Gravitational waves from \( f(R) \) theory and its detection using spherical antenna

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Abstract. Gravitational waves generated from a metric \( f(R) \) theory of the form \( f(R) = R + \lambda R^2 \) are studied. Two solutions for the metric perturbation, one corresponding to massless mode as in General Theory of Relativity and the other a massive mode, are obtained. The massive mode is found to follow non-null geodesics indicating that the gravitational wave arising from such a mode travels with a speed less than that of light. The detection of the massive mode of the gravitational waves using spherical antenna is studied. The response of the antenna to massive modes show better energy sensitivity and high directionality compared to the ordinary gravitational waves from General Theory of Relativity.

1. Introduction

The fact that General Theory of Relativity(GTR) is not renormalizable and the current scenario of the accelerating universe call for searching new theories of gravitation. \( f(R) \) theories are one such attempt[1]. \( f(R) \) theories come out as a straightforward generalization of the Einstein-Hilbert action of GTR and can be written as,

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R),
\]

where \( \kappa = 8\pi G(c=1) \), \( G \) is the gravitational constant, \( g \) is the determinant of the metric \( g_{\mu\nu} \) and \( R \) is the Ricci scalar. \( f(R) \) is a function of \( R \) and in the present study we write \( f(R) \) as \( f(R) = R + \lambda R^2 \), where \( \lambda \) is a constant.

To be compatible with special relativity, gravity must be causal, i.e., any change to a gravitating source must be communicated to distant observers with a speed no faster than the speed of light. This leads immediately to the idea that there must exist some notion of “gravitational radiation”[2]. The objects massive and relativistic enough to generate detectable Gravitational Waves(GWs) are astrophysical. There are no direct observations of GWs till today. Of the detectors designed to search for GWs, interferometers and resonators are the simplest ones. “Spherical antenna detectors” whose origin date back to the 1970s[3] are receiving lot of attention even now.

The theoretical framework for the detection of GWs from GTR using spherical antennas had already been well developed. This paper deals with the production of GWs as a result of using the modified action (1). A massive mode in addition to the massless mode appears here. The massive mode of Gravitational Waves(GWs) could be thought of as a suitable candidate for
dark matter present in galaxies. The feasibility of spherical antenna detection of massive GWs is also discussed.

2. Production of GWs
In order to find out the GWs arising from \( f(R) \) theory, the modified action (1) is varied with respect to the metric alone and the resulting field equation is given by

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) f'(R) = \kappa T_{\mu\nu}. \tag{2}
\]

Substituting for \( f(R) \) in (2) and taking trace we find,

\[6\lambda \Box R - R = \kappa T. \tag{3}\]

Using the linearization condition \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and also making use of trace of the reversed perturbation \( \bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h \) and using the De Donder gauge \( \partial^2 \bar{h}_{\alpha\beta} = 0 \), we get

\[-6\lambda \Box^2 \bar{h}_{\mu\nu} + \Box \bar{h}_{\mu\nu} = \kappa T_{\mu\nu}. \tag{4}\]

Considering the vacuum situation, we get two solutions

\[\bar{h}_{\mu\nu} = \int A(k) \exp(-i(k.r - \omega t))dk, \tag{5}\]

\[\bar{h}_{\mu\nu} = \int A(k) \exp(-i(k.r - \omega_m t))dk. \tag{6}\]

The general solution is a sum of solutions (5) and (6) and \( \omega \neq \omega_m \). Equation (5) is the one that we usually get from general theory of relativity. Thus we have the usual general relativity solution and an additional solution containing the term \( m^2 \). Also in this case we have \( k^\alpha k_\mu = m^2 \).

Thus \( k^\mu \), the propagation vector, is no longer a null vector. Hence we cannot expect the massive GWs to travel with the speed of light.

3. Detection of massive modes using spherical antenna

Antennas generally consisted of an elastic body which may become deformed when gravitational radiation falls on it and thereby its normal modes get excited. Such an antenna measures precisely the Fourier transform of certain components of the Riemann curvature tensor averaged over its volume [3]. The present calculations of spherical antenna detection are done following Lobo [4] and Gasparini [5].

The five fundamental quadrupole modes of a spherical antenna are given by

\[
\bar{h}^\alpha(t) = \frac{8 \pi}{15} M^\alpha_{ij} R_{0ij0}, \tag{7}
\]

with \( M^\alpha_{ij} \) given by

\[M^{1c}_{ij} = \sqrt{\frac{15}{16\pi}} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right), \quad M^{2c}_{ij} = \sqrt{\frac{15}{16\pi}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right), \]

\[M^{1s}_{ij} = \sqrt{\frac{15}{16\pi}} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right), \quad M^{2s}_{ij} = \sqrt{\frac{15}{16\pi}} \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad M^{0}_{ij} = \sqrt{\frac{15}{16\pi}} \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{array} \right). \]

By finding \( R_{0ij0} \), \( \bar{h}^\alpha(t) \) for each mode \( \alpha \) can be obtained. Then, the energy stored in the mode \( \alpha \) at frequency \( \omega_{nl} \) is given by [5]

\[E_s(n, l, \alpha) = \frac{1}{2} a_n^2 c \int |\bar{h}^\alpha(t)e^{-i\omega_{nl}t}|^2 dt, \tag{8}\]

where \( a_n \) is some quantity independent of \( t \). \( n \) is a positive integer which represents the energy level for a fixed angular momentum \( l \). It is found that the mode with \( l = 2 \) alone contributes. If we fix the value of \( n \) and replace \( \bar{h}^\alpha \), the energy stored for each mode \( \alpha \) and hence the average energy stored can be found out. The rate of energy stored can be found out by calculating the ratio of energy stored in each mode to the average energy stored. This gives the energy sensitivity of the detector (antenna).
The metric perturbation from (5) and (6) can be written as [6]
\[ h_{\mu\nu}(t, z) = h_+(t - z)\epsilon_{\mu\nu}^{(+)} + h_\times(t - z)\epsilon_{\mu\nu}^{(\times)} + h_3(t - v_g z)\epsilon_{\mu\nu}^{s}, \]
where \( \epsilon_{\mu\nu}^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \epsilon_{\mu\nu}^{(\times)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \epsilon_{\mu\nu}^{s} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Riemann tensor from our modified theory may be written as [7]
\[ R_{ij00} = \frac{1}{2} \begin{pmatrix} \tilde{h}_+ & \tilde{h}_x & 0 \\ -\tilde{h}_x & 0 & 0 \\ 0 & 0 & m_g^2 h_s \end{pmatrix}. \]

Converting the Riemann tensor to the detector frame, for a generic arrival direction given by the angle \((\theta, \phi)\), the components of \( \tilde{h}_{\alpha}(t) \) can be written as [5]
\[ \tilde{h}_0(t) = \sqrt{\frac{2}{3}}(\sin^2 \theta \tilde{h}_+ + m_g^2 \cos^2 \theta \tilde{h}_s), \]
\[ \tilde{h}_1(t) = \sqrt{\frac{2}{3}}(-\sin \theta \cos \theta \cos \phi \tilde{h}_+ + \sin \theta \sin \phi \tilde{h}_x - m_g^2 \sin \theta \cos \theta \cos \phi \tilde{h}_s), \]
\[ \tilde{h}_2(t) = \sqrt{\frac{2}{3}}(\sin \theta \cos \theta \sin \phi \tilde{h}_+ - \sin \theta \cos \phi \tilde{h}_x + m_g^2 \sin \theta \cos \theta \sin \phi \tilde{h}_s), \]
\[ \tilde{h}_3(t) = \sqrt{\frac{2}{3}}(\cos^2 \theta \sin^2 \phi \tilde{h}_+ - \cos \theta \sin \phi \sin \tilde{h}_x + \sin \phi \cos \phi \tilde{h}_s), \]
\[ \tilde{h}_4(t) = \sqrt{\frac{2}{3}}(\cos^2 \theta \sin^2 \phi \cos \phi \tilde{h}_+ + \cos \theta \sin \phi \cos \tilde{h}_x + m_g^2 \sin^2 \theta \sin \phi \cos \phi \tilde{h}_s). \] (9)

Then for a source emitting randomly polarized radiation, the average energy for each mode can be written as
\[ E_α(n, 2, 0) = \frac{π a_2^2}{15} \left[ \frac{3}{2} \sin^4 \theta \omega_{n2}^4 (|\tilde{e}_+|^2 + |\tilde{e}_x|^2) + m_g^4 \cos^4 \theta |\tilde{e}_s|^2 \right], \]
\[ E_α(n, 2, 1c) = \frac{π a_2^2}{15} \left[ \omega_{n2}^4 \sin^2 \theta (\cos^2 \theta \cos^2 \phi + \sin^2 \phi (|\tilde{e}_+|^2 + |\tilde{e}_x|^2)) (m_g^4 \sin^2 \theta \cos^2 \theta \cos^2 \phi) \right], \]
\[ E_α(n, 2, 1s) = \frac{π a_2^2}{15} \left[ \omega_{n2}^4 (\cos^2 \theta \sin^2 \phi + |\tilde{e}_+|^2 + |\tilde{e}_x|^2) + (m_g^4 \sin^2 \theta \cos^2 \theta \sin^2 \phi |\tilde{e}_s|^2) \right], \]
\[ E_α(n, 2, 2c) = \frac{π a_2^2}{15} \left[ \omega_{n2}^4 ((1 + \cos^2 \theta)^2 (\cos^2 \phi - \frac{1}{2})^2 + 4 \cos^2 \theta \sin^2 \phi \cos^2 \phi) \right], \]
\[ \tilde{E}_α(n, 2, 2s) = \frac{π a_2^2}{15} \left[ \omega_{n2}^4 ((1 + \cos^2 \theta)^2 \cos^2 \phi \cos^2 \phi + \cos^2 \theta \sin^2 \theta (\cos^2 \phi - \sin^2 \phi) \right) \left(|\tilde{e}_+|^2 + |\tilde{e}_x|^2\right) + (m_g^4 \sin^4 \theta |\tilde{e}_s|^2) \right], \] (10)

with \( h_+ = \cos(\psi) e_+ + \sin(\psi) e_\times, \)
\[ h_\times = -\sin(\psi) e_+ + \cos(\psi) e_\times, \]
\[ h_3 = \epsilon_3, (h_3 has no \psi dependence). \] (11)

It is clear from these equations that the total average energy is not a constant [8]. It varies with \( \theta \) and \( \phi \). Considering only those contributions from the massive mode, the total average energy is given by
\[ \tilde{E}_α(n, 2, tot) = \frac{2π a_2^2 m_g^4}{15} \left[ \frac{3}{4} \cos^2 \theta + \cos^2 \phi \sin^2 \phi (\frac{1}{4} - \cos^2 \phi) + \sin^2 \theta \cos^2 \phi \right] |\tilde{h}_s(\omega_{n2})|^2. \] (12)

This indicates that the total sensitivity of the detector does not remain the same always and it depends on the GW direction. The variation of energy stored in each mode \( \alpha \) with \( \theta \) and \( \phi \) are shown in the plots (Fig. 1). The energy sensitivity is defined as \( \epsilon_α = \frac{\tilde{E}_α(n, 2, α)}{\tilde{E}_α(n, 2, tot)} \). The plots show a comparison of total energy transferred from GWs to the sphere between the + and \times polarizations and the scalar polarization in the order of modes 1c, 1s, 2c, 2s, 0. The light portion(yellow) indicates a maximum and the dark portion(blue) indicates minimum. The case of + and \times polarizations have already been done in the literature. Here, in the massive case, the maximum has been shifted. The region where it showed a maximum before is no more a region of maximum sensitivity. For instance, the rate \( \epsilon_α \) for the mode \( α = 0 \) has a maximum value \( \frac{3}{4} \) in the massless case, whereas for the massive component, the maximum value is 1. It is then clear that the rate has been increased by 25% which can be recognized easily if GWs
are detected using a spherical antenna. The total sensitivity for + and × polarizations was a constant no matter what the direction of the GWs is. But here the total sensitivity becomes direction dependent. We can see from the plots that the energy sensitivity of each mode has been increased compared to those of GTR.

Figure 1. Comparision of $\epsilon_\alpha$ of GW from GR(left) with that from f(R)(right) with $\epsilon_{1c}$, $\epsilon_{1s}$, $\epsilon_{2c}$, $\epsilon_{2s}$, $\epsilon_0$ respectively from top left

4. Conclusion
In this paper, GWs from $f(R)$ gravity with $f(R)$ of the form $R + \lambda R^2$, $\lambda$ being some unknown constant, is obtained. A massive mode in addition to the massless mode as in the case of GTR is obtained. This mode is found to propagate with a speed less than that of light. The detection of massive mode using spherical antenna detector has been analyzed. It showed an increase in energy sensitivity of the antenna. Also, the antenna is found to become highly directional towards such a massive component and if so, such a massive component can be easily recognizable and will be detectable if ever possible.

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6. References
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