SYMBOLIC BLOWUP ALGEBRAS AND INVARIANTS OF CERTAIN MONOMIAL CURVES IN AN AFFINE SPACE

CLARE D’CRUZ AND SHREEDEVI K. MASUTI

Abstract. Let $d \geq 2$ and $m \geq 1$ be integers such that $\gcd(d,m) = 1$. Let $p$ be the defining ideal of the monomial curve in $\mathbb{A}^d_\mathbb{k}$ parametrized by $(t^{n_1}, \ldots, t^{n_d})$ where $n_i = d + (i - 1)m$ for all $i = 1, \ldots, d$. In this paper, we describe the symbolic powers $p(n)$ for all $n \geq 1$. As a consequence we show that the symbolic blowup algebras $R_s(p)$ and $G_s(p)$ are Cohen-Macaulay. This gives a positive answer to a question posed by S. Goto in [15]. We also discuss when these blowup algebras are Gorenstein. Moreover, for $d = 3$, considering $p$ as a weighted homogeneous ideal, we compute the resurgence, the Waldschmidt constant and the Castelnuovo-Mumford regularity of $p(n)$ for all $n \geq 1$. The techniques of this paper for computing $p(n)$ are new and we hope that these will be useful to study the symbolic powers of other prime ideals.

1. Introduction

Let $I$ be an ideal in a Noetherian ring $A$. Then for all $n \geq 1$, the $n$-th symbolic power of $I$ is the ideal $I^{(n)} := \bigcap_{p \in \operatorname{MinAss}(A/I)} (IA_p \cap A)$. In this paper we are interested in the symbolic powers of certain prime ideals in the polynomial ring and power series ring. In particular, let $T := \mathbb{k}[x_1, \ldots, x_d]$ and $p := p_{(n_1, \ldots, n_d)} \subseteq T$ be the defining ideal of the monomial curve in $\mathbb{A}^d_\mathbb{k}$ parametrized by $(t^{n_1}, t^{n_2}, \ldots, t^{n_d})$, where $t \in \mathbb{k}$ and $n_i = d + (i - 1)m$ for all $i = 1, \ldots, d$. If $R := \mathbb{k}[[x_1, \ldots, x_d]]$ denotes the $\mathfrak{m}$-adic completion of $T$, where $\mathfrak{m} = (x_1, \ldots, x_d)$, then $p^{(n)}R = p^{(n)}T \otimes_T R$. The first part of our paper concerns the Cohen-Macaulay and Gorenstein property of the symbolic Rees algebra $R_s(p) := \oplus_{n \geq 0} (p(n)R)t^n$ and symbolic associated graded ring $G_s(p) := \oplus_{n \geq 0} p(n)R/p(n+1)R$ when $R_s(p)$ is Noetherian. The second part of our paper concerns the computation of resurgence and Waldschmidt constant of $pT$. We also give a formula for the Castelnuovo-Mumford regularity of $p^{(n)}T$ for all $n \geq 1$.

The $n$-th symbolic power $p^{(n)}$ is of interest for several reasons. It is related to an open question which goes back to the work of L. Kronecker [26], where he showed that every irreducible curve in $\mathbb{A}^d_\mathbb{k}$ can be defined by $(d + 1)$-equations. In 1981, R. Cowsik gave a striking relationship between the Noetherianness of the symbolic Rees algebra and the problem of set-theoretic complete intersection. He showed that if $p$ is a prime ideal in a regular local ring $R$ such that $\dim(R/p) = 1$ and $R_s(p)$ is Noetherian, then $p$ is a set-theoretic complete intersection [6]. Motivated by Cowsik’s result, in 1987, C. Huneke gave necessary and sufficient conditions for $R_s(p)$ to be Noetherian when $\dim \ R = 3$ [24]. Huneke’s result was generalised for $\dim \ R \geq 3$ by M. Morales [29]. In general, the symbolic Rees algebra need not be Noetherian even for an affine monomial curve in $\mathbb{A}^3_\mathbb{k}$ and depends on the characteristic of $\mathbb{k}$ [18]. This unpredictable behaviour

2010 Mathematics Subject Classification. Primary: 13A30, 13D05, 13H15, 13P10.

Key words and phrases. Symbolic Rees algebra, Cohen-Macaulay, Gorenstein.

Both authors are partially funded by a grant from Infosys Foundation.

SKM is supported by INSPIRE faculty award funded by Department of Science and Technology, Govt. of India.

1
attracted the attention of several researchers and properties of the Noetherian symbolic Rees algebra was studied in several cases (for example see [12], [23], [16], [17], [18], [20], [27], [30], [25], [31] and [35]).

The main difficulty in the study of the symbolic Rees algebra is describing the generators of the symbolic powers. The symbolic powers \( p^{(2)} \) and \( p^{(3)} \) for monomial curves in \( \mathbb{A}^3 \) has been studied extensively ([12], [23], [25], [31] and [35]). In fact, using the ideas in [34] and [35], J. Herzog and B. Ulrich gave a characterization for \( R_s(p) = R[p, p^{(2)} t^2] \) [20, Corollary 2.12]. However, for \( d \geq 4 \), there are very few results on \( p^{(n)} \), \( n \geq 1 \) ([15] and [32]). In 1994, S. Goto gave necessary and sufficient conditions for the Cohen-Macaulayness and Gorensteinness of the symbolic blowup algebras when \( R_s(p) \) is Noetherian [15]. The Gorenstein property of \( R_s(p) \) for monomial curves has also been studied in [33], [16] and [17].

In the last decade, motivated by the work in [11], [10] and [21], there has been a great interest in the relation between the symbolic powers and ordinary powers of ideals. Since symbolic powers are hard to describe, in order to compare the ordinary and symbolic powers of a homogenous ideal \( I \subset T \), C. Bocci and B. Harbourne defined an asymptotic quantity called the resurgence of \( I \) which is defined as \( \rho(I) = \sup\{m/r : I^{(m)} \not\subset I^r \} \) [3]. They observed that it exists for radical ideals. The resurgence is hard to compute in general and is a challenging problem. Hence, in order to give a bound for \( \rho(I) \), in the same paper they defined another invariant \( \gamma(I) \) called it the Waldschmidt constant. The Waldschmidt constant of \( I \), denoted as \( \gamma(I) \), is defined as \( \gamma(I) = \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n} \), where \( \alpha(I) := \min\{n | I_n \neq 0 \} \). They showed that if \( I \) is a homogenous ideal, then \( \alpha(I)/\gamma(I) \leq \rho(I) \) and in addition if \( I \) defines a zero dimensional subscheme in a projective space, then \( \rho(I) \leq \reg(I)/\gamma(I) \) where \( \reg(I) \) denotes the Castelnuovo-Mumford regularity of \( I \) [3, Theorem 1.2.1]. The resurgence and the Waldschmidt constant has been studied in a few cases: for certain general points in \( \mathbb{P}^2 \) [2], smooth subschemes [19], fat linear subspaces [14], special point configurations [9] and monomial ideals [4].

We now describe the work in this paper. Let \( \gcd(d, m) = 1 \), \( n_i := d + (i - 1)m \) for \( i = 1, \ldots, d \) and \( p := \mathbb{C}(n_1, \ldots, n_d) \subset R = \mathbb{K}[x_1, \ldots, x_d] \). In 1994, S. Goto showed that \( R_s(p) \) is Noetherian for all \( d \geq 2 \) and is Cohen-Macaulay if \( d \leq 4 \) [15, Proposition 7.6]. In the same article he raised the question whether \( R_s(p) \) is Cohen-Macaulay if \( d \geq 5 \) [15, page 58]. In this paper we explicitly describe \( p^{(n)} \) for all \( n \geq 1 \). For this, we first define the ideal \( I_n \subseteq p^{(n)} \) (see 2.14). One important observation is that \( I_n T' \) is a homogenous ideal (Proposition 2.15) where \( T' = T/x_1T \). The new idea of this paper is to give a monomial order on the monomials in \( T' = T/(x_1) \) (Definition 3.1) and compute the leading ideal of \( p^{(n)} T' \) (Theorem 6.8). More precisely, we define monomial ideals \( I_n \subseteq T' \) (3.3) and show that \( I_n = LI(p^{(n)} T') \). As a consequence, we show that \( p^{(n)} R = I_n R \) for all \( n \geq 1 \).

As a first application, we show that \( R_s(p) \) is Cohen-Macaulay for all \( d \geq 2 \) (Theorem 7.2(2)). This gives a positive answer to Goto’s question. We also show that \( G_s(p) \) is Gorenstein for all \( d \geq 2 \) (Theorem 7.1) and \( R_s(p) \) is Gorenstein if and only if \( d = 3 \) (Theorem 7.2(3)).

We elucidate other applications in this paper. Let \( d = 3 \). We put weights \( w(x_i) = n_i \) where \( n_i := 3 + (i - 1)m \) and \( i = 1, 2, 3 \). With these weights, \( p^{(n)} = (p^n)_{sat} \) defines a fat point for all \( n \geq 1 \) in the weighted projective space \( \mathbb{P} := \mathbb{P}^2(n_1, n_2, n_3) := \text{Proj}(T) \). Since \( p \) is a weighted homogenous ideal, we extend the definition of resurgence and Waldschmidt constant to \( p \). Moreover, we observe that Theorem 1.2.1 of [3] holds true for \( p \) (Theorem 7.9 and Theorem 7.19).
In [8, Theorem 1.1] Cutkosky and Kurano showed that \( \lim_{n \to \infty} \frac{\text{reg}((p^n)_{\text{sat}})}{n} \) exists and \( \text{reg}(T/p^n) \) is eventually periodic [8, Corollary 4.9]. We give an explicit formula for \( \text{reg}(T/(p^n)_{\text{sat}}) \) for \( d = 3 \) and for all \( n \geq 1 \). In particular \( \lim_{n \to \infty} \frac{\text{reg}((p^n)_{\text{sat}})}{n} = \frac{3c(T/p)}{2} + 3m \) (Theorem 7.18).

We remark that using the techniques of this paper one can compute the resurgence, Waldschmidt constant and regularity of \( p^{(n)} \) for \( d \geq 4 \). However, this involves tedious computation and hence we restrict ourselves to \( d = 3 \) in this paper.

We now describe the organisation of this paper. In Section 2 we prove some preliminary results which will be needed in the subsequent sections. In Section 3 we describe the monomial order we are using and describe the ideals \( I_n T' \subseteq LI(T_n T') \). Section 4 is mainly devoted to show that the associated graded ring corresponding to the filtration \( \{I_n \}_{n \geq 0} \) is Cohen-Macaulay. In Section 5 we explicitly describe the monomials which span \( I_{n-1} \) modulo \( (I_n : x_d) \). In Section 6 we explicitly describe all the symbolic powers \( p^{(n)} \). The main results of this paper are in Section 7. In this section we study the Cohen-Macaulay and Gorenstein property of \( R_s(p) \) and \( G_s(p) \) for \( d \geq 2 \), and give an explicit formula for the resurgence and Waldschmidt constant. Moreover, we also compute the Castelnuovo-Mumford regularity of the symbolic powers.

Acknowledgements

The authors would like to thank Prof. J. K. Verma for suggesting the problem and for many useful conversations. The second author would like to thank the Department of Atomic Energy, Government of India, and the Chennai Mathematical Institute (CMI) for providing financial support for her post doctoral studies at the Institute of Mathematical Sciences (IMSc) and CMI, respectively, during which part of the work is done. She also thanks INdAM cofunded by Marie Curie actions, Italy, for her research in Genova, during which part of the work is done.

2. Preliminaries

In this paper, we consider the following class of monomial curves: Let \( d \geq 2 \). Let \( R = \mathbb{k}[[x_1, \ldots, x_d]] \) and \( S = \mathbb{k}[[t]] \) be formal power series rings over \( \mathbb{k} \). For any positive integer \( m \geq 1 \), with \( \gcd(d, m) = 1 \), we put \( n_i := d + (i-1)m \) for \( i = 1, \ldots, d \). Let \( C(n_1, \ldots, n_d) \) be the affine curve parameterised by \( (t^{n_1}, \ldots, t^{n_d}) \) and let \( I_C(n_1, \ldots, n_d) \) be the ideal defining this monomial curve. In other words, let \( \phi : R \to \mathbb{k}[[t]] \) denote the homomorphism defined by \( \phi(x_i) = t^{n_i} \) for \( 1 \leq i \leq d \) and \( p := \ker(\phi) = I_C(n_1, \ldots, n_d) \).

Throughout this paper \( p = I_C(n_1, \ldots, n_d) \) unless otherwise specified. It is well known that \( p \) is generated by the \( 2 \times 2 \) minors of the matrix described in (2.1). In [15, Proposition 7.6], Goto described \( p^{(n)} \) for \( d = 4 \) and \( n = 2, 3 \). It is not easy to describe the ideals \( p^{(n)} \) in general. To achieve this, we define ideals \( I_n R \subseteq p^{(n)} \) (see (2.14)). We exploit the fact that the ideals \( (I_n, x_1)T \) are homogeneous ideals (Proposition 2.15).

2.1. Computation of multiplicity. Let \( R = \mathbb{k}[[x_1, \ldots, x_d]] \) and \( X = [X_{ij}] \) be the \( d \times d \) matrix given by

\[
X_{ij} := \begin{cases} 
  x_{i+j-1} & \text{if } 1 \leq i \leq d \text{ and } 1 \leq j \leq d - i + 1 \\
  x_1^{n_i} x_{i+j-d-1} & \text{if } 2 \leq i \leq d \text{ and } d - i + 2 \leq j \leq d.
\end{cases}
\] (2.1)
For each $1 \leq i, k \leq d - 1$, we define:

\[
X(i) := \text{The matrix consisting of the first } i + 1 \text{ rows and } i + 1 \text{ columns of } X, \quad (2.2)
\]
\[
f_i := \det(X(i)) \quad \text{and } f_k := f_1, \ldots, f_k. \quad (2.3)
\]

Goto showed that $f_{d-1}$ satisfies Huneke’s criterion for the Noetherianness of $\mathcal{R}_s(p)$ ([15, Theorem 7.4]).

In this section we give a lower bound for the length of the modules $R/(p^n + (x_1, f_k))$ where $p = I_{c(n_1, \ldots, n_d)}$, $1 \leq k \leq d - 1$ and $n \geq 1$. We need a few preliminary results.

Let $(A, n)$ be a Noetherian local ring of positive dimension $d$ and $\mathfrak{a}$ an $n$-primary ideal. Let $\mathcal{F} = \{\mathcal{F}(n)\}_{n \in \mathbb{Z}}$ be a Noetherian filtration of ideals, i.e., $\mathcal{F}(0) = A$, $\mathcal{F}(1) \neq A$, $\mathcal{F}(n + 1) \subseteq \mathcal{F}(n)$, $\mathcal{F}(n) \cdot \mathcal{F}(m) \subseteq \mathcal{F}(n + m)$ for all $n, m \in \mathbb{Z}$ and the Rees ring $\mathcal{R}(\mathcal{F}) := \oplus_{n \geq 0} \mathcal{F}(n)t^n$ is Noetherian. Let $1 \leq k \leq d$ and $z_i \in \mathcal{F}(a_i) \setminus \mathcal{F}(a_i + 1)$ for all $i = 1, \ldots, k$. Put $z_k = z_{i_1}, \ldots, z_k$. For all $n \in \mathbb{Z}$, using the mapping cone construction, similar to that in [22], we construct the complex $C_\bullet(z_k; n)$ which has the form:

\[
0 \rightarrow \mathcal{F}(n - (a_1 + \ldots + a_k)) \rightarrow \bigoplus_{1 \leq i < j \leq k} \mathcal{F}(n - a_i - a_j) \rightarrow \bigoplus_{i=1}^k \mathcal{F}(n - a_i) \rightarrow A \rightarrow \mathcal{F}(n) \rightarrow 0. \quad (2.4)
\]

The maps are from the Koszul complex $K_\bullet(z_k, A)$. Let $H_i(C_\bullet(z_k, n))$ denote the $i$-th homology of the complex $C_\bullet(z_k; n)$.

For any element $z \in \mathcal{F}(n) \setminus \mathcal{F}(n + 1)$, let $z^*$ denote the image of $z$ in $G(\mathcal{F}) := \oplus_{n \in \mathbb{N}} \mathcal{F}(n)/\mathcal{F}(n + 1)$. Let $z_k^* := z_1^*, \ldots, z_k^*$.

**Proposition 2.5.** Let $\{\mathcal{F}(n)\}_{n \geq 0}$ be a filtration of $m$-primary ideals. For $1 \leq i \leq k$, let $z_i \in \mathcal{F}(a_i) \setminus \mathcal{F}(a_i + 1)$. Suppose $z_k^*$ is a regular sequence in $G(\mathcal{F})$. Then

1. $H_i(C_\bullet(z_k, n)) = 0$ for all $i \geq 1$ and all $n \in \mathbb{Z}$.
2. $\ell \left( \frac{A}{\mathcal{F}(n) + (z_k)} \right) = \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \ldots < j_i \leq k} \ell \left( \frac{A}{\mathcal{F}(n) + (a_j)} \right) \right].$

**Proof.** (1) Let $K_\bullet(z_k^*, G(\mathcal{F}))$ denote the Koszul complex of $G(\mathcal{F})$ with respect to $z_k^*$. Then we have the short exact sequence of complexes:

\[
0 \rightarrow K_\bullet(z_k^*, G(\mathcal{F}))_{n-1} \rightarrow C_\bullet(z_k, n) \rightarrow C_\bullet(z_k, n - 1) \rightarrow 0. \quad (2.6)
\]

Since $z_k^*$ is a regular sequence in $G(\mathcal{F})$, $H_i(K_\bullet(z_k^*, G(\mathcal{F}))) = 0$ for all $i \geq 1$ [28, Theorem 16.5]. Hence from (2.6) for all $n \in \mathbb{Z}$ we have:

$H_i(C_\bullet(z_k, n)) \cong H_i(C_\bullet(z_k, n - 1))$ \quad \text{for all } i \geq 2

and the short exact sequence

\[
0 \rightarrow H_1(C_\bullet(z_k, n)) \rightarrow H_1(C_\bullet(z_k, n - 1)).
\]

As $H_1(C_\bullet(z_k, n)) = 0$ for all $n \leq 0$, we conclude that $H_i(C_\bullet(z_k, n)) = 0$ for all $n$ and for all $i \geq 1$. This proves (1).
(2) As \(H_0(C_\bullet(z_k, n)) = A/(\mathcal{F}(n) + (z_k))\), from the complex (2.4) we get
\[
\ell \left( \frac{A}{\mathcal{F}(n) + (z_k)} \right) + \sum_{i \geq 1} (-1)^i \ell(H_i(C_\bullet(z_k, n))) = \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \ell \left( \frac{A}{\mathcal{F}(n) - [a_{j_1} + \cdots + a_{j_i}]} \right) \right].
\]
Applying (1) we get the result. \(\square\)

**Corollary 2.7.** Let \((A, n)\) be a Cohen-Macaulay local ring of dimension \(d\). Let \(p\) be a prime ideal of height \(d - 1\) and \(x \notin p\). Let \(1 \leq k \leq d - 1\) and \(z_i \in p^{(n)} \setminus p^{(n+1)}\). Suppose \(z_k^d\) is a regular sequence in \(G(pA_p)\). Then
\[
\ell \left( \frac{A}{p^{(n)} + (z_k)} \right) \geq \ell \left( \frac{A}{(p, x)} \right) \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \ell \left( \frac{A_p}{p^{n-[a_{j_1} + \cdots + a_{j_i}]} A_p} \right) \right].
\]

**Proof.** (1) As \(p^{(n)} \subseteq p^{(n)} + (z_k) \subseteq p\), taking radicals we get \(\sqrt{p^{(n)} + (z_k)} = p\). Hence \(p\) is the only minimal prime of \(p^{(n)} + (z_k)\). From the associativity formula for multiplicities [28, Theorem 14.7] we get
\[
e \left( x; \frac{A}{p^{(n)} + (z_k)} \right) = e \left( x; \frac{A}{p} \right) \ell \left( \frac{A_p}{(p^{(n)} + (z_k)) A_p} \right).
\]
As \(x\) is a nonzero divisor on \(A/p\), \(e(x; A/p) = \ell(A/(p, x))\). Replacing \(A\) by \(A_p\) and \(G(F)\) by \(G(pA_p)\) in Proposition 2.5(2) we get the result.

(2) From [28, Theorem 14.10], we get
\[
\ell \left( \frac{A}{p^{(n)} + (z_k) + (x)} \right) \geq e \left( x; \frac{A}{p^{(n)} + (z_k)} \right). \text{ Now apply (1).} \quad \square
\]

**Theorem 2.8.** Let \(R = \mathbb{k}[[x_1, \ldots, x_d]]\) and \(p = I_{C(n_1, \ldots, n_d)}\). For \(1 \leq i, k \leq d - 1\), let \(f_i\) and \(f_k\) be as in (2.3). Then
\[
\ell \left( \frac{R}{p^{(n)} + (f_k) + (x_1)} \right) \geq \ell \left( \frac{R}{(p, x)} \right) \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \ell \left( \frac{R_p}{p^{n-[j_1 + \cdots + j_i]} R_p} \right) \right].
\]

**Proof.** By [15, Lemma 7.5], \(f_i \in p^{(i)}\). As \(G(pR_p)\) is a regular ring and \(f_d^* \subseteq (f_{d-1}^* \cap R)\) is a regular sequence [15, Proposition 5.3(3)], from Corollary 2.7(2), we get the result. \(\square\)

### 2.2. The power series ring and the polynomial ring.

From now on \(R = \mathbb{k}[[x_1, \ldots, x_d]]\) and \(T = \mathbb{k}[x_1, \ldots, x_d]\). The following lemma gives us a way to compute the length of an \(R\)-module in terms of the length of the corresponding \(T\)-module.

**Lemma 2.9.** Let \(m = (x_1, \ldots, x_d)T\) and \(M\) a finitely generated \(T\)-module such that \(\text{Supp}(M) = \{m\}\). Then
\[
\ell_R(M \otimes_T R) = \ell_T(M).
\]

**Proof.** We prove by induction on \(\ell_T(M)\). If \(\ell_T(M) = 1\), then \(M \cong T/m\). Therefore,
\[
\ell_R(M \otimes_T R) = \ell_R \left( \frac{R}{mR} \right) = 1 \quad \text{(as } mR \text{ is the maximal ideal of } R).\]
If $\ell_T(M) > 1$, then as the minimal primes of $\text{Supp}(M)$ and $\text{Ass}(M)$ are the same, $m \in \text{Ass}(M)$. This gives the exact sequence

$$0 \rightarrow \frac{T}{m} \rightarrow M \rightarrow C \rightarrow 0,$$

where $C \cong M/(T/m)$. As $R$ is $T$-flat, tensoring (2.10) with $R$ we get:

$$0 \rightarrow \frac{T}{m} \otimes_T R \cong \frac{R}{mR} \rightarrow M \otimes_T R \rightarrow C \otimes_T R \rightarrow 0.$$  

From the exact sequence (2.10), we get $\text{Supp}(C) = \{m\}$ and $\ell_T(C) < \ell_T(M)$. Therefore by induction hypothesis $\ell_R(C \otimes_T R) = \ell_T(C)$. Hence

$$\ell_R(M \otimes_T R) = \ell_R(C \otimes_T R) + \ell_R \left( \frac{R}{mR} \right) = \ell_T(C) + \ell_T \left( \frac{T}{m} \right) = \ell_T(M).$$

\[\square\]

Let

$$X_{i+1,(j_1, \ldots, j_{i+1})} := \text{the matrix obtained by choosing the first } i + 1 \text{ rows and } j_1, \ldots, j_{i+1} \text{ columns of } X$$

\[\text{(2.11)}\]

$$J_i := \{ \det(X_{i+1,(j_1, \ldots, j_{i+1})}) | 1 \leq j_1 < \cdots < j_{i+1} \leq d \}.\]

\[\text{(2.12)}\]

**Notation 2.13.** If $A_1, \ldots, A_n$ are $n$ sets of monomials we define the set $A_1 \cdots A_n$ by $A_1 \cdots A_n := \{a_1 \cdots a_n : a_i \in A_i\}$.

Let $I_n$ denote the set

$$I_n := \sum_{a_1+2a_2+\cdots+(d-1)a_{d-1}=n} J_1^{a_1} \cdots J_{d-1}^{a_{d-1}}.\]

\[\text{(2.14)}\]

As $R$ is a flat $T$-module, $I_nR = I_nT \otimes_T R$.

**Proposition 2.15.** Let $n \geq 1$. Then

1. $I_nR \subseteq p^{(n)}$.
2. $(I_n + (x_1))T$ is a homogeneous ideal.
3. $(I_n + (x_1))T$ is an $m$-primary ideal.

**Proof.** (1) By [15, Lemma 7.5], $J_i \subseteq p^{(i)}$ for all $i = 1, \ldots, d - 1$. Hence for all $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\geq 0}$,

$$J_1^{a_1} \cdots J_{d-1}^{a_{d-1}} \subseteq p^{a_1} (p^{(2)})^{a_2} \cdots (p^{(d-1)})^{a_{d-1}} \subseteq p^{a_1+2a_2+\cdots+(d-1)a_{d-1}}.\]

\[\text{(2.16)}\]

Summing over all $a_1 + 2a_2 + \cdots + (d-1)a_{d-1} = n$ and applying (2.16) to (2.14) we get (1).

(2) Fix $1 \leq j_1 < j_2 < \cdots < j_{i+1} \leq d$. Then $\det(X_{i+1,(j_1, \ldots, j_{i+1})})$ is a sum of distinct monomials and the monomials which do not contain $x_1$ are homogeneous of degree $i + 1$. Hence $(J_i + (x_1))T$ is a homogeneous ideal. From (2.14) we get (2).

(3) By (2.14), $J_1^n \subseteq I_n$ and $J_1^n + (x_1) = (x_2, \ldots, x_d)^{2n} + (x_1)$ which implies that $m = \sqrt{J_1^n + (x_1)} \subseteq \sqrt{I_n + (x_1)} \subseteq m.$
3. Monomial ordering and Initial ideals

Using the description of $p^{(n)}$ for $d = 4$ and $n = 2, 3$, Goto proved that the rings $R/(p^{(n)} + (f_1, f_2, f_3))$ are Cohen-Macaulay, where the $f_i \in p^{(i)}$ ($1 \leq i \leq 3$) are as described in [15, page 57]. However, from their method it is not easy to prove a similar result for $d \geq 5$. The new idea in this paper is to give an ordering on $T' = T/(x_1)$ which we call the grevlex which is described in Definition 3.1.

**Definition 3.1.** Let $a = (a_2, \ldots, a_d)$ and $x^a := \prod_{i=2}^{d} x_i^{a_i}$. We say that $x^a > x^b$ if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and in the ordered tuple $(a_2 - b_2, \ldots, a_d - b_d)$ the left-most nonzero entry is negative.

Note that with respect to this order we have $x_2 < x_3 < \ldots < x_d$. For any polynomial $f \in T'$, let $LM(f)$ denote the initial term of $f$ and for any ideal $I \subset T'$, let $LI(I)$ be the initial ideal of the ideal $I$ with respect to the grevlex order. We define monomial ideals $I_n \subseteq LI(\mathcal{I}_nT')$ ((3.3), Proposition 3.4) and consider the filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$. In fact $I_n = LI(\mathcal{I}_nT')$ (Theorem 6.8(3)).

For $2 \leq r < s \leq d$ and $l \geq 1$, let $M_{r,s}^l$ denote the set of monomials of degree $l$ in the variables $x_r, \ldots, x_s$. We set $M_{r,s} := M_{r,s}^1$.

Let $1 \leq i \leq d - 1$ and $n \geq 1$. We define the ideals $J_i$ and $I_n$ in $T'$ as follows:

\[ J_i := (M_{i+1,d})^{i+1}, \quad I_n := \sum_{a_1+2a_2+\cdots+(d-1)a_{d-1}=n} J_1^{a_1} \cdots J_{d-1}^{a_{d-1}}. \tag{3.3} \]

**Proposition 3.4.** For all $n \geq 1$, $I_n \subseteq LI(\mathcal{I}_nT')$.

To prove Proposition 3.4, we first need to consider $LM(\det(X_{i+1,(j_1,\ldots,j_{i+1}))})$ for all $1 \leq j_1 < \cdots < j_{i+1} \leq d$. This is done in Proposition 3.6.

**Notation 3.5.** For any $n \times n$ matrix $M = (m_{ij})$, let $p(M) := \prod_{i+j=n+1} m_{ij}$ denote the product of anti-diagonal elements of the matrix $M$.

**Proposition 3.6.** For $1 \leq i \leq d$,

1. $p(X_{i+1,(j_1,\ldots,j_{i+1}))} = LM(\det(X_{i+1,(j_1,\ldots,j_{i+1)})T')) = \prod_{k=1}^{i+1} x_{j_k + (i-k+1)}$.
2. $J_i \subseteq LI(J_iT')$.

**Proof.** (1) By definition, $p(X_{i+1,(j_1,\ldots,j_{i+1}))} = \prod_{k=1}^{i+1} X_{i-k+2,j_k}$. 


We claim that \(X_{i-k+2,j_k} = x_{j_k+(i-k+1)}\) for all \(k = 1, \ldots, i+1\). Since \(j_k \leq j_{i+1} - (i - k + 1)\) for all \(1 \leq k \leq i+1\), it follows that \(j_k + (i - k + 2) \leq j_{i+1} + 1 \leq d + 1\). Hence the matrix \(X_{i+1,(j_1,\ldots,j_{i+1})}\) is

\[
\begin{pmatrix}
  x_{j_1} & \cdots & x_{j_k} & \cdots & X_{i,j_{i+1}} = x_{j_{i+1}} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  x_{j_1+(i-k+1)} & \cdots & X_{i-k+2,j_k} = x_{j_k+(i-k+1)} & \cdots & \vdots \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  X_{i+1,j_1} = x_{j_1+i} & \cdots & \cdots & \cdots & * 
\end{pmatrix}.
\] (3.7)

This proves the claim. Hence \(p(X_{i+1,(j_1,\ldots,j_{i+1})}) = \prod_{k=1}^{i+1} x_{j_k+(i-k+1)}\).

To complete the proof of (1) we need to show that:

\[
LM(\det(X_{i+1,(j_1,\ldots,j_{i+1})}')) = \prod_{k=1}^{i+1} x_{j_k+(i-k+1)}.
\] (3.8)

We prove (3.8) by induction on \(i\). Note that in the matrix \(X\) defined in (2.1), \(x_1\) divides \(X_{11}\) and \(X_{ij}\) for \(i+j \geq d+2\). Let \(i = 1\). Then

\[
\det(X_{2,(j_1,j_2)})' = \begin{cases} 
  x_{j_1+1}x_{j_2} & \text{if } j_1 = 1 \text{ or } j_2 = d \\
  x_{j_1}x_{j_2+1} - x_{j_1+1}x_{j_2} & \text{if } 1 < j_1 < j_2 < d .
\end{cases}
\]

Hence \(LM(\det(X_{2,(j_1,j_2)})') = x_{j_1+1}x_{j_2}\) if \(j_1 = 1\) or \(j_2 = d\). If \(1 < j_1 < j_2 < d\), then \(j_1 < j_1+1 \leq j_2 < j_2+1\) and hence \(x_{j_1+1}x_{j_2} > x_{j_1}x_{j_2+1}\) which implies that \(LM(\det(X_{2,(j_1,j_2)}')) = x_{j_1+1}x_{j_2}\). Hence (3.8) is true for \(i = 1\).

Now let \(i > 1\). Expanding the matrix in (3.7) along the last row we get:

\[
\det(X_{i+1,(j_1,\ldots,j_{i+1})})' = \left(\sum_{k=1}^{t} (-1)^{k+i+1} x_{j_k+i} \det(X_{i,(j_1,\ldots,j_k,\ldots,j_{i+1})})\right)' \tag{3.9}
\]

where \(t = \max\{k|j_k+i \leq d\}\). As \(X_{i,(j_1,\ldots,j_k,\ldots,j_{i+1})}\) has the form as the matrix described in (3.7), by induction hypothesis,

\[
LM(\det(X_{i,(j_1,\ldots,j_k,\ldots,j_{i+1})}')) = \prod_{\alpha=1}^{k-1} x_{j_{\alpha+i-\alpha}} \prod_{\alpha=k+1}^{i+1} x_{j_{\alpha}+(i-\alpha+1)}
\]

\[
= \begin{cases} 
  \prod_{\alpha=2}^{i+1} x_{j_{\alpha}+(i-\alpha+1)} & \text{if } k = 1 \\
  \prod_{\alpha=2}^{k-1} x_{j_{\alpha}+(i-\alpha+1)} \prod_{\alpha=k+1}^{i+1} x_{j_{\alpha}+(i-\alpha+1)} & \text{if } k = 2, \ldots, t.
\end{cases} \tag{3.10}
\]
Hence for all \( k = 2, \ldots, t \)
\[
x_{jk+1}LM(\det(X_{i,(j_1,\ldots,j_{k-1})})T')
= x_{j1+i}LM(\prod_{a=2}^{k-1} x_{j_0+i-a} x_{jk+i}) \prod_{a=k+1}^{i+1} x_{j_0+(i-a+1)} 
\quad [\text{by (3.10)}]
< x_{j1+i}x_{j2+i-1} \cdots x_{j_{k+1}+1} 
= x_{j1+i}LM(\det(X_{i,(j_1,\ldots,j_{k+1})})T') 
\quad [\text{by (3.10)}]. \quad (3.11)
\]

Therefore
\[
LM(\det(X_{i+1,(j_1,\ldots,j_{k+1})})T') = LM(\sum_{k=1}^{t} (-1)^{k+i+1} x_{jk+i}LM(\det(X_{i,(j_1,\ldots,j_{k-1})})T') 
\quad [\text{by (3.9)}]
= x_{j1+i}LM(\det(X_{i,(\hat{j}_1,\hat{j}_2,\ldots,j_{k+1})})T') 
\quad [\text{by (3.11)}]
= x_{j1+i}x_{j2+i-1} \cdots x_{j_{k+1}+1} 
\quad [\text{by (3.10)}].
\]

This proves (1).

(2) Let \( x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}} \in M_{i+1,d}^{+1} \) where \( 1 \leq k_1 < k_2 < \cdots < k_s \leq d-i \) and \( \alpha_{i+k_1}, \ldots, \alpha_{i+k_s} \neq 0 \) such that \( \alpha_{i+k_1} + \cdots + \alpha_{i+k_s} = i + 1 \). Set \( \beta_0 = 0 \) and \( \beta_r = \alpha_{i+k_1} + \cdots + \alpha_{i+k_r} \) for \( 1 \leq r \leq s \). Define
\[
S_r = \{ \beta_{r-1} + 1, \beta_{r-1} + 2, \ldots, \beta_r \} \text{ for } 1 \leq r \leq s.
\]

Then \( \bigcup_{r=1}^{s} S_r = \{ 1, \ldots, i + 1 \} \). Let \( 1 \leq t \leq i + 1 \). If \( t \in S_r \) define
\[
\hat{j}_t = k_r + (t - 1).
\]
With this choice of \( j_1, \ldots, j_{i+1} \), \( p(X_{i+1,(j_1,\ldots,j_{i+1})}) = x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}} \). By (1) \( x_{i+k_1}^{\alpha_{i+k_1}} \cdots x_{i+k_s}^{\alpha_{i+k_s}} \in LI(J_t T') \).

Hence \( J_t \subseteq LI(J_t T') \). \( \square \)

**Proof of Proposition 3.4:** The proof follows from (2.14), Proposition 3.6(2) and (3.3).  

4. **The Associated Graded Ring Corresponding to the Filtration** \( \mathcal{F} := \{ I_n \}_{n \geq 0} \)

Let \( G(\mathcal{F}) := \oplus_{n \geq 0} I_n/I_{n+1} \) be the associated graded ring corresponding to the filtration \( \mathcal{F} = \{ I_n \}_{n \geq 0} \), where \( I_n \) are ideals defined in (3.3). By definition of \( I_n \), \( G(\mathcal{F}) \) is Noetherian (Theorem 4.6). One of the key steps is to prove that the associated graded ring \( G(\mathcal{F}) = \oplus_{n \geq 0} (I_n/I_{n+1}) \) is Cohen-Macaulay. In particular we show that \( (x_2^2)^*, \ldots, (x_d^2)^* \) is a regular sequence in \( G(\mathcal{F}) \) (Theorem 4.6). As an immediate consequence, we give a formula for \( \ell \left( \frac{T'}{I_n + (x_2^2, \ldots, x_d^2)} \right) \) (Proposition 4.7) which is useful in the subsequent sections. The following proposition is crucial to prove Theorem 4.6.

**Proposition 4.1.** For all \( n \geq 1 \) and \( i = 2, \ldots, d \),
\[
(I_n : (x_i^1)) = \begin{cases} T' & \text{if } n < i \\ I_{n-i+1} & \text{if } n \geq i. \end{cases}
\]
Proof. If \( n < i \), then \( x_i^i \in J_{i-1} \subseteq I_n \) which implies that \( (I_n : (x_i^i)) = T' \). Therefore, for the rest of the proof we will assume that \( n \geq i \).

As \( I_n = \sum_{a_1+2a_2+\ldots+(d-1)a_{d-1}=n} J_{a_1}^1 \cdots J_{a_{d-1}}^{a_{d-1}} \), by [13, Proposition 1.14], we only need to consider \( M_j \in J_{a_j}^a \) with \( \text{deg}(M_j) = (j+1)a_j \) and show that \((\prod_{j=1}^{d-1} M_j) : (x_i^i)) \subseteq I_{n-i+1} \). Note that

\[
\left( \left( \prod_{j=1}^{d-1} M_j \right) : (x_i^i) \right) = \left( \frac{\left( \prod_{j=1}^{d-1} M_j \right)}{\text{gcd}(\prod_{j=1}^{d-1} M_j, x_i^i)} \right) = \left( \frac{\left( \prod_{j=1}^{i-1} M_j \right)}{x_i^g} \left[ \prod_{j=i}^{d-1} M_j \right] \right)
\]

where \( g = \min\{i, \sum_{j=1}^{i-1} b_j\} \) and \( b_j := \max\{t|x_i^i \text{ divides } M_j\} \).

If \( b_j = 0 \) for all \( j = 1, \ldots, i-1 \), then \( g = 0 \) and

\[
\left( \prod_{j=1}^{d-1} M_j \right) : (x_i^i) = \left( \prod_{j=1}^{d-1} M_j \right) \subseteq I_n \subseteq I_{n-i+1} \text{.}
\]

Hence, for the rest of the proof we will assume that \( b_j \neq 0 \) for some \( j = 1, \ldots, i-1 \).

Claim: For \( j = 1, \ldots, i-1 \), there exist integers \( a'_j \) and monomials \( M'_j \) such that:

1. \( M'_j \in J'^{a'_j}_j \) for all \( j = 1, \ldots, i-1 \).
2. \( \left( \prod_{j=1}^{i-1} M'_j \right) \frac{x_i^g}{x_i^i} = \left( \prod_{j=1}^{i-1} M'_j \right) N \), for some monomial \( N \) in \( T' \).
3. \[ \sum_{j=1}^{i-1} ja'_j + \sum_{j=i}^{d-1} ja_j \geq n - i + 1. \]

Put \( k := \min \left\{ l \left| \sum_{j=l}^{i-1} b_j \leq i - 1 \right. \right\} \). For \( k \leq j \leq i - 1 \) we define \( q_j \) and \( r_j \) using the following algorithm:

Define non-negative integers \( q_{k-1} \) and \( r_{k-1} \) as follows: Put

\[
c := g - \sum_{j=k}^{i-1} b_j. \tag{4.3}\]

If \( c = 0 \), then put \( q_{k-1} := 0 \) and \( r_{k-1} := r_k \). If \( c > 0 \), then choose \( q_{k-1} \geq 0 \) and \( 0 \leq r_{k-1} \leq k - 1 \) such that

\[
c - r_k = kq_{k-1} - r_{k-1}. \tag{4.4}\]

For \( j = 1, \ldots, i-1 \) we define \( a'_j \) as follows:

\[
a'_j := \begin{cases} a_j - q_j & \text{if } j \in \{k-1, \ldots, i-1\} \setminus \{r_{k-1} - 1\} \\ a_j & \text{if } j \in \{1, \ldots, k-2\} \setminus \{r_{k-1} - 1\} \\ a_{r_{k-1} - 1} + 1 & \text{if } j = r_{k-1} - 1 \text{ and } r_{k-1} \geq 2. \end{cases} \tag{4.5}\]
Algorithm

Output: Defines $q_j, r_j$ for $k \leq j \leq i - 1$.

Initialize: $r_i = 0$ and $j = i - 1$

1: while $j \geq k$ do
2:     if $b_j = 0$ then
3:         define $q_j = 0$ and $r_j = r_{j+1}$
4:     else
5:         find integers $q_j$ and $0 \leq r_j \leq j$ such that
6:             \[ b_j - r_{j+1} = (j+1)q_j - r_j. \]
7:     end if
8:     return $q_j, r_j$
9:     $j \leftarrow j - 1$
10: end while

Set $M_0 = 1$ and define $N_j$ for $j = k - 1, \ldots, i$ as follows:

\[
N_j = \begin{cases} 
1 & \text{if } j = i \\
\text{a monomial of degree } r_j \text{ that divides } \frac{M_jN_{j+1}}{x_i^{b_j}} & \text{if } b_j \neq 0 \text{ and } k \leq j < i \\
N_{j+1} & \text{if } b_j = 0 \text{ and } k \leq j < i \\
\text{a monomial of degree } r_k - 1 \text{ that divides } \frac{M_k - 1N_k}{x_i^{c_i}} & \text{if } j = k - 1.
\end{cases}
\]

For $j = 1, \ldots, i - 1$ we define $M_j'$ as follows:

\[
M_j' := \begin{cases} 
\frac{M_jN_{j+1}}{x_i^{b_j}N_i} & \text{if } j \in \{k, \ldots, i - 1\} \setminus \{r_k - 1 - 1\} \\
\frac{M_k - 1N_k}{x_i^{c_i}N_{k-1}} & \text{if } j = k - 1 \\
M_j & \text{if } j \in \{1, \ldots, k - 2\} \setminus \{r_k - 1 - 1\} \\
M_{r_k - 1}N_{k-1} & \text{if } j = r_k - 1 - 1 \text{ and } 2 \leq r_k - 1.
\end{cases}
\]

By our definition of $M_j'$, $\deg(M_j') = (j + 1)a_j'$ for all $j = 1, \ldots, i - 1$. Hence $M_j' \in J_j^{a_j'}$. This proves (1) of the Claim.

Let

\[
N = \begin{cases} 
N_{k-1} & \text{if } r_{k-1} \leq 1 \\
1 & \text{if } r_{k-1} > 1.
\end{cases}
\]

Then we can express $\prod_{j=1}^{i-1} M_j$ as in (2) of the Claim.
We now prove (3) of the Claim. To complete the proof it suffices to show that $\sum_{j=1}^{i-1} j a_j' + \sum_{j=i}^{d-1} j a_j \geq n - i + 1$. Put
\[
\alpha(r_{k-1}) := \begin{cases} 
0 & \text{if } r_{k-1} = 0 \\
r_{k-1} - 1 & \text{if } r_{k-1} \neq 0.
\end{cases}
\]
Then
\[
\sum_{j=1}^{i-1} j a_j' + \sum_{j=i}^{d-1} j a_j = n - \sum_{j=k-1}^{i-1} [(j+1)q_j] + \sum_{j=k-1}^{i-1} q_j + \alpha(r_{k-1}) \quad \text{[by (4.5)]}
\]
\[
= n - [c - r_k + r_{k-1}] - \sum_{j=k}^{i-1} [b_j - r_{j+1} + r_j] + \sum_{j=k-1}^{i-1} q_j + \alpha(r_{k-1}) \quad \text{[by (4.4) and (4.2)]}
\]
\[
= n - g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j \quad \text{[by (4.3)],}
\]
We claim that:
(a) $\sum_{j=k-1}^{i-1} q_j \geq 1$.
(b) If $g = i$ and $r_{k-1} > 0$, then $\sum_{j=k-1}^{i-1} q_j \geq 2$.

Suppose $q_j = 0$ for all $j = k - 1, \ldots, i - 1$. Then
\[
0 \leq g = \sum_{j=k}^{i-1} b_j + c = \sum_{j=k}^{i-1} [r_{j+1} - r_j] + r_k - r_{k-1} = -r_{k-1} \leq 0,
\]
which implies that $g = 0$. Hence $\sum_{j=k}^{i-1} b_j = 0$ which leads to a contradiction on our assumption of $b_j$’s. This proves (a) of the claim.

Now suppose $g = i$ and $r_{k-1} > 0$. By (a), $\sum_{j=k-1}^{i-1} q_j \geq 1$. If $\sum_{j=k-1}^{i-1} q_j = 1$, then $q_l = 1$ for some $k-1 \leq l \leq i-1$ and $q_j = 0$ for $j \neq l$. Hence
\[
i = g = \sum_{j=k}^{i-1} b_j + c = (l + 1) - r_{k-1} \leq i - r_{k-1} \leq i - 1
\]
which leads to a contradiction.

If $g \leq i-1$ or $g = i$ and $r_{k-1} = 0$, then by Claim (a), $-g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j \geq -i + 1$. If $g = i$ and $r_{k-1} \neq 0$, then by Claim (b) $-g + [\alpha(r_{k-1}) - r_{k-1}] + \sum_{j=k-1}^{i-1} q_j \geq -i + 1$. This completes the proof of (3) of the Claim. \qed
Theorem 4.6. The associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay.

Proof. Let $a^*$ denote the image of $a$ in $G(\mathcal{F})$. Since $x_i^i \in I_{i-1} \setminus I_i$ it follows that $(x_i^i)^* \in [G(\mathcal{F})]_{i-1}$. To prove the theorem it is enough to show that $(x_2^2)^*, \ldots, (x_i^i)^*$ is a regular sequence in $G(\mathcal{F})$ for all $2 \leq i \leq d$. We prove by induction on $i$. If $i = 2$, then by Proposition 4.1, $(x_2^2)^*$ is a regular element in $G(\mathcal{F})$. Now let $i > 2$ and assume that $(x_2^2)^*, \ldots, (x_{i-1}^{i-1})^*$ is a regular sequence in $G(\mathcal{F})$. Then

$$
\frac{G(\mathcal{F})}{((x_2^2)^*, \ldots, (x_{i-1}^{i-1})^*)} \cong \bigoplus_{n \geq 0} I_{n+1} + \sum_{j=2}^{i-1} x_j^j I_{n+1-j}.
$$

One can verify that

$$
((I_{n+i} + \sum_{j=2}^{i-1} x_j^j I_{n+i-j}) : (x_i^i)) = (I_{n+i} : (x_i^i)) + \sum_{j=2}^{i-1} x_j^j (I_{n+i-j} : (x_i^i)) \quad \text{[13, Proposition 1.14]}
$$

$$
= (I_{n+i} : (x_i^i)) + \sum_{j=2}^{i-1} x_j^j (I_{n+i-j} : (x_i^i))
$$

$$
= I_{n+1} + \sum_{j=2}^{i-1} x_j^j I_{n+1-j} \quad \text{[by Proposition 4.1].}
$$

Hence $(x_i^i)^*$ is $G(\mathcal{F})/((x_2^2)^*, \ldots, (x_{i-1}^{i-1})^*)$-regular. $\square$

Proposition 4.7. Let $2 \leq k \leq d$. Then for all $n \geq 1$,

$$
\ell \left( \frac{T'}{(I_n + (x_2^2, \ldots, x_k^k)) T'} \right) = \sum_{i=0}^{k-1} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k-1} \ell \left( \frac{T'}{(I_n-(j_1, \ldots, j_i)) T'} \right) \right].
$$

Proof. The proof follows from Proposition 2.5 and Theorem 4.6. $\square$

5. Monomial generators of $I_{n-1}$ modulo $(I_n : (x_d))$ as a $k$-vector space

In this section we first show that $(I_n : (x_d)) \subseteq I_{n-1}$. Next we describe the generators of $I_{n-1}$ modulo $(I_n : (x_d))$ (Proposition 5.3). This will be used to compute $\ell(T'/I_n)$ and $\ell(T'/I_n + (x_2^2, \ldots, x_{k+1}^k))$.

The following lemma is simple, but we state it as it is crucially used to prove Lemma 5.2.

Lemma 5.1. (1) Let $1 \leq j \leq d - 1$ and $a \geq 1$. Then

$$
(M_{j+1,d})^{(j+1)a} = x_{j+1}^{(j+1)a-j} (M_{j+1,d})^j + (M_{j+2,d})^{j+1} (M_{j+1,d})^{(j+1)(a-1)}.
$$

(2) Let $1 \leq k < j \leq d - 1$ and $a, b \geq 1$. Then

$$
(M_{k+1,d})^a (M_{j+1,d})^b = (M_{k+1,j+1})^a (M_{j+1,d})^b + (M_{k+1,d})^a (M_{j+2,d})^{b+1}.
$$

Proof. (1) The proof follows by induction on $a$. (2) The proof follows by induction on $a + b$. $\square$
Before we proceed we set up some notation. For $1 \leq j \leq d - 1$, let $a_j \neq 0$ and $a_j := (a_1, \ldots, a_j) \in \mathbb{N}^j$. We inductively define the set $S(a_j)$ as follows:

$$S(a_1) := \{x_2^{2a_1-1}\}$$

$$S(a_j) := \begin{cases} 
  \{x_{j+1}^{(j+1)a_j-j}\} & \text{if } \{i < j|a_i \neq 0\} = \emptyset \\
  \{x_{j+1}^{(j+1)a_j-j}\}S(a_k)M_{k+1,j+1}^j & \text{if } \{i < j|a_i \neq 0\} \neq \emptyset \text{ and } k = \max\{i < j|a_i \neq 0\}.
\end{cases}$$

We set $J^a_j := J_1^{a_1} \cdots J^{a_j}_j$. Let $w(a_j) := a_1 + 2a_2 + \cdots + ja_j$ be the weight of $a_j$. For all $n \in \mathbb{N}$ we define $\Lambda_{j,n} := \{a_j \in \mathbb{N}^j : w(a_j) = n, \ a_j \neq 0\}$.

**Lemma 5.2.** Let $n \geq 2$. Then

1. $(I_n : (x_d)) \subseteq I_{n-1}$.
2. For all $1 \leq j \leq d - 1$ and for all $a_j \in \Lambda_{j,n-1}$
   a. $S(a_j)M_{j+1,d}^j \subseteq J^{a_j}_j \setminus m'J^{a_j}_j$ where $m' = (x_2, \ldots, x_d)T'$.
   b. For all $1 \leq j \leq d - 1$, $J^{a_j}_j \subseteq (S(a_j)M_{j+1,d}^j) + (I_n : (x_d))$.
3. $I_{n-1} = \sum_{j=1}^{d-1} \sum_{a_j \in \Lambda_{j,n-1}} (S(a_j)M_{j+1,d}^j) + (I_n : (x_d))$.

**Proof.** (1) By [13, Proposition 1.14] it is enough to show that for all $j = 1 \ldots d - 1$ and $a_j \in \Lambda_{j,n}$, $(J^{a_j}_j : (x_d)) \subseteq I_{n-1}$. One can verify that

$$\begin{align*}
(J^{a_j}_j : (x_d)) &= (M_{j+1,d}^{a_1} \cdots (M_{j,d})^{ja_{j-1}}(M_{j+1,d})^{(j+1)a_j-1} \\
&= (M_{j+1,d}^{a_1} \cdots [(M_{j,d})^{ja_{j-1}}(M_{j+1,d})^j](M_{j+1,d})^{(j+1)a_j-(j+1)} \\
&\subseteq (M_{j+1,d}^{a_1} \cdots (M_{j,d})^{ja_{j-1}+1}(M_{j+1,d})^{(j+1)a_j-1}) & \text{[as } (M_{j+1,d}) \subseteq (M_{j,d})]\n&\subseteq I_{n-1},
\end{align*}$$

since $a_1 + \cdots + (j - 2)a_{j-2} + (j - 1)(a_{j-1} + 1) + ja_j - 1 = n - 1$. This proves (1).

(2) Set $r(a_j) = \#\{i : 1 \leq i \leq j \text{ and } a_i \neq 0\}$. We prove by induction on $r(a_j)$.

(2a) If $r(a_j) = 1$, then $S(a_j) = \{x_{j+1}^{(j+1)a_j-j}\}$. Hence $S(a_j)M_{j+1,d}^j = \{x_{j+1}^{(j+1)a_j-j}\}M_{j+1,d}^j \subseteq J^{a_j}_j = J^{a_j}_j$.

If $r(a_j) > 1$ and $k = \max\{i|1 \leq i < j \text{ and } a_i \neq 0\}$, then $S(a_j)M_{j+1,d}^j = S(a_k)M_{k+1,j+1}^k \left[x_{j+1}^{(j+1)a_j-j}M_{j+1,d}^j\right]$ and by induction hypothesis,

$$S(a_k)M_{k+1,j+1}^k \left[x_{j+1}^{(j+1)a_j-j}M_{j+1,d}^j\right] \subseteq J^{a_k}j^{a_j}_j = J^{a_j}_j.$$ 

Comparing the degree of the monomials in $S(a_j)M_{j+1,d}^j$ we conclude that these monomials are not in $m'J^{a_j}_j$.

(2b) If $r(a_j) = 1$, then

$$J^{a_j}_j = (M_{j+1,d})^{(j+1)a_j}$$

$$= x_{j+1}^{(j+1)a_j-j}(M_{j+1,d})^j + (M_{j+2,d})^{j+1}(M_{j+1,d})^{(j+1)a_j-1} & \text{[by Lemma 5.1(1)]}\n\subseteq S(a_j)(M_{j+1,d})^j + (I_n : (x_d))$$

as $x_d(M_{j+2,d})^{j+1}(M_{j+1,d})^{(j+1)a_j-1} \subseteq J_{j+1}J^{a_j}_j-1$ and $(j+1) + j(a_j - 1) = ja_j + 1 = (n - 1) + 1 = n$. Hence (2b) is true for $r(a_j) = 1$. 


Now let \( r(a_j) > 1 \) and \( k = \max\{i | 1 \leq i < j \text{ and } a_i \neq 0\} \). Then
\[
J^a_j = J^{a_k} J^{a_j}_j \\
\subseteq (S(a_k)M^k_{j+1,d}) + (I_{n-j a_j} : (x_d)) J^{a_j}_j \\
\subseteq x_d^{(j+1)a_{j-1}} (S(a_k)M^k_{j+1,d})(M^j_{j+1,d})^2 + (S(a_k)M^k_{j+1,d})(I_{j a_j+1} : (x_d)) + (I_{n-j a_j} : (x_d)) J^{a_j}_j \\
\subseteq x_d^{(j+1)a _{j-1}} (S(a_k)M^k_{j+1,d})(M^j_{j+1,d})^2 + (I_n : (x_d)) \\
\subseteq x_d^{(j+1)a _{j-1}} (S(a_k)M^k_{j+1,d})^2 (M^j_{j+1,d})^2 + (I_n : (x_d)) \\
\subseteq (S(a_j)(M^j_{j+1,d})^2 + (I_n : (x_d)) \\
as \[
= (S(a_j)(M^j_{j+1,d})^2 + (I_n : (x_d)) \\
\subseteq x_d^{(j+1)a_{j-1}} (S(a_k)) (M^k_{j+1,d})^{j-1} (M^j_{j+1,d})^{j+1} \\
\subseteq [x_d^{(j+1)a_{j-1}} (S(a_k)) (M^k_{j+1,d})^{j-1} x_d (M^j_{j+1,d})^{j+1} x_d((j+1)a_{j-1})) \\
\subseteq J^{a_k} J^{a_{j-1}} J^{a_j}_j \\
\subseteq I_n. 
\]
This proves (2b) for all \( r(a_j) \geq 1 \).

(3) The proof follows from (1) and (2). \( \square \)

**Proposition 5.3.** The set \( \{M + (I_n : (x_d)) | M \in \bigcup_{j=1}^{d-1} A_{j,n-1} \{S(a_j)M^j_{j+1,d}\} \} \) generates \( I_{n-1} / (I_n : (x_d)) \) as a \( k \)-vector space.

**Proof.** Let \( M \) be a monomial in \( S(a_j)M^j_{j+1,d} \). By Lemma 5.2(2a), \( M \in J^a_j \). Thus \( x_d x_i M \in (m^r)^2 J^{a_1}_1 \cdots J^{a_j}_j \subseteq I_n \). This implies that \( x_i M \in (I_n : (x_d)) \) for all \( i = 2, \ldots, d \). Hence from Lemma 5.2(3), the monomials in \( S(a_j)M^j_{j+1,d} \) generate \( I_{n-1} \) modulo \( (I_n : (x_d)) \) as a \( k \)-vector space. \( \square \)

From Proposition 5.3, giving an upper bound for the length of the vector space \( I_{n-1} / (I_n : (x_d)) \) involves counting monomials and hence it is combinatorial in nature. Hence we prove some preliminary results before we arrive at the main result of this section. We state the well known Vandermonde’s identity which will be needed in our proofs.

**Lemma 5.4.** [Vandermonde’s identity] Let \( n, r, s \in \mathbb{N} \). Then
\[
\sum_{i \geq 0} \binom{n}{i} \binom{s}{r-i} = \binom{n+s}{r}
\]

The next lemma is the main step in proving our main result.
Lemma 5.5. Fix $1 \leq j \leq d - 1$ and $n > 1$. Then  
\[ \sum_{a_j \in \Lambda_{j,n-1}} \#S(a_j) = \binom{n - 2}{j - 1}. \]

Proof. We prove by induction on $j$. If $j = 1$ then $S(a_1) = \{2a_1 - 1\}$, and hence the assertion is true for $j = 1$.

Now let $j > 1$. Then

\[
\sum_{a_j \in \Lambda_{j,n-1}} \#S(a_j) = \begin{cases} 
\sum_{a_j = 1}^{n-2} \sum_{i=1}^{j-1} \left[ \sum_{a_i \in \Lambda_{i,n-1-ja_j}} \#S(a_i) \right] \#M_{i+1,j+1} & \text{if } j \nmid (n - 1) \\
\sum_{a_j = 1}^{n-2} \sum_{i=1}^{j-1} \left[ \sum_{a_i \in \Lambda_{i,n-1-ja_j}} \#S(a_i) \right] \#M_{i+1,j+1} & \text{if } j \mid (n - 1) 
\end{cases}.
\]

(5.6)

Define $\alpha_{j,n} := \begin{cases} 
0 & \text{if } j \nmid (n - 1) \\
1 & \text{if } j \mid (n - 1) 
\end{cases}$. Then (5.6) can be written as

\[
\sum_{a_j \in \Lambda_{j,n-1}} \#S(a_j) = \alpha_{j,n} + \sum_{a_j = 1}^{n-2} \sum_{i=1}^{j-1} \left[ \sum_{a_i \in \Lambda_{i,n-1-ja_j}} \#S(a_i) \right] \#M_{i+1,j+1} 
\]

[by induction hypothesis]

\[
= \alpha_{j,n} + \sum_{a_j = 1}^{n-2} \sum_{i=1}^{j-1} \left[ \sum_{i=0}^{j-1} \binom{n - ja_j - 2}{i} \binom{j}{j-i} \right] - \binom{n - ja_j - 2}{j-1} 
\]

[replacing $i - 1$ by $i$]

\[
= \alpha_{j,n} + \sum_{a_j = 1}^{n-2} \left[ \sum_{a_i \in \Lambda_{i,n-1-ja_j}} \#S(a_i) \right] \#M_{i+1,j+1} 
\]

[by Lemma 5.4]

\[
= \alpha_{j,n} + \binom{n-2}{j-1} - \alpha_{j,n} 
\]

\[
= \binom{n-2}{j-1}. 
\]

\[ \square \]

We are now ready to prove the main result in this section.

Proposition 5.7. Let $d, n \geq 2$. Then

\[ \ell \left( \frac{I_{n-1}}{(I_n : (x_d))} \right) \leq \binom{n + d - 3}{d - 2}. \]
Proof. By Proposition 5.3 we get

\[
\ell \left( \frac{I_{n-1}}{(I_n : (x_d))} \right) \leq \sum_{j=1}^{d-1} \sum_{a_j \in \Lambda_{j,n-1}} \#S(a_j) \#M_{j+1,d}^{j}
\]

\[
= \sum_{j=1}^{d-1} \binom{n-2}{j-1} \frac{d-1}{d-j-1}
\quad \text{[by Lemma 5.5].}
\]

\[
= \sum_{i=0}^{d-2} \binom{n-2}{i} \frac{d-1}{d-i-2}
\quad \text{[put } i = j - 1]\]

\[
= \binom{n+d-3}{d-2}
\quad \text{[by Lemma 5.4].}
\]

\[
\square
\]

6. Cohen-Macaulayness of \( R/(p^{(n)} + (f_k)) \)

In [15, Proposition 7.6] Goto showed that \( R/(p^{(n)} + (f_{d-1})) \) is Cohen-Macaulay for \( d = 3, 4 \) and \( n \leq \binom{d-1}{2} \). This was done by explicitly describing \( p^{(n)} \) for \( d \leq 4 \) and \( n \leq \binom{d-1}{2} \). Using the techniques developed in this paper, we generalise Goto’s result for all \( d \geq 2 \) and \( n \geq 1 \). A lower bound for \( \ell(R/(p^{(n)} + (f_k, x_1))) \) was given using the multiplicity formula (Theorem 2.8). In this section, we show that the inequality in Theorem 2.8 is indeed an equality (Theorem 6.5). This implies that for all \( n \geq 1 \) and \( 1 \leq k \leq d-1 \), the rings \( R/(p^{(n)} + (f_k)) \) are Cohen-Macaulay. As a consequence, we describe \( p^{(n)} \) for all \( d \geq 2 \) and all \( n \geq 1 \). In particular we prove that \( p^{(n)} = \mathcal{I}_nR \) and \( LI(p^{(n)}T') = I_nT' \) for all \( d \geq 2 \) and \( n \geq 1 \).

We first give an upper bound on \( \ell(T'/I_n) \). This is crucial to prove an interesting result which shows that the the equality of the lengths of the various modules (over different rings) in Theorem 6.2.

Proposition 6.1. Let \( d \geq 2 \). Then for all \( n \geq 1 \),

\[
\ell \left( \frac{T'}{I_n} \right) \leq d \binom{n+d-2}{d-1}.
\]

Proof. We prove by double induction on \( n \) and \( d \). If \( n = 1 \), then

\[
\ell \left( \frac{T'}{I_1} \right) = \ell \left( \frac{k[x_2, \ldots, x_d]}{(x_2, \ldots, x_d)^2} \right) = d.
\]

If \( d = 2 \), then

\[
\ell \left( \frac{T'}{I_1} \right) = \ell \left( \frac{k[x_2]}{(x_2)^2n} \right) = 2n.
\]

Now let \( n > 1 \) and \( d > 2 \). From the exact sequence

\[
0 \to \frac{T'}{(I_n : (x_d))} \xrightarrow{x_d} \frac{T'}{I_n} \xrightarrow{I_n + (x_d)} 0
\]
we get

\[
\ell \left( \frac{T'}{I_n} \right) = \ell \left( \frac{T'}{I_n + (x_d)} \right) + \ell \left( \frac{T'}{(I_n : (x_d))} \right) \\
= \ell \left( \frac{T'}{I_n + (x_d)} \right) + \ell \left( \frac{T'}{I_{n-1}} \right) + \ell \left( \frac{I_{n-1}}{(I_n : (x_d))} \right) \tag{Lemma 5.2(1)}
\]

\[
\leq (d-1) \binom{n+d-3}{d-2} + d \binom{n-1+d-2}{d-1} + \binom{n+d-3}{d-2} \quad \text{[by induction hypothesis and Proposition 5.7]}
\]

\[
= d \binom{n+d-3}{d-2} + d \binom{n+d-3}{d-1}
\]

\[
= d \binom{n+d-2}{d-1}.
\]

\[
\square
\]

Theorem 6.2. Let \( d \geq 2 \). Then for all \( n \geq 1 \),

\[
e \left( x_1; \frac{R}{p^{(n)}} \right) = \ell \left( \frac{R}{p^{(n)} + (x_1)} \right) = \ell_R \left( \frac{R}{(I_n, x_1)R} \right) = \ell_T \left( \frac{T'}{I_n} \right) = \ell_T \left( \frac{T'}{I_n} \right) = d \binom{n+d-2}{d-1}.
\]

Proof. From Proposition 2.15(1) \( I_n R \subseteq p^{(n)} \). Since \( R/p^{(n)} \) is Cohen-Macaulay,

\[
e \left( x_1; \frac{R}{p^{(n)}} \right) = \ell_R \left( \frac{R}{p^{(n)} + (x_1)} \right) \leq \ell_R \left( \frac{R}{(I_n, x_1)R} \right).
\]

(6.3)

By Proposition 2.15(3), for any prime \( q \neq m \), \( ((I_n, x_1)T)_q = T \). This implies that \( \text{Supp}_T \left( \frac{T}{(I_n, x_1)T} \right) = \{ m \} \).

Hence we get
Thus equality holds in (6.3) and (6.4) which proves the theorem.

**Theorem 6.5.** Let $d \geq 2$ and $1 \leq k \leq d - 1$. Let $f_k$ be as in (2.3). Then for all $n \geq 1$,

\[
e \left( x_1; \frac{R}{p(n) + (f_k)} \right) = \ell_R \left( \frac{R}{p(n) + (x_1, f_k)} \right) = \ell_{T'} \left( \frac{T'}{T(I_n + f_k)} \right) = \ell_{T'} \left( \frac{T'}{I_n + (x_1, f_k)} \right) \]

\[
= d \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \binom{n - (j_1 + \cdots + j_i) + d - 2}{d - 1} \right].
\]

In particular, $R/(p(n) + (f_k))$ is Cohen-Macaulay.

**Proof.** From Proposition 2.15(1) $(I_n, x_1, f_k)R \subseteq (p(n), x_1, f_k)R$. Hence

\[
e \left( x_1; \frac{R}{p(n) + (f_k)} \right) \leq \ell_R \left( \frac{R}{p(n) + (x_1, f_k)} \right) \leq \ell_R \left( \frac{R}{(I_n, x_1, f_k)R} \right). \tag{6.6}
\]
Since \((I_n, x_1)^T \subseteq (I_n, f_k, x_1)^T\) by Proposition 2.15(3), for any prime \(q \neq m\), \(((I_n, f_k, x_1)^T)_q = T\). This implies that \(\operatorname{Supp}_T\left(\frac{R}{(I_n, f_k, x_1)^T}\right) = \{m\}\). Hence we get

\[
\ell_R\left(\frac{R}{(I_n, x_1, f_k)R}\right)
= \ell_R\left(\frac{T}{(I_n, x_1, f_k)^T \otimes_T R}\right)
= \ell_T\left(\frac{T}{(I_n, x_1, f_k)^T}\right)
= \ell_{T'}\left(\frac{T'}{(I_n, f_k)^{T'}}\right)
\leq \ell_{T'}\left(\frac{T'}{I_n + (x_2^2, \ldots, x_{k+1})}\right)
\]

[Propositions 3.4 and 3.6(1)]

\[
= \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \ell\left(\frac{T'}{I_n - (j_1 + \cdots + j_i)T'}\right) \right] \quad \text{[Proposition 4.7]}
\]

\[
= d \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \left( n - (j_1 + \cdots + j_i) + d - 2 \right) \right]
\]

[Theorem 6.2]

\[
= d \sum_{i=0}^{k} (-1)^i \left[ \sum_{1 \leq j_1 < \cdots < j_i \leq k} \ell\left(\frac{R_p}{p^n - (j_1 + \cdots + j_i)R_p}\right) \right]
\]

\[
= e\left(x_1; \frac{R}{p^{(n)}(f_k)}\right) \quad \text{[[15, Proposition 5.3(3) and Corollary 2.7(1)], (6.7)]}
\]

Hence equality holds in (6.6) and (6.7) which proves the theorem.

We end this section by explicitly describing the generators of \(p^{(n)}\) for all \(n \geq 1\). We also describe the leading ideal \(LI(p^{(n)})T'\).

**Theorem 6.8.**

1. For all \(n \geq 1\), \(p^{(n)} = I_n R\).
2. For all \(n \geq d\), \(p^{(n)} = \sum_{a_1 + 2a_2 + \cdots + (d-1)a_{d-1} = n} p^{a_1}(p^{(2)})^{a_2} \cdots (p^{(d-1)})^{a_{d-1}}\).
3. For all \(n \geq 1\), \(LI(p^{(n)}T') = I_n = LI(I_n T')\).

**Proof.**

1. By Theorem 6.2 we get

\[
\ell\left(\frac{R}{p^{(n)} + (x_1)}\right) = \ell\left(\frac{R}{I_n R + (x_1)}\right)
\]

This implies that \(p^{(n)} = I_n R + x_1(p^{(n)} : (x_1))\). As \(x_1\) is a nonzerodivisor on \(R/p^{(n)}\), \((p^{(n)} : (x_1)) = p^{(n)}\). By Nakayama’s lemma, \(p^{(n)} = I_n R\).
(2) For all \( n \geq d \),

\[
\mathfrak{p}^{(n)} = \mathcal{I}_n R = \sum_{a_1 + 2a_2 + \cdots + (d-1)a_{d-1} = n} \mathcal{J}_{a_1}^{a_1} \mathcal{J}_{a_2}^{a_2} \cdots \mathcal{J}_{a_{d-1}}^{a_{d-1}} R \subseteq \sum_{a_1 + 2a_2 + \cdots + (d-1)a_{d-1} = n} \mathfrak{p}^{a_1} (\mathfrak{p}(2))^{a_2} \cdots (\mathfrak{p}(d-1))^{a_{d-1}} \quad \text{[by Proposition 2.15(1)]}
\]

Hence equality holds.

(3) The proof follows from Proposition 3.4 and Theorem 6.2. \( \square \)

7. Applications

7.1. Cohen-Macaulayness and Gorensteinness of symbolic blowup algebras.

In [16], Goto et al. studied the Gorenstein property of the symbolic Rees algebra. If \( d = 3 \), then \( \text{ht}(\mathfrak{p}) = 2 \) and hence, if \( \mathcal{R}_s(\mathfrak{p}) \) is Cohen-Macaulay, then it is also Gorenstein ([33, Corollary 3.4]). From [15, Theorem 6.7(4)] and Theorem 6.5, it follows that \( G_s(\mathfrak{p}) \) is Cohen-Macaulay. In this paper we give an alternate argument for \( G_s(\mathfrak{p}) \) to be Cohen-Macaulay. In fact, we show that \( G_s(\mathfrak{p}) := \oplus_{n \geq 0} \mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)} \) is Gorenstein for all \( d \geq 2 \) (Theorem 7.1). We also prove that \( \mathcal{R}_s(\mathfrak{p}) \) is Cohen-Macaulay for all \( d \geq 2 \) (Theorem 7.2(2)). Moreover, \( \mathcal{R}_s(\mathfrak{p}) \) is Gorenstein if and only if \( d = 3 \) (Theorem 7.2(3)).

Put \( f_0 = x_1 \). Let \( f_i \)'s be as in (2.3) and let \( f^*_i \) denotes the image of \( f_i \) in \( \mathfrak{p}^{(i)}/\mathfrak{p}^{(i+1)} \). In [15, Proposition 5.3], Goto showed that \( f_{d-1} \) is a homogenous system of parameters in \( G_s(\mathfrak{p}) \). In Theorem 7.1, we show that \( f^*_0, f^*_{d-1} \) is a regular sequence in \( G_s(\mathfrak{p}) \).

**Theorem 7.1.** Let \( d \geq 2 \). Then

1. For all \( d \geq 2 \), \( f^*_0, f^*_{d-1} \) is a regular sequence in \( G_s(\mathfrak{p}) \).
2. \( G_s(\mathfrak{p}) \) is Gorenstein.

**Proof.** We first show that \( G_s(\mathfrak{p}) \) is Cohen-Macaulay. By induction on \( k \), we prove that \( f^*_0, f^*_k \) is a regular sequence in \( G_s(\mathfrak{p}) \) for all \( k = 0, \ldots, d-1 \). Let \( k = 0 \). Then as \( x_1 \) is a nonzerodivisor on \( R/\mathfrak{p}^{(n)} \) for all \( n \), we conclude that \( f^*_0 \) is a nonzerodivisor in \( G_s(\mathfrak{p}) \). Now let \( k \geq 1 \) and assume that \( f^*_0, f^*_{k-1} \) is a regular sequence in \( G_s(\mathfrak{p}) \). Then

\[
\frac{G_s(\mathfrak{p})}{(f^*_0, f^*_{k-1})} \cong \bigoplus_{n \geq 0} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}} + \sum_{j=0}^{k-1} f_j \mathfrak{p}^{(n-j)} \cong \bigoplus_{n \geq 0} \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^{(n+1)}} + (f_0, f_{k-1}).
\]
Hence, to show that $f_k^*$ is a nonzerodivisor on $G_s(p)/(f_0^*, f_{k-1}^*)$ it is enough to show that $((p^{(n+1)}, f_0, f_{k-1}^*) : (f_k)) = (p^{(n+1-k)}, f_0, f_{k-1})$ for all $n \geq k$. Since

$$\ell\left(\frac{R}{(p^{(n+1)}, f_0, f_{k-1}^*) : (f_k)}\right) = \ell\left(\frac{R}{(p^{(n+1)}, f_0, f_k)}\right) - \ell\left(\frac{R}{(p^{(n+1)}, f_0, f_{k-1})}\right)$$

[Theorem 6.5]

$$= \ell\left(\frac{T'}{I_{n+1} + (x_2^2, \ldots, x_k)}\right) - \ell\left(\frac{T'}{I_{n+1} + (x_2^2, \ldots, x_{k+1})}\right)$$

[Proposition 4.1 and [13, Proposition 1.14]]

$$= \ell\left(\frac{T'}{(I_{n+1} + (x_2^2, \ldots, x_k^k)) : (x_{k+1}^{k+1})}\right)$$

$$= \ell\left(\frac{R}{(p^{(n+1-k)}, f_0, f_{k-1})}\right)$$

[Theorem 6.5],

we get $((p^{(n+1)}, f_0, f_{k-1}^*) : (f_k)) = (p^{(n+1-k)}, f_0, f_{k-1})$. This implies that $f_k$ is a nonzerodivisor in $G_s(p)/(f_0^*, f_{k-1}^*)$. Hence $G_s(p)$ is Cohen-Macaulay.

As $G(pR_p)$ is a polynomial ring, it is Gorenstein. Hence by Theorem 6.5 and [15, Corollary 5.8] $G_s(p)$ is Gorenstein.

**Theorem 7.2.** Let $d \geq 2$. Then

1. $\mathcal{R}_s(p) = R[pt, J_2t^2, \ldots, J_{d-1}t^{d-1}]$.
2. $\mathcal{R}_s(p)$ is Cohen-Macaulay.
3. $\mathcal{R}_s(p)$ is Gorenstein if and only if $d = 3$.

**Proof.** (1) The proof follows from Theorem 6.8(2).

(2) By [15, Theorem 6.7], it suffices to show that $\frac{R}{p^{(n)} + (f_{d-1})}$ is Cohen-Macaulay for $1 \leq n \leq \binom{d-1}{2}$. This holds true by Theorem 6.5.

(3) By [15, Lemma 6.1], the a-invariant of $(G_s(p))$, $a(G_s(p)) = -(d-1)$. By [15, Theorem 6.6] and Theorem 7.1, $\mathcal{R}_s(p)$ is Gorenstein if and only if $d = 3$.

**7.2. Computation of resurgence.**

In [3] C. Bocci and B. Harbourne defined the *resurgence* of an ideal $I$ in $R$ as

$$\rho(I) := \sup \left\{ \frac{n}{r} : f^{(n)} \notin I^r \right\}.$$
We can also compute the resurgence in the following way: For any ideal $I \subseteq R$ let $\rho_n(I) := \min \{ r : I^{(n)} \nsubseteq I^r \}$. Then

$$\rho(I) := \sup \left\{ \frac{n}{\rho_n(I)} : n \geq 1 \right\}.$$ 

In this subsection we explicitly describe the resurgence of $\mathfrak{p} = \mathcal{I}_{(n_1,n_2,n_3)}$.

From (2.1) we have

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1^{m+1} \\ x_3 & x_1^{m+1} & x_1^m x_2 \end{pmatrix}.$$ Put

$$\Delta_1 := \det(X_{2,(2,3)}), \quad \Delta_2 := \det(X_{2,(1,3)}) \quad \text{and} \quad \Delta_3 := \det(X_{2,(1,2)}).$$

(7.3)

Let $f_2$ be as in (2.3).

**Lemma 7.4.** With the above notation:

1. For all $i = 1, 2, 3$, $x_if_2 \in \mathfrak{p}^2$.
2. $f_2^2 \in \mathfrak{p}^3$.

**Proof.** (1) One can verify that

$$x_1f_2 = -\Delta_2^2 + \Delta_1 \Delta_3$$
$$x_2f_2 = -x_1^m \Delta_2^2 - \Delta_1 \Delta_2$$
$$x_3f_2 = -\Delta_1^2 - x_1^m \Delta_2 \Delta_3.$$ As $\Delta_j \in \mathfrak{p}$ for all $j = 1, 2, 3$, we get $x_if_2 \in \mathfrak{p}^2$ for all $i = 1, 2, 3$.

(2) We have

$$f_2^2 = (x_3 \Delta_1 - x_1^{m+1} \Delta_2 + x_1^m x_2 \Delta_3)f_2$$
$$= \Delta_1(x_3f_2) - \Delta_2(x_1^{m+1}f_2) + \Delta_3(x_1^m x_2 f_2)$$
$$\in \mathfrak{p}^2$$ \hspace{1cm} \text{[from (1)]}$$
$$= \mathfrak{p}^3.$$ \hfill \Box

**Proposition 7.5.** Let $k \geq 0$. Then

$$\rho_n(\mathfrak{p}) = \begin{cases} 3k + 1 & \text{if } n = 4k \\ 3k + 2 & \text{if } n = 4k + 1 \\ 3k + 2 & \text{if } n = 4k + 2 \\ 3k + 3 & \text{if } n = 4k + 3 \end{cases}$$

**Proof.** From Theorem 6.8(1) and Theorem 6.8(2) we get

$$\mathfrak{p}^{(2)} = \mathfrak{p}^2 + (f_2), \quad \mathfrak{p}^{(2n)} = (\mathfrak{p}^{(2)})^n \quad \text{and} \quad \mathfrak{p}^{(2n+1)} = \mathfrak{p}\mathfrak{p}^{(2n)}. \hspace{1cm} (7.6)$$
From (7.6) and Lemma 7.4 we get
\[ p^{(4k)} = (p^{(4)})^k = ((p^2 + (f_2))^2)^k = (p^4 + f_2p^2 + (f_2)^2)^k \subseteq (p^3)^k = p^{3k} \]
\[ p^{(4k+1)} = pp^{(4k)} \subseteq pp^{3k} = p^{3k+1} \]
\[ p^{(4k+2)} = p^{(2)}p^{(4k)} \subseteq pp^{3k} = p^{3k+1} \]
\[ p^{(4k+3)} = p^{(2)}p^{(4k)} \subseteq p^2p^{3k} = p^{3k+2} \].

As \( f_2 \equiv x_3^3( \mod x_1), \Delta_3 \equiv x_3^2( \mod x_1) \) and \( p \equiv (x_2, x_3)^2( \mod x_1) \)
\[ f_2^{2k} \equiv x_3^{6k} \in p^{(4k)} \setminus p^{3k+1}( \mod x_1) \]
\[ \Delta_1 f_2^{2k} \equiv x_3^{6k+2} \in p^{(4k+1)} \setminus p^{3k+2}( \mod x_1) \]
\[ f_2^{2k+1} \equiv x_3^{6k+3} \in p^{(4k+2)} \setminus p^{3k+2}( \mod x_1) \]
\[ \Delta_1 f_2^{2k+1} \equiv x_3^{6k+5} \in p^{(4k+3)} \setminus p^{3k+3}( \mod x_1) \].

This completes the proof.

**Theorem 7.7.** \( \rho(p) = \frac{4}{3} \).

**Proof.** By Proposition 7.5
\[ \rho(p) = \sup \left\{ \frac{4k}{3k+1}, \frac{4k+1}{3k+2}, \frac{4k+2}{3k+2}, \frac{4k+3}{3k+3} : k \geq 0 \right\} = \frac{4}{3} \]

\[ 7.3. \textbf{Waldschmidt Constant}. \text{ Consider the polynomial ring } T = \mathbb{k}[x_1, x_2, x_3] \text{ with weights } d_i = \text{wt}(x_i) \text{ where } d_1 = 3, d_2 = 3 + m \text{ and } d_3 = 3 + 2m. \text{ With these weights, } p^n \text{ and } p^{(n)} \text{ are weighted homogenous ideals. For any weighted homogenous ideal } I \subseteq T, \text{ let } \alpha(I) := \min \{n | I_n \neq 0 \}. \text{ Recall that the Waldschmidt constant is defined as } \]
\[ \gamma(I) = \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n}. \]

In this section we compute \( \alpha(p)/\gamma(p) \) and compare it with \( \rho(p) \). We obtain similar results as in [3, Theorem 1.2.1] and [3, Lemma 2.3.2].

**Theorem 7.8.**
1. \( \alpha(p) = 2m + 6 \)
2. \( \gamma(p) = \begin{cases} 
15/2 & \text{if } m = 1 \\
2m + 6 & \text{if } m > 1.
\end{cases} \)

**Proof.** Note that \( p = (\Delta_1, \Delta_2, \Delta_3) \) where \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) are as defined in (7.3). Then \( \text{deg}(\Delta_1) = 4m + 6, \text{deg}(\Delta_2) = 3m + 6, \text{deg}(\Delta_3) = 2m + 6, \text{and } \text{deg}(f_2) = 6m + 9. \text{ Hence } \alpha(p) = 2m + 6. \)

We now compute \( \alpha(p^{(n)}) \). Then from (7.6) we get
\[ \alpha(p^{(2n)}) = \begin{cases} 
n \text{deg}(f_2) = 15 \text{ if } m = 1 \\
2n \text{deg}(\Delta_3) = 2n(2m + 6) \text{ if } m > 1
\end{cases} \]
\[ \alpha(p^{(2n+1)}) = \begin{cases} 
15n + 8 = n \text{deg}(f_2) + \text{deg}(\Delta_3) \text{ if } m = 1 \\
(2n + 1) \text{deg}(\Delta_3) = (2n + 1)(2m + 6) \text{ if } m > 1.
\end{cases} \]
Hence
\[
\gamma(p) = \lim_{n \to \infty} \frac{\alpha(p^{(n)})}{n} = \begin{cases} 
15/2 & \text{if } m = 1 \\
2m + 6 & \text{if } m > 1.
\end{cases}
\]

\[\square\]

**Theorem 7.9.**
\[1 \leq \frac{\alpha(p)}{\gamma(p)} \leq \rho(p).\]

**Proof.** By Theorem 7.8,
\[
\frac{\alpha(p)}{\gamma(p)} = \begin{cases} 
\frac{8}{15/2} = \frac{16}{15} & \text{if } m = 1 \\
\frac{2m+6}{2m+6} = 1 & \text{if } m > 1.
\end{cases}
\]

By Theorem 7.7, the result follows. \[\square\]

7.4. **Regularity.** In [8], S. D. Cutkosky and K. Kurano studied the regularity of saturated ideals in a weighted projective space.

In this subsection we consider the polynomial ring \( T = \mathbb{k}[x_1, x_2, x_3] \) with weights \( d_i = wt(x_i) \), where \( d_1 = 3, d_2 = 3 + m \) and \( d_3 = 3 + 2m \). With these weights, \( p^{(n)} \) is a weighted homogenous ideal. We compute the regularity of \( p^{(n)} \) for all \( n \geq 1 \).

We begin with some basic results comparing \( p^{(n)}T' \) and \( I_nT' \).

**Lemma 7.10.** For all \( n \geq 1 \),

1. \( p^{(n)}T' = I_nT' \).
2. \( I_{2n}T' = I_2^{n}T' \).
3. \( I_{2n+1}T' = I_2I_{2n}T' \).

**Proof.** Since \( J_iT' = J_i' \) for \( i = 1, 2 \), we get \( I_nT' = I_nT' \) for all \( n \geq 1 \). Hence from Theorem 6.8(1), \( p^{(n)}T' = I_nT' = I_nT' \).

(2) and (3) follow from (1) and (7.6) \[\square\]

**Lemma 7.11.** For all \( n \geq 1 \), \( \text{reg}(T/p^{(n)}T) = \text{reg}(T'/I_n) \).

**Proof.** As \( x_1 \) is a nonzerodivisor on \( T/p^{(n)} \) and \( T/I_nT \),
\[
\text{reg} \left( \frac{T'}{I_n} \right) = \text{reg} \left( \frac{T}{I_n} \right) = \text{reg} \left( \frac{T}{I_n + (x_1)} \right) - 2 \quad [5, \text{Remark 4.1}] \\
= \text{reg} \left( \frac{T}{p^{(n)} + (x_1)} \right) - 2 \\
= \text{reg} \left( \frac{T}{p^{(n)}} \right) \quad [5, \text{Remark 4.1}].
\]
Let $F_*$ be a minimal free resolution of $T'/p^{(n)}T'$. Since $T$ is a free $T'$-module, $F_* \otimes_{T'} T$ is a minimal free resolution of $T'/p^{(n)}T' \otimes_{T'} T \cong T/p^{(n)}T$. Hence $\text{reg}(T/p^{(n)}T) = \text{reg}(T'/p^{(n)}T')$. By Lemma 7.10(1), $p^{(n)}T' = I_nT'$.

It follows from Lemma 7.11, that in order to compute the regularity of $T/p^{(n)}$, it is enough to compute the regularity of $T'/I_n$.

**Lemma 7.12.** Let $n \geq 1$. Then

$$\text{reg} \left( \frac{T'}{I_n + (x_2^3)} \right) = \begin{cases} \frac{3d}{2} n + 2d_2 - 2 & \text{if } n \text{ is even} \\ \frac{3d}{2} n + d_2 + \frac{d_3}{2} - 2 & \text{if } n \text{ is odd} \end{cases}$$

**Proof.** Let $n = 2r$, where $r \geq 1$. Then by Lemma 7.10(2),

$$(I_{2r} + (x_2^3))T' = I_{2r}T' + (x_2^3)T' = (x_2^4, x_2^3x_3, x_2^2x_3^2, x_3^3)T' + (x_2^3)T' = (x_2^2, x_3^3)T'.$$

Hence

$$\text{reg} \left( \frac{T'}{I_{2r} + (x_2^3)} \right) = 3rd_3 + 2d_2 - 2 = \frac{3d_3}{2} n + 2d_2 - 2.$$

Let $n = 2r - 1$, where $r \geq 1$. Then by Lemma 7.10(3) we get

$$(I_{2r-1} + (x_2^3))T' = (I_1 + (x_2^3))(I_{2(r-1)} + (x_2^3))T' + (x_2^3)T'
= (x_2^2, x_2x_3, x_3^2)(x_2^2, x_3^{2(r-1)})T' + (x_2^3)T' \quad \text{[by (7.13)]}
= (x_2^2, x_2x_3^{3r-2}, x_3^{3r-1})T'.$$

By Hilbert-Burch theorem the minimal free resolution of $(I_{2r-1} + (x_2^3))T'$ is

$$0 \rightarrow T'[-2d_2 - (3r - 2)d_3] \oplus T'[-d_2 - (3r - 1)d_3] \rightarrow T'[-2d_2] \oplus T'[-d_2 - (3r - 2)d_3] \rightarrow T' \rightarrow I_{2r-1} + (x_2^3) \rightarrow 0.$$

This gives

$$\text{reg} \left( \frac{T'}{I_{2r-1} + (x_2^3)} \right) = (3r - 1)d_3 + d_2 - 2 = \frac{3d_3}{2} n + d_2 + \frac{d_3}{2} - 2.$$

**Lemma 7.14.** For all $n \geq 1$,

$$\text{reg} \left( \frac{T'}{I_{2n} + (x_3^3)} \right) = 2d_2(2n) - 2d_2 + 3d_3 - 2.$$

**Proof.** By Lemma 7.10(2) we get

$$I_{2n} + (x_3^3) = I_2^n + (x_3^3) = (x_2, x_3)^{4n} + (x_3^3) = (x_2^{4n}, x_2^{4n-1}x_3, x_2^{4n-2}x_3^2, x_3^3).$$
Hence by Hilbert-Burch theorem the minimal free resolution of $I_{2n} + (x_3^3)$ is

$$
T'[-(4n - 1)d_2 - 2d_3] \oplus T'[-4nd_2] \oplus T'[-(4n - 1)d_2 - d_3] \oplus T'[-(4n - 2)d_2 - 2d_3] \oplus T'[-3d_3] \rightarrow T'[-(4n - 1)d_2 - 2d_3] \oplus T'[-4nd_2] \oplus T'[-(4n - 2)d_2 - 2d_3] \oplus T'[-3d_3] \rightarrow T' \rightarrow I_{2n} + (x_3^3) \rightarrow 0.
$$

This gives $\text{reg}(T/I_{2n} + (x_3^3)) = 3d_3 + (4n - 2)d_2 - 2 = 2d_2(2n) - 2d_2 + 3d_3 - 2.$ \hfill \Box

**Proposition 7.15.** Let $n \geq 1$. Then

$$
\text{reg} \left( \frac{T'}{I_{2n}} \right) = \begin{cases} 
(2d_2)(2n) - 2d_2 + 3d_3 - 2 & \text{if } m = 1, \\
\frac{3d_3}{2}(2n) + 2d_2 - 2 & \text{if } m \geq 2.
\end{cases}
$$

**Proof.** For all $n \geq 1$, the sequence

$$
0 \rightarrow T'[-3d_3] \rightarrow T'[-6d_3] \rightarrow T' \rightarrow I_{2n} + (x_3^3) \rightarrow 0
$$

is exact by Proposition 4.1. Hence

$$
\text{reg} \left( \frac{T'}{I_{2n}} \right) = \max \left\{ \text{reg} \left( \frac{T'}{I_{2n} - 2i} \right) + 3d_3, \text{reg} \left( \frac{T'}{I_{2n} + (x_3^3)} \right) \right\}
$$

$$
= \max \left\{ \text{reg} \left( \frac{T'}{I_{2n} - 4} \right) + 6d_3, \text{reg} \left( \frac{T'}{I_{2n} - 2 + (x_3^3)} \right) + 3d_3, \text{reg} \left( \frac{T'}{I_{2n} + (x_3^3)} \right) \right\}
$$

$$
= \vdots
$$

$$
= \max \left\{ \text{reg} \left( \frac{T'}{I_{2n - 2i}} \right) + 3id_3 \mid i = 0, \ldots, n - 1 \right\}
$$

$$
= \max \{ 2d_2(2n - 2i) - 2d_2 + 3d_3 - 2 + 3id_3 \mid i = 0, \ldots, n - 1 \} \ \text{[ by Lemma 7.14]}
$$

$$
= \max \{ 4nd_2 - 2d_2 + 3d_3 - 2 + i(-4d_2 + 3d_3) \mid i = 0, \ldots, n - 1 \}
$$

$$
= \begin{cases} 
(2d_2)(2n) - 2d_2 + 3d_3 - 2 & \text{if } m = 1, \\
(2d_2)(2n) - 2d_2 + 3d_3 - 2 + (n - 1)(-4d_2 + 3d_3) & \text{if } m \geq 2.
\end{cases}
$$

$$
= \begin{cases} 
(2d_2)(2n) - 2d_2 + 3d_3 - 2 & \text{if } m = 1, \\
\frac{3d_3}{2}(2n) + 2d_2 - 2 & \text{if } m \geq 2.
\end{cases}
$$

\hfill \Box

**Proposition 7.16.** Let $n \geq 1$. Then

$$
\text{reg} \left( \frac{T'}{I_{2n + 1}} \right) = \begin{cases} 
(2d_2)(2n + 1) - 2d_2 + 3d_3 - 2 & \text{if } m = 1, \\
\frac{3d_3}{2}(2n + 1) + 4d_2 - 3d_3 - 2 & \text{if } m = 2, \\
\frac{3d_3}{2}(2n + 1) + d_2 + \frac{3d_3}{2} - 2 & \text{if } m \geq 3.
\end{cases}
$$
Proof. For all \( n \geq 1 \), the sequence

\[
\begin{array}{c}
0 \longrightarrow T'/I_{2n} \xrightarrow{\frac{x_2^2}{2}} T'/I_{2n+1} \longrightarrow T'/I_{2n+1} + (x_2^2) \longrightarrow 0
\end{array}
\]

is exact by Proposition 4.1. Hence

\[
\text{reg} \left( \frac{T'}{I_{2n+1}} \right) = \max \left\{ \text{reg} \left( \frac{T'}{I_{2n}} \right) + 2d_2, \text{reg} \left( \frac{T'}{I_{2n+1} + (x_2^2)} \right) \right\}.
\]

(7.17)

Using Proposition 7.15 and Lemma 7.12 in (7.17) we get

\[
\text{reg} \left( \frac{T'}{I_{2n+1}} \right) = \begin{cases} 
\max \left\{ (2d_2)(2n+1) - 2d_2 + 3d_3 - 2, \frac{3d_3}{2}(2n+1) + d_2 + \frac{d_3}{2} - 2 \right\} & \text{if } m = 1 \\
\max \left\{ \frac{3d_3}{2}(2n+1) + 4d_2 - \frac{3d_3}{2} - 2, \frac{3d_3}{2}(2n+1) + d_2 + \frac{d_3}{2} - 2 \right\} & \text{if } m \geq 2 \\
(2d_2)(2n+1) - 2d_2 + 3d_3 - 2 & \text{if } m = 1 \\
\frac{3d_3}{2}(2n+1) + 4d_2 - \frac{3d_3}{2} - 2 & \text{if } m = 2 \\
\frac{3d_3}{2}(2n+1) + d_2 + \frac{d_3}{2} - 2 & \text{if } m \geq 3
\end{cases}
\]

In particular, \( \lim_{n \to \infty} \text{reg}((p^n)^{sat})/n = \frac{3\epsilon(T/p)}{2} + 3m \).

Proof. By Lemma 7.11, \( \text{reg}(T/p^{(n)}) = \text{reg}(T'/I_n) \). Since \( I_1 + (x_2^2) = I_1 \), (1) from Lemma 7.12. (2) follows from Proposition 7.15 and Proposition 7.16.

Finally, \( \lim_{n \to \infty} \text{reg}((p^n)^{sat})/n = \frac{3d_3}{2} = \frac{3\epsilon(T/p) + 2m}{2} = \frac{3\epsilon(T/p)}{2} + 3m \)

(7.18)

Theorem 7.18. (1) \( \text{reg}(T/p) = d_2 + 2d_3 - 2 \).

(2) Let \( n \geq 2 \).

(a) If \( m = 1 \), then \( \text{reg}(T/p^{(n)}) = (2d_2)n - 2d_2 + 3d_3 - 2 \).

(b) If \( m = 2 \), then \( \text{reg} \left( \frac{T}{p^{(n)}} \right) = \begin{cases} 
\frac{3d_3}{2}n + 4d_2 - \frac{3d_3}{2} - 2 & \text{if } n \text{ is odd,} \\
\frac{3d_3}{2}n + 2d_2 - 2 & \text{if } n \text{ is even.}
\end{cases} 
\]

(c) If \( m \geq 3 \), then \( \text{reg} \left( \frac{T}{p^{(n)}} \right) = \begin{cases} 
\frac{3d_3}{2}n + d_2 + \frac{d_3}{2} - 2 & \text{if } n \text{ is odd} \\
\frac{3d_3}{2}n + 2d_2 - 2 & \text{if } n \text{ is even.}
\end{cases} 
\]

Proof. By Theorem 7.18 and Theorem 7.8(2)

\[
\frac{\text{reg}(p)}{\gamma(p)} = \begin{cases} 
\frac{13}{15}/2 \geq \frac{26}{15} \geq \frac{4}{3} = \text{reg}(p) & \text{if } m = 1 \\
\frac{3m + (9/2)}{2m + 6} \geq \frac{4}{3} = \text{reg}(p) & \text{if } m \geq 2.
\end{cases}
\]
References

[1] D. Bayer and M. Stillman, *Computation of Hilbert functions*, J. Symbolic Comput. 14 (1992), 31-50. 19, 20

[2] C. Bocci and B. Harbourne, *The resurgence of ideals of points and the containment problem*, Proc. Amer. Math. Soc. 138 (2010), no. 4, 1175-1190. 2

[3] C. Bocci and B. Harbourne, *Comparing powers and symbolic powers of ideals*, J. Algebraic Geom. 19 (2010), no. 3, 399-417. 2, 22, 24

[4] C. Bocci, S. Cooper, E. Guardo, B. Harbourne, M. Janssen, U. Nagel, A. Seceleanu, A. Van Tuyl and Thanh Vu, *The Waldschmidt constant for squarefree monomial ideals*, J. Algebraic Combin. 44 (2016), no. 4, 875-904. 2

[5] M. Chardin, *Some results and questions on Castelnuovo-Mumford regularity, Syzygies and Hilbert functions*, Lect. Notes Pure Appl. Math., 254, Chapman & Hall/CRC, Boca Raton, FL, 2007, 1-40. 25

[6] R. C. Cowsik, *Symbolic powers and number of defining equations*, Algebra and its applications (New Delhi, 1981), 13-14, Lecture Notes in Pure and Appl. Math., 91, Dekker, New York, 1984. 1

[7] S. D. Cutkosky, *Irrational asymptotic behaviour of Castelnuovo-Mumford regularity*, J. Reine Angew. Math. 522 (2000), 93-103.

[8] S. D. Cutkosky and K. Kurano, *Asymptotic regularity of powers of ideals of points in a weighted projective plane*, Kyoto J. Math. 51 (2011), no. 1, 25-45. 3, 25

[9] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg and H. Tutaj-Gasińska, *Resurgences for ideals of special point configurations in $\mathbb{P}^N$ coming from hyperplane arrangements*, J. Algebra 443 (2015), 383-394. 2

[10] L. Ein, R. Lazarsfeld and K. E. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. 144 (2001), no. 2, 241-252. 2

[11] D. Eisenbud and B. Mazur, *Evolutions, symbolic squares, and Fitting ideals*, J. Reine Angew. Math. 488 (1997), 189-201. 2

[12] S. Eliahou, *Courbes monomiales et algebre de Rees Symbolique*, Thèse, Université de Genève, 1983. 2

[13] V. Ene and J. Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, 130 American Mathematical Society, (2012). 10, 13, 14, 22

[14] G. Fatabbi, B. Harbourne and A. Lorenzini, *Inductively computable unions of fat linear subspaces*, J. Pure Appl. Algebra 219 (2015), no. 12, 5413-5425. 2

[15] S. Goto, *The Gorensteinness of symbolic Rees algebras for curve singularities. The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*, Mem. Amer. Math. Soc. 110 (1994), no. 526, 1-68. 1, 2, 3, 4, 5, 6, 7, 17, 20, 21, 22

[16] S. Goto, K. Nishida and Y. Shimoda, *The Gorensteinness of symbolic Rees algebras for space curves*, J. Math. Soc. Japan 43 (1991), 465-481. 2, 21

[17] S. Goto, K. Nishida and Y. Shimoda, *Topics on symbolic Rees algebras for space monomial curves*, Nagoya Math. J. 124 (1991), 99-132. 2

[18] S. Goto, K. Nishida and K. Watanabe, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik’s question*, Proc. Amer. Math. Soc. 120 (1994), no. 2, 383-392. 1, 2

[19] E. Guardo, B. Harbourne, and A. Van Tuyl, *Asymptotic resurgences for ideals of positive dimensional subschemes of projective space*, Adv. Math. 246 (2013), 114-127. 2

[20] J. Herzog and B. Ulrich, *Self linked curve singularities*, Nagoya Math. J. 120 (1990), 129-153. 2

[21] M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), no. 2, 349-369. 2

[22] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. 56 (1997), no. 1, 64-76. 4

[23] C. Huneke, *The Primary components of and integral closures of ideals in 3-dimensional regular local rings*, Math. Ann. 275 (1986), no. 4, 617-635. 2

[24] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. 34 (1987), no. 2, 293-318. 1

[25] G. Knödel, P. Schenzel and R. Zonsarow, *Explicit computations on symbolic powers of monomial curves in affine space*, Comm. Algebra 20 (1992), no. 7, 2113-2126. 2
[26] L. Kronecker, Grundzüge einer arithmetischen Theorie der algebraische Grössen, J. Reine Angew. Math. 92 (1882), 1-122.

[27] K. Kurano and N. Matsuoka, On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves, J. Algebra 322 (2009), no. 9, 3268-3290.

[28] H. Matsumura, Commutative ring theory, translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, 8 Cambridge University Press, Cambridge, (1989).

[29] M. Morales, Noetherian symbolic blow-ups, J. Algebra 140 (1991), no. 1, 12-25.

[30] M. Reed, Generation in degree four of symbolic blowups of self-linked monomial space curves, Comm. Algebra 37 (2009), no. 12, 4346-4365.

[31] P. Schenzel, Examples of Noetherian symbolic blow-up rings, Rev. Roumaine Math. Pures Appl. 33 (1988), no. 4, 375-383.

[32] Š. Solčan, On the computation of symbolic powers of some curves in \( \mathbb{A}^4 \), Acta Math. Univ. Comenian. (N.S.) 69 (2000), no. 1, 85-95.

[33] A. Simis and N. V. Trung, The divisor class group of ordinary and symbolic blow-ups, Math. Z. 198 (1988), 479-491.

[34] W. V. Vasconcelos, The structure of certain ideal transforms, Math. Z. 198 (1988), 435-448.

[35] W. V. Vasconcelos, Symmetric algebras and factoriality, Commutative Algebra (M. Hochster, C. Huneke and J. D. Sally, Eds.), Springer-Verlag, New York, 1989, 467-496.

Chennai Mathematical Institute, Plot H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, Tamil Nadu, India

E-mail address: clare@cmi.ac.in

Chennai Mathematical Institute, H1-SIPCOT IT Park, Siruseri, Kelambakkam - 603 103, India

E-mail address: shreedevikm@cmi.ac.in