ON A NEW SUPERSYMMETRIC KdV HIERARCHY
IN 2−d QUANTUM SUPERGRAVITY

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ABSTRACT

Recently a new supersymmetric extension of the KdV hierarchy has appeared in a matrix-model-inspired approach to 2−d quantum supergravity. Here we prove that this hierarchy is essentially the KdV hierarchy, where the KdV field is now replaced by an even superfield. This allows us to find the conserved charges and the bihamiltonian structure, and to prove its integrability. We also extend the hierarchy by odd flows in a supersymmetric fashion.

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Introduction

One of the most pleasant surprises that noncritical string theory so far had in store for us, is its relation with classical integrable hierarchies of the KP type. For example, the KdV hierarchy appeared unsuspectedly in the double scaling limit of the one-matrix model—a fact which recurs in the multimatrix models for the generalized KdV hierarchies, and which allows one to exactly compute correlation functions on arbitrary topology. This success notwithstanding, the generalization of these techniques to the supersymmetric case is still an open problem and the precise relation, if any, with supersymmetric integrable hierarchies remains elusive.

In order to circumvent the problems encountered in an earlier unsuccessful attempt ([1]) to define a theory of noncritical superstrings using supermatrices, a model was proposed in [2] in which one does away with the matrices all together, and takes as a starting point the integral over the would-be eigenvalues. By imposing superVirasoro constraints—in analogy with the Virasoro constraints in the one-matrix model—correlation functions and critical exponents were calculated to first order in the topological expansion. Remarkably, they were found to coincide with those of certain superconformal matter coupled to $2-d$ supergravity. Recently, in [3], the model was solved for arbitrary genus and, in the double scaling limit, a new supersymmetric extension of the KdV hierarchy appeared.

It is the purpose of this note to identify the hierarchy and mention some of its immediate properties: conserved charges, bihamiltonian structure, integrability,... The punch line is that the hierarchy is simply a supersymmetric covariantization of the KdV hierarchy, and as such not very different from it. In this sense, the model solved in [3] seems once again not to be too distantly related to the bosonic one-matrix model, a fact already remarked in [4] as well as in [3].

We should remark, parenthetically, that the existence of this new supersymmetric extension of KdV does not contradict the Painlevé analysis of [5] which suggested that the only two integrable fermionic extensions of the KdV equation are the nonsupersymmetric extension of [6] and the one of [7]. This is due to the fact that the hierarchy we study in this paper does not fit the Ansatz in [5]. This hierarchy contains the KdV hierarchy as a subhierarchy and not as a reduction after setting the fermionic field to zero.

The SKdV-B hierarchy

The hierarchy of [3] is a hierarchy of flows on two variables $u$ and $\tau$—the “body” of the two-point function of the puncture operator and the first fermionic scaling variable, respectively; although their physical interpretation is of no relevance to our discussion. We introduce an infinite number of even times \{t_0, t_1, t_2, \ldots\} and an infinite number of odd times \{\tau_0, \tau_1, \tau_2, \ldots\}. On $u$ the odd flows are trivial

$$\frac{\partial u}{\partial \tau_k} = 0 \quad \forall k$$

(1)
whereas the even flows are those of the KdV hierarchy:

\[
\frac{\partial u}{\partial t_k} = R'_{k+1} = \left[\kappa^2 \partial^3 + 2u\partial + 2\partial u\right] \cdot R_k ,
\]

where the Gel’fand–Dickey polynomials \( R_k = R_k(u) \) are the gradients of the conserved charges of the KdV hierarchy and \( \kappa \) is the renormalized string coupling constant. The equality of (2) and (3) imply the celebrated Lenard relations between the \( R_k \), which can be translated into a recursion relation for the flows:

\[
\frac{\partial u}{\partial t_{n+1}} = \left[\kappa^2 \partial^2 + 2u + 2\partial u\partial^{-1}\right] \cdot \frac{\partial u}{\partial t_{n}} .
\]

Normalizing \( R_0 = \frac{1}{2} \), we can compute all the other \( R_k \) recursively: \( R_1 = u \), \( R_2 = \kappa^2 u'' + 3u^2 \), \( \text{ad nauseam} \). In terms of the \( R_k \), the commutativity of the KdV flows translates into

\[
\frac{\partial R'_k}{\partial t_n} = \frac{\partial R'_{n+1}}{\partial t_{k-1}} ,
\]

an identity that, as we will see shortly, implies the invariance of the even flows under supersymmetry. From the analysis in [3], \( \tau \) is given by

\[
\tau = -\sum_{k \geq 0} \tau_k R_k ,
\]

wherefrom we can read how it evolves along the flows

\[
\frac{\partial \tau}{\partial \tau_k} = -R_k \quad \text{and} \quad \frac{\partial \tau}{\partial t_{n}} = -\sum_{k \geq 0} \tau_k \frac{\partial R_k}{\partial t_{n}} .
\]

The first nontrivial even flows were found in [3] to be

\[
\frac{\partial u}{\partial t_1} = \kappa^2 u'' + 6uu' \quad \text{and} \quad \frac{\partial \tau}{\partial t_1} = \kappa^2 \tau''' + 6u\tau' ,
\]

whereas the odd flows were found to be

\[
\frac{\partial u}{\partial \tau_1} = 0 \quad \text{and} \quad \frac{\partial \tau}{\partial \tau_1} = -u .
\]

Notice that the first equation in (8) is nothing but the KdV equation for \( u \).

It was moreover observed in [3] that (8) is invariant under the (global) supersymmetric transformations

\[
\delta u = \tau' \quad \text{and} \quad \delta \tau = u .
\]

In fact, as we will show in a moment, this continues to be the case for all the even flows. On the other hand, the odd flows are not supersymmetric, for whereas \( \tau \) evolves, its supersymmetric partner \( u \) does not. Nevertheless, one can modify the odd flows to make them supersymmetric. We will comment on this further on.
Proposition 11. The even flows are invariant under (10), while the odd flows satisfy
\[
\left[ \delta, \frac{\partial}{\partial \tau_n} \right] u = -\frac{\partial}{\partial t_{n-1}} .
\] (12)

Proof: We first consider the even flows:
\[
\left[ \delta, \frac{\partial}{\partial t_n} \right] u = \left( \delta R_{n+1} \frac{\partial}{\partial u} . \tau' \right)' - \frac{\partial \tau'}{\partial t_n} \\
= -\left( \sum_{k \geq 0} \tau_k \frac{\partial R'_{n+1}}{\partial t_{k-1}} \right)' + \left( \sum_{k \geq 0} \tau_k \frac{\partial R'_k}{\partial t_n} \right)' = 0 ,
\]
where we have used (5). Notice that this allows us to rewrite the flow on \( \tau \) in a simpler way:
\[
\frac{\partial \tau}{\partial t_n} = \delta R_{n+1} .
\] (13)

From this, the analog result for \( \tau \) follows trivially, because
\[
\delta \frac{\partial \tau}{\partial t_n} = R'_{n+1} = \frac{\partial u}{\partial t_n} = \frac{\partial}{\partial t_n} \delta \tau .
\] (14)

On the other hand, for the odd flows we obtain for \( u \)
\[
\left[ \delta, \frac{\partial}{\partial \tau_n} \right] u = \frac{\partial \tau'}{\partial \tau_n} = -\frac{\partial u}{\partial t_{n-1}} ,
\] (15)

whereas for \( \tau \) one has
\[
\left[ \delta, \frac{\partial}{\partial \tau_n} \right] \tau = -\delta R_n = -\frac{\partial \tau}{\partial t_{n-1}} ,
\] (16)

where we have once again used (13). \( \blacksquare \)

In summary, the subhierarchy defined by the even flows is a supersymmetric extension of the KdV hierarchy, to which we refer in what follows as the SKdV-B hierarchy.

The SKdV-B Hierarchy is the KdV Hierarchy

We now begin the analysis of this hierarchy. We will find it convenient to employ superfields in order to preserve manifest supersymmetry and we also use freely the formalism of the formal calculus of variations and of pseudodifferential operators, for which we refer the reader to Dickey’s book [8], and, for instance, to [9] for the supersymmetric case.

Since the SKdV-B hierarchy is supersymmetric, one can express its flows in a way that makes this manifest, wherefore we introduce the superfield \( T = \tau + \theta u \), a function in a \((1|1)\) superspace. In superspace, the supersymmetry algebra is realized as supertranslations, which on superfields look like \( \delta T = QT \), where \( Q = \frac{\partial}{\partial \theta} - \theta \partial \). We will denote by \( D \) the supercovariant derivative \( D = \frac{\partial}{\partial \theta} + \theta \partial \), which anticommutes with \( Q \). One can recover the fields \( u \) and \( \tau \) by taking the appropriate projections: \( u = DT|_{\theta=0} , \tau = T|_{\theta=0} \).
Rewriting both equations in (8) as a single equation on the superfield $T$, we find
\[ \frac{\partial T}{\partial t_1} = \kappa^2 T^{[6]} + 6T'T'' , \] (17)
where $\nabla$ denotes differentiation with respect to $D$. Now notice that if we differentiate both sides of the equation once more with respect to $D$, we get
\[ \frac{\partial T'}{\partial t_1} = \kappa^2 T^{[7]} + 6T'T''' , \] (18)
which is nothing but the KdV equation (cf. the first equation in (8)) for the superfield $T' = u + \theta \tau'$. In fact, as we now show, this continues to be the case for all the other equations of the hierarchy; whence we will be able to conclude that the SKdV-B hierarchy is essentially equivalent to the KdV hierarchy.

This may require some explanation. The abstract KdV hierarchy is defined as the hierarchy of isospectral deformations of the Lax operator $L = \kappa^2 \partial^2 + u$, where $u$ is simply a commuting variable generating a differential ring. Particular representations of this abstract KdV hierarchy are obtained by letting $u$ be, for instance, a smooth function on the circle or a rapidly decaying smooth function on the real line. A more exotic representation can be defined by taking $u$ to be an even superfield, e.g., $T'$. We claim that the hierarchy so obtained is precisely SKdV-B. For notational convenience we will denote by KdV($T'$) the KdV hierarchy with $T'$ as the basic variable, and reserve KdV for when the basic variable is $u$.

Consecutive flows in both the KdV($T'$) and SKdV-B hierarchies are related by a recursion relation. This means that knowing the first flow one can obtain all the others by repeated application of a recursion operator. We have seen that the first flows of both hierarchies agree, thus all we need to show in order to prove the equivalence is that the recursion operators are the same.

The recursion relation for the flows of the KdV($T'$) hierarchy can be read off from (4) and is given by
\[ \frac{\partial T'}{\partial t_{n+1}} = \left[ \kappa^2 \partial^2 + 2 \partial T' \partial^{-1} + 2T' \right] \cdot \frac{\partial T'}{\partial t_n} . \] (19)
Stripping off a $D$ from both sides, we can rewrite this as
\[ \frac{\partial T}{\partial t_{n+1}} = \left[ \kappa^2 \partial^2 + 2DT' D^{-1} + 2D^{-1}T'D \right] \cdot \frac{\partial T}{\partial t_n} , \] (20)
which in components reads
\[
\left( \frac{\partial \tau}{\partial t_{n+1}} \right) = \begin{pmatrix}
\kappa^2 \partial^2 + 2u + 2 \partial^{-1} u \partial & 2\partial \tau \partial^{-1} - 2 \partial^{-1} \tau \partial \\
0 & \kappa^2 \partial^2 + 2u + 2 \partial u \partial^{-1}
\end{pmatrix}
\left( \frac{\partial \tau}{\partial t_n} \right) ,
\] (21)
and this, in turn, agrees with the recursion relation (40) in [3]. Thus, we conclude that the flows of the two hierarchies agree.

\footnotemark[1] Please note that on a superfield $'$ denotes derivative with respect to $D$, whereas on components it denotes derivative with respect to $\partial$. This should cause no confusion.
Conserved Charges

As it is well known, the conserved charges of the KdV hierarchy are given by the traces of the fractional powers of the Lax operator, namely (up to normalization)

\[ H_n = \text{Tr} L^{n-1/2} = \int h_n \quad \text{for } n = 1, 2, \ldots , \]  

(22)

where \( h_n \) is the residue of \( L^{n-1/2} \). For the SKdV-B hierarchy, the relevant Lax operator is \( L = \kappa^2 \partial^2 + T' \). For such a Lax operator, \( h_n \) is a superfield, whence \( H_n \) still has \( \theta \) dependence:

\[ H_n[u, \tau] = A_n[u, \tau] + \theta B_n[u, \tau] . \]  

(23)

Notice that since \( T'|_{\theta=0} = u \), \( A_n \) is simply the \( n \)th conserved charge \( H_{KdV}^n \) of the KdV hierarchy—in particular, it is independent of \( \tau \). It is clear that both \( A_n \) and \( B_n \) are conserved under the SKdV-B flows, but in order to consider them as conserved charges of a supersymmetric hierarchy, one has to take into account the requirement that they be invariant under supersymmetry. Making use of the above remark, let us rewrite (23) as follows:

\[ \int h_n(T') = \int h_n(u) + \theta \int b_n(u, \tau) . \]  

(24)

Because \( h_n(T') \) is a differential polynomial in \( T' \), it transforms under supersymmetry as a superfield. In other words,

\[ \delta h_n(u) = b_n(u, \tau) \quad \text{and} \quad \delta b_n(u, \tau) = h_n(u)' , \]  

(25)

whence \( \delta B_n[u, \tau] = \int \delta b_n(u, \tau) = \int h_n(u)' = 0 \). That is, the charges \( B_n \) are both supersymmetric and conserved. Because they are supersymmetric they can be written as integrals over superspace. In fact,

\[ B_n[u, \tau] = \delta \int h_n(u) = \int \frac{\delta h}{\delta u} \cdot \tau' = \int D h_n(T') |_{\theta=0} = \int_B h_n(T') \]  

(26)

by definition of the Berezin integral. It remains to show that these charges are nontrivial. Now, it is a classic result \([10]\) that the \( h_n(u) \) have the form \( h_n(u) \propto u^n + \cdots \), where \( \cdots \) stand for terms with derivatives and hence a smaller number of \( u \)'s. Therefore,

\[ \frac{\delta B_n}{\delta T} \propto n(n-1)(T')^{n-2}T'' + \cdots \]  

(27)

which is nonzero for \( n \geq 2 \). For \( n = 1 \), \( A_1 = \int u = \int_B T \) is already supersymmetric. In summary, we have proven the following result.

**Proposition 28.** The conserved charges of the SKdV-B hierarchy are given by

\[ H_{n}^{SKdv-B} = \delta H_{KdV}^{n} \quad \text{for } n \geq 2 , \]

\[ H_{1}^{SKdv-B} = H_{KdV}^{1} , \]

and they obey the following relation (for \( n \geq 2 \))

\[ \text{Tr} \, L^{n-1/2} = H_{n}^{KdV} [u] + \theta H_{n}^{SKdv-B} [u, \tau] , \]  

(29)

where \( L = \kappa^2 \partial^2 + T' \).
One can nevertheless ask the question whether these are in fact all the conserved charges of the SKdV-B hierarchy. We have found out by hand one “exotic” charge

\[ H^{\text{exotic}} = \int_B T T' , \]  

and we have verified that there are no other exotic charges up to weight \( \frac{15}{2} \), where we say that \( T \) has weight \( \frac{3}{2} \) and \( D \) has weight \( \frac{1}{2} \).

**SKdV-B as a Reduction of SKP-type Hierarchies**

Since the SKdV-B flows are given by the isospectral deformations of the Lax operator \( L = \kappa^2 \partial^2 + T' \), it is easy to see that SKdV-B is but a particular reduction of the SKP\(_2\) hierarchy introduced in [11]. First of all it is clear that \( L \) has a unique square root of the form

\[ L^{1/2} = \kappa \partial + \sum_{i \geq 1} A_i(T') \partial^{1-i} , \]  

where the \( A_i(T') \) are \( \partial \)-differential polynomials in \( T' \). In terms of \( L^{1/2} \), the flows defining SKdV-B are given by

\[ \frac{\partial L^{1/2}}{\partial t_n} \propto \left[ L_{+}^{n-1/2} , L^{1/2} \right] . \]  

Now, \( L^{1/2} \) is but a special case of the general SKP\(_2\) operator \( \Lambda = \kappa \partial + \sum_{k \geq 1} B_k(T) D^{2-k} \) treated (\( \kappa \) aside) in [11], where the \( B_k(T) \) are \( D \)-differential polynomials in \( T \). Moreover the SKP\(_2\) flows are given by \( \frac{\partial \Lambda}{\partial t_n} = \left[ \Lambda^n , \Lambda \right] \), which agree (after relabeling and rescaling the times) with (32). In other words, the submanifold of SKP\(_2\) operators of the form (31) is preserved by the SKP\(_2\) flows and, moreover, these flows agree with the ones defining SKdV-B.

Moreover, since the Lax operator \( L = \kappa^2 \partial^2 + T' \) can be “undressed”, one can map the SKdV-B hierarchy into the even part of the SKP hierarchy of [7] or, equivalently, the Jacobian SKP hierarchy of [12]. To this effect, let us define an element \( S \) of the Volterra group by

\[ L = S \kappa^2 \partial^2 S^{-1} . \]  

In terms of \( S \), the SKdV-B flows can be written as (up to \( \kappa \) factors)

\[ \frac{\partial S}{\partial t_n} \propto -(S \partial^{2n+1} S^{-1}) S . \]  

This equation is then the one defining the even flows of the SKP hierarchy, when we think of \( S \) as an element of the larger superVolterra group.

**Bihamiltonian Structure and Integrability**

It was shown in [11] that SKdV-type reductions of the SKP\(_2\) hierarchy are bihamiltonian: the two structures being given by the supersymmetric analogs of the Gel’fand–Dickey brackets constructed in [13]. In particular, the hierarchy associated to the operator \( D^4 + U_1 D^3 + U_2 D^2 + U_3 D + U_4 \) is bihamiltonian, and so is its reduction \( U_1 = U_2 = U_4 = 0 \) to SKdV. It would thus seem reasonable to expect that the SKdV-B hierarchy, which is obtained as the reduction \( U_1 = U_2 = U_3 = 0 \) and \( U_4 = T' \), would inherit a bihamiltonian structure in this fashion. However, this turns out not to be the case: it is easy to show that setting \( U_1 = U_2 = U_3 = 0 \) collapses the rest of the phase space.
We can nevertheless exhibit a bihamiltonian structure for SKdV-B exploiting its equivalence with KdV($T'$). We first rewrite the analogs of (2) and (3) for KdV($T'$):

\[
\frac{\partial T'}{\partial t_k} = \partial \cdot \frac{\delta H_{k+1}^{KdV}}{\delta u} \bigg|_{u=T'} = \left[ \kappa^2 \partial^3 + 2T' \partial + 2\partial T' \right] \cdot \frac{\delta H_{k+1}^{KdV}}{\delta u} \bigg|_{u=T'}.
\]

(34)

For $H_{k}^{KdV} = \int h_k(u)$, we have that

\[
\frac{\delta H_{k}^{KdV}}{\delta u} \bigg|_{u=T'} = \sum_{i \geq 0} (\partial^i)^* \cdot \frac{\partial h_k}{\partial u^{(i)}} \bigg|_{u=T'} = \sum_{i \geq 0} (D^{2i})^* \cdot \frac{\partial h_k}{\partial T^{[2i+1]}} = -D^{-1} \sum_{i \geq 0} (D^{2i+1})^* \cdot \frac{\partial h_k}{\partial T^{[2i+1]}}.
\]

(35)

Since $h_k(T')$ only depends on the odd $D$-derivatives of $T$ we may add for free the contribution of the even derivatives, and we obtain \(^2\)

\[
\frac{\delta H_{k}^{KdV}}{\delta u} \bigg|_{u=T'} = -D^{-1} \cdot \sum_{i \geq 0} (D^i)^* \cdot \frac{\partial h_k}{\partial T^{[i]}} = D^{-1} \cdot \frac{\delta H_{k}^{SKdV-B}}{\delta T}.
\]

(36)

for $H_{k}^{SKdV-B} = \int_B h_k(T')$. We can thus rewrite (34) and (35) as follows

\[
\frac{\partial T'}{\partial t_k} = \frac{\delta H_{k+1}^{KdV}}{\delta T} = \left[ \kappa^2 \partial^2 + 2D^{-1}T'D + 2DT'D^{-1} \right] \cdot \frac{\delta H_{k}^{SKdV-B}}{\delta T}.
\]

(37)

These equations look already to be in hamiltonian form, with Poisson structures $J_1 = 1$ and $J_2 = \kappa^2 \partial^2 + 2D^{-1}T'D + 2DT'D^{-1}$. Notice that $J_1$ satisfies the Jacobi identities trivially, since it is constant. It may seem at first odd that it is not antisymmetric—but this is nothing new in supersymmetric hierarchies, which can have both even and odd Poisson structures \([15]\). The second structure $J_2$ may not seem obviously Poisson, but it is not hard to show that the Jacobi identities are satisfied. Notice that $J_2$ also defines odd Poisson brackets which are moreover nonlocal. This is again nothing new in supersymmetric hierarchies: the first Poisson structure of SKdV is also nonlocal; although the flows, just like the ones here, are local. Notice, parenthetically, that as expected $J_2 J_1^{-1}$ coincides with the recursion operator (20) for SKdV-B.

\(^2\) It should be noticed that the above formula does not mix grading. This follows from the definition of the variational derivative in the formal calculus of variations, in which integration is simply the operation of dropping total derivatives. This point seems to have caused some confusion in the literature \([14]\).
Finally, notice that \( J_1 \) can be obtained from \( J_2 \) by shifting \( T' \mapsto T' + \lambda \). Since \( J_2 \) is Poisson for any \( T \), it follows that \( J_1 \) and \( J_2 \) are coordinated. Usual arguments now imply that the conserved charges are in involution relative to both Poisson structures. In summary, SKdV-B is an integrable bihamiltonian supersymmetric hierarchy.

Some Remarks on Odd flows

Although as proven in Proposition 11 the odd flows are not supersymmetric, it is possible to modify them in such a way that they are. First of all notice that the explicit expression (6) of \( \tau \) as a function of the odd times and the \( R_k \) can only be reconciled with its transformation law (10) under supersymmetry, if \( \tau_1 \) transforms under supersymmetry. To see this, let us plug (6) into the second equation of (10):

\[
\begin{align*}
    u &= -\sum_{k \geq 0} (\delta \tau_k) R_k + \sum_{k \geq 0} \tau_k \delta R_k \\
    &= -\sum_{k \geq 0} (\delta \tau_k) R_k - \sum_{k \geq 0} \tau_k \frac{\partial}{\partial t_{n-1}} \sum_{\ell \geq 0} \tau_\ell R_\ell \\
    &= -\sum_{k \geq 0} (\delta \tau_k) R_k - \sum_{k, \ell \geq 0} \tau_k \tau_\ell \frac{\partial R_\ell}{\partial t_{n-1}} \\
    &= -\sum_{k \geq 0} (\delta \tau_k) R_k 
\end{align*}
\]

by (13) and (6)

which implies that

\[
\delta \tau_k = -\delta_{k,1} . \tag{38}
\]

Consider now the flows given by

\[
D_n \equiv \frac{\partial}{\partial \tau_n} - \tau_1 \frac{\partial}{\partial t_{n-1}} . \tag{39}
\]

From (38) and (12) it follows that these flows are supersymmetric. It is moreover obvious that they commute with the even flows, and that all \( D_{n \neq 1} \) (anti)commute among themselves. The remaining algebra of flows is

\[
D_1^2 = -\partial \quad \text{and} \quad [D_1, D_n] = -\frac{\partial}{\partial t_{n-1}} \quad \forall n > 1 , \tag{40}
\]

where we have used the fact that \( \frac{\partial}{\partial \tau_0} = \partial \). This defines a supersymmetric extension of the SKdV-B hierarchy by odd flows.

It now remains to find a representation of the above algebra of flows in superspace. The main obstacle lies in that the \( D_n \) explicitly depend on \( \tau_1 \) which, as (38) suggests, should be represented as \(-\theta\). It is easy to check that the representation induced from \( \delta \mapsto Q \) and \( \tau_1 \mapsto -\theta \) is inconsistent, and we have thus far been unable to find a consistent superspace representation for the odd flows.

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Alternatively one could try to induce odd flows via the embedding of the SKdV-B hierarchy into the even part of (a reduction of) the SKP hierarchy hierarchy of [12]. The even flows of both the SKP hierarchy of [7] and the Jacobian SKP hierarchy of [12] agree, but the odd ones don’t. Nevertheless, via (33), we can understand these flows as flows in the superVolterra group. It is easy to see that the flows of neither of the two hierarchies preserve the Volterra subgroup where $S$ lives.

Epilogue

We can thus understand the SKdV-B hierarchy as a (perhaps somewhat naive) supersymmetrization of the KdV hierarchy. It thus behooves us to ask whether other reductions of the KP hierarchy may be supersymmetrized in this fashion. It turns out that of the generalized KdV hierarchies, only the Boussinesque admits this supersymmetrization.\(^3\) Brevity demands that we omit the details, which will appear somewhere else [16]. As a closing remark, let us simply add that if, as expected, the supersymmetrized Boussinesque hierarchy alluded to above plays a role in the double scaling limit of the supersymmetric analog of the two-matrix model, then something interesting should happen in the supersymmetric three-matrix model.

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\(^3\) Of course, the naive supersymmetrization of substituting the bosonic fields by even superfields which are not derivatives of anything can always be achieved, but these don’t seem to be the ones that appear in the supersymmetric matrix models.
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