Equilibration and Generalized GGE in the Lieb Liniger gas

G. Goldstein and N. Andrei
Department of Physics, Rutgers University, Piscataway, New Jersey 08854

We study the nonequilibrium properties of the 1-D Lieb-Liniger model in the thermodynamic limit for finite repulsive coupling. For this purpose we introduce a new version of the Yudson representation applicable to finite size systems and obtain the thermodynamic limit by appropriately taking the infinite volume - at constant density - limit. We provide a formalism to compute various correlation functions for highly non-equilibrium initial states. In the strong coupling limit we are able to find explicit analytic expressions for the expectation of the density, density density and related correlation functions at arbitrary times. We present our result as a power series expansion in inverse coupling strength. We show that the gas equilibrates to a steady state from arbitrary initial states with "smooth" correlation functions. For nearly translationally invariant states the gas equilibrates to a diagonal ensemble which we show is equivalent to a generalized version of the GGE for sufficiently simple correlation functions, which in particular include density density correlations.

Introduction. Non-equilibrium processes can be found in many diverse fields ranging from biology to metallurgy to quantum chemistry. In all these instances it is of interest to study the evolution of a system away from thermodynamic equilibrium. In theoretical physics non-equilibrium processes play an important role, underlying statistical mechanics, transport theory, linear and nonlinear response theory. One of the key questions under study is the equilibration of a macroscopic system initiated far from thermal equilibrium, see e.g. \cite{1,2}. The study of equilibration phenomena has recently received a boost from the field of cold atoms, where quench experiments can be carried out in a highly controlled manner: by a judicious use of external lasers one creates a system in well defined state $|\Phi (t = 0)\rangle$. Then through a rapid change in the parameters of the system, in the cold atom setup this would typically correspond to an adjustment of the external lasers and magnetic fields, the experimenter is able to modify the dynamics of the system, create a new effective Hamiltonian $H$ and induce time evolution: $|\Phi (t = 0)\rangle \rightarrow e^{-iHt} |\Phi (t = 0)\rangle$. This way one may study interesting correlation effects and equilibration properties. Furthermore the final state of the system, say its density density correlation function, can be effectively measured through time of flight experiments and absorption imaging \cite{3,4}.

Many of the systems currently under such study may be well described by integrable models, ones with an infinite number of conserved quantities \cite{5,6}. The eigenstates $|k\rangle$ of such models are exactly known and are parameterized by a a set of quantum variables, rapidities, $\{k_i\}$. The equilibration or lack thereof of some local observable $\Theta$ is captured by the time evolved expectation value \cite{7,21}:

$$\langle \Theta (t) \rangle \equiv \langle \Phi (t = 0) | e^{itH} \Theta e^{-itH} | \Phi (t = 0) \rangle = \sum_q \sum_k \langle \Phi (t = 0) | k \rangle \langle k | \Theta | q \rangle \times \langle q | \Phi (t = 0) \rangle e^{i(E_q - E_k) t}$$  \hspace{1cm} (1)

Here $|k\rangle$ and $|q\rangle$ are complete sets of states and $E_k$ and $E_q$ are their respective energies \cite{22}.

For translationally invariant systems, be they integrable or not, it has been conjectured \cite{23,24,25} that at large times, $t \to \infty$, the expectation value in Eq. (1) equilibrates to a diagonal ensemble, where only entries with $|k\rangle = |q\rangle$ contribute. For non-integrable systems it has been further conjectured, the ETH conjecture \cite{24,25}, that the expectation value depends smoothly on the energy of the state $\{k\}$ and on no other parameters, $\langle k | \Theta | k \rangle = F(E_k)$. This conjecture leads directly to the fact that the long time limit of $\langle \Theta \rangle$ may be computed in the microcanonical ensemble. For integrable systems there is a different conjecture, the GGE hypothesis \cite{13}, that the correlation function may be computed using the GGE density matrix: $\rho_{GGE} = Z^{-1} \exp \left[ - \sum_m \alpha_m I_m \right]$. Here $I_m$ is the full set of commuting integrals of motion with $|k\rangle$ being their common eigenstates, $I_m |k\rangle = \sum_n k_n^m |k\rangle$, $Z = Tr[\exp (- \sum \alpha_m I_m)]$ is the partition function and $\{\alpha_m\}$ are Lagrange multipliers fixed by the initial conditions $Tr[I_m \rho_{GGE}] = \langle I_m \rangle (t = 0)$. The expectation value of any local observable at large times is conjectured to be given by $\langle \Theta (t \to \infty) \rangle = Tr[\rho_{GGE} \Theta]$.

In this paper we examine these issues in the context of the Lieb-Liniger model and provide explicit results for finite strong coupling as an expansion in $1/c$, where $c$ is the lieb-liniger coupling constant, see Eq. (3). We shall study the system in the thermodynamic limit - where the system size $L \to \infty$, the number of particles $N$ scales with the system size $N/L = const$, and for times much less then the system size, $t < L/v_{typ}$ ($v_{typ}$ is a typical velocity). We show explicitly (1) that the system equilibrates at long times, (2) that it is then described by a diagonal ensemble and (3) that this ensemble is a generalized GGE, defined as:

$$\hat{\rho}_{GGE} = \tilde{Z}^{-1} \exp \left[ - \sum_{m_1 m_2 \ldots} \alpha_{m_1 m_2 \ldots} I_{m_1} I_{m_2} \ldots \right] \hspace{1cm} (2)$$

Here $\tilde{Z} = Tr[\exp (- \sum \alpha_{m_1 m_2 \ldots} I_{m_1} I_{m_2} \ldots)]$ with the Lagrange multipliers $\{\alpha_{m_1 m_2 \ldots}\}$ fixed by the initial conditions $Tr[I_{m_1} I_{m_2} \ldots \hat{\rho}_{GGE}] = \langle I_{m_1} I_{m_2} \ldots \rangle (t = 0)$. We note that products of the $I_m$ as used in Eq. (2) are also conserved albeit nonlocal quantities. We show below that when the initial state contains only short range correlations the generalized GGE reduces to the usual GGE.
The nonlocality of the generalized GGE thus reflects the long range correlation of the initial state when present. As we show below the generalized GGE appears when the long time limit of ensemble is diagonal and the expectation value of a generic operator \( \Theta \) may be Taylor expanded in the rapidities:

\[
\langle k| \Theta |k \rangle = c_0 + c_1 \sum k_i + c_{1,1} \sum k_i k_j + c_2 \sum k_i^2 + ... 
\]

We establish the validity of this Taylor expansion for specific operators below. We note that no such expansion may be made for non-integrable models as there is no convenient set of rapidities to parameterize them, see Fig. 1. For example for an electron gas with coulomb interactions the eigenstates are complex combinations of products of single particle states whose only constraint is to have the same total momentum and energy. We finally note that for strongly non-translationally invariant systems, such as those with domain walls, the system never equilibrates in the thermodynamic limit and in particular it does not attain the diagonal ensemble. Such a state will be studied below as an example.

The simplest model that describes the dynamics of strongly correlated 1-d bosons, and which can also be realized in the lab is the Lieb-Liniger Hamiltonian,

\[
H_{LL} = \int_{-L/2}^{L/2} dx \left\{ \partial_x b^\dagger(x) \partial_x b(x) + c \left( b^\dagger(x) b(x) \right)^2 \right\} 
\]

Here \( b^\dagger(x) \) is the bosonic creation operator at the point \( x \). The model is integrable \(^{27}\) and has infinitely many conserved quantities. For the purposes of this work we shall assume repulsive interactions, \( c > 0 \). This parameter may be experimentally tuned in real time via a Feshbach resonance. The equilibration of this system starting from a highly excited state is one of the most interesting properties of interacting bosonic many body systems and it has been extensively studied \(^{3\,21}\).

There are varied techniques to study analytically the quench dynamics and equilibration of such models. Most of these are based on the fact that it is possible to find exact eigenstates of the many body Hamiltonian, decompose initial states in terms of these eigenstates and then time evolve, see Eq. (1). As such, from the theory side, the study of the evolution and the equilibration, of correlation functions for an initial state may be decomposed into four steps. (1) The computation of the eigenstates \( |k \rangle \) of an exactly integrable system; which may be done using co-ordinate Bethe Ansatz techniques \(^{5\,25}\). (2) The computation of various overlaps \( \langle \Phi(t=0)|k \rangle \); the Yudson representation, which we shall extend to finite size systems, is an efficient technique for accomplishing almost that. (3) The computation of various matrix elements for local operators \( \langle k|\Theta|q \rangle \). (4) Summation (or integration, in the thermodynamic limit) over various intermediate states which for the strong coupling limit, as we shall show, may be converted into a computation of an appropriate correlation function with respect to the initial state.

In this paper we describe how to accomplish these four steps for generic initial states, be they translationally invariant or not, with short or long range correlations. We present generic formulas for the correlation functions for these states at arbitrary times in terms of correlation functions of the initial state. We show that in the long time thermodynamic limit the correlation functions equilibrate and for translationally invariant initial states they equilibrate to a diagonal ensemble, that is \( |k \rangle = |q \rangle \) in Eq. (1) above. We shall also show the validity of the GGGE hypothesis. As far as the authors are aware this is the first analytic proof of equilibration, diagonal ensemble or GGGE for the interacting case see however \(^{29}\).

**Yudson decomposition** provides an efficient way to compute overlaps. We extend previous studies \(^{30\,32}\) to finite size systems, an extension that is necessary to reach the thermodynamic limit. The Yudson decomposition for a \( N \)-particles state on a ring of length \( L \) may be described as follows: supposing that our initial state may be written as an integral over a wave function localized in the first quadrant of our co-ordinate space \( x_1 < x_2 < ... < x_N \) (from Bose symmetry it follows that every function may be written like that):

\[
|\Phi_N(t=0)\rangle = \int_{-L/2}^{L/2} dx_N \int_{-L/2}^{L/2} dx_{N-1}... \int_{-L/2}^{L/2} dx_1 \times \Phi(x_1, x_2, ..., x_N) b^\dagger(x_N) ... b^\dagger(x_1) |0 \rangle,
\]

then we claim this can be written as a sum of Bethe ansatz eigenstates of the form:

\[
|\Phi_N(t=0)\rangle = \sum_{\ell_1=0}^{\infty} ... \sum_{\ell_N=0}^{\infty} N^{-1} (k_{\ell_1} ... k_{\ell_N})^{-1} \times |k_{\ell_1} ... k_{\ell_N}\rangle |k_{\ell_1} ... k_{\ell_N}\rangle |\Phi_N(t=0)\rangle.
\]

Here \( |k_{\ell_1} ... k_{\ell_N}\rangle \) is a Bethe ansatz eigenstate:

\[
\Phi_{\ell_1}^{\ell_2} ... \Phi_{\ell_N}^{\ell_1} \times \prod_{i < j} Z_{\ell_i - \ell_j} (k_i - k_j) \cdot \prod e^{ik_i y_i} \cdot \prod b^\dagger(y_i) |0 \rangle
\]

where the scattering factor \( Z_{\ell_i} (K) = \frac{K+ic(1-2\alpha Y_i)}{K-ic} \) incorporates the S-matrix, \( S_{ij} = \frac{k_i-k_j+ic}{k_i-k_j-ic} \). With
periodic boundary conditions the momenta \(|k_n|\) must satisfy the Bethe ansatz equations: \(k_n = \frac{2\pi}{L} n_i - 2 \sum_{i=1}^N \arctan \left( \frac{2\pi}{L} \frac{k_i - k_{i-1}}{c} \right), \forall i, \) with \(n_i\) (half) integers. We denote: \(|k_{n_1},...,k_{n_N}| = \prod \sum_{i} \prod \sum_{b(y)|}\). The inner product is taken over one quadrant only (\(| \langle \rangle | = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \int_{-L/2}^{L/2} dx'' \text{ Normalization of the wave function is } N(k_{n_1},...,k_{n_N}) = \det (M_{jk}) \text{ with } M_{jk} = \delta_{jk} \left( L + \sum_{i=1}^N \frac{2c}{c+(k_{j}-k_{i})^2} \right) - \frac{2c}{c+(k_{j}-k_{i})^2}, \text{ see } [5]. \) In the limit \( L \to \infty \) for a finite number of particles this simplifies to \( M_{jk} = L \delta_{jk} \) and the identity: \( N(k_{n_1},...,k_{n_N}) = L^{N} \). The density of states becomes \( \left( \frac{4\pi}{2L} \right)^N \). The proof of the Yudson resolution of the identity \( \mathbb{I}_N = \sum_{n_1,n_2,...,n_N} \frac{1}{N(k_{n_1},...,k_{n_N})} \langle k_{n_1},...,k_{n_N} \rangle \langle k_{n_1},...,k_{n_N} \rangle \) follows from the standard resolution: \( \mathbb{I}_N = \sum_{n_1 < n_2 < ... < n_N} N(k_{n_1},...,k_{n_N}) \langle k_{n_1},...,k_{n_N} \rangle \langle k_{n_1},...,k_{n_N} \rangle \) and the identity: \( \langle k_{n_1},...,k_{n_N} \rangle = \sum_{S \in S_N} \prod_{i,j \in P} S' \langle k_i, k_j \rangle \) valid in the first quadrant. \[ (6) \]

Greens Functions. In order to calculate the expectation values of \( \langle \Theta(t) I \rangle \), see Eq. (1), we wish to calculate the value of operator Greens functions in the basis states:

\[ G(\Theta, t; x_1,x_2,...,x_N; y_1,...,y_N) = \langle 0| \theta(y_1) \theta(y_2)... \theta(y_N) \Theta(t) b^\dagger(x_1) ... b^\dagger(x_N) |0 \rangle \]

This allows a basis for calculating the expectation \( \langle \Theta(t) I \rangle \) with any initial and final states \( \Psi, \Phi \) since:

\[ \langle \Psi | \Theta(t) | \Phi \rangle = \int \sum dx_1 ... dx_N dy_1 ... dy_N \times G(\Theta, t; x_1,x_2,...,x_N; y_1,...,y_N) \Psi(x_1,...,x_N) \Phi^* (y_1...y_N) \]

Using the Yudson representation we may rewrite the Green’s functions in the form:

\[ G^{LL}(\Theta, t; x_1,x_2,...,x_N; y_1,...,y_N) = \sum_{n_1,n_2,...,n_N} \frac{1}{N(k_{n_1},...,k_{n_N})} \times \prod_{i,j} \langle k_{n_i} - k_{n_j} \rangle \times \langle 0| b(y_1) b(y_2)...b(y_N) | k_{n_1},...,k_{n_N} \rangle \times \langle B(k_{n_1}) ... B(k_{n_N}) \rangle \Theta \langle B(q_{n_1}) ... B(q_{n_N}) \rangle \times \sum_{n_1,n_2,...,n_N} \frac{1}{N(q_{n_1},...,q_{n_N})} \times \prod_{i,j} \langle q_{n_i} - q_{n_j} \rangle \times \langle q_{n_1},...,q_{n_N}| b^\dagger(x_1) ... b^\dagger(x_N) |0 \rangle \prod e^{i(q_{n_i} - q_{n_j})t} \]

(7)

We note that \( \langle 0| b(y_1) b(y_2)...b(y_N) | k_{n_1},...,k_{n_N} \rangle = \prod \exp(ik_{n_i}y_i) \) with \( x_1,...,x_N \) and \( y_1,...,y_N \) in the first quadrant. We will consider the operator \( \Theta = \exp(\alpha Q_{xy}) \equiv \exp(\alpha \int dV \theta(z)b^\dagger b(z)dz) \), from which all local density correlations functions can be obtained, e.g. \( \rho(x) \rho(y) = -\frac{1}{2\pi L} \exp(\alpha q_{xy}) \). As \( \langle B(k_{n_1}) ... B(k_{n_N}) \rangle \Theta \langle B(q_{n_1}) ... B(q_{n_N}) \rangle \) may be efficiently calculated see [5]. It is possible to know the exact correlators using only a finite Taylor expansion with respect to \( \alpha \). Expanding the expectation value with, after a considerable amount of algebra, in the thermodynamic limit the generating function is given by:

\[ \langle \exp(\alpha Q_{xy}) \rangle = 1 + \int dX \int dY F_{\alpha,xy}(X,Y,t) \exp(\alpha Q_{xy}) \exp(\alpha Q_{xy}) \langle b(x) b^\dagger(y) \rangle \]

(8)

Here we have introduced \( F_{\alpha,xy}(X,Y,t) \equiv \int \sum dx dy \exp(-\alpha q_{xy}) \times \frac{\exp(\alpha i(X-Y)\cdot x)}{2\pi X-Y} - \exp(\alpha i(X-Y)\cdot y) \times \frac{\exp(\alpha i(X-Y)\cdot y)}{2\pi X-Y} \]

(9)

We will like to show that for any initial state, for which various field correlation functions have convergent smooth Fourier transforms, the expectation value of \( \langle \exp(\alpha Q_{xy}) \rangle \) converges to a constant value in the thermodynamic limit. For this we will consider Eq. (9), and note that the Fourier transform of all the functions \( F/G_{\alpha,xy} \times G_{\alpha,xy} \) with respect to \( X,Y \) are proportional to \( \exp(\alpha i q_{xy}t) \). These terms are multiplied by the expectation values of the Fourier transforms of some correlations functions, e.g. the Fourier transform of the terms in the correlation functions \( \langle \rangle \) in Eq. (9). Call these Fourier transforms \( O_n(\{k_i\}, \{q_i\}) \). If \( O_n(\{k_i\}, \{q_i\}) \) is smooth then from the method of stationary phase we know that any integral containing this term is dominated by the point \( k_i = q_i = 0 \) and is proportional \( \alpha^\frac{n_i}{2} \).
it disappears in the long time limit. Therefore to get a non-zero result at long times we need to assume that the Fourier transform $\mathcal{O}_n \{ \{ k_i \} \}$ has singularities proportional to delta functions with the support of these singularities being the set where $\sum k_i^2 - \sum q_j^2 = \text{const}$. However when this condition is satisfied the expression for $\langle \exp(\alpha Q_{xy}(t)) \rangle$ explicitly has no time dependence and therefore equilibrates. Singularities in Fourier space are determined by the initial state and correspond to some order in it. For most initial states, e.g. translationally invariant states, lattices, superconducting order etc., these orderings are localized to $q$-vectors that lie in hyperplanes - not curved manifolds. For hyperplanes the condition $\sum k_i^2 - \sum q_j^2 = \text{const}$ simplifies to $k_i = \pm q_j$. Furthermore for states that have no ordering at non-zero total momentum, such as translationally invariant states, this constraint simplifies to $k_i = q_j$. In this case the long time thermodynamic limit is given by the diagonal ensemble. Indeed the condition $k_i = q_j$ implies that $\{ k_i \} = \{ q_j \}$ or the rapidities of the states used in the Yudson decomposition in Eq. 8 are the same. Our derivation applies also to states with crystal order, these do have ordering near the edge of the insulator (B) leading order correlation functions: (A) For a the Tonks Gas ($c \to \infty$) exact correlations near the edge of the insulator (B) leading order 1/c corrections for the edge of the insulator, the entire graph (B) should be multiplied by $\frac{1}{c}$. The initial state is chosen to have parameters $l = 1 \sigma = 0.01$ and the times are 0.001, 0.01, 0.02 and 0.05 for the Tonks Girardeau gas and 0.1, 0.2, 0.4 and 1 for the 1/c corrections.

Figure 2: For initial state a Mott insulator on the half line: correlation functions: (A) For a the Tonks Gas ($c \to \infty$) exact correlations near the edge of the insulator (B) leading order 1/c corrections for the edge of the insulator, the entire graph (B) should be multiplied by $\frac{1}{c}$. The initial state is chosen to have parameters $l = 1 \sigma = 0.01$ and the times are 0.001, 0.01, 0.02 and 0.05 for the Tonks Girardeau gas and 0.1, 0.2, 0.4 and 1 for the 1/c corrections.

The expectation value of any Taylor expandable operator - which includes all correlation functions of densities $\rho$ may be computed in GGGE. It is clear that the GGGE density operator takes the form $\hat{\rho} = \int d k f(k) |k\rangle \langle k|$ while the GGE density matrix corresponds to a single eigenvalue of the Lieb-Liniger gas $\sigma$. Thus for the GGE $\langle I_m, I_{m_2}, ..., I_{m_l} \rangle = \langle I_m \rangle \langle I_{m_2} \rangle ...$ and in particular

$$\langle \Theta (t \to \infty) \rangle - \langle \Theta_{\text{GGGE}} \rangle = c_{1,1} \left( \langle I_2^2 \rangle - \langle I_1 \rangle^2 \right) + c_{1,2} \left( \langle I_1 I_2 \rangle - \langle I_1 \rangle \langle I_2 \rangle \right) + ...$$

(10)

implying that long range correlations are not captured by GGE. An example of an initial state that does not have $\langle I_2 \rangle = 0$ is $\hat{\rho}_T = \frac{1}{T} \int_0^T \frac{e^{-H_{LL}/\varepsilon}}{Z_t} dt$ and it is also possible to construct examples involving pure states$[37]$. However for many initial states $\lim_{|x-y| \to \infty} \langle J_i (x) \rangle$ for $|x-y| \to \infty$, $\langle J_i (x) \rangle$, $\langle J_i (y) \rangle$, etc. in which case the GGGE reduces to the GGE. Here $J_i (x)$ are the local densities corresponding to $I_i$, $I_i = \int J_i (x)$. As such we have shown that for translationally invariant systems correlations of densities equilibrate to the diagonal ensemble so they automatically equilibrate to the GGGE.

Examples of initial states. We would like to apply our formalism to some interesting initial states. We shall study a quench from an initial state whose average density is of the form of a domain wall, see Fig. 2, and study the density evolution as a function of time $\rho (x, t)$. Choosing an initial state that is a Mott insulator on the half line: $|\Psi (t = 0)\rangle = \prod_{j=0}^{\infty} \varphi (x + j) b^\dagger (x) |0\rangle$, with $\varphi (x) = \frac{e^{-x^2/\sigma}}{(\pi \sigma / 2)^{1/4}}$, will allow us to demonstrate both equilibration by considering $x \to -\infty$ and nonequi-
librium ballistic transport physics for $|x| \sim t$. For $x \to -\infty$ in the Tomk's Girardeau regime it is possible to compute the density exactly. It is given by $\rho(x,t) = \frac{1}{t} \left( 1 + \sum_{i=1}^{\infty} e^{-\pi^2 (8t^2/\sigma + \sigma/2) x^2 / t^2} \cos(2\pi x \sigma / t) \right)$. At large times this corresponds to a constant density plus exponentially decaying small oscillations, see Fig. 1. The $1/t$ correction corresponds to exponentially decaying small oscillations. For $x \sim t$, ignoring exponentially decaying small oscillating terms, we have that the density is given by:

$$\rho(x,t) = \frac{1}{t} \left( \frac{1}{2} \text{Erfc} \left( \frac{x - A}{\sqrt{A}} \right) + \frac{16}{\pi s} e^{-x^2 / A} \left( \frac{1}{2} \sqrt{\pi} \frac{s}{A} \text{Erfc} \left( \frac{x - A}{\sqrt{A}} \right) - \frac{1}{2} e^{-x^2 / A} \right) \right)$$

(11)

with $A \equiv 8t^2/\sigma + \sigma/2$. We notice that for $x \ll -\sqrt{8t^2/\sigma}$ the density becomes $1/l$ or its equilibrium value while for $x \gg \sqrt{8t^2/\sigma}$ the density becomes zero corresponding to no particles having reached our observation point and for $|x| \ll t$ the density becomes $\frac{1}{t} \left( \frac{1}{2} + \frac{x^2}{2A} \right)$. From this we see that ballistic transport with signal velocity $\sim \sqrt{A}$. We note that technically this system never equilibrates, for large enough $x$ there is always time dependance, so in particular it does not attain the GGE. A translationally invariant initial state, e. g. $|\Psi(t=0)\rangle = \prod_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x + il) b^\dagger(x) |0\rangle$ equilibrates to a GGE, since $\langle I_i I_j \rangle = \langle I_i \rangle \langle I_j \rangle$.

Conclusions. By studying the relation between differently ordered Bethe eigenstates and decomposing a Bethe eigenstate appropriately in terms of plane waves we have introduced a new type of Yudson representation, valid for finite sized systems. We have shown how it can be used to study the quench dynamics, in particular equilibration, of macroscopic systems in the thermodynamic limit. We have introduced some techniques for writing time dependent observables in terms of Green’s functions and initial correlation functions and we have demonstrated how to take the long time thermodynamic limit of these functions. We have used this technique to study the dynamics of the Lieb Liniger gas and showed that for translationally invariant systems the gas equilibrates to GGE. We are currently applying our approach to the XXZ model, the Gaudin-Yang model and to the Hubbard models.

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matrix elements of the ABA transfer matrix. Recall that we may write that the transfer matrix between \(-L/2\) and \(L/2\) in the form \(T(k) \equiv \)

\[
\begin{pmatrix}
A(k) & B(k) \\
C(k) & D(k)
\end{pmatrix} = L(n \mid k) L(n-1 \mid k) \ldots L(1 \mid k)
\]

(12)

Here \(L(m \mid k) =: \)

\[
\begin{pmatrix}
1 - \frac{\Delta}{\Delta} & -i \sqrt{\Delta} (\nu \nu - L/2) \\
i \sqrt{\Delta} (\nu \nu - L/2) & 1 + \frac{\Delta}{\Delta}
\end{pmatrix}
\]

(13)

with \(\Delta = L/n\) and take the limit \(n \to \infty\). Using this we can show that the normalization ratio between the two types of eigenstates is given by:

\[
\frac{B(k_1) \ldots B(k_N) \langle 0 \rangle}{\langle k_1, \ldots, k_N \rangle} = (-i \sqrt{\Delta} \nu \nu \prod_{i<j} (k_j - k_i - i \nu )) \prod_{i<j} (k_j - k_i)
\]

(14)

\[\text{[35]}\text{This is also a feature of all higher order terms in the expansion.}\]

\[\text{[36]}\text{N. G. de Bruijn, Asymptotic methods in analysis, (Dover Publication Inc. 1981).}\]

\[\text{[37]}\text{G. Goldstein and N. Andrei in preparation}\]

\[\text{[38]}\text{J. Mossel, J.-S. Caux, J. Phys. A: Math. Theor. 45, 255001, (2012).}\]

### Supplementary online information

We now wish to extend the Yudson decomposition for repulsive bosons to systems of a finite size. That is we would like to prove that for a finite sized system with periodic boundary conditions there is an alternative resolution of unity:

\[
\mathbb{I}_N = \sum_{n_1, n_2, \ldots, n_N} \frac{1}{\mathcal{N}(k_{n_1}, \ldots, k_{n_N})} \langle k_{n_1}, \ldots, k_{n_N} \rangle \langle k_{n_1}, \ldots, k_{n_N} \rangle | k_{n_1}, \ldots, k_{n_N} \rangle \]

(15)

Here the normalization factor \(\mathcal{N}(k_{n_1}, \ldots, k_{n_N}) = \langle k_{n_1}, \ldots, k_{n_N} \rangle | k_{n_1}, \ldots, k_{n_N} \rangle\), the rapidities \(k_i\) are assumed to satisfy the Bethe ansatz equation, the ket \(| k_{n_1}, \ldots, k_{n_N} \rangle\) is an exact eigenstate of the Lieb Linger hamiltonian, the bra \(\langle k_{n_1}, \ldots, k_{n_N} \rangle\) is a plane wave state with \(\langle k_{n_1}, \ldots, k_{n_N} \rangle | x_1, \ldots, x_N \rangle = \prod e^{-ik_i x_i}\) and the inner product is over the first quadrant e.g. \(\langle \Phi | \Psi \rangle = \int_{-L/2}^{x_N} \int_{-L/2}^{x_N} \Phi^* (x_1 \ldots x_N) \Psi (x_1 \ldots x_N)\). The representation will be based on the following resolution of identity:

\[
\mathbb{I}_N = \sum_{n_1 < n_2 < \ldots < n_N} \frac{1}{\mathcal{N}(k_{n_1}, \ldots, k_{n_N})} | k_{n_1}, \ldots, k_{n_N} \rangle \langle k_{n_1}, \ldots, k_{n_N} | \]

(16)

which comes from the completeness of the bethe ansatz eigenstates. To show that the two resolutions Eqs. (15) and (16) are equal we will need to recall some facts about the eigenstates given in Eq. (5). First in the quadrant \(x_1 < x_2 < \ldots < x_N\) they have a simple form:

\[
| k_{n_1}, \ldots, k_{n_N} \rangle = \sum_{P \in S_N} \prod_{(i,j) \in P} S(k_i, k_j) | k_{P_1}, \ldots, k_{P_N} \rangle
\]

(17)

The second observation we need to make is that

\[
\frac{| k_{P_1}, \ldots, k_{P_N} \rangle}{| k_1, \ldots, k_N \rangle} = \prod_{i,j \in P} S^{-1}(k_i, k_j) = \prod_{i,j \in P} S^*(k_i, k_j)
\]

(18)

which may be derived by considering the ratio of the components of \(| k_{P_1}, \ldots, k_{P_N} \rangle\) of the two states \(| k_1, \ldots, k_N \rangle\) and \(| k_{P_1}, \ldots, k_{P_N} \rangle\). Now in order to prove that the two resolutions of unity in Eqs. (15) and (16) are equal we need to show that:

\[
| k_{n_1}, \ldots, k_{n_N} \rangle \langle k_{n_1}, \ldots, k_{n_N} | = \sum_{P \in S_N} | k_{P_1}, \ldots, k_{P_N} \rangle \langle k_{P_1}, \ldots, k_{P_N} |
\]

(19)

in the quadrant \(x_1 < \ldots < x_N\). However this follows directly from Eqs. (17) and (18) and the resolution of unity follows.