On the Zero Defect Conjecture

Sébastien Labbé∗1, Edita Pelantová2, and Štěpán Starosta†3

1 Université de Liège, Bât. B37 Institut de Mathématiques, Grande Traverse 12, 4000 Liège, Belgium
2 Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Czech Republic
3 Faculty of Information Technology, Czech Technical University in Prague, Czech Republic

Abstract

Brlek et al. conjectured in 2008 that any fixed point of a primitive morphism with finite palindromic defect is either periodic or its palindromic defect is zero. Bucci and Vaslet disproved this conjecture in 2012 by a counterexample over ternary alphabet. We prove that the conjecture is valid on binary alphabet. We also describe a class of morphisms over multiliteral alphabet for which the conjecture still holds.

Keywords: palindromic defect, marked morphism, special factor
2000 MSC: 68R15, 37B10

1 Introduction

Palindromes — words read the same from the left as from the right — are a favorite pun in various languages. For instance, the words ressasser, ťahať, and šílíš are palindromic words in the first languages of the authors of this paper. The reason for a study of palindromes in formal languages is not only to deepen the theory, but it has also applications.

The theoretical reasons include the fact that a Sturmian word, i.e., an infinite aperiodic word with the least factor complexity, can be characterized using the number of palindromic factors of given length that occur in a word, see [10]. The application motives include the study of the spectra of discrete Schrödinger operators, see [12,13].

In [9], the authors provide an elementary observation that a finite word of length $n$ cannot contain more than $n + 1$ (distinct) palindromic factors, including the empty word as a palindromic factor. We illustrate this on the following 2 examples of words of length 9:

\[ w^{(1)} = 010010100 \quad \text{and} \quad w^{(2)} = 011010011. \]

The word $w^{(1)}$ is a prefix of the famous Fibonacci word and $w^{(2)}$ is a prefix of (also famous) Thue–Morse word. There are 10 palindromic factors of $w^{(1)}$: 0, 1, 00, 010, 101, 1001, 01010, 010010, 0010100, and the empty word. The word $w^{(2)}$ contains only 9 palindromes: 0, 1, 11, 0110, 101, 010, 00, 1001, and the empty word.

∗Electronic address: slabbe@ulg.ac.be
†Electronic address: stepan.starosta@fit.cvut.cz
The existence of the upper bound on the number of distinct palindromic factors lead to the definition of palindromic defect of a finite word \( w \), see [5], as the value

\[
D(w) = n + 1 - \text{the number of palindromic factors of } w
\]

with \( n \) being the length of \( w \). Our examples satisfy \( D(w^{(1)}) = 0 \), i.e., the upper bound is attained, and \( D(w^{(2)}) = 1 \). The notion of palindromic defect can be naturally extended to infinite words. For an infinite word \( u \) we set

\[
D(u) = \sup \{ D(w) : w \text{ is a factor of } u \}.
\]

In this paper, we deal with infinite words that are generated by a primitive morphism of a free monoid \( A^* \) with \( A \) being a finite alphabet. A morphism \( \varphi \) is completely determined by the images of all letters \( a \in A \): \( a \mapsto \varphi(a) \in A^* \). A morphism is primitive if there exists a power \( k \) such that any letter \( b \in A \) appears in the word \( \varphi^k(a) \) for any letter \( b \in A \).

The two mentioned infinite words can be generated using a primitive morphism. Consider the morphism \( \varphi_F \) over \( \{0, 1\}^* \) determined by \( 0 \mapsto 01 \) and \( 1 \mapsto 0 \). By repeated application of \( \varphi_F \), starting from 0, we obtain

\[
0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto 01001010 \ldots
\]

Since \( \varphi_F^n(0) \) is a prefix of \( \varphi_F^{n+1}(0) \) for all \( n \in \mathbb{N} \), there exists an infinite word \( u_F \), called the Fibonacci word, such that \( \varphi_F^n(0) \) is its prefix for all \( n \). Consider a natural extension of \( \varphi_F \) to infinite words, we obtain that \( u_F \) is a fixed point of \( u_F \) since

\[
u_F = \varphi_F(u_F) = \varphi_F(u_0u_1u_2\ldots) = \varphi_F(u_0)\varphi_F(u_1)\varphi(u_2)\ldots
\]

where \( u_i \in \{0, 1\} \).

Similarly, let \( \varphi_{TM} \) be a morphism determined by \( 0 \mapsto 01 \) and \( 1 \mapsto 10 \). By repeated application of \( \varphi_{TM} \), starting again from 0, we obtain

\[
0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto 0110100110010110 \ldots
\]

The infinite word having \( \varphi_{TM}^n(0) \) as a prefix for each \( n \) is the Thue–Morse word, sometimes also called Prouhet–Thue–Morse word.

The present article focuses on palindromic defect of infinite words which are fixed points of primitive morphisms. In order for the palindromic defect of such an infinite word to be finite, the word must contain an infinite number of palindromic factors. This property is satisfied by the two mentioned words \( u_F \) and \( u_{TM} \). However, for their palindromic defect, we have \( D(u_F) = 0 \), whilst \( D(u_{TM}) = +\infty \).

There exist fixed points \( u \) of primitive morphisms with \( 0 < D(u) < +\infty \), but on a two-letter alphabet, only ultimately periodic words are known. In [5], examples of such words are given by Brlek, Hamel, Nivat and Reutenauer as follows: for any \( k \in \mathbb{Z}, k \geq 2 \) denote by \( z \) the finite word

\[
z = 01^k01^{k-1}001^{k-1}01^k0.
\]

Then the infinite periodic word \( z^\omega \) has palindromic defect \( k \). Of course, the periodic word \( z^\omega \) is fixed by the primitive morphism \( 0 \mapsto z, 1 \mapsto z \). In [4], the authors stated the following conjecture:

**Conjecture** (Zero Defect Conjecture). If \( u \) is a fixed point of a primitive morphism such that \( D(u) < +\infty \), then \( u \) is periodic or \( D(u) = 0 \).
In 2012, Bucci and Vaslet [7] found a counterexample to this conjecture on a ternary alphabet. They showed that the fixed point of the primitive morphism determined by

\[
a \mapsto aabcacba, \quad b \mapsto aa, \quad c \mapsto a
\]

has finite positive palindromic defect and is not periodic.

In this article, we show that the conjecture is valid on a binary alphabet. Then we generalize the method used for morphisms on a binary alphabet to marked morphisms on a multiliteral alphabet. The main result of the article is the following theorem.

**Theorem 1.** Let \( \varphi \) be a primitive marked morphism and let \( u \) be its fixed point with finite defect. If all complete return words of all letters in \( u \) are palindromes or there exists a conjugate of \( \varphi \) distinct from \( \varphi \) itself, then \( D(u) = 0 \).

Observe that in the case of primitive marked morphisms, as it was noted in [15, Cor. 30, Cor. 32], there is no ultimately periodic infinite word \( u \) fixed point of a primitive marked morphisms such that \( 0 < D(u) < \infty \).

The main ingredients for the presented proofs of Theorem 1 and Theorem 14 are the following:

1. description of bilateral multiplicities of factors in a word with finite palindromic defect ([1]),
2. description of the form of marked morphisms with their fixed points containing infinitely many palindromic factors ([15]).
3. observation that non-zero palindromic defect of a word implies an existence of a factor with a specific property, see Lemma 13 for the binary case and Theorem 23 for the multiliteral case.

The paper is organized as follows: First we recall notions from combinatorics on words and we list known results that we use in the sequel. In Section 3, the properties of marked morphisms are studied. The validity of the Zero Defect Conjecture on binary alphabet is demonstrated in Section 4 (Theorem 14). In Section 5, we introduce the notion of graph of a factor and we interpret bilateral multiplicity of factors in the language of the graph theory. Section 6 links the palindromic defect to the properties of these graphs. Section 7 contains the proof of Theorem 1.

## 2 Preliminaries

### 2.1 Language of an infinite word

Consider an infinite word \( u = (u_n)_{n \in \mathbb{N}} \) over the alphabet \( A \). An index \( i \in \mathbb{N} \) is an occurrence of a factor \( w = w_0w_1 \cdots w_{n-1} \) of \( u \) if \( u_iu_{i+1} \cdots u_{i+n-1} = w \), in other words \( w \) is a prefix of the infinite word \( u_iu_{i+1}u_{i+2} \cdots \). The set of all factors of \( u \) is referred to as the **language** of \( u \) and denoted \( L(u) \). A word \( u \) is called recurrent if any factor \( w \in L(u) \) has infinitely many occurrences. If \( i < j \) are two consecutive occurrences of the factors \( w \), then the factor \( u_iu_{i+i} \cdots u_ju_{j+1} \cdots u_{j+n-1} \) is the **complete return word** to \( w \) in \( u \). If any factor of a recurrent word \( u \) has only finitely many complete return words, then \( u \) is called **uniformly recurrent**.

**Reversal** of a finite word \( w = w_0w_1 \cdots , w_{n-1} \) is the word \( \tilde{w} = w_{n-1}w_{n-2} \cdots w_0 \). A word \( w \) is a **palindrome** if \( w = \tilde{w} \). The language of \( u \) is said to be **closed under reversal** if \( w \in L(u) \) implies...
\( \tilde{w} \in L(u) \); the language of \( u \) is said to be palindromic if \( L(u) \) contains infinitely many palindromes. If a uniformly recurrent word \( u \) is palindromic, then its language is closed under reversal.

A letter \( b \in \mathcal{A} \) is called the right (resp. left) extension of \( w \) in \( L(u) \) if \( wb \) (resp. \( bw \)) belongs to \( L(u) \). In a recurrent word \( u \) any factor has at least one right and at least one left extension. A factor \( w \) which is simultaneously left and right special is bispecial. To describe one-sided and both-sided extensions of a factor \( w \) we put

\[
E^+(w) = \{ b \in \mathcal{A} : wb \in L(u) \}, \quad E^-(w) = \{ a \in \mathcal{A} : aw \in L(u) \},
\]

and \( E(w) = \{ (a, b) \in \mathcal{A}^2 : awb \in L(u) \} \).

The bilateral multiplicity \( m(w) \) of a factor \( w \in L(u) \) is defined as

\[
m(w) = \#E(w) - \#E^+(w) - \#E^-(w) + 1
\]

(see [8] for a recent reference on factor complexity). A bispecial factor \( w \in L(u) \) is said to be strong if \( m(w) > 0 \), weak if \( m(w) < 0 \) and neutral if \( m(w) = 0 \).

### 2.2 Palindromic defect

As shown in [9] finite words with zero defect can be characterized using palindromic suffixes. More specifically, a word \( w = w_0w_1 \cdots w_{n-1} \) has defect 0 if and only if for any \( i = 0, 1, \ldots, n - 1 \) the longest palindromic suffix of \( w_0w_1 \cdots w_i \) is unioccurrent in \( w \). To illustrate this important property, consider the words

\[
w^{(1)} = 010010100 \quad \text{and} \quad w^{(2)} = 011010011.
\]

mentioned in Introduction. The longest palindromic suffix of \( w^{(1)} \) is 0010100 and it is unioccurrent in \( w^{(1)} \), whereas the longest palindromic suffix of \( w^{(2)} \) is 11 and occurs in \( w^{(2)} \) twice. The index \( i \) for which the longest palindromic suffix is not unioccurrent is called a lacuna and the number of lacunas equals the palindromic defect of \( w \).

Since the number of palindromes in \( w \) and in its reverse \( \tilde{w} \) is the same, we have \( D(w) = D(\tilde{w}) \). Therefore, instead of the longest palindromic suffix one can consider the longest palindromic prefix as well.

The complete return words were applied in [11] to characterize infinite words with zero defect.

**Theorem 2 ([11]).** \( D(u) = 0 \) if and only if for all palindromes \( q \in L(u) \) all complete return words to \( q \) in \( u \) are palindromes.

Before stating a generalization of the previous result we need a new notion.

**Definition 3.** Let \( u \in \mathcal{A}^\mathbb{N} \) and \( q \in L(u) \). A word \( c = c_1c_2 \cdots c_n \) is a complete mirror return to \( q \) in \( u \) if

1. neither \( q \) nor \( \tilde{q} \) is a factor of \( c_2 \cdots c_{n-1} \), and
2. either \( q \) is a prefix of \( c \) and \( \tilde{q} \) is suffix of \( c \), or \( \tilde{q} \) is a prefix of \( c \) and \( q \) is a suffix of \( c \).

Note that \( c \) is a complete mirror return to \( q \) if and only if it is a complete mirror return to \( \tilde{q} \).
Example 4. We illustrate the notion of complete mirror return word on the Fibonacci word $u_F$. The factors $r_1$, $r_2$ and $r_3$ are complete mirror returns to $q_1 = 0101$, $q_2 = 001$ and $q_3 = 00$ respectively.

$$u_F = 010\overbrace{010100101100101001010010100100}^{r_1}0101001010010010100\cdots$$

Note that if $q = \tilde{q}$, then the complete mirror return words of $q$ and $\tilde{q}$ coincide with complete return words of $q$.

Using the notion of complete mirror return word we can reformulate Proposition 2.3 from [6].

**Proposition 5 ([6]).** Let $u \in \mathcal{A}^\mathbb{N}$. Then $D(u) = 0$ if and only if, for each factor $q \in \mathcal{L}(u)$ any complete mirror return word to $q$ in $u$ is a palindrome.

A generalization of the previous statement to finite defect was given in [2].

**Theorem 6 ([2]).** Let $u \in \mathcal{A}^\mathbb{N}$ have its language closed under reversal. $D(u) < +\infty$ if and only if there exists a positive integer $K$ such that for every bispecial factor $q$ of length at least $K$ the occurrences of $q$ and $\tilde{q}$ alternate and every complete mirror return to $q$ in $u$ is a palindrome.

The next important corollary of the previous theorem was proved in [1]. It uses the both-sided symmetric extensions of a factor that we note

$$E^\pm(w) = \{a \in \mathcal{A}: awa \in \mathcal{L}(u)\}.$$

**Theorem 7 ([1]).** Let $u \in \mathcal{A}^\mathbb{N}$ have its language closed under reversal. If $D(u) < +\infty$, then there exists an integer $K$ such that each bispecial factor $q \in \mathcal{L}(u)$ with $|q| \geq K$ satisfies

$$m(q) = \begin{cases} 0 & \text{if } q \neq \tilde{q}, \\ #E^\pm(q) - 1 & \text{if } q = \tilde{q}. \end{cases}$$

2.3 Morphisms

In this section we concentrate on primitive morphisms. For a morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ consider the maps $\text{Fst}(\varphi), \text{Lst}(\varphi) : \mathcal{A} \to \mathcal{A}$ defined by the formula

$$\text{Fst}(\varphi)(a) = \text{the first letter of } \varphi(a) \quad \text{and} \quad \text{Lst}(\varphi)(a) = \text{the last letter of } \varphi(a)$$

for all $a \in \mathcal{A}$. A morphism $\varphi$ may have more fixed points, see for example the Thue–Morse morphism. The number of fixed points of a primitive morphism $\varphi$ is the number of letters for which $\text{Fst}(\varphi)(a) = a$. It is easy to see that the languages of all fixed points of a primitive morphism coincide and therefore all its fixed points have the same defect.

Recall from Lothaire [17] (Section 2.3.4) that a morphism $\psi$ is a left conjugate of $\varphi$, or that $\varphi$ is a right conjugate of $\psi$, denoted $\psi \triangleright \varphi$, if there exists $w \in \mathcal{A}^*$ such that

$$\varphi(x)w = w\psi(x), \quad \text{for all words } x \in \mathcal{A}^*, \quad (1)$$

or equivalently that $\varphi(a)w = w\psi(a)$, for all letters $a \in \mathcal{A}$. We say that the word $w$ is the conjugate word of the relation $\psi \triangleright \varphi$. If, moreover, the map $\text{Fst}(\psi)$ is not constant, then $\psi$ is the leftmost conjugate of $\varphi$. Analogously, if $\text{Lst}(\varphi)$ is not constant, then $\varphi$ is the rightmost conjugate of $\psi$. 

5
Example 8. Let
\[ \varphi : a \mapsto abab \quad \text{and} \quad \psi : a \mapsto baba . \]
We have \( \psi \vartriangleright \varphi \) and the conjugate word of the relation is \( w = a \). The leftmost conjugate of \( \varphi \) (and of \( \psi \)) is the morphism
\[ a \mapsto abab \quad \text{and} \quad b \mapsto bab . \]

If \( \varphi \) is a primitive morphism, then any of its (left or right) conjugate is primitive as well and the languages of their fixed points coincide.

A morphism \( \varphi : \mathcal{A}^* \rightarrow \mathcal{A}^* \) is cyclic \([16]\) if there exists a word \( w \in \mathcal{A}^* \) such that \( \varphi(a) \in w^* \) for all \( a \in \mathcal{A} \). Otherwise, we say that \( \varphi \) is acyclic. If \( \varphi \) is cyclic, then the fixed point of \( \varphi \) is \( \ldots w w w \ldots \) and is periodic. Remark that the converse does not hold. For example, the morphism determined by \( a \mapsto aba \) and \( b \mapsto bab \) is acyclic but its fixed point is periodic. Obviously, a morphism is cyclic if and only if it is conjugate to itself with a nonempty conjugate word. If a morphism is acyclic, it has a leftmost and a rightmost conjugate. See [15] for a proof of these statements on cyclic morphisms.

3 Marked morphisms

A morphism \( \varphi \) over binary alphabet has a trivial but important property: the leftmost conjugate of \( \varphi \) maps both letters to words with a distinct starting letter and analogously for the rightmost conjugate. The notion of marked morphism extends this important property to any alphabet.

Definition 9. Let \( \varphi \) be an acyclic morphism. We say that \( \varphi \) is marked if
\[ \text{Fst}(\varphi_L) \text{ and Lst}(\varphi_R) \text{ are injective} \]
and that \( \varphi \) is well-marked if
\[ \text{it is marked and if Fst}(\varphi_L) = \text{Lst}(\varphi_R) \]
where \( \varphi_L \) (resp. \( \varphi_R \)) is the leftmost (resp. rightmost) conjugate of \( \varphi \).

Remark 10. Any injective mapping \( f \) on a finite set is a permutation and thus there exists a power \( k \) such that \( f^k \) is the identity map. It implies that for any marked morphism \( \varphi \) there exists a power \( k \) such that \( \varphi^k \) is well-marked and moreover \( \text{Fst}(\varphi_L^k) = \text{Lst}(\varphi_R^k) = \text{Id} \).

Theorem 11 ([15]). Let \( \varphi \) be a primitive well-marked morphism and \( u \) its fixed point such that \( \mathcal{L}(u) \) contains infinitely many palindromes. The conjugacy word \( w \) of \( \varphi_L \vartriangleright \varphi_R \) is a palindrome and
\[ \overline{\varphi_R(a)} = \varphi_L(a) \quad \text{for all} \quad a \in \mathcal{A} . \]

We are interested in the defect of fixed points of primitive marked morphisms. We can consider, instead of the marked morphism \( \varphi \), a suitable power of \( \varphi \). Thus, without loss of generality we assume that \( \varphi \) is well marked and that \( \text{Fst}(\varphi_L) = \text{Lst}(\varphi_R) = \text{Id} \). For such \( \varphi \) with the conjugacy word \( w \) of \( \varphi_L \vartriangleright \varphi_R \) we define the mapping \( \Phi : \mathcal{A}^* \rightarrow \mathcal{A}^* \) by
\[ \Phi(u) = \varphi_R(u)w \quad \text{for all} \quad u \in \mathcal{A}^* . \]

As \( \varphi \) is primitive, each of its power and also each of its conjugate have the same language. Moreover, if we assume that \( \mathcal{L}(u) \) contains infinitely many palindromes we can deduce using [15, Lemma 15, Lemma 27, Prop. 28] remarkable properties of the mapping \( \Phi \).
Lemma 12 ([15]). Let $u \in A^\mathbb{N}$ and $u \in A^*$. If $\varphi$ satisfies assumptions of Theorem 11, we have

(I) If $u \in L(u)$, then $\Phi(u) \in L(u)$.

(II) $\overline{\Phi(u)} = \Phi(\overline{u})$.

(III) The word $u$ is a palindrome if and only if $\Phi(u)$ is a palindrome.

(IV) For any $a, b \in A$, $aub \in L(u)$ implies $a\Phi(u)b \in L(u)$.

(V) If $u$ is a palindromic (respectively non-palindromic) bispecial factor, then $\Phi(u)$ is a palindrome (respectively non-palindromic) bispecial factor.

Proof. (I) Let us find $v$ such that $uv \in L(u)$ with $|\varphi_L(v)| \geq w$. We have

$$\varphi_R(uv)w = \varphi_R(u)w\varphi_L(v).$$

Since $\varphi_R(uv) \in L(u)$, by erasing a suffix of length greater than or equal to $|w|$ from $\varphi_R(u)w\varphi_L(v)$ we obtain a factor of $L(u)$, in particular $\varphi_R(u)w \in L(u)$.

(II) Let $u = u_1u_2 \cdots u_n$ with $u_i \in A$. We obtain

$$\Phi(u) = w\varphi_L(u_1) \cdots \varphi_L(u_n) = \varphi_R(u_1) \cdots \varphi_R(u_n)w.$$

Using Theorem 11 we obtain

$$\overline{\Phi(u)} = \overline{w}\varphi_R(u_n) \cdots \varphi_R(u_1) = w\varphi_L(u_n) \cdots \varphi_L(u_1) = \Phi(\overline{u}).$$

(III) Let us note that any marked morphism is injective and thus $\Phi$ is injective as well. If $u$ is a palindrome, then $\overline{\Phi(u)} = \Phi(\overline{u}) = \Phi(u)$ from Item (II), therefore $\Phi(u)$ is a palindrome. Conversely, if $\Phi(u)$ is a palindrome, then $\overline{\Phi(u)} = \Phi(\overline{u}) = \Phi(u)$. As $\varphi_L$ is injective, $\Phi$ is injective and the claim follows.

(IV) Let $aub \in L(u)$. We have $\Phi(aub) \in L(u)$ and

$$\Phi(aub) = \varphi_R(a)\varphi_R(u)w\varphi_L(b) = \varphi_R(a)\Phi(u)\varphi_L(b).$$

By our assumption, $LST(\varphi_R)(c) = FST(\varphi_L)(c) = Id(c) = c$ for any $c \in A$. Thus, $a\Phi(u)b$ is a factor $\Phi(aub) \in L(u)$.

(V) The statement follows from the previous properties. \qed

4 Proof of Zero Defect Conjecture for binary alphabet

Binary alphabet offers less variability for the construction of a strange phenomenon. The recent counterexamples for two conjectures concerning palindromes in fixed points of primitive morphisms — namely the Buccí-Vaslet counterexample to the Zero Defect Conjecture and the Labbé counterexample to the HKS conjecture — use ternary alphabet. On binary alphabet, Bo Tan demonstrated the validity of the HKS conjecture, see [20]. Here we prove the Zero Defect Conjecture.

Lemma 13. Let $A = \{0, 1\}$ and $u \in A^\mathbb{N}$. If $L(u)$ is closed under reversal and $D(u) > 0$, then there exists a non-empty and non-palindromic factor $q \in L(u)$ such that $0q0, 01q1, 1q0, 1q1 \in L(u)$. 
Proof. By proposition 5, as \( D(u) > 0 \), there exist factors \( v \) and \( w \) in \( \mathcal{L}(u) \) such that \( v \) is a complete mirror return word to \( w \) and \( v \) is not a palindrome. Let us consider the shortest \( v \) with this property. For this fixed \( v \) we find the longest \( w \) such that \( v \) is a complete mirror return word to \( w \). It means that \( v \) has a prefix \( wa \) and a suffix \( \tilde{w}\tilde{a} \) where \( a, b \in \mathcal{A} \) and \( a \neq b \). Since on a binary alphabet every complete mirror return word to a letter is always a palindrome, we have \( |w| > 1 \).
Without loss of generality we can write \( w = 0q \) with \( q \neq \varepsilon \). Consequently \( v = 0u0 \). Clearly \( u \) has a prefix \( q \), the word \( u \) has a suffix \( \tilde{q} \) and \( u \) is not a palindrome. Our choice of \( v \) (to be the shortest non-palindromic mirror return to a factor) implies that \( u \) is not a complete mirror return word to \( q \) and thus \( q \) or \( \tilde{q} \) has another occurrence inside \( u \). Since \( v \) is a complete mirror return word to \( w = 0q \),

\[
0q \text{ and } \tilde{q}0 \text{ do not occur in } u. \quad (2)
\]

Let us suppose that \( q = \tilde{q} \). Consider the shortest prefix of \( u \) which has exactly two occurrences of \( q \). It is a palindrome. Since \( v \) has a prefix \( wa = 0qa \) the second occurrence of \( q \) is extended to the left as \( aq \). Analogously, consider the shortest suffix of \( u \) which contains exactly two occurrences of \( q \). It is a palindrome and thus the penultimate occurrence of \( q \) is extended to the right as \( qb \). This contradicts (2) as \( a \neq b \). We conclude that \( q \) is not a palindrome.

Now we show that occurrences of \( q \) and \( \tilde{q} \) in \( u \) alternate. Assume that there exists a factor of \( u \), denoted by \( u' \), such that \( q \) is a prefix and a suffix of \( u' \) and \( u' \) does not contain \( \tilde{q} \). It follows that the longest palindromic suffix of \( u' \) is not unicooccurent in \( u' \). Therefore \( D(u') \geq 1 \) (see Section 2.2), which contradicts the minimality of \( |v| \).

The minimality of \( |v| \) implies that all mirror return words to \( q \) in \( u \) are palindromes. Therefore, the leftmost occurrence of \( \tilde{q} \) in \( u \) is extended to the left as \( a\tilde{q} \) and the rightmost occurrence of \( q \) in \( u \) is extended to the right as \( qb \). From (2) we deduce that \( 0qa, a\tilde{q}1, 1qb, \) and \( b\tilde{q}0 \) belong to \( \mathcal{L}(u) \). The assumption that \( \mathcal{L}(u) \) is closed under reversal and the fact that \( a \neq b \) finish the proof. \( \square \)

**Theorem 14.** Let \( u \in \mathcal{A}^N \) be a fixed point of a primitive morphism \( \varphi \) over a binary alphabet \( \mathcal{A} \). If \( D(u) < +\infty \), then \( D(u) = 0 \) or \( u \) is periodic.

**Proof.** Assume the contrary, i.e., \( u \) is not periodic and \( D(u) > 0 \) and let \( \mathcal{A} = \{0, 1\} \).

Since \( D(u) \) is finite, \( \mathcal{L}(u) \) contains infinitely many palindromes. As \( \varphi \) is primitive, \( \mathcal{L}(u) \) is uniformly recurrent. Any uniformly recurrent word which contains infinitely many palindromes has its language closed under reversal. Due to Lemma 13 there exists a strong bispecial non-palindromic factor \( q \) with \( m(q) = 1 \).

Since \( u \) is not periodic, the morphism \( \varphi \) is acyclic. On the binary alphabet, it means that \( \varphi \) is well-marked. Applying repeatedly Lemma 12 (IV) and (V), we can construct an infinite sequence of strong bispecial factors \( q, \Phi(q), \Phi^2(q), \Phi^3(q), \ldots \), each with bilateral multiplicity 1. By Lemma 12 (III), all these bispecial factors are not palindromic. This contradicts Theorem 7. \( \square \)

## 5 The graph of a factor

To study the Zero Defect Conjecture on a multiliteral alphabet, we assign graphs to palindromic and non-palindromic bispecial factors. These graphs were used already in the proof of Theorem 7 in [1] where only words with zero defect are considered. These graphs are often used to represent extensions of a bispecial factor in the context of factor complexity and appear in a more general context in [3]. We use these graphs to demonstrate that the definition of bilateral multiplicity of bispecial factors is related to basic notions of graph theory which we use later in the proofs.
**Definition 15** ($\Gamma(w)$). Let $u \in A^n$. We assign to a factor $w \in L(u)$ the bipartite graph $\Gamma(w) = (V, U)$ whose vertices $V$ consist of the disjoint union of $E^-(w)$ and $E^+(w)$

$$V = (E^-(w) \times \{-1\}) \cup (E^+(w) \times \{+1\})$$

and whose edges $U$ are essentially the elements of $E(w)$:

$$U = \{\{(a,-1), (b,+1)\} : (a,b) \in E(w)\}.$$

**Lemma 16.** If $\Gamma(w)$ is connected, then $m(w) \geq 0$ and

- $m(w) > 0$ if and only if $\Gamma(w)$ contains a cycle,
- $m(w) = 0$ if and only if $\Gamma(w)$ is a tree.

*Proof.* Let $G = (V, U)$ be a graph with vertices $V$ and edges $U$. If $G$ is connected then $\#U - \#V + 1 \geq 0$. A connected graph $G = (V, U)$ is a tree if and only if $\#U - \#V# + 1 = 0$ and it contains a cycle if and only if $\#U - \#V + 1 > 0$. In the case of the graph $\Gamma(w)$, it turns out that

$$\#U - \#V + 1 = \#E(w) - \#E^-(w) - \#E^+(w) + 1 = m(w).$$

Another graph will be useful in the case when $w = \tilde{w}$ and when the language $L(u)$ is closed under reversal. These two hypothesis imply that $E^-(w) = E^+(w)$ and that $E(w)$ is symmetric, i.e. $(a,b) \in E(w)$ if and only if $(b,a) \in E(w)$.

**Definition 17** ($\Theta(w)$). Let $u \in A^n$. To a palindromic factor $w \in L(u)$ we assign a graph $\Theta(w) = (V, U)$ whose vertices $V = E^-(w) = E^+(w)$ are exactly the right (or left) extensions of $w$ and whose edges $U$ are unordered pairs of distinct elements of $E(w)$:

$$U = \{\{a,b\} : (a,b) \in E(w), a \neq b\}.$$

In particular, $\Theta(w)$ does not contain loops.

**Lemma 18.** Suppose that the language $L(u)$ is closed under reversal and $w = \tilde{w}$. If $\Theta(w)$ is connected, then $m(w) \geq \#E^-(w) - 1$ and

- $m(w) > \#E^-(w) - 1$ if and only if $\Theta(w)$ contains a cycle,
- $m(w) = \#E^-(w) - 1$ if and only if $\Theta(w)$ is a tree.

*Proof.* Using the same argument as for the previous lemma, we compute that

$$\#U = \frac{1}{2}(\#E(w) - \#E^-(w)) \quad \text{and} \quad \#V = \#E^-(w) = \#E^+(w).$$

Therefore,

$$\#U - \#V + 1 = \frac{1}{2}(\#E(w) - \#E^-(w) - \#E^-(w) - \#E^+(w)) + 1$$

$$= \frac{1}{2}(m(w) - \#E^-(w) + 1) \quad \Box.$$
Example 19. Let $u$ be the fixed point of the substitution $\eta : a \mapsto aabcaacb, b \mapsto aa, c \mapsto a$ used by Bucci and Vaslet. The list of all factors of length 2 is:

$$aa, ab, ac, ba, ca, bc, cb.$$  

The list of all factors of length 3 is:

$$aaa, aab, abc, acb, baa, bca, cac, cba.$$  

This allows to construct the graphs $\Theta(w)$ and $\Gamma(w)$ for $w \in \{\varepsilon, a, b, c\}$ (see Fig. 1) and the following table of values for the bilateral multiplicity:

| $w$     | $\varepsilon$ | $a$ | $b$ | $c$ |
|---------|---------------|-----|-----|-----|
| $m(w)$  | 2             | -1  | -1  | -1  |
| $\#E^= (w) - 1$ | 0     | 0   | -1  | -1  |

Figure 1: Example of graphs $\Theta(w)$ and $\Gamma(w)$ for $w \in \{\varepsilon, a, b, c, aaab\}$ in the language of the fixed point of the morphism $a \mapsto aabcaacb, b \mapsto aa, c \mapsto a$.

1. The graph $\Theta(\varepsilon)$ has vertices $V = \{a, b, c\}$ and edges $U = \{(a, b), (a, c), (b, c)\}$. The graph $\Theta(\varepsilon)$ contains a cycle. The bilateral multiplicity equals $m(\varepsilon) = 2 > 0 = \#E^= (\varepsilon) - 1$.

2. The graph $\Theta(a)$ has vertices $V = \{a, b, c\}$ and edges $U = \{(a, b)\}$. The graph $\Theta(a)$ is not connected. The bilateral multiplicity equals $m(a) = -1 < 0 = \#E^= (a) - 1$.

3. The graph $\Theta(b)$ has vertices $V = \{a, c\}$ and edges $U = \{(a, c)\}$. The graph $\Theta(b)$ is a tree. The bilateral multiplicity equals $m(b) = -1 = \#E^= (b) - 1$.

It is easy to see that the graph $\Theta(c)$ is isomorphic to $\Theta(b)$. The construction of the graphs $\Gamma(w)$ is analogous. From the extension set $E(aaab) = \{(a, c), (b, c)\}$ of the non-palindromic left special word $w = aaab$, the graph $\Gamma(aaab)$ can be constructed (see Fig. 1). Notice that it is a tree.
6 The graphs for finite positive palindromic defect

The graphs introduced in the previous section allow to interpret notions on the palindromic defect in terms of graph theory (Theorem 22 and Theorem 23). In particular, Theorem 22 is a translation of known results presented in the preliminaries. Namely Lemma 16 and Lemma 18 give an interpretation of Theorem 7 while Lemma 21 given below builds upon Theorem 6.

The next two lemmas explain the link between complete mirror return to a factor \( q \) and the connectedness of its associated graphs.

Lemma 20. Let \( u \in A^N \) have its language closed under reversal. Suppose that \( v \) is a palindromic complete mirror return word to \( q \in \mathcal{L}(u) \) such that \( \tilde{q} \) is a suffix of \( v \) and \( av \in \mathcal{L}(u) \) for some letters \( a, b \in A \). Then \( \{(a, -1), (b, +1)\} \) is an edge of the graph \( \Gamma(q) \). If \( q \) is a palindrome and \( a \neq b \), then \( \{a, b\} \) is an edge of the graph \( \Theta(q) \).

Proof. Let \( s \in A^* \) such that \( v = sb\tilde{q} \). Since \( v \) is a palindrome, we get \( v = qb\tilde{s} \). Therefore, \( aq \in \mathcal{L}(u) \) being a prefix of \( av \) and \((a, b) \in E(q). \) We conclude that \( \{(a, -1), (b, +1)\} \) is an edge of the graph \( \Gamma(q) \). Also if \( q = \tilde{q} \) and \( a \neq b \), we conclude that \( \{a, b\} \) is an edge of the graph \( \Theta(q) \). \( \square \)

Lemma 21. Let \( u \in A^N \) have its language closed under reversal, \( q \in \mathcal{L}(u) \) and suppose that occurrences of \( q \) and \( \tilde{q} \) alternate in \( u \). Suppose that all complete mirror return words to \( q \) are palindromes. Then \( \Gamma(q) \) is connected. If \( q \) is a palindrome, then \( \Theta(q) \) is connected.

Proof. It suffices to show that there is a path from any vertex \((a, -1)\) to any vertex \((b, +1)\) in \( \Gamma(q) \). Let \((a, -1)\) and \((b, +1)\) be two vertices of \( \Gamma(q) \). Then \( aq, qb \in \mathcal{L}(u) \). Among the occurrences of factors in \( Aq \cup A\tilde{q} \), if there exist two consecutive occurrences of \( aq \) and \( b\tilde{q} \), then \( \{(a, -1), (b, +1)\} \) is an edge of the graph \( \Gamma(q) \) from Lemma 20. Otherwise, we conclude that there exists a path from \((a, -1)\) to \((b, +1)\) by transitivity.

Assume \( q = \tilde{q} \). Let \( a, b \in E^{-}(q) = E^{+}(q) \) be two distinct vertices of \( \Theta(q) \). Then \( aq, bq \in \mathcal{L}(u) \). We want to show that there exists a path from \( a \) to \( b \) in \( \Theta(q) \). Among the occurrences of factors in \( Aq \), if there exist two consecutive occurrences of \( aq \) and \( bq \), then \( \{a, b\} \) is an edge of \( \Theta(q) \) from Lemma 20. Otherwise, we conclude that there exists a path from \( a \) to \( b \) by transitivity. \( \square \)

The next result that restates Theorem 7 in terms of \( \Gamma(q) \) and \( \Theta(q) \) is in fact hidden in [1, Lemma 12 and proof of Theorem 11].

Theorem 22. Let \( u \in A^N \) be an infinite word with its language closed under reversal and \( D(u) < +\infty \). There exists a positive integer \( K \) such that for every \( q \in \mathcal{L}(u) \) of length at least \( K \)

- if \( q \) is not a palindrome, then the graph \( \Gamma(q) \) is a tree,
- if \( q \) is a palindrome, then the graph \( \Theta(q) \) is a tree.

Proof. Theorem 6 implies that there exists a positive integer \( K_1 \) such that for every \( q \in \mathcal{L}(u) \) longer than \( K_1 \), the occurrences of \( q \) and \( \tilde{q} \) alternate and every complete mirror return to \( q \) in \( u \) is a palindrome. We conclude from Lemma 21 that the graph \( \Gamma(q) \) is connected. Also if \( q \) is a palindrome, then \( \Theta(q) \) is connected.

From Theorem 7, it follows that there exists a constant \( K_2 \) such that every factor \( q \) longer than \( K_2 \) satisfies

\[
m(q) = \begin{cases} 
0 & \text{if } q \neq \tilde{q}, \\
\#E^{-}(q) - 1 & \text{if } q = \tilde{q}.
\end{cases}
\]

Let \( K = \max\{K_1, K_2\} \) and \( q \) be a factor of \( \mathcal{L}(u) \) such that \( |q| > K \). If \( q \neq \tilde{q} \), Lemma 16 implies that \( \Gamma(q) \) is a tree. If \( q = \tilde{q} \), Lemma 18 implies that \( \Theta(q) \) is a tree. \( \square \)
We now prove a multiliteral analogy of Lemma 13 for word with its language closed under reversal and with positive palindromic defect.

**Theorem 23.** Let \( u \in \mathcal{A}^N \) have its language closed under reversal. If \( D(u) > 0 \), then either

1. there exists a non-palindrome \( q \in \mathcal{L}(u) \) such that \( \Gamma(q) \) contains a cycle or

2. there exists a palindrome \( q \in \mathcal{L}(u) \) such that \( \Theta(q) \) contains a cycle.

Moreover, if the empty word is the unique factor \( q \) with the above property, then there exists a letter with a non-palindromic complete return word.

**Proof.** Since \( D(u) > 0 \), there exists a word \( v = v_0v_1 \ldots v_n \) such that \( w \) is a prefix of \( v \), \( \tilde{w} \) is a suffix of \( v \), \( v \) does not contain other occurrences of \( w \) or \( \tilde{w} \), \( v \) is not a palindrome and \( |w| \geq 1 \). Suppose that \( v \) is a word of minimal length with this property and suppose that \( w \) is the longest prefix of \( v \) such that \( \tilde{w} \) is a suffix of \( v \). Then there exist letters \( \alpha \neq \beta \) such that \( w\alpha \) is a prefix and \( \beta\tilde{w} \) is a suffix of \( v \). Let us define \( t \in \mathcal{A} \) and \( q \in \mathcal{A}^* \) to satisfy \( w = tq \) (see Figure 2).

![Figure 2: The complete mirror return word \( v \) to the factor \( w \).](image)

We discuss three cases:

1. Let us suppose \( q = \tilde{q} \neq \varepsilon \). Due to the minimality of \( v = v_0v_1 \ldots v_n = tq\alpha \ldots \beta qt \), the non-palindromic factor \( v_1v_2 \ldots v_{n-1} = q\alpha \ldots \beta q \) cannot be a complete return word to \( q \) and thus contains at least 3 occurrences of \( q \). Let \( k \) be the number of occurrences \( q \) in \( v \). For \( i = 1, 2, \ldots, k \), denote by \( \gamma_i \) the letter which precedes the \( i^{th} \) occurrence of \( q \) and by \( \delta_i \) the letter which succeeds the \( i^{th} \) occurrence of \( q \).
   - Obviously, \( \gamma_1 = t, \delta_1 = \alpha, \) and \( \gamma_k = \beta \) and \( \delta_k = t \).
   - Since \( v \) is a complete mirror return word to the factor \( w = tq \), necessarily \( t \neq \gamma_i \) for \( i = 2, \ldots, k \) and \( t \neq \delta_i \) for \( i = 1, \ldots, k-1 \). In particular, \( \alpha \neq t \) and \( \beta \neq t \).
   - Since each complete return word to \( q \) in \( v \) is a palindrome, \( \delta_i = \gamma_{i+1} \) for \( i = 1, 2, \ldots, k-1 \). We artificially put \( \gamma_{k+1} = \delta_k = t \).

According to the definition of \( \Theta(q) \), if \( \gamma_i \neq \gamma_{i+1} = \delta_i \), then the pair \( \{\gamma_i, \gamma_{i+1}\} \) forms an edge. We want to find a cycle in \( \Theta(q) \). For this purpose we modify the sequence of letters \( \gamma_1, \gamma_2, \ldots, \gamma_k, \gamma_{k+1} \) as follows: If \( \gamma_{j+1} = \gamma_j \) for some index \( j = 1, \ldots, k \), then we erase from the sequence the \( (j+1)^{th} \) entry \( \gamma_{j+1} \). Then the modified sequence is a path in \( \Theta(q) \) which starts and ends at \( t \). The second vertex on the path is \( \alpha \), the penultimate vertex is \( \beta \). As \( \alpha \neq \beta \), the graph \( \Theta(q) \) contains a cycle.

2. Let us suppose that \( q = \varepsilon \). Now \( v = v_0v_1 \ldots v_n = t\alpha v_2v_3 \ldots \beta t \). It means that \( v \) is a complete mirror return to the letter \( t \) which is non-palindromic. If \( v_i \neq v_{i+1} \), the pair of consecutive letters \( \{v_i, v_{i+1}\} \) is an edge in the graph \( \Theta(\varepsilon) \) connecting vertices \( v_i \) and \( v_{i+1} \). If we erase
from the sequence $v_0, v_1, \ldots, v_n$ each vertex $v_{j+1}$ which coincides with its predecessor $v_j$, we get a path starting and ending in the vertex $t$. The first edge on this path is $\{t, \alpha\}$, the last one is $\{t, \beta\}$. As $\alpha \neq \beta$, the graph $\Theta(\varepsilon)$ contains a cycle.

3. Now we assume that $q \neq \tilde{q}$. Note that occurrences of $q$ and $\tilde{q}$ alternate inside $v$. Indeed, suppose the contrary, that is there exists a complete return word $z$ of $q$ that has no occurrences of $\tilde{q}$ and $z$ is a factor of $v$. The longest palindrome suffix of $z$ must be shorter than $q$. Therefore the longest palindromic suffix of $z$ is not unicoherent in $z$. This contradicts the minimality of $v$. Note also that $v$ must contain other occurrences of $q$ or $\tilde{q}$ inside or otherwise we get a contradiction on minimality of $v$. Let us denote $k$ the number of occurrences of $q$ in $v$. Clearly $k$ equals to the number of occurrences of $\tilde{q}$ as well.

Again we denote by $\gamma_i$ the letter which precedes the $i^{th}$ occurrence of $q$ and by $\delta_i$ the letter which succeeds the $i^{th}$ occurrence of $q$. In particular, $\gamma_1 = t$ and $\delta_1 = \alpha$. Analogously, we denote by $\tilde{\gamma}_i$ the letter which precedes the $i^{th}$ occurrence of $\tilde{q}$ and by $\tilde{\delta}_i$ the letter which succeeds the $i^{th}$ occurrence of $\tilde{q}$. In particular, $\tilde{\gamma}_k = \beta$ and $\tilde{\delta}_k = t$. Point out three important facts:

- $\gamma_i q \delta_i \in L(u)$ implies $\{(\gamma_i, -1), (\delta_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$.
- As the language $L(u)$ is closed under reversal, $\tilde{\gamma}_i q \tilde{\delta}_i \in L(u)$ implies $\{(\tilde{\delta}_i, -1), (\tilde{\gamma}_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$.
- Due to minimality of $v$, any mirror return to $q$ in $v$ is a palindrome. Thus $\tilde{\delta}_i = \tilde{\gamma}_i$ for $i = 1, 2, \ldots, k$ and $\delta_i = \gamma_{i+1}$ for $i = 1, 2, \ldots, k-1$.

Therefore, $\{(\gamma_i, -1), (\tilde{\gamma}_i, +1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k$, $\{(\tilde{\gamma}_i, +1), (\gamma_{i+1}, -1)\}$ is an edge in $\Gamma(q)$ for $i = 1, 2, \ldots, k-1$ and $\{(\tilde{\gamma}_k, +1), (\delta_k, -1)\}$ is an edge in $\Gamma(q)$. We can summarize that the sequence of vertices

$$(\gamma_1, -1), (\tilde{\gamma}_1, +1), (\gamma_2, -1), (\tilde{\gamma}_2, +1), \ldots, (\gamma_k, -1), (\tilde{\gamma}_k, +1), (\delta_k, -1)$$

forms a path in the bipartite graph $\Gamma(q)$ with $\gamma_1 = \tilde{\delta}_k = t$ and $\tilde{\gamma}_1 = \alpha \neq \beta = \gamma_k = t$. In this path the first and the last vertices coincide and the second and the penultimate vertices are distinct. Thus the graph $\Gamma(q)$ contains a cycle.

As we have seen in Example 19 for the fixed point $u$ of the morphism $\eta : a \mapsto aabcacba, b \mapsto aa, c \mapsto a$ for which the defect is known to be positive, the graph $\Theta(\varepsilon)$ contains a cycle. Since the defect of $u$ is finite, Theorem 22 also applies. Thus there are no arbitrarily large palindromic factors $w$ containing a cycle in their graph $\Theta(w)$ nor non-palindromic factors $w$ containing a cycle in their graph $\Gamma(w)$. This is readily seen on the conjugacy word of $\eta_L \triangleright \eta_R$ which is $aaa$ (see Fig. 3).

![Figure 3: $\Gamma(\text{aaa})$ contains a cycle but $\Theta(\text{aaa})$ is a tree in the language of the fixed point of the morphism $a \mapsto aabcacba, b \mapsto aa, c \mapsto a$.](image-url)

13
7 Proof of Zero Defect Conjecture for marked morphisms

We have prepared all ingredients needed for proving our main theorem. At first we have to stress that unlike the binary version, the statement of Theorem 1 does not speak about periodic fixed points. The following result from [15] allows to deduce that on a larger alphabet there is no ultimately periodic infinite word \( u \) fixed point of a primitive marked morphisms such that \( 0 < D(u) < \infty \).

**Proposition 24.** [15, Cor. 30, Cor. 32] Let \( u \) be an eventually periodic fixed point of a primitive marked morphism \( \varphi \) over an alphabet \( A \). If \( u \) is palindromic, then \( A = \{0, 1\} \) is a binary alphabet and \( u \) equals to \((01)^\omega\) or \((10)^\omega\).

Due to the previous proposition, a fixed point of a marked morphisms on binary alphabet is either not eventually periodic or equals to \((01)^\omega\) or \((10)^\omega\). Since both words \((01)^\omega\) and \((10)^\omega\) have defect zero and the Zero Defect Conjecture for binary alphabet is proven by Theorem 14, we may restrict ourselves to alphabets with cardinality at least three.

**Proof of Theorem 1.** As the languages of the fixed points of \( \varphi \) and \( \varphi^k \) coincide, we may assume without loss of generality that the marked morphism \( \varphi \) has already the property \( L_{st}(\varphi_{R}) = F_{st}(\varphi_{L}) = \text{Id} \).

Proving that the Zero Defect Conjecture holds in the case of marked morphism amounts to prove that the defect is either zero or \(+\infty\). Let us assume on the contrary that \( 0 < D(u) < +\infty \).

It follows that \( L(u) \) contains infinitely many palindromes. The primitiveness of \( \varphi \) implies that \( L(u) \) is closed under reversal. Theorem 23 implies that there exists a factor \( q \) such that if \( q \neq \tilde{q} \) the graph \( \Gamma(q) \) contains a cycle, or if \( q = \tilde{q} \), the graph \( \Theta(q) \) contains a cycle. Lemma 12, property (IV), implies that for all \( n \), there is a cycle in the graph of \( \Phi^n(q) \).

If \( q \neq \varepsilon \), then the primitiveness of \( \varphi \) implies that \( \lim_{n \to +\infty} |\Phi^n(q)| = +\infty \). If \( q = \varepsilon \), then, again by Theorem 23, there exists a letter having non-palindromic complete return word. By the assumption of the theorem, there must exist a conjugate of \( \varphi \) distinct from \( \varphi \) itself. It implies that the conjugacy word of \( \varphi_L \triangleright \varphi_R \) is nonempty, i.e., \( \tilde{\Phi}(\varepsilon) \neq \varepsilon \). Moreover, \( \lim_{n \to +\infty} |\Phi^n(q)| = +\infty \).

To conclude, we have that \( \lim_{n \to +\infty} |\Phi^n(q)| = +\infty \) and there is a cycle in the graph of \( \Phi^n(q) \) for all \( n \). This is a contradiction with Theorem 22. \( \square \)

8 Comments

Let us comment two conjectures concerning palindromes in languages of fixed points of primitive morphisms.

- The counterexample to the Zero Defect Conjecture in full generality was already mentioned in the Introduction. It is taken from [7]. The fixed point of \( \varphi : a \mapsto aabcacba, b \mapsto aa, c \mapsto a \)

has finite positive palindromic defect and is not periodic. There is a remarkable property of the fixed point \( u = \varphi(u) \).

Let \( \mu : a \mapsto ap, p \mapsto apaaaaapaaap \) be a morphism over the binary alphabet \( \{a, p\} \). Let us denote \( v \) the fixed point of \( \mu \). Then one can easily verify that \( u = \pi(v) \), where \( \pi : a \mapsto a, p \mapsto abcacba \). Moreover, \( v \) has zero defect.
In other words, the counterexample word is just an image under $\pi$ of a binary purely morphic binary word with zero defect.

- The counterexample to the question Hof, Knill and Simon given in [14] by the first author is

$$\psi : a \mapsto aca, b \mapsto cab, c \mapsto b.$$ 

As mentioned in [18], the fixed point $u = \psi(u)$ is again an image of a Sturmian word $v$ under a morphism $\pi : \{0, 1\} \mapsto \{a, b, c\}$ and the Sturmian word $v$ itself is a fixed point of a morphism over binary alphabet $\{0, 1\}$. Since $v$ is Sturmian, its defect is zero.

- Both counterexamples are in some sense degenerate. Both words are on ternary alphabet, but the binary alphabet is hidden in their structure. For further research in this area, it would be instructive to find another kind of counterexamples to both mentioned conjectures. In this context we mention that the second and third authors shown in [19] that any infinite word $u$ with a finite defect is a morphic image of a word $v$ with defect 0.

Acknowledgements

The first author is supported by a postdoctoral Marie Curie fellowship (BeIPD-COFUND) co-funded by the European Commission. The second and the third authors acknowledge support of GAČR 13-03538S (Czech Republic).

References

[1] L. Balková, E. Pelantová, and Š. Starosta, Sturmian jungle (or garden?) on multi-literal alphabets, RAIRO-Theor. Inf. Appl., 44 (2010), pp. 443–470.

[2] ———, Proof of the Brlek-Reutenauer conjecture, Theor. Comput. Sci., 475 (2013), pp. 120–125.

[3] V. Berthé, C. Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer, and G. Rindone, Acyclic, connected and tree sets, Monatsh. Math., 176 (2014), pp. 521–550.

[4] A. Blondin Massé, S. Brlek, A. Garon, and S. Labbé, Combinatorial properties of $f$-palindromes in the Thue-Morse sequence, Pure Math. Appl., 19 (2008), pp. 39–52.

[5] S. Brlek, S. Hamel, M. Nivat, and C. Reutenauer, On the palindromic complexity of infinite words, Int. J. Found. Comput. Sci., 15 (2004), pp. 293–306.

[6] M. Bucci, A. De Luca, A. Glen, and L. Q. Zamboni, A connection between palindromic and factor complexity using return words, Adv. in Appl. Math., 42 (2009), pp. 60–74.

[7] M. Bucci and E. Vaslet, Palindromic defect of pure morphic aperiodic words, in Proceedings of the 14th Mons Days of Theoretical Computer Science, 2012.

[8] J. Cassaigne and F. Nicolas, Factor complexity, in Combinatorics, automata and number theory, vol. 135 of Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, 2010, pp. 163–247.
[9] X. Droubay, J. Justin, and G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy, Theor. Comput. Sci., 255 (2001), pp. 539–553.

[10] X. Droubay and G. Pirillo, Palindromes and Sturmian words, Theor. Comput. Sci., 223 (1999), pp. 73–85.

[11] A. Glen, J. Justin, S. Widmer, and L. Q. Zamboni, Palindromic richness, European J. Combin., 30 (2009), pp. 510–531.

[12] A. Hof, O. Knill, and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, Comm. Math. Phys., 174 (1995), pp. 149–159.

[13] S. Jitomirskaya and B. Simon, Operators with singular continuous spectrum: III. almost periodic Schrödinger operators, Commun. Math. Phys., 165 (1994), pp. 201–205.

[14] S. Labbé, A counterexample to a question of Hof, Knill and Simon, Electron. J. Comb., 21 (2014).

[15] S. Labbé and E. Pelantová, Palindromic sequences generated from marked morphisms, Eur. J. Comb., 51 (2016), pp. 200–214.

[16] M. Lothaire, Combinatorics on Words, vol. 17 of Encyclopaedia of Mathematics and its Applications, Addison-Wesley, Reading, Mass., 1983. Reprinted in the Cambridge Mathematical Library, Cambridge University Press, 1997.

[17] ———, Algebraic combinatorics on words, no. 90 in Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2002.

[18] Z. Masáková, E. Pelantová, and Š. Starosta, Exchange of three intervals: itineraries, substitutions and palindromicity, preprint available at, (2015).

[19] E. Pelantová and Š. Starosta, Almost rich words as morphic images of rich words, Int. J. Found. Comput. Sci., 23 (2012), pp. 1067–1083.

[20] B. Tan, Mirror substitutions and palindromic sequences, Theor. Comput. Sci., 389 (2007), pp. 118–124.