Energy-Momentum Distribution in Weyl Metrics

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Abstract

In this paper, we evaluate energy and momentum density distributions for the Weyl metric by using the well-known prescriptions of Einstein, Landau-Lifshitz, Papaterou and Möller. The metric under consideration is the static axisymmetric vacuum solution to the Einstein field equations and one of the field equations represents the Laplace equation. Curzon metric is the special case of this spacetime. We find that the energy density is different for each prescription. However, momentum turns out to be constant in each case.

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1 Introduction

The concept of energy, momentum and angular momentum has always an important character in both Classical and Quantum Physics. In Special Relativity, the density of energy and momentum form a second rank tensor field $T^a_b$ whose divergence vanishes. Serious difficulties in connection with its notion arise in the theory of General Relativity (GR). The energy problem is the oldest [1] and one of the most difficult problems of classical GR.

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The law of conservation of energy in GR, $T_{a,b}^b = 0$, can be written in the form

$$(T_a^b + t_a^b)_b = 0,$$

where $t_a^b$ is the energy-momentum pseudo tensor of the gravitational field. Because of the way we defined $t_a^b$, it is not a tensor - the procedure of picking out a partial derivative from a covariant one is not invariant. By using a superpotential $H^{bc}_a$, $H^{bc}_{a,c}$, $t_a^b$ is defined non-covariantly from the very beginning. In the same way, the canonical energy-momentum pseudo tensor is not a tensor, because to form it we take a partial (non-covariant) derivative of the Lagrangian.

There have been many attempts to resolve the energy problem in GR. Penrose [2] introduced a definition of quasi-local mass in order to discuss gravitational energy in the framework of GR. The definition applies to space-like surfaces of spherical topology and, through construction intrinsic to the two-surface, produces a quantity to be termed as the Penrose mass within the two surface. In linearised theory this is the same as the norm of the energy-momentum of the sources within the two-surface. So far most of the applications have been made in asymptotic regions, null and space-like infinity for asymptotically flat spaces and time-like infinity for asymptotically anti-de Sitter spaces [2,3].

Jeffryes [4] applied the construction of quasi-local mass in the Newtonian limit of GR with perfect fluids using the formulation of Futamase et al. [5]. Bartnik [6] presented a new definition of quasi-local mass. Dirac and Arnowitt et al. [7] succeeded in putting the exact theory of gravitation in Hamiltonian form.

There have been series of some other attempts [8]-[13] to evaluate the energy-momentum distribution. In this series, the first attempt was made by Einstein [8] who suggested an expression for energy-momentum density. After this, many prescriptions, followed by Landau-Lifshitz [9], Papapetrou [10], Bergman [11], Tolman [12] and Weinberg [13], were proposed. The main problem with these definitions is that they are coordinate dependent. These prescriptions give meaningful results only when calculations are performed in Cartesian coordinates.

Möller [14,15] proposed an expression which is the best to make calculations in any coordinate system. He claimed that his expression would give the same values for the total energy and momentum as the Einstein’s energy-momentum complex for a closed system. However, Möller’s energy-
momentum complex was subjected to some criticism [2], [15]-[17]. Komar’s [16] prescription, though not restricted to the use of Cartesian coordinates, is not applicable to non-static spacetimes. Thus each of these energy-momentum complexes has its own drawbacks. As a result, these ideas of the energy-momentum complex were severally criticized.

Virbhadra et al. [18]-[22] and Xulu [23,24] explored several examples of the spacetimes and found that different prescriptions could provide exactly the same energy-momentum distribution. Virbhadra concluded that Einstein’s prescription might provide the best results among all the known prescriptions for the energy-momentum distribution of a given spacetime. In a recent paper, Lessner [25] pointed out that the Möller’s energy-momentum prescription is a powerful concept of energy and momentum in GR.

In recent papers, Sharif [26-28] has explored some examples of spacetimes in which energy-momentum density components turn out to be different for different prescriptions. In this paper, we use energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Möller to evaluate energy-momentum density components of the Weyl metric. The paper is formulated as follows. In the next section, we shall describe the Weyl metric and transform it into Cartesian coordinates. In sections 3, 4, 5 and 6, we shall evaluate energy and momentum densities using the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller respectively. In the last section, we shall discuss the results.

2 The Weyl Metrics

Static axisymmetric solutions to the Einstein field equations are given by the Weyl metric [30]

\[ ds^2 = e^{2\psi} dt^2 - e^{2(\gamma-\psi)} (d\rho^2 + dz^2) - \rho^2 e^{-2\psi} d\phi^2 \]  

in the cylindrical coordinates \((\rho, \phi, z)\). Here \(\psi\) and \(\gamma\) are functions of coordinates \(\rho\) and \(z\). The metric functions satisfy the following differential equations

\[ \psi_{\rho\rho} + \frac{1}{\rho} \psi_{\rho} + \psi_{zz} = 0, \]  

\[ \gamma_{\rho} = \rho(\psi_\rho^2 - \psi_z^2), \quad \gamma_z = 2\rho\psi_\rho\psi_z. \]
It is obvious that Eq.(3) represents the Laplace equation for $\psi$. Its general solution, yielding an asymptotically flat behaviour, will be

$$\psi = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos \theta),$$

where $r = \sqrt{\rho^2 + z^2}$, $\cos \theta = z/r$ are Weyl spherical coordinates and $P_n(\cos \theta)$ are Legendre Polynomials. The coefficients $a_n$ are arbitrary real constants which are called Weyl moments. It is mentioned here that if we take

$$\psi = -\frac{m}{r}, \quad \gamma = -\frac{m^2 \rho^2}{2r^4}, \quad r = \sqrt{\rho^2 + z^2}$$

then the Weyl metric reduces to special solution of Curzon metric [30]. In order to have meaningful results in the prescriptions of Einstein, Landau-Lifshitz and Papapetrou, it is necessary to transform the metric in Cartesian coordinates. We transform this metric in Cartesian coordinates by using

$$x = \rho \cos \phi, \quad y = \rho \sin \phi.$$  

The resulting metric in these coordinates will become

$$ds^2 = e^{2\psi} dt^2 - \frac{e^{2(\gamma-\psi)}}{\rho^2} (x dx + y dy)^2 - \frac{e^{-2\psi}}{\rho^2} (x dy - y dx)^2 - e^{2(\gamma-\psi)} dz^2.$$  

3 Energy and Momentum in Einstein’s Prescription

The energy-momentum complex of Einstein [8] is given by

$$\Theta^b_a = \frac{1}{16\pi} H^{bc}_{a,c},$$

where

$$H^{bc}_{a} = \frac{g^{ad}}{\sqrt{-g}} [-g^{bd} g^{ce} - g^{be} g^{cd})]_e, \quad a, b, c, d, e = 0, 1, 2, 3.$$  

Here $\Theta^0_0$ is the energy density, $\Theta^i_i$ ($i = 1, 2, 3$) are the momentum density components and $\Theta^i_i$ are the energy current density components. The Einstein energy-momentum satisfies the local conservation laws

$$\frac{\partial \Theta^b_a}{\partial x^b} = 0.$$  

The required non-vanishing components of $H^{bc}_a$ are

$$H^{01}_0 = \frac{x}{\rho^2}(e^{2\gamma} - 1) - \frac{2x}{\rho}(\gamma_{\rho} - 2\psi_{\rho}), \quad (12)$$

$$H^{02}_0 = \frac{y}{\rho^2}(e^{2\gamma} - 1) - \frac{2y}{\rho}(\gamma_{\rho} - 2\psi_{\rho}), \quad (13)$$

$$H^{03}_0 = 2(\gamma_z - 2\psi_z). \quad (14)$$

Using Eqs. (12)-(14) in Eq. (9), we obtain the energy and momentum densities in Einstein’s prescription

$$\Theta^0_0 = \frac{1}{8\pi\rho}[\gamma_{\rho}(e^{2\gamma} - 1) - \rho\gamma_{\rho\rho} + 2\psi_{\rho} + 2\rho\psi_{\rho\rho} + \rho\gamma_{zz} - 2\rho\psi_{zz}], \quad (15)$$

$$\Theta^i_0 = 0. \quad (16)$$

This gives momentum density zero and consequently momentum is constant.

## 4 Energy and Momentum in Landau-Lifshitz’s Prescription

The Landau-Lifshitz [9] energy-momentum complex can be written as

$$L^{ab} = \frac{1}{16\pi} \ell^{acbd}_{,cd}, \quad (17)$$

where

$$\ell^{acbd} = -g(g^{ab}g^{cd} - g^{ad}g^{cb}). \quad (18)$$

$L^{ab}$ is symmetric with respect to its indices. $L^{00}$ is the energy density and $L^{0i}$ are the momentum (energy current) density components. $\ell^{acbd}$ has symmetries of the Riemann curvature tensor. The local conservation laws for Landau-Lifshitz energy-momentum complex turn out to be

$$\frac{\partial L^{ab}}{\partial x^b} = 0. \quad (19)$$

The required non-vanishing components of $\ell^{acbd}$ are

$$\ell^{0101} = -\frac{1}{\rho^2}(y^2e^{4(\gamma-\psi)} + x^2e^{2(\gamma-2\psi)}), \quad (20)$$
\[
\ell^{0202} = -\frac{1}{\rho^2}(x^2 e^{4(\gamma - \psi)} + y^2 e^{2(\gamma - 2\psi)}), \tag{21}
\]
\[
\ell^{0102} = \frac{x y}{\rho^2} (e^{4(\gamma - \psi)} - e^{2(\gamma - 2\psi)}), \tag{22}
\]
\[
\ell^{0303} = -e^{2(\gamma - 2\psi)}. \tag{23}
\]
When we substitute these values in Eq. (17), it follows that the energy density remains non-zero while momentum density components vanish. These are given as follows
\[
L^{00} = \frac{-e^{2(\gamma - 2\psi)}}{8\pi\rho^2} \left\{ \left\{ 2(\gamma_\rho - \psi_\rho) \rho + 1 \right\} e^{2\gamma} - 1 + \rho^2(\gamma_{\rho\rho} - 2\psi_{\rho\rho}
+ \gamma_{zz} - 2\psi_{zz}) + 2\rho^2 \left\{ (\gamma_\rho - 2\psi_\rho)^2 + (\gamma_z - 2\psi_z)^2 \right\} \right\},
\]
\[
L^{0i} = 0. \tag{24}
\]

## 5 Energy and Momentum in Papapetrou’s Prescription

We can write the prescription of Papapetrou [10] energy-momentum distribution in the following way
\[
\Omega^{ab} = \frac{1}{16\pi} N^{abcd}, \tag{25}
\]
where
\[
N^{abcd} = \sqrt{-g} (g^{ab} \eta^{cd} - g^{ac} \eta^{bd} + g^{cd} \eta^{ab} - g^{bd} \eta^{ac}), \tag{26}
\]
and \(\eta^{ab}\) is the Minkowski spacetime. It follows that the energy-momentum complex satisfies the following local conservation laws
\[
\frac{\partial \Omega^{ab}}{\partial x^b} = 0. \tag{27}
\]
\(\Omega^{00}\) and \(\Omega^{0i}\) represent the energy and momentum (energy current) density components respectively. The required non-vanishing components of \(N^{abcd}\) are given by
\[
N^{0011} = -\frac{1}{\rho^2} [x^2 + y^2 e^{2\gamma} + \rho^2 e^{2(\gamma - 2\psi)}], \tag{28}
\]
\[
N^{0022} = -\frac{1}{\rho^2} [x^2 e^{2\gamma} + y^2 + \rho^2 e^{2(\gamma - 2\psi)}], \quad (29)
\]
\[
N^{0012} = \frac{xy}{\rho^2} (e^{2\gamma} - 1), \quad (30)
\]
\[
N^{0033} = -1 - e^{2(\gamma - 2\psi)}. \quad (31)
\]

Substituting Eqs. (28)-(31) in Eq. (25), we obtain the following energy density and momentum density components

\[
\Omega^{00} = \frac{e^{2\gamma}}{8\pi \rho} \left[ (1 - e^{-4\psi}) \gamma_\rho + \{2\psi_\rho - \rho (\gamma_\rho - 2\psi_\rho + \gamma_{zz} - 2\psi_{zz}) \right. \\
\left. - 2\rho \{(\gamma_\rho - 2\psi_\rho)^2 + (\gamma_z - 2\psi_z)^2\} \} e^{-4\psi} \right], \\
\Omega^{0i} = 0. \quad (32)
\]

6 Energy and Momentum in Möller’s Prescription

The energy-momentum density components in Möller’s prescription [14,15] are given as

\[
M^b_a = \frac{1}{8\pi} K^{bc}_{a,c}, \quad (33)
\]

where

\[
K^{bc}_{a} = \sqrt{-g} (g_{ad,e} - g_{ae,d}) g^{be} g^{cd}. \quad (34)
\]

Here \( K^{bc}_{a} \) is symmetric with respect to the indices. \( M^0_0 \) is the energy density, \( M^i_0 \) are momentum density components, and \( M^0_i \) are the components of energy current density. The Möller energy-momentum satisfies the following local conservation laws

\[
\frac{\partial M^b_a}{\partial x^b} = 0. \quad (35)
\]

Notice that Möller’s energy-momentum complex is independent of coordinates. For the Weyl metric, we obtain the following non-vanishing components of \( K^{bc}_{a} \)

\[
K^{01}_0 = 2\rho \psi_\rho, \quad (36)
\]
\[
K^{03}_0 = 2\rho \psi_z. \quad (37)
\]
When we make use of Eqs.(36) and (37) in Eq.(33), the energy and momentum density components turn out to be

\[ M^0_0 = \frac{1}{4\pi} (\psi_\rho + \rho\psi_{\rho\rho} + \rho\psi_{\omega\omega}), \]  

(38)

\[ M^i_0 = 0. \]  

(39)

7 Discussion

The debate on the localization of energy-momentum is an interesting and a controversial problem. According to Misner et al [31], energy can only be localized for spherical systems. In a series of papers [32] Cooperstock et al. has presented a hypothesis which says that, in a curved spacetime, energy and momentum are confined to the regions of non-vanishing energy-momentum tensor \( T^b_a \) of the matter and all non-gravitational fields. The results of Xulu [23,24] and the recent results of Bringley [33] support this hypothesis. Also, in the recent work, Virbhadra and his collaborators [18-22] have shown that different energy-momentum complexes can provide meaningful results. Keeping these points in mind, we have explored the Weyl spacetime for the energy-momentum distribution.

In this paper, we are using prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller to evaluate the energy-momentum density components for the Weyl metric. From Eqs.(14), (23), (32) and (38), it can be seen that the energy-momentum densities are finite and well defined. We also note that the energy density is different for the four different prescriptions. However, momentum density components turn out to be zero in all the prescriptions and consequently we obtain constant momentum for this spacetime.

In recent papers [26-28], we have used Einstein and Papapetrou’s prescriptions to determine the energy-momentum distribution of Gödel and Gödel type spacetimes. These results do not coincide for the two different prescriptions. Here we have found the spacetime in which the energy density is different for the four prescriptions but the momentum become constant. We know many examples for which the energy-momentum complexes can yield the same results. It is mentioned here that these results turn out to be the same [34] under the limiting case of the Curzon metric which is a special solution of the Weyl metric. It would be interesting to look for the
reasons of this difference. It seems that the basic problem of definition of energy-momentum in GR is still there which needs to be resolved.
References

[1] Lorentz, H.A.: Veral. Kon. Akad. Wet. Amsterdam 25(1916)1380.

[2] Penrose, R.: Proc. Roy. Soc. London A388(1982)457; GR10 Conference, eds. Bertotti, B., de Felice, F. and Pascolini, A. Padova 1 (1983)607.

[3] Shaw, W.: Proc. Roy. Soc. London A390(1983)191; in Asymptotic Behaviour of Mass and Spacetime Geometry, ed. Flaherty, F.J. Springer Lecture Notes in Physics 202 (Berlin, 1984b); Dray, T.: Class. Quant. Grav. L7(1985)2; Kelly, R.: Twistor Newsletter 20(1985)11.

[4] Jeffryes, B.P.: it GR11 Conference Abstracts, Jena (1986)539.

[5] Futamase, T. and Schutz, B.: Phys. Rev. D20(1983)10.

[6] Bartnik, R.: Proc. of the Fifth Marcel Grossmann Meeting on General Relativity ed. Blair, D.G. and Buckingham, M.J. (World Scientific, 1989)399.

[7] Dirac, P.A.M.: Proc. Roy. Soc. London A246(1958)326; Arnowitt, R. Deser, S. and Misner, C.W.: Phys. Rev. 117(1960)1595.

[8] Trautman, A.: Gravitation: An Introduction to Current Research ed. Witten, L. (Wiley, New York, 1962)169.

[9] Landau, L.D. and Lifshitz, E.M.: The Classical Theory of Fields (Addison-Wesley Press, 1962).

[10] Papapetrou, A.: Proc. R. Irish Acad A52(1948)11.

[11] Bergman P.G: and Thompson R. Phys. Rev. 89(1958)400.

[12] Tolman R. C: Relativity, Thermodynamics and Cosmology, (Oxford University Press, Oxford) p.227(1934).

[13] Weinberg, S.: Gravitation and Cosmology (Wiley, New York, 1972).

[14] Möller, C.: Ann. Phys. (NY) 4(1958)347.

[15] Möller, C.: Ann. Phys. (NY) 12(1961)118.
[16] Komar, A. Phys. Rev. 113(1959)934.

[17] Kovacs, D. Gen. Relatv. and Grav. 17, 927 (1985); Novotny, J. Gen. Relatv. and Grav. 19, 1043 (1987).

[18] Virbhadra, K.S.: Phys. Rev. D42(1990)2919.

[19] Virbhadra, K.S.: Phys. Rev. D60(1999)104041.

[20] Rosen, N. and Virbhadra, K.S.: Gen. Relati. Gravi. 25(1993)429.

[21] Virbhadra, K.S. and Parikh, J.C.: Phys. Lett. B317(1993)312.

[22] Virbhadra, K.S. and Parikh, J.C.: Phys. Lett. B331(1994)302.

[23] Xulu, S.S.: Int. J. of Mod. Phys. A15(2000)2979; Mod. Phys. Lett. A15(2000)1151 and reference therein.

[24] Xulu, S.S.: Astrophys. Space Sci. 283(2003)23-32.

[25] Lessner, G. Gen. Relativ. Grav. 28(1996)527.

[26] Sharif, M.: Int. J. of Mod. Phys. A18(2003)4361; Errata A19(2004)1495.

[27] Sharif, M.: Int. J. of Mod. Phys. D13(2004)1019; Nuovo Cimento B119(2004)463.

[28] Sharif, M. and Fatima, Tasnim: Int. J. of Mod. Phys. A(2005).

[29] Weyl, H.: Ann. Phys. (Leipzig) 54(1917)117; 59(1919)185; Civita, Levi, L.: Atti. Acad. Naz. Lince Rend. Classe Sci. Fis. Mat. e Nat., 28(1919)101; Synge, J.L.: Relativity, the General Theory (North-Holland Pub. Co. Amsterdam, 1960); Kramer, D., Stephani, H., MacCallum, M.A.H. and Hearlt, E.: Exact Solutions of Einstein’s Field Equations (Cambridge University Press, 2003).

[30] Curzon, H.E.J.: Proc. Math. Soc. London 23(1924)477.

[31] Misner,C.W., Thorne, K.S. and Wheeler, J.A. Gravitation (W.H. Freeman, New York, 1973)603.
[32] Cooperstock, F.I. and Sarracino, R.S. *J. Phys. A.* *Math. Gen.* 11(1978)877.
Cooperstock, F.I.: in *Topics on Quantum Gravity and Beyond*, Essays in honour of Witten, L. on his retirement, ed. Mansouri, F. and Scanio, J.J. (World Scientific, Singapore, 1993); Mod. Phys. Lett. A14(1999)1531; Annals of Phys. 282(2000)115;
Cooperstock, F.I. and Tieu, S.: Found. Phys. 33(2003)1033.

[33] Bringley, T.: Mod. Phys. Lett. A17(2002)157.

[34] Gad, R.M.: Mod. Phys. Lett. A19(2004)1847.