Nuclear recoil corrections to the Lamb shift of hydrogen and light hydrogen-like ions

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Accurate calculations of the nuclear recoil effect to the Lamb shift of hydrogen-like atoms are presented. Numerical results are reported for the $\ell s$ states with $\ell \leq 5$ and for the $2p_{3/2}$ and $2p_{1/2}$ states. The calculations are performed to the first order in the electron-nucleus mass ratio and to all orders in the nuclear binding strength parameter $Z\alpha$ (where $Z$ is the nuclear charge number and $\alpha$ is the fine structure constant). The obtained results provide accurate predictions for the higher-order remainder beyond the known $Z\alpha$-expansion terms. In the case of hydrogen, the remainder was found to be much larger than anticipated. This result resolves the previously reported disagreement between the numerical all-order and the analytical $Z\alpha$-expansion approaches for the nuclear recoil effect in the hydrogen Lamb shift.

I. INTRODUCTION

Hydrogen atom and hydrogen-like ions are the examples of the most fundamental physical systems. Their simplicity makes them an ideal testing ground for extending the theory based on the principles of quantum electrodynamics (QED) up to the utmost precision \[1\]. The theory of the hydrogen Lamb shift is of particular importance because of its connections to the utmost precision \[1\]. The theory of the hydrogen Lamb shift of energy levels in hydrogen-like atoms with $Z \leq 10$. Our first results for $n = 1$ and $n = 2$ states were published in Ref. \[19\]. In the present paper we extend our calculations to a larger range of the nuclear charge numbers and to higher excited states and describe details of the calculation.

Relativistic units ($\hbar = c = 1$) are used throughout the paper.

II. THEORY

The recoil correction to the Lamb shift of hydrogen-like atoms, to first order in $m/M$ but to all orders in $Z\alpha$ can be represented as a sum of four terms,

$$\Delta E_{\text{rec}} = \Delta E_L + \Delta E_C + \Delta E_{\text{tr}(1)} + \Delta E_{\text{tr}(2)},$$

(1)

where $\Delta E_L$ (the low-energy part) is the recoil correction as can be derived from the Breit equation, $\Delta E_C$ (the Coulomb part) is the QED recoil correction induced by the exchange of arbitrary number of virtual Coulomb photons between the electron and the nucleus, $\Delta E_{\text{tr}(1)}$ and $\Delta E_{\text{tr}(2)}$ (the one-transverse-photon and two-transverse-photons parts, respectively) are the QED recoil corrections induced by the exchange of one (respectively, two) transverse photon(s) and arbitrary number of virtual Coulomb photons between the electron and the nucleus. The low-energy part $\Delta E_L$ contains the complete result to orders $(Z\alpha)^2 m/M$ and $(Z\alpha)^4 m/M$ and partial results for the higher-order (in $Z\alpha$) corrections. The remaining terms $\Delta E_C$, $\Delta E_{\text{tr}(1)}$, and $\Delta E_{\text{tr}(2)}$ induce contributions to orders $(Z\alpha)^5 m/M$ and higher.

In the following, we will first consider the case where the nucleus is considered to be the point source of the Coulomb field. Additional corrections arising because of the finite nucleus charge distribution will be addressed separately in the second part of the section.
A. Point nucleus

For the point nucleus, the low-energy part of the recoil effect $\Delta E_L$ can be derived from the Breit equation. It is given by [7]

$$\Delta E_L = \frac{1}{2M} \langle a | p^2 - D(0) \cdot p - p \cdot D(0) | a \rangle, \quad (2)$$

where $p$ is the electron momentum operator, $D_j(\omega) = -4\pi\alpha Z_\alpha \delta_{ij}(\omega, r)$, $\alpha_i$ are the Dirac matrices, and $D_{ij}(\omega, r)$ is the transverse part of the photon propagator in the Coulomb gauge,

$$D_{ij}(\omega, r) = -\frac{1}{4\pi} \left[ \exp(i|\omega|r) \delta_{ij} + \nabla_i \nabla_j \exp(i|\omega|r) - 1 \right]. \quad (3)$$

Equation (3) can be calculated analytically and cast in a very simple form [7],

$$\Delta E_L = \frac{m^2 - \varepsilon_0^2}{2M}, \quad (4)$$

where $\varepsilon_0$ is the Dirac energy of the reference state.

The corrections $\Delta E_C$, $\Delta E_{1r,1}$, and $\Delta E_{1r,2}$ in Eq. (1) are derived within the QED theory [7,12]. The result for the Coulomb part is

$$\Delta E_C = \frac{2\pi i}{M} \int_{-\infty}^{\infty} d\omega \delta_+^2(\omega) \langle a | [p, V] G(\omega + \varepsilon_0) [p, V] | a \rangle, \quad (5)$$

where $\delta_+(\omega) = i/(2\pi)/(\omega + i0)$, $V(r) = -Z\alpha/r$ is the nuclear Coulomb potential, $G(\omega) = 1/[\omega - H(1 - i0)]$ is the relativistic Coulomb Green function, $H = \alpha \cdot p + \beta m + V$ is the Dirac-Coulomb Hamiltonian, and $[\ldots]$ denotes commutator. The integration over $\omega$ in Eq. (5) can be carried out analytically, yielding

$$\Delta E_C = -\frac{1}{M} \sum_{\varepsilon_0 < 0} \langle a | p \rangle \langle n | p | a \rangle, \quad (6)$$

where the summation over $n$ is extended over the negative-energy part of the Dirac spectrum and the scalar product is implicit.

The one-transverse-photon part $\Delta E_{1r(1)}$ is induced by the exchange of one transverse and arbitrary number of Coulomb photons between the electron and the nucleus. The result is

$$\Delta E_{1r(1)} = -\frac{1}{M} \int_{-\infty}^{\infty} d\omega \delta_+(\omega) \langle a | [p, V] G(\omega + \varepsilon_0) D(\omega)$$

$$- D(\omega) G(\omega + \varepsilon_0) [p, V] | a \rangle. \quad (7)$$

The two-transverse-photons part $\Delta E_{1r(2)}$ is induced by the exchange of two transverse and arbitrary number of Coulomb photons between the electron and the nucleus. The result is

$$\Delta E_{1r(2)} = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \langle a | D(\omega) G(\omega + \varepsilon_0) D(\omega) | a \rangle. \quad (8)$$

The QED part of the recoil effect can be conveniently parameterized in terms of a slowly-varying dimensionless functions $P(Z\alpha)$,

$$\Delta E_C + \Delta E_{1r(1)} + \Delta E_{1r(2)} = \frac{m^2}{M} \frac{(Z\alpha)^5}{\pi n^3} P(Z\alpha), \quad (9)$$

where $n$ is the principal quantum number of the state under consideration. For low-$Z$ atoms, the function $P(Z\alpha)$ can be expanded in a series over the parameter $Z\alpha$, which is of the form

$$P(Z\alpha) = \ln(Z\alpha)^{-2} D_{51} + D_{50}$$

$$+ (Z\alpha) D_{60} + (Z\alpha)^2 G_{\text{rec}}(Z\alpha), \quad (10)$$

where $D_{ij}$ are the coefficients and $G_{\text{rec}}(Z\alpha)$ is the higher-order remainder containing all higher orders in $Z\alpha$. The coefficients of the $Z\alpha$ expansion are [6,11,16]

$$D_{51} = \frac{1}{3} \delta_{i,0}, \quad (11)$$

$$D_{50} = \left[ -\frac{8}{3} \ln k_0(n, l) + d_{50} \right], \quad (12)$$

$$D_{60} = \left( 4 \ln 2 - \frac{7}{2} \right) \pi \delta_{i,0}$$

$$+ \left[ 3 - \frac{l(l+1)}{n^2} \right] \frac{2\pi (1 - \delta_{i,0})}{(4l^2 - 1)(2l + 3)}, \quad (13)$$

where $\ln k_0(n, l)$ is the Bethe logarithm, whose numerical values for the states of the current interest are [20]

$$\ln k_0(1s) = 2.984 128 556, \quad (14)$$

$$\ln k_0(2s) = 2.811 769 893, \quad (15)$$

$$\ln k_0(3s) = 2.767 663 612, \quad (16)$$

$$\ln k_0(4s) = 2.749 811 840, \quad (17)$$

$$\ln k_0(5s) = 2.740 823 728, \quad (18)$$

$$\ln k_0(2p) = -0.030 016 709. \quad (19)$$

The values of the coefficients $d_{50}(n, l)$ for these states are

$$d_{50}(ns) = \frac{41}{9} + \frac{14}{3} \left[ \frac{2}{n} + \sum_{i=1}^{n} \frac{1}{l} - \frac{1}{2n} \right], \quad (20)$$

$$d_{50}(2p) = -\frac{7}{18}. \quad (21)$$

The $Z\alpha$ expansion of the higher-order remainder reads

$$G_{\text{rec}}(Z\alpha) = \ln^2(Z\alpha)^{-2} D_{72} + \ln(Z\alpha)^{-2} D_{71} + D_{70} + \ldots, \quad (22)$$

where only the double logarithmic contribution is presently known [17,18]

$$D_{72} = -\frac{11}{60} \delta_{i,0}. \quad (23)$$
B. Extended nucleus

The low-energy part of the recoil correction for the case of an extended nuclear charge was derived in Refs. [21, 22] (see also [23]),

\[ \Delta E_L = \frac{1}{2M} \langle a \mid [\varepsilon_a^2 - m^2] - 2m\beta V(r) - W'(r) V'(r) - V^2(r) \rangle |a \rangle, \]  

(24)

where

\[ V(r) = -Z\alpha \int dr' \frac{\rho(r')}{|r - r'|}, \]

(25)

\[ W(r) = -Z\alpha \int dr' \rho(r') |r - r'|, \]

(26)

\[ V'(r) = dV(r)/dr, \quad W'(r) = dW(r)/dr, \] and \( \rho(r) \) is the density of the nuclear charge distribution \( \{ \int dr \rho(r) = 1 \} \).

The Coulomb part of the QED recoil correction \( \Delta E_{C} \) for an extended nuclear charge is given by the same formula [3] as for the point-nucleus case, with \( V(r) \) being the extended-nucleus potential [24]. Exact expressions for the one-transverse-photon part \( \Delta E_{C(1)} \) and the two-transverse-photon part \( \Delta E_{C(2)} \) for the extended nucleus case are not yet known. In the present work, we will use the expressions [7] and [8] derived for the point nucleus but evaluate these expressions with the extended-nucleus wave functions, energies, electron propagators, and nuclear potential. The same treatment was presented earlier in Refs. [24, 25]. The uncertainty introduced by this approximation will be discussed in Sec. [13].

We are interested in the recoil correction induced by the finite nuclear size (fns), so we take a difference between the results obtained with an extended nucleus and with the point nucleus,

\[ E_{\text{fns,rec}} = E_{\text{rec}(\text{ext})} - E_{\text{rec}(\text{pnt})}, \]

(27)

where ext and pnt refer to the extended and the point nuclear distributions, respectively.

For the low-\( Z \) atoms, it is customary [11, 12] to account for a part of the recoil fns effects by introducing the reduced mass prefactor \( (M/(m + M))^3 \) in the expression for the fns correction. To the first order in \( m/M \), such correction is given by

\[ E_{\text{fns,rm}} = -\frac{3}{2} \frac{m}{M} [\varepsilon_a(\text{ext}) - \varepsilon_a(\text{pnt})], \]

(28)

where \( \varepsilon_a(\text{ext}) \) and \( \varepsilon_a(\text{pnt}) \) are the eigenvalues of the Dirac equation with the extended and the point nuclear potentials, respectively.

In the present work, we will identify the higher-order fns recoil correction that is beyond the reduced-mass part [28] and parameterize it in terms of the function \( \delta_{\text{fns}} P \),

\[ E_{\text{fns,rec}} = E_{\text{fns,rm}} + \frac{m^2}{M} \frac{(Z\alpha)^5}{\pi n^3} \delta_{\text{fns}} P. \]

(29)

We would like to draw the reader’s attention to the fact that \( \delta_{\text{fns}} P \) includes the fns contribution from \( \Delta E_L \).

III. NUMERICAL CALCULATION

The general scheme of the calculation was described previously in Ref. [13]. An important issue in the computation of Eqs. [5]-[8] is the adequate numerical representation of the Dirac-Coulomb Green function \( G(\omega) \). The Dirac-Coulomb Green function is known in a form of the partial-wave expansion over the angular momentum-parity quantum number \( \kappa \) (see, e.g., Ref. [26] for details). After all angular-momentum selection rules are taken into account, only a few of the partial-wave contributions of the Green function survive (two for the \( j = 1/2 \) reference states and three for the \( j = 3/2 \) reference states). The resulting expressions were evaluated by summing over the spectrum of the radial Dirac equation with the help of the finite basis set constructed with \( B \)-splines. For the point nuclear model we used the standard variant of the \( B \)-spline method [27]. The calculations for the extended nucleus were performed by the Dual kinetic balance method [28].

The main technical problem of the previous calculations [13, 15] that limited the numerical accuracy in the low-\( Z \) region was lack of convergence with increase of the size of the basis set \( N \). In the present investigation, we found out that this effect was caused by numerical instabilities associated with limitations of the standard double-precision (approximately 16 digits) arithmetics. In the present work we implemented the procedure of solving the Dirac equation with the \( B \)-splines basis set in the quadruple-precision (approximately 32-digit) arithmetics. After that we were able to achieve a clear convergence pattern of the calculated results when the size of the basis set was increased. The largest basis size used in actual calculations was \( N = 250 \). The numerical uncertainty of the obtained results was estimated by changing the size of the basis set by 30-50\% and by increasing the number of integration points in numerical quadratures.

Calculations for the extended nucleus were performed with two models of the nuclear charge distribution, the Gauss model and the homogeneously charged sphere model. We did not use the Fermi model, which is commonly used in calculations of heavy and medium-\( Z \) atoms, since this model is not suitable for very light nuclei. The Gauss distribution of the nuclear charge reads

\[ \rho_{\text{Gauss}}(r) = \left( \frac{3}{2\pi R^2} \right)^{3/2} \exp \left( -\frac{3r^2}{2R^2} \right), \]

(30)

where \( R \) is the root-mean-square radius of the nuclear charge distribution. The homogeneously charged sphere distribution is given by

\[ \rho_{\text{Sph}}(r) = \frac{3}{4\pi R_{\text{Sph}}^3} \theta(R_{\text{Sph}} - r), \]

(31)

where \( \theta(r) \) is the Heaviside step function and \( R_{\text{Sph}} = \sqrt{5/3} R \). We estimate the nuclear model dependence of our calculations by comparing the results obtained for these two nuclear models.

In our calculations, we had to numerically evaluate inte-
grals of the form
\[ F(\Delta) = \int_0^\infty dy \frac{\Delta}{\Delta^2 + y^2} f(y), \quad (32) \]
where \( \Delta = \epsilon_a - \epsilon_n \) is the energy difference of the reference state and the virtual state and \( f(y) \) is a smooth function of \( y \). The integration over \( y \) was performed numerically by splitting the interval \((0, \infty)\) into subintervals, making suitable change of variables and applying the Gauss-Legendre quadratures. Special care had to be taken in performing numerical integrations when \( |\Delta| \) happens to be small, since the integrand has a rapidly changing structure at \( y \sim |\Delta| \). In such cases, we represent \( F(y) \) as a sum of 3 terms,
\[ F(\Delta) = f(0) \frac{\Delta}{|\Delta|} \arctan \frac{Z \alpha}{|\Delta|} \]
\[ + \int_0^{Z \alpha} dy \frac{\Delta}{\Delta^2 + y^2} [f(y) - f(0)] \]
\[ + \int_{Z \alpha}^\infty dy \frac{\Delta}{\Delta^2 + y^2} f(y). \quad (33) \]
Taking into account that for small \( x \), \( \arctan(1/x) = \pi/2 - x + \ldots \), it is easy to see that Eq. (33) has a smooth and numerically safe transition to the limit \( \Delta \to 0 \), in contrast to the original expression (32). In order to calculate the second term in the right-hand side of Eq. (33), we first store the (slowly varying) function \( f(y) - f(0) \) on a grid and then compute the integral over \( y \) numerically with obtaining function \( f(y) \) by a polynomial interpolation. The third term in the right-hand side of Eq. (33) does not represent any problems and is evaluated in the standard way.

IV. RESULTS AND DISCUSSION

For the point nuclear model, our numerical results for the \( n = 1 \) and \( n = 2 \) states are presented in Table I and those for the \( ns \) states with \( n = 3 \ldots 5 \) in Table II. Table II presents also a comparison with the previous numerical and \( Z \alpha \)-expansion calculations. Generally, we find very good agreement with previous numerical results [13,15]. The only exception is the \( 2s \) state and \( Z = 1 \), for which a small deviation is found that was caused by a minor mistake in the previous calculation. At the same time, we observe a strong contrast between the all-order results for the higher-order remainders \( G_{\text{rec}}(1s) \) and \( G_{\text{rec}}(2s) \) and the corresponding \( Z \alpha \)-expansion values [17,18]. We recall that the \( Z \alpha \)-expansion results for \( G_{\text{rec}} \) include only the double-log contribution \( D_{72} \ln^2(Z \alpha)^{-2} \) and neglect the higher-order terms. For hydrogen, \( \ln(1 \alpha)^{-2} \approx 10 \) is a large parameter. For this reason, the leading logarithmic approximation is often used for estimating the tail of the \( Z \alpha \) expansion, with a typical estimate of uncertainty of 50% [29].

In order to perform a detailed analysis of the seeming discrepancy with the \( Z \alpha \) expansion results, we performed our calculations for a series of nuclear charges including fractional \( Z \) values as low as \( Z = 0.5 \). The results obtained for the higher-order remainder \( G_{\text{rec}}(Z \alpha) \) are plotted in Fig. 1.

We discover a rapidly changing structure at very low values of \( Z \). Most remarkably, the bending of the curve is practically undetectable for \( Z \geq 2 \). In order to access such a structure in an all-order calculation, one needs to achieve a very high numerical accuracy at very low (and fractional) values of \( Z \).

We now analyse our numerical results obtained for the higher-order remainder \( G_{\text{rec}}(Z \alpha) \) by fitting them to the following ansatz that incorporates the known form of the \( Z \alpha \) expansion to order \( (Z \alpha)^7 \),
\[ G_{\text{rec}}(Z \alpha) = d_{7,2} \ln^2(Z \alpha)^{-2} + d_{7,1} \ln(Z \alpha)^{-2} + \sum_{i=0}^{n} d_{7+i,0}(Z \alpha)^i, \quad (34) \]
where \( n = 2 \ldots 4 \) and \( d_{i,k} \) are fitting coefficients. We use the fitted values of \( d_{7,j} \) coefficients as approximations for the \( D_{7,j} \) coefficients in the \( Z \alpha \) expansion (22). The uncertainties are determined by changing of the length of the anzatz (i.e., \( n \)), by varying the number of fitted data points, and also by changing between using the analytical value of \( D_{72} \) and fitting it as a free parameter. For the squared logarithmic contributions we find the fitting results \( D_{72}(1s) = -0.183(1) \) and \( D_{72}(2s) = -0.185(1) \) which perfectly agree with the analytical value \(-11/60 = -0.18333 \ldots \). The next two coefficients are:
\[ D_{71}(1s) = 2.919(10), \quad D_{70}(1s) = -1.32(10), \quad (35) \]
\[ D_{71}(2s) = 3.335(10), \quad D_{70}(2s) = -0.26(6), \quad (36) \]
\[ D_{71}(2p_{1/2}) = 0.149(5), \quad D_{70}(2p_{1/2}) = -0.035(15), \quad (37) \]
\[ D_{71}(2p_{3/2}) = -0.283(5), \quad D_{70}(2p_{3/2}) = 0.685(20). \quad (38) \]

The results obtained for the \( D_{71} \) coefficients rely on the equivalence of the nonperturbative and the \( Z \alpha \) expansion approaches. This equivalence follows from the systematic derivation of the \( Z \alpha \) expansion from the full QED within the formalism of nonrelativistic quantum electrodynamics (NRQED) [30] and was also confirmed by explicit calculations in different physical contexts [2].

We thus conclude that our all-order results are consistent with all known coefficients of the \( Z \alpha \) expansion. The deviation observed for the higher-order remainder of the \( s \) states in Table I comes from the higher-order terms, whose contribution turns out to be unexpectedly large. Specifically, the single-log coefficient \( D_{71} \) is found to be 16 times larger than the double-log coefficient \( D_{72} \). As a result, the inclusion of the single-log contribution changes drastically the \( Z \alpha \) expansion result for the higher-order recoil effect.

We now turn to the correction to the nuclear recoil effect induced by the finite nuclear size. The numerical results for the higher-order fs recoil effect are presented in Table III. The values of the rms radii of the nuclear charge distribution \( R \) used in the calculations [31] are listed in the second column of the table. For \( Z = 1 \) we performed calculations for two values of \( R \), one corresponding to hydrogen and another, to deuterium. Numerical results obtained for two different models of the nuclear charge distribution are listed for each \( Z \) in
The errors specified in Table III are estimations of the uncertainty of the approximation for the low-order part, the absolute value of
\[ \Delta E_L = \Delta E_{\text{app}}. \]
We then estimate the approximation error as the absolute value of
\[ 2 \Delta E_L - \Delta E_{\text{app}} = \frac{E_{\text{ins-rec}} - E_{\text{ins,rm}}}{E_{\text{ins-rec}} - E_{\text{ins,rm}}} \left[ \Delta E_{\text{tr}(1),\text{fns}} + \Delta E_{\text{tr}(2),\text{fns}} \right], \]
where \( \Delta E_{\text{tr}(1),\text{fns}} \) and \( \Delta E_{\text{tr}(2),\text{fns}} \) are the fns corrections to the one-transverse-photon and the two-transverse-photons parts, respectively. In Eq. [39], the numerator \( \Delta E_L - \Delta E_{\text{app}} \) is the error of the approximation for the low-order part, the denominator \( E_{\text{ins-rec}} - E_{\text{ins,rm}} \) is the total value of the recoil fns correction, whereas 2 is a conservative factor. We note that we cannot use \( \Delta E_L \) in the denominator of Eq. [39] because of...
large cancellations of spurious terms between $\Delta E_1$ and $\Delta E_C$ [24]. It might be also mentioned that the full expressions for the two-transverse-photons $\text{fn}$s correction should contain contributions induced by virtual nuclear excitations [24, 25]. These terms need to be considered together with the nuclear polarization effect [32, 33] and are beyond the scope of the present investigation.

V. SUMMARY

In the present investigation we calculated the nuclear recoil correction to the Lamb shift of light hydrogen-like atoms. The calculation is performed to the first order in the electron-nucleus mass ratio $m/M$ and to all orders in the nuclear binding strength parameter $Z\alpha$, both for the point and the extended nuclear models. The results were found to be in excellent agreement with those obtained previously within the $Z\alpha$ expansion approach. The higher-order recoil contribution beyond the previously known $Z\alpha$-expansion terms was identified.

Our calculation resolves the previously reported disagreement between the numerical all-order and the analytical $Z\alpha$-expansion approaches and eliminates the second-largest theoretical uncertainty in the hydrogen Lamb shift of the $1S$ and $2S$ states. The calculated values of the higher-order recoil correction beyond the previously known $Z\alpha$-expansion terms for hydrogen are 0.65 kHz for the $1S$ state and 0.08 kHz for the $2S$ state, for the point nuclear model. The finite nuclear size effect beyond the reduced mass shifts the above values by $-0.08$ and $-0.01$ kHz, respectively. These results may be compared with the experimental uncertainty of 0.01 kHz for the $1S$-$2S$ transition [34].

The higher-order recoil corrections calculated in the present work influence the interpretation of experimental results for the hydrogen-deuterium isotope shift [35]. Specifically, our results increase the theoretical value of the hydrogen-deuterium $1S$-$2S$ isotope shift as reported in Ref. [36] by 0.36 kHz (including 0.28 kHz from the point nucleus and 0.08 kHz from the finite nuclear size). These results may be compared to the experimental uncertainty of 0.015 kHz [35] and the total theoretical uncertainty of 0.6 kHz. The change of the theoretical value increases the deuteron-hydrogen mean-square charge-radii difference as obtained in Ref. [56] by 0.00026 fm$^2$.

The results obtained in the present work demonstrate the importance of the non-perturbative (in $Z\alpha$) calculations as an alternative to the traditional $Z\alpha$-expansion approach. Despite the smallness of the parameter $Z\alpha$ for hydrogen, $1\alpha \approx 0.0073$, the convergence of the (semi-analytical) $Z\alpha$ expansion is complicated by the presence of powers of logarithms. Moreover, the predictive power of the $Z\alpha$ expansion calculations is limited by the difficulty to reliably estimate contributions of the uncalculated tail of the expansion. The non-perturbative calculations, while clearly preferable over the perturbative ones, are often hampered by technical difficulties associated with large numerical cancellations occurring in the low-$Z$ region. In particular, technical difficulties prevented so far a direct numerical calculation of the two-loop electron self-energy for hydrogen [57], which is badly needed as this effect presently determines the theoretical uncertainty in the hydrogen $1S$ and $2S$ Lamb shifts [2].

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TABLE III: Finite nuclear size recoil correction, expressed in terms of $\delta_{\text{rms}} P$. For each $Z$, the upper line corresponds to the Gauss nuclear model, whereas the second line corresponds to the homogeneously charged sphere nuclear model. The specified uncertainty is the estimated error of the approximation.

| $Z$ | $R$ [fm] | $1s$ | $2s$ | $2p_{1/2}$ | $2p_{3/2}$ |
|-----|---------|------|------|-----------|-----------|
| 1   | 0.8775  | $-0.000\,1840$ | $-0.000\,1840$ | $-0.000\,000\,01$ | $-0.000\,000\,01$ |
|     |         | $-0.000\,1851$ | $-0.000\,1852$ | $-0.000\,000\,01$ | $-0.000\,000\,01$ |
|     |         | $0.000\,0\,(2)^a$ | | | |
| 2.1424 | $-0.000\,7861$ | $-0.000\,7866$ | $-0.000\,000\,03$ | $-0.000\,000\,04$ |
|     |         | $-0.000\,7918$ | $-0.000\,7923$ | $-0.000\,000\,03$ | $-0.000\,000\,04$ |
| 2   | 1.6755  | $-0.000\,628$ | $-0.000\,629$ | $-0.000\,000\,06$ | $-0.000\,000\,04$ |
|     |         | $-0.000\,632$ | $-0.000\,633$ | $-0.000\,000\,06$ | $-0.000\,000\,04$ |
|     |         | $-0.000\,0\,(2)^a$ | | | |
| 3   | 2.4440  | $-0.001\,282$ | $-0.001\,285$ | $-0.000\,000\,2\,(4)$ | $-0.000\,000\,1\,(1)$ |
|     |         | $-0.001\,292$ | $-0.001\,294$ | $-0.000\,000\,2\,(4)$ | $-0.000\,000\,1\,(1)$ |
| 4   | 2.5190  | $-0.001\,502$ | $-0.001\,504$ | $-0.000\,000\,3\,(7)$ | $-0.000\,000\,2\,(2)$ |
|     |         | $-0.001\,512$ | $-0.001\,513$ | $-0.000\,000\,3\,(7)$ | $-0.000\,000\,2\,(2)$ |
| 5   | 2.4060  | $-0.001\,562$ | $-0.001\,562$ | $-0.000\,000\,4\,(10)$ | $-0.000\,000\,2\,(2)$ |
|     |         | $-0.001\,572$ | $-0.001\,574$ | $-0.000\,000\,4\,(10)$ | $-0.000\,000\,2\,(2)$ |
|     |         | $-0.001\,5\,(2)^a$ | | | |
| 6   | 2.4702  | $-0.001\,774$ | $-0.001\,784$ | $-0.000\,001\,(1)$ | $-0.000\,000\,2\,(4)$ |
|     |         | $-0.001\,794$ | $-0.001\,794$ | $-0.000\,001\,(1)$ | $-0.000\,000\,2\,(4)$ |
| 7   | 2.5582  | $-0.002\,024$ | $-0.002\,024$ | $-0.000\,001\,(1)$ | $-0.000\,000\,2\,(6)$ |
|     |         | $-0.002\,044$ | $-0.002\,045$ | $-0.000\,001\,(1)$ | $-0.000\,000\,2\,(6)$ |
| 8   | 2.6991  | $-0.002\,356$ | $-0.002\,376$ | $-0.000\,001\,(2)$ | $-0.000\,000\,3\,(8)$ |
|     |         | $-0.002\,386$ | $-0.002\,386$ | $-0.000\,001\,(2)$ | $-0.000\,000\,3\,(8)$ |
| 9   | 2.8976  | $-0.002\,808$ | $-0.002\,828$ | $-0.000\,002\,(3)$ | $-0.000\,000\,3\,(12)$ |
|     |         | $-0.002\,828$ | $-0.002\,828$ | $-0.000\,002\,(3)$ | $-0.000\,000\,3\,(12)$ |
| 10  | 3.0055  | $-0.003\,128$ | $-0.003\,148$ | $-0.000\,003\,(3)$ | $-0.000\,000\,4\,(16)$ |
|     |         | $-0.003\,148$ | $-0.003\,178$ | $-0.000\,003\,(3)$ | $-0.000\,000\,4\,(16)$ |
|     |         | $-0.003\,0\,(2)^a$ | | | |

$^a$ Shabaev et al. 1998 [24]
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