Modelling, simulation and inference for multivariate time series of counts

ALMUT E. D. VERAART
Department of Mathematics, Imperial College London
180 Queen’s Gate, London, SW7 2AZ, UK
a.veraart@imperial.ac.uk
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Abstract

This article presents a new continuous-time modelling framework for multivariate time series of counts which have an infinitely divisible marginal distribution. The model is based on a mixed moving average process driven by Lévy noise – called a trawl process – where the serial correlation and the cross-sectional dependence are modelled independently of each other. Such processes can exhibit short or long memory. We derive a stochastic simulation algorithm and a statistical inference method for such processes. The new methodology is then applied to high frequency financial data, where we investigate the relationship between the number of limit order submissions and deletions in a limit order book.

Keywords: Count data, continuous time modelling of multivariate time series, trawl processes, infinitely divisible, Poisson mixtures, multivariate negative binomial law, limit order book

Mathematics Subject Classification: 60G10, 60G55, 60E07, 62M10, 62P05

1 Introduction

Time series of counts can be viewed as realisations of non-negative integer-valued stochastic processes and arise in various applications in the natural, life and social sciences. As such there has been very active research in the various fields and recent textbooks treatments can be found in Cameron & Trivedi (1998), Davis et al. (2015), Kedem & Fokianos (2002), Winkelmann (2003) and we refer to Cui & Lund (2009), Davis et al. (1999), Davis & Wu (2009), Ferland et al. (2006), Jung & Tremayne (2011), McKenzie (2003), Weiß (2008) for recent surveys and some new developments of the literature.

However, most of these previous works focus on univariate time series of counts and the literature on multivariate extensions is rather sparse and almost exclusively deals with models formulated in discrete time and borrow ideas from traditional autoregressive time series models. E.g. Franke & Rad (1995) and Latour (1997) introduced the first-order integer-valued autoregression model, which is based on the generalised Steutel and van Harn (1979) thinning operator. Recently, Boudreault & Charpentier (2011) applied such models to earthquake
counts. Also, the recent handbook on discrete-valued time series by Davis et al. (2015) contains the chapter by Karlis (2015) who surveys recent developments in multivariate count time series models.

One challenge in handling multivariate time series is the modelling of the cross-sectional dependence. While for continuous distributions the theory of copulas presents a powerful toolbox, it has been pointed out by Genest & Nešlehová (2007) that a problem arises in the discrete context due to the non-uniqueness of the associated copula. This can be addressed by using the continuous extension approach by Denuit & Lambert (2005). Indeed, for instance Heinen & Rengifo (2007) introduce a multivariate time series model for counts based on copulas applied to continuously extended discrete random variables and fit the model to the numbers of trades of various assets at the New York stock exchange. Also, Koopman et al. (2015) study discrete copula distributions with time-varying marginals and dependence structure in financial econometrics. Motivated by the reliability literature, Lindskog & McNeil (2003) introduced the so-called common Poisson shock model to describe the arrival of insurance claims in multiple locations or losses due to credit defaults of various types of counterparty.

While the models mentioned above are interesting in their own right, the goal of this article is more ambitious since it formulates a more general modelling framework which can handle a variety of marginal distributions as well as different types of serial dependence including, in particular, both short and long memory specifications. That said, motivated by an application in financial econometrics and recognising the success the class of Lévy processes has in such settings, we focus exclusively on models whose marginal distribution is infinitely divisible. This assumption puts a restriction on the cross-sectional dependence due to the well-known result by Feller (1968), which says that a random vector with infinitely divisible distribution on \( \mathbb{N}^n \) always has non-negatively correlated components. Moreover, any non-degenerate distribution on \( \mathbb{N}^n \) is infinitely divisible if and only if it can be expressed as a discrete compound Poisson distribution. We will see that this is nevertheless a very rich class of distributions and suitable for our application to high frequency financial data.

The new modelling framework is based on so-called multivariate integer-valued trawl processes, which are special cases of multivariate mixed moving average processes where the driving noise is given by an integer-valued Lévy basis.

In the univariate case, trawl processes – not necessarily restricted to the integer-valued case – have been introduced by Barndorff-Nielsen (2011). Also, Noven et al. (2015) used such processes in an hierarchical model in the context of extreme value theory. The univariate integer-valued case has been developed in detail in Barndorff-Nielsen et al. (2014). Shephard & Yang (2016a, 2016b) studied likelihood inference for a particular subclass of an integer-valued trawl process and, more recently, Shephard & Yang (2016d) used such processes to build an econometric model for fleeting discrete price moves. While the multivariate extension was already briefly mentioned in Barndorff-Nielsen et al. (2014), this article develops the theory of multivariate integer-valued trawl (MIVT) processes in detail and presents new methodology for stochastic simulation and statistical inference for such processes and applies the new results to high frequency financial data from a limit order book. The key feature of MIVT processes, which makes them powerful for a wide range of applications is the fact that the serial dependence and the marginal distribution can be modelled independently of each other, which is for instance not the case in the famous DARMA models, see Jacobs & Lewis (1978a, 1978b, 1978c, 1978d). As such we will present parsimonious ways of parameterising the serial correlation and will show that we can accommodate both short and long memory processes as well as seasonal fluctuations. Moreover, since MITV processes are formulated in continuous time,
we can handle both asynchronous and not necessarily equally spaced observations, which is particularly important in a multivariate set-up.

The motivation for this study comes from high frequency financial econometrics where discrete data arise in a variety of scenarios, e.g. high frequent price moves for stocks with fixed tick size resemble step functions supported on a fixed grid. Also, the number of trades can give us an indication of market activity and is widely analysed in the industry. In this article, we will apply our new methodology to model the relationship between the number of submitted and deleted limit orders in a limit order book, which are key quantities in high frequency trading.

The outline of this article is as follows. Section 2 introduces the class of multivariate integer-valued trawl processes and presents its probabilistic properties. Section 3 gives a detailed overview of parametric model specifications focusing on a variety of different cases for modelling the serial correlation. Moreover, we present relevant examples of multivariate marginal distributions which fall into the infinitely divisible framework. In particular, as pointed out by Nikoloulopoulos & Karlis (2008), the negative binomial distribution often appears to be a suitable candidate for various applications. Hence we will derive several approaches to defining a multivariate infinitely divisible distribution which allows for univariate negative binomial marginal law. In Section 4 we will derive an algorithm to simulate from MIVT processes and develop a statistical inference methodology which we will also test in a simulation study. Section 5 applies the new methodology to limit order book data. Finally, Section 6 concludes. The proofs of the theoretical results are relegated to the Appendix, Section A, and Section B provides more details on the algorithms used in the simulation study.

2 Multivariate integer-valued trawl processes

2.1 Integer-valued Lévy bases as driving noise

Throughout the paper, we denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ the underlying filtered probability space satisfying the usual conditions. Also, we choose a set $E \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) and let the corresponding Borel $\sigma$-algebra be denoted by $\mathcal{E} = \mathcal{B}(E)$. Next we define a Radon measure $\mu$ on $(E, \mathcal{E})$, which by definition satisfies $\mu(B) < \infty$ for every compact measurable set $B \in \mathcal{E}$.

In the following, we will always assume that the Assumption (A1) stated below holds.

**Assumption (A1)** Let $E = \mathbb{R}^n \times [0,1] \times \mathbb{R}$ for $n \in \mathbb{N}$ and let $N$ be a homogeneous Poisson random measure on $E$ with intensity measure $\mu(dy, dx, ds) = \mathbb{E}(N(dy, dx, ds)) = \nu(dy)dxds$, where $\nu$ is a Lévy measure concentrated on $\mathbb{Z}^n \setminus \{0\}$ and satisfying

$$\int_{\mathbb{R}^n} \min(1, ||y||)\nu(dy) < \infty.$$ 

Using the Poisson random measure, we can define an integer-valued Lévy basis as follows.

**Definition 1.** Suppose that $N$ is a homogeneous Poisson random measure on $(E, \mathcal{E})$ satisfying Assumption (A1). An $\mathbb{Z}^n$-valued, homogeneous Lévy basis on $([0,1] \times \mathbb{R}, \mathcal{B}([0,1] \times \mathbb{R}))$ is defined as

$$L(dx, ds) = (L^{(1)}(dx, ds), \ldots, L^{(n)}(dx, ds))^\top = \int_{\mathbb{R}^n} yN(dy, dx, ds).$$

(1)
From the definition, we can immediately see that $L$ is infinitely divisible with characteristic function given by

$$
E(\exp(i\theta^\top L(dx,ds))) = \exp(C_{L(dx,ds)}(\theta)), \quad \theta \in \mathbb{R}^n.
$$

Here, $C$ denotes the associated cumulant function, which is the (distinguished) logarithm of the characteristic function. It can we written as

$$
C_{L(dx,ds)}(\theta) = \text{C}_{L'}(\theta) dx ds,
$$

where the random vector $L'$ denotes the corresponding *Lévy seed* with cumulant function given by

$$
C_{L'}(\theta) = \int_{\mathbb{R}^n} \left( e^{i\theta^\top y} - 1 \right) \nu(dy),
$$

where $\nu$ denotes the corresponding Lévy measure defined above.

**Remark 1.** It is important to note that the Lévy seed specifies the homogeneous Lévy basis uniquely, and vice versa, with any homogeneous Lévy basis we can associate a unique Lévy seed. Hence, in modelling terms, it will later be sufficient to discuss various modelling choices for the corresponding Lévy seed, since this will fully characterise the associated Lévy basis.

**Remark 2.** Based on the Lévy seed, we can define a Lévy process denoted by $(L'_t)_{t \geq 0}$, when setting $L'_1 = L'$. Clearly, in this case, we get $C_{L'_t}(\theta) = tC_{L'}(\theta)$.

Following the construction in [Satd (1999), Theorem 4.3], we model the Lévy seed by an $n$-dimensional compound Poisson random variable given by

$$
L' = \sum_{j=1}^{N_1} Z_j,
$$

where $N = (N_t)_{t \geq 0}$ is an homogeneous Poisson process of rate $v > 0$ and the $(Z_j)_{j \in \mathbb{N}}$ form a sequence of i.i.d. random variables independent of $N$ and which have no atom in 0, i.e. not all components are simultaneously equal to zero, more precisely, $\mathbb{P}(Z_j = 0) = 0$ for all $j$.

**Remark 3.** Recall that by modelling the Lévy seed by a multivariate compound Poisson process we can only allow for positive correlations between the components.

### 2.2 The trawls

Following the approach presented in [Barndorff-Nielsen (2011)], see also [Barndorff-Nielsen et al. (2014)], we now define the so-called trawls.

**Definition 2.** We call a Borel set $A \subset [0,1] \times (-\infty,0]$ such that $\text{Leb}(A) < \infty$ a trawl. Further, we set

$$
A_t = A + (0,t), \quad t \in \mathbb{R}.
$$

The above definition implies that the trawl at time $t$ is just the shifted trawl from time 0.
Remark 4. Note that the size of the trawl does not change over time, i.e. we have $\text{Leb}(A_t) = \text{Leb}(A)$ for all $t$.

Clearly, there is a wide class of sets which can be considered as trawls. Throughout the paper, we will hence narrow down our focus, and will concentrate on a particular subclass of trawls which can be written as

$$A = \{(x, s) : s \leq 0, \ 0 \leq x \leq d(s)\}, \tag{4}$$

where $d : (-\infty, 0] \to [0, 1]$ is a continuous function such that $\text{Leb}(A) < \infty$. Typically we refer to $d$ as the trawl function. In such a semi-parametric setting, we can easily deduce that

$$\text{Leb}(A) = \int_{-\infty}^{0} d(s)ds. \tag{5}$$

Moreover, the corresponding trawl at time $t$ is given by

$$A_t = A + (0, t) = \{(x, s) : s \leq t, \ 0 \leq x \leq d(s-t)\}.$$

Definition 3. Let $A$ denote a trawl given by (4). If $d(0) = 1$ and $d$ is monotonically non-decreasing, then we call $A$ a monotonic trawl.

Example 1. Let $d(s) = \exp(\lambda s)$ for $\lambda > 0, s \leq 0$. Then the corresponding trawl is monotonic with $A_t = A + (0, t) = \{(x, s) : s \leq t, \ 0 \leq x \leq \exp(\lambda(s-t))\}$.

In our multivariate framework, we will choose $n$ trawls denoted by $A^{(i)} = A^{(i)}_0$. Then we set $A^{(i)}_t = A^{(i)} + (0, t)$ for $i \in \{1, \ldots, n\}$. When we work with trawls of the type (4), we will denote by $d^{(i)}$ the corresponding trawl functions.

2.3 The multivariate integer-valued trawl process and its properties

Definition 4. The stationary multivariate integer-valued trawl (MIVT) process is defined by

$$Y_t = \left( L^{(1)}(A^{(1)}_t), \ldots, L^{(n)}(A^{(n)}_t) \right)^\top, \quad t \in \mathbb{R},$$

where each component is given by

$$Y^{(i)}_t = L^{(i)}(A^{(i)}_t) = \int_{[0,1] \times \mathbb{R}} I_{A^{(i)}}(x, s-t)L^{(i)}(dx, ds), \quad i \in \{1, \ldots, n\},$$

where $I$ denotes the indicator function.

Since the trawls have finite Lebesgue measure, the integrals above are well-defined in the sense of Rajput & Rosinski (1989).

When we define $I_A(x, s-t) = \text{diag}(I_{A^{(1)}}(x, s-t), \ldots, I_{A^{(n)}}(x, s-t))$, then we can represent the MIVT process as

$$Y_t = \int_{\mathbb{R}^n \times [0,1] \times \mathbb{R}} yI_A(x, s-t)N(dy, dx, ds), \quad t \in \mathbb{R},$$

which shows that we are dealing with a special case of a multivariate mixed moving average process.

The law of the MIVT process is fully characterised by its characteristic function, which we shall present next.
Proposition 1. For any $\theta \in \mathbb{R}^n$, the characteristic function of $Y_t$ is given by $\mathbb{E}(\exp(i\theta^\top Y_t)) = \exp(C_Y(\theta))$, where the corresponding cumulant function is given by

$$C_Y(\theta) = \sum_{k=1}^{n} \sum_{1 \leq i_1, \ldots, i_k \leq n: i_v \neq i_\mu, \text{ for } v \neq \mu} \text{Leb} \left( \bigcap_{l=1}^{k} A^{(i_l)} \setminus \bigcup_{1 \leq j \leq n, j \notin \{i_1, \ldots, i_k\}} A^{(j)} \right) C_{(L^{(i_1)}, \ldots, L^{(i_k)})}((\theta_{i_1}, \ldots, \theta_{i_k})^\top).$$

Corollary 1. In the special case when $A^{(1)} = \cdots = A^{(n)} = A$, the characteristic function simplifies to $\mathbb{E}(\exp(i\theta^\top Y_t)) = \exp(\text{Leb}(A) C_L(\theta)).$

This is an important result, which implies that to any infinitely divisible integer-valued law $\pi$, say, there exists a stationary integer-valued trawl process having $\pi$ as its marginal law.

2.3.1 Cross-sectional and serial dependence

Let us now focus on the cross-sectional and the serial dependence of multivariate integer-valued trawl processes.

First, the cross-sectional dependence is entirely characterised through the multivariate Lévy measure $\nu$. For instance, when we focus on the pair of the $i$th and the $j$th component for $i, j \in \{1, \ldots, n\}$, we define the corresponding joint Lévy measure by

$$\nu^{(i,j)}(d\cdot, d\cdot) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \nu(dy_1, \ldots, dy_{i-1}, d\cdot, dy_{i+1}, \ldots, dy_{j-1}, d\cdot, dy_{j+1}, \ldots, dy_n).$$

Then the covariance between the $i$th and the $j$th Lévy seed is given by

$$\kappa_{i,j} := \int_{\mathbb{R}} \int_{\mathbb{R}} y_i y_j \nu^{(i,j)}(dy_i, dy_j).$$

Relevant specifications of $\nu$ will be discussed in Section 3.2.

Second, the serial dependence is determined through the trawls. More precisely, following Barndorff-Nielsen (2011), we introduce the so-called autocorrelator between the $i$th and the $j$th component, which is defined as

$$R_{ij}(h) = \text{Leb}(A_0^{(i)} \cap A_h^{(j)}), \quad h \geq 0.$$

Let us now focus on the autocorrelators for trawls of type (4).

Proposition 2. Suppose the trawls $A^{(i)}$, $i \in \{1, \ldots, n\}$ are of type (4). Then for $h \geq 0$ the intersection of two trawls is given by

$$A^{(i)} \cap A_h^{(j)} = \{(x, s) : s \leq 0, 0 \leq x \leq \min\{d^{(i)}(s), d^{(j)}(s-h)\}\}.$$

I.e. the autocorrelator satisfies

$$R_{ij}(h) = \int_{-\infty}^{0} \min\{d^{(i)}(s), d^{(j)}(s-h)\} ds.$$

The proof is straightforward and hence omitted.
Remark 5. Note that the autocorrelators can be computed as soon as the corresponding trawl functions and their parameters are known. We will come back to this aspect when we discuss inference for trawl processes in Section 4.2.

Let us consider a canonical example when the trawl functions are given by exponential functions.

Example 2. Let \( d^{(i)}(s) = \exp(\lambda_i s) \). For \( i, j \in \{1, \ldots, n\} \) suppose that \( \lambda_i < \lambda_j \). Then for \( s \leq 0 \) we have that \( e^{\lambda_i s} \geq e^{\lambda_j s} \) and hence \( A^{(i)} \cap A^{(j)} = A^{(j)} \). Hence \( \text{Leb}(A^{(i)} \cap A^{(j)}) = \text{Leb}(A^{(j)}) = 1/\lambda_j \). Similarly, we get that \( R_{ij}(h) = \text{Leb}(A^{(i)} \cap A^{(j)}_h) = \frac{1}{\lambda_j} e^{-\lambda_j h} \), for \( h \geq 0 \).

For monotonic trawl functions we observe that there are two possible scenarios: Either, one trawl function is always ‘below’ the other one, which implies that \( R_{ij}(h) = \min(\text{Leb}(A^{(i)}), \text{Leb}(A^{(j)})) \), see e.g. Example 2, or the trawl functions intersect each other. In the latter case, suppose there is one intersection of \( d^{(i)} \) and \( d^{(j)} \) at time \( s^* < 0 \), say. Consider the scenario when \( d^{(i)}(s) \leq d^{(j)}(s) \) for \( s \leq s^* \) and \( d^{(j)}(s) \leq d^{(i)}(s) \) for \( s^* \leq s \leq 0 \). Then

\[
R_{ij}(0) = \text{Leb}(A^{(i)} \cap A^{(j)}) = \int_{-\infty}^{s^*} d^{(i)}(s) ds + \int_{s^*}^{0} d^{(j)}(s) ds.
\]

Extensions to a multi-root scenario are straightforward.

Clearly, the autocorrelators are closely related to the autocorrelation function. More precisely, we have the following result, which follows directly from the expression of the cumulant function of the multivariate trawl process.

Proposition 3. The covariance between two (possibly shifted) components \( 1 \leq i \leq j \leq n \) for \( t, h \geq 0 \) is given by

\[
\rho_{ij}(h) = \text{Cov}\left(L^{(i)}(A^{(i)}_t), L^{(j)}(A^{(j)}_{t+h})\right) = \text{Leb}\left(A^{(i)} \cap A^{(j)}_h\right) \left( \int_{\mathbb{R}} \int_{\mathbb{R}} y_i y_j \nu^{(i,j)}(dy_i, dy_j) \right)
\]

\[
= R_{ij}(h) \kappa_{i,j}.
\]

Also, the corresponding auto- and cross-correlation function is given by

\[
r_{ij}(h) := \text{Cor}\left(L^{(i)}(A^{(i)}_t), L^{(j)}(A^{(j)}_{t+h})\right) = \frac{\text{Leb}(A^{(i)} \cap A^{(j)}_h) \left( \int_{\mathbb{R}} \int_{\mathbb{R}} y_i y_j \nu^{(i,j)}(dy_i, dy_j) \right)}{\sqrt{\text{Leb}(A^{(i)}) \text{Var}(L^{(i)}) \text{Leb}(A^{(j)}) \text{Var}(L^{(j)})}}
\]

\[
= \frac{R_{ij}(h) \kappa_{i,j}}{\sqrt{\text{Leb}(A^{(i)}) \text{Leb}(A^{(j)}) / \text{Var}(L^{(i)}) \text{Var}(L^{(j)})}},
\]

i.e. the autocorrelation function is proportional to the autocorrelators.

We will come back to the above result when we turn our attention to parametric inference for MIVT processes in Section 4.2.

3 Parametric specifications

In order to showcase the flexibility of the new modelling framework, we will discuss various parametric model specifications in this section, where we start off by considering specifications of the trawl, followed by models for the multivariate Lévy seed.
3.1 Specifying the trawl function

We have already covered the case of an exponential trawl function above and will now present alternative choices for the trawl functions and their corresponding autocorrelators, see also Barndorff-Nielsen et al. (2014) for other examples.

While an exponential trawl leads to an exponentially decaying autocorrelation function, we sometimes need model specifications which exhibit a more slowly decaying autocorrelation function. Such trawl functions can be constructed from the exponential trawl function by randomising the memory parameter as we will describe in the following example.

To simplify the notation we will in the following suppress the indices $i$ for the corresponding component in the multivariate construction, i.e. we set $d = d^{(i)}$ and do not write the sub-/superscripts for the corresponding parameters.

**Example 3.** Define the trawl function by

$$d(z) = \int_0^\infty e^{\lambda z} \pi(d\lambda), \quad \text{for } z \leq 0,$$

for a probability measure $\pi$ on $(0, \infty)$. Suppose that $\pi$ is absolutely continuous with density $f_\pi$, then the corresponding trawl function can be written as

$$d(z) = \int_0^\infty e^{\lambda z} f_\pi(\lambda)d\lambda,$$

which again leads to a monotonic trawl function. The corresponding autocorrelation function is given by

$$r(h) = \frac{\int_0^\infty \frac{1}{2}e^{-\lambda h} \pi(d\lambda)}{\int_0^\infty \frac{1}{2}\pi(d\lambda)},$$

assuming that $\int_0^\infty \frac{1}{2}\pi(d\lambda) < \infty$.

Barndorff-Nielsen et al. (2014) discuss various constructions of that type depending on different choices of the probability measure $\pi$ and we refer to that article for more details on the computations.

In applications, we often assume that $\pi$ is absolutely continuous with respect to the Lebesgue measure and we denote its density by $f_\pi$. A very flexible parametric framework can be obtained by choosing $f_\pi$ to be a generalised inverse Gaussian (GIG) density as we shall discuss in the next example.

**Example 4.** Suppose that $f_\pi$ is the density of the GIG distribution, i.e.

$$f_\pi(x) = \frac{(\gamma/\delta)^\nu}{2K_\nu(\delta\gamma)} x^{\nu-1} \exp\left(-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right),$$

where $\nu \in \mathbb{R}$ and $\gamma$ and $\delta$ are both nonnegative and not simultaneously equal to zero. Here we denote by $K_\nu(\cdot)$ the modified Bessel function of the third kind. Straightforward computation show that the corresponding trawl function is given by

$$d(z) = \left(1 - \frac{2z}{\gamma^2}\right)^{-\frac{\nu}{2}} \frac{K_\nu(\delta\gamma \sqrt{1 - \frac{2z}{\gamma^2}})}{K_\nu(\delta\gamma)},$$
and the corresponding size of the trawl set equals
\[ \text{Leb}(A) = \frac{(\gamma/\delta)K_{\nu-1}(\delta \gamma)}{K_{\nu}(\delta \gamma)}. \]

Moreover, the autocorrelation function is given by
\[ r(h) = \frac{K_{\nu-1}(\delta \sqrt{\gamma^2 + 2h})}{K_{\nu}(\delta \gamma)} \left(1 + \frac{2h}{\gamma^2}\right)^{\frac{1}{2}(1-\nu)}. \]

Some special cases of the GIG distribution include the inverse Gaussian and the gamma distribution, which lead to interesting parametric examples which we shall study next.

**Example 5.** Suppose we choose an inverse Gaussian (IG) density function for \( f_\pi \). Then we obtain the so-called sup-IG trawl function, which can be written as
\[ d(z) = \left(1 - \frac{2z}{\gamma^2}\right)^{-1/2} \exp\left(\delta \gamma \left(1 - \sqrt{1 - \frac{2z}{\gamma^2}}\right)\right), \]
for nonnegative parameters \( \delta, \gamma \) which are assumed not to be simultaneously equal to zero. Then we have that \( \text{Leb}(A) = \frac{\delta}{\gamma} \) and the corresponding autocorrelation function is given by
\[ r(h) = \exp\left(\delta \gamma \left(1 - \sqrt{1 + \frac{2h}{\gamma^2}}\right)\right), \quad h \geq 0. \]

Next, we consider an example where the trawl function decays according to a power law.

**Example 6.** A long memory specification can be obtained when the probability measure \( \pi \) is chosen to have Gamma distribution. In that case, we obtain a trawl function given by
\[ d(z) = \left(1 - \frac{z}{\alpha}\right)^{-H}, \quad \alpha > 0, H > 1. \]
Then \( \text{Leb}(A) = \frac{\alpha}{H-1} \) and the corresponding autocorrelation function is given by
\[ r(h) = \left(1 + \frac{h}{\alpha}\right)^{1-H}. \]
I.e. when \( H \in (1, 2] \) we have a stationary long memory model, and when when \( H > 2 \) we obtain a stationary short memory model.

Finally, we consider the case of a seasonal trawl function.

**Example 7.** A seasonally varying trawl function can be obtained by setting \( d(z) = d_m(z)d_s(z) \), where \( d_m \) is a monotonic trawl function and \( d_s \) is a periodic seasonal function. E.g. as discussed in [Barndorff-Nielsen et al. 2014, Example 9], we can consider the following functional form
\[ d(z) = \frac{1}{2} \exp(\lambda x) [\cos(ax) + 1], \quad \text{where } a = 2\pi \psi. \]
Here \( \lambda > 0 \) determines how quickly the function decays, whereas \( \psi \in \mathbb{R} \) denotes the period of the season. In this case, we obtain \( \text{Leb}(A) = (2\lambda^2 + a^2)/(2\lambda(\lambda^2 + a^2)) \) and
\[ r(h) = \frac{e^{-\lambda h}}{2\lambda(\lambda^2 + a^2)} \left(\lambda^2 \cos(ah) - a\lambda \sin(ah) + \lambda^2 + a^2\right). \]
Note that this construction leads to a seasonal autocorrelation function, but not to seasonality in the levels of the trawl process.
3.2 Modelling the cross-sectional dependence

The trawl process is completely specified, as soon as both the trawls and the marginal distribution of the multivariate Lévy seed are specified. When it comes to infinitely divisible discrete distributions, the Poisson distribution is the natural starting point and we will review multivariate extensions in Section 3.2.1. However, since many count data exhibit overdispersion, it is crucial that we go beyond the Poisson framework. In the univariate context, there have been a variety of articles on suitable discrete distributions, see e.g. [17] Puig & Valero (2006) and [18] Nikoloulopoulos & Karlis (2008) amongst others. However, the literature on parametric classes of multivariate infinitely divisible discrete distributions with support on \( \mathbb{N}^n \) is rather sparse. We know that any such distribution necessarily is of discrete compound Poisson type, see Feller (1968), Sundt (2000), Valderrama Ospina & Gerber (1987), and always has non-negative correlation. We know that any such distribution necessarily is of discrete compound Poisson type, see Feller (1968), Sundt (2000), Valderrama Ospina & Gerber (1987), and always has non-negative correlation. We know that any such distribution necessarily is of discrete compound Poisson type, see Feller (1968), Sundt (2000), Valderrama Ospina & Gerber (1987), and always has non-negative correlation. We know that any such distribution necessarily is of discrete compound Poisson type, see Feller (1968), Sundt (2000), Valderrama Ospina & Gerber (1987), and always has non-negative correlation.

Example 8. An \( n \)-dimensional model with one common factor between all components can be obtained by choosing \( m = n + 1 \), and

\[
\begin{align*}
  A & = 
  \begin{pmatrix}
    1 & 0 & \cdots & \cdots & 1 \\
    0 & 1 & 0 & \cdots & 1 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 1 & 1
  \end{pmatrix}, \\
  X & = 
  \begin{pmatrix}
    X^{(1)} \\
    \vdots \\
    X^{(n)} \\
    X^{(0)}
  \end{pmatrix},
\end{align*}
\]

and independent Poisson random variables \( X^{(i)} \sim \text{Poi}(\theta_i) \), for \( i = 0, 1, \ldots, n \). Then we have

\[
L^{(1)} = X^{(1)} + X^{(0)}, \quad L^{(2)} = X^{(2)} + X^{(0)}, \quad \cdots, \quad L^{(n)} = X^{(n)} + X^{(0)}.
\]

Here each component has marginal Poisson distribution, i.e. \( L^{(i)} \sim \text{Poi}(\theta_i + \theta_0) \) and for \( i \neq j \) we have that \( \text{Cov}(L^{(i)}, L^{(j)}) = \theta_0 \).
Beyond the bivariate case, the example above presents a rather restrictive model for applications since it only allows for one common factor. A less sparse choice of \( \mathbf{A} \) would allow for more flexible model specifications. Let us consider a more realistic example in the trivariate case next.

**Example 9.** Consider a model of the type

\[
L^{(1)} = X^{(1)} + X^{(12)} + X^{(13)} + X^{(123)},
L^{(2)} = X^{(2)} + X^{(12)} + X^{(23)} + X^{(123)},
L^{(3)} = X^{(3)} + X^{(13)} + X^{(23)} + X^{(123)}
\]

for independent Poisson random variables \( X^{(i)} \) with parameters \( \theta_i \), for \( i \in \{\{1\}, \{2\}, \{3\}, \{12\}, \{13\}, \{23\}, \{123\} \} \). Such a model specification corresponds to the choice of

\[
\mathbf{A} = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix},
\quad
\mathbf{X} = \begin{pmatrix}
X^{(1)}, X^{(2)}, X^{(3)}, X^{(12)}, X^{(13)}, X^{(23)}, X^{(123)}
\end{pmatrix}^\top.
\]

Here we have that \( L^{(1)} \sim \text{Poi}(\theta_1 + \theta_{12} + \theta_{13} + \theta_{123}) \), \( L^{(2)} \sim \text{Poi}(\theta_2 + \theta_{12} + \theta_{23} + \theta_{123}) \) and \( L^{(3)} \sim \text{Poi}(\theta_3 + \theta_{13} + \theta_{23} + \theta_{123}) \).

The above example treats a very general case which allows for all possible bivariate as well as a trivariate covariation effect. A slightly simpler specification is given in the next example, which only considers pairwise interaction terms.

**Example 10.** Choosing

\[
\mathbf{A} = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix},
\quad
\mathbf{X} = \begin{pmatrix}
X^{(1)}, X^{(2)}, X^{(3)}, X^{(12)}, X^{(13)}, X^{(23)}
\end{pmatrix}^\top,
\]

results in a trivariate model of the form

\[
L^{(1)} = X^{(1)} + X^{(12)} + X^{(13)},
L^{(2)} = X^{(2)} + X^{(12)} + X^{(23)},
L^{(3)} = X^{(3)} + X^{(13)} + X^{(23)}
\]

for independent Poisson random variables \( X^{(i)} \) with parameters \( \theta_i \), for \( i \in \{\{1\}, \{2\}, \{3\}, \{12\}, \{13\}, \{23\} \} \). Then we have that \( L^{(1)} \sim \text{Poi}(\theta_1 + \theta_{12} + \theta_{13}) \), \( L^{(2)} \sim \text{Poi}(\theta_2 + \theta_{12} + \theta_{23}) \) and \( L^{(3)} \sim \text{Poi}(\theta_3 + \theta_{13} + \theta_{23}) \); also,

\[
\text{Var}(L') = \begin{pmatrix}
\theta_1 + \theta_{12} + \theta_{13} & \theta_{12} & \theta_{13} \\
\theta_{12} & \theta_2 + \theta_{12} + \theta_{23} & \theta_{23} \\
\theta_{13} & \theta_{23} & \theta_3 + \theta_{13} + \theta_{23}
\end{pmatrix}.
\]

### 3.2.2 Multivariate discrete compound Poisson marginal distribution obtained from Poisson mixtures

While the Poisson distribution is a good starting point in the context of modelling count data, for many applications it might be too restrictive. In particular, often one needs to work with distributions which allow for overdispersion, i.e. that the variance is bigger than the mean.
Since we are interested in staying within the class of discrete infinitely divisible stochastic processes, the most general class of distributions we can consider are the discrete compound Poisson distributions. To this end, we model the Lévy seed by an $n$-dimensional compound Poisson random variable, see e.g. Sato (1999, Theorem 4.3), given by

$$L' = \sum_{j=1}^{N_1} C_j,$$

where $N = (N_t)_{t \geq 0}$ is an homogeneous Poisson process of rate $\nu > 0$ and the $(C_j)_{j \in \mathbb{N}}$ form a sequence of i.i.d. random variables independent of $N$ and which have no atom in 0, i.e. not all components are simultaneously equal to zero, more precisely, $\mathbb{P}(C_j = 0) = 0$ for all $j$.

**General Poisson mixtures**

Previous research has clearly documented that Poisson mixture distributions provide a flexible class of distributions which are suitable for various applications, see e.g. Karlis & Xekalaki (2005) for a review.

In this section, we are going to introduce a parsimonious parametric model class for the $n$-dimensional Lévy seed $L'$, which uses Poisson mixtures and is based on the results in Section 5 of Barndorff-Nielsen et al. (1992). To this end, consider random variables $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$ for $n \in \mathbb{N}$ and assume that conditionally on $\{Z_1, \ldots, Z_n\}$ the $X_1, \ldots, X_n$ are independent and Poisson distributed with means given by the $\{Z_1, \ldots, Z_n\}$.

We then model the joint distribution of the $\{Z_1, \ldots, Z_n\}$ by a so-called additive effect model as follows:

$$Z_i = \alpha_i U + V_i, \quad i = 1, \ldots, n,$$

where the random variables $U, V_1, \ldots, V_n$ are independent and the $\alpha_1, \ldots, \alpha_n$ are nonnegative parameters.

We can easily derive the probability generating function of the joint distribution of $X_1, \ldots, X_n$, cf. Barndorff-Nielsen et al. (1992, Section 5):

$$\mathbb{E}(t_1^{X_1} \cdots t_n^{X_n}) = M_U \left( \sum_{i=1}^{n} \alpha_i (t_i - 1) \right) \prod_{i=1}^{n} M_{V_i}(t_i - 1),$$

where we denote by $M_X(\theta) = \mathbb{E}(e^{\theta X})$ the moment generating function of a random variable $X$ with parameter $\theta$.

Also, we can compute the means and the covariance function of the $Y_i$s and find that

$$\mathbb{E}(X_i) = \alpha_i \mathbb{E}(U) + \mathbb{E}(V_i), \quad i = 1, \ldots, n,$$

and

$$\text{Cov}(X_i, X_j) = \begin{cases} \alpha_i^2 \text{Var}(U) + \text{Var}(V_i) + \alpha_i \mathbb{E}(U) + \mathbb{E}(V_i), & \text{if } i = j, \\ \alpha_i \alpha_j \text{Var}(U), & \text{if } i \neq j. \end{cases}$$

Next we derive the joint law of $(X_1, \ldots, X_n)$, see Barndorff-Nielsen et al. (1992) for the bivariate case.
Proposition 4. In the additive random effect model the joint law of \((X_1, \ldots, X_n)\) is given by
\[
P(X_1 = x_1, \ldots, X_n = x_n) = \frac{1}{x_1! \cdots x_n!} \sum_{j_1=0}^{x_1} \cdots \sum_{j_n=0}^{x_n} (x_1^{j_1} \cdots x_n^{j_n} \alpha_1^{j_1} \cdots \alpha_n^{j_n}) \cdot \mathbb{E}(U_1^{j_1} + \cdots + U_n^{j_n} e^{-(\alpha_1 + \cdots + \alpha_n)U}) \prod_{k=1}^{n} \mathbb{E}(V_k^{y_k} - V_k).
\]

Next, we establish the key result of this section, which links the Poisson mixture distribution based on an additive effect model to a discrete compound Poisson distribution. Recall, see e.g. Sato (1999, p. 18), that an \(n\)-dimensional compound Poisson random variable \(L' = \sum_{i=1}^{N} C_i\) has Laplace transform given by
\[
\mathcal{L}_{L'}(\theta) = \mathbb{E}(e^{-\theta^\top L'}) = \exp(\nu(\mathcal{L}_C(\theta) - 1)),
\]
where \(\nu > 0\) is the intensity of the Poisson process \(N\) and \(\mathcal{L}_C(\theta)\) is the Laplace transform of the i.i.d. jump sizes.

Proposition 5. The Poisson mixture model of random-additive-effect type can be represented as a discrete compound Poisson distribution with rate
\[
v = -\left(\mathcal{K}_U(\alpha) + \sum_{i=1}^{n} \mathcal{K}_V_i(1)\right),
\]
where \(\alpha = \sum_{i=1}^{n} \alpha_i\) and \(\mathcal{K}\) denotes the kumulant function, i.e. the logarithm of the Laplace transform, and the jump size distribution has Laplace transform given by
\[
\mathcal{L}_C(\theta) = \frac{1}{v} \left\{ \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k(U) + \sum_{i=1}^{n} \sum_{k=1}^{\infty} e^{-\theta_i k} q_k(V_i) \right\},
\]
where
\[
q_k^{(U)} = \frac{1}{k!} \int_{\mathbb{R}} e^{-\alpha x} x^k \nu_U(dx), \quad q_k^{(V_i)} = \int_{\mathbb{R}} \frac{x^k}{k!} e^{-x} \nu_{V_i}(dx), \quad \text{for } i \in \{1, \ldots, n\},
\]
where \(\nu_U\) and \(\nu_{V_i}\) denotes the Lévy measure of \(U\) and \(V_i\), respectively.

The above result is very important since we need the compound Poisson representation to efficiently simulate the trawl process, as we shall discuss in Section 4.1.

Multivariate negative binomial distribution
In situations where the count data are overdispersed and call for distributions other than the Poisson one, we can in principle choose from a great variety of discrete compound Poisson distributions. Motivated by our empirical study, see Section 5 and also the results in Barndorff-Nielsen et al. (2014), we investigate the case of a negative binomial marginal law in more detail since this is one of the infinitely divisible distributions which can cope with overdispersion.
Recall that we say that a random variable $X$ has negative binomial law with parameters $\kappa > 0, 0 < p < 1$, i.e. $X \sim NB(\kappa, p)$ if its probability mass function is given by

$$
P(X = x) = \binom{\kappa + x - 1}{x} p^x (1 - p)^\kappa, \quad x \in \{0, 1, \ldots\}.
$$

Its probability generating function is given by $G(t) = \mathbb{E}(t^X) = \left(1 - \frac{p}{1 - t}\right)^{-\kappa}$. Also, recall that a random variable $X$ is said to be gamma distributed with parameters $\alpha, \beta > 0,$ i.e. $X \sim \Gamma(\alpha)$ if its probability density is given by

$$
f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.
$$

Now, we set $U \sim \Gamma(\kappa, 1)$ and $V_i \sim \Gamma(\kappa_i, \beta_i^{-1})$ in the Poisson mixture model. Then the probability generating function of $(X_1, \ldots, X_n)$ is given by

$$
\mathbb{E}(t_1^{X_1} \cdots t_n^{X_n}) = \left(1 + \sum_{i=1}^n \alpha_i(t_i - 1)\right)^{-\kappa} \prod_{i=1}^n (1 - \beta_i(t_i - 1))^{-\kappa_i}.
$$

Next we are going to describe three examples, see Barndorff-Nielsen et al. (1992, Example 5.3), which lead to negative binomial marginals. The first example, Example 11 covers the case of independent components, in the second example, Example 12 the fully dependent case is achieved through the presence of a common factor, and the third example, Example 13 combines the previous two cases by allowing for both a common (dependent) factor and additional independent components.

**Example 11** (Independence case). We set $\alpha_i \equiv 0$, for $i = 1, \ldots, n$ and choose $V_i \sim \Gamma(\kappa_i, 1/\beta_i)$. Then $\mathbb{E}(t_1^{X_1} \cdots t_n^{X_n}) = \prod_{i=1}^n (1 - \beta_i(t_i - 1))^{-\kappa_i}$, which implies that the $X_i$ are independent and satisfy $X_i \sim NB(\kappa_i, \beta_i/(1 + \beta_i))$.

**Example 12** (Dependence through common factor). Choose $U \sim \Gamma(\kappa, 1)$ and $V_i \equiv 0$, for $i = 1, \ldots, n$. Note that such a construction extends the bivariate case considered in Arbous & Kerrich (1951). Then $\mathbb{E}(t_1^{X_1} \cdots t_n^{X_n}) = (1 + \sum_{i=1}^n \alpha_i(t_i - 1))^{-\kappa}$, which implies that $X_i \sim NB(\kappa, \alpha_i/(1 + \alpha_i))$ and also $\sum_{i=1}^n X_i \sim NB\left(\kappa, \frac{\sum_{i=1}^n \alpha_i}{1 + \sum_{i=1}^n \alpha_i}\right)$.

**Example 13** (Dependence through common factor and additional independent factors). Suppose that $U \sim \Gamma(\kappa, 1)$ and $V_i \sim \Gamma(\kappa_i, 1/\alpha_i)$. Then one can write $Z_i = \alpha_i(U + W_i)$, for $U \sim \Gamma(\kappa, 1)$ and $W_i \sim \Gamma(\kappa_i, 1)$. Then we can deduce that $\mathbb{E}(t_1^{X_1} \cdots t_n^{X_n}) = (1 + \sum_{i=1}^n \alpha_i(t_i - 1))^{-\kappa} \prod_{i=1}^n (1 - \alpha_i(t_i - 1))^{-\kappa_i}$. Hence $X_i \sim NB(\kappa + \kappa_i, \alpha_i/(1 + \alpha_i))$.

**Remark.** The dependence concepts used here can be considered as Poisson mixtures of the first kind, see Karlis & Xekalaki (2003).

We conclude this section by deriving the compound Poisson representation of the multivariate negative binomial distribution.

**Example 14.** As before, let $U \sim \Gamma(\kappa, 1), V_i \sim \Gamma(\kappa_i, 1/\beta_i)$. Recall that for $X \sim \Gamma(a, b)$, $\mathbb{E}(e^{\theta X}) = (1 - i\theta/b)^{-a}$. Hence $\mathcal{L}_U(\theta) = (1 + \theta)^{-\kappa}$, and $\mathcal{L}_V(\theta) = (1 + \theta \beta_i)^{-\kappa_i}$. Also, $\mathcal{F}_U(\theta) = -\kappa \log(1 + \theta)$, and $\mathcal{F}_V(\theta) = -\kappa_i \log(1 + \theta \beta_i)$. Then the rate in the compound Poisson representation is given by $r = \kappa \log(1 + \alpha) + \sum_{i=1}^n \kappa_i \log(1 + \beta_i)$. Further, we have $\nu_U(dx) = \kappa x^{-1} e^{-x} dx$, and $\nu_V(dx) = \kappa_i x^{-1} e^{-x/\beta_i} dx$. Then we can compute

$$
q_k^{(U)} = \frac{1}{k!} \int_{\mathbb{R}} e^{-ax} x^k \nu_U(dx) = \frac{1}{k!} \int_{\mathbb{R}} e^{-ax} x^k \kappa x^{-1} e^{-x} dx = \frac{\kappa}{k!} \int_{\mathbb{R}} e^{-(\alpha+1)x} x^{k-1} dx
$$

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\[ q_k^{(V_i)} = \int_{\mathbb{R}} \frac{x^k}{k!} e^{-x} \nu_i(dx) = \int_{\mathbb{R}} \frac{x^k}{k!} e^{-x} \kappa_i x^{-1} e^{-x/\beta_i} dx = \frac{\kappa_i}{k!} \int_{\mathbb{R}} e^{-(1+1/\beta_i)x} x^{k-1} dx = \frac{\kappa_i}{k} (1 + 1/\beta_i)^{-k}. \]

Recall the series expansion of the logarithm: \[ \sum_{k=1}^{\infty} \frac{x^k}{k!} = -\log(1 - x), \text{ for } x \leq 1 \text{ and } x \neq 1. \] Hence we conclude that

\[ \mathcal{L}(\theta) = \frac{1}{v} \left\{ \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k^{(U)} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} e^{-\theta_i} q_k^{(V_i)} \right\} \]

\[ = \frac{1}{v} \left\{ \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k \frac{\kappa_i}{k} (\alpha + 1)^{-k} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} e^{-\theta_i} \kappa_i \frac{k}{k} (1 + 1/\beta_i)^{-k} \right\} \]

\[ = \frac{1}{v} \left\{ -\kappa \log \left( 1 - \sum_{i=1}^{n} \frac{\alpha_i}{\alpha + 1} e^{-\theta_i} \right) - \sum_{i=1}^{n} \kappa_i \log \left( 1 - e^{-\theta_i} (1 + 1/\beta_i)^{-1} \right) \right\}. \]

I.e. we can either represent the distribution by one discrete compound Poisson distribution. Alternatively, we can write it as convolution of \( n + 1 \) independent compound Poisson laws, where one component has the multivariate logarithmic distribution with parameters \( (p_1, \ldots, p_n) \) for \( p_i = \alpha_i/\alpha + 1 \) as the jump size distribution, see e.g. Patil & Bildikar (1967) and Remark 7 below. The remaining components have a one-dimensional logarithmic distribution in one component of the jump sizes and the other components are set to zero, more precisely, we can write

\[ \mathbf{L}' = \sum_{i=1}^{n} \mathbf{C}_{j}^{(i)\perp} + \sum_{i=1}^{n} \sum_{j=1}^{N_i} (0, \ldots, 0, C_{j}^{(i)\perp}, 0, \ldots, 0)^\top, \]

where the component \( C_{j}^{(i)\perp} \) is in the \( i \)th row in the \( n \)-dimensional column vector. The Poisson random variable \( N_i^{\parallel} \) has intensity \( \kappa_i \log(1 + \alpha) \) and the Poisson random variables \( N_j^{(i)\perp} \) have rates \( \kappa_i \log(1 + \beta_i) \). Further, \( C_{j}^{(i)\perp} \sim \text{Log}(\frac{\beta_i}{1 + \beta_i}) \).

Remark 7. Recall the following properties of the multivariate logarithmic series distribution, see e.g. Patil & Bildikar (1967). \( \mathbf{C}^{\parallel} \sim \text{Log}(p_1, \ldots, p_n) \), where \( 0 < p_i < 1, p = \sum_{i=1}^{n} p_i < 1 \) if for \( c \in \mathbb{N}^n \setminus \{0\} \),

\[ \mathbb{P}(\mathbf{C}^{\parallel} = c) = \frac{\Gamma(c_1 + \cdots + c_n)}{c_1! \cdots c_n!} \frac{p_1^{c_1} \cdots p_n^{c_n}}{[-\log(1 - p)]}. \]

Each component \( C_l^{(i)} \) follows the modified univariate logarithmic distribution with parameters \( \tilde{p}_i = p_i/(1 - p + p_i) \) and \( \delta_i = \log(1 - p + p_i)/\log(1 - p) \), i.e.

\[ \mathbb{P}(C_l^{(i)}) = c_i) = \begin{cases} \delta_i, & \text{for } c_i = 0 \\ (1 - \delta_i) \frac{1}{c_i} \frac{\tilde{p}_i^{c_i}}{[-\log(1 - p_i)]} & \text{for } c_i \in \mathbb{N}. \end{cases} \]
4 Simulation and inference

We will now turn our attention to simulation and inference for trawl processes. We will start off by deriving a simulation algorithm which is based on the compound-Poisson-type representation of MIVTs. This will enable us to simulate sample paths from our new class of processes, which can be used for model-based parametric bootstrapping in parametric inference. The inference procedure itself will be based on the (generalised) method of moments, since the cumulants of the multivariate trawl process are readily available.

4.1 Simulation algorithm

First of all, we discuss how to simulate a univariate MIVT process. For each component \( i \in \{1, \ldots, n\} \), we have the following representation.

\[
Y^{(i)}_t = L^{(i)}(A^{(i)}_t) = X^{(i)}_{0,t} + X^{(i)}_t,
\]

where \( X^{(i)}_{0,t} = L^{(i)}(\{(x,s) : s \leq 0, 0 \leq x \leq d^{(i)}(s-t)\}) \) and \( X^{(i)}_t = L^{(i)}(\{(x,s) : 0 < s \leq t, 0 \leq x \leq d^{(i)}(s-t)\}) \), for a trawl function \( d^{(i)} \).

We would like to argue that the term \( X^{(i)}_{0,t} \) is asymptotically negligible in the sense that it converges to zero as \( t \to \infty \), which will allow us to concentrate on the term \( X^{(i)}_t \) in the following. Indeed, this conjecture holds as the following proposition shows.

**Proposition 6.** For a trawl function \( d^{(i)} \), we have that \( X^{(i)}_{0,t} \to 0 \) in probability, as \( t \to \infty \).

Hence, we will focus on simulating \( X^{(i)}_t \) and will work with a burn-in period in the simulation such that the effect of \( X^{(i)}_{0,t} \) is negligible.

A realisation of \( L \) consists of a countable set \( R \) of points \((y,x,s)\) in \( \mathbb{N}_n \times \{0\} \times [0,1] \times \mathbb{R} \). When we project the point pattern to the time axis, we obtain the arrival times of a Poisson process \( N_t \) with intensity \( v = \nu(\mathbb{R}^n) \). The corresponding arrival times are denoted by \( t_1, \ldots, t_{N_t} \) and we associate uniform heights \( U_1, \ldots, U_{N_t} \) with them, see Barndorff-Nielsen et al. (2014) for a detailed discussion in the univariate case. So as soon as we have specified the jump size distribution of the \( C \), we can use the representation

\[
X^{(i)}_t = \sum_{j=1}^{N_t} C^{(i)}_j I\{U_j \leq d^{(i)}(t_j-t)\},
\]

to simulate each component.

**Algorithm 5.** In this algorithm we suppress the dependence on the superscript \( (i) \) and describe how to simulate from the one-dimensional of components of the form

\[
X_t := \sum_{j=1}^{N_t} C_j I\{U_j \leq d(t_j-t)\} \quad (8)
\]

We want to simulate \( X \) on a \( \Delta \)-grid of \([0,t]\), where \( \Delta > 0 \), i.e. we want to find \((X_0, X_{1\Delta}, \ldots, X_{\lfloor t/\Delta \rfloor \Delta})\).

1. Generate a realisation \( n_t \) of the Poisson random variable \( N_t \) with mean \( vt \) for \( v > 0 \).
2. Generate the pairs \((t_j, U_j)\) where the series \((t_1, \ldots, t_n)\) consists of realisations of ordered i.i.d. uniform random variables on \([0, t]\). The \((U_1, \ldots, U_n)\) are i.i.d. and uniformly distributed on \([0, 1]\) and independent of the arrival times \((t_1, \ldots, t_n)\).

3. Simulate the i.i.d. jump sizes \(C_1, \ldots, C_n\).

4. Construct the trawl process on a \(\Delta\)-grid, where \(\Delta > 0\), by setting \(X_0 = 0\) and

\[
X_{k\Delta} := \sum_{j=1}^{\text{card}(t_1, t_t \leq k\Delta)} C_j I\{U_j \leq d(t_j - k\Delta)\}, \quad k = 1, \ldots, \lfloor t/\Delta \rfloor. \tag{9}
\]

**Remark 8.** Note that the condition in the indicator function in (9) can be expressed in a vectorised form, which allows a fast implementation of the simulation algorithm, see Section B.2 for details.

In order to generate samples from the multivariate process, it is easiest to split the compound Poisson seed into dependent and independent components and simulate the components separately as we shall describe in more detail in the following example.

**Example 15.** Suppose we want to simulate from the multivariate trawl process with negative binomial marginal law as described in Example 13. Then we split each component into a dependent and an independent component as follows:

\[
X_t^{(i)} = \sum_{j=1}^{N_i} C_j^{||} I\{U_j \leq d^{(i)}(t_j - t)\} = \sum_{j=1}^{N_i^{||}} C_j^{||} I\{U_j \leq d^{(i)}(t_j - t)\} + \sum_{j=1}^{N_i^{(i)}} C_j^{(i)} I\{U_j \leq d^{(i)}(t_j - t)\},
\]

where \(C^{||} = (C_j^{||(1)}, \ldots, C_j^{||(n)})^\top \sim \text{Log}(p_1, \ldots, p_n)\), where \(p_i = \alpha_i/(1 + \alpha)\), \(C_j^{(i)} \sim \text{Log}(p_i)\). Note that \(C_j^{||}\) and \(C_j^{(1)}, \ldots, C_j^{(n)}\) are independent for all \(j\) and the intensities of the independent Poisson processes \(N^{||}, N_j^{(1)} \downarrow, \ldots, N_j^{(n)} \downarrow\) are given by \(\kappa \log(1 + \alpha)\) and \(\kappa_1 \log(1 + \alpha_1), \ldots, \kappa_n \log(1 + \alpha_n)\), respectively. Then we can use the algorithm above to simulate each component separately.

**Remark 9.** Since the above scheme ignores the initial value \(X_{0,t}\), it is advisable to work with a burn-in period in a practical implementation. In the situation when the support of the trawl function \(d\) is bounded, then an exact simulation of the trawl process is possible since its initial value can be generated precisely.

### 4.2 Inference

We propose to estimate the model parameters using a two stage equation-by-equation procedure, where the marginal parameters for each component are estimated first, and the parameter determining the dependence are estimated in a second step. Recent research on inference in multivariate models, see e.g. Joe (2005) and, more recently, Francq & Zakoïan (2016), has highlighted that such a procedure is very powerful in a high-dimensional set-up.

Motivated by the results in Barndorff-Nielsen et al. (2014), we propose to work with the (generalised) method of moments to infer the model parameters since the cumulants are readily available and the procedure works well in our simulation study. Full maximum likelihood
estimation is numerically rather intractable, whereas composite likelihood methods based on
pairwise observations also seem to work well in the univariate case, as ongoing work not
reported here, reveals.

In the following, we shall assume that we have decided on a parametric model for the
multivariate trawl process with trawl functions $d^{(i)}$.

**Step 1**: We can use the time series for each component to estimate the marginal pa-
rameters. We will estimate the parameter of $d^{(i)}$ in Step a) and the ones of $L^{(i)}$ in Step
b).

a) Recall that for each component we have the following representation for the autocor-
relation function:

$$ r_{ii}(h) = \text{Cor} \left( L^{(i)}(A_t^{(i)}), L^{(i)}(A_{t+h}) \right) = \frac{\text{Leb}(A_t^{(i)} \cap A_{t+h}^{(i)})}{\text{Leb}(A_t^{(i)})} = \frac{R_{ii}(h)}{\text{Leb}(A_t^{(i)})}, \quad \text{for } i = 1, \ldots, n. $$

I.e. the autocorrelation function only depends on the parameters of the trawl function $d^{(i)}$. These parameters can hence be estimated by using the method of moments or generalised
method of moments (depending on the model specification) by matching the empirical and
the theoretical autocorrelation function. This will, in particular, provide us with an estimate
of $\text{Leb}(A_t^{(i)})$.

b) In a second step, we can then estimate the parameters determining the marginal distri-
bution of $L^{(i)}$, again using a method of moments, by using a sufficient number of cumulants
of the observed trawl process. Note that the cumulant function for an individual component
has the form

$$ C(\xi \dagger Y^{(i)}_t) = \text{Leb}(A_t^{(i)}) C(\xi \dagger L^{(i)}), \quad \text{for } i = 1, \ldots, n. \tag{10} $$

I.e. the cumulants of the trawl process can be easily derived. We denote by $\kappa_k$ the $k$th
cumulant for $k \in \mathbb{N}$. Then we have that

$$ \kappa_k(Y^{(i)}_t) = \text{Leb}(A_t^{(i)}) \kappa_k(L^{(i)}), \quad \text{for } i = 1, \ldots, n. $$

I.e. as long as the parameters are identified through the cumulants, we can estimated them
after having estimated the trawl parameters by setting

$$ \hat{\kappa}_k(L^{(i)}) = \frac{\kappa_k(Y^{(i)}_t)}{\text{Leb}(A_t^{(i)})}, \quad \text{for } i = 1, \ldots, n, $$

where $k^e_k$ stands for the corresponding empirical $k$th cumulant. We then just need to solve
the equations for the corresponding parameters. If a direct matching does not work, then one
can use the generalised method of moments.

**Step 2**: After the marginal parameters have been identified, we turn to estimating the
parameters describing the dependence. We note that as soon as the trawl parameters have
been estimated, the corresponding autocorrelators can be computed. I.e. we then obtain
estimates $\hat{R}_{ij}(h) = \text{Leb}(A_t^{(i)} \cap A_{t+h}^{(j)})$. Then we get that

$$ \kappa_{i,j} = \frac{\rho_{i,j}^e(h)}{\hat{R}_{ij}(h)}, \quad \text{for } i \neq j, i, j \in \{1, \ldots, n\}, $$

18
where $\rho_{ij}(h)$ denote the empirical autocovariance function between the $i$th and $j$th component evaluated at lag $h$. In fact, it will be sufficient to set $h = 0$ when we estimate the parameters $\kappa_{ij}$. Note here that while we can clearly estimate the pairwise covariance parameter $\kappa_{ij}$ using this method, depending on the parametric model chosen, there might be more than one parameter describing the dependence structure. As such, not all parameters might be identified through this procedure in which case additional moment conditions need to be considered. However, since this scenario did not arise in the model specifications we studied in relation to our empirical work, we shall refer this aspect to future research.

Let us briefly comment on the validity of this estimation method: According to Fuchs & Stelzer (2013) multivariate mixed moving-average processes are mixing as long as they exist. This result implies that our stationary multivariate trawl processes are mixing and hence also weakly mixing and ergodic. Hence we can deduce that moment-based estimation methods are consistent, see e.g. Mátyás (1999).

In order to construct confidence bounds for the various parameters, we proceed by implementing a parametric bootstrap procedure, where we plug in the estimated parameters into the model specification, simulate from the model as described in the previous section, and then report the corresponding 95% confidence bounds.

4.2.1 Simulation study

In order to check how well the inference procedure works in finite samples, we conduct a Monte Carlo study, where we choose the model setting which describes our empirical data well, see Section 5. To this end, we simulate samples consisting of 3960 observations each from the bivariate version of the negative binomial model with common factor as described in Example 12. The distribution of the corresponding Lévy seed is determined by three parameters: $\alpha_1, \alpha_2$ and $\kappa$. In addition, we choose an exponential trawl function for both components, which are parametrised by $\lambda_1$ and $\lambda_2$, respectively. The parameters are set to their empirical counterparts, see Table 2 below. Some of the technical details regarding the simulation study can be found in Section 13 in the Appendix.

We draw 5000 samples from the model using the simulation algorithm described above and estimate the parameters for each sample using the method of moments. In Figure 1, we present the boxplots for the estimates for each of the five parameters. The true values are highlighted by a vertical red line. We observe that all five estimates center around the true values.

5 Empirical illustration

In this section, we apply our new modelling framework to high frequency financial data. More precisely, we study limit order book data from the database LOBSTER.

We have downloaded the limit order book data for Bank of America (ticker: BAC) for one day (21st April 2016). We are interested in investigating the joint behaviour between the time series of the number of newly submitted limit orders versus the number of fully deleted limit orders. Note that the trading day starts at 9:30am and ends at 16:00. For our analysis, we discard the first and last 30 minutes of the data which typically have a peculiar

1LOBSTER: Limit Order Book System - The Efficient Reconstructor at Humboldt Universität zu Berlin, Germany. http://LOBSTER.wiwi.hu-berlin.de
Figure 1: Boxplots of the five parameter estimates from a bivariate trawl model with exponential trawl function and negative binomial Lévy seed, see Example 12. The results are based on 5000 Monte Carlo runs, where each sample contains 3960 observations. The true values are indicated by a red vertical line.

As such we analyse data for a time period of 5.5 hours. We split this time period into intervals of length five seconds, resulting in 3960 intervals. In each interval we count the number of newly submitted limit orders and the ones which have been fully deleted.

|                         | Min | 1st Quartile | Median | Mean    | 3rd Quartile | Max  |
|-------------------------|-----|--------------|--------|---------|--------------|------|
| No. of new submissions  | 0   | 7            | 13     | 34.06   | 28           | 646  |
| No. of full deletions   | 0   | 5.75         | 12     | 29.13   | 27.25        | 567  |

Table 1: Summary statistics of the BAC data from 21st April 2016 based on intervals of length five seconds. Also, we find that the correlation between the two time series is equal to 0.984.

The summary statistics of these two count series are provided in Table 1. Moreover, Figure 2 depicts the corresponding time series plot, which also includes a picture of the difference of the two time series (in the middle), and Figure 3 presents histograms of the joint and the marginal distribution of the data.

We observe that there is a very strong correlation and co-movement between the two time series, which confirms the well-known fact that, for highly traded stocks such as BAC, the majority of newly submitted limit orders gets deleted rather than executed.

Since the empirical autocorrelation function decays rather quickly for both time series, see Figure 4, we fit an exponential trawl function in both cases and get a good fit. Based on the estimated trawl parameters, we compute \( \text{Leb}(A^{(1)}), \text{Leb}(A^{(2)}), \) and \( \text{Leb}(A^{(1)} \cap A^{(2)}) \).

Next, we estimate the parameters \( \alpha_1, \alpha_2 \) from the marginal law and finally infer \( \kappa \) from the
Figure 2: Time series plots of the BAC data from 21st April 2016 based on intervals of length five seconds. Black (top): number of submitted orders; light grey (middle): number of submitted - fully deleted orders; dark grey (bottom): number of fully deleted orders.

Figure 3: Histograms depicting the joint distribution and the marginal distributions of the new submissions (top) and the full cancellations (right).
empirical cross-covariance. All parameter estimates are summarised in Table 2. In addition, we provide the corresponding 95% confidence intervals, which are based on a parametric bootstrap, where we simulated 5000 samples from the estimated model using the estimated parameters as the plug-in values.

|          | $\lambda_1$  | $\lambda_2$  | $\alpha_1$  | $\alpha_2$  | $\kappa$  |
|----------|---------------|---------------|--------------|--------------|------------|
| Estimates| 2.157         | 1.919         | 95.161       | 73.055       | 0.812      |
| CB       | (1.771, 2.673)| (1.597, 2.322)| (85.321, 106.147)| (65.797, 81.222)| (0.741, 0.885)|

Table 2: Estimated parameters and estimates of the 95% confidence bounds (CB) from the moment-based estimates. The CB estimates have been computed using a model-based bootstrap, where 5,000 bootstrapped samples were drawn.

In addition to checking the goodness-of-fit of the trawl function, see Figure 4 we also need to assess whether the parametric model for the bivariate Lévy seed is appropriate. To this end, we first check the marginal fit, which corresponds to a univariate negative binomial law for each component. Figure 5 shows the empirical and the estimated probability densities and the corresponding quantile-quantile plots. While the fit seems to be acceptable overall, we note that the fit appears to be better for the time series of the cancelled orders, where the quantile-quantile plot is closer to a straight line, than in the case of the newly submitted orders, where we observe a mildly wiggly line. Finally, we investigate the goodness-of-fit of the joint law. For this, we draw the bivariate law from one of our bootstrap samples and the corresponding univariate laws, see Figure 6. We observe that the histogram of the simulated joint law resembles the one from the empirical data well, cf. Figure 4. Also a visual inspection of the simulated sample paths, see Figure 7 for one example, shows that the empirical data and the simulated data have indeed very similar features, which supports our hypothesis that a bivariate trawl process can describe the number of order submissions and cancellations in a limit order book well.
Figure 5: Empirical and fitted densities and quantile-quantile plots of the negative binomial marginal law for the new submissions (top) and the full cancellations (bottom).

Figure 6: Histograms depicting the joint distribution and the marginal distributions of one path of the simulated bivariate time series in our bootstrap procedure.
Figure 7: Time series plots of one simulated sample path. Black (top): first component; light grey (middle): number of first - second component; dark grey (bottom): - second component. This is the same path as the one used to generate Figure 6.

6 Conclusion

We propose a new modelling framework for multivariate time series of counts, which is based on so-called multivariate integer-valued trawl (MVIT) processes. Such processes are highly analytically tractable and enjoy useful properties, such as stationarity, infinitely divisibility, ergodicity and a mixing property. A variety of serial dependence patterns, including short and long memory, as well as all discrete infinitely divisible marginal distributions can be achieved within this novel framework. In this article, we focused in particular on various specifications of a multivariate infinitely divisible negative binomial distribution, since its univariate counterpart has been widely used in empirical work. Moreover, since the MVIT process is defined in continuous time, it can be applied to non-equidistant and asynchronous data, which increases its broad applicability. Further contributions of this article include a simulation algorithm for MVIT processes and a suitable inference procedure which is based on the two-stage equation-by-equation approach, where the parameters describing the univariate marginal distributions are estimated in the first step, followed by the estimation of the dependence parameters in the second step. A simulation study confirms the effectiveness of this inference method in finite samples. The estimation itself is based on the generalised method of moments and suitable confidence bounds are obtained through a parametric bootstrap procedure. In an empirical illustration, a bivariate version of an MVIT process has been used to successfully describe the relationship between the number of order submissions and cancellations in a limit order book.
A Proofs

Proof of Proposition 1. Using the properties of the Lévy basis, we immediately obtain that

$$
E(\exp(i\theta^T Y_t)) = \exp \left( \int_{\mathbb{R}^n \times [0,1] \times \mathbb{R}} \exp \left( i \sum_{j=1}^{n} \theta_j I_{A(j)}(x, s - t) y_j \right) - 1 \right) \nu(dy)dxds.
$$

The expression for the characteristic function can be further simplified by using a partition $S = \{S_1, \ldots, S_{2^n - 1}\}$ of $A^{\cup,n} = \bigcup_{i=1}^{n} A^{(i)}$, see Noven et al. (2015). More precisely, we have that

$$
A^{\cup,n} = \bigcup_{k=1}^{n} \bigcup_{1 \leq i_1, \ldots, i_k \leq n: \atop i_v \neq i_{v'}, \text{ for } v \neq v'}^{n} \left( \bigcup_{l=1}^{k} A^{(i_l)} \right) \setminus \bigcup_{1 \leq j \leq n, \atop j \not\in \{i_1, \ldots, i_k\}} A^{(j)}.
$$

Note that

$$
\theta^T X_t = \sum_{j=1}^{n} \theta_j L^{(j)}(A_t^{(j)}) = \sum_{j=1}^{n} \theta_j \sum_{k: S_k \subset A^{(j)}} L^{(j)}(S_k) = \sum_{k=1}^{2^n - 1} \sum_{1 \leq j \leq n: \atop A^{(j)} \supset S_k} \theta_j L^{(j)}(S_k).
$$

Finally, combining (12) with the representation (11) and using the fact that a Lévy basis is independently scattered, we obtain the result.

Proof of Proposition 2. The joint law is given by

$$
P(X_1 = x_1, \ldots, X_n = x_n) = \int_{[0,\infty)^{n+1}} P(X_1 = x_1, \ldots, X_n = x_n | U = u, V_1 = v_1, \ldots, V_n = v_n) \times f_U(u) f_{V_1}(v_1) \cdots f_{V_n}(v_n) du dv_1 \cdots dv_n
$$

$$
= \int_{[0,\infty)^{n+1}} \prod_{i=1}^{n} e^{-\alpha_i u + v_i} (\alpha_i u + v_i)^{x_i} / x_i! f_U(u) f_{V_i}(v_i) du dv_i
$$

$$
= \int_{[0,\infty)^{n+1}} f_U(u) \prod_{i=1}^{n} e^{-\alpha_i u + v_i} 1 / x_i! \sum_{j=1}^{x_i} \left( \begin{array}{c} x_i \\ j_i \end{array} \right) \alpha_j \cdot v_i^{x_i-j_i} f_{V_i}(v_i) du dv_i
$$

$$
= \frac{1}{x_1! \cdots x_n!} \sum_{j_1=0}^{x_1} \cdots \sum_{j_n=0}^{x_n} \left( \begin{array}{c} x_1 \alpha_1^j \cdots \alpha_n^j \cdot e^{Uj_1+\cdots+j_n} e^{-(\alpha_1+\cdots+\alpha_n)U} \end{array} \right)
$$

$$
\cdots \prod_{k=1}^{n} E(V_k^{x_k-j_k} e^{-V_k}).
$$

Proof of Proposition 3. Let $M_U$ and $M_{V_i}$ denote the moment generating functions of $U$ and $V_i$, respectively. According to Barndorff-Nielsen et al. (1992, equation (5.1)), the probability generating function of $(X_1, \ldots, X_n)$ is given by

$$
G(t_1, \ldots, t_n) = E(t_1^{X_1} \cdots t_n^{X_n}) = M_U \left( \sum_{i=1}^{n} \alpha_i (t_i - 1) \right) \prod_{i=1}^{n} M_{V_i}(t_i - 1).
$$
Hence, the corresponding Laplace transform for positive $\theta$ is given by

$$
\mathcal{L}(\theta_1, \ldots, \theta_n) = G(e^{-\theta_1}, \ldots, e^{-\theta_n}) = M_U \left( \sum_{i=1}^{n} \alpha_i (e^{-\theta_i} - 1) \right) \prod_{i=1}^{n} M_{V_i}(e^{-\theta_i} - 1). \quad (13)
$$

The aim is to find $v$ and $\mathcal{L}_C(\theta)$ by equating (7) and (13). Using the relation between the Laplace and the moment generating function, we deduce that

$$
\mathcal{L}(\theta_1, \ldots, \theta_n) = M_U \left( \sum_{i=1}^{n} \alpha_i (e^{-\theta_i} - 1) \right) \prod_{i=1}^{n} M_{V_i}(e^{-\theta_i} - 1)
= \mathcal{L}_U \left( \sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) \right) \prod_{i=1}^{n} \mathcal{L}_{V_i}(1 - e^{-\theta_i})
= \exp \left( \log \mathcal{L}_U \left( \sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) \right) + \sum_{i=1}^{n} \log \mathcal{L}_{V_i}(1 - e^{-\theta_i}) \right).
$$

We use the notation $\mathcal{R} = \log \mathcal{L}$ for the so-called kumulant function. Since $U$ is a subordinator without drift, we have that

$$
\mathcal{R}_U \left( \sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) \right) = \int_{\mathbb{R}} (e^{-\sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) x} - 1) \nu_U(dx)
= \int_{\mathbb{R}} (e^{-\sum_{i=1}^{n} \alpha_i x} - e^{-\sum_{i=1}^{n} \alpha_i x} + e^{\sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) x} - 1) \nu_U(dx)
= \int_{\mathbb{R}} (e^{-\sum_{i=1}^{n} \alpha_i x} - 1) \nu_U(dx) + \int_{\mathbb{R}} e^{-\sum_{i=1}^{n} \alpha_i x} (e^{\sum_{i=1}^{n} \alpha_i e^{-\theta_i} x} - 1) \nu_U(dx).
$$

Note that

$$
e^{\sum_{i=1}^{n} \alpha_i e^{-\theta_i} x} - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} x \right)^k = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k x^k.
$$

We set $\alpha := \sum_{i=1}^{n} \alpha_i$. Then

$$
\int_{\mathbb{R}} e^{-\sum_{i=1}^{n} \alpha_i x} \left( e^{\sum_{i=1}^{n} \alpha_i e^{-\theta_i} x} - 1 \right) \nu_U(dx) = \int_{\mathbb{R}} e^{-\alpha x} \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k x^k \nu_U(dx)
= \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right) \frac{1}{k!} \int_{\mathbb{R}} e^{-\alpha x} x^k \nu_U(dx). \quad \text{(14)}
$$

I.e.

$$
\mathcal{R}_U \left( \sum_{i=1}^{n} \alpha_i (1 - e^{-\theta_i}) \right) = \int_{\mathbb{R}} (e^{-\alpha x} - 1) \nu_U(dx) + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k^{(U)}
$$

26
\[
= \overline{K}_U(\alpha) + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k^{(U)}.
\]

Similarly,
\[
\sum_{i=1}^{n} K_{V_i}(1 - e^{-\theta_i}) = \sum_{i=1}^{n} \left( \overline{K}_{V_i}(1) + \sum_{k=1}^{\infty} e^{-\theta_i k} q_k^{(V_i)} \right),
\]
where \( q_k^{(V_i)} = \int_{\mathbb{R}} \frac{x^k}{k!} e^{-x} \nu_{V_i}(dx). \)

So, overall we have
\[
\overline{K}_X(\theta) = \log \mathcal{L}(\theta_1, \ldots, \theta_n)
= \left( \overline{K}_U(\alpha) + \sum_{i=1}^{n} \overline{K}_{V_i}(1) \right) + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k^{(U)} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} e^{-\theta_i k} q_k^{(V_i)}
\]
if and only if
\[
v = -\left( \overline{K}_U(\alpha) + \sum_{i=1}^{n} \overline{K}_{V_i}(1) \right),
\]

\[
\mathcal{L}_C(\theta) = \frac{1}{v} \left\{ \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \alpha_i e^{-\theta_i} \right)^k q_k^{(U)} + \sum_{i=1}^{n} \sum_{k=1}^{\infty} e^{-\theta_i k} q_k^{(V_i)} \right\}.
\]

**Proof of Proposition 6.** The requirement that \( \text{Leb}(A^{(i)}) < \infty \) implies that \( \text{Leb}(\{(x, s) : s \leq 0, 0 \leq x \leq d(s-t)\}) \rightarrow 0 \) as \( t \rightarrow \infty \). Since a Lévy basis is countably additive (in the sense that for any sequence \( A_n \downarrow \emptyset \) of Borel sets with bounded Lebesgue measure, \( L^{(i)}(A_n) \rightarrow 0 \) in probability as \( n \rightarrow \infty \), see Barndorff-Nielsen et al. (2011)), we can deduce that \( X^{(i)}_{0,t} \rightarrow 0 \) in probability as \( t \rightarrow \infty \).

---

**B Details regarding the simulation study**

In Section 4.2.1, we simulate from a bivariate negative binomial trawl process, where both components have an exponential trawl function and their joint law is given by the bivariate negative binomial distribution as described in Example 12. In the simulation of the trawl process, we work with the compound-Poisson-type representation (8) and specify the jump size distribution as the bivariate logarithmic series distribution (BLSD) as in Example 14.

**B.1 Simulating from the bivariate logarithmic series distribution**

First of all, we describe how we can generate random samples \( C = (C_1, C_2)^T \) from the BLSD with parameters \( p_1, p_2 \). The algorithm is based on the idea that we can simulate \( C_1 \) from the modified logarithmic series distribution (ModLSD) (with parameters \( \tilde{p}_1 = p_1/(1 - p_2) \) and \( \delta_1 = \log(1 - p_2)/\log(1 - p_1 - p_2) \)) in a first step, and then \( C_2 \) can be simulated from the conditional distribution, given \( C_1 \), see e.g. Kemp & Loukas (1978). We note here, that
if \( C_1 \equiv 0 \), then \( C_2|C_1 \) follows the logarithmic distribution (with parameter \( p_2 \)), and when \( C_1 > 0 \), then \( C_2|C_1 \) follows the negative binomial distribution with parameters \( C_1 \) and \( p_2 \), see e.g. [Kocherlakota & Kocherlakota (1990)]. We describe the simulation algorithm for the BLSD using pseudo code tailored to the \( R \) language. Throughout the section we use the abbreviation rv for random variable.

**Algorithm 6** (Simulation from the bivariate logarithmic series distribution).

1: library(VGAM) \( \triangleright \) Load the VGAM package in \( R \).

2: function Sim-BLSD\((N,p_1,p_2)\) \( \triangleright \) Calculate the parameters of the modified LSD.

3: \( \tilde{p}_1 \leftarrow p_1/(1 - p_2) \)

4: \( \delta_1 \leftarrow \log(1 - p_2)/\log(1 - p_1 - p_2) \)

5: \( L \leftarrow \text{rlog}(N,p_1) \) \( \triangleright \) Simulate \( N \) i.i.d. \( \text{Log}(p_1) \) rvs.

6: \( B \leftarrow \text{rbinom}(N,1,1 - \delta_1) \) \( \triangleright \) Simulate \( N \) i.i.d. \( \text{Bernoulli}(1 - \delta_1) \) rvs.

7: \( C_1 \leftarrow L \ast B \) \( \triangleright \) Generate \( N \) i.i.d. \( \text{ModLog}(\tilde{p}_1,\delta_1) \) rvs.

8: \( C_2 \leftarrow \text{numeric}(N) \)

9: for \( i \text{ in } 1:N \) do \( c_1 \leftarrow C_1[i] \)

10: if \( c_1 == 0 \) then

11: \( C_2[i] \leftarrow \text{rlog}(1,p_2) \) \( \triangleright \) Simulate a \( \text{Log}(p_2) \) rv.

12: end if

13: if \( c_1 > 0 \) then

14: \( C_2[i] \leftarrow \text{rnbinom}(1,\text{size }= c_1,\text{prob }= 1 - p_2) \) \( \triangleright \) Simulate a \( \text{NB}(c_1,p_2) \) rv.

15: end if

16: end for

17: \( C \leftarrow \text{cbind}(C_1,C_2) \) \( \triangleright \) Combine the component vectors to an \( N \times 2 \) matrix.

18: return \( C \)

19: end function

**B.2 Simulating the bivariate trawl process**

Next, we provide the pseudo code tailored to the \( R \) language which has been used to simulate the bivariate trawl process with exponential trawl function and bivariate negative binomial law (as in Example 12). Here we are using the same notation as in the general description of Algorithm 5. In addition, we denote by \( bi \) the length of the burn-in period. I.e. we will simulate the process over the time interval \([0,t]\) for \( t = T + bi \) and then remove the initial burn-in period, i.e. we return the paths over the interval \((bi,bi + T]\).

**Algorithm 7** (Simulation from the bivariate trawl process).

1: library(VGAM) \( \triangleright \) Load the VGAM \( R \) package and the function Sim-BLSD defined above.

2: function Expfct\((x,\lambda)\) \( \triangleright \) Choose an exponential trawl function.

3: return \( \exp(\lambda \ast x) \)

4: end function

5: procedure Sim-Trawl\((\Delta,T,bi,\lambda_1,\lambda_2,\alpha_1,\alpha_2,\kappa)\) \( \triangleright \) Intensity of the driving Poisson process.

6: \( v \leftarrow \kappa \ast \log(1 + \alpha_1 + \alpha_2) \)

7: \( p_1 \leftarrow \alpha_1/(\alpha_1 + \alpha_2 + 1); p_2 \leftarrow \alpha_2/(\alpha_1 + \alpha_2 + 1) \) \( \triangleright \) Parameters in the BLSD

8: \( N_t \leftarrow \text{rpois}(1,v \ast t) \) \( \triangleright \) Draw the number of jumps in \([0,t]\) from \( \text{Pois}(vt) \).
9:  \( \tau \leftarrow \text{sort}\left(\text{runif}(N_t, \min = 0, \max = t)\right) \) \quad \triangleright \quad \text{simulate the } N_t \text{ jump times from the ordered uniform distribution on } [0, t].
10: \( h \leftarrow \text{runif}(N_t, \min = 0, \max = 1) \) \quad \triangleright \quad \text{Simulate the } N_t \text{ jump heights of the abstract spatial parameter of the Poisson basis from the uniform distribution on } [0, 1].
11: \( m \leftarrow \text{Sim}\_\text{BLSD}(N_t, p_1, p_2) \) \quad \triangleright \quad \text{Draw the jump marks from the BLSD}
12: \( C_1 \leftarrow m[1:1]; C_2 \leftarrow m[2] \) \quad \triangleright \quad \text{Assign the jump marks to } C_1 \text{ and } C_2.
13: \triangleright \quad \text{Determine the number of jumps up to each grid point } k\Delta \text{ and store them in the vector } V.
14: \( V \leftarrow \text{vector}(\text{mode} = \text{"numeric"}, \text{length} = \text{floor}(t/\Delta)) \)
15: \( c \leftarrow \text{table}(\text{cut}(\text{jumptimes}, \text{seq}(0, t, 1), \text{include.lowest} = \text{TRUE})) \)
16: \( V[1] < -\text{as.integer}(c[1]) \)
17: \( \text{for } k \text{ in } 2 : \text{floor}(t/\Delta) \) \text{ do}
18: \( V[k] \leftarrow V[k - 1] + \text{as.integer}(c[k]) \)
19: \text{end for}
20: \text{for } i \text{ in } 1 : 2 \quad \triangleright \quad \text{Simulate the } i\text{th trawl process}
21: \( TP_i \leftarrow \text{vector}(\text{mode} = \text{"numeric"}, \text{length} = \text{floor}(t/\Delta)) \)
22: \( \text{for } k \text{ in } 1 : \text{floor}(t/\Delta) \) \text{ do}
23: \( N_{k\Delta} \leftarrow V[k] \) \quad \triangleright \quad \text{Number of jumps until time } k\Delta.
24: \( \text{if } N_{k\Delta} > 0 \text{ then} \)
25: \( d \leftarrow k \cdot \Delta - \tau[1:N_{k\Delta}] \) \quad \triangleright \quad \text{Compute the time differences between } k\Delta \text{ and each jump time up to } k\Delta.
26: \( \text{cond}_i \leftarrow 1 - \text{ceiling}(h[1:N_{k\Delta}] - \text{Expfct}(-d, \lambda_i)) \) \quad \triangleright \quad \text{Check which points are in the trawl.}
27: \( TP_i[k] \leftarrow \text{sum}(\text{cond}_i \ast C_i[1:N_{k\Delta}]) \) \quad \triangleright \quad \text{Sum up the marks in the trawl.}
28: \text{end if}
29: \text{end for}
30: \text{end for}
31: \( b_1 \leftarrow bi/\Delta, b_2 = bi/\Delta + T/\Delta \)
32: \text{for } i \text{ in } 1 : 2 \text{ do}
33: \( \text{TrawlProcess}_i \leftarrow TP_i[(b_1 + 1) : b_2] \) \quad \triangleright \quad \text{Cut off burn-in period.}
34: \text{end for}
35: \text{end procedure}

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