Quantum tomography of pure states with projective measurements generated by MUBs and SIC-POVM

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The article undertakes the problem of pure state estimation based on projective measurements. Two generic frames for qubit tomography are considered – one defined by the vectors from the mutually unbiased bases (MUBs) and the other composed of the elements of the SIC-POVM. Both frames are combined with the method of least squares in order to reconstruct a sample of input qubits. The accuracy of each frame is quantified by the average fidelity and purity. The efficiency of the frames is compared and discussed. The method can be generalized to higher-dimensional states and transferred to other fields where the problem of complex vectors reconstruction appears.

Keywords: quantum state tomography, mutually unbiased bases, complex vector reconstruction

I. INTRODUCTION

Quantum state tomography (QST) originated in 1852, when G. G. Stokes derived the polarization state of a light beam based on intensity measurements [1]. Apparently, there are many approaches to QST which differ from one another in the kind of measurement(s) and the number of their repetitions [2, 3]. One popular method involves polarization measurements and can be applied to recovering the quantum state of photons [4]. Other frameworks connected with photonic state tomography rely on registering Hong-Ou-Mandel interference [5, 6].

A fundamental problem of QST has always concerned determining a set of measurement operators sufficient to identify an unknown state [7]. Some proposals aim at reducing the number of operators to the necessary minimum [8–10], while experimental frameworks tend to apply over-complete sets of operators in order to overcome noise and errors [11, 12].

To clarify the problem investigated in the article, let us postulate that the achievable information about a \(d\)-level pure quantum system is encoded in a complex vector – called the state vector and denoted by \(|\psi\rangle\), which belongs to the Hilbert space \(\mathcal{H} \cong \mathbb{C}^d\) such that \(\dim \mathcal{H} = d < \infty\). Moreover, \(|\psi\rangle\) is normalized, i.e. \(\langle \psi | \psi \rangle = 1\), where \(\langle . | . \rangle\) denotes the inner product in the Hilbert space. Then, the goal of QST is to reconstruct the accurate representation of \(|\psi\rangle\) on the basis of data accessible from an experiment. Naturally, multiplying the state vector by a scalar of unit modulus does not change the measurement results. Thus, the state vector can be determined up to a global phase factor.

In the case of pure states tomography, we are analyzing the problem of recovering a complex vector from intensity measurements – the very same kind of problem is considered in many other areas of science, from pure mathematics to speech recognition or signal processing [13]. Thus, there is vast literature concerning phase retrieval, see a review article Ref. [14]. In particular, in recent years, a lot of attention has been paid to the connection between complex vector reconstruction and the theory of frames, e.g. Ref. [15, 16].

Let us recall a definition. By an \(N\)-element complex frame in \(\mathbb{C}^d\), denoted \(\Xi = \{ |\xi_1\rangle, \ldots, |\xi_N\rangle \}\) (where \(|\xi_i\rangle \in \mathbb{C}^d\), one should understand a set of complex vectors that span \(\mathbb{C}^d\). In articles not connected to quantum tomography, authors usually consider in general the problem of reconstructing an unknown complex vector \(|x\rangle \in \mathbb{C}^d\) from its intensity measurements, i.e. it is discussed whether the knowledge about the non-linear map:

\[
\mathcal{J}_\Xi : |x\rangle \rightarrow (|\langle \xi_i |x\rangle|^2)_{i=1,\ldots,N}
\]

is sufficient to determine the complex vector \(|x\rangle\). In Physics, this type of measurement is referred to as projective measurement.

To formulate a sufficient condition for complex vector reconstruction, let us first revise general definitions involving the question when phase retrieval is possible. In Ref. [17], the authors propose to assume that phase retrieval is possible when any two vectors \(|\psi\rangle\) and \(|\psi'\rangle\) with identical intensity measurements differ only by a scalar of norm one, i.e. \(|\psi\rangle = e^{i\phi} |\psi'\rangle\). In other words, the same postulate can be stated that it is possible to reconstruct the complex vector \(|\psi\rangle\) if and only if the non-linear map \(\mathcal{J}_\Xi\) is injective and \(\Xi\) is a frame. Thus, henceforth, in the situations when the phase retrieval is possible, we shall say that the frame \(\Xi\) generates (or defines) injective measurements.

In Ref. [18], Bandeira, Cahill, Mixton and Nelson postulated a conjecture according to which if one wants to reconstruct a vector \(|x\rangle\) such that \(|x\rangle \in \mathbb{C}^d\), then a frame that contains less than \(4d - 4\) vectors cannot generate injective intensity measurements, i.e. according to the authors fewer than \(4d - 4\) modulus of inner product of \(|x\rangle\) with other vectors is not sufficient to obtain the structure of \(|x\rangle\). Furthermore, in the same paper the authors postulated the second part of the conjecture that a generic frame with \(4d - 4\) vectors (or more) generates injective measurements on \(\mathbb{C}^d\). The second part of the conjecture...
has been proved in [17], where the authors explained the notion of a generic frame and proved that for a generic frame $\Xi$ which contains at least $4d - 4$ elements the corresponding map $\mathcal{F}_\Xi$ is injective.

Another recent paper [19] proves a result that contradicts the first part of the conjecture from [18]. C. Vinzant proposed a frame in $\mathbb{C}^4$ which consists of 11 vectors and proved that it defines injective measurements on $\mathbb{C}^4$. Therefore, the current knowledge about the phase retrieval problem does not give an answer to the question what is the minimal number of elements of the frame $\Xi$ so that the map $\mathcal{F}_\Xi$ is injective, i.e. so far it remains unknown, in general, how many intensity measurements are required to reconstruct an unknown complex vector. However, in Ref. [18], the authors proposed a relatively efficient way to verify whether a given frame $\Xi$ generates injective measurements. Their approach is presented below as a theorem.

**Theorem I.1 (Bandeiraa et al. 2014)** A frame $\Xi = \{|\xi_1\rangle, \ldots, |\xi_N\rangle\}$ (where $|\xi_i\rangle \in \mathbb{C}^d$) defines injective measurements, i.e. one can reconstruct an unknown vector $|x\rangle \in \mathbb{C}^d$ from intensity measurements $|\langle \xi_i | x \rangle|^2$ for $i = 1, \ldots, N$, if and only if the linear space

$$
\mathcal{L}_\Xi := \{ Q \in \mathbb{C}^{d \times d} : |\langle \xi_i | Q | \xi_i \rangle| = \cdots = |\langle \xi_N | Q | \xi_N \rangle| = 0 \}
$$

(2)
does not contain any non-zero Hermitian matrix of the rank $\leq 2$.

Theorem 1.2 states clearly a sufficient condition that needs to be satisfied so that the frame $\Xi$ defines injective measurements and, therefore, it is possible to reconstruct a complex vector on the basis of the intensity measurements. For a given frame, one can relatively easy verify whether the condition stated in Theorem I.1 is fulfilled or not. However, so far there has been no concrete proposal concerning a feasible procedure to obtain such a sufficient frame.

In Sec. III we present the framework of QST for pure states and assumptions concerning experimental noise. Then, in Sec. III the results are introduced and discussed. We investigate qubit state reconstruction with two frames – one consisting of the vectors from MUBs [20] and the other defined by the elements from the SIC-POVM [21]. The accuracy of both frames is quantified by two figures of merit – the average fidelity and purity, which are computed and presented on graphs. The results demonstrate how efficient the frames are at overcoming the experimental noise.

## II. STATE RECONSTRUCTION FRAMEWORK

In this work, we assume that the initial state of a qubit can be presented as a vector:

$$
|\psi_{in}\rangle = \left( \begin{array}{c} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{array} \right),
$$

(3)

where $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$. An unknown quantum state of the form Eq. 3 can be reconstructed from projective measurements, which in the case of photons are based on photon counting. Let us denote a frame by $\Xi = \{|\xi_1\rangle, \ldots\}$, where $|\xi_i\rangle \in \mathbb{C}^2$. Next, we take into account experimental noise connected with each measurement. In particular, we impose the Poisson noise, which is a typical kind of uncertainty arising in photon counting [22]. Thus, assuming that the total number of photons equals $\mathcal{N}$, we shall introduce a formula for the measured photon count associated with the $k$-th projective measurement:

$$
n_k^M = \mathcal{N}_k \langle |\xi_k|\psi_{in}\rangle^2,
$$

(4)

where $\mathcal{N}_k$ stands for a number generated randomly from the Poisson distribution characterized by the expected value $\mathcal{N}$.

For a given frame $\Xi$, we can numerically generate experimental data for any specific input state Eq. 3 (we shall consider a sample consisting of 400 input states). However, in quantum state reconstruction problem, we postulate that there is no a priori knowledge about an unknown state. For this reason, an output state, which results from the tomographic algorithm, is assumed to be represented by a density matrix [4, 23]:

$$
\rho_{out} = \frac{T^\dagger T}{\text{Tr} \{T^\dagger T\}}, \quad \text{where} \quad T = \begin{pmatrix} t_1 & 0 \\ t_3 + i t_4 & t_2 \end{pmatrix},
$$

(5)

which is equivalent to the Cholesky factorization. In other words, the problem of state reconstruction of qubits means that we strive to estimate the values of the four parameters $t_1, t_2, t_3, t_4$ which fully characterize the density matrix. Consequently, the expected photon count for the $k$-th frame vector takes the form:

$$
n_k^E = \mathcal{N} \text{Tr} \{ |\xi_k\rangle \langle \xi_k| \rho_{out} \}. 
$$

(6)

In order to estimate the parameters $t_1, t_2, t_3, t_4$, we shall apply the method of least squares (LS) [24]. This method has been used to study the performance of QST frameworks with simulated measurement results [25, 26]. According to the LS method, the minimum value of the following function needs to be determined:

$$
f_{LS}(t_1, t_2, t_3, t_4) = \sum_k \left( n_k^E - n_k^M \right)^2 =
$$

$$
= \sum_k \left( \mathcal{N} \text{Tr} \{ |\xi_k\rangle \langle \xi_k| \rho_{out} \} - \mathcal{N}_k \langle |\xi_k|\psi_{in}\rangle^2 \right)^2,
$$

(7)

which allows one to perform QST for any input state $|\psi_{in}\rangle$ and a given frame $\Xi$.

Next, a scenario with extended measurement errors will be considered. Apart from the Poisson noise, dark counts shall be taken into account. In practice, it means that the detector receives not only the desirable signal, but also some number of photons which come from the background. Mathematically speaking, the background
noise shall be modeled by adding to the input state a component proportional to the maximally mixed state (perturbation term), i.e.

\[ \rho_{in} = (1 - \epsilon) |\psi_{in}\rangle \langle \psi_{in}| + \frac{\epsilon}{2} 1_2, \]  

where \( 1_2 \) denotes the \( 2 \times 2 \) identity matrix and \( \epsilon \) shall be referred to as the noise parameter (naturally: \( 0 \leq \epsilon \leq 1 \)). This gives a modified formula for the measured photon counts:

\[ n_k^{M'} = N_k \text{Tr} (|\xi_k\rangle \langle \xi_k| \rho_{in} = (1 - \epsilon) N_k |\xi_k| |\psi_{in}\rangle|^2 + N_k \frac{\epsilon}{2}, \]

which can be substituted into the function Eq. (7) in order to estimate the state in the other scenario.

In this article, we shall compare the quality of qubit estimation with two frames: one defined by the SIC-POVM and the other by the MUBs. First, the scenario with the Poisson noise alone will be considered. In the next step, the perturbation term will be added to the input states. In order to evaluate the efficiency of the frames, we utilize the notion of quantum fidelity \( F \) and purity \( \pi \) which are defined as \[\text{[27]}\]:

\[ F := \left( \text{Tr} \sqrt{\rho_{out} |\psi_{in}\rangle \langle \psi_{in}| \sqrt{\rho_{out}}} \right)^2 \]

\[ \pi := \text{Tr} (\rho_{out}^2). \]

The first figure, \( F \), quantifies the overlap between the actual state \( |\psi_{in}\rangle \) and the result of estimation \( \rho_{out} \) \[\text{[28]}\], whereas \( \pi \) measures how close the reconstructed density matrix is to the pure state. In the case of both measurement scenarios, we perform QST with each frame for a sample of 400 qubits defined as Eq. \[\text{[3]}\] with \( \phi \) and \( \theta \) covering the full range. Then, the performance of the frames is discussed based on the average fidelity \( F_{av} \) and purity \( \pi_{av} \), computed over the sample, cf. Ref. \[\text{[29]}\].

III. RESULTS AND DISCUSSION

The goal of this section is to compare the performance of two frames in QST of pure states. The first frame, denoted by \( \Xi^{MUB} \) consists of six vectors which correspond to the elements of the mutually unbiased bases (MUB) for \( \text{dim } \mathcal{H} = 2 \), i.e.:

\[ |\xi_1^{MUB}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\xi_2^{MUB}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

\[ |\xi_3^{MUB}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad |\xi_4^{MUB}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \]

\[ |\xi_5^{MUB}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |\xi_6^{MUB}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}. \]

Mathematically speaking, the vectors Eq. (11) constitute a frame which defines injective measurements according to Theorem \[\text{[11]}\]. From the physical point of view, intensity measurements associated with the frame \( \Xi^{MUB} \) can be realized on photons through polarization measurements since the vectors are commonly used to represent: vertical/horizontal, diagonal/anti-diagonal, right/left circular polarization states, respectively, e.g. Ref. \[\text{[20]}\].

Instead of the vectors belonging to the MUBs, one can utilize the SIC-POVM in order to reconstruct an unknown quantum state \[\text{[30]}\]. For qubits, the SIC-POVM consists of four projectors defined by the vectors:

\[ |\xi_1^{SIC}\rangle = |0\rangle, \quad |\xi_2^{SIC}\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle, \]

\[ |\xi_3^{SIC}\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{1}{\sqrt{3}} e^{i\frac{2\pi}{3}} |1\rangle, \quad |\xi_4^{SIC}\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{1}{\sqrt{3}} e^{i\frac{4\pi}{3}} |1\rangle, \]

which also generate injective measurements. The frame comprising the SIC-POVM shall be denoted by \( \Xi^{SIC} \).

\[\text{TABLE I. Average fidelity } F_{av} \text{ and purity } \pi_{av} \text{ in pure state estimation with two distinct frames. The method of least squares was applied with the function Eq. (7). Each value was computed as the mean for a sample of 400 input qubits of the form Eq. (3)}\]

| Frame | \( N \) | 10 | 50 | 100 | 1000 | 10000 |
|-------|-------|-----|-----|-----|------|-------|
| \( \Xi^{MUB} \) | \( F_{av} \) | 0.9334 | 0.9721 | 0.9793 | 0.9940 | 0.9981 |
| | \( \pi_{av} \) | 0.9354 | 0.9597 | 0.9679 | 0.9890 | 0.9964 |
| \( \Xi^{SIC} \) | \( F_{av} \) | 0.9080 | 0.9655 | 0.9761 | 0.9925 | 0.9979 |
| | \( \pi_{av} \) | 0.9128 | 0.9550 | 0.9652 | 0.9864 | 0.9959 |

In order to investigate the efficiency of each frame in pure state reconstruction, numerical simulations were conducted, assuming different number of photons involved in measurements. A sample of 400 input states of the form Eq. (3) was considered and each state was reconstructed with the measured photon counts distorted by the Poisson noise Eq. (4). The results are gathered in Tab. I.

It was expected that the impact of the Poisson noise should be greater if we utilize fewer photons. Thus, we can observe that the accuracy of both frames in QST increases along with the number of photons. For \( N = 10000 \), both frames lead exactly to the unknown state. However, if the number of photons drops down to 10, one can observe a substantial discrepancy between the state obtained from the algorithm \( \rho_{out} \) and the original state \( |\psi_{in}\rangle \). It is worth noting that, for \( N = 10 \), \( \Xi^{MUB} \) results in better quality of pure state estimation than \( \Xi^{SIC} \). This feature is in agreement with the common practice in QST to employ over-complete sets of
measurement operators in order to combat experimental noise. When we increase the number of photons, the figures of merit for both frames converge.

Further insight into the efficiency of the frames can be provided by investigating input states influenced by the error parameter $\epsilon$, as in Eq. [8]. Let us assume that the number of photons is fixed: $N = 1000$. Then, for each frame, we can consider the average fidelity and purity as functions of $\epsilon$, denoted by $F_{av}(\epsilon)$ and $\pi_{av}(\epsilon)$, respectively. The plots of the functions are presented in Fig. I.

![Figure 1](image-url)

**FIG. 1.** Plots present the average fidelity $F_{av}(\epsilon)$ (the upper graph) and the purity $\pi_{av}(\epsilon)$ (the lower graph). Each point was obtained by the method of least squares for a sample of 400 input states including a noise parameter $\epsilon$. The formula Eq. [8] for the measured photon counts was applied, assuming that $N = 1000$.

For the number of photons $N = 1000$, the results demonstrate that if we consider the input states distorted by the perturbation term Eq. [8] (while still keeping the Poisson noise), then both frames lead to very similar accuracy. The whole range of the noise parameter $\epsilon$ was considered and there is no significant difference between the efficiency of the frames. One can notice that the function $F_{av}(\epsilon)$ is linear, whereas $\pi_{av}(\epsilon)$ is convex. The plots allow one to observe how the accuracy of state estimation degenerates as we increase the amount of noise (dark counts).

| $\epsilon$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|------------|-----|-----|-----|-----|-----|
| $\Xi_{MUB}$ | 0.8960 | 0.8632 | 0.8231 | 0.7867 | 0.7476 |
| $\Xi_{SIC}$ | 0.8919 | 0.8584 | 0.8298 | 0.7900 | 0.7457 |

**TABLE II.** Average fidelity $F_{av}(\epsilon)$ in pure state estimation with two distinct frames. Each value was computed as the mean for a sample of 400 input qubits of the form Eq. [8]. The formula Eq. [9] for the measured photon counts was applied with $N = 10$.

Finally, let us investigate the difference between the frames in the case of few photons and the presence of dark counts Eq. [8]. If we decrease the number of photons down to $N = 10$, then from Tab. II, we observe that $\Xi_{MUB}$ has no significant advantage over $\Xi_{SIC}$ for non-zero values of the error parameter. These results prove that for noisy measurements, distorted by both the Poisson noise and dark counts, $\Xi_{SIC}$ and $\Xi_{MUB}$ deliver the same quality. Though there are tiny differences between the figures in Tab. II, they should be considered negligible.

**IV. SUMMARY AND OUTLOOK**

In this article, two frames have been compared in terms of their applicability in quantum tomography of qubits. One frame, $\Xi_{MUB}$, contained the vectors from mutually unbiased bases (MUBs) and the other, $\Xi_{SIC}$, comprised the elements of the SIC-POVM. Based on the numerical simulations, we have demonstrated that $\Xi_{MUB}$ outperforms $\Xi_{SIC}$ only if we consider a measurement scenario which involves single-photon counting along with the Poisson noise as the only source of experimental uncertainty. Then, the over-complete frame, $\Xi_{MUB}$, has an advantage over the minimal frame, $\Xi_{SIC}$. However, the improvement in quality due to the over-complete frame appears to be rather moderate.

Furthermore, if we increase the number of photons involved in measurements or include dark counts as another source of experimental noise, both frames deliver the same quality of pure state estimation. In such cases, the four-element minimal frame, $\Xi_{SIC}$, which consists of the vectors generating the SIC-POVM, fully suffices for QST of qubits.

In conclusion, one can agree that an over-complete set of measurement operators is advisable only if we consider experiments with single photons and perturbation term can be neglected. This outcome is in line with other results which demonstrate that over-complete measurements can improve the efficiency of QST frameworks [31].

In the future, a similar approach can be applied to investigate the accuracy of different frames in higher-dimensional cases. In particular, the problem of recon-
Structuring 4-dimensional complex vectors shall be studied since this case includes entangled photons.

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[1] G. G. Stokes, *Trans. Cambridge Philos. Soc.* 9, 399 (1852) doi:10.1017/CBO9780511702266.010
[2] G. M. D’Ariano, M. G. A. Paris, M. F. Sacchi, *Adv. Imaging Electron Phys.* 128, 205 (2003) doi:10.1016/S1076-5670(03)80065-4
[3] M. G. A. Paris, J. Řeháček (eds.), *Quantum State Estimation* (Lecture Notes in Physics), Springer, Berlin-Heidelberg (2004) doi:10.1007/b98673
[4] D. P. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, *Phys. Rev. A* 64, 052312 (2001) doi:10.1103/PhysRevA.64.052312
[5] W. Wasilewski, P. Kolenderski, R. Frankowski, *Phys. Rev. Lett.* 99, 123601 (2007) doi:10.1103/PhysRevLett.99.123601
[6] P. Kolenderski, W. Wasilewski, *Phys. Rev. A* 80, 015801 (2009) doi:10.1103/PhysRevA.80.015801
[7] G. M. D’Ariano, L. Maccone, and M. G. A. Paris, *J. Phys. A: Math. Gen.* 34, 93 (2001) doi:10.1088/0305-4470/34/1/307
[8] A. Czerwinski, *Int. J. Theor. Phys.* 55, 658 (2016) doi:10.1007/s10773-015-2703-2
[9] A. Czerwinski, *J. Phys. A: Math. Theor.* 49, 075301 (2016) doi:10.1088/1751-8113/49/7/075301
[10] A. Czerwinski, *Int. J. Theor. Phys.* 59, 3646 (2020) doi:10.1007/s10773-020-04625-8
[11] R. T. Horn, P. Kolenderski, D. Kang, P. Abolghasem, C. Scarcella, A. D. Frera, A. Tosi, L. G. Helt, S. V. Zhukovsky, J. E. Sipe, G. Weihs, A. S. Helmy, and T. Jennewein, *Sci. Rep.* 3, 2314 (2013) doi:10.1038/srep02314
[12] R. Horn and T. Jennewein, *Opt. Express* 27, 17369 (2019) doi:10.1364/OE.27.017369
[13] K. Jaganathan, Y. C. Eldar, B. Hassibi, *Phase Retrieval: An Overview of Recent Developments*, in: A. Stern (eds.), *Optical Compressive Imaging*, CRC Press, Boca Raton, pp. 263-296 (2016)
[14] P. G. Casazza, L. M. Woodland, *Contemp. Math.* 626, 1 (2014) doi:10.1090/conm/626/12501
[15] R. Balan, P. G. Casazza, D. Edidin, *Appl. Comput. Harmon. Anal.* 20, 345 (2006) doi:10.1016/j.acha.2005.07.001
[16] A. Jamiołkowski, *J. Phys.: Conf. Ser.* 213, 012002 (2010) doi:10.1088/1742-6596/213/1/012002
[17] A. Conca, D. Edidin, M. Hering and C. Vinzant, *Appl. Comput. Harmon. Anal.* 38, 346 (2015) doi:10.1016/j.acha.2014.06.006
[18] A. S. Bandeira, J. Cahill, D. G. Mixon and A. A. Nelson, *Appl. Comput. Harmon. Anal.* 37, 106 (2014) doi:10.1016/j.acha.2013.10.002
[19] C. Vinzant, ”A small frame and a certificate of its injectivity”, *International Conference on Sampling Theory and Applications (SampTA)* 2015, pp. 197-200 (2015) doi:10.1109/SAMPTA.2015.7148879
[20] W. K. Wootters and B. D. Fields, *Ann. Phys.* 191, 363 (1989) doi:10.1016/0003-4916(89)90322-9
[21] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *J. Math. Phys.* 45, 2171 (2004) doi:10.1063/1.1737053
[22] S. W. Hasinoff, *Photon, poisson noise*, in: K. Ikeuchi (eds.), *Computer Vision*, Springer, Boston, MA, pp. 608-610 (2014) doi:10.1007/978-0-387-31439-6-482
[23] J. Altepeter, E. Jerey, and P. Kwiat, *Adv. At. Mol. Opt. Phys.* 52, 105 (2005) doi:10.1016/S1049-250X(05)52003-2
[24] T. Opatrný, D.-G. Welsch, and W. Vogel, *Phys. Rev. A* 56, 1788 (1997) doi:10.1103/PhysRevA.56.1788
[25] A. Acharya, T. Kypraios, and M. Gută, *J. Phys. A: Math. Theor.* 52, 234001 (2019) doi:10.1088/1751-8121/ab1958
[26] K. Sedziak-Kacprowicz, A. Czerwinski, and P. Kolenderski, *Phys. Rev. A* 102, 052420 (2020) doi:10.1103/PhysRevA.102.052420
[27] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2000) doi:10.1017/CBO9780511976667
[28] R. Jozsa, *J. Mod. Opt.* 41, 2315 (1994) doi:10.1080/0950034941552171
[29] O. Bayraktar, M. Swillo, C. Canalias, and G. Bjork, *Phys. Rev. A* 94, 020105(R) (2016) doi:10.1103/PhysRevA.94.020105
[30] J. Řeháček, B.-G. Englert, and D. Kaszlikowski, *Phys. Rev. A* 70, 052321 (2004) doi:10.1103/PhysRevA.70.052321
[31] H. Zhu, *Phys. Rev. A* 90, 012115 (2014) doi:10.1103/PhysRevA.90.012115