On the growth of linear perturbations

David Polarski* and Radouane Gannouji†

Lab. de Physique Théorique et Astroparticules, CNRS
Université Montpellier II, France

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Abstract

We consider the linear growth of matter perturbations in various dark energy (DE) models. We show the existence of a constraint valid at $z = 0$ between the background and dark energy parameters and the matter perturbations growth parameters. For ΛCDM $\gamma_0 \equiv \frac{d\gamma}{dz}|_0$ lies in a very narrow interval $-0.0195 \leq \gamma_0 \leq -0.0157$ for $0.2 \leq \Omega_{m,0} \leq 0.35$. Models with a constant equation of state inside General Relativity (GR) are characterized by a quasi-constant $\gamma_0$, for $\Omega_{m,0} = 0.3$ for example we have $\gamma_0 \approx -0.02$ while $\gamma_0$ can have a nonnegligible variation. A smoothly varying equation of state inside GR does not produce either $|\gamma_0| > 0.02$. A measurement of $\gamma(z)$ on small redshifts could help discriminate between various DE models even if their $\gamma_0$ is close, a possibility interesting for DE models outside GR for which a significant $\gamma_0$ can be obtained.

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*email:polarski@lpta.univ-montp2.fr
†email:gannouji@lpta.univ-montp2.fr
1 Introduction

There is growing observational evidence for the late-time accelerated expansion of our universe \[2\]. This radical departure from conventional decelerated expansion is certainly a major challenge to cosmology. This non standard expansion could be due to an exotic non clustered component yet to be determined with a sufficiently negative pressure called Dark Energy (DE). By analogy all models trying to explain the accelerated expansion are called DE models but many models go now well beyond this simple picture. While the usual Friedmann equations in the presence of a cosmological constant term \(\Lambda\) seem to be in good agreement with the data, it is clear that other models with a variable equation of state are allowed as well \[2\]. While a cosmological constant universe is appealing because of its simplicity it nonetheless poses the problem of the magnitude of the cosmological constant \(\Lambda\). This is the basic incentive to look for other models where DE has a variable equation of state. An additional incentive comes from the possibility to have phantom dark energy at low redshifts as this excludes the quintessence models, models with a minimally coupled scalar field inside GR \[3\]. It might also be that one should change the theory of gravity, as for example in scalar-tensor models \[4, 5\], and a lot of research has focused recently on other modified gravity models and higher dimensional models. Sometimes, the background expansion going back to high redshifts is enough to rule out some models \[7\], but typically this is not the case: models of a very different kind will be able to have a viable background expansion where the low redshift expansion is in accordance with SNIa data.

Depending on the gravity theory one is considering, the growth of the perturbations, even at the linear level, will be affected. Indeed, while distance luminosity measurements probe the cosmic expansion, matter perturbations probe in an independent way (see e.g. \[8\]) the gravity theory responsible for their growth (and of course also for the cosmic expansion). The growth rate of matter perturbations could be probed with three dimensional weak lensing surveys (see e.g. \[9\]). Hence two DE models based on different gravitation theories can give the same late-time accelerated expansion and still differ in the matter perturbations they produce \[10\]. This fact could provide an additional important way to discriminate between various models (see e.g. \[11\]) and it is therefore important to characterize as accurately as possible the growth of matter perturbations which is the aim of the present work.

2 Linear growth of perturbations

Let us consider the dynamics of the linear matter perturbations. These perturbations satisfy a modified equation of the type

\[
\ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G_{\text{eff}} \rho_m \delta_m = 0 ,
\]

(1)

The gravitational constant \(G_{\text{eff}}\) depends on the specific model under consideration and the corresponding modification of gravity. For example, as was shown in \[5\], for
scalar-tensor DE models we have
\[ G_{\text{eff}} = G_N \frac{F + 2(dF/d\Phi)^2}{F + \frac{2}{3}(dF/d\Phi)^2} = G_N \frac{1 + 2\omega_{BD}^{-1}}{1 + \frac{3}{2}\omega_{BD}}. \]  
(2)

An equation similar to (1) is also found for example in DGP models. The physics behind it is a modification of Poisson’s equation (see e.g. [6]) according to (we drop the subscript \( m \))
\[ \frac{k^2}{a^2} \phi = -4\pi G \rho \delta \rightarrow \frac{k^2}{a^2} \phi = -4\pi G_{\text{eff}} \rho \delta. \]  
(3)

Of course, more drastic modifications are possible as well. In particular more elaborate DE models can be considered that could further increase the degeneracy between models inside and outside GR (see e.g. [12]). It is convenient to introduce the quantity \( f = \frac{d\ln \delta}{d\ln a} \). Then the linear perturbations obey the equation
\[ \frac{df}{dx} + f^2 + \frac{1}{2} \left( 1 - \frac{d\ln \Omega_m}{dx} \right) f = \frac{3}{2} G_{\text{eff}} \frac{G_N}{\Omega_m}. \]  
(4)

with \( x \equiv \ln a \). Equation (4) reduces to eq.(B7) given in [13] for \( G_{\text{eff}} G_N = 1 \). The quantity \( \delta \) is easily recovered using \( f \) as follows
\[ \delta(a) = \delta_i \exp \left[ \int_{x_i}^{x} f(x')dx' \right]. \]  
(5)

We see that \( f = p \) when \( \delta \propto a^p \), in particular \( f \rightarrow 1 \) in ΛCDM for large \( z \) and \( f = 1 \) in an Einstein-de Sitter universe.

An important issue is to characterize departures on small redshifts for different models. It is well known that for in a ΛCDM universe one can write
\[ f \simeq \Omega_m^\gamma, \]  
(6)

with \( \gamma = \text{constant} \sim 0.6 \), an approach pioneered some time ago [14] and generalised in [15]. The characterization of the growth of matter perturbations using a parametrization of the form (6) has attracted a lot of interest in the hope to discriminate between DE models based on different gravity theories.

Of course it is possible to write in full generality
\[ f = \Omega_m(z)^{\gamma(z)}. \]  
(7)

Let us consider the quantity \( \gamma' \equiv \frac{d\gamma}{dz} \). For many models it turns out that
\[ \gamma(z) \approx \gamma_0 + \gamma'_0 z \quad 0 \leq z \leq 0.5. \]  
(8)

As we will see later, this could have interesting observational consequences.

We now derive a constraint which is valid in general for any \( \gamma(z) \). It is easy to obtain the following equation
\[ - (1 + z) \ln \Omega_m \gamma' + \Omega_m^\gamma + \frac{1}{2} (1 + 3(2\gamma - 1) w_{\text{eff}}) = \frac{3}{2} \frac{G_{\text{eff}}(z)}{G_{N,0}} \Omega_m^{1-\gamma}, \]  
(9)
where \( w_{\text{eff}} \equiv w_{DE} \Omega_{DE} \). From (10), it is easy to derive the following equation

\[
\gamma'_0 = [\ln \Omega_{m,0}^{-1}]^{-1} \left[ -\Omega_{m,0}^{\gamma_0} - 3(\gamma_0 - \frac{1}{2}) w_{\text{eff},0} + \frac{3}{2} \frac{G_{\text{eff},0}}{G_{N,0}} \Omega_{m,0}^{1-\gamma_0} - \frac{1}{2} \right].
\]  

(10)

Equation (10) is further simplified in models for which \( \frac{G_{\text{eff},0}}{G_{N,0}} = 1 \) to very high accuracy. An example where this is the case is provided by scalar-tensor DE models for which \( 0 < \frac{G_{\text{eff},0}}{G_{N,0}} < 1 < 1.25 \times 10^{-5} \). We then obtain

\[
\gamma'_0 = [\ln \Omega_{m,0}^{-1}]^{-1} \left[ -\Omega_{m,0}^{\gamma_0} - 3(\gamma_0 - \frac{1}{2}) w_{\text{eff},0} + \frac{3}{2} \Omega_{m,0}^{1-\gamma_0} - \frac{1}{2} \right].
\]  

(11)

This does not mean that equation (11) cannot differentiate between different gravitation theories satisfying \( \frac{G_{\text{eff},0}}{G_{N,0}} = 1 \) but rather that if it does so it is through the value of \( \gamma_0 \). This value is of course affected by the function \( G_{\text{eff}}(z) \). We will assume below \( \frac{G_{\text{eff},0}}{G_{N,0}} = 1 \) to very high accuracy. As we see from (11), we have \( \gamma'_0 = \gamma'_0(\gamma_0, \Omega_{m,0}, w_{DE,0}) \) which is clearly equivalent to a constraint of the form

\[
f(\gamma_0, \gamma'_0, \Omega_{m,0}, w_{DE,0}) = 0.
\]  

(12)

In this connection one should note that fitting functions of \( \gamma(z) \) proposed in the literature, even though they give a satisfactory fit for \( f(z) \) in models satisfying some assumptions, generically will not satisfy the constraint (12). In contrast the constraint (12) does not depend on any assumption about \( w(z) \). For fixed \( \Omega_{m,0}, w_{DE,0}, \) there will be a value \( \gamma_{0,cr} \) for which \( \gamma'_0 = 0 \). However we will have generically \( \gamma_0 \neq \gamma_{0,cr} \) and therefore \( \gamma'_0 \neq 0 \).

Very generally, in any model for which the parameters \( \Omega_{m,0} \) and \( w_{DE} \) (and hence \( w_{DE,0} \)) are given, one can compute numerically the function \( \gamma(z) \) from the linear growth of the matter perturbations. Using (12) it is then possible to obtain \( \gamma'_0 \). We will do this in the next Section for various models inside GR.

Before considering specific DE models, it is possible to derive some general consequences from the constraint (12). Generically \( \gamma'_0 \) will not vanish, it needs not even be small. Let us consider \( \gamma'_0 \) in function of \( \gamma_0 \) for \( \Omega_{m,0} \) and \( w_{DE,0} \) fixed. As we can see from Figure 1a, the constraint (12) implies in excellent approximation a linear relation as follows

\[
\gamma'_0 \simeq c + b (\gamma_0 - 0.5) \quad \quad \quad \quad b \sim 3.
\]  

(13)

The coefficients \( c, b \) depend on the background parameters \( b = b(w_{DE,0}, \Omega_{m,0}) \) (remembering that we take \( \frac{G_{\text{eff},0}}{G_{N,0}} = 1 \)). The coefficient \( b \) decreases while \( c \) increases when \( \Omega_{m,0} \) decreases from 0.35 to 0.20 (see Figure 1b). In contrast, \( c \) increases from \(-0.19 \) for \( \Omega_{m,0} = 0.3 \), to \(-0.17 \) for \( \Omega_{m,0} = 0.2 \).

For \( \Omega_{m,0} = 0.3 \) we have \( c = -0.19 \). We stress that relation (13) will hold independently of any particular model and is a consequence of the constraint (12).

Depending on the specific model under consideration, for given background parameters \( \Omega_{m,0} \) and \( w_{DE,0}, \gamma'_0 \) will take the value \( \gamma'_0(\gamma_0) \) corresponding to the value \( \gamma_0 \) “realized” by the model. Generically we will have \( \gamma'_0 \neq 0 \).
Figure 1: a) The left panel shows the constraint (11) for $\Omega_{m,0} = 0.3$ and various values of $w_{DE,0}$. We have from top to bottom: $w_{DE,0} = -1.4$, $-1.3$, $-1.2$, $-1$, $-0.8$. For given $\Omega_{m,0}$ and $w_{DE,0}$, the couple $\gamma_0$, $\gamma_0'$ is on the corresponding line for any model while $\gamma_0'$ will depend on the value $\gamma_0$ realized in a particular model. b) On the right panel the constraint (11) is shown in function of $\Omega_{m,0}$. From top to bottom we have $w_{DE,0} = -1.2$, $-1$, $-0.8$. We see that the coefficient $b$ defined in (13) increases for increasing $\Omega_{m,0}$ and decreasing $w_{DE,0}$. It is also seen from Figure 1a that a small variation of $\gamma_0$, for fixed parameters $w_{DE,0}$, $\Omega_{m,0}$, can induce a non negligible variation of $\gamma_0'$ in accordance with eq. (13). In particular the relative change in $\gamma_0'$ can be very large. We will show below that for $w_{DE} = \text{constant}$, the $\gamma_0'$ values are restricted to a very narrow range with $\gamma_0' \approx -0.02$. Even when one consider a smoothly varying equation of state, we still have $-0.02 \lesssim \gamma_0' \lesssim 0.005$ (see below) for $0.20 \leq \Omega_{m,0} \leq 0.35$. In other words a smooth change in the equation of state of DE is not able to produce $\gamma_0' < -0.02$ for viable cosmological parameters. Therefore, a measurement of $\gamma_0'$ outside this range could be a characteristic signature of a DE model where gravity is modified. Moreover, a precise determination of $\gamma_0'$ could help to better discriminate between various modified gravity models.

When $\Omega_{m,0} = 0.3$ we have $b = 3.13$ for $w_{DE,0} = -1$, while $b$ becomes smaller for $w_{DE,0} > -1$ and larger for $w_{DE,0} < -1$ (phantom DE today). Hence for $w_{DE,0} < -1$, we get a larger variation $\Delta \gamma_0'$ for a given variation $\Delta \gamma_0$. When $\Omega_{m,0}$ decreases, so does the coefficient $b$ however this decrease is rather small for relevant cosmological values. It would be most interesting to investigate whether a precise determination of $\gamma_0'$ is observationally accessible. In view of this it means that one should measure precisely $\gamma(z)$ on $0 \leq z \leq 0.5$. Another aspect concerns the extraction of one, or both, of the parameters $\Omega_{m,0}$ or $w_{DE,0}$. If we assume erroneously that $\gamma_0' = 0$, a large error can result in the determination of $\Omega_{m,0}$ or $w_{DE,0}$ from the knowledge of $\gamma_0$. This is illustrated in Figure 2a.
Figure 2: a) On the left panel, the blue line shows the degeneracies in the $\Omega_{m,0}$, $w_{DE,0}$ plane for $\gamma_0 = 0.555$ assuming $\gamma_0' = 0$. The red, resp. green, dashed lines correspond to $\gamma_0' = -0.02$ (top) and $\gamma_0' = 0.02$ (bottom), resp. $\gamma_0' = -0.05$ (top) and $\gamma_0' = 0.05$ (bottom). Ignoring the true non-vanishing value of $\gamma_0'$ increases significantly the uncertainty on the couples $\Omega_{m,0}$, $w_{DE,0}$. b) On the right panel it is seen that models with very close $\gamma_0$ can be discriminated if $\gamma$ is measured for $0 \leq z \leq 0.5$ assuming $\gamma$ is linear on small $z$, as often is the case. The lower the values of $\gamma_0$, the easier it is to discriminate these models through the difference in their slope $\gamma_0'$. For illustration, we have assumed here an error of 1%.

3 Some specific models

We now turn our attention to specific models inside General Relativity where DE has a known equation of state.

3.1 $\Lambda$CDM

Because of its simplicity and of the recent data that seem to imply that viable DE models should not be too far from $\Lambda$CDM (see however [16]), this model plays a central role. We find for $\Lambda$CDM $0.554 \leq \gamma_0 \leq 0.558$ (see Figure 3b) and $-0.0195 \leq \gamma_0' \leq -0.0157$ for $0.2 \leq \Omega_{m,0} \leq 0.35$. Hence $\gamma_0$ varies very little in function of $\Omega_{m,0}$ while $\gamma_0'$ is negative with $|\gamma_0'| < 0.02$. An observation outside these values, in particular a positive value for $\gamma_0'$, or a large negative $\gamma_0'$, would signal a departure from $\Lambda$CDM.

3.2 Constant equation of state

We consider now a constant equation of state which includes of course the $\Lambda$CDM model. For the conservative ranges $0.2 \leq \Omega_{m,0} \leq 0.35$ and $-1.5 \leq w_{DE,0} \leq -0.5$, we find $0.542 < \gamma_0 < 0.583$ and $-0.021 < \gamma_0' < -0.013$. However, as can be seen from Figure 3a, for fixed parameter $\Omega_{m,0}$, the value of $\gamma_0'$ is practically constant with $\gamma_0' \approx -0.02$ for different constant $w_{DE}$ despite a non-negligible variation of $\gamma_0$. To summarize, for constant $w_{DE}$, $\gamma_0'$ lies in the restricted range $-0.024 < \gamma_0' < 0.01$ while it is practically constant if $\Omega_{m,0}$ is fixed. However, as emphasized above (see Figure
2a), even in that case neglecting the true (nonzero) value of $\gamma_0'$ can induce a significant error in the determination of $\Omega_{m,0}$ or $w_{DE,0}$ from $\gamma_0$. Finally, it is interesting to note that for given $\Omega_{m,0}$ all these models have essentially the same $\gamma_0'$ while the parameter $\gamma_0$ can vary by about 4\% (see Figure 3a).

Figure 3: a) The lines in colour on the left panel are the same as in Figure 1. The black line gives the true value of $\gamma_0$ realised in models with $w_{DE} = w_{DE,0} = \text{constant}$ and $\Omega_{m,0} = 0.3$. It is seen that all models with $w_{DE} = \text{constant}$ shown here have practically the same non vanishing $\gamma_0'$, $\gamma_0' \approx -0.02$. Note that $\gamma_0$ increases when $w_{DE}$ increases. b) On the right, $\gamma_0$ is displayed in function of $\Omega_{m,0}$ for the $\Lambda$CDM model.

### 3.3 Variable equation of state

Our analysis can be repeated for DE with a variable equation of state. To be specific, we take a smoothly varying equation of state of the type [17, 18]

$$w_{DE}(z) = (-1 + \alpha) + \beta (1 - x) \equiv w_0 + w_1 \frac{z}{1+z},$$

(14)

where $x \equiv \frac{a}{a_0}$. The corresponding evolution of the DE energy density can be computed analytically and yields [17]

$$\rho_{DE}(z) = \rho_{DE,0} (1 + z)^{3(\alpha + \beta)} e^{-3\beta \frac{z}{1+z}}.$$  

(15)

The results are displayed in Figure 4 for models with a negligible $\Omega_{DE}$ for $z \gg 1$. For example, if we fix $w_0 = -1.2$, we can compute the values of $\gamma_0$ and $\gamma_0'$ in function of $\beta \equiv w_1$ and $\Omega_{m,0}$. We find $0.55 \lesssim \gamma_0 \lesssim 0.56$ and $-0.022 \lesssim \gamma_0' \lesssim 0.005$ for $0.20 \leq \Omega_{m,0} \leq 0.35$ and $0 \leq \beta \leq 1$.\footnote{Note that slightly lower values for $\gamma_0'$ can be obtained for less interesting models with substantial phantomness in the asymptotic past} To summarize, a smoothly varying equation of state does not seem able to generate $|\gamma_0'| > 0.02$.\footnote{Note that slightly lower values for $\gamma_0'$ can be obtained for less interesting models with substantial phantomness in the asymptotic past}
Figure 4: The parameters $\gamma_0$ (left) and $\gamma'_0$ (right) are shown in function of $\beta \equiv w_1$ and $\Omega_{m,0}$ for a model with variable equation of state parameter $w_{DE} = -1.2 + \beta \frac{z}{1+z}$. Hence all the points on the two surfaces have $w_{DE,0} = -1.2$. The results for $w_{DE} = -1.2$ are recovered for $\beta = 0$. We note for the left figure that $\gamma'_0 = 0$ is obtained for some particular combinations $\beta$, $\Omega_{m,0}$.

4 Summary and conclusions

Considering the linear growth of matter perturbations in various models, we give a constraint at $z = 0$, eq.(11) valid for all models, including modified gravity DE models that satisfy $\frac{G_{eff,0}}{G_{N,0}} = 1$. This constraint implies that the quantity $\gamma'_0$ is completely fixed by the remaining parameters $\gamma_0$, $w_{DE,0}$ and $\Omega_{m,0}$. For the models considered here inside GR, $|\gamma'_0| \lesssim 0.02$. Interestingly for models inside GR with constant $w_{DE}$, $\gamma'_0$ is quasi-constant with $\gamma'_0 \approx -0.02$ as the variation of $w_{DE,0}$ is compensated by a simultaneous variation of $\gamma_0$ (for given $\Omega_{m,0}$).

We have generically $\gamma'_0 \neq 0$ and we emphasize that a significant $\gamma'_0$ could help discriminate between models, even if their $\gamma_0$ values are close. We have illustrated this schematically on Figure 2b. This potential resolution improves as $\Omega_{m,0}$ goes up and/or $w_{DE,0}$ goes down and could be important when dealing with DE models outside General Relativity. We will give elsewhere specific models where this is the case [19]. Generally, this approach could be very fruitful whenever $\gamma(z)$ is close to linear on small redshifts $0 \leq z \leq 0.5$ so that the slope is essentially given by $\gamma'_0$. So we feel it would be useful to try to measure $\gamma(z)$ on small redshifts, and not just $\gamma_0$. Finally it is important to realize that neglecting a small but nonvanishing $\gamma'_0$ can induce a large error on the parameters $\Omega_{m,0}$, $w_{DE,0}$ that one could extract from the growth of matter perturbations.

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