An Optimal Transmission Policy for Energy Harvesting Systems with Continuous Energy and Data Arrivals

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Abstract—Energy harvesting has been developed as an effective technology for communication systems in order to extend the lifetime of these systems. In this work, we consider a single-user energy harvesting wireless communication system, in which arrival data and harvested energy curves are modeled as continuous functions. For the single-user model, our first goal is to find an offline algorithm, which maximizes the amount of data which is transmitted to the receiver node by a given deadline. If more than one scheme exists that transmits the maximum data, we choose the one with minimum utilized energy at the transmitter node. Next, we propose an online algorithm for this system. We also consider a multi-hop energy harvesting wireless communication system in a full-duplex mode and find the optimal offline algorithm to maximize the throughput.

Index Terms—Energy harvesting, continuous arrivals, Throughput maximization, Optimal scheduling, Online optimization.

I. INTRODUCTION

Energy Harvesting (EH) has been appeared as an approach in order to make the green communications possible. In EH systems, nodes extract energy from the nature to extend their lifetimes. The harvested energy can also be used for the purpose of communication and specially for the transmission process. Compared to the conventional battery-powered systems, the EH systems have access to an unbounded source of energy (like vibration absorption devices, water mills, wind turbines, microbial fuel cells, solar cells, thermo-electric generators, piezoelectric cells, etc). However, the diffused nature of this energy makes it difficult to be used for communication.

From information-theoretic point of view, the capacity of channels with EH nodes has been investigated. The basic results on the capacity of EH systems have been presented in [1], which are continued in other works such as [2]–[4]. Another important research field in this area has focused on optimum schemes to maximize the throughput in a given deadline [5]–[7]. Most of the existing works have considered the EH nodes with discrete energy and/or data arrivals. This problem for a single-user fading channel with additive Gaussian noise is considered in [5]. The authors in [6] consider a single-user communication with battery imperfections while the harvested energy curve is continuous. Throughput is maximized in [7] while the battery is assumed to be limited. Moreover, two throughput maximization problems for single-user and multiple access channels are investigated in [8] with EH transmitters and receiver while the receiver utilizes the harvested energy for decoding process.

Also some researches try to minimize the completion time to transmit a given amount of data [9]–[11]. In [9], an algorithm is proposed to minimize the transmission period for a specific amount of given data in two scenarios. A broadcast channel is considered in [10], where the goal is to investigate the minimum of the transmission completion time. A single-user communication system is considered in [11], in which the receiver (Rx) is not supplied by an external source and its energy is provided by harvesting, resulting in a limited energy at the Rx. One of the very interesting problems in the EH systems is to consider the scenarios where some relays participate in the transmission process. A two-hop EH system is considered in [12]–[15]. The authors in [12] investigate the two-hop relay channel with EH transmitter (Tx) and relay, and one-way energy transfer from Tx to the relay. Throughput maximization and transmission completion time minimization problems in a half-duplex two-hop relay channel with EH only at the source have been considered in [13]. In [14], it has been assumed that the relay and the source harvest energy from the environment and the problem of throughput maximization in a given deadline in both full-duplex and half-duplex cases is investigated. [15] only considers half-duplex mode in a throughput maximization problem by a given deadline. In [16], the authors consider a throughput maximization problem for a diamond channel with one-way energy transfer from Tx node to the relays node. Moreover, there are some researches on Gaussian relay channel with direct link with EH at both of Tx and the relay, such as [17] and [18]. An energy minimization problem in order to transmit a given amount of data subject to quality-of-service in a given deadline is investigated in [19].

EH systems with online algorithms are less investigated, two examples are [20] and [21]. [20] considers the design of an online algorithm to maximize the throughput of a wireless communication channel with arbitrary fading coefficients. In [21], the author finds a lower bound and an upper bound for the completion time of optimal online algorithm to the completion time of optimal offline algorithm, in order to transfer a given data in single-user and multiple-access channels with Gaussian additive noise and EH nodes.

In general in a throughput maximization problem at a multi-hop relay channel, the arrival data curve at the relay node may...
be continuous even when the data arrival is discrete at Tx. In Section V, we provide an example to show that a throughput maximization problem in a two-hop channel with continuous energy arrivals in Tx and the relay and buffered data in Tx reduces to a throughput maximization problem in a single-user channel with continuous data arrival and harvested energy in the relay. Therefore, investigating a system with continuous data arrival curve is crucial in analyzing EH systems. In addition, there are very limited results for the EH systems with continuous energy arrivals [6]. The authors in [20] investigate an EH system with a degrading battery of finite capacity by convex analysis tools for a continuous harvested energy function. Although, in [6] and [20], the harvested energy curve is continuous but all data is stored in information buffer at the beginning of the transmission and the arrival data curve is not a continuous function.

By considering the continuous energy and data arrivals, the problem enters a new space where the existing discrete-space proofs are not applicable. Noting this fact, the central question is how the existing discrete-space results change in this new continuous-space. We answer this question in this paper by providing the proofs which fit the continuous-space.

In this paper, we consider a single-user communication channel with an EH Tx with random data arrivals. We model harvested energy curve and arrival data curve with continuous functions in time and assume that the size of the energy and data buffers at Tx and Rx are infinite. We focus on a throughput maximization problem. However, it is possible that there exist more than one scheme which maximize the throughput when the harvested energy is more than the needed energy to transmit all of arrival data till the deadline (which may happen depending the harvested energy curve). Hence, we need an extra condition for our model. We need a constraint on the utilized energy when this situation occurs. For this setup, we propose an offline algorithm to obtain the optimal policy which maximizes the amount of data transmitted to Rx by a given deadline. If more than one scheme exist that transmit the maximum data, we choose the one with minimum utilized energy at the Tx.

However, in practice, we may have no information about the future of the harvested energy and data arrival in Tx. Thus, we need an algorithm which without knowing the future amounts of harvested energy and data arrival determines the power in Tx. We propose an online algorithm in Section VI. We prove that the online algorithm uses all of the energy or send all of the data in the data buffer and the transmitted power curve is a nondecreasing function similar to optimal offline algorithm. Then, we derive a lower bound on the ratio of the amount of transmitted data in the online algorithm to the optimal offline algorithm.

Our model motivation mainly comes from its possible extension to a multi-hop relay channel in which the transmitted data from the source to the relay node is continuous in time. The most challenging parts in this paper are to apply data and energy causalities, which need totally different approaches from the discrete model in [9] and the continuous model in [20], in which only harvested energy is a continuous function. Another difference between our work and [9] is that: [9] founds optimal policy between piecewise linear functions for the discrete harvested energy and arrival data curves; however, considering our continuous energy and data arrival curves, we search among the set of general functions (detailed in Section II). Our method for finding the algorithm considers both continuous and discrete arrival data functions as well as both continuous and discrete harvested energy functions. Moreover, there is a basic difference: [9] investigates a completion time minimization problem while we investigate a throughput maximization problem.

The rest of the paper is organized as follow. In Section II, we formulate the main problem by an optimization problem. In Section III, we state the properties of the optimal energy and data transmitted curves for the optimal offline algorithm. In Section IV, we first prove a theorem for a simpler special case of the main problem; then, we propose the offline algorithm which gives us the optimal transmitted data curve. In Section V, we investigate a multi-hop channel by a throughput maximization problem. In Section VI, we propose an online algorithm for optimization problem of Section II, and in Section VII, we provide the simulation results. Finally, Section VIII concludes the paper.

II. SYSTEM MODEL

We consider a single-user wireless communication system, where the Tx is a node that harvests energy from an external source in a continuous fashion. We assume that at time \( t \), we have \( B_s(t) \) bits of data (where \( B_s(t) \) is a continuous function) available at the Tx. The Rx is assumed to have enough energy to provide adequate power for decoding at any rate that can be achieved by the Tx. Also, we have the following assumptions:

The harvested energy curve \( E_s(t) \) and arrival data curve \( B_s(t) \) are bounded differentiable functions of time \( t \), for \( t \in [0, \infty) \) (except probably in finite points because of discontinuity in these points or not equal right and left derivatives) which denote the amount of energy harvested and the amount of data arrived at the Tx in the interval \( [0, t] \) respectively. Also we assume that derivative of \( B_s(t) \) and \( E_s(t) \) are piecewise continuous. The transmitted energy curve \( E(t) \) and the transmitted data curve \( B(t) \) are continuous functions. We assume that \( E(t) \) and \( B(t) \) are differentiable functions of time, \( t \), for \( t \in [0, \infty) \) (except probably in finite points) and these denote the amount of energy utilized and the amount of data that transmitted at the Tx in the interval \( [0, t] \) for \( t \in [0, \infty) \). The transmitted power curve \( p(t) \) is a piecewise continuous function, that denotes the amount of power used at the Tx for \( t \in [0, \infty) \). Also, we use \( B^*(t) \), \( E^*(t) \) and \( p^*(t) \) as the optimal transmitted data, energy and power curves respectively.

We assume that the instantaneous transmission rate relates to the power of transmission through a continuous function \( r(p) \), which satisfies the following properties: i) \( r(0) = 0 \), ii) \( r(p) \) is a non-negative strictly concave function in \( p \), iii) \( r(p) \) is differentiable, iv) \( r(p) \) increases monotonically in \( p \), and v) \( \lim_{p \to \infty} r(p) = \infty \).

It can easily be seen that the above conditions are satisfied in many systems with practical encoding/decoding schemes, such as single-user additive white Gaussian noise channel with optimal random coding, i.e., \( r(p) = \frac{B}{2} \log(1 + p) \).

According to the above assumptions, we have:

\[
B(t) = \int_0^t r(E'(t'))dt',
\]

where \( E'(t) \) is the derivative of function \( E(t) \) at time \( t \), and it is clear that \( p(t) = E'(t) \).

In Sections III and IV, our goal is to find an offline algorithm, which maximizes the amount of data transmitted to the Rx in a given deadline. If more than one scheme
exist which maximize the transmitted data, we impose another constraint: the algorithm must give the scheme, in which the amount of data is maximized, while the utilized energy is minimized at the Tx. Therefore, the optimization problem is:

\[
D(T) = \max_{p(t)} \int_0^T r(p(t)) dt \tag{2}
\]

\[
s.t. \int_0^t p'(t) dt' \leq E_s(t), \quad 0 \leq t \leq T \tag{3}
\]

\[
\int_0^t r(p(t)) dt' \leq B_s(t), \quad 0 \leq t \leq T, \tag{4}
\]

and if there exist more than one \(p(t)\) such that \(D(T) = B_s(T)\), then the one is selected which uses the minimum energy. Properties \(3\) and \(4\) hold due to the causality of energy and data, respectively. The causality of energy means that one cannot use the energy which is not harvested and the causality of data means that the data which has not arrived yet, cannot be sent. We remark that we investigate both continuous and discrete harvested energy and arrival data curves in our model.

### III. Properties of the Optimal Policy

In this section, we state the properties of the optimal policy by some useful lemmas.

**Lemma III.1.** (Jensen’s inequality). Let \(f(x)\) be a function defined in \(a \leq x \leq b\) such that \(\alpha \leq f(x) \leq \beta\). Let \(\phi(u)\) be a concave function defined on the interval \(\alpha \leq u \leq \beta\); then

\[
\phi \left( \frac{\int_a^b f(t) dt}{b-a} \right) \geq \frac{\int_a^b \phi(f(t)) dt}{b-a}. \tag{5}
\]

with strict inequality if \(\phi(.)\) is strictly concave, \(a \neq b\) and \(f(.)\) is not constant over the interval \([a, b]\).

Now, substitute \(f(.) = p(.)\) and \(\phi(.) = r(.)\). Assuming that \(p(.)\) is not a constant function in \([a, b]\) and \(a \neq b\), because of \(r(.)\) is strictly concave we obtain:

\[
r \left( \frac{E(b) - E(a)}{b-a} \right) (b-a) > \int_a^b r(p(t)) dt. \tag{6}
\]

Hence, it can be concluded that if \(E(t)\) is nonlinear in \(t\) over \([a, b]\), the straight line which passes through the two points \((a, E(a))\) and \((b, E(b))\), transmits more data than \(E(t)\). In other words, if we can find two points on the \(E(t)\), such that the line passing through these points does not violate the energy and data causality conditions, then \(E(t)\) is not the optimal transmitted energy curve. It can be similarly shown that if there exist two points on \(B(t)\), such that the straight line passing through these points does not violate the data and energy causality conditions, then \(B(t)\) is not the optimal transmitted data curve. These intuitions are stated in the following theorem.

**Theorem III.2.** Let \(E(t)\) be a feasible transmitted energy curve, \(m(t)\) be a straight line fragment over \([a, b]\) that passes through points \((a, E(a))\) and \((b, E(b))\), then \(E(t)\) is not the optimal transmitted energy curve if there exist a curve \(E_{new}(t) \neq E(t)\) which satisfies the causality conditions \(3\) and \(4\). Also

\[
E_{new}(t) = \begin{cases} E(t) & 0 \leq t < a \\ m(t) & a \leq t \leq b \\ E(t) & b < t \leq T \end{cases} \tag{7}
\]

*Proof:* We have to show that \(E_{new}(t)\) transmits more data than \(E(t)\) while it uses the same energy. First, it is clear that \(E_{new}(t)\) and \(E(t)\) use equal energy to send equal amount of data in \((0, a)\) and \((b, T)\). \(7\) implies that \(E(t)\) transmits smaller amount of data than \(E_{new}(t)\), using the same energy in \((a, b)\). Thus, overall \(E_{new}(t)\) transmits more amount of data and \(E(t)\) is not optimal.

Similarly, it can be shown that Theorem III.2 is valid if we replace energy transmitted curve with data transmitted curve. Using the above theorem we conclude that in the optimal policy, there are no two points on \(E(t)\) such that the line passing through these points satisfies the causality conditions and is not equal to \(E(t)\). The same results holds for \(B(t)\).

**Lemma III.3.** \(p^*(t)\) is not decreasing.

*Proof:* We prove this lemma by contradiction. We propose a transmitted energy curve \((E(t))\) such that transmitted data curve \((B(t))\) is equal to \(B^*(t)\) at \(t = T\), while \(E(t)\) uses smaller energy than \(E(t)\). Assume that \(p^*(t)\) is decreasing. Then there exists an interval such that \(p^*(t)\) is strictly decreasing in it. Thus, we assume that \(p^*(t)\) is strictly decreasing in \([t_c, t_c + \delta]\), then \(E(t)\) is strictly concave in this interval. \(\delta\) implies that the straight line which passes through \(t_c\) and \(t_c + \delta\) transmits more amount of data than \(E(t)\) in this interval. Now, we decrease the slope of this line to \(p_0\) such that this line transmits equal data with \(E^*(t)\) in \([t_c, t_c + \delta]\). We assume that \(E(t)\) is a curve such that: (i) it is equal to \(E^*(t)\) except at \([t_c, T]\); (ii) it is a line with slope of \(p_0\), that proposed above, at \([t_c, t_c + \delta]\); (iii) it is equal to \(E^*(t) - \varepsilon\) where \(\varepsilon = E^*(t_c + \delta) - E(t_c + \delta)\) at \(t_c + \delta, T\). It is obvious that \(E(t)\) satisfies the energy causality condition, because \(E(t) \leq E^*(t)\). There exists \(t_0 \in (t_c, t_c + \delta)\) such that \(p^*(t) > p_0\) for \(t_c < t < t_0\), and \(p^*(t) < p_0\) in \(t_0 < t < t_c + \delta\). Otherwise, \(p^*(t) > p_0\) or \(p^*(t) < p_0\), which both of them are contradictions: because they result in \(\int_{t_c}^{t_c + \delta} r(p^*(t)) dt > \delta r(p_0)\) or \(\int_{t_c}^{t_c + \delta} r(p^*(t)) dt > \delta r(p_0)\), while

\[
\int_{t_c}^{t_c + \delta} r(p^*(t)) dt > \delta r(p_0) \tag{8}
\]

must hold.

It is enough to prove that the transmitted data curve that obtains from \(E(t)\), i.e., \(B(t)\), satisfies the data causality condition. In \([0, t_c]\), since \(E^*(t) = E(t)\), then \(B^*(t) = B(t)\). We prove if \(B(t) \leq B^*(t)\) in \([t_c, t_c + \delta]\) then \(B(t) \leq B^*(t)\) in \([t_c + \delta, T]\) which means that \(B(t)\) satisfies causality of data, hence we will show that \(B(t) \leq B^*(t)\) in \([t_c + \delta, T]\). Since \(p^*(x) > p_0\) and \(r(.)\) increases monotonically, we have \(\int_{t_c}^{t_c + \delta} r(p^*(t)) dt > \int_{t_c}^{t_c + \delta} (x - t_c) r(p_0)\) for \(t_c < x < t_0\). Now, we use contradiction to prove that \(B(t) \leq B^*(t)\) in \([t_c, t_c + \delta]\). To do this we assume that, there exists a point \(a\), such that \(t_0 \leq a \leq t_c + \delta\) and \(B(a) > B^*(a)\). Hence,

\[
\int_a^{t_c} r(p^*(t)) dt < (a - t_c) r(p_0) \tag{9}
\]

which using \(8\) we obtain:

\[
\int_{t_c}^{t_c + \delta} r(p^*(t)) dt - \int_{t_c}^a r(p^*(t)) dt > \delta r(p_0) - (a - t_c) r(p_0)
\]

\[
\Rightarrow \int_{t_c}^{t_c + \delta} r(p^*(t)) dt > (t_c + \delta - a) r(p_0) \tag{10}
\]
However, for every \( x \in [a, t_c + \delta] \), we have \( p^*(x) \leq p_0 \) and since \( r(.) \) increases monotonically, we obtain
\[
\int_a^{t_c+\delta} r(p^*(\tau))d\tau \leq (t_c+\delta-a)r(p_0).
\]
(11)
which is inconsistent with (10). Thus, \( E(T) < E^*(T) \), but \( B(T) = B^*(T) \). Now, if \( B(T) = B_s(T) \), then we have contradiction due to the assumption that \( p^*(t) \) is optimal and proof is completed. Otherwise, if \( B(T) < B_s(T) \), we can increase \( p(t) \) slightly in \( \tilde{T} \cup \{1\} \), which implies that \( p^*(t) \) is not optimal. This completes the proof.

**Lemma III.4.** Under the optimal policy, if there exists an epoch that no energy and no data are received i.e., if \( B_s(t) \) and \( E_s(t) \) are constant, then \( p^*(t) \) is constant in this epoch.

**Proof:** Similar to the above lemma, we prove this lemma by proposing a suitable transmitted energy curve \( E(t) \). We assume that \( p^*(t) \) is not constant while \( E_s(t) \) and \( B_s(t) \) are constant in \( [a, b] \). Similar to the proof of the above lemma, there exists a point \( c \) such that, we can replace \( E^*(t) \) in \( (a, c) \) with a straight line, where the new curves satisfy both data and energy causality conditions, \( E(T) < E^*(T) \) and \( B(T) = B^*(T) \). Hence, \( E^*(t) \) is not optimal.

**Lemma III.5.** Under the optimal policy, whenever \( p^*(t) \) changes (at instant \( t_0 \)), at least one of the followings holds: (i) \( E^*(t_0) = E_s(t_0) \), (ii) \( B^*(t_0) = B_s(t_0) \).

**Proof:** The proof is similar to the proof of Lemma III.4 by proposing an appropriate transmitted energy curve \( E(t) \).

**Corollary 1.** Based on Lemmas III.4 and III.5, the optimal transmitted data/energy curve must be linear except probably in the epochs in which \( E^*(t) \) equals \( E_s(t) \) or \( B^*(t) \) equals \( B_s(t) \). Also based on Lemma III.3, since \( p^*(t) \) is an increasing function, \( E^*(t) \) and \( B^*(t) \) are convex functions.

**Remark 1.** Though similar results to the ones in Lemmas III.3 III.4 and III.5 have been proposed in [9] for discrete energy and data arrivals, we show these results for the continuous case using different proof techniques from the proofs in [9], noting that the proofs of [9] cannot easily be extended to the continuous model.

**Lemma III.6.** Assume that \( f(t) \) and \( g(t) \neq f(t) \) are continuous piecewise differentiable functions in \([a, b]\) such that
\[
g(t) \leq f(t) \text{ in } (a, b) \text{ and } f(a) = g(a) \text{ and } f(b) = g(b).
\]
Then \( \text{length}_{[a,b]}(f(t)) < \text{length}_{[a,b]}(g(t)) \), where \( \text{length}_{[a,b]}(f(t)) \) means length curve \( f(t) \) in interval \([a, b]\).

**Proof:** We choose arbitrarily points \( t_1, t_2, \ldots, t_{n-1} \) in interval \([a, b]\) and we draw the lines that connect any two adjacent points on \( f(t) \) and we denote these line segments by \( l_1, l_2, \ldots, l_n \). Then, we draw the perpendicular lines on \( l_i \) at both ends to collide \( g(t) \) and draw line segments \( k_1, k_2, \ldots, k_n \) on \( g(t) \) as illustrated in Fig. 1. Also assume that curves corresponding to \( l_i \) and \( k_i \) is \( l_i' \) and \( k_i' \), respectively on \( f(t) \) and \( g(t) \) in Fig. 1. According to Fig. 1, \( \forall i: l_i \leq k_i \leq k_i' \), so, we can write,
\[
\sum_{i=1}^{n} l_i \leq \sum_{i=1}^{n} k_i \leq \sum_{i=1}^{n} k_i' \leq \text{length}_{[a,b]}(g(t)).
\]
(12)
Since for all partitions \( t_1, t_2, \ldots, t_{n-1} \) in interval \([a, b]\), in-
\[ \hat{t}_1 + \hat{t}_2 < k_1 + k_2 + k_3 \Rightarrow \text{length}_{[a,b]}(f(t)) < \text{length}_{[a,b]}(g(t)). \]  
\[ (17) \]

Lemma III.7. If in an interval we have \( p'(t) \neq 0 \), then \( B(t) \leq B^*(t) \).

**Proof:** To prove this lemma we use contradiction. Assume that \( t_0 \) is a point such that \( p'(t_0) \neq 0 \) and there exists a transmitted data curve \( B(t) \) that \( B^*(t) < B(t_0) \) which results in \( B^*(t_0) < B_s(t_0) \). Since \( p'(t_0) \neq 0 \), from Lemma III.5 we have \( E^*(t_0) = E_s(t_0) \) which means \( E(t_0) \leq E^*(t_0) \). Consider the transmitted power curve \( p_1(t) \) as below,

\[ p_1(t) = \begin{cases} p(t) & 0 \leq t < t_0 \\ p'(t) & t_0 \leq t < t_0 + \epsilon \end{cases}. \]  
\[ (18) \]

Now we want to show that \( p_1(t) \) is more efficient than \( p^*(t) \), thus we want to find an \( \epsilon \) such that \( 0 < \epsilon \leq T - t_0 \) in \( (18) \) such that,

\[ \int_{t_0}^{t_0+\epsilon} r(p^*(t))dt = B(t_0) - B^*(t_0). \]  
\[ (19) \]

If there exists an \( \epsilon \) that satisfies \( (19) \), then \( p_1(t) \) transmits \( B_1(T) = B^*(T) \) data using \( E_1(T) < E^*(T) \) energy and satisfies causality conditions: this means that we have a contradiction, and if there does not exist any \( \epsilon \) in interval \( (0,T - t_0) \), we assume that \( p_1(t) \) is as follows,

\[ p_1(t) = \begin{cases} p(t) & 0 \leq t < t_0 \\ 0 & t_0 \leq t \leq T \end{cases}. \]  
\[ (20) \]

From above, which results in \( B^*(T) < B_1(t_0) = B_1(T) \): this means we have a contradiction, too.

**Conjecture 1.** If in the optimal policy \( E_s(T) \) is totally used, i.e., \( E_1(T) = E^*(T) \), then, curve \( E^*(t) \) has minimum length among the feasible transmitted energy curves that use \( E_s(T) \) totally, that is, \( E^*(t) \) minimizes the metric,

\[ \text{length}_{[0,T]}(E(t)) = \int_0^T \sqrt{1 + (E'(t))^2} dt. \]  
\[ (21) \]

**Theorem III.8.** If in the optimal policy \( B_s(T) \) is totally transmitted i.e., \( B_s(T) = B^*(T) \), then curve \( B^*(t) \) has minimum length among the feasible transmitted data curves that transmit all of \( B_s(T) \), that is, \( B^*(t) \) minimizes the metric,

\[ \text{length}_{[0,T]}(B(t)) = \int_0^T \frac{1}{1 + (B'(t))^2} dt \]  
\[ (22) \]

among feasible curves which connects origin to \( (T, B_s(T)) \).

**Proof:** Assume that \( B(t) \) is a feasible curve such that \( B(T) = B_s(T) \) and \( B(t) \neq B^*(t) \). Based on Lemma III.7 whenever in an interval \( B(t) > B^*(t) \), then \( B^*(t) \) must be linear in this interval, we can divide the interval \( [0,T] \) as follow:

1- Intervals \([c_i, d_i], 1 \leq i \leq n\), in which we have \( B(t) \leq B^*(t), B(c_i) = B^*(c_i) \) and \( B(d_i) = B^*(d_i) \), \( \forall i \).
2- Intervals \([e_i, f_i], 1 \leq i \leq m\), in which we have \( B^*(t) < B(t) \) in \((e_i, f_i), B(e_i) = B^*(e_i) \) and \( B(f_i) = B^*(f_i) \), \( \forall i \).

In the intervals of part 1 based on Lemma III.6, \( \text{length}_{[c_i, d_i]} B^*(t) \leq \text{length}_{[c_i, d_i]} B(t) \) and if \( B^*(t) \neq B(t) \) in \([c_i, d_i] \) then \( \text{length}_{[c_i, d_i]} B^*(t) < \text{length}_{[c_i, d_i]} B(t) \).

In the intervals of part 2, as declared above, \( B^*(t) \) in intervals \([e_i, f_i], 1 \leq i \leq m\) is linear and it conclude \( \text{length}_{[e_i, f_i]} B^*(t) < \text{length}_{[e_i, f_i]} B(t) \). Hence, we have \( \text{length}_{[c_i, d_i]} B^*(t) < \text{length}_{[c_i, d_i]} B(t) \).

**Theorem III.9.** If in the optimal policy \( B^*(T) = B_s(T) \) then,

\[ B^*(t) \leq \max_{[0, T]} B^*(t), \]  
\[ (23) \]

where \( B^*(T) \) is the left derivative of \( B^*(t) \) in \( T \), and \( B(t) \) is any arbitrary feasible transmitted data curve which connects the origin to \((0, B_s(T)) \).

**Proof:** If \( B^*(T) \leq B^*(t) \) then proof is complete. Hence we assume that \( B^*(T) > B^*(t) \). Since \( B^*(T) < B^*(t) \) and \( B(T) = B^*(T) \), there exists an \( \epsilon \) such that for any \( t \in (T - e, T) \), \( B^*(t) < B(t) \). Also, we assume that \( \epsilon = \min_{t \in [T - \epsilon, T]} B^*(t), t \in [0, T] \) Then based on Lemma III.7 for \( t \in [T - \epsilon, T] \), \( B^*(t) \) is linear \((B^*(T) = B^*(T - \epsilon)) \). Thus due to \( B^*(t) \leq B(t) \) for \( t \in [T - \epsilon, T] \) and \( B^*(T - \epsilon) = B(T - \epsilon) \) then \( B^*(T - \epsilon) \leq B^*(T - \epsilon) \). This completes the proof.

**Remark 2.** The importance of the Conjecture 1 and Theorem III.8 comes from the fact that if we can prove that Conjecture 1 we may propose the optimal offline algorithm as the shortest path curve among all admissible policies which use all the energy or send all the data in data buffer till \( T \). As a result, we have a method to describe the optimal offline algorithm.

**Remark 3.** Theorem III.9 becomes very significant if we impose an additional maximum power constraint in the optimization problem in (2)-(4). In this case, we first solve the problem without considering the maximum power. If the optimal policy satisfies the maximum power constraint, we are done; otherwise, we can deduce that for the amount of less than \( E^*(T) \) in the maximum power neither all of data in data buffer is sent, nor all of energy till \( T \) is used for the optimal policy. As a result we can determine the cases where the maximum power constraint is a limiting element.

**IV. OPTIMAL OFFLINE ALGORITHM**

In this section, we propose the optimal offline algorithm for the optimization problem (2)-(4). First, for simplicity, we explain the main idea of this algorithm for the discrete arrival data and discrete harvested energy curves, i.e., we assume that at instants \( 0, t_1^E, ... \) Tx harvests energy in amount of \( E_0, E_1, ... \) and at instants \( 0, t_1^B, ... \) the data arrives in amount of \( B_0, B_1, ... \) bits.

**Definition IV.1.** (Event point): every time in which the energy is harvested or the data is arrived is an event point.

**Theorem IV.1.** Let \( u_i \) be the \((i+1) - \text{th} \) event point. Assume that there exist \( m - 1 \) event points before \( T \), and also assume that \( u_m = T, u_0 = 0, u_{i-1} = 0 \) and \( n \in N \). Then the optimal policy structure of the transmitted rate is as follows.

\[ i_n = \arg \min_{i} \min_{u_{i-1} \leq u_i \leq u_m} \frac{r(E(u_{i-1}) - E(u_{i-1}))}{u_i - u_{i-1}} + \min_{u_{i-1} \leq u_i \leq u_m} \frac{r(B(u_{i-1}) - B(u_{i-1}))}{u_i - u_{i-1}} \]  
\[ (24) \]
\[ r_n = \min \left\{ r \left( \frac{E_s(u_{i_n}) - E(u_{i_{n-1}})}{u_{i_n} - u_{i_{n-1}}} \right), \frac{B_s(u_{i_n}) - B(u_{i_{n-1}})}{u_{i_n} - u_{i_{n-1}}} \right\} \tag{25} \]

and
\[ B(u_{i_n}) = B(u_{i_{n-1}}) + (u_{i_n} - u_{i_{n-1}})r_n, \tag{26} \]
where \( r_n \) is the transmitted rate in the interval \((u_{i_{n-1}}, u_{i_n})\).

**Proof:** We again use contradiction. First, it is concluded from Lemma III.5 that the transmitted power and rate curves must be constant between any two event points. Thus, the transmitted data and energy curves are piecewise linear and we must find the optimal curves among the piecewise linear functions. Without loss of generality, we assume that \( n = 1 \):
\[ r_1 = \min \left\{ r \left( \frac{E_s(u_{i_1})}{u_{i_1}}, \frac{B_s(u_{i_1})}{u_{i_1}} \right) \right\}. \tag{27} \]

Noting the optimal rate in interval \((0, u_{k})\) by \( r_1^* \), if \( r_1 \) is not the optimal rate, based on Lemma III.5 we must have:
\[ r_1^* > \min \left\{ r \left( \frac{E_s(u_{i_1})}{u_{i_1}}, \frac{B_s(u_{i_1})}{u_{i_1}} \right) \right\}. \tag{28} \]

Based on Lemma III.5 under the optimal policy, at least one of two following equations holds,
\[ r_1^* = r \left( \frac{E_s(u_{k})}{u_k} \right) \quad \text{or} \quad r_1^* = \frac{B_s(u_{k})}{u_k}. \tag{29} \]

We consider two cases: (i) if \( u_k > u_{i_1} \), then at least one of the causality conditions is violated due to \( r_1^* < r_1^* \); (ii) if \( u_k < u_{i_1} \), then the transmitted rate curve must be decreased at least in an interval which is inconsistent with Lemma III.5. Therefore, (27) gives \( r_1^* \). A similar argument proves that \( r_2, r_3, \ldots \), also satisfy this structure.

**Remark 4.** In [9], the purpose is to find the best scheme among piecewise linear functions for the transmitted data curve. However, we first prove that the optimal transmitted data curve among all functions, assumed in Section II, is a piecewise linear function, and we prove this result for our general model. Moreover, [9] investigates a minimization completion time problem however, we investigate a throughput maximization problem. The algorithm of [9] is as follows: it first calculates the minimum needed energy and a lower bound \( T_1 \) for completion time to transmit a given amount of data. Next, it uses (24) and (25) this algorithm from 0 to \( T_1 \) to find the next point \( u_k \), where it again calculates minimum needed energy and another lower bound for completion time to transmit the remaining data. Then, this procedure is repeated from \( u_k \) to the newest lower bound finally to deliver all of given data. However, we use (24) and (25) repeatedly to compare all the event points (i.e., \( u_i : i = 0, 1, \ldots, m \)).

To describe the algorithm we define two variable \( r_a \) and \( r_b \) as follow:
\[ r_b = \lim_{u \to x^+} r \left( \frac{E_s(u) - E(x)}{u - x} \right), \quad r_a = \lim_{u \to x^+} \frac{B_s(u) - B(x)}{u - x}. \tag{30} \]

Now, we can describe our proposed algorithm for the continuous model. Unlike the optimal algorithm in [9], our algorithm has three parts which depends on the positions of two curves \( B_s(t) \) and \( E_s(t) \). This algorithm is the extension of the previous algorithm that has been presented in Theorem IV.1. The difference of these two algorithms is that in the algorithm of Theorem IV.1 the optimal transmitted energy/data curve is piecewise linear; while in this algorithm, in some intervals, the energy/data transmitted curve could be equal to the harvested energy curve/arrival data curve (which are continuous in general).

Using Tables I, II, III and Fig. 3, we apply (24) and (25) for every event point, which are now a continuum, i.e., every point in \( t \in [0, T] \) is an event point. Thus at first we use (24) and (25), to find the next point in order to execute the algorithm with this new point. If the output point of the algorithm is the same as its input point, in this point either the transmitted data curve is equal to the arrival data curve, or the transmitted energy curve is equal to the harvested energy curve. This process repeats till instant \( T \). In this algorithm for detecting the intervals in which transmitted data/energy curve is equal to arrival data/harvested energy curve, we first must check that whether \( r_a \) and \( r_b \) are bounded or not. Therefore we have the three following items:

**Item 1:** Both \( r_a \) and \( r_b \) are unbounded, then we have a linear part to the next point (which is found by the algorithm in \( F_3 - F_6 \)).

**Item 2:** Only one of \( r_a \), \( r_b \) is bounded, the algorithm determines which is unbounded (in \( F_9 \)) and if none of conditions in \( F_{10}, F_{11} \) and \( F_{12} \) or \( F_{25}, F_{26} \) and \( F_{27} \) is satisfied, we have \( B(t) = B_s(t) \) (or \( E(t) = E_s(t) \)) for \( x \leq t < x + \epsilon \) and \( \epsilon \) is determined in \( F_{15} - F_{24} \) (or \( F_{30} - F_{39} \)). Otherwise, if one of conditions in \( F_{10}, F_{11} \) and \( F_{12} \) or \( F_{25}, F_{26} \) and \( F_{27} \) is satisfied we have a linear part to the next point that found by the algorithm in \( F_3 - F_6 \).

**Item 3:** Both \( r_a \) and \( r_b \) are bounded we have the third part of the algorithm in \( F_{40} - F_{45} \); if none of \( F_{40} - F_{47} \) is not satisfied we jump to B or C, which implies that \( B(t) = B_s(t) \) or \( E(t) = E_s(t) \) for \( x \leq t < x + \epsilon \). Otherwise, we have a linear part as item 1.

For the sake of simplicity, we define some variables as follows:
\[ \tilde{r} = \min \left\{ \min_{x < u \leq T} r \left( \frac{E_s(u) - E(x)}{u - x} \right), \min_{x < u \leq T} \frac{B_s(u) - B(x)}{u - x} \right\} \tag{31} \]
\[ \tilde{u} = \max \left\{ \min_u \left\{ r \left( \frac{E_s(u) - E(x)}{u - x} \right), \min_u \frac{B_s(u) - B(x)}{u - x} \right\} \right\} \tag{32} \]
\[ r_e(x) = \inf_{x < u \leq T} \left\{ \frac{B_s(u) - B(x)}{u - x} \right\}, \quad r_d(x) = \inf_{x < u \leq T} \left\{ r \left( \frac{E_s(u) - E(x)}{u - x} \right) \right\}. \tag{33} \]

Also, we define the following notations: we use the notation \( L > f(t, x, \epsilon) \) to show that there exists an \( \epsilon \) such that the straight line \( L \) is above the function \( f(t) \) in \( (x, x + \epsilon) \) and...
Definition

\[ L > f(t), x, \epsilon \] to show that there exists no \( \epsilon \) such that the straight line \( L \) is above the function \( f(t) \) in \((x, x+\epsilon)\). Also, we use the notation \( \{ L \times f(t), x \} \) to show that the straight line \( L \) collides with the function \( f(t) \) for \( t > x \) and \( \{ L \times f(t), x \} \) to show that the straight line \( L \) does not collide with the function \( f(t) \) for \( t > x \) opposite.

In addition, assume that \( L_1, L_2, L_3 \) and \( L_4 \) are four straight lines which pass through the points \( x, h, x \) and \( h \) with the slopes of \( E_\epsilon(x), E_\epsilon(h), B_\epsilon(x) \) and \( B_\epsilon(h) \), respectively and

| Function | Definition |
|----------|------------|
| \( F_1 \) | Calculate \( r_a, r_b \). |
| \( F_2 \) | Are both of \( r_a, r_b \) unbounded? |
| \( F_3 \) | \( s = r \circ r_{\delta}(x) \). |
| \( F_4 \) | \( s^*(t) = r_{\epsilon}(x), p^*(t) = r^{-1}(r_{\epsilon}(x)), f o r x < t < u \), \( x = u \). |
| \( F_5 \) | \( s^*(t) = r_{\delta}(x), p^*(t) = r^{-1}(r_{\delta}(x)), f o r x < t < u \), \( x = u \). |
| \( F_6 \) | \( s = x \) is \( T \)? |
| \( F_7 \) | \( s = x \) is \( T \)? |
| \( F_8 \) | \( s \) are both of \( r_a, r_b \) bounded? |
| \( F_9 \) | \( s \) are \( r \)-bounded? |
| \( F_{10} \) | \( \{ L_3 > E_\epsilon(t), x, \epsilon \} \)? |
| \( F_{11} \) | \( \{ L_3 \times B_\epsilon(t), h, \epsilon \} \)? |
| \( F_{12} \) | \( m = r_{\epsilon}(x) \)? |
| \( F_{13} \) | \( h = x \). |
| \( F_{14} \) | Calculate \( E_\epsilon(h) \). |
| \( F_{15} \) | \( \{ L_3 > B_\epsilon(t), x, \epsilon \} \)? |
| \( F_{16} \) | \( x = h \). |
| \( F_{17} \) | \( x \) are \( B \)-distinct transmitted data curves and \( B_1(t) > B_2(t) \) in the interval \((a, b)\) and \( B_1(t) = B_2(t) \) at \( t = a \) and \( t = b \). If \( B_1(t) \) is a convex function and \( B_1(t) > B_2(t) \) increase monotonically in \( t \), then:

\[
m = \min \left\{ r_{\epsilon}(x), r(E_\epsilon(x)) \right\} \quad (35)
\]

\[
m' = \min \left\{ r_{\delta}(x), B_\epsilon(x) \right\} \quad (36)
\]

\[
m(t) = \min \left\{ r_{\epsilon}(x), r(E_\epsilon(x)) \right\} \quad (37)
\]

\[
m'(t) = \min \left\{ r_{\delta}(x), B_\epsilon(t) \right\} \quad (38)
\]

**Lemma IV.2.** Let \( B_1(t) \) and \( B_2(t) \) be two distinct transmitted data curves and \( B_1(t) > B_2(t) \) in the interval \((a, b)\) and \( B_1(t) = B_2(t) \) at \( t = a \) and \( t = b \). If \( B_1(t) \) is a convex function and \( B_1(t) > B_2(t) \) increase monotonically in \( t \), then:

\[
\int_a^b r^{-1}(B_1(t))dt < \int_a^b r^{-1}(B_2(t))dt. \quad (39)
\]

**Proof:** If we assume that \( A(t) = B_1(t) \) and \( D_{min}(t) = B_2(t) \) in \([19]\), based on \([19]\) Theorem IV it concludes that the curve which uses the minimum energy has shortest length. Also, based on Lemma [19], \( B_1(t) \) has minimum length among the feasible transmitted curves, assuming \( A(t) = B_1(t) \) and \( D_{min}(t) \leq B_2(t) \). Thus \( B_1(t) \) uses minimum energy and the proof is complete.

**Lemma IV.3.** In our proposed algorithm, there are not any two points on the transmitted data curve such that the line passing through these points satisfies the causality conditions.

**Proof:** The proof is based on contradiction. Hence, we assume that there exist two points \( s \) and \( l \) such that the transmitted data curve of our proposed algorithm, replaced with the straight line that passes through these two points in \([s, l] \), does not violate the causality conditions. We denote this new
Fig. 3. Flowchart of the optimal algorithm

curve with $B_{new}(t)$. As explained, our proposed algorithm gives a transmitted data curve $B(t)$, that has at most three parts: 1- linear part, 2- some parts in which, $B(t) = B_1(t)$, 3- some parts in which $E(t) = E_1(t)$. Moreover, since the algorithm obtains a convex transmitted data curve, the straight line that passes through $s$ and $l$ is above of $B(t)$ in $(s, l)$, i.e., $B_{new}(t) > B(t)$ in $(s, l)$. It is clear that $s$ and $l$ are not on the linear part. Hence, there exists a point $x$ in $(s, l)$ such that $B(x) = B_s(x)$ or $E(x) = E_s(x)$; otherwise, $s$ and $l$ are on a straight line in our algorithm and $B_{new}(t) = B(t)$. We assume $M = \{t : B(t) = B_s(t) or E(t) = E_s(t), t \in (s, l)\}$ and $v = \inf(M)$. If $v = s$ there exists an $\epsilon$ such that $B(t) = B_s(t)$ or $E(t) = E_s(t)$ for $s < t < s + \epsilon$. Thus one of the causality conditions is violated: because, if $B(t) = B_s(t)$ in $s < t < s + \epsilon$, we have $B(t) = B_s(t)$ and $B_{new}(t) > B(t)$ which result in $B_{new}(t) > B_s(t)$. Therefore, the data causality condition is violated. If $E(t) = E_s(t)$ in $s < t < s + \epsilon$, there exists an $\epsilon_1$ such that in $s < t < s + \epsilon_1$, we have $p(t) < p_{new}(t)$. Thus, $E(t) < E_{new}(t)$ which results in $E_s(t) < E_{new}(t)$ in $s < t < s + \epsilon_1$ and the energy causality condition violated. Hence, it must be that $v \neq s$ and it results that $t = v$ is the first instant, in which $r(t) = B(t)$ can be changing in $s < t < l$. If $B(v) = B_s(v)$ then the data causality condition is violated. Now assume that $E(v) = E_s(v)$: since $B(t)$ is linear in $(s, v)$ and its slope is smaller than the slope of the straight line in curve $B_{new}(t)$ in $s < t < l$. Therefore, $p(t) < p_{new}(t)$ which results in $E(t) < E_{new}(t)$ in $(s, v)$. This implies that if we replace this line with the curve in the interval $(s, l)$, $E_s(v) < E_{new}(v)$ and thus the causality of energy is violated at $t = v$. This results in a contradiction which completes the proof.

**Lemma IV.4.** If there exists a convex transmitted data curve, $B(t)$ such that $B(T) = B_s(T)$ or $E(T) = E_s(T)$ and there do not exist any two points on $B(t)$ such that the line passing through these points satisfies the causality conditions, then $B(t)$ is optimal.

**Proof:** Again we use contradiction. Assume that there exists a transmitted data curve $B_1(t)$, such that $B_1(T) > B(T)$ or $(B_1(T) = B(T)$ and $E_1(T) \leq E(T))$. We assume that $B(t)$ is not optimal and $a = \max\{t : (\forall x : B_1(x) = B(x)), 0 \leq x \leq t\}$. Thus, in $[0, a]$, we have $B(t) = B_1(t)$. If in an interval we have $B(t) < B_1(t)$, and $B(t)$ is not linear in this interval, then we can choose two points in this interval such that the line passing through these two points does not violate the causality conditions, which is inconsistent with the assumptions of lemma. Therefore, if in an interval $B(t) < B_1(t)$ holds, then $B(t)$ must be linear. Moreover, based on extending of Lemma [III.1] $B(t)$ uses smaller amount of energy than $B_1(t)$ in this interval. From above, it is concluded that $B_1(t) \leq B(t)$ or $B(t) < B_1(t)$, where if $B(t) < B_1(t)$, then $B(t)$ is linear in this interval. If $B_1(t) \leq B(t)$ for $a < t$ then $B_1(T) = B(T)$. $B(t)$ is convex and $B_1(t)$ and $B_1(t)$ increase in $t$, then based on Lemma [IV.2] $E(T) < E_1(T)$ which is a contradiction. If for $a < t$ always $B_1(t) \leq B(t)$ does not hold, then we define $t_c$ as:

$$t_c = \max\{t : B_1(t) = B(t) for t < T\} \quad (40)$$

Based on extending Lemmas [III.1] and [IV.2] we obtain $E(t_c) \leq E_1(t_c)$. Now, if $B(T) > B_1(t)$ for $t_c < t < T$, we have a contradiction and the proof is completed. If $B(t) < B_1(t)$, then $B(t)$ should be linear in this interval. For this case if
B(T) = B_1(T) then should B_1(T) = B_2(T). Since the curve \( B(T) \) uses smaller amount of energy than \( B_1(T) \) in \( t_c < t < T \) and \( E(t_c) < E_1(t_c) \), we get \( E(T) < E_1(T) \) which is a contradiction. Having \( E(T) = E_s(T) \) implies that \( E_1(T) < E_1(T) \) which is a contradiction, too. Therefore, the proof is complete.

**Theorem IV.5.** The presented algorithm is optimal.

**Proof:** The proof is directly obtained from Lemmas \[\text{[IV.3]}\] and \[\text{[IV.4]}\].

V. THROUGHPUT MAXIMIZATION AND COMPLETION TIME MINIMIZATION WITH MULTI-HOPPING

In this section, we consider a multi-hop channel with one Tx, one Rx and some relays and we want to investigate a throughput maximization and a completion time minimization problem in an offline model in a full-duplex mode. For simplicity we first assume that we have a two-hop communication channel which is illustrated in Fig. 4. Then we extend the results to \( n \) relays in Corollaries 2 and 3.

### A. Throughput Maximization

Following our assumption in Section II, in this model both harvested energy curves \( E_s(t) \) and \( E_r(t) \) and arrival data curve \( B_s(t) \) are continuous. Similar to the single-user channel in Tx and Rx the power of transmission through continuous functions \( r_s(p) \) and \( r_r(p) \), respectively. \( B_{sr}(t) \) and \( B_{rd}(t) \) are the amount of data which are transmitted from the Tx to the relay and from the relay to the Rx, respectively. \( E_{sr}(t) \) and \( E_{rd}(t) \) are the amount of energy that are utilized in the Tx and the Rx to transmit data from the Tx to the relay, and the relay to Rx in \([0, t]\) respectively. \( p_{sr}(t) \) and \( p_{rd}(t) \) are the amount of power used in Tx, and the relay for data transmission. We assume that \( B_{sr}(t) \) and \( E_{sr}(t) \) are, respectively the optimal transmitted data and energy curves which are obtained from problem \[\text{[2]}\] and \[\text{[4]}\].

When we consider the causality conditions in Tx and \( B_{rd,B_{sr}}(t), E_{rd,B_{sr}}(t) \) and \( p_{rd,B_{sr}}(t) \) are respectively the optimal transmitted data curve, optimal transmitted energy curve and optimal transmitted power curve which are obtained from problem \[\text{[2]}\] and \[\text{[4]}\].

We substitute \( B_s(t) = B_{sr}(t) \) and \( E_s(t) = E_r(t) \). Now, we can formulate our problem as follows:

$$
D^{(MH)}(T) = \max_{p_{sr}(t), p_{rd}(t)} \int_0^T r_{rd}(p_{rd}(t)) dt \quad (41)
$$

s.t.

$$
\int_0^t p_{sr}(t') \leq E_s(t), \ 0 \leq t \leq T \quad (42)
$$

$$
\int_0^t p_{rd}(t') \leq E_r(t), \ 0 \leq t \leq T \quad (43)
$$

$$
\int_0^t r_{sr}(p_{sr}(t')) dt' \leq B_s(t), \ 0 \leq t \leq T \quad (44)
$$

$$
\int_0^t r_{rd}(p_{rd}(t')) dt' \leq B_{rd,B_{sr}}(t), \ 0 \leq t \leq T \quad (45)
$$

\[\text{[42]}\] and \[\text{[43]}\] are the energy causality conditions in Tx and the relay, \[\text{[44]}\] and \[\text{[45]}\] are the data causality conditions in Tx and the relay. Also we assume that \( B_{sr}(t) \) and \( E_{sr}(t) \) are the optimal transmitted data and energy curves in Tx, \( B_{rd}(t) \) and \( E_{rd}(t) \) are the optimal transmitted data and energy curves in relay for problem \[\text{[31]}\] and \[\text{[33]}\].

**Theorem V.1.** In the optimal policy we have \( B_{sr}^*(t) = B_{sr}^*(t) \) and \( B_{rd}^*(t) = B_{rd}^*(t) \).

**Proof:** It is clear that for any fixed \( E_s(t) \) and \( B_{sr}(t) \), the throughput is maximized when the amount of data from the relay to Rx is maximized. Thus, it is enough to prove that for any feasible \( B_{sr}(t) \), we have: \( B_{rd,B_{sr}}^*(t) \leq B_{rd,B_{sr}}^*(t) \).

We know that the instant 0 is the first point in which \( B_{rd,B_{sr}}^*(t) = B_{rd,B_{sr}}^*(t) \). From \[\text{[31]}\] in Section IV, we have the following equation in instant 0:

$$
p_{rd,B_{sr}}^*(0) = \min_x \left\{ r_{rd}^{-1} \left( \frac{B_{sr}^*(x)}{x} \right) \right\} \quad (46)
$$

Also, we assume that:

$$
t_s = \min \{ t : B_{rd,B_{sr}}^*(t) = B_{rd,B_{sr}}^*(t), 0 < t < T \} \quad (47)
$$

Based on Lemma \[\text{[11.7]}\] if in time \( x_c \) we have \( B_{sr,B_{sr}}^*(x_c) < B_{sr}(x_c) \), then there exist \( \epsilon_1 \) and \( \epsilon_2 \) such that \( B_{sr}(t) \) is linear and \( B_{sr,B_{sr}}^*(t) < B_{sr}(t) \) in \((x_c - \epsilon_1.x_c + \epsilon_2)\). \( B_{sr,B_{sr}}(x_c - \epsilon_1) = B_{sr}(x_c - \epsilon_1) \) and \( B_{sr,B_{sr}}(x_c - \epsilon_2) = B_{sr}(x_c - \epsilon_2) \). Thus, \( m = \arg \min_x \left( \frac{B_{sr,B_{sr}}(x)}{x} \right) \) is not in \((x_c - \epsilon_1.x_c + \epsilon_2)\). Therefore, we have \( B_{sr}(m) \leq B_{sr,B_{sr}}^*(m) \) and this concludes \( \min_x \left( \frac{B_{sr,B_{sr}}(x)}{x} \right) \leq \min_x \left( \frac{B_{sr,B_{sr}}(x)}{x} \right) \). Hence we have \( p_{rd,B_{sr}}^*(0) \leq p_{rd,B_{sr}}(0) \), which means \( B_{rd,B_{sr}}^*(t) \leq B_{rd,B_{sr}}(t) \) in interval \([0, t_s]\). Similarly, we can use this argument for any point in which \( B_{sr,B_{sr}}^*(t) = B_{sr,B_{sr}}^*(t) \) and this concludes, we have \( B_{rd,B_{sr}}^*(t) \leq B_{rd,B_{sr}}(t) \) for every \( t \in [0, T] \) and this completes the proof.

**Corollary 2.** Theorem \[\text{[V.1]}\] can be extended to \( n \) relays: Tx transmits maximum amount of data by proposed algorithm in Section IV, the first relay sends maximum amount of data to the second relay by the same algorithm and this procedure repeats till the Rx.

**An example:** Assume that harvested energy curves in Tx and the relay nodes are \( E_s(t) = e^t - 1 \), \( E_r(t) = 2e^t - 2 \), respectively and \( r_{sr}(p) = r_{rd}(p) = \frac{1}{2} \log(1 + p) \) in which...
logarithm is in base 2. We want to maximize the throughput from Tx to the destination. Using energy causality and convexity of \( E_r(t) \) based on Section IV instantaneous arrival data at the relay is maximized in every \( t \in [0, 1] \) if \( E_{sr}(t) = E_s(t) \). Thus, the optimal arrival data at the relay is \( B_{sr, s}(t) = \int_0^t \frac{1}{2} \log(1 + E_s(t')) dt' \) which is a continuous curve. Now, the problem reduces to a single-user throughput maximization problem in the relay node with harvested energy curve \( E_r(t) \), and arrival data curve \( B_{sr, s}(t) \).

**B. Completion Time Minimization**

In this subsection we investigate a completion time minimization problem to transmit \( B_0 \) amount of data to Rx for a multi-hop channel. We can formulate the problem as follows:

\[
T_{\text{offline}} = \min T \quad \text{s.t.} \quad \int_0^T r_{rd}(p_{rd}(t)) dt = B_0, \quad \text{[42], [43], [44] and [45].} \tag{48}
\]

**Lemma V.2.** \( D^{(MH)}(t) \) in [41] is nondecreasing. Also if \( \lim_{p \to \infty} \frac{r_{rd}(p)}{p} = 0 \), then \( D^{(MH)}(t) \) is continuous.

**Proof:** The proof of the first part of the above lemma is straightforward and omitted for brevity. For the second part of the lemma, we have to prove that for each instant \( t_0 \), \( D^{(MH)}(t) \) is continuous, then for \( t \in (t_0, t_0 + \epsilon) \) we have,

\[
D^{(MH)}(t_0) \leq D^{(MH)}(t) \leq D^{(MH)}(t_0) + (t - t_0)r_{rd}(A(t)) \tag{50}
\]

where \( A(t) \) is bounded and \( \lim_{t \to t_0} A(t) = A_0 \). \( A_0 \) is a finite number. Based on \( \lim_{p \to \infty} \frac{r_{rd}(p)}{p} = 0 \) we have,

\[
\lim_{t \to t_0} \left( D^{(MH)}(t_0) + (t - t_0)r_{rd}(A(t)) \right) = D^{(MH)}(t_0) + A_0 \lim_{p \to \infty} \frac{r_{rd}(p)}{p} = D^{(MH)}(t_0). \tag{51}
\]

From above, it is concluded \( \lim_{t \to t_0} D^{(MH)}(t) = D^{(MH)}(t_0) \). We can similarly prove that \( \lim_{t \to t_0} D^{(MH)}(t) = D^{(MH)}(t_0) \). Thus \( D^{(MH)}(t) \) is continuous.

**Theorem V.3.** Assume that \( C = \{ t : D^{(MH)}(t) = B_0 \} \) and \( \lim_{p \to \infty} \frac{r_{rd}(p)}{p} = 0 \) if \( C \neq \emptyset \) and \( T_{\text{min}} = \min C \), then \( T_{\text{offline}} = T_{\text{min}} \), and optimal offline algorithm is given by the proposed algorithm in Section IV for given deadline \( T_{\text{offline}} \) and if \( C = \emptyset \) there is not any policy to transmit the amount of \( B_0 \) data.

**Proof:** Since \( C \neq \emptyset \), for \( T_{\text{min}} \) exists a method to transmit amount of \( B_0 \) data. On the other hand, if \( T_{\text{offline}} < T_{\text{min}} \) hold, based on Lemma V.2 we have \( D^{(MH)}(T_{\text{offline}}) < B_0 \). Thus \( T_{\text{offline}} = T_{\text{min}} \). If we assume that \( C = \emptyset \), we can conclude the amount of \( B_0 \) data cannot be transmitted by any time, because if there exist a time \( T_{\text{sr}} \) such that we can transmit \( B_0 \) amount of data till \( T_{\text{sr}} \) results in \( B_0 \leq D^{(MH)}(T_{\text{sr}}) \). Thus, Based on Lemma V.2 concluded \( C \neq \emptyset \).

**Corollary 3.** We can extend Theorem V.3 to \( n \) relays with defining \( D^{(MH)}(t) \) as maximum amount of data curve in Rx in Theorem V.3

**VI. AN ONLINE ALGORITHM**

In this section, we want to propose an online algorithm for the optimization problem proposed in Section II. In our online algorithm, we do not have any information about the future of two curves \( B_s(t), E_s(t) \), (even the distributions of two processes \( B_s(t), E_s(t) \) are unknown). First, we prove that the proposed online algorithm uses all of the energy or sends all of the data in the data buffer, and the transmitted power curve is a nondecreasing function similar to optimal offline algorithm. Then, we derive a lower bound on the ratio of the amount of transmitted data in the online algorithm to the optimal offline algorithm.

We express the online algorithm in the following.

\[
p_{\text{online}}(t) = \min \left\{ r^{-1} \left( \frac{B_{\text{rem}}(t)}{T - t + \epsilon}, \frac{E_{\text{rem}}(t)}{T - t + \epsilon} \right) \right\}, \tag{52}
\]

where \( B_{\text{rem}}(t) = B_s(t) - B_{\text{online}}(t), E_{\text{rem}}(t) = E_s(t) - E_{\text{online}}(t) \) and \( \epsilon \) is chosen to make the \( p_{\text{online}}(t) \) a bounded curve. Note that \( \epsilon \) is a sufficiently small real number.

According to above, in our algorithm, if in time \( t \) the amount of energy is the limiting element, then \( p_{\text{online}}(t) \) is determined such that all of remaining energy in \( t \) is utilized with the fixed power till time \( T \) and if in time \( t \) the amount of information is the limiting element, then \( p_{\text{online}}(t) \) is determined such that all of the remaining bits in \( t \) are transmitted with the fixed rate till time \( T \). In the following we want to obtain equation \( p_{\text{online}}(t) \) in parameters \( B_s(t), E_s(t), T \). We assume that \( t_1, t_2, \ldots, t_n \) are instants in which the equation \( p_{\text{online}}(t) \) switches from \( r^{-1} \left( \frac{B_{\text{rem}}(t)}{T - t + \epsilon} \right) \) to \( \frac{E_{\text{rem}}(t)}{T - t + \epsilon} \) or vice versa. We assume that in interval \( (t_{i-1}, t_i) \) we have \( p_{\text{online}}(t) = \frac{E_{\text{rem}}(t)}{T - t + \epsilon} \). Hence, in \( (t_i, t_{i+1}) \) we have \( p_{\text{online}}(t) = r^{-1} \left( \frac{B_{\text{rem}}(t) - B_{\text{online}}(t)}{T - t_i + \epsilon} \right) \), then we have

\[
p_{\text{online}}(t_i^+) = r^{-1} \left( \frac{B_s(t_i^+) - B_{\text{online}}(t_i)}{T - t_i + \epsilon} \right), \tag{53}
\]

and after some algebraic calculation we obtain

\[
p_{\text{online}}(t) = r^{-1} \left( \int_{t_i}^t \frac{B_s'(t')}{T - t' + \epsilon} dt' + r(p_{\text{online}}(t_i^+)) \right) \quad \text{for } t_i < t < t_{i+1}. \tag{54}
\]

Also we can easily show that if in \( (t_{i-1}, t_i) \) we have, \( p_{\text{online}}(t) = \frac{E_{\text{rem}}(t)}{T - t + \epsilon} \) holds, then in interval \( (t_i, t_{i+1}) \) we have \( p_{\text{online}}(t) = \frac{E_s(t)}{T - t + \epsilon} \) and we can write the following equations,

\[
p(t_i^+) = \frac{E_s(t_i^+)}{T - t_i + \epsilon} \tag{55}
\]

\[
p_{\text{online}}(t) = \int_{t_i}^t \frac{E_s'(t')}{T - t' + \epsilon} dt' + p_{\text{online}}(t_i^+) \quad \text{for } t_i < t < t_{i+1} \tag{56}
\]

**Lemma VI.1.** \( p_{\text{online}}(t) \) is a nondecreasing function.
Proof: \([53, 56]\) conclude that \(p_{online}(t)\) is nondecreasing in all intervals \((t_i, t_{i+1})\) for \(0 \leq i \leq n\) with \(t_0 = 0, t_{n+1} = T\). Thus, we must only prove that \(p(t^-_i) \leq p(t^+_i)\) for \(0 < i < n + 1\). If in \((t_{i-1}, t_i)\), \(p_{online}(t) = B_{rem}(t)\) holds, then in \((t_i, t_{i+1})\) we have \(p_{online}(t) = r^{-1} \left( \frac{T - t + \epsilon}{B_{rem}(t')} \right)\). Therefore, \([52]\) concludes that \(T - t + \epsilon \leq \frac{B_{rem}(t')}{T - t + \epsilon}\) and since \(B_{online}(t^-_i) = B_{online}(t^+_i)\) but \(B_{s}(t^-_i) \leq B_{s}(t^+_i)\), we have \(B_{rem}(t^-_i) \leq B_{rem}(t^+_i)\) which results in \(p(t^-_i) \leq p(t^+_i)\). If in \((t_{i-1}, t_i)\) we have \(p_{online}(t) = r^{-1} \left( \frac{B_{rem}(t)}{T - t + \epsilon} \right)\), similarly we can show that \(p(t^-_i) \leq p(t^+_i)\), this completes the proof.

**Lemma VI.2.** In our online algorithm either \(\lim_{\epsilon \to 0} E_{online}(T) = E_s(T)\) or \(\lim_{\epsilon \to 0} B_{online}(T) = B_s(T)\). Moreover, if in interval \((t_n, T)\) we have \(p_{online}(t) = r^{-1} \left( \frac{B_{rem}(t)}{T - t + \epsilon} \right)\), then \(\lim_{\epsilon \to 0} B_{online}(T) = B_s(T)\); otherwise, \(\lim_{\epsilon \to 0} E_{online}(T) = E_s(T)\).

**Proof:** We assume that in \((t_n, T)\) we have \(p_{online}(t) = r^{-1} \left( \frac{B_{rem}(t)}{T - t + \epsilon} \right)\), the other condition can be proved similarly.

\[
B_{online}(T) = \int_0^T r(p_{online}(t))dt = \int_0^{t_n} r(p_{online}(t))dt + \int_{t_n}^T r(p_{online}(t))dt = B_{online}(t_n) + \int_{t_n}^T r(p_{online}(t))dt.
\]

From \([53], [54]\):

\[
r(p_{online}(t)) = \int_{t_n}^{T} \frac{B'_s(t') - B_{online}(t)}{T - t' + \epsilon} dt' + \frac{B_s(t_n^+) - B_{online}(t_n)}{T - t_n + \epsilon}
\]

for \(t_n < t < T\).

So,

\[
B_{online}(T) = \int_0^T \int_{t_n}^{T} \frac{B'_s(t')}{T - t' + \epsilon} dt' dt + \frac{B_s(t_n^+) - B_{online}(t_n)}{T - t_n + \epsilon}
\]

\[
+ \int_{t_n}^{T} \frac{B_s(t_n^+) - B_{online}(t_n)}{T - t' + \epsilon} dt' + \frac{B_{online}(t)_n}{T - t_n + \epsilon}
\]

\[
\text{for } t_n < t < T.
\]

Now, it is enough to prove that \(\lim_{\epsilon \to 0}(B_s(T) - B_{online}(T)) = 0\).

\[
B_s(T) - B_{on}(T) = B_s(T) - B_s(t_n^+) + B_s(t_n^+) - B_{on}(T)
\]

\[
= \int_{t_n}^{T} \left( B'_s(t') - \frac{(T - t + \epsilon)B_s(t')}{T - t' + \epsilon} \right) dt + \frac{\epsilon}{T - t_n + \epsilon} B_{online}(t_n) = \int_{t_n}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt + \frac{\epsilon}{T - t_n + \epsilon} B_{online}(t_n).
\]

\[
\text{It is clear that } \lim_{\epsilon \to 0} \frac{\epsilon}{T - t_n + \epsilon} B_s(t_n^+) = 0 \text{ and } \lim_{\epsilon \to 0} \frac{\epsilon}{T - t_n + \epsilon} B_{online}(t_n) = 0, \text{ because, } B_s(t) \text{ is bounded and as a result } B_{online}(t) \text{ is bounded, too. Now, assume that } t_{j_1}, t_{j_2}, ..., t_{j_m} \text{ are all of instants in interval } (t_n, T) \text{ such that } B_s(t_{j_i}) \neq B_s(t_{j_i}^+) \text{ for } 1 \leq i \leq m. \text{ Thus,}
\]

\[
\int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt' = \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt' + \epsilon \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt' + \epsilon \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt' + \epsilon \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt'.
\]

\[
(61)
\]

It is clear that \(\lim_{\epsilon \to 0} \frac{\epsilon}{T - t_n + \epsilon} B_s(t_n^+) = 0, \text{ for } 0 < i < m. \text{ From assumptions in Section II } B_s(t) \text{ we conclude that } B_s(t) \text{ is bounded in intervals } (t_{j_i}, t_{j_{i+1}}) \text{ for } 0 \leq i \leq m \text{ with } t_{j_0} = t_n \text{ and } t_{j_{m+1}} = T. \text{ Thus,}
\]

\[
0 \leq \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} B'_s(t') dt' \leq \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} M dt'
\]

\[
\text{for } 0 \leq i \leq m,
\]

\[
(62)
\]

where \(B'_s(t^+) \leq M \text{ and } B'_s(t^-) \leq M \text{ for } 0 \leq t \leq T \text{ in which } B'_s(t), B'_s(t^+) \text{ mean left and right derivatives of } B_s(t) \text{ in } t. \text{ Also it can be shown easily that,}
\]

\[
\lim_{\epsilon \to 0} \int_{t_{j_i}}^{T} \frac{\epsilon}{T - t' + \epsilon} M dt' = 0 \text{ for } 0 \leq i \leq m.
\]

Hence the proof is completed.

**Theorem VI.3.** Assume \(l\) and \(k\) are two real numbers such that \(t_i < \frac{T}{l} < t_{i+1}\) and \(E_s(T) = E_s(T)\). If

\[
p_{online}(T) = \frac{E_{rem}(T)}{T - \frac{T}{l} + \epsilon}, \text{ then } \frac{1}{k} (1 - \frac{1}{l}) \leq \frac{B_{online}(T)}{B_{offline}(T)}.
\]

\[
\text{otherwise } \frac{B_s(T)}{B_s(T)} \leq \frac{B_{online}(T)}{B_{offline}(T)}.
\]

**Proof:** Note that if \(p_{online}(T) = \frac{E_{rem}(T)}{T - \frac{T}{l} + \epsilon}\), then

\[
E_s(T) \leq p_{online}(T),
\]

because if \(63\) does not hold then should

\[
E_s(T) - E_{online}(T) \leq E_s(T) - E_s(T) \leq E_s(T).
\]

Also, due to the fact that \(p_{online}(t)\) is a nondecreasing function
(based on Lemma \ref{lem:VI.2}, we have,)
\[ \frac{E_{\text{online}}(T)}{T} - \frac{E_{\text{online}}(T)}{T} \leq \frac{E_s(T) - E_{\text{online}}(T)}{T - T + \epsilon}. \]  
(65)

After some algebraic calculation, it can be shown that \ref{eq:65} and \ref{eq:65} contradict together. On the other hand, we have,
\[ B_{\text{online}}(T) = B_{\text{online}}(T) + \int_{0}^{T} r(p_{\text{online}}(t))dt \geq \int_{0}^{T} r(p_{\text{online}}(t))(T - T + \epsilon) = \frac{(T - T + \epsilon)E_s(T)}{T - T + \epsilon} = 1 \cdot \frac{E_s(T)}{T} + \epsilon \]
Also, we have from Lemma \ref{lem:III.1}
\[ B_{\text{offline}}(T) \leq T r\left(E_s(T)\right). \]  
(67)

Thus, it is concluded from \ref{eq:63}, \ref{eq:66} and \ref{eq:65} and assuming that \( \epsilon \) is small sufficiently,
\[ \frac{B_{\text{online}}(T)}{B_{\text{offline}}(T)} > \frac{1}{k(1 - l)}. \]  
(68)

Now, since \( p_{\text{online}}(t) \) is a nondecreasing function in \( t \), if
\[ p_{\text{online}}(T) > r^{-1}\left( \frac{B_{\text{online}}(T)}{T - T + \epsilon} \right) \]
then we have,
\[ B_{\text{online}}(T) \geq B_{\text{online}}(T) + \frac{B_s(T)}{T - T + \epsilon} (T - T) \]
\[ \geq \frac{T - T}{T - T + \epsilon} B_s(T) + \frac{\epsilon}{T - T + \epsilon} B_{\text{online}}(T) \geq \frac{T - T}{T - T + \epsilon} B_s(T). \]
(69)

Also we know \( B_{\text{offline}}(T) \leq B_s(T) \) and since \( \epsilon \) is sufficiently small we obtain,
\[ \frac{B_{\text{online}}(T)}{B_{\text{offline}}(T)} > \frac{B_s(T)}{B_{\text{offline}}(T)}, \]  
(70)

and the proof of the theorem is complete.

Theorem VI.3 expresses that if in (\ref{thm:VI.3}, \( p_{\text{online}}(t) = \frac{B_{\text{rem}}(t)}{T - t + \epsilon} \), then \( \frac{B_{\text{online}}(t)}{B_{\text{offline}}(t)} \approx 1 \), because we can make the number \( l \) (in Theorem VI.3) sufficiently close to 1 which means that the online algorithm transmits all of data bits that offline algorithm transmits. Moreover, if \( E_s(T) = \frac{T}{2} \), from Theorem VI.3 we conclude that \( \frac{B_{\text{online}}(T)}{B_{\text{offline}}(T)} \geq \frac{1}{2} \), which means that the online algorithm transmits at least half of data bits that offline algorithm transmits. Although, there are many examples that this bound is good for them but the authors believe that the above lower bound is not tight enough for all the arbitrary two curves \( E_s(t), B_s(t) \), and the algorithm is more efficient than the bound in these examples. Another advantage of this online algorithm is that it does not require any information about the distributions of the two processes \( B_s(t) \) and \( E_s(t) \).

VII. NUMERICAL RESULTS

In this section, we provide a simple numerical example to explain our results. Consider a band-limited additive white Gaussian noise channel with bandwidth \( W = 1 \) Hz, also we assume that the actual channel gain divided by the noise power spectral density multiplied by the bandwidth is 1. So we have, \( r(p) = \log(1 + p) \) where the logarithm is in base 2. Assume that \( E_s(t) = 100t^2 \) J and \( B_s(t) = 100t^2 \) bits. Moreover, we assume that \( T = 1s \). On the left of Fig. 5, are illustrated curves \( E_s(t), E_{\text{offline}}(t), B_{\text{offline}}(t) \) and \( B_s(t) \) derived using our proposed offline algorithm. As can be seen, approximately, in the interval \((0, .34)\), \( E_{\text{offline}}(t) \) is nonlinear; \( E_{\text{offline}}(t) \neq E_s(t) \) and this means that \( B(t) = B_s(t) \), according to Lemma \ref{lem:III.3}. For interval \((.34, .54)\), \( E_{\text{offline}}(t) \) is linear that it can be seen in Fig. 5. Also, it can be seen \( p(t) \) is a nondecreasing function because \( E_{\text{offline}}(t) \geq 0 \) (Lemma \ref{lem:III.3}). In instant \( t = 1s \) we have \( E_{\text{offline}}(1) = E_s(1) \) and this algorithm can transmit approximately 6 bits at the end of 1s that is illustrated in Fig. 5, "energy constrained".

For the online algorithm we assume the same \( E_s(t), B_s(t), r(p) \) and \( T \) and assume that \( \epsilon = .001 \). It is seen that \( E_{\text{online}}(t) \) and \( B_{\text{online}}(t) \) are convex hence, \( p_{\text{online}}(t) \) is nondecreasing (Lemma VI.1). Also, \( E_{\text{online}}(1) \approx E_s(1) \) (Lemma VI.2). In this example we have \( \frac{B_{\text{online}}(T)}{B_{\text{offline}}(T)} = .7 \) which means that approximately 70 percent of data that transmitted by offline algorithm is transmitted by online algorithm. On the right of Fig. 5 are illustrated \( E_{\text{online}}(t) \) and \( B_{\text{online}}(t) \).

VIII. DISCUSSION AND CONCLUSION

In this paper, we considered an EH system with continuous arrival data and continuous harvested energy curves; while, most of the research in this area considered a discrete model due to the mathematical tractability of the ensuing system optimization. More practical models considering the energy buffer and/or data buffer (to provide quality-of-service) at the Tx and the relay node may be investigated in future works. Comparing this work with \ref{ref:20}, we deduce that in \ref{ref:20} a model with continuous harvested energy curve is investigated, while it is assumed that the large amount of data exists to transmit with no limitation on transmitting data with any harvested energy curve, that is, in \ref{ref:20} causality condition of \ref{eq:4} does not exist. Thus, the model in this paper is more general than \ref{ref:20}. If we compare this work with \ref{ref:9}, we deduce that in \ref{ref:9} only the model with discrete \( E_s(t) \) and \( B_s(t) \) curves is investigated and its goal is to find the optimal policy that minimizes the completion time for transmitting a given amount of data among...
piecewise linear curves; while, in this paper we consider a model that includes both discrete and continuous models for $E_s(t)$ and $B_s(t)$, and find optimal policy among all of curves assumed in Section II as can be seen in Fig. 5, therefore, the considered model of this paper is more general than [9].

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