Geometric convexity of an operator mean

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Abstract
Let \( \sigma \) be an operator mean in the sense of Kubo and Ando. If the representation function \( f_\sigma \) of \( \sigma \) satisfies
\[
f_\sigma(t)^p \leq f_\sigma(t^p) \quad \text{for all } p > 1,
\]
then \( \sigma \) is called a pmi mean. Our main interest is the class of pmi means (denoted by PMI). To study PMI, the operator mean \( \sigma \), wherein
\[
f_\sigma(\sqrt{xy}) \leq \sqrt{f_\sigma(x)f_\sigma(y)} \quad (x, y > 0)
\]
is considered in this paper. The set of such means (denoted by GCV) includes certain significant examples and is contained in PMI. The main result presented in this paper is that GCV is a proper subset of PMI. In addition, we investigate certain operator-mean classes, which contain PMI.

1 Introduction
Let \( \mathcal{H} \) be a Hilbert space with an inner product \( \langle \cdot | \cdot \rangle \). A bounded linear operator \( A \) on \( \mathcal{H} \) is said to be positive (denoted by \( A \geq 0 \)) if \( \langle Ax | x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). We denote the set of positive operators on \( \mathcal{H} \) by \( B(\mathcal{H})_+ \). If an operator \( A \in B(\mathcal{H})_+ \) is invertible, we denote \( A > 0 \).

A continuous real function \( f \) from \( (0, \infty) \) is said to be operator monotone on \( (0, \infty) \), if the inequality \( A \geq B > 0 \) implies \( f(A) \geq f(B) \).

An operator monotone function \( f \) on \( (0, \infty) \) is called normal, if \( f(1) = 1 \). In this paper, \( OM_1^+ \) denotes the set of normalized operator monotone functions on \( (0, \infty) \) into itself.

In [7], Kubo and Ando provide the following axiom for operator means. A binary operation \( \sigma \) among \( B(\mathcal{H})_+ \) is called an operator mean, if it satisfies the following:

\[
(i) \quad A \leq C, B \leq D \Rightarrow A\sigma B \leq C\sigma D,
\]
If $f$ is in $OM_+^1$, then the binary operation $\sigma_f$ on $B(\mathcal{H})_+$ defined by
\[
A \sigma_f B = \lim_{\epsilon \downarrow 0} A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B^\epsilon A^{-\frac{1}{2}}) A^{\frac{1}{2}}
\]
is an operator mean, where $A_\epsilon = A + \epsilon 1$ and $B_\epsilon = B + \epsilon 1$. Kubo and Ando show that the function $f \mapsto \sigma_f$ is an order isomorphism from $OM_+^1$ onto the set of operator means [7]. In this paper, we call $\sigma_f$ an operator mean corresponding to $f$ and at times identify $\sigma_f$ as $f$.

The following theorem is referred to as the Ando-Hiai inequality [2]. $A, B > 0, A \#_\alpha B \geq I \Rightarrow A^p \#_\alpha B^p \geq I$ (p $\geq 1$), where $\alpha \in [0, 1]$ and $\#_\alpha$ is an operator mean corresponding to a power function $t \mapsto t^\alpha$. In [14], it is shown that the generalized inequality
\[
A, B > 0, A \sigma_f B \geq I \Rightarrow A^p \sigma_f B^p \geq I \quad (p \geq 1)
\]
holds if and only if $f$ is power monotone increasing (pmi for short), i.e.,
\[
f(t)^p \leq f(t^p) \quad (p \geq 1, t > 0).
\]

Our main interest is the class of pmi means (denoted by $PMI$). To study $PMI$, in this paper, we consider an operator mean $\sigma_f$, wherein
\[
f(\sqrt{xy}) \leq \sqrt{f(x)f(y)} \quad (x, y > 0)
\]
holds. A positive valued function with (1.2) is called geometrically convex or multiplicatively convex [3, 11]; hence, we denote the set of functions $f \in OM_+^1$ with (1.2) by $GCV$.

In Section 3, we present some of the basic properties of $GCV$ and its adjoint (denoted by $GCC$). From this argument, we conclude that several significant $PMI$ means are contained in $GCV$.

It is conjectured that $GCV$ is a proper subset of $PMI$, i.e.,
\[
GCV \subsetneq PMI.
\]

In Section 4, we characterize a pmi mean and a gcv mean by using Hansen’s integral representaion of an operator mean [5]. Using this, we prove the above conjecture, which is our main result.

In Section 5, we consider $PMI_{\infty}$ defined by
\[
PMI_{\infty} := \{ f \in OM_+^1 \mid f(t) \geq t^{p'(1)}\}
\]
and prove that $PMI$ is a proper subset of this class.
Combining the above arguments, we finally obtain the following relationships: Let $\sigma$ be an operator mean and $f_\sigma$ be the representation function of $\sigma$. Consider the following statements:

(I) $f_\sigma$ is geometrically convex.
(II) $A, B > 0, A\sigma B \geq I \Rightarrow A^p \sigma B^p \geq I$ $(p \geq 1)$.
(III) $\sigma \geq \#_\alpha$ for certain $\alpha \in [0, 1]$.

Then

(1) I implies II; II implies III,
(2) III does not imply II; II does not imply I.

2 Geodesic mean

As per the theory of Kubo and Ando [7], the set $OM_1^+$ of normalized positive operator monotone functions on $(0, \infty)$ is identified with the set of operator means. Hence, the following classes

$$PMI := \{ f \in OM_1^+ | f(t)^r \leq f(t') \quad (\forall r > 1)\}$$

and

$$PMD := \{ f \in OM_1^+ | f(t)^r \geq f(t') \quad (\forall r > 1)\}$$

can be viewed as subsets of the set of operator means. The function $f \in PMI$ (resp. $f \in PMD$) is referred to as a pmi (resp. pmd) mean. As stated in [14], for any probability measure $p$ on $[0, 1]$, the function

$$x \mapsto \int_0^1 x^\alpha dp(\alpha)$$

is in $PMI$. Such a function $f$ is called a geodesic mean and the set of geodesic means is denoted by $GM$. Several examples of a pmi mean can be obtained by using the fact that $GM \subset PMI$.

Although there are a number of functions belonging to PMI, it is not easy to show the pmi property [1.1] of a certain operator mean because the verification of condition [1.1] or [2.1] requires considerable calculation. Bourin and Hiai [3] mention that a positive operator monotone function $f$ on $[0, \infty)$ belongs to $GM$ if and only if $\frac{1}{n} f(e^n) \geq 0$ for all $n \geq 0$. Thus we need to determine a technique for obtaining pmi means and evaluate this technique.

3 Geometrically convex mean

3.1 Definitions and basic properties

A positive function $f$ on $(0, \infty)$ is called geometrically convex (resp. geometrically concave), if

$$f(\sqrt{xy}) \leq \sqrt{f(x)f(y)} \quad (\text{resp. } f(\sqrt{xy}) \geq \sqrt{f(x)f(y)})$$
holds for all \(x, y > 0\). Let \(gcv\) (resp. \(gcc\)) be the set of monotone increasing continuous functions that are geometrically convex (resp. geometrically concave) on \((0, \infty)\). We also define \(GCV\) (resp. \(GCC\)) as follows:

\[
GCV := \{ f \in OM^1_+ \mid f \in gcv \} \quad (resp. \ GCC := \{ f \in OM^1_+ \mid f \in gcc \}).
\]

As stated in [3], the convexity of the function \(t \mapsto \log f(e^t)\) is a necessary and sufficient condition for \(f \in OM^1_+\) to be in \(GCV\). Using this, we have some inclusions among the subclasses of \(OM^1_+\). The second inclusion in the following is proved in [4].

**Proposition 3.1.** \(GM \subseteq GCV \subseteq PMI\).

**Lemma 3.1.** Let \(f\) and \(g\) be in \(gcv\) and let \(h \in GCV\). Then we have the following.

1. \(f \cdot g\) and \(f^\alpha\) are in \(gcv\) for all \(\alpha > 0\);
2. The function \(t \mapsto (f \sigma_h g)(t) := f(t) \cdot h(\frac{g(t)}{f(t)})\) is in \(gcv\);
3. If \(f\) is bijective, the inverse function of \(f\) is in \(gcc\).

**Proof.** The geometric convexity of \(f \cdot g\) and \(f^\alpha\) is immediate.

From the definition of \(GCV\), inequalities

\[
(f \sigma_h g)(\sqrt{xy}) = f(\sqrt{xy})g(\sqrt{xy})
\]

\[
\leq \sqrt{f(x)f(y)}\sigma_h \sqrt{g(x)g(y)}
\]

\[
= \sqrt{f(x)f(y)}h(\sqrt{\frac{g(x)}{f(x)}} \cdot \frac{g(y)}{f(y)})
\]

\[
\leq \sqrt{f(x)f(y)}h(g(x) \cdot \frac{g(y)}{f(x)})
\]

\[
= \sqrt{(f \sigma_h g)(x)(f \sigma_h g)(y)}
\]

hold for all \(x, y > 0\). This implies (2).

Let \(f^{(-1)}\) be the inverse function of \(f\). Then for every \(x, y > 0\),

\[
f\left(\sqrt{f^{(-1)}(x)f^{(-1)}(y)}\right) \leq \sqrt{xy},
\]

signifying that \(f^{(-1)}\) is geometrically concave. \(\square\)

Using (2) of the above lemma, for \(f, g \in gcv\), the weighted arithmetic mean of \(f\) and \(g\) is in \(gcv\), which signifies that \(gcv\) is a convex set. This implies that \(GCV\) is a convex set.
Proof of Proposition 3.1. From the above lemma, $GCV$ is a convex set and has a power function $x^\alpha$ ($0 \leq \alpha \leq 1$), which implies the first inclusion $GM \subseteq GCV$.

Next, we prove the second inclusion. As stated above, if $f \in GCV$ if and only if function $t \mapsto F(t) := \log f(e^t)$ is convex on $(-\infty, \infty)$. Thus

$$F((1-\alpha)t + \alpha s) \leq (1-\alpha)F(t) + \alpha F(s)$$

holds for all $\alpha \in [0,1]$. Considering $t = 0$, $F(\alpha s) \leq \alpha F(s)$, which implies that $f(x^\alpha) \leq f(x)\alpha$ for all $x > 0$.

Recall that $f(t) \mapsto f^*_p(t) := \frac{1}{f(1/t)^p}$ is an idempotent mapping on $OM_+^1$ and $PMD = PMI^* := \{f^* | f \in PMI\}$. The following is obtained.

Corollary 3.2. $GM^* \subseteq GCC \subseteq PMD$.

Proof. From the preceding result,

$$GM^* \subseteq GCV^* = GCC \subseteq PMI^* = PMD.$$

Remark 3.1. From the above lemma, the set gcv is closed under the sum, i.e., $f_1, f_2 \in gcv \Rightarrow f_1 + f_2 \in gcv$. However, the same does not hold for gcc. For example, $\frac{2}{t+1}$ and $t^2$ are in gcc and $\frac{2}{t+1} + t^2$ is not in gcc.

Before closing this section, we note that there are some counterexamples for $GM = GCV$.

Example 3.1. (3) Let $p \in [-1,1]$ and $b_p(t) := \left(\frac{t^{p+1}}{e}\right)^{1/p}$. Then $b_p \in GCV \setminus GM$ if and only if $p \in (0,1) \setminus \{\frac{1}{n} | n \in \mathbb{N}\}$.

Example 3.2. (3) Let $\alpha \in [-1,2]$ and $u_{\alpha}(t) := \frac{\alpha-1}{\alpha} t^{\alpha-1} \frac{e}{e^\alpha - 1}$. Then $u_{\alpha} \in GCV \setminus GM$ if and only if $\alpha \in [1/2,2] \setminus \{1, \frac{m+1}{2}, \frac{m}{m+1} | m \in \mathbb{N}\}$.

3.2 Functions in GCV

In this section, we present a few examples of a function in $GCV \subseteq PMI$. We first consider the function $u_{\alpha}$ defined in the previous section. The geometric convexity of $u_{\alpha}$ is characterized as follows [3]:

$$u_{\alpha} \in GCV \ (\text{resp. } u_{\alpha} \in GCC) \iff 1/2 \leq \alpha \leq 2 \ (\text{resp. } -1 \leq \alpha \leq 1/2).$$

The function $u_{\alpha}$ is generalized as $u_{a,b}$ defined by

$$u_{a,b}(t) := \frac{b t^a - 1}{a t^b - 1} \quad (a, b \in [-2, 2], (a, b) \neq (0,0)),$$
where \( t_{a}^{-1} \) is defined as \( \log t \), when \( a = 0 \). In [9, Example 3.4(1)], it is proved that \( u_{a,b} \in OM_{1}^{+} \) if and only if \((a, b)\) is in \( \Gamma \), where

\[
\Gamma := \{(a, b) \mid 0 < a - b \leq 1, 2 \geq a \geq -1, -2 \leq b \leq 1 \}
\cup \{(0, 1) \times [0, 1) \}\n\cup \{(a, a) \mid a \neq 0 \}.
\]

**Proposition 3.3.** \( u_{a,b} \in GCV \) if and only if \((a,b)\) is in \( \Gamma \) and \(|a| \geq |b| \).

**Proof.** We first consider the case, where \( ab = 0 \). If \( a = 0 \) and \( b \neq 0 \), then \( u_{a,b}(t) = b t_{b}^{-1} \log t \) and

\[
\frac{d^2}{dx^2} \log u_{a,b}(e^{x}) = \frac{-1}{x^2} + b^2(e^{bx} + e^{-bx} - 2)^{-1} < 0
\]

for \( x \neq 0 \). Thus we obtain

\[
\frac{d^2}{dx^2} \log u_{b,a}(e^{x}) = -\frac{d^2}{dx^2} \log u_{a,b}(e^{x}) \geq 0,
\]

which implies the desired result.

We next consider the case, where \( a \neq 0, b \neq 0 \). Then there exists \( \alpha \in \mathbb{R} \) such that

\[
u_{a,b}(t) = \frac{|b|}{|a|} \left( \frac{|a|}{|b|} - 1 \right) t^{\alpha}
\]

and

\[
\frac{d^2}{dx^2} \log u_{a,b}(e^{x}) = \frac{(|a|x)^2 \psi(|a|x) - (|b|x)^2 \psi(|b|x)}{x^2},
\]

where \( \psi(x) = -(e^{x} + e^{-x} - 2)^{-1} \). The function \( \Psi(y) := y^2 \psi(y) \) is a negative valued function on \((\infty, \infty) \setminus \{0\}\) and

\[
\Psi(-y) = \Psi(y), \quad \Psi(x) < \Psi(y)
\]

for \( 0 < x < y \). Thus

\[
\frac{d^2}{dx^2} \log u_{a,b}(e^{x}) \geq 0 \iff \Psi(|a|x) \geq \Psi(|b|x) \quad \text{for all} \quad x \neq 0 \iff |a| \geq |b|.
\]

From

\[
\frac{d^2}{dx^2} \log u_{a,b}^*(e^{x}) = \frac{d^2}{dx^2} \log u_{b,a}(e^{x}) = -\frac{d^2}{dx^2} \log u_{a,b}(e^{x}),
\]

the following is obtained.

**Corollary 3.4.** \( u_{a,b} \in GCC \) if and only if \((a,b)\) is in \( \Gamma \) and \(|a| \leq |b| \).

From the above results, we have \( u_{a,b} \in GCV \cup GCC \) for all \((a,b)\) in \( \Gamma \), which implies a condition for \( u_{a,b} \) to be in \( PMI \).
Corollary 3.5. Let \((a, b) \in \Gamma\). Then the following are equivalent:

1. \(|a| \geq |b|\) (resp. \(|a| \leq |b|\));
2. \(u_{a,b} \in GCV\) (resp. \(u_{a,b} \in GCC\));
3. \(u_{a,b} \in PMI\) (resp. \(u_{a,b} \in PMD\)).

The Stolarsky mean is defined as

\[
S_{\alpha}(s, t) := \left( \frac{s^\alpha - t^\alpha}{\alpha(s - t)} \right)^{1/\alpha-1},
\]

for \(\alpha \in [-2, 2] \setminus \{0, 1\}\),

\[
S_0(s, t) := \lim_{\alpha \to 0} S_{\alpha}(s, t) \quad \text{and} \quad S_1(s, t) := \lim_{\alpha \to 1} S_{\alpha}(s, t).
\]

It is known that \(S_{\alpha}(1, t)\) is operator monotone, if \(-2 \leq \alpha \leq 2\) \[10\]. Using

\[
\log S_{\alpha}(1, e^x) = \frac{1}{\alpha - 1} \log u_{\alpha,1}(e^x),
\]

we have a condition for \(S_{\alpha}(1, t)\) to be in \(GCV\).

Corollary 3.6. \(S_{\alpha}(1, t) \in GCV\) (resp. \(S_{\alpha}(1, t) \in GCC\)) if and only if \(\alpha \in [-1, 2]\) (resp. \(\alpha \in [-2, -1]\)).

Proof. By simple calculation, we have

\[
\frac{d^2}{dx^2} \log S_1(1, e^x) = \frac{d^2}{dx^2} \left( \frac{xe^x}{e^x - 1} \right) \geq 0.
\]

Thus \(S_1(1, t) \in GCV\).

We next consider the case, where \(\alpha \neq 1\). By Proposition 3.3,

\[
\frac{d^2}{dx^2} \log u_{a,b}(e^x) \leq 0 \quad \text{(resp.} \quad \geq 0) \iff \frac{d^2}{dx^2} \log u_{a,b}(e^x) \leq 0 \quad \text{(resp.} \quad \geq 0).
\]

Thus

\[
\frac{d^2}{dx^2} \log S_{\alpha}(1, e^x) = \frac{1}{|1 - \alpha|} \frac{d^2}{dx^2} \log u_{|\alpha|,1}(e^x) \geq 0 \quad (1 < \alpha \leq 2),
\]

\[
\frac{d^2}{dx^2} \log S_{\alpha}(1, e^x) = \frac{1}{|1 - \alpha|} \frac{d^2}{dx^2} \log u_{1,|\alpha|}(e^x) \geq 0 \quad (-1 \leq \alpha < 1)
\]

and

\[
\frac{d^2}{dx^2} \log S_{\alpha}(1, e^x) = \frac{1}{|1 - \alpha|} \frac{d^2}{dx^2} \log u_{1,|\alpha|}(e^x) \leq 0 \quad (-2 \leq \alpha \leq -1).
\]
3.3 Inverses

In [1], Ando proves that for every \( f \in OM_1^+ \), the function \( t \mapsto tf(t) \) has the inverse function \( (tf)^{(-1)} \) which is in \( OM_1^+ \), i.e.,

\[
f \in OM_1^+ \Rightarrow (tf)^{(-1)} \in OM_1^+.
\]

In this section, we investigate this result with respect to the theory of geometrically convex functions.

Let \( P \) be the set of nonnegative operator monotone functions on \([0, \infty)\) and \( P^{-1} := \{ h \in P \mid h([0, \infty)) = [0, \infty), \; h^{(-1)} \in P \} \).

In [13], Uchiyama proves the product formula

\[
P \cdot P^{-1} \subset P^{-1}. \tag{3.1}
\]

Using this, Ando’s result stated above can be extended as

\[
f \in OM_1^+ \Rightarrow (t^\alpha f)^{(-1)} \in OM_1^+ \quad (\alpha \geq 1).
\]

The following is immediate from the above argument.

**Proposition 3.7.** Let \( \alpha \geq 1 \) and \( f \in OM_1^+ \). Then

\[
f \in GCV \iff (t^\alpha f)^{(-1)} \in GCC.
\]

**Proof.** Assume \( f \in GCV \). Then it is evident that

\[
\frac{d^2}{dt^2} \log(t^\alpha f)(e^t) = \frac{d^2}{dt^2} \log f(e^t) \geq 0,
\]

which implies that \( t^\alpha f \in gcv \) and \( (t^\alpha f)^{(-1)} \in gcc \). The operator monotonicity of \( (t^\alpha f)^{(-1)} \) comes from (3.1).

Conversely, if \( (t^\alpha f)^{(-1)} \in GCC \), \( (t^\alpha f) \) is in \( gcv \). Thus \( \frac{d^2}{dt^2} \log f(e^t) = \frac{d^2}{dt^2} \log(t^\alpha f)(e^t) \geq 0 \). \( \square \)

From \( ((t^\alpha f)^{(-1)})^* = (t^\alpha f^*)^{(-1)} \) and \( GCC^* = GCV \), the preceding proposition can be rewritten as follows:

**Corollary 3.8.** Let \( \alpha \geq 1 \) and \( f \in OM_1^+ \). Then

\[
f \in GCC \iff (t^\alpha f)^{(-1)} \in GCV.
\]

We next consider a function \( u(t) \) defined by

\[
u(t) := \beta \prod_{i=1}^{n} (t + a_i)^{\gamma_i}, \tag{3.2}
\]

where \( 0 = a_1 < a_2 < \cdots a_n \), \( 1 \leq \gamma_1 \), \( 0 < \gamma_i \) and \( 0 < \beta \). Uchiyama shows that \( u \) is in \( P^{-1} \) and this result can be derived using the above product formula ([12], [13]). Additionally, we show the following.
Proposition 3.9. If \( f \in GCV \) and \( u(1) = 1 \), then \((u \cdot f)^{(-1)} \in GCC\).

Proof. From Lemma 3.1, we have \((u \cdot f) \in gcc\) and \((u \cdot f)^{(-1)} \in gcc\). The operator monotonicity of \((u \cdot f)^{(-1)}\) comes from (3.1).

As the constant function 1 is in \( GCV \), the following is evident:

Corollary 3.10. If \( u(1) = 1 \), then \( u^{(-1)} \in GCC \).

Example 3.3. For \( \alpha \in (0, 1) \), a function \( u(t) := t(1 - \alpha + \alpha t) \) has the inverse \( u^{(-1)}(s) = \frac{\alpha - 1 + \sqrt{(1 - \alpha)^2 + 4s\alpha}}{2\alpha} \) and \( u^{(-1)} \) is in \( GCC \).

4 Main results

In this section, we present an integral representation of an element of \( GCV \). In [5], Hansen considers a class of real valued continuous functions defined as

\[ E := \{ F \mid F : \mathbb{R} \to \mathbb{R} \text{ is continuous and } e^A \leq e^B \Rightarrow e^{F(A)} \leq e^{F(B)} \}, \]

and proves the following:

1. A function \( F : \mathbb{R} \to \mathbb{R} \) is in \( E \) if and only if there exists \( \beta \in \mathbb{R} \) and a measurable function, \( h : (-\infty, 0] \to [0, 1] \) such that

\[ F(x) = \beta + \int_{-\infty}^{0} \left( \frac{1}{\lambda - e^x} - \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda)d\lambda, \]

where \( d\lambda \) is the Lebesgue measure on \((-\infty, 0]\);

2. the preceding measurable function \( h \) is uniquely determined by \( F \);

3. the function \( F \mapsto \exp F(\log t) \) is a bijection from \( E \) onto \( P \).

From this result, for \( f \in OM_1^1 \), \( F(x) := \log f(e^x) \) can be expressed as

\[ F(x) = \beta + \int_{-\infty}^{0} \left( \frac{1}{\lambda - e^x} - \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda)d\lambda. \]

As \( F(0) = 0 \),

\[ \beta = \int_{-\infty}^{0} \left( \frac{-1}{\lambda - 1} + \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda)d\lambda. \]

Thus

\[ f(t) = \exp \int_{-\infty}^{0} \left( \frac{1}{\lambda - t} - \frac{1}{\lambda - 1} \right) h(\lambda)d\lambda. \]

Using this, we obtain the following:
Proposition 4.1. Let \( f \in OM^+_1 \) and let \( h \) be a measurable function determined using the above method. Then \( f \in PMI \) if and only if
\[
\int_{-\infty}^{0} \left( \frac{1}{\lambda - t^r} - \frac{r}{\lambda - t} + \frac{r - 1}{\lambda - 1} \right) h(\lambda) d\lambda \geq 0
\] (4.1)
for all \( t > 0 \) and \( r \geq 1 \). Moreover, \( f \in GCV \) if and only if
\[
\int_{-\infty}^{0} \left( \frac{\lambda + t}{(\lambda - t)^3} \right) h(\lambda) d\lambda \geq 0
\] (4.2)
for all \( t > 0 \).

Proof. From the above argument,
\[
F(x) = \log f(e^x) = \int_{-\infty}^{0} \left( \frac{1}{\lambda - e^x} - \frac{1}{\lambda - 1} \right) h(\lambda) d\lambda.
\]
Hence, the condition \( f(e^{rx})/f(e^x)^r \geq 1 \) can be expressed as
\[
0 \leq F(rx) - rF(x) = \int_{-\infty}^{0} \left( \frac{1}{\lambda - e^{rx}} - \frac{r}{\lambda - e^x} + \frac{r - 1}{\lambda - 1} \right) h(\lambda) d\lambda
\]
for all \( x \in \mathbb{R} \) and \( r \geq 1 \).

From Lebesgue’s dominated convergence theorem, the condition
\[
\frac{d^2}{dx^2} \log f(e^x) \geq 0
\]
can be expressed as
\[
F''(x) = \int_{-\infty}^{0} \frac{d^2}{dx^2} \left( \frac{1}{\lambda - e^x} - \frac{1}{\lambda - 1} \right) h(\lambda) d\lambda
\]
\[
= \int_{-\infty}^{0} \left( e^x \lambda + e^{2x} \right) \frac{h(\lambda)}{(\lambda - e^x)^3} d\lambda \geq 0,
\]
which implies the desired result. \( \square \)

Remark 4.1. Let \( 0 < a < \infty \). Considering \( h = I_{(-\infty,-a)} \) (resp. \( h = I_{(-a,0)} \)), we have
\[
f(t) = \frac{a + t}{a + 1} \in GCV \quad (\text{resp.} \quad f(t) = \frac{(a + 1)t}{a + t} \in GCC).
\]

Remark 4.2. Let \( \alpha \in [0,1] \). Considering \( h = \alpha I_{(-\infty,0)} \),
\[
f(t) = t^\alpha \in GCV \cap GCC.
\]
4.1 Conjecture and theorem

It is conjectured that $GCV$ is a proper subset of $PMI$. To prove this, we use the argument of the preceding section. We set $\alpha = \frac{9}{14}$ and $h(\lambda) := \alpha I_{(-\infty,-2)} + (1 - \alpha)I_{(-1,0)}$ and show that inequality (4.2) does not hold, but (4.1) holds.

Theorem 1.

$GCV \subsetneq PMI$.

Proof. Let us show that (4.1) holds. For $r > 1$, we have

$$
\int_{-\infty}^{0} \left( \frac{1}{\lambda - t^r} - \frac{r}{\lambda - t} + \frac{r-1}{\lambda-1} \right) h(\lambda) d\lambda = (1 - \alpha) \left( \frac{\alpha}{1 - \alpha} \log \frac{3^{r-1}(t^r+2)}{(t+2)^r} - \log \frac{2^{r-1}(t^r+1)}{(t+1)^r} \right).
$$

Let us set $\beta := \frac{\alpha}{1 - \alpha}$ and

$$
\varphi(t) := \beta \log \frac{3^{r-1}(t^r+2)}{(t+2)^r} - \log \frac{2^{r-1}(t^r+1)}{(t+1)^r}.
$$

Then

$$
\frac{d\varphi}{dt} = \frac{2r t (t^{r-1} - 1) \psi_{\beta,r}(t)}{(t^r+2)(t^r+1)(t+1)^2},
$$

where $\psi_{\beta,r}(t) = \{(\beta - \frac{1}{2}) t^{r+1} + (\beta - 1) (t^r + t) - (2 - \beta)\}$. Here, $\psi_{\beta,r}$ is strictly monotone increasing and equation $\psi_{\beta,r}(t) = 0$ has a unique solution in $(0,1)$. Thus

$$
\min_{t \geq 0} \varphi(t) = \min \{ \varphi(0), \varphi(1) \} = \varphi(1) = 0.
$$

We next show that inequality (4.2) does not hold. By simple calculation,

$$
\int_{-\infty}^{0} \left( \frac{\lambda + t}{(\lambda - t)^3} \right) h(\lambda) d\lambda = (1 - \alpha) \frac{2}{(t+2)^2} \left( \beta - \frac{(t+2)^2}{2(t+1)^2} \right),
$$

where $\beta = \frac{\alpha}{1 - \alpha} \left( = \frac{9}{8} \right)$. This takes a negative value, if $t$ is sufficiently small. \(\square\)

Corollary 4.2.

$GCC \subsetneq PMD$.

5 Related results

In this section, we consider some operator-mean classes containing $PMI$ and prove certain relationships among them. For $r > 1$, we define

$$
PMI_r := \{ f \in OM^1_+ \mid f(t^r) \geq f(t)^r \}
$$

and

$$
PMI_\infty := \{ f \in OM^1_+ \mid f(t) \geq t^{\alpha} \text{ for some } \alpha \in [0,1] \}.
$$
We first note a property of $PMI_r$. It follows from [4, Corollary 4.7] that the Ando-Hiai type inequality,

$$A, B > 0, \quad A \sigma f B \geq 1 \Rightarrow A^r \sigma f B^r \geq 1$$

is a necessary and sufficient condition for $f \in OM_1^+$ to be in $PMI_r$. In addition, from the proof of [14, Lemma 2.1], we have

$$PMI = \bigcap_{x \in (1, 2]} PMI_x \subset PMI_r.$$  

We next consider the case, where $r = \infty$. Let $f \in OM_1^+$ such that $f(t) \geq t^\alpha$. Then,

$$\frac{f(t) - f(1)}{t - 1} \geq \frac{t^\alpha - 1}{t - 1} \quad (t > 1)$$

and

$$\frac{f(t) - f(1)}{t - 1} \leq \frac{t^\alpha - 1}{t - 1} \quad (t < 1),$$

which implies that $f'(1) = \alpha$. Thus the definition of $PMI_\infty$ can be rewritten as follows:

$$PMI_\infty = \{ f \in OM_1^+ \mid f(t) \geq tf'(1) \}.$$  

As stated in [15], the following relationship among $PMI_r$ and $PMI_\infty$ is known.

**Proposition 5.1.** ([15]) For $r > 1$,

$$PMI_r \subset PMI_\infty.$$  

**Proof.** Let $f \in PMI_r$. It is evident from the definition that

$$f(t^{s_n})^{1/s_n} \leq f(t)$$

for $s_n := 1/r^n$. Thus

$$\lim_{n \to \infty} \exp(\log(f(t^{s_n})^{1/s_n})) = \exp(\lim_{n \to \infty} \frac{\log(f(t^{s_n}))}{s_n}) = t^{f'(1)} \leq f(t).$$  

From the above discussion, the problem whether $PMI$ is a proper subset of $PMI_\infty$ arises (cf. [15]). We provide an answer to this problem.

**Proposition 5.2.**

$$\bigcup_{r > 1} PMI_r \subsetneq PMI_\infty.$$
Proof. We show that
\[
    f(t) := \frac{(1/3)t + (2/3)t^{1/3}}{(1/3) + (2/3)t^{1/3}}
\]
is in $\text{PMI}_\infty \setminus \text{PMI}_r$ for all $r > 1$.

Let us show $f \in \text{PMI}_\infty$. As the operator monotonicity of $f$ comes from [8], it is sufficient to show $f(t) \geq t^f(1)$. Set $g(t) := f(t) - t^{1/3}$, then
\[
    g'(t) = t^{1/3} \frac{(t^{1/3} - 1) (t^{1/3} + 1)}{24 t^2 + 36 t^2 + 18 t + 3 t}
\]
and $g(0) = g(1) = 0$, which implies that $g(t) \geq 0$ and $f(t) \geq t^{1/3}$.

In addition, from
\[
    \lim_{t \to 0} \frac{f(t^r)}{f(t)^r} = \lim_{t \to 0} \left( \frac{(1/3)t^r + (2/3)t^{r/3}}{(1/3) + (2/3)t^{r/3}} \frac{(1/3) + (2/3)t^{1/3})^r}{(1/3)t + (2/3)t^{1/3})^r} \right) = 2^{1-r} < 1,
\]
we have $f \notin \text{PMI}_r$.

Combining all the results stated above, we obtain the following:

Corollary 5.3. Let $\sigma$ be an operator mean and $f_\sigma$ be the representation function of $\sigma$. Consider the statements:

(I) $f_\sigma$ is geometrically convex.
(II) $A,B > 0, A\sigma B \geq I \Rightarrow A^r \sigma B^r \geq I$ for all $r > 1$.
(III) $A,B > 0, A\sigma B \geq I \Rightarrow A^r \sigma B^r \geq I$ for some $r > 1$.
(IV) $\sigma \geq \#_\alpha$ for some $\alpha \in [0,1]$.

Then

(1) I implies II; II implies III; III implies IV,
(2) IV does not imply III; II does not imply I.

Thus a problem arises.

Problem 1. Let $r > 1$. Then, $\text{PMI} = \text{PMI}_r$?

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