The running maximum of the Cox-Ingersoll-Ross process with some properties of the Kummer function

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Abstract

We derive tail asymptotics for the running maximum of the Cox-Ingersoll-Ross process. The main result is proved by the saddle point method, where the tail estimate uses a new monotonicity property of the Kummer function. This auxiliary result is established by a computer algebra assisted proof. Moreover, we analyse the coefficients of the eigenfunction expansion of the running maximum distribution asymptotically.

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1 Introduction

The Cox-Ingersoll-Ross (CIR) process, also known as Feller diffusion, is defined by the stochastic differential equation
\[ dX_t = (\alpha - \beta X_t)dt + \sigma \sqrt{X_t}dW_t, \]  
where \( W \) is a standard Brownian motion, and \( \alpha, \beta, \sigma, X_0 > 0 \). This process has been intensively studied and is of particular interest in mathematical finance, where its mean-reversion property, non-negativity and explicit transition density make it a popular choice for modelling stock volatility and other quantities [7, 20]. The main results of this paper, Corollaries 2.6 and 2.7, give asymptotics for \( \mathbb{P}[\max_{0 \leq s \leq t} X_s \geq z] \) for fixed \( t \) and large \( z \). This is achieved by a saddle point approximation of an integral representation involving the Kummer function
\[ M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n, \]  
where \((a)_n = a(a+1) \cdots (a+n-1)\) is the Pochhammer symbol. This function satisfies the confluent hypergeometric ODE
\[ x \frac{d^2M}{dx^2} + (b - x) \frac{dM}{dx} - aM = 0. \]  
We refer to 13.7 and 13.8 in [5, 15] for many asymptotic results about this function. In the proof of our main theorems we apply two auxiliary results that may be of independent interest. In Appendix A we obtain asymptotics of the Kummer function \( M(a, b, x) \), where \( x \uparrow \infty \) and the parameter \( a \) is proportional to \( x \). This result is known, but we give a new proof, again using the saddle point method. Next, we give a computer algebra assisted proof of the monotonicity of \( |M(a, b, x)| \) with respect to \( \text{Im}(a) \) in Appendix B which is also needed for one of our main results (Corollary 2.6). In Section 3 we analyse the coefficients of the eigenfunction expansion of the running maximum distribution. Appendix C contains a new proof of the known fact that the \( a \)-zeros of \( M(a, b, x) \) are negative and simple for \( b, x > 0 \).

2 Tail asymptotics for the running maximum of the CIR process

For any \( \varepsilon \in (0, 1) \) the scaled CIR process \( \varepsilon X \) satisfies
\[ d(\varepsilon X) = (\varepsilon \alpha - \varepsilon \beta X_t)dt + \sigma \sqrt{\varepsilon X_t}dW_t. \]
Since $\varepsilon \alpha < \alpha$, it follows from a standard comparison result (Proposition 5.2.18 in \[9\]) that $\varepsilon X$ is almost surely dominated by $Z^{(\varepsilon)}$ with dynamics

$$dZ^{(\varepsilon)}_t = (\alpha - \beta Z^{(\varepsilon)}_t)dt + \sigma \sqrt{\varepsilon Z^{(\varepsilon)}_t}dW_t, \quad Z^{(\varepsilon)}_0 = X_0.$$  

The family of processes $Z^{(\varepsilon)}$ converges to the deterministic solution of $dZ^{(0)}_t = (\alpha - \beta Z^{(0)}_t)dt$ and satisfies a large deviations principle for $\varepsilon \downarrow 0$ (Theorem 1.2 in \[2\]). From the contraction principle (Theorem 4.2.1 in \[4\]) applied to the functional $f \mapsto \max_{0 \leq s \leq t} f$ it easily follows that

$$\mathbb{P}\left[ \max_{0 \leq s \leq t} X_s \geq z \right] = \mathbb{P}\left[ \max_{0 \leq s \leq t} z^{-1} X_s \geq 1 \right] \leq \mathbb{P}\left[ \max_{0 \leq s \leq t} Z^{(1/z)}_s \geq 1 \right] \leq \exp \left( -c \sigma^{-2} z (1 + o(1)) \right), \quad z \uparrow \infty,$$

(2.1)

where $c > 0$ depends on $t, \alpha, \beta$. This exponential bound was used recently in \[6\], and prompted us to analyse the tail of the running maximum of the CIR process in more detail, i.e., to determine the asymptotic behavior of the left-hand side of (2.1).

Define the hitting time of level $z$ by

$$\tau_{X_0 \to z} := \inf \{ t \geq 0 : X_t = z \}.$$  

It is a classical fact that the Laplace transform of a diffusion hitting time can be expressed by the eigenfunctions of the infinitesimal generator; see pp. 128–130 in \[8\]. For the CIR process, these eigenfunctions are Kummer functions; we refer to \[3\] for details. Corollary 4 of that paper states that

$$\mathbb{E}\left[ e^{-s \tau_{X_0 \to z}} \right] = \frac{M(s/\beta, 2\alpha/\sigma^2, 2\beta X_0/\sigma^2)}{M(s/\beta, 2\alpha/\sigma^2, 2\beta z/\sigma^2)}$$

for $0 < X_0 < z$. By Laplace inversion, the law of the running maximum of the CIR process can be expressed as

$$\mathbb{P}\left[ \max_{0 \leq s \leq t} X_s \geq z \right] = \mathbb{P}[\tau_{X_0 \to z} \leq t] \leq \frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} e^{ts} \frac{M(s/\beta, 2\alpha/\sigma^2, 2\beta X_0/\sigma^2)}{s M(s/\beta, 2\alpha/\sigma^2, 2\beta z/\sigma^2)} ds.$$  

(2.3)

The main results of the present paper, namely Corollaries 2.6 and 2.7 below, give asymptotics of this probability for fixed $t$ and large $z$. To simplify notation, we define

$$I(\lambda, b, x, y) := \frac{1}{2\pi i} \int_{1-\infty}^{1+\infty} e^{\lambda s} \frac{M(s, b, y)}{s M(s, b, x)} ds.$$

(2.5)
for $0 < y < x$ and $b, \lambda > 0$, so that

$$P\left[ \max_{0 \leq s \leq t} X_s \geq z \right] = I\left( \beta t, \frac{2\alpha}{\sigma^2}, \frac{2\beta z}{\sigma^2}, \frac{2\beta X_0}{\sigma^2} \right). \quad (2.6)$$

One of our main results (Theorem 2.3 and its Corollary 2.6) will be proven conditionally, assuming the following statement.

**Conjecture 2.1.** Let $b, u_0 > 0$ and $x > y > 0$. Then

$$v \in \mathbb{R}_+ \mapsto \left| \frac{M((u_0 + iv)x, b, y)}{M((u_0 + iv)x, b, x)} \right| \text{ decreases.} \quad (2.7)$$

While we did not succeed in proving this conjecture, note that a related inequality is established in Corollary B.2, namely that the denominator of (2.7) increases with respect to $v > 0$. Define

$$\phi(u) := \lambda u - \psi(t_0(u)) \quad (2.8)$$

as well as

$$\phi(u_0) = -\frac{1}{2}(1 + \coth(\frac{1}{2}\lambda)), \quad \phi''(u_0) = u_0^{-1} \tanh(\frac{1}{2}\lambda), \quad \sqrt{1 + 4u_0} = \coth(\frac{1}{2}\lambda).$$

In addition, we define

$$C_1 := \frac{u_0^{b-3/2}}{\Gamma(b)\sqrt{\phi''(u_0)}} \left( \frac{\sqrt{1 + 4u_0} - 1}{2} \right)^{-b} (1 + 4u_0)^{1/4} \quad (2.10)$$

and

$$C_2 := \frac{u_0^{b/2-5/4}}{\sqrt{2\pi\phi''(u_0)}} \left( \frac{\sqrt{1 + 4u_0} - 1}{2} \right)^{-b} (1 + 4u_0)^{1/4} e^{y/2} y^{-b/2+1/4} \times \exp \left( \frac{1}{2} \phi''(u_0) y u_0 (1 + 4u_0) - y \sqrt{1 + 4u_0} \right).$$

Both of these quantities are constants, because $C_2$ is used in a result where $y > 0$ is constant.
Theorem 2.2. Let $\lambda, b > 0$, and let $y = y(x) > 0$ be a function of $x$ satisfying $y(x) = o((x \log x)^{-1})$ for $x \uparrow \infty$. Then the integral in (2.5) satisfies

$$I(\lambda, b, x, y) \sim C_1 x^{b-1} \exp \left( x \phi(u_0) \right)$$

as $x \uparrow \infty$.

The proof of Theorem 2.2 will be given towards the end of this section. It uses the main results of the appendices (Theorem A.1 and Corollary B.2). We first prove the following result, where the parameter $y$ is constant. Theorems 2.2 and 2.3 give first-order asymptotics. As usual when applying the saddle point method, providing further terms of the asymptotic expansion would be a matter of straightforward, but cumbersome calculations.

Theorem 2.3. Suppose that Conjecture 2.1 is true. Let $\lambda, b, y > 0$. Then

$$I(\lambda, b, x, y) \sim C_2 x^{b/2-3/4} \exp \left( x \phi(u_0) + 2\sqrt{yu_0x} \right)$$

(2.11)
as $x \uparrow \infty$.

Proof. We rewrite the integral as

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{\lambda s} M(s, b, y) \frac{ds}{s} = \frac{1}{2\pi i} \int_{\hat{u}-i\infty}^{\hat{u}+i\infty} e^{\lambda xu} \frac{M(ux, b, y)}{u} \frac{du}{u}$$

$$= \frac{1}{2\pi i} \left( \int_{\hat{u}-i\infty}^{\hat{u}+i\infty} e^{\lambda xu} M(ux, b, y) \frac{du}{u} + \int_{|\text{Re}(u)|=\hat{u}}^{Re(\hat{u})=\hat{u}} e^{\lambda xu} M(ux, b, y) \frac{du}{u} \right),$$

(2.12)

where $\hat{u} = \hat{u}(x) > 0$ satisfies $\hat{u}(x) \uparrow u_0$ and will be fixed later. We will show in Lemmas 2.4 and 2.5 that the second integral in (2.12) is negligible, and focus now on the first integral. For $u$ in its integration range and $x \uparrow \infty$, the first term of the expansion (10.3.51) in [17] yields

$$M(ux, b, y) \sim \frac{\Gamma(b)}{\sqrt{2\pi}} \exp \left( \frac{1}{2} y + 2\sqrt{uxy} \right)(uxy)^{-b/2+1/4}.$$ (2.13)

As for the denominator, Theorem A.1 implies

$$M(ux, b, x) \sim \frac{\Gamma(b)}{\sqrt{2\pi}} (1 + 4u)^{-1/4} \left( \frac{\sqrt{1 + 4u} - 1}{2} \right)^b (ux)^{1/2-b} e^{x\psi(t_0)}.$$ (2.14)

From these estimates, we obtain

$$\frac{1}{2\pi i} \int_{\hat{u}-i\infty}^{\hat{u}+i\infty} e^{\lambda xu} \frac{M(ux, b, y)}{u} \frac{du}{u} \sim u_0^{b/2-5/4} \left( \frac{\sqrt{1 + 4u_0} - 1}{2} \right)^{-b} (1 + 4u_0)^{1/4} e^{y/2} y^{-b/2+1/4} \times \frac{1}{2\pi i} \int_{\hat{u}-i\infty}^{\hat{u}+i\infty} \exp \left( \lambda xu + 2\sqrt{uxy} - x\psi(t_0) \right) du.$$ (2.15)
We put 
\[ \chi(u, x) := x\phi(u) + 2\sqrt{uxy}, \]
so that \( e^\chi \) is the integrand on the right-hand side of (2.15). We now define \( \hat{u}(x) \) as the saddle point of this integrand, i.e. as the positive solution of
\[ 0 = \frac{1}{x} \frac{\partial}{\partial u} \chi(u, x) = \lambda - \log \frac{\sqrt{1 + 4u} + 1}{\sqrt{1 + 4u} - 1} + \sqrt{\frac{y}{ux}}. \tag{2.16} \]
It is easy to see that there is a unique solution for large \( x \), and that it converges to the (constant) saddle point \( u_0 \) of \( e^{x\phi(u)} \) as \( x \uparrow \infty \). If we write the integration parameter as \( u = \hat{u} + iv \), then the local expansion of \( \chi \) is
\[ \chi(u, x) = \chi(\hat{u}, x) - \frac{1}{2} \chi''(\hat{u}, x)v^2 + O(x^{-1/5}), \]
where the derivative is with respect to \( u \), and the error term follows from \( \chi''(\hat{u}, x) = O(x) \) and \( v^3 = O(x^{-6/5}) \). Now we can evaluate the integral in (2.15) asymptotically:
\[ \frac{1}{2\pi i} \int_{\hat{u}-ix^{-2/5}}^{\hat{u}+ix^{-2/5}} \exp (\chi(u, x)) du \]
\[ \sim \exp \left( \chi(\hat{u}, x) \right) \frac{1}{2\pi} \int_{-x^{-2/5}}^{x^{-2/5}} \exp \left( -\frac{1}{2} \chi''(\hat{u}, x)v^2 \right) dv \]
\[ \sim \frac{\exp \left( \chi(\hat{u}, x) \right)}{2\pi \chi''(\hat{u}, x)} \int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi x} \chi''(\hat{u}, x). \tag{2.17} \]
By inserting an ansatz \( \hat{u} = u_0 + w \) with \( w = o(1) \) into (2.16), it is easy to see that
\[ \hat{u} = u_0 - \sqrt{\frac{yu_0(1 + 4u_0)}{x}} \left( 1 + O(x^{-1/2}) \right). \]
This implies
\[ \sqrt{\hat{u}x} = \sqrt{u_0x} - \frac{1}{2} \sqrt{y(1 + 4u_0)} + O(x^{-1/2}) \]
and (recall that \( u_0 \) satisfies \( \phi'(u_0) = 0 \))
\[ x\phi'(\hat{u}) = x\phi(u_0) + \frac{1}{2} \phi''(u_0) yu_0(1 + 4u_0) + O(x^{-1/2}). \]
We conclude
\[ \chi(\hat{u}, x) = x\phi(u_0) + 2\sqrt{u_0xy} + \frac{1}{2} \phi''(u_0) yu_0(1 + 4u_0) - y\sqrt{1 + 4u_0} + O(x^{-1/2}). \]
We insert this and \( \chi''(\hat{u}, x) \sim x\phi''(u_0) \) into (2.17), and then use the resulting asymptotics in (2.15). Estimation of the second integral in (2.12) by Lemmas 2.4 and 2.5 below completes the proof. Clearly, it suffices to do the tail estimate for the upper half of the integration path.

\[ \square \]
Lemma 2.4 (Tail estimate for large $\text{Im}(u)$). Let $\lambda, b, y > 0$. Then
\[
\left| \int_{\hat{u}+i\log x}^{\hat{u}+i\infty} \frac{e^{\lambda xu} M(ux, b, y)}{u M(ux, b, x)} du \right| \leq \exp \left( -x \log x \left( 1 + o(1) \right) \right).
\]

Proof. Recall that the saddle point $\hat{u} = \hat{u}(x)$ was defined in (2.16). By (10.3.51) in [17], we have
\[
M(ux, b, x) = v - b/2 \exp (x \sqrt{2v} + O(x))
\]
where we write $u = \hat{u} + iv$ again. From (2.13), we get
\[
|M(ux, b, y)| = \exp (2 \text{Re} \sqrt{uxy} + O(\log |ux|))
\]
for large $x$. We can thus estimate the integral by
\[
e^{O(x)} \int_{\log x}^{\infty} v^{b/2} \exp \left( -x \sqrt{2v} + 3 \sqrt{xyv} \right) dv \leq e^{O(x)} \int_{\log x}^{\infty} e^{-x\sqrt{v}} dv
\]
\[
= e^{O(x)} \int_{\log(x/2)}^{\infty} e^{-xz} dz
\]
\[
\leq e^{O(x)} \int_{\log(x/2)}^{\infty} e^{-z(x-1)} dz
\]
\[
= \exp \left( -x \log x (1 + o(1)) \right). \quad \square
\]

The final estimate for the proof of Theorem 2.3 is provided by the following lemma. Note that the exponential factor $\exp(-cx^{1/5})$ is negligible compared to the power of $x$ in (2.11).

Lemma 2.5 (Tail estimate for intermediate $\text{Im}(u)$). Suppose that Conjecture 2.1 is true. Let $\lambda, b, y > 0$. Then there is a positive constant $c$ such that
\[
\left| \int_{\hat{u}+ix-2/5}^{\hat{u}+i\log x} \frac{e^{\lambda xu} M(ux, b, y)}{u M(ux, b, x)} du \right| \leq \exp \left( \chi(u_0, x) - cx^{1/5} + o(x^{1/5}) \right).
\]

Proof. By assumption (Conjecture 2.1), $|M(ux, b, y)/M(ux, b, x)|$ is a decreasing function of $\text{Im}(u)$. The integrand thus satisfies
\[
\left| \frac{e^{\lambda xu} M(ux, b, y)}{u M(ux, b, x)} \right| \leq e^{\lambda \hat{u} + O(1)} \left| \frac{M(ux, b, y)}{M(ux, b, x)} \right|_{u=\hat{u}+ix-2/5}. \quad (2.18)
\]
We know from (2.14), the definition of \( \phi \), and \( \phi'(u_0) = 0 \) that

\[
|M(ux, b, x)|_{u = \hat{u} + ix^{-2/5}} = \exp \left( \lambda \hat{u}x - x \text{Re}(u + ix^{-2/5}) + O(\log x) \right)
= \exp \left( \lambda \hat{u}x - x\phi(u_0) + \frac{1}{2} \phi''(u_0)x^{1/5} + O(1) \right).
\]

(2.19)

By (2.13), we have

\[
|M(ux, b, y)|_{u = \hat{u} + ix^{-2/5}} = \exp \left( 2\text{Re}(\sqrt{uxy} + O(\log x)) \right)
= \exp \left( 2\sqrt{u_0 xy} + O(\log x) \right).
\]

(2.20)

Formulas (2.18)–(2.20) imply

\[
\left| e^{\lambda xu} M(ux, b, y) \right|_{u = \hat{u} + ix^{-2/5}} \leq \exp \left( \chi(u_0, x) - \frac{1}{2} \phi''(u_0)x^{1/5} + O(x^{1/10}) \right).
\]

The assertion is established, with \( c = \frac{1}{2} \phi''(u_0) > 0 \), by multiplying this estimate for the integrand with the length of the integration path.

**Proof of Theorem 2.2.** This proof is a simplified variant of the proof of Theorem 2.3, which we have just completed. Instead of \( \text{Re}(u) = \hat{u}(x) \), we integrate over the line \( \text{Re}(u) = u_0 \), which does not depend on \( x \). Lemma 2.4 and its proof need no modification except replacing \( \hat{u} \) by \( u_0 \). By (10.3.51) in [17], we have

\[
M(ux, b, y) \sim \Gamma(b)(uxy)^{(1-b)/2}I_{b-1}(2\sqrt{uxy}), \quad x \uparrow \infty,
\]

where \( I_\nu \) is the modified Bessel function. For \( 0 \leq \text{Im}(u) \leq \log x \), this implies \( M(ux, b, y) \sim 1 \), since \( I_{b-1}(z) \sim 2^{1-b}z^{b-1}/\Gamma(b) \) for \( z \to 0 \). This can be used to adapt the proof of Lemma 2.5. The estimate (2.18) becomes

\[
\left| e^{\lambda xu} M(ux, b, y) \right|_{u = \hat{u} + ix^{-2/5}} \leq \exp \left( \chi(u_0, x) - \frac{1}{2} \phi''(u_0)x^{1/5} + O(x^{1/10}) \right),
\]

where we have applied the main result of Appendix B (Corollary B.2). From this, it easily follows that this part of the tail satisfies

\[
\left| \int_{u_0 + i x^{-2/5}}^{u_0 + i \log x} e^{\lambda xu} M(ux, b, y) \frac{du}{M(ux, b, x)} \right| \leq \exp \left( x\phi(u_0) - cx^{1/5} + o(x^{1/5}) \right).
\]
It remains to approximate the central part of the integral. Using \( M(ux, b, y) \sim 1 \) again, we obtain
\[
\frac{1}{2\pi i} \int_{u_0 - ix - 2/5}^{u_0 + ix - 2/5} \frac{e^{\lambda xu} M(ux, b, y)}{u M(ux, b, x)} du \\
\sim \sqrt{2\pi} u_0^{-3/2} \left( \frac{\sqrt{1 + 4u_0} - 1}{2} \right)^{-b} (1 + 4u_0)^{1/4} x^{b-1/2} \\
\times \frac{1}{2\pi i} \int_{u_0 - ix - 2/5}^{u_0 + ix - 2/5} \exp \left( \lambda xu - x\psi(t_0) \right) du.
\]
The proof is now completed analogously to the proof of Theorem 2.3. By (2.8) and (2.9), the exponent of the integrand has the expansion
\[
\lambda xu - x\psi(t_0(u)) = x\phi(u) \\
= x\phi(u_0) - \frac{1}{2} \phi''(u_0)xv^2 + O(x^{-1/5}),
\]
which implies
\[
\frac{1}{2\pi i} \int_{u_0 - ix - 2/5}^{u_0 + ix - 2/5} \exp \left( \lambda xu - x\psi(t_0) \right) du \sim \frac{\exp \left( x\phi(u_0) \right)}{\sqrt{2\pi \phi''(u_0)x}}.
\]

Now we return to the problem on CIR processes raised at the beginning of this section. Define (see (2.10))
\[
\hat{C}_1 := C_1 \bigg|_{\lambda = \beta t, b = 2\alpha/\sigma^2}
\]
and
\[
\hat{C}_2 := C_2 \bigg|_{\lambda = \beta t, b = 2\alpha/\sigma^2, y = 2\beta X_0/\sigma^2}.
\]

**Corollary 2.6.** Let \( \alpha, \beta, \sigma, t > 0 \), and let \( X_0 = X_0(z) > 0 \) be a function of \( z \) that satisfies \( X_0 = o\left((z \log z)^{-1}\right) \) as \( z \uparrow \infty \). Then the CIR process defined in (1.1) satisfies
\[
P \left[ \max_{0 \leq s \leq t} X_s \geq z \right] \sim \hat{C}_1 \left( \frac{2\beta z}{\sigma^2} \right)^{2\alpha/\sigma^2 - 1} \exp \left( -\frac{\beta}{\sigma^2} \left( 1 + \coth \left( \frac{1}{2}\beta t \right) \right) z \right), \quad z \uparrow \infty.
\]

**Proof.** Immediate from (2.6) and Theorem 2.2. \( \square \)

Analogously, using Theorem 2.3, we get the following result.
Corollary 2.7. Suppose that Conjecture 2.1 is true. Let $\alpha, \beta, \sigma, X_0, t > 0$. Then the CIR process satisfies

$$\Pr \left[ \max_{0 \leq s \leq t} X_s \geq z \right] \sim \hat{C}_2 \left( \frac{2\beta}{\sigma^2} \right)^{\alpha/\sigma^2 - 3/4} z^{\alpha/\sigma^2 - 3/4} \times \exp \left( \frac{\beta}{\sigma^2} \left( 1 + \coth \left( \frac{1}{2} \beta t \right) \right) z + \frac{2\beta \sqrt{X_0}}{\sigma^2 \sinh \left( \frac{1}{2} \beta t \right) \sqrt{z}} \right), \quad z \uparrow \infty.$$ 

Note that the cruder LDP bound (2.2) correctly captures the dependence of the exponential factor of the tail asymptotics on $z$ and $\sigma$. As a consistency check, we compare our results with the tail of the CIR marginal distribution. From the well-known explicit transition density (see (4) in [3]), we obtain, for fixed $t > 0$,

$$\Pr \left[ X_t \geq z \right] = e^{O(\log z)} \int_z^{\infty} \exp \left( - \frac{2\beta e^{\beta t} y}{\sigma^2 (e^{\beta t} - 1)} \right) I_{2\alpha/\sigma^2 - 1} \left( \frac{4\beta \sqrt{X_0} e^{\beta t} y}{\sigma^2 (e^{\beta t} - 1)} \right) dy$$

$$= e^{O(\sqrt{z})} \int_z^{\infty} \exp \left( - \frac{2\beta e^{\beta t} y}{\sigma^2 (e^{\beta t} - 1)} \right) dy$$

$$= \exp \left( - \frac{2\beta e^{\beta t}}{\sigma^2 (e^{\beta t} - 1)} z + O(\sqrt{z}) \right)$$

$$= \exp \left( - \frac{\beta}{\sigma^2} \left( 1 + \coth \left( \frac{1}{2} \beta t \right) \right) z + O(\sqrt{z}) \right), \quad z \uparrow \infty.$$ 

Therefore, logarithmic tail asymptotics of the marginal and the running maximum agree. This is not surprising, because for paths having a very large running maximum $\max_{0 \leq s \leq t} X_s$, this maximum is typically realized close to time $t$, where the process has had the most time to deviate from its initial value.

While there is a considerable literature on tail asymptotics of diffusion hitting times (also known as first-passage times), asymptotics with respect to the level have received less attention. By (2.3), level asymptotics are equivalent to tail asymptotics of the running maximum. Some results related to ours are given in [12, 13]. However, Assumption (1.3) of [12] is not satisfied for the CIR process. In [13], Corollary 1 is of interest for our work. It gives level asymptotics for the density of the hitting time, for a rather general diffusion that has an invariant distribution, which is the case for the CIR process. By integrating this density approximation, we can formally get asymptotics for the cumulative distribution function, which translates into running maximum tail asymptotics by (2.3). However, the result of this heuristic argument does not agree with our findings. While this may simply be a case where integration and asymptotics (with respect to a parameter)
do not commute, we note that several steps in [13] appear to be non-rigorous. For instance, no argument is given for the interchange of limit and summation in the proof of Corollary 1.

3 Eigenfunction expansion

The integrand in (2.5) has infinitely many poles, all of which are simple and in \((-\infty, 0]\) (see Proposition C.1 for a new proof of the latter two properties). We denote them by

\[ 0 > -s_0 > -s_1 > \cdots. \]

Using the residue theorem and some asymptotic properties of the Kummer function and its \(a\)-zeros, it is not hard to show that

\[ I(\lambda, b, x, y) = 1 + \sum_{k=0}^{\infty} \text{res}_{s=-s_k} \frac{e^{\lambda s} M(s, b, y)}{s} \frac{M(s, b, x)}{M(s, b, x)} \]

\[ = 1 - \sum_{k=0}^{\infty} \frac{e^{-\lambda s_k}}{s_k} \frac{M(-s_k, b, y)}{M'(-s_k, b, x)}. \]

(3.1)

Throughout this section, \(M'\) denotes the derivative with respect to the first parameter. By (2.3) and (2.6), this gives an expansion of the cumulative distribution function of the CIR hitting time, which is well known. We refer to Propositions 1 and 2 in [11], and to [10] for a classical reference on such expansions for general diffusions. Proposition 2 in [11] also gives the asymptotic behavior of the expansion coefficients for large \(k\). In the spirit of the above results, we analyse the coefficients as \(x \uparrow \infty\).

From 13.2.39 in [5, 15] and the expansion for \(M\) in Theorem 1 of [16], it follows that

\[ M(s, b, x) = e^{x} M(b - s, b, -x) \]

\[ = \frac{e^{x} \Gamma(b)}{\Gamma(b - s)} \left( \frac{x^{s-b} \Gamma(b - s)}{\Gamma(s)} + x^{-s} e^{-x} \cos \pi s + O(x^{s-b-1}/\Gamma(s)) \right). \]

(3.2)

From this we easily see that \(s_k = s_k(x)\) converges to \(k\) for \(x \uparrow \infty\). (In contrast to that, for fixed \(x\) and large \(k\) the behavior of \(s_k \sim \pi^2 k^2/(4x)\) is quadratic, by 13.9.10 in [5, 15].) The asymptotics of \(s_k\) for large \(x\) can be found by setting the leading term of (3.2) to zero, namely

\[ x^{s-b} \Gamma(b - s)/\Gamma(s) + x^{-s} e^{-x} \cos \pi s = 0. \]
Since

$$\frac{\Gamma(b-s)}{\Gamma(s)} = \frac{(s)_{k+1} \Gamma(b-s)}{\Gamma(s+k+1)}$$

$$\sim \frac{\Gamma(b+k)(s+k)(-1)^k(-s)(-s-1)\ldots(-s-k+1)}{\Gamma(b+k)(s+k)(-1)^kk!},$$

we have

$$s + k = -\frac{x^{b+2k}e^{-x}}{k!\Gamma(b+k)}.$$ 

We have proved:

**Lemma 3.1.** For $k \in \mathbb{N}_0$, we have

$$s_k = k + \frac{x^{b+2k}e^{-x}}{k!\Gamma(b+k)}(1 + o(1)), \quad x \uparrow \infty. \quad (3.3)$$

**Definition 3.2.** The harmonic numbers of order $\nu$ are defined by

$$H_n^{(\nu)} := \sum_{k=1}^{n} \frac{1}{k^\nu},$$

and $H_n := H_n^{(1)}$.

**Theorem 3.3.** Fix $k \geq 0$. For $x \uparrow \infty$, we have

$$M'(-s_k, b, x) = (-1)^k k! \sum_{r=k+1}^{\infty} \frac{x^r(r-k-1)!}{(b)_r r!} + O(x^k)$$

$$= \frac{(-1)^k x^{k+1}}{(b)_{k+1}(k+1)} \binom{1,1}{b+k+1,k+2|x} + O(x^k).$$

For $k = 0$, we have, more precisely:

$$M'(-s_0, b, x) = \frac{x}{b} \binom{1,1}{b+1,2|x} - 2s_0 \sum_{r=1}^{\infty} \frac{H_{r-1} x^r}{(b)_r r} + O(x^b e^{-x}).$$

**Proof.** By (1.2),

$$M'(s, b, x) = \sum_{r=1}^{\infty} \frac{(s)_r x^r}{(b)_r r!} (\psi(s+r)-\psi(r)) = \sum_{r=1}^{\infty} \frac{(s)_r x^r}{(b)_r r!} \sum_{m=1}^{\infty} \frac{1}{m-1+s}, \quad (3.4)$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function. This formula, as well as many others concerning the derivatives of $M$ with respect to its parameters,
also appears in [1]. If we put \( s = -s_k \) in (3.4), then the sum \( \sum_{r=1}^k \) is zero for \( k = 0 \) and \( O(x^k) \) for \( k \geq 1 \), and can thus be ignored in the following. For \( r > k \), we write the Pochhammer symbol as

\[
(s)_r = (s)_k (s + k)(s + k + 1)_{r-k-1}.
\]

We may assume \(|s + k| < 1\), as we intend to put \( s = -s_k \to -k \). Then, the last factor is

\[
(s + k + 1)_{r-k-1} = \prod_{j=0}^{r-k-2} (j + 1) \left( 1 + \frac{s + k}{j + 1} \right)
\]

\[
= (r - k - 1)! \exp \left( \sum_{j=0}^{r-k-2} \log \left( 1 + \frac{s + k}{j + 1} \right) \right)
\]

\[
= (r - k - 1)! \exp \left( - \sum_{j=0}^{r-k-2} \sum_{\nu=1}^{\infty} \frac{1}{\nu} (\frac{-s - k}{j + 1})^\nu \right).
\]

Since

\[
\sum_{j=0}^{r-k-2} \sum_{\nu=2}^{\infty} \frac{1}{\nu} (\frac{s + k}{j + 1})^\nu = \sum_{\nu=2}^{\infty} \frac{H_{r-k-1}^{(\nu)}}{\nu} (s + k)^\nu = O((s + k)^2)
\]

as \( s + k \to 0 \) (recall that \( k \) is fixed throughout), uniformly with respect to \( r \), we obtain

\[
(s + k + 1)_{r-k-1} = (r - k - 1)! \exp \left( - \sum_{j=0}^{r-k-2} \frac{s + k}{j + 1} + O((s + k)^2) \right)
\]

\[
= (r - k - 1)! \left( 1 + H_{r-k-1}(s + k) + O((s + k)^2) \right).
\]

We proceed with the first factor in (3.5):

\[
(s)_k = (-1)^k k! \exp \left( \sum_{j=0}^{k-1} \log \left( 1 + \frac{-s - k}{k - j} \right) \right)
\]

\[
= (-1)^k k! \exp \left( - \sum_{j=0}^{k-1} \sum_{\nu=1}^{\infty} \frac{1}{\nu} (\frac{s + k}{k - j})^\nu \right)
\]

\[
= (-1)^k k! \exp \left( - \sum_{j=0}^{k-1} \frac{-s - k}{k - j} + O((s + k)^2) \right)
\]

\[
= (-1)^k k! \left( 1 - H_k(s + k) + O((s + k)^2) \right).
\]
It is easy to see that the last sum in (3.4) satisfies
\[ \sum_{m=1}^{r} \frac{1}{m - 1 + s} = \frac{1}{s + k} + H_{r-k-1} - H_k + O((s + k)) , \]
as \( s \to -k \), uniformly with respect to \( r > k \). Using this and the estimate we
found for \( (s) \), yields
\[
\sum_{r=k+1}^{\infty} \frac{1}{m - 1 + s} = (-1)^{k}k! \sum_{r=k+1}^{\infty} \frac{x^{r}(r - k - 1)!}{(b)_{r} r!} \left( 1 - 2(s + k)(H_k - H_{r-k-1}) + O((s + k)^{2}) \right).
\]
Since
\[
\sum_{r=k+1}^{\infty} \frac{x^{r}(r - k - 1)!}{(b)_{r} r!} = \frac{x^{k+1}}{(b)_{k+1}(k+1)!} 2F_{2}\left( \begin{array}{c} 1, 1 \\ b + k + 1, k + 2 \end{array} \right|x \right) 
\sim \frac{x^{k+1}}{(b)_{k+1}(k+1)!} \frac{\Gamma(b + k + 1)(k+1)!}{x^{2k+b+1}} e^{x} 
= O(x^{-k-b}e^{x}), \quad x \uparrow \infty,
\]
where we have used 16.11.7 in [5, 15], it follows that
\[
\sum_{r=k+1}^{\infty} \frac{1}{m - 1 + s} = \frac{(-1)^{k}k!}{(b)_{k+1}(k+1)!} 2F_{2}\left( \begin{array}{c} 1, 1 \\ b + k + 1, k + 2 \end{array} \right|x \right) - 2(-1)^{k}k!(s + k) \sum_{r=k+1}^{\infty} \frac{x^{r}(r - k - 1)!}{(b)_{r} r!} \left( H_k - H_{r-k-1} \right) 
+ O((s + k)^{2}x^{-k-b}e^{x}).
\]
As mentioned at the beginning of the proof, it suffices to estimate the sum in (3.6), with \( s = -s_{k} \). For \( k = 0 \), the result now follows from (3.3). For \( k \geq 1 \), the claim follows from (3.3), the fact that \( H_k - H_{r-k-1} = O(r) \) and the expansion of \( 2F_{2} \) (see 16.11.7 in [5, 15]).

**Corollary 3.4.** The asymptotic behavior of the summands in (3.1) for \( x \uparrow \infty \) is
\[
\frac{e^{-\lambda x}}{s_{k}M(-s_{k}, b, y)} = \begin{cases} 1 + o(1), & k = 0, \\ (-1)^{k}M(-k, b, y) e^{x+b} e^{-x} + o(x^{k+b}e^{-x}), & k \geq 1. \end{cases}
\]
By (3.7), the summand \( k = 0 \) almost cancels with 1 on the right-hand side of (3.1). With a bit of extra work, it can be shown that the net contribution of these two summands satisfies

\[
1 - \frac{e^{-\lambda s_0}}{s_0} \cdot \frac{M(-s_0, b, y)}{M'(-s_0, b, x)} \sim s_0 \left( \lambda + \frac{y}{b} \right) {}_2F_2 \left( \frac{1, 1}{b + 1, 2}, y \right), \quad x \uparrow \infty.
\]

### A Asymptotics of \( M(a, b, x) \) for \( a \approx x \)

We use the saddle point method to analyse the Kummer function \( M(a, b, x) \) for \( x \uparrow \infty \), with \( b \) fixed and \( a \) of the same growth order as \( x \). It is important to note that this result is not new, as it can be obtained from the expansion (27.4.64) in [17] by putting \( a = ux, c = b \) and \( z = 1/u \). There, a different method was used, and \( z \) is assumed to be real and positive, but the latter constraint can be easily relaxed to \( \text{Re}(z) > 0 \), by inspection of the proof in [17].

**Theorem A.1.** Let \( b \in \mathbb{C} \setminus \{0, -1, \ldots\} \) and \( \text{Re}(u) > 0 \). Then

\[
M(ux, b, x) \sim \frac{\Gamma(b)}{\sqrt{2\pi(1 + 4u)^{1/4}}} \left( \frac{\sqrt{1 + 4u} - 1}{2} \right)^b (ux)^{1/2 - b} e^{x\psi(t_0)}
\]

as \( x \uparrow \infty \), where

\[
\psi(t_0) = \frac{1 + \sqrt{1 + 4u}}{2} + u \log \left( \frac{\sqrt{1 + 4u} + 1}{\sqrt{1 + 4u} - 1} \right).
\]  

This holds uniformly with respect to \( u \) if \( u \) is bounded and bounded away from zero, and \( |\arg u| \leq \frac{1}{2}\pi - \varepsilon \) for some \( \varepsilon > 0 \).

**Proof.** By (10.1.6) in [17], we have

\[
M(a, b, x) = \frac{\Gamma(1 + a - b)\Gamma(b)}{\Gamma(a)} \frac{1}{2\pi i} \int_0^{(1+)} e^{xt} t^{a-1} (t - 1)^{b-a-1} dt,
\]  

where \( a = ux \). The integration path starts and ends at zero and goes around \( t = 1 \) counterclockwise. Defining

\[
\psi(t) := t + u \log \frac{t}{t - 1} \quad \text{and} \quad f(t) := (t - 1)^{b-1}/t,
\]

we can write the integral as

\[
\frac{1}{2\pi i} \int_0^{(1+)} e^{x\psi(t)} f(t) dt.
\]
Equating the first derivative $\psi'(t) = 1 - \frac{u}{t(t-1)}$ to zero, we find a saddle point at
\[ t_0 = t_0(u) := \frac{1 + \sqrt{1 + 4u}}{2}. \] (A.4)

The second derivative of $\psi$ at the saddle point is
\[ \psi''(t_0) = \frac{u(2t - 1)}{t^2(t - 1)^2} \bigg|_{t=t_0} = \frac{\sqrt{1 + 4u} - 1}{u} = |\psi''(t_0)|e^{i\theta}. \]

The integration contour is deformed in order to pass through $t_0$. If $u \in (0, \infty)$, then $t_0 > 1$ is real, and the contour is vertical at $t_0$. For general $u$, we let the contour be such that $\arg(t - t_0) = \frac{1}{2}\pi - \frac{1}{2}\theta$ holds for $|t - t_0|$ small after $t$ traverses the saddle point. Now we can apply Theorem 4.7.1 in [14]. It is straightforward to see that the contour can be chosen such that the inequality before that theorem is satisfied. Its other assumptions are clearly satisfied as well, and we obtain
\[ \frac{1}{2\pi i} \int_{|t|=t_0} e^{x\psi(t)} f(t)dt \sim (1 + 4u)^{-1/4} \left( \frac{\sqrt{1 + 4u} - 1}{2} \right)^{b} (2\pi ux)^{-1/2} e^{x\psi(t_0)}. \] (A.5)

By inspecting the proof of Theorem 4.7.1 in [14], uniformity with respect to $u$ is easy to verify. From Stirling’s formula, we have
\[ \frac{\Gamma(1 + a - b)\Gamma(b)}{\Gamma(a)} \sim \Gamma(b)a^{1-b}. \]

Combination of this with (A.2) and (A.5) yields the assertion.

**B** Monotonicity of $|M(a, b, x)|$ with respect to Im($a$)

Let
\[ f(t, x) := |M(a + it, b, x)|^2 = M(a + it, b, x)M(a - it, b, x), \]
which is an entire function of $t$ and $x$ with a power series expansion
\[ f(t, x) = \sum_{m \geq 0} \sum_{n \geq 0} v_{mn} \frac{t^m x^n}{m! n!}, \]

where $v_{mn} = f^{(m,n)}(0,0)$. Using the power series of $M$ and the Cauchy product we obtain
\[ f(t, x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} \frac{(a + it)_k (a - it)_{n-k}}{(b)_k (b)_{n-k}} \right) \frac{x^n}{n!}. \]
Since $f(t, x)$ is an even function of $t$, $v_{mn} = 0$ when $m > 2[n/2]$ or $m \equiv 1 \mod 2$.

**Theorem B.1.** Suppose $a \geq b > 0$. Then $v_{mn} \geq 0$ for all $m \geq 0$, $n \geq 0$.

We give the proof of the theorem after some lemmas at the end of this section. It immediately implies the following corollary, which is the main result of the section.

**Corollary B.2.** Suppose $a \geq b > 0$ and $x > 0$. Then

$$t \in \mathbb{R}_+ \mapsto |M(a + it, b, x)|$$

is increasing.

**Lemma B.3.** The function $f$ solves the differential equation

$$-4t^2 x f(t, x) + \sum_{k=0}^{4} p_k(x) f^{(0,k)}(t, x) = 0,$$

where

\begin{align*}
p_0(x) &= -2a(1 - 3b + 2b^2) - 2a(1 - 4b)x - 4ax^2, \\
p_1(x) &= b - 3b^2 + 2b^3 + (2a + b - 8ab - 6b^2)x + (2 + 8a + 6b)x^2 - 2x^3, \\
p_2(x) &= (5b^2 - b)x + (-3 - 4a - 10b)x^2 + 5x^3, \\
p_3(x) &= (1 + 4b)x^2 - 4x^3, \\
p_4(x) &= x^3.
\end{align*}

**Proof.** Note that both $M(a + it, b, x)$ and $M(a - it, b, x)$ satisfy second-order differential equations with polynomial coefficients, namely the corresponding confluent hypergeometric differential equations. Thus $f(t, x)$ also satisfies an ODE (with respect to $x$) with polynomial coefficients. In the combinatorial and symbolic computation literature, such functions are called holonomic, or D-finite [19]. The ODE for $f$ can be computed with Mathematica by the command `DifferentialRootReduce`.

Some computations in the following proofs are not given in detail, because they can be easily done with a computer algebra system. For ease of notation we allow negative indices and set $v_{mn} = 0$ for $m < 0$ or $n < 0$.

**Lemma B.4.** The power series coefficients of $f$ satisfy the recursion

$$A_{-1,n}v_{m,n+1} + A_{0,n}v_{m,n} + A_{1,n}v_{m,n-1} + A_{2,n}v_{m,n-2} = 4nm(m - 1)v_{m-2,n-1}$$

\[(B.2)\]
with

\[ A_{-1,n} = b - 3b^2 + 2b^3 + (1 - 5b + 5b^2)n + (-2 + 4b)n^2 + n^3 \]
\[ A_{0,n} = 6ab - 4ab^2 - 2a + (6a + 11b - 5 - 8ab - 6b^2)n \]
\[ + (9 - 4a - 10b)n^2 - 4n^3 \]
\[ A_{1,n} = (8 - 10a - 6b + 8ab)n + (-13 + 8a + 6b)n^2 + 5n^3 \]
\[ A_{2,n} = (-4 + 4a)n + (6 - 4a)n^2 - 2n^3. \]

Use of our negative index convention shows that the recursion holds for \( m \geq 0, n \geq 0 \).

**Proof.** We extract coefficients from the differential equation by

\[
\left[ \frac{t^m x^n}{m! n!} \right] t^x f^{(\ell)}(t, x) = (m - \ell + 1)\ell (n - k + 1)k \ v_{m - \ell, n - k + j}
\]

and with our convention for negative indices this equation is true for all \( m \geq 0, n \geq 0, \ell \geq 0, k \geq 0, j \geq 0 \). Then we collect terms.

Let us introduce the differences

\[ v'_m,n = v_{m,n} - v_{m,n-1}, \quad v''_m,n = v'_m,n - v'_{m,n-1}, \quad v'''_m,n = v''_m,n - v''_{m,n-1}, \]

and

\[ u'_m,n = 4nm(m-1)v_{m-2,n-1} - 4(n-1)m(m-1)v_{m-2,n-2}. \quad (B.3) \]

Conversely

\[ v''_m,n = v''_{m,n-1} + v''_{m,n}, \quad v'_m,n = v'_{m,n-1} + v'_{m,n}, \quad v_m,n = v'_m,n + v_{m,n-1}. \quad (B.4) \]

**Lemma B.5.** The differences satisfy the recursion

\[ G_{-1,n}v'''_{m,n+1} = G_{0,n}v''_{m,n} + G_{1,n}v''_{m,n-1} + G_{2,n}v'_{m,n-2} + G_{3,n}v_{m,n-3} + u'_m,n \quad (B.5) \]

with

\[ G_{-1,n} = b - 3b^2 + 2b^3 + (1 - 5b + 5b^2)n + (-2 + 4b)n^2 + n^3 \]
\[ G_{0,n} = 2a + 7b - 4 - 6ab + b^2 + 4ab^2 - 4b^3 + (10 - 6a - 9b + 8ab - 4b^2)n \]
\[ + (4a + 2b - 8)n^2 + 2n^3 \]
\[ G_{1,n} = 6 + 6a + 3b - 4ab + 8ab^2 - 6b^3 + (6a - 5b + 8ab - 3b^2 - 10)n \]
\[ + (4 + 2b)n^2 \]
\[ G_{2,n} = -2 - 4b + 2ab + 4ab^2 - 2b^3 + (2 + 4b + b^2)n \]
\[ G_{3,n} = b^2. \]

With our negative index convention this recursion is valid for \( m \geq 0, n \geq 1 \).
Proof. Take the difference of (B.2) for \( n \) and \( n - 1 \) and rearrange terms.

**Lemma B.6.** Suppose \( a \geq b > 0 \) and \( n \geq 2 \). Then

\[
G_{-1,n} \geq 0, \quad G_{0,n} \geq 0, \quad G_{1,n} \geq 0, \quad G_{2,n} \geq 0, \quad G_{3,n} \geq 0. \tag{B.6}
\]

**Proof.** This follows from elementary analysis of the polynomials, or mechanically using the Mathematica commands *Simplify* and *Reduce*; see also *CyclicDecomposition*.

We can now prove the main results of this section (Theorem B.1 and its corollary).

**Proof of Theorem B.1.** To show \( v_{m,n} \geq 0 \) for all \( m \geq 0, n \geq 0 \) when \( a \geq b > 0 \) we perform a nested induction. The outer induction is with respect to \( m \geq 0 \), and the inner one with respect to \( n \geq 0 \). There is a little difficulty involved concerning

\[
v''_{0,1} = \frac{2a}{b} - 3,
\]

which is the only term in the induction that can be negative. For ease of notation let

\[
\tilde{v}''_{m,n} = \begin{cases} 0 & m = 0, n = 1 \\ v''_{m,n} & \text{otherwise.} \end{cases}
\]

Let us define the statement

\[
\mathcal{B}(m,n) \equiv (\tilde{v}''_{m,k} \geq 0, \ v''_{m,k} \geq 0, \ v'_{m,k} \geq 0, \ v_{m,k} \geq 0, \ \forall k \leq n).
\]

Recall that we work under the assumption \( a \geq b > 0 \). If \( m < 0 \) or \( n < 0 \) or \( m \equiv 1 \pmod{2} \), then \( \mathcal{B}(m,n) \) is trivially true due to our negative index convention.

**Step** \( m = 0 \): For \( n < 0 \)

\[
\tilde{v}''_{0,n} = 0, \quad v''_{0,n} = 0, \quad v'_{0,n} = 0, \quad v_{0,n} = 0,
\]

trivially. For \( n = 0 \) we have

\[
\tilde{v}''_{0,0} = 1, \quad v''_{0,0} = 1, \quad v'_{0,0} = 1, \quad v_{0,0} = 1.
\]

For \( n = 1 \) we have

\[
\tilde{v}''_{0,1} = 0, \quad v''_{0,1} = \frac{2a}{b} - 2 \geq 0, \quad v'_{0,1} = \frac{2a}{b} - 1 \geq 0, \quad v_{0,1} = \frac{2a}{b} \geq 0.
\]
For \( n = 2 \) we have
\[
\begin{align*}
v'''_{0,2} &= \frac{2a(2ab + a + b)}{b^2(b + 1)} - \frac{6a}{b} + 3 \geq 0, \\
v''_{0,2} &= \frac{2a(2ab + a + b)}{b^2(b + 1)} - \frac{4a}{b} + 1 \geq 0,
\end{align*}
\]
and
\[
\begin{align*}
v'_{0,2} &= \frac{2a(2ab + a + b)}{b^2(b + 1)} - \frac{2a}{b} \geq 0, \\
v_{0,2} &= \frac{2a(2ab + a + b)}{b^2(b + 1)} \geq 0.
\end{align*}
\]
So far we have shown \( B(0, n) \) for \( n \leq 2 \). Now the recursion can be applied.

Note that \( u'_{0,n} = 0 \) for all \( n \in \mathbb{Z} \). Suppose the hypothesis \( B(0, n) \) is true. Then we can show \( B(0, n + 1) \) by the recursion (B.5), property (B.6) and (B.4).

We are at the basis of the outer induction, namely we have shown \( B(m, n) \) for all \( m \leq 0 \) and \( n \in \mathbb{Z} \). Recall that all coefficients are zero when \( m \equiv 1 \pmod{2} \) and we must show it for even \( m \geq 2 \).

Suppose the outer induction hypothesis \( B(m - 2, n) \) holds for some \( m \geq 2 \) and all \( n \in \mathbb{Z} \). Note that \( B(m, n) \) holds trivially for \( n < 0 \). Now we have
\[
\begin{align*}
v'''_{m,n} &= 0, \\
v''_{m,n-1} &= 0, \\
v'_{m,n-2} &= 0, \\
v_{m,n-3} &= 0, \\
n &= 0, \ldots, m - 1.
\end{align*}
\]
So we can use \( B(m, n) \) for \( n \leq m - 1 \) as inner induction basis.

Suppose the inner induction hypothesis \( B(m, n) \) is true. Then inspection of (B.3) shows that
\[
v'_{m-2,n-1} \geq 0 \Rightarrow v_{m-2,n-1} \geq v_{m-2,n-2} \Rightarrow u'_{m,n} \geq 0.
\]
Thus all terms in (B.5) are non-negative and \( B(m, n + 1) \) is true. This concludes the induction. \( \square \)

Remark B.7. We conjecture that the conclusions of Theorem B.1 and Corollary B.2 are true for all \( a \geq 0, b > 0 \) and \( x > 0 \). Some partial results can be obtained by similar methods as above for the case \( 0 \leq a < b \), but the statements we could obtain so far are involved and unsatisfactory.

C The \( a \)-zeros are simple and negative

For \( b, x > 0 \), the \( a \)-zeros of \( M(a, b, x) \) are simple and located on the negative real line. This follows from applying Sturm-Liouville theory to the ODE (1.3); see Propositions 1 and 2 in [11] and the references given there. We now give an alternative proof of this fact, which is inspired by a similar proof concerning the Bessel function \( J_{\nu} \) (see p. 482 in [18]).
Proposition C.1. Let $b, x > 0$. Then all $a$-zeros of $M(a, b, x)$ are negative real and simple.

Proof. First, observe that (1.2) is an increasing function of $a \geq 0$, and since $M(0, b, x) = 1$, we see that there are no $a$-zeros in $[0, \infty)$. The function $y(x) := x^{b/2}e^{-x/2}M(a, b, x)$ satisfies the differential equation

$$y'' + Py = 0, \quad P := -\frac{1}{4} + \frac{\frac{1}{2}b - a}{x} + \frac{2b - b^2}{x^2},$$

and $\eta(x) := x^{b/2}e^{-x/2}M(\bar{a}, b, x)$, where $\bar{a}$ denotes the complex conjugate, satisfies

$$\eta'' + Q\eta = 0, \quad Q := -\frac{1}{4} + \frac{\frac{1}{2}b - \bar{a}}{x} + \frac{2b - b^2}{x^2}.$$  

Then (see p. 133 in [18])

$$\int_{x}^{\infty} (P - Q)y\eta dx = y \frac{d\eta}{dx} - \eta \frac{dy}{dx},$$

and so, with $E := x^{b/2}e^{-x/2}$,

$$(\bar{a} - a) \int_{x}^{\infty} t^{b-1}e^{-t}M(a, b, t)M(\bar{a}, b, t)dt$$

$$= EM(a, b, x)\left(EM(\bar{a}, b, x)\right)' - EM(\bar{a}, b, x)\left(EM(a, b, x)\right)'$$

$$= E^2 \left( M(a, b, x) \frac{d}{dx}M(\bar{a}, b, x) - M(\bar{a}, b, x) \frac{d}{dx}M(a, b, x) \right).$$

Therefore, for $a \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{0}^{x} t^{b-1}e^{-t}M(a, b, t)M(\bar{a}, b, t)dt$$

$$= \frac{x^b e^{-x}}{\bar{a} - a} \left( M(a, b, x) \frac{d}{dx}M(\bar{a}, b, x) - M(\bar{a}, b, x) \frac{d}{dx}M(a, b, x) \right). \quad (C.1)$$

Now let $a$ be a zero of $M(a, b, x)$. If $a$ is non-real, then the right-hand side of (C.1) vanishes, since $a \neq \bar{a}$ and $\bar{a}$ is a zero as well, but the left-hand side becomes

$$\int_{0}^{x} t^{b-1}e^{-t}|M(a, b, t)|^2 dt > 0,$$

a contradiction.

We now show that the $a$-zeros are simple. Analogously to (C.1), we have

$$\int_{0}^{x} t^{b-1}e^{-t}M(a, b, t)M(a', b, t)dt$$

$$= \frac{x^b e^{-x}}{a' - a} \left( M(a, b, x) \frac{d}{dx}M(a', b, x) - M(a', b, x) \frac{d}{dx}M(a, b, x) \right)$$

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for $a \neq a'$. Let $a < 0$ be a zero, and $a' = a + h$ with $h \to 0$. Then
\[
\int_0^x t^{b-1}e^{-t}M(a,b,t)dt = -\lim_{h \to 0} \frac{x^b e^{-x}}{h} M(a + h, b, x) \frac{d}{dx} M(a, b, x)
\]
\[
= -x^b e^{-x} \frac{d}{da} M(a, b, x) \frac{d}{dx} M(a, b, x),
\]
which shows that $(d/da)M(a, b, x)$ does not vanish. 

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