Hopf Term, Loop Algebras and Three Dimensional Navier-Stokes Equation

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Abstract

The dynamics of the 3 dimensional perfect fluid is equivalent to the motion of vortex filaments or “strings”. We study the action principle and find that it is described by the Hopf term of the nonlinear sigma model. The Poisson bracket structure is described by the loop algebra, for example, the Virasoro algebra or the analogue of O(3) current algebra. As a string theory, it is quite different from the standard Nambu-Goto string in its coupling to the extrinsic geometry. We also analyze briefly the two dimensonal case and give some emphasis on the $w_{1+\infty}$ structure.
1 Introduction

As Polyakov emphasized in his recent lecture\cite{1}, the string theory\cite{2} may be one of the unifying concepts in physics, which is quite essential to the understanding of QCD, quantum gravity, cosmology or even three dimensional Ising model. The application of string theory to these fields are enormously successful, due to its robust mathematical structure, characterized by the Virasoro algebra, the Kac-Moody algebras, BRST symmetry or topological methods. In some occasions, we could derive exact results just as the consequence of their infinite dimensional symmetries.

Some months ago, Polyakov\cite{3} (See also \cite{4}) made a bold proposal that the conformal symmetry may determine the critical exponent of the two dimensional turbulence exactly. In his argument, only some simple hypothesis together with the conformal invariance gives an discrete series of permitted spectrum of the exponent of the turbulence. For three dimensional case, Migdal\cite{5} proposed to use the loop expansion method of QCD may give a nice tool to analyze turbulence. It seems that serious attempt to use string technique to the fluid dynamics has just begun.

In this letter, we would like to propose another “stringy” viewpoints to the Navier-Stokes equation. In everyday life, virtually everybody knowns the existence of “string”—the vortex filament. It may be observed everywhere, in the motion of tornado, cigaret smoke, or in the coffee cup. Usually, this extended object is dissipated after a while. However, in the limit where the viscosity vanishes, these filaments can keep their identities due to the conservation of circulation. In this case, the field theoretical description of fluid reduces to the dynamics of vortex filament. Although the classical motion of this “string” has been studied in every detail \cite{6}, we do not seem to have enough knowledge from the viewpoint of “string theory”. In this letter, we derive the action principle and the Poisson bracket structure for the vortex filament. The kinetic action does not have the standard form but is described by the Hopf term of the $O(3)$ non-linear sigma model\cite{7}. Since our action is first order with respect to time, we need to use the canonical formalism for the singular system. The algebraic structure, thus obtained, is given by loop algebras, such as the Virasoro algebra or an analogue of the $O(3)$ current algebra. We leave the application of our formalism in the future issues.

2 Two-dimensional vortices and $w_{1+\infty}$ algebra

As a warm-up, let us think of the dynamics of two-dimensional vortices. It is interesting in its own sake since it gives a nice representation of $w_{1+\infty}$ algebra.\cite{8} In two dimensions, the Navier-Stokes equation takes the following form,

$$\frac{\partial \omega}{\partial t} = -\nabla_\alpha G_\alpha,$$

(1)
where $\omega(x) = \epsilon_{\alpha\beta} \partial_{\alpha} u_{\beta}(x)$ is the two dimensional vorticity field and
\[ G_{\alpha}(x) = \omega u_{\alpha}(x) - \nu \partial_{\alpha} \omega. \] (2)

$\nu$ is the viscosity.

It is very important that if we start from the delta-function distribution for the vorticity field,
\[ \omega(x) = \sum_{k=1}^{N} \Gamma_{k} \delta(x - x_{k}(t)), \] (3)
it keeps its form under time evolution as long as the viscosity is vanishing. The dynamics of the Navier-Stokes equation (field theory equation) thus reduces to the motion of vortex “particles”, i.e. the equation of motion can be written by $x_{k}(t)$ alone. To see this, it is convenient to combine $x$ and $y$ coordinate by $z = x + iy$ and $\bar{z} = x - iy$. With these variables, the equation of motion for the vortex center becomes,
\[ \frac{\partial z_{k}(t)}{\partial t} = -\frac{1}{2\pi i} \sum_{j(\neq k)}^{N} \frac{\Gamma_{j}}{z_{k} - z_{j}} \quad \frac{\partial \bar{z}_{k}(t)}{\partial t} = \frac{1}{2\pi i} \sum_{j(\neq k)}^{N} \frac{\Gamma_{j}}{z_{k} - \bar{z}_{j}}. \] (4)

Although these equations seem quite innocent, it is known that they are far from integrable. Even for the N=3 case, the motion of the vortices become chaotic.

These equations reminds us strongly the motion of anyon. Indeed, the Lagrangian of the system is in some sense related to the $U(1)$ Chern-Simon term. To derive it, let us discuss its Hamiltonian dynamics. The energy functional of the system is given by
\[ H = \frac{1}{2} \int d^{2}x \sum_{\alpha=1,2} u_{\alpha}(x) u_{\alpha}(x) \]
\[ = -\frac{1}{8\pi} \sum_{j \neq k}^{N} \Gamma_{j} \Gamma_{k} \ln \left[ (z_{j} - z_{k})(\bar{z}_{j} - \bar{z}_{k}) \right]. \] (5)

The sympletic structure which gives the Hamiltonian equation of motion is
\[ [f, g]_{PB} = -2i \sum_{j=1}^{N} \frac{1}{\Gamma_{j}} \frac{\partial (f, g)}{\partial (z_{j}, \bar{z}_{j})}. \] (6)

The equations of motion Eq.(4) are rewritten as $\frac{\partial z_{i}}{\partial t} = [z_{i}, H]_{PB}$ and $\frac{\partial \bar{z}_{i}}{\partial t} = [\bar{z}_{i}, H]_{PB}$. In this system, the coordinates $z$ and $\bar{z}$ are the canonical conjugate with each other. The Lagrangian which gives this sympletic structure and the equation of motion is given by,
\[ \mathcal{L} = -i \sum_{i=1}^{N} \Gamma_{i} \frac{\partial z_{i}}{\partial t} \bar{z}_{i} + \frac{1}{8\pi} \sum_{i \neq j}^{N} \Gamma_{i} \Gamma_{j} \ln \left[ (z_{i} - z_{j})(\bar{z}_{i} - \bar{z}_{j}) \right]. \] (7)

The motion of incompressive fluid gives the diffeomorphism of two dimensional surface which preserve the area element (area preserving diffeomorphism=APD). In the string theory, this symmetry is sometimes called $w_{1+\infty}$ algebra and plays essential role in several
Dynamical variables of vortices give a representation of this algebra through
Poisson bracket. In general, an element of APD is given by the diffeomorphism,
\[
\delta z = \frac{\partial \epsilon(z, \bar{z})}{\partial \bar{z}}, \quad \delta \bar{z} = -\frac{\partial \epsilon(z, \bar{z})}{\partial z}.
\] (8)
Since
\[
[z_i, \epsilon(z_j, \bar{z})]_{PB} = \delta_{ij} \frac{2}{i\Gamma} \frac{\partial \epsilon(z, \bar{z})}{\partial \bar{z}}\quad [\bar{z}_i, \epsilon(z_i, \bar{z}_i)]_{PB} = -\delta_{ij} \frac{2}{i\Gamma} \frac{\partial \epsilon(z, \bar{z})}{\partial z},
\] (9)
the generator of the transformation Eq. (8) is given by
\[
\hat{\epsilon} = \frac{i}{2} \sum_{j=1}^{N} \Gamma_j \epsilon(z_i, \bar{z}_i) = \frac{i}{2} \int d^2 x \omega(x) \epsilon(x).
\] (10)
In a sense, it gives the relation between the operator of the quantum mechanics \(z\) and \(\bar{z}\) and the operator of the quantum field theory, \(\omega\). This correspondence reminds us of the collective field theory method.\[3\]\[9\] The Poisson bracket between \(\omega\) is
\[
[\omega(x), \omega(y)]_{PB} = -2 \epsilon_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} (\omega(y) \delta(x-y)).
\] (11)
This is a standard Poisson bracket which defines the APD.

Recently, Polyakov\[3\] proposed an approach to the two dimensional turbulence based on the conformal field theory. In his method, \(\psi = \Delta^{-1} \omega\) is identified with a primary field in a minimal model of CFT. The operator production of the \(\psi\) field is essential to deal with the non-linear term in Navie-Stokes equation. The proposed OPE for \(\psi\) field is that
\[
[\psi] \times [\psi] \sim [\phi] + \cdots.
\] (12)
On the right hand side of this OPE, there should not appear the \(\psi\) field itself. Since \(\psi\) field is the pseudoscalar, it should be true as long as the symmetry of the system is \(Vir \times \overline{Vir}\). This statement is somehow in contradiction to our Poisson bracket Eq. (11). This is possible since our symmetry is APD and the structure constants themselves become parity-odd.

Of course, it is dangerous to think that one may directly apply the \(w_{1+\infty}\) symmetry to the two dimensional turbulence. In that case, we take the limit when the Reynolds number approaches infinity. As a consequence, the viscosity disappears in the final expression. However, even if the viscosity is infinitely small, there appears finite amount of outcome. It is because the essential nature of the turbulence comes from the balance between the dissipation due to the viscosity and the enstrophy flow from outside. As Polyakov\[3\] remarked in his paper, the situation seems to be similar to the anomaly in the quantum field theory.

In this way, the \(w_{1+\infty}\) symmetry may be broken in the turbulence even if we take the limit of vanishing viscosity. It is extremely interesting to investigate how \(w_{1+\infty}\) is broken and possibly the Virasoro structure emerges.

\[1\] I would like to thank Dr. S. Iso for pointing out this fact.
3 Action for vortex filament

For the three dimensional case, the computation becomes more involved but basic strategy remains the same. In this case, instead of thinking about point vortices, we need to consider the vortex string. The Navier-Stokes equation which is convenient for us is

\[
\frac{\partial \vec{\omega}}{\partial t} = -\vec{\nabla} \times \vec{G}, \quad \vec{G} = \vec{\omega} \times \vec{u}.
\] (13)

The vortex filament is described by the vortex field,

\[
\vec{\omega}(x) = \sum_{i=1}^{N} \Gamma_i \int d\sigma_i \frac{\partial X_i^j(\sigma_i, t)}{\partial \sigma_i} \delta^{(3)}(\vec{x} - \vec{X}_i(\sigma_i, t))
\] (14)

\[
\vec{\omega} = \vec{\nabla} \times \vec{u}
\]

\[
\vec{u}(x) = \sum_{i=1}^{N} \frac{\Gamma_i}{4\pi} \int d\sigma_i \frac{\partial X_i^j}{\partial \sigma_i} \times \frac{\vec{x} - \vec{X}_i}{|\vec{x} - \vec{X}_i|^3}.
\] (15)

One puts these formulae into the Navier-Stokes equation. After some computations, one may find the time evolution of the “string field” \( \vec{X}(\sigma, t) \),

\[
\frac{\partial \vec{X}_j}{\partial \sigma_j} \times \frac{\partial \vec{X}_j}{\partial t} = \frac{\sum_{k} \Gamma_k}{4\pi} \int d\sigma_k \frac{\partial \vec{X}_k}{\partial \sigma_k} \times \frac{\vec{X}_j - \vec{X}_k}{|\vec{X}_j - \vec{X}_k|^3}.
\] (16)

This is equivalent to

\[
\frac{\partial \vec{X}_j}{\partial t} = \frac{\sum_{k} \Gamma_k}{4\pi} \int d\sigma_k \frac{\partial \vec{X}_k}{\partial \sigma_k} \times \frac{\vec{X}_j - \vec{X}_k}{|\vec{X}_j - \vec{X}_k|^3} + \alpha \frac{\partial \vec{X}_j}{\partial \sigma_j}.
\] (17)

\( \alpha \) can not be determined from the Navier-Stokes equation alone. This term is present because of the freedom of the parametrization of the string \( \sigma_j \).

We claim that the action of the vortex filament is given by,

\[
S = S_0 - H
\]

\[
S_0 = \sum_{j=1}^{N} \frac{\Gamma_j}{3} \int dt \int d\sigma_j \vec{X}_j \cdot \left( \frac{\partial \vec{X}_j}{\partial \sigma_j} \times \frac{\partial \vec{X}_j}{\partial t} \right)
\]

\[
H = \frac{1}{2} \int d^2 x |\vec{u}(x)|^2
\]

\[
= \frac{1}{8\pi} \sum_{jk} \Gamma_j \Gamma_k \int d\sigma_j \int d\sigma_k \left( \frac{\partial \vec{X}_j}{\partial \sigma_j} \cdot \frac{\partial \vec{X}_k}{\partial \sigma_k} \right) \frac{1}{|\vec{X}_j - \vec{X}_k|}.
\] (18)

As far as we know, this action seems to be new. The second part of the action is identical to the Hamiltonian of the system. Together with its variation,

\[
\delta H = \frac{1}{4\pi} \sum_{jk} \Gamma_j \Gamma_k \int d\sigma_j \int d\sigma_k \delta \vec{X}_j \cdot \left( \frac{\partial \vec{X}_j}{\partial \sigma_j} \times \frac{\partial \vec{X}_k}{\partial \sigma_k} \right) \left( \frac{1}{|\vec{X}_j - \vec{X}_k|^3} \right).
\] (19)
and the variation of the first term, Eq.(18) gives equation of motion in the form Eq.(16).

The first term of the action \( S_0 \) is known as the Hopf term in the theory of O(3)-invariant nonlinear sigma model.\[^7\] It counts the instanton numbers and describes the structure of the \( \theta \) vacuum. It is obvious that our \( S_0 \) has also topological nature. Unlike the Nambu-Goto string, it does not couple to the two dimensional metric on the world sheet. However, it is invariant under the reparametrization of world sheet variables because of its Chern-Simon type structure.

This situation strongly reminds us of the Chern-Simon approach to the braid group \[^{10}\] where Witten has shown that the Jones Polynomial can be described by the correlation functions of \( su(2) \) Chern-Simon theory. In our case, the fundamental conserved quantities of the 3d Navier-Stokes Equation are the energy and the helicity.\[^{11}\] The latter has definite topological meaning since it is expressed by,

\[
h = \int d^3x \bar{\omega}(x) \bar{u}(x) = 2 \sum_{i<j} \beta_{ij} \Gamma_i \Gamma_j, \tag{20}\]

where

\[
\beta_{ij} = \frac{1}{4\pi} \int d\sigma_i \int d\sigma_j \frac{\partial X_i}{\partial \sigma_i} \cdot \left( \frac{\partial X_j}{\partial \sigma_j} \times \frac{\bar{X}_i - \bar{X}_j}{|\bar{X}_i - \bar{X}_j|^3} \right). \tag{21}\]

We remark that \( \beta_{ij} \) is the linking number of two filaments \( \bar{X}_i \) and \( \bar{X}_j \). Also, the interesting analogy is the Chern-Simon gravity theory in three dimensions.\[^{13}\] Whereas the three dimensional gravity theory is the gauge theory of diffeomorphism of three dimensional space-time, the Poisson bracket of Navier-Stoke equation is volume preserving diffeomorphism.\[^{17}\]

Finally, we would like to make a comment on the difference between the vortex motion and that of Nambu-Goto string. The issue is their coupling to the extrinsic curvature.\[^{4}\] To see this, we should first remark that our formulae Eqs.(16-19) are not well-defined due to the divergence in the \( \sigma \) integral. For example, the integral which appear in the left hand side of Eq.(17) diverges logarithmically. We put \( j = k \) and write \( \sigma_j = \sigma \) and \( \sigma_k = \sigma + \theta \). Collecting the most divergent peace gives,

\[
\left. \frac{\partial \bar{X}}{\partial t} \right|_{\sigma=\sigma_0} \approx \left[ \Gamma \frac{1}{4\pi} \int d\theta \frac{1}{2|\theta|} \right] \frac{\partial X}{\partial \sigma_0} \times \frac{\partial^2 X}{\partial \sigma_0^2}. \tag{22}\]

In this form, one recognizes that one may absorb the divergent factor by the redenition of time variable,

\[
t \longrightarrow \tilde{t} = \Gamma \Lambda t, \quad \Lambda = \frac{1}{4} \log(L/\epsilon), \tag{23}\]

\[^{2}\] This part is a short review. Readers who are familiar with vortex motion can skip the rest of this section.
where \( L \) is the length of the vortex and \( \epsilon \) is the ultraviolet cutoff which may be identified with the diameter of vortex filament. In the following, we rewrite \( \tilde{t} \) as just \( t \) for the simplicity.

The remaining term is directly related to the extrinsic geometry of the curve. Since the parametrization is arbitrary, we take a special choice of \( \sigma \), \( \left| \partial \vec{X} / \partial s \right|^2 = 1 \), i.e. \( s \) gives the length of the filament. With this parametrization, we may use the Frenet-Serret formula,

\[
\frac{\partial \vec{X}}{\partial s} = \vec{t}, \quad \frac{\partial \vec{t}}{\partial s} = \kappa(s)\vec{n}, \quad \frac{\partial \vec{n}}{\partial s} = \tau(s)\vec{b} - \kappa(s)\vec{t}, \quad \frac{\partial \vec{b}}{\partial s} = -\tau(x)\vec{n}.
\] (24)

One may readily prove that

\[
\frac{\partial \vec{X}}{\partial s} \times \frac{\partial^2 \vec{X}}{\partial s^2} = \kappa \vec{b} \left( \frac{\partial \vec{X}}{\partial t} \right). \quad (25)
\]

This is well-known equation which describes that the vortex filament moves to the binormal direction with the velocity proportional to its curvature at that point. Very interestingly, Hashimoto derived that this equation is exactly solvable.\footnote{The Frenet-Serret formula of the world sheet appeared previously in the context of W-gravity theory.} It was proved that by changing the variable,

\[
\psi(s) = \kappa(s) \exp \left( i \int_0^s \tau(s') ds' \right), \quad (26)
\]

Eq.(25) becomes the non-linear Schrödinger equation,

\[
\frac{1}{i} \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2} (|\psi|^2 + A) \psi, \quad (27)
\]

which is integrable.

### 4 Canonical formalism

The action derived in the previous section is the first order in the time variable. Furthermore, there is the first class constraint because of the reparametrization invariance of \( \sigma \). This is an ideal example to learn the canonical formalism of the singular system. In the following, we consider the special case \( N = 1 \) for the simplicity. The extension to general case is straightforward.

The canonical conjugate of the string field is given by,

\[
\Pi_\rho(\sigma,t) \equiv \frac{\delta S_0}{\delta X^\rho} = \Gamma^\nu_{\mu\rho} \frac{\partial X^\nu}{\partial \sigma}. \quad (28)
\]
On the right hand side, there is no time derivative. Hence the definition of momentum already gives constraints. Let us denote them as \( \phi_\rho \),
\[
\phi_\rho(\sigma, t) \equiv \Pi_\rho(\sigma) - \frac{\Gamma}{3} \epsilon_{\mu\nu\rho} X^\mu \frac{\partial X^\nu}{\partial \sigma} \approx 0.
\] (29)

To see the structure of these constraints, we take the Poisson bracket of them,
\[
[\phi_\mu(\sigma), \phi_\nu(\sigma')]_{PB} = -\Gamma \epsilon_{\mu\nu\rho} \frac{\partial X^\rho}{\partial \sigma} \delta(\sigma - \sigma') \equiv M_{\mu\nu}(X) \delta(\sigma - \sigma').
\] (30)

Following the general procedure, we add terms proportional to the constraint to the Hamiltonian,
\[
\delta H = \sum_{\rho=1}^{3} f_\rho(X, \Pi) \phi_\rho.
\] (31)

We need to check whether the time evolution of \( \phi \) with respect to new Hamiltonian gives rise to any new constraints. The Poisson bracket of the constraint with the extended Hamiltonian is,
\[
[\phi_\rho, H + \delta H]_{PB} \approx \epsilon_{\rho\mu\nu} \frac{\partial X^\mu}{\partial \sigma} u^\nu(X(\sigma)) - \Gamma \epsilon_{\rho\mu\nu} f^\mu \frac{\partial X^\nu}{\partial \sigma}.
\] (32)

If we choose \( f^\mu = \frac{1}{4} u^\mu(X) \), the right hand side vanishes weakly. Hence, there are no secondary constraints.

Since the rank of matrix \( M_{\mu\nu} \) is two, one linear combination of the constraint fields \( \phi_\mu \) should be the first class. It corresponds to the reparametrization of the arbitrary parameter \( \sigma \). Explicitly, we define,
\[
T(\sigma) \equiv -\frac{\partial X^\mu}{\partial \sigma} \phi_\mu = -\frac{\partial X^\mu}{\partial \sigma} \Pi_\mu.
\] (33)

It satisfies the classical version of the Virasoro algebra,
\[
[T(\sigma), T(\sigma')]_{PB} = 2T(\sigma') \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') + \frac{\partial T(\sigma')}{\partial \sigma'} \delta(\sigma - \sigma').
\] (34)

Poisson brackets with other fields are given by
\[
[T(\sigma), X^\mu(\sigma')]_{PB} = \frac{\partial X^\mu}{\partial \sigma} \delta(\sigma - \sigma').
\] (35)
\[
[T(\sigma), \Pi_\mu(\sigma')]_{PB} = -\Pi_\mu(\sigma) \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma').
\] (36)
\[
[T(\sigma), \phi_\mu]_{PB} = -\phi_\mu(\sigma) \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma').
\] (37)

The last equation shows that \( T(\sigma) \) is indeed the first class constraint.

In order to define the Dirac bracket, we need to make gauge fixing of the parametrization \( \sigma \). A natural choice is to use the length of the vortex filament as \( \sigma \),
\[
\chi \equiv \left| \frac{\partial X}{\partial \sigma} \right|^2 - 1 \approx 0.
\] (38)
The Poisson brackets of $\chi$ with other constraints are
\[
[\phi_{\mu}(\sigma), \chi(\sigma')]_{PB} = \frac{\partial X^{\mu}}{\partial \sigma'} \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma')
\] (39)

Constraints $\phi$ together with the gauge fixing condition $\chi$ becomes second class. In general, if the second class constraint $\xi_{\alpha}$ satisfies Poisson bracket, $[\xi_{\alpha}, \xi_{\beta}]_{PB} = C_{\alpha \beta}$, the Dirac bracket is given by
\[
[F, G]_D = [F, G]_{PB} - \sum_{\alpha, \beta} [F, \xi_{\alpha}]_{PB} \left( C^{-1} \right)_{\alpha \beta} [\xi_{\beta}, G]_{PB}.
\] (40)

In our case, from Eq.(30) and Eq.(39) the matrix $C$ is given by
\[
C_{AB}(\sigma - \sigma') = \begin{pmatrix}
0 & a_3 & -a_2 & b_1 \\
-a_3 & 0 & a_1 & b_2 \\
a_2 & -a_1 & 0 & b_3 \\
-b_1 & -b_2 & -b_3 & 0
\end{pmatrix}
\]
\[
\begin{cases}
    a_i = -\Gamma^{X_i} \frac{\partial X}{\partial \sigma} \delta(\sigma - \sigma') \\
    b_i = \frac{\partial X_i}{\partial \sigma} \delta(\sigma - \sigma')
\end{cases}
\] (41)

The index $A$ stands for $\phi^A$ for $A=1,2,3$, and $\chi$ for $A=4$. Because of the constraint $\chi \approx 0$, the inverse of this matrix becomes very simple,
\[
(C^{-1})^{AB}(\sigma - \sigma') = \begin{pmatrix}
0 & c_3 & -c_2 & d_1 \\
-c_3 & 0 & c_1 & d_2 \\
c_2 & -c_1 & 0 & d_3 \\
-d_1 & -d_2 & -d_3 & 0
\end{pmatrix}
\]
\[
\begin{cases}
    c_i = -\Gamma^{X_i} \frac{\partial X}{\partial \sigma} \delta(\sigma - \sigma') \\
    d_i = \frac{\partial X_i}{\partial \sigma} \theta(\sigma - \sigma')
\end{cases}
\] (42)

with $\theta$ is the step function. The Dirac bracket may be computed by using the general formula Eq.(41) and, for special case, it gives,
\[
[X^{\mu}(\sigma), X^{\nu}(\sigma)]_D = \Gamma^{\rho \nu \rho} \frac{\partial X}{\partial \sigma} \delta(\sigma - \sigma').
\] (43)

This algebra resembles $O(3)$ current algebra. However, there appear the derivative of $\vec{X}$ instead of $\vec{X}$ itself. Furthermore, we should not forget that there is a constraint Eq.(38).

5 Discussion

There are several issues that we could not discuss well in this letter. Let us make a list to illustrate the future issues.

- **Relativistic Formulation:** In this letter, our consideration is restricted to the non-relativistic situation. Because of this restriction, only one Virasoro algebra could appear. If we formulate it relativistically, we expect we have the Virasoro algebras both in right and left moving sectors.
Quantization and BRST formalism: Since our algebra Eq. (38) is slightly different from the current algebra, the quantization and the BRST analysis seems a little bit non-trivial. Anyway, what is “the critical dimension” of our string?

Description of Turbulence: As Migdal observed [5], there seems to be a relation between the viscosity and the plank constant. In this sense, the quantized version of our analysis should give some aspects of the turbulence.

Relation with the quantum gravity: As we discussed in section 3, there are some analogies between three dimensional gravity theory and the Navier-Stokes equation. It seems very important to investigate their relation more explicitly. Also, in two dimensions, the area preserving diffeomorphism describe some aspect of the quantum gravity [8]. The appearance of APD in two dimensional Navier Stokes seems to establish another link.

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Note Added: After we submitted this paper, we learned that substantial part of this work have been done previously in beautiful papers [15], [16]. See also [17] where some material on two dimensional case was discussed. We would like to give our thanks to Drs. T. J. Allen, M. Genna, S. Yahikozawa for indicating these references.
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