Red’kov V.A., George J. Spix

On the different forms of the Maxwell’s electromagnetic equations in a uniform media

Institute of Physics, National Academy of Sciences of Belarus¹
Bachelor of Science in Electrical Engineering, Illinois Institute of Technology²

Two known, alternative to each other, forms of the Maxwell’s electromagnetic equations in a moving uniform media are investigated and discussed. Approach commonly used after Minkowski is based on the two tensors: \( H^{ab} = (D, H/c) \) and \( F^{ab} = (E, cB) \) which transform independently of each other at Lorentz transitions; relationships between fields \( D = \epsilon_0 E, B = \mu_0 \mu H \) change their form at Lorentz transformations and have the form of the Minkowski equations depending on the 4-velocity \( u^a \) of the moving media under an inertial reference frame. In this approach, the wave equation for electromagnetic potential involve explicitly the \( u^a \)-velocity of the moving media. So, the electrodynamics by Minkowski implies the absolute nature of the mechanical motion. An alternative formalism (Rosen and others) may be developed in the new variables:

\[
\begin{align*}
q & = 1/\sqrt{\epsilon_0 \mu_0}, \quad k = 1/\sqrt{\epsilon \mu}, \quad x^0 = kct, \quad j^0 = J^0, \quad j = J/kc, \quad d = \epsilon_0 \mu_0, \quad E, \quad h = H/kc.
\end{align*}
\]

In these variables, the Maxwell’s equations can be written in terms of a single tensor \( f^{BC} = (d, h) \). This form of the Maxwell’s equations exhibits symmetry under modified Lorentz transformations in which everywhere instead of the vacuum speed of light \( c \) is used the speed of light in the media, \( kc \). In virtue of this symmetry we might consider such a formulation of the Maxwell theory in the media as invariant under the mechanical motion of the reference frame. In connection with these two theoretical schemes, a point of principle must be stressed: it might seem well-taken the requirement to perform Poincaré-Einstein clock synchronization in the uniform medias with the help of real light signals influenced by the media, which leads us to the modified Lorentz symmetry.

Keywords: Maxwell, Minkowski, electromagnetic field, media, relativity principle, clock synchronization.

1 Maxwell equations in a media, transition to new variables

Maxwell’s equations in a uniform media with two characteristics \( \epsilon > 1 \) and \( \mu > 1 \) (dielectric and magnetic penetrability) have the form [1]

\[
\begin{align*}
\text{div } D &= \rho, \quad \text{div } B = 0, \quad \text{rot } H = J + \frac{\partial D}{\partial t}, \quad \text{rot } E = -\frac{\partial B}{\partial t}.
\end{align*}
\]

The notation is used:

\[
\begin{align*}
B &= \mu \mu_0 H, \quad \epsilon_0 E = D, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \quad k = \frac{1}{\sqrt{\epsilon \mu}} < 1.
\end{align*}
\]

¹redkov@dragon.bas-net.by
²gjspix@msn.com
The speed of light in the media \( c_{\text{media}} \) is less than that in vacuum \( c \) and the coefficient \( k \) describes this decreasing: \( c_{\text{media}} = k c \). Now a principal point in further analysis is that eqs. (1) may be re-written as

\[
\text{div } \mathbf{D} = \rho , \quad \text{div } \frac{\mathbf{H}}{kc} = 0 , \quad \text{rot } \frac{\mathbf{H}}{kc} = \frac{\mathbf{J}}{kc} + \frac{\partial}{\partial (kct)} \mathbf{D} , \quad \text{rot } \mathbf{D} = -\frac{\partial}{\partial (kct)} \frac{\mathbf{H}}{kc} .
\] (3)

Instead of variables \((t, x^i)\), \((\rho, \mathbf{J})\), \((\mathbf{D}, \mathbf{H})\), you may define new ones by means of the formulas

\[
x^0 = kc t , \quad j^0 = \rho , \quad \mathbf{j} = \frac{\mathbf{J}}{kc} , \quad \mathbf{d} = \mathbf{D} , \quad \mathbf{h} = \frac{\mathbf{H}}{kc} .
\] (4)

On making so, Maxwell’s equations (3) will take the form

\[
\text{div } \mathbf{d} = j^0 , \quad \text{div } \mathbf{h} = 0 , \quad \text{rot } \mathbf{h} = \mathbf{j} + \frac{\partial}{\partial x^0} \mathbf{d} , \quad \text{rot } \mathbf{d} = -\frac{\partial}{\partial x^0} \mathbf{h} .
\] (5)

## 2 Maxwell’s equations and Lorentz transformations in the media

Maxwell’s equations (5) can be written down in the explicit comprehensive form:

\[
\partial_1 d^1 + \partial_2 d^2 + \partial_3 d^3 = j^0 , \quad \partial_1 h^1 + \partial_2 h^2 + \partial_3 h^3 = 0 ,
\]
\[
\partial_2 h^3 - \partial_3 h^2 = j^1 + \partial_0 d^1 , \quad \partial_3 h^1 - \partial_1 h^3 = j^2 + \partial_0 d^2 , \quad \partial_1 h^2 - \partial_2 h^1 = j^1 + \partial_0 d^1 ,
\]
\[
\partial_2 d^3 - \partial_3 d^2 = -\partial_0 h^1 , \quad \partial_3 d^1 - \partial_1 d^3 = -\partial_0 h^2 , \quad \partial_1 d^2 - \partial_2 d^1 = -\partial_0 h^1 .
\] (6)

Now let us introduce certain linear transformation over quantities entering the Maxwell’s equations (modified Lorentz transformation):

\[
x^0 = \text{ch} \sigma x^0 - \text{sh} \sigma x^1 , \quad x^1 = -\text{sh} \sigma x^0 + \text{ch} \sigma x^1 , \quad x^2 = x^2 , \quad x^3 = x^3 ,
\]
\[
j^0 = \text{ch} \sigma j^0 - \text{sh} \sigma j^1 , \quad j^1 = -\text{sh} \sigma j^0 + \text{ch} \sigma j^1 , \quad j^2 = j^2 , \quad j^3 = j^3 ,
\]
\[
d^1 = +d^1 , \quad d^2 = \text{ch} \sigma d^2 - \text{sh} \sigma h^3 , \quad h^3 = -\text{sh} \sigma d^2 + \text{ch} \sigma h^3 ,
\]
\[
h^1 = +h^1 , \quad d^3 = \text{ch} \sigma d^3 + \text{sh} \sigma h^2 , \quad h^2 = +\text{sh} \sigma d^3 + \text{ch} \sigma h^2 .
\] (7)

Now the task is to show that if one transforms equations (7) to new (primed) variables, then as a result one again will obtain equations of the form (6) with a single difference: all quantities become primed ones (see also [11-14]):

\[
\text{div' } \mathbf{d}' = j'^0 , \quad \text{div' } \mathbf{h}' = 0 , \quad \text{rot' } \mathbf{h}' = j' + \frac{\partial}{\partial x'^0} \mathbf{d}' , \quad \text{rot' } \mathbf{d}' = -\frac{\partial}{\partial x'^0} \mathbf{h}' .
\] (8)

## 3 On physical sense of the modified Lorentz transformations

What is the physical sense of the dimensionless parameter \( \sigma \) in the above Lorentz formulas? In fact, from the very beginning the question was: how Maxwell’s equations behave themselves
when reference frame is changed from $K$ to a moving $K'$. For the situation when velocity is small enough we must obtain a simple and "evident" solution in the form of Galilei formula for a coordinate transform:

$$t' = t, \quad x' = x - Vt, \quad y' = y, \quad z' = z.$$  \hspace{1cm} (9)

The Lorentz transformation at small $\sigma$ will take the form

$$kc t' \approx kct - \sigma x \approx kct \Rightarrow t' = t, \quad x' = -\sigma kct + x = x - Vt, \quad \text{if} \quad \sigma = \frac{V}{kc}.$$  

So that the physical sense of the parameter $\sigma$ (at its small values $\sigma << 1$) is found:

$$\sigma << 1 : \quad \Rightarrow \quad \sigma = \frac{V}{kc} = \frac{V}{c_{\text{media}}}. \hspace{1cm} (10)$$

One needs to generalize eq. (10) for arbitrary values of $V$. This is achieved by the following relations:

$$0 < |V| < kc : \quad \text{ch} \sigma = \frac{1}{\sqrt{1 - (V/kc)^2}}, \quad \text{sh} \sigma = \frac{(V/kc)}{\sqrt{1 - (V/kc)^2}}.$$ \hspace{1cm} (11)

or

$$t' = \frac{t - Vx/k^2c^2}{\sqrt{1 - (V/kc)^2}}, \quad x' = \frac{x - Vt}{\sqrt{1 - (V/kc)^2}}. \hspace{1cm} (12)$$

4 The speed of light and the modified Lorentz transformations

On finding the modified Lorentz transformations a simple kinematical problem of modified Lorentz formulas for velocity may be immediately solved. It provides us with a postulate on light velocity ($kc$) constancy, the crucial logical element in Einstein’s construction of Special Relativity [4].

So it may be readily derived the modified rule for transforming the velocity vector $\mathbf{W} = (W_x, W_y, W_z)$:

$$W'_{x} = \frac{W_x - V}{1 - VW_x/k^2c^2}, \quad W'_{y} = \sqrt{1 - V^2/k^2c^2} W_y, \quad W'_{z} = \sqrt{1 - V^2/k^2c^2} W_z. \hspace{1cm} (13)$$

This is a modified version of the famous rule for velocity summing by Lorentz-Poincaré-Einstein. This formulas results in some consequences.

$Lorentz \text{ transformation along the axis } x \text{ does not change the value of the speed of light propagating along this direction } x$. Lorentz transformation along the axis $x$ does not change the modulus of the light velocity vector.
5 Lorentz-Poincaré-Einstein, controversy and misunderstanding

It was Lorentz [2] who first established a remarkable property of the Maxwell’s equations: its symmetry under special mathematical transformations when entering these equations quantities – time and space coordinates, charge-current density, and electromagnetic fields. Poincaré introduced clarity [3] into Lorentz initial formulas and revealed its mathematical (so-called) group structure. Undoubtedly the first deciding steps on the road to Special relativity theory were made by Lorentz and this was stressed by Poincaré more than once. In the same time, Lorentz never ascribes to himself everything on this road and very highly appreciated the role and contribution of Poincaré.

Unfortunately, afterwards in connection with Special relativity there arose controversy and misunderstanding on the question – who is creator of this theory: Lorentz, Poincaré, or Einstein. To the present day this dispute is still with us (reviewing of the situation, for example, see in [15]). We will not join the debaters. In our opinion, all three, Lorentz, Poincaré, and Einstein, are creators of the theory.

The first was Lorentz, then Poincaré sided with him, and next Einstein started his work on creating Special relativity [4], mainly on its physical interpretation, comprehension, and logical reconstruction. The question – who is the main creator – is false. Lorentz formulated his view on the matter concisely and definitely [5]: the same what we had deduced from Maxwell’s equations Einstein has postulated ....

So, creating Special relativity has proceeded along two lines, and it is absurd to consider them as absolutely independent. One line goes upward to Special relativity from symmetry property of Maxwell’s theory in moving bodies. This is inductive way and it is historically first one. The second line, though logically independent in appearance, is deductive construction of the theory by going down from a special postulate [4]. But the postulate itself can be regarded as a logical mathematical result of the Lorentz-Poincaré analysis of the Maxwell’s theory.

Logical treatment suggested by Einstein seems for many simple and clear, so that it may be explained even to a person without any special education. This circumstance assists in the promotion of Einstein treatment of Special relativity and its notability in general public.

However by creating the deductive way to construct Special relativity does not provide the ground for assigning to A. Einstein the main or say single creator of the theory. Lorentz and Poincaré provided us with inductive way to this theory, and this way was historically the first. Both lines to Special relativity are legitimate and mutually complementary.

6 On the Maxwell theory in the media, Minkowski approach

Although the main work on Special relativity of A. Einstein [4] in 1905 is titled ”On the Electrodynamics of Moving Bodies”, in this paper Maxwell equations only in vacuum (a media with trivial values $\epsilon = 1, \mu = 1$ ) had been considered in fact and the symmetry properties

\footnote{Also we should not ignore a great number of physicists, owing to their enormous and laborious work we could go up mounts where only three great names are present.}
of these equations had been used. In accordance with this in all theoretical building from the very beginning only a universal light velocity in vacuum was used and just for that velocity was advanced a postulate of its constancy irrespective of the motion of the reference frame.

Later in 1908 H. Minkowski gave [6] a more detailed and accurate treatment of the Maxwell theory in a uniform medium ($\epsilon \neq 1, \mu \neq 1$) with respect to requirements of Special relativity. Two points of his study should be emphasized:

Minkowski had elaborated a very convenient and still actively exploited mathematical technique – so called 4-dimensional tensor formalism\(^4\).

Minkowski had found the way to describe symmetry properties of the Maxwell equations in a uniform medium with the use of Lorentz formulas on the base of the light velocity $c$ in vacuum\(^5\).

In this work Minkowski had achieved some unification between Einstein earlier analysis and electrodynamics in media in fact.

Here might be mentioned specially that logical construction of Special relativity by Einstein formally does not depend on the numerical values of the light velocity – this might be 300000 km/sec as well as 3 sm/sec. Essential is only the existence of a (light) signal which goes through the space with the same velocity for all inertial observers. And in this context we must recognize that operating with a light signal of velocity $c$ in a medium is only a mental fiction; in fact any real light can move through a uniform medium with velocity $\tilde{c} = kc$. So it might seem well-taken the requirement to perform clock synchronization in the uniform medias with the help of real light signals influenced by the media. However this was not done by Einstein and also it was not done by Minkowski. On the contrary, Minkowski found the way to speak about relativistic symmetry of the Maxwell’s theory in a media and to use only the Lorentz formulas with the vacuum light velocity $c$. Below we will introduce the Minkowski’s approach [6] without following it in detail (see also [16,17,18]).

7 Standard Lorentz symmetry of the Maxwell theory in a media

Let us start from the Maxwell’s equations in the form ($x^0 = ct, J^0 = \rho$)

$$
\text{div} \, D = J^0, \quad \text{rot} \, \frac{H}{c} = \frac{J}{c} + \frac{\partial D}{\partial ct}, \quad \text{div} \, cB = 0, \quad \text{rot} \, E = -\frac{\partial cB}{\partial ct}.
$$

(14)

Here equations are divided into two groups: for vectors $(D, H/c)$ and for vectors $(E, cB)$. Note that the source fields $(J^0 = \rho, J/c)$ enter only the first group. Also one point to emphasize is that eqs. (14) do not include parameters of dielectric and magnetic penetrability, however as a peculiar compensation for this we need to use concurrently two sets of electromagnetic vectors: $(D, H/c)$ and $(E, cB)$.

It is readily established that if eqs. (14) are subjected to the (ordinary) Lorentz transfor-

\(^4\)To be exact, H.Poincaré had proposed and developed in some aspects the same technique before Minkowski [4].

\(^5\)This point is the most significant in the context of the above established the symmetry of the Maxwell theory in a media under modified Lorentz transformations involving the light velocity in the media $kc$. 

mation (with the light velocity \(c\) in the vacuum and correspondingly with the variable \(x^0 = ct\)):

\[
\begin{align*}
x^0 &= \cosh \beta x^0 - \sinh \beta x^1, \\
x^1 &= -\sinh \beta x^0 + \cosh \beta x^1, \\
x^2 &= x^2, \\
x^3 &= x^3,
\end{align*}
\]

\[
\begin{align*}
J^0 &= \cosh \beta J^0 - \sinh \beta c^{-1} J^1, \\
J^1 &= -\sinh \beta J^0 + \cosh \beta c^{-1} J^1, \\
J^2 &= J^2, \\
J^3 &= J^3,
\end{align*}
\]

\[
\begin{align*}
D^1 &= +D^1, \\
D^2 &= \cosh \beta D^2 - \sinh \beta c^{-1} H^3, \\
H^1 &= +H^1, \\
H^3 &= \cosh \beta D^3 + \sinh \beta c^{-1} H^2, \\
E^1 &= +E^1, \\
E^2 &= \cosh \beta E^2 - \sinh \beta c B^3, \\
B^1 &= +B^1, \\
E^3 &= \cosh \beta E^3 + \sinh \beta c B^2, \\
B^2 &= +\sinh \beta E^3 + \cosh \beta c B^2,
\end{align*}
\]

we again will obtain equations in the Maxwell’s form:

\[
\begin{align*}
\nabla' \cdot D' &= J^0, \\
\nabla' \times \frac{H'}{c} &= \frac{J'}{c} + \frac{\partial D'}{\partial ct'}, \\
\nabla' \cdot cB' &= 0, \\
\nabla' \times E' &= -\frac{\partial cB'}{\partial ct'}.
\end{align*}
\]

The point of first importance is that the modified Lorentz transformations used in (10) generate much different formulas: from formal view point all the difference is reduced to appearance of the modified quantity \(kc\) everywhere instead of \(c\):

So, with Maxwell’s equation we have faced a rather peculiar situation when at the same time two different symmetries are revealed:

I) a symmetry with respect to the ordinary Lorentz transformations in which there is presented a universal constant – the light velocity in the vacuum;

II) another symmetry with respect to the modified Lorentz transformations in which there is presented a media dependent constant – the light velocity in the media;

III) explicit transforms both for space-time coordinates and for electromagnetic quantities differ for these two cases.

Which symmetry of these two is more correct or adequate? What attitude should we have to the fact itself of existence of two symmetries for Maxwell equations in a media? Which of them corresponds more closely to the physical reality? Exist there any criteria to pick out only one of two logical possibilities? All these questions should be answered.

From purely theoretical view point, taking seriously the need to synchronize clocks with the help of real light signals in a media, one must use the modified version of the Lorentz transformations in the media (also see [13]).

8 Field restrain conditions and the Minkowski equations

Now let us examine the following problem: what form will the field relations

\[
D^i = \epsilon_0 \epsilon E^i, \quad H^i = \frac{1}{\mu_0 \mu} B^i,
\]

(17)
take after the Lorentz transformation to a moving reference frame? Firstly, this problem was considered and solved by H. Minkowski in 1908 [6]. For simplicity we will take the most simple Lorentz formulas that correspond to a moving reference frame along axis \(x\).
In the first place consider the ordinary Lorentz transforms. With the use of (15) from eqs. (17) it follows

\[ D'_i = \epsilon_0 \epsilon E'_i, \quad \implies D'^1 = \epsilon_0 \epsilon E'^1, \]

\[ \text{ch} \beta D'^2 + \text{sh} \beta \frac{H'^3}{c} = \epsilon_0 \epsilon (\text{ch} \beta E'^2 + \text{sh} \beta c B'^3), \]

\[ \text{ch} \beta D'^3 - \text{sh} \beta \frac{H'^2}{c} = \epsilon_0 \epsilon (\text{ch} \beta E'^3 - \text{sh} \beta c B'^2); \quad (18) \]

\[ H^i = \frac{1}{\mu_0 \mu} B^i, \quad \implies H'^1 = \frac{1}{\mu_0 \mu} B^1, \]

\[ \text{sh} \beta D'^3 - \text{ch} \beta \frac{H'^2}{c} = \frac{1}{\mu_0 \mu c^2} (\text{sh} \beta E'^3 - \text{ch} \beta c B'^2), \]

\[ \text{sh} \beta D'^2 + \text{ch} \beta \frac{H'^3}{c} = \frac{1}{\mu_0 \mu c^2} (\text{sh} \beta E'^2 + \text{ch} \beta c B'^3). \quad (19) \]

Relations (18) and (19) are just what we call Minkowski equations [6] written down in a particular simple case. Let us change them to another form. To this end, they should be rewritten as three pairs of linear systems under the variables \((D'^1, H'^1), (D'^2, H'^3/c), (D'^3, H'^2/c)\). Solutions of which look as follows

\[ D'^1 = \epsilon_0 \epsilon E'^1, \]

\[ D'^2 = \epsilon_0 \epsilon \left[ (\text{ch}^2 \beta - k^2 \text{sh}^2 \beta) E'^2 + \text{sh} \beta \text{ch} \beta (1 - k^2) c B'^3 \right], \]

\[ D'^3 = \epsilon_0 \epsilon \left[ (\text{ch}^2 \beta - k^2 \text{sh}^2 \beta) E'^3 - \text{sh} \beta \text{ch} \beta (1 - k^2) c B'^2 \right], \]

\[ H'^1/c = \epsilon_0 \epsilon k^2 c B^1, \]

\[ H'^2/c = \epsilon_0 \epsilon \left[ (k^2 \text{ch}^2 \beta - \text{sh}^2 \beta) c B'^2 - \text{sh} \beta \text{ch} \beta (k^2 - 1) E'^3 \right], \]

\[ H'^3/c = \epsilon_0 \epsilon \left[ (k^2 \text{ch}^2 \beta - \text{sh}^2 \beta) c B'^3 + \text{sh} \beta \text{ch} \beta (k^2 - 1) E'^2 \right]. \quad (20) \]

Equations (20) say that simple connections (17) between electromagnetic vectors in initial (unmoving) frame after translating to a moving reference frame become rather complex ones – they involve now the velocity as a parameter. In other terms, this means that the field relations (17) are not Lorentz invariant. However, we can see that in the vacuum case when \(k = 1\) the formulas (20) will take the same most simple form from which we have started initially:

\[ k = 1, \quad D'^i = \epsilon_0 \epsilon E'^i, \quad H'^i = \frac{1}{\mu_0} B^i. \quad (21) \]

Now, let us proceed to the consideration of the field relations when we use the modified Lorentz transformations. We will easily see that properties of these relations under that modified Lorentz theory become different and much more attractive: they turn out to be Lorentz
invariant\textsuperscript{6}. Indeed, let us start with (everywhere instead of $c$ there appears $kc$):

\[ D^i = \epsilon_0 \epsilon E^i, \quad \Rightarrow \quad D'^i = \epsilon_0 \epsilon E'^i, \]

\[ \text{ch} \sigma D'^2 + \text{sh} \sigma \frac{H'^3}{kc} = \epsilon_0 \epsilon (\text{ch} \sigma E'^2 + \text{sh} \sigma kcB'^3), \]

\[ \text{ch} \sigma D'^3 - \text{sh} \sigma \frac{H'^2}{kc} = \epsilon_0 \epsilon (\text{ch} \sigma E'^3 - \text{sh} \sigma kcB'^2), \quad (22) \]

\[ H^i = \frac{1}{\mu_0 \mu} B^i, \quad \Rightarrow \quad H'^i = \frac{1}{\mu_0 \mu} B'^i, \]

\[ -\text{sh} \sigma D'^3 + \text{ch} \sigma \frac{H'^2}{kc} = \frac{1}{\mu_0 \mu} \frac{1}{k^2 c^2} (-\text{sh} \sigma E'^3 + \text{ch} \sigma kcB'^2), \]

\[ \text{sh} \sigma D'^2 + \text{ch} \sigma \frac{H'^3}{kc} = \frac{1}{\mu_0 \mu} \frac{1}{k^2 c^2} (\text{sh} \sigma E'^2 + \text{ch} \sigma kcB'^3). \quad (23) \]

Their solutions are

\[ D'^i = \epsilon_0 \epsilon E'^i, \quad H'^i = \frac{1}{\mu_0 \mu} B'^i. \quad (24) \]

So we have arrived at the most attractive result from theoretical viewpoint: The Maxwell’s in the media together with the field restrain conditions turn out to be invariant under modified Lorentz transformations. This is a significant theoretical argument in favor of the Lorentz symmetry involving the light velocity in a media $kc$ instead of the light velocity in the vacuum $c$. Such a modified theoretical scheme looks simpler and more attractive then commonly exploited one.

9 4-tensor formalism

The two Maxwell equations with sources

\[ \text{div} \ D = J^0, \quad \text{rot} \frac{H}{c} = \frac{J}{c} + \frac{\partial D}{\partial ct} \]

can be presented in a very compact and simple form if one introduces a special notation with the use of indices taking over four values. Let us introduce the tensor

\[ (H^{ab}) = \begin{vmatrix} 0 & -D^1 & -D^2 & -D^3 \\ +D^1 & 0 & -H^3/c & +H^2/c \\ +D^1 & +H^3/c & 0 & -H^1/c \\ +D^3 & -H^2/c & +H^1/c & 0 \end{vmatrix}. \quad (25) \]

The main assertion is that the above two equations are equivalent to the tensor one (the notation $x^a = (ct; x^i)$, $j^a = (J^0, J^i/c)$ is used)

\[ \partial_b H^{ba} = j^a. \quad (26) \]

\textsuperscript{6}This is a long time known result, see for instance the Rosen’s work [13]
Now let us consider the two remaining Maxwell equations
\[
\text{div } c\mathbf{B} = 0, \quad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}.
\]
To deal with these two equations Minkowski had introduced another tensor $F^{ab}$:
\[
(F^{ab}) = \begin{vmatrix}
0 & -E^1 & -E^2 & -E^3 \\
+E^1 & 0 & -cB^3 & +cB^2 \\
+E^1 & +cB^3 & 0 & -cB^1 \\
+E^3 & -cB^2 & +cB^1 & 0
\end{vmatrix}. \quad (27)
\]
The main assertion here is that the remaining Maxwell’s equations are equivalent to the tensor one
\[
\partial_c F_{ab} + \partial_a F_{bc} + \partial_b F_{ca} = 0. \quad (28)
\]
Thus, we have arrived at the compact tensor form of the Maxwell equations:
\[
\partial_b H^{ba} = j^a, \quad \partial_c F_{ab} + \partial_a F_{bc} + \partial_b F_{ca} = 0. \quad (29)
\]
The Maxwell’s equations in other variables, in which they exhibit symmetry under modified Lorentz transformations, may be rewritten with the use of only one electromagnetic tensor $(f^{AB}) = (d, h)$ (appearance here and in the following of the capital letters to stand for tensor’s indexes means that such quantities transform in accordance with the modified Lorentz symmetry) in the form of two tensor equations
\[
\partial_B f^{BC} = j^C, \quad \partial_C f_{AB} + \partial_A f_{BC} + \partial_B f_{CA} = 0. \quad (30)
\]
Besides, the Maxwell’s equations, invariant under modified Lorentz transformations, may be rewritten with the help of two tensors as well. Indeed, equations (5) may be taken as
\[
\text{div } kc\mathbf{B} = 0, \quad \text{rot } \mathbf{E} = -\frac{\partial}{\partial kct} kc\mathbf{B}, \quad \text{div } \mathbf{D} = J^0, \quad \text{rot } \frac{H}{kc} = \frac{J}{kc} + \frac{\partial}{\partial kct} \mathbf{D}. \quad (31)
\]
From where, introducing (modified) electromagnetic tensors:
\[
(H^{AB}) = (\mathbf{D}, \mathbf{H}/kc), \quad (F^{AB}) = (\mathbf{E}, kc\mathbf{B}) \quad (32)
\]
eqs. (31) can be readily presented as follows:
\[
\partial_B H^{BA} = j^A, \quad \partial_C F_{AB} + \partial_A F_{BC} + \partial_B F_{CA} = 0. \quad (33)
\]

10 Minkowski relations in covariant tensor form

The Minkowski’s equations may be quite easily rewritten in a special form that remains the same for any Lorentz transformations, including arbitrary rotations and (uniform) movings. Trick that is employed below is simple but useful and often applicable. It is based on the following property of the tensor formalism: if we think (know) that a certain physical equation
must be Lorentz invariant and an explicit form of the equation is given only in some particular reference frame then its invariant form may be found with the help of Lorentz transformations. The same may be achieved if we can see in the particular equation its general tensor form.

To make use of this trick, one special notion, 4-vector of velocity that can be related to the moving media, is needed

\[ u^a = \frac{d x^a}{d s} = \left( \frac{1}{\sqrt{1-v^2/c^2}}, \frac{v^i/c}{\sqrt{1-v^2/c^2}} \right). \] (34)

To obtain a tensor form of Minkowski equations, we will take a particular 4-velocity vector:

\[ u^a = \left( \frac{1}{\sqrt{1-v^2/c^2}}, \frac{-v/c}{\sqrt{1-v^2/c^2}}, 0, 0 \right) = \left( \text{ch} \beta, -\text{sh} \beta, 0, 0 \right). \] (35)

It is the matter of simple calculation to see [6] that the all six Minkowski equations are equivalent to the tensor ones:

\[ H_{ab} u^b = \epsilon_0 \epsilon F^{ab} u^b, \] (36)
\[ H_{ab} u^c + H^{bc} u^a + H^{ca} u^b = \frac{1}{\mu \mu_0} \left( F^{ab} u^c + F^{bc} u^a + F^{ca} u^b \right). \] (37)

In the vacuum case, when \( \epsilon = 1, \mu = 1 \), eqs. (36) and (37) may be rewritten differently

\[ H_{ab} u^b = \epsilon_0 \epsilon F^{ab} u^b, \] (38)
\[ H_{ab} u^c + H^{bc} u^a + H^{ca} u^b = \epsilon_0 (F^{ab} u^c + F^{bc} u^a + F^{ca} u^b). \] (39)

These tensor equations admit a simple solution. Indeed, let us multiply eq. (39) by \( u_c \) (considering \( u^c u_c = +1 \)), then

\[ H_{ab} = \epsilon_0 (F^{ab} + F^{bc} u_c u^a + F^{ca} u_c u^b) - H^{bc} u_c u^a - H^{ca} u_c u^b; \]

from where, bearing in mind (38), we arrive at

\[ H_{ab} = \epsilon_0 F^{ab}. \] (40)

This tensor condition, in component form will look as six relations (just the same ones were obtained earlier in (21))

\[ D^i = \epsilon_0 E^i, \quad H^i = \frac{1}{\mu_0} B^i. \]

However, for any media, analogous calculation leads to a very different result. Indeed, let us multiply (37) by \( u_c \):

\[ H_{ab} + H^{bc} u_c u^a + H^{ca} u_c u^b = \frac{1}{\epsilon \mu \mu_0} \left( F^{ab} + F^{bc} u_c u^a + F^{ca} u_c u^b \right). \]

From this, taking (36), we get to

\[ H_{ab} = \epsilon_0 \epsilon k^2 F^{ab} + \epsilon_0 \epsilon (k^2 - 1) \left[ F^{bc} u_c u^a - F^{ac} u_c u^b \right]. \] (41)
Evidently, these are a covariant tensor form the earlier found in (20) – also see in [7-10,13,14,16-18].

Take notice that while using the Maxwell theory with only one (modified) tensor \( f_{AB} \), no additional condition between electromagnetic tensors is needed at all. Some clarifying analysis can be easily done. In this case, there arise modified Minkowski relations (\( c \) is changed by \( kc \)):

\[
D^i = \epsilon_0 \epsilon_0 E^i : \quad \implies \quad D'^i = \epsilon_0 \epsilon'E'^i,
\]

\[
\text{ch} \sigma D'^2 + \text{sh} \sigma \frac{H'^3}{kc} = \epsilon_0 \epsilon (\text{ch} \sigma E'^2 + \text{sh} \sigma kcB'^3),
\]

\[
\text{ch} \sigma D'^3 - \text{sh} \sigma \frac{H'^2}{kc} = \epsilon_0 \epsilon (\text{ch} \sigma E'^3 - \text{sh} \sigma kcB'^2),
\]

\[
H'^i = \frac{1}{\mu \mu_{0}} B^i : \quad \implies \quad H'^1 = \frac{1}{\mu \mu_{0}} B^1,
\]

\[
-\text{sh} \sigma D'^3 + \text{ch} \sigma \frac{H'^2}{kc} = \frac{1}{\sigma \mu_{0}^2} \frac{1}{k^2 c^2} (-\text{sh} \sigma E'^3 + \text{ch} \sigma kcB'^2),
\]

\[
\text{sh} \sigma D'^2 + \text{ch} \sigma \frac{H'^3}{kc} = \frac{1}{\mu \mu_{0}^2} \frac{1}{k^2 c^2} (\text{sh} \sigma E'^2 + \text{ch} \sigma kcB'^3).
\]

(42)

(43)

Now a modified 4-velocity \( U^A \) is needed:

\[
U^A = \frac{dx^a}{ds} = \left( \frac{1}{\sqrt{1-v^2/k^2 c^2}}, \frac{v^i/kc}{\sqrt{1-v^2/k^2 c^2}} \right),
\]

(44)

and its particular form

\[
U^A = \left( \frac{1}{\sqrt{1-v^2/k^2 c^2}}, \frac{-v/kc}{\sqrt{1-v^2/k^2 c^2}}, 0, 0 \right) = (\text{ch} \sigma, -\text{sh} \sigma, 0, 0).
\]

Tensor representation of eqs. (42) and (43) is

\[
H^{AB} U_B = \epsilon_0 \epsilon F^{AB} U_B,
\]

(45)

\[
H^{AB} U^C + H^{BC} U^A + H^{CA} U^B = \frac{1}{k^2 c^2 \mu \mu_{0}} (F^{AB} U^C + F^{BC} U^A + F^{CA} U^B).
\]

(46)

The latter equation may be rewritten as

\[
H^{AB} U^C + H^{BC} U^A + H^{CA} U^B = \epsilon_0 \epsilon (F^{AB} U^C + F^{BC} U^A + F^{CA} U^B).
\]

(47)

Multiplying it by \( U_C \):

\[
H^{AB} + H^{BC} U_C U^A + H^{CA} U_C U^B = \epsilon_0 \epsilon (F^{AB} + F^{BC} U_C U^A + F^{CA} U_C U^B),
\]

from where, with (45), it follows

\[
H^{AB} = \epsilon_0 \epsilon F^{AB}, \quad \text{or} \quad D^i = \epsilon_0 \epsilon E^i, \quad H^i = \frac{1}{\mu \mu_{0}} B^i.
\]

(48)
11 Potentials in a media

In remains to be seen which peculiarities arise from the presence of a uniform media when we try do describe electromagnetic fields in terms of the scalar and vector potentials \( \varphi, A \):

\[
\{ E, B, D, H \} \implies \{ \varphi, A \}.
\]

We might anticipate some difficulties because the Maxwell theory in the media exhibits symmetry under ordinary, the vacuum light velocity based, Lorentz transformations only if the two electromagnetic tensors, \( H^{ab} \) and \( F^{ab} \), are used. There is no ground for feeling enthusiastic about existence of relativistically invariant formulas (36) and (37). These formulas involve 4-velocity vector, characteristic of the state of moving of the media under the inertial reference frame. In other words, these formulas may be understood as explicit dependence of the basic electrodynamic equations upon the absolute velocity of a moving body. It somehow contradicts to the initial principles and main claims of Special relativity theory.

Let us write down the Maxwell’s equations again:

\[
\begin{align*}
\nabla \cdot B &= 0, \\
\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\nabla \cdot E &= \frac{1}{\varepsilon \varepsilon_0} \rho, \\
\nabla \times B &= J + \frac{1}{\mu \mu_0} \nabla \cdot E.
\end{align*}
\]

The most general substitution for potentials \( \varphi, A \) which makes two first equations (49) identities has the form

\[
B = d \nabla \times A, \quad E = -n \nabla \varphi - d \frac{\partial A}{\partial t},
\]

where \( d, n \) are some yet unknown parameters. The first equation with the source \( \rho \) in (49) gives

\[
(-\nabla^2 \varphi + \frac{dm}{n} \frac{\partial^2 \varphi}{\partial t^2}) = \frac{1}{n \varepsilon \varepsilon_0} \rho + \frac{d}{n \mu \mu_0} (\nabla \cdot A + m \frac{\partial \varphi}{\partial t}).
\]

The second equations in (49) with the source \( J \) leads to

\[
\nabla \times (\nabla \times A) = \frac{\mu \mu_0}{d} J + \frac{\varepsilon \varepsilon_0 \mu \mu_0}{d} \frac{\partial}{\partial t} (-n \nabla \varphi - d \frac{\partial A}{\partial t}).
\]

From this, with the help of the identity \( \nabla \times (\nabla \times A) = -\nabla^2 A + \nabla (\nabla \cdot A) \), we get to

\[
(-\nabla^2 A + \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 A}{\partial t^2}) = \frac{\mu \mu_0}{d} J - \nabla (\nabla \cdot A + \frac{\varepsilon \varepsilon_0 \mu \mu_0 n}{d} \frac{\partial \varphi}{\partial t}).
\]

Comparing (51) and (52), we see that it suffices to demand

\[
\frac{dm}{n} = \varepsilon \varepsilon_0 \mu \mu_0, \quad m = \frac{\varepsilon \varepsilon_0 \mu \mu_0 n}{d}
\]

so that eqs. (51) and (52) will have quite symmetrical form with the same wave operator on the left:

\[
\begin{align*}
(-\nabla^2 \varphi + \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 \varphi}{\partial t^2}) &= \frac{1}{n \varepsilon \varepsilon_0} \rho + \frac{d}{n \mu \mu_0} (\nabla \cdot A + m \frac{\partial \varphi}{\partial t}), \\
(-\nabla^2 A + \varepsilon \varepsilon_0 \mu \mu_0 \frac{\partial^2 A}{\partial t^2}) &= \frac{\mu \mu_0}{d} J - \nabla (\nabla \cdot A + m \frac{\partial \varphi}{\partial t}).
\end{align*}
\]
With the use of \( c \) and \( k \), the previous equations become

\[
\frac{1}{k^2 c^2} = \frac{d m}{n},
\]

(56)

\[
(-\nabla^2 \varphi + \frac{1}{k^2 c^2} \frac{\partial^2 \varphi}{\partial t^2}) = \frac{1}{n \varepsilon_0} \rho + \frac{d}{n} \frac{\partial}{\partial t} \left( \nabla \cdot A + m \frac{\partial \varphi}{\partial t} \right),
\]

(57)

\[
(-\nabla^2 A + \frac{1}{k^2 c^2} \frac{\partial^2 A}{\partial t^2}) = \frac{\mu \mu_0}{d} J - \nabla \left( \nabla \cdot A + m \frac{\partial \varphi}{\partial t} \right).
\]

(58)

Relationship (56) allows different solutions. The most symmetrical and simplified all formulas seems to be

\[
m = \frac{1}{k c}, \quad n = \frac{1}{\varepsilon_0 \varepsilon}, \quad d = \frac{n}{k c \varepsilon_0}, \quad \frac{\mu \mu_0}{d} = \mu \mu_0 \varepsilon_0 k c = \frac{1}{k c}.
\]

(59)

at this eqs. (57) and (58) provide us with these

\[
(-\nabla^2 \varphi + \frac{1}{k^2 c^2} \frac{\partial^2 \varphi}{\partial t^2}) = \rho + \frac{1}{k c} \frac{\partial}{\partial t} \left( \nabla \cdot A + \frac{1}{k c} \frac{\partial \varphi}{\partial t} \right),
\]

\[
(-\nabla^2 A + \frac{1}{k^2 c^2} \frac{\partial^2 A}{\partial t^2}) = \frac{J}{k c} - \nabla \left( \nabla \cdot A + \frac{1}{k c} \frac{\partial \varphi}{\partial t} \right).
\]

(60)

Eqs. (60) may be rewritten as a (modified) tensor equation:

\[
\partial^B \partial_B A^C = j^C + \partial^C (\partial_B A^B),
\]

(61)

where

\[
x^C = (k c t, x^i), \quad A^C = (\varphi, A^i), \quad j^C = (\rho, \frac{J^i}{k c}).
\]

(62)

Eq. (61) proves its invariance under modified Lorentz transformations constructed on the base of the light velocity \( k c \) in the media. Initial relations (50) introducing electromagnetic potentials:

\[
\nabla \times A = \frac{B}{\mu_0 \mu k c} = h, \quad -\nabla \varphi - \frac{1}{k c} \frac{\partial A}{\partial t} = \varepsilon_0 \varepsilon E = d
\]

(63)

may be readily translated to tensor form:

\[
f_{BC} = \partial_B A_C - \partial_C A_B,
\]

(64)

where \( f_{ab} \) is the electromagnetic tensor for Maxwell equations in modified variables. Tensor relationship (64) permits quite easily reveal a gauge freedom in determining of electromagnetic potentials:

\[
A'_B = A_B + \partial_B \Lambda, \quad \Rightarrow \quad f'_{BC} = f_{BC}.
\]

(65)

Often the Lorentz gauge condition is taken to be the most convenient \( \partial_B A^B = 0 \). Thus, as a result of the change of variables: \( E, D, B, H \Rightarrow (\varphi, A) = A^C \) the simplicity has been
achieved: all the Maxwell’s electrodynamics formally is equivalent to the single equation for 1-rank tensor $A^B$. In the Lorentz gauge, the Maxwell’s electrodynamics looks the most simple and beautiful: namely it reduces to the wave equation:

$$\partial \partial_B A^A = j^C, \quad \partial_B A^B = 0. \quad (66)$$

One notice again: solutions of the equation (66) there correspond to wave processes propagating with the speed of light in the media, not in the vacuum, and this velocity is invariant under modified Lorentz formulas.

The choice of solution in (56) that was taken above is not unique. Equally, starting from (56), (57), (58), we may chose the more traditional one:

$$d = 1, \quad n = 1, \quad m = \frac{1}{k^2 c^2}, \quad (67)$$

$$\left(-\nabla^2 \varphi + \frac{1}{k^2 c^2} \frac{\partial^2 \varphi}{\partial t^2}\right) = \frac{1}{\varepsilon \varepsilon_0} \rho + \frac{\partial}{\partial t} (\nabla \cdot A + \frac{1}{k^2 c^2} \frac{\partial \varphi}{\partial t}),$$

$$\left(-\nabla^2 A + \frac{1}{k^2 c^2} \frac{\partial^2 A}{\partial t^2}\right) = \mu \mu_0 J - \nabla (\nabla \cdot A + \frac{1}{k^2 c^2} \frac{\partial \varphi}{\partial t})$$

which coincides with that used in the the known handbook on electrodynamics by Stratton [1]. Evidently, two variants are totally equivalent, because they differ only in determining units for potentials – see (50).

12 Potentials in a media, ordinary Lorentz symmetry treatment

Now we consider an alternative way of introducing potentials into electrodynamics in presence of a media which has its origin in earlier investigation by Minkowski [6] (just this variant is being used mainly; see for instance the thorough review [16].)

The most noticeable feature of this method consists in the following: in this approach we are able to formulate the Maxwell’s electrodynamics in a media in terms of potentials with the use of the ordinary Lorentz symmetry based on the vacuum light velocity $c$ only. Concurrently, the mathematical equations achieved look more complicated, and also these equations involve explicitly the velocity of the media under the reference frame. The latter might be considered as return to prehistory of Special relativity with all searching some absolute velocities. However, we are not going to be submerged in so metaphysical subtleties. Nevertheless, one point should be emphasized: there exist two alternative ways to develop potential approach for electrodynamics in media – one developed in previous Section, and another exposed below. The ways are completely equivalent in mathematical sense. The first is much more simpler technically but it presumes invariance under modified Lorentz symmetry based on the light velocity $kc$. Why must we employ the more complicated technique – only because of its concomitant ordinary Lorentz symmetry treatment?

Let us write down the Minkowski’s equations are equivalent to

$$H_{ab} = \Delta_{abmn} F^{mn}, \quad \Delta_{abmn} = \epsilon_0 \epsilon k^2 g_{am} g_{bn} + \epsilon_0 (\epsilon k^2 - 1) u_n (g_{bm} u_a - g_{am} u_b). \quad (68)$$
Firstly, the 4-rank tensor connecting $H^{ab}$ and $F^{ab}$ was introduced (for a more general case of anisotropic media) by Tamm and Mandel’stam [7,8]. For a uniform media, accordingly Watson-Yauch-Riazanov [9,10] the tensor $\Delta_{abmn}$ may be taken in another form:

$$H^{ab} = \Delta_{abmn} F^{mn}, \quad \Delta_{abmn} = A (g_{am} + Bu_{a}u_{m}) (g_{bn} + Bu_{b}u_{n}).$$  \hfill (69)

Although, this $\Delta_{abmn}$ contains a term of fourth order in velocity, only terms of second order give non-zero contribution into the formula (68). Let us demonstrate that (68) and (69) are the same at specially given $A$ and $B$. From (69) it follows

$$H^{ab} = A g_{am} g_{bn} F^{mn} + AB u_{a}u_{m}g_{bn} F^{mn} = A g_{am} g_{bn} F^{mn} + AB (g_{am}u_{b} - g_{bm}u_{a}) F^{mn}.$$  \hfill (70)

Comparing this with (68), we get

$$A = \epsilon_0 \epsilon k^2, \quad AB = \epsilon_0 \epsilon (k^2 - 1),$$

from where it follows

$$A = \epsilon_0 \epsilon k^2, \quad B = \frac{1 - k^2}{k^2} = \epsilon \mu - 1.$$  \hfill (70)

Therefore, equations (69) become

$$H^{ab} = \Delta_{abmn} F^{mn}, \quad \Delta_{abmn} = \epsilon_0 \epsilon k^2 \left[ (g_{am} + (\epsilon \mu - 1) u_{a}u_{m}) \right] \left[ (g_{bn} + (\epsilon \mu - 1) u_{b}u_{n}) \right].$$  \hfill (71)

Just this representation for 4-rank tensor relating $H^{ab}$ to $F^{ab}$ in a uniform media is given in the review [16]. A fresh review of the history of different electrodynamics constitutive equations is given in recent work [19].

Now we are ready to introduce potentials. The Maxwell equations are

$$\partial_{a} H^{ab} = j^{b}, \quad \partial_{a} \left( \Delta^{abmn} F_{mn} \right) = j^{b},$$  \hfill (72)

$$\partial_{c} F_{ab} + \partial_{a} F_{bc} + \partial_{b} F_{ca} = 0.$$  \hfill (73)

Potentials $A_{b}$ are defined in such a way that equations (73) turn to identities:

$$F_{ab} = \partial_{a} A_{b} - \partial_{b} A_{a};$$  \hfill (74)

With the help of (68), the $H^{ab}$ may be rewritten as

$$H^{ab} = \epsilon_0 \epsilon k^2 (\partial^{a} A^{b} - \partial^{b} A^{a}) + \epsilon_0 \epsilon (k^2 - 1)(u^{a} \partial^{b} - u^{b} \partial^{a})(u^{n} A_{n}).$$  \hfill (75)

Therefore, eq. (72) leads us to

$$\epsilon_0 \epsilon k^2 \partial_{a}(\partial^{a} A^{b} - \partial^{b} A^{a}) + \epsilon_0 \epsilon (k^2 - 1)\partial_{a}(u^{a} \partial^{b} - u^{b} \partial^{a})(u^{n} A_{n}) = j^{b}.$$  \hfill (76)

This is the main equation for electromagnetic 4-potentials in a media, this equation is invariant under the ordinary Lorentz transformations based on the vacuum light velocity. For purely vacuum case, the factor $(k^2 - 1)$ equals to zero and (76) takes the more familiar form

$$\partial_{a}\partial^{a} A^{b} - \partial^{b}(\partial_{a} A^{a}) = \mu_0 j^{b}.$$  \hfill (77)

The scheme with modified Lorentz symmetry, after transition to 4-potential $f_{CB} = \partial_{C} A_{B} - \partial_{B} A_{C}$ leads to the simple wave equation (compare with (76))
\[ \partial^B \partial_B A^C - \partial_C \partial_B A^B = j^C. \] 

(78)

No additional 4-velocity parameter enters this equation, so this form of the electrodynamics presumes a relative nature of the mechanical motion; also this equation describes waves propagating in space with the light velocity \( kc \), which is invariant under modified Lorentz formulas. In connection with these two theoretical schemes, a point of principle must be stressed: it might seem well-taken the requirement to perform Poincaré-Einstein clock synchronization in the uniform medias with the help of real light signals influenced by the media, which leads us to the modified Lorentz symmetry.

13 Discussion

From formal mathematical standpoint the all situation with the Maxwell’s theory in uniform media and its properties under the mechanical motion of the inertial reference frames looks rather peculiar and designing. Two alternative possibilities exist.

I.

One may start with usual Maxwell’s equation in terms of four electromagnetic vectors \( \mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H} \) with two additional restrain conditions \( \mathbf{D} = \varepsilon_0 \mathbf{E} \) and \( \mathbf{B} = \mu_0 \mu \mathbf{H} \), then to translate the Maxwell equations in term of new electromagnetic variables \( d, h \) without any additional restrain conditions. It is mathematically correct procedure evidently without any serious objection. Such a form of the Maxwell’s theory is remarkable because it allows us to reveal the existence of very simple symmetry in this theory; namely, it is invariant with respect to the group of modified Lorentz transformations, based on the use of the speed of light \( kc \) in the media instead of the speed of light in the vacuum applied in the conventional Lorentz symmetry. This mathematical result might to be seen as an ideal realization of the relativity principle in the presence of the uniform media. One might expect that just such a symmetry for electrodynamics in the uniform media must be taken as a base for describing the properties of all electromagnetic quantities under the mechanical motion of the reference frame.

II.

The second theoretical possibility is realized in the Minkowski approach to electrodynamics in uniform media. In this approach only conventional Lorentz symmetry with vacuum speed of light is used. Maxwell equation are formulated in covariant form at the rest reference frame tied with the media, in terms of two electromagnetic tensors \( F^{ab} \) and \( H^{ab} \). At the Lorentz translation they transform independently, the Maxwell’s equations are Lorentz invariant, but the additional restrain conditions change their form. To reach the formal relativistic invariance of the theory, the new modified form of the above restrain condition by definition is taken as genuine and true. In virtue of the construction procedure itself this new form of the restrain conditions is automatically Lorentz invariant. At this we must understand that almost any equation, through the mathematical trick of that type, can be translated to some new form that will be formally Lorentz invariant. In this connection the question may be posed – what is the new knowledge gained here. Therefore, the formal relativistical invariance is achieved in the Minkowski electrodynamics, but his equations contain explicitly an additional physical parameter – 4-velocity of the moving media (or differently, of the reference frame).
This means, that electrodynamics by Minkowski in the media presumes the absolute nature of the mechanical motion. In this connection, we might recall that Special relativity theory had started many years ago somehow from the requirement – to mechanical motion in electrodynamics should be a relative concept, no physical experiment is able to reveal the inertial motion of the reference frame. In a sense, the time has played a joke with our previous, and being taken seriously, theoretical arguments about symmetry of electromagnetic equations and relativity principle. The special relativity had been constructed to avoid an absolute velocity concept, but this absolute velocity concept again has returned in the frame of electrodynamics by Minkowski.

In any case simply ignoring the existence of the modified Lorentz symmetry in the Maxwell theory is not a correct attitude. It is hard to believe that this modified symmetry for the Maxwell’s theory in the uniform media is of no physical meaning and value.
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