MULTIVARIATE DIAGONAL COINVARIANT SPACES FOR COMPLEX REFLECTION GROUPS

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Abstract. For finite complex reflection groups, we consider the graded $W$-modules of diagonally harmonic polynomials in $r$ sets of variables, and show that associated Hilbert series may be described in a global manner, independent of the value of $r$.

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1. Introduction

For finite complex reflection groups $W = G(m, p, n)$, we study the diagonal coinvariant space $C_W(r)$ for $W$, in several (say $r$) sets of $n$ variables. Here, the use of the term diagonal refers to the fact that $W$ is considered as a diagonal subgroup of $W^r$, acting on the $r^{th}$-tensor power $R_n(r)$ of the symmetric algebra of the defining representation of $W$. The space considered, $C_W(r)$, is simply the quotient of $R_n(r)$ by the ideal generated by constant-term-free (diagonal) $W$-invariants. We shall see that the associated multigraded Hilbert series, denoted $C_W(r)(q_1, \ldots, q_r)$ (which is symmetric in the variables $q := (q_1, \ldots, q_r)$, can be described in an uniform manner as a positive coefficient linear combination of Schur polynomials

$$C_W(r)(q) = \sum_\mu c_\mu s_\mu(q), \quad (1)$$

with the $c_\mu$ independent of $r$, and $\mu$ running through a finite set of integer partitions that depend only on the group $W$. This expression has the striking feature that it gives one global formula that specializes to the dimension of $C_W(r)$, for each individual $r$. To better see what is striking here, it is worth recalling that, although the case $r = 1$ has a long history [4, 13, 14], it is only recently that the special case $r = 2$ has been somewhat settled [3, 6, 7, 10]. However much still needs to be done along these lines as discussed.

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in [8]. Some headway has recently been made in the case \( r = 3 \) (see [2, 12]), but the general case is still wide open.

## 2. Definitions and Discussion

For a rank \( n \) complex reflection group \( W \), we may consider its “diagonal” action on the \( \mathbb{N}^r \)-graded space \( \mathcal{R}_n^{(r)} := \mathbb{C}[X] \), of polynomials in the \( r \times n \) variables

\[
X = \begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{r1} & x_{r2} & \cdots & x_{rn}
\end{pmatrix}.
\]

We consider each row of \( X \) as a set of \( n \) variables. Thinking of elements \( w \in W \) as \( n \times n \) matrices, the action of \( w \) is simply the map sending \( f(X) \) to \( f(Xw) \). Naturally, \( W \)-invariant polynomials in \( \mathbb{C}[X] \) are those that fixed by any element of \( W \), i.e.: 

\[
f(X \cdot w) = f(X), \quad \text{for all } w \in W.
\]

The **diagonal coinvariant space** \( \mathcal{C}_W^{(r)} \) is defined to be the quotient

\[
\mathcal{C}_W^{(r)} := \mathcal{R}_n^{(r)}/\mathcal{J}_W^{(r)},
\]

where \( \mathcal{J}_W^{(r)} \) is the ideal generated by constant-term-free \( W \)-invariant polynomials in \( \mathcal{R}_n^{(r)} \).

For an integer \( r \times n \) matrix \( A = (a_{ij}) \), we denote by \( X^A \) the monomial

\[
X^A := X_1^{A_1} X_2^{A_2} \cdots X_n^{A_n},
\]

where \( X_j \) and \( A_j \) respectively stand for the \( j^{th} \) column of \( X \) and \( A \), and where

\[
X_j^{A_j} := \prod_{i=1}^r x_{ij}^{a_{ij}}.
\]

The ideal \( \mathcal{J}_W \) is homogeneous with respect to the (vector) **degree**

\[
\deg(X^A) := \sum_{j=1}^n A_j,
\]

and the **total degree**, \( \text{tdeg}(X^A) \), is the sum of the components of \( \deg(X^A) \in \mathbb{N}^r \). The diagonal \( W \)-action is clearly degree preserving.

The space \( \mathcal{C}_W^{(r)} \) may be turned into a polynomial representation of \( GL_r \), using the fact that the usual action of \( GL_r \) on \( \mathcal{R}_n^{(r)} \) (that sends \( f(X) \) to \( f(MX) \), when \( M \in GL_r \)) commutes with the \( W \)-action. Hence the \( GL_r \)-action preserves the ideal \( \mathcal{J}_W^{(r)} \). Recall that the **\( GL_r \)-character** of \( \mathcal{C}_W^{(r)} \) is the symmetric polynomial \( \mathcal{C}_W^{(r)}(q) \), in the variables \( q \), obtained by taking the trace of the linear transform

\[
f(X) \mapsto f(QX),
\]
where $Q$ is the diagonal matrix $[q_1, \ldots, q_n]$. Since characters of irreducible polynomial $GL_r$-modules correspond precisely to the Schur polynomials $\sigma_{\mu}(q)$, we get an expansion of form (1) for $C_W^{(r)}(q)$, with the $c_{\mu}$ giving the multiplicity of the representation having character $\sigma_{\mu}(q)$. Moreover, the ideal $\mathcal{I}_W$ being homogeneous, the variable $q_i$ serves as a “degree counter” for the variables $x_{ij}, 1 \leq j \leq n$, so that we may also consider $C_W^{(r)}(q)$ as the Hilbert series of $C_W^{(r)}$.

**Harmonic polynomials.** Some properties of $C_W^{(r)}$ are better formulated in the context of the isomorphic space $\mathcal{H}_W^{(r)}$, of “diagonally harmonic polynomials”. In fact, we think of this new space as canonical representatives for elements of $C_W^{(r)}$. More precisely, for each of the variables $x_{ij} \in X$, consider the partial derivation denoted by $\partial x_{ij}$. Given polynomial $f(X)$, we denote by $f(\partial X)$ the differential operator obtained by replacing the variables in $X$ by the corresponding derivation. The space $\mathcal{H}_W^{(r)}$ of diagonally harmonic polynomials is the set of polynomial solutions, $g(X)$, of the system of partial differential equations

$$f(\partial X)(g(X)) = 0, \quad (5)$$

with one equation for each $f(X) \in \mathcal{I}_W$. Evidently, we need only consider a generating set of $\mathcal{I}_W$ for these equations to characterize all solutions. An elementary proof (see [1]) that $C_W^{(r)}$ and $\mathcal{H}_W^{(r)}$ are isomorphic relies on the fact that $\mathcal{H}_W^{(r)}$ appears as the orthogonal complement of $\mathcal{I}_W$ for the scalar product

$$\langle f(X), g(X) \rangle = f(\partial X)g(X)|_{X=0}, \quad (6)$$

where $p(X)|_{X=0} := p(0)$ is the constant term of $p(X)$. It is easy to see that the set of monomials forms an orthogonal basis for this scalar product, and hence deduce that it is $W$-invariant. One also checks readily that

**Lemma 2.1.** $\mathcal{H}_W^{(r)}$ is the orthogonal complement of $\mathcal{I}_W^{(r)}$:

$$\mathcal{H}_W^{(r)} = \left( \mathcal{I}_W^{(r)} \right) ^{\perp}. \quad (7)$$

**Proof.** Indeed, let $g(X)$ be in $\mathcal{I}_W^{(r)}$, and consider the leading term (for any suitable term order) of $f(\partial X)g(X) = cX^A + \ldots$, if any, for some $f(X)$ in $\mathcal{I}_W$. Since $X^A f(X)$ also lies in $\mathcal{I}_W$, we must have $\langle X^A f_i(X), g(X) \rangle = 0$, but this means precisely that $c = 0$. Hence we must have $f(\partial X)G(X) = 0$. □

The case $r = 1$, i.e.: $\mathcal{H}_W^{(1)}$, gives rise to the classical theorems of [13, 15] regarding $W$-harmonic polynomials and the coinvariant space for $W$. The case $r = 2$, for the symmetric group $W = S_n$, corresponds to the space of diagonal harmonic polynomials of Haiman-Garsia, which is of dimension $(n + 1)^{n-1}$. Its alternating part has bigraded Hilbert series given by the now famous $q,t$-Catalan polynomials $C_{n+1}(q,t)$. Again for $r = 2$, the case of other reflection groups has also been studied for other groups $W$ (see [6] for instance).
Any \( f(X) \) lying in \( \mathcal{H}_W^{(r-1)} \) also lies in \( \mathcal{H}_W^{(r)} \), so that we have

\[
\mathcal{H}_W^{(1)} \subseteq \mathcal{H}_W^{(2)} \subseteq \ldots \subseteq \mathcal{H}_W^{(r)} \subseteq \ldots .
\]

In particular,

\[
\mathcal{H}_W^{(r-1)}(q_1, \ldots, q_{r-1}) = \mathcal{H}_W^{(r)}(q_1, \ldots, q_{r-1}, 0) .
\]

It is easy to adapt an argument of \[5\], for the case \( r = 2 \), to show that the total degree of elements of \( \mathcal{H}_W^{(r)} \) is bounded by the maximal total degree of an element of \( \mathcal{H}_W^{(1)} \) (which is well known to be the degree of the Jacobian of \( W \)). Hence, only a finite number of Schur functions may appear in the expansions of the \( \mathcal{H}_W^{(r)}(q), r \geq 1 \). This implies that \( \mathcal{H}_W^{(r)}(q) \) affords an expansion in the \( s_\mu(q_1, \ldots, q_r) \), with \( \mu \) independent of \( r \). For this reason, we say that this expansion is \textit{universal}, and we drop the \( "(r)" \) superscript and write simply \( \mathcal{H}_W(q) \).

Theorem 2.1. For any rank \( n \) complex reflection group \( W = G(p, 1, n) \), the Hilbert series of the diagonal coinvariant space in \( r \) sets of variables affords an expansion, in terms of Schur functions \( s_\mu(q_1, \ldots, q_r) \), with positive integer coefficients \( c_\mu \) that are independent of \( r \), the sum being over the set of partitions \( \mu \) of integers \( d \):

\[
0 \leq d \leq \frac{n(r n + r - 2)}{2} ,
\]

and having at most \( n \) parts.

In many of the cases considered here, the graded Hilbert series of \( \mathcal{H}_W \) seems to take the form

\[
\mathcal{H}_W(q) = \sum_\mu a_\mu h_\mu(q) ,
\]

with the sum being over a finite set of partitions, and the \( a_\mu \) positive integers. When this last property holds, one says that \( \mathcal{H}_W(q) \) is \textit{h-positive}. In the case of the symmetric group (at least), writing \( \mathcal{H}_n \) for \( \mathcal{H}_{S_n} \), an even finer \( h \)-positivity phenomenon seems to occur. It involves the decomposition into irreducibles of the homogeneous components of \( \mathcal{H}_n \). This is all encompassed into the \textit{graded Frobenius characteristic} of \( \mathcal{H}_n \):

\[
\mathcal{H}_n(w; q) := \sum_{d \in \mathbb{N}^3} q^d F_{\mathcal{H}_{n,d}}(w) .
\]

Recall that the Frobenius characteristic \( F_\mathcal{V}(w) \), of a \( S_n \)-module \( \mathcal{V} \), is the symmetric function (in auxiliary variables \( w = w_1, w_2, \ldots \)) whose expansion in terms of the Schur functions \( S_\lambda(w) \) records the multiplicity of irreducibles in \( \mathcal{V} \). This is to say that we have

\[
F_\mathcal{V}(w) = \sum_{\lambda \vdash n} b_\lambda S_\lambda(w) ,
\]

with the sum being over partitions of \( n \), and \( b_\lambda \) giving the multiplicity of the irreducible representation classified by \( S_\lambda(w) \). Our previous arguments show that there is a universal expansion of \( \mathcal{H}_n(w; q) \) in terms of the \( S_\lambda(w) \), having Schur positive coefficients in the \( s_\mu(q) \). This is to say that
Theorem 2.2.

\[ H_n(w; q) = \sum_{\lambda \vdash n} \left( \sum_{\mu} b_{\lambda,\mu} s_{\mu}(q) \right) S_{\lambda}(w), \quad n_{\lambda,\mu} \in \mathbb{N}. \quad (11) \]

where the \( n_{\lambda,\mu} \) are independent of \( r \), with \( \mu \) running over partitions of \( d \) \((0 \leq d \leq \binom{n}{2})\), having at most \( n \) parts.

We underline that there are two kinds of Schur function at play here. Those in the \( q \)-variables (denoted by a lower case “s” and with the \( q \)-variables dropped), that account for graded multiplicities, and those in the \( w \)-variables, that account for the decomposition into \( S_n \)-irreducibles. For \( n \) up to 5, the expansion of \( H_n(w; q) \) in terms of the monomial symmetric functions \( m_{\lambda}(w) \) is \( h \)-positive in the \( q \) variables. In formula,

\[ H_n(w; q) = \sum_{\lambda \vdash n} \left( \sum_{\mu} a_{\lambda,\mu} h_{\mu}(q) \right) m_{\lambda}(w), \quad (12) \]

with \( a_{\lambda,\mu} \in \mathbb{N} \) independent of \( r \). For example, we have

\[ H_3(w; q) = m_3(w) + (1 + h_1 + h_2) m_{21}(w) + (1 + 2 h_1 + h_2 + h_3 + h_{11}) m_{111}(w), \quad (13) \]

once again with the \( q \)-variables dropped. Since the coefficient of \( m_{11\cdots1}(w) \) is the Hilbert series of the underlying representation, the \( h \)-positivity of \((12)\) implies that \( H_n(q) \) is also \( h \)-positive, on top of being universal. Hence, in such cases the Hilbert series of \( H_n \) would have to take the form

\[ H_n(q) = \sum_{\sigma \in S_n} h_{\mu(\sigma)}(q), \quad (14) \]

with \( \mu(\sigma) \) some partition of the number of inversions of \( \sigma \). Direct calculations give the universal expansions

\[
H_1(q) = 1, \\
H_2(q) = 1 + h_1, \\
H_3(q) = 1 + 2 h_1 + h_2 + h_{11} + h_3, \\
H_4(q) = 1 + 3 h_1 + 2 h_2 + 3 h_{11} + 2 h_3 + 3 h_{21} + h_{111}, \\
\quad + h_4 + 4 h_{31} + 2 h_5 + h_{41} + h_6, \\
H_5(q) = 1 + 4 h_1 + 3 h_2 + 6 h_{11} + 3 h_3 + 8 h_{21} + 4 h_{111} \\
\quad + 2 h_4 + 9 h_{31} + 2 h_{22} + 6 h_{211} + h_{1111} \\
\quad + 3 h_5 + 4 h_{41} + 5 h_{32} + 10 h_{311} \\
\quad + h_6 + 9 h_{51} + h_{42} + 5 h_{411} + 4 h_{33} \\
\quad + 2 h_7 + 9 h_{61} + 2 h_{52} + h_{511} + h_{43} \\
\quad + 4 h_8 + 4 h_{71} + h_{62} + 3 h_9 + h_{81} + h_{71},
\]

with \( h_{\overline{11}} \) being indexed by a one part partition. From the above data, we might expect (as was conjectured in first drafts of this paper) that we always have \( h \)-positivity, but
this fails to hold in general. Indeed\footnote{As recently calculated by M. Haiman.}, the degree 9 term of the Hilbert series $\mathcal{H}_6(q)$ is

$$
-h_9 + 18h_1h_8 + 2h_2h_7 + 17h_3h_6 + 7h_4h_5 + 28h_1^2h_7 + 12h_1h_2h_6 + 5h_1h_3h_5 + h_4^2 + h_1^3h_6.
$$

Still, as discussed in section 4, low degree terms of $\mathcal{H}_n(w, q)$ are $h$-positive.

Specializing the universal formula (2.2), we get the following polynomial formulas (in the parameter $r$) for the dimension of the spaces $\mathcal{H}_r^{(r)}$:

\begin{align*}
\dim \mathcal{H}_1 &= 1 \\
\dim \mathcal{H}_2 &= 1 + r \\
\dim \mathcal{H}_3 &= (1 + r)^2 + \binom{r+1}{2} + \binom{r+2}{3} \\
\dim \mathcal{H}_4 &= (1 + r)^3 + 2\binom{r+1}{2} + 3r\binom{r+1}{2} + 2\binom{r+2}{3} \\
&\quad + 4r\binom{r+2}{3} + \binom{r+3}{4} + r\binom{r+3}{4} + 2\binom{r+4}{5} + \binom{r+5}{6}
\end{align*}

(16)

In particular, at $r = 1$ these expressions evaluate to $n!$, and at $r = 2$ they evaluate to $(n + 1)^{n-1}$. In [2], we discuss the apparent fact that, at $r = 3$, formulas (16) should further specialize to $2^n(n + 1)^{n-2}$. There is apparently no such nice formula for $r > 3$.

To better see how we may verify the above formulas, let us observe that we have the following basis for $\mathcal{H}_2$

$$
\{1, x_{12} - x_{11}, x_{22} - x_{21}, \ldots, x_{r2} - x_{r1}\}.
$$

Taking into account the action of $S_2$, we can then easily calculate that

\begin{align*}
\mathcal{H}_2(w; q) &= S_2(w) + (q_1 + q_2 + \ldots + q_r) S_{11}(w) \\
&= m_2(w) + (1 + h_1) m_{11}(w),
\end{align*}

(17)

(18)

as well as

$$
\mathcal{H}_2(q) = 1 + q_1 + q_2 + \ldots + q_r.
$$
At it appears to be the number of intervals in the Tamari Lattice (see [19]), where, once again, we write $\lambda = \sum_{i \geq 0} w_i q^i$ may also specialize to 1 all the expansions. Similar explicit calculations (with some help from the computer) give the universal expansions

\[
\begin{align*}
\mathcal{H}_3(w; q) &= S_3(w) + (s_2 + s_1) \cdot S_{21}(w) + (s_3 + s_{11}) \cdot S_{111}(w), \\
\mathcal{H}_4(w; q) &= S_4(w) + (s_3 + s_2 + s_1) \cdot S_{31}(w) \\
&\quad + (s_4 + s_{21} + s_2) \cdot S_{22}(w) \\
&\quad + (s_5 + s_4 + s_{31} + s_3 + s_{21} + s_{11}) \cdot S_{211}(w) \\
&\quad + (s_6 + s_{41} + s_{31} + s_{111}) \cdot S_{1111}(w), \\
\mathcal{H}_5(w; q) &= S_5(w) + (s_4 + s_3 + s_2 + s_1) S_{41}(w) \\
&\quad + (s_6 + s_5 + s_{41} + s_4 + s_{31} + s_{22} + s_3 + s_{21} + s_2) S_{32}(w) \\
&\quad + (s_7 + s_6 + s_{51} + 2 s_5 + s_{32} + s_{41} + s_4 + 2 s_{31} + s_3 + s_{21} + s_{11}) S_{311}(w) \\
&\quad + (s_8 + s_7 + s_6 + s_{51} + s_{42} + s_5 + 2 s_41 + s_{32} + s_{31} + s_{22} + s_{21} + s_{11}) S_{221}(w) \\
&\quad + (s_9 + s_8 + s_7 + s_6 + s_{51} + s_{42} + s_6 + 2 s_{51} + s_{42} + s_{41} + s_{32} + s_{31} + s_{31} + s_{31} + s_{21} + s_{111}) S_{2111}(w) \\
&\quad + (s_{10} + s_9 + s_8 + s_7 + s_6 + s_6 + s_{51} + s_{51} + s_{43} + s_{42} + s_{41} + s_{31} + s_{1111}) S_{11111}(w),
\end{align*}
\]

where, once again, we write $s_{10}$ to make clear that the index is a one part partition. We may also specialize to 1 all the $q_i$ in $\mathcal{H}_n(w; q)$, using the well known evaluation

\[
s_\mu(1^k) = s_\mu(1 \cdots 1) = \prod_{(i,j) \in \mu} \frac{r + j - 1}{h_{ij}(\mu)},
\]

with $h_{ij}(\mu)$ denoting the hook length associated to the cell $(i,j)$ in the diagram of $\mu$. The resulting expressions have coefficients that are polynomials in $r$. For example,

\[
\begin{align*}
\mathcal{H}_2(w; 1^r) &= S_2(w) + r S_{11}(w), \\
\mathcal{H}_3(w; 1^r) &= S_3(w) + \frac{1}{2} r (r + 3) S_{21}(w) + \frac{1}{6} r (r^2 + 6 r - 1) S_{111}(w), \\
\mathcal{H}_4(w; 1^r) &= S_4(w) + \frac{1}{6} r (r^3 + 6 r + 11) S_{31}(w) \\
&\quad + \frac{1}{24} r (r + 1) (r^2 + 3 r + 10) S_{22}(w), \\
&\quad + \frac{1}{120} r (r + 3) (r^3 + 27 r^2 + 74 r - 12) S_{211}(w), \\
&\quad + \frac{1}{720} r (r^4 + 39 r^4 + 295 r^4 + 645 r^2 - 296 r + 36) S_{1111}(w),
\end{align*}
\]

At $r = 2$ the coefficient of $S_{11\cdots 1}$, in $\mathcal{H}_n(w; 1^r)$, is the $n^{th}$ Catalan number, and at $r = 3$ it appears to be the number of intervals in the Tamari Lattice (see [2]).

We may also specialize formulas (19) by setting $q_1 = t$, and $q_i = 0$ for $i \geq 2$. From classical results on the coinvariant space for $S_n$, as well as typical calculations on symmetric
functions, we get
\[ H_n(w; t) = \sum_{\lambda \vdash n} \left[ \begin{array}{c} n \\ \lambda \end{array} \right] m_\lambda(w), \tag{22} \]
where
\[ \left[ \begin{array}{c} n \\ \lambda \end{array} \right] := \frac{n!}{\lambda_1! \cdots \lambda_k!}, \]
with
\[ n! := (1 + t)(1 + t^2) \cdots (1 + t + \ldots + t^{n-1}). \]
In view of results in [9], we must also have that
\[ H_n(w; q, t) = \nabla(e_{n+1}(w)) \tag{23} \]
where \( \nabla \) is an operator on symmetric functions, having Macdonald symmetric functions as eigenfunctions. This imposes further constraints on the form of Formula (2.2).

Similar situations are settled by the following for two infinite families of groups.

**Theorem 2.3.** For the dihedral groups \( I_2(m) = G(m, m, 2) \), the cyclic groups \( C_m = G(m, 1, 1) \), and the groups \( G(m, 1, 2) \), we have the respective universal \( h \)-positive Hilbert series
\[ H_{C_m}(q) = \sum_{j=0}^{n} h_j(q), \tag{24} \]
\[ H_{I_2(m)}(q) = 1 + 2 h_1(q) + h_{11}(q) + h_2(q) + 2 \sum_{j=3}^{m-1} h_j(q) + h_m(q), \tag{25} \]
\[ H_{G(m,1,2)}(q) = \left( \sum_{k=0}^{m-1} h_k \right)^2 + \sum_{k=0}^{m-1} (k+1) h_{m+k} + \sum_{k=1}^{m-1} (m-k) h_{2m-1+k}. \tag{26} \]

3. Proofs

**Proof of Theorem 2.1 and 2.2.** In both theorems, it remains only to show that the \( q \)-variables Schur polynomials involved are indexed by partitions having at most \( n \)-parts, since every such Schur polynomial occurs as the character of an irreducible \( GL_r \)-sub-representation of the space \( \mathcal{R}_n^{(r)} \). Indeed, this property holds the Hilbert series (or \( GL_r \)-character) of the space \( \mathcal{R}_n^{(r)} \) which is given by the following formula
\[ \mathcal{R}_n^{(r)}(q) = h_n[H(q)], \tag{27} \]
where
\[ H(q) = \sum_{k=0}^{\infty} h_k(q) = \prod_{i=1}^{r} \frac{1}{1-q_i}. \tag{28} \]

\[ \text{For more on this, see [1].} \]
In Equation (27), we use a plethystic substitution notation. This means that, in order to calculate the right-hand side, we simply expand every symmetric functions in terms of power sum symmetric functions $p_k = p_k(q)$ and apply the following rules:

- $p_k[c] = c$ if $c$ is a constant;
- $p_k[p_j] = p_{k-j}$;
- $F[f + g] = F[f] + F[g]$, and $(F + G)[f] = F[f] + G[f]$;
- $F[f \cdot g] = F[f] \cdot F[g]$, and $(F \cdot G)[f] = F[f] \cdot G[f]$.

For the purpose of such calculations, $q$ is identified with the sum $q_1 + \ldots + q_r = p_1$. Using classical calculations on symmetric functions (see [11], we may then check that the only Schur polynomials that occur in the expansion of (27) are indexed by partitions having at most $n$ parts.

Proof of (14). To prove Proposition ??, we observe that, for $r = 1$, $\mathcal{H}_W(q)$ specializes to the well known Poincaré polynomial of $W$, in the variable $t = q_1$. Thus we must have

$$\mathcal{H}_W(t) := \sum_{w \in W} t^{\ell(w)} = \prod_{i=1}^n \frac{t^{d_i} - 1}{t - 1},$$

with the $d_i$'s standing for the degrees of the group $W$, and $\ell(w)$ is the length function. Perforce, if the universal Hilbert series $\mathcal{H}_W(q)$ is $h$-positive, such as in (10), then the positive integers $a_\mu$ must be such that

$$\sum_\mu a_\mu t^{|\mu|} = \sum_{w \in W} t^{\ell(w)}.$$

Indeed, the evaluation of $h_\mu$ in one variable $t$ is exactly $t^{|\mu|}$, where $|\mu|$ stands for the sum of the parts of $\mu$. Moreover, for the symmetric group, the “length” function $\ell(\sigma)$ corresponds to the number of inversion in $\sigma$.

Proof of Theorem 2.3. Each of the formula is obtained by constructing an explicit basis of the associated space. The cyclic group $C_m = G(m, 1, 1)$ case is almost immediate, since there is but one variable in each of the $r$-sets. The ring of diagonal $C_m$-invariants is easily seen to be spanned by the set of monomials of total degree $km$, in the $r$-variables $x_i = x_{i1}$, with $k \in N$. It follows that the associated diagonal harmonics are all monomials of degree at most $m - 1$, hence the formula.

For both the Dihedral groups and $G(m, 1, 2)$, each set of variables contains two variables, which we denote by $x_i$ and $y_i$. Just as for the cyclic group case, the construction of an explicit basis of $\mathcal{H}_W$ is relatively easy. We illustrate for the dihedral case. By polarization (see [15]), we get that the ring of diagonal invariant of $I_2(m)$ is generated by the polynomials appearing as coefficients (themselves polynomials in the $x, y$-variables) of the polynomials in the $t_i$-variables:

$$\left(\sum_{i=1}^r x_i t_i\right)^m + \left(\sum_{i=1}^r y_i t_i\right)^m, \quad \text{and} \quad \left(\sum_{i=1}^r x_i t_i\right) \cdot \left(\sum_{i=1}^r y_i t_i\right).$$
It is easy to describe explicitly all the polynomial solutions of the resulting partial differential equations.

Now, expanded in terms of Schur polynomials, Formula (25) takes the form

$$H_{I_2(m)}(q) = 1 + s_{11} + s_m + 2 \sum_{k=1}^{m-1} s_k.$$ 

(30)

We need only identify the highest weight vectors in $H_{I_2(m)}$ associated to each term of this last expansion. All of the terms having one index are easy to identify, they correspond to classical harmonic polynomials for $I_2(m)$. The only term of exception is $s_{11}$. It is readily seen to account for the irreducible component spanned by the polynomials in the set

$$\{x_iy_j - x_jy_i \mid 1 \leq i < j \leq m\}.$$  

One then checks directly that all diagonal harmonics are accounted in this manner. □

4. Low degree components

Even tough we restrict the discussion in this section to the case $W = S_n$, much of it holds in generality. Our intent here is to compare the spaces $R_n$ and $\mathcal{H}_n \otimes R_n^{S_n}$, where $R_n^{S_n}$ is the ring of diagonal $S_n$-invariants in $r$ sets of variables. In the case $r = 1$, it is well known that these two spaces are isomorphic as graded $S_n$-modules (see [4]). In other words, $R_n$ is a free module over the ring of symmetric polynomials. This immediately implies that we have the explicit formula for

$$\mathcal{H}_n(w; q, 0, 0, \ldots) = h_n(w(1 - q)^{-1}) \prod_{k=1}^{n} (1 - q^k).$$ 

(31)

For $r \geq 2$, the space $R_n$ is not a free module over the ring of diagonally symmetric polynomials. However, we do have an isomorphism between the low degree homogeneous components of $R_n$ and $\mathcal{H}_n \otimes R_n^{S_n}$. This would be made more precise if one could calculate explicitly a free resolution of the quotient involved. This is the subject of future work. For the time being, let us only observe that this leads to polynomial expressions in $n$ for the low degree coefficients of the Frobenius characteristic $\mathcal{H}_n(w; q)$ and the Hilbert series $\mathcal{H}_n(q) = \mathcal{H}_n(q)$. These are derived using the following explicit expressions for $R_n(w; q)$ and $R_n^{S_n}(q)$:

$$R_n(w; q) = h_n[wH(q)], \quad \text{and} \quad R_n^{S_n}(q) = h_n[H(q)].$$ 

(32)

We thus get an approximation

$$\mathcal{H}_n(w; q) \simeq_n \frac{h_n[wH(q)]}{h_n[H(q)]},$$ 

(33)

that experimentally seems to hold for all terms of $q$-degree less or equal to $n$. Equivalently, using one of the Cauchy identities (expressing the fact that $\{h_\lambda\}_\lambda$ and $\{m_\lambda\}_\lambda$ are dual bases):

$$h_n[xy] = \sum_{\lambda \vdash n} h_\lambda(x)m_\lambda(y),$$ 

(34)
the right-hand side of (33) may be expanded as
\[ \sum_{\lambda \vdash n} \frac{h_\lambda[H(q)]}{h_n[H(q)]} m_\lambda(w). \] (35)

In particular, we may approximate the Hilbert series of \( H_n \) by the expansion
\[ \frac{H(q)^n}{h_n[H(q)]} = 1 + (n-1)h_1 + (n-2)h_2 + \binom{n-1}{2} h_1^2 + \ldots \] (36)

Observe here that the right-hand side of (36) gives explicit expressions, as polynomials
in \( n \), for the coefficient of \( h_\mu \) in \( H_n(q) \). These hold for \( n \geq 3 \). Observe that, using
the other Cauchy identity
\[ h_n[xy] = \sum_{\lambda \vdash n} s_\lambda(x)s_\lambda(y), \] (37)

we may get a similar formula, describing the small degree isotopic \( S_n \)-components of \( H_n \),
in the form
\[ H_n(w; q) \simeq \sum_{\lambda \vdash n} \frac{s_\lambda[H(q)]}{h_n[H(q)]} S_\lambda(w). \] (38)

In this way, we may calculate the following order 4 approximation
\[ H_4(w; q) \simeq S_4(w) + (s_1 + s_2 + s_3) S_31(w) + (s_2 + s_21 + s_4) S_{22}(w) + (s_4 + s_31 + s_3 + s_21 + s_{11}) S_{211}(w) + (s_31 + s_{111}) S_{1111}(w), \] (39)
in which we miss only (see (19)) the terms \( s_5 S_{211}(w) + (s_{41} + s_6) S_{1111}(w) \). For sure,
many more terms will be missing for larger \( n \), since the maximal degree is \( \binom{n}{2} \).

5. Concluding remarks and thanks

The proof of Theorems 2.2 may readily be adapted to show that all isotypic components
of the space \( H_W \) afford a universal Schur-positive expansion, for all finite complex
reflection group \( W \).

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