High temperature AdS black holes are low temperature quantum phonon gases

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Abstract

We report a precise match between the high temperature $(D+2)$-dimensional Tangherlini-AdS black hole and the low temperature quantum phonon gas in $D$-dimensional nonmetallic crystals residing in $(D + 1)$-dimensional flat spacetime. The match is realized by use of the recently proposed restricted phase space formalism for black hole thermodynamics, and the result can be viewed as a novel contribution to the AdS/CMT correspondence on a quantitative level.

Key words: AdS black hole, phonon gas, thermodynamics, restricted phase space formalism

1 Introduction

Black holes are the most important objects predicted by the modern relativistic theories of gravity. Since the 1970s it has been commonly believed that black holes are thermal objects \cite{1–5}, and as such they must contain a large number of microscopic degrees of freedom, and understanding the nature of these microscopic degrees of freedom might provide a window for inspecting the quantum nature of gravitation. Although the goal for understanding quantum gravity is still far beyond the scope of our human’s sight, there indeed has been a number of progresses and speculations toward the understanding of the microscopic structure of black holes, among which the most notable ideas come from the AdS/CFT correspondence \cite{6–8}.

In the recent years, another line of thinking pumps up which attempts to understand the black hole microstructure purely from thermodynamic perspective, especially following the so-called extended phase space (EPS) formalism \cite{9–19}. The EPS formalism to black hole thermodynamics is an approach which takes the negative cosmological constant as a
thermodynamic variable which is proportional to the pressure and is accompanied by an associated conjugate variable known as the thermodynamic volume. A significant amount of works have been published about the behavior of various black holes under the EPS approach, mostly concentrated in the critical phase transitions. Some authors also attempted to explore the interaction potential between the microscopic degrees of freedom (also called black hole molecules) of the black holes starting from the thermal equation of states that arises from the EPS formalism [20–22].

Since the end of the last year, we proposed and kept working on an alternative formalism of black hole thermodynamics called the restricted phase space (RPS) formalism [23–26]. The RPS formalism differs from the EPS formalism in that the cosmological constant is no longer taken as a thermodynamic variable but rather kept as a constant. Instead, we allow the gravitational constant $G$ to be variable and introduced a pair of new thermodynamic quantities $N = L^D/G, \mu = GT I_E L^D$ which are respectively interpreted as the effective number of microscopic degrees of freedom (or alternatively the number of black hole molecules) and the chemical potential of the black hole, wherein $D$ is the dimension of the black hole bifurcation horizon, $I_E$ is the Euclidean action of the black hole spacetime and $T$ is the black hole temperature. The constant length scale $L$ corresponds to the maximum value for the radius of the event horizon in any thermodynamic processes of interests, which can be chosen arbitrarily besides the above requirement. The arbitrariness of $L$ may be fixed in several ways, e.g. by taking some specific convention for the unit of the chemical potential, or by comparison to low temperature quantum phonon gases when considering the high temperature limit. We shall come back on this last possibility later.

The RPS formalism is a restricted version of the holographic thermodynamics for AdS/CFT systems proposed by Visser [28], but with the removal of cosmological constant from the list of thermodynamic variables. This removal of cosmological constant resolves a number of issues, including, but not limited to, the following points: 1) The Euler homogeneity holds perfectly without introducing rational coefficients in the mass formula; 2) The theory-changing problem which we call the ensemble of theories issue which existed in the EPS formalism is completely avoided, and the black hole mass restores its original interpretation as internal energy rather than as enthalpy as did in the EPS formalism; 3) The definition of $N, \mu$ are now independent of the holographic duality, and thus the RPS formalism applies perfectly to the cases of non-AdS black holes. There are some arguments [29] which can avoid the theory-changing problem in the presence of variable cosmological constant, but the lack of complete Euler homogeneity and the inability for generalizations to non-AdS cases remain to be critical issues in other existing formalisms.

Subsequent works [27,30,31] revealed that the RPS formalism works perfectly for black holes with different asymptotics in Einstein gravity and certain higher curvature gravity models in diverse spacetime dimensions. Some universal behaviors have also been found for
black holes under this formalism, e.g. the black holes in Einstein-Hilbert and Born-Infeld like gravities behave qualitatively the same, while black holes in Chern-Simons like gravities behave completely different. Moreover, for certain AdS black holes, the high temperature limit of the heat capacity has a power law dependence on the temperature which is identical to the low temperature limit of the Debye heat capacity of nonmetallic crystals. We believe that this similarity is not a coincidence, and this letter is a further exploration on this similarity. As will be shown in the main text below, the similarity between the high temperature AdS black holes and the low temperature nonmetallic crystals is much more profound. Not only the heat capacities, but also the Helmholtz free energies, the internal energies and the entropies of the two drastically different types of systems (high temperature AdS black hole and low temperature nonmetallic crystals) behave almost identically in certain temperature ranges. Since the microscopic description of nonmetallic crystal is identical to a quantum phonon gas, we conclude that the high temperature AdS black holes are actually equivalent to low temperature quantum phonon gases. Although this link still does not reveal the nature of individual black hole molecules, it indeed reveals their collective effects, i.e. the black hole molecules are quantum, and their collective motion behaves like phonons in certain temperature limit. We believe that this observation is a meaningful progress towards the ultimate understanding of the black hole microstructure as well as the quantum nature of gravitation.

2 Thermodynamics of Tangherlini-AdS black holes and the high temperature limit

We will exemplify the connection between high temperature AdS black holes and low temperature Bose gases by studying in detail the thermodynamics of Tangherlini-AdS black holes in the RPS formalism. Throughout this letter we work in units $c = 1, k_B = 1, \hbar = 1$ but keep the Newton constant $G$ intact.

We take the spacetime dimension to be $D+2$, so that the bifurcation horizon for the black hole is $D$. The $(D+2)$-dimensional Einstein-Hilbert action with Gibbons-Hawking boundary term is given by

$$I = \frac{1}{16\pi G} \int_M (R - 2\Lambda) \sqrt{g} \, d^{D+2}x + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{h} \, d^{D+1}x,$$

where $g = |\det(g_{\mu\nu})|$, $h = |\det(h_{ab})|$, with $h_{ab}$ being the induced metric on the boundary $\partial M$ of the spacetime $M$ and $K$ being the trace of the extrinsic curvature of $\partial M$ in $M$. The inclusion of the Gibbons-Hawking boundary term is important in order to obtain the correct value for the Euclidean action $I_E$.

The metric of the $(D+2)$-dimensional Tangherlini-AdS black hole written in spherical
coordinates takes the form
\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_D^2, \]
where \( d\Omega_D^2 \) is the line element on a unit \( D \)-sphere with area \( \mathcal{A}_D = \frac{2\pi(D+1)/2}{\Gamma(D+1)/2} \), and \( \ell \) is the AdS radius which is related to the negative cosmological constant via
\[ \Lambda = -\frac{D(D+1)}{2\ell^2}. \]

The black hole event horizon is located at \( r = r_h \) which is a root of the equation \( f(r) = 0 \). Accordingly, the mass of the black hole can be expressed in terms of \( r_h \) and \( G \) as
\[ M = \frac{D\mathcal{A}_D r_h^{D-1}}{16\pi G} \left( 1 + \frac{r_h^2}{\ell^2} \right) = \frac{D\pi(D-1/2)r_h^{D-1} (\ell^2 + r_h^2)}{8\ell^2 \Gamma(D+1)/2 G}. \]

In the RPS formalism, the Tangherlini-AdS black hole has two independent extensive variables, i.e. the entropy \( S \) and the effective number of microscopic degrees of freedom \( N \),
\[ S = \frac{\mathcal{A}_D r_h^D}{4G} = \frac{\pi(D+1)/2 r_h^D}{2\Gamma(D+1)/2 G}, \]
\[ N = \frac{L^D}{G}, \]
where \( r_h \) is the radius of the event horizon of the black hole which is a root of the equation \( f(r) = 0 \). The corresponding intensive variables are
\[ T = \frac{1}{4\pi} \left( \frac{\partial f}{\partial r} \right)_{r=r_h} = \frac{(D-1)\ell^2 + (D+1)r_h^2}{4\pi \ell^2 r_h}, \]
\[ \mu = \frac{GT I_E}{L^D} = \frac{\pi(D-1)/2 r_h^{D-1} (\ell^2 - r_h^2)}{8\ell^2 L^D \Gamma(D+1)/2}, \]
where the Euclidean on-shell action \( I_E \) can be calculated following the standard procedures as given in, e.g. [32–35].

As expected, the two intensive variables are not independent of each other, because the first law
\[ dM = TdS + \mu dN \]
and the Euler homogeneity relation
\[ M = TS + \mu N. \]
hold simultaneously, which implies the Gibbs-Duhem relation
\[ SdT + Nd\mu = 0. \]
The complete Euler homogeneity is an outstanding feature of the RPS formalism in contrast to alternative proposals, see, e.g. [28, 36], which included additional variables $P, V$, with the sacrifice of complete Euler homogeneity ($M$ is first order homogeneous in $S, N$ but not in $V$). The framework introduced in [28] could be meaningful for the dual CFT in the context of holographic duality, however, in the absence of complete Euler homogeneity, it may cause some problems in understanding black hole phase transitions in a scale independent way, because the law of corresponding states cannot be realized in the presence of $P−V$ variables. In the RPS formalism, the law of corresponding states holds perfectly, making the $T−S$ phase transitions scale independent, as we have shown in Refs. [23, 24, 27] for different black hole solutions. Please be reminded that scale independence is a characteristic property that any phase transition should have.

Let us proceed to analyze the high temperature limit for the Tangherlini-AdS black hole. It is preferable to replace the geometric parameter $r_h$ and the coupling coefficient $G$ in the above expressions for thermodynamic quantities by a set of independent macro state parameters. The standard practice is to take the extensive parameters $(S, N)$ as independent variables and re-express $M$ (understood as the internal energy) and the intensive variables $T, \mu$ as functions in $(S, N)$: $M = M(S, N), T = T(S, N), \mu = \mu(S, N)$. However, since the major goal of this work is to analyze the high temperature limit of various thermodynamic quantities, we prefer to take $(T, N)$ as independent variables and rewrite the other macro state functions in terms of these. From eqs. (6) and (7) it is straightforward to get

$$G = \frac{L^D}{N},$$

$$r_h = \frac{\ell \left( 2\pi \ell T \pm \sqrt{4\pi^2 \ell^2 T^2 - D^2 + 1} \right)}{D + 1}.$$  

The two branched values for $r_h$ indicate that there are two black hole states at the same temperature, of which the smaller one (i.e. the negative branch) is unstable because it has smaller entropy. One can easily check that, in the high temperature limit, the value of the negative branch of $r_h$ goes to zero, and so are the mass and entropy. Therefore, in order to study the high temperature limit, we only need to consider the positive branch of $r_h$. In this branch, the entropy and the internal energy of the black hole can be rewritten as

$$S = \frac{N \pi \ell D^{(D+1)/2}}{2 \Gamma \left( \frac{D+1}{2} \right)} \left( \frac{\ell \left( \sqrt{4\pi^2 \ell^2 T^2 - D^2 + 1} + 2\pi \ell T \right)}{(D + 1)L} \right)^D,$$

$$M = \frac{ND \pi \ell D^{-1}}{8L^D \Gamma \left( \frac{D+1}{2} \right)} \left( \frac{\ell \left( \sqrt{4\pi^2 \ell^2 T^2 - D^2 + 1} + 2\pi \ell T \right)}{D + 1} + 1 \right),$$

both tend to be divergent at $T \to \infty$. Notice that this branch of states is specific to AdS black holes and does not exist for asymptotically flat and de Sitter cases. The Helmholtz free
energy $F$ and the heat capacity $C_N$ of the black hole can be written as

$$F = M - TS$$

$$C_N = T \left( \frac{\partial S}{\partial T} \right)_N = \frac{N \pi^{(D+3)/2} \ell T}{\Gamma \left( \frac{D+1}{2} \right) \sqrt{4 \pi^2 \ell^2 T^2 - D^2 + 1}}. \tag{15}$$

The temperature dependence of the above results appears to be very complicated. However, if we consider the high temperature limit $T \rightarrow \infty$, the results become much simplified,

$$\lim_{T \rightarrow \infty} M = \frac{N \pi^{1/2} D}{2(D + 1) \Gamma \left( \frac{D+1}{2} \right)} \left( \pi^{3/2} (2\ell)^2 \right)^D \frac{T^{D+1}}{(D + 1) L}, \tag{16}$$

$$\lim_{T \rightarrow \infty} F = -\frac{N \pi^{1/2}}{2(D + 1) \Gamma \left( \frac{D+1}{2} \right)} \left( \pi^{3/2} (2\ell)^2 \right)^D \frac{T^{D+1}}{(D + 1) L}, \tag{17}$$

$$\lim_{T \rightarrow \infty} S = \frac{N \pi^{1/2}}{2 \Gamma \left( \frac{D+1}{2} \right)} \left( \pi^{3/2} (2\ell)^2 \right)^D \frac{T^D}{(D + 1) L}, \tag{18}$$

$$\lim_{T \rightarrow \infty} C_N = \frac{N \pi^{1/2} D}{2 \Gamma \left( \frac{D+1}{2} \right)} \left( \pi^{3/2} (2\ell)^2 \right)^D \frac{T^D}{(D + 1) L}. \tag{19}$$

At this point we need to make it clear what is meant by the high temperature limit $T \rightarrow \infty$. Since $T$ is a dimensionful quantity, it does not make sense to say the temperature is high or low without comparing to a constant characteristic temperature. It is evident that in the present case, the characteristic temperature $T_{bh}$ can be chosen as

$$T_{bh} = \frac{(D + 1) L}{\pi^{3/2} (2\ell)^2}. \tag{20}$$

With this choice we can rearrange the above high temperature (i.e. $T \gg T_{bh}$) values of thermodynamic quantities in the form

$$M \approx \frac{\pi^{1/2}}{2(D + 1) \Gamma \left( \frac{D+1}{2} \right)} D N T \left( \frac{T}{T_{bh}} \right)^D, \tag{20}$$

$$F \approx -\frac{\pi^{1/2}}{2(D + 1) \Gamma \left( \frac{D+1}{2} \right)} N T \left( \frac{T}{T_{bh}} \right)^D, \tag{21}$$

$$S \approx \frac{\pi^{1/2}}{2 \Gamma \left( \frac{D+1}{2} \right)} N \left( \frac{T}{T_{bh}} \right)^D, \tag{22}$$

$$C_N \approx \frac{\pi^{1/2}}{2 \Gamma \left( \frac{D+1}{2} \right)} D N \left( \frac{T}{T_{bh}} \right)^D. \tag{23}$$
3 Relation to low temperature phonon gases

The power law temperature dependence of the black hole thermodynamic quantities $M, F, S, C_N$ reminds us of the quantum phonon gases that appear in nonmetallic crystals. Let us recall the following known results [37] for the internal energy $E$, Helmholtz free energy $F$, entropy $S$ and isochoric heat capacity $C_V$ of the $D$-dimensional phonon gases in nonmetallic crystals residing in $(D + 1)$-dimensional flat spacetime, which can be obtained straightforwardly by use of (grand)canonical ensemble and the Debye’s linear dispersion relation $\epsilon(k) = v_s |k|$ for phonons,

\begin{align}
E &= DNT\mathcal{D}_D \left( \frac{T_D}{T} \right), \\
F &= DNT \log \left( 1 - e^{-T_D/T} \right) - NT\mathcal{D}_D \left( \frac{T_D}{T} \right), \\
S &= -DN \log \left( 1 - e^{-T_D/T} \right) + (D + 1)N\mathcal{D}_D \left( \frac{T_D}{T} \right), \\
C_V &= DN\mathcal{L}_D \left( \frac{T_D}{T} \right),
\end{align}

where $N$ is the number of crystal lattice atoms, $T_D$ is the Debye temperature, $\mathcal{D}_D(x)$ is the $D$-dimensional Debye function

\[ \mathcal{D}_D(x) \equiv Dx^{-D} \int_0^x \frac{y^D}{e^y - 1} dy, \]

and $\mathcal{L}_D(x)$ is the $D$-dimensional Langevin function,

\[ \mathcal{L}_D(x) \equiv \mathcal{D}_D(x) - x \frac{d}{dx} \mathcal{D}_D(x). \]

At low temperature $T \ll T_D$, the above phonon gas functions behave as

\begin{align}
E &\approx f(D)DNT \left( \frac{T}{T_D} \right)^D, \\
F &\approx -f(D)NT \left( \frac{T}{T_D} \right)^D, \\
S &\approx f(D)(D + 1)N \left( \frac{T}{T_D} \right)^D, \\
C_V &\approx f(D)D(D + 1)N \left( \frac{T}{T_D} \right)^D,
\end{align}

where $f(D) = D\zeta(D + 1)\Gamma(D + 1)$.

We can see a surprising similarity between the high temperature results (20)-(23) for black holes and the low temperature results (28)-(31) for quantum phonon gases. In fact, if we make the identification

\[ T_D = \left( \frac{2(D + 1)\Gamma\left( \frac{D+1}{2} \right) f(D)}{\pi^{1/2}} \right)^{1/D} T_{bh}, \]

where \( f(D) = D\zeta(D + 1)\Gamma(D + 1). \)
then there will be a precise quantitative match between eqs. (20)-(23) and eqs. (28)-(31), provided we further identify $M$ with $E$ and $C_N$ with $C_V$. We can even absorb the constant factor appearing on the right hand side of eq.(32) by a simple redefinition of the length scale $L$,

$$L \to \left( \frac{2(D+1)\Gamma \left( \frac{D+1}{2} \right) f(D)}{\pi^{1/2}} \right)^{1/D} L,$$

which is allowed because of the arbitrariness in the choice of $L$. Then the Debye temperature $T_D$ for the phonon gas will be identical to the black hole characteristic temperature $T_{bh}$. We thus conclude that the Tangherlini-AdS black holes at high temperature are nothing but quantum phonon gases in nonmetallic crystals at low temperature. In this correspondence, the number of black hole molecules is identified with the number of crystal lattice atoms which are both denoted as $N$.

4 Concluding remarks and discussions

The major conclusion of this work can be summarized in a single sentence: *Tangherlini-AdS black holes at high temperature are equivalent to quantum phonon gases in nonmetallic crystals at low temperature.* Since we have already evidence about the same power law dependences of heat capacities for different black hole solutions in different gravity models [27], it is natural to expect that this *AdS/phonon gas correspondence* may also hold for other asymptotically AdS black holes. The AdS asymptotics is a necessary condition for the above correspondence to hold, because for non-AdS black holes the heat capacities are identically negative [26], which could not have direct relationship to normal phonon gases.

Asymptotically AdS black holes have been extensively studied during the past 25 years or so, mostly in connection with AdS/CFT correspondence and its various extensions such as AdS/CMT (condensed matter theory) [38–40], AdS/QCD [41–43], etc. Although a countless number of theoretical results have been obtained in these fields, most of the correspondences remain qualitative. The present work adds some extra contribution to the field of AdS/CMT correspondence with a precise quantitative match between the two sides. Let us remark that the AdS/phonon gas correspondence described in this work holds in generic spacetime dimension $D + 2$, where $D$ is the dimension of the bifurcation horizon of the black hole and $D+1$ is the dimension of the boundary of the black hole spacetime in which the $D$-dimensional nonmetallic crystal resides. Let us stress that the AdS/phonon gas correspondence that we report here is established purely on the level of thermodynamics. No attempts on the microscopic description were made here, and there is no need to assume broken translational symmetries from the beginning, as did in previous works on similar topics in the context of AdS/CFT correspondence [44–47]. On the level of thermodynamics, the correspondence is complete, because, according to the famous Massieu's theorem, the thermodynamic properties
of any macroscopic system are completely determined by a single thermodynamic potential with adapted independent variables. In the present context, the Helmholtz free energy \( F = F(T, N) \) plays the role of the thermodynamic potential on both sides.

From the point of view of statistical physics, the power law dependence of thermodynamic quantities for the phonon gas is solely determined by Debye's linear dispersion relation. The AdS/phonon gas correspondence seems to indicate that the dispersion relation for black hole molecules at high temperature is also linear, provided the underlying statistical physics principles are the same. Although the black hole microstructure remains unclear as we still cannot say anything about individual black hole molecules, the result of the present work does reveal that the black hole molecules are quantum, and collectively behave as phonons at sufficiently high temperature.

One may wonder why the AdS/phonon gas correspondence could hold, given that the AdS black hole has a chemical potential whilst the phonon gas has not. This is actually a misreading of the role of black hole chemical potential. As we have already pointed out in the end of the last section, the number of black hole molecules corresponds to the number of crystal lattice sites, rather than the number of phonons — the latter is not conserved, and hence the phonon gas has a vanishing chemical potential. On the other hand, the number of lattice atoms is conserved and hence the crystal background does have a nonvanishing chemical potential which is connected with the binding energy of the crystal and is responsible for the lattice growth. Likewise, the black hole chemical potential should also be connected with the binding energy of the black hole. Using the Euler homogeneity relation (9), it is straightforward to write down the high temperature behavior of the black hole chemical potential,

\[
\mu = \frac{F}{N} \approx -\frac{\pi^{1/2}}{2(D+1)\Gamma\left(\frac{D+1}{2}\right)} T^{D} \left(\frac{T}{T_{bh}}\right)^D.
\]

The negativity of the chemical potential implies that the black hole molecules are attractive, which is required in order to make the black hole the stable.

**Acknowledgement**

This work is supported by the National Natural Science Foundation of China under the grant No. 12275138 and 11575088.

**Data Availability Statement**

No data was used for the research described in the article.
Declaration of competing interest

The authors declare no competing interest.

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