On The Deser-Siegel-Townsend Notivarg

M. Bakalarska, W. Tybor*
Department of Theoretical Physics I
University of Łódź
ul. Pomorska 149/153, 90-236 Łódź, Poland

Abstract

The interaction of the notivarg with an external Weyl current is discussed. The continuity equation for the Weyl current is obtained. The canonical analysis of the theory of the notivarg interacting with the external Weyl current is performed. The covariant propagator of the notivarg is found.

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1 Introduction

The notion of the notivarg has been introduced by Deser, Siegel and Townsend [1] as a parallel to the Ogievetsky-Polubarinov notoph [2]. The notivarg is a scalar particle described by the gauge theory. The notivarg field is a twenty component tensor $K^{\mu \nu \alpha \beta}$ with symmetries of the Riemann tensor. The Lagrangian density for the Deser-Siegel-Townsend theory of the free notivarg is [1]

\[ L_0 = -\frac{1}{2}(\partial_\mu K^{\mu \nu \alpha \beta} \partial_\nu K_{\nu \alpha \beta} - \frac{1}{3} \partial_\mu K^{\mu \nu \alpha \beta} \partial_\alpha K^{\sigma \lambda \sigma \lambda}). \] (1)

There exists another description of the notivarg [3,4] given by the Lagrangian density

\[ L_0 = -(\partial_\sigma K^{\sigma \nu \alpha \beta})^2 + (\partial_\sigma K^{\sigma \nu \alpha \beta})^2. \] (2)

The descriptions (1) and (2) are not connected by the point transformation [3,4]. The notivarg theory based on the Lagrangian (2) has been investigated with some details:

(i) the interaction of the notivarg with the external Weyl current has been discussed in Ref. [5];
(ii) the canonical analysis of the free theory and the theory of the notivarg interacting with the external Weyl current has been performed in Ref. [6];
(iii) the covariant form of the notivarg propagator has been fixed in Ref. [7].

In the present paper the similar program of investigations is performed for the Deser-Siegel-Townsend notivarg. In Section 2 we obtain the conservation law for the external Weyl current. In Section 3 we carry out the canonical analysis of the theory. Its gauge invariance is discussed in Section 4. In Section 5 we obtain the physical Lagrangian demonstrating the pure spin-0 content of the theory. In Section 6 we fix the covariant form of the notivarg propagator.

2 Interaction with external Weyl current

Let us discuss the notivarg theory in the Deser-Siegel-Townsend description. We take into account the interaction of the notivarg with an external Weyl current $j^{\mu \nu \alpha \beta}$. The action integral has the form

\[ I = \int d^4x (L_0 + L_{int}) = \int d^4x L, \] (3)

where the free Lagrangian density $L_0$ is given by Eq. (1) and the interaction term is

\[ L_{int} = \frac{1}{4} j^{\mu \nu \alpha \beta} K^{\mu \nu \alpha \beta} = \frac{1}{4} j^{\mu \nu \alpha \beta} C^{\mu \nu \alpha \beta}. \] (4)

The 20-component field $K^{\mu \nu \alpha \beta}$ has the symmetry of the Riemann tensor, i.e.

$K^{\mu \nu \alpha \beta} = -K^{\nu \mu \alpha \beta} = K^{\alpha \beta \mu \nu}, \varepsilon_{\mu \nu \alpha \beta} K^{\mu \nu \alpha \beta} = 0$.
the 10-component current $j^\mu\nu\alpha\beta$ has the symmetry of the Weyl tensor, i.e. it is the Riemann tensor with $j^\mu\nu\alpha\nu = 0$. $C^\mu\nu\alpha\beta$ is the Weyl part of $K^\mu\nu\alpha\beta$ (see Appendix I). The free part of the action

$$I_0 = \int d^4x \mathcal{L}_0$$

is invariant under the following gauge transformations

$$\delta K^\mu\nu\alpha\beta = \varepsilon^{\mu\nu\lambda\eta} \partial_{\lambda} \omega^\eta \lambda_k \partial_{\eta} \omega^\lambda \eta;$$

$$\delta K^\mu\nu\alpha\beta = g^\mu\alpha \partial^\nu \eta^\alpha + g^\nu\beta \partial^\mu \eta^\beta - g^\mu\beta \partial^\nu \eta^\alpha - g^\nu\alpha \partial^\mu \eta^\beta,$$

where the gauge tensor $\omega^\alpha\beta$ is symmetric $\omega^\alpha\beta = \omega^\beta\alpha$. Not all components of $\omega^\alpha\beta$ act effectively because the transformation (6) is invariant under

$$\delta \omega^\alpha\beta = \partial^\alpha \lambda^\beta + \partial^\beta \lambda^\alpha$$

where $\lambda^\alpha$ is an arbitrary vector. We note that the transformation (6) varies some components of the Weyl part of $K^\mu\nu\alpha\beta$, and the transformation (7) varies some components of the other parts of the field $K^\mu\nu\alpha\beta$. The action integral describing the interaction with the external Weyl current

$$I_{int} = \int d^3x \mathcal{L}_{int}$$

is invariant under the gauge transformation (6) if the source obeys the following condition

$$\varepsilon^{\alpha\lambda\mu\nu} \partial_{\lambda} \partial_{\alpha} j^\sigma\beta_{\mu\nu} = 0$$

where the dual properties of the Weyl tensor are taken into account. Using the decomposition of the Weyl tensor (see Appendix II)

$$j^\mu\nu\alpha\beta = (\lambda^i, \sigma^i)$$

we can rewrite the conservation law (6) in the form

$$\partial_i \partial_j \sigma^i = 0,$$

$$\partial^i \sigma^i + \varepsilon^{ikp} \partial_k \lambda_p = 0,$$

$$[(\partial^0)^2 + \Delta] \sigma^i + \partial^i \sigma^j + \partial^j \sigma^i + \partial^i [\varepsilon^{ikp} \partial_k \lambda_p^j + \varepsilon^{ikp} \partial_k \lambda_p^i] = 0,$$

where

$$\sigma^i \equiv \partial_i \sigma^i,$$

$$\lambda^i \equiv \partial_i \lambda^i.$$
In the helicity components (see Appendix IV) we get
\[
\sigma_L = 0, \\
\partial^0 \sigma_T + \varepsilon^{kp} \partial_k \lambda_T p = 0, \\
\left[ (\partial^0)^2 + \Delta \right] \sigma ij (\pm 2) + \partial^0 [\varepsilon^{kp} \partial_k \lambda_T (\pm 2) + \varepsilon^{kp} \partial_k \lambda_T (\pm 2)] = 0.
\]
(11)

The conservation law (9) can be obtained as well from the field equation following from the variational principle \(\delta I = 0\). We do not write down this field equation. For further aim we write the field equation in the covariant gauge
\[
K_{\mu \nu} = 0, \\
\partial_\mu K_{\mu \nu} = 0, \\
\partial_\mu K_{\mu \alpha \beta} = -\frac{1}{2} (\partial^\alpha K_{\mu \mu \beta} - \partial^\beta K_{\mu \mu \alpha})
\]
(12)

It has the following form
\[
\square C^{\mu \nu \alpha \beta} = 4 j^{\mu \nu \alpha \beta}
\]
(13)

We note that the gauge conditions (12) lead to
\[
\varepsilon^{\alpha \mu \lambda \nu} \partial_\lambda \partial_\sigma C^{\sigma \beta \mu \nu} = 0.
\]
(14)

So, the current conservation law (9) follows from Eq. (13).

3 Canonical analysis

Using the decomposition of the Weyl tensor (see Appendix II)
\[
j^{\mu \nu \alpha \beta} = (\lambda^{ij}, \sigma^{ij})
\]
and the Riemann tensor (see Appendix III)
\[
K^{\mu \nu \alpha \beta} = (T^{ij}, R^{ij}, S^{ij}, A^i, T, R)
\]
we can rewrite the action (3) in the component form. After some integrations by parts we remove the velocities \(\partial^0 A^i, \partial^0 S^{ij}\) and \(\partial^0 R\) from the action. Performing the Legendre transformation we obtain
\[
I = \int d^4x \left( P^{ij} \partial^0 T_{ij} + \Pi^{ij} \partial^0 R_{ij} + P \partial^0 T - H_c \right),
\]
(15)

where the canonical momenta are
\[
P^{ms} = \frac{\partial L}{\partial \partial^0 T_{ms}} = -\left\{ \partial^0 T^{ms} + \frac{1}{2} \partial^0 R^{ms} + \frac{3}{2} (\partial^s A^m + \partial^m A^s) + \right\}.
\]
The canonical Hamiltonian density is
\[ \Pi^{ms} = \frac{\partial L}{\partial \dot{\Theta}^m R_{ms}} = -\left\{ \frac{1}{2} \dot{\dot{\Theta}}^m T^{ms} + \frac{1}{2} (\dot{\Theta}^s A^m + \dot{\Theta}^m A^s) - \frac{1}{3} \dot{\Theta}^m \dot{\Theta}^k A_k + \right. \\
\left. - \frac{1}{2} [\varepsilon^{mnp} \partial_n S_p^s + \varepsilon^{snp} \partial_n S_p^m] \right\}, \tag{16} \]
and the canonical Hamiltonian density is
\[ H_c = 2(\Pi^{ij})^2 - 2\Pi_{ij} P^{ij} + \frac{3}{2} P^2 + \frac{1}{6} T^i \dot{\Theta}^i + \frac{1}{18} (\dot{\Theta}^i T^i)^2 + \frac{1}{2} (\dot{\Theta}^i R^{ij})^2 + \]
\[ + \frac{1}{2} \dot{\Theta}^i R^{ij} \dot{\Theta}^i T_j - (R^i)^2 - R^i T_j - \frac{1}{6} R^i \text{curl} T - \lambda_{ij} (T^{ij} - R^{ij}) + \]
\[ + 2 (P^i + \Pi^{i} - 2 \dot{\Theta}^i P) A_i - \left( \frac{1}{3} \Delta T - \frac{1}{12} \partial_i R^i - \frac{1}{4} \partial_i T_i \right) R + \]
\[ + 2 [\varepsilon^{pmn} \partial_m (P_{ps}^{n} - \Pi^m_{ps}) + \sigma_{ps}] S_{ps}, \tag{17} \]
where the following abbreviations are introduced
\[ T^i = \partial_j T^{ji}, \quad R^i = \partial_j R^{ji}, \quad P^i = \partial_j P^{ji}, \quad \Pi^i = \partial_j \Pi^{ji}. \]
The momenta conjugated to \( A_i \), \( S^{ij} \) and \( R \) vanish because the Lagrangian density is independent of the corresponding velocities
\[ p^i_A \equiv \frac{\partial L}{\partial \dot{\Theta}^i A_i} = 0, \quad p^{ij}_S \equiv \frac{\partial L}{\partial \dot{\Theta}^i S_{ij}} = 0, \quad p_R \equiv \frac{\partial L}{\partial \dot{\Theta}^0 R} = 0. \]
So, there are the following primary constraints
\[ \Phi_{(1)} = p^i_A, \quad \Phi_{(2)} = p^{ij}_S, \quad \Phi_{(3)} = p_R. \tag{18} \]
We introduce the total Hamiltonian \[ H_{\text{tot}} = \int \dot{\delta}^a \left( H_c + \lambda_i \Phi_{(1)} + \lambda_{ij} \Phi_{(2)} + \lambda \Phi_{(3)} \right), \tag{19} \]
where \( \lambda_i \), \( \lambda_{ij} \) and \( \lambda \) are Lagrange multipliers. The dynamics is expressed by
\[ \partial^0 a = \{ a, H_{\text{tot}} \} \varphi_{(1)} = \Phi_{(2)} = \Phi_{(3)} = 0, \tag{20} \]
where \{........\} is the Poisson bracket and \( a \) is a function of dynamical variables. The theory is consistent if constraints hold for all times. This leads to the secondary constraints:
\[ \Phi_{(4)} = P^i + \Pi^i - 2 \dot{\Theta}^i P, \]
\[ \Phi_{(5)} = \left[ \varepsilon^{inp} \partial_n \left( P^i_p - \Pi^i_p \right) + \varepsilon^{inr} \partial_i \left( P^j_p - \Pi^j_p \right) \right] + 2 \sigma^{ij}, \]
\[ \Phi_{(6)} = \Delta T - \frac{3}{4} \partial_i R^i - \frac{9}{4} \partial_i T_i, \tag{21} \]
\[ \Phi_{(7)} = \varepsilon^{inm} \partial_n \left[ \Delta \left( T^{jm} + R^{jm} \right) + \partial^i \left( T^{jm} + R^{jm} \right) \right] + \varepsilon^{inm} \partial_n \left[ \Delta \left( T^{im} + R^{im} \right) + \partial^j \left( T^{im} + R^{im} \right) \right] + \]
\[ 4 \left[ \dot{\Theta}^i \sigma^{ij} + \varepsilon^{inm} \partial_n \lambda^j_m + \varepsilon^{jnmm} \partial_n \lambda^j_m \right], \]
where the conservation law \[9\] is taken into account. We note that
\[
\Phi_{(5)}^{ij} = \Phi_{(5)}^{ij}(\pm 2) + \Phi_{(5)}^{ij}(\pm 1),
\]
\[
\Phi_{(7)}^{ij} = \Phi_{(7)}^{ij}(\pm 2),
\]
because \(\partial_i \partial_j \Phi_{(5)}^{ij} = 0\) and \(\partial_i \Phi_{(7)}^{ij} = 0\). The dynamics of the constraints is
\[
\partial^0 \Phi_{(1)}^i = -2\Phi_{(4)}^i,
\]
\[
\partial^0 \Phi_{(2)}^{ij} = -\Phi_{(5)}^{ij},
\]
\[
\partial^0 \Phi_{(3)} = \frac{1}{9} \Phi_{(6)},
\]
\[
\partial^0 \Phi_{(4)}^i = \frac{2}{9} \partial^i \Phi_{(6)},
\]
\[
\partial^0 \Phi_{(6)} = \frac{3}{2} \partial_t \Phi_{(4)}^i,
\]
\[
\partial^0 \Phi_{(5)}^{ij} = \frac{1}{2} \Phi_{(7)}^{ij},
\]
\[
\partial^0 \Phi_{(7)}^{ij} = -2(\Delta \Phi_{(5)}^{ij} + \partial^i \partial_k \Phi_{(5)}^{kj} + \partial^i \partial_k \Phi_{(5)}^{ki}).
\]

Because the constraints can be added to Hamiltonian \[9\], we construct the new Hamiltonian density
\[
H_{\text{new}} = H_0 + V_{(4)}^{ij} \Phi_{(4)}^i + V_{(5)}^{ij} \Phi_{(5)}^{ij} + V_{(6)} \Phi_{(6)} + V_{(7)}^{ij} \Phi_{(7)}^{ij},
\]
where \(V_{(4)}^{ij}, V_{(5)}^{ij}, V_{(6)}^{ij}\) and \(V_{(7)}^{ij}\) are the Lagrange multipliers, and \(H_0\) has the following form
\[
H_0 = 2(\Pi_{ij})^2 - 2\Pi_{ij} P^{ij} + \frac{3}{2} P^2 + \frac{1}{6} T^i \partial_i T +
\]
\[
+ \frac{1}{18} (\partial^i T)^2 + \frac{1}{2} (\partial^k R^{ij})^2 + \frac{1}{2} \partial^k R^{ij} \partial_k T_{ij} +
\]
\[
- (R^i)^2 - R^i T_i - \frac{1}{6} R^i \partial_i T - \lambda_{ij} (T^{ij} - R^{ij}).
\]

We observe that the variables \(A^i, S^{ij}\) and \(R\) disappear in the new description. Let us note that according to \[22\] we have (see Appendix IV)
\[
V_{ij}^{(5)} = V_{ij}^{(5)}(\pm 2) + V_{ij}^{(5)}(\pm 1),
\]
\[
V_{ij}^{(7)} = V_{ij}^{(7)}(\pm 2).
\]

The Hamiltonian density \(H_{\text{new}}\) is derivable \[10\] from the phase-space Lagrangian density
\[
L_{\text{new}} = P_{ij} \partial^0 T^{ij} + \Pi_{ij} \partial^0 R^{ij} + P \partial^0 T - H_{\text{new}},
\]
where the Lagrangian multipliers \( V_i^{(4)} \), \( V_{ij}^{(5)} \), \( V^{(6)} \) and \( V_{ij}^{(7)} \) are treated as dynamical variables. So, passing to the canonical formalism, we find the primary constraints

\[
\pi_i^{(4)} = 0, \quad \pi_{ij}^{(5)} = 0, \quad \pi^{(6)} = 0, \quad \pi_{ij}^{(7)} = 0,
\]

(27)

where \( \pi_i^{(4)} \), \( \pi_{ij}^{(5)} \), \( \pi^{(6)} \) and \( \pi_{ij}^{(7)} \) are the canonical momenta conjugated to \( V_i^{(4)} \), \( V_{ij}^{(5)} \), \( V^{(6)} \) and \( V_{ij}^{(7)} \) respectively. Thus the new total Hamiltonian is

\[
H_{\text{tot}}^{\text{new}} = \int d^3x (H^{\text{new}} + \lambda_i^{(4)} \pi_i^{(4)} + \lambda_{ij}^{(5)} \pi_{ij}^{(5)} + \lambda^{(6)} \pi^{(6)} + \lambda_{ij}^{(7)} \pi_{ij}^{(7)}),
\]

(28)

where \( \lambda_i \)‘s are the Lagrange multipliers. The dynamics is expressed by

\[
\partial^0 a = \{ a, H_{\text{tot}}^{\text{new}} \} \big|_{\pi=0}.
\]

(29)

In particular we have

\[
\partial^0 \pi_i^{(4)} = -\Phi_i^{(4)}, \quad \partial^0 \pi_{ij}^{(5)} = -\Phi_{ij}^{(5)},
\]

\[
\partial^0 \pi^{(6)} = -\Phi^{(6)}, \quad \partial^0 \pi_{ij}^{(7)} = -\Phi_{ij}^{(7)}.
\]

(30)

The time derivatives of \( \Phi^{(4)} \), \( \Phi^{(5)} \), \( \Phi^{(6)} \) and \( \Phi^{(7)} \) are given by Eqs. (23).

### 4 Gauge transformations

Let us discuss the gauge transformations of the free notivarg theory described by the action integral

\[
I_{\text{free}} = \int d^4x L_{\text{free}}^{\text{new}},
\]

(31)

where \( L_{\text{free}}^{\text{new}} \) is obtained from \( L^{\text{new}} \) putting \( \lambda^{ij} = \sigma^{ij} = 0 \). In this limit the constraints are

\[
\Phi_i^{(4)} = P^i + \Pi^i - 2\partial^i P,
\]

\[
\Phi_{ij}^{(5)} = \varepsilon^{imn} \partial_n (P^j_p - \Pi^j_p) + \varepsilon^{jm} \partial_n (P^i_p - \Pi^i_p),
\]

\[
\Phi^{(6)} = \Delta T - \frac{3}{4} \partial^i R^i - \frac{9}{4} \partial^i T^i,
\]

\[
\Phi_{ij}^{(7)} = \varepsilon^{imn} \partial_n [\Delta(T^j_m + R^j_m) + \partial^j(T_m + R_m)] + \varepsilon^{jm} \partial_n [\Delta(T^i_m + R^i_m) + \partial^i(T_m + R_m)],
\]

(32)

and they obey the relations

\[
\{ \Phi_{(a)}, \Phi_{(b)} \} = 0, \quad a, b = 4, 5, 6, 7.
\]
So, we have the theory with the first class constraints.
The generator of the gauge transformations is
\[
G = \int d^3 x \left( \alpha_i^{(4)} \pi_i^{(4)} + \alpha_{ij}^{(5)} \pi_{ij}^{(5)} + \alpha_i^{(6)} \pi_i^{(6)} + \alpha_{ij}^{(7)} \pi_{ij}^{(7)} \right) + \eta_i^{(4)} \Phi_i^{(4)} + \eta_{ij}^{(5)} \Phi_{ij}^{(5)} + \eta_i^{(6)} \Phi_i^{(6)} + \eta_{ij}^{(7)} \Phi_{ij}^{(7)},
\]
where \(\alpha_i's\) and \(\eta_i's\) are gauge functions. They have the helicity structure as the corresponding constraints. The generator (33) obeys the consistency condition
\[
\frac{d}{dt} G \bigg|_{\pi=0} = 0.
\]
Using Eq. (29) we obtain
\[
\begin{align*}
\alpha_i^{(4)} &= \partial^a \eta_i^{(4)} - \frac{3}{2} \partial_i \eta^{(6)}, \\
\alpha_i^{(6)} &= \partial^a \eta_i^{(6)} - \frac{2}{9} \partial_i \eta_a^{(4)}, \\
\alpha_{ij}^{(5)}(\pm 2) &= \partial^a \eta_{ij}^{(5)}(\pm 2) - 2 \Delta \eta_i^{(7)}(\pm 2), \\
\alpha_{ij}^{(5)}(\pm 1) &= \partial^a \eta_{ij}^{(5)}(\pm 1), \\
\alpha_{ij}^{(7)} &= \partial^a \eta_{ij}^{(7)} + \frac{1}{2} \eta_i^{(5)}(\pm 2).
\end{align*}
\]
The gauge transformations are
\[
\begin{align*}
\delta V_i^{(4)} &= \left\{ V_i^{(4)}, G \right\} = \alpha_i^{(4)}, \\
\delta V_{ij}^{(5)} &= \alpha_{ij}^{(5)}, \\
\delta V_{ij}^{(6)} &= \alpha_{ij}^{(6)}, \\
\delta V_{ij}^{(7)} &= \alpha_{ij}^{(7)}, \\
\delta T &= 2 \partial^a \eta_a^{(4)}, \\
\delta P &= -\Delta \eta^{(6)}, \\
\delta T^{ij} &= -\frac{1}{2} \left( \partial^i \eta^{(4)}_j + \partial^j \eta^{(4)}_i \right) + \frac{1}{3} g^{ij} \partial^k \eta_k^{(4)} + \\
&+ \epsilon^{inm} \partial_n \eta_m^{(5)_j} + \epsilon^{jnm} \partial_n \eta_m^{(5)_i}, \\
\delta R^{ij} &= -\frac{1}{2} \left( \partial^i \eta^{(4)}_j + \partial^j \eta^{(4)}_i \right) + \frac{1}{3} g^{ij} \partial^k \eta_k^{(4)} + \\
&- \left[ \epsilon^{inm} \partial_n \eta_m^{(5)_j} + \epsilon^{jnm} \partial_n \eta_m^{(5)_i} \right], \\
\delta P^{ij} &= \frac{9}{4} \left( \partial^i \partial^j + \frac{1}{3} g^{ij} \Delta \right) \eta^{(6)} - \Delta \left[ \epsilon^{inm} \partial_n \eta_m^{(7)_j} + \epsilon^{jnm} \partial_n \eta_m^{(7)_i} \right], \\
\delta \Pi^{ij} &= \frac{3}{4} \left( \partial^i \partial^j + \frac{1}{3} g^{ij} \Delta \right) \eta^{(6)} - \Delta \left[ \epsilon^{inm} \partial_n \eta_m^{(7)_j} + \epsilon^{jnm} \partial_n \eta_m^{(7)_i} \right].
\end{align*}
\]
In Appendix V we impose the noncovariant conditions to remove completely
the gauge freedom.
Using the conservation law of the current (9) we can verify that the action
\[ I = \int d^4 x L^{\text{new}} \] (37)
is invariant under the gauge transformations (36).

5 Physical Lagrangian

Solving the constraints (21) we obtain
from \( \Phi_{(4)} = 0 \):
\[ P_L + \Pi_L + 2P = 0, \quad P_T + \Pi_T = 0, \]
from \( \Phi_{(5)} = 0 \):
\[ P^i_T - \Pi^i_T = \frac{2}{\Delta} \varepsilon^{ikp} \partial_k \sigma_{T^p}, \]
\[ P^{ij}(\pm 2) - \Pi^{ij}(\pm 2) = \frac{1}{2\Delta} \left[ \varepsilon^{ikp} \partial_k \sigma^j_p(\pm 2) + \varepsilon^{jkp} \partial_k \sigma^i_p(\pm 2) \right], \quad (38) \]
from \( \Phi_{(6)} = 0 \):
\[ T - \frac{9}{4} T_L - \frac{3}{4} R_L = 0, \]
from \( \Phi_{(7)} = 0 \):
\[ T^{ij}(\pm 2) + R^{ij}(\pm 2) = \frac{1}{\Delta^2} \delta_{[i}^{[j} \varepsilon^{k]p} \partial_k \sigma^{[i}_p(\pm 2) + \varepsilon^{k]p} \partial_k \sigma^{[i}_p(\pm 2)] - \frac{4}{\Delta} \lambda^{ij}(\pm 2). \]
Inserting these solutions to the action (37) we get
\[ I = \int d^4 x L_{\text{phys}}, \quad (39) \]
where
\[ L_{\text{phys}} = p \partial^0 \varphi - \mathcal{H}_{\text{free}} - \mathcal{H}_{\text{int}} \] (40)
and
\[ (p, \varphi) = \left( \frac{\sqrt{3}}{2} (P_L - 3\Pi_L), \frac{\sqrt{3}}{4} (T_L - R_L) \right), \quad (41) \]
or other 8 pairs that can be obtained from (41) using the constraints:
\[ P_L + \Pi_L + 2P = 0, \quad \text{and} \quad T - \frac{9}{4} T_L - \frac{3}{4} R_L = 0. \]
The free Hamiltonian density is

$$H_{\text{free}} = \frac{1}{2} p^2 - \frac{1}{2} (\partial^i \varphi)^2 \tag{42}$$

and the interacting one is

$$H_{\text{int}} = -2\sqrt{3} \lambda L \varphi + 4 \lambda_{ij}(\pm 2) \frac{1}{\Delta^2} \lambda^{ij}(\pm 2) + (\partial^0 \sigma^{ij}(\pm 2)) \frac{1}{\Delta^2} (\sigma^0 \sigma_{ij}(\pm 2)) - 3 \sigma^{ij}(\pm 2) \frac{1}{\Delta^2} \sigma_{ij}(\pm 2) - 8 \delta^i \frac{1}{\Delta^2} \sigma T_i. \tag{43}$$

In the momentum space we get

$$H_{\text{int}} = -2\sqrt{3} \lambda L (-k) \varphi(k) + k^2 (k^0)^{-2} |\vec{k}|^{-2} \sigma^{ij}(\pm 2, -k) \sigma_{ij}(\pm 2, k) + 8 |\vec{k}|^{-4} \sigma^i \sigma(-k) \sigma T_i(k). \tag{44}$$

### 6 Notivarg propagator

Let us consider the exchange of the notivarg between two external currents. The general structure of the amplitude describing the process in the second order of the perturbation theory is [7]

$$A = -\left( a \frac{j_{\mu\nu\alpha\beta}(-k) j_{\mu\nu\alpha\beta}(k)}{k^2} + \frac{b}{k^4} k_\mu j_{\mu\nu\alpha\beta}(-k) k^\sigma j_{\sigma\nu\alpha\beta} + \frac{c}{k^4} k_\mu j_{\mu\nu\alpha\beta}(-k) k^\sigma k^\kappa j_{\sigma\nu\kappa\beta} \right), \tag{45}$$

where $a, b, c$ are number factors. Due to the conservation law [8] we have

$$k_\mu k_{\alpha j} j_{\mu\nu\alpha\beta}^\mu k^\kappa j_{\sigma\nu\kappa\beta} = \frac{1}{2} k^2 k_{\mu j} j_{\mu\nu\alpha\beta}^\mu k^\sigma j_{\sigma\nu\alpha\beta}.$$ 

Using the following identity for the Weyl tensor

$$j_{\mu\nu\alpha\beta}^\mu j_{\sigma\nu\alpha\beta} = \frac{1}{4} \delta_{\mu j} j_{\mu\nu\alpha\beta}^\mu j_{\nu\alpha\beta} \tag{46}$$

we observe only the first term in Eq. (45) is independent. Assuming the following form of the notivarg propagator

$$D_{\mu\nu\alpha\beta, \sigma\gamma\delta}(k) = -\frac{1}{8 k^2} \left( g_{\mu\sigma} g_{\nu\lambda} g_{\alpha \gamma} g_{\beta \delta} + g_{\mu \lambda} g_{\nu\sigma} g_{\alpha \delta} g_{\beta \gamma} + g_{\mu \gamma} g_{\nu\delta} g_{\alpha \sigma} g_{\beta \lambda} + g_{\mu \delta} g_{\nu\gamma} g_{\alpha \sigma} g_{\beta \lambda} - g_{\mu \gamma} g_{\nu\delta} g_{\alpha \lambda} g_{\beta \sigma} - g_{\mu \delta} g_{\nu\gamma} g_{\alpha \lambda} g_{\beta \sigma} + g_{\mu \lambda} g_{\nu\sigma} g_{\alpha \gamma} g_{\beta \delta} + g_{\mu \gamma} g_{\nu\delta} g_{\alpha \lambda} g_{\beta \sigma} - g_{\mu \delta} g_{\nu\gamma} g_{\alpha \lambda} g_{\beta \sigma} - g_{\mu \lambda} g_{\nu\sigma} g_{\alpha \gamma} g_{\beta \delta} \right)$$

we obtain the amplitude

$$A = -j_{\mu\nu\alpha\beta}(-k) D_{\mu\nu\alpha\beta, \sigma\gamma\delta}(k) j_{\sigma\gamma\delta}(k) \tag{47}$$
The number factor 1 = 1 follows from Eqs (4), (13), and (46)
\[ \frac{1}{4} \mu_{\alpha \beta} C_{\mu \nu \alpha \beta} \rightarrow \frac{1}{4} \mu_{\alpha \beta} D_{\mu \nu \alpha \beta, \gamma \lambda \delta} (4 j^\gamma \gamma^\delta) . \]
Using the current conservation law (4) we obtain
\[ A = 12 \lambda \lambda (k) + 2 k^2 (k^0)^{-2} \sigma^j (\sigma^j) \sigma^j (\sigma^j) + (48) \]
\[ + 16 | \vec{k} |^{-4} \sigma_\gamma (-k) \sigma T (k) . \]
We note that the amplitude of the current - current interaction via one notivarg exchange can be calculated with the help of the Hamiltonian (44) using standard methods of the S - matrix formalism [11]. Following this way we get exactly the amplitude (48). So, the form (46) of the notivarg propagator is confirmed.

7 Final remarks
We finish with the following remarks:
(i) substituting \[ \lambda \rightarrow \frac{1}{2 \sqrt{2}} \sigma, \sigma \rightarrow - \frac{1}{2 \sqrt{2}} \lambda \] (see Appendix II) in Eq. (43) we get the physical Hamiltonian in the notivarg theory based on the Lagrangian (2) (see Refs [6,7]);
(ii) in the Deser-Siegel-Townsend description the scalar field is not necessarily a component of the Weyl tensor. So, we can expect the notivarg interacting with other parts of the external Riemann current, not only with the Weyl one.

8 Acknowledgment
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Appendix I
The decomposition of the Riemann tensor \( K^{\mu \nu \alpha \beta} \) in the irreducible Lorentz parts is:
\[ K^{\mu \nu \alpha \beta} = C^{\mu \nu \alpha \beta} + E^{\mu \nu \alpha \beta} + G^{\mu \nu \alpha \beta} , \]
where \( C^{\mu \nu \alpha \beta} \) is the Weyl tensor and
\[ E^{\mu \nu \alpha \beta} = \frac{1}{2} (g^{\mu \alpha} K^{\nu \beta} + g^{\nu \beta} K^{\mu \alpha} - g^{\mu \beta} K^{\nu \alpha} - g^{\nu \alpha} K^{\mu \beta} ) + \]
\[ - \frac{1}{4} (g^{\mu \alpha} g^{\nu \beta} - g^{\mu \beta} g^{\nu \alpha}) K , \]
\[ G^{\mu \nu \alpha \beta} = \frac{1}{12} (g^{\mu \alpha} g^{\nu \beta} - g^{\mu \beta} g^{\nu \alpha}) K , \]
\[ K^{\mu \alpha} = K^{\mu \alpha \nu}, \quad K = K^\mu_\mu . \]
The dual property of the Weyl tensor reads
\[ \varepsilon^{\mu\nu}_{\sigma\lambda} C^{\sigma\lambda\alpha\beta} = \varepsilon^{\lambda\alpha}_{\sigma\lambda} C^{\mu\nu\sigma\lambda}. \]

**Appendix II**

Let us consider the Weyl tensor \( j^{\mu\nu\alpha\beta} \). We introduce the new variables
\[ \lambda^i = \lambda^i, \quad \lambda^i_0 = 0, \quad \sigma^{ij} = \sigma^{ij}, \quad \sigma^i_i = 0 \]
defined by
\[
\begin{align*}
j_{0i0j} &= \lambda^i_j, \\
j_{0ijk} &= \varepsilon^{kp}_{\sigma^p}, \\
j_{ijkl} &= -\left(g^{ik}\lambda^j_l + g^{jl}\lambda^i_k - g^{il}\lambda^j_k - g^{jk}\lambda^i_l\right).
\end{align*}
\]
So, we get the following decomposition of the Weyl tensor
\[ j^{\mu\nu\alpha\beta} = (\lambda^i_j, \sigma^{ij}). \]
The dual transformation
\[ j^{\mu\nu\alpha\beta} \rightarrow \frac{1}{2} \varepsilon^{\mu\nu}_{\sigma\lambda} j^{\sigma\lambda\alpha\beta} \]
in the component form is
\[ \lambda^i_j \rightarrow -\sigma^{ij}, \quad \sigma^{ij} \rightarrow \lambda^i_j. \]

**Appendix III**

Let us consider the Riemann tensor \( K^{\mu\nu\alpha\beta} \). We introduce the new variables
\[ T^{ij} = T^{ji}, \quad T^i_i = 0, \quad R^{ij} = R^{ji}, \quad R^i_i = 0, \]
\[ S^{ij} = S^{ji}, \quad S^i_i = 0, \quad A^i, \quad T, \quad R \]
defined by
\[
\begin{align*}
K^{00ij} &= T^{ij} + \frac{1}{3} g^{ij} T, \\
K^{0ijk} &= \varepsilon^{kp}_{\sigma^p} S^i_i + g^{ij} A^k - g^{ik} A^j, \\
K^{ijmn} &= g^{im} R^{jn} + g^{jn} R^{im} - g^{in} R^{jm} - g^{jm} R^{in} + \\
&\quad + \frac{1}{6} (g^{im} g^{jn} - g^{in} g^{jm}) R.
\end{align*}
\]
So, we get the following decomposition of the Riemann tensor
\[ K^{\mu\nu\alpha\beta} = (T^{ij}, R^{ij}, S^{ij}, A^i, T, R). \]
Appendix IV

The well known decomposition of a vector into transversal and longitudinal parts is

\[ V^i = V^i_T + V^i_L \]

where

\[ V^i_T = V^i + \frac{1}{\Delta} \partial^i \partial_j V^j, \]
\[ V^i_L = -\frac{1}{\Delta} \partial^i \partial_j V^j, \]
\[ \Delta = -\partial_i \partial^i. \]

The analogous decomposition of a symmetric traceless tensor \( a^{ij} \) is

\[ a^{ij} = a^{ij}(\pm 2) + a^{ij}(\pm 1) + a^{ij}(0) \]

where

\[ a^{ij}(\pm 1) = -\frac{1}{\Delta} (\partial^i a^j_T + \partial^j a^i_T), \]
\[ a^{ij}(0) = \frac{3}{2} \left( \frac{1}{\Delta} \partial^i \partial^j + \frac{1}{3} g^{ij} \right) a_L, \]
\[ a_T = a^i + \frac{1}{\Delta} \partial^i \partial_j a^j, \]
\[ a_L = \frac{1}{\Delta} \partial_i a^i, \]
\[ a^i = \partial_j a^{ij}. \]

Appendix V

The gauge transformations (38) in the component form are

\[ \delta T^{ij}(\pm 2) = -\delta R^{ij}(\pm 2) = \varepsilon^{ikp} \partial_k \eta^{(5)}_p (\pm 2) + \varepsilon^{ijk} \partial_k \eta^{(5)}_p (\pm 2), \]
\[ \delta T^i_T = \frac{1}{2} \Delta \eta^{(4)}_T + \varepsilon^{ikp} \partial_k \eta^{(5)}_T p, \]
\[ \delta R^i_T = \frac{1}{2} \Delta \eta^{(4)}_T - \varepsilon^{ikp} \partial_k \eta^{(5)}_T p, \]
\[ \delta R_L = \frac{2}{3} \Delta \eta^{(4)}_L, \]
\[ \delta T_L = \frac{2}{3} \Delta \eta^{(4)}_L, \]
\[ \delta T = 2 \Delta \eta^{(4)}_L, \]
\[ \delta P^{ij}(\pm 2) = \delta \Pi^{ij}(\pm 2) = -\Delta \left( \varepsilon^{ikp} \partial_k \eta^{(7)}_p (\pm 2) + \varepsilon^{ijk} \partial_k \eta^{(7)}_p (\pm 2) \right). \]
\[ \delta P_T^i = \delta \Pi_T^i = 0, \]
\[ \delta P_L = \frac{3}{2} \Delta \eta^{(6)}, \]
\[ \delta \Pi_L = \frac{1}{2} \Delta \eta^{(6)}, \]
\[ \delta P = -\Delta \eta^{(6)}. \]

To remove completely the gauge freedom we impose the following noncovariant conditions \{\chi_i\}:

\begin{align*}
T^{ij}(\pm 2) - R^{ij}(\pm 2) &= 0, & T_T^i &= R_T^i = 0, \\
P^{ij}(\pm 2) + \Pi^{ij}(\pm 2) &= 0, & aT - \frac{3}{8}(3T_L + R_L) &= 0, \\
P + a(P_L + \Pi_L) &= 0, & a &\neq \frac{1}{2}, \\
\{\chi_i, \chi_j\} &= 0.
\end{align*}

The constraints \(\Phi\)'s and the gauge condition \(\chi\)'s form the set of the second class constraints.

References

[1] S. Deser, W. Siegel, P. K. Townsend, Nucl. Phys. B184, 333(1981).
[2] V. I. Ogievetsky, I. V. Polubarinov, Yad. Fiz. 4, 216(1966).
[3] W. Tybor, Acta Phys. Pol. B18, 69(1987).
[4] W. Tybor, Acta Phys. Pol. B18, 369(1987).
[5] J. Rembielinski, W. Tybor, Acta Phys. Pol. B22, 439(1991).
[6] J. Rembielinski, W. Tybor, Acta Phys. Pol. B22, 447(1991).
[7] W. Tybor, Preprint KFT 1/95, University of Lodz, (1995).
[8] P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, New York 1964.
[9] D. M. Gitman, I. V. Tyutin, Kanonicheskoye kvantovaniye poley so sviazami, (Canonical quantization of fields with constraints), Nauka, Moscow 1986.
[10] R. Marnelius, Acta Phys. Pol. B13, 669(1982).
[11] J. D. Bjorken, S. D. Drell, Relatywistyczna teoria kwantow, (Relativistic Quantum Mechanics, Relativistic Quantum Fields), PWN, Warszawa (1985), p.433.