Modulo factors with bounded degrees

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Abstract

Let $G$ be a bipartite graph with bipartition $(X, Y)$, let $k$ be a positive integer, and let $f : V(G) \to \{-1, \ldots, k - 2\}$ be a mapping with $\sum_{v \in X} f(v) \equiv k \sum_{v \in Y} f(v)$. In this paper, we show that if $G$ is essentially $(3k - 3)$-edge-connected and for each vertex $v$, $d_G(v) \geq 2k - 1 + f(v)$, then it admits a factor $H$ such that for each vertex $v$, $d_H(v) \equiv f(v)$, and

$$\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + k - 1.$$ 

Next, we generalize this result to general graphs and derive sufficient conditions for a highly edge-connected general graph $G$ to have a factor $H$ such that for each vertex $v$, $d_H(v) \in \{f(v), f(v) + k\}$. Finally, we show that every $(4k - 1)$-edge-connected essentially $(6k - 7)$-edge-connected graph admits a bipartite factor whose degrees are positive and divisible by $k$.

Keywords: Modulo factor; edge-connected; partition-connected; bipartite graph; vertex degree.

1 Introduction

In this article, graphs may have loops and multiple edges. Let $G$ be a graph. The vertex set, the edge set, and the minimum degree of the vertices of $G$ are denoted by $V(G)$, $E(G)$, and $\delta(G)$, respectively. We denote by $d_G(v)$ the degree of a vertex $v$ in the graph $G$. If $G$ has an orientation, the out-degree and in-degree of $v$ are denoted by $d^+_G(v)$ and $d^-_G(v)$. For a vertex set $A$ of $G$ with at least two vertices, the number of edges of $G$ with exactly one end in $A$ is denoted by $d_G(A)$. Also, we denote by $e_G(A)$ the number of edges with both ends in $A$ and denote by $d_G(A, B)$ the number of edges with one end in $A$ and one end in $B$, where $B$ is a vertex set. We denote by $G[A]$ the induced subgraph of $G$ with the vertex set $A$ containing precisely those edges of $G$ whose ends lie in $A$, and denote by $G[A, B]$ the induced bipartite factor of $G$ with the bipartition $(A, B)$. A graph $G$ is called $m$-tree-connected if it contains $m$ edge-disjoint spanning trees. Note that by the result of Nash-Williams [18] and Tutte [24] every $2m$-edge-connected graph is $m$-tree-connected. In fact, the result of them says that a loopless graph $G$ is $m$-tree-connected if and only if $e_G(P) \geq m(|P| - 1)$ for every partition $P$ of $V(G)$, where $e_G(P)$ denotes the number of edges of $G$ joining different parts of $P$. A graph
is termed essentially $\lambda$-edge-connected, if all edges of any edge cut of size strictly less than $\lambda$ are incident with a common vertex. Let $k$ be a positive integer. The cyclic group of order $k$ is denoted by $\mathbb{Z}_k$. For two integers $x$ and $y$, we say that $x \equiv y \pmod{k}$, if $x - y$ is divisible by $k$. For any integer $n$, we denote by $[n]_k$ the unique integer $n_0$ such that $n_0 \equiv n \pmod{k}$ and $n_0 \in \{-1, 0, \ldots, k-2\}$. An orientation of $G$ is said to be $p$-orientation if for each vertex $v$, $\frac{d^+_G(v)}{k} \equiv \frac{p(v)}{k}$, where $p : V(G) \to \mathbb{Z}_k$ is a mapping. Likewise, an $f$-factor refers to a spanning subgraph $H$ such that for each vertex $v$, $\frac{d_H^+(v)}{k} \equiv f(v)$, where $f : V(G) \to \mathbb{Z}_k$. For a graph $G$, we say that a mapping $f : V(G) \to \mathbb{Z}_k$ is compatible with $G$ with respect to a bipartition $X, Y$ of $V(G)$ if $\sum_{v \in X} f(v) - 2x \equiv \sum_{v \in Y} f(v) - 2y$ for two integers $x$ with $0 \leq x \leq e_G(X)$ and $0 \leq y \leq e_G(Y)$. Note that one of $x$ and $y$ can be zero. Likewise, we say that a mapping $f$ is compatible with $G$ if it is compatible with $G$ with respect to every bipartition $X, Y$ of $V(G)$. We will show that when $G$ is a $(2k - 3)$-edge-connected bipartite graph with bipartition $(X, Y)$, $f$ is compatible with $G$ if and only if $f$ is compatible with respect to the bipartition $X, Y$ of $V(G)$; see Theorem 2.6. It is easy to see that if $G$ has an $f$-factor, then $f$ must be compatible with $G$. The bipartite index $bi(G)$ of a graph $G$ is the smallest number of all $|E(G) \setminus E(H)|$ taken over all bipartite factors $H$. A modulo $k$-regular graph refers to a graph whose degrees are positive and divisible by $k$. A proper coloring of a graph $G$ is to assign colors to vertices such that any two adjacent vertices receive different colors. The chromatic number $\chi(G)$ refers to the minimum number of necessary colors among all proper colorings. Throughout this article, all variables $m$ are nonnegative integers and all variables $k$ are positive integers.

In 2008 Shirazi and Verstraëte [19] introduced the concept of modulo factors and established a sufficient condition for the existence of $f$-factors modulo $k$ in graphs with average degree $2k - 2$.

**Theorem 1.1.** ([19]) Let $G$ be a graph, let $k$ be a prime number, and let $f : V(G) \to \mathbb{Z}_k$ be an arbitrary mapping. If the number of orientations with out-degrees $k - 1$ is not divisible by $k$, then $G$ has an $f$-factor.

In 2014 Thomassen formulated the following result about the existence of $f$-factors modulo $k$ in $(3k - 3)$-edge-connected bipartite graphs. In this paper, we improve Theorem 1.2 by giving a sharp bound on degrees as mentioned in the abstract. Our result is based on a new improvement of the main result in [16] about the existence of modulo orientations with bounded out-degrees [9].

**Theorem 1.2.** ([22]) Let $G$ be a bipartite graph with partition $(X, Y)$, let $k$ be an integer, $k \geq 2$, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$. If $G$ is $(3k - 3)$-edge-connected, then it has an $f$-factor.

In 2016 Thomassen, Wu, and Zhang [23] generalized Theorem 1.2 to $(6k - 7)$-edge-connected graphs with bipartite index at least $k - 1$ provided that $k$ is odd. In Subsection 5.1, we extend their result to even integers $k$ and improve it to the following bounded-degree version.

**Theorem 1.3.** Let $G$ be a graph, let $k$ be a positive integer, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping. Assume that $bi(G) \geq k - 1$ and $(k-1) \sum_{v \in V(G)} f(v)$ is even. If $G$ is $(6k - 7)$-edge-connected, then it has an $f$-factor.
such that for each vertex \( v \),

\[
\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + k.
\]

As an application, we derive a sufficient condition for a highly edge-connected graph \( G \) to have a factor whose degrees fall in predetermined integer sets as the following corollary. In Subsection 5.2, we refine degree bounds for graphs with higher edge-connectivity.

**Corollary 1.4.** Let \( G \) be a graph, let \( k \) be a positive integer, and let \( f \) be a positive integer-valued function on \( V(G) \) satisfying \( f(v) \leq \frac{1}{2}d_G(v) < f(v) + k \) for each vertex \( v \). Assume that \( bi(G) \geq k - 1 \) and \( (k - 1) \sum_{v \in V(G)} f(v) \) is even. If \( G \) is \((6k - 7)\)-edge-connected, then it has a factor \( H \) such that for each vertex \( v \),

\[
d_H(v) \in \{f(v), f(v) + k\}.
\]

In [22], Thomassen established a sufficient edge-connectivity condition for the existence of a special type of bipartite modulo \( k \)-regular factors by concluding the following result from Theorem 1.2. In Section 6, we push down the needed edge-connectivity to \( 10k - 3 \) in essentially \((12k - 7)\)-edge-connected graphs. Moreover, we show that every \((4k - 1)\)-edge-connected essentially \((6k - 7)\)-edge-connected graph admits a bipartite modulo \( k \)-regular factor.

**Theorem 1.5.**([22]) Every \((12k - 7)\)-edge-connected graph of even order has a bipartite modulo \( k \)-regular factor whose degrees are not divisible by \( 2k \).

In 1984 Alon, Friedland, and Kalai proposed the following elegant conjecture and confirmed it for the case that \( k \) is a prime power.

**Conjecture 1.6.**([1]) Let \( G \) be a loopless graph of order \( n \) and let \( k \) be a positive integer. If \(|E(G)| > (k - 1)n\), then \( G \) admits a modulo \( k \)-regular subgraph.

Recently, Botler, Colucci, and Kohayakawa (2020) [6, Lemma 3] proved a weaker version of this conjecture by replacing the lower bound by \((24k - 12)n\) based on Theorem 1.5, and applied it for their purpose. In this paper, we introduce a simpler technique to improve their result by replacing a better lower bound less than \( 4(k - 1)n \).

2 Basic tools and preliminary results

2.1 Tools: Orientations modulo \( k \)

For making some results on the existence of modulo factors with bounded degree, we need to apply some results on the existence of modulo orientations with bounded out-degrees.
Theorem 2.1. ([9], see also Theorem 3.1 in [16]) Let $G$ be a loopless graph with $z_0 \in V(G)$, let $k$ be an integer, $k \geq 3$, and let $p : V(G) \to \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. Let $D_{z_0}$ be an orientation of the set of edges incident with $z_0$. Assume that $G$ is essentially $(3k - 3)$-edge-connected, and

1. $d_G(z_0) \leq 2k - 1 + p(z_0)$, and the edges incident with $z_0$ are directed such that $d^+_G(z_0) \equiv p(z_0)$.

2. $d_G(v) \geq 2k - 1 + [p(v)]_k$, for each $v \in V(G) \setminus \{z_0\}$.

Then the orientation $D_{z_0}$ can be extended to a $p$-orientation $D$ of $G$ such that for each vertex $v$,

$$\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d^+_G(v) \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + (k - 1).$$

Corollary 2.2. Let $G$ be a loopless graph with $z_0 \in V(G)$, let $k$ be an integer, $k \geq 3$, and let $p : V(G) \to \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. Let $D_{z_0}$ be an orientation of the set of edges incident with $z_0$. Let $s$ and $s_0$ be two nonnegative integer-valued functions on $V(G)$ satisfying $s_0(v) + s(v) < 2k$ for each vertex $v$, and $s(z_0) = s_0(z_0) = 0$. Assume that $G$ is essentially $(3k - 3)$-edge-connected, and

1. $d_G(z_0) + \sum_{v \in V(G)} \max\{s(v), s_0(v)\} < 2k$, and the edges incident with $z_0$ are directed such that $d^+_G(z_0) \equiv p(z_0)$.

2. $d_G(v) \geq 2k - 1 + [p(v)]_k$, for each $v \in V(G) \setminus \{z_0\}$.

Then the orientation $D_{z_0}$ can be extended to a $p$-orientation $D$ of $G$ such that for each vertex $v$,

$$\left\lfloor \frac{d_G(v) + s(v)}{2} \right\rfloor - (k - 1) \leq d^+_G(v) \leq \left\lceil \frac{d_G(v) - s_0(v)}{2} \right\rceil + (k - 1).$$

Proof. Define $S$ to be the set of all $v \in V(G) \setminus \{z_0\}$ such that there is an integer $q(v)$ satisfying $q(v) \equiv p(v) \mod k$ and $d_G(v)/2 < q(v) \leq d_G(v)/2 + s(v)/2$. Likewise, define $S_0$ to be the set of all $v \in V(G) \setminus \{z_0\}$ such that there is an integer $q(v)$ satisfying $q(v) \equiv p(v) \mod k$ and $d_G(v)/2 - s_0(v)/2 \leq q(v) < d_G(v)/2$. Since $s(v) + s_0(v)/2 < 2k$, we must have $S \cap S_0 = \emptyset$. Let $G'$ be the graph obtained from $G$ by adding $|2q(v) - d_G(v)|$ new parallel edges $vz_0$ for all $v \in S \cup S_0$. We orient these new edges toward $v$ when $v \in S$, and orient them toward $z_0$ when $v \in S_0$. Define $p'(v) = p(v)$ for each $v \in V(G) \setminus (S_0 \cup \{z_0\})$, $p'(v) = p(v) + |2q(v) - d_G(v)|$ for each $v \in S_0$, and $p'(z_0) = p(z_0) + \sum_{v \in S} |2q(v) - d_G(v)|$. It is easy to check that $|E(G')| \equiv \sum_{v \in V(G')} p'(v)$, and $d_{G'}(v) \geq 2k - 1 + [p'(v)]_k$ for each $v \in V(G) \setminus \{z_0\}$, and also $d_{G'}(z_0) = d_G(z_0) + \sum_{v \in S \cup S_0} |2q(v) - d_G(v)| \leq d_G(z_0) + \sum_{v \in V(G)} \max\{s(v), s_0(v)\} \leq 2k - 1 \leq 2k - 1 + p'(z_0)$. Therefore, by Theorem 2.1, the graph $G'$ has a $p'$-orientation modulo $k$ such that for each vertex $v$, $|d^+_{G'}(v) - d_{G'}(v)/2| < k$. Since for each $v \in S \cup S_0$, $p'(v) \equiv d_{G'}(v)/2$, we must have $d^+_{G'}(v) = d_{G'}(v)/2$. Thus this orientation induces a $p$-orientation for $G$ such that for each $v \in S \cup S_0$, $d^+_G(v) = q(v)$. In addition, $|d^+_{G'}(v) - d_{G'}(v)/2| < k$ for each $v \in V(G) \setminus (S \cup S_0)$. According to the definition of $S$ and $S_0$, we must therefore have $d_G(v)/2 - s(v)/2 - k < d^+_G(v) < d_G(v)/2 - s_0(v)/2 + k$ for all $v \in V(G)$. Hence the proof is completed. \qed
Corollary 2.3. (Corollary 3.3 in [9]) Let $G$ be a loopless graph, let $k$ be an integer, $k \geq 3$, let $p : V(G) \to \mathbb{Z}_k$ be a mapping with $|E(G)| = k \sum_{v \in V(G)} p(v)$. If $G$ is essentially $(3k - 3)$-edge-connected and for each vertex $v$, $d_G(v) \geq 2k - 1 + |p(v)|_k$, then $G$ has a $p$-orientation such that for each vertex $v$,

$$\left\lceil \frac{d_G(v)}{2} \right\rceil - (k - 1) \leq d_G^+(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + (k - 1).$$

Furthermore, for an arbitrary vertex $z$, $d_G^+(z)$ can be assigned to any plausible integer value in whose interval.

Proof. It is enough to apply Corollary 2.2 on the graph obtained from $G$ by adding a new artificial vertex $z_0$ with degree zero. For the desired restriction on $d_G^+(z)$, we only need to set \{s(z), s_0(z)\} = \{0, 2k - 1\} and $s(v) = s_0(v) = 0$ for all vertices with $v$ with $v \neq z$. $\square$

2.2 Compatible mappings

In the following, we shall provide some sufficient conditions for a mapping to be compatible. Before doing so, let us make the following lemma which can conclude that every $(2bi(G) + 1)$-edge-connected graph $G$ has a unique bipartition $X, Y$ of $V(G)$ satisfying $e_G(X) + e_G(Y) = bi(G)$. In particular, every connected bipartite graph has a unique bipartition.

Lemma 2.4. If $G$ is a $(2bi(G) + 1 + i)$-edge-connected graph and $i \geq 0$, then there is a unique bipartition $X, Y$ of $V(G)$ satisfying $e_G(X) + e_G(Y) \leq bi(G) + i$.

Proof. Let $X, Y$ be a bipartition of $V(G)$ satisfying $e_G(X) + e_G(Y) = bi(G)$. Let $X', Y'$ be another bipartition of $V(G)$ with $\{X', Y'\} \neq \{X, Y\}$. Set $A = (X' \cap X) \cup (Y' \cap Y)$. If $A$ is empty, then it is easy to check that $X' = Y$ and $Y' = X$. Likewise, if $V(G) \setminus A$ is empty, then we must have $X' = X$ and $Y' = Y$. Thus $A$ is a nonempty proper subset of $V(G)$. Therefore, $e_G(X') + e_G(Y') \geq d_G(A) - (e_G(X) + e_G(Y)) \geq (2bi(G) + 1 + i) - bi(G) > bi(G) + i$. Hence the proof is completed. $\square$

The following lemma introduces a lower bound on the bipartite index of $k$-tree-connected graphs.

Lemma 2.5. Let $G$ be a graph. If $G[X, Y]$ is $k$-tree-connected and $e_G(X) + e_G(Y) \geq k$ for a bipartition $X, Y$ of $V(G)$, then $bi(G) \geq k$.

Proof. Let $X', Y'$ be a partition of $V(G)$ with $e_G(X') + e_G(Y') = bi(G)$. Let $T_1, \ldots, T_k$ be $k$ edge-disjoint spanning trees of $G[X, Y]$ and let $e_1, \ldots, e_k$ be $k$ distinct edges of the graph $G[X] \cup G[Y]$. Since $T_i$ is a bipartite graph with the bipartition $(X, Y)$, the graph $T_i + e_i$ must contain an odd cycle $C_i$ which implies that $e_C(X') + e_C(Y') \geq 1$. Therefore, $bi(G) = e_G(X') + e_G(Y') \geq \sum_{1 \leq i \leq k} (e_{C_i}(X') + e_{C_i}(Y')) \geq k$. $\square$

The following theorem gives sufficient conditions for a mapping to be compatible with the main graph.
Theorem 2.6. Let $G$ be a graph, let $k$ be a positive integer, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $(k - 1) \sum_{v \in V(G)} f(v)$ even. Then $f$ is compatible with $G$ if one of the following conditions holds:

1. $k$ is even and $\text{bi}(G) \geq k/2 - 1$.
2. $k$ is odd and $\text{bi}(G) \geq k - 1$; see [23].
3. $G[X, Y]$ is $(2k - 2)$-edge-connected and $e_G(X) + e_G(Y) \geq k - 1$ for a bipartition $X, Y$ of $V(G)$.
4. $G$ is $(2k - 3)$-edge-connected and $f$ is compatible with $G$ with respect to a bipartition $X, Y$ of $V(G)$ satisfying $e_G(X) + e_G(Y) \leq k - 2$.

Proof. Let $X, Y$ be a bipartition of $V(G)$. First assume that $k$ is even and $e_G(X) + e_G(Y) \geq k/2 - 1$. By the assumption, $\sum_{v \in X} f(v) - \sum_{v \in Y} f(v)$ must be even. Thus there are two integers $x, y \in \{0, \ldots, k/2 - 1\}$ such that $\frac{1}{2}(\sum_{v \in X} f(v) - \sum_{v \in Y} f(v)) \equiv x$ and $\frac{1}{2}(\sum_{v \in X} f(v) - \sum_{v \in Y} f(v)) \equiv y$. Therefore, $x + y \equiv 0$ which can conclude that $x \leq e_G(X)$ or $y \leq e_G(Y)$. Now assume that $k$ is odd and $e_G(X) + e_G(Y) \geq k - 1$. Let $x, y \in \{0, \ldots, k - 1\}$ be two integers such that $\sum_{v \in X} f(v) - \sum_{v \in Y} f(v) \equiv 2x$ and $\sum_{v \in Y} f(v) - \sum_{v \in X} f(v) \equiv 2y$. Therefore, $x + y \equiv 0$ which can conclude that $x \leq e_G(X)$ or $y \leq e_G(Y)$. These imply that $f$ is compatible with $G$ with respect to $X, Y$. Hence the first two assertions hold. To prove the third assertion, it is enough to apply Lemma 2.5 together with the first two assertions.

Now, assume that $G$ is $(2k - 3)$-edge-connected and $f$ is compatible with $G$ with respect to a bipartition $X, Y$ of $V(G)$ satisfying $e_G(X) + e_G(Y) \leq k - 2 = \text{bi}(G) + i$ and $i \geq 0$. Let $X', Y'$ be another bipartition of $V(G)$. Since $G$ is $(2\text{bi}(G) + 1 + i)$-edge-connected, by Lemma 2.4, we must have $e_G(X') + e_G(Y') \geq \text{bi}(G) + i + 1 \geq k - 1$. Thus one can conclude that $f$ must be compatible with $G$ with respect to $X', Y'$ by repeating the proof of items (1) and (2). This can complete the proof. \qed

An immediate consequence of Theorem 1.1 and the following corollary says that if $G$ is a graph satisfying $\text{bi}(G) \leq k - 2$, then the number of orientations with out-degrees $k - 1$ must be divisible by $k$, provided that $k$ is a prime number; for example, see [19, Theorem 4]. Recall that if a graph $G$ contains an $f$-factor modulo $k$, then $f$ must be compatible with $G$.

Corollary 2.7. Let $G$ be a graph, let $k$ be an integer with $k \geq 2$. Let $k_0 = k$ when $k$ is odd, and let $k_0 = k/2$ when $k$ is even. Then $\text{bi}(G) \geq k_0 - 1$ if and only if every mapping $f : V(G) \to \mathbb{Z}_k$ with $(k - 1) \sum_{v \in V(G)} f(v)$ even is compatible with $G$.

Proof. Let $X, Y$ be a bipartition of $V(G)$ with $e_G(X) + e_G(Y) = \text{bi}(G)$. For a vertex $z \in X$, define $f(z) = 2e_G(X) + 2 \, (\text{mod } k)$, and define $f(v) = 0$ for all $v \in V(G) \setminus \{z\}$. If $\text{bi}(G) < k_0 - 1$, then $f$ is not compatible with $G$. Otherwise, there are two integers $x$ and $y$ with $0 \leq x \leq e_G(X)$ and $0 \leq y \leq e_G(Y)$ such that $\sum_{v \in X} f(v) - 2x \equiv k \sum_{v \in Y} f(v) - 2y$ which implies that $2(e_G(X) - x) + 2y \equiv 2(k - 1)$. Therefore, $(e_G(X) - x) + y \equiv (k_0 - 1)$ and so $k_0 - 1 \leq e_G(X) - x + y \leq e_G(X) + e_G(Y)$, which is a contradiction. Hence the proof can be completed using Theorem 2.6. \qed
2.3 Bipartite index and tree-connectivity

A well-known observation, attributed to Erdős (1965), says that every loopless graph with minimum degree at least $2m - 1$ contains a bipartite factor with minimum degree at least $m$, see [5, Theorem 2.4]. This result is developed to an edge-connected version by Thomassen (2008) as the following theorem.

**Theorem 2.8.** ([21]) Every $(2m - 1)$-edge-connected graph has an $m$-edge-connected bipartite factor.

In the following, we shall provide a tree-connected version for it which will be used several times in this paper. This theorem is also generalized in [11] for finding factors with bounded chromatic numbers.

**Theorem 2.9.** Every $2m$-tree-connected loopless graph $G$ has an $m$-tree-connected bipartite factor $H$ such that for every vertex set $A$, $d_H(A) \geq \lceil d_G(A)/2 \rceil$.

**Proof.** Let $H$ be a bipartite factor of $G$ with the maximum $|E(H)|$. We claim that $H$ is the desired factor. Suppose, to the contrary, that $d_H(A) < d_G(A)/2$ for a vertex set $A$. Define $X_0 = (X \setminus A) \cup (A \cap Y)$ and $Y_0 = (Y \setminus A) \cup (A \cap X)$, where $(X, Y)$ is the bipartition of $H$. It is not difficult to see that the graph $G[X_0, Y_0]$ is a bipartite factor of $G$ with more edges than $H$ which is a contradiction. Now, let $P$ be a partition of $V(G)$. Since $G$ is $2m$-tree-connected, we must have $e_G(P) \geq 2m(|P| - 1)$, where $e_G(P)$ denotes the number of edges of $G$ joining different parts of $G$. Therefore, $e_H(P) = \sum_{A \in P} \frac{1}{2}d_H(A) \geq \sum_{A \in P} \frac{1}{2}d_G(A) = \frac{1}{2}e_G(P) \geq m(|P| - 1)$. Hence by the well-known result of Nash-Williams [18] and Tutte [24], the graph $H$ must be $m$-tree-connected. \hfill \Box

The following corollary gives a criterion for the existence of edge-disjoint odd cycles in $2k$-tree-connected graphs in terms of bipartite index. The edge-connected version of this corollary is mentioned in [20, Section 7].

**Corollary 2.10.** Let $G$ be a $2k$-tree-connected graph. Then $\text{bi}(G) \geq k$ if and only if $G$ contains $k$ edge-disjoint odd cycles.

**Proof.** Let $X, Y$ be a partition of $V(G)$. If $C$ is an odd cycle, then obviously $e_C(X) + e_C(Y) \geq 1$. This implies that $e_G(X) + e_G(Y) \geq k$ provided that $G$ contains $k$ edge-disjoint odd cycles. Now, assume that $\text{bi}(G) \geq k$. By Theorem 2.9, there exists a bipartition $X, Y$ of $V(G)$ such that $G[X, Y]$ is $k$-tree-connected. Since $e_G(X) + e_G(Y) \geq \text{bi}(G) \geq k$, by Lemma 2.5, the graph $G$ contains $k$ edge-disjoint odd cycles. \hfill \Box

**Corollary 2.11.** Every $2k$-tree-connected graph $G$ satisfying $\text{bi}(G) \geq k$ contains a subgraph $H$ with maximum degree at most $2k$ satisfying $\text{bi}(H) \geq k$.

**Proof.** By Corollary 2.10, the graph $G$ contains $k$ edge-disjoint odd cycles which the union of them is the desired subgraph. \hfill \Box

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Corollary 2.12. Every $(2m + 4)$-tree-connected graph $G$ can be decomposed into two factors $G_1$ and $G_2$ such that $G_1$ is Eulerian, $G_2[X, Y]$ is $m$-tree-connected for a bipartition $X, Y$ of $V(G)$, and
\[ e_{G_2}(X) + e_{G_2}(Y) = \min\{k, bi(G)\}, \]
where $k$ is an arbitrary nonnegative integer.

Proof. By Theorem 2.9, there exists a bipartition $X, Y$ of $V(G)$ such that $G[X, Y]$ is $(m + 2)$-tree-connected. Decompose $G[X, Y]$ into two spanning trees $T_0$ and $T$ and an $m$-tree-connected factor $H$. Since $e_G(X) + e_G(Y) \geq bi(G)$, we can decompose $G[X] \cup G[Y]$ into two factors $M_0$ and $M$ such that $|E(M)| = \min\{k, bi(G)\}$. Let $F$ be a spanning forest of $T$ such that for each vertex $v$, $d_F(v)$ and $d_{T_0}(v) + d_{M_0}(v)$ have the same parity. It is enough to set $G_1 = T_0 \cup M_0 \cup F$ and $G_2 = G \setminus E(G_1)$ to complete the proof.

Corollary 2.13. Let $G$ be a graph with $bi(G) \geq k$. If $G$ is $4k$-tree-connected, then it has $k$ edge-disjoint spanning Eulerian subgraphs with odd size.

Proof. By Theorem 2.9, $G[X, Y]$ is $2k$-tree-connected for a partition $X, Y$ of $V(G)$. Thus $G[X, Y]$ contains $2k$ edge-disjoint spanning trees $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$. By the assumption, $e_G(X) + e_G(Y) \geq k$. Thus we can take $e_1, \ldots, e_k$ to be $k$ edges of $G[X] \cup G[Y]$. Define $H_i = T'_i + e_i$. Let $F_i$ be a spanning forest of $T_i$ such that for each vertex $v$, $d_{F_i}(v)$ and $d_{H_i}(v)$ have the same parity. It is enough to set $G_i = F_i \cup H_i$ to construct the desired parity factors.

3 Factors modulo 2

In this section, we consider the existence of parity factors. Our results are based on the following lemma which is a special case of a result due to Lovász (1970) who gave a criterion for the existence of parity $(g, f)$-factors. Here, we denote by $\omega(G)$ the number of components of a graph $G$, and a parity $(g, f)$-factor refers to a spanning subgraph $H$ such that for each vertex $v$, $g(v) \leq d_H(v) \leq f(v)$ and $d_H(v) \equiv g(v) \equiv f(v)$.

Lemma 3.1. (14; see also [10, Lemma 6.1]) Let $G$ be a connected graph and let $g$ and $f$ be two integer-valued functions on $V(G)$ with $g \leq f$ satisfying $\sum_{v \in V(G)} f(v) = 0$, and $f(v) \equiv g(v)$ for each vertex $v$. Then $G$ has a parity $(g, f)$-factor, if for any two disjoint subsets $A$ and $B$ of $V(G)$ with $A \cup B \neq \emptyset$,
\[ \omega(G \setminus (A \cup B)) \leq 1 + \sum_{v \in A} f(v) + \sum_{v \in B} (d_G(v) - g(v)) - d_G(A, B). \]

It is known that edge-connectivity 1 is sufficient for a graph to have an $f$-factor modulo 2, see [8, 23]. The following theorem shows that edge-connectivity 2 is sufficient for a graph to have an $f$-factor modulo 2 whose degrees fall in predetermined short intervals.
Theorem 3.2. Let $G$ be a graph and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \equiv 0$. If $G$ is 2-edge-connected, then it has an $f$-factor $H$ such that for each vertex $v$,

$$\left\lfloor \frac{d_G(v)}{2} \right\rfloor - 1 \leq d_H(v) \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + 1.$$ 

Furthermore, for an arbitrary vertex $z$, $d_H(z)$ can be assigned to any plausible integer value in whose interval.

Proof. For each vertex $v$, define $g'(v) \in \{ \lfloor d_G(v)/2 \rfloor, \lceil d_G(v)/2 \rceil + 1 \}$ such that $g'(v) \equiv f(v) \equiv f'(v)$. Obviously, $g'(v) \leq f'(v)$. Note that if $g'(z) < f'(z)$, we allow to replace $g'(z)$ by $g'(z) + 2$ or replace $f'(z)$ by $f'(z) - 2$ with respect to our purpose related to $z$. For the first choice, we have $|d_G(z)/2 - g'(z)| \leq 3/2$ and for the second choice, we have $|d_G(z)/2 - f'(z)| \leq 3/2$. Let $A$ and $B$ be two disjoint subsets of $V(G)$ with $A \cup B \neq \emptyset$. Since $G$ is 2-edge-connected, it is not hard to check that

$$\omega(G \setminus (A \cup B)) \leq \sum_{A \cup B} \frac{1}{2} d_G(v) - e_G(A \cup B) \leq \sum_{A \cup B} \frac{1}{2} d_G(v) - d_G(A, B).$$

Therefore,

$$\omega(G \setminus (A \cup B)) < 2 + \sum_{v \in A} f'(v) + \sum_{v \in B} (d_G(v) - g'(v)) - d_G(A, B),$$

whether $z \in A \cup B$ or not. Thus by Lemma 3.1, the graph $G$ has a parity $(g', f')$-factor and the proof is completed.

The following well-known result on Eulerian graphs is a special case of Theorem 3.2.

Corollary 3.3. Every Eulerian graph $G$ with $z \in V(G)$ has a factor $H$ such that for each $v \in V(G) \setminus \{z\}$, $d_H(v) = d_G(v)/2$, and $d_H(z) \in \{d_G(z)/2, d_G(z)/2 + 1\}$.

Proof. For each $v \in V(G) \setminus \{z\}$, define $f(v) = d_G(v)/2 \pmod{2}$, and also define $f(z) = \sum_{v \in V(G) \setminus \{z\}} f(v) \pmod{2}$. Since $G$ is 2-edge-connected, by Theorem 3.2, the graph $G$ has an $f$-factor $H$ such that for each $v \in V(G) \setminus \{z\}$, $d_G(v)/2 - 1 \leq d_H(v) \leq d_G(v)/2 + 1$, and $d_G(z)/2 \leq d_H(z) \leq d_G(z)/2 + 1$. Hence $H$ is the desired factor.

An interesting application of Theorem 3.2 is given in the following corollary.

Corollary 3.4. Every connected 2$r$-regular graph $G$ with $(r + 1)|V(G)|$ even can be decomposed into two factors whose degrees lie in the set $\{r - 1, r + 1\}$.

Proof. For each vertex $v$, define $f(v) = r + 1 \pmod{2}$. By Theorem 3.2, the graph $G$ has an $f$-factor such that for each vertex $v$, $r - 1 \leq d_G(v)/2 - 1 \leq d_H(v) \leq d_G(v)/2 + 1 \leq r + 1$. Hence $H$ and its complement are the desired factors whose degrees lie in the set $\{r - 1, r + 1\}$.

Remark 3.5. Note that Theorem 3.2 can conclude Theorem 11 in [7]. A generalization of it is formulated in [10, Theorem 6.2].
In the following theorem, we develop Theorem 3.2 to a partition-connected version. A graph $G$ is said to be $(m, l_0)$-partition-connected, if it can be decomposed into an $m$-tree-connected factor and a factor $F$ which admits an orientation such that for each vertex $v$, $d_F(v) \geq l_0(v)$, where $l_0$ is a nonnegative integer-valued function on $V(G)$.

**Theorem 3.6.** Let $G$ be a graph and let $f : V(G) \to \mathbb{Z}_2$ be a mapping with $\sum_{v \in V(G)} f(v) \equiv 2 \pmod{2}$. Let $s$, $s_0$, and $l_0$ be three integer-valued functions on $V(G)$ satisfying $s(v) + s_0(v) < d_G(v)$ and $\max\{s(v), s_0(v)\} \leq l_0(v)$ for each vertex $v$. If $G$ is $(1, l_0)$-partition-connected, then it has an $f$-factor $H$ such that for each vertex $v$,

$$s(v) \leq d_H(v) \leq d_G(v) - s_0(v).$$

**Proof.** For each vertex $v$, define $g'(v) \in \{s(v), s(v) + 1\}$ and $f'(v) \in \{d_G(v) - s_0(v) - 1, d_G(v) - s_0(v)\}$ such that $g'(v) \equiv f(v) \equiv f'(v)$. Note that the condition $s(v) + s_0(v) < d_G(v)$ implies that $g'(v) \leq f'(v) + 1$ and hence $g'(v) \leq f'(v)$, because those have the same parity. By the assumption, the graph $G$ can be decomposed into two factors $T$ and $F$ such that $T$ is a spanning tree and $F$ admits an orientation such that for each vertex $v$, $d_F(v) \geq l_0(v)$. Let $A$ and $B$ be two disjoint subsets of $V(G)$. It is not hard to check that

$$\omega(G \setminus (A \cup B)) \leq \omega(T \setminus (A \cup B)) = \sum_{A \cup B} (d_T(v) - 1) + 1 - e_T(A \cup B) \leq \sum_{A \cup B} (d_T(v) - 1) + 1 - d_T(A, B).$$

Since $d_F(v) \geq l_0(v) \geq \max\{s(v), s_0(v)\}$ for each vertex $v$, we must have

$$0 \leq \sum_{v \in A \cup B} d_F(v) - d_F(A, B) \leq \sum_{v \in A} (d_F(v) - s_0(v)) + \sum_{v \in B} (d_F(v) - s(v)) - d_F(A, B).$$

Therefore,

$$\omega(G \setminus (A \cup B)) \leq \sum_{v \in A} (d_G(v) - s_0(v) - 1) + \sum_{v \in B} (d_G(v) - s(v) - 1) - d_G(A, B) + 1,$$

which implies that

$$\omega(G \setminus (A \cup B)) \leq \sum_{v \in A} f'(v) + \sum_{v \in B} (d_G(v) - g'(v)) - d_G(A, B) + 1.$$

Thus by Lemma 3.1, the graph $G$ has a parity $(g', f')$-factor and the proof is completed.

\[\square\]

4 Factors modulo $k$: Almost bipartite graphs

4.1 Bipartite graphs

There is a special one-to-one mapping between orientations and factors of any bipartite graph, which was utilized by Thomassen in [22] in order to establish Theorem 1.2. Using the same arguments, we derive the following strengthened version.
Theorem 4.1. Let $G$ be a bipartite graph with bipartition $(X,Y)$, let $k$ be a positive integer, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$. If $G$ is essentially $(3k-3)$-edge-connected and $d_G(v) \geq 2k - 1 + \lfloor f(v) \rfloor_k$ for each vertex $v$, then $G$ has an $f$-factor $H$ such that for each vertex $v$,

$$\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k-1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + (k-1).$$

Furthermore, for an arbitrary vertex $z$, $d_H(z)$ can be assigned to any plausible integer value in whose interval.

Proof. The special case $k = 2$ follows from Theorem 3.2. Assume $k \geq 3$. For each $v \in X$, define $p(v) = f(v)$, and for each $v \in Y$, define $p(v) = d_G(v) - f(v)$. Since $f$ is compatible with $G$, $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$ which can conclude that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. By the assumption for each $v \in X$, $d_G(v) \geq 2k - 1 + \lfloor f(v) \rfloor_k = 2k - 1 + [p(v)]_k$. Also, for each $v \in Y$, $d_G(v) \geq 2k - 1 + [f(v)]_k = 2k - 1 + [d_G(v) - p(v)]_k$ which can imply that $d_G(v) \geq 2k - 1 + [p(v)]_k$. More precisely, if we let $i = d_G(v) - 2k$, then since $i - [f(v)]_k \geq -1$, we must have $d_G(v) \geq 2k + i - [f(v)]_k - 1 \geq 2k - 1 + [i - f(v)]_k = 2k - 1 + [p(v)]_k$. Thus by Corollary 2.3, the graph $G$ has a $p$-orientation modulo $k$ such that for each vertex $v$, $\lfloor d_G(v)/2 \rfloor - (k-1) \leq d^+_G(v) \leq \lfloor d_G(v)/2 \rfloor + (k-1)$. Take $H$ to be the factor of $G$ consisting of all edges directed from $X$ to $Y$. Since for all vertices $v \in X$, $d_H(v) = d^+_G(v)$, and for all vertices $v \in Y$, $d_H(v) = d_G(v) - d^+_G(v)$, the graph $H$ is the desired $f$-factor. The remaining case $k = 1$ can be proved similarly, because it is known that every graph $G$ has an orientation such that for each vertex $v$, $\lfloor d_G(v)/2 \rfloor \leq d^+_G(v) \leq \lfloor d_G(v)/2 \rfloor$; in particular, we can arbitrarily have $d^+_G(z) = \lfloor d_G(z)/2 \rfloor$ or $d^+_G(z) = \lceil d_G(z)/2 \rceil$ by reversing the orientation (if necessary). \hfill \Box

Bensmail, Merker, and Thomassen (2017) [4] applied a weaker version of the following corollary to deduce that every 16-edge-connected bipartite graph admits a decomposition into at most two locally irregular subgraphs. Their proof is based on Theorem 5.2 in [4] for the special case $k = 6$. By replacing the following result in their proof, this number can be pushed down to 15.

Corollary 4.2. Let $G$ be a bipartite graph with bipartition $(X,Y)$, let $k$ be a positive integer, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$. If $G$ is $(3k-3)$-edge-connected, then it has an $f$-factor $H$ such that for each vertex $v$,

$$\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k-1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + (k-1).$$

Furthermore, for an arbitrary vertex $z$, $d_H(z)$ can be assigned to any plausible integer value in whose interval.

Proof. Apply Theorem 4.1 with the fact that for each vertex $v$, $d_G(v) \geq 3k - 3 \geq 2k - 1 + [f(v)]_k$. \hfill \Box

We have the following immediate conclusions similar to Corollaries 3.8 and 3.9 in [9]. Note the required edge-connectivity $3k - 3$ of the first one can be replaced by odd edge-connectivity $3k - 2$ and the second one can be replaced by the condition $d_G(A) \geq 6k - 2$, where $A$ is a subset of $V(G)$ with odd size.
Corollary 4.3. Let $G$ be a bipartite graph and let $k$ be an odd positive integer. If $G$ is $(3k - 3)$-edge-connected, then it has a factor $H$ such that for each vertex $v$,

$$d_H(v) \in \{ \frac{d_G(v)}{2} - \frac{k}{2}, \frac{d_G(v)}{2}, \frac{d_G(v)}{2} + \frac{k}{2}\}.$$ 

Proof. For each vertex $v$, define $f(v) = d_G(v)/2 \pmod{k}$ when $d_G(v)$ is even, and define $f(v) = (d_G(v) + k)/2 \pmod{k}$ when $d_G(v)$ is odd. Let $(X, Y)$ be the bipartition of $G$. Obviously, $\sum_{v \in X} f(v) = \frac{|E(G)|}{2} + n_xk/2$ and $\sum_{v \in Y} f(v) = \frac{|E(G)|}{2} + n_yk/2$, where $n_x$ and $n_y$ are the number of vertices in $X$ and $Y$ with odd degrees, respectively. Since $n_x$ and $n_y$ have the same parity, $\sum_{v \in X} f(v) = \sum_{v \in Y} f(v)$. Thus by Corollary 4.2, the graph $G$ has an $f$-factor such that for each vertex $v$, $\lfloor d_G(v)/2 \rfloor - (k - 1) \leq d_H(v) \leq \lceil d_G(v)/2 \rceil + (k - 1)$. If $d_G(v)$ is even, then $d_H(v) = d_G(v)/2$. Otherwise, since $d_G(v)/2 - 3k/2 < d_H(v) < d_G(v)/2 + 3k/2$, we must have $d_H(v) \in \{d_G(v)/2 - k/2, d_G(v)/2 + k/2\}$. Hence $H$ is the desired factor. □

Corollary 4.4. Let $G$ be a bipartite Eulerian graph of even order and let $k$ be a positive integer. If $G$ is $(6k - 2)$-edge-connected, then it has a factor $H$ such that for each vertex $v$,

$$d_H(v) \in \{ \frac{d_G(v)}{2} - k, \frac{d_G(v)}{2}, \frac{d_G(v)}{2} + k\}.$$ 

Proof. For each vertex $v$, define $f(v) = d_G(v)/2 + k \pmod{2k}$, and let $(X, Y)$ be the bipartition of $G$. Obviously, $\sum_{v \in X} d_G(v)/2 = |E(G)|/2 = \sum_{v \in Y} d_G(v)/2$. Since $|V(G)|$ is even, we must have $\sum_{v \in X} f(v) = \sum_{v \in Y} f(v)$. Thus by Corollary 4.2, the graph $G$ has an $f$-factor $H$ such that for each vertex $v$, $d_G(v)/2 - 3k < d_G(v)/2 - (2k - 1) \leq d_H(v) \leq d_G(v)/2 + 2k - 1 < d_G(v)/2 + 3k$. Hence $H$ is the desired factor. □

The following corollary is an improved version of Lemma 4.1 in [17]. Note the required edge-connectivity $3k$-3 can be replaced by the condition $d_G(A) \geq 3k - 3$, where $A$ is a subset of $V(G)$ satisfying $0 < |A \cap Y| < |Y|$; by replacing Corollary 3.8 in [9] in the proof.

Corollary 4.5. Let $G$ be a $(3k - 3)$-edge-connected bipartite graph on classes $X$ and $Y$, where each vertex in $X$ has even degree. For every function $f : Y \to \mathbb{Z}_k$ satisfying $\sum_{v \in Y} f(v) = \frac{k}{2} |E(G)|$, there exists a factor $H$ of $G$ such that

1. $d_H(v) = \frac{1}{2} d_G(v)$ for each $v \in X$.

2. $|d_H(v) - \frac{1}{2} d_G(v)| < k$ and $d_H(v) = \frac{k}{2} f(v)$ for each $v \in Y$.

Proof. For each $v \in X$, define $f(v) = d_G(v)/2 \pmod{k}$. According to the assumption, $\sum_{v \in X} f(v) = \frac{k}{2} |E(G)| = \sum_{v \in X} d_G(v)/2 = \frac{1}{2} |E(G)| = \sum_{v \in Y} f(v)$. Thus by Corollary 4.2, the graph $G$ has an $f$-factor $H$ such that for each vertex $v$, $d_G(v)/2 + k < |d_G(v)/2| - (k - 1) \leq d_H(v) \leq |d_G(v)/2| + (k - 1) < d_G(v)/2 + k$ which implies that $|d_H(v) - \frac{1}{2} d_G(v)| < k$. For each $v \in X$, we therefore have $d_H(v) = d_G(v)/2$. This completes the proof. □
For proving the remaining two theorems, we need the following two lemmas. Recall that a graph $G$ is said to be $(m,l_0)$-partition-connected, if it can be decomposed into an $m$-tree-connected factor and a factor $F$ which admits an orientation such that for each vertex $v$, $d^+_F(v) \geq l_0(v)$, where $l_0$ is a nonnegative integer-valued function on $V(G)$.

**Lemma 4.6.** (9) Let $G$ be a graph, let $k$ be an integer, $k \geq 3$, and let $p : V(G) \to \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. Let $s, s_0,$ and $l_0$ be three integer-valued functions on $V(G)$ satisfying $s(v) + s_0(v) + k - 1 \leq d_G(v)$ and $\max\{s(v), s_0(v)\} \leq l_0(v) + (k-1)$ for each vertex $v$, and $\max\{s(z), s_0(z)\} \leq l_0(z)$ for a vertex $z$. If $G$ is $(2k-2, l_0)$-partition-connected, then it has a $p$-orientation such that for each vertex $v$,

$$s(v) \leq d^+_G(v) \leq d_G(v) - s_0(v).$$

**Lemma 4.7.** (9) Let $G$ be a graph of order at least two, let $k$ be an integer, $k \geq 3$, and let $p : V(G) \to \mathbb{Z}_k$ be a mapping with $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is $(2k-2)$-tree-connected, then it has a $p$-orientation such that for each vertex $v$,

$$k/2 - 1 \leq d^+_G(v) \leq d_G(v) - k/2 + 1.$$  

The following theorem provides a partition-connected version for Theorem 4.1.

**Theorem 4.8.** Let $G$ be a bipartite graph with bipartition $(X,Y)$, let $k$ be an integer, $k \geq 3$, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$. Let $s, s_0,$ and $l_0$ be three integer-valued functions on $V(G)$ satisfying $s(v) + s_0(v) + k - 1 \leq d_G(v)$ and $\max\{s(v), s_0(v)\} \leq l_0(v) + (k-1)$ for each vertex $v$, and $\max\{s(z), s_0(z)\} \leq l_0(z)$ for a vertex $z$. If $G$ is $(2k-2, l_0)$-partition-connected, then it admits an $f$-factor $H$ such that for each vertex $v$,

$$s(v) \leq d_H(v) \leq d_G(v) - s_0(v).$$

**Proof.** For each $v \in X$, define $p(v) = f(v)$, $s'(v) = s(v)$, $s'_0(v) = s_0(v)$, and for each $v \in Y$, define $p(v) = d_G(v) - f(v)$, $s'(v) = s_0(v)$, $s'_0(v) = s(v)$. By the assumption, we must have $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. By Lemma 4.6, the graph $G$ has a $p$-orientation modulo $k$ such that for each vertex $v$, $s'(v) \leq d^+_G(v) \leq d_G(v) - s'_0(v)$. Take $H$ to be the factor of $G$ consisting of all edges directed from $X$ to $Y$. Since for all vertices $v \in X$, $d_H(v) = d^+_G(v)$, and for all vertices $v \in Y$, $d_H(v) = d_G(v) - d^+_G(v)$, the graph $H$ is the desired $f$-factor we are looking for. \qed

An application of the following theorem is stated in Section 6.

**Theorem 4.9.** Let $G$ be a bipartite graph of order at least two with bipartition $(X,Y)$, let $k$ be an integer, $k \geq 3$, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping with $\sum_{v \in X} f(v) \equiv \sum_{v \in Y} f(v)$. If $G$ is $(2k-2)$-tree-connected, then it has an $f$-factor $H$ such that for each vertex $v$, $k/2 - 1 \leq d_H(v) \leq d_G(v) - k/2 + 1$.  

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Proof. For each \( v \in X \), define \( p(v) = f(v) \), and for each \( v \in Y \), define \( p(v) = d_G(v) - f(v) \). By the assumption, we must have \(|E(G)| \leq \sum_{v \in V(G)} p(v)\). By Lemma 4.7, the graph \( G \) has a \( \rho \)-orientation such that for each vertex \( v \), \( k/2 - 1 \leq d_G^+(v) \leq d_G(v) - k/2 + 1 \). Define \( H \) to be the factor of \( G \) consisting of all edges directed from \( X \) to \( Y \). It is easy to check that \( H \) is the desired factor we are looking for. \( \square \)

4.2 Graphs with small bipartite index

The following theorem establishes a non-bipartite version for Theorem 4.1 with a stronger version.

Theorem 4.10. Let \( G \) be a graph, let \( k \) be an integer, \( k \geq 3 \), and let \( f : V(G) \to \mathbb{Z}_k \) be a compatible mapping. If \( G \) is essentially \((3k - 3)\)-edge-connected and \( d_G(v) \geq 2k-1 + [f(v)]_k \) for each vertex \( v \), and \( e_G(X) + e_G(Y) \leq k-1 \) for a bipartition of \( X, Y \) of \( V(G) \), then \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \),

\[
\left\lfloor \frac{d_G(v) + s(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v) - s_0(v)}{2} \right\rfloor + (k - 1),
\]

where \( s \) and \( s_0 \) are two nonnegative integer-valued functions on \( V(G) \) satisfying \( s_0(v) + s(v) < 2k \) for each vertex \( v \), and \( e_G(X) + e_G(Y) + \frac{1}{2} \sum_{v \in V(G)} \max\{s(v), s_0(v)\} < k \).

Proof. Let \( M \) be the graph \( G[X] \cup G[Y] \). Since \( f \) is compatible, there are two integers \( x \) and \( y \) with \( 0 \leq x \leq e_G(X) = e_M(X) \) and \( 0 \leq y \leq e_G(Y) = e_M(Y) \) such that \( 2y - 2x \equiv \sum_{v \in Y} f(v) - \sum_{v \in X} f(v) \). Let \( M_1 \) be a factor of \( M \) such that \( e_{M_1}(X) = x \) and \( e_{M_1}(Y) = y \), and let \( M_0 = M \setminus E(M_1) \). We are going to construct a new graph \( L \) that plays an important role in the proof. First, we add a new vertex \( z_0 \) to \( G \setminus E(M) \). Next, for each edge \( uv \in E(M) \), we add two edges \( z_0u \) and \( z_0v \) directed as follows. Both them are directed toward \( z_0 \), if either \( uv \in E(M_1) \cap E(G[X]) \) or \( uv \in E(M_0) \cap E(G[Y]) \), and also directed away from \( z_0 \), if either \( uv \in E(M_0) \cap E(G[X]) \) or \( uv \in E(M_1) \cap E(G[Y]) \). Note that we might have some multiple edges incident with \( z_0 \). Call the resulting loopless graph \( L \). Note also that \( d_G(v) = d_L(v) \) for all \( v \in V(G) \).

![Figure 1: An orientation of all edges incident with \( z_0 \).](image)

Define \( p(z_0) = d_L^+(z_0) = 2e_{M_0}(X) + 2e_{M_1}(Y) \), and for each \( v \in V(L) \setminus \{z_0\} \), define

\[
p(v) = \begin{cases} 
  d_L(v) - f(v), & \text{if } v \in Y; \\
  f(v), & \text{if } v \in X.
\end{cases}
\]
Thus
\[ \sum_{v \in V(L)} p(v) \equiv k p(z_0) + \sum_{v \in V(G)} p(v) \equiv 2e_{M_0}(X) + 2e_{M_1}(Y) + \sum_{v \in X} f(v) + \sum_{v \in Y} (d_L(v) - f(v)), \]
and so
\[ \sum_{v \in V(L)} p(v) \equiv 2e_{M_1}(Y) - 2e_{M_1}(X) + \sum_{v \in X} f(v) - \sum_{v \in Y} f(v) + |E(L)| \equiv |E(L)|. \]

Obviously, \( L \) is essentially \((3k - 3)\)-edge-connected, \( d_L(v) = d_G(v) \geq 2k - 1 + [f(v)]_k = 2k - 1 + [p(v)]_k \) for each \( v \in X \), and \( d_L(v) = d_G(v) \geq 2k - 1 + [f(v)]_k \) for each \( v \in Y \) which can imply that \( d_L(v) \geq 2k - 1 + [d_L(v) - f(v)]_k = 2k - 1 + [p(v)]_k \). In addition, \( d_L(z_0) + \sum_{v \in V(G)} \max\{s'(v), s'_0(v)\} = 2|E(M)| + \sum_{v \in V(G)} \max\{s'(v), s'_0(v)\} < 2k \), where \( s'(v) = s(v) \) and \( s'_0(v) = s_0(v) \) when \( v \in X \) and \( s'(v) = s_0(v) \) and \( s'_0(v) = s(v) \) when \( v \in Y \). Therefore, by Theorem 2.1, the orientation of the edges of \( L \) incident with \( z_0 \) can be extended to a \( p \)-orientation of \( L \) such that for each vertex \( v \), \( [(d_L(v) + s'(v))/2] - (k - 1) \leq d_L^+(v) \leq [(d_L(v) - s'_0(v))/2] + (k - 1) \). Let \( F \) be the factor of \( G \) consisting of all edges of \( L - z_0 \) directed from \( X \) to \( Y \). Define \( H = M_1 \cup F \). According to the construction of \( H \), for each \( v \in V(H) \), we have
\[ d_H(v) = d_{M_1}(v) + d_F(v) = \begin{cases} d_L^+(v), & \text{if } v \in Y; \\ d_L^+(v), & \text{if } v \in X. \end{cases} \]
Thus \( [(d_L(v) + s'(v))/2] - (k - 1) \leq d_H(v) \leq [(d_L(v) - s'_0(v))/2] + (k - 1) \). Hence it is not hard to check that \( H \) is the desired \( f \)-factor we are looking for. \( \square \)

The following corollary plays an essential role in the proof of Theorem 5.6 in Subsection 5.2.

**Corollary 4.11.** Let \( G \) be a graph, let \( k \) be an integer, \( k \geq 3 \), and let \( f : V(G) \to \mathbb{Z}_k \) be a compatible mapping. If \( G \) is \((3k - 3)\)-edge-connected and \( e_G(X) + e_G(Y) \leq k - 1 \) for a bipartition of \( X, Y \) of \( V(G) \), then \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \),
\[ \left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lceil \frac{d_G(v)}{2} \right\rceil + (k - 1). \]
Furthermore, for an arbitrary given vertex \( z \) of odd degree, the upper bound and the lower bound can be reduced by one.

**Proof.** Apply Theorem 4.10 by setting \( s(z) = s_0(z) = 1 \) and \( s(v) = s_0(v) = 0 \) for all \( v \in V(G) \setminus \{z\} \). \( \square \)

The following theorem establishes a non-bipartite version for Corollary 4.2 and plays an essential role in the subsequent section. Let \( Q \) be a trail-decomposition of the edges of a graph \( G \) and let \( X \subseteq V(G) \). Here, we say that \( Q \) is \( X \)-parity trail-decomposition, if every trail in \( Q \) of odd size has exactly one end in \( X \) and every trail in \( Q \) of even size has both ends in either \( X \) or \( V(G) \setminus X \).

**Theorem 4.12.** Let \( G \) be a graph, let \( k \) be an integer, \( k \geq 3 \), and let \( f : V(G) \to \mathbb{Z}_k \) be a compatible mapping. Let \( G_0 \) be a factor of \( G \) such that its complement admits an \( X \)-parity trail-decomposition and
Let \( e_G(X) + e_G(Y) = k - 1 \), where \( X, Y \) is a bipartition of \( V(G) \). If \( G_0 \) is \((3k-3)\)-edge-connected, then \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \),
\[
\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k-1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + (k-1).
\]

Furthermore, for an arbitrary given vertex \( z \) of odd degree, the upper bound and the lower bound can be reduced by one.

**Proof.** Let \( Q \) be an \( X \)-parity trail-decomposition of \( G \setminus E(G_0) \). Let \( T \) be a graph with \( V(T) = V(G) \) and for each trail in \( Q \) having different end vertices \( v \) and \( u \) add the edge \( uv \) in \( T \); adding parallel edges if necessary. Note that \( T \) is loopless. For each vertex \( v \), we denote by \( t(v) \) the number of times trails pass through \( v \) but not finish and start at \( v \) plus the number of closed trails started at \( v \). Let \( M = G_0[X] \cup G_0[Y] \) so that \( |E(M)| = k - 1 \). Since \( f \) is compatible, \( \sum_{v \in X} f(v) - \sum_{v \in Y} f(v) \) is even when \( k \) is even. On the other hand, since \( Q \) is an \( X \)-parity trail-decomposition, one can conclude that \( \sum_{v \in Y} t(v) - \sum_{v \in X} t(v) + e_T(Y) - e_T(X) \) must be even (by splitting into similar expressions corresponding to trails of \( Q \)). Thus there is a factor \( M_1 \) of \( M \) such that \( 2e_{M_1}(Y) - 2e_{M_1}(X) = \frac{k}{2} \sum_{v \in Y} (f(v) - t(v)) - \frac{k}{2} \sum_{v \in X} (f(v) - t(v)) - e_T(Y) + e_T(X) \). Let \( M_0 = M \setminus E(M_1) \).

We are going to construct a new graph \( L \) similarly to the proof of Theorem 4.10. First, we add a new vertex \( z_0 \) to \( G_0 \setminus E(M) \). Next, for each edge \( uv \in E(M) \), we add two edges \( z_0u \) and \( z_0v \) directed as follows. Both are directed toward \( z_0 \), if either \( uv \in E(M_1) \cap E(G[X]) \) or \( uv \in E(M_0) \cap E(G[Y]) \), and also directed away from \( z_0 \), if either \( uv \in E(M_0) \cap E(G[X]) \) or \( uv \in E(M_1) \cap E(G[Y]) \). Call the resulting loopless graph \( L \). Note that for each \( v \in V(G) \), \( d_L(v) = d_L(v) + d_T(v) + 2t(v) \). Define \( p(z_0) = d_L^k(z_0) = 2e_{M_0}(X) + 2e_{M_1}(Y) \), and for each \( v \in V(L) \setminus \{z_0\} \), define
\[
p(v) = \begin{cases} d_L(v) + d_T(v) - (f(v) - t(v)), & \text{if } v \in Y; \\ f(v) - t(v), & \text{if } v \in X. \end{cases}
\]

Thus
\[
\sum_{v \in V(L)} p(v) = p(z_0) + \sum_{v \in V(G)} p(v) = 2e_{M_0}(X) + 2e_{M_1}(Y) + \sum_{v \in X} (f(v) - t(v)) + \sum_{v \in Y} (d_L(v) + d_T(v) - (f(v) - t(v))),
\]

and so
\[
\sum_{v \in V(L)} p(v) = 2e_{M_1}(Y) - 2e_{M_1}(X) + e_T(Y) - e_T(X) + \sum_{v \in X} (f(v) - t(v)) - \sum_{v \in Y} (f(v) - t(v)) + |E(L)| + |E(T)|.
\]

This implies that \( \sum_{v \in V(L)} p(v) \geq |E(L)| + |E(T)| \). Obviously, \( L \) is essentially \((3k-3)\)-edge-connected, \( d_L(v) = d_G(v) \geq 3k-3 \geq 2k-1+|p(v)|k \) for all \( v \in V(L) \setminus \{z_0\} \), and \( d_L(z_0) + d_T(z_0) + 1 \leq 2|E(M)| + 1 = 2k - 1 \). Thus by Theorem 2.1, the orientation of the edges of \( L \) incident with \( z_0 \) can be extended to a \( p \)-orientation of \( L \cup T \) such that for each vertex \( v \),
\[
\left| \frac{1}{2} (d_L(v) + d_T(v)) \right| - (k-1) \leq d_T^-(v) + d_T^+(v) \leq \left| \frac{1}{2} (d_L(v) + d_T(v)) \right| + (k-1).
\]

In addition, we can have \( \left| \frac{1}{2} (d_L(z) + d_T(z) + 1) \right| - (k-1) \leq d_T^-(z) + d_T^+(z) \leq \left| \frac{1}{2} (d_L(z) + d_T(z) - 1) \right| + (k-1) \).
Define \( F \) to be the factor of \( G_0 \) consisting of all edges of \( L - z_0 \) directed from \( X \) to \( Y \). Let \( v_0, \ldots, v_n \) be an arbitrary trail in \( Q \) such that the edge \( v_0v_n \) directed from \( v_0 \) to \( v_n \) in \( T \). If \( v_0 \in X \), we select all edges \( v_2v_3, \ldots, v_{2i+1}v_{2i+2} \) of this trail, and if \( v_0 \in Y \), we select all edges \( v_2v_3, \ldots, v_{2i+1}v_{2i+2} \). Let \( T' \) be the factor of \( T \) consists of all selected edges. Since \( Q \) is an \( X \)-parity trail-decomposition, we must have \( d_{T'}(v) = d_{T'}^+(v) + t(v) \) for each \( v \in X \), and \( d_{T'}(v) = d_{T'}^-(v) + t(v) \) for each \( v \in Y \). Define \( H = M_1 \cup F \cup T' \). According to the construction of \( H \), for each \( v \in V(H) \), we have

\[
d_H(v) = d_{M_1}(v) + d_F(v) + d_{T'}(v) = \begin{cases} 
d_L^+(v) + d_T^+(v) + t(v), & \text{if } v \in Y; \\
d_L^-(v) + d_T^-(v) + t(v), & \text{if } v \in X.
\end{cases}
\]

Therefore, \([d_G(v)/2] - (k - 1) \leq d_H(v) \leq [d_G(v)/2] + (k - 1)\), and also \([d_G(z)/2] - (k - 1) \leq d_H(z) \leq [d_G(z)/2] + (k - 1)\). Hence it is not hard to check that \( H \) is the desired \( f \)-factor we are looking for. \( \square \)

5 Factors modulo \( k \): General graphs

Our aim in this section is to generalize Corollary 4.2 to general graphs using the same degree bounds and characterize the exceptional graphs with high enough edge-connectivity. We begin with a similar version by increasing the upper bound \([d_G(v)/2] + k - 1\) to \([d_G(v)/2] + k\).

5.1 General graphs

The following theorem improves the condition of edge-connectivity of Theorem 1.3.

**Theorem 5.1.** Let \( G \) be a graph, let \( k \) be a positive integer, and let \( f : V(G) \rightarrow \mathbb{Z}_k \) be a compatible mapping. If \( G \) contains a \((3k - 3)\)-edge-connected bipartite factor, then \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \),

\[
\left[ \frac{d_G(v)}{2} \right] - (k - 1) \leq d_H(v) \leq \left[ \frac{d_G(v)}{2} \right] + k.
\]

**Proof.** The special case \( k = 1 \) can be proved by Corollary 3.3. More precisely, when \( G \) is not Eulerian, we need to add an artificial vertex \( z \) and join it to all vertices with odd degrees. Moreover, Theorem 3.2 confirms the case \( k = 2 \). So, suppose \( k \geq 3 \). Let \( X, Y \) be a bipartition of \( V(G) \) such that \( G[X, Y] \) is \((3k - 3)\)-edge-connected. Define \( G_0 \) to be a factor of \( G \) containing all edges of \( G[X, Y] \) such that \( e_{G_0}(X) + e_{G_0}(Y) = \min\{k - 1, e_G(X) + e_G(Y)\} \). If \( G_0 = G \), then the assertion follows from Corollary 4.11. Thus we may assume that \( e_{G_0}(X) + e_{G_0}(Y) \geq k - 1 \). Let \( T \) be a factor of \( G \) obtained by selecting edge-disjoint trails of length two from \( G \setminus E(G_0) \) as long as possible. Since both ends of each selected trails lie either in \( X \) or in \( Y \), the graph \( T \) admits an \( X \)-parity trail-decomposition. Let \( M = G \setminus (E(G_0) \cup E(T)) \). According to the construction of \( T \), the graph \( M \) must be a matching. For each vertex \( v \), define \( f'(v) = f(v) - d_{M_1}(v) \) (mod \( k \)). Since \( f \) is compatible with \( G \), \( \sum_{v \in V(G)} f'(v) \) must be even when \( k \) is even. Therefore, \( f' \) must be compatible with \( G_0 \cup T \) by applying Theorem 2.6 (iii). Thus by Theorem 4.12, the graph \( G_0 \cup T \) has an \( f' \)-factor \( F \) such
that for each vertex \( v \), \( [(d_G(v) - d_M(v))/2] - (k - 1) \leq d_H(v) \leq [(d_G(v) - d_M(v))/2] + (k - 1) \). Hence it is not hard to check that \( F \cup M \) is the desired \( f \)-factor we are looking for.

**Corollary 5.2.** Let \( G \) be a graph, let \( k \) be a positive integer, and let \( f : V(G) \to \mathbb{Z}_k \) be a mapping. Assume that \( (k - 1) \sum_{v \in V(G)} f(v) \) is even and \( b_i(G) \geq k_0 - 1 \), where \( k_0 = k \) when \( k \) is odd, and \( k_0 = k/2 \) when \( k \) is even. If \( G \) is \( (6k - 7) \)-edge-connected, then it has an \( f \)-factor \( H \) such that for each vertex \( v \),

\[
\left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + k.
\]

**Proof.** By Theorem 2.8, the graph \( G \) has a bipartite \((3k-3)\)-edge-connected factor. Also, by Theorem 2.6, the mapping \( f \) is compatible with \( G \). Thus the assertion follows from Theorem 5.1.

The following corollary is a supplement of Corollary 4.4 for the special case \( k = 2 \).

**Corollary 5.3.** Every non-bipartite 18-edge-connected Eulerian graph \( G \) of even size admits a factor \( H \) such that for each vertex \( v \),

\[
d_H(v) \in \{d_G(v)/2 - 2, d_G(v)/2 + 2\}.
\]

**Proof.** For each \( v \), define \( f(v) = d_G(v)/2 + 2 \) (mod 4). Since \( G \) is non-bipartite and \( |E(G)| \) is even, \( bi(G) \geq 1 = 4/2 - 1 \) and \( \sum_{v \in V(G)} f(v) \equiv |E(G)| \equiv 0 \) (mod 4). This implies that \( f \) is compatible with \( G \) (modulo 4) using Theorem 2.6. Thus by applying Corollary 5.2 with \( k = 4 \), the graph \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \), \( d_G(v)/2 - 3 \leq d_H(v) \leq d_G(v)/2 + 4 \). Since \( d_H(v) \equiv d_G(v)/2 + 2 \), we must have \( d_H(v) \in \{d_G(v)/2 - 2, d_G(v)/2 + 2\} \). Hence the proof is completed.

The following result is an interesting application of Theorem 5.1. This result will be refined for 6\( k \)-tree-connected graphs by replacing Theorem 5.6 in the proof.

**Corollary 5.4.** Let \( G \) be a graph, let \( k \) be a positive integer, and let \( f \) be a positive integer-valued function on \( V(G) \) satisfying \( f(v) \leq \frac{1}{2} d_G(v) < f(v) + k \) for each vertex \( v \). Assume that \( f \) is compatible with \( G \) (modulo \( k \)). If \( G \) is \((6k-7)\)-edge-connected, then \( G \) has a factor \( H \) such that for each vertex \( v \),

\[
d_H(v) \in \{f(v), f(v) + k\}.
\]

**Proof.** By Theorem 2.8, the graph \( G \) has a bipartite \((3k-3)\)-edge-connected factor. Thus by Theorem 5.1, the graph \( G \) has an \( f \)-factor \( H \) (mod \( k \)) such that for each vertex \( v \), \( f(v) - (k - 1) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor - (k - 1) \leq d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + k \leq f(v) + 2k - 1 \). Hence \( H \) is the desired factor.

We can also formulate the following theorem similar to Theorem 5.1.

**Theorem 5.5.** Let \( G \) be a graph of order at least two, let \( k \) be an integer, \( k \geq 3 \), and let \( f : V(G) \to \mathbb{Z}_k \) be a compatible mapping. If \( G \) contains a \((2k-2)\)-tree-connected bipartite factor, then it has an \( f \)-factor \( H \) such that for each vertex \( v \), \( k/2 - 1 \leq d_H(v) \leq d_G(v) - k/2 + 1 \).
Proof. Let $X, Y$ be a bipartition of $V(G)$ such that $G[X, Y]$ is $(2k-2)$-tree-connected. Let $G_0 = G[X, Y]$. Since $f$ is compatible, there are two nonnegative integers $x$ and $y$ with $x \leq e_G(X)$ and $y \leq e_G(Y)$ and $\sum_{v \in X} f(v) - 2x + k \leq \sum_{v \in Y} f(v) - 2y$. Let $M$ be a factor of $G[X] \cup G[Y]$ such that $e_M(X) = x$ and $e_M(Y) = y$. For each vertex $v$, define $f'(v) = f(v) - d_M(v)$. Obviously, $\sum_{v \in X} f'(v) = \sum_{v \in Y} f'(v) + k$. Thus by Theorem 4.9, the graph $G_0$ has an $f'$-factor $F$ such that for each vertex $v$, $k/2 - 1 \leq d_F(v) \leq d_{G_0}(v) - k/2 + 1$. Hence it is not hard to check that $F \cup M$ is the desired $f$-factor we are looking for.

\[ \blacksquare \]

### 5.2 Improving degree bounds: highly edge-connected graphs

The following result improves the upper bound stated in Theorem 5.1 for highly edge-connected graphs.

**Theorem 5.6.** Let $G$ be a graph with $z \in V(G)$, let $k$ be a positive integer, and let $f : V(G) \rightarrow \mathbb{Z}_k$ be a compatible mapping. If $G$ is $(6k-2)$-tree-connected, then $G$ has an $f$-factor $H$ such that for each vertex $v$,

$$\frac{d_G(v)}{2} - (k - 1) \leq d_H(v) \leq \begin{cases} \lfloor \frac{d_G(v)}{2} \rfloor + k, & \text{when } v = z; \\ \lceil \frac{d_G(v)}{2} \rceil + (k - 1), & \text{otherwise.} \end{cases}$$

Furthermore, for the vertex $z$, the upper bound can be reduced to $\lfloor d_G(z)/2 \rfloor + (k - 1)$ if and only if one of the following conditions hold:

- $k$ is even.
- $G$ has a vertex of odd degree.
- $G$ is an Eulerian graph of even size.
- There is a vertex $v$ for which $f(v) \equiv d_G(v)/2.$

**Proof.** The special case $k = 1$ can be proved by Corollary 3.3; see [15, Problem 41, Page 61]. More precisely, when $G$ is not Eulerian, we need to add an artificial vertex $z'$ and join it to all vertices with odd degrees, and this new vertex should play the role of the vertex $z$ in Corollary 3.3. Moreover, Theorem 3.2 confirms the case $k = 2$. So, suppose $k \geq 3$. If $bi(G) \leq k - 1$, then the statement follows from Corollary 4.11 and Theorem 2.8. We may assume that $bi(G) \geq k$. By Corollary 2.12, the graph $G$ can be decomposed into two factors $G_1$ and $G_2$ such that $G_1$ is Eulerian and $G_2$ is $(3k-3)$-edge-connected, and $e_{G_1}(X) + e_{G_2}(Y) = k - 1$ for a bipartition $X, Y$ of $V(G)$. According to Corollary 3.3, the graph $G_1$ has a factor $F_1$ such that for each $v \in V(G) \setminus \{z\}$, $d_{F_1}(v) = d_{G_1}(v)/2$, and $d_{F_1}(z) \in \{d_{G_1}(z)/2, d_{G_1}(z)/2 + 1\}$. For each vertex $v$, define $f'(v) = f(v) - d_{F_1}(v) \pmod{k}$. Since $f$ is compatible with $G$, $(k - 1)\sum_{v \in V(G)} f(v)$ is even and hence $(k - 1)\sum_{v \in V(G)} f'(v)$ is even. Thus by Theorem 2.6, $f'$ must be compatible with $G_2$. According to Corollary 4.11, the graph $G_2$ has an $f'$-factor $F_2$ such that for each vertex $v$, $\lfloor d_{G_2}(v)/2 \rfloor - (k - 1) \leq d_{F_2}(v) \leq \lfloor d_{G_2}(v)/2 \rfloor + (k - 1)$, and also $d_{F_2}(z) \leq \lfloor d_{G_2}(z)/2 \rfloor + (k - 1)$. Hence $H = F_1 \cup F_2$ is the desired factor we are looking for.
Assume that the upper bound cannot be reduced to \( \lceil d_G(z)/2 \rceil + (k - 1) \). If \( d_G(z) \) is odd, then obviously the upper bound is the same number \( \lceil d_G(z)/2 \rceil + (k - 1) \). Thus we may assume that all vertices of \( G \) have even degree (otherwise, we can select a vertex of odd degree for playing the role of \( z \)). If \( f(z) \neq d_G(z)/2 \), then obviously the upper bound can also be reduced to \( \lceil d_G(z)/2 \rceil + (k - 1) \). Thus we may assume that \( f(v) \equiv d_G(v)/2 \) for all vertices \( v \). On the other hand, by applying Corollary 3.3, the graph \( G \) has a factor \( H \) such that for each \( v \in V(G) \setminus \{z\} \), \( d_H(v) = d_G(v)/2 \), and \( d_H(z) \in \{d_G(z)/2, d_G(z)/2 + 1\} \). If \( |E(G)| \) is even, then \( \sum_{v \in V(G)} d_G(v)/2 \) is even which implies that \( d_H(z) = d_G(z)/2 \). Thus \( |E(G)| \) must be odd. If \( k \) is even, then \( \sum_{v \in V(G)} f(v) \) is even and hence \( |E(G)| \) is even, which is a contradiction. Therefore, \( k \) must be odd.

Conversely, assume that \( G \) is an Eulerian graph of odd size and \( G \) has an \( f \)-factor \( H \) such that for each vertex \( v \), \( f(v) \equiv d_G(v)/2 \) and \( \lceil d_G(v)/2 \rceil - (k - 1) \leq d_H(v) \leq \lceil d_G(v)/2 \rceil + (k - 1) \). This implies that \( d_H(v) = d_G(v)/2 \) for each vertex \( v \). Since \( \sum_{v \in V(G)} d_H(v) \) is even, \( |E(G)| \) must be even, which is a contradiction. Hence the proof is completed. \( \square \)

**Remark 5.7.** Note that the needed tree-connectivity of Theorem 5.6 can be reduced by one and two for odd and even integers \( k \) using a little extra effort.

The following corollary is a refined version of Corollary 5.4 for highly tree-connected graphs.

**Corollary 5.8.** Let \( G \) be a graph, let \( k \) be a positive integer, and let \( f \) be a positive integer-valued function on \( V(G) \) satisfying \( f(v) \leq \frac{1}{2} d_G(v) \leq f(v) + k \) for each vertex \( v \). Assume that \( f \) is compatible with \( G \) (modulo \( k \)). If \( G \) is \((6k - 2)\)-tree-connected, then \( G \) has a factor \( H \) such that for each vertex \( v \),

\[ d_H(v) \in \{f(v), f(v) + k\}. \]

**Proof.** Let \( z \in V(G) \). By Theorem 5.6, the graph \( G \) has an \( f \)-factor \( H \) (mod \( k \)) such that for each \( v \in V(G) \setminus \{z\} \), \( f(v) - (k - 1) \leq \lfloor d_G(v)/2 \rfloor - (k - 1) \leq d_H(v) \leq \lceil d_G(v)/2 \rceil + (k - 1) \leq f(v) + 2k - 1 \) which implies that \( d_H(v) \in \{f(v), f(v) + k\} \). In addition, we can have \( f(z) - (k - 1) \leq \lfloor d_G(z)/2 \rfloor - (k - 1) \leq d_H(z) \leq \lceil d_G(z)/2 \rceil + k \). If the last upper bound can be improved by one or \( d_G(z)/2 \leq f(z) + k - 1 \), then we must also have \( d_H(z) \in \{f(z), f(z) + k\} \). Otherwise, \( G \) must be Eulerian and \( d_G(v)/2 = f(v) + k \) for all vertices \( v \). In this case, we first apply Theorem 5.6 to find a factor \( H_0 \) such that for each vertex \( v \),

\[ d_{H_0}(v) \in \{d_G(v) - f(v) - k, d_G(v) - f(v)\}. \]

Next, we set \( H = G \setminus E(H_0) \). Note that the function \( d_G(v) - f(v) \) is compatible with \( G \) as well. More precisely, if there are two integers \( x \) and \( y \) satisfying \( 0 \leq x \leq e_G(X), 0 \leq y \leq e_G(Y) \), and \( \sum_{v \in X} f(v) - 2x \equiv \sum_{v \in Y} f(v) - 2y \), then \( \sum_{v \in X} (d_G(v) - f(v)) - 2(e_G(X) - x) \equiv \sum_{v \in Y} (d_G(v) - f(v)) - 2(e_G(Y) - y) \), where \( X, Y \) is an arbitrary bipartition of \( V(G) \). Hence the proof is completed. \( \square \)
6 Modulo $k$-regular factors and subgraphs

6.1 Bipartite modulo $k$-regular factors

The following well-known theorem gives a sufficient condition for the existence of even factors. In this subsection, we develop this result for the existence of bipartite modulo $k$-regular factors.

**Theorem 6.1.** (Lovász [15, Problem 42, Page 61]) Every 2-edge-connected loopless graph $G$ with $\delta(G) \geq 3$ admits a modulo 2-regular factor.

We begin with the following corollary which provides a bipartite version for Theorem 6.1. Note that this result is sharp by considering that there exits a class of 4-edge-connected graphs without bipartite modulo 2-regular factors. (For example, consider a number of copies of the complete graph of order four and add a new vertex and join it to all other vertices).

**Corollary 6.2.** Every 3-edge-connected loopless graph $G$ with $\delta(G) \geq 5$ admits a bipartite modulo 2-regular factor.

**Proof.** By Theorem 2.9, the graph $G$ has a bipartite factor $H$ such that for every vertex set $X$, $d_H(X) \geq \lceil d_G(X)/2 \rceil$. Since $G$ is 3-edge-connected and $\delta(G) \geq 5$, the graph $H$ must 2-edge-connected and $\delta(H) \geq 3$. Hence by Theorem 6.1 the graph $H$ admits a modulo 2-regular factor. □

The following theorem gives sufficient edge-connectivity conditions for the existence of bipartite modulo $k$-regular factors.

**Theorem 6.3.** Every $(4k - 1)$-edge-connected essentially $(6k - 7)$-edge-connected graph with $k \geq 3$ admits a bipartite modulo $k$-regular factor. In addition, this result is true for bipartite $2k$-edge-connected essentially $(3k - 3)$-edge-connected graphs.

**Proof.** By Theorem 2.9, the graph $G$ has a bipartite factor $G_0$ such that for every vertex set $X$, $d_{G_0}(X) \geq \lceil d_G(X)/2 \rceil$. Since $G$ is $(4k - 1)$-edge-connected and essentially $(6k - 7)$-edge-connected, the graph $G_0$ must be $2k$-edge-connected and essentially $(3k - 3)$-edge-connected. For each vertex $v$, define $f(v) = 0$. Obviously, $f$ is compatible with $G$. Thus by Theorem 4.1, the graph $G_0$ has an $f$-factor $F$ modulo $k$ such that for each vertex $v$, $d_F(v) \geq \lceil d_{G_0}(v)/2 \rceil - (k - 1) > 0$. Thus $F$ is a bipartite modulo $k$-regular factor of $G$. □

The following theorem provides a tree-connected version for Theorem 6.3.

**Theorem 6.4.** Every $(4k - 4)$-tree-connected graph with $k \geq 3$ admits a bipartite modulo $k$-regular factor. In addition, this result is true for bipartite $(2k - 2)$-tree-connected graphs.
Proof. If $G$ is $(4k-4)$-tree-connected, then by Theorem 2.9, the graph $G$ contains a $(2k-2)$-tree-connected bipartite factor $G_0$. By Theorem 4.9, the bipartite graph $G_0$ contains a modulo $k$-regular factor, which can complete the proof. □

Finally, we formulate the following improved version of Theorem 3 in [22].

**Theorem 6.5.** Every $(10k-3)$-edge-connected essentially $(12k-7)$-edge-connected graph of even order admits a modulo $k$-regular factor whose degrees are not divisible by $2k$. In addition, this result is true for bipartite $(5k-1)$-edge-connected essentially $(6k-3)$-edge-connected graphs of even order.

Proof. By Theorem 2.9, the graph $G$ has a bipartite factor $G_0$ such that for every vertex set $X$, $d_{G_0}(X) \geq \lceil d_G(X)/2 \rceil$. Since $G$ is $(10k-3)$-edge-connected and essentially $(12k-7)$-edge-connected, the graph $G_0$ must be $(5k-1)$-edge-connected and essentially $(6k-3)$-edge-connected. For each vertex $v$, define $f(v) = k \pmod{2k}$. Since $G_0$ has even order, $f$ must be compatible with $G_0$. Thus by Theorem 4.1, the graph $G_0$ has an $f$-factor $F$ modulo $2k$. Note that for each vertex $v$, we have $d_{G_0}(v) \geq 5k-1 = 2(2k-1) + \lceil f(v) \rceil$. Thus $F$ is the desired factor of $G$. □

### 6.2 Bipartite modulo $k$-regular subgraphs

The following theorem completely confirms Conjecture 1.6 for prime powers. We shall below replace another condition for the existence of bipartite modulo $k$-regular subgraphs.

**Theorem 6.6.** ([1]) Let $G$ be a loopless graph of order $n$ and let $q$ be a prime power. Then $G$ admits a modulo $q$-regular subgraph if

\[ |E(G)| > (q-1)n. \]

In addition, the lower bound can be improved to $(q-1)(n-1)$ when $G$ is bipartite.

**Lemma 6.7.** ([12]) Let $k$ and $q$ be two positive integers with $k \leq q$. Let $G$ be a bipartite graph and let $f$ be a nonnegative integer-valued function on $V(G)$. If $G$ has a factor $H$ satisfying $d_H(v) = qf(v)$ for each vertex $v$, then $G$ has a factor $F$ satisfying $d_F(v) = kf(v)$ for each vertex $v$.

Proof. We split every vertex $v$ of $H$ into $f(v)$ vertices such that the resulting graph $H_0$ would be a $q$-regular bipartite graph. Thus $H_0$ has a $k$-regular factor $F_0$ using König’s Theorem [13]. Obviously, this factor $F_0$ induces a factor $F$ of $H$ satisfying $d_F(v) = kf(v)$ for each vertex $v$. Hence the proof is completed. □

The following theorem is an improved version of Lemma 3 in [6]. By a computer search, we observed that for small positive integers $k$ (at most nine digits), there are some prime powers less than $(1 + 1/10)k + 1$. 22
On the other hand, Baker, Harman, and Pintz (2001) [3] proved that for every sufficiently large integer \( x \), there is a prime number \( p \in [x - x^{0.525}, x] \). This shows that the following lower bound can be replaced by \((2 + \varepsilon)(k - 1)(n - 1)\) for sufficiently large \( k \).

**Theorem 6.8.** A loopless graph \( G \) of order \( n \) has a bipartite modulo \( k \)-regular subgraph if \( \chi(G) \leq 2t \) and

\[
|E(G)| > (2 - \frac{1}{t})(q(k) - 1)(n - 1),
\]

where \( q(k) \) denotes the smallest prime power with \( q(k) \geq k \). Consequently, if \( |E(G)| > (2q(k) - \frac{5}{2})(n - 1) \) or \( |E(G)| > (4k - 6)(n - 1) \), then \( G \) has a bipartite modulo \( k \)-regular subgraph.

**Proof.** We may assume that \( k \geq 2 \). We first show that the graph \( G \) has a bipartite factor \( H \) such that

\[
|E(H)| \geq \frac{4}{2t - 1}|E(G)|,
\]

see [2]. Let \( X_1, \ldots, X_{2t} \) be a partition of \( V(G) \) such that every \( G[X_i] \) has no edge. Let \( S \) be a subset of \( 1, \ldots, 2t \) with size \( t \), and let \( H_S \) be the bipartite factor of \( G \) with one partite set \( \cup_{i \in S} X_i \). The number of such factors is obviously \( \binom{2t}{t} \). On the other hand, every edge is contained in exactly \( 2\binom{2t - 2}{t - 2} \) such factors. Among all such factors, we consider \( H \) with the maximum size. Thus

\[
\binom{2t}{t}|E(H)| \geq \frac{4}{2t - 1}|E(G)| > (q(k) - 1)(n - 1),
\]

which implies that \( |E(H)| \geq \frac{4}{2t - 1}|E(G)| > (q(k) - 1)(n - 1) \). Thus by Theorem 6.6, the graph \( H \) has a modulo \( q(k) \)-regular subgraph \( F \). According to Lemma 6.7, the bipartite graph \( F \) has a modulo \( k \)-regular factor and so the graph \( G \) has a bipartite modulo \( k \)-regular subgraph.

Let us prove the first conclusion. Suppose, to the contrary, \( G \) has no bipartite modulo \( k \)-regular subgraph and \( |E(G)| > (2q(k) - 5/2)(n - 1) \). Thus every subgraph \( G' \) of \( G \) contains at most \( 2(q(k) - 1)|V(G')| - 1 \) edges, and so it has minimum degree at most \( 4q(k) - 5 \). Therefore, one can conclude that the graph \( G \) its chromatic number is at most \( 4q(k) - 4 \) (using a greedy algorithm with the fact that \( G \) is \((4q(k) - 5)\)-degenerate). Thus the graph \( G \) contains at most \( (2 - \frac{1}{2q(k) - 2})(q(k) - 1)(n - 1) \) edges, which is a contradiction. For proving the second conclusion, it is enough to check that \( q(k) \leq 2^{i+1} \leq 2k - 2 \) when \( k = 2^i + r \) and \( 0 < r < 2^i \). Hence the proof is completed.

When \( k = 3 \) and \( t = 2 \), Theorem 6.8 becomes simpler as the following version. Note that the graph \( G \) obtained from the Cartesian product of two cycles of order three by joining its vertices to a new vertex is a 4-chromatic simple graph of size \( 3(|V(G)| - 1) \) having no bipartite modulo 3-regular subgraph.

**Corollary 6.9.** A loopless graph \( G \) of order \( n \) has a bipartite modulo 3-regular subgraph if \( |E(G)| > (3 + \frac{1}{2})(n - 1), \) or \( \chi(G) \leq 4 \) and \( |E(G)| > 3(n - 1) \).

Finally, we propose the following conjecture to introduce a sharp version of Corollary 6.9. Note also the graph \( G \) obtained from the complete graph of order five by inserting a new copy of a triangle is a graph of size \( (3 + \frac{1}{2})(|V(G)| - 1) \) having no bipartite modulo 3-regular subgraph.

**Conjecture 6.10.** A loopless graph \( G \) of order \( n \) has a bipartite modulo 3-regular subgraph if \( |E(G)| > (3 + \frac{1}{2})(n - 1), \) or \( G \) is simple and \( |E(G)| > 3(n - 1) \).
References

[1] N. Alon, S. Friedland, and G. Kalai, Regular subgraphs of almost regular graphs, J. Combin. Theory Ser. B 37 (1984) 79–91.

[2] L.D. Andersen, D.D. Grant, and N. Linial, Extremal $k$-colorable subgraphs, Ars Combinatoria 16 (1983) 259–270.

[3] R.C. Baker, G. Harman, and J. Pintz, The difference between consecutive primes. II, Proc. London Math. Soc. 83 (2001) 532–562.

[4] J. Bensmail, M. Merker, and C. Thomassen, Decomposing graphs into a constant number of locally irregular subgraphs, European J. Combin. 60 (2017) 124–134.

[5] J.A. Bondy and U.S.R. Murty, Graph theory, Springer, London, 2008.

[6] F. Botler, L. Colucci, and Y. Kohayakawa, The mod $k$ chromatic index of graphs is $O(k)$, J. Graph Theory 102 (2023) 197–200.

[7] C. Bujtás, S. Jendrol’, and Z. Tuza, On specific factors in graphs, Graphs Combin. 36 (2020) 1391–1399.

[8] J. Edmonds and E.L. Johnson, Matching, Euler tours and the Chinese postman, Mathematical Programming 5 (1973) 88–124.

[9] M. Hasanvand, Modulo orientations with bounded out-degrees, Discrete Math. 347 (2024) 113634. arXiv:1702.07039.

[10] M. Hasanvand, Equitable factorizations of edge-connected graphs, Discrete Appl. Math. 317 (2022) 136–145.

[11] M. Hasanvand, Bipartite partition-connected factors with small degrees, arXiv:1905.12161.

[12] A.J. Hoffman, Generalization of a theorem of König, J. Washington Acad. Sci. 46 (1956) 211–212.

[13] D. König, Über graphen und ihre anwendung auf determinantentheorie und mengenlehre, Mathematische Annalen, 77 (1916) 453–465.

[14] L. Lovász, The factorization of graphs II, Acta Math. Acad. Sci. Hungar. 23 (1972) 223–246.

[15] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam (1979).

[16] L.M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang, Nowhere-zero 3-flows and modulo $k$-orientations, J. Combin. Theory Ser. B 103 (2013) 587–598.

[17] M. Merker, Decomposing highly edge-connected graphs into homomorphic copies of a fixed tree, J. Combin. Theory Ser. B 122 (2017) 91–108.
[18] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.

[19] H. Shirazi and J. Verstraëte, A note on polynomials and $f$-factors of graphs, Electron. J. Combin. 15 (2008) Note 22, 5.

[20] C. Thomassen, The Erdős-Pósa property for odd cycles in graphs of large connectivity, Combinatorica 21 (2001) 321–333.

[21] C. Thomassen, Edge-decompositions of highly connected graphs into paths, Abh. Math. Semin. Univ. Hambg. 78 (2008) 17–26.

[22] C. Thomassen, Graph factors modulo $k$, J. Combin. Theory Ser. B 106 (2014) 174–177.

[23] C. Thomassen, Y. Wu, and C.-Q. Zhang, The 3-flow conjecture, factors modulo $k$, and the 1-2-3-conjecture, J. Combin. Theory Ser. B 121 (2016) 308–325.

[24] W.T. Tutte, On the problem of decomposing a graph into $n$ connected factors, J. London Math. Soc. 36 (1961) 221–230.