PSEUHOLOMORPHIC SIMPLE HARNACK CURVES

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Abstract. We give a short proof of Mikhalkin’s Theorem on the topological classification of simple Harnack curves, which in particular extends Mikhalkin’s result to real pseudoholomorphic curves.

We consider the complex projective plane equipped with the standard complex conjugation \( \text{conj} \). Given a real algebraic curve \( C \), we denote by \( R C \) its real part, i.e. the set of its real points. The genus of \( C \) is denoted by \( g(C) \).

1. Introduction

Let \( L_0, L_1, \) and \( L_2 \) be three distinct real lines in \( \mathbb{C}P^2 \). A simple Harnack curve is a real algebraic map \( \phi : C \to \mathbb{C}P^2 \) satisfying the two following conditions:

- \( C \) is a non-singular maximal real algebraic curve (i.e. \( R C \) has \( g(C) + 1 \) connected components);
- there exist a connected component \( O \) of \( R C \), and three disjoint arcs \( l_0, l_1, l_2 \) on \( O \) such that \( \phi^{-1}(L_i) \subset l_i \).

We depict in Figure 1 examples of simple Harnack curves with a non-singular image in \( \mathbb{C}P^2 \) and intersecting transversely all lines \( L_i \). Theorem 1 below says that these are essentially the only simple Harnack curves.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{simple_harnack_curves.png}
\caption{Simple Harnack curves of degree \( d \) and genus \( \frac{(d-1)(d-2)}{2} \).}
\end{figure}

Let \( \phi : C \to \mathbb{C}P^2 \) be a simple Harnack curve, and choose an orientation of \( O \). This induces an ordering of the intersection points of \( O \) (or \( C \)) with \( L_i \), and we denote by \( s_i \) the corresponding sequence of intersection multiplicities. Let \( s \) be the sequence \( (s_0, s_1, s_2) \) considered up to the

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equivalence relation generated by
\[(s_0, s_1, s_2) \sim (s_0, \overline{s}_1, \overline{s}_2), \quad (s_0, s_1, s_2) \sim (s_2, s_0, s_1), \quad \text{and} \quad (s_0, s_1, s_2) \sim (s_0, \overline{s}_2, \overline{s}_1),\]
where \((u_1)_{1 \leq i \leq n} = (u_{n-i})_{1 \leq i \leq n} \). This equivalence relation is so that \(s\) does not depend on the chosen orientation on \(O\), nor on the labeling of the three lines \(L_i\).

**Theorem 1** (Mikhalkin \[Mik00\], Mikhalkin-Rullgärd \[MR01\]). Let \(\phi : C \to \mathbb{C}P^2\) be a simple Harnack curve of degree \(d\). Then the curve \(\phi(C)\) is nodal with solitary nodes (i.e. real nodes with local equation \(x^2 + y^2 = 0\)) as only singularities. Moreover if \(g(C) = 0\) or \(g(C) = \frac{(d-1)(d-2)}{2}\), then the topological type of the pair \((\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^{d-1} \mathbb{R}L_i)\) only depends on \(d\), \(g(C)\), and \(s\).

It follows from Theorem 1 that Figure 1 is enough to recover all topological types of pairs \((\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^{d-1} \mathbb{R}L_i)\), where \(\phi : C \to \mathbb{C}P^2\) is a simple Harnack curve. Mikhalkin actually proved Theorem 1 for simple Harnack curves in any toric surface, nevertheless this a priori more general statement can be deduced from the particular case of \(\mathbb{C}P^2\) (one may use for example Viro’s patchworking, see \[Vir84a, Vir89\]). Existence of simple Harnack curves of maximal genus with any Newton polygon, and intersecting transversely all toric divisors, was first established by Itenberg (see \[IV96\]). Simple Harnack curves of any degree, genus, and sequence \(s\) were first constructed by Kenyon and Okounkov in \[KO06\].

Simple Harnack curves are important, because extremal, objects in real algebraic geometry, and Theorem 1 had a deep impact on subsequent developments in this field. However their importance goes beyond real geometry, as showed their connection to dimers discovered by Kenyon, Okounkov, and Sheffield in \[KOS06\].

The goal of this note is to give a short alternative proof of Theorem 1. Moreover, our proof is also valid for real pseudoholomorphic curves, which are also very important objects in real algebraic and symplectic geometry. Note that a real algebraic curve is a real pseudoholomorphic curve, however the converse is not true in general. Mikhalkin’s original proof of Theorem 1 uses amoebas of algebraic curves, and does not a priori apply to real pseudoholomorphic curves which are not algebraic.

It is nevertheless possible to read our proof of Theorem 1 in the algebraic category, by going directly to Section 2 and defining the map \(\pi_1 : C \to L_i\) as the composition of \(\phi\) with the linear projection \(\mathbb{C}P^2 \setminus (L_j \cap L_k) \to L_i\), with \(\{i, j, k\} = \{0, 1, 2\}\).

We consider \(\mathbb{C}P^2\) equipped with the standard Fubini-Study symplectic form \(\omega_{FS}\). A real almost complex structure \(J\) on \(\mathbb{C}P^2\) is an almost complex structure tamed by \(\omega_{FS}\) (i.e. \(\omega_{FS}(v, Jv) > 0\) for any non-null vector \(v \in T_{\mathbb{C}P^2}\) for which the standard complex conjugation \(\text{conj}\) on \(\mathbb{C}P^2\) is \(J\)-antiholomorphic (i.e. \(\text{conj} \circ J = J^{-1} \circ \text{conj}\)). For example, the standard complex structure on \(\mathbb{C}P^2\) is a real almost complex structure.

Let \((C, \omega)\) be a compact symplectic surface equipped with a complex structure \(J_C\) tamed by \(\omega\), and a \(J\)-antiholomorphic involution \(\text{conj}_C\), and let \(J\) be a real almost complex structure on \(\mathbb{C}P^2\). A symplectomorphism \(\phi : C \to \mathbb{C}P^2\) is a real \(J\)-holomorphic map if
\[d\phi \circ J_C = J \circ d\phi \quad \text{and} \quad \phi \circ \text{conj}_C = \text{conj} \circ \phi.\]

It is of degree \(d\) if \(\phi_*([C]) = d[C]\) in \(H_2(\mathbb{C}P^2; \mathbb{Z})\). Recall that any intersection of two \(J\)-holomorphic curves is positive (see \[LS12\] Appendix E]).

The definition of simple Harnack curves extends immediately to the real \(J\)-holomorphic case. Given three distinct real \(J\)-holomorphic lines \(L_0\), \(L_1\), and \(L_2\) in \(\mathbb{C}P^2\), a real \(J\)-holomorphic curve \(\phi : C \to \mathbb{C}P^2\) is a simple Harnack curve if \(C\) is maximal, and if there exists a connected component \(O\) of \(\mathbb{R}C\), and three disjoint arcs \(l_0, l_1, l_2\) on \(O\) such that \(\phi^{-1}(L_i) \subset l_i\).
Theorem 2. Theorem \[\text{[2]}\] holds for \(J\)-holomorphic simple Harnack curves. In particular if \(g(C) = 0\) or \(g(C) = \frac{(d-1)(d-2)}{2}\), then the topological type of the pair \((\mathbb{R}P^2, \mathbb{R}\phi(C) \cup \bigcup_{i=0}^{2} \mathbb{R}L_i)\) does not depend on \(J\), once \(d\) and \(s\) are fixed.

As in the case of algebraic curves, one may use patchworking of real pseudoholomorphic curves (see \[\text{[IS02]}\]) to generalize Theorem \[\text{[2]}\] to \(J\)-holomorphic simple Harnack curves in any toric surface.

The proof of Theorem \[\text{[2]}\] is quite simple: the three projections from \(\mathbb{C}P^2 \setminus (L_j \cap L_k)\) to \(L_i\) induce three holomorphic maps \(\pi_i : C \to L_i\); by considering the arrangement of the real Dessins d’enfants \(\pi_i^{-1}(\mathbb{R}L_j)\) on \(C/\text{conj}_C\), we deduce the number of connected components of \(\mathbb{R}\phi(C)\) in each quadrant of \(\mathbb{R}P^2 \setminus (\bigcup_{i=0}^{2} \mathbb{R}L_i)\), as well as its complex orientation; the mutual position of all these connected components is then deduced from Rokhlin’s complex orientation formula.

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2. PROOF OF THEOREM \[\text{[2]}\]

Let \(\phi : C \to \mathbb{C}P^2\) be a \(J\)-holomorphic simple Harnack curve in \(\mathbb{C}P^2\) of degree \(d\) and genus \(g\). We define \(p_{i,j} = L_i \cap L_j\).

2.1. Construction of the maps \(\pi_i : C \to L_i\). Gromov proved in \[\text{[Gro85]}\] that there exists a unique \(J\)-holomorphic line passing through two distinct points in \(\mathbb{C}P^2\). By uniqueness, this line is real if the two points are in \(\mathbb{R}P^2\), hence there exists a real pencil of \(J\)-holomorphic lines through any point of \(\mathbb{R}P^2\). In particular if \(\{i, j, k\} = \{0, 1, 2\}\), the map \(\mathbb{C}P^2 \setminus \{p_{j,k}\} \to L_i\), which associates to each point \(p\) the unique intersection point of \(L_i\) with the \(J\)-holomorphic line passing through \(p\) and \(p_{j,k}\), is a real \(J\)-holomorphic map. We define \(\pi_i : C \to L_i\) as the composition of \(\phi\) with this projection. This is a real holomorphic map (recall that any almost complex structure on a topological surface is integrable).

2.2. Dessins d’Enfants on \(C\). We denote by \(\tilde{C}\) the quotient of \(C\) by \(\text{conj}_C\). Since \(C\) is maximal, the surface \(\tilde{C}\) is a disk with \(g\) holes.

Let \(\Gamma_i \subset \tilde{C}\) be the graph \(\pi_i^{-1}(\mathbb{R}L_i)/\text{conj}_C\). Note that \(\Gamma_j \cap \Gamma_k = \bigcap_{i=0}^{d-2} \Gamma_i\) if \(j \neq k\), and we call a triple point an isolated point in \(\bigcap_{i=0}^{d-2} \Gamma_i\). By construction, a triple point corresponds to a singular point of \(\phi(C)\) in \(\mathbb{R}P^2\), where at least two complex conjugated non-real branches intersect. By the adjunction Formula (see \[\text{[MS12]}\] Chapter 2] in the case of \(J\)-holomorphic curves), the graph \(\bigcup_{i=0}^{d-2} \Gamma_i\) has no more than \(\frac{(d-1)(d-2)}{2} - g\) triple points, and \(\phi(C)\) is nodal with only solitary nodes in case of equality.

Let \(\{i, j, k\} = \{0, 1, 2\}\). We label by \(+\) (resp. \(-\)) the connected component of \(\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}\) containing (resp. disjoint from) \(\phi(O) \cap L_i\). We endow each connected component of \(\Gamma_i \setminus \pi_i^{-1}(\{p_{i,j}, p_{i,k}\})\) with the sign of the corresponding component of \(\mathbb{R}L_i \setminus \{p_{i,j}, p_{i,k}\}\). We also denote by \((\varepsilon_0, \varepsilon_1) \in \{+, -\}^2\) the connected component of \(\mathbb{R}P^2 \setminus \left(\bigcup_0^{d-1} \mathbb{R}L_i\right)\) which project to the components labeled by \(\varepsilon_0\) and \(\varepsilon_1\) of \(\mathbb{R}L_0\) and \(\mathbb{R}L_1\) by the projections of center \(p_{1,2}\) and \(p_{0,2}\) respectively.

The map \(\pi_i : C \to L_i\) is holomorphic of degree \(d\), so by the Riemann-Hurwitz formula it has exactly \(2(d - g - 1)\) ramification points (counted with multiplicity). Since a subarc of \(L_i\) connecting two (possibly infinitesimally) consecutive points in \(L_i \cap \phi^{-1}(L_i)\) has to contain such a ramification point, the arc \(L_i\) contains at least \(2(d - 1)\) ramification points of \(\pi_i\) (counted with multiplicity). Moreover a connected component of \(\mathbb{R}C\) distinct from \(O\) contains at least two ramification points of \(\pi_i\). Since \(C\) has \(g + 1\) connected components, if follows that these two previous lower bounds are in fact equality, in particular all ramification points of \(\pi_i\) are real. This implies that each connected
component of $\tilde{C} \setminus \Gamma_i$ is a disk, and that the restriction of $\pi_i$ on this disk is a homeomorphism to one of the two hemispheres of $L_i \setminus \mathbb{R}L_i$.

**Lemma 3.** If $g = 0$, then the arrangement of $\bigcup_{i=0}^2 \Gamma_i$ in $\tilde{C}$ only depends, up to orientation preserving homeomorphism, on $d$ and $s$. In particular it has exactly $\frac{(d-1)(d-2)}{2}$ triple points.

**Proof.** Since $\pi_i$ has no ramification point outside $O$, the graph $\Gamma_i$ decomposes $\tilde{C}$ into a chain of disks, where two adjacent disks intersect along (the closure of) a connected component of $\Gamma_i \setminus O$. See Figure 2 in the case when $d = 6$ and $\phi^{-1}(L_i)$ consists of 6 distinct points. By definition, the points of $\Gamma_i$ in $L_i$ are endowed with the sign $+$. 

![Figure 2. The graph $\Gamma_0$](image)

By the adjunction formula, the number of intersection points of the graphs $\Gamma_i$ and $\Gamma_j$, with $i \neq j$, is not more than $\frac{(d-1)(d-2)}{2} = 1 + 1 + \ldots + d - 2$. However, this number is clearly the minimal number of intersection point of $\Gamma_i$ and $\Gamma_j$, and there exists a unique mutual position of those graphs that achieves this lower bound (see Figure 3a). The lemma follows immediately by symmetry (see Figure 3b).

![Figure 3.](image)

In case of positive genus, we have the following lemma.
Lemma 4. The arrangement of $\bigcup_{i=0}^{2} \Gamma_i$ has exactly $(d-1)(d-2) - g$ triple points. Moreover if $d = 2k$ (resp. $d = 2k + 1$), then $\mathbb{R}\phi(C)$ has exactly $\frac{(k-1)(k-2)}{2}$ (resp. $\frac{k(k+1)}{2}$) connected components in the quadrant $(+, +)$ (resp. $(-, -)$), and $\frac{k(k-1)}{2}$ connected components in each of the other quadrants.

Proof. Locally around each boundary component of $\tilde{C}$ distinct from $O$, the graph $\bigcup_{i=0}^{2} \Gamma_i$ looks like in Figure 4a. In particular, we may glue a disk as depicted in Figure 4b. Performing this operation to each boundary component of $\tilde{C}$ distinct from $O$, the lemma is proved with the same arguments than Lemma 3.

![Figure 4](image)

Even if this will eventually follow from Theorem 2, we do not claim that the disk gluing in the proof of Lemma 4 has any interpretation in terms of degenerations of $\phi(C)$. Note that when $g = \frac{(d-1)(d-2)}{2}$, the arrangement $\bigcup_{i=0}^{2} \Gamma_i$ only depends, up to orientation preserving homeomorphism, on $d$ and $s$. See Figure 4c in the case $d = 6$.

2.3. Application of Rokhlin’s complex orientation Formula. To end the proof of Theorem 2 in the case $d = 2k$, it remains us to prove the following lemma. The case of curves of odd degree is entirely similar, and is left to the reader.

Lemma 5. The following hold:

1. $\phi(\gamma)$ bounds a disk in $\mathbb{R}P^2$ disjoint from $\mathbb{R}\phi(C \setminus \gamma)$ for any connected component $\gamma$ of $\mathbb{R}C \setminus O$;
2. a connected component of $\mathbb{R}\phi(C \setminus O)$ is contained in the disk bounded by $\phi(O)$ in $\mathbb{R}P^2$ if and only if it is contained in the quadrant $(+, +)$.

Proof. These two facts will be a consequence of Rokhlin’s complex orientation Formula ([Rok74] see also [Vir84b]). Since there exists a smoothing $\phi'(C')$ of $\phi(C)$ where $\phi' : C \to \mathbb{C}P^2$ is a real $J'$-holomorphic curve of degree $d$ and genus $\frac{(d-1)(d-2)}{2}$, we may assume that from now on that $C$ has genus $\frac{(d-1)(d-2)}{2}$.

Recall that since $C$ is maximal, the set $C \setminus \mathbb{R}C$ has two connected components. Moreover the choice of one of these components induces an orientation of $\mathbb{R}C$ (as boundary). The effect of choosing the other component of $C \setminus \mathbb{R}C$ is to reverse the orientation of $\mathbb{R}C$. Hence there is a canonical orientation,

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1This assumption is aimed to simplify the exposition, and is not formally needed for our purposes. Indeed, there exists a generalization of Rokhlin’s Formula for nodal curves that we could also have used here ([Zvo83] see also [Vir96]).
up to a global change of orientation of $\mathbb{R}C$, of all connected components of $\mathbb{R}C$. This orientation is called the \textit{complex orientation of} $\mathbb{R}C$.

Recall also that a disjoint pair of embedded circles in $\mathbb{R}P^2$ is said to be \textit{injective} if their union bounds an annulus $A$. If the two circles are oriented and form an injective pair, this latter is said to be \textit{positive} if the two orientations is induced by some orientation of $A$, and is said to be \textit{negative} otherwise, see Figure 5a and b. We denote respectively by $\Pi_+$ and $\Pi_-$ the number of positive and negative injective pairs of connected components of $\phi(\mathbb{R}C)$ equipped with their complex orientation. Rokhlin’s complex orientation Formula reduces in our case to

\begin{equation}
\Pi_+ - \Pi_- = \frac{(k-1)(k-2)}{2}.
\end{equation}

By considering separately the three projections $\pi_i : C \to L_i$, it follows from Fiedler’s orientation formula (\cite{Fie83} see also \cite{Vir84b}) and the (unique) graph $\bigcup_{i=0}^{2} \Gamma_i \subset \tilde{C}$ that the complex orientation of the curve $\phi(C)$ is as depicted in Figure 5c. In particular if $\gamma_1$ and $\gamma_2$ are two distinct connected components of $\phi(\mathbb{R}C)$ which form an injective pair, we see that this pair contributes to $\Pi_+$ if and only if $\gamma_i = \phi(O)$ and $\gamma_{2-i}$ is in the quadrant $(+,+)$. Hence we deduce from Lemma 4 that

$$\Pi_+ \leq \frac{(k-1)(k-2)}{2} \quad \text{and} \quad \Pi_- \geq 0,$$

with equality if and only if the conclusion of the lemma hold. Now the result follows from Equation (1).

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