On superactivation of one-shot zero-error quantum capacity and the related property of quantum measurements

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Abstract

We begin with a detailed description of a low dimensional quantum channel \((d_A = 4, d_E = 3)\) demonstrating the symmetric form of superactivation of one-shot zero-error quantum capacity. This means appearance of a noiseless (perfectly reversible) subchannel in the tensor square of a channel having no noiseless subchannels.

Then we describe a quantum channel \(\Phi\) such that \(\bar{Q}_0(\Phi) = 0\) and \(\bar{Q}_0(\Phi \otimes \Phi) \geq \log n\) for any \(n \leq +\infty\).

We also show that the superactivation of one-shot zero-error quantum capacity of a channel can be reformulated in terms of quantum measurements theory as appearance of an indistinguishable subspace for tensor product of two observables having no undistinguishable subspaces.

1 Introduction

The phenomenon of superactivation of quantum channel capacities has been intensively studied since 2008 when G. Smith and J. Yard established this property for the case of quantum capacity [15].

This phenomenon means that the particular capacity of the tensor product of two quantum channels may be positive despite the same capacity of each of these channels is zero. During the last five years it was shown that

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superactivation holds for different quantum channel capacities, in particular, for (one-shot and asymptotic) zero-error classical and quantum capacities \( [1, 8] \).

In this paper we focus attention on the superactivation of one-shot zero-error quantum capacity which means that

\[ \bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) > 0 \quad (1) \]

for some channels \( \Phi_1 \) and \( \Phi_2 \), where \( \bar{Q}_0 \) denotes the one-shot zero-error quantum capacity (described in Section 2).

This effect can be reformulated with no use the term ”capacity” as appearance of a noiseless (i.e. perfectly reversible) subchannel in the tensor product of two channels each of which has no noiseless subchannels. This reformulation seems more adequate for specialists in functional analysis and operator algebras theory.

The existence of quantum channels, for which (1) holds, follows from the existence of quantum channels demonstrating so called extreme superactivation of asymptotic zero-error capacities shown in \([5]\) by rather in explicit way in sufficiently high dimensions. So, this result neither gives an explicit form of channels demonstrating the superactivation of one-shot zero-error quantum capacity, nor says anything about their minimal dimensions.

In our recent paper \([14]\) we explicitly describe low dimensional channels \( \Phi_1 \neq \Phi_2 \) (\( \dim \mathcal{H}_A = 8, \dim \mathcal{H}_E = 5 \)) demonstrating the extreme superactivation of one-shot zero-error capacity which means (1) with the condition \( \bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0 \) replaced by the stronger condition \( \bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0 \) (where \( \bar{C}_0 \) is the one-shot zero-error classical capacity). For these channels superactivation (1) obviously holds.

In this paper we use the same approach to construct more simple example of superactivation (1). It turns out that the change

\[ \bar{C}_0(\Phi_1) = \bar{C}_0(\Phi_2) = 0 \quad \rightarrow \quad \bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0 \]

of prerequisites makes it possible to essentially decrease dimensions (\( \dim \mathcal{H}_A = 4, \dim \mathcal{H}_E = 3 \)) and to construct a symmetrical example \( \Phi_1 = \Phi_2 \), i.e. a such channel \( \Phi \) that

\[ \bar{Q}_0(\Phi) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi \otimes \Phi) > 0. \]

Moreover, this channel \( \Phi \) is defined via so simple noncommutative graph, which gives possibility to write a minimal Kraus representation of \( \Phi \) in explicit (numerical) form.
Then we describe a quantum channel \( \Phi \) such that
\[
\bar{Q}_0(\Phi) = 0, \quad \text{but} \quad \bar{Q}_0(\Phi \otimes \Phi) \geq \log n,
\]
where \( n \) is any natural number or \( +\infty \) (in the last case \( \Phi \) is an infinite-dimensional channel: \( \dim \mathcal{H}_A = \dim \mathcal{H}_B = +\infty \)).

In the last part of the paper (Section 3) we show that the superactivation of one-shot zero-error quantum capacity (1) has a counterpart in the theory of quantum measurements. Namely, it can be reformulated as appearance of an indistinguishable subspace for the tensor product of two quantum observables having no indistinguishable subspaces. This observation is quite simple but seems interesting for specialists in quantum measurements theory.

A general way to write the Kraus representation of a channel with given noncommutative graph is considered in the Appendix.

### 2 Superactivation of one-shot zero-error quantum capacity

Let \( \mathcal{H} \) be a separable Hilbert space, \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{S}(\mathcal{H}) \) – the Banach spaces of all bounded operators in \( \mathcal{H} \) and of all trace-class operators in \( \mathcal{H} \) correspondingly, \( \mathcal{S}(\mathcal{H}) \) – the closed convex subset of \( \mathcal{S}(\mathcal{H}) \) consisting of positive operators with unit trace called states \([10, 12]\). If \( \dim \mathcal{H} = n < +\infty \) we may identify \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{S}(\mathcal{H}) \) with the space \( \mathcal{M}_n \) of all \( n \times n \) matrices (equipped with the appropriate norm).

Let \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) be a quantum channel, i.e. a completely positive trace-preserving linear map \([10, 12]\). Stinespring’s theorem implies the existence of a Hilbert space \( \mathcal{H}_E \) and of an isometry \( V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E \) such that
\[
\Phi(\rho) = \text{Tr}_{\mathcal{H}_E} V \rho V^*, \quad \rho \in \mathcal{S}(\mathcal{H}_A).
\]

(2)
The quantum channel
\[
\mathcal{S}(\mathcal{H}_A) \ni \rho \mapsto \tilde{\Phi}(\rho) = \text{Tr}_{\mathcal{H}_B} V \rho V^* \in \mathcal{S}(\mathcal{H}_E)
\]

is called complementary to the channel \( \Phi \) \([10, 11]\). The complementary channel is defined uniquely up to isometrical equivalence \([11, \text{the Appendix}]\).

The one-shot zero-error quantum capacity \( \bar{Q}_0(\Phi) \) of a channel \( \Phi \) can be defined as \( \sup_{\mathcal{H} \in q_0(\Phi)} \log \dim \mathcal{H} \), where \( q_0(\Phi) \) is the set of all subspaces \( \mathcal{H}_0 \)}
of $\mathcal{H}_A$ on which the channel $\Phi$ is perfectly reversible (in the sense that there is a channel $\Theta$ such that $\Theta(\Phi(\rho)) = \rho$ for all states $\rho$ supported by $\mathcal{H}_0$). The (asymptotic) zero-error quantum capacity is defined by regularization: $Q_0(\Phi) = \sup_n n^{-1} Q_0(\Phi^{\otimes n})$ \cite{2, 4, 5, 8, 9}.

It is well known that a channel $\Phi$ is perfectly reversible on a subspace $\mathcal{H}_0$ if and only if the restriction of the complementary channel $\hat{\Phi}$ to the subset $\mathcal{G}(\mathcal{H}_0)$ is completely depolarizing, i.e. $\hat{\Phi}(\rho_1) = \hat{\Phi}(\rho_2)$ for all states $\rho_1$ and $\rho_2$ supported by $\mathcal{H}_0$ \cite[Ch.10]{10}. It follows that the one-shot zero-error quantum capacity $\bar{Q}_0(\Phi)$ of a channel $\Phi$ is completely determined by the set $\mathcal{G}(\Phi) = \hat{\Phi}^*(\mathcal{B}(\mathcal{H}_E))$ called the noncommutative graph of $\Phi$ \cite{9}.

**Lemma 1.** A channel $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is perfectly reversible on the subspace $\mathcal{H}_0 \subseteq \mathcal{H}_A$ spanned by the family $\{\varphi_i\}_{i=1}^n$, $n \leq +\infty$, of orthogonal unit vectors (which means that $\bar{Q}_0(\Phi) \geq \log n$) if and only if

$$\langle \varphi_i | A \varphi_j \rangle = 0 \quad \text{and} \quad \langle \varphi_i | A \varphi_i \rangle = \langle \varphi_j | A \varphi_j \rangle \quad \forall i, j \forall A \in \mathcal{L},$$

(4)

where $\mathcal{L} = \mathcal{G}(\Phi)$ or, equivalently, $\mathcal{L}$ is any subset of $\mathcal{B}(\mathcal{H}_A)$ such that

the weak operator closure of $\text{lin} \mathcal{L} = \text{the weak operator closure of } \mathcal{G}(\Phi)$.

**Proof.** Relations (4) mean that the complementary channel $\hat{\Phi}$ has completely depolarizing restriction to the subset $\mathcal{G}(\mathcal{H}_0)$. \hfill $\square$

**Remark 1.** Since a subspace $\mathcal{L}$ of the algebra $\mathcal{M}_n$ of $n \times n$-matrices is a noncommutative graph of a particular channel if and only if

$$\mathcal{L} \text{ is symmetric } (\mathcal{L} = \mathcal{L}^*) \text{ and contains the unit matrix }$$

(5)

(see Lemma 2 in \cite{8} or Proposition 2 in \cite{14}), Lemma 1 shows that one can "construct" a channel $\Phi$ with $\dim \mathcal{H}_A = n$ having positive (correspondingly, zero) one-shot zero-error quantum capacity by taking a subspace $\mathcal{L} \subset \mathcal{M}_n$ satisfying (5) for which the following condition is valid (correspondingly, not valid)

$$\exists \varphi, \psi \in [\mathbb{C}^n]_1 \text{ s.t. } \langle \psi | A \varphi \rangle = 0 \text{ and } \langle \varphi | A \varphi \rangle = \langle \psi | A \psi \rangle \forall A \in \mathcal{L},$$

(6)

where $[\mathbb{C}^n]_1$ is the unit sphere of $\mathbb{C}^n$.

If $m$ is a natural number such that $\dim \mathcal{L} \leq m^2$, then Corollary 1 in \cite{14} and Proposition 3 in the Appendix give explicit expressions of a channel $\Phi$ such that $\mathcal{G}(\Phi) = \mathcal{L}$ and $\dim \mathcal{H}_E \leq m$. \hfill $\square$
Superactivation of one-shot zero-error quantum capacity means that

\[ Q_0(\Phi_1) = Q_0(\Phi_2) = 0, \quad \text{but} \quad Q_0(\Phi_1 \otimes \Phi_2) > 0. \quad (7) \]

for some channels \( \Phi_1 \) and \( \Phi_2 \). As mentioned in the Introduction the existence of channels \( \Phi_1 \) and \( \Phi_2 \) for which (7) holds follows from the results in [5], but explicit examples of such channels with minimal dimensions are not known (as far as we know).

Below we will construct a channel \( \Phi \) with \( \dim H_A = 4, \dim H_E = 3, \dim H_B = 12 \) such that (7) holds with \( \Phi_1 = \Phi_2 = \Phi \).

By Remark 1 the problem of finding channels, for which (7) holds, is reduced to the problem of finding subspaces \( L_1 \) and \( L_2 \) satisfying (5) such that condition (6) is not valid for \( L = L_1 \) and for \( L = L_2 \) but it is valid for \( L = L_1 \otimes L_2 \). Now we will consider a symmetrical example (\( L_1 = L_2 \)) of such subspaces in \( \mathcal{M}_4 \).

Let \( U \) be the unitary operator in \( \mathbb{C}^2 \) determined (in the canonical basis) by the matrix

\[ U = \begin{bmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{bmatrix}, \]

where \( \eta = \exp[i \frac{\pi}{4}] \). Consider the subspace

\[ L_0 = \left\{ M = \begin{bmatrix} A & \lambda U^* \\ \lambda U & A \end{bmatrix}, \ A \in \mathcal{M}_2, \ \lambda \in \mathbb{C} \right\} \]

of \( \mathcal{M}_4 \). It obviously satisfies condition (5).

**Theorem 1.** Condition (7) is not valid for \( L = L_0 \) but it is valid for \( L = L_0 \otimes L_0 \) with the vectors

\[ |\varphi_t\rangle = \frac{1}{\sqrt{2}} \left[ |1\rangle \otimes |1\rangle + e^{i t} |2\rangle \otimes |2\rangle \right], \quad |\psi_t\rangle = \frac{1}{\sqrt{2}} \left[ |3\rangle \otimes |3\rangle + e^{i t} |4\rangle \otimes |4\rangle \right], \quad (8) \]

where \( \{|k\rangle\}_{k=1}^4 \) is the canonical basis in \( \mathbb{C}^4 \) and \( t \) is a fixed number in \([0, 2\pi)\).

**Proof.** Throughout the proof we will identify \( \mathbb{C}^4 \) with \( \mathbb{C}^2 \oplus \mathbb{C}^2 \).

Assume there exist unit vectors \( \varphi = [x_1, x_2] \) and \( \psi = [y_1, y_2] \), \( x_i, y_i \in \mathbb{C}^2 \) such that \( \langle \psi | M \varphi \rangle = 0 \) and \( \langle \psi | M \psi \rangle = \langle \varphi | M \varphi \rangle \) for all \( M \in \mathcal{L}_0 \). It follows that

\[ \langle y_1 | A x_1 \rangle + \langle y_2 | A x_2 \rangle = 0 \quad \forall A \in \mathcal{M}_2, \quad (9) \]

\[ \langle y_1 | U^* x_2 \rangle + \langle y_2 | U x_1 \rangle = 0, \quad (10) \]
\[ \langle y_1 | A y_1 \rangle + \langle y_2 | A y_2 \rangle = \langle x_1 | A x_1 \rangle + \langle x_2 | A x_2 \rangle \quad \forall A \in \mathcal{M}_2 \quad (11) \]

and

\[ \langle y_1 | U^* y_2 \rangle + \langle y_2 | U y_1 \rangle = \langle x_1 | U^* x_2 \rangle + \langle x_2 | U x_1 \rangle. \quad (12) \]

If \( x_1 \parallel x_2 \) then, by 2-transitivity of \( \mathcal{M}_2 \), there is \( A_0 \in \mathcal{M}_2 \) such that \( y_1 = A_0 x_1 \) and \( y_2 = A_0 x_2 \). So, (9) implies \( \langle y_1 | y_1 \rangle + \langle y_2 | y_2 \rangle = 0 \), i.e. \( y_1 = y_2 = 0 \). Similarly, if \( x_1 \parallel y_2 \) then (9) implies \( x_1 = x_2 = 0 \).

Thus, we necessarily have \( x_1 \parallel x_2 \) and \( y_1 \parallel y_2 \). Now we will obtain a contradiction to (9)-(12) by considering the following cases.

Case 1: \( x_2 = 0, x_1 \neq 0 \). In this case (9) implies \( \langle y_1 | A x_1 \rangle = 0 \) for all \( A \in \mathcal{M}_2 \), which can be valid only if \( y_1 = 0 \). Then (11) implies \( \langle x_1 | A x_1 \rangle = \langle y_2 | A y_2 \rangle \) for all \( A \in \mathcal{M}_2 \), which can be valid only if \( x_1 \parallel y_2 \). By Lemma 2 below this and (10) show that \( y_2 = 0 \). So, we obtain \( y_1 = y_2 = 0 \).

Case 2: \( y_2 = 0, y_1 \neq 0 \). Similar to Case 1 we obtain \( x_1 = x_2 = 0 \).

Case 3: \( x_2 \neq 0, y_2 \neq 0 \). In this case \( x_1 = \mu x_2, y_1 = \nu y_2 \) and (11) implies

\[ (1 + |\mu|^2) \langle x_2 | A x_2 \rangle = (1 + |\nu|^2) \langle y_2 | A y_2 \rangle \quad \forall A \in \mathcal{M}_2, \]

which can be valid only if \( x_2 \parallel y_2 \). Hence we have \( x_1 = \alpha y_2 \) and \( x_2 = \beta y_2 \) (in addition to \( y_1 = \nu y_2 \)). We may assume that \( x_1 \neq 0 \) and \( y_1 \neq 0 \), since otherwise (9) implies \( \langle y_2 | A x_2 \rangle = 0 \) for all \( A \in \mathcal{M}_2 \), which can be valid only if either \( x_2 = 0 \) or \( y_2 = 0 \).

It follows from (9) that \( (\bar{\nu} \alpha + \beta) \langle y_2 | y_2 \rangle = 0 \) and hence

\[ \beta = -\bar{\nu} \alpha. \quad (13) \]

By the below Lemma 2 \( z_0 = \langle y_2 | U y_2 \rangle \) is a nonzero complex number. So, (12) and (13) imply \( \text{Re}(\nu z_0) = \text{Re}(\alpha \bar{\beta} z_0) = -|\alpha|^2 \text{Re}(\nu z_0) \) and hence

\[ \text{Re}(\nu z_0) = 0. \quad (14) \]

It follows from (10) and (13) that

\[ \bar{\nu} \beta z_0 + \alpha z_0 = \alpha (\bar{\nu}^2 z_0 + z_0) = 0. \]

Since \( \alpha \neq 0 (x_1 \neq 0) \) we have \( \nu^2 z_0 = \bar{z}_0 \). This equality implies that \( \nu z_0 \) is a real number. So, (14) shows that \( \nu = 0 \) contradicting to \( y_1 \neq 0 \).

Thus, condition (6) is not valid for \( \mathcal{L} = \mathcal{L}_0 \).

Now we will show that

\[ \langle \psi_l | M_1 \otimes M_2 \varphi_l \rangle = 0 \quad \forall M_1, M_2 \in \mathcal{L}_0, \quad (15) \]
and
\[ \langle \psi_t | M_1 \otimes M_2 \psi_t \rangle = \langle \varphi_t | M_1 \otimes M_2 \varphi_t \rangle \quad \forall M_1, M_2 \in \mathcal{L}_0, \] (16)
where \( \varphi_t \) and \( \psi_t \) are vectors defined in (8). Since we identify \( \mathbb{C}^4 \) with \( \mathbb{C}^2 \oplus \mathbb{C}^2 \), these vectors are represented as follows
\[
| \varphi_t \rangle = \frac{1}{\sqrt{2}} \left[ | e_1, 0 \rangle \otimes | e_1, 0 \rangle + e^{it} | e_2, 0 \rangle \otimes | e_2, 0 \rangle \right]
\]
\[
| \psi_t \rangle = \frac{1}{\sqrt{2}} \left[ | 0, e_1 \rangle \otimes | 0, e_1 \rangle + e^{it} | 0, e_2 \rangle \otimes | 0, e_2 \rangle \right],
\]
where \( \{ | e_i \rangle \} \) is the canonical basis in \( \mathbb{C}^2 \).

By setting \( \alpha_1 = 1 \) and \( \alpha_2 = e^{it} \) we have
\[
M_1 \otimes M_2 | \varphi_t \rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \alpha_i | A_1 e_i, \lambda_1 U e_i \rangle \otimes | A_2 e_i, \lambda_2 U e_i \rangle,
\] (17)
and hence
\[
\langle \psi_t | M_1 \otimes M_2 | \psi_t \rangle = \frac{1}{\sqrt{2}} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | U e_j \rangle \langle e_i | e_j \rangle = \frac{1}{2} \lambda_1 \lambda_2 \left[ \eta^2 | \alpha_1 |^2 + \bar{\eta}^2 | \alpha_2 |^2 \right] = 0,
\]
Thus (15) is valid. It follows from (17) that
\[
\langle \varphi_t | M_1 \otimes M_2 | \varphi_t \rangle
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | 0 \rangle \otimes \langle e_i | 0 \rangle \cdot | A_1 e_j, \lambda_1 U e_j \rangle \otimes | A_2 e_j, \lambda_2 U e_j \rangle
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | A_1 e_j \rangle \langle e_i | e_j \rangle.
\] (18)

Since
\[
M_1 \otimes M_2 | \psi_t \rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \alpha_i | \lambda_1 U^* e_i, A_1 e_i \rangle \otimes | \lambda_2 U^* e_i, A_2 e_i \rangle
\]
we have
\[
\langle \psi_t | M_1 \otimes M_2 | \psi_t \rangle = \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle 0, e_i | \otimes \langle 0, e_i | \cdot | \lambda_1 U^* e_j, A_1 e_j \rangle \otimes | \lambda_2 U^* e_j, A_2 e_j \rangle
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | A_1 e_j \rangle \langle e_i | e_j \rangle.
\]
This equality and (18) imply (16). □

**Lemma 2.** If $y$ is a nonzero vector in $\mathbb{C}^2$ then $\langle y | U y \rangle \neq 0$.

**Proof.** Let $y = [y_1, y_2]$ then $U y = [\eta y_1, \bar{\eta} y_2]$ and $\langle y | U y \rangle = |y_1|^2 \eta + |y_2|^2 \bar{\eta} \neq 0$ (since $\eta = \exp[\frac{i \pi}{4}]$). □

By Proposition 2 in [14] Theorem 1 implies the following assertion.

**Corollary 1.** There is a pseudo-diagonal channel $\Phi$ with $\dim \mathcal{H}_A = 4$, $\dim \mathcal{H}_E = 3$, $\dim \mathcal{H}_B = 12$ such that $G(\Phi) = L_0$ and hence $\bar{Q}_0(\Phi) = 0$, but $\bar{Q}_0(\Phi \otimes \Phi) > 0$.

The channel $\Phi \otimes \Phi$ is perfectly reversible on the subspace $\mathcal{H}_t = \text{lin}\{ |\varphi_t \rangle, |\psi_t \rangle \}$, where $\varphi_t, \psi_t$ are vectors defined in (8), for each given $t \in [0, 2\pi)$.

**Remark 2.** It is easy to see that the above subspace $L_0$ is not transitive. So, by Lemma 2 in [14], the corresponding channel $\Phi$ has positive one-shot zero-error classical capacity and hence this channel does not demonstrate the extreme superactivation of one-shot zero-error capacity.

To obtain a minimal Kraus representation of one of the channels having properties stated in Corollary 1 we have to find a basis $\{A_i\}_{i=1}^5$ of $L_0$ such that $A_i \geq 0$ for all $i$ and $\sum_{i=1}^5 A_i = I_4$. Such basis can be easily found, for example:

\[
A_1 = \frac{1}{6} \begin{bmatrix}
1 & 0 & \bar{\eta} & 0 \\
0 & 2 & 0 & \eta \\
\eta & 0 & 1 & 0 \\
0 & \bar{\eta} & 0 & 2
\end{bmatrix},
A_2 = \frac{1}{6} \begin{bmatrix}
1 & 0 & -\bar{\eta} & 0 \\
0 & 2 & 0 & -\eta \\
-\eta & 0 & 1 & 0 \\
0 & -\bar{\eta} & 0 & 2
\end{bmatrix},
A_3 = \frac{5}{9} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
A_4 = \frac{1}{18} \begin{bmatrix}
1 & \sqrt{3} & 0 & 0 \\
\sqrt{3} & 3 & 0 & 0 \\
0 & 0 & 1 & \sqrt{3} \\
0 & 0 & \sqrt{3} & 3
\end{bmatrix},
A_5 = \frac{1}{18} \begin{bmatrix}
1 & -\sqrt{3} & 0 & 0 \\
-\sqrt{3} & 3 & 0 & 0 \\
0 & 0 & 1 & -\sqrt{3} \\
0 & 0 & -\sqrt{3} & 3
\end{bmatrix}.
\]

1 A channel $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ is called pseudo-diagonal if it has the representation

\[
\Phi(\rho) = \sum_{i,j} c_{ij} \langle \psi_i | \rho | \psi_j \rangle |i\rangle \langle j|, \quad \rho \in \mathcal{S}(\mathcal{H}_A),
\]

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i\rangle\}$ is a collection of vectors in $\mathcal{H}_A$ such that $\sum_i |\psi_i\rangle \langle \psi_i| = I_{\mathcal{H}_A}$ and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_B$. [3]
We also have to choose a collection $\{ |\psi_i\rangle \}_{i=1}^5$ of unit vectors in $\mathbb{C}^3$ such that $\{ |\psi_i\rangle \langle \psi_i| \}_{i=1}^5$ is a linearly independent subset of $\mathcal{M}_3$. Let

$$|\psi_1\rangle = |1\rangle, \ |\psi_2\rangle = |2\rangle, \ |\psi_3\rangle = |3\rangle, \ |\psi_4\rangle = \frac{1}{\sqrt{2}}|1 + 3\rangle, \ |\psi_5\rangle = \frac{1}{\sqrt{2}}|2 + 3\rangle,$$

where $\{ |1\rangle, |2\rangle, |3\rangle \}$ is the canonical basis in $\mathbb{C}^3$.

Now, by noting that $r_i = \text{rank} A_i = 3$ for $i = 1, 2$ and $r_i = \text{rank} A_i = 2$ for $i = 3, 4, 5$, we can apply Proposition 3 in the Appendix to obtain a minimal Kraus representation for pseudo-diagonal channel $\Phi$ having properties stated in Corollary 1. Direct calculation gives the following Kraus operators

$$V_1 = \frac{1}{6} \begin{bmatrix} \sqrt{6} & 0 & \sqrt{6}\eta & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & \bar{\beta} & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V_2 = \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{6} & 0 & -\sqrt{6}\eta & 0 \\ 0 & \alpha & 0 & -\beta \\ 0 & -\bar{\beta} & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \end{bmatrix},$$

$$V_3 = \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{5} & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \end{bmatrix},$$

where $\alpha = \frac{3 + \sqrt{3}}{\sqrt{2}}$ and $\beta = \eta \frac{3 - \sqrt{3}}{\sqrt{2}}$ ($\eta = e^{i\pi/4}$). Thus, $\Phi(\rho) = \sum_{k=1}^{3} V_k \rho V_k^*$. 

9
3 Superactivation with $\bar{Q}_0(\Phi \otimes \Phi) \geq \log n$

By generalizing the above construction one can obtain the following result.

**Theorem 2.** Let $\dim \mathcal{H}_A = 2n \leq +\infty$, \{\ket{k}\}$_{k=1}^{2n}$ an orthonormal basis in $\mathcal{H}_A$, and $m$ the minimal natural number such that $n^2 - n + 4 \leq m^2$ if $n < +\infty$ and $m = +\infty$ otherwise.

There exists a pseudo-diagonal channel $\Phi : \mathcal{G}(\mathcal{H}_A) \rightarrow \mathcal{G}(\mathcal{H}_B)$ with $\dim \mathcal{H}_E = m$ such that $\bar{Q}_0(\Phi) = 0$ while the channel $\Phi \otimes \Phi$ is perfectly reversible on the subspace of $\mathcal{H}_A \otimes \mathcal{H}_A$ spanned by the vectors

$$|\varphi_k^t\rangle = \frac{1}{\sqrt{2}} \left[ |2k-1\rangle \otimes |2k-1\rangle + e^{it}|2k\rangle \otimes |2k\rangle \right], \quad k = 1, 2, \ldots, n,$$

where $t$ is a fixed number in $[0, 2\pi)$, and hence $\bar{Q}_0(\Phi \otimes \Phi) \geq \log n$.

**Proof.** Assume first that $n < +\infty$. Consider the subspace $\Lambda_n = \left\{ M = \begin{bmatrix} A & \lambda_1 U^* & \cdots & \lambda_{1n} U^* \\ \lambda_2 U & A & \cdots & \lambda_{2n} U \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} U & \lambda_{n2} U & \cdots & A \end{bmatrix}, A \in \mathcal{M}_2, \lambda_{ij} \in \mathbb{C} \right\}$ of $\mathcal{M}_{2n}$, where $U$ is the unitary operator in $\mathbb{C}^2$ defined in the previous section (it has the matrix $\text{diag}\{\eta, \bar{\eta}\}$ in the canonical basis in $\mathbb{C}^2$, $\eta = \exp[i\pi/4]$).

The subspace $\Lambda_n$ satisfies condition (5) and $\dim \Lambda_n = n^2 - n + 4$. So, by Proposition 2 in [14], there is a pseudo-diagonal channel $\Phi$ with $\dim \mathcal{H}_A = 2n$ and $\dim \mathcal{H}_E = m$, where $m$ is the minimal number satisfying the inequality $n^2 - n + 4 \leq m^2$, such that $\mathcal{G}(\Phi) = \Lambda_n$.

We will prove that $\bar{Q}_0(\Phi) = 0$ by showing that condition (6) is not valid for $\mathcal{G} = \Lambda_n$.

Assume there exist unit vectors $\varphi = [x_1, x_2, \ldots, x_n]$ and $\psi = [y_1, y_2, \ldots, y_n]$, $x_i, y_i \in \mathbb{C}^2$, such that $\langle \psi | M \varphi \rangle = 0$ and $\langle \psi | M \psi \rangle = \langle \varphi | M \varphi \rangle$ for all $M \in \Lambda_n$. It follows that

$$\sum_{i=1}^{n} \langle y_i | Ax_i \rangle = 0 \quad \forall A \in \mathcal{M}_2, \quad (21)$$

$$\langle y_i | U^* x_k \rangle = 0, \quad \forall k > 1, i < k, \quad (22)$$

$$\langle y_i | U x_k \rangle = 0, \quad \forall k < n, i > k. \quad (23)$$
and

\[ \sum_{i=1}^{n} \langle y_i | A y_i \rangle = \sum_{i=1}^{n} \langle x_i | A x_i \rangle \quad \forall A \in \mathcal{M}_2. \]  \hspace{1cm} (24)

Note that (24) means that

\[ \sum_{i=1}^{n} |y_i\rangle\langle y_i| = \sum_{i=1}^{n} |x_i\rangle\langle x_i|. \]  \hspace{1cm} (25)

It suffices to show that

either \( x_1 \parallel x_2 \parallel x_3 \parallel \ldots \parallel x_n \) or \( y_1 \parallel y_2 \parallel y_3 \parallel \ldots \parallel y_n \), \hspace{1cm} (26)

since this and (25) imply \( x_i \parallel y_j \) for all \( i, j \), which, by Lemma 2 in Section 2, contradicts to (22) and (23) (if \( x_i = y_i = 0 \) for all \( i \neq k \) then \( \langle y_k | x_k \rangle = \langle \psi | \varphi \rangle = 0 \)).

We will consider that the both vectors \( \varphi \) and \( \psi \) have at least two nonzero components (since otherwise (26) obviously holds).

Let \( k \) be the minimal number such that \( x_i = y_i = 0 \) for all \( i < k \) and either \( x_k \) or \( y_k \) is nonzero.

By symmetry we may assume that \( x_k \neq 0 \). Then (23) implies

\[ y_{k+1} \parallel y_{k+2} \parallel \ldots \parallel y_n. \]  \hspace{1cm} (27)

If \( y_k = 0 \) then this means (26). If \( y_k \neq 0 \) then we have the following three cases.

Case 1: \( x_i \neq 0 \) and \( y_j \neq 0 \), where \( i > j > k \). In this case (22) with \( k = i \) shows that

\[ y_k \parallel y_{k+1} \parallel \ldots \parallel y_{i-1}. \]

Since \( y_j \neq 0 \) and \( i \geq k + 2 \), this and (27) imply (26).

Case 2: \( x_i \neq 0 \) and \( y_j \neq 0 \), where \( j > i > k \). Since \( x_k \neq 0 \) and \( y_k \neq 0 \), this case is reduced to the previous one by permuting \( \varphi \) and \( \psi \).

Case 3: \( x_i = y_i = 0 \) for all \( i > k \) excepting \( i = l > k \). In this case (21) implies

\[ \langle y_k | A x_k \rangle + \langle y_l | A x_l \rangle = 0 \quad \forall A \in \mathcal{M}_2. \]

If \( x_k \nparallel x_l \) then, by 2-transitivity of \( \mathcal{M}_2 \), there is \( A_0 \in \mathcal{M}_2 \) such that \( y_k = A_0 x_k \) and \( y_l = A_0 x_l \) \cite{7}. So, the above equality implies \( \langle y_k | y_k \rangle + \langle y_l | y_l \rangle = 0 \), which contradicts to the assumption \( y_k \neq 0 \). Thus \( x_k \parallel x_l \) and (26) holds.
Now we will show that
\[ \langle \varphi_k^t | M_1 \otimes M_2 \varphi_l^t \rangle = 0 \quad \forall M_1, M_2 \in \mathcal{L}_n, \; k \neq l \]  
(28)
and
\[ \langle \varphi_k^t | M_1 \otimes M_2 \varphi_k^t \rangle = \langle \varphi_l^t | M_1 \otimes M_2 \varphi_l^t \rangle \quad \forall M_1, M_2 \in \mathcal{L}_n, \; k \neq l, \]  
(29)
for the family \( \{ \varphi_k^t \}_{k=1}^n \) of vectors defined in (19). By Lemma 1 these relations mean perfect reversibility of the channel \( \Phi \otimes \Phi \) on the subspace spanned by this family, which implies \( \bar{Q}_0(\Phi \otimes \Phi) \geq \log n \).

Let \( |\xi_k^t\rangle = |0, \ldots, 0, e_i, 0, \ldots, 0\rangle \) be a vector in \( \mathbb{C}^{2n} = [\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2] \), where \( e_i \) is in the \( k \)-th position (\( \{ e_1, e_2 \} \) is the canonical basis in \( \mathbb{C}^2 \)). Then
\[ |\varphi_k^t\rangle = \frac{1}{\sqrt{2}} \left[ |\xi_k^t\rangle \otimes |\xi_k^t\rangle + e^{it}|\xi_2^t\rangle \otimes |\xi_2^t\rangle \right], \quad k = 1, 2, \ldots, n. \]

By setting \( \alpha_1 = 1 \) and \( \alpha_2 = e^{it} \), we have
\[ M_1 \otimes M_2 |\varphi_k^t\rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^2 \alpha_j |\psi(1, k, j)\rangle \otimes |\psi(2, k, j)\rangle, \]  
(30)
where
\[ |\psi(r, k, j)\rangle = |\lambda_1^r U^* e_j, \lambda_2^r U^* e_j, \ldots, \lambda_{k-1}^r U^* e_j, A^r e_j, \lambda_{k+1}^r U e_j, \ldots, \lambda_n^r U e_j\rangle, \]
\( r = 1, 2 \) (\( A^r, \lambda_{ij}^r \) correspond to the matrix \( M_r \)). If \( l > k \) then
\[ \langle \varphi_k^t | M_1 \otimes M_2 \varphi_k^t \rangle = \frac{1}{2} \sum_{i,j=1}^2 \tilde{\alpha}_i \alpha_j \langle \xi_i^t | \otimes (\xi_i^t \cdot |\psi(1, k, j)\rangle \otimes |\psi(2, k, j)\rangle) \]
\[ = \frac{1}{2} \lambda_{1k}^1 \lambda_{2k}^2 \sum_{i,j=1}^2 \tilde{\alpha}_i \alpha_j \langle e_i | U e_j \rangle \langle e_i | U e_j \rangle = \frac{1}{2} \lambda_{1k}^1 \lambda_{2k}^2 \left[ |\eta^2| \alpha_1|^2 + |\bar{\eta}^2| \alpha_2|^2 \right] = 0, \]
Thus (28) is valid for \( l > k \) and hence for all \( l \neq k \). It follows from (30) that
\[ \langle \varphi_k^t | M_1 \otimes M_2 \varphi_k^t \rangle = \frac{1}{2} \sum_{i,j=1}^2 \tilde{\alpha}_i \alpha_j \langle \xi_i^t | \otimes (\xi_i^t \cdot |\psi(1, k, j)\rangle \otimes |\psi(2, k, j)\rangle) \]
\[ = \frac{1}{2} \sum_{i,j=1}^2 \tilde{\alpha}_i \alpha_j \langle e_i | A^1 e_j \rangle \langle e_i | A^2 e_j \rangle. \]  
(31)
and that
\[
\langle \varphi_t^1 | M_1 \otimes M_2 \varphi_t^2 \rangle = \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle \xi^1_l | \otimes \langle \xi^2_l | \cdot |\psi(1, l, j)\rangle \otimes |\psi(2, l, j)\rangle
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{2} \bar{\alpha}_i \alpha_j \langle e_i | A^1 e_j \rangle \langle e_i | A^2 e_j \rangle.
\]

This equality and \((31)\) imply \((29)\).

Consider the case \(n = +\infty\). Let \(\mathcal{H}_A\) be a separable Hilbert space represented as a countable direct sum of 2-D Hilbert spaces \(\mathbb{C}^2\). So, each operator in \(\mathfrak{B}(\mathcal{H}_A)\) can be identified with infinite block matrix satisfying a particular "boundedness" condition.

Let \(\mathcal{L}_*\) be the set of all infinite block matrices \(M\) defined in \((20)\) with \(n = +\infty\) satisfying the condition
\[
\Lambda^2 = \sum_{i=1}^{+\infty} \sum_{j \neq i} |\lambda_{ij}|^2 < +\infty.
\]

This condition guarantees boundedness of the corresponding operator due to the following easily-derived inequality
\[
\|M\|^2_{\mathfrak{B}(\mathcal{H}_A)} \leq 2 \left[ \|A\|^2_{\mathfrak{B}(\mathbb{C}^2)} + \Lambda^2 \right].
\]

Let \(\overline{\mathcal{L}}_*\) be the operator norm closure of \(\mathcal{L}_*\). It is clear that \(\overline{\mathcal{L}}_*\) is a symmetric subspace of \(\mathfrak{B}(\mathcal{H}_A)\) containing the unit operator \(I_{\mathcal{H}_A}\). By using inequality \((33)\) it is easy to show separability of the subspace \(\overline{\mathcal{L}}_*\) in the operator norm topology (as a countable dense subset of \(\overline{\mathcal{L}}_*\) one can take the set of all matrices \(M\) in which \(A\) and all \(\lambda_{ij}\) have rational components).

Symmetricity and separability of \(\overline{\mathcal{L}}_*\) imply (by the proof of Proposition 2 in \([14]\)) existence of a countable subset \(\{\tilde{M}_i\}_{i=2}^{+\infty} \subset \overline{\mathcal{L}}_*\) of positive operators generating \(\overline{\mathcal{L}}_*\) (i.e. such that the operator norm closure of all linear combinations of the operators \(\tilde{M}_i\) coincides with \(\overline{\mathcal{L}}_*\)). Let \(M_i = 2^{-i}\|\tilde{M}_i\|^{-1}\tilde{M}_i, i = 2, 3, \ldots\). Since \(I_{\mathcal{H}_A} \in \overline{\mathcal{L}}_*\) and the series \(\sum_{i=2}^{+\infty} M_i\) converges in the operator norm topology, the positive operator \(M_1 = I_{\mathcal{H}_A} - \sum_{i=2}^{+\infty} M_i\) lies in \(\overline{\mathcal{L}}_*\). Thus, \(\{M_i\}_{i=1}^{+\infty}\) is a countable subset of \(\overline{\mathcal{L}}_* \cap \mathfrak{B}_+(\mathcal{H}_A)\) generating the subspace \(\overline{\mathcal{L}}_*\) such that
\[
\sum_{i=1}^{+\infty} M_i = I_{\mathcal{H}_A},
\]
\(\overline{\mathcal{L}}_*\) is a}
where the series converges in the operator norm topology.

Let \( \{|e_i\rangle\}_{i=1}^{+\infty} \) be an orthonormal basis in a separable Hilbert space \( H_B \). Consider the unital completely positive map

\[
\mathcal{B}(H_B) \ni X \mapsto \Psi(X) = \sum_{i=1}^{+\infty} \langle e_i | X e_i \rangle M_i \in \mathcal{B}(H_A).
\]

Apparently all \( M_i \) lie in \( \text{Ran} \Psi^* \). Since the series in (34) converges in the operator norm topology, \( \text{Ran} \Psi^* \subseteq \mathcal{L}_* \). Hence \( \text{Ran} \Psi^* \) is a dense subset of \( \mathcal{L}_* \).

The predual map

\[
\mathcal{F}(H_A) \ni \rho \mapsto \Psi(\rho) = \sum_{i=1}^{+\infty} \text{Tr} M_i \rho |e_i \rangle \langle e_i| \in \mathcal{F}(H_B)
\]

is an entanglement-breaking quantum channel. Let \( \Phi \) be the complementary channel to \( \Psi \), so that \( \Phi \) is pseudo-diagonal channel and \( \mathcal{G}(\Phi) = \text{Ran} \Psi^* \).

To prove that \( \bar{Q}_0(\Phi) = 0 \) it suffices to show, by Lemma \( \mathbb{H} \) that condition (10) is not valid for \( \mathcal{L} = \mathcal{L}_* \) (since \( \mathcal{L}_* \) and \( \text{Ran} \Psi^* \) are dense in \( \mathcal{L}_* \)). This can be done by repeating the arguments from the proof of the same assertion in the case \( n < +\infty \).

The vectors defined in (19) with \( n = +\infty \) are represented as follows

\[
|\varphi'_k \rangle = \frac{1}{\sqrt{2}} \left[ |\xi^k_1 \rangle \otimes |\xi^k_1 \rangle + e^{i\theta_k} |\xi^k_2 \rangle \otimes |\xi^k_2 \rangle \right], \quad k = 1, 2, 3, ...
\]

where \( |\xi^k_i \rangle = |0, \ldots, 0, e_i, 0, 0, \ldots \rangle \) is a vector in \( H_A = [C^2 \oplus C^2 \oplus \ldots \oplus C^2 \oplus \ldots] \) containing \( e_i \) in the \( k \)-th position (\( \{e_1, e_2\} \) is the canonical basis in \( C^2 \)).

Since \( \text{Ran} \Psi^* \) is a dense subset of \( \mathcal{L}_* \), \( \text{Ran} [\Psi^* \otimes \Psi^*] \) is a dense subset of \( \mathcal{L}_* \otimes \mathcal{L}_* \) (where \( \otimes \) denotes the spacial tensor product). So, to prove that the channel \( \Phi \otimes \Phi \) is perfectly reversible on the subspace spanned by the family \( \{|\varphi'_k \rangle\}_{k=1}^{+\infty} \) it suffices to show, by Lemma \( \mathbb{H} \) that that relations (11) hold for any pair \( |\varphi'_k \rangle, |\varphi'_l \rangle \) and \( \mathcal{L} = \{M_1 \otimes M_2 | M_1, M_2 \in \mathcal{L}_*\} \). This can be done by the same way as in the proof of the similar relations in the case \( n < +\infty \). \( \square \)

4 One property of quantum measurements

In this section we will show that the effect of superactivation of one-shot zero-error quantum capacity has a counterpart in the theory of quantum measurements.
In accordance with the basic postulates of quantum mechanics any measurement of a quantum system associated with a Hilbert space $\mathcal{H}$ corresponds to a Positive Operator Valued Measure (POVM) also called (generalized) quantum observable \cite{[10],[12]}. A quantum observable with finite or countable set of outcomes is a discrete resolution of the identity in $\mathcal{B}(\mathcal{H})$, i.e., a set $\{M_i\}_{i=1}^m$, $m \leq +\infty$, of positive operators in $\mathcal{H}$ such that $\sum_{i=1}^m M_i = I_\mathcal{H}$. An observable is called sharp if it corresponds to an orthogonal resolution of the identity (in this case $\{M_i\}_{i=1}^m$ consists of mutually orthogonal projectors).

If an observable $\mathcal{M} = \{M_i\}_{i=1}^m$ is applied to a quantum system in a given state $\rho$ then the probability of $i$-th outcome is equal to $\text{Tr}M_i\rho$. So, we may consider the observable $\mathcal{M}$ as the quantum-classical channel

$$\mathcal{S}(\mathcal{H}) \ni \rho \mapsto \pi_{\mathcal{M}}(\rho) = \{\text{Tr}M_i\rho\}_{i=1}^m \in \mathcal{P}_m,$$

where $\mathcal{P}_m$ is the set of all probability distributions with $m$ outcomes.

In the theory of quantum measurements the notion of informational completeness of an observable and its modifications are widely used \cite{[1],[3],[13]}. An observable $\mathcal{M}$ is called informational complete if for any two different states $\rho_1$ and $\rho_2$ the probability distributions $\pi_{\mathcal{M}}(\rho_1)$ and $\pi_{\mathcal{M}}(\rho_2)$ are different.

Informational non-completeness of an observable can be characterized by the following notion\footnote{We would be grateful for any references concerning original definition of this notion.}

**Definition 1.** A subspace $\mathcal{H}_0 \subset \mathcal{H}$ is called indistinguishable for an observable $\mathcal{M}$ if $\pi_{\mathcal{M}}(\rho_1) = \pi_{\mathcal{M}}(\rho_2)$ for any states $\rho_1$ and $\rho_2$ supported by $\mathcal{H}_0$.

If $\mathcal{M} = \{M_i\}$ is a sharp observable then all its indistinguishable subspaces coincide with the ranges of the projectors $M_i$ of rank $\geq 2$. So, a sharp observable has no indistinguishable subspaces if and only if it consists of one rank projectors. This is not true for unsharp observables (see the example at the end of this section).

To describe indistinguishable subspaces of a given observable one can use the following characterization of such subspaces.

**Proposition 1.** Let $\mathcal{M} = \{M_i\}_{i=1}^m$, $m \leq +\infty$, be an observable in a Hilbert space $\mathcal{H}$ and $\mathcal{H}_0$ a subspace of $\mathcal{H}$. The following statements are equivalent:

(i) $\mathcal{H}_0$ is an indistinguishable subspace for the observable $\mathcal{M}$;
(ii) $\langle \psi | M_i \varphi \rangle = 0$ for all $i$ and any orthogonal vectors $\varphi, \psi \in \mathcal{H}_0$;

(iii) there exists an orthonormal basis $\{ | \varphi_k \rangle \}$ in $\mathcal{H}_0$ such that

$$
\langle \varphi_k | M_i \varphi_j \rangle = 0 \quad \text{and} \quad \langle \varphi_k | M_i \varphi_k \rangle = \langle \varphi_j | M_i \varphi_j \rangle \quad \forall i,j,k.
$$

**Proof.** It suffices to note that the subspace $\mathcal{H}_0$ is indistinguishable for the observable $M$ if and only if the quantum channel $S(\mathcal{H}) \ni \rho \mapsto \sum_{i=1}^{m} [\text{Tr} \ M_i \rho] | i \rangle \langle i | \in S(\mathcal{H}_m)$, where $\{ | i \rangle \}$ is an orthonormal basis in the $m$-dimensional Hilbert space $\mathcal{H}_m$, has completely depolarizing restriction to the subset $\mathcal{S}(\mathcal{H}_0) \subset \mathcal{S}(\mathcal{H})$ and to use the well known characterizations of completely depolarizing channels. \qed

Nonexistence of indistinguishable subspaces for a quantum observable can be treated as recognition quality of this observable. So, if we have two observables $M_1$ and $M_2$ having no indistinguishable subspaces it is natural to ask about the existence of indistinguishable subspaces for their tensor product $M_1 \otimes M_2$. It turns out that this question is closely related to the superactivation of one-shot zero-error quantum capacity.

**Proposition 2.** Let $\mathcal{H}_A^1, \mathcal{H}_A^2$ be finite-dimensional Hilbert spaces. The following statements are equivalent:

(i) there exist channels $\Phi_1 : \mathcal{G}(\mathcal{H}_A^1) \to \mathcal{G}(\mathcal{H}_B^1)$ and $\Phi_2 : \mathcal{G}(\mathcal{H}_A^2) \to \mathcal{G}(\mathcal{H}_B^2)$ with $\dim \mathcal{G}(\Phi_1) = m_1$ and $\dim \mathcal{G}(\Phi_2) = m_2$ such that

$$
\bar{Q}_0(\Phi_1) = \bar{Q}_0(\Phi_2) = 0 \quad \text{and} \quad \bar{Q}_0(\Phi_1 \otimes \Phi_2) \geq \log n;
$$

(ii) there exist observables $M_1 = \{ M_1^i \}_{i=1}^{m_1}$ and $M_2 = \{ M_2^i \}_{i=1}^{m_2}$ in spaces $\mathcal{H}_A^1$ and $\mathcal{H}_A^2$ having no indistinguishable subspaces such that the observable $M_1 \otimes M_2$ has an $n$-dimensional indistinguishable subspace.

If $\Phi_1 = \Phi_2$ in (i) then $M_1 = M_2$ in (ii) and vice versa.

---

3If $\mathcal{H}_1$ and $\mathcal{H}_2$ are indistinguishable subspaces for observables $M_1$ and $M_2$, then it is easy to see that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an indistinguishable subspaces for the observable $M_1 \otimes M_2$, but there is a possibility of existence of entangled indistinguishable subspaces for the observable $M_1 \otimes M_2$. 
Proof. An observable $\mathcal{M} = \{M_i\}_{i=1}^m$ has a $n$-dimensional indistinguishable subspace if and only if the one-shot zero-error quantum capacity of the channel complementary to channel (35) is not less than $\log n$, this observable $\mathcal{M}$ has no indistinguishable subspaces if and only if the above capacity is zero. This follows from Lemma 1 and Proposition 1, since the output set of the channel dual to channel (35) coincides with the subspace of $\mathcal{B}(\mathcal{H}_A)$ generated by the family $\{M_i\}_{i=1}^m$.

(ii) ⇒ (i). This directly follows from the above remark.

(i) ⇒ (ii). By the proof of Proposition 2 in [14] there exist base $\{A_i^1\}_{i=1}^{m_1}$ and $\{A_i^2\}_{i=1}^{m_2}$ of the subspaces $\mathcal{G}(\Phi_1)$ and $\mathcal{G}(\Phi_2)$ consisting of positive operators such that $\sum_{i=1}^{m_1} A_i^1 = I_{\mathcal{H}_A^1}$ and $\sum_{i=1}^{m_2} A_i^2 = I_{\mathcal{H}_A^2}$. If we consider these base as observables $\mathcal{M}_1$ and $\mathcal{M}_2$ then validity of (ii) can be shown by using the remark at the begin of this proof.

Remark 3. By the above proof the implication (ii) ⇒ (i) in Proposition 2 holds for infinite-dimensional Hilbert spaces $\mathcal{H}_A$ and $n \leq \infty$. The implication (i) ⇒ (ii) can be generalized to this case if the noncommutative graphs $\mathcal{G}(\Phi_1), \mathcal{G}(\Phi_2)$ are separable. This can be done by using the arguments at the end of the proof of Theorem 2 instead of Proposition 2 in [14].

Proposition 2 and Corollary 1 imply the following result.

Corollary 2. There exists a quantum observable $\mathcal{M} = \{M_i\}_{i=1}^5$ in 4-D Hilbert space with no indistinguishable subspaces such that the observable $\mathcal{M} \otimes \mathcal{M}$ has a continuous family of 2-D indistinguishable subspaces.

As a concrete example of such observable $\mathcal{M}$ on can take the resolution of the identity $\{A_i\}_{i=1}^5$ described after Corollary 1 in Section 2. In this case each 2-D subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4$ spanned by the vectors (8) is indistinguishable for $\mathcal{M} \otimes \mathcal{M}$.

Proposition 2 (with Remark 3) and Theorem 2 imply the following observation.

Corollary 3. Let $n \in \mathbb{N}$ or $n = +\infty$. There exists a quantum observable $\mathcal{M} = \{M_i\}_{i=1}^{n^2-n+4}$ in $2n$-dimensional Hilbert space with no indistinguishable subspaces such that the observable $\mathcal{M} \otimes \mathcal{M}$ has a continuous family of $n$-dimensional indistinguishable subspaces.

Remark 4. The above effect of appearance of (entangled) indistinguishable subspace for tensor product of two observables $\mathcal{M}_1$ and $\mathcal{M}_2$ having no

\footnote{If $n = +\infty$ then $n^2 - n + 4 = +\infty$ and the $n$-dimensional Hilbert space (subspace) means a separable Hilbert space (subspace).}
indistinguishable subspaces does not hold for sharp observables $\mathcal{M}_1$ and $\mathcal{M}_2$ (since the tensor product of two observables consisting of mutually orthogonal 1-rank projectors is an observable consisting of mutually orthogonal 1-rank projectors as well).

**Appendix: The Kraus representation of a channel with given noncommutative graph**

The following proposition is a modification of Corollary 1 in [14].

**Proposition 3.** Let $\mathcal{L}$ be a subspace of $\mathcal{M}_n$, $n \geq 2$, satisfying condition (5) and $\{A_i\}_{i=1}^d$ a basis of $\mathcal{L}$ such that $A_i \geq 0$ for all $i$ and $\sum_{i=1}^d A_i = I_n$. Let $m$ be a natural number such that $d = \dim \mathcal{L} \leq m^2$ and $\{|\psi_i\rangle\}_{i=1}^d$ a collection of unit vectors in $\mathbb{C}^m$ such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$ is a linearly independent subset of $\mathcal{M}_m$.

For each $k = 1, \ldots, m$ let $V_k$ be the linear operator from $\mathcal{H}_A = \mathbb{C}^n$ into $\mathcal{H}_B = \bigoplus_{i=1}^d \mathbb{C}^{r_i}$, where $r_i = \text{rank} A_i$, defined as follows

$$V_k = \sum_{i=1}^d \langle k | \psi_i \rangle W_i A_i^{1/2},$$

where $\{|k\rangle\}$ is the canonical basis in $\mathbb{C}^m$ and $W_i$ is a partial isometry from $\mathcal{H}_A$ into $\mathcal{H}_B$ with the initial subspace $\text{Ran} A_i$ and the final subspace $\mathbb{C}^{r_i}$. Then the channel

$$\mathcal{M}_n \ni \rho \mapsto \Phi(\rho) = \sum_{k=1}^m V_k \rho V_k^* \in \mathcal{M}_{r_1 + \ldots + r_d},$$

is pseudo-diagonal and its noncommutative graph $\mathcal{G}(\Phi)$ coincides with $\mathcal{L}$.

**Proof.** In the proof of Corollary 1 in [14] it is shown that the channel

$$\mathcal{M}_n \ni \rho \mapsto \Psi(\rho) = \sum_{i=1}^d [\text{Tr} A_i \rho] |\psi_i\rangle \langle\psi_i| \in \mathcal{M}_m$$

has the Stinespring representation

$$\Psi(\rho) = \text{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^d} \rho V V^*$$

The existence of a basis $\{A_i\}_{i=1}^d$ with the stated properties for any subspace $\mathcal{L}$ satisfying condition (5) is shown in the proof of Proposition 2 in [14].
where

\[ V : |\varphi\rangle \mapsto \sum_{i=1}^{d} A_1^{1/2} |\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle \]

is an isometry from \( \mathbb{C}^n \) into \( \mathbb{C}^n \otimes \mathbb{C}^d \otimes \mathbb{C}^m \) (here \( \{|i\rangle\} \) is the canonical basis in \( \mathbb{C}^d \)).

Since the channel \( \Psi \) is entanglement-breaking and \( \Psi^*(\mathcal{M}_m) = \mathcal{L} \), its complementary channel

\[ \hat{\Psi}(\rho) = \text{Tr}_{\mathbb{C}^m} V \rho V^* \]

is pseudo-diagonal and \( \mathcal{G}(\hat{\Psi}) = \mathcal{L} \). Its Kraus representation is \( \hat{\Psi}(\rho) = \sum_{k=1}^{m} \hat{V}_k \rho \hat{V}_k^* \), where the operators \( \hat{V}_k \) are defined by the relation

\[ \langle \phi | \hat{V}_k | \varphi \rangle = \langle \phi \otimes k | V | \varphi \rangle, \quad \varphi \in \mathbb{C}^n, \phi \in \mathbb{C}^n \otimes \mathbb{C}^d, \]

so that

\[ \hat{V}_k | \varphi \rangle = \sum_{i=1}^{d} \langle k | \psi_i \rangle A_1^{1/2} | \varphi \rangle \otimes |i\rangle. \]

By identifying \( \mathbb{C}^n \otimes \mathbb{C}^d \) with \( \bigoplus_{i=1}^{d} \mathbb{C}^n \), it is easy to show that the channel \( \Phi \) defined by (36) is isometrically equivalent to the channel \( \hat{\Psi} \) (see [11, the Appendix]) and hence \( \mathcal{G}(\Phi) = \mathcal{G}(\hat{\Psi}) = \mathcal{L} \).

We are grateful to A.S.Holevo and to the participants of his seminar "Quantum probability, statistic, information" (the Steklov Mathematical Institute) for useful discussion.

The work of the first author is partially supported by RFBR grant 12-01-00319 and by the RAS research program. The work of the second author is partially supported by the Danish Research Council through the Centre for Symmetry and Deformation at the University of Copenhagen.

References

[1] P.Busch, "Informationally complete sets of physical quantities", Internat. J. Theoret. Phys., 30:9, P.1217-1227, 1991.

[2] R.A.C.Medeiros and F.M. de Assis, "Quantum zero-error capacity", Int. J. Quant. Inf., 3, P.135, 2005.
[3] C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, "Tasks and premises in quantum state determination", arXiv:1308.5502.

[4] T. S. Cubitt, J. Chen, and A. W. Harrow, "Superactivation of the asymptotic zero-error classical capacity of a quantum channel", IEEE Trans. Inf. Theory 57:2, P. 8114, 2011; arXiv:0906.2547.

[5] T. S. Cubitt, G. Smith "An Extreme form of Superactivation for Quantum Zero-Error Capacities", arXiv:0912.2737 [quant-ph], 2009.

[6] T. S. Cubitt, M. B. Ruskai, G. Smith, "The structure of degradable quantum channels", J. Math. Phys., V. 49, 102104, 2008; arXiv:0802.1360.

[7] K. R. Davidson, L. E. Marcoux, and H. Radjavi, "Transitive spaces of operators", Integ. equ. oper. theory 61:187, 2008.

[8] R. Duan, "Superactivation of zero-error capacity of noisy quantum channels", arXiv:0906.2527 [quant-ph], 2009.

[9] R. Duan, S. Severini, A. Winter, "Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovasz theta function", IEEE Trans. Inf. Theory 59(2):1164-1174, 2013; arXiv:1002.2514 [quant-ph].

[10] A. S. Holevo "Quantum systems, channels, information. A mathematical introduction", Berlin, DeGruyter, 2012.

[11] A. S. Holevo "On complementary channels and the additivity problem", Probability Theory and Applications. 2006. V. 51. N. 1. P. 134-143; arXiv:quant-ph/0509101.

[12] M. A. Nielsen, I. L. Chuang "Quantum Computation and Quantum Information", Cambridge University Press, 2000.

[13] E. Prugovecki, "Information-theoretical aspects of quantum measurement", Int. J. Theor. Phys., 16, P. 321-331, 1977.

[14] M. E. Shirokov, T. V. Shulman, "On superactivation of zero-error capacities and reversibility of a quantum channel", arXiv:1309.2610.

[15] G. Smith, J. Yard, "Quantum communication with zero-capacity channels", Science, 321, P. 1812, 2008; arXiv:0807.4935 [quant-ph].