Twisted $T$-adic exponential sums are studied. The Hodge bound for the $T$-adic Newton polygon of the $C$-function is established. As an application, the behavior of the $L$-function under diagonal base change is explicitly given.

1. Introduction

1.1. Preliminaries. Let $F_q$ be the field of characteristic $p$ with $q$ elements, and $\mathbb{Z}_q = W(F_q)$. Let $T$ and $s$ be two independent variables. In this subsection we are concerned with the ring $\mathbb{Z}_q[[T]][[s]]$, elements of which are regarded as power series in $s$ with coefficients in $\mathbb{Z}_q[[T]]$.

Let $Q_p = \mathbb{Z}_p[\frac{1}{p}]$, $\overline{Q}_p$ the algebraic closure of $Q_p$, and $\widehat{Q}_p$ the $p$-adic completion of $\overline{Q}_p$.

Definition 1.1. A (vertical) specialization is a morphism $T \mapsto t$ from $\mathbb{Z}_q[[T]]$ into $\widehat{Q}_p$ with $0 \neq |t|_p < 1$.

We shall prove the vertical specialization theorem.

Theorem 1.2 (Vertical specialization). Let $A(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]]$ be a $T$-adic entire series in $s$. If $0 \neq |t|_p < 1$, then

$$t - \text{adic NP of } A(s, t) \geq T - \text{adic NP of } A(s, T),$$

where NP is the short for Newton polygon. Moreover, the equality holds for one $t$ iff it holds for all $t$.

By the vertical specialization, the Newton polygon of a $T$-adic entire series in $1 + s\mathbb{Z}_q[[T]][[s]]$ goes up under vertical specialization, and is stable under all specializations if it is stable under one specialization.

Definition 1.3. A $T$-adic entire series in $1 + s\mathbb{Z}_q[[T]][[s]]$ is said to be stable if its Newton polygon is stable under specialization.

Definition 1.4 (Tensor product). If

$$A(s, T) = \exp(-\sum_{k=1}^{+\infty} a_k(T) \frac{s^k}{k}),$$

and

$$B(s, T) = \exp(-\sum_{k=1}^{+\infty} b_k(T) \frac{s^k}{k}),$$

then
we define
\[ A \otimes B(s, T) = \exp\left(-\sum_{k=1}^{+\infty} a_k(T)b_k(T)\frac{s^k}{k}\right). \]

We have the distribution law
\[ (A_1A_2) \otimes B = (A_1 \otimes B)(A \otimes B). \]

So, equipped with the usual multiplication and the new tensor operation, the set of \( T \)-adic entire series in \( 1 + s\mathbb{Z}_q[[T]][[s]] \) becomes a ring. We shall prove that the stable \( T \)-adic entire series form a subring.

**Theorem 1.5.** The set of stable \( T \)-adic entire series \( 1 + s\mathbb{Z}_q[[T]][[s]] \) is closed under multiplication and tensor operation.

1.2. **Twisted \( T \)-adic exponential sums.** In this subsection we introduce \( L \)-functions of twisted \( T \)-adic exponential sums. The theory of \( T \)-adic exponential sums without twists was developed by Liu-Wan [LW].

Let \( \mu_{q-1} \) be the group of \((q - 1)\)-th roots of unity in \( \mathbb{Z}_q \), \( \omega : x \mapsto \hat{x} \) the Teichmüller character of \( \mathbb{F}_q^\times \) into \( \mu_{q-1} \), \( \chi = \omega^{-d} \) with \( s \in \mathbb{Z}^n/(q-1) \) a character of \( (\mathbb{F}_q^\times)^n \) into \( \mu_{q-1} \), and \( \chi_k = \chi \circ \text{Norm}_{\mathbb{F}_q^k/\mathbb{F}_q} \). Let \( \psi(x) = (1 + T)^x \) be the quasi-character from \( \mathbb{Z}_p \) to \( \mathbb{Z}_p[[T]][[s]] \), and \( \psi_q = \psi \circ \text{Tr}_{\mathbb{Z}_q/\mathbb{Z}_p} \). Let \( f \in \mu_{q-1}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) be a non-constant polynomial in \( n \)-variables with coefficients in \( \mu_{q-1} \).

**Definition 1.6.** The sum
\[ S_f(T, \chi) = S_f(T, \chi, \mathbb{F}_q) = \sum_{x \in \mu_{q-1}^n} x^{-d}\psi_q \circ f(x) \]
is called a twisted \( T \)-adic exponential sum. And the function
\[ L_f,\chi(s, T) = L_f,\chi(s, T, \mathbb{F}_q) = \exp\left(\sum_{k=1}^{+\infty} S_f(T, \chi_k, \mathbb{F}_q^k)\frac{s^k}{k}\right) \]
is called an \( L \)-function of twisted exponential sums.

We have
\[ L_f,\chi(s, T) = \prod_{x \in [G_m^\times \otimes \mathbb{F}_q]} \frac{1}{1 - \chi_{\deg(x)}(x)\psi_{\deg(x)} \circ f(\hat{x})s^{\deg(x)}}, \]
where \( G_m \) is the multiplicative group \( xy = 1 \).

That Euler product formula gives
\[ L_f,\chi(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]]. \]

Define
\[ C_f,\chi(s, T) = \exp\left(\sum_{k=1}^{+\infty} -\frac{1}{(q^k - 1)^n} S_f(T, \chi_k)\frac{s^k}{k}\right). \]

Call it a \( C \)-function of twisted \( T \)-adic exponential sums. We have
\[ L_f,\chi(s, T) = \prod_{i=0}^{n} C_f,\chi(q^{i}s, T)^{(-1)^{n-i+1}(n)}, \]
and
\[ C_{f,\chi}(s, T) = \prod_{j=0}^{+\infty} L_{f,\chi}(q^j s, T)^{-1^{n-j}+1^{-j}}. \]

So we have
\[ C_{f,\chi}(s, T) \in 1 + s\mathbb{Z}[T][[s]]. \]

We shall prove the analytic continuation of \( C_{f,\chi}(s, T) \).

**Theorem 1.7** (Analytic continuation). The series \( C_{f,\chi}(s, T) \) is \( T \)-adic entire in \( s \).

The analytic continuation of \( C_{f,\chi}(s, T) \) immediately gives the meromorphic continuation of \( L_{f,\chi}(s, T) \).

**Theorem 1.8** (Meromorphic continuation). The series \( L_{f,\chi}(s, T) \) is \( T \)-adic meromorphic.

**Definition 1.9.** The Laurent polynomial \( f \) is said to be \( \chi \)-twisted stable if \( C_{f,\chi}(s, T) \) is stable \( T \)-adic entire series in \( s \).

Let \( \zeta_p^m \) denote a primitive \( p^m \)-th root of unity. The specialization \( L_{f,\chi}(s, \zeta_p^m - 1) \) is the \( L \)-function of twisted algebraic exponential sums \( S_f(\zeta_p^m, \chi_k) \). These sums were studied by Liu [L], with the \( m = 1 \) case studied by Adolphson-Sperber [AS].

**Definition 1.10.** Let \( f(x) = \sum_{u \in I} a_u x^u \) with \( I \subseteq \mathbb{Z} \) and \( a_u^{-1} = 1 \). We define \( \triangle(f) \) to be the convex polytope in \( \mathbb{R}^n \) generated by the origin and the vectors \( u \in I \).

**Definition 1.11.** We call \( f \) non-degenerate if \( \triangle(f) \) is of dimension \( n \), and for every closed face \( \sigma \not\ni 0 \) of \( \triangle(f) \), the system
\[ \frac{\partial f_\sigma}{\partial x_1} \equiv \cdots \equiv \frac{\partial f_\sigma}{\partial x_n} \equiv 0 \pmod{p} \]
has no common zeros in \( (\mathbb{F}_q^\times)^n \), where \( f_\sigma = \sum_{u \in \sigma} a_u x^u \).

Gelfand-Kapranov-Zelevinsky proved the following.

**Theorem 1.12** ([GKZ]). Let \( \triangle \ni 0 \) be an integral convex polytope in \( \mathbb{R}^n \). If \( p \) is sufficiently large, and \( f \) is a generic Laurent polynomial in \( \triangle(f) = \triangle \), then \( f \) is non-degenerate.

By the above theorem, we are mainly concerned with non-degenerate \( f \). We have the following.

**Theorem 1.13** ([L]). If \( f \) is non-degenerate, then \( L_{f,\chi}(s, \zeta_p^m - 1, \mathbb{F}_q)^{-1^{n-1}} \) is a polynomial of degree \( p^{n(m-1)}\text{Vol}(\triangle(f)) \).

For non-degenerate \( f \), the determination of the Newton polygon of \( L_{f,\chi}(s, \zeta_p^m - 1, \mathbb{F}_q)^{-1^{n-1}} \) is a challenging problem. The case \( m = 1 \) is already very difficult, let alone the case \( m > 1 \). However, from the vertical specialization theorem, one can prove the following.

**Theorem 1.14** (Newton polygon for stable Laurent polynomials). Suppose that \( f \) is non-degenerate and \( \chi \)-twisted stable. Let \( \lambda_1, \cdots, \lambda_r \) be the slopes of the \( q \)-adic Newton polygon of
Then the slopes of the $q$-adic Newton polygon of $L_{f,\chi}(s, \zeta_p - 1)^{(-1)^{n-1}}$ are the numbers

$$\lambda_i + j_1 + j_2 + \cdots + j_n,$$

where $i = 1, \cdots, r$, and each $j_k = 0, 1, \cdots, p^{m-1} - 1$.

The non-degenerate condition for $f$ can be replaced by the condition that the functions $L_{f,\chi}(s, \zeta_p - 1)^{(-1)^{n-1}}$ and $L_{f,\chi}(s, \zeta_p^{m-1} - 1)^{(-1)^{n-1}}$ are polynomials. The above theorem reduces the determination of the Newton polygon of $L_{f,\chi}(s, \zeta_p - 1)^{(-1)^{n-1}}$ to the $m = 1$ case, provided that $f$ is non-degenerate and $\chi$-twisted stable. So, for non-degenerate $f$, we are mainly concerned with the stability of $f$ and the determination of the Newton polygon of $L_{f,\chi}(s, \zeta_p - 1)^{(-1)^{n-1}}$.

We shall prove the following stability criterion.

**Theorem 1.15** (Stability of ordinary Laurent polynomials). If $f$ is $\chi$-twisted ordinary, then it is $\chi$-twisted stable.

We now recall the notion of $\chi$-twisted ordinary Laurent polynomial. Let $\Delta \ni 0$ be an integral convex polytope in $\mathbb{R}^n$, $C(\Delta)$ the cone generated by $\Delta$, $M(\Delta) = C(\Delta) \cap \mathbb{Z}^n$, and $\deg_{\Delta}$ the degree function on $C(\Delta)$, which is $\mathbb{R}^+$ linear and takes the value 1 on each face $\delta \ni 0$. Let $d \in \mathbb{Z}^n / (q - 1)$, and

$$M_d(\Delta) := \frac{1}{q - 1}(M(\Delta) \cap d).$$

**Definition 1.16.** Let $b$ be the least positive integer such that $p^b d = d$. Order elements of $\cup_{i=0}^{b-1} M_{p^i d}(\Delta)$ so that

$$\deg_{\Delta}(x_1) \leq \deg_{\Delta}(x_2) \leq \cdots .$$

The infinite $d$-twisted Hodge polygon $H_{\Delta,d}^\infty$ of $\Delta$ is the convex function on $\mathbb{R}^+$ with initial value 0 which is linear between consecutive integers and whose slopes (between consecutive integers) are

$$\frac{\deg_{\Delta}(x_{b(i+1)}) + \deg_{\Delta}(x_{b(i+2)}) + \cdots + \deg_{\Delta}(x_{b(i+1)})}{b}, \ i = 0, 1, \cdots .$$

**Definition 1.17.** If $T - \text{adic NP of } C_{f,\omega^{-d}}(s, T, \mathbb{F}_q) = \text{ord}_p(q)(p - 1)H_{\Delta(f),d}^\infty$, then $f$ is called $\omega^{-d}$-twisted ordinary.

### 1.3. Exponential sums under diagonal base change

In this subsection we introduce the exponential sums associated to the tensor product of two Laurent polynomials.

**Definition 1.18.** If $g = \sum_v b_v y^v \in \mu_{q-1}^n [y_1^\pm 1, \cdots, y_m^\pm 1]$, we define

$$f \otimes g = \sum_{u,v} a_u b_v z_{u \otimes v} \in \mu_{q-1} [z_{ij}^\pm 1, i = 1, \cdots, n, j = 1, \cdots, m],$$

and call it a diagonal base change of $f$. 
The number \( q \) acts on the \( m \)-tuples \( u = (u_1, \ldots, u_m) \) of vectors in \( \mathbb{Z}^n_p / \mathbb{Z}^n \) by multiplication. The length of the orbit \( u \) is denoted by \( |u| \). The congruences \( u_1 \otimes v_1 + \cdots + u_m \otimes v_m \equiv \frac{d}{q-1} \), and \( (q^k - 1)u_j \equiv 0 \) are defined on the orbit space \( q \setminus (\mathbb{Z}^n_p / \mathbb{Z}^n)^m \).

We shall prove the following.

**Theorem 1.19.** Let \( v_1, \ldots, v_m \) be an integral basis of \( \mathbb{R}^m \), and \( g(y) = \sum_j b_j y^{v_j} \) with \( b_j \in \mu_{q-1}^n \). Then

\[
C_{f \otimes g, \omega^a}(s, T) = \prod_{(u_1, \ldots, u_m) \in q^{n}(\mathbb{Z}^n_p / \mathbb{Z}^n)^m} \otimes_{j=1}^m C_{f, \omega^{u_j(q^{j}|u|-1)}}(s^{u_j}, T, \mathbb{F}_q).
\]

As a function of \((u_1, \ldots, u_m)\), the tensor product on the right-hand side of the equality is defined on the orbit space. And, as the solutions of \( u_1 \otimes v_1 + \cdots + u_m \otimes v_m \equiv 0 \), under the map

\[
(u_1, \ldots, u_m) \mapsto u_1 \otimes v_1 + \cdots + u_m \otimes v_m,
\]

can be embedded into \( \mathbb{Z}^n \otimes \mathbb{Z}^m / \mathbb{Z}^n \otimes \sum_{j=1}^m \mathbb{Z}v_j \), the product on the right-hand side of the equality is a finite product.

The above theorem has the following equivalent form.

**Theorem 1.20.** Let \( v_1, \ldots, v_m \) be an integral basis of \( \mathbb{R}^m \), and \( g(y) = \sum_j b_j y^{v_j} \) with \( b_j \in \mu_{q-1}^n \). Then

\[
L_{f \otimes g, \omega^d}(s, T)^{(-1)^{mn-1}} = \prod_{(u_1, \ldots, u_m) \in q^{n}(\mathbb{Z}^n_p / \mathbb{Z}^n)^m} \otimes_{j=1}^m L_{f, \omega^{u_j(q^{j}|u|-1)}}(s^{u_j}, T, \mathbb{F}_q)^{(-1)^{n-1}}.
\]

By the above theorem, the Newton polygon of the \( L \)-function or \( C \)-function of \( T \)-adic (resp. algebraic) exponential sums of \( f \otimes g \) is determined by that of \( f \).

Combine the above with theorem the fact that the set of stable \( T \)-adic entire series in \( 1 + s\mathbb{Z}_q[[T]][[s]] \) is closed under multiplication and tensor operation, we get the following.

**Corollary 1.21.** Let \( v_1, \ldots, v_m \) be an integral basis of \( \mathbb{R}^m \), and \( g(x) = \sum_j b_j x^{v_j} \) with \( b_j \in \mu_{q-1}^n \). If \( f \) is \( \chi \)-twisted stable for all \( \chi \), then so is \( f \otimes g \).

### 2. Analytic continuation

In this section, we prove the analytic continuation of \( C_{f, \psi}(s, T, \mathbb{F}_q) \).

Define a new variable \( \pi \) by the relation \( E(\pi) = 1 + T \), where

\[
E(\pi) = \exp(\sum_{i=0}^{\infty} \pi^{p^i} \overline{p^i}) \in 1 + \pi \mathbb{Z}[[\pi]]
\]

is the Artin-Hasse exponential series. Thus, \( \pi \) is also a \( T \)-adic uniformizer of \( \mathbb{Q}_p((T)) \).
Let $\Delta = \Delta(f)$, and $D$ the least common multiple of the denominators of $\deg(\Delta)$. Write

$$L_d(\Delta) = \{ \sum_{u \in M_d(\Delta)} c_u \pi^{\deg(u)} x^u : c_u \in \mathbb{Z}_q[[\pi^{(q-1)}]] \},$$

and

$$B_d(\Delta) = \{ \sum_{u \in M_d(\Delta)} c_u \pi^{\deg(u)} x^u : c_u \in \mathbb{Z}_q[[\pi^{(q-1)}]], \ \text{ord}_T(c_u) \to +\infty \text{ if } \deg(u) \to +\infty \}.$$

Note that $L_d(\Delta)$ is stable under multiplication by elements of $L_0(\Delta)$, and

$$E_f(x) = \prod_{a_u \neq 0} E(\pi a_u \hat{x}^u) \in L_0(\Delta).$$

Define

$$\phi : L_d(\Delta) \to L_{dp^{-1}}(\Delta), \quad \sum_{u \in M_d(\Delta)} c_u x^u \mapsto \sum_{u \in M_{dp^{-1}}(\Delta)} c_{pu} x^u.$$

Then the map $\phi \circ E_f$ sends $L_d$ to $B_{dp^{-1}}$.

**Lemma 2.1.** If $x^{p^a-1} = 1$, then

$$E(\pi)^{x+x^p+\cdots+x^{p^{a-1}}} = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}).$$

**Proof.** Since

$$\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+1}},$$

we have

$$E(\pi)^{x+x^p+\cdots+x^{p^{a-1}}} = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \sum_{j=0}^{a-1} x^{p^{j+i}} \right) = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}).$$

The Galois group $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is generated by the Frobenius element $\sigma$, whose restriction to $(q-1)$-th roots of unity is the $p$-power map. That Galois group can act on $L(\Delta)$ by fixing $\pi^{(q-1)}$ and $x_1, \cdots, x_n$.

**Lemma 2.2** (Dwork’s splitting lemma). If $q = p^a$, and $x \in (\mathbb{F}_q^\times)^n$, then

$$E(\pi)^{\text{Tr}_{q^k/q}(\sum a_u \hat{x}^u)} = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(\hat{x}^{p^i}).$$

**Proof.** We have

$$E(\pi)^{\text{Tr}_{q^k/q}(\sum a_u \hat{x}^u)} = \prod_{a_u \neq 0} E(\pi)^{\text{Tr}_{q^k/q}(a_u \hat{x}^u)}$$

$$= \prod_{a_u \neq 0} \prod_{i=0}^{ak-1} E(\pi(\hat{a}_u \hat{x}^{p^i})) = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(\hat{x}^{p^i}).$$

□
Define \( c(u, v) = \deg(u) + \deg(v) - \deg(u + v) \) if \( u, v \in C(\Delta) \). Then \( c(u, v) \geq 0 \), and is zero if and only if \( u \) and \( v \) are cofacial. We call \( c(u, v) \) the cofacial defect of \( u \) and \( v \).

**Lemma 2.3.** Write

\[
E_f(x) = \sum_{u \in M(\Delta)} \alpha_u(f) \pi^{\deg(u)} x^u.
\]

Then, for \( u \in M_d(\Delta) \), we have

\[
\phi \circ E_f(\pi^{\deg(u)} x^u) = \sum_{u \in M_{d-1}(\Delta)} \alpha_{pu-u}(f) \pi^{c(pu-u,u)} \pi^{(p-1) \deg(u)} \pi^{\deg(u)} x^u.
\]

**Proof.** Obvious. \( \square \)

Define \( \phi_p := \sigma^{-1} \circ \phi \circ E_f \), and \( \phi_{p^a} = \phi_p^a \). Then \( \phi_{p^a} \) sends \( B_d \) to \( B_{d-p^a} \), and

\[
\phi_{p^a} = \sigma^{-a} \circ \phi^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}).
\]

It follows that \( \phi_q \) operates on \( B_d \), and is linear over \( \mathbb{Z}_q[[\pi^{1/(q-1)}]] \). Moreover, by the last lemma, it is completely continuous in the sense of [?].

**Theorem 2.4** (Dwork’s trace formula). Suppose that \( \chi = \omega^{-d} \). Then

\[
S_{f,\chi}(T, \mathbb{F}_q) = (q^k - 1)^n \text{Tr}_{B_d/\mathbb{Q}_q[\pi^{1/(q-1)}]}(\phi_q^k), \quad k = 1, 2, \ldots.
\]

**Proof.** Suppose that \( q = p^a \). Let \( g(x) \in B_d \). We have

\[
\phi_q^k(g) = \phi^k(g \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i})).
\]

Write \( \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_u x^u \). Then

\[
\phi_q^k(\pi^{\deg(v)} x^v) = \sum_{u \in M_d(\Delta)} \beta_{q^k u - v} \pi^{\deg(v)} x^u.
\]

So the trace of \( \phi_q^k \) on \( B_d \) over \( \mathbb{Z}_q[[\pi^{1/(q-1)}]] \) equals \( \sum_{u \in M_d(\Delta)} \beta_{(q^k-1)u} \). But, by Dwork’s splitting lemma, we have

\[
S_{f,\chi}(T, \mathbb{F}_q) = \sum_{x_1^{q-1} = 1, \ldots, x_n^{q-1} = 1} x^{-d(1+q+\cdots+q^{k-1})} \prod_{i=0}^{k-1} E_f^{\sigma^i}(x^{p^i}) = (q^k - 1)^n \sum_{u \in M_d(\Delta)} \beta_{(q^k-1)u}.
\]

The theorem now follows. \( \square \)

**Theorem 2.5** (Analytic trace formula). If \( \chi = \omega^{-d} \), then

\[
C_{f,\chi}(s, T, \mathbb{F}_q) = \det_{\mathbb{Z}_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | B_d).
\]

In particular, \( C_{f,\chi}(s, T, \mathbb{F}_q) \) is \( T \)-adic analytic in \( s \).
Proof. This follows from the last theorem and the identity
\[
det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | B_d) = \exp\left(-\sum_{k=1}^{+\infty} \text{Tr}_{B_d/Z_q[[\pi^{1/(q-1)}]]} (\phi_q^k) \frac{s^k}{k} \right).
\]

3. Hodge bound

In this section, we prove the Hodge bound for the Newton polygon of \( C_{f,\chi}(s, T, \mathbb{F}_q) \). It will play an important role in establishing the stability of ordinary Laurent polynomial.

Let the Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}_p) \) act on \( Z_q[[T]][[s]] \) by fixing \( s \) and \( T \).

Lemma 3.1. We have
\[
C_{f,\chi}(s, T, \mathbb{F}_q)^p = C_{f,\chi}(s, T, \mathbb{F}_q).
\]

Proof. Obvious.

Corollary 3.2. Suppose that \( q = p^a \) and \( \chi = \omega^{-d} \). Let \( b \) be the least positive integer such that \( p^b d = d \). Then, as power series in \( s \) with coefficients in \( Z_q[[T]] \),
\[
\text{NP of } C_{f,\chi}(s, T, \mathbb{F}_q)^{ab} = \text{NP of } \det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | \oplus_{i=0}^{b-1} B_{p^d}).
\]

Proof. In fact, we have
\[
\det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | \oplus_{i=0}^{b-1} B_{p^d}) = \prod_{j=0}^{a-1} \det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | \oplus_{i=0}^{b-1} B_{p^d})^{\sigma^j} = \prod_{j=0}^{a-1} \prod_{i=0}^{b-1} \det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_q s | B_d)^{\sigma^j}.
\]
The corollary now follows.

Corollary 3.3. Suppose that \( q = p^a \) and \( \chi = \omega^{-d} \). Let \( b \) be the least positive integer such that \( p^b d = d \). Then, as power series in \( s \) with coefficients in \( Z_q[[T]] \),
\[
\text{NP of } C_{f,\chi}(s^a, T, \mathbb{F}_q)^b = \text{NP of } \det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_p s | \oplus_{i=0}^{b-1} B_{p^d}).
\]

Proof. This follows from the identity
\[
\det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_p s^a | \oplus_{i=0}^{b-1} B_{p^d}) = \prod_{\zeta^a = 1} \det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_p \zeta s | \oplus_{i=0}^{b-1} B_{p^d}).
\]

Theorem 3.4. Suppose that \( q = p^a \) and \( b \) is the least positive integer such that \( p^b d = d \). Then, as a power series in \( s \) with coefficients in \( Z_q[[T]] \), the \( T \)-adic Newton polygon of
\[
\det_{Z_q[[\pi^{1/(q-1)}]]} (1 - \phi_p s | \oplus_{i=0}^{b-1} B_{p^d})
\]
lies above the convex polygon with initial point \((0,0)\) and slopes \((p-1)\text{deg}(w)\), where \( w \) runs through elements of \( \cup_{i=0}^{b-1} M_{p^d} \) with multiplicity \( a \).
Proof. Choose \( \zeta \in \mathbb{Z}_q^\times \) such that \( \zeta^{a_i} \), \( i = 0, 1, \ldots, a - 1 \) be a basis of \( \mathbb{Z}_q \) over \( \mathbb{Z}_p \). Write
\[
\alpha_u(f) = \sum_{i=0}^{a-1} \alpha_{u,i}(f) \zeta^{a_i}, \quad \alpha_{u,i}(f) \in \mathbb{Z}_p[[\pi^{1/D}]].
\]
Then, for \( u \in M_{p'}d(\Delta) \), we have
\[
\phi_p(\zeta^{\sigma_i} \pi^{\deg(u)}x^u) = \sum_{i=0}^{a-1} \sum_{w \in M_{d^{p'-1}}(\Delta)} \alpha_{pw-u,i-1}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg(w)} \zeta^{\sigma_i} \pi^{\deg(w)} x^w.
\]
So, the matrix of \( \phi_p \) over \( \mathbb{Z}_p[[\pi^{1/(p-1)}]] \) with respect to the basis \( \{ \zeta^{q^i} \pi^{\deg(u)}x^u \}_{0 \leq i < a, u \in \bigoplus_{i=0}^{d-1} B_{p'}d(\Delta)} \) is
\[
A = (\alpha_{pw-u,i-1}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg(w)}) (1 - \phi_p s | \bigoplus_{i=0}^{d-1} B_{p'}d(\Delta))
\]
It follows that, the \( T \)-adic Newton polygon of \( \det_{\mathbb{Z}_p[[\pi^{1/(p-1)}]]}(1 - \phi_p s | \bigoplus_{i=0}^{d-1} B_{p'}d(\Delta)) \) lies above the convex polygon with initial point \((-1,0)\) and slopes \((p-1)\deg(w)\), where \( w \) runs through elements of \( \bigcup_{i=0}^{d-1} M_{p'}d \) with multiplicity \( a \).

Corollary 3.5. Suppose that \( q = p^a \) and \( \chi = \omega^{-d} \). Let \( b \) be the least positive integer such that \( p^b d = d \). Then, as a power series in \( s \) with coefficients in \( \mathbb{Z}_q[[T]] \), the \( T \)-adic Newton polygon of \( C_f,\chi(s,T,\mathbb{F}_q)^b \) lies above the convex polygon with initial point \((-1,0)\) and slopes \((p-1)\deg(w)\), \( w \in \bigcup_{i=0}^{d-1} M_{p'}d \).

Proof. Obvious.

Theorem 3.6 (Hodge bound). Suppose that \( \chi = \omega^{-d} \), and \( q = p^a \). Then
\[
T - \text{adic NP of } C_f,\chi(s,T) \geq a(p-1)H_{\Delta(f),d}^\infty,
\]
where \( NP \) is the short for Newton polygon, and \( H_{\Delta(f),d}^\infty \) is the infinite \( d \)-twisted Hodge polygon of \( \Delta(f) \).

Proof. Obvious.

Definition 3.7. If \( \chi = \omega^{-d} \), \( q = p^a \), \( 0 \neq |t|_p < 1 \), and
\[
|t|_p \leq 1,
\]
then \( f \) is said to be \( \chi \)-twisted ordinary.

4. Vertical specialization and stability

In this section we prove the vertical specialization theorem, the theorem for the Newton polygon of stable Laurent polynomials, and the stability of ordinary Laurent polynomials.

Theorem 4.1 (Vertical specialization). Let \( A(s,T) \in 1 + s\mathbb{Z}_q[[T]][[s]] \) be a \( T \)-adic entrie series in \( s \). If \( 0 \neq |t|_p < 1 \), then
\[
t - \text{adic NP of } A(s,T) \geq T - \text{adic NP of } A(s,T),
\]
where \( NP \) is the short for Newton polygon. Moreover, the equality holds for one \( t \) iff it holds for all \( t \).
Proof. Write
\[ A(s, T) = \sum_{i=0}^{\infty} a_i(T)s^i. \]
The inequality follows from the fact that \( a_i(T) \in \mathbb{Z}_q[[T]] \). Moreover,
\[ t - \text{adic NP of } A(s, t) = T - \text{adic NP of } A(s, T) \]
if and only if
\[ a_i(T) \in T^e \mathbb{Z}_q[[T]]^\times \]
for every turning point \((i, e)\) of the \(T\)-adic Newton polygon of \(A(s, T)\). It follows that the equality holds for one \(t\) iff it holds for all \(t\).

\[ \square \]

**Theorem 4.2** (Newton polygon for stable Laurent polynomials). Suppose that \(f\) is non-degenerate and \(\chi\)-twisted stable. Let \(\lambda_1, \cdots, \lambda_r\) be the slopes of the \(q\)-adic Newton polygon of \(L_{f,\chi}(s, \zeta_p - 1)(-1)^{n-1}\). Then the \(q\)-adic orders of the reciprocal roots of \(L_{f,\chi}(s, \zeta_p m - 1)(-1)^{n-1}\) are the numbers
\[ \frac{\lambda_i + j_1 + j_2 + \cdots + j_n}{p^{m-1}}, \]
where \(i = 1, \cdots, r\), and each \(j_k = 0, 1, \cdots, p^{m-1} - 1\).

**Proof.** Apply the relationship between the \(L\)-function and the \(C\)-function, we see that the \(q\)-adic orders of the reciprocal roots of \(C_{f,\chi}(s, \zeta_p - 1)\) are the numbers
\[ \lambda_i + j_1 + j_2 + \cdots + j_n, \]
where \(i = 1, \cdots, r\), and each \(j_k = 0, 1, \cdots, (\zeta_p - 1)\)-adic orders of the reciprocal roots of \(C_{f,\chi}(s, \zeta_p m - 1)\) are the numbers
\[ \text{ord}_{\zeta_p - 1}(q)(\lambda_i + j_1 + j_2 + \cdots + j_n), \]
where \(i = 1, \cdots, r\), and each \(j_k = 0, 1, \cdots, (\zeta_p m - 1)\)-adic orders of the reciprocal roots of \(C_{f,\chi}(s, \zeta_p m - 1)\) are the numbers
\[ \text{ord}_{\zeta_p - 1}(q)(\lambda_i + j_1 + j_2 + \cdots + j_n), \]
where \(i = 1, \cdots, r\), and each \(j_k = 0, 1, \cdots, (\zeta_p m - 1)\)-adic orders of the reciprocal roots of \(L_{f,\chi}(s, \zeta_p m - 1)\) are the numbers
\[ \frac{\lambda_i + j_1 + j_2 + \cdots + j_n}{p^{m-1}}, \]
where \(i = 1, \cdots, r\), and each \(j_k = 0, 1, \cdots, p^{m-1} - 1\).

\[ \square \]

**Theorem 4.3** (Specialization of the Hodge bound). If \(\chi = \omega^{-d}\), \(q = p^a\), and \(0 \neq |t|_p < 1\), then
\[ t - \text{adic NP of } C_{f,\chi}(s, t) \geq T - \text{adic NP of } C_{f,\chi}(s, T) \geq a(p - 1)H^{\infty}_{\Delta(f), d}. \]
Moreover, the equalities hold for one \(t\) iff they hold for all \(t\).
Proof. Just combine the Hodge bound for the Newton polygon of $C_{f,\chi}(s, T)$ with the vertical specialization theorem.

**Theorem 4.4** (Stability of ordinary Laurent polynomials). If $f$ is $\chi$-twisted ordinary, then it is $\chi$-twisted stable, and $\chi$-twisted $T$-adic ordinary.

**Proof.** Obvious. □

**Definition 4.5.** Let $\alpha_1, \alpha_2, \cdots$ be the slopes of the infinite $d$-twisted Hodge polygon of $\triangle$. Then

$$(1 - t)^n \sum_i t^{\alpha_i} = \sum_{i=1}^{n! \text{Vol}(\triangle)} t^{w_i}.$$  

The $d$-twisted Hodge polygon $H_{\triangle, d}$ of $\triangle$ is the convex function on $[0, n! \text{Vol}(\triangle)]$ with initial value 0 which is linear between consecutive integers and whose slopes (between consecutive integers) are $w_i$, $i = 1, \cdots, n! \text{Vol}(\triangle)$.

**Theorem 4.6** (Newton polygon for ordinary Laurent polynomials). Let $f$ be non-degenerate, $\chi = \omega^{-d}$ and $m \geq 1$. Then $f$ is $\chi$-twisted ordinary if and only if

$q - \text{adic NP of } L_{f,\chi}(s, \zeta_p m - 1)^{(-1)^{n-1}} = H_{p^m-1 \triangle(f), d}.$  

The non-degenerate condition for $f$ can be replaced by the condition that the function $L_{f,\chi}(s, \zeta_p m - 1)^{(-1)^{n-1}}$ is a polynomial.

**Proof.** In fact, $f$ is $\chi$-twisted ordinary if and only if

$(\zeta_p - 1) - \text{adic NP of } C_{f,\chi}(s, \zeta_p - 1) = \text{ord}_p(q)(p - 1)H_{\triangle(f), d}^{\infty},$  

if and only if

$(\zeta_p m - 1) - \text{adic NP of } C_{f,\chi}(s, \zeta_p m - 1) = \text{ord}_p(q)(p - 1)H_{\triangle(f), d}^{\infty},$  

if and only if

$q - \text{adic NP of } C_{f,\chi}(s, \zeta_p m - 1) = \frac{1}{p^m-1}H_{\triangle(f), d}^{\infty} = H_{p^m-1 \triangle(f), d}^{\infty},$  

if and only if

$q - \text{adic NP of } L_{f,\chi}(s, \zeta_p m - 1)^{(-1)^{n-1}} = H_{p^m-1 \triangle(f), d}.$  

□

**Lemma 4.7** (Hasse-Davenport relation).

$L_{x,\chi}(s, \zeta_p, \overline{\mathbb{F}}_q) = 1 + G(\zeta_p, \chi, \overline{\mathbb{F}}_q)s.$

**Proof.** This follows from the the following classical formulation:

$G(\zeta_p, \chi_k, \overline{\mathbb{F}}_q^k) = (-1)^{k-1}G(\zeta_p, \chi, \overline{\mathbb{F}}_q)^k.$  

□

**Definition 4.8.** Let $q = p^a$, and $d \in \mathbb{Z}/(q - 1)$. We define

$\sigma_q(d) = (p - 1) \sum_{i=0}^{a-1} \left\lfloor \frac{pd}{q - 1} \right\rfloor.$
Theorem 4.9 (Stickelberger theorem for Gauss sums). The polynomial \( f(x) = x \) is \( \chi \)-twisted ordinary for all \( \chi \).

Proof. We have \( \triangle = \triangle(f) = [0, 1] \), \( C(\triangle) = \mathbb{R}_+ \), and \( \deg \_\triangle(u) = u \). Let \( q = p^a \), and \( \chi = \omega^{-d} \). We have \( M(\triangle) = \mathbb{N} \), and \( M_{\mu^d}(\triangle) = \{ \frac{p^d}{q-1} \} + \mathbb{N} \). It follows that the infinite \( d \)-twisted Hodge polygon of \( \triangle \) has slopes \( \frac{\sigma_q(d)}{a(p-1)} + k \), \( k = 0, 1, \ldots \).

So the finite \( d \)-twisted Hodge polygon of \( \triangle \) has only one slope \( \frac{\sigma_q(d)}{a(p-1)} \). By the Hasse-Davenport relation and the classical Stickelberger theorem, the Newton polygon of the \( L \)-function \( L_{x,\chi}(s, \zeta_{p^{m-1}}) \) also has only one slope \( \frac{\sigma_q(d)}{a(p-1)} \). Therefore the polynomial \( f(x) = x \) is \( \chi \)-twisted ordinary.

\[ \square \]

Theorem 4.10 (Stickelberger theorem for Gauss-Heilbronn sums). Let \( q = p^a \), and \( \chi = \omega^{-d} \). Then \( q \)-adic orders of the reciprocal zeros of the \( L \)-function \( L_{x,\chi}(s, \zeta_{p^{m-1}}) \) of the Gauss-Heilbronn sums \( G(\zeta_{p^m-1}, \chi_k, \mathbb{F}_{q_k}) \) are

\[
\frac{\sigma_q(d)}{a(p-1)p^{m-1}} + \frac{k}{p^{m-1}}, \quad k = 0, 1, \ldots, p^{m-1} - 1.
\]

The above theorem was proved by Blache [B], and Liu [L]. But the proof here is much simpler.

Proof. Apply the Stickelberger theorem for Gauss sums and the theorem on the Newton polygon for ordinary \( p \)-power order exponential sums, we get

\( q \)-adic NP of \( L_{x,\chi}(s, \zeta_{p^{m-1}}) = H_{p^{m-1}\_\triangle,d} \)

with \( \triangle = [0, 1] \). In the proof of the Stickelberger theorem for Gauss sums, we show that

\[ M_{p^d}(\triangle) = \{ \frac{p^d}{q-1} \} + \mathbb{N}. \]

It follows that the infinite \( d \)-twisted Hodge polygon of \( p^{m-1}\_\triangle \) has slopes

\[
\frac{\sigma_q(d)}{a(p-1)p^{m-1}} + \frac{k}{p^{m-1}}, \quad k = 0, 1, \ldots.
\]

So the finite \( d \)-twisted Hodge polygon of \( p^{m-1}\_\triangle \) has slopes

\[
\frac{\sigma_q(d)}{a(p-1)p^{m-1}} + \frac{k}{p^{m-1}}, \quad k = 0, 1, \ldots, p^{m-1} - 1.
\]

The theorem now follows. \[ \square \]

5. Stable \( T \)-adic entire series

In this section we prove that the set of stable \( T \)-adic entire series in \( 1 + s\mathbb{Z}_q[[T]][[s]] \) is closed under multiplication and tensor operation.
Lemma 5.1 (Weierstrass preparation theorem). Let $A(s, T) \in \mathbb{Z}_q[[T]]\langle s \rangle$ be a $T$-adically strictly convergent power series in $s$ with unitary constant term. Suppose that $A(s, T) \mod T \in \mathbb{Z}_q[s]$ is a unitary polynomial of degree $n$. Then

$$A(s, T) = u(s, T)B(s, T),$$

where $u(s, T) \in 1 + T\mathbb{Z}_q[[T]]\langle s \rangle$, and $B(s, T) \in \mathbb{Z}_q[[T]][s]$ is a monic polynomial of degree $n$ with unitary constant term.

Proof. Let $A_0(s) = A(s, T) \mod T$, and $\alpha = \text{ord}_T(A(s, T) - A_0(s))$. Then

$$\mathbb{Z}_q[[T]]\langle s \rangle/(T^n, A(s, T)) = \mathbb{Z}_q[[T]]\langle s \rangle/(T^n, A_0(s)),$$

and is generated as $\mathbb{Z}_q[[T]]$-module by $1, s, \ldots, s^{n-1}$. In particular.

$$v_0 = s^n = \sum_{j=0}^{n-1} a_{ij} s^j + v_1 T^n + w_1 A(s, T), \quad v_1 \in \mathbb{Z}_q[[T]]\langle s \rangle, \quad w_1 \in 1 + T\mathbb{Z}_q[[T]]\langle s \rangle.$$

By induction, we can construct sequences $v_i, w_i \in \mathbb{Z}_q[[T]]\langle s \rangle$ so that

$$v_{i-1} = \sum_{j=0}^{n-1} a_{ij} s^j + v_i T^n + w_i A(s, T).$$

We have

$$\sum_{i=1}^{\infty} v_{i-1} T^{(i-1)\alpha} = \sum_{j=0}^{n-1} s^j \sum_{i=1}^{\infty} a_{ij} T^{(i-1)\alpha} + \sum_{i=1}^{\infty} v_i T^{i\alpha} + A(s, T) \sum_{i=1}^{\infty} w_i T^{(i-1)\alpha}.$$

So

$$s^n - \sum_{j=0}^{n-1} s^j \sum_{i=1}^{\infty} a_{ij} T^{(i-1)\alpha} = A(s, T) \sum_{i=1}^{\infty} w_i T^{(i-1)\alpha}.$$

Since $w = \sum_{i=1}^{\infty} w_i T^{(i-1)\alpha} \in 1 + T\mathbb{Z}_q[[T]]\langle s \rangle$, we have $u(s, T) = w^{-1} \in 1 + T\mathbb{Z}_q[[T]]\langle s \rangle$. Set

$$B(s, T) = s^n - \sum_{j=0}^{n-1} s^j \sum_{i=1}^{\infty} a_{ij} T^{(i-1)\alpha} \in \mathbb{Z}_q[[T]][s],$$

we get

$$A(s, T) = u(s, T)B(s, T).$$

\[\square\]

Theorem 5.2 (Weierstrass factorization theorem). Let $A(s, T) \in 1 + s\mathbb{Z}_q[[T]]\langle s \rangle$ be a $T$-adically entire power series in $s$, whose Newton polygon has slopes $\lambda_i$ of horizontal length $n_i$. Then

$$A(s, T) = \prod_{i=1}^{\infty} A_i(s),$$

where $A_i(s) \in \mathbb{Z}_q[[T^{\lambda_1}, \ldots, T^{\lambda_i}]][s]$ is polynomial of degree $n_i$ with unitary constant term and linear Newton polygon.

Proof. Apply the Weierstrass preparation theorem to construct $A_i(s)$ inductively.  \[\square\]
Lemma 5.3. Let $A(s, T)$ and $B(s, T)$ be two $T$-adic entire power series in $1 + s\mathbb{Z}_q[[T]][[s]]$. Suppose that

$$A(s, T) = \prod_{\alpha \in I} (1 - \alpha s),$$

and

$$B(s, T) = \prod_{\beta \in J} (1 - \beta s).$$

Then

$$A \otimes B(s, T) = \prod_{\alpha \in I, \beta \in J} (1 - \alpha \beta s).$$

Proof. We have

$$A(s, T) = \exp\left(-\sum_{k=1}^{+\infty} \frac{s^k}{k} \sum_{\alpha \in I} \alpha^k\right),$$

and

$$B(s, T) = \exp\left(-\sum_{k=1}^{+\infty} \frac{s^k}{k} \sum_{\beta \in J} \beta^k\right).$$

So

$$A \otimes B(s, T) = \exp\left(-\sum_{k=1}^{+\infty} \frac{s^k}{k} \sum_{\alpha \in I, \beta \in J} \alpha^k \beta^k\right) = \prod_{\alpha \in I, \beta \in J} (1 - \alpha \beta s).$$

□

Lemma 5.4. Let $A(s, T)$ and $B(s, T)$ be two $T$-adic entire power series in $1 + s\mathbb{Z}_q[[T]][[s]]$. Then $A(s, T)B(s, T)$ is stable iff both $A(s, T)$ and $B(s, T)$ are stable.

Proof. Write

$$A(s, T) = \sum_n A_n(T)s^n,$$

$$B(s, T) = \sum_n B_n(T)s^n,$$

and

$$A(s, T)B(s, T) = \sum_n C_n(T)s^n.$$

Let $\{\alpha\}$ be the set of the reciprocal zeros of $A(s, T)$, and $\{\beta\}$ the set of reciprocal zeros of $B(s, T)$. Let $(n, e)$ be a turning points of the Newton polygon of $A(s, T)B(s, T)$. Then $n = \sum_{\text{ord}_T(\alpha) \leq r} 1 + \sum_{\text{ord}_T(\beta) \leq r} 1$ for some $r \in \mathbb{R}_+$, and

$$C_n(T) \equiv (-1)^n \prod_{\text{ord}_T(\alpha) \leq r} \alpha \prod_{\text{ord}_T(\beta) \leq r} \beta \equiv A_{n_1}(T)B_{n_2}(T) \pmod{T^{>e}},$$

where $n_1 = \sum_{\text{ord}_T(\alpha) \leq r} 1$, and $n_2 = \sum_{\text{ord}_T(\beta) \leq r} 1$. Note that the $T$-adic order of $C_n(T)$ is stable under specialization iff both the $T$-adic orders of $A_{n_1}(T)$ and $B_{n_2}(T)$ are stable under specialization. The lemma now follows. □
Theorem 5.5. The set of stable $T$-adic entire series in $1 + s\mathbb{Z}_q[[T]][[s]]$ is closed under tensor operation.

Proof. Let $A(s, T)$ and $B(s, T)$ be two stable $T$-adic entire power series in $1 + s\mathbb{Z}_q[[T]][[s]]$. By the Weierstrass factorization theorem, and the last lemma, we may assume that $A$ and $B$ are polynomials with linear Newton polygon. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the set of the reciprocal zeros of $A(s, T)$, and $\{\beta_1, \ldots, \beta_n\}$ the set of reciprocal zeros of $B(s, T)$. Then the leading term of $A \otimes B$ is

$$C_{mn}(T) = \prod_{i=1}^{m} \prod_{j=1}^{n} (-\alpha_i\beta_j) = (-1)^{mn} A_m(T)^n B_n(T),$$

where $A_m$ is the leading coefficient of $A(s, T)$, and $B_n$ is the leading coefficient of $B(s, T)$. Since the $T$-adic orders of $A_m(T)$ and $B_n(T)$ do not go up under specialization, so does the $T$-adic order of $C_{mn}(T)$. The theorem is proved. \qed

6. Exponential sums under the tensor operation

In this section we explore Wan’s method [W], and study the exponential sums associated to the tensor product of two Laurent polynomials.

Lemma 6.1. Let $v_1, \ldots, v_m$ be an integral basis of $\mathbb{R}^m$, and $g(y) = \sum_j b_j y^{u_j}$ with $b_j \in \mu_q^n - 1$. Then

$$S_{f \otimes g}(T, \omega^{-d}) = \sum_{u_1, \ldots, u_m \in \mathbb{Z}^n/(q-1)} \prod_{j=1}^{m} b_j^{u_j} \prod_{j=1}^{m} S_f(T, \omega^{-u_j}, \mathbb{F}_q).$$

Proof. Since

$$S_f(T, \omega^{-u}, \mathbb{F}_q) = \sum_{\alpha \in \mu_q^n} \alpha^{-u} \psi_q \circ f(\alpha),$$

we have

$$\psi_q \circ f(\alpha) = \frac{1}{(q-1)^n} \sum_{u \in \mathbb{Z}^n/(q-1)} \alpha^u S_f(T, \omega^{-u}, \mathbb{F}_q).$$

So

$$\psi_q(f \otimes b_j y^{u_j}) = \frac{1}{(q-1)^n} \sum_{u \in \mathbb{Z}^n/(q-1)} b_j^u z^{u \otimes u_j} S_f(T, \omega^{-u}, \mathbb{F}_q).$$

Thus,

$$\psi_q(f \otimes g(z)) = \frac{1}{(q-1)^{mn}} \prod_{j=1}^{m} \sum_{u \in \mathbb{Z}^n/(q-1)} b_j^u z^{u \otimes u_j} S_f(T, \omega^{-u}, \mathbb{F}_q).$$

Therefore

$$S_{f \otimes g}(T, \omega^{-d}) = \sum_{z \in \mu_q^{mn} - 1} \frac{z^{-d}}{(q-1)^{mn}} \sum_{u_1, \ldots, u_m \in \mathbb{Z}^n/(q-1)} z^{u_1 \otimes u_1 + \cdots + u_m \otimes u_m} \prod_{j=1}^{m} b_j^{u_j} S_f(T, \omega^{-u_j}).$$

Change the order of summation, we get the desired formula. \qed
Corollary 6.2. Let $v_1, \cdots, v_m$ be an integral basis of $\mathbb{R}^m$, and $g(y) = \sum_j b_j y^u_j$ with $b_j \in \mu^n_{q-1}$. Then

$$S_{f \otimes g}(T, \omega^{\frac{d}{m}(q^k-1)}, \mathbb{F}_{q^k}) = \sum_{u_1 \cdot \cdots \cdot u_m \in \mathbb{Z}_{(p)}/\mathbb{Z}^n} \prod_{j=1}^m b_j^{u_j(q^k-1)} \prod_{j=1}^m S_f(T, \omega^{-u_j(q^k-1)}, \mathbb{F}_{q^k}).$$

Proof. Just scale the variables $u_j$ in the last lemma.

Lemma 6.3. We have

$$S_f(T, \chi^q, \mathbb{F}_{q^k}) = S_f(T, \chi, \mathbb{F}_{q^k}).$$

Proof. Since $\sigma : x \mapsto x^q$ is an automorphism of $\mu^n_{q-1}$, and extends to be an element of $\text{Gal}(\mathbb{Q}_{q^k}/\mathbb{Q})$, we have

$$S_f(T, \chi^q, \mathbb{F}_{q^k}) = \sum_{x \in \mu^n_{q-1}} \chi(x^q) \psi_{q^k} \circ f(x) = \sum_{x \in \mu^n_{q-1}} \chi(x) \psi_{q^k} \circ f(x^{q-1}).$$

Note that

$$\psi_{q^k} \circ f(x^{q-1}) = \psi_{q^k} \circ \text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}}(f(x)^{q-1}) = \psi_{q^k} \circ \text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}}(f(x)) = \psi_{q^k} \circ f(x).$$

The lemma now follows.

By the above lemma, the product

$$\prod_{j=1}^m b_j^{-u_j(q^k-1)} \prod_{j=1}^m S_f(T, \omega^{-u_j(q^k-1)}, \mathbb{F}_{q^k}),$$

as a function of $(u_1, \cdots, u_m)$, is also defined on the orbit space $q \setminus (\mathbb{Z}_p^n)/\mathbb{Z}^n$. So we can restate the last corollary as follows.

Corollary 6.4. Let $v_1, \cdots, v_m$ be an integral basis of $\mathbb{R}^m$, and $g(y) = \sum_j b_j y^u_j$ with $b_j \in \mu^n_{q-1}$. Then

$$S_{f \otimes g}(T, \omega^{\frac{d}{m}(q^k-1)}, \mathbb{F}_{q^k}) = \sum_{(u_1, \cdots, u_m) \in q \setminus (\mathbb{Z}_p^n)/\mathbb{Z}^n} \prod_{j=1}^m b_j^{u_j(q^k-1)} \prod_{j=1}^m S_f(T, \omega^{-u_j(q^k-1)}, \mathbb{F}_{q^k}).$$

Theorem 6.5. Let $v_1, \cdots, v_m$ be an integral basis of $\mathbb{R}^m$, and $g(y) = \sum_j b_j y^u_j$ with $b_j \in \mu^n_{q-1}$. Then

$$C_{f \otimes g, \omega^{-d}}(s, T) = \prod_{(u_1, \cdots, u_m) \in q \setminus (\mathbb{Z}_p^n)/\mathbb{Z}^n} \times_{(u_1, \cdots, u_m) \in q \setminus (\mathbb{Z}_p^n)/\mathbb{Z}^n} \prod_{j=1}^m b_j^{-u_j(q^{|u|}-1)} C_{f, \omega^{-u_j(q^{|u|}-1)}(s^u)} \prod_{j=1}^m b_j^{-u_j(q^{|u|}-1)} T, \mathbb{F}_{q^{|u|}}).$$
Proof. We have

\[ \sum_{k=1}^{\infty} \frac{-1}{(q^k - 1)^{mn} k} S_f \otimes g(T, \omega^{-\frac{d}{q-1}(q^k-1)}, \mathbb{F}_{q^k})^{S_k} \]

\[ = \sum_{k=1}^{\infty} \frac{-1}{(q^k - 1)^{mn} k} \sum_{(u_1, \ldots, u_m) \in \mathbb{F}^{(Z^n_p/Z^n_m)} m} |u| \prod_{j=1}^{m} b_j^{u_j(q^k-1)} \prod_{j=1}^{m} S_f(T, \omega^{-u_j(q^k-1)}, \mathbb{F}_{q^k}) \]

So

\[ C_f \otimes g, -d(s, T) = \exp\left( \sum_{k=1}^{\infty} \frac{-1}{(q^k - 1)^{mn} k} S_f(T, \omega^{-\frac{d}{q-1}(q^k-1)})^{S_k} \right) \]

\[ = \prod_{(u_1, \ldots, u_m) \in \mathbb{F}^{(Z^n_p/Z^n_m)} m} C_f, -d(s^{u_1}, \ldots, u_m, \omega^{-d(q^k-1)}, T, \mathbb{F}_{q^k}). \]

□

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