FUKAYA CATEGORIES AS CATEGORICAL MORSE HOMOLOGY

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ABSTRACT. The Fukaya category of a Weinstein manifold is an intricate symplectic invariant of high interest in mirror symmetry and geometric representation theory. We show in analogy with Morse homology that the Fukaya category can be obtained by gluing together Fukaya categories of Weinstein cells. Our main technical result is a dévissage pattern for Lagrangian branes parallel to that for constructible sheaves. As an application, we exhibit the Fukaya category as the global sections of a sheaf on the conic topology of the Weinstein manifold. This can be viewed as a symplectic analogue of the well-known algebraic and topological theories of (micro)localization.

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1. Introduction

To realize “compact, smooth” global objects as glued together from simpler local pieces, one often pays the price that the local pieces are “noncompact” or “singular”. For several representative examples, one could think about compact manifolds versus cells and simplices, smooth projective varieties versus smooth affine varieties and singular hyperplane sections, vector bundles with flat connection versus regular holonomic $\mathcal{D}$-modules, or perhaps most universally of all, irreducible modules versus induced modules.

In this paper, we explain how a similar pattern holds for exact Lagrangian branes in Weinstein manifolds: complicated branes can be systematically glued together (in the homological sense) from simple but noncompact branes. If the initial branes are compact, then Seidel’s theory of iterated Lefschetz fibrations and their vanishing cycles and thimbles [56] provides a complete solution, both theoretically and computationally. For applications to areas such as mirror symmetry and geometric representation theory, it is useful to go further and consider noncompact branes from the start. When the Weinstein manifold is a cotangent bundle, constructible sheaves capture the structure of all branes, compact and noncompact alike, and in particular, organize the gluing relations between local and global calculations [44, 42, 43]. Our aim here is to turn to another basic structure of a Weinstein manifold and decompose branes along its unstable coisotropic Morse cells. One can interpret the resulting presentation of the Fukaya category as a categorical form of Morse homology.

As a consequence, we obtain a canonical localization of branes: we construct a sheaf of categories whose global sections recover the Fukaya category. This provides a symplectic counterpart to the algebraic theory of Beilinson-Bernstein localization [9] and the topological microlocalization of Kashiwara-Schapira [27]. Though we are exclusively occupied in this paper with general theory, we anticipate applications of potentially broad appeal. The localization of branes should lead to the calculation of many Fukaya categories, in particular in terms of algebraic quantizations when the Weinstein manifold is the Hamiltonian reduction of a cotangent bundle. For the simplest example, one could consider Slodowy slices and their smoothings: on the one hand, their compact branes have been studied extensively in the context of knot homologies [29, 61, 38]; on the other hand, their noncommutative modules globalize to modules over Lie algebras and finite $\mathcal{W}$-algebras (see [31, 49, 33, 34] for origins and up-to-date lists of references). More broadly, one could also consider other symplectic settings for geometric representation theory, for example for localizations of Cherednik-type algebras and hypertoric enveloping algebras [24, 25, 26, 16, 41]. This paper provides useful tools to see that such targets admit a single categorical quantization which can be described in terms of localized branes or alternatively noncommutative modules.

From a concrete geometric perspective, we have been particularly inspired by Seidel’s theory of exact symplectic Dehn twists [52, 53, 54, 56]. For technical foundations or at least moral
reassurance, we have also leaned heavily upon the cornerstones set by Eliashberg, Gromov, and Weinstein [21, 73, 20], Fukaya, Oh, Ohta, and Ono [22], Seidel [56], Fukaya and Oh [23], Wehrheim and Woodward [67, 68, 69, 70, 71, 72], Kashiwara and Schapira [27] and Lurie [35, 36, 37]. There is a wealth of results in closely aligned directions, in particular in the analysis of the homology of closed strings. We are familiar with only a small part of this growing literature, in particular, the guiding advances of Seidel [55, 57], and the striking results of Bourgeois, Cieliebak, Eckholm, and Eliashberg [17, 14, 15].

In what follows, we first outline the specific setup adopted in this paper, then go on to describe our main results and their immediate precursors.

1.1. Setup. Let $(M, \theta)$ be a Weinstein manifold, so a manifold $M$ equipped with a one-form $\theta$ whose differential $\omega = d\theta$ is a symplectic form satisfying standard axioms (recalled in Section 2 below). We will always assume that $M$ is real analytic, and all subsets and functions are subanalytic (or definable within some fixed o-minimal context). The basic source of examples are Stein manifolds, or more specifically, smooth affine complex varieties.

Let $Z$ be the Liouville vector field on $M$ characterized by $\theta = i_Z \omega$, and let $\mathcal{C} \subset M$ be the finite subset of zeros of $Z$. For generic data, the flow of $Z$ provides a stratification $\mathcal{S} = \{C_p\}_{p \in \mathcal{C}}$ into coisotropic unstable cells $C_p \subset M$ contracting to the zeros $p \in \mathcal{C}$. Hamiltonian reduction along each coisotropic unstable cell $C_p \subset M$ produces a contractible Weinstein manifold $(M_p, \theta_p)$ which we refer to as a Weinstein cell.

Let $F(M)$ denote the Fukaya category of not necessarily compact exact Lagrangian branes, and let $\text{Perf}\ F(M)$ denote the stable Fukaya category of perfect modules over $F(M)$. Here we fix a coefficient field $k$, and by perfect modules, mean summands of finite complexes of representable modules with values in $k$-chain complexes. Let us briefly orient the reader as to what version of the Fukaya category we work with. (See Section 3 below for a more detailed discussion of all of the following notions.)

First, recall that any symplectic manifold $M$ is canonically oriented, and one can unambiguously speak about its Chern classes. To work with graded Lagrangian submanifolds, we will make the standard assumption that the characteristic class $2c_1(M)$ is trivialized. For simplicity, we will also assume that $M$ comes equipped with a spin structure. Then by Lagrangian brane, we will mean a graded Lagrangian submanifold equipped with a finite-dimensional local system and pin structure. The restriction to spin manifolds provides sufficient duality so that we may geometrically access functors between Fukaya categories via Lagrangian branes on products. It is a categorified analogue of the familiar assumption that manifolds are oriented allowing cochains on their products to give homomorphisms on their cohomology.

Second, by the Fukaya category $F(M)$, we mean the infinitesimal (as describes the setting of [52, 56, 44, 42]) rather than wrapped variant (as found in [4]); this is one extreme of the partially wrapped paradigm [6, 7]. Our perturbation framework involves Hamiltonian isotopies (in the direction of the rotated Liouville vector field) of constant size with respect to a radial coordinate near infinity rather than of linear growth. This has the advantage of a more evident functoriality due to a clear-cut distinction between local versus global aspects. If one specifies a conic Lagrangian support $\Lambda \subset M$ for branes, the resulting full subcategory $\text{Perf}_\Lambda\ F(M) \subset \text{Perf}\ F(M)$ becomes a close “Verdier dual” cousin to the corresponding partially wrapped category (the relation is analogous to that of cohomology and homology, or more immediately, perfect and coherent $O$-modules). A basic and concrete aspect of this relation is that there is a canonical fully faithful embedding of the partially wrapped category into the category of modules over $\text{Perf}_\Lambda\ F(M)$. Thus by coupling results for the infinitesimal category with notions of support and homological bounds, one can obtain a parallel picture for the partially wrapped category.
It is beyond the scope of this paper to develop this picture in detail, but we have included a precise assertion in the form of a conjecture in Remark 1.9 below.

1.2. Dévissage for branes. Broadly speaking, we show that the stable Fukaya category Perf$F(M)$ can be recovered from the stable Fukaya categories Perf$F(M_p)$ of its Weinstein cells together with gluing data in the form of natural adjunctions. Moreover, we show that this presentation of Perf$F(M)$ can be localized in the conic topology of $M$.

In one formulation of our first main result, we construct a semiorthogonal decomposition of Perf$F(M)$ associated to a closed coisotropic submanifold $i : C \to M$ that is a union of unstable cells. On the one hand, the formalism of Hamiltonian reduction provides a Lagrangian correspondence

$$
N \leftarrow \begin{array}{c}
\pi \\
\downarrow
\end{array} C \leftarrow \begin{array}{c}
i \\
\uparrow
\end{array} M
$$

where $N$ is a Weinstein manifold, and $q$ is the quotient along the integrable isotropic foliation determined by $i$. On the other hand, the open complement $j : M^c = M \setminus C \to M$ naturally inherits the structure of a Weinstein manifold.

One can view the following result as a dévissage pattern for Lagrangian branes analogous to standard gluings for constructible sheaves (as recalled in Section 1.4.2 below). From a symplectic perspective, one can view the functors involved as resulting from “Dehn twists around cells”.

**Theorem 1.1.** There is a natural diagram of adjunctions

$$
\begin{array}{ccc}
\text{Perf } F(N) & \xrightarrow{i^*} & \text{Perf } F(M) \\
\downarrow & & \downarrow i^! \\
\text{Perf } F(M^c) & \xrightarrow{j^*} & \text{Perf } F(M^c) \\
\downarrow & & \downarrow j^! \\
\end{array}
$$

with $i! \simeq i_*, j!, j_*$ fully faithful embeddings, and exact triangles of functors

$$
\begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow u \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow c \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow u \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow c \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow u \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow c \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{\text{Perf } F(M)} \\
\downarrow u \downarrow \\
\text{id}_{\text{Perf } F(M)}
\end{array}
$$

where $c$ denotes the counits of adjunctions and $u$ the units.

**Remark 1.2.** If $M$ admits an anti-symplectomorphism compatible with other structures, then one can obtain an analogue of Verdier duality as well.

**Remark 1.3.** The exact triangles of the theorem are categorical versions of the long exact sequences of cohomology of a pair. For technical reasons, it is useful to observe that they are formally equivalent to exact triangles

$$
\begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
\quad \begin{array}{c}
\text{id}_{F(M)} \\
\downarrow \downarrow \\
\text{id}_{F(M)}
\end{array}
$$

where the first and second maps are the units and counits of adjunctions. These are categorical versions of the Mayer-Vietoris sequences of cohomology of a covering pair.

**Remark 1.4.** We mention here an attractive way to package the theorem. It requires a bit more care to verify but at the end of the day encodes the same amount of geometry.

Consider the differential graded derived category $\text{Sh}_S(M)$ of complexes of sheaves constructible along the stratification $S = \{C_p\}_{p \in \epsilon}$ by coisotropic unstable cells $C_p \subset M$. We equip $\text{Sh}_S(M)$ with its natural symmetric monoidal structure given by tensor product. The monoidal unit is the constant sheaf $k_M \in \text{Sh}_S(M)$. More generally, a union of unstable coisotropic cells $i : C \to M$, the extension by zero $k_C! = i_! k_C \in \text{Sh}_S(M)$ is an idempotent.
Now one can organize the functors of Theorem 1.1 to show that there is a natural fully faithful monoidal embedding

\[ \text{Sh}_S(M) \longrightarrow \text{End}_{st_k}(\text{Perf}(M)) \]

characterized by the property that \( k_{C!} \mapsto i_!^* \). This allows one to analyze \( \text{Perf}(F(M)) \) as a module over the “spectrum” of the commutative algebra \( \text{Sh}_S(M) \). We will not develop this picture further, though it guides our thinking in the discussion to follow.

Theorem 1.1 provides various frameworks for gluing together \( \text{Perf}(F(M)) \) from the constituent pieces \( \text{Perf}(F(N)), \text{Perf}(F(M^a)) \). To pursue gluing, it is useful to regard each of the above categories as a small stable idempotent-complete \( k \)-linear \( \infty \)-category. Then we can adopt the foundations of [35, 36], and work within the \( \infty \)-category \( st_k \) of small stable idempotent-complete \( k \)-linear \( \infty \)-categories.

According to the reinterpretation of Remark 1.3, we can view \( \text{Perf}(F(M)) \) as classifying triples of data \( L^o \in \text{Perf}(F(M^o)), L_N \in \text{Perf}(F(N)) \) together with a morphism

\[ r \in \text{Hom}_{\text{Perf}(F(N))}(i^1_!L^o, L_N). \]

Let us recast this in monadic terms by considering the adjunction

\[ L = j_! \oplus i_! : \text{Perf}(F(M^o)) \oplus \text{Perf}(F(N)) \longrightarrow \text{Perf}(F(M)) : R = j^1 \oplus i^! \]

We obtain a resulting monad, or in other words, algebra object in endomorphisms

\[ T = RL \in \text{End}(\text{Perf}(F(M^o)) \oplus \text{Perf}(F(N))). \]

The \( \infty \)-categorical Barr-Beck Theorem provides a canonical equivalence

\[ \text{Perf}(F(M)) \simeq \text{Mod}_T(\text{Perf}(F(M^o)) \oplus \text{Perf}(F(N))) \]

where the right hand side denotes module objects over the monad \( T \).

We can inductively apply the above considerations by successively taking the coisotropic submanifold \( i : C \rightarrow M \) to be a single closed unstable cell. Recall that \( \epsilon \subset M \) denotes the finite subset of zeros of the Liouville vector field \( Z \). For each \( p \in \epsilon \), we have the Hamiltonian reduction of the corresponding unstable cell

\[ M_p \xrightarrow{g_p} C_p \xrightarrow{i_p} M \]

By induction, Theorem 1.1 provides an adjunction

\[ L = \oplus_{p \in \epsilon} i^!_p : \oplus_{p \in \epsilon} \text{Perf}(F(M_p)) \longrightarrow \text{Perf}(F(M)) : R = \oplus_{p \in \epsilon} i^1_p \]

We obtain a resulting upper-triangular monad

\[ T = RL \in \text{End}(\oplus_{p \in \epsilon} \text{Perf}(F(M_p))) \]

with matrix entries \( i^!_p, i^1_p \) for unstable cells \( C_q \subset C_p \). Finally, the \( \infty \)-categorical Barr-Beck Theorem provides the following corollary.

**Corollary 1.5.** There is a canonical equivalence

\[ \text{Perf}(F(M)) \simeq \text{Mod}_T(\oplus_{p \in \epsilon} \text{Perf}(F(M_p))) \]

One can view this presentation of \( \text{Perf}(F(M)) \) as a categorified version of the Morse homology of \( M \). First, to each critical point \( p \in \epsilon \), we assign the stable Fukaya category \( \text{Perf}(F(M_p)) \) of the Weinstein cell \( M_p \). From the perspective of Morse homology, one can view \( F(M_p) \) as the analogue of a scalar vector space \( k[\deg p] \) shifted by the index of \( p \). (Since the \( \infty \)-category \( st_k \)}
of small stable idempotent-complete \( k \)-linear \( \infty \)-categories is not itself stable, it is unsurprising that “suspension” in the current setting is not invertible as it is in the traditional setting.)

Second, to pairs of critical points \( p, q \in c \), we assign the individual term \( i_p \circ i_q \) of the gluing monad \( T \). More generally, to collections of critical points, we assign the corresponding terms of the gluing monad \( T \) together with the monadic structure maps among them. From the perspective of Morse homology, one can view the individual terms and their monadic structure as the analogue of boundary maps and their higher relations.

Part of the appeal of Morse homology is that the boundary maps and their higher relations localize along the spaces of flow lines connecting the relevant critical points. Furthermore, the boundary maps and their higher relations admit simple descriptions in terms of the spaces of flow lines. For instance, the boundary maps themselves are given by counts of isolated flow lines. We will develop an analogue of this picture in the section immediately following.

1.3. Localization of branes. An analysis of the constructions of Theorem 1.1 allow one to go further and localize the stable Fukaya category \( \text{Perf} F(M) \) over the conic topology of \( M \). There are many inspiring precedents [1, 4, 58, 59, 63] and informed assertions [30] that suggest such a construction should be possible. In our own thinking, we have often returned to the setting of cotangent bundles as a guide: combining the equivalence of Lagrangian branes with constructible sheaves [44, 42, 43] and the formalism of microlocalization [27] leads to a complete solution in that case.

To describe the localization for a general Weinstein manifold \( (M, \theta) \), let us return to the geometry of the Liouville vector field \( Z \). We say a subset of \( M \) is conic if it is invariant under the flow of \( Z \), and use the term core to refer to the compact conic isotropic subvariety \( K \subset M \) of points that do not escape to infinity under the flow. We use the term ether to refer to the complement \( E = M \setminus K \), and define the projectivization \( M^\infty = E/\mathbb{R}_+ \) to be the compact contact manifold obtained by quotienting the ether by the flow. More generally, given any conic subset \( A \subset M \), we can consider its projectivization \( A^\infty = (A \cap E)/\mathbb{R}_+ \subset M^\infty \).

To any object \( L \in \text{Perf} F(M) \), we assign its singular support \( \text{ss}(L) \subset M \) which is a closed conic isotropic subvariety depending only on the isomorphism class of the object (see Section 3.7 below). The singular support records the coarse homological nontriviality of objects, and for a Lagrangian brane is a subset of its limiting dilation to a conic Lagrangian subvariety. For example, if \( L \) is a compact Lagrangian brane, then \( \text{ss}(L) \subset K \).

Let us fix a (most likely singular and noncompact) conic Lagrangian subvariety \( \Lambda \subset M \) which we refer to as the characteristic cone. We will assume that \( \Lambda \) contains the core \( K \subset M \), and hence is completely determined by its projectivization \( \Lambda^\infty \subset M^\infty \). Note that the inclusions \( K \subset \Lambda \subset M \) are all homotopy equivalences, so what will interest us most is their local geometry.

Let us consider the full subcategory \( \text{Perf}_\Lambda(M) \subset \text{Perf} F(M) \) of objects \( L \in \text{Perf} F(M) \) with singular support satisfying \( \text{ss}(L) \subset \Lambda \). So for example, if \( L \) is a compact Lagrangian brane, then it provides an object of \( \text{Perf}_\Lambda F(M) \) irrespective of the choice of \( \Lambda \). Note that \( \Lambda_1 \subset \Lambda_2 \) implies \( \text{Perf}_{\Lambda_1} F(M) \subset \text{Perf}_{\Lambda_2} F(M) \), and \( \text{Perf} F(M) = \cup_\Lambda \text{Perf}_\Lambda F(M) \).

Recall that \( \mathfrak{c} \subset M \) denotes the finite subset of zeros of the Liouville vector field \( Z \). For each \( p \in \mathfrak{c} \), we have the Hamiltonian reduction of the corresponding unstable cell

\[
M_p \xrightarrow{q_p} C'_p \xrightarrow{i_p} M
\]

We obtain a conic Lagrangian subvariety \( \Lambda_p \subset M_p \) by setting \( \Lambda_p = q_p(i_p^{-1}(\Lambda)) \), and similarly a full subcategory \( \text{Perf}_{\Lambda_p} F(M_p) \subset \text{Perf} F(M_p) \).

Now by localizing \( \text{Perf}_{\Lambda} F(M) \) with respect to singular support, we obtain the following theorem. Its verification appeals to the dévissage results outlined above.
Theorem 1.6. There exists a $\stk$-valued sheaf $\mathcal{F}_\Lambda$ on the conic topology of $M$ with the following properties:

1. The support of $\mathcal{F}_\Lambda$ is the characteristic cone $\Lambda \subset M$.
2. The global sections of $\mathcal{F}_\Lambda$ are canonically equivalent to $\Perf_{\Lambda_0} F(M)$.
3. The restriction of $\mathcal{F}_\Lambda$ to an open Weinstein manifold $M^\circ \subset M$ is canonically equivalent to the sheaf $\mathcal{F}_{\Lambda^\circ}$ constructed with respect to $\Lambda^\circ = \Lambda \cap M^\circ$.
4. For each zero $p \in \mathfrak{c}$, the sections of $\mathcal{F}_\Lambda$ lying strictly above the unstable cell $C_p \subset M$ are canonically equivalent to $\Perf_{\Lambda_p} F(M_p)$.

Example 1.7. Here is a description of $\mathcal{F}_\Lambda$ in the simplest example.

Consider the two-dimensional Weinstein cell $M = \mathbb{C}$ with standard Liouville form $\theta$ and projectivization $M^\infty \simeq S^3$. Its core is the single point $K = \{0\} \subset \mathbb{C}$, and its ether is the complement $E = \mathbb{C}^* \subset \mathbb{C}$. Any characteristic cone $\Lambda \subset \mathbb{C}$ will be the union of $K = \{0\}$ with finitely many rays. For $n = 0, 1, 2, \ldots$, let $\Lambda_n \subset \mathbb{C}$ denote the characteristic cone with $n$ rays.

Then $\Perf_{\Lambda_n} F(M)$ is equivalent to finite-dimensional modules over the $A_n$-quiver (in particular, for $n = 0$, it is the zero category). This is also the stalk of the sheaf $\mathcal{F}_{\Lambda_n}$ at the point $0 \in \mathbb{C}$. Its stalk at other points $x \in \mathbb{C}$ is either (not necessarily canonically) equivalent to $\Perf k$ when $x \in \Lambda_n$, and is the zero category otherwise.

Remark 1.8. It is possible to say much more about $\mathcal{F}_\Lambda$ though we will reserve a detailed analysis for another time. Let us content ourselves here with mentioning that when $M$ is a cotangent bundle, sections of $\mathcal{F}_\Lambda$ are equivalent to (the sheafification of) microlocal sheaves. In general, a similar description holds for the restriction of $\mathcal{F}_\Lambda$ to the complement $M \setminus \mathfrak{c}$ of the zeros of the Liouville vector field $Z$. Namely, the quotient $(M \setminus \mathfrak{c})/\mathbb{R}_+$ by the Liouville flow is naturally a (non-Hausdorff) contact manifold, and the restriction of $\mathcal{F}_\Lambda$ to the complement $M \setminus \mathfrak{c}$ can be obtained by pulling back microlocal sheaves from $(M \setminus \mathfrak{c})/\mathbb{R}_+$.

Remark 1.9. This remark is devoted to a conjectural parallel picture for the partially wrapped Fukaya category. Let us continue with the setting of Theorem 1.6, and write $WF\mathcal{F}_\Lambda(M)$ for the partially wrapped Fukaya category as developed in [6, 7]. It specializes to the fully wrapped variant of [4] when the characteristic cone $\Lambda \subset M$ coincides with the compact core $K \subset M$.

Let $WF\mathcal{F}_\Lambda(M)$ denote the stable category of perfect modules over $WF\mathcal{F}_\Lambda(M)$, and let $ModWF\mathcal{F}_\Lambda(M)$ denote the stable category of all modules. Note that the two stabilizations are formally the same amount of information: $ModWF\mathcal{F}_\Lambda(M)$ is the ind-category $\text{Ind}(WF\mathcal{F}_\Lambda(M))$, and $WF\mathcal{F}_\Lambda(M)$ is the full subcategory of compact objects $(ModWF\mathcal{F}_\Lambda(M))^c$.

Given a conic open subset $U \subset M$, let us imagine a “mirror” picture of the category $\mathcal{F}_\Lambda(U)$ as perfect quasicoherent sheaves over a scheme $X_U$ proper over Spec $k$. The statement that $X_U$ is proper can be formalized by the conjecture that $\mathcal{F}_\Lambda(U)$ is hom-finite. Given conic open subsets $V \subset U \subset M$, let us also imagine a “mirror” picture of the restriction map $\mathcal{F}_\Lambda(U) \to \mathcal{F}_\Lambda(V)$ as the pullback of perfect quasicoherent sheaves under a morphism $X_U \to X_V$.

Now given a conic open subset $U \subset M$, consider the category of finite functionals

$$\text{Coh} \mathcal{F}_\Lambda(U) = \text{Hom}_{\stk}(\mathcal{F}_\Lambda(U)^{\text{op}}, \Perf k)$$

From the above “mirror” perspective, $\text{Coh} \mathcal{F}_\Lambda(U)$ corresponds to coherent sheaves over the proper scheme $X_U$. Given conic open subsets $V \subset U \subset M$, the restriction map $\mathcal{F}_\Lambda(U) \to \mathcal{F}_\Lambda(V)$ induces a corestriction map

$$\text{Coh} \mathcal{F}_\Lambda(V) \longrightarrow \text{Coh} \mathcal{F}_\Lambda(U)$$

From the above “mirror” perspective, the corestriction map corresponds to the pushforward (right adjoint to pullback) of coherent sheaves under the morphism $X_U \to X_V$. 
The above constructions equip $\text{Coh} \mathcal{F}_A$ with the structure of a pre-cosheaf. Let us write $\text{Ind}(\text{Coh} \mathcal{F}_A)$ for the corresponding pre-cosheaf of ind-categories, and $\text{Ind}(\text{Coh} \mathcal{F}_A)^+$ for its cosheafification. (If one prefers the intuitions of sheaves over cosheaves, one could pass to the right adjoints of the corestriction maps of $\text{Ind}(\text{Coh} \mathcal{F}_A)^+$ and turn it into a sheaf.) We conjecture that there is a natural equivalence

$$\Gamma(M, \text{Ind}(\text{Coh} \mathcal{F}_A)^+) \sim \text{Mod} WF_A(M)$$

In other words, the stable partially wrapped Fukaya category $WF_A(M)$ consists of the compact objects of the global sections of the cosheaf $\text{Ind}(\text{Coh} \mathcal{F}_A)^+$. Finally, let us mention some corroborating evidence for the above conjecture. First, using the results of [2, 44, 42], one can check it for $M$ a cotangent bundle and $\Lambda$ the zero section. Second, recent calculations of Fukaya categories of Riemann surfaces, pairs-of-pants and their generalizations [60, 3, 62] reveal a mirror symmetry with matrix factorizations. In the situations considered, the wrapped Fukaya category corresponds to “coherent” matrix factorizations and the full subcategory of compact branes corresponds to “perfect” matrix factorizations.

We have briefly mentioned anticipated applications of Theorems 1.1 and 1.6 earlier in the introduction. Let us conclude here with a useful technical application: Theorem 1.6 allows us to define the stable Fukaya category of an “inexhaustible, singular Weinstein manifold”. Namely, given a union of coisotropic cells $C \subset M$, we can take the sections of the sheaf $\mathcal{F}_A$ lying strictly above $C$ for increasing $\Lambda$. Rather than leaving it in such an abstract form, let us explain what results by appealing directly to the concrete geometry of Theorem 1.1.

To begin, let us rotate our viewpoint on the exact triangles appearing in Theorem 1.1 and switch the roles of the known and unknown categories in our discussion. Recall that we assumed that the closed coisotropic subvariety $i : C \to M$ is both a smooth submanifold and a union of unstable cells. Let us relax these two requirements in turn.

First, suppose the closed coisotropic subvariety $i : C \to M$ is no longer necessarily smooth, but still a union of unstable cells. On the one hand, the formalism of Hamiltonian reduction still provides a correspondence

$$N \xrightarrow{q} C \xrightarrow{i} M$$

though $N$ is now a “singular Weinstein manifold”, whatever that might mean. On the other hand, the open complement $j : M^o = M \setminus C \to M$ continues to inherit the structure of a Weinstein manifold. With this setup, the arguments of Theorem 1.1 provide adjunctions

$$j^* : \text{Perf} F(M) \xrightarrow{\sim} \text{Perf} F(M^o) : j_* \quad \quad j_! : \text{Perf} F(M^o) \xrightarrow{\sim} \text{Perf} F(M) : j!$$

with $j_!, j_*$ fully faithful, and $j! \simeq j^*$. We do not have an a priori definition of a Fukaya category $\text{Perf} F(N)$, but Theorem 1.1 tells us what it should be. Namely, we should define $F(N)$ to classify triples of data $L \in \text{Perf} F(M)$, $L^o \in \text{Perf} F(M^o)$, together with a morphism $u \in \text{Hom}_{\text{Perf} F(M)}(L, j_* L^o)$. When $N$ is smooth, Theorem 1.1 confirms that we recover precisely $\text{Perf} F(N)$ by taking the kernel of morphisms $u$ appearing in such data.

**Remark** 1.10. One should view the preceding as more than a formal analogue of the situation for $\mathcal{D}$-modules on singular varieties. There Kashiwara’s theorem confirms that such an approach provides an unambiguous notion of $\mathcal{D}$-module.

Second, suppose the closed coisotropic subvariety $i : C \to M$ is a smooth submanifold invariant under the Liouville flow, but not necessarily a union of unstable cells. From this starting point, the Hamiltonian reduction $N$ continues to be a Weinstein manifold, but the open complement $j : M^o = M \setminus C \to M$ may be an “inexhaustible Weinstein manifold”, or
roughly speaking, a complete exact symplectic manifold such that the Liouville vector field is gradient-like for a possibly inexhausting Morse function. With this setup, the arguments of Theorem 1.1 provide adjunctions

\[ i^* : \text{Perf} F(N) \longrightarrow \text{Perf} F(M) : i_* \quad i : \text{Perf} F(M) \longrightarrow \text{Perf} F(N) : i! \]

with \( i! \simeq i_* \) fully faithful. We do not have an \textit{a priori} definition of a Fukaya category \( \text{Perf} F(M^\circ) \), but Theorem 1.1 tells us what it should be. Namely, we should define \( \text{Perf} F(M^\circ) \) to classify triples of data

\[ L_N \in \text{Perf} F(N), L \in \text{Perf} F(M), \text{hom} \in \text{Hom}_{\text{Perf} F(M)}(i! L_N, L). \]

When \( N \) is smooth, Theorem 1.1 confirms that we recover precisely \( \text{Perf} F(M^\circ) \) by taking the cokernel of morphisms \( c \) appearing in such data.

Putting together the above generalizations, we obtain an unambiguous infinitesimal Fukaya category of an "inexhaustible, singular Weinstein manifold" so that it is compatible with familiar notions whenever they apply.

1.4. \textbf{Influences.} In the remainder of the introduction, we recount some influences on our thinking, in particular the symplectic geometry of Dehn twists, the d\'evisage pattern for constructible sheaves, and the Morse theory of integral kernels.

1.4.1. \textit{Dehn twists.} One can view our geometric constructions as simple elaborations on the fundamental notion of exact symplectic Dehn twists. In turn, the many incarnations of this notion (spherical twists, mutations, braid actions, Hecke correspondences,...) play a prominent role in mirror symmetry and geometric representation theory.

We sketch here an informal picture of our geometric constructions from the perspective of Dehn twists. The basic mantra could be: \textit{Dehn twists around spheres provide mutations; Dehn twists around cells provide semiorthogonal decompositions.} (There are also extensive relations between mutations and semiorthogonal decompositions, often arising from exceptional collections, thanks to the fact that spheres themselves can be cut into cells.)

Let us recall Seidel’s long exact sequence in Floer cohomology [52, 53, 54]. There are also highly relevant relative sequences found in the work of Perutz [46, 47, 48] and Wehrheim-Woodward [72]. We will ignore technical issues and proceed as quickly as possible to the statement.

Let \( S \subset M \) be an exact Lagrangian sphere in an exact symplectic manifold. Let \( \tau_S : M \rightarrow M \) be the associated exact symplectic Dehn twist around \( S \). Then for any two exact Lagrangian submanifolds \( L_0, L_1 \subset M \), there is a long exact sequence of \((\mathbb{Z}/2\mathbb{Z}-\text{graded, } \mathbb{Z}/2\mathbb{Z}-\text{linear})\) Floer cohomology groups

\[ HF(\tau_S(L_0), L_1) \longrightarrow HF(L_0, L_1) \longrightarrow HF(S, L_1) \otimes HF(L_0, S) \]

The sequence admits a straightforward categorical interpretation. Assume \( M \) is equipped with appropriate background structures, and the exact Lagrangian submanifolds \( S, L_0, L_1 \) are all equipped with appropriate brane structures. Let \( \text{Perf} F(M) \) denote the resulting \( \mathbb{Z} \)-graded, \( k \)-linear stable Fukaya category (of summands of finite complexes of representable modules), and let \( S, L_0, L_1 \) denote the corresponding objects. Thanks to the functoriality of the sequence in the variable \( L_1 \), we can rewrite it as an exact triangle of objects

\[ S \otimes \text{Hom}_{\text{Perf} F(M)}(S, L_0) \longrightarrow L_0 \longrightarrow \tau_S(L_0) \]

Let us introduce the (presently elaborate but ultimately justified) notation

\[ i^! : \text{Perf} F_S(M) \longrightarrow \text{Perf} F(M) : i_* \]
for the fully faithful embedding $i_S$ of the subcategory $\mathbf{Perf}_S(M)$ generated by $S$, and its right adjoint $i^*_S = \text{Hom}_{\mathbf{Perf}_M}(S, -)$. Then thanks to the functoriality of sequence in the variable $L_0$, we can view it as an exact triangle of functors

$$
\begin{array}{ccc}
i^*_S & \longrightarrow & \text{id}_{\mathbf{Perf}(M)} \longrightarrow \tau S [1] \\
\end{array}
$$

where the first map is the counit of the adjunction.

Now suppose the exact Lagrangian sphere $S \subset M$ were rather a closed but noncompact exact Lagrangian cell $C \subset M$. To place it in a categorical context, let us now allow the Fukaya category $F(M)$ to contain closed but noncompact Lagrangian branes. Then one should have a similar exact triangle of functors

$$
\begin{array}{ccc}
i_C \tau_C' & \longrightarrow & \text{id}_{\mathbf{Perf}(M)} \longrightarrow \tau C [1] \\
\end{array}
$$

where $\tau_C$ denotes the “Dehn twist” around the cell $C \subset M$. It takes compact Lagrangian branes to noncompact Lagrangian branes which are asymptotically close to $C$ near infinity.

Let us go one step further and consider the open exact symplectic submanifold $C \subset M$. Assuming $C$ is in good position with respect to the exact symplectic structure, we should be able to realize $\tau_C$ as the monad of an adjunction

$$
\begin{array}{ccc}
j^{*}_M : \mathbf{Perf}(M) & \longrightarrow & \mathbf{Perf}(M^\circ) : j^*_M \\
\end{array}
$$

where $j^*_M$ is a fully faithful embedding. Here the functor $j^{*}_M$ is geometric, built out of the inclusion $j_M$ and the Dehn twist $\tau_C$ near infinity. Its left adjoint $j^*_M$ is of a categorical origin, just as the left adjoint $i'_C$ is of a categorical origin.

Putting the above together, we obtain an exact triangle of functors

$$
\begin{array}{ccc}
i_C j'_C & \longrightarrow & \text{id}_{\mathbf{Perf}(M)} \longrightarrow j^*_M j^*_M [1] \\
\end{array}
$$

where the initial map is the counit of the adjunction, and the middle map is the unit of the adjunction. Thus we have a semiorthogonal decomposition of $\mathbf{Perf}(M)$ by the two full subcategories $\mathbf{Perf}_F(C(M))$, $\mathbf{Perf}(M^\circ)$ with the semiorthogonality $\mathbf{Perf}_F(C(M))^\perp \simeq \mathbf{Perf}(M^\circ)$, $\mathbf{Perf}_F(C(M)) \simeq \perp \mathbf{Perf}(M^\circ)$.

We will develop a generalization of the above picture where we allow the cell $C \subset M$ to be coisotropic rather than Lagrangian. The geometric constructions and formal consequences are similar, with the main new development that the full subcategory $\mathbf{Perf}_F(C(M))$ will no longer be generated by a single object. In the section immediately following, we motivate the general pattern with the formalism of dévissage for constructible sheaves.

1.4.2. Dévissage pattern. We recall here the dévissage pattern for constructible sheaves. Our ultimate aim is to reproduce this pattern for Lagrangian branes in Weinstein manifolds.

In what follows, we will only consider subanalytic sets $X$ and subanalytic maps $f : X \rightarrow Y$. We write $\mathbf{Sh}(X)$ for the differential graded category of constructible complexes of sheaves on $X$, and have the standard adjunctions

$$
\begin{array}{ccc}
f^* : \mathbf{Sh}(Y) & \longrightarrow & \mathbf{Sh}(X) : f_* \\
\end{array}
\quad f_! : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y) : f^! \\
$$

Verdier duality provides an anti-involution

$$
\mathbb{D}_X : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(X)^{op} \\
$$

intertwining the preceding adjunctions

$$
\begin{array}{ccc}
f^! \simeq \mathbb{D}_X f^* \mathbb{D}_Y \\
f_! = \mathbb{D}_Y f_* \mathbb{D}_X \\
\end{array}
$$
Suppose we have a partition of $X$ into an open subset and its closed complement
\[ j : U^c \longrightarrow X \longleftarrow Y = X \setminus U : i \]
Then the standard functors provide a diagram
\[
\begin{array}{ccc}
\text{Sh}(U) & \xrightarrow{j^! \simeq j^*} & \text{Sh}(X) \\
\downarrow j_* & & \downarrow i_* \\
\text{Sh}(Y) & \xleftarrow{i^* \simeq i_*} & \text{Sh}(X)
\end{array}
\]

The fact that $j$ is an open embedding (hence smooth of relative dimension zero) provides a canonical identification $j^! \simeq j^*$, and the fact that $i$ is a closed embedding (hence proper) provides a canonical identification $i^* \simeq i_*$. The fact that $U$ and $Y$ are disjoint provide canonical identifications $j^*i_* \simeq 0 \simeq i_*j^*$ and $i^*j^* \simeq 0 \simeq i_*j_*$. The additional fact that $U$ and $Y$ cover $X$ lead to dual exact triangles
\[
\begin{array}{ccc}
i_*j_* & \longrightarrow & \text{id}_{\text{Sh}(X)} & \longrightarrow & j_!j^* & \longrightarrow & i_*i^* \quad [1] \\
\end{array}
\]
where the first and middle morphisms are respectively units and counits of adjunctions. These triangles are generalizations of the Mayer-Vietoris long exact sequences of pairs in cohomology.

Alternatively, we also have the dual exact triangles
\[
\begin{array}{ccc}
\text{id}_{\text{Sh}(X)} & \longrightarrow & j_*j^* \oplus i_*i^* & \longrightarrow & i_*i^*j_*j^* \quad [1] & \longrightarrow & \text{id}_{\text{Sh}(X)} \\
\end{array}
\]
where the first and middle morphisms result from units and counits of adjunctions respectively. These triangles are generalizations of the Mayer-Vietoris long exact sequences of cohomology.

Now let us find the above formalism as a special case of Theorem 1.1. Suppose $X$ is a compact manifold, and consider the Weinstein manifold $M = T^*X$. Recall from [44, 42, 43] the microlocalization equivalence
\[
\mu_X : \text{Sh}(X) \longrightarrow \text{Perf}(T^*X)
\]
Suppose $i : Y \rightarrow X$ is a closed submanifold, with open complement $j : U = X \setminus Y \rightarrow X$. Then from Theorem 1.1 and the construction of microlocalization, we obtain commutative diagrams with horizontal maps fully faithful embeddings and vertical maps equivalences
\[
\begin{array}{ccc}
\text{Sh}(Y) & \longrightarrow & \text{Sh}(X) \\
\downarrow \mu_Y & & \downarrow \mu_X \\
\text{Perf}(T^*Y) & \longrightarrow & \text{Perf}(T^*X)
\end{array}
\quad \begin{array}{ccc}
\text{Sh}(U) & \longrightarrow & \text{Sh}(X) \\
\downarrow \mu_U & & \downarrow \mu_X \\
\text{Perf}(T^*U) & \longrightarrow & \text{Perf}(T^*X)
\end{array}
\]
Thus we can view Theorem 1.1 as a generalization of the usual dévissage for constructible sheaves. Strictly speaking, a direct application of Theorem 1.1 would require $Y$ to be a smooth submanifold, but iterated applications allows one to reach similar conclusions in general.

1.4.3. Morse theory of integral kernels. Our arguments exploit the well-known strategy: to prove universal statements about objects of a category, one should realize endofunctors of the category as integral transforms and establish canonical identities among them.

We will illustrate this with the toy case of Morse theory since our later arguments are direct analogues of it. For simplicity, let us work with a compact oriented manifold $M$. Consider a generic pair consisting of a Morse function $f : M \rightarrow \mathbb{R}$ and Riemannian metric $g$ on $M$. Let
\( \mathcal{c} \subset M \) denote the critical locus of \( f \), and \( \Phi_t : M \to M \) the flow along the gradient \( \nabla_g f \). To each critical point \( p \in \mathcal{c} \), associate the stable and unstable cells
\[
S_p = \{ x \in M \mid \lim_{t \to \infty} \Phi_t(x) = p \} \quad U_p = \{ x \in M \mid \lim_{t \to -\infty} \Phi_t(x) = p \}
\]
Finally, orient \( S_p \) and \( U_p \) so that the orientation of \( U_p \times S_p \) agrees with that of \( X \) at \( p \).

One formulation of Morse theory is that every cohomology class \( c \in H^*(M; \mathbb{R}) \) can be expressed in the form
\[
(1.1) \quad c = \sum_{p \in \mathcal{c}} \langle c, U_p \rangle S_p
\]
where \( \langle c, U_p \rangle \) denotes the natural pairing, and we regard \( S_p \) as a cohomology class via Poincaré duality. In other words, the stable and unstable cells form dual bases in cohomology.

To establish equation (1.1), it is useful to recast it as an equation in the cohomology of the product \( X \times X \). Namely, each cohomology class \( k \in H^*(X \times X; \mathbb{R}) \) can be regarded as an integral kernel providing an endomorphism
\[
\Phi_k : H^*(X; \mathbb{R}) \to H^*(X; \mathbb{R}) \quad \Phi_k(c) = p_{2!}(p_1^*(c) \cap k)
\]
where we use Poincaré duality to integrate. Then equation (1.1) follows from the identity of cohomology classes
\[
(1.2) \quad \Delta_X = \sum_{p \in \mathcal{c}} U_p \times S_p
\]
where \( \Delta_X \subset X \times X \) is the diagonal, and hence provides the identity endomorphism. Finally, to establish equation (1.2), one observes that the gradient flow of the Morse function
\[
\Phi_k : H^*(X; \mathbb{R}) \to H^*(X; \mathbb{R})
\]
provides a homotopy between the diagonal \( \Delta_X \) and the sum of external products \( \sum_{p \in \mathcal{c}} U_p \times S_p \).

All in all, a pleasant aspect of the above argument is that it makes no reference to an arbitrary cohomology class \( c \in H^*(X; \mathbb{R}) \), but only involves highly structured cohomology classes such as the diagonal \( \Delta_X \) and the sum of external products \( \sum_{p \in \mathcal{c}} U_p \times S_p \). Furthermore, we do not need to know a precise relation between cohomology classes \( k \in H^*(X \times X; \mathbb{R}) \) and arbitrary endomorphisms \( \Phi : H^*(X; \mathbb{R}) \to H^*(X; \mathbb{R}) \), only that there is a linear map
\[
H^*(X \times X; \mathbb{R}) \to \text{End} H^*(X; \mathbb{R}) \quad k \to \Phi_k
\]

Turning from vector spaces to the setting of linear categories, one finds many examples of the above argument. Most prominently, there is Beilinson’s universal resolution [8] of coherent sheaves on projective space \( \mathbb{P}^n \) by vector bundles. It suffices once and for all to introduce the Koszul resolution of the structure sheaf \( \mathcal{O}_{\Delta_{p^n}} \) of the diagonal \( \Delta_{p^n} \subset \mathbb{P}^n \times \mathbb{P}^n \). Then for any coherent sheaf on \( \mathbb{P}^n \), convolution with the Koszul resolution produces the desired resolution by vector bundles. What results is a concrete description of all coherent sheaves in terms of the quiver of constituent vector bundles in the Koszul resolution.

Our argument for Theorem 1.1 applies the above version of Morse theory in the setting of linear categories. We consider the product Weinstein manifold \( M^{op} \times M \) where we write \( M^{op} \) to denote the opposite symplectic structure. The formalism of bimodules allows us to view Lagrangian branes \( L \subset M^{op} \times M \) as endofunctors. In particular, the diagonal brane \( \Delta_M \subset M^{op} \times M \) represents the identity functor. The endofunctors of adjunctions appearing in the exact triangle of Theorem 1.1 are also represented by natural correspondences.

The key to our arguments begins with the familiar observation that the geometry of a neighborhood of the diagonal \( \Delta_M \) looks like the geometry of the cotangent bundle \( T^*M \). We
exploit a useful elaboration: the geometry of natural correspondences matches the geometry of the conormal bundles to the coisotropic unstable cells.

1.5. Acknowledgements. I am deeply indebted to D. Ben-Zvi, P. Seidel, and E. Zaslow for the impact they have had on my thinking about symplectic and homotopical geometry. I am grateful to T. Perutz and D. Treumann for many stimulating discussions, both of a technical and philosophical nature. I am grateful to M. Abouzaid and D. Auroux for their patient explanations of foundational issues and related questions in mirror symmetry. I would like to thank A. Preygel for sharing his perspective on ind-coherent sheaves. I am also pleased to acknowledge the motivating influence of a question asked by C. Telemann at ESI in Vienna in January 2007. Finally, I am grateful to the participants of the June 2011 MIT RTG Geometry retreat for their inspiring interest in this topic.

This work was supported by NSF grant DMS-0600909.

2. Weinstein manifolds

Most of the material of this section is well-known and available in many beautiful sources [21, 73, 20, 18]. Unless otherwise stated, we will assume that all manifolds are real analytic, and all subsets and maps are subanalytic [11, 66].

2.1. Basic notions.

Definition 2.1. An exact symplectic manifold $(M, \theta)$ is a manifold $M$ with a one-form $\theta$ such that $\omega = d\theta$ is a symplectic form. We use the term Liouville form to refer to $\theta$.

One defines the Liouville vector field $Z$ by the formula $i_Z \omega = \theta$. The symplectic form $\omega$ is an eigenvector for the Lie derivative $L_Z \omega = i_Z d\omega + di_Z \omega = 0 + d\theta = \omega$. An exact symplectic manifold is equivalently a triple $(M, \omega, Z)$ consisting of a manifold $M$ with symplectic form $\omega$ and fixed vector field $Z$ such that $L_Z \omega = \omega$.

We will always assume that $M$ is complete with respect to $Z$ in the sense that $Z$ integrates for all time to provide an expanding action

$$\Phi_t : \mathbb{R}_+ \times M \longrightarrow M$$

Definition 2.2. A subset $A \subset M$ is said to be conic if it is invariant under the expanding action in the sense that $\Phi_t(A) = A$, for all $t \in \mathbb{R}_+$.

Example 2.3. Any manifold $X$ provides an exact symplectic manifold $(T^* X, \theta_X)$ given by the cotangent bundle $\pi_X : T^* X \rightarrow X$ equipped with its canonical exact structure $\theta_X$. The Liouville vector field $Z_X$ generates the standard linear scaling $\Phi_{X,t}$ along the fibers of $\pi_X$.

More generally, any conic open subset $A \subset T^* X$ in particular the complement of the zero section $T^* X \setminus X$, provides an exact symplectic manifold $(A, \theta_X|_A)$.

By a Morse function $h : M \rightarrow \mathbb{R}$, we will always mean an exhausting (proper and bounded below) function whose critical points are nondegenerate and finite in number. A vector field $V$ is gradient-like with respect to $h$ if away from the critical points of $h$, we have $dh(V) > 0$, and in some neighborhood of the critical points, $V$ is the gradient of $h$ with respect to some Riemannian metric.

Definition 2.4. By a Weinstein manifold, we will mean an exact symplectic manifold $(M, \theta)$ that admits a Morse function $h : M \rightarrow \mathbb{R}$ such that the Liouville vector field $Z$ is gradient-like with respect to $h$. 
Example 2.5. A Weinstein cell is a Weinstein manifold $(M, \theta)$ such that $\theta$ has a single zero. It follows that any Morse function $h : M \to \mathbb{R}$ such that the Liouville vector field $Z$ is gradient-like with respect to $h$ will have a single critical point which is a minimum.

Example 2.6. Suppose $X$ is a compact manifold. Choose a generic Morse function $f_X : X \to \mathbb{R}$ and Riemannian metric $g$, and let $\nabla_g f_X$ denote the resulting gradient of $f_X$. We obtain a fiberwise linear function $F_X : T^*X \to \mathbb{R}$ by setting $F_X(x, \xi) = \xi(\nabla_g f_X|_x)$.

Then for $\epsilon > 0$ sufficiently small, the pair $(T^*X, \theta_X + \epsilon dF_X)$ forms a Weinstein manifold, exhibited by the Morse function $h = g + \pi^*_X f_X : T^*X \to \mathbb{R}$ given by $h(x, \xi) = |\xi|_g^2 + f_X(x)$.

2.2. Cell decompositions. For the statements of this section, in addition to the previously mentioned works, we recommend the excellent sources [12, 13].

Let $(M, \theta)$ be a complete exact symplectic manifold with Liouville vector field $Z$ and expanding action $\Phi^t : \mathbb{R}_+ \times M \to M$.

Definition 2.7. The critical locus $c \subset M$ is the zero-locus of the exact structure $c = \{x \in M | \theta|_{T_xM} = 0\}$.

The core $K \subset M$ is the subset $K = \{x \in M | \lim_{t \to \infty} \Phi^t(x) \in c\}$.

The ether $E \subset M$ is the complement $E = M \setminus K$.

Remark 2.8. All three of the above subsets are evidently conic, and the expanding action on the complement $M \setminus c$, and in particular the ether $E$, is free.

The critical locus $c \subset M$ is equivalently the zero-locus of the Liouville vector field $Z$, or the fixed points of the expanding action $\Phi^t$.

Example 2.9. For the exact symplectic manifold $(T^*B, \theta_B)$, the core and critical locus are the zero section $c_B = K_B = B$, and the ether is its complement $E_B = T^*B \setminus B$.

More generally, for any conic open subset $A \subset T^*B$ and resulting exact symplectic manifold $(A, \theta_B|_A)$, its critical locus and core are the intersection $c = K = A \cap B$, and its ether is the complement $E = A \cap (T^*B \setminus B)$.

Now suppose $(M, \theta)$ is a Weinstein manifold. Then the critical locus $c \subset M$ consists of the finitely many critical points of a Morse function $h : M \to \mathbb{R}$ for which $Z$ is gradient-like.

Definition 2.10. Given a Weinstein manifold $(M, \theta)$, to each critical point $p \in c$, we associate the stable and unstable manifolds $c_p = \{x \in M | \lim_{t \to \infty} \Phi^t(x) = p\}$, $C_p = \{x \in M | \lim_{t \to -\infty} \Phi^t(x) = p\}$

Lemma 2.11. Given a Weinstein manifold $(M, \theta)$, for each critical point $p \in c$, the stable manifold $c_p$ is an isotropic cell and the unstable manifold $C_p$ is a coisotropic cell.

The coisotropic cells provide a partition $M = \bigsqcup_{p \in c} C_p$ while the isotropic cells partition the core $K = \bigsqcup_{p \in c} c_p \subset M$. 

We will always perturb our Liouville form $\theta$ to be generic. Then it follows that the partition by coisotropic cells is a Whitney stratification.

**Lemma 2.12.** Given a Weinstein manifold $(M, \theta)$, the core $K \subset M$ is compact and its inclusion is a homotopy-equivalence.

We will use the coisotropic cells to view the finite critical locus $\mathfrak{c} \subset M$ as a poset: we say critical points $p, q \in \mathfrak{c}$ satisfy $p \leq q$ if and only if $\overline{C_p} \cap C_q \neq \emptyset$.

In particular, maxima $p \in \mathfrak{c}$ correspond to closed coisotropic cells $C_p \subset M$, and subsets $s \subset \mathfrak{c}$ such that $p \in s$ and $q \in \mathfrak{c}$ with $q \leq p$ implies $q \in s$ correspond to open unions $M_s = \bigcup_{p \in s} C_p \subset M$.

### 2.3. Markings

Let us first list some useful notions about conic sets.

**Definition 2.13.** Let $(M, \theta)$ be a complete exact symplectic manifold.

1. The projectivization of $M$ is the contact manifold $M^\infty = E/\mathbb{R}_+$ equipped with its canonical contact structure $\xi$.

   More generally, the projectivization of a conic subset $A \subset M$ is the subset $A^\infty = (A \cap E)/\mathbb{R}_+ \subset M^\infty$.

2. The cone over a subset $A^\infty \subset M^\infty$ is the conic subset $cA^\infty = \{ x \in E_M \mid \lim_{t \to \infty} \Phi_t(x) \in A^\infty \} \subset M$.

**Remark 2.14.** When $(M, \theta)$ is a Weinstein manifold, its projectivization $M^\infty$ is compact.

Next let us list some useful notions about closed sets.

**Definition 2.15.** Let $(M, \theta)$ be a complete exact symplectic manifold.

1. The compactification at infinity of $M$ is the manifold with boundary $\overline{M} = ((M \times (0, \infty)) \setminus (K \times \{\infty\}))/\mathbb{R}_+ \simeq M \cup M^\infty$.

   More generally, the compactification at infinity of a closed subset $A \subset M$ is the closure $\overline{A} \subset \overline{M}$.

2. The boundary at infinity of a closed subset $A \subset M$ is the frontier $\partial^\infty A = \overline{A} \setminus A \subset M^\infty = \overline{M} \setminus M$.

**Remark 2.16.** If $A \subset M$ is closed and conic, then our notation agrees in that $A^\infty = \partial^\infty A$ as subsets of $M^\infty$.

**Definition 2.17.** By a marked exact symplectic manifold $(M, \theta, \Lambda)$, we will mean an exact symplectic manifold $(M, \theta)$ together with a closed conic isotropic subvariety $\Lambda \subset M$ containing the core $K \subset \Lambda$. We use the term characteristic cone to refer to $\Lambda$.

By a marked Weinstein manifold $(M, \theta, \Lambda)$, we will mean a marked exact symplectic manifold such that the underlying exact symplectic manifold $(M, \theta)$ is a Weinstein manifold.

**Remark 2.18.** Given a marked exact symplectic manifold $(M, \theta, \Lambda)$, we can alternatively encode the characteristic cone $\Lambda \subset M$ by taking its projectivization $\Lambda^\infty \subset M^\infty$. We recover the characteristic cone $\Lambda \subset M$ by taking the union of the core and the cone over the projectivization $\Lambda = K \cup c\Lambda^\infty \subset M$. 


2.4. Coisotropic cells. Suppose \((M, \theta, \Lambda)\) is a marked Weinstein manifold. Fix a critical point \(p \in \mathfrak{c}\), and consider the inclusion of the coisotropic cell \(i_p : C_p \to M\).

The linear geometry of the inclusion of the coisotropic cell \(i_p : C_p \to M\) provides a useful guide to keep in mind.

First, we have the normal bundle \(\mathfrak{N}i_p \to C_p\) appearing in the exact sequence

\[
0 \to TC_p \to i_p^*TM \to \mathfrak{N}i_p \to 0
\]

Dually, we have the conormal bundle \(\mathfrak{N}^*i_p = T^*_C M \to C_p\) appearing in the exact sequence

\[
0 \to T^*C_p \to i_p^*T^*M \to \mathfrak{N}^*i_p \to 0
\]

The symplectic form \(\omega = d\theta\) provides an integrable foliation \(f_p \subset TC_p\) by either taking the symplectic orthogonal \(f_p = (TC_p)^{\perp}\) or equivalently, the symplectic partner \(i_{f_p}\omega = \mathfrak{N}^*i_p\). Thus the symplectic form identifies the partial flags

\[
f_p \leftarrow TC_p \leftarrow i_p^*TM \quad \mathfrak{N}^*i_p \leftarrow T^*_p M \leftarrow i_p^*T^*M
\]

and dually, it identifies the quotient sequences

\[
f_p \leftarrow T^*C_p \leftarrow i_p^*T^*M \quad \mathfrak{N}i_p \leftarrow i_p^TM/\mathfrak{i}_p \leftarrow i_p^*TM
\]

Now consider the Hamiltonian reduction diagram

\[
M_p \xrightarrow{q_p} C_p \xleftarrow{i_p} M
\]

where \(i_p\) is the inclusion of the coisotropic cell, and \(q_p\) is the quotient by the integrable isotropic foliation \(f_p \subset TC_p\) determined by \(i_p\). By construction, we have the following statement.

**Lemma 2.19.** We have a natural marked Weinstein cell \((M_p, \theta_p, \Lambda_p)\) characterized by

\[
q_p^*\theta_p = \theta|_{C_p} \quad \Lambda_p = q_p(\Lambda \cap C_p)
\]

**Remark 2.20.** In the case when \(\Lambda^\infty = \emptyset\) so that \(\Lambda = K\), we have \(\Lambda_p = q_p(K \cap C_p)\). If in addition \(p \in \mathfrak{c}\) is maximal, so that \(C_p \subset M\) is closed, we have \(K \cap C_p = \{p\}\), and hence \(\Lambda_p = \{q_p(p)\}\), and so \(\Lambda_p^\infty = \emptyset\).

2.5. Dévissage cone. We continue with \((M, \theta, \Lambda)\) a marked Weinstein manifold.

Fix a critical point \(p \in \mathfrak{c}\), and return to the Hamiltonian reduction diagram

\[
M_p \xrightarrow{q_p} C_p \xleftarrow{i_p} M
\]

where \(i_p\) is the inclusion of the coisotropic cell, and \(q_p\) is the quotient by the integrable isotropic foliation \(f_p \subset TC_p\) determined by \(i_p\).

Recall the natural marked Weinstein cell \((M_p, \theta_p, \Lambda_p)\) characterized by

\[
q_p^*\theta_p = \theta|_{C_p} \quad \Lambda_p = q_p(\Lambda \cap C_p)
\]

Let us observe that the inverse-image

\[
\Lambda_p = q_p^{-1}(\Lambda_p) \subset C_p
\]

is a conic isotropic subvariety such that

\[
\Lambda \cap C_p \subset \Lambda_p
\]
Definition 2.21. The local dévissage cone $\Lambda_{p+} \subset M$ is the conic isotropic subvariety
\[ \Lambda_{p+} = \Lambda \cup \Lambda_{\tilde{p}} \subset M \]
The global dévissage cone $\Lambda_+ \subset M$ is the conic isotropic subvariety
\[ \Lambda_+ = \Lambda \cup \bigcup_{p \in \Sigma} \Lambda_{p+} \subset M \]

Remark 2.22. In the spirit of the adjunctions to come, one could note that we can rewrite the definition of $\Lambda_{\tilde{p}} \subset C_p$ in the evidently equivalent forms
\[ \Lambda_{\tilde{p}} = q^{-1}_p q_p (\Lambda \cap C_p) = i_p q^{-1}_p q_p i_p^{-1}(\Lambda) \subset C_p \]

2.6. Open submanifolds. We continue with $(M, \theta, \Lambda)$ a marked Weinstein manifold.

Suppose that $p \in \Sigma$ is maximal, so that the coisotropic cell $i_p : C_p \to M$ is closed. It will be useful to introduce some further structure on its normal geometry.

Definition 2.23. Given a marked Weinstein manifold $(M, \theta, \Lambda^\infty)$, and a maximal critical point
\[ p \in \Sigma, \text{ a defining function } m_p : M \to [0,1] \text{ for the corresponding closed coisotropic cell } C_p \subset M, \]
is a subanalytic function such that:
\begin{enumerate}
\item $m_p^{-1}(0) = C_p$,
\item $dm_p(Z) \leq 0$,
\item $0, 1 \in [0,1]$ are the only critical values of $m_p$,
\item there is an open subset $U \subset M$ containing $\Sigma \setminus \{p\} \subset M$ such that $U \subset m_p^{-1}(1)$,
\item there is a compact subset $W \subset M$ such that $dm_p(Z) = 0$ over the complement $M \setminus W$.
\end{enumerate}

Lemma 2.24. Defining functions always exist.

Proof. This is easily obtained from the basic properties of subanalytic functions [11, 66]. \qed

We will say that a subset $s \subset \Sigma$ is open if $p \in s$ and $q \in \Sigma$ with $q \leq p$ implies $q \in s$. Open subsets $s \subset \Sigma$ correspond to open unions of coisotropic cells
\[ M_s = \bigcup_{p \in s} C_p \]

Lemma 2.25. For any open subset $s \subset \Sigma$, we obtain a natural marked Weinstein manifold
$(M_s, \theta_s, \Lambda_s)$ by restriction
\[ \theta_s = \theta|_{M_s} \quad \Lambda_s = \Lambda \cap M_s \]

Proof. We must produce a Morse function $h_s : M_s \to \mathbb{R}$ such that $Z_s = Z|_{M_s}$ is gradient-like with respect to $h_s$. By induction, it suffices to assume that $\Sigma \setminus s$ is a single maximal critical point $p$, so the corresponding coisotropic cell $C_p \subset M$ is closed.

Let $h : M \to \mathbb{R}$ be a Morse function such that $Z$ is gradient-like with respect to $h$. Choose a defining function $m_p : M \to [0,1]$ for the closed coisotropic cell $C_p = M \setminus M_s$. Consider the new function
\[ h_s = h + 1/m_p : M_s \to \mathbb{R} \]
By construction, we have
\[ dh_s(Z_s) = dh(Z) - dm_p(Z)/m_p \geq 0 \]
with equality if and only if $dh = 0$ hence if and only if we are at a point of $s$. Furthermore, there is a neighborhood of $s$ on which $dm_p = 0$, and so since $Z$ is gradient-like with respect to $h$, we conclude $Z_s$ is gradient-like with respect to $h_s$. \qed

Remark 2.26. In the case when $\Lambda^\infty = \emptyset$ so that $\Lambda = K$, we have $\Lambda_s = K \cap M_s$. Note that the core $K_s \subset M_s$ is contained in $K \cap M_s$, with equality if and only if $M_s = M$. \qed
3. Fukaya category

In this section, we survey the construction and basic properties of the stable infinitesimal Fukaya category of a Weinstein manifold [52, 56, 44, 42, 43].

3.1. Background structures. Let \((M, \theta)\) be a Weinstein manifold. We briefly review the standard additional data needed to consider Lagrangian branes and the Fukaya category.

We will work with compatible almost complex structures \(J \in \text{End}(TM)\) that are invariant under dilations near infinity. The corresponding Riemannian metrics present \(M\) near infinity as a metric cone over the projectivization \(M^\infty\). The space of all such compatible almost complex structures is nonempty and convex.

Given an almost complex structure \(J \in \text{End}(TM)\), we can speak about the complex canonical line bundle \(\kappa_M = (\wedge \dim M/2 T^{\text{hol}} M)^{-1}\). A bicanonical trivialization \(\eta\) is an identification of the tensor-square \(\kappa_M \otimes 2\) with the trivial complex line bundle. The obstruction to a bicanonical trivialization is twice the first Chern class \(2c_1(M) \in H^2(M, \mathbb{Z})\), and all bicanonical trivializations form a torsor over the gauge group \(\text{Map}(M, S^1)\). Forgetting the specific almost complex structure \(J \in \text{End}(TM)\), we will use the term bicanonical trivialization to refer to a section of bicanonical trivializations over the space of compatible almost complex structures.

Any symplectic manifold \(M\) is canonically oriented, or in other words, after choosing a Riemannian metric, the structure group of \(TM\) is the special orthogonal group. A spin structure \(\sigma\) is a further lift of the structure group to the spin group. The obstruction to a spin structure is the second Stiefel-Whitney class \(w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})\), and all spin structures form a torsor over the group \(H^1(M, \mathbb{Z}/2\mathbb{Z})\).

**Definition 3.1.** A Weinstein target \((M, \theta, \eta, \sigma)\) is a Weinstein manifold \((M, \theta)\) together with a bicanonical trivialization \(\eta\) and spin structure \(\sigma\). We will often suppress mention of the latter structures when they are fixed throughout.

3.2. Lagrangian branes. By an exact Lagrangian submanifold \(L \subset M\), we mean a closed but not necessarily compact submanifold of dimension \(\dim M/2\) such that the restriction \(\theta|_L\) is an exact one-form, so in particular, \(\omega|_L = 0\) where \(\omega = d\theta\).

To ensure reasonable behavior near infinity, we place two assumptions on our exact Lagrangian submanifolds \(L \subset M\). First, we insist that the compactification \(\overline{L} \subset \overline{M}\) is a subanalytic subset. Along with other nice properties, this implies the following two facts:

1. The boundary at infinity \(\partial^\infty L \subset M^\infty\) is an isotropic subvariety.
2. For \(h : M \to \mathbb{R}\) a Morse function such that \(Z\) is gradient-like with respect to \(h\), there is a real number \(r > 0\) such that the restricted function

\[
L \cap h^{-1}(r, \infty) \to \mathbb{R}
\]

has no critical points.

Second, we also assume the existence of a perturbation \(\psi\) that moves \(L\) to a nearby exact Lagrangian submanifold that is tame (in the sense of [64]) with respect to a conic metric.

**Definition 3.2.** Fix a field \(k\).

A brane structure on an exact Lagrangian submanifold \(L \subset M\) is a three-tuple \((E, \alpha, \beta)\) consisting of a flat finite-dimensional \(k\)-vector bundle \(E \to L\), along with a grading \(\alpha\) (with respect to the given compatible class of bicanonical trivializations) and a pin structure \(\beta\).

A Lagrangian brane in \(M\) is a four-tuple \((L, E, \alpha, \beta)\) of a Lagrangian submanifold \(L \subset M\) equipped with a brane structure \((E, \alpha, \beta)\). When there is no chance for confusion, we often write \(L\) alone to signify the Lagrangian brane.

The objects of the Fukaya category \(\text{Ob} F(M)\) comprise all Lagrangian branes \(L \subset M\).
Here is a brief reminder on what a grading and pin structure entail. First, consider the bundle of Lagrangian planes $\text{Lag}_M \to M$, and the squared phase map

$$\overline{\alpha}: \text{Lag}_M \longrightarrow \mathbb{C}^\times \quad \overline{\alpha}(L) = \eta(\Lambda^{\dim M/2} L)^2$$

Given a Lagrangian submanifold $L \subset M$, we obtain the restricted map

$$\overline{\alpha}: L \longrightarrow \mathbb{C}^\times \quad \overline{\alpha}(x) = \overline{\alpha}(T_x L)$$

A grading of $L$ is a lift

$$\alpha: L \longrightarrow \mathbb{C} \quad \overline{\alpha} = \exp \circ \alpha$$

The obstruction to a grading is the second Stiefel-Whitney class $[\overline{\alpha}] \in H^1(L, \mathbb{Z})$, and all gradings form a torsor over the group $H^0(L, \mathbb{Z})$.

Second, and recall that the (positive) pin group $\text{Pin}^+(n)$ is the double cover of the orthogonal group $O(n)$ with center $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A pin structure $\flat$ on a Riemannian manifold $L$ of dimension $n$ is a lift of the structure group of $TL$ along the map

$$\text{Pin}^+(n) \longrightarrow O(n).$$

The obstruction to a pin structure is the second Stiefel-Whitney class $w_2(L) \in H^2(L, \mathbb{Z}/2\mathbb{Z})$, and all possible pin structures form a torsor over the group $H^1(L, \mathbb{Z}/2\mathbb{Z})$.

3.3. Intersections. Graded linear spans of intersection points provide the morphisms in the Fukaya category $F(M)$. Given a finite collection of Lagrangian branes $L_0, \ldots, L_d \in \text{Ob} F(M)$, we must perturb them so that their intersections occur in some bounded domain. To organize the perturbations, we recall the inductive notion of a fringed set.

A fringed set $R_1 \subset \mathbb{R}^*_+$ is any interval of the form $(0, r)$ for some $r > 0$. A fringed set $R_{d+1} \subset \mathbb{R}^{d+1}$ is a subset satisfying the following:

1. $R_{d+1}$ is open in $\mathbb{R}^{d+1}$.
2. Under the projection $\pi: \mathbb{R}^{d+1} \to \mathbb{R}^d$ forgetting the last coordinate, the image $\pi(R_{d+1})$ is a fringed set.
3. If $(r_1, \ldots, r_d, r_{d+1}) \in R_{d+1}$, then $(r_1, \ldots, r_d, r_{d+1}) \in R_{d+1}$ for $0 < r_{d+1} < r_d$.

A Hamiltonian function $H: M \to \mathbb{R}$ is said to be controlled if near infinity it is equal to a conical coordinate. Given a finite collection of Lagrangian branes $L_0, \ldots, L_d \in \text{Ob} F(M)$, and controlled Hamiltonian functions $H_0, \ldots, H_d$, we may choose a fringed set $R \subset \mathbb{R}^{d+1}$ such that for $(\delta_0, \ldots, \delta_0) \in R$, there is a compact region $W \subset M$ such that for any $i \neq j$, we have

$$\varphi_{H_i, \delta_i}(T_i) \cap \varphi_{H_j, \delta_j}(T_j) \text{ lies in } W.$$

By a further compactly supported Hamiltonian perturbation, we may also arrange so that the intersections are transverse.

We consider finite collections of Lagrangian branes $L_0, \ldots, L_d \in \text{Ob} F(M)$ to come equipped with such perturbation data, with the brane structures $(E_i, \alpha_i, \psi_i)$ and taming perturbations $\psi_i$ transported via the perturbations.

**Definition 3.3.** Given a finite collection of Lagrangian branes $L_0, \ldots, L_d \in \text{Ob} F(M)$, for branes $L_i, L_j$ with $i < j$, the graded vector space of morphisms between them is the direct sum

$$\text{hom}_{F(M)}(L_i, L_j) = \bigoplus_{p \in H^1(\varphi_{H_i, \delta_i}(L_i) \cap \varphi_{H_j, \delta_j}(L_j))} \text{Hom}(E_i[p], E_j[p])[-\deg(p)].$$

where the integer $\deg(p)$ denotes the Maslov grading at the intersection.
It is worth emphasizing that the salient aspect of the above perturbation procedure is the relative position of the perturbed branes rather than their absolute position. The following informal viewpoint can be a useful mnemonic to keep the conventions straight. In general, we always think of morphisms as “propagating forward in time”. Thus to calculate the morphisms \( \text{hom}_{F(M)}(L_0, L_1) \), we have required that \( L_0, L_1 \) are perturbed near infinity so that \( L_1 \) is further in the future than \( L_0 \). But what is important is not that they are both perturbed forward in time, only that \( L_1 \) is further along the timeline than \( L_0 \). So for example, we could perturb \( L_0, L_1 \) near infinity in the opposite direction as long as \( L_0 \) is further in the past than \( L_1 \).

3.4. Compositions. Signed counts of pseudoholomorphic polygons provide the differential and higher composition maps of the \( A_\infty \)-structure of the Fukaya category \( F(M) \). We use the following approach of Sikorav [64] (or equivalently, Audin-Lalonde-Polterovich [5]) to ensure that the relevant moduli spaces are compact, and hence the corresponding counts are finite.

First, a Weinstein manifold \((M, \theta)\) equipped with a compatible almost complex structure conic near infinity is tame in the sense of [64]. To see this, it is easy to derive an upper bound on its curvature and a positive lower bound on its injectivity radius.

Next, given a finite collection of branes \( L_0,\ldots, L_d \in \text{Ob} F(M) \), denote by \( L \) the union of their perturbations \( \psi_i(\varphi_{H_i, \delta}(L_i)) \) as described above. By construction, the intersection of \( L \) with the region \( M \setminus W \) is a tame submanifold (in the sense of [64]). Namely, there exists \( \rho_L > 0 \) such that for every \( x \in L \), the set of points \( y \in L \) of distance \( d(x, y) \leq \rho_L \) is contractible, and there exists \( C_L \) giving a two-point distance condition \( d_L(x, y) \leq C_L d(x, y) \) whenever \( x, y \in L \) with \( d(x, y) < \rho_L \).

Now, consider a fixed topological type of pseudoholomorphic map

\[
u: (D, \partial D) \longrightarrow (M, L).
\]

Assume that all \( u(D) \) intersect a fixed compact region, and there is an a priori area bound \( \text{Area}(u(D)) < A \). Then as proven in [64], one has compactness of the moduli space of such maps \( u \). In fact, one has a diameter bound (depending only on the given constants) constraining how far the image \( u(D) \) can stretch from the compact set.

In the situation at hand, for a given \( A_\infty \)-structure constant, we must consider pseudoholomorphic maps \( u \) from polygons with labeled boundary edges. In particular, all such maps \( u \) have image intersecting the compact set given by a single intersection point. The area of the image \( u(D) \) can be expressed as the contour integral

\[
\text{Area}(u(D)) = \int_{u(\partial D)} \theta.
\]

Since each of the individual Lagrangian branes making up \( L \) is exact, the contour integral only depends upon the integral of \( \theta \) along minimal paths between intersection points. Thus such maps \( u \) satisfy an a priori area bound. We conclude that for each \( A_\infty \)-structure constant, the moduli space defining the structure constant is compact, and its points are represented by maps \( u \) with image bounded by a fixed distance from any of the intersection points.

**Definition 3.4.** Given a finite collection of Lagrangian branes \( L_0,\ldots, L_d \in \text{Ob} F(M) \), the composition map

\[
m^d : \text{hom}_{F(M)}(L_0, L_1) \otimes \cdots \otimes \text{hom}_{F(M)}(L_{d-1}, L_d) \rightarrow \text{hom}_{F(M)}(L_0, L_d)[2 - d]
\]

is defined as follows. Consider elements \( p_i \in \text{hom}(L_i, L_{i+1}) \), for \( i = 0,\ldots, d - 1 \), and \( p_d \in \text{hom}(L_0, L_d) \). Then the coefficient of \( p_d \) in \( m^d(p_0,\ldots, p_{d-1}) \) is defined to be the signed sum over pseudoholomorphic maps from a disk with \( d + 1 \) counterclockwise cyclically ordered marked points mapping to the \( p_i \) and corresponding boundary arcs mapping to the perturbations of
Each map contributes according to the holonomy of its boundary, where adjacent perturbed components $L_i$ and $L_{i+1}$ are glued with $p_i$.

3.5. **Coherence.** In the preceding, we have described the objects, morphisms, and compositions of the Fukaya category $F(M)$. As explained in the fundamental sources [22, 56], there are a large number of details to organize to be sure to obtain an honest $A_\infty$-category. In particular, calculations require branes be in general position, and hence must be invariant under suitable perturbations. In the setting of noncompact branes, we have additional perturbations near infinity to keep track of. In particular, at first pass, the constructions given only provide what might be called a compatible collection of directed $A_n$-categories, for all $n$. Here $A_n$ denotes Stasheff’s operad of partial associative operations [65], and we use the term directed as in [56]. The former arises since we only define finitely many composition coefficients at one time, and the latter since our perturbations near infinity are directed “forward in time”. To confirm the coherence of the definitions, one can appeal to a refined version of the well-known invariance of Floer calculations under Hamiltonian isotopies. We include a brief discussion here (largely borrowed from [43]) to explain the key ideas behind this approach.

Let $h : M \to \mathbb{R}$ be a Morse function compatible with the Weinstein structure, in particular, providing a conical coordinate near infinity.

**Definition 3.5.** By a one-parameter family of closed (but not necessarily compact) submanifolds (without boundary) of $M$, we mean a closed submanifold $\mathcal{L} \hookrightarrow \mathbb{R} \times T^* X$ satisfying the following:

1. The restriction of the projection $p_{\mathbb{R}} : \mathbb{R} \times M \to \mathbb{R}$ to the submanifold $\mathcal{L}$ is nonsingular.
2. There is a real number $r > 0$, such that the restriction of the product $p_{\mathbb{R}} \times h : \mathbb{R} \times M \to \mathbb{R} \times [0, \infty)$ to the subset $\{h > r\} \cap \mathcal{L}$ is proper and nonsingular.
3. There is a compact interval $[a, b] \hookrightarrow \mathbb{R}$ such that the restriction of the projection $p_M : \mathbb{R} \times M \to M$ to the submanifold $p_{\mathbb{R}}^{-1}([\mathbb{R} \setminus [a, b]) \cap \mathcal{L}$ is locally constant.

**Remark 3.6.** Conditions (1) and (2) will be satisfied if the restriction of the projection $\overline{p}_\mathbb{R} : \mathbb{R} \times \overline{M} \to \mathbb{R}$ to the closure $\overline{\mathcal{L}} \hookrightarrow \overline{M}$ is nonsingular as a stratified map, but the weaker condition stated is a useful generalization. It implies in particular that the fibers $\mathcal{L}_s = p_{\mathbb{R}}^{-1}(s) \cap \mathcal{L} \hookrightarrow M$ are all diffeomorphic, but imposes no requirement that their boundaries at infinity should all be homeomorphic as well.

**Definition 3.7.** By a one-parameter family of tame Lagrangian branes in $M$, we mean a one-parameter family of closed submanifolds $\mathcal{L} \hookrightarrow \mathbb{R} \times M$ such that the fibers $\mathcal{L}_s = p_{\mathbb{R}}^{-1}(s) \cap \mathcal{L} \hookrightarrow M$ also satisfy:

1. The fibers $\mathcal{L}_s$ are exact tame Lagrangians with respect to the usual symplectic structure and any almost complex structure conical near infinity.
2. The fibers $\mathcal{L}_s$ are equipped with a locally constant brane structure $(\mathcal{E}_s, \alpha_s, \beta_s)$ with respect to the given background classes.

**Remark 3.8.** Note that if we assume that $\mathcal{L}_0$ is an exact Lagrangian, then $\mathcal{L}_s$, being an exact Lagrangian is equivalent to the family $\mathcal{L}$ being given by the flow $\varphi_H$ of the vector field of a time-dependent Hamiltonian $H_s : T^* X \to \mathbb{R}$. Note as well that a brane structure consists of topological data, so can be transported unambiguously along the fibers of such a family.

The rest of this section will be devoted to the following statement of Floer invariance. It is the basic instance (going beyond the foundational results of [22, 56]) of the general pattern that confirms $F(M)$ is a well-defined $A_\infty$-category.
**Proposition 3.9.** Suppose $\mathcal{L}_s$ is a one-parameter family of tame Lagrangian branes in $M$. Suppose $L'$ is a fixed test brane which is disjoint from $\mathcal{L}_s$ near infinity for all $s$. Suppose $\mathcal{L}_s$ is transverse to $L'$ except for finitely many points.

Then for any $a,b$ with $\mathcal{L}_a$ and $\mathcal{L}_b$ transverse to $L'$, the Floer chain complexes $\text{CF}(\mathcal{L}_a,L')$ and $\text{CF}(\mathcal{L}_b,L')$ are quasi-isomorphic.

Before proving the proposition in general, it is convenient to first prove the following special case.

**Lemma 3.10.** Suppose $\mathcal{L}_s$ is one-parameter family of tame Lagrangian branes in $M$. Suppose $L'$ is a fixed test object which is disjoint from $\mathcal{L}_s$ near infinity for all $s$.

Fix $s_0$ and assume $\mathcal{L}_{s_0}$ is transverse to $L'$. Then there is an $\epsilon > 0$ so that for all $s_1 \in (s_0 - \epsilon, s_0 + \epsilon)$, the Floer chain complexes $\text{CF}(\mathcal{L}_{s_0}, L')$ and $\text{CF}(\mathcal{L}_{s_1}, L')$ are quasi-isomorphic.

**Proof.** By our assumptions on the tame behavior (in the sense of [64]) of $\mathcal{L}_{s_0}$ and $L'$ near infinity, the moduli spaces giving the differential of $\text{CF}(\mathcal{L}_{s_0}, L')$ are compact. This follows from the a priori $C^0$-bound: there is some $r_0 > 0$, such that no disk in the moduli space leaves the region $h < r_0$, where $h : M \to \mathbb{R}$ is the fixed Morse function.

Choose some $r_1 > r_0$. Then for very small $\epsilon > 0$ and any $s_1 \in (s_0 - \epsilon, s_0 + \epsilon)$, we may decompose the motion $\mathcal{L}_{s_0} \leadsto \mathcal{L}_{s_1}$ into two parts: first, a motion $\mathcal{L}_{s_0} \leadsto L$ supported in the region $h > r_0$; and then second, a compactly supported motion $L \leadsto \mathcal{L}_{s_1}$. We must show that each of the above two motions leads to a quasi-isomorphism.

First, for the motion $\mathcal{L}_{s_0} \leadsto L$, since we have not changed $\mathcal{L}_{s_0}$ or $L'$ in the region $h < r_0$, the same a priori $C^0$-bounds of [64] hold (they only depend on the Lagrangians in the region $h < r_0$), and the pseudoholomorphic strips for the pair $(\mathcal{L}_{s_0}, L')$ and for the pair $(L, L')$ are in fact exactly the same (we could perversely attach “wild” non-intersecting ends to either and it would not make a difference.) Thus we can take the “continuation map” to be the identity.

(One should probably not use the term “continuation map” for such a construction. Rather, it is an example of the more general setup of parameterized moduli spaces. In the above setting one can obtain a uniform $C^0$-bound over the family, so the parameterized moduli space is compact, and hence one can apply standard cobordism arguments to prove the matrix coefficients at the initial and final time are the same.)

Second, the motion $L \leadsto \mathcal{L}_{s_1}$ is compactly supported, so standard PDE techniques provide a continuation map.

**Proof of Proposition 3.9.** By the previous lemma, it suffices to show that for any $s_0$ with $\mathcal{L}_{s_0}$ not (necessarily) transverse to $L'$, there is a small $\epsilon > 0$ such that the Floer chain complexes $\text{CF}(\mathcal{L}_{s_0-\epsilon}, L')$ and $\text{CF}(\mathcal{L}_{s_0+\epsilon}, L')$ are quasi-isomorphic.

To see this, let $H_s(x, \xi)$ be a (time-dependent) Hamiltonian giving the motion $\mathcal{L}_s$. Choose a bump function $b(\xi)$ which is 0 near infinity and 1 on a compact set containing all of the (possibly non-transverse) intersection points $\mathcal{L}_{s_0} \cap L'$.

The product Hamiltonian $\tilde{H}(x, \xi) = b(\xi)H_s(x, \xi)$ gives a family $\tilde{\mathcal{L}}_s$ through the base object $\mathcal{L}_{s_0}$ satisfying: (1) $\tilde{\mathcal{L}}_s$ is transverse to $L'$ whenever $|s - s_0|$ is small and nonzero, and (2) $\tilde{\mathcal{L}}_s$ is equal to $\mathcal{L}_{s_0}$ near infinity. Therefore since the motion of $\tilde{\mathcal{L}}_s$ is compactly supported, standard PDE techniques provide a continuation map giving a quasi-isomorphism between $\text{CF}(\tilde{\mathcal{L}}_{s_0-\epsilon}, L')$ and $\text{CF}(\tilde{\mathcal{L}}_{s_0+\epsilon}, L')$, for small enough $\epsilon > 0$.

Finally, returning to the bump function $b(\xi)$, one can construct motions $\mathcal{L}_{s_0-\epsilon} \leadsto \tilde{\mathcal{L}}_{s_0-\epsilon}$ and $\tilde{\mathcal{L}}_{s_0+\epsilon} \leadsto \mathcal{L}_{s_0+\epsilon}$ which are supported near infinity and thus in particular always transverse to $L'$. Thus we may apply the previous lemma to obtain quasi-isomorphisms between $\text{CF}(\mathcal{L}_{s_0-\epsilon}, L')$.
and $CF(\Sigma_{s_0-\epsilon}, L')$, and similarly, between $CF(\Sigma_{s_0+\epsilon}, L')$ and $CF(\Sigma_{s_0+\epsilon}, L')$. Putting together the above, we obtain a quasi-isomorphism between $CF(\Sigma_{s_0-\epsilon}, L')$ and $CF(\Sigma_{s_0+\epsilon}, L')$. □

Remark 3.11. The above proposition (which is a condensed form of arguments of [42, 44] and appears explicitly in [43] for cotangent bundles) is closely related to Question 1.3 of Oh’s paper [45] which asks whether a homology-level continuation map constructed by a careful limiting argument with PDE techniques is induced by a chain-level morphism. While we have not investigated this, it is not hard to believe that the quasi-isomorphism of the above proposition provides the desired lift.

3.6. Stabilization. It is convenient to work interchangeably with small idempotent-complete pre-triangulated $k$-linear $A_\infty$-categories [22, 56] when thinking about Fukaya categories and small stable idempotent-complete $k$-linear $\infty$-categories [35, 36] when thinking about abstract constructions. They have equivalent homotopy theories, and we lose nothing by going back and forth. In what follows, all of the specific assertions we will use can be found in [10].

Definition 3.12. Let $\mathcal{C}, \mathcal{C}'$ be stable $\infty$-categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be
1. continuous if it preserves coproducts,
2. proper if it preserves compact objects,
3. exact if it preserves zero objects and exact triangles (equivalently, finite colimits).

It is convenient to work alternatively within two related $k$-linear contexts.

Definition 3.13. We denote by $St_k$ the symmetric monoidal $\infty$-category of stable presentable $k$-linear $\infty$-categories with morphisms continuous functors. The monoidal unit is the $\infty$-category $Mod_k$ of $k$-chain complexes.

We denote by $st_k$ the symmetric monoidal $\infty$-category of small stable idempotent-complete $k$-linear $\infty$-categories with morphisms exact functors. The monoidal unit is the $\infty$-category $Perf_k$ of perfect $k$-chain complexes.

Definition 3.14. The big stabilization $Mod\mathcal{C} \in St_k$ of a small $k$-linear $\infty$-category $\mathcal{C}$ is the stable presentable $k$-linear differential graded category of $A_\infty$-right modules

\[ \mathcal{C}^{op} \longrightarrow Mod_k. \]

The small stabilization $Perf\mathcal{C} \in st_k$ is the small $k$-linear full $\infty$-subcategory of $Mod\mathcal{C}$ consisting of compact objects (summands of finite colimits of representable objects).

Lemma 3.15. For a small $k$-linear $\infty$-category $\mathcal{C}$, the Yoneda embedding is fully faithful

\[ \mathcal{Y} : \mathcal{C}^{op} \longrightarrow Perf\mathcal{C} \] \[ \mathcal{Y}_L(P) = hom_{\mathcal{C}}(P, L) \]

If $\mathcal{C} \in st_k$, then the Yoneda embedding is an equivalence.

Corollary 3.16. For a small $k$-linear $\infty$-category $\mathcal{C}$, forming its stabilization canonically commutes with forming its opposite category

\[ Perf(\mathcal{C}^{op}) \simeq Perf(\mathcal{C})^{op}. \]

Lemma 3.17. Forming big stabilizations is a faithful symmetric monoidal functor

\[ Mod : st_k \longrightarrow St_k \]

The monoidal dual of $\mathcal{C} \in st_k$ is the opposite category $\mathcal{C}^{op}$.

The monoidal dual of $Mod\mathcal{C} \in St_k$ is the restricted opposite category $Mod(\mathcal{C}^{op})$. 
Remark 3.18. We can recover \( \mathcal{C} \in \text{st}_k \) from \( \text{Mod}\mathcal{C} \in \text{St}_k \) by passing to compact objects \( \mathcal{C} = (\text{Mod}\mathcal{C})^c \). The image of the morphism \( \text{Mod} : \text{hom}_{\text{st}_k}(\mathcal{C}, \mathcal{C}') \to \text{hom}_{\text{St}_k}(\text{Mod}\mathcal{C}, \text{Mod}\mathcal{C}') \) comprises proper functors.

Corollary 3.19. For \( \mathcal{C}, \mathcal{C}' \in \text{st}_k \), there are canonical equivalences

\[
\mathcal{C}^\text{op} \otimes \mathcal{C}' \overset{\sim}{\longrightarrow} \text{Hom}_{\text{st}_k}(\mathcal{C}, \mathcal{C}')
\]

\[
\text{Mod}(\mathcal{C}^\text{op} \otimes \mathcal{C}') \cong \text{Mod}(\mathcal{C}^\text{op}) \otimes \text{Mod}(\mathcal{C}') \overset{\sim}{\longrightarrow} \text{Hom}_{\text{St}_k}(\text{Mod}\mathcal{C}, \text{Mod}\mathcal{C}')
\]

Definition 3.20. The perfect Fukaya category \( \text{Perf} F(M) \) is the small stabilization of \( F(M) \).

The stable Fukaya category \( \text{Mod} F(M) \) is the big stabilization of \( F(M) \).

3.7. Singular support. While calculations among Lagrangian branes reflect quantum topology, we nevertheless have access to their underlying Lagrangian submanifolds. We take advantage of this in the following definition.

Definition 3.21. Fix an object \( L \in \text{Perf} F(M) \).

(1) The null locus \( n(L) \subset M \) is the conic open subset of points \( x \in M \) for which there exists a conic open set \( U \subset M \) containing \( x \) such that we have the vanishing

\[
\text{hom}_{F(M)}(L, P) \simeq 0, \quad \text{for any } P \in F(M) \text{ with } P \subset U, \vartheta^\infty P \subset U^\infty.
\]

(2) The singular support \( \text{ss}(L) \subset M \) is the conic closed complement

\[
\text{ss}(L) = M \setminus n(L).
\]

Remark 3.22. For cotangent bundles, under the equivalence of branes and constructible sheaves recalled in the next section, the above notion of singular support for branes coincides with the traditional notion of singular support of sheaves.

Definition 3.23. Let \( (M, \vartheta, \Lambda) \) be a marked Weinstein manifold.

We define the full subcategory \( \text{Perf}_\Lambda F(M) \subset \text{Perf} F(M) \) to comprise objects \( L \in \text{Perf} F(M) \) with singular support satisfying \( \text{ss}(L) \subset \Lambda \).

Lemma 3.24. \( \text{Perf}_\Lambda F(M) \) is a small stable idempotent-complete \( k \)-linear \( \infty \)-category.

If \( \Lambda_1 \subset \Lambda_2 \), then \( \text{Perf}_{\Lambda_1} F(M) \subset \text{Perf}_{\Lambda_2} F(M) \), and \( \text{Perf} F(M) = \bigcup \text{Perf}_\Lambda F(M) \).

Proof. The singular support condition is clearly preserved by extensions and summands. \( \square \)

In the remainder of this section, we explain (without proof) how to calculate the projectivization of the singular support. By induction using dévissage, this provides a complete picture of the singular support. We will not need this material for any further developments, but include it to help orient the reader.

Fix a finite collection of Lagrangian branes \( L_i \in F(M) \), for \( i \in I \), and let \( V = \cup_{i \in I} L_i \subset M \) denote the Lagrangian subvariety given by their union.

Lemma 3.25. The boundary at infinity \( \partial^\infty V \subset M^\infty \) is a closed Legendrian subvariety.

Proof. By dilation, we can contract \( V \subset M \) to a conical Lagrangian subvariety \( V^c \subset M \). Then we need only observe that \( \partial^\infty V = \partial^\infty V^c \).

\( \square \)

Let \( \partial^\infty_{sm} V \subset \partial^\infty V \) denote the smooth locus. Given a point \( x \in \partial^\infty_{sm} V \), we can find a small Legendrian sphere \( S(x) \subset M^\infty \) centered at \( x \), and simply linked around \( \partial^\infty_{sm} V \). Then we can find a Lagrangian brane \( B(x) \subset M \) diffeomorphic to a ball, and with boundary at infinity \( \partial^\infty B(x) = S(x) \). The particular grading on \( B(x) \) will play no role.
Proposition 3.26. For an object \( L \in \text{Perf} F(M) \) in the perfect envelope of the finite collection \( L_i \in F(M) \), for \( i \in I \), the projectivization its singular support \( \text{ss}(L)^\infty \subset M^\infty \) is the closure of the subset
\[
\{ x \in \partial_{\text{sm}}^\infty V \mid \text{hom}_{F(M)}(L, B(x)) \neq 0 \}.
\]

Corollary 3.27. The projectivization of the singular support \( \text{ss}(L)^\infty \subset M^\infty \) is a Legendrian subvariety.

3.8. Cotangent bundles. We briefly remind the reader of the equivalence between branes in a cotangent bundle and constructible sheaves on the base manifold.

Let \( X \) be a compact manifold with cotangent bundle \( \pi : T^*X \to X \) and projectivization \( \pi^\infty : T^\infty X \to X \). For simplicity, let us assume that \( X \) is equipped with an orientation and spin structure. Then \( (T^*X, \theta_X) \) is a Weinstein manifold with a canonically trivial canonical bundle and canonical spin structure.

Definition 3.28. Let \( \text{Sh}(X) \) denote the differential graded category of complexes of sheaves on \( X \) with constructible cohomology.

Definition 3.29. Given an object \( \mathcal{F} \in \text{Sh}(X) \), we write \( \text{ss}\mathcal{F} \subset T^*X \) for its singular support, and \( \text{ss}^\infty \mathcal{F} \subset T^\infty X \) for the projectivization of its singular support.

Given a Whitney stratification \( \mathcal{S} = \{ X_\alpha \}_{\alpha \in A} \), we define its conormal bundle and projectivized conormal bundle to be the unions
\[
T^\alpha_X X = \bigsqcup_{\alpha \in A} T^\ast_{X_\alpha} X \subset T^\ast X \quad T^\infty_X X = \bigsqcup_{\alpha \in A} T^\infty_{X_\alpha} X \subset T^\infty X
\]

Definition 3.30. Suppose \( \Lambda \subset T^*X \) is a conical Lagrangian subvariety. Let \( \text{Sh}_\Lambda(X) \subset \text{Sh}(X) \) denote the full subcategory of complexes of sheaves with \( \text{ss}^\infty \mathcal{F} \subset \Lambda \).

Suppose \( \mathcal{S} = \{ X_\alpha \}_{\alpha \in A} \) is a Whitney stratification of \( X \). Let \( \text{Sh}_\mathcal{S}(X) \subset \text{Sh}(X) \) denote the full subcategory of complexes of sheaves with \( \mathcal{S} \)-constructible cohomology.

Lemma 3.31. For a Whitney stratification \( \mathcal{S} \), we have \( \text{Sh}_\mathcal{S}(X) = \text{Sh}_{T^\mathcal{S}X}(X) \).

Now let \( i : Y \to X \) be a locally closed submanifold with frontier \( \partial Y = \overline{Y} \setminus Y \). On the one hand, we have the standard and costandard extensions \( i_*k_Y, i!k_Y \in \text{Sh}(X) \).

On the other hand, we correspondingly Lagrangian branes constructed as follows. Choose a non-negative function \( m : X \to \mathbb{R}_{\geq 0} \) with zero-set precisely \( \partial Y \subset X \).

Definition 3.32. We define the standard and costandard Lagrangians \( L_Y^*, L_Y! \in F(T^*X) \) to be the fiberwise translations
\[
L_Y^* = \Gamma_{\text{dlog}} m + T^\ast_Y X \quad L_Y! = \Gamma_{\text{dlog}} m + T^\gamma_Y X
\]
equipped with the orientation bundle \( o_Y \), and canonical gradings and spin structures.

We have the following from [44, 42, 43].

Theorem 3.33. There is a canonical equivalence
\[
\mu_X : \text{Sh}(X) \xrightarrow{\sim} \text{Perf} F(T^*X)
\]
such that \( \mu_X(i_*k_Y) \simeq L_Y^*, \mu_X(i!k_Y) \simeq L_Y! \).

Furthermore, we have \( \text{ss}^\infty \mathcal{F} = \text{ss}^\infty \mu_X(F) \), and hence for a conical Lagrangian subvariety \( \Lambda \subset T^*X \) containing the zero section, \( \mu_X \) restricts to an equivalence
\[
\mu_X : \text{Sh}_\Lambda(X) \xrightarrow{\sim} \text{Perf}_\Lambda F(T^*X)
\]
There are various extensions of the above result to noncompact manifolds $X$, but we will only need the following mild partial generalization.

Suppose $X$ is a manifold whose noncompactness is concentrated at single conical end. In other words, we have a manifold $Y$ with boundary $\partial Y$ such that $\overline{Y} = Y \bigsqcup \partial Y$ is compact, and an identification

$$X \simeq \overline{Y} \bigsqcup (\partial Y \times [0, \infty))$$

We can equivalently assume $X$ is equipped with a Morse function $f_X : X \to \mathbb{R}$ and Riemannian metric $g$ such that the flow of the gradient $\nabla g f_X$ exhibits the conical end as a metric product.

Consider the fiberwise linear function $F_X : T^*X \to \mathbb{R}$ defined by $F_X(x, \xi) = \xi(\nabla g f_X|_x)$, and the Morse function $h = g + \pi^*_X f_X : T^*X \to \mathbb{R}$ defined by $h(x, \xi) = |\xi|^2_g + f_X(x)$. The following is elementary to verify.

**Lemma 3.34.** For $\epsilon > 0$ sufficiently small, the pair $(T^*X, \theta + \epsilon dF_X)$ forms a Weinstein manifold exhibited by the Morse function $h$.

Now suppose in addition that $\mathcal{S} = \{X_\alpha\}_{\alpha \in A}$ is a Whitney stratification with strata $X_\alpha \subset X$ that are conical near the end. In other words, we assume that the strata $X_\alpha$ are transverse to $\partial Y$ inside of $X$, and the above identification restricts to an identification

$$X_\alpha = (X_\alpha \cap \overline{Y}) \bigsqcup ((X_\alpha \cap \partial Y) \times [0, \infty))$$

We can equivalently assume the strata are invariant under the flow of the gradient of our Morse function along the conical end.

Now let $i : Y \to X$ be a union of strata. On the one hand, we have the standard and costandard extensions $i_* k_Y, i! k_Y \in \text{Sh}_S(X)$.

On the other hand, we have the corresponding standard and costandard branes $L_{Y*}, L_{Y!} \in F(T^*X)$ constructed with a non-negative function $m : X \to \mathbb{R}_{\geq 0}$ invariant under the flow of the gradient of our Morse function along the conical end.

With no significant modifications, the arguments used to establish Theorem 3.33 provide the following.

**Proposition 3.35.** There is a canonical functor

$$\mu_X : \text{Sh}_S(X) \rightarrow \text{Perf} F(T^*X)$$

such that $\mu_X(i_* k_Y) \simeq L_{Y*}$, $\mu_X(i! k_Y) \simeq L_{Y!}$.

**Remark 3.36.** We will apply the above in the special situation when $Y$ itself is a Weinstein manifold, with $f_X$ a Morse function compatible with its Liouville vector field, and $\mathcal{S}$ the stratification by coisotropic cells.

### 4. Adjunctions for branes

Suppose throughout this section that $(M, \theta, \eta, \sigma)$ is a fixed Weinstein target, so a Weinstein manifold $(M, \theta)$ with a bicanonical trivialization $\eta$ and spin structure $\sigma$. Suppose as well that each of its Weinstein cells $(M_p, \theta_p)$ is equipped with a bicanonical trivialization $\eta_p$ and the (necessarily) trivial spin structure $\sigma_p$.

#### 4.1. Bimodules via correspondences

**Definition 4.1.** Let $(M, \theta)$ be an exact symplectic manifold. The opposite exact symplectic manifold $(M^{op}, -\theta)$ is the same underlying manifold $M$ equipped with the negative Liouville form, and hence negative symplectic form $-\omega = -d\theta$. 
When \((M, \theta)\) is equipped with background structures, we transport them by the identity to obtain background structures on \((M^{op}, -\theta)\).

**Lemma 4.2.** If \((M, \theta)\) is a Weinstein manifold, its opposite \((M^{op}, -\theta)\) is a Weinstein manifold with the same Liouville vector field.

**Proof.** If \(\theta = \iota_Z \omega\), then \(-\theta = \iota_Z (-\omega)\). \(\square\)

**Proposition 4.3.** There is a canonical identification

\[ F(M) \xrightarrow{\sim} F(M^{op})^{op} \]

given on Lagrangian branes by the duality

\[ L = (L, \mathcal{E}, \alpha, \flat) \xrightarrow{\sim} L' = (L, \mathcal{E}', -\alpha, \flat) \]

**Proof.** Our perturbation framework is compatible with the assertion. \(\square\)

**Definition 4.4.** Suppose \(M, N\) are Weinstein targets.

A Lagrangian correspondence is an object \(K \in F(M^{op} \times N)\). The dual correspondence \(K' \in F(N^{op} \times M)\) is the matched object under the equivalence \(F(M^{op} \times N) \simeq F(N^{op} \times M)^{op}\).

**Definition 4.5.** Suppose \(M, N\) are Weinstein targets, and \(K \in F(M^{op} \times N)\) a Lagrangian correspondence. Let \(L \in F(M), P \in F(N)\) be test branes.

We obtain a morphism (continuous functor)

\[ f_K \in \text{Hom}_{\text{St}}(\text{Mod} F(M), \text{Mod} F(N)) \simeq \text{Mod}(F(M^{op}) \otimes F(N)) \]

characterized by the functorial identification

\[ \text{hom}_{F(N)}(P, f_K(L)) \simeq \text{hom}_{F(M^{op} \times N)}(L' \times P, K). \]

**Remark 4.6.** There are (at least) two approaches to confirm that the above definition makes sense. On the one hand, we can adopt the geometric formalism of Lagrangian correspondences as developed by Wehrheim and Woodward [67, 68, 69, 70, 71, 72] and Ma’u [40] and count pseudoholomorphic quilts to provide the structure constants of an \(A_\infty\)-functor. On the other hand, we can adopt the algebraic approach of bimodules and count pseudoholomorphic disks in each factor separately. To enact the latter approach, we first must be careful with transversality to be sure that the moduli spaces behave as expected, and then we can appeal to the now established fact [50, 51, 39, 32] that the tensor product of categories is governed by the tensor product of \(A_\infty\)-structures. In both approaches, we must take into account our perturbation framework and apply elaborations on the invariance arguments of Section 3.5.

The following assertion can be checked by hand, and similar assertions in parallel settings can be found in many places including [67, 68, 69, 70, 71, 72, 42].

**Proposition 4.7.** The diagonal \(\Delta_M \subset M^{op} \times M\) Lagrangian correspondence gives an endo-functor \(f_{\Delta_M}\) of \(F(M)\) canonically equivalent to the identity \(\text{id}_{F(M)}\).

### 4.2. Closed cell correspondences

Fix a maximal critical point \(p \in c\), so that the corresponding coisotropic cell is closed, and consider the Hamiltonian reduction diagram

\[ M_p \xleftarrow{q_p} C_p \xrightarrow{i_p} M \]

where \(i_p\) is the inclusion of the coisotropic cell, and \(q_p\) is the quotient by the integrable isotropic foliation determined by \(i_p\).
**Definition 4.8.** We define the *closed cell correspondence* $$K_p \in F(M_{op}^p \times M)$$ to be the Lagrangian submanifold $$C_p \subset M_{op}^p \times M$$ equipped with the trivial local system, arbitrary fixed grading, and trivial pin structure.

**Theorem 4.9.** There is a fully faithful functor $$F(M_p) \to F(M)$$ such that its continuous extension $i : \text{Mod } F(M_p) \to \text{Mod } F(M)$ is given by the correspondence construction $f_{K_p}$.

**Proof.** The representability of the functor $f_{K_p}$ can be seen by elementary methods, or as a non-compact variation on the theory of Lagrangian correspondences of Wehrheim and Wood ward [67, 68, 69, 70, 71, 72]. The geometric composition of a Lagrangian brane $L \subset M_p$ with the correspondence $K_p$ is nothing more than the pullback $i_p^{-1}L \subset M$. For any fixed $A_\infty$-structure constant, one can compare moduli spaces to exhibit the representability.

To see that the functor is fully faithful, fix two branes $L_0, L_1 \in F(M_p)$, and consider $iL_0, iL_1 \in F(M)$. It suffices to show that for some open neighborhood $\mathcal{N}(C_p) \subset M$, all disks involved in the calculation of $\text{hom}_{F(M)}(iL_0, iL_1)$ lie in $\mathcal{N}(C_p)$. Following for example [44, 42], this can be accomplished as follows. First, by contracting the original branes $L_0, L_1 \subset M_p$ with the Liouville vector field of $M_p$, we can ensure arbitrarily small energy bounds on the disks. Then we can invoke [64] so that the energy bounds provide sufficient diameter bounds. $\square$

**Remark 4.10.** For a fixed marking $\Lambda_p \subset M_p$, consider the induced marking $$\Lambda_p^+ = K \cup q_p^{-1}(\Lambda_p) \subset M$$

The functor $i$ restricts to a functor $$i : F_{\Lambda_p}(M_p) \to F_{\Lambda_p^+}(M)$$

**Proposition 4.11.** There exists a continuous right adjoint $$i^! : \text{Mod } F(M) \to \text{Mod } F(M_p)$$ coinciding with the correspondence construction $f_{K_p^*}$.

**Proof.** By construction, $\text{Mod } F(M_p)$ is presentable, and $i$ is continuous and proper, hence we have a continuous right adjoint characterized by $$\text{hom}_{F(M)}(P, i^!L) \simeq \text{hom}_{F(M_p)}(i^!P, L).$$ It is elementary to unwind the definitions to see that $i^!$ coincides with $f_{K_p^*}$. $\square$
4.3. Open complements. We continue with a maximal critical point \( p \in c \), so that the corresponding coisotropic cell \( C_p \subseteq M \) is closed. Now let us consider the open subset \( s = c \setminus \{ p \} \), and the corresponding open union of coisotropic cells

\[
M_s = \bigsqcup_{q \in s} C_q = M \setminus C_p
\]

Choose a defining function \( m_p : M \to [0, 1] \) for the closed coisotropic cell \( C_p \subseteq M \). Consider the Hamiltonian function \( \log m_p : M_s \to \mathbb{R} \), and the symplectomorphisms

\[
\Xi_{s!} : M_s \longrightarrow M_s \quad \Xi_{s*} : M_s \longrightarrow M_s
\]

resulting from the Hamiltonian flow of \( \log m_p \) for negative unit time and unit time respectively.

**Theorem 4.12.** For \( \dagger = !, * \), there is a fully faithful functor

\[
F(M_s) \longrightarrow F(M)
\]

with continuous extension

\[
j_\dagger : \text{Mod } F(M_s) \longrightarrow \text{Mod } F(M)
\]

given on Lagrangian branes \( L \in F(M_s) \) by the pushforward

\[
j_\dagger L = \Xi_{s!}(L).
\]

**Proof.** For \( \dagger = !, * \), the existence of the functor \( j_\dagger \) and the fact that it is fully faithful all follow from some simple observations. Consider a finite collection of branes in \( M_s \) and finite collection of \( A_\infty \)-compositions. First, observe that the symplectomorphism \( \Xi_\dagger \) is compatible with our perturbation framework. Second, observe that the relevant intersections and disks lie in a fixed compact subset of \( M_s \) thanks to diameter bounds depending only on the compact subset. Thus changes to the target and branes outside of this subset are immaterial. In particular, we can alter the almost complex structure away from the compact set to obtain an almost complex structure extending over all of \( M \) without affecting any of the calculations. \( \square \)

**Remark 4.13.** In the language of Lagrangian correspondences, the functor \( j_\dagger \) is given by the correspondence construction \( f_{\Gamma_s} \) where we write \( \Gamma_{s!} \subseteq M_s \times M \) for the graph of \( \Xi_{s!} \) equipped with its canonical brane structure.

**Remark 4.14.** Recall the induced markings

\[
\Lambda_s = \Lambda \cap M_s \subseteq M_s \quad \Lambda_p = q_p(\Lambda \cap C_p) \subseteq M_p \quad \Lambda_{p+} = \Lambda \cup q_p^{-1}(\Lambda_p) \subseteq M
\]

The functor \( j_\dagger \) restricts to a functor

\[
j_\dagger : F_{\Lambda_s}(M_s) \longrightarrow F_{\Lambda_{p+}}(M)
\]

**Proposition 4.15.** There is a continuous right adjoint

\[
j^\dagger : \text{Mod } F(M) \longrightarrow \text{Mod } F(M_s)
\]

given by the correspondence construction \( f_{\Gamma'_s} \).

**Proof.** By construction, \( \text{Mod } F(M_s) \) is presentable, and \( j_\dagger \) is continuous and proper, hence there is a continuous right adjoint \( j^\dagger \) characterized by

\[
\text{hom}_{F(M_s)}(P, j^\dagger L) \simeq \text{hom}_{F(M)}(j_\dagger P, L).
\]

It is elementary to unwind the definitions to see that \( j^\dagger \) coincides with \( f_{\Gamma'_s} \). \( \square \)
Remark 4.16. The same argument applies to $i_!$ to produce a continuous right adjoint. We have not introduced this object since it will have bad representability properties.

4.4. Simple identities. We collect here some simple relations between the functors constructed in the previous sections.

Lemma 4.17. The counit maps provide equivalences of functors

$$\text{id}_{F(M_p)} \sim i_! \quad \text{id}_{F(M_s)} \sim j_!$$

Proof. A functor admitting a right (respectively left) adjoint is fully faithful if and only if the unit (respectively counit) of the adjunction is an equivalence. □

Lemma 4.18. There are canonical equivalences of functors

$$j_! i_! \simeq 0 : F(M_p) \to F(M_s) \quad 0 \simeq i_! j_! : F(M_s) \to F(M_p)$$

Proof. Thanks to our perturbation framework, given test branes, each of the compositions will result in a configuration with non-intersecting branes. □

4.5. Geometry of diagonal. We continue with the previous setup: a maximal critical point $p \in \mathcal{C}$ corresponding to a closed coisotropic cell $C_p \subset M$, and open subset $s = \mathcal{C} \setminus \{p\}$ corresponding to the open complement

$$M_s = \bigsqcup_{q \in s} C_q = M \setminus C_p$$

Proposition 4.19. There exists an open neighborhood $N(\Delta_M) \subset M \times M$, an open neighborhood $N(M) \subset T^* M$, and a symplectomorphism

$$\psi : N(\Delta_M) \to N(M)$$

such that in the domains of definition we have identifications

$$\psi(\Delta_M) = M \quad \psi(C_p \times_{M_p} C_p) = T^{c_{C_p}} \quad \psi(\Gamma_s) = -\Gamma_{d \log f_p} \quad \psi(\Gamma_{s^*}) = \Gamma_{d \log f_p}$$

Definition 4.20. We define the closed cell projector

$$K_p \in F(M \times M)$$

to be the Lagrangian submanifold $C_p \times_{M_p} C_p \subset M \times M$ equipped with the trivial local system, trivial grading, and trivial pin structure.

For $\dagger =!, \ast$, we define the open complement projector

$$K_{s!} \in F(M \times M)$$

to be the Lagrangian graph $\Gamma_{\Xi!} \subset M \times M$ equipped with the trivial local system, trivial grading, and trivial pin structure.

Theorem 4.21. Inside of $F(M \times M)$, we have exact triangles of branes

$$K_{s!} \to \Delta_M \to K_p \quad [1] \quad K_p \otimes \omega_{C_p/M_p} \to \Delta_M \to K_{s^*} \quad [1]$$

Proof. On the one hand, given any open neighborhood $N(\Delta_M) \subset M \times M$, and any finite number of $A_\infty$-compositions among the branes in the assertion of the theorem, we can arrange so that all relevant geometry lies in $N(\Delta_M)$.

On the other hand, using the microlocalization functor

$$\mu_M : \text{Sh}(M) \to F(T^* M)$$
we can transport the standard exact triangles

\[
j_k M_s \longrightarrow k_M \longrightarrow i_* k_{C_p} \longrightarrow [1]
\]

\[
i! \omega_{C_p/M_p} \longrightarrow k_M \longrightarrow j_* k_{M_s} \longrightarrow [1]
\]
to exact triangles of branes

\[
L_{M_s} \longrightarrow L_M \longrightarrow L_{C_p} \longrightarrow [1]
\]
\[
L_{C_p} \otimes \omega_{C_p/M_p} \longrightarrow L_M \longrightarrow L_{M_s} \longrightarrow [1]
\]

Furthermore, given any neighborhood \( \mathcal{N}(M) \subset T^* M \), and any finite number of \( A_\infty \)-compositions among the branes, we can arrange so that all relevant geometry lies in \( \mathcal{N}(M) \).

Thus choosing a symplectomorphism as in Proposition 4.19, we obtain a matching of all calculations.

\[\square\]

Corollary 4.22. Any object \( L \in \text{Mod} F(M) \) fits into exact triangles

\[
f_{K_s}(L) \longrightarrow L \longrightarrow f_{K_p}(L) \longrightarrow [1]
\]
\[
f_{K_p} \otimes \omega_{C_p/M_p} \longrightarrow L \longrightarrow f_{K_s}(L) \longrightarrow [1]
\]

Proposition 4.23. There are equivalence of functors

\[
f_{K_p} \simeq f_{K_p} \circ f_{K_p} = i \circ i^! \in \text{Fun}(\text{Mod} F(M), \text{Mod} F(M))
\]
\[
f_{K_s} \simeq f_{K_s} \circ f_{K_s} = j \circ j^* \in \text{Fun}(\text{Mod} F(M), \text{Mod} F(M))
\]

Proof. This can be seen by elementary methods, or as a non-compact variation on the theory of Lagrangian correspondences of Wehrheim Woodward \([67, 68, 69, 70, 71, 72]\). In particular, for any fixed \( A_\infty \)-structure constant, established methods produce an identification, and then we can apply elaborations on the invariance arguments of Section 3.5.

\[\square\]

4.6. Representability. Recall that by construction the functors \( i^!, j_!, j_* \) are proper, or in other words, preserve perfect modules.

Proposition 4.24. The right adjoints \( i^!, j_! \) are proper.

Proof. It suffices to check the assertion on objects in the image of the respective left adjoints \( i, j \). But we have already seen that the left adjoints are fully faithful and hence the right adjoints restricted to the respective images are simply the inverse equivalences.

\[\square\]

Proposition 4.25. The right adjoint \( j^* \) is also the left adjoint of \( j_* \).

Proof. We seek a natural equivalence

\[
\text{hom}_{F(M_x)}(i^! L, P) \simeq \text{hom}_{F(M)}(L, j_* P)
\]

Since both sides vanish for objects of the form \( L = iL' \), it suffices to assume that \( L = j^* P' \), and to establish a natural equivalence

\[
\text{hom}_{F(M_x)}(P', P) \simeq \text{hom}_{F(M)}(j_* P', j_* P)
\]

Recall the construction of the functors in Theorem 4.12. Observe that the above left hand side can be calculated in a fixed compact subset of \( M_x \). Our perturbation framework guarantees a similar assertion for the right hand side. Thus by construction, one can identify the two.

\[\square\]

Proposition 4.26. There is a proper left adjoint \( i^* \) such that \( i^! j_! \simeq 0 \).

Proof. It suffices to define \( i^* \) on the image of the fully faithful functor \( i \) to be the inverse equivalence.
5. Localization

We continue to suppose throughout this section that \((M, \theta, \eta, \sigma)\) is a fixed Weinstein target, so a Weinstein manifold \((M, \theta)\) with a bicanonical trivialization \(\eta\) and spin structure \(\sigma\). We will also suppose as well that each of its Weinstein cells \((M_p, \theta_p)\) is equipped with a bicanonical trivialization \(\eta_p\) and the (necessarily) trivial spin structure \(\sigma_p\).

5.1. Preliminaries.

Definition 5.1. The conic topology of \(M\) is the category \(M_{\text{con}}\) with objects conic open subanalytic subsets of \(M\) and morphisms inclusions.

Recall that \(\text{st}_k\) denotes the \(\infty\)-category of small stable idempotent-complete \(k\)-linear \(\infty\)-categories.

Definition 5.2. (1) A \(\text{st}_k\)-valued presheaf on the conic topology of \(M\) is an \(\infty\)-functor \(\mathcal{F}_{\text{pre}} : M_{\text{con}}^{\text{op}} \to \mathcal{C}\).

(2) A \(\text{st}_k\)-valued sheaf on the conic topology of \(M\) is a continuous \(\text{st}_k\)-valued presheaf.

Definition 5.3. The sheafification \(\mathcal{F} = (\mathcal{F}_{\text{pre}})^+\) of a presheaf \(\mathcal{F}_{\text{pre}}\) is a sheaf equipped with a universal presheaf morphism \(\mathcal{F}_{\text{pre}} \to \mathcal{F}\).

In our argument, we will only need the following elementary functoriality. Throughout what follows, \(\mathcal{F}\) always denotes a \(\text{st}_k\)-valued presheaf or sheaf on the conic topology of \(M\).

Definition 5.4. Let \(j : U \to M\) be the inclusion of a conic open set.

The restriction \(j^* \mathcal{F} = \mathcal{F}|_U\) is obtained by pullback along the induced functor \(j : U_{\text{con}} \to M_{\text{con}}\). In other words, it assigns \(j^* \mathcal{F}(V) = \mathcal{F}(V)\) to any conic open set \(V \subset U\).

Definition 5.5. Let \(\pi : M \to N\) be a conic map.

The pushforward \(\pi_* \mathcal{F}\) is obtained by pullback along the induced functor \(\pi^{-1} : N_{\text{con}} \to M_{\text{con}}\). In other words, it assigns \(\pi_* \mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U))\) to any conic open set \(U \subset N\).

Remark 5.6. Restriction evidently commutes with sheafification, while pushforward does not in general.

Remark 5.7. Note that the global sections of a presheaf or sheaf are simply its pushforward to a point.

Definition 5.8. The support of \(\mathcal{F}\) is the smallest conic closed set \(S \subset M\) such that \(\mathcal{F}|_{M\setminus S} \simeq 0\).

Remark 5.9. To see that the support is well-defined, note that if \(S_1, S_2 \subset M\) are conic closed sets such that \(\mathcal{F}|_{M\setminus S_1} \simeq \mathcal{F}|_{M\setminus S_2} \simeq 0\), then the sheaf property implies \(\mathcal{F}|_{M\setminus (S_1 \cap S_2)} \simeq 0\).

5.2. Construction of sheaf.

Definition 5.10. Let \(U \subset M\) be a conic open subset.

We define the full subcategory of \(U\)-null branes \(\text{Null}(M, U) \subset \text{Perf} F(M)\) to comprise objects \(L \in \text{Perf} F(M)\) with singular support satisfying \(\text{ss}(L) \cap U = \emptyset\), or equivalently, null locus satisfying \(U \subset n(L)\).

Remark 5.11. Note that \(\text{Null}(M, \emptyset) = \text{Perf} F(M)\), and \(U_1 \subset U_2\) implies \(\text{Null}(M, U_2) \subset \text{Null}(M, U_1)\).
Remark 5.12. As a consequence of our main results, we will deduce the nontrivial identity
\[ \text{Null}(M, M) \simeq 0. \]
In other words, if a brane \( L \in \text{Perf}(M) \) has empty singular support \( ss(L) = \emptyset \), then it itself is trivial \( L \simeq 0 \).

**Definition 5.13.** (1) We define the \( st_k \)-valued presheaf \( \mathcal{F}_M^{pre} \) on the conic topology of \( M \) by the assignment
\[ \mathcal{F}_M^{pre}(U) = \text{Perf}(M)/\text{Null}(M, U) \]
for conic open subsets \( U \subset M \).

(2) We define the \( st_k \)-valued sheaf \( \mathcal{F}_M \) of localized branes to be the sheafification of \( \mathcal{F}_M^{pre} \).

For conic open subsets \( U \subset M \), we have the canonical localization morphism
\[ \text{Loc}_U : \text{Perf}(M) \longrightarrow \mathcal{F}_M^{pre}(U) \longrightarrow \mathcal{F}_M(U) \]

**Lemma 5.14.** To each conic open subset \( U \subset M \), and localized brane \( L \in \mathcal{F}_M(U) \), there is a unique conic closed subvariety \( \text{ss}_U(L) \subset U \) called the localized singular support characterized by the properties:

1. For conic open subsets \( V \subset U \subset M \), we have compatibility with restriction
\[ \text{ss}_V(L|_V) = \text{ss}_U(L) \cap V \]
2. For a brane \( L \in \text{Perf}(M) \), we have compatibility with global singular support
\[ \text{ss}_U(\text{Loc}_U(L)) = \text{ss}(L) \cap U. \]

**Proof.** The assertion is evident for sections of the presheaf \( \mathcal{F}_M^{pre} \). Since conic closed subvarieties form a sheaf, the assertion follows for sections of the sheaf \( \mathcal{F}_M \). \( \square \)

5.3. **Case of Weinstein cells.** Let \( (N, \theta) \) be a Weinstein cell.

**Lemma 5.15.** If \( L \in \text{Perf}(N) \) has empty singular support \( ss(L) = \emptyset \), then it itself is trivial \( L \simeq 0 \). In other words, we have \( \text{Null}(N, N) \simeq 0 \).

**Proof.** By assumption, the unique zero \( p \in N \) of the Liouville form lies in the null locus \( n(L) \subset N \). Hence by definition, there exists a conic open set \( U \subset N \) containing \( p \) such that for any test brane \( P \in \text{Perf}(N) \) with \( P \subset N \), we have \( \text{hom}_{\text{Perf}(M)}(L, P) \simeq 0 \). But \( N \) itself is the unique conic open set \( U \subset N \) containing \( p \). Thus \( \text{hom}_{\text{Perf}(M)}(L, P) \simeq 0 \) for any \( P \in \text{Perf}(M) \), and hence \( L \simeq 0 \). \( \square \)

**Proposition 5.16.** Global localization is an equivalence
\[ \text{Loc}_N : \text{Perf}(N) \longrightarrow \mathcal{F}_N(N) \]

**Proof.** Note that \( N \) itself is the unique conic open set containing the unique zero \( p \in N \) of the Liouville form. Hence any cover of \( N \) by conic open sets must contain \( N \) itself as a constituent. Thus the canonical map is an equivalence
\[ \mathcal{F}_N^{pre}(N) \longrightarrow \mathcal{F}_N(N). \]

Finally, by the previous lemma \( \text{Null}(N, N) \simeq 0 \), hence the canonical map is an equivalence
\[ \text{Perf}(N) \longrightarrow \mathcal{F}_N^{pre}(N) \]
\( \square \)
Example 5.17. Let us see the impact of sheafifying via the easiest example. Consider the two-dimensional Weinstein cell $M = \mathbb{C}$ with standard Liouville form $\theta$ and projectivization $M^\infty \simeq S^1$. Its core is the single point $K = \{0\} \subset \mathbb{C}$, and its ether is the complement $E = \mathbb{C}^* \subset \mathbb{C}$.

On the one hand, one can check that $\text{Null}(\mathbb{C}, \mathbb{C}^*) \simeq 0$, and hence $\mathcal{F}_M^\mathfrak{pr}(\mathbb{C}^*) \simeq \text{Perf } F(\mathbb{C})$. On the other hand, one can check that $\mathcal{F}_M(\mathbb{C}^*) \simeq \prod_{x \in S^1} \text{Perf } k$. The image of the canonical morphism $\mathcal{F}_M^\mathfrak{pr}(\mathbb{C}^*) \to \mathcal{F}_M(\mathbb{C}^*)$ consists of sequences of almost everywhere zero objects whose sum is of the form $V \oplus V[1] \in \text{Perf } k$.

5.4. Compatibility with dévissage. Let $(M, \theta)$ be a Weinstein manifold.

Let $C \subset M$ be a closed coisotropic cell and consider the Hamiltonian reduction diagram

$$N \xrightarrow{q} C^\infty \xrightarrow{i} M$$

where $i$ is the inclusion of the coisotropic cell, and $q$ is the quotient by the integrable isotropic foliation determined by $i$.

Consider as well the complementary open

$$j : M^\circ = M \setminus C^\infty \longrightarrow M$$

Proposition 5.18. Let $U^\circ \subset M^\circ \subset M$ be a conic open subset.

Then the dévissage functors restrict to a diagram of adjunctions

$$\begin{align*}
\text{Perf } F(N) & \xrightarrow{i^*} \text{Null}(M, U^\circ) \\
& \xleftarrow{i_!} \text{Null}(M^\circ, U^\circ)
\end{align*}$$

Proof. To avoid possible confusion, given a conic subset $A \subset M^\circ \subset M$, we will write $A_M^\infty$ and $A_M^{\infty\circ}$ for its projectivizations as a subset of $M$ and $M^\circ$ respectively. Similarly, given a closed subset $A \subset M^\circ \subset M$, we will write $\partial_M^\infty A$ and $\partial_M^{\infty\circ} A$ for its boundaries at infinity as a subset of $M$ and $M^\circ$ respectively.

To see that the functor $i^*$ lands in $\text{Null}(M, U^\circ)$, note first that $U^\circ \cap C = \emptyset$ and $(U^\circ)^\infty_M \cap C^\infty = \emptyset$. Then any test brane $P \in F(M)$ with $P \subset U^\circ$ and $\partial_M^\infty P \subset (U^\circ)^\infty_M$ will not intersect a brane of the form $i(L) \in F(M)$, for any $L \in F(N)$.

From here, it suffices to see that the functor $j^*$ takes $\text{Null}(M, U^\circ)$ to $\text{Null}(M^\circ, U^\circ)$. Fix a brane $L \in \text{Null}(M, U^\circ)$, and any point $x \in U^\circ$. We must confirm that $x \in n(j^*(L))$.

Fix a conic open set $V \subset U^\circ$ containing $x$ that exhibits $x \in n(L)$. We will show that $V$ regarded as a subset of $M^\circ$ also exhibits $x \in n(j^*(L))$.

Fix a test brane $P \in F(M^\circ)$ with $P \subset V$ and $\partial_M^\infty V \subset V_{M^\circ}^\infty$. Then we seek to show that

$$\text{hom}_{\text{Perf } F(M^\circ)}(j^*(L), P) \simeq 0.$$ 

By adjunction, this is the same as to show that

$$\text{hom}_{\text{Perf } F(M)}(L, j_*(P)) \simeq 0.$$ 

By construction, we have $j_*(P) \subset V$. After the small perturbation required to compute the above morphisms, we also have $\partial_M^\infty (j_*(P)) \subset V_{M^\circ}^\infty$. Thus since $V$ exhibits $x \in n(L)$, the above morphisms vanish, and hence $V$ also exhibits $x \in n(j^*(L))$. 

Corollary 5.19. Restriction induces a canonical equivalence

$$j^! \simeq j^* : \mathcal{F}_M|_{M^\circ} \longrightarrow \mathcal{F}_M^{\infty\circ}.$$
Proof. It suffices to show the analogous statement for presheaves

\[ F_{M}^{pre} |_{M^o} \sim \sim F_{M}^{pre}. \]

In other words, for conic open subsets \( U^o \subset M^o \), it suffices to show that the restriction descends to compatible equivalences

\[ \text{Perf } F(M)/\text{Null}(M,U^o) \sim \sim \text{Perf } F(M^o)/\text{Null}(M^o,U^o) \]

This follows immediately from the dévissage compatibility of Proposition 5.18.

**Proposition 5.20.** Let \( U \subset M \) be a conic open subset containing \( C \), and let \( U^o \subset M^o \) denote the conic open subset \( U^o = U \cap M^o \).

Then the dévissage functors restrict to a diagram of equivalences

\[
\begin{array}{ccc}
\text{Null}(M,U) & \xrightarrow{i^! \simeq j^*} & \text{Null}(M^o,U^o) \\
\sim & \sim & \sim
\end{array}
\]

Proof. It is evident that \( j^! \) takes \( \text{Null}(M^o,U^o) \) to \( \text{Null}(M,U) \), and also that \( \text{Null}(M,U) \cap i(\text{Perf } F(M)) = 0 \).

**Corollary 5.21.** Let \( U \subset M \) be a conic open subset containing \( C \), and let \( U^o \subset M^o \) denote the conic open subset \( U^o = U \cap M^o \).

Then the dévissage functors induce a diagram of adjunctions

\[
\begin{array}{ccc}
\text{Perf } F(N) & \xrightarrow{i^! \simeq j^*} & F_{M}^{pre}(U) \\
\sim & \sim & \sim
\end{array}
\]

Consequently, we have an equivalence

\[ F_{M}^{pre}(U) \simeq \text{Mod}_T(F_{M}^{pre}(U^o) \oplus \text{Perf } F(N)) \]

where \( T = RL \in \text{End}(F_{M}^{pre}(U) \oplus \text{Perf } F(N)) \) is the monad of the adjunction

\[ L = j^! \oplus i^! : F_{M}^{pre}(U^o) \oplus \text{Perf } F(N) \longrightarrow F_{M}^{pre}(U) : R = j^! \oplus i^! \]

Proof. The first part follows immediately from the dévissage compatibility of Proposition 5.20. The second part is an immediate application of the Barr-Beck Theorem.

**Remark 5.22.** While the abstract monadic language is convenient, little of the sophisticated theory it represents is needed. More simply, we can say that \( F_{M}^{pre}(U) \) is equivalent to the \( \infty \)-category of triples \( L^o \in F_{M}^{pre}(U^o), L_N \in \text{Perf } F(N) \) and a morphism

\[ r \in \text{Hom}_{\text{Perf } F(N)}(i^! j^! L^o, L_N). \]
5.5. **Global sections.** Let \((M, \theta)\) be a Weinstein manifold.

Let \(C \subset M\) be a closed coisotropic cell and consider the Hamiltonian reduction diagram

\[
\begin{array}{ccc}
N & \overset{q}{\leftarrow} & C^\circ \overset{i}{\to} M
\end{array}
\]

where \(i\) is the inclusion of the coisotropic cell, and \(q\) is the quotient by the integrable isotropic foliation determined by \(i\).

Consider as well the complementary open

\[
j : M^\circ = M \setminus C^\circ \to M
\]

To find a natural context for Corollaries 5.19 and 5.21, let us consider the conic quotient

\[
\pi : M \to M^\sim = M^\circ \cup \ast
\]

where we collapse \(C \subset M\) to a point denoted by \(\ast\). Observe that the inverse-image under \(\pi\) provides an equivalence from the category of conic open sets \(U^\sim \subset M^\sim\) to the category of conic open sets \(U \subset M\) such that \(U \cap C\) is either all of \(C\) or empty.

Let us introduce the pushforward presheaves \(\pi_* F_{M^\circ}^{pre}\) and \(\pi_* j_* F_{M^\circ}^{pre}\), and denote by \(\text{Perf} F(N)\ast\) the skyscraper sheaf with fiber \(\text{Perf} F(N)\) supported at \(\ast\). Then we can reformulate Corollaries 5.19 and 5.21 as a diagram of adjunctions of presheaves

\[
\begin{array}{ccc}
\text{Perf} F(N)\ast & \overset{i^*}{\leftarrow} & \pi_* F_{M^\circ}^{pre} \overset{i_{\sim}!}{\to} \pi_* j_* F_{M^\circ}^{pre}
\end{array}
\]

Consequently, we have an equivalence

\[
\pi_* F_{M}^{pre} \cong \text{Mod}_T(\pi_* j_* F_{M^\circ}^{pre} \oplus \text{Perf} F(N)\ast)
\]

where \(T = RL \in \text{End}(\pi_* j_* F_{M^\circ}^{pre} \oplus \text{Perf} F(N)\ast)\) is the monad of the adjunction

\[
L = j! \oplus i! : \pi_* j_* F_{M^\circ}^{pre} \oplus \text{Perf} F(N)\ast \longrightarrow \pi_* F_{M}^{pre} ; R = i^! \oplus i^!
\]

In concrete terms, to any conic open set \(U^\sim \subset M^\sim\), we have that \(\pi_* F_{M}^{pre}(U^\sim)\) is equivalent to the \(\infty\)-category of triples \(L^\circ \in F_{M^\circ}^{pre}(U^\circ)\) where \(U^\circ = U^\sim \cap M^\circ\), \(L_N \in \text{Perf} F(N)\) nonzero only if \(\ast \in U^\sim\), and a morphism

\[
r \in \text{Hom}_{\text{Perf} F(N)}(i^! L^\circ, L_N).
\]

Now we will check that the above description is compatible with sheafification.

**Lemma 5.23.** The canonical morphism is an equivalence of sheaves

\[
(\pi_* F_{M}^{pre})^+ \overset{\sim}{\longrightarrow} \pi_* F_{M}
\]

**Proof.** This is evident over the open set \(M^\circ \subset M^\sim\). It suffices to check that the canonical morphism induces an equivalence on stalks at \(\ast\). This follows from the further observation that any conic open set \(U \subset M\) containing the zero \(p \in \pi^{-1}(\ast) = C\) in fact contains all of \(C\). \(\square\)

**Theorem 5.24.** The pushforward \(\pi_* F_{M}\) admits the canonical description

\[
\pi_* F_{M} \cong \text{Mod}_T(\pi_* j_* F_{M^\circ} \oplus \text{Perf} F(N)\ast)
\]

where \(T = RL \in \text{End}(\pi_* j_* F_{M^\circ} \oplus \text{Perf} F(N)\ast)\) is the monad of the adjunction

\[
L = j! \oplus i! : \pi_* j_* F_{M^\circ} \oplus \text{Perf} F(N)\ast \longrightarrow \pi_* F_{M} ; R = i^! \oplus i^!
\]
Proof. By our previous results reformulated above, the sheafification \((\pi_*\mathcal{F}_M)^+\) clearly admits the asserted description. Thus by the previous lemma, the pushforward \(\pi_*\mathcal{F}_M\) does as well. □

**Corollary 5.25.** *Global localization is an equivalence*

\[ \text{Loc}_M : \text{Perf} F(M) \overset{\sim}{\longrightarrow} \mathcal{F}_M(M) \]

**Proof.** Note that \(\pi_*\mathcal{F}_M(M^\sim) \simeq \mathcal{F}_M(M)\) and \(\pi_* j_* \mathcal{F}_M^\sim(M^\sim) \simeq \mathcal{F}_M^\sim(M^\circ)\). By induction, global localization is an equivalence on the open Weinstein manifold

\[ \text{Loc}_{M^\circ} : \text{Perf} F(M^\circ) \overset{\sim}{\longrightarrow} \mathcal{F}_M^\circ(M^\circ) \]

Hence by Theorem 5.24, we have an equivalence on global sections

\[ \mathcal{F}_M(M) \simeq \text{Mod}_T(\text{Perf} F(M^\circ) \oplus \text{Perf} F(N)) \]

where \(T = RL \in \text{End}(\text{Perf} F(M^\circ) \oplus \text{Perf} F(N))\) is the monad of the adjunction

\[ L = j! \oplus i! : \text{Perf} F(M^\circ) \oplus \text{Perf} F(N) \overset{\sim}{\longrightarrow} \mathcal{F}_M(M) : R = j^! \oplus i^! \]

Comparison with the similar monadic description of \(\text{Perf} F(M)\) yields the theorem. □

**Corollary 5.26.** *For \(L \in \text{Perf} F(M)\), if \(ss(L) = \emptyset\), then \(L \simeq 0\).*

**Proof.** The localization of \(L\) is a null brane for any conic open set. □

**Remark 5.27.** We will not need the following discussion but include it to help further orient the interested reader.

One might ask whether a dévissage description similar to that of Theorem 5.24 might exist for the sheaf \(\mathcal{F}_M\) itself rather than its pushforward \(\pi_*\mathcal{F}_M\). The immediate answer is negative since the key functors \(j_!, j_*\) are not local. But one need not pass all the way to the quotient \(M \to M^\sim\) induced by the collapse \(C \to \ast\). Rather it is possible to pass to the intermediate quotient \(M \to \langle M^\sim\rangle\) induced by the natural collapse \(C \to N\). More broadly, the sheaf which admits a natural dévissage pattern is the pushforward of \(\mathcal{F}_M\) along the quotient of \(M\) where each coisotropic cell is collapsed to its corresponding Weinstein cell.

In another direction, one might also ask which aspects of the dévissage pattern can be lifted to the sheaf \(\mathcal{F}_M\) itself. First, we can consider the full subsheaf \(i q^* \mathcal{F}_N \subset \mathcal{F}_M\) generated by objects of the form \(i(L) \in \text{Perf} F(M)\), for objects \(L \in \text{Perf} F(N)\). There is a canonical equivalence on global sections

\[ \text{Perf} F(N) \simeq \mathcal{F}_N(N) \overset{\sim}{\longrightarrow} i q^* \mathcal{F}_N(M) \]

Second, we have seen that there is a canonical morphism \(\mathcal{F}_M \to j_* \mathcal{F}_M^\circ\) that induces an equivalence

\[ \mathcal{F}_M|_{M^\circ} \overset{\sim}{\longrightarrow} \mathcal{F}_M^\circ. \]

Unfortunately, as mentioned above, there are no evident adjoint maps of sheaves. The following related construction provides a partial solution. For \(\uparrow=\downarrow\) or \(\ast\), we can consider the full subsheaf \(\mathcal{F}_{M^\circ} \subset \mathcal{F}_M\) generated by objects of the form \(j_! L \in \text{Perf} F(M)\), for objects \(L \in \text{Perf} F(M^\circ)\). We caution the reader that \(\mathcal{F}_{M^\circ} \uparrow\) is not the same as the pushforward \(j_! \mathcal{F}_M\). Then restriction induces a canonical equivalence

\[ \mathcal{F}_{M^\circ} \uparrow|_{M^\circ} \overset{\sim}{\longrightarrow} \mathcal{F}_{M^\circ}. \]

Furthermore, it also induces a canonical equivalence on global sections

\[ \mathcal{F}_{M^\circ}(M) \overset{\sim}{\longrightarrow} \mathcal{F}_{M^\circ}(M^\circ). \]
5.6. **Prescribed support.** Now let us introduce a characteristic cone $\Lambda \subset M$ so that we have a marked Weinstein manifold $(M, \theta, \Lambda)$. Recall by Lemma 5.14, we have the notion of singular support for localized branes.

**Definition 5.28.** We define the full subsheaf $\mathcal{F}_\Lambda \subset \mathcal{F}_M$ to consist of those localized branes $L \subset \mathcal{F}_M(U)$ such that

$$ss_U(L) \subset \Lambda \cap U$$

for any conic open set $U \subset M$.

We now can confirm the following assertion from the introduction.

**Theorem 5.29.** Let $(M, \theta, \Lambda)$ be a marked Weinstein manifold.

The $st_k$-valued sheaf $\mathcal{F}_\Lambda$ on the conic topology of $M$ has the following properties:

1. The support of $\mathcal{F}_\Lambda$ is the characteristic cone $\Lambda \subset M$.
2. The global sections of $\mathcal{F}_\Lambda$ are canonically equivalent to $\text{Perf}_\Lambda(M)$.
3. The restriction of $\mathcal{F}_\Lambda$ to an open Weinstein submanifold $M^\circ \subset M$ is canonically equivalent to the sheaf $\mathcal{F}_\Lambda^\circ$ constructed with respect to $\Lambda^\circ = \Lambda \cap M^\circ$.
4. For each zero $p \in \mathbb{C}$, the sections of $\mathcal{F}_\Lambda$ lying strictly above the unstable cell $C_p \subset M$ are canonically equivalent to $\text{Perf}_{\Lambda_p}(M_p)$.

**Proof.** All follow from our previous constructions and results. $\Box$

**Example 5.30.** Let us return to the setting of Example 5.17 and add a characteristic cone to the mix.

Recall the two-dimensional Weinstein cell $M = \mathbb{C}$ with standard Liouville form $\theta$ and projectivization $M^\infty \simeq S^1$. Its core is the single point $K = \{0\} \subset \mathbb{C}$, and its ether is the complement $E = \mathbb{C}^* \subset \mathbb{C}$. Any characteristic cone $\Lambda \subset \mathbb{C}$ will be the union of $K = \{0\}$ with finitely many rays. Let us denote by $\Lambda \subset \mathbb{C}$ the characteristic cone with $n$ rays, for $n = 0, 1, 2, \ldots$.

Then as is well known, $\text{Perf}_{\Lambda_n}(\mathbb{C})$ is equivalent to finite-dimensional modules over the $A_{n-1}$-quiver (in particular, for $n = 0$, it is the zero category). This is also the stalk of the sheaf $\mathcal{F}_{\Lambda_n}$ at the point $0 \in \mathbb{C}$. Its stalk at other points $x \in \mathbb{C}$ is either (not necessarily canonically) equivalent to $\text{Perf}_k$ when $x \in \Lambda_n$, and is the zero category otherwise.

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