On super polyharmonic property of high-order fractional Laplacian

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Abstract
Let $0 < \alpha < 2$, $p \geq 1$, $m \in \mathbb{N}_+$. Consider $u$ to be the positive solution of the PDE
\[
(\Delta)^{\frac{\alpha}{2}} + mu(x) = u^p(x) \quad \text{in } \mathbb{R}^n.
\]
(0.1)

In [2] (Transactions of the American mathematical society, 2021), Cao, Dai and Qin showed that, under the condition $u \in L^{\alpha}$, (0.1) possesses super polyharmonic property $(\Delta)^k + \frac{\alpha}{2} u \geq 0$ for $k = 0, 1, \ldots, m - 1$. In this paper, we show another kind of super polyharmonic property $(\Delta)^k u > 0$ for $k = 1, \ldots, m$ under different conditions $(\Delta)^m u \in L^{\alpha}$ and $(\Delta)^m u \geq 0$. Both kinds of super polyharmonic properties can lead to the equivalence between (0.1) and the integral equation $u(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+\alpha}} dy$. One can classify solutions to (0.1) following the work of [6] by Chen, Li, Ou.

Keywords: Super polyharmonic, fractional Laplacian, equivalence, classification.

1 Introduction

In this paper we consider the partial differential equation
\[
(\Delta)^{\frac{\alpha}{2}} + mu(x) = u^p(x) \quad \text{in } \mathbb{R}^n,
\]
(1.1)
under conditions that $n \geq 2$, $0 < \alpha < 2$, $p \geq 1$, $m \in \mathbb{N}_+$.

For $\alpha$ taking any real number between 0 and 2, $(\Delta)^{\frac{\alpha}{2}}$ is a nonlocal differential operator defined by
\[
(\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy.
\]
(1.2)
where PV is the Cauchy principal value. (1.2) is valid for $u \in C^{(\alpha)}_{loc}(\mathbb{R}^n) \cap L^{\alpha}(\mathbb{R}^n)$, where
\[
L^{\alpha}(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.
\]
(1.3)
The super polyharmonic property $(\Delta)^k + \frac{\alpha}{2} u(x) \geq 0$ for $k = 0, 1, \ldots, m - 1$ of (1.1) was first proved in [2] (Transactions of the American mathematical society, 2021) by Cao, Dai and Qin under the condition that $u \in L^{\alpha}$. In fact they worked on more general equations of the form $(\Delta)^m u(x) = f(x, u, Du, \ldots)$. Their idea is to use Green’s formula and Poisson kernel to represent the PDE.

Inspired by their method, here in this paper we obtain another kind of super polyharmonic property under a quite different definition. We write $(\Delta)^{\frac{\alpha}{2}} + mu(x)$ as $(\Delta)^{\frac{\alpha}{2}} (\Delta)^m u(x)$, and therefore it is nature to require $(\Delta)^m u(x) \in L^{\alpha}$. Moreover we require $(\Delta)^m u(x) \geq 0$. And the result is a different kind of super polyharmonic property $(\Delta)^k u(x) \geq 0$ for $k = 1, \ldots, m$.
Theorem 1.1. Let $n \geq 2$, $0 < \alpha < 2$, $p \geq 1$, $m \in \mathbb{N}_+$. Suppose $(-\Delta)^m u \in L_\alpha(\mathbb{R}^n)$, $u \in C^\infty(\mathbb{R}^n)$ and $u$ is a positive solution of (1.1). Moreover, suppose $(-\Delta)^m u \geq 0$ in $\mathbb{R}^n$. Then for $k = 1, \ldots, m$, $(-\Delta)^k u(x) > 0$ in $\mathbb{R}^n$. \hfill (1.4)

The proof for $(-\Delta)^m u \geq 0$ stays unsolved and will be worked on in the future.

In theorem (1.1) we require $u \in C^\infty(\mathbb{R}^n)$, but it is not necessary. In fact one only needs $C^{2m+|\alpha|, \alpha}_\text{loc}(\mathbb{R}^n)$ (see [13], proposition 2.4) to make sure $(-\Delta)^{\frac{n}{2}+m} u$ is a continuous functions and its value is given by (1.2). Nevertheless, $u \in C^\infty(\mathbb{R}^n)$ is enough since the precise regularity is not the main goal in this paper.

(1.1) is closely related to the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{w^p(y)}{|x-y|^{n-2m-\alpha}} dy.$$ \hfill (1.5)

Once super polyharmonic property holds, then it is quick to obtain the equivalence between (1.5) and (1.1). Then one can use Chen, Li, Ou’s work [6] [7] about classification of solutions to integral equations and therefore obtain the classification of high-order Laplacian.

Theorem 1.2. Suppose conditions in theorem (1.1) hold, and

$$\int_{\mathbb{R}^n} \frac{w^p(y)}{|x-y|^{n-2m-\alpha}} dy < +\infty \text{ for any } x \in \mathbb{R}^n.$$ \hfill (1.6)

Let $u$ be a positive solution of (1.1). Then $u$ also solves (1.3), and vice versa.

Corollary 1.3. The solution $u$ as given in theorem (1.3) satisfies

1. For $0 < 2m + \alpha < n$, in the critical case $p = \frac{n+2m+\alpha}{n-2m-\alpha}$, $u$ must have the form

$$u(x) = c \left( \frac{t}{t^2 + |x-x_0|^2} \right)^{\frac{n-2m-\alpha}{2}},$$ \hfill (1.7)

with some constant $c = c(n, m, \alpha)$ and for some $t > 0$ and $x_0 \in \mathbb{R}^n$.

2. For $0 < 2m + \alpha < n$, in the subcritical case $1 \leq p \leq \frac{n+2m}{n-2m-\alpha}$, $u$ does not exist.

Corollary (1.3) is the immediate consequence of theorem (1.2) and classification results in [6] [7].

Super polyharmonic property of high-order Laplacian has long been studied. It leads directly to Liouville theorem and equivalence between integral equations like (1.5) and PDEs like (1.1). For integer high-order Laplacian, Wei and Xu [15] first gave the super polyharmonic property for even order equations. After a dozen years Chen and Li [4] proved the general case and covered Wei and Xu’s work. There are also much work on other types of super polyharmonic property, see [11] [3] [10] [9] [19].

For more results about fractional equation $(-\Delta)^{\frac{n}{2}} = u^p$, polyharmonic equations and maximum principle about fractional Laplacian, please refer to [8] [18] [16] [12].

2 Proof of theorem (1.1)

Proof. Denote $(-\Delta)^k u(x)$ as $u_k(x)$ for $k = 1, \ldots, m$. Then (1.1) can be rewritten as

$$\begin{cases}
-\Delta u = u_1 \\
-\Delta u_1 = u_2 \\
\vdots \\
-\Delta u_{m-1} = u_m \\
(-\Delta)^{\frac{n}{2}} u_m = u^p
\end{cases} \text{ in } \mathbb{R}^n. \hfill (2.1)$$

$$\begin{cases}
-\Delta u_1 = u_2 \\
-\Delta u_2 = u_3 \\
\vdots \\
-\Delta u_{m-2} = u_{m-1} \\
(-\Delta)^{\frac{n}{2}} u_{m-1} = u^p
\end{cases} \text{ in } \mathbb{R}^n.$$ \hfill (2.2)
First show that if \((-\Delta)^m u\) is nonnegative, then it must be positive. Indeed, if there exists \(x_0\) such that \((-\Delta)^m u(x_0) = u_m(x_0) = 0\), then \(u_m(x_0) - u_m(y) \leq 0\) for any \(y \in \mathbb{R}^n\). So
\[
(-\Delta)^m u_m(x_0) = \int_{\mathbb{R}^n} \frac{u_m(x_0) - u_m(y)}{|x-y|^{n+m}} dy \leq 0, \tag{2.2}
\]
contradicted with \((-\Delta)^m u_m(x_0) = u^p(x_0) > 0\).

Now prove \(u_k(x) > 0\) for \(k = 1, ..., m\) by contradiction. Suppose NOT, then there exists a largest integer \(k \in \{1, 2, ..., m-1\}\) such that \(u_k\) is less or equal to 0 somewhere. Without loss of generality suppose \(u_k(0) \leq 0\). Then there are two cases:

1. \(u_k(0) = 0, \text{ and } u_k \geq 0 \text{ in } \mathbb{R}^n\).
2. \(u_k(0) < 0\).

Case 1 is impossible. The reason is, under this case, 0 is a minimum point of \(u_k\). However \(-\Delta u_k = u_{k+1} \geq 0\), by maximum principle \(u_k\) cannot obtain minimum. So only need to consider case 2.

We argue by two steps to see such \(k\) does not exist.

Step 1, \(k\) cannot be even. Denote by \(\overline{v}\) the spherical average of \(v\) centered at 0, i.e.
\[
\overline{v}(r) = \frac{1}{|\partial B(0, r)|} \int_{\partial B(0, r)} v(y) dS_y. \tag{2.3}
\]
Use the well-known property \(\Delta \overline{v} = \overline{\Delta v}\) and \(-\Delta \overline{v}(r) = -\frac{1}{r^{n-1}} (r^{n-1} \overline{v}')'\) (here ' means taking derivative about \(r\)) to get
\[
-\Delta \overline{u_k}(r) = -\frac{1}{r^{n-1}} (r^{n-1} \overline{u_k}')' = \overline{u_{k+1}}(r) > 0 \quad \text{for any } r > 0. \tag{2.4}
\]
Then \((r^{n-1} \overline{u_k})' < 0\). Integrate on both sides to get \(r^{n-1} \overline{u_k}' < 0\). So
\[
\overline{u_k}'(r) < 0 \quad \text{for any } r > 0. \tag{2.5}
\]
Thus
\[
\overline{u_k}(r) < \overline{u_k}(0) = -a_k < 0 \quad \text{for any } r > 0. \tag{2.6}
\]
Using \(-\Delta \overline{u_{k-1}} = \overline{u_k}\) one obtains
\[
-\frac{1}{r^{n-1}} (r^{n-1} \overline{u_{k-1}})' < -a_k \quad \text{for any } r > 0, \tag{2.7}
\]
and in turn implies
\[
\overline{u_{k-1}}(r) > \overline{u_{k-1}}(0) + \frac{a_k}{2n} r^2 \quad \text{for any } r > 0. \tag{2.8}
\]
If \(k\) is even, repeat the process above to get
\[
\overline{v}(r) < \overline{v}(0) + c_1 r^2 - c_2 r^4 + ... + c_k r^{2k} \quad \text{for any } r > 0, \tag{2.9}
\]
where we know \(c_k > 0\) but do not know the sign of other \(c_i, i = 1, ..., k-1\). For \(r\) sufficiently large, \(\overline{v}(r)\) attains negative value, contradicted to \(u > 0\).

Step 2, \(k\) cannot be odd. If \(k\) is odd, repeat the similar process above to get
\[
\overline{v}(r) > \overline{v}(0) - c_1 r^2 + c_2 r^4 - ... + c_k r^{2k} \quad \text{for any } r > 0, \tag{2.10}
\]
where we know $c_k > 0$ but do not know the sign of other $c_i$, $i = 1, ..., k - 1$. Hence there exists $r_0 > 0$ and a positive constant $M$ such that
\[ \pi(r) > M > 0, \quad \text{for any } r > r_0. \] (2.11)

Let $u_m = v_1 + v_2$ with $v_1, v_2$ satisfying
\[ \begin{cases} (-\Delta)^\frac{\alpha}{2} v_1 = u^p, & \text{in } B(0, R), \\ v_1 = 0, & \text{on } \mathbb{R}^n \setminus B(0, R), \end{cases} \] (2.12)
and
\[ \begin{cases} (-\Delta)^\frac{\alpha}{2} v_2 = 0, & \text{in } B(0, R), \\ v_2 = u_m, & \text{on } \mathbb{R}^n \setminus B(0, R), \end{cases} \] (2.13)
for any $R > 0$. By [1] it follows
\[ u_m(x) = \int_{B(0,R)} G_R^\alpha(x,y)u^p(y)dy + \int_{\mathbb{R}^n \setminus B(0,R)} P_R^\alpha(y,x)u_m(y)dy. \] (2.14)

Here $G_R^\alpha(x,y)$ and $P_R^\alpha(y,x)$ represent the Green’s function and Poisson kernel of $\alpha$-Laplacian in $B(0,R)$, respectively. The formulas are
\[ G_R^\alpha(x,y) = \frac{\kappa(n, \alpha)}{|y-x|^{n-\alpha}} \int_0^{R_0(x,y)} \frac{t^{2\alpha-1}}{(t+1)^{\frac{\alpha}{2}}} dt, \] (2.15)
\[ P_R^\alpha(y,x) = c(n, \alpha) \frac{(R^2 - |x|^2)^{\frac{\alpha}{2}}}{|y|^2 - R^2} \frac{1}{|x-y|^n} \quad \text{for } x \in B(0, R) \text{ and } y \in \mathbb{R}^n \setminus \overline{B}(0, R) \] (2.16)
with
\[ R_0(x,y) = \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2|x-y|^2} \] (2.17)
and $\kappa(n, \alpha), c(n, \alpha)$ are constants depending only on $\alpha$ and $n$. Then
\[ u_m(0) = \int_{B(0,R)} \frac{\kappa(n, \alpha)}{|y|^{n-\alpha}} \left( \int_0^{R_0(x,y)} \frac{t^{\frac{\alpha}{2}-1}}{(1+t)^{\frac{\alpha}{2}}} dt \right) u^p(y)dy + c(n, \alpha) \int_{\mathbb{R}^n \setminus B(0,R)} \frac{R^\alpha}{(|y|^2 - R^2)^{\frac{\alpha}{2}}} \cdot \frac{u_m(y)}{|y|^n} dy \]
\[ = \kappa(n, \alpha) \int_0^R r^{\alpha-1} \left( \int_0^{R_0(x,y)} \frac{t^{\frac{\alpha}{2}-1}}{(1+t)^{\frac{\alpha}{2}}} dt \right) u^p(r)dr + c(n, \alpha) \int_{R}^{+\infty} \frac{R^\alpha}{r(r^2 - R^2)^{\frac{\alpha}{2}}} \pi_m(r)dr \]
\[ \geq \kappa(n, \alpha) \int_0^R r^{\alpha-1} \left( \int_0^{R_0(x,y)} \frac{t^{\frac{\alpha}{2}-1}}{(1+t)^{\frac{\alpha}{2}}} dt \right) \pi^p(r)dr. \] (2.18)

For $0 < r < \frac{R}{2}$,
\[ \int_0^{R_0(x,y)} \frac{t^{\frac{\alpha}{2}-1}}{(1+t)^{\frac{\alpha}{2}}} dt \geq \int_0^3 \frac{t^{\frac{\alpha}{2}-1}}{(1+t)^{\frac{\alpha}{2}}} dt, \] (2.19)
and the right hand side is a constant. Choose \( R \) satisfying \( R > 2r_0 \). Then

\[
\int_0^{r_0} r^{\alpha-1} \left( \int_0^{R^2} t^{\frac{\alpha-1}{2}} \frac{dt}{(1+t)^{\frac{\alpha}{2}}} \right) \mu^p(r) dr \geq 0,
\]

\[
\int_{r_0}^{R} r^{\alpha-1} \left( \int_0^{R^2} t^{\frac{\alpha-1}{2}} \frac{dt}{(1+t)^{\frac{\alpha}{2}}} \right) \mu^p(r) dr \geq C \int_{r_0}^{R} r^{\alpha-1} dr
\]

\[
= C_1 R^\alpha - C_2 \rightarrow +\infty \quad \text{as} \quad R \rightarrow +\infty,
\]

Combining (2.18) and (2.20), it follows that \( u_m(0) = \infty \), contradicted.

3 Proof of theorem 1.2

To prove the equivalence, one needs to show that any solution of one equation also solves another equation. It is easy that any solution of integral equation (1.5) also solves PDE (1.1). So it suffices to show any solution of PDE solves integral equation.

Denote the Riesz potential as

\[
(I_\alpha f)(x) = \frac{1}{\gamma(\alpha, n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,
\]

where

\[
\gamma(\alpha, n) = \frac{\pi^{\frac{n}{2}}2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.
\]

The proof relies on two properties of Riesz potential (see [14], chapter V, section 1.1).

1. \((-\Delta)^{\frac{\alpha}{2}}(I_\alpha f) = f\) for \(0 < \alpha < n\).

2. \(I_\alpha(I_\beta f) = I_{\alpha+\beta} f\) for \(\alpha > 0, \beta > 0\) and \(\alpha + \beta < n\).

A brief outline is given here. Rewrite (1.1) as

\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u_m &= u^p \\
-\Delta u_{m-1} &= u_m \\
& \quad \text{in } \mathbb{R}^n \\
-\Delta u_1 &= u_2 \\
-\Delta u &= u_1
\end{aligned}
\]

and then prove by Liouville theorem and induction that

\[
\begin{aligned}
u_m(x) &= f_1(x) := C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy \\
u_{m-1}(x) &= f_2(x) := C \int_{\mathbb{R}^n} \frac{f_1(y)}{|x-y|^{n-\alpha}} dy \\
& \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]

Finally by property 2 of Riesz potential it follows that

\[
u(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-2m-\alpha}} dy.
\]
Proof. Set \( v_0^R(x) = \int_{B(0,R)} G_R^\alpha(x,y)u^p(y)dy \), where \( G_R^\alpha(x,y) \) is the Green’s function for fractional Laplacian as stated in (2.15). Then

\[
\begin{aligned}
\begin{cases}
(-\Delta)^\frac{\alpha}{2} v_0^R = u^p, & \text{in } B(0,R), \\
v_0^R = 0, & \text{on } \mathbb{R}^n \setminus B(0,R).
\end{cases}
\end{aligned}
\tag{3.3}
\]

Set \( w_0^R = u_m - v_0^R \). It follows that

\[
\begin{aligned}
\begin{cases}
(-\Delta)^\frac{\alpha}{2} w_0^R = 0, & \text{in } B(0,R), \\
w_0^R \geq 0, & \text{on } \mathbb{R}^n \setminus B(0,R).
\end{cases}
\end{aligned}
\tag{3.4}
\]

By maximum principle (see [13], proposition 2.17), \( w_0^R \geq 0 \). Sending \( R \to \infty \), then \( u_m(x) \geq C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy \). Let \( w_0 = u_m - C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy \). Then

\[
\begin{aligned}
\begin{cases}
(-\Delta)^\frac{\alpha}{2} w_0 = 0, & \text{in } \mathbb{R}^n, \\
w_0 \geq 0, & \text{in } \mathbb{R}^n.
\end{cases}
\end{aligned}
\tag{3.5}
\]

By Liouville theorem (see [17], theorem 1), \( w_0 \equiv C_0 \geq 0 \), i.e

\[
u_m(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy + C_0.
\tag{3.6}
\]

In fact \( C_0 \) can only be 0 thus \( u_m(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy \), which will be proved later.

Let

\[
f_1(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy + C_0,
\tag{3.7}
\]

\[
v_1^R(x) = \int_{\mathbb{R}^n} G_R^2(x,y)f_1(y)dy,
\tag{3.8}
\]

\[
w_1^R = u_{m-1} - v_1^R.
\tag{3.9}
\]

where \( G_R^2(x,y) \) is the Green’s function of Laplacian \( \Delta \) in \( B(0,R) \). Then

\[
\begin{aligned}
\begin{cases}
-\Delta w_1^R = 0 & \text{in } B(0,R), \\
w_1^R \geq 0, & \text{on } \mathbb{R}^n \setminus B(0,R).
\end{cases}
\end{aligned}
\tag{3.10}
\]

By maximum principle of harmonic functions, \( w_1^R \geq 0 \). Sending \( R \to \infty \), then \( u_{m-1}(x) \geq C \int_{\mathbb{R}^n} \frac{f_1(y)}{|x-y|^{n-\alpha}}dy \). Let \( w_1 = u_{m-1} - C \int_{\mathbb{R}^n} \frac{f_1(y)}{|x-y|^{n-\alpha}}dy \). Then

\[
\begin{aligned}
\begin{cases}
(-\Delta)^\frac{\alpha}{2} w_1 = 0, & \text{in } \mathbb{R}^n, \\
w_1 \geq 0, & \text{in } \mathbb{R}^n.
\end{cases}
\end{aligned}
\tag{3.11}
\]

By Liouville theorem by ones for harmonic functions, one obtains

\[
u_{m-1}(x) = C \int_{\mathbb{R}^n} \frac{f_1(y)}{|x-y|^{n-\alpha}}dy + C_1
\tag{3.12}
\]

for some \( C_1 \geq 0 \). Moreover, (3.12) implies \( C_0 = 0 \), otherwise

\[
+\infty > C \int_{\mathbb{R}^n} \frac{f_1(y)}{|x-y|^{n-\alpha}}dy \geq \int_{\mathbb{R}^n} \frac{C_0}{|x-y|^{n-\alpha}}dy = +\infty.
\tag{3.13}
\]
Let
\[ f_k(x) = C \int_{\mathbb{R}^n} \frac{f_{k-1}(y)}{|x-y|^{n-\alpha}} \, dy + C_{k-1}, \quad (3.14) \]
\[ v_k^R(x) = \int_{\mathbb{R}^n} G_k^2(x,y)f_k(y) \, dy, \quad (3.15) \]
for \( k = 2, 3, \ldots, m-1 \). Use similar process and induction, one proves that
\[ u_{m-k}(x) = C \int_{\mathbb{R}^n} \frac{f_k(y)}{|x-y|^{n-\alpha}} \, dy \text{ and } C_k = 0 \text{ for } k = 1, 3, \ldots, m-1, \quad (3.16) \]
\[ u(x) = C \int_{\mathbb{R}^n} \frac{f_m(y)}{|x-y|^{n-\alpha}} \, dy + C_m \text{ for some } C_m \geq 0. \quad (3.17) \]
Apply property 2 to \( \{f_k\} \) one obtains that
\[ u(x) = C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-2m-\alpha}} \, dy + C_m. \quad (3.18) \]
Note that \( u \geq C_m \). If \( C_m > 0 \), then
\[ + \infty > u(x) > C \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-2m-\alpha}} \, dy \geq \int_{\mathbb{R}^n} \frac{C_m}{|x-y|^{n-2m-\alpha}} \, dy = + \infty. \quad (3.19) \]
Thus \( C_m = 0 \), and \( u \) indeed solves integral equation (1.5).

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