On full Souslin trees

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Abstract
In the present note we answer a question of Kunen (15.13 in [Mi91]) showing (in 1.7) that
it is consistent that there are full Souslin trees.

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0 Introduction

In the present paper we answer a combinatorial question of Kunen listed in Arnie Miller’s Problem List. We force, e.g. for the first strongly inaccessible Mahlo cardinal $\lambda$, a full (see 1.1(2)) $\lambda$–Souslin tree and we remark that the existence of such trees follows from $V = L$ (if $\lambda$ is Mahlo strongly inaccessible). This answers [Mi91, Problem 15.13].

Our notation is rather standard and compatible with those of classical textbooks on Set Theory. However, in forcing considerations, we keep the older tradition that

\[ \text{a stronger condition is the larger one.} \]

We will keep the following conventions concerning use of symbols.

**Notation 0.1**

1. $\lambda, \mu$ will denote cardinal numbers and $\alpha, \beta, \gamma, \delta, \xi, \zeta$ will be used to denote ordinals.

2. Sequences (not necessarily finite) of ordinals are denoted by $\nu, \eta, \rho$ (with possible indexes).

3. The length of a sequence $\eta$ is $\ell g(\eta)$.

4. For a sequence $\eta$ and an ordinal $\alpha \leq \ell g(\eta)$, $\eta|\alpha$ is the restriction of the sequence $\eta$ to $\alpha$ (so $\ell g(\eta|\alpha) = \alpha$). If a sequence $\nu$ is a proper initial segment of a sequence $\eta$ then we write $\nu < \eta$ (and $\nu \leq \eta$ has the obvious meaning).

5. A tilde indicates that we are dealing with a name for an object in forcing extension (like $\bar{x}$).

1 Full $\lambda$–Souslin trees

A subset $T$ of $\alpha > 2$ is an $\alpha$–tree whenever ($\alpha$ is a limit ordinal and) the following three conditions are satisfied:

- $\langle \rangle \in T$, if $\nu < \eta \in T$ then $\nu \in T$,
- $\eta \in T$ implies $\eta^{-}(0), \eta^{-}(1) \in T$, and
- for every $\eta \in T$ and $\beta < \alpha$ such that $\ell g(\eta) \leq \beta$ there is $\nu \in T$ such that $\eta \preceq \nu$ and $\ell g(\eta) = \beta$.

A $\lambda$–Souslin tree is a $\lambda$–tree $T \subseteq \lambda > 2$ in which every antichain is of size less than $\lambda$. 
Definition 1.1 1. For a tree $T \subseteq \alpha > 2$ and an ordinal $\beta \leq \alpha$ we let
\[
T[\beta] \overset{\text{def}}{=} T \cap \beta^2 \quad \text{and} \quad T[<\beta] \overset{\text{def}}{=} T \cap \beta^> 2.
\]
If $\delta \leq \alpha$ is limit then we define
\[
\lim_\delta T[<\delta] \overset{\text{def}}{=} \{ \eta \in \delta^2 : (\forall \beta < \delta)(\eta|\beta \in T) \}.
\]

2. An $\alpha$–tree $T$ is full if for every limit ordinal $\delta < \alpha$ the set $\lim_\delta T[<\delta] \setminus T[\delta]$ has at most one element.

3. An $\alpha$–tree $T \subseteq \alpha > 2$ has true height $\alpha$ if for every $\eta \in T$ there is $\nu \in \alpha^2$ such that
\[
\eta < \nu \quad \text{and} \quad (\forall \beta < \alpha)(\nu|\beta \in T).
\]

We will show that the existence of full $\lambda$–Souslin trees is consistent assuming the cardinal $\lambda$ satisfies the following hypothesis.

Hypothesis 1.2 (a) $\lambda$ is strongly inaccessible (Mahlo) cardinal,
(b) $S \subseteq \{ \mu < \lambda : \mu$ is a strongly inaccessible cardinal $\}$ is a stationary set,
(c) $S_0 \subseteq \lambda$ is a set of limit ordinals,
(d) for every cardinal $\mu \in S$, $\Diamond_{S_0 \cap \mu}$ holds true.

Further in this section we will assume that $\lambda$, $S_0$ and $S$ are as above and we may forget to repeat these assumptions.

Let as recall that the diamond principle $\Diamond_{S_0 \cap \mu}$ postulates the existence of a sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$ (called a $\Diamond_{S_0 \cap \mu}$–sequence) such that $\nu_\delta \in \delta^2$ (for $\delta \in S_0 \cap \mu$) and
\[
(\forall \nu \in \mu^2)[ \text{ the set } \{ \delta \in S_0 \cap \mu : \nu| \delta = \nu_\delta \} \text{ is stationary in } \mu].
\]

Now we introduce a forcing notion $Q$ and its relative $Q^*$ which will be used in our proof.

Definition 1.3 1. A condition in $Q$ is a tree $T \subseteq \alpha > 2$ of a true height $\alpha = \alpha(T) < \lambda$ (see (1.1)(2); so $\alpha$ is a limit ordinal) such that $\| \lim_\delta T[<\delta] \setminus T[\delta] \| \leq 1$ for every limit ordinal $\delta < \alpha$.

the order on $Q$ is defined by $T_1 \leq T_2$ if and only if $T_1 = T_2 \cap \alpha(T_1)^> 2$ (so it is the end–extension order).
2. For a condition $T \in \mathbb{Q}$ and a limit ordinal $\delta < \alpha(T)$, let $\eta_{\delta}(T)$ be the unique member of $\lim_{\delta}(T_{|<\delta}) \setminus T_{[\delta]}$ if there is one, otherwise $\eta_{\delta}(T)$ is not defined.

3. Let $T \in \mathbb{Q}$. A function $f : T \rightarrow \lim_{\alpha(T)}(T)$ is called a witness for $T$ if $\forall \eta \in T(\eta < f(\eta))$.

4. A condition in $\mathbb{Q}^*$ is a pair $(T, f)$ such that $T \in \mathbb{Q}$ and $f : T \rightarrow \lim_{\alpha(T)}(T)$ is a witness for $T$.

The order on $\mathbb{Q}^*$ is defined by $(T_1, f_1) \leq (T_2, f_2)$ if and only if $T_1 \leq_T T_2$ and $(\forall \eta \in T_1)(f_1(\eta) \leq f_2(\eta))$.

**Proposition 1.4** 1. If $(T_1, f_1) \in \mathbb{Q}^*$, $T_1 \leq_T T_2$ and

(*) either $n_{\alpha(T_1)}(T_2)$ is not defined or it does not belong to $\text{rang}(f_1)$

then there is $f_2 : T_2 \rightarrow \lim_{\alpha(T_2)}(T_2)$ such that $(T_1, f_1) \leq (T_2, f_2) \in \mathbb{Q}^*$.

2. For every $T \in \mathbb{Q}$ there is a witness $f$ for $T$.

**Proof** Should be clear.

**Proposition 1.5** 1. The forcing notion $\mathbb{Q}^*$ is $(< \lambda)$–complete, in fact any increasing chain of length $< \lambda$ has the least upper bound in $\mathbb{Q}^*$.

2. The forcing notion $\mathbb{Q}$ is strategically $\gamma$-complete for each $\gamma < \lambda$.

3. Forcing with $\mathbb{Q}$ adds no new sequences of length $< \lambda$. Since $\|\mathbb{Q}\| = \lambda$, forcing with $\mathbb{Q}$ preserves cardinal numbers, cofinalities and cardinal arithmetic.

**Proof** 1) It is straightforward: suppose that $\langle (T_\xi, f_\xi) : \xi < \zeta \rangle$ is an increasing sequence of elements of $\mathbb{Q}^*$. Clearly we may assume that $\xi < \lambda$ is a limit ordinal and $\zeta_1 < \zeta_2 < \xi$ $\Rightarrow$ $\alpha(T_{\zeta_1}) < \alpha(T_{\zeta_2})$. Let $T_\zeta = \bigcup_{\zeta_1 < \zeta} T_\zeta$ and $\alpha = \sup_{\zeta_1 < \zeta} \alpha(T_{\zeta_1})$. Easily, the union is increasing and the $T_\zeta$ is a full $\alpha$–tree. For $\eta \in T_\zeta$ let $\zeta_0(\eta)$ be the first $\zeta < \xi$ such that $\eta \in T_\zeta$ and let $f_\xi(\eta) = \bigcup \{f_{\xi}(\eta) : \zeta_0(\eta) \leq \zeta < \xi\}$. By the definition of the order on $\mathbb{Q}^*$ we get that the sequence $\langle f_\xi(\eta) : \zeta_0(\eta) \leq \zeta < \xi \rangle$ is $\leq$-increasing and hence $f_\xi(\eta) \in \lim_{\alpha}(T_\zeta)$. Plainly, the function $f_\xi$ witnesses that $T_\zeta$ has a true height $\alpha$, and thus $(T_\zeta, f_\xi) \in \mathbb{Q}^*$. It should be clear that $(T_\zeta, f_\xi)$ is the least upper bound of the sequence $\langle (T_\xi, f_\xi) : \xi < \zeta \rangle$. 


2) For our purpose it is enough to show that for each ordinal \( \gamma < \lambda \) and a condition \( T \in \mathcal{Q} \) the second player has a winning strategy in the following game \( G_\gamma(T, \mathcal{Q}) \). (Also we can let Player I choose \( T_\xi \) for \( \xi \) odd.)

The game lasts \( \gamma \) moves and during a play the players, called I and II, choose successively open dense subsets \( D_\xi \) of \( \mathcal{Q} \) and conditions \( T_\xi \in \mathcal{Q} \). At stage \( \xi < \gamma \) of the game:

- Player I chooses an open dense subset \( D_\xi \) of \( \mathcal{Q} \) and
- Player II answers playing a condition \( T_\xi \in \mathcal{Q} \) such that

\[
T \leq_q T_\xi, \quad (\forall \zeta < \xi)(T_\zeta \leq_q T_\xi), \quad \text{and} \quad T_\xi \in D_\xi.
\]

The second player wins if he has always legal moves during the play.

Let us describe the winning strategy for Player II. At each stage \( \xi < \gamma \) of the game he plays a condition \( T_\xi \) and writes down on a side a function \( f_\xi \) such that \( (T_\xi, f_\xi) \in \mathcal{Q}^* \). Moreover, he keeps an extra obligation that \( (T_\zeta, f_\zeta) \leq_q^* (T_\xi, f_\xi) \) for each \( \zeta < \xi < \gamma \).

So arriving to a non-limit stage of the game he takes the condition \( (T_\zeta, f_\zeta) \) he constructed before (or just \( (T, f) \), where \( f \) is a witness for \( T \), if this is the first move; by 1.4(2) we can always find a witness). Then he chooses \( T^*_\xi \geq_q T_\xi \) such that \( \alpha(T^*_\xi) = \alpha(T_\xi) + \omega \) and \( (T^*_\xi)[\alpha(T_\xi)] = \lim\alpha(T_\xi)(T_\xi) \).

Thus \( \eta_{\alpha(T_\xi)}(T^*_\xi) \) is not defined. Now Player II takes \( T^*_\xi \geq_q T^*_\zeta \) from the open dense set \( D_{\xi+1} \) played by his opponent at this stage. Clearly \( \eta_{\alpha(T_\zeta)}(T^*_\xi) \) is not defined, so Player II may use 1.4(1) to choose \( f_{\xi+1} \) such that \( (T^*_\xi, f_\xi) \leq_q^* (T^*_\xi+1, f_{\xi+1}) \in \mathcal{Q}^* \).

At a limit stage \( \xi \) of the game, the second player may take the least upper bound \( (T^*_\xi, f^*_\xi) \in \mathcal{Q}^* \) of the sequence \( ((T_\zeta, f_\zeta) : \zeta < \xi) \) (exists by 1)) and then apply the procedure described above.

3) Follows from 2) above.

**Definition 1.6** Let \( \mathbf{T} \) be the canonical \( \mathcal{Q} \)-name for a generic tree added by forcing with \( \mathcal{Q} \):

\[
\models_{\mathcal{Q}} \mathbf{T} = \bigcup \{ T : T \in G_\mathcal{Q} \}.
\]

It should be clear that \( \mathbf{T} \) is (forced to be) a full \( \lambda \)-tree. The main point is to show that it is \( \lambda \)-Souslin and this is done in the following theorem.

**Theorem 1.7** \( \models_{\mathcal{Q}} \text{“ } \mathbf{T} \text{ is a } \lambda \text{-Souslin tree”} \).
Proof Suppose that $A$ is a $Q$–name such that
\[ \models_Q " A \subseteq T \text{ is an antichain } " , \]
and let $T_0$ be a condition in $Q$. We will show that there are $\mu < \lambda$ and a condition $T^* \in Q$ stronger than $T_0$ such that $T^* \models_Q " A \subseteq T \] \leq \mu ] " (and thus it forces that the size of $A$ is less than $\lambda$).
Let $A$ be a $Q$–name such that
\[ \models_Q " A = \{ \eta \in T : (\exists \nu \in A)(\nu \leq \eta) \text{ or } \neg(\exists \nu \in A)(\eta \leq \nu) \} " . \]
Clearly, $\models_Q " A \subseteq T \text{ is dense open } "$.
Let $\chi$ be a sufficiently large regular cardinal ($\beth_7(\lambda^+) + is enough).

Claim 1.7.1 There are $\mu \in S$ and $B \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that:
(a) $A, A, S, S_0, Q, Q^*, T_0 \in B,$
(b) $\|B\| = \mu$ and $\mu > B \subseteq B,$
(c) $B \cap \lambda = \mu.$

Proof of the claim: First construct inductively an increasing continuous sequence $\langle B_\xi : \xi < \lambda \rangle$ of elementary submodels of $\langle \mathcal{H}(\chi), \in, <^*_\chi \rangle$ such that $A, A, S, S_0, Q, Q^*, T_0 \in B_0$ and for every $\xi < \lambda$
\[ \|B_\xi\| = \mu_\xi < \lambda, \quad B_\xi \cap \lambda \in \lambda, \quad \text{and} \quad \mu_\xi > B_\xi \subseteq B_{\xi+1}. \]
Note that for a club $E$ of $\lambda$, for every $\mu \in S \cap E$ we have
\[ \|B_\mu\| = \mu, \quad \mu > B_\mu \subseteq B_\mu, \quad \text{and} \quad B \cap \lambda = \mu. \]

Let $\mu \in S$ and $B \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be given by 1.7.1. We know that $\diamond_{S_0 \cap \mu}$ holds, so fix a $\diamond_{S_0 \cap \mu}$–sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$.
Let
\[ I \overset{\text{def}}{=} \{ T \in Q : T \text{ is incompatible (in } Q \text{) with } T_0 \quad \text{or:} \]
\[ T \geq T_0 \text{ and } T \text{ decides the value of } A \cap \alpha(T) >_2 \text{ and} \]
\[ (\forall \eta \in T)(\exists \rho \in T)(\eta \leq \rho \& T \models_Q \rho \in A) \}. \]

Claim 1.7.2 $I$ is a dense subset of $Q$. 
Proof of the claim: Should be clear (remember 1.3(2)).

Now we choose by induction on $\xi < \mu$ a continuous increasing sequence $\langle (T_\xi, f_\xi) : \xi < \mu \rangle \subseteq Q^* \cap B$.

**Step:** $i = 0$

$T_0$ is already chosen and it belongs to $Q \cap B$. We take any $f_0$ such that $(T_0, f_0) \in Q^* \cap B$ (exists by 1.4(2)).

**Step:** limit $\xi$ since $\mu > B \subseteq B$, the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is in $B$. By 1.5(1) it has the least upper bound $(T_\xi, f_\xi)$ (which belongs to $B$).

**Step:** $\xi = \zeta + 1$

First we take (the unique) tree $T^*_\xi$ of true height $\alpha(T^*_\xi) = \alpha(T^*_\zeta) + \omega$ such that $T^*_\xi \cap \alpha(T^*_\zeta) \geq 2 = T^*_\zeta$ and:

if $\alpha(T^*_\xi) \in S_0$ and $\nu_\alpha(T^*_\xi) \not\in \text{rang}(f_\zeta)$ then $(T^*_\xi)_{[\alpha(T^*_\xi)]} = \lim_{\alpha(T^*_\xi)}(T^*_\xi) \setminus \{\nu_\alpha(T^*_\zeta)\}$,

otherwise $(T^*_\xi)_{[\alpha(T^*_\xi)]} = \lim_{\alpha(T^*_\xi)}(T^*_\xi)$.

Let $T_\xi \in Q \cap I$ be strictly above $T^*_\xi$ (exists by 1.7.2). Clearly we may choose such $T_\xi$ in $B$. Now we have to define $f_\xi$. We do it by 1.4, but additionally we require that

if $\eta \in T_\xi$ then $(\exists \rho \in T_\xi)(\rho \prec f_\xi(\eta) \& T \models \text{ " } \mu \rho \in A \text{ " } )$.

Plainly the additional requirement causes no problems (remember the definition of $I$ and the choice of $\bar{\nu}$) and the choice can be done in $B$.

There are no difficulties in carrying out the induction. Finally we let

$$T_\mu \overset{\text{def}}{=} \bigcup_{\xi < \mu} T_\xi \quad \text{and} \quad f_\mu = \bigcup_{\xi < \mu} f_\xi.$$  

By the choice of $B$ and $\mu$ we are sure that $T_\mu$ is a $\mu$–tree. It follows from 1.5(1) that $(T_\mu, f_\mu) \in Q^*$, so in particular the tree $T_\mu$ has enough $\mu$ branches (and belongs to $Q$).

**Claim 1.7.3** For every $\rho \in \lim_\mu(T_\mu)$ there is $\xi < \mu$ such that

$$(\exists \beta < \alpha(T^*_{\xi+1}))(T^*_{\xi+1} \models \text{ " } \mu \beta \in A \text{ " } ).$$

**Proof of the claim:** Fix $\rho \in \lim_\mu(T_\mu)$ and let

$$S^*_\rho \overset{\text{def}}{=} \{ \delta \in S_0 \cap \mu : \alpha(T^*_\delta) = \delta \text{ and } \nu_\delta = \rho|\delta \}.$$  

Plainly, the set $S^*_\rho$ is stationary in $\mu$ (remember the choice of $\bar{\nu}$). By the definition of the $T^*_\xi$’s (and by $\rho \in \lim_\mu(T_\mu)$) we conclude that for every $\delta \in S^*_\rho$
if $\eta_\delta(T_{\delta+1})$ is defined then $\rho|\delta \neq \eta_\delta(T_\mu) = \eta_\delta(T_{\delta+1})$.

But $\rho|\delta = \nu_\delta$ (as $\delta \in S_\nu^*$). So look at the inductive definition: necessarily for some $\rho_\delta^* \in T_\delta$ we have $\nu_\delta = f_\delta(\rho_\delta^*)$, i.e. $\rho|\delta = f_\delta(\rho_\delta^*)$. Now, $\rho_\delta^* \in T_\delta = \bigcup_{\xi < \delta} T_\xi$ and hence for some $\xi(\delta) < \delta$, we have $\rho_\delta^* \in T_{\xi(\delta)}$. By Fodor lemma we find $\xi^* < \mu$ such that the set

$$S_\nu^* \overset{\text{def}}{=} \{ \delta \in S_\nu^* : \xi(\delta) = \xi^* \}$$

is stationary in $\mu$. Consequently we find $\rho^*$ such that the set

$$S_\nu^+ \overset{\text{def}}{=} \{ \delta \in S_\nu^* : \rho^* = \rho_\delta^* \}$$

is stationary (in $\mu$). But the sequence $(f_\xi(\rho^*) : \xi^* \leq \xi < \mu)$ is $\leq$–increasing, and hence the sequence $\rho$ is its limit. Now we easily conclude the claim using the inductive definition of the $(T_\xi, f_\xi)$’s.

It follows from the definition of $A$ and $1.7.3$ that

$$T_\mu \forces \langle A \subseteq T_\mu \rangle$$

(remember that $A$ is a name for an antichain of $T$), and hence

$$T_\mu \forces \langle \|A\| < \lambda \rangle,$$

finishing the proof of the theorem. □

**Definition 1.8** A $\lambda$–tree $T$ is $S_0$–full, where $S_0 \subseteq \lambda$, if for every limit $\delta < \lambda$

- if $\delta \in \lambda \setminus S_0$ then $T|\delta = \lim_\delta(T)$,
- if $\delta \in S_0$ then $\|T|\delta \| \setminus \lim_\delta(T)\| \leq 1$.

**Corollary 1.9** Assuming Hypothesis $1.2$:

1. The forcing notion $Q$ preserves cardinal numbers, cofinalities and cardinal arithmetic.

2. $\forces \langle T \subseteq \lambda > 2 \rangle$ is a $\lambda$–Souslin tree which is full and even $S_0$–full ".

[So, in $V^Q$, in particular we have:

- for every $\alpha < \beta < \mu$, for all $\eta \in T \cap \alpha^2$ there is $\nu \in T \cap \beta^2$ such that $\eta < \nu$, and for a limit ordinal $\delta < \lambda$, $\lim_\delta(T|_{<\delta}) \setminus T|\delta$ is either empty or has a unique element (and then $\delta \in S_0$).]
Proof. By 1.5 and 1.7.

Of course, we do not need to force.

Definition 1.10. Let $S_0, S \subseteq \lambda$. A sequence $\langle (C_\alpha, \nu_\alpha) : \alpha < \lambda \text{ limit} \rangle$ is called a squared diamond sequence for $(S, S_0)$ if for each limit ordinal $\alpha < \lambda$

(i) $C_\alpha$ a club of $\alpha$ disjoint to $S$, 

(ii) $\nu_\alpha \in \alpha^2$, 

(iii) if $\beta \in \text{acc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ and $\nu_\beta < \nu_\alpha$, 

(iv) if $\mu \in S$ then $\langle \nu_\alpha : \alpha \in C_\mu \cap S_0 \rangle$ is a diamond sequence.

Proposition 1.11. Assume (in addition to 1.2)

(e) there exist a squared diamond sequence for $(S, S_0)$.

Then there is a $\lambda$–Souslin tree $T \subseteq \lambda^{> 2}$ which is $S_0$–full.

Proof. Look carefully at the proof of 1.7.

Corollary 1.12. Assume that $V = L$ and $\lambda$ is Mahlo strongly inaccessible. Then there is a full $\lambda$–Souslin tree.

Proof. Let $S \subseteq \{ \mu < \lambda : \mu$ is strongly inaccessible} be a stationary non-reflecting set. By Beller and Litman [BeLi80], there is a square $\langle C_\delta : \delta < \lambda \text{ limit} \rangle$ such that $C_\delta \cap S = \emptyset$ for each limit $\delta < \lambda$. As in Abraham Shelah Solovay [AShS 221, §1] we can have also the squared diamond sequence.

References

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