BOUNDARY VALUE PROBLEM: WEAK SOLUTIONS INDUCED BY FUZZY PARTITIONS

LINH NGUYEN, IRINAPerfilieva* AND MICHAL HOLČAPEK

Institute for Research and Applications of Fuzzy Modelling, NSC IT4Innovations
University of Ostrava
30. dubna 22, 701 03 Ostrava 1, Czech Republic

Abstract. The aim of this paper is to propose a new methodology in the construction of spaces of test functions used in a weak formulation of the Boundary Value Problem. The proposed construction is based on the so called “two dimensional” approach where at first, we select a partition of a domain and second, a dimension of an approximating functional subspace on each partition element. The main advantage consists in the independent selection of the key parameters, aiming at achieving a requested quality of approximation with a reasonable complexity. We give theoretical justification and illustration on examples that confirm our methodology.

1. Introduction. The laws of physics or economics for space and time dependent problems are usually described in terms of ordinal or partial differential equations (ODEs or PDEs). Finding exact solution of such a problem with boundary conditions is difficult (often impossible), because its parameters, describing real phenomena, are beyond the constraints that guarantee solvability. Therefore, instead of finding analytical solutions to ODEs as well as PDEs, it is common to apply numerical techniques and find their approximations. The generic idea of all numerical methods is to transform a given problem into a simpler one such that its solution approximates the solution to the original problem.

The commonly used technique consists in transforming a boundary value problem (BVP) into its variational (or a weak) form in a Hilbert space, where a solution to the original problem is replaced by a solution of its variational form accordingly (see, e.g., [3, 5, 20]). An approximate solution to the weak form of a BVP can be found by an appropriate reduction of the infinite dimensional Hilbert space (where the weak form of the BVP is formulated) into its finite dimensional subspace. In this direction, the most popular methods belong to the group of the Ritz-Galerkin methods [9, 20]. For a better picture of the Ritz-Galerkin methods as well as the ideas important to our proposal, we restrict our consideration to the following standard boundary value problem (BVP) with Dirichlet conditions:

\[-(p(x)u'(x))' + q(x)u(x) = f(x), \quad x \in (a, b), \tag{1}\]

2010 Mathematics Subject Classification. Primary: 34L99, 65N30; Secondary: 65N99, 03E72.

Key words and phrases. Fuzzy transform, fuzzy partition, boundary value problem, Ritz approximation, Ritz method, Galerkin method, finite element method.

This research was supported by the Czech Ministry of Education, Youth and Sports, project OP VVV (AI-Met4AI): No. CZ.02.1.01/0.0/0.0/17-049/0008414. Additional support was given by the Grant Agency of the Czech Republic, project 18-06915S.

* Corresponding author: Irina Perfilieva.
A continuous function \( u \) on \([a,b]\) is said to be a (classical) solution to the standard BVP, if

- function \( p(x)u'(x) \) is continuous,
- function \( u \) fulfills boundary conditions and differential equation (1) in each point of \((a,b)\).

It is known that if \( p \in C^1[a,b] \), \( q, f \in C[a,b] \), and for any \( x \in [a,b] \), \( 0 < p_L \leq p(x) \leq p_R \), \( 0 \leq q_L \leq q(x) \leq q_R \), then the standard BVP has a unique solution (see, e.g., [3]). It is also known that these conditions are sufficient in the sense that there exists a BVP that does not fulfill them, but has a unique solution. Moreover, the requirement of a differentiability of \( u \) (even in the combination with \( p \)) makes the problem of BVP rather complicated.

All mentioned facts significantly reduce the class of BVPs with classical solutions. To get this around, the notion of a weak solution has been proposed. The importance of this notion is confirmed by the fact that the great majority of differential equations encountered in modeling of real world phenomena do not admit classical solutions, so that the only way of solving them is to use the weak formulation. The latter means that in (1), the unknown function \( u \) is considered as a generalized function, acting on an appropriate set \( V \) of test functions, i.e.

\[
\int_a^b \left( -(p(x)u'(x))' + q(x)u(x) - f(x) \right)v(x)dx = 0, \tag{4}
\]

where \( v \in V \). If \( u \) fulfills (4) together with (2) and (3), and has a generalized derivative, then it is a weak solution to the standard BVP. The way how the spaces of test functions are introduced specifies a particular methodology in the theory of BVP. Among them, let us mention the conventional spaces of trigonometric functions [19], finite elements (FEM) [3], shape functions in the meshless and generalized FEM [2, 13, 22], etc. All FEMs use only one technical trick to reach the required level of precision - increase the number of finite elements. In complicated cases, this leads to high dimensional stiffness matrices and as a result, requires high computational complexity.

The aim of this paper is to propose a new methodology in the construction of spaces of test functions. It is based on the so called “two dimensional” approach where at first, we select a partition of a domain and second, a dimension of an approximating functional subspace on each partition element. As a result, we come with the construction of a system of finite dimensional subspaces \( V_1, \ldots, V_N, \ldots \) of the space \( V \) of test functions that approximate \( V \) in the following sense: for any \( f \in V, \epsilon > 0, \) and any \( v \in V, \) there exists \( N(\epsilon) > 0, \) such that

\[
\inf \{ \|f - v\|_{1,2} \mid v \in V \} \leq \epsilon, \quad N \geq N(\epsilon). \tag{5}
\]

Our innovation consists in a smart selection of finite dimensional spaces \( V_1, \ldots, V_N, \ldots \) of test functions that allows adjusting the two free parameters: dimension and basis of \( V_N \) to the specificity (complexity) of the BVP, given by its inputs. By this “two dimensional” trick, we are able to guarantee an exponential error decrease (see details in Section 3).

The proposed approach is motivated by the theory of fuzzy partitions [16] and higher degree fuzzy transforms [17]. Recall that the ordinary theory of fuzzy transform (F-transform) was introduced byPerfilieva in [15] with the purpose to include
fuzzy models into the approximation theory. At that time this idea was independently elaborated by many authors, and many interesting scientific paper appeared (see, e.g., [8, 12]) where the most important fuzzy model, based on IF-THEN rules, was developed on a rigorous mathematical platform. However, only in [15], the development of fuzzy models was raised on the level of transforms, and the position of fuzzy modeling appeared in the same line with other classical (integral) transforms.

Later on in [17], the notion of the F-transform was extended to a higher degree fuzzy transform $(F^m$-transform), $m \geq 0$. The $F^m$-transform consists of two phases: direct and inverse. The direct $F^m$-transform maps a function from the $L^2$ space on a bounded interval to a finite vector of polynomials up to degree $m$, called the $F^m$-transform components. They are determined with respect to basic functions of a fuzzy partition of the given bounded interval. Note that the $F^m$-transform components are best approximations of the original function with respect to weights determined by basic functions. The inverse $F^m$-transform uses the $F^m$-transform components as "coefficients" in the linear combination with the basic functions to obtain a locally optimal approximation of the original function. The key parameters of the $F^m$-transform are: a fuzzy partition of a bounded interval and degree $m$ of polynomials that are used as components. Both parameters significantly influence the approximation quality provided by the $F^m$-transform.

In this contribution, we apply only the inverse $F^m$-transform (with arbitrary polynomial components), assuming that the approximate weak solution to the BVP can be expressed in this form, i.e. as a linear-like combination of polynomials up to degree $m$ and basic functions of a fuzzy partition. In general, this combination is not of a polynomial type, so that it opens a novel, wide class of approximation spaces suitable for weak solutions. We show that the proposed approach can be used in numerical applications with reasonable complexity. In our opinion, the main advantage of the proposed approach consists in the independent selection of the key parameters, aiming at achieving a requested quality of approximation with a reasonable complexity. By this we mean a dynamical "shift of focus" from approximate representations with a fixed degree of polynomials and varying partitions to those with a fixed partition and varying degrees of polynomials. Moreover, since the notion of a fuzzy partition is also based on a selection of parameters, we can use them as additional "degree of freedom" in the construction of approximation spaces. This will be a matter of our further research.

The paper is structured as follows. Section 2 is devoted to the basic concepts related to the Ritz-Galerkin method. The third main Section 3 provides with the details of the proposed construction and its theoretical justification. Section 4 is devoted to various illustrations. Then, conclusion follows.

2. **The Ritz-Galerkin method.** In this section, we recall the abstract formulation of the boundary value problem in a weak form. In the succeeding sections, we introduce another interpretation of a weak solution to the standard BVP problem and discuss how a weak solution can be obtained by a newly proposed method. Throughout this section, we use the denotation introduced above and give necessary details regarding the space $W^{1,2}(a, b)$.

2.1. **Sobolev space.** The space $W^{1,2}(a, b)$ is known as a *Sobolev space* of functions that are characterized to be *weakly differentiable* in $(a, b)$ (see [4, 19] and references therein). It is also known that $W^{1,2}(a, b)$ is a Hilbert space with the inner product
\langle \cdot, \cdot \rangle$, where 
\[ \langle f, g \rangle = \int_a^b fgdx + \int_a^b f'g'dx, \]
and the corresponding norm \( \| \cdot \|_{1,2} \), where \( \| f \|_{1,2} = \sqrt{\langle f, f \rangle} \). Let \( V \) be a subspace of \( W^{1,2}(a, b) \) and \( f \in W^{1,2}(a, b) \). The distance of \( f \) and \( V \) in \( W^{1,2}(a, b) \) is denoted by \( \text{dist}(f, V) \) and defined as
\[ \text{dist}(f, V) = \inf\{ \| f - v \|_{1,2} \mid v \in V \}. \]

Let \( C_c^{\infty}(a, b) \) be the set of infinitely-differentiable functions of compact support in \((a, b)\). Then, the completion of \( C_c^{\infty}(a, b) \) with respect to the norm \( \| \cdot \|_{1,2} \) is known to be \( W^{1,2}_0(a, b) \), where

\[ W^{1,2}_0(a, b) = \{ f \in W^{1,2}(a, b) \mid f(a) = f(b) = 0 \} \]
(see Theorem 8.12 in [4]). Obviously, \( W^{1,2}_0(a, b) \) is a subspace of \( W^{1,2}(a, b) \).

2.2. Abstract formulation of the BVP in a weak form. Let us rewrite equation (4) as follows:
\[ \mathcal{A}(u, v) = \mathcal{L}(v), \tag{7} \]
where \( \mathcal{A} : W^{1,2}(a, b) \times W^{1,2}_0(a, b) \to \mathbb{R} \) is a bilinear form
\[ \mathcal{A}(u, v) = \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx, \]
and \( \mathcal{L} : W^{1,2}_0(a, b) \to \mathbb{R} \) is a linear functional
\[ \mathcal{L}(v) = \int_a^b f(x)v(x)dx. \]

A function \( u \in W^{1,2}(a, b) \) is a weak solution to the standard BVP (1) - (3), if for every \( v \in W^{1,2}_0(a, b) \), \( u \) fulfills equation (7) and boundary conditions (2) - (3). \(^1\)

2.3. The Ritz-Galerkin approximation. Consider \( V_g = W^{1,2}(a, b) \) and \( V = W^{1,2}_0(a, b) \) and assume that the standard BVP (1) - (3) fulfills the following conditions:

(i) functions \( p, q \) are bounded and measurable in \((a, b)\);
(ii) function \( f \in L^2(a, b) \);
(iii) \( 0 < p_L \leq p(x) \leq p_R, \ 0 \leq q(x) \).

Then, it can be proved [3, 14] that a weak solution \( u \in V_g \) exists and it is unique. The problem is in the computation of this solution.

It is assumed that spaces \( V_g \) and \( V \) coincide. This takes place, if e.g., boundary conditions are homogeneous, i.e. \( u(a) = u(b) = 0 \). Let \( V \) have a countable (orthonormal) basis \( \{ v_1, \ldots, v_n, \ldots \} \) in the above explained sense. Then, a weak solution (as an element of \( V \)) can be represented by a linear combination of basis elements. However, this representation is not finite, because the dimension of \( V \) is infinite. In this case, a suitable approximation to a weak solution is to be found [21, 19]. Below, we shortly remind the principles of the Ritz-Galerkin approximation.

\(^1\)Let us remark that these requirements are correct because functions in \( W^{1,2}(a, b) \) are continuous.
The focus is to find finite-dimensional subspaces $V_1, \ldots, V_N$ of $V$ with $\dim V_N = N$ (not necessary to have $V_N \subset V_{N+1}$), which fulfill: for each $u \in V$, and each $\varepsilon > 0$, there is a $N(\varepsilon)$ and a $u^{(N)} \in V_N$ with

$$\|u - u^{(N)}\|_V \leq \varepsilon, N \geq N(\varepsilon),$$

and to replace $W^{1,2}(a, b) = V$ by an appropriate $V_N$ in equation in (7). Function $u^{(N)} \in V_N$ such that

$$\mathcal{A}(u^{(N)}, v) = \mathcal{L}(v), \quad v \in V_N,$$

is called a Ritz-Galerkin approximation of a weak solution of (7). It is known that there exists exactly one solution of (8), which moreover, is the best approximation of $u$ in $V_N$, i.e.

$$\|u - u^{(N)}\|_V = \inf_{v \in V_N} \|u - v\|_V.$$

Moreover, if the system of subspaces $V_1, \ldots, V_N, \ldots$ approximates $V$ in the sense of (5), then the sequence of Ritz approximations $u^{(N)}$ converges to $u$ so that

$$\lim_{N \to \infty} \|u - u^{(N)}\|_V = 0.$$

### 2.4. Ritz and Galerkin methods.

We proceed with the numerical methods of computation the Ritz-Galerkin approximation $u^{(N)}$. Let $\{v_1, \ldots, v_N\}$ be a basis of $V_N$. As an element of $V_N$, $u^{(N)}$ can be expressed in the following form:

$$u^{(N)}(x) = \sum_{i=1}^{N} c_i v_i. \quad (9)$$

Moreover, (8) is satisfied for all $v \in V_N$, if and only if it is satisfied for each basis function $v_i$. Therefore, we come to the following system of linear equations:

$$\sum_{i=1}^{N} c_i \mathcal{A}(v_i, v_j) = \mathcal{L}(v_j), \quad j = 1, \ldots, N,$$

with respect to unknown coefficients $c_1, \ldots, c_N$. It is easy to see that the matrix of coefficients $A^2$, where $A(i, j) = \mathcal{A}(v_i, v_j)$, is symmetric. By the above accepted assumptions, this system is uniquely solvable and its solution determines the Ritz-Galerkin approximation $u^{(N)}$.

The principal difference between the Ritz method and the Galerkin method consists in the assumption on matrix $A$, which need not be symmetric in the latter case. In this sense, the Galerkin method generalizes the Ritz method, and both methods coincide for a symmetric matrix $A$.

### 3. Spaces with a fuzzy partition.

There are many numerical realizations of the Ritz-Galerkin method that differ one from another one by the choice of finite dimensional spaces of test functions. The most frequently used realization is known as the finite element method, [7] where the spaces $V_N$ are spanned by piecewise constant or piecewise linear functions. These methods have restricted qualities of approximation (linear or quadratic) in the related to them spaces $V_N$. Both of them use only one technical trick to reach the required level of precision - increase the number of finite elements. In complicated cases, this leads to high dimensional stiffness matrices and as a result, requires high computational complexity. We propose a new method of the construction of spaces of test functions, based on the so called “two dimensional” approach: first, we select a partition of a domain and second,

---

2This matrix is called stiffness matrix.
a dimension of an approximating functional subspace (spanned by polynomials) on each partition element. As a result, we obtain a flexible system of finite dimensional spaces $V_1, \ldots, V_N, \ldots$ of test functions with reasonable dimensions.

In this section, we give technical details to our main proposal. We start with the explanation of the notion of a fuzzy partition$^3$ and then, show how it can be used in the construction of (actually) two spaces of test functions.

3.1. Fuzzy partition. In this paper, we restrict our analysis to a uniform fuzzy partition of a closed interval of real line, which is defined as a family of fuzzy sets that are determined by a specific generating function (see, e.g., [10] and references therein). In contrast to the standard definition, we assume that each generating function is weakly differentiable and its weak derivative is a bounded function.

Definition 3.1. A real-valued function $K : \mathbb{R} \to [0, 1]$ is said to be a generating function of a fuzzy partition if it is continuous, even, non-increasing on $[0, 1]$, vanishing outside of $(-1, 1)$, weakly differentiable on $\mathbb{R}$, and the function of weak derivatives of $K$ is bounded in $\mathbb{R}$.

The most applicable generating functions are the so called “triangle” and “raised cosine” ones. Their description is given below:

1. the triangle generating function
   \[ K_{tr}(x) = \max(1 - |x|, 0), \quad (10) \]

2. the raised cosine generating function
   \[
   K_{rc}(x) = \begin{cases} 
   \frac{1}{2}(1 + \cos(\pi x)), & -1 \leq x \leq 1; \\
   0, & \text{otherwise}.
   \end{cases}
   \]

The corresponding weak derivatives of them are as follows:

\[
(K_{tr})'(x) = \begin{cases} 
1, & -1 \leq x < 0; \\
-1, & 0 \leq x \leq 1; \\
0, & \text{otherwise},
\end{cases}
\]

\[
(K_{rc})'(x) = \begin{cases} 
-\frac{\pi}{2} \sin(\pi x), & -1 \leq x \leq 1; \\
0, & \text{otherwise}.
\end{cases}
\]

Definition 3.2. Let $K$ be a generating function, let $h$ be a positive real constant, and let $x_0 \in \mathbb{R}$. For any $k \in \mathbb{Z}$, let
\[ A_k(x) = K\left(\frac{x - x_0 - c_k}{h}\right), \]
where $c_k = kh$. The family $A = \{A_k \mid k \in \mathbb{Z}\}$ is said to be a uniform fuzzy partition of the real line determined by the triplet $(K, h, x_0)$ if
\[ \sum_{k \in \mathbb{Z}} A_k(x) = 1, \quad x \in \mathbb{R}. \]

The parameters $h$ and $x_0$ are called the bandwidth and the central node of the fuzzy partition, respectively. Moreover, $A_k$ and $c_k$ are called the $k$-th basic function and the $k$-th node of $A$, respectively.

$^3$A uniform fuzzy partition can be characterized with the help of a kernel function, see e.g., [8, 12, 18]. However, we prefer this notion due to its historical origin and because, this is the only connection with fuzzy modeling.
In what follows, we always deal with a uniform fuzzy partition of a closed interval \([a, b]\), denoted by \(A_N = \{A_0, A_1, \ldots, A_N\}\), where \(x_0 = a\) and \(h = (b - a)/N\). For the sake of simplicity, we omit the reference to the parameters of the triplet \((K, h, x_0)\) and simply say that \(A_N\) is a uniform fuzzy partition of \([a, b]\) determined by a generating function \(K\).

3.2. Approximating spaces of test functions. In this section, we give technical details of our methodology that was referred to as based on the “two dimensional” approach. By this, we mean a construction of a system of approximating spaces to \(V\) (the space of test functions in the weak form of BVP (4)) using two independent parameters: \(N\) – a number of elements in a fuzzy partition, and \(m\) – a highest degree polynomial in its span.

For any integer \(N \geq 2\), let \(A_N = \{A_0, A_1, \ldots, A_N\}\) be a uniform fuzzy partition of \([a, b]\). For any integer \(m \geq 1\), let \(B^m(A_N)\) be a set of functions determined as follows:

\[
B^m(A_N) = \Phi^m_0 \cup \Phi^m_N \cup \bigcup_{k=1}^{N-1} \Phi^m_k,
\]

where

\[
\Phi^m_0 = \{\phi_{j,0} = (x - c_0)^j A_0(x), \quad j = 1, \ldots, m\},
\]

\[
\Phi^m_N = \{\phi_{j,N} = (x - c_N)^j A_N(x), \quad j = 1, \ldots, m\},
\]

\[
\Phi^m_k = \{\phi_{j,k} = (x - c_k)^j A_k(x), \quad j = 0, 1, \ldots, m\}.
\]

Let \(D^m(A_N)\) be a linear space spanned by \(B^m(A_N)\). It is easy to see that \(D^m(A_N)\) is a linear \(d\)-dimensional subspace of the Sobolev space \(W_0^{1,2}(a, b)\) where \(d = (m + 1)(N + 1) - 2^4\). This space will be used as a source of sequences of finite dimensional approximating spaces of test functions.

Let \(2 \leq N_1 \leq \ldots \leq N_p \leq \ldots\) and \(0 \leq m_1 \leq \ldots \leq m_p \leq \ldots\) be two sequences of integers. We propose to consider the following sequence of finite dimensional spaces of test functions:

\[
S_{N_1, \ldots, N_p, \ldots}^{m_1, \ldots, m_q, \ldots} = (D^{m_1}(A_{N_1}), \ldots, D^{m_q}(A_{N_p}), \ldots)
\]  

(11)

Below, we will be working with the two particular sequences:

\[
S_{N_1, \ldots, N_p, \ldots}^{m_1, \ldots, m_q, \ldots} = D^{m_1}(A_{N_1}), \ldots, D^{m_q}(A_{N_p}), \ldots
\]

(12)

\[
S_{N_1, \ldots, N_p, \ldots}^{m_1, \ldots, m_q, \ldots} = D^{m_1}(A_N), \ldots, D^{m_q}(A_N), \ldots
\]

(13)

The first particular sequence has the highest degree \(m\) of polynomials fixed, while the second one has a fixed uniform fuzzy partition with \(N\) partition elements. The proposed Ritz-Galerkin approximation process will be based on making a fuzzy partition denser or enlarging the degree of polynomials for a fixed partition or simultaneous enlarging both parameters \(N\) and \(m\). In order to justify the proposed construction in (11), we will show that any function from \(W_0^{1,2}(a, b)\) can be approximated by a certain function from \(D^m(A_N)\) with respect to the increasing number \(N\) of partition elements (first particular sequence), or the increasing degree \(m\) of polynomials for a fixed partition (second particular sequence), or with respect to both

\[\text{Let us remark that } pu \in W_0^{1,2}(a, b) \text{ for any polynomial } p \text{ and any } u \in W_0^{1,2}(a, b). \text{ This is a straightforward consequence of the fact that } p \in C^\infty(a, b) \text{ and } p^{(i)}, i = 0, 1, \text{ is bounded on } (a, b). \text{ Furthermore, } (pu)' = p'u + pu' \text{ (see [1]).}\]
indicated parameters. This will be done by estimating qualities of approximation in the corresponding approximating spaces.

3.3. Sequence of finite dimensional spaces of test functions with the fixed degree polynomials. Let \( m \geq 1 \) be fixed. We start our analysis with the estimation of the distance between arbitrary \( f \in C_c^\infty(a, b) \) and the subspace \( \mathcal{D}^m(A_N) \) of \( W_{1,2}(a, b) \).

**Lemma 3.3.** Let \( m \geq 1 \) be fixed, \( f \in C_c^\infty(a, b) \), \( K \) a generating function, and \( A_N = \{A_0, A_1, \ldots, A_N\} \) a uniform fuzzy partition of \([a, b] \) determined by \( K \) with \( N \geq 2 \). Then, the following estimate is true

\[
\text{dist}(f, \mathcal{D}^m(A_N)) \leq \mathcal{E}(m, f, K) \cdot N^{-m+1/2},
\]

where \( \mathcal{E}(m, f, K) \) is as follows:

\[
\mathcal{E}(m, f, K) = \sqrt{3} \cdot \|f(m+1)\|_{\infty} \cdot \sqrt{(b-a)^2 + 8((m+1)^2 + C_K^2)} / 2(m+1)!
\]

with \( C_K = \sup\{|K'(x)| \mid x \in \mathbb{R}\} \).

**Proof.** Let \( m \geq 1 \) be fixed, and let \( f \in C_c^\infty(a, b) \). It is easy to see that \( f \in C^{(m+1)}[a, b] \) if we extend \( f \) by \( f^{(i)}(a) = f^{(i)}(b) = 0 \) for any \( i = 0, \ldots, m \). Note that \( f^{(i)}(a) = \lim_{x \to a^+} f^{(i)}(x) \) and similarly for \( f^{(i)}(b) \). To apply Taylor’s expansion theorem in a unified way, from now on, we consider \( f \) to belong to the space \( C^{(m+1)}[a, b] \) instead of \( C_c^\infty(a, b) \).

Put \( h = b-a \) and define \( c_k = a + k \cdot h \) for any \( k = 0, \ldots, N \). Moreover, consider \( c_{-1} = a \) and \( c_{N+1} = b \). By Taylor’s theorem, we can expand the function \( f \) with respect to the first \( m \) derivatives for any \( x \in [c_{k-1}, c_{k+1}] \), \( k = 0, \ldots, N \), as follows

\[
f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(c_k)}{j!} (x - c_k)^j + \frac{f^{(m+1)}(c_k + \theta(x - c_k))}{(m+1)!} (x - c_k)^{m+1},
\]

where \( \theta_{x,k} \in (0, 1) \). Note that the value \( \theta_{x,k} \) depends on \( k \) and \( x \). For any \( k = 0, \ldots, N \), define

\[
v_k(x) = \sum_{j=0}^{m} \frac{f^{(j)}(c_k)}{j!} (x - c_k)^j, \quad x \in [a, b].
\]

Since \( f^{(i)}(c_0) = f^{(i)}(c_N) = 0 \) for any \( i = 0, \ldots, m \), we obtain

\[
v_0^{(i)}(x) = v_N^{(i)}(x) = 0
\]

for any \( x \in [a, b] \) and \( i = 0, \ldots, m \). From (15), one can see that

\[
|f(x) - v_k(x)| \leq \|f^{(m+1)}\|_{\infty} \cdot h^{m+1}
\]

holds for any \( k = 0, \ldots, N \) and \( x \in [c_{k-1}, c_{k+1}] \). Note that \( \|f^{(m+1)}\|_{\infty} \) is the supremum of the absolute values of \( f^{(m+1)} \) over \([a, b]\), since \( f^{(m+1)} \) is a continuous function. Similarly, using Taylor’s theorem, for any \( k = 0, \ldots, N \), we can expand \( f' \in C^m[a, b] \) for any \( x \in [c_{k-1}, c_{k+1}] \) as follows

\[
f'(x) = \sum_{j=0}^{m-1} \frac{f^{(j+1)}(c_k)}{j!} (x - c_k)^j + \frac{f^{(m+1)}(c_k + \eta_{x,k}(x - c_k))}{m!} (x - c_k)^m
\]

\[
= v_k'(x) + \frac{f^{(m+1)}(c_k + \eta_{x,k}(x - c_k))}{m!} (x - c_k)^m,
\]
where \( \eta_{k,x} \in (0, 1) \). Again, for any \( k = 0, \ldots, N \), we find that
\[
|f'(x) - v'_k(x)| \leq \frac{\|f^{(m+1)}\|_\infty}{m!} \cdot h^m,
\] (19)
for any \( x \in [c_{k-1}, c_{k+1}] \). Consider
\[
u_{m,N}(x) = \sum_{k=0}^{N} v_k(x) A_k(x), \quad x \in [a, b].
\]
From (17) and the definition of fuzzy partition, one can see easily that \( u_{m,N} \in D^m(A_N) \). Then, for any \( x \in [a, b] \), we have
\[
|f(x) - u_{m,N}(x)|^2 = \left| f(x) - \sum_{k=0}^{N} v_k(x) \cdot A_k(x) \right|^2
\]
\[
= \left| \sum_{k=0}^{N} (f(x) - v_k(x)) \cdot A_k(x) \right|^2 \leq \left( \sum_{k=0}^{N} |f(x) - v_k(x)| \cdot |A_k(x)| \right)^2
\]
\[
\leq \sum_{k=0}^{N} |f(x) - v_k(x)|^2 \cdot \sum_{k=0}^{N} |A_k(x)|^2 \leq 2 \cdot \sum_{k=0}^{N} |f(x) - v_k(x)|^2,
\]
where we used the Bunyakovsky-Cauchy-Schwarz (BCS for short) inequality, i.e.,
\[
\left( \sum_{k=0}^{N} a_k \cdot b_k \right)^2 \leq \sum_{k=0}^{N} a_k^2 \cdot \sum_{k=0}^{N} b_k^2,
\]
and the fact that merely two basis functions are overlapped and \( |A_k| \leq 1 \); therefore, \( \sum_{k=0}^{N} |A_k(x)|^2 \leq 2 \) for any \( x \in [a, b] \). According to (18) and since \( N \geq 2 \), we have
\[
|f(x) - u_{m,N}(x)|^2 \leq \frac{2(N + 1) \cdot \|f^{(m+1)}\|_\infty^2}{((m + 1)!)^2} \cdot h^{2m+2}
\]
\[
= \frac{2(b - a)^{2m+2} \cdot \|f^{(m+1)}\|_\infty^2}{((m + 1)!)^2} \cdot \left( 1 + \frac{1}{N} \right) N^{-2m-1}
\]
\[
\leq \frac{3(b - a)^{2m+2} \cdot \|f^{(m+1)}\|_\infty^2}{((m + 1)!)^2} \cdot N^{-2m-1}
\]
\[
\leq \frac{3(b - a)^{2m+2} \cdot \|f^{(m+1)}\|_\infty^2}{4 \cdot ((m + 1)!)^2} \cdot N^{-2m+1}.
\] (20)
Further, we evaluate \( |f'(x) - u'_{m,N}(x)|^2 \) for any \( x \in [a, b] \), where \( u'_{m,N} \) is the weak derivative of \( u_{m,N} \) over \( [a, b] \). Note the existence of the weak derivatives of \( u_{m,N} \) follows from \( D^m(A_N) \subset W_0^{1,2}(a, b) \) and the fact that \( u'_{m,N}(a) = v'_0(a)A_0(a) + v_0(a)A'_0(a) = 0 \cdot 1 + 0 \cdot 0 = 0 \) and similarly \( u'_{m,N}(b) = 0 \). Using the BCS inequality, we obtain
\[
|f'(x) - u'_{m,N}(x)|^2 = \left| f'(x) - \sum_{k=0}^{N} v'_k(x) \cdot A_k(x) \right|^2
\]
\[
\leq 2 \cdot \left| f'(x) - \sum_{k=0}^{N} v'_k(x) \cdot A_k(x) \right|^2 + 2 \cdot \left| \sum_{k=0}^{N} v_k(x) \cdot A'_k(x) \right|^2,
\] (21)
and moreover,

$$\left| f'(x) - \sum_{k=0}^{N} v'_k(x) \cdot A_k(x) \right|^2 = \left| \sum_{k=0}^{N} (f'(x) - v'_k(x)) \cdot A_k(x) \right|^2$$

$$\leq \sum_{k=0}^{N} |f'(x) - v'_k(x)|^2 \cdot \sum_{k=0}^{N} |A_k(x)|^2$$

$$\leq 2 \cdot \sum_{k=0}^{N} |f'(x) - v'_k(x)|^2$$  \hspace{1cm} (22)

holds or any \( x \in [a, b] \), where we again used \( \sum_{k=0}^{N} |A_k(x)|^2 \leq 2 \). Since \( \sum_{k=0}^{N} A_k(x) = 1 \) for any \( x \in [a, b] \), we find that

$$\sum_{k=0}^{N} f(x) \cdot A'_k(x) = f(x) \cdot \sum_{k=0}^{N} A'_k(x) = 0$$

holds almost everywhere in \([a, b]\). Using the BCS inequality, we now get

$$\left| \sum_{k=0}^{N} v_k(x) \cdot A'_k(x) \right|^2 = \left| \sum_{k=0}^{N} (v_k(x) - f(x)) \cdot A'_k(x) \right|^2$$

$$\leq \sum_{k=0}^{N} |v_k(x) - f(x)|^2 \cdot \sum_{k=0}^{N} |A'_k(x)|^2$$

$$\leq \sum_{k=0}^{N} |v_k(x) - f(x)|^2 \cdot h^{-2} \cdot (|A'_k(x)|^2 + |A'_{k+1}(x)|^2)$$

$$= \sum_{k=0}^{N} |v_k(x) - f(x)|^2 \cdot h^{-2} \cdot \left( \left| K' \left( \frac{x - c_k}{h} \right) \right|^2 + \left| K' \left( \frac{x - c_{k+1}}{h} \right) \right|^2 \right)$$

for a suitable \( k_i = 0, \ldots, N - 1 \), where we used the fact that at most two weak derivatives of basic functions are nonzero at the point \( x \). Since the weak derivatives of \( K \) are bounded on \( \mathbb{R} \), there exists a constant \( C_K \in \mathbb{R} \) dependent on \( K \) such that \( \left| K' \left( \frac{x - c_k}{h} \right) \right| \leq C_K \) for any \( k = 0, \ldots, N \). Then, we obtain that

$$\left| \sum_{k=0}^{N} v_k(x) \cdot A'_k(x) \right|^2 \leq \frac{2 \cdot C_K^2}{h^2} \cdot \sum_{k=0}^{N} |v_k(x) - f(x)|^2$$  \hspace{1cm} (23)

for any \( x \in [a, b] \). From (21), (22) and (23), we obtain

$$|f'(x) - u'_{m,N}(x)|^2 \leq 4 \cdot \sum_{k=0}^{N} |f'(x) - v'_k(x)|^2 + \frac{4 \cdot C_K^2}{h^2} \cdot \sum_{k=0}^{N} |v_k(x) - f(x)|^2$$  \hspace{1cm} (24)

for any \( x \in [a, b] \). From (18), (19) and the assumption that \( N \geq 2 \), we find that

$$|f'(x) - u'_{m,N}(x)|^2 \leq \frac{4(N + 1) \cdot \left| f^{(m+1)} \right|_\infty^2}{((m + 1)!)^2} \left[ (m + 1)^2 + C_K^2 \right] \cdot h^{2m}$$

$$= \frac{4(b - a)^{2m} \cdot \left| f^{(m+1)} \right|_\infty^2}{((m + 1)!)^2} \left[ (m + 1)^2 + C_K^2 \right] \cdot \left( 1 + \frac{1}{N} \right) \cdot N^{-2m+1}$$  \hspace{1cm} (25)
\[ \leq \frac{6(b-a)^{2m} \| f^{(m+1)} \|_\infty^2}{((m+1)!)^2} \left[ (m+1)^2 + C_K^2 \right] \cdot N^{-2m+1} \]

holds for any \( x \in [a,b] \). By integrating both sides in inequalities (20) and (25) with respect to \( x \) over \([a,b]\) and using the property of the distance, we find that

\[ \text{dist}(f, \mathcal{D}^m(A_N))^2 \leq \| f - u_{m,N} \|_{1,2}^2 \leq \mathcal{E}^2(m, f, K) \cdot N^{-2m+1}, \]

where

\[ \mathcal{E}(m, f, K) = \sqrt{3(b-a)^{m+\frac{1}{2}}} \cdot \| f^{(m+1)} \|_\infty \cdot \sqrt{(b-a)^2 + 8((m+1)^2 + C_K^2)}/2(m+1)! \],

which concludes the proof.

By Lemma 3.3 and because \( C_c^\infty(a,b) \) is dense in \( W_0^{1,2}(a,b) \), we prove below that the spaces in the first particular sequence (12) approximate \( V = W_0^{1,2}(a,b) \) in the sense of (5).

**Theorem 3.4.** Let \( m \geq 1 \) be fixed, \( f \in W_0^{1,2}(a,b) \), and \( K \) a generating function of a uniform fuzzy partition \( A_N = \{A_0, A_1, \ldots, A_N\} \) of \([a,b]\). For any \( \epsilon > 0 \), there exists \( N(\epsilon) > 2 \) such that for any \( N \geq N(\epsilon) \), the following inequality holds

\[ \text{dist}(f, \mathcal{D}^m(A_N)) < \epsilon. \] (26)

**Proof.** Let \( \epsilon > 0 \) be arbitrary. Since \( C_c^\infty(a,b) \) is dense in \( W_0^{1,2}(a,b) \), there exists a function \( g \in C_c^\infty(a,b) \) such that

\[ \| f - g \|_{1,2} < \frac{\epsilon}{2}. \] (27)

Let \( N(\epsilon) \) be a positive integer such that

\[ (N(\epsilon))^{-m+1/2} < \frac{\epsilon}{2 \cdot \mathcal{E}(m, g, K)}, \]

where \( \mathcal{E} \) is defined in Lemma 3.3. Let \( N \geq N(\epsilon) \), and let \( A_N = \{A_0, A_1, \ldots, A_N\} \) be a uniform fuzzy partition of \([a,b]\) determined by \( K \). From Lemma 3.3, we find that

\[ \text{dist}(g, \mathcal{D}^m(A_N)) \leq \mathcal{E}(m, g, K) \cdot N^{-m+1/2} \leq \mathcal{E}(m, g, K) \cdot (N(\epsilon))^{-m+1/2} < \frac{\epsilon}{2}. \]

Hence, there exists a function \( u_{m,N} \in \mathcal{D}^m(A_N) \) such that

\[ \| g - u_{m,N} \|_{1,2} < \frac{\epsilon}{2}. \]

From the triangle inequality, we obtain

\[ \| f - u_{m,N} \|_{1,2} \leq \| f - g \|_{1,2} + \| g - u_{m,N} \|_{1,2} < \epsilon, \] (28)

and hence,

\[ \text{dist}(f, \mathcal{D}^m(A_N)) \leq \| f - u_{m,N} \|_{1,2} < \epsilon, \]

which concludes the proof. \( \square \)
3.4. Sequence of finite dimensional spaces of test functions with the fixed fuzzy partitions. Let $\mathcal{P}_c(a, b)$ be a set of functions in $C_c^\infty(a, b)$ whose $m$-th derivative, $m \geq 1$, grows no faster than the $m$-th power of some fixed positive number. Namely, for any $f \in \mathcal{P}_c(a, b)$, there exists a positive constant $M$ such that $\|f^{(m)}\|_\infty \leq M^m$ holds for any $m \geq 1$. Let $\mathcal{E}[\mathcal{P}_c(a, b)]$ denote the completion of $\mathcal{P}_c(a, b)$ with respect to the Sobolev norm $\| \cdot \|_{1,2}$. It is easy to see that $\mathcal{E}[\mathcal{P}_c(a, b)]$ is a subspace of the Sobolev space $W^{1,2}_0(a, b)$.

Let $N \geq 2$ be fixed. In the two subsequent claims, we show that the spaces in the second particular sequence (13) approximate $\mathcal{E}[\mathcal{P}_c(a, b)]$ in the sense of (5).

**Lemma 3.5.** Let $N \geq 2$ be fixed, $f \in \mathcal{P}_c(a, b)$, $K$ a generating function of a uniform fuzzy partition $A_N = \{A_0, A_1, \ldots, A_N\}$ of $[a, b]$. Then, for all $m \geq 1$, the following estimate holds:

$$\text{dist}(f, \mathcal{D}^m(A_N)) \leq \mathcal{E}(m, f, K) \cdot N^{-m+1/2},$$

(29)

where $\mathcal{E}(m, f, K)$ is determined as in Lemma 3.3. Furthermore, it holds that

$$\lim_{m \to \infty} \mathcal{E}(m, f, K) = 0.$$

(30)

**Proof.** The first statement of this lemma is a straightforward consequence of Lemma 3.3. Since $f \in \mathcal{P}_c(a, b)$, there exists a positive constant $M$ such that for all $m \geq 1$,

$$\|f^{(m+1)}\|_\infty \leq M^m.$$ 

Put $L_1 = \sqrt{\frac{3(b-a)}{2}}$ and $L_2 = \sqrt{(b-a)^2 + 8 \cdot C_K^2}$. Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$, we have that

$$0 \leq \lim_{m \to \infty} \mathcal{E}(m, f, K) \leq \lim_{m \to \infty} \frac{L_1 \cdot ((b-a)M)^m(L_2 + (m+1))}{(m+1)!} = \lim_{m \to \infty} \frac{L_1 \cdot L_2 ((b-a)M)^m}{(m+1)!} + \lim_{m \to \infty} \frac{L_1 \cdot ((b-a)M)^m}{m!} = 0,$$

where we made use of the fact that for all $A \in \mathbb{R}$, $\lim_{m \to \infty} \frac{A^m}{m!} = 0$. This completes the proof. 

By Lemma 3.5 and because $\mathcal{P}_c(a, b)$ is dense in $\mathcal{E}[\mathcal{P}_c(a, b)]$, we prove below that the spaces in the second particular sequence (13) approximate $\mathcal{E}[\mathcal{P}_c(a, b)]$ in the sense of (5).

**Theorem 3.6.** Let $N \geq 2$ be fixed, $f \in \mathcal{E}[\mathcal{P}_c(a, b)]$, and $K$ a generating function of a uniform fuzzy partition $A_N = \{A_0, A_1, \ldots, A_N\}$ of $[a, b]$. Then, for any $\epsilon > 0$, there exists $m(\epsilon) \geq 1$, such that for all $m \geq m(\epsilon)$, the following inequality holds:

$$\text{dist}(f, \mathcal{D}^m(A_N)) < \epsilon.$$ 

(31)

**Proof.** Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{E}[\mathcal{P}_c(a, b)]$, there exists a function $g \in \mathcal{P}_c(a, b)$ such that

$$\|f - g\|_{1,2} < \frac{\epsilon}{2}.$$ 

(32)

From (30) in Lemma 3.5, there exists $m(\epsilon) \geq 1$, such that for all $m \geq m(\epsilon)$,

$$\mathcal{E}(m, g, K) \cdot N^{-m+1/2} < \frac{\epsilon}{2},$$

holds. By Lemma 3.5, we have that for all $m \geq m(\epsilon)$,

$$\text{dist}(g, \mathcal{D}^m(A_N)) \leq \mathcal{E}(m, g, K) \cdot N^{-m+1/2} < \frac{\epsilon}{2}. $$

(33)
Hence, there exists \( u_{m,N} \in \mathcal{D}^m(\mathcal{A}_N) \) such that
\[
\| g - u_{m,N} \|_{1,2} < \frac{\epsilon}{2}
\]
From (32) and (33), we obtain that
\[
\text{dist}(f, \mathcal{D}^m(\mathcal{A}_N)) \leq \| f - u_{m,N} \|_{1,2} \leq \| f - g \|_{1,2} + \| g - u_{m,N} \|_{1,2} < \epsilon,
\]
and the proof is completed.

3.5. Discussion. In the above given lemmas and theorems, we showed that the proposed finite dimensional spaces of test functions in the first and second particular sequences (12) and (13) approximate a certain infinite dimensional spaces of test functions in the sense of (5). The estimations of distances (14) and (29) coincide. Therefore, the general two-parametric sequence of finite dimensional spaces of test functions (11) approximates an infinite dimensional space of test functions with the same level of precision. Moreover, being two-parametric, the sequence (11) converges with exponential speed. Comparing with the polynomial speed of various FEMs, we demonstrate significant better quality. On the other hand, the computational costs of the proposed and FEM methods are similar because both of them find a solution through a corresponding system of linear equation with a similar structure. From the theoretical estimation of distances (14) and (29) we see that the proposed method shows exponential convergence rate in the specified cases of source functions. Other details and comparison results are discussed in Section 4.

4. Numerical examples. This section aims to demonstrate and confirm the above considered theoretical results. We apply the proposed methodology to various BVPs, and compare the obtained results with those, obtained by the finite element method (FEM). The FEM is selected, because both methods are in the same group of Galerkin-type methods. Although the selected examples are particular, they represent various classes of BVP with smooth and non-smooth coefficients and with and without oscillation in the right-hand side function. To estimate the quality of approximation, we selected BVPs with known analytic solutions. This selection puts a limitation on the demonstration but still allows us to make relevant conclusions.

The following four examples were selected.

**Example 1** (BVP with smooth coefficients).
\[
-(k(x)u'(x))' = 10x, \quad x \in (0,1),
\]
where \( k(x) = 1 + 0.1x \). This problem has a unique solution given by
\[
u(x) = x(500 - 25x) + (10c_1 - 5000) \log(10 + x) + c_2,
\]
where
\[
c_1 = 500 - 47.5/\log(11/10),
\]
\[
c_2 = 475 \log 10/\log(11/10)).
\]

**Example 2** (BVP with non-smooth coefficients).
\[-(p(x)u'(x))' = 1, \quad x \in (0,1),
\]
\[
u(0) = u(1) = 0,
\]
where
\[ p(x) = \begin{cases} 
1, & 0 \leq x < 1/2, \\
2, & 1/2 \leq x \leq 1.
\end{cases} \]

Despite the coefficient \( p(x) \) is not continuous, the problem has unique solution
\[ u(x) = \begin{cases} 
\frac{1}{12}(5x - 6x^2), & 0 \leq x < 1/2, \\
\frac{1}{24}(1 + 5x - 6x^2), & 1/2 \leq x \leq 1.
\end{cases} \]

**Example 3** (BVP with oscillating source function and extended domain).
\[-u''(x) + 8xu(x) = f(x), \quad x \in (0, 5),
\]
\[ u(0) = u(5) = 0, \]
where
\[ f(x) = -16 \left[(8x^4 - 36x^3 - 20x^2 - 1) \cos 2x^2 + 10x(x - 3) \sin 2x^2 \right]. \]
The exact solution is the oscillating function \( u(x) = 8x(5 - x) \cos 2x^2 \).

**Example 4** (BVP with the rapidly increasing (decreasing) source function).
\[ u''(x) - 2u'(x) = 6e^{3x}, \quad x \in (0, 1), \]
\[ u(0) = u(1) = 0, \]
The analytical solution is
\[ u(x) = 2e^{3x} - \frac{2(e^3 - 1)}{e^2 - 1}(e^{2x} - 1) - 2. \]

The quality of approximation is expressed by the relative error, i.e.,
\[ \text{Error} = \frac{\|\tilde{u} - u\|_2}{\|u\|_2}, \tag{34} \]
where \( u \) and \( \tilde{u} \) are the exact and numerical solutions, respectively.

In Table 1, the approximating sequence (11) of spaces of test functions is as follows: \( S_{4,8,...}^{1,2,...} \) and the computation is performed for \( m = 1, \ldots, 4 \) and \( N = 4i, \quad i = 1, \ldots, 4 \).

In Table 2, another approximating sequence \( S_{32,64,...}^{1,2,...} \) is used. This is because a solution of Example 3 is to be found on longer interval than other considered examples so that the exponential convergence rate can be confirmed for larger values of \( N \) and \( m \) only, see (14). The computation is performed for \( m = 1, \ldots, 4 \) and \( N = 2^i, \quad i = 5, \ldots, 8 \).

In all computations, we use uniform fuzzy partitions with the triangular generating function (10).

In Table 3, the proposed method denoted by “FPP” (Fuzzy Partition with Polynomials) is compared with the piecewise linear FEM. To have a fair comparison, we choose \( m = 1 \) and consider the approximating sequence of test functions in the form \( S_{8,16,...}^{1,2,...} \). The computation is performed for \( N = 2^i, \quad i = 3, \ldots, 8 \).

Besides the approximation error (34) we estimate the convergence rates of numerical solutions. They are computed using [6]:
\[ r_i = \frac{\log (E_i/E_{i+1})}{\log (N_{i+1}/N_i)}, \tag{35} \]
where \( E_i \) is the relative error corresponding to the \( i \)-th numerical solution with \( N = 2^i \). In Table 3, we see that the proposed approach shows better results in all
Table 1. The approximation quality with respect to the degree of polynomials \( m \) and the number of basic functions \( N \).

| \( m \setminus N \) | 4        | 8        | 12       | 16       |
|---------------------|----------|----------|----------|----------|
| 1                   | \( 4.24 \times 10^{-5} \) | \( 3.34 \times 10^{-4} \) | \( 9.64 \times 10^{-4} \) | \( 3.39 \times 10^{-3} \) |
| 2                   | \( 2.34 \times 10^{-5} \) | \( 7.92 \times 10^{-7} \) | \( 1.30 \times 10^{-7} \) | \( 3.70 \times 10^{-9} \) |
| 3                   | \( 1.44 \times 10^{-9} \) | \( 2.09 \times 10^{-9} \) | \( 2.17 \times 10^{-10} \) | \( 4.55 \times 10^{-11} \) |
| 4                   | \( 1.10 \times 10^{-9} \) | \( 1.27 \times 10^{-11} \) | \( 1.24 \times 10^{-11} \) | \( 9.41 \times 10^{-12} \) |

Table 2. The approximation quality with respect to the degree of polynomials \( m \) and the number of basic functions \( N \).

| \( m \setminus N \) | 4        | 8        | 12       | 16       |
|---------------------|----------|----------|----------|----------|
| 1                   | \( 7.30 \times 10^{-3} \) | \( 1.64 \times 10^{-3} \) | \( 8.83 \times 10^{-4} \) | \( 6.24 \times 10^{-4} \) |
| 2                   | \( 4.21 \times 10^{-3} \) | \( 1.52 \times 10^{-3} \) | \( 8.80 \times 10^{-4} \) | \( 6.23 \times 10^{-4} \) |
| 3                   | \( 2.25 \times 10^{-4} \) | \( 7.87 \times 10^{-4} \) | \( 4.86 \times 10^{-4} \) | \( 3.54 \times 10^{-4} \) |
| 4                   | \( 2.13 \times 10^{-4} \) | \( 7.85 \times 10^{-4} \) | \( 4.75 \times 10^{-4} \) | \( 3.52 \times 10^{-4} \) |

| \( m \setminus N \) | 4        | 8        | 12       | 16       |
|---------------------|----------|----------|----------|----------|
| 1                   | \( 1.0308 \) | \( 0.7448 \) | \( 0.5683 \) | \( 0.2451 \) |
| 2                   | \( 0.8802 \) | \( 0.5818 \) | \( 0.1846 \) | \( 9.83 \times 10^{-2} \) |
| 3                   | \( 0.8501 \) | \( 0.3572 \) | \( 9.60 \times 10^{-2} \) | \( 2.03 \times 10^{-2} \) |
| 4                   | \( 0.8338 \) | \( 0.2113 \) | \( 3.67 \times 10^{-2} \) | \( 7.41 \times 10^{-3} \) |

| \( m \setminus N \) | 4        | 8        | 12       | 16       |
|---------------------|----------|----------|----------|----------|
| 1                   | \( 2.07 \times 10^{-3} \) | \( 1.71 \times 10^{-3} \) | \( 4.36 \times 10^{-4} \) | \( 1.72 \times 10^{-4} \) |
| 2                   | \( 1.40 \times 10^{-3} \) | \( 4.99 \times 10^{-5} \) | \( 8.26 \times 10^{-5} \) | \( 2.37 \times 10^{-6} \) |
| 3                   | \( 1.52 \times 10^{-4} \) | \( 1.15 \times 10^{-4} \) | \( 1.21 \times 10^{-5} \) | \( 2.49 \times 10^{-6} \) |
| 4                   | \( 2.26 \times 10^{-5} \) | \( 2.15 \times 10^{-8} \) | \( 1.44 \times 10^{-9} \) | \( 2.16 \times 10^{-10} \) |

four selected examples than the piecewise linear FEM. This conclusion is supported by smaller errors and by higher convergence rates.

Last, but not least, the computational complexity of the used FPP is the same as that of the piecewise linear FEM.

4.1. Discussion. Let us comment results in Tables 1-3 and discuss the stability issue of the proposed method with respect to various source functions. By the theoretical estimation of distances (14) and (29), the exponential rate of convergence has bounded multiplier that tends to zero with the growth of \( m \). This estimation was obtained for those classes of source functions that are considered in Examples 1, 3, 4. It has been confirmed in Tables 1, 2. In more detail, the rapid error decrease along columns with sufficient large values of \( N \) is observed in Table 1 for Examples 1, 4, and in Table 2 for Example 3. The sufficient large value of \( N \) should be used in order to suppress the influence of the multiplier in (14) or (29). The
Table 3. The comparison of the proposed method (FPP) and the piecewise linear FEM based on the approximation errors and the convergence rates

| Example 1 | # N | FPP      | FEM      |
|-----------|-----|----------|----------|
|           |     | Error    | Error    |
|           |     | Rate     | Rate     |
| 8         | 1.6 x 10^{-3} | 2.3 x 10^{-3} | 2.23 |
| 16        | 2.2 x 10^{-4} | 4.9 x 10^{-3} | 2.16 |
| 32        | 2.8 x 10^{-5} | 1.1 x 10^{-3} | 1.97 |
| 64        | 3.3 x 10^{-6} | 2.8 x 10^{-4} | 2.04 |
| 128       | 4.5 x 10^{-7} | 6.8 x 10^{-5} | 2.00 |
| 256       | 5.1 x 10^{-8} | 1.7 x 10^{-5} | 2.00 |

| Example 2 | # N | FPP      | FEM      |
|-----------|-----|----------|----------|
|           |     | Error    | Error    |
|           |     | Rate     | Rate     |
| 8         | 7.1 x 10^{-3} | 2.2 x 10^{-3} | 2.03 |
| 16        | 1.6 x 10^{-3} | 5.4 x 10^{-3} | 1.67 |
| 32        | 6.2 x 10^{-4} | 1.7 x 10^{-3} | 1.39 |
| 64        | 3.0 x 10^{-4} | 6.5 x 10^{-4} | 1.39 |
| 128       | 1.5 x 10^{-4} | 2.9 x 10^{-4} | 1.16 |
| 256       | 8.6 x 10^{-5} | 1.4 x 10^{-4} | 1.05 |

| Example 3 | # N | FPP      | FEM      |
|-----------|-----|----------|----------|
|           |     | Error    | Error    |
|           |     | Rate     | Rate     |
| 8         | 0.9785 | 1.0055 | 1.32 |
| 16        | 0.8787 | 0.8559 | 2.32 |
| 32        | 0.2447 | 0.2764 | 1.32 |
| 64        | 3.9 x 10^{-2} | 7.1 x 10^{-2} | 1.96 |
| 128       | 5.4 x 10^{-3} | 1.8 x 10^{-2} | 1.98 |
| 256       | 6.9 x 10^{-4} | 4.4 x 10^{-3} | 2.03 |

| Example 4 | # N | FPP      | FEM      |
|-----------|-----|----------|----------|
|           |     | Error    | Error    |
|           |     | Rate     | Rate     |
| 8         | 1.1 x 10^{-3} | 4.9 x 10^{-4} | 2.08 |
| 16        | 1.7 x 10^{-3} | 1.1 x 10^{-2} | 2.16 |
| 32        | 1.7 x 10^{-4} | 2.6 x 10^{-3} | 2.08 |
| 64        | 1.9 x 10^{-5} | 6.2 x 10^{-4} | 2.07 |
| 128       | 2.4 x 10^{-6} | 1.5 x 10^{-4} | 2.05 |
| 256       | 3.1 x 10^{-7} | 3.8 x 10^{-5} | 1.98 |

most rapid error decrease is observed along the diagonals in Tables 1, 2. This corresponds to the simultaneous increase of both parameters N and m. In Example 2 with the piecewise constant source function with breaks, the polynomial degree has no significant influence. Moreover, by Table 3, on this particular Example both methods FPP and FEM demonstrate similar behavior. The Example 3 with oscillating (noisy) source function demonstrates the evident advantage of the proposed method against FEM: the relative error decreases quicker with the growth of m than with the growth of N (see Tables 2, 3 with the same values of N). In Table 3, the comparison against FEM on Examples 1, 4, is in favor of the proposed FPP. It is observed from the corresponding convergence rates that are at least one order higher.

5. Conclusion. We proposed a new methodology in the construction of approximating spaces used in a weak formulation of the Boundary Value Problem. It is based on a fuzzy partition of a universe and use test functions in the form of the
inverse $F^m$-transform (with arbitrary polynomial components). We propose one
general and two particular sequences of approximating spaces, where all of them
are based on a uniform fuzzy partition of the problem domain. These sequences
arise as results of: making a fuzzy partition denser, or enlarging the degree of poly-
nomials for a fixed partition, or simultaneous enlarging both parameters $N$ and $m$.
In order to justify the proposed construction in (11), we proved that any function
from $W^{1,2}_0(a,b)$ can be approximated by a certain function from $D^m(A_N)$ with re-
spect to the increasing number $N$ of partition elements (first particular sequence),
or the increasing degree $m$ of polynomials for a fixed partition (second particular
sequence), or with respect to both indicated parameters. This has been done by
estimating qualities of approximation in the corresponding approximating spaces.

We showed that the proposed approach can be efficiently used in numerical appli-
cations. The main advantage of this method consists in the independent selection
of the key parameters, aiming at achieving a requested quality of approximation
with a reasonable complexity. By the theoretical estimations, the generic sequence
of approximating spaces shows an exponential speed of convergence. Comparing
with the polynomial speed of various FEMs, this is a significant better quality.

On the other hand, the computational costs of the proposed and FEM methods
are similar because both of them use corresponding systems of linear equation.
Last but not least, we remarked that the influence of the BVP source function
$f$ is restricted. Therefore, the test functions from the proposed approximation
spaces are not influenced by (possible) noise and the related numerical method is
computationally stable.

We illustrated theoretical results on representative examples (with smooth and
non-smooth coefficients and with and without oscillation in the right-hand side
function) and compared with the technique of FEM. In all cases, our method showed
better quality of approximation and higher convergence rate than that of FEM.

The future research will be focused on using non-uniform fuzzy partitions and
selection of the relevant to a partition element degree of the polynomial.

Acknowledgments. This research was supported by the Czech Ministry of Educa-
tion, Youth and Sports, project OP VVV (AI-Met4AI): No. CZ.02.1.01/0.0/0.0/17-
049/0008414. Additional support was given by the Grant Agency of the Czech
Republic, project 18-06915S.

The authors are grateful to anonymous reviewers for their valuable comments
and recommendations, which helped to study some relevant articles and improve
the content of the manuscript.

REFERENCES

[1] K. W. Anthony, Advanced Real Analysis, Birkhäuser, 2005.
[2] I. Babuška, U. Banerjee and J. E. Osborn, Survey of meshless and generalized finite element
methods: A unified approach, Acta Numer., 12 (2003), 1–125.
[3] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, Texts
in Applied Mathematics, 15, Springer, New York, 2008.
[4] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer,
New York, 2011.
[5] K. W. Cassel, Variational Methods With Applications in Science and Engineering, Cambridge
University Press, Cambridge, 2013.
[6] D. Čena, Cubic spline wavelets with four vanishing moments on the interval and their appli-
cations to option pricing under Kou model, Int. J. Wavelets Multiresolut. Inf. Process., 17
(2019), 1850061, 27 pp.
[7] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, Mathematics and its Applications, 4, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[8] B. C. Cuong, N. C. Luong and H. V. Long, Approximation properties of fuzzy systems for multi-variables functions, *PanAmer. Math. J.*, 20 (2010), 97–113.

[9] C. A. J. Fletcher, *Computational Galerkin Methods*, Springer Series in Computational Physics, Springer-Verlag, New York, 1984.

[10] M. Holčapek, I.Perfilieva, V. Novák and V. Kreinovich, Necessary and sufficient conditions for generalized uniform fuzzy partitions, *Fuzzy Sets and Systems*, 277 (2015), 97–121.

[11] K. Höllig, R. Ulrich and W. Joachim, Weighted extended B-spline approximation of Dirichlet problems, *SIAM J. Numer. Anal.*, 39 (2001), 442–462.

[12] H. V. Long, A note on the rates of uniform approximation of fuzzy systems, *Internat. J. Comput. Intelligence Systems*, 4 (2011), 712–727.

[13] J. M. Melenk, On approximation in meshless methods, in *Frontiers of Numerical Analysis*, Universitext, Springer, Berlin, 2005, 65–141.

[14] L. Nguyen, I. Perfilieva and M. Holčapek, Weak boundary value problem: Fuzzy partition in Galerkin method, *World Scientific Proceedings Series on Computer Engineering and Information Science*, (2018), 1478–1485.

[15] I. Perfilieva, Fuzzy transforms: Theory and applications, *Fuzzy Sets and Systems*, 157 (2006), 993–1023.

[16] I. Perfilieva, A. P. Singh and S. P. Tiwari, On the relationship among $F$-transform, fuzzy rough set and fuzzy topology, *Soft Computing*, 21 (2017), 3513–3523.

[17] I. Perfilieva, M. Daňková and B. Bede, Towards a higher degree $F$-transform, *Fuzzy Sets and Systems*, 180 (2011), 3–19.

[18] I. Perfilieva and P. Vlašánek, F-transform and discrete convolution, *Proc. of the EUSFLAT Conf.*, (2015), 1054–1059.

[19] D. B. Reddy, *Introductory Functional Analysis: With Applications to Boundary Value Problems and Finite Elements*, Texts in Applied Mathematics, 27, Springer-Verlag, New York, 1998.

[20] K. Rektorys, *Variational Methods in Mathematics, Science and Engineering*, D. Reidel Publishing Co., Dordrecht-Boston, MA, 1977.

[21] J. Volker, *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics, 51, Springer, Cham, 2016.

[22] J. G. Wang and G. Liu, A point interpolation meshless method based on radial basis functions, *Internat. J. Numerical Methods in Engineering*, 54 (2002), 1623–1648.

Received January 2019; revised August 2019.

E-mail address: Linh.Nguyen@osu.cz
E-mail address: Irina.Perfilieva@osu.cz
E-mail address: Michal.Holcapek@osu.cz