AN UNORTHODOX INTRODUCTION TO
SUPERSYMMETRIC GAUGE THEORY

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Numerous topics in three and four dimensional supersymmetric gauge theories are covered. The organizing principle in this presentation is scaling (Wilsonian renormalization group flow.) A brief introduction to scaling and to supersymmetric field theory, with examples, is followed by discussions of nonrenormalization theorems and beta functions. Abelian gauge theories are discussed in some detail, with special focus on three-dimensional versions of supersymmetric QED, which exhibit solitons, dimensional antitransmutation, duality, and other interesting phenomena. Many of the same features are seen in four-dimensional non-abelian gauge theories, which are discussed in the final sections. These notes are based on lectures given at TASI 2001.

1. Introduction

These lectures are the briefest possible introduction to some important physical ideas in supersymmetric field theory.

I have specifically avoided restating what the textbooks already contain, and instead have sought to provide a unique view of the subject. The usual presentation on superfields is sidestepped; no introduction to the supersymmetric algebra is given; superconformal invariance is used but not explained carefully. Instead the focus is on the renormalization group, and the special and not-so-special qualitative features that it displays in supersymmetric theories. This is done by discussing the often-ignored classical renormalization group (the best way to introduce beta functions, in my opinion) which is then generalized to include quantum corrections. This approach is most effective using theories in both three and four dimensions. Initially, models with only scalars and fermions are studied, and the classic nonrenormalization theorem is presented. Then I turn to abelian gauge theories, and
finally non-abelian gauge theories. Fixed points, unitarity theorems, duality, exactly marginal operators, and a few other amusing concepts surface along the way. Enormously important subjects are left out, merit ing only a sad mention in my conclusions or a brief discussion in the appendix.

Clearly these lectures are an introduction to many things and a proper summary of none. I have tried to avoid being too cryptic, and in some sections I feel I have failed. I hope that the reader can still make something useful of the offending passages. I have also completely failed to compile a decent bibliography. My apologies.

My final advice before beginning: Go with the flow. If you aren’t sure why, read the lectures.

2. Classical theory

2.1. Free massless fields

Consider classical free scalar and spinor fields in $d$ space-time dimensions.

$$S_\phi = \int d^d x \, \partial_\mu \phi^\dagger \partial^\mu \phi$$

$$S_\psi = \int d^d x \, i \bar{\psi} \partial / \psi$$

By simple dimension counting, since space-time coordinates have mass dimension $-1$ and space-time derivatives have mass dimension $+1$, the dimensions of these free scalar and spinor fields are

$$\dim \phi = \frac{d - 2}{2} \quad ; \quad \dim \psi = \frac{d - 1}{2}$$

so in $d = 3$ scalars (fermions) have dimension $1/2$ (1) while in $d = 4$ they have dimension $1$ ($\frac{3}{2}$). These free theories are scale-invariant — for any $s > 0$,

$$x \to sx \; ; \; \phi \to s^{-(d-2)/2} \phi \; ; \psi \to s^{-(d-1)/2} \psi \Rightarrow S_\phi \to S_\phi \; ; \; S_\psi \to S_\psi .$$

Less obviously, they are conformally invariant. (If you don’t already know anything about conformal invariance, don’t worry; for the purposes of these lectures you can just think about scale invariance, and can separately study conformal symmetry at another time.)
2.2. Free massive fields

Now let’s consider adding some mass terms.

\[ S = \int d^d x \left[ \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \right] \]

Scalar masses always have dimension \( \text{dim} \, m = 1 \) for all \( d \). The propagator

\[ \langle \phi(k) \phi(-k) \rangle = \frac{i}{k^2 - m^2} \]

is a power law \( 1/|x-y|^{d-2} \) (the Fourier transform of the propagator \( i/k^2 \)) for \( |x-y| \ll m^{-1} \), and acts like a delta function \( \delta^4(x) \) (the Fourier transform of the propagator \( -i/m^2 \)) at \( |x-y| \gg m^{-1} \). The scale-dependent propagator interpolates between these two scale-invariant limits.

**Exercise:** Compute the propagator in position space and show that it does indeed interpolate between two scale-invariant limits.

What is the renormalization group flow associated with this theory? It can’t be completely trivial even though the theory is free and therefore soluble. The number of degrees of freedom is two (one complex scalar equals two real scalars) in the ultraviolet and zero in the infrared, so obviously the theory is scale dependent. In particular, Green’s functions are not pure power laws, as we just saw for the propagator. But how are we to properly discuss this given that the one coupling constant in the theory, \( m \), does not itself change with scale? The approach we will use is to define a *dimensionless* coupling \( \nu^2 \equiv m^2/\mu^2 \) where \( \mu \) is the renormalization-group scale – the scale at which we observe the theory. We can think of this theory as transitioning, as in Fig. 1, between two even simpler theories: the scale-invariant theory at \( \mu \to \infty \) where the mass of \( \phi \) is negligible and \( \nu \to 0 \), and the empty though scale-invariant \( \nu \to \infty \) theory in the infrared, where the scalar does not propagate.

![Figure 1](Figure 1. The effect of a mass term grows in the infrared.)
Now consider a massive fermion.

\[ S = \int d^dx \left[ i \bar{\psi} \frac{\partial}{\partial x} \psi - m \psi \bar{\psi} \right] \] (4)

Again, fermion masses always have dimension \( \text{dim} m = 1 \) for all \( d \). Better yet, consider a theory with two free fermions.

\[ S = \int d^dx \sum_{n=1}^{2} \left[ i \bar{\psi}_n \frac{\partial}{\partial x} \psi_n - m_n \psi_n \bar{\psi}_n \right] \] (5)

Now we have a dimensionless coupling constant \( \rho = m_1/m_2 \).

Consider the classical scaling behavior (renormalization group flow) shown in Fig. 2 below. Note there are four scale-invariant field theories in this picture: one with two massless fermions, two with one massless and one infinitely massive fermion, and one with no matter content (indeed, no content at all!) We will call such scale-invariant theories “conformal fixed points”, or simply “fixed points”, to indicate that the dimensionless couplings of the theory, if placed exactly at such a point, do not change with scale.

Whatever are \( m_1 \) and \( m_2 \), or for our purposes the dimensionless couplings \( \nu_1 \) and \( \nu_2 \), scale transformations take the theory from the \( \nu_1 = \nu_2 = 0 \) conformal fixed point to one of the other fixed points. The parameter \( \rho \equiv \nu_1/\nu_2 \), which is scale-invariant, parameterizes the “flows” shown in the graph. The arrows indicate the change in the theory as one considers it at larger and larger length scales. In addition to flows which actually end at \( \nu_1 = 0, \nu_2 = \infty \), note there are also interesting flows from \( \nu_1 = \nu_2 = 0 \) to \( \nu_1 = \nu_2 = \infty \) which pass arbitrarily close to the fixed point \( \nu_1 = 0, \nu_2 = \infty \). These can remain close to the intermediate fixed point for an arbitrarily large range of energy, (namely between the scales \( \mu = m_2 \).
Figure 2. The scaling flow of two masses, with $\rho = m_1/m_2 = \nu_1/\nu_2$.  

and $\mu = m_1 = \rho m_2$) although this is not obvious from the graph of the flow; it is something one must keep separately in mind.

A mass term is known as a “relevant” operator, where the relevance in question is at long distances (low energies.) Although the mass term has no effect in the ultraviolet — at short distance — it dominates the infrared (in this case by removing degrees of freedom.) We can see this from the fact that the dimensionless coupling $\nu$ grows as we scale from the ultraviolet toward the infrared. In fact we can define a beta function for $\nu = m/\mu$ as follows:

$$
\beta_\nu \equiv \mu \frac{\partial \nu}{\partial \mu} = -\nu
$$

(6)

That $\nu$ grows in the infrared is indicated by the negative beta function. More specifically, the fact that the coefficient is $-1$ indicates that $\nu$ scales like $1/\mu$. This tells us that the mass $m$ has dimension 1. We will see examples of irrelevant operators shortly.

2.3. Supersymmetry! The Wess-Zumino model

Now let’s add some “interactions” (more precisely, let’s consider non-quadratic theories.)

$$
S_{\text{Yukawa}} = \int d^d x \left[ \partial_\mu \phi^\dagger \partial^\mu \phi + i \bar{\psi} \gamma^\mu \psi - y \phi \psi \psi - y^* \phi^\dagger \bar{\psi} \bar{\psi} - \lambda \phi^\dagger \phi \phi \right] 
$$

(7)

The coefficients $\lambda$ and $y$ are dimensionful for $d < 4$ but dimensionless for $d = 4$. Appropriate dimensionless coefficients are $\lambda \mu^{d-4}$ and $y \mu^{(d-4)/2}$, 

with classical beta functions

$$\beta_{\lambda \mu}^{d-4} = (d-4)\lambda^{d-4} \quad \text{and} \quad \beta_{y \mu}^{(d-4)/2} = \frac{1}{2}(d-4)y\mu^{(d-4)/2}$$

showing the interactions are classically relevant for \(d < 4\) (and irrelevant for \(d > 4\)) but are “marginal” for \(d = 4\) — they do not scale. The theories in question are thus classically scale invariant for \(d = 4\) (and in fact conformally invariant.)

This Yukawa-type theory can be conveniently written in the following form

$$S_{\text{Yukawa}} = S_{\text{kin}} + S_{\text{int}} + S_{\text{int}}^*$$

$$S_{\text{kin}} = \int d^d x \left[ \partial_{\mu} \phi^i \partial^{\mu} \phi + i\bar{\psi} \gamma_{\mu} \psi + F^i F^i \right]$$

$$S_{\text{int}} = \int d^d x \left[ -y\phi \psi \psi + h^2 F^i F^i \right]$$

(8)

Here \(F\) is just another complex scalar field, except for one thing – it has no ordinary kinetic term, just a wrong-sign mass term. This looks sick at first, but it isn’t. \(F\) is what is known as an “auxiliary field”, introduced simply to induce the \(|\phi^i \phi|2 \right.\) interaction in (7). The classical equation of motion for \(F\) is simply \(F^* \propto \phi^2\).

We can break the conformal invariance of the \(d = 4\) theory by adding mass terms and cubic scalar terms:

$$S_{\text{Yukawa}} \rightarrow S_{\text{Yukawa}} - \int d^d x \left[ M^2 \phi^i \phi + \frac{1}{2}m\psi \psi + h^* \phi |\phi|2 + h^2 |\phi|2 \right] .$$

(9)

If \(M\), \(h\) and \(\lambda\) are related so that the scalar potential is a perfect square, then we may write this in the form (8),

$$S_{\text{int}} = \int d^d x \left[ -\frac{1}{2}m\psi \psi - y\phi \psi \psi + (M\phi + s\phi^2)F \right]$$

(10)

where \(s\) is a constant, and with \(S_{\text{kin}}\) as before.

When \(y\) and \(s\) are equal, and \(M\) and \(m\) are equal, the theory has supersymmetry. Given any spinor \(\zeta\), the transformations

$$\delta\phi = \sqrt{2}\zeta \psi \quad \delta\psi_a = i\sqrt{2} \gamma_a \bar{\psi} \gamma_a \partial_{\mu} \phi + \sqrt{2}\zeta \psi, \delta F = i\sqrt{2} \gamma_a \partial_{\mu} \phi \psi^a$$

(11)

change the Lagrangian only by a total derivative, which integrates to nothing in the action. I leave it to you to check this. This supersymmetric theory is called the “Wess-Zumino model”; it dates to 1974.
At this point there is a huge amount of supersymmetric superfield and superspace formalism one can introduce. There are lots of books on this subject and many good review articles. You don’t need me to write another (and many of you have already read one or more of them) so I will not cover this at all. Instead I will state without proof how one may construct supersymmetric theories in a simple way. Those of you who haven’t seen this before can take this on faith and learn it later. Those of you who have seen it will recognize it quickly.

Let us define a chiral multiplet, which we will represent using something we will call a chiral superfield Φ. Φ contains three “component” fields: φ, ψ, F. Note that Φ is complex: φ is a complex scalar (with two real components), ψ is a Weyl fermion (two complex components, reduced to one by the Dirac equation) and F is a complex auxiliary field (no propagating components.) Thus before accounting for the equations of motion there are four real bosonic and four real fermionic degrees of freedom; after using the equations of motion, there are two real bosonic propagating degrees of freedom and two fermionic ones. That the numbers of bosonic and fermionic degrees of freedom are equal is a requirement (outside two dimensions) for any supersymmetric theory.

Since Φ is complex we can distinguish holomorphic functions W(Φ), and antiholomorphic functions, from general functions K(Φ, Φ†). As we will see, this fact is the essential feature which explains why we know so much more about supersymmetric theories than nonsupersymmetric ones — the difference is the power of complex analysis compared with real analysis.

Consider any holomorphic function (for now let it be polynomial) W(Φ). A well-behaved supersymmetric classical field theory can be written in the form of Eqs. (7)-(10) with

$$S_{\text{int}} = \int d^d x \left[ -\frac{1}{2} \frac{\partial^2 W(\phi)}{\partial \phi^2} \psi \psi + \frac{\partial W(\phi)}{\partial \phi} F \right]$$

(In this expression the function W(Φ) is evaluated at Φ = φ.) The function W is called the “superpotential.” (Note that dim W = d – 1.)

**Exercise:** Check that we recover the previous case by taking

$$W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{4} y \Phi^3.$$  

The potential for the scalar field φ is always

$$V(\phi) = |F|^2 = \left| \frac{\partial W(\phi)}{\partial \phi} \right|^2$$  

(13)
Notice this is positive or zero for all values of $\phi$.

It is a theorem that supersymmetry is broken if $\langle F \rangle \neq 0$; you can see this from the transformation laws (11), in which $F$ appears explicitly. (By contrast, $\langle \phi \rangle$ does not appear in the transformation laws, so it may be nonzero without breaking supersymmetry automatically.) From (13) we see that $V(\phi) = 0$ in a supersymmetry-preserving vacuum. This also follows from the fact that the square of one of the supersymmetry generators is the Hamiltonian; if the Hamiltonian, acting on the vacuum, does not vanish, than the supersymmetry transformation will not leave the vacuum invariant, and the vacuum thus will not preserve the supersymmetry generator.

To find the vacua of a supersymmetric theory is therefore much easier than in a nonsupersymmetric theory. In the nonsupersymmetric case we must find all $\phi$ for which $\frac{\partial W}{\partial \phi} = 0$, and check which of these extrema is a local or global minimum. In the supersymmetric case, we need only find those $\Phi$ for which $\frac{\partial W}{\partial \Phi} = 0$; we are guaranteed that any solution of these equations has $V = 0$ and is therefore a supersymmetric, global minimum (although it may be one of many).

For example, if we take the theory $W = \frac{1}{3} y \Phi^3$, then $V(\phi) = |\phi|^2$; we can see it has only one supersymmetric minimum, at $\phi = 0$.

**Exercise:** Find the scalar potential and the supersymmetric minima for $W = \frac{1}{3} y \Phi^3 + \frac{1}{2} m \Phi^2$, $W = \frac{1}{3} y \Phi^3 + \xi \Phi$, $W = \frac{1}{3} y \Phi^3 + c$, where $\xi$ and $c$ are constants. In each case, note and carefully interpret what happens as $y$ goes to zero.

### 2.4. The XYZ model

Let us consider a theory with three chiral superfields $X, Y, Z$ with scalars $x, y, z$, and a superpotential $W = hXYZ$. Then the potential $V(x, y, z) = |h|^2((|xy|^2 + |xz|^2 + |yz|^2)$ has minima whenever two of the three fields are zero. In other words, there are three complex planes worth of vacua (any $x$ if $y = z = 0$, any $y$ if $z = x = 0$, or any $z$ if $x = y = 0$) which intersect at the point $x = y = z = 0$. This rather elaborate space of degenerate vacua — noncompact, continuous, and consisting of three intersecting branches — is called a “moduli space”. The massless complex fields whose expectation values parameterize the vacua in question ($x$ on the X-branch, etc.) are

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bUsually! we will return to a subtlety later.
called “moduli”. It is useful to represent the three complex planes as three intersecting cones, as in Fig. 3. The point where the cones intersect, and all fields have vanishing expectation values, is called the “origin of moduli space.”

\[ \langle x \rangle = \langle y \rangle = 0 \]
\[ \langle y \rangle \neq 0, \langle z \rangle = 0 \]
\[ \langle z \rangle \neq 0, \langle x \rangle = \langle y \rangle = 0 \]

Figure 3. The moduli space of the XYZ model.

**Exercise:** Prove that these vacua are not all physically equivalent (easy — compute the masses of the various fields.) Note that there are extra massless fields at the singular point \( x = y = z = 0 \) where the three branches intersect. Then look at the symmetries — discrete and continuous — of the theory and determine which of the vacua are isomorphic to one another.

2.5. **XYZ with a mass**

Let’s quickly consider what happens in the theory \( W = hXYZ + \frac{1}{2} mX^2 \). Then the \( Y \) and \( Z \) branches remain while the \( X \) branch is removed, as in Fig. 4. It is interesting to consider the classical renormalization group flow. Let’s think of this in \( d = 4 \); then the theory with \( m = 0 \) is classically conformal, with \( h \) a marginal coupling, and \( m \) a relevant one.

What is happening in the far infrared? The theory satisfies the equation

\[ \nabla^2 X^\dagger = m^\dagger (mX + hYZ) \]

which for momenta much lower than \( m \) (where \( \nabla^2 X \ll \vert m^2 \vert X \)) simply becomes \( mX = -hYZ \). In this limit the kinetic term for \( X \) plays no role, and \( X \) acts like an auxiliary field. We may therefore substitute its equation
of motion back into the Lagrangian, obtaining a low-energy effective theory for \( Y \) and \( Z \) with superpotential

\[
W_L(Y, Z) = \kappa Y^2 Z^2 \quad \kappa = -h^2/2m
\]

At the origin of moduli space, where \( \langle y \rangle = 0 = \langle z \rangle \), \( Y \) and \( Z \) are massless.

The superpotential \( Y^2 Z^2 \) leads to interactions such as \( \kappa^2 |y|^2 |z|^2 \) and \( \kappa y \bar{y} z \bar{z} \) which have dimension higher than 4; equivalently, \( \kappa \) has negative mass dimension \(-1\). Defining a dimensionless quantity \( k = \kappa \mu = -h^2/2 \nu \), we see it has a positive beta function \( \beta_k = 2 \beta_h - \beta_\nu = +k \) so it becomes unimportant in the infrared. We therefore call \( \kappa \), or \( k \), an “irrelevant coupling”, and \( Y^2 Z^2 \) an “irrelevant operator.” More precisely, at the origin of moduli space there is a fixed point in the infrared at which the massless chiral superfields \( Y, Z \) have \( W(Y, Z) = 0 \) (since the physical coupling \( k \to 0 \) in the infrared.) We say that \( \kappa \) (or \( k \)) is an irrelevant coupling, and \( Y^2 Z^2 \) is an irrelevant operator\(^c \) \textit{with respect to this infrared fixed point}. The renormalization group flow for \( \langle y \rangle = \langle z \rangle = 0 \) is shown in Fig. 5.

Thus, we may think of this field theory as a flow from a (classically) conformal fixed point with \( W(X, Y, Z) = h X Y Z \) — one of a continuous class of fixed points with coupling \( h \) — to the isolated conformal field theory with \( W(Y, Z) = 0 \). In this flow, the mass term \( X^2 \) acts as a relevant operator on the ultraviolet fixed point, causing the flow to begin, and the flow into the infrared fixed point occurs along the direction given by the irrelevant operator \((Y Z)^2\).

However, this description is only correct if \( \langle y \rangle = \langle z \rangle = 0 \), that is, exactly

\(^c\)Still more precisely, the irrelevant operators are those which appear in the Lagrangian; it is not \( W \) but \( |\partial W/\partial Y|^2 \) which has dimension higher than 4. This shorthand is a convenient but sometimes confusing abuse of language. The assertion that \( \kappa \) is an irrelevant coupling is not subject to this ambiguity.
at the origin of moduli space. Suppose \( \langle y \rangle \) is not zero; then \( Z \) is massive at the scale \( hy^2 \) and we must re-analyze the flow below this scale. The far-low-energy theory in this case would have only \( Y \) in it. Here lies a key subtlety. The existence of the fixed point with vanishing superpotential \( W(Y, Z) \) at the origin of moduli space might seem, naively, to imply that somehow \( Z \) would remain massless even when \( \langle y \rangle \neq 0 \). But this is inconsistent with the original classical analysis, using either the \( XYZ + \frac{1}{2}mX^2 \) superpotential or the \( kY^2Z^2 \) superpotential. How is this contradiction resolved? To understand this better, consider more carefully the effect of the nonvanishing superpotential for finite \( \langle y \rangle \). At momenta \( \mu \gg \kappa \langle y^2 \rangle \) the theory has light fields \( Y, Z \), as does the conformal point, but at momenta \( \mu \ll \kappa \langle y^2 \rangle \) only \( Y \) is light; see Fig. 6. *Thus the limits \( \mu \to 0 \) and \( \langle y \rangle \to 0 \) do not commute!*

This is one way to see that the infrared conformal theory at the origin of moduli space is effectively disconnected from the infrared theory at \( \langle y \rangle \neq 0 \).

**Exercise:** Show that the theory with \( W(X, Y, Z) = hXY^2 + mYZ + \xi X \) has no supersymmetric vacuum. Supersymmetry is spontaneously broken. Find the supersymmetry-breaking minimum and check that there is a massless fermion in the spectrum (the so-called Goldstino.)
2.6. Solitons in $XYZ$

What happens to the classical theory if we take the superpotential $W = hXYZ + \xi X$? The equations $\frac{\partial W}{\partial X} = hYZ + \xi X = 0$, $\frac{\partial W}{\partial Y} = hXZ = 0$, $\frac{\partial W}{\partial Z} = hXY = 0$, have solutions $X = 0$, $YZ = \xi/h$. (Henceforth we will usually not distinguish the chiral superfield’s expectation value from that of its component scalar field; thus $X = 0$ means $\langle x \rangle = 0$.) Now the equation $YZ = c$, $c$ a constant, is a hyperbola. To see this consider $c$ real (without loss of generality) and now note that for $Y$ real the equation $YZ = c$ gives a real hyperbola; rotating the phase of $Y$ gives a complex hyperbola. What has happened to the three branches is this: the X-branch has been removed, while the two cones of the Y and Z-branches have been joined together as in the figure below. The moduli space $YZ = c$ (Fig. 7) can be parameterized by a single modulus $\frac{Y}{\sqrt{c}}$ which lives on the complex plane with the point at zero removed.

This seemingly innocent theory hides something highly nontrivial. Let us take the case of $d = 3$, and use polar coordinates $r, \theta$ on the two spatial directions. Now suppose that we can find a time-independent circularly-symmetric solution to the classical equations, of characteristic size $r_0$, in which

$$Y(r, \theta) = \sqrt{c}f(r)e^{i\theta}, \quad Z(r, \theta) = \sqrt{c}f(r)e^{-i\theta}$$
where $f(r \to 0) = 0$ (to avoid multivalued fields at the origin) and $f(r \gg r_0) = 1$ (so that $YZ = c$ at large $r$, meaning the potential energy density in this solution is locally zero for $r \gg r_0$.) This object would be a candidate for a "vortex" soliton, a composite particle-like object, in which $Y$ and $Z$ wind (in opposite directions, maintaining $YZ = c$) around the circle at spatial infinity, and in which the energy density is large only in a "core" inside $r = r_0$ (Fig. 8).

However, although the energy density falls to zero rapidly for $r \gg r_0$, the total energy of this vortex diverges. The kinetic energy from the winding of $Y$ and $Z$ is logarithmically divergent — so any soliton of this type would have infinite energy. Why even consider it? Well, a vortex and antivortex pair, a distance $\Delta$ apart, will have finite energy, and this even if they are quite far apart, with $\Delta \gg r_0$, as in Fig. 9. For such a pair, $Y$ and $Z$ would wind locally but not at spatial infinity. Consequently there would be no logarithmic divergence in the energy; instead the energy would go as
twice the core energy plus a term proportional to $\log \Delta$. Since we cannot pull the solitons infinitely far apart, we should think of these objects as logarithmically confined solitons.

![Figure 9. A finite-energy vortex/antivortex configuration; there is no winding of $\arg Y$ at infinity.](image)

This is not as strange as it sounds. In three dimensions, massive electrons exchanging photons are logarithmically confined! In other words, these vortices are no worse than electrons! and indeed they behave as though they are exchanging some massless particle. We’ll come back to this idea later.

In four dimensions, we may repeat the same analysis. Instead of particle-like vortices, we would find logarithmically-confined vortex *strings*, extended in the third spatial dimension. Again, these really can be pair-produced in the theory (or rather, as in Fig. 10, a closed loop of this string can be created at finite energy cost.)

### 2.7. A more general form

Finally, I should point out that I have by no means written the most general supersymmetric theory. Taking any real function $K(\Phi^i, \Phi^{i\dagger})$ of chiral superfields $\Phi^i$ and their conjugates, called the “Kähler” potential, and a superpotential $W(\Phi)$, one can construct a supersymmetric theory by writing

$$S_{\text{general}} = S_{\text{kin}} + S_{\text{int}} + S_{\text{int}}^\dagger$$
we may write

\[
S_{\text{kin}} = \int d^d x \left[ \frac{\partial^2 K(\phi, \phi^\dagger)}{\partial \phi^\dagger \partial \phi} \left( \partial_\mu \phi^\dagger \partial^\mu \phi^i + i \bar{\psi}^i \phi^j + F^{ij} F^i \right) + \text{higher order terms} \right]
\]

\[
S_{\text{int}} = \int d^d x \left[ -\frac{1}{2} \frac{\partial^2 W(\phi)}{\partial \phi^\dagger \partial \phi} \bar{\psi}^i \psi^j + \frac{\partial W(\phi)}{\partial \phi^i} F^i \right]
\]

where repeated indices are to be summed over. (The higher order terms in \( S_{\text{kin}} \) are given in Wess and Bagger; we will not need them but they are quite interesting.) More compactly, through the definition of the “Kähler metric”

\[
K_{ij} = \frac{\partial^2 K(\phi, \phi^\dagger)}{\partial \phi^\dagger \partial \phi^j}
\]

and

\[
W_i = \frac{\partial W(\phi)}{\partial \phi^i} ; \quad W_{ij} = \frac{\partial^2 W(\phi)}{\partial \phi^i \partial \phi^j},
\]

we may write

\[
S_{\text{kin}} = \int d^d x \left[ K_{ij} \left( \partial_\mu \phi^\dagger \partial^\mu \phi^j + i \bar{\psi}^i \phi^j + F^{ij} F^j \right) + \ldots \right]
\]

\[
S_{\text{int}} = \int d^d x \left[ -\frac{1}{2} W_{ij} \bar{\psi}^i \psi^j + W_i F^i \right]. \tag{15}
\]

Note that the scalar potential of the theory is now modified! It is now

\[
V(\phi_i) = W_j^\dagger (K^{-1})^{ij} W_i
\]
(Previously we considered only a “canonical” Kähler potential \( K = \sum_i \Phi_i^\dag \Phi_i \), with a metric \( K_{i\bar{j}} = \delta_{i\bar{j}} \).) The above potential is still positive or zero, however, so the condition for a supersymmetric vacuum, which was \( W_i = 0 \) for all \( i \), remains true unless the metric \( K_{i\bar{j}} \) is singular! Such singularities do occur (and have definite physical origins) so one must not neglect this possibility.

**Exercise:** Take the theory of a single chiral superfield \( \Phi \), with \( W = y \Phi^3 \), and rewrite it by defining a new chiral superfield \( \Sigma \equiv \Phi^3 \). The superpotential is now \( W = y \Sigma \), for which \( dW/d\Sigma \neq 0 \), even for \( \Sigma = 0 \). Compute the Kähler potential and the Kähler metric, and show that the theory does have a supersymmetric vacuum at \( \Sigma = 0 \). The moral: one cannot determine from the superpotential alone whether a theory breaks supersymmetry! At a minimum, additional qualitative information about the Kähler metric is required.

Even this set of supersymmetric theories is a small subset of the whole. For example, we have considered only theories with two-derivative terms. However, there is no reason to restrict ourselves in this way. For example, in the theory \( W(X,Y,Z) = hXYZ + \frac{1}{2}mX^2 \) and a canonical Kähler potential, the classical effective theory at scales \( \mu \ll m \) for the fields \( Y \) and \( Z \) most certainly has terms in its Lagrangian with four or more derivatives of \( y \) and/or \( z \), suppressed by inverse powers of \( m \). For these more general cases, the Lagrangian is not fully specified by the Kähler and superpotential alone.

**Exercise:** Check this last claim by substituting the equation of motion for \( X \) into the action of the original theory and expanding in \( 1/m^2 \).

### 3. Perturbation Theory

#### 3.1. The quantum Wess-Zumino model

Now let’s return to the theory with a single field \( \Phi \) and a superpotential \( W = \frac{1}{4}y\Phi^3 + \frac{1}{2}m\Phi^2 \), with Lagrangian obtained using (12). Classically \( m \) is a relevant coupling; when \( m \) is zero, \( y \) is scale-invariant and the theory is conformal. What happens quantum mechanically?

Since most such theories are divergent, we must regulate them. We can do this by putting in a cutoff at a scale \( \Lambda_{UV} \) (though this is difficult in supersymmetric theories since most cutoffs violate supersymmetry) or by
introducing ghost fields (called Pauli-Villars regulators) of mass $\Lambda_{UV}$ which cancel the degrees of freedom at very high momentum while leaving those at low momentum.

Having done this, we can guess the form of perturbative corrections to any coupling constant using dimensional analysis. If the theory has one or more dimensionless coupling constants, then we expect any coupling of dimension $p$ to get a correction of order $(\Lambda_{UV})^p$. (For $p = 0$ we expect a log $\Lambda_{UV}$ correction.) In particular, in four dimensions a $\lambda \phi^4$ interaction gives a quadratic divergence ($\Lambda_{UV}^2$ times a function of $\lambda$) to $M^2 |\phi|^2$. This is not so for $\phi^4$ theory in $d = 3$; its coupling $\lambda$ has dimension 1, and this reduces the degree of divergence possible in any diagram. In fact the divergence in $M^2$ is now proportional to $\lambda \Lambda_{UV}$. Furthermore, while $\lambda$ itself gets a logarithmic divergence in $d = 4$, in $d = 3$ it gets only finite corrections.

**Exercise:** Check these statements about $\phi^4$ theory.

However, even correct dimensional analysis can overestimate the degree of divergence if there are symmetries around. A coupling constant which breaks a symmetry cannot get an additive divergence, but only a multiplicative one. For example, in the Yukawa theory in Eq. (7), the fermion mass term might be expected to get a linear divergence of order $|y|^2 \Lambda_{UV}$. However, the theory (7) has an explicitly broken chiral symmetry $\psi \rightarrow \psi e^{i\alpha}$; it is broken by both $m$ and the coupling $y$. It also has a symmetry $\phi \rightarrow \phi e^{i\beta}$ broken by $y$ and $h$. But what can you do with broken symmetries? Just ask our teachers, who understood the chiral Lagrangian of QCD! Replace the broken symmetries with “spurious” ones, under which the symmetry-breaking couplings $m, y, h$ — thought of as though they were background scalar fields, called “spurions” — transform with definite charges. Here’s a table of dimensions and of spurion charges under two spurio us symmetries:

|       | $\phi$ | $\psi$ | $M^2$ | $h$ | $\lambda$ | $m$ | $y$ |
|-------|---------|--------|-------|-----|-----------|-----|-----|
| $d = 4$ dimension | 1       | $\frac{4}{3}$ | 2      | 1   | 0         | 1   | 0   |
| $d = 3$ dimension | $\frac{7}{2}$ | 1      | 2      | $\frac{7}{2}$ | 1   | 1   | $\frac{7}{4}$ |
| $U(1)_\alpha$ | 0       | 1      | 0      | 0   | 0         | $-2$ | $-2$ |
| $U(1)_\beta$ | 1       | 0      | 0      | 1   | 0         | 0   | $-1$ |

(16)

For simplicity suppose for the moment that $h = 0$; now let us see why $m$ cannot have a linear divergence in four dimensions and is finite in three. We want to know $\Delta m$, the quantum mechanical corrections to the fermion
mass. These must be in the form of polynomials in the coupling constants times a possible power of $\Lambda_{UV}$. But by the above spurious symmetries, $\Delta m$ must be proportional to $m$; otherwise $m$ and $\Delta m$ could not possibly have the same charge under the spurious symmetries. (Equivalently, this is required by the fact that when $m = 0$ the spurious symmetry $U(1)_{2\alpha-\beta}$ becomes a real one, and this true symmetry then forbids a non-zero $\Delta m$.) Therefore the linear dimension of $\Delta m$ has already been soaked up by the factor of $m$, and we can therefore have only a logarithm of $\Lambda_{UV}$ appearing in $\Delta m$. In $d = 3$ in this theory, it is even better; $\Delta m$ comes from interactions, but all of the couplings have positive mass dimension, making even a logarithmic divergence impossible.

**Exercise:** Show that in this theory $M^2$ has a divergence $\Lambda_{UV}^{d-2}$ while all other couplings, as well as the wave-function renormalization factors for $\phi$ and $\psi$, are log divergent in $d = 4$ and finite in $d = 3$ (once $M^2$ has been renormalized.)

Once you’ve gone through this exercise, you’re ready to see why supersymmetry is so powerful. Supersymmetry requires $M^2 = |m^2|$; but this must hold even quantum mechanically (assuming supersymmetry is preserved) so the divergences in $M^2$ must be reduced down to those for $m$! Thus in $d = 4$ the above Wess-Zumino model has at most log divergences; and in $d = 3$ it is completely finite!

How can this happen? Let’s look at the diagrams in Fig. 11.

![Diagrams](image.png)

Figure 11. The quadratic divergences to the scalar mass cancel.
Now, spurious symmetries can do more; they can constrain finite as well as infinite quantum corrections. For example, if $h = 0$ and $y = 0$ then the $U(1)_\beta$ symmetry is genuine; therefore the effective potential for $\phi$ can only be a function of $\phi^4$, and a quintic $\phi^5$ is clearly forbidden. Once $h \neq 0$ (still with $y = 0$ for simplicity) then that symmetry is lost. But it is easy to see that the coefficient of $\phi^2 \phi^2$ must be proportional to $h^*$ (times a polynomial in $\lambda$) since otherwise there is no way for the quantum effective Lagrangian to respect the spurious symmetries!

Now comes the astounding part. The spurious symmetries of supersymmetric theories profoundly constrain the finite as well as infinite corrections to supersymmetric theories. In particular, the superpotential cannot be renormalized at any order in perturbation theory! All quantum corrections must appear in the Kähler potential or in higher-derivative operators.$^d$

Let us prove this, using modern methods, in the model with $W = \frac{1}{3} y \Phi^3 + \frac{1}{2} m \Phi^2$; the generalization is straightforward though tedious. The key is that the renormalized effective superpotential (which must be carefully defined, in a way that I am avoiding getting into here) is itself a homomorphic function of the chiral fields $\Phi$ (and not their conjugates) and of the coupling constants $y$ and $m$ (and not their conjugates). In fact, one should think of $y$ and $m$ as additional “background” chiral fields! That is, they act as though they are the expectation values of scalar components of other, nonpropagating, chiral fields. The mathematics of supersymmetry, and in particular the Feynman graph expansion, automatically treats them this way.

We start with two special cases. First, suppose $y = 0$ and $m \neq 0$. In this case there is obviously no renormalization since there are no quantum effects. Next, suppose $m = 0$ and $y \neq 0$. This is more interesting. The theory has one real symmetry and one spurious symmetry. The real symmetry is especially curious, in that it takes

$$\phi \rightarrow \phi e^{2i\alpha/3} \; ; \; \psi \rightarrow \psi e^{-i\alpha/3} \; ; \; F \rightarrow F e^{-4i\alpha/3} \; ; \; y \rightarrow y \; ; \; W \rightarrow e^{2i\alpha} W$$

which means that different parts of the $\Phi$ superfield transform differently. In fact the charge of $\phi$ is one unit greater than that of $\psi$ and two units greater than that of $F$. Such a symmetry is called an R-symmetry, and it plays a special role since it does not commute with supersymmetry transformations.

$^d$The original proofs of this, made in the 1980s, did not use spurious, so this language is somewhat ahistorical. It was Seiberg’s great insights in 1993 which led him to this proof.
Note that the superpotential, thought of as a function of $\phi$, transforms with charge 2; this is a requirement of any R-symmetry, as can be seen from the action (14). The spurious symmetry is more ordinary, except that $y$ transforms:

$$
\phi \rightarrow \phi e^{i\beta} ; \quad \psi \rightarrow \psi e^{i\beta} ; \quad F \rightarrow Fe^{i\beta} ; \quad y \rightarrow ye^{-3i\beta} ; \quad W \rightarrow W
$$

leaves the action invariant. Now, what terms can we write in the effective superpotential? We can only write objects which carry charge 2 under the R-symmetry and are neutral under the spurious symmetry. In perturbation theory, every term in the superpotential must be of the form $y^p\Phi^q$, $p,q$ integers; since $W$ is holomorphic we cannot write any powers of $y^*$ or $\Phi^\dagger$. This is very important. We cannot have any functions of $|y|^2$, in contrast to what would have occurred in a nonsupersymmetric theory where holomorphy is not an issue. Clearly $y\Phi^3$ is the unique choice, and therefore the superpotential remains, even quantum mechanically, $W = \frac{1}{3}y\Phi^3$. This is remarkable; no mass term can be generated in the effective superpotential. Moreover, no $\Phi^4$ term can be generated either. This can be understood by thinking about the component fields; the resulting term $\phi^2\psi^2$ in the Lagrangian is simply forbidden by the chiral symmetries. Clearly this is also true for any $\Phi^k, k > 3$.

Now finally let us consider the more complicated case $W = \frac{1}{3}y\Phi^3 + \frac{1}{2}m\Phi^2$. In this case there are no real symmetries. However there are two spurious symmetry, one of them an R-symmetry. Under the ordinary symmetry, $\Phi, y, m$ have charge $1, -3, -2$ respectively. Invariance under this symmetry requires the superpotential depend only on $m\Phi^2$ and $u \equiv y\Phi/m$. The choice of R-symmetry is a bit arbitrary (since we may take linear combinations of any R-symmetry and the spurious but ordinary symmetry to get a new R-symmetry) but a simple choice is to assign, as before, charges $2/3, 0, 2/3$ to $\Phi, y, m$. This means $\phi, \psi, F$ have charge $2/3, -1/3, -4/3$ as before.

| $d = 4$ dimension | $\phi(\Phi)$ | $\psi$ | $F$ | $W$ | $m$ | $y$ | $u = y\Phi/m$ |
|------------------|--------------|--------|-----|-----|-----|-----|----------------|
| $d = 3$ dimension | $1$          | $\frac{2}{3}$ | $2$ | $3$ | $1$ | $0$ | $0$            |
| $U(1)_R$         | $\frac{1}{2}$ | $1$ | $\frac{3}{2}$ | $2$ | $1$ | $\frac{1}{2}$ | $0$            |
| $U(1)_\Phi$      | $\frac{3}{4}$ | $-\frac{4}{3}$ | $-\frac{1}{4}$ | $2$ | $\frac{1}{4}$ | $0$ | $0$            |

Notice that $u$ is invariant under this as well, while $m\phi^2$ has R-charge 2,
so the superpotential must take the form

\[ W_{\text{eff}} = \frac{1}{2} m \Phi^2 f \left( \frac{y \Phi}{m} \right) \]

where \( f \) is a holomorphic function of its argument. This fact is crucial. We know that when \( y = 0 \), \( W_{\text{eff}} = \frac{1}{2} m \Phi^2 \), so \( f(0) = 1 \). Therefore the mass term in the superpotential (we will soon see how important it is to say “in the superpotential”) cannot be renormalized even when it is nonzero; any attempt to correct \( m \) with a factor of \( y \) is always accompanied by a field \( \Phi \), which means this correction does not give a contribution to \( \Phi^2 \).

We also determined just a moment ago that when \( m \to 0 \), \( W = \frac{1}{3} y \Phi^3 \), so \( f(|u| \to \infty) = 2u/3 \). This in turn guarantees there can be no correction to the coefficient of \( \Phi^3 \).

What about \( \Phi^4 \)? Its coefficient is \( y^2/m \) times a number, which might be zero. Let’s consider first what graphs might lead to a corresponding \( \phi^2 \psi^2 \) term in the Lagrangian. The mass term means that we can draw a non-vanishing tree graph, the first in Fig. 12, which is proportional to \( m \), with two \( \phi \psi \psi \) vertices and one chirality-flipping mass insertion, with a factor of \( 1/m^2 \) coming from the propagator of the virtual \( \psi \). However, we are interested in quantum effective actions, to which tree graphs do not contribute. To get a quantum contribution to a \( \phi^2 \psi^2 \) term, we see that we need more than two \( \phi \psi \psi \) vertices. But there’s the rub; this means that the coefficient of this term in the quantum superpotential is proportional at least to \( y^4 \), or more precisely (if you look at the second diagram in Fig. 12 carefully) \( y^2 |y|^2 \). This is in contradiction to the general form of the superpotential; therefore this term must vanish. And so on, for all perturbative contributions to terms in the superpotential. To all orders in perturbation theory, \( f(u) = f_{\text{classical}}(u) = 1 + \frac{2}{3} u \), and \( W_{\text{eff}} = W_{\text{classical}} \).

![Figure 12](image-url)  
**Figure 12.** The first diagram has the right form, but is classical; the second diagram contributes to the quantum effective action, but has the wrong form.

But how far can we carry this argument? What about non-perturbative corrections? We know these corrections must be very small in the limit that
$y$ is very small and $m$ is finite; therefore $f(u) \approx 1 + \frac{2}{3}u$, to all orders in $u$, even nonperturbatively near $u = 0$. And we still know that $f(u) \rightarrow \frac{2}{3}u$ as $u \rightarrow \infty$, because our arguments using the real symmetry of the $m = 0$ case left no room, in that case, for an unknown function even non-perturbatively. Now, the claim is that there are no holomorphic functions except $f_{\text{classical}}$ which have these properties — and therefore $f = f_{\text{classical}}$ exactly.

Note holomorphy is essential here, as is the fact that we know the superpotential when $|u|$ is large but has arbitrary phase. For instance, holomorphy rules out functions such as $e^{-1/|u|^2} + \frac{2}{3}u$ whereas our constraint on $u \rightarrow \infty$ rules out functions such as $e^{-1/u^2} + \frac{2}{3}u$ which has the wrong behavior for small imaginary $u$.

Fantastic. The superpotential for this theory gets no quantum corrections; the coupling constants appearing there are unaltered. It would seem, then, naively, that the coupling constants of this theory do not run. But this sounds wrong. We have already argued that all the coupling constants of the theory should have logarithmic divergences in $d = 4$; has supersymmetry eliminated them? And should there be no finite renormalizations whatsoever? Indeed, this is far too facile. The effective superpotential is well under control, but the effective Kähler potential is not. The latter potential is real, so it can contain real functions of $y^*y$ and $m^*m$ appearing all over the place. Consequently we cannot make any strong statements about its renormalization. But how does this affect the coupling constants?

The resolution of this puzzle is that the coupling $y$ appearing in $W(\Phi)$ is not a physical quantity. Let us rename it $\hat{y}$. By construction it is a holomorphic quantity. Note that we can change it by redefining our fields by $\Phi \rightarrow a\Phi$, where $a$ is any complex constant. This changes $\hat{y}$, and it also changes the Kähler potential, making the kinetic terms noncanonical. Physical quantities (such as the running coupling constants, as measured, say, in scattering amplitudes at particular scales) must be independent of such field redefinitions. To define physical quantities, we should be more careful. Let us take the Kähler potential to have the form

$$K(\Phi^\dagger, \Phi) = Z\Phi^\dagger \Phi.$$  

$Z$ gives the normalization of the wave-function of $\Phi$. Note that a propagator representing an incoming or outgoing particle state should be $i/(k^2 - m^2)$.  

Thus the presence of the factor $Z$, giving $i/Z(k^2 - m^2)$ for the propagator, implies that we will have to take care in normalizing Green functions, a point we will return to shortly.

The rescaling of $\Phi$ by a factor $a$ changes $\hat{y}$ by $a^{-3}$ and $Z$ by $|a|^{-2}$. A natural definition of an invariant coupling is then the quantity $|y|^2 \equiv \hat{y}^1 Z^{-3} \hat{y}$, which is clearly invariant under field redefinitions of the sort we were just considering. The coupling $|y|$ is physical but non-holomorphic, in contrast to the unphysical but holomorphic $\hat{y}$.

But now we see how renormalization of the wave function — divergences or even finite renormalizations which affect the kinetic terms in the Kähler potential — can in turn renormalize physical coupling constants. While $\hat{y}$ cannot be renormalized and become a scale-dependent function, $Z$ can indeed become a scale-dependent function of $\mu$. In fact, we may expect that the graph in Fig. 13 will renormalize the wave function of $\Phi$ by

\[ Z(\mu) = 1 + \tilde{c}_0 \frac{|y|^2}{16 \pi^2} \log(\mu/\Lambda_{UV}) = Z(\mu_0) + \tilde{c}_0 \frac{|y|^2}{16 \pi^2} \log(\mu/\mu_0) \]  

where $\tilde{c}_0$ is a constant of order 1 and $\mu_0$ is an arbitrary scale. Shortly we will determine the sign of $\tilde{c}_0$.

We may now determine the scaling behavior of the physical coupling $y$ (again, to be distinguished from the holomorphic coupling $\hat{y}$ appearing in the superpotential.)

\[ \beta_{|y|^2} = y^* \beta_y + y \beta_{y^*} = -3|y|^2 \frac{\partial \ln Z}{\partial \ln \mu}. \]

Let us define the anomalous mass dimension $\gamma(y)$ of the field $\Phi$ by

\[ \gamma = \frac{\partial \ln Z}{\partial \ln \mu} \]

which tells us how $Z$ renormalizes with energy scale.
Exercise: Why should we think of this as an “anomalous dimension”? What is the relation between $Z(\mu)$ and the dimension of a field? Show $\dim \Phi = 1 + \frac{1}{2} \gamma$.

Then we have an exact relation
$$\beta_y = \frac{3}{2} y \gamma(y)$$

We can use Eq. (18) at small $y$ to find the approximate result
$$\gamma = -\tilde{c}_0 \left| y \right|^2 \frac{1}{16\pi^2} + \text{order } (\left| y \right|^4) \Rightarrow \beta_y = -\frac{3}{2} \tilde{c}_0 y \left| y \right|^2 \frac{1}{16\pi^2} + \text{order } (\left| y \right|^4),$$

but when $y$ is larger we have no hope of computing $\gamma(y)$, and therefore none of computing $\beta_y$. Fortunately, even though $\gamma(y)$ itself is an unknown function, relations such as (19) can be extremely powerful in and of themselves, as we will see shortly.

Let us understand where this relation (19) came from by looking at diagrams. Fig. 14 shows the full propagator, which is proportional to $Z^{-1}$; therefore we must normalize the external fields in any physical process by a factor of $1/\sqrt{Z}$. The graphs contributing to the physical $|y|^2$ take the form of Fig. 15; but supersymmetry eliminates all corrections to the holomorphic $\Phi^3$ vertex, making the graphs much simpler — and (remembering that we must normalize the fields) proportional to $Z^{-3}$.

As an important aside, let me note that chiral superfields have a very special property, namely that products of chiral superfields have no short-distance singularities! In contrast to expectations from non-supersymmetric field theories, composite operators built from chiral fields (but no antichiral or real fields) have the property that they may defined without a short-distance subtraction. The dimension of such a composite is the sum of the dimensions of its components.
Figure 15. The physical coupling $|y|^2$ gets quantum corrections only from $Z$ (grey circles); the vertex factors (black circles) get no quantum contributions, since the holomorphic coupling $\hat{y}$ is unrenormalized.

**Exercise:** Show that $\beta_y = y[\dim (\Phi^3) - \dim W] = y[3 \dim (\Phi) - \dim W]$.

Now, there is a very important theorem which we may put to use. Near any conformal fixed point (including a free field theory) all gauge invariant operators $\mathcal{O}$ must have dimension greater than or equal to 1 (or more generally, $(d-2)/2$). If its dimension is 1 (or more generally, $(d-2)/2$), then $\nabla^2 \mathcal{O} = 0$ (i.e., the operator satisfies the Klein-Gordon equation.) This is true without any appeal to supersymmetry!

The theorem applies to the scalar field $\phi$ which is the lowest component of $\Phi$. Therefore, $\gamma \geq 0$; and $\gamma = 0$ if and only if $y = 0$. From this we may conclude that

$$\gamma(y) = c_0 \frac{|y|^2}{16\pi^2} + \text{order } (|y|^4)$$

where $c_0$ is a positive constant. This in turn implies $\beta_y > 0$, and so $y$ flows to zero in the far infrared.

**Exercise:** Calculate $c_0$.

Thus, rather than being a conformal field theory, as it was classically, with an exactly marginal coupling $y$, the quantum $W = y\Phi^3$ theory flows logarithmically to a free conformal field theory with $y = 0$. We refer to $y$ as a marginally irrelevant operator; it is marginal to zeroth order in $y$, but when $y$ is nonzero then $\beta_y > 0$. The quantum renormalization group flow of the theory with nonzero $y$ and $m$ is shown in Fig. 16.
3.2. Wess-Zumino model in $d = 3$

Now, what do we expect to happen in three dimensions? Here the formula (19) is not really appropriate, because it leaves out the classical dimension of $y$. Perturbation theory can only be done in dimensionless quantities, so we should study not $y$ but $\omega = y/\sqrt{\mu}$. Already we notice a problem. The coupling $\omega$ is large, classically, when $\mu \ll 1/y^2$, so perturbation theory can’t possibly work in the infrared! At long distances this theory will automatically be strongly coupled, unless large quantum effects change the scaling of $\omega$ drastically. But quantum effects will generally be small unless $\omega$ is large — so this can’t happen self-consistently.

Let’s be more explicit. The beta function for $\omega$ is

$$\beta_\omega = \omega \left[ -\frac{1}{2} + \frac{3}{2} \gamma(\omega) \right]$$

Again $\gamma$ must be positive (by the above theorem) and a perturbative power series in $\omega$, beginning at order $\omega^2/16\pi^2$. It has a large negative beta function (meaning it grows toward the infrared) and will only stop growing if $\gamma(\omega) = \frac{1}{3}$. However, this can only occur if $\omega/4\pi \sim 1$, so a one-loop analysis will be insufficient by the time this occurs. Consequently, the most important behavior of the theory will occur in regimes where the perturbative expansion is breaking down, and nonperturbative effects might be important. We cannot expect perturbation theory to tell us everything about
this theory, and specifically we cannot reliably calculate $\gamma(\omega)$.

However, suppose that there is some coupling $\omega_*$ for which $\gamma(\omega_*) = \frac{1}{3}$. This need not be the case; it could be that $\gamma < \frac{1}{3}$ for all values of $\omega$. But if it is the case, then at $\omega_*$ the beta function of the dimensionless coupling $\omega$ vanishes, and the theory becomes truly scale invariant. (Notice that the beta function for $y$ is nonzero there; but scale invariance requires that *dimensionless* couplings not run.) In fact, since $\gamma < \frac{1}{3}$ for $\omega < \omega_*$, and since (barring a special cancellation) we may therefore expect $\gamma > \frac{1}{3}$ for $\omega > \omega_*$, the beta function for $\omega$ is negative below $\omega_*$ and positive above it. The renormalization group flow for $\omega$ then is illustrated in Fig. 17. The point $\omega = \omega_*$ is a stable infrared fixed point; if at some scale $\mu$ the physical coupling $\omega$ takes a value near $\omega_*$, then, at smaller $\mu$, $\omega$ will approach $\omega_*$.  

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node[above] at (0,0) {Classical};
    \draw[->] (0,0) -- (3,0);
    \node[above] at (0,0) {$\gamma = 0$ everywhere};
    \node[circle,fill,inner sep=2pt] at (0,0) {}; \node[above] at (0,0) {$\omega = 0$};
    \node[circle,fill,inner sep=2pt] at (3,0) {}; \node[above] at (3,0) {$\omega = \infty$};
    \draw[->] (0,-0.5) -- (3,-0.5);
    \node[above] at (0,-0.5) {Quantum};
    \draw[->] (0,-1) -- (3,-1);
    \node[above] at (0,-1) {$\gamma < \frac{1}{3}$};
    \node[circle,fill,inner sep=2pt] at (0,-1) {}; \node[above] at (0,-1) {$\omega = 0$};
    \node[circle,fill,inner sep=2pt] at (1.5,-1) {}; \node[above] at (1.5,-1) {$\omega_*$};
    \node[circle,fill,inner sep=2pt] at (3,-1) {}; \node[above] at (3,-1) {$\omega = \infty$};
    \node[above] at (1.5,-1) {$\gamma > \frac{1}{3}$};
    \node[above] at (0,-2) {Figure 17. The scale-dependence of the coupling $\omega = y/\sqrt{\mu}$ is lost, due to large quantum corrections, at the value $\omega_*$.

**Exercise:** What is the behavior of the wave function $Z(\mu)$ once this fixed point is reached?

At this conformal fixed point, the field $\Phi$ will have dimension $\frac{2}{3}$. Thus

\footnote{We know a scale can be generated in a classically scale-invariant theory; this is called "dimensional transmutation" and is familiar from QCD. Here we have a theory with a classical scale $\tilde{g}^2$; but this scale is lost quantum mechanically. This equally common phenomenon is often called "dimensional antitransmutation."}
the operator $\Phi^4$, when appearing in the superpotential, represents not a marginal operator of dimension 2 (as in the free theory) but an irrelevant operator of dimension $\frac{8}{3} > 2$.

**Exercise:** For the mass term $m\Phi^2$, what is the dimension of $m$ at the fixed point?

Should we skeptical of the existence of this fixed point? No; massless nonsupersymmetric $\lambda\phi^4$ theory has a fixed point for $d = 4 - \epsilon$ dimensions, with $\lambda \sim \epsilon$. This “Wilson-Fisher” fixed point is easy to find in perturbation theory. It has been verified that it continues all the way to $d = 3$ dimensions. A similar analysis can be done for a complex scalar and two-component complex fermion coupled as in (7); this too shows a similar fixed point.

As another example, a theory (the $O(N)$ model) with $N$ massless scalar fields $\phi_i$ and a potential $V = \lambda(\sum_i \phi_i^2)^2$ can be shown, directly in $d = 3$ and at leading order in a $1/N$ expansion, to have a nontrivial fixed point $\lambda \sim 1/N$. A corresponding analysis can be done for a supersymmetric theory with $N$ chiral fields $\Phi_i$ and a single chiral field $X$, with superpotential $hX(\sum_i \Phi_i^2)$.

**Exercise:** Verify the claims of the previous paragraph for the $O(N)$ model. Then try the supersymmetric theory with $W(X, \Phi_i) = hX(\sum_i \Phi_i^2)$; show it has a fixed point. Compute the anomalous dimensions of $X$ and $\Phi$ at the fixed point (hint — do the easiest one, then use conformal invariance to determine the other.) Work only to leading nonvanishing order in $1/N$. You may want to see Coleman’s Erice lectures on the $O(N)$ model.

Thus the putative fixed point at $\omega = \omega_*$ is very plausible, and we will assume henceforth that it exists. Note that at this fixed point the effective theory is highly nonlocal. It has no particle states (the propagator $\langle \phi^\dagger(x)\phi(0) \rangle = x^{-4/3}$ does not look like a propagating particle or set of particles, which would have to have integer or half-integer dimension.) In this and all similar cases the superpotential and Kähler potential together are insufficient. Indeed it is very difficult to imagine writing down an explicit Lagrangian for this fixed point theory. Even in two dimensions it is rarely known how to write a Lagrangian for a nontrivial fixed point, although in two dimensions there are direct constructive techniques for determining the properties of many conformal field theories in detail. In $d = 3$ there are no
such techniques known; we have only a few pieces of information, including the dimension of $\Phi$.

### 3.3. Dimensions and R-charge

In fact there is another way (which at first will appear trivial) to determine the dimension of $\Phi$ at the fixed point. This requires looking a bit more closely at the symmetries of the Lagrangian (7). In particular, for $W = \frac{1}{3} y \Phi^3$, we saw earlier that there is an R-symmetry

$$
\phi \rightarrow \phi e^{2i\alpha/3} ; \quad \psi \rightarrow \psi e^{-i\alpha/3} ; \quad F \rightarrow F e^{-4i\alpha/3}
$$

which is the unique R-symmetry of the theory. Under this symmetry $\Phi$ has charge $2/3$ and the superpotential has charge 2.

At a conformal fixed point, there is a close relation between the dimensions of many chiral operators and the R-charges that they carry. The energy-momentum tensor (of which the scale-changing operator, the “dilation” or “dilatation generator”, is a moment) and the current of the R-charge are actually part of a single supermultiplet of currents. At a conformal fixed point, the dilation current and the R current are both conserved quantities, and the superconformal algebra can then be used to show that the dimension of a chiral operator is simply $(d - 1)/2$ times its $R$ charge. This implies that at any fixed point with $W \propto \Phi^3$, the dimension of $\Phi$ is 1 in $d = 4$, 2/3 in $d = 3$, and 1/3 in $d = 2$. (In all cases, dim $\Phi = \text{dim} \phi = \text{dim} \psi - \frac{1}{2} = \text{dim} F - 1$.) Since the dimension of $\Phi$ must be strictly greater than 1 (1/2) at any nontrivial fixed point in $d = 4(3)[2]$, it follows that there can be nontrivial fixed points in two or three dimensions, but not for $d = 4$.

### 3.4. The quantum XYZ model in $d = 3$

Now let us turn to some other fixed points. If we have three fields $X,Y,Z$ and superpotential $W = \hat{h}XYZ$, then by symmetry $\gamma_X = \gamma_Y = \gamma_Z = \gamma_0$. In four dimensions there can again be no nontrivial fixed point; the coupling $h$ is marginally irrelevant. But in three dimensions, it is relevant and interesting dynamics can be expected. In particular, the coupling $\eta = h/\sqrt{\mu}$ has

$$
\beta_\eta = \frac{1}{2} \eta[-1 + \gamma_X(\eta) + \gamma_Y(\eta) + \gamma_Z(\eta)] = \frac{1}{2} \eta[-1 + 3\gamma_0(\eta)] .
$$

As in the previous theory we see that if there exists some $\eta_*$ for which $\gamma_0(\eta_*) = \frac{1}{3}$, then the theory is conformal there. Let us assume $\eta_*$ exists
and the flow is as given in Fig. 18.

\[ \eta = 0 \quad \eta^* \quad \eta = \infty \]

Figure 18. A fixed point for \( \eta \).

Exercise: List the marginal and relevant operators at this fixed point. Note that some apparently relevant operators are actually just field redefinitions and are not actually present; for example, the addition of \( mXY \) to the superpotential \( W = hXYZ \) has no effect, since we may simply redefine \( Z \rightarrow Z - m \) and eliminate the coupling altogether. Such eliminable operators are called “redundant.”

Now we can consider the effect of adding other operators to the theory. For example what happens if we add \( \omega_x \sqrt{\mu X^3} \) to the theory?

Exercise: Prove that \( \omega_x \) is a marginally irrelevant coupling. To do this use the facts that (1) at \( \eta = 0, \omega_x \neq 0 \), we know \( \gamma_x > 0 = \gamma_y = \gamma_z \), and (2) \( \gamma_x - \gamma_y \), a continuous real function of the couplings, is known to be zero when \( \omega_x = 0 \).

Since \( \omega_x \) (and similarly \( \omega_y \) and \( \omega_z \), when we add them in as well) are marginally irrelevant couplings, we may wonder if this fixed point, located at \( (\eta, \omega_x, \omega_y, \omega_z) = (\eta^*, 0, 0, 0) \), is an isolated point in the space of the four coupling constants. In fact, the answer is no. Examine the four beta functions

\[ \beta_\eta = \frac{1}{2} \eta(-1 + \gamma_x + \gamma_y + \gamma_z) \]
\[ \beta_{\omega_x} = \frac{1}{2} \omega_x(-1 + 3\gamma_x) \]
\[ \beta_{\omega_y} = \frac{1}{2} \omega_y(-1 + 3\gamma_y) \]
\[ \beta_{\omega_z} = \frac{1}{2} \omega_z (-1 + 3\gamma_Z) \]  

where \( \gamma_X, \gamma_Y, \gamma_Z \) are functions of the four couplings. We see that a condition for all four beta functions to vanish simultaneously puts only three conditions on the anomalous dimensions \( \gamma_X, \gamma_Y, \gamma_Z \). Specifically, the conditions are \( \gamma_X(\eta, \omega_x, \omega_y, \omega_z) = \frac{1}{3} \) and similarly for \( \gamma_Y \) and \( \gamma_Z \). Three conditions on four couplings imply that any solutions occur generally on one-dimensional subspaces (and since these couplings are complex, the subspace is one-complex-dimensional in extent.) Since the three anomalous dimensions must be equal on this subspace, the symmetry permuting the three fields is presumably unbroken on it. Let us therefore take \( \omega_x = \omega_y = \omega_z = \omega_0 \) and examine the anomalous dimension \( \gamma_0(\eta, \omega_0) \).

Fig. 19 indicates the renormalization group flow of the couplings. Notice that there is a line of conformal field theories ending at \( \eta = \eta^*, \omega_0 = 0 \) and extending into the \( \eta, \omega_0 \) plane. The line ends at \( \eta = 0, \omega_0 = \omega_*, \) clearly the same \( \omega_* \) as in the \( W = \frac{1}{3} \tilde{g} \Phi^3 \) model (since for \( \eta = 0 \) we have three noninteracting copies of the latter model) The precise location of this line is totally unknown, since we do not know \( \gamma_0(\eta, \omega_0) \); but if \( \omega_* \) and/or \( \eta_* \) exists, then the line must exist also. We can define a new coupling \( \rho(\eta, \omega_0) \) which tells us where we are along this line. This coupling is called an “exactly marginal coupling,” and the operator to which it couples is called an “exactly marginal operator.”

![Figure 19. The (complex) line of fixed points lies at \( \gamma_0(\omega, \eta) = \frac{1}{3} \) and may be parameterized by a single (complex) variable \( \rho \).](image-url)
Exercise: Argue there are no nontrivial fixed points in $d = 4$ in supersymmetric theories with only chiral superfields (and no gauge interactions.) Use the fact that since all chiral superfields are gauge invariant operators, their dimensions are greater than one. Then consider all possible interactions which are relevant at the free fixed point and might drive the theory to a nontrivial fixed point. Note loopholes in your proof.

4. Abelian gauge theories

4.1. The classical theory

Time to turn to gauge theories. Gauge bosons are contained in vector supermultiplets, given by superfields $V$, which are real and contain, in $d = 4$, a real vector potential $A_{\mu}$, a Majorana fermion $\lambda^a$ called a “gaugino”, and a real auxiliary field $D$. In three dimensions, the only change is that the vector potential has one less component, which is made up by the presence of a single real scalar field $\varphi$. All of these are in the adjoint representation of the gauge group. In the case of $U(1)$, which we now turn to, they are all neutral.$^1$

The kinetic terms of the pure $U(1)$ theory are

$$S_{\text{gauge}} = \frac{1}{e^2} \int d^d x \left[ -\frac{1}{4} F_{\mu \nu}^2 - i \bar{\lambda} \partial \lambda + \frac{1}{2} D^2 + \frac{1}{2} (\partial_{\mu} \varphi)^2 \right]$$

The last term is absent in four dimensions. Notice I have normalized all of the fields with a $1/e^2$ out front; this is convenient for many purposes. However, since $e^2$ has mass dimension $4 - d$, this means that I have made the dimension of the gauge field somewhat unusual. In four dimensions, it has dimension 1, like any free bosonic field, but in $d = 3$, the gauge field and scalar also have dimension 1, in contrast to the scalars in the chiral multiplet which were normalized with dimension $\frac{1}{2}$. This choice is arbitrary; but we will see soon why this is physically convenient.

It is instructive to count degrees of freedom. Accounting for gauge invariance but not the equations of motion, the gauge boson has $d - 1$ degrees of freedom, the Majorana fermion 4 real degrees of freedom (we will write them as 2 complex), the auxiliary field has 1 and the scalar field has $4 - d$; thus there are four bosonic and four fermionic degrees of freedom.

$^1$Instead of $V$, it is often convenient to use $W_{\alpha}$, a superfield containing the gaugino $\lambda$, the field strength $F^{\mu \nu}$, the auxiliary field $D$, and (in $d = 3$) $\varphi$. This object transforms homogeneously under gauge transformations (and is gauge invariant in the abelian case.) There is yet another useful superfield in $d = 3$ but I’ll skip that here.
After the equations of motion, the gauge boson has $d - 2$, the fermion $2$, and the scalar $4 - d$, so there are two bosonic and two fermionic propagating degrees of freedom.

What we have defined above is the vector supermultiplet of a theory with four supersymmetry generators. The chiral multiplet also comes from such a theory. The fact that the Majorana fermion has four real degrees of freedom is related to the number of supersymmetry generators. Confusingly, this much supersymmetry is known as $\mathcal{N} = 1$ in $d = 4$ and $\mathcal{N} = 2$ in $d = 3$. This is because in four dimensions there is one gaugino, while in $d = 3$ the gaugino defined above is actually a reducible spinor, representing two copies of the smallest possible spinor.

### 4.2. Extended supersymmetry

There are other supersymmetries, each with their own vector and matter multiplets. A theory with *eight* supersymmetry generators has two Majorana spinors in $d = 4$ and four of the smallest spinors in $d = 3$; it is therefore called $\mathcal{N} = 2$ in $d = 4$ and $\mathcal{N} = 4$ in $d = 3$. Its vector multiplet contains one vector multiplet plus one chiral multiplet (both in the adjoint) from the case of four generators. Altogether it contains a gauge boson $A_\mu$, two Majorana fermions $\lambda, \psi$, a complex scalar $\Phi$, and three real auxiliary fields $D, \text{Re} F, \text{Im} F$, as well as (in three dimensions only) a real scalar $\phi$. (With this much supersymmetry there is another multiplet, called a hypermultiplet, consisting of two chiral multiplets of opposite charge under the gauge symmetry; more on this below.) A theory with 16 supersymmetry generators — the maximum allowed without introducing gravity — has only vector multiplets, each of which contains one vector multiplet and three chiral multiplets of the 4-generator case (*i.e.*, one vector multiplet and one hypermultiplet of the 8-generator case,) all in the adjoint representation. It is called $\mathcal{N} = 4$ in $d = 4$ and $\mathcal{N} = 8$ in $d = 3$. Finally, in $d = 3$ there are some cases with 2, 6 and 12 supersymmetry generators, which we will not have time to discuss.

In these lectures we will use the language of $d = 4 \mathcal{N} = 1$ (which is almost the same as $d = 3 \mathcal{N} = 2$) even to describe the other cases. This is common practise, since there is little convenient superfield notation for more than four supersymmetry generators.
34

| Number of SUSY generators | $d = 3$ | $d = 4$ |
|---------------------------|--------|--------|
| 4                        | $\mathcal{N} = 2$ | $\mathcal{N} = 1$ |
| 8                        | $\mathcal{N} = 4$ | $\mathcal{N} = 2$ |
| 16                       | $\mathcal{N} = 8$ | $\mathcal{N} = 4$ |

(21)

4.3. The gauge kinetic function

Before proceeding further it is important to mention the $\theta$ angle in $d = 4$ gauge theories. In the $d = 4$ action, we should also include a term

$$
\int d^4x \frac{\theta}{32\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu} = \int d^4x \frac{\theta}{32\pi^2} F^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma}
$$

a term to which electrically charged objects are insensitive but which strongly affects magnetically charged objects. In fact we should define a generalized holomorphic gauge coupling

$$
\tau \equiv \frac{1}{2\pi} \left[ \theta + \frac{8\pi^2}{e^2} \right].
$$

(22)

Then we may write the action as

$$
S_{gauge} = \frac{i\tau}{8\pi} \int d^4x \left[ \frac{1}{4} (F^2 + iF\tilde{F}) + i\tilde{\lambda}\bar{\phi}\lambda - \frac{1}{2} D^2 \right] + \text{hermitean conjugate.}
$$

Even this is not sufficiently general. Consider, for example, adding a neutral chiral multiplet $\Phi$ to a theory with a $U(1)$ vector multiplet $V$. The complex scalar $\phi$ can have an expectation value. In principle, just as the low-energy QED coupling in nature depends on the Higgs expectation value through radiative effects, the gauge coupling for $V$ could depend functionally on $\langle \phi \rangle$. In other words, we could write a theory of the form

$$
\frac{i}{8\pi} \int d^4x \tau(\phi) \left[ \frac{1}{4} (F^2 + iF\tilde{F}) + i\tilde{\lambda}\phi\lambda - \frac{1}{2} D^2 \right] + \text{hermitean conjugate.} + \cdots
$$

(23)

where the dots indicate the presence of many other terms required by supersymmetry, which I will neglect here. Since $\tau$ is a holomorphic quantity, it must be a holomorphic function of the chiral superfield $\Phi$. We will refer to this new holomorphic function as the “gauge kinetic function.” Thus, to define our gauge theory, we need to specify at least a superpotential, a gauge

\footnote{We will not discuss the dimensional reduction of this object to three dimensions.}
kinetic function, and a Kähler potential; the first two are holomorphic, and
the latter is real.

We can modify the theory of $V$ and $\Phi$ to have $d=4$ $\mathcal{N}=2$ invariance.
To do this, we must make sure that the Kähler potential $K(\Phi, \Phi^\dagger)$ and
the gauge kinetic function $\tau(\Phi)$ are related such that the two gauginos of
the $\mathcal{N}=2$ vector multiplet (one of which, in the above notation, is in the
$\mathcal{N}=1$ vector multiplet, while the other is in the multiplet $\Phi$) have the
same kinetic term. The simplest theory with $\mathcal{N}=2$ has $\tau$ a constant and
$K = (1/g^2)\Phi^\dagger \Phi$. There is no superpotential in this theory; the moduli
space is simply the complex $\phi$ plane.

**Exercise:** Derive the above-mentioned condition!

The $\mathcal{N}=4$ $U(1)$ gauge theory in four dimensions has three complex
scalars $\phi_i$, $i=1,2,3$, from its three $\mathcal{N}=1$ chiral multiplets, and no superpotential.
This means it has a moduli space which is simply six dimensional unconstrained flat space, with an $SO(6)$ symmetry rotating the six
real scalars into each other. This is an R-symmetry, since the four fermions of $\mathcal{N}=4$ are spinors of $SO(6)$, while the vector boson is a singlet and the
scalars are in the $6$ representation.

What about in three dimensions? The $\mathcal{N}=2$ vector multiplet has
a single real scalar, so the classical moduli space is simply the real line.
Similarly, the $\mathcal{N}=4$ vector multiplet has three real scalars, and the $\mathcal{N}=8$
vector multiplet has seven. But quantum mechanically this will not be the
whole story. To see why, we must discuss duality.

### 4.4. Dualities in three and four dimensions

The pure Maxwell theory in $d=4$ has a famous symmetry between its
electric and magnetic fields. One may phrase this as follows: given physical electric and magnetic fields $E$ and $B$, one may find a gauge potential $A_\mu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and one may also find a potential $C_\mu$ with $F_{\mu\nu} = \epsilon_{\mu\nu}^{\rho\sigma} (\partial_\rho C_\sigma - \partial_\sigma C_\rho)$. The electric fields of one potential are the magnetic fields of the other. Notice both gauge potentials have separate $U(1)$
gauge invariances, $A_\mu \to A_\mu + \partial_\mu \varphi$ and $C_\mu \to C_\mu + \partial_\mu \chi$; both invariances
are unphysical, since the physical fields $E$ and $B$ are unaffected by them.
Always remember that gauge symmetries are not physical symmetries; they are *redundancies* introduced only when we simply our calculations by re-
placing the physical $E$ and $B$ by the partly unphysical potential $A_\mu$ (or $C_\mu$).
The two gauge invariances simply remove the unphysical longitudinal parts of $A$ and $C$. “Electric-magnetic” duality exchanges the electric currents (to which $A$ couples simply) with magnetic currents (to which $C$ couples simply) and as such exchanges electrically charged particles with magnetic monopoles. We know from Dirac that the charge of a monopole is $2\pi/e$, so this transformation must exchange $e^2$ with $4\pi^2/e^2$ — weak coupling with strong coupling — and more generally $\tau \rightarrow -1/\tau$. (In modern parlance, this kind of duality, whose quantum version is discussed in more detail in my Trieste 2001 lectures, is often called an S-duality.)

In $d = 3$ there is also an electric-magnetic duality, but it does not exchange a gauge potential with another gauge potential. $E$ and $B$ are both three-vectors in $d = 4$, but in $d = 3$ the electric field is a two-vector and $B$ is a spatial scalar. We may exchange the two-vector with the gradient of a scalar $\sigma$, and $B$ with its time derivative; thus $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \epsilon_{\mu\nu\rho}V^\rho$ in three dimensions. Now $\sigma \rightarrow \sigma + \epsilon$ is a global symmetry. But the expectation value of the scalar $\sigma$ represents a new degree of freedom, one which spontaneously breaks this symmetry. The Goldstone boson of this breaking, $\partial \sigma$, is just the photon that we started with. It can be shown that $\sigma$ is periodic and takes values only between 0 and $2\pi e^2$. Thus, when determining the moduli space of the pure $d = 3$ $\mathcal{N} = 2 U(1)$ gauge theory, we need to include not only the scalar $\varphi$ but also $\sigma$, and in particular these two combine as $\varphi + i\sigma$ into a complex field $\Sigma$. Since $\sigma$ is periodic, the moduli space of the theory — the allowed values for $\langle \Sigma \rangle$ — is a cylinder, shown in Fig. 20. (Similarly, the $\mathcal{N} = 4$ theory in $d = 3$ actually has a four-dimensional moduli space, while the $\mathcal{N} = 8$ theory has an eight-dimensional moduli space.) Note the cylinder becomes the entire complex plane in the limit $e \rightarrow \infty$, which in $d = 3$ (since $e^2$ has mass dimension 1) is equivalent to the far infrared limit; thus the theory acquires an accidental $SO(2)$ symmetry, as in figure Fig. 20. Similarly, the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ cases have $SO(3)$ enhanced to $SO(4)$ and $SO(7)$ enhanced to $SO(8)$ in the infrared.$^b$



$^b$There is one more duality transformation in $d = 3$ that does exchange one gauge potential with another. Identify the gauge field strength as the dual of another gauge field $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \epsilon_{\mu\nu\rho}V^\rho$ (which looks gauge non-invariant, but read on.) The equation of motion $\partial_{\mu}F^{\mu\nu} = J^{\nu}_e$, where $J_e$ is the conserved electric current, tells us that $J^{\nu}_e = \epsilon^{\mu\nu\rho}F_{\rho\nu}$ (and conservation of $J_e$ is the Bianchi identity for $F^{\nu}_\rho$.) An electric charge (nonzero $J^0_e$) corresponds to localized nonzero magnetic field $F^{\nu}_2$ for $V$. An object carrying nonzero magnetic field is a magnetic vortex. Thus unlike electric-magnetic duality, which exchanges electrically charged particles and magnetic monopoles, which are particles in 3+1 dimensions, this duality transformation exchanges electrically

I have discussed these duality transformations in the context of pure $U(1)$ gauge theories. However, it is easy to extend them to the supersymmetric case, since all the superpartners in an abelian vector multiplet are gauge-neutral. I will do so as we need them.

### 4.5. Classical $d = 4$, $N = 1$ SQED

Now we are ready to add matter to the theory. Let us add $N_f$ chiral multiplets $Q_r$ of charge $k_r$ and $N_f \tilde{Q}^s$ of charge $-\tilde{k}_s$. The superpotential $W(Q_r, \tilde{Q}^s)$ and the gauge kinetic function $\tau(Q_r, \tilde{Q}^s)$ must be holomorphic functions of gauge-invariant combinations of the chiral multiplets. For now, classically, we will take $W = 0$ and $\tau = 4\pi i / e^2$. The kinetic terms for the charged fields are modified by the gauge interactions. Taking a canonical Kähler potential for simplicity, we have

$$S_{kin} = \sum_r \int d^4x \left[ D_\mu q_r^\dagger D^\mu q_r + i \bar{\psi}_r \slashed{D} \psi_r + F_r^\dagger F_r + k_r (\lambda \psi_r q_r^\dagger + \bar{\lambda} \bar{\psi}_r q_r + q_r^\dagger D q_r) \right]$$

$$+ \sum_s \int d^4x \left[ D_\mu \tilde{q}^s \dagger D^\mu \tilde{q}^s + i \bar{\tilde{\psi}}^s \slashed{D} \tilde{\psi}^s + \tilde{F}^s \dagger \tilde{F}^s - \tilde{k}_s (\lambda \psi^s q^s \dagger + \bar{\lambda} \bar{\psi}^s \tilde{q}^s + \tilde{q}^s \dagger D\tilde{q}^s) \right]$$

(24)

Here $Q_r$ contains the superfields $q_r, \psi_r, F_r$ and similarly for $\tilde{Q}^s$; the covariant derivative is $D_\mu = \partial_\mu + ikA_\mu$ acting on a particle of charge $k$ (remember that the coupling $e$ appears not here but in the kinetic term of the gauge boson); and $D$ with no index is the auxiliary field in the vector multiplet.

charged particles and magnetic vortices, which are particles in $2 + 1$ dimensions. For this reason we will call it “particle-vortex” duality; see Intriligator and Seiberg (1996).
In the special case where all the $k_r = k_s = 1$, then the $Q$'s are rotated by an $U(N_f)$ global symmetry and the $\tilde{Q}$'s are rotated by a different $U(N_f)$ symmetry; let us call then $U(N_f)_L$ and $U(N_f)_R$ in analogy with terminology in QCD. The diagonal $U(1)_V$ which rotates $Q$ and $\tilde{Q}$ oppositely is the symmetry which is gauged (and so is a redundancy, not a true symmetry.) The gauge-invariant chiral operators take the form $M^r_s = Q_r \tilde{Q}_s$; all other gauge-invariant combinations of chiral superfields reduce to products of the $M^r_s$ fields.

Since there is a term $D^2$ in the vector multiplet kinetic terms (23), we see that there will be a new contribution to the potential energy of the theory. In particular, for a canonical Kähler potential and $\tau$ a constant the potential will be

$$V(q_r, \tilde{q}_s) = \frac{1}{2g^2} D^2 + \sum_r |F_r|^2 + \sum_s |\tilde{F}_s|^2$$

where the equation of motion for $D$ reads

$$D = \sum_r k_r |q_r|^2 - \sum_s k_s |\tilde{q}_s|^2$$

Again, a supersymmetric vacuum must have $V = 0$, and therefore all the auxiliary fields must separately vanish.

The condition $D = 0$ is very special. Let us consider first the simplest possible case, namely $N_f = 1$, with $Q$ and $\tilde{Q}$ having charge 1 and -1. Although there are two complex fields, with four degrees of freedom, the gauge symmetry removes one of these, since we may use it to give $q$ and $\tilde{q}$ the same phase. The real condition $D = |q|^2 - |\tilde{q}|^2 = 0$ removes one more degree of freedom and ensures that both $q$ and $\tilde{q}$ have the same magnitude. In fact, it acts as though the gauge invariance of the theory were complexified! at least as far as the moduli space of the theory is concerned. The moduli space is then given by one complex parameter $v = \langle q \rangle = \langle \tilde{q} \rangle$, which we may also write in gauge invariant form as $v^2 = \langle M \rangle = \langle Q \tilde{Q} \rangle$. In short, the moduli space is simply the complex $M$ plane. We began with two chiral multiplets; only one is needed to describe the moduli space.

Why did one chiral multiplet of freedom have to disappear? Well, when $M$ is nonzero, the gauge group is broken, and as we know very well, the

---

1 We have assumed so far that the superpotential is zero. A superpotential $W = mQ\tilde{Q}$ simply gives the chiral multiplets masses, leaving only the massless vector multiplet and its unique vacuum at $q = \tilde{q} = 0$. If we add a superpotential $W = y(Q\tilde{Q})^2$, the resulting potential again has a vacuum only at $q = \tilde{q} = 0$, but $Q$ and $\tilde{Q}$ are massless there.
photon is massive. But a massive gauge boson has to absorb a scalar field to generate its third polarization state, as we know from the electroweak Higgs mechanism. However, in supersymmetry it must be that a vector multiplet must absorb an entire chiral multiplet; otherwise there would be partial multiplets left over, which would violate supersymmetry. One massive photon means that one of the two massless chiral multiplets has paired up with the massless vector multiplet; the remaining fields form the massless and neutral chiral multiplet $M$.

How about $N_f = 2$, with $Q_1, Q_2$ of charge 1 and $\check{Q}_1, \check{Q}_2$ of charge $-1$, and no superpotential? In this case the condition $D = |q_1|^2 + |q_2|^2 - |\check{q}_1|^2 - |\check{q}_2|^2 = 0$ (combined with gauge invariance) leaves three massless chiral multiplets. It turns out that the solution to this equation is $M_1^1 M_2^2 = M_2^1 M_1^2$. Thus the four gauge invariant operators $M^s_r$, subject to the constraint $\det M = 0$, give us the three-complex-dimensional moduli space.

**Exercise:** Verify that $\det M = 0$ is the solution to the above equation. Hint: use the $SU(2) \times SU(2)$ flavor symmetry to rotate the vevs into a convenient form.

**Exercise:** Verify that for $N_f > 2$ the D-term constraints imply that the gauge-invariant operators $M^s_r$, subject to the constraint that $M$ be a matrix of rank zero or one, parameterize the moduli space.

### 4.6. $\mathcal{N} = 2 \, d = 4$ SQED

Now let us slightly complicate the story by considering the $\mathcal{N} = 2 \, d = 4$ gauge theory. The $\mathcal{N} = 2$ vector multiplet has an extra chiral multiplet $\Phi$. The fields $Q_r$ and $\check{Q}^r$ can be organized into $N_f$ hypermultiplets in which the indices $r$ and $s$ should now be identified. The global $U(N_f) \times U(N_f)$ symmetry will now be reduced to a single $U(N_f)$, because of the superpotential

$$W(\Phi, Q_r, \check{Q}^r) = \sqrt{2} \Phi \sum_r Q_r \check{Q}^r$$

required by the $\mathcal{N} = 2$ invariance. In normalizing the superpotential this way, I have assumed that the kinetic terms for $\Phi$ are normalized

$$K = \frac{1}{e^2} \Phi^4$$
to agree with the normalization of the kinetic terms of the $\mathcal{N} = 1$ vector multiplet $V$. Sometimes it is more convenient to normalize $\Phi$ canonically; then a factor of the gauge coupling $e$ appears in front of the superpotential.

Let us begin with the case $N_f = 1$. Now we have several conditions:

\begin{align*}
D &= |Q|^2 - |\tilde{Q}|^2 = 0; \quad F_\Phi^\dagger = Q\tilde{Q} = 0; \quad F^\dagger = \Phi Q = 0; \quad \tilde{F}^\dagger = \Phi \tilde{Q} = 0,
\end{align*}

which clearly have no solution with nonzero $Q$ and/or $\tilde{Q}$. In fact, in the language of the operator $M = Q\tilde{Q}$, which we know satisfies the D-term conditions, the $F_\Phi$ equation explicitly says $M = 0$, while $Q\tilde{F}^\dagger$ gives $\Phi M = 0$. This allows any nonzero $\Phi$. When only $\Phi$ is nonzero, the gauge group is unbroken, so the photon is massless and the electric potential of a point charge is $1/r$ at large $r$. For this reason, this branch of moduli space is called the “Coulomb branch.”

Does this structure make sense? Suppose $Q$ and $\tilde{Q}$ were nonzero, so that the Higgs mechanism were operative; would this be consistent? As before, were the vector multiplet to become massive it would have to absorb an entire charged multiplet, which in this case would have to be the entire hypermultiplet. This would leave no massless fields to serve as moduli. Therefore this theory cannot have a branch of moduli space on which the photon is massive. By contrast, the vector multiplet scalars $\Phi$ can have expectation values without breaking the gauge symmetry at all; instead $\Phi$ simply makes $Q$ and $\tilde{Q}$ massive, while itself remaining massless. Thus the only branch of moduli space in this theory is the Coulomb branch, in the form of the complex $\Phi$ plane, with a special point at the origin where the hypermultiplet is massless.

Now consider $N_f = 2$. We can expect that there will again be no obstruction to having $\langle \Phi \rangle \neq 0$; such an expectation value will make the hypermultiplets massive, preventing them from having expectation values. We can also expect that if the charged scalars do have expectation values, they will make the vector multiplet massive, preventing $\Phi$ from having an expectation value; and since only one hypermultiplet will be eaten by the vector multiplet, there should be an entire hypermultiplet — two chiral superfields — describing the moduli space. Is this true? As in the case of $\mathcal{N} = 1 N_f = 2$ SQED, the D-term conditions are satisfied by using the

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Henceforth we will not distinguish between superfields and their scalar components, since it is generally clear from context which is relevant; also we will generally write $\Phi$ to represent $\langle \Phi \rangle$. 
operators $M^a_r$ with $\det M = 0$. The conditions

$$F^\dagger_{\Phi} = Q_1 \bar{Q}_1 + Q_2 \bar{Q}_2 = 0 ; \ F^\dagger = \Phi Q_r = 0; \  \bar{F}^\dagger_r = \Phi \bar{Q}_r = 0$$

can be rewritten as

$$F^\dagger_{\Phi} = \text{tr} \ M = 0 ; \ Q_s \bar{F}^\dagger_r = \Phi M^a_r = 0$$

which indeed imply that either $\Phi$ or $M$ can be nonzero, but not both simultaneously, and that there are two complex degrees of freedom in $M$ which can be nonzero, making a single neutral hypermultiplet.

![Diagram](image.png)

Figure 21. The classical two-complex-dimensional Higgs branch (H) and the one-complex-dimensional Coulomb branch (C) of $d = 4 \ N = 2 \ U(1)$ with two charged hypermultiplets.

This last example illustrates the branch structure of these theories, indicated schematically in Fig. 21. Either $\Phi$ is nonzero, with the hypermultiplets massive and the gauge group unbroken (the Coulomb phase), or $M$ is nonzero, with the gauge group broken (the Higgs phase) and only some massless neutral hypermultiplets remaining. In this case the Coulomb branch C has complex dimension 1 while the “Higgs branch” H has complex dimension 2 (in fact quaternionic dimension 1.) The two branches meet at the point where all the fields are massless.

**Exercise:** Show that for $N_f > 2 \ N = 2 \ U(1)$ gauge theory, this Higgs and Coulomb branch structure continues to be found, with the quaternionic dimension of the Higgs branch being $N_f - 1$. 
4.7. Classical $d = 3 \mathcal{N} = 2, 4$ SQED

In three dimensions, the physics is slightly more elaborate, because even the $\mathcal{N} = 2$ multiplet has a real scalar field $\varphi$. This means that even for $\mathcal{N} = 2$ $N_f = 1$ $U(1)$ gauge theory with no superpotential, there is a Coulomb branch with $\langle \varphi \rangle$ nonzero, in addition to the Higgs branch with nonzero $\langle M \rangle = \langle Q\tilde{Q} \rangle$. This is in contrast to the $d = 4 \mathcal{N} = 1$ $N_f = 1$ $U(1)$ gauge theory, which has only a Higgs branch. The Coulomb branch of the $d = 3 \mathcal{N} = 2$ theory is similar, classically, to the one we encountered in Sec. 4.6 when we considered the pure $d = 4 \mathcal{N} = 2$ abelian gauge theory.

The details of the moduli space are controlled partly by a new term in the Lagrangian

$$\varphi^2(|Q|^2 + |\tilde{Q}|^2).$$

(25)

The existence of this term can be inferred from the $d = 4 \mathcal{N} = 1$ gauge theory as follows. When we go from four dimension to three, the component of the photon $A_3$ becomes the scalar $\varphi$. All derivatives $\partial_3$ are to be discarded in this dimensional reduction, but covariant derivatives $D_3 = \partial_3 + iA_3$ become $i\varphi$. The interaction (25) is simply the dimensional reduction of the $d = 4$ kinetic term $|D_3Q|^2 + |D_3\tilde{Q}|^2$. For (25) to be zero, it must always be that either $\varphi = 0$ or $Q = \tilde{Q} = 0$; this gives two branches. Note that the charged chiral multiplets are massive on the Coulomb branch, as usual, because of this quartic potential. The classical moduli space is shown in Fig. 22.

![Figure 22](image-url)

Figure 22. The classical one-complex-dimensional Higgs branch and the one-real-dimensional Coulomb branch of $d = 3 \mathcal{N} = 2$ $U(1)$ with $N_f = 1$. 

What about the case of $\mathcal{N} = 4$ supersymmetry? Here the $N_f = 1$ case is not so confusing, because there is no Higgs branch; in $d = 4$ $\mathcal{N} = 2$ there was a one-complex-dimensional Coulomb branch, whereas here there should be (classically!) a three-real-dimensional Coulomb branch, made from the scalars $\varphi, \text{Re} \phi, \text{Im} \phi$. There is an $SO(3)$ symmetry acting on the scalars (which, as usual for theories with more than four supercharges, is an extended R-symmetry.)

4.8. Quantum SQED in $d = 4$

To go further, we have to do some quantum mechanics. Let’s begin with perturbation theory.

In four dimensions, perturbation theory is familiar. Just as electrons generate a positive logarithmic running for the electromagnetic coupling, via the one-loop graph above, so do scalar charged particles; and the combination of $N_f$ chiral multiplets $Q_r$ of charge 1 and $N_f \tilde{Q}_s$ of charge $-1$ gives a one-loop beta function

$$\beta_e = \frac{e^3}{16\pi^2} N_f$$

It is often convenient to write formulas not for $e$ but for $1/e^2$:

$$\beta_e = \frac{N_f}{8\pi^2} \Rightarrow \frac{1}{e^2(\mu)} = \frac{1}{e^2(\mu_0)} + \frac{N_f}{8\pi^2} \ln \left( \frac{\mu_0}{\mu} \right)$$

so $e$ shrinks as we head toward the infrared. This means that $e$ becomes large in the ultraviolet, which means that perturbation theory breaks down there, making it difficult to define the theory. We can avoid this problem by defining the theory with some additional Pauli-Villars regulator fields, $N_f$ ghost chiral superfields of charge 1 and $N_f$ of charge -1, all of mass $M$. In this case there are no charged fields above the scale $M$, so $\beta_e = 0$; thus for $\mu > M$ the gauge coupling is a constant $e_0$, and

$$\frac{1}{e^2(\mu)} = \frac{1}{e_0^2} + \frac{N_f}{8\pi^2} \ln \left( \frac{M}{\mu} \right)$$

for $\mu < M$. (Remember this is only accurate at one-loop, so it only makes sense if $e_0 \ll 1$) If the fields $Q$ and $\tilde{Q}$ have masses $m$ then (as in real-world QED) the gauge coupling will stop running at the scale $m$, as illustrated in Fig. 23.

Now, the scale $M$ was just put in to regulate the theory, while $e^2(\mu)$ is physical for low $\mu$ and should not depend on $M$. We therefore should define a physical scale $\Lambda$ by taking it to be the value of $M$ where the one-loop
Figure 23. The coupling runs (approximately) logarithmically below $M$ and above $m$.

coupling $e_0$ is formally infinite (though remember it will differ from the real coupling at that scale due to higher-loop effects)

$$\frac{1}{e^2(\mu)} = \frac{N_f}{8\pi^2} \ln \left( \frac{\Lambda}{\mu} \right)$$

where

$$\Lambda^{N_f} = \mu^{N_f}e^{8\pi^2/e^2(\mu)}$$

I’ve been careless here: $m$ and $M$ are complex parameters (while $\mu$ is real$^k$) so $\Lambda$ should be complex also; but $e^2$ is real. How can we make the above expressions sensible? Clearly, we should introduce the $\theta$ angle, and rewrite the previous equation in its final form as

$$-2\pi i \tau(\mu) = \frac{8\pi^2}{e^2(\mu)} - i \theta = N_f \ln \left( \frac{\Lambda}{\mu} \right)$$

which is a one-loop formula that is only sensible for $\mu \ll \Lambda$.

Note that $\tau$ is a holomorphic coupling constant. Let us verify that in perturbation theory this one-loop formula is exact! In perturbation theory the expression for $\tau$ must be a perturbative series in $e^2 \propto 1/\text{Im } \tau$ and cannot contain $\theta \propto \text{Re } \tau$; but that’s impossible if it is to be a holomorphic expression. The only term which can appear in quantum corrections to $\tau$ must then be $\tau$-independent, namely the one we see above.

$^k$You can make it complex, actually — this is itself interesting but beyond what I can cover here.
But as before, the fact that we have found a simple formula for a holomorphic quantity by no means indicates that the physical coupling is so simple. The physical coupling gets corrections from higher loop effects (caution: as we will see, these are cancelled for theories with eight or more supersymmetry generators). As before, these can only occur in the non-holomorphic part of the theory: the \( \text{Kähler potential} \). As we did for the coupling \( y \) in Sec. 3.1, we should find a definition of the coupling constant which is independent of field redefinitions. We’ll come back to this point soon.

Let’s first examine \( d = 4 \) \( \mathcal{N} = 2 \) \( U(1) \) gauge theory with \( N_f \) massless hypermultiplets. We’ve already discussed its branch structure in Sec. 4.6; there is a Higgs branch on which some of the gauge-invariant operators \( M^I = Q_i \tilde{Q}^I \) act as massless fields, with the others massive. There is also a Coulomb branch on which \( \Phi \) has an expectation value and the term \( \sqrt{2\Phi} Q_i \tilde{Q}_j \) in the superpotential gives the charged fields masses. The branch structure for \( N_f > 1 \) massless hypermultiplets was shown in Fig. 21. On the Coulomb branch, we can ask an interesting physical question: how does the infrared limit

\[
\tau_L \equiv \lim_{\mu \to 0} \tau(\mu)
\]

(26)

of the coupling constant \( \tau \) depend on \( \Phi \)? Since (1) the theory has a \( \Phi \)-independent value of \( \Lambda \), and (2) for any value of \( \Phi \), the coupling constant stops running at the scale \( \Phi \) at which the charged fields are massive,

\[
-2\pi i \tau_L = N_f \ln \left( \frac{\Lambda}{\Phi} \right) \quad [\Phi \ll \Lambda].
\]

This is singular only at \( \Phi = 0 \), where the Higgs and Coulomb branches meet and the charged fields are massless. (Recall that charged massless fields always drive the electric coupling \( e \) to zero, and thus \( \tau \to i\infty \), in the infrared.) Note we cannot take \( |\Phi| \) to be larger than \( |\Lambda| \), since our description of the theory is not reliable there. The behavior of \( \tau_L \) on the moduli space is sketched in Fig. 24.

**Exercise:** For \( N_f = 4 \), if two hypermultiplets have mass \( m \) and two have mass \( m' \), show that

\[
-2\pi i \tau_L = 2 \ln \left( \frac{\Lambda}{\Phi + m} \right) + 2 \ln \left( \frac{\Lambda}{\Phi + m'} \right)
\]

so that there are two singular points; at each singular point there are two massless hypermultiplets, which have a Higgs branch intersecting the
Figure 24. The quantum version of Fig. 21; the low-energy coupling $e (\mu \to 0)$ grows from 0 at the origin to $\infty$ at the dashed line, where the description of the theory breaks down.

Coulomb branch at that point. If each hypermultiplet has its own mass, then show that there are four singular points but no Higgs branches anywhere.

What does this function $\tau_L$ really tell us? Away from the singular points, for any value of $\Phi$, the charged fields are all massive, and there is simply a pure $U(1)$ $\mathcal{N} = 2$ gauge theory in the infrared. Its effective action is of the form (23) with a nontrivial gauge kinetic function $\tau_L (\Phi)$. But like any pure abelian gauge theory, it has duality symmetries. In particular, there is the electric-magnetic transformation $\tau \to -\frac{1}{\tau}$. There is also the obvious symmetry $\tau \to \tau + 1$, which represents a shift by $2\pi$ of the $\theta$ angle. These two symmetry transformations generate a group of duality transformations of the form $SL(2,\mathbb{Z})$, the symmetry group of a torus.

A torus can be defined by taking a parallelogram and identifying opposite sides, as in Fig. 25. Ignore the size of the parallelogram by taking one side to have length 1; then the other size has a length and angle with respect to the first that can be specified by a parameter $\tau$ that lives in the upper-half of the complex plane. (For the gauge theory, the gauge coupling must be positive, so $\text{Im} \, \tau > 0$.) However, exchanging the two sides obviously leaves the torus unchanged ($\tau \to -\frac{1}{\tau}$) as does shifting one side by a unit of the other side ($\tau \to \tau + 1$) and any combination of these transformations.

One can therefore take the point of view that the low-energy $\mathcal{N} = 2$ $U(1)$ gauge theory should not be specified by $\tau$. In fact, we can see this by
taking $\Phi \rightarrow \Phi e^{2\pi i}$, that is, let us circle the singular point at $\Phi = 0$. The theory obviously must come back to itself, since the physics depends only on $\Phi$; but $\tau_L$ shifts by $N_f$ as we make the circle. Thus $\tau_L$ does not properly characterize the theory; all values of $\tau_L$ related by $SL(2,\mathbb{Z})$ transformations are actually giving the same theory. We should therefore characterize the low-energy theory by specifying a torus! For each value of $\Phi$, there should be a torus with parameter $\tau_L(\Phi)$ which tells us the properties of the low-energy theory. More precisely, this is a fiber bundle, with a torus fibered over the complex $\Phi$ plane, as expressed in Fig. 26. This torus is invariant under $\Phi \rightarrow \Phi e^{2\pi i}$, and becomes singular at $\Phi = 0$.

4.9. Quantum SQED in $d = 3$

Now let's move back to three dimensions. The gauge coupling is now dimensionful, so classically it has a negative beta function. Due to the wonders of gauge symmetry, the diagram in Fig. 27 is ultraviolet finite! But not trivial. In fact, if the fermion is massless, and the momentum flowing through the photon line is $p^\mu$, this graph is proportional to $\frac{1}{\sqrt{p^2}}$!

**Exercise:** Calculate the one-loop correction to ordinary nonsupersymmetric QED in three dimensions for $N_f$ massless electrons.

This means that the one-loop gauge coupling in three dimensions has the form

$$\frac{1}{e^2(\mu)} = \frac{1}{e_0^2} + c_3 \frac{N_f}{\mu}$$
Figure 26. The low-energy gauge coupling and its dualities are best understood using a torus fibered over the Coulomb branch; at the origin, where $\tau \to i\infty$, the torus degenerates.

Figure 27. This one-loop graph is finite in three dimensions.

where $c_3$ is a positive constant, of order one, which depends on the specific theory. Notice that there is no divergence in $e$ as $\mu \to \infty$; $e$ goes to a constant $e_0$ in the ultraviolet, so in $d = 3$ supersymmetric QED is well-defined in the ultraviolet.

As always, to define a beta function we should employ a dimensionless coupling

$$\zeta \equiv \frac{c^2(\mu)}{\mu} = \left( \frac{\mu}{e_0^2 + c_3 N_f} \right)^{-1}$$

which is infinite for large $\mu$ but — interestingly — goes (at one loop) to a
constant at small $\mu$. In other words,

$$
\beta_\zeta = \frac{\mu}{\epsilon_0} \left( \frac{\mu}{\epsilon_0^2} + c_3 N_f \right)^{-2} = \zeta(1 - c_3 N_f \zeta)
$$

so there is a fixed point in the one-loop formula at $\zeta_* = 1/c_3 N_f$. This is illustrated in Fig. 28 and Fig. 29.

Figure 28. $\zeta$ as a function of scale $\mu$; the dashed line shows its classical flow.

Classical
\[ \zeta = 0 \quad \text{---} \quad \zeta = \infty \]

Quantum
\[ \zeta = 0 \quad \text{---} \quad \zeta_* \quad \text{---} \quad \zeta = \infty \]

Figure 29. The coupling $\xi$ has a quantum fixed point.

This is very interesting. Remembering that perturbation theory is an expansion in the parameter $e^2/\mu = \zeta$, we see that the one-loop formula has a fixed point at weak coupling if $N_f$ is large. If this is true, then for large $N_f$ two-loop effects such as those in Fig. 30 are always suppressed by factors of $1/N_f$ and can be neglected. Thus the large-$N_f$ behavior of the
theory is indeed given by the one-loop formula, with a fixed point at small \( \zeta \) and all diagrams calculable. The theory is soluble!

\[ \gamma \quad \gamma \quad \gamma \quad \gamma \]

Figure 30. In \( d = 3 \) \( N_f \gg 1 \) (S)QED the one-loop correction \( e^2 N_f \) dominates the propagator; the higher-loop corrections are suppressed by extra powers of \( e^2 \propto 1/N_f \), and can be dropped.

**Exercise:** Show nonsupersymmetric QED in \( d = 3 \) is soluble and has a conformal fixed point at large \( N_f \).

In nonsupersymmetric QED, it is believed that there is a value of \( N_f \) below which the fixed point disappears and other nonperturbative phenomena take place. In \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) QED, however, the fixed point visible at one loop survives for all \( N_f \). In fact, in the latter case, there are no higher-loop corrections to the gauge coupling, so the above beta function is exact and the fixed point at small \( N_f \) is completely reliable.

Let us examine the \( N_f = 1 \) case in both \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) supersymmetry. Both of these theories are truly remarkable. As we noted, for \( d = 3 \) a vector boson has an electric-magnetic dual pseudoscalar \( \sigma \). In \( \mathcal{N} = 2 \), this scalar combines with the scalar \( \varphi \) to make a complex field \( \Sigma = \varphi + i\sigma \). The scalar \( \sigma \) is compact, with radius \( 2\pi e^2 \), so classically the field \( \Sigma \) takes values on a cylinder of radius \( e^2 \). As \( e^2 \to 0 \) the field \( \sigma \) disappears and the cylinder becomes the \( \varphi \) line; when \( e^2 \to \infty \) the cylinder expands to become the entire plane, as we saw in Fig. 20.

But in the presence of charged matter, \( e^2 \) is not so simple. In particular, the terms

\[ \varphi^2(|Q|^2 + |\bar{Q}|^2) + \varphi(\bar{\psi}\psi - \bar{\psi}\bar{\psi}) \]

imply that the charged matter has mass \( \varphi \). Since \( e^2(\mu) \) stops running below this scale, the low energy value \( e_L^2 \) of the gauge coupling is

\[ \frac{1}{e_L^2} = \frac{1}{e_0^2} + \frac{1}{\varphi} \]

For very large \( |\varphi| \) we have \( e_L \approx e_0 \), so the radius of the cylinder on which \( \Sigma \) lives is of order \( e_0^2 \) far from \( \varphi = 0 \). However, for very small \( \varphi \) the \( 1/e_0^2 \) term
can be neglected and $e^{2}\sim \phi$; thus the cylinder shrinks in size. At $\phi = 0$ — where the Higgs branch meets the Coulomb branch — the cylinder shrinks to zero radius. Thus, the moduli space has the form of Fig. 31.

Figure 31. The quantum version of Fig. 22, which combines it with Fig. 20.

This picture should look familiar. At the meeting point of the three branches, there is a conformal field theory which looks remarkably like the $W = hXYZ$ conformal fixed point that we considered earlier (Fig. 3 and Sec. 3.4). And in fact, it is the same! In the XYZ model, the three branches had nonzero expectation values for $X, Y$ and $Z$ respectively. Here, the branches are the three complex planes labeled by the expectation values for $M = \bar{Q}\bar{Q}, e^{\Sigma},$ and $e^{-\Sigma}$. Thus we have another example of “duality”: a single conformal fixed point is the infrared physics of two different field theories, one the $W = hXYZ$ model, the other $\mathcal{N} = 2$ super-QED. The theories are different in the ultraviolet but gradually approach each other, becoming identical in the infrared, as shown in Fig. 32. This is called an “infrared” duality. Notice the $Z_3$ symmetry between the branches is exact in the XYZ model but is an “quantum accidental” symmetry (a property only of the infrared physics) in super-QED.

**Exercise:** Calculate the anomalous dimension of $Q$. Note the sign! Why is it allowed here?

This duality is a particle-vortex duality. Along the Higgs branch, where $\langle M \rangle \neq 0$, there are vortex solitons of finite mass; these are similar to the
vortices discussed in Sec. 2.6. The phases of the fields $Q$ and $\tilde{Q}$ wind once around the circle at infinity; however, the presence of the gauge field cuts off the logarithmically divergent energy that was a feature we dwelt on in Sec. 2.6. (You can read about how this works in Nielsen and Olesen (1973).) These solitons correspond to the fields $Y$ and $Z$, which have (finite) mass when $\langle X \rangle \neq 0$. Thus the vortices of the one theory correspond to the particles of the other.

**Exercise:** Since $X$ and $M = Q\tilde{Q}$ are to be identified, a mass term $W = mQQ$ should correspond to changing the dual theory to $W = hXYZ + mX$. The massive fields $Q, \tilde{Q}$ are logarithmically confined (as always for weakly-coupled electrically-charged particles in $d = 3$) by the light photon which remains massless. Look back at section 1.6, where we showed there are vortex solitons in the dual $W = hXYZ + mX$ theory which are logarithmically confined, and argue that it is consistent to identify $Q$ and $\tilde{Q}$ with these solitons. Using the relation between the gauge field and $\sigma$, try to show that the electric field surrounding the electrons $Q, \tilde{Q}$ corresponds to a variation in $\Sigma$ which agrees with the properties of $Y$ and $Z$ near these solitons.

Now let us examine the theory with $\mathcal{N} = 4$ supersymmetry, which is even more amazing. We can obtain it from the $\mathcal{N} = 2$ case by adding a neutral chiral superfield $\Phi$ to the theory and coupling it to the other fields.
via the superpotential $W = \sqrt{2} \Phi Q \bar{Q}$. This will destabilize the $\mathcal{N} = 2$ fixed point and cause it to flow to a new one, as in Fig. 33.

Figure 33. There is a flow linking the two theories.

**Exercise:** Check that the operator $\Phi Q \bar{Q}$ is a relevant operator both at the free $\mathcal{N} = 2$ fixed point and at the infrared $\mathcal{N} = 2$ conformal fixed point.

Since the XYZ model is the same as $\mathcal{N} = 2$ SQED in the infrared, we may obtain the $\mathcal{N} = 4$ theory another way. Let us go to the far infrared of the XYZ model. We just studied what happens when we add a single field $\Phi$ and couple it to $M = Q \bar{Q}$ in the superpotential. But we can simply change variables from SQED to XYZ; from this dual point of view, what we did was couple $\Phi$ to $X$. The low-energy physics of a model with $W = h XYZ + \Phi X$ should be the same as that of $\mathcal{N} = 4$ SQED. But $\Phi X$ is just a mass term which removes $\Phi$ and $X$ from the theory, leaving $Y$ and $Z$, with no superpotential. *Thus the dual description of the $\mathcal{N} = 4$ SQED fixed point is a free theory!*

In short, the low-energy limit of $\mathcal{N} = 4$ SQED is a conformal fixed point which can be rewritten as a free theory — a theory whose massless particles are the vortices of SQED.

From this astonishing observation, a huge number of additional duality transformations of other abelian gauge theories can be obtained. In this sense, it plays a role similar to “bosonization” (boson-fermion duality) in
two dimensions, which can be used to study and solve many field theories. These three-dimensions “mirror” duality transformations, first uncovered by Intriligator and Seiberg and much studied by many other authors, are a simple yet classic example of dualities, and I strongly encourage you to study them. A summary of previous work and a number of new results on this subject appear in work I did with Kapustin (1999).

An important aside: it is essential to realize that we have here an example of a nontrivial exact duality, which is not merely an infrared duality. We noted that the flow from the weakly coupled XYZ to the \( \mathcal{N} = 2 \) fixed point is different from the flow from weakly coupled SQED to the \( \mathcal{N} = 2 \) fixed point. However, at the \( \mathcal{N} = 2 \) fixed point the two flows reach the same theory, and the operators \( \tilde{Q}Q \) and \( X \) are identical there. The relevant perturbations \( \Phi_{\tilde{Q}Q} \) and \( \Phi_X \) may be added with arbitrarily tiny couplings; in this case the two different flows approach and nearly reach the \( \mathcal{N} = 2 \) fixed point, stay there for a long range of energy, and then flow out, together, along the same direction, heading for the \( \mathcal{N} = 4 \) fixed point. This is shown schematically in Fig. 34. In the limit where the \( \mathcal{N} = 2 \) fixed point is reached at arbitrarily high energies, the flow to the \( \mathcal{N} = 4 \) fixed point is described exactly by two different descriptions, one using the XYZ variables, the other using those of SQED. One will sometimes read in the string theory literature that “field theory has infrared dualities, but duality in string theory is exact.” Clearly this is not true; as we have seen in this example, infrared dualities always imply the existence of exact dualities. You can look at my work with Kapustin (1999) for some very explicit examples.

5. Non-Abelian Four-Dimensional Gauge Theory

We now turn to nonabelian gauge theories in four dimensions. This is a huge subject and we shall just scratch the surface, but hopefully this lecture will give you some sense of the immensity of this field and teach you a few of the key ideas you need to read the already existing review articles.

5.1. The classical theory

Let us first consider the classical pure gauge theory. The only difference from the abelian case (aside from some complications in the superfield formalism) is that the kinetic terms reflect the fact that the pure vector multiplet is self-interacting. The gauge group, a Lie group such as \( SU(N) \), is generated by a Lie algebra with generators \( T^A \), \( A \) an index running from
Figure 34. By adjusting couplings and scales we may obtain two exactly equivalent descriptions of the flow in Fig. 33.

1 to the dimension of the group.\(^1\) Gauge bosons \(A_\mu = A_\mu^A T^A\), gauginos \(\lambda = \lambda^A T^A\), and auxiliary fields \(D = D^A T^A\) are all in the adjoint representation, and the kinetic terms are the minimal ones

\[
S_{\text{gauge}} = \frac{i\tau}{4\pi} \int d^4x \text{tr} \left[ -\frac{1}{4} (F^2 + iF i\tilde{F}) + i\lambda \partial \lambda + \frac{1}{2} D^2 \right] + \text{hermitean conjugate.}
\]

where \(\tau\) is again defined in Eq. (22) (with \(e \to g\)) and

\[
F_{\mu\nu} = F_{\mu\nu}^A T^A = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]
\]

\[
= \left( \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f^{ABC} A_\mu^B A_\nu^C \right) T^A ,
\]

\[
D_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha + i[A_\mu, \lambda_\alpha] , \quad \text{and} \quad D^2 \equiv \sum_A |D^A|^2 . \quad \text{We will often choose to represent fields in the adjoint representation using matrices} \ T^A \ \text{that are in the fundamental representation; this is usually the easiest representation}
\]

---

\(^1\)For example, for \(SU(N)\), \(A\) runs from 1 to \(N^2 - 1\). The generators may themselves appear as \(N^2 - 1\) matrices in any representation of the group. If we take \(T^A\) in the fundamental representation, then each \(T^A\) is an \(N \times N\) matrix \((T^A)^i_j\), where \(i\) and \(j\) are indices in the fundamental and antifundamental representation of \(SU(N)\). The matrices are normalized by the condition \(\text{tr}(T^A T^B) = \delta^{AB}\). In the adjoint representation, \((T^A)^j_i\) is the matrix \(f^{ABC} T^C\), the structure constants of the group. In any representation, \([T^A, T^B] = if^{ABC} T^C\).
to work with. In $SU(N)$, doing so allows us to write $A_\mu, \lambda, D$ as $N \times N$ hermitean traceless matrices.

The addition of charged chiral fields involves a fairly minimal change in the kinetic terms from the abelian case. If we add chiral fields $Q_r$ and $\tilde{Q}_s$, $r, s = 1, \ldots, N_f$, in the fundamental and antifundamental representation of $SU(N)$, we obtain

$$S_{\text{kin}} = \sum_r \int d^4x \left[ D_\mu q_r^\dagger D^\mu q_r + i \bar{\psi}_r D \psi_r + F_r^\dagger F_r + \lambda \bar{\psi}_r q_r + \bar{\lambda} \psi_r q_r^\dagger + q_r^\dagger D q_r \right] + \sum_s \int d^4x \left[ D_\mu \tilde{q}_s^\dagger D^\mu \tilde{q}_s + i \bar{\tilde{\psi}}_s D \tilde{\psi}_s + \tilde{F}_s^\dagger \tilde{F}_s + \lambda \bar{\tilde{\psi}}_s \tilde{q}_s + \bar{\lambda} \tilde{\psi}_s \tilde{q}_s^\dagger + \tilde{q}_s^\dagger D \tilde{q}_s \right]$$

where the contraction of gauge indices is in each case unique: for example in the term $q_r^\dagger D q_r$ the indices are contracted as

$$(q_r^\dagger)_j^i D^A(T^A)_i^j (q_r)_r$$

If we add a chiral superfield $\Phi$ in the adjoint representation, the kinetic terms take the same form as above, but we should interpret $\phi^\dagger D \phi$ as

$$\text{tr} \phi^\dagger [D, \phi] = - \text{tr} D[\phi^\dagger, \phi] = - i f^{ABC} D^A \phi^B \phi^C$$

and similarly for the scalar-fermion-fermion terms. (Be careful not to confuse the derivative $D_\mu$ and the auxiliary field $D^A$!)

As in the abelian case,

- We may obtain $\mathcal{N} = 1$ gauge theories by adding arbitrary charged (and neutral) matter to the theory with arbitrary gauge-invariant holomorphic gauge kinetic and superpotential functions and an arbitrary gauge-invariant Kähler potential.
- We may obtain a pure $\mathcal{N} = 2$ gauge theory by writing an $\mathcal{N} = 1$ gauge theory with a single chiral multiplet $\Phi$ in the adjoint representation, a gauge kinetic term and Kähler potential term for $\Phi$ which must be related, and zero superpotential.
- We may add matter to the $\mathcal{N} = 2$ gauge theory in the form of a hypermultiplet (two chiral multiplets $Q$ and $\tilde{Q}$ in conjugate representations) coupled in the superpotential $W = \sqrt{2} Q \Phi \tilde{Q}$, with gauge indices contracted in the unique way. We may also add mass terms for the hypermultiplets, obtaining $W = \sqrt{2} Q \Phi \tilde{Q} + m Q \tilde{Q}.$
Finally, if we have a massless hypermultiplet in the adjoint representation, so that the theory has a total of three chiral multiplets \( \Phi_1 = \Phi, \Phi_2 = Q, \Phi_3 = \tilde{Q} \) in the adjoint, with superpotential
\[
W = \sqrt{2} \text{tr} \Phi_1 [\Phi_2, \Phi_3],
\]
then the theory has \( \mathcal{N} = 4 \) supersymmetry.

Now the condition for a supersymmetric vacuum requires that
\[
D^A = 0, \quad F_f = 0, \quad F_\bar{f} j = 0.
\]
If the superpotential is zero, then the constraints all come from
\[
0 = D^A \propto (q^I_r)_j (T^A)^i_j (\tilde{q}^s_\ell)^i_j (T^A)^s_j (\tilde{q}_r)_j = (T^A)^i_j [(q^I_r)(q_r) - \tilde{q}^s_\ell \tilde{q}^s_\ell]^i_j
\]
\[
= \text{tr} T^A [(q^I_r)(q_r) - \tilde{q}^s_\ell \tilde{q}^s_\ell]
\]
(29)

(I have written a proportional sign since the precise relation depends on the Kähler potential and gauge kinetic term, while the proportionality relation does not!)

These equations are beautifully solved in the case of \( SU(N) \) with fields \( N_f Q \) in the \( \mathbf{N} \) representation and \( \tilde{Q} \) in the \( \mathbf{\bar{N}} \) representation.\(^m\) The hermitian matrix \([ (q^I_r)(q_r) - \tilde{q}^s_\ell \tilde{q}^s_\ell]^i_j \) can be uniquely expanded as
\[
[(q^I_r)(q_r) - \tilde{q}^s_\ell \tilde{q}^s_\ell]^i_j = c_0 \delta^i_j + \sum_B c_B (T^B)^i_j
\]

Then, using the fact that
\[
\text{tr} T^A T^B = \frac{1}{2} \delta^{AB}; \quad \text{tr} T^B = 0
\]
we find that the conditions (29) become simply
\[
[(q^I_r)(q_r) - \tilde{q}^s_\ell \tilde{q}^s_\ell]^i_j = c_0 \delta^i_j
\]
(30)
for \( \text{any} \) \( c_0 \). Before writing any solutions to these equations, we note the following: when any expectation values of \( q \) and \( \tilde{q} \) which are a solution to these equations, a continuously infinite class of solutions is generated by multiplying all of the \( q \) and \( \tilde{q} \) fields by a complex constant. Thus there will generally be, as in the abelian case, noncompact, continuous moduli spaces

\(^m\)In this case there is an \( SU(N_f)_L \) and an \( SU(N_f)_R \) symmetry acting on the \( Q \) and \( \tilde{Q} \) fields respectively; there is also a baryon number under which \( Q \) and \( \tilde{Q} \) have charges 1 and \(-1\), and an anomalous axial symmetry (present classically but explicitly violated quantum mechanically) under which \( Q \) and \( \tilde{Q} \) both have charge 1.
of vacua. As before, these vacua are not related by any symmetry, in that they have fields with different masses.\textsuperscript{n}

**Exercise:** The scalars $q_r^i$ and $q_s^{\bar{j}}$ are $N_f \times N$ and $N \times N_f$ matrices respectively. Being careful with the indices, show that the only solutions for $N_f < N$ are gauge and global symmetry transformations of the particular solution $q_r^i = v \delta_r^i$, $q_s^{\bar{j}} = v \delta_s^{\bar{j}}$ for $i, \bar{j} \leq N_f$. Then show that the only solutions for $N_f \geq N$ are gauge and global symmetry transformations of the particular solution $q_r^i = v \delta_r^i$, $q_s^{\bar{j}} = v \delta_s^{\bar{j}}$ for $r, s \leq N$. Note that for $N_f \geq N$, $v$ and $\tilde{v}$ are in general different and the constant $c_0$ in (30) is nonzero, while for $N_f < N$ $c_0$ must be zero.

As another example, consider $SU(2)$ with fields $\Phi_n (n = 1, 2, \ldots, N_a)$ in the adjoint representation (the 3). Representing $(\Phi_n)^j_i$ as a traceless $2 \times 2$ complex matrix, the D-term conditions are the matrix equation

\[
\sum_n [\Phi_n^\dagger, \Phi_n]_{ij} = 0.
\]

In the case of $N_a = 1$, appropriate to pure $\mathcal{N} = 2$ gauge theory, the solution is clearly that $\Phi$ is diagonal

\[
\langle \Phi \rangle = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}
\]

This breaks the $SU(2)$ gauge group to $U(1)$. In terms of the one independent gauge invariant operator which can be built from $\Phi$\n
\[
\langle u \rangle \equiv \langle \text{tr } \Phi^2 \rangle = 2a^2
\]

the moduli space of the theory is the complex $u$ plane, classically. The Kähler potential for $u$ has a singularity at $u = 0$, where the gauge group is unbroken. Let’s check the counting: $\Phi$ is a triplet, and two of its components are eaten when $SU(2)$ breaks to $U(1)$, leaving one — the chiral multiplet $u$.

Now suppose there are three fields $\Phi_n$, $n = 1, 2, 3$, in the 3 of $SU(2)$, and also a superpotential $W = \text{tr } \Phi_1[\Phi_2, \Phi_3]$. This is the $\mathcal{N} = 4$ $SU(2)$

\textsuperscript{n}More precisely, they are related by scale invariance (since the vevs are the only scales in the classical $d = 4$ gauge theory) but since scale invariance will be broken quantum mechanically, while the D-term conditions generally will not be altered, we will see that the vacua really are physically very different.
gauge theory. The easiest way to establish the solutions to the D-term and F-term constraints is to rewrite the fields as

\[
\begin{align*}
\Phi_1 &= X_4 + iX_7, \\
\Phi_2 &= X_5 + iX_8, \\
\Phi_3 &= X_6 + iX_9
\end{align*}
\]

in terms of which the potential \( V(X_m) = \sum_A (D^A)^2 + \sum_n F_n^2 \) can be rewritten as

\[
V(X_p) \propto \sum_{p,q=4}^9 \text{tr} ([X_p, X_q])^2
\]

This formulation has the advantage that the \( SO(6) \) symmetry rotating the \( X_p \) is manifest; the superpotential only exhibits a \( U(3) \) subgroup of that symmetry. The condition \( V = 0 \) implies all of the \( X_p \) are simultaneously diagonalizable. In short, the solutions are

\[
X_p = \begin{bmatrix} c_p & 0 \\ 0 & -c_p \end{bmatrix}
\]

with \((c_4, \ldots, c_9)\) forming a real six-vector living in a flat six-real-dimensional moduli space. Again, with the exception of the point \( c_p = 0 \), all of the vacua have \( SU(2) \) broken to \( U(1) \).

Let’s add a mass \( m \text{ tr } \Phi_2 \Phi_3 \) (which breaks \( \mathcal{N} = 4 \) but preserves \( \mathcal{N} = 2 \)) and integrate out the massive fields. Their equations of motion, which at low momenta reduce simply to

\[
\begin{align*}
\frac{\partial W}{\partial \Phi_2} &= \sqrt{2} [\Phi_1, \Phi_3] + m \Phi_3 = 0; \\
\frac{\partial W}{\partial \Phi_3} &= -\sqrt{2} [\Phi_1, \Phi_2] + m \Phi_2 = 0;
\end{align*}
\]

only have a solution \( \Phi_2 = \Phi_3 = 0 \). When substituted back into the superpotential, this solution gives a low-energy theory with one adjoint \( \Phi_1 \) and \( W = 0 \) — the pure \( SU(2) \) \( \mathcal{N} = 2 \) theory. Thus it is easy to flow from the \( \mathcal{N} = 4 \) theory to the pure \( \mathcal{N} = 2 \) theory, and this was studied in the \( SU(2) \) case by Seiberg and Witten (1994).

By contrast, consider adding a mass \( \frac{1}{2} m \text{ tr } \Phi_3^2 \). The situation here is much like the XYZ model with a mass for \( X \); the low-energy theory there had \( W = Y^2 Z^2 \). Here the equation of motion for \( \Phi_3 \) reduces to

\[
\sqrt{2} [\Phi_1, \Phi_2] = m \Phi_3
\]

leaving a superpotential

\[
W_L = p ([\Phi_1, \Phi_2])^2
\]

This generalizes: for \( SU(N) \), we have \( N - 1 \) such real six-vectors, corresponding to the \( N - 1 \) sextuplets of eigenvalues of the traceless matrices \( X_p \).
where \( p \propto 1/m \) is a coupling with classical dimension \(-1\). Like \((YZ)^2\), it is an irrelevant operator in four dimensions, scaling to zero in the infrared.

### 5.2. Beta functions and fixed points

Now we turn to the quantum mechanics of these theories. We have already studied the abelian case in great detail at the one-loop level, and we know that the main difference in the nonabelian case will be that gauge boson loops will give a negative contribution to the beta function. The contributions to the beta function of various particles are shown in Fig. 35, from which one can see that a vector multiplet contributes a factor of \(-3N\)

\[
\beta_g = -\frac{g^3}{8\pi^2} (3N - N_f) \Rightarrow \frac{1}{g^2(\mu)} = \frac{3N - N_f}{8\pi^2} \log \left( \frac{\Lambda}{\mu} \right) \tag{32}
\]

where (defining \( b_0 = 3N - N_f \))

\[
\Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2 / g^2 + i\theta} = \mu^{b_0} e^{2\pi i \tau} \tag{33}
\]

For \( N_a \) adjoint fields the one-loop beta function has \( b_0 = (3 - N_a)N \); note it vanishes for \( N' = 4 \) Yang-Mills. Just as in the abelian case, the entry of
the theta angle into $\Lambda$ implies there can be no perturbative corrections to these formulas.

As an aside, let me stress that the existence of the quantum mechanical holomorphic parameter $\Lambda$ is very important. Although the effective superpotential is still constrained by the perturbative nonrenormalization theorem, as in non-gauge theories, the presence of $\Lambda$ permits, in many cases, a nonperturbative renormalization of the superpotential. There is a lot of literature on this subject; the classic paper is that of Affleck, Dine and Seiberg (1984), although a number of others obtained similar results using somewhat less reliable techniques. There are good reviews on this subject by other authors, including ones by Intriligator and Seiberg and one by Argyres. For lack of time, in these lectures we will only discuss cases where there is no nonperturbative correction to the effective superpotential (or at least no qualitative change in its structure.) Large classes of interesting theories have this property, but we are leaving out other large classes; see the Appendix for some examples.

As before, the formula for the holomorphic coupling $g(\mu)$, and the holomorphic renormalization scale $\Lambda$, can’t represent the physical properties of the theory. This is obvious from the fact that $g^2(\mu)$ blows up at small $\mu$ and can’t make sense below $\Lambda$. Higher loop effects, and possibly nonperturbative effects, appearing in the nonholomorphic parts of the theory can change this formula significantly. How can we define a physical gauge coupling? A natural approach is to find a more physical definition of $\Lambda$, so let the holomorphic object in Eq. (33) be renamed $\hat{\Lambda}$, and let us attempt to define a $\Lambda$ independent of field redefinitions. (The presentation here is related to recent work of Arkani-Hamed and Rattazzi, although the resulting formula is due to Novikov, Shifman, Vainshtein and Zakharov from the early 1980s.)

Recall how we defined a physical version of $y$, the coupling in the Wess-Zumino model. We noted that if we sent $\Phi \rightarrow a\Phi$, $a$ a constant, this would affect both $\hat{y}$ in the superpotential and $Z$ in the Kähler potential. Let’s do the same here for the charged fields $Q$ and $\tilde{Q}$. Suppose we multiply them all of them by $a$, where $a$ is a phase $e^{i\alpha}$. This is equivalent to a transformation by an anomalous “axial” $U(1)$ global symmetry, under which quarks and antiquarks have the same charge. As happens in QCD, this kind of transformation is an anomalous symmetry, and shifts the $\theta$ angle; it therefore rotates $\Lambda^{\text{hol}}$ by a phase. The phase by which $\Lambda^{\text{hol}}$ rotates is $a^{2N_f}$. But since $Q$ and $\Lambda$ are holomorphic, it must still be true that $\Lambda$ changes by $a^{2N_f}$ even if $a$ is not a phase but has $|a| \neq 1$! Therefore, since $Z \rightarrow |a|^{-2}Z$ under this
transformation, only
\[ (\hat{\Lambda}^b_0)^\dagger \left[ \prod_{r=1}^{N_f} Z_r \right] ^{N_f} \left[ \prod_{s=1}^{N_f} \tilde{Z}_s \right] \hat{\Lambda}^b_0, \]

where \( Z_r \) and \( \tilde{Z}_s \) are the wave function factors for \( Q_r \) and \( \tilde{Q}_s \), can be invariant under these field redefinitions.

We can go further by considering R-symmetry transformations. Suppose we rotate the gluino fields \( \lambda \) by \( e^{i\alpha} \) and the fields \( q, \tilde{q} \) by \( e^{i\alpha} \), so that the quarks \( \psi_q \) and antiquarks \( \tilde{\psi}_q \) do not rotate at all (recall there is a \( \lambda \psi_q \tilde{\psi}_{\tilde{q}}^\dagger \) interaction which fixes the charge of one field in terms of the other two.) Then the anomaly in this transformation involves only the gluinos, and the \( \theta \) angle shifts by \( e^{2N_i\alpha} \). (The factor \( 2N \) is the group theory “index” of the adjoint representation in \( SU(N) \); it determines the size of the anomaly.) Again, we can generalize this by taking \( \alpha \) to be complex, so that \( |e^{i\alpha}| \neq 1 \). An invariant which is unchanged by both this and the previous transformation is

\[ \mu^{2b_0} e^{-16\pi^2/g^2(\mu)} = |\Lambda^2(\mu)| = (\hat{\Lambda}^b_0)^\dagger \left[ \prod_{r=1}^{N_f} Z_r \right] ^{2N} \left[ \prod_{s=1}^{N_f} \tilde{Z}_s \right] \hat{\Lambda}^b_0 \]

Taking a derivative with respect to \( \mu \) of both sides, we obtain

\[ \beta_{\frac{8\pi^2}{g^2}} = b_0 + \frac{1}{2} \sum_r \gamma_r + \frac{1}{2} \sum_s \gamma_s + N\gamma_\lambda \]

But from the kinetic terms of the gauginos it is evident that \( Z_\lambda = 1/g^2(\mu) \) itself! Therefore

\[ \gamma_\lambda = -\frac{\partial \ln Z_\lambda}{\partial \ln \mu} = \frac{\beta_{8\pi^2/g^2}}{8\pi^2/g^2} \]

from which we obtain the exact NSVZ beta function

\[ \beta_{\frac{8\pi^2}{g^2}} = \frac{b_0 + \frac{1}{2} \sum_r \gamma_r + \frac{1}{2} \sum_s \gamma_s}{1 - g^2 N/8\pi^2} = \frac{16\pi^2}{g^2} \beta_g \]

(Remember to keep track of the difference in sign between \( \beta_g \) and \( \beta_{8\pi^2/g^2} \).)

In supersymmetric QCD, where in the absence of a superpotential all charged fields are related by symmetry, and therefore have the same anomalous dimension \( \gamma_0 \), we may write

\[ \beta_{\frac{8\pi^2}{g^2}} = \frac{3N - N_f[1 - \gamma_0]}{1 - g^2 N/8\pi^2} \]
In a general theory with charged fields \( \phi_i \) in representations \( R_i \) with \( \text{tr} \, T^A T^B = T_{R_i} \delta^{AB} \), and with anomalous dimensions \( \gamma_i \), we have

\[
\beta_{\frac{g^2}{\pi}} = \frac{3N - \sum_i T_{R_i} [1 - \gamma_i]}{1 - g^2 N/8\pi^2} = b_0 + \sum_i T_{R_i} \gamma_i
\]

These formulas continue to hold even when there are other gauge and matter couplings in the theory. This formula, which has a Taylor expansion in \( g^2 \), summarizes all higher-loop corrections. If there is matter in the theory, then we do not know \( \gamma_i \) exactly; but in the absence of matter (the pure \( \mathcal{N} = 1 \) gauge theory) the formula is a definite function of \( g \).

We will now use this exact beta function to prove a few things. Before doing so, let’s consider the one-loop contributions to the anomalous dimensions of charged fields. As we saw, trilinear terms in the superpotential give one-loop graphs as in Fig. 36 which must give a positive contribution, if there are no gauge couplings around. To leading order in the gauge coupling, this must still be true; the one-loop graph from a superpotential term must be positive. However, there is no such constraint for the one-loop diagram in Fig. 37 involving the gauge interactions and it is a determining, crucial feature of supersymmetric gauge theories that the coefficient of this diagram has the opposite sign. (I don’t know of an argument which explains this fact in physical terms.) Consequently the sign of \( \gamma \) will flip as the coupling constants are varied. For a gauge theory with no superpotential, the charged fields have negative anomalous dimensions.

First, let us prove that for large \( N_f \), slightly less than \( 3N \), SQCD has nontrivial conformal fixed points (and does not for \( N_f \geq 3N \)). These are

---

Footnote: The pole in the denominator is still not fully understood, even after 20 years; notice that it becomes dominant when \( g^2 N \sim 8\pi^2 \), an important issue for those studying large-\( N \) gauge theories at large ‘t Hooft coupling!
Figure 37. Contributions to $Z_\Phi$ from the gauge interactions.

sometimes called “Banks-Zaks” fixed points although they were discussed earlier; they exist also in the nonsupersymmetric case!

**Exercise:** By examining the one- and two-loop beta functions (given in Ellis’s lectures at this school) verify that ordinary QCD has conformal fixed points when $N_f$ and $N_c$ are large and the theory is just barely asymptotically free. Remember to check that all higher-loop terms in the beta-function can be neglected!

To see this, note that the loop expansion is really an expansion in $g^2N$, so if we can trap $g^2$ in a region where it is of order $1/N^2$, then the loop expansion can be terminated at leading nonvanishing order. The anomalous dimension $\gamma_0$ of the superfields $Q_r, \tilde{Q}_s$ will be of the form, on general grounds,

$$-\hat{c}g^2N/8\pi^2 + \text{order } [(g^2N)^2]$$

where $\hat{c} > 0$. For $N_f = 3N - k$, where $k$ is order 1, the beta function takes the form

$$\beta_{\frac{\Delta x^2}{2\sigma^2}} = \frac{k + \frac{N}{8\pi^2} (-\hat{c}g^2N + \text{order } [(g^2N)^2])}{1 - g^2N/8\pi^2} \approx k - 3N\hat{c}g^2N/8\pi^2$$

where in the last expression we have dropped terms of order $g^2Nk$ and $N(g^2N)^2$. For $k \leq 0$, that is, $N_f \geq 3N$, the beta function $\beta_g$ is positive (i.e. $\beta_{g^2}/g^2 < 0$) for small $g$, so the gauge coupling $g$ flows back to zero in the *infrared* and is not asymptotically free in the ultraviolet. However, if $k > 0$, and thus for $N_f < 3N$, the beta function $\beta_g$ is negative at small $g$ but has a zero at

$$g_*^2 = \frac{8\pi^2k}{3\hat{c}N^2}$$
and therefore the coupling $g^2$ never gets larger than of order $1/N^2$. The above formula is therefore self-consistent in predicting that the gauge coupling flows from zero to the above fixed point value. (Note that the holomorphic coupling has no such fixed point! thus the physical properties and holomorphic properties of the theory are vastly different!) The flow is shown in Fig. 38.

$$N_f \geq 3N$$

$$g = 0$$

$$N_f < 3N$$

$$g = 0 \quad \Rightarrow \quad g_*$$

Figure 38. The gauge coupling is marginally relevant for $N_f < 3N$, with a nearby fixed point at $g_*$. Next, let’s prove that in SQCD for $N_f \leq \frac{3}{2}N$ there can be no such fixed points — specifically, ones in which no fields are free, and which are located at the origin of moduli space, where no fields have expectation values. In SQCD a fixed point requires

$$\frac{\beta_{g^2}}{g^2} \propto b_0 + N_f \gamma_0 = 0 \Rightarrow \gamma_0 = \frac{3N - N_f}{N_f}.$$  

(This in turn implies that the R-charge of $Q$ and $\tilde{Q}$, using the earlier formula that $\dim Q = 1 + \frac{1}{2} \gamma_0 = \frac{3}{2} R_Q$, must be

$$R_Q = 1 - \frac{N}{N_f}.$$  

one may check that this particular R-symmetry is the unique nonanomalous chiral symmetry of the theory!)

However, if $\gamma_0 \leq -1$, then the gauge-invariant operator $Q, \tilde{Q}$ would have dimension $2(1 + \frac{1}{2} \gamma_0) \leq 1$. This is not allowed at a nontrivial fixed point, so to have such a fixed point (at least one in the simple class we have been discussing) it must be that

$$\gamma_0 > -1 \Rightarrow b_0 < N_f \Rightarrow N_f > \frac{3}{2}N.$$
What happens for adjoint fields? If the number of fields is \( N_a \), then we have
\[
\beta_{2g^2} \propto (3 - N_a)N + N_a \gamma_0 = 0 \Rightarrow \gamma_0 = 1 - \frac{3}{N_a}.
\]
This is too negative a \( \gamma_0 \) for \( N_a = 1 \), so there is no ordinary \( SU(N) \) fixed point in the pure \( \mathcal{N} = 2 \) theory.\(^8\) For \( N_a = 2 \), the anomalous dimension at any fixed point must be \(-\frac{1}{2}\); that is, the dimension of \( \Phi_1 \) and \( \Phi_2 \) must be \( \frac{3}{4} \). For \( N_a = 3 \) the anomalous dimension at any fixed point must be zero. If \( \Phi_n \) were gauge-invariant, then by the earlier theorem this could only happen if \( \Phi \) were free; but outside of the abelian case, \( \Phi_n \) is not gauge-invariant and this is not a requirement.

**Exercise:** In an \( \mathcal{N} = 2 \) gauge theory, the one-loop formula \( \beta_{8\pi^2/g^2} = b_0 \) is exact. Using the facts that (a) the anomalous dimension of the adjoint field \( \Phi \) is related to that of the gauge bosons by \( \mathcal{N} = 2 \) supersymmetry, and (b) \( \mathcal{N} = 2 \) forbids hypermultiplets to have anomalous dimensions, prove that the NSVZ beta function is consistent with this statement.

Now, Seiberg has suggested that there are fixed points in \( \mathcal{N} = 1 \) SQCD for \( 3N > N_f > \frac{3}{2}N \). That is, he conjectured in 1994 that in this range there is some value \( g_* \) of the gauge coupling for which \( \gamma_0(g_*) = \frac{3N - N_f}{N_f} \). (Again, we know this is true for \( 3N - N_f \ll N \); Seiberg’s conjecture is that this continues down to much lower \( N_f \).) If he is right, as most experts think that he is, then some remarkable and exciting phenomena immediately follow. These can be found by combining our minimal knowledge concerning the properties of these putative fixed points with the approach to the renormalization group outlined in the previous lectures.

For example, consider adding the superpotential
\[
W = \hat{p} \sum_{r,s=1}^{N_f} (Q_r \bar{Q}_s^*)(Q_s \bar{Q}_r^*)
\]
with gauge indices contracted inside the parentheses. Very importantly, this superpotential preserves a diagonal \( SU(N_f) \) global symmetry and charge conjugation; this is enough symmetry to ensure that all of the fields share the same anomalous dimension \( \gamma_0(p, \tau) \), as was true for \( p = 0 \). As always

\(^8\)There are however some much more subtle fixed points discovered by Argyres and Douglas in 1995.
we should study the dimensionless coupling constant \( \pi \equiv p\mu \) and ask how it scales. Classically it scales like \( \mu \) (and thus has \( \beta_{\pi} = \pi \)) but quantum mechanically, —

Wait a minute. This theory, whose potential contains (scalar)\(^6\) terms, is nonrenormalizable. Can we even discuss it?

Well, nonrenormalizable simply means that the operator in the superpotential is irrelevant, so in the ultraviolet regime the effective coupling is blowing up and perturbative diagrams in the theory don’t make sense. But we’re interested in the infrared anyway. We’ll deal with the ultraviolet later; for now will think of \( 1/\hat{p} \) as setting a cutoff on the theory. Perturbation theory may not converge, but we are asking perfectly valid nonperturbative infrared questions which do not depend on the details of the ultraviolet cutoff.

In particular, we know that we need to define a physical coupling \( \pi \), of the form

\[
|\pi^2| = |\hat{p}\mu|^2Z_0^{-4}.
\]

We see it has a beta function

\[
\beta_{\pi} = \pi[1 + 2\gamma_0]
\]

Now, this means \( \pi \) is irrelevant if \( \gamma_0 > -\frac{1}{2} \) and is relevant if \( \gamma_0 < -\frac{1}{2} \). The formula for the gauge coupling is unchanged

\[
\beta_{g_{\pi}} \propto 3N - N_f + N_f\gamma_0.
\]

Now, remember that \( \gamma_0 \) is a function of \( \tau \) and \( \pi \) with the following properties: (1) if \( g = 0 \), \( \pi \neq 0 \) then \( \gamma_0 > 0 \); (2) if \( \pi = 0 \), \( 0 \neq g < 1 \) then \( \gamma_0 > 0 \); and (3) there is at least one nontrivial fixed point at \( g = g_* \), \( \pi = 0 \) with \( \gamma_0 = \frac{3N-N_f}{N_f} \). Notice that at this fixed point \( \pi \) is irrelevant (as it is classically) if \( N > 2N_f \), marginal if \( N = 2N_f \), and relevant if \( N < 2N_f \).

From this we can guess the qualitative features of the renormalization group flow. For \( N > 2N_f \), the qualitative picture is given in Fig. 39. Even if \( \pi \neq 0 \), we still end up at the Seiberg fixed point. For \( N < 2N_f \), however, there is a very different picture, as in Fig. 40. Notice that if we start at weak gauge coupling initially, \( \pi \) is irrelevant and flows toward zero as we would expect classically; but as we flow toward the infrared, the gauge coupling grows, \( \gamma_0 \) becomes more negative, and eventually the coupling \( \pi \) turns around and becomes relevant. Although at first it seems as though it will be negligible in the infrared, it in fact dominates. This is called a

\[\text{Note that we did essentially the same thing with SQED in four dimensions, which is perturbatively renormalizable but nonperturbatively nonrenormalizable, since we cannot take the cutoff on the theory to infinity without the gauge coupling diverging in the ultraviolet.}\]
For $N_f > 2N$ the coupling $\pi$ is irrelevant both at $g = 0$ and at $g = g_*$. 

"dangerous irrelevant" operator, since although it is initially irrelevant it is dangerous to forget about it! In the infrared it becomes large, and we must be more precise about what happens when it gets there. 

What about $N_f = 2N$? You’ll do this as an exercise, after I’ve done a bit more.

5.3. Using $\mathcal{N} = 1$ language to understand $\mathcal{N} = 4$

As another application of these ideas, let’s argue that $\mathcal{N} = 4$ Yang-Mills is finite. Consider an $\mathcal{N} = 1$ gauge theory with three chiral superfields and a superpotential $W = h \text{tr} \Phi_1 [\Phi_2, \Phi_3]$. I will use canonical normalization here for the $\Phi_n$, so $h = g$ is the $\mathcal{N} = 4$ supersymmetric theory. But let’s not assume that $h = g$. For any $g, h$, the symmetry relating the three fields ensures they all have the same anomalous dimension $\gamma_0$, which is a single function of two couplings. The beta functions for the couplings are 

$$
\beta_h = \frac{3}{2} h \gamma_0; \quad \beta_g^2 = \frac{-g^2}{16\pi^2} \frac{\gamma_0}{1 - g^2 N/8\pi^2}
$$

These are proportional to one another, so the conditions for a fixed point ($\beta_h = 0$ and $\beta_g = 0$) reduce to a single equation, $\gamma_0(g, h) = 0$. But this is

---

*What happens is fascinating — the theory flows to a different Seiberg fixed point, that given by $SU(N_f - N)$ SQCD with $N_f$ flavors. See Leigh and Strassler (1995) for an understanding of how to treat the limit where the coupling is large.*
For $N_f < 2N$ the coupling $\pi$ is relevant at $g_*$; its initial decrease is reversed once $g$ is sufficiently large.

one equation on two variables, so if a solution exists, it will be a part of a one-dimensional space of such solutions.

Now, does a solution exist? We know that $\gamma(g, h = 0) < 0$ and that $\gamma_0(g = 0, h) > 0$; so yes, by continuity, there must be a curve, passing through $g = h = 0$, along which $\gamma_0 = 0$ and $\beta_h = \beta_g = 0$ (and thus perturbation theory has no infinities along this line.) The renormalization group flow must look like the graph in Fig. 41. Both the theory with $h = 0$ and the theory with $g = 0$ are infrared free; yet a set of nontrivial field theories lies between. Notice that we do not know, however, the precise position of the curve $\gamma_0 = 0$. In particular, we have not shown that $g = h$ gives $\gamma_0 = 0$. However, the existence of a finite theory (which is renormalization-group stable in the infrared) requires only arguments using $\mathcal{N} = 1$ symmetry. Of course, since the theory at $g = h$ has more symmetry (namely $\mathcal{N} = 4$) it is natural to expect $g = h$ to be the solution to $\gamma_0(g, h) = 0$.

The motivation for introducing this $\mathcal{N} = 1$-based reasoning is there are many $\mathcal{N} = 1$ field theories which are also finite, as one can show using similar arguments. (For example, replace the $\mathcal{N} = 4$ superpotential with $W = h \text{ tr } \Phi_1\{\Phi_2, \Phi_3\}$; the discussion is almost unchanged, except that $g = h$ is not the solution to $\gamma_0 = 0$.) The existence of these theories was discovered in the 1980s; the slick proof presented above is in Leigh and
Figure 41. In some $\mathcal{N} = 1$ theories one can argue for a line of fixed points indexed by an exactly marginal coupling $\rho$; perturbation theory has no divergences on this line. Only in $\mathcal{N} = 4$ is the equation for this line $g = h$.

Strassler (1995).

As mentioned in Sec. 3.4, the coupling which parametrizes the line of conformal fixed points (which is actually a complex line, since the couplings are complex) is called an “exactly marginal coupling.” Let’s call this complex coupling $\rho$. (In the $\mathcal{N} = 4$ case we can identify $\rho$ as equal to the gauge coupling $i/\tau$, but in a more general $\mathcal{N} = 1$ finite theory these will not be simply related.) Unlike $\lambda$ in $\lambda \phi^4$, which is marginal at $\lambda = 0$ but irrelevant at $\lambda \neq 0$, $\rho$ is marginal at $\rho = 0$, and remains marginal for any value of $\rho$. Thus $\rho$ is a truly dimensionless coupling, indexing a continuous class of scale-invariant theories. It is very common for such classes of theories to be acted upon by duality transformations. In fact, for $\mathcal{N} = 4$ electric-magnetic duality (S-duality) acts on this coupling $\rho$ as in Fig. 42, identifying those theories at large $\rho$ with those at small $\rho$.

5.4. The two-adjoint model

We conclude with a discussion of a theory with two adjoint chiral multiplets, obtained from the $\mathcal{N} = 4$ gauge theory by adding a mass for the third adjoint $\Phi_3$. It has a superpotential (31):

$$W_L = p(\Phi_1, \Phi_2)^2.$$
This quartic superpotential is nonrenormalizable, but low-energy effective theories often are. We are interested (for the moment) in the infrared behavior, so the fact that \( \pi = p\mu \propto \nu^{-1} \) (where \( \nu = \frac{m}{\mu} \)) blows up in the ultraviolet is not our immediate concern.

What are the beta functions for \( \pi \) and for \( g \)? We have

\[
\beta_\pi = \pi [1 + 2\gamma_0] ; \beta_{8\pi^2/g^2} \propto 3N - 2N(1 - \gamma_0) = N + 2N\gamma_0
\]

and thus \( \beta_\pi \propto \beta_g \). This means that, as before, the conditions for a fixed point to exist, namely \( \beta_\pi = 0 = \beta_g \), reduce to a single condition (except at \( g = \pi = 0 \)):

\[
1 + 2\gamma_0 = 0 \Rightarrow \gamma_0 = -\frac{1}{2}.
\]

Again, this is one condition on two couplings, so any solution will be part of a one-dimensional space of solutions. Following Seiberg, we might well expect that the theory with two adjoints and \( W = 0 \) has a fixed point at some \( g_* \) where \( \gamma_0(g_*) = -\frac{1}{2} \). If this is true, then the renormalization group flow of the theory will look like Fig. 43.

So here our irrelevant operator has been converted into an exactly marginal one! The coupling \( \rho \) which parametrizes the line of fixed points, and on which duality symmetries might act, now has nothing to do with the gauge coupling. In fact \( \rho = 0 \) corresponds to \( g = g_* \). Nowhere are these
In the two-adjoint theory, $\pi$ is marginal at $g = g_*$ and there is a line of fixed points emerging from the fixed point at $(g, \pi) = (g_*, 0)$.

conformal field theories near $g = \pi = 0$, so we have no hope of seeing them in any perturbation expansion.

Now, since the theory is nonrenormalizable, we probably should at least say something about the ultraviolet. But we know a perfectly good ultraviolet theory into which we can embed this theory, namely the $\mathcal{N} = 4$ gauge theory with $\Phi_3$ massive. Classically, we know what this flow would look like. Just as in the XYZ model with $X$ massive, shown in Fig. 5, the theory would start from a conformal field theory indexed by $\tau$ and flow into a classical fixed point, with a nonzero gauge coupling and $W = 0$, along the irrelevant operator $(\Phi_1, \Phi_2)^2$. But quantum mechanically the endpoint of the theory is not $W = 0$; instead, it is one of the conformal field theories we found above in the two-adjoint theory. In fact, we can expect that each $\mathcal{N} = 4$ field theory flows to a unique two-adjoint theory, along a flow which looks schematically like Fig. 44. It is natural therefore to identify $\rho$ with $i/\tau$, as we did in the $\mathcal{N} = 4$ case, but this "$\tau$" is not the gauge coupling of the two-adjoint theory. Rather, we have defined here a physical mechanism for using the label $\tau$ of the $\mathcal{N} = 4$ fixed points as a label for the two-adjoint fixed points. Since S-duality acts on $\tau$ in $\mathcal{N} = 4$ we are essentially guaranteed that duality will also act on $\rho$ in the two-adjoint theory.

**Exercise:** Examine the beta functions and sketch the renormalization group flow for SQCD with $N_f = 2N$ and the quartic superpotential (35).
6. Regrets

What didn’t I talk about in these lectures? The list seems to be infinite. There was nothing on the phases of gauge theories; nothing on confinement; nothing on nonperturbative renormalizations of superpotentials; nothing on Seiberg duality; nothing on two, five or six dimensions; nothing on spontaneous dynamical supersymmetry breaking; nothing on exact methods for studying renormalization in theories with broken supersymmetry; nothing on D-brane or other stringy constructions of these theories; and above all, nothing on applications of this material to real-world physics! But there are good reviews on almost all of these subjects. By contrast, there are no reviews on the material I have discussed here, which is necessary for an understanding of duality, plays a very important role in the AdS/CFT correspondance, and (as Ann Nelson and I have suggested) may even be responsible for the pattern of quark and lepton masses and for the low rate of proton decay. So I hope that this somewhat unorthodox introduction to this subject will serve you well; and I hope I have convinced you that this is a profound and fascinating subject, where much is to be learned and much remains to be understood.

Figure 44. The two-adjoint theory inherits S-duality from the $\mathcal{N} = 4$ theory through the flow which takes one to the other.
7. Appendix: Comments on $\mathcal{N} = 1$ SU(2) SQCD

We will focus our attention on the case of SU(2). This case is a bit special because the 2 and $\bar{2}$ representation are identical [you already know this from quantum mechanics: there is no conjugate-spin-1/2 representation of SU(2)] so we actually should combine the $Q_r$ and $\tilde{Q}^r$ into $2N_f$ fields $Q_u$, with an SU($2N_f$) global symmetry, and a D-term condition

$$\left[ \sum_{u=1}^{2N_f} (q_u^i)(q_u) \right]^i_j = c_0 \delta^i_j \quad (36)$$

One solution to this condition is $q_1 = (\sqrt{c_0/2}, 0), q_2 = (0, \sqrt{c_0/2})$, with all others zero. As in the abelian case, it is most convenient to express this result using gauge invariant combinations of the chiral superfields. The $2N_f \times 2N_f$ antisymmetric matrix of gauge-invariants $M_{uv} = Q_u^i Q_v^j \epsilon_{ij}$ has $M_{12} = -M_{21} = c_0/2$, with all other components zero.

In fact, all solutions to the condition (36) can be written as SU($2N_f$)-flavor and SU(2)-gauge rotations of the above particular solution. The gauge rotations leave $M_{uv}$ invariant, and the flavor rotations leave invariant the fact that it has rank at most two, with either zero or two equal non-vanishing eigenvalues.

Note that unless $M_{uv} = 0$, the gauge group is completely broken. Let’s check this is the case for $N_f = 1$. There are two chiral fields $Q_1$ and $Q_2$, each in the 2 of SU(2), for a total of four complex fields. Three of these must be eaten by the three gauge bosons if SU(2) is completely broken. Consequently, there should be one remaining. Indeed, there is only one (unconstrained) field $M_{12}$. Let’s check it for $N_f = 2$: in this case there are six fields $M_{uv}$ ($u, v = 1, 2, 3, 4$) but also a single constraint that the rank must be 2, not 4, which is the condition Pf($M$) = 0. (The Pfaffian is just the square root of the determinant, and is defined as $\epsilon_{uvwz} M^{uv} M^{wx}$ in this case.) This leaves five unconstrained fields. Initially there are four doublets $Q_1, Q_2, Q_3, Q_4$ for a total of eight fields, with three being eaten when the gauge group is broken; this too leaves five.

**Exercise:** For SU(3) the $Q_r$ and $\tilde{Q}^r$ are in distinct representations. The allowed operators are $M_r^s = Q_r \tilde{Q}^s$, $B = QQQ$ and $\tilde{B} = \tilde{Q} \tilde{Q} \tilde{Q}$ (indices suppressed.) Show that the conditions we have just obtained from the SU($N$) D-terms imply that for $N_f < N$ the rank of $M$ is $N_f$ or less; for $N_f = N$ det $M = B\tilde{B}$; and for $N_f > N$ the rank of $M$ is $N$ or less. Show
also that for $N_f \geq N$ there are branches with $B \neq 0$ but $M = 0$, and that $B\hat{B}$ always equals a subdeterminant of rank $N$ of $M$.

What then happens quantum mechanically for $SU(2)$? Let’s note that for $N_f = 1, 2, 3, 4, 5$, the nonanomalous R-charge of the $Q_r$ is $-1, 0, 1, 3, 5$, for $N_f \geq 6$ the theory is no longer asymptotically free. For $N_f = 4, 5$ we might well have a conformal fixed point in the infrared, but not for $N_f < 4$. We should check for renormalizations. The classical superpotential is zero; the one-loop holomorphic gauge coupling is renormalized; but neither can get any further perturbative renormalization for the reasons we discussed earlier. All of the higher-loop effects in the NSVZ beta function come through the Kähler potential. However, we did not check that nonperturbative effects were absent. In particular, while the perturbative superpotential cannot depend on the theta angle, this is not true nonperturbatively. We should therefore look for a superpotential of the form

$$W_{\text{nonpert}}(M_{uv}, \hat{\Lambda})$$

which is invariant under all of the global symmetries. The only globally-symmetric holomorphic object which we can build from $M$ is its Pfaffian $\text{Pf} M$, which has dimension $2N_f$ and has R-charge $2(N_f - 2)$, and its powers. The superpotential has dimension 3 and R-charge 2, so its form is very highly constrained; in fact

$$W_{\text{nonpert}} = c \left( \frac{\text{Pf} M}{\hat{\Lambda}^{6-N_f}} \right)^{1/(N_f-2)}$$

where $c$ is a constant, is the only possibility. (This was pointed out by Affleck, Dine and Seiberg in 1984.) You should check that this formula is also consistent with the anomalous $U(1)$ symmetries which we used to write the physical version of $\Lambda$. For this reason, the above formula even holds for $N_f = 0$, where there is no anomaly-free R-symmetry.

Now let us examine whether the coefficient $c$ can ever be nonzero. Affleck, Dine and Seiberg pointed out that $c$ is in fact nonzero in the case $N_f = 1$; they showed that an instanton effect does indeed give a mass to the fermion in the multiplet $M \equiv M_{12}$, and calculated it, showing that

$$W_{N_f=1} = \frac{\Lambda^5}{M}$$

This is rather strange; the potential

$$V(M) \sim \frac{1}{|M|^2}$$
blows up at small $M$ (though the Kähler potential cannot be calculated there) and runs gradually to zero as $M \to \infty$ (where the gauge theory is broken at a very high scale, and thus at weak coupling, where the Kähler potential is easy to calculate.) In short, this theory has no supersymmetric vacuum except at $M = \infty$; it has a runaway instability!

However, if we add a mass for the two doublets

$$W_{\text{classical}} = mQ_1Q_2$$

then the effective superpotential becomes

$$W_{\text{full}} = \frac{\Lambda^5}{M} + mM$$

which has supersymmetric minima

$$M^2 = m\Lambda^5$$

or in other words two vacua, $M = \pm \sqrt{m\Lambda^5}$. Notice that the superpotential in Eq. (37), for $N_f = 0$, gives $W = \pm c\sqrt{\Lambda_6^0}$, which, for $c = 2$ and the matching condition $\Lambda^6_0 = m\Lambda^5_1$, is consistent with (38). The interpretation of this result, originally due to Witten (1980), is that the pure $\mathcal{N} = 1$ $SU(2)$ gauge theory has a fermion bilinear condensate

$$\langle \lambda\lambda \rangle \propto \sqrt{\Lambda_6^0}$$

which breaks a discrete chiral symmetry, somewhat analogous to QCD’s breaking of chiral symmetries, and generates a nonzero superpotential $W \propto \lambda\lambda$.

What about $N_f = 2$? In this case the theory has six mesons $M_{uv}$ subject to the constraint $\text{Pf} M = 0$. There can be no nontrivial superpotential here built just from $M$, but Seiberg (1994) pointed out that it was useful to implement this constraint using a Lagrange multiplier field $X$, of R-charge 2 and dimension -1, in the tree-level superpotential:

$$W_{\text{classical}} = X(\text{Pf} M)$$

Then $\frac{\partial W}{\partial X} = \text{Pf} M = 0$ defines the classical moduli space. However, quantum mechanically we are allowed by the symmetries to add

$$W_{\text{nonpert}} = cX\Lambda^4_2$$

which means

$$\frac{\partial W}{\partial X} = \text{Pf} M + c\Lambda^4_2 = 0$$
so the classical moduli space is modified quantum mechanically. In particular, the symmetric point \( \text{Pf} \, M = 0 \) is removed! This means that the chiral \( SU(4) \) symmetry is nowhere restored on the moduli space — there is quantum breaking of a chiral symmetry in this theory!

**Exercise**: By adding mass terms for the fields \( Q_3, Q_4 \) and comparing with the \( N_f = 1 \) case, show that \( c \) cannot be zero.

And for \( N_f = 3 \)? Here the proposed quantum superpotential

\[
W = \frac{\text{Pf} \, M}{\Lambda_3^3}
\]

is exactly right for imposing the classical constraint that \( M \) have rank 2. The interpretation Seiberg gave is that the \( M \) fields are mesons built from confined quarks, and they have an \( XYZ \)-like superpotential quantum mechanically, one which is marginally irrelevant. In the infrared, the \( M \) fields are free, and at the origin, the \( SU(6) \) symmetry is unbroken. This is the first example known of confinement *without* chiral symmetry breaking.

For \( N_f = 4, 5 \), the proposed superpotential is singular at \( M = 0 \), and cannot be valid there. Seiberg (1994) therefore suggested that there are nontrivial infrared fixed points at \( M = 0 \) for \( N_f = 4, 5 \). The evidence in favor of this suggestion is now very strong, although it is still not really proven beyond a shadow of doubt. Personally I don’t doubt it, but I would love to see a conclusive proof someday.

8. Suggested Reading

There are many great papers, and many excellent reviews, for you to look at in your further explorations of this subject. I learned supersymmetry, the Wess-Zumino model, non-renormalization theorems, and so forth from Wess and Bagger and from West; both books have advantages and problems. Philip Argyres has a set of very useful lectures; they can be accessed from his website. Renormalization you must learn from many places; no one book does it all well. The two classic papers of Seiberg and Witten (1994) on duality in \( \mathcal{N} = 2 \) and the various papers of Seiberg on holomorphy and on duality (1993-1994) in \( \mathcal{N} = 1 \) are must-reads for everyone. There are pedagogical reviews (try Bilal (1995) for \( \mathcal{N} = 2 \), Intriligator and Seiberg (1995) for \( \mathcal{N} = 1 \) that unpack these papers somewhat. Three-dimensional supersymmetric abelian gauge theories were first studied in papers by Seiberg and Witten (1996) and by Intriligator and Seiberg (1996); see also de Boer...
et al. (1996, 1997), and Aharony et al. (1997). Vortex solutions appear in Nielsen and Olesen, and earlier in work of Abrikosov in the context of superconductivity. Duality is best understood by first studying the classic work on the Ising model, and by reading a lovely paper on bosonization by Burgess and Quevedo (1995). The work of the author and Kapustin (1998) follows in this spirit and points in new directions. You can also get a quick tour of duality (though not as quick as in these lectures) and vortices in my Trieste 2001 lectures. The papers of Shifman and Vainshtein, many cowritten with Novikov and Zakharov (1980-1988), painstakingly explored and finally drained the swamp surrounding the distinction between the holomorphic and physical gauge couplings. The work of Leigh and the author (1995) on exactly marginal couplings builds on their results, as well as on related results in two dimensions (see for example Martinac (1989) and Lerche, Vafa and Warner (1989)). A summary and list of references concerning recent refinements in the study of beta functions can be found in an appendix of a paper by Nelson and the author (2002).

This is as sketchy a bibliography as can be imagined; there are literally hundreds of interesting papers which are relevant to these lecture notes. Well, such is the fate of most papers that we write; we may love them dearly, but it is wise to remember that the next generation of students will never read them.