\beta(\sigma') + \alpha(\sigma')\beta(\sigma)]\epsilon(\sigma - \sigma') + (\partial_\sigma - \partial_{\sigma'})[\xi(\sigma')\delta(\sigma - \sigma')]$$  \tag{3.29}$$

where $\alpha$ and $\beta$ are functionals of $f_\pm$ and have (naive) dimension 1 and $\xi$ (naive) dimension 0. This leads to a term $\sim \delta'''(\sigma - \sigma')$ in $\{V^-(\sigma), V^-(\sigma')\}$ unless $\xi = 0$. It is also easily seen that in order to reproduce $\{V^-(\sigma), V^-(\sigma')\}$ correctly one must identify $\alpha = \beta = \frac{\gamma}{\sqrt{2}} O^-$, hence

$$\{O^-(\sigma), O^-(\sigma')\} = \gamma^2 \epsilon(\sigma - \sigma')O^-(\sigma)O^-(\sigma').$$  \tag{3.30}$$

This looks identical to the bracket of $V^-$ with itself, but $O^- = e^2 \int^\sigma B f'_+ $ is a simpler object than $V^-$. One has

$$\{e^2 \int^\sigma B f'_+(\sigma), e^2 \int^{\sigma'} B f'_+(\sigma')\} = e^2 \int^\sigma B e^2 \int^{\sigma'} B \left[\{f'_+(\sigma), f'_+(\sigma')\} 
+ 2f'_+(\sigma)\{\int B, f'_+(\sigma')\} + 2\{f'_+(\sigma), \int B\} f'_+(\sigma') 
+ 4f'_+(\sigma)\{\int B, \int B\} f'_+(\sigma') \right]. \tag{3.31}$$
A natural Ansatz is

\[
\{ \int_{B}^{\sigma} B, \int_{B}^{\sigma'} B \} = \frac{a}{4} \gamma^2 \epsilon(\sigma - \sigma') \tag{3.32}
\]

\[
\{ \int_{B}^{\sigma} B, f_+^{\prime}(\sigma') \} = \frac{b}{4} \gamma^2 f_+^{\prime}(\sigma') \epsilon(\sigma - \sigma')
\]

\[
\{ f_+^{\prime}(\sigma), f_+^{\prime}(\sigma') \} = d \gamma^2 f_+^{\prime}(\sigma) f_+^{\prime}(\sigma') \epsilon(\sigma - \sigma').
\]

Then (3.30) is satisfied if \( a + b + d = 1 \). Taking derivatives of the Poisson brackets (3.32) one obtains the brackets of \( B \equiv f_- f_+ \) with itself and with \( f_+^{\prime} \). From these one then also deduces the Poisson brackets of \( f_- \) with itself and with \( f_+^{\prime} \). Let me anticipate that \( a \) must vanish. With \( a = 0 \) one has

\[
\{ f_-^{\prime}(\sigma), f_-^{\prime}(\sigma') \} = d \gamma^2 f_-^{\prime}(\sigma) f_-^{\prime}(\sigma') \epsilon(\sigma - \sigma')
\]

\[
\{ f_-^{\prime}(\sigma), f_+^{\prime}(\sigma') \} = -d \gamma^2 f_-^{\prime}(\sigma) f_+^{\prime}(\sigma') \epsilon(\sigma - \sigma') + \frac{b}{2} \gamma^2 \delta(\sigma - \sigma')
\]

\[
\{ f_+^{\prime}(\sigma), f_+^{\prime}(\sigma') \} = d \gamma^2 f_+^{\prime}(\sigma) f_+^{\prime}(\sigma') \epsilon(\sigma - \sigma').
\]

From these Poisson brackets it follows

\[
\{(f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma), (f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma') \} = d \gamma^2 (f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma)(f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma') \epsilon(\sigma - \sigma')
\]

\[
- \left( \frac{b}{2} + d \right) \gamma^2 \left( \partial_\sigma - \partial_{\sigma'} \right) [f_-^2(\sigma') \delta(\sigma - \sigma')]
\]

\[
\{ B(\sigma), (f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma') \} = \frac{b}{2} \gamma^2 f_-^{\prime}(\sigma') \delta(\sigma - \sigma')
\]

\[
- \frac{b}{2} \gamma^2 (f_-^{\prime} - f_-^2 f_+^{\prime})(\sigma') \delta(\sigma - \sigma').
\]

(3.34)

Exactly as eq. (3.30) was derived, one must have the same Poisson bracket with \( O^- \to O^+ \). Using the definition (3.26) of \( O^+ \) and the Poisson brackets (3.34) (recalling that \( \{ B, B \} \sim a = 0 \)) one finds

\[
\{ O^+(\sigma), O^+(\sigma') \} = (d + b) \gamma^2 \epsilon(\sigma - \sigma') O^+(\sigma) O^+(\sigma')
\]

\[
- \left( \frac{b}{2} + d \right) \gamma^2 \left( \partial_\sigma - \partial_{\sigma'} \right) \left[ -4 f_-^{\prime} B f_-^2(\sigma') \delta(\sigma - \sigma') \right].
\]

(3.35)

Had \( a \) not been zero, a term \( \sim \delta''(\sigma - \sigma')/(f_+^{\prime}(\sigma)f_+^{\prime}(\sigma')) \) would have appeared,
among others, on the right hand side. Hence one must choose \( a = 0, \quad \frac{b}{2} + d = 0, \quad d + b = 1 \), or

\[
d = -1, \quad b = 2, \quad a = 0.
\]

It is then straightforward to verify that the Poisson brackets

\[
\{ f_1(\sigma), f_1(\sigma') \} = 2\gamma^2 \delta'(\sigma - \sigma')
\]

\[
\{ f_1(\sigma), f_\pm(\sigma') \} = 0
\]

\[
\{ f_-(\sigma), f_-(\sigma') \} = -\gamma^2 f_-(\sigma)f_-(\sigma')\epsilon(\sigma - \sigma')
\]

\[
\{ f'_+(\sigma), f'_+(\sigma') \} = -\gamma^2 f'_+(\sigma)f'_+(\sigma')\epsilon(\sigma - \sigma')
\]

\[
\{ f_-(\sigma), f'_+(\sigma') \} = \gamma^2 f_-(\sigma)f'_+(\sigma')\epsilon(\sigma - \sigma') + \gamma^2 \delta(\sigma - \sigma')
\]

do imply all six Poisson brackets (3.22) of the \( T, V^+ \) and \( V^- \). Note that I have not completely proven the converse, i.e. I have not proven that (3.37) is the only possible choice of \( f_i \) Poisson brackets that implies the \( T, V^\pm \) Poisson brackets. On the other hand it is hard to see how this could fail to be true. Indeed, I deduced (3.37) from only two Poisson brackets \( \{ V^\pm, V^\pm \} \) and a simple Ansatz. The fact that all other four Poisson brackets \( \{ T, T \}, \{ T, V^\pm \} \) and \( \{ V^+, V^- \} \) then are satisfied, very strongly supports the conjecture, that (3.37) is indeed the only possible Poisson bracket of the \( f_i \). Obviously, the same Poisson brackets are valid if one replaces \( f_i \rightarrow g_i \) (star operation).

It also follows that

\[
\{ T(\sigma), f_\pm(\sigma') \} = -f_\pm(\sigma')\delta(\sigma - \sigma')
\]

\[
\{ T(\sigma), e^{f_1(\sigma')} \} = -\partial_{\sigma'} \left[ e^{f_1(\sigma')} \delta(\sigma - \sigma') \right]
\]

so that \( f_+ \) and \( f_- \) are conformal primaries with conformal dimension 0 and \( e^{f_1} \) is primary with dimension 1. This is indeed needed to ensure that the functions \( F_i \)

* In particular, it is easy to obtain \( \{ V^+, V^- \} \) using these \( f \) Poisson brackets while it was not at all easy using the canonical \( r, t, \phi \) and \( \nu \) Poisson brackets.
defined in section 2 have well-defined conformal properties: they are all conformal primaries of dimension zero, i.e. scalar functions. It then also follows that the “building blocks” for the classical solutions \( V, W, X, Y, Z \) are conformal primaries with dimensions zero with respect to \( T_{++} \) and \( T_{--} \). Moreover, from (2.43) and (2.46) one sees that \( r \) and \( t \) are also dimension zero primaries (scalars) while \( e^{2\phi} \) has conformal dimension \((1, 1)\) (i.e. is a scalar density) as it should in order that the action and equations of motion be conformally invariant.

### 3.3. Transformation to free fields (harmonic oscillators)

While the Poisson bracket of \( f_1 \) with itself is a free field bracket, i.e. its Fourier modes \((f_1)_n\) have harmonic oscillator Poisson brackets, this is not true for the brackets (3.37) of \( f_+, f_- \). The bracket of \( f_- \) with itself implies

\[
\{ \partial_\sigma \log f_-(\sigma), \partial_{\sigma'} \log f_-(\sigma') \} = 2\gamma^2 \delta'(\sigma - \sigma')
\]  
(3.39)

which is a free field Poisson bracket, just as for \( f_1 \). The same is true if one replaces \( f_- \) by \( f'_+ \). However, \( f_- \) has conformal dimension zero while \( f'_+ \) has conformal dimension one. Also the Poisson bracket of \( f_- \) with \( f'_+ \) is non-vanishing. It is then natural to set

\[
f_- = e^{\sqrt{2} \phi_1}
\]
\[
f'_+ = \frac{1}{\sqrt{2}} e^{-\sqrt{2} \phi_1}(\partial_\sigma \phi_1 + i \partial_\sigma \phi_2)
\]
\[
f_1 = \sqrt{2} \phi_3
\]  
(3.40)

A straightforward computation then shows that the Poisson brackets (3.37) are equivalent† to

\[
\{ \partial_\sigma \phi_i(\sigma), \partial_{\sigma'} \phi_j(\sigma') \} = \gamma^2 \delta_{ij} \delta'(\sigma - \sigma')
\]  
(3.41)

† Of course, \( \{ \phi'_i, \phi'_j \} \) follows from \( \{ \phi_i, \phi_j \} \) while the converse is only true up to a function of integration. However as already discussed earlier (see the discussion of \( \{ \nu, \nu \} \) in section 3.1) this ambiguity is only related to the zero-modes of \( \phi_j \) and can be fixed in a natural way, thus replacing e.g. \( \epsilon(\sigma - \sigma') \rightarrow \epsilon_1(\sigma - \sigma') \) if \( \sigma, \sigma' \in S^1 \).
or
\[
\{\varphi_i(\sigma), \varphi_j(\sigma')\} = -\frac{\gamma^2}{2} \delta_{ij} \epsilon(\sigma - \sigma').
\] (3.42)

If one considers \(\sigma \in S^1\), the mode expansion
\[\partial_\sigma \varphi_j(\tau, \sigma) = \frac{\gamma}{\sqrt{2\pi}} \sum_n \varphi_n^j e^{-in(\tau+\sigma)}\] (3.43)
leads to
\[i\{\varphi_n^j, \varphi_m^k\} = n\delta^{ij}\delta_{n+m,0}\] (3.44)
as appropriate for three sets of harmonic oscillators.

The conserved quantities \(T, V^\pm\) are easily expressed in terms of the \(\varphi_j\) as
\[
T = \frac{1}{2} \sum_{j=1}^{3} (\partial_\sigma \varphi_j)^2 - \frac{1}{\sqrt{2}} \partial_\sigma^2 \varphi_3
\]
\[
V^\pm = \frac{1}{2} \left[ (\sqrt{2}\partial_\sigma \varphi_3 - \partial_\sigma) \left[ e^{\mp i\sqrt{2}\varphi_2}(\partial_\sigma \varphi_1 \mp i\partial_\sigma \varphi_2) \right] \right].
\] (3.45)

Thus in terms of the \(\varphi_j\) the \(V^\pm\) are local expressions of the fields, analogous to standard vertex operators. (Of course, their Poisson brackets exhibit the non-local \(\epsilon(\sigma - \sigma')\)-function.)

Again, the same formulas apply if one replaces \(f_j \rightarrow g_j, \varphi_j \rightarrow \bar{\varphi}_j\) and \(T = T_{++}, V^\pm = V^\pm_{++} \rightarrow T_{--}, V^\pm_{--}\) (and \(i \rightarrow -i\)).

\[\dagger\] Note that this is a Fourier expansion in \(\sigma\). The factor \(e^{-in\tau}\) is only extracted from the \(\varphi_n^j\) for convenience. The \(\varphi_n^j\) are still functions of \(\tau\). It is only if one imposes the equations of motion that the \(\varphi_n^j\) are constant.
4. Conclusions and open problems

In this final section, after summarizing of what has been done, I will briefly discuss a couple of issues which should be addressed but for which the solution has yet to be worked out. These are: periodicity in $\sigma$ and zero-modes, periodicity in the target space, the associated linear differential equation and the hierarchy of non-linear integrable partial differential equations, the screening fields and the whole issue of quantization, in particular the structure of the quantum constraint algebra, and last, but not least, the relevance to black hole physics.

4.1. Conclusions

In this article, I have considered a $1+1$ dimensional field theory that describes a string propagating on a two-dimensional (euclidean) black hole background. This string also has an “internal” degree of freedom, which is coupled through a non-trivial tachyon potential to the radial target-space field. Alternatively one can view this as a string propagating on a three-dimensional background which is the product of a two-dimensional blackhole and a flat direction. I have investigated the corresponding classical theory. Using the formal solution of refs. 4 and 3, I have given the general explicit solution to the equations of motion. It expresses the interacting fields through a rather involved transformation in terms of functions $f_i(u)$ and $g_i(v)$ of one light-cone variable only. The theory has three left-moving and three right-moving conserved quantities. The right (left)-moving conserved quantities form a closed non-linear, non-local Poisson bracket algebra. This algebra is a Virasoro algebra extended by two conformal dimension-two primaries. The name “V-algebra” seems appropriate, stressing the similarities (non-linearity) and differences (non-locality) with the well known $W$-algebras (see e.g. ref. 5). From these Poisson brackets I obtained the Poisson brackets of the $f_i$ and $g_i$, and finally, after an ultimate transformation those of $\varphi_i$ and $\bar{\varphi}_i$. The $\varphi_i$ ($\bar{\varphi}_i$) are left (right)-moving free fields with harmonic oscillator Poisson brackets.
4.2. Periodicity in $\sigma$ and zero-modes

If one considers the theory as given by (1.3), (2.1) as a 1 + 1 dimensional field theory on the infinite real line, i.e. $\sigma \in R$, then nothing special has to be observed and in particular the $\epsilon(\sigma - \sigma')$-function is just the usual one ($= +1$ for $\sigma > \sigma'$ and $= -1$ if $\sigma' < \sigma$). If however, one thinks of the theory as a theory of closed strings, then $\sigma \in S^1$ and one must have periodicity in $\sigma$ of all physical quantities. For example, the field $\nu$ itself is not physical, and hence need not be periodic. On the other hand, the conserved quantities (constraints) $T$ and $V^\pm$ are physical and should be periodic. Hence $e^{\pm i\nu}$ and $\partial_u \nu, \partial_v \nu$ must be periodic, and under $\sigma \rightarrow \sigma + 2\pi$, one must have $\nu \rightarrow \nu + 2\pi n$, $n \in Z$. Moreover, the fields $r, t, \phi$ certainly are physical and should be periodic under $\sigma \rightarrow \sigma + 2\pi$. The issue of periodicity is probably best discussed when the fields are expressed in terms of the $\varphi_j$ and $\bar{\varphi}_j$. The non-zero modes of $\varphi_j$ and $\bar{\varphi}_j$ are automatically periodic, while the zero-mode piece $(\varphi_j)_0 = q_j + p_j \frac{\tau + \sigma}{2\pi}$ gives $\varphi_j \rightarrow \varphi_j + p_j$, and similarly $\bar{\varphi}_j \rightarrow \bar{\varphi}_j + \bar{p}_j$. One can then work out how the fields $r, t$ and $\phi$ change under $\sigma \rightarrow \sigma + 2\pi$, and what conditions have to be imposed on the $p_j, \bar{p}_j$ to make $r, t, \phi$ periodic.

4.3. Periodicity in the target space

Another related issue is periodicity in the target space. Given that the action (1.3) describes a $\sigma$-model for a string on the euclidean black hole manifold, one should have $r \geq 0$ and $t$ should have period $2\pi$, i.e. $t$ and $t + 2\pi$ should be identified. $r \geq 0$ can be easily achieved by choosing the appropriate branch of the square root of $sh^2 r$. Identifying $t$ with $t + 2\pi$ allows for more freedom in the zero-mode discussion of the previous subsection. One can (and has to) include winding modes: $t \rightarrow t + 2\pi m$ as $\sigma \rightarrow \sigma + 2\pi$ corresponds to a string winding $m$ times around the semi-infinite cigar-shaped 2D black hole manifold.

$\star$ Actually it does not seem to be a priori guaranteed that one can achieve periodicity of $r, t, \phi$. If one could not, then all the developments in this paper would only make sense as a field theory with $\sigma \in R$. However, this would be most surprising since one expects a string theory to make sense on the black hole background, at least classically.
4.4. Associated linear differential equation and hierarchy of integrable non-linear partial differential equations

From experience with integrable models, in particular the (non-affine, conformally invariant) Toda models [6] one expects that the conserved quantities appear as coefficients of an ordinary linear differential equation, e.g. for the \( A_{n-1} \) Toda model

\[
\left[ \partial_u^n - \sum_{k=2}^{n} W^{(k)}(u) \partial_u^{n-k} \right] \psi(u) = 0 . \tag{4.1}
\]

The \( W^{(k)}(u) \) are the conserved quantities which form the \( W_n \)-algebra, while the solutions \( \psi_j(u) \) of this equation, together with the solutions \( \chi_j(v) \) of a similar equation in \( v \), are the building blocks of the general solution to the Toda equations of motion. The \( W^{(k)}(u) \) have (naive) dimension \( k \).

In the present theory all conserved quantities have dimension 2. The theory being non-abelian in character, one may guess a linear differential equation of the form

\[
\left[ \partial_u^2 - \begin{pmatrix} \alpha T(u) & \beta^+ V^+(u) \\
\beta^- V^-(u) & \delta T(u) \end{pmatrix} \right] \Psi(u) = 0 . \tag{4.2}
\]

It should be stressed that, at present, this should only be regarded as a guess about the form of the fundamental differential equation. If (4.2) or some similar differential equation is true, one may explore the whole hierarchy of non-linear partial differential equations associated with it, the same way as the KdV hierarchy is associated with \((\partial^2 - T) \psi = 0\) or the KP hierarchy is associated with (4.1).

Actually, one can at least find one differential equation of type (4.2) and its solution. From experience with Toda theories one can try a simple Ansatz:

\[
\psi_1 = \exp(a \varphi_1 + ib \varphi_2 + d \varphi_3) \\
\psi_2 = \exp(a \varphi_1 - ib \varphi_2 + d \varphi_3) . \tag{4.3}
\]

Then using the form (3.45) of \( T \) and \( V^\pm \) one finds that the above differential
equation (4.2) is satisfied if and only if

\[ a = \frac{1}{\sqrt{2}} , \quad b = d = -\frac{1}{\sqrt{2}} \]
\[ \alpha = \delta = 1 , \quad \beta_+ = \beta_- = -\sqrt{2} \]  \hspace{1cm} (4.4)

4.5. Screening fields and quantization

A natural starting point for quantization are the Poisson brackets (3.42) of the free fields \( \varphi_j, \bar{\varphi}_j \) or, if \( \sigma \in S^1 \) of their Fourier modes (3.44). Let’s suppose \( \sigma \in S^1 \). Quantization then gives

\[
[\varphi_n^j, \varphi_m^k] = [\bar{\varphi}_n^j, \bar{\varphi}_m^k] = n\delta^{jk}\delta_{n+m,0} \]
\[
[\varphi_n^j, \bar{\varphi}_m^k] = 0 \] \hspace{1cm} (4.5)

together with the appropriate commutators with possible zero-modes \( q^j, \bar{q}^j \). Before bothering about zero-modes one can already write down the stress tensor:

\[ T(\sigma) = \frac{1}{2} \sum_{j=1}^{3} : (\partial_\sigma \varphi_j)^2 : - \frac{1}{\sqrt{2}} \partial_\sigma^2 \varphi_3 \] \hspace{1cm} (4.6)

(which is normalized as in section 3, and thus differs from the usual normalization by a factor of \( \gamma^2/(2\pi) \)) or the (correctly normalized) Virasoro generators (recall (3.12) and (3.43))

\[ L_n = \frac{1}{2} \sum_m \sum_{j=1}^{3} : \varphi_m^j \varphi_{n-m}^j : + \alpha_0 n \varphi_n^3 - \frac{\alpha_0^2}{2} \delta_{n,0} \]
\[
\alpha_0 = i\sqrt{\frac{\pi}{\gamma}} \] \hspace{1cm} (4.7)

which is of the standard* Feigin-Fuchs form with background charge \( \alpha_0 \). Without further computation one knows that the \( L_n \) satisfy a Virasoro algebra with

* Usually one redefines \( a_n^j = \varphi_n^j - \alpha_0 \delta_{n,0} \delta^{j,3} \) so that the Virasoro generators take the more familiar form \( L_n = \frac{1}{2} \sum_m \sum_{j=1}^{3} : a_m^j a_{n-m}^j : + \alpha_0 (n+1) a_n^3. \)
quantum central charge given by

\[ c = 3 - 12\alpha_0^2 = 3 + \frac{12\pi}{\gamma^2} \]  

(4.8)

where the contribution \(-12\alpha_0^2 = \frac{12\pi}{\gamma^2}\) was already present at the classical level (cf. (3.14)) while the 3 is the quantum contribution of the three fields.

In order to be able to construct a quantum version of the classical transformations from the \(\varphi^j, \bar{\varphi}^j\) to the fields \(f_j, g_j\) and then to \(r, t, \phi\) one needs to ensure, among other things, that the integrals defining the \(F_i, G_i\) (eqs. (2.30)-(2.32)) have conformal dimension zero, i.e. are screening operators. One possibility could be to replace

\[ e^{f_i} \equiv e^{\sqrt{2}\varphi_3} \rightarrow :e^{\sqrt{2}\alpha\varphi_3}: \quad \text{with} \quad \alpha - \frac{\gamma^2}{2\pi}\alpha^2 = 1 \]  

(4.9)

so that it has conformal dimension one, and to modify \(f_\pm\) so that they keep conformal dimension zero and commute with each other and with \( :e^{\sqrt{2}\alpha\varphi_3} :\). Whether this or a similar possibility can be successfully implemented has still to be worked out.

4.6. Structure of the quantum constraint algebra

As just discussed, upon quantization, the \(L_n\) will satisfy the usual (quantum) Virasoro algebra, obtained from the classical Poisson bracket relation (3.24) by the simple substitution \(i\{,\} \rightarrow [\,]\), only the value of the central charge \(c\) is changed with respect to its classical value by a quantum contribution. What about the other commutators corresponding to (3.24)? Clearly, one expects \(V^\pm\) to remain a conformal dimension 2 primary operator so that

\[ [L_n, V^\pm_m] = (n - m)V^\pm_{n+m}. \]  

(4.10)

The commutators of \(V^\pm\) with \(V^\pm\) and \(V^\mp\) are expected to contain a similar bilinear term as in (3.24). However, one has to introduce some normal-ordering, e.g. with
respect to the mode index of the $V_k^\pm$. This has to be consistent with the Jacobi identities. Note that for the linear part of the algebra, the Jacobi identities are always satisfied if they are for the corresponding Poisson brackets. For the non-linear terms however, the proof of the Jacobi identities for the Poisson brackets (3.24) used the fact that $V_{n+k}$ and $V_{m-k}$ on the right hand side commute. This is no longer true at the quantum level where the ordering is important. For this reason, it is not clear to me at present what should be the correct quantum commutator replacing $i\{V^\pm, V^\pm\}$ and $i\{V^\pm, V^\mp\}$.

4.7. Relevance to black hole physics

One can consider all the developments in this paper merely from the point of view of integrable models and conformal field theory. In particular, I believe that the new $V$-algebra (3.24) and the possible hierarchy associated with the linear differential equation (4.2) may lead to exciting developments.

On the other hand, one may ask whether anything has been gained for our understanding of strings propagating on the black hole background. At this point it is too early to give a definite answer. Clearly, first one has to solve the above-mentioned issue of periodicity and winding-modes. This does not seem too difficult. Then one should actually quantize the theory. This may be quite non-trivial, but the rigid structure of integrable models probably will prove helpful. In particular, the spectrum is expected to fall into representations of a possible quantum version of the $V$-algebra (3.24). This in turn gives information about the zero-modes, making it eventually possible to compute the one-loop partition function (and maybe the string-theoretic one-loop entropy). Quantization of the free fields $\varphi_j$ is straightforward (and exact to all orders in $\hbar \sim \gamma^2 \sim \alpha'$ where $\alpha'$ is the usual inverse string tension). When formulating the quantum version of the transformation to the original fields $r, t$ and $f$, one might be forced into a semiclassical expansion in $\hbar \sim \alpha'$ which can be compared to the well-known $\beta$-function equations of ref. 1. If however, one is able to formulate this transformation exactly one would have
succeeded in providing exact, non-perturbative\textsuperscript{*} information about a string on the black hole background. It would be most interesting to compare these results to those of ref. 7 for the \( Sl(2, \mathbb{R})/U(1) \) black hole.

Acknowledgements:

It is a pleasure to acknowledge discussions with Curtis Callan, Jean-Loup Gervais, David Gross and Ed Witten.

APPENDIX A

The Lie algebra \( B_2 \) has two Cartan generators, \( h_1 \) and \( h_2 \) and eight step operators \( E_{e_1}, E_{e_2}, E_{e_1\pm e_2}, E_{e_1\mp e_2} \). Section 2 makes use of the following realization [3] of the generators in terms of five fermionic oscillators \( b_0, b_{\pm 1}, b_{\pm 2} \) with 
\[
[b_j, b_k^+] = \delta_{jk}.
\]

\[
h_1 = b_1^+ b_1 - b_{-1}^- b_{-1} - b_2^+ b_2 + b_{-2}^- b_{-2}
\]

\[
h_2 = 2(b_2^+ b_2 - b_{-2}^- b_{-2})
\]

\[
E_{e_1} = \sqrt{2}(b_1^+ b_0 - b_{0}^- b_{-1})
\]

\[
E_{e_2} = \sqrt{2}(b_2^+ b_0 - b_{0}^- b_{-2})
\]

\[
E_{e_1 + e_2} = b_1^+ b_{-2} - b_2^+ b_{-1}
\]

\[
E_{e_1 - e_2} = b_1^+ b_2 - b_{-2}^- b_{-1}
\]

\[
\text{together with}
\]

\[
E_{-\alpha} = E_{\alpha}^+.
\]

The grading element \( H \) is given by

\[
H = 2h_1 + h_2 = 2(b_1^+ b_1 - b_{-1}^- b_{-1})
\]

and simply counts (twice) the oscillators of type 1 minus those of type \(-1\).

\textsuperscript{*} Here the term “non-perturbative” refers to the \( \alpha' \) expansion. The string world sheet still has cylindrical (or spherical) topology.
APPENDIX B

In this appendix, I collect some useful relations between the functions $V, W, X, Y, Z$ and their derivatives used in section 2.

From
\[
W = -g_+ + F_1 G_- - F_2 G_+ - f_- X \\
Y = 1 + g_+ g_- + F_1 (G_1 - g_- G_-) + F_2 (G_2 + g_- G_+) - f_- V
\] (B.1)

one gets after some algebra
\[
W \partial_v V - X \partial_v Y = -\frac{1}{2} \partial_v Z - g_- g'_+ Z
\] (B.2)

and using the star operation also
\[
X \partial_u Y - V \partial_u W = \frac{1}{2} \partial_u Z + f_+ f'_+ Z .
\] (B.3)

Another relation needed to prove the $a_1$ (or $\phi$) equation of motion is
\[
Z \partial_u \partial_v Z - \partial_u Z \partial_v Z - 2 e^{f_1 + g_1} Z = -4 e^{f_1 + g_1} X Y
\] (B.4)

which is verified by direct computation, recalling that $\partial_u$ only acts on $F$’s and $f$’s while $\partial_v$ only acts on $G$’s and $g$’s, and that $\partial_u F_1 = -e^{f_1}(1 + 2 f_+ f_-)$ etc. as seen from the definitions (2.30).

When computing $T_{--}$ the following relation are needed
\[
X \partial_v W - W \partial_v X = -g'_+ Z + \alpha
\] (B.5)
\[
Y \partial_v V - V \partial_v Y = -(g'_- - g_2 g'_+) Z + \beta
\] (B.6)
\[
\alpha \beta - \frac{1}{4} (\partial_v Z)^2 = -(F_1^2 + F_2 F_3) e^{2g_1} Z
\] (B.7)
\[-\frac{1}{2}g'_1 \partial_v Z + \frac{1}{2} \partial^2_v Z - g'_+ \beta - (g'_- - g^2 g'_+) \alpha = \frac{(\partial_v Z)^2}{4Z} - \frac{\alpha \beta}{Z} \quad (B.8)\]

where

\[\alpha = e^{g_1} \left[ 2g_+ F_1 - F_2 + g^2_+ F_3 - (F^2_1 + F_2 F_3)(G_- - g_+ G_+) \right] \]
\[\beta = e^{g_1} \left\{ 2g_-(1 + g_+) F_1 - g^2_- F_2 + (1 + g_+) g_- F_3 \right. \]
\[\left. + (F^2_1 + F_2 F_3) [(1 + g_+) G_2 + g_- G_+ + g_- (G_1 - g_- G_-)] \right\}. \quad (B.9)\]

Relations (B.5) and (B.6) are derived much as (B.2) using (B.1). Equation (B.7) is verified by straightforward algebra, as is also (B.8) (using (B.7)). When evaluating $V_{++}^+$ one needs

\[(\partial_u - f'_1 - 2f_- f'_+) \beta^* + (f'_- - f^2_- f'_+) \partial_u Z = 0 \quad (B.10)\]

which is also not too difficult to be checked.

REFERENCES

1. C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B262 (1985) 593.
2. E. Witten, Phys. Rev. D44 (1991) 314.
3. J.-L. Gervais and M.V. Saveliev, Phys. Lett. B286 (1992) 271.
4. A.N. Leznov and M.V. Saveliev, Comm. Math. Phys. 89 (1983) 59.
5. A. Bilal, Introduction to W-algebras, in: Proc. Trieste Spring School on String Theory and Quantum Gravity, 1991, J. Harvey et al eds., World Scientific.
6. A. Bilal and J.-L. Gervais, Phys. Lett. B206 (1988) 412; Nucl. Phys. B314 (1989) 646, B318 (1989) 579.
7. R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B371 (1992) 269.