Translation Covers Among Triangular Billiards Surfaces

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Abstract

We identify all translation covers among triangular billiards surfaces. Our main tools are the J-invariant of Kenyon and Smillie and a property of triangular billiards surfaces, which we call fingerprint type, that is invariant under balanced translation covers.
1 Introduction

An unfolding construction, already described in [3] and furthered in [7], associates a translation surface to each rational-angled triangle; we call such a surface a *triangular billiards surface*. Translation covers of such surfaces have been used by Hubert and Schmidt [6], among others, to gain information about the affine symmetry groups of the surfaces. We determine all translation covers among triangular billiards surfaces. It is well known (see Subsection 4.1) that a flat torus admits translation covers of arbitrarily high degree by scalar multiples of itself, and that there are three rational triangles which correspond to triangular billiards surfaces of genus 1. However, other translation covers are rare; in fact, our main result is the following.

**Theorem 1.** Let \( f : X \to Y \) be a nontrivial translation cover of triangular billiards surfaces, where \( X \) has genus greater than 1. Then each of \( X \) and \( Y \) is either a right triangular billiards surface or an isosceles triangular billiards surface, and \( f \) is of degree at most 2.

We give explicit formulas for all such covers in Lemma 6. To prove Theorem 1, we use two main tools: the \( J \)-invariant of Kenyon and Smillie [8], and what we call the **fingerprint** of a point \( P \) on a translation surface. The fingerprint of \( P \) depends on the configuration of the shortest geodesics connecting \( P \) to singularities. We show that every point on a triangular billiards surface which corresponds to a vertex of the triangular billiard table has a fingerprint of one of two distinct types, which we call Type I and Type II (see Section 3 for definitions). We establish the following invariance results:

**Proposition 1.** Suppose \( X \) is a triangular billiards surface with a point \( P \) of Type II fingerprint. Then \( X \) is uniquely determined by that fingerprint, up to an action of \( O(2, \mathbb{R}) \).

**Lemma 4.** The fingerprint of a point is invariant under balanced translation covers, up to a doubling of cone angle.

Balanced translation covers are translation covers which map singularities to singularities; that is, they do not ramify over nonsingular points. This restriction ensures that they preserve a great deal symmetry; this fact has been used by Gutkin and Judge [4], Vorobets [10], Hubert and Schmidt [5], and others to gain information about affine symmetry groups of translation surfaces. Because of this, we first show that the main theorem can be proven for balanced translation covers using only the fingerprint as the primary tool. To extend the result to all translation covers, we require the \( J \)-invariant of Kenyon and Smillie [8], as well as work of Calta and Smillie [2] regarding the holonomy fields of \( J \)-invariants of special surfaces.
1.1 Outline

In Section 2 we review the construction of triangular billiards surfaces and record some simple combinatorial formulas for their canonical triangulations, following Aurell and Itzykson [1]. In Section 3 we define the fingerprint of a point and prove results about its behavior under translation covers. In 4.1 we explicitly identify all possible translation covers among triangular billiards surfaces. The goal of the remainder of Section 4 is to show that this list of covers is complete. In 4.2 we prove the main theorem for balanced covers. In 4.3 we discuss the J-invariant and holonomy field of Kenyon and Smillie; here we use results of Calta and Smillie [2] to show that the holonomy fields of triangular billiards surfaces related by a translation cover are equal. After some combinatorial lemmas in 4.5, we prove the main theorem in 4.6.

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2 The Rational Billiards Construction

Let \( R \) be a polygonal region whose interior angles are rational multiples of \( \pi \). Let \( D_{2Q} \) be the dihedral group of order \( 2Q \) generated by Euclidean reflections in the sides of \( R \). Suppose a particle moves within this region at constant speed and with initial direction vector \( v \), changing directions only when it reflects off the sides of \( R \), with the angle of incidence equaling the angle of reflection. Every subsequent direction vector for the particle is of the form \( \delta \cdot v \), for some element \( \delta \in D_{2Q} \), where \( D_{2Q} \) acts on \( \mathbb{R}^2 \) via Euclidean reflections.

The rational billiards construction consists of a flat surface corresponding to this physical system. Consider the set \( D_{2Q} \cdot R \) of \( 2Q \) copies of \( R \) transformed by the elements of \( D_{2Q} \). For each edge \( e \) of \( R \), we consider the corresponding element \( \rho_e \in D_{2Q} \) which represents reflection across \( e \). For each \( \delta \in D_{2Q} \), we glue \( \rho_e \cdot \delta \cdot R \) and \( \delta \cdot R \) together along their copies of \( e \). The result is a closed Riemann surface with flat structure induced by the tiling by \( 2Q \) copies of \( R \). This construction is described by Fox and Kershner in [3] and by Katok and Zemlyakov in [7].

In fact this surface is an example of a compact translation surface.

Definition 1. Let \( X \) be a flat surface with conical singularities. Let \( \hat{X} \) be the flat surface obtained by puncturing all singularities of \( X \). If all transition functions of \( \hat{X} \) are translations, then \( X \) is a translation surface.

The natural map between translation surfaces is one which respects this translation structure:
Definition 2. A translation cover is a holomorphic (possibly ramified) cover of translation surfaces \( f : X \to Y \) such that, for each pair of coordinate maps \( \phi_X \) and \( \phi_Y \) on \( X \) and \( Y \), respectively, the map \( \phi_Y \circ f \circ \phi_X^{-1} \) is a translation when \( \phi_X \) and \( \phi_Y \) are restricted to open sets not containing singular points. We say that \( f \) is balanced if \( f \) does not map singular points to nonsingular points.

Definition 3. We say that \( X \) and \( Y \) are translation equivalent if there exists a degree 1 translation cover \( f : X \to Y \).

We focus on billiards in rational triangles. We fix the notation \( T(a_1, a_2, a_3) \) to refer to a triangle with internal angles \( \frac{a_1 \pi}{Q} \), \( \frac{a_2 \pi}{Q} \), and \( \frac{a_3 \pi}{Q} \), where \( Q := a_1 + a_2 + a_3 \) and \( \gcd(a_1, a_2, a_3) = 1 \). We use the notation \( X(a_1, a_2, a_3) \) to refer to the translation surface arising from billiards in \( T(a_1, a_2, a_3) \) via the Fox-Kershner construction. We call such a surface a triangular billiards surface. If the triangle is isosceles or right, we call the corresponding surface an isosceles triangular billiards surface or a right triangular billiards surface.

Definition 4. Note that the Fox-Kershner construction gives a natural “tiling by flips” of the surface by copies of \( T \). A billiards triangulation is a triangulation \( \tau \) of \( X \) whose triangles are the various elements of \( D_{2Q} \cdot T \) described above.

Remark 1. Some data can be gained about \( \tau \) via simple combinatorics. Letting \( T := T(a_1, a_2, a_3) \), label the vertices of \( T \) as \( v_1, v_2, \) and \( v_3 \), where \( v_i \) corresponds to \( a_i \). It is not hard to check that the total number of triangles in \( \tau \) is \( 2Q \), that the number of vertices of \( \tau \) corresponding to \( v_i \) is \( \gcd(a_i, Q) \), and that each member of this set has a cone angle of \( \frac{a_i}{\gcd(a_i, Q)} \cdot 2\pi \). A good reference for these matters is [1].

Definition 5. Let \( v_1, v_2, v_3 \) be the vertices of \( T(a_1, a_2, a_3) \), and let \( \pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3) \) be the standard projection. A vertex class of \( X(a_1, a_2, a_3) \) is any of the three sets \( \pi_X^{-1}(v_1) \), \( \pi_X^{-1}(v_2) \), or \( \pi_X^{-1}(v_3) \). Note that for a given vertex class, either all the elements are singular or all are nonsingular; hence we call a vertex class singular if its elements are singularities and nonsingular if its elements are nonsingular.

Clearly, a vertex class \( \pi_X^{-1}(v_i) \) is nonsingular if and only if \( a_i \) divides \( Q \). Furthermore, the sum of the cone angles of the elements of \( \pi_X^{-1}(v_i) \) is \( 2\pi \). The following lemma shows how we will use Remark 1 to analyze translation covers.

Lemma 1. Suppose \( f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3) \) is a translation cover of triangular billiards surfaces. Let \( \pi_X : X(a_1, a_2, a_3) \to T(a_1, a_2, a_3) \) and \( \pi_Y : X(a_1, a_2, a_3) \to T(b_1, b_2, b_3) \) be the canonical projections to triangles with vertices \( v_1, v_2, v_3 \) and \( w_1, w_2, w_3 \) respectively. Suppose that \( P \in \pi_Y^{-1}(v_j) \), \( P' \in \pi_X^{-1}(v_j) \), and \( f(P') = P \) with a ramification index of \( m \) at \( P' \). Then

\[
\frac{mb_i}{\gcd(b_i, b_1 + b_2 + b_3)} = \frac{a_j}{\gcd(a_j, a_1 + a_2 + a_3)}.
\]
Proof. The cone angle at \( P' \) is \( m \) times the cone angle at \( P \). Therefore the result follows from Remark 1.

Of course, the translation structure of \( X(a_1, a_2, a_3) \) depends on the chosen area and direction of \( T(a_1, a_2, a_3) \). A translation surface \( X \) can represented as a pair \((S, \omega)\), where \( S \) is a Riemann surface and \( \omega \) is a holomorphic 1-form on \( S \) which induces the translation structure of \( X \). Using this language, suppose that \((S, \omega)\) is a triangular billiards surface arising from billiards in some \( T(a_1, a_2, a_3) \), and that \( \alpha \) is a nonzero complex number. The notation \( X(a_1, a_2, a_3) \) does not distinguish the pairs \((S, \omega)\) and \((S, \alpha \omega)\). The following lemma shows that this ambiguity will not affect our classification of translation covers.

**Lemma 2.** Suppose that \((S, \omega)\) is a triangular billiards surface of genus greater than one, and let \( \alpha \in \mathbb{C} \setminus \{0\} \). Then any translation cover \( f : (S, \omega) \to (S, \alpha \omega) \) is of degree 1.

**Proof.** This is a simple application of the Riemann-Hurwitz formula. Let \((S, \omega)\) have genus \( g \), and let \( \text{deg} \, f = n \). The 1-form \( \omega \) which gives \((S, \omega)\) its translation structure has \( 2g - 2 \) zeros (counting multiplicities). Clearly \( \alpha \omega \) has the same zeros as \( \omega \). The Riemann-Hurwitz formula then gives us that

\[
g = n(g - 1) + 1 + \frac{R}{2},
\]

where \( R \) is the total ramification number of \( f \). Since \( R \geq 0 \), Equation (1) is only satisfied if \( n = 1 \).

3 The Fingerprint

Consider a point \( P \) on a translation surface \( X \), along with the set of \( S \) all shortest geodesic segments on \( X \) which connect \( P \) to a singularity. Let \( s_1 \) and \( s_2 \) be two of these segments. We say that \( s_1 \) and \( s_2 \) are adjacent if \( s_1 \) can be rotated continuously about \( P \) onto \( s_2 \) without first coinciding with any other elements of \( S \).

**Definition 6.** A fingerprint of a point \( P \in \tau \) is the data \( \{\theta_i\}, \phi, L \), where \( \{\theta_i\} \) contains the distinct angle measures separating adjacent pairs of shortest geodesic segments connecting \( P \) to singularities, \( \phi \) is the total cone angle at \( P \), and \( L \) is the length of each of the shortest geodesic segments. We shall say that \( P \) has a Type I fingerprint if \( \{\theta_i\} \) has one element, and that \( P \) has a Type II fingerprint if \( \{\theta_i\} \) has two elements. We call \( \{\theta_i\} \) the angle set of a fingerprint.

Note that the angle set (and hence the fingerprint type) of the fingerprint of a point \( P \in X \) is invariant under the scaling of the flat structure of \( X \) by a nonzero complex number.
Lemma 3. Let $X$ be a surface of genus greater than one, arising from billiards in a rational triangle $T$. Fix a billiards triangulation $\tau$ of $X$ by $T_X$. Let $P$ be a vertex of $\tau$. Let $s$ be a shortest geodesic segment connecting $P$ to a singularity of $X$. Then either $s$ is an edge of $\tau$, or else $s$ is perpendicularly bisected by an edge of $\tau$.

Proof. Let $X$, $T_X$, $s$, $P$ and $\tau$ be as above. Let $\pi_X : X \to T$ be the natural projection induced by $\tau$.

Since singularities in the translation structure of $X$ can only occur at vertices of $\tau$, we only examine geodesics connecting vertices of $\tau$. This is equivalent to considering billiard paths between corners of the triangular billiard table $T_X$ in the original dynamical system.

Let $v = \pi_X(P)$; since $P$ is a vertex of $\tau$, $v$ is a corner of $T_X$. The shortest billiard path within $T_X$ from $v$ to a different corner $w$ of $T$ cannot be as short as the table edge connecting $v$ and $w$. This proves the claim if $s$ connects $P$ to a singularity which is not in the vertex class $\pi_X^{-1}(v)$.

Now suppose that $s$ connects $P$ to a singularity in $\pi_X^{-1}(v)$. Then $s$ corresponds to a billiard path from $v$ back to itself. If both of the other two corners of $T_X$
are acute, then the shortest billiard path from \(v\) to itself is accomplished via a single reflection by choosing the initial direction to be perpendicular to the side opposite \(v\); hence here an edge of \(\tau\) bisects \(s\). If one of the two other corners \(w\) is obtuse, then \(\pi_X^{-1}(w)\) must be a singular vertex class. But the distance from \(v\) to an obtuse corner of \(T_X\) is less than twice the distance from \(v\) to the opposite side of \(T_X\). Thus if \(w\) is obtuse then there is a geodesic segment \(s'\) in \(X\) connecting an element of \(\pi_X^{-1}(v)\) to a singular element of \(\pi_X^{-1}(w)\) such that \(s'\) is shorter than \(s\); this is a contradiction.

Lemma 3 allows us to relate fingerprints of points on \(X\) to the angle measures of vertices of \(T_X\). We summarize these relations in the following Corollary; see Figures 1 and 2 for illustrations.

**Corollary 1.** Let \(\tau\) be a billiards triangulation of a triangular billiards surface \(X\). For a given point \(P \in \tau\), let \(v\) be the projection of \(P\) onto the triangle \(T\) generating \(X\). Then one of three situations exists:

1) \(P\) has a Type I fingerprint with angle set \(\{\theta\}\), and \(\theta = \angle v\).
   
   This occurs if and only if \(T\) is isosceles and \(v\) is the apex of \(T\).

2) \(P\) has a Type I fingerprint with angle set \(\{\theta\}\), and \(\theta = 2\angle v\).
   
   This occurs if \(P\) has a Type I fingerprint and \(v\) is not the apex of an isosceles triangle.

3) \(P\) has a Type II fingerprint with angle set \(\{\theta_1, \theta_2\}\), and \(\theta_1 + \theta_2 = 2\angle v\).

**Proposition:** Suppose the billiards triangulation of a triangular billiards surface \(X\) contains a point with a Type II fingerprint. Then \(X\) is uniquely determined by that fingerprint, up to an action of \(O(2, \mathbb{R})\). Indeed, if the fingerprint has angle set \(\{\theta_1, \theta_2\}\), then \(X\) is the billiards surface for the triangle of angles \(\frac{\theta_1 + \theta_2}{2}\), \(\pi - \frac{\theta_1}{2}\), and \(\pi - \frac{\theta_2}{2}\).

**Proof.** The proof is evident from Figure 3 which illustrates the fingerprint of the singularity on \(X(3, 4, 5)\) (since \(X(3, 4, 5)\) is not isosceles and has only one singularity \(P\), it follows that \(P\) has a Type II fingerprint. In the figure, the geodesics defining the fingerprint are the thicker lines, whereas the edges of the billiards triangulation are the thinner lines.) Let the angle set be \(\{\theta_1, \theta_2\}\). Each \(\theta_i\) is an interior angle of a quadrilateral whose other three angles include two right angles and an angle which has twice the measure of an angle of the triangular billiard table \(T\) for \(X\). Therefore two of the angles of \(T\) have the form \(\frac{1}{2}(2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \theta_i) = \frac{\pi - \theta_i}{2}\). The length of the geodesics defining the fingerprint of \(P\) determines the scaling of \(T\). Thus \(T\) (and hence \(X\)) is uniquely identified, up to an action of \(O(2, \mathbb{R})\).
Lemma 4': Suppose that \( f : X \rightarrow Y \) is a balanced translation cover, that \( P' \in X \) and \( P \in Y \) are vertices of billiards triangulations on their respective surfaces, and that \( f(P') = P \). Then either: 1) \( P' \) and \( P \) have the same fingerprint, or 2) their fingerprints differ only in the cone angle, \( P \) has half the cone angle of \( P' \), \( X \) arises from billiards in an isosceles triangle, and \( P' \) corresponds to the apex of that triangle.

Proof. Let \( d \) be the length of a shortest geodesic which connects \( P \) to a singularity. Let \( B \subset Y \) be the set of points of distance less than \( d \) from \( P \). Let \( B' \subset X \) be the maximal connected component of \( f^{-1}(B) \) which contains \( P' \). Since \( f \) is a balanced translation cover, \( B' \) consists of all points of distance less than \( d \) from \( P' \), and \( B' \) contains no singularities other than possibly \( P' \) (\( P' \) is singular if and only if \( P \) is singular). We have that \( f \) is locally an \( m \)-to-one cover at \( P \) for some integer \( m \).

Now consider a pair of adjacent geodesics \( e_1 \) and \( e_2 \), each of length \( d \), connecting \( P \) to singularities. Label the angle between them \( \theta \). The union of these two edges with a portion of the boundary of \( B \) bounds a wedge-shaped region \( W \) which contains singularities only at the endpoints of \( e_1 \) and \( e_2 \) (see Figure 3). Since \( f \) is a translation cover, the \( f \)-preimage of \( W \) is \( m \) copies of \( W \), each of which is bounded by part of the boundary of \( B' \) and two shortest geodesics \( e_1' \) and \( e_2' \) of length \( d \) connecting \( P' \) to singularities of \( X \). The interior angle measure between \( e_1' \) and \( e_2' \) is \( \theta \). Because \( f \) is balanced, we know that \( e_1' \) and \( e_2' \) are adjacent; otherwise, the wedge they bound would have a geodesic \( e' \) in its interior such that \( f(e') \) lies in the interior of \( W \) and connects \( P \) to a singularity, a contradiction to the adjacency of \( e_1 \) and \( e_2 \). Therefore we have established that the fingerprints of \( P \) and \( P' \) have the same angle sets.

Because \( f \) is a translation cover, the cone angle at \( P' \) is \( m \) times the cone angle.
at $P$. We claim that $m \leq 2$. Let $v$ and $v'$ be the vertices of the triangles $T$ and $T'$ corresponding to $P$ and $P'$. By Remark 1, the cone angle at $P$ is completely determined by $\angle v$. But Corollary 1 tells us that $\angle v$ is determined, up to a factor of 2, by the angle set of the fingerprint of $P$. Hence, since the fingerprints of $P$ and $P'$ have the same angle set, we see that $m \in \{1, 2\}$, and our claim is proven.

Furthermore, note that if $m = 2$, then since the cone angle at $P'$ is greater than the cone angle at $P$ and cone angle is completely determined by the corresponding vertex of the triangular billiard table, Corollary 1 implies that $T_X$ is isosceles and $v'$ is the apex of $T_X$.

**Corollary 2.** Fingerprint type is invariant under balanced translation covers.

**Corollary 3.** Any rational triangular billiards surface with a Type II singularity cannot be a part of any composition of nontrivial balanced covers.

**Proof.** This follows directly from Proposition 1. Suppose we have $f : X \to Y$ a balanced cover with either $X$ or $Y$ possessing a singularity with a Type II fingerprint. By Corollary 2, $X$ and $Y$ must both have singularities with Type II fingerprints. Since a Type II fingerprint identifies the triangular billiards table of a surface, $X$ and $Y$ must be the same surface.  

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Figure 4: A balanced cover ramified above $P$. Here, $m = 2$. 
Definition 7. A saddle connection on a translation surface is a geodesic with singular endpoints and no singularities in its interior.

Corollary 4. Let $X(a_1,a_2,a_3)$ be a non-isosceles triangular billiards surface with only one singular vertex class. Then $X$ cannot be a part of any composition of nontrivial balanced covers.

Proof. Let $v$ be the vertex of $T(a_1,a_2,a_3)$ that unfolds to a singular vertex class. Let $P \in \pi_X^{-1}(v)$. All saddle connections on $X$ have endpoints in $\pi_X^{-1}(v)$, so by Lemma 3 the geodesics defining the fingerprint of $P$ are realized via single reflections of $P$ across the opposite sides of the copies of $T(a_1,a_2,a_3)$ of which $P$ is a vertex. Thus, since $T(a_1,a_2,a_3)$ is not isosceles, $P$ has a Type II fingerprint. Therefore Proposition 1 applies. \qed

As we shall see, the preceding results allow us to quickly classify all balanced covers in the category of triangular billiards surfaces. However, since not all of the material (such as Corollary 3) extends to unbalanced translation covers, we shall refine our use of the fingerprint with the following lemma.

Lemma 5. Let $X$ be a triangular billiards surface with more than one singular vertex class. Let $\tilde{X}$ be the surface obtained from $X$ by puncturing either one entire singular vertex class or two entire singular vertex classes such that neither deleted class corresponds to an obtuse angle of the triangular billiard table and such that at least one singular vertex class remains. Let $\pi_X^{-1}(v_i)$ be a singular vertex class not deleted. Let $P \in \pi_X^{-1}(v_i)$. If $P$ has Type II fingerprint on $\tilde{X}$ with angle set $\{\theta_1, \theta_2\}$, then $X$ arises from billiards in the triangle with angles $\frac{\pi - \theta_1}{2}$, $\frac{\theta_1 + \theta_2}{2}$. If $P$ has a Type I fingerprint on $\tilde{X}$ with angle set $\{\theta_1\}$, then $\angle v_i \in \{\theta_1, \frac{\theta_1}{2}\}$.

Proof. If none of the punctured points are endpoints of shortest separatrices through $P$, then $P$ has the same fingerprint on $\tilde{X}$ as on $X$, and we are done.

Suppose a singular vertex class has been punctured which contained endpoints of shortest separatrices through $P$. Then there is a new “closest” vertex class to $P$; call it $C$. If $C$ does not contain $P$ then the shortest geodesics connecting $P$ to $C$ are edges of the billiards triangulation of $X$. If $C$ does contain $P$ then, since a vertex class corresponding to an obtuse angle of the billiard table must be singular (by Remark 1) and we have assumed that no such classes have been deleted, it follows that the shortest geodesics from $P$ to $C$ correspond to a single reflection in the original dynamical system. Thus the same reasoning holds as in Lemma 3.

The only potential difficulty would be if the new “closest” vertex class was the one containing $P$, for in that case, since the shortest geodesics from $P$ to elements of its own class pass through more than one triangle, we must consider
the possibility that our punctures obstruct these geodesics. However, since the shortest geodesics are perpendicular to the sides of the triangles opposite \( P \), this is only a problem if the vertex class punctured is \( \pi^{-1}(v_j) \) with \( \angle v_j = \frac{\pi}{2} \). But such a class is nonsingular. \( \square \)

4 All Translation Covers

4.1 The Possible Covers

Any isosceles triangle is naturally "tiled by flips" by a right triangle. The following lemma demonstrates how to use this tiling to create nontrivial translation covers in the category of triangular billiards surfaces. In fact, our main theorem is that the covers of Lemma 6 are the only nontrivial translation covers among triangular billiards surfaces.

**Lemma 6.** Let \( a_1 \) and \( a_2 \) be relatively prime positive integers, not both equal to one. The right triangular billiards surface \( Y := X(a_1 + a_2, a_1, a_2) \) is related to two isosceles triangular billiards surfaces

\[
X_1 = \begin{cases} 
X(2a_2, a_1, a_1) & a_1 \text{ odd} \\
X(a_2, \frac{a_1}{2}, \frac{a_1}{2}) & a_1 \text{ even}
\end{cases}
\]

and

\[
X_2 = \begin{cases} 
X(2a_1, a_2, a_2) & a_2 \text{ odd} \\
X(a_1, \frac{a_2}{2}, \frac{a_2}{2}) & a_2 \text{ even}
\end{cases}
\]

via balanced covers \( f_1 : X_1 \to Y \) and \( f_2 : X_2 \to Y \). The maps have degrees

\[
\text{deg}(f_i) = \begin{cases} 
2 & a_i \text{ odd} \\
1 & a_i \text{ even}
\end{cases}
\]. Furthermore at least one of the \( f_i \) has degree 2.

**Proof.** It suffices to prove the result for \( X_1 \) and \( f_1 \). Write \( Q := 2a_1 + 2a_2 \). We reflect the triangle \( T = T(a_1 + a_2, a_1, a_2) \) across the edge connecting the \( a_2 \) and \( a_1 + a_2 \) vertices, to obtain its mirror image \( T' \). By joining \( T \) and \( T' \) along the edge of reflection we create an isosceles triangle \( T \) which can be written as either \( T(2a_2, a_1, a_1) \) (if \( a_1 \) is odd) or \( T(a_2, \frac{a_1}{2}, \frac{a_1}{2}) \) (if \( a_1 \) is even). Note that since \( (a_1 + a_2, a_1, a_2) \) must be a reduced triple, \( a_1 \) and \( a_2 \) cannot both be even. It also follows that \( \text{gcd}(a_i, Q) \leq \text{gcd}(2a_i, Q) = 2 \).
Suppose $a_1$ is even. Consider the translation surface $S$ (with boundary) obtained by developing $T$ around its $a_2$ vertex. Since $a_2$ is odd we have $\gcd(a_2, Q) = 1$, so $S$ is tiled (by reflection) by $2Q$ copies of $T$, and hence after appropriate identifications along the boundary we will have $X(a_1 + a_2, a_1, a_2)$. Let $\hat{S}$ be the surface obtained by developing $\hat{T}$ around the corresponding vertex; it is tiled via reflection by $Q$ copies of $\hat{T}$, so appropriate boundary identifications will yield $Y_1$. Because $\hat{T}$ is tiled via reflection by two copies of $T$, it follows that $S$ and $\hat{S}$ are translation equivalent. Finally, note that the boundary identifications are the same for $S$ and $\hat{S}$. Therefore $Y$ and $X_1$ are translation equivalent.

Now suppose that $a_1$ is odd and $a_2$ is even. We then have $\hat{T} = T(2a_2, a_1, a_1)$. Since $\gcd(2a_2, Q) = 2$, we again have that $\hat{S}$ is tiled by $Q$ copies of $\hat{T}$. Since $a_2$ is even, $\gcd(a_2, Q) = 2$, implying that $S$ is tiled by $Q$ copies of $T$. Thus if $a_2$ is even then $\hat{S}$ is a double cover of $S$, ramified over a single point. Furthermore, in this case $X_1$ and $Y$ are obtained by identifying appropriate edges of two copies of $\hat{S}$ and $S$, respectively. It follows that if $a_2$ is even then $X_1$ is a ramified double cover of $Y$.

Finally, suppose that $a_1$ and $a_2$ are both odd. We have $\hat{T} = T(2a_2, a_1, a_1)$, $\gcd(2a_2, Q) = 2$, and $\gcd(a_2, Q) = 1$. In this case we have that $S$ and $\hat{S}$ are translation equivalent surfaces; however, $X_1$ is obtained from two copies of $\hat{S}$ whereas $Y$ is obtained from a single copy of $S$. Thus again we have that $X_1$ is a double cover of $Y$, this time unramified.

Remark 2. If we allow $a_1 = a_2 = 1$ in the statement of Lemma then we arrive at $Y = X_1 = X_2 = X(1, 1, 2)$. This is because $T(1, 1, 2)$ is the unique right isoceles triangle.

Because the location of singularities is such a major tool in analyzing translation surfaces, it is worth identifying the triangular billiards surfaces which have no singularities. As detailed in [1], there are only three of these surfaces: $X(1, 1, 2)$, $X(1, 2, 3)$, and $X(1, 1, 1)$. These are also the only three triangular billiards surfaces of genus 1; furthermore $X(1, 2, 3)$ and $X(1, 1, 1)$ are actually translation equivalent. Each of these surfaces admits balanced translation covers of itself by itself of arbitrarily high degree; this fact is related to the fact that $T(1, 1, 2)$, $T(1, 2, 3)$, and $T(1, 1, 1)$ are the only Euclidean triangles which tile the Euclidean plane by flips. Note that any such cover must be unramified, since flat ramified covers are locally of the form $z \mapsto z^{1/n}$ for some $n > 1$, implying that the cone angle of the ramification point is greater than $2\pi$.

4.2 Balanced Covers

Balanced translation covers $f : X \to Y$ of translation surfaces are of interest because they imply an especially strong relationship between the affine symmetry groups of $X$ and $Y$; in particular, these groups must have finite-index subgroups
which are conjugate. We shall prove Theorem for balanced covers using only the machinery built up thus far.

**Lemma 7.** Let $X$ and $Y$ be surfaces arising from billiards in the rational triangles $T_X$ and $T_Y$, respectively. Let $f : X \to Y$ be a balanced translation cover which ramifies over a point of $Y$. Then $f$ is a cover described in Lemma. In particular, $T_X$ is isosceles and $f$ is of degree two.

**Proof.** Suppose that $f$ ramifies over a point $P$ in $Y$, at a point $P'$ in $X$. Then the cone angle of $P'$ is greater than the cone angle of $P$, and Lemma tells us that the cone angle of $P'$ is twice the cone angle of $P$. Hence $P'$ is a singular point, and since $f$ is balanced, $P$ is also singular.

Lemma also tells us that $T_X$ is isosceles, so we write $T_X = T(a_1, a_2, a_2)$ and $Q = a_1 + 2a_2$, with $\gcd(a_1, a_2) = 1$. Label the vertices of $T_X$ as $v_1, v_2, v_3$, with $v_1$ the apex of $T_X$. Let $\pi_X$ be the standard projection of $X$ onto $T_X$, and let $C_1$ denote the vertex class $\pi_X^{-1}(v_1)$. We have that $P' \in C_1$ by Lemma Since $\gcd(a_2, Q) = 1$ and $\gcd(a_1, Q) \in \{1, 2\}$, $C_2$ and $C_3$ each have one element and $C_1$ has one or two elements.

Since $P'$ has twice the cone angle of $P$, $\deg f \geq 2$. Therefore by Corollary $X$ $P$ has a Type I fingerprint.

Suppose that $Y$ has only one singular vertex class $C$. Then Corollary tells us that $Y$ is an isosceles triangular billiards surface. Let $T_Y$ be the isosceles triangle which unfolds to the billiards triangulation of $Y$, and let $\pi_Y : Y \to T_Y$ be the induced projection. Since $T_X \neq T_Y$, $\pi_Y(P)$ cannot be the apex of $T_Y$. Therefore $T_Y = T(a_1, a_2, a_2)$. But this contradicts the supposition that $Y$ has only one singular vertex class.

Now suppose instead that $Y$ has more than one singular vertex class. Let $R$ be a singular vertex class distinct from that of $P$. If $R \in f(C_1)$ then $f$ also ramifies above $R$, so $\angle \pi_Y(R) = \angle \pi_Y(P)$ and $Y = X(a_1, a_1, 2a_2)$, which is a surface described in Lemma If $R \notin f(C_1)$ then since $f$ cannot ramify at points outside of $C_1$, $f$ does not ramify above $R$. Therefore $f^{-1}(R)$ has $\deg f$ elements, all singular. But $X$ has at most two singularities not in $C_1$: they are the two elements of $C_2 \cup C_3$. Hence since $\deg f \geq 2$, we have $\deg f = 2$ and $f^{-1}(R) = C_2 \cup C_3$.

Thus $\angle \pi_Y(R) \in \{\angle \pi_X(C_2), 2\angle \pi_X(C_2), \frac{1}{2}\angle \pi_X(C_2)\}$. If $\angle \pi_Y(R) = \angle \pi_X(C_2)$ then $T_Y$ is a right triangle as in Lemma If $\angle \pi_Y(R) = 2\angle \pi_X(C_2)$ then $T_Y$ is an isosceles triangle as in Lemma If $\angle \pi_Y(R) = \frac{1}{2}\angle \pi_X(C_2)$, then $T_Y$ has a vertex with angle $\frac{a_1 + 3a_2}{2Q}$; but since $\gcd(a_1 + 3a_2, 2Q) \in \{1, 2\}$, this vertex corresponds to a vertex class on $Y$ of total cone angle at least $(a_1 + 3a_2)\pi$. Since $\deg f \geq 2$, the $f$-preimage of this vertex class must have total cone angle at least $(a_1 + 3a_2)2\pi$, which contradicts the fact that the sum of the cone angles of all singularities on $X$ cannot exceed $(a_1 + 2a_2)2\pi$. 

\[\Box\]
The results of Lemma 7 allow us to quickly handle all balanced covers by considering only unramified covers.

**Lemma 8.** Let $X$ and $Y$ be triangular billiards surfaces such that the genus of $X$ is greater than 1. Suppose that $f : X \to Y$ is a nontrivial balanced translation cover. Then $f$ is of the form described in Lemma 6.

**Proof.** By Lemma 7 if $f$ is ramified then we are done. Therefore we may assume that $f$ is unramified. Let $T_X$ and $T_Y$ denote the rational triangles corresponding to $X$ and $Y$.

By Corollary 3, we may assume that $X$ and $Y$ both contain only Type I singularities, since otherwise $f$ must be trivial. Since $f$ is unramified, we know that if $P'$ and $P$ are vertices of the billiards triangulations of $X$ and $Y$ respectively such that $f(P') = P$ then $P'$ and $P$ have identical fingerprints; in particular, $\angle \pi_X(P') = \angle \pi_Y(P)$. Thus if $X$ has more than one singular vertex class, and if the same is true of $Y$, then in fact $T_X = T_Y$. Therefore there are two remaining cases to check.

**Case 1.** $X$ has two singular vertex classes, and $Y$ has one singular vertex class.

In this case, Remark 1 implies that the two vertices of $T_X$ with singular preimages must have the same angle measure, since $f$ takes both their preimages to the $\pi_Y$-preimage of a single vertex of $T_Y$. Thus $T_X$ is an isosceles triangle and $X$ has exactly two singularities. Let $m$ be the degree of $f$. Since $f$ is unramified, each point of $Y$ has $m$ $f$-preimages on $X$. But $X$ has exactly two singularities, and the $f$-preimage of each singularity of $Y$ consists only of singularities, so $f$ has degree at most 2.

**Case 2.** $X$ has only one singular vertex class.

By Corollary 4, $X$ must be an isosceles triangular billiards surface. Hence its singular vertex class can have at most two elements. Therefore, as in Case 1, $f$ has degree at most 2.

Note that in addition to relating right and isosceles triangles, Lemma 6 also gives a way to construct covers between isosceles triangular billiards surfaces. In the language of Lemma 6, if $a_2$ is even, then $f_2^{-1} \circ f_1$ is a degree two translation cover of $X_2$ by $X_1$.

### 4.3 The $J$-invariant and holonomy field

To prove Theorem 1 for all translation covers, we introduce another tool: the $J$-invariant of Kenyon and Smillie [8].

**Definition 8.** Let $P$ be a polygon in the plane. Let $w_1, w_2, ..., w_n$ be the vertices of $P$. The $J$-**invariant of $P$** is the element of $\mathbb{R}^2 \wedge \mathbb{Q} \mathbb{R}^2$ given by $J(P) := w_1 \wedge w_2 + w_2 \wedge w_3 + ... + w_{n-1} \wedge w_n + w_n \wedge w_1$. 

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It is easily shown that the $J$-invariant of a polygon is invariant under translations of the polygon, and that it is a “scissors invariant” in the sense that cut-and-paste operations do not affect its $J$-invariant. Furthermore, it is well known that any compact translation surface can be constructed by identifying parallel edges of a finite set of polygons in the plane. For these reasons the definition naturally extends to translation surfaces.

**Definition 9.** Let $X$ be a compact translation surface. Let $\{P_1, \ldots, P_n\}$ be a collection of planar polygons such that appropriate identification of sides yields the surface $X$. Then the $J$-invariant of $X$ is $J(X) := \sum_{i=1}^{n} J(P_i)$.

The following lemma is presumably well-known.

**Lemma 9.** Let $f : X \to Y$ be a degree $n$ translation cover of translation surfaces. Then $J(X) = nJ(Y)$.

**Proof.** We can triangulate $Y$ by Euclidean triangles in such a way that the ramification points of $f$ are among the vertices of the triangulation. Let $Y'$ be the set of triangles obtained by cutting open $Y$ along all the edges of our triangulation. Lifting our triangulation to $X$ via $f$, we let $X'$ be the corresponding decomposition of $X$. Since $J$ is a scissors invariant, we have $J(Y) = J(Y')$ and $J(X) = J(X')$. Furthermore, since each triangle in $Y'$ lifts to $n$ identical copies in $X'$, we have that $J(X') = nJ(Y')$. Thus $J(X) = J(X') = nJ(Y') = nJ(Y)$. \qed

**Definition 10.** The rational absolute holonomy of a translation surface $X$ is the image of the map $\text{hol} : H_1(X; \mathbb{Q}) \to \mathbb{C}$ defined by $\text{hol} : \sigma \mapsto \int_{\sigma} \omega$, where the 1-form $\omega$ is locally the differential $dz$ in each chart not containing a singular point.

Now we define a property of translation surfaces which will be useful in classifying triangular billiards surfaces. This definition is due to Kenyon and Smillie [8].

**Definition 11.** The holonomy field of a translation surface $X$, denoted $k_X$, is the smallest field $k_X$ such that the absolute holonomy of $X$ is contained in a two-dimensional vector space over $k_X$.

Calta and Smillie [2] discuss the algebraically periodic directions of a translation surface, which they define to be those directions in which a certain projection of the $J$-invariant is zero.

**Definition 12.** Fix coordinates for a translation surface $S$ such that 0, 1, and $\infty$ are all slopes of algebraically periodic directions. The periodic direction field of $S$ is the collection of slopes of algebraically periodic directions in this coordinate system.

It is shown in [2] that this definition is well-defined, and that the periodic direction field is a number field whose degree is bounded by the genus of $S$. The following lemma relies on the results of Kenyon and Smillie [8] and Calta and Smillie [2].
Lemma 10. Let \( f : X(a_1, a_2, a_3) \to Y \) be a degree \( n \) translation cover. Write \( Q := a_1 + a_2 + a_3 \). Then \( X \) and \( Y \) have the same holonomy field \( k \), and \( k = \mathbb{Q}(\zeta_Q + \zeta_Q^{-1}) \), where \( \zeta_Q \) is a primitive \( Q \)th root of unity.

Proof. By Lemma 9, \( J(X) = nJ(Y) \). Assume that \( Y \) has area 1; thus \( X \) has area \( n \). Let \( X' \) be the surface of area 1 obtained by uniformly scaling \( X \). We have that \( J(X') = \frac{1}{n} J(X) = J(Y) \). Since uniformly scaling a surface clearly does not affect its periodic direction field, \( X \) and \( X' \) have the same periodic direction field. Calta and Smillie note that their work in Section 6 of [2] implies that the periodic direction field of a surface depends only on the \( J \)-invariant of that surface; hence \( X' \) and \( Y \) have the same periodic direction field. Corollary 5.21 of [2] states that a translation surface is completely algebraically periodic if and only if its holonomy field equals its periodic direction field. Furthermore, Theorem 1.4 of [2] states that triangular billiards surfaces are algebraically periodic. Therefore \( X \) and \( Y \) have the same periodic direction field. Finally, Kenyon and Smillie [8] calculate this holonomy field to be \( k = \mathbb{Q}(\zeta_Q + \zeta_Q^{-1}) \).

The proof of the algebraic periodicity of triangular billiards surfaces in [2] contains a small error which could be corrected by applying a normalization outlined in [8]. We also offer a different proof of this result in [9].

4.4 Some elementary number theory

Note that the holonomy field \( k_X := \mathbb{Q}(\zeta_Q + \zeta_Q^{-1}) \) is a degree two subfield of the cyclotomic field \( \mathbb{Q}(\zeta_Q) \), since it is the maximal subfield fixed by complex conjugation. In light of this, we list some classical results about these two fields as recorded in Washington’s text [11].

Lemma 11. If \( Q \) is odd then \( \mathbb{Q}(\zeta_Q) = \mathbb{Q}(\zeta_{2Q}) \).

Lemma 12. (Prop 2.3 in [11]) Assume that \( Q \neq 2 \mod 4 \). A prime \( p \) ramifies in \( \mathbb{Q}(\zeta_Q) \) if and only if \( p|Q \).

Lemma 13. (Prop 2.15 in [11]) Let \( p \) be a prime, and assume that \( n \neq 2 \mod 4 \). If \( n = p^m \) then \( \mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) is ramified only at the prime above \( p \) and at the archimedean primes. If \( n \) is not a prime power, then \( \mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) is unramified except at the archimedean primes.

Remark 3. Washington’s proofs of Lemmas 12 and 13 make clear that the results carry through to the case \( Q = 2 \mod 4 \) except that in that case, the prime 2 does not ramify in \( \mathbb{Q}(\zeta_Q) \).

For a triangular billiards surface \( X = X(a_1, a_2, a_3) \), it is tempting to define a “\( Q \)-value” for the surface by \( Q_X := a_1 + a_2 + a_3 \). Unfortunately this notion is not
Lemma 14. Distinct cyclotomic fields have distinct maximal totally real subfields.

Proof. This is an exercise in elementary algebraic number theory, and is presumably well known. Let \( k \) be the maximal real subfield of the cyclotomic fields \( \mathbb{Q}(\zeta_m) \) and \( \mathbb{Q}(\zeta_n) \) for positive integers \( m, n > 2 \). Let \( p \) be an odd prime dividing \( m \). By Lemma 12, \( p \) ramifies in \( \mathbb{Q}(\zeta_m) \). If \( m \) is a power of \( p \), then \( p \) is totally ramified in \( \mathbb{Q}(\zeta_m) \). Since \( \mathbb{Q} < k < \mathbb{Q}(\zeta_m) \), if \( m \) is a power of \( p \) then \( p \) must ramify in \( k \).

If \( m \) is not a power of \( p \), then Lemma 13 tells us that the extension \( \mathbb{Q}(\zeta_m)/k \) is not ramified at the prime above \( p \); thus again \( p \) must ramify in \( k \). But also \( \mathbb{Q} \leq k < \mathbb{Q}(\zeta_n) \), so \( p \) must ramify in \( \mathbb{Q}(\zeta_n) \). By Lemma 12 this implies that \( p \) divides \( n \). Therefore \( m \) and \( n \) have the same odd prime divisors; furthermore, by Remark 3 these arguments extend to show that either \( 4 \) divides both \( m \) and \( n \) or it divides neither.

The degrees of \( \mathbb{Q}(\zeta_m) \) and \( \mathbb{Q}(\zeta_n) \) as field extensions of \( \mathbb{Q} \) are \( \phi(m) \) and \( \phi(n) \) respectively, where \( \phi \) is the Euler totient function. Since \( \mathbb{Q}(\zeta_m) \) and \( \mathbb{Q}(\zeta_n) \) are each degree 2 extensions of \( k \), we have that \( \phi(m) = \phi(n) \).

First suppose that \( m \) and \( n \) are congruent modulo 2. Let \( m = \Pi p_i^{e_i} \) and \( n = \Pi p_i^{f_i} \) be the prime factorizations of \( m \) and \( n \). Then we have

\[
1 = \frac{\phi(m)}{\phi(n)} = \frac{\prod (p_i - 1)p_i^{e_i - 1}}{\prod (p_i - 1)p_i^{f_i - 1}} = \prod p_i^{e_i - f_i}.
\]

Therefore \( e_i = f_i \) for each \( i \), and \( m = n \). Hence in this case \( \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_n) \).

If \( m \) and \( n \) are not congruent modulo 2, then we may assume that \( m \) is odd and \( n \) is congruent to 2 modulo 4. Since \( \phi(m) = \phi(2m) \) when \( m \) is odd, we can repeat the calculation with \( 2m \) and \( n \), and get that \( 2m = n \). But it is well known that for any odd \( m \), \( \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{2m}) \). Therefore in fact \( k \) is the maximal totally real subfield of only one cyclotomic field.

Corollary 5. Suppose that \( X(a_1, a_2, a_3) \) and \( X(b_1, b_2, b_3) \) have the same holonomy field, and that \( b_1 + b_2 + b_3 < a_1 + a_2 + a_3 \). Then \( b_1 + b_2 + b_3 \) is odd, and \( a_1 + a_2 + a_3 = 2(b_1 + b_2 + b_3) \).

Proof. Suppose \( X(a_1, a_2, a_3) \) and \( X(b_1, b_2, b_3) \) have the same holonomy field \( k \). Write \( Q_X = a_1 + a_2 + a_3 \) and \( Q_Y = b_1 + b_2 + b_3 \). Then by Lemma 14 we have that \( k \) is the maximal totally real subfield of \( \mathbb{Q}(\zeta_Q) \) and of \( \mathbb{Q}(\zeta_{Q_Y}) \). The result then follows directly from Lemma 14.

\[\square\]
4.5 Combinatorial Lemmas

Lemma 15. Let \( f : X(a_1, a_2, a_3) \rightarrow X(b_1, b_2, b_3) \) be a translation cover of triangular billiards surfaces such that the genus of \( X(a_1, a_2, a_3) \) is greater than 1. If \( a_1 + a_2 + a_3 = b_1 + b_2 + b_3 \) and \( f \) is not a composition of covers from Lemma 6, then \( f \) is of degree 1.

Proof. Write \( Q := a_1 + a_2 + a_3 = b_1 + b_2 + b_3 \). Let \( n \) be the degree of \( f \), and suppose that \( n \geq 2 \). Clearly the sum of the cone angles of the singular points of \( X(a_1, a_2, a_3) \) is at least \( n \) times the sum of the cone angles of the singular points of \( X(b_1, b_2, b_3) \). In light of Remark \( \text{ii} \), this gives the inequality

\[
\sum_{a_i \mid (a_1 + a_2 + a_3)} a_i \geq n \sum_{b_i \mid (b_1 + b_2 + b_3)} b_i. \tag{3}
\]

Therefore, since \( Q := a_1 + a_2 + a_3 = b_1 + b_2 + b_3 \), we must have \( \sum b_i \leq \frac{Q}{n} \).

Hence, since \( n \geq 2 \), we have

\[
\sum_{b_i \mid Q} b_i \geq \frac{Q}{2}. \tag{4}
\]

Finally, writing \( q_i = Q \frac{b_i}{b_i} \), we have the equivalent expression

\[
\sum_{b_i \mid Q} \frac{1}{q_i} \geq \frac{Q}{2}. \tag{5}
\]

Note that if \( b_i \mid Q \) then \( q_i \) is an integer. Of course, Equation \( \text{ii} \) is always satisfied if \( T(b_1, b_2, b_3) \) is a right triangle. If \( T(b_1, b_2, b_3) \) is not a right triangle, the equation is rarely satisfied. Thus we will reduce the problem to three cases (up to permutation of vertices).

Case 1. \( T(b_1, b_2, b_3) \) is not a right triangle.

In this case, recalling that \( \gcd(b_1, b_2, b_3) = 1 \), there are only three possibilities for the \( b_i \) which satisfy Equation \( \text{iv} \).

If all three \( b_i \) divide \( Q \) then \( Y \) is nonsingular. The only non-right triangle which unfolds to a nonsingular surface is \( T(1, 1, 1) \); but since this is also the only triangle with \( Q = 3 \), if \( Y = X(1, 1, 1) \) then \( X = X(1, 1, 1) \), contradicting our assumption that \( X \) has a singularity.

Hence we can assume for this case that \( b_3 \nmid Q \). Therefore to satisfy Equation \( \text{v} \) we seek integers \( q_1, q_2 > 2 \) such that

\[
\frac{1}{q_1} + \frac{1}{q_2} > \frac{1}{2} \tag{6}
\]
Without loss of generality we assume \( q_1 \leq q_2 \). If \( q_1 \geq 4 \), Equation \([3]\) is impossible. If \( q_1 = 3 \) then Equation \([3]\) is satisfied if \( q_2 \leq 5 \). Thus the remaining candidates for \( Y \) are \( X(3, 4, 5) \) and \( X(3, 5, 7) \). By Equation \([3]\), \( X(3, 4, 5) \) admits at most a degree 2 cover; by Lemma \([4]\) the degree 2 covers satisfying the hypotheses of the lemma could only be \( f : X(2, 5, 5) \to X(3, 4, 5) \) or \( X(1, 1, 10) \to X(3, 4, 5) \). However, these maps would have to be balanced covers, and \( X(3, 4, 5) \) has a singularity with a Type II fingerprint. Thus by Corollary \([3]\) these maps do not exist. Similarly, the only feasible cover of \( X(3, 5, 7) \) of degree greater than 1 is \( f : X(1, 7, 7) \to X(3, 5, 7) \); again, this would be a balanced cover, and \( X(3, 5, 7) \) has a singularity with a Type II fingerprint.

**Case 2.** \( T(b_1, b_2, b_3) \) is a right triangle, with \( b_1 = \frac{Q}{2} \) and neither \( b_2 \) nor \( b_3 \) dividing \( Q \).

Here Equation \([3]\) implies that the degree of \( f \) is at most two. The sum of the cone angles of the singularities of \( Y = X(b_2 + b_3) \). Thus if \( n = 2 \) then the sum of the cone angles of the singularities of \( X \) is \( 2(b_2 + b_3) = Q = a_1 + a_2 + a_3 \). Therefore \( T(a_1, a_2, a_3) \) must be either \( T(b_2, b_2, 2b_3) \) or \( T(2b_2, b_3, b_3) \). Both these possibilities are accounted for by the covers of Lemma \([4]\).

**Case 3.** \( T(b_1, b_2, b_3) \) is a right triangle, with \( b_1 = \frac{Q}{2} \) and \( b_2 | Q \).

Hence the triangle has angles \( \frac{\pi}{2}, \frac{\pi}{q}, \) and \( \frac{q - 2}{2q} \) for some integer \( q \) dividing \( Q \). We have

\[
T(b_1, b_2, b_3) = \begin{cases} 
T(2, q - 2, q) & \text{if } q \text{ odd} \\
T(1, \frac{q}{2} - 1, \frac{q}{2}) & \text{if } q \text{ even}
\end{cases}
\]

First suppose that \( q \) is odd. Then \( Y = X(2, q - 2, q) \). If \( q = 3 \) then \( Y = X(1, 2, 3) \) and \( X \) is either \( X(1, 2, 3) \) (ruled out because it is genus 1) or \( X(1, 1, 4) \) (already listed in Lemma \([4]\)). If \( q = 5 \) then by Lemma \([1]\) \( X \) is either \( X(3, 3, 4) \) (already listed in Lemma \([4]\) or \( X(1, 3, 6) \). A translation cover \( X(1, 3, 6) \to X(2, 3, 5) \) would have to be a balanced triple cover, and the fingerprints would not match. For \( q \geq 7 \), only double covers are possible, by Equation \([3]\). Since \( \gcd(q - 2, q) = 1 \), there is only one singularity on \( Y \) and it has cone angle \( 2(q - 2)\pi \). Thus by Lemma \([4]\) possible double covers are \( X(4, q - 2, q - 2) \) and \( X(1, 3, 2q - 4) \). The covers \( X(4, q - 2, q - 2) \) are accounted for by Lemma \([6]\). The covers \( X(1, 3, 2q - 4) \) have one singular vertex class when \( 3|q \); in this case \( f \) must be balanced. But if \( 3 \nmid q \) then \( X \) would have a conical singularity with cone angle \( 6\pi \) mapping to a nonsingular point of \( Y \), which is impossible since the degree of the cover is at most 2. Now suppose that \( q \) is even. If \( q = 4 \) then \( Y = X(1, 1, 2) \), but the lemma assumes that \( X \) has a singularity. If \( q = 6 \) then \( Y = X(1, 2, 3) \),
but we have already dealt with this surface. If \( q \geq 8 \) then \( \gcd(q, \frac{q}{2} - 1) < \frac{q}{2} - 1 \), so \( Y \) has a singular vertex class and the total cone angle of the singularities in that class is \( 2(q - 2)\pi \). Thus the only possible covers are \( X(2, \frac{q}{2} - 1, \frac{q}{2} - 1) \) and \( X(1, 1, q - 2) \); but both these possibilities are accounted for by Lemma 6.

Remark 4. Note that Equation (3) holds for all translation covers \( f : X(a_1, a_2, a_3) \to X(b_1, b_2, b_3) \), even if \( a_1 + a_2 + a_3 \neq b_1 + b_2 + b_3 \). We use this fact extensively in the proof of Theorem 1.

Lemma 16. Let \( f : X \to Y \) be a translation cover of triangular billiards surfaces. Let \( m \) be the smallest integer such that all singularities of \( Y \) have cone angle at least \( 2m\pi \). Suppose that \( \deg f < m \). Then for each vertex class \( C_i \) on \( X \), \( f(C_i) \) consists entirely of singular points entirely of nonsingular points.

Proof. Let \( m \) be as above and assume that \( \deg(f) < m \). Suppose for contradiction that for some \( j \), \( f(C_j) \) contains singular points and nonsingular points. Each member of \( C_j \) has the same cone angle, and this cone angle must be at least \( 2m\pi \), since some of the members are mapped by a translation cover to a singularity of cone angle \( 2m\pi \). Thus, for those elements of \( C_j \) which are mapped to nonsingular points, the definition of a ramified cover requires that \( f \) be locally of degree at least \( m \), which contradicts our assumption that \( \deg(f) < m \). This completes the proof.

4.6 Proof of the Main Theorem

Now we can prove Theorem 1 which for our ease we restate in the following way.

Theorem 1. Suppose \( f : X \to Y \) is a translation cover of triangular billiards surfaces of degree greater than 1. Then \( f \) is of degree 2, and is a composition of one or two of the covers \( f_i \) described in Lemma 6.

Proof. Suppose \( X := X(a_1, a_2, a_3) \), \( Y := X(b_1, b_2, b_3) \), and \( f : X \to Y \) is a translation cover of degree \( \deg f > 1 \). Assume that the genus of \( X \) is greater than 1. Write \( Q_X := a_1 + a_2 + a_3 \) and \( Q_Y := b_1 + b_2 + b_3 \). Let \( v_1, v_2, v_3 \) and \( w_1, w_2, w_3 \) be the corresponding vertices of \( T(a_1, a_2, a_3) \) and \( T(b_1, b_2, b_3) \) respectively. By Corollary 1, \( X \) and \( Y \) have the same holonomy field \( k \). By Corollary 5 we have \( Q_Y \in \{2Q_X, Q_X, \frac{1}{2}Q_X\} \). If \( Q_Y = 2Q_X \), then by Equation (3), we must have
Lemma 16 tells us that if \( \deg b \leq \frac{Q_X}{2} = \frac{Q_Y}{4} \), then \( \sum_{i \in Q_Y} b_i \geq \frac{3}{4} Q_Y \), which is only the case for the following surfaces with even \( Q \)-value: \( X(1, 1, 2), X(1, 2, 3), X(3, 4, 5) \). Of course, \( Q_X \geq 3 \), so \( Y \neq X(1, 1, 2) \). If \( Y = X(1, 2, 3) \) then \( X = X(1, 1, 1) \), which is of genus 1, a contradiction. If \( Y = X(3, 4, 5) \), then \( Y \) has a singularity with cone angle \( 10\pi \). But, no surface \( X \) with \( Q_X = 6 \) could have a cone angle of at least \( 10\pi \).

If \( Q_Y = Q_X \), then we are done by Lemma 16. Thus, appealing to Corollary 8, we shall assume for the remainder of the proof that \( Q_X = 2Q_Y \).

If \( Y \) has no singular vertex classes, then since \( Q \) is odd, we must have \( Y = X(1, 1, 1) \). There are only two surfaces with a \( Q \)-value of 6: they are \( X(1, 1, 4) \) and \( X(1, 2, 3) \), and each of these surfaces covers \( X(1, 1, 1) \) as described in Lemma 8. If \( Y \) has three singular vertex classes, then Equation (3) implies that \( f \) can only be a degree 2 balanced cover. Thus we are done by Lemma 8.

There are two cases remaining: \( Y \) may have either one or two singular vertex classes.

**Case 1.** \( Y \) has one singular vertex class.

In this case we have, without loss of generality, \( b_1|Q_Y \), \( b_2|Q_Y \), and \( b_3 \nmid Q_Y \). Since \( b_1 \) and \( b_2 \) are divisors of the odd number \( Q_Y := b_1 + b_2 + b_3 \), \( b_3 \) must also be odd. Therefore \( \frac{b_3}{\gcd(b_3, Q)} \geq 3 \). The cone angle at each of the singularities of \( Y \) corresponding to \( b_3 \) is \( \frac{b_3}{\gcd(b_3, Q)} \cdot 2\pi \geq 6\pi \).

Equation (2) eliminates all possible \( Y \) for \( \deg f \geq 4 \) except \( Y = X(3, 5, 7) \). But, again by Equation (3), the only possible degree 4 covering surface would be \( X(1, 1, 28) \), and such a cover would have to be balanced, contradicting Lemma 8.

If \( \deg f = 2 \), Lemma 10 tells us that if \( \deg f = 2 \) then for each \( j = 1, 2, 3 \), we must have that \( f(\pi_X^{-1}(v_j)) \cap \pi_Y^{-1}(w_j) \) is either empty or all of \( f(\pi_X^{-1}(v_j)) \).

Suppose that \( Y = X(3, 5, 7) \). Lemma 10 restricts the possible degree 2 covers to surfaces of the form \( X(14, a_2, a_3) \), where each of \( a_2 \) and \( a_3 \) is either a divisor of 30 or twice a divisor of 30. The only possibility this leaves is \( X(15, 14, 1) \). But any translation cover \( X(15, 14, 1) \rightarrow X(3, 5, 7) \) would have to be balanced, so Lemma 8 applies.

Now suppose that \( Y \neq X(3, 5, 7) \). Let \( C \) be the singular vertex class of \( Y \). We must have \( \frac{b_3}{Q} > \frac{1}{2} \), and so by Remark 4, \( C \) must correspond to an obtuse angle \( \theta \) of the billiard table. Let \( \tilde{X} \) be the surface obtained from \( X \) by puncturing all singular vertex classes of \( X \) which are not contained in \( f^{-1}(C) \). Since \( \frac{b_3}{Q} > \frac{1}{2} \) and \( f \) is degree 2, the sum of the angles of the billiard table corresponding to the vertex classes in the \( f \)-preimage of \( C \) must be obtuse. Thus we can apply Lemma 5 to \( \tilde{X} \). The restriction of \( f \) to \( X \) is balanced. Since \( Y \) has only one singular vertex class, elements of \( C \) must have Type II fingerprints unless \( T(b_1, b_2, b_3) \) is isosceles.
If the fingerprints are Type II, then Proposition 1 and Lemma 5 demonstrate that
$X$ and $Y$ are translation equivalent. So the only possibility is that the fingerprints
are Type I. In that case $Y$ is an isosceles triangular billiards surface. Let $C'$ be
a vertex class on $X$ that is in $f^{-1}(C)$, and write $\theta = \frac{b_3 \pi}{Q}$. The billiard table
angle that $C'$ corresponds to is either $\theta$ or $\theta/2$. If the angle is $\theta$, then $X$ and $Y$ are
translation equivalent. If the angle is $\theta/2$, then there is another vertex class on
$X$ which is also mapped to $C$. But then that vertex class would also correspond
to an angle of $\theta/2$, and we would have that $X$ is an isosceles triangular billiards
surface cover $Y$ via a cover described in Lemma 6.

If $\text{deg}(f) = 3$: Then Equation (3) allows only the following possibilities for $Y$:
the surfaces

$$
Y_n = \begin{cases} 
X(3, n, 2n - 3) & 3 \nmid n \\
X(1, \frac{n}{3}, \frac{2n}{3} - 1) & 3 \nmid n 
\end{cases}
$$

Note that $\gcd(2n - 3, 3n) \in \{1, 3\}$. First suppose that $\gcd(2n - 3, 3n) = 1$.
Then $Q = 3n$ (thus $n$ is odd), $3 \nmid n$, and we have $Y_n = X(3, n, 2n - 3)$. We
have that $n \geq 5$ and hence that $2n - 3 \geq 7$. On $Y_n$, there is only one singular
vertex class and the cone angle of each singular point is $(2n - 3)2\pi$. Thus Lemma
16 applies here. Since $Y_n$ is never isosceles, each singular point has a Type II
fingerprint. Let $\tilde{X}$ be the surface obtained from $X$ by deleting all singularities of
$X$ which $f$ maps to nonsingular points, and let $\tilde{f}$ be the restriction of $f$ to $X$. By
Lemma 16, the elements of $X - \tilde{X}$ are the union of entire vertex classes. Thus a
Type II fingerprint on $\tilde{X}$ will uniquely identify the triangular billiards table used
to generate $X$, by Lemma 5. Because $\tilde{f}$ is a balanced map, each singular point of
$\tilde{X}$ must have the same Type II fingerprint (on $\tilde{X}$) as its $\tilde{f}$-image on $Y$. But,
a Type II fingerprint uniquely identifies the triangle used to generate the surface
(this works for $\tilde{X}$ as well); hence $X$ and $Y_n$ are the same billiards surface, and
Lemma 2 says that a triple cover is impossible.

Now suppose that $\gcd(2n - 3, 3n) = 3$. Then the cone angle of each singular
point on $Y_n$ is $\frac{2n - 3}{3}2\pi$. If $n > 6$ then $\frac{2n - 3}{3} > 3$, so that again we can apply
Lemma 16 and Lemma 5 and the same fingerprint argument goes through. The
remaining cases are $n = 3, 6$. We have $Y_3 = X(1, 1, 1)$ and $Y_6 = X(1, 2, 3)$, neither
of which have singularities.

**Case 2. $Y$ has two singular vertex classes.**

Assume $b_1 | Q$ and $b_2, b_3 \nmid Q$. Since $Q$ is odd, $\frac{b_1}{Q} \leq \frac{1}{3}$, so Equation (3) implies
that $\text{deg}(f) \leq 3$. But, if $\text{deg}(f) = 3$, Equation (3) also implies that $f$ is balanced,
contradicting the result of Lemma 8 that balanced covers are of degree at most 2.
Thus $\text{deg}(f) = 2$. 

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Note that $b_2$ and $b_3$ must have the same parity.

Subcase 2A. Both $b_2$ and $b_3$ are odd.

Then $\frac{b_i}{\gcd(b_i, Q)} \geq 3$, so by Lemma 16 each vertex class of $X$ maps to all singular points or all nonsingular points.

If one vertex class of $X$ maps to nonsingular points: Say the vertex class $C_1$ corresponding to $a_1$ maps to nonsingular points. Then $a = 2b_1$, and $2b_1|2Q$, so $C_1$ is nonsingular, so $f$ is balanced.

If two vertex classes of $X$ map to nonsingular points: Let them be $C_1$ and $C_2$, corresponding to $a_1$ and $a_2$. If $C_1$ is singular, then by Lemma 16 we have $a_1 = 2d$ for some $d|Q$. But since $a_3 = 2(b_2 + b_3)$, this would mean that all the $a_i$ are even, contradicting the fact that $\gcd(a_1, a_2, a_3) = 1$.

Subcase 2B. Both $b_2$ and $b_3$ are even.

If one vertex class of $X$ maps to nonsingular points: Let it be $C_1$. We have $a_2 + a_3 = 2(b_2 + b_3)$, so $a_1$ must be even. But also $a_2$ and $a_3$ must be even, since $2\frac{b_i}{\gcd(b_i, Q)}$ and $\frac{b_i}{\gcd(b_i, Q)} \frac{a_j}{\gcd(a_j, Q)}$ for each $i,j \in \{2,3\}$. Again, this is a contradiction.

If two vertex classes of $X$ map to nonsingular points: Let them be $C_1$ and $C_2$. We have that $a_3 = 2(b_2 + b_3)$ is even. If $C_1$ is singular then again we have that $a_1$ (and hence $a_2$) is even, once more contradicting that $\gcd(a_1, a_2, a_3) = 1$. Hence $C_1$ and $C_2$ are nonsingular, and $f$ is balanced.

\[\square\]

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