Research Article

Delta Shocks and Vacuums to the Isentropic Euler Equations with the Flux Perturbation for van der Waals Gas

Jinhuan Wang and Yongbin Nie

1Department of Mathematics and Information, Tangshan Normal University, Tangshan 063000, China
2Beijing Aerospace Unmanned Vehicles System Engineering Research Institute, Beijing 100094, China

Correspondence should be addressed to Jinhuan Wang; jinhuanwn@163.com

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In this paper, we study the isentropic Euler equations with the flux perturbation for van der Waals gas, in which the density has both lower and upper bounds due to the introduction of the flux approximation and the molecular excluded volume. First, we solve the Riemann problem of this system and construct the Riemann solutions. Second, the formation mechanisms of delta shocks and vacuums are analyzed for the Riemann solutions as the pressure, the flux approximation, and the molecular excluded volume all vanish. Finally, some numerical simulations are demonstrated to verify the theoretical analysis.

1. Introduction

In this paper, we consider the isentropic Euler equations with the flux perturbation:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho - \epsilon)u^2 + P_x &= 0,
\end{align*}
\]

where \(u\) denotes the velocity, \(P\) denotes the pressure, and \(\rho\) denotes the density satisfying \(\rho \geq 2\epsilon\). Here, \(\epsilon\) is a small positive perturbation parameter.

When the flux perturbation and pressure both vanish, system (1) turns to be the transport equations:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_x + (\rho u^2)_x &= 0,
\end{align*}
\]

which can describe the formation of large-scale structures in the universe [1, 2] and the motion of free particles which stick under collision [3]. We also call this system the zero-pressure gas dynamics. For more details, readers can see [4–6].

In recent years, a lot of research work and achievement for the formation mechanisms of delta shocks and vacuums have been done by many scholars. Li [7] started the research on the isentropic Euler equations for perfect fluids in 2001, and then Chen and Liu [8, 9] made an in-depth study on isentropic and nonisentropic Euler equations for polytropic gas in 2003 and 2004. In [8, 9], formation mechanisms of delta shocks and vacuums were discussed as the polytropic gas pressure vanishes. Following that, the results were extended to various systems for different gas state equations in [10–16]. Furthermore, readers can also see [17] for the perturbed system of generalized pressureless gas dynamics model and [18] for the case of triangular conservation law system arising from “generalized pressureless gas dynamics model.” In addition, Yang and Liu [19] considered the limit behaviors of Riemann solutions of system (1) for polytropic gas as the flux approximation and the polytropic gas pressure both vanish. Flux perturbation can be regarded as the external shear force that causes the deformation of fluid particles and can be used to control some dynamic behaviors of fluid. Further, the flux approximation including the pressure perturbation portion is a more general physical consideration.

Recently, Wang et al. [20] considered the isentropic Euler equations for van der Waals gas:

\[
P = k \left( \frac{\rho}{1 - bp} \right)^\gamma,
\]

where \(b\) and \(k\) are constants, and \(\gamma\) is a parameter that depends on the type of gas. The van der Waals gas model is a classical example of a nonideal gas, and it is widely used in various fields such as chemistry, physics, and engineering. The van der Waals gas equation of state is given by:

\[
P = \frac{\rho R T}{\nu - b \rho} - \frac{a \rho^2}{\nu^2}
\]

where \(R\) is the universal gas constant, and \(\nu\) is the molar volume of the gas. The terms \(b\) and \(a\) are the van der Waals constants, which are related to the intermolecular forces and the size of the molecules, respectively.

In this paper, we study the isentropic Euler equations with the flux perturbation for van der Waals gas, in which the density has both lower and upper bounds due to the introduction of the flux approximation and the molecular excluded volume. First, we solve the Riemann problem of this system and construct the Riemann solutions. Second, the formation mechanisms of delta shocks and vacuums are analyzed for the Riemann solutions as the pressure, the flux approximation, and the molecular excluded volume all vanish. Finally, some numerical simulations are demonstrated to verify the theoretical analysis.
where $b$ denotes the molecular excluded volume satisfying $0 \leq bp \ll 1$, $k$ denotes a positive constant, and $\gamma$ denotes the adiabatic exponent with $1 < \gamma < 2$. It is easy for us to find that when $b = 0$, the state equation (3) just corresponds to the ideal gas. As mentioned in, at a high pressure or a low temperature, the behavior of real gas is not consistent with the ideal polytropic gas model but accords with the van der Waals gas one. Therefore, it is natural for us to explore the limit behavior of Riemann solutions of system (1) with the van der Waals gas (3) as the corresponding pressure vanishes.

Compared to the works in [8, 19], the flux approximation and the molecular excluded volume are considered simultaneously in this paper, which makes the density has both lower and upper bounds. Further, we find that if only the flux approximation and the pressure tend to zero at the same time and the molecular excluded volume does not tend to zero, then the density always has an upper bound, so only vacuums may be generated, and delta shocks may not occur. This is the motivation for us to study the case where all of the pressure, the flux approximation, and the molecular excluded volume tend to zero simultaneously; that is, the triple parameters $c, k, b \to 0$.

In this paper, we rigorously prove that as $c, k, b \to 0$, any Riemann solution to the perturbation isentropic Euler equations (1) for van der Waals gas (3) containing two shocks converges to the delta shock solution of system (2) and any Riemann solution of equations (1) and (3) containing two rarefaction waves converges to the vacuum solution of system (2). By theory analysis, it is found that the introduction of the flux perturbation and the van der Waal gas pressure does not affect the formation of delta shocks and vacuums when the triple parameters $c, k, b \to 0$ simultaneously, which means that our work can also be regarded as the extension of that in [8, 19].

The remainder of this paper is arranged as follows. In Section 2, we solve the Riemann problem of equations (1) and (3). In Section 3, the limit behaviors of Riemann solution of equations (1) and (3) are considered as the flux perturbation, the van der Waals gas pressure, and the molecular excluded volume all vanish. In Section 4, some numerical results are shown to verify the theoretical analysis of the formation of delta shocks and vacuum states.

For the details of delta shocks and vacuums for the transport equations (2), we will not repeat them here. Readers can refer to Section 2 in [8, 11].

### 2. Riemann Solutions to the Perturbation Euler Equations for van der Waals Gas

In this section, we solve the Riemann problem of the perturbation isentropic Euler equations (1) for the van der Waals gas (3) with Riemann initial data:

$$(u, \rho) (0, x) = (u_s, \rho_s),$$

where $(u_s, \rho_s)$ are arbitrary constants and then we construct Riemann solutions of this system.

For equations (1) and (3), the eigenvalues and corresponding right eigenvectors are

$$\lambda_1 = u - \sqrt{\frac{k \rho^{\gamma-2}}{(1 - bp)^{\gamma+1}}(\rho - 2e)} \xi,$$

$$\lambda_2 = u + \sqrt{\frac{k \rho^{\gamma-2}}{(1 - bp)^{\gamma+1}}(\rho - 2e)} \xi,$$

$$\eta_1 = \left(\frac{(\rho - 2e)}{(1 - bp)^{\gamma+1}} \xi, 2e - \rho\right),$$

$$\eta_2 = \left(\frac{(\rho - 2e)}{(1 - bp)^{\gamma+1}} \xi, \rho - 2e\right),$$

respectively. By direct calculation, we can obtain that $\forall\lambda_i, \eta_i \neq 0 (i = 1, 2)$, which implies that both eigenvalues $\lambda_i (i = 1, 2)$ are genuinely nonlinear. Thus, the elementary waves of this system contain rarefaction waves and shock waves.

We look for the self-similar solution $(u, \rho) (\xi) (\xi = x/t)$ of equations (1) and (3) with (4) and obtain the two-point boundary value problem as follows:

$$\begin{cases}
-\xi \rho_x + ((\rho - 2e)u)_\xi = 0, \\
-\xi (\rho u)_\xi + ((\rho - e)u^2 + p)_\xi = 0,
\end{cases} \quad P = k\left(\frac{\rho}{1 - bp}\right)^\gamma,$$

$$(u, \rho) (\pm \infty) = (u_s, \rho_s).$$

Now, we consider the smooth solution of (6) and get either the constant state solution $(u, \rho) (\xi) = \text{constant}$, or the backward rarefaction wave

$$\begin{cases}
\xi = \lambda_1 = u - \sqrt{\frac{k \rho^{\gamma-2}(\rho - 2e)}{(1 - bp)^{\gamma+1}}}, \\
(\rho - 2e)du + \sqrt{\frac{k \rho^{\gamma-2}(\rho - 2e)}{(1 - bp)^{\gamma+1}}} \, dp = 0,
\end{cases} \quad R_1;$$

or the forward rarefaction wave

$$\begin{cases}
\xi = \lambda_2 = u + \sqrt{\frac{k \rho^{\gamma-2}(\rho - 2e)}{(1 - bp)^{\gamma+1}}}, \\
(\rho - 2e)du - \sqrt{\frac{k \rho^{\gamma-2}(\rho - 2e)}{(1 - bp)^{\gamma+1}}} \, dp = 0.
\end{cases} \quad R_2;$$
By (8) and (9), we calculate that when $\epsilon$ is small enough,

$$\frac{d\lambda_1}{d\rho} = \frac{k\rho^{r-3}((y+1)\rho - 2ey + 4e - 6eb\rho)}{2(1 - b\rho)^{r/2}((k\rho^{r/2}(\rho - 2e))'(1 - b\rho)^{r+1})} < 0,$$

(10)

$$\frac{d\lambda_2}{d\rho} = \frac{k\rho^{r-3}((y+1)\rho - 2ey + 4e - 6eb\rho)}{2(1 - b\rho)^{r/2}((k\rho^{r/2}(\rho - 2e))'(1 - b\rho)^{r+1})} > 0.$$

(11)

From these two conditions, we find that the velocity of the backward rarefaction wave $\lambda_1$ is monotonically decreasing with respect to $\rho$, while the forward one $\lambda_2$ is increasing. Furthermore, integrating the second equations of (8) and (9), respectively, by $\lambda_i(u_\rho, \rho_\rho) > \lambda_i(u_\rho, \rho_-)$ ($i = 1, 2$) yields

$$\begin{align*}
\xi &= \lambda_1 = u - \sqrt{\frac{k\rho^{r/2}(\rho - 2e)}{(1 - b\rho)^{r+1}}}s, \\
R_1: \\
&\quad u = u_- - \int_\rho^{\rho_-} \frac{k\rho^{r/2}}{(1 - b\rho)^{r+1}}(s - 2e)ds, \quad \rho < \rho_-,
\end{align*}$$

(12)

$$\begin{align*}
\xi &= \lambda_2 = u + \sqrt{\frac{k\rho^{r/2}(\rho - 2e)}{(1 - b\rho)^{r+1}}}s, \\
R_2: \\
&\quad u = u_- + \int_\rho^{\rho_-} \frac{k\rho^{r/2}}{(1 - b\rho)^{r+1}}(s - 2e)ds, \quad \rho > \rho_-.
\end{align*}$$

(13)

In the $(u, \rho)$-plane, we call the curve of the second equation of (12) the forward rarefaction wave curve, which is monotonically decreasing with respect to $\rho$ by a direct calculation from the second equation of (8). Further, for this curve, we obtain that

$$\lim_{\rho \to 2e} u = u_- + \int_\rho^{\rho_-} \sqrt{\frac{(k\rho^{r/2})(1 - bs)^{r+1}(s - 2e)}{1(1 - b\rho)^{r+1}}}ds.$$

In fact,

$$\lim_{\rho \to 2e} \frac{(s - 2e)^{r+1}}{1(1 - b\rho)^{r+1}} \frac{1}{(s - 2e)}ds \to +\infty$$

and the integral

$$\int_\rho^{\rho_-} \sqrt{\frac{(k\rho^{r/2})(1 - bs)^{r+1}(s - 2e)}{1(1 - b\rho)^{r+1}}}ds < \int_\rho^{\rho_-} \sqrt{\frac{(k\rho^{r/2})(1 - bs)^{r+1}(s - 2e)}{1(1 - b\rho)^{r+1}}}ds,$$

obtain that

$$\lim_{\rho \to 2e} u = +\infty$$

from the second equation of (13).

Now, we turn to a bounded discontinuity at $\xi = \sigma$. For equations (1) and (3), the Rankine–Hugoniot compatibility conditions

$$\begin{align*}
\sigma_\rho &= [(\rho - 2e)u], \\
\sigma_{pu} &= \left[\frac{(\rho - 2e)u^2 + k(\frac{\rho}{1 - b\rho})^\gamma}{\rho}ight]
\end{align*}$$

(14)

hold, where $[F] = F - F_-$ denotes the jump of function $F$ across the discontinuity. By solving (14), in terms of the stability conditions

$$\begin{align*}
\lambda_2(u_\rho, \rho_-) > \lambda_1(u_\rho, \rho_-) > \sigma_1, \\
\lambda_2(u, \rho) > \sigma_1 > \lambda_1(u, \rho),
\end{align*}$$

(15)

for $\lambda_1$, and

$$\begin{align*}
\lambda_2(u_\rho, \rho_-) > \sigma_2 > \lambda_1(u_\rho, \rho_-), \\
\sigma_2 > \lambda_2(u, \rho) > \lambda_1(u, \rho),
\end{align*}$$

(16)

for $\lambda_2$, we obtain the backward shock $S_1$ and the forward shock $S_2$ as follows:

$$\begin{align*}
S_1: \\
\sigma_1 &= u_- - (\rho - 2e)\sqrt{k\left[\left(\frac{\rho}{1 - b\rho}\right)^\gamma - \left(\frac{\rho_-}{1 - b\rho_-}\right)^\gamma\right]}\frac{1}{(\rho - \rho_-)(\rho_- - \epsilon)(\rho + \rho_-)}, \\
u &= u_- - \sqrt{k\left[\left(\frac{\rho}{1 - b\rho}\right)^\gamma - \left(\frac{\rho_-}{1 - b\rho_-}\right)^\gamma\right]}\frac{\rho - \rho_-}{\rho_- - \epsilon}(\rho + \rho_-), \quad \rho > \rho_-,
\end{align*}$$

(17)

$$\begin{align*}
S_2: \\
\sigma_2 &= u_- + (\rho - 2e)\sqrt{k\left[\left(\frac{\rho}{1 - b\rho}\right)^\gamma - \left(\frac{\rho_-}{1 - b\rho_-}\right)^\gamma\right]}\frac{1}{(\rho - \rho_-)(\rho_- - \epsilon)(\rho + \rho_-)}, \\
u &= u_- - \sqrt{k\left[\left(\frac{\rho}{1 - b\rho}\right)^\gamma - \left(\frac{\rho_-}{1 - b\rho_-}\right)^\gamma\right]}\frac{\rho - \rho_-}{\rho_- - \epsilon}(\rho + \rho_-), \quad \rho < \rho_-.
\end{align*}$$

(18)
In the \((u, \rho)-\)plane, we call the curve of the second equations of (17) and (18) the backward shock curve (forward shock curve). From the second equation of (17), we can obtain

\[
u = \rho_\ast \left( (\rho_\ast - 2\epsilon_\ast)^{-1} - (\rho_\ast - 2\epsilon_\ast)^{-1} \right) + ((\rho_\ast - 2\epsilon_\ast)^{-1} - (\rho_\ast - 2\epsilon_\ast)^{-1}) (\rho_\ast - 2\epsilon_\ast) < 0,
\]

which means that the backward shock curve is monotonically decreasing. Similarly, we have \(u_\ast > 0\) for the forward shock wave, which implies that the forward shock curve is monotonically increasing. Further, a direct calculation gives that \(\lim_{\rho_\ast \to \infty} u_\ast = \infty\) for the backward shock curve and \(\lim_{\rho_\ast \to \infty} u_\ast = \infty\) for the forward shock curve, which implies that this forward shock wave curve and the straight line \(\rho = 2\epsilon_\ast\) have an intersection point \((u_\ast, 2\epsilon_\ast) = (\rho_\ast - \sqrt{k((\rho_\ast - 2\epsilon_\ast)^{-1} - (2\epsilon_\ast)^{-1})) e^{-1}}\) for the forward shock curve.

Based on the aforementioned analysis, we conclude that the elementary waves of equations (1) and (3) contain two rarefaction waves \((R_1, R_2)\) and two shock waves \((S_1, S_2)\). Using the curves of these elementary waves, we can divide the \((u, \rho)-\)plane \((\rho \geq 2\epsilon_\ast, 0 < b < 2\epsilon_\ast)\) into four regions \(S_1, S_2, S_1, S_2)\), \(R_1, R_2, S_1, S_2, S_1, S_2\), and \(R_1, R_2, S_1, S_2, S_1, S_2\) for any fixed left state \((u_\ast, \rho_\ast)\). Furthermore, for any fixed right state \((u_\ast, \rho_\ast)\), we can construct a unique Riemann solution, no matter which of the four regions the right state belongs to. Specifically, when \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\), the Riemann solution can be constructed with two shocks \((S_1, S_2)\) and \((S_1, S_2)\) besides a nonvacuum constant state in between. When \((u_\ast, \rho_\ast) \in R_1, R_2(u_\ast, \rho_\ast)\), the Riemann solution can be constructed with two rarefaction waves \((R_1, R_2)\) besides an intermediate constant state which may be a constant-density solution \(\rho = 2\epsilon_\ast\). The discussion for the other two cases \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\) and \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\) is trivial, so in this paper we only consider the limit process for the cases \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\) and \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\).

3. Formation of Delta Shocks and Vacuums as \(\epsilon, k, b \to 0\)

3.1. Formation of Delta Shocks. In this subsection, the formation of delta shocks is considered in the Riemann solutions of equations (1) and (3) in the case \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\) as \(u_\ast > u_\ast\) as \(\epsilon, k, b \to 0\).

3.1.1. Limit Behavior of the Riemann Solutions as \(\epsilon, k, b \to 0\). For the case \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\), we assume that \((u_\ast, \rho_\ast) \in S_1, S_2(u_\ast, \rho_\ast)\) is the intermediate state. Then the left and right states of the backward shock \(S_1\) (the forward shock \(S_2\)) are \((u_\ast, \rho_\ast)\) \((u_\ast, \rho_\ast)\) \((u_\ast, \rho_\ast)\) \((u_\ast, \rho_\ast)\), respectively. Thus, we have

\[
\begin{align*}
\sigma_1^\Pi &= u_\ast - (\rho_\ast - 2\epsilon_\ast) \left( \frac{1}{(\rho_\ast - \epsilon_\ast)(\rho_\ast + \epsilon_\ast)} \left( \frac{\rho_\ast}{1 - b(\rho_\ast - \epsilon_\ast)^{-1}} - \frac{\rho_\ast}{1 - b(\rho_\ast - \epsilon_\ast)^{-1}} \right) \right),
\end{align*}
\]

for \(S_1\), and

\[
\begin{align*}
\sigma_2^\Pi &= u_\ast + (\rho_\ast - 2\epsilon_\ast) \left( \frac{1}{(\rho_\ast - \epsilon_\ast)(\rho_\ast + \epsilon_\ast)} \left( \frac{\rho_\ast}{1 - b(\rho_\ast - \epsilon_\ast)^{-1}} - \frac{\rho_\ast}{1 - b(\rho_\ast - \epsilon_\ast)^{-1}} \right) \right),
\end{align*}
\]

for \(S_2\). Here \(\sigma^\Pi_1 (i = 1, 2)\) denotes the speed of \(S_i\).
Now, we are ready to present our main results for the formation of delta shocks.

**Theorem 1.** Let \( u_- > u_+ \) and \((u_-, \rho_-) \in S_1S_2(u_-, \rho_-)\). For any fixed \( \epsilon, k, b > 0 \), suppose that \((u^{\Pi}, \rho^{\Pi})\) is a Riemann solution of equations (1) and (3) with (4) containing two shocks \( S_1 \) and \( S_2 \). Then as \( \epsilon, k, b \to 0 \), \( \rho^{\Pi} \) and \( \rho^{\Pi}u^{\Pi} \) converge, in the sense of distributions, to the sums of a step function and a \( \delta \)-measure with weights
\[
(1 + \sigma^2)^{-1/2} \begin{pmatrix} \sigma [\rho] - [\rho u] t \\ \sigma [\rho u] - [\rho u^2] t \end{pmatrix},
\]
respectively, which just form the delta shock solution of system (2) with (4).

Before we prove this theorem, some lemmas should be presented. The proof process of these lemmas is similar to that in [10], so we do not repeat it here.

**Lemma 1.** \( \lim_{\epsilon, k, b \to 0} \rho^{\Pi}_* = +\infty \).

**Lemma 2.** \( \lim_{\epsilon, k, b \to 0} \rho^\Pi k ((\rho^{\Pi}_*/(1-b^{\Pi}))/\sqrt{\rho^{\Pi}_* + \sqrt{\rho^{\Pi}_*}})^t = (\rho^{\Pi}_*/(1-\rho^{\Pi}_* u^2))/(\sqrt{\rho^{\Pi}_* + \sqrt{\rho^{\Pi}_*}})^t \).

**Lemma 3.** Set \( \lim_{\epsilon, k, b \to 0} \rho^{\Pi}_* u^{\Pi}_* = \sigma \); then \( \lim_{\epsilon, k, b \to 0} \rho^{\Pi}_* = \lim_{\epsilon, k, b \to 0} u^{\Pi}_* = \sigma \in (u_+, u_-) \).

**Lemma 4.**
\[
\begin{align*}
\lim_{\epsilon, k, b \to 0} \int_{0}^{\rho^{\Pi}_*} \rho^{\Pi}_* d\xi & = \sigma [\rho] - [\rho u], \\
\lim_{\epsilon, k, b \to 0} \int_{0}^{\rho^{\Pi}_*} \rho^{\Pi}_* u^{\Pi}_* d\xi & = \sigma [\rho u] - [\rho u^2].
\end{align*}
\]

**Lemma 5.** For above quantity \( \sigma \), we can obtain
\[
\sigma = (\sqrt{\rho^{\Pi}_* + \sqrt{\rho^{\Pi}_*}})^{-1} (\sqrt{\rho^{\Pi}_*} u_- + \sqrt{\rho^{\Pi}_*} u_+).
\]

On the basis of the above results, we find that as \( \epsilon, k, b \to 0 \), the velocity of forward shock \( u^{\Pi}_* \), and the intermediate velocity \( \rho^{\Pi}_* \) of equations (1) and (3) tend to the same quantity \( \sigma \), which is proposed for the delta shock solution of system (2), and \( \rho^{\Pi}_* \) becomes singular simultaneously.

3.1.2. Proof of Theorem 1. Now, we give the rigorous proof of Theorem 1.

**Proof of Theorem 1.**

**Step 1.** Set \( \xi = xt \). By the two-shock Riemann solution
\[
(u^{\Pi}_*, \rho^{\Pi}_*) (\xi) = \begin{cases} (u_-, \rho_-), & \xi < \sigma_1, \\
(u^{\Pi}_*, \rho^{\Pi}_*) (\xi), & \sigma_1 < \xi < \sigma_2, \\
(u_+, \rho_+), & \xi > \sigma_2,
\end{cases}
\]
we can obtain the weak formulations
\[
-\int_{-\infty}^{\infty} (\rho^{\Pi}_* (u^{\Pi}_* - \xi) - 2\epsilon u^{\Pi}_*) \phi \, d\xi + \int_{-\infty}^{\infty} \rho^{\Pi}_* \phi \, d\xi = 0,
\]
\[
-\int_{-\infty}^{\infty} \left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi - \int_{-\infty}^{\infty} k
\]
\[
\cdot \left( \rho^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi = 0,
\]
for any test function \( \phi \in C^1_c (-\infty, +\infty) \).

**Step 2.** Here, we prove that the limit functions of \( \rho^{\Pi}_* u^{\Pi}_* \) and \( \rho^{\Pi}_* \) are the sums of a step function and a \( \delta \)-measure. We rewrite the first term on the left of (27) and obtain
\[
\int_{-\infty}^{\infty} \left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi = \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{\infty} \right)
\]
\[
\cdot \left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi.
\]
(28)

With the analysis of integration by part, noticing Lemmas 3 and 4, we have

\[
\lim_{\epsilon, k, b \to 0} \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{\infty} \right)
\]
\[
\left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi
\]
\[
= \lim_{\epsilon, k, b \to 0} \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{\infty} \right)
\]
\[
\left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi
\]
\[
= \lim_{\epsilon, k, b \to 0} \left( \rho^{\Pi}_* u^{\Pi}_* (u^{\Pi}_* - \xi) - \epsilon (u^{\Pi}_*)^2 \right) \phi \, d\xi
\]
\[
= \lim_{\epsilon, k, b \to 0} \epsilon (u^{\Pi}_*)^2 \left( \phi (u^{\Pi}_*) - \phi (\sigma^{\Pi}_*) \right) \phi (u^{\Pi}_*) + \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*) - \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*)
\]
\[
= \lim_{\epsilon, k, b \to 0} \epsilon (u^{\Pi}_*)^2 \left( \phi (u^{\Pi}_*) - \phi (\sigma^{\Pi}_*) \right) \phi (u^{\Pi}_*) + \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*) - \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*)
\]
\[
= \lim_{\epsilon, k, b \to 0} \epsilon (u^{\Pi}_*)^2 \left( \phi (u^{\Pi}_*) - \phi (\sigma^{\Pi}_*) \right) \phi (u^{\Pi}_*) + \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*) - \epsilon (u^{\Pi}_*)^2 \phi (\sigma^{\Pi}_*)
\]
\[
= 0.
\]
A combination of (29) and (30) gives that
\[
\lim_{\epsilon,k,b \to 0} \int_{-\infty}^{+\infty} \left( \rho^{\Pi} u^{\Pi} \right)^Y \phi \, d\xi = \phi(\sigma) \left( \sigma [pu] - [pu^2] \right) + \int_{-\infty}^{+\infty} H(\xi - \sigma) \phi \, d\xi,
\]
(31)
where
\[
H(\xi - \sigma) = \begin{cases} 
\rho_{-u_+}, & \xi < \sigma, \\
\rho_{+u_+}, & \xi > \sigma.
\end{cases}
\]

Similarly, we calculate the second integral on the left side of (27), noticing Lemmas 1–3, and obtain that
\[
\int_{0}^{+\infty} \psi(t,\sigma) \left( \sigma [pu] - [pu^2] \right) t \, dt = \langle \psi(\cdot,\cdot), w_1(\cdot)\delta_\xi \rangle,
\]
(38)
where
\[
w_1(t) = \left( 1 + \sigma^2 \right)^{-(1/2)} \left( \sigma [pu] - [pu^2] \right) t.
\]

For this case, we assume that
\[
\lim_{\epsilon,k,b \to 0} \int_{-\infty}^{+\infty} \rho^{\Pi} u^{\Pi} (H(\xi - \sigma) \phi) \, d\xi = \lim_{\epsilon,k,b \to 0} \int_{-\infty}^{+\infty} \left( \rho^{\Pi} \frac{1}{1 - b\rho^{\Pi}} \right)^Y \phi(\sigma) \, d\xi
\]
which together with definition (2.3) in [11] yields that
\[
\int_{0}^{+\infty} \psi(t,\sigma) \left( \sigma [pu] - [pu^2] \right) t \, dt = \langle \psi(\cdot,\cdot), w_1(\cdot)\delta_\xi \rangle,
\]
(39)
where
\[
w_1(t) = \left( 1 + \sigma^2 \right)^{-(1/2)} \left( \sigma [pu] - [pu^2] \right) t.
\]

Step 3. We turn to prove the weights of the \(\delta\)-measures. For any test function \(\psi \in C^0_c(R^+ \times \mathbb{R})\), by (34), we obtain
\[
\lim_{\epsilon,k,b \to 0} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \psi(t,\xi) \rho^{\Pi} \left( \frac{\xi}{t} \right)^2 \, d\xi \, dt
\]
(37)
where
\[
\psi(t,\xi) = \begin{cases} 
\rho_{+u_+}, & \xi < x, \\
\rho_{-u_+}, & \xi > x.
\end{cases}
\]

3.2. Formation of Vacuums. In this subsection, the limit of Riemann solutions of equations (1) and (3) is considered in the case \((u_+, \rho_+) \in R_1 R_2 (u_+, \rho_-) \) with \(u_+ < u_- \) as \(c, k, b \to 0\). For this case, we assume that \(\left( \rho^{\Pi}_{-u_+}, \rho^{\Pi}_{+u_-} \right) \) is the intermediate state. Then the left and right states of the backward rarefaction wave \(R_2\) (the forward rarefaction wave \(R_1\)) are \((u_+, \rho_+)\) and \((u^{\Pi}_{-u_+}, \rho^{\Pi}_{+u_-})\), respectively. Then, for the backward rarefaction wave \(R_1\), we have
\[
\xi = u_1^\Pi - \sqrt{\frac{ky(\rho_1^\Pi)^{\gamma-2}(\rho_1^\Pi - 2\varepsilon)}{(1 - b\rho_1^\Pi)^{\gamma+1}}},
\]
\[
u - \sqrt{\frac{ky(\rho_-)^{\gamma-2}(\rho_- - 2\varepsilon)}{(1 - b\rho_-)^{\gamma+1}}}, \quad < \xi < u_1^\Pi - \sqrt{\frac{ky(\rho^*_-)^{\gamma-2}(\rho^*_- - 2\varepsilon)}{(1 - b\rho^*_-)^{\gamma+1}}}, \quad \rho_- > \rho^*_-.
\]

For the forward rarefaction wave \( R_2 \), we have
\[
\xi = u_1^\Pi + \sqrt{\frac{ky(\rho^*_+)^{\gamma-2}(\rho^*_+ - 2\varepsilon)}{(1 - b\rho^*_+)^{\gamma+1}}}, \quad u^*_- + \sqrt{\frac{ky(\rho_-)^{\gamma-2}(\rho_- - 2\varepsilon)}{(1 - b\rho_-)^{\gamma+1}}}, \quad \rho_- > \rho^*_-.
\]

Now, we can get the following theorem.

**Theorem 2.** Let \( u_- < u_+ \) and \( (u_+, \rho_+) \in R_1 \cup R_2 \). For any fixed \( \varepsilon, k, b > 0 \), suppose that \( (u_1^\Pi, \rho_1^\Pi) \) is a Riemann solution of equations (1) and (3) with (4) containing two rarefaction waves \( R_1 \) and \( R_2 \). Then, there exists a positive constant \( \alpha \), and the constant-density solution \( \rho = 2\varepsilon \) occurs in the solution when \( 0 < \varepsilon, k, b < a \). As \( \varepsilon, k, b \to 0 \), \( R_1 \) and \( R_2 \) become two contact discontinuities, which connect the vacuum \( \rho = 0 \) and constant states \( (u_+, \rho_+) \).

**Proof.** Set \( k = b = \varepsilon = a \). Since \( (u^\Pi_+, \rho^\Pi_+) \) is on the curve \( R_1 \), we obtain
\[
0 \leq \nu_1^\Pi - \int_{\rho^*_-}^{\rho_-} \frac{ays^{\gamma-2}}{(1 - as)^{\gamma+1}(s - 2a)} \, ds \equiv A^\Pi.
\]

When \( u_- < u_+ \), the constant-density state does not appear in the solution, which means that there exists a positive constant \( \alpha \), and when \( u_- < u_+ < A^\Pi \), the Riemann solution only contains two rarefaction waves \( R_1 \) and \( R_2 \) and the two given constant states \( (u_+, \rho_+) \). Meanwhile, when \( A^\Pi < u_- \), the constant-density state occurs, which means that there exists a positive constant \( \alpha \), and when \( A^\Pi < u_- \), the intermediate state between the backward rarefaction wave \( R_1 \) and the forward rarefaction wave \( R_2 \) is just a constant-density state.

We set \( g(a) = \int_{2a}^{\rho_-} \frac{ays^{\gamma-2}}{(1 - as)^{\gamma+1}(s - 2a)} \, ds + u_- - u_+, \quad a \in [a_2, a_1] \). Since
\[
\int_{2a}^{\rho_-} \frac{ays^{\gamma-2}}{(1 - as)^{\gamma+1}(s - 2a)} \, ds \leq \frac{1}{2a}, \quad u_- - u_+ \leq A^\Pi.
\]

\[
\int_{2a}^{\rho_-} \frac{ays^{\gamma-2}}{(1 - as)^{\gamma+1}(s - 2a)} \, ds < \frac{1}{2a}, \quad \frac{1}{2a} \leq \frac{1}{a_2}.
\]

**4. Numerical Results**

In this section, we simulate the formation process of delta shocks and vacuums. In order to discretize equations (1) and...
we use the fifth-order weighted essentially non-oscillatory scheme and third-order Runge-Kutta method with 150 × 150 cells. For the sake of convenience, we take \( c = 1.4 \).

To illustrate the formation of the delta shock, we take the following initial data:

\[
\begin{align*}
\rho^-, u^- &= (3.0, 2.0), \\
\rho^+, u^+ &= (2.0, 0.5).
\end{align*}
\]  

Here, we begin with \( \epsilon = 0.1, k = 1, \) and \( b = 0.05 \) and then choose \( \epsilon = 0.05, k = 0.09, \) and \( b = 0.001 \), and finally we choose \( \epsilon = 0.00001, k = 0.00001, \) and \( b = 0.00005 \). The corresponding numerical results at \( t = 2.0 \) are listed in Figures 1–3, which present the formation process of a delta shock in the two-shock solution of equations (1) and (3) as the van der Waals gas pressure, the molecular excluded volume, and the flux approximation all vanish.

From Figures 1–3, it is easy to find that when the values of \( \epsilon, k, \) and \( b \) become smaller and smaller, the locations of \( S_1 \) and \( S_2 \) get closer and closer, and the intermediate density \( \rho^H \) increases dramatically, and at the same time the velocity turns to be a step function. When \( \epsilon, k, \) and \( b \) vanish, two shocks \( S_1 \) and \( S_2 \) coincide with each other and a delta shock wave develops, while the velocity remains a step function.

**Figure 1:** (a) Density and (b) velocity of the delta shock for \( \epsilon = 0.1, k = 1, \) and \( b = 0.05 \).

**Figure 2:** (a) Density and (b) velocity of the delta shock for \( \epsilon = 0.05, k = 0.09, \) and \( b = 0.001 \).
For the formation of vacuum states, we take the following initial data:
\[
\begin{align*}
\rho^-, u^- &= (1.0, 0.5), \\
\rho^+, u^+ &= (2.0, 2.0).
\end{align*}
\] (51)

In this case, we start with \(\epsilon = 0.1, k = 1, b = 0.05\) and then choose \(\epsilon = 0.05, k = 0.09, b = 0.001\), and finally we choose \(\epsilon = 0.00001, k = 0.00001, b = 0.00005\). The corresponding numerical results at \(t = 2.0\) are shown in Figures 4–6, which describe the formation of a vacuum state in the Riemann solution containing two rarefaction waves \((R_1\) and \(R_2\)) and a nonvacuum intermediate state for equations (1) and (3) as the van der Waals gas pressure, the molecular excluded volume, and the flux approximation vanish.

Figures 4–6 show that as \(\epsilon, k, b\) decrease, the locations of \(R_1\) and \(R_2\) get closer and closer, and \(\rho^\Pi\) tends to zero leading to an inside vacuum state as \(\epsilon, k, b \to 0\), and at the same time the velocity tends to a linear function. These numerical results illustrate that Riemann solution of equations (1) and (3) consisting of \(R_1\) and \(R_2\) converges to the vacuum solution of system (2) including a vacuum state \((\rho = 0)\) as \(\epsilon, k, b \to 0\).
To sum up, all of above numerical simulations completely support the theoretical analysis.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

**Authors’ Contributions**

All authors read and approved the final manuscript.

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