Many worlds and modality in the interpretation of quantum mechanics: an algebraic approach

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Abstract

Many worlds interpretations (MWI) of quantum mechanics avoid the measurement problem by considering every term in the quantum superposition as actual. A seemingly opposed solution is proposed by modal interpretations (MI) which state that quantum mechanics does not provide an account of what ‘actually is the case’, but rather deals with what ‘might be the case’, i.e. with possibilities. In this paper we provide an algebraic framework which allows us to analyze in depth the modal aspects of MWI. Within our general formal scheme we also provide a formal comparison between MWI and MI, in particular, we provide a formal understanding of why —even though both interpretations share the same formal structure— MI fall prey of Kochen-Specker (KS) type contradictions while MWI escape them.

Key words: Contextuality, Kochen Specker, orthomodular lattices, modality, many worlds.

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1 INTRODUCTION: MANY WORLDS AND MODALITY

Today, almost 50 years after its birth in 1957, the many worlds interpretation (MWI) of quantum mechanics has become one of the most important lines of investigation within the many interpretations of quantum theory. MWI is considered to be a direct conclusion from Everett’s first proposal in terms of ‘relative states’ [1]. Everett’s idea was to let quantum mechanics find its own interpretation, making justice to the symmetries inherent in the Hilbert space formalism in a simple and convincing way [2]. In this paper we will not address the main argumentative lines of discussion raised for and against MWI (see for example [3, 4, 5]). Rather, we shall concentrate in its relation to the formal structure of quantum mechanics and provide an algebraic frame which will allow us to discuss the notion of logical possibility within it.

The main idea behind MWI is that superpositions refer to collections of worlds, in each of which exactly one value of an observable, which corresponds to one of the terms in the superposition, is realized. Apart from being simple, the claim is that it possesses a natural fit to the formalism, respecting its symmetries. This provides a solution to the measurement problem by assuming that each one of the terms in the superposition is actual in its own correspondent world. Thus, it is not only the single value which we see in ‘our world’ which gets actualized but rather, that a branching of worlds takes place in every measurement, giving rise to a multiplicity of worlds with their corresponding actual values. The possible splits of the worlds are determined by the laws of quantum mechanics.

Another proposed solution to the so called measurement problem has been developed in the frame of modal interpretations (MI) [6, 7, 8]. According to these interpretations “the quantum formalism does not tell us what actually is the case in the physical world, but rather provides us with a list of possibilities and their probabilities. The modal viewpoint is therefore that quantum theory is about what may be the case, in philosophical jargon, quantum theory is about modalities” [5]. Instead of actualizing every term in the superposition, MI claim that each term remains possible, evolving with the Schrödinger equation of motion.

Although MWI and MI share the same formal orthodox scheme, there are but few comparisons in the literature [5, 9]. In this paper we develop an algebraic framework which allows us to analyze and discuss the modal
aspects of MWI. Within this new formal account, we can also provide a rigorous comparison between MWI and MI. In particular, we can give a formal understanding of why MI fall prey of KS-type contradictions [10, 11] while MWI escape them.

In Section 2, we introduce basic notions about lattice theory that will be necessary later. In section 3, we provide a general discussion on contextuality and modality in quantum mechanics. In section 4, we develop a new algebraic frame for MWI. In section 5, we formally compare MWI to MI.

2 BASIC NOTIONS

Now we recall from [12] and [13] some notions of lattice theory that will play an important role in what follows. Let $L$ be a lattice and $a, b \in L$. We say that $b$ covers $a$ iff $a < b$ and moreover there exists no $x \in L$ such that $a < x < b$ for any $x$. Suppose that $L$ is a bounded lattice with $0$ the minimum element and $1$ the maximum element. An element $p \in L$ is called an atom iff $p$ covers $0$ and a coatom iff $1$ covers $p$. $L$ is said to be an atomistic lattice iff for each $x \in L - \{0\}$, $x = \bigvee \{p \leq x : p \text{ is an atom}\}$. An element $c \in L$ is said to be a complement of $a$ iff $a \land c = 0$ and $a \lor c = 1$.

Let $L = \langle L, \lor, \land, 0, 1 \rangle$ be a bounded lattice. Given $a, b, c$ in $L$, we write: $(a, b, c)D$ iff $(a \lor b) \land c = (a \land c) \lor (b \land c)$; $(a, b, c)D^*$ iff $(a \land b) \lor c = (a \lor c) \land (b \lor c)$ and $(a, b, c)T$ iff $(a, b, c)D$, $(a, b, c)D^*$ hold for all permutations of $a, b, c$. An element $z$ of a lattice $L$ is called central iff for all elements $a, b \in L$ we have $(a, b, z)T$ and $z$ is complemented. We denote by $Z(L)$ the set of all central elements of $L$ and it is called the center of $L$.

A lattice with involution [14] is an algebra $\langle L, \lor, \land, \neg \rangle$ such that $\langle L, \lor, \land \rangle$ is a lattice and $\neg$ is a unary operation on $L$ that fulfills the following conditions: $\neg \neg x = x$ and $\neg (x \lor y) = \neg x \land \neg y$. An orthomodular lattice is an algebra $\langle L, \land, \lor, \neg, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$ that satisfies the following conditions

1. $\langle L, \land, \lor, \neg, 0, 1 \rangle$ is a bounded lattice with involution,
2. $x \land \neg x = 0$.
3. $x \lor (\neg x \land (x \lor y)) = x \lor y$

We denote by $O\mathcal{ML}$ the variety of orthomodular lattices. It is well known that if $\mathcal{H}$ is a Hilbert space then $L(\mathcal{H})$, the lattice of closed subspaces
of \( \mathcal{H} \), is an atomistic orthomodular lattice. Boolean algebras are orthomodular lattices satisfying the distributive law \( x \land (y \lor z) = (x \land y) \lor (x \land z) \). We denote by \( 2 \) the Boolean algebra of two elements. If \( \mathcal{L} \) is a bounded lattice then \( Z(\mathcal{L}) \) is a Boolean sublattice of \( \mathcal{L} \) [13, Theorem 4.15].

Let \( A \) be a Boolean algebra. A subset \( F \) of \( A \) is called a filter iff it satisfies: if \( a \in F \) and \( a \leq x \) then \( x \in F \) and if \( a, b \in F \) then \( a \land b \in F \). \( F \) is a proper filter iff \( F \neq A \) or, equivalently, \( 0 \notin F \). If \( X \subseteq A \), the filter \( F_X \) generated by \( X \) is the minimum filter containing \( X \). It is well known that \( F_X = \{ x \in A : \exists x_1 \cdots x_n \in X \text{ with } x_1 \land \cdots \land x_n \leq x \} \). Each filter \( F \) in \( A \) determines univocally a congruence in which the equivalence classes are given by \( [x] = \{ y \in A : \neg x \lor y \in F \text{ and } x \lor \neg y \in F \} \). In this case the quotient set \( A/\sim \), noted as \( A/F \), is a Boolean algebra and the natural application \( x \mapsto [x] \) is a Boolean homomorphism from \( A \) to \( A/F \). A proper filter \( F \) is maximal iff the quotient algebra \( A/F \) is isomorphic to \( 2 \). It is well known that each proper filter can be extended to a maximal one. A very important property associated with maximal filters is the following: suppose that \( x \nleq y \). Then there exists a maximal filter \( F \) in \( A \) such that \( x \in F \) and \( y \notin F \). We will refer to this result as the maximal filter theorem.

3 CONTEXTUALITY AND MODALITY IN QUANTUM SYSTEMS

In the usual terms of quantum logic [15, 16], a property of a system is related to a subspace of the Hilbert space \( \mathcal{H} \) of its (pure) states or, analogously, to the projector operator onto that subspace. A physical magnitude \( \mathcal{M} \) is represented by an operator \( \mathbf{M} \) acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition \( \mathbf{M} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i><a_i| \) are satisfied. The real numbers \( a_i \) are related to the outcomes of measurements of the magnitude \( \mathcal{M} \) and projectors \( |a_i><a_i| \) to the mentioned properties. Thus, the physical properties of the system are organized in the lattice of closed subspaces \( \mathcal{L}(\mathcal{H}) \). Moreover, each self-adjoint operator \( \mathbf{M} \) has associated a Boolean sublattice \( W_\mathbf{M} \) of \( L(\mathcal{H}) \) which we will refer to as the spectral algebra of the operator \( \mathbf{M} \).

Assigning values to a physical quantity \( \mathcal{M} \) is equivalent to establishing a Boolean homomorphism \( v : W_\mathbf{M} \to 2 \). Thus, we can say that it makes sense to use the “classical discourse” —this is, the classical logical laws are valid— within the context given by \( \mathcal{M} \).

One may define a global valuation of the physical magnitudes over \( \mathcal{L}(\mathcal{H}) \)
as a family of Boolean homomorphisms \((v_i : W_i \to \mathcal{2})_{i \in I}\) such that 
\(v_i | W_i \cap W_j = v_j | W_i \cap W_j\) for each \(i, j \in I\), being \((W_i)_{i \in I}\) the family of Boolean sublattices of \(\mathcal{L}(\mathcal{H})\). This global valuation would give the values of all magnitudes at the same time maintaining a compatibility condition in the sense that whenever two magnitudes shear one or more projectors, the values assigned to those projectors are the same from every context. As we have proved in [17], the KS theorem in the algebraic terms of the previous definition rules out this possibility:

\[\textbf{Theorem 3.1} \quad \text{If } \mathcal{H} \text{ is a Hilbert space such that } \dim(\mathcal{H}) > 2, \text{ then a global valuation over } \mathcal{L}(\mathcal{H}) \text{ is not possible.} \]

This impossibility to assign values to the properties at the same time satisfying compatibility conditions is a weighty obstacle for the interpretation of the formalism.

We have introduced elsewhere [18, 19] a general modal scheme which extends the expressive power of the orthomodular structure to provide a rigorous framework for the Born rule and mainly, to discuss the restrictions posed by the KS theorem to possible properties. We recall here some notions that will be useful in our development.

First, we enriched the orthomodular structure with a modal operator taking into account the following considerations:

1. Propositions about the properties of the physical system are interpreted in the orthomodular lattice of closed subspaces of \(\mathcal{H}\). Thus, we retain this structure in our extension.

2. Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones (Born rule). In other words, one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition \(P\), if we refer with \(\Diamond P\) to the possibility of \(P\), then \(\Diamond P\) will be a central element of a orthomodular structure.

3. If \(P\) is a proposition about the system and \(P\) occurs, then it is trivially possible that \(P\) occurs. This is expressed as \(P \leq \Diamond P\).
4. Assuming an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shall the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible one. This is interpreted in the following way: if \( P \) is a property of the system, \( \Diamond P \) is the smallest central element greater than \( P \).

From consideration 1, it follows that the original orthomodular structure is maintained. The other considerations are satisfied if we consider a modal operator \( \Diamond \) over an orthomodular lattice \( L \) defined as

\[
\Diamond a = \text{Min}\{z \in Z(L) : a \leq z\}
\]

with \( Z(L) \) the center of \( L \). When this minimum exists for each \( a \in L \) we say that \( L \) is a *Boolean saturated orthomodular lattice*. On each Boolean saturated orthomodular lattice we can define the necessity operator as a unary operation \( \Box \) given by \( \Box x = \neg \Diamond \neg x \). We have shown that this enriched orthomodular structure can be axiomatized by equations conforming a variety denoted by \( \mathcal{OML}^\Diamond \) [18]. More precisely, each element of \( \mathcal{OML}^\Diamond \) is an algebra \( \langle L, \land, \lor, \neg, \Box, 0, 1 \rangle \) of type \( (2,2,1,1,0,0) \) satisfying the following equations:

\[
\begin{align*}
\text{S1 } & \Box x \leq x & \text{S5 } & y = (y \land \Box x) \lor (y \land \neg \Box x) \\
\text{S2 } & \Box 1 = 1 & \text{S6 } & \Box(x \lor \Box y) = \Box x \lor \Box y \\
\text{S3 } & \Box \Box x = \Box x & \text{S7 } & \Box(\neg x \lor (y \land x)) \leq \neg \Box x \lor \Box y \\
\text{S4 } & \Box(x \land y) = \Box(x) \land \Box(y)
\end{align*}
\]

Orthomodular complete lattices are examples of Boolean saturated orthomodular lattices and we can embed each orthomodular lattice \( L \) in an element \( L^\Diamond \in \mathcal{OML}^\Diamond \). In general, \( L^\Diamond \) is referred as a *modal extension of \( L \)*. In this case we may see the lattice \( L \) as a subset of \( L^\Diamond \) (see [18]).

**Definition 3.2** Let \( L \) be an orthomodular lattice and \( L^\Diamond \in \mathcal{OML}^\Diamond \) be a modal extension of \( L \). We define the *possibility space* of \( L \) in \( L^\Diamond \) as

\[
\Diamond L = \{\Diamond p : p \in L\}_{L^\Diamond}
\]
The possibility space represents the modal content added to the discourse about properties of the system.

**Proposition 3.3** [18, Proposition 14] Let \( \mathcal{L} \) be an orthomodular lattice, \( W \) a Boolean sublattice of \( \mathcal{L} \) and \( \mathcal{L}^\diamond \in OML^\diamond \) a modal extension of \( \mathcal{L} \). Then \( \langle W \cup \Diamond \mathcal{L} \rangle_{\mathcal{L}^\diamond} \) is a Boolean sublattice of \( \mathcal{L}^\diamond \). In particular \( \Diamond \mathcal{L} \) is a Boolean sublattice of \( Z(\mathcal{L}^\diamond) \). \( \square \)

Now, we develop the algebraic counterpart of the classical notion of consequence which will be useful when formalizing the concept of possibility in MWI. As will become clear below, Proposition 3.3 allows to establish a deep relation between this concept and the possibility space.

**Definition 3.4** Let \( \mathcal{L} \) be an orthomodular lattice, \( p \in \mathcal{L} \) and \( \mathcal{L}^\diamond \in OML^\diamond \) a modal extension of \( \mathcal{L} \). Then \( x \in \mathcal{L}^\diamond \) is said to be a classical consequence of \( p \) iff for each Boolean sublattice \( W \) in \( \mathcal{L} \) (with \( p \in W \)) and each Boolean valuation \( v : W \to 2 \), \( v(x) = 1 \) whenever \( v(p) = 1 \). We denote by \( Cons_{\mathcal{L}^\diamond}(p) \) the set of classical consequences of \( \mathcal{L} \).

**Proposition 3.5** Let \( \mathcal{L} \) be an orthomodular lattice, \( p \in \mathcal{L} \) and \( \mathcal{L}^\diamond \in OML^\diamond \) a modal extension of \( \mathcal{L} \). Then we have that \( Cons_{\mathcal{L}^\diamond}(p) = \{ x \in \mathcal{L}^\diamond : p \leq x \} = \{ x \in \mathcal{L}^\diamond : \Diamond p \leq x \} \)

**Proof:** By definition of \( \Diamond \) it is clear that \( \{ x \in \mathcal{L}^\diamond : p \leq x \} = \{ x \in \mathcal{L} : \Diamond p \leq x \} \) and the inclusion \( \{ x \in \mathcal{L}^\diamond : \Diamond p \leq x \} \subseteq Cons_{\mathcal{L}^\diamond}(p) \) is trivial. Let \( x \in Cons_{\mathcal{L}^\diamond}(p) \). Assume that \( p \not\leq x \). Consider the Boolean subalgebra of \( \mathcal{L} \) given by \( W = \{ p, \neg p, 0, 1 \} \). By Proposition 3.3, \( W^\diamond = \langle W \cup \Diamond \mathcal{L} \rangle_{\mathcal{L}^\diamond} \) is a Boolean sublattice of \( \mathcal{L}^\diamond \). By the maximal filter theorem, there exists a maximal filter \( F \) in \( W^\diamond \) such that \( p \in F \) and \( x \notin F \). If we consider the quotient Boolean algebra \( W^\diamond / F \) and the natural Boolean homomorphism \( f : W^\diamond \to W^\diamond / F = 2 \), then \( f(p) = 1 \) and \( f(x) = 0 \), which is a contradiction. \( \square \)

Let \( \mathcal{L} \) be an orthomodular lattice, \( (W_i)_{i \in I} \) the family of Boolean sublattices of \( \mathcal{L} \) and \( \mathcal{L}^\diamond \) a modal extension of \( \mathcal{L} \). If \( f : \Diamond \mathcal{L} \to 2 \) is a Boolean homomorphism, an actualization compatible with \( f \) is a global valuation \( (v_i : W_i \to 2)_{i \in I} \) such that \( v_i \mid W_i \cap \Diamond = f \mid W_i \cap \Diamond \mathcal{L} \) for each \( i \in I \). Compatible actualizations represent the passage from possibility to actuality.

**Theorem 3.6** [18, Theorem 19] Let \( \mathcal{L} \) be an orthomodular lattice. Then \( \mathcal{L} \) admits a global valuation iff for each possibility space there exists a Boolean homomorphism \( f : \Diamond \mathcal{L} \to 2 \) that admits a compatible actualization. \( \square \)
The addition of modalities to the discourse about the properties of a quantum system enlarges its expressive power. At first sight it may be thought that this could help to circumvent contextuality, allowing to refer to physical properties belonging to the system in an objective way that resembles the classical picture. Since the possibility space is a Boolean algebra, there exists a Boolean valuation of the possible properties. But in view of the last theorem, a global actualization that would correspond to a family of compatible valuations is prohibited. Thus, the theorem states that the contextual character is maintained even when the discourse is enriched with modalities.

4 AN ALGEBRAIC FRAME FOR MANY WORLDS

In the MWI, all possibilities encoded in the wave function take place, but in different worlds. More precisely: let $\mathcal{M}$ be a physical magnitude represented by an operator $\mathbf{M}$ with spectral decomposition $\mathbf{M} = \sum_i a_i \mathbf{P}_i$. If a measurement of $\mathbf{M}$ is performed and $a_1$ occurs, then in another world $a_2$ occurs, and in some other world $a_3$ occurs, etc. Let us now see how we can introduce our modal algebraic frame for MWI.

Let $\mathcal{H}$ be a Hilbert space and suppose that $\mathbf{M}$ has associated a Boolean sublattice $W_\mathbf{M}$ of $L(\mathcal{H})$. The family $(\mathbf{P}_i)_i$ is identified as elements of $W_\mathbf{M}$. If a measurement is performed and its result is $a_i$, this means that we can establish a Boolean homomorphism

$$v_i : W_\mathbf{M} \rightarrow 2 \quad s.t. \quad v_i(\mathbf{P}_i) = 1$$

4.1 $\text{OML}^\diamond$-CONSEQUENCES

In a possible world where $v_i(\mathbf{P}_i) = 1$ we will have classical consequences. We can take an arbitrary modal extension $\mathcal{L}^\diamond$ of $\mathcal{L}(\mathcal{H})$ and consider the set $Cons_{\mathcal{L}^\diamond}(\mathbf{P}_i)$. The modal extension does not depend on the valuation over the family $(\mathbf{P}_i)_i$. Thus, it is clear that the modal extension is independent of any possible world. Modal extensions are simple algebraic extensions of an orthomodular structure. By Proposition 3.5 we have that $Cons_{\mathcal{L}^\diamond}(\mathbf{P}_i) = \{x \in \Diamond \mathcal{L}(\mathcal{H}) : \Diamond \mathbf{P}_i \leq x\}$. Thus, for any arbitrary modal extension $\mathcal{L}^\diamond$ of $\mathcal{L}(\mathcal{H})$ in terms of classical consequences, the classical consequences of $v_i(\mathbf{P}_i) = 1$ are exactly the same ones as $\Diamond \mathbf{P}_i$ (independently of any possible splitting). In terms of classical consequences which refer to a property $\mathbf{P}_i$,
it is the same to consider the classical consequences in the possible world
where \( v_i(P_i) = 1 \), than to study the classical consequences of \( \Diamond P_i \) before
the splitting.

MWI maintains that in each respective \( i \)-world, \( v_i(P_i) = 1 \) for each \( i \). Thus, a family of valuations \( (v_i(P_i) = 1)_i \) may be simultaneously con-
sidered, each member being realized in each different \( i \)-world. From an
algebraic perspective, this would be equivalent to have a family of pairs
\( \langle L(H), v_i(P_i) = 1 \rangle_i \), each pair being the orthomodular structure \( L(H) \) with
a distinguished Boolean valuation \( v_i \) over a spectral sub-algebra containing
\( P_i \) such that \( v_i(P_i) = 1 \). In what follows, we will show that the \( \text{OML}^\diamond \)
structure is able to capture this fact in terms of classical consequences. For
this purpose, the following proposition is needed.

\textbf{Proposition 4.1} Let \( H \) be Hilbert space such that \( \dim(H) > 2 \) and \( a, b \) be
a two distinct atoms in \( L(H) \). If we consider a modal extension \( \text{L} \diamond \) of \( L(H) \),
then \( \Diamond(a) = \Diamond(b) \).

\textbf{Proof:} We first note that there exists a coatom \( c \) such that \( c \) is not com-
parable with \( a \) and \( b \). In fact, let \( (c_i)_{i \in I} \) be the family of coatoms of
\( L(H) \) and suppose that \( a \leq c_i \) and \( b \leq c_i \) for each \( i \in I \). We then get
\[ a \lor b \leq \bigcap_{i \in I} c_i = 0, \] which is a contradiction. Since \( a \) and \( b \) are atoms
and \( c \) is a coatom not comparable with \( a \) and \( b \) then \( 0 = a \land c = b \land c \) and
\[ 1 = a \lor c = b \lor c. \] Hence \( c \) is a common complement of \( a, b \). Since
\( L(H) \hookrightarrow \text{L} \diamond \) is an \( \text{OML}^\diamond \)-embedding, \( c \) is a common complement of \( a, b \) in
\( \text{L} \diamond \). We first note \( \neg c = a \land c \leq c \), then \( a \land \neg c \leq a \land c = 0 \). Since
\( \neg c \) is a central element, \( a \leq \Diamond c \) and \( \Diamond a \leq \Diamond c \). Since \( a \lor c = 1 \) then
\[ \neg a \land c = 0. \] Therefore \( a \land c \leq \neg a \land \neg c = 0. \] Since \( a \) is central
element then \( c \leq \Diamond a \) and \( \Diamond c \leq \Diamond a \). With the same argument we can
prove that \( \Diamond(b) = \Diamond(c) \).

\( \square \)

The following theorem is crucial in order to relate MWI with modality
in terms of valuations and classical consequences.

\textbf{Theorem 4.2} Let \( H \) be Hilbert space such that \( \dim(H) > 2 \) and \( (p_i)_{i \in I} \) be
a family of elements of \( L(H) \). If we consider a modal extension \( L(H) \hookrightarrow \text{L} \diamond \)
then there exists a Boolean homomorphism \( v: \Diamond L \rightarrow 2 \) such that \( v(\Diamond p_i) = 1 \)
for each \( i \in I \).

\textbf{Proof:} Since \( L(H) \) is an atomic lattice, for each \( p_i \) there exists an atom
\( a_i \) such that \( a_i \leq p_i \). Let \( I_0 \) be a finite subfamily of \( I \). By Proposition 4.1,
we have that $0 < \bigwedge_{i \in I_0} \diamond (a_i) \leq \bigwedge_{i \in I_0} \diamond (p_i)$. Therefore the family $(\diamond p_i)_{i \in I}$ generates a proper filter $F$ in the Boolean algebra $\diamond \mathcal{L}$. Extending $F$ to a maximal filter $F_M$, the natural Boolean homomorphism $v : \diamond \mathcal{L} \rightarrow 2$ satisfies that for each $i \in I$, $v(\diamond p_i) = 1$.

While MWI considers a family of pairs $\langle \mathcal{L}(\mathcal{H}), v_i(P_i) = 1 \rangle_i$ for each possible i-world and the classical consequences of $v_i(P_i) = 1$ in the i-world, the $\mathcal{OMCL}^\diamond$ structure, by Proposition 3.5, considers classical consequences of each $v_i(P_i) = 1$ coexisting simultaneously in one and the same structure, what is possible in view of Theorem 4.2. More precisely, as a valuation $v : \diamond \mathcal{L} \rightarrow 2$ exists such that $v(\diamond P_i) = 1$ for each $i$, each element $x \in \diamond \mathcal{L}$ such that $P_i \leq x$ necessarily satisfies $v(x) = 1$.

4.2 MANY WORLDS AND KOCHEN-SPECKER TYPE THEOREMS

KS theorem does not impose conditions on both the family of valuations $v_i(P_i) = 1$, considered as a family of pairs $\langle \mathcal{L}(\mathcal{H}), v_i(P_i) = 1 \rangle_i$ in MWI nor on the Boolean valuation $v : \diamond \mathcal{L}(\mathcal{H}) \rightarrow 2$ satisfying $v(\diamond P_i) = 1$ for each $i$ in the $\mathcal{OMCL}^\diamond$ structure (Theorem 4.2). In fact, by Theorem 3.6 KS only prevents from extending the valuation $v : \diamond \mathcal{L}(\mathcal{H}) \rightarrow 2$ to $\mathcal{L}(\mathcal{H})$ in a compatible manner. In the wording previous to Theorem 3.6, KS theorem prohibits to pass from the realm of possibility to that of actuality in the sense that it precludes to establish a compatible actualization for $v : \diamond \mathcal{L}(\mathcal{H}) \rightarrow 2$ to $\mathcal{L}(\mathcal{H})$ when $\dim(\mathcal{H}) > 2$ (see Theorem 3.1).

In its algebraic version given in Theorem 3.1, KS only imposes a limit to the possibility of establishing compatible valuations over $\mathcal{L}(\mathcal{H})$ when $\dim(\mathcal{H}) > 2$ but does not cause incompatibilities when reference is made to possible global valuations in the realm of possibilities considered in the $\mathcal{OMCL}^\diamond$ structure or in the family $\langle \mathcal{L}(\mathcal{H}), v_i(P_i) = 1 \rangle_i$ from MWI. Thus, valuations over different i-worlds are admitted.

5 CONCLUSIONS

In this paper we have analyzed the orthomodular formal structure of quantum mechanics in relation to both MWI and MI. In order to deal with logical possibility in these interpretations, we considered two different algebraic approaches which were characterized in Sec. 3 and 4. In the case of MWI, the structure is the family of pairs $\langle \mathcal{L}(\mathcal{H}), v_i(P_i) = 1 \rangle_i$ of orthomodular
There exists a Boolean valuation \( v : \Diamond(\mathcal{L}(\mathcal{H})) \rightarrow 2 \) such that \( v(\Diamond P_i) = 1 \) for each \( i \).

Families of Boolean valuations \( (v_i(P_i) = 1)_i \), may be simultaneously considered, each member being realized in each different \( i \)-world.

KS theorem precludes to establish compatible actualizations for \( v : \Diamond(\mathcal{L}(\mathcal{H})) \rightarrow 2 \) to \( \mathcal{L}(\mathcal{H}) \). does not cause incompatibility when each member of a family of valuations \( (v_i(P_i) = 1)_i \) is considered.

### Table 1: Modality and MWI under \( OML^\circ \)-structure

| Modality | MWI |
|----------|-----|
| Valuations | Families of Boolean valuations \( (v_i(P_i) = 1)_i \), may be simultaneously considered, each member being realized in each different \( i \)-world. |
| KS theorem | does not cause incompatibility when each member of a family of valuations \( (v_i(P_i) = 1)_i \) is considered. |

Lattices with a distinguished Boolean valuation that assigns “true” to a projector of a spectral algebra in each one of them. For MI, we have the Boolean saturated orthomodular lattice. Both structures allows us to compare the role of contextuality in relation to the formal account of actual and possible properties in a rigorous way as it is shown in Table 1.

The modal scheme we developed in [18], i.e. the Boolean saturated orthomodular lattice, is also adequate to consider the notion of possibility within MWI. The whole set of possible worlds, each one with an actualized value of a property, is algebraically equivalent to the set of valuations to “true” of the possible properties in \( \mathcal{L}^\circ \). That is to say, the actualization of each value of a given property in each \( i \)-world is analogous to the assignment of the value “true” to all possible properties in the scheme of MI.

We have shown that the KS theorem only imposes a limit to the possibility of establishing compatible valuations over \( \mathcal{L}(\mathcal{H}) \). However, there is no incompatibility when reference is only made to valuations in the realm of possibilities, i.e., in the \( OML^\circ \) structure for MI or in the family \( (\mathcal{L}(\mathcal{H}), v_i(P_i) = 1)_i \) for MWI.

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