Smooth Gowdy-symmetric generalized Taub–NUT solutions

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Abstract
We study a class of $S^3$-Gowdy vacuum models with a regular past Cauchy horizon which we call smooth Gowdy-symmetric generalized Taub–NUT solutions. In particular, we prove the existence of such solutions by formulating a singular initial value problem with asymptotic data on the past Cauchy horizon. We prove that also a future Cauchy horizon exists for generic asymptotic data, and derive an explicit expression for the metric on the future Cauchy horizon in terms of the asymptotic data on the past horizon. This complements earlier results about $S^1 \times S^2$-Gowdy models.

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1. Introduction

Studies of cosmological solutions of Einstein’s field equations have a long tradition and led to astonishing results about our own Universe. In particular, observations indicate that there was a big bang in the distant past, and indeed, the simplest cosmological models, namely the Friedmann solutions for reasonable matter fields, predict precisely this behavior. The question arises as to whether such curvature singularities occur for generic solutions of Einstein’s field equations or if strong symmetry assumptions, e.g. those made for the Friedmann models, are necessary for this. The Hawking–Penrose singularity theorems [26] shed some light on this question. They predict incompleteness of causal geodesics in a wide class of situations. However, the information about the geometric reason for the incompleteness which is provided by these theorems is very limited, and it is indeed not always caused by a geometric singularity.

Let us restrict all of our investigations to vacuum with a vanishing cosmological constant and to four spacetime dimensions. The corresponding Cauchy problem for Einstein’s field equations is well posed and leads to the notion of the maximal globally hyperbolic development (MGHD) of a given Cauchy data set [3, 45]. The example of the Taub solution [47], which we discuss in more detail below, shows that incompleteness of causal geodesics, as predicted by the singularity theorems, can signal a different kind of phenomenon; in particular, it is possible to extend the MGHD. The extended solutions are, however, not globally hyperbolic. There exist closed causal curves and indeed, there are several non-equivalent extensions. This unexpected property has caused an ongoing debate in the literature. Are such pathological phenomena a typical feature of Einstein’s theory of gravity—in which case it could not be considered as a ‘proper’ physical theory—or, again, do such phenomena only occur under very strong and special assumptions, for example the high symmetry of the Taub solutions?

An interesting hypothesis in this context is the strong cosmic censorship conjecture whose widely accepted formulation was given for the first time in [18], based on ideas by Eardly and Moncrief [31] and Penrose [41]. More details and references can be found in [45, 43]. This conjecture states that, for spatially compact or asymptotically flat spacetimes, the MGHD of generic Cauchy data is inextendible. If this were true, it would imply that, in the generic situation, incompleteness of causal geodesics is indeed caused by a geometric singularity in some sense, while the pathologies encountered for the Taub spacetimes only occur in special circumstances. At this stage, however, this conjecture has not been confirmed in general situations; see e.g. [45].

In his effort to generalize the family of Taub solutions and hence to show that there is a large (but presumably still non-generic) class of solutions of the field equations with similar ‘undesired’ properties, Moncrief defines the family of generalized Taub–NUT spacetimes in [32]. He is able to prove an existence result under an analyticity assumption. The motivation in our paper is twofold. First, we want to extend this existence result to the smooth case and formulate it as a singular initial value problem with ‘asymptotic data’ on the Cauchy horizon. Then our second motivation is to study the global dynamics of such solutions. To make this

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1 This conjecture, of course, only makes sense if one is able to give a precise and reasonable meaning to the terms ‘generic’ and ‘inextendible’.

2 We use the term ‘smooth’ for infinitely differentiable objects, as opposed to (real) ‘analytic’, which entails the convergence of the Taylor series.
feasible, we restrict to the case with Gowdy symmetry. By the means of so-called soliton methods, we find that the existence of a past Cauchy horizon generically implies the existence of a future Cauchy horizon, at least under certain topological restrictions on the horizons. The reader should compare these results to the $S^1 \times S^2$ case in [28].

Note that the global existence of Gowdy-symmetric solutions with Cauchy horizons, which is our main result, does not necessarily imply that they are generic (and thus violate strong cosmic censorship). Indeed, cosmic censorship remains a fundamental open problem in GR and studying existence and properties of solutions with acausal extensions is of crucial importance for a deeper understanding of the mathematical structure of the Einstein equations and their solutions. Our considerations are meant to contribute some insights in that direction.

The paper is organized as follows. In section 2 we discuss some background material, in particular the symmetry reduction introduced by Geroch which is later applied to write the metric and the field equations in a useful way. Section 3 is devoted to the definition and discussion of our class of smooth Gowdy-symmetric generalized Taub–NUT solutions based on Moncrief’s earlier mentioned class. The existence and uniqueness theory and the corresponding singular initial value problem are considered in section 3.3. The basic ingredients for these investigations are Fuchsian methods which we describe in the appendix. The next main part is section 4 where we discuss the global-in-time properties of smooth Gowdy-symmetric generalized Taub–NUT solutions. We finish the paper with conclusions in section 5.

2. Geometric preliminaries

2.1. General symmetry reduction by Geroch

We briefly present here the symmetry reduction introduced by Geroch in [24]. Let $M = \mathbb{R} \times H$ be an oriented and time-oriented globally hyperbolic four-dimensional spacetime endowed with a metric $g_{ab}$ of signature $(-, +, +, +)$, a global time-function $t$ and a Cauchy surface $H$. We denote the volume form of $g_{ab}$ by $\epsilon_{abcd}$ and the hypersurfaces given by $t = t_0$ for any constant $t_0$ by $H_{t_0}$. Each $H_{t_0}$ is homeomorphic to $H$.

Now, let $\xi$ be a smooth space-like Killing vector field which is tangent to the hypersurfaces $H_t$ and set

$$\lambda := g(\xi, \xi).$$

The twist 1-form of $\xi$ is

$$\Omega^a := \epsilon_{abcd} \xi^b \nabla^c \xi^d,$$

where $\nabla$ is the covariant derivative compatible with $g$. The field $\xi$ is hypersurface orthogonal if and only if $\Omega_a \equiv 0$, which will, however, not be assumed in the following. We define the ‘3-metric’:

$$h_{ab} := g_{ab} - \frac{1}{\lambda} \xi_a \xi_b,$$

on $M$, and, by raising indices with the inverse of $g$, we define also $h^a_b$ and $h^{ab}$ on $M$. The first of these tensors is the projector to the space orthogonal to $\xi$ in $T_p M$, $p \in M$. From the volume form $\epsilon_{abcd}$ of $g$, we furthermore introduce

$$\epsilon_{abc} := \frac{1}{\sqrt{\lambda}} \epsilon_{abcd} \xi^d.$$

3 Depending on the context, we denote any tensor field either by a symbol like $T$, or we use the abstract index notation $\Gamma^{ab}_{cd}$, where $a, b, c, d, \ldots$ denote abstract indices.

4 Geroch considers the case of a time-like Killing vector field. As a consequence of this, some signs in certain expressions are different than those in Geroch’s reference.
Let $\alpha_a$ be any 1-form. One defines the derivative operator $D$ as

$$D_{a\theta}\alpha_b := h^{\alpha}_{\phantom{\alpha}ab} h^{\beta}_{\phantom{\beta}\theta\gamma} \nabla_\gamma \alpha_\beta.$$  

Note that at this stage we are only interested in local patches of $M$. Then, the flow generated by $\xi$ induces a map $\pi$ from $M$ to the space of orbits $S$, i.e. $\pi$ maps every $p \in M$ to the (locally) uniquely determined integral curve of $\xi$ starting at $p$. The requirement that $\pi$ is a smooth map induces a differentiable structure on $S$, and hence $S$ can be considered as a smooth manifold. The relations $L\xi^a h = 0$ and $h(\xi, \cdot) = 0$, are used by Geroch to show that there is a unique metric on $S$ which pulls back to $h_{ab}$ along $\pi$. We write this metric on $S$ again as $h_{ab}$. The same can be done for $h^a_b$ and $h^{ab}$, which are henceforth considered as objects on the quotient space $S$. The first becomes the identity operator on $T_pS$, while the second is the inverse of the 3-metric $h_{ab}$ on $S$. We can proceed in the same way with the function $\lambda$, the 1-form $\Omega\pi$, the tensor $\epsilon_{abc}$ and the covariant derivative operator $D_a$, which can henceforth all be considered as objects on $S$. Then $\epsilon_{abc}$ becomes the volume form of $h_{ab}$ and $D_a$ the covariant derivative operator compatible with $h_{ab}$. Geroch shows that Einstein's vacuum field equations on $(M, g)$ imply that the 1-form $\Omega$ is closed, $d\Omega = 0$. At this stage we focus on local considerations, which allows us to introduce a twist potential $\omega$ so that $\Omega = d\omega$. The quantities $\lambda, \omega$ and $h_{ab}$ on $S$ therefore completely characterize the local geometry of $(M, g)$.

Now Geroch introduces a conformal rescaling

$$\hat{h}_{ab} := \lambda h_{ab}.$$  

We refer to the associated covariant derivative operator as $\hat{D}_a$, Ricci tensor as $\hat{R}_{ab}$ etc. Geroch shows that the vacuum field equations for $(M, g)$ (and certain geometric identities) are equivalent to the following set of equations:

$$\hat{D}_\lambda \hat{D}^\lambda = \frac{1}{\lambda} (\hat{D}^\rho \hat{D}_\rho \lambda - \hat{D}^\rho \omega \hat{D}_\rho \omega),$$  

$$\hat{D}_\lambda \hat{D}^\omega = \frac{2}{\lambda} \hat{D}^\rho \lambda \hat{D}_\rho \omega,  

(1)$$  

$$\hat{S}_{ab} = \frac{1}{2\lambda^2} (\hat{D}_a \lambda \hat{D}_b \omega + \hat{D}_a \omega \hat{D}_b \lambda).$$  

(2)$$  

(3)$$  

These equations are the Euler–Lagrange equations of the Lagrangian density

$$\mathcal{L} = \sqrt{-\hat{h}} \left[ \hat{S} + \frac{1}{2\lambda^2} (\hat{D}^\rho \lambda \hat{D}_\rho \omega + \hat{D}^\rho \omega \hat{D}_\rho \lambda) \right].$$  

Hence, the equations can be interpreted as $(2 + 1)$-dimensional gravity on $S$ coupled to a wave map $u : S \to \mathcal{H}$ where $\mathcal{H}$ is the two-dimensional hyperbolic space represented by the components $(\lambda, \omega)$.

2.2. Symmetry reduction for spacetimes of spatial $3$-sphere topology

We think of $S^3$ as the submanifold of $\mathbb{R}^4$ determined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The Euler coordinates $(\theta, \lambda_1, \lambda_2)$ of $S^3$ are

$$x_1 = \cos \frac{\theta}{2} \cos \lambda_1, \quad x_2 = \cos \frac{\theta}{2} \sin \lambda_1, \quad x_3 = \sin \frac{\theta}{2} \cos \lambda_2, \quad x_4 = \sin \frac{\theta}{2} \sin \lambda_2,$$

with $\theta \in (0, \pi)$ and $\lambda_1, \lambda_2 \in (0, 2\pi)$. Clearly, these coordinates break down at the points $\theta = 0$ and $\pi$ which we refer to as 'poles' or 'axes' of $S^3$ in the following. We also make use of the coordinates $(\rho_1, \rho_2)$ (which we also call Euler coordinates) with $\theta$ as above and

$$\lambda_1 =: (\rho_1 + \rho_2)/2, \quad \lambda_2 =: (\rho_1 - \rho_2)/2.$$  

(4)
Note that the coordinate fields $\partial_{\rho}$ and $\partial_{\phi}$ are smooth non-vanishing vector fields on $S^3$, while the smooth fields $\partial_{\lambda}$ and $\partial_{\theta}$ vanish at certain places. We remark that these fields can be characterized geometrically (without making reference to coordinates) in terms of left- and right-invariant vector fields of the standard action of $SU(2)$ on $S^3$, see for example [10, 11].

Now we specialize the general discussion of section 2.1 to the case $M = \mathbb{R} \times S^3$ with $H = S^1$ and assume that the smooth space-like Killing vector field $\xi$ generates a Hopf bundle on every Cauchy surface $H_t$. This means in particular that the quotient manifold $S$ is $\mathbb{R} \times S^2$ (in particular, it is a smooth manifold globally). Moreover, the group $U(1)$ acts on $H_t$ effectively, and the integral curves of $\xi$, which are in $H_t$, are closed. The quotient map $\pi$, which was introduced for the general case in the previous section, becomes the Hopf map $M \to S$ whose explicit coordinate representation can be given as follows. In every $H_t$, we introduce Euler coordinates and assume that $\xi = \partial_{\rho_1}$ on every $H_t$. (This is a gauge condition for the coordinates, which certainly must be justified in our later discussion.) Then the map $\pi$ can be written as

$$\pi : \mathbb{R} \times S^3 \to \mathbb{R} \times S^2,$$

$$(t, x_1, x_2, x_3, x_4) \mapsto (t, y_1, y_2, y_3) = (t, 2(x_1x_3 + x_2x_4), 2(x_2x_3 - x_1x_4), x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

$$= (t, \sin \theta \cos \rho_2, \sin \theta \sin \rho_2, \cos \theta).$$

(5)

Here we consider $S^2$ as the submanifold of $\mathbb{R}^3$ determined by $y_1^2 + y_2^2 + y_3^2 = 1$. When we introduce standard polar coordinates on $S^2$, namely

$$y_1 = \sin \vartheta \cos \phi, \quad y_2 = \sin \vartheta \sin \phi, \quad y_3 = \cos \vartheta,$$

(6)

then $\pi$ reduces to

$$\pi : (t, \rho_1, \rho_2) \mapsto (t, \vartheta, \phi) = (t, \rho_1, \rho_2).$$

(7)

We see explicitly that the push-forward of $\xi = \partial_{\rho_1}$ to $\mathbb{R} \times S^2$ along $\pi$ vanishes (which must be the case of course). Moreover, the push-forward of $\partial_{\phi}$ equals the coordinate vector field $\partial_{\vartheta}$ on $S^2$.

Let us now consider the case of an additional smooth space-like Killing vector field $\eta$ with closed orbits which commutes with $\xi$. We assume that $\xi$ and $\eta$ together generate a global smooth effective action of the direct product group $U(1) \times U(1)$. It can be shown that all smooth effective actions of $U(1) \times U(1)$ on $S^3$ are equivalent in the sense that any other smooth effective action of $U(1) \times U(1)$ on $S^3$ can be reduced to a canonical one by applying a diffeomorphism of $S^3$ into itself and an automorphism of $U(1) \times U(1)$ to itself [17]. On any $H_t$, we can assume that the Euler coordinates above have been chosen so that $\eta = \partial_{\rho_2}$, in addition to our previous assumption that $\xi = \partial_{\rho_1}$. The assumption that $\xi$ and $\eta$ can be identified with these coordinate vector fields for all $t$ is again a restriction on the coordinate gauge which we choose later. According to the definition of Euler coordinates, we see that the two Killing fields generate two families of ‘conjugate circles’ in $S^3$ which yield a foliation of a dense subset of $S^3$ in terms of 2-tori; this is related to the Clifford parallelism discussed in [7, 8]. Our symmetry action (and hence the foliation in terms of 2-tori) degenerates, in the sense that the group orbits become one-dimensional, precisely at the ‘poles’ $\theta = 0$ and at $\theta = \pi$. When $\theta = 0$, the vector $\partial_{\rho_1}$ vanishes, while $\partial_{\rho_2}$ vanishes at $\theta = \pi$. The vectors $\partial_{\rho_1}$ and $\partial_{\rho_2}$ on the other hand never vanish, but both become parallel at $\theta = 0$ and $\theta = \pi$. Since $\xi$ and $\eta$ commute, we can apply Geroch’s symmetry reduction successively for both fields. However, the result is then not a smooth manifold, but rather a manifold with boundary. While our discussion in section 4 is not affected by this and is therefore carried out in this fully reduced picture, we would run into serious problems for the PDE analysis in section 3.3. In that section we therefore only perform the reduction with respect to $\xi$ and hence obtain the smooth orbit manifold $S = \mathbb{R} \times S^2$; the push-forward of the other Killing vector field $\eta$ along
\( \pi \), which we denote again by \( \eta \), is then a smooth Killing vector field of the 3-metric \( h \) and we have \( \eta = \partial_{\phi} \) (in the standard polar coordinates on \( \mathbb{S}^2 \) given by equation (6) associated with the Hopf map by equation (5)).

According to [25, 17], Einstein’s vacuum field equations imply that the twist quantities
\[
\kappa_1 := \epsilon_{abcd} \eta^a \xi^b \nabla^c \xi^d, \quad \kappa_2 := \epsilon_{abcd} \eta^a \xi^b \nabla^c \eta^d,
\]
vanish for spatial \( \mathbb{S}^3 \)-topology. The geometrical interpretation is that the 2-space orthogonal to the 2-space spanned by \( \xi \) and \( \eta \) in \( T_p M \) \((p \in M)\) is integrable and hence forms a 2-surface everywhere. This suggests the following ansatz for the metric
\[
g = g_{AB} \, dx^A \, dx^B + R \Big[ e^\lambda \left( d\rho_1 + Q \, d\rho_2 \right)^2 + e^{-\lambda} \, d\rho_2^2 \Big],
\]
where \( A, B = 0, 1 \) label coordinates \( t \) and \( \theta \) on the submanifolds orthogonal to the Killing vector fields; the metric \( g_{AB} \) is so far unspecified. The functions \( R, L \) and \( Q \) only depend on \( t \) and \( \theta \), i.e. are constant along the Killing vector fields. Chruściel [19] shows that under a certain genericity condition, \( U(1) \times U(1) \)-symmetric vacuum solutions imply the existence of a smooth function \( M \), a constant \( R_0 > 0 \), and functions \( Q \) and \( L \) as above such that the spacetime \((0, \pi) \times \mathbb{S}^3 \) with \( g \) of the form equation (9) and
\[
R = R_0 \sin t \sin \theta, \quad (g_{AB}) = \delta^M \text{diag} (-1, 1),
\]
can be isometrically embedded into a maximally extended globally hyperbolic vacuum spacetime. One calls such a spacetime a Gowdy spacetime and the coordinate condition equation (10) for \( t \in (0, \pi) \) the areal gauge.

By performing the symmetry reduction of a Gowdy spacetime with respect to the Killing field \( \xi \) as mentioned above, it follows that \( \lambda \) and \( \omega \), as objects on \( S \), are constant along \( \eta \). We can compute from equation (9) that
\[
\lambda = R e^L,
\]
and
\[
\partial_t \omega = -R e^{2L} \sqrt{\det(g_{CD})} g^{dA} \partial_d Q, \quad \partial_{\theta} \omega = R e^{2L} \sqrt{\det(g_{CD})} g^{dA} \partial_d Q,
\]
which simplifies to
\[
\partial_t \omega = -R_0 e^{2L} \sin t \sin \theta \partial_{\theta} Q, \quad \partial_{\theta} \omega = -R_0 e^{2L} \sin t \sin \theta \partial_{\theta} Q
\]
for the diagonal metric (10). The 3-metric is
\[
h = g_{AB} \, dx^A \, dx^B + R e^{-L} \, d\rho_2^2 = g_{AB} \, dx^A \, dx^B + \frac{R^2}{\lambda} \, d\rho_2^2.
\]

2.3. Smoothness conditions for the metric components

We have found above that the 3-metric \( h \) is a smooth Lorentzian metric on \( \mathbb{R} \times \mathbb{S}^2 \) with Killing field \( \eta \). The condition that the twist constants (equation (8)) vanish implies in addition that \( \eta \) is hypersurface orthogonal with respect to \( h \). Of particular importance now is the behavior of the metric components in standard polar coordinates, with \( \pi \) given\(^6\) by equation (7), at the poles of the 2-sphere at \( \vartheta = 0, \pi \). Recall that \( \eta = \partial_{\phi} \) in this representation.

Any metric \( l_{ab} \) on \( \mathbb{S}^2 \), for which \( \eta \) is a hypersurface orthogonal Killing vector field, must be of the form
\[
l = F(\vartheta) \, d\vartheta^2 + G(\vartheta) \, d\phi^2,
\]
\(^5\) The function \( M \) must not be confused with the symbol for the manifold \( M \).
\(^6\) In many of the following expressions, we will therefore replace \( \rho \) by \( \phi \) and \( \theta \) by \( \vartheta \), and vice versa, in accordance with equation (7).
for some functions $F$ and $G$ at all points except $\theta = 0, \pi$. In order to find necessary and sufficient conditions on those functions that imply the smoothness of the metric at the poles $\theta = 0, \pi$, we introduce further regular local coordinate patches in a neighborhood of each pole: at the north pole $\theta = 0$, we introduce coordinates $(y_1, y_2)$ by

$$y_1 = \sin \theta \cos \phi, \quad y_2 = \sin \theta \sin \phi,$$

so that the north pole corresponds to $(y_1, y_2) = (0, 0)$ and at the south pole $\theta = \pi$, $(\tilde{y}_1, \tilde{y}_2)$ by

$$\tilde{y}_1 = \sin \theta \cos \phi, \quad \tilde{y}_2 = \sin \theta \sin \phi,$$

so that the south pole corresponds to $(\tilde{y}_1, \tilde{y}_2) = (0, 0)$. Both of these local patches break down at the equator given by $\theta = \pi/2$. Close to the north pole, every smooth metric can be written as

$$l = l_{11} \, dy_1^2 + 2 l_{12} \, dy_1 \, dy_2 + l_{22} \, dy_2^2,$$  

(16)

where $l_{11}, l_{12}$ and $l_{22}$ are smooth functions of the coordinates $(y_1, y_2)$. When we transform this metric to polar coordinates and use the conditions that (i) the resulting metric should be diagonal and (ii) the metric components should be independent of $\phi$, we find that there must be smooth functions $f_1$ and $f_2$ so that

$$l_{11}(y_1(\theta, \phi), y_2(\theta, \phi)) = f_1(\cos \theta) - f_2(\cos \theta) \sin^2 \theta \cos(2\phi),$$

$$l_{22}(y_1(\theta, \phi), y_2(\theta, \phi)) = f_1(\cos \theta) + f_2(\cos \theta) \sin^2 \theta \cos(2\phi),$$

(17)

$$l_{12}(y_1(\theta, \phi), y_2(\theta, \phi)) = -f_2(\cos \theta) \sin^2 \theta \sin(2\phi).$$

By transforming the metric equation (16) with coefficients equation (17) to polar coordinates $(\theta, \phi)$, and comparing the result with equation (15), we find that $F$ and $G$ must be smooth functions of $\cos \theta$ so that

$$F(\cos \theta) = O(1), \quad G(\cos \theta) = \sin^2 \theta \left( F(\cos \theta) + O(\sin^2 \theta) \right),$$

(18)

at $\theta = 0$. The same can be carried out for the south pole, and we find that equation (18) must also hold at $\theta = \pi$.

For the following, we rather parametrize the metric $l$ as

$$l = e^M \, d\theta^2 + \sin^2 \theta e^{2U} \, d\phi^2,$$

for functions $M$ and $U$ instead of the functions $F$ and $G$ in equation (15). According to our above result, the functions $M$ and $U$ must hence be two smooth functions of $\cos \theta$ for which the smoothness condition

$$e^M = e^{2U} + \dot{M}(\cos \theta) \sin^2 \theta,$$

(19)

holds with some smooth function $\dot{M}$. Using this together with equation (10), we can rewrite equation (14) for the general metric $h$ on $S = \mathbb{R} \times S^2$ as

$$h = e^M (-dr^2 + d\theta^2) + \sin^2 \theta e^{2U} \, d\phi^2,$$

(20)

where

$$U = \ln R_0 + \ln \sin \tau - \frac{1}{2} \ln \lambda = (\ln R_0 - L - \ln \sin \theta + \ln \sin \tau)/2.$$  

This is a bounded function for every fixed $\tau \in (0, \pi)$ under the previous assumptions because the quantity $\lambda$ is finite and bounded away from zero (including the poles). Clearly equation (11) yields that

$$e^L = O(R^{-1}) = O(\sin^{-1} \theta)$$

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at the poles, and the smoothness condition (equation (19)) translates to
\[
e^M = \frac{R^2}{\lambda \sin^2 \vartheta} + \hat{M}(\cos \vartheta) \sin^2 \vartheta.
\] (22)
A consequence is that \(e^M\) is bounded and non-vanishing at the poles for every fixed \(t \in (0, \pi)\).

Let us also derive a smoothness restriction for the function \(Q\) in equation (9). Our choice of Killing basis \(\{\xi, \eta\} = \{\partial_{\rho_1}, \partial_{\rho_2}\}\), for which \(\partial_{\rho_1} = \partial_{\rho_2}\) at \(\theta = 0\) and \(\partial_{\rho_1} = -\partial_{\rho_2}\) at \(\theta = \pi\) holds, has the consequence that \(g(\xi, \xi) = g(\eta, \eta) = \pm g(\xi, \eta)\) at the poles for every fixed \(t \in (0, \pi)\). Therefore there must exist a smooth function \(\hat{Q}\) which only depends on \(t\) and \(\cos \theta\) so that
\[
Q(t, \theta) = \cos \theta + \hat{Q}(t, \cos \theta) \sin^2 \theta.
\] In particular, it follows from equation (13) that for each fixed \(t \in (0, \pi)\),
\[
-2 = Q(t, \pi) - Q(t, 0) = \int_0^\pi Q_0 \, d\theta = -\int_0^\pi R^{-1} e^{-2\omega} \partial_\omega \, d\theta = -\int_0^\pi R \lambda^{-2} \partial_\omega \, d\theta.
\]

2.4. Reparametrizations of the Gowdy orbits

All of our discussions so far are based on the choice \(\{\partial_{\rho_1}, \partial_{\rho_2}\}\) as the Gowdy Killing basis on \(M\). Now we study general reparametrizations of the Gowdy Killing orbits in \(M\), i.e. arbitrary bases of the same Gowdy Killing algebra.

Let \((\phi_1, \phi_2) \in \mathbb{R}^2\) be coordinates on the Killing orbits so that a Gowdy invariant metric has the form analogous to equation (9), i.e.
\[
g = e^M(-dr^2 + d\vartheta^2) + R[e^\hat{\omega}(d\phi_1 + Q d\phi_2)^2 + e^{-\hat{\omega}} d\phi_2^2].
\]
We are allowed to reparametrize the orbits by means of constants \(a, b, c, d \in \mathbb{R}\), so that
\[ad - bc \neq 0\]
and
\[
\phi_1 = a\tilde{\phi}_1 + b\tilde{\phi}_2, \quad \phi_2 = c\tilde{\phi}_1 + d\tilde{\phi}_2.
\]
The coordinates \(t\) and \(\theta\) are not changed. In terms of the new coordinates, we want to write the metric as
\[
g = e^M(-dr^2 + d\vartheta^2) + \tilde{R}[e^{\tilde{\omega}}(d\tilde{\phi}_1 + \tilde{Q} d\tilde{\phi}_2)^2 + e^{-\tilde{\omega}} d\tilde{\phi}_2^2].
\]
One finds that
\[
\tilde{R} = |ad - bc|R, \quad e^{\tilde{\omega}} = \frac{(a + cQ)^2 e^\omega + c^2 e^{-L}}{|ad - bc|}, \quad \tilde{Q} = \frac{(a + cQ)(b + dQ)e^\omega + c \, d \, e^{-L}}{(a + cQ)^2 e^\omega + c^2 e^{-L}}.
\] (24-26)
A particularly useful transformation is the inversion, i.e. the interchange of the Killing basis fields. Then we have \(a = d = 0, b = c = 1\), and hence
\[
\tilde{R} = R, \quad e^{\tilde{\omega}} = e^\omega Q^2 + e^{-L}, \quad \tilde{Q} = \frac{e\tilde{Q}}{e^\tilde{\omega} Q^2 + e^{-L}}.
\]

Another useful reparametrization is the following. Let us consider a metric in the parametrization \((\rho_1, \rho_2)\) of the Killing orbits as given by equation (9), i.e. we pick \(\phi_1 = \rho_1\) and \(\phi_2 = \rho_2\). Now let \(\phi_1 = \lambda_1\) and \(\phi_2 = \lambda_2\) and hence
\[
g = e^M(-dr^2 + d\vartheta^2) + \tilde{R}[e^{\hat{\omega}}(d\lambda_1 + Q d\lambda_2)^2 + e^{-\hat{\omega}} d\lambda_2^2].
\]
For this we must choose $a = 1$, $b = 1$, $c = 1$, $d = -1$ from equation (4). It follows that

$$
\hat{R} = 2R, \quad e^\ell = \frac{(1 + Q) e^\ell + e^{-\ell}}{2}, \quad \hat{Q} = \frac{-(1 - Q^2) e^\ell + e^{-\ell}}{(1 + Q)^2 e^\ell + e^{-\ell}}.
$$

(27)

The inverse of this reparametrization is

$$
\hat{R} = \frac{\hat{R}}{2}, \quad e^\ell = \frac{(1 + \hat{Q})^2 e^\ell + e^{-\ell}}{2}, \quad Q = \frac{(1 - \hat{Q})^2 e^\ell - e^{-\ell}}{(1 + \hat{Q})^2 e^\ell + e^{-\ell}}.
$$

From this and the discussion in section 2.3, we can easily derive the behavior of the functions $\hat{R}, \hat{L}, \hat{Q}$ at the poles for $\ell \in (0, \pi)$ in areal coordinates,

$$
\hat{R} = \hat{R} \sin \theta, \quad e^\ell = e^\ell \cos^2 \frac{\theta}{2}, \quad \hat{Q} = (1 - \cos \theta) \hat{Q},
$$

(28)

$$
e^M = \hat{R}^2 \left[ e^\ell (1 - \cos \theta) + e^{-\ell} (1 + \cos \theta) \right] + \hat{M} \sin^2 \theta,
$$

with smooth functions $\hat{R}, \hat{L}, \hat{Q}$ and $\hat{M}$ which only depend on $\ell$ and $\cos \theta$.

A particular consequence is the following interesting fact about polarized Gowdy spacetimes. A Gowdy spacetime is called polarized if there exists an everywhere orthogonal basis of Gowdy Killing fields. Hence, with respect to this basis, the function $Q$ must vanish identically. Now, equation (23) shows that this can never happen for the Killing basis $[\partial_{\rho}, \partial_{\omega_3}]$, but it is possible for the basis $[\partial_{\rho_1}, \partial_{\omega_2}]$, according to equation (28). Indeed, one can show that $Q$ can only vanish identically for a smooth Gowdy-symmetric metric on $\mathbb{S}^3$ if the Killing basis is chosen such that one of the two fields is proportional to $\partial_{\rho_1}$ and the other to $\partial_{\omega_2}$. This fact will be used in our later discussion.

### 3. The class of smooth Gowdy-symmetric generalized Taub–NUT solutions

#### 3.1. The Taub solutions

The Taub solutions were discovered by Taub [47] as a family of cosmological solutions of the vacuum field equation with spatial $\mathbb{S}^3$-topology. They are a family of spacetimes

$$
g = l^2 \left( -\frac{4(1 + \tau^2)}{V(\tau)} \, d\tau^2 + (1 + \tau^2)(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2) + \frac{V(\tau)}{1 + \tau^2} \omega_3 \otimes \omega_3 \right),
$$

with two free parameters $l > 0$ and $m \in \mathbb{R}$, and where

$$
V(\tau) := -4\tau^2 - 8\frac{m}{l} \tau + 4.
$$

Here,

$$
\omega_1 = \sin \rho_1 \, d\theta - \cos \rho_1 \sin \theta \, d\rho_2,
$$

$$
\omega_2 = \cos \rho_1 \, d\theta + \sin \rho_1 \sin \theta \, d\rho_2,
$$

$$
\omega_3 = d\rho_1 + \cos \theta \, d\rho_2,
$$

are the standard invariant 1-forms with respect to the standard transitive action of $SU(2)$ on $\mathbb{S}^3$. In particular, it follows that $SU(2)$ is a three-dimensional subgroup of the isometry group—hence the Taub solutions are spatially homogeneous. The full isometry group is four-dimensional where the fourth symmetry is generated by the smooth right-invariant vector field $\partial_{\rho_1}$; the details can be found e.g. in [9, 10]. The total symmetry group is the direct product $U(1) \times SU(2)$ and therefore these spacetimes can be classified as LRS–Bianchi IX [48]. Taking the $U(1)$ subgroup of $SU(2)$ generated by $\partial_{\rho_1}$, it follows that Taub solutions are
particular Gowdy solutions. They are not polarized, and we can bring them to the form (9) with (10)
\[ g = e^\tau (-dt^2 + d\theta^2) + R^2 (d\rho_1 + Q d\rho_2)^2 + e^{-\tau} d\rho_2^2 \].
For arbitrary parameters \( l > 0 \) and \( m \in \mathbb{R} \), the Taub solutions are then given by
\[ R = 2l/\sqrt{l^2 + m^2 \sin^2 \theta}, \quad e^\tau = l^2 + (m + \sqrt{l^2 + m^2 \cos^2 \theta})^2, \quad Q = \cos \theta. \]
The Taub metric is smooth and globally hyperbolic where \( V \) is positive, i.e. for
\[ \tau \in (\tau_-, \tau_+), \quad \tau_{\pm} := -m/l \pm \sqrt{1 + m^2/l^2}. \]
It was demonstrated for the first time in [39] that the solutions can be extended analytically through the apparently singular times \( \tau_{\pm} \). By this we mean the following [45]. A spacetime \((M, g)\) is called extendible (in which case we say that ‘it can be extended’) if there exists a spacetime \((\tilde{M}, \tilde{g})\) of the same dimension and an isometric embedding \( \Psi : M \to \tilde{M} \) which is not surjective. If \((M, g)\) and \((\tilde{M}, \tilde{g})\) are analytic (smooth) manifolds and \( \Psi \) is an analytic (smooth) map, then we say that the extension is analytic (smooth).

In the case of the Taub spacetimes, we can find \((\tilde{M}, \tilde{g})\) and \( \Psi \) by appropriate coordinate transformations. We discuss more of these issues in section 3.2. The extensions are not globally hyperbolic and the surface corresponding to \( \tau = \tau_- \) in the extended spacetime is a smooth null hypersurface with a closed null generator; in particular, this implies that there exist closed causal curves. This surface is therefore a past Cauchy horizon. In the same way, there is a future Cauchy horizon at \( \tau = \tau_+ \). It has turned out that there are several non-equivalent analytic extensions of the Taub solutions. All these extended spacetimes were christened Taub–NUT solutions.

### 3.2. Generalizations of the Taub–NUT solutions

Motivated by these intriguing properties of the Taub solutions above, Moncrief [32] introduces the family of generalized Taub–NUT solutions. The idea is to obtain a family of solutions of Einstein’s vacuum equations—which is not restricted to spatial homogeneity—with similar properties as for the Taub solutions; in particular, there should exist smooth compact Cauchy horizons. It was shown in [33] that if the spacetime is an analytic solution of the vacuum field equations and if the Cauchy horizon is ruled by closed null generators in the sense of an \( S^1 \)-bundle, in particular the null generator coincides with the generators of the bundle, then the spacetime necessarily has a one-dimensional isometry group and the corresponding Killing field is proportional to the null generators of the Cauchy horizon on the horizon. The result was generalized to the case of non-analytic (i.e. smooth) solutions in [22].

Motivated by all this, Moncrief restricts to the class of analytic vacuum solutions \((M, g)\) with \( M = (0, \delta) \times S^3 \) for some sufficiently small \( \delta > 0 \) with one spatial Killing vector field with the following properties. The spacetime is globally hyperbolic and the level sets of a global time function \( t \) are Cauchy surfaces homeomorphic to \( S^3 \). The spacetime can be extended analytically through \( t = 0 \) (in the sense described before) so that the points corresponding to \( t = 0 \) in the extended spacetime form an analytic null hypersurface with \( S^3 \)-topology. In particular, this null hypersurface is then a compact Cauchy horizon and hence global hyperbolicity breaks down at \( t = 0 \). Additionally, he assumes that the Cauchy horizon is a Hopf bundle whose \( U(1) \)-generator is a Killing field.

Let us study the consequences of these assumptions. Introducing Euler coordinates as before, Moncrief shows that the metrics of all such spacetimes can be written as
\[ g = e^{-2\tau} (-N^2 dt^2 + \tilde{g}_{ab} dx^a dx^b) + \sin^2 \tau e^{2\tau} [k(\bar{d}\rho + \cos \theta \bar{d}\rho_2) + \bar{\beta}_a dx^a]^2, \]
(29)
for all \( t \in (0, \delta] \), where the \( U(1) \)-Killing field is \( \partial_{\rho} \). The functions \( \gamma \) and \( \tilde{N} \) only depend on \( t, \theta \) and \( \rho_2 \). The index \( \alpha \) takes the values 1 (corresponding to the coordinate \( \theta \)) and 3 (corresponding to the coordinate \( \rho_2 \)). The field \( \tilde{g}_{\alpha\beta} \) is a symmetric 2-tensor field and \( \beta_{\alpha} \) a 1-form. The function \( \tilde{N} \) is supposed to be uniformly positive. Moreover, \( k > 0 \) is a constant.

Moncrief assumes that all fields \( \gamma, \tilde{N}, \tilde{g}_{\alpha\beta} \) and \( \beta_{\alpha} \) are analytic on \( (0, \delta] \times S^3 \) for some small \( \delta > 0 \).

However, these assumptions are not yet sufficient to guarantee that the spacetime can be extended through \( t = 0 \). In order to write down the conditions for this, we here restrict to the case of interest, namely to the case of Gowdy symmetry; the details for the general \( U(1) \)-symmetric case can be found in Moncrief’s paper. For Gowdy symmetry, the metric becomes

\[
e^L = \frac{R e^{-M}}{\sin^2 \theta} N^2.
\]

(30)

In order to identify Moncrief’s metric with the metric given by equations (9)–(10) for \( t \in (0, \delta] \) and \( \delta \in (0, \pi) \), we set

\[
\tilde{g}_{\theta\theta} = e^M, \quad \tilde{g}_{\theta\rho_1} = 0, \quad \tilde{g}_{\rho_1\rho_1} = \frac{e^M \sin^2 \theta}{N^2}, \quad \beta_\theta = 0, \quad \beta_\rho = R_0 (Q - \cos \theta),
\]

\[
\tilde{N} = \frac{R_0 N}{k}, \quad \gamma = -\frac{M}{2} + \ln \frac{R_0 N}{k}.
\]

Now, in order to find the extensions through \( t = 0 \), let us introduce new coordinates \((t', \theta', \rho_1', \rho_2')\) by

\[
t = \arcsin \sqrt{t'}, \quad \theta = \theta', \quad \rho_1 = \rho_1' + \frac{k}{R_0} \ln t', \quad \rho_2 = \rho_2'.
\]

(31)

The quantity \( \kappa \) is a constant which has, so far, not been fixed. In these new coordinates, the metric becomes

\[
g = -\left( \frac{e^M}{4(1 - t') t'} - \frac{e^{-M} N^2 \kappa^2}{t'} \right) dt'^2 + e^M d\theta'^2
\]

\[
+ e^{-M} N^2 \left[ 2R_0 \kappa (d\rho_1' + Q d\rho_2') dt' + R_0^2 t' \left( d\rho_1' + Q d\rho_2' \right)^2 \right] + \frac{e^M \sin^2 \theta}{N^2} d\rho_2'^2.
\]

The metric extends analytically through \( t' = 0 \) if all the functions \( M, N^2, Q \) and \((4\kappa^2 N^2 - e^2M)/t'\)—for some choice of the constant \( \kappa \)—extend analytically through \( t' = 0 \) when expressed in terms of the new coordinates, and if \( N^3 \) extends as a strictly positive function. Note that it is necessary for this that \( M, N^2, Q \)—expressed in terms of the original coordinates—extend as analytic functions to the manifold \([-\delta, \delta] \times S^3\), which are even in \( t \), for some sufficiently small \( \delta > 0 \).

If the spacetime can be extended through \( t = 0 \) in this way, then the field \( \partial_{\rho_1} \) (which equals \( \partial_{\rho} \) wherever the latter is defined) is a null generator of the surface given by \( t' = 0 \). This surface is therefore an analytic null hypersurface with \( S^3 \)-topology whose null generators are closed, and so is an analytic past Cauchy horizon.

As an example, the Taub solutions satisfy all the above assumptions when we choose

\[
\kappa = \pm (t^2 + m(m + \sqrt{t^2 + m^2})).
\]

As we have mentioned above, Moncrief restricts us to the analytic case. One of our main contributions here is a generalization of his results to the non-analytic (i.e. smooth) case; this means that the reader must replace every occurrence of ‘analytic’ in the previous discussion

\footnote{We have \( g_{\rho_1'\rho_1'} = R_0^2 t' e^{-M} N^2 \) which vanishes at \( t' = 0 \).}
by ‘smooth’. Due to our restriction to Gowdy symmetry here, we call spacetimes with all the above properties smooth Gowdy-symmetric generalized Taub–NUT spacetimes. The name is supposed to reflect the fact that this family of spacetimes is motivated by Moncrief’s generalized Taub–NUT spacetimes.

Note that if an analytic spacetime as above solves Einstein’s field equations in vacuum for $t > 0$, then the analytically extended spacetimes are also necessarily the solutions of the vacuum field equations. In the non-analytic smooth case, we do not know in general whether the extensions are vacuum solutions. We will not address this problem in this paper. As another interesting side-remark: Chruściel et al note in [19] that there are no smooth extensions through a Cauchy horizon of $S^3$-topology—solution of the field equations or not—in the polarized Gowdy case. We find here that none of the spacetimes which we consider are polarized, and hence there is no contradiction.

3.3. Existence of smooth Gowdy-symmetric generalized Taub–NUT solutions

3.3.1. The main existence result. In this section we show the existence of a non-trivial family of smooth Gowdy-symmetric generalized Taub–NUT spacetimes, which solve Einstein’s vacuum field equations. A central technique here is the Fuchsian method introduced in [13, 12, 14, 2, 1], which we must reformulate for the particular spatial topology used here. This is done in the appendix.

In the following we call a function on $S^2$ rotationally symmetric if it does not depend on the azimuthal angle $\phi$ in standard spherical coordinates equation (6). The Hopf map allows us to lift any such function to a smooth $U(1) \times U(1)$-invariant function on $S^3$.

**Theorem 3.1.** Let $S_{ss}$ and $Q_{s}$ be rotationally symmetric functions in $C^\infty(S^2)$ with the property

$$S_{ss}(0) = S_{ss}(\pi),$$

(32)

and $R_0 > 0$ a constant. Then there exists a unique smooth Gowdy-symmetric generalized Taub–NUT solution of the form equations (9)–(10) for all $t \in (0, \pi)$ which has the following uniform expansions at $t = 0$:

$$R(t, \theta)e^{L(t, \theta)} = t^2 e^{S_{ss}(\theta)} + O(t^4),$$

$$Q(t, \theta) = \cos \theta + Q_{s}(\theta) \sin^2 \theta + O(t^2),$$

$$M(t, \theta) = S_{ss}(\theta) - 2S_{ss}(0) + 2 \ln R_0 + O(t^2).$$

Corresponding expansions hold for all derivatives.

Let us make a couple of comments before we proceed with the proof of this theorem in section 3.3.2. Theorem 3.1 implies that we can prescribe arbitrary smooth asymptotic data functions $S_{ss}$ and $Q_{s}$ subject to the condition (32) and find a unique smooth Gowdy–symmetric generalized Taub–NUT solution of the vacuum equations so that the leading-order behavior at $t = 0$ is determined by these data functions. In this sense, the functions $S_{ss}$ and $Q_{s}$ can be considered as ‘data on the Cauchy horizon’ at $t = 0$. Hence we solve here a singular initial value problem with leading-order terms as above; this is discussed in greater depth later. In particular, we find the same number of free functions as in Moncrief’s class of the generalized Taub–NUT solution (after factoring out gauge transformations in his class [32]).

We stress that theorem 3.1 implies only the existence of a past Cauchy horizon at $t = 0$, but says nothing about the properties of the solution at $t = \pi$, where the areal coordinates break down; section 4 is devoted to the question of what happens at $t = \pi$ for the solutions of theorem 3.1.
Particular examples are the Taub solutions (section 3.1), which correspond to the asymptotic data
\[ R_0 = 2t \sqrt{I^2 + m^2}, \quad S_* = 2 \ln R_0 - \ln (I^2 + (m + \sqrt{I^2 + m^2})^2), \quad Q_* = 0. \]
We can therefore interpret theorem 3.1 as a statement about the ‘stability’ of the Cauchy horizon of the Taub solutions at \( t = 0 \) with respect to smooth Gowdy-symmetric perturbations.

It is interesting to compare our theorem with earlier results by various authors. For general spacetimes which are ‘singular’ in some sense at \( t = 0 \), one expects that the full degree of freedom corresponds to four free data functions. For the case of Gowdy symmetry with spatial \( \mathbb{T}^3 \)-topology \([30, 42, 12]\), one can indeed show the well-posedness of a singular initial value problem with the full number of free functions. One obtains a large variety of solutions: on the one hand, solutions whose curvature blows up at \( t = 0 \), and, on the other hand, solutions with Cauchy horizons at \( t = 0 \), which can be both compact or non-compact.

An earlier attempt to obtain a similarly general result in the case of spatial \( S^3 \)-topology and \( S^1 \times S^2 \)-topology \([46]\) has partly failed: the author successfully constructs a general family of singular solutions of some of Einstein’s vacuum equations, but the remaining ‘constraints’, which require certain matching conditions to hold, see below, are ignored. Stahl’s statement ‘In what follows we will assume without further comment that the solutions are chosen such that the matching conditions hold.’ on page 4489 of \([46]\) is vacuous: while it is clear how to choose Cauchy data so that the matching conditions are satisfied \([17]\), this is in general not clear for asymptotic data of the singular initial value problem, which we here—and Stahl—consider. Indeed, these matching conditions turn out to be a major difficulty in our proof of theorem 3.1, see proposition 3.4. There are reasons to believe that the matching conditions can never be satisfied for this singular initial value problem in more general situations, in particular if a neighborhood of the axes of the \( t = 0 \)-surface is supposed to represent a curvature singularity. This case is, however, not covered in this paper.

It is interesting to note that for the Gowdy case with spatial \( \mathbb{T}^3 \)-topology, the asymptotic data have to satisfy an integral constraint. Such a constraint must be imposed in order to guarantee that the function \( M \) in the metric, as a solution of the field equations, is consistent with the periodic topology. Here, it is rather the smoothness condition (equation (22)) for \( M \) at the ‘poles’ of the 3-sphere which gives rise to the non-integral constraint equation (32).

We also point out that none of the solutions of theorem 3.1 is polarized. Recall from section 2.4 that a smooth Gowdy-symmetric spacetime is polarized if and only if \( Q \) vanishes with respect to the \((\lambda_1, \lambda_2)\)-parametrization of the symmetry orbits and hence if \( 1 - (1 - Q^2)e^{2L} = 0 \) with respect to the \((\rho_1, \rho_2)\)-parametrization according to equation (27). However, for our solutions, \( Q \) is bounded in a neighborhood of \( t = 0 \) and \( e^{2L} \) is \( O(t^4) \); from this we find that \( 1 - (1 - Q^2)e^{2L} = 1 + O(t^4) \).

As a last comment, let us point out that theorem 3.1 can be generalized to asymptotic data with only finitely many derivatives. We, however, do not discuss this here.

### 3.3.2. Equations and unknowns

For our proof of theorem 3.1, let us make the following convenient choices. We consider Geroch’s reduction with respect to the field \( \xi = \partial_\rho \) and hence with the projection map \( \pi \) of the form equation (7) as discussed before. In areal coordinates, the 3-metric \( h_{ab} \) on the quotient manifold \((0, \delta] \times S^2 \) is therefore given by equations (20)–(21). When we define
\[ S := \ln \lambda = L + \ln R, \]
which we expect to be a smooth function wherever the spacetime is defined in \((0, \delta] \times S^2 \), it becomes
\[ h = e^M (-dr^2 + d\theta^2) + R^2_0 \sin^2 t \sin^2 \theta \, e^{-S} \, d\phi^2. \]
As outlined before, the geometry of the Gowdy spacetimes is completely determined by the quantities $S$, $\omega$ and $h$ on the quotient manifold $(0, \delta] \times S^2$. Then, equations (1)–(2), as equations on the quotient manifold, imply

$$D^2 S - i^2 \Delta_{S^2} S = (1 - t \cot t) DS - e^{-2s} ((D\omega)^2 - (t \partial_\phi \omega)^2),$$  \hspace{1cm} (35)$$

$$D^2 \omega - 4D\omega - i^2 \Delta_{S^2} \omega = (1 - t \cot t) D\omega + 2(DS - 2) D\omega - 2(t \partial_\phi S)(t \partial_\phi \omega).$$  \hspace{1cm} (36)$$

We use the notation $D := i \partial_t$ and $D^2 := i \partial_t (i \partial_t)$. Note that we have added a term $-4D\omega$ to both sides of the second equation for later convenience. The operator $\Delta_{S^2}$ is the Laplace operator of the standard metric on the unit sphere

$$\Delta_{S^2} := \partial_\rho^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2.$$  

In our case, we restrict ourselves to solutions which are independent of the azimuthal angle $\phi$ and hence the $\partial_\phi^2$-term in this operator does not appear. In this case, all terms in the equations above have a geometric coordinate independent meaning with respect to the scalar product of the standard Riemannian metric on $S^2$:

$$\tilde{h} = d\theta^2 + \sin^2 \theta \, d\phi^2.$$  

Equations (35) and (36) therefore constitute a coupled semilinear system of geometric wave equations for the scalar quantities $S$ and $\omega$ with respect to the standard metric on the unit sphere as long as $t \in (0, \pi)$.

The remaining equations are found from the $(2 + 1)$-dimensional Einstein equations coupled to the wave map given by $S$ and $\omega$ equation (3). On the one hand, the above form of the 3-metric $h$ implies

$$0 = -M_{tt} + \Delta_{S^2} M + \cot t (M_{tt} + 2S_t) - S_t^2 - e^{-2s} \omega_\phi^2 + 2 =: H,$$  \hspace{1cm} (37)$$

and on the other hand,

$$0 = 4R_{\pm \phi} + R(S_{\pm}^2 + e^{-2s} \omega_{\phi \phi}^2) - 2R_{\pm} (S_\pm + M_\pm) =: 2R_{\pm} C_{\pm},$$  \hspace{1cm} (38)$$

with $\partial_\pm := \partial_\rho \pm i \partial_\phi$. While for the first of these equations all terms have a geometric coordinate independent meaning as before, this is not the case for equations (38). This is not a major problem, as we will be able to analyze equations (38) completely in the particular coordinate system.

3.3.3. Steps of the proof of theorem 3.1. The strategy for the proof of theorem 3.1 is to solve only equations (35)–(36) in a first step; we refer to the latter as the main evolution system or the Gowdy equations. Given any such solution $(S, \omega)$ of that system with the ‘correct’ behavior at $t = 0$, the remaining equations, equations (37)–(38), form an overdetermined system for the other unknown $M$. We must therefore ask for the conditions for which these can be solved for $M$. As already discussed in [17, 23, 46], one encounters certain ‘matching conditions’ which must be satisfied in order to obtain smooth solutions for $M$. The quantities $H$, $C_+$ and $C_-$ introduced above are of particular importance since they measure the violation of equations (37)–(38).

The first step is the construction of such solutions of equations (35)–(36) in a small time neighborhood of $t = 0$, which are compatible with the notion of smooth Gowdy-symmetric generalized Taub–NUT solutions given in section 3.2. We find the following central result.

8 Derivatives of functions along coordinate vector fields are written either as e.g. $\partial_\phi \omega$ or as e.g. $\omega,\phi$ in all of what follows.
Proposition 3.2. Let $\omega_\omega \in \mathbb{R}$ and $\omega_{s\omega}, S_{s\omega} \in C^\infty(S^2)$ be rotationally symmetric functions. Choose an exponent vector $\mu = (\mu_1, \mu_2)$ with $1 < \mu_1(x) < 2$ and $(4 + \sqrt{17})/2 < \mu_2(x) < 4 + \mu_1(x)$ for all $x \in S^2$. Then there exists a unique solution $(S, \omega)$ of the Gowdy equations (35)–(36) with

$$S(t, \vartheta) = 2 \ln t + S_{s\omega}(\vartheta) + w_1(t, \vartheta), \quad \omega(t, \vartheta) = \omega_\omega + \omega_{s\omega}(\vartheta) t^4 + w_2(t, \vartheta),$$

where $w := (w_1, w_2) \in \tilde{X}_{s, \mu, \infty}(S^2)$ and $D^2 w \in X_{s, \mu, \infty}(S^2)$ for a sufficiently small $\delta > 0$. In particular, the functions $w_1, w_2$ are rotationally symmetric for each $t \in (0, \delta]$.

Before we prove this proposition let us make the following remarks. The meaning of this proposition can be summarized as follows. (The technical details of the spaces $X_{s, \mu, \infty}(S^2)$ and $\tilde{X}_{s, \mu, \infty}(S^2)$ are listed in the appendix.) Suppose that $\omega_{s\omega}$ and $S_{s\omega}$ are smooth rotationally symmetric asymptotic data functions on $S^2$, and $\omega_\omega$ is a constant. Then there exists a smooth solution of equations (35)–(36) of the form (39)–(40) on the time interval $(0, \delta]$ with $w_1 \in \tilde{X}_{s, \mu, \infty}(S^2)$ and $w_2 \in \tilde{X}_{s, \mu, \infty}(S^2)$. The fact that the remainder $w = (w_1, w_2)$ is in such a space implies that it, together with all of its spatial derivatives, decays at a rate $t^{\mu_1}$ and $t^{\mu_2}$, respectively, uniformly at each spatial point in the limit $t \to 0$. Moreover, the second time derivative $D^2 w$, and in the same way all higher time derivatives of the remainder, decay with the same rate at $t = 0$. Notice here that $(4 + \sqrt{17})/2 \approx 4.06$. All this shows that equations (39)–(40) can be interpreted as describing the leading-order behavior, which we may prescribe by means of the asymptotic data $\omega_{s\omega}, \omega_{s\omega}$, and $S_{s\omega}$. We stress that this singular initial value problem differs significantly from a standard Cauchy initial value problem. The equations are singular at $t = 0$ (of semilinear Fuchsian wave equation type, see the appendix). A consequence is the fourth power of $t$ in the leading-order term of $\omega$ and the logarithm in the expansion of $S$ on the one hand, and, on the other hand, we do not obtain four free data functions. Note in particular that $\omega_{s\omega}$ must be a constant. All this is justified below.

As mentioned before, the quantities $\mu_1$ and $\mu_2$ control the decay of the remainders $w_1$ and $w_2$ at $t = 0$. We argue below that the bounds for these constants in proposition 3.2 are not yet optimal; in fact we find below that $\mu_1 \leq 2$ and $\mu_2 \leq 6$ are upper bounds. Moreover, the uniqueness result, as it is given by the lower bounds, is not yet optimal. It guarantees uniqueness only among remainders in the spaces $X_{s, \mu, \infty}$ with $\mu_1 > 1$ and $\mu_2 > (4 + \sqrt{17})/2$. The general class of remainders which is of interest here, however, is given by the larger space $X_{s, \mu, \infty}$ with $\mu_1 > 0$ and $\mu_2 > 4$. At this stage, we cannot rule out possible further solutions in this larger space. However, by computing expansions of higher order below, we can show that the solutions of proposition 3.2 are indeed unique in the full space.

The form of the leading-order term (equations (39)–(40)) has been chosen to be compatible with the notion of smooth Gowdy-symmetric generalized Taub–NUT solutions. For example, the term $2 \ln t$ in equation (39) guarantees that the coefficient of $d \rho_1^2$ in the metric is $O(t^2)$ at $t = 0$ in agreement with equation (29).

We see in the proof of proposition 3.2 that the same result can be obtained under much less stringent regularity assumptions. Indeed, we only require that the asymptotic data have a certain finite number of derivatives. This will, however, no longer be discussed in detail.

Proof of proposition 3.2. The proof is an application of theorem A.2 in the appendix. First, we realize that our equations are a system of the form equation (A.2) with $u = (S, \omega)^2$, and $d = 2$. The coefficient matrix $A$ is

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$
From this, we compute the energy dissipation matrix equation (A.5) which is positive definite—and hence condition (i) of theorem A.2 is satisfied—if \( \mu_1 > 1 \) and \( \mu_2 > (4 + \sqrt{17})/2 \). The leading-order term is
\[
t_0 = (2 \ln t + S_{ss}\omega_\ast + \omega_\ast t^4)^T, \tag{41}
\]
so that
\[
L[u_0] = (-t^2 \Delta_{\theta^2} S_{s\ast}, -t^4 \Delta_{\theta^2} \omega_\ast)^T,
\]
where the operator \( L \) is given by equation (A.3). The expressions for the operators \( F(u_0) \) and \( F_{\text{red}}(u_0) \), see equations (A.4) and (A.6), can then be found from the right-hand sides of equations (35) and (36). Suppose that \( q > 1 \) and \( 0 < \mu_1 < 2 \) and \( 4 < \mu_2 < 4 + \mu_1 \) (the lower bound 4 is sufficient here and we do not need to require the slightly larger value \( (4 + \sqrt{17})/2 \) for this part of the argument). Then one can find easily that condition (ii) of theorem A.2 is satisfied. Notice that this would not be true if \( \omega_\ast \) was not a constant due to terms in the equation that are proportional to \( \partial_\theta \omega_\ast \) which would be too singular at \( t = 0 \). The remaining condition (iii) follows automatically; one can argue in the same way as in the proof of lemma 3.4 in [2].

In order to prove that the solutions, which we have just found, are rotationally symmetric at every \( t \), we can take a \( \phi \)-derivative of both sides of equations (35) and (36). One obtains a system of evolution equations for the unknowns \( \partial_\theta S \) and \( \partial_\theta \omega_\ast \), which has exactly the same form as the original system. We can hence solve the same singular initial value problem, but now with a vanishing leading-order term. Then, uniqueness implies that \( \partial_\theta S(t, x) = 0 \) and \( \partial_\theta \omega_\ast(t, x) = 0 \) for all \( (t, x) \in (0, \delta] \times \mathbb{S}^2 \). \( \square \)

Let us gather further information about the solutions \((S, \omega)\) of proposition 3.2.

**Lemma 3.3.** Let \((S, \omega)\) be the functions \((0, \delta] \times \mathbb{S}^2 \to \mathbb{R}\) determined in proposition 3.2 as solutions of equations (35) and (36) from smooth rotationally symmetric asymptotic data functions \(\omega_\ast S_{s\ast}\) and \(S_{s\ast}\), and the constant \(\omega_\ast\). Let \(\lambda := e^\lambda\) (cf equation (33)). Then \(\lambda\) and \(\omega\) are solutions of
\[
D^2 \lambda - t^2 \Delta_{\theta^2} \lambda = (1 - t \cot t)D\lambda - \lambda^{-1}(t \partial_\theta \lambda)^2 - \lambda^{-1}(D\omega)^2 - (t \partial_\theta \omega)^2, \tag{42}
\]
\[
D^2 \omega - 4D\omega - t^2 \Delta_{\theta^2} \omega = (1 - t \cot t)D\omega + 2\lambda^{-1}(D\lambda - 2\lambda)D\omega - 2\lambda^{-1}(t \partial_\theta \lambda)(t \partial_\theta \omega), \tag{43}
\]
derived from equations (35) and (36), on \((0, \delta] \times \mathbb{S}^2\). The expansions of \(\lambda\) and \(\omega\) at \(t = 0\) are
\[
\lambda(t, \theta) = t^2 S_{ss}(\theta) + O(t^4),
\]
\[
\omega(t, \theta) = \omega_\ast + \omega_\ast \theta + O(t^4),
\]
and all higher terms are proportional to even powers of \(t\). Thus, \(\lambda\) and \(\omega\) can be extended to functions in \(C^\infty([-\delta, \delta] \times \mathbb{S}^2)\) which are even in \(t\): \(\lambda(t, x) = \lambda(-t, x)\), \(\omega(t, x) = \omega(-t, x)\). These extended functions satisfy equations (42)–(43) on \([-\delta, \delta] \times \mathbb{S}^2\) in the sense of uniform limits at \(t = 0\).

**Proof.** The main technical tool here is the theory of \((\text{order } n)\)-leading order terms, see [2] which can be generalized directly to our case. This is an algorithm to compute leading-order expansions of arbitrarily high order for solutions with a given leading-order term. (Here the leading-order term is of the form equation (41).) When we use this method to compute expansions of \((S, \omega)\) of any order in \(t\) at \(t \searrow 0\), we find that only the first term in the expansion of \(S\) is a log-term, and all other terms in the expansions of both \(S\) and \(\omega\) are positive even integer powers of \(t\). Hence \((\lambda, \omega)\) can be extended as smooth, even functions as claimed above. Moreover, it is easy to see that equations (42) and (43) for \((\lambda, \omega)\) are invariant under
the transformation $t \mapsto -t$. This proves that the extended functions $(\lambda, \omega)$ satisfy the equations on both intervals $t \in [-\delta, 0)$ and $t \in (0, \delta]$. Both limits $t \searrow 0$ and $t \nearrow 0$ of the equations, which involve formally singular terms at $t = 0$, exist uniformly in space, and as a result, the equations are also satisfied at $t = 0$.

In the following, we often consider functions, whose definition involves the quantities $S$ and $\omega$, and we attempt to extend those to the time interval $[-\delta, \delta]$. Then we understand that we first replace $S$ by $\ln \lambda$, where $\lambda$ is the extended function in lemma 3.3. If it is possible to find a smooth extension in this sense, then we say that the function is extendible to $[-\delta, \delta]$.

Given a solution of the main evolution equations as above, we now attempt to solve the remaining equations implied by Einstein’s field equations thereby determining the remaining unknown $M$. We repeat that while there is a general result for existence of solutions of the main evolution equations also in [46], the problem of solving these remaining equation has not been considered there. We address this problem now and find the following.

**Proposition 3.4.** Let $R_0 > 0$ and $\omega_0$ be constants, and $\omega_0$ and $S_0$ be rotationally symmetric functions in $C^\infty(S^2)$ satisfying

$$S_0(0) = S_0(\pi).$$

Suppose that $S(\omega)$ is the corresponding solution of equations (35)–(36) according to proposition 3.2 on $(0, \delta] \times S^2$, and $(\lambda, \omega)$ the smooth continuation to $[-\delta, \delta] \times S^2$ according to lemma 3.3. Then there is a unique smooth function $M \in C^\infty((0, \delta] \times S^2)$, which is rotationally symmetric and satisfies the smoothness condition (22) at each $t \in (0, \delta]$, so that the 3-metric $h$ given by equation (34) is smooth and so that the $(2 + 1)$-Einstein equations (3), which are represented by equations (37) and (38), are satisfied on $(0, \delta] \times S^2$. This function $M$ can be extended as a smooth function to $[-\delta, \delta] \times S^2$ which is even in $t$ and satisfies the smoothness condition on the extended time interval. The expansion of $M$ at $t = 0$ is

$$M(t, \vartheta) = S_0(\vartheta) - 2S_0(0) + 2\ln R_0 + O(t^2).$$

We note that the smoothness condition equation (22), which takes the form $e^M = (R_0^2 \sin^2 t) e^{-\delta}$ at $\vartheta = 0, \pi$, is non-trivial in the limit $t \to 0$ due to the presence of the singular factor $e^{-\delta}$. However, the limit turns out to be well defined since the singular behavior of $e^{-\delta} = O(t^{-2})$ is canceled exactly by the factor $\sin^2 t$.

The steps of the proof are as follows. The details will be discussed afterward. We consider equations (38) and first recall that those equations are not in a coordinate independent geometric form, and are, in particular, not well defined at $\vartheta = 0, \pi$. However, each equation of the system (37)–(38) is a linear combination of the components of these $(2 + 1)$-dimensional Einstein equations (3), which can be understood as the difference of the Einstein tensor of the 3-metric $h$ and the energy–momentum tensor corresponding to the scalar fields $\lambda$ and $\omega$. Therefore, as long as the 3-metric $h$ is a smooth metric on the quotient manifold $(0, \delta] \times S^2$, and since the functions $\lambda$ and $\omega$ as solutions of the previous proposition are smooth, the $(2 + 1)$-dimensional Einstein equations are satisfied on $(0, \delta] \times S^2$ if and only if they are satisfied on the dense subset $(0, \delta] \times S^2$ with $\bar{S}^2 := S^2 \setminus \{ \text{north pole, south pole} \}$, that is, precisely where all terms in equations (38) are well defined. This insight allows us to treat equations (37)–(38) as follows. For given smooth solutions $\lambda$ and $\omega$ above, we look for a function $M$ which (i) is smooth on $(0, \delta] \times S^2$, (ii) satisfies the smoothness condition equation (22) at the poles $\vartheta = 0$ and $\vartheta = \pi$ and (iii) satisfies equations (38) on $(0, \delta] \times S^2$. In a subsequent step, we can then show that this implies that equation (37), and hence the full set of $(2 + 1)$-Einstein equations is satisfied on $(0, \delta] \times S^2$, and therefore on the full domain $(0, \delta] \times S^2$. When this is achieved, we show that this function $M$ can be extended as a smooth function to $[-\delta, \delta] \times S^2$ which is even in
t at \( t = 0 \) and therefore satisfies the smoothness condition equation (22) on this extended time interval. In total, this gives a smooth 3-metric \( h \), which, on the one hand, satisfies the \((2 + 1)\)-dimensional Einstein equations on \((0, \delta] \times S^2 \), and which, on the other hand, turns out to be compatible with the hypotheses of smooth Gowdy-symmetric generalized Taub–NUT solutions.

We use the following technical trick. We consider the map

\[
\Psi : \mathbb{T}^2 \rightarrow S^2 : (\alpha, \beta) \mapsto (y_1, y_2, y_3) = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha) \in S^2 \subset \mathbb{R}^3,
\]

where \( \alpha \) and \( \beta \) are \( 2\pi \)-periodic coordinates on \( \mathbb{T}^2 \). This allows us to pull back all smooth rotationally symmetric functions on \( S^2 \) to functions on \( \mathbb{T}^2 \), which are (a) \( 2\pi \)-periodic with respect to \( \alpha \), (b) even in \( \alpha \) at \( \alpha = 0 \), and (c) independent of \( \beta \). On the other hand, since the above implies that we only need to solve equations (38) on the spatial domain \( \mathbb{T}^2 \) where \( \Psi^{-1} \) is well defined, we can also push forward the coordinate vector field \( \partial_\vartheta \) to \( \mathbb{T}^2 \) and extend the result as a global smooth vector field which then coincides with \( \partial_\vartheta \). In summary, in order to satisfy conditions (i) and (iii) in the previous paragraph for the function \( M \), we replace each function in equations (38) by its composition with \( \Psi \) (this can be understood as extending the variable \( \vartheta \) in each function \( 2\pi \)-periodically) and each \( \partial_\vartheta \)-derivative by an \( \alpha \)-derivative. Then we look for a solution of this version of equations (38) for the unknown \( M \circ \Psi \) which is even in \( \alpha \) at \( \vartheta = 0 \) for every time \( t \in (0, \delta] \). The corresponding function \( M \), which at first is defined only on \((0, \delta] \times \mathbb{T}^2 \), then extends as a smooth function to \((0, \delta] \times S^2 \) and is a solution of the equation equations (38) on \((0, \delta] \times \mathbb{T}^2 \).

We follow this approach now; in order to simplify the notation, however, we write \( S, \omega, \lambda, M \) and \( R \) for the functions defined on \( \mathbb{T}^2 \) as opposed to \( S \circ \Psi, \omega \circ \Psi \) etc., and we write \( \partial_\vartheta \) instead of \( \partial_\varphi \) for the vector field on \( \mathbb{T}^2 \), as long as there is no confusion with the corresponding quantities on \( \mathbb{T}^2 \). We then refer to any function \( f(t, \vartheta) \) on \([-\delta, \delta] \times \mathbb{T}^2 \), which does not depend on \( \beta \), as \( t \)-even (or \( \vartheta \)-even) if \( f(t, \vartheta) = f(-t, \vartheta) \) at every \( \vartheta \) (or \( f(t, \vartheta) = f(t, -\vartheta) \) at every \( t \)) similar we define the notion of \( t \)-odd and \( \vartheta \)-odd functions.

The most difficult step of the proof of proposition 3.4 is summarized in the following lemma, which we prove first.

**Lemma 3.5.** Let \( R_0 > 0 \) and \( \omega_0 \) be constants, and \( \omega_\pm \) and \( S_\pm \) be rotationally symmetric functions in \( C^\infty(S^2) \). Suppose that \((S, \omega)\) is the corresponding solution of equations (35)–(36) according to proposition 3.2 on \((0, \delta] \times \mathbb{T}^2 \), and \((\lambda, \omega)\) the smooth continuation to \([-\delta, \delta] \times \mathbb{T}^2 \) according to lemma 3.3. Then the two functions

\[
F_0 := -\frac{R_+ \mu_- - R_- \mu_+}{4R_+ R_-} - \frac{2}{t}, \quad F_1 := \frac{R_+ \mu_- + R_- \mu_+}{4R_+ R_-} \sin \vartheta.
\]

are smooth \( \vartheta \)-even functions on \([-\delta, \delta] \times \mathbb{T}^2 \), where \( F_0 \) is \( t \)-odd and \( F_1 \) is \( t \)-even. We have

\[
F_0(0, \cdot) = 0, \quad F_1(0, \cdot) = \frac{2}{\sin \vartheta} \partial_\vartheta S_\pm.
\]

The issue which is addressed in this lemma is the following. Since \( R_\pm = R_0 \sin(t \pm \vartheta) \), the functions \( F_0 \) and \( F_1 \) could, \textit{a priori}, be singular along the diagonals of the Gowdy square where one of the functions \( R_\pm \) vanishes. Hence, when equations (38) are written as

\[
M_t = -(S_t - 2/t) + F_0, \quad M_\vartheta = -S_\vartheta + F_1 \sin \vartheta,
\]

it may then follow that there exists no smooth solution for the remaining metric function \( M \), even if the functions \( S \) and \( \omega \) are smooth. Since the functions \( F_0 \) and \( F_1 \) are smooth, however, as
asserted by the previous lemma, we briefly see that a smooth solution for $M$ exists. In this case, we say that equations (47) satisfy matching conditions. As shown by Chruściel [17], there is a generic subset of Cauchy data for $S$ and $ω$ prescribed at any $t ∈ (0, π)$ which guarantee that the matching conditions are satisfied on that Cauchy hypersurface. It is then a consequence of the main evolution equations that the matching conditions are satisfied at every time $t$ where the solution is defined. In our case, however, we do not solve a Cauchy problem for $S$ and $ω$ and hence this fact does not help. Indeed, it is a rather surprising result of the previous lemma that all choices of asymptotic data for $S$ and $ω$ are compatible with the matching conditions.

We also see from equations (47) that the term $2/t$ in the definition of $F_0$ has been introduced to cancel precisely the most singular term of $S_t$; we see in particular that $S_t - 2/t$ can be extended as a smooth function to $[-δ, δ] × T^2$ which is $t$-even and $\vartheta$-even.

Another remark about lemma 3.5 is that the term $2/t$ in equation (46) is indeed a smooth function on $T^2$ because $S_{++}$ is $\vartheta$-even.

**Proof of lemma 3.5.** Let us define the functions

$$f_0 := t(R_+\mu_- - R_-\mu_+), \quad f_1 := \frac{R_+\mu_- + R_-\mu_+}{\sin \vartheta},$$

with $\mu_{±}$ given by equation (45), so that

$$F_0 = -\frac{1}{t} \frac{f_0}{4R_+^2 - 2} - \frac{2}{t}, \quad F_1 = \frac{f_1}{4R_+^2 - 2}.$$ 

In a first step, it is straightforward to show that $f_0$ and $f_1$ are smooth $\vartheta$-even functions on $T^2$. for every $t ∈ (0, δ]$ using the definitions of the functions $\mu_{±}$ in equation (45); notice that this is neither the case for $\mu_+$, $\mu_-$, $R_+$ nor $R_-$ individually. Moreover, by using our knowledge about the behavior of $S$ and $ω$ as solutions of proposition 3.2 and the corresponding extensions $(λ, ω)$ according to lemma 3.3 and hence by expressing the quantities above by $λ$ and $ω$ and their derivatives, lengthy computations confirm that $f_0$ and $f_1$ extend as smooth functions to $[-δ, δ] × T^2$ which are $t$-even and $\vartheta$-even.

We now show that the smooth functions $f_0$ and $f_1$ vanish precisely where the product $R_+ R_-$ vanishes. After some computations, we find that the main evolution equations lead to the following equations for $\mu_+$ and $\mu_-:

$$\partial_+ \mu_- = -R_-(S_+ S_- + e^{-2S}\omega_+ \omega_-), \quad \partial_- \mu_+ = -R_+(S_+ S_- + e^{-2S}\omega_+ \omega_-).$$  

(48)

This implies that along the diagonal $\vartheta = t$, where $R_- = 0$, we have that $\mu_- = \mu_-^{(0)} = \text{constant}$ and hence

$$f_0|_{\vartheta = t} = t R_0 \sin(2t)\mu_-^{(0)}, \quad f_1|_{\vartheta = t} = \frac{R_0 \sin(2t)\mu_-^{(0)}}{\sin t},$$

for all $t ∈ (0, δ]$. Since both are smooth functions, their expansions at $\vartheta = t = 0$ along the curve $\vartheta = t$ are

$$f_0|_{\vartheta = t} = 2R_0\mu_-^{(0)} t^2 + O(t^3), \quad f_1|_{\vartheta = t} = 2R_0\mu_-^{(0)} + O(t).$$

On the other hand, the definitions of $f_0$ and $f_1$ and the uniform leading-order expansions of $λ$ and $ω$ at $t = 0$,

$$λ = t^2 e^{S_0} + O(t^3), \quad ω = ω_0 + t^2 ω_1 + O(t^3),$$

imply

$$f_0|_{\vartheta = t} = O(t^3), \quad f_1|_{\vartheta = t} = O(t).$$

We conclude that $\mu_-^{(0)} = 0$ and hence that

$$f_0|_{\vartheta = t} = 0, \quad f_1|_{\vartheta = t} = 0,$$
for all $t \in [-\delta, \delta]$. The same argument applied at $\vartheta = \pi - t$, where $R_+ = 0$ and hence $\mu_+ = \text{const}$, leads to the similar result

$$f_0|_{\vartheta = \pi - t} = 0, \quad f_1|_{\vartheta = \pi - t} = 0,$$

for all $t \in [-\delta, \delta]$.

Let us consider a small neighborhood of the point $(t, \vartheta) = (0, 0)$ where we now introduce coordinates $x = t + \vartheta$ and $y = t - \vartheta$. With our analysis above, we have found that there exist smooth functions $f_0(x, y)$ and $f_1(x, y)$ so that

$$f_0(x, y) = xy \tilde{f}_0(x, y), \quad f_1(x, y) = xy \tilde{f}_1(x, y).$$

Since $xy = t^2 - \vartheta^2$, the functions $\tilde{f}_0, \tilde{f}_1$ must therefore be $\vartheta$-even and $t$-even. Now we note that the product

$$R_+ R_- = R_0^2 (\cos \vartheta - \cos t)(\cos \vartheta + \cos t) = R_0^2 \sin x \sin y.$$ 

Hence, the quotients $f_0/(R_+ R_-)$ and $f_1/(R_+ R_-)$, which appear in the definitions of $F_1$ and $F_0$, are smooth functions on our small neighborhood of $(0, 0)$. The same argument applies to a neighborhood of the point $(t, \vartheta) = (0, \pi)$, and hence $f_0/(R_+ R_-)$ and $f_1/(R_+ R_-)$ are smooth functions everywhere on $[-\delta, \delta] \times \mathbb{T}^2$ which are $\vartheta$-even and $t$-even. As a consequence, $F_1$ is a smooth function on $[-\delta, \delta] \times \mathbb{T}^2$ which is $\vartheta$-odd and $t$-even, and $F_0$ a smooth function which is $t$-even and $\vartheta$-even.

It only remains to compute the values of the extended functions $F_0$ and $F_1$ at $t = 0$, i.e. equation (46). This can be done directly using the leading-order behavior of $\lambda$ and $\omega$. \hfill $\square$

**Proof of proposition 3.4.** We have proven in lemma 3.5 that all terms in equations (47) are well defined when they are considered as functions with spatial domain $\mathbb{T}^2$ as discussed above. The integrability condition is satisfied, and we can hence conclude that there exists a unique solution $M$ on $[-\delta, \delta] \times \mathbb{T}^2$ which is $\vartheta$-even, as soon as the value $M_0$ of $M$ has been fixed somewhere, say, at $(t, \vartheta) = (0, 0)$. This solution can be written as

$$M(t, \vartheta) = M_* - (S(t, \vartheta) - 2 \ln t) + \int_0^t F_0(0, x) \sin x \, dx + \int_0^t F_0(\tau, \vartheta) \, d\tau,$$

for some constant $M_*$. Since $F_0$ vanishes at $t = 0$ and is $t$-odd, the function $M$ must be $t$-even.

Now let us consider the smoothness condition equation (22), i.e. we want to show that

$$M(t, \vartheta) = 2 \ln R_0 + S(t, \vartheta) - 2 \ln \sin t|_{\vartheta = \pi, t} = 0,$$

for all $t \in [-\delta, \delta]$. Using the expression for $M$ above, we compute

$$M(t, \vartheta) = 2 \ln R_0 + S(t, \vartheta) - 2 \ln \sin t = M_* + 2(\ln t - \ln \sin t) + 2S_* - 2S_0(0) - 2 \ln R_0 + \int_0^t F_0(\tau, \vartheta) \, d\tau.$$

On the other hand, we can evaluate $F_0$ at $\vartheta = 0$ (and below at $\vartheta = \pi$) using equations (44)–(45) (recall that $R = 0$ at $\vartheta = 0, \pi$). We find

$$F_0|_{\vartheta = 0} := -\frac{R_+ R_- - R_- R_+}{R_+ R_-} - \frac{2}{t} \bigg|_{\vartheta = 0} = -\sin t \cos t - \sin t \cos t - \frac{2}{t} = 2 \left(\frac{\cos t}{\sin t} - \frac{1}{t}\right).$$
The integral of this expression is
\[ \int_0^1 F_0(\tau, 0) \, d\tau = -2(\ln t - \ln \sin t). \]

Hence,
\[ M(t, 0) - 2 \ln R_0 + S(t, 0) - 2 \ln \sin t = M_* - 2 \ln R_0, \]
which vanishes for all \( t \) if and only if
\[ M_* = 2 \ln R_0. \]

Now we perform the same analysis at \( \vartheta = \pi \). We find
\[ F_0|_{\vartheta=\pi} = 2 \left( \frac{\cos t}{\sin t} - \frac{1}{t} \right). \]

Therefore,
\[ M(t, \pi) - 2 \ln R_0 + S(t, \pi) - 2 \ln \sin t = M_* + 2(S_+ - S_+ - (0)) - 2 \ln R_0. \]

Given the value for \( M_* \) above, the smoothness condition at \( \vartheta = \pi \) is satisfied if and only if
\[ S_+ = S_+ - (0). \]

Now it remains to show that equation (37) is satisfied on \((0, \delta] \times S^2\). We define
\[ C_1 := (C_1 + C_2)/2 \quad \text{and} \quad C_2 := (C_1 - C_2)/2 \]
from equations (38), and \( H \) as defined in equations (37). The system (35) and (36) implies the subsidiary system
\[ \partial_\lambda C_1 - \partial_\lambda C_2 = 0, \quad \partial_\lambda C_2 - \partial_\lambda C_1 = H + \cot C_2 + \cot \vartheta C_1. \]

Since \( C_1 \) and \( C_2 \) vanish identically on \((0, \delta] \times S^2\), so must \( H \). We conclude that our solutions do solve equation (37); indeed, the full set of Geroch’s equations is satisfied on \((0, \delta] \times S^2\).

All functions constructed so far can now be lifted to smooth functions on \([-\delta, \delta] \times S^3\) which are invariant along \( \partial_\rho \) and \( \partial_\varphi \). In particular, the quantities \( \lambda, \omega \) and \( h_{ab} \) allow us to determine the metric \( g_{ab} \) on \((0, \delta] \times S^3\). It remains to compute \( Q \) on \((0, \delta] \times S^3\) and to show that it can be extended as a smooth function to \([-\delta, \delta] \times S^3\), which is even in time. We must therefore find a smooth function \( Q \) which satisfies
\[ \partial_\vartheta Q = -R\lambda^{-2}\partial_\lambda \omega, \quad \partial_\varphi Q = -R\lambda^{-2}\partial_\lambda \omega, \quad (49) \]
from equation (13). In the same the spirit as above, we can ignore the fact that, strictly speaking, these equations are not defined at \( \vartheta = 0, \vartheta = \pi \). Moreover, \( Q \) must also satisfy the smoothness condition equation (23). Given a solution \((\delta, \omega)\) of proposition 3.2, or equivalently the extensions \( (\lambda, \omega) \) as above, the integrability condition of equation (49) follows for both time intervals \((0, \delta]\) and \([-\delta, 0]\). The expansions of \( \lambda \) and \( \omega \) can be used to show that the right-hand sides of equations (49) are in fact smooth functions on \([-\delta, \delta] \times S^2\) which can be lifted to a dense subset of \([-\delta, \delta] \times S^3\). It can be concluded that, once the value of \( Q \) is fixed at some point, there is a unique smooth function \( Q \) on \([-\delta, \delta] \times S^3\) which satisfies the equations above and the smoothness condition. The expansion for \( Q \) at \( t = 0 \) has no \( \log t \)-terms and all powers in \( t \) are positive even integers. From
\[ \partial_\vartheta Q(0, \vartheta) = -4R\omega_\omega \omega_\omega(0)e^{-2S_+ - (0)} \sin \vartheta, \]
it follows that
\[ Q(0, \vartheta) = Q_0 - 4R\vartheta \int_0^{\vartheta} \omega_\omega(x)e^{-2S_+ - (x)} \sin x \, dx. \]
for some constant $Q_0$. In order to guarantee the smoothness condition equation (23), namely $Q(t,0) = 1, Q(t,\pi) = -1$ for all $t > 0$, we must choose $Q_0 = 1$ and the asymptotic data must satisfy

$$
\int_0^\pi \omega_+ (x) e^{-2S_+ (x)} \sin x \, dx = \frac{1}{2R_0},
$$

(50)

Then it follows that $Q(0,0) = 1, Q(0,\pi) = -1$, and the second part of equations (49) implies that $Q(t,0) = 1, Q(t,\pi) = -1$ for all $t \in [-\delta, \delta]$. We have found the expansion

$$Q(t, \theta) = 1 - 4R_0 \int_0^\theta \omega_+ (x) e^{-2S_+ (x)} \sin x \, dx + \frac{1}{2} R_0 \partial_\theta \omega_+ (\theta) e^{-2S_+ (\theta)} \sin \theta t^2 + O(t^4).
$$

Equivalent to prescribing the data functions $S_+$ and $\omega_+$ which obey equation (50), we can prescribe free smooth rotationally symmetric data functions $S_+$ and $\omega_+$, and set

$$w_+ (\theta) = e^{2S_+ (\theta)} \frac{1 - \partial_\theta Q_+ (\theta) \sin \theta - 2Q_+ (\theta) \cos \theta}{4R_0}.
$$

Then, equation (50) follows automatically. The above expansion for $Q$ simplifies to

$$Q(t, \theta) = \cos \theta + Q_+ (\theta) \sin^2 \theta + O(t^2),
$$

and the smoothness conditions for $Q$ at $\theta = 0, \pi$ become manifest.

Hence, we obtain smooth Gowdy vacuum solutions on the time interval $(0, \delta]$. In order to show that these solutions are indeed smooth Gowdy-symmetric generalized Taub–NUT solutions and hence extendible through $t = 0$, we must perform the coordinate transformation equation (31) near $t' = 0$. Since all the metric functions are smooth through $t = 0$ and even in $t$ and independent of $\rho_1$ in particular, they can be extended as smooth functions in terms of the new coordinates through $t' = 0$. It remains to show that the expression $(4\kappa^2N^2 - e^{2M})/t'$ is also a smooth function through $t' = 0$ for some choice of $\kappa$. For this, we compute the uniform limit of the quantity $N$ (defined in equation (30)) at $t = 0$,

$$N(0, \theta) = e^{2(S_+ (\theta) - S_+ (0))}.
$$

From that it is easy to determine that the quotient above is extendible if

$$\kappa = \pm \frac{R_0^2}{2} e^{-S_+ (0)}.
$$

We have thus obtained smooth Gowdy-symmetric generalized Taub–NUT solutions in a (possibly small) time interval $(0, \delta]$. The global existence theorem of Chruściel [17] (theorem 6.3 in [17], which makes essential use of results by Christodoulou and Tahvildar–Zadeh [16]) can now be used to extend the spacetimes to the time interval $(0, \pi)$ as smooth globally hyperbolic Gowdy solutions.

4. The linear problem and global-in-time properties

We have seen above that for given smooth asymptotic data at $t = 0$ (e.g. the values of $S_+$ and $Q_+$), a smooth Gowdy-symmetric generalized Taub–NUT solution exists in the vicinity of $t = 0$. Moreover, using Chruściel’s theorem, we see that this solution can even be extended smoothly to the whole time interval $(0, \pi)$. However, the surface $t = \pi$ itself is expected to contain either singularities or Cauchy horizons. On the other hand, Chruściel’s result also allows the case that the $t = \pi$-surface is regular, but just the coordinates break down there. It is the purpose of the following considerations to find out what happens at $t = \pi$. In particular, we construct explicitly the metric potentials at this boundary (as well as on the axes $\theta = 0, \pi$) in terms of the asymptotic data. For that purpose, we apply the so-called soliton methods,
which were used in [28] for the investigation of $S^2 \times S^1$ Gowdy spacetimes$^9$, to the present $S^3$-symmetric case.

In all of what follows we make the same hypotheses as in theorem 3.1. These assumptions are consistent with those listed in section 3.2, and hence we consider ‘smooth Gowdy-symmetric generalized Taub–NUT solutions’.

4.1. Einstein’s field equations and the Ernst formulation

The first important step for the following considerations is the introduction of the complex Ernst formulation of the Einstein equations which will be described in this subsection.

Again we start from the metric
\[ g = e^{2u}(-dr^2 + d\theta^2) + R_0 \sin t \sin \theta \left[e^{2t} (d\rho_1 + Q d\rho_2)^2 + e^{-t} d\rho_3^2 \right] \]  \hspace{1cm} (51)

in the Killing basis $\{\partial_{\rho_1}, \partial_{\rho_2}\}$. Here, we express $L$ in terms of a metric potential $u$ via
\[ e^L = \frac{\sin t}{\sin \theta} e^u. \]  \hspace{1cm} (52)

In this way, we arrive at
\[ g = e^{2u}(-dr^2 + d\theta^2) + R_0 \left[\sin^2 t e^{2t} (d\rho_1 + Q d\rho_2)^2 + \sin^2 \theta e^{-t} d\rho_3^2 \right]. \]  \hspace{1cm} (53)

Note that $u$ is related to the quantity $S$ (defined in section 3.3.2) via
\[ u(t, \theta) = S(t, \theta) - \ln(R_0) - 2 \ln \sin t, \]
i.e. the singularity of $S$ at $t = 0$ ($S$ behaves as $2 \ln t$ for $t \to 0$, see proposition 3.2) is removed by subtracting the term $2 \ln \sin t$.

Now we reformulate the Einstein equations as equations for $u$, $Q$ and $M$. We obtain two second-order equations for the metric potentials $u$ and $Q$.

\[ -\partial_t^2 u - \cot t \partial_u u + \partial_u^2 u + \cot \theta \partial_\theta u + e^{2u} \frac{\sin^2 t}{\sin^2 \theta} [(\partial_t Q)^2 - (\partial_\theta Q)^2] + 2 = 0, \]  \hspace{1cm} (54)

\[ -\partial_t^2 Q - 3 \cot t \partial_u Q + \partial_u^2 Q - \cot \theta \partial_\theta Q - 2[(\partial_t u) (\partial_t Q) - (\partial_u u) (\partial_\theta Q)] = 0 \]  \hspace{1cm} (55)

and two first-order equations for $M$.

\[ (\cos^2 t - \cos^2 \theta) \partial_u M = \frac{1}{2} e^{2u} \sin^3 t \frac{1}{\sin \theta} \left[\cos t \sin \theta [(\partial_u Q)^2 + (\partial_\theta Q)^2] - 2 \sin t \cos \theta (\partial_u Q)(\partial_\theta Q) \right] \]
\[ + \frac{1}{2} \sin t \sin \theta \left[\cos t \sin \theta [(\partial_u u)^2 + (\partial_\theta u)^2] - 2 \sin t \cos \theta (\partial_u u)(\partial_\theta u) \right] \]
\[ - (2 \cos^2 t \cos^2 \theta - \cos^2 t - \cos^2 \theta) \partial_u u \]
\[ - 2 \sin t \cos t \sin \theta \cos \theta (\partial_u u + \tan \theta), \]  \hspace{1cm} (56)

\[ (\cos^2 t - \cos^2 \theta) \partial_\theta M = \frac{1}{2} e^{2u} \sin^3 t \frac{1}{\sin \theta} \left[\sin t \cos \theta [(\partial_u Q)^2 + (\partial_\theta Q)^2] - 2 \cos t \sin \theta (\partial_u Q)(\partial_\theta Q) \right] \]
\[- \frac{1}{2} \sin t \sin \theta \left[\sin t \cos \theta [(\partial_u u)^2 + (\partial_\theta u)^2] - 2 \cos t \sin \theta (\partial_u u)(\partial_\theta u) \right] \]
\[- 2 \sin t \cos t \sin \theta \cos \theta (\partial_\theta u + \tan t) \]
\[- (2 \cos^2 t \cos^2 \theta - \cos^2 t - \cos^2 \theta) \partial_\theta u. \]  \hspace{1cm} (57)

$^9$ The methods described [28] have also been applied to studying the interior region of axisymmetric and stationary black holes with surrounding matter, see [4, 5, 27].
Since $M$ does not appear in (54) and (55) and since we assume the genericity condition of Chruściel, these equations may be solved as a first step. Afterward, (56) and (57) can be used to calculate $M$ via a line integral. Note that the integrability condition $\partial_r \partial_t M = \partial_r \partial_t M$ of the system (56), (57) is satisfied as a consequence of (54), (55). Hence, $M$ does not depend on the path of integration.

It turns out that the two Einstein equations (54) and (55) are equivalent to a single complex equation, namely to the Ernst equation

$$\Re(\mathcal{E}) = \partial_t \mathcal{E} + \cot \theta \partial_\theta \mathcal{E} + \psi \partial_\psi \mathcal{E} = -(\partial_r \mathcal{E})^2 + \partial_\theta \mathcal{E}^2$$

(58)

for the complex Ernst potential $\mathcal{E} = f + ib$. Here, the real part $f$ of $\mathcal{E}$ is defined in terms of the Killing vector $\partial_\rho$ by

$$f := \frac{1}{R_0} g(\partial_\rho, \partial_\rho) = Q^2 e^u \sin^2 t + e^{-u} \sin^2 \theta$$

(59)

and the imaginary part $b$ is given by

$$\partial_\rho a = \frac{1}{f^2} \sin t \sin \theta \partial_\rho b, \quad \partial_\rho b = \frac{1}{f^2} \sin t \sin \theta \partial_\rho b$$

(60)

with

$$a := \frac{g(\partial_\rho, \partial_\rho)}{g(\partial_\rho, \partial_\rho)} = \frac{Q}{f} e^u \sin^2 t.$$

(61)

Note that for smooth functions $u$ and $Q$ the Ernst potential $\mathcal{E}$ is also smooth: For the real part $f$, smoothness is clear from definition (59). In the case of the imaginary part $b$, it can be shown by solving the two equations in (60) for $\partial_\rho b$ and $\partial_\rho b$ and replacing $a$ and $f$ via (61) and (59). The resulting expressions for $\partial_\rho b$ and $\partial_\rho b$ in terms of $Q$ and $u$ (and their first order derivatives) turn out to be smooth functions, if we use the fact that $Q$ behaves as given in (23). Hence, integration will lead to a smooth function $b$. Therefore, we can conclude from the previous local existence results and Chruściel’s global existence theorem that for any given set of asymptotic data (as described in theorem 3.1) the corresponding Ernst potential $\mathcal{E}$ is a smooth complex function on $(-T, \pi) \times S^2$ for some $T > 0$. In the following we investigate under which conditions $\mathcal{E}$ can be extended smoothly to the boundary $t = \pi$ and beyond. Note that our assumptions imply that $f > 0$ holds in the entire Gowdy square with the exception of the points $A$ and $B$ and with the possible exception of the future boundary $\mathcal{H}$ (see figure 1) which is important since we will divide by $f$ in some of the following formulae.

Once we have obtained an Ernst potential $\mathcal{E}$ as a solution to the Ernst equation (58), we can calculate the corresponding metric potentials from it. It turns out that the integrability condition $\partial_t \partial_r a = \partial_t \partial_r a$ of (60) is satisfied as a consequence of the Ernst equation. Therefore, $a$ may be calculated via line integration from $\mathcal{E}$. The metric potentials $u$ and $Q$ can then be obtained from $a$ and $f$. With (59) and (61) we find

$$e^u = \frac{f^2 a^2}{\sin^2 t} + \frac{\sin^2 \theta}{f}, \quad Q = \frac{f^2 a}{f^2 a^2 + \sin^2 t \sin^2 \theta}.$$

(62)

Finally, $M$ may be calculated using (56) and (57), as mentioned earlier.

As an example, we give the Ernst potential for the Taub solution:

$$f = \frac{2l}{X} \sin^2 t \cos^2 \theta + \frac{X}{2l} \sin^2 \theta,$$

(63)

$$b = \frac{1}{X} [\cos t (\cos^2 t - 3) \sqrt{m^2 + l^2 - 2m} \cos^2 \theta + \cos t$$

(64)

with $X := (1 + \cos^2 t) \sqrt{m^2 + l^2} + 2m \cos t$. (Here we have set an arbitrary additive integration constant in $b$ to zero.)
We integrate the LP along the boundaries of the Gowdy square (dashed path) in order to investigate for which asymptotic data the solution can be regularly extended up to the future boundary $H_f (t = \pi)$. 

### 4.2. The linear problem

Interestingly, the Ernst equation (58) belongs to a remarkable class of nonlinear partial differential equations for which an associated linear problem (LP) exists which is equivalent to the nonlinear equation via its integrability condition. For applications of this LP in the context of axisymmetric and stationary spacetimes we refer the reader to, e.g. [38, 37]. In the Gowdy setting, we use the LP in the form [34, 35], which reads in our coordinates\(^{10}\) as

\[
\begin{align*}
\partial_x \Phi &= \left[ \begin{pmatrix} B_x & 0 \\ 0 & A_x \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} 0 & B_x \\ A_x & 0 \end{pmatrix} \right] \Phi, \\
\partial_y \Phi &= \left[ \begin{pmatrix} B_y & 0 \\ 0 & A_y \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & B_y \\ A_y & 0 \end{pmatrix} \right] \Phi,
\end{align*}
\]

where the pseudopotential \( \Phi = \Phi(x, y, K) \) is a 2 x 2 matrix depending on the coordinates

\[
x = \cos(t - \theta), \quad y = \cos(t + \theta)
\]

as well as on the spectral parameter \( K \in \mathbb{C} \). The remaining ingredients of the LP are the function \( \lambda \),

\[
\lambda(x, y, K) := \sqrt{\frac{K - y}{K - x}}
\]

and the matrix elements \( A_x, A_y, B_x \) and \( B_y \), defined in terms of the Ernst potential as

\[
A_i = \frac{\partial_i \mathcal{E}}{2f}, \quad B_i = \frac{\partial_i \bar{\mathcal{E}}}{2f}, \quad i = x, y.
\]

Due to the two possible signs of the square root in (67), \( \lambda : \mathbb{C} \rightarrow \mathbb{C}, K \mapsto \lambda \) describes, for fixed values \( x, y \), a mapping from a two-sheeted Riemann surface (K-plane) onto the complex \( \lambda \)-plane. The two \( K \)-sheets are connected at the branch points

\[
K_1 = x \quad (\lambda = \infty), \quad K_2 = y \quad (\lambda = 0).
\]

\(^{10}\)The formal relation between our coordinates (describing Gowdy spacetimes with two space-like Killing vectors) and the Weyl–Lewis–Papapetrou coordinates \((\rho, \zeta, \varphi, \bar{t})\) as used by Neugebauer (describing axisymmetric and stationary spacetimes with one space-like and one time-like Killing field) is given by \( \rho = iR_0 \sin t \sin \theta, \quad \zeta = R_0 \cos t \cos \theta, \quad \psi = \rho_1, \quad \bar{t} = \rho_2. \)
In general, the pseudopotential $\Phi$ will take on different values on the two $K$-sheets. Only at the branch points it has to be unique, since both Riemannian sheets coincide there. We will see below that this observation plays an important role for the calculation of the Ernst potential from the solution of the LP.

As already mentioned, the integrability condition $\partial_q \partial_\theta \Phi = \partial_\theta \partial_q \Phi$ of (65) is equivalent to the Ernst equation (58). Hence, the Ernst equation is a consequence of the LP and, on the other hand, for a given potential $\mathcal{E}$ as a solution to the Ernst equation, the matrix $\Phi$ does not depend on the path of integration.

Finally, we note that for any solution $\Phi$ to the LP (65), the product $\Phi C(K)$, where $C(K)$ is an arbitrary $2 \times 2$ matrix, is also a solution (corresponding to the same Ernst potential). As shown by Neugebauer [38], it is always possible to choose $C(K)$ in such a way that the transformed pseudopotential takes the form

$$
\Phi^+(x, y, K) = \begin{pmatrix} \psi^+(x, y, K) & \psi^+(x, y, K) \\ \psi^+(x, y, K) & -\psi^+(x, y, K) \end{pmatrix},
$$

$$
\Phi^-(x, y, K) = \begin{pmatrix} \psi^-(x, y, K) & \psi^-(x, y, K) \\ \psi^-(x, y, K) & -\psi^-(x, y, K) \end{pmatrix},
$$

(70)

where the superscripts ‘$>$’ or ‘$<$’ indicate whether the functions are evaluated on the ‘upper’ ($\lambda = 1$ for $K = \infty$) or ‘lower’ ($\lambda = -1$ for $K = \infty$) $K$-sheet. Hence, $\Phi$ is completely determined by the values of two functions $\psi^1$ and $\psi^2$ on both $K$-sheets. In all of what follows we assume that we have already achieved this form for $\Phi$.

4.3. Solution of the linear problem

4.3.1. Coordinate transformation. In the following we intend to integrate the LP along the boundaries of the Gowdy square. For that purpose, it turns out to be useful to study the situation not only in the coordinate system $\Sigma$, corresponding to the Killing basis $\{\partial_{r_1}, \partial_{r_2}\}$, but also in a coordinate frame $\tilde{\Sigma}$,

$$
\tilde{\Sigma}: \quad \tilde{t} = t, \quad \tilde{\theta} = \theta, \quad \tilde{\rho}_1 = \rho_1 + q \rho_2, \quad \tilde{\rho}_2 = \rho_2
$$

(71)

with $q = \text{constant}$. According to (24)–(26) and (52), the transformed metric potentials are

$$
\tilde{R}_0 = R_0, \quad \tilde{u} = u, \quad \tilde{Q} = Q - q,
$$

(72)

i.e. only $Q$ is changed by subtracting a constant. In particular, we will choose the two systems $\tilde{\Sigma}$ with $q = 1$ or $q = -1$, in which $\tilde{Q}|_{A_1} = 0$ or $\tilde{Q}|_{A_2} = 0$ holds, respectively.

Since the coordinate transformation (71) is merely a change of the Killing basis, the Ernst equation (58) retains its form in $\tilde{\Sigma}$. This implies the existence of an LP (65) for a pseudopotential $\Phi$ in this frame. As shown by Neugebauer [36, 38], the matrices $\Phi$ and $\Phi^+$ are connected by the transformation

$$
\Phi = \left[ \begin{array}{cc} c_- & 0 \\ 0 & c_+ \end{array} \right] + \frac{i}{f}(K - \lambda) \left( \begin{array}{cc} 1 & \lambda \\ -\lambda & -1 \end{array} \right) \Phi
$$

(73)

with

$$
c_\pm := 1 - q \left( a \pm \frac{i}{f} \sin \theta \sin \theta \right),
$$

(74)

where all quantities on the right-hand side of equation (73) belong to the original frame $\Sigma$. As we will see, this transformation becomes particularly simple at the boundaries of the Gowdy square for our choices $q = \pm 1$. 
4.3.2. The LP on $\mathcal{H}_0$, $A_1$ and $A_2$. From our previous discussions, namely from the local investigation of the singular initial value problem for the Einstein equations with Fuchsian methods (see section 3.3) and from Chruściel’s global existence theorem, we know that for any smooth set of asymptotic data on $\mathcal{H}_0$ a corresponding smooth Gowdy-symmetric generalized Taub–NUT solution exists. Moreover, this solution is smooth both on the axes of symmetry $A_1$, $A_2$ and in the interior of the Gowdy square. The goal of this subsection is to find explicit expressions for the values of the solution on $A_1$ and $A_2$, which are determined by the data on $\mathcal{H}_0$. Afterwards we will study the behavior as $t \to \pi$ and investigate whether a continuation of the solution to $\mathcal{H}_f$ is possible.

Along the entire integration path, we have $x = y$ and therefore $\lambda = \pm 1$, cf (67). However, it suffices to study the case $\lambda = 1$ alone, since the solution on the Riemannian sheet with $\lambda = -1$ can easily be obtained from the solution with $\lambda = 1$ using (70).

For $x = y$ and $\lambda = 1$, the LP (65) reduces to the ODE
\[
\frac{\partial_t \Phi}{2f} = \left( \frac{\partial_x \tilde{E}}{\partial_x \tilde{E}} \frac{\partial_x \tilde{E}}{\partial_x \tilde{E}} \right) \Phi
\]
with the general solution\(^{11}\)
\[
\Phi = EC(K), \quad E := \begin{pmatrix} \tilde{E} & 1 \\ \tilde{E} & -1 \end{pmatrix}
\]
in terms of the Ernst potential on the boundary, where the $2 \times 2$ matrix $C$ is a $K$-dependent ‘integration constant’. The solutions on all parts of the integration path have the form (76), but with different integration constants:

\[
t = 0 : \Phi = EC, \quad C = \begin{pmatrix} C_1 & C_3 \\ C_2 & C_4 \end{pmatrix},
\]
\[
\theta = 0 : \Phi = ED, \quad D = \begin{pmatrix} D_1 & D_3 \\ D_2 & D_4 \end{pmatrix},
\]
\[
\theta = \pi : \Phi = E\tilde{D}, \quad \tilde{D} = \begin{pmatrix} \tilde{D}_1 & \tilde{D}_3 \\ \tilde{D}_2 & \tilde{D}_4 \end{pmatrix}.
\]

A further simplification can be achieved by normalizing $\Phi$ at $t = 0$ via
\[
t = 0 : \quad \psi_1^\omega = \psi_2^\omega = \psi(K),
\]
where $\psi(K)$ is an arbitrary gauge function (which will later be specified in such a way that the LP has a regular solution, see equation (91) below). This is possible since the form (70) of $\Phi$ is invariant under the transformation\(^3\)
\[
\Phi \to \Phi \cdot \begin{pmatrix} \alpha(K) & \beta(K) \\ \beta(K) & \alpha(K) \end{pmatrix}.
\]

The two degrees of freedom $\alpha$, $\beta$ can be used to achieve the two conditions in (80). As a consequence, we obtain
\[
C_3 = 0, \quad C_4 = \psi
\]
in this gauge.

\(^{11}\) By plugging the solution (76) into (75), we see that the matrix on the right-hand side of (75) is proportional to $f$, i.e. the factor $1/f$ is canceled and the solution extends smoothly over points with $f = 0$. However, it follows from (59) that $f$ does not vanish on $\mathcal{H}_0$, $A_1$, $A_2$ with the exception of the corners $A$, $B$ of the Gowdy square (as well as $C$ and $D$, provided $Q^2 e^\psi$ is bounded for $t \to \pi$).
From (77) to (79), we can now calculate the solution of the LP in the frame $\tilde{\Sigma}$ (cf (71)) using the transformation formula (73). It follows from (61) that $a$ takes on the boundary values

$$\mathcal{H}_p: \quad a = 0, \quad \mathcal{A}_1: \quad a = \frac{1}{Q} = 1, \quad \mathcal{A}_2: \quad a = \frac{1}{Q} = -1. \quad (83)$$

Plugging this into (73), we obtain for $\lambda = 1$:

$$t = 0: \quad \Phi = \begin{pmatrix} \xi \pm 2i(K - x) & 1 \\ \xi + 2i(K - x) & -1 \end{pmatrix} C \quad \text{in } \tilde{\Sigma} \text{ with } q = \pm 1, \quad (84)$$

$$\theta = 0: \quad \Phi = +2i(K - x) \begin{pmatrix} D_1 & D_2 \\ -D_1 & -D_3 \end{pmatrix} \quad \text{in } \tilde{\Sigma} \text{ with } q = 1, \quad (85)$$

$$\theta = \pi: \quad \Phi = -2i(K - x) \begin{pmatrix} \tilde{D}_1 & \tilde{D}_2 \\ -\tilde{D}_1 & -\tilde{D}_3 \end{pmatrix} \quad \text{in } \tilde{\Sigma} \text{ with } q = -1. \quad (86)$$

As we will see below, $C_i(K)$ and $C_2(K)$ are determined completely by the data at $t = 0$. Now we intend to express the components of the matrices $D$ and $\tilde{D}$ in terms of $C_1, C_2$. For that purpose, we use that $\Phi$ has to be continuous at the corners A and B of the Gowdy square (see figure 1). This condition leads to an algebraic system of 4 equations which, however, is not sufficient to calculate the eight unknowns $D_1, \ldots, D_4, \tilde{D}_1, \ldots, \tilde{D}_4$. This is the reason for introducing the coordinate frame $\tilde{\Sigma}$. From the requirement that also $\Phi$ (in $\tilde{\Sigma}$ with $q = 1$) is continuous at A and B (in $\tilde{\Sigma}$ with $q = -1$) is continuous at B we find other four algebraic equations. In this way we obtain an algebraic system of 8 equations for the 8 unknowns with the following solution for the matrices $D(K)$ and $\tilde{D}(K)$ in terms of $C(K)$:

$$D_1 = C_1 - \frac{b_A C_1 + iC_2}{2(K - 1)}, \quad D_2 = C_2 - \frac{ib_A (b_A C_1 + iC_2)}{2(K - 1)},$$

$$D_3 = -\frac{i\psi}{2(K - 1)}, \quad D_4 = \psi \left( 1 + \frac{b_A}{2(K - 1)} \right),$$

$$\tilde{D}_1 = C_1 + \frac{b_B C_1 + iC_2}{2(K + 1)}, \quad \tilde{D}_2 = C_2 + \frac{ib_B (b_B C_1 + iC_2)}{2(K + 1)},$$

$$\tilde{D}_3 = \frac{i\psi}{2(K + 1)}, \quad \tilde{D}_4 = \psi \left( 1 - \frac{b_B}{2(K + 1)} \right),$$

where $b_A$ and $b_B$ denote the values of $b = 3\xi$ at the points A and B respectively.

In the following subsection we utilize these results to determine the Ernst potential and the metric potentials on $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the data on $\mathcal{H}_p$.

4.3.3. Ernst potential on $\mathcal{A}_1$ and $\mathcal{A}_2$. From the solution of the LP obtained in the previous subsection we may, as a first step, calculate the Ernst potential on $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the initial potential on $\mathcal{H}_p$. To this end, we start by expressing $C_1$ and $C_2$ in terms of $\xi$ on $\mathcal{H}_p$.

As mentioned in section 4.2, the mapping $K \mapsto \lambda$ in (67) defines a two-sheeted Riemannian $K$-surface. At the branch points $K_1$ and $K_2$, where both sheets are connected,

12 Note that $a$ is automatically discontinuous at the points A, B, C, D as a consequence of the definition (61). In contrast, $a$ is a smooth function in the remaining part of the Gowdy square.

13 The reason why just the usage of a different coordinate system can lead to independent algebraic equations is the following. The boundary values of the quantity $a$ enter the transformation law (73). Therefore, the additional algebraic equations found in $\tilde{\Sigma}$ ensure that $a$ indeed takes on the boundary values (83) (and, as a consequence, also $Q$ takes on the correct boundary values $Q = 1$ on $A_1$ and $Q = -1$ on $A_2$). From (60) alone it would only follow that $a$ is constant on the boundaries without specification of these constants.
any function of \( K \) has to be unique, i.e. the values on the upper and lower sheet have to be the same. For \( r-\theta \)-values on the boundaries of the Gowdy square, we have confluent branch points, i.e. \( K_1 = K_2 = x \). The uniqueness of \( \Phi \) at \( K = K_1 = K_2 \) leads to the conditions (see (70))

\[
\mathcal{H}_p, A_1, A_2 : \quad \psi_1^c = \psi_1^c \quad \text{and} \quad \psi_2^c = \psi_2^c \quad \text{for} \quad K = x.
\]

(87)

In particular, on \( \mathcal{H}_p \) we obtain the two equations

\[
\tilde{E}_p C_1 + C_2 = \psi, \quad \tilde{E}_p C_1 - C_2 = \psi
\]

with the solution

\[
C_1(x) = \frac{2\psi(x)}{\tilde{E}_p(x) + \tilde{E}_p(x)} = \frac{\psi(x)}{f_p(x)}, \quad C_2(x) = \frac{\tilde{E}_p(x) - \tilde{E}_p(x)}{\tilde{E}_p(x) + \tilde{E}_p(x)} \psi \equiv \frac{ib_p(x)}{f_p(x)} \psi(x),
\]

where

\[
\tilde{E}_p(x) = f_p(x) + ib_p(x) = \mathcal{E}(t = 0, \theta = \arccos x).
\]

(90)

Now we can suggest a possible choice for the gauge function \( \psi \), which was introduced in (80). If we set

\[
\psi(K) = (K^2 - 1)^2,
\]

then the solution \( \Phi \) (as well as \( \tilde{\Phi} \)) is regular, because \( \psi \) compensates for the poles that the matrices \( \tilde{D} \) and \( \tilde{D} \) would otherwise have at \( K = \pm 1 \). Note that \( C_1 \) and \( C_2 \) are also regular because \( f_p(x) = e^{-u}(1-x^2) \), cf (59). But of course, as we will see below, the Ernst potential on \( A_1, A_2 \) and \( \mathcal{H}_p \) is independent of this gauge choice.

With these expressions for \( C_1 \) and \( C_2 \) together with the solution of the LP from the previous section, we can also evaluate condition (87) on \( A_1 \) and \( A_2 \) to obtain explicit formulae for the Ernst potential there. The result that we find independently of the particular choice for the gauge function \( \psi \) is

\[
A_1 : \quad E_1(x) := \mathcal{E}(t = \arccos x, \theta = 0) = \frac{i[b_A - 2(1 - x)]E_p(x) + b_A^2}{E_p(x) - i[b_A + 2(1 - x)]},
\]

(92)

\[
A_2 : \quad E_2(x) := \mathcal{E}(t = \arccos(-x), \theta = \pi) = \frac{i[b_B - 2(1 + x)]E_p(x) + b_B^2}{E_p(x) - i[b_B + 2(1 + x)]}.
\]

(93)

From the latter equations we can conclude that \( E_1 \) and \( E_2 \) are smooth functions of \( x \). To see this, recall that we assume smooth data at \( t = 0 \). For smooth initial functions \( u(0, \theta) \) and \( Q(0, \theta) \), the initial Ernst potential \( E_1 \) will also be smooth, cf equations (94), (95) below. As a consequence, the numerators and denominators of the fractions in (92), (93) are also smooth and an irregularity in the Ernst potentials could only occur if the denominators became zero for some \( x \in [-1, 1] \). However, it follows from (94), (95) together with equation (23) that the only zeros are at \( x = 1 \) or at \( x = -1 \). Moreover, these equations show that the numerators have zeros of at least the same multiplicity at these \( x \)-values. Hence, the zeros in the numerators and denominators cancel each other out and the fractions are smooth functions of \( x \) for all \( x \in [-1, 1] \). The only exceptional cases occur for asymptotic data with \( b_B = b_A + 4 \) or \( b_B = b_A - 4 \). In the first case, \( E_1 \) diverges at point \( C \) and in the second case \( E_2 \) diverges at \( D \), cf figure 1.

4.3.4. Metric potentials on \( A_1 \) and \( A_2 \). In the previous subsection we have provided explicit formulae for the Ernst potential on the axes \( A_1 \) and \( A_2 \) in terms of the initial potential on \( \mathcal{H}_p \). Now we will see how the metric potentials \( u, Q \) and \( M \) can be obtained from the Ernst potential on these boundaries.
We assume that asymptotic data \( u(0, \theta) \) and \( Q(0, \theta) \) (or, equivalently, \( S_a(\theta) \) and \( Q_a(\theta) \)) and a constant \( R_0 > 0 \) are given. From these data, we may calculate the initial Ernst potential \( \mathcal{E}_P = f_p + i b_p \). The real part can be obtained from (59),

\[
f_p(\theta) = e^{-u(0,\theta^\prime)} \sin^2 \theta^\prime,
\]

and the imaginary part can be calculated by integrating the first equation in (60) with respect to \( \theta \), using (61). We obtain

\[
b_p(\theta) = b_A + 2 \int_0^\theta Q(0, \theta^\prime) \sin \theta^\prime \, d\theta^\prime,
\]

where \( b_A = b(0, 0) \) is an arbitrary integration constant.

From \( \mathcal{E}_P \) we may calculate \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) via (92) and (93). Afterward, we can use these results to determine the potentials \( u \), \( Q \) and \( M \) on \( A_1 \) and \( A_2 \). Using again (59)–(61) together with (19), (23) and (52), we find

\[
A_1 : \quad e^{u(t,0)} = \frac{f_1(t)}{\sin^2 \theta}, \quad e^{M(t,0)} = \frac{R_0 \sin^2 t}{f_1(t)}, \quad Q(t,0) = 1,
\]

\[
A_2 : \quad e^{u(t,\pi)} = \frac{f_2(t)}{\sin^2 \theta}, \quad e^{M(t,0)} = \frac{R_0 \sin^2 t}{f_2(t)}, \quad Q(t,\pi) = -1.
\]

### 4.4. Situation on \( \mathcal{H}_1 \)

So far we have seen that we can prescribe arbitrary smooth data at \( t = 0 \) and will always find smooth potentials for \( t < \pi \) as solutions to the field equations. In particular, we have derived explicit formulae for the Ernst potential and the metric potentials on the axes \( A_1 \) and \( A_2 \). It remains to study under which conditions the solution can even be extended smoothly to the future boundary \( \mathcal{H}_1 \). In order to answer this question, we tentatively solve the LP on \( \mathcal{H}_1 \) and investigate whether this solution can be attached continuously to the solutions on \( A_1 \) and \( A_2 \).

The LP on \( \mathcal{H}_1 \) reduces to the same ODE as on the other boundaries of the Gowdy square, namely to equation (75). We write the solution in \( \Sigma \) as

\[
t = \pi : \quad \Phi = E\hat{C}, \quad \hat{C} = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \\ \hat{C}_3 \\ \hat{C}_4 \end{pmatrix}.
\]

In order to obtain the solution in the coordinate frame \( \hat{\Sigma} \) too, we need to calculate the quantity \( a \) on \( \mathcal{H}_1 \) so we can apply the transformation formula (73). It follows from (61) that, if the metric potentials \( u \) and \( Q \) remain bounded for \( t \to \pi \), then \( a = 0 \) holds on \( \mathcal{H}_1 \) (provided \( f \) does not vanish on \( \mathcal{H}_1 \) with exception of the boundary points \( C, D \)). However, it is not yet clear how \( u \) and \( Q \) behave as \( t \to \pi \). Therefore, so far we can only say that \( a \) is constant on \( \mathcal{H}_1 \), cf

\[
t = \pi : \quad a = a_0 = \text{constant}.
\]

Using (73), we find therefore in \( \hat{\Sigma} \)

\[
t = \pi : \quad \hat{\Phi} = \begin{pmatrix} (1 \mp a_0)\hat{C} \pm 2i(K-x) \\ (1 \mp a_0)\hat{C} \pm 2i(K-x) \\ \mp a_0 \end{pmatrix} \hat{C} \text{ in } \hat{\Sigma} \text{ with } q = \pm 1,
\]

Now we can investigate whether \( \Phi \) in (98) and \( \hat{\Phi} \) in (100) can be attached continuously to the corresponding solutions on \( A_1 \) and \( A_2 \), i.e. whether \( \Phi \) and \( \hat{\Phi} \) are continuous at the points \( C \) and \( D \), see figure 1. This question is equivalent to the solvability of an algebraic system of 8 equations, which involves the matrix components \( C_1, \ldots, C_4 \) as well as the constant \( a_0 \) and
the values $b_C$ and $b_D$ of the imaginary part of the Ernst potential at $C$ and $D$. It turns out that this system can be solved if and only if the initial parameters $b_A$ and $b_B$ satisfy

$$b_B \neq b_A + 4, \quad \text{and} \quad b_B \neq b_A - 4.$$  \hfill (101)

(It was already discussed at the end of section 4.3.3 that the Ernst potential diverges at $C$ or $D$ if one of these conditions is violated.) The algebraic equations then fix the functions $\tilde{C}_1(K), \ldots, \tilde{C}_4(K)$ in dependence of our initial data as well as the values of $b_C$, $b_D$ and $a_0$ in terms of the initial quantities $b_A$ and $b_B$. In particular, we obtain

$$b_C = \frac{4b_B + b_A(b_A - b_B)}{b_A - b_B + 4},$$  \hfill (102)

$$b_D = \frac{-4b_A + b_B(b_A - b_B)}{b_A - b_B - 4},$$  \hfill (103)

$$a_0 = \frac{8(b_B - b_A)}{16 + (b_B - b_A)^2}. \hfill (104)$$

With these results we can calculate the Ernst potential on $\mathcal{H}_f$. With the same considerations as in section 4.3.3 we obtain

$$\mathcal{E}_f := \mathcal{E}(t = \pi, \theta = \arccos(-x)) = \frac{a_1(x)\mathcal{E}_p(x) + a_2(x)}{b_1(x)\mathcal{E}_p(x) + b_2(x)}, \hfill (105)$$

where

$$a_1 = -i\{(b_A - b_B)^2 + 16\}x^2 - 2(b_A - b_B)(b_A + b_B - 4)x + (b_A - b_B)^2 $$
$$+ 8(b_A + b_B - 2)\}, \hfill (106)$$

$$a_2 = 4(b_A - b_B)(b_A b_B - 2b_A - 2b_B)x - 8(b_A^2 + b_B^2), \hfill (107)$$

$$b_1 = 4[(b_A - b_B)x - 4], \hfill (108)$$

$$b_2 = -i\{(b_A - b_B)^2 + 16\}x^2 + 2(b_A - b_B)(b_A + b_B - 4)x + (b_A - b_B)^2 - 8(b_A + b_B + 2)\}. \hfill (109)$$

Similarly to the discussion in section 4.3.3 we find that $\mathcal{E}_f$ is a smooth function on the entire boundary $\mathcal{H}_f$ (with our assumption (101)).

As already mentioned, the auxiliary quantity $a$ would satisfy the boundary condition $a = 0$ on $\mathcal{H}_f$ if the metric potentials $u$ and $Q$ were bounded for $t \to \pi$. From (104) we can read off that this is only the case if the initial parameters $b_A$ and $b_B$ satisfy the condition

$$b_A = b_B, \quad (110)$$

which can also be expressed in terms of the metric potential $Q$ as

$$\int_0^\pi Q(0, \theta) \sin \theta \, d\theta = 0, \quad (111)$$

cf (95). In the following subsections, we study separately the cases $b_A = b_B$ and $b_A \neq b_B$.  

31
4.4.1. Initial data with $b_A = b_B$. Such data lead to a solution of the field equations with $a = 0$ on $\mathcal{H}_t$. As a consequence, the metric potentials $u$ and $Q$ are regular at $\mathcal{H}_t$.

The formula (105) for $\mathcal{E}_t$ simplifies in this case to

$$\mathcal{H}_t: \quad \mathcal{E}_t(x) = \frac{i(b_A - 1 + x^2)\mathcal{E}_t(x) + b_A^2}{\mathcal{E}_t(x) - i(b_A + 1 - x^2)}$$

and in terms of this Ernst potential, the metric potentials are given by

$$\mathcal{H}_t: \quad e^{u(\pi, \theta)} = \frac{\sin^2\theta}{f_i(\theta)}, \quad e^{M(\pi, \theta)} = R_0e^{2u}\frac{\sin^2\theta}{f_i(\theta)}, \quad Q(\pi, \theta) = -\frac{\partial_u b_i(\theta)}{2\sin\theta},$$

where $f_i = \Re E_i$, $b_i = \Im E_i$. Here we have used that $M - u$ is constant on $\mathcal{H}_t$ as a discussion of equation (57) in the limit $t \to \pi$ reveals.

It follows from these results that $\mathcal{H}_t$ is a regular Cauchy horizon, generated by the Killing vector $\partial_{\rho}$ (just like the past horizon $\mathcal{H}_0$). To see this, we can use a modification of the transformation (31) to regular coordinates in a vicinity of this boundary,

$$\pi - t = \arcsin \sqrt{t'}, \quad \theta = \theta', \quad \rho_1 = \rho_1' + \frac{\kappa}{R_0} \ln t', \quad \rho_2 = \rho_2'.$$

As a consequence of (113), the constant $\kappa$ can always be chosen such that the metric is regular in terms of $t'$, $\theta'$, $\rho_1'$, $\rho_2'$. Moreover, $g(\partial_{\rho_1}, \partial_{\rho_2}) = R_0 e^{u} \sin^2 t'$ tends to zero for $t \to \pi$, i.e. $\mathcal{H}_t$ is indeed a regular null hypersurface and therefore a Cauchy horizon.

4.4.2. Initial data with $b_A \neq b_B$. Now we study the case $b_A \neq b_B$ and assume that in addition $b_B \neq b_A \pm 4$ holds. In this case, the auxiliary quantity $a$ tends to $a_0 \neq 0$ as given in (104) as $t \to \pi$. As a consequence of (61), we see that at least one of the metric potentials $Q$ and $u$ cannot be bounded in this limit. Indeed, we can read off from (62) that $e^u$ diverges as $1/\sin^2 t$ as $t \to \pi$ for all $\theta \in (0, \pi)$.

However, it turns out that this divergence is only a peculiarity of our special choice of metric potentials. A better quantity for discussing regularity is the Ernst potential $E$ which is defined invariantly in terms of the Killing vectors. And indeed, the Ernst potential also cannot be bounded in this limit. Indeed, we can read off from (62) that $e^u$ diverges as $1/\sin^2 t$ as $t \to \pi$ for all $\theta \in (0, \pi)$.

In order to obtain the metric potentials on $\mathcal{H}_t$ in terms of the Ernst potential and the constant $a_0$, we replace the potential $u$ by a potential $v$ in a neighborhood of $\mathcal{H}_t$ via

$$e^{u(\pi, \theta)} = \frac{e^{v(\pi, \theta)}}{\sin^2 t}. \quad (115)$$

Then we find

$$\mathcal{H}_t: \quad Q = \frac{1}{a_0}, \quad e^{v(\pi, \theta)} = a_0^2 f_i(\theta), \quad e^{M(\pi, \theta)} = c - \frac{\sin^2\theta}{f_i},$$

where the integration constant $c$ in the expression for $M$ can be determined from the requirement of a continuous transition to the axes.

In the present case $b_A \neq b_B$, it turns out that $\mathcal{H}_t$ is a regular Cauchy horizon, generated by the linear combination $\partial_{\rho_1} - a_0 \partial_{\rho_2}$ of the two Killing vectors. Regular coordinates can be introduced via

$$\pi - t = \arcsin \sqrt{t'}, \quad \rho_1 = \rho_1' + \frac{\kappa_1}{R_0} \ln t', \quad \rho_2 = \rho_2' + \frac{\kappa_2}{R_0} \ln t'. \quad (117)$$
where $\kappa_1$ and $\kappa_2$ are two constants that can always be chosen such that the resulting metric potentials are regular.

Finally, we may look again at the singular cases $b_B = b_A \pm 4$. As mentioned earlier, the corresponding Ernst potential and the Kretschmann scalar on $A_1$ (for $b_B = b_A + 4$) or $A_2$ (for $b_B = b_A - 4$) diverge in the limit $t \to \pi$. Since we therefore cannot find a solution of the LP on $H_f$ that is continuously connected to the axes, it is not possible to construct the Ernst potential on $H_f$ in these two singular cases directly. However, in order to study the situation on $H_f$ in these cases too, we can consider a sequence of solutions with $b_B \neq b_A \pm 4$ that approaches a solution with $b_B = b_A \pm 4$. Then, for each element of the sequence, the LP can be solved along all four boundaries of the Gowdy square and the corresponding expression for the Ernst potential $E_f$ on $H_f$, constructed from this solution, is valid. It turns out that the limit $t \to \pi$ of $E_f$ remains regular for $0 < \theta < \pi$, whereas $E_f$ diverges as expected at $C$ or $D$. Hence we can conclude that only the boundary points $C$ or $D$ of $H_f$ become singular and the interior of $H_f$ is still a regular null hypersurface.

5. Discussion

In this paper we have studied the class of smooth Gowdy-symmetric generalized Taub–NUT solutions as interesting examples of Gowdy spacetimes with spatial $S^3$ topology. This class is characterized by a special behavior of the metric potentials in a vicinity of the initial surface $H_p$ ($t = 0$), see figure 1, which, in particular, implies that $H_p$ is a smooth (past) Cauchy horizon. Utilizing Fuchsian methods, we were able to show that for smooth asymptotic data, describing the spacetime at $H_p$, there always exists a unique smooth Gowdy-symmetric generalized Taub–NUT spacetime as a solution to the Einstein equations for $t \in (0, \pi)$. In the second step, we have investigated the behavior of these solutions on the symmetry axes $A_1$ ($\theta = 0$) and $A_2$ ($\theta = \pi$). Using the complex Ernst formulation of the field equations and its reformulation in terms of an equivalent LP, we have constructed explicit formulae for the metric potentials on $A_1$ and $A_2$ in terms of the data on $H_p$. Afterward, it was possible to extend the solution to the future boundary $H_f$ ($t = \pi$) of the Gowdy square and to find explicit expressions for the metric potentials there, too. It followed from these expressions, that we have to distinguish between four types of asymptotic data, which are characterized by the values $b_A$ and $b_B$ of the imaginary part $b$ of the Ernst potential at the points $A$ ($t = \theta = 0$) and $B$ ($t = 0, \theta = \pi$) and which lead to solutions with a completely different behavior on $H_f$:

(i) $b_B = b_A + 4$:
In this case a scalar curvature singularity occurs at the point $C$ ($t = \pi, \theta = 0$).

(ii) $b_B = b_A - 4$:
Here, a scalar curvature singularity occurs at the point $D$ ($t = \theta = \pi$).

(iii) $b_B = b_3$:
The spacetime is regular in the entire Gowdy square. In particular, the Ernst potential $E$ and the metric potentials $u$, $Q$ and $M$ are smooth. Moreover, $H_f$ is a smooth Cauchy horizon, generated by the Killing vector $\partial_{\rho_1}$.

(iv) $b_B \neq b_A$ and $b_B \neq b_A \pm 4$:
The spacetime is regular in the entire Gowdy square and the Ernst potential $E$ is smooth, but the metric potential $u$ is not well adapted to describing this case and blows up at $H_f$. However, there is no physical singularity at $H_f$. Instead, $H_f$ is a smooth Cauchy horizon, generated by the null vector $\partial_{\rho_1} - a_0 \partial_{\rho_2}$.

This shows that—with the exception of the two singular cases (i) and (ii)—smooth Gowdy-symmetric generalized Taub–NUT solutions (with a past Cauchy horizon at $t = 0$) always
develop a second Cauchy horizon at $t = \pi$. This future Cauchy horizon, in the same way as the one in the past, is homeomorphic to $S^3$ and its null generator has closed integral curves. Hence our results can in particular be understood as a partial resolution of a problem which remained open in [17]. Namely, at least in our class of spacetimes, the Gowdy square is isometric to the closure of the MGHD of corresponding Cauchy data.

It is interesting to compare these results with the situation of spatial $S^2 \times S^1$ topology as investigated in [28]. For this case, it was shown that Gowdy spacetimes with a regular past Cauchy horizon $\mathcal{H}_p$ develop a regular future horizon $\mathcal{H}_f$ if and only if a particular quantity $J$, which can be read-off from the asymptotic data\(^ {14}\), does not vanish. In the limit $J \to 0$, $\mathcal{H}_f$ transforms into a scalar curvature singularity. Hence, the behavior is similar to the $S^3$ case: with the exception of non-generic singular cases, spacetimes with a past Cauchy horizon generically develop a future Cauchy horizon. However, the nature of the singular cases is slightly different: In the $S^2 \times S^1$ case, the curvature blows up along the entire future boundary $\mathcal{H}_f$, whereas we find only singularities at the isolated points $C$ or $D$ on $\mathcal{H}_f$ for $S^3$ topology.

Do our assumptions rule out important cases? This is not clear and probably difficult to answer. For example our assumptions do not allow solutions with past Cauchy horizons ruled by non-closed generators. There is no reason why such solutions should not exist. Also solutions with non-compact or incomplete Cauchy horizons at $t = 0$ are excluded by our assumptions so far. Indeed, such solutions have been constructed in the polarized case by the techniques of Moncrief and Isenberg in [29]. Since the geometry of $S^1$-Gowdy spacetimes away from the axes is locally the same as the geometry of $T^3$-Gowdy spacetimes, it seems possible to construct $S^3$-Gowdy solutions which are smooth concatenations of solutions with pieces of Cauchy horizons covering a neighborhood of the axes of the $t = 0$-surface (by means of the same Fuchsian method which we have employed in this paper; in particular the matching conditions would be satisfied) and pieces of solutions which are singular away from the axes at $t = 0$ (by means of the Fuchsian method for $T^3$-Gowdy solutions). The question of how such solutions evolve globally in time is of course completely open.

Can our results be generalized to situations with less symmetry and eventually maybe even to generic solutions with Cauchy horizons? We have employed two, in principle, independent techniques for the two main steps of our discussion: the Fuchsian method for the basic existence proof and the soliton method for the study of the global properties of the solutions. As far as the Fuchsian method and the underlying singular initial value problem and hence the existence and uniqueness of solutions with prescribed ‘data’ on a Cauchy horizon are concerned, we can say the following. In general, it cannot be expected that such an ‘initial value problem’ for equations of hyperbolic type is well-posed. It seems at least necessary that the generator of the horizon, being a null hypersurface, is a Killing field. This is the case here and also in the more general $U(1)$-symmetric case discussed by Moncrief [32]. The results in [33, 22] suggest that there must be a $U(1)$-symmetry in a neighborhood of a Cauchy horizon in general vacuum spacetimes, at least if the horizon is compact and the generator has closed integral curves. So, there is hope that a similar singular initial value problem can be formulated under quite general assumptions and that the existence and uniqueness proof based on Fuchsian methods goes through.

As far as the Ernst formulation and the associated LP are concerned there is probably little hope of a generalization to situations with fewer symmetries. The introduction of the complex Ernst potential relies essentially on the existence of two Killing vectors, cf (59), (61), and the LP makes use of the special structure of the Ernst equation. However, it should be

\(^ {14}\) In terms of the Ernst potential at $\mathcal{H}_p$, $J$ is defined as $J = -\frac{1}{8Q_H^2}(b_A - b_B - 4Q_H)$, where $Q_H$ denotes the constant value of the metric potential $Q$ on $\mathcal{H}_p$ in the $S^2 \times S^1$ case.
quite straightforward to apply the methods to Gowdy-symmetric spacetimes with additional electromagnetic fields. In that case one has to study the coupled system of the Einstein–Maxwell equations, for which, remarkably, a complex Ernst formulation and an associated LP exist as well. The corresponding calculations would follow closely the investigation of axisymmetric and stationary black hole spacetimes with electromagnetic fields as presented in [27].

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Appendix. Semilinear Fuchsian wave equations on Riemannian manifolds

One of the main tasks of section 3.3.3 is to analyze a singular initial value problem of a coupled system of nonlinear wave equations (35)–(36). The idea is to use the Fuchsian method which is introduced in [2] for general quasi-linear symmetric hyperbolic Fuchsian equations and which was based on the Fuchsian theory for semi-linear second-order hyperbolic Fuchsian equations in [13, 12]. However, these earlier discussions are restricted to the case where the spatial topology is a one-dimensional circle; the straightforward generalization to the case of an $n$-torus has been considered in [1]. Here we discuss the case of more general smooth compact Riemannian manifolds of which the 2-sphere $S^2$ in our applications is a particular example. In order to make the presentation as short as possible we restrict ourselves to that particular class of equations which is of interest in section 3.3.3: semilinear wave equations. In summary, the main difference to the case of spatial $n$-torus topology will be the definition of the function spaces and the way, the energy estimates are obtained. Despite the fact that we deal with second-order equations here, while [2, 1] focuses on first-order equations, almost all of the other details of the proofs stay the same.

An alternative approach is the one by Stähl [46], who has modified the Fuchsian theory in [30, 42] to treat the spatial manifold $S^2$. There are several reasons why we do not follow his approach here. One of them is that we do know how to go from semilinear equations to quasilinear equations in our theoretical framework above. It is therefore likely that our discussion here can also be generalized to the quasilinear case. This will be of interest in future work when we study solutions of Einstein’s field equations under more general conditions and more general topologies than before. On the other hand, our technique above gives rise to a numerical approximation scheme for the singular initial value problem with practical error estimates. In future work, we plan to compute Gowdy-symmetric generalized Taub NUT solutions numerically using our framework.

Notation and function spaces. We use the definition of Sobolev spaces on Riemannian manifolds given in appendix I of [15], which we quickly summarize as follows; further details can be found in [6]. Let $(H, h_{ab})$ be a smooth orientable $n$-dimensional Riemannian manifold. As before, the indices $a, b$ etc. are abstract tensor indices with respect to $H$. Let $f : H \to \mathbb{R}^d$ be a $d$-vector-valued function for some integer $d \geq 1$. For any integer $s \geq 0$, we can define $\nabla_{a_1} \cdots \nabla_{a_s} f$ by considering the usual covariant derivative $\nabla$ compatible with $h_{ab}$ which acts on each of the $d$ components of $f$ individually. The result $\nabla_{a_1} \cdots \nabla_{a_s} f$ is then understood as a $d$-vector-valued $(0, s)$-tensor field. In the following, when we write expressions where two such objects are multiplied, e.g. $\nabla_{a_1} \cdots \nabla_{a_s} f \nabla_{a'_1} \cdots \nabla_{a'_s} f$, we, first, evaluate each of the two factors.
componentwise and then take the Euclidean scalar product of the resulting $d$-vector-valued quantities, i.e.

$$\nabla a_1 \cdots \nabla a_n f \nabla a_1' \cdots \nabla a_n' f : = \sum_{i=1}^d \nabla a_1 \cdots \nabla a_n f / \nabla a_1' \cdots \nabla a_n' f ,$$

the result being a scalar-valued $(0, 2s)$-tensor field. Our notation in the following will hence mimic the notation for scalar-valued quantities as much as possible, but it should always be kept in mind that many of the following quantities will actually be $d$-vector-valued.

Now pick a non-negative integer $q$. Let $f$ and $g$ be $d$-vector-valued functions such that the $d$-vector-valued tensor fields $\nabla a_1 \cdots \nabla a_n f$, $\nabla a_1' \cdots \nabla a_n' g$ exist in the distributional sense and are square-integrable for all $s = 0, \ldots, q$, in the sense that

$$\langle f, g \rangle_{H^q(H)} := \left( \int_0^q \sum_{a=0}^q \int_{s=0}^{a} \int_{t=0}^{a} f_1(a) g_2(a) \nabla a_1 \cdots \nabla a_n f / \nabla a_1' \cdots \nabla a_n' g \right)^{1/2},$$

is well defined and finite. Here $\epsilon = \epsilon_{a_1 \cdots a_n}$ is a volume form corresponding to $h_{ab}$. The summand $s = 0$ is understood to involve no covariant derivatives and hence no contractions with the contravariant metric. We also set

$$\|f\|_{H^q(H)} := (\langle f, f \rangle_{H^q(H)})^{1/2},$$

whenever this is finite. Then, the Sobolev space $H^q(H)$ of functions $f$ with $\|f\|_{H^q(H)} < \infty$ equipped with the norm $\|\cdot\|_{H^q(H)}$ is a Banach space. In the case $q = 0$, we write $L^2(H)$ instead of $H^0(H)$. We note that this definition of Sobolev spaces is purely geometric, i.e. it does not depend on the choice of coordinates. If $(H, h)$ is complete, then $H^q(H)$ equals the completion of the space of smooth functions with compact support with the norm $\|\cdot\|_{H^q(H)}$. We also point the reader to the following properties [15] which we will use frequently below.

**Lemma A.1.** Suppose that $(H, h_{ab})$ is a complete orientable smooth Riemannian manifold of dimension $n$ and that $q$ is an integer larger than $n/2$. Then:

(i) $H^q(H)$ is continuously imbedded into $C^0(H)$, i.e. the Banach space of continuous and bounded functions on $H$ with the supremum norm, and there exists a constant $C > 0$ so that

$$\sup_{x \in H} |f| \leq C \|f\|_{H^q(H)},$$

for all $f \in H^q(H)$.

(ii) For all $f, g \in H^q(H)$, the pointwise product $f \cdot g$ is in $H^q(H)$ and

$$\|f \cdot g\|_{H^q(H)} \leq C \|f\|_{H^q(H)} \|g\|_{H^q(H)},$$

for a constant $C > 0$. Hence $H^q(H)$ is a Banach algebra.

Now, in accordance with [2, 1], we define spaces $X_{\delta, \mu, q}$ of time-dependent $d$-vector-valued functions on $H$ as follows. We do this parametrically, specifying (i) a non-negative constant $\delta$, (ii) a non-negative integer $q$, and (iii) a smooth $d$-vector-valued function $\mu : H \rightarrow \mathbb{R}^d$ with the corresponding diagonal matrix-valued function

$$R[\mu](t, x) := \text{diag}(\mu_1(x), \ldots, \mu_d(x)),$$

and then defining the norm

$$\|w\|_{\delta, \mu, q} := \sup_{t \in (0, \delta)} \|R[\mu](t, \cdot)w(t, \cdot)\|_{H^q(H)},$$

for...
for appropriate $d$-vector-valued time-dependent functions $w(t,x)$. Based on the norm $\|\cdot\|_{h,\mu,q}$, we define the Banach space $X_{h,\mu,q}(H)$—also simply written as $X_{h,\mu,q}$—as the completion of the set of all $d$-vector-valued functions $w \in C^\infty((0,\delta) \times H)$ for which this norm is finite. We use $B_{h,\mu,q,r} \subset X_{h,\mu,q}$ to denote the closed ball of radius $r > 0$ (measured using the norm $\|\cdot\|_{h,\mu,q}$) and center 0. To handle the class of functions which are infinitely differentiable, we define the space

$$X_{h,\mu,\infty} := \bigcap_{q=0}^\infty X_{h,\mu,q}.$$  

In comparing a pair of function spaces $X_{h,\mu,q}$ and $X_{h,\nu,q}$ for functions with the same dimension $d$, we find it useful to write $v > \mu$ to denote the condition that for each index $i = 1, \ldots, d$ and for all $x \in H$, the components of $v$ and $\mu$ satisfy the inequality $v_i(x) > \mu_i(x)$. Note that if $v > \mu$, then $X_{h,v,q} \subset X_{h,\mu,q}$. It is often useful to consider functions $f$ in any of the spaces $X_{h,\mu,q}$ as maps $f : (0,\delta) \to H^q(H)$. In this context, we often use both notations $f(t)$ and $f(t,\cdot)$ to denote the corresponding element in $H^q(H)$ at any $t \in (0,\delta)$.

We also point the reader to the appendix in [2], where we list further properties of the above function spaces. All those results there easily generalize to the case of $n$-dimensional Riemannian manifolds. Only at places, where part (ii) of lemma A.1 above is used, i.e. in Lemmas B.1–B.4 in the appendix of [2], we must replace the lower bound $\nu > \mu$ by $\nu > \frac{\mu}{2}$.

In our context of second-order wave equations, we will see that the following spaces are also convenient. Let $u$ be a time-dependent $d$-vector-valued function on $(0,\delta) \times H$ such that $u$, $Du$, $\iota \nabla u$ are defined and are elements of $X_{h,\mu,q}$ for some exponent vector $\mu$ and some integer $q \geq 0$. Then we say that $u \in \hat{X}_{h,\mu,q}$ and we set

$$\|u\|_{\hat{h},\mu,q} := \|u\|_{h,\mu,q} + \|Du\|_{h,\mu,q} + \|\iota \nabla u\|_{h,\mu,q}.$$  

Again, we find that $\hat{X}_{h,\mu,q}$, also written as $\hat{X}_{h,\mu,q}(H)$, equipped with the norm $\|\cdot\|_{\hat{h},\mu,q}$, is a Banach space. We use the notation analogous to that before for closed bounded subsets $\hat{B}_{h,\mu,q,r}$, and for the spaces $\hat{X}_{h,\mu,\infty}$.

Semilinear Fuchsian wave equations on $H$ and singular initial value problems. From now on, we assume that $(H,h_{ab})$ is compact in addition to the assumptions before. It is a consequence of the Hopf Rinow Theorem, see theorem 21 and corollary 23 in [40], that compact Riemannian manifolds are complete. We restrict ourselves to equations of the form

$$D^2u + 2ADu - \iota^2 \Delta_{h} u = f(t,x,u,Du,\iota \nabla u),$$  

(A.2)

in the following—the system equations (35)–(36) is a particular example—where $u : (0,\delta) \times H \to \mathbb{R}^d$ is the time-dependent $d$-vector-valued unknown, $A$ is a $d \times d$-diagonal matrix—which we assume to be a constant here for simplicity (and which acts by matrix multiplication on the vector-valued function $Du$). The Laplace operator is $\Delta_{h} := h^{ab}\nabla_a \nabla_b$, and $f$ is the source term which depends as a $q$-times continuously differentiable vector-valued function on $t$, $x$, $u$, $Du$ and the 1-form $\iota \nabla u$; the integer $q$ will be fixed later. The latter means that, with respect to any local coordinate patch, $f$ depends as a $q$-times continuously differentiable function in particular on the $n$ coordinate components of $\iota \nabla u$. Such a system of equation is referred to as a *semilinear Fuchsian wave equation on $H$*, and we write

$$L[u] := D^2u + 2ADu - \iota^2 \Delta_{h} u,$$  

(A.3)

for its principal part.

Clearly, these equations are singular at $t = 0$ and we wish to discuss the properties of solutions at this singular time. Given the parameters $\delta$, $\mu$, $q$ as in the above discussion of the
function space $X_{\delta,\mu,q}$, and a chosen function $u_0 : (0, \delta] \times H \to \mathbb{R}^d$ (with so far unspecified regularity), the singular initial value problem consists of seeking a solution $u = u_0 + w$ to equation (A.2) whose remainder function $w$ belongs to $X_{\delta,\mu,q}$. This problem is said to be well-posed if, for each choice of $u_0$ in some given function space (discussed below), there exists a unique remainder solution $w \in X_{\delta,\mu,q}$. As soon as a leading-order term $u_0$ has been fixed, we use the operator notation

$$F(u_0)[w](t, x) := f(t, x, u_0 + w, D(u_0 + w), t\nabla_x(u_0 + w)).$$  \hfill (A.4)

The idea is therefore to prescribe the leading-order behavior of a solution at $t = 0$ by fixing the leading-order term $u_0$ and then to construct solutions which approach $u_0$ at some specified rate $\mu$, i.e. whose remainder $w$ is in $X_{\delta,\mu,q}$.

Our aim is to prove the following central theorem.

**Theorem A.2.** (Well–posedness for semilinear Fuchsian wave equations on $S^2$). Let $H = S^2$ and $h = d\delta^2 + \sin^2 \theta \, d\phi^2$. Choose a leading-order term $u_0$, a constant $\delta > 0$, an exponent vector $\mu$, and an integer $q > 2$. Then there exists a unique solution $u$ to equation (A.2) whose remainder $w := u - u_0$ belongs to $\tilde{X}_{\delta,\mu,q}$ with $D^q w \in X_{\delta,\mu,q-1}$ for some $\delta \in (0, \delta)$, provided the following structural conditions are satisfied:

(i) The energy dissipation matrix

$$M_0(x) := \begin{pmatrix} M(x) & -1/2 & 0 \\ -1/2 & M(x) + 2A & 0 \\ 0 & 0 & M(x) - 1 \end{pmatrix},$$  \hfill (A.5)

with $M(x) = \text{diag}(\mu_1(x), \ldots, \mu_d(x))$, is positive definite for all $x \in H$.

(ii) The map

$$F_{\text{red}}(u_0) : w \mapsto F(u_0)[w] - L[u_0]$$  \hfill (A.6)

with $F(u_0)$ given by equation (A.4) is well defined, and maps $w \in \tilde{X}_{\delta,\mu,q}$ to $X_{\delta,\mu,q}$ for some exponent vector $\nu > \mu$.

(iii) For each $s > 0$ and for all $\delta' \in (0, \delta)$, there exists a constant $C > 0$ with

$$\|F_{\text{red}}(u_0)[w] - F_{\text{red}}(u_0)[\tilde{w}]\|_{\delta',\nu,q} \leq C \|w - \tilde{w}\|_{\delta',\mu,q},$$

for all $w, \tilde{w} \in \tilde{X}_{\delta',\mu,q,s}$. The constant $C$ may depend on $s$, but, in particular, not on $\delta'$.

If all of the previous conditions are satisfied for all values of $q > 2$, then there exists a unique solution $u$ of equation (A.2) with $u - u_0 \in \tilde{X}_{\delta,\mu,\infty}$.

The result actually holds for general smooth oriented connected compact Riemannian $n$-dimensional manifolds; the condition $q > 2$ above must then be replaced by $q > n/2 + 1$.

The reader may want to compare this to theorem 2.4 in [2] and its $n$-dimensional version in [1]. The hypotheses simplify here thanks to our restriction to semilinear equations with smooth coefficients. A further difference is that we consider second-order equations here. Apart from the necessity of introducing the spaces $\tilde{X}_{\delta,\mu,q}$, this, however, does not lead to any more complications than for the theory for first-order equations.

We remark that in its applications, this theorem often allows one to find an open set of values for the exponent vector $\mu$ for which the singular initial value problem is well-posed. A lower bound for this set can originate in condition (i), while an upper bound is usually determined by condition (ii). Although those bounds on the set of allowed values for $\mu$ may not be sharp, both bounds provide useful information on the problem. The upper bound for $\mu$ specifies the smallest regularity space and, hence, the most precise description of the behavior of $w$ (in the limit $t \searrow 0$), while the lower bound for $\mu$ determines the largest space in which
the solution \( u \) is guaranteed to be unique. We note that this uniqueness property must be interpreted with care: under the conditions of our theorem, there is exactly one solution \( w \) in the space \( X_{b,\mu,q} \), although we do not exclude the possibility that another solution may exist in a larger space, for example, in \( X_{\tilde{b},\tilde{\mu},q} \) with \( \tilde{\mu} < \mu \).

We also remark that results analogous to those stated in theorem A.2 for \( D^2 (u - u_0) \) can be derived for higher time derivatives if the regularity given by \( q \) is sufficiently large.

**Outline of the proof of theorem A.2.** We follow the strategy outlined in detail in [2, 1]; the proof hence consists of the following steps. First, we consider linear equations and derive energy estimates for the Cauchy problem with data at any \( t > 0 \). Secondly, we use these energy estimates to prove convergence of a certain sequence of approximate solutions with respect to the spaces defined before, which implies the existence of solutions of the singular initial value problem, and then to improve the regularity of the solutions. Uniqueness is obtained by employing the energy estimates one more time. In a third main step, we use these results for linear equations to set up a fixed point iteration where the nonlinear equations are linearized. The resulting limit is then the unique solution of the singular initial value problem for general nonlinear equations of the form equation (A.2). We discuss all this in more detail now.

**Step 1. Energy estimates for the Cauchy problem of linear tensorial Fuchsian wave equations.** We start by considering particular linear Fuchsian wave equations derived from equation (A.2). We see below that it is useful to consider linear equations not only for time-dependent \( d \)-vector-valued scalar quantities, as in equation (A.2), but equations for time-dependent \( d \)-vector-valued \((0, s)\)-tensor fields \( u_{a_1...a_s} \) for some \( s \geq 0 \); the scalar case is recovered for \( s = 0 \). The class of equations, which is relevant for this step of the proof, is

\[
D^2 u_{a_1...a_s} + 2 AD u_{a_1...a_s} - r^2 \Delta_b u_{a_1...a_s} = f_{a_1...a_s} + r^2 G(u)_{a_1...a_s},
\]

where \( f_{a_1...a_s} \) is some time-dependent vector-valued \((0, s)\)-tensor field. The map \( G \) is linear and built from contractions of \( u_{a_1...a_s} \) with other smooth scalar-valued time-independent tensor fields; in our particular case those are the Riemann tensor of the metric \( h_{ab} \) and its covariant derivatives, as we see below. For any \( u_{a_1...a_s} \), it follows that \( G(u)_{a_1...a_s} \) is a time-dependent \( d \)-vector-valued \((0, s)\)-tensor field with the same regularity as \( u_{a_1...a_s} \). Under these assumptions, we refer to equation (A.7) as a linear tensorial Fuchsian wave equation.

Before we turn our attention to singular initial value problems, we focus on the Cauchy problem of equation (A.7). If \( f_{a_1...a_s} \) is smooth, then it is a consequence of \( G \) mapping smooth to smooth and theorem 12.17 in [45] that the Cauchy problem is well-posed: if we prescribe smooth fields \( u^*_{a_1...a_s} \) and \( u^{**}_{a_1...a_s} \) at some \( t_0 \in (0, \delta) \), then there exists a unique smooth solution \( u_{a_1...a_s} \) on \( [0, \delta] \times H \) of equation (A.7) such that \( u_{a_1...a_s}(t_0) = u^*_{a_1...a_s} \) and \( \partial_t u_{a_1...a_s}(t_0) = u^{**}_{a_1...a_s} \).

Next we derive energy estimates for such smooth solutions of the Cauchy problem of equation (A.7). We fix a non-negative integer \( s \) and assume that \( u_{a_1...a_s} \) is a time-dependent smooth vector-valued tensor field. Let us choose two positive constants \( \kappa, \gamma \) and define, for a given exponent vector \( \mu \), the following energy

\[
E_{\kappa,\gamma,\mu}[u_{a_1...a_s}]:= \frac{1}{2} e^{-\kappa t} \langle [R][u]u_{a_1...a_s}, [R][\mu]u_{a_1...a_s}\rangle_{L^2(H)} + \langle [R][\mu]D u_{a_1...a_s}, [R][\mu]D u_{a_1...a_s}\rangle_{L^2(H)} + \langle [R][\mu]D u_{a_1...a_s}, [R][\mu]D u_{a_1...a_s}\rangle_{L^2(H)}.
\]

recall the definition of \( [R][\mu] \) in equation (A.1).

**Lemma A.3.** (Basic energy estimates for the Cauchy initial value problem) Suppose that \( (H, h_{ab}) \) is a smooth connected orientable compact Riemannian manifold. Pick a constant \( \delta > 0 \) and an exponent vector \( \mu \). Suppose that equation (A.7) is a linear tensorial Fuchsian wave equation for smooth \( f_{a_1...a_s} \), so that the energy dissipation matrix given by equation (A.5) is positive definite for all \( x \in H \). Then there exist positive constants \( \kappa, \gamma, \) and \( C \) such that for
any smooth initial data \( u_{a_{n1}}^{*} \) and \( u_{a_{n1}}^{**} \) at \( t = t_0 \in (0, \delta) \), the solution of the Cauchy problem \( u_{a_{n1}}^{*} \), for this system and this initial data satisfies the energy estimate

\[
\sqrt{E_{\mu, \kappa, \gamma}[u_{a_{n1}}^{*}]}(t) \leq \sqrt{E_{\mu, \kappa, \gamma}[u_{a_{n1}}^{*}]}(t_0) + C \int_{t_0}^{t} s^{-1} \| \mathcal{R}[\mu](s) f_{a_{n1}}(s) \|_{L^2(T^1)} \, ds, \tag{A.8}
\]

for all \( t \in [t_0, \delta] \). The constants \( C, \kappa \), and \( \gamma \) may be chosen independently of \( f_{a_{n1}} \). If one replaces the data \( u_{a_{n1}}^{*} \) and \( u_{a_{n1}}^{**} \) at \( t = t_0 \) by other smooth data specified at any time \( t_1 \in (0, t_0] \), then the energy estimate holds for the same constants \( C, \kappa, \gamma \).

Before proving this lemma, we make a few remarks. First, this lemma does not imply that the energy estimate equation (A.8) holds for \( t < t_0 \), in particular not for the limit \( t \searrow 0 \). Finally, we point out that the map \( G \) does not appear explicitly in the energy estimate; the constants, however, will in general depend on \( G \).

Note that we can clearly find constants \( C_1 \) and \( C_2 \), which do not depend on \( t, \mu \) and \( u_{a_{n1}}^{*} \), so that at every \( t \in (0, \delta) \)

\[
C_2(\| \mathcal{R}[\mu](t) u_{a_{n1}}^{*} \|_{L^2(H)} + \| \mathcal{R}[\mu](t) D u_{a_{n1}}^{*} \|_{L^2(H)} + \| \mathcal{R}[\mu](t) \nabla u_{a_{n1}}^{*} \|_{L^2(H)})
\]

\[
\leq \sqrt{E_{\mu, \kappa, \gamma}[u_{a_{n1}}^{*}]}(t)
\]

\[
\leq C_1(\| \mathcal{R}[\mu](t) u_{a_{n1}}^{*} \|_{L^2(H)} + \| \mathcal{R}[\mu](t) D u_{a_{n1}}^{*} \|_{L^2(H)}
\]

\[
+ \| \mathcal{R}[\mu](t) \nabla u_{a_{n1}}^{*} \|_{L^2(H)}).
\tag{A.9}
\]

Hence, we can reformulate equation (A.8) purely in terms of spatial \( L^2 \)-norms.

**Proof.** Under our assumptions, we are allowed to differentiate under the integral and thereby obtain (writing \( E \) instead of \( E_{\mu, \kappa, \gamma}[u_{a_{n1}}^{*}] \))

\[
DE = -\kappa \gamma t^4 E + e^{-\gamma t} ((D \mathcal{R}[\mu]) u_{a_{n1}}^{*} + \mathcal{R}[\mu] D u_{a_{n1}}^{*}, \mathcal{R}[\mu] u_{a_{n1}}^{*})_{L^2(H)}
\]

\[
+ (D \mathcal{R}[\mu]) D u_{a_{n1}}^{*} + \mathcal{R}[\mu] D^2 u_{a_{n1}}^{*}, \mathcal{R}[\mu] D u_{a_{n1}}^{*})_{L^2(H)}
\]

\[
+ (D \mathcal{R}[\mu] \nabla u_{a_{n1}}^{*} + \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*}, \mathcal{R}[\mu] \nabla u_{a_{n1}}^{*})_{L^2(H)}
\]

Since u_{a_{n1}}^{*} is a solution of our equation, we have

\[
(\mathcal{R}[\mu] D^2 u_{a_{n1}}^{*} + \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*}, \mathcal{R}[\mu] D u_{a_{n1}}^{*})_{L^2(H)} = (\mathcal{R}[\mu](-2AD u_{a_{n1}}^{*} + r^2 \Delta u_{a_{n1}}^{*} + f_{a_{n1}} +
\]

\[
+ r^2 G(u_{a_{n1}}^{*}), \mathcal{R}[\mu] D u_{a_{n1}}^{*} )_{L^2(H)}.
\]

Before we proceed, let us now consider the following expression:

\[
\int (\mathcal{R}[\mu] \nabla u_{a_{n1}}^{*} + \mathcal{R}[\mu] D u_{a_{n1}}^{*} - 2 \nabla (\mathcal{R}[\mu] D u_{a_{n1}}^{*} +\n\]

\[
\times (\mathcal{R}[\mu] \nabla u_{a_{n1}}^{*} + \mathcal{R}[\mu] D u_{a_{n1}}^{*})_{L^2(H)}.
\]

When this is multiplied with the volume form and integrated over \( H \), the first term drops out due to Stokes theorem (see for example [6]). We have therefore found

\[
(\mathcal{R}[\mu](-2AD u_{a_{n1}}^{*} + \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*}, \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*})_{L^2(H)}
\]

\[
= -2 \| \nabla (\mathcal{R}[\mu] D u_{a_{n1}}^{*} + \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*})_{L^2(H)}.
\]

We get

\[
DE = e^{-\gamma t} (\mathcal{R}[\mu] f_{a_{n1}}^{*}, \mathcal{R}[\mu] D u_{a_{n1}}^{*})_{L^2(H)} - e^{-\gamma t} K(U, U),
\]

where \( K \) is a bilinear form on the space of smooth time-dependent vector-valued tensor fields generated by

\[
U := (\mathcal{R}[\mu] u_{a_{n1}}^{*}, \mathcal{R}[\mu] D u_{a_{n1}}^{*}, \mathcal{R}[\mu] \nabla D u_{a_{n1}}^{*})^T = (U_1, U_2, U_3).
\]
The expression of $K(U, V) = 2\kappa t^t ((U_1, V_1)_{L^2(H)} + (U_2, V_2)_{L^2(H)} + (U_3, V_3)_{L^2(H)}) - t^2 \tilde{G}(U, V)$

$$+ 2\kappa t^t |R[U]| |R[U]|^{-1} U_1, U_1)_{L^2(H)}$$

$$- (D[R[U]]) R[U]^{-1} U_1, V_1)_{L^2(H)} - (U_2, V_1)_{L^2(H)}$$

$$- (D[R[U]]) R[U]^{-1} V_2, V_1)_{L^2(H)}$$

$$- (D[R[U]]) R[U]^{-1} U_3, V_3)_{L^2(H)} + (2A U_2, V_2)_{L^2(H)},$$

where $\tilde{G}$ is a bilinear form defined by the map $G$. The bilinear form $K$ is positive definite uniformly on $(0, \delta)$ if $\kappa$ is sufficiently large and $\gamma$ is sufficiently small and if the $3d \times 3d$ energy dissipation matrix given by equation (A.5) is positive definite at each spatial point $x \in H$; note that $(D[R[U]]) R[U]^{-1} = -\text{diag}(\mu_1, \ldots, \mu_d)$, as follows directly from the definition of $R[U]$.

If this is the case, then

$$D E \leq e^{-t^t} \|R[U]\|_{L^2(H)} \|D[U]\|_{L^2(H)}$$

$$\leq e^{-t^t} \|R[U]\|_{L^2(H)} \|D[U]\|_{L^2(H)}$$

$$\leq C e^{-t^t} \|R[U]\|_{L^2(H)} \sqrt{E},$$

for some constant $C > 0$. Using the Grönwall inequality, this can be integrated as usual, and we obtain equation (A.8). \hfill \square

Now we want to obtain similar estimates for higher derivatives of the unknown. First, we show that the form of equation (A.7) is retained, when we take a covariant derivative of both sides and then interpret the result as an equation for $\nabla_b u_{a_1 \ldots a_s}$:

$$D^2 \nabla_b u_{a_1 \ldots a_s} + 2AD \nabla_b u_{a_1 \ldots a_s} = 2h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} = \nabla_b f_{a_1 \ldots a_s} + t^2 G_1(u) h_{a_1 \ldots a_s}$$

$$+ t^2 G_2(\nabla u) h_{a_1 \ldots a_s}.$$}

Here $G_1$ and $G_2$ are maps obtained from $G$ by covariant differentiation. Using the definition of the Riemann and Ricci tensors of $b_{a_1 b_{a_2 \ldots a_s}}$, we compute

$$h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} = h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} = \sum_{l=1}^s h^d e^{\gamma} R_{bca_l a_s} \nabla_d u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s}$$

$$= h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} = \sum_{l=1}^s h^d e^{\gamma} (R_{bca_l a_s} u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s})$$

$$+ R_b d \nabla_d u_{a_1 \ldots a_s} = \sum_{l=1}^s h^d e^{\gamma} R_{bca_l a_s} \nabla_d u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s}$$

$$= h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} = \sum_{l=1}^s h^d e^{\gamma} (R_{bca_l a_s} u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s})$$

$$+ R_b d \nabla_d u_{a_1 \ldots a_s} = \sum_{l=1}^s h^d e^{\gamma} R_{bca_l a_s} \nabla_d u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s}$$

$$= h^d e^{\gamma} \nabla_b \nabla_d u_{a_1 \ldots a_s} + R_b d \nabla_d u_{a_1 \ldots a_s}$$

$$- \sum_{l=1}^s (\nabla_d R_{bca_l a_s} u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s} + 2R_{bca_l a_s} \nabla_d u_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_s}).$$

Hence, for the time-dependent $d$-vector-valued $(0, s + 1)$-tensor field $\hat{u}_{a_1 a_2 \ldots a_s} := \nabla_a u_{a_1 \ldots a_s}$, we have found the system

$$D^2 \hat{u}_{a_1 a_2 \ldots a_s} + 2AD \hat{u}_{a_1 a_2 \ldots a_s} = t^2 \Delta \hat{u}_{a_1 a_2 \ldots a_s} = \hat{f}_{a_1 a_2 \ldots a_s} + t^2 \tilde{G}(\hat{u})_{a_1 a_2 \ldots a_s}$$

(A.10)
approximation scheme which works as follows: We first choose a monotonically decreasing \( \parallel \rho \) then the energy estimate holds for the same constants \( C, \epsilon > 0 \), leading-order term \( a u \) is singular.

Hence, when \( u(a_{0}, \ldots, a_{s}) \) is considered as a fixed known field and equation (A.10) is understood as an equation for \( \hat{u}(a_{0}, \ldots, a_{s}) \), then it is of the same form as equation (A.7) with \( s \) replaced by \( s + 1 \). In fact, we can apply lemma A.3 to equation (A.10)

\[
\sqrt{\mathcal{R}[\mu, \gamma]} \parallel \nabla a_{0} f(a_{0}, \ldots, a_{s}) \parallel_{L^2(H)}(t) - \sqrt{\mathcal{R}[\mu, \gamma]} \parallel \nabla a_{0} u(a_{0}, \ldots, a_{s}) \parallel_{L^2(H)}(t_0)
\leq C \int_{t_0}^{t} s^{-1} \parallel \mathcal{R}[\mu](s) f(a_{0}, \ldots, a_{s})(s) \parallel_{L^2(H)} ds,
\]

where we have used equation (A.11) and the constant \( C \) has been adapted in the last step in order to incorporate contributions from the map \( G_1 \) and from the covariant derivatives of the Riemann tensor. This can be understood as an estimate for the first spatial derivatives of the unknown. When we add this inequality to the estimate equation (A.8) for the undifferentiated unknown, we obtain an estimate for the \( H^1(H) \)-norm of the unknown by a relation which is similar to the equivalence estimate equation (A.9). However, we must be cautious when \( \mu \) depends on space: then the spatial derivatives of the function \( \mathcal{R}[\mu] \) introduce additional \( \ln t \)-factors which must be controlled by introducing an appropriate constant \( \epsilon > 0 \) as shown below. We have hence proved the following lemma for the case \( q = 1 \); the case \( q > 1 \) be can obtained iteratively.

**Lemma A.4.** (Energy estimates of higher order). Pick an integer \( q \geq 1 \) and any sufficiently small constant \( \epsilon > 0 \). Under the otherwise same conditions as lemma A.3, there exist positive constants \( C, \rho \) such that for any smooth initial data \( u(a_{0}, \ldots, a_{s}) \) and \( u^{\ast\ast}(a_{0}, \ldots, a_{s}) \) at \( t = t_0 \in (0, \delta), \) the solution \( u(a_{0}, \ldots, a_{s}) \) of the Cauchy problem for this system and this initial data satisfies the estimate

\[
\parallel \mathcal{R}[\mu - \epsilon] u(a_{0}, \ldots, a_{s})(t) \parallel_{L^2(H)} + \parallel \mathcal{R}[\mu - \epsilon] Du(a_{0}, \ldots, a_{s})(t) \parallel_{H^1(H)}(t)
\leq C(\parallel \mathcal{R}[\mu] u(a_{0}, \ldots, a_{s})(t_0) \parallel_{L^2(H)} + \parallel \mathcal{R}[\mu] Du(a_{0}, \ldots, a_{s})(t_0) \parallel_{H^1(H)}(t_0)
\]

\[\quad + \parallel \mathcal{R}[\mu] \nabla b a_{0}, \ldots, a_{s}(t_0) \parallel_{H^1(H)} + \int_{t_0}^{t} s^{-1} \parallel \mathcal{R}[\mu](s) f(a_{0}, \ldots, a_{s})(s) \parallel_{H^1(H)} ds,
\]

for all \( t \in [t_0, \delta) \). The constants \( C, \rho \) may be chosen independently of \( f(a_{0}, \ldots, a_{s}) \). If one replaces the data \( u(a_{0}, \ldots, a_{s}) \) by other smooth data specified at any time \( t_1 \in (0, t_0] \), then the energy estimate holds for the same constants \( C, \rho \).

From now on, we restrict ourselves to the scalar case \( s = 0 \); it is, however, clear from these energy estimates that the tensorial case works in exactly the same way.

**Step 2. Existence and uniqueness of the singular initial value problem for linear Fuchsian wave equations.** We now use the same arguments as in [2] to show existence of solutions of the singular initial value problem for scalar equations of the form equation (A.7) with leading-order term \( u_0 = 0 \). We set out to use solutions of the Cauchy problem via an approximation scheme which works as follows: We first choose a monotonically decreasing
sequence of times \( t_n \in (0, \delta] \) which converges to zero. Then for each \( n \), we construct a function \( v_n : (0, \delta] \times H \rightarrow \mathbb{R}^n \) which vanishes on \((0, t_n]\), and which is equal on \((t_n, \delta] \) to the solution of the Cauchy problem with zero initial data at \( t_n \). One readily verifies for smooth source-term functions \( f \) that for every choice of \( \mu \), one has \( v_n \in C^1((0, \delta] \times H) \cap \bar{X}_{\delta,\mu,0} \) for all integers \( q \geq 0 \). The central result of this section is that if certain hypotheses hold, then the sequence \( (v_n) \)—whose elements we label approximate solutions—converges to a solution of the singular initial value problem for the linear system with vanishing leading term.

**Proposition A.5.** (Existence of solutions of the singular initial value problem). Suppose that 
\( (H, h_{ab}) \) is a smooth connected orientable compact Riemannian manifold of dimension \( n \). Pick a constant \( \delta > 0 \), an exponent vector \( \mu \) and an integer \( q \geq 1 \). Suppose that equation (A.7) is a linear Fuchsian wave equation with \( s = 0 \) and \( f \) is an element of \( X_{\delta,\nu,0} \) for some exponent vector \( \nu > \mu \) so that the energy dissipation matrix given by equation (A.5) is positive definite for all \( x \in H \). Then, there exists a solution \( u = w : (0, \delta] \times H \rightarrow \mathbb{R}^d \) to the singular initial value problem with vanishing leading term \( u_0 = 0 \) which is an element of \( \bar{X}_{\delta,\mu,q} \). If \( q > n/2 + 1 \), then the solution is unique in the space \( \bar{X}_{\delta,\mu,q} \). The solution operator \( \mathbb{H} : X_{\delta,\nu,q} \rightarrow \bar{X}_{\delta,\mu,q} \) \( f \rightarrow w \) satisfies
\[
\|\mathbb{H}[f]\|_{\bar{X}_{\mu,q}} \leq C\|f\|_{X_{\nu,q}},
\]
for all \( f \in X_{\delta,\nu,0} \) for constants \( C > 0 \) and \( \rho > 0 \).

The reader should be aware of the fact that we do not require the source-term \( f \) to be smooth here, while this was a condition in lemma A.3 and lemma A.4. In particular, we do not restrict ourselves to smooth solutions here; in fact the regularity of the solutions is determined by the integer \( q \). Only if the conditions of proposition A.5 hold for all integers \( q \geq 1 \), then the remainder of the solution \( w \) is in \( \bar{X}_{\delta,\mu,\infty} \) and hence \( w \) has infinitely many spatial derivatives. In general one finds that the constants in equation (A.12) may depend on \( q \). In particular, \( C \) may be unbounded and \( \rho \) may go to zero in the limit \( q \rightarrow \infty \). Concerning time derivatives, we note that the remainder \( w \) is a solution of the equations and hence \( w \) is at least twice differentiable in \( t \) under the hypothesis of the above proposition with \( D^2 w \in X_{\delta,\mu,q-1} \). If \( q \) is sufficiently large, then corresponding statements can be derived for higher time derivatives of \( w \). In the case \( q \rightarrow \infty \), it follows that \( w \) has infinitely many time derivatives and all time derivatives are in \( X_{\delta,\mu,\infty} \).

**Proof of proposition A.5.** The proof follows the arguments in [2] where all the details can be found. We only list the steps here and give short discussions of each. In a first step we consider the case \( q = 0 \) and assume that the source term \( f \) is in \( C^\infty((0, \delta] \times H) \cap \bar{X}_{\delta,\nu,0} \). The sequence of approximate solutions \( (v_n) \) defined above is in the Banach space \( \bar{X}_{\delta,\mu,0} \) as mentioned above. Using the energy estimates of lemma A.3, our hypothesis is sufficient to guarantee that this is a Cauchy sequence; this can be shown in exactly the same way as in the proof of proposition 2.10 in [2]. Next we could show, as done in that aforementioned proposition in [2], that the corresponding limit \( w \) in \( \bar{X}_{\delta,\mu,0} \) is a solution of the equation in a weak sense; however, we will not consider weak solutions here.

Next, following the argument of proposition 2.12 in [2], we consider the case \( q = 1 \), i.e. that \( (v_n) \) is a sequence in \( \bar{X}_{\delta,\mu,1} \) when we assume that \( f \in C^\infty((0, \delta] \times H) \cap \bar{X}_{\delta,\nu,1} \). By combining the energy estimates in lemma A.4 for \( q = 1 \) and by estimating the \( q = 0 \)-terms by means of the estimates which were obtained in the previous step, we find that \( (v_n) \) is a Cauchy sequence in \( \bar{X}_{\delta,\mu,1} \). If \( f \in C^\infty((0, \delta] \times H) \cap X_{\delta,\nu,q} \) for any integer \( q \geq 1 \) as given in the hypothesis, then we prove that \( (v_n) \) is a Cauchy sequence in \( \bar{X}_{\delta,\mu,0} \) by successively using lemma A.4 and by estimating the terms with lower derivatives by the estimates obtained from the previous step. We call the limit \( w \).
We have hence gathered the following information about \( w \): since \( w \in \tilde{X}_{k,\mu,q} \), (i) the function \( w \) is in \( X_{k,\mu,q} \), (ii) the function \( w \) is differentiable in time (see more details about this in the appendix of [2]) with \( Dw \in X_{k,\mu,q} \), and (iii) \( \triangledown_t w \in X_{k,\mu,q} \). In order to check whether \( u = w \) is a solution of equation (A.7) with vanishing leading-order term \( u_0 \), we must first guarantee that all terms in equation (A.7) are well defined at each \( \ell \in (0, \delta) \) when evaluated on \( w \). Since we suppose that \( q \geq 1 \), it remains to show that \( w \) is in fact twice time differentiable. We do this in the same way as in the proof of proposition 2.12 in [2]: we solve equation (A.7) algebraically for \( D^2 w \) and replace \( u = w \) on the right-hand side by \( v_n \). Then we define

\[
\hat{v}_n := -2ADv_n + i^2 \Delta_t v_n + f + i^2 G \cdot v_n.
\]

Let us fix a compact subinterval \([\delta_0, \delta]\). By choosing \( n \) sufficiently large, we can assume without loss of generality that \( \hat{v}_n(t, x) = D^2v_n(t, x) \) for all \((t, x) \in [\delta_0, \delta] \times H\), and we find that \( \hat{v}_n \in X_{k,\mu,q} \) for all \( n \). In fact \((\hat{v}_n)\) is a Cauchy sequence in \( X_{k,\mu,q} \) and therefore has a limit \( \hat{v} \). Using standard arguments of uniform convergence, we can then show that \( w \) is twice time differentiable and that \( D^2w = \hat{v} \in X_{k,\mu,q} \). It is therefore clear that \( w \) is really a solution (in the strong sense) of equation (A.7).

The same arguments as in proposition 2.11 in [2] lead to the conclusion that the thus constructed solution of the singular initial value problem does not depend on the choice of sequence \((v_n)\) which we have used above to define the sequence \((v_n)\). We can thus define a solution operator \( H \) which maps the source-term \( f \) to the thus constructed solution \( w \) (but we do not claim uniqueness of the solution yet). It is, at this stage, a map \( C^\infty((0, \delta] \times H) \cap X_{k,\mu,q} \rightarrow \tilde{X}_{k,\mu,q} \). In the same way as in the proof of Propositions 2.11 and 2.12 in [2], we derive the continuity estimate equation (A.12) for \( H \). It is a standard result that a linear operator with an estimate of the form equation (A.12) can be uniquely extended from the dense subspace \( C^\infty((0, \delta] \times H) \cap X_{k,\mu,q} \) to the full space \( \tilde{X}_{k,\mu,q} \). It follows easily that this extended operator, which we refer to with the same name \( H \), maps any source-term \( f \in X_{k,\mu,q} \) to the remainder of solution \( w \in \tilde{X}_{k,\mu,q} \).

The last step is to show uniqueness of the solution in \( \tilde{X}_{k,\mu,q} \). Suppose that there are two solutions \( w_1, w_2 \in \tilde{X}_{k,\mu,q} \) of the same equation. Hence, their difference \( \xi := w_1 - w_2 \in \tilde{X}_{k,\mu,q} \) satisfies the same equation with vanishing source-term function \( f \). We notice that the same computation as in the proof of lemma A.3 is valid if the solution is not necessarily smooth, but, say, if \( w, Dw, \triangledown_t w, D^2w, \triangledown_t Dw \) and \( \Delta_t w \) are all continuous on \((0, \delta] \times H\). According to lemma A.1, this is the case if \( q - 1 > n/2 \). Now, we can consider the function \( \xi \) as a solution of the Cauchy problem of the equation with vanishing source term corresponding to data \( \xi^\ast := \xi(t_0, \cdot), \xi^{**} := \partial_\xi(t_0, \cdot) \). Then lemma A.3 implies that

\[
\|R[\mu]\xi(t_0, \cdot)\|_{L^2(H)} + \|R[\mu]D\xi(t_0, \cdot)\|_{L^2(H)} + \|R[\mu]D^2\xi(t_0, \cdot)\|_{L^2(H)} \\
\leq C(\|R[\mu]\xi(t_0, \cdot)\|_{L^2(H)} + \|R[\mu]D\xi(t_0, \cdot)\|_{L^2(H)} + \|R[\mu]D^2\xi(t_0, \cdot)\|_{L^2(H)}).
\]

In the same way as in the proof of proposition 2.14 in [2] we take the limit \( t_0 \) of such an estimate and find that \( \xi \) must vanish. It follows that \( w_1 = w_2 \).

**Step 3. Existence and uniqueness for nonlinear equations.** We now have the essential tools needed to prove theorem A.2 for general semilinear Fuchsian wave equations of the form equation (A.2). Again, we follow the approach of [2] and we refer to that work for the details.

**Proof of theorem A.2.** We restrict ourselves to \( H = S^2 \) and \( h = d\theta^2 + \sin^2 \theta \ d\phi^2 \) here, but we remark that the result holds under more general assumptions.

The idea is to construct the following iteration scheme. We start with some seed function \( u_{[0]} = u_0 + w_{[0]} \) where \( u_0 \) is the fixed leading-order term (which does not need to vanish
here), and where \( w_0 \) is some element in \( \tilde{X}_{\delta,\mu,q} \). Without loss of generality, we can assume that \( w_0 = 0 \). We can linearize the equation around \( u_0 \), i.e. consider the equation
\[
L[w_{[1]}] = F_{\text{red}}(u_0)[w_0] = F(u_0)[w_0] - L[u_0],
\]
for an unknown \( w_{[1]} \). Because of the hypothesis of theorem A.2, the source-term \( F(u_0)[w_0] - L[u_0] \) satisfies the hypothesis of proposition A.5, and hence there is a unique \( w_{[1]} \) in \( \tilde{X}_{\delta,\mu,q} \) which solves this equation. In a next step, we solve the linear equation
\[
L[w_{[2]}] = F_{\text{red}}(u_0)[w_{[1]}],
\]
and obtain a unique solution \( w_{[2]} \) in \( \tilde{X}_{\delta,\mu,q} \) etc. This yields a sequence \( (w_{[i]}) \) in \( \tilde{X}_{\delta,\mu,q} \) and we wish to show that this sequence converges to the unique solution of the nonlinear equation. The operator \( G = \tilde{H} \circ F_{\text{red}}(u_0) \) maps \( w_{[i]} \) to \( w_{[i+1]} \), and solutions of the nonlinear equations are precisely the fixed points of this operator. As in the proof of theorem 2.4 in [2] we find that the Lipschitz requirement on \( F_{\text{red}} \) in the hypothesis of theorem A.2 and the estimate equation (A.12) for a sufficiently small value of \( \delta \) is enough to show that the sequence is bounded and \( G \) is a contraction. The result follows therefore from the Banach fixed point theorem.

Note that in the semilinear case, which we are considering here, we do not lose a derivative by this argument; this is an issue which occurs for general quasi-linear equations as discussed in [2] and which needs to be fixed by a duality argument.

Now we consider the ‘smooth case’ \( q = \infty \). Hence, we assume that the hypothesis of theorem A.2 holds for all integers \( q > 2 \). In particular, there exists a sequence of constants \( (C_q) \) in condition (iii) of theorem A.2 for each value of \( q \), which may be unbounded in the limit \( q \to \infty \). While it allows us to choose a sufficiently small \( \delta_q \) to make the above argument work for every finite value of \( q \), it may turn out that the sequence \( (\delta_q) \) of these numbers has to approach 0 in the limit \( q \to \infty \). Hence there would be no smooth solution since the time interval of existence would be empty. However, this problem can be circumvented by means of standard continuation arguments for wave equations (or more general hyperbolic equations). In the literature one finds these arguments mainly for nonlinear wave equations on the spatial domain \( \mathbb{R}^n \); see for example lemma 9.14 in [45]. The point is that for this class of equations, the energy of all higher derivatives of the solutions is bounded by the time integral of a norm of the solution (a function called \( m \) in [45]). This function can be bounded by the \( H^p \)-norm of the solution for a sufficiently large \( p \) (dependent on the dimension; the particular value is not important in the smooth case here). In particular, if the \( H^p \)-norm is bounded, then the solution can be extended as an \( H^q \)-solution for every integer \( q > p \) which is compatible with the coefficients of the equations and the data. This argument can easily be adapted to our compact spatial manifold \( H = S^2 \): first, cover the manifold \( H \) by a finite number of open coordinate patches \( U \subset H \), and hence use the coordinate maps \( \Phi \) to transfer the solution and the equation to the corresponding open patches \( V \subset \mathbb{R}^n \) (with \( n = 2 \) in order case). If the \( H^p(H) \)-norm is finite at a certain time, then the \( H^p(V) \)-norm is finite for any of those open subsets \( V \subset \mathbb{R}^n \). According to the above theorem, the solution can be extended for a sufficiently small time interval inside the domain of dependence of each of these local domains as a \( H^q(\tilde{V}) \)-solution where \( \tilde{V} \subset V \) is sufficiently small. A standard ‘patching’ argument then implies that we obtain a \( H^q(H) \) solution on a sufficiently small extension of the time interval. Such patching arguments have been used often in the literature in general relativity, see for example [20, 21, 44]. Now, in our case, we know that there is a solution with remainder in \( X_{\delta,\mu,p} \) for an arbitrary sufficiently large \( p \). So far, we know that it is also in \( X_{\delta,\mu,q} \) for all \( q > p \) where, possibly, \( \delta_q < \delta_p \). Thanks to this extension argument, we can, however, always choose \( \delta_q = \delta_p \). After we have applied this argument to all \( q > p \), we find that there exists a solution with remainder \( w \in X_{\delta_p,\mu,\infty} \). \( \square \)
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