Thermal Decay of the Cosmological Constant into Black Holes

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We show that the cosmological constant may be reduced by thermal production of membranes by the cosmological horizon, analogous to a particle “going over the top of the potential barrier”, rather than tunneling through it. The membranes are endowed with charge associated with the gauge invariance of an antisymmetric gauge potential. In this new process, the membrane collapses into a black hole, thus the net effect is to produce black holes out of the vacuum energy associated with the cosmological constant. We study here the corresponding Euclidean configurations (“thermalons”), and calculate the probability for the process in the leading semiclassical approximation.

I. INTRODUCTION

One of the outstanding open problems of theoretical physics is to reconcile the very small observational bound on the cosmological constant, \( \Lambda \), with the very large values that standard high energy physics theory predicts for it \(^{[1]}\). This challenge has led to consider the cosmological constant as a dynamical variable, whose evolution is governed by equations of motion. In that context one has looked for mechanisms that would enable \( \Lambda \) to relax from a large initial value to a small one during the course of the evolution of the universe. The simplest context in which this idea may be analyzed, is through the introduction of an antisymmetric gauge potential, a three–form \(^{[2]}\) in four spacetime dimensions. The three–form couples to the gravitational field with a term proportional to the square of the field strength. In the absence of sources, the field strength is constant in space and time and provides a contribution to the cosmological term, which then becomes a constant of the motion rather than a universal constant.

Changes in the cosmological constant occur when one brings in sources for the three–form potential. These sources are two–dimensional membranes which sweep a three dimensional history during their evolution (“domain walls”). The membranes carry charge associated with the gauge invariance of the three–form field, and they divide spacetime into two regions with different values of the cosmological term.

The membranes may be produced spontaneously in two physically different ways. One way is by tunneling through a potential barrier as it happens in pair production in two–dimensional spacetime. The other, is by a thermal excitation of the vacuum analogous to going “over the top” of the potential barrier rather than tunneling through it.

The tunneling process was originally studied in \(^{[3]}\), and was further explored in \(^{[4]}\). The purpose of this article is to study the spontaneous decay of the cosmological constant through the other process, namely, the production of membranes due to the thermal effects of the cosmological horizon.

It is useful to visualize the decay process in terms of its simplest context, which is a particle in a one dimensional potential barrier, as recalled in Fig. \( \text{I} \). When the barrier is in a thermal environment, the particle can go from one side of the barrier to the other by “climbing over the top” rather than tunneling. There is a probability given by the Boltzmann factor \( e^{-\beta E} \) for the particle to be in a state of energy \( E \). If \( E \) is greater than the height of the barrier the particle will move from one side to the other even classically. The effect is optimized when the energy is just enough for the particle to be at the top of the barrier and roll down to the other side. In this case, the Boltzmann factor is as large as possible while still allowing for the process without quantum mechanical tunneling. It turns out that, in the leading approximation, the probability is given by the exponential of the Euclidean action evaluated on an appropriate classical solution, just as for tunneling \(^{[5]}\). In the case of tunneling, the classical solution is called an instanton and it is time dependent \(^{[6]}\). In the present case, the classical solution corresponds to the configuration in which the particle sits at the top of the barrier, and thus it is time independent. Since the solution is unstable, when slightly perturbed the particle will fall half of the time to the left side and half of the time to the right side.
FIG. 1: The figure shows a one-dimensional potential barrier. If the particle is initially at the minimum $x = 0$ it may end up in the other side of the barrier, reaching the lower minimum $x = x_T$ by two different mechanisms. It can either (i) tunnel through the potential quantum mechanically or (ii) it can jump over it by a thermal fluctuation when the barrier is in a thermal environment.

In more complex situations, “sitting at the top of the barrier” is replaced by a “time independent classical solution with one instability mode”[3]. In the context of gauge theories such solutions appear in the analysis of violation of baryon-number conservation and have been called “sphalerons”[4]. In the present case we will use the name “thermalon”, to emphasize that the static solutions will be intimately connected with the intrinsic thermal properties of event horizons in gravitational theory. We show below that there exists a thermalon which reduces the cosmological constant through production of membranes in de Sitter space due to the thermal properties of the cosmological horizon. In this thermalon, a membrane is emitted by the cosmological horizon and collapses forming a black hole.

II. THE THERMALON

A. Basic Geometry and Matching Equations

Once produced, the membrane divides space into two regions with the interior having a lower value of $\Lambda$. Subsequently, the membrane evolves with the region of lower $\Lambda$ filling more and more of the space, thus lowering the value of the cosmological constant.

In ref. [5] we gave the equations of motions of a charged membrane of tension $\mu$ and charge $q$ coupled to the gravitational field, and employed them to analyze instantons associated with tunneling. The same equations will be used here for the study of thermalons. We reproduce them verbatim here, together with the corresponding explanations, to make the discussion self-contained.

We consider an Euclidean spacetime element of the form

$$ds^2 = f^2(r)dt^2 + f^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$  \hspace{1cm} (1)

The antisymmetric field strength tensor takes the form

$$F_{\mu\nu\lambda\rho} = (dA)_{\mu\nu\lambda\rho} = E\sqrt{g}e^{\mu\nu\lambda\rho} .$$  \hspace{1cm} (2)

The history of the membrane will divide the spacetime in two regions, one which will be called the interior, labeled by the suffix “-” and the other the exterior, labeled by the suffix “+”. The exterior is the initial region and defines the “background”, while the interior is the final region. The boundary may be described by the parametric equations

$$r = R(\tau) \quad t_{\pm} = T_{\pm}(\tau) ,$$  \hspace{1cm} (3)
where $\tau$ is the proper length in the $r-t$ sector, so that its line element reads
\[
 ds^2 = d\tau^2 + R^2(\tau) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]  
with
\[
 1 = f^2_\pm (R(\tau)) \dot{T}^2_\pm + f^2_\mp (R(\tau)) \dot{R}^2.
\]

In the “+” and “−” regions the solution of the field equations read
\[
 f^2_\pm = 1 - \frac{2M_\pm}{r} - \frac{r^2}{l^2_\pm},
\]
\[
 E^2_\pm = \frac{1}{4\pi} \left( \frac{3}{l^2_\pm} - \lambda \right).
\]

The actual cosmological constant $\Lambda = 3/l^2$ is thus obtained by adding $\lambda$, normally taken to be negative, coming from “the rest of physics” and not subject to change, and the contribution $4\pi E^2$, which is subject to dynamical equations. The discontinuities in the functions $f^2$ and $E$ across the membrane are given by
\[
 f^2 T_+ - f^2 T_- = \mu R,
\]
\[
 E_+ - E_- = q.
\]

Here $\mu$ and $q$ are the tension and charge on the membrane respectively. Eq. (4) follows from integrating the Gauss law for the antisymmetric tensor across the membrane, whereas Eq. (5) represents the discontinuity in the extrinsic curvature of the membrane when it is regarded as embedded in either the “−” or the “+” spaces. In writing these equations, the following orientation conventions have been adopted, and will be maintained from here on: (i) The coordinate $t$ increases anticlockwise around the cosmological horizon, (ii) the variable $\tau$ increases when the curve is traveled along leaving the interior on its right side.

Equation (5) may be thought of as the first integral of the equation of motion for the membrane, which is thus obtained by differentiating it with respect to $\tau$ (“equations of motion from field equations”). Hence, satisfying (6 - 9) amounts to solving all the equations of motion and, therefore, finding an extremum of the action. More explicitly, the first integral of the equation of motion for the membrane may be written as,
\[
 \Delta M = \frac{1}{2} (\alpha^2 - \mu^2) R^3 - \mu f^2 T R^2 = \frac{1}{2} (\alpha^2 + \mu^2) R^3 - \mu f^2 T R^2,
\]
where $\Delta M = M_+ - M_-$ is the mass difference between the initial and final geometries and,
\[
 \alpha^2 = \frac{1}{l^2_+} - \frac{1}{l^2_-}.
\]

The Euclidean evolution of the membrane lies between two turning points. Once the initial mass $M_+$ and the initial cosmological constant $\Lambda = 3/l^2_+$ are given, the turning points, $R$, are determined through,
\[
 \Delta M = \frac{1}{2} (\alpha^2 - \mu^2) R^3 - \varepsilon_+ f R^3 = \frac{1}{2} (\alpha^2 + \mu^2) R^3 - \varepsilon_- f R^3,
\]
by setting $\dot{R} = 0$ in (10). Here we have defined $\varepsilon_{\pm} = \text{sgn} \dot{T}_{\pm}$.

The graph of the function (12) for fixed $M_+$, shown in Fig. 2, will be referred to as the “mass diagram” for the decay of Schwarzschild–de Sitter space. There are two branches which merge smoothly, corresponding to taking both signs in Eq. (12), much as $x = \pm \sqrt{1 - y^2}$ gives the smooth circle $x^2 + y^2 = 1$. The lower and upper branches merge at the intersections with the curve $\Delta M = (1/2)(\alpha^2 - \mu^2) R^3$ which determines the sign of $T_+$, which is positive below the curve and negative above it. This curve, the “$T_+ = 0$ curve”, plays an important role because it determines which side of the “+” geometry must be retained: According to the conventions established above, if $T_+$ is positive, one must retain the side of the membrane history with (locally) greater values of the radial coordinate; if $T_+$ is negative, one must keep the other side. The curve $T_+ = 0$ is also shown on the mass diagram, as is the curve $\dot{T}_+ = 0$, which, as can be seen from (10), has for equation $\Delta M = (1/2)(\alpha^2 + \mu^2) R^3$. A similar rule applies for determining which side of the minus geometry is retained: it is the side of increasing $r$ if $\dot{T}_-$ is negative and the other side if it is positive.
B. Black Hole and Cosmological Thermalons

There are four distinguished points in the mass diagram, two for which the tangent is vertical and two for which it is horizontal. The former, located at $r_+$ and $r_{++}$, correspond to membrane creation through tunneling and they give rise to the instantons discussed in [8]. The latter are thermalons. The thermalon at the top of the diagram will be called “cosmological thermalon” because it is associated with the cosmological event horizon. The thermalon at the bottom of the mass diagram will be called “black hole thermalon”. It needs an initial black hole to provide the thermal environment.

For the particular values of $\Delta M$ corresponding to the thermalons, the two turning points coalesce and the membrane trajectory is a circle. The geometry of the thermalons is depicted in Fig. 3.

For the cosmological thermalon, the sign of $\dot{T}_-$ may be either positive or negative, depending on the values of the parameters. If $\dot{T}_-$ is negative, one must glue the region of the minus geometry that contains the cosmological horizon $r_{--}$ to the original background geometry. The thermalon has then one black hole horizon ($r_+$) and one cosmological horizon ($r_{--}$). If, on the other hand, $\dot{T}_-$ is positive, one must glue the other side of the membrane history to the original background geometry. This produces a solution with two black hole horizons, one at $r_+$ and one at $r_-$. The inversion of the sign of $\dot{T}_-$ happens when $r_-=r_{--}$, that is, when the “-” geometry becomes the Nariai geometry. Therefore we will call this particular case the “Nariai threshold” and will discuss it in Section IV below. Note that $\dot{T}_+$ is always negative, hence it is always the $r_+$-side of the plus geometry that must be kept.

For the black hole thermalon $\dot{T}_-$ is always positive. This is because $\dot{T}_-$ is greater than $\dot{T}_+$, which is positive in this case.

For both thermalons, the Minkowskian solution is unstable and the Euclidean solution is stable. When slightly perturbed, the Euclidean solution oscillates around the thermalon. For the Minkowskian case, the membrane can evolve in two ways: it can start accelerating (rolling down) either towards the exterior or the interior. If the acceleration is directed toward the exterior, spacetime will become filled with the interior (“-”) geometry, and the cosmological term will decrease. If, on the other hand, it accelerates toward the interior, then, and as time increases, the exterior will become the whole geometry and the cosmological constant will return to its original value. In this case the process will not have changed anything.

The minimum and the maximum of the mass curve are Euclidean stable points, because both correspond to maxima of the potential on the sense of Fig. 1. The cosmological thermalon corresponds to a maximum of the potential ultimately because, as established in [10, 13], the internal energy of the cosmological horizon is $-M$ rather than $M$. Thus, in the analogy with the potential problem one should plot $-\Delta M$ in the vertical axis, so that the potential in Fig. 2 is upside down and the analogy holds. The instability of both thermalons is proven explicitly in Appendix A.

Lastly, there is another important distinction between black hole and cosmological thermalons which resides in how they behave when gravity is decoupled, that is, when Newton’s constant $G$ is taken to vanish. In that case the black hole thermalon becomes the standard nucleation of a bubble of a stable phase within a metastable medium, whereas the cosmological thermalon no longer exists. The decoupling of gravity is discussed in Appendix B.

FIG. 2: The closed curve shows the points $R$ where $\dot{R} = 0$ for a given mass gap $\Delta M$. The thermalons are located at the maximum (cosmological thermalon) and minimum (black hole thermalon) of the curve. The curves where $R$ is such that for a given $\Delta M$, $\dot{T}_\pm = 0$ are also shown. All the “+” parameters are held fixed.
FIG. 3: Thermalon Geometry. Figures (a) and (b) represent the geometry of the cosmological thermalon below and above the Nariai threshold respectively, while (c) depicts the black hole thermalon. Only the $r-t$ section is shown. In each figure, the radial coordinate increases if one moves upwards.

III. LORENTZIAN CONTINUATION

The prescription for obtaining the Lorentzian signature solution, which describes the decay process in actual space-time, is to find a surface of time symmetry in the Euclidean signature solution and evolve the Cauchy data on that surface in Lorentzian time. The fact that the surface chosen is one of time symmetry, ensures that the Lorentzian signature solution will be real. Another way of describing the same statement is to say that one matches the Euclidean and Lorentzian signature solutions on a surface of time symmetry.

For the cosmological thermalon that induces decay of de Sitter space we will take the surface of time symmetry as the line in $r-t$ space which, described from the Euclidean side, starts from $t = t_0, \ r = 0$, proceeds increasing $r$, keeping $t = t_0$ until it reaches the cosmological horizon, and then descends back to $r = 0$ along the line $t = t_0 + \beta/2$. Thus, the surface of time symmetry crosses the membrane formation radius, $R$, twice, which implies that actually two membranes, of opposite polarities, are formed. The cosmological constant is decreased in the finite-volume region between them. The Penrose diagram for the Lorentzian section, below the Nariai threshold, is given in Fig. 4. To obtain the diagram above the Nariai threshold one should replace, in Fig. 4, $r_-$ by $r_-$ and $r = \infty$ by the black hole singularity $r = 0$ of the “-” geometry.

IV. ACTION AND PROBABILITY

We will be interested in the probability $\Gamma$ per unit of time and unit of spatial volume for the production of a thermalon. In the leading semiclassical approximation, that probability is given by

$$\Gamma = Ae^{-B/\hbar}(1 + \mathcal{O}(h)),$$

(13)

where $-B = I_{\text{thermalon}}$ is the value of the Euclidean action on the thermalon solution (in our conventions, the sign of the Euclidean action is such that, in the semiclassical limit, it corresponds to $-\beta F$, the inverse temperature times the free energy). The prefactor $A$ is a slowly varying function with dimensions of a length to the negative fourth power build out of $l_+, \alpha^2$ and $\mu$. The choice of the action depends on the boundary conditions used, and also involves “background subtractions”, which ensure that when the coupling with the membrane is turned off ($\mu = q = 0$), the probability (13) is equal to unity, since, in that case, $P$ becomes the probability for things to remain as they are when nothing is available to provoke a change.

To address the issue of boundary conditions we refer the readers to figures 3 (a), (b) and (c). There we have drawn small circles around the exterior black hole and cosmological horizons respectively, to indicate that the corresponding point is treated as a boundary, as discussed in [10, 13]. The point in question takes in each case the place of spatial infinity for a black hole in asymptotically flat space. It represents the “platform” on which the “external observer” sits. On the boundary there is no demand that the equations of motion should hold, and thus it is permissible for a conical singularity to appear there, as it indeed happens.

Once a boundary is chosen one sums in the path integral over all possible configurations elsewhere. As a consequence, no conical singularity is allowed anywhere else but at the boundary. Therefore, the gravitational action should include a contribution equal to one fourth of the area of that horizon which is not at the boundary.
FIG. 4: Cosmological Thermalon Penrose diagrams. Figure (b) is the Penrose diagram for the Lorentzian geometry of the cosmological thermalon when the initial mass is zero. The generic case $M_\pm \neq 0$, below the Nariai threshold is shown for completeness in figure (a). The dotted lines are the points, $r = r_+$, where the boundary is located in the Euclidean version of this geometries. Each diagram has two static membranes with opposite polarities at $r = R$.

To be definite, consider the cosmological thermalon. In that case the boundary is placed at $r_+$, and therefore, one is including the thermodynamical effects of the cosmological horizon (which is the reason for the term “cosmological thermalon”). We then fix at the boundary the value of $r_+$ itself or, equivalently, the mass $M_+$, in addition to fixing the cosmological constant $\Lambda_+$. With these boundary conditions the total action for the problem is just the standard “bulk” Hamiltonian action of the coupled system formed by the gravitational field, the antisymmetric tensor field and the membrane, with one fourth of the corresponding horizon area added\[14\]. This action includes the minimal coupling term of the three–form field to the membrane, by which the canonical momentum of the membrane differs from the purely kinetic (“mass times the velocity”) term, and whose evaluation needs a definition of the potential on the membrane. Such a definition requires a mild form of regularization which is dictated by the problem itself, and which we pass to analyze now.

The minimal coupling term is

$$ q \int_{V_3} A , $$

where the integral extends over the membrane history $V_3$ and $A$ is evaluated on $V_3$. The potential $A$ is such that the magnitude $E$ of its exterior derivative jumps by $q$ when passing from the interior (“-” region) of $V_3$ to the exterior (“+” region) as stated in Eq. \[14\]. We will impose the following conditions on $A$: (i) $A$ must be regular at that origin (horizon) which is in the interior of $V_3$. More precisely, $A_-$ should be equal to zero up to a regular gauge transformation, in order for the integral over a very small loop enclosing the horizon to vanish as the loop shrinks to a point. This is quite straightforward. The subtlety comes in with $A_+$, the value of $A$ in the exterior of $V_3$. The fields in the exterior should be those corresponding to the solution of the equations of motion that would hold everywhere if the transition where never to occur. Therefore, we will demand that: (ii) the function $A_+$ will be the same, for the given value of $E_+$, as when the membrane is absent. This means that $A_+$ should be regular at the horizon in the absence of the membrane, which implies – as one may show – that $A$ is discontinuous across the membrane. With the above definition of $A$ one may now rewrite the minimal coupling term \[14\] as an integral over the interior $V_3$ of...
the membrane, by means of Stoke’s formula. This gives,
\[ \int_{V_3} A = qE_{av} = (E_+ - E_-) \frac{1}{2} (E_+ + E_-) = \frac{1}{2} (E^2_+ - E_-^2) = \frac{3}{8\pi} \alpha^2 , \]
with \( \alpha^2 \) given by (14). The appearance of the average field, \( E_{av} \), may be thought of as coming from defining the integral over \( V_4 \) as the average of the integrals obtained when the boundary of the \( V_4 \) is displaced infinitesimally towards the interior and exterior of the membrane worldsheet. This is equivalent to “thickening” the membrane and taking the boundary half way inside.

It is interesting to point out that, in the case of membrane production in flat space, the definition of the potential \( A \) on the membrane through the “thickening” employed in (15) is equivalent to subtracting the background field action \( \int F_+^2 \). Thus, we will assume that when (15) is used there remains only to subtract the gravitational background action, which is equal to one fourth of the horizon area in the absence of the membrane (the background Hamiltonian is zero!). The difference in the horizon areas with and without membrane may be thought of as the change in the available phase space of horizon states induced by the creation of the membrane.

To be able to write explicitly the form of \( I \), one further notices that, since on-shell the Hamiltonian constraints hold, the bulk Hamiltonian action reduces to the “\( pq\)” term. Furthermore, for both the gravitational and antisymmetric fields, which are time independent in the exterior and interior of the membrane, \( \dot{q} \) vanishes, and, therefore, only the membrane contribution which contains both the membrane kinetic term and the minimal coupling terms, remains.

With all these observations taken into account, the action which appears in the probability of the cosmological thermalon becomes,
\[ I_{\text{thermalon}} = \frac{1}{4} [A(r_--) - A(r_{++})] - \frac{\mu}{4\pi} V_3 + \alpha^2 \frac{3}{8\pi} V_4 , \]
below the Nariai threshold, and
\[ I_{\text{thermalon}} = \frac{1}{4} [A(r_--) - A(r_{++})] - \frac{\mu}{4\pi} V_3 + \alpha^2 \frac{3}{8\pi} V_4 . \]
above it.

For the central case of interest in this paper, namely, the decay of de Sitter space through the cosmological thermalon, we set the initial mass \( M_i \) equal to zero. Note that the cosmological thermalon lies in the upper branch of the mass curve, while the tunneling decay investigated in (2) lies on the lower branch. Therefore, the thermalon process discussed here and the tunneling of (2) are not to be thought of as happening in the same potential barrier in the sense of the analogy illustrated by Fig. (3). Yet, the two probabilities may be compared, and the comparison is of interest. Since there is no black hole in the initial state, the initial geometry has the full \( O(5) \)-symmetry and the nucleation process can occur anywhere in spacetime. Hence, the computed probability is a probability per unit of spacetime volume. Note that the thermalon solution breaks the symmetry down to \( O(3) \times O(2) \), while the tunneling solution of (2) breaks it to the larger symmetry group \( O(4) \). Hence it is expected to have higher probability \( \frac{1}{3} \), which is indeed confirmed by the analysis given below.

After de Sitter decays through the thermalon once, the process may happen again, and again. If the black hole formed after the decay is small, on may, to a first approximation, ignore its presence and use the same formula for the probability, taking the final cosmological radius \( l_+ \) of the previous step as the \( l_+ \) of the new step. It is quite alright to ignore the presence of the black hole when \( r_- \) is small because the probability of the second black hole to be near the first one will be very small, since it is a probability per unit of volume. The approximation will break down when the Nariai bound is in sight.

V. NARIAI THRESHOLD

For given membrane parameters \( \mu, q \), bare cosmological constant \( \lambda \) and initial mass \( M_i \), there is a value \( l_N \) of \( l_+ \) for which the final geometry becomes the Nariai solution. In that case, \( R = r_- = r_{++} \), since there is nowhere else for \( R \) to be, and thus the curve \( \tilde{T}_- = 0 \), which always crosses the mass curve at a root of \( f_2^2 \), does it now precisely at the cosmological thermalon nucleation radius \( R \). If one starts from a small \( l_- \) one finds the situation illustrated in Fig. 3a. As \( l_+ \) increases, the size of the black hole present in the “-” region also increases, and at \( l_+ = l_N \), the “-” geometry becomes the Nariai solution. If \( l_+ \) increases further, and we will refer to this further increase as “crossing the Nariai threshold”, \( \tilde{T}_- \) becomes positive, which means that, as seen from the “-” side, the orientation of the membrane is reversed. This implies that one must glue the part of the “-” region which one was discarding below the Nariai threshold to the “+” region, thus giving rise to the situation described in Fig. 3b.
It should be emphasized that crossing the Nariai threshold is not a violent operation. The bound
\[ M_- \leq \frac{l_-}{3\sqrt{3}} \]
(18)
for the existence of a de Sitter black hole is maintained throughout \((f_+(R)\) is real since \(R < l_+\), which implies, using (12), that \(f_-(R)\) is also real and hence the function \(f_-^2\) has two positive real roots). The action, and hence the probability, remains continuous. However, the nature of the thermalon geometry is now somewhat different, since the \(\sim\) geometry has a black hole horizon instead of a cosmological horizon. As one moves from the boundary \(r_+\) towards the membrane, their radius \(r\) of the attached \(S^2\) increases. But, after crossing the membrane, it starts decreasing – in contradistinction to what happens below the threshold, – until it reaches its minimum value at the black hole radius of the \(\sim\) geometry.

VI. DECAY OF DE SITTER SPACE THROUGH THE COSMOLOGICAL THERMALON

A. Nucleation radius and mass of final state black hole

We now focus on the central case of interest in this paper, namely, the decay of de Sitter space through the cosmological thermalon. The cosmological thermalon radius of nucleation, \(R\), may be obtained by differentiating Eq. (12) in its plus sign version,
\[
\frac{3}{2}(\alpha^2 - \mu^2)R + 2\mu f_+ + \mu f_+'R = 0 .
\]
(19)
When \(M_+ = 0\), Eq. (19) can be rewritten in terms of the dimensionless auxiliary variable \(x = f_+ l_+/R\), or, equivalently,
\[
R^2 = \frac{l_+^2}{1 + x^2} .
\]
(20)
Then, (19) gives,
\[
x = \frac{3}{4} \left[ -\gamma + \left( \gamma^2 + \frac{8}{9} \right)^{1/2} \right],
\]
(21)
where,
\[
\gamma = \frac{l_+(\alpha^2 - \mu^2)}{2\mu} .
\]
(22)
From Eq. (12) we obtain for the mass of the black hole appearing in the final state,
\[
M_- = \frac{\mu l_+^2}{3x} \left( 1 + x^2 \right)^{-1/2} .
\]
(23)
The Nariai threshold radius \(l_+ = l_N\) may be also evaluated explicitly. It is given by
\[
\frac{1}{l_N^2} = \frac{1}{2} \left( \mu^2 + 8\pi q^2 + \sqrt{8\pi q^2(6\pi q^2 - 2\lambda + 3\mu^2)} \right) .
\]
(24)

B. Response of the final geometry to changes in the initial cosmological constant

For \(l_+ < l_N\) one easily verifies that, at \(R\), \(l_- < \Delta M\) and hence \(\dot{T}_- < 0\). Therefore the thermalon geometry is the one pictured in Fig. 3(a). As \(l_+\) increases the final state approaches the Nariai solution. We have plotted in Fig. 3 the quantity \(l_- - 3\sqrt{3}M_-\) as a function of \(l_+\). As \(l_+\) crosses the Nariai value \(l_N\), the situation becomes the one depicted in Fig. 3(b). If \(l_+\) increases further, the function \(l_-\) and thus also \(l_- - 3\sqrt{3}M_-\) blows up for some value \(l_\infty > l_N\) of \(l_+\); for that value, the final cosmological constant \(\Lambda_-\) is infinite and \(\Delta_-\) vanishes . Above \(l_\infty\) the final cosmological constant is negative and thus a transition from de Sitter space to Schwarzschild-anti-de Sitter space takes place. The
transition probability is well defined because, since $\dot{T}_-$ is still positive, a finite-volume piece of Schwarzschild-anti-de Sitter space enters in the action, namely the one between the membrane and the black hole horizon $r_-$. The radius $l_+ = l_\infty$, for which the final cosmological constant vanishes it is given by

$$\frac{1}{l_\infty^2} = \frac{4}{3} q \left( \pi q + \sqrt{-\pi \lambda} \right).$$

(25)

Because the horizons $r_-, r_-$ and inverse temperature $\beta_-$ of the final state are determined by algebraic equations of a high degree, we have found it necessary to compute the action $I(l_+)$ by direct numerical attack. The result gives a curve of the form shown in figure 6. One sees that the probability decreases very quickly as $l_+$ increases. Since, as the graph shows, the action is very small for large cosmological constant the process is not exponentially suppressed at
the beginning. As the process goes on, the action becomes monotonously more negative and the probability becomes exponentially suppressed. The action is continuous both at the Nariai value $l_N$ and at the critical point $l_\infty$ where the final cosmological constant becomes negative.

### C. Small charge and tension limit

In order to avoid fine tuning it is necessary to assume that the jumps in the cosmological constant are of the order of the currently observed value $\Lambda_{obs}$, which is, in Planck units adopted from now on, $\Lambda_{obs} \sim 10^{-120}$. Thus we take $\alpha^2 \sim 10^{-120}$ ("small jump condition"). The bare cosmological constant itself could be as big as the Planck scale ($|\lambda| \sim 1$) unless protected by some broken symmetry, e.g. $|\lambda| \sim 10^{-60}$ (SUSY). From (7) we find that the $E$ field is of the order of $q \sim 10^{-120}/\sqrt{-\lambda}$: the small jump condition requires a small charge $q$. As the decay of the cosmological constant proceeds, the radius of the universe $l_+\mu$ goes from “small” values $\sim \sqrt{|\lambda|}$ to large values $\sim 10^{60}$ and the dimensionless product $\alpha^2 l_+^2$ varies from $\sim 10^{-120}$ (Planck) or $\sim 10^{-60}$ (SUSY) to a quantity of order unity. We will assume that,

$$l_+^2 \alpha^2 \ll 1 \ , \quad (26)$$

which holds during the whole decay of the cosmological constant, beginning from early stages $l_+ \sim 1$ (Planck), or $l_+ \sim 10^{30}$ (SUSY), all the way up to “almost” the late stages $l_+ \sim 10^{58}$, say. We shall call $l_+^2 \alpha^2$ the “small charge limit”.

We will also assume

$$l_+ \mu \ll 1 \ , \quad (27)$$

which we shall call the “small tension limit”.

Within the small charge and tension limits there are two interesting subcases which are amenable to analytical treatment:

(a) First, we may assume that

$$\frac{l_+ \alpha^2}{\mu} \ll 1 \ , \quad (28)$$

This limit may be achieved, for instance, by taking $\mu$ to be of the order of $\alpha$, that is, $\mu \sim 10^{-60}$, again, up late stages of the evolution of $l_+$. In this limit,

$$\gamma \ll 1 \ . \quad (29)$$

Direct computations yield

$$R^2 \approx \frac{2}{3} l_+^2 \left(1 + \frac{\gamma}{\sqrt{2}}\right) \ , \quad \frac{M_-}{l_+} \approx \frac{2\sqrt{3}}{9} \mu l_+ \left(1 + \sqrt{2} \gamma\right) \ . \quad (30)$$

The nucleation radius is roughly $\sqrt{2/3} l_+$ and the initial and final universe radii $l_+$ and $l_-$ are equal to leading order. Since both $\alpha^2 l_+^2$ and $\mu^2 l_+^2$ are small, Eq. (24) implies $l_+ \ll l_N$ so that the final geometry contains the cosmological horizon $r_{\infty}$. We then get for the action $I_{Thermalon}$

$$-\frac{8\pi \sqrt{3}}{9} l_+^2 (\mu l_+) \ , \quad (31)$$

which can be compared to the tunneling–process action with the same parameters

$$-\frac{\pi}{2} l_+^2 (\mu l_+) \ . \quad (32)$$

We conclude that in this regime the tunneling–process is more probable than the cosmological thermalon.

(b) Conversely, we may assume, namely,

$$\frac{\mu}{l_+ \alpha^2} \ll 1 \ , \quad (33)$$
which is equivalent to,
\[ \gamma \gg 1 . \] (34)

This limit may be achieved, for instance, by setting \( q \sim \mu \) (“BPS condition”) for late stages of the evolution of the cosmological constant \((l_+ > 10 \text{ (Planck)}, l_+ > 10^{31} \text{ (SUSY)})\). Here we get for the nucleation radius and final mass,
\[ R^2 \approx l_+^2 \left(1 - \frac{1}{9\gamma^2}\right), \quad \frac{M_-}{l_+} \approx \mu l_+ \gamma \approx \alpha^2 l_+^2, \] (35)

which shows that, in this limit, the nucleation radius gets close to the cosmological horizon. The value of the action for this process is, to leading order,
\[ -\frac{4\pi l_+^2}{9} \left(\frac{\mu}{\alpha^2 l_+}\right)^2, \] (36)

which may be compared with the action for the tunneling process in the same limit,
\[ -4\pi l_+^2 \left(\frac{\mu}{\alpha^2 l_+}\right)^4 \left(\alpha^2 l_+^2\right). \] (37)

We again see in this regime that the tunneling process is more probable than the cosmological thermalon.

VII. CAN THE THERMALON ACCOUNT FOR THE SMALL PRESENT VALUE OF THE COSMOLOGICAL TERM?

The cosmological thermalon has a distinct advantage over the instanton proposed in [3] as a mechanism for relaxing the cosmological constant in that it does not have the so-called “horizon problem”. Indeed, since for the thermalon the cosmological constant is reduced in the region with bigger values of the radial coordinate, and then the membrane proceeds to collapse, one is sure that, even though the universe is expanding (and even more because it is so!), the whole of the universe will relax its cosmological constant.

The other problem that the instanton has is that the rate of membrane nucleation was too small to account for the present small value of the cosmological term. It is not clear at the moment of this writing whether the thermalon will also be an improvement in this regard. Indeed, to make a mild assessment of whether the rate is sufficiently strong, we first recall that, in Planck units, the observed value of the cosmological constant now is \(10^{-120}\) and the age of the universe \(10^{60}\). Even though the process of relaxation of the cosmological constant may have not occurred through the entire life span of the post Big-Bang universe, we may use for very rough estimates \(10^{60}\) as the time available for the process to keep occurring. If we assume that the cosmological radius started at the Planck scale, \(l_+ = 1\) and at present has the value \(10^{60}\), and recall that each bubble nucleation reduces \(l_+^2\) by \(\alpha^2\), one needs \(10^{120}\) events to occur during \(10^{60}\) Planck time units. Furthermore, since at any given time the volume of space available for the location of the center of the bubble is of the order of \(l_+^3\) (spatial volume in the comoving frame of de Sitter space), we conclude that the rate \(\Gamma\) is given by
\[ \Gamma \sim \frac{10^{120}}{10^{60} l_+^3} = \frac{10^{60}}{l_+^3} \text{ (Planck units)} . \] (38)

In order for Eq. (38) to be realizable for \(l_+\) in the range \(1 < l_+ < 10^{60}\) we get, using (13),
\[ 10^{-120} < Ae^{-B} < 10^{60} . \] (39)

One must keep in mind that as stated above, the parameter \(\alpha^2\) is fixed to be \(10^{-120}\) and that the parameter \(B\) should not be too small in order for the semiclassical approximation to be valid, which imposes a constraint relating the other two parameters \(\mu\) and \(l_+\). It is therefore necessary to perform a careful analysis of the prefactor \(A\) to see whether the inequality (39) can be satisfied with an \(l_+\) within the range available in the history of the universe. If this is not so, one could not argue that the thermalon can account for all of the relaxation of the cosmological constant and, a fortiori, one could not argue that all of the vacuum energy was condensed into black holes.
VIII. CONCLUSIONS

In this paper, we have exhibited a new process through which the cosmological constant can decay. At the same time, a black hole is created. The classical solution describing the process is an unstable (“ready-to-fall”) static solution which we have called “cosmological thermalon”; it is an analog of the sphaleron of gauge theories. Gravity and, in particular, the thermal effects of the cosmological horizon is essential for the existence of the solution, which disappears in the flat space limit.

The net effect of the process is thus to transform non localized dark energy into localized dark matter, thus providing a possible link between the small present value of the cosmological constant and the observed lack of matter in the universe. Of course, the emergence in this way of (nearly) flat space as a natural endpoint of dynamical evolution is most intriguing in view of cosmological observations.

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After this paper was finished we became aware of the interesting work of J. Garriga and A. Megevand [15], where “...a ‘static’instanton, representing pair creation of critical bubbles - a process somewhat analogous to thermal activation in flat space...” is discussed.

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APPENDIX A: INSTABILITY OF THERMALONS

It is illuminating to see explicitly that both the minimum and the maximum of the mass curve are Euclidean stable points, since “unthoughtful” analogy with a standard potential problem might have suggested that only the minimum of the mass curve should be a stable Euclidean equilibrium.

Consider first the black hole thermalon and perturb it, $R_{\text{BHT}} \rightarrow R_{\text{BHT}} + \eta(\tau)$. The mass equation yields,

$$\Delta M(R_{\text{BHT}}) + \delta m = \frac{1}{2}(\alpha^2 - \mu^2)R^3 - \mu \sqrt{f'_+ - R^2} \frac{\hat{R}}{R}$$  \hspace{1cm} (A1)

where $\delta m$ is the mass perturbation. Using that $\Delta M$ is extremum at the thermalon, one finds,

$$\delta m = \frac{1}{2} \frac{\partial^2 \Delta M}{\partial R^2} \bigg|_{\text{BHT}} \eta^2 + \frac{\mu R^2}{2f'_+} \eta^2$$  \hspace{1cm} (A2)

Because the second derivative of $\Delta M$ is positive at the minimum $R_{\text{BHT}}$, the right-hand side of [A2] is non-negative, so that the perturbation $\eta(\tau)$ is forced to remain in the bounded range $|\eta| \leq \sqrt{2\delta m (\partial^2 \Delta M/\partial R^2)^{-1}}$, which implies stability.

For the cosmological thermalon, it is now the upper branch of the mass curve which is relevant, and (A1) is replaced by

$$\Delta M(R_{\text{CT}}) + \delta m = \frac{1}{2}(\alpha^2 - \mu^2)R^3 + \mu \sqrt{f'_+ - R^2} \frac{\hat{R}}{R}$$  \hspace{1cm} (A3)

so that, instead of [A2] one has

$$\delta m = \frac{1}{2} \frac{\partial^2 \Delta M}{\partial R^2} \bigg|_{\text{CT}} \eta^2 - \frac{\mu R^2}{2f'_+} \eta^2$$  \hspace{1cm} (A4)
At the cosmological thermalon, $\Delta M$ is maximum and so its second derivative is negative. Again the perturbation $\eta$ is bounded and the Euclidean solution is stable.

The signs in (A4) are in agreement with the proof given in [10, 13], that the internal energy of the cosmological horizon is $-M$, thus the “thoughtful” analogy with the potential problem simply amounts to realize that for the cosmological thermalon one should plot $-\Delta M$ in the vertical axis, so that the “potential” in Fig. 2 is upside down and the analogy holds.

We end this appendix computing the frequency of oscillations around the cosmological thermalon. From (A4) we find that those are given by

$$\omega^2 = \left( \frac{\partial^2 \Delta M}{\partial R^2} \right) \frac{f_+}{\mu R^2} \bigg|_{\text{CT}},$$

which may be evaluated in terms of the nucleations radius $R$ of the cosmological thermalon,

$$\omega^2 = \frac{2l_+^2 - R^2}{R^2(l_+^2 - R^2)}. \quad (A5)$$

Note that there is a resonance in the limit when $R$ approaches the cosmological radius $l_+$. This may be achieved when the BPS condition of Eq. (33) is satisfied ($\mu/l_+ \alpha^2 < 1$, which holds, for instance, setting $q \sim \mu$). In that case we get,

$$\omega = \frac{3 \alpha^2}{2 \mu}, \quad (A6)$$

and, from (B3) we see that, in fact, $\omega >> l_+^{-1}$.

**APPENDIX B: GRAVITATION ESSENTIAL FOR EXISTENCE OF COSMOLOGICAL THERMALONS.**

An interesting feature of the cosmological thermalon is that it does not exist in the limit of no gravity, where Newton’s constant $G$ is taken to vanish. Indeed, when $G$ is explicitly written, the mass equation becomes

$$G\Delta M = \frac{1}{2} (\alpha^2 - G^2 \mu^2) R^3 \pm G \mu f_+ R^2 \quad (B1)$$

with

$$\alpha^2 = \frac{4\pi G}{3}(E_+^2 - E_-^2) \quad (B2)$$

and

$$f_+^2 = 1 - \frac{2GM_+}{R} - \frac{R^2}{l_+^2}, \quad \frac{3}{l_+^2} = \lambda + 4\pi G E_+^2 \quad (B3)$$

Note that the bare cosmological constant $\lambda$ depends on $G$ through

$$\lambda = 8\pi G \rho_{\text{vac}}, \quad (B4)$$

where $\rho_{\text{vac}}$ is the vacuum energy density coming from “the rest of physics”. Therefore, in the limit $G \to 0$, the lower branch of the mass curve becomes,

$$\Delta M = -V(R) = - (\mu R^2 - \nu R^3), \quad \nu = \frac{2\pi}{3}(E_+^2 - E_-^2) > 0 \quad (B5)$$

The (Minkowskian) potential $V(R)$ appearing in (B5) exhibits the competition between surface and volume effects which gives rise to the nucleation of a bubble of a stable phase within a metastable medium. Thus, in the no-gravity limit, the distinguished points in the lower branch of the mass curve have a clear interpretation: the black hole thermalon is the unstable static solution sitting at the maximum of the potential, while the instanton becomes the standard bounce solution for metastable vacuum decay.

On the other hand, the upper branch of the mass curve yields

$$V(R) = \mu R^2 + \nu R^3, \quad (B6)$$
and has no maximum. More precisely, the maximum, $R$, has gone to infinity ($R \sim 1/\sqrt{G}$). There is therefore no zero-gravity limit of the cosmological thermalon. One may understand the behavior of the potential by enclosing the bubble in a sphere of radius $L$ and recalling that in this case, the change from the metastable to the stable phase occurs in the *exterior* of the bubble (i.e., on the side with bigger values of the radial coordinates). This yields the potential $V(R) = \mu R^2 - \nu (L^3 - R^3)$ which has the correct volume dependence for bubble nucleation. The extra term $-\nu L^3$ is constant, and infinite in the limit $L \to \infty$. Gravity changes the shape of the potential and makes it finite, so that there is an unstable static solution, the cosmological thermalon.

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