Generalized wall function and its application to compressible turbulent boundary layer over a flat plate

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Abstract. Wall function boundary conditions including the effects of compressibility and heat transfer are improved for compressible turbulent boundary flows. Generalized wall function formulation at zero-pressure gradient is proposed based on coupled velocity and temperature profiles in the entire near-wall region. The parameters in the generalized wall function are well revised. The proposed boundary conditions are integrated into Navier-Stokes computational fluid dynamics code that includes the shear stress transport turbulence model. Numerical results are presented for a compressible boundary layer over a flat plate at zero-pressure gradient. Compared with experimental data, the computational results show that the generalized wall function reduces the first grid spacing in the directed normal to the wall and proves the feasibility and effectivity of the generalized wall function method.

1. Introduction

The high-Reynolds-number turbulence models, which do not include wall correction terms in the differential equations, require modification for near-wall flow to account for fluid viscous effects and for damping of velocity fluctuations normal to the wall. Two main approaches for near-wall flows are near-wall simpler models and wall functions. The first approach is to patch in a simpler model near the wall, such as an algebraic model or a one-equation model. This method requires fine grid resolution near the wall, resulting in a relatively large total number of grid points. Although this method has been shown to yield good results for some flowfields[1], more numerical experiments are needed before it may be fully assessed. A disadvantage of the patched simpler models is that they also require fine grid spacing at the wall.

The second approach is to choose wall functions. Empirically derived algebraic models of the near-wall region of the turbulent boundary layer, high-Reynolds-number turbulence models provide boundary conditions to the mean flow Navier-Stokes equations at the first point off the wall. Researchers[2,3] demonstrated that the computational results of friction coefficient and heat transfer highly rely on the divided ways of grid in the directed normal to the wall, especially the grid spacing. These models of the near-wall region are called wall functions. Because the high-gradient near-wall region for velocity and temperature is modeled with these empirical relationships, the first point off
the wall may be placed much farther away from the viscous sublayer. Wall functions that do not constraint the position of the first grid point between the wall and logarithmic layer are called generalized wall function. Generalized wall function method reduces the number of points required to discretize a flowfield and increases the maximum allowable time step. Both of these advantages are especially useful in modeling unsteady viscous flows.

Based on analysis of Millikan[4] for turbulent channel flows, most of the early wall function boundary conditions did not consider the pressure gradient or heat transfer. Numerous formulas have been proposed to describe the universal turbulent velocity profile, called by Coles[5] “the law-of-the-wall”. Spalding[6] introduced a single formula over the whole range of the wall which represented adequately the experimental data for the universal turbulent velocity profile when the viscosity and density of the fluid are uniform. White and Christoph[7] proposed a new approach for analyzing the compressible turbulent boundary layer by using the Crocco-Busemann energy approximation. Nichols and Nelson[8] described a wall function boundary condition upon coupled velocity and temperature boundary-layer profiles based on White’s work. The novel wall function considering heat transfer and compressibility was applicable to incompressible flows, compressible flows and flows with heat transfer.

Based on Nichols’s work, the paper includes mainly three sections. In Section 2, wall function boundary conditions are improved in the aspect of compressibility and heat transfer. The proposed wall functions are formulated for zero-pressure gradient flows. The validation of the proposed generalized wall function for the compressible turbulent boundary layer over a flat plate is performed in section 3. By integrating the wall function boundary conditions with the shear stress transport turbulence model, detailed numerical results for flow over a flat plate at zero-pressure gradient are also presented in this section. The conclusions of this paper are given in section 4.

2. Generalized wall function including heat transfer and compressibility

For turbulent flow conditions with constant molecular viscosity, the velocity profile can be split into three distinguished region: the viscous sublayer, the logarithmic layer and the defect layer. In a quasi-equilibrium boundary layer, such as flow over a flat plate at zero-pressure gradient, the region between the wall and the outer edge of the logarithmic layer is universal.

Near the wall, derivatives in streamwise direction can be neglected and the turbulence variables depend only on the coordinate $y$, which is directed normal to the wall. Essentially, this is a turbulent Couette flow approximation. To derive solutions for the viscous sublayer and logarithmic layer, the equations are recast in dimensionless form:

$$u^+ = \frac{u}{u_w}, \quad \gamma^+ = \frac{\rho_w u_w y^+}{\mu_w}, \quad \gamma = \left( \frac{\tau_u}{\mu_w} \right)^{1/2}$$

(2.1)

$u$, velocity parallel to the wall; $\tau_u$, wall shear stress; $\rho_w$, wall density; $\mu_w$, wall molecular viscosity; $y$, the distance to the wall; $u_w$, friction velocity, and subscript $w$ is a label meaning the wall. Nichols proposed the velocity profiles near the wall concerning heat transfer and compressibility:

$$\gamma^+ = u^+ + \gamma_{\text{wall}}^+ + e^{-\beta} \left[ 1 - \kappa u^+ - \left( \frac{\kappa u^+}{2} \right) - \left( \frac{\kappa u^+}{3} \right) \right]$$

(2.2)

where

$$\gamma_{\text{wall}}^+ = \exp \left( \frac{\kappa}{\gamma} [\sin^{-1}(2\gamma u^+ - \beta/Q) - \phi] - KB \right)$$

(2.3)

$$\gamma = \frac{\mu_w^2}{2C_p T_w}, \quad \beta = \frac{q_w \mu_w}{\phi T_w k_w u_w}, \quad Q = (\beta^2 + 4\gamma)^{1/2}, \quad \phi = \sin^{-1} \left( \frac{\beta}{Q} \right)$$

(2.4)

with $r$, recovery factor; $C_p$, pressure coefficient; $T_w$, wall temperature; $q_w$, wall heat transfer; $k_w$, thermal conductivity; heat transfer $q_w = -k_w \partial T_w / \partial y$; the constant $\kappa$ and $B$ are generally taken as 0.41 and 5.5, respectively. Non-dimensional parameter $\gamma$ and $\beta$ model compressibility effects and heat transfer effects. The wall heat transfer $q_w$ appears only in the definition of $\beta$. The two parameters $\gamma$
and $\beta$ arise from writing the Crocco-Busemann assumption for the energy equation $T = a + bu + cu^2$ [9]. The Crocco assumption appears to be quite reasonable for arbitrary compressible turbulent flows. For example, Lee [10] expressed the density distribution in terms of velocity, hence uncoupling the energy and momentum equations. The constants $(a, b, c)$ can be related to wall conditions as follows. First, at the wall, $u = 0$, which is the no-slip condition. Hence $a = T_w$. Second, the temperature gradient at the wall must reflect the wall heat flux. Hence $\beta = -r / 2C_p$ for the adiabatic wall. It is desirable to use the wall variable $u' = u / u_c$ in the final form of the Crocco-Busemann relation. With the above considerations, for a perfect gas Crocco-Busemann relation can now be written as:

$$\frac{\rho_u}{\rho} = \frac{T}{T_u} = 1 + \beta u' - \gamma u'^2 \Rightarrow$$

$$T = T_u (1 + \beta u' - \gamma (u')^2) \quad (2.5)$$

Equation (2.5) is used to formulate the equation (2.3). For adiabatic wall cases $q_w = 0$, that is $\beta = 0$, with the definition of $\gamma$ in equation (2.4) and the wall variable $u' = u / u_c$ in equation (2.1), the Crocco-Busemann equation reduces to:

$$T = T_u - ru^2 / (2C_p) \quad (2.6)$$

2.1. Improved wall function formulation

Equation (2.5) is derived from the Crocco-Busemann energy approximation. For the compressible turbulent boundary layer at zero-pressure gradient, the expression of Crocco-Busemann equation is:

$$T = T_u + (T_{aw} - T_u) \frac{u}{U_c} - \frac{ru^2}{2C_p} \quad (2.7)$$

where

$$T_{aw} = T_e (1 + \frac{\Gamma - 1}{2} rMa^2_e) \quad (2.8)$$

$U_c$, the free stream velocity; $T_e$, the free stream temperature and $Ma^2_e$, the free stream Mach number; $T_{aw}$, adiabatic wall temperature; $\Gamma = 1.4$ for the perfect gas; subscripts of $(.)_w$ and $(.)_c$ mean the adiabatic wall and the free stream. $T_u$ has a special relation with $q_w$ as follows[11]:

$$\frac{T_{aw} - T_u}{U_c} = \frac{q_w \mu_u}{k_u \tau_w} \quad (2.9)$$

Integrating equation (2.7) and equation (2.9), we obtain:

$$T = T_u + \frac{q_w \mu_u}{k_u \tau_w} u - \frac{ru^2}{2C_p} \quad (2.10)$$

Equation (2.10) becomes equation (2.5) by supposing the parameters $\beta$ and $\gamma$. Moreover, the above derivation process shows that parameter $\beta$ comes from equation (2.9). In other words, the formulation of $\beta$ is expressed implicitly in the equation (2.9). Utilizing $Pr = \mu C_f / k_u$, Stanton number $St$ [12] is formulated by:

$$St = \frac{q_w}{\rho_c U_c C_f (T_{aw} - T_u)} = \frac{2C_f}{Pr} \quad (2.11)$$

$$\Rightarrow \frac{St}{2C_f} = \frac{1}{Pr} \approx 1.39 \quad (2.12)$$

where $C_f$, skin friction coefficient; $\rho_c$, free stream density; $Pr = 0.72$, laminar Prandtl number. Therefore the equation (2.9) is rewritten to:
According to a large number of supersonic experimental data with skin friction $C_f \approx 3 \times 10^{-4}$ and CFD numerical simulation results, the empirical Reynolds analogy factor $St/(2C_f)$ [13] should be 1.16. Hence the equation (2.10) must be revised by:

$$T = T_w + \frac{1.2q_w \mu_w}{k_w \tau_w} u - \frac{ru^2}{2C_p}$$

(2.14)

with $1.2 \approx 1.39/1.16$.

We denominate the improved wall function “Method 1”, with heat transfer parameter $\beta$:

$$\beta = \frac{1.2q_w \mu_w}{\rho_w T_w k_w u}$$

(2.15)

### 2.2. Generalized wall function based on Walz relation

In order to revise the Crocco-Busemann energy approximation on condition of $Pr \approx 1$, Walz[14] proposed:

$$\frac{T}{T_e} = \frac{T_w}{T_e} + (\frac{T_m - T_e}{T_e}) f\left(\frac{u}{U_e}\right) + \frac{T_e - T_w}{T_e} \left(\frac{u}{U_e}\right)^2$$

(2.16)

where

$$f\left(\frac{u}{U_e}\right) = (1-\alpha)\left(\frac{u}{U_e}\right)^2 + \alpha \left(\frac{u}{U_e}\right), \quad \alpha \approx 0.82$$

(2.17)

Equation (2.16) is called Walz function. According to equation (2.17), we have:

$$\frac{T}{T_e} = \frac{T_w}{T_e} + \alpha\left(\frac{T_m - T_e}{T_e}\right) \frac{u}{U_e} + \left[\frac{T_e - T_m}{T_e} + (1-\alpha)\frac{T_m - T_w}{T_e}\right] \left(\frac{u}{U_e}\right)^2$$

(2.18)

Then

$$T = T_w + \frac{\alpha q_w \mu_w}{k_w \tau_w} u - \frac{ru^2}{2C_p} - (1-\alpha)\frac{q_w \mu_w}{k_w \tau_w} \frac{u^2}{U_e^2}$$

(2.19)

The new dimensionless parameters are in the following form:

$$\beta = \frac{\alpha q_w \mu_w}{\rho_w T_w k_w u}, \quad \gamma = \frac{ru^2}{2C_p T_w} \frac{(1-\alpha)q_w \mu_w}{\rho_w T_w k_w U_e}, \quad \alpha \approx 0.82$$

(2.20)

We dominate the generalized wall function method based on Walz equation “Method 2”.

In this section, the generalized wall function, as a semi-empirical approach, has been corrected and proposed by two different ways. In the section 3, we will compare the improved wall function “Method 1” and the generalized wall function method based on Walz relation “Method 2” by means of numerical simulation of compressible turbulent boundary layer over a flat plate.

### 3. Implication to compressible boundary layer over a flat plate

The generalized wall functions described above are implemented in a two-dimensional compressible RANS flow and SST turbulence model. Inflow Conditions[15] of flow over a flat plate are defined by $M_e = 3$, $T_e = 288K$, $\rho_e = 0.17527kg/m^3$, wall temperature $T_w = 294.44K$. In the numerical simulation, discretized format is Roe scheme[16]. SST model[17] is solved on different first grid spacing $\Delta y = 5 \times 10^{-7}$, and $2.5 \times 10^{-5}, 5 \times 10^{-5}$, which corresponds the dimensionless distance $y^* = 0.1, 25, 50$ by making use of the relation $y^* = \rho_e u^2 / \mu_w$. The numerical simulation is performed using a finite difference CFD code developed by us. Reliability of the code has been validated and verified by a variety of simulations[18,19]. The computational domain is a rectangle with a length of 1m and a width of 0.8m.
The main work for CFD code coupled with wall function boundary condition is to replace the corresponding terms of momentum and energy equations in CFD code respectively by wall shear stress \( \tau_w \) and heat transfer \( q_w \) obtained from the wall function. Firstly, we use velocity law-of-the-wall of the momentum equation in CFD viscous equals at the first point off the wall computed by CFD code to solve \( \tau_w \) iteratively. Then, we replace the shear stress term \( \tau_{ij} \) of the momentum equation in CFD viscous subroutine by \( \tau_w \). Based on the constant stress assumption in the lower part of the turbulent boundary layer[20], it is reasonable to let \( \tau_{ij} \) be equal to \( \tau_w \). Secondly, we utilize \( u^+ \) obtained from the above velocity law-of-the-wall and the temperature \( T_i \) at the first point off the wall computed by CFD code to calculate parameter \( \beta \) by temperature law-of-the-wall equation (2.5) and further solve the wall heat transfer \( q_w \). Similarly, we replace the heat transfer term \( q_{ij} \) of the energy equation in CFD viscous subroutine by \( q_w \). According to the assumption by Nichols[8] that heat transfer term keeps constant in the lower part of the turbulent boundary layer, in other words \( q_{ij} \) equals \( q_w \). Finally, the results of \( \tau_w \) and \( q_w \) can be returned back to main CFD code.

Nichols’s wall function formulation is described from equation (2.2) to (2.4). The improved wall function “Method 1”, equation (2.15) and the generalized wall function method based on Walz function “Method 2”, equation (2.20) are integrated into de boundary conditions of the flow over a flat plate.

Figure 1 shows the numerical simulation solutions for \( u^+ \) and \( y^+ \) in logarithmic coordinates when \( y^+ = 0.1, 25, 50 \). Linear in the legend represents \( u^+ = y^+ \) for the viscous sublayer near the wall, and log-law of wall stands for \( u^+ = \kappa^{-1} \cdot \ln(y^+) + B \), the logarithmic layer[21].

![Figure 1. Comparison of near-wall velocity profiles among different grid spacing in directed normal to the wall](image)

Three results could be obtained from the figure 1. When \( y^+ = 0.1 \), the first grid notes in the directed normal to the wall are located in viscous sublayer, the simulation do not need use the wall function; when the first grid spacing is broadening to \( y^+ = 50 \), the generalized wall function become crucial in order to get the better results compared with the theoretical solutions. Generalized wall functions are designed to be used with coarse near-wall grids; the results of Method 1 and Method 2 presented with dashed line are much better than the Nichols’ wall function method with solid lines.
Figure 2 demonstrates comparisons of skin friction coefficient $C_f = 2\tau_w / (\rho C U^2)$ and Van Driest theoretical data[13] under the above inflow conditions and different grid spacing. Figure 3 presents the computational results for heat transfer $q_w$ and $x$. It can be seen from the two figures that computational results agree with the theoretical solutions well, which illustrates that the simulations are reliable.

![Figure 2. Near-wall skin friction profiles of different grid spacing](image1)

![Figure 3. Near-wall heat transfer profiles of different grid spacing](image2)

We define the deviation[22] between the numerical result and experimental data $\epsilon$ by

$$
\epsilon = \frac{1}{N} \sum_{j=1}^{N} |\phi(r)_j - \phi(d)_j|
$$

(3.1)

with $N$, number of discrete points; $\phi$, turbulent quantity; $r$ and $d$ mean the numerical result and theoretical data. The deviation defined can indicate well the reliability of method. Finally, we can compare the deviation between Method 1 and Method 2 according to equation (3.1) with the same first grid spacing $y^+ = 50$.

| Method | $\epsilon(u^+)$ | $\epsilon(C_f \times 10^{-5})$ | $\epsilon(q_w \times 10^{-3})$ |
|--------|-----------------|------------------|-----------------|
| Method 1 | 1.240 | 3.912 | 9.789 |
| Method 2 | 1.149 | 3.267 | 9.142 |

From the table, the results of Method 2, $C_f$ and $q_w$, are more reliable than Method 1 on account of the much smaller deviation of Method 2. Therefore Method 2 is chosen as the generalized wall function. We formulate finally the generalized wall function based on Walz function as follows:

$$
y^+ = u^+ + y_{White}^+ + e^{-\beta}\{1 - ku^+ - \left(\frac{ku^+}{2!}\right)^2 - \left(\frac{ku^+}{3!}\right)^3\}
$$

(3.2)

with

$$
y_{White}^+ = \exp\left(\frac{\beta}{\gamma}\right)\sin^{-1}\left(\frac{\beta}{\gamma}\right)\Phi[\sin^{-1}\left(\frac{\beta}{\gamma}\right)]
$$

$$
\beta = \frac{\alpha q_w u_c}{\rho u_c k_w T_w}, \quad \gamma = \frac{\mu^2}{2C_p T_w}, \quad \alpha \approx 0.82, \quad Q = (\beta^2 + 4\gamma)^{1/2}, \quad \Phi = \sin^{-1}\left(\frac{-\beta}{\gamma}\right)
$$

(3.3)

4. Conclusions
A general set of wall function boundary conditions has been developed and corrected for compressible flows with heat transfer near the wall. The generalized wall functions are based on coupled velocity profiles and Walz relation in the near-wall of the boundary layer. The proposed boundary conditions
are integrated into Navier-Stokes computational fluid dynamics code that includes the shear stress transport turbulence model. The formula of the heat transfer term in the near-wall region was derived so that the coupling between the wall function boundary condition and CFD code is realized more accurately.

Through the analysis of numerical results of the near-wall region, we obtained the velocity, skin friction and heat transfer focusing on compressible turbulent boundary layer over a flat plate at zero-pressure gradient. Moreover, the results verify that the proposed generalized wall functions are designed to be used with coarse near-wall grids. For the flat plate case, the modified compressible wall function can produce reasonable solutions of skin friction and heat transfer. The successful numerical results powerfully show that generalized wall function and near-wall grid adaptation ensure both an appropriate resolution of near-wall flow physics and take into account the range of validity and stability of the wall-function formulation.

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