QUADRATIC NON-RESIDUES IN SHORT INTERVALS

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Abstract. We use the Burgess bound and combinatorial sieve to obtain an upper bound on the number of primes \( p \) in a dyadic interval \([Q, 2Q]\) for which a given interval \([u+1, u+\psi(Q)]\) does not contain a quadratic non-residue modulo \( p \). The bound is nontrivial for any function \( \psi(Q) \to \infty \) as \( Q \to \infty \). This is an analogue of the well known estimates on the smallest quadratic non-residue modulo \( p \) on average over primes \( p \), which corresponds to the choice \( u = 0 \).

1. Introduction

1.1. Motivation and background. For a prime \( p \geq 3 \) we denote by \( n(p) \) the smallest quadratic non-residue modulo \( p \). The best known upper bound \( n(p) \leq p^{1/4e^{-1/2}+o(1)} \) is due to Burgess [1], while it is expected that \( n(p) = p^{o(1)} \), which is widely known as a Conjecture of Vinogradov.

Bound of this type, and in fact much more precise, are also known. For example, conditionally on the Generalised Riemann Conjecture, we have \( n(p) = O(\log^2 p) \) for any prime \( p \), see [8, Theorem 13.11].

Furthermore, unconditionally, using the large sieve method, Erdős [3] has established that

\[
\frac{1}{\pi(x)} \sum_{p \leq x} n(p) \to \sum_{k=1}^{\infty} \frac{p_k}{2^k}, \quad x \to \infty,
\]

where, as usual \( \pi(x) \) denotes the number of primes \( p \leq x \) and \( p_k \) denotes the \( k \)-th prime. This instantly implies that the inequality \( n(p) \leq \psi(p) \) holds for almost all primes \( p \) (that is, for all but \( o(x/\log x) \) primes \( p \leq x \), as \( x \to \infty \)), where \( \psi \) is an arbitrary function with \( \psi(z) \to \infty \) as \( z \to \infty \).

On the other hand, by a result of Graham and Ringrose [6], there is an absolute constant \( C > 0 \) such that for infinitely many primes \( p \) all nonnegative integers \( z \leq C \log p \log \log \log p \) are quadratic residues modulo \( p \).

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Another Conjecture of Vinogradov is the bound \(d(p) = p^{o(1)}\), where \(d(p)\) is the longest sequence of consecutive quadratic residues modulo \(p\). It seems that this conjecture received less attention than the one about the smallest quadratic non-residue. In particular, the only known result about \(d(p)\) is the bound \(d(p) \leq p^{1/4+o(1)}\), which is due to Burgess [1] as well. It is still unknown whether the Generalised Riemann Conjecture or the large sieve method (or any other standard methods and conjectures) can lead to a better estimate on \(d(p)\) for at least almost all primes. This naturally leads to the following:

**Problem 1.** Assuming the Generalised Riemann Conjecture, show that for some constant \(\gamma < 1/4\) the bound \(d(p) < p^\gamma\) holds for almost all primes \(p\).

In fact, it is still unknown whether \(d(p) = o(p^{1/4})\) for an infinite sequence of primes.

Our main goal here is to attract more attention to the function \(d(p)\) and also make a modest step towards better understanding the distribution of quadratic non-residues.

We also denote by \(n_k(p)\) the \(k\)th quadratic non-residue modulo \(p\), and consider the gaps \(\Delta_k(p) = n_{k+1}(p) - n_k(p)\), \(k = 1, \ldots, (p-3)/2\).

It is shown in [2, Lemma 2] that for any fixed \(\varepsilon > 0\) and \(h \geq p^\varepsilon\)

\[
\#\{k = 1, \ldots, (p-3)/2 : \Delta_k(p) \geq h\} \leq p^{1/2+o(1)}h^{-2}.
\]

which, via partial summation, leads to the estimate

\[
S(h, p) = \sum_{\Delta_k(p) \geq h}^{(p-3)/2} \Delta_k(p) \leq p^{1/2+o(1)}h^{-1}.
\]

We also note that a result of Garaev, Konyagin and Malykhin [5, Theorem 2], in particular, gives an asymptotic formula for the average values of the \(\gamma\)-powers of gaps between quadratic residues modulo \(p\) for \(0 < \gamma < 4\). This can easily be extended to the same estimate for the gaps between quadratic non-residues modulo \(p\).

1.2. **Main result.** Let \(d_u(p)\) be smallest \(h\) such that there exist a quadratic non-residue in the interval \(I = [u+1, u+h]\). Clearly

\[
n(p) = d_u(p) \quad \text{and} \quad d(p) = \max_{u \in \mathbb{Z}} d_u(p).
\]

So estimating \(d_u(p)\) for a given \(u\) can be considered as an intermediate question between estimating \(n(p)\) and \(d(p)\).

Here we estimate \(d_u(p)\), uniformly over \(u\), for almost all primes \(p\). It is more convenient to work with primes from dyadic intervals \([Q, 2Q]\).
Theorem 2. Let $\psi$ be an arbitrary function with $\psi(z) \to \infty$ as $z \to \infty$. For any sufficiently large real positive $Q$, for any integer $u \leq 2Q$, for the set $E_u(\psi, Q)$ of primes $p \in [Q, 2Q]$ with $d_u(p) > \psi(p)$ we have $E_u(\psi, Q) = o(Q/\log Q)$ uniformly in $u$.

2. Preliminaries

2.1. General notation. Throughout the paper, the implied constants in the symbols “$O$”, “$\ll$” and “$\gg$” may occasionally, where obvious, depend on the real positive parameters $\varepsilon$ and $\eta$ and are absolute otherwise. We recall that the expressions $A = O(B)$, $A \ll B$ and $B \gg A$ are each equivalent to the statement that $|A| \leq cB$ for some constant $c$.

We always use the letter $p$, with or without subscripts, to denote a prime number, while $k$, $m$, $n$ and $q$ always denote positive integer numbers.

As usual, we use $\varphi(k)$ is the Euler function.

2.2. Burgess bound. We now recall the Burgess bound for some of multiplicative characters modulo arbitrary integers, see [7, Theorems 12.5 and 12.6]. In fact we only need it for sums of Jacobi symbols.

Lemma 3. For any integers $q \geq M \geq 1$, where $q \geq 2$ is not a perfect square, we have

$$\left| \sum_{m \leq M} \left( \frac{m}{q} \right) \right| \leq M^{1-1/\nu} q^{(\nu+1)/4\nu^2+o(1)},$$

with $\nu = 1, 2, 3$.

In particular, Lemma 3 implies:

Corollary 4. For any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any integers $M \geq q^{1/3+\varepsilon}$, where $q \geq 2$ is not a perfect square, we have

$$\left| \sum_{m \leq M} \left( \frac{m}{q} \right) \right| \leq M^{1-\delta}$$

2.3. Integers with a prescribed multiplicative structure. Now given some $\eta > 0$ we denote by $\mathcal{P}(\eta, M)$ the set of positive integers $m \leq M$ which do not have prime divisors $p \leq M^\eta$. It is well known that for any fixed $\eta > 0$ we have

$$|\mathcal{P}(\eta, M)| \leq c_0 \frac{M}{\eta \log M}$$
for some absolute constants $c_0 > 0$, see, for example, [9, Section III.6.2, Theorem 3].

We now recall the so-called fundamental lemma of the combinatorial sieve, see, for example, [9, Section I.4.2, Theorem 3].

For a finite set of integers $A$ and a set of primes $P$ we denote

$$P(y) = \prod_{p \leq y, p \in P} p,$$

and

$$S(A, P, y) = \# \{ a \in A : \gcd(a, P(y)) = 1 \}.$$

**Lemma 5.** Assume that for a finite set of integers $A$ and a set of primes $P$ there exist a non-negative multiplicative function $\omega(d)$, a real $X$ and positive constants $\alpha$ and $A$ such that:

- for any $d \mid P(y)$, we have
  $$\# \{ a \in A : a \equiv 0 \pmod{d} \} = X \frac{\omega(d)}{d} + R_d;$$

- for any real $v > w \geq 2$ we have
  $$\prod_{w \leq p \leq v} \left( 1 - \frac{\omega(p)}{p} \right) < \left( \frac{\log v}{\log w} \right)^\alpha \left( 1 + \frac{A}{\log w} \right).$$

Then uniformly for $A, X, y$ and $u \geq 1$

$$S(A, P, y) = X \prod_{p \mid P(y)} \left( 1 - \frac{\omega(p)}{p} \right) \left( 1 + O\left( u^{-u/2} \right) \right) + O \left( \sum_{{d \mid P(y) \atop d \leq y^u}} |R_d| \right).$$

We also need the following well-known statement which follows from the standard inclusion-exclusion argument and the classical bound on the number of integer divisors of $q$.

**Lemma 6.** For any integers $q \geq M \geq 1$, we have

$$\# \{ 1 \leq m \leq M : \gcd(m, q) = 1 \} = \frac{\varphi(q)}{q} M + O\left( q^{o(1)} \right).$$

The following asymptotic formula for the number of square-free integers in a short interval is a very special case of a much more general result of Tolev [10, Theorem 1.3] (which we apply with $r = 2$, $l_1 = 1$, $l_2 = 2$), which in turn extends and generalises a result of Filaseta and Trifonov [4].
Lemma 7. For any fixed $\varepsilon > 0$ and real $h \geq u^{1/5+\varepsilon}$, the interval $[u+1, u+h]$ contains $(A + o(1))h$ square-free integers $n$ for which $n+1$ is also square-free, where

$$A = \prod_{p \text{ prime}} \left(1 - \frac{2}{p^2}\right).$$

Corollary 8. For any fixed $\varepsilon > 0$ and real $u \geq h \geq u^{1/5+\varepsilon}$, the interval $[u+1, u+h]$ contains at least $(A + o(1))h$ odd square-free integers $n$.

Note, that Corollary 8 is much stronger than what we actually need. Namely, any result with $\alpha < 1/2$ instead of $1/5$ and arbitrary $A > 0$ is sufficient for our purposes.

2.4. Character sums with integers from $\mathcal{P}(\eta, M)$. We now consider the sets

$$\mathcal{P}_\pm(\eta, M, q) = \left\{ m \in \mathcal{P}(\eta, M) : \left(\frac{m}{q}\right) = \pm 1 \right\}.$$

Lemma 9. For any $\varepsilon > 0$ there exists some $\eta_0 > 0$ such that for any positive $\eta < \eta_0$ and integers $M \geq q^{1/3+\varepsilon}$, where $q \geq 2$ is not a perfect square, we have

$$\left| \mathcal{P}_\pm(\eta, M, q) - \frac{1}{2}M \prod_{p \leq M^\eta} \left(1 - \frac{1}{p}\right) \right| \leq C\eta^{\varepsilon^{-1/4}} + O\left(M^{1-\eta}\right),$$

where $C$ is an absolute constant.

Proof. We see from Corollary 4 and Lemma 6 that for any positive integer $d < q^{\varepsilon/2}$ with $\gcd(d, q) = 1$ we have

$$\# \left\{ 1 \leq m \leq M : d \mid m \text{ and } \left(\frac{m}{q}\right) = \pm 1 \right\} = \frac{\varphi(q)}{2dq} M + R(q, M, d),$$

where

$$R(q, M, d) = O((M/d)^{-\delta})$$

for some $\delta > 0$ depending only on $\varepsilon$.

We now set $\eta_0 = \delta^2/4$ and apply Lemma 5 with $u = \eta^{-1/2}$, $y = M^\eta$ and

$$\omega(d) = \begin{cases} 1, & \text{if } \gcd(d, q) = 1; \\ 0, & \text{if } \gcd(d, q) > 1. \end{cases}$$

We also assume that $\eta$ is small enough so that

$$y^u = M^{\eta^{1/2}} \leq q^{\varepsilon/2}.$$
so \( (2) \) applies to all positive integers \( d \leq y^u \). This implies,

\[
(4) \quad \left| P_{\pm}(\eta, M, q) - \frac{\varphi(q)}{2q} M \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right) \right| \leq \Delta_1 + \Delta_2,
\]

where

\[
\Delta_1 = C u^{-u/2} \frac{\varphi(q)}{q} M \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right)
\]

for some absolute constant \( C \), and

\[
\Delta_2 \ll \sum_{d \leq y^u} |R(q, M, d)|
\]

with \( R(q, M, d) \) defined by \( (2) \).

For \( \Delta_1 \), recalling the choice of \( u \) and \( y \), we derive

\[
(5) \quad \Delta_1 \leq C \eta^{-1/4} \frac{\varphi(q)}{q} M \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right).
\]

For \( \Delta_2 \), using \( (3) \) and assuming that \( \eta \leq \delta/2 \), we obtain

\[
(6) \quad \Delta_2 \ll \sum_{d \leq y^u} (M/d)^{1-\delta} \ll M^{1-\delta/2} \leq M^{1-\eta}.
\]

We also note that

\[
\frac{\varphi(q)}{q} \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right) = \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right) \prod_{\substack{p > M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right)
\]

\[
= \left( 1 + O(M^{-\eta}) \right) \prod_{\substack{p \leq M^\eta \\ p \nmid q}} \left( 1 - \frac{1}{p} \right).
\]

Thus substituting \( (5) \), \( (6) \) and \( (7) \) in \( (4) \) and recalling that by the Mertens formula, see [9, Section I.1.6, Theorem 11], we have

\[
\prod_{p \leq M^\eta} \left( 1 - \frac{1}{p} \right) = \frac{e^{-\gamma} + o(1)}{\eta \log M},
\]

where \( \gamma = 0.57721 \ldots \) is the Euler constant, we conclude the proof. \( \square \)

**Corollary 10.** For any \( \varepsilon > 0 \) there exists some \( \eta_0 > 0 \) such that for any positive \( \eta < \eta_0 \), integers \( M \geq q^{1/3+\varepsilon} \), where \( q \geq 2 \) is not a perfect
square, we have
\[
\left| \sum_{m \in \mathcal{P}(\eta,M)} \left( \frac{m}{q} \right) \right| \leq C_0 \eta^{-1/2} \frac{M}{\log M} + O \left( M^{1-\eta} \right),
\]
where \( C_0 \) is an absolute constant.

3. Proof of Theorem 2

Let
\[ h = \min_{z \in [Q,2Q]} \psi(z). \]
We consider the interval \( I = [u + 1, u + h] \). Without loss of generality we can assume that, say, \( \psi(z) \leq \log z \), so that \( h = o(Q) \).

Let us fix some arbitrary \( \kappa > 0 \), we show that for all but at most \( \kappa Q / \log Q \) primes \( p \in [Q,2Q] \) there is a quadratic non-residue in \( I \).

Let \( \mathcal{N} \) be an arbitrary set of integers \( n \in I \) with either \( n \equiv 1 \) (mod 4) or \( n \equiv 3 \) (mod 4). So we observe that

\[
\sum_{n \in \mathcal{N}} \left( \frac{n}{p} \right) \geq \# \mathcal{N} - 1.
\]

Thus

\[
\#\left\{ p \in [Q,2Q] : d_a(p) \geq h \right\} \leq \frac{S}{(\# \mathcal{N} - 1)^2}.
\]

We now choose yet another real parameter \( \eta > 0 \).

Expanding the summation from primes \( p \in [Q,2Q] \), squaring and extending the summation to all integers \( m \in \mathcal{P}(\eta,M) \), we obtain

\[
S \leq \sum_{m \in \mathcal{P}(\eta,M)} \left| \sum_{n \in \mathcal{N}} \left( \frac{n}{m} \right) \right|^2.
\]

Squaring and changing the order of summation, we obtain

\[
S \leq \sum_{n_1,n_2 \in \mathcal{N}} \sum_{m \in \mathcal{P}(\eta,M)} \left( \frac{n_1n_2}{m} \right).
\]
Finally, using (8), we derive

\[ S \leq \sum_{n_1, n_2 \in \mathcal{N}} \sum_{m \in \mathcal{P}(\eta, M)} \left( \frac{m}{m_1 m_2} \right). \]

If \( n_1 n_2 \) is not a perfect square, we apply Corollary 10 with

\[ q = n_1 n_2 \leq (u + h)^2 \leq 5Q^2 \]

(provided that \( Q \) is large enough) to estimate the inner sum. Otherwise, that is, when \( n_1 n_2 \) is a perfect square, we use the trivial bound \( \#\mathcal{P}(\eta, M) \) for the inner sum, getting

\[ S \leq T \#\mathcal{P}(\eta, 2Q) + h^2 \left( C_0 \eta^{\gamma - 1/2 - 1} \frac{Q}{\log(2Q)} + O \left( Q^{1-\eta} \right) \right), \]

where \( T \) is the number of products \( n_1 n_2 \) with \( n_1, n_2 \in \mathcal{N} \) that are perfect squares. Thus using (1), we see from (9) that

\[ \# \{ p \in [Q, 2Q] : d_u(p) \geq h \} \]

\[ \leq c_0 \eta (\#\mathcal{N} - 1)^2 \log Q \]

\[ + \frac{h^2}{(\#\mathcal{N} - 1)^2} \left( C_0 \eta^{\gamma - 1/2 - 1} \frac{Q}{\log(2Q)} + O \left( Q^{1-\eta} \right) \right), \]

(10)

We now consider two different choices of the set \( \mathcal{N} \) depending on the relative size of \( u \) and \( h \).

If \( h \geq u^{1/2} / \log u \), we consider the sets of \( \mathcal{N}_1 \) and \( \mathcal{N}_3 \) of square-free integers \( n \in \mathcal{I} \) with \( n \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{4} \) respectively. We now define \( \mathcal{N} \) as the largest set out of \( \mathcal{N}_1 \) and \( \mathcal{N}_3 \). We see from Corollary 8 that there are

\[ \#\mathcal{N}_1 + \#\mathcal{N}_3 \geq (A + o(1))h. \]

Hence \( \#\mathcal{N} \geq (A/2 + o(1))h \). Clearly for two square-free integers \( n_1 \) and \( n_2 \) their product is a perfect square only if \( n_1 = n_2 \). Hence, \( T = \#\mathcal{N} \) and we see from (9) and (10) that in this case

\[ \# \{ p \in [Q, 2Q] : d_u(p) \geq h \} \]

\[ \leq C_1 \eta^{-1} \frac{Q}{h \log Q} + C_2 \eta^{\gamma - 1/2 - 1} \frac{Q}{\log Q} + C_3 Q^{1-\eta} \]

for some absolute constants \( C_1, C_2, C_3 \).

We now assume that \( h < u^{1/2} / \log u \). If \( n_1 n_2 = m^2 \) for an integer \( m \) then, writing \( n_1 = k_1 d, n_2 = k_2 d, \) with \( d = \gcd(n_1, n_2) \), we see that

\[ k_1 = m_1^2 \quad \text{and} \quad k_2 = m_2^2 \]
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for some integers $m_1, m_2$. Assume $m_1 < m_2$. Thus

$$u/d \leq m_1^2 < m_2^2 \leq u/d + h/d.$$  

Therefore

$$(u/d)^{1/2} \ll h/d$$

or

$$h \gg (du)^{1/2} \geq u^{1/2},$$

which contradicts our choice of $h$. So taking $\mathcal{N}$ as the set of all integer $n \in I$ with $n \equiv 1 \pmod{4}$ we see that $T = \#\mathcal{N}$ and we obtain (11) again.

We not choose $\eta$ small enough to satisfy

$$C_{2\eta}^{-1/2} \leq \frac{1}{3} \kappa$$

then we choose $Q$ large enough to satisfy

$$C_1 \eta^{-1} \leq \frac{1}{3} \kappa \quad \text{and} \quad C_3 Q^{-\eta} \leq \frac{1}{3} \kappa.$$

With these parameters, we derive from (11) that

$$\# \{ p \in [Q, 2Q] : d_u(p) \geq h \} \leq \kappa \frac{Q}{\log Q}.$$  

Since $\kappa > 0$ is arbitrary, the result now follows.

4. Comments

Note that the inequality $u \leq 2Q$ in Theorem 2 is a natural restriction with respect to primes $p \in [Q, 2Q]$. On the other hand, it is also interesting to remove this condition. It is easy to see that the limit $u \leq 2Q$ in Theorem 2 can be increased a little if one uses the full power of the Burgess bound. In fact it is easy to see that for quadratic characters only the square-free part of the modulus $q$ matters so one can actually use Lemma 3 with any integer $\nu \geq 1$, see [7, Theorem 12.6]. However for large $u$ one needs some new ideas.

Furthermore, obtaining a version of Theorem 2 with an unlimited $u$ is essentially equivalent to estimating $d(p)$ for almost all primes $p$. Indeed, assume there are $N$ “exceptional” primes $\ell_1, \ldots, \ell_N \in [Q, 2Q]$ with $d(\ell_i) \geq \psi(\ell_i), i = 1, \ldots, N$, for some function $\psi(z)$. This means that there are integers $u_i$ with

$$d_{u_i}(\ell_i) \geq \psi(\ell_i), \quad i = 1, \ldots, N.$$  

Let us choose an integer $u$ satisfying

$$u \equiv u_i \pmod{\ell_i}, \quad i = 1, \ldots, N.$$
Then we have
\[ d_u(\ell_i) = d_{u_i}(\ell_i) \geq \psi(\ell_i), \quad i = 1, \ldots, N. \]

So a version of Theorem 2 with an unlimited \( u \) immediately implies an upper bound on \( N \).

Similar questions are also interesting to study for the gaps between primitive roots modulo \( p \).

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