A Generalized Occupation Time Formula
For Continuous Semimartingales

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Abstract

We show that for a wide class of functions $F$ that:

$$
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ F(s, X_s) - F(s, X_s - \varepsilon) \right\} d\langle X, X \rangle_s = -\int_0^t \int_\mathbb{R} F(s, x) dL^x_s
$$

where $X_t$ is a continuous semi-martingale, $(L^x_t, x \in \mathbb{R}, t \geq 0)$ its local time process and $(\langle X, X \rangle_t, t \geq 0)$ its quadratic variation process.

Key words and phrases: Continuous semimartingale, local time, occupation time formula.

MSC2000: 60H05, 60J65.

1 Introduction

Recently Feng and Zhao [1] define the integral of local time $\int_0^t \int_\mathbb{R} g(s, x) dL^x_s$ pathwise and then they derived a generalized Itô’s formula when $\nabla^- F(s, x)$ is only of bounded $p, q$-variation in $(s, x)$. In the case that $g(s, x) = \nabla^- F(s, x)$ is of locally bounded variation in $(s, x)$, the integral $\int_0^t \int_\mathbb{R} \nabla^- F(s, x) dL^x_s$ is the Lebesgue-Stieltjes integral. When $g(s, x) = \nabla^- F(s, x)$ is of only locally $p, q$-variation, where $p \geq 1$, $q \geq 1$, and $2q + 1 > 2pq$, the integral is a two-parameter rough path integral rather than a Lebesgue-Stieltjes integral.

In section 2, we first study the time-independent case and establish a formula which in particular unify the expression of local time of a continuous semimartingales defined as

$$
L^a_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[a, a+\varepsilon]}(X_s) d\langle X, X \rangle_s
$$

and the expression of the quadratic variation process in terms of contributions coming from fluctuations in the process that occur in the vicinity of different spatial points $a \in (-\infty, \infty)$:

$$
\langle X, X \rangle_t = \int_\mathbb{R} L^a_t da
$$

we then deal with the time-dependent case. A recent survey of semimartingales local time and occupation density concepts is given by I. Serot in [2].
2 Main Results

2.1 Time independent Case

Using Lyons-Youngs integration of one parameter \( p \)-variation, Feng and Zhao \[1\] defined \( \int_{\mathbb{R}} F(x) \, dx \, L_t^x \) as a rough path integral if \( F(x) \) is of bounded \( p \)-variation \((1 \leq p < 2)\). They also proved a dominated convergence theorem \([1]\) Theorem 2.1) for the rough path integral and then extended Meyer’s formula to \( F(x) \) is of bounded \( p \)-variation \((1 \leq p < 2)\). We shall use their results in order to establish the following theorem.

**Theorem 1** Let \( F \) be a left continuous function with bounded \( p \)-variation \((1 \leq p < 2)\), we have the following:

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \int_0^t \left\{ F(X_s) - F(X_s + \varepsilon) \right\} d\langle X, X \rangle_s = -\int_{\mathbb{R}} F(x) \, dx \, L_t^x
\]

\(2.1\)

**Remark 2.1**

1. If we take \( F(t, x) = 1_{(x \leq a)} \) in \((2.1)\) we have the very definition of \( L_t^a \).
2. If we take \( F(x) = x \) in \((2.1)\) we have \( \langle X, X \rangle_t = \int_{\mathbb{R}} L_t^x \, dx \).

**Proof:** Let us associate to \( F \) the following function:

\[
H_\varepsilon(x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} F(y) \, dy
\]

\(2.2\)

On the one hand we have:

\[
H_\varepsilon(x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} F(y) \, dy \to F(x) \quad \text{for} \quad \varepsilon \to 0
\]

\(2.3\)

On the other hand

\[
\frac{\partial}{\partial x} H_\varepsilon(x) := \frac{1}{\varepsilon} \{ F(x + \varepsilon) - F(x) \}
\]

\(2.4\)

We note that the function \( H_\varepsilon(x) \) in \((2.2)\) is of bounded \( p \)-variation \((1 \leq p < 2)\) for any fixed \( \varepsilon > 0 \). This is may be easily proved by checking the definition of \( p \)-variation or as communicated to the author by Prof. Lyons:

(i) the property of having finite \( p \)-variation \((p < 2)\) can be expressed in terms of a norm being bounded.

(ii) the property is preserved under translation.

(iii) the ball in any norm is convex.

(iv) the function \( H_\varepsilon \) is defined as an integral with a convex combination of translates of the original path.

It follows, from Theorem 2.1 in \([1]\)

\[
\int_{\mathbb{R}} H_\varepsilon(x) \, dx \, L_t^x \to \int_{\mathbb{R}} F(x) \, dx \, L_t^x
\]

\(2.5\)

and

\[
\int_{\mathbb{R}} H_\varepsilon(x) \, dx \, L_t^x = \frac{1}{\varepsilon} \int_0^t \left\{ F(X_s) - F(X_s + \varepsilon) \right\} d\langle X, X \rangle_s
\]

\(2.6\)

We use here Feng-Zhao theorem (2.1) \([1]\)
2.2 Time-dependent Case

**Theorem 2** Let \( F : [0, t] \times \mathbb{R} \rightarrow \mathbb{R} \) be a left continuous, locally bounded with bounded \( \gamma \)-variation in \( x \) uniformly in \( s \) and of bounded \( p, q \)-variation in \( (s, x) \), where \( 1 \leq \gamma < 2 \) and \( p, q \geq 1, 2q + 1 > 2pq \), Then

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ F(s, X_s) - F(s, X_s - \varepsilon) \right\} d\langle X, X \rangle_s = -\int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s
\]

(2.7)

and also,

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \left\{ F(s, X_s - \varepsilon) - F(s, X_s + \varepsilon) \right\} d\langle X, X \rangle_s = \int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s
\]

(2.8)

**Proof:** By Remark 4.1 and Theorem 4.2 in Feng and Zhao (Two-parameter \( p, q \)-variation Paths and Integrations of Local Times): Let us associate to \( F \) the following function:

\[
H_\varepsilon(t, x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} F(t, y) dy
\]

(2.9)

On the one hand we have:

\[
H_\varepsilon(t, x) := \frac{1}{\varepsilon} \int_x^{x+\varepsilon} F(t, y) dy \rightarrow F(t, x) \text{ for } \varepsilon \rightarrow 0
\]

(2.10)

On the other hand

\[
\frac{\partial}{\partial x} H_\varepsilon(t, x) := \frac{1}{\varepsilon} \{ F(t, x + \varepsilon) - F(t, x) \}
\]

(2.11)

We check easily that the function \( H_\varepsilon(s, x) \) is of bounded \( \gamma \)-variation in \( x \) uniformly in \( s \) and of bounded \( p, q \)-variation in \( (s, x) \), where \( 1 \leq \gamma < 2 \) and \( p, q \geq 1, 2q + 1 > 2pq \) for any fixed \( \varepsilon > 0 \) (see Proof of Theorem 1 above for similar arguments). It follows:

\[
\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL^x_s \rightarrow \int_0^t \int_{\mathbb{R}} F(s, x) dL^x_s
\]

and

\[
\int_0^t \int_{\mathbb{R}} H_\varepsilon(s, x) dL^x_s = \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}} \left\{ F(s, X_s) - F(s, X_s + \varepsilon) \right\} d\langle X, X \rangle_s
\]

3 Occupation Time Formula

When \( F_\varepsilon(t, x) = f(t, x) \) exists, (1.1) becomes the classical occupation time formula for continuous semimartingales:

\[
\int_0^t f(s, X_s) d\langle X, X \rangle_s = \int_{\mathbb{R}} \int_0^t f(s, x) d_s L^x_s dx
\]

(3.1)

**References**

[1] C. Feng and H. Zhao, *Two-parameter p, q-variation Paths and Integrations of Local Times, Potential Analysis*, Vol. 25 (2006).

[2] I. Serot, *Temps local et densités d’occupation: panorama* [A survey of local time and occupation density] Ann. I.S.U.P. 46 (2002), no. 3, 21–41.