Weak* topology on modular space and some properties

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Abstract: This study aims to redefine the weak topology \(\sigma(X_M, X_M^{**})\) on a specific topological dual space (modular dual space \(X_M^{**}\)) this is weak topology generated by all linear bounded functional on \(X_M\), but we interested in a subspace of this topology generated by \(X_M\) called weak* topology on \(X_M\), it follows from that the modular space \(X_M\) over the field \(K\) can be embedded in \(X_M^{**}\) by using the canonical map, we denoted to this topology by \(\sigma(X_M^{**}, X_M)\). After that, we checked the weak* topology is Hausdorff and investigated some properties, finally, we showed that under which condition the strong topology and the weak* topology coincided.

Keyword: weak topology on modular space, weak-star topology, weak* topology on modular space, modular space, weak topology,

1. Introduction

The initials definitions and basic concepts of modular space, weak topology and weak topology on modular spaces were indicated in preliminaries. Nakano's assumption of modular functions appeared in the 1950s [13], who introduced a family of functions from any vector space \(V\) over a field \(K\) (where \(K = R\) or \(C\)) into the interval \([0, \infty]\) \(M: V \to [0, \infty]\) with conditions, the vector space \(V\) with modular \(M\) is called modular space [4]. It will be metric space when the distance between any two points \(v, u\) in \(V\) is defined by \(d(v, u) = M(v - u)\) [2]. That is, \(d\) generates a topology for \(V\). After that, The preliminaries introduced weak topology on any set in a general view, see coherent topology as well as [7,16]. Moreover, there is research that especially talks about weak topology on modular space which has recently been published 2020, see [12]. Finally, the weak topology on a specific topological dual space (modular dual space \(X_M^{**}\)) is redefined by the family \(X_M^{**}\) of all linear bounded functions from \(X_M\) into \(R\). But, the interest will be focused on weak topology generated by a subspace \(X_M\) of \(X_M^{**}\); this weak-star topology of \(X_M^{**}\).

2. Preliminaries:

This section is divided into three parts, let's begin with:

2.1. Modular space

In this part, basic definitions and descriptions of the concept of the modular space

2.1.1 Definition : [4]
Let \(X\) be a linear space over afield \(K\). A map \(M: X \to [0, \infty]\) called a modular if 1. \(M(0) = 0\) if and only if \(m = 0\).
2. $M(\lambda m) = M(m)$ with $|\lambda| = 1$, for $\lambda \in K$, $m \in X$
3. $M(\alpha m + \beta r) \leq M(m) + M(r)$ when $\alpha, \beta \geq 0$, $m \in X$ and $\alpha + \beta = 1$
   Space is given by $X_M = \{m \in X : M(\lambda m) \to 0 \text{ when } \lambda \to 0\}$ is called modular space, as follows.
   If condition 3 above replaced by $M(\alpha m + \beta r) \leq \alpha M(m) + \beta M(r)$, for $\alpha + \beta = 1, \alpha, \beta \geq 0$ for all $m, r \in M$, then $M$ called a convex modular.
   If $r = 0$ then $M(\alpha x) = M\left(\frac{\alpha x}{\beta} \beta m\right) \leq M(\beta x), \alpha, \beta \in K, 0 < \alpha < \beta$. Thus $M$ increasing map.

2.1.2. Remarks:
1. If $X_M$ is a modular space, then $X_M$ is a metric space by defined the distance function as follow
   $d(m, r) = M(m - r)$, for all $m, r \in X$. See [2-4]
2. every modular space is topological vector space and it is Hausdorff [11]

Now, For the definition of topological vector space

2.1.3. Definition: [1,2,10]
In the modular space $X_M$
1- The $M$-open ball $B_\varepsilon(m)$ with centre $m \in X_M$ and radius $\varepsilon > 0$ as $B_\varepsilon(m) = \{r \in X_M; M(m - r) < \varepsilon\}$.
2- The $M$-closed ball $B_\varepsilon(m)$ centred $m \in X_M$ with radius $\varepsilon > 0$ as $B_\varepsilon(m) = \{r \in X_M; M(m - r) \leq \varepsilon\}$.
3- The family of all $M$-balls in $X_M$ generates the topology makes $X_M$ Hausdorff
4- Since every $M$-ball is convex, then every modular space is locally convex topological linear space.
5- Let $X_M$ be a modular space and $E \subseteq X_M$ we say that $E$ is $M$-open set if for every $m \in E$ there exist $\varepsilon > 0, \exists B_\varepsilon(m) \subseteq E$.
6- A subset $E$ of $X_M$ is said to be $M$-closed if its complement is $M$-open, that is, $E^c = X_M - E$ is $M$-open.

2.1.4. Definition:
Let $X_M$ be a modular space over the field $K$, then the space of all continuous linear functional from $X_M$
into the field $K$ called the dual modular space and denoted by $X_M^*$

2.1.5. Remark:
The space $X_M^*$ is also modular space.
By defining $M^": X_M^* \to [0, \infty]$ as $M^*(f) = sup\{M(f(m)); M(m) = 1, m \in X_M\}$

2.2. The weak topology
In this part, introduced notion of weak topology and some properties we needed it

2.2.1. Definition:[9]
Let $A$ be a nonempty set and let $\{(A_\alpha, \tau_\alpha) : \alpha \in A\}$ be a nonempty family of topological spaces. For each $\alpha \in A$, let $f_\alpha$ be a map of $A$ into $A_\alpha$. Then the topology $\tau$ on $A$ generated by the family $G = \{f_\alpha^{-1}(G) : G \in \tau_\alpha, \alpha \in A\}$ is called the initial (weak) topology on $A$ determined by the family $\{f_\alpha : \alpha \in A\}$.
$G$ is defining subbase of $\tau$ and the family $\beta$ of all finite intersections of members of $G$ is called a basis of $\tau$.

2.2.2. Remark:[7]
Let $A$ a nonempty set with $\{(A_\alpha, \tau_\alpha) : \alpha \in A\}$ be a nonempty collection of topological spaces indexed
by $A$. The weak (initial) topology generated by a collection of functions $F = \{f_\alpha : A \to A_\alpha, \alpha \in A\}$ is
the topology generated by the subbasis $G = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in A\}$. Denoted to the topology
generated by $F$ on $A$ by $\sigma(A, F)$.

2.2.3. Definition: [8]
A set $G$ in $A$ is said to be open in a topology $\sigma(A,F)$ if for all $z \in G$, there exists a finite subset $I$ of $\Delta$ and open sets $\{G_\alpha\}_{\alpha \in I}$ such that $G_\alpha \subseteq A_\alpha$ for all $\alpha \in I$, $z \in \bigcap_{i=1}^n f_i^{-1}(G_i)$ that means that $\forall i \in I, f_i(z) \in G_i$.

### 2.2.4. Definition [7]

In this part, $X_M$ modular space over the field $K$ where, $K = R$ or $K = C$, we don't assume that it is $X_M$ complete.

suppose $f_\alpha: X_M \rightarrow X_\alpha$ be a function and $X_\alpha = K$ and let $F = \{f_\alpha: \alpha \in \Delta\}$, and let $G = \{I \subseteq \Delta: I$ finite $\}$

Then the weak topology on $X_M$ denoted by $\sigma(X_M,X_M^*)$ such that generated by $F$ has the defining

$$\beta = \{\alpha \in \Delta \mid \|f_\alpha^{-1}(-\varepsilon,\varepsilon)\| \leq \varepsilon > 0 \}$$

So, a set $E$ is a weak open in $X_M$ if and only if given $E$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ with $x \in \bigcap_{i=1}^n f_i^{-1}(-\varepsilon,\varepsilon)$ \iff $|f_\alpha(x) - f_\alpha(x_0)| < \varepsilon$ for all $i = 1, 2, \ldots, n$

A subsbasis of the weak open set containing $x_0 \in X_M$ is of the form

$$f_\alpha^{-1}(f_\alpha(x_0) - \varepsilon, f_\alpha(x) + \varepsilon)$$

for all $\alpha \in \Delta$ and each $\varepsilon > 0$. Hence it can be as the form

$$\beta(f_\alpha, f_1, f_2, \ldots, f_n; \varepsilon) = \{x \in F \mid |f_\alpha(x) - f_\alpha(x_0)| < \varepsilon\}$$

### 3. The main result

Let $X_M$ be any modular space over a field $K$ (where $K = R$ or $K = C$), then by definition of modular in $X_M^*$ and by remark (2.1.5) $X_M^*$ is a modular space. Therefore, a weak topology can be defined on $X_M^*$ and generated by the family of all bounded linear function from $X_M^*$ to the field $K$; that's nothing but the weak topology $\sigma(X_M,X_M^*)$.

But, we interested in a weak topology generated by $X_M$ i.e. the topology $\sigma(X_M^*,X_M)$, where $X_M$ is a subspace embedded in $X_M^*$ such that every element of $X_M$ is written as a bounded linear function from $\sigma(X_M^*,X_M^*)$ into $K$ by the canonical map $\Psi: X_M \rightarrow X_M^*$ and given by $\Psi(x) = p_x$ where $p_x(f) = f(x)$ for every $f \in X_M$ with $M(p_x) = \sup\{|f(x)|: f \in S_M\} = M(x)$ for each $x \in X_M$. Since $\Psi$ is an isometry, then can be concluded that $X_M$ is isometrically-isomorphic $\Psi(X_M)$.

#### 3.1. Definition

A set $E$ in the modular space $X_M^*$ is said to be weak-star open set ($W^*$-open) if and only if for each function $f \in E$ there is $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n \in X_M$ such that $\{g \in X_M^* \mid |g(x_i) - f(x_i)| < \varepsilon\} \subseteq E$ where $i = 1, 2, \ldots, n$ and $n \geq 1$. A set $E$ is called $W^*$-closed if the complement is $W^*$ open set.

#### 3.2. Definition

Let $X_M$ be a modular space, the weak topology $\sigma(X_M^*,X_M)$ consist of all weak-star open sets in $X_M^*$ is called weak-star topology ($W^*$-topology) on $X_M^*$ and denoted by $\sigma(X_M^*,X_M)$.

**Note that:** Since $X_M \subseteq X_M^*$, then the $W^*$-topology $\sigma(X_M^*,X_M)$ is weaker than the topology $\sigma(X_M^*,X_M^*)$.

3.3. Remark: If $X_M$ is reflexive, then then the weak topology on $X_M$ and the weak-star topology of $X_M^*$ are the same; $\sigma(X_M^*,X_M^*) = \sigma(X_M,X_M^*)$.

Now we introduce the local base of the weak-topology of $X_M^*$ in next theorem

#### 3.4. Theorem

Let $f_0 \in X_M^*$. A local base of $f_0$ for the weak-star topology of $X^*_M$ is given by the collection of open balls of the form

$$\beta(\varepsilon, x_1, x_2, \ldots, x_n) = \{f \in X_M^* \mid \forall i = 1, 2, \ldots, n, |f(x_i) - f_0(x_i)| < \varepsilon\}$$

**Proof:** Since the weak-star topology of $X_M^*$ generated by $X_M$ has the basis

$$\beta = \{\bigcap_{\alpha \in \Delta} f_\alpha^{-1}(-\varepsilon,\varepsilon) \mid I \subseteq \Delta, \varepsilon > 0\}$$

where $G = \{I \subseteq \Delta: I$ finite $\}$ and $\Delta$ any index for $X_M$.

Thus a set $E$ is $W^*$-open in $X_M^*$ iff given $E$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ with

$$f \in \bigcap_{i=1}^n f_i^{-1}(-\varepsilon_i,\varepsilon_i) \subseteq E \implies |f_{\alpha_i}(f)| < \varepsilon_i$$

for $i = 1, 2, \ldots, n$. A sub basis open set
containing a point \( f_0 \in X_M \) is of the form \( x_\alpha^{-1}(x_\alpha(f_0) - \varepsilon, x_\alpha(f_0) + \varepsilon) \) for all \( \alpha \in \Lambda \) and each \( \varepsilon > 0 \).

Hence it can be of the form \( \beta(\varepsilon, x_1, x_2, ..., x_n) = \{ f \in X_M : f \text{ for all } i = 1, 2, ..., n, |f(x_i) - f_0(x_i)| < \varepsilon \} \), \( \varepsilon > 0, |x_1, x_2, ..., x_n \in X_M \).

The following theorem is very important to study the properties of the topology \( \sigma(X_M^*, X_M) \) because it shows whether the limit point is unique or not.

3.5. Theorem: Let \( X_M \) be a modular space over the field \( K \), then the \( w^*\)-topology of \( X_M^* \) is Hausdorff.

Proof: Let \( f, g \in X_M^* \) with \( f \neq g \), then \( f(x) \neq g(x) \) for some \( x \in X_M \).

Let \( x \in \{ x \in X_M^* : f(x) \neq g(x) \} \), then either \( f(y) < g(y) \) or \( f(y) > g(y) \) and in both cases can be found \( y \in R \) such that either \( f \in y^{-1}((-\infty, y)) \) and \( g \in y^{-1}((y, \infty)) \) or converse.

Thus there are two disjoint sets in \( \sigma(X_M^*, X_M) \) separate \( f \) and \( g \), hence the weak-star topology is Hausdorff space.

3.6. Definition: a sequence \( \{f_n\} \) in the dual modular space \( X_M^* \) is \( w^*\)-convergent to a function \( f \) and denoted by \( f_n \to^w f \) if it converges to \( f \) in the topology \( \sigma(X_M^*, X_M) \).

The next theorem is to redefine the convergence property in \( w^*\)-topology of \( X_M^* \).

3.7. Theorem: Let \( X_M \) be a modular space, a sequence \( \{f_n\} \) in the dual space of \( X_M \) is said to be \( \omega^*\)-convergent to a function \( f \) if and only if for every \( \varepsilon > 0 \) and for each element \( x \) in \( X_M \), there exists \( k \in Z^+ \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n > k \); this \( f_n \omega^* \to f \) if and only if \( f_n(x) = f(x) \).

Proof: Suppose that \( \{f_n\} \) is a sequence in \( X_M^* \).

Firstly, take \( f_n \to^w f \). Let \( \varepsilon > 0 \) and \( E \in \sigma(X_M^*, X_M) \) s.t. \( E = \{ h \in X_M^* : |h(x) - f(x)| < \varepsilon \} \) for each \( x \in X_M \). Since \( f_n \to^w f \), then by definition of \( w^* \)-convergent can be shown there is \( k \in Z^+ \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n > k \) and each element in \( X_M \). Thus for each \( \omega^* \)-open set \( E \) containing \( f(x) \), there is \( k \in Z^+ \) with \( f_n(x) \in E \) for all \( n > k \); \( f_n(x) = f(x) \).

Conversely, when \( f_n(x) = f(x) \) for all \( x \) in \( X_M \). Let \( E \in \sigma(X_M^*, X_M) \) such that \( E \) containing \( f(x) \).

There exists \( \varepsilon > 0 \) and a finite number of elements \( x_1, x_2, ..., x_r \) of \( X_M \) with \( h \in X_M^* : |h(x_i) - f(x_i)| < \varepsilon, i = 1, 2, ..., r \) \( \supseteq E \).

Since \( f_n(x_i) \to f(x_i) \) for \( i = 1, 2, ..., r \), then there exists \( k_i \in Z^+ \) where \( i = 1, 2, ..., r \) with \( |f_n(x_i) - f(x_i)| < \varepsilon \) for all \( n > k_i \).

By choosing \( k = \max\{k_1, k_2, ..., k_r\} \). Then for each \( i = 1, 2, ..., r \), we have \( |f_n(x_i) - f(x_i)| < \varepsilon \) for all \( n > k \).

Thus \( f_n \in E \) for all \( n > k \): that is \( f_n \to^w f \).

Here some properties of \( w^*\)-topology of \( X_M^* \)

3.8. Proposition: let \( X_M \) be a modular linear space and \( \{f_n\} \) be a sequence in \( X_M \) then the following properties are holding:

1. if \( f_n \to^w f \) in \( X_M \), then \( f_n \to^\omega f \) in \( w^*\)-topology.
2. if \( f_n \omega^* \to f \) and \( \{x_n\} \) a sequence in \( X_M \) with \( x_n \to^w x \in X_M \), then \( f_n(x_n) \to f(x) \) as \( n \to \infty \).

Proof:

1. Suppose that \( \{f_n\} \) be a sequence in the dual modular space \( X_M^* \) with \( f_n \to^w f \), that’s mean for all \( x \in X_M \) the limit point by \( \{f_n\} \) is exists, unique and equal to \( f(x) \). Thus by (3.6) \( f_n \to^w f \).

2. Suppose that \( \{f_n\} \subseteq X_M^* \) and \( \{x_n\} \subseteq X_M \) such that \( f_n \omega^* \to f \) and \( x_n \to^w x \). Let \( \varepsilon > 0 \) then there exists \( k_1 \) and \( k_2 \in Z^+ \) such that \( |x_n - x| < \varepsilon, n > k_1 \) and for all \( x \in X_M \) \( |f_n(x) - f(x)| < \varepsilon \). Choose \( k = \max\{k_1, k_2\} \), then we have \( |f_n(x_n) - f(x)| < \varepsilon \).

And the next theorem showed that under which condition the strong modular topology and the \( w^*\)-topology are coincided, as following

3.9. Theorem: let \( X_M \) be a modular space, if \( X_M \) finite-dimensional, then the weak-star topology of \( X_M^* \) and the modular topology on \( X_M^* \) are coinciding.
Proof: Since the weak–star topology of $X_M^*$ is weaker than the modular topology on $X_M^*$, The proof: will be limited to proving the opposite side: every open set in the modular space $X_M^*$ is $w^*$-open set.

Let $E$ be open in the modular space $X_M^*$ and $g \in E$. Then can be founded $\varepsilon > 0$ such that $g + B_{X_M^*}(\varepsilon) \subseteq E$ where $B_{X_M^*}(\varepsilon)$ is an open ball at the origin with radius $\varepsilon > 0$ in $X_M^*$. Since $X_M^*$ finite-dimensional, then it has a basis $\beta$ consists of an only finite number of elements. Now define $M^*(f) = \max\{|f(\beta)|, e \in \beta\}$ for all $f \in X_M^*$, then $M^*: X_M^* \to [0, \infty)$ is a modular space. Since all modulars on a finite-dimensional are equivalent, there is $\delta > 0$ with $M^*(f) < \delta$, we have $M(f) < \varepsilon$. Then the $w^*$-open $\{f \in X_M^* : \max\{|f(\beta) - g(\beta)| < \varepsilon, e \in \beta\}\}$ is contained in $\{f \in X_M^*: M(f - g) < \varepsilon\}$. Hence $E$ is $w^*$-open.

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