FREELY DECAYING TURBULENCE IN FORCE-FREE ELECTRODYNAMICS

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ABSTRACT

Freely decaying, relativistic force-free turbulence is studied for the first time. We initiate the magnetic field at a short wavelength and simulate its relaxation toward equilibrium on two- and three-dimensional periodic domains in both helical and nonhelical settings. Force-free turbulent relaxation is found to exhibit an inverse cascade in all settings and in three dimensions to have a magnetic energy spectrum consistent with the Kolmogorov 5/3 power law. Three-dimensional relaxations also obey the Taylor hypothesis; they settle promptly into the lowest-energy configuration allowed by conservation of the total magnetic helicity. However, in two dimensions, the relaxed state is a force-free equilibrium whose energy greatly exceeds the Taylor minimum and that contains persistent force-free current layers and isolated flux tubes. We explain this behavior in terms of additional topological invariants that exist only in two dimensions, namely the helicity enclosed within each level surface of the magnetic potential function. The speed and completeness of turbulent magnetic free-energy discharge could help account for rapidly variable gamma-ray emission from the Crab Nebula, gamma-ray bursts, blazars, and radio galaxies.

Key words: gamma rays–general – magnetic fields – magnetic reconnection – magnetohydrodynamics (MHD) – plasmas – turbulence

1. INTRODUCTION

The most extreme sources of high-energy astrophysical radiation are widely believed to exist in magnetically dominated, relativistic environments. Jets powered by supermassive black holes, plasma winds driven by pulsars, and gamma-ray bursts are prime examples. The violent intermittency of gamma-ray production by these systems could be taken as strong evidence that turbulence is critically linked to their radiative output. And yet, the physics of magnetically dominated, relativistic turbulence remains nearly unexplored.

The importance of understanding turbulence in this new regime is underscored by the discovery of powerful gamma-ray flares originating within the Crab Nebula (Abdo et al. 2011; Tavani et al. 2011). Moreover, rapid time variability seems to be ubiquitous among gamma-ray emitters; the blazars PKS 2155-304 (Aharonian et al. 2007), 1510-089 (Saito et al. 2013), and 3C 279 (Hayashida et al. 2015), as well as radio galaxies such as M87 (Aharonian et al. 2006) and IC 310 (Aleksić et al. 2014), have each been observed to produce sporadic, high-intensity outbursts of gamma radiation. Such dramatic enhancements of synchrotron or inverse Compton emissivity require a reservoir of free energy to spontaneously energize the active region’s electron population. If that free energy resides in magnetic fields, then its discharge could be triggered by magnetic reconnection, the general picture of which has been rendered in many different ways (Lazarai et al. 2003; Lyutikov & Uzdensky 2003; Zhang & Yan 2011; McKinney & Uzdensky 2012; Sironi & Spitkovsky 2014; Blandford et al. 2015).

In this paper we intend to demonstrate that magnetic free-energy discharge can proceed from field geometries that are far more general than those typically considered in reconnection models, and on a timescale that is not limited by the rate with which microphysical or anomalous (e.g., Lazarian & Vishniac 1999) resistivity can destroy magnetic flux. This amounts to extending the historical problem of magnetic relaxation (e.g., Chandrasekhar & Fermi 1953) to relativistic, magnetically dominated conditions. We focus on only a few of the many aspects of this topic that could be studied. Briefly, they are (1) the rate and completeness of magnetic free-energy discharge in various topological settings, (2) a characterization of persistent nonlinear structures, and (3) the spectral energy distribution of freely decaying, relativistic force-free turbulence. To be most relevant for astrophysical gamma-ray emission, we are interested in regions far from any solid boundaries that could anchor the magnetic field (so periodic domains are appropriate) and where the plasma is nearly perfectly conducting, inviscid, and magnetically dominated—conditions that are the domain of force-free electrodynamics (FFE) theory.

FFE forms the basis for historical theories of pulsar magnetospheres (Goldreich & Julian 1969; Spitkovsky 2006) and angular momentum extraction from black holes (Blandford & Znajek 1977) and continues to be a widely used description for studying these highly relativistic settings (Palenzuela et al. 2010; Gralla et al. 2015; Yang et al. 2015). It can be derived from relativistic magnetohydrodynamics (MHD) when the electromagnetic contribution to the stress-energy tensor greatly exceeds contributions from matter, and hence it captures the essential nonlinear dynamics of relativistic MHD for the regime of interest. It also admits a numerical approach that is more robust and efficient than relativistic MHD solution schemes.

Turbulence in FFE has only been considered in a few previous studies. The theory of Alfvén wave turbulence in the presence of a strong guide field, originally formulated for Newtonian MHD by Goldreich & Sridhar (1995), has been extended to the magnetically dominated, relativistic regime by Thompson & Blaes (1998). Alfvén wave turbulence has since been studied numerically in both the momentum balanced (Cho 2005) and unbalanced (Cho & Lazarian 2014) situations. Even the study of mildly relativistic MHD turbulence is in its infancy, having only been treated so far in a handful of studies (Zhang et al. 2009; Inoue et al. 2011; Zrake & MacFadyen 2011, 2013; Zrake 2014). There are, by comparison, a great number of Newtonian MHD turbulence studies (see, e.g., Tobias et al. 2011, for a review) treating all different
circumstances, including turbulent relaxation. Comparisons with them will be made wherever possible.

One of the issues we will explore in this paper is the applicability of the Taylor (1974) hypothesis to magnetic relaxation in FFE. Taylor’s original conjecture was that magnetic relaxation would universally settle in the lowest-energy configuration allowed by the conservation of total magnetic helicity:

$$H = \int A \cdot B d^3x.$$  \hfill (1)

These so-called Taylor states are linear force-free equilibria, having an electric current density \(J\) that is not only aligned with the magnetic field, but is also uniformly proportional to it; that is, they solve the constraint

$$\nabla \times B = \alpha B$$  \hfill (2)

for a global inverse length scale \(\alpha\). Such field configurations are monochromatic; all their magnetic energy is concentrated around the spatial frequency \(\alpha\). The converse of Taylor’s conjecture is that relaxation may, in some circumstances, end in a more general force-free equilibrium in which \(\alpha\) could vary from one magnetic field line to another. In such nonlinear equilibria, the highest values of \(\alpha\) are associated with the smallest-scale coherent structures, which may be current layers or flux tubes, and are associated with peaks in the intensity of electrical current flow.

Counterexamples to Taylor’s conjecture do exist, but those identified so far apply to settings in which gas pressure plays a role. For example, hydromagnetic relaxed states with nonuniform \(\alpha\) were reported by Amari & Luciani (2000) and Pontin et al. (2013) where the magnetic field lines terminate on conducting plates, a boundary condition that is motivated by the physics of the solar corona. More general hydromagnetic equilibria have also been found in simulations of stratified environments such as stellar interiors, a setting that has been extensively explored by Braithwaite (2006, 2008, 2009) and Duez et al. (2010). Gruzinov (2009) followed incompressible MHD relaxation of a nonhelical magnetic field in two dimensions\(^1\) and found that it did not decay toward the Taylor minimum (with total annihilation of the field in this case), but instead was halted in an approximate equilibrium with many current layers, beyond which further decay was only made possible by slow resistive evolution.

Our study makes frequent use of the periodic short-wavelength Taylor states as initial conditions. A Taylor state of frequency \(\omega_0\) and helicity \(H\) has an energy \(\omega_0 H/2\), a fraction \(1 - \alpha_1/\omega_0\) of which could be dissipated without changing the total helicity (where \(\alpha_1 = 2 \pi / L\) is the lowest allowed frequency, although we will use \(L = 2 \pi\) so that \(\alpha_1 = 1\)). This implies that their free energy supply can be arbitrarily large and so raises the question of their mechanical stability. Very recently, East et al. (2015) found that in FFE, as well as in relativistic MHD, generic examples of the three-dimensional (3D), periodic \(\omega_0 > 1\) Taylor states are unstable to small, ideal perturbations, with a growth rate that is proportional to the inverse Alfvén time. Upon saturation of the linear instability, decay enters a turbulent stage that lasts until the remaining energy \(\alpha_1 H/2\) resides at the lowest allowed frequency \(\alpha_1\). This behavior bears out the predictions of Frisch et al. (1975), which were based on the prediction that turbulence would generically shift magnetic helicity toward large scales.

Conventionally, this so-called inverse cascade has been thought to operate efficiently only when the field is strongly helical, a belief that has dramatic consequences for largescale dynamo theory (Blackman & Field 2004), as well as the evolution of cosmic magnetic fields since the early universe (Olesen 1997; Son 1999; Banerjee & Jedamzik 2004). However, an efficient inverse cascade was recently observed to occur even when the field was fully nonhelical, in both Newtonian (Brandenburg et al. 2015) and relativistic (Zrake & East 2014) MHD settings. Although the magnetic energy eventually decays toward zero, the relaxation evolves in a self-similar manner, depositing energy in structures larger than the coherence scale \(k\rho^{-1}\), which increases over time until it attains the system size. In this study, we will show that all settings of freely decaying turbulence in FFE, 2D and 3D, helical and nonhelical, exhibit inverse cascading. The 2D and helical case is particularly fast and nearly conservative; as time goes on, magnetic energy is shifted toward ever-increasing scales while suffering a diminishing rate of dissipative losses.

Our paper is organized as follows. We briefly describe the theory of FFE and its invariants in Section 2. There we also discuss the special case of two dimensions and define the additional topological invariants that it imposes. We then outline the numerical scheme that is used to solve the FFE equations in Section 3 and describe our numerical implementation of various diagnostics, such as power spectra, characteristic scales, and the helicity invariants. Our simulation results, including the energy of relaxed magnetic configurations, an analysis of coherent structures, spectral energy distributions, and details of the inverse cascade, are presented in Section 4.

We discuss the implications of these results for astrophysical gamma-ray sources in Section 5 and also point out how our results might aid in the interpretation of 2D (including axisymmetric) calculations. The Appendix contains some details on the numerical convergence of our scheme. Throughout our paper, we use units in which the speed of light \(c = 1\). The domain scale \(L\) is set to \(2 \pi\) so that the smallest spatial frequency is 1, and time is reported in units of the light-crossing time \(L/c\).

### 2. FFE AND ITS INVARIANTS

FFE describes the flow of electromagnetic energy in a charge-supplied medium with vanishing inertia. This approximation is useful for plasma environments in which the energy density of the magnetic field greatly exceeds contributions from the matter. Under such conditions, the flow of electrical current responds rapidly to changes in \(E\) and \(B\) in order to cancel the Lorentz force density \(\rho E + J \times B\), where \(\rho = \nabla \cdot E\) is the net electrical charge per unit volume. FFE thus admits an Ohm’s law that is a function of \(E\) and \(B\) alone (e.g., McKinney 2006a):

$$J = \frac{B}{B^2} (B \cdot \nabla \times B - E \cdot \nabla \times E) + \frac{E \times B}{B^2} \rho, \hfill (3)$$

\(^1\) Gruzinov’s two-dimensional (2D) simulations followed only the in-plane magnetic field. In the rest of this paper, “2D” means that translational symmetry is enforced along the \(z\) axis, but \(B\) need not vanish. This setting is sometimes referred to as 2.5D.
which may be coupled to the Maxwell equations
\[ \partial_t E = - \nabla \times B - J \]
\[ \partial_t B = - \nabla \times E \]
(4)
in order to yield a hyperbolic system of partial differential equations governing the evolution of the six components of the electromagnetic field (Pfeiffer & MacFadyen 2013).

The evolution of ideal MHD systems in general is constrained by three quadratic invariants: the total energy \( U \), the magnetic helicity \( H \), and the cross helicity \( W = \int \mathbf{v} \cdot \mathbf{BD} \) \( dx \)
where \( \mathbf{v} \) is the fluid velocity (Bekenstein 1987). As a limiting case of MHD, FFE shares these invariants, with the exception of the cross helicity, since FFE does not define the fluid velocity in the direction of \( \mathbf{B} \). Most relevant to our study of magnetic relaxation is conservation of magnetic helicity, which is a topological invariant of the magnetic field alone and thus functions in the same way for MHD as it does for FFE. In both MHD and FFE, \( H \) is a robust invariant because it is generally found to be conserved even in the presence of small nonideal effects (see, e.g., Blackman 2014).

2.1. Energy

Since the Lorentz force density vanishes in FFE, no \( E \cdot \mathbf{J} \) work is done on the charge carriers, and the system is formally energy conserving. Nevertheless, time-dependent solutions may develop regions in which the condition \( E < B \) is violated, so no frame exists in which the electric field vanishes. This indicates a breakdown of the ideal force-free assumption, and the Ohm’s law given by Equation (3) must be modified. Although nonideal force-free Ohm’s laws have been proposed in the literature (Gruzinov 2007; Li et al. 2012), it is still commonplace to evolve the ideal magnetic field until such a breakdown occurs and when it does to simply reduce the magnitude of \( E \) to prevent \( E > B \). The physical motivation for this prescription, which we use here, is that energy is being radiated away when charges are accelerated to short out the electric field and restore \( E < B \). The numerical evolution scheme is described in more detail in Section 3.1, and in the Appendix we confirm that it yields numerical convergence of the energy dissipation rate.

2.2. Topology

FFE shares the same magnetic topological invariants as Newtonian and relativistic MHD, the lowest order of which is the total magnetic helicity given by Equation (1). Its invariance can be seen as stemming from the conservation of magnetic flux through a closed-field loop, which is why it is commonly referred to as characterizing the interlocking of the magnetic field. Although the helicity density \( \mathbf{A} \cdot \mathbf{B} \) depends on the choice of gauge, its integral \( \mathcal{H}_v \) over any volume \( V \) bounded by magnetic surfaces is well defined and also invariant under ideal evolutions, such as Faraday’s law, when \( \mathbf{E} \cdot \mathbf{B} = 0 \) (e.g., Brandenburg & Subramanian 2005):
\[ \partial_t \mathcal{H}_v = -2 \int_V \mathbf{E} \cdot \mathbf{B} d^3x. \]
(5)
Equation (5) implies that \( \mathcal{H}_v \) can still be conserved approximately in the presence of nonideal processes, as long as the volume in which they occur is small.

In principle, a domain that admits a partitioning by magnetic surfaces has an independently conserved helicity associated with each smaller volume. But, in practice, 3D field geometries are too complex for such a partitioning to be possible. This is what led Taylor to conclude that relaxation is only constrained by a single topological invariant: the total helicity. The story is different in two dimensions since the magnetic surfaces are far simpler; their cross sections are nested, closed curves in the \( x-y \) plane, and the helicity enclosed by each functions as an independently conserved quantity for as long as that surface retains its identity. But nonideal effects, however small, permit the surfaces to merge with one another, erasing their identities and shuffling up their conserved helicities. Nevertheless, in two dimensions we can still construct a robust topological invariant that is far stricter than the total helicity.

We recall that, in the presence of \( \varepsilon \)-translational symmetry, the in-plane magnetic field is tangent to the isocontours of the magnetic flux function \( A_x \), so each subvolume \( V_i(\psi) \) in which \( A_x > \psi \) is associated with a conserved helicity \( \mathcal{H}_i(\psi) = \mathcal{H}_i(\phi) \). To whatever extent the helicities of such volumes remain additive with respect to reconnections between their bounding surfaces, the sum \( \mathcal{H}(\psi) = \sum_i \mathcal{H}_i(\psi) \) must be a robust topological invariant. This assumption is justified by Equation (5) when the nonideal regions (where \( \mathbf{E} \cdot \mathbf{B} \neq 0 \)) are small. Alternatively, consider the helicity (e.g., Berger & Field 1984)
\[ \mathcal{H}_{12} = 2\Phi_1\Phi_2 \]
(6)
of two flux tubes that are once-linked and independently nonhelical. An example is a cylindrical structure whose external, toroidal flux \( \Phi_2 \) wraps an interior, poloidal flux \( \Phi_1 \). Two such structures that were joined to share an outer surface would then enclose a poloidal flux \( 2\Phi_1 \), which is wrapped by the same toroidal flux \( \Phi_2 \). So, to whatever extent the “joining” is done without destroying magnetic flux, the resulting arrangement has a helicity \( 2\mathcal{H}_{12} \).

The presence of additional invariants \( \mathcal{H}(\psi) \) is expected to place a more restrictive lower bound on the magnetic energy than would the total helicity \( H \) alone. In particular, given that Taylor states of the same helicity but different wavelength will generally span a distinct range of \( \psi \) values, there may be no way for one 2D linear equilibrium to evolve into another of a different wavelength while leaving \( \mathcal{H}(\psi) \) unchanged. With this in mind, we anticipate that fully relaxed 2D states will have an energy that exceeds the Taylor minimum of \( \alpha_0 H/2 \). We will describe the numerical determination of \( \mathcal{H}(\psi) \) in Section 3.3, and in Section 4.3 we confirm that it is conserved by directly measuring it in our 2D simulations.

There is also the question of what should happen to configurations in which \( H = 0 \) but \( \mathcal{H}(\psi) \neq 0 \) since such a configuration could not attain arbitrarily low energy while respecting each of the invariants \( \mathcal{H}(\psi) \). Although behavior like this may be entirely possible, the particular initial conditions used in this study (described in Section 3.2 and summarized in Table 1) generally have values of \( |\mathcal{H}(\psi)| \) that are much smaller than \( 2U_B/\alpha_1 \), so we have not been able to observe it yet. Indeed, we will see in Section 4.2 that our 2D states with zero net helicity still decay toward very small energy.

3. METHODS

We simulate magnetically dominated, relativistic turbulence on a periodic domain of length \( L = 2\pi \) in either two or three
dimensions, using solutions of the ideal FFE equations given by Equations (3) and (4).

### 3.1. Numerical Scheme

We evolve the FFE system using a fourth-order finite difference scheme. We use standard fourth-order difference operators to evaluate all the field gradients and standard fourth-order Runge–Kutta time stepping to advance the solution in time.

The FFE system requires three vector constraints to be maintained: no monopoles, \( \nabla \cdot \mathbf{B} = 0 \), perfect conductivity, \( \mathbf{E} : \mathbf{B} = 0 \), and the existence of a frame in which \( \mathbf{E} \) vanishes, \( \mathbf{E} < \mathbf{B} \). The first two constraints are formally preserved by FFE but can be violated numerically at the level of truncation error. Our scheme maintains the solenoidal constraint using the hyperbolic divergence cleaning scheme proposed by Dedner et al. (2002) and later used in FFE simulations (e.g., Palenzuela et al. 2010). This amounts to supplementing Faraday’s law with a magnetic monopole current \( \mathbf{J}_\text{B} = -\nabla \Psi \), where the scalar field \( \Psi \) evolves according to the damped wave equation \( \partial_t \Psi = -\nabla \cdot \mathbf{B} - \tau^{-1} \Psi \), with \( \tau \) being a nonphysical timescale for quenching the magnetic monopole. Here, \( \mathbf{E} : \mathbf{B} = 0 \) is maintained exactly by disregarding the part of the truncation error that would give rise to a component of \( \mathbf{E} \) in the direction of \( \mathbf{B} \). Numerical noise introduced by finite difference operations can lead to the unphysical growth of modes whose wavelengths are comparable to the numerical grid spacing. Our scheme suppresses these unphysical modes using Kreiss–Oliger dissipation, a form of low-pass filtering. Each of the procedures just mentioned supplements the FFE equations with terms at or below the level of the truncation error, so they do not modify the formal convergence order of our numerical scheme.

### 3.2. Initial Conditions

We start our simulations with a monochromatic magnetic field, where all of the power is at a single wavenumber magnitude \( \alpha_0 \), and with a vanishing electric field. The general expression for our initial conditions is

\[
\mathbf{B}(\mathbf{x}) = \sum_{|\mathbf{k}| = \alpha_0} (\alpha_0 \Psi_\mathbf{k} + e^{i\mathbf{k} \cdot \mathbf{x}}) e^{i\mathbf{k} \cdot \mathbf{x}}
\]

\[
\mathbf{k} \cdot \dot{\Psi}_\mathbf{k} = 0
\]

\[
\dot{\Psi}_\mathbf{k} = \dot{\Psi}_\mathbf{k}^* - \mathbf{k}
\]

(7)

where the parameter \( \epsilon \) is chosen to be either one or zero, corresponding to helical or nonhelical configurations, respectively. Helical initial configurations where \( \alpha_0 > 1 \) are unstable equilibria (see Section 4.1), whereas the nonhelical configurations are out of equilibrium. Some of our initial conditions are randomized, having \( \Psi_\mathbf{k} = \hat{\mathbf{e}}_k e^{i\phi_0} \) where \( \hat{\mathbf{e}}_k \) is a random unit vector in the plane orthogonal to \( \mathbf{k} \), and \( \phi_0 \) is a random phase.

We also make use of a special case of Equation (7) known as the “ABC” solution (Arnold 1965; Dombre et al. 1986). In general this is given by

\[
\mathbf{B}^{\text{ABC}}(\mathbf{x}) = \begin{pmatrix} B_3 \cos \alpha_0 z - B_2 \sin \alpha_0 y \\ B_1 \cos \alpha_0 x \end{pmatrix}
\]

\[
\begin{pmatrix} B_1 \cos \alpha_0 y & -B_3 \sin \alpha_0 x \\ B_2 \cos \alpha_0 y & B_1 \sin \alpha_0 x \end{pmatrix}
\]

(8)

which is highly ordered and fully helical, meaning that \( \mathbf{B} = \alpha_0 \mathbf{A} \) (in the Coulomb gauge). In this study we will make frequent use of the case with \( B_1 = B_2 = 1 \) and \( B_3 = 0 \), which we refer to as the 2D ABC configuration.

Our results are based on simulations having a range of initial frequencies \( \alpha_0 \) and numerical resolutions, which we will refer to by the number of grid points in each linear dimension \( N \). In general, the quality of our results improves when we are able to simulate larger values of \( \alpha_0 \) with more separation between the initial length scale and the domain length scale. However, features (of size \( \alpha_0^{-1} \)) in our initial condition need to be resolved by a certain number of grid points in order to obtain robust solutions. In the Appendix we show that 32 cells per \( \alpha_0^{-1} \) are sufficient to keep the error in the global helicity conservation smaller than 1%. In 2D, we will present simulations with \( \alpha_0 \) as large as 256 with resolutions up to 16384\(^2\). In 3D, we will present simulations with \( \alpha_0 \) as large as 48 and resolution 1024\(^3\).
3.3. Diagnostics

We define the power spectral density of the electric, magnetic, and helicity fields, respectively, as

\[ P_E(k_i) = \frac{1}{\Delta k_i} \sum_{k_i<|q|<k_i+\Delta k_i} E_q \cdot E_q^*/2, \]

\[ P_B(k_i) = \frac{1}{\Delta k_i} \sum_{k_i<|q|<k_i+\Delta k_i} B_q \cdot B_q^*/2, \]

\[ P_H(k_i) = \frac{1}{\Delta k_i} \sum_{k_i<|q|<k_i+\Delta k_i} A_q \cdot B_q^* \]

where \( E_q, B_q, \) and \( A_q \) are, respectively, the electric field, the magnetic field, and the vector potential Fourier harmonics of wavenumber \( q \). We normalize the Fourier harmonics so that the volume-integrated electric and magnetic field energies \( U_E \) and \( U_B \) and the magnetic helicity \( H \) are given by

\[ U_E = \sum_i P_E(k_i) \Delta k_i, \]

\[ U_B = \sum_i P_B(k_i) \Delta k_i, \]

\[ H = \sum_i P_H(k_i) \Delta k_i. \]

We also define the characteristic frequency of each field \( k_E, k_B, \) and \( k_H \) as

\[ k_X = \frac{\sum_i P_X(k_i) k_i \Delta k_i}{\sum_i P_X(k_i) \Delta k_i} \]

where \( X = E, B, \) or \( H \). The most probable wavenumber, where \( P_X(k) \) is maximal, is denoted by \( k_X \).

In two dimensions, we track the “helicity mass” function discussed in Section 2.2:

\[ \mathcal{H}(\psi) = \int \Theta(A_z(x) - \psi) A \cdot B d^3 x, \]

where \( \Theta \) is the Heaviside step function. In practice, this diagnostic is more easily computed as the “helicity density” function \( d\mathcal{H}/d\psi \), which we calculate by binning the lattice points according to their value of \( A_z \) and assigning the weight \( A \cdot B \). We also create the volume distribution \( dV/d\psi \) by binning points according to \( A_z \) with uniform weights and the helicity distribution over volume \( d\mathcal{H}/dV = \frac{d\mathcal{H}}{d\psi} \frac{d\psi}{dV} \).

4. RESULTS

Figure 1 shows the evolution of both 2D and 3D freely decaying, force-free magnetic turbulence. Both of these calculations are initiated in the 2D ABC state, but the one on top takes place in a 2D domain where translational symmetry is assumed in the \( z \) direction, and the bottom one was given a low-level white-noise perturbation to break the \( z \) symmetry. The left-most image shows the solution shortly after saturation of the linear instability that was recently observed in East et al. (2015), an overview of which is provided in Section 4.1. The difference between the two runs is visually evident. While the 3D solution becomes increasingly smooth at late times, the 2D one maintains a network of abrupt field reversals. These structures are force-free rotational current layers and are examined in depth in Section 4.4. As we will see in Section 4.2,
the total magnetic energy dissipated is dramatically greater in the 3D case than in the 2D case. Both runs show evidence of the inverse cascade; large-scale coherency of the magnetic field must result from dynamical transfer of some magnetic energy toward longer wavelengths since the initial spectrum is monochromatic around \( k = \alpha_0 \). The inverse cascade will be examined in detail in Section 4.6.

4.1. Linear Instability of the Excited Taylor States

Our helical initial conditions are linear force-free equilibria. Clearly they are stable when \( \alpha_0 = 1 \) since such states are global energy minima for a given magnetic helicity. The question of the ideal stability of the periodic shorter-wavelength (\( \alpha_0 > 1 \)) Taylor states has a conflicted history (e.g., Moffatt 1986; Galloway & Frisch 1987; Er-Riani et al. 2014), which has recently been resolved in East et al. (2015). In that study, numerical simulations of both FFE and relativistic MHD revealed that \( \alpha_0 > 1 \) Taylor states are linearly unstable to ideal perturbations. The instability is marked by exponential growth of the electric field on roughly the Alfvén wave crossing time of the initial structure size \( \alpha_0^{-1} \) and saturates when the medium attains the Alfvén speed, which for the magnetically dominated case is \( c \), implying the existence of regions where \( E \approx B \). This instability affects generic states, and the only counterexamples that were found were one-dimensional ABC states (e.g., \( B_1 = 1, B_2 = 0, B_3 = 0 \)) that are stable for all values of \( \alpha_0 \). Such states are pure plane waves having circular polarization and are force-free by virtue of having uniform magnetic pressure. All of our helical initial conditions are short wavelength and either 2D or 3D, and turbulent relaxation begins after the saturation of the ideal instability.

4.2. Energy of Fully Relaxed Configurations

Here we discuss the magnetic energy associated with the end-state magnetic configurations. Since the Taylor states have \( B = \alpha_0 A \), their energy is given simply by \( \alpha_0 H/2 \). In other words, their energy is \( \alpha_0 \) times larger than the theoretical lower limit imposed by assuming the state reaches \( \alpha = 1 \) at constant \( H \). Whether or not dynamical relaxation processes settle with the field in this global energy minimum remains an open question, but here we provide some evidence to support the following conjecture: “Force-free magnetic relaxation starting from periodic Taylor states ends in the lowest-energy configuration allowed by helicity conservation if and only if the domain is 3D.”

We have carried out a suite of calculations belonging to one of four categories, being either 2D or 3D and either helical or nonhelical. Those that are nonhelical are, by construction, out of equilibrium at \( t = 0 \) and could decay until they reach zero energy because helicity conservation does not place any lower bound on their magnetic energy. The helical ones are initially at an unstable equilibrium, enter a period of turbulent relaxation, and settle in a force-free equilibrium of lower energy. We considered initial conditions that are both of the randomized type given by Equation (7) and the ABC type given by Equation (8).

Our results are summarized in Figure 2, which shows the time series \( U_B \) for each of six different runs. As expected, the nonhelical configurations decay toward zero energy in both 2D and 3D settings. In three dimensions, the helical configurations all terminate in the lowest-energy state allowed by helicity conservation.\(^2\) Both a randomized setup with \( \alpha_0^2 = 257 \) and the 2D ABC setup with \( \alpha_0^2 = 256 \) showed the same general behavior. The latter setup was intentionally chosen to be identical with the 2D setup apart from the inclusion of a low-level (one part in \( 10^7 \)) white-noise perturbation introduced to break the \( z \)-translational symmetry.

Both of the helical 2D runs terminate their relaxation with an energy that is decreased to only 30% of its initial value. For example, a randomized 2D initial condition with \( \alpha_0 \approx 256 \) settles in a state that has roughly 77 times more magnetic energy than the Taylor minimum-energy state. This is not unique because actually all of our helical 2D runs where \( \alpha_0 \gg 1 \) (including \( \alpha_0 = 16 \)) settle in a state whose energy is decreased to roughly 30%. In the Appendix we show that, for the 2D ABC \( \alpha_0 = 256 \) setup, the final energy is numerically converging to a value very near 30%. This means that the terminal states in two dimensions are not linear equilibria. In other words, they do approximately solve the force-free condition of Equation (2), but the value of \( \alpha \) may vary from one field line to another.

4.3. Helicity Distribution

The fact that the 2D configurations do not relax into linear equilibria stems from invariance of the helicity distribution \( H(\psi) \) that we introduced in Section 2.2. Figure 3 confirms that it does indeed remain constant over time to a very good approximation, even while the volume distribution over the magnetic potential \( dV/d\psi \) changes significantly. The feature evident in the bottom panel of Figure 3, where in the relaxed state most of the volume is occupied by the extreme values of the magnetic potential, can be connected to the formation of current layers, which we discuss in the next section.

\(^2\) The state of lowest allowed energy can be referred to (e.g., Shats et al. 2005) as the spectral condensate.
As an illustration that invariance of the helicity distribution might prevent 2D relaxation from attaining longer-wavelength linear equilibria, consider that the magnetic potential function $A_z$ of any 2D linear equilibrium satisfies
\[
A_z = \left( \frac{H}{2 \alpha \Omega L^3} \right)^{1/2} \frac{B_z}{\bar{B}_z}
\]
where $\bar{B}_z$ is the root mean square value of $B_z$, and $H$ is the total helicity. Note that $H(\psi)$ can be characterized by its domain, namely the global maximum of $|A_z|$, which we denote by $\psi_{max}$. For two different linear equilibria with frequencies $\alpha_0$ and $\alpha_0'$ to have the same helicity distribution, it would be necessary for them to at least share the same values of $H$ and $\psi_{max}$, requiring that
\[
\left( \frac{\alpha_0}{\alpha_0'} \right)^{1/2} = \frac{B_{z,\text{max}}/\bar{B}_z}{B'_{z,\text{max}}/\bar{B}'_z}, \quad (12)
\]
where $B_{z,\text{max}}$ is the global maximum of $|B_z|$. Note that, in general, $\bar{B}_z \leq B_{z,\text{max}}$. For the particular case of the 2D ABC state in which $B_1 = B_2 = 1$, Equation (12) leads to the requirement that $\alpha_0' \geq \alpha_0/2$, and therefore its relaxation into a linear state with two times smaller energy is impossible. In other words, there is no way for the 2D ABC state represented in Figure 3, whose $\alpha_0 = 32$, to evolve into another linear equilibrium whose frequency is smaller than 16 while preserving $H(\psi)$. We suspect this argument could be generalized further, but for now we leave it as a conjecture that the wavelength of a linear 2D equilibrium may be uniquely specified by its helicity distribution, which if true would render it impossible for one linear 2D equilibrium to relax into another of lower energy.

4.4. The Current Layers

During the turbulent relaxation of helical 2D configurations, the solution consists of opposingly signed flux domains separated from one another by a network of rotational force-free current layers. The flux domains (black and white regions of Figure 1) have nearly uniform $B_z$, and are thus relatively current-free. Across the current layers, the magnetic field direction rotates through approximately 180°, while its magnitude (and thus magnetic pressure) remains fixed. One current layer is shown in Figure 4, where a one-dimensional profile has been taken along the $x$ axis, passing through the layer where it is aligned with the $y$ axis.

It is evident that the current layers have a characteristic frequency, which we denote by $\alpha_c$, and determine empirically as follows. Since the solution is near a force-free equilibrium, the current is $\mathbf{J} \approx \alpha \mathbf{B}$ for some spatially dependent frequency $\mathbf{B} = \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{B}/\mathbf{B}^2$.

We anticipate that the probability density function $P(\alpha)$ will have two local maxima: one at $\alpha = 0$, corresponding to the potential flux domain interiors, and the other at the frequency $\alpha_c$, marking the frequency of the current layers. Figure 5 confirms this to be the case. It shows $P(\alpha)$ at logarithmically spaced times throughout the relaxation and reveals the location of the second peak once the solution is sufficiently close to a force-free equilibrium. For this particular run, with $\alpha_0 = 256$ and $N = 3072$, the value of $\alpha_c \approx 128$.

It turns out to be a coincidence that in this case $\alpha_c \approx \alpha_0/2$. We have performed a family of calculations, varying $\alpha_0$ between 8 and 128 and varying $N$ between 128 and 8192. For each run, we recorded the value of $\alpha_c$, time-averaged over roughly 100 snapshots between time $t = 10$ and $t = 16$, which was late enough that the second peak in $P(\alpha)$ had emerged in each run. There was no secular evolution of $\alpha_c$. Figure 6 reveals that current layers become increasingly narrow with higher resolution but also with increasing initial frequency $\alpha_0$. The scaling is consistent with the expression
\[
\alpha_c = k_1^{3/4} \alpha_0^{1/4} \quad \text{(13)}
\]
where $k_1 = N/30$ is the turbulence cutoff frequency, which has been determined by modeling the magnetic energy spectrum (see Figure 7) and is insensitive to initial conditions, depending only on the numerical scheme and grid resolution. We emphasize that Equation (13) is strictly empirical, in that it matches the data shown in Figure 6.

The scaling given by Equation (13) indicates that, in the infinite resolution limit, the current layers will have zero characteristic length. We are not able to say whether the scaling could be associated with a physical property of 2D FFE at a finite Reynolds number or if it might depend on the details of the numerical scheme. This question could be resolved by imposing the turbulence cutoff frequency $k_1$ explicitly and then varying the numerical resolution. In other words, solutions to a resistive FFE system at a given conductivity parameter will be necessary.

Figure 3. Top: the helicity distribution $d\mathcal{H} = d\psi$ vs. the level surface value of the magnetic potential function, shown at 16 evenly spaced times up to $t = 16$ in a 2D ABC run with $\alpha_0 = 32$ and resolution 4096$^2$. As anticipated in Section 2.2, $d\mathcal{H} = d\psi$ is essentially constant in time. Middle: the volume distribution $d\psi = d\phi$, indicating the volume between level surfaces at a given $\psi$. The separatrix surfaces $\phi = 0$ initially occupy the greatest volume, become the current layers, and end up with the smallest share of the volume. Bottom: the helicity per volume $d\mathcal{H} = d\psi$. Lighter, wider curves indicate earlier times, whereas darker, thinner curves indicate later times. The distributions are each normalized to unity.
4.5. Solitary Magnetic Bubbles

The flux domain interiors contain another type of coherent structure. They are long-lived bubbles of toroidal magnetic field that are confined by ambient magnetic pressure. The circular object to the left of the current layer in Figure 4 is an example. They may be referred to as flux tubes or magnetic vortices, but we will refer to them as magnetic bubbles because we believe they are objects similar to those studied by Gruzinov (2010). The bubbles are helical, force-free magnetic field structures, having \( \mathbf{J} \) very well aligned with \( \mathbf{B} \). But they are not linear equilibria: the value of \( \alpha \) varies with distance \( r \) from the axis, and all the bubbles we examined had similar radial profiles \( \alpha(r) \). The current flow is parallel to the background magnetic field near the core, but an equal and opposite return current flows through the sheath, so they could also be thought of as coaxial electric current channels oriented along the symmetry axis.

![Image of Electric current density and Magnetic field](image)

**Figure 4.** Amplification of the small red rectangle overlying the right-most panel of Figure 1, showing relief plots of the \( x, y, \) and \( z \) components of the magnetic field (on the top three panels). The bottom panel shows one-dimensional profiles taken along the horizontal centerline (dashed magenta).

![Image of Probability density function](image)

**Figure 5.** Probability density function \( P(\alpha) \) at logarithmically spaced times in 2D, helical FFE turbulence, where \( \alpha = \mathbf{B} \cdot \nabla \times \mathbf{B}/B^2 \). The right-most vertical dashed line is the frequency \( \alpha_0 = 256 \) of the initial field configuration. The local maximum at \( \alpha \approx 128 \) is the frequency \( \alpha_c \) of the current sheets (which is resolution dependent), and the maximum at \( \alpha = 0 \) corresponds to the zero current characterizing the flux domain interiors.

![Image of Dependence of the current layer frequency \( \alpha_c \) on the frequency \( \alpha_0 \) of the initial field configuration](image)

**Figure 6.** Dependence of the current layer frequency \( \alpha_c \) on the frequency \( \alpha_0 \) of the initial field configuration (top) and the grid resolution \( N \) (bottom) in 2D, helical, freely decaying FFE turbulence. Here, \( \alpha_c \) is defined to be the second local maximum (other than \( \alpha = 0 \)) in the probability density function \( P(\alpha) \), where \( \alpha = \mathbf{B} \cdot \nabla \times \mathbf{B}/B^2 \).
Force-free configurations with axial and $z$ symmetry satisfy the ordinary differential equations
\[
\begin{align*}
B_{\phi}(r) + B_{\phi}'(r) &= 0 \\
B_{\phi}(r) - B_{z}(r) - B_{\phi}'(r) &= 0
\end{align*}
\] (14)
where $\alpha(r)$ needs to be specified to make the solution unique, as well as the boundary conditions $B_{\phi}(r) \rightarrow B_\infty$ and $B_{\phi}(r) \rightarrow 0$ as $r \rightarrow \infty$. Requiring the total current $I = \int B_{\phi} r d\phi$ to be zero only forces $B_{\phi}(r)$ to decrease faster than $1/r$. Alternatively, the magnetic pressure $u_B(r)$ may be specified instead of $\alpha(r)$. One simple assumption, consistent with measurements of the bubbles, is that the magnetic pressure enhancement, relative to the ambient pressure $B_\infty^2/2$, is Gaussian. This gives a pressure profile
\[
u_B(\tilde{r}) = \frac{1}{2} B_\infty^2 (1 + f e^{-\tilde{r}^2/2})
\] (15)
in the dimensionless radius $\tilde{r} = \alpha_B^{-1} r$ for a bubble of radius $\alpha_B^{-1}$, where $f$ is the magnetic pressure enhancement relative to the background and could be any positive number. The corresponding solution of Equation (14) is
\[
\begin{align*}
B_{\phi}(\tilde{r}) &= B_\infty f^{1/2} \tilde{r} e^{-\tilde{r}^2/2} \\
B_z(\tilde{r}) &= B_\infty \sqrt{1 + f e^{-\tilde{r}^2/2}} (1 - \tilde{r}^2)
\end{align*}
\]
which has the profile
\[
\alpha(\tilde{r}) = \alpha_B \frac{(2 - \tilde{r}^2)e^{-\tilde{r}^2/2}}{\sqrt{(1 - \tilde{r}^2)e^{-\tilde{r}^2/2} + f^{-1}}}
\] (16)

In Figure 8 we show the radial profile of magnetic pressure, azimuthally averaged around the bubble’s axis. This is the same object depicted to the left of the current layer in Figure 4. We also show the expression given in Equation (15) with its best-fit model parameters. For this particular object, the best-fit fractional magnetic energy enhancement was $f = 0.41$, and the best-fit inverse radius was $\alpha_B = 162$. The lower panel of Figure 8 shows the radial profile of $\alpha$ and also the model-predicted value given by Equation (16) with the best-fit parameters.

So far we have not detected these structures in 3D simulations, but that does not mean they never happen. It is crucial to address whether they are even stable in three dimensions, and if they are, then under what conditions they may be attractors.

4.6. The Inverse Cascade

The inverse cascade can be characterized by the rate with which the magnetic coherence scale $k_B^{-1}$ (Equation (10)) migrates toward longer wavelengths. We observe inverse cascading in FFE for every setting we have considered, whether the turbulence is 2D or 3D, helical or nonhelical. Figure 9 shows the evolution of the magnetic energy spectrum $P_B(k)$ over time, for representative helical and nonhelical runs in 2D. We note how, in both cases, the remaining magnetic energy resides at an increasing scale over time. We also note that the spectral energy density at long wavelengths is an increasing function of time, indicating that selective decay of the short-wavelength modes alone cannot explain growth of the spatial coherency. This further generalizes the observations made of nonhelical 3D MHD turbulence in both Newtonian (Brandenburg et al. 2015) and relativistic (Zrake 2014) settings.

Figure 10 shows the evolution of the characteristic magnetic frequency $k_B$ for one model from each of the four categories. For the helical runs we also show the evolution of the helicity frequency $k_H$. Except for the general feature that, over time, both $k_B$ and $k_H$ move to smaller frequency, there is no single

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.pdf}
\caption{Example best fit (shown in red) to the magnetic power spectrum data (+s) for 2D, helical FFE turbulence with $\alpha_0 = 256$ and resolution 16384$^2$. Here, $P_B(k)$ is shown compensated for by the best-fit spectral index $s = 1.94$ (top) and uncompensated (bottom). Vertical dashed lines indicate the window used for the fit, between the peak magnetic frequency $k_B$ and the spectral cutoff frequency $k_3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.pdf}
\caption{Top: the magnetic pressure profile of a magnetic bubble (appearing in 2D FFE turbulence) in the dimensionless cylindrical radius $\tilde{r} = \alpha_B r$. The blue-shaded region indicates the azimuthal standard deviation at each radius from the center, and the dashed line shows the best-fit model parameters for the Gaussian magnetic pressure enhancement given by Equation (15). Bottom: the azimuthally averaged value of $\alpha$ along with its predicted value (dashed line) given by Equation (16). The top and bottom insets show 2D relief plots of $u_B$ and $\alpha$, respectively.}
\end{figure}
In general, freely decaying magnetic turbulence must exhibit an inverse cascade when the magnetic helicity is near maximal \((H \approx U_B/k_B)\) if \(\partial_t H = 0\) is to be satisfied (Frisch et al. 1975; Christensson et al. 2001; Cho 2011). However, the same conclusion cannot be drawn from magnetic helicity conservation alone when \(H \ll U_B/k_B\), as is the case for our nonhelical runs. Observation of the nonhelical inverse cascade was thus seen as a surprise (Zrake 2014; Brandenburg et al. 2015). It was suggested in Zrake (2014) that the nonhelical inverse cascade may reflect the tendency for aligned current structures to attract one another. Brandenburg et al. (2015) found that the net transfer of energy from small to large scales was about twice larger in helical than in nonhelical, freely decaying, Newtonian MHD turbulence.

### 4.7. Power Spectrum of Electric and Magnetic Energy

In all of our initial conditions, the magnetic energy is concentrated around a single frequency, \(P_B(k) \propto \delta(k - \alpha_0)\). As turbulence develops, the energy redistributes itself over all available scales, with the bulk of the energy around \(k_B\) (which decreases over time, as discussed in Section 4.6) and a power-law tail extending to the spectral cutoff frequency \(k_1\), which lies consistently near \(N/30\) when the grid resolution is \(N\). We have determined the index \(s\) of the power-law tail by fitting the logarithm of the spectral distributions \(P_B(k)\) and \(P_f(k)\) to the model function

\[
f(x) = Ae^{-(x-x_0)^2/2\sigma^2} + xs \tag{17}
\]

where \(x = \log k\), and \(A, x_0, \sigma, s\) are model parameters. We have found it expedient not to try to model the high-frequency cutoffs; we just use frequency bins between the peak frequency \(k_B\) and the cutoff \(k_1\) to obtain our fit. A representative spectral fit is shown in Figure 7.

Figure 11 shows \(P_B(k)\) and \(P_f(k)\) for one model from each of the categories 2D/3D and helical/nonhelical along with the best-fit model given by Equation (17). We use randomized initial conditions for each model, and the resolution is 16384\(^2\) in 2D and 1024\(^3\) in 3D (coinciding dash-dotted curves are taken from simulations whose resolution is 12288\(^2\) and 768\(^3\) to indicate numerical consistency). The spectra represent the solution at an intermediate time, when \(k_B \approx 8\), and we find the power-law indices to be quite stable\(^3\) during a window of time beginning on the snapshot chosen for each model. The power-law index of the electric field spectral energy distribution \(P_E(k)\) is between 0.96 and 1.18 for the various models, with both of the helical models having an index of 1.18. The power-law index of the magnetic field spectral energy distribution \(P_B(k)\) is different in two and three dimensions. In both helical and nonhelical 3D settings, it is notably close to the Kolmogorov value of 5/3. Note that Zrake (2014) and Brandenburg et al. (2015) both measured an index near 2 for freely decaying nonhelical MHD turbulence. The helical 2D model has an index of 1.93, and it is tempting to conclude that the true value is 2, but actually the best-fit parameter for all resolutions up to 16384\(^2\) is significantly smaller than 2, always being between 1.92 and 1.96. The nonhelical 2D model is found to have an index of 1.41.

\(^3\) Between \(t = 0.6\) and \(t = 1.2\), the spectral indices show negligible secular evolution and a standard deviation with respect to different time levels that is below the level of 1%. The spectral indices reported are the instantaneous values, which are within a standard deviation of the mean.
Using high-resolution simulations of the FFE equations, we have studied freely decaying, magnetically dominated relativistic turbulence for the first time. We focused on various differences in the magnetic relaxation process in settings where the domain is 2D and 3D, and where the magnetic field is helical and nonhelical. We found that helical, 2D relaxation terminates in a state whose energy is far above the theoretical minimum imposed by helicity conservation and whose volume is predominantly current-free but is punctuated by coherent structures, namely, current layers and solitary magnetic bubbles. We tried to determine what sets the width of the current layers, and we determined that it depends not only on the turbulence cutoff frequency $\alpha_0$ (or grid resolution), but also on the frequency $\alpha_0$ of the initial condition. The solitary magnetic bubbles are axisymmetric, nonlinear force-free equilibria and are consistent with Gaussian magnetic pressure enhancement relative to the surrounding relatively current-free volume. The 3D stability of these structures remains an open question.

The unusual behavior of 2D relaxations can be understood in terms of additional topological constraints that are imposed by the extra symmetry. We proposed that, unlike generic 3D relaxation, whose topology is only constrained by the total helicity invariant $H$, 2D relaxations are subject to a whole spectrum of helicity invariants $\mathcal{H}(\psi)$, one associated with each value of the magnetic potential $\psi$. Although this invariance is only guaranteed when magnetic reconnections are restricted to regions of zero volume, our simulation data showed that $\mathcal{H}(\psi)$ remains unchanged throughout the evolution to a very good approximation.

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All of the settings we considered exhibited inverse cascading, in which some magnetic energy is redistributed toward progressively longer wavelengths. The rate of inverse cascading was characterized by the time evolution of the magnetic frequency $k_B$, which was found to decrease faster when the field is helical than when nonhelical, and also faster in 3D than in 2D. The inverse cascade of 2D helical turbulence is nearly conservative; merging of magnetic flux domains moves...
energy to larger scales while suffering a diminishing rate of dissipative losses. The nonhelical inverse cascade has $k_B \propto r^{-0.38}$ in 2D and $k_B \propto r^{-0.48}$ in 3D, in rough agreement with the results of Zrake (2014) for 3D turbulent relaxation in relativistic MHD. Better scale separation (larger $\alpha_0$) and thus higher numerical resolution are needed to confirm the 3D scaling measurement.

5.1. Astrophysical Gamma-Ray Sources and Magnetoluminescence

We have examined turbulent relaxation in FFE with the motivation of elucidating the physical mechanism behind the extremely fast time variability that is characteristic of astrophysical gamma-ray emitters, including the Crab Nebula, many blazars, and nearly all gamma-ray bursts. The extreme energetics and temporal intermittency of gamma radiation from these sources require a mechanism in which plasma promptly converts the majority of its magnetic energy into high-energy particles and radiation. Furthermore, the emitting regions are thought to be strongly magnetized and are known to lie a great distance from the primary mover (pulsar, progenitor star, or black hole). These facts are highly suggestive that electromagnetic outflows may contain persistent magnetic structures with copious free energy supplies, whose spontaneous disruption could be linked to the observed flaring events. A scenario like this was referred to in Blandford et al. (2014) as magnetoluminescence. For it to be plausible, it is necessary (1) that metastable, force-free (or hydromagnetic) equilibrium can exist far from any supporting boundaries, (2) that such objects can form under realistic astrophysical conditions, and (3) that upon their disruption, magnetic energy is promptly and completely dissipated.

In this paper, we have begun to address points (1) and (3) and found results that are at least partially encouraging. All of our periodic 3D simulations exhibit prompt relaxation into the Taylor minimum energy state, supporting the idea that magnetic energy can be dissipated completely in a light-travel time. But the same result suggests that persistent, metastable structures are not a generic outcome of turbulence in FFE. This is not to say that such behavior is impossible, as we have only considered a small class of initial conditions. In fact, Smiet et al. (2015) very recently identified magnetic arrangements that, in 3D periodic domains, in full MHD, relax to nonlinear hydromagnetic equilibria. So (1) is possible, at least in the hydromagnetic case. The volatility of such objects and the generality of conditions under which they may arise remain important questions for the future.

5.2. Comparison with Other Studies of Magnetic Relaxation

In this study we have begun to address the question of whether relativistic, force-free magnetic relaxation in periodic domains generically ends in a Taylor state, and we found evidence to support the view that it does, provided the domain is 3D. But thus far we have only considered a restricted class of initial conditions, namely, isotropic, monochromatic fields that are either linear force-free equilibria (helical) or completely nonhelical.

Several studies in full MHD have now identified settings that relax to more general hydromagnetic equilibria for which $\mathbf{J} \times \mathbf{B} = \nabla p$, where the Lorentz force density balances the gradient of gas pressure $p$. Examples include stratified 3D environments (Braithwaite 2006, 2008) where the field is helical and 2D periodic settings where the field is incompressible and nonhelical (Gruzinov 2009). Amari & Luciani (2000) and Braithwaite (2015) have both provided examples of 3D hydromagnetic relaxation that first develop current layers and then proceed to a smooth configuration via resistive processes. Both studies used boundary conditions where at least one of the directions was not periodic. Very recently, Smiet et al. (2015) has found instances of 3D hydromagnetic relaxation ending in smooth, nonlinear equilibria with nonuniform pressure, even when the boundary is periodic or open. Such boundary conditions are highly relevant for the astrophysical processes mentioned in Section 5.1, and an important question is whether their results possess any force-free analogs. In particular, if stable and nonlinear force-free equilibria do exist in three dimensions away from boundaries, then how likely are they to arise under realistic astrophysical conditions?

5.3. Effects of Imposing Extra Symmetries in Astrophysical Simulations

Many studies of relativistic plasma and MHD processes are carried out assuming either translational or rotational symmetry, including simulations of FFE (McKinney 2006b; Tchekhovskoy et al. 2008), relativistic MHD (Barkov & Komissarov 2008; Komissarov et al. 2009; Komissarov & Barkov 2009; Mizuno et al. 2011), and particle-in-cell (PIC) simulations (Spitkovsky 2008; Keshet et al. 2009). As we have seen, 2D magnetic relaxations are far more topologically constrained and persist in configurations of much higher energy than equivalent 3D relaxations. Furthermore, axisymmetric calculations are expected to exhibit artificialities similar to the slab-symmetric ones studied here because they share the same topological simplifications. In particular, the axisymmetric magnetic surfaces, now toroidal shells that are labeled by their value of the azimuthal vector potential component $A_\phi$, each enclose a conserved magnetic helicity.

Our results indicate that as a field’s complexity increases, so does the discrepancy between the energy of its most relaxed state in 2D and 3D. So, axisymmetry may be appropriate when the field is near a stable force-free equilibrium, when little or no energy resides in higher-order radial or angular modes. But when these modes are populated, the imposed symmetry could make them artificially persistent.

Numerical studies of pulsar magnetospheres and magnetar flares are quite challenging, and much of the progress in this field has been obtained by restricting them to axisymmetry. Simulations using both FFE (Parfrey et al. 2012; Yu 2012; Parfrey et al. 2013) and PIC (Cerutti et al. 2015) show the development of a complex structure in the meridional plane. Similarly, a short-wavelength toroidal magnetic structure has been found in axisymmetric FFE simulations of black hole accretion. For example, the results of Parfrey et al. (2014) suggest that even disordered (as opposed to large-scale) magnetic fields advected inward by an accretion disk could facilitate angular momentum extraction from a black hole. The differences in 2D versus 3D found here suggest that extending these studies to be fully 3D, as that becomes computationally feasible, will be an exciting frontier and will likely reveal qualitatively new dynamics.

Another setting where 2D and 3D calculations are expected to differ is shock-generated turbulence, which may account for the relatively high magnetizations inferred from nonthermal
emission spectra of astrophysical shock fronts—both in nonrelativistic settings such as supernova remnants and in relativistic settings such as gamma-ray burst afterglows. Amplification of the magnetic field by turbulence in and around the shock and its subsequent decay in the postshock flow have been studied extensively in both two (Mizuno et al. 2011) and three (Inoue et al. 2011) dimensions with relativistic MHD and also in 2D and 3D PIC simulations (Sironi & Spitkovsky 2009; Sironi et al. 2013). In this study we have observed a power-law decay of the magnetic energy in all settings, but with a steeper index in 3D than in 2D. This should be kept in mind because even minor differences in the decay law can have an impact on the efficiency of first-order Fermi processes (Lemoine et al. 2006; Niemiec & Ostrowski 2006; Niemiec et al. 2006; Pelletier et al. 2009), as well as on interpretations of gamma-ray burst afterglows (Gruzinov & Waxman 1999; Rossi & Rees 2003; Lemoine 2014).

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**APPENDIX**

**NUMERICAL CONVERGENCE**

Here we demonstrate some numerical convergence properties of our scheme. We have chosen the conservation of magnetic helicity $\Delta H(t)$ and the time series of magnetic energy $U_B(t)$ as diagnostics. Convergence properties are reported for the 2D helical runs with $\alpha_2 = 256$ and grid resolutions of $4096^2$, $6144^2$, $8192^2$, $12288^2$, and $16384^2$. This configuration was found to be representative of convergence properties in other settings reported in this paper. All runs were evolved for at least two light-crossing times. In order to establish the numerical convergence order $n$, we model the error of the numerical solution $y_h$ with grid spacing $h$ as $y_h = y_0 + Eh^n$, where $E$ is a constant and $y_0$ is the extrapolated solution. This is similar to a Richardson extrapolation, but instead of fitting for the coefficient $E_n$ for each integer power of $h$, we have fit for the single error coefficient $E$ and convergence order $n$.

The upper left panel of Figure 12 shows the fractional change in magnetic helicity $H(t)/H_0 - 1$ as a function of time for each resolution. For resolutions $\geq4096^2$, the helicity change is never worse than $\pm10\%$. For $>8192^2$ it is never worse than $\pm1\%$, and for $>12288^2$ it is never worse than $\pm0.1\%$. The extrapolated value of the helicity change consistently has a gain of about 0.1%, and the convergence order (shown on the lower left panel of Figure 12) is between 2.8 and 2.9. The right panel shows the evolution of magnetic energy $U_B(t)$ at each resolution. Less dissipation occurs for each higher resolution, but the sequence converges consistently at first order. The extrapolated value of the magnetic energy at $t = 2$ is 0.30, and it changes by less than 1% between $t = 1$ and $t = 2$.

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