Anti-localisation and its relation to localisation of non-stationary linear waves in an infinite 1D system with a point inclusion

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Abstract

We introduce a new wave phenomenon, namely, the linear waves anti-localisation, which is a tendency for propagating non-stationary waves caused by an impulse source at an inclusion (or a defect) to avoid a neighbourhood of the inclusion. In the framework of an illustrative problem considered in the paper, we have demonstrated that the anti-localisation exists for all cases excepting the boundary of the domain in the parameter space where the wave localisation occurs. Thus, the nature of the anti-localisation is deeply related with the wave localisation.

Keywords: anti-localisation, linear wave localisation, trapped mode, non-stationary waves, inclusion, defect

1. Introduction

It is well known that in infinite (continuum or discrete) systems with inclusions or defects, provided that there is a stop-band in the dispersion characteristics for the corresponding homogeneous system, one can observe linear wave localisation (see, e.g., studies [1, 2] and references there). The wave localisation is usually related to the formation of the discrete spectrum of natural frequencies inside a stop-band under certain conditions, which are fulfilled in some domain of the parameter space (the localisation

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domain). The present paper demonstrates that in such a system one can generally expect another new wave phenomenon, that we suggest to call the waves anti-localisation. This is a tendency for non-stationary waves to propagate avoiding a neighbourhood of an inclusion. In the framework of an illustrative problem considered in the paper, we show that the anti-localisation exists for all cases excepting the boundary of the localisation domain. Thus, we first time demonstrate that the nature of the anti-localisation is deeply related with the wave localisation.

2. The illustrative problem and its solution

We consider transverse oscillation of an infinite taut string on the Winkler elastic foundation. The string is equipped with a discrete mass-spring oscillator, which is subjected to a pulse loading. The governing equation in the dimensionless form is

\[ u'' - \ddot{u} - u = (M\ddot{u} + Ku - \delta(t))\delta(x). \] (1)

Here and in what follows, we denote by prime the derivative with respect to spatial coordinate \( x \) and by overdot the derivative with respect to time \( t \), non-negative constants \( M \) and \( K \) (the problem parameters) are the dimensionless mass and the stiffness characterizing the oscillator. Zero initial conditions are assumed.

The solution can be represented in the following form:

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(x, \Omega) \exp(-i\Omega t) \, d\Omega = \frac{1}{2\pi} \int_{0}^{\Omega_*} G^{\text{stop}}(x, \Omega) \exp(-i\Omega t) \, d\Omega \]

\[ + \frac{1}{2\pi} \int_{\Omega_*}^{+\infty} G^{\text{pass}}(x, \Omega) \exp(-i\Omega t) \, d\Omega + \text{c.c.} = I^{\text{stop}} + I^{\text{pass}} + \text{c.c.} \] (2)

Here \( G(x, \Omega) \) is the corresponding Green function in the frequency domain [3]:

\[ G(x, \Omega) = G^{\text{stop}}(x, \Omega) \overset{\text{def}}{=} \frac{\exp(-\sqrt{1 - \Omega^2|x|})}{2\sqrt{1 - \Omega^2 + K - M\Omega^2}}, \quad \Omega \in \mathbb{S}; \] (3)

\[ G(x, \Omega) = G^{\text{pass}}(x, \Omega) \overset{\text{def}}{=} -\frac{\exp(i\sqrt{\Omega^2 - 1}|x| \text{sign}(\Omega))}{2i \text{sign}(\Omega)\sqrt{\Omega^2 - 1} - K + M\Omega^2}, \quad \Omega \in \mathbb{P}. \] (4)

Here \( \mathbb{S} = (\Omega_*, \Omega_*) \) and \( \mathbb{P} = (\infty, -\Omega_*) \cup (\Omega_*, \infty) \) are the stop-band and the pass-band, respectively; \( \Omega_* = 1 \) is the cut-off frequency, which separates the bands.
The integral \( I_{\text{stop}} \) describes, in particular, a localized non-vanishing oscillation that one can observe in the system under certain conditions. Namely, in the case \( K \geq M \) a rough estimate \( I_{\text{stop}} = O(t^{-1}) \) \((t \to \infty)\) can be obtained from the Erdélyi lemma [4] (see [5]). In the case

\[ K < M \]  

a localized (trapped) mode exists in the system [1, 3]. Inequality (5) defines the localisation domain in the 2D parameter space. If this inequality is true, in the interval \((0, \Omega^\ast)\) there exists a simple root \( \Omega_0 \) of the denominator for \( G \) such that

\[ \Omega^2_0 = \frac{2}{M^2} \left( \sqrt{M^2 - MK + 1} + \frac{MK}{2} - 1 \right), \]  

and, therefore, integral \( I_{\text{stop}} \) does not exist in the classical sense. In the latter case, integral \( I_{\text{stop}} \) should be considered as the Fourier transform for a generalized function. To regularize this we can apply [5] the limit absorption principle. Finally, for \( t \to \infty \) one gets:

\[ I_{\text{stop}} + \text{c.c.} = H(M - K)L(x, t) + O(t^{-1}), \]  

\[ L(x, t) = \frac{\sqrt{1 - \Omega^2_0} e^{-\sqrt{1 - \Omega^2_0}|x|}}{\Omega_0(M \sqrt{1 - \Omega^2_0} + 1)} \sin \Omega_0 t. \]  

where \( L(x, t) \) is the localized non-vanishing oscillation, \( H \) is the Heaviside function. An alternative approach [6] to calculate the contribution from the stop-band, which is more popular in the literature, is to modify integration path to a closed one using branch cuts. In the framework of the latter approach a trapped mode (if exists) can be taken into account by the residue theorem. The integral over the branch cuts should be estimated by the Erdélyi lemma. Both approaches lead to the identical results.

In this paper we are mostly interested in the evaluation of the integral \( I_{\text{pass}} \), which describes the propagating part of the wave-field. Following to [5], we estimate it on a moving at an arbitrary sub-critical speed \( w \) point of observation. Put \(|x| = wt, 0 \leq w < 1\) in the expression for \( I_{\text{pass}} \). The obtained integral can be estimated for \( t \to \infty \) using the method of stationary phase [4]. The only stationary point is \( \Omega_s = 1/\sqrt{1 - w^2} \). Applying the formula [4] for a contribution from a stationary point

\[ c = \ldots \]
yields:

\[ I^{\text{pass}} + \text{c.c.} = -\frac{A(w)}{\sqrt{t}} \cos \left( \sqrt{1 - w^2} t + \frac{\pi}{4} + \psi \right) + O(t^{-1}), \quad (9) \]

\[ A(w) = \frac{\sqrt{2} w (1 - w^2)^{1/4} H(1 - w)}{\sqrt{\pi} \sqrt{4 w^2 (1 - w^2) + (M - K + K w^2)^2}}, \quad (10) \]

\[ \psi = \arctan \frac{2 w (1 - w^2)^{1/2}}{M - K + K w^2}. \quad (11) \]

Note that for \( w > 1 \) there is no stationary point and the corresponding term of order \( t^{-1/2} \) is zero, this is taken into account by introducing the multiplier \( H(1 - w) \) in the numerator of the right-hand side of Eq. (10). Formulae (9)–(11) describe the wave-field quite well in the case when the trapped mode does not exist, see Fig. 1. The numerical solution in Fig. 1 is obtained using an approach based on finite difference schemes. In the case \( K < M \) when the trapped mode exists we additionally should take into account contribution (8) from the trapped mode frequency \( \Omega_0 \in S \), see Eq. (7).

For \( K \neq M \), i.e., everywhere in the parameter space excepting the boundary of the localisation domain (5), the term of order \( t^{-1/2} \) in expansion of \( I^{\text{pass}} \) is zero at \( w = 0 \) (or \( x = 0 \)):

\[ A(w) = \frac{\sqrt{2 w}}{\sqrt{\pi |K - M|}} + O(w^3), \quad w \to 0; \quad A(0) = 0; \quad (12) \]

and, thus, the amplitude \( A(w) \) of the propagating part \( I^{\text{pass}} + \text{c.c.} \) for the string displacements is small in a certain neighbourhood of zero. From the physical point of view, this means that the amplitude of the string displacement (as well as the particle velocity) is small in a certain expanding (since \( w = |x|/t \)) neighbourhood of the inclusion. We call this phenomenon the anti-localisation. The greater the quantity \( |K - M| \) for given \( M \), the wider the anti-localisation zone for the propagating component of the wave-field and more energy concentrates closer to the leading wave-fronts (see Fig. 2).

The expansion for the right-hand side of Eq. (2) at \( x = 0 \) has the following form

\[ u(0, t) = H(M - K) L(0, t) + \frac{2 \sqrt{2} \cos \left( t + \frac{\pi}{4} \right)}{\sqrt{\pi} (K - M)^2} t^{3/2} + o(t^{-3/2}). \quad (13) \]

The first term in the right-hand side of Eq. (13) describes localised oscillation, which exists if and only if \( K < M \). The second term is the anti-localised part of the wave-field at the inclusion expressed as the total contribution from the cut-off frequency \( \Omega_* \),
Figure 1: Comparing of the asymptotic solution for $u$ in the form of the right-hand side of Eq. (9), wherein $w = |x|/t$, and the corresponding numerical solution.

Figure 2: The amplitude $A(w)$ defined by Eq. (10) for various system parameters (note that $A(1) \to \infty$ if $M = 0$ and $A(1) = 0$ otherwise.)
for both integrals $I_{\text{stop}}$ and $I_{\text{pass}}$ (see [5], where a similar problem is considered). The calculations show that the terms of order $O(t^{-1})$ in the right-hand sides of Eqs. (7), (9) compensate each other. Equivalently, the second term can be obtained by applying the Erdélyi lemma to calculate the contribution from the branch cuts [6].

At the boundary of the localisation domain $K = M$, one has instead of Eq. (12):

$$A(w) = \frac{1}{\sqrt{2\pi}} + O(w^2), \quad w \to 0.$$  \hspace{1cm} (14)

This is the only case when the propagating wave-field defined by the integral $I_{\text{pass}}$ is not anti-localized.

3. Discussion

In the framework of the illustrative problem considered in the paper, we have demonstrated that the anti-localisation exists for all cases excepting the boundary ($K = M$) of the localisation domain (5), see Eq. (12) and the subsequent explanations. For the best of our knowledge, the anti-localization was never treated before as a general wave phenomenon. The term “anti-localization” was introduced by Shishkina & Gavrilov in recent study [5]. Our results are in agreement with particular observations in studies, where non-stationary oscillation caused by an impulse source at an inclusion in some infinite (discrete [7, 8, 9] or continuum [6]) systems have been considered. In these previous studies (as well as in [5]) pure inertial inclusions were considered, and asymptotic expansions analogous to our Eq. (13) were obtained in the cases, where the boundary of the localisation domain always corresponds to a homogeneous system without any inclusion ($M = 0, \ K = 0$ for our illustrative problem).

Thus, in this paper we first time have demonstrated that the emergence and the intensity of the anti-localisation in a system are related not with its non-uniformity itself, but with the position in the parameter space outside the boundary of the localisation domain. The analysis of Eqs. (10), (12) shows that the greater the quantity $|K - M|$ for given $M \neq K$, the wider the anti-localisation zone for the propagating component of the wave-field, and more energy concentrates closer to the leading wave-fronts (see the solid lines in Fig. 2). Note that the behaviour of the anti-localised part of the wave-field...
at the inclusion, i.e., of the second term in the right-hand side of Eq. (13), is determined by the total contribution from the cut-off frequency for both integrals $I_{\text{stop}}$ and $I_{\text{pass}}$.

The observability of the anti-localization and its relation to the localisation in systems with more than one stop-bands and cut-off frequencies, as well as in systems characterized by more than one dispersion curves, 2D–3D systems, systems with a trapped mode embedded into the continuous spectrum remain to be the open questions. The case when a source and an inclusion are located at the different positions also requires an additional investigation. The possible practical application of the phenomenon is an acoustical isolation and a meta-material construction. The anti-localisation can probably be an explanation for the phenomenon of “a cold point” discussed in [10].

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Declaration of competing interest

None to declare.

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References

[1] D. Indeitsev, N. Kuznetsov, O. Motygin, Y. Mochalova, Lokalizatsia lineynykh voln [Localisation of linear waves], Izdatelstvo Sankt-Peterburgskogo universiteta [St. Petersburg University publishing house], St. Petersburg, 2007, (in Russian).
[2] I. Andrianov, V. Danishevs’kyy, A. Kalamkarov, Vibration localization in one-dimensional linear and nonlinear lattices: discrete and continuum models, Nonlinear Dynamics 72 (2012) 37–48. https://doi.org/10.1007/s11071-012-0688-4.

[3] S. Gavrilov, E. Shishkina, I. Poroshin, Non-stationary oscillation of a string on the Winkler foundation subjected to a discrete mass–spring system non-uniformly moving at a sub-critical speed, Journal of Sound and Vibration 522 (2022) 116673. https://doi.org/10.1016/j.jsv.2021.116673.

[4] A. Erdélyi, Asymptotic expansions, Dover Publications, New York, 1956.

[5] E. Shishkina, S. Gavrilov, Unsteady ballistic heat transport in a 1D harmonic crystal due to a source on an isotopic defect, arXiv preprint 2206.08079. https://doi.org/10.48550/arXiv.2206.08079.

[6] J. Kaplunov, Krutil’niye kolebaniya sterzhnya na deformiruemom osnovanii pri deystvii dvizhuscheysia inertsionnoy nagruzki [The torsional oscillations of a rod on a deformable foundation under the action of a moving inertial load], Izvestiya Akademii Nauk SSSR, MTT [Mechanics of solids] 6 (1986) 174–177, (in Russian).

[7] S. Kashiwamura, Statistical dynamical behaviors of a one-dimensional lattice with an isotopic impurity, Progress of Theoretical Physics 27 (3) (1962) 571–588. https://doi.org/10.1143/PTP.27.571.

[8] R. Rubin, Momentum autocorrelation functions and energy transport in harmonic crystals containing isotopic defects, Physical Review 131 (3) (1963) 964–989. https://doi.org/10.1103/PhysRev.131.964.

[9] I. Müller, W. Weiss, Thermodynamics of irreversible processes — past and present, The European Physical Journal H 37 (2) (2012) 139–236. https://doi.org/10.1140/epjh/e2012-20029-1.
[10] O. Gendelman, J. Paul, Kapitza thermal resistance in linear and nonlinear chain models: Isotopic defect, Physical Review E 103 (5) (2021) 052113. https://doi.org/10.1103/PhysRevE.103.052113.