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Nef cones of some Quot schemes on a Smooth Projective Curve

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Abstract. Let $C$ be a smooth projective curve over $\mathbb{C}$. Let $n, d \geq 1$. Let $\mathcal{Q}$ be the Quot scheme parameterizing torsion quotients of the vector bundle $\mathcal{O}_C^n$ of degree $d$. In this article we study the nef cone of $\mathcal{Q}$. We give a complete description of the nef cone in the case of elliptic curves. We compute it in the case when $d = 2$ and $C$ very general, in terms of the nef cone of the second symmetric product of $C$. In the case when $n \geq d$ and $C$ very general, we give upper and lower bounds for the Nef cone. In general, we give a necessary and sufficient criterion for a divisor on $\mathcal{Q}$ to be nef.

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1. Introduction

Throughout this article we assume that the base field to be $\mathbb{C}$. Let $X$ be a smooth projective variety and let $N^1(X)$ be the $\mathbb{R}$-vector space of $\mathbb{R}$-divisors modulo numerical equivalence. It is known that $N^1(X)$ is a finite dimensional vector space. The closed cone $\text{Nef}(X) \subset N^1(X)$ is the cone of all $\mathbb{R}$-divisors whose intersection product with any curve in $X$ is non-negative. It has been an interesting problem to compute $\text{Nef}(X)$. For example, when $X = \mathbb{P}(E)$ where $E$ is a semistable vector bundle over a smooth projective curve, Miyaoka computed the $\text{Nef}(X)$ in [14]. In [4], $\text{Nef}(X)$ was computed in the case when $X$ is the Grassmann bundle associated to a vector bundle $E$ on a smooth projective curve $C$, in terms of the Harder Narasimhan filtration of $E$. Let $C^{(d)}$ denote the $d$th symmetric product. In [15], the author computed the $\text{Nef}(C^{(d)})$ in the case when $C$ is a very general curve of even genus and $d = \text{gon}(C) - 1$. In [11] $\text{Nef}(C^{(2)})$ is computed in the case when $C$ is very general and $g$ is a perfect square. In [5] $\text{Nef}(C^{(2)})$ was computed assuming the Nagata conjecture. We refer the reader to [12, Section 1.5] for more such examples and details.

The reader is referred to [6] for the definition and details on Quot schemes. Let $E$ be a vector bundle over a smooth projective curve $C$. Fix a polynomial $P \in \mathbb{Q}[t]$. Let $\mathcal{Q}(E, P)$ denote the Quot
scheme parametrizing quotients of $E$ with Hilbert polynomial $P$. In [16], when $C = \mathbb{P}^1$, the quot scheme $\mathcal{Q}(\mathcal{O}_C^N, P)$ is studied as a natural compactification of the set of all maps from $C$ to some Grassmannians of a fixed degree. In this article we will consider the case when $P = d$ a constant, that is, when $\mathcal{Q}(E, d)$ parametrizes torsion quotients of $E$ of degree $d$. For notational convenience, we will denote $\mathcal{Q}(E, d)$ by $\mathcal{Q}$, when there is no possibility of confusion. It is known that $\mathcal{Q}$ is a smooth projective variety. Many properties of $\mathcal{Q}$ have been studied. In [1], the Betti cohomologies of $\mathcal{Q}(\mathcal{O}_C^N, d)$ are computed, $\mathcal{Q}(\mathcal{O}_C^N, d)$ has been interpreted as the space of higher rank divisors of rank $n$, and an analogue of the Abel–Jacobi map was constructed. In [2] the automorphism group scheme of $\mathcal{Q}(\mathcal{O}_C^N, d)$ was computed in the case when the genus of $C$ satisfies $g(C) > 1$ and a Torelli theorem for these Quot schemes was proved. In [3] the Brauer group of $\mathcal{Q}(\mathcal{O}_C^N, d)$ is computed. In [7], the automorphism group scheme of $\mathcal{Q}(E, d)$ was computed in the case when either $rk E \geq 3$ or $E$ is semistable and genus of $C$ satisfies $g(C) > 1$. In [8], the S-fundamental group scheme of $\mathcal{Q}(E, d)$ was computed.

In this article, we address the question of computing $\text{Nef}(\mathcal{Q})$. Recall that we have a Hilbert–Chow map $\Phi : \mathcal{Q} \to C^{(d)}$ (this map is explained after Definition 9. A precise definition can be found, for example, in [8]). For notational convenience, for a divisor $D \in N^1(C^{(d)})$ we will denote its pullback $\Phi^* D \in N^1(\mathcal{Q})$ by $D$, when there is no possibility of confusion. The line bundle $\mathcal{O}_D(1)$ is defined in Definition 9. In Section 2 we recall the results we need on $\text{Nef}(C^{(d)})$. In Section 3 we compute $\text{Pic}(\mathcal{Q})$.

**Theorem (Theorem 11).** $\text{Pic}(\mathcal{Q}) = \Phi^* \text{Pic}(C^{(d)}) \oplus \mathbb{Z}[\mathcal{O}_D(1)]$.

As a corollary (Corollary 13) we get that $N^1(\mathcal{Q}) \cong N^1(C^{(d)}) \oplus \mathbb{R}[\mathcal{O}_D(1)]$. The computation of $N^1(\mathcal{Q})$ can also be found in [3]. As a result, when $C \cong \mathbb{P}^1$, since $C^{(d)} \cong \mathbb{P}^d$, we have that the $N^1(\mathcal{Q})$ is 2-dimensional and we prove that its nef cone is given as follows.

**Theorem (Theorem 34).** Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^k \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Then

$$\text{Nef}(\mathcal{Q}(E, d)) = \mathbb{R}_{\geq 0} \left( [\mathcal{O}_D(E, d)(1)] + (-a_1 + d - 1) [\mathcal{O}_P(1)] \right) + \mathbb{R}_{\geq 0} [\mathcal{O}_P(1)].$$

Note that this theorem was already known in the case when $E = V \otimes \mathcal{O}_C$, for a vector space $V$ over $k$ ([16, Theorem 6.2]).

For the rest of the introduction, we will assume $E = V \otimes \mathcal{O}_C$ with $\dim_k V = n$ and denote by $\mathcal{Q} = \mathcal{Q}(n, d)$ the Quot scheme $\mathcal{Q}(E, d)$. Let us consider the case $g = 1$. In this case, $N^1(\mathcal{Q})$ is three-dimensional (see Proposition 14), and we prove that its nef cone is given as follows (see Definition 4 for notations).

**Theorem (Theorem 43).** Let $g = 1$, $n \geq 1$ and $\mathcal{Q} = \mathcal{Q}(n, d)$. Then the class $[\mathcal{O}_D(1)] + [\Delta_d/2] \in N^1(\mathcal{Q})$ is nef. Moreover,

$$\text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} ([\mathcal{O}_D(1)] + [\Delta_d/2]) + \mathbb{R}_{\geq 0} [\mathcal{O}_C] + \mathbb{R}_{\geq 0} [\Delta_d/2].$$

From now on assume that $g \geq 2$ and $C$ is very general. See Definition 9 for the definition of $t$ and $\alpha_t$. When $d = 2$ we have the following result.

**Theorem (Theorem 37).** Let $g \geq 2$ and $C$ be very general. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then

$$\text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} \left( [\mathcal{O}_D(1)] + \frac{t + 1}{g + t} [L_0] \right) + \mathbb{R}_{\geq 0} [L_0] + \mathbb{R}_{\geq 0} [\alpha_t].$$

Precise values of $t$ are known for small genus. When $g \geq 9$ it is conjectured that $t = \sqrt{g}$. This is known when $g$ is a perfect square. The precise statements have been mentioned after Theorem 37.
In general (without any assumptions on \( n \) and \( d \)), we give a criterion for certain line bundle on \( \mathcal{O} \) to be nef in terms of its pullback along certain natural maps from products \( \prod_i \mathbb{C}(d_i) \), see Subsection 7.1 for notation.

**Theorem (Theorem 39).** Let \( \beta \in N^1(\mathcal{C}(d)) \). Then the class \([\mathcal{O}(1)] + \beta \in N^1(\mathcal{O})\) is nef iff the class \([\mathcal{O}(-\Delta d/2)] + \pi_d^* \beta \in N^1(\mathbb{C}(d))\) is nef for all \( \mathcal{O} \in \mathcal{O}_d \).

Using the above we show that certain classes are in \( \text{Nef}(\mathcal{O}) \). Define

\[
\kappa_1 := [\mathcal{O}(1)] + \mu_0[L_0] + \frac{d + g - 2}{dg} [\theta_d] \quad \kappa_2 := [\mathcal{O}(1)] + \frac{g + 1}{2g} [L_0] \in N^1(\mathcal{O}).
\]

**Proposition (Proposition 41).** Let \( g \geq 1, n \geq 1 \) and \( \mathcal{O} = \mathcal{O}(n, d) \). Then

\[
\text{Nef}(\mathcal{O}) \supset \mathbb{R}_{\geq 0} \kappa_1 + \mathbb{R}_{\geq 0} \kappa_2 + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [L_0].
\]

Now consider the case when \( n \geq d \geq \text{gon}(C) \). Then \( \text{Nef}(\mathbb{C}(d)) \) is generated by \( \theta_d \) and \( L_0 \) (see Definitions 1 and 4). In this case we give the following upper bound for the nef cone in Proposition 20. Let \( \mu_0 := \frac{d + g - 1}{dg} \). Then

\[
\text{Nef}(\mathcal{O}) \subset \mathbb{R}_{\geq 0} \left( [\mathcal{O}(1)] + \mu_0[L_0] \right) + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [L_0].
\]

When \( d \geq \text{gon}(C) \), in Lemma 30 we show that any convex linear combination of the \( \kappa_1 \) and \( \theta_d \) is nef but not ample. In particular, any such class lies on the boundary of \( \text{Nef}(\mathcal{O}) \). Similarly, in Corollary 42 we show when \( n \geq d \), any convex linear combination of the class \( \kappa_2 \) and \( L_0(d) \) is nef but not ample. So any such class lies on the boundary of \( \text{Nef}(\mathcal{O}) \).

In terms of the above diagram, we have that when \( n \geq d \geq \text{gon}(C) \)

\[
\left\langle \overline{OD}, \overline{OE}, \overline{OC}, \overline{OB} \right\rangle \subset \text{Nef}(\mathcal{O}) \subset \left\langle \overline{OA}, \overline{OC}, \overline{OB} \right\rangle.
\]

We do not know if the inclusion in the right is an equality when \( n \geq d \geq \text{gon}(C) \). This is same as saying that \([\mathcal{O}(1)] + \mu_0[L_0] \) is nef when \( n \geq d \geq \text{gon}(C) \). In Section 8 we give a sufficient condition for when the pullback of \([\mathcal{O}(1)] + \mu_0[L_0] \) along a map \( D \to \mathcal{O} \) is nef. However, when \( d = 3 \) we have the following result.
Theorem (Theorem 49). Let $C$ be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{Q} = \mathcal{Q}(n,3)$. Let $\mu_0 = \frac{g+2}{3g}$ Then
\[ \text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} \left[ (\theta_{\mathcal{Q}}(1)) + \rho_0[\mathcal{L}_0] \right] + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[\mathcal{L}_0]. \]

Some of the results above can be improved in the case when $g = 2k$ using the results in [15].
(See Proposition 32.)

2. Nef cone of $C^{(d)}$

We follow [15, § 2] for this section. Assume that either $C$ is an elliptic curve or is a very general curve of genus $g \geq 2$. Then it is known that the Neron–Severi space is 2-dimensional. So in this case, to compute the nef cone, it is enough to give two classes in $N^1(C)$ which are nef but not ample.

For any smooth projective curve and $d \geq 2$ (not just a very general curve) there is a natural line bundle $L_0$ on $C^{(d)}$ which is nef but not ample. This line bundle is constructed in the following manner. Consider the map
\[ \phi : C^d \to J(C)^{d^2}, \]
\[ (x_i) \mapsto (x_i - x_j)_{i<j}. \]

Let $p_{ij}$ denote the projections from $J(C)^{d^2}$. Since $\phi$ is not finite, as it contracts the diagonal, the line bundle $\phi^* (\Theta p^*_{ij} \Theta)$ is nef but not ample. This line bundle is invariant under the action of $S_d$ on $C^d$. This follows from the fact that $\Theta$ in $J(C)$ is invariant under the involution $L \to L^{-1}$.

Definition 1. $\phi^* (\Theta p^*_{ij} \Theta)$ descends to a line bundle $L_0$ on $C^{(d)}$.

Since $\phi$ contracts the small diagonal $\delta : C \to C^{(d)}$, we have $\delta^* [L_0] = 0$. Hence $L_0$ is nef but not ample [15, Lemma 2.2]. Therefore, in the case when $C$ is very general, computing the nef cone of $C^{(d)}$ boils down to finding another class which is nef but not ample.

In the case when $d \geq \text{gon}(C) = e$, [15, Lemma 2.3] we can easily construct another line bundle which is nef but not ample: Then we have a map $g_e : C \to \mathbb{P}^1$ of degree $e$. This induces a closed immersion $\mathbb{P}^1 \to C^{(e)}$ with $v \mapsto ([g_e]^{-1}(v)) \in C^{(e)}$. This in turn gives a closed immersion $\mathbb{P}^1 \to C^{(d)}$ with $v \mapsto ([g_e]^{-1}(v) + (d-e)x)$ for some point $x \in C$.

Definition 2. Denote the class of this $\mathbb{P}^1$ in $N^1_1(C^{(d)})$ by $[l']$.

The composition $\mathbb{P}^1 \to C^{(d)} \xrightarrow{u_d} J(C)$ is constant, since there can be no non-constant maps from $\mathbb{P}^1 \to J(C)$. Hence $u_d : C^{(d)} \to J(C)$ is not finite and we get that $u_d^* \Theta$ is nef but not ample.

Definition 3. Define $\theta_d := u_d^* \Theta$.

Recall that over $C^{(d)}$ we have natural divisors [15, § 2]:

Definition 4. Define
(1) $\theta_d$
(2) the big diagonal $\Delta_d \to C^{(d)}$
(3) If $i_{d-1} : C^{(d-1)} \to C^{(d)}$ is the map given by $D \mapsto D + x$ for a point $x \in C$, then the image $i_{d-1}(C^{(d-1)})$. This divisor will be denoted $[x]$.

It is known that when $g = 1$ or $C$ is very general of $g \geq 2$, then $N^1(C^{(d)})$ is of dimension 2 and any two of the above three forms a basis.

By abuse of notation, let us denote the class ($\delta$ is the small diagonal) $[\delta_* (C)] \in N_1(C^{(d)})$ by $\delta$.

We summarise the above discussion in the following theorem.
Proposition 5 ([15, Proposition 2.4]). When \( d \geq \text{gon}(C) \), we have:

1. \( \text{Nef}(C^{(d)}) = \mathbb{R}_{\geq 0}[L_0] \oplus \mathbb{R}_{\geq 0}[\theta_d] \),
2. \( \overline{\text{NE}}(C^{(d)}) = \mathbb{R}_{\geq 0}[L'] \oplus \mathbb{R}_{\geq 0}[\delta] \).

The above basis are dual to each other.

We will need to write \([L_0]\) in terms of \([x]\) and \([\theta_d]\), for which we need the following computations. Define

\[ \delta^*: C^{(d)} \to C^{(d)} \]

where the first map is given by \( x \mapsto (x, x_1, \ldots, x_{d-1}) \).

Lemma 6. Let \( d \geq 1 \). We have the following

1. \( \deg(\delta^*[\theta_d]) = d^2 g \)
2. \( \deg(\delta^*[\theta_d]) = g \)
3. \( \deg(\delta^*[x]) = d \)
4. \( \deg(\delta^*[x]) = 1 \)

Proof. Recall that \( \theta_d = u^g \Theta \), where \( u_d : C^{(d)} \to J(C) \) is given by \( D \mapsto \Theta(D - dx_0) \) for a fixed point \( x_0 \in C \). Therefore the composition \( u_d \circ \delta : C \to J(C) \) is given by \( x \mapsto dx \mapsto \Theta(dx - dx_0) \), which is the map

\[ C \xrightarrow{\delta} J(C) \xrightarrow{\Theta} J(C). \]

The pullback of \( \Theta \) under the map \( J(C) \xrightarrow{\delta'} J(C) \) is \( \Theta^d \) and the degree of the pullback of \( \Theta \) under the map \( u_1 : C \to J(C) \) is \( g \). Hence degree of \( \delta^* \theta_d = d^2 g \). This proves (1).

The composition \( u_d \circ \delta' : C \to J(C) \) is given by \( C \xrightarrow{\delta} C^{(d)} \xrightarrow{\Theta} J(C) \)

\[ x \mapsto x + \sum_{i=1}^{d-1} x_i \mapsto \Theta \left( x + \sum_{i=1}^{d-1} x_i - dx_i \right) \]

which is the composition \( C \xrightarrow{\delta} J(C) \xrightarrow{\Theta} J(C), \) where \( \Theta \) is translation by an element in \( J(C) \). Hence degree of \( \delta^* \theta_d = g \). This proves (2).

For a line bundle \( L \) on \( C \), we will denote by \( L^\otimes d \) to be the unique line bundle on \( C^{(d)} \), whose pullback under the quotient map \( \pi : C^{d} \to C^{(d)} \) is \( \bigotimes_{i=1}^d \pi^i_L \). Recall that by [15, § 2], we have that \([x] = \Theta(x)^\otimes d \) for a point \( x \in C \). By definition under the map \( \pi : C^{d} \to C^{(d)} \) the pullback of \( \Theta(x)^\otimes d \) is \( \bigotimes_{i=1}^d \pi^i \Theta(x) \). Now \( \delta : C \to C^{(d)} \) is the composition \( C \xrightarrow{\delta} C^{(d)} \xrightarrow{\Theta} C^{(d)} \)

\[ x \mapsto (x, x, \ldots, x) \mapsto dx. \]

Hence we get that the pullback of \( \Theta(x)^\otimes d \) to \( \delta \) is \( \Theta(dx) \). Therefore degree of \( \delta^*[x] = d \). This proves (3).

We know \( \delta' \) is the composition \( C \xrightarrow{\delta} C^{(d)} \xrightarrow{\Theta} C^{(d)} \)

\[ x \mapsto (x, x, \ldots, x_{d-1}) \mapsto x + x_1 + \ldots + x_{d-1}. \]

Hence we get that \( \delta'^*[x] = \Theta(x) \). Therefore degree of \( \delta'^*[x] = 1 \). This proves (4).

Lemma 7. Let \( g, d \geq 1 \). Let \( \mu_0 := \frac{d + g - 1}{d g} \). Then

\[ [L_0] = d g [x] - [\theta_d] \]

\[ = (d g - d - g + 1) \cdot [x] + [\Delta_d/2] \]

\[ = \left( \frac{1}{\mu_0} - 1 \right) [\theta_d] + \frac{1}{\mu_0} [\Delta_d/2]. \]
Proof. Let \([L_0] = a[\theta_d] + b[x]\). We need two equations to solve for \(a\) and \(b\). The first equation is \(\delta^*[L_0] = 0\). Recall

\[ \delta' : C \overset{f}{\to} C^d \to C^{(d)} \]

where the first map is given by \(x \mapsto (x, x_1, \ldots, x_d)\). Hence

\[ \delta'^*[L_0] = f^* \phi^* \left( \otimes p_i^* \Theta \right). \]

Now the composition

\[ C \overset{f}{\to} C^d \overset{\Phi}{\to} J(C)^{(d)} \]

is given by \(x \mapsto (x - x_1, x - x_2, \ldots, x - x_{d-1}, x_i - x_j)_{i < j}\). Hence

\[ \deg(\delta'^*[L_0]) = \sum_{i=1}^{d-1} \deg(\theta_1) = (d - 1)g. \]

This will be our second equation.

We use these two equations and the preceding computations to compute \(a\) and \(b\).

\[
0 = \deg(\delta^*[L_0]) \\
= a \cdot \deg(\delta^*[\theta_d]) + b \cdot \deg(\delta^*[x]) \\
= ad^2g + bd.
\]

Therefore

\[ b = -adg. \]

Now using the second equation we get

\[
(d - 1)g = \deg(\delta'^*[L_0]) \\
= a \cdot \deg(\delta'^*[\theta_d]) + b \cdot \deg(\delta'^*[x]) \\
= ag + b \\
= ag - adg = ag(1 - d).
\]

Therefore

\[ a = -1, \quad b = dg. \]

Hence we get \([L_0] = dg[x] - [\theta_d]\). For the other two equalities, we use the relation

\[ [\theta_d] = (d + g - 1)[x] - [\Delta_d/2] \]

between \([x]\), \([\Delta_d/2]\) and \([\theta_d]\) [15, Lemma 2.1].

3. Picard group and Neron–Severi group of \(\mathcal{Q}\)

Let \(E\) be a locally free sheaf over \(C\). Throughout this section \(\mathcal{Q}\) will denote the Quot scheme \(\mathcal{Q}(E, d)\) which parametrizes torsion quotients of \(E\) of degree \(d\). In this section we compute the Picard group of \(\mathcal{Q}\), and the vector spaces \(N_1(\mathcal{Q})\) and \(N_1(\mathcal{Q})\).

Lemma 8. Let \(S\) be a scheme over \(k\). Let \(F\) be a coherent sheaf over \(C \times S\) which is \(S\)-flat and for all \(s \in S\), \(F|_{C \times s}\) is a torsion sheaf over \(C\) of degree \(d\). Let \(p_S : C \times S \to S\) be the projection. Then

(i) \(p_S^*(F)\) is locally free of rank \(d\) and \(\forall s \in S\) the natural map \(p_S^*(F)|_s \to H^0(C, F|_{C \times s})\) is an isomorphism.
(ii) Assume that we are given a morphism $\phi: T \to S$. We have the following diagram:

$$
\begin{array}{ccc}
C \times T & \xrightarrow{id \times \phi} & C \times S \\
\downarrow{p_T} & & \downarrow{p_S} \\
T & \xrightarrow{\phi} & S
\end{array}
$$

Then the natural morphism

$$
\phi^* p_{S*}(F) \to (p_T)_*(id \times \phi)^* F
$$

is an isomorphism.

**Proof.** Since $F|_{C \times S}$ is a torsion sheaf for all $s \in S$, we have $H^1(C, F|_{C \times S}) = 0$. By [9, Chapter III, Theorem 12.11 (a)] we get $R^1 p_{S*}(F) = 0$. Using [9, Chapter III, Theorem 12.11 (b)] (ii) with $i = 1$ we get that the morphism $p_{S*}(F)|_S \to H^0(C, F|_{C \times S})$ is surjective. Again using the same with $i = 0$ we get that $p_{S*}(F)$ is locally free of rank $d$ and the map $p_{S*}(F)|_S \to H^0(C, F|_{C \times S})$ is an isomorphism.

Since $F$ is $S$-flat it follows that $(id \times \phi)^* F$ is $T$-flat. Applying the above we see $\phi^* p_{S*}(F)$ and $(p_T)_*(id \times \phi)^* F$ are locally free of rank $d$. For each $t \in T$ we have the commutative diagram:

$$
\begin{array}{ccc}
\phi^* p_{S*}(F)|_t = p_{S*}(F)|_{\phi(t)} & \to & (p_T)_*(id \times \phi)^* F|_t \\
\downarrow & & \downarrow \\
H^0(C, F|_{C \times \phi(t)}) & \to & H^0(C, (id \times \phi)^* F|_{C \times t})
\end{array}
$$

By the first part we get that the vertical arrows are isomorphisms. Hence we get that the first row of the diagram is an isomorphism. Therefore

$$
\phi^* p_{S*}(F) \to (p_T)_* (id \times \phi)^* F
$$

is a surjective morphism of vector bundles of same rank and hence an isomorphism. \qed

We define a line bundle on $\mathcal{O}$. Let us denote the projections $C \times \mathcal{O}$ to $C$ and $\mathcal{O}$ by $p_C$ and $p_Q$ respectively. Then we have the universal quotient $p_C^* E \to \mathcal{O}|_{\mathcal{O}^1}$ over $C \times \mathcal{O}$. By Lemma 8, $p_{\mathcal{O}^1}(\mathcal{O}|_{\mathcal{O}^1})$ is a vector bundle of rank $d$.

**Definition 9.** Denote the line bundle $\det(p_{\mathcal{O}^1}(\mathcal{O}|_{\mathcal{O}^1}))$ by $\mathcal{O}|_{\mathcal{O}^1}$.

Denote the $d^{th}$ symmetric product of $C$ by $C^{(d)}$. Recall the Hilbert–Chow map $\Phi: \mathcal{O} \to C^{(d)}$ which sends $[E \to B]$ to $\sum i(B_p)p$, where $i(B_p)$ is the length of the $\mathcal{O}_{C,p}$-module $B_p$. Therefore, we have the pullback $\Phi^*: \text{Pic}(C^{(d)}) \to \text{Pic}(\mathcal{O})$ which is in fact an inclusion. To see this, recall that the fibres of $\Phi$ are projective integral varieties [8, Corollary 6.6] and $\Phi$ is flat [8, Corollary 6.3]. Hence $\Phi^*(\mathcal{O}|_{\mathcal{O}^1}) = \mathcal{O}_{C^{(d)}}$. Now by projection formula $\Phi_* \Phi^* L \equiv L$ for all $L \in \text{Pic}(C^{(d)})$ and the statement follows.

The big diagonal is the image of the map $C \times C \to C^{(d)}$ given by $(x, A) \to 2x + A$. Let us denote the big diagonal in $C^{(d)}$ by $\Delta$. Let $U_C := C^{(d)} \setminus \Delta$ and $\mathcal{U} := \Phi^{-1}(U_C)$. Then $\mathcal{U} \subset \mathcal{O}$.

**Lemma 10.** For any line bundle $\mathcal{L} \in \text{Pic}(\mathcal{O})$, $\exists$ an unique $n \in \mathbb{Z}$ such that $(\mathcal{L} \otimes \mathcal{O}|_{\mathcal{O}^1})|_{\Phi^{-1}(p)} \equiv \mathcal{O}_{\Phi^{-1}(p)}$ for all $p \in U_C$.

**Proof.** Let $\pi: \mathbb{P}(E) \to C$ be the projective bundle associated to $E$ and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the universal line bundle over $\mathbb{P}(E)$. Let $Z = \mathbb{P}(E)^d$. Let $p_i: Z \to \mathbb{P}(E)$ be the $i^{th}$ projection. Let $\pi_d: Z \to C^d$ be the product map. The symmetric group $S_d$ acts on $Z$ and the map $\pi_d$ is equivariant for this action. Let $\psi: C^d \to C^{(d)}$ be the quotient map. Define $U_Z := (\psi \circ \pi_d)^{-1}(U)$.

Let $c \in C$ be a closed point and let $k_c$ denote the skyscraper sheaf supported at $c$. A closed point of $\mathbb{P}(E)$ which maps to $c \in C$ corresponds to a quotient $E \to E_c \to k_c$. Recall that we have a map [7, Theorem 2.2(a)]

$$
\tilde{\psi}: U_Z \to \mathcal{U}
$$

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which sends a closed point

\[(E_{c_1} \to k_{c_1})^d_{i=1} \in U_Z\]

to the quotient

\[E = \bigoplus_i E_{c_i} \to \bigoplus_i k_{c_i} \in \mathcal{U} .\]

So we have a commutative diagram:

\[
\begin{array}{ccc}
U_Z & \xrightarrow{\psi} & \mathcal{U} \\
\downarrow{\pi_d} & & \downarrow{\Phi} \\
\psi^{-1}(U_C) & \xrightarrow{\psi} & U_C
\end{array}
\]

Moreover, if \(\underline{c} = (c_1, \ldots, c_d) \in \psi^{-1}(U_C)\), then by [8, Lemma 6.5] \(\tilde{\psi}\) induces an isomorphism

\[
\prod P(E_{c_i}) = \pi_d^{-1}(\underline{c}) \cong \Phi^{-1}(\psi(\underline{c})) .
\]

Applying Lemma 8 by taking \(T = U_Z, S = \mathcal{U}\) and \(\phi = \tilde{\psi}\) and the definition of the map \(\tilde{\psi}\) (see the proof of [7, Theorem 2.2(a)]) we see that

\[
\tilde{\psi}^* \mathcal{O}(1) = \bigotimes_{i=1}^d p_i^* \mathcal{O}(E_{c_i})|_{U_Z} .
\]

Hence it is enough to show that \(\exists n \in \mathbb{Z}\) such that \(\forall \underline{c} \in \psi^{-1}(U_C)\)

\[
\tilde{\psi}^* \mathcal{L} \mid_{\pi_d^{-1}(\underline{c})} \cong \bigotimes_{i=1}^d p_i^* \mathcal{O}(n) \mid_{\pi_d^{-1}(\underline{c})} .
\]

For \(\underline{c} \in \psi^{-1}(U_C)\) define \(n_i(\underline{c}) \in \mathbb{Z}\) using the equation

\[
\tilde{\psi}^* \mathcal{L} \mid_{\pi_d^{-1}(\underline{c})} \cong \bigotimes_{i=1}^d p_i^* \mathcal{O}(n_i(\underline{c})) .
\]

We may view the \(n_i\) as functions \(n_i : \psi^{-1}(U_C) \to \mathbb{Z}\). Since the line bundle \(\tilde{\psi}^* \mathcal{L}\) is invariant under the action of the group \(S_d\), it follows that

\[
n_{\sigma(i)}(\underline{c}) = n_i(\sigma(\underline{c})) .\tag{3}
\]

Here \(\sigma(\underline{c}) := (c_{\sigma(1)}, \ldots, c_{\sigma(d)})\). Hence it suffices to show that \(n_1\) is a constant function.

Let \(c_2, \ldots, c_d\) be distinct points in \(C\). Define \(V := C \setminus \{c_2, \ldots, c_d\}\) and a map

\[i : V \to \psi^{-1}(U_C) \quad i(c) := (c, c_2, \ldots, c_d) .\]

Then \(\pi_d^{-1}(V)\) is equal to \(\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})\). The restriction of \(\tilde{\psi}^* \mathcal{L}\) to \(\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})\) is isomorphic to

\[
\pi_* M \otimes p_1^* \mathcal{O}(a_1) \otimes p_2^* \mathcal{O}(a_2) \ldots \otimes p_d^* \mathcal{O}(a_d) ,
\]

where \(M\) is a line bundle on \(V\). Further restricting to \((c, c_2, \ldots, c_d)\) and \((c', c_2, \ldots, c_d)\), where \(c, c' \in V\), we see that

\[
n_i(c, c_2, \ldots, c_d) = n_i(c', c_2, \ldots, c_d) \quad \forall \ i .\tag{4}
\]

This proves that for distinct points \(c, c', c_2, \ldots, c_d \in C\) we have

\[
n_i(c, c_2, \ldots, c_d) = n_i(c', c_2, \ldots, c_d) \quad \forall \ i .\tag{5}
\]
Choose $2d$ distinct points $c_1, \ldots, c_d, c'_1, \ldots, c'_d$ in $C$. Then using equations (4) and (5) we get

$$n_1(c_1, c_2, \ldots, c_d) = n_1(c'_1, c_2, \ldots, c_d)$$
$$= n_2(c_2, c'_1, \ldots, c_d)$$
$$= n_2(c'_2, c'_1, c_3, \ldots, c_d)$$
$$= n_1(c'_1, c'_2, c_3, \ldots, c_d)$$
$$= \ldots$$
$$= n_1(c'_1, c'_2, \ldots, c'_d).$$

Finally, for any two points $c, c' \in \psi^{-1}(U_C)$ choose a third point $c''$ such that the coordinates of $c''$ are distinct from those of $c$ and $c'$. Then we see that $n_1(c) = n_1(c'') = n_1(c')$. This proves that $n_1$ is the constant function. Therefore, $\psi^* L|_{\pi^{-1}(C)}$ is of the form $\bigotimes p_1^* \Theta_{P(E_i)}(n)$, $\forall c \in \psi^{-1}(U_C)$. The uniqueness of $n$ is obvious.

**Theorem 11.** $\text{Pic}(C) = \Phi^* \text{Pic}(C^{(d)}) \oplus \mathbb{Z}[\Theta_{\mathcal{O}}(1)].$

**Proof.** Let $L \in \text{Pic}(C)$. By [8, Corollary 6.3] and [8, Corollary 6.4] the morphism $\Phi$ is flat and fibres of $\Phi$ are integral. Then by [13, Lemma 2.1.2] and Lemma 10 we get that $L \otimes \Theta_{\mathcal{O}}(-n) = \Phi^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(C^{(d)})$. Hence $L = \Phi^* \mathcal{M} \otimes \Theta_{\mathcal{O}}(n)$. The uniqueness of such an expression follows from the statement on uniqueness in Lemma 10.

For a projective variety $X$ over $k$ recall that $N^1(X)$ (respectively, $N_1(X)$) is the vector space of $\mathbb{R}$-divisors (respectively, 1-cycles) modulo numerical equivalences [12, § 1.4]. It is known that $N^1(X)$ and $N_1(X)$ are finite dimensional and the intersection product defines a non-degenerate pairing $N^1(X) \times N_1(X) \to \mathbb{R}$ such that $[\beta], [\gamma] \mapsto [\beta] \cdot [\gamma]$.

We will compute $N^1(\mathcal{O})$ and $N_1(\mathcal{O})$. Let $c \in U_C \subset C^{(d)}$. As we saw in the proof of Theorem 11,

$$\Phi^{-1}(c) \equiv \prod \mathcal{P}(E_{c_i}).$$

Let $\mathbb{P}^1 \to \mathcal{P}(E_{c_i})$ be a line and let $v_i \in \mathcal{P}(E_{c_i})$ for $i \geq 2$. Then we have an embedding:

$$\mathbb{P}^1 \cong \mathbb{P}^1 \times v_2 \times \ldots \times v_d \to \mathcal{P}(E_{c_1}) \times \prod_{i \geq 2} \mathcal{P}(E_{c_i}) = \Phi^{-1}(c) \subset \mathcal{O}.$$

**Definition 12.** Let us denote the class of this curve in $N_1(\mathcal{O})$ by $[l]$.

**Corollary 13.** $N^1(\mathcal{O}) = \Phi^* N^1(C^{(d)}) \oplus \mathbb{R}[\Theta_{\mathcal{O}}(1)].$

**Proof.** Since $\Phi$ is surjective, $N^1(C^{(d)}) \to N^1(\mathcal{O})$ is an inclusion [12, Example 1.4.4]. Note that $\Theta_{\mathcal{O}}(1) \neq 0$ in $N^1(\mathcal{O})$ since $[\Theta_{\mathcal{O}}(1)] \cdot [l] = 1$. Hence $\Theta_{\mathcal{O}}(1) \neq 0$ in $N^1(\mathcal{O})$. This also shows that $\Theta_{\mathcal{O}}(1) \notin \Phi^* N^1(C^{(d)})$.

By Theorem 11, we know that any $N^1(\mathcal{O})$ is generated by $\Phi^* N^1(C^{(d)})$ and $[\Theta_{\mathcal{O}}(1)]$. The only thing left is to show that $\Phi^* N^1(C^{(d)}) \cap \mathbb{R}[\Theta_{\mathcal{O}}(1)] = 0$.

For $a \in \mathbb{R}$ if $a[\Theta_{\mathcal{O}}(1)] \in N^1(C^{(d)})$, then $a[\Theta_{\mathcal{O}}(1)] \cdot [l] = a = 0$. Hence the result follows.

Hence, it follows from Corollary 13 that

**Proposition 14.** If $g = 1$ or $C$ is very general with $g \geq 2$, then $\dim_{\mathbb{R}} N^1(\mathcal{O}) = 3$.

**Proof.** We already saw that $N^1(C^{(d)})$ is of dimension 2. The Proposition follows.
To compute \( N_1(\mathcal{Q}) \) we first construct a section of \( \Phi : \mathcal{Q} \rightarrow C^{(d)} \). Over \( C \times C^{(d)} \) we have the universal divisor \( \Sigma \) which gives us the universal quotient \( \mathcal{O}_{C \times C^{(d)}} \rightarrow \mathcal{O}_{\Sigma} \). Choose a surjection \( E \rightarrow L \) over \( C \), where \( L \) is a line bundle on \( C \). This induces a surjection \( E \otimes \mathcal{O}_{C \times C^{(d)}} \rightarrow L \otimes \mathcal{O}_{C \times C^{(d)}} \). Then the composition
\[
E \otimes \mathcal{O}_{C \times C^{(d)}} \rightarrow L \otimes \mathcal{O}_{C \times C^{(d)}} \rightarrow L \otimes \mathcal{O}_{\Sigma}
\]
gives us a morphism
\[
\eta : C^{(d)} \rightarrow \mathcal{Q}
\]
which is easily seen to be a section of \( \Phi \).

**Corollary 15.** \( N_1(\mathcal{Q}) = N_1(C^{(d)}) \oplus \mathbb{R}[l] \) where \( N_1(C^{(d)}) \hookrightarrow N_1(\mathcal{Q}) \) is the morphism given by the pushforward \( \eta_* \).

**Proof.** Since \( \Phi \circ \eta = id_{C^{(d)}} \) we have that \( \eta_* \) is an injection. Also since \( [\mathcal{Q}(1)] \cdot [l] = 1 \), we have \( [l] \neq 0 \). We claim that \( [l] \notin N_1(C^{(d)}) \). If not, assume that \( [l] = \eta_*[\gamma] \) for \( \gamma \in N_1(C^{(d)}) \). Then for every \( \beta \in N^1(C^{(d)}) \) we have
\[
[l] \cdot \Phi^* \beta = \Phi_*([l]) \cdot \beta = 0 = \gamma \cdot \beta.
\]

This proves that \( \gamma = 0 \).

Let \( \gamma \in N_1(\mathcal{Q}) \). Then we claim that
\[
\gamma = \eta_* \Phi_* \gamma + ([\mathcal{Q}(1)] \cdot [\gamma \cdot \Phi_* \gamma]) [l].
\]

This can be seen as follows. It is enough to show that \( \forall \ D \in N^1(\mathcal{Q}) \),
\[
[D] \cdot \gamma = [D] \cdot (\eta_* \Phi_* \gamma) + ([\mathcal{Q}(1)] \cdot [\gamma] [D] \cdot [l]).
\]

By Corollary 13, it is enough to consider the case when \( D = \Phi^* D' \) where \( D' \in N^1(C^{(d)}) \) or \( D = \mathcal{Q}(1) \). In the first case the statement follows from projection formula and the second case is by definition. This completes the proof of the Corollary 15. \( \square \)

Let \( p_C : C \times \mathcal{Q} \rightarrow \mathcal{Q} \) and \( p_{\mathcal{Q}} : C \times \mathcal{Q} \rightarrow C \) be the projections. Let \( \mathcal{Q} \) denote the universal quotient on \( C \times \mathcal{Q} \). For a vector bundle \( F \) over \( C \), we define
\[
B_{F,\mathcal{Q}} := \det(p_{\mathcal{Q}}^* (\mathcal{Q} \otimes p_C^* F)).
\]

**Lemma 16.** Suppose we are given a map \( f : T \rightarrow \mathcal{Q} \). Let \( (id \times f)^* \mathcal{Q} = \mathcal{Q}_T \). Let \( p_T : C \times T \rightarrow T \) and \( p_{1,T} : C \times T \rightarrow C \) be the projections.

\[
\begin{array}{ccc}
C \times T & \xrightarrow{id \times f} & C \times \mathcal{Q} \\
\downarrow p_T & & \downarrow p_{\mathcal{Q}} \\
T & \xrightarrow{f} & \mathcal{Q}
\end{array}
\]

(i) \( f^* p_{\mathcal{Q}}^* (\mathcal{Q} \otimes p_C^* F) \rightarrow p_T^* (\mathcal{Q}_T \otimes p_{1,T}^* F) \) is an isomorphism.

(ii) For a vector bundle \( F \) on \( C \) define \( \alpha_{F,T} := \det(p_{1,T}^* (\mathcal{Q}_T \otimes p_{1,T}^* F)). \) Then \( f^* \alpha_{F,\mathcal{Q}} = \alpha_{F,T} \).

**Proof.** For (i) take \( \mathcal{Q} \otimes p_C^* F \) and use Lemma 8. The assertion (ii) follows from (i) by applying determinant to the isomorphism
\[
f^* p_{\mathcal{Q}}^* (\mathcal{Q} \otimes p_C^* F) \xrightarrow{\sim} p_T^* (\mathcal{Q}_T \otimes p_{1,T}^* F).
\]

Recall the definition of \( \eta \) from equation (7), this is a section of \( \Phi \). For a line bundle \( L \) on \( C \) we have a line bundle \( \mathcal{Q}_{d,L} \) over \( C^{(d)} \) (see [15, page 8] for notation).

**Lemma 17.** Let \( \eta \) be defined by a quotient \( E \rightarrow M \rightarrow 0 \). Then
\[
\eta^* \alpha_{\mathcal{Q},\mathcal{Q}} \equiv \mathcal{Q}_{d,L \otimes M}.
\]
Thus, using the remark in the preceding para, we get that the same as that of 

By definition, the sheaf $B$ the $c$ section ($v$ $\leq$ $d$).

Hence by Lemma 16, we have

$$\eta^* B_{L,\mathcal{O}} \cong \mathcal{Q}_{d,L \otimes M}.$$

**Proposition 18.** For any two line bundles $L, L'$ over $C$

$$B_{L,\mathcal{O}} \otimes B_{L',\mathcal{O}}^{-1} = \Phi^* \left( \left( L \otimes L'^{-1} \right)^{\otimes d} \right).$$

**Proof.** First we show that $B_{L,\mathcal{O}} \otimes B_{L',\mathcal{O}}^{-1} \in \Phi^* \text{Pic}(C^{(d)})$. Since any line bundle over $\mathcal{O}$ is of the form $\mathcal{O}_\mathcal{O}(a) \otimes \mathcal{O}_\mathcal{O} \mathcal{L}$, where $\mathcal{L} \in \text{Pic}(C^{(d)})$, it is enough to show that both $B_{L,\mathcal{O}}$ and $B_{L',\mathcal{O}}$ have the same $\mathcal{O}_\mathcal{O}(1)^{th}$ coefficient.

To compute the coefficient of this component of any line bundle over $\mathcal{O}$, we can do the following. Fix $d$ distinct points $c_1, \ldots, c_d \in C$. These define a point $c \in C^{(d)}$. As we saw in the proof of Theorem 11,

$$\Phi^{-1}(c) \cong \prod_{i=1}^d \mathbb{P}(E_{c_i}).$$

Let $v_i \in \mathbb{P}(E_{c_i})$ for $i \geq 2$. Then we have an embedding:

$$f : \mathbb{P}(E_{c_i}) \times v_2 \times \ldots \times v_d \hookrightarrow \mathbb{P}(E_{c_1}) \times \prod_{i=2}^d \mathbb{P}(E_{c_i}) = \Phi^{-1}(c).$$

Then the $\mathcal{O}_\mathcal{O}(1)^{th}$ coefficient of a line bundle $\mathcal{M}$ over $\mathcal{O}$ is the degree of $f^* \mathcal{M}$ with respect to $\mathcal{O}_\mathcal{O}(E_{c_j})(1)$. Let $Y = \mathbb{P}(E_{c_1})$. Using Lemma 16, $f^* B_{L,\mathcal{O}} = \det(p_{Y*}(\mathcal{B}_Y \otimes p_{L,Y}^* L)).$

The $v_j \in \mathbb{P}(E_{c_j})$ correspond to quotients $v_j : E \rightarrow E_{c_j} \rightarrow k_{c_j}$, for $2 \leq j \leq d$. Over $C \times Y$ we have the inclusions $i_j : Y \cong c_j \times Y \rightarrow C \times Y$ for every $1 \leq j \leq d$. We have a map

$$p_{1,Y}^* E \rightarrow \bigoplus_{j=1}^d i_j \left( p_{1,Y}^* E_{c_j} \times Y \right).$$

The bundle $p_{1,Y}^* E_{c_j} \times Y$ is just the trivial bundle on $Y$, and using $v_j$ we can get quotients $p_{1,Y}^* E_{c_j} \times Y \rightarrow \mathcal{O}_Y$ for $2 \leq j \leq d$. For $j = 1$ we have the quotient $p_{1,Y}^* E_{c_1} \times Y \rightarrow i_{1*}(\mathcal{O}_Y(1))$. Since the $c_j \times Y$ are disjoint we can put these together to get a quotient on $C \times Y$

$$p_{1,Y}^* E \rightarrow \left( \bigoplus_{j=2}^d i_j \mathcal{O}_Y \right) \bigoplus i_{1*} \mathcal{O}_Y(1).$$

By definition, the sheaf $\mathcal{B}_Y$ is the sheaf in the RHS. Then

$$\mathcal{B}_Y \otimes p_{1,Y}^* L = \left( \bigoplus_{j=2}^d i_j \mathcal{O}_Y \right) \otimes p_{1,Y}^* L \bigoplus i_{1*} \mathcal{O}_Y(1) \otimes p_{1,Y}^* L$$

$$= \left( \bigoplus_{j=2}^d i_j \mathcal{O}_Y \right) \bigoplus i_{1*} \mathcal{O}_Y(1)$$

$$= \mathcal{B}_Y.$$}

Thus, using the remark in the preceding para, we get that the $\mathcal{O}_\mathcal{O}(1)^{th}$ coefficient of $B_{L,\mathcal{O}}$ is the same as that of $B_{L',\mathcal{O}}$. Hence $B_{L,\mathcal{O}} \otimes B_{L',\mathcal{O}}^{-1} = \Phi^* \mathcal{L}$.
Recall the section $\eta$ of $\Phi$ from equation (7), constructed using some line bundle quotient $E \to M$. Then $\eta^*(B_{L,\mathcal{Q}} \otimes B_{L',\mathcal{Q}})^{-1} = s^* \Phi^* L = L$. Now using Lemma 17, we get that $\eta^* B_{L,\mathcal{Q}} = \mathcal{Q}_{d, L \otimes M}$.

By Göttsche’s theorem (15, page 9) we get that $\eta^* B_{L,\mathcal{Q}} = \mathcal{Q}_{d, L \otimes M} = (L \otimes M)^{\otimes d} \otimes \mathcal{O}(-\Delta_d/2)$. Therefore, we get

$$L = \eta^* \left( B_{L,\mathcal{Q}} \otimes B_{L',\mathcal{Q}}^{-1} \right) = (L \otimes L')^{\otimes d}.$$ 

This completes the proof of the Proposition 18.

**Corollary 19.** $[B_{L,\mathcal{Q}}] = [\mathcal{Q}_{\mathcal{Q}}(1)] + \deg(L)[x]$ in $N^1(\mathcal{Q})$.

## 4. Upper bound on NEF cone

Let $V$ be a vector space of dimension $n$. From now, unless mentioned otherwise, the notation $\mathcal{Q}$ will be reserved for the space $\mathcal{Q}(V \otimes \mathcal{O}_C, d)$. Sometimes we will also denote this space by $\mathcal{Q}(n, d)$ when we want to emphasize $n$ and $d$.

**Notation**

For the rest of this article, except in section 6, the genus of the curve $C$ will be $g(C) \geq 1$. If $g(C) \geq 2$ then we will also assume that $C$ is very general.

Our aim is to compute the NEF cone of $\mathcal{Q}$. Since this cone is dual to the cone of effective curves, it follows that if we take effective curves $C_1, C_2, \ldots, C_r$, take the cone generated by these in $N_1(\mathcal{Q})$, and take the dual cone $T$ in $N^1(\mathcal{Q})$, then $\text{Nef}(\mathcal{Q})$ is contained in $T$. This gives us an upper bound on $\text{Nef}(\mathcal{Q})$. We already know two curves in $\mathcal{Q}$. The first being a line in the fiber of $\Phi : \mathcal{Q} \to C^{(d)}$, see Definition 12, which was denoted $[l]$. Recall the section $\eta$ of $\Phi$ from equation (7), taking $L$ to be the trivial bundle. The second curve is $\eta_*([l'])$, where $[l']$ is from Definition 2. Now we will construct a third curve in $\mathcal{Q}$.

Define a morphism

$$\tilde{\delta} : C \to \mathcal{Q}$$

as follows. Let $p_1, p_2 : C \times C \to C$ be the first and second projections respectively. Let $i : C \to C \times C$ be the diagonal. Fix a surjection $k^n \to k^d$ of vector spaces. Then define the quotient over $C \times C$

$$\mathcal{O}_{C \times C}^n \to \mathcal{O}_{C \times C}^d \to i^* i^* \mathcal{O}_{C \times C}^d.$$

This induces a morphism $\tilde{\delta} : C \to \mathcal{Q}$ which sends $c \to [\mathcal{O}_C^n \to k^d \to 0]$. We will abuse notation and denote the class $[\tilde{\delta}_*(C)] \in N_1(\mathcal{Q})$ by $[\tilde{\delta}]$.

We now give an upper bound for the NEF cone when $n \geq d \geq \text{gon}(C)$.

**Proposition 20.** Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Assume $n \geq d \geq \text{gon}(C)$. Let $\mu_0 := \frac{d+g-1}{d}$. Then

$$\text{Nef}(\mathcal{Q}) \subseteq \mathbb{R}_{\geq 0} \left[ [\mathcal{Q}_{\mathcal{Q}}(1)] + \mu_0 [L_0] \right] + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [L_0].$$

**Proof.** We claim that the cone dual to $\langle [l], \eta_*([l']) \rangle$ is precisely

$$\langle [\mathcal{Q}_{\mathcal{Q}}(1)] + \mu_0 [L_0], [L_0], [\theta_d] \rangle.$$

We have the following equalities:

1. $([\mathcal{Q}_{\mathcal{Q}}(1)] + \mu_0 [L_0]) \cdot [l] = 1$. This is clear.
2. $([\mathcal{Q}_{\mathcal{Q}}(1)] + \mu_0 [L_0]) \cdot \eta_*([l']) = 0$. By projection formula and Lemma 17, we get that $([\mathcal{Q}_{\mathcal{Q}}(1)] + \mu_0 [L_0]) \cdot [\eta_*([l']) = \{[-\Delta_d/2] + \mu_0 [L_0]\} \cdot [l']$.

By Lemma 7 we get that $[-\Delta_d/2] + \mu_0 [L_0] = (1 - \mu_0) [\theta_d]$. But as we saw earlier, $[\theta_d] \cdot [l'] = 0$. 

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(3) \(|\mathcal{O}_\mathcal{Z}(1) + \mu_0[L_0]| \cdot |\bar{\delta}| = 0\). By Lemma 8, it is easy to see that \(|\mathcal{O}_\mathcal{Z}(1)| \cdot |\bar{\delta}| = 0\). By projection formula, we get
\[\left((\mathcal{O}_\mathcal{Z}(1) + \mu_0[L_0]) \cdot |\bar{\delta}| = \left|\mu_0L_0\right| \cdot |\Phi_\ast \bar{\delta}| = \left|\mu_0L_0\right| \cdot |\delta| = 0.\]

(4) \(|\theta_d\mid \cdot |l| = |L_0| \cdot |l| = 0\) follows using the projection formula.

Now the claim follows from Proposition 5. As explained before, since \(\text{Nef}(\mathcal{Z})\) is contained in the dual to the cone \(\langle |l|, \eta_\ast (|l'|), \bar{\delta}\rangle\), the proposition follows.

When the genus \(g = 1\), we have the following improvement of Proposition 20.

**Proposition 21.** Let \(C\) be a smooth projective curve of genus \(g = 1\). Consider the Quot scheme \(\mathcal{Z} = \mathcal{Z}(n, d)\). Assume \(d \geq \text{gon}(C) = 2\). Then
\[\text{Nef}(\mathcal{Z}) \subset \mathbb{R}_{\geq 0} \left(\left(\mathcal{O}_\mathcal{Z}(1) + |L_0|\right) \ast \mathbb{R}_{\geq 0} \mid \theta_d\right) + \mathbb{R}_{\geq 0} |L_0| = 0.\]

**Proof.** We claim that the cone dual to \(\langle |l|, \eta_\ast (|l'|), \eta \ast |\delta|\rangle\) is precisely
\[\langle |\mathcal{O}_\mathcal{Z}(1)| \ast |L_0|, |L_0| \mid \theta_d\rangle\].

Let us check that \((|\mathcal{O}_\mathcal{Z}(1)| \ast |L_0|) \ast |\eta \ast |\delta| = 0\). Since \(|L_0| \ast |\delta| = 0\) it is clear that it suffices to check that \(|\mathcal{O}_\mathcal{Z}(1)| \ast |\eta \ast |\delta| = 0\). Applying the definition of the map \(\eta \ast |\delta : C \to \mathcal{Z}\) we see that \(|\mathcal{O}_\mathcal{Z}(1)| \ast |\eta \ast |\delta| = \text{deg}(p_{2 \ast} (\mathcal{O} / \mathcal{J}^d))\), where \(\mathcal{J}\) is the ideal sheaf of the diagonal in \(E \times E\). Since \(\mathcal{J} \ast \mathcal{J}^2\) is trivial and \(\mathcal{J} \ast \mathcal{J}^2 = 0\), it follows that \(\text{deg}(p_{2 \ast} (\mathcal{O} / \mathcal{J}^d)) = 0\). The rest of the proof is the same as that of Proposition 20.

\[\square\]

5. **Lower bound on NEF cone**

In this section we obtain a lower bound for \(\text{Nef}(\mathcal{Z}) = \mathcal{Z}(n, d)\).

**Lemma 22.** Let \(f : D \to \mathcal{Z}\) be a morphism, where \(D\) is a smooth projective curve. Fix a point \(q \in f(D)\) and an effective divisor \(A\) on \(C\) containing the scheme theoretic support of \(\mathcal{R}_q\). If there is a line bundle \(L\) on \(C\) such that \(H^0(L) \to H^0(L) A\) is surjective then \([B_L, \mathcal{Z}] \ast |D| \geq 0\).

**Proof.** Consider the map
\[p_{\mathcal{Z} \ast} (p_{C \ast} (V \ast \mathcal{O}_C) \ast p_{C \ast} L) \to p_{\mathcal{Z} \ast} (\mathcal{R}_L \ast p_{C \ast} L)\]
on \(\mathcal{Z}\). We claim that this map is surjective at the point \(q\). In view of Lemma 8 when we restrict this map to \(q\), it becomes equal to the map
\[H^0 (V \ast L) \to H^0 (\mathcal{R}_q \ast L) .\]
The map \(V \ast L \to \mathcal{R}_q \ast L\) on \(C\) factors as
\[V \ast L \to V \ast L | A \to \mathcal{R}_q \ast L.\]
Taking global sections we see that the map \(H^0 (V \ast L) \to H^0 (\mathcal{R}_q \ast L)\) factors as
\[H^0 (V \ast L) \to H^0 (V \ast L | A) \to H^0 (\mathcal{R}_q \ast L) .\]
The second arrow is surjective since these are coherent sheaves on a zero dimensional scheme. The first arrow is simply
\[V \ast H^0 (L) \to V \ast H^0 (L | A) .\]
Since \(H^0 (L) \to H^0 (L | A)\) is surjective by our choice of \(L\), it follows that \(H^0 (V \ast L) \to H^0 (\mathcal{R}_q \ast L)\) is surjective, and so it follows that \(p_{\mathcal{Z} \ast} (V \ast p_{C \ast} L) \to p_{\mathcal{Z} \ast} (\mathcal{R}_L \ast p_{C \ast} L)\) is surjective at the point \(q\).

The rank of the vector bundle \(p_{\mathcal{Z} \ast} (\mathcal{R}_L \ast p_{C \ast} L)\) on \(\mathcal{Z}\) is \(d\). Taking the \(d\)th exterior of \(p_{\mathcal{Z} \ast} (V \ast p_{C \ast} L) \to p_{\mathcal{Z} \ast} (\mathcal{R}_L \ast p_{C \ast} L)\) we get a map
\[\bigwedge^d (V \ast H^0 (L) \to B_{L, \mathcal{Z}} .\]
This map is nonzero and that can be seen by looking at the restriction to the point \( q \). This shows that there is a global section of \( B_{L,2} \) whose restriction to \( q \) does not vanish. It follows that \( |B_{L,2}| \cdot |D| \geq 0 \). This completes the proof of the Lemma 22.

**Lemma 23.** Let \( A \) be an effective divisor on \( C \) of degree \( d \). Then there is a line bundle \( L \) of degree \( d + g - 1 \) such that the natural map
\[
H^0(L) \to H^0(L|_A)
\]
is surjective.

**Proof.** It suffices to find a line bundle of degree \( d + g - 1 \) such that \( H^1(L \otimes \mathcal{O}_C(-A)) = 0 \). By Serre duality this is same as saying that \( H^0(L^\vee \otimes K_C \otimes \mathcal{O}_C(A)) = 0 \). The degree of \( L^\vee \otimes K_C \otimes \mathcal{O}_C(A) \) is \( g - 1 \). Thus, fixing \( A \) we may choose a general \( L \) such that \( L^\vee \otimes K_C \otimes \mathcal{O}_C(A) \) line bundle has no global sections. \( \square \)

**Definition 24.** Define \( U \subset \mathcal{Q} \) to be the set of quotients of the form
\[
\underline{\mathcal{O}}_C^n \to \frac{\mathcal{O}_C}{\prod_{i=1}^r m_{C,c_i}^{d_i}} \cong \bigoplus m_{C,c_i}^{d_i} \quad c_i \neq c_j.
\]

We now prove a lemma, which is implicitly contained [8, Section 5]. Let \( \Sigma \subset C \times C^{(d)} \) denote the closed sub-scheme which is the universal divisor. In the following Lemma we work more generally with \( \mathcal{Q}(E, d) \).

**Lemma 25.** Let \( E \) be a locally free sheaf of rank \( r \) on \( C \). Let \( \mathcal{Q} = \mathcal{Q}(E, d) \) denote the Quot scheme of torsion quotients of length \( d \). The universal quotient \( \mathcal{B}_\mathcal{Q} \) is supported on \( \Phi^* \Sigma \subset C \times \mathcal{Q} \). The set \( U \) is open in \( \mathcal{Q} \). On \( C \times U \) the sheaf \( \mathcal{B}_\mathcal{Q} \) is a line bundle supported on the scheme \( \Phi^* \Sigma \cap (C \times U) \).

**Proof.** Let \( A \) denote the kernel of the universal quotient on \( C \times \mathcal{Q} \)
\[
0 \to A^h \to p^*_C E \to \mathcal{B}_\mathcal{Q} \to 0.
\]
The map \( \Phi \) is defined taking the determinant of \( h \), that is, using the quotient
\[
0 \to \det(A) \to p^*_C \det(E) \to \mathcal{F} \to 0.
\]
If \( \mathcal{I}_\Sigma \) denotes the ideal sheaf of \( \Sigma \) then this shows that
\[
\Phi^* \mathcal{I}_\Sigma = \det(A) \otimes p^*_C \det(E)^{-1}.
\]
Let \( 0 \to E^h \to E \) be locally free sheaves of the same rank on a scheme \( Y \). Let \( \mathcal{F} \) denote the ideal sheaf determined by \( \det(h) \). Then it is easy to see that \( \mathcal{F} \subset h(E^h) \subset E \). Applying this we get that \( (\Phi^* \mathcal{I}_\Sigma) p^*_C E \subset A \). This proves that \( \mathcal{B}_\mathcal{Q} \) is supported on \( \Phi^* \Sigma \). Let us denote by \( Z := \Phi^* \Sigma \subset C \times \mathcal{Q} \). Consider the closed subset \( Z_2 \subset Z \) defined as follows
\[
Z_2 := \{ z = (c, q) \in Z \mid \text{rank}_k (\mathcal{B}_\mathcal{Q} \otimes k(z)) \geq 2 \}.
\]
Then the image of \( Z_2 \) in \( \mathcal{Q} \) is closed and \( U \) is precisely the complement of \( Z_2 \). This proves that \( U \) is open in \( \mathcal{Q} \).

Let \( R \) be a local ring with maximal ideal \( m \) and let \( R \to S \) be a finite map. Let \( M \) be a finite \( S \) module, which is flat over \( R \) and such that \( M/mM \cong S/mS \). Then it follows easily that \( M \cong S \).

Let \( q \in U \subset \mathcal{Q} \) be a point. The sheaf \( \mathcal{B}_\mathcal{Q} \) is a coherent sheaf supported on \( Z \), the map \( Z \to \mathcal{Q} \) is finite, the fiber
\[
\mathcal{B}_q = \bigoplus m_{C,c_i}^{d_i} \cong \mathcal{O}_Z|_q \cong \mathcal{O}_Z|_q.
\]
From the preceding remark it follows that \( \mathcal{B}_\mathcal{Q} \) is a line bundle over \( Z \cap (C \times U) \). \( \square \)
Lemma 26. Consider the Quot scheme \( \mathcal{Q} = \mathcal{Q}(n, d) \). Let \( D \) be a smooth projective curve and let \( D \to \mathcal{Q} \) be a morphism such that its image intersects \( U \). Then \( ([\sigma_{\mathcal{Q}}(1)] + [\Delta_d/2]) \cdot [D] \geq 0 \).

Proof. Denote by \( \mathcal{R}_D \) the pullback of the universal quotient over \( C \times \mathcal{Q} \) to \( C \times D \). Denote by \( i_D : \Gamma \to C \times D \) the pullback of the universal subscheme \( \Sigma \to C \times C(d) \) to \( C \times D \). Then \( \mathcal{R}_D \) is supported on \( \Gamma \).

Let \( \Gamma_i \) be the irreducible components of \( \Gamma \). Since \( \Gamma \to D \) is flat each \( \Gamma_i \) dominates \( D \). Let \( f : \Gamma \to D \) denote the projection. There is an open subset \( U_i \subset D \) such that

\[
\Gamma_i \cap U_i = \bigcup_i f^{-1}(U_i)
\]

and \( \mathcal{R}_D \) restricted to \( f^{-1}(U_i) \) is a line bundle. Note that by \( \Gamma_i \cap f^{-1}(U_i) \) we mean this open subscheme of \( \Gamma \). Fix a closed point \( x_i \in \Gamma_i \cap f^{-1}(U_i) \). Consider the quotient

\[
\mathcal{V} \otimes \mathcal{O}_{C \times D} \to \mathcal{R}_D
\]

and restrict it to the point \( x_i \). We get a quotient

\[
\mathcal{V} \to \mathcal{R}_D \otimes k(x_i) \to 0.
\]

If we pick a general line in \( \mathcal{V} \), then it surjects onto \( \mathcal{R}_D \otimes k(x_i) \). Thus, for the general element \( s \in \mathcal{V} \), \( s \otimes \mathcal{O}_{C \times D} \) surjects onto \( \mathcal{R}_D \otimes k(x_i) \). This map factors through \( \mathcal{O}_{\Gamma} \), and we get an exact sequence

\[
0 \to \mathcal{O}_{\Gamma} \to \mathcal{R}_D \to F \to 0
\]

where \( F \) is supported on a 0 dimensional scheme. Then we have

\[
0 \to f_* \mathcal{O}_{\Gamma} \to f_* \mathcal{R}_D \to f_* F \to 0.
\]

Since \( f_* F \) is again supported on finitely many points, hence we have

\[
\deg(f_* \mathcal{R}_D) - \deg(f_* \mathcal{O}_{\Gamma}) \geq 0
\]

By Lemma 8, \( \deg(f_* \mathcal{R}_D) = [\sigma_{\mathcal{Q}}(1)] \cdot [D] \) and by [15, § 3] we have

\[
\deg(f_* \mathcal{O}_{\Gamma}) = [\sigma(-\Delta_d/2)] \cdot [D].
\]

Hence the result follows. \( \square \)

Corollary 27. If the image of \( f : D \to \mathcal{Q} \) intersects \( U \), then \( ([\sigma_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [D] \geq 0 \).

Proof. If its image intersects \( U \), then by Lemma 26,

\[
([\sigma_{\mathcal{Q}}(1)] + [\Delta_d/2]) \cdot [D] \geq 0.
\]

By Lemma 7,

\[
[\Delta_d/2] = \mu_0[L_0] - (1 - \mu_0)[\theta_d].
\]

Since \( \theta_d \) is nef, we have that

\[
([\sigma_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [D] \geq 0.
\]

\( \square \)

Lemma 28. Consider the Quot scheme \( \mathcal{Q} = \mathcal{Q}(n, d) \). Let \( D \) be a smooth projective curve and let \( f : D \to (\mathcal{Q} \setminus U) \subset \mathcal{Q} \) be a morphism. Then \( ([\sigma_{\mathcal{Q}}(1)] + (d + g - 2)[x]) \cdot [D] \geq 0 \).

Proof. Fix a point \( q \in f(D) \). Let \( A \) be the scheme theoretic support of the quotient \( \mathcal{R}_q \) on \( C \). Let \( \deg(A) = d' \). Since \( q \in \mathcal{Q} \), we have \( d' < d \). By Lemma 23 we have a line bundle \( L \) of degree \( d' + g - 1 \) such that \( H^0(L) \to H^0(L_A) \) is surjective. By Lemma 22 and Corollary 19 we get that \( [B_{L,\mathcal{Q}}] \cdot [D] = ([\sigma_{\mathcal{Q}}(1)] + (d' + g - 1)[x]) \cdot [D] \geq 0 \). Since \( [x] \) is nef on \( \mathcal{Q} \) and \( d' \leq d - 1 \) we get that \( ([\sigma_{\mathcal{Q}}(1)] + (d + g - 2)[x]) \cdot [D] \geq 0 \). \( \square \)

Proposition 29. Consider the Quot scheme \( \mathcal{Q} = \mathcal{Q}(n, d) \). Let \( \mu_0 = \frac{d + g - 1}{dg} \). Then the class \( \kappa_1 := ([\sigma_{\mathcal{Q}}(1)] + \mu_0[L_0] + \frac{d + g - 2}{dg}[\theta_d]) \) is nef.
Proof. Let $D \to \mathcal{D}$ be a morphism, where $D$ is a smooth projective curve. If the image of this morphism intersects $U$ then by Lemma 26 we have $([\mathcal{O}_\mathcal{D}(1)] + [\Delta_d/2]) \cdot [D] \geq 0$. By Lemma 7 we have $[\Delta_d/2] = \mu_0[L_0] - (1 - \mu_0)[\theta_d]$. Hence we get

$$([\mathcal{O}_\mathcal{D}(1)] + \mu_0[L_0]) \cdot [D] \geq (1 - \mu_0)\mu_0[L_0] \cdot [D] \geq 0.$$ 

Since $[\theta_d]$ is nef, we get

$$([\mathcal{O}_\mathcal{D}(1)] + \mu_0[L_0]) \cdot [D] + \frac{d + g - 2}{d g}[\theta_d] \cdot [D] \geq 0.$$ 

Now assume $D \to \mathcal{D}$ does not intersect $U$. Then by Lemma 28 we get

$$([\mathcal{O}_\mathcal{D}(1)] + (d + g - 2)[x]) \cdot [D] \geq 0.$$ 

By Lemma 7 we have $[x] = \frac{1}{d g}[L_0] + \frac{1}{d g}[\theta_d]$. Therefore

$$(d + g - 2)[x] = \frac{d + g - 2}{d g}[L_0] + \frac{d + g - 2}{d g}[\theta_d]$$

and

$$= \mu_0[L_0] - \frac{1}{d g}[L_0] + \frac{d + g - 2}{d g}[\theta_d].$$

Since $L_0$ is nef we get that

$$([\mathcal{O}_\mathcal{D}(1)] + \mu_0[L_0] + \frac{d + g - 2}{d g}[\theta_d]) \cdot [D] \geq 0.$$

□

Lemma 30. Let $L$ be a line bundle on $C$ of degree $d + g - 1$. If $d \geq \gamma(C)$ then the line bundle $B_{L, \mathcal{D}}$ is not ample. Moreover, for any $t \in [0, 1]$ the class $t[B_{L, \mathcal{D}}] + (1 - t)[\theta_d]$ is nef but not ample.

Proof. We saw in the last para of the proof of Proposition 18 that $\eta^* B_{L, \mathcal{D}} = L \mathcal{D} \otimes \mathcal{O}(-\Delta_d/2)$. Its class in the nef cone is $(d + g - 1)x - [\Delta_d/2]$. It follows from Lemma 7 that this is equal to $[\theta_d]$. Since $d \geq \gamma(C)$ we have $\theta_d$ is not ample on $C^{(d)}$. That $t[B_{L, \mathcal{D}}] + (1 - t)[\theta_d]$ is nef is clear since both $[B_{L, \mathcal{D}}]$ and $[\theta_d]$ are nef. This is not ample since $\eta^*$ of this class is $[\theta_d]$ on $C^{(d)}$, which is not ample. □

Proposition 31. Consider the Quot scheme $\mathcal{D} = \mathcal{D}(n, d)$. Then the class $[\mathcal{O}_\mathcal{D}(1)] + (d + g - 1)[x] \in N^1(\mathcal{D})$ is nef.

Proof. It is easily checked that the class $[\mathcal{O}_\mathcal{D}(1)] + (d + g - 1)[x]$ can be written as a positive linear combination of $[\theta_d]$ and the class in Proposition 29. □

We may slightly improve Proposition 31 in a special case using the results in [15]. For this we first recall the main results in [15, §4]. Let $C$ be a very general curve of genus $g(C) = 2k$. Since the gonality is given by $[\theta_{g(C)}], in this case it is $k + 1$. Let $C'$ denote the finitely many $g_{k+1}$'s on $C$ and define $L_i = K_C - L_i$. Then $\deg(L_i) = 3(k - 1)$. It is proved in [15, Proposition 3.6, Theorem 4.1] that $\mathcal{D}_{k, L_i}$ is nef but not ample.

Proposition 32. Let $C$ be a very general curve of genus $g(C) = 2k$. Consider the Quot scheme $\mathcal{D} = \mathcal{D}(n, k)$. The line bundle $B_{L, \mathcal{D}}$ is nef when $\deg(L) \geq 3(k - 1)$. When $\deg(L) = 3(k - 1)$ the class $t[B_{L, \mathcal{D}}] + (1 - t)[\mathcal{D}_{k, L_i}]$ is nef but not ample for any $t \in [0, 1]$.

We remark that this is an improvement since Proposition 31 only shows that $B_{L, \mathcal{D}}$ is nef when $\deg(L) \geq 3(k - 1)$.

Proof. It follows from Proposition 18 that the class of $B_{L, \mathcal{D}}$ in $N^1(\mathcal{D})$ is $[\mathcal{O}_\mathcal{D}(1)] + \deg(L)[x]$, since $B_{\theta_{g(C)}, \mathcal{D}} = \mathcal{O}_\mathcal{D}(1)$. Notice that this class only depends on the degree of $L$. Since the sum of nef line bundles is nef, it suffices to show that $[B_{L, \mathcal{D}}] = [\mathcal{O}_\mathcal{D}(1)] + \deg(L)[x]$ is nef when $\deg(L) = 3(k - 1)$.

The set $V(\mathcal{O}_L)$ is defined in equation [15, equation (18)]. Then (A) in [15, Theorem 4.1] says that for every $A \in C^{(k)}$ there is an $L_i$ such that $H^0(C, L_i) \to H^0(C, L_i|A)$ is surjective.
Let $f : D \to \mathcal{Z}$ be morphism, where $D$ is a smooth projective curve. Fix a point $q \in f(D)$. Let $A$ be the divisor corresponding to $\Phi(q)$, then $A$ is an effective divisor of degree $k$. For this $A$, choose a line bundle $L_i$ such that
\[ H^0(C, L_i) \to H^0(C, L_i|_A) \]
is surjective. The scheme theoretic support of $\mathcal{B}_q$ is contained in $A$. It follows from Lemma 22 that
\[ f^* B_{L_i, \mathcal{Z}} = f^* (\mathcal{O}_\mathcal{Z}(1)) + 3(k-1)[x] \geq 0. \]
It follows that $B_{L_i, \mathcal{Z}}$ is nef.

6. The genus 0 case

Throughout this section we will work with $C = \mathbb{P}^1$. Let us first compute the nef cone of $\mathcal{Z}(n, d)$.

Note that we have $C^{(d)} = \mathbb{P}^d$. Hence $N^1(C^{(d)}) = \mathbb{R}[\mathcal{O}_{\mathbb{P}^d}(1)]$. By Corollary 13 it follows that $N^1(\mathcal{Z})$ is two dimensional. Hence, it suffices to find a line bundle on $\mathcal{Z}$ which is different from the pullback of $\mathcal{O}_{\mathbb{P}^d}(1)$ and which is nef but not ample. The following result is proved in [16, Theorem 6.2], but we include it for the benefit of the reader.

**Proposition 33.**

\[ \text{Nef(} \mathcal{Z}(n, d) \text{)} = \mathbb{R}_{\geq 0} \left[ B_{\mathcal{O}(d-1), \mathcal{Z}} \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{\mathbb{P}^d}(1) \right] \]
\[ = \mathbb{R}_{\geq 0} \left[ (\mathcal{O}_\mathcal{Z}(1) + (d-1) \mathcal{O}_{\mathbb{P}^d}(1)) \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{\mathbb{P}^d}(1) \right]. \]

**Proof.** Let $W := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. There is a natural isomorphism $\mathbb{P}W^* \xrightarrow{\sim} C^{(d)}$. The universal subscheme $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}W^*$ is given by the tautological section
\[ p_2^* \mathcal{O}_{\mathbb{P}W^*}(-1) \to p_2^* W = p_1^* W \to p_1^* \mathcal{O}_{\mathbb{P}^1}(d). \]

By Lemma 22 and Lemma 23 we get that $B_{\mathcal{O}(d-1), \mathcal{Z}}$ is nef. To show $B_{\mathcal{O}(d-1), \mathcal{Z}}$ is not ample, consider a section $\eta : C^{(d)} \to \mathcal{Z}$ constructed as in (7) with $L$ the trivial bundle. Let $p_i$ denote the two projections from $\mathbb{P}^1 \times \mathbb{P}W^*$. By definition and Lemma 16 it follows that $\eta^* B_{\mathcal{O}(d-1), \mathcal{Z}} = \det(p_2) \mathcal{O}_{\mathcal{Z}} \oplus p_1^* \mathcal{O}_{\mathbb{P}^1}(d-1))$. Tensoring the exact sequence
\[ 0 \to p_1^* \mathcal{O}_{\mathbb{P}^1}(d-1) \to p_2^* \mathcal{O}_{\mathbb{P}W^*}(-1) \to \mathcal{O}_{\mathcal{Z}} \xrightarrow{\eta} \mathcal{O}_{\mathcal{Z}} \to 0 \]
with $p_1^* \mathcal{O}_{\mathbb{P}^1}(d-1)$ and applying $p_2$ it easily follows that $p_2 \mathcal{O}_{\mathcal{Z}} \oplus p_1^* \mathcal{O}_{\mathbb{P}^1}(d-1)$ is the trivial bundle and so $\eta^* B_{\mathcal{O}(d-1), \mathcal{Z}}$ is trivial. This proves that $B_{\mathcal{O}(d-1), \mathcal{Z}}$ is nef but not ample.

By restricting to a fiber of $\Phi$ and using Corollary 19 we see that $[B_{\mathcal{O}(d-1), \mathcal{Z}}]$ is linearly independent from $[\mathcal{O}_{\mathbb{P}^d}(1)]$. This completes the proof of the first equality. The second equality will follow from the first equality once we show that
\[ \left[ B_{\mathcal{O}(d-1), \mathcal{Z}} \right] = \left[ \mathcal{O}_\mathcal{Z}(1) \right] + (d-1) \left[ \mathcal{O}_{\mathbb{P}^d}(1) \right]. \]

By Corollary 19, we have that $[B_{\mathcal{O}(d-1), \mathcal{Z}}] = [\mathcal{O}_\mathcal{Z}(1)] + (d-1)[x]$. Now recall that given $x \in \mathbb{P}^1$, $[x]$ is the class of the divisor in $C^{(d)}$ whose underlying set consists of effective divisors of degree $d$ containing $x$ (see (4)). Hence, $[x]$ is the class of the hyperplane section
\[ \mathbb{P} \left( H^0(\mathbb{P}^1, \mathcal{O}(d) \otimes \mathcal{O}(-x)) \right)^* \subset \mathbb{P} \left( H^0(\mathbb{P}^1, \mathcal{O}(d)) \right)^* = C^{(d)}. \]

Therefore $[x] = [\mathcal{O}_{\mathbb{P}^1}(1)]$ and this completes the proof of the second equality. \qed
Theorem 34. Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Let $L = \mathcal{O}(-a_1 + d - 1)$. Then
\[
\text{Nef}(\mathcal{Q}(E, d)) = \mathbb{R}_{\geq 0} \left[ B_{L, \mathcal{Q}(E, d)} \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right]
\]
\[
= \mathbb{R}_{\geq 0} \left[ \left( \mathcal{O}_{\mathcal{Q}(E, d)}(1) \right) + (-a_1 + d - 1) \left[ \mathcal{O}_{pd}(1) \right] \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right].
\]

Proof. By Corollary 13 we get that $N^1(\mathcal{Q}(E, d))$ is 2-dimensional. Hence it is enough to give two line bundles which are nef but not ample. Clearly $\Phi^*_n(\mathcal{Q}(E, d)) \mathcal{O}_{pd}(1)$ is nef but not ample. So it is enough to show that $B_{L, \mathcal{Q}(E, d)}$ is nef but not ample.

Since $a_j - a_1 \geq 0 \quad \forall \quad j \geq 1$, we get that $\mathcal{O}(a_1)$ is globally generated. Let $V := H^0(C, \mathcal{O}(-a_1))$ and let $\dim V = n$. Then we have a surjection $V \otimes \mathcal{O}_C \rightarrow E(-a_1)$. Then gives us a surjection
\[
V \otimes \mathcal{O}_C \rightarrow p^n_\mathcal{C}E(-a_1) \rightarrow B_{\mathcal{Q}(E, d)} \otimes p^n_\mathcal{C}(\mathcal{O}(a_1)) \rightarrow 0.
\]
This defines a map $f : \mathcal{Q}(E, d) \rightarrow \mathcal{Q}_C(n, d)$. By Lemma 16 we get that
\[
f^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)} = B_{L, \mathcal{Q}(E, d)} = \det \left( p_\mathcal{Q}(E, d)^* \left( \mathcal{Q}(E, d) \otimes p^n_\mathcal{C}L \right) \right).
\]
Since $B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$ is nef we get that $B_{L, \mathcal{Q}(E, d)}$ is nef. We next show that the $B_{L, \mathcal{Q}(E, d)}$ is not ample. Consider the section $\eta_{\mathcal{Q}(E, d)}$ of $\mathcal{Q}(E, d) \rightarrow C(d)$ defined by the quotient $p^n_\mathcal{C}E \rightarrow p^n_\mathcal{C}(\mathcal{O}(a_1)) \otimes \mathcal{O}_\Sigma$ on $C \times C(d)$ (see (7)). Then $f \circ \eta_{\mathcal{Q}(E, d)}$ is a section of $\mathcal{Q}(n, d) \rightarrow C(d)$ defined by a quotient $\mathcal{O}_n \rightarrow \mathcal{O}_\Sigma \rightarrow 0$ on $C \times C(d)$. Therefore $\eta_{\mathcal{Q}(E, d)}B_{\mathcal{Q}(E, d)} = \eta^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$. As $\eta^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$ is not ample, we get that $B_{L, \mathcal{Q}(E, d)}$ is not ample. The second equality follows again from the fact that $[x] = [\mathcal{O}_{pd}(1)]$. \hfill \Box

7. Some cases of equality

Now we are back to the assumption that the genus of the curve satisfies $g(C) \geq 1$ and if $g(C) \geq 2$ then we also assume that $C$ is very general.

Definition 35. Let $U' \subset \mathcal{Q}$ be the open set consisting of quotients $\mathcal{Q}_C^n \rightarrow B \rightarrow 0$ such that the induced map $H^0(C, \mathcal{Q}_C^n) \rightarrow H^0(C, B)$ is surjective.

Lemma 36. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Let $D$ be a smooth projective curve and let $D \rightarrow \mathcal{Q}$ be a morphism such that its image intersects $U'$. Then $[\mathcal{O}_\mathcal{Q}(1)] \cdot [D] \geq 0$.

Proof. We continue with the notations of Lemma 26. Let $p_D : C \times D \rightarrow D$ be the projection. Then applying $(p_D)^*$ to the quotient $\mathcal{Q}_C^n \rightarrow \mathcal{Q}_D$ we get that the morphism
\[
(p_D)^* \mathcal{Q}_C^n \rightarrow \mathcal{Q}_D = (p_D)^* \mathcal{Q}_D
\]
is generically surjective by our assumption and Lemma 8. Hence we get that
\[
[\mathcal{O}_\mathcal{Q}(1)] \cdot [D] = \deg \left( (p_D)^* \mathcal{Q}_D \right) \geq 0.
\]

One extremal ray in Nef($\mathcal{Q}^2$) is given by $L_0$. Let other extremal ray of Nef($\mathcal{Q}^2$) be given by
\[
\alpha_t = (t+1)x - \Delta_2/2,
\]
(see [12, page 75]). Then using Lemma 7, we get that
\[
\Delta_2/2 = \frac{t+1}{g+t}L_0 - \frac{g-1}{g+t} \alpha_t.
\]
\hfill (10)

Theorem 37. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then
\[
\text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} \left( [\mathcal{Q}_1] + \frac{t+1}{g+t} [L_0] \right) + \mathbb{R}_{\geq 0} [L_0] + \mathbb{R}_{\geq 0} [\alpha_t].
\]
Proof. We first prove that \([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]\) is nef. Since \(d = 2\), then there are only three types of quotients:

1. \(\mathcal{O}_C^m \rightarrow \frac{C_{c_1}}{m_{c_1}} \oplus \frac{C_{c_2}}{m_{c_2}} \) with \(c_1 \neq c_2\),
2. \(\mathcal{O}_C^m \rightarrow \frac{C_{c_1}}{m_{c_1}}\),
3. \(\mathcal{O}_C^m \rightarrow \frac{C_{c}}{m_{c}} \oplus \frac{C_{c}}{m_{c}}\).

The first two quotients are in \(U\) while the third one is in \(U'\), that is, we get \(U \cup U' = \mathcal{O}\). Now let \(D\) be a smooth projective curve and \(D \rightarrow \mathcal{O}\) be a morphism. If its image intersects \(U\), then by Corollary 27, \([\mathcal{O}_\mathbb{P}(1)] + \Delta_2/2 \cdot [D] \geq 0\). Using (10) and the fact that \(\alpha_t\) is nef, we get that \(([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]) \cdot [D] \geq 0\). If \(D\) does not intersect \(U\) then \(D \subset U'\). Hence by Lemma 36, we have

\([\mathcal{O}_\mathbb{P}(1)] \cdot [D] \geq 0\).

Since \([L_0]\) is nef we have that

\(\left([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]\right) \cdot [D] \geq 0\).

Also \(([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]) \cdot [\delta] = 0\). Hence any convex linear combination of \([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]\) and \([L_0]\) is nef but not ample. By (10) \(\eta^*([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]) = \frac{g-1}{g+t} \alpha_t\). Hence any convex linear combination of \([\mathcal{O}_\mathbb{P}(1)] + \frac{t+1}{g+t} [L_0]\) and \([\alpha_t]\) is not ample. Hence the result follows.

Precise values for \(t\) depending on \(g\) are known when

1. When \(g = 1, t = 1\).
2. When \(g = 2, t = 2\).
3. When \(g = 3, t = 9/5\).
4. When \(g = \text{a perfect square } t = \sqrt{g}\), see [11, Theorem 2].
5. In [5, Proposition 3.2], when \(g \geq 9\), assuming the Nagata conjecture, they prove that \(t = \sqrt{g}\).

Thus, in all these cases using Theorem 37 we get the Nef cone of \(\mathcal{O}(n, 2)\).

7.1. Criterion for nefness

In the remainder of this section, we will need to work with \(C^{(d)}\) for different values of \(d\). The line bundles \(L_0\) on \(C^{(d)}\) will therefore be denoted by \(L_0^{(d)}\) when we want to emphasize the \(d\). Similarly, we will denote \(\mu_0^{(d)} = \frac{d+g-1}{dg}\). Let \(\mathcal{S}^n\) be the set of all partitions \((d_1, d_2, \ldots, d_k)\) of \(d\) of length at most \(n\). Given an element \(d \in \mathcal{S}^n\) define

\(C^{(d)} := C^{(d_1)} \times C^{(d_2)} \times \ldots \times C^{(d_k)}\)

and if \(p_i : C^{(d)} \rightarrow C^{(d_i)}\) is the \(i\)th projection we define a class

\([\mathcal{O}(-\Delta_d/2)] := \left[\sum p_i^* \mathcal{O}(-\Delta_d/2)\right] \in N^1\left(C^{(d)}\right)\).

Note that we have a natural addition

\(\pi_d : C^{(d)} \rightarrow C^{(d)}\).

For a partition \(d \in \mathcal{S}^n\) define a morphism

\(\eta_d : C^{(d)} \rightarrow \mathcal{O}\).
as follows. For any \( l \geq 1 \), we define the universal subscheme of \( C^{(l)} \) over \( C \times C^{(l)} \) by \( \Sigma_l \). Then over \( C \times C^{(d)} \) we have the subschemes \((id \times p_i)\ast \Sigma_l \). We have a quotient
\[
q_d: \theta^n_{C \times C^{(d)}} \rightarrow \bigoplus_i \theta((id \times p_i)\ast \Sigma_l)
\]
defined by taking direct sum of morphisms \( \theta_{C \times C^{(d)}} \rightarrow \theta((id \times p_i)\ast \Sigma_l) \). Then \( q_d \) defines a map \( C^{(d)} \rightarrow \mathfrak{D} \). By Lemma 16, we have
\[
[\eta_d^*\Theta_{\mathfrak{D}}(1)] = [\theta(-\Delta_d/2)] .
\]

**Lemma 38.** Let \( D \) be a smooth projective curve. Let \( D \rightarrow \mathfrak{D} \) be a morphism. Then there exists a partition \( d \in \mathfrak{D}^{\geq n} \) such that the composition \( D \rightarrow \mathfrak{D} \rightarrow C^{(d)} \) factors as \( D \rightarrow C^{(d)} \rightarrow C^{(d')} \) and \( [\Theta_{\mathfrak{D}}(1)] \cdot [D] \geq [\theta(-\Delta_d/2)] \cdot [D] \).

**Proof.** We will proceed by induction on \( d \). When \( d = 1 \) the statement is obvious.

Let us denote the pullback of the universal quotient on \( C \times \mathfrak{D} \) to \( C \times D \) by \( \mathcal{B}_D \) and let \( f: C \times D \rightarrow D \) be the natural projection. Consider a section such that the composite \( \mathcal{B}_{C \times D} \rightarrow \mathcal{B}^{\geq n}_{C \times D} \rightarrow \mathcal{B}_D \) is non-zero and let \( \mathcal{F} \) denote the cokernel of the composite map. We have a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \theta_{C \times D} & \rightarrow & \theta_{C \times D}^{\geq n} & \rightarrow & \theta_{C \times D}^{\geq n-1} & \rightarrow & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & \theta_{\Gamma'} & \rightarrow & \mathcal{B}_D & \rightarrow & \mathcal{F} & \rightarrow & 0
\end{array}
\]
(12)

Let \( T_0(\mathcal{F}) \subset \mathcal{F} \) denote the maximal subsheaf of dimension 0, see [10, Definition 1.1.4]. Define \( \mathcal{F'} := \mathcal{F} / T_0(\mathcal{F}) \). Now, either \( \mathcal{F'} = 0 \) or \( \mathcal{F'} \) is torsion free over \( D \), and hence, flat over \( D \). In the first case, it follows that \( D \) meets the open set \( U \) in Lemma 26. Then we take \( d = (d) \) and the statement follows from Lemma 26. So we assume \( \mathcal{F'} \) is flat over \( D \) and let \( d' \) be the degree of \( \mathcal{F'}|_{C \times x} \), for \( x \in D \). So \( 0 < d' < d \). By (12) we have
\[
\deg f_*\mathcal{B}_D = \deg f_*\theta_{\Gamma'} + \deg f_*\mathcal{F}.
\]
Since \( T_0(\mathcal{F}) \) is supported on finitely many points, we have \( \deg \mathcal{F} \geq \deg \mathcal{F'} \). In other words, we have
\[
\deg f_*\mathcal{B}_D \geq \deg f_*\theta_{\Gamma'} + f_*\mathcal{F}'.
\]
(13)

Now \( \Gamma' \) defines a morphism \( D \rightarrow C^{(d-d')} \) and note that
\[
\deg f_*\theta_{\Gamma'} = [\theta(-\Delta_d/2)] \cdot [D].
\]
The quotient \( \theta^{\geq n-1}_{C \times D} \rightarrow \mathcal{F'} \rightarrow 0 \) defines a map \( D \rightarrow \mathfrak{D}(n-1, d') \). By induction hypothesis, we get that there exists a partition \( d' \in \mathfrak{D}_d^{\geq n-1} \) such that the composition \( D \rightarrow \mathfrak{D}(n-1, d') \rightarrow C^{(d')} \) factors as \( D \rightarrow C^{(d')} \rightarrow C^{(d')} \) and
\[
[\Theta_{\mathfrak{D}(n-1,d')}(1)] \cdot [D] \geq [\theta(-\Delta_d/2)] \cdot [D].
\]
Since \( \deg f_*\mathcal{F}' = [\Theta_{\mathfrak{D}(n-1,d')}(1)] \cdot [D] \) we have that \( \deg f_*\mathcal{F}' \geq [\theta(-\Delta_d/2)] \cdot [D] \). From (13) we get that
\[
[\Theta_{\mathfrak{D}}(1)] \cdot [D] \geq [\theta(-\Delta_d/2)] \cdot D + [\theta(-\Delta_d/2)] \cdot [D].
\]
Now we define \( d := (d - d', d') \) and the statement follows from the above inequality. \( \square \)

**Theorem 39.** Let \( \beta \in N^1(C^{(d)}) \). Then the class \( [\Theta_{\mathfrak{D}}(1)] + \beta \in N^1(\mathfrak{D}) \) is nef iff the class \( [\theta(-\Delta_d/2)] + \pi_d^*\beta \in N^1(C^{(d)}) \) is nef for all \( d \in \mathfrak{D}_d^{\geq n} \).
Theorem 43. Let $g \in \mathcal{O}(\mathbb{P}^n_d)$. Let $\Phi \circ \tilde{\Phi}$ be a morphism. By Lemma 38 we have that there exists $d \in \mathcal{O}(\mathbb{P}^n_d)$ such that $D \to C^{(d)} \to C^{(d)}$ and

$$[\mathcal{O} \cdot \mathcal{O}(\mathbb{P}^n_d)] \cdot [D] \geq [\mathcal{O}(\mathbb{P}^n_d^2)] \cdot [D].$$

Now by assumption we have that

$$[\mathcal{O}(\mathbb{P}^n_d^2)] \cdot [D] \geq -\beta \cdot [D].$$

Therefore we get

$$[\mathcal{O}(\mathbb{P}^n_d)] \cdot [D] \geq -\beta \cdot [D].$$

Hence we get that the class $[\mathcal{O}(\mathbb{P}^n_d)] + \beta$ is nef.

Proof. From (11) it is clear that if $[\mathcal{O}(\mathbb{P}^n_d)] + \beta$ is nef, then $\eta^*_d([\mathcal{O}(\mathbb{P}^n_d)] + \beta) = [\mathcal{O}(\mathbb{P}^n_d^2)] + \pi^*_d\beta$ is nef.

For the converse, we assume $[\mathcal{O}(\mathbb{P}^n_d^2)] + \pi^*_d\beta$ is nef for all $d \in \mathcal{O}(\mathbb{P}^n_d^2)$. Let $D$ be a smooth projective curve and $D \to \mathcal{O}$ be a morphism. By Lemma 38 we have that there exists $d \in \mathcal{O}(\mathbb{P}^n_d^2)$ such that $D \to C^{(d)} \to C^{(d)}$ and

$$[\mathcal{O} \cdot \mathcal{O}(\mathbb{P}^n_d)] \cdot [D] \geq [\mathcal{O}(\mathbb{P}^n_d^2)] \cdot [D].$$

As a corollary we get the following result. When $g = 1$ note that $\mu^{(2)}_d = 1$.

Corollary 42. Let $n \geq d$. Then the class $[\mathcal{O}(\mathbb{P}^n_d)] + \mu^{(2)}_d [L^{(d)}_0] \in N^1(\mathcal{O})$ is nef but not ample.

Proof. By Proposition 41 we have that $[\mathcal{O}(\mathbb{P}^n_d)] + \mu^{(2)}_d [L^{(d)}_0]$ is nef. Now recall that when $n \geq d$ we have the curve $\delta \to \mathcal{O}$ (8). From the definition of $\delta$ and Lemma 16 we have $[\mathcal{O}(\mathbb{P}^n_d)] \cdot [\delta] = 0$. Also $\Phi \circ \delta = \delta$. Hence $[L^{(d)}_0] \cdot [\delta] = [L^{(d)}_0] \cdot [\delta] = 0$. From this we get $[\mathcal{O}(\mathbb{P}^n_d)] + \mu^{(2)}_d [L^{(d)}_0] \cdot [\delta] = 0$ and hence $[\mathcal{O}(\mathbb{P}^n_d)] + \mu^{(2)}_d [L^{(d)}_0]$ is not ample.

As a corollary we get the following result. When $g = 1$ note that $\mu^{(2)}_d = 1$.

Theorem 43. Let $g = 1$, $n \geq 1$ and $\mathcal{O} = \mathcal{O}(\mathbb{P}^n_d)$. Then the class $[\mathcal{O}(\mathbb{P}^n_d)] + [\Delta_d/2] \in N^1(\mathcal{O})$ is nef. Moreover,

$$\text{Nef}(\mathcal{O}) = \mathbb{R}_{\geq 0} [\mathcal{O}(\mathbb{P}^n_d)] + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0} [\Delta_d/2].$$
8. Curves over the small diagonal

Throughout this section the genus of the curve $C$ will be $g(C) \geq 2$ and $C$ is a very general curve. Recall that $\Phi : \mathcal{Z} \to C^{(d)}$ is the Hilbert–Chow map.

**Proposition 44.** Let $f : D \to \mathcal{Z}(n, d)$ be such that $\Phi \circ f$ factors through the small diagonal. Then $|\mathcal{O}_\mathcal{Z}(1)| \cdot [D] \geq 0$.

**Proof.** Since $\Phi \circ f$ factors through the small diagonal, there is a map $g : D \to C$ such that if $\Gamma := \Gamma_g$ denotes the graph of $g$ in $C \times D$, and $\mathcal{O}_C^n \to \mathcal{B}_D$ is the quotient on $C \times D$, then $\mathcal{B}_D$ is supported on $\mathcal{O}_{C} \times D/\mathcal{I}(\Gamma)^d$. Denote $\mathcal{I} := \mathcal{I}(\Gamma)$. Then $\mathcal{B}_D/\mathcal{I}\mathcal{B}_D$ is a globally generated sheaf on $D$ and so its determinant has degree $\geq 0$. Now consider the sheaf

$$\mathcal{I}^i \mathcal{B}_D/\mathcal{I}^{i+1}\mathcal{B}_D \cong \left(\mathcal{I}/\mathcal{I}^2\right)^{\otimes i} \otimes \mathcal{B}_D/\mathcal{I}\mathcal{B}_D.$$  

Using adjunction it is easily seen that $\mathcal{I}/\mathcal{I}^2 \cong g^*\omega_C$. Since $\text{det}(\mathcal{B}_D/\mathcal{I}\mathcal{B}_D)$ has degree $\geq 0$, it follows that $\text{det}(\mathcal{I}^{i}\mathcal{B}_D/\mathcal{I}^{i+1}\mathcal{B}_D)$ has degree $\geq 0$. From the filtration

$$\mathcal{B}_D \supset \mathcal{I}\mathcal{B}_D \supset \mathcal{I}^2\mathcal{B}_D \supset \ldots \supset \mathcal{I}^d\mathcal{B}_D = 0$$

we easily conclude that $|\mathcal{O}_\mathcal{Z}(1)| \cdot [D] \geq 0$. □

**Lemma 45.** Let $D \to C^{(d)}$ be a morphism. Then we can find a cover $\tilde{D} \to D$ such that the composite $\tilde{D} \to D \to C^{(d)}$ factors through $C^d$.

**Proof.** Let $D_1$ be a component of $D \times C^{(d)} C^d$ which dominates $D$. Take $\tilde{D}$ to be a resolution of $D_1$. □

**Corollary 46.** Let $D \to \mathcal{Z}$ be a morphism. Replacing $D$ by a cover $\tilde{D}$ we may assume that the map $\tilde{D} \to D \to \mathcal{Z} \to C^{(d)}$ factors through $C^d$.

In view of the above, given a map $D \to Q$ we may assume that there is a sequence of maps $D \to \mathcal{Z} \to C^{(d)}$ factors through $C^d$. Let each component be given by a map $f_i : D \to C$. Denote by $i_D : \Gamma := \Gamma_f \to C \times D$ the pullback of the universal subscheme $\Sigma := C \times C^{(d)}$ to $C \times D$. The ideal sheaf of $\Gamma$ is the product $\mathcal{I}(\Gamma_f)$, the ideal sheaves of the graphs $\Gamma_f \subset C \times D$. Moreover, $\mathcal{B}_D$ is supported on $\Gamma$. Let $g_1, g_2, \ldots, g_r$ be the distinct maps in the set $\{f_1, f_2, \ldots, f_d\}$ and assume that $g_i$ occurs $d_i$ many times. Then we have $\mathcal{I}(\Gamma) = \prod_{i=1}^r \mathcal{I}(\Gamma_{g_i})^{d_i}$. There is a natural map

$$\psi : \mathcal{B}_D \to \bigoplus \mathcal{B}_D/\mathcal{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D.$$  

**Lemma 47.** Let $f : D \to \mathcal{Z}$ be such that $\Phi \circ f$ factors through $C^d \to C^{(d)}$. If $\psi$ is an isomorphism then $|\mathcal{O}_\mathcal{Z}(1)| \cdot [D] \geq 0$.

**Proof.** Since $\mathcal{B}_D$ is a quotient of $\mathcal{O}_C^n \times D$ it follows that each $\mathcal{B}_D/\mathcal{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D$ is a quotient of $\mathcal{O}_C^n \times D$. Thus, each $\mathcal{B}_D/\mathcal{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D$ defines a map $D \to \mathcal{Z}(n, d_i)$ such that the image under the map $\Phi : \mathcal{Z}(n, d_i) \to C^{(d_i)}$ is the small diagonal. By Proposition 44 it follows that degree of $\text{det}(\mathcal{B}_D/\mathcal{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D)$ is $\geq 0$. Since $\psi$ is an isomorphism it follows that degree of $\text{det}(\mathcal{B}_D)$ is $\geq 0$. □

We can use the above method to prove a result similar to Theorem 37 when $d = 3$.

**Corollary 48.** Let $d = 3$. Consider the Quot scheme $\mathcal{Z} = \mathcal{Z}(n, 3)$. Let $\mu_0^{(3)} = \frac{g+2}{3g}$. Then $|\mathcal{O}_\mathcal{Z}(1)| + \mu_0^{(3)}[L^{(3)}_0]$ is nef.

**Proof.** If $d = 3$ there are only these types of quotients:

1. $\mathcal{O}_C^n \to \mathcal{O}_C/m_{C,c_1}m_{C,c_2}m_{C,c_3}$,
Let $f : D \to \mathcal{D}$ be a map. If $D$ contains a quotient of type (1) or (3) then $D$ meets $U$ or $U'$ (see Definition 24 and Definition 35). Thus, in these cases $([\mathcal{O}_{\mathcal{D}}(1)] + \mu_0^{(3)} [L^{(3)}_0]) \cdot [D] \geq 0$ by Corollary 27 and Lemma 36.

Now consider the case when all points in the image of $D$ are of type (2). After replacing $D$ by a cover, using Corollary 46, we may assume that the map $D \to \mathcal{D}$ looks like $d \to (g_1(d), g_2(d))$. Now consider a general section $\mathcal{O}_{C \times D} \to \mathcal{B}_D$. Arguing as in the proof of Lemma 26 we get a diagram as in equation (12), such that $\mathcal{O}_{\mathcal{D}}$ defines a map $D \to C^{(2)}$ and $\mathcal{F} = \mathcal{F}/T_0(\mathcal{F})$ is a line bundle on $D$ which is globally generated. Hence

$$([\mathcal{O}_{\mathcal{D}}(1)] \cdot [D] \geq [\mathcal{O}(-\Delta_2/2)] \cdot [D] + [c_1(p_{D*,}(\mathcal{F}))] \cdot [D]$$

$$\geq -\mu_0^{(2)} [L^{(2)}_0] \cdot [D].$$

One easily checks using the definition of $L_0$ that in this case $[L^{(3)}_0] \cdot [D] = 2[L^{(2)}_0] \cdot [D]$. Thus,

$$([\mathcal{O}_{\mathcal{D}}(1)] + \mu_0^{(3)} [L^{(3)}_0]) \cdot [D] \geq 2 \mu_0^{(3)} - \mu_0^{(2)} [L^{(2)}_0] \cdot [D] \geq 0.$$ 

This completes the proof of the Corollary 48.

Combining this with Proposition 20 we get the following result.

**Theorem 49.** Let $C$ be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{D} = \mathcal{D}(n, 3)$. Let $\mu_0 = \frac{g + 2}{3g}$. Then

$$\text{Nef}(\mathcal{D}) = \mathbb{R}_{\geq 0} \left( [\mathcal{O}_{\mathcal{D}}(1)] + \mu_0 [L^{(3)}_0] \right) + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [L^{(3)}_0].$$

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