Asymptotic formula for the tail of the maximum of smooth Gaussian fields on non locally convex sets

Jean-Marc Azaïs and Viet-Hung Pham
Institut de mathématiques de Toulouse
Université Paul Sabatier (Toulouse III)
31062 Toulouse Cedex 09, France

May 7, 2014

Abstract

In this paper we consider the maximum of Gaussian processes defined on non locally convex sets. Adler and Taylor or Azaïs and Wschebor give the expansions in the locally convex case. The present paper generalizes their results giving a full expansion in dimension 2. The main tools are (1) the Steiner formula for the index set and (2) a recent result of Azaïs and Wschebor. Various examples, including example in larger dimension are given. They correspond to new results.

Key-words: Stochastic processes, Gaussian fields, Rice formula, distribution of the maximum, non locally convex indexed set.

Classifications: 60G15, 60G60, 60G70.

1 Introduction

The Euler characteristic method of Adler and Taylor \cite{1} or the direct method of Azaïs and Wschebor \cite{5} give the super exponentially precise expansion for the tail of the maximum of a sufficiently regular random field $X(t)$ defined on a sufficiently regular set $S$.

An important example of such sets are the convex bodies in $\mathbb{R}^2$ (compact, convex with non-empty interior). If $S$ has a finite number of irregular points, it is proved in \cite{1} that if $X(t)$ is a centered Gaussian field with variance 1 defined on a neighborhood of $S$ and if

$$M_S = \max_{t \in S} X(t),$$

then

$$P(M_S \geq u) = \Phi(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} u \varphi(u) + o(u^{-1} \varphi(u)), \quad (1)$$

where $\Phi(u)$ and $\varphi(u)$ are the tail distribution and the density of a standard normal variable and $\sigma_i$ is the Hausdorff measure of dimension $i$. It is well-known that a formula of the form \cite{1} can be extended to a much wider class of sets.
Basically, Adler and Taylor use the local convexity that can be defined as the fact that for every point \( t \in S \), the contact cone \( C_t \) generated by the set of directions

\[
\left\{ \sigma \in \mathbb{R}^2 : \|\sigma\| = 1, \exists s_n \in S \text{ such that } s_n \to t \text{ and } \frac{s_n - t}{\|s_n - t\|} \to \sigma \right\},
\]

is convex, plus some regularity conditions (see, for example [1]).

Azaïs and Wschebor [6, p. 231] use the condition

\[
\kappa(S) = \sup_{t \in S} \sup_{s \neq t} \frac{\text{dist}(s - t, C_t)}{\|s - t\|^2} < \infty
\]

plus some additional ones.

But none of these methods is able, for example, to deal with the very simple case of \( S \) being "the angle" that is the union of two segments with the angle \( \beta \in (0, \pi) \), see Figure 1.

![Figure 1: The angle- an example of non-local convexity.](image)

which is presented in [1] as a kind of benchmark (see Subsection 3.1).

The aim of this paper is to consider sets as "the angle" and to give a full expansion of the tail in dimension 2, see Theorem 1. Additionally, we give an expansion with three terms in dimension 3, see Subsection 3.7.

Our main tools are the Steiner formula that gives the volume of the tube around \( S \), and a result of Azaïs and Wschebor [7] that shows that except some negligible events, the excursion set is close to a ball. The present paper extends their result in the sense that it gives an extra term.

Another main result is given by the examples of Section 3 that are all new and give rather unexpected results. In particular, it appears that in dimension 2, the coefficient of \( \Phi(u) \) in [1] is not always the Euler characteristic of the parameter set.

The organization of the paper is the following: in Section 2 we give the fundamental lemma and the main results in dimension 2. In Section 3 we consider some examples in dimension 2 and some remarks and examples in dimension 3. Some additional proofs are given in the Appendix.

Notation and hypotheses

We use the following notation.

- \( S \) is a set of \( \mathbb{R}^n \) with some regularity properties (see Definition 2).
- \( B \) is a ball in \( \mathbb{R}^n \) containing \( S \) such that \( \text{dist}(S, \partial B) > 0 \).
- \( M_Z \) is the maximum (or the supremum) of \( X(t) \) on the set \( Z \subset \mathbb{R}^n \).
- \( \sigma_i \) is the Hausdorff measure of dimension \( i \).
- \( S^{+\epsilon} \) is the tube around \( S \) defined as

\[
S^{+\epsilon} = \{ t \in \mathbb{R}^n : \text{dist}(t, S) \leq \epsilon \}.
\]
In all the paper we will use the following assumption on the random field \(X(t)\):

**Assumption A:** \(X(t)\) is a random field defined on a neighborhood \(NS\) of \(S \subset \mathbb{R}^n\) satisfying

i. \(X(t)\) is a centered stationary Gaussian field.

ii. \(\text{Var}(X(t)) = 1\) and \(\text{Var}(X'(t)) = I_n\).

iii. The paths of \(X(t)\) are of class \(C^3\).

iv. For all \(s \neq t \in B\), the distribution of \((X(s), X(t), X'(s), X'(t))\) does not degenerate.

v. For all \(t \in B\), \(\sigma \in S^{n-1}\), the distribution of \((X(t), X'(t), X''(t)\sigma)\) does not degenerate.

### 2 Main results

Our main tool is the following lemma.

**Lemma 1.** Let \(S_1, \ldots, S_m\) be \(m\) subsets of \(S\). Assume that there exist two constants \(C > 0\) and \(d \geq 0\) such that

\[
\sigma_n(S_1^{+\epsilon} \cap \ldots \cap S_m^{+\epsilon}) = (C + o(1))e^{n-d} \text{ as } \epsilon \to 0.
\]

Then, as \(u \to +\infty\),

\[
P \left( \min \{M_{S_i} \} \geq u \right) = u^{d-1} \varphi(u) \left( \frac{C}{2^{d/2}(\pi)^{n/2}} \Gamma(1 + (n - d)/2) + o(1) \right),
\]

where \(\Gamma\) is the Gamma function.

**Remark.** We observe that the order of the main term in (2) is \(u^{d-1}\varphi(u)\) for non negative \(d\). Moreover, a classical result shows that the order of \(F(u)\) is \(u^{-1}\varphi(u)\). Then, in the proof, an event is said to be "negligible" if its probability is \(o(u^{-1}\varphi(u))\).

**Proof.** Here we essentially follow in Azaïs and Wschebor [7] where it is proven that:

Except some "negligible events", there exists only one local maximum \(t\) inside \(B\) with value in the interval \([u, u+1]\); and the excursion set \(K_u\) above \(u\) satisfies

\[
B(t, \overline{\tau}) \subset K_u \subset B(t, \tau),
\]

where

\[
\overline{\tau} = \sqrt{\frac{2(X(t) - u)}{X(t) + u^{\alpha}}}, \quad \tau = \sqrt{\frac{2(X(t) - u)}{u - u^{\alpha}}},
\]

with \(\alpha\) is a constant \(0 < \alpha < 1\) that can be chosen close to zero.

From the fact

\[
P \left( \min \{M_{S_i} \} \geq u \right) = P \left( \{ \exists t \in B, X(.) \text{ has a local maximum at } t, X(t) \geq u \} \cap \{ \forall i = 1 \ldots m : K_u \cap S_i \neq \emptyset \} \right) + o(u^{-1}\varphi(u))
\]

and the above observations, we have the upper bound

\[
P \left( \min \{M_{S_i} \} \geq u \right) \leq o(u^{-1}\varphi(u)) + P \left( \exists t \in \hat{B}, X(.) \text{ has a local maximum at } t, u \leq X(t) \leq u + 1, t \in \bigcap_{i=1}^{m} S_i^{+\tau} \right)
\]

\[
\leq o(u^{-1}\varphi(u)) + \mathbb{E} \left( \| \exists t \in \hat{B}, X(.) \text{ has a local maximum at } t, u \leq X(t) \leq u + 1, t \in \bigcap_{i=1}^{m} S_i^{+\tau} \| \right).
\]
To compute the expectation, we use the Rice formula to get
\[ E := E \left( \mathbb{1} \{ \exists t \in \hat{B}, \ X(t) \text{ has a local maximum at } t, \ u < X(t) < u + 1, \ t \in \bigcap_{i=1}^{T} S^{\uparrow}_{i} \} \right) \]
\[ = \int_{u}^{u+1} dx \int_{\hat{B}} E \left( |X^{\prime}(t)|_{1_{\{X(t) \leq 0\}} \big| X(t) = x, \ X^{\prime}(t) = 0 \right) p_{X(t), X^{\prime}(t)}(x, 0) \sigma_{n}(dt) \]
\[ = \frac{1}{(2\pi)^{n/2}} \int_{u}^{u+1} \sigma_{n}(\bigcap_{i=1}^{T} S^{\uparrow}_{i}) E \left( |X^{\prime}(0)|_{1_{\{X^{\prime}(0) \leq 0\}} \big| X(0) = x, \ X^{\prime}(0) = 0 \right) \varphi(x) \ dx, \]
where \( \varphi \) is the value of \( \varphi \) when \( X(t) = x \). Here we use the stationary property of the field and the fact that \( X(t) \) and \( X^{\prime}(t) \) are two independent Gaussian vectors.

Using the well-known result (see Delmas [11]),
\[ E \left( |X^{\prime}(0)|_{1_{\{X^{\prime}(0) \leq 0\}}} \right| X(0) = x, \ X^{\prime}(0) = 0 = x^{\circ} + O(x^{n-2}) \text{ as } x \to \infty, \]
and the hypothesis
\[ \sigma_{n}(S^{\uparrow}_{1} \cap \ldots \cap S^{\uparrow}_{m}) \approx C \epsilon^{n-d} \text{ as } \epsilon \to 0, \]
we have
\[ E = \frac{1}{(2\pi)^{n/2}} \int_{u}^{u+1} x^{n} \varphi(x) C \left[ \frac{x - u}{u^{n}} \right]^{(n-d)/2} \ dx + o(u^{d-1} \varphi(u)) \]
\[ = \frac{C_{u}^{n+d)/2}{2d/2(n/2)} \int_{u}^{u+1} \varphi(x)(x - u)^{(n-d)/2} dx + o(u^{d-1} \varphi(u)). \]
By the change of variable \( x = u + y/u \), we obtain
\[ E = \frac{C}{2d/2(n/2)} u^{d-1} \varphi(u) \int_{0}^{u} \exp \left( -\frac{y^{2}}{2u^{2}} \right) y^{(n-d)/2} dy + o(u^{d-1} \varphi(u)) \]
\[ = u^{d-1} \varphi(u) \left( \frac{C}{2d/2(n/2)} \Gamma(1 + (n-d)/2) + o(1) \right). \]

For the lower bound, we have
\[ P \left( \min \{ M_{S_{i}} \} \geq u \right) \]
\[ \geq o(u^{-1} \varphi(u)) + P \left( \exists t \in \hat{B}, \ X(t) \text{ has a local maximum at } t, \ u < X(t) < u + 1, \ t \in \bigcap_{i=1}^{T} S^{\uparrow}_{i} \right). \]

Denote
\[ M = \mathbb{1} \{ \exists t \in \hat{B}, \ X(t) \text{ has a local maximum at } t, \ u < X(t) < u + 1, \ t \in \bigcap_{i=1}^{T} S^{\uparrow}_{i} \}. \]

It is proven in [10] or [7],
\[ 0 \leq E(M) - P(M \geq 1) \leq E(M(M - 1))/2 \leq E(M(M - 1))/2 = o(u^{-1} \varphi(u)), \]
where
\[ M_{u} = \mathbb{1} \{ \exists t \in \hat{B}, \ X(t) \text{ has a local maximum at } t, \ X(t) \geq u \}. \]

Then
\[ P \left( \min \{ M_{S_{i}} \} \geq u \right) \geq o(u^{-1} \varphi(u)) + E(M). \]

Here, using again the Rice formula and by the same arguments, we obtain that the upper and lower bounds have the same equivalent formula and the result follows.

The main object that we consider is the collection of the subsets of \( \mathbb{R}^{2} \) that satisfy the Steiner formula heuristic defined as follows.
Definition 1. A compact subset \( S \subset \mathbb{R}^2 \) is said to satisfy the Steiner formula heuristic (SFH) if it satisfies the following conditions

- As \( \epsilon \) tends to 0,
  \[
  \sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon L_1(S) + \pi \epsilon^2 L_0(S) + o(\epsilon^2).
  \]  
  (3)

- For all processes \( X(t) \) satisfying Assumption A,
  \[
  P(M_S \geq u) = L_0(S) \varphi(u) + \sigma_2(S) \frac{\varphi(u)}{\sqrt{2\pi}} + o(u^{-1/2})
  \]  
  (4)

Remarks.

1. If \( S \) is a convex body then (3) becomes the Steiner formula
  \[
  \sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon L_1(S) + \pi \epsilon^2 L_0(S),
  \]  
  (5)
  that holds true for all \( \epsilon \geq 0 \). \( L_1(S) \) is just the Hausdorff measure of the boundary of \( S \): \( \sigma_1(\partial S) \) and \( L_0(S) \) is the Euler characteristic of \( S \) which is equal to 1.

If in addition, the number of irregular points of \( S \) is finite, then from the result of Adler and Taylor, we have (4).

2. If \( S \) has a positive reach in the sense that there exists a positive constant \( r \) such that for all \( t \in S^{+r} \), there is only one projection of \( t \) on \( S \), then (5) is true for all \( \epsilon < r \) (see [2], [9]). Moreover if \( S \) is a domain with piecewise-\( C^2 \) boundary in \( \mathbb{R}^2 \) in the sense of Definition 2 hereunder and satisfies \( \kappa(S) < \infty \), then (4) still holds true (see Appendix).

3. In the most general cases, the constant \( L_1(S) \) is the outer Minkowski content of \( S \) (\( \text{OMC}(S) \)), for more details see [2], which is defined by
  \[
  \sigma_2(S^{+\epsilon}) = \sigma_2(S) + \epsilon \text{OMC}(S) + o(\epsilon).
  \]
  It can differ from \( \sigma_1(\partial S) \), for example in the case of "the square with whiskers", see Figure 2

![Figure 2: The square with whiskers.](image)

In this case, \( \sigma_1(\partial S) \) is equal to the perimeter of the square plus the length of the whiskers, while \( \text{OMC}(S) \) is equal to the perimeter of the square plus two times the length of the whiskers.

In addition it should be noticed that \( L_0(S) \) is not always equal to the Euler characteristic, see Subsection 3.4.

Definition 2. Domains with piecewise-\( C^2 \) boundary.

We assume that the boundary of \( S \) consists of a finite union of \( C^2 \) curves that will be called "edges".

The edge \( E_i \) of length \( L_i \) can be parametrized on \([0, L_i]\) in a \( C^2 \) manner by its arc length. To introduce the case of angle in the plane or the case of whiskers, we consider two kinds of edges:

- Edges that are included in \( S \): non isolated edges.
- Edges such that the intersection with \( \overline{S} \) is at most a point: this is the case of whiskers or of the angle.
To limit the number of configurations to consider, we exclude more complicated cases. Irregular points are the points where the parametrization is no more $C^2$. We assume that these points belong to four categories:

- **Convex binary points**: the intersection of two non isolated edges and the contact cone is convex.
- **Concave binary points**: as above but the contact cone is not convex. Denote $\beta \in [0, \pi]$ by the discontinuity of the angle of the tangent at this point when we choose the orientation for the boundary such that the interior is always on the left.
- **Angle points**: the intersection of two isolated edges. Denote $\beta \in [0, \pi]$ by the discontinuity of the angle is in Figure 7.
- **Concave ternary points**: the intersection of two non isolated edges $E_1, E_2$ and one isolated one $E_3$. In the main result, these points will be considered with multiplicity two. We associate to these points two angles:
  - $\beta_1$ which is the discontinuity of the angle of the tangent when we turn from $E_1$ to $E_3$.
  - $\beta_2$ which is the discontinuity of the angle of the tangent when we turn from $E_3$ to $E_2$.

To calculate explicitly, we only consider the concave ternary points such that $\beta_1 + \beta_2 \leq \pi$ and we exclude more complicated situations.

![Figure 3: Convex, concave binary and concave ternary points, respectively.](image)

Our next lemma shows a way to construct a class of compact subsets of $\mathbb{R}^2$ satisfying the SFH.

**Lemma 2.** Let $S_1$, $S_2$, $S_3$ and $S_4$ be four compact sets such that

1.) For every $i = 1, 2, 3, 4$, $S_i$ satisfies the SFH.
2.) $S_1 \cup S_2$, $S_2 \cup S_3$, $S_3 \cup S_4$, and $S_4 \cup S_1$ satisfy the SFH.
3.) $S_2 \cap S_4 = \emptyset$ and $S_1 \cap S_3 \cap S_4 = \emptyset$.
4.) As $\epsilon$ tends to 0, there exist two positive constants $C_{13}$ and $C_{123}$ such that

$$\sigma_2(S_1^{++} \cap S_3^{++}) \simeq C_{13}\epsilon^2 \quad \text{and} \quad \sigma_2(S_1^{+} \cap S_2^{++} \cap S_3^{++}) \simeq C_{123}\epsilon^2.$$  \[(6)\]

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ also satisfies the SFH and

- $L_1(S) = L_1(S_1 \cup S_2) + L_1(S_2 \cup S_3) + L_1(S_3 \cup S_4) + L_1(S_4 \cup S_1) - \sum_{i=1}^{4} L_1(S_i)$,
- $L_0(S) = L_0(S_1 \cup S_2) + L_0(S_2 \cup S_3) + L_0(S_3 \cup S_4) + L_0(S_4 \cup S_1) - \sum_{i=1}^{4} L_0(S_i) + \frac{C_{123} - C_{13}}{\pi}$. 

6
Proof. First, we consider the tube formula of $S$. By the inclusion-exclusion principle,

$$
\sigma_2(S^+) = \sigma_2((S_1 \cup S_2 \cup S_3 \cup S_4)^+) 
\begin{aligned}
&\sigma_2((S_1 \cup S_2)^+) + \sigma_2((S_3 \cup S_4)^+) - \sigma_2((S_1 \cup S_2) \cap (S_3 \cup S_4)^+) \\
&= \sigma_2((S_1 \cup S_2)^+) + \sigma_2((S_3 \cup S_4)^+) - \sigma_2((S_1 \cup S_2)^+ \cap (S_3 \cup S_4)^+) \\
&= \sigma_2((S_1 \cup S_2)^+) + \sigma_2((S_3 \cup S_4)^+) - \sigma_2((S_1^+ \cup S_2^+) \cap (S_3^+ \cup S_4^+)) \\
&= \sigma_2((S_1^+ \cup S_2^+ \cap S_3^+ \cup S_4^+) \cup (S_1^+ \cup S_2^+ \cap S_3^+ \cup S_4^+)) \\
&= \sigma_2((S_1^+ \cup S_2^+ \cap S_3^+ \cup S_4^+) - \sigma_2(S_3^+ \cap S_4^+)
= \sigma_2(S_3^+ \cap S_4^+)
\end{aligned}
$$

Thus we have

$$
\sigma_2(S^+) = \sigma_2(S) + \epsilon L_1(S) + \pi \epsilon^2 L_0(S) + o(\epsilon^2),
$$

with

$$
L_1(S) = \frac{1}{4} L_1(S_1 \cup S_2) + \frac{1}{4} L_1(S_2 \cup S_3) + \frac{1}{4} L_1(S_3 \cup S_4) + \frac{1}{4} L_1(S_4 \cup S_1) - \sum_{i=1}^{4} L_1(S_i),
$$

$$
L_0(S) = \frac{1}{4} L_0(S_1 \cup S_2) + \frac{1}{4} L_0(S_2 \cup S_3) + \frac{1}{4} L_0(S_3 \cup S_4) + \frac{1}{4} L_0(S_4 \cup S_1) - \sum_{i=1}^{4} L_0(S_i) + \frac{C_{123} - C_{13}}{\pi}.
$$

For the excursion probability on $S$, using again the inclusion-exclusion principle,

$$
P(M_S \geq u) = P(M_{S_1 \cup S_2 \cup S_3 \cup S_4} \geq u) 
= \sum_{i=1}^{4} P(M_{S_i} \geq u) - \sum_{1 \leq i < j \leq 4} P(M_{S_i} \geq u, M_{S_j} \geq u) + \sum_{1 \leq i < j < k \leq 4} P(M_{S_i} \geq u, M_{S_j} \geq u, M_{S_k} \geq u) - P(M_{S_i} \geq u, \forall i = 1, 2, 3, 4).
$$

By the Borel-Sudakov-T$$\text{t}$reilson inequality, it is easy to check that $\{M_{S_2} \geq u, M_{S_1} \geq u\}$ and $\{M_{S_1} \geq u, M_{S_3} \geq u, M_{S_4} \geq u\}$ have a negligible probability. Then,

$$
P(M_S \geq u) = \sum_{i=1}^{4} P(M_{S_i} \geq u) - P(M_{S_1} \geq u, M_{S_2} \geq u) - P(M_{S_2} \geq u, M_{S_3} \geq u) \\
- P(M_{S_3} \geq u, M_{S_4} \geq u) - P(M_{S_4} \geq u, M_{S_1} \geq u) + P(M_{S_1} \geq u, M_{S_2} \geq u, M_{S_3} \geq u) + o(u^{-1}\varphi(u))
$$

$$
= P(M_{S_1} \geq u, M_{S_2} \geq u) + P(M_{S_2} \geq u, M_{S_3} \geq u) + P(M_{S_3} \geq u, M_{S_4} \geq u)
$$

$$
+ P(M_{S_4} \geq u, M_{S_1} \geq u) - \sum_{i=1}^{4} P(M_{S_i} \geq u) - P(M_{S_1} \geq u, M_{S_3} \geq u)
$$

$$
+ P(M_{S_3} \geq u, M_{S_2} \geq u, M_{S_4} \geq u) + o(u^{-1}\varphi(u)).
$$

Now, using the SFH property in 1.) and 2.) and applying Lemma for two probabilities $P(M_{S_1} \geq u, M_{S_2} \geq u)$ and $P(M_{S_2} \geq u, M_{S_3} \geq u, M_{S_4} \geq u)$, we can deduce that

$$
P(M_S \geq u) = L_0(S) \frac{\varphi(u)}{2\sqrt{2\pi}} + L_1(S) \frac{\varphi(u)}{2\pi} + \sigma_2(S) \frac{u\varphi(u)}{2\pi} + o(u^{-1}\varphi(u)).
$$
An introducing example to understand the method

To introduce our method, we consider the simple case of a non-convex polygon as in Figure 4. \( S \) is decomposed into three polygons \( S_1, S_2 \) and \( S_3 \) as indicated in Figure 4. These polygons are convex so they satisfy the SFH.

To apply Lemma 2, it remains to compute the area of \( (S_1^+ \cap S_3^+) \) and \( (S_1^+ \cap S_2^+ \cap S_3^+) \).

Elementary geometry shows that \( (S_1^+ \cap S_3^+) \) consists of: two sections of disc with angle \( (\pi - \beta) \) and two quadrilaterals of area \( \epsilon^2 \tan(\beta/2) \); while in \( (S_1^+ \cap S_2^+ \cap S_3^+) \) one quadrilateral is replaced by a section of disc of angle \( \beta \), see Figure 5.

Thus

\[
\sigma_2(S_1^+ \cap S_3^+) = \left( (\pi - \beta) + 2 \tan\left(\frac{\beta}{2}\right) \right) \epsilon^2,
\]

\[
\sigma_2(S_1^+ \cap S_2^+ \cap S_3^+) = \left( (\pi - \beta) + \frac{\beta}{2} + \tan\left(\frac{\beta}{2}\right) \right) \epsilon^2.
\]

As a consequence,

\[
C_{123} - C_{13} = \frac{\beta}{2} - \tan\left(\frac{\beta}{2}\right).
\]

This quantity measures the non convexity of the concave binary point. An application of Lemma 2 shows that the coefficient of \( \Phi(u) \) is now \( 1 + \frac{\beta/2 - \tan(\beta/2)}{\pi} \).

Our main result is the following theorem.

**Theorem 1.** Let \( S \) be a compact domain of \( \mathbb{R}^2 \) with piecewise-\( C^2 \) boundary and with concave angles \( \beta_1, \ldots, \beta_m \). Let \( X(t) \) be a random field satisfying assumption A. Let \( M_S \) be the maximum of \( X(t) \) on \( S \). Then

\[
P(M_S \geq u) = \left[ \chi(S) + \frac{1}{\pi} \sum_{j=1}^k \left( \frac{\beta_j}{2} - \tan\left(\frac{\beta_j}{2}\right) \right) \right] \Phi(u) + \frac{\text{OMC}(S)}{2\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} u \varphi(u) + o(u^{-1} \varphi(u)),
\]

where \( \chi(S) \) is the Euler characteristic of \( S \).

In addition the outer Minkowski content \( \text{OMC}(S) \) is equal to the length of the non isolated edges plus twice the length of the isolated edges.
Proof. By using the classical inequalities as Borel-Sudakov-Tsirelson Theorem, it is easy to prove that if $S$ consists of several connected components then the tail of these components can be summed with an error which is $o(u^{-1} \varphi(u))$. So we can assume that $S$ is connected.

We will prove by induction on the number of concave points that $S$ satisfies the SFH.

Suppose that $S$ has no concave point. $S$ is whether a $C^2$ curve in $\mathbb{R}^2$ or $\partial \bar{S} = S$.

In the first case, using the parameterization of the unique edge, we see that $M_S$ is just the maximum of a smooth random process (with parameter of dimension 1). In that case, the result by Piterbarg, using Rice method for up-crossings see [10] shows that $S$ satisfies the SFH.

In the second case $S$ it has clearly a positive reach in the sense of Federer [9] and in that case,

$$\sigma_2(S^{++}) = \chi(S) \pi^2 + \text{OMC}(S) \epsilon + \sigma_2(S).$$

(8)

On the other hand from Theorem 8.12 of Azaïs and Wschebor [6], one can deduce the excursion probability (see Appendix for details).

The induction is based on a "destruction" of the concave points as in the introducing example.

Let $P$ be a concave point. There are four possibilities regarding $P$:

- **Concave binary point on the exterior boundary of $S$.** We decompose $S$ into three subsets $S_1$, $S_2$ and $S_3$ as in Figure 6. The decomposition is as follows: at $P$ we prolong inward the two tangents and construct to $C^2$ paths that avoid hole and touch the outside boundary and define $S_1$, $S_2$ and $S_3$ as in Figure 6. To apply Lemma 2 we set $S_4 = \emptyset$ and remark that to compute $\sigma_2(S_1^{++} \cap S_3^{++})$ and $\sigma_2(S_1^{++} \cap S_2^{++} \cap S_3^{++})$ we can replace, locally, with an error which is $O(\epsilon^3)$ the two portions of edges starting from $P$ by their tangent. In that case the computation is exactly the same as in the introducing example.

- **It is a concave binary point on the boundary of a hole inside $S$.** Using the two curves as above, we decompose $S$ into four subsets as follows: we also choose two regular points on the boundary of the hole, and two corresponding regular points on the exterior boundary of $S$ and construct two smooth curves that connect one regular point on the boundary of the hole with the corresponding one on the exterior boundary, and do not intersect themselves or two curves from the irregular point or additional holes. Then $S_1$, $S_2$, $S_3$, $S_4$ are constructed as Figure 7.

The proof is essentially the same as in the preceding case.

- **A concave ternary point.** We put $S_1$ as the isolated edge, $S_3$ as its complement, $S_2 = P$ and $S_4 = \emptyset$ as in Figure 8.

- **An angle point,** we do the same as in the concave ternary point case, see Figure 9.
Figure 7: Decomposition at a concave point on the interior boundary.

Figure 8: Decomposition at a concave ternary point.

Then the result follows. □

3 Examples and remarks

We give examples that are direct applications or direct generalizations of Theorem 1. All these results are new.

3.1 The angle

Let $S$ be the angle as in Figure 7. Then $S$ satisfies the SFH and

$$P(M_S \geq u) = \left(1 + \frac{\beta/2 - \tan(\beta/2)}{\pi}\right) \Phi(u) + \frac{\sigma_1(S_1) + \sigma_1(S_2)}{\sqrt{2\pi}} \varphi(u) + o(u^{-1} \varphi(u)).$$

3.2 The multi-angle

This is an extension of the "angle" case. Let $S$ be a self-avoiding curve that is union of $n + 1$ segments with the discontinuity of the angles $\{\beta_1, \ldots, \beta_k\}$. We have

$$P(M_S \geq u) = \left(1 + \frac{\sum_{i=1}^{k} (\beta_i - 2 \tan(\beta_i/2))}{2\pi}\right) \Phi(u) + \frac{\sigma_1(S)}{\sqrt{2\pi}} \varphi(u) + o(u^{-1} \varphi(u)).$$

3.3 The empty square

Let $S$ be the empty square, i.e. the boundary of a square in $\mathbb{R}^2$, then applying the Lemma three times, each time adding one more edge, becomes

$$P(M_S \geq u) = \frac{\pi - 4}{\pi} \Phi(u) + \frac{\sigma_1(S)}{\sqrt{2\pi}} \varphi(u) + o(u^{-1} \varphi(u)).$$

In conclusion, when $S$ is a union of some segments in a space of arbitrary dimension, we can give an exact asymptotic expansion with two terms corresponding to $\Phi(u)$ and $\varphi(u)$ from the tube formula of $S$ as in the above examples. More general, $S$ can be a union of non tangent curves.
3.4 The full square with whiskers

We consider "the square with whiskers" as in Figure 2. In this case, the domain has two concave ternary points. From the main theorem,

\[
P(M_S \geq u) = \frac{2\pi}{\pi} - \frac{4\pi}{\tilde{\Phi}(u)} + \frac{\text{OMC}(S)}{\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} u \varphi(u) + o(u^{-1} \varphi(u)).
\]

3.5 An apparent counter-example

In some strange cases, the condition (6) is not satisfied. This can happen when we consider two tangent curves, see Figure 10.

Here, \(S_1\) is a section of a circle of radius \(R\) and \(S_3\) is a tangent segment. We see that for \(\epsilon\) small enough, the area of the intersection between two tubes is

\[
\frac{\pi}{2} \epsilon^2 + \frac{(R + \epsilon)^2}{2} \arcsin \frac{2\sqrt{R\epsilon}}{R + \epsilon} - (R - \epsilon)\sqrt{R\epsilon} = \frac{\pi}{2} \epsilon^2 + \frac{8}{3} \sqrt{R\epsilon}^{3/2} + O(\epsilon^{5/2}).
\]

In the above equation, we use the fact that as \(x\) is small enough,

\[
\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1.3}{5} \frac{x^5}{5} + \ldots.
\]

It is clear that the order of the area of the intersection is not of 2 as in the condition (6), so we cannot apply the Lemma directly. However, with a careful examination in the proof of Lemma we can choose \(\alpha\) such that the difference between the upper bound and the lower one of the probability \(P(M_S \geq u, M_{S_1} \geq u)\) is "negligible". Thus, we have

\[
P(M_{S_1 \cup S_3} \geq u) = \frac{3\tilde{\Phi}(u)}{2} - \frac{8\sqrt{R}}{2^{7/3}3\pi} (7/4) u^{-1/2} \varphi(u) + \frac{\sigma_1(S_1) + \sigma_1(S_3)}{\sqrt{2\pi}} \varphi(u) + o(u^{-1} \varphi(u)).
\]  

This example is an apparent counter-example to the results of Adler and Taylor. More precisely, \(S\) is clearly a piecewise smooth locally convex manifold: it is easy to check that at the intersection of the circle and the straight line, the contact cone is limited to one direction thus convex. So if \(X(t)\) is sufficiently smooth, it seems that Theorem 14.3.3 of [1] implies the validity of the Euler characteristic heuristic and Theorem 12.4.2 of [1] gives an expansion of the Euler characteristic function should apply. This would be clearly in contradiction with the term \(u^{-1/2} \varphi(u)\) in [9].
In fact, there is no contradiction: Theorem 14.3.3 demands also the manifold to be regular in the sense of definition 9.22 of [1] and the present set is not a cone space in the sense of definition 8.3.1 of [1]. This shows that the local convexity itself is not sufficient.

3.6 Other domains in dimension 2

The result of Theorem 1 can be extended to more general domains for example domains with ternary points. For $\beta_1 + \beta_2 \geq \pi$ or domains with four intersecting edges but it is difficult to give a general simple formula as (7).

3.7 Some remarks and examples in dimension 3

The procedure that we have done in dimension 2 can be also used in dimension 3 or more. However, we do not obtain a full expansion. In fact, the coefficient of $\Phi(u)$ is not determined when $S$ is not locally convex. Here we give some examples.

- $S$ is a dihedral that is the union of two non coplanar rectangles $S_1$ and $S_2$ with a common edge, see Figure 11.

![Figure 11: Example of dihedral.](image)

Then, by using Lemma 2

$$P(M_S \geq u) = \frac{\sigma_1(\partial S_1) + \sigma_1(\partial S_2) - \sigma_1(S_1 \cap S_2)((\pi + \alpha)/2 + \cot(\alpha/2))/\pi \varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S_1) + \sigma_2(S_2)}{2\pi} u \varphi(u) + o(\varphi(u)).$$

- $S$ has the $L$–shape, see Figure 12.

![Figure 12: L-shape.](image)

Then, by decomposing $S$ into three subsets $S_1$, $S_2$ and $S_3$, we have

$$P(M_S \geq u) = \frac{\varphi(u)L_1(S)}{2\sqrt{2\pi}} + \frac{L_2(S)u\varphi(u)}{2\pi} + \frac{L_3(S)(u^2 - 1)\varphi(u)}{(2\pi)^{3/2}} + o(\varphi(u)), \quad (10)$$

where the coefficients $\{L_i(S), i = 1, \ldots, 3\}$ are given by the Steiner formula.
In a more complicated case, that is nonconvex trihedral, see Figure 13. We extend the planes of the non-convex trihedral so that they cut $S$ into smaller subsets with disjoint interiors. Then, we repeatedly use the inclusion-exclusion principle and Lemma 2 to obtain (10).

In general, by the same arguments and using the induction, when $S$ is a polytope,

$$P(M_S \geq u) = \frac{\varphi(u)L_1(S)\varphi(u)}{2\sqrt{2\pi}} + \frac{L_2(S)\varphi(u)}{2\pi} + \frac{L_3(S)(u^2 - 1)\varphi(u)}{(2\pi)^{3/2}} + o(\varphi(u)),$$

where

- $L_3(S)$ is the volume of $S$.
- $L_2(S)$ is one half of the surface area.
- To calculate $L_1(S)$, we consider two kinds of edge: convex and concave. Denote $\{(\alpha_i, l_{1i}), i = 1, \ldots, h\}$ by the set of couples of convex inside angle and the length of the corresponding edge and $\{(\beta_j, l_{2j}), j = 1, \ldots, k\}$ by the set of couples of concave inside angle and the length of the corresponding edge. Then,

$$L_1(S) = \sum_{i=1}^{h} \frac{(\pi - \alpha_i)}{2\pi} l_{1i} + \sum_{j=1}^{k} \frac{\cot(\beta_j/2)}{\pi} l_{2j}.$$

**Conclusion**

In all the examples considered, the Steiner formula for the tube governs the expansion of the tail of the maximum as if the excursion set were exactly a unique ball with random radius. We have found no counter-example to that principle and a conjecture is that the result is true for a much wider class of sets as those considered in this paper.

**4 Appendix**

We will prove that a compact connected domain in $\mathbb{R}^2$ with piecewise-$C^2$ boundary and without concave irregular point satisfies the SFH. Firstly the Steiner formula (8) has already been established. Now, we consider the excursion probability. We recall the following definitions

- Let $S_2$ be the interior of $S$; $S_1$ by the union of the $C^2$ edges and $S_0$ by the union of the convex irregular points.
- For $t \in S_j$, $X_j'(t)$ and $X_j''(t)$ are respectively the first and second derivatives of $X$ along $S_j$; $X_j'N(t)$ denotes the outward normal derivative.

In our case, it is easy to see that

$$\kappa(S) = \sup_{t \in S} \sup_{x \neq t} \frac{\text{dist}(s - t, C)}{\|s - t\|^2} < \infty.$$
In order to apply Theorem 8.12 and Corollary 8.13 of Azaïs and Wschebor[6], we have to check the conditions (A1) to (A5), page 185 in [6]. The first three ones are easy. Note that since the edges are of dimension 1, a direct proof of Rice formula can be performed without assuming that they are of class $C^3$ as in (A1).

- The condition (A4) states that the maximum is attained at a single point. It can be deduced from the Bulinskaya lemma (Proposition 6.11 in [6]) since for $s \neq t$, $(X(s), X(t), X'(s), X'(t))$ has a non-degenerate distribution.

- The condition (A5) that states that almost surely there is no point $t \in S$ such that $X'(t) = 0$ and $\det(X''(t)) = 0$, can be deduced from Proposition 6.5 in [6] applied to the process $X'(t)$ which is $C^2$.

Since all the required conditions are met, we have

$$\liminf_{u \to +\infty} -2u^{-2} \log \left[ \int_u^{\infty} p^E(x)dx - P\{M_S \geq u\} \right] \geq 1 + \inf_{t \in S} \frac{1}{\sigma_t^2 + \kappa_t^2} > 1,$$

(11)

where

- $p^E(x)$ is the approximation of the density of the maximum given by the Euler Characteristic method. More precisely

$$p^E(x) = \sum_{t \in S_0} E \left( \mathbb{1}_{x_0(t) \in \mathcal{C}_{t,0}} \mid X(t) = x \right) \varphi(x)$$

$$+ \sum_{j=1}^2 (-1)^j \int_{S_j} E \left( \det \left( X_j''(t) \right) \mathbb{1}_{x_j(t) \in \mathcal{C}_{t,j}} \mid X(t) = x, X_j'(t) = 0 \right) \frac{\varphi(x)}{(2\pi)^{1/2}} dt,$$

(12)

with $\mathcal{C}_{t,j}$ is the dual cone of the contact cone $C_t$,

$$\mathcal{C}_{t,j} = \{ z \in \mathbb{R}^2 : \langle z, x \rangle \geq 0, \forall x \in C_t \}.$$

- $$\sigma_t^2 = \sup_{s \in S(t)} \frac{\text{Var} \left( X(s) \mid X(t), X'(t) \right)}{(1 - \text{Cov}(X(s), X'(t)))^2}.$$

- $$\kappa_t = \sup_{s \in X(t)} \frac{\text{dist} \left( \frac{\sigma_t}{\text{Cov}(X(s), X(t)); C_t} \right)}{1 - \text{Cov}(X(s), X(t))}.$$

We compute $p^E(x)$ as follows:

- When $j = 2$, there is no normal space and $X_{2,N}'(t)$ makes no sense. It is easy to see that (see Azais and Wschebor [6] p. 244)

$$\int_{S_2} E \left( \det \left( X_2''(t) \right) \mid X(t) = x, X_2'(t) = 0 \right) dt = \sigma_2(S)(x^2 - 1).$$

- When $j = 0$, $X_{0,N}'(t) = X'(t)$ and

$$E \left( \mathbb{1}_{X'(t) \in \mathcal{C}_{t,0}} \mid X(t) = x \right) = \frac{\mathcal{A}(\mathcal{C}_t)}{2\pi},$$

where $\mathcal{A}(\mathcal{C}_{t,0})$ is the angle of the cone that is equal to the discontinuity of the angle of the tangent at the irregular point $t$.

- When $j = 1$, we consider a point $t$ on an edge $L$ on the exterior boundary. At this point, the second derivative along this curve can be expressed as

$$X_1''(t) = X_1''(t) + C(t)X_1',N(t).$$
where \( X''_T \) is the tangent projection and \( C(t) \) is the signed curvature at the point \( t \).

It is easy to check that the covariance function of the vector \((X''_T, X'_1, N, X, X'_1)\) is

\[
\begin{pmatrix}
\text{Var}(X''_T) & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Therefore, for such edge \( L \),

\[
\begin{align*}
\mathbb{E}\left( X''_T(t) \mathbb{I}_{X'_1(t) \in \mathcal{C}_t, t} \mid X(t) = x, X'_1(t) = 0 \right) &= \mathbb{E}\left( -x + C(t)X'_1(t) \right) \mathbb{I}_{X'_1(t) \in \mathcal{C}_t, t} \\
&= \frac{-x}{2} + \frac{C(t)}{\sqrt{2\pi}} \\
- \int_L \mathbb{E}\left( X''_T(t)X'_1(t) \mathbb{I}_{X'_1(t) \in \mathcal{C}_t, t} \mid X(t) = x, X'_1(t) = 0 \right) \frac{\varphi(x)}{\sqrt{2\pi}} dt &= \frac{\sigma_1(L)x}{2\sqrt{2\pi}} \varphi(x) - \frac{\varphi(x)}{2\pi} \int_L C(t) dt.
\end{align*}
\]

the quantity \(- \int_L C(t) dt\) can be viewed as the variation of the angle of the tangent from the beginning to the end of this edge.

Since we complete a whole turn in the positive orientation:

\[
\sum_{\text{irregular points of the exterior boundary}} \mathcal{A}(\hat{C}_t) + \sum_{\text{edges of the exterior boundary}} - \int_{L_1} C(t) dt = 2\pi.
\]

For a point \( t \) on an edge \( L_i \) of the interior boundary (holes), the interpretation of the second derivative changes into

\[
X''_1(t) = X''_T(t) - C(t)X'_1(t).
\]

Therefore,

\[
- \int_{L_i} \mathbb{E}\left( X''_T(t)X'_1(t) \mathbb{I}_{X'_1(t) \in \mathcal{C}_t, t} \mid X(t) = x, X'_1(t) = 0 \right) \frac{\varphi(x)}{\sqrt{2\pi}} dt &= \frac{\sigma_1(L_i)x}{2\sqrt{2\pi}} \varphi(x) + \frac{\varphi(x)}{2\pi} \int_{L_i} C(t) dt.
\]

For the boundary of a hole inside \( S \),

\[
\sum_{\text{irregular points}} \mathcal{A}(\hat{C}_t) + \sum_{\text{edges}} \int_{L_i} C(t) dt = -2\pi.
\]

In conclusion, substituting into (12),

\[
p^E(x) = \chi(S)\varphi(x) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} x\varphi(x) + \frac{\sigma_2(S)}{2\pi} (x^2 - 1)\varphi(x),
\]

and we obtain the asymptotic expansion

\[
P(M_S \geq u) = \chi(S)\Phi(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} u\varphi(u) + \frac{\sigma_2(S)}{2\pi} u^2\varphi(u) + \text{Rest},
\]

where \( \text{Rest} \) is super exponentially smaller in the sense of (11).

That implies the correspondence between the asymptotic expansion and the Steiner formula.
References

[1] R.J. Adler and J. Taylor, Random fields and Geometry, Springer, New York, 2007.

[2] L. Ambrosio, A. Colesanti and E. Villa, Outer Minkowski content for some classes of closed sets, Math. Ann. 342 (2008), no. 4, 727-748.

[3] J.M. Azaïs and C. Delmas, Asymptotic expansions for the distribution of the maximum of Gaussian random fields, Extremes 5 (2002), no. 2, 181-212.

[4] J.M. Azaïs and V.H. Pham, The record method for two and three dimensional parameters random fields, arXiv:1302.1017v1.

[5] J.M. Azaïs and M. Wschebor, A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail, Stochastic Process. Appl. 118 (2008), no. 7, 1190-1218.

[6] J.M. Azaïs and M. Wschebor, Level sets and extrema of random processes and fields, John Wiley and Sons, 2009.

[7] J.M. Azaïs and M. Wschebor, The tail of the maximum of smooth Gaussian fields on fractal sets, Journal of Theoretical Probability (2012).

[8] C. Delmas, Distribution du maximum d’un champ aléatoire et application statistiques, Ph.D. thesis, Université Paul Sabatier (2001).

[9] H. Federer, Curvature measures, Trans. Amer. Math. Soc. 93 (1959), 418-481.

[10] V.I. Piterbarg, Asymptotic methods in the theory of Gaussian processes and fields, Translated from the Russian by V. V. Piterbarg. Revised by the author. Translations of Mathematical Monographs, 148. American Mathematical Society, Providence, RI, 1996.

[11] V. I. Piterbarg, Rice’s Method for Large Excursions of Gaussian Random Fields, Technical Report No. 478, University of North Carolina. Translation of Rice’s method for Gaussian random fields.

[12] I. Rychlik, New bounds for the first passage, wave-length and amplitude densities, Stochastic Process. Appl. 34 (1990), no. 2, 313-339.

[13] J. Sun, Tail probabilities of the maxima of Gaussian random fields, Ann. Prob. 21 (1993), 34-71.

[14] A. Takemura and S. Kuriki, On the equivalence of the tube and Euler characteristic methods for the distribution of the maximum of Gaussian fields over piecewise smooth domains, Ann. Appl. Prob. 12 (2002), 768-796.

[15] A. Takemura and S. Kuriki, Tail probability via the tube formula when the critical radius is zero, Bernoulli 9 (2003), no. 3, 535-558.

[16] J. Taylor, A. Takemura and R.J. Adler, Validity of the expected Euler characteristic heuristic, Ann. Probab. 33 (2005), no. 4, 1362-1396.

jean-marc.azais@math.univ-toulouse.fr
viet-hung.pham@math.univ-toulouse.fr