LINEAR BIMETRIC GRAVITATION THEORY

M.I. Piso, N. Ionescu-Pallas, S. Onofrei
Gravitational Researches Laboratory 71111 Bucharest, Romania

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Abstract

A general bimetric theory of gravitation is described as a linear in the second approximation. This is allowed due to the small experimental significance of the higher order terms. Solar System tests are satisfied. The theory allows black holes, which are physical singularities. The predicted black hole radius is equal to the one resulting from Einstein’s theory, if the compatibility between the gauge conditions and the existence of gravitational waves is required (the Rosen-Fock metric). The quadrupolar gravitational radiation formula is regained.

Keywords:
general relativity - bimetric theories - gravitational radiation

1 Introduction

Bimetric theories are considered as a main result of Nathan Rosen’s work in the early 40’s [1]. The direct interest in this way of space-time and gravity interpretation is related to the possibility to describe the gravitational interaction as a physical field, acting directly onto the standard flat space-time as a result of some ”geometric” properties of a locally attached general relativistic manifold. The resulting field description of gravity may suggest standard field theory methods in order to quantize the gravitational field. However, the efforts in this direction are discouraged by some non convenient features of the Rosen’s theory: i) black holes do not exist and ii) the interpretation of the observational
gravitational radiation emission results from the PSR 1913+16 pulsar fails in comparison with general relativity.

The present paper deals with some opinions on the physical possibilities in order to make from bimetric theories good gravitational field theories, by removing the incompatibilities with general relativity and observational data.

2 Bimetric theory

In the sequel, we shall try to give a physical description of the basic bimetric concepts. To conciliate both general relativity and the standard Minkowski description of quantum field theories, two four-dimensional "Universes" are simultaneously considered: one of them is flat and the other curved. The flat Universe is the "real" one, in which experiments and observations are performed. The curved Universe is considered as a necessary geometric construction in order to satisfy the inertial and gravitational mass equivalence principle.

For both "Universes", we adopt the same arbitrary coordinate system. To evidence the separation between the "Universes", we denote by the sign $R$ the quantities defined on the curved Universe. The metrics in the two "Universes" are:

$$(dS)^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (dS_R)^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta$$

$$(\alpha, \beta) = (0, 1, 2, 3) \quad dS > 0, \quad dS_R > 0 \quad (1)$$

where $g_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ are the metric tensors for the considered cases. Contravariant equivalent metric tensors are defined by means of the relations:

$$g_{\alpha\lambda} h^{\lambda\beta} = \delta_\beta^\beta, \quad \gamma_{\alpha\lambda} \chi^{\lambda\beta} = \delta_\alpha^\beta \quad (2)$$

where $\delta_\alpha^\beta$ are the Kronecker symbols.

Since bimetric concepts, the physical "meaning" of the curved Universe is established by giving a relation between the contravariant metric tensor $\chi^{\alpha\beta}$ and a necessary tensor potential $\Psi^{\alpha\beta}$ of the gravitational field, which lies on the flat Universe. In this paper, we shall restrict to the second order terms of the general relation, due to the small experimental significance of the higher order terms:

$$\chi^{\alpha\beta} = h^{\alpha\beta} - \epsilon \Psi^{\alpha\beta} + \epsilon^2 (b_1 \Psi^{\alpha\beta} + b_2 \Psi^{\alpha\lambda} \Psi^{\lambda\beta} + b_3 \Psi^2 h^{\alpha\beta} + b_4 \Psi_{\mu\nu} \Psi^{\mu\nu} h^{\alpha\beta}) \quad (3)$$
where $\Psi = \Psi^{\alpha\beta}\Psi_{\alpha\beta}$; $b_j$ ($j = 1, 2, 3, 4$) are dimensionless quantities to be determined in the sequel, and $\epsilon$ is a parameter whose physical dimensions are the inverse of the ones of the gravitational potential $\Psi$. Outside the sources, it seems obvious from 3 that $\epsilon \Psi \ll 1$. It is convenient to adopt for $\epsilon$ the expression:

$$\epsilon = \frac{GM_0}{c^2}$$

which holds for a finite physical system with rest mass $M_0$; $G$ is the Newtonian gravity constant and $c$ - the speed of light.

To resume, geometry on the curved space-time is determined by the metric tensor $\gamma^{\alpha\beta}$, the curvature being produced by the flat space gravitational tensor potential $\Psi^{\alpha\beta}$ as seen from 3.

A basic principle of the bimetric theories is the existence of a variational principle of the action $A$:

$$A = \frac{1}{c} \int L \sqrt{-g}(d^4x)$$

where the Lagrangeian density $L$ is composed of a sources part, formulated in the curved Universe, and a field part, formulated in the flat Universe [5][6]:

$$L = S \left[ \rho c^2 + (\rho H - p) \right] R - \frac{c^4}{64\pi G} c^2 \left( \Psi_{\alpha\beta\lambda\mu} \Psi^{\alpha\beta\mu\lambda} - \frac{1}{2} \Psi_{\lambda\mu} \Psi^{\lambda\mu} \right)$$

The quantities $S$ and $H$ are as follows:

$$S = \sqrt{-\gamma} \sqrt{-g} \quad , \quad H = \int_0^{p(p)} \frac{dp}{\rho(p)} \quad , \quad \gamma = \text{Det} \parallel \gamma_{\alpha\beta} \parallel \quad , \quad g = \text{Det} \parallel g_{\alpha\beta} \parallel$$

By means of 6 and 7, the physical system to be studied is composed of a perfect fluid, described by the invariant mass density $\rho$, the invariant pressure $p$ and the tensor potential of the system’s proper gravity $\Psi^{\alpha\beta}$.

Performing the variation of the action 3 with respect to $\Psi^{\alpha\beta}$ and vanishing it, we get the field equations [7]:

$$\frac{1}{2} \epsilon \Box \xi_{\alpha\beta} = - \frac{8\pi G}{c^4} \sigma_{\alpha\beta}$$

where

$$\xi_{\alpha\beta} \equiv \Psi_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \Psi \quad , \quad \sigma_{\alpha\beta} \equiv - \frac{1}{\epsilon} ST_{\mu\nu} \frac{\partial \Psi_{\alpha\beta}}{\partial \Psi_{\mu\nu}}$$

and $T_{\mu\nu}$ is the energy tensor defined in the curved Universe:

$$T_{\alpha\beta} = \left[ (c^2 + H) \rho u_{\alpha} u_{\beta} - p g_{\alpha\beta} \right] R$$
Explicitly from 8 and 9, \(\sigma_{\alpha\beta}\) - the tensor of the gravitational sources - has the form:

\[
\sigma_{\alpha\beta} = S \{ T_{\alpha\beta} - e b_1 (\Psi T_{\alpha\beta} + \Psi^{\mu\nu} T_{\mu\nu} g_{\alpha\beta}) - e b_2 \left( \Psi^{\lambda} T_{\lambda\alpha} + \Psi^{\lambda} T_{\lambda\beta} \right) \\
- 2 e b_3 \Psi (h^{\mu\nu} T_{\mu\nu}) g_{\alpha\beta} - 2 e b_4 \Psi_{\alpha\beta} (h^{\mu\nu} T_{\mu\nu}) \}
\tag{11}
\]

One considers the next two approximations: the first one, derived from the first terms expansion of 10, is

\[
ST_{\alpha\beta} \approx \left[ (c^2 + H) \rho u_{\alpha} u_{\beta} - pg_{\alpha\beta} \right] + \epsilon \rho c^2 \left( u_{\alpha} u^{\lambda} \Psi_{\lambda\beta} + u_{\beta} u^{\lambda} \Psi_{\lambda\alpha} - \frac{1}{2} u_{\alpha} u_{\beta} \Psi_{\mu\nu} u^{\mu} u^{\nu} \right)
\tag{12}
\]

and the second stands only for the terms proportional to \(\epsilon\) in 11

\[
ST_{\alpha\beta} \approx \rho c^2 u_{\alpha} u_{\beta}
\tag{13}
\]

In this way, we get for the gravitational field sources tensor an approximative form, which is covariant in the flat Universe:

\[
\sigma_{\alpha\beta} \approx \left[ (c^2 + H) \rho u_{\alpha} u_{\beta} - pg_{\alpha\beta} \right] + \epsilon \rho c^2 \left\{ - b_1 \left[ \Psi u_{\alpha} u_{\beta} + (u_{\mu} u_{\nu} \Psi^{\mu\nu}) g_{\alpha\beta} \right] \\
+ (1 - b_2) \left[ u_{\alpha} u^{\lambda} \Psi_{\lambda\beta} + u_{\beta} u^{\lambda} \Psi_{\lambda\alpha} \right] - 2 b_3 \Psi g_{\alpha\beta} - 2 b_4 \Psi_{\alpha\beta} - \frac{1}{2} (u_{\mu} u_{\nu} \Psi^{\mu\nu}) u_{\alpha} u_{\beta} \right\}
\tag{14}
\]

Further, considering a covariant approximation for the potential and introducing a new denotation instead:

\[
\Psi_{\alpha\beta} \approx \Psi \left( \frac{1}{2} g_{\alpha\beta} - u_{\alpha} u_{\beta} \right), \quad \Psi = \frac{4}{\epsilon c^2} \Phi
\tag{15}
\]

we get for 14

\[
\sigma_{\alpha\beta} \approx \left[ (c^2 + H) \rho u_{\alpha} u_{\beta} - pg_{\alpha\beta} \right] + 4 \rho \Phi \left\{ \left[ \frac{1}{2} b_1 - 2 b_3 - b_4 \right] g_{\alpha\beta} + \left[ 2 b_4 + b_2 - b_1 - \frac{3}{4} \right] u_{\alpha} u_{\beta} \right\}
\tag{16}
\]

In order to satisfy the equivalence principle for the one body problem, in the case

\[
g_{\alpha\beta} \rightarrow 2 \delta_{0\alpha} \delta_{0\beta} - \delta_{\alpha\beta}, \quad u_{\alpha} \rightarrow \delta_{0\alpha}
\tag{17}
\]

the following expression should be fulfilled

\[
\sigma_{00} \approx c^2 \rho + (\rho H - p) - \frac{1}{2} \rho \Phi .
\tag{18}
\]

From 16, 17 and 18 we get the first relation between the coefficients:
\[ 4b_4 - 8b_3 + 4b_2 - 2b_1 = \frac{5}{2} \]  \hfill (19)

In the static approximation, \( \gamma_{\alpha\beta} \) becomes

\[
\gamma_{\alpha\beta} = (2\delta_{0\alpha}\delta_{0\beta} - \delta_{\alpha\beta}) - \frac{1}{2} \epsilon \Psi \delta_{\alpha\beta} + \epsilon^2 \Psi^2 \left\{ \left( \frac{1}{4} - \frac{1}{3} b_2 - 2b_3 - 2b_4 \right) \delta_{0\alpha}\delta_{0\beta} - \\
\left( \frac{1}{4} - \frac{1}{2} b_1 - \frac{1}{4} b_2 - b_3 - b_4 \right) \delta_{\alpha\beta} \right\} + O(\epsilon^3 \Psi^3) \]
\hfill (20)

To satisfy the equivalence principle in the two-body problem, one should have

\[
\gamma_{00} = 1 - \frac{1}{2} \epsilon \Psi + \frac{1}{8} \epsilon^2 \Psi^2 \]  \hfill (21)

From (20) and (21) we get the second condition for the coefficients \( b_i \)

\[ 4b_4 + 4b_3 + b_2 - 2b_1 = \frac{1}{2} \]  \hfill (22)

and, using also (19), we get the solutions:

\[
b_1 = \frac{1}{12} + (4b_3 + 2b_4) , \quad b_2 = \left( \frac{2}{3} + 4b_3 \right) . \]  \hfill (23)

In the static case, \( \chi^{\alpha\beta} \) is a diagonal tensor which may be exactly inverted in order to get \( \gamma^{\alpha\beta} \) and get for the metric the Schwarzschild type form

\[
(dS_R)^2 = \frac{(cdt)^2}{1 + \frac{3}{2} \epsilon \Psi + \frac{1}{8} \epsilon^2 \Psi^2} - \frac{(dx)^2 + (dy)^2 + (dz)^2}{1 - \frac{3}{2} \epsilon \Psi + \epsilon^2 \Psi^2 \left( \frac{1}{8} + 2\eta \right)} \]  \hfill (24)

\[
\eta \equiv b_4 + 2b_3 + \frac{1}{24} \]

Performing the variation of the action with respect to the coordinates of the fluid particle we get that, outside the field sources, in the curved Universe metric, the probe particle movement is geodetic. On the other hand, if we take in (24) the \( \epsilon \) expansions in the second order for \( \gamma_{00} \) and \( \gamma_{jk} \) \( (j, k = 1, 2, 3) \), we get an expression which, in the same approximation, is to coincide to the homologous expression in standard general relativity. This correspondence implies that the Solar system general relativistic tests are satisfied.

If we take \( \eta = 0 \), i.e. from (24)

\[
\left( b_4 + 2b_3 + \frac{1}{24} = 0 \right) \]  \hfill (25)
the metric $\mathcal{M}_2$ may be considered as an approximation of the Rosen’s metric $\mathcal{M}_4$ written as following:

\[
(dS_R)^2 = \frac{(cdt)^2}{\exp\left(\frac{1}{4}c\Psi\right)} - \frac{(dx)^2 + (dy)^2 + (dz)^2}{\exp\left(\frac{1}{4}c\Psi\right)} \tag{26}
\]

However, for $r \to 0$, $\Psi \propto \frac{1}{r}$ in the case of a pointlike source, and the metric $\mathcal{M}_2$ manifests exponentially divergent in the origin, different from the behaviour of the metric $\mathcal{M}_4$. If we take in $\mathcal{M}_2$ the values of $\eta > -\frac{1}{32}$, then the denominators of the metric coefficients do not vanish for real values of $r$, i.e. no Schwarzschild-like black-holes exist. But, it is to emphasize that the resultant metric $\mathcal{M}_4$ does not exclude the existence of black holes. It is obvious that, for

\[
\eta = -\frac{1}{32}, \text{ i.e. } \left(b_4 + 2b_3 + \frac{7}{96} = 0\right) \tag{27}
\]

the denominator of the spacelike part of the metric $\mathcal{M}_2$ becomes $(1 - \frac{1}{4}c\Psi)^2$ and the resulting radius of the singularity is:

\[
r = \epsilon \tag{28}
\]

The important conclusion is that bimetric theories may provide a more general formalism for the description of gravity. It is interesting to remark that Rosen himself believed that bimetrism is incompatible with black holes [6]. Is is to be mentioned that the predicted black hole radius $\mathcal{M}_2$ is equal with the one resulting from Einstein’s theory, if the compatibility between the gauge conditions and the existence of gravitational waves is required (the Rosen-Fock metric) [9].
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3 Appendix

To study the gravitational radiation, we start from the following approximations:

\[ \epsilon \xi^{\alpha \beta} \approx \frac{4G}{c^2} \int \frac{\tau^{\alpha \beta}}{R} dxdydz, \quad \xi^{\alpha \beta, \beta} \approx 0, \quad \tau^{\alpha \beta} \approx \rho c^2 u^\alpha u^\beta \]  

(29)

Performing a standard procedure of approximation [10] we reach to the following formula:

\[ \epsilon \xi^{jk} \approx \frac{2G}{c^2 R_0} \frac{d^2}{dt^2} \int \rho x^j x^k dxdydz \]  

(30)

With the standard definition for the mass quadrupole momentum:

\[ D^{jk} = \int \rho \left( 3x^j x^k - \delta^{jk} \delta_{lm} x^l x^m \right) dxdydz \]  

(31)

we get the known relation:

\[ \epsilon \left( 3\Psi^{jk} - \delta^{jk} \delta_{lm} \Psi^{lm} \right) = \frac{2G}{c^2 R_0} \, D^{jk} \]  

(32)

In the following, we introduce the denotations:

\[ \alpha = \frac{2G}{c^2 R_0} \, D^{j11}, \quad \beta = \frac{2G}{c^2 R_0} \, D^{j22}, \quad \gamma = \frac{2G}{c^2 R_0} \, D^{j33} \]  

(33)

and the approximate

\[ \Psi^{\alpha \beta} = A^{\alpha \beta} \sin \omega \left( t - \frac{x}{c} \right) \]  

(35)

\[ \phi'^\alpha = q^\alpha \left[ 1 - \cos \omega \left( t - \frac{x}{c} \right) \right] \]  

(36)

\[ \Psi'_{\mu \nu} = \Psi_{\mu \nu} + \left( g_{\mu \nu} D_\nu + g_{\nu \lambda} D_\mu \right) \phi^\lambda \]  

(37)

(approximate gauge transformation for avoiding longitudinal waves)

\[ c \theta_0^{-1} = \frac{c^3}{32\pi G} \, \epsilon^2 \frac{\partial \Psi'}{\partial t} \frac{\partial \Psi'}{\partial t} \]  

(38)

(supplemented by the condition \( \Psi' = 0 \) accomplished as a result of the gauge transformation)

\[ P = -c \int \theta_0^{-1} R_0^2 d\Omega = -\frac{G}{45c^5} \, D^{jk} D^{jk} \]  

(39)