THIN-SHELL CONCENTRATION FOR ZERO CELLS OF
STATIONARY POISSON MOSAICS

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Abstract. We study the concentration of the norm of a random vector $Y$ uniformly sampled in the centered zero cell of two types of stationary and isotropic random mosaics in $\mathbb{R}^n$ for large dimensions $n$. For a stationary and isotropic Poisson-Voronoi mosaic, $Y$ has a radial and log-concave distribution, implying that $|Y|/E(|Y|^2)^{1/2}$ approaches one for large $n$. If we assume that the centroids of the mosaic have intensity scaling like $e^{n\lambda}$, then $|Y|$ is on the order of $\sqrt{n}$ for large $n$. For the Poisson-Voronoi mosaic, we show that $|Y|/\sqrt{n}$ concentrates to $e^{-\lambda}(2\pi e)^{-1/2}$ as $n$ increases, and for a stationary and isotropic Poisson hyperplane mosaic, we show there is a range $(R_\ell, R_u)$ such that $|Y|/\sqrt{n}$ will be within this range with high probability for large $n$. The rates of convergence are also computed in both cases.

1. Introduction

Random mosaics, also called random tessellations, have long been studied in stochastic geometry and give rise to interesting classes of random polytopes. Recently, there has been more interest in high dimensional tessellations, partially due to applications in signal processing [15] and information theory [2]. For these applications, it is important to understand the asymptotic geometric properties of the polytopes induced by random tessellations, in order to decode and reconstruct signals with small error.

Some well-known classes of random mosaics are built from Poisson point processes, either in $\mathbb{R}^n$ or in the space of hyperplanes in $\mathbb{R}^n$. Statistics of the cells of these random mosaics have been well-studied, particularly in dimensions $n = 2$ and $n = 3$. See [17] and [8] for more background and further references. Some recent work has focused on high dimensional Poisson mosaics, particularly on the volume and shape of the zero cell and typical cell as dimension $n$ tends to infinity ([1], [13], and [12]). The zero cell is the cell of the tessellation containing the origin, and the distribution of the typical cell is obtained by averaging over all cells in a large bounded subset and then increasing this subset to the entire space. The volume of these cells has been studied in [1] and [13], as well as some analysis of their shape in high dimensions. For example, in [1], it is proved that the volume of the intersection of the typical cell of a Poisson-Voronoi

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tessellation with intensity \( \lambda \) and a co-centered ball of volume \( u \) tends to \( \lambda^{-1}(1 - e^{-\lambda u}) \) as the space dimension tends to infinity.

In this paper, we aim to better understand the nature of the zero cell of stationary Poisson mosaics in high dimensions by considering the random vector that, conditioned on the random mosaic, is uniformly distributed in the centered zero cell. By “centered”, we mean that an appropriately chosen center of the cell is located at the origin, for instance, the center of the largest ball contained within the cell.

It is well-known (see e.g. [3]) that a random vector \( Y \) in \( \mathbb{R}^n \) with density \( f(x) := g(|x|) \), where \( g : \mathbb{R} \to \mathbb{R} \) is log-concave, satisfies the following: For some absolute constant \( C > 0 \), for all \( t \geq 0 \),

\[
E \left( \frac{|Y|}{(E|Y|^2)^{\frac{1}{2}}} - 1 \right)^2 \leq \frac{C}{n}.
\]

This says that the norm of \( Y \) will be concentrated near its expectation for large dimension \( n \). This phenomenon is known as thin-shell concentration, and has been shown to occur for all log-concave random vectors \( Y \) in \( \mathbb{R}^n \) normalized so that \( E(Y) = 0 \) and \( E(Y_i Y_j) = \delta_{ij} \), for \( i, j = 1, \ldots, n \). The best currently known estimate is: for absolute constants \( C, c > 0 \),

\[
P(||Y| - \sqrt{n}| \geq t\sqrt{n}) \leq Ce^{-c\min\{t^2, t\}\sqrt{n}} \text{ for all } t \geq 0,
\]

and was proved by Guédon and Milman in [10]. We refer to the monograph [4] for more on thin-shell estimates and log-concave random vectors. This paper studies to what extent the phenomenon of thin-shell concentration occurs for the random vector described in the previous paragraph for specific models. If this random vector is concentrated around its mean in high dimensions, then most of the volume of the zero cell of the random mosaic will be contained within a narrow annulus.

One motivation for the study of the norm of this random vector is data compression. Random mosaics can be used to compress data in \( \mathbb{R}^n \) such that all data contained in the same cell of the tessellation will have the same encoding. This is the case, for instance, in one-bit compressed sensing using hyperplane tessellations, see [15] and [16]. Reconstructing the original data with small error requires that all data within the same cell of the tessellation are close together. The volume of the cell is not a useful metric in this case, since a very thin cell could have small volume and also contain signals that lie very far apart. The norm of the random vector studied in this paper is a more useful metric to ensure the mass of the cell does not lie far away from the center.

The distribution of the vector described above is shown to depend on the typical cell, as shown in Lemma 2.1 due to the fact that the distribution of the zero cell has a Radon-Nikodym derivative with respect to the distribution of the typical cell. Here, we restrict to studying two types of stationary random mosaics, a stationary Poisson-Voronoi mosaic and a stationary and isotropic Poisson hyperplane mosaic, since in both cases there exists an explicit representation for the distribution of the typical cell that allows for computations. Both of these random mosaics are isotropic, that is, their distribution is invariant under rotations about the origin. This implies that the random
vector chosen uniformly from the centered zero cell will be radially symmetric. In the Poisson-Voronoi case, we show that this random vector is also log-concave, and thus satisfies the thin-shell estimate \([1]\). Also, we will show that even stronger estimates can be obtained directly.

The main complementary results can be stated as follows. For each \(n\), let \(X_n\) be a stationary random mosaic in \(\mathbb{R}^n\) where the intensity of cell centroids is \(e^{n\lambda_n}\) and assume \(\lim_{n \to \infty} \lambda_n = \lambda \in \mathbb{R}\). Let \(Y_n\) denote a random vector in \(\mathbb{R}^n\) such that, conditionally on \(X_n\), \(Y_n\) is uniformly distributed in the centered zero cell of \(X_n\). For the Poisson-Voronoi mosaic, we show that \(|Y_n|/\sqrt{n}\) concentrates to \(e^{-\lambda(2\pi e)^{-1}}\) as the dimension \(n\) increases. Exponential rates of convergence are also computed, as shown in Theorem 3.2. In the case of the zero cell of a Poisson hyperplane tessellation, we show there exists an interval \((R_\ell, R_u)\) such that \(|Y_n|/\sqrt{n}\) will be contained in this interval with high probability in high dimensions. Rates of convergence are also computed in this case as shown in Theorem 4.4.

2. Preliminaries and notation

Let \(\mathcal{F}\) denote the set of closed sets in \(\mathbb{R}^n\) and define \(\mathcal{F}' := \mathcal{F} \setminus \emptyset\). Similarly, define \(\mathcal{C}\), \(\mathcal{K}\), \(\mathcal{C}'\), and \(\mathcal{K}'\) as the set of compact sets in \(\mathbb{R}^n\), the set of compact and convex sets of \(\mathbb{R}^n\), and their non-empty counterparts, respectively. Random sets will be studied using hitting probabilities using the following notation. For \(A \subset \mathbb{R}^n\), define \(\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A = \emptyset\}\).

The open ball in \(\mathbb{R}^n\) of radius \(R\) centered at the origin is denoted by \(B_n(R)\) and the unit sphere by \(S^{n-1}\). The measure \(\sigma_{n-1}\) will denote the uniform probability measure on \(S^{n-1}\), i.e., the normalized spherical Lesbegue measure. Also, let \(\kappa_n\) denote the volume of the unit ball \(B_n(1)\), and \(\omega_n\) denote the surface area of the unit sphere \(S^{n-1}\). Note that \(\omega_n = n\kappa_n\), and \(\kappa_n = \frac{n^{n/2}}{\Gamma(n/2+1)}\), where \(\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt\) is the gamma function. Stirling’s formula implies that as \(n \to \infty\),

\[
\kappa_n \sim \frac{1}{\sqrt{\pi n}} \left(\frac{2\pi e}{n}\right)^{n/2}.
\]

A particle process is a point process in \(\mathcal{C}'\). A mosaic is defined to be a collection of convex polytopes in \(\mathbb{R}^n\) such that the union is the entire space and no two polytopes in the collection share interior points. Letting \(\mathcal{M}\) denote the set of all face-to-face mosaics (see [17]), a random mosaic in \(\mathbb{R}^n\) is defined to be a particle process in \(\mathbb{R}^n\) such that \(X \in \mathcal{M}\) almost surely. The polytopes contained in the mosaic will be called the cells of the mosaic.

The intensity measure of a stationary particle process \(X\) is defined as \(\Theta(\cdot) = \mathbb{E}[X(\cdot)]\) and can be decomposed in the following way. Let \(c : \mathcal{C}' \to \mathbb{R}^n\) be a center function, a measurable map which is compatible with translations, i.e. \(c(C + x) = c(C) + x\) for all \(x \in \mathbb{R}^n\). Define the grain space
\[
\mathcal{C}_0 := \{C \in \mathcal{C}' : c(C) = 0\},
\]
and the homeomorphism (see [17] for more details)
\[ \Phi : \mathbb{R}^n \times C_0 \to C' ; \quad (x, C) \to x + C. \]

**Theorem 2.1.** (Theorem 4.1.1 in [17]) Let \( X \) be a stationary particle process in \( \mathbb{R}^n \) with intensity measure \( \Theta \neq 0 \). Then there exist a number \( \lambda \in (0, \infty) \) and a probability measure \( Q \) on \( C_0 \) such that
\[ \Theta = \lambda \Phi(\nu \otimes Q). \]

The number \( \lambda \) is called the intensity of the particle process and it will also be referred to as the cell intensity. \( Q \) is called the grain distribution. The point process of centers of the cells of a stationary mosaic is a stationary point process with intensity \( \lambda \). For a stationary random mosaic \( X \), a random set with distribution \( Q \) is called the typical cell of \( X \). This cell can be considered as a spatial average of the cells in the mosaic, as it can also be defined as follows.

**Definition 2.1.** The typical cell \( Z \) of a random mosaic \( X \) with intensity \( \lambda \) is the random polytope with the following distribution. For all Borel sets \( A \in \mathcal{B}(K) \),
\[ Q(A) = \lim_{r \to \infty} \frac{\mathbb{E} \left[ \sum_{K \in X} 1_A\{ K - c(K) \} 1_{B_n(r)}(c(K)) \right]}{\mathbb{E} \left[ \sum_{K \in X} 1_{B_n(r)}(c(K)) \right]} , \quad a.s. \]

It is known that that the expected volume of the typical cell is given by the reciprocal of the cell intensity, i.e.,
\[ \mathbb{E}[V(Z)] = \int V(K)Q(K) = \frac{1}{\lambda}. \]

The zero cell of a random mosaic is defined to be the cell the origin is contained in. It will be denoted \( Z_0 \). The following result shows an important relationship between the distribution of the zero cell and the typical cell of a stationary random mosaic, that the distribution of \( Z_0 - c(Z_0) \) has a Radon-Nikodym derivative with respect to the distribution of \( Z \) given by \( V(\cdot)/\mathbb{E}[V(Z)] \).

**Theorem 2.2.** (Theorem 10.4.1 in [17]) Let \( X \) be a stationary random mosaic in \( \mathbb{R}^n \) with intensity \( \gamma \). Denote its typical cell by \( Z \) and zero cell by \( Z_0 \). For any non-negative measurable and translation-invariant function \( g : K' \to \mathbb{R} \),
\[ \mathbb{E}[g(Z_0)] = \frac{1}{\mathbb{E}[V(Z)]} \mathbb{E}[g(Z)V(Z)]. \]

An application of the above result gives the density of a vector uniformly sampled in the zero cell of a random mosaic.

**Lemma 2.1.** Let \( X \) be a stationary random mosaic in \( \mathbb{R}^n \) with zero cell \( Z_0 \) and typical cell \( Z \) with respect to the center function \( c : C' \to \mathbb{R}^n \) as previously defined. Let \( Y \) be a random vector in \( \mathbb{R}^n \) such that conditioned on \( X \),
\[ Y \sim \text{Uniform}(Z_0 - c(Z_0)). \]
Then, for all measurable $g$,
\[ E[g(Y)] = \int_{\mathbb{R}^n} g(x) \frac{\mathbb{P}(x \in Z)}{E[V(Z)]} dx, \]
i.e., $Y \in \mathbb{R}^n$ has density $f_Y(x) = \frac{\mathbb{P}(x \in Z)}{E[V(Z)]}.$

**Proof.** First, by the definition of $Y$,
\[ E[g(Y)] = E\left[ E[g(Y)|Z_0] \right] = E\left[ \frac{1}{V(Z_0)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in Z_0 - c(Z_0)\}} dx \right]. \]

Note that the function $f(\cdot) = \frac{1}{V(\cdot)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in \cdot - c(\cdot)\}} dx$ is invariant under translations, since for any $t \in \mathbb{R}$ and $K \in \mathcal{K}$,
\[ f(K + t) = \frac{1}{V(K + t)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in K + t - c(K + t)\}} dx = \frac{1}{V(K)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in K - c(K)\}} dx = f(K). \]

Thus, by Theorem 2.2 and since $c(Z) = 0$,
\[ E[g(Y)] = E\left[ \frac{1}{V(Z_0)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in Z_0 - c(Z_0)\}} dx \right] = \frac{1}{E[V(Z)]} E\left[ V(Z) \frac{1}{V(Z)} \int_{\mathbb{R}^n} g(x) 1_{\{x \in Z - c(Z)\}} dx \right] = E\left[ \int_{\mathbb{R}^n} g(x) \frac{1_{\{x \in Z\}}}{E[V(Z)]} dx \right]. \]

Finally, applying Fubini’s Theorem gives the result. \[ \square \]

### 3. Poisson-Voronoi Mosaic

A special type of random mosaic comes from the Voronoi cells of a Poisson point process in $\mathbb{R}^n$. Let $N$ be a stationary Poisson point process with intensity $\lambda$ and, for $x \in N$, define the Voronoi cell of $N$ with center $x$ by
\[ C(x, N) := \{ z \in \mathbb{R}^n : |z - x| \leq |z - y| \text{ for all } y \in N \}. \]

The collection $X := \{ C(x, N) : x \in N \}$ is a stationary random mosaic and is called the Poisson-Voronoi mosaic induced by $N$. The intensity $\lambda$ of the underlying Poisson point process is the cell intensity of the induced mosaic.

First we show that for a Poisson-Voronoi mosaic $X$, the density of the random vector $Y$ that is uniformly distributed in $Z_0$ conditioned on $X$, is log-concave, and we then compute the moments of its norm.
Lemma 3.1. Let \( Z_0 \) be the zero cell of the stationary and isotropic Poisson-Voronoi tessellation associated to \( N \sim \text{PPP}(\lambda) \) in \( \mathbb{R}^n \). Define the random vector \( Y \), such that conditioned on \( Z_0 \),

\[
Y \sim \text{Uniform}(Z_0 - c(Z_0)).
\]

Then, \( Y \) has a log-concave density and for all \( k \in \mathbb{N} \),

\[
\mathbb{E}[|Y|^k] = \frac{\Gamma(1 + \frac{k}{n})}{(\lambda \kappa_n)^{\frac{k}{n}}} = O\left(\lambda^{-\frac{k}{n}}n^\frac{k}{2}\right).
\]

Proof. By Lemma 2.1 the density of \( Y \) is \( \mathbb{P}_{x(Z)} \left[ x \in \text{Bin}(Z) \right] \). By the fact that \( \mathbb{E}[V(Z)] = \frac{1}{\lambda} \) and by Slivnyak’s theorem [17],

\[
\mathbb{P}(x \in Z) = \lambda \mathbb{P}_0 \left[ \text{Bin}(x, |x|) \right] = \lambda e^{-\lambda \kappa_n |x|^n},
\]

and this is clearly log-concave. Thus, the density of \( Y \) is log-concave.

For the moments, switching to polar coordinates and using another change of variables \( (y = \lambda \kappa_n r^n) \) gives

\[
\mathbb{E}[|Y|^k] = \lambda \int_{\mathbb{R}^n} |x|^k e^{-\lambda \kappa_n |x|^n} \, dx = \lambda n \kappa_n \int_0^{\infty} r^{n+k} e^{-\lambda \kappa_n r^n} \, dr
\]

\[
= \lambda n \kappa_n \int_0^{\infty} y \left( \frac{y}{\lambda \kappa_n} \right)^{1+\frac{k}{n}} e^{-y} \frac{1}{n \lambda \kappa_n} \left( \frac{y}{\lambda \kappa_n} \right)^{\frac{k}{n} - 1} \, dy
\]

\[
= (\lambda \kappa_n)^{-\frac{k}{n}} \int_0^{\infty} y^{\frac{k}{n}} e^{-y} \, dy = \frac{\Gamma(1 + \frac{k}{n})}{(\lambda \kappa_n)^{\frac{k}{n}}}.
\]

Then, by (2), as \( n \to \infty \),

\[
\mathbb{E}[|Y|^k] \sim \frac{n^{\frac{k}{2}}}{\lambda^{\frac{k}{2}}(2\pi e)^{\frac{k}{2}}}.
\]

The fact that \( Y \) has a radial and log-concave density already implies that \( |Y| \) concentrates to

\[
\mathbb{E}[|Y|^2]^{\frac{1}{2}} = \frac{\Gamma(1 + \frac{1}{n})^{\frac{1}{2}}}{(\lambda \kappa_n)^{\frac{1}{2}}} \sim \frac{\sqrt{n}}{\lambda^{\frac{1}{2}} \sqrt{2\pi e}},
\]

in high dimensions by the thin-shell estimate [17]. However, we can prove strong concentration inequalities by direct computation.

Theorem 3.1. Let \( X \) be a stationary Poisson-Voronoi mosaic in \( \mathbb{R}^n \) with intensity \( \lambda \) and \( Y \) a random vector such that, conditioned on \( X \), \( Y \sim \text{Uniform}(Z_0 - c(Z_0)) \). Let \( \sigma^2 = \mathbb{E}[|Y|^2] \). Then, there exists \( c > 0 \) such that for all \( t > 0 \),

\[
\mathbb{P}(|Y| \geq (1 + t)\sigma) \leq e^{-c n \ln(1+t)},
\]

and for all \( t \in (0,1) \),

\[
\mathbb{P}(|Y| \leq (1 - t)\sigma) \leq e^{n \ln(1-t)}.
\]
Proof. By Lemma 3.1,
\[
P(|Y| \leq R) = \lambda \int_{B(R)} e^{-\lambda n |x|^n} dx = \lambda n \kappa_n \int_0^R r^{n-1} e^{-\lambda \kappa_n r^n} dr
\]
\[
= \int_0^{\lambda \kappa_n R^n} e^{-y} dy = 1 - e^{-\lambda \kappa_n R^n},
\]
and by (3),
\[
P(|Y| \leq (1-t)\sigma) = 1 - e^{-\Gamma(1+\frac{1}{n})\frac{\pi}{2}(1-t)^n}.
\]
By the inequality \(1 - e^{-x} \leq x\) for all \(x \geq 0\),
\[
P(|Y| \leq (1-t)\sigma) \leq \Gamma \left( 1 + \frac{1}{n} \right)^{\frac{\pi}{2}} (1-t)^n \leq \Gamma(2)^{\frac{n}{2}} e^{n\ln(1-t)} = e^{n\ln(1-t)}.
\]
Similarly,
\[
P(|Y| \geq (1+t)\sigma) = \int_{\lambda \kappa_n R^n}^{\infty} e^{-y} dy = e^{-\Gamma(1+\frac{1}{n})\frac{\pi}{2}(1+t)^n},
\]
and by the fact that \(\Gamma(1+\frac{1}{n})\frac{\pi}{2} \geq c := \min_{s>0} \Gamma(s),\)
\[
P(|Y| \geq (1+t)\sigma) \leq e^{-ce^{n\ln(1+t)}}.
\]
\[\square\]

Considering a sequence of these vectors in increasing dimensions, we obtain the following threshold result when the cell intensity of the random mosaics grows exponentially with dimension.

**Theorem 3.2.** Let \(Y_n \sim \text{Uniform}(Z_{0,n} - c(Z_{0,n}))\), where \(Z_{0,n}\) is the zero cell of a stationary Poisson-Voronoi tessellation in \(\mathbb{R}^n\) with intensity \(e^{n\lambda_n}\). Assume \(\lim_{n \to \infty} \lambda_n = \lambda \in \mathbb{R}\). Then,
\[
\lim_{n \to \infty} P(|Y_n| \leq \sqrt{n}R) = \begin{cases} 0, & R < e^{-\lambda} (2\pi e)^{-\frac{1}{2}} \\ 1, & R > e^{-\lambda} (2\pi e)^{-\frac{1}{2}} \end{cases}
\]
For \(R < e^{-\lambda} (2\pi e)^{-\frac{1}{2}}\),
\[
\lim_{n \to \infty} \frac{1}{n} \ln P(|Y_n| \leq \sqrt{n}R) = \lambda + \frac{1}{2} \ln(2\pi e) + \ln R,
\]
and for \(R > e^{-\lambda} (2\pi e)^{-\frac{1}{2}}\),
\[
\lim_{n \to \infty} \frac{1}{n} \ln \left( -\frac{1}{n} \ln P(|Y_n| \geq \sqrt{n}R) \right) = \lambda + \frac{1}{2} \ln(2\pi e) + \ln R.
\]

Proof. Switching to polar coordinates and then using another change of variables gives
\[
P(|Y_n| \leq \sqrt{n}R) = e^{n\lambda_n} \int_{B(\sqrt{n}R)} e^{-e^{n\lambda_n} \kappa_n |x|^n} dx = e^{n\lambda_n} n \kappa_n \int_0^{\sqrt{n}R} r^{n-1} e^{-e^{n\lambda_n} \kappa_n r^n} dr
\]
\[
= \int_0^{e^{n\lambda_n} \kappa_n (\sqrt{n}R)^n} e^{-y} dy = 1 - e^{-e^{n\lambda_n} \kappa_n (\sqrt{n}R)^n}.
\]
For $R < e^{-\lambda (2\pi e)^{-\frac{1}{2}}}$, by (2),
\[ e^{n\lambda \kappa_n} (\sqrt{n}R)^n \sim \frac{1}{\sqrt{n} \pi} \left( e^{\lambda_n (2\pi e)^{\frac{1}{2}} R} \right)^n \to 0, \text{ as } n \to \infty. \]

implying that $P(|Y_n| \leq \sqrt{n}R) \to 0$ as $n \to \infty$. This also implies that there is an $\alpha > 0$ such that for all $n$ large enough,
\[ \alpha(e^{n\lambda \kappa_n} (\sqrt{n}R)^n) \leq 1 - e^{-e^{n\lambda \kappa_n} (\sqrt{n}R)^n} \leq e^{n\lambda \kappa_n} (\sqrt{n}R)^n. \]

Thus,
\[ \lim_{n \to \infty} -\frac{1}{n} \ln P(|Y_n| \leq \sqrt{n}R) = \lambda + \frac{1}{2} \ln(2\pi e) + \ln R. \]

Also, similarly to above, for $R > e^{-\lambda (2\pi e)^{-\frac{1}{2}}}$,
\[ P(|Y_n| \geq \sqrt{n}R) = \int_0^\infty e^{-y} dy = e^{-e^{n\lambda \kappa_n} (\sqrt{n}R)^n} \to 0, \text{ as } n \to \infty, \]

and
\[ \lim_{n \to \infty} -\frac{1}{n} \ln \left( -\frac{1}{n} \ln P(|X| \geq \sqrt{n}R) \right) = \lambda + \frac{1}{2} \ln(2\pi e) + \ln R. \]

\[ \square \]

4. Poisson Hyperplane Mosaic

The second type of random mosaic we consider is the mosaic induced by a stationary and isotropic Poisson hyperplane process $X$ in $\mathbb{R}^n$. A hyperplane process in $\mathbb{R}^n$ is a point process in the space of $n-1$ dimensional affine subspaces in $\mathbb{R}^n$, denoted by $\mathcal{H}^n$. The following theorem provides a decomposition for the intensity measure of stationary hyperplane processes. Note that elements of the space $\mathcal{H}^n$ are of the form $H(u, \tau) := \{ x \in \mathbb{R}^n : \langle x, u \rangle = \tau \}$, where $u \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$.

**Theorem 4.1.** (Theorem 4.4.2 and (4.33) in [17]) Let $X$ be a stationary and isotropic hyperplane process in $\mathbb{R}^n$ with intensity measure $\Theta \neq 0$. Then, there is a unique number $\gamma \in (0, \infty)$ such that for all nonnegative measurable functions $f$ on $\mathcal{H}^n$,
\[ \int_{\mathcal{H}^n} f \, d\Theta = 2\gamma \int_{S^{n-1}} \int_0^\infty f(H(u, \tau)) d\tau \sigma_{n-1}(du). \]

The parameter $\gamma$ is called the intensity of the hyperplane process. Its relation to the cell intensity $\lambda$ of the induced random mosaic is given by
\[ \lambda = \kappa_n \left( \frac{\gamma \kappa_{n-1}}{n \kappa_n} \right)^n. \]

If the hyperplane process is stationary, then the induced random mosaic is stationary. The typical cell of this random mosaic has an inradius, that is, the radius of the largest ball completely contained in the cell, given by the following theorem.
Theorem 4.2. (Theorem 10.4.8 in [17]) Let $X$ be a nondegenerate stationary Poisson hyperplane process in $\mathbb{R}^n$ with intensity $\gamma$. Let $Z$ be the typical cell. Then,

$$P(r(Z) \leq a) = 1 - e^{-2\gamma a}, \quad a \geq 0.$$ 

If we define the center function $c(C)$ to be the center of the inball of $C$, where the inball is the largest ball included in the cell $C$, Calka [5] showed that the distribution of the typical cell can be described in the following way. Let $R \in \mathbb{R}^+$ and $(U_0, ..., U_n) \in (S^{n-1})^{n+1}$ be independent random variables with the following distributions. Let $R$ is exponentially distributed with parameter $2\gamma$, as $r(Z)$ is in the above theorem. Let $(U_0, ..., U_n)$ have a density with respect to the uniform measure that is proportional to the volume of the simplex constructed with these $n + 1$ vectors multiplied by the indicator that this simplex contains the origin. Then, let $X_R$ be the hyperplane process $X$ restricted to $\mathbb{R}^n \setminus B_r(0)$. Letting $C_1$ be the polyhedron containing the origin obtained as the intersection of the $(n + 1)$ half spaces bounded by the hyperplanes $H_{RU_i}, \; i = 0, \ldots, n$, and $C_2$ be the zero-cell of the hyperplane tessellation associated with $X_R$, the typical cell of the stationary and isotropic Poisson hyperplane tessellation is distributed as $C_1 \cap C_2$. In other words,

Theorem 4.3. (Theorem 10.4.6 in [17]) Let $X$ be a stationary and isotropic Poisson hyperplane process in $\mathbb{R}^n$ with intensity $\gamma$. If $Q$ is the probability distribution of the typical cell $Z$ with respect to the inball center as the center function, then for all Borel sets $A \in \mathcal{B}(\mathbb{K}),$

$$Q(A) = E[V(Z)] \frac{\gamma^{n+1}}{(n+1)} \int_0^{\infty} \int_{(S^{n-1})^{n+1}} e^{-2\gamma r} P\left(\bigcap_{H \in X \cap F_{B_0(0,r)}} H_0^+ \cap \bigcap_{j=0}^n H^-(u_j, r) \in A\right) \cdot \Delta_n(u_0, ..., u_n) P(du_0) \ldots P(du_n) dr.$$ 

The triangle notation $\Delta_n(u_0, ..., u_n)$ is the $n$-dimensional volume of the convex hull of the vectors, $u_0, \ldots, u_n$. The set $P \subset (S^{n-1})^{n+1}$ is the set of all $(n + 1)$-tuples of unit vectors such that the origin is contained in their convex hull.

In the case where the stationary mosaic is induced by a stationary and isotropic Poisson hyperplane process, we first have the following proposition. For $C \in \mathcal{K}',$ let $c(C)$ be the center of the inball of $C$.

Proposition 4.1. Let $X$ be a stationary and isotropic Poisson hyperplane mosaic in $\mathbb{R}^n$ with cell intensity $\lambda$. Let $Y$ be the random vector such that, conditional on $X,$

$$Y \sim \text{Uniform}(Z_0 - c(Z_0)).$$

Then, for all $k \geq 0,$

$$E[|Y|^k] \leq \frac{\Gamma(n + k + 1)}{\Gamma(n + 1)2^k} \left(\frac{k_n}{\lambda}\right)^{k/n}. $$

Next, we consider a sequence of these random vectors in increasing dimensions $n$, and obtain the following result.
Theorem 4.4. For each \( n \), let \( X_n \) be a stationary and isotropic Poisson hyperplane process with cell intensity \( e^{n\lambda_n} \). Assume \( \lim_{n \to \infty} \lambda_n = \lambda \in \mathbb{R} \). Let \( Z_{0,n} \) be the zero cell of \( X_n \), and define the random vectors \( Y_n \) such that, conditional on \( X_n \),

\[
Y_n \sim \text{Uniform}(Z_{0,n} - c(Z_{0,n})).
\]

Then, for all \( R > e^{-\lambda \sqrt{\frac{\pi}{2}}} \),

\[
\lim_{n \to \infty} P(|Y_n| \geq \sqrt{n}R) = 0,
\]

and there is a \( R_\ell \) such that \( 0 < R_\ell < e^{-\lambda \sqrt{\frac{\pi}{2}}} \) and for all \( R < R_\ell \),

\[
\lim_{n \to \infty} P(|Y_n| \leq \sqrt{n}R) = 0.
\]

Also, for \( R > e^{-\lambda \sqrt{\frac{\pi}{2}}} \),

\[
\limsup_{n \to \infty} \frac{1}{n} \ln P(|Y_n| \geq \sqrt{n}R) \leq \lambda + \frac{1}{2} \ln \left( \frac{2e}{\pi} \right) + \ln R - \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi e}},
\]

and for \( R < e^{-\lambda \sqrt{\frac{\pi}{2}}} \),

\[
\limsup_{n \to \infty} \frac{1}{n} \ln P(|Y_n| \leq \sqrt{n}R) \leq \lambda + \frac{1}{2} \ln (2\pi e) + \ln R - \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi e}} - \ln 2.
\]

Remark 4.1. The lower bound \( R_\ell \) is the radius satisfying

\[
\lambda + \frac{1}{2} \ln (2\pi e) + \ln R_\ell - \frac{e^\lambda R_\ell \sqrt{2}}{\sqrt{\pi e}} - \ln 2 = 0.
\]

Then, for all \( R > 0 \),

\[
\lambda + \frac{1}{2} \ln (2\pi e) + \ln R - \frac{e^\lambda R_\ell \sqrt{2}}{\sqrt{\pi e}} - \ln 2 < \lambda + \frac{1}{2} \ln (2\pi e) + \ln R,
\]

and thus, \( \lambda + \frac{1}{2} \ln (2\pi e) + \ln R_\ell > 0 \), implying that

\[
R_\ell > e^{-\lambda (2\pi e)^{-\frac{1}{2}}}.
\]

This implies that that vector \( Y_n \) chosen with respect to the Poisson-Voronoi zero cell will have a smaller norm in high dimensions than the vector \( Y_n \) chosen with respect to the zero cell of the Poisson hyperplane mosaic.

Remark 4.2. The assumption on \( \lambda_n \) can be generalized to \( \lambda_n \sim e^{n\lambda_n^{\alpha}} \) for some \( \alpha \in \mathbb{R} \). By \( (\natural) \), this implies that the scaling for the intensity of hyperplanes is \( \gamma_n = O(n^{\alpha+1}) \). Then a similar result holds with the probabilities

\[
P(|Y_n| \leq R_n) \quad \text{and} \quad P(|Y_n| \geq R_n),
\]

where \( R_n = Rn^{\frac{1}{2} - \alpha} \). This requirement gives two special cases: \( \lambda_n \sim e^{n\lambda}, \gamma_n = O(n) \), \( R_n = O(\sqrt{n}) \) and \( \lambda_n \sim e^{n\lambda} n^{\frac{1}{2}}, \gamma_n = O(n^{3/2}), R_n = O(1) \).
Before proving the theorem, recall the following special functions. The beta function is defined by

$$B(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1} dt.$$ 

The incomplete beta function is defined as $B(x; a, b) := \int_0^x t^{a-1}(1 - t)^{b-1} dt$, and the regularized incomplete beta function is

$$I_x(a, b) := \frac{B(x; a, b)}{B(a, b)}.$$ 

Recall that the Gamma function is defined by $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t} dt$, and we define the upper and lower incomplete gamma functions by

$$\Gamma(x, R) := \int_R^\infty t^{x-1}e^{-t} dt, \text{ and } \Gamma_l(x, R) := \int_0^R t^{x-1}e^{-t} dt,$$

respectively. The following series of lemmas are needed before proving the results.

**Lemma 4.1.** Let $X$ be a stationary and isotropic Poisson hyperplane process with intensity $\gamma$. Then, letting $[0, x]$ denote the line segment between $0$ and the point $x$,

$$\Theta\left(\mathcal{F}^{B(r)}_{[0, x]}\right) = \gamma|x| \left[ \frac{2\kappa_{n-1}}{n\kappa_n} \left(1 - \frac{r^2}{|x|^2}\right)^{\frac{n-1}{2}} - \frac{r}{|x|} \int_{|x|}^{\infty} \left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-1}{2}} dt \right] 1_{\{|x| \geq r\}}.$$ 

**Proof.** Note that if $r > |x|$, then a hyperplane cannot hit $[0, x]$ and not hit the open ball $B(r)$ at the same time and thus $\Theta\left(\mathcal{F}^{B(r)}_{[0, x]}\right)$ is zero. If $r \leq |x|$, then by Theorem 4.1,

$$\Theta\left(\mathcal{F}^{B(r)}_{[0, x]}\right) = 2\gamma \int_{S^{n-1}} \int_0^\infty 1_{\{H(u, t) \cap [0, x] \neq \emptyset\}} 1_{\{H(u, t) \cap B(r) = \emptyset\}} dt \sigma_{n-1}(du)$$

$$= 2\gamma \int_{S^{n-1}} \int_0^\infty 1_{\{|r| \leq \langle x, u \rangle \}} dt \sigma_{n-1}(du)$$

$$= 2\gamma \int_{\{v \in S^{n-1} : \langle v, x \rangle \geq r\}} \langle \langle x, u \rangle, -r \rangle \sigma_{n-1}(du)$$

$$= 2\gamma|x| \int_{\{v \in S^{n-1} : \langle v, x \rangle \geq r\}} \langle \frac{x}{|x|}, u \rangle \sigma_{n-1}(du) - 2r \gamma \sigma_{n-1}(\{v \in S^{n-1} : \langle v, x \rangle \geq r\}),$$

where $a_+ = \max\{a, 0\}$. To compute the first integral, first note that the integral does not depend on the direction of $x$, only on the norm $|x|$. We can then assume $x = |x|e_n$, where $e_n = (0, ..., 0, 1)$, and

$$\int_{\{v \in S^{n-1} : \langle v, x \rangle \geq r\}} \langle e_n, u \rangle \sigma_{n-1}(du) = \int_{\{v \in S^{n-1} : \langle v, x \rangle \geq r\}} u_n \sigma_{n-1}(du)$$

$$= \frac{\omega_{n-2}}{\omega_{n-1}} \int_{|x|}^1 \int_{|v|}^1 t(1 - t^2)^{\frac{n-3}{2}} \sigma_{n-2}(du) dt.$$
Then, for 

First, we see that

By Lemma 2.1, the density of

Proof. Let 

The fractional area of a spherical cap is given by

Then, 

\[
\Theta \left( F_{B(r)}^{[0,x]} \right) = \frac{2\gamma |x| \kappa_{n-1}^{\frac{n+1}{2}}}{n \kappa_n} \left( 1 - \frac{r^2}{|x|^2} \right)^{\frac{n+1}{2}} - r \gamma I_{1 - \frac{r^2}{|x|^2}} \left( \frac{n-1}{2}, \frac{1}{2} \right) \right] \mathbb{1}_{\{|x| \geq r\}} \]

\[
= \gamma |x| \left[ \frac{2\kappa_{n-1}}{n \kappa_n} \left( 1 - \frac{r^2}{|x|^2} \right)^{\frac{n+1}{2}} - \frac{r}{|x|} I_{1 - \frac{r^2}{|x|^2}} \left( \frac{n-1}{2}, \frac{1}{2} \right) \right] \mathbb{1}_{\{|x| \geq r\}}. \]

Lemma 4.2. Let \( Z_0 \) be the zero cell of a stationary and isotropic Poisson hyperplane tessellation with intensity \( \gamma \) in \( \mathbb{R}^n \). Conditioned on \( Z_0 \), let \( Y \sim \text{Uniform}(Z_0 - c(Z_0)) \). Then, for \( R > 0 \),

\[
P(|Y| \geq R) \leq \frac{\Gamma_u \left( n + 1, 2\gamma R^{n-1} \frac{\kappa_{n-1}}{n \kappa_n} \right)}{\Gamma(n+1)},
\]

and

\[
P(|Y| \leq R) \leq \frac{n \kappa_n^2}{4^n} \left( \frac{n \kappa_n}{2 \kappa_{n-1}} \right) \left[ \Gamma \left( n + 1, 2\gamma R^{n-1} \frac{\kappa_{n-1}}{n \kappa_n} \right) + \Gamma(n) \left( \frac{\kappa_{n-1}}{n \kappa_n} \right)^{n+1} \right].
\]

Proof. By Lemma 2.1, the density of \( Y \) is

\[
f_Y(x) = \frac{P(x \in Z)}{E[V(Z)]}.
\]

Using the representation of \( Z \) in Theorem 4.3

\[
P(x \in Z) = E[V(Z)] \frac{\gamma^{n+1}}{(n+1)} \int_0^\infty \int_{S^{n-1}} e^{-2\gamma r} P \left( x \in \bigcap_{H \in X \cap \mathcal{F}_B(r)} H^+ \cap \bigcap_{j=0}^n H^-(u_j, r) \right) \]

\[
\cdot \Delta_n(u_0, ..., u_n) 1_P(u_0, ..., u_n) \prod_{i=0}^n \phi(du_i) dr.
\]

First, we see that

\[
P \left( x \in \bigcap_{H \in X \cap \mathcal{F}_B(r)} H^+ \cap \bigcap_{j=0}^n H^-(u_j, r) \right) = \prod_{j=0}^n 1_{\{x \in H^-(u_j, r)\}} P \left( X \left( F_{B(r)}^{[0,x]} \right) = 0 \right)
\]
Then,
\[
P(|Y| \geq R) = \int_{B(R) \cap \mathbb{E}[V(Z)]} P(x \in Z) \, dx
\]
\[
= \int_{B(R)^c} \frac{\gamma^{n+1}}{(n+1)} \int_0^\infty \int_{(S^{n-1})^{n+1}} e^{-2\gamma r} P \left( x \in \bigcap_{j=0}^n H^+ \cap \bigcap_{j=0}^n H^- (u_j, r) \right) \cdot \Delta_n(u_0, \ldots, u_n) P(u_0, \ldots, u_n) \prod_{i=0}^n \sigma_n-1(du_i) \, dr
\]
\[
= \frac{\gamma^{n+1}}{(n+1)} \int_{B(R)^c} \int_0^\infty \int_{(S^{n-1})^{n+1}} e^{-2\gamma r} \prod_{j=0}^n 1\{x \in H^- (u_j, r)\} e^{-\Theta(F_B^{(0)}, 0, x)} \bigtriangleup_n 1 P \prod_{i=0}^n \sigma_n-1(du_i) \, dr \, dx
\]
\[
= \frac{\gamma^{n+1}}{(n+1)} \int_{B(R)^c} \int_0^\infty e^{-2\gamma r} e^{-\Theta(F_B^{(0)}, 0, x)} \bigtriangleup_n 1 P \prod_{i=0}^n \sigma_n-1(du_i) \, dr \, dx.
\]
Making the change of variables \( t = \frac{r}{|x|} \), observing that the innermost integral does not depend on the direction of \( x \), and using Fubini’s Theorem,
\[
P(|Y| \geq R)
\]
\[
= \frac{\gamma^{n+1}}{(n+1)} \int_{B(R)^c} \int_0^\infty |x| e^{-2\gamma |x|t} e^{-\Theta(F_B^{(0)}, 0, x)} \bigtriangleup_n 1 P \prod_{i=0}^n \sigma_n-1(du_i) \, dt \, dx
\]
\[
= \frac{\gamma^{n+1}}{(n+1)} \int_0^\infty \int_{B(R)^c} |x| e^{-2\gamma |x|t} e^{-\Theta(F_B^{(0)}, 0, x)} \, dx \prod_{j=0}^n 1\{x \in H^- (u_j, t)\} \bigtriangleup_n 1 P \prod_{i=0}^n \sigma_n-1(du_i) \, dt
\]
By Lemma 4.1 and a change to polar coordinates,
\[
\int_{B(R)^c} |x| e^{-2\gamma |x|t} e^{-\Theta(F_B^{(0)}, 0, x)} \, dx = \int_{B(R)^c} |x| e^{-2\gamma |x|t} e^{\frac{2\kappa_n-1}{n+1} (1-t^2) \left[ t_1 t_2 \left( \frac{n-1}{2} + \frac{n-1}{2} \right) \right] } \, dx
\]
\[
= n\kappa_n \int_R r^n e^{-\gamma r \left[ 2t + \left( \frac{2\kappa_n-1}{n+1} \right) \left( \frac{n-1}{2} + \frac{n-1}{2} \right) \right] } \, dr.
\]
Now, using the identity \( I_{1-t}(a, b) = 1 - I_t(b, a) \),
\[
\int_{B(R)^c} |x| e^{-2\gamma |x|t} e^{-\Theta(F_B^{(0)}, 0, x)} \, dx = n\kappa_n \int_R r^n e^{-\gamma r f_n(t)} \, dx,
\]
where
\[
f_n(t) := \begin{cases} t + \frac{2\kappa_n-1}{n+1} (1-t^2) \frac{n-1}{2} + t I_{1-t} \left( \frac{1}{2}, \frac{n-1}{2} \right), & 0 \leq t \leq 1 \\ 2t, & t \geq 1. \end{cases}
\]
Note that \( f_n \) is differentiable and that

\[
(5) \quad f_n(0) = \frac{2\kappa_{n-1}}{n\kappa_n}.
\]

Then, by the change of variables \( y = \gamma f_n(t) \),

\[
n\kappa_n \int_{\mathbb{R}} r^n e^{-\gamma f_n(t)r} \, dr = \frac{n\kappa_n}{(\gamma f_n(t))^{n+1}} \int_{\mathbb{R}} y^n e^{-y} \, dy = \frac{n\kappa_n}{(\gamma f_n(t))^{n+1}} \Gamma_u(n+1, \gamma f_n(t)R).
\]

This gives us that

\[
\mathbb{P}(|Y| \geq R) = \frac{n\kappa_n}{(n+1)} \int_0^\infty \frac{\Gamma_u(n+1, \gamma f_n(t)R)}{f_n(t)^{n+1}} \left( \int_{(S^{n-1})^{n+1}} \Delta_n 1_P \prod_{i=0}^{n} \sigma_{n-1}(du_i) \right) dt.
\]

Since the upper incomplete gamma function is decreasing in its second argument, for all \( t \geq 0 \),

\[
\Gamma_u(n+1, \gamma f_n(t)R) \leq \Gamma_u \left( n+1, \gamma \frac{2\kappa_{n-1}}{n\kappa_n} R \right),
\]

where we have used (5). So, we have the upper bound

\[
\mathbb{P}(|Y| \geq R) \leq \frac{\Gamma_u(n+1, \gamma \frac{2\kappa_{n-1}}{n\kappa_n} R)}{\Gamma(n+1)} \cdot \left[ \frac{n\kappa_n}{(n+1)} \int_0^\infty \frac{\Gamma(n+1)}{f_n(t)^{n+1}} \left( \int_{(S^{n-1})^{n+1}} \Delta_n 1_P \prod_{i=0}^{n} \sigma_{n-1}(du_i) \right) dt \right].
\]

The term in the parentheses is the value of the integral \( \int_{\mathbb{R}^n} \mathbb{P}(x \in \mathbb{Z}) dx \) and is thus equal to 1. Hence,

\[
\mathbb{P}(|Y| \geq R) \leq \frac{\Gamma_u \left( n+1, \gamma \frac{2\kappa_{n-1}}{n\kappa_n} R \right)}{\Gamma(n+1)}.
\]

To obtain the upper bound for \( \mathbb{P}(|Y| \leq R) \), we can follow a similar procedure up to the equality

\[
\mathbb{P}(|Y| \leq R) = \frac{n\kappa_n}{(n+1)} \int_0^\infty \frac{\Gamma_u(n+1, \gamma f_n(t)R)}{f_n(t)^{n+1}} \left( \int_{(S^{n-1})^{n+1}} \prod_{j=0}^{n} 1_{\{e_j, u_j \leq t\}} \Delta_n 1_P \prod_{i=0}^{n} \sigma_{n-1}(du_i) \right) dt.
\]

The lower incomplete gamma function is not decreasing in \( t \) like the upper incomplete gamma function, so we cannot proceed exactly as above.

First we use the upper bound

\[
\int_{(S^{n-1})^{n+1}} \prod_{j=0}^{n} 1_{\{e_j, u_j \leq t\}} \Delta_n(u_0, \ldots, u_n) 1_P \prod_{i=0}^{n} \sigma_{n-1}(du_i) \leq \int_{(S^{n-1})^{n+1}} \prod_{j=0}^{n} \Delta_n(u_0, \ldots, u_n) 1_P \prod_{i=0}^{n} \sigma_{n-1}(du_i).
\]

Then, by the fact that

\[
\frac{n 2^n}{(n+1)(\omega_n)^2(n-1)} \Delta(u_0, \ldots, u_n) 1_P(u_0, \ldots, u_n) \prod_{i=0}^{n} d\sigma_{n-1}(u_i)
\]
is a joint density (see equation (11) in [6]),

\[ \int_{(S^n)^{n+1}} \frac{n!}{n!} \Delta_n(u_0, \ldots, u_n) \prod_{i=0}^{n} \sigma_n-1(du_i) = \frac{\kappa_n(n+1)}{2^n} \left( \frac{\kappa_{n-1}}{n \kappa_n} \right)^n, \]

and thus

\[ P(|Y| \leq R) \leq \frac{n \kappa_n^2}{2^n} \left( \frac{\kappa_{n-1}}{n \kappa_n} \right)^n \int_0^\infty \frac{\Gamma(n+1, \gamma f_n(R))}{f_n(t)^{n+1}} \, dt. \]

Then, note that for \( t \geq 1 \), \( f_n(t) = 2t \), so

\[ \int_1^\infty \frac{\Gamma(n+1, \gamma f_n(t))}{f_n(t)^{n+1}} \, dt \leq \frac{1}{2^{n+1}} \int_1^\infty \frac{\Gamma(n+1)}{t^{n+1}} \, dt = \frac{\Gamma(n)}{2^{n+1}}. \]

Thus,

\[ P(|Y| \leq R) \leq \frac{n \kappa_n^2}{2^n} \left( \frac{\kappa_{n-1}}{n \kappa_n} \right)^n \left[ \int_0^1 \frac{\Gamma(n+1, \gamma f_n(t))}{f_n(t)^{n+1}} \, dt + \frac{\Gamma(n)}{2^{n+1}} \right]. \]

Now, we note that the function

\[ h_n(t) := \frac{\Gamma(n+1, \gamma f_n(t))}{f_n(t)^{n+1}} \]

is decreasing and thus reaches its maximum at \( t = 0 \). It suffices to show \( h_n'(t) \leq 0 \). Indeed, we first note that the derivative of \( f_n'(t) \),

\[ f_n'(t) = \begin{cases} 1 + I_2 \left( \frac{1}{2}, \frac{n-1}{2} \right), & 0 \leq t \leq 1 \\ 2, & t \geq 1, \end{cases} \]

is positive. Then, by the Fundamental Theorem of Calculus,

\[ \frac{d}{dt} \Gamma(n+1, \gamma f_n(t)) = e^{-\gamma f_n(t)} R \left( \gamma f_n(t) R \right)^n (\gamma R) f_n'(t), \]

and by the quotient rule,

\[ h_n'(t) = \frac{1}{f_n(t)^{2n+2}} \left( f_n(t)^{n+1} \frac{d}{dt} \Gamma(n+1, \gamma f_n(t)) - (n+1) f_n(t)^n f_n'(t) \Gamma(n+1, \gamma f_n(t)) \right) \]

\[ = -\frac{1}{f_n(t)^{n+2}} \left( (n+1) f_n'(t) \Gamma(n+1, \gamma f_n(t)) - e^{-\gamma f_n(t)} R \left( \gamma f_n(t) R \right)^{n+1} f_n'(t) \right) \]

\[ = -\frac{f_n'(t)}{f_n(t)^{n+2}} \left( \Gamma(n+1, \gamma R) (n+1) - e^{-\gamma f_n(t)} R \left( \gamma f_n(t) R \right)^{n+1} \right). \]

Since \( f_n \) and \( f_n' \) are positive, it suffices to show the following inequality for \( h' \) to be negative:

\[ e^{-\gamma f_n(t)} R \left( \gamma f_n(t) R \right)^{n+1} \leq \Gamma(n+1, \gamma f_n(t)) (n+1). \]

Indeed, since \( e^{-t} \geq e^{-x} \) for all \( t \in [0, x] \),

\[ \Gamma(n+1, x) = \int_0^x e^{-t} t^n \, dt \geq e^{-x} \int_0^x t^n \, dt = e^{-x} \frac{x^{n+1}}{n+1}. \]
Letting \( x = 2\gamma f(t)R \) gives (6), and hence \( h'(t) \leq 0 \).

Thus, by (5),

\[
P(|Y| \leq R) \leq \frac{n\kappa_n^2}{4n} \left( \frac{n\kappa_n}{2\kappa_{n-1}} \right)^2 \left[ \Gamma_{\ell} \left( n + 1, 2\gamma R \frac{\kappa_{n-1}}{n\kappa_n} \right) + \Gamma(n) \left( \frac{\kappa_{n-1}}{n\kappa_n} \right)^{n+1} \right].
\]

\[\square\]

We can now prove the main results.

4.1. **Proof of Proposition 4.1.**

By Lemma 4.2,

\[
E[|Y|^k] = k \int_0^\infty y^{k-1} P(|Y| \geq y) dy \\
\leq k \int_0^\infty y^{k-1} \frac{\Gamma_u(n+1, 2\gamma \frac{\kappa_{n-1}}{n\kappa_n} y)}{\Gamma(n+1)} dy \\
= \frac{k}{\Gamma(n+1)} \int_0^\infty y^{k-1} \left( \int_{2\gamma \frac{\kappa_{n-1}}{n\kappa_n} y}^\infty t^ne^{-t} dt \right) dy.
\]

Then, by Fubini’s Theorem,

\[
E[|Y|^k] \leq \frac{k}{\Gamma(n+1)} \int_0^\infty t^ne^{-t} \left( \int_{2\gamma \frac{\kappa_{n-1}}{n\kappa_n} y}^\infty y^{k-1} dy \right) dt \\
= \frac{1}{\Gamma(n+1)} \int_0^\infty t^ne^{-t} \left( \frac{n\kappa_n}{2\gamma \kappa_{n-1}} t \right)^k dt = \frac{\Gamma(n+k+1)}{\Gamma(n+1)} \left( \frac{n\kappa_n}{2\gamma \kappa_{n-1}} \right)^k.
\]

Then, by (4),

\[
E[|Y|^k] \leq \frac{\Gamma(n+k+1)}{\Gamma(n+1)} \left( \frac{\kappa_n}{\lambda} \right)^{k/n}.
\]

4.2. **Proof of Theorem 4.4.**

By the assumption on \( \lambda_n \) and (4), the intensity \( \gamma_n \) of \( X_n \) satisfies

\[
\gamma_n \sim \frac{e^\lambda}{\sqrt{e}} n \quad \text{as} \quad n \to \infty.
\]

Then, by (2),

\[
\lim_{n \to \infty} \gamma_n \sqrt{n} R \frac{2\kappa_{n-1}}{n^2\kappa_n} = \lim_{n \to \infty} \frac{e^\lambda \sqrt{n} R \frac{2\kappa_{n-1}}{n\kappa_n}}{\sqrt{e}} = \lim_{n \to \infty} 2e^\lambda \sqrt{n} \frac{R}{\sqrt{2\pi n}} = \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi e}}.
\]

Let \( c_n = \frac{e^\lambda \sqrt{n} R}{\kappa_n} \), and \( c = \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi e}} \). Then, by a modified application of Laplace’s method (see A.2), for \( \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi e}} > 1 \),

\[
\Gamma_u(n+1, 2\gamma_n \sqrt{n} R \frac{\kappa_{n-1}}{n\kappa_n}) = \Gamma_u(n+1, c_n n) \sim n^n e^{-n(c_n - \log c_n)} \frac{c^n (c_n - \log c_n)}{(c-1)}.
\]
and for $\frac{e^{R\sqrt{2}}}{\sqrt{\pi} e} < 1$,

$$\Gamma_{\ell}(n + 1, 2\gamma_n \sqrt{n} R^{\frac{\kappa_n - 1}{n\kappa_n}}) = \Gamma_{\ell}(n + 1, c_n n) \sim n^\ell e^{-n(c_n - \log c_n)} (1 - c).$$

Then, by Lemma 4.2, (7), and Stirling’s formula, as $n \to \infty$.

$$P(|Y_n| \geq \sqrt{n} R) \leq \frac{\Gamma_{\ell}(n + 1, c_n n)}{\Gamma(n + 1)} \sim \frac{1}{\sqrt{2\pi n}} \frac{\left(\frac{e}{n}\right)^n n^\ell e^{-n(c_n - \log c_n)}}{(c - 1)} = \frac{c e^{-n(c_n - \log c_n - 1)}}{\sqrt{2\pi n}(c - 1)}.$$

Then, for $R > e^{-\lambda \sqrt{\pi} e}$,

$$\limsup_{n \to \infty} \frac{1}{n} \ln P(|Y_n| \geq \sqrt{n} R) \leq \limsup_{n \to \infty} 1 - c_n + \ln c_n = 1 - c + \ln c = -\frac{e^\lambda R \sqrt{2}}{\sqrt{\pi} e} + \ln \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi}}$$

$$= \lambda + \frac{1}{2} \ln \left(\frac{2e}{\pi}\right) + \ln R - \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi} e}.$$

Similarly, for $R < e^{-\lambda \sqrt{\pi} e}$,

$$P(|Y_n| \leq \sqrt{n} R) \leq \frac{n^{\kappa_n^2}}{4^n} \frac{\left(\frac{\kappa_n}{2\kappa_n - 1}\right)^n}{\Gamma(n + 1, c_n n)} \left(1 + \frac{\Gamma(n) \left(\frac{\kappa_n - 1}{n\kappa_n}\right)^{n+1}}{\Gamma_{\ell}(n + 1, c_n n)}\right).$$

Then, for all $\lambda$ and $R$,

$$\lim_{n \to \infty} \frac{\Gamma(n) \left(\frac{\kappa_n - 1}{n\kappa_n}\right)^{n+1}}{\Gamma_{\ell}(n + 1, c_n n)} = \lim_{n \to \infty} \frac{1}{n} \left(\frac{\kappa_n - 1}{n\kappa_n}\right)^{n+1} \frac{\Gamma(n + 1)}{\Gamma_{\ell}(n + 1, c_n n)} = 0.$$

Hence, by (8) and Stirling’s formula, as $n \to \infty$,

$$P(|Y_n| \leq \sqrt{n} R) \leq \frac{n^{\kappa_n^2}}{4^n} \frac{\left(\frac{\kappa_n}{2\kappa_n - 1}\right)^n}{\Gamma(n + 1, c_n n)} \left(\frac{\pi}{2}\right)^n \frac{c e^{-n(c_n - \log c_n - 1)}}{\sqrt{2\pi n}(1 - c)},$$

and thus

$$\limsup_{n \to \infty} \frac{1}{n} \ln P(|Y_n| \leq \sqrt{n} R) \leq \lambda + \frac{1}{2} \ln \left(\frac{2e}{\pi}\right) + \ln R - \frac{e^\lambda R \sqrt{2}}{\sqrt{\pi} e} - \ln 2 + \ln \pi.$$

**Appendix A. Laplace Method**

**Lemma A.1.** Let $f(t)$ be a function such that $f(t)$ achieves its minimum at $t = a$ on the interval $[a, b]$ and $f'(t)$ is continuous. Also assume $\lim_{n \to \infty} a_n = a \in \mathbb{R}$. If $f'(a) > 0$, then as $n \to \infty$

$$\int_{a_n}^{b} e^{-nf(t)} dt \sim \frac{e^{-nf(a_n)}}{nf'(a)}.$$
If $f(t)$ achieves its minimum at $t = b$ over the interval $(a, b)$, $\lim_{n \to \infty} b_n = b$, and $f'(b) < 0$, then

$$\int_{a}^{b_n} e^{-nf(t)} \, dt \sim -\frac{e^{-nf(b_n)}}{nf'(b)}.$$ 

Proof. Let $\varepsilon > 0$. By the continuity of $f'$, there exists $\min_{n}(b - a_n) > \delta > 0$ such that $|t - a| < 2\delta$ implies $f'(t) \leq f'(a) + \varepsilon$. By Taylor’s theorem, for each $t \in [a_n, b]$, there is some $\xi_t \in (a_n, t)$ such that

$$f(t) = f(a_n) + f'(...(t - a_n).$$

Then, for $t$ such that $|t - a_n| < \delta$ and $n$ large enough such that $|a_n - a| < \delta$, we have that $|\xi_t - a_n| < \delta$, and thus by the triangle inequality, $|\xi_t - a| < 2\delta$, which implies that for all $n$ large enough,

$$f(t) \leq f(a_n) + (f'(a) + \varepsilon)(t - a_n).$$

Since the integrand is positive,

$$\int_{a_n}^{b} e^{-nf(t)} \, dt \geq \int_{a_n}^{a_n + \delta} e^{-nf(t)} \, dt \geq \int_{a_n}^{a_n + \delta} e^{-n(f(a_n) + (f'(a) + \varepsilon)(t - a_n))} \, dt$$

$$= e^{-n(f(a_n))} \int_{0}^{\delta n(f'(a) + \varepsilon)} e^{-\eta d}\, dy = \frac{e^{-nf(a_n)}}{n(f'(a) + \varepsilon)} \left(1 - e^{-\delta n(f'(a) + \varepsilon)}\right).$$

Then,

$$\liminf_{n \to \infty} \frac{\int_{a_n}^{b} e^{-nf(t)} \, dt}{e^{-nf(a_n)} n(f'(a) + \varepsilon)} \geq \liminf_{n \to \infty} \left(1 - e^{-\delta n(f'(a) + \varepsilon)}\right) = 1.$$ 

For $\delta > 0$, let $N$ be such that for all $n > N$, $a_n + \delta > a + \frac{\delta}{2}$. Then, for the upper bound, define

$$C := \inf_{t\in[a + \frac{\delta}{2}, b]} f(t) > f(a),$$

where the last inequality follows from the hypothesis that $f$ achieves its minimum at $a$ on the interval $[a, b)$. By a similar Taylor series argument, we have for all $|t - a_n| < \delta$, and $n$ large enough,

$$f(t) \geq f(a_n) + (f'(a) - \varepsilon)(t - a_n).$$

Define $\eta := C - f(a) > 0$. Then, for all $t \in [a + \frac{\delta}{2}, b]$, $f(t) > f(a) + \eta$. Then, for all $n$ large enough,

$$\int_{a_n}^{b} e^{-nf(t)} \, dt = \int_{a_n}^{a_n + \delta} e^{-nf(t)} \, dt + \int_{a_n + \delta}^{b} e^{-nf(t)} \, dt$$

$$\leq \int_{a_n}^{a_n + \delta} e^{-n(f(a_n) + (f'(a) - \varepsilon)(t - a_n))} \, dt + \int_{a_n + \delta}^{b} e^{-nC} \, dt$$

$$< (b - a)e^{-nC} + \frac{e^{-nf(a_n)}}{n(f'(a) - \varepsilon)} \int_{0}^{\delta n(f'(a) - \varepsilon)} e^{-\eta d}\, dy.$$
\[(b - a)e^{-\eta c} + \frac{e^{-nf(a_n)}}{n(f'(a) - \varepsilon)} \left(1 - e^{-\delta n(f'(a) - \varepsilon)}\right)\].

Then,
\[
\limsup_{n \to \infty} \int_{a_n}^{b} \frac{e^{-nf(t)}}{n(f'(a) - \varepsilon)} dt \leq \limsup_{n \to \infty} \{(b - a)n(f'(a) - \varepsilon)e^{-\eta n} + 1 - e^{-\delta n(f'(a) + \varepsilon)}\} = 1,
\]
since \(\eta > 0\). These limits hold for all \(\varepsilon\), and thus,
\[
\lim_{n \to \infty} \int_{a_n}^{b} \frac{e^{-nf(t)}}{n f'(a)} dt = 1.
\]

**Lemma A.2.** Assume that \(\lim_{n \to \infty} c_n = c \in (0, \infty)\). Then, as \(n \to \infty\), if \(c > 1\),
\[
\Gamma_u(n + 1, c_n n) \sim n^n c e^{n(ln c_n - c_n)} \frac{e^{-n(\ln c_n - c_n)}}{n(c - 1)},
\]
and if \(c < 1\),
\[
\Gamma_\ell(n + 1, c_n n) \sim n^n c e^{n(ln c_n - c_n)} \frac{e^{-n(\ln c_n - c_n)}}{n(1 - c)}.
\]

**Proof.** By a change of variables,
\[
\Gamma_u(n + 1, c_n n) = \int_{c_n n}^{\infty} e^{-t n} dt = n^{n+1} \int_{c_n}^{\infty} e^{-n(y - \log y)} dy.
\]

For \(c > 1\), letting \(f(y) = y - \log(y)\), \(f\) has a minimum at \(c\) on the interval \([c, \infty)\). Also observe that \(f'(y) = 1 - \frac{1}{y}\), and for \(c > 1\), \(f'(c) = 1 - \frac{1}{c} > 0\). Thus, by Lemma \[\text{A.1}\]
\[
\Gamma_u(n + 1, c_n n) \sim n^{n+1} \frac{e^{-n(\ln c_n - c_n)}}{n(1 - \frac{1}{c})} = n^n c e^{n(ln c_n - c_n)} \frac{e^{-n(\ln c_n - c_n)}}{(c - 1)}.
\]

Similarly,
\[
\Gamma_\ell(n + 1, c_n n) = \int_{0}^{c_n n} e^{-t n} dt = n^{n+1} \int_{0}^{c_n} e^{-n(y - \log y)} dy,
\]
and for \(c < 1\), \(f(y) = y - \log(y)\) hits its minimum at \(c\) on the interval \((0, c]\) and \(f'(c) = 1 - \frac{1}{c} < 0\). Then, by Lemma \[\text{A.1}\]
\[
\Gamma_\ell(n + 1, c_n n) \sim n^n c e^{n(ln c_n - c_n)} \frac{e^{-n(\ln c_n - c_n)}}{(1 - c)}.
\]
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