A Brief Overview of Bipartite and Multipartite Entanglement Measures

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Abstract

Measuring entanglement is a demanding task in the field of quantum computation and quantum information theory. Very recently some authors [J. C. Loredo et al. Phys. Rev. Lett. 116, 070503 (2016)] experimentally demonstrate an embedding quantum simulator, using it to efficiently measure two-qubit entanglement. Here, we review some measures of entanglement (pure and mixed states). Furthermore, we report the efficient measures of bipartite and multipartite entanglement.

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I. INTRODUCTION

The entanglement is a key ingredient of quantum computation and quantum information theory \cite{1, 2}. The study of entanglement properties of such states is essential and recently has been a field of intense research \cite{3}. Quantum entanglement represents one of the most striking features of quantum systems that has no classical counterpart \cite{4}. There are a lot of measures of entanglement that exist nowadays, but we have to focus on the meaning of the term efficient. It depends if we want to qualify or quantify the entanglement. For example, for pure states, we have some algebraic varieties that describe a certain type or class of entanglement, under stochastic local operations with classical communication (SLOCC) \cite{5, 6}. Also by using invariant theory, some polynomial can help us to distinguish different classes of entanglement. We can also study the entanglement of pure state by associating, to each quantum state, a geometrical singularity, that can helps to have an information on the entanglement of the state. There are also some numerical measurement of entanglement that can help us to quantify entanglement, such as concurrence \cite{7, 8}, negativity and logarithmic negativity \cite{9}, Von Neumann entropy, etc \cite{10}. In this Letter, we demonstrate the efficient measurement of bipartite and multipartite entanglement. For pure or mixed states, we have some numerical measures of entanglement, but each of them give us certain information about the entanglement, but are not known that a particular measure surpasses all the others and is applicable to any quantum system. This paper is organized as follows: in Section II we provide bipartite systems and then we introduce multipartite systems in the Section III. Section IV is dedicated to discussion and conclusions.

II. BIPARTITE SYSTEMS

When a system consists of two subsystems we say it is a bipartite system. The Hilbert space of the composite system is the tensor product of the Hilbert space that describes Alice’s system and the Hilbert space that describes Bob’s system. If we denote these as $H_A$ and $H_B$, respectively, then the Hilbert space of the composite system is $H = H_A \otimes H_B$. Not all states $|\psi\rangle \in H_A \otimes H_B$ are entangled. When two systems are entangled, the state of each composite system can only be described with reference to the other state. If $|\psi\rangle \in H_A$ and $|\phi\rangle \in H_B$ and $|\xi\rangle = |\psi\rangle \otimes |\phi\rangle$, then $|\xi\rangle$ is a product state or separable.
In this Section we will review a variety of such measures. For bipartite systems, several measures of entanglement have been proposed [7–12]. Entanglement measure (EM) quantifies how much entanglement is contained in a quantum state. Formally it is any nonnegative real function of a state which cannot increase under local operations and classical communication (LOCC), and is zero for product state or separable. One of typical applications of abstract EM’s is to show that certain task can not be achieved by means of LOCC. One does it by showing that if the task could be done, then some EM would increase. Entanglement measures are also studied and classified according to their properties, e.g. additivity, convexity and continuity. This approach to entanglement measures is known as axiomatic approach [13]. An EM for a bipartite system is a state functional that vanishes on separable states and that does not increase under separable operations. It is well-known that for pure states, essentially all entanglement measures are equal to the Von Neumann entropy of the reduced state, but for mixed states, this uniqueness is lost [10].

III. MULTIPARTITE SYSTEMS

The study of multipartite entanglement has attracted much attention in the last years [14, 15]. From the theoretical side, multipartite entanglement may be a key element to improve various applications like quantum information processing or quantum metrology, or to understand and simulate physical systems, such as quantum spin chains undergoing a quantum phase transition. For multipartite systems, several measures of entanglement have been proposed. For example, generalized concurrence [14, 16], global entanglement [17], Scott measure (or generalized Meyer-Wallach measure) [18–21], geometric measures [22], etc.

Multipartite entanglement has been extensively investigated as a resource for quantum enhanced measurements. In the multipartite setting there are EM that simply are functions of sums of bipartite entanglement measures. For these multipartite entanglement measures the monotonicity under LOCC is simply inherited from the bipartite measures. But there are also EM that were constructed specifically for multipartite states. As required by a good measure of entanglement, it can be checked that both bipartite and multipartite entanglement are two non-increasing EM’s under LOCCs. Hence it is natural to ask how much entanglement can be obtained from the imperfectly entangled states which arise, for
example, during the sharing of a perfectly entangled state between two observers using only LOCC. Concerning fundamental questions of quantum information theory, of which tasks such as the characterization and general understanding of entanglement belong to, LOCC operations are of importance because of their locality. As the concept of entanglement is strongly related to the nonlocal properties of a physical state, LOCC operations cannot affect the intrinsic nature of entanglement. By using LOCC operations, different equivalence classes of states can be defined; representatives of each class can be used in experiments to perform the same tasks, but with a different probability.

IV. CONCLUSIONS

The amount of entanglement does not really have a meaning apart from a well-defined measure. An ideal measure of entanglement should have the following characteristics: it is non-vanishing if and only if the state is entangled; it is maximized by some recognizably maximally-entangled states; it has an operational interpretation (i.e., it quantifies the ability to carry out some quantum information protocol); it is monotonic (non-increasing under local operations and classical communication); and it is easy to calculate. For bipartite pure states there is a measure that satisfies all of those requirements: the entropy of entanglement, which is monotonic, straightforward to calculate, nonzero for all entangled states and zero for all product states, and which quantifies the number of maximally entangled pairs that can be produced asymptotically from many copies of the given state. But for mixed states, and multipartite states, no measure that we know of satisfies all of these requirements. There are a variety of different measures that may satisfy some of these requirements but not others. Some (like negativity) are widely used in numerical modeling because they are easy to calculate, but in general do not have a direct operational interpretation, and may not be nonzero for all entangled states. Others have great theoretical importance (like the entanglement of formation) but cannot generally be calculated in closed form for most states. They require difficult optimizations, or regularized expressions, or both.
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