Modified tetrahedron equations and related 3D integrable models

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Abstract

Using a modified version of the tetrahedron equations we construct a new family of $N$-state three-dimensional integrable models with commuting two-layer transfer-matrices. We investigate a particular class of solutions to these equations and parameterize them in terms of elliptic functions. The corresponding models contain one free parameter $k$ – an elliptic modulus.
1. Introduction

Recent progress in the construction of solvable lattice models in higher dimensions has stimulated efforts to find new interesting examples “living” in \( D > 2 \) dimensions. As is known the main ingredient of practically all two-dimensional solvable models is a Yang-Baxter equation (YBE) which provides an existence of a family of commuting transfer-matrices. As a rule, the presence of such a family permits us to calculate the partition function per site in the thermodynamical limit. Unlike two-dimensional conformal field theories the theory of the YBE can be easily generalized for the cases \( D > 2 \) [1–5] and it will lead us to the tetrahedron equations (\( D = 3 \)) and \( D \)-simplex equations. The main problem, of course, is to solve them, because the number of unknowns and equations grows with increasing dimension in a catastrophic way.

Recently Bazhanov and Baxter in their paper [7] have generalized the trigonometric Zamolodchikov model [1–3] for the case of \( N \) colours. The weight functions of their model satisfy tetrahedron equations ([8]) and possess some remarkable symmetry properties. But the essential shortcoming of Bazhanov-Baxter \( N \)-color model is an absence any temperature-like parameter.

Authors of [9, 14] have proposed a new class of three-dimensional models containing a temperature-like parameter \( k \). In the limit \( k \to 0 \) the weight functions of these models reduce to the “static limit” case of the Bazhanov-Baxter model. The main idea of papers [9, 14] is to introduce a couple of modified tetrahedron equations (MTE). In two dimensions this idea have been considered in [6]. Then using a couple of modified tetrahedron equations, the authors of [9, 14] have constructed two-layer commuting transfer-matrices for the checkerboard lattice of weights.

This paper, in fact, is a continuation of [9, 14]. Our main purpose is to remove the “static limit” condition which is a constraint on three “angle” parameters for weight functions. The result is that each elementary weight function of the model will depend on three spectral parameters and one “modulus”. Unfortunately, to remove the “static limit” condition we were forced to refuse from the second modified tetrahedron equation. It leads to a more complicated commuting family of two-layer transfer-matrices.

The paper is organized as follows: In Section 2 we give necessary definitions, the explicit expression for weight functions and write out the modified
tetrahedron equations. In Section 3 we recall the proof of the modified tetrahedron equations based on the Star-Square relation, give all necessary algebraic constraints on the parameters and consider an important submanifold of “equal” moduli in the space of parameters (in next sections we restrict ourselves to this case only). In Section 4 we introduce a parameterization of weight functions in terms of angle-like variables. In fact, all the formulae of Section 4 can be rewritten in terms of elliptic functions and we give the corresponding formulae in the Appendix. Section 5 contains a discussion of the symmetry properties of the weight functions under the group of transformations of the elementary cube. In Section 6 we construct a commuting family of two-layer transfer-matrices using the solutions obtained from the modified tetrahedron equations and discuss the structure of the lattice model. Finally, the Appendix contains all the elliptic formulae and a detailed consideration of the $N = 2$ case.

2. Modified tetrahedron equations and Body-Centered-Cube (BCC) ansatz for weight functions

In this section we present the modified tetrahedron equations (MTE) and briefly discuss their properties. We recall necessary definitions from [8, 11, 14] and give the symmetrical form of the weight functions $W$. Further we will follow a statistical mechanics interpretation and use the notations introduced by Baxter in [3] for the Zamolodchikov model. Namely, consider a simple cubic lattice $\mathcal{L}$ and at each site of $\mathcal{L}$ place a spin variable taking its values in $\mathbb{Z}_N$, for any integer $N \geq 2$.

To each elementary cube of the lattice we can assign some weight function $W(a|efg|bcd|h)$, depending on eight surrounding spin variables (see Fig. 1).
Note that for different elementary cubes of the lattice \( \mathcal{L} \) we can use different weight functions \( W^{(1)}, W^{(2)} \) etc. Later we will fix the explicit structure of the lattice \( \mathcal{L} \), demanding that the two-layer transfer-matrices commute between themselves.

Now let us consider two sets of weight functions: \( W, W', W'' \) and \( \overline{W}, \overline{W}', \overline{W}'' \). Suppose that they satisfy the following equations:

\[
\begin{align*}
\sum_d W(a_4|c_2, c_1, c_3|b_1, b_3, b_2|d)\overline{W'}(c_1|b_2, a_3, b_1|c_4, d, c_6|b_4) \\
\times W''(b_1|d, c_4, c_3|a_2, b_3, b_4|c_5)\overline{W}''(d|b_2, b_4, b_3|c_5, c_2, c_6|a_1) \\
= \sum_d W''(b_1|c_1, c_4, c_3|a_2, a_4, a_3|d)\overline{W}'(c_1|b_2, b_3, a_4|d, c_2, a_6|a_1) \\
\times W'(a_4|c_2, d, c_3|a_2, b_3, a_1|c_5)\overline{W}(d|a_1, a_3, a_2|c_4, c_5, c_6|b_4),
\end{align*}
\]

(2.1)

where \( a_i, b_i, c_j \in \mathbb{Z}_N, \ i = 1, \ldots, 4, \ j = 1, \ldots, 6 \). We will call relations (2.1) the modified tetrahedron equations. Note that all eight weights in (2.1) are independent, but we placed \( W \) and \( \overline{W} \) in relations (2.1) so as to follow the notation of papers [9, 14]. For the case \( W = \overline{W} \) we come to the usual tetrahedron equations (see, for example, [3]). We would like to stress that contrary to [9, 14] we do not demand the validity of the dual variant of (2.1) (with all \( W \)'s replaced by \( \overline{W} \)'s and vice versa). The absence of the dual variant of (2.1) permits us to overcome the static limit condition (see [9, 14]) but leads to a more complicated commutativity of the two-layer transfer-matrices.

Let us recall some definitions used in papers [8, 11, 14]. First denote

\[
\omega = \exp(2\pi i/N), \quad \omega^{1/2} = \exp(\pi i/N).
\]

(2.2)

Further, taking \( x, y, z \) to be complex parameters constrained by the Fermat
equation
\[ x^N + y^N = z^N \]  (2.3)
and \( l \) to be an element of \( \mathbb{Z}^N \), define
\[ w(x, y, z|l) = \prod_{s=1}^{l} \frac{y}{z - x^{\omega^s}} \]  (2.4)

In addition, define the function with one more argument
\[ w(x, y, z|k, l) = w(x, y, z|k - l)\Phi(l), \quad k, l \in \mathbb{Z}^N, \]  (2.5)
where
\[ \Phi(l) = \omega^{l(l+N)/2}. \]  (2.6)

Let us mention also two formulae for the \( w \) functions, which are useful for calculations:
\[ w(x, y, z|l + k) = w(x, y, z|k)w(x\omega^k, y, z|l), \]  (2.7)
\[ w(x, y, z|k, l) = \omega^{kl}/w(z, \omega^{1/2}y, \omega x|l, k). \]  (2.8)

Now introduce the set of homogeneous variables \( x_i, i = 1, \ldots, 8 \) and \( x_{13}, x_{24}, x_{58}, x_{67} \) satisfying
\[ x_{13}^N = x_1^N - x_3^N, \quad x_{24}^N = x_2^N - x_4^N, \quad x_{58}^N = x_5^N - x_8^N, \quad x_{67}^N = x_6^N - x_7^N. \]  (2.9)

We will also need six additional variables to ensure the necessary transformation properties of the weight functions under the group \( G \) of transformations of a three-dimensional cube. Define them as
\[
\begin{align*}
  u^N &= (x_3x_5/x_1)^N - x_8^N, \quad v^N = x_7^N - (x_4x_6/x_2)^N, \\
  \xi^N &= (x_1x_7/x_3)^N - x_6^N, \quad \lambda^N = x_5^N - (x_2x_8/x_4)^N, \\
  \mu^N &= (x_{13}x_{24}x_7/x_3x_4)^N - (x_{58}x_{67}/x_8)^N, \\
  \nu^N &= (x_{13}x_{24}x_6/x_1x_2)^N - (x_{58}x_{67}/x_5)^N. 
\end{align*}
\]  (2.10)

Using these definitions we define the weight function \( W(a|efg|bcd|h) \) as
\[
W(a|efg|bcd|h) = \left[ \frac{w(x_{58}x_{67}, x_8\mu, x_{13}x_{24}x_7x_8/x_3x_4|a + d, e + f)}{w(x_{58}x_{67}, x_5\nu, x_{13}x_{24}x_5x_6/x_1x_2|g + h, b + c)} \right]^{1/2}
\]
where the lower index “0” after the curly brackets implies that the expression in the curly brackets is divided by itself with all exterior spin variables equated to zero.

Note that all gauge multipliers (face and edge types) before curly brackets in (2.11) are necessary in order to provide the correct symmetry properties of the weight functions under the transformations of the group \( G \).

Formula (2.11) generalizes the weight functions of the model proposed by Bazhanov and Baxter [7, 11]. In fact, it coincides with the weight functions from [14] up to gauge multipliers and we will also call (2.11) the Body-Centered-Cube (BCC) ansatz for weight functions. But in contrast to [14] we will introduce a new parameterization for \( x_i \) in such a way that each weight function will depend on three independent spectral parameters and modulus parameter \( k \).

3. The proof of the modified tetrahedron equations

In this section we recall the proof of the modified tetrahedron equations (2.1) for weight functions (2.11). In fact, it reproduces the method of papers [8, 14] and we give it here for completeness.

The main idea of papers [8, 14] is to reduce complicated modified tetrahedron equations (2.1) to a couple of much more simple relations: namely “inversion” and Star-Square ones.

The “inversion” relation for functions \( w(x, y, z|k, l) \) has the form:

\[
\frac{w(x_1 x_8, x_1 u, x_3 x_5|e + h, d + c)}{w(x_4 x_6, x_2 v, x_2 x_7|a + b, f + g)} \frac{w(x_2 x_8, x_4 \lambda, x_4 x_5|e + g, a + c)}{w(x_3 x_6, x_3 \xi, x_1 x_7|b + d, f + h)} \\
\times \omega^{bf} \omega^{(ag + gb + bh)/2} \sum_{\sigma \in \mathbb{Z}^N} \frac{w(x_3, x_{13}, x_1|d, h + \sigma)w(x_4, x_{24}, x_2|a, g + \sigma)}{w(x_8, x_{58}, x_5|e, c + \sigma)w(x_7/\omega, x_{67}, x_6|f, b + \sigma)} \right)_{0}^{1/2}
\]

(2.11)
The Star-Square relation permits us to calculate the sum over the one spin variable from the product of four $w$ functions provided that their arguments satisfy some algebraic constraint. To avoid the introduction of additional variables for ensuring the cyclic property of all $w$'s modulo $N$, we will use a non-cyclic analog of $w$ function, defined recurrently as follows:

$$w(x|0) = 1, \quad \frac{w(x|l)}{w(x|l-1)} = \frac{1}{1-x\omega^l}, \quad l \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers. It is obvious that

$$w(x, y, z|l) = (\frac{y}{z})^l w(x/z|l), \quad l \in \mathbb{Z}_N,$$

where index $l$, being considered modulo $N$, is interpreted as an element of $\mathbb{Z}_N$.

Then the Star-Square relation can be written as:

$$\left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{w(x_1, y_1, z_1|a+\sigma)w(x_2, y_2, z_2|b+\sigma)}{w(x_3, y_3, z_3|c+\sigma)w(x_4, y_4, z_4|d+\sigma)} \right\}_0 =$$

$$\frac{(x_2z_1/x_1z_2)^{a-b}(x_1y_2/x_2z_1)^{b}(z_3/y_3)^{c}(z_4/y_4)^{d}}{\Phi(a-b)\omega^{(a+b)/2}}$$

$$\times \frac{w(\omega x_3x_4z_1z_2/x_1x_2z_3z_4|c+d-a-b)}{w(x_3z_1x_4z_2|d-a)w(x_3z_1x_4z_2|c-b)w(x_3z_1x_4z_2|c-a)w(x_3z_1x_4z_2|d-b)},$$

where the lower index “0” after the curly brackets indicates that the l.h.s. of (3.4) is normalized to unity at zero exterior spins, and the following constraint is imposed

$$\frac{y_1 y_2 z_3 z_4}{z_1 z_2 y_3 y_4} = \omega.$$

Note that the separate $w$’s in the r.h.s. of (3.4) are not single-valued functions on $\mathbb{Z}_N$, while the whole expression is cyclic in the exterior spins $a, b, c, d$.

Relations (3.1), (3.4) are proved by using properties of Fourier transformation over spin variables and the detailed proof was given in [8].

Now we will prove relations (2.1) for ansatz (2.11) with some appropriate choice of constant multiplier $R$. Instead of weights $W, W', W'', W'''$ let us substitute into (2.1) explicit formula (2.11) with corresponding sets of
parameters:
\[ W, W', W'', W''' \rightarrow W(x_i, x_{ij}), W(x'_i, x'_{ij}), W(x''_i, x''_{ij}), W(x'''_i, x'''_{ij}), \]
\[ \overline{W}, \overline{W}', \overline{W}'', \overline{W}''' \rightarrow W(\overline{x}_i, \overline{x}_{ij}), W(\overline{x}'_i, \overline{x}'_{ij}), W(\overline{x}''_i, \overline{x}''_{ij}), W(\overline{x}'''_i, \overline{x}'''_{ij}). \]  

(3.6)

Now we will show that all equations (2.1) are equivalent to some algebraic system of nonlinear equations for parameters \( x_i, \overline{x}_i \) etc.

Hereafter we will fix a normalization of all parameters \( x' \)'s that enter in the definition (2.11) as
\[ x_3 = 1, \quad x_4 = 1, \quad x_7 = 1, \quad x_8 = 1, \]  

and similarly for all the \( \overline{x} \)'s.

It is easy to see that after substituting of (2.11) in relations (2.1) we will explicitly obtain 24 pairs of \( w \) functions coming from the multipliers before the curly brackets in (2.11). Let us demand that all these factors cancel each other (for each \( w \) function there is one partner with the same dependence from spin variables). Then we obtain 24 relations for variables \( x_i, x_{ij} \) and \( \overline{x}_i, \overline{x}_{ij} \) (only 21 of the constraints are independent) and 24 relations for the additional variables defined by (2.10). In fact, they relate the set of rapidity like parameters between different weight functions.

Let us multiply both sides of (2.1) by the following product of \( w \) weights:
\[
\frac{w(x_7', x_6', x_5'|c_4, a_2 + l_2)}{w(x_7'/\omega, x_{13}/\omega, x_1|c_6, a_1 + l_1)} \frac{w(\overline{x}_7, \omega\overline{x}_6, \overline{x}_0|a_3, c_4 + l_3)}{w(x_4, x_{24}, \omega x_2|a_4, c_3 + l_4)} \times \\
\frac{w(\overline{x}_8', \overline{x}_{58}', \overline{x}_5'/\omega|b_2, c_2 + m_2)}{w(x_8'/\omega, x_{24}/\omega, x_1|b_1, c_3 + m_1)} \frac{w(\omega x'_5, x'_{58}, x'_5|c_2, b_3 + m_3)}{w(x_3'/\omega, x_{13}/\omega, x_1|b_4 + m_4)}
\]  

(3.8)

and sum over \( a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \). Note that due to the “inversion” relation (3.1) we do not lose any information after such transition of spins \( a_i, b_i \) to \( l_i, m_i \).

The functions \( w \) in expression (3.7) are chosen in such a way that using relation (3.1) we can calculate the sums over the spins \( a_1, a_2, a_3, a_4 \) in the l.h.s. and those over \( b_1, b_2, b_3, b_4 \) in the r.h.s. of the obtained equation and cancel the summations over the spins \( \sigma \)'s, which come from expression (2.11) for the functions \( W \)'s and \( \overline{W} \)'s. Now let us consider the applicability conditions (3.5) of Star-Square relation (3.4) for the sums over \( a_1, a_2, a_3, a_4 \) in the r.h.s. and over \( b_1, b_2, b_3, b_4 \) in the l.h.s. of the obtained equation. We
obtain eight conditions on the $x$’s and $\bar{x}$’s. Applying relation (3.4) eight times and calculating the sums over $d$ spin in the l.h.s. and r.h.s (the spin structure of the sums over $d$ has the form of relation (3.1) and we demand that corresponding variables $x, y, z$ entering in the arguments of functions $w(x, y, z | k, l)$ are constrained in such a way that relation (3.1) can be used), we come to the equation without any summation. The l.h.s. and r.h.s. of this equation consists of the products of ten $w$ functions with the same spin structure and expressions like $x_1 x_2$ coming from relation (3.4). Let us impose all the necessary constraints on parameters $x_i, x_{ij}, \bar{x}_i$ and $\bar{x}_{ij}$ to satisfy this equation. The explicit calculations show that we obtain 11 more independent constraints on parameters. Taking into account $21 + 24 = 45$ constraints coming from a cancellation of “face” factors in (2.11) we have an algebraic system of 56 nonlinear equations.

Taking into account relations (2.9,2.10) it is easy to understand that only 23 of the constraints are independent on the level of $N$-th powers of coordinates $x’$s and $\bar{x}$’s. Since each weight of form (2.11) depends only on four independent parameters, we have a nine parameter solution of relations (2.1).

To order all these constraints let us introduce the following rapidity like parameters:

$$a = \frac{x_5 x_6}{x_1 x_2}, \quad i_1 = \omega x_6 x_2, \quad i_2 = \frac{x_5}{x_2}, \quad i_3 = a \cdot \frac{x_{13} x_{24}}{x_{58} x_{67}},$$

$$j_1 = \omega \frac{x_1}{x_5}, \quad j_2 = \frac{x_1}{x_6}, \quad j_3 = \frac{x_{13} x_{24}}{x_{58} x_{67}}.$$

and the same for the $\bar{x}$’s. Then the set of 24 relations following from a cancellation of “face” factors (which do not contain additional variables defined in (2.10)) has the form:

$$i_1 = i''_2, \quad i_2 = i'_2, \quad i_3 = i''_3, \quad i'_1 = i'''_3, \quad i''_1 = i'''_1,$$

$$\bar{j}_1 = \bar{j}''_2, \quad \bar{j}_2 = \bar{j}_2, \quad \bar{j}_3 = \bar{j}'_2, \quad \bar{j}'_1 = \bar{j}'''_3, \quad \bar{j}''_1 = \bar{j}'''_1,$$

$$j_1 = j''_2, \quad j_2 = j'_2, \quad j_3 = j''_3, \quad j'_1 = j'''_3, \quad j''_1 = j'''_1,$$

$$\bar{i}_1 = \bar{j}''_2, \quad \bar{i}_2 = \bar{j}_2, \quad \bar{i}_3 = \bar{j}'_2, \quad \bar{i}'_1 = \bar{j}'''_3, \quad \bar{i}''_1 = \bar{j}'''_1.$$

(3.9) (3.10) (3.11) (3.12) (3.13)
Only 21 of the 24 relations (3.11-3.13) are independent. A simple consequence of (3.11-3.12) is
\[ \Delta \equiv \frac{a}{a'} = \frac{a''}{a'''} = \frac{a'''}{a''} \]
(3.14)
and we can conclude that a parameter \( \Delta \) is an absolute invariant of relations (2.1).

We will also define rapidity variables of the mixed type:
\[ \alpha_1 = \frac{x_{13}x_{67}}{x_{67}x_{13}}, \quad \alpha_2 = \frac{x_{13}x_{58}}{x_{58}x_{13}}, \quad \alpha_3 = \frac{x_1}{x_1}. \]
(3.15)
Then six of the eleven additional relations coming from the coincidence of the structures of l.h.s. and r.h.s. of the modified tetrahedron equations (after using of “inversion” and Star-Square relations) take the form
\[ \alpha_1 = \alpha_2'', \quad \alpha_2 = \alpha_2', \quad \alpha_3 = \alpha_3'', \quad \alpha_1' = \alpha_3'''', \quad \alpha_2' = \alpha_3''', \quad \alpha_3' = \alpha_3'''. \]
(3.16)
Now we can similarly rewrite 24 of the constraints containing additional variables defined in (2.10). Define
\[ \beta_1 = \omega^{1/2}u, \quad \beta_2 = \frac{\lambda}{x_2}, \quad \beta_3 = \frac{\nu x_5}{x_{58}x_{67}}, \]
(3.17)
\[ \gamma_1 = \omega^{1/2}u x_1, \quad \gamma_2 = \frac{\xi}{x_6}, \quad \gamma_3 = \frac{\mu}{x_{58}x_{67}} \]
(3.18)
and similarly for \( \bar{\beta}_i, \bar{\gamma}_i \). Then we have
\[ \beta_1 = \beta_2'', \quad \beta_2 = \beta_2', \quad \beta_3 = \beta_2'', \quad \beta_1' = \beta_3'''', \quad \beta_2' = \beta_3''', \quad \beta_3' = \beta_3''. \]
(3.19)
\[ \bar{\gamma}_1 = \bar{\gamma}_2'', \quad \bar{\gamma}_2 = \bar{\gamma}_2', \quad \bar{\gamma}_3 = \bar{\gamma}_2'', \quad \bar{\gamma}_1' = \bar{\gamma}_3'''', \quad \bar{\gamma}_2' = \bar{\gamma}_3''', \quad \bar{\gamma}_3' = \bar{\gamma}_3'''. \]
(3.20)
and
\[ \gamma_1 = \bar{\gamma}_2'', \quad \gamma_2 = \bar{\gamma}_2', \quad \gamma_3 = \bar{\gamma}_2'', \quad \gamma_1' = \bar{\gamma}_3'''', \quad \gamma_2' = \bar{\gamma}_3''', \quad \gamma_3' = \bar{\gamma}_3'''. \]
(3.21)
The remaining five constraints include parameters of all four sets of weight functions and correspond to a generalization of tetrahedron quadrilateral constraint for the Zamolodchikov model. We can choose them as
\[ x_{24}^2 = x_{24}'x_{24}''' \]
and
\[ \frac{x_{13} x_{1}'' x_{13}'}{x_{1}} = 1, \quad \frac{x_{67} x_{24}'}{x_{24}'} = 1, \quad \frac{x_{67} x_{m}'}{x_{m}'} = 1, \quad \frac{x_{58} x_{m}'}{x_{m}'} = 1. \]

(3.23)

Now let us briefly discuss the main properties of relations (3.11-3.23). As we mention all these equations have a nine parametric solution. We can construct it in the following way. Relations (3.11,3.22) give 7 restrictions on 16 parameters \(x_i, x_i', x_i''\) and \(x_i'''\). Then all parameters \(\pi_i\) can be calculated in a unique way using relations (3.12,3.13,3.16,3.22). All remaining constraints will be satisfied automatically up to some \(N\)-th root of unity. And choosing the degrees of the \(\omega\)'s in a corresponding way we will satisfy all relations (3.11-3.23) and therefore the modified tetrahedron equation (2.1).

Now let us consider an important particular case of relations (3.11-3.23):
\[ a = a' = a'' = a'''. \]

(3.24)

Constraints (3.24) reduce the number of parameters to six. In this paper we will consider only a class of solutions for the modified tetrahedron equations restricted by conditions (3.24). Then the parameter \(a\) can be interpreted as an invariant. Detailed analysis of equations (3.11,3.22) shows that on the surface (3.24) we have two solutions. Only the first one contains the Zamolodchikov – Bazhanov – Baxter model in the limit \(a \to 1\). So we will investigate this case only. Then one can show that
\[ \Delta = a^2 = \frac{1}{\alpha}. \]

(3.25)

It is easy also to show that the following relations between the \(N\)-th powers of the parameters \(i_k, j_k, \alpha_k, \beta_k\) and \(\gamma_k\) are valid:
\[ j_k^N = \frac{i_k^N}{\alpha^N}, \quad \alpha_k^N = \frac{j_k^N - 1}{i_k^N - 1}, \quad \beta_k^N = i_k^N - 1, \quad \gamma_k^N = j_k^N - 1, \quad k = 1, 2, 3. \]

(3.26)

Note that formulae (3.26) for the \(N\)-th powers of parameters \(\alpha_k\) give a convenient way to find all parameters \(\pi\)'s in terms of \(x\)'s.

In the next section we will give a convenient parameterization of all parameters in terms of variables resembling tetrahedron angles.
4. Parameterization

Define $m, T_1, T_2, T_3$ through the relations

\[ \frac{x_N^2}{x_6^2} = -mT_1^2; \quad \frac{x_N^2}{x_5^2} = -mT_2^2; \]
\[ \frac{x_3^4 x_4^4}{x_{58}^4 x_{67}^4} = -mT_3^2; \quad \frac{x_1^N x_2^N}{x_5^N x_6^N} = m^2. \]  \hspace{1cm} (4.1)

The advantage of this parameterization is that when we put $m = 1$ and $T_i = \tan(\theta_i/2)$, we obtain the conventional parameterization of the Zamolodchikov – Bazhanov – Baxter model.

Solving (4.1) with respect to the $x_i^N$'s, we obtain

\[ x_1^N = \frac{1}{T_1 T_2} \sqrt{\frac{1 + mT_3^2}{1 + m^{-1}T_3^2}} \exp(i\alpha_3), \]
\[ x_2^N = T_1 T_2 \sqrt{\frac{1 + mT_3^2}{1 + m^{-1}T_3^2}} \exp(i\alpha_3), \]
\[ x_5^N = -m^{-1}T_1 \sqrt{\frac{1 + mT_3^2}{1 + m^{-1}T_3^2}} \exp(i\alpha_3), \]
\[ x_6^N = -m^{-1}T_2 \sqrt{\frac{1 + mT_3^2}{1 + m^{-1}T_3^2}} \exp(i\alpha_3), \]  \hspace{1cm} (4.2)

where $\alpha_3$ is defined from its cosine:

\[ \cos(\alpha_3) = \frac{1 + T_1^2 T_2^2 - T_1^2 T_3^2 - T_2^2 T_3^2}{2T_1 T_2 \sqrt{(1 + mT_3^2)(1 + m^{-1}T_3^2)}}. \]  \hspace{1cm} (4.3)

The formulae for the $x_i$’s can be obtained from these ones by changing $m \to m^{-1}$, $\alpha_3$ being the same.

The formula (4.3) can be rewritten in another form by the substitution

\[ T_p^2 = \frac{D_p - C_p}{D_p + C_p} = \left( \frac{k_p S_p}{D_p + C_p} \right)^2 \]  \hspace{1cm} (4.4)
where

\[ D_p^2 = 1 - k^2 S_p^2, \quad C_p^2 = 1 - S_p^2, \]
\[ m = \frac{1 - k}{1 + k}, \quad k^2 + k' = 1. \]  \hspace{1cm} (4.5)

and \( p = 1, 2, 3 \). Then we have (defining the angles \( a_1 \) and \( a_2 \) at once)

\[ \cos(a_r) = \frac{D_p D_q C_r + C_p C_q D_r}{k^2 S_p S_q}, \]  \hspace{1cm} (4.6)

where \( \{p, q, r\} = \{1, 2, 3\} \). Introduce else the angle \( a_0 \) as follows:

\[ \cos(a_0) = \frac{D_1 D_2 D_3 + k^2 C_1 C_2 C_3}{k^2}. \]  \hspace{1cm} (4.7)

\( a_0 \) is symmetric under the permutation of \( T_1, T_2, T_3 \), and the sign of \( a_0 \) we choose as follows

\[ \sin(a_0) = k \sin(a_r) S_p S_q. \]  \hspace{1cm} (4.8)

To write down \( x_{ij}^N \) we need to introduce a “linear excesses”

\[ \beta_0 = \pi - \frac{a_1 + a_2 + a_3 + a_0}{2}, \quad \beta_r = \frac{a_p + a_q - a_r - a_0}{2}, \]
\[ \overline{\beta}_0 = \pi - \frac{a_1 + a_2 + a_3 - a_0}{2}, \quad \overline{\beta}_r = \frac{a_p + a_q - a_r + a_0}{2}. \]  \hspace{1cm} (4.9)

Then

\[ x_{13}^N = -m^{1/2} \frac{T_3}{T_1 T_2} \sqrt{\frac{(1 + m^{-1} T_1^2)(1 + m^{-1} T_2^2)}{(1 + m^{-1} T_3^2)}} \exp(-i\beta_0), \]
\[ x_{24}^N = -m^{-1/2} T_3 \sqrt{\frac{(1 + m T_1^2)(1 + m T_2^2)}{(1 + m^{-1} T_3^2)}} \exp(-i\beta_3), \]
\[ x_{58}^N = -m^{-1/2} \frac{1}{T_2} \sqrt{\frac{(1 + m^{-1} T_1^2)(1 + m T_2^2)}{(1 + m^{-1} T_3^2)}} \exp(i\overline{\beta}_2), \]
\[ x_{67}^N = -m^{-1/2} \frac{1}{T_1} \sqrt{\frac{(1 + m T_1^2)(1 + m^{-1} T_2^2)}{(1 + m^{-1} T_3^2)}} \exp(i\overline{\beta}_1). \]  \hspace{1cm} (4.10)
We mention again that the $x_i^N$’s and $x_{ij}^N$’s can be obtained by the same formulae as $x_i^N$ and $x_{ij}^N$ with $m$ replaced by $m^{-1}$ (hence $k$ replaced by $-k$ and $a_0$ replaced by $-a_0$), $a_{1,2,3}$ being unchanged.

One can take the $N$-th roots of the expressions (4.2,4.10) so that when $m \to 1$ there appears the parameterization of the Zamolodchikov – Bazhanov – Baxter model. Namely, if $m$, $T_r$ and $a_r$, $r = 1,2,3$, are real positive then we can choose the phases as follows

$$x_1 = |x_1| \exp(i a_3/N), \quad x_{13} = |x_{13}| \omega^{1/2} \exp(-i \beta_0/N);$$

$$x_2 = |x_2| \exp(i a_3/N), \quad x_{24} = |x_{24}| \omega^{1/2} \exp(-i \beta_3/N);$$

$$x_5 = |x_5| \omega^{1/2} \exp(i a_3/N), \quad x_{58} = |x_{58}| \omega^{1/2} \exp(i \beta_2/N);$$

$$x_6 = |x_6| \omega^{-1/2} \exp(i a_3/N), \quad x_{67} = |x_{67}| \omega^{-1/2} \exp(i \beta_1/N). \quad (4.11)$$

In addition we reproduce some exotic formulae for the “spherical sides”.

$$\sqrt{\sin \beta_0 \sin \beta_1 \sin \beta_2 \sin \beta_3} = m,$$

$$m T_r^2 = \frac{\sin \beta_p \sin \beta_q}{\sin \beta_0 \sin \beta_r},$$

$$k S_r^2 = \frac{\sin a_0 \sin a_r}{\sin a_p \sin a_q}. \quad (4.12)$$

Now rewrite the scheme of arguments of weights $(W, \overline{W})$ in the modified tetrahedron equations. Exhibiting the dependence on $T_{1,2,3}$ and corresponding $a_{0,1,2,3}$ as

$$W = W(T_1, T_2, T_3) \to a_0, a_1, a_2, a_3, \quad (4.13)$$

the arguments of the weights in the modified tetrahedron equations can be written as follows

$$W = W(T_1, T_2, T_3) \to a_0, a_1, a_2, a_3;$$

$$W' = W(T_4, T_2, T_5) \to a'_{0}, a'_1, a'_2, a'_3;$$

$$W'' = W(T_6, 1/T_3, T_5) \to a''_{0}, a''_1, a''_2, a''_3;$$

$$W''' = W(T_6, T_1, 1/T_4) \to a'''_{0}, a'''_1, a'''_2, a'''_3. \quad (4.14)$$

In terms of the “spherical sides” of the “tetrahedron”, the modified tetrahedron equations correspond to the set of equivalent relations

$$a_0 + a'_2 + a'''_2 = a''_2, \quad a'_0 + a_2 + a'''_3 = a''_3,$$
\[ a''_0 + a_3 + a'''_1 = a'_3, \quad a''''_0 + a_1 + a''_1 = a'_1 \]  \quad (4.15)

Concluding this section we note that some of the formulae resemble those for the elliptic functions. Surely one can put \( S = \text{sn}(\theta, k) \) for \( S \) and \( k \) defined by (4.4,4.5). Such version of the formulae we shall collect in the Appendix, where we also give the explicit form of the Boltzmann weight functions for the case \( N = 2 \).

5. Symmetry properties

In this section we give all the necessary formulae for the transformation of parameters from which the weight function (2.11) depends on under the action of elements of the group \( G \) of all symmetry transformations of the three-dimensional cube. The main advantage of introducing parameters (2.9-2.10) is that any element from group \( G \) induces a multiplicative law of the transformation of parameters \( x_i, x_{ij}, u, v, \xi, \lambda, \mu \) and \( \nu \).

Any element of the group \( G \) can be represented as a composition of two generating elements: \( \tau \) and \( \rho \) (see [11, 13]). The action of these two elements on the set of spins \( \{a|e, f, g|b, c, d|h\} \) (see Fig. 1) can be described as follows

\[ \tau\{a|e, f, g|b, c, d|h\} = \{a|f, e, g|c, b, d|h\} \]  \quad (5.1)

and

\[ \rho\{a|e, f, g|b, c, d|h\} = \{g|c, a, b|f, h, e, d\}. \]  \quad (5.2)

In fact, \( \tau \) corresponds to the reflection of a three-dimensional cube in the plane \( \{aghd\} \) and \( \rho \) – to a rotation of \( \pi/2 \) around the vertical axis of the cube. Further it will be convenient to remove normalization conditions (3.10) and to restore the homogeneity of the parameters \( x \)'s.

The action of the element \( \rho \) can be obtained by use of the Fourier transformation (see [11]) and the action of \( \tau \) is a quite obvious. Here we give only the resulting formulae for their action on the coordinates. Note that all coordinates can be split into four multiplets: \( (x_3, x_{13}, x_1), (x_4, x_{24}, x_2), (x_8, x_{58}, x_5, u, \lambda) \) and \( (x_7, x_{67}, x_6, v, \xi, \mu, \nu) \) and that any group element acts homogeneously on these four sets. Also, we must demand a trivial action of the elements: \( \tau^2, \rho^4 \) and \( (\tau\rho)^6 \) (trivial transformations) on all coordinates. This will fix some uncertainties in the \( N \)-th roots of unity in the transformation laws.
Then a correct action of the elements $\tau$ and $\rho$ looks like:

$$(x_3, x_{13}, x_1) \xrightarrow{\tau} (x_3, x_{13}, x_1), \quad (x_4, x_{24}, x_2) \xrightarrow{\tau} (x_4, x_{24}, x_2),$$

$$(x_8, x_{58}, x_5, u, \lambda) \xrightarrow{\tau} (x_7/\omega, x_{67}, x_6, \omega^{-1/2}x_3^\xi, \omega^{-1/2}x_4^\mu), \quad (5.3)$$

$$(x_7, x_{67}, x_6, \xi, \mu, \nu) \xrightarrow{\tau} (\omega x_8, x_{58}, x_5, \omega^{1/2}x_4^\lambda x_2, \omega^{1/2}x_1^u x_3, \omega x_8 x_1^\mu x_5^\nu x_4^\xi).$$

And

$$(x_3, x_{13}, x_1) \xrightarrow{\rho} (x_8 x_{13}, x_1 u, x_3 x_{58}), \quad (x_4, x_{24}, x_2) \xrightarrow{\rho} (x_2 x_{67}, x_2 v, x_6 x_{24}),$$

$$(x_8, x_{58}, x_5, u, \lambda) \xrightarrow{\rho} (x_5 x_{13}, \omega x_1 u, \omega x_1 x_{58}, \omega^{1/2}x_1 x_{13}^u, \omega^{1/2}x_1 x_{13}^v x_{67}), \quad (5.4)$$

$$(x_7, x_{67}, x_6, \xi, \mu, \nu) \xrightarrow{\rho} (\omega x_4 x_{67}, x_2 v, x_7 x_{24}, \omega^{1/2}x_3 x_{67}^v x_6^\xi, \omega^{1/2}x_3 x_{13}^u x_6^\mu, \omega x_1 x_4 u v x_5^\lambda x_6^\xi, \omega x_5 x_{68} x_{13}^v x_6^\mu x_{13}^u).$$

Substituting formulae (5.1-5.4) in (2.11) one can show that the whole $W$ function will be invariant under $\tau$ and $\rho$ transformations from the group $G$. The gauge factors before curly brackets in (2.11) are chosen in such a way to cancel multipliers coming after the Fourier transformation of the sum in (2.11). The action of any element from $G$ on coordinates can be easily calculated by a composition of formulae (5.3,5.4).

Let us briefly comment on these formulae. It is easy to see that after the $\rho$ transformation, the modulus parameter $a$ (see 3.9) transforms as: $a \xrightarrow{\rho} a^{-1}$. So it is not invariant under elements from the group $G$ containing an odd number of $\rho$ transformations. In some sense this is a pseudo-invariant (it is obvious that any combination like $a + a^{-1}$ will be invariant under all transformations from $G$). Introducing “angles” as arguments of the elliptic functions (see Appendix) we can define a “crossing” transformation which in fact corresponds to some spin permutation like \{a|efg|bcd|h\} $\rightarrow$ \{a|fg|cde|h\} and etc.

In the next section we will show that the weight functions of the lattice should have alternating values of moduli: $a$ and $a^{-1}$ in all directions in order to provide a commuting family of two-layer transfer-matrices.

6. Two-layer model

In this section we will construct a commuting family of two-layer transfer-matrices from solutions of the modified tetrahedron equations (2.1). More
explicitly, we will show that a composite weight function consisting of eight elementary $W$ satisfying (2.1) can be chosen in a such way that it will satisfy the tetrahedron equations of mixed type with spin variables placed in the corner sites, in the middles of edges and in the centers of all faces of the composite cube.

Consider a composite weight $W$ which consists of eight independent weight functions $W$, $W'$, $W''$, $W'''$ and the same for $W'$, $W''$, $W'''$.

Let us demand that four weights $W$, $W'$, $W''$ and $W'''$ should satisfy to usual tetrahedron equations of mixed type:

$$W W' W'' W''' = W''' W'' W' W.$$  (6.1)

So constructing from $W$ and $W'$ two-layer transfer matrices $T(W)$ and $T(W')$ in a usual way we come to a commuting relation:

$$[T(W), T(W')] = 0$$  (6.2)

Then one can show that it is equivalent to the validity of the following 16 equations of type (2.1):

$$W_h W_e W_a W''' = W_h W_e W'_a W''' ,$$  

$$W_a W_f W_b W_h' = W_a W' f W_b W_h ,$$  

$$W_c W_e W_c W_g = W_c W_c W_e W_g ,$$  

$$W_f W_f W_d W'' = W_f W_f W'' W_d ,$$
An arrangement of spin variables in relations (6.1) is the same as in the modified tetrahedron equations (2.1). Note that we must introduce a set of additional weights $\mathbb{W}_a, \ldots, \mathbb{W}_h$ which appear only in intermediate steps when proving relations (6.1) and cancel after 16-fold application of relations (6.3).

Not let us analyze relations (6.3). The simplest way to satisfy them is to put $W_g = W_e = W_f = W_h = W$ and $W_d = W_b = W_c = W_a = \mathbb{W}$. In this case we must demand that $W$ and $\mathbb{W}$ should satisfy relations (2.1) and its dual version. This is the case considered in paper [14] where it was shown that then we have only two solutions: namely the Bazhanov-Baxter model and the “static limit” elliptic model. So we should try to avoid this situation.

But using results of the Section 3 we can directly analyze solutions of system (6.3). To simplify our discussion we will omit below all gauge factors before curly brackets entering in formulae for weight function (2.11) and relations for corresponding variables $u, v, \lambda, \xi, \mu, \nu$. They can be easily restored by using relations like (3.19-3.21).

In Section 3 we have seen that all relations containing the coordinates $\mathbf{\tau}$ follow (up to some roots of unity) from relations (3.11.3.22). So we will analyze only those constraints which contain variables $x_{a_1}, x_{a_2}, \ldots, x_{h_1}, x_{h_2}$. Introducing notations like (3.9-3.10) for the rapidity variables for the letters: $a, \ldots, h$ we come to the following relations:

\begin{align*}
  &j_{c1} = i_{g1}, \quad j_{h1} = i_{b1}, \quad j_{e1} = i_{a1}, \quad j_{d1} = i_{f1}, \quad j_{b2} = i_{g2}, \quad j_{f2} = i_{a2},
  &j_{h2} = i_{c2}, \quad j_{d2} = i_{c2}, \quad j_{a3} = i_{g3}, \quad j_{f3} = i_{b3}, \quad j_{v3} = i_{c3}, \quad j_{d3} = i_{h3} (6.4) \\
  &a_g = a_e = a_f = a_h = a_d^{-1} = a_b^{-1} = a_c^{-1} = a_a^{-1} = a = a' = a'' = a''' (6.5)
\end{align*}

and the same for all $i, i', i'', j, j', j'', j'''$.

Relations (6.5) show that modulus parameters $a_i$ associated with elementary cubes of the lattice have a checkerboard structure in all directions.
We also have 24 relations like (3.11):
\[
\begin{align*}
  i_1 &= i_2'', \quad i_2 = i_2', \quad i_3 = i_3'', \quad i_4 = i_3', \quad i_5 = i_5'', \quad i_6 = i_5', \\
  \dot{j}_1 &= j_2'', \quad \dot{j}_2 = j_2', \quad \dot{j}_3 = j_3'', \quad \dot{j}_4 = j_3', \quad \dot{j}_5 = j_5'', \quad \dot{j}_6 = j_5'.
\end{align*}
\]
(6.6)

and
\[
\begin{align*}
  i_1 &= i_1'', \quad i_2 = i_2'', \quad i_3 = i_3', \quad i_4 = i_3'', \quad i_5 = i_5', \quad i_6 = i_6''.
\end{align*}
\]
(6.7)

At last, we have 16 tetrahedron constraints like (3.22):
\[
\begin{align*}
  \frac{x_{c58}^i x_{c5}^i x_{c24}^i}{x_{h13} x_{c58} x_{c2}} &= \omega, \quad \frac{x_{a24}^i x_{f6}^i x_{b67}^i}{x_{h13} x_{a2} x_{f67}} = 1, \quad \frac{x_{f6}^i x_{f6}^i x_{f67}^i x_{b67}^i}{x_{h13} x_{f67} x_{f67} x_{a2}} = 1, \\
  \frac{x_{e57}^1 x_{a2}^i x_{a2}^i x_{e58}^i}{x_{a2} x_{a2} x_{a2} x_{e58}} = 1, \quad \frac{x_{f6}^i x_{f6}^i x_{f67}^i x_{f67}^i}{x_{h13} x_{f67} x_{f67} x_{h13}} = 1, \\
  \frac{x_{e58}^1 x_{e58}^i x_{e58}^i}{x_{h13} x_{d1} x_{f67}} = 1, \quad \frac{x_{f6}^i x_{f6}^i x_{f67}^i x_{f67}^i}{x_{e58} x_{e58} x_{e58} x_{e58}} = 1, \quad \frac{x_{b67}^i x_{b67}^i x_{b67}^i}{x_{b67} x_{b67} x_{b67}} = 1.
\end{align*}
\]
(6.10)

Since relations (6.4-6.5) leave 13 degrees of freedom in each of the composite weights (12 “spectral” and one modulus parameters), we have 40 relations (2.3-2.7) on \(12 \times 4 + 1 = 49\) variables.

One can show that all other relations constrained \(x\) and \(\tau\) are satisfied automatically up to some roots of unity provided that relations (6.4-6.10) are valid. The proof of this fact is quite tedious and we will not give it here. But we must be sure that there exists a consistent choice of degrees of \(\omega\) in all parameters so that all 16 equations (6.3) will be valid. Below we will give a simplified solution of system (6.4-6.10) and the whole system of equations (6.3). It shows we should find a general solution only for relations (6.4-6.10) and then all degrees of \(\omega\) in parameters \(x\) and \(\tau\) can be restored by a continuous deformation of the parameters for the simplified solution.
Now let us turn to system (6.4-6.10). Unfortunately, up to now we did not succeed in obtaining a complete understanding of the underlying geometrical picture (if any). If we put $a = 1$ then we come to the inhomogeneous variant of the $N$-state generalization of the Zamolodchikov model proposed by Bazhanov and Baxter. In this case we have a natural geometrical picture. Each elementary weight is described by the three angles between the three planes. Then a two-layer composite cube is defined by six intersecting planes and their mutual orientation depends on nine angle variables. All system (6.4-6.10) has 13 degrees of freedom and therefore we have a four-parametric commuting family of two-layer transfer-matrices.

Unfortunately, the case $a \neq 1$ looks much more complicated due to the absence of a geometrical analogy. Nevertheless, we have analyzed it quite carefully. The results can be formulated as follows: system (6.4-6.10) has 12 degrees of freedom (11 “angle” parameters and one modulus). A composite two-layer cube depends from only from eight independent “angles” and modulus. So we have an additional constraint among nine “angles” and this constraint will disappear in the limit $a \to 1$. We also obtain only a three-parametric family of commuting two-layer transfer-matrices. We will not write here these formulae since they are rather cumbersome and therefore practically useless. Nevertheless, we are sure that there should exist a homogeneous solution to the system (6.4-6.10), so that all weights $W_a, \ldots, W_h$ depend on the same three spectral parameters and modulus) which will generalize the model of Bazhanov-Baxter and “static limit” elliptic model of [14].

Now we will consider another limit case of system (6.4-6.10). Demand that all parameters $x_a, \ldots, x_h$ are expressed in terms of $x_g$ by the following formulae:

$$
\begin{align*}
  x_{a1} &= \frac{\omega}{x_{g1}}, \quad x_{a2} = \frac{\omega}{x_{g2}}, \quad x_{a5} = \frac{\omega^2}{x_{g5}}, \quad x_{a6} = \frac{1}{x_{g6}}, \quad (6.11)
  x_{a13} = \omega^{1/2} \frac{x_{g13}}{x_{g1}}, \quad x_{a24} = \omega^{-1/2} \frac{x_{g24}}{x_{g2}}, \quad x_{a58} = \omega^{1/2} \frac{x_{g58}}{x_{g5}}, \quad x_{a67} = \omega^{-1/2} \frac{x_{g67}}{x_{g6}},
  x_{e1} &= \frac{1}{x_{g6}}, \quad x_{e2} = \frac{\omega}{x_{g5}}, \quad x_{e5} = \frac{\omega}{x_{g2}}, \quad x_{e6} = \frac{1}{x_{g1}}, \quad (6.12)
  x_{e13} = \omega^{1/2} \frac{x_{g13}}{x_{g6}}, \quad x_{e24} = \omega^{1/2} \frac{x_{g24}}{x_{g5}}, \quad x_{e58} = \omega^{1/2} \frac{x_{g58}}{x_{g2}}, \quad x_{e67} = \omega^{-1/2} \frac{x_{g67}}{x_{g1}},
  x_{f1} &= \frac{\omega}{x_{g5}}, \quad x_{f2} = \frac{1}{x_{g6}}, \quad x_{f5} = \frac{\omega}{x_{g1}}, \quad x_{f6} = \frac{1}{x_{g2}}, \quad (6.13)
  x_{f13} = \omega^{1/2} \frac{x_{g13}}{x_{g5}}, \quad x_{f24} = \omega^{1/2} \frac{x_{g24}}{x_{g6}}, \quad x_{f58} = \omega^{1/2} \frac{x_{g58}}{x_{g1}}, \quad x_{f67} = \omega^{-1/2} \frac{x_{g67}}{x_{g2}},
\end{align*}
$$

\[20\]
\[ x_{d1} = 1/x_{g2}, \quad x_{d2} = 1/x_{g1}, \quad x_{d5} = 1/x_{g6}, \quad x_{d6} = 1/x_{g5}, \quad (6.14) \]

\[ x_{d13} = \omega^{-1/2}x_{g24}^x, \quad x_{d24} = \omega^1x_{g13}^x, \quad x_{d58} = \omega^1x_{g67}^x, \quad x_{d67} = \omega^{-1/2}x_{g58}^x, \]

\[ x_{b1} = x_{g5}/\omega, \quad x_{b2} = x_{g6}, \quad x_{b5} = x_{g1}, \quad x_{b6} = x_{g2}/\omega, \quad (6.15) \]

\[ x_{b13} = x_{g58}/\omega, \quad x_{b24} = \omega x_{g67}, \quad x_{b58} = x_{g13}, \quad x_{b67} = x_{g24}/\omega, \]

\[ x_{c1} = x_{g6}, \quad x_{c2} = x_{g5}/\omega, \quad x_{c5} = x_{g2}, \quad x_{c6} = x_{g1}/\omega, \quad (6.16) \]

\[ x_{c13} = x_{g67}, \quad x_{c24} = x_{g58}, \quad x_{c58} = x_{g24}, \quad x_{c67} = x_{g13}/\omega, \]

\[ x_{h1} = x_{g2}/\omega, \quad x_{h2} = x_{g1}/\omega, \quad x_{h5} = x_{g6}, \quad x_{h6} = x_{g5}/\omega^2, \quad (6.17) \]

\[ x_{h13} = x_{g24}/\omega^2, \quad x_{h24} = x_{g13}, \quad x_{h58} = x_{g67}, \quad x_{h67} = x_{g58}/\omega^2 \]

and same for \( x', x'', x''' \) and \( \vec{x}, \vec{x}', \vec{x}, \vec{x}'' \). Then all relations (6.4-6.10) will be satisfied automatically as a consequence of the relations for \( x_g \) (6.6) and the tetrahedron constraints for \( x_g \) (see (3.22)). Moreover, all relations containing \( \vec{x} \) will be satisfied also. So we can use formulae (2.8-3.4) in order to determine all \( \omega \) multipliers. Now we will show that transformations (6.11-3.4) have a natural interpretation.

First consider a transformation of \( x \) given by (6.14). Let us denote a weight function from this set of parameters as \( W^{(i)} \). Then one can show that \( W \) and \( W^{(i)} \) satisfy to the following inversion relation:

\[ \sum_{\sigma \in \mathbb{Z}_N} W(a|efg|bcd|\sigma)W^{(i)}(\sigma|bcd|efg|h) = \Phi(x)\delta_{a,h}, \quad (6.18) \]

where \( \Phi(x) \) is some scalar factor which can be calculated.

Define also the following automorphism \( \theta \) on \( x \)'s as:

\[ \theta(x_1, x_2, x_5, x_6, x_{13}, x_{24}, x_{58}, x_{67}) = \]

\[ (x_1/\omega, x_2/\omega, x_5/\omega, x_6/\omega, x_{13}/\omega^2, x_{24}, x_{58}/\omega, x_{67}/\omega). \quad (6.19) \]

Then all relations (6.11-6.17) can be rewritten in the following compact form:

\[ W_f = W_g^{\rho^2}, \quad W_e = W_g^{\tau \rho^2 \tau}, \quad W_h = W_g^{\rho_\tau \rho^2 \tau^2}, \]

\[ W_a = W_h^{(i)}, \quad W_b = W_e^{(i)}, \quad W_c = W_f^{(i)}, \quad W_d = W_g^{(i)}, \quad (6.20) \]
where the action of the transformations \( \tau \) and \( \rho \) on \( x' \)'s was defined in Section 5 and the action of their compositions can be easily calculated. We call the choice of weights in the form (6.20) as the “inverse” variant of two-layer model.

Note that we were forced to introduce automorphism \( \theta \) in order to satisfy all the systems of relations for \( x' \)'s and \( \pi \)'s coming from (6.3). This is a very overdetermined system of scalar equations and it is remarkable that we have any solution at all.

7. Conclusion

In this paper we have found a new solution to the modified tetrahedron equations in the framework of the so called “Body-Centered-Cube” ansatz. The weight functions depend on three angle-like parameter \( \theta_i \) and one elliptic modulus \( k \). We have also constructed the elliptic parameterization of the weight function. Note that in the limit \( k \to 0 \) we obtain the Baxter-Bazhanov model. If the “static limit” condition \( \theta_1 + \theta_2 + \theta_3 = 2K \) with \( K \) being the full elliptic integral is satisfied than we obtain the model proposed in [9, 14]. As was mentioned above we were forced to achieve the integrability of the model without the validity of “dual” version of MTE to be able to generalize the Baxter-Bazhanov model and not to restrict ourself onto “static-limit” condition. The constructed integrable two-layer model has the spin variables in the cube vertices, the middle points of the cube edges and the centers of the cube faces as was shown in fig. 6.1. So, this model is of mixed type. The weight function consist of eight weights \( W_a, W_b, \ldots, W_h \) which must satisfy the system of the sixteen sets of equations (6.3). The solution to these equations appeared to have twelve free parameters (including the modulus \( k \)) but the formulae are rather cumbersome and we do not write them. We have also considered one particular case when all four pairs of the opposite weights (a-h, b-e, c-f, d-g) are connected with each other by the inversion. Unfortunately, we did not succeed to find a geometrical picture like in the case of the Baxter-Bazhanov model. Nevertheless some of the formulae presented in Appendix resemble the formulae for the spherical triangle taking place for the angles \( \theta_1, \theta_2, \theta_3 \) in the Baxter-Bazhanov model. So, we hope that some appropriate geometrical picture does exist. We also hope that besides the “inverse” model some another two-layer “homogeneous” model
which might have a rich physical content can be found. We expect to do this elsewhere.

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8. Appendix

The formulae for "modified spherical triangle" are more aesthetic if we introduce a parameterization for $T_1, T_2, T_3$ in terms of the elliptic functions of the modulus $k = (1 - m)/(1 + m)$. Besides, some of the formulae below resemble corresponding formulae in the spherical trigonometry. We shall use the ordinary Jacobi’s elliptic functions sn, cn and dn, theta – and eta – functions $H, \Theta$. Let us also introduce the following notations:

$$\text{tn}(u, k) = k' \frac{\text{sn}(u, k)}{\text{cn}(u, k)\text{dn}(u, k)} \quad (8.1)$$

and

$$\text{tn}_L(u, k) = \text{tn}((1 + k)u, \frac{2k^{1/2}}{1 + k}), \quad \text{tn}_L(u, k) = \text{tn}((1 - k)u, \frac{2ik^{1/2}}{1 - k})$$

$$\text{tn}_L(u, k)\text{tn}_L(u, k) = \text{tn}^2(u, k). \quad (8.2)$$

The last two functions represent the Landen’s transformation $\tau \rightarrow \tau/2$ and $\tau \rightarrow \tau/2 + 1$. The parameters $k$ and $m$ we regard to be fixed forever and so we shall omit them everywhere. Also we shall use the indexes $p, q, r$ for $1, 2, 3$.

Defining $\theta_r$ so as

$$T_r = \text{tn}(\theta_r/2), \quad (8.3)$$

then we obtain

$$S_r = \text{sn}(\theta_r), \quad C_r = \text{cn}(\theta_r), \quad D_r = \text{dn}(\theta_r). \quad (8.4)$$

Having got the angles $\theta_r$, define the spherical excesses as usual:

$$\alpha_0 = \frac{\theta_1 + \theta_2 + \theta_3}{2} - K, \quad \alpha_r = \theta_r - \alpha_0, \quad (8.5)$$

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where \( K \) is the complete elliptic integral of the first kind of the modulus \( k \). It can be shown that the following expressions are valid:

\[
\sin^2 a_0/2 = \frac{H(\alpha_0)H(\alpha_1)H(\alpha_2)H(\alpha_3)}{\Theta(\theta_0)\Theta(\theta_1)\Theta(\theta_2)\Theta(\theta_3)},
\]

\[
\cos^2 a_0/2 = \frac{\Theta(\alpha_0)\Theta(\alpha_1)\Theta(\alpha_2)\Theta(\alpha_3)}{\Theta(0)\Theta(\theta_1)\Theta(\theta_2)\Theta(\theta_3)},
\]

\[
\tan \frac{a_0}{2} = k\sqrt{\text{sn}(\alpha_0)\text{sn}(\alpha_1)\text{sn}(\alpha_2)\text{sn}(\alpha_3)}; \quad (8.6)
\]

\[
\sin^2 a_r/2 = \frac{H(\alpha_0)H(\alpha_r)\Theta(\alpha_p)\Theta(\alpha_q)}{\Theta(\theta_0)\Theta(\theta_r)\Theta(\theta_p)\Theta(\theta_q)},
\]

\[
\cos^2 a_r/2 = \frac{\Theta(\alpha_0)\Theta(\alpha_r)H(\alpha_p)H(\alpha_q)}{\Theta(0)\Theta(\theta_r)H(\theta_p)H(\theta_q)},
\]

\[
\tan \frac{a_r}{2} = \sqrt{\frac{\text{sn}(\alpha_0)\text{sn}(\alpha_r)}{\text{sn}(\alpha_p)\text{sn}(\alpha_q)}} \quad (8.7)
\]

Similarly one can obtain for the linear excesses:

\[
\tan^2 \frac{\beta_0}{2} = \frac{tn_L(\alpha_1/2)tn_L(\alpha_2/2)tn_L(\alpha_3/2)}{tn_L(\alpha_0/2)},
\]

\[
\tan^2 \frac{\beta_r}{2} = \frac{tn_L(\alpha_0/2)tn_L(\alpha_p/2)tn_L(\alpha_q/2)}{tn_L(\alpha_r/2)}. \quad (8.8)
\]

The formulae for the \( \overline{\beta} \) can be obtained from (8.8) by replacing all the \( tn_L \) by \( \overline{tn_L} \) (it is the common rule: replacing \( k \) by \( -k \) is equivalent to the modular transformation \( \tau \) to \( \tau + 2 \)). We complete our list of the formulae by

\[
\frac{\sin \beta_r}{\sin \beta_0} = tn_L(\theta_r/2)tn_L(\theta_p/2),
\]

\[
\frac{\sin \beta_p \sin \beta_q}{\sin \beta_0 \sin \beta_r} = tn_L^2(\theta_r/2), \quad \frac{\sin \beta_p \sin \beta_q}{\sin \beta_0 \sin \beta_r} = mtn_L^2(\theta_r/2). \quad (8.9)
\]

Now let us give the explicit form of the Boltzmann weight function for the case \( N = 2 \). The calculations resembles ones from the Appendix of [14] and we give here only the answer. In our previous sections all the spins belong
to $Z_N$ and take the values $0, 1, ..., N - 1$. For the case $N = 2$ it is convenient to deal with multiplicative spins $(-1)^a = \pm 1$ instead of $a = 0, 1$. Further we will imply the multiplicative spins as the arguments of the Boltzmann weight functions.

Introduce for shortness some conventional notations:

$$s_i = \sqrt{\text{sn}(\alpha_i/2)}, \quad c_i = \sqrt{\text{cd}(\alpha_i/2)}, \quad i = 0, 1, 2, 3; \quad (8.10)$$

and

$$P_0 = c_0c_1c_2c_3, \quad P_r = c_0c_r s_ps_q;$$
$$Q_0 = s_0s_1s_2s_3, \quad Q_r = s_0s_r c_pc_q;$$
$$R_i = s_ic_i, \quad S_i = k \frac{s_0c_0s_1c_1s_2c_2s_3c_3}{s_ic_i}, \quad i = 0, 1, 2, 3. \quad (8.11)$$

Then for the weight $W$ given by (2.11) there exists the following identity:

$$W = \frac{1}{P_0 - Q_0} W_{SPQR} \quad (8.12)$$

and $W_{SPQR}$ is given by the following table:

| abeh | acfh | adgh | $W_{SPQR}(a|efg|bcd|h)$ |
|-------|------|------|-------------------|
| +     | +    | +    | $P_0 - abcdQ_0$  |
| -     | +    | +    | $R_1 + abcdS_1$  |
| +     | -    | +    | $R_2 + abcdS_2$  |
| +     | +    | -    | $R_3 + abcdS_3$  |
| +     | -    | -    | $abP_1 + cdQ_1$  |
| -     | +    | -    | $acP_2 + bdQ_2$  |
| -     | -    | +    | $adP_3 + bcQ_3$  |
| -     | -    | -    | $R_0 + abcdS_0$  |

Note that for $W_{SPQR}$ all the symmetry properties are quite evident.
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