Padé approximant related to asymptotics for the gamma function

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Abstract
Based on the Padé approximation method, we determine the coefficients $a_j$ and $b_j$ ($1 \leq j \leq k$) such that
\[ \frac{\Gamma(x+1)}{\sqrt{2\pi x \frac{x}{e}}^x} = \sum_{j=1}^{k} a_j x^{k-j} + \sum_{j=1}^{k} b_j x^{k-j} + \frac{1}{x^{2k+1}}, \quad x \to \infty, \]
where $k \geq 1$ is any given integer. Based on the obtained result, we establish new bounds for the gamma function.

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1 Introduction
Stirling’s formula
\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad n \in \mathbb{N} := \{1, 2, \ldots\} \] (1.1)
has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667-1754) in the form
\[ n! \sim \text{constant} \cdot \sqrt{n} (n/e)^n \]
when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692-1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling’s series for the gamma function is given (see [1], p.257, Eq. (6.1.40)) by
\[ \Gamma(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \exp \left( \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}} \right) \] (1.2)
as $x \to \infty$, where $B_n (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \quad |z| < 2\pi.$$  \quad (1.3)

The following asymptotic formula is due to Laplace:

$$\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots \right)$$  \quad (1.4)

as $x \to \infty$ (see [1], p.257, Eq. (6.1.37)).

The expression (1.4) is sometimes incorrectly called Stirling’s series (see [2], pp.2-3). Stirling’s formula is in fact the first approximation to the asymptotic formula (1.4). Stirling’s formula has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [3–54] and the references cited therein). It is interesting to note that the aforementioned mathematicians represent many nationalities. So the topic is of interest for mathematicians from diverse cultural background.

Using the Maple software, we find, as $x \to \infty$,

$$\frac{\Gamma(x + 1)}{\sqrt{2\pi x}(x/e)^x} = \frac{x + \frac{1}{24}}{x - \frac{1}{24}} + O\left( \frac{1}{x^5} \right)$$  \quad (1.5)

and

$$\frac{\Gamma(x + 1)}{\sqrt{2\pi x}(x/e)^x} = \frac{x^2 + \frac{1}{24}x + \frac{293}{8640}}{x^2 - \frac{1}{24}x + \frac{293}{8640}} + O\left( \frac{1}{x^5} \right).$$  \quad (1.6)

Based on the Padé approximation method, in this paper we develop the approximation formulas (1.5) and (1.6) to produce a general result. More precisely, we determine the coefficients $a_j$ and $b_j$ ($1 \leq j \leq k$) such that

$$\frac{\Gamma(x + 1)}{\sqrt{2\pi x}(x/e)^x} = \frac{x^k + a_1 x^{k-1} + \cdots + a_k}{x^k + b_1 x^{k-1} + \cdots + b_k} + O\left( \frac{1}{x^{2k+1}} \right), \quad x \to \infty,$$

where $k \geq 1$ is any given integer. Based on the obtained result, we establish new bounds for the gamma function.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2 Lemmas

The following lemmas are required in our present investigation.

**Lemma 2.1** ([9]) Let $r$ be a given nonzero real number. The gamma function has the following asymptotic formula:

$$\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \sum_{j=1}^{\infty} \frac{b_j}{x^j} \right)^{1/r}, \quad x \to \infty,$$

\quad (2.1)
with the coefficients \( b_j = b_j(r) \) \((j = 1, 2, \ldots)\) given by
\[
b_j = \sum_{k_1 + 2k_2 + \cdots + jk_j = j} r^{k_1 + k_2 + \cdots + k_j} \left( \frac{B_2}{1 \cdot 2} \right)^{k_1} \left( \frac{B_3}{2 \cdot 3} \right)^{k_2} \cdots \left( \frac{B_{j+1}}{j(j+1)} \right)^{k_j},
\]
(2.2)
where \( B_n \) \((n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})\) are the Bernoulli numbers defined in (1.3), summed over all nonnegative integers \( k_j \) satisfying the equation \( k_1 + 2k_2 + \cdots + jk_j = j \).

Laplace formula (1.4) can be rewritten as
\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( \sum_{j=0}^{\infty} \frac{c_j}{x^j} \right), \quad x \to \infty,
\]
with the coefficients \( c_j \) given by
\[
c_0 = 1,
\]
\[
c_j = \sum_{k_1 + 2k_2 + \cdots + jk_j = j} \frac{1}{k_1!k_2!\cdots k_j!} \left( \frac{B_2}{1 \cdot 2} \right)^{k_1} \left( \frac{B_3}{2 \cdot 3} \right)^{k_2} \cdots \left( \frac{B_{j+1}}{j(j+1)} \right)^{k_j} \text{ for } j \geq 1.
\]
(2.4)

Lemma 2.2 ([55], Theorem 8) Let \( n \geq 0 \) be an integer. The functions
\[
F_n(x) = \ln \Gamma(x) - \left( x - \frac{1}{2} \right) \ln x + x - \frac{1}{2} \ln(2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}
\]
and
\[
G_n(x) = -\ln \Gamma(x) + \left( x - \frac{1}{2} \right) \ln x - x + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}
\]
are completely monotonic on \((0, \infty)\). Here \( B_n \) \((n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})\) are the Bernoulli numbers.

Remark 2.1 Lemma 2.2 can be stated as follows: for every \( m \in \mathbb{N}_0 \), the function
\[
R_m(x) = (-1)^m \left[ \ln \Gamma(x) - \left( x - \frac{1}{2} \right) \ln x + x - \ln \sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right]
\]
is completely monotonic on \((0, \infty)\).

In 2006, Koumandos [56] presented a simpler proof of complete monotonicity of the functions \( R_m(x) \). In 2009, Koumandos and Pedersen [57], Theorem 2.1, strengthened this result.

From \( F_n'(x) < 0 \) and \( G_n'(x) < 0 \) for \( x > 0 \), we obtain
\[
\sum_{j=1}^{2n} \frac{B_{2j}}{2jx^{2j}} < \ln x - \psi(x) - \frac{1}{2x} < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2jx^{2j}}, \quad x > 0,
\]
(2.5)
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the psi (or digamma) function. Noting that
\[
\psi(x + 1) = \psi(x) + \frac{1}{x}
\]
holds, we obtain from (2.5) that for \( x > 0 \),
\[
\begin{align*}
- \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} &< \psi(x + 1) - \ln x - \frac{1}{2x} \\
< - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \frac{691}{32760x^{12}}.
\end{align*}
\]

(2.6)

3 Approximations to the gamma function

For our later use, we introduce Padé approximant (see [58–61]). Let \( f \) be a formal power series
\[
f(t) = c_0 + c_1 t + c_2 t^2 + \cdots .
\]

(3.1)

The Padé approximation of order \((p, q)\) of the function \( f \) is the rational function, denoted by
\[
[p/q]_f(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j},
\]

(3.2)

where \( p \geq 0 \) and \( q \geq 1 \) are two given integers, the coefficients \( a_j \) and \( b_j \) are given by (see [58–60])

\[
\begin{align*}
a_0 &= c_0, \\
a_1 &= c_0 b_1 + c_1, \\
a_2 &= c_0 b_2 + c_1 b_1 + c_2, \\
a_p &= c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\
0 &= c_{p+1} + c_p b_1 + \cdots + c_{p+q} b_q, \\
&\vdots \\
0 &= c_{p+q} + c_{p+1} b_1 + \cdots + c_p b_q,
\end{align*}
\]

(3.3)

and the following holds:
\[
[p/q]_f(t) - f(t) = O(t^{p+q+1}).
\]

(3.4)

Thus, the first \( p + q + 1 \) coefficients of the series expansion of \([p/q]_f\) are identical to those of \( f \). Moreover, we have (see [61])
\[
[p/q]_f(t) = \begin{pmatrix}
\epsilon_p & \epsilon_p & \cdots & \epsilon_{p-q} & 1 \\
\epsilon_{p-q+1} & \epsilon_{p-q+2} & \cdots & \epsilon_{p+1} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{p+q-1} & \epsilon_{p+q-2} & \cdots & \epsilon_{p+q} \\
\epsilon_{p+q+1} & \epsilon_{p+q+2} & \cdots & \epsilon_{p+q+1} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_p & \epsilon_{p+1} & \cdots & \epsilon_{p+q}
\end{pmatrix}
\]

(3.5)
with \( f_n(x) = c_0 + c_1 x + \cdots + c_n x^n \), the \( n \)th partial sum of the series \( f \) in (3.1) \((f_n \) is identically zero for \( n < 0 \)).

Let

\[
f(x) = \frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^x}}.
\]

(3.6)

It follows from (2.3) that, as \( x \to \infty \),

\[
f(x) \sim \sum_{j=0}^{\infty} c_j x^j = 1 + \frac{1}{12} x + \frac{1}{288} x^2 - \frac{139}{51,840} x^3 - \frac{571}{2,488,320} x^4 + \cdots,
\]

(3.7)

with the coefficients \( c_j \) given by (2.4). In what follows, the function \( f \) is given in (3.6).

Based on the Padé approximation method, we now give a derivation of formula (1.5). To this end, we consider

\[
[1/1]_f(x) = \frac{\sum_{j=0}^{1} a_j x^{-j}}{1 + \sum_{j=1}^{1} b_j x^{-j}}.
\]

Noting that

\[
c_0 = 1, \quad c_1 = \frac{1}{12}, \quad c_2 = \frac{1}{288}, \quad c_3 = -\frac{139}{51,840}, \quad c_4 = -\frac{571}{2,488,320}
\]

(3.8)

holds, we have, by (3.3),

\[
\begin{align*}
    a_0 &= 1, \\
    a_1 &= b_1 + \frac{1}{12}, \\
    0 &= \frac{1}{288} + \frac{1}{12} b_1,
\end{align*}
\]

that is,

\[
a_0 = 1, \quad a_1 = \frac{1}{24}, \quad b_1 = -\frac{1}{24}.
\]

We thus obtain that

\[
[1/1]_f(x) = \frac{1 + \frac{1}{24} x}{1 - \frac{1}{24} x} = \frac{x + \frac{1}{24} x}{x - \frac{1}{24} x},
\]

(3.9)

and we have, by (3.4),

\[
\frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^x}} = \frac{x + \frac{1}{24} x}{x - \frac{1}{24} x} + O \left( \frac{1}{x^2} \right).
\]

We now give a derivation of formula (1.6). To this end, we consider

\[
[2/2]_f(x) = \frac{\sum_{j=0}^{2} a_j x^{-j}}{1 + \sum_{j=1}^{2} b_j x^{-j}}.
\]
Noting that (3.8) holds, we have, by (3.3),

\[
\begin{align*}
& a_0 = 1, \\
& a_1 = b_1 + \frac{1}{12}, \\
& a_2 = b_2 + \frac{1}{12} b_1 + \frac{1}{288}, \\
& 0 = -\frac{139}{51,840} b_1 + \frac{1}{12} b_2, \\
& 0 = -\frac{571}{4,488,320} - \frac{139}{51,840} b_1 + \frac{1}{288} b_2,
\end{align*}
\]

that is,

\[
\begin{align*}
a_0 &= 1, \\
a_1 &= \frac{1}{24}, \\
a_2 &= \frac{293}{8,640}, \\
b_1 &= -\frac{1}{24}, \\
b_2 &= \frac{293}{8,640}.
\end{align*}
\]

We thus obtain that

\[
\frac{[2/2]}{f(x)} = \frac{1 + \frac{1}{24} x + \frac{293}{8,640} x^2}{1 - \frac{1}{24} x + \frac{293}{8,640} x^2} = \frac{x^2 + \frac{1}{24} x + \frac{293}{8,640}}{x^2 - \frac{1}{24} x + \frac{293}{8,640}},
\]

and we have, by (3.4),

\[
\frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^x}} = \frac{x^2 + \frac{1}{24} x + \frac{293}{8,640}}{x^2 - \frac{1}{24} x + \frac{293}{8,640}} + O\left(\frac{1}{x^3}\right).
\]

From the Padé approximation method and the expansion (3.7), we now present a general result given by Theorem 3.1.

**Theorem 3.1** The Padé approximation of order \((p, q)\) of the Laplace asymptotic formula of the function \(f(x) = \frac{\Gamma(x+1)}{\sqrt{2\pi x(x/e)^x}}\) at the point \(x = \infty\) is the following rational function:

\[
[p/q]_f(x) = \frac{1 + \sum_{j=1}^{q} a_j x^{-j}}{1 + \sum_{j=1}^{p} b_j x^{-j}} = x^{p-q} \left( \frac{x^p + a_1 x^{p-1} + \cdots + a_p}{x^q + b_1 x^{q-1} + \cdots + b_q} \right),
\]

where \(p \geq 1\) and \(q \geq 1\) are any given integers, the coefficients \(a_j\) and \(b_j\) are given by

\[
\begin{align*}
a_1 &= b_1 + c_1, \\
a_2 &= b_2 + c_1 b_1 + c_2, \\
& \vdots \\
a_p &= b_p + \cdots + c_{p-1} b_1 + c_p, \\
0 &= c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\
& \vdots \\
0 &= c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q,
\end{align*}
\]

and \(c_j\) is given in (2.4), and the following holds:

\[
f(x) - [p/q]_f(x) = O\left(\frac{1}{x^{p+q+1}}\right), \quad x \to \infty.
\]
Moreover, we have

\[
[p/q](x) = \frac{\frac{1}{p} f_p(x) \frac{1}{q} f_{p+q}(x) - f_p(x)}{c_p \frac{1}{p} f_p(x) - \frac{1}{q} f_{p+q}(x) - c_p + \frac{1}{q} c_p + \cdots + c_p} = (3.14)
\]

with \( f_n(x) = \sum_{j=0}^{n} \frac{c_j}{x^j} \), the \( n \)th partial sum of the asymptotic series (3.7).

**Remark 3.1** Using (3.14), we can also derive (3.9) and (3.10). Indeed, we have

\[
[1/1](x) = \frac{\frac{1}{p} f_0(x) f_1(x)}{c_1 c_2} = \frac{\frac{1}{1} + \frac{1}{p}}{\frac{1}{1} + \frac{1}{p}} = x + \frac{1}{24}
\]

and

\[
[2/2](x) = \frac{\frac{1}{p} f_0(x) \frac{1}{q} f_1(x) f_2(x)}{c_1 c_2 c_3 c_4} = \frac{\frac{1}{1} + \frac{1}{p} + \frac{1}{q} + \frac{1}{pq}}{\frac{1}{1} + \frac{1}{p} + \frac{1}{q} + \frac{1}{pq}} = x^2 + \frac{23}{24} x + \frac{293}{8640}
\]

Setting \( (p, q) = (k, k) \) in (3.13), we obtain the following corollary.

**Corollary 3.1** As \( x \to \infty \),

\[
\Gamma(x+1) \sqrt{2\pi x(x/e)^x} = x^k + a_1 x^{k-1} + \cdots + a_k + O\left(\frac{1}{x^{2k+1}}\right),
\]

(3.15)

where \( k \geq 1 \) is any given integer, the coefficients \( a_j \) and \( b_j \) \((1 \leq j \leq k)\) are given by

\[
\begin{align*}
a_1 &= b_1 + c_1, \\
a_2 &= b_2 + c_1 b_1 + c_2, \\
&\vdots \\
a_k &= b_k + \cdots + c_{k-1} b_1 + c_k, \\
0 &= c_{k+1} + c_k b_1 + \cdots + c_1 b_k, \\
&\vdots \\
0 &= c_{2k} + c_{2k-1} b_1 + \cdots + c_k b_k,
\end{align*}
\]

(3.16)

and \( c_i \) is given in (2.4).
Setting $k = 3$ and $k = 4$ in (3.15), respectively, yields

$$\frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^{x}}} = \frac{x^3 + \frac{1}{2}x^2 + \frac{169,903}{590,688}x + \frac{4,406,147}{425,295,360}}{x^3 - \frac{1}{2}x^2 + \frac{169,903}{590,688}x - \frac{4,406,147}{425,295,360}} + O\left(\frac{1}{x^7}\right)$$

(3.17)

and

$$\frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^{x}}} = \frac{x^4 + \frac{1}{24}x^3 + \frac{685,893,605}{845,980,224}x^2 + \frac{14,787,105,577}{456,829,320,960}x + \frac{2,749,505,046,083}{153,499,631,842,560}}{x^4 - \frac{1}{24}x^3 + \frac{685,893,605}{845,980,224}x^2 - \frac{14,787,105,577}{456,829,320,960}x + \frac{2,749,505,046,083}{153,499,631,842,560}} + O\left(\frac{1}{x^9}\right).$$

(3.18)

In view of (1.5), (1.6), (3.17) and (3.18), we pose the following conjecture.

**Conjecture 3.1** The coefficients $a_j$ and $b_j$ ($1 \leq j \leq k$) in (3.15) satisfy the following relation:

$$a_j = (-1)^j b_j, \quad j = 1, 2, \ldots, k.$$  

(3.19)

### 4 Inequalities for the gamma function

Formulas (3.17) and (3.18) motivate us to establish the following theorem.

**Theorem 4.1** The following inequalities hold:

$$U(x) < \frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^{x}}} < V(x),$$

(4.1)

where

$$U(x) = \frac{x^4 + \frac{1}{24}x^3 + \frac{685,893,605}{845,980,224}x^2 + \frac{14,787,105,577}{456,829,320,960}x + \frac{2,749,505,046,083}{153,499,631,842,560}}{x^4 - \frac{1}{24}x^3 + \frac{685,893,605}{845,980,224}x^2 - \frac{14,787,105,577}{456,829,320,960}x + \frac{2,749,505,046,083}{153,499,631,842,560}}$$

(4.2)

and

$$V(x) = \frac{x^3 + \frac{1}{2}x^2 + \frac{169,903}{590,688}x + \frac{4,406,147}{425,295,360}}{x^3 - \frac{1}{2}x^2 + \frac{169,903}{590,688}x - \frac{4,406,147}{425,295,360}}.$$  

(4.3)

The left-hand side inequality holds for $x \geq 3$, while the right-hand side inequality is valid for $x \geq 2$.

**Proof** It suffices to show that

$$F(x) > 0 \quad \text{for} \quad x \geq 3 \quad \text{and} \quad G(x) < 0 \quad \text{for} \quad x \geq 2,$$

where

$$F(x) = \ln \Gamma(x + 1) - \left(x + \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} - \ln U(x)$$

and

$$G(x) = \ln \Gamma(x + 1) - \left(x + \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} - \ln V(x).$$
Differentiating $F(x)$ and applying the second inequality in (2.6) yield

\[
F'(x) = \psi(x + 1) - \ln x - \frac{1}{2x} - \frac{U'(x)}{U(x)} < - \frac{1}{2x^2 + 12x} - \frac{1}{252x^6 + 120x^4} + \frac{1}{240x^8} - \frac{1}{132x^{10}} - \frac{691}{32760x^{12}} - \frac{U'(x)}{U(x)}
\]

\[
= - \frac{P_{10}(x - 3)}{720,720x^{12}P_6(x)},
\]

where

\[
P_{10}(x) = 1,698,313,885,002,591,369,403,376,359,237,155,137
\]

\[
+ 7,041,090,100,510,955,203,400,650,726,407,309,444x
\]

\[
+ 12,215,302,599,727,743,342,615,877,184,100,329,802x^2
\]

\[
+ 12,025,928,200,234,176,519,514,968,711,811,967,964x^3
\]

\[
+ 7,551,739,592,924,831,815,437,063,682,435,942,293x^4
\]

\[
+ 3,187,338,342,726,357,084,428,868,676,747,628,952x^5
\]

\[
+ 920,408,575,975,851,494,996,412,447,435,781,084x^6
\]

\[
+ 180,133,255,608,389,118,267,365,601,710,648,784x^7
\]

\[
+ 22,910,271,532,985,226,283,927,122,357,066,246x^8
\]

\[
+ 1,711,635,468,441,001,446,976,320,994,717,320x^9
\]

\[
+ 57,054,515,614,700,048,232,544,033,157,244x^{10}
\]

and

\[
P_6(x) = (153,494,651,842,560x^4 + 6,395,610,493,440x^3
\]

\[
+ 124,448,535,691,200x^2 + 4,968,467,473,872x
\]

\[
+ 2,749,505,046,083)(153,494,651,842,560x^4
\]

\[
- 6,395,610,493,440x^3 + 124,448,535,691,200x^2
\]

\[
- 4,968,467,473,872x + 2,749,505,046,083).
\]

Hence, $F'(x) < 0$ for $x \geq 3$, and we have

\[
F(x) > \lim_{t \to \infty} F(t) = 0 \text{ for } x \geq 3.
\]

Differentiating $G(x)$ and applying the first inequality in (2.6) yield

\[
G'(x) = \psi(x + 1) - \ln x - \frac{1}{2x} - \frac{V'(x)}{V(x)} < - \frac{1}{2x^2 + 12x} - \frac{1}{252x^6 + 120x^4} + \frac{1}{240x^8} - \frac{1}{132x^{10}} - \frac{V'(x)}{V(x)}
\]

\[
= \frac{Q_8(x - 2)}{55,440x^{10}Q_6(x)},
\]
where
\[
Q_b(x) = 2,456,573,428,493,290,077,832 + 14,719,278,306,954,453,533,828x \\
+ 32,394,299,960,322,640,776,801x^2 \\
+ 37,478,643,384,199,534,772,000x^3 + 25,805,343,259,499,481,612,340x^4 \\
+ 11,004,898,939,796,249,295,384x^5 + 2,862,385,365,338,807,176,962x^6 \\
+ 416,852,240,076,239,943,360x^7 + 26,053,265,004,764,996,460x^8
\]

and
\[
Q_a(x) = (425,295,360x^3 + 17,720,640x^2 + 120,170,160x + 4,406,147) \\
\times (425,295,360x^3 - 17,720,640x^2 + 120,170,160x - 4,406,147).
\]

Hence, \( G'(x) > 0 \) for \( x \geq 2 \), and we have
\[
G(x) < \lim_{t \to \infty} G(t) = 0 \quad \text{for} \quad x \geq 2.
\]
The proof is complete. \( \square \)

**Remark 4.1** Following the same method as the one used in the proof of Theorem 4.1, we can prove the double inequality
\[
\frac{x^2 + \frac{1}{24}x + \frac{293}{8,640}}{x^2 - \frac{1}{24}x + \frac{293}{8,640}} < \frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^x}} < \frac{x + \frac{1}{24}}{x - \frac{1}{24}} \quad (4.4)
\]
for \( x \geq 2 \). We here omit it. Some computer experiments indicate that inequalities (4.1) and (4.4) are valid for \( x \geq 1 \).

In view of (4.1) and (4.4), we pose the following conjecture.

**Conjecture 4.1** If \( k \) is odd, then for \( x \geq 1 \),
\[
\frac{\Gamma(x + 1)}{\sqrt{2\pi x(x/e)^x}} < \frac{x^k + a_1x^{k-1} + \cdots + a_k}{x^k + b_1x^{k-1} + \cdots + b_k}, \quad (4.5)
\]
where the coefficients \( a_j \) and \( b_j \) (\( 1 \leq j \leq k \)) are determined in (3.16). If \( k \) is even, then inequality (4.5) is reversed.

**5 Comparison**

In 2011, Mortici [47] showed by numerical computations that his formula
\[
n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \exp \left( \frac{1}{12n + \frac{1}{2}} \right) = \mu_n \quad (5.1)
\]
is much stronger than other known formulas such as:
\[
n! \sim \sqrt{2\pi} \left( \frac{n + 1/2}{e} \right)^{n+1/2} = \beta_n \quad \text{(Burnside [8])}, \quad (5.2)
\]
Table 1 Comparison among approximation formulas (5.6)-(5.8)

| n     | $\lambda_n / n$ | $U_n / n$ | $V_n / n$ |
|-------|-----------------|-----------|-----------|
| 10    | $1.7686 \times 10^{-9}$ | $3.6355 \times 10^{-11}$ | $3.5843 \times 10^{-13}$ |
| 100   | $1.7855 \times 10^{-14}$ | $3.7108 \times 10^{-18}$ | $3.7317 \times 10^{-22}$ |
| 1,000 | $1.7857 \times 10^{-19}$ | $3.7115 \times 10^{-25}$ | $3.7332 \times 10^{-31}$ |
| 10,000| $1.7857 \times 10^{-24}$ | $3.7115 \times 10^{-32}$ | $3.7333 \times 10^{-40}$ |



$n! \sim \frac{\sqrt{2\pi \cdot e^{-n} \cdot n^{n+1}}}{\sqrt{n - \frac{1}{6}}} = \delta_n$ (Batir [4]),

$n! \sim \sqrt{2\pi \left(n + \frac{1}{6}\right)^n \left(\frac{n}{e}\right)^n} = \gamma_n$ (Gosper [19]),

$n! \sim \sqrt{\pi \left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6}} = \rho_n$ (Ramanujan [62], p.339).

In 2012, Mahmoud et al. [31] showed numerically that their formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{20n} + \frac{1}{30} \zeta(2, n + 1/2)\right) = \lambda_n$$

has a superiority over Mortici’s formula (5.1). Here $\zeta(s, a)$ denotes the Hurwitz (or generalized) zeta function defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k + a)^s} \quad (\Re(s) > 1; a \notin Z)$

$Z$ being the set of nonpositive integers.

From (3.17) and (3.18), we obtain

$$n! \sim \sqrt{2\pi \cdot n} \left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6} = U_n$$

and

$$n! \sim \sqrt{2\pi \cdot n} \left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6} = V_n.$$

We here offer some numerical computations (see Table 1) to show the superiority of our sequences $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ over the sequence $(\lambda_n)_{n \geq 1}$.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors read and approved the final manuscript.

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