On a morphism of compactifications of moduli scheme of vector bundles

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A morphism of the (possibly, nonreduced) Gieseker – Maruyama functor (of semistable coherent torsion-free sheaves) on the surface to the nonreduced functor of admissible semistable pairs with the same Hilbert polynomial, is constructed. This leads to the morphism of moduli schemes. As usually, we consider subfunctors corresponding to main components of moduli schemes.

Bibliography: 11 items.

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The purpose of the present paper is to construct a morphism of Gieseker – Maruyama moduli scheme $\overline{M}$ to the moduli scheme of semistable admissible pairs $\tilde{M}$ which was built up in a series of papers of the author [5] - [11]. In [8] the construction of a morphism $\kappa_{\text{red}} : \overline{M}_{\text{red}} \to \tilde{M}_{\text{red}}$ of same schemes was constricted but both schemes were considered with reduced scheme structures. This restriction (the absence of nilpotent elements in structure sheaves) is essential for the construction performed in the cited paper. In the present article we remove this restriction and prove the existence of a morphism $\kappa : \overline{M} \to \tilde{M}$. The morphism $\kappa_{\text{red}}$ from [8] is the reduction of $\kappa$.

For this purpose we develop the version of the standard resolution for the family of semistable coherent torsion-free sheaves for the case when the base scheme is nonreduced. For our considerations it is enough to restrict ourselves by the class of schemes $T$ such that their reductions $T_{\text{red}}$ are irreducible schemes.

We construct the natural transformation of the Gieseker – Maruyama functor to the functor of admissible semistable pairs. The natural transformation leads to the morphism of moduli schemes. In [11] it is shown that reductions $\overline{M}_{\text{red}}$ and $\tilde{M}_{\text{red}}$ are isomorphic.

We work on a smooth irreducible projective algebraic surface $S$ over a field $k = \overline{k}$ of characteristic zero. If the opposite is not declared, the term ”point” means closed point. On $S$ an ample invertible sheaf $L$ is chosen and fixed. It is called a polarization.
The article consists of three sections. In sect. 1 we recall the basic notions which are necessary for the further considerations. Sect. 2 contains the construction of the natural transformation \( f^{GM} \rightarrow f \). At last, in sect. 3 we prove the existence of the morphism of moduli schemes of functors of interest. The morphism is induced by the natural transformation constructed. In the present article we prove the following result.

**Theorem 1.** The Gieseker – Maruyama functor \( f^{GM} \) of semistable torsion-free coherent sheaves of rank \( r \) and with Hilbert polynomial \( r p(t) \) on the surface \((S, L)\), has a natural transformation to the functor \( f \) of admissible semistable pairs \((\tilde{S}, \tilde{L}), \tilde{E})\) where the locally free sheaf \( \tilde{E} \) on the projective scheme \((\tilde{S}, \tilde{L})\) has same rank and Hilbert polynomial. In particular, there is a morphism of moduli schemes \( \tilde{M} \rightarrow M \) associated with this natural transformation.

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1 **Objects and functors**

We use the classical definition of semistability due to Gieseker [1].

**Definition 1.** Coherent \( \mathcal{O}_S \)-sheaf \( E \) is stable (resp., semistable) if for any proper subsheaf \( F \subset E \) of rank \( r' = \text{rank} F \) for \( n \gg 0 \)

\[
\frac{\chi(E \otimes L^n)}{r} > \frac{\chi(F \otimes L^n)}{r'}, \quad \text{(resp.,} \quad \frac{\chi(E \otimes L^n)}{r} \geq \frac{\chi(F \otimes L^n)}{r'} \text{)}.
\]

Consider the Gieseker – Maruyama functor \( f^{GM} : (\text{Schemes}_k)^o \rightarrow \text{Sets} \) attaching to any scheme \( T \) the set \( \mathcal{F}^{GM}_T \) where

\[
\mathcal{F}^{GM}_T = \left\{ \begin{array}{l}
\text{E– sheaf of } \mathcal{O}_{T \times S} \text{ – modules, flat over } T; \\
\mathcal{L} \text{– invertible sheaf of } \mathcal{O}_{T \times S} \text{ – modules,} \\
\text{ample relative to } T; \\
E_t := E|_{t \times S} \text{ torsion-free and semistable} \\
due to Gieseker relative to } L_t := \mathcal{L}|_{t \times S}; \\
\chi(E_t \otimes \mathcal{L}_t^n) = r p(n).
\end{array} \right\}
\]

**Definition 2.** [7, 8] Polarized algebraic scheme \((\tilde{S}, \tilde{L})\) is admissible if the scheme \((\tilde{S}, \tilde{L})\) satisfies one of the conditions

i) \((\tilde{S}, \tilde{L}) \cong (S, L),\

ii) \( S \cong \text{Proj } \bigoplus_{t \geq 0} (I[t] + (t))^*/(t^{s+1}), \) where \( I = \mathcal{F} \text{itt}^0 \mathcal{E}xt^2(\mathcal{E}, \mathcal{O}_S) \) for Artinian quotient sheaf \( \mathcal{E} : \bigoplus I^s \mathcal{O}_S \rightarrow \mathcal{E} \) of length \( l(\mathcal{E}) \leq c_2. \) There is a canonical morphism \( \sigma : \tilde{S} \rightarrow S \) and \( \tilde{L} = L \otimes (\sigma^{-1} I \cdot \mathcal{O}_S) \) — ample invertible sheaf on \( \tilde{S}; \) this polarization \( \tilde{L} \) is called the distinguished polarization.

**Definition 3.** [8, 9] S-stable (semistable) pair \((\tilde{S}, \tilde{L}), \tilde{E})\) is the following data:

\[
\begin{align*}
&\text{\begin{tabular}{l}
\end{tabular}}
\end{align*}
\]
• \( \widetilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \) — admissible scheme, \( \sigma : \tilde{S} \to S \) — its canonical morphism, 
  \( \sigma_i : \tilde{S}_i \to S \) — restrictions of \( \sigma \) to components \( \tilde{S}_i, i \geq 0; \)

• \( \widetilde{E} \) — vector bundle on the scheme \( \tilde{S} \);

• \( \tilde{L} \in \text{Pic} \tilde{S} \) — distinguished polarization;

such that

• \( \chi(\tilde{E} \otimes \tilde{L}^m) = rp(m), \) the polynomial \( p(m) \) and the rank \( r \) of the sheaf \( \tilde{E} \) are fixed;

• on the scheme \( \tilde{S} \) the sheaf \( \tilde{E} \) is stable (resp., semistable) due to Gieseker i.e. for any proper subsheaf \( \tilde{F} \subset \tilde{E} \) for \( m \gg 0 \)

\[
\frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank } \tilde{F}} < \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank } \tilde{E}},
\]

(resp.,
\[
\frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank } \tilde{F}} \leq \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank } \tilde{E}};
\]

• on any of additional components \( \tilde{S}_i, i > 0 \), the sheaf \( \tilde{E}_i := \tilde{E}|_{\tilde{S}_i} \) is quasi-ideal, i.e. it has a description

\[
\tilde{E}_i = \sigma_i^* \ker q_0 / \text{tors}.
\]

for some \( q_0 \in \bigsqcup_{i \leq 2} \text{Quot}^1 \bigoplus \mathcal{O}_S \).

Analogously, we consider families of semistable pairs

\[
\tilde{S}_T = \left\{ \begin{array}{ll}
\pi : \tilde{\Sigma} \to T, & \tilde{L} \in \text{Pic} \tilde{\Sigma} \text{ flat over } T, \\
\forall t \in T \; \tilde{E}_t = \tilde{L}|_{\pi^{-1}(t)} \text{ ample};& \\
(\pi^{-1}(t), \tilde{L}_t) \text{ admissible scheme with distinguished polarization}; & \\
\chi(\tilde{L}_t^n) \text{ does not depend on } t; & \\
\tilde{E} - \text{locally free } \mathcal{O}_{\tilde{\Sigma}} - \text{sheaf, flat over } T; & \\
\chi(\tilde{E} \otimes \tilde{L}_t^n)|_{\pi^{-1}(t)} = rp(n); & \\
((\pi^{-1}(t), \tilde{L}_t), \tilde{E}|_{\pi^{-1}(t)}) - \text{stable (semistable) pair} \end{array} \right\}
\]

and a functor \( \tilde{f} : (\text{Schemes}_k)^{\circ} \to (\text{Sets}) \) from the category of \( k \)-schemes to the category of sets. It attaches to a scheme \( T \) the set \( \tilde{S}_T \).

Following \cite{1} ch. 2, sect. 2.2] we recall some definitions. Let \( \mathcal{C} \) be a category, \( \mathcal{C}^\circ \) its dual category, \( \mathcal{C}' = \mathcal{Funct}(\mathcal{C}^\circ, \text{Sets}) \) a category of functors to the category of sets. By Yoneda’s lemma, the functor \( \mathcal{C} \to \mathcal{C}' : F \mapsto (\mathcal{F} : X \mapsto \text{Hom}_\mathcal{C}(X, F)) \) includes \( \mathcal{C} \) as full subcategory in \( \mathcal{C}' \).

**Definition 4.** \cite{1} ch. 2, definition 2.2.1] The functor \( \tilde{f} \in \text{Ob} \mathcal{C}' \) is *corepresented by the object* \( F \in \text{Ob} \mathcal{C} \) if there exists a \( \mathcal{C}' \)-morphism \( \psi : \tilde{f} \to F \) such that any morphism \( \psi' : \tilde{f} \to F' \) factors through the unique morphism \( \omega : F \to F' \).
Definition 5. The scheme $M$ is a coarse moduli space for the functor $\mathcal{F}$ if $\mathcal{F}$ is corepresented by $M$.

In this article $\tilde{M}$ is moduli space for $\mathcal{F}^GM$, and $\tilde{M}$ is moduli space for $\mathcal{F}$.

2 Morphism of nonreduced functors

The aim of this section is construction of natural transformation $\mathcal{F}^GM \to \mathcal{F}$ given by the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{F}^GM} & \{\mathcal{F}^GM\} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mathcal{F}} & \{\mathcal{F}\}
\end{array}
\]

We are given a (possibly, nonreduced) scheme $X$; let $\Sigma := X \times S$ and $p : \Sigma \to X$ a projection onto the factor. We are also given a sheaf of $\mathcal{O}_\Sigma$-modules $E$, flat over $X$, invertible sheaf $L$ of $\mathcal{O}_\Sigma$-modules which is ample relative to $X$. We know that for any point $x \in X$ the sheaf $E|_{x \times S}$ is semistable due to Gieseker with respect to the polarization $L|_{x \times S}$. The sheaf $E|_{x \times S}$ is torsion-free $\mathcal{O}_{x \times S}$-sheaf of rank $r$ and with Hilbert polynomial $\chi(E \otimes L^n|_{x \times S}) = rp(n)$.

Since the subject of our consideration is the relation between compactifications $\tilde{M}$ and $\tilde{M}$ of the scheme $M_0$ of moduli of stable vector bundles, we restrict ourselves by such families of coherent sheaves that the scheme $X$ contains nonempty open subset $X_0$ corresponding to semistable locally free sheaves. This means that the sheaf $E_0 := E|_{X_0 \times S}$ is locally free. In particular this means that we consider points of the moduli scheme which are deformed to locally free sheaves, and components of moduli scheme consisting of such points. This circumstance is stressed out when main components of moduli schemes are spoken of. Denote $\Sigma_0 := X_0 \times S$, $L_0 := L|_{\Sigma_0}$.

Choose and fix a positive integer $m \gg 0$ such that $\mathcal{O}_X$-module $p_*E \otimes L^m$ is locally free (of rank $rp(m)$) and an evaluation morphism $p^*p_*E \otimes L^m \to E \otimes L^m$ is surjective. Besides we assume that the sheaf $L$ is trivial along $X$ in the sense that $\det p_*L^m = \mathcal{O}_X$.

Form a Grassmannian bundle $\text{Grass}(p_*E \otimes L^m, r) \to X$ of $r$-quotient spaces of vector bundle $p_*E \otimes L^m$; then $\text{Grass}(p_*E_0 \otimes L^m_0, r) \to X_0$ is its restriction to open subscheme $X_0$. The epimorphism of locally free sheaves $p^*p_*E_0 \otimes L^m_0 \to E_0 \otimes L^m_0$ defines a closed immersion $\tilde{j}_0 : \Sigma_0 \hookrightarrow \text{Grass}(p_*E_0 \otimes L^m_0, r)$ include into the commutative diagram

\[
\begin{array}{ccc}
\Sigma_0 & \xrightarrow{j_0} & \text{Grass}(p_*E_0 \otimes L^m_0, r) \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{p} & X_0
\end{array}
\]
We introduce shorthand notations \( \text{Grass}(m) := \text{Grass}(p_*, \mathcal{E} \otimes L^m, r) \) and \( \text{Grass}_0(m) := \text{Grass}(p_*, \mathcal{E}_0 \otimes L^m_0, r) \). Let \( \mathcal{O}_{\text{Grass}(m)}(1) \) be the positive generator in relative Picard group of \( \text{Grass}(m) \), then the subscheme \( j_0(\Sigma_0) \subset \text{Grass}_0(m) \) is flat over \( X_0 \) and Hilbert polynomial \( P(t) = \chi(\tilde{j}_0 \mathcal{O}_{\text{Grass}_0(m)}(t)|_{p^{-1}(x)}) \) does not depend on the choice of the point \( x \in X_0 \). Consider (relative) Hilbert schemes \( \text{Hilb}^P(t) \text{Grass}(m) \) and \( \text{Hilb}^P(t) \text{Grass}_0(m) \) of subschemes with Hilbert polynomials equal to \( P(t) \), in fibred spaces \( \text{Grass}(m) \) and \( \text{Grass}_0(m) \) respectively. Let \( \text{Univ}^P(t) \text{Grass}(m) \) and \( \text{Univ}^P(t) \text{Grass}_0(m) \) be their universal families.

There are immersions \( \mu_0 : X_0 \hookrightarrow \text{Hilb}^P(t) \text{Grass}_0(m) \) and \( \Sigma_0 \hookrightarrow \text{Univ}^P(t) \text{Grass}_0(m) \) include in the commutative diagram

\[
\begin{array}{c}
\Sigma_0 \leftarrow \text{Univ}^P(t) \text{Grass}_0(m) \\
p \\
X_0 \leftarrow \mu_0 : \text{Hilb}^P(t) \text{Grass}_0(m) \rightarrow X_0 \\
\end{array}
\]

(2.2)

Now consider the fibred square

\[
\begin{array}{c}
\text{Hilb}^P(t) \text{Grass}_0(m) \\
\downarrow \\
\text{Hilb}^P(t) \text{Grass}(m) \\
\downarrow \\
X_0 \leftarrow \mu_0 : \text{Hilb}^P(t) \text{Grass}_0(m) \rightarrow X \\
\end{array}
\]

(2.3)

induced by the open immersion \( X_0 \hookrightarrow X \). It allows to form a closure \( \overline{\mu_0(X_0)} \) of locally closed subscheme \( \mu_0(X_0) \subset \text{Hilb}^P(t) \text{Grass}(m) \).

**Remark 1.** In the diagram

\[
\begin{array}{c}
\overline{\mu_0(X_0)} \leftarrow \text{Hilb}^P(t) \text{Grass}(m) \\
u \\
\text{X} \\
\end{array}
\]

the morphism \( u \) is birational and it is isomorphism over \( X_0 \). Since \( u \) is a composite of closed immersion and projective morphism then the image of \( u \) is closed in \( X \). Although it may happen that \( u(\overline{\mu_0(X_0)}) \neq X \) since \( X \) carries nonreduced scheme structure.

The result of [11] implies that \( \overline{\mu_0(X_0)}_{\text{red}} = X_{\text{red}} \), i.e. the restriction of the morphism \( u \) to reduced schemes (reduction) \( u_{\text{red}} : \overline{\mu_0(X_0)}_{\text{red}} \rightarrow X_{\text{red}} \) is an isomorphism. Choose such a scheme structure \( \widetilde{X} \) on a subset \( \overline{\mu_0(X_0)} \) that in
\[ \tilde{\Sigma} := \tilde{X} \times_{\text{Hilb}^P(t)\text{Grass}(m)} \text{Univ}^P(t)\text{Grass}(m). \]

Let \( \tilde{\mu} : \tilde{\Sigma} \hookrightarrow \text{Univ}^P(t)\text{Grass}(m) \) be the corresponding closed immersion. By the construction the projection \( \pi : \tilde{\Sigma} \to \tilde{X} \) is flat morphism and the scheme \( \tilde{\Sigma} \) is supplied with a locally free sheaf \( \tilde{\mu}^*(\mathcal{O}_{\text{Hilb}^P(t)\text{Grass}(m)} \boxtimes X^m|_{\text{Univ}^P(t)\text{Grass}(m)}) \), where \( S^m \) is a universal quotient bundle on \( \text{Grass}(m) \).

To compute an invertible sheaf \( \tilde{L} \) on the scheme \( \tilde{\Sigma} \) note that all constructions can be repeat with the sheaf \( E \otimes L^m \). The inclusion \( E \otimes \mathbb{L}^m \hookrightarrow E \otimes \mathbb{L}^{m+1} \) induces a closed immersion \( \text{Grass}(m) \to \text{Grass}(m+1) \) of \( X \)-schemes. Let \( P'(t) = \chi(j_0^*\mathcal{O}_{\text{Grass}(m+1)}(t)|_{p^{-1}(x)}) \), then there is a closed immersion of \( X \)-schemes

\[ \text{Hilb}^P(t)\text{Grass}(m) \hookrightarrow \text{Hilb}^{P'}(t)\text{Grass}(m+1). \]

Repeat the construction with immersion \( X_0 \hookrightarrow \text{Hilb}^{P'}(t)\text{Grass}_0(m+1) \), closure of the image of this immersion in \( \text{Hilb}^{P'}(t)\text{Grass}(m+1) \), and with choice of such a scheme structure \( \tilde{X}' \) on it that \( \tilde{X}' \cong X \).

The immersion \( \text{Hilb}^P(t)\text{Grass}(m) \hookrightarrow \text{Hilb}^{P'}(t)\text{Grass}(m+1) \) provides the immersion of the universal subscheme \( \text{Univ}^P(t)\text{Grass}(m) \hookrightarrow \text{Univ}^{P'}(t)\text{Grass}(m+1) \) and a subscheme \( \tilde{\Sigma}' := \tilde{X}' \times_{\text{Hilb}^{P'}(t)\text{Grass}(m+1)} \text{Univ}^{P'}(t)\text{Grass}(m+1) \) includes into the fibred diagram

\[ \text{Univ}^P(t)\text{Grass}(m) \hookrightarrow \text{Univ}^{P'}(t)\text{Grass}(m+1) \tag{2.5} \]

\[ \text{Hilb}^P(t)\text{Grass}(m) \hookrightarrow \text{Hilb}^{P'}(t)\text{Grass}_0(m+1) \]

\[ \tilde{X}' \]

It implies that since \( \tilde{\Sigma}' \) is a scheme-theoretic image of \( \tilde{\Sigma} \), that the morphism \( \tilde{\Sigma}' \to \tilde{\Sigma} \) is an isomorphism.
Now set
\[ \tilde{L} := \det \bar{\mu}^* (\mathcal{O}_{\text{Hilb} P(t) \text{Grass}(m+1)} \boxtimes X S^{(m+1)}|_{\text{Univ} P(t) \text{Grass}(m+1)}) \]
\[ \otimes \det \bar{\mu}^* (\mathcal{O}_{\text{Hilb} P(t) \text{Grass}(m)} \boxtimes X S^{(m)}|_{\text{Univ} P(t) \text{Grass}(m)})^\vee. \]

It is clear that this is an invertible $\mathcal{O}_X$-sheaf which is flat over $X$. It induces distinguished polarizations on fibres of the scheme $\hat{X}$ over closed points $x \in X$. Below we examine this sheaf and convince that it is ample relative to $X$.

Also set
\[ \tilde{E} := \bar{\mu}^* (\mathcal{O}_{\text{Hilb} P(t) \text{Grass}(m)} \boxtimes X S^{(m)}|_{\text{Univ} P(t) \text{Grass}(m)}) \otimes \tilde{L}^{-m}. \]

This is a locally free $\mathcal{O}_X$-sheaf which is flat over $X$. It is clear that $\tilde{E} \otimes \tilde{L}^{-m} = \bar{\mu}^* (\mathcal{O}_{\text{Hilb} P(t) \text{Grass}(m)} \boxtimes X S^{(m)}|_{\text{Univ} P(t) \text{Grass}(m)})$, and

\[ \chi(\tilde{E} \otimes \tilde{L}^{-m}|_{\pi^{-1}(x)}) = \chi(\bar{\mu}^* (\mathcal{O}_{\text{Hilb} P(t) \text{Grass}(m)} \boxtimes X S^{(m)}|_{\text{Univ} P(t) \text{Grass}(m)})|_{\pi^{-1}(x)}) = rp(m) \]

for the value of $m$ taken in the construction. For an arbitrary $n$ one has

\[ \chi(\tilde{E} \otimes \tilde{L}^{n}|_{\pi^{-1}(x)}) = \chi(\tilde{E}_{\text{red}} \otimes \tilde{L}^{n}_{\text{red}}|_{\pi^{-1}(x)}) = rp(n), \]

since sets of closed points of schemes $X$ and $X_{\text{red}}$ coincide.

**Proposition 1.** Let $X$ be a complete scheme, $L$ be a linear bundle on $X$. $L$ is ample on $X$ if and only if $L_{\text{red}} = L|_{X_{\text{red}}}$ is ample on $X_{\text{red}}$.

We need a relative version of this proposition.

**Proposition 2.** Let $f : Y \to X$ be a proper morphism of Noetherian schemes, $L$ be a linear bundle on $Y$. $L$ is ample relative to $X$ if and only if $L_{\text{red}} := L|_{f^{-1}(X_{\text{red}})}$ is ample relative to $X_{\text{red}}$.

**Proof.** We prove part "if"; the part "only if" is obvious. The proof is relative version of the proof presented in [3, 1.2.12(i)]. Consider a closed subscheme $f^{-1}(X_{\text{red}}) \subset Y$ and its sheaf of ideals $N \subset \mathcal{O}_Y$. It is clear that it is nilpotent, i.e. there exists a positive integer $s$ such that $N^s = 0$. Let $\mathcal{F}$ be a coherent sheaf on $Y$; consider its filtration

\[ \mathcal{F} \supset \mathcal{F} N \supset \cdots \supset \mathcal{F} N^s = 0. \]

The quotients of the filtration $\mathcal{F} N^i/\mathcal{F} N^{i+1}$ are coherent $\mathcal{O}_{f^{-1}(X_{\text{red}})}$-modules. Since $L_{\text{red}}$ is ample relative to $X_{\text{red}}$, then for $m \gg 0$ we have

\[ R^j f_* (\mathcal{F} N^i/\mathcal{F} N^{i+1} \otimes L^m) = R^j f_* (f^{-1}(X_{\text{red}}), (\mathcal{F} N^i/\mathcal{F} N^{i+1}) \otimes L^m_{\text{red}}) = 0 \]

for $j > 0$ ([3, §2, Proposition 2.6.1]). Consider a series of exact sequences

\[ 0 \to \mathcal{F} N^i+1 \to \mathcal{F} N^i \to \mathcal{F} N^i/\mathcal{F} N^{i+1} \to 0, \quad i = 0, \ldots, s - 1. \]
The adjoint sequences of direct images together with an equality
\[ R^i f_*(Y, \mathcal{F} \mathcal{N}^{i-1} \otimes L^m) = R^i f_*(Y, (\mathcal{F} \mathcal{N}^{i-1}/\mathcal{F} \mathcal{N}^i) \otimes L^m) = 0 \]
allows us to organize descending induction on \( i \). This yields \( R^i f_*(Y, \mathcal{F} \mathcal{N}^i \otimes L^m) = 0 \), in particular, for \( i = 0 \) \( R^i f_*(Y, \mathcal{F} \otimes L^m) = 0 \). Applying [4, \S2, Proposition 2.6.1] we conclude the amplitude of \( L \) relative to \( X \).

Applying the proposition to the morphism \( \pi : \tilde{\Sigma} \to X \) and to the sheaf \( \tilde{L} \) we get that \( \tilde{L} \) is ample relative to \( X \).

### 3 Morphism of moduli

In this section we construct the morphism \( \kappa : \overline{M} \to \tilde{M} \) of moduli schemes which is induced by the natural transformation of functors \( f^{GM} \to f \), done in \S2.

According to the classical construction of Gieseker – Maruyama moduli scheme \( \overline{M} \) choose an integer \( m > 0 \) such that for each semistable coherent sheaf \( E \) the morphism \( H^0(S, E \otimes L^m) \otimes L^{-m} \to E \) is surjective, \( H^i(S, E \otimes L^m) \) is a \( k \)-vector space of dimension \( rp(m) \) and \( H^i(S, E \otimes L^m) = 0 \) for all \( i > 0 \).

Then take a \( k \)-vector space \( V \) of dimension \( rp(m) \) and form a Grothendieck scheme \( \text{Quot}^{rp(t)} V \otimes L^{-m} \) of coherent \( \mathcal{O}_S \)-quotient sheaves of the form \( V \otimes L^{-m} \to E \). Consider its subscheme \( Q \subset \text{Quot}^{rp(t)} V \otimes L^{-m} \), corresponding to those \( E \) which are Gieseker-semistable and torsion-free. Let \( \mathcal{O}_{\text{Quot}} \) be the structure sheaf of the scheme \( \text{Quot}^{rp(t)} V \otimes L^{-m} \) and \( V \otimes L^{-m} \boxtimes \mathcal{O}_{\text{Quot}} \to \mathcal{E}_{\text{Quot}} \) be the universal quotient sheaf. Then we denote as \( \mathcal{E}_Q \) the sheaf \( \mathcal{E}_{\text{Quot}} |_{Q \times S} \), and as \( L_Q \) the sheaf \( \mathcal{O}_{\text{Quot}} |_Q \boxtimes L \). Consider the collection of data \( (Q, \Sigma = Q \times S, L_Q, \mathcal{E}_Q) \). Set \( X = Q \) and apply the transformation which is described in \S2. This leads to the data \( (Q, \tilde{\Sigma}_Q, \tilde{L}_Q, \tilde{\mathcal{E}}_Q) \) (the transformation does not change the base \( Q \)).

Following [2] we also consider the Grassmannian variety \( G(V, r) \) of \( r \)-quotient spaces of the vector space \( V \), and the Hilbert scheme \( \text{Hilb}^{P(t)} G(V, r) \) (cf. \S2). Let \( H_0 \subset \text{Hilb}^{P(t)} G(V, r) \) be the quasiprojective scheme whose points correspond to semistable admissible pairs. Then by the universal property of Hilbert scheme the data \( (Q, \Sigma_Q, L_Q, \mathcal{E}_Q) \) yields to the morphism \( H_0 \to \text{Hilb}^{P(t)} G(V, r) \). Since the data \( (Q, \Sigma_Q, L_Q, \mathcal{E}_Q) \) provides the family of semistable admissible pairs (with possibly nonreduced base scheme \( Q \)), then there is a commutative triangle of morphisms

\[
\begin{array}{ccc}
Q & \xrightarrow{\kappa_Q} & \text{Hilb}^{P(t)} G(V, r) \\
\downarrow & & \downarrow \\
H_0 & \xrightarrow{\beta} & \\
\end{array}
\]

Moduli schemes \( \overline{M} \) and \( \tilde{M} \) are built up as (good) \( \text{PGL}(V) \)-quotients of schemes \( Q \) and \( H_0 \) respectively, i.e. \( \overline{M} = Q/\text{PGL}(V) \) and \( \tilde{M} = H_0/\text{PGL}(V) \). Let \( \beta : \)
$PGL(V) \times Q \to Q$ and $\alpha : PGL(V) \times H_0 \to H_0$ be morphisms of actions of group $PGL(V)$ upon subschemes $Q \subset \text{Quot}^{rP(t)}V \times \mathcal{L}^{-m}$ and $H_0 \subset \text{Hilb}^{P(t)}G(V,r)$ respectively. Both actions are induced by linear transformations of the vector space $V$. Then the following diagram commutes

$$
\begin{array}{ccc}
PGL(V) \times Q & \xrightarrow{\beta} & Q \\
\downarrow{\text{id} \times \kappa_Q} & & \downarrow{\kappa_Q} \\
PGL(V) \times H_0 & \xrightarrow{\alpha} & H_0
\end{array}
$$

This leads to required morphism of $PGL(V)$-quotients $\kappa : \overline{M} \to \widetilde{M}$.

Remark 2. According to the result of [11], the morphism of reduced schemes $\kappa_{\text{red}} : \overline{M}_{\text{red}} \to \widetilde{M}_{\text{red}}$ which corresponds to the morphism $\kappa$, is an isomorphism.

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