Storage and retrieval of \((3 + 1)\)-dimensional weak-light bullets and vortices in a coherent atomic gas

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A robust light storage and retrieval (LSR) in high dimensions is highly desirable for light and quantum information processing. However, most schemes on LSR realized up to now encounter problems due to not only dissipation, but also dispersion and diffraction, which make LSR with a very low fidelity. Here we propose a scheme to achieve a robust storage and retrieval of weak nonlinear high-dimensional light pulses in a coherent atomic gas via electromagnetically induced transparency. We show that it is available to produce stable \((3 + 1)\)-dimensional light bullets and vortices, which have very attractive physical property and are suitable to obtain a robust LSR in high dimensions.

The investigation of light storage and retrieval (LSR), a key technique for realizing optical quantum memory, has received much attention in recent years\(^1\)–\(^3\). One of important techniques for LSR is electromagnetically induced transparency (EIT)\(^4\), a quantum interference effect typical occurring in a three-level atomic system interacting with a probe and a control laser fields. The origination of EIT is the existence of dark state, which makes not only the absorption (dissipation) of the probe field largely suppressed but also the LSR possible through an adiabatical manipulation of the control field.

Up to now, nearly all studies on LSR have been carried out in various schemes working in linear regime\(^5\),\(^6\). Such schemes are simple but encounter the inevitable problem of pulse spreading due to the existence of dispersion, which may result in a serious distortion for retrieved pulse. Recently, the EIT-based LSR has been generalized to weak nonlinear regime, where the storage and retrieval of a \((1 + 1)\)-dimensional \([1 + 1]D\) (i.e., the first '1' refers to one spatial dimension, and the second '1' refers to time) soliton pulse is suggested\(^7\),\(^8\). However, because the \((1 + 1)D\) soliton pulse is unstable in high dimensions due to the existence of diffraction, such scheme is still not realistic or quite limited. For practical applications of optical quantum memory, a challenged problem is to obtain a light pulse that is robust (i.e., with a high fidelity) during storage and retrieval in \((3 + 1)D\).

Before proceeding, we note that in recent years there is much effort focused on high-dimensional optical solitons due to their rich nonlinear physics and important applications\(^9\),\(^10\). Although in recent works\(^11\)–\(^13\) \((3 + 1)D\) light bullets and vortices in coherent atomic systems have been studied, the possibility of their storage and retrieval is not explored yet to the best of our knowledge.

Here we propose an EIT-based new scheme to realize a robust LSR for \((3 + 1)D\) light pulses in a coherent atomic ensemble working in a free space. Based on Maxwell-Bloch equations governing the evolution of atoms and light field we derive a nonlinear equation controlling the motion of the envelope of a probe field. We show the possibility for obtaining \((3 + 1)D\) light bullets (or called \((3 + 1)D\) spatiotemporal optical solitons\(^9\),\(^10\)) and vortices, which have ultraslow propagating velocity and extremely low generation power. We further show that these high-dimensional light pulses can be stabilized by using the balance between dispersion, diffraction, nonlinearity, and by a far-detuned laser field. We demonstrate that these high-dimensional light pulses can be stored and retrieved very stably by switching off and on a control field.

Results

Model. We consider a cold, lifetime-broadened \(\Lambda\)-type three-level atomic gas interacting with a probe field (with pulse length \(t_0\), center angular frequency \(\omega_p\), and half Rabi frequency \(\Omega_p\)) that drives the \(|1\rangle \leftrightarrow |3\rangle\) transition, and a continuous-wave control field (with the center angular frequency \(\omega_c\) and half Rabi frequency \(\Omega_c\)) that drives \(|2\rangle \leftrightarrow |3\rangle\) transition; see the inset of Fig. 1(a).

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For simplicity, we assume the electric field propagates along $z$ direction with the form $E = \sum_{l=p,c} e_l(t) e^{i(kz - \omega_c t) + c.c.}$, where $e_l(t)$ is the unit polarization vector (envelope). A far-detuned laser field (Stark field) used to stabilize (3 + 1)D light bullets and vortices (see below) is applied to the system [as shown in Fig. 1(a)] with the form $E_{\text{Stark}}(x,y,z) = e_{\text{Stark}} e^{i\sqrt{2}E_c(x,y)\cos(\omega_s t)}$, where $e_{\text{Stark}}$, $E_c$, and $\omega_s$ are the unit polarization vector, field amplitude, and angular frequency, respectively. Due to the existence of the Stark field, an energy shift for the level $|j\rangle$ occurs, i.e., $\Delta E_{j,\text{Stark}} = -\frac{1}{2} j_{x}(E_{\text{Stark}}^{2})_{x}/2 = -\frac{1}{2} j_{x}(E_{c}(x,y))^{2}/2$. Here $j_x$ is the scalar polarizability of the level $|j\rangle$, and $\langle \cdots \rangle$ denotes the time average in one oscillating cycle.

Under electric-dipole and rotating-wave approximations, the Hamiltonian of the system in the interaction picture reads

$$\hat{H}_{\text{int}} = -\sum_{j=1}^{3} \hbar \Delta'_{j}|j\rangle\langle j| - \hbar \left[ \Omega_p|3\rangle\langle 1| + \Omega_c|1\rangle\langle 2| + \text{H.c.}\right],$$

with $\Delta'_{j} = \Delta_{j} - \frac{1}{2} j_{x}^{2}(2\hbar)|E_{c}|^{2}$, $\Omega_p = \langle e_{p}|\rho_{13}\rangle E_{p}/\hbar$, and $\Omega_c = \langle e_{c}|\rho_{13}\rangle E_{c}/\hbar$. Here $\Delta_{j} = \omega_p - \omega_c - \omega_{21}$ and $\Delta_{3} = \omega_p - \omega_{31}$ are respectively the two- and one-photon detunings, $p_{j}$ is the electric-dipole matrix element related to the levels $|j\rangle$ and $|0\rangle$, $\rho_{0,j} = E_j - E_0$ is the energy difference between the level $|j\rangle$ and the level $|0\rangle$ with $E_j$ the eigenenergy of the level $|j\rangle$.

The equation of motion for density matrix $\sigma$ in the interaction picture reads

$$\left( \frac{\partial}{\partial t} + \Gamma \right) \sigma = -i \hbar \left[ \hat{H}_{\text{int}}, \sigma \right],$$

where $\sigma$ is a $3 \times 3$ density matrix, $\Gamma$ is a $3 \times 3$ relaxation matrix denoting the spontaneous emission and dephasing. The explicit expressions of Eq. (1) are presented in Methods.

The equation of motion for $\Omega_p$ can be obtained by the Maxwell equation $\nabla \times \vec{E} = -(1/c^2) \partial^2 \vec{E}/\partial t^2 + [1/(\epsilon_0 c^2)] \partial^2 \vec{P}/\partial t^2$, where $\vec{P} = N_{a} \{ p_{13} \sigma_{31} \exp[i(kz - \omega_p t)] + p_{23} \sigma_{32} \exp[i(kz - \omega_c t)] + c.c. \}$ with $N_{a}$ the atomic concentration. Under slowly varying envelope approximation, the Maxwell equation is reduced to ref. 14

$$i \left( \frac{\partial^2}{\partial t^2} + \frac{1}{2} \frac{c^2}{c^2} \right) \Omega_p + \frac{c}{2} \omega_p \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Omega_p + \kappa_{13} \sigma_{31} = 0,$$

where $\kappa_{13} = N_{a} \omega_p |p_{13}|^2/(2\epsilon_0 c^2)$, with $c$ the light speed in vacuum.

Our model can be realized by selecting realistic physical systems. One of them is the ultracold $^{87}$Rb atomic gas with the energy levels selected as $|1\rangle = [5^2S_{1/2}, F = 1]$, $|2\rangle = [5^2S_{1/2}, F = 2]$, and $|3\rangle = [5^2P_{3/2}, F = 2]$, respectively. The decay rates are given by $\Gamma_{2} \approx 2\pi \times 1.0 \text{ kHz}$, and $\Gamma_{3} \approx 2\pi \times 5.75 \text{ MHz}$, and $p_{13} \approx p_{23} = 2.54 \times 10^{-27} \text{ C cm}^2$. If atomic density $N_{a} = 1.1 \times 10^{11} \text{ cm}^{-3}$, $\kappa_{13}$ takes the value of $3.0 \times 10^{6} \text{ s}^{-1}$.

**Nonlinear envelope equation.** We use the standard method of multiple scales developed for EIT system to derive the nonlinear envelope equation for the probe field based on the asymptotic expansion of the Maxwell-Bloch (MB) Eqs. (1) and (2)(see Methods).
The first-order solution of the asymptotic expansion reads \( \Omega_2^{(1)} = \text{Fe}^{0}, \sigma_2^{(1)} = \left\{ \delta_3 \left( \omega + \Delta_2 + i \gamma_2 \right) - \delta_2 \Omega_2^{(2)} \right\} / D \text{Fe}^{(1)}. \) Here \( D = \Omega_2^{(2)} - \left( \omega + \Delta_2 + i \gamma_2 \right) \left( \omega + \Delta_1 + i \gamma_1 \right) \) and \( \theta = K \left( \omega \right) \omega_0 - \omega_0, \) with \( K \left( \omega \right) = \omega \left( \omega + \kappa_3 \right) + \Delta_2 + i \gamma_2 \left( \omega \right) / D \) (linear dispersion relation). Note that the frequency and wave number of the probe field are respectively given by \( \omega_0 + \omega \) and \( k_0 + K \left( \omega \right), \) so \( \omega = 0 \) corresponds to the center frequency of probe field.

Fig. 1(b) shows the imaginary and real parts of \( K \left( \omega \right), \) i.e., \( \text{Im} \left( K \right) \) and \( \text{Re} \left( K \right). \) The dashed and solid lines are for \( \Omega = 0 \) and for \( \Omega = 1.0 \times 10^9 \text{ s}^{-1}, \) respectively. From the upper panel we see that for \( \Omega = 0 \) (no EIT) the probe pulse suffers a large absorption (the dashed line), whereas for \( \Omega = 1.0 \times 10^9 \text{ s}^{-1} \) (with EIT) a transparency window opens and hence the probe pulse is nearly free of absorption (the solid line). The lower panel of Fig. 1(b) shows the drastic change of dispersion due to EIT, which results in a significant reduction of the group velocity of the probe pulse.

The solvability condition at the second order of the asymptotic expansion is \( i \partial \Phi \left( t \right) / \partial t + \left( K \left( \omega \right) \omega_0 \right) \partial \Phi \left( t \right) = 0, \) which means that the probe-pulse envelope \( F \) travels with the group velocity \( \gamma_2 \) (the solid line), and hence we have \( F \left( t \right) \) obtained from the solvability condition at the third order, i.e.,

\[
i \partial F / \partial \xi - 1 / 2 \left( \partial^2 K / \partial \xi^2 \right) F + c \left( \partial^2 \bar{c} / \partial \xi^2 + \partial^2 \bar{d} / \partial \xi^2 \right) F + W_1 \left| F \right| \left| \text{Fe} \right|^2 F = 0,
\]

where \( W_1 \) is the self-phase modulation coefficient of the probe field and \( W_2 \) is the cross-phase modulation coefficient contributed by the Stark field. The explicit expressions of \( W_1 \) and \( W_2 \) are given in Methods.

Combining the solvability conditions (i.e., the equations for \( F \)) at the all orders, we obtain the unified equation for \( F \), which can be written into the dimensionless form

\[
i \partial u / \partial s + \left[ g_1 \partial^2 u / \partial \xi^2 + g_2 \partial^2 u / \partial \eta^2 \right] + g_3 V \left( \xi, \eta \right) u = 0,
\]

with \( u = \text{Fe} \left( \omega_0 \right), s = z / L_{\text{diff}} + \tau \left[ \tau_0 - \text{Re} \left( \text{V} \left( \xi, \eta \right) \right) \right], \) \( \tau_0 = \left( \omega_0 \right) / \left( \gamma_0 \right), \) \( R = 1 / \left( \gamma_0 \right), \) \( \text{L}_{\text{diff}} = \left[ \text{L}_{\text{diff}} \right] / \left( \text{L}_{\text{nonl}} \right) = \left( \text{k} / \text{c} \right) \), \( \text{L}_{\text{diff}} = \left( \omega \text{Re} \left( \text{W} \left( \xi \right) \right) \right) / \left( \gamma_0 \right), \) and \( \text{L}_{\text{nonl}} = 1 / \left( \text{Re} \left( \text{W} \left( \xi \right) \right) \right) \).

Note that the envelope equation (4) includes dispersion, diffraction, nonlinearity, and “external” potential. When obtaining Eq. (4) we have neglected the imaginary parts of \( \partial \Phi \) \( \omega \) (\( \left( \gamma_2 \right) \), \( W_1 \), and \( W_2 \). This is reasonable because the system works under the EIT condition \( \left| \Omega_2 \right| \gg \left| \gamma_2 \right| \) so that their imaginary parts are much smaller than their real parts. In addition, the diffraction, dispersion, and nonlinearity are assumed to be balanced, i.e., \( \text{L}_{\text{diff}} = \text{L}_{\text{disp}} = \text{L}_{\text{nonl}} \).

By choosing the realistic system parameters \( \Omega_2 = 9.0 \times 10^6 \text{ Hz}, \Delta_2 = -6.0 \times 10^6 \text{ Hz}, \Delta_3 = -2.0 \times 10^6 \text{ Hz}, R = 40 \mu \text{m}, \tau_0 = 2.0 \times 10^{-7} \text{ s}, U_0 = 2.87 \times 10^5 \text{ Hz}, \text{ and } E_0 = 3.04 \times 10^4 \text{ V/cm}, \) we have \( \text{L}_{\text{diff}} \approx \text{L}_{\text{disp}} \approx \text{L}_{\text{nonl}} = 1.26 \text{ cm}, \) and then

\[
\text{Re} \left( \text{V} \right) \approx 6.5 \times 10^{-5} \text{c}.
\]

We see that the probe pulse propagates with an ultrasonic group velocity.

### Solutions of (3 + 1)D Weak-light bullets and vortices.

In order to obtain high-dimensional nonlinear localized solutions of the system, we assume the Stark field has the form of Bessel function, i.e.,

\[
E_\xi \left( \xi, \eta \right) = E_\theta \left( l / \sqrt{2} \right),
\]

where \( l \) is an integer; \( r = \sqrt{\xi^2 + \eta^2} \). Then Eq. (4) becomes

\[
i \partial u / \partial s + \left[ \partial^2 u / \partial \xi^2 + \partial^2 u / \partial \eta^2 \right] + u + \left| u \right|^2 u + \left| u \right|^2 \psi \left( l / \sqrt{2} \right) \psi = \psi \left( l / \sqrt{2} \right),
\]

with \( \psi = \text{EIT} \). Note that Eq. (6) is similar to that obtained in Ref. 16. However, the physics here is different from that in Ref. 16 because Eq. (6) describes the nonlinear evolution of the probe-field envelope in the EIT system whereas the equation in Ref. 16 governs the dynamics of a Bose-Einstein condensate. Using the transformation \( u = \psi \exp \left( i \mu \right) \), Eq. (6) is reduced into

\[
i \partial \tilde{u} / \partial s + \left[ \partial^2 \tilde{u} / \partial \xi^2 + \partial^2 \tilde{u} / \partial \eta^2 \right] \tilde{u} + \left| \tilde{u} \right|^2 \psi + \left| \psi \right|^2 \tilde{u} \left( l / \sqrt{2} \right) \psi = \psi \left( l / \sqrt{2} \right),
\]

where \( \mu \) is a propagation constant.

### Storage and retrieval of (3 + 1)D light solitons and vortices.

The principle of EIT-based LSR is well known\(^1\). When switching on the control field, probe pulse propagates in the atomic medium with nearly vanishing absorption; by slowly switching off the control light, the light bullets and vortices stored in the medium are retrieved.
field the probe pulse disappears and gets stored in the form of atomic coherence; when the control field is switched on again the probe pulse reappears. However, this principle is usually applied for linear optical pulses, which may suffer serious distortion due to the dispersion and/or diffraction. In the following we show that it is available to realize the LSR of the (3+1)D light bullets and vortices in our present system.

To this end, we consider the solution of the MB Eqs. (1) and (2) by using a control field that is adiabatically changed with time to realize the function of its turning on and off. The switching-on and switching-off of the control field is modeled by the following function

$$\Omega = \Omega_0 \left\{ 1 - \frac{1}{2} \tanh \left( \frac{t - T_{\text{off}}}{T_s} \right) + \frac{1}{2} \tanh \left( \frac{t - T_{\text{on}}}{T_s} \right) \right\} , \quad (8)$$

where $T_{\text{off}}$ and $T_{\text{on}}$ are respectively the times of switching-off and the switching-on of the control field with a switching time $T_s$. The storage time of the light bullets and vortices is approximately given by $T_{\text{on}} - T_{\text{off}}$.

We first consider the LSR of the (1+1)D soliton pulse, corresponding the case $\xi^2 \partial_t^2 \psi^2 = \xi^2 \partial_t \partial_t \psi^2 = 0$ and $g_1 = 0$ in Eq. (4). The result of numerical simulation on the time evolution of $|\Omega_p(z)\psi(t)|$ and atomic coherence $\sigma_{21}$ as functions of $z$ and $t$ is presented in Fig. 3. The red solid line shown in the upper part of each panel represents the control field $\Omega_p(z)\psi(t)$. Here we choose $T_{\text{off}}/T_0 = 0.2$, $T_{\text{on}}/T_0 = 5.0$, $T_{\text{on}}/T_0 = 15.0$, and the other system parameters are mentioned above. The wave shape of the input probe pulse is taken as a hyperbolic secant, i.e., $\Omega_p(0, t) = 7.0 \text{sech}(t/t_0)$. Lines 1 to 4 are for propagation distance $z = 0, 1.5, 3.0,$ and 4.5 cm, respectively.

Shown in Fig. 3(a) is the result of $|\Omega_p(z)\psi(t)|$. We see that the retrieved pulse has nearly the same shape with the one before the storage. The physical reason of the shape-preservation of the probe pulse before and after the storage is due to a balance between dispersion and nonlinearity, i.e., the pulse is indeed a soliton that is rather stable during the storage and retrieval. Fig. 3(b) shows the atomic coherence $\sigma_{21}$, which has been amplified by 20 times for a better visualization. We see that $\sigma_{21}$ is nonzero during the switch-off of the control field, which is a manifestation of the information transfer (i.e., storage) from the light field to the atomic ensemble.

We now turn to investigate the LSR of the (3+1)D light pulses. Fig. 4 shows the storage and retrieval of the light pulses with the Stark field taken to be the zero-order Bessel function (the left side of each column) and the light pulses with the Stark field taken to be the first-order Bessel function (the right side of each column) for different probe-field intensities, with the other parameters the same as used above. Isosurfaces ($|\Omega_p(z)\psi(t)| = 0.5$) for $\Omega_p(0, t)$ = 2.0, 7.0, 10.0 at $z = 0$ (before the storage), 2.25 cm (during the storage), and 4.5 cm (after the storage) are illustrated, respectively. The results are the following: (i) For the case of weak probe-field intensity (the first line in the figure), the probe pulse broadens before and after the storage; (ii) For the case of moderate probe-field intensity (the second line in the figure), the retrieved probe pulse has nearly the same shape with the one before the storage; (iii) For the case of strong probe-field intensity (the third line in the figure), the retrieved probe pulse is quite different from the original one.
We show the numerical result of the evolution of the probe field (i.e., the case (ii) described above), in Fig. 5(a), Fig. 5(b), and Fig. 5(c). The second line corresponds to the storage and retrieval of stable light bullets and vortices. All figures are isosurface plots.

In order to illustrate more clearly the evolution process of the storage and retrieval of the stable (3 + 1)D light bullets and vortex (i.e., the case (ii) described above), in Fig. 5(a), Fig. 5(b), and Fig. 5(c) we show the numerical result of the evolution of the probe field (\(|\Omega_p t_0|\)) and the control field (\(|\Omega_c t_0|\)) as functions of time at \(z = 0, 2.25\) cm, and \(4.5\) cm, respectively. We see that the light bullet and vortex undergo steps of appearance, disappearance, and reappearance. Presented in the first (second) column of Fig. 5(d) is the light-intensity distribution in \(x-y\) plane of the bullet (vortex) for \(t/t_0 = 5.0, 10.0,\) and \(15.0\), respectively. The third column is the phase distribution of the light vortex. The result shows that the light bullet and vortex can be stored around \(t/t_0 = 5.0\) when the control field is switched off, and be retrieved around \(t/t_0 = 15.0\) when the control field is switched on again. Interestingly, the phase distribution of the vortex can also be stored and retrieved, which means that the memory of the light vortex can bring more information than that of the light bullet.

We have also studied the storage and retrieval of vortices for \(m = 2\). The numerical result shows that these vortices are unstable during the propagation, and hence a robust storage and retrieval of them are not available.

**Discussion**

From the results described above, a robust SLR for the (3 + 1)D weak-light bullets and vortices is possible by using the cold \(\Lambda\)-type three-level atomic system. These results can be easily generalized to other types of EIT systems with different (such as ladder-type\(^5\) level configurations. Furthermore, our theory can also be used to study the (3 + 1)D LSR with a Raman scheme\(^{20,21}\), which has been suggested to obtain a broadband quantum memory of linear light pulses and has been realized recently by experiment using the atomic ensemble working at room temperature\(^{22,23}\).

In conclusion, we have proposed an EIT-based new scheme to realize a robust LSR for (3 + 1)D light pulses in a coherent atomic ensemble. Based on MB equations we have derived a nonlinear equation controlling the evolution of the probe-field envelope. We have shown that it is possible to obtain (3 + 1)D light bullets and vortices, which have very slow propagating velocity and ultra low generation power. We have further shown that these high-dimensional light pulses can be stabilized by using the balance between dispersion, diffraction, nonlinearity, and by a Stark laser field. We have demonstrated that these high-dimensional light pulses can be stored and retrieved very stably by switching off and on a control field. Our
study raise the possibility of guiding a related experiment and have potential applications in the area of light and quantum information processing.

**Methods**

Maxell-Bloch equations. In our semi-classical approach, MB equations are used to describe the motion of light field and atoms. Explicit expressions of the Bloch equation in the interaction picture are

\[ i \frac{\partial}{\partial t} \sigma_{13} + i \Gamma_{13} \sigma_{33} + \Omega_{e} \sigma_{33} - \Omega_{e} \sigma_{33} = 0, \]  

\[ i \frac{\partial}{\partial t} \sigma_{32} - i \Gamma_{13} \sigma_{33} + \Omega_{e} \sigma_{32} - \Omega_{e} \sigma_{32} = 0, \]  

\[ i \frac{\partial}{\partial t} \sigma_{31} + i \Gamma_{13} \sigma_{33} - \Omega_{e} \sigma_{31} + \Omega_{e} \sigma_{31} = 0, \]  

\[ \left( \frac{\partial}{\partial t} + d_{11} \right) \sigma_{31} - \Omega_{e} \sigma_{32} + \Omega_{e} \sigma_{32} = 0, \]  

\[ \left( \frac{\partial}{\partial t} + d_{12} \right) \sigma_{32} - \Omega_{e} \sigma_{32} + \Omega_{e} \sigma_{32} = 0, \]  

\[ \frac{\partial}{\partial t} \sigma_{31} + i \Gamma_{13} \sigma_{33} - \Omega_{e} \sigma_{31} + \Omega_{e} \sigma_{31} = 0, \]  

where \( d_{1} = \Delta_{1} - \Delta_{1} + i \gamma_{1} \). Dephasing rates are defined as \( \gamma_{1} = (\Gamma_{1} + \Gamma_{2})/2 + \gamma_{1}^{0} \) with \( \Gamma_{1} = \sum_{\ell < \ell} \Gamma_{\ell} \) being the spontaneous emission rate from the state \( |\ell\rangle \) to all lower energy states \( |j\rangle \) and \( \gamma_{1}^{0} \) being the dephasing rate reflecting the loss of phase coherence between \( |j\rangle \) and \( |\ell\rangle \).

Asymptotic expansion. Assume \( \sigma_{j} = \sum_{\ell = 0}^{\infty} \sigma_{j}^{(\ell)} e^{i \ell \Phi} \), with \( \sigma_{j}^{(0)} = \delta_{j} \delta_{j}, \Omega_{j} = \sum_{\ell = 1}^{\infty} e^{i \ell \Phi} \), and \( E_{j} = E_{j}^{(0)} \). Thus \( d_{j} = d_{j}^{(0)} + \gamma_{1}^{0} \). With \( d_{j}^{(0)} = \Delta_{j} + i \gamma_{1} \) and \( d_{j}^{(0)} = i \gamma_{1} / 2 \Delta_{j} \). Here \( e \) is the dimensionless small parameter characterizing the typical amplitude of the probe pulse. To obtain a divergence-free expansion, all the quantities on the right-hand side of the expansion are considered as functions of the multi-scale variables \( x_{1} = x_{1}, y_{1} = y_{1}, z_{1} = z_{1}^{2}, \) and \( t_{1} = t_{1}^{2} \). Substituting the expansions into Eqs. (1) and (2) and comparing the coefficients of \( e^{\ell} \), we obtain a set of linear but inhomogeneous equations which can be solved order by order.

The first order \( (q = 1) \) solution is given by \( \sigma_{j}^{(1)} = F^{(0)} \) and \( \sigma_{j}^{(1)} = \left[ \sigma_{j}^{(0)}(\omega + \Delta_{j} + \gamma_{1} / \Delta_{j}) - F^{(0)} \right] / \Delta_{j} \), with \( D = \Omega_{j}^{2} - (\omega + \Delta_{j} + \gamma_{1} / \Delta_{j} + i \gamma_{1} / \Delta_{j}) \) and \( \theta = K(\omega) - \text{coth} \omega \). The linear dispersion relation reads \( K(\omega) = \text{coth} \omega + \text{coth} (\omega + \Delta_{j} + \gamma_{1} / \Delta_{j}) / D \). D is a yet to be determined envelope function depending on the slow variables \( x_{1}, y_{1}, z_{1}, t_{1}, \) and \( \omega \).

At the second order \( (q = 2) \), a solvability condition gives \( \{ c F^{2} c_{2} + (K^{2}(\omega) c_{2}) F / \Delta_{j} \} = 0 \), with \( c_{2} = (K^{2}(\omega)/c_{2}) \). The approximation solution at this order reads

\[ \sigma_{j}^{(2)} = \left( d_{j}^{(0)} / \Delta_{j} \right)^{2} F^{(0)} e^{-2 \omega t_{1}} \left( j = 1, 2, 3 \right) \], and

\[ \sigma_{j}^{(2)} = \left( d_{j}^{(0)} / \Delta_{j} \right)^{2} e^{-2 \omega t_{1}} \],

where

\[ d_{j}^{(0)} = \left( \Gamma_{13} - 2 \Omega_{j}^{2} \right) \left( 1 / d_{j}^{(0)} - 1 / d_{j}^{(0)} \right) \left( G - i \Gamma_{13} \Omega_{j}^{2} / D \right) \square \],

\[ d_{j}^{(0)} = \left( \Omega_{j}^{2} / D \right) \left( 2 \omega + d_{j}^{(0)} + d_{j}^{(0)} \right), \]

\[ d_{j}^{(0)} = \left( d_{j}^{(0)} + 2 \omega \right) + \left| \Omega_{j}^{2} \right| / D, \]

\[ d_{j}^{(0)} = \left( 2 \omega + d_{j}^{(0)} + d_{j}^{(0)} \right)^{2} / D, \]

\[ d_{j}^{(0)} = \left( d_{j}^{(0)} + d_{j}^{(0)} \right)^{2} / D. \]

At the third order \( (q = 3) \), a solvability condition yields the equation \( (3) \). The explicit expressions of the self- and cross-phase modulation coefficients \( W_{11} \) and \( W_{12} \) are given by

\[ W_{11} = \kappa_{11} \left( \Omega_{e} d_{j}^{(0)} + \left( \omega + d_{j}^{(0)} \right) \left( d_{j}^{(0)} + d_{j}^{(0)} \right) / D \right). \]

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**Author contributions**

Z.C. carried out the analytical and numerical calculations. Z.B., H.-L. and C.H. developed primary calculating code and helped the numerical calculation. Z.C. and C.H. wrote the manuscript. G.H. conceived the idea, conducted the calculation and revised the manuscript.

**Additional information**

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