Incomplete sets in $\mathbf{P}$ for logspace reduction

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Abstract

In this article, we investigate the behaviour of TMs with time limit and tape space limit. This problem is in $\mathbf{P}$ when the time limit is unary coded. When both limits go to infinity, it is undecidable which limit is exceeded first. Thus logspace incomplete sets can be constructed in $\mathbf{P}$. This implies $\mathbf{L} \neq \mathbf{P}$.

1 Introduction

It is known that the set

$$A = \{(M, x, 1^t) \mid \text{TM } M \text{ accepts } x \text{ within } t \text{ steps}\}$$

is logspace-complete for $\mathbf{P}$ [HS11, page 176, Th. 7.24]. This means $\forall X : X \leq^m \log A$.

In this paper, we look for logspace-incomplete sets in $\mathbf{P}$. We observe the following sets:

$$U(f) := \{(M, x, 1^t, 1^s) \mid \text{TM } M \text{ accepts } x \text{ within } t \text{ steps and within } f(s) \text{ tape space}\}$$

For this reason, we use the sets:

$$B(f) := \{(M, x, 1^s) \mid \text{TM } M \text{ accepts } x \text{ within } f(s) \text{ tape space}\}$$

The sets $U(f)$ are in $\mathbf{P}$, but the sets $B(f)$ are easier to analyse. All these sets are connected:

**Lemma 1.** $U(f) \leq^T B(f)$ and $U(f) \leq^m A$
Proof. immediately

If \( f \) is a time- and space-constructible function then

\[
\text{DTIME}(f) \subseteq \text{DSpace}(f) \subseteq \text{DTIME}(f \cdot 2^O(f)) \quad [\text{HPV75}]
\]

Let \( f : \mathbb{N} \to \mathbb{R}_+ \) a sublinear function with \( f \in \Omega(\log) \). In section 2 we will show that it is undecidable for a TM \( M \) if \( s_M \in O(f(t_M)) \) when \( s_M \) is the used tape space of \( M \) and \( t_M \) is the computation time. See also [Cze21].

In section 3 we will use these results to show that the sets \( U(f) \), where \( f \) is sub-linear and \( f \in \Omega(\log) \), are incomplete in \( \text{P} \).

Baker, Gill, and Solovay have proved the relativization barrier for P versus NP[BS75]. Like the P versus NP problem, the L versus P problem cannot be solved with diagonalization alone. Nor can it be solved by computability methods alone. One also needs methods from complexity theory. In this paper, we combine these concepts. In section 4, we explain why we can circumvent the relativization barrier.

## 2 Tape Space of polynomial time TMs and the Arithmetical Hierarchy

Let \( K_0 := \{ M \mid \text{TM } M \text{ terminates on empty input} \} \) and \( \overline{K_0} := \{ M \mid \text{TM } M \not\in K_0 \} \). Both sets are not computable, due to the halting problem[Tur36]. It is known, that \( K_0 \) is \( \Sigma_1 \)-complete and \( \overline{K_0} \) is \( \Pi_1 \)-complete[Soa87].

For a TM \( M \) we define:

\[
\tilde{S}_M(t) := \text{tape space used by } M \text{ to compute on empty input within } t \text{ steps}
\]

and

\[
\tilde{P}(f) := \{ M \mid \forall t \\tilde{S}_M(t) < f(t) \}
\]

**Lemma 2.** \( K_0 \leq_1 \tilde{P}(f) \) if \( f \in o(t) \) and \( f \in \Omega(\log(t)) \)

**Proof.** There is a one-one reduction that transforms \( M \) to a multi-taped TM \( h(M) \) with one tape more than \( M \). \( h(M) \) does not terminate and \( S_{h(M)} \) grows linearly if \( M \) does not terminate, otherwise it is logarithmic.

If \( M \) does not terminate, the tape space of \( h(M) \) grows linearly with time, so that

\[
M \in K_0 \iff h(M) \in \tilde{P}(f)
\]

\( \square \)
Algorithm 1 $h(M)$:

**Require:** multi-taped TM $M$ with empty input  
**Ensure:** algorithm runs on a TM that has one more tape than $M$

while $M$ is not terminated do
  calculate one step on $M$
  write ‘1’ on the last tape
  move the head of last the tape one position to the left
end while

if $M$ has terminated then
  apply binary counter to the last tape
end if

Lemma 3. $\overline{K}_0 \leq \tilde{P}(f)$ if $f \in o(t)$ and $f \in \Omega(\log(t))$

*Proof.* Similar to lemma 2, we will construct a one-one reduction, that transforms the TM $M$ to a non-terminating multi-taped TM $g(M)$. In contrast to lemma 2, $S_{g(M)}$ grows logarithmically when $M$ is non-terminating, and linearly otherwise.

Algorithm 2 $g(M)$:

**Require:** multi-taped TM $M$ with empty input  
**Ensure:** algorithm runs on a TM that has one more tape than $M$

while $M$ not terminated do
  calculate one step on $M$
  Reserve as much space on the last tape as is used by $M$
  Overwrite reserved space with ’0’$'s
  repeat
    apply binary counter on the last tape
  until reserved space is filled with ’1’$'s
end while

if $M$ has terminated then
  loop
    write ‘1’ on the last tape
    move the head of the last tape one position to the left
  end loop
end if

If $M$ does not terminate, the tape space of $g(M)$ grows with the loga-
rithm of time, so that

\[ M \notin K_0 \iff g(M) \in \bar{P}(f) \]

\[ \square \]

**Corollary 1.** \( K_0 <_1 \bar{P}(f) \) if \( f \in o(t) \) and \( f \in \Omega(\log(t)) \)

**Proof.** \( K_0 \leq_1 \bar{P}(f) \) and \( K_0 \leq_1 \bar{P}(f) \). But \( \bar{P}(f) \) is neither in \( \Sigma_1 \) nor in \( \Pi_1 \), due to the Hierarchy Theorem [Soa87, page 65]. So \( K_0 <_1 \bar{P}(f) \) and \( K_0 <_1 \bar{P}(f) \).

\[ \square \]

**Theorem 1.** \( \bar{P}(f) \in \Sigma_2 \)

**Proof.** \( M \in \bar{P}(f) \Rightarrow \exists k \forall t : \bar{S}_M(t) < k \ast l(t) \)

\[ \square \]

**3 Space Hierarchy within \( P \)**

In this section we use Turing-reduction instead of many-one-reduction because for any sets \( X \leq_T Y \Rightarrow X \leq_m Y \).

**Theorem 2.** If \( h \) and \( l \) are space-constructible functions with \( l \in o(h) \) and \( l \in \Omega(\log) \) then \( B(h) \not\leq_T B(l) \)

**Proof.** Due to the space hierarchy theorem [SHI65], there is a set \( X \in \text{DSPACE}(h) \) with \( X \not\in \text{DSPACE}(l) \). Thus \( X \leq_T B(h) \) and \( X \not\leq_T B(l) \) are valid. This implies \( B(h) \not\leq_T B(l) \).

\[ \square \]

**Lemma 4.** If \( h \) and \( l \) are space-constructible functions with \( l \in o(h) \) and \( l(n) \in \Omega(\log(n)) \) and \( h(n) \in o(n) \) then \( U(h) \not\leq_T B(l) \)

**Proof.** Let \( M \) be a TM and \( k > 0 \), both arbitrary but fixed. There is a function \( t : \mathbb{N} \mapsto \mathbb{N} \) with \( M \) accepting \( x \) within \( t(s) \) steps and within \( h(s) \) tape space and \(|x| < k \). This means

\[ \forall s \in \mathbb{N} \forall |x| < k : (M, x, 1^{t(s)}, 1^s) \in U(h) \Rightarrow (M, x, 1^s) \in B(h) \]

. In this case, there is no reduction function \( f \) with the space complexity \( \log(s) \) such that \( (M, x, 1^{t(s)}, 1^s) \in U(h) \Leftrightarrow f((M, x, 1^{t(s)}, 1^s)) \in B(l) \), because \( B(l) \not\leq_T B(h) \). Due to corollary 1 it is undecidable whether there exists a function \( t \) with \( \forall s \forall |x| < k : (M, x, 1^{t(s)}, 1^s) \in U(h) \Rightarrow (M, x, 1^s) \in B(h) \) and \( t \in O(s) \). So \( U(h) \not\leq_T B(l) \).

\[ \square \]
Theorem 3. If $h$ and $l$ are space-constructible functions with $l \in o(h)$ and $\ln(n) \in \Omega(\log(n))$ and $h(n) \in o(n)$ then $U(h) \not\leq_T U(l)$

Proof. $U(h) \not\leq_T B(l)$, but $U(l) \leq_T B(l)$ \hfill \Box

Corollary 2. If $k < n$ then $U(\log^k) \not<_{m} U(\log^n)$

Proof. $U(\log^k) \not<_{T} U(\log^n) \Rightarrow U(\log^k) \not<_{m} U(\log^n)$ \hfill \Box

This means $\forall n : U(\log^n) \not<_{m} A$, so $U(\log^n)$ is incomplete in $P$. From the existence of incomplete sets follows:

Corollary 3. $L \neq P$

4 Notes on the Relativization Barrier

To separate $L$ and $P$, the relativization barrier applies as for $P$ versus $NP$ [BGS75]. Therefore, here we have mentioned the reason why the proof in this paper does not violate this barrier such as diagonalization. Trivially, $U(\log)^P = A^P$, so $L^P = P^P$. But from theorem 2 follows $U(\log) \not<_{T} U(\log^2)$, so $U(\log)^L \neq A^L$.

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