The lamb shift

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Abstract. In this paper I have returned to the coordinate representation of Coulomb propagator and wave function significantly simplified thanks to spiral choice of Dirac matrices (see Appendix). The series in $Z\alpha$ appeared as hypergeometric functions, division to soft and hard photons and their sewing became superfluous etc. The nonrelativistic part of shift is shown completely due to contribution of only one partial wave $P_{1/2}$ of propagator.

To memory of my friends.

1. Introduction
The Lamb shift of a given energetic level is usually written as decomposition in powers of $Z\alpha$ [1]

$$\Delta E = \frac{\alpha}{\pi} m \sum_n C_n (Z\alpha)^n. \quad (1)$$

At present we know three first coefficients $C_4$, $C_5$, $C_6$ along with corrections to them of various order in $\alpha/\pi$ [2].

The necessity of this decomposition has a deep cause – apart of nice expression $G = (\hat{p} - \hat{V} - m)^{-1}$ there is no other formulation for the electron propagator in momentum representation [3]. Thus, $Z\alpha$-decomposition is necessary.

The usual exit from this situation is to divide contributions of soft and hard photons, following with appropriate sewing of them.

I have returned to the coordinate representation of propagator because it has a well-known partial amplitudes which were significantly simplified by me (see Appendix). As a conclusion all the series in $Z\alpha$ were appeared as hypergeometric functions and main calculation was reduced to integration over energy of intermediate electron. I have considered only 1-loop contribution.

The paper is built as following: in 1-st part the angular and radial integrals are calculated. In 2-nd part I integrate over electron energy. In Conclusion the results are collected. In Appendix the wave function and electron propagator are obtained.

I. Angular and Radial integrals
The calculation of energy shift will be doing for the main state $1S_{1/2}$ of hydrogen atom, whose wave function

$$\Psi_0 = Ne^{-\lambda r} r^{\gamma_0-1} \Phi(\vec{\alpha})$$

has energy $E_0 = m\gamma_0$, parameters $\lambda = Z\alpha m$, $\gamma_0 = \sqrt{1 - (Z\alpha)^2}$, $N^2 = \frac{(2\lambda)^{2\gamma_0+1}}{\Gamma(2\gamma_0+1)}$. 

Expression of energy shift in 1-loop approximation is as following:
\[
\Delta E = \frac{e}{\pi} \int d\vec{r}d\vec{r}' \Psi^* \mathcal{Y}_0(\vec{r}) \gamma_\mu G(\vec{r}, \vec{r}'|E) \gamma_\nu \Psi_0(\vec{r}') D_{\mu\nu}(\vec{r} - \vec{r}') \omega
\]  
(3)
Along with \( \Psi_0(\vec{r}) \) and \( G(\vec{r}, \vec{r}'|E) \) (both are given in Appendix) it contains the photon propagator
\[
D_{\mu\nu}(\vec{R}|\omega) = \frac{\delta_{\mu\nu}}{2\pi^2} \int \omega d\omega \sin qR \frac{qR}{\omega} ,
\]
(4)
having a retardation factor \( \sin qR = 1 - \frac{q^2 R^2}{6} + \cdots \). It depends over all the variables, that considerably complicates the computations. We omit the second additive because it is of order \((\Delta a)^2\) and return later along with other corrections of same order.

Inserting all these in \( \Delta E \) we obtain a combination
\[
\Delta E = \frac{e^2}{\pi} \int \omega d\omega \sum_j R_j \mathcal{O}_j 
\]
(5)
of a radial \( R_j \) \( = C \mathcal{N}^2 \int (rr')^2 d\vec{r}d\vec{r}' \Psi^*_0(\vec{r}) G(r, r') \Psi_0(\vec{r}') \) and spin-angular integral \( \mathcal{O}_j = \int d\Omega d\Omega' \bar{\gamma}_\mu \mathcal{T}_\gamma(\vec{n}, \vec{n}') \gamma_\mu u \), which must be integrated over photon (or electron) energy.

2. Angular integral
This integral contains a matrix element \( u(\vec{n}) \gamma_\mu \mathcal{T}_\gamma(\vec{n}, \vec{n}') \gamma_\mu u(\vec{n}') \). Inserting the Tamm matrix \( \mathcal{T}_\gamma = \frac{1}{2 \chi_{\gamma\omega}} \left( e^{i\omega T_3} \begin{array}{cc} T_1 & T_2 \\ T_2 & T_3 \end{array} \right) \) with its «edging» \( \gamma_\mu \mathcal{T}_\gamma \gamma_\mu \), we receive
\[
\gamma_\mu \mathcal{T}_\gamma \gamma_\mu = \frac{-4}{\chi_{\gamma\omega}} \left( e^{-i\omega T_4} ; 0 _\gamma \right) \left( 0 ; e^{i\omega T_1} _\gamma \right).
\]
(6)
The point is that «edging» nullified all the elements containing \( \bar{\sigma} \) (i.e. \( \gamma_\mu \bar{\sigma} \gamma_\mu = 0 \)) so that non-diagonal elements are disappeared. So, the computation of \( \bar{\gamma}_\mu \mathcal{T}_\gamma \gamma_\mu u \) is reduced to taking the trace
\[
Sp \left( a T_4 ; 0 _\gamma \right) \left( 1 + \gamma_0 \right) \left( 1 - \gamma_0 \right).
\]
where \( a = e^{-i\omega}, b = e^{i\omega} \). The polynomials \( t_{1,4} \) are similar to \( T_{1,4} \) with \( \vec{n} \) and \( \vec{n}' \) exchanged. The trace is equal to
\[
\text{ASp} T_{4} t_{1} + \text{BSp} T_{4} t_{4} = \frac{1}{(4\pi)^2} P_{1} \left[ (2j + 2) A P_{j+1/2} + 2j B P_{j-1/2} \right],
\]
(8)
\( A, B = a, b(1 \pm \gamma_0) \).
(8a)
After angular integration we obtain (A and B are given up to \((ZA)^4\))
\[
\mathcal{O}_j = \frac{(Za)^2}{3} \left[ \mu_j^2 + 4 \delta_j, j, 3/2 \right].
\]
(9)

3. Radial integral
Radial integral is defined as
\[
\mathcal{R}_j = N^2 \int_0^\infty dr dr' (rr')^2 e^{-\lambda (r+r')} (rr')^{\gamma_\nu - 1} \mathcal{M}_{\nu\mu}(\rho_{<\nu}) \mathcal{W}_{\nu\mu}(\rho_{>\nu}) (rr')^{-1}.
\]
(10)
Asymmetry of this expression in \( \rho_{\nu} \)s may be removed by combining the Whittaker functions. According to Hostler [4], we have
\[
C_{\gamma} \mathcal{M}_{\nu\mu}(\rho_{<\nu}) \mathcal{W}_{\nu\mu}(\rho_{>\nu}) = \sqrt{rr'} \int_x^\infty \frac{dx}{\sqrt{x^2 - 1}} \left( \frac{x+1}{x-1} \right)^y e^{ik(r+r')} r I_{2\mu}(\mu x) \mu (2\mu + 1 | X),
\]
and radial integration becomes trivial
\[
\int_0^\infty dr dr' e^{-A(r+r')} (rr')^{\gamma_\nu + 1/2} I_{2\mu}(x) = \frac{\Gamma^2(\beta)}{\Gamma(2\mu + 1)} \frac{[-k^2(x^2 - 1)]_\mu}{A^2} F(\beta, \beta; 2\mu + 1 | X),
\]
\[
A = \lambda - ik x, \quad \beta = \mu + \gamma_\nu + \frac{3}{2}, \quad X = -k^2(x^2 - 1) 
\]
(11)
and (12)
The computation ends with transformation of parametric integral
\[
\mathcal{R}_j = N^2 \frac{\Gamma^2(\beta)}{\Gamma(2\mu + 1)} \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \left( \frac{x+1}{x-1} \right)^y [-k^2(x^2 - 1)]^{y+1/2} A^{-2\beta} F(\beta, \beta; 2\mu + 1 | X)
\]
(13)
to following form
\[
\mathcal{R}_j = C \int_1^\infty dx \frac{x^j}{(x+1)^{y+v}(x-1)^{y-v}(x+\eta)^{-2\beta} F\left[\beta, \beta; 2\gamma + 2; \frac{x^2-1}{(x+\eta)^2}\right]},
\]

Here
\[
C = \frac{(2\eta)^{2\gamma v + 3}}{(2\xi)^2 \Gamma(2\gamma v + 1) \Gamma(2\mu + 1)}; \quad \eta = \frac{\lambda}{-\mu}; \quad \beta = \gamma + \gamma_0 + 2
\]

II. Separation of the nonrelativistic contribution

4. H. Bethe’s contribution

According to results of previous sections – see (9) and (13) – the shift is

\[
\Delta E = \frac{4\pi m (Z\alpha)^2 \int \omega d\omega \left(\mathcal{R}_1 + 4\mathcal{R}_3\right)}{Z^2},
\]

Let us begin with the first term

\[
\mathcal{R}_1 = C \int_1^\infty dx \frac{x^j}{(x+1)^{y+v}(x-1)^{y-v}(x+\eta)^{-2\beta} F\left[\beta, \beta; 2\gamma + 2; \frac{x^2-1}{(x+\eta)^2}\right]},
\]

Inserting \( j = 1/2 \) into (13) we obtain \( \beta = 2\gamma + 2 \) and hypergeometric function (HGF) becomes the elementary one \((1 - X)^{-\beta}\), and the parametric integral

\[
I = \int_1^\infty dx \frac{x^j}{(x+1)^{y+v}(x-1)^{y-v}(x+\eta)^{-2\beta} \left[1 - \frac{x^2-1}{(x+\eta)^2}\right]^{-\beta}}
\]

may be computed with a result

\[
I = 2^{2\gamma_0 + 1}(1 + \eta)^{-2\beta} \frac{F(\beta, \beta; b + 1; \xi^2)}{b}, \quad b = \gamma_0 - \nu + 1
\]

Altogether we have

\[
\mathcal{R}_{1/2} = 2^{2\gamma_0 + 1} C (1 + \eta)^{-2\beta} \frac{F(\beta, \beta; b + 1; \xi^2)}{b}.
\]

The coefficient \( C \) is equal to \( \frac{(2\eta)^{2\gamma v + 3}}{(2\xi)^2 \Gamma(2\gamma v + 1) \Gamma(2\mu + 1)} \), so finally we obtain

\[
\mathcal{R}_{1/2} = \frac{1}{\lambda^2} \frac{(2\eta)^{2\gamma v + 3}}{(1 + \eta)^{2\beta}} \frac{F(\beta, \beta; b + 1; \xi^2)}{b}
\]

\[
\beta = 2\gamma + 2; \quad b = \gamma_0 - \nu + 1.
\]

Integral \( \mathcal{R}_{1/2} \) is a generalization of corresponding nonrelativistic integral [5]

\[
\mathcal{R}_{\text{nonrel}} = 128 \frac{x^5}{(1 + x)^8} \frac{F(4, 2 - \nu; 3 - \nu; \xi^2)}{2 - \nu}, \quad \xi = \frac{\eta - 1}{\eta + 1}.
\]

It is easy to see – replace all the parameters by their approximate values

\[
\gamma_0 \approx 1; \quad \beta \approx 4; \quad b \approx 2 - \nu.
\]

As is well known [5], \( \mathcal{R}_{\text{nonrel}} \) being integrated over photon energy gives a nonrelativistic formula for Lamb shift (obtained by H.Bethe)

\[
\Delta E_B = \frac{4\pi m (Z\alpha)^4}{3\pi} \left[\ln(Z\alpha)^2 - \ln k_0\right].
\]

In the same paper [5] it was also obtained the analytic expression for \( \ln k_0 \) (Bethe logarithm) (1)

\[
\ln k_0 = 2ln2 + \frac{11}{6} - 16 \int_0^1 dx \frac{x(x-1)^2}{(1+x)^6} f(1, 2 - x; 3 - x; \xi^2) = 2.984 128 555 765 498
\]

Thus, the \( \mathcal{R}_{1/2} \) may be named relativistic Bethe - logarithm.

Thus, the nonrelativistic part is separated from corrections to it. This number for \( \ln k_0 \) was calculated from (23) by P. Gliva. Remarkably enough, it is equal to contribution of only one of P-waves, namely \( P_{1/2} \) – all the relativistic corrections are concentrated at another wave, \( P_{3/2} \).

5. “Final” integration

Now we are going to the second term of formula (15)

\[
\Delta E_{3/2} = \frac{4\pi m (Z\alpha)^2}{3\pi} \int \omega d\omega \mathcal{R}_{3/2}.
\]
which concentrate all the correction to $\Delta E_B$. According to (13)

$$\mathcal{R}_{3/2} = \int_0^\infty \frac{(2\eta)^{2\gamma_0+3}}{4\lambda^2(2\gamma_0+1)} \frac{\Gamma(\eta)}{\Gamma(2\gamma_0+2)} \frac{\Gamma(3/2)(2\gamma_0+3)}{\Gamma(3)} \sum_n \frac{\Gamma(\gamma_0+3/2)}{\Gamma(n+3/2)} \eta^n \tag{25}$$

The series allow to compute the parametric integral

$$(1 + \eta)^{-2\gamma_0-3} \sum_n a_n \frac{\eta^{-1}}{n+1} \eta^n. \tag{26}$$

The coefficients $a_n$ are quickly decreasing and argument $\frac{\eta^{-1}}{n+1}$ is lesser than 1. Replacing the sum with its first term we have

$$\mathcal{R}_{3/2} \approx \frac{1}{\lambda^2(2\gamma_0+1)} \frac{\Gamma(3/2)(2\gamma_0+3)}{\Gamma(3)} \frac{\Gamma(\eta)}{\Gamma(2\gamma_0+2)} \frac{\Gamma(3/2)(2\gamma_0+3)}{\Gamma(3)}.$$

In (24) let us going to variable $\eta$ instead of because of $E = m\sqrt{1 - 3^2/\eta^2}$, the $d\omega = -dE = \frac{-m^2}{\sqrt{1-3^2/\eta^2}} \frac{dn}{\eta^3}$. So $\omega d\omega = -\lambda^2 \gamma_0 - \frac{4}{\sqrt{1-3^2/\eta^2}} \frac{dn}{\eta^3}$ and after replacing radical by 1 we obtain an expression

$$\omega d\omega = \frac{4\lambda^2(\eta)(2\gamma_0+3)}{\eta^3}. \tag{28}$$

Now we get the «final» integral

$$\int \omega d\omega \mathcal{R}_{3/2} = \frac{\lambda^2(\eta)(2\gamma_0+3)}{4(\gamma_0+1)} \frac{\Gamma(3/2)(2\gamma_0+3)}{\Gamma(3)} \frac{\Gamma(\eta)}{\Gamma(2\gamma_0+2)} \frac{\Gamma(3/2)(2\gamma_0+3)}{\Gamma(3)} = (\Lambda^2)^2 f(2\gamma_0 + 3) \gamma \frac{11}{24}, \tag{29}$$

where

$$f(t) = \frac{2t^2 - 2t}{4(t-1)(t-2)}.$$

Thus,

$$\Delta E_{3/2} = \frac{4\Lambda^2}{3\pi} m(\Lambda^2)^4 \frac{11}{24} + O(3^2). \tag{30}$$

The term 11/24 is the known correction to the Bethe logarithm. The discarded terms are corrections of higher orders as can be seen from preceding formula where $O(3^2) = -\frac{(\Lambda^2)^2}{3} \left(21n2 - \frac{89}{96}\right)$.

6. “Forgotten” potential

Speaking «1-loop approximation» we always had meant an electron self-energy. But really this concept includes also the vacuum polarization (VP).

Action of VP is taken into account by adding -1/5 to main approximation. L=(4$\alpha m(\Lambda^2)^4$)/3$\pi$=1083,4 MHz is Lamb constant:

$$\Delta E_{SE} = \mathcal{L} \left[ \ln(\Lambda^2) - \ln k_0 + 11/24 \right]. \tag{31}$$

The origin of VP was understood at early thirties: after W.Heisenberg [6] an atomic nucleus have induced in vacuum a charge of opposite sign whose density is

$$\rho_{ind} = -\mathcal{L} \Delta \rho_0 / 15\pi m^2 \tag{32}$$

Year later Uehling applied Poisson equation to this density and discovered the potential $V_0 = -4\alpha \rho_0 / 15m^2$. An action of this potential on the nuclear charge shifts atomic energy to $\Delta E = \langle V_0 \rangle = \mathcal{L}(-1/5) [7]$.

The simplest derivation of Heisenberg equation was given by F.Dyson in his lectures at Cornell University (1951). His result is more general:

$$\rho_{ind}(q) = -\frac{\alpha \rho_0}{4\sqrt{\pi}} \frac{d}{dt} F(1,1/2|-t) \frac{t}{(t/2)}, \quad t = \frac{q^2}{4m^2} \tag{33}$$

but for $q^2 \ll 4m^2$ it gives that of Heisenberg.

Dyson’s formula is obviously equivalent to the known dispersion integral $\int_{-\infty}^{\infty} dt f(t) (t + q^2)^{-1}$, whose denominator $(t + q^2)^{-1}$ is Fourier-transform of Yukawa potential $\exp(-2m t r) r^{-1}$ – limiting form of Uehling potential at large distances.

For $q^2 > -4m^2$ denominator obtains imaginary part – vacuum gives birth to electron-positron pairs. Dyson considers this effect as something similar to “heating of electric cable”.

7. Conclusion

What a conclusion could be made from the results of this investigation?
First – an application of coordinate representation (instead of momentum) gave a possibility to get rid of $2\alpha$ decomposition. Generally, it was due to application of spiral representation of matrix, wave functions and electron propagator

Second – computation is free of infrared divergence, dividing photons on hard and soft and procedure of sewing of this two methods of calculation. All this is due to exact account of Coulomb interaction.

Third – it was confirmed the defining role of partial P-waves - in particular, the $P_{1/2}$ component completely defined the value of Bethe logarithm and correction to it.

Finally - it should be noted the concentration of all the relativistic corrections in the only expression, contribution of $P_{3/2}$ - wave.

The 2-loop problem stands apart – its contribution being small $\sim 10^{-4}$, nonetheless it is beyond of experimental accuracy. Unfortunately, analysis of this problem is connected with great complications,

I didn’t touch the question of the nucleus motion – it goes beyond the method of external field. A fortiori for its computation the existing methods are completely sufficient.

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To their bright memory dedicate I this article!

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Appendix. Coulomb functions

A. Wave function

Dirac equation reads

\[(\hat{\Pi} - m)\Psi = 0, \quad \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}.\]  \hspace{1cm} (A.1)

Here $\hat{\Pi} = \gamma_\mu \Pi_\mu$. $\Pi_\mu = p_\mu - A_\mu$ – gauge-invariant momentum.

In Coulomb’s field $\Pi_0 = E + \frac{Ze^2}{r}$ and $\hat{\Pi} = \hat{p} - e\hat{A} = -i\nabla$. Neglecting proton motion, magnetic moment, dimensions and structure.

Matrices are choosing as $\gamma_0 = \rho_1$, $\gamma^\dagger = i\rho_2\sigma$, $(\rho_a \sigma_b – Pauli$ matrices). This choice leads to the following equations

\[(\Pi_0 - \sigma\hat{p})\phi = m\chi \quad \text{and} \quad (\Pi_0 + \sigma\hat{p})\chi = m\phi,\]  \hspace{1cm} (A.2)

which bind spinors $\phi$ and $\chi$. The former satisfies 2-nd order equation

\[\left(\Pi_0 + \sigma\hat{p}\right)\left(\Pi_0 - \sigma\hat{p}\right)\phi = m^2\phi\]  \hspace{1cm} (A.3)

or, explicitly
Denoting \( k^2 = E^2 - m^2 \), \( \tilde{\Omega} = \tilde{\Omega}^2 - (Z\alpha)^2 - iZ\alpha\bar{\sigma}\tilde{n} \), we get
\[
\left( \frac{\partial^2}{r \partial r^2} + \frac{2ZeE}{r} - \frac{\alpha}{r^2} \right) \Psi = 0.
\] (A.4)

The angles are contained only in \( \tilde{\Omega}^2 \) and \( \bar{\sigma}\tilde{n} \), that allows us to split \( r \) and \( \tilde{n} \):
\[
\Psi'(\tilde{r}) = f (-2ikr) \Phi(\tilde{n}).
\]

With \( \tilde{\Omega} \Phi(\tilde{n}) = \gamma (\gamma + 1) \Phi(\tilde{n}) \), we obtain the Whittaker equation
\[
\left( \frac{\partial^2}{\rho^2} + \frac{2}{\rho} + \frac{\gamma + 1}{\rho^2} \right) \rho f(\rho) = 0,
\] (A.5)

\( \nu = \frac{ZeE}{ik} \) – dimensionless Coulomb parameter). Its solution is \( \rho f(\rho) = M_{\nu\gamma}(\rho) \).

Thus, common radial function of both spinors is Whittaker’s \( M_{\nu\gamma}(-2ikr)r^{-1} \).

The parameter \( \gamma = \sqrt{\kappa^2 - (Z\alpha)^2} \) is an eigenvalue of Martin-Glauber-Johnson (MGD) operator \( \Gamma = \kappa \rho_3 + iZ\rho_2\bar{\sigma}\tilde{n} \). It is connected to \( \tilde{\Omega} \)-operator by \( \tilde{\Omega} = \Gamma^2 + \Gamma \) and has the common eigen functions with it

\[
\Phi_{\pm}(\tilde{n}) = \frac{1}{\sqrt{2\kappa}} \left[ \sqrt{\kappa + \gamma} \Omega_{jml}(\tilde{n}) \pm \sqrt{\kappa - \gamma} \Omega_{jml}(\tilde{n}) \right].
\] (A.6)

Here \( \Omega_{jml}(\tilde{n}) \) – spherical spinor having angular momentum \( j \), projection \( m \) and parity \( P = (-1)^l \).

Remark that \( \Omega_{jml}(\tilde{n}) = -\bar{\sigma} \tilde{n} \Omega_{jml}(\tilde{n}) \).

Taking \( \kappa = 3\epsilon\omega \), \( \gamma = 3\epsilon\omega \), we shall write

\[
\Phi_{\pm}(\tilde{n}) = \frac{1}{\sqrt{2\epsilon\omega}} \left[ e^{\epsilon\omega/2} \Omega_{jml}(\tilde{n}) \pm e^{-\epsilon\omega/2} \Omega_{jml}(\tilde{n}) \right].
\] (A.7)

**B. Electron propagator**

Propagator \( G(\tilde{r}, \tilde{r}'|E) \) satisfies Dirac equation with \( \delta \)-shaped right-hand side
\[
(\hat{n} - m)G(\tilde{r}, \tilde{r}'|E) = i\delta(\tilde{r} - \tilde{r}').
\] (B.1)

The same transformations as in \( A \) are giving
\[
\left[ (\Pi_0 + \bar{\sigma}\Pi) (\Pi_0 - \bar{\sigma}\Pi) - m^2 \right] G_{11} = m \delta(\tilde{r} - \tilde{r}').
\] (B.2)

and analogous equation for another diagonal component \( G_{22} \). The nondiagonal \( G_{12} \) and \( G_{21} \) are having zero at their right-hand side just as in unit matrix I.

We can decompose diagonal components of \( G(\tilde{r}, \tilde{r}'|E) \) over solutions found in Appendix \( A \) (more details see at [9])

\[
G(\tilde{r}, \tilde{r}'|E) = \sum_{jml} c_j \mathcal{M}_{\nu\gamma}(\rho) \mathcal{W}_{\nu\gamma}(\rho')(r'r')^{-1} T_{j}(\tilde{n}, \tilde{n}'), \quad r < r'
\] (B.3)

(summation is carried out over angular parameters \( j, m \) and parity \( P = (-1)^l \)). Both solutions of Dirac equation are used in order to satisfy boundary conditions at zero and infinity independently. Thanks to coefficient \( c_j = \frac{r'(\gamma + 1)}{r(2\gamma + 1)} \) the jump of derivative at \( r = r' \) is equal to unity.

Matrix \( T_{j}(\tilde{n}, \tilde{n}') \) is eigen matrix of MGD operator and can be received from angular functions \( \Phi_{\pm}(\tilde{n}) \) by summing their product \( \Phi_{\pm}(\tilde{n}) \Phi_{\pm}(\tilde{n}') \) over quantum number \( m \).

**Tamm matrix**

Propagator’s angular part is matrix \( T_{\gamma} = \sum_m \Phi(\tilde{n}) \tilde{\Phi}(\tilde{n})^\dagger \). According to (A.7) it is equal to

\[
T_{\gamma} = \frac{1}{2\epsilon\omega} \begin{pmatrix} e^{\epsilon\omega/2} & -T_2 \\ -T_3 & e^{-\epsilon\omega/2} \end{pmatrix},
\] (B.4)

Here we introduced designations

\[
T_2 = T_1 \bar{\sigma} \tilde{n}', \quad T_3 = T_4 \bar{\sigma} \tilde{n}' \quad and \quad T_4 = \bar{\sigma} \tilde{n} T_1 \bar{\sigma} \tilde{n}',
\] (B.5)

(factors \( \bar{\sigma} \tilde{n} \) have appeared due to connection \( \Omega_{jml}(\tilde{n}) = -\bar{\sigma} \tilde{n} \Omega_{jml}(\tilde{n}) \)).

The \( T_1 \) is simplest to obtain: thanks to the theorem of adding Legendre polynomials
\[ \sum_{j\ell m} \Omega_{jm\ell} (\vec{n}) \Pi_{jm\ell} (\vec{n}') = (2l + 1)P_l (\vec{n} \vec{n}') / 4\pi. \]  
(B.6)

For \( l = j - 1/2 \) the left-hand side of this equation is equal to \( T_1 \), and because
\[ T_1 = 2jP_{j-1/2} (\cos \theta) / 4\pi. \]  
(B.7)

Other components of matrix \( T_\gamma \) are following from \( T_1 \) according to (B.5), but for \( T_4 \) we must use \( \bar{l} = j + 1/2 \):
\[ T_4 = (2j + 2)P_{j+1/2} (\cos \theta) / 4\pi. \]  
(B.8)

Polynomials \( T_1 \) and \( T_4 \) were introduced by Ig. Tamm and carry his name [10].