Optimal Prandtl expansion around concave boundary layer

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Abstract

We provide an optimal Gevrey stability result for general boundary layer expansions, under a mild concavity condition on the boundary layer profile. Our result generalizes (and even improves in the non strictly concave case) the one obtained in [7], restricted to expansions of shear flow type.

1 Introduction

We are interested in the high Reynolds number dynamics of the Navier-Stokes equation in a half-plane:

$$\begin{align*}
\partial_t u^\nu - \nu \Delta u^\nu + \nabla p^\nu + u^\nu \cdot \nabla u^\nu &= 0, & t > 0, \ x \in \mathbb{T}, \ y > 0, \\
\nabla \cdot u^\nu &= 0, & t \geq 0, \ x \in \mathbb{T}, \ y > 0,
\end{align*}$$

where $\nu$ stands for the inverse Reynolds number. Note that we consider periodic boundary conditions in $x$, but could consider decay conditions as well. As is well-known, the Navier-Stokes solution $u^\nu$ exhibits a boundary layer near $y = 0$, that is a region of high velocity gradients generated by the no-slip condition. A famous modelling of this boundary layer was provided by Prandtl. In modern language, he provided approximate solutions of Navier-Stokes in the form of multiscale asymptotic expansions:

$$v = \sum_{i=0}^{N} \sqrt{\nu} U^{E,i}(t,x,y) + \sum_{i=0}^{N} \sqrt{\nu} \left( V^{bl,i}_{1}(t,x,y/\sqrt{\nu}), \sqrt{\nu} V^{bl,2}_{2}(t,x,y/\sqrt{\nu}) \right)$$

where the profiles $U^{E,i} = U^{E,i}(t,x,y)$ describe the flow away from the boundary, and the profiles $V^{bl,i} = V^{bl,i}(t,x,Y)$ are boundary layer correctors, that go to zero exponentially fast in variable $Y = y/\sqrt{\nu}$. We stress that there is a factor $\sqrt{\nu}$ between the amplitudes of the horizontal and vertical components of the boundary layer profiles: this is consistent with the divergence-free condition. In particular, the leading order term $U^{E} := U^{E,0}$ solves the Euler
equation, while the leading order boundary corrector $V^{bl} := V^{bl,0}$ solves the modified Prandtl equation

$$
\begin{align*}
\partial_t V^{bl}_1 + (U^E_1|_{y=0} + V^{bl}_1)\partial_x V^{bl,1} + V^{bl}_1 \partial_x U^E_1|_{y=0} + (Y \partial_y U^E_2|_{y=0} + V^{bl}_2) \partial_y V^{bl}_1 - \partial^2_Y V^{bl}_1 &= 0 \\
\partial_x V^{bl,1} + \partial_Y V^{bl}_2 &= 0 \\
V^{bl}_1|_{Y=0} = -U^E_1|_{y=0}, \quad V^{bl} \to 0, \quad Y \to +\infty
\end{align*}
$$

Prandtl boundary layer theory has revealed illuminating about the mechanism of vorticity generation in fluids, and successful in the quantitative understanding of some model problems, notably the description of the Blasius flow near a flat plate. Still, Navier-Stokes flows of type (1.2) are known to experience instabilities, due to two main mechanisms:

- **Boundary layer separation**, which corresponds to a loss of monotonicity and concavity of the boundary layer profile $V^{BL}_1$, under an adverse pressure gradient. Mathematically, it corresponds to some ill-posedness or blow-up of the Prandtl model.

- **Hydrodynamic instabilities of Tollmien-Schlichting type**, experienced by concave boundary layer flows.

These phenomena have crucial consequences in hydrodynamics and aerodynamics. From the mathematical point of view, describing the stability/instability properties of flows $v$ of type (1.2) is a difficult topic. The evolution of the perturbation $w = w^\nu - v$ obeys the perturbed Navier-Stokes system

$$
\begin{align*}
\partial_t w - \nu \Delta w + \nabla q + v \cdot \nabla w + w \cdot \nabla v &= -w \cdot \nabla w + r, \quad t > 0, \quad x \in \mathbb{T}, \quad y > 0, \\
\nabla \cdot w &= 0, \quad t \geq 0, \quad x \in \mathbb{T}, \quad y > 0, \\
w|_{y=0} &= 0, \quad w|_{t=0} = w_0.
\end{align*}
$$

(1.3)

Here, $r$ represents a remainder term due to the approximation $v$, while $w_0$ is a given initial perturbation of the velocity. We will assume that $r$ and $w_0$ are of the order $O(\nu^n)$ in some norm with $n \gg 1$. In the case of $r$, this is realized by taking $N$ large enough in (1.2). More precisely, one has to consider functional frameworks such that the equations of both Prandtl type and Euler type are uniquely solvable at least locally in time. Then, the point is to understand under which conditions one can obtain uniform (in $\nu$) estimates of $w$ in a suitable norm, that is justification of the Prandtl theory.

An important result in this direction is due to Caflisch and Sammartino [22, 23], who proved local well-posedness of Euler and Prandtl equations, as well as stability results for (1.3) in the case of analytic data. The stability result is then extended by [21, 25, 5, 24, 19], where all of them requires the analyticity near the boundary. This general analytic stability result is somehow optimal, in view of a work of Grenier [10], see also [14]. Grenier studied the case where the Prandtl expansion $v$ in (1.2) is a shear flow: this means that

$$
v = \left( V^{bl}_1(t, x, y/\sqrt{\nu}), 0 \right)
$$

(1.4)

where $V^{bl}_1$ solves the heat equation

$$
\begin{align*}
\partial_t V^{bl}_1 - \partial^2_Y V^{bl}_1 &= 0, \quad V^{bl}_1|_{Y=0} = 0.
\end{align*}
$$

(1.5)
He proved that for some profiles \( V_{1}^{bl} \) that have initially inflexion points, the linearized version of (1.3) admits growing perturbations of the form 

\[
w^{\nu}(t, x, y) \approx e^{\alpha t/\nu^2} e^{ix/\nu^2} \tilde{w}^{\nu}(y),
\]

with fixed \( \alpha > 0 \). This shows that high frequencies \( k \approx 1/\nu^{1/2} \) in variable \( x \) may be amplified by \( e^{\alpha k t} \). In other words, to obtain a bound independent of \( \nu \) over a time \( T = O(1) \) will only be possible if those modes \( k \) have amplitude less than \( e^{-\delta k} \), \( \delta \leq \alpha T \). This necessary exponential decay of the frequency spectrum corresponds to analytic perturbations. Let us note that the result of Grenier relies on the so-called Rayleigh instability, which is an inviscid instability mechanism for shear flows with inflexion points. In terms of hydrodynamics of the boundary layer, the appearance of inflexion points corresponds to the separation phenomenon. Hence, it is a framework in which various negative results exist for the Prandtl equation itself [4, 6, 9, 18].

The case without inflexion points, corresponding to the nicer situation where the boundary layer profile \( V_{1}^{bl} \) is concave in variable \( Y \), is much more involved. Again, the natural first step is to consider the shear flow situation (1.4). The stability of shear flows within the Navier-Stokes equation is an old topic of hydrodynamics, notably studied by Tollmien and Schlichting. See [3] for a detailed account. They showed that generic concave shear flows, although stable in the Euler evolution, exhibit instability in the Navier-Stokes one (albeit with a growth rate vanishing with viscosity). This is the so-called Tollmien-Schlichting instability, revisited on a rigorous basis by Grenier, Guo and Nguyen [12]. Roughly, by using a proper rescaling of these unstable eigenmodes, one can construct for the linearization of (1.3) solutions of the type 

\[
w^{\nu}(t, x, y) \approx e^{\alpha t/\nu^2} e^{ix/\nu^2} \tilde{w}^{\nu}(y).
\]

This time, high frequencies \( k \approx 1/\nu^{3/8} \) may be amplified by \( e^{\alpha k^2 t} \). This is still not compatible with Sobolev uniform bounds. More precisely, under the assumption that the spectral radius of the linearized Navier-Stokes operator is given by the growth rate of the Tollmien-Schlichting instability, one can obtain exponential bounds on the semigroup and from there show nonlinear Sobolev instability of Prandtl expansions of shear flow type: cf [13, 15].

Nevertheless, in the setting of concave boundary layer flows, the class of data \( w_{0} \) for which one can hope uniform (in \( \nu \)) local (in time) control of \( w \) is larger than analytic: namely, one may expect control for data whose Fourier spectrum in \( x \) decays like \( O(e^{-k^{2/3}}) \). This corresponds to the so-called Gevrey class of exponent 3/2.

To show such optimal stability result for general “concave” Prandtl expansions is the main goal of the present paper. It extends the result established in [7], limited to the case when the boundary layer is the shear type like (1.4). See also the recent development [1], still on shear flow expansions. Precise statements will be given in next Section 2. Three preliminary remarks are in order:

- The approach in [7] was very much based on Fourier transform in \( x \), made easy because (1.4) is independent of \( x \). It does not adapt to general Prandtl expansions. The approach in the present paper relies on strongly different ideas.

- The main step in our approach is the derivation of stability estimates for the linearized
equations:
\[
\partial_t w - \nu \Delta v + v \cdot \nabla w + w \cdot \nabla v = f, \quad t > 0, \quad x \in \mathbb{T}, \quad y > 0,
\]
\[
\nabla \cdot w = 0, \quad t \geq 0, \quad x \in \mathbb{T}, \quad y > 0,
\]
\[
w|_{y=0} = 0, \quad w|_{t=0} = w_0.
\]

But to derive such bounds, we do not make any assumption on the spectral radius of the linearized operator, in contrast with works [13, 15].

- A strong point of our analysis is that it applies to boundary layer profiles $V_{bl}^1$ that are concave in $Y$, but not necessarily strictly concave. See Section 2 for detailed hypotheses.

This is important for applications, as can be seen from (1.5): there, $\partial^2 Y V_{bl}^1$ vanishes at the boundary for $Y = 0$ at positive times. Despite such possible degeneracies, we are able to reach Gevrey $\frac{3}{2}$ stability: this was not the case in our previous paper [7], where our Gevrey exponent for stability was less than $\frac{3}{2}$ for non strictly concave flows.

Let us insist that our result is the first one justifying boundary layer theory beyond the analytic scale.

2 Statements of the results

To state our stability result, we first introduce our functional framework. Let $p \in [1, \infty]$, $K \geq 1$, and $\nu \in (0, 1]$. For simplicity we assume $\nu^{-\frac{1}{2}} \in \mathbb{N}$, but it is not at all essential to our argument. We set

\[
\|f\|_p = \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{(j!)^{\frac{1}{2}}} \sup_{j_2=0,\cdots,j} \|e^{-Kt(j+1)}\beta_{j_2} \partial_x^{j_2} f\|_{L^p_T(0,1;L^2_{x,y})},
\]

where

\[
\beta_{j_2} = \chi^{j_2} \partial_y^{j_2}, \quad \chi(y) = 1 - e^{-\kappa y}.
\]

Here $\kappa \in (0,1]$ is a fixed number, which will be taken small enough. We note that $\|f\|_p$ depends on $\nu, \kappa \in (0,1]$ and $K \geq 1$, though we drop this dependence to simplify the notation. Note that for each fixed $\nu$ the norm $\|f\|_p$ is of Sobolev type, but if $\|f\|_p$ is uniformly bounded in $\nu$, it implies a usual Gevrey $\frac{3}{2}$ regularity for the $C^\infty$ function $f$. The reason we can restrict to $j \leq \nu^{-\frac{1}{2}}$ in the sum above is that in (1.3), the stretching term $\nabla v = O(\nu^{-\frac{1}{2}})$ creates at most an amplification $O(e^{C\nu^{-\frac{1}{2}}t})$. For $j \sim \nu^{-\frac{1}{2}}$, it is therefore balanced by the factor $e^{-K(j+1)}$ for large enough $K$. This means that we will be able to close an estimate considering only derivatives up to order $\nu^{-\frac{3}{2}}$.

Our main theorem is the following. Let us set $H^1_{0,\sigma}(T \times \mathbb{R}_+) = \{ f \in H^1_0(T \times \mathbb{R}_+) \mid \text{div } f = 0 \text{ in } T \times \mathbb{R}_+ \}$, the space of all $H^1$ solenoidal vector fields satisfying the noslip boundary condition at $Y = 0$.

**Theorem 2.1.** (Nonlinear stability of concave Prandtl expansions) Let $v = v(t, x, y)$ a divergence-free vector field that fulfills the regularity and concavity conditions gathered in Assumption 1 below, not necessarily of type (1.2). There exists $\kappa_0 > 0$, such that the following
statement holds for any $\kappa \in (0, \kappa_0)$: there exist $C > 0$, $K > 0$, $\delta_0 > 0$ such that: for all $\nu \leq K^{-2}$, if $r \in L^2(0, \frac{1}{K}; L^2(\mathbb{T} \times \mathbb{R}_+)^2)$ and $w_0 \in H^1_{1,\sigma}(\mathbb{T} \times \mathbb{R}_+)$ satisfy

$$||w_0|| + ||\text{rot} \ w_0|| \leq \delta_0 \nu^{\frac{3}{2}}, \quad ||r||_2 \leq \delta_0 \nu^{\frac{13}{4}},$$

then the system (1.3) has a unique solution $w \in C([0, \frac{1}{K}], \mathcal{H} \mathcal{1}_0, \sigma(\mathbb{T} \times \mathbb{R}_+) \cup \mathbb{R}_+^2)$ satisfying

$$\|w\| \infty + \nu^{\frac{1}{2}} \|\text{rot} \ w\| \infty \leq C \nu^{-\frac{1}{2}} (||w_0|| + ||\text{rot} \ w_0|| + \nu^{-\frac{3}{4}} ||r||_2).$$

Here $\text{rot} \ w = \partial_x w_2 - \partial_y w_1$ and $||w_0|| = \sum_{j=0}^{\nu^{\frac{1}{2}}} \frac{1}{(j!)^{\frac{3}{2}}} \sup_{j_2=0, \ldots, j} \|\beta_{j_2} \partial_x^{j_2} w_0\|_{L^2_x,y}.$

To complete the statement of our theorem, it remains to describe the set of assumptions on $\nu$ that yield Theorem 2.1. Of course, these assumptions are designed to be satisfied by Prandtl expansions of type (1.2), when $V^{\nu}_{bl}$ has some mild concavity. Due to the boundary layer variable $Y$, it is more convenient to work with rescaled variables $(\tau, X, Y) := \nu^{-\frac{1}{2}} (t, x, y).$ Accordingly, we shall express our assumptions directly on $V(\tau, X, Y) := v(t, x, y)$, $\tau > 0$, $X \in T_\nu$, $Y > 0$.

Here, $T_\nu := \nu^{-\frac{1}{2}} T$. We set

$$\Omega = \partial_X V_2 - \partial_Y V_1,$$

which describes the vorticity field of the approximation in the rescaled variables. We also set

$$\chi_\nu = \chi(\nu^{\frac{1}{2}} Y) = 1 - e^{-\kappa \nu^{\frac{1}{2}} Y}.$$  

Note that $\kappa \in (0, 1]$ is fixed, but taken small enough. Also, in the rescaled variables our almost Gevrey norm becomes

$$\|F\|_p = \sum_{j=0}^{\nu^{\frac{1}{2}}} \frac{1}{(j!)^{\frac{3}{2}}} \sup_{j_2=0, \ldots, j} \|e^{-K \nu^{\frac{1}{2}} (j+1)} B_{j_2} \partial_x^{j_2} F\|_{L^p_\nu(0, \frac{1}{K^{\frac{1}{2}}}; L^2_{X, Y})}, \quad B_{j_2} = \chi_\nu \partial_Y^{j_2}.$$  

We state our key assumptions in terms of $V$ and $\Omega$.

**Assumption 1.**

(i) Divergence-free and Dirichlet condition on $V$:

$$\partial_X V_1 + \partial_Y V_2 = 0, \quad V|_{Y=0} = 0$$  

Moreover, there exist constants $C_* \geq 1$ and $C_0^*, C_1^*, C_2^* > 0$ such that the following statements hold for any $\nu \in (0, 1]$ and $K \geq 1$. 

(ii) Almost Gevrey $L^\infty$ bounds for $V$ and $\nabla \Omega$: For any $\kappa \in (0, 1]$ we have

\[
\sum_{j=0}^{\nu} \frac{1}{(j!)^{1/2} \nu^{1/2}} \sup_{j_2=0,\cdots,j} \left( \|e^{-K\nu\tau^j_j} B_{j_2} \partial_X^{j-j_2} V_i \|_{L^\infty_{r,X,Y}} + \kappa \|e^{-K\nu\tau^j_j} \partial_X^{j_2} V_i \|_{L^\infty_{r,X,Y}} \right) \\
+ \nu^{-1/2} (j+1) \|e^{-K\nu\tau^j_j} B_{j_2} \partial_X^{j-j_2} \partial_Y V_i \|_{L^\infty_{r,X,Y}} + (j+1) \|e^{-K\nu\tau^j_j} B_{j_2} \partial_X^{j-j_2} \partial_Y V_i \|_{L^\infty_{r,X,Y}} \\
+ \nu^{-1/2} \left\| \frac{1+Y}{1+\nu^2 Y} e^{-K\nu\tau^j_j} B_{j_2} \partial_X^{j-j_2} \partial_Y \Omega \|_{L^\infty_{r,X,Y}} \right\| \\
\leq C_0^* \tag{2.8}
\]

Here $L^\infty_{r,X,Y} = L^\infty_r(0, \frac{1}{K\nu^2}; L^\infty_{X,Y})$.

(iii) Derivative bounds for $V$ and $\Omega$: We have

\[
\|V\|_{L^\infty_{r,X,Y}} + \nu^{-1/2} \|\partial_X V\|_{L^\infty_{r,X,Y}} + \frac{1+Y}{1+\nu^2 Y} \|\partial_Y V\|_{L^\infty_{r,X,Y}} \\
+ \nu^{-1/2} \left\| \frac{1+Y}{1+\nu^2 Y} \partial_X \Omega \|_{L^\infty_{r,X,Y}} \right\| + \left\| \frac{(1+Y)}{1+\nu^2 Y} \right\| \partial_Y \Omega \|_{L^\infty_{r,X,Y}} \\
+ \nu^{-1/2} \left\| \frac{Y}{1+\nu^2 Y} \right\| \partial_X \partial_Y \Omega \|_{L^\infty_{r,X,Y}} + \nu^{-1/2} \left\| \frac{(1+Y)}{(1+\nu^2 Y)2} \partial_X \partial_Y \Omega \|_{L^\infty_{r,X,Y}} \right\| \\
\leq C_1^* \tag{2.9}
\]

(iv) Monotonicity of $\Omega$: Set $\rho(Y) = C_\tau (1 + \frac{Y}{\nu^2})^{-2} + \nu^{1/2} (1+Y)^{-2} + \nu$. Then we have

\[
\partial_Y \Omega + \rho \geq 0, \tag{2.10}
\]

and

\[
\nu^{-1/2} \left\| \frac{Y}{1+\nu^2 Y} \right\| \partial_X \partial_Y \Omega \|_{L^\infty_{r,X,Y}} + \left\| \frac{(1+Y)}{(1+\nu^2 Y)2} \right\| \partial_X \partial_Y \Omega \|_{L^\infty_{r,X,Y}} \leq C_2^* \tag{2.11}
\]

Remark 2.1 (Link between the Prandtl expansions and the assumptions).

Let us explain how the set of assumptions above relates to Prandtl expansions as given in (1.2).

i) The divergence-free and Dirichlet conditions are satisfied by Prandtl expansions of type (1.2). Fields $U^{E,i}$ solve Euler or linearized Euler equations, while fields $V^{Bl,i}$ solve Prandtl or linearized Prandtl equations: in both cases, they are divergence-free. Moreover, they are constructed alternatively in order to satisfy the Dirichlet boundary condition: once $U^{E,i}$ is constructed, $V^{Bl,i}$ is constructed so that

\[
U^{E,i}_{1,0} = 0 + V^{Bl,i}_{1,0} = 0.
\]
Then, $U_{E,i+1}$ is constructed by solving an Euler type equation with the non-penetration condition

$$U_{E,i+1}^{y=0} + V_{E,i+1}^{y=0} = 0.$$ 
More precisely, one can construct $(U_{E,i}, V_{bl,i})$ in this way for $i \leq N - 1$, and conclude by

$$U^{E,N}(t, x, y) := \left(0, -V_{2}^{bl,N-1}(t, x, 0)\right), \quad V^{BL,N} := 0.$$ 

ii) Assumption ii) amounts essentially to a Gevrey $\frac{3}{2}$ bound on solutions $U_{E,i}$, resp. $V_{bl,i}$, of Euler like and Prandtl like equations. Such solutions exist locally in time. For the Euler equations, we refer to [16] and references therein. For the Prandtl equations, as mentioned before, the works [22, 17] provide local in time solutions for analytic data. These local solutions being analytic, they belong to the Gevrey class $\frac{3}{2}$. More recently, Gevrey local in time well-posedness of the Prandtl equation has been established in [2] (see [8, 20] for preliminary partial results). Also, if $v$ is given by (1.2), as $V_{2}(\tau, X, Y) = v_{2}(t, x, y)$ is zero at the boundary $Y = 0$, we can write

$$V_{2} = \int_{0}^{Y} \partial_{Y}V_{2} \simeq \int_{0}^{Y} \left(\nu^{1/2}(\partial_{y}V_{E,0}^{E,0} + \partial_{y}V_{2}^{bl,0}) + \ldots\right) = O(\nu^{2}Y) = O(1)$$

at $Y = 0$ so that $\frac{1}{\nu^{2}}$ is under control as required in ii).

iii) Again, assumption iii) is satisfied by classical Prandtl expansions of type (1.2). To check that, one has to keep in mind that $\partial_{x} \sim \nu^{2} \partial_{y}$, $\partial_{Y} \sim \nu^{2} \partial_{x}$, so that for Prandtl expansions, which depend smoothly on $t$ and $x$, any $\tau$- or $X$-derivative allows to gain $\nu^{1/2}$. This explains for instance the factor $\nu^{1/2}$ in front of the second and fourth terms of (2.9), related to $\partial_{X}V$ and $\partial_{Y}V$. In the same spirit, as $\partial_{Y} \sim \nu^{2} \partial_{x}$, for the Euler part of the Prandtl expansion (which depends smoothly on $y$), any $Y$-derivative allows to gain $\nu^{1/2}$. This remark does not apply to the boundary layer part of the expansion, as it depends genuinely on $Y$. Still, this part has good decay in $Y$ (typically like $e^{-Y}$ or $(1 + Y)^{-N}$ for large $Y$). This is coherent with the weights $(1 + Y)/(1 + \nu^{2}Y)$ or $Y/(1 + \nu^{2}Y)$ that can be found in (2.9) in front of terms with $Y$ derivatives: outside the boundary layer ($Y \gg 1$), it yields a gain of $\nu^{2}$, but in the boundary layer ($Y \sim 1$), it yields some decay information on the boundary layer terms.

iv) In the case $v$ is given by Prandtl expansions of type (1.2),

$$\partial_{Y}V = \partial_{XY}V_{2} - \partial_{Y}^{2}V_{1} = -\partial_{Y}^{2}V_{1}^{bl} + O(\nu) + O(\sqrt{\nu}(1 + Y)^{-2})$$

Here, the $O(\nu)$ comes from the Euler part of the Prandtl expansion. The $O(\sqrt{\nu}(1 + Y)^{-2})$ corresponds to the boundary layer profiles $V_{bl,i}$, $i \geq 1$. The last two terms in the definition of the weight $\rho$ allow to control them for $C_{\ast}$ large enough. Hence, condition (2.10) is essentially a (non strict) concavity condition on the leading term of the Prandtl boundary layer, $V_{bl} := V_{bl,0}$. Moreover, by the addition of the sublayer term $(1 + (Y/\nu^{2}))^{-2}$ in the definition of $\rho$, we allow any sign for $\partial_{Y}^{2}V_{0,1}^{bl}$ in the sublayer $0 \leq Y \leq O(\nu^{2})$, and the concavity is only needed for $Y \geq O(\nu^{2})$. In the original variables this sublayer is of the order $O(\nu^{2})$, which is typical order of Kolmogorov dissipation length in the theory of turbulence.

As regards (2.11), we notice that for Prandtl expansions:

$$\partial_{XY}^{2}V = -\partial_{X}^{2}V_{1}^{bl} + O(\nu^{2}) + O(\nu(1 + Y)^{-2}),$$

$$\partial_{X}^{2}V = -\partial_{X}^{2}V_{1}^{bl} + +O(\nu^{2}) + O(\nu^{2}(1 + Y)^{-2})$$
Hence, by taking into account the bound \( \frac{1}{\sqrt{\partial_y V_1^{bl} + 2\dot{\rho}}} \leq \frac{1}{C\nu^2} \), the condition (2.11) is essentially verified if \( V_1^{bl} \) satisfies
\[
\nu^{-\frac{2}{3}} \| Y \partial_x \partial_x^2 V_1^{bl} \|_{L^\infty_{\tau,X,Y}} + \| Y (1 + Y) \partial_Y V_1^{bl} \|_{L^\infty_{\tau,X,Y}} \leq C < \infty.
\]

In the next section, we will explain the general strategy for the proof of our main stability theorem. More precisely, we will briefly describe our stability analysis of the linearized equation (1.6), for \( f \) a given force. This is the core of our paper: the transition from linear to nonlinear stability is more standard. As explained before, we shall work with the rescaled variables \((\tau, X, Y)\). We set
\[
W(\tau, X, Y) := w(t, x, y), \quad F(\tau, X, Y) := \sqrt{\nu} f(t, x, y), \quad W_0(\tau, X, Y) := w_0(x, y)
\]
(and still \( V(\tau, X, Y) = v(t, x, y) \)). System (1.6) becomes
\[
\begin{align*}
\partial_\tau W - \nu^{\frac{2}{3}} \Delta W + \nabla Q + V : \nabla W + W \cdot \nabla V &= F, & \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\
\nabla \cdot W &= 0, & \tau \geq 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \quad (2.12)
\end{align*}
\]

The main result on this linear system is

**Theorem 2.2.** Suppose that Assumption 1 holds. Then there exists \( \kappa_0 \in (0, 1] \) such that the following statement holds for any \( \kappa \in (0, \kappa_0] \). There exists \( K_0 = K_0(\kappa, C_\nu, C_\nu^*) \geq 1 \) such that if \( K \geq K_0 \) then the system (2.12) admits a unique solution \( W \in C([0, \infty); H^1_{0,\sigma}(\mathbb{T}_\nu \times \mathbb{R}_+)) \) satisfying
\[
|| W ||_{L^\infty} + || \text{rot } W ||_{L^\infty} \leq C \left( (\nu^{\frac{2}{3}} + K^{\frac{2}{3}} \nu^{-\frac{2}{3}}) || W_0 || + \nu^{-1} || \text{rot } W_0 || + \nu^{-\frac{2}{3}} || F ||_2 \right). \quad (2.13)
\]

Here \( \text{rot } W = \partial_X W_2 - \partial_Y W_1 \) and \( || W_0 || = \sum_{j=0}^{\nu^{\frac{2}{3}}} \frac{1}{j!} \sup_{j_2=0,\ldots,\nu^{\frac{2}{3}}} || B_{j_2} \partial_X^{j_2} W_0 ||_{L^2_{X,Y}}, \) and \( C \) is a universal constant.

As a consequence, we have the following result in the original variables. Note that, from \( F(\tau, X, Y) = \nu^{\frac{2}{3}} f(t, x, y) \), we have \( \nu^{-\frac{2}{3}} || F ||_2 = \nu^{-\frac{2}{3}} || f ||_2 \).

**Theorem 2.3.** Suppose that Assumption 1 holds. Then there exists \( \kappa_0 \in (0, 1] \) such that the following statement holds for any \( \kappa \in (0, \kappa_0] \). There exists \( K_0 = K_0(\kappa, C_\nu, C_\nu^*) \geq 1 \) such that if \( K \geq K_0 \) then the system (1.6) admits a unique solution \( w \in C([0, \infty); H^1_{0,\sigma}(\mathbb{T} \times \mathbb{R}_+)) \) satisfying
\[
|| w ||_{L^\infty} + \nu^{\frac{2}{3}} || \text{rot } w ||_{L^\infty} \leq C \nu^{\frac{2}{3}} \left( (1 + K^{\frac{2}{3}} \nu^{\frac{2}{3}}) || w_0 || + || \text{rot } w_0 || + \nu^{-\frac{2}{3}} || f ||_2 \right). \quad (2.14)
\]

Here \( \text{rot } w = \partial_X w_2 - \partial_Y w_1 \) and \( || w_0 || = \sum_{j=0}^{\nu^{\frac{2}{3}}} \frac{1}{j!} \sup_{j_2=0,\ldots,\nu^{\frac{2}{3}}} || \beta_{j_2} \partial_X^{j_2} w_0 ||_{L^2_{X,Y}}, \) and \( C \) is a universal constant.
3 General strategy

Estimates on system (2.12) will be performed at the level of the vorticity, through the Orr-Sommerfeld formulation:

\[
(\partial \tau + V \cdot \nabla - \nu^{\frac{1}{2}} \nabla) \omega + \nabla \phi \cdot \nabla \Omega = \text{rot } F, \quad \tau > 0, \quad X \in T_\nu, \quad Y > 0, \quad (3.1)
\]

Here, \( \omega = \text{rot } W := \partial_X W_2 - \partial_Y W_1 \) is the vorticity, and \( \phi \) is the stream function, satisfying \( W = \nabla \phi := \left( \frac{\partial_Y \phi}{-\partial_X \phi} \right) \) and \(-\Delta \phi = \omega\). We recall that \( \tau = \nu^{\frac{1}{2}} t \): the point is to get estimates that are valid over time intervals of size \( \nu^{\frac{1}{2}} \), which is difficult due to the stretching term \( \nabla \phi \cdot \nabla \Omega \). Classical estimates and Gronwall lemma would only yield a control on time intervals \( O(1) \). We have to use both our Gevrey functional framework and concavity condition.

Actually, several difficulties are already captured by the toy model

\[
(\partial \tau - \nu^{\frac{1}{2}} \Delta) \omega + \partial_X \phi \partial_Y \Omega = 0, \quad \tau > 0, \quad X \in T_\nu, \quad Y > 0, \quad (3.2)
\]

where \( \Omega = \Omega(Y) \) (for simplicity, we assume no dependence on \( \tau \) and \( X \)). We shall stick to this model for what follows.

In the case of the inviscid equation

\[
\partial \omega + \partial_X \phi \partial_Y \Omega = 0, \quad \phi|_{Y=0} = 0
\]

under the strict sign condition \( \partial_Y \Omega \geq C > 0 \), a trick that goes back to \([11]\) is to test the equation against \( \frac{\omega}{\partial_Y \Omega} \). By the cancellation

\[
\int \partial_X \phi \partial_Y \Omega \frac{\omega}{\partial_Y \Omega} = - \int \partial_X \phi \Delta \phi = \frac{1}{2} \int \partial_X |\nabla \phi|^2 = 0
\]

one can obtain a uniform in time control on the weighted quantity \( ||\frac{\omega}{\partial_Y \Omega}||_{L^2} \sim ||\omega||_{L^2} \). However, back to the model (3.2), we are facing two difficulties:

1. Inspired by the case of Prandtl layers, we must consider situations where \( \partial_Y \Omega \) vanishes or even becomes slightly negative : see iv) in Assumption 1.
2. Even in the simpler case \( \partial_Y \Omega \geq C > 0 \), the weighted estimate above is not compatible with the introduction of viscosity and no-slip conditions.

We recall that these difficulties are not purely technical, as no uniform in \( \nu \) stability estimate is expected below Gevrey \( \frac{3}{2} \) regularity. To overcome these issues, we shall proceed in two steps.

3.1 First step : Gevrey estimates for artificial boundary conditions

The first step consists in deriving Gevrey bounds for the same equation, but with pure slip instead of no-slip conditions. For the real Orr-Sommerfeld equation, it will be performed in Section 4. For our toy model, this means that we consider

\[
(\partial \tau - \nu^{\frac{1}{2}} \Delta) \omega + \partial_X \phi \partial_Y \Omega = 0, \quad \tau > 0, \quad X \in T_\nu, \quad Y > 0, \quad (3.3)
\]

\[
\phi|_{Y=0} = \omega|_{Y=0} = 0.
\]
The main point in this change of boundary conditions is that the difficulty 2. mentioned above disappears: the Dirichlet condition on \( \omega \) goes well with integration by parts, and in the case \( \partial Y \Omega \geq C > 0 \), one can achieve again some good control on \( \| \frac{\omega^j}{\sqrt{\partial Y \Omega}} \|_{L^2} \). Still, we have to explain how to obtain stability under the less stringent condition iv) in Assumption 1. Here, we need Gevrey regularity. Let us for simplicity forget about \( Y \)-derivatives, which are not important for the toy model, and set

\[
\omega^j := e^{-K\nu^{1/2}(j+1)} \partial_X^j \omega, \quad \phi^j := e^{-K\nu^{1/2}(j+1)} \partial_X^j \phi, \ldots
\]

The point is to obtain a bound on \( \sum_{j \leq \nu - 1/2} \frac{1}{(j!)^{1/2}2^{j/2}} \| \omega^j \|_{L^2_{X,Y}} \). As \( \Omega = \Omega(Y) \), equation satisfied by \( \omega^j \) is:

\[
(K\nu^{1/2}(j + 1) + \partial_Y - \nu^{1/2}\Delta)\omega^j + \partial_X \phi^j \partial_Y \Omega = 0. \tag{3.4}
\]

Roughly, the idea is to control a weighted Gevrey norm of the form

\[
\sum_{j \leq \nu - 1/2} \frac{1}{(j!)^{1/2}2^{j/2}} \frac{\omega^j}{\sqrt{\partial Y \Omega + 2\rho_j}} \| \omega^j \|_{L^2_{X,Y}},
\]

where \( \rho_j \) is added to compensate for possible degeneracies of \( \partial Y \Omega \). Testing (3.4) against \( \partial_y \Omega + 2\rho_j \), we find

\[
K\nu^{1/2}(j + 1) \| \frac{\omega^j}{\sqrt{\partial Y \Omega + 2\rho_j}} \|_{L^2}^2 + \frac{1}{2} \frac{d}{d\tau} \| \omega^j \|_{L^2}^2 + \nu^{1/2} \| \frac{\nabla \omega^j}{\sqrt{\partial Y \Omega + 2\rho_j}} \|_{L^2}^2
\]

\[
= -\nu^{1/2} \int \frac{1}{\partial_Y \Omega + 2\rho_j} \cdot \nabla \omega^j \cdot \omega^j - \int \partial_X \phi^j \partial_Y \Omega \frac{\omega^j}{\partial_Y \Omega + 2\rho_j}
\]

\[
= \nu^{1/2} \int \frac{\nabla \partial_Y \Omega}{(\partial_Y \Omega + 2\rho_j)^2} \cdot \nabla \omega^j \omega^j + \nu^{1/2} \int \frac{\nabla \omega^j}{(\partial_Y \Omega + 2\rho_j)^2} \cdot \nabla \omega^j \omega^j
\]

\[
+ \int \partial_X \phi^j \frac{2\rho_j}{\partial_Y \Omega + 2\rho_j} \omega^j
\]

where we used again the cancellation property \( \int \partial_X \phi^j \omega^j = 0 \). One must then choose \( \rho_j \) so that the three terms at the right are controlled by the left-hand side, for \( K \) large enough. Roughly, this can be achieved by taking \( \rho_j \) in the form \( \rho_j(Y) \approx \rho + (1 + \lambda_j Y)^{-2} \), \( \lambda_j := (j + 1)^{1/2} \). To give an idea of why it works, let us consider the first and last terms. As regards the first one, we write

\[
\nu^{1/2} \int \frac{\nabla \partial_Y \Omega}{(\partial_Y \Omega + 2\rho_j)^2} \cdot \nabla \omega^j \omega^j
\]

\[
= \nu^{1/2} \int \{Y \geq \frac{1}{\lambda_j} \} \frac{1}{Y \sqrt{\partial_Y \Omega + 2\rho_j}} \cdot \frac{Y \nabla \partial_Y \Omega}{\sqrt{\partial_Y \Omega + 2\rho_j}} \cdot \frac{\nabla \omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \omega^j
\]

\[
+ \nu^{1/2} O \left( \| \frac{\nabla \omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \|_{L^2} \| \omega^j \|_{L^2} \right)
\]

The second term at the right corresponds to the contribution of the region \( Y \leq \frac{1}{\lambda_j} \), for which the weight \( \partial_Y \Omega + 2\rho_j \) is bounded from below and raises no issue (we further assumed here that \( \partial_Y \nabla \Omega \) for the sake of brevity). As regards the first term, we use the bounds

\[
\forall Y \geq \frac{1}{\lambda_j}, \quad \frac{1}{Y \sqrt{\partial_Y \Omega + 2\rho_j}} \leq \frac{1}{Y \sqrt{2\rho_j}} \leq C\lambda_j,
\]

10
and

\[ \frac{|Y\nabla \partial_Y \Omega|}{\sqrt{\partial_Y \Omega + 2\rho_j}} \leq \frac{|Y\nabla \partial_Y \Omega|}{\sqrt{\partial_Y \Omega + 2\rho_j}} \leq C \]

where we used Assumption 1 iv). We end up with

\[ \nu^\frac{1}{2} \int \frac{\nabla \partial_Y \Omega}{(\partial_Y \Omega + 2\rho_j)^2} \cdot \nabla \omega^j \omega^j \leq C \nu^\frac{1}{2} \lambda_j \left\| \frac{\nabla \omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \right\|_{L^2} \left\| \frac{\omega^j}{\partial_Y \Omega + 2\rho_j} \right\|_{L^2} \]

which is absorbed by the left-hand side under the constraint \( \lambda_j \lesssim (j + 1)^{\frac{3}{2}} \). As regards the third term at the right of (3.5), we use the inequality

\[ \frac{\rho_j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \leq \frac{\sqrt{\rho_j}}{\sqrt{2}} \leq C \left( \sqrt{\nu} + \frac{1}{\lambda_j} \right) \]

to obtain

\[ \int \partial_X \phi^j \frac{2\rho_j}{\partial_Y \Omega + 2\rho_j} \omega^j \leq C \sqrt{\nu} \left\| \partial_X \phi^j \right\|_{L^2} \left\| \frac{\omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \right\|_{L^2} + \frac{C}{\lambda_j} \left\| \partial_X \phi^j \right\|_{L^2} \left\| \frac{\omega^j}{\partial_Y \Omega + 2\rho_j} \right\|_{L^2} \]

\[ \leq C \left( \sqrt{\nu} \left\| \partial_X \phi^j \right\|_{L^2} + \frac{1}{\lambda_j} \left\| \partial_Y \partial_X \phi^j \right\|_{L^2} \right) \left\| \frac{\omega^j}{\partial_Y \Omega + 2\rho_j} \right\|_{L^2} \]

where the second line comes from Hardy’s inequality. Using that \( \left\| \partial_X \phi^j \right\|_{L^2} \approx \left\| \phi^{j+1} \right\|_{L^2} \), we have for any sequence \((a_j)\)

\[ \sum_j \frac{1}{(j!)^{3/2} \nu^{j/2}} a_j \left\| \partial_Y \partial_X \phi_j \right\|_{L^2} \approx \sum_j \frac{1}{(j!)^{3/2} \nu^{j/2}} \nu^{1/2} (j + 1)^{3/2} a_{j-1} \left\| \partial_Y \phi^j \right\|_{L^2} \]

In other words, at Gevrey \( \frac{3}{2} \) regularity, \( a_j \left\| \partial_Y \partial_X \phi^j \right\|_{L^2} \) behaves like \( \nu^{1/2} (j + 1)^{3/2} a_{j-1} \left\| \partial_Y \phi^j \right\|_{L^2} \). Combining this with a control of \( \left\| \nabla \phi^j \right\|_{L^2} \) by \( \left\| \omega^j / \sqrt{\partial_Y \Omega + 2\rho_j} \right\|_{L^2} \), with a precise statement to be given in Section 4, the previous bound is in the same spirit as

\[ “ \int \partial_X \phi^j \frac{2\rho_j}{\partial_Y \Omega + 2\rho_j} \omega^j \leq C \nu^{1/2} (j + 1)^{3/2} \frac{\omega^j}{\lambda_{j-1}} \left\| \frac{\omega^j}{\partial_Y \Omega + 2\rho_j} \right\|_{L^2} “ \]

which allows a control by the left-hand side of (3.5) as soon as \( (j + 1)^{1/2} \lesssim \lambda_j \). Hence, the choice \( \lambda_j = (j + 1)^{1/2} \).

Of course, the elements above provide only glimpses of the approach carried in the first step of our stability study. The full study of the Orr-Sommerfeld equation with artificial boundary conditions is given in Section 4.

### 3.2 Recovery of the right boundary conditions

We give again a few elements on the toy model (3.2). The analysis of the complete model is carried in Section 5. After the first step, one has a solution of system (3.3), with the same initial condition and same boundary condition \( \phi|_{Y=0} = 0 \) as in (3.2), but not the same boundary condition on the derivative: \( h := \partial_Y \phi|_{Y=0} \neq 0 \). Note that by the first step and
trace theorem, one is able to get a Gevrey bound for $h$: as shown rigorously in the next sections, one may get an estimate of the form

$$\| h \|_{bc} := \sum_{j \leq \nu - 1/2} \frac{1}{(j!)^{2} \nu^{2}} \| h_{j} \|_{L^{2}((0, K^{2}/2); \mathbb{R}^{3})} \leq \frac{C}{K^{1/4}} \left( \| \nabla \phi_{0} \|_{L^{2}} + C \sum_{j \leq \nu - 1/2} \frac{1}{(j!)^{2} \nu^{2}} \| \omega_{0} \|_{L^{2}(X,Y)} \right)$$

where $\phi_{0}$ and $\omega_{0} := -\Delta \phi_{0}$ are the initial data for the stream function and vorticity.

Working in Gevrey, regularity, the point is then to solve:

$$\left( \partial_{\tau} - \frac{1}{2} \Delta \right) \omega + \partial_{X} \phi \partial_{Y} \Omega = 0, \quad \tau > 0, \ X \in \mathbb{T}_{\nu}, \ Y > 0,$$

$$\phi_{|Y=0} = 0 \quad \partial_{Y} \phi_{|Y=0} = h, \quad \phi_{|\tau=0} = 0. \quad (3.6)$$

The main idea is to use the following scheme:

a) We solve the approximate Stokes equation

$$\left( \partial_{\tau} - \frac{1}{2} \Delta \right) \Delta \phi = 0, \quad \tau > 0, \ X \in \mathbb{T}_{\nu}, \ Y > 0,$$

$$\phi_{|Y=0} = 0 \quad \partial_{Y} \phi_{|Y=0} = h, \quad \phi_{|\tau=0} = 0. \quad (3.7)$$

and obtain in this way a solution $\phi_{a} = \phi_{a}[h]$.

b) We correct the stretching term created by the previous approximation, by considering the full equation with artificial boundary condition:

$$\left( \partial_{\tau} - \frac{1}{2} \Delta \right) \omega + \partial_{X} \phi \partial_{Y} \Omega = -\partial_{X} \phi_{a} \partial_{Y} \Omega, \quad \tau > 0, \ X \in \mathbb{T}_{\nu}, \ Y > 0,$$

$$\phi_{|Y=0} = 0 \quad \Delta \phi_{|Y=0} = 0, \quad \phi_{|\tau=0} = 0. \quad (3.8)$$

We denote by $\phi_{b} = \phi_{b}[h]$ the solution of such system. It can be seen as a functional of $h$ through $\phi_{a}$.

c) At the end of the steps a) and b), the function $\phi - \phi_{a} - \phi_{b}$ solves formally the same system as $\phi$, replacing $h$ by $R_{bc}h := -\partial_{Y} \phi_{b}[h]|_{Y=0}$. The point is to show that for $K$ large enough,

$$\| R_{bc}h \|_{bc} \leq \frac{1}{2} \| h \|_{bc}. \quad (3.9)$$

which allows to solve (3.6) by iteration.

Obviously, to establish (3.9), one must have careful Gevrey stability estimates for systems (3.7) and (3.8). The estimates for (3.8) follows from the same ideas as those described in the first step to treat (3.3) (the initial condition is just replaced by a source term). As regards (3.7), the initial data being zero, one can take Laplace transform in $\tau$ and Fourier transform in $X$ and solve explicitly the resulting ordinary differential equation in $Y$. It leads to sharp $L^{2}$ estimates on $\phi$ and its derivatives on the Fourier-Laplace side, which transfer to $L^{2}$ estimates in the physical space by Plancherel theorem.

All the analysis in the framework of the Orr-Sommerfeld equation is provided in Section 5. In this setting, the iteration scheme mentioned above has to be modified, because the advection
term creates extra difficulties. Namely, one has to add an intermediate step between steps a) and b) above, see Section 5 for details.

Of course, we have indicated here key ideas for the stability analysis of the linearized system (1.6). One has then to go from these estimates to the nonlinear Theorem 2.1. This will be achieved in Section 7. Finally we introduce the simplified notation

\[
\|f\| = \|f\|_{L^2_{X,Y}}, \quad \langle f, g \rangle = \langle f, g \rangle_{L^2_{X,Y}}
\]

for convenience.

4 Vorticity estimate under artificial boundary condition

In accordance with the strategy described in the previous section, we consider here the solution to the system

\[
\begin{align*}
\nu^2 \Delta^2 \phi - \partial_x \Delta \phi - V \cdot \nabla \Delta \phi + \nabla^j \phi \cdot \nabla \Omega &= \text{rot} F + G, \quad \tau > 0, \ X \in \mathbb{T}_\nu, \ Y > 0, \\
\phi|_{Y=0} &= \Delta \phi|_{Y=0} = 0, \quad \phi|_{r=0} = \phi_0.
\end{align*}
\]

The goal of this section is to establish estimates for the vorticity \( \omega = -\Delta \phi \), where \( \phi \) is the streamfunction uniquely determined in the class \( \phi \in \mathcal{H}^1_0(\mathbb{T} \times \mathbb{R}_+) \). For \( j = (j_1, j_2) \) with \( j_1 + j_2 = j \), we set

\[
\omega^j = e^{-K\tau \nu^2 (j+1)} B_{j_2} \partial_{X}^{j_1} \omega, \quad (\nabla \phi)^j = e^{-K\tau \nu^2 (j+1)} B_{j_2} \partial_{X}^{j_1} \nabla \phi,
\]

and similarly, \( (\Delta \omega)^j = e^{-K\tau \nu^2 (j+1)} B_{j_2} \partial_{X}^{j_1} \Delta \omega \). We also set

\[
V^j = e^{-K\tau \nu^2 j} B_{j_2} \partial_{X}^{j_1} V, \quad (\nabla \Omega)^j = e^{-K\tau \nu^2 j} B_{j_2} \partial_{X}^{j_1} \nabla \Omega.
\]

From the first equation of (4.1) we observe that \( \omega^j \) satisfies, by setting \( l = (l - l_2, l_2) \),

\[
- \nu^2 (\Delta \omega)^j + (\partial_x + K\nu^2 (j + 1) + V \cdot \nabla) \omega^j + (\nabla^j \phi) \cdot \nabla \Omega = -V^j |_{l_2} [B_{j_2}, \partial_{x}] e^{-K\tau \nu^2 (j+1)} \partial_{X}^{j_1} \omega
\]

\[
- \sum_{l=0}^{j-1} \max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\} \bigg( \frac{j_2}{l_2} \bigg) \bigg( \frac{j-j_2}{l-l_2} \bigg) V^{j-1} \cdot (\nabla \omega)^j
\]

\[
- \sum_{l=0}^{j-1} \max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\} \bigg( \frac{j_2}{l_2} \bigg) \bigg( \frac{j-j_2}{l-l_2} \bigg) (\nabla^j \phi) \cdot (\nabla \Omega)^{j-1}
\]

\[+ \text{rot } F^j - [B_{j_2}, \partial_{x}] \partial_{X}^{j_1} e^{-K\tau \nu^2 (j+1)} F_1 + G^j.
\]

Here the sum \( \sum_{l=0}^{j-1} \) is defined as 0 for \( j = 0 \), and the definitions of \( F^j \) and \( G^j \) are straightforward.

To simplify notations let us introduce weighted seminorms; for a given nonnegative smooth function \( \xi_j = \xi_j(\tau, X, Y) \), we set

\[
M_{p,j,\xi_j}[\omega] = \sup_{j_2=0,\cdots,j} \|\xi_j \omega^{(j-j_2,j_2)}\|_{L^p_\nu(0,1^{1/\nu^2}; L^2_{X,Y})}.
\]
and also set with the definition $\xi = (\xi_j)_{j=0}^\infty$,

$$
\|F\|_{\nu, \xi}^\prime = \sum_{j=0}^{\nu - \frac{1}{2}} \frac{\nu^{\frac{3}{2}} (j + 1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}} \nu^{\frac{3}{2}} \rho_j \nu^{\frac{3}{2}} M_{\rho_j, \xi_j} [F].
$$

(4.6)

Note that

$$
\|F\|_{\infty, 1} = \|F\|_{\infty, 1}, \quad 1 = (1, 1, \cdots).
$$

(4.7)

The choice of $\xi_j$ is essential in the stability estimate for $\omega^1$. We will take

$$
\xi_j = \frac{1}{\sqrt{\nu_Y \Omega + 2 \rho_j}},
$$

(4.8)

where

$$
\rho_j = K_{\nu}^{\frac{1}{4}} C_{\nu} (1 + (j + 1)^{\frac{3}{2}} Y)^{-2} + C_{\nu} \left((1 + \nu)^{-2} + \nu^2 (1 + Y)^{-2} + \nu\right).
$$

(4.9)

See Section 3 for more on the origin of this weight. We also introduce the norm of the boundary trace as

$$
\|\partial_Y \phi|_{Y=0}\|_{bc} = \sum_{j=0}^{\nu - \frac{1}{2}} \frac{\nu^{\frac{3}{2}} (j + 1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}} \nu^{\frac{3}{2}} \rho_j \nu^{\frac{3}{2}} \|e^{-K_{\nu}^{\frac{1}{4}} (j + 1)^{\frac{3}{2}} \nu_{Y}} \partial_Y \phi|_{Y=0}\|_{L^2(\nu, \nu_{Y}, \nu_{X})},
$$

(4.10)

and we denote by $H^{-1}$ the dual space of the homogeneous Sobolev space $H^1_0(T_{\nu} \times \mathbb{R}_+)$ (here, the subscript 0 means the zero boundary trace).

The main result of this section is:

**Proposition 4.1.** There exists $\kappa_1 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_1]$. There exists $K_1 = K_1(\kappa, C_{\nu}, C_{\nu}^*) \geq 1$ such that if $K \geq K_1$ then the system (4.1) admits a unique solution $\phi \in C([0, \infty); H^1_0(T_{\nu} \times \mathbb{R}_+))$ with $\omega = -\Delta \phi \in C([0, \infty); L^2(T_{\nu} \times \mathbb{R}_+))$ satisfying

$$
\|\omega\|_{\infty, \xi}^\prime + K_{\nu}^{\frac{1}{4}} \|\omega\|_{2, \xi}^\prime + K_{\nu}^{\frac{1}{4}} \|\nabla \phi\|_{2, X, Y}^\prime + K_{\nu}^{\frac{1}{4}} \|\partial_Y \phi|_{Y=0}\|_{bc}
\leq C \left(\|\nabla \phi_0\|_{L^2_{X, Y}} + \nu^{-\frac{1}{2}} \|\Delta \phi_0\|\right)
+ \frac{1}{K^{\frac{1}{2}} \nu^{\frac{3}{2}}} \|F\|_{2, \xi, 1}^\prime + \frac{1}{K^{\frac{1}{2}} \nu^{\frac{3}{2}}} \|G\|_{2, \xi, 1}^\prime.
$$

(4.11)

Here $C > 0$ is a universal constant, while the weight $\tilde{\xi}^{(k)}$ is defined as

$$
\tilde{\xi}^{(k)} = \left(\frac{\xi_j}{(j + 1)^{\frac{3}{2}}}\right)^{\infty}_{j=0}.
$$

**Remark 4.1.** (1) From the bound $\frac{1}{\xi_j} \leq (C_{\nu}^* + 8 K_{\nu}^{\frac{1}{4}} C_{\nu}^*)^\frac{1}{2}$ in (4.17) below, we have

$$
K_{\nu}^{\frac{1}{4}} \|\omega\|_{2, 1} \leq K_{\nu}^{\frac{1}{4}} (C_{\nu}^* + 8 K_{\nu}^{\frac{1}{4}} C_{\nu}^*)^\frac{1}{2} \|\omega\|_{2, \xi}^\prime \leq K_{\nu}^{\frac{1}{4}} \|\omega\|_{2, \xi}^\prime
$$

(4.12)
if $K$ is large enough further depending only on $C_1^*$ and $C_*$. Estimates (4.12) and (4.11) gives the estimate of $K \pi \| \omega \|_{2.1}^\prime$.

(2) By the definition of (4.6), we have

$$\nu^{-\frac{1}{2}} \| F \|_{2, \xi}^{(1)} = \nu^{-\frac{1}{2}} \sum_{j=0}^{\nu^{-\frac{1}{2}}} M_{2,j} \xi_j [F] \quad \nu^{-\frac{1}{2}} \| G \|_{2, \xi}^{(2)} = \nu^{-\frac{1}{2}} \sum_{j=0}^{\nu^{-\frac{1}{2}}} M_{2,j} \xi_j [G].$$

Since $\xi_j \leq \frac{1}{\sqrt{\rho_j}} \leq \frac{1}{C_* \nu^\frac{1}{2}}$ holds by the definitions (4.8)-(4.9) with the monotonicity condition (2.10), we have

$$\nu^{-\frac{1}{2}} \| F \|_{2, \xi}^{(1)} \leq \frac{\| F \|_{2}^2}{C_* \nu^\frac{1}{2}}. \quad (4.13)$$

Before going into the details of the proof of Proposition 4.1, let us give a lemma for the weight $\xi_j$ and $\rho_j$, which will be used frequently. By the concavity condition on $\partial_Y \Omega$ in Assumption 1 (iv) and the definition of $\rho_j$ we have

**Lemma 4.1.** There exists $C > 0$ such that the following estimates hold for any $j \geq 0$.

$$\xi_j^2 \leq \frac{1}{\rho_j} \leq \frac{1}{C_* \max\{K\pi(1 + (j + 1)^\frac{1}{2} Y)^{-2}, \nu\}} \quad \text{for} \quad Y \geq 0,$$

$$\frac{1}{\rho_j} \leq \frac{4}{K\pi C_*} \quad \text{for} \quad 0 \leq Y \leq (j + 1)^{-\frac{1}{2}}. \quad (4.14)$$

In particular,

$$\| 1 + \nu^\frac{1}{2} Y \|_{L^\infty} + \| \frac{1 + \nu^\frac{1}{2} Y}{Y} \xi_j \|_{L^\infty((Y \geq (j + 1)^{-\frac{1}{2}}))} \leq C(j + 1)^{\frac{1}{2}}. \quad (4.15)$$

Moreover,

$$\| \rho_j \|_{L^\infty} \leq 4K\pi C_* \quad \text{and} \quad \| \frac{Y \partial_Y \rho_j}{\rho_j} \|_{L^\infty} \leq 2. \quad (4.16)$$

and

$$\| \frac{1}{\xi_j} \|_{L^\infty} \leq (C_1^* + 8K\pi C_*)^\frac{1}{2} \quad \text{and} \quad \sup_{j \geq 1} \| \frac{\xi_j}{\xi_{j-1}} \|_{L^\infty} \leq C. \quad (4.17)$$

The proof of Lemma 4.1 is straightforward from the definitions of $\xi_j$ and $\rho_j$, so we omit the details.

**4.1 Vorticity estimate for the modified system**

In this subsection we collect lemmas for the solution to (4.4) and give the estimate for the vorticity. The main result of this subsection is as follows.
Proposition 4.2. There exists $\kappa_1' \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_1']$. There exists $K_1' = K_1'(\kappa, C_\ast, C_1') \geq 1$ such that if $K \geq K_1'$ then the system (4.1) admits a unique solution $\phi \in C([0, \infty); \mathcal{H}_0^1(\mathbb{T}_\nu \times \mathbb{R}_+))$ with $\omega = -\Delta \phi \in L^2(\mathbb{T}_\nu \times \mathbb{R}_+)$ satisfying

$$
\|\nabla \omega\|_{L^2(\xi_1)}' + \|\nabla \omega\|_{L^2(\xi_2)}' + K_1'\|\nabla \omega\|_{L^2(\xi_2)}' 
\leq C \left( \nu^{-\frac{1}{2}}\|\Delta \phi_0\| + \frac{C_2 + 1}{\nu^2} \|F\|_{L^2(\xi_1)}' + \frac{1}{K_1'\nu^2} \|G\|_{L^2(\xi_2)}' + \|\nabla \phi\|_{L^2(\xi_1)}' \right). 
$$

(4.18)

Here $C > 0$ is a universal constant.

Since the unique solvability of the linear system (4.1) itself follows from the standard theory of parabolic equations, we focus on establishing the estimate (4.18). Then the core part of the proof of Proposition 4.2 consists of the calculation of the inner product for each term in (4.4) with $\xi_j^2\omega_j$, where $j = (j_1, j_2)$ with $j_1 + j_2 = j$ and the weight $\xi_j$ is defined as in (4.8). Let us start from the following lemma. The number $\tau_0 \in (0, \frac{1}{K_1'\nu^2})$ is taken arbitrary below.

Lemma 4.2. There exists $K_{1,1} = K_{1,1}(C_1', C_\ast) \geq 1$ such that if $K \geq K_{1,1}$ then we have

$$
\int_0^{\tau_0} \left( -\nu^\frac{1}{2}(\Delta \omega)^j, \xi_j^2\omega_j \right) d\tau 
\geq \nu^\frac{1}{2} \|\xi_j(\nabla \omega)^j\|^2_{L^2((0, \tau_0); L^2_X, Y)} 
- C\nu^\frac{1}{2}(\kappa\nu^\frac{1}{2}j_2)^2 M_{2, j_1, j_2-1}[\partial_Y \omega]^2
- C(C_2 + 1)\nu^\frac{1}{2}(j_1 + 1)\|\xi_j \omega_j\|^2_{L^2((0, \tau_0); L^2_X, Y)}.
$$

Here $C > 0$ is a universal constant.

Proof. Let us write $\chi^{\prime}_\nu = (\chi)'(\nu^\frac{1}{2}Y) = \kappa e^{-\nu^\frac{1}{2}Y}$. We will frequently use the identity

$$
[B_{j_2}, \partial_Y] = -\nu^\frac{1}{2} j_2 \chi^{\prime}_\nu B_{j_2-1} \partial_Y = -\nu^\frac{1}{2} j_2 \chi^{\prime}_\nu B_{j_2}. 
$$

(4.19)

Then we observe that

$$
(\Delta \omega)^j = e^{-K\nu^\frac{1}{2}(j_1+1)} B_{j_2} \partial_Y^j \Delta \omega = \nabla \cdot (\nabla \omega)^j - \nu^\frac{1}{2} j_2 \chi^{\prime}_\nu (\partial_Y \omega)^j 
$$

(4.20)

and

$$
\nabla \omega^j = (\nabla \omega)^j + \nu^\frac{1}{2} j_2 \chi^{\prime}_\nu e^{-K\nu^\frac{1}{2}(j_1+1)} (\partial_Y \omega)^j (j_1, j_2-1) e_2, \quad \omega^j = \chi^{\prime}_\nu e^{-K\nu^\frac{1}{2}(j_1+1)} (\partial_Y \omega)^j (j_1, j_2-1). 
$$

(4.21)

Here $e_2 = (0, 1)$. Hence the integration by parts gives

$$
\int_0^{\tau_0} -\nu^\frac{1}{2}(\Delta \omega)^j, \xi_j^2\omega_j \right) d\tau 
\geq \nu^\frac{1}{2} \int_0^{\tau_0} \left( \|\xi_j(\nabla \omega)^j\|^2 + 2\nu^\frac{1}{2} j_2 e^{-K\nu^\frac{1}{2}(j_1, j_2-1)} \chi^{\prime}_\nu \xi_j (\partial_Y \omega)^j (j_1, j_2-1) \right) d\tau 
- \nu^\frac{1}{2} \int_0^{\tau_0} \|\xi_j(\nabla \omega)^j \cdot \nabla (\xi_j^2, \omega_j)\|^2_{L^2((0, \tau_0); L^2_X, Y)}.
$$

(4.22)
Here we have used $\|\xi_i\|_{L^\infty} \leq C$ in the last line as stated in Lemma 4.1. When $j_2 = 0$ the term $\partial_Y Y^{(j_1,j_2-1)}$ is defined as 0 for convenience. It suffices to estimate $\langle (\nabla \omega)^j \cdot \nabla (\xi_j^2) \omega \rangle^j$. We have

$$\nabla (\xi_j^2) = -\frac{\nabla \partial_Y \Omega + 2 \nabla \rho_j}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2,$$

which yields

$$\langle (\nabla \omega)^j \cdot \nabla (\xi_j^2) \omega \rangle^j \leq \|\xi_j (\nabla \omega)^j\| \left(\|\frac{\nabla \partial_Y \Omega}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \| + \|\frac{2 \partial_Y \rho_j}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \|\right).$$

To estimate $\|\frac{\nabla \partial_Y \Omega}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \|$ we decompose the integral about $Y$ into $0 \leq Y \leq (j + 1)^{-\frac{1}{4}}$ and $Y \geq (j + 1)^{-\frac{1}{4}}$. Then we see from Lemma 4.1 with $\frac{\xi_j^2}{\sqrt{\partial_Y \Omega + 2 \rho_j}} = \xi_j^3 \leq \frac{1}{\rho_j^2}$,

$$\|\frac{\nabla \partial_Y \Omega}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \|_{L^2((0 < Y < (j + 1)^{-\frac{1}{4}}))} \leq \left\|\frac{1}{\rho_j^2}\right\|_{L^\infty((0 < Y < (j + 1)^{-\frac{1}{4}}))} \|\nabla \partial_Y \Omega \omega \|_{L^2((0 < Y < (j + 1)^{-\frac{1}{4}}))} \leq \frac{2}{(K^2 C_4)^2} \left\|\frac{Y}{1 + \nu \nabla Y} \nabla \partial_Y \Omega \omega \right\|_{L^\infty(\omega)} \leq \frac{CC_j^2}{(K^2 C_4)^2} \|\partial_Y \omega \|.$$

Here we have used Assumption 1 (iii) and the Hardy inequality $\|\omega\|_{L^2} \leq 4\|\partial_Y \omega\|$. Then by using (4.21) for $\partial_Y \omega$ and (4.17) we have

$$\|\partial_Y \omega\| \leq \|\partial_Y \omega\| + \kappa \nu\frac{j_2}{j_2+1} \|\partial_Y \omega\|^{(j_1,j_2-1)} \leq \frac{1}{\xi_j} \|\partial_Y \omega\| \leq \|\partial_Y \omega\| \|\partial_Y \omega\|^{(j_1,j_2-1)} \leq C(C_1^2 + K^2 C_4^2)\left(\|\xi_j \partial_Y \omega\|^j \|\kappa \nu\frac{j_2}{j_2+1}(\partial_Y \omega)^{(j_1,j_2-1)}\right).$$  \hspace{1cm} (4.23)

On the other hand, we have from Assumption 1 (iv) and (4.15) in Lemma 4.1,

$$\|\frac{\nabla \partial_Y \Omega}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \|_{L^2((Y \geq (j + 1)^{-\frac{1}{4}}))} \leq \|\frac{Y \nabla \partial_Y \Omega}{1 + \nu \nabla Y} \sqrt{\partial_Y \Omega + 2 \rho_j} \|\frac{1 + \nu \frac{j_2}{j_2+1} Y}{Y^{2(j + 1)^{-\frac{1}{4}}} } \|\xi_j \omega \| \leq CC_j^2 (j + 1)^{\frac{1}{2}} \|\xi_j \omega \|.$$

Next we estimate the term $\|\frac{2 \partial_Y \rho_j}{\sqrt{\partial_Y \Omega + 2 \rho_j}} \xi_j^2 \omega \|$. To this end we observe that

$$|\partial_Y \rho_j| \leq 2(j + 1)^{\frac{1}{2}} K^2 C_4 (1 + (j + 1)^{\frac{1}{4}} Y)^{-3} + 2C_4 \nu^2 (1 + Y)^{-3} + 2C_4 \nu (1 + \frac{Y}{\nu^2})^{-3} \leq \left\{ \begin{array}{ll} 2(j + 1)^{\frac{1}{2}} \rho_j + \frac{2C_4}{Y}, & 0 < Y < (j + 1)^{-\frac{1}{4}}, \\ 2(j + 1)^{\frac{1}{2}} \rho_j + \frac{2 \rho_j}{Y}, & Y \geq (j + 1)^{-\frac{1}{4}}. \end{array} \right.$$
which gives from Lemma 4.1,
\[
\| \frac{2\partial_Y \rho_j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \xi_j^2 \omega^j \|_{L^2}
\]
\[
\leq 4(j + 1)^{\frac{1}{2}} \| \xi_j \omega^j \| + 2C_* \| \frac{\xi_j^3 \omega^j}{Y} \|_{L^2 \{0 < Y < (j + 1)^{-\frac{1}{2}}\}} + 2 \| \frac{\rho_j \xi_j^3 \omega^j}{Y} \|_{L^2 \{Y \geq (j + 1)^{-\frac{1}{2}}\}}
\]
\[
\leq 4(j + 1)^{\frac{1}{2}} \| \xi_j \omega^j \| + \frac{2C_*}{(K^4 C_s)^{\frac{1}{2}}} \| \omega^j \| + 2(j + 1)^{\frac{3}{2}} \| \xi_j \omega^j \|.
\]

Then we apply the Hardy inequality \( \| \omega^j \| \leq 4 \| \partial_Y \omega^j \| \) and then use (4.23). Collecting these, we obtain
\[
\| \langle (\nabla \omega)^j \cdot \nabla (\xi_j^2), \omega^j \rangle \|
\]
\[
\leq \| \xi_j (\nabla \omega)^j \| \left( \frac{C(C_*^4 + 1)(C_*^4 + K^4 C_s)^{\frac{1}{2}}}{(K^4 C_s)^{\frac{1}{2}}} \left( \| \xi_j (\partial_Y \omega)^j \| + \kappa^j \nu^j_2 \| \xi_{j-1} (\partial_Y \omega)^{(j_1, j_2-1)} \| \right) 
\right.
\]
\[
+ C(C_*^4 + 1)(j + 1)^{\frac{1}{2}} \| \xi_j \omega^j \| \right).
\]

Thus, by taking \( K \) large enough depending only on \( C_*^4 \) and \( C_* \), we obtain the desired estimate as stated in Lemma 4.3. The proof is complete. \( \square \)

**Lemma 4.3.** There exists \( K_{1, 2} = K_{1, 2}(C_*^4, C_*) \geq 1 \) such that if \( K \geq K_{1, 2} \) then we have
\[
\int_0^{\tau_0} \langle (\partial_\tau + K \nu^\frac{1}{2}(j + 1) + V \cdot \nabla) \omega^j, \xi_j^2 \omega^j \rangle \, d\tau
\]
\[
\geq \frac{1}{2} \| \xi_j \omega^j(\tau_0) \|_{L^2_{X,Y}}^2 - \frac{1}{2} \| \xi_j \omega^j(0) \|_{L^2_{X,Y}}^2 + \frac{K}{2} \nu^\frac{1}{2}(j + 1) \| \xi_j \omega^j \|_{L^2(0, \tau_0; L^2_{X,Y})}^2 
\]
\[
- \frac{CC_*^4 K^{\frac{1}{2}}}{K^4 C_*} \left( \| \xi_j (\partial_Y \omega)^j \|_{L^2(0, \tau_0; L^2_{X,Y})}^2 + (\kappa^j \nu^j_2)^2 M_{2, j-1, j-1} [\partial_Y \omega]^2 \right).
\]

Here \( C > 0 \) is a universal constant.

**Proof.** The integration by parts yields
\[
\int_0^{\tau_0} \langle (\partial_\tau + K \nu^\frac{1}{2}(j + 1) + V \cdot \nabla) \omega^j, \xi_j^2 \omega^j \rangle \, d\tau
\]
\[
= \frac{1}{2} \| \xi_j \omega^j(\tau_0) \|_{L^2_{X,Y}}^2 - \frac{1}{2} \| \xi_j \omega^j(0) \|_{L^2_{X,Y}}^2 + K \nu^\frac{1}{2}(j + 1) \| \xi_j \omega^j \|_{L^2(0, \tau_0; L^2_{X,Y})}^2 
\]
\[
- \frac{1}{2} \int_0^{\tau_0} \langle \partial_\tau (\xi_j^2) + V \cdot \nabla (\xi_j^2), (\omega^j)^2 \rangle \, d\tau.
\]

As for the term \( \langle \partial_\tau (\xi_j^2), (\omega^j)^2 \rangle \), we decompose the integral about \( Y \) into \( \{0 < Y < (j + 1)^{-\frac{1}{2}}\} \)
and \( \{Y \geq (j+1)^{-\frac{1}{2}}\} \) and compute as follows:

\[
|\langle \partial_r (\xi_j^2), (\omega^j)^2 \rangle | \\
\leq \|(1 + \frac{1}{2}Y)^2 \partial_r \partial_Y \Omega \|_{L^\infty} \| (1 + \frac{1}{2}Y) \xi_j^2 \omega^j \|_2 \\
\leq C_1^* \nu^\frac{1}{2} \left( \| (1 + \frac{1}{2}Y) \xi_j^2 \omega^j \|_2 \right) \left( \| (1 + \frac{1}{2}Y) \xi_j^2 \omega^j \|_2 \right) \\
\leq C_1^* \nu^\frac{1}{2} \left( \frac{C}{(K+C)^2} \| \partial_Y \omega^j \|_2 + C(j+1) \| \xi_j \omega^j \|_2 \right). \\
\text{(by the Hardy inequality and Lemma 4.1)}
\]

(4.24)

Next we have

\[
|\langle V \cdot \nabla (\xi_j^2), (\omega^j)^2 \rangle | \leq \| V \cdot \nabla (\partial_r \Omega + 2\rho_j) \|_{L^\infty} \| \xi_j \omega^j \|_2
\]

Then we have from Assumption 1 (iii) and Lemma 4.1,

\[
\left\| \frac{V_1 \partial_Y \partial_X \Omega}{\partial_Y \Omega + 2\rho_j} \right\|_{L^\infty} \leq \left\| \frac{V_1 (1 + \frac{1}{2}Y)^2}{Y(1 + \frac{1}{2}Y)} \partial_X \partial_Y \Omega \right\|_{L^\infty} \leq \frac{C_1^* \nu^\frac{1}{2}}{(1 + \frac{1}{2}Y)^2} \left( 2 \frac{V_1}{Y(1 + \frac{1}{2}Y)^2} \| \partial_Y V_1 \|_{L^\infty} \right) \leq C(j+1).
\]

Here we have computed as, using \( V_1 \mid_{Y=0} = 0, \)

\[
\left\| \frac{V_1}{Y(1 + \frac{1}{2}Y)^2} \right\|_{L^\infty} \leq \| \partial_Y V_1 \|_{L^\infty} \left( \frac{1}{\rho_j} \| \omega^j \|_{L^\infty((Y \geq (j+1)^{-\frac{1}{2}})^+)} + \| V_1 \|_{L^\infty} \frac{1}{Y(1 + \frac{1}{2}Y)^2} \right) \\
\leq C_1^*(j+1).
\]

Similarly,

\[
\left\| \frac{V_1 \partial_Y \partial_X \Omega}{\partial_Y \Omega + 2\rho_j} \right\|_{L^\infty} \leq \left\| \frac{V_1 (1 + \frac{1}{2}Y)^2}{Y(1 + \frac{1}{2}Y)^2} \partial_Y \partial_X \Omega \right\|_{L^\infty} \left( \frac{V_1 (1 + \frac{1}{2}Y)^3}{Y(1 + \frac{1}{2}Y)^2} \right) \leq \frac{CC_1^* (V_1 \| \partial_Y V \|_{L^\infty} \| \partial_Y \rho \|_{L^\infty}) + 2C_1^* \nu^\frac{1}{2}}{(1 + \frac{1}{2}Y)^2} \\
\leq CC_1^*( \| \partial_Y V \|_{L^\infty} \left( \frac{1}{(1 + \frac{1}{2}Y)^2} \right) \| \partial_Y V \|_{L^\infty} \| \partial_Y \rho \|_{L^\infty} + \| V_2 \|_{L^\infty} \nu^\frac{3}{2} \| \partial_Y \rho \|_{L^\infty} + \| V_2 \|_{L^\infty} \nu^\frac{1}{2} \| \partial_Y \rho \|_{L^\infty} ) + 2C_1^* \nu^\frac{1}{2} \\
\leq CC_1^*(C_1^* + 1) \nu^\frac{3}{2} (j+1). \\
\text{(by Lemma 4.1)}
\]

Note that we have also used \( \| \partial_Y V_2 \|_{L^\infty} = \| \partial_X V_1 \|_{L^\infty} \leq C_1^* \nu^\frac{1}{2}. \) Collecting these and applying the identity (4.21) for \( \partial_Y \omega^j \) in (4.24) (that is, we use (4.23)), we obtain the desired estimate by taking \( K \) large enough depending only on \( C_1^* \) and \( C_\star \). The proof is complete. \( \Box \)
Lemma 4.4. It follows that

\[
\int_0^{t_0} |\langle (\nabla^\perp \phi)^j, \nabla \Omega, \xi^j_\omega \rangle| \, d\tau \leq \frac{C(R_{j, \text{Lem4.4}}[\nabla \phi])^2}{\nu^j_\perp (j + 1)} + \frac{K}{8} \nu^j_\perp (j + 1) \|\xi_j \omega_j\|^2_{L^2(0, \tau_0; L^2_{X,Y})}, \tag{4.25}
\]

where

\[
R_{j, \text{Lem4.4}}[\nabla \phi]
\]

\[
: = \left( \frac{C^*_1}{K^{\frac{1}{2}}} + \frac{(K^{\frac{1}{2}} C_\perp)^{\frac{1}{2}}}{K^{\frac{1}{2}}} + \kappa^{\frac{1}{2}} \nu^{\frac{1}{2}} (j + 1) M_{2,j} [\nabla \phi] + \frac{(K^{\frac{1}{2}} C_\perp)^{\frac{1}{2}}}{K^{\frac{1}{2}}} \delta_{j \leq \nu^{-\frac{1}{2}} - 1} \frac{M_{2,j+1} [\partial_Y \phi]}{(j + 1)^{\frac{1}{2}}}. \right)
\]

Here \( \delta_{j \leq \nu^{-\frac{1}{2}} - 1} = 1 \) for \( 0 \leq j \leq \nu^{-\frac{1}{2}} - 1 \) and 0 for \( j = \nu^{-\frac{1}{2}} \). Moreover, there exists \( K_{1,3} = K_{1,3}(C^*_1, C_\perp) \geq 1 \) such that if \( K \geq K_{1,3} \) then

\[
\sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{R_{j, \text{Lem4.4}}[\nabla \phi]}{(j!)^\frac{1}{2} \nu^j_\perp (j + 1)^{\frac{1}{2}}} \leq C \|\nabla \phi\|_{2,1}. \tag{4.26}
\]

Here \( C > 0 \) is a universal constant.

Proof. It suffices to show

\[
\int_0^{t_0} |\langle (\partial_X \phi)^j, \omega_j \rangle| \, d\tau \leq 2 \kappa \nu^{\frac{1}{2}} j_2 (M_{2,j} [\nabla \phi])^2, \tag{4.27}
\]

\[
\int_0^{t_0} |\langle \rho_j (\partial_X \phi)^j, \xi^j_\omega \rangle| \, d\tau \leq \left\{
\begin{aligned}
& C(K^{\frac{1}{2}} C_\perp)^{\frac{1}{2}} \left( \frac{M_{2,j+1} [\partial_Y \phi]}{(j + 1)^{\frac{1}{2}}} \right) \|\xi_j \omega_j\|_{L^2(0, \tau_0; L^2_{X,Y})}, \\
& 0 \leq j \leq \nu^{-\frac{1}{2}} - 1,
\end{aligned}
\right. \tag{4.29}
\]

and

\[
\int_0^{t_0} |\langle (\partial_Y \phi)^j \partial_X \phi, \xi^j_\omega \rangle| \, d\tau \leq CC^*_1 \nu^{\frac{1}{2}} j_2 (M_{2,j} [\partial_Y \phi]) \|\xi_j \omega_j\|_{L^2(0, \tau_0; L^2_{X,Y})}. \tag{4.30}
\]

Let us start from (4.27). To compute \( \langle (\partial_X \phi)^j, \omega_j \rangle \) we firstly observe that

\[
\omega_j = \nabla : (\nabla \phi)^j - \frac{\nu^{\frac{1}{2}} j_{2\nu} \chi_{\nu}^j}{\chi_{\nu}^j} (\partial_Y \phi)^j. \tag{4.31}
\]

Then we have from the integration by parts and \( [B_{j_2}, \partial_Y] = -\frac{\nu^{\frac{1}{2}} j_{2\nu} \chi_{\nu}^j}{\chi_{\nu}^j} B_{j_2} \),

\[
\langle (\partial_X \phi)^j, \omega_j \rangle = - \langle \nabla (\partial_X \phi)^j, (\nabla \phi)^j \rangle - \nu^{\frac{1}{2}} j_2 \langle ((\partial_Y \phi)^{(j_1+1,j_2-1)}, \chi_{\nu}^j (\partial_Y \phi)^j \rangle
\]

\[
= - \langle \partial_X (\nabla \phi)^j, (\nabla \phi)^j \rangle - 2 \nu^{\frac{1}{2}} j_2 \langle \chi_{\nu}^j (\partial_Y \phi)^{(j_1+1,j_2-1)}, (\partial_Y \phi)^j \rangle
\]

\[
= -2 \nu^{\frac{1}{2}} j_2 \langle \chi_{\nu}^j (\partial_Y \phi)^{(j_1+1,j_2-1)}, (\partial_Y \phi)^j \rangle.
\]
Hence we have from \(\|\chi_j^\prime\|_{L^\infty} = \kappa,\)
\[
\int_0^{\tau_0} |\langle (\partial_X\phi)^j,\omega_j^\prime \rangle| \, d\tau \leq 2K\nu^{j}j_2M_{2,j} |\partial_Y\phi|^2.
\] (4.32)

To estimate \(\int_0^{\tau_0} |\langle \rho_j(\partial_X\phi)^j,\xi_j^2\omega_j^\prime \rangle| \, d\tau\) the key inequality from the definition (4.9) is
\[
\xi_j\rho_j \leq \sqrt{\rho_j} \leq C(K_4^jC_4)^{\frac{j}{2}}(1 + (j + 1)^{1/2}Y)^{-1} + C\nu^{\frac{j}{2}},
\] (4.33)

where \(\nu^{\frac{j}{2}}(j + 1) \leq 2\) is used. Thus we have from the Hardy inequality,
\[
\int_0^{\tau_0} |\langle \rho_j(\partial_X\phi)^j,\xi_j^2\omega_j^\prime \rangle| \, d\tau \leq \int_0^{\tau_0} \|\xi_j\rho_j(\partial_X\phi)^j\|\|\xi_j\omega_j^\prime\| \, d\tau
\]
\[
\leq \frac{C(K_4^jC_4)^{\frac{j}{2}}}{(j + 1)^{\frac{1}{2}}} \int_0^{\tau_0} \|\frac{\partial_X\phi)^j}{Y}\|\|\xi_j\omega_j^\prime\| \, d\tau
\]
\[
+ C\nu^{\frac{j}{2}}\|\partial_Y(\partial_X\phi)^j\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega_j^\prime\|_{L^2(0,\tau_0;L^2)}
\]
\[
\leq \frac{C(K_4^jC_4)^{\frac{j}{2}}\|\partial_Y(\partial_X\phi)^j\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega_j^\prime\|_{L^2(0,\tau_0;L^2)}}{(j + 1)^{\frac{1}{2}}}
\]
\[
+ C\nu^{\frac{j}{2}}\|\partial_Y(\partial_X\phi)^j\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega_j^\prime\|_{L^2(0,\tau_0;L^2)}
\] (4.34)

Then the desired estimate for \(0 \leq j \leq \nu^{\frac{1}{2}} - 1\) follows from \(K\tau\nu^{\frac{1}{2}} \leq 1\) and
\[
\partial_Y(\partial_X\phi)^j = e^{K\tau\nu^{\frac{1}{2}}} (\partial_Y\phi)^{(j+1,j_2)} + \nu^{\frac{j}{2}}j_2\chi_j^\prime(\partial_Y\phi)^{(j+1,j_2-1)}.
\] (4.35)

On the other hand, the estimate for \(j = \nu^{\frac{1}{2}} - 1\) easily follows from
\[
\|\xi_j\rho_j(\partial_X\phi)^j\| \leq \|\sqrt{\rho_j}\|_{L^\infty}\|\partial_X\phi)^j\| \leq C(K_4^jC_4)^{\frac{j}{2}}\|\partial_X\phi)^j\|.
\]
Finally we have from Assumption 1 (iii) and Lemma 4.1,
\[
\|\xi_j(\partial_Y\phi)^j\partial_X\Omega\| \leq \frac{1 + Y}{1 + \nu^{\frac{1}{2}}Y} \|\partial_X\Omega\|_{L^\infty}\|\|\partial_Y\phi)^j\| \frac{1 + \nu^{\frac{1}{2}}Y}{1 + Y} \|\xi_j\omega_j^\prime\|_{L^\infty}\|\partial_Y\phi)^j\|
\]
\[
\leq CC_1^j\nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}}\|(\partial_Y\phi)^j\|,
\]
which gives
\[
\int_0^{\tau_0} \|\langle (\partial_Y\phi)^j\partial_X\Omega,\xi_j^2\omega_j^\prime \rangle\| \, d\tau \leq CC_1^j\nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}}M_{2,j}\|\partial_Y\phi\|\|\xi_j\omega_j^\prime\|_{L^2(0,\tau_0;L^2)}.
\]

Collecting these, we obtain (4.25), for the identity \(\partial_Y\Omega \xi_j^2 = \frac{\partial_Y\Omega}{\partial_Y\Omega_{\Omega}} = 1 - 2\rho_j\xi_j^2\) holds. The estimate (4.26) is verified from the definition \(\|\nabla\phi\|_{L^2}\) and
\[
\sum_{j=0}^{\nu^{-\frac{1}{2}} - 1} \frac{M_{2,j+1}[\partial_Y\phi]}{(j!)^{1/2}\nu^{\frac{j}{2}}(j + 1)} = \sum_{j=0}^{\nu^{-\frac{1}{2}} - 1} \frac{\nu^{\frac{1}{2}}(j + 1)^{1/2}M_{2,j+1}[\nabla\phi]}{(j + 1)!(j!)^{1/2}\nu^{\frac{j}{2}}(j + 1)}
\]
\[
\leq \sum_{j=0}^{\nu^{-\frac{1}{2}} - 1} \frac{\nu^{\frac{1}{2}}j^{1/2}M_{2,j}[\nabla\phi]}{(j!)^{1/2}\nu^{\frac{j}{2}}} \leq \|\nabla\phi\|_{L^2}.
\]

The proof is complete. \(\Box\)
Lemma 4.5. Let $j_2 \geq 1$. Then it follows that
\[ \int_0^{\tau_0} |\langle V_2[B_{j_2} \partial_\nu] e^{-K \tau_2^{1/2}(j+1)} \partial_{X}^j \omega, \xi^j \rangle| \, d\tau \leq CC^* \nu_2^{j_2} \| \xi_j \omega \|_{L^2(0, \tau_0; L^2_X)}^2, \] (4.36)

Here $C > 0$ is a universal constant.

Proof. The estimate directly follows from (4.19) and
\[ |V_2 \chi'_\nu| \leq \| V_2 Y \|_{L^\infty(0, \tau_0; L^\infty_X)} \leq C^* \nu_2 \| \partial_X^j \omega \|_{L^\infty(0, \tau_0; L^\infty_X)}, \]

by Assumption 1 (iii) and $\kappa \nu_2^2 Y e^{-\kappa \nu_2^2 Y} \leq C \chi_\nu$ for a universal constant $C > 0$. The proof is complete. \qed

Lemma 4.6. Let $j \geq 1$. It follows that
\[ \int_0^{\tau_0} |\sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\}} \left( \begin{array}{c} j_2 \\ l_2 \\ l \\ j_2 \\ l - l_2 \\ j - j_2 \\ j_2 - l_2 \end{array} \right) \cdot (\nabla \omega^1, \xi_j \omega^j) | \, d\tau \leq C \kappa R_{j, \text{Lem4.6}}[\omega] \| \xi_j \omega^j \|_{L^2(0, \tau_0; L^2_X)}, \]

where
\[ R_{j, \text{Lem4.6}}[\omega] := \sum_{l=0}^{j-1} (j - l + 1) \frac{1}{l_2} \min\{l + 1, j - l + 1\} \left( \begin{array}{c} j \\ l \\ j \\ l \end{array} \right) N_{\infty, j-1}[V] M_{2, l+1, j}[\omega], \]

and
\[ N_{\infty, j}[V] := \sup_{j_2=0}^{\cdots j} \left( \| B_{j_2} \partial_X^{j-j_2} V_1 \|_{L^\infty(0, \kappa \nu_2^2 \tau_2^2 \tau, L^\infty_X)} + \kappa \| \partial_X \omega \|_{L^\infty(0, \kappa \nu_2^2 \tau, L^\infty_X)} \right). \]

Moreover,
\[ \sum_{j=0}^{\nu_2^{-1}} \frac{R_{j, \text{Lem4.6}}[\omega]}{(j!)^{3/4} \nu_2^{3/4} (j+1)^{3/4}} \leq CC^*_0 \| \omega \|_{2, \xi}. \] (4.37)

Here $C > 0$ is a universal constant.

Proof. We first observe that
\[ \left( \begin{array}{c} j_2 \\ l_2 \\ l \\ j_2 \\ l - l_2 \\ j - j_2 \\ j_2 - l_2 \end{array} \right) \leq \left( \begin{array}{c} j \\ l \end{array} \right), \quad 0 \leq j_2 \leq l_2 \leq l \leq j, \] (4.38)

and
\[ \# \{ l_2 \in \mathbb{N} \cup \{0\} \mid \max\{0, l + j_2 - j\} \leq l_2 \leq \min\{l, j_2\} \} \leq \min\{l + 1, j - l + 1\}. \] (4.39)
Hence we have
\[
\int_0^{\tau_0} \left| \sum_{l=0}^{j-1} \sum_{l_1=0}^{\min\left\{0, l+j_2-j\right\}} \frac{j_2}{l_2} \left( j - j_2 \right) \left( l_2 \right) \right| \left| \xi_j V^{j-1} \cdot \left( \nabla \omega \right) \right|^1_{L^2(0, \tau_0 \cdot L^2)} \leq \sum_{l=0}^{j-1} \left( \frac{j}{l} \right) \min\{l+1, j-l+1\} \left| \xi_j V^{j-1} \cdot \left( \nabla \omega \right) \right|^1_{L^2(0, \tau_0 \cdot L^2)} \| \xi_j \omega^j \|_{L^2(0, \tau_0 \cdot L^2)}.
\]

From the definition of \( \xi_j \), we see that \( 0 \leq l \leq j-1, \)
\[
\frac{\xi_j}{\xi_l} \leq \sqrt{1 + \frac{(1 + (j + 1)l \omega)^2}{(1 + (l + 1)l \omega)^{-2}}} \leq C(j + l - 1)\frac{l}{2},
\]
where \( C > 0 \) is a universal constant, and thus,
\[
\| \xi_j V^{j-1} \cdot \left( \nabla \omega \right) \|^1_{L^2(0, \tau_0 \cdot L^2)} \leq C(j + l - 1)^{\frac{1}{2}} \| \xi_l V^{j-1} \cdot \left( \nabla \omega \right) \|^1_{L^2(0, \tau_0 \cdot L^2)}.
\]

Next we have
\[
\| \xi_l V^{j-1} \cdot \left( \partial_X \omega \right) \|^1_{L^2(0, \tau_0 \cdot L^2)} \leq \| \xi_l \|_{L^\infty} \| V^{j-1} \|^1_{L^\infty} \| \xi_l \|_{L^\infty} \| \omega \|_{L^\infty}(l_1, l_2) \| L^2(0, \tau_0 \cdot L^2) \leq C N_{\infty, j-l}[V] M_{2, l+1, \xi_l} \| \omega \|_{L^\infty},
\]
and similarly,
\[
\| \xi_l V^{j-1} \cdot \left( \partial_Y \omega \right) \|^1_{L^2(0, \tau_0 \cdot L^2)} \leq \| \xi_l \|_{L^\infty} \| V^{j-1} \|^1_{L^\infty} \| \xi_l \|_{L^\infty} \| \omega \|_{L^\infty}(l_1, l_2) \| L^2(0, \tau_0 \cdot L^2) \leq C \frac{N_{\infty, j-l}[V]}{\kappa} M_{2, l+1, \xi_l} \| \omega \|_{L^\infty}.
\]

Here we have used from \( \partial_X V_1 + \partial_Y V_2 = 0 \) that \( V^{j-1}_{\frac{l}{\omega}} = (\partial_Y V_2)(j_1 - 1, j_2 - 1) = -V_{\frac{1}{j_1 - 1, j_2 - 1 - 1}} \) for \( j_2 - l_2 \geq 1 \), which verifies \( \| V^{j-1}_{\frac{l}{\omega}} \|_{L^\infty} \leq C N_{\infty, j-l}[V] \). The estimate (4.37) follows from
\[
\sum_{j=0}^{\nu - \frac{1}{2}} \frac{1}{(j!) \nu \frac{1}{2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac{1}{2}} \min\{l+1, j-l+1\} \left( \frac{j}{l} \right) \frac{(j-l)!(l+1)!}{(l+1)!} \nu^\frac{1}{2} \left( \frac{j}{l} \right)^\frac{1}{2} \times \frac{N_{\infty, j-l}[V]}{(j-l)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}} \frac{M_{2, l+1, \xi_l}[\omega]}{(l+1)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}}
\]
\[
\leq \sum_{j=0}^{\nu - \frac{1}{2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac{1}{2}} \min\{l+1, j-l+1\} \frac{(j-l)!(l+1)!}{(l+1)!} \nu^\frac{1}{2} \left( \frac{j-l}{l} \right)^\frac{1}{2} \times \frac{N_{\infty, j-l}[V]}{(j-l)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}} \frac{\nu^\frac{1}{2} \left( l+2 \right)^\frac{1}{2} M_{2, l+1, \xi_l}[\omega]}{(l+1)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}}
\]
\[
\leq C \sum_{j=0}^{\nu - \frac{1}{2}} \sum_{l=0}^{j-1} \frac{N_{\infty, j-l}[V]}{(j-l)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}} \frac{\nu^\frac{1}{2} \left( l+2 \right)^\frac{1}{2} M_{2, l+1, \xi_l}[\omega]}{(l+1)! \nu^\frac{1}{2} \nu^\frac{1}{2} \frac{1}{2}}.
\]
Here we have used for $j \geq 1$,
\[
(j - l + 1)^{\frac{3}{2}} \min\{l + 1, j - l + 1\} \frac{(l + 1)^{3/2}}{(j + 1)^{3/2}(l + 2)^{3/2}} \frac{(j - l)!!}{j!} \leq C, \quad 0 \leq l \leq j - 1,
\]  
(4.40)
with a universal constant $C > 0$. Here the key is the following estimate for each $k = 0, 1, 2, 3$:
\[
\frac{(j - l)!!}{j!} \leq \frac{C}{(j + 1)^{1+k}} \quad \text{for} \quad 1 + k \leq j - 1 - k.
\]  
(4.41)
Then we obtain (4.37) from the Young inequality by convolution in the $l^1$ space. The proof is complete.

**Lemma 4.7.** Let $j \geq 1$. It follows that
\[
\int_0^\tau |\langle \sum_{l=\max\{0,l+j_2-j\} \leq l_2 \leq \min\{l,j_2\}} \frac{j_2}{l_2} \left( \frac{j - j_2}{l - l_2} \right) (\nabla \phi)^1 \rangle| d\tau 
\]
\[
\leq CR_{j,\text{Lem}4.7} |\langle \xi_j \omega^j \rangle|_{L^2(0,\tau;L^2_x,Y)},
\]
where
\[
R_{j,\text{Lem}4.7} |\langle \nabla \phi \rangle| := C_0^* \nu^{\frac{1}{2}} j (M_{2,j} |\langle \nabla \phi \rangle| + \nu^{\frac{1}{2}} j M_{2,j-1} |\langle \nabla \phi \rangle|)
\]
\[
\quad + (j + 1)^{3/2} \sum_{l=0}^{j-2} \min\{l + 1, j - l + 1\} \left( \frac{j}{l} \right) N_{\infty,j-l}[\langle \nabla \phi \rangle]
\]
\[
\quad \times (M_{2,l+1} |\langle \partial_Y \phi \rangle| + \nu^{\frac{1}{2}} (l + 1) M_{2,l} |\langle \nabla \phi \rangle|)
\]
\[
\quad + \nu^{\frac{1}{2}} (j + 1)^{3/2} N_{\infty,1}[\langle \nabla \phi \rangle] M_{2,j-1} |\langle \partial_Y \phi \rangle|,
\]
and
\[
N_{\infty,j-l}[\langle \nabla \phi \rangle] := \sup_{j_2=0,\ldots,j} \left( \| \frac{1}{1 + \nu^{1/2}Y} (\partial_Y \Omega)^j \|_{L^2(0,1/\kappa;L^2_x,Y)} + \nu^{-\frac{1}{2}} \| \frac{1}{1 + \nu^{1/2}Y} (\partial_X \Omega)^j \|_{L^2(0,1/\kappa;L^2_x,Y)} \right).
\]
Here the second term in the right-hand side is defined as zero when $j = 1$. Moreover,
\[
\sum_{j=0}^{\nu^{-\frac{1}{2}}} R_{j,\text{Lem}4.7} |\langle \nabla \phi \rangle| \leq C(C_0^* + C_2^*) ||\langle \nabla \phi \rangle||^j_{2,1}.
\]  
(4.42)

**Proof.** As in the proof of Lemma 4.6, we have from (4.38) and (4.39),
\[
\int_0^\tau |\langle \sum_{l=\max\{0,l+j_2-j\} \leq l_2 \leq \min\{l,j_2\}} \frac{j_2}{l_2} \left( \frac{j - j_2}{l - l_2} \right) (\nabla \phi)^1 \rangle| d\tau
\]
\[
\leq \sum_{l=0}^{j-1} C_l \min\{l + 1, j - l + 1\} ||\langle \xi_j (\nabla \phi)^1 \rangle|_{L^2(0,\tau;L^2)} ||\langle \xi_j \omega^j \rangle||_{L^2(0,\tau;L^2)}. 
\]
Then we have from Lemma 4.1,

\[ \|\xi_j(\partial_Y \phi)^1(\partial_X \Omega)^{j-1}\|_{L^2(0, \tau_0; L^2)} \leq \| \frac{(1 + \nu^\frac{j}{2} Y)}{1 + Y} \xi_j \|_{L^\infty} \| \frac{1 + Y}{1 + \nu^\frac{j}{2} Y} (\partial_X \Omega)^{j-1}\|_{L^\infty} \| (\partial_Y \phi)^1\|_{L^2(0, \tau_0; L^2)} \]

\[ \leq C \nu^\frac{j}{2} (j + 1)^\frac{1}{2} \lambda_{\infty,j-l}^{\frac{1}{2}} \| \nabla \Omega \|_M \| \partial_Y \phi \|. \]

Let \( j \geq 2 \) and \( 0 \leq l \leq j - 2 \). Then,

\[ \|\xi_j(\partial_X \phi)^1(\partial_Y \Omega)^{j-1}\|_{L^2(0, \tau_0; L^2)} \]

\[ \leq C(j + 1)^\frac{1}{2} \lambda_{\infty,j-l}^{\frac{1}{2}} \| \nabla \Omega \|_M \| \partial_Y \phi \|_{L^2(0, \tau_0; L^2)} + \nu^\frac{j}{2} (l + 1) \| \partial_X \phi \|_{L^2(0, \tau_0; L^2)}, \quad 0 \leq l \leq j - 2. \]

As for the case \( l = j - 1 \), we rather compute as, by recalling \( \xi_j \leq \frac{\nu^\frac{j}{2} Y}{1 + \nu^\frac{j}{2} Y} \chi_{\nu} \chi_{\nu} \leq C \nu^\frac{j}{2}, \) Assumption 1 (iii), and (4.35), we have

\[ \|\xi_j(\partial_X \phi)^1(\partial_Y \Omega)^{j-1}\|_{L^2(0, \tau_0; L^2)} \leq C \| \frac{\chi_{\nu} \chi_{\nu}}{1 + \nu^\frac{j}{2} Y} \chi_{\nu} \partial_Y \Omega \|_{L^\infty} \| \frac{\nu^\frac{j}{2} Y}{1 + \nu^\frac{j}{2} Y} \chi_{\nu} \chi_{\nu} \partial_X \phi^1\|_{L^2(0, \tau_0; L^2)} \]

\[ \leq C(\| \chi_{\nu} \chi_{\nu} \partial_Y \Omega \|_{L^\infty} + \| \frac{\chi_{\nu} \chi_{\nu} \partial_Y \Omega}{1 + \nu^\frac{j}{2} Y} \partial_X \phi^1\|_{L^2(0, \tau_0; L^2)}) \]

\[ \times \| \frac{\nu^\frac{j}{2} Y}{1 + \nu^\frac{j}{2} Y} \chi_{\nu} \chi_{\nu} \partial_X \phi^1\|_{L^2(0, \tau_0; L^2)} \]

Here we have used that, when \( l = j - 1 \), either \( \partial_Y \Omega^{j-1} = \partial_X \Omega^\chi \) or \( \chi_{\nu} \partial_Y \Omega^\chi \) holds, and that the Hardy inequality. Then, by using \( \| \nu^\frac{j}{2} Y \chi_{\nu} \operatorname{ch}_{\nu} \|_{L^\infty} \leq C \nu^\frac{j}{2} \), Assumption 1 (iii), and (4.35), we have

\[ \|\xi_j(\partial_X \phi)^1(\partial_Y \Omega)^{j-1}\|_{L^2(0, \tau_0; L^2)} \leq CC \nu^\frac{j}{2} (M_{2,l+1} \| \partial_Y \phi \| + \nu^\frac{j}{2} (l + 1) M_{2,l} \| \nabla \phi \|), \quad l = j - 1. \]

Collecting these, we obtain the term \( R_{j, \text{lem} 4.7} \| \nabla \phi \| \) by noticing \( j C_l = j \) for \( l = j - 1 \), as desired. The estimate (4.42) is proved as in (4.37) but by also using the Young inequality for convolution in the \( l^1 \) space together with the estimates for \( j \geq 2 \),

\[ (j + 1)^\frac{1}{2} \min\{l + 1, j - l + 1\} \frac{(j + 1)^{\frac{1}{2}}}{(j + 1)^{\frac{1}{2}}}(j - l)! \frac{l!}{j!} \leq C, \quad 0 \leq l \leq j - 2, \]

\[ (j + 1)^\frac{1}{2} \min\{l + 1, j - l + 1\} \frac{l + 1}{(j + 1)^{\frac{1}{2}}(l + 1)^{\frac{1}{2}}}(j - l)! \frac{l!}{j!} \leq C, \quad 0 \leq l \leq j - 2. \]

Note that the condition \( l \leq j - 2 \) is crucial here, for we apply (4.41). We omit the details. The proof is complete. \( \square \)
Lemma 4.8. There exists $K_{1,4} = K_{1,4}(C_1', C_*) \geq 1$ such that for $K \geq K_{1,4}$,
\[
\int_0^{\tau_0} \langle \text{rot } F^j - [B_{j2}, \partial Y] \partial_X^j \frac{e^{-K\tau \frac{1}{2}(j+1)}}{\nu(j+1)^\frac{1}{2}} F_1, \xi_j^2 \omega^j \rangle \, d\tau \\
\leq C(C_1' + 1)M_{2,j,\xi_j}[F] \\
\times \left( (\|\xi_j(\nabla Y)^j\|_{L^2(0,\tau_0; L^2_X, Y)}) + \nu\frac{1}{2} j M_{2,j-1,\xi_j-1} [\partial_Y \omega] + (j + 1)\frac{1}{2} \|\xi_j \omega^j\|_{L^2(0,\tau_0; L^2_X, Y)} \right),
\]
and
\[
\int_0^{\tau_0} \langle G^j, \xi_j^2 \omega^j \rangle \, d\tau \leq M_{2,j,\xi_j}[G] \|\xi_j \omega^j\|_{L^2(0,\tau_0; L^2_X, Y)}.
\]
Here $C > 0$ is a universal constant.

Proof. The estimate about $G^j$ is straightforward and we focus on the estimate about $F^j$. The integration by parts and also (4.19) yield
\[
\int_0^{\tau_0} \langle \text{rot } F^j - [B_{j2}, \partial Y] \partial_X^j \frac{e^{-K\tau \frac{1}{2}(j+1)}}{\nu(j+1)^\frac{1}{2}} F_1, \xi_j^2 \omega^j \rangle \, d\tau \\
= \int_0^{\tau_0} \langle F^j, \nabla \cdot (\xi_j^2 \omega^j) \rangle \, d\tau + \int_0^{\tau_0} \langle F^j, \xi_j(\nabla \cdot \omega^j) \rangle \, d\tau + \nu\frac{1}{2} j M_{2,j-1,\xi_j-1} [\partial_Y \omega] (j, j_2-1) \, d\tau \\
\leq \int_0^{\tau_0} \langle F^j, \nabla \cdot (\xi_j^2 \omega^j) \rangle \, d\tau + M_{2,j,\xi_j}[F] \|\xi_j(\nabla \omega^j)\|_{L^2(0,\tau_0; L^2)} \\
+ C\nu\frac{1}{2} j M_{2,j,\xi_j}[F] M_{2,j-1,\xi_j-1} [\partial_Y \omega].
\]
Then we have from Assumption 1 (iv) and Lemma 4.1, by recalling $\nabla \cdot (\xi_j^2 \omega^j) = -\frac{\nabla \cdot (\partial_Y \omega^j + 2\rho_j)}{\partial_Y \omega^j + 2\rho_j} \xi_j^3 = -\frac{\nabla \cdot (\partial_Y \omega^j + 2\rho_j)}{\partial_Y \omega^j + 2\rho_j} \xi_j^3 \omega^j$
\[
\langle F^j, \nabla \cdot (\xi_j^2 \omega^j) \rangle \omega^j \\
\leq \|\xi_j F^j\| \left( \left\| \frac{Y \nabla (\partial_Y \Omega + 2\rho_j)}{(1 + \nu\frac{1}{2} Y)} \xi_j^2 \|_{L^\infty((0 < Y < (j+1) - \frac{1}{8}))} \right\| \frac{1 + \nu\frac{1}{2} Y}{Y} \|\omega^j\|_{L^2((0 < Y < (j+1) - \frac{1}{8}))} \right) \\
+ \left\| \frac{Y \nabla \partial_Y \Omega}{(1 + \nu Y)} \|_{L^\infty((Y > (j+1) - 1\frac{1}{8}))} \right\| \frac{1 + \nu\frac{1}{2} Y}{Y} \|\partial_Y \omega^j\|_{L^\infty((Y > (j+1) - \frac{1}{8}))} \|\xi_j^2 \omega^j\| \\
+ \left\| \frac{1}{Y} \|_{L^\infty((Y > (j+1) - 1\frac{1}{8}))} \right\| \|\xi_j^2 \omega^j\| \\
\leq C\|\xi_j F^j\| \left( \left\| \frac{C_2}{K\frac{1}{2} C_s} \|_{L^\infty((0 < Y < (j+1) - \frac{1}{8}))} + \right\| \nabla \partial_Y \rho_j \xi_j^2 \|_{L^\infty((0 < Y < (j+1) - \frac{1}{8}))} \right) \|\omega^j\| \\
+ (C_2 + 1)(j + 1)^\frac{3}{2} \|\xi_j^2 \omega^j\| \\
\leq C\|\xi_j F^j\| \left( \left\| \frac{C_2}{K\frac{1}{2} C_s} + \frac{1}{(K\frac{1}{2} C_s)^{\frac{1}{2}}} \right\| \|\partial_Y \omega^j\| + (C_2 + 1)(j + 1)^\frac{3}{2} \|\xi_j^2 \omega^j\| \right).
Thus, the estimate (4.23) for \( \partial_t \omega \) yields the desired estimate by taking \( K \) large enough depending only on \( C^*_1 \) and \( C_\ast \). The proof is complete.

We are now in position to prove Proposition 4.2. Lemmas 4.2-4.8 imply that, by taking the supremum over \( j_2 = 0, \cdots, j \),

\[
\nu^{\frac{1}{2}} M_{2,j,\xi_j} [\nabla \omega] + M_{\infty,j,\xi_j} [\omega] + (K \nu^{\frac{1}{2}} (j + 1))^{\frac{1}{2}} M_{2,j,\xi_j} [\omega]
\]

\[
\leq C \left( \sup_{j_2=0, \cdots, j} \| \xi_j \omega^j (0) \| + \kappa \nu^{\frac{1}{2}} \int \frac{M_{2,j-1,\xi_j} [\nabla \omega]}{(K \nu^{\frac{1}{2}} (j + 1))^{\frac{1}{2}}} + (C^*_2 + 1) \nu^{-\frac{1}{4}} \| M_{2,j,\xi_j} [F] \| \right)
\]

for \( j = 0, 1, \cdots, \nu^{-\frac{1}{2}} \). Here \( K \geq 1 \) is taken large enough depending only on \( C_\ast \) and \( C^*_j \), while \( C > 0 \) is a universal constant. Hence, by taking the sum \( \sum_{j=0}^{\nu^{-\frac{1}{2}}} \) with the factor \( \frac{1}{(j!)^{\frac{1}{2}} \nu^{\frac{1}{2}}} \), we obtain

\[
\| \nabla \omega \|_{2, \xi (1)} + \| \omega \|_{2, \xi \ast} + K^{\frac{1}{2}} \| \omega \|_{2, \xi}
\]

\[
\leq C \left( \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{(j!)^{\frac{1}{2}} \nu^{\frac{1}{2}}} \sup_{j_2=0, \cdots, j} \| \xi_j \omega^j (0) \| + \kappa \| \nabla \omega \|_{2, \xi (1)} + \frac{C^*_0}{K^{\frac{1}{2}}} \| \omega \|_{2, \xi}
\]

\[
+ (1 + \frac{C^*_2 + C^*_2}{K^{\frac{1}{2}}}) \| \nabla \phi \|_{2, 1} + \frac{1}{K^{\frac{1}{2}} \nu^{\frac{1}{2}}} \| G \|_{2, \xi (2)} + \frac{C^*_2 + 1}{\nu^{\frac{1}{2}}} \| F \|_{2, \xi (1)} \right).
\]

Thus we obtain (4.18) by first taking \( \kappa > 0 \) small enough and then by taking \( K \) large enough, and also by using \( \xi_j \leq \frac{1}{C^*_0 \nu^{\frac{1}{2}}} \leq \frac{1}{\nu^{\frac{1}{2}}} \) to bound \( \| \xi_j \omega^j (0) \| \). Note that the required smallness on \( \kappa \) is independent of \( \nu, K, C_\ast, \) and \( C^*_j \), while the required largeness of \( K \) depends only on \( \kappa, C_\ast, C^*_j \). The proof of Proposition 4.2 is complete.

### 4.2 Estimate for the velocity in terms of the vorticity

In this subsection we give the estimate of the streamfunction \( \phi \) in terms of the vorticity \( \omega \). We remind that \( \omega = -\Delta \phi \) with the boundary condition \( \phi|_{\Sigma} = 0 \).

**Proposition 4.3.** There exists \( \kappa_2 \in (0, 1) \) such that for any \( K \geq 1 \), \( \kappa \in (0, \kappa_2) \), and \( p \in [1, \infty) \),

\[
\| \nabla \phi \|_{p, 1} \leq C (K^{\frac{1}{2}} C_\ast + C^*_1)^{\frac{1}{2}} \| \omega \|_{p, \xi} + C \nu^{\frac{1}{2}} \| \nabla \phi (0, 0) \|_{L^p \left(K^{\frac{1}{2}} \nu^{\frac{1}{2}} L^2 \right)}.
\]

Here \( C > 0 \) is a universal constant.

**Proof.** It suffices to show

\[
\sum_{j=1}^{\nu^{-\frac{1}{2}}} \frac{\nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}}}{(j!)^{\frac{1}{2}} \nu^{\frac{1}{2}}} M_{p,j} [\nabla \phi] \leq C (K^{\frac{1}{2}} C_\ast + C^*_1)^{\frac{1}{2}} \| \omega \|_{p, \xi} + C \nu^{\frac{1}{2}} \| \nabla \phi (0, 0) \|_{L^p \left(K^{\frac{1}{2}} \nu^{\frac{1}{2}} L^2 \right)}.
\]

(4.43)
Let $j \geq 1$ and let us recall that $\omega^j = e^{-K\nu \partial_j^2(j+1)}B_{j\nu} \partial_j^{j-j_2} \omega$ with $\omega = -\Delta \phi$. Computations similar to those in (4.20) imply $\omega^j = -\nabla \cdot (\nabla \phi)^j + \frac{\nu^2 j_2 \chi'}{2\nu}(\partial_Y \phi)^j$. Then the integration by parts together with the identity $\nabla \phi^j = (\nabla \phi)^j + \nu \frac{1}{2} j_2 e^{-K\nu \partial_j^2} (\partial_Y \phi)^{(j-j_2,j_2-1)} e_2$ yields

$$\langle \omega^j, \phi^j \rangle = \| (\nabla \phi)^j \|^2 + 2 \nu \frac{1}{2} j_2 e^{-K\nu \partial_j^2} (\chi' (\partial_Y \phi)^j, (\partial_Y \phi)^{(j-j_2,j_2-1)}).$$

(4.44)

Then $\langle \omega^j, \phi^j \rangle \leq \| \xi_j \omega^j \| \| \phi^j \|$ and the definition of $\xi_j$ in (4.8) gives

$$\| \phi^j \| \| \phi^j \| = \| \sqrt{\partial_Y \Omega + 2 \nu \phi^j} \| \| (\frac{1+Y}{1+\nu Y})^2 \partial_Y \Omega \| \| \| (\frac{1+\nu Y}{1+Y})^{-\frac{1}{2}} \phi^j \| + \sqrt{2} \| \phi^j \| \leq (C^*_1)^{\frac{1}{2}} (C \| \partial_Y \phi^j \| + \nu \frac{1}{2} \| \phi^j \|) + \sqrt{2} \| \phi^j \|.$$

Here we have used Assumption 1 (iii) and the Hardy inequality. Next the definition of $\rho_j$ in (4.9) implies

$$\sqrt{\rho_j} \leq K^\frac{1}{2} C^*_2 (1 + (j + 1) \frac{1}{2} Y)^{-1} + C^*_2 \left( \left( \frac{1+Y}{\nu^2} \right)^{-1} + \nu \frac{1}{2} \left( 1 + Y \right)^{-1} + \nu \frac{1}{2} \right),$$

which gives from the Hardy inequality and $\nu \frac{1}{2} (j + 1) \leq 2$ and $K \geq 1$,\n
$$\| \sqrt{\rho_j} \phi^j \| \leq C K^\frac{1}{2} C^*_2 (j + 1)^{-\frac{1}{2}} \| \partial_Y \phi^j \| + C^*_2 \nu \frac{1}{2} \| \phi^j \|.$$

Thus we have

$$\| \phi^j \| \| \phi^j \| \leq C (C^*_1 + (K^\frac{1}{2} C^*_4)) \| \partial_Y \phi^j \| + C (C^*_1 + C^*_4)^{\frac{1}{2}} \nu \frac{1}{2} \| \phi^j \|.$$

Thus (4.44) and the identity $\partial_Y \phi^j = (\partial_Y \phi)^j + \nu \frac{1}{2} j_2 \chi' e^{-K\nu \partial_j^2} (\partial_Y \phi)^{(j-j_2,j_2-1)}$ finally give

$$\| (\nabla \phi)^j \|^2 \leq C (C^*_1 + (K^\frac{1}{2} C^*_4))^\frac{1}{2} \| \xi_j \omega^j \| + C K \nu \frac{1}{2} j_2 \| (\partial_Y \phi)^{(j-j_2,j_2-1)} \| + \frac{1}{16} \nu \frac{1}{2} \| \phi^j \|.$$

Here $C > 0$ is a universal constant. Taking the supremum about $j_2 = 0, \cdots, j$ yields

$$M_{p,j,1}[\nabla \phi] \leq C (C^*_1 + (K^\frac{1}{2} C^*_4))^\frac{1}{2} M_{p,j,\xi_j}[\omega] + C K \nu \frac{1}{2} j M_{p,j-1,1}[\nabla \phi] + \frac{1}{16} \nu \frac{1}{2} M_{p,j,1}[\phi].$$

Thus we have from $M_{p,j,1}[\phi] \leq M_{p,j-1,1}[\nabla \phi]$ and $\frac{j+1}{j} \leq 2$ for $j \geq 1$,

$$\sum_{j=1}^{\nu \frac{1}{2}} \frac{\nu \frac{1}{2} (j+1)}{(j+1)^{\frac{1}{2}} \nu \frac{1}{2}} M_{p,j,1}[\nabla \phi] \leq C (K^\frac{1}{2} C^*_4 + C^*_1)^{\frac{1}{2}} \| \omega \|_{p,\xi} + \left( C K + \frac{1}{8} \right) \sum_{j=0}^{\nu \frac{1}{2}} \nu \frac{1}{2} (j+1) \frac{1}{(j+1)^{\frac{1}{2}} \nu \frac{1}{2}} M_{p,j,1}[\nabla \phi].$$

Here $C > 0$ is a universal constant. By taking $\kappa$ small enough we obtain (4.43). The proof is complete. □
In view of the estimate in Proposition 4.3 our next task is to show the estimate of the zero-th order term $\nabla \phi^{(0,0)}$.

**Proposition 4.4.** Let $\kappa_2 \in (0,1]$ be the number in Proposition 4.3. There exists $K_2 = K_2(C_*, C_1^*) \geq 1$ such that for any $K \geq K_2$ and $\kappa \in (0, \kappa_2)$,

$$
\nu^\frac{1}{2} \|\omega^{(0,0)}\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + \|\nabla \phi^{(0,0)}\|_{L^\infty(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + K_2 \nu^\frac{1}{2} \|\nabla \phi^{(0,0)}\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} \\
\leq C \left( \|\nabla \phi(0)\|_{L^2_{X,Y}} + \frac{1}{K_2 \nu^2} \|F\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + \frac{1}{K_2 \nu^2} \|G\|_{L^2(0, \frac{1}{\kappa \nu^2}; H^{-1})} + \|\omega\|_{L^2_{X,Y,Y}} \right),
$$

(4.45)

Here $C > 0$ is a universal constant.

**Proof.** It suffices to show

$$
\nu^\frac{1}{2} \|\omega^{(0,0)}\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + \|\nabla \phi^{(0,0)}\|_{L^\infty(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + K_2 \nu^\frac{1}{2} \|\nabla \phi^{(0,0)}\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} \\
\leq C \left( \|\nabla \phi(0)\|_{L^2_{X,Y}} + \frac{1}{K_2 \nu^2} \|F\|_{L^2(0, \frac{1}{\kappa \nu^2}; L^2_{X,Y})} + \frac{1}{K_2 \nu^2} \|G\|_{L^2(0, \frac{1}{\kappa \nu^2}; H^{-1})} \\
+ \frac{C_1^*}{K_2} \|\partial_\gamma \phi\|_{L^2_{X,Y}} \right). \tag{4.46}
$$

Indeed, estimate (4.45) is a direct consequence of (4.46) and Proposition 4.3 by taking $K$ large enough depending only on $C_1^*$ and $C_*$. To prove (4.45) let us go back to (4.1), and we take the inner product with $\eta_R \phi$ for (4.1), where $\eta_R = \eta(Y/R)$ with a smooth cut-off $\eta$ such that $\eta = 1$ for $0 \leq Y \leq 1$ and $\eta = 0$ for $Y \geq 1$. Then, taking the limit $R \to \infty$ after the integration by parts verifies the identity

$$
\nu^\frac{1}{2} \|\omega^{(0,0)}\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \phi^{(0,0)}\|^2 + K_2 \nu^\frac{1}{2} \|\nabla \phi^{(0,0)}\|^2 \\
= -\langle \Delta \phi^{(0,0)}, V \cdot \nabla \phi^{(0,0)} \rangle + \langle F^{(0,0)}, \nabla \phi^{(0,0)} \rangle + \langle G^{(0,0)}, \phi^{(0,0)} \rangle, \quad \tau > 0. \tag{4.47}
$$

Note that $|\langle F^{(0,0)}, \nabla \phi^{(0,0)} \rangle| \leq \|F\| \|\nabla \phi^{(0,0)}\|$ and $|\langle G^{(0,0)}, \phi^{(0,0)} \rangle| \leq \|G\|_{H^{-1}} \|\nabla \phi^{(0,0)}\|$. Thus it suffices to focus on the term $-\langle \Delta \phi^{(0,0)}, V \cdot \nabla \phi^{(0,0)} \rangle$. The integration by parts and $\nabla \cdot V = 0$ imply

$$
-\langle \Delta \phi^{(0,0)}, V \cdot \nabla \phi^{(0,0)} \rangle = \langle \partial_X \phi^{(0,0)}, (\partial_X V) \cdot \nabla \phi^{(0,0)} \rangle + \langle \partial_Y \phi^{(0,0)}, (\partial_Y V) \cdot \nabla \phi^{(0,0)} \rangle \\
= \langle \partial_X \phi^{(0,0)}, (\partial_X V) \cdot \nabla \phi^{(0,0)} \rangle - \langle \partial_Y \phi^{(0,0)}, (\partial_Y V) \partial_X \phi^{(0,0)} \rangle \\
+ \langle \partial_Y \phi^{(0,0)}, (\partial_Y V) \partial_X \phi^{(0,0)} \rangle \\
\leq 2C_1^* \nu^\frac{1}{2} \|\nabla \phi^{(0,0)}\|^2 + \langle \partial_Y \phi^{(0,0)}, (\partial_Y V) \partial_X \phi^{(0,0)} \rangle.
$$

Here we have used Assumption 1 (ii). Then the last term is estimated as

$$
\langle \partial_Y \phi^{(0,0)}, (\partial_Y V) \partial_X \phi^{(0,0)} \rangle \leq \left\| \frac{1+ Y}{1+\nu^\frac{1}{2} Y} \partial_Y V \right\|_{L^\infty} \|\partial_Y \phi^{(0,0)}\| \left\| \frac{1+ \nu^\frac{1}{2} Y}{1+ Y} \partial_X \phi^{(0,0)} \right\| \\
\leq C_1^* \|\partial_Y \phi^{(0,0)}\| \left( C \|\partial_X^2 \phi^{(0,0)}\| + \nu^\frac{1}{2} \|\partial_X \phi^{(0,0)}\| \right).
$$
Here we have used Assumption 1 (ii) and the Hardy inequality. Hence by taking $K$ large enough depending only on $C^*_1$ we obtain
\[
\nu^{\frac{1}{2}} \|\omega(0,0)\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \phi(0,0)\|^2 + K \nu^{\frac{1}{2}} \|\nabla \phi(0,0)\|^2 \leq \frac{C(C^*_1)^2}{K \nu^{\frac{1}{2}}} \|\partial_X \partial_Y \phi(0,0)\|^2 + C(\|F\|^2 + \|G\|^2_{H^{-1}}).
\]
Integrating about $\tau$ shows (4.46), for $\nu^{-\frac{1}{2}} \|\partial_X \partial_Y \phi(0,0)\|^2_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \leq (\|\partial_Y \phi(0,0)\|_{2,1})^2$ holds. The proof is complete. \(\square\)

### 4.3 Proof of Proposition 4.1

Propositions 4.3 and 4.4 yield
\[
K \nu^{\frac{1}{2}} \|\nabla \phi\|_{2,1}^2 \leq C \left( K \nu^{\frac{1}{2}} (K \nu^{\frac{1}{2}} C^*_1 + C^*_1)^{\frac{1}{2}} \|\omega\|_{2,\xi}^2 + \left\| \nabla \phi(0) \right\|_{L^2_{X,Y}}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| F \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| G \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; H^{-1})}^2 \right).
\]

Then (4.48) and Proposition 4.2 give
\[
\|\omega\|_{2,\xi; \xi} + K \nu^{\frac{1}{2}} \|\omega\|_{2,\xi} + K \nu^{\frac{1}{2}} \|\nabla \phi\|_{2,1} \leq C \left( \left\| \nabla \phi_0 \right\|_{L^2_{X,Y}} + \nu^{\frac{1}{2}} \left\| \Delta \phi_0 \right\| ight) + \left( C^*_2 + 1 \right) \nu^{-\frac{1}{2}} \left\| F \right\|_{2,\xi; (1)}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| G \right\|_{2,\xi; (2)}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| G \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; H^{-1})}^2 \right).
\]

It remains to estimate the boundary trace $\|\partial_Y \phi|_{Y=0}\|_{bc}$. By the interpolation inequality we have
\[
|\partial^j_X \partial_Y \phi(\tau, X, 0)| \leq C \left\| \partial^j_X \partial_Y \phi(\tau, X, \cdot) \right\|_{L^2_{Y}} \left\| \partial^j_X \partial_Y \phi(\tau, X, \cdot) \right\|_{L^2_{Y}}^2,
\]
which implies
\[
K \nu^{\frac{1}{2}} \left\| \partial_Y \phi(\cdot, 0) \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \leq C \left( K \nu^{\frac{1}{2}} \left\| \partial^2_Y \phi(\cdot, 0) \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \left\| \partial_Y \phi(\cdot, 0) \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \right)^2 \leq C \left( K \nu^{\frac{1}{2}} \left\| \omega(\cdot, 0) \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \right)^2 \left( K \nu^{\frac{1}{2}} \left\| \partial_Y \phi(\cdot, 0) \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \right)^2.
\]

Here we used the Calderón-Zygmund inequality. Since (4.17) yields $\|\omega(\cdot, 0)\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; L^2_{X,Y})} \leq (C^*_1 + 8K \nu^{\frac{1}{2}} C^*_1)^{\frac{1}{2}} M_{2,j; \xi}(\omega)$, we have from (4.49) that, by taking $K$ further large enough if necessary,
\[
K \nu^{\frac{1}{2}} \left\| \partial_Y \phi|_{Y=0} \right\|_{bc} \leq C \left( K \nu^{\frac{1}{2}} \left\| \omega \right\|_{2,\xi} \right)^2 \left( K \nu^{\frac{1}{2}} \left\| \nabla \phi \right\|_{2,1} \right)^2 \leq C \left( \left\| \nabla \phi_0 \right\|_{L^2_{X,Y}} + \nu^{\frac{1}{2}} \left\| \Delta \phi_0 \right\| \right) + \left( C^*_2 + 1 \right) \nu^{-\frac{1}{2}} \left\| F \right\|_{2,\xi; (1)}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| G \right\|_{2,\xi; (2)}^2 + \frac{1}{K \nu^{\frac{1}{2}}} \left\| G \right\|_{L^2(0, \frac{1}{K \nu^{\frac{1}{2}}}; H^{-1})}^2 \right).
\]

The proof of Proposition 4.1 is complete.
5 Construction of the boundary corrector

In the previous section, we constructed a solution to the Orr-Sommerfeld equation with arbitrary initial data, but artificial boundary conditions: we replaced condition \( \partial Y \phi |_{Y=0} = 0 \) by \( \Delta \phi |_{Y=0} = 0 \). Hence, to prove Theorem 2.2, we still need to understand how to correct the Neumann condition, that is how to construct solutions for systems of the following type

\[
\nu^{\frac{1}{2}} \Delta^2 \phi - \partial_x \Delta \phi - V \cdot \nabla \Delta \phi + \nabla^\perp \phi \cdot \nabla \Omega = 0, \quad \tau > 0, \ X \in T_\nu, \ Y > 0, \\
\phi|_{Y=0} = 0, \quad \partial_Y \phi|_{Y=0} = h, \quad \phi|_{\tau=0} = 0.
\] (5.1)

Such construction will be performed through an iteration, with first approximation given by the Stokes equation.

5.1 Stokes estimate

In this subsection we consider the solution to the Stokes equations (in terms of the stream-function):

\[
\nu^{\frac{1}{2}} \Delta^2 \phi - \partial_x \Delta \phi = 0, \quad \tau > 0, \ X \in T_\nu, \ Y > 0, \\
\phi|_{Y=0} = 0, \quad \partial_Y \phi|_{Y=0} = h, \quad \phi|_{\tau=0} = 0.
\] (5.2)

Here \( h \) is a given boundary data satisfying \( h(\tau) = 0 \) for \( \tau = 0 \) and \( \tau \geq \frac{1}{K\nu^{\frac{1}{2}}} \), and the bound

\[
\|h\|_{bc} = \sum_{j=0}^{\nu^{\frac{1}{2}}} \frac{\nu^{\frac{1}{2}}(j+1)^{\frac{1}{2}}}{(j!)^{\frac{1}{2}}\nu^{\frac{1}{2}}} \|e^{-K\tau\nu^{\frac{1}{2}}(j+1)}\partial^n h\|_{L^2(0, \frac{1}{K\nu^{\frac{1}{2}}}; L^2_X)} < \infty.
\] (5.3)

Set \( \psi = e^{-K\tau\nu^{\frac{1}{2}}(j+1)}\partial^n_X \phi, \ 0 \leq j_1 \leq j \), with the zero extension for \( \tau \leq 0 \) and let \( \hat{\psi} = \hat{\psi}(\lambda, \alpha, Y) \) be the Fourier (in \( X \) and \( \tau \)) transform of \( \psi \). Then \( \hat{\psi} \) obeys the ODE

\[
\nu^{\frac{1}{2}}(\partial^2_Y - \alpha^2)\hat{\psi} - (i\lambda + K\nu^{\frac{1}{2}}(j+1))(\partial^2_Y - \alpha^2)\hat{\psi} = 0, \quad Y > 0, \\
\hat{\psi}|_{Y=0} = 0, \quad \partial_Y \hat{\psi}|_{Y=0} = \hat{g}^{(j_1)},
\] (5.4)

where \( \lambda \in \mathbb{R} \) and \( \hat{g}^{(j_1)} \) is the Fourier transform of \( g^{(j_1)} := e^{-K\tau\nu^{\frac{1}{2}}(j+1)}\partial^n_X h \). We note that

\[
\alpha = \nu^{\frac{1}{2}} n,
\] (5.5)

where \( n \) is the \( n \)th Fourier mode in the original variable \( x \in T \). Assuming the decay of \( (|\alpha|\hat{\psi}, \partial_Y \hat{\psi}) \) and the boundedness of \( \hat{\psi} \), we obtain the formula

\[
\hat{\psi}(\lambda, \alpha, Y) = \frac{e^{-\gamma Y} - e^{-|\alpha|Y}}{\gamma - |\alpha|} g^{(j_1)}(\lambda, \alpha),
\]

\[
\gamma = \gamma_j(\lambda, \alpha, \nu, K) = \sqrt{\alpha^2 + K(j+1) + \frac{i\lambda}{\nu^{\frac{1}{2}}}}.
\] (5.6)

where the square root is taken so that the real part is positive, and it follows that

\[
|\alpha| \leq \sqrt{\alpha^2 + K(j+1)} \leq Re(\gamma) \leq |\gamma| \leq \sqrt{2} Re(\gamma).
\] (5.7)
This inequality will be freely used. We can also check the identity
\[ \partial_Y \hat{\psi}(\lambda, \alpha, Y) = -e^{-\gamma Y} \hat{g}(j_1)(\lambda, \alpha) + \text{sgn}(\alpha) \alpha \hat{\psi}(\lambda, \alpha, Y). \]  
(5.8)

We also have from (5.6),
\[ -(\partial_Y^2 - \alpha^2) \hat{\psi} = (\gamma + |\alpha|) e^{-\gamma Y} \hat{g}(j_1). \]  
(5.9)

This formula will be used in estimating the vorticity field.

**Lemma 5.1.** There exists \( \kappa' \in (0, 1] \) such that the following statement holds for any \( \kappa \in \{0, \kappa'\} \). Let \( j_1 = 0, \ldots, j \) and \( j_2 = j - j_1 \). Then
\[
|B_{j_2} i \alpha \hat{\psi}(\lambda, \alpha, Y)| \leq \frac{C \nu^{\frac{j_2}{2}} j_2! |\alpha \hat{g}(j_1)|}{j_2 + 1} \left( Ye^{-\frac{\nu \kappa(j_1)}{2}} e^{-\frac{\nu}{2} Y} \left| 1 - e^{-\left(\gamma - |\alpha|\right) Y} \right| \right),
\]  
(5.10)

\[
|B_{j_2} \partial_Y \hat{\psi}(\lambda, \alpha, Y)| \leq \frac{C \nu^{\frac{j_2}{2}} j_2! |\hat{g}(j_1)|}{j_2 + 1} e^{-\frac{\nu \kappa(j_1)}{2} Y}.
\]  
(5.11)

As a consequence,
\[
\left( \sum_{\alpha \in \nu^{\frac{j_2}{2}} Z} \|B_{j_2} i \alpha \hat{\psi}(\cdot, \alpha, \cdot)\|_{L^2_{\lambda,Y}}^2 + \|B_{j_2} \partial_Y \hat{\psi}(\cdot, \alpha, \cdot)\|_{L^2_{\lambda,Y}}^2 \right)^{\frac{1}{2}}
\leq \frac{C \nu^{\frac{j_2}{2}} j_2!}{K^{\frac{1}{4}} (j + 1)^{\frac{1}{2}} (j_2 + 1)} \left( \sum_{\alpha \in \nu^{\frac{j_2}{2}} Z} \|\hat{g}(j_1)(\cdot, \alpha)\|_{L^2_{\lambda,Y}}^2 \right)^{\frac{1}{2}}.
\]  
(5.12)

We also have
\[
\left( \sum_{\alpha \in \nu^{\frac{j_2}{2}} Z} \left\| \frac{1}{1 + Y} B_{j_2} i \alpha \hat{\psi}(\cdot, \alpha, \cdot) \right\|_{L^2_{\lambda,Y}}^2 \right)^{\frac{1}{2}} \leq \frac{C \nu^{\frac{j_2}{2}} j_2!}{K^{\frac{1}{4}} (j + 1)^{\frac{1}{2}} (j_2 + 1)} \left( \sum_{\alpha \in \nu^{\frac{j_2}{2}} Z} \|\alpha \hat{g}(j_1)(\cdot, \alpha)\|_{L^2_{\lambda,Y}}^2 \right)^{\frac{1}{2}}.
\]  
(5.13)

Here \( C > 0 \) is a universal constant.

**Proof.** We first show (5.10) for \( B_{j_2} i \alpha \hat{\psi} \). It suffices to consider the case \( j_2 \geq 1 \), for the case \( j_2 = 0 \) is trivial from (5.6). We observe from (5.6) that
\[
B_{j_2} \hat{\psi} = \frac{\hat{g}(j_1) \chi_{j_2}^{j_2}}{\gamma - |\alpha|} \left( (-\gamma)^{j_2} e^{-\gamma Y} - (-|\alpha|)^{j_2} e^{-|\alpha| Y} \right)
= \frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|} \chi_{j_2}^{j_2} e^{-\gamma Y} \hat{g}(j_1) + (-|\alpha|)^{j_2} \chi_{j_2}^{j_2} e^{-|\alpha| Y} \hat{g}(j_1) \frac{1 - e^{(-\gamma - |\alpha|) Y}}{\gamma - |\alpha|}.
\]  
(5.14)

Since
\[
(-\gamma)^{j_2} - (-|\alpha|)^{j_2} = (-1)^{j_2} \sum_{l_2=0}^{j_2} \binom{j_2}{l_2} (\gamma - |\alpha|)^{j_2-l_2} |\alpha|^{l_2},
\]
we have from \( \frac{j_2}{l_2} \leq j_2 \left( \frac{j_2-1}{l_2} \right) \) for \( 0 \leq l_2 \leq j_2 - 1, \)
\[
\left| \frac{(-\gamma)^j - (-|\alpha|)^j}{\gamma - |\alpha|} \right| \leq \sum_{l_2=0}^{j_2-1} \left( \frac{j_2}{l_2} \right) |\gamma - |\alpha||^{j_2-l_2-1} |\alpha|^{l_2} \leq j_2 \sum_{l_2=0}^{j_2-1} \left( \frac{j_2-1}{l_2} \right) |\gamma - |\alpha||^{j_2-l_2-1} |\alpha|^{l_2} \\
= j_2 (|\gamma - |\alpha|| + |\alpha|)^{j_2-1} \\
\leq j_2 (3|\gamma||j_2-1|).
\]

Here we have used \(|\alpha| \leq |\gamma|\) by (5.7). Then the inequality \( \chi_{\nu} = 1 - e^{-\kappa \frac{1}{2} Y} \leq \kappa \frac{1}{2} Y \) implies
\[
\left| \frac{(-\gamma)^j - (-|\alpha|)^j}{\gamma - |\alpha|} \right| \chi_{\nu}^{j_2} e^{-\gamma Y} \leq j_2 \kappa \frac{1}{2} Y \left( 3 \kappa \frac{1}{2} |\gamma| Y \right)^{j_2-1} e^{-\Re(\gamma) Y} \\
\leq j_2 \kappa \frac{1}{2} Y \left( 3 \sqrt{2} \kappa \Re(\gamma) Y \right)^{j_2-1} e^{-\Re(\gamma) Y} \quad \text{(by (5.7)).}
\]

From the bound \( e^k e^{-r} \leq \left( \frac{k}{r} \right)^k \) and the Stirling bound \( \left( \frac{k}{r} \right)^k \leq (2\pi)^{\frac{1}{2}} k^{-\frac{1}{2}} k! \) for \( k \in \mathbb{N} \), we have
\[
\left( \frac{1}{2} \Re(\gamma) Y \right)^{j_2-1} e^{-\frac{1}{2} \Re(\gamma) Y} \leq \frac{(j_2-1)!}{\sqrt{2\pi(j_2-1)^{\frac{1}{2}}}}, \quad j_2 \geq 2.
\]

This gives when \( 6\sqrt{2} \kappa \leq \frac{1}{2} \),
\[
\left| \frac{(-\gamma)^j - (-|\alpha|)^j}{\gamma - |\alpha|} \right| \chi_{\nu}^{j_2} e^{-\gamma Y} \leq \frac{\nu^{j_2} j_2!}{(j_2+1)} \gamma e^{-\frac{1}{2} \Re(\gamma) Y}, \quad j_2 \geq 1.
\]

Similarly, we have for \( j_2 \geq 1,
\[
\left| (-|\alpha|)^j \chi_{\nu}^{j_2} e^{-|\alpha| Y} \right| \leq \frac{\nu^{j_2} j_2!}{j_2+1} e^{-\frac{1}{2} |\alpha| Y}.
\]

Hence (5.10) for \( B_{j_2} \partial_Y \hat{\psi} \) follows by collecting these with (5.14). The estimate for \( B_{j_2} \partial_Y \hat{\psi} \) is proved in the same manner in view of (5.8), and we omit the details. Estimate (5.12) follows from (5.10) and the Plancherel theorem, by observing the estimates for the multipliers
\[
\| \alpha Y e^{-\Re(\gamma) Y} \|_{L_2^2} \leq \frac{C}{K^\frac{1}{2}(j+1)^{\frac{1}{2}}},
\]
\[
\| \alpha e^{-\frac{|\alpha|}{2} Y} \|_{L_2^2} \leq \frac{C}{K^\frac{1}{2}(j+1)^{\frac{1}{2}}}
\]

Here \( C > 0 \) is a universal constant. Estimate (5.15) is a consequence of (5.7). As for (5.16), we devide into two cases. (i) The case \(|\alpha| \leq \frac{1}{2} K^\frac{1}{2}(j+1)^{\frac{1}{2}}\): in this case we have from (5.7),
\[
|\gamma - |\alpha|| \geq |\gamma| - |\alpha| \geq \frac{|\alpha| + K^\frac{1}{2}(j+1)^{\frac{1}{2}}}{C}
\]

with a universal constant \( C > 0 \), which gives
\[
\| \alpha e^{-\frac{|\alpha|}{2} Y} \|_{L_2^2} \leq \frac{C}{|\alpha| + K^\frac{1}{2}(j+1)^{\frac{1}{2}}} \| \alpha e^{-\frac{|\alpha|}{2} Y} \|_{L_2^2} \leq \frac{C|\alpha|^\frac{1}{2}}{|\alpha| + K^\frac{1}{2}(j+1)^{\frac{1}{2}}}
\]
\[
\leq \frac{C}{K^\frac{1}{2}(j+1)^{\frac{1}{2}}}.
\]
(ii) The case $|\alpha| \geq \frac{1}{2} K^\frac{1}{2}(j+1)^{\frac{1}{2}}$: In this case we used the bound
\[
\sup_{\mathbb{R}^2(z) > 0} \left| \frac{1 - e^{-z}}{z} \right| \leq C,
\]
which gives
\[
\|\alpha e^{-|\alpha|Y} \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \|_{L^2_Y} \leq C \|\alpha Y e^{-|\alpha|Y} \|_{L^2_Y} \leq \frac{C}{|\alpha|^\frac{1}{2}} \leq \frac{C}{K^\frac{1}{2}(j + 1)^{\frac{1}{2}}}.
\]

The proof of (5.16) is complete, and (5.12) is proved. Estimate (5.13) is proved similarly by using (5.10), the Plancherel theorem, and
\[
\begin{align*}
\| \frac{Y}{1 + Y} e^{-\frac{\alpha Y}{j}} \|_{L^2_Y} &\leq \frac{C}{K^\frac{1}{2}(j + 1)^{\frac{1}{2}}}, \tag{5.17} \\
\| \frac{1}{1 + Y} e^{-\frac{|\alpha|Y}{j}} \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \|_{L^2_Y} &\leq \frac{C}{K^\frac{1}{2}(j + 1)^{\frac{1}{2}}}. \tag{5.18}
\end{align*}
\]
Here $C > 0$ is a universal constant. Indeed, (5.17) is straightforward, while in (5.18), the reason why the estimate becomes worse is due to the case $|\alpha| \leq \frac{1}{2} K^\frac{1}{2}(j + 1)^{\frac{1}{2}}$ with $|\alpha| \ll 1$, where we compute as
\[
\begin{align*}
\| \frac{1}{1 + Y} e^{-\frac{|\alpha|Y}{j}} \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \|_{L^2_Y} &\leq \frac{C}{|\alpha| + K^\frac{1}{2}(j + 1)^{\frac{1}{2}}} \| \frac{1}{1 + Y} e^{-\frac{|\alpha|Y}{j}} \|_{L^2_Y} \leq \frac{C}{K^\frac{1}{2}(j + 1)^{\frac{1}{2}}}.
\end{align*}
\]
Here we essentially use the factor $\frac{1}{1 + Y}$ to obtain the uniform estimate in $\alpha$. The proof is complete. \hfill \Box

In Propositions 5.1 and 5.2 below we give estimates for the solution to (5.1) given by the formula as above in terms of the Fourier transform. We always take $\kappa$ small enough so that $\kappa \in (0, \kappa')$ as in Lemma 5.1.

**Proposition 5.1 (Estimate for velocity).** It follows that
\[
\begin{align*}
\sum_{j=0}^{\nu^{\frac{1}{2}}} \frac{\nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2}} \nu^{\frac{3}{2}}} M_{2,j,1} [\nabla \phi] + \sum_{j=0}^{\nu^{\frac{1}{2}}} \frac{1}{(j!)^{\frac{3}{2}} \nu^{\frac{3}{2} + \frac{1}{2}(j + 1)^{\frac{1}{2}}}} M_{2,j,1^{\frac{1}{2}}} [\partial X \phi] &\leq \frac{C}{K^\frac{1}{2}} \| h \|_{bc}.
\end{align*}
\] (5.19)
Here $C > 0$ is a universal constant.

**Proof.** Assume that $M_{2,j,1} [\nabla \phi] = \| (\nabla \phi)^j \|_{L^2(I, \mathbb{R}^3)}$ for some $j = (j_1, j_2)$ with $j_1 + j_2 = j$. Note that this $j_1$ depends on $j$, and we write $j_1[j]$ if necessary. By the Plancherel theorem the estimate (5.12) implies
\[
\begin{align*}
\| (\nabla \phi)^{j_1[j]} \|_{L^2(I, \mathbb{R}^3)} &\leq \frac{C \nu^{\frac{1}{2} - \frac{1}{2} j_1[j]}}{K^\frac{1}{2}(j + 1)^{\frac{1}{2}}(j - j_1[j] + 1)} \| h^{(j_1)} \|_{L^2(I, \mathbb{R}^3)} ^{j_1[j]} h.
\end{align*}
\]

Thus we have
\[
\sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{\nu^{\frac{1}{2}} (j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2} + \frac{1}{2}}} M_{2,j,1} [\nabla \phi]
\leq \frac{C}{K^2} \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{(j - j_1[j])! j_1[j]! j_1[j]!}{j!} \left( \frac{j + 1}{j_1[j] + 1} \right)^{\frac{1}{2}} \left( \frac{\nu^{\frac{1}{2}} (j_1[j] + 1)^{\frac{1}{2}}}{(j_1[j]! \nu^{\frac{1}{2}})} \right) \left\| h^{(j_1[j])} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)}.
\]

We decompose the summation in the right-hand side as \( \sum_{j_1[j] = j} \) (i.e., \( j \)’s such that \( 0 \leq j \leq \nu^{-\frac{1}{2}} \) and \( j_1[j] = j \)) and \( \sum_{j_1[j] \leq j - 1} \) (i.e., \( j \)’s such that \( 0 \leq j \leq \nu^{-\frac{1}{2}} \) and \( j_1[j] \leq j - 1 \)). Then the sum of \( \sum_{j_1[j] = j} \) is bounded from above by \( \left\| h \right\|_{bc} \), while the sum of \( \sum_{j_1[j] \leq j - 1} \) is bounded as
\[
\sum_{j_1[j] \leq j - 1} \frac{(j - j_1[j])! j_1[j]! j_1[j]!}{j!} \left( \frac{j + 1}{j_1[j] + 1} \right)^{\frac{1}{2}} \left( \frac{\nu^{\frac{1}{2}} (j_1[j] + 1)^{\frac{1}{2}}}{(j_1[j]! \nu^{\frac{1}{2}})} \right) \left\| h^{(j_1[j])} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)}
\leq \sum_{j_1[j] \leq j - 1} \frac{(j - j_1[j])! j_1[j]! j_1[j]!}{j!} \left( \frac{j + 1}{j_1[j] + 1} \right)^{\frac{1}{2}} \sup_{0 \leq k \leq \nu^{-\frac{1}{2}}} \left( \frac{\nu^{\frac{1}{2}} (k + 1)^{\frac{1}{2}}}{(k!) \nu^{\frac{1}{2}}} \right) \left\| h^{(k)} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)}
\leq C \left\| h \right\|_{bc}.
\]

Indeed, it suffices to use
\[
\sum_{j_1[j] \leq j - 1} \frac{(j - j_1[j])! j_1[j]! j_1[j]!}{j!} \left( \frac{j + 1}{j_1[j] + 1} \right)^{\frac{1}{2}} \left( \frac{\nu^{\frac{1}{2}} (j_1[j] + 1)^{\frac{1}{2}}}{(j_1[j]! \nu^{\frac{1}{2}})} \right) \left\| h^{(j_1[j])} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)} \leq C \sum_{j_1[j] \leq j - 1} (j + 1)^{\frac{3}{2}} \leq C. \tag{5.20}
\]

Next we prove the estimate about \( M_{2,j,1}^{1/\nu} [\partial_X \phi] \). Arguing as above, we have from (5.13) that, for \( 0 \leq j \leq \nu^{-\frac{1}{2}} - 1 \),
\[
M_{2,j,1}^{1/\nu} [\partial_X \phi] \leq \frac{C \nu^{-\frac{1}{2}} (j - j_1[j])!}{K^2 (j + 1)^{\frac{1}{2}} (j - j_1[j] + 1)^{\frac{1}{2}}} \left\| \partial_X h^{(j_1[j])} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)},
\]
where \( j_1[j] \) is taken similarly as in the above argument. Thus we have
\[
\nu^{-\frac{1}{2}} \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{(j!)^{\frac{3}{2} + \frac{1}{2}}} \left( \frac{\nu^{\frac{1}{2}} (j_1[j] + 1)^{\frac{1}{2}}}{(j_1[j]! \nu^{\frac{1}{2}})} \right) M_{2,j,1}^{1/\nu} [\partial_X \phi]
\leq \frac{C}{K^2} \nu^{-\frac{1}{2}} \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{(j - j_1[j])!}{(j!)^{\frac{3}{2} + \frac{1}{2}}} \left\| \partial_X h^{(j_1[j])} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)}
\leq \frac{C}{K^2} \nu^{-\frac{1}{2}} \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{(j - j_1[j])!}{(j!)^{\frac{3}{2} + \frac{1}{2}}} \left\| h^{(j_1+1)} \right\|_{L^2(0, \frac{1}{K^{\nu^2}} L^2_X)} + \frac{\nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,1} [\partial_X \phi]}{(j!)^{\frac{3}{2} + \frac{1}{2}}} \bigg|_{j=\nu^{-\frac{1}{2}}}
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The second term is bounded from above by \( \frac{C}{K^{\frac{1}{2}}} \| h \|_{bc} \), as we have shown as above. As for the first term, we again decompose the summation \( \sum_{j=0}^{\nu-\frac{1}{2}} \) into \( \sum_{j_{21}=[j]+1} \) and \( \sum_{j_{21}[j] \leq j-1} \), as we have done previously. Then the sum of \( \sum_{j_{21}[j] \leq j-1} \) is bounded from above by \( C \| h \|_{bc} \), while the sum of \( \sum_{j_{21}[j] \leq j} \) is estimated as

\[
\sum_{j_{21}[j] \leq j-1} \frac{(j - j_{21}[j])!}{(j!)^2 \nu_j \nu_j^2 + (j + 1) (j - j_{21}[j] + 1)} J_{21}[j+1] L(0, \nu_j, \nu_j, L_j)^2 \leq \sum_{j_{21}[j] \leq j-1} \frac{(j - j_{21}[j])!}{(j!)^2 \nu_j \nu_j^2 + (j + 1) (j - j_{21}[j] + 1)} \frac{1}{(j+1) (j_{21}[j] + 1)^2 (j - j_{21}[j] + 1)}
\]

\[
\times \sup_{0 \leq k \leq \nu - \frac{1}{2}} \left( \frac{\nu_j^2 (k + 1)^2}{(k!)^2 \nu_j^2} \| h^{(k)} \|_{L^2(0, \nu_j, \nu_j, L_j)} \right) \leq C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{1}{(j+1)^2} \sup_{0 \leq k \leq \nu - \frac{1}{2}} \left( \frac{\nu_j^2 (k + 1)^2}{(k!)^2 \nu_j^2} \| h^{(k)} \|_{L^2(0, \nu_j, \nu_j, L_j)} \right) \leq C \| h \|_{bc}.
\]

The proof is complete.

Next we show the estimate for the vorticity field. The argument is similar to the one for the velocity.

**Lemma 5.2.** There exists \( \kappa'' \in (0, 1) \) such that the following statement holds for any \( \kappa \in (0, \kappa'') \). Let \( j_1 = 0, \cdots, j \) and \( j_2 = j - j_1 \). Then

\[
|B_{j_2}(\partial_Y^2 \alpha \psi(\lambda, \alpha, Y)) + |Y B_{j_2} \partial_Y (\partial_Y^2 \alpha - \alpha^2) \psi(\lambda, \alpha, Y)| \leq \frac{C \nu_j^{\frac{3}{2}} j_2^1}{j_2 + 1} |\gamma| e^{-\frac{R \kappa'' Y}{s}} |\hat{g} (j_1)|.
\]

(5.21)

As a consequence, for \( \theta' \in [-\frac{1}{2}, 2] \),

\[
\left( \sum_{\alpha \in \nu_j^{\frac{1}{2}} \mathbb{Z}} \| Y^{1+\theta'} B_{j_2} (\partial_Y^2 \alpha \psi(\cdot, \alpha, \cdot)) \|^2_{L_{\lambda, Y}} + \| Y^{2+\theta'} B_{j_2} \partial_Y (\partial_Y^2 \alpha - \alpha^2) \psi(\cdot, \alpha, \cdot) \|^2_{L_{\lambda, Y}} \right)^{\frac{1}{2}} \leq \frac{C \nu_j^{\frac{3}{2}} j_2^1}{K^{\frac{\theta'}{2} + \frac{1}{2}} (j_2 + 1)^{\theta'} + \frac{1}{2} (j_2 + 1)} \left( \sum_{\alpha \in \nu_j^{\frac{1}{2}} \mathbb{Z}} \| \hat{g}^{(j_1)}(\cdot, \alpha) \|^2_{L_{\lambda, Y}} \right)^{\frac{1}{2}}.
\]

(5.22)

Here \( C > 0 \) is a universal constant.

**Proof.** Estimate (5.21) follows from (5.9) by arguing as in the proof of (5.10). Estimates (5.22) then follows from (5.22), the Plancherel theorem, and

\[
\| Y^{1+m} |\gamma| e^{-\frac{R \kappa'' Y}{s}} \|_{L_{\lambda, Y}}^m \leq \frac{C}{(R \kappa'' (\gamma))^{m+\frac{1}{2}}} \leq \frac{C}{(|\alpha| + K^{\frac{1}{2}} (j_2 + 1)^{\frac{1}{2}} (j_2 + 1))^{m+\frac{1}{2}}}
\]

by (5.7) for \( m \in [-\frac{1}{2}, 3] \). The details are omitted here. The proof is complete. □
Proposition 5.2 (Estimate for vorticity). Let $\theta \in [0, 2]$. It follows that

$$\sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} \left( M_{2,j,Y} [\Delta \phi] + M_{2,j,Y} [\nabla \Delta \phi] \right) \leq \frac{C}{K^{\frac{1}{2}}} \| h \|_{bc}. \quad (5.23)$$

$$\sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} \left( M_{2,j,Y} [\partial_X \Delta \phi] + \nu^{\frac{1}{2}} M_{2,j,Y} [\partial_Y \Delta \phi] \right) \leq \frac{C}{K^{\frac{1}{2}}} \| h \|_{bc}. \quad (5.24)$$

Here $C > 0$ is a universal constant.

Proof. Estimate (5.23) is a consequence of (5.22) with $\theta' = 0$, by introducing $j_1[j]$ as in the proof of Proposition 5.1. As for (5.24), we have from (5.22) with $\theta' = \theta - \frac{1}{2}$ that

$$\sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\partial_X \Delta \phi] \leq C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\Delta \phi]$$

$$= C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\Delta \phi]$$

$$= C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\Delta \phi].$$

By arguing as in the proof of Proposition 5.1, the application of (5.22) gives

$$C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\Delta \phi] \leq \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} \nu^{\frac{1}{2}} (j + 1)^{\frac{1}{2}} M_{2,j,Y} [\partial_X \Delta \phi] \leq \frac{C}{K^{\frac{1}{2}}} \| h \|_{bc},$$

where the smoothing factor $(j + 1)^{\frac{\theta - 1}{2}}$ with $\theta' = \theta + \frac{1}{2}$ in (5.22) plays a key role. When $j = \nu - \frac{1}{2}$ we have

$$M_{2,j,Y} [\partial_X \Delta \phi] \leq \frac{(j + 1)^{\frac{\theta - 1}{2}}}{(j!)^{\frac{\theta}{2} + \frac{1}{4}}} M_{2,j,Y} [\partial_X \Delta \phi] \leq \frac{C}{K^{\frac{1}{2}}} \| h \|_{bc},$$

by (5.22) with $\theta' = \theta - \frac{1}{2}$.

The proof is complete. □
5.2 Vorticity transport estimate

Propositions 5.1 and 5.2 of the previous paragraph reflect a strong difference between the weighted fields $\nabla \phi$ and $(\Delta \phi)$ associated to the Stokes solution $\phi$ of (5.1): the former is not localized near the boundary, while the latter is, at scale $(K(j + 1))^{-\frac{1}{2}}$. This is due to a harmonic non-localized part in $\phi$, see expression (5.6). As a consequence, as shown in Proposition 5.2, for the vorticity field the weight $Y^\theta$ gives a gain $(j + 1)^{-\frac{6}{2}}$. In particular, the transport term $V \cdot \nabla \Delta \phi$ shares similar properties. When working in the Gevrey class $\frac{3}{2}$, this term can be seen to be formally of the same size as the Stokes term $\nu^\frac{1}{2} \Delta \phi - \partial_\tau \Delta \phi$. Hence, we need to add one step to our iteration in which we solve the heat-transport equations:

$$\nu^\frac{1}{2} \Delta \phi - \partial_\tau \Delta \phi - V \cdot \nabla \Delta \phi = H, \quad \tau > 0, \quad X \in T_\nu, \quad Y > 0,$$

with $H$ will be the transport term created by the Stokes approximation. A key point in dealing with this equation rather than with the full Orr-Sommerfeld equation is that we will be able to propagate weighted estimates with weight $Y^\theta$, which is crucial to have sharp bounds. In the last step of our iteration, we will correct non-local stretching terms using the Orr-Sommerfeld equation with artificial boundary conditions, using the bounds of Section 4. The main result of this paragraph is

**Proposition 5.3.** There exists $K_3 = K_3(C_1^+)$ such that if $K \geq K_3$ then the system (5.25) admits a unique solution $\phi \in C([0, \infty); H_0^2(T_\nu \times \mathbb{R}_+))$ with $\omega = -\Delta \phi \in C([0, \infty); L^2(T_\nu \times \mathbb{R}_+))$ satisfying, for $0 \leq j \leq \nu^{-\frac{3}{2}}, \kappa \in (0, 1]$, and $\theta = 0, 1, 2$,

$$\nu^\frac{1}{2} M_{2,j,Y}^\theta [\nabla \omega] + M_{\infty,j,Y}^\theta [\omega] + K^\frac{1}{4} \nu^\frac{1}{2} (j + 1)^\frac{3}{2} M_{2,j,Y}^\theta [\omega] \leq C \left( \kappa \nu^\frac{3}{4} j M_{2,j-1,Y}^\theta [\nabla \omega] + \nu^\frac{1}{2} \theta M_{2,j,Y}^\theta [\omega] + \frac{1}{K^\frac{1}{4} \nu^\frac{3}{4} (j + 1)^\frac{3}{4}} M_{2,j,Y}^\theta [H] \right)$$

$$+ \frac{1}{\kappa K^\frac{1}{4} \nu^\frac{1}{2} (j + 1)^\frac{3}{4}} \sum_{l=0}^{j-1} \min\{l + 1, j - l + 1\} \left( j^l \right) N_{\infty,j-l}[V] M_{2,l+1,Y}^\theta [\omega].$$

Here $C > 0$ is a universal constant.

**Remark 5.1.** The solution $\omega = -\Delta \phi$ to (5.25) in Proposition 5.3 has the regularity $(\partial_\tau - \nu^\frac{1}{2} \Delta) Y^\theta \omega \in L^2_{loc}([0, \infty); L^2(T_\nu \times \mathbb{R}_+))$, $\theta = 0, 1, 2$, with the Dirichlet boundary condition. Hence, the maximal regularity for the heat equation implies

$$\partial_\tau Y^\theta \omega, \Delta(Y^\theta \omega) \in L^2_{loc}([0, \infty); L^2(T_\nu \times \mathbb{R}_+)).$$

To prove Proposition 5.3 let us recall that $\omega^j = e^{-K^\nu^\frac{1}{2} (j+1)} B_{2j} \partial^j_\chi \omega$ satisfies

$$- \nu^\frac{1}{2} \Delta \omega^j + \partial_\tau \omega^j + K^\frac{1}{4} \nu^\frac{1}{2} (j + 1) \omega^j + V \cdot \nabla \omega^j = -V_2[B_{2j}, \partial_Y] e^{-K^\nu^\frac{1}{2} (j+1)} \partial^j_\chi \omega$$

$$- \sum_{l=0}^{j-1} \sum_{\max\{0, l+j-2-j\} \leq l_2 \leq \min\{l, j_2\}} \binom{j_2}{l_2} \binom{j-j_2}{l-l_2} V^{j-1} \cdot (\nabla \omega)^l$$

$$+ H^j.$$

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Then (5.26) is proved by taking the inner product in (5.27) with $Y^{2\theta_j}$ for each $\theta = 0, 1, 2,$ and then by taking the supremum about $j_2 = 0, \cdots, j$ and about $\tau_0 \in (0, \frac{1}{K\nu^2}]$. Hence the proof proceeds as in the proof of Proposition 4.2.

**Lemma 5.3.** There exists $C > 0$ such that for any $K \geq 1$ and $\kappa \in (0, 1)$,

$$\int_0^{\tau_0} \langle -\nu^{\frac{1}{2}}(\Delta \omega)_j, Y^{2\theta_j} \rangle \ d\tau \geq \frac{3}{4} \nu^{\frac{1}{2}} \| Y^\theta (\nabla \omega)^j \|_{L^2(0, \tau_0; L^2_{X, \gamma})}^2 - C \nu^{\frac{1}{2}} (K\nu^{\frac{1}{2}} j_2)^2 M_{2,j-1,Y^\kappa (\partial Y \omega)^j}.$$

**Proof.** The proof is similar (and much simpler) to the one of Lemma 4.2. Indeed, the only difference is the presence of the weight $Y^{2\theta_j}$ with $\theta = 0, 1, 2$, which creates the term $2\nu^{\frac{1}{2}} \int_0^{\tau_0} \langle Y^\theta (\partial Y \omega)^j, Y^{2\theta_j - 1} \omega^j \rangle \ d\tau$ after the integration by parts. This is responsible for the last term in the estimate of this lemma. The details are omitted. The proof is complete.

**Lemma 5.4.** There exists $K_{3,2} = K_{3,2}(C_1) \geq 1$ such that if $K \geq K_{3,2}$ then

$$\int_0^{\tau_0} \langle V \cdot \nabla \omega^j, Y^{2\theta_j} \rangle \ d\tau \leq \frac{1}{2} \| Y^\theta (\tau_0) \|_{L^\infty}^2 + \frac{3}{4} K \nu^{\frac{1}{2}} (j+1) \| Y^\theta \omega^j \|_{L^2(0, \tau_0; L^2_{X, \gamma})}^2.$$

**Proof.** The proof is a simple modification of the one of Lemma 4.3. We note that the initial data is taken as zero, and the integration by parts gives

$$\int_0^{\tau_0} \langle V \cdot \nabla \omega^j, Y^{2\theta_j} \rangle \ d\tau \leq \theta \| V^2 \|_{L^\infty} \| Y^\theta \omega^j \|_{L^2(0, \tau_0; L^2)}^2.$$

Then the desired estimate follows by taking $K$ large enough depending only on $C_1^*$, for $\| V^2 \|_{L^\infty} \leq \| \partial_y V^2 \|_{L^\infty} = \| \partial_X V^2 \|_{L^\infty} \leq C_1^* \nu^{\frac{1}{2}}$. The details are omitted. The proof is complete.

**Lemma 5.5.** Let $j_2 \geq 1$. It follows that

$$\int_0^{\tau_0} \langle -V_2[B_{j_2}, \partial Y] e^{-K\nu^{\frac{1}{2}} (j+1) \partial_X^{j_1} \omega}, Y^{2\theta_j} \omega^j \rangle \ d\tau \leq CC_1^* \nu^{\frac{1}{2}} j_2 \| Y^\theta \omega^j \|_{L^2(0, \tau_0; L^2_{X, \gamma})}^2.$$

Here $C > 0$ is a universal constant.

**Proof.** The proof is similar to the one of Lemma 4.4. The details are omitted here. The proof is complete.
Lemma 5.6. Let \( j \geq 1 \). It follows that
\[
\int_0^\tau \left( - \sum_{l=0}^{j-1} \sum_{\max\{0,l+j_2-j\} \leq l_2 \leq \min\{l,j_2\}} \frac{j_2}{l_2} \frac{j-j_2}{l-l_2} V^{l_2-1} \cdot (\nabla \omega)^1, Y^{2\theta} \omega^j \right) d\tau 
\leq \frac{C}{\kappa} R_{j, \text{Lem 5.6}}[\omega] M_{2,j,Y^\theta}[\omega],
\]
where
\[
R_{j, \text{Lem 5.6}}[\omega] = \sum_{l=0}^{j-1} \min\{l+1,j-l+1\} \left\{ \frac{j}{l} \right\} N_{\infty,j-l}[V] M_{2,l+1,Y^\theta}[\omega].
\]
Here \( C > 0 \) is a universal constant, and \( N_{\infty,j-l}[V] \) is defined as in Lemma 4.6.

Proof. The proof is similar to the one of Lemma 4.6. The details are omitted here. The proof is complete.

Lemma 5.7. It follows that
\[
\int_0^\tau \langle \mathcal{H}^j, Y^{2\theta} \omega^j \rangle d\tau 
\leq \begin{cases} 
CM_{2,j,Y^\theta} \left[ H \right] \left( M_{2,j,Y^\theta} \left[ \partial Y^\omega \right] + \kappa \nu^j \right) M_{2,j-1,Y^\theta} \left[ \nabla \omega \right] \left( M_{2,j,Y^\theta} \left[ \omega \right] \right)^{\frac{1}{2}}, & \theta = 0, \\
CM_{2,j,Y^\theta} \left[ H \right] \left( M_{2,j,Y^\theta-1} \left[ \omega \right] \right)^{\frac{1}{2}} \left( M_{2,j,Y^\theta} \left[ \omega \right] \right)^{\frac{1}{2}}, & \theta = 1, 2.
\end{cases}
\]

Here \( C > 0 \) is a universal constant.

Proof. The estimate follows from the inequality
\[
\langle \mathcal{H}^j, Y^{2\theta} \omega^j \rangle \leq \left\| Y^{\theta+\frac{1}{2}} \mathcal{H}^j \right\| \left\| Y^{\theta-\frac{1}{2}} \omega^j \right\| \leq \left\| Y^{\theta+\frac{1}{2}} \mathcal{H}^j \right\| \left\| Y^{\theta-1} \omega^j \right\| \left\| Y^\theta \omega^j \right\| \left\| Y^\theta \omega^j \right\| \left\| Y^\theta \omega^j \right\| \left\| Y^\theta \omega^j \right\|
\]
and the Hardy inequality for \( \theta = 0 \):
\[
\left\| Y^{-1} \omega^j \right\| \leq C \left\| \partial Y^\omega \right\| \leq C \left( \left\| \left( \partial Y^\omega \right)^j \right\| + \kappa \nu^j \right) \left\| \left( \partial Y^\omega \right)^{(j_1+j_2-1)} \right\|.
\]
The proof is complete.

Proof of Proposition 5.3. It suffices to show the estimate (5.26), but it follows from Lemmas 5.3–5.7 by dividing into the case \( \theta = 0 \) and the case \( \theta = 1, 2 \). The details are omitted here. The proof is complete.
Corollary 5.1. There exists $\kappa_3 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_3]$. There exists $K_3' = K_3'(\kappa, C_0', C_1') \geq 1$ such that if $K \geq K_3'$ then the system (5.25) admits a unique solution $\phi \in C([0, \infty); H_0^1(\mathbb{T} \times \mathbb{R}))$ with $\omega = -\Delta \phi \in C([0, \infty); L^2(\mathbb{T} \times \mathbb{R}_+))$ satisfying, for $\theta = 0, 1, 2$,

$$
\sum_{j=0}^{\nu \frac{1}{2}} \frac{(j+1)^{\frac{3}{2} - \frac{1}{4}}}{(j!)^{\frac{3}{2} \nu \frac{1}{2}}} \left( \nu \frac{1}{2} M_{2,j,Y^0}[\nabla \omega] + M_{\infty,j,Y^0}[\omega] + K^2 \nu \frac{1}{2} (j+1)^{\frac{1}{2}} M_{2,j,Y^0}[\omega] \right)
\leq C \frac{1}{K'^2} \sum_{\nu'=0}^{\nu} \sum_{j=0}^{\nu \frac{1}{2}} \frac{1}{(j!)^\frac{3}{2} \nu \frac{1}{2} (j+1)^{\frac{1}{2} - \frac{1}{4}} M_{2,j,Y^0}^{\nu'+\frac{1}{2}}[H],}
$$

(5.28)

and

$$
||\nabla \phi||_{2,1} + ||\partial_\nu \phi||_{Y=0} ||_{bc} \leq C \frac{1}{K'^2} \sum_{\nu'=0}^{\nu} \sum_{j=0}^{\nu \frac{1}{2}} \frac{1}{(j!)^\frac{3}{2} \nu \frac{1}{2} (j+1)^{\frac{1}{2} - \frac{1}{4}} M_{2,j,Y^0}^{\nu'+\frac{1}{2}}[H].}
$$

(5.29)

Here $C > 0$ is a universal constant.

Proof. Let us first show (5.28). In virtue of Proposition 5.3 we have for $\theta = 0, 1, 2$,

$$
\sum_{\nu'=0}^{\nu} \sum_{j=0}^{\nu \frac{1}{2}} \frac{(j+1)^{\frac{3}{2} - \frac{1}{4}}}{(j!)^\frac{3}{2} \nu \frac{1}{2}} \left( \nu \frac{1}{2} M_{2,j,Y^0}[\nabla \omega] + M_{\infty,j,Y^0}[\omega] + K^2 \nu \frac{1}{2} (j+1)^{\frac{1}{2}} M_{2,j,Y^0}[\omega] \right)
\leq C \frac{1}{K'^2} \sum_{\nu'=0}^{\nu} \sum_{j=0}^{\nu \frac{1}{2}} \frac{1}{(j!)^\frac{3}{2} \nu \frac{1}{2} (j+1)^{\frac{1}{2} - \frac{1}{4}} M_{2,j,Y^0}^{\nu'+\frac{1}{2}}[H],}
$$

(5.30)
Here \( C > 0 \) is a universal constant. As for the last term in (5.30), arguing as at the end of the proof of Lemma 4.6, we find that

\[
\sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{3}{2} - \frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \frac{1}{K^{\frac{3}{2} - \frac{1}{2}}(j + 1)^{\frac{1}{2}}} \sum_{l=0}^{j-1} \min\{l + 1, j - l + 1\} \left( \nu \right)_{l} N_{\infty, j-l}[V] M_{2,l+1,Y \nu}[\omega] \\
\leq C C_{\nu}^{\frac{1}{2}} \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{3}{2} - \frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}} M_{2,j,Y \nu}[\omega].
\]

Hence (5.28) follows by taking \( \kappa \) small enough so that \( C \kappa \leq \frac{1}{2} \), and then by taking \( K \) large enough so that \( \frac{C C_{\kappa}}{K} \leq \frac{1}{2} \).

To show (5.29) it suffices to prove the embedding inequality

\[
\| \nabla \phi \|_{2,1} \leq \nu^{\frac{1}{2}} \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}} M_{2,j,1}[\nabla \phi] \\
\leq C \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}} M_{2,j,Y}[\omega]
\]

and the interpolation inequality

\[
\| \partial Y \phi |_{Y=0} \|_{bc} := \nu^{\frac{1}{2}} \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \| e^{-K \tau \nu^{\frac{3}{2}}(j+1)} \partial \chi_{Y} \partial Y \phi |_{Y=0} \|_{L^{2}(0,\frac{1}{\nu \kappa K},L^{2})} \\
\leq C \left( \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}} M_{2,j,1}[\omega] \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\nu - \frac{1}{2}} \frac{(j + 1)^{\frac{1}{2}}}{(j!)^{\frac{3}{2} - \frac{1}{2}} \nu^{\frac{3}{2}}} \nu^{\frac{1}{2}}(j + 1)^{\frac{1}{2}} M_{2,j,Y}[\omega] \right)^{\frac{1}{2}}.
\]

Then (5.29) follows from (5.28) with (5.31) and (5.32). The proof of (5.31) proceeds as in the proof of Proposition 4.3. Indeed, we have from \( \omega^{j} = - \nabla \cdot (\nabla \phi)^{j} + \frac{\nu^{j} j_{2} \kappa}{\gamma_{e}} \partial Y \phi \) and from the integration by parts,

\[
\|(\nabla \phi)^{j}\|^{2} = \langle \omega^{j}, \phi^{j} \rangle - 2 \nu^{j} j_{2} e^{-K \rho \nu^{\frac{3}{2}}}(\chi_{Y} \partial Y \phi)^{j} (\partial Y \phi)^{j} (j+1)^{2j-1-2}) \\
\leq \| Y \omega^{j} \| \| \partial Y \phi^{j} \| + 2 \nu^{j} j_{2} \kappa \| (\partial Y \phi)^{j} \| \| (\partial Y \phi)^{j} \| (j+1)^{2j-1-2}) \\
\leq C \| Y \omega^{j} \| \| \partial Y \phi^{j} \| + 2 \nu^{j} j_{2} \kappa \| (\partial Y \phi)^{j} \| \| (\partial Y \phi)^{j} \| (j+1)^{2j-1-2})
\]

Here the Hardy inequality is used in the last line. Then the identity \( \partial Y \phi^{j} = (\partial Y \phi)^{j} + \nu^{j} j_{2} \kappa e^{-K \rho \nu^{\frac{3}{2}}}(\partial Y \phi)^{j} (j+1)^{2j-1-2} \) yields

\[
\| (\nabla \phi)^{j} \| \leq C \left( \| Y \omega^{j} \| + \nu^{j} j_{2} \kappa \| (\partial Y \phi)^{j} \| (j+1)^{2j-1-2} \| \right).
\]
This estimate gives

\[
\sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} M_{2,j,1}[\nabla \phi]
\leq C \sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} (M_{2,j,Y}[\omega] + \nu^\frac{1}{4} j K M_{2,j-1,1}[\nabla \phi])
\leq C \sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} M_{2,j,Y}[\omega] + C K \sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} M_{2,j,1}[\nabla \phi],
\]

where \( C > 0 \) is a universal constant. This proves (5.31) if \( \kappa \) is small enough so that \( C \kappa \leq \frac{1}{2} \). As for (5.32), we observe from (4.50) that

\[
\| e^{-K \nu^\frac{1}{2} (j+1) \partial_X \partial_Y \phi|_{Y=0} \| L^2(0, \frac{1}{K \nu^\gamma}; L^2_{\kappa}) \)
\leq C \left( (j+1)^{-\frac{1}{2}} \| \omega^{(j,0)} \| L^2(0, \frac{1}{K \nu^\gamma}; L^2_{\kappa}) \right)^{\frac{1}{2}} \left( (j+1)^{\frac{1}{2}} \| \partial_Y \phi^{(j,0)} \| L^2(0, \frac{1}{K \nu^\gamma}; L^2_{\kappa}) \right)^{\frac{1}{2}},
\]

which implies from the Schwarz inequality,

\[
\| \partial_Y \phi|_{Y=0} \| \leq C \left( \sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} M_{2,j,1}[\omega] \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \nu^\frac{1}{4}(j+1)^{\frac{3}{2}} M_{2,j,1}[\nabla \phi] \right)^{\frac{1}{2}}.
\]

Then (5.31) shows (5.32). The proof is complete.

\[
\text{Corollary 5.2. In Corollary 5.1, let } H = V \cdot \nabla \Delta \phi_{1,1}[h], \text{ where } \phi_{1,1}[h] \text{ is the solution to (5.2) in Propositions 5.1-5.2. Then}
\]

\[
\sum_{j=0}^{\nu} \frac{(j+1)^{\frac{3}{2}}}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2}} \left( \nu^\frac{1}{4} M_{2,j,Y}[\nabla \omega] + M_{\infty,j,Y}[\omega] + K \nu^\frac{1}{4} (j+1)^{\frac{3}{2}} M_{2,j,Y}[\omega] \right) \leq \frac{CC_\kappa}{K^\frac{1}{4}} \| h \|_{bc},
\]

and

\[
\| \nabla \phi \|_{2,1} + \| \partial_Y \phi|_{Y=0} \| \leq \frac{CC_\kappa}{K^\frac{1}{4}} \| h \|_{bc}.
\]

Moreover, we have

\[
\sum_{j=0}^{\nu} \frac{1}{(j!)^{\frac{3}{2}}\nu^\frac{3}{2} (j+1)^{\frac{3}{2}} \nu^\frac{1}{4}} M_{2,j,1}[\partial_X \phi] \leq \frac{CC_\kappa}{K^\frac{1}{4}} \| h \|_{bc}.
\]

Here \( C > 0 \) is a universal constant.
Proof. To show (5.33) and (5.34) it suffices to prove for $\theta' = 0, 1, 2$,

$$
\sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{(j!)^2 \nu^2 (j+1)^{\nu^2}} M_{2,j,Y^{\theta'}+\frac{1}{2}} [H] 
\leq CC_\kappa \sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{(j!)^2 \nu^2 (j+1)^{\nu^2}} (M_{2,j,Y^{\theta'}+\frac{1}{2}} [\partial X \Delta \phi_{1,1}] + \nu^{\frac{1}{2}} M_{2,j,Y^{\theta'}+\frac{1}{2}} [\partial Y \Delta \phi_{1,1}] ).
$$

(5.36)

Then (5.33) and (5.34) follow from (5.28), (5.29), (5.24) and (5.36). To show (5.36) we observe that

$$
H^1 = \sum_{l=0}^{j} \max\{0,l+j+2,j\} \leq \min\{l,j\} \sum_{l=0}^{j} \sum_{l\leq \min\{l,j\}} (\frac{j}{j-j_2}) (\frac{j-j_2}{l-l_2}) V^{j-l} \cdot (\nabla \Delta \phi_{1,1})^1.
$$

Thus we have

$$
\|Y^{\theta'}+\frac{1}{2} H^1\| \leq \sum_{l=0}^{j} \max\{0,l+j+2,j\} \leq \min\{l,j\} \sum_{l=0}^{j} \sum_{l\leq \min\{l,j\}} (\|\partial Y V^{-1}\|_{L^\infty} \|Y^{\theta'}+\theta' (\partial X \Delta \phi_{1,1})^1\|)
+ \|\partial Y V^{-1}\|_{L^\infty} \|Y^{\theta'}+\theta' (\partial Y \Delta \phi_{1,1})^1\|).
$$

Set

$$
N_{\infty,j} [\nabla V_1] = (j+1)^{\frac{3}{2}} \sup_{j_2=0,\ldots,j} \nu^{-\frac{1}{2}} \|\partial X V_1\|_{L^\infty} \|\partial Y V_1\|_{L^\infty} + \|\partial X V_1\|_{L^\infty}.
$$

(5.37)

Since

$$
\|\partial Y V^{-1}\|_{L^\infty} \leq \|\partial Y V\|_{L^\infty} + \kappa \nu^{\frac{1}{2}} (j_2 - l_2) \|\partial Y V\|_{L^\infty} (j_2 - l_2 + l_2 - 1) \leq (j - l - 1)^{\frac{3}{2}} N_{\infty,j-l} [\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j - l)^{\frac{1}{2}} N_{\infty,j-l} [\nabla V_1]
$$

and similarly

$$
\|\partial Y V^{-1}\|_{L^\infty} \leq \|\partial Y V\|_{L^\infty} + \kappa \nu^{\frac{1}{2}} (j_2 - l_2) \|\partial Y V\|_{L^\infty} (j_2 - l_2 + l_2 - 1) \leq (j - l - 1)^{\frac{3}{2}} N_{\infty,j-l} [\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j - l)^{\frac{1}{2}} N_{\infty,j-l} [\nabla V_1]
$$

we obtain

$$
M_{2,j,Y^{\theta'}+\frac{1}{2}} [H] \leq \sum_{l=0}^{j} \min\{l+1, j-l+1\}
\times (j - l - 1)^{\frac{3}{2}} N_{\infty,j-l} [\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j - l)^{\frac{1}{2}} N_{\infty,j-l} [\nabla V_1]
\times (M_{2,l,Y^{\theta'}+\frac{1}{2}} [\partial X \Delta \phi_{1,1}] + \nu^{\frac{1}{2}} M_{2,l,Y^{\theta'}+\frac{1}{2}} [\partial Y \Delta \phi_{1,1}] )
$$

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Then (5.36) follows from the Young inequality for convolution in the $l^1$ space. For example, we have, by using $\frac{(l+1)^{\frac{1}{2}}}{(j+1)^{\frac{1}{2}}(l+1)^{\frac{1}{2}}} \leq C$ for $\theta' = 0, 1, 2$ and $(\frac{j}{j+1})^\frac{\theta}{2} \min\{l+1, j-l+1\} \leq C$,

$$
\sum_{j=0}^{\nu-\frac{1}{2}} \sum_{l=0}^{j} \frac{1}{(j+1)^{\frac{1}{2}}(l+1)^{\frac{1}{2}}} \frac{(j-l)!}{j!} \min\{l+1, j-l+1\} (j-l+1)^{-\frac{1}{2}} (l+1)^{1-\frac{\theta'}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{l=0}^{j} \frac{1}{((j-l)!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (l+1)^{\frac{1}{2}}} N_{\infty,j-l} |\nabla V_1| \left( \frac{1}{(l!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (l+1)^{\frac{1}{2}}} M_{2,l,y^\frac{\theta}{2}+\nu'} [\partial X \Delta \phi_{1,1}] \right) \\
\leq C \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{l=0}^{j} \frac{1}{((j-l)!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (l+1)^{\frac{1}{2}}} N_{\infty,j-l} |\nabla V_1| \left( \frac{1}{(l!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (l+1)^{\frac{1}{2}}} M_{2,l,y^\frac{\theta}{2}+\nu'} [\partial X \Delta \phi_{1,1}] \right) \\
\leq CC_\kappa \sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} M_{2,j,y^\frac{\theta}{2}+\nu'} [\partial X \phi_{1,1}].
$$

The other terms are handled in the same manner and we omit the details. The proof of (5.33)-(5.34) is complete. Finally let us prove (5.35). The key is to apply the interpolation-type inequality proved in Proposition A.1. Indeed, Proposition A.1 implies

$$
\sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} M_{2,j+1,y^\frac{\theta}{2}+\nu'} [\partial X \phi] \\
\leq C \sum_{\theta=0}^{1} \sum_{j=0}^{\nu-\frac{1}{2}-1} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} (j+1)^{\frac{\theta}{2}-\frac{1}{2}} M_{2,j+1,y^\frac{\theta}{2}+\nu'} [\phi] \\
+ C \sum_{j=0}^{\nu-\frac{1}{2}-1} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} K \nu^\frac{1}{2} \left( M_{2,j-1,y^\frac{\theta}{2}+\nu} [\phi] + M_{2,j-1,1} |\nabla \phi| \right) \\
+ \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} M_{2,j,y^\frac{\theta}{2}+\nu'} [\partial X \phi] \bigg|_{j=\nu-\frac{1}{2}} \\
\leq C \sum_{\theta=0}^{1} \sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} M_{2,j,y^\frac{\theta}{2}+\nu'} [\phi] \\
+ C \sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{((j!)^\frac{\theta}{2} \nu^\frac{\theta}{2} (j+1)^{\frac{1}{2}})} M_{2,j,y^\frac{\theta}{2}+\nu'} [\phi] + C \|\nabla \phi\|_{2,1} \\
\leq CC_\kappa \frac{\|h\|_{bc}}{K^{\frac{\theta}{2}}}.
$$

Here we have used (5.33) and (5.34) in the last line. The proof is complete. \qed

### 5.3 Full construction of boundary corrector

We set $\phi_{app,1} = \phi_{app,1}[h] = \phi_{1,1}[h] + \phi_{1,2}[h]$, where $\phi_{1,1}[h]$ is the solution to (5.2) in Propositions 5.1-5.2, and $\phi_{1,2}[h]$ is the solution to (5.25) with $H = V \cdot \nabla \Delta \phi_{1,1}[h]$ as in Corollary 5.2. Then
the approximate solution \( \phi_{\text{app}} \) to the full system (5.1) is constructed in the form \( \phi_{\text{app}} = \phi_{\text{app},1} + \phi_1 \), which leads to the equations for \( \phi_1 = \tilde{\phi}_1[h] \) as

\[
\nu^2 \Delta^2 \tilde{\phi}_1 - \partial_\tau \Delta \tilde{\phi}_1 - V \cdot \nabla \Delta \tilde{\phi}_1 + \nabla^\perp \tilde{\phi}_1 \cdot \nabla \Omega = -\nabla^\perp \phi_{\text{app},1} \cdot \nabla \Omega, \quad \tau > 0, \ X \in \mathbb{T}_\nu, \ Y > 0, \quad \tilde{\phi}_1|_{y=0} = \Delta \tilde{\phi}_1|_{y=0} = 0, \quad \tilde{\phi}_1|_{\tau=0} = 0.
\]

(5.38)

Let us first give the estimate for the force term \( -\nabla^\perp \phi_{\text{app},1} \cdot \nabla \Omega \).

**Proposition 5.4.** Let \( \kappa_3 \in (0, 1] \) be the number in Corollary 5.1. For any \( \kappa \in (0, \kappa_3] \) there exists \( K'_3 = K'_3(\kappa, C_s, C_s^*) \geq 1 \) such that for any \( K \geq K'_3 \),

\[
\frac{1}{K^2 \nu^2} \| \nabla \phi_{\text{app},1} \cdot \nabla \Omega \|_{L^2(0, \kappa \sigma^2, H^{-1})}^2 + \frac{1}{K^2 \nu^2} \| \nabla \phi_{\text{app},1} \cdot \nabla \Omega \|_{L^2(0, \kappa \sigma^2, H^{-1})}^2 \leq \left( \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{(j!)^2 \nu^2 \nu^{\frac{1}{2}} (j + 1)^2} M_{2, j, \nu, \tau} [\partial_X \phi_{\text{app},1}] + 2 \| \nabla \phi_{\text{app},1} \|_{L^2(0, \kappa \sigma^2, H^{-1})} \right). \tag{5.39}
\]

**Proof.** Let us recall that

\[
\frac{1}{\nu^2} \| \nabla \phi_{\text{app},1} \cdot \nabla \Omega \|_{L^2(0, \kappa \sigma^2, H^{-1})}^2 = \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{(j!)^2 \nu^2 \nu^{\frac{1}{2}} (j + 1)^2} \sup \| \xi_j e^{-\kappa \nu^2 (j+1) B_{j2} \partial_{X \partial_{j2}} (\nabla^\perp \phi_{\text{app},1} \cdot \nabla \Omega) \|_{L^2(0, \kappa \sigma^2, H^{-1})}. \]

Thus we consider the estimate of

\[
e^{-\kappa \nu^2 (j+1) B_{j2} \partial_{X \partial_{j2}} (\nabla^\perp \phi_{\text{app},1} \cdot \nabla \Omega) \end{align}

\[
= (\nabla^\perp \phi_{\text{app},1})^1 \cdot \nabla \Omega + \sum_{j=0}^{\nu^{-\frac{1}{2}}} \sum_{l=0}^{\min\{l, j\}} \left( j \frac{j}{l} \right) \left( j - l \right) \left( j - l \right) \left( j - l \right) (\nabla^\perp \phi_{\text{app},1})^1 \cdot (\nabla \Omega)^{-1},
\]

where \( j = (j - j_2, j_2) \) and \( l = (l - l_2, l_2) \). We observe that, from the definition of \( \rho_j \) in (4.9), point iii) in Assumption 1 and \( K \geq 1 \),

\[
\| \xi_j \partial_X \phi_{\text{app},1} \partial_Y \Omega \| = \| \frac{\partial_Y \Omega}{\sqrt{\partial_Y \Omega^2 + 2 \rho_j}} \partial_X \phi_{\text{app},1} \|
\leq C \| (\partial_Y \| \frac{1}{1 + \nu^2 Y} \partial_X \phi_{\text{app},1} \|
\leq C (\frac{1 + Y}{1 + \nu^2 Y})^2 \partial_Y \Omega \|_{L^2(0, \kappa \sigma^2, H^{-1})}^2 \| (1 + \nu^2 Y) \partial_X \phi_{\text{app},1} \|
\leq C (\nu^2 C_s)^{\frac{1}{2}} \| \partial_X \phi_{\text{app},1} \| + C C_s^\frac{1}{2} \nu^2 \| \partial_X \phi_{\text{app},1} \|
\leq C (C_1^* + K^\frac{1}{2} C_s)^{\frac{1}{2}} \| \frac{1}{1 + \nu^2 Y} \partial_X \phi_{\text{app},1} \| + C (C_1^* + C_s^\frac{1}{2} \nu^2 \| \partial_X \phi_{\text{app},1} \|.
\]

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Here we have On the other hand,
\[
\|\xi_j (\partial_Y \phi_{app,1})^j \partial_X \Omega \| \leq \| \frac{1}{1 + \nu^2 Y} \partial_X \Omega \|_{L^\infty} \| \frac{1 + \nu^2 Y}{1 + Y} \xi_j \|_{L^\infty} \| (\partial_Y \phi_{app,1})^j \| \\
\leq CC_1^j \nu^2 (j + 1)^{3/2} \| (\partial_Y \phi_{app,1})^j \|.
\]

Here we have used (4.15) and point iii) in Assumption 1. Thus we have from \(C_* \geq 1\),
\[
\|\xi_j (\nabla \phi_{app,1})^j \cdot \nabla \Omega\|_{L^2(0, \frac{1}{k_*}, \tilde{L}_x^2)} \\
\leq C(C_*^j + \tilde{K}^j C_*^j)^j M_{2,j, \tilde{\nu}} \sum \Omega \partial_X \phi_{app,1} + C(C_*^j + \tilde{C}_*^j) \nu^2 (j + 1)^{3/2} M_{2,j,1} \partial_Y \phi_{app,1}.
\]

Next we see
\[
\|\xi_j \sum_{l=0}^{j-1} \sum_{l_2 \leq \min\{l, l_2\}} (j_2 \choose l_2) (j - j_2 \choose l - l_2) (\nabla \phi_{app,1})^j \cdot (\nabla \Omega)^{j-1} \|
\leq \sum_{l=0}^{j-1} \sum_{l_2 \leq \min\{l, l_2\}} \|\xi_j (\nabla \phi_{app,1})^j \cdot (\nabla \Omega)^{j-1} \|,
\]

and
\[
\|\xi_j (\nabla \phi_{app,1})^j \cdot (\nabla \Omega)^{j-1} \| \leq \| (\frac{1}{1 + \nu^2 Y} (\partial_Y \Omega)^{j-1}) \|_{L^\infty} \| \frac{1 + \nu^2 Y}{1 + Y} \xi_j \|_{L^\infty} \| \frac{1 + \nu^2 Y}{1 + Y} \partial_X \phi_{app,1} \|
\]
\[+ \| \frac{1 + \nu^2 Y}{1 + Y} \partial_X \phi_{app,1} \|_{L^\infty} \| \frac{1 + \nu^2 Y}{1 + Y} \partial_X \phi_{app,1} \| \\\n\leq C(j + 1)^{3/2} N_{\infty, j-l}(\frac{1 + \nu^2 Y}{1 + Y})^2 \| \partial_Y \Omega \| \| \frac{1 + \nu^2 Y}{1 + Y} \partial_X \phi_{app,1} \| \\
+ C\nu^2 (j + 1)^{3/2} N_{\infty, j-l} \| \nabla \Omega \| \| (\nabla \phi_{app,1})^j \|.
\]

Thus we have
\[
\|\xi_j \sum_{l=0}^{j-1} \sum_{l_2 \leq \min\{l, l_2\}} (j_2 \choose l_2) (j - j_2 \choose l - l_2) (\nabla \phi_{app,1})^j \cdot (\nabla \Omega)^{j-1} \|_{L^2(0, \frac{1}{k_*}, \tilde{L}_x^2)} \\
\leq C(j + 1)^{3/2} \sum_{l=0}^{j-1} \min\{l + 1, j - l + 1\} \left( \frac{1}{l} \right) N_{\infty, j-l} \| \nabla \Omega \| \\
\times \left( M_{2,l, \tilde{\nu}} \partial_X \phi_{app,1} + \nu^2 M_{2,l,1} \| \nabla \phi_{app,1} \| \right)
\]

We note that
\[
(j + 1)^{3/2} \min\{l + 1, j - l + 1\} \left( \frac{1}{l!} \right) \frac{1}{j!} \leq C, \quad 1 \leq l \leq j - 1.
\]

Taking into account this uniform bound (by decomposing the sum \(\sum_{l=0}^{j-1}\) into the term of \(\|l = 0\|\) and \(\sum_{l=1}^{j-1}\)) and collecting (5.40) and (5.41), we obtain from the Young inequality for
convolution in the $l^1$ space,

$$
\frac{1}{K^{\frac{1}{2}}\nu Y} \| \nabla \phi_{app,1} \cdot \nabla \Omega \|_{2,\xi(2)}'
\leq \frac{1}{K^{\frac{1}{2}}} \left( \sum_{j=0}^{\nu Y} \frac{1}{(j+1)^{\frac{1}{2}}} M_{2,j,\nabla X,\nabla Y} \| \partial_X \phi_{app,1} \| + \| \nabla \phi_{app,1} \|_{2,1}' \right),
$$

where $K$ has been taken large enough depending on $C_\nu$, $C_\kappa^*$, and $C_\kappa$. As for the estimate of $\| \nabla \phi_{app,1} \cdot \nabla \Omega \|_{L^2(0, \frac{1}{\nu Y} ; H^{-1})}$, let us take any $\eta \in H_{\nu}^0(\mathbb{T} \times \mathbb{R}_+ )$. Then we have

$$
\langle \nabla \phi_{app,1} \cdot \nabla \Omega, \eta \rangle = \langle \frac{1 + \nu Y}{1 + \nu^2 Y} \nabla \phi_{app,1} \cdot \nabla \Omega, \frac{\eta}{1 + \nu^2 Y} \rangle + \langle \nabla \phi_{app,1} \cdot \nabla \Omega, \frac{\nu Y \eta}{1 + \nu^2 Y} \rangle
\leq \frac{1 + \nu Y}{1 + \nu^2 Y} \nabla \phi_{app,1} \cdot \nabla \Omega \| \| \nabla \eta \| + C \nu Y \| \nabla \phi_{app,1} \| \| \nabla \eta \|,
$$

This implies

$$
\| \nabla \phi_{app,1} \cdot \nabla \Omega \|_{H^{-1}} \leq C \| \frac{1 + \nu Y}{1 + \nu^2 Y} \nabla \phi_{app,1} \cdot \nabla \Omega \| \frac{\eta}{1 + \nu^2 Y} \| + \| \frac{1 + \nu Y}{1 + \nu^2 Y} \nabla \phi_{app,1} \cdot \nabla \Omega \| \| \nabla \eta \|,
$$

where the Hardy inequality was used several times. Hence we obtain

$$
\| \nabla \phi_{app,1} \cdot \nabla \Omega \|_{H^{-1}} \leq C \| \frac{1 + \nu Y}{1 + \nu^2 Y} \nabla \phi_{app,1} \cdot \nabla \Omega \| + C \nu Y \frac{1 + \nu Y}{1 + \nu^2 Y} \| \nabla \phi_{app,1} \| \| \nabla \eta \|,
$$

Then

$$
\frac{1}{K^{\frac{1}{2}}\nu Y} \| \nabla \phi_{app,1} \cdot \nabla \Omega \|_{L^2(0, \frac{1}{\nu Y} ; H^{-1})} \leq C \nu Y \| \nabla \phi_{app,1} \| \| \nabla \partial_X \phi_{app,1} \|_{L^2(0, \frac{1}{\nu Y}; L^2_{X,Y})} \| \partial_X \partial_Y \phi_{app,1} \|_{L^2(0, \frac{1}{\nu Y}; L^2_{X,Y})} \| \| \nabla \phi_{app,1} \|_{2,1}'
\leq \frac{1}{K^{\frac{1}{2}}} \| \nabla \phi_{app,1} \|_{2,1}.
$$

The proof is complete. \qed

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Corollary 5.3. There exists $\kappa_4 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_4]$. There exists $K_4 = K_4(\kappa, C_\ast, C_j^\ast) \geq 1$ such that if $K \geq K_4$ then the system (5.38) admits a unique solution $\tilde{\phi}_1 \in C([0, \infty); \dot{H}^1(\mathbb{T}_\nu \times \mathbb{R}_+))$ with $\tilde{\omega}_1 = -\Delta \phi \in C([0, \infty); L^2(\mathbb{T}_\nu \times \mathbb{R}_+))$ satisfying

$$\| \tilde{\omega}_1 \|_{2, \xi} + K^{\frac{1}{2}} \| \tilde{\varphi} \|_{2, \xi} + K^{\frac{1}{2}} \| \tilde{\varphi} \|_{2, 1} + K^{\frac{1}{2}} \| Y \tilde{\varphi} |_{Y=0} \|_{bc} \leq \frac{1}{K^{\frac{1}{2}}} \| h \|_{bc}. \quad (5.43)$$

Proof. Propositions 4.1 and 5.4 give

$$\| \omega \|_{2, \xi} + K^{\frac{1}{2}} \| \omega \|_{2, \xi} + K^{\frac{1}{2}} \| \varphi \|_{2, 1} + K^{\frac{1}{2}} \| Y \varphi \|_{Y=0} \|_{bc} \leq \frac{C}{K^{\frac{1}{2}}} \left( \sum_{j=0}^{\nu - \frac{1}{2}} \frac{1}{\nu^{2} \tau} M_{2,j} \left[ \partial X \phi_{app,1} \right] + \| \nabla \phi_{app,1} \|_{2, 1} \right).$$

Here $C > 0$ is a universal constant. Recall that $\phi_{app,1}[h] = \phi_{1,1}[h] + \phi_{1,2}[h]$. Then the assertion follows from Proposition 5.1 for $\phi_{1,1}[h]$ and Corollary 5.2 for $\phi_{1,2}[h]$. The proof is complete.

From the construction, $\phi_{app} = \phi_{app}[h] = \phi_{app,1}[h] + \tilde{\phi}_1[h]$ satisfies

$$\nu^{\frac{1}{2}} \Delta^2 \phi_{app} - \partial_\tau \Delta \phi_{app} - V \cdot \nabla \Delta \phi_{app} + \nabla \phi_{app} \cdot \nabla \Omega = 0, \quad \tau > 0, \; X \in \mathbb{T}_\nu, \; Y > 0, \; \phi_{app}|_{Y=0} = 0, \quad \partial_\tau \phi_{app}|_{Y=0} = h + R_{bc}[h], \quad \phi_{app}|_{\tau=0} = 0. \quad (5.44)$$

Here $R_{bc}[h]$ is the linear operator defined as

$$R_{bc}[h] = \partial_\tau \Phi_{1,2}[h]|_{Y=0} + \partial_\tau \tilde{\varphi}_1[h]|_{Y=0}. \quad (5.45)$$

We note that the operator $R_{bc}$ is well-defined on the Banach space

$$Z_{bc} = \{ h \in L^2(0, \frac{1}{K^{\frac{1}{2}}}) : \| h \|_{Z_{bc}} := \| h \|_{bc} < \infty \}. \quad (5.46)$$

Proposition 5.5. There exists $\kappa_5 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_5]$. There exists $K_5 = K_5(\kappa, C_\ast, C_j^\ast) \geq 1$ such that if $K \geq K_5$ then the map $R_{bc} : Z_{bc} \to Z_{bc}$ defined by (5.45) satisfies

$$\| R_{bc}[h] \|_{bc} \leq \frac{1}{2} \| h \|_{bc}. \quad (5.47)$$

Hence, the operator $I + R_{bc}$ is invertible in $Z_{bc}$, and the map

$$\Phi_{bc}[h] := \phi_{app}[(I + R_{bc})^{-1} h], \quad h \in Z_{bc} \quad (5.48)$$

gives the solution to (5.1) and satisfies

$$\| \nabla \Phi_{bc}[h] \|_{2, 1} \leq C \| h \|_{bc}. \quad (5.49)$$

Here $C > 0$ is a universal constant.
Proof. By the definition of $R_{bc}$ in (5.45), estimate (5.47) is a consequence of Corollaries 5.2 and 5.3, by taking $\kappa$ small first and then $K$ large enough depending only on $C_*, C_j^*$, and $C_\kappa$. In particular, we have
\[
\|(I + R_{bc})^{-1}h\|_{bc} \leq 2\|h\|_{bc}, \quad h \in Z_{bc}. \tag{5.50}
\]
Then Proposition 5.1 and Corollaries 5.2-5.3 give (5.49). The proof is complete. □

6 Full estimate for linearization

We have constructed the solution to (2.12) of the form
\[
\nabla^\perp \phi = \nabla^\perp \Phi_{slip} + \nabla^\perp \Phi_{bc}[h], \quad h = -\partial_Y \Phi_{slip}|_{Y=0} \in Z_{bc}, \tag{6.1}
\]
and
\[
\Phi_{bc}[h] = \phi_{app,1}[(I + R_{bc})^{-1}h] + \tilde{\phi}_1[(I + R_{bc})^{-1}h], \quad \phi_{app,1} = \phi_{1,1} + \phi_{1,2}.
\]
To simplify the notation we will write $\phi_{app,1}$ for $\phi_{app,1}[(I + R_{bc})^{-1}h]$ below. So far we have the bound of $\nabla^\perp \phi_{1,1}$ only in the norm $\||\cdot||_{2,1}$. To obtain the estimates of $\||\nabla \phi||_\infty$ and $\||\Delta \phi||_\infty$ we need the extra work.

**Proposition 6.1.** There exists $\kappa_6 \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa_6]$. There exists $K_6 = K_6(C_0^*, C_1^*) \geq 1$ such that if $K \geq K_6$ then the solution to (2.12) constructed as (6.1) satisfies
\[
\nu \frac{1}{2} \||\omega||_\infty + K^2 \nu \frac{1}{2} \||\nabla \phi||_\infty
\leq C(C_0^* + C_1^*) \left(\||\nabla \phi||'_{2,1} + \||\Delta(\phi - \phi_{app,1})||'_{2,1} + \||\Delta \phi_{app,1}||'_{2,Y}\right)
+ C \left(K^2 \||\nabla \phi_0|| + \nu \frac{1}{2} \||\Delta \phi_0|| + \||F||_2\right).
\]
Here $C > 0$ is a universal constant.

The proof of Proposition 6.1 is similar to the one of Proposition 4.2, and we postpone it to the appendix. Admitting Proposition 6.1, we will now complete the proof of Theorem 2.2. Let us recall (6.1). We first observe from Proposition 4.1 and Remark 4.1 that
\[
\||\Delta \Phi_{slip}||'_{2,1} + \||\nabla \Phi_{slip}||'_{2,1} + \||\partial_Y \Phi_{slip}|_{Y=0}||_{bc} \leq \frac{1}{K^\frac{1}{2}} \left(\||\nabla \phi_0||_{L^2_{X,Y}} + \nu^{-\frac{1}{2}} \||\Delta \phi_0|| \right) + \nu^{-\frac{3}{2}} \||F||_2\right), \tag{6.2}
\]
by taking $K$ large enough. On the other hand, Proposition 5.5 (for $\nabla \Phi_{bc}$), Corollary 5.3 and (1) of Remark 4.1 (for $\Delta(\Phi_{bc} - \phi_{app,1}) = \Delta \phi_1$), Proposition 5.2 and Corollary 5.2 and (for $\Delta \phi_{app,1} = \Delta \phi_{1,1} + \Delta \phi_{1,2}$), and (6.2) give
\[
\||\nabla \Phi_{bc}||'_{2,1} + \||\Delta(\Phi_{bc} - \phi_{app,1})||'_{2,1} + \||\Delta \phi_{app,1}||'_{2,Y}
\leq C \left(\||\partial_Y \Phi_{slip}|_{Y=0}||_{bc}
\leq \frac{C}{K^\frac{1}{2}} \left(\||\nabla \phi_0||_{L^2_{X,Y}} + \nu^{-\frac{1}{2}} \||\Delta \phi_0|| + \nu^{-\frac{3}{2}} \||F||_2\right). \tag{6.3}
\]
Here $C > 0$ is a universal constant. By applying the estimate in Proposition 6.1 and by taking $K$ large enough, the proof of Theorem 2.2 is complete.
7 Nonlinear stability: Proof of Theorem 2.1

Let us recall the nonlinear system (1.3). Theorem 2.1 is a consequence of Theorem 2.3 for the linear system (1.6) and the bilinear estimate in Lemma 7.1 stated below. We observe that 
\[ -w \cdot \nabla w = w \operatorname{rot} w + \nabla \tilde{q} \]
for any solenoidal vector field \( w \). To this end we fix \( K \geq 1 \) and \( \nu \in (0, 1] \), and let \( X \) be the Banach space of solenoidal vector fields \( f = (f_1, f_2) \) on \( [0, \frac{1}{K}] \times \mathbb{R}_+^2 \) defined as
\[
X = \{ f \in C([0, \frac{1}{K}]; H^1_{0, \sigma}(\mathbb{T} \times \mathbb{R}_+)) \mid \| f \|_X = \| f \|_\infty + \nu^{\frac{1}{2}} \| \operatorname{rot} f \|_\infty < \infty \},
\]
where \( \| \cdot \|_\infty \) is defined as (2.1) with \( p = \infty \).

**Lemma 7.1.** There exists a universal constant \( C > 0 \) such that for any \( f, g \in X \),
\[
\| f \operatorname{rot} g \|_2 \leq C \frac{\nu^{-\frac{1}{2}}}{K^2} \| f \|_X \| g \|_X.
\]  

**Proof.** We compute
\[
\| f \operatorname{rot} g \|_2 \leq C \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{j!^2} \sup_{|j| = j} \sum_{1 \leq l \leq j} \left( \frac{j}{l} \right) \| f^l (\operatorname{rot} g)^{j-l} \|_{L^2(0, \frac{1}{K}; L^2_{x,y})} \]
\[
\leq C \frac{\nu^{-\frac{1}{2}}}{K^2} \sum_{j=0}^{\nu^{-\frac{1}{2}}} \frac{1}{j!^2} \sup_{|j| = j} \sum_{1 \leq l \leq j} \left( \frac{j}{l} \right) \| f^l (\operatorname{rot} g)^{j-l} \|_{L^\infty(0, \frac{1}{K}; L^2_{x,y})}.
\]

As \( \left( \frac{j}{l} \right) \leq \left( \frac{j}{|j|} \right) \) and as for all \( l \in \mathbb{N}_0 \),
\[
\sharp \{ l, |l| = l, 1 \leq j \} = \sharp \{ l_2, \max(0, l - j + j_2) \leq l_2 \leq \min(j_2, l) \} \leq \min(l + 1, j - l + 1)
\]
we end up with

\[
\| \text{rot } g \|_2 \leq \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu} \frac{1}{j!^2} \sum_{l=0}^{j} \min(l + 1, j - l + 1) \left( \frac{j}{l} \right) \sup_{|\ell| = l} \sup_{|k| = j - l} \| f^4 (\text{rot } g)^k \|_{L_2^\infty L_2^y}
\]

\[
\leq \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu} \sum_{0 \leq |l| \leq j/2} (l + 1) \left( \frac{j}{l} \right)^{\frac{1}{2}} \frac{1}{l!^2} \sup_{|\ell| = l} \| f^4 \|_{L_2^\infty L_2^y} \frac{1}{(j - l)!^2} \sup_{|k| = j - l} \| (\text{rot } g)^k \|_{L_2^\infty L_2^y}
\]

\[
+ \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu} \sum_{j/2 < |l| \leq j} (j - l + 1) \left( \frac{j}{l} \right)^{\frac{1}{2}} \frac{1}{l!^2} \sup_{|\ell| = l} \| f^4 \|_{L_2^\infty L_2^y} \frac{1}{(j - l)!^2} \sup_{|k| = j - l} \| (\text{rot } g)^k \|_{L_2^\infty L_2^y}
\]

\[
\leq \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu} \sum_{0 \leq |l| \leq j/2} (l + 1) \left( \frac{j}{l} \right)^{\frac{1}{2}} \frac{1}{l!^2} \sup_{|\ell| = l} \| \partial_x f^4 \|_{L_2^\infty L_2^y L_2^y} + \| f^4 \|_{L_2^\infty L_2^y L_2^y} \frac{1}{(j - l)!^2} \sup_{|k| = j - l} \| (\text{rot } g)^k \|_{L_2^\infty L_2^y L_2^y}
\]

\[
\times \sup_{|\ell| = l} \| \partial_x \partial_y f^4 \|_{L_2^\infty L_2^y L_2^y} + \| \partial_y f^4 \|_{L_2^\infty L_2^y L_2^y}
\]

\[
+ \frac{C}{K^{\frac{1}{2}}} \sum_{j=0}^{\nu} \sum_{j/2 < |l| \leq j} (j - l + 1) \left( \frac{j}{l} \right)^{\frac{1}{2}} \frac{1}{l!^2} \sup_{|\ell| = l} \| f^4 \|_{L_2^\infty L_2^y L_2^y} \frac{1}{(j - l)!^2} \sup_{|k| = j - l} \| (\text{rot } g)^k \|_{L_2^\infty L_2^y L_2^y}
\]

\[
\times \sup_{|\ell| = l} \| \partial_y (\text{rot } g)^k \|_{L_2^\infty L_2^y L_2^y} + \| (\text{rot } g)^k \|_{L_2^\infty L_2^y L_2^y}.
\]

Here we have used the Sobolev embedding type inequality. By using the bound

\[
\sup_{|\ell| = l} (\| \partial_x f^4 \|_{L_2^\infty L_2^y L_2^y} + \| f^4 \|_{L_2^\infty L_2^y L_2^y}) \frac{1}{2} (\| \partial_x \partial_y f^4 \|_{L_2^\infty L_2^y L_2^y} + \| \partial_y f^4 \|_{L_2^\infty L_2^y L_2^y}) \frac{1}{2}
\]

\[
\leq \nu^\frac{1}{4} \sup_{l \leq |\ell| \leq l+1} \| f^4 \|_{L_2^\infty L_2^y L_2^y} + \nu^\frac{1}{4} \sup_{l \leq |\ell| \leq l+1} \| \partial_y f^4 \|_{L_2^\infty L_2^y L_2^y}
\]

and by observing that there exists \( C > 0 \) such that for \( \left( \frac{j}{l} \right)^{\frac{1}{2}} (l + 1)^{\frac{1}{2}} \leq C \) for \( 0 \leq l \leq j/2 \), we
have

\[
\frac{C}{K^\frac{1}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{0 \leq l \leq j/2} (l + 1)^{\frac{1}{2}} \left( \begin{array}{c} j \\ l \end{array} \right)^{-\frac{1}{2}} \frac{1}{(l + 1)!^\frac{1}{2}} \\
\times \sup_{|l|=l} \left( \|\partial_x f_l\|_{L^\infty L^2_x L^2_y} + \|f_l\|_{L^\infty L^2_x L^2_y} \right)^{\frac{1}{2}} \left( \|\partial_x \partial_y f_l^l\|_{L^\infty L^2_x L^2_y} + \|\partial_y f_l^l\|_{L^\infty L^2_x L^2_y} \right)^{\frac{1}{2}} \\
\times \frac{1}{(j - l)!^\frac{1}{2}} \sup_{|k|=j-l} \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y}
\leq \frac{C}{K^{\frac{1}{2}} \nu^\frac{1}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{0 \leq l \leq j/2} (l + 1)!^\frac{1}{2} \sup_{|l|=l} \frac{1}{(l + 1)!^\frac{1}{2}} \sup_{|k|=j-l} \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y}
\leq \frac{C}{K^{\frac{1}{2}} \nu^\frac{1}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{0 \leq l \leq j/2} (l + 1)!^\frac{1}{2} \sup_{|l|=l} \frac{1}{(l + 1)!^\frac{1}{2}} \sup_{|k|=j-l} \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y}
\leq \frac{C}{K^{\frac{1}{2}} \nu^\frac{1}{2}} \|f\|_\infty \|\text{rot } g\|_\infty + \frac{C \nu^\frac{1}{2}}{K^{\frac{1}{2}}} \|\partial_y f\|_\infty \|\text{rot } g\|_\infty.
\]

where the discrete Young’s convolution inequality is applied in the last line together with the estimate

\[
\sum_{j=0}^{\nu-\frac{1}{2}} \frac{1}{j!^\frac{1}{2}} \sup_{|l|=l} \|\partial_y f^j\|_{L^\infty L^2_x L^2_y} \leq C \|\partial_y f\|_\infty.
\]

Similarly, since \((j - l + 1)^{\frac{1}{2}} \left( \begin{array}{c} j \\ l \end{array} \right)^{-\frac{1}{2}} \leq C\) for \(j/2 \leq l \leq j\), we have

\[
\frac{C}{K^\frac{1}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{j/2 \leq l \leq j} (j - l + 1)^{\frac{1}{2}} \left( \begin{array}{c} j \\ l \end{array} \right)^{-\frac{1}{2}} \frac{1}{l!^\frac{1}{2}} \sup_{|l|=l} \left( \|\partial_x f_l\|_{L^\infty L^2_x L^2_y} + \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y} \right)^{\frac{1}{2}} \left( \|\partial_x \partial_y f_l^l\|_{L^\infty L^2_x L^2_y} + \|\partial_y f_l^l\|_{L^\infty L^2_x L^2_y} \right)^{\frac{1}{2}} \\
\times \frac{1}{(j - l + 1)!^\frac{1}{2}} \sup_{|k|=j-l} \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y}
\leq \frac{C}{K^\frac{1}{2}} \sum_{j=0}^{\nu-\frac{1}{2}} \sum_{j/2 \leq l \leq j} \frac{1}{l!^\frac{1}{2}} \sup_{|l|=l} (\nu^{-\frac{1}{2}} \|f_l\|_{L^\infty L^2_x L^2_y} + \nu^{\frac{1}{2}} \|\partial_y f_l^l\|_{L^\infty L^2_x L^2_y}) \\
\times \frac{1}{(j - l + 1)!^\frac{1}{2}} \sup_{j - l \leq |k| \leq j + 1} \|\text{rot } g^k\|_{L^\infty L^2_x L^2_y}
\leq \frac{C}{K^{\frac{1}{2}} \nu^\frac{1}{2}} \|f\|_\infty \|\text{rot } g\|_\infty + \frac{C \nu^\frac{1}{2}}{K^{\frac{1}{2}}} \|\partial_y f\|_\infty \|\text{rot } g\|_\infty.
\]

Hence the result follows from Lemma C.1. The proof is complete.

\[\square\]

**Proof of Theorem 2.1.** Let \(C\) be the universal constant in Theorem 2.3. Then the standard fixed point theorem in the closed convex set

\[X_R = \{f \in C([0, \frac{1}{K}] ; H^1_{0,\sigma}(\mathbb{T} \times \mathbb{R}_+)) \mid \|f\|_X = \|f\|_\infty + \nu^{\frac{1}{2}} \|\text{rot } f\|_\infty \leq R\}, \quad R = 4C\delta_0 \nu^\frac{1}{2}\]

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is applied by using Theorem 2.3 and Lemma 7.1, if $\nu \leq K^{-2}$ holds and if $\delta_0$ is sufficiently small. We note that the smallness condition $||w_0|| + ||\text{rot } w_0|| \leq \delta_0 \nu^{\frac{2}{3}}, \|r\|_2 \leq \delta_0 \nu^{\frac{1}{3}}$ is needed to close the estimate. Since the argument is standard we omit the details. The proof is complete.

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A Interpolation estimate for solutions to the Poisson equation

Lemma A.1. Assume that $Y^k \omega \in L^2(T_\nu \times \mathbb{R}^+_+)$ for $k = 0, 1, 2$. Let $\phi \in \dot{H}^1_0(T_\nu \times \mathbb{R}^+_+)$ be the solution to the Poisson equation $-\Delta \phi = \omega$ in $T_\nu \times \mathbb{R}^+_+$ with $\phi|_{Y=0} = 0$. Then there exists $C > 0$ such that for any $j \geq 0$ we have

$$\sup_{Y>0} ||\phi(\cdot,Y)||_{L^2(T_\nu)} \leq C \left( (j+1)^{-\frac{1}{2}} ||\omega||_{L^2(T_\nu \times \mathbb{R}^+_+)} + (j+1)^{\frac{1}{2}} ||Y^2 \omega||_{L^2(T_\nu \times \mathbb{R}^+_+)} \right). \quad (A.1)$$

Proof. The solution is given by the formula

$$\phi(X,Y) = \int_0^Y e^{-(Y-Y')(\partial^2_X)^{\frac{1}{2}}} \int_{Y'}^\infty e^{-(Y''-Y')(\partial^2_X)^{\frac{1}{2}}} \omega(\cdot,Y'') dY'' dY'.$$

Here $e^{-(Y-(\partial^2_X)^{\frac{1}{2}})}$ is the Poisson semigroup. Then we have

$$||\phi(\cdot,Y)||_{L^2(T_\nu)} \leq \int_0^Y \int_{Y'}^\infty ||\omega(\cdot,Y'')||_{L^2(T_\nu)} dY'' dY'.$$

By decomposing the integral $\int_0^Y$ into $\int_{0}^{\min(Y,(j+1)^{-\frac{1}{2}})}$ and $\int_{\min(Y,(j+1)^{-\frac{1}{2}})}^Y$, we have from the Hölder inequality,

$$\sup_{Y>0} ||\phi(\cdot,Y)||_{L^2(T_\nu)} \leq C(j+1)^{-\frac{1}{2}} ||\omega||_{L^2(T_\nu \times \mathbb{R}^+_+)} + C(j+1)^{\frac{1}{2}} ||Y^2 \omega||_{L^2(T_\nu \times \mathbb{R}^+_+)}.$$

The proof is complete.

Lemma A.1 yields the following

Proposition A.1. Let $\phi \in \dot{H}^1_0(T_\nu \times \mathbb{R}^+_+)$ be the solution to the Poisson equation $-\Delta \phi = \omega$ in $T_\nu \times \mathbb{R}^+_+$ with $\phi|_{Y=0} = 0$. Then for any $j \geq 0$ we have

$$M_{2,j} \frac{1}{1+x} \partial X \phi \leq C(j+1)^{-\frac{j}{2}} M_{2,j+1,Y}[\omega] + C(j+1)^{\frac{j}{2}} M_{2,j+1,Y^2}[\omega] + CKD_{j} \frac{1}{j}(M_{2,j-1,Y}[\omega] + M_{2,j-1,1} [\nabla \phi]). \quad (A.2)$$

Here $C > 0$ is a universal constant.
Proof. Since $-\Delta \partial_X \phi = \partial_X \omega$ we have $-(\Delta \partial_X \phi)^j = \partial_X \omega^j$. Then we use the commutator relation
\[
-(\Delta \phi)^j = -\nabla \cdot (\nabla \phi)^j + \nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j
\]
\[
= -\Delta \phi^j + \partial_Y (\nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j) + \nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j.
\]
Thus we have the Poisson equation for $\phi^j$:
\[
-\Delta \phi^j = \omega^j - \partial_Y (\nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j) - \nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j.
\]
Then we decompose $\phi^j$ into $\phi_1 + \phi_{2,1} + \phi_{2,2}$ so that
\[
-\Delta \phi_1 = \omega^j, \quad -\Delta \phi_{2,1} = -\partial_Y (\nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j), \quad -\Delta \phi_{2,2} = -\nu^j j_2 \frac{\chi'}{\chi} (\partial_Y \phi)^j,
\]
subject to the Dirichlet boundary condition. Then Lemma A.1 implies for $\partial_X \phi_1$,
\[
\sup_{Y > 0} \|\partial_X \phi_1(\cdot, Y)\|_{L^2(T_\nu)} \leq C (j + 1)^{-\frac 1 2} \|Y \partial_X \omega^j\|_{L^2(T_\nu \times \mathbb{R}_+)} + (j + 1) \|\partial_Y \omega^j\|_{L^2(T_\nu \times \mathbb{R}_+)}.
\] (A.3)

On the other hand, the simple energy estimate gives
\[
\|\nabla \phi_{2,1}\| \leq \nu^j j_2 \frac{\chi'}{\chi} \|\phi^j\| \leq \kappa^j \nu^j j_2 \|\partial_Y \phi(\partial_Y \phi)^{(j_1, j_2 - 1)}\|.
\]
As for $\phi_{2,2}$, we have from
\[
\frac{1}{K} (\partial_Y \phi)^j = e^{-K \nu^j \frac{1}{2} (\partial_Y \phi)^{(j_1, j_2 - 1)}} = e^{-K \nu^j \frac{1}{2} (-\omega(\partial_Y \phi)^{(j_1, j_2 - 1)} - \partial_Y \phi(\partial_Y \phi)^{(j_1, j_2 - 1)})},
\]
the Hardy inequality and the integration by parts imply
\[
\|\nabla \phi_{2,2}\| \leq C \kappa^j \nu^j j_2 \left(\|\omega(\partial_Y \phi)^{(j_1, j_2 - 1)}\| + \|\partial_X \phi(\partial_Y \phi)^{(j_1, j_2 - 1)}\|\right).
\]
Hence we obtain the desired estimate by taking the $L^2$ norm in time and by taking the supremum about $j$ such that $j = j$. The proof is complete. \qed

B \ Proof of Proposition 6.1

Let us go back to (4.1) with $G = 0$, but we impose the noslip boundary condition $\phi|_{Y=0} = \partial_Y \phi|_{Y=0} = 0$ in this appendix. Then we have
\[
-\nu^j (\Delta \omega)^j + (\partial_r + K \nu^j (j + 1)) \omega^j = -(V \cdot \nabla \omega)^j - (\nabla^\perp \phi \cdot \nabla \Omega)^j + (\text{rot } F)^j
\]
\[
= (\text{div } H)^j,
\] (B.1)
where
\[
H = -V \omega - \Omega \nabla^\perp \phi + (F_2, -F_1).
\]
The idea is to take the $L^2$ inner product with $\partial_r \phi^j$, which gives the estimates of $\|\nabla \phi\|_\infty$ and $\|\Delta \phi\|_\infty$ in terms of $\|\nabla \phi\|_{L^2}$.

The most technical part is the computation of the viscous term $\langle (\Delta \omega)^j, \partial_r \phi^j \rangle$ when $j_2 \neq 0$, for which one needs to convert the vertical derivative $\partial_Y^j \omega$ into the tangential ones by using the equation.
Lemma B.1. For any $\kappa \in (0,1]$ and $K \geq 1$ we have
\[
\int_0^{\tau_0} \langle (\partial_\tau + K\nu^{\frac{1}{2}}(j+1))\omega^1, (\partial_\tau \phi) \rangle \, d\tau \\
\geq \frac{1}{2} \| (\partial_\tau (\nabla \phi)^1) \|_{L^2(\Omega_{t_0};L^2)}^2 + \frac{K\nu^{\frac{1}{2}}(j+1)}{2} \| (\nabla \phi)^1(\tau_0) \|^2 - \| (\nabla \phi)^1(0) \|^2 \\
- C\kappa^2 K\nu^{\frac{1}{2}}j (\nu^{\frac{1}{2}}j^2)^2 M_{\infty,j-1,1}[\nabla \phi]^2 - C(\kappa \nu^{\frac{1}{2}}j)^2 M_{2,j-1,1}[\partial_\tau \nabla \phi]^2.
\]
Here $C$ is a universal constant.

Proof. Let us recall the identity
\[
\omega^1 = -(\Delta \phi)^1 = -\nabla \cdot (\nabla \phi)^1 + \nu^{\frac{1}{2}}j^2 \chi^\nu \frac{\chi^\nu}{\nu} (\partial_\tau \phi)^1,
\]
which implies
\[
\langle (\partial_\tau + K\nu^{\frac{1}{2}}(j+1))\omega^1, (\partial_\tau \phi) \rangle
= \| (\partial_\tau (\nabla \phi)^1) \|^2 + 2\nu^{\frac{1}{2}}j^2 \langle \frac{\chi^\nu}{\nu} (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau \phi) \rangle + \frac{K\nu^{\frac{1}{2}}(j+1)}{2} \| (\nabla \phi)^1 \|^2 \\
+ 2\nu^{\frac{1}{2}}j^2 K\nu^{\frac{1}{2}}(j+1) \langle \frac{\chi^\nu}{\nu} (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau \phi) \rangle.
\]
Then from $\partial_\tau \phi = \chi^\nu \partial_\tau (e^{-K\nu^{\frac{1}{2}}(\partial_\tau \phi)^1(j_1,j_2-1)})$ for $j_2 \geq 1$ we have
\[
\int_0^{\tau_0} 2\nu^{\frac{1}{2}}j^2 \langle \frac{\chi^\nu}{\nu} (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau \phi) \rangle \, d\tau \\
\geq -\frac{1}{4} \| (\partial_\tau (\nabla \phi)^1) \|_{L^2(\Omega_{t_0};L^2)}^2 - C(\kappa \nu^{\frac{1}{2}}j)^2 (M_{2,j-1,1}[\partial_\tau \nabla \phi]^2 + (K\nu^{\frac{1}{2}})^2 M_{2,j-1,1}[\nabla \phi]^2),
\]
while we have from the integration by parts in time,
\[
\int_0^{\tau_0} 2\nu^{\frac{1}{2}}j^2 K\nu^{\frac{1}{2}}(j+1) \langle \frac{\chi^\nu}{\nu} (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau \phi) \rangle \, d\tau \\
= 2\nu^{\frac{1}{2}}j^2 K\nu^{\frac{1}{2}}(j+1) \left( e^{-K\nu^{\frac{1}{2}}(\nu^{\frac{1}{2}}j+1)} \langle \chi^\nu (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle(\tau_0) - \langle \chi^\nu (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle(0) \right) \\
- 2\nu^{\frac{1}{2}}j^2 K\nu^{\frac{1}{2}}(j+1) \int_0^{\tau_0} e^{-K\nu^{\frac{1}{2}}(\nu^{\frac{1}{2}}j+1)} \langle \partial_\tau (\partial_\tau (\partial_\tau \phi)^1), \chi^\nu (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle \, d\tau \\
\geq 2\nu^{\frac{1}{2}}j^2 K\nu^{\frac{1}{2}}(j+1) \left( e^{-K\nu^{\frac{1}{2}}(\nu^{\frac{1}{2}}j+1)} \langle \chi^\nu (\partial_\tau (\partial_\tau (\partial_\tau \phi)^1)) (\tau_0), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle(\tau_0) - \langle \chi^\nu (\partial_\tau \phi)^1(0), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle(0) \right) \\
- \frac{1}{4} \| (\partial_\tau (\partial_\tau \phi)^1) \|_{L^2(\Omega_{t_0};L^2)}^2 - C(\kappa \nu^{\frac{1}{2}}j)^2 M_{2,j-1,1}[\nabla \phi]^2.
\]
We also observe that for $j_2 \geq 1$,
\[
\langle \chi^\nu (\partial_\tau (\partial_\tau \phi)^1), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle = e^{-K\nu^{\frac{1}{2}}(\nu^{\frac{1}{2}}j+1)} \langle \chi^\nu (\partial_\tau \phi)^1(j_1,j_2-1), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle \\
= -\frac{e^{-K\nu^{\frac{1}{2}}}(\nu^{\frac{1}{2}}j+1)}{2} \langle \partial_\tau (\chi^\nu (\partial_\tau \phi)^1(j_1,j_2-1)), (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \rangle \\
- e^{-K\nu^{\frac{1}{2}}}(\nu^{\frac{1}{2}}j+1-1) \| (\chi^\nu (\partial_\tau (\partial_\tau \phi)^1(j_1,j_2-1)) \|^2.
\]
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Thus we conclude also from $K^{\tau} \nu^{\frac{1}{2}} \leq 1$ that
\[
\int_0^{\tau_0} 2\nu^{\frac{1}{2}} j_2 K^{\tau} (j + 1) \left( \frac{\lambda'}{\lambda} (\partial_Y^\nu \phi)^j, \partial_r \phi^j \right) d\tau \\
\geq -CK^{\tau} (K\nu^{\frac{1}{2}} j_2^2 (j) \| (\partial_Y^\nu \phi)^{(j+1)}(\tau_0) \|^2 + \| (\partial_Y^\nu \phi)^{(j+1)}(0) \|^2) \\
- \frac{1}{4} \| \partial_r (\partial_Y^\nu \phi)^j \|^2_{L^2(0, \tau_0; L^2)} - C(K\nu^{\frac{1}{2}})^2 M_{2,j-1,1} |\nabla \phi|^2.
\]
Collecting these above and $M_{2,j-1,1} |\nabla \phi|^2 \leq (K\nu^{\frac{1}{2}})^{-1} M_{\infty,j-1,1} |\nabla \phi|^2$, we obtain the desired estimate. The proof is complete. \hfill \Box

**Lemma B.2.** For any $\kappa \in (0, 1]$ and $K \geq 1$ we have
\[
\int_0^{\tau_0} \langle -\nu^{\frac{1}{2}} (\Delta^\nu)^{\frac{1}{2}}, \partial_r \phi^j \rangle d\tau \\
\geq \frac{\nu^{\frac{1}{2}}}{2} \left( \| \omega^\nu (0) \|^2 - \| \omega^\nu (\tau_0) \|^2 \right) - \frac{1}{4} M_{2,j,1} [\partial_r \nabla \phi]^2 \\
- C(K\nu^{\frac{1}{2}} j)^2 \left( M_{2,j-1,1} |\partial_r \nabla \phi|^2 + (\nu^{\frac{1}{2}} (j - 1))^2 M_{2,j-2,1} |\partial_r \nabla \phi|^2 \right) \\
- C\nu^{\frac{1}{2}} \left( M_{\infty,j-1,1} |\omega|^2 + (\nu^{\frac{1}{2}} j)^2 M_{\infty,j-1,1} |\omega|^2 \right) \\
- C \left( M_{2,j,1} [H]^2 + (\nu^{\frac{1}{2}} j)^2 M_{2,j-1,1} [H]^2 \right).
\]

Here $C$ is a universal constant.

**Proof.** We observe from
\[
(\Delta^\nu)^{\frac{1}{2}} = \nabla \cdot (\nabla^\nu)^{\frac{1}{2}} - \nu^{\frac{1}{2}} j_2 \frac{\lambda'}{\lambda} (\partial_Y^\nu \omega)^{\frac{1}{2}}, \quad \lambda' = \kappa e^{-\nu^{\frac{1}{2}} Y},
\]
(B.3)
\[
\nabla \partial_r \phi^j = \partial_r (\nabla^\nu)^{\frac{1}{2}} + \nu^{\frac{1}{2}} j_2 \frac{\lambda'}{\lambda} \partial_r \phi^j e_2
\]
and the integration by parts that
\[
\langle -\nu^{\frac{1}{2}} (\Delta^\nu)^{\frac{1}{2}}, \partial_r \phi^j \rangle = \nu^{\frac{1}{2}} \langle (\nabla^\nu)^{\frac{1}{2}}, \partial_r (\nabla^\nu)^{\frac{1}{2}} \rangle + 2\nu j_2 \frac{\lambda'}{\lambda} (\partial_Y^\nu \omega)^{\frac{1}{2}}, \partial_r \phi^j \rangle.
\]
Then the similar identities
\[
(\nabla^\nu)^{\frac{1}{2}} = \nabla^\nu - \nu^{\frac{1}{2}} j_2 \frac{\lambda'}{\lambda} \omega^j e_2,
\]
(B.4)
\[
\nabla \cdot \partial_r (\nabla^\nu)^{\frac{1}{2}} = \partial_r (\Delta^\nu)^{\frac{1}{2}} + \nu^{\frac{1}{2}} j_2 \frac{\lambda'}{\lambda} \partial_r (\partial_Y \phi)^j
\]
together with the integration by parts yield
\[
\langle -\nu^{\frac{1}{2}} (\Delta^\nu)^{\frac{1}{2}}, \partial_r \phi^j \rangle = \nu^{\frac{1}{2}} \langle \omega^j, \partial_r \omega^j \rangle - 2\nu j_2 \frac{\lambda'}{\lambda} \omega^j, \partial_r (\partial_Y \phi)^j \rangle + 2\nu j_2 \frac{\lambda'}{\lambda} (\partial_Y^\nu \omega)^{\frac{1}{2}}, \partial_r \phi^j \rangle.
\]
(B.5)
Again from the above identities about the commutators we have for \( j_2 \geq 1, \)
\[
\left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau}(\partial_{Y} \phi) \right) = -\left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) - \nu \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right) - \nu \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right) - \nu \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right) \cdot (2j_2 - 1) \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right).
\]

Here \( \chi'' = -\kappa^2 e^{-\kappa \nu^2} Y. \) Thus (B.5) is written as
\[
\begin{align*}
\left( -\nu \left( \Delta \omega \right)^j, \partial_{\tau} \phi \right) &= \nu \left( \omega, \partial_{\tau} \omega \right) + 4\nu j_2 \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) + 2\nu j_2 (2j_2 - 1) \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right),
\end{align*}
\]

Let us compute the term \( \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right). \) From the identity
\[
\frac{1}{\chi} \left( \partial_{\tau} \omega \right) = e^{-K\nu \frac{1}{2}} \left( \partial_{\tau} \omega \right) = e^{-K\nu \frac{1}{2}} \left( \partial_{\tau} \omega \right) = e^{-K\nu \frac{1}{2}} \left( \partial_{\tau} \omega \right),
\]
we have
\[
\left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) = e^{-K\nu \frac{1}{2}} \left( \partial_{\tau} \omega \right) = e^{-K\nu \frac{1}{2}} \left( \partial_{\tau} \omega \right).
\]

Next we compute the term \( \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right). \) From the identities as in (B.4) we have
\[
e^{-K\nu \frac{1}{2}} \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) = e^{-K\nu \frac{1}{2}} \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) = e^{-K\nu \frac{1}{2}} \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) = e^{-K\nu \frac{1}{2}} \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right).
\]

By setting \( \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right) = e^{-K\nu \frac{1}{2}} \left( \frac{\chi''}{\chi} \omega^j, \partial_{\tau} \phi \right) \) for simplicity, we have
\[
e^{-K\nu \frac{1}{2}} \left( \frac{\chi'}{\chi} \omega^j, \partial_{\tau} \phi \right) = \left( \frac{\chi'}{\chi} \partial_{\tau} \phi \right) + 2\nu j_2 \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right) + \nu \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right) + \nu \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right) = \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right) + \nu \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right) + \nu \left( \frac{\chi''}{\chi} \partial_{\tau} \phi \right).
\]

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we then arrive at

\[ \nu^2 j_2 e^{-Kj_2} \left\langle \chi'_{\nu}(\partial_\tau + Kj_2)\omega^{(j_1,j_2-1)}, \partial_\tau \phi \right\rangle \]

\[ = \nu^2 j_2 \left\{ -\frac{1}{2} \left( \partial_\gamma (\chi'_{\nu} \chi_\nu) \partial_\tau (\nabla \phi)^{\bar{1}} - \partial_\tau (\nabla \phi)^{\bar{1}} \right) \right. \]

\[ - \nu^2 (j_2 - 1) \left( (\chi'_{\nu})^2 \partial_\tau (\nabla \phi)^{\bar{1}} - \partial_\tau (\nabla \phi)^{\bar{1}} \right) \]

\[ + 2\nu^2 j_2 (\chi'_{\nu})^2 \partial_\tau (\nabla \phi)^{\bar{1}} - \partial_\tau (\nabla \phi)^{\bar{1}} - \nu^2 (\chi''_{\nu} \partial_\tau (\nabla \phi)^{\bar{1}}, \chi_\nu \partial_\tau (\nabla \phi)^{\bar{1}}) \]

\[ + K\nu j_2 \left( (\chi'_{\nu}(\nabla \phi)^{\bar{1}}, \partial_\tau (\nabla \phi)^{\bar{1}}) \right) \]

\[ + 2\nu^2 j_2 (\chi''_{\nu}(\partial_\tau (\nabla \phi)^{\bar{1}}, \chi_\nu (\partial_\tau (\nabla \phi)^{\bar{1}})) \]

\[ \geq -C(\nu^2 j_2)^2 \left\| \partial_\tau (\nabla \phi)^{\bar{1}} \right\|^2 \]

\[ + K\nu j_2 \left( (\chi'_{\nu}(\nabla \phi)^{\bar{1}}, \partial_\tau (\nabla \phi)^{\bar{1}}) \right) + \nu^2 j_2 \left\| \chi''_{\nu}(\partial_\tau (\nabla \phi)^{\bar{1}}, \chi_\nu (\partial_\tau (\nabla \phi)^{\bar{1}}) \right\|^2 \]

\[ + \frac{\nu^2}{2} \partial_\tau (\chi''_{\nu}(\partial_\tau (\nabla \phi)^{\bar{1}}, \chi_\nu (\partial_\tau (\nabla \phi)^{\bar{1}}) \right). \]

(B.8)

Here we have used the fact that it suffices to consider the case \( j_2 \geq 1 \), and \( C \) is a universal constant. Hence, by going back to (B.7), we have

\[ \langle -\nu^2 (\Delta \omega)^{\bar{1}}, \partial_\tau \phi \rangle \geq \nu^2 \langle \omega^{\bar{1}}, \partial_\tau \phi \rangle - C(\nu^2 j_2)^2 \left\| \partial_\tau (\nabla \phi)^{\bar{1}} \right\|^2 \]

\[ + K\nu j_2 \left( (\chi'_{\nu}(\nabla \phi)^{\bar{1}}, \partial_\tau (\nabla \phi)^{\bar{1}}) \right) \]

\[ + \nu^2 j_2 \left\| \chi''_{\nu}(\partial_\tau (\nabla \phi)^{\bar{1}}, \chi_\nu (\partial_\tau (\nabla \phi)^{\bar{1}}) \right\|^2 \]

\[ - 4\nu^2 j_2 e^{-Kj_2} (\chi'_{\nu}(\text{div} \, H)^{(j_1,j_2-1)}, \partial_\tau \phi) \]

\[ + 4\nu j_2 \chi''_{\nu}(\omega^{(j_1+1,j_2-1)}, \partial_\tau \omega^{\bar{1}}) \]

\[ + 2\nu\chi''_{\nu}(\omega^{\bar{1}}, \partial_\tau \phi) + 2\nu\chi''_{\nu}(\omega^{\bar{1}}, \partial_\tau \phi) \]

(B.9)

Here \( C \) is a universal constant. Next we observe from \( \partial_\tau \phi = \chi_\nu \partial_\tau (\partial_\nu \phi)^{\bar{1}} \) that

\[ -4\nu^2 j_2 e^{-Kj_2} (\chi'_{\nu}(\text{div} \, H)^{(j_1,j_2-1)}, \partial_\tau \phi) \geq -CK\nu j_2 (\|H_1^{(j_1+1,j_2-1)}\| + \|H_2^{(j_1+1,j_2-1)}\|) \|\partial_\tau (\partial_\nu \phi)^{\bar{1}}\|, \]

(B.10)

and also

\[ 4\nu j_2 (\chi''_{\nu}(\omega^{(j_1+1,j_2-1)}, \partial_\tau \omega^{\bar{1}}) \geq -CK\nu j_2 \|\omega^{(j_1+1,j_2-1)}\| \|\partial_\tau \omega^{\bar{1}}\|, \]

(B.11)

\[ 2\nu\chi''_{\nu}(\omega^{\bar{1}}, \partial_\tau \phi) \geq -CK\nu j_2 \|\omega^{\bar{1}}\| \|\partial_\tau (\partial_\nu \phi)^{\bar{1}}\|. \]

(B.12)
Finally let us compute the term \( \nu^{\frac{1}{2}} (\frac{\nu}{\chi_\nu})^2 \omega J, \partial_\tau \phi J \) when \( j_2 \geq 1 \). If \( j_2 = 1 \) then
\[
\nu^{\frac{1}{2}} \langle (\frac{\nu}{\chi_\nu})^2 \omega J, \partial_\tau \phi J \rangle = \nu^{\frac{1}{2}} \langle (\chi_\nu')^2 e^{-K \nu^{\frac{1}{2}} \tau} (\partial_Y \omega)(J_1, 0), \partial_\tau (e^{-K \nu^{\frac{1}{2}} \tau} (\partial_Y \phi)(J_1, 0)) \rangle
\]
\[
= \nu^{\frac{1}{2}} \langle e^{-K \nu^{\frac{1}{2}} \tau} \nabla \partial_Y \phi(J_1, 0), \nabla (\chi_\nu')^2 \partial_\tau (e^{-K \nu^{\frac{1}{2}} \tau} (\partial_Y \phi)(J_1, 0)) \rangle
\]
\[
= \frac{1}{2} \nu^{\frac{1}{2}} \partial_\tau (\chi_\nu^2 e^{-K \nu^{\frac{1}{2}} \tau} \nabla \partial_Y \phi(J_1, 0)) \|^2
\]
\[
+ 2 \nu (\chi_\nu'^2 e^{-K \nu^{\frac{1}{2}} \tau} \partial_\nu^2 \phi(J_1, 0), \partial_\tau (e^{-K \nu^{\frac{1}{2}} \tau} (\partial_Y \phi)(J_1, 0)) \rangle
\]
\[
\geq \frac{1}{2} \nu^{\frac{1}{2}} \partial_\tau (\chi_\nu^2 e^{-K \nu^{\frac{1}{2}} \tau} \nabla \partial_Y \phi(J_1, 0)) \|^2 - C \nu^{3 \nu} \| (\omega)(J_1, 0) \| \| \partial_\tau (\partial_Y \phi) \| - 1 \|. \quad (B.13)
\]
If \( j_2 \geq 2 \) then
\[
\nu^{\frac{1}{2}} \langle (\frac{\nu}{\chi_\nu})^2 \omega J, \partial_\tau \phi J \rangle = e^{-2K \nu^{\frac{1}{2}} \tau} \nu^{\frac{1}{2}} \langle (\chi_\nu')^2 (\partial_\nu^2 \omega)(J_1, j_2 - 2), \partial_\tau \phi J \rangle,
\]
and then by using the identity \( \nu^{\frac{1}{2}} (\Delta \omega)(J_1, j_2 - 2) = (\partial_\tau + K \nu^{\frac{1}{2}} (j - 1))(\omega)(J_1, j_2 - 2) - (\text{div } H)(J_1, j_2 - 2) \), we have
\[
\nu^{\frac{1}{2}} \langle (\frac{\nu}{\chi_\nu})^2 \omega J, \partial_\tau \phi J \rangle = -\nu^{\frac{1}{2}} \langle (\chi_\nu')^2 \omega(J_1 + 2, j_2 - 2), \partial_\tau \phi J \rangle
\]
\[
+ e^{-2K \nu^{\frac{1}{2}} \tau} \langle (\chi_\nu')^2 (\partial_\tau + K \nu^{\frac{1}{2}} (j - 1))(\omega)(J_1, j_2 - 2), \partial_\tau \phi J \rangle
\]
\[
- e^{-2K \nu^{\frac{1}{2}} \tau} \langle (\chi_\nu')^2 (\text{div } H)(J_1, j_2 - 2), \partial_\tau \phi J \rangle. \quad (B.15)
\]
As for the second term of the right-hand side of (B), we have for \( j \geq j_2 \geq 2 \),
\[
e^{-2K \nu^{\frac{1}{2}} \tau} \langle (\chi_\nu')^2 (\partial_\tau + K \nu^{\frac{1}{2}} (j - 1))(\omega)(J_1, j_2 - 2), \partial_\tau \phi J \rangle
\]
\[
e^{-2K \nu^{\frac{1}{2}} \tau} \langle (\chi_\nu')^2 (\partial_\tau + K \nu^{\frac{1}{2}} (j - 1))(\partial_X \phi)(J_1, j_2 - 2), \partial_\tau \partial_X \phi J \rangle
\]
\[
e^{-2K \nu^{\frac{1}{2}} \tau} \langle (\chi_\nu')^2 (\partial_\tau + K \nu^{\frac{1}{2}} (j - 1))(e^{2K \nu^{\frac{1}{2}} \tau} (\partial_Y \phi)^{j - 1}), \partial_\tau (\partial_Y \phi)^{j - 1} \rangle
\]
\[
\geq -\kappa^2 \| (\partial_\tau \partial_X \phi)(J_1, j_2 - 2) \| + K \nu^{\frac{1}{2}} \| \partial_X \phi(J_1, j_2 - 2) \| \| \partial_\tau \partial_X \phi J \| \| \partial_\tau (\partial_Y \phi)^{j - 1} \| - \kappa^2 \| (\partial_\tau (\partial_Y \phi)^{j - 1} \| + K \nu^{\frac{1}{2}} \| (\partial_Y \phi)^{j - 1} \| \| \partial_\tau (\partial_Y \phi)^{j - 1} \|.
\]
Since it is straightforward to see that
\[
-\nu^{\frac{1}{2}} \langle (\chi_\nu')^2 \omega(J_1 + 1, j_2 - 2), \partial_\tau \phi J \rangle \geq -\kappa^2 \nu^{\frac{1}{2}} \| \omega(J_1 + 1, j_2 - 2) \| \| \partial_\tau (\partial_X \phi) J \|
\]
\[
-\kappa^2 \nu^{\frac{1}{2}} \| \omega(J_1 + 1, j_2 - 2) \| \| \partial_\tau (\partial_Y \phi)^{j - 1} \|, \quad (B.16)
\]

\[60\]
Collecting (B.9)-(B.12) with (B.13) for $j_2 = 1$ and (B.16) for $j_2 \geq 2$, we conclude the desired estimate by using the bound such as
\[
M_{2,j,1}[f]^2 = \sup_{|j|=j} \|f^j\|_{L^2(0, \frac{1}{\kappa v^2} ; \Omega) \setminus \Omega}^2 \leq \frac{1}{Kv^2} \sup_{|j|=j} \|f^j\|_{L^2(0, \frac{1}{\kappa v^2} ; \Omega)}^2 = \frac{1}{Kv^2} M_{\infty,j,1}[f]^2.
\]
The proof is complete.

As a consequence of Lemmas B.1 and B.2, we obtain

**Corollary B.1.** There exists $\kappa_B \in (0,1]$ such that for any $\kappa \in (0,\kappa_B]$ and $K \geq 1$,
\[
\nu^\frac{1}{2} \sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} M_{\infty,j,1}[\omega] + K^\frac{1}{2} \nu^\frac{1}{2} \sum_{j=0}^{\nu^\frac{1}{2}} \frac{(j+1)^\frac{1}{2}}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} M_{\infty,j,1}[\nabla \phi] + \sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} \nu^\frac{1}{2} M_{2,j,1}[\partial_r \nabla \phi]
\leq C \left( \nu^\frac{1}{2} \sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} \|\omega^j(0)\| + K^\frac{1}{2} \nu^\frac{1}{2} \sum_{j=0}^{\nu^\frac{1}{2}} \frac{(j+1)^\frac{1}{2}}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} \|\nabla \phi^j(0)\| + \sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} \nu^\frac{1}{2} M_{2,j,1}[H] \right).
\]
Here $C$ is a universal constant.

We note that
\[
\sum_{j=0}^{\nu^\frac{1}{2}} \nu^\frac{1}{2} (j+1)^\frac{1}{2} \|\nabla \phi^j(0)\| \leq C \sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} \|\nabla \phi^j(0)\| = C \|\nabla \phi(0)\|
\]
since $j \leq \nu^\frac{1}{2}$. In virtue of Corollary B.1 it remains to estimate $\sum_{j=1}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} M_{2,j,1}[H]$.

Recall that $H = -V \omega - \Omega \nabla^\perp \phi + (F_2, -F_1)$. Hence it suffices to show

**Lemma B.3.** For any $\kappa \in (0,1]$ and $K \geq 1$ we have
\[
\sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} M_{2,j,1}[\Omega \nabla \phi] \leq \frac{C(C_0^\perp + C_1^\perp)}{\nu^\frac{1}{2}} \|\nabla \phi\|_{L^2}, \tag{B.17}
\]
\[
\sum_{j=0}^{\nu^\frac{1}{2}} \frac{1}{(j!)^{\frac{1}{2}} \nu^\frac{1}{2}} M_{2,j,1}[V \omega] \leq \frac{C(C_0^\perp + C_1^\perp)}{\nu^\frac{1}{2}} \left( \|\Delta (\phi - \phi_{app,1})\|_{L^2} + \|\Delta \phi_{app,1}\|_{L^2} \right). \tag{B.18}
\]
Here $\phi_{app,1} = (\phi_{1,1} + \phi_{1,2})([I + R_{bc}]^{-1} h)$ with $h = -\partial_r \Phi_{slip}|_{y=0}$ and $C$ is a universal constant.

**Proof.** We give a sketch of the proof only for (B.18), for (B.17) is proved in the similar manner. Let $|j| = j$. Then
\[
\sum_{j=1}^{\nu^\frac{1}{2}} \frac{1}{j!^{\frac{1}{2}} \nu^\frac{1}{2}} M_{2,j,1}[V \omega] \leq \frac{1}{j!^{\frac{1}{2}} \nu^\frac{1}{2}} \max_{|j|=j} \sum_{1 \leq |j| \leq j} \left( \frac{1}{j!} \right) \|V^j \omega^j\|_{L^2(0, 1/\kappa v^2 ; L^2)}.
\]
Here $V^j = e^{-K\nu\frac{4}{3}j}B_j,\phi^j_1V$, while $\omega^j = e^{-K\nu\frac{4}{3}(j+1)}B_j,\phi^j_1\omega$. Since $\omega = -\Delta (\phi - \phi_{app,1}) - \Delta \phi_{app,1}$ in virtue of the construction, we have

$$
\|V^1_{\omega^j}1\|_{L^2(0, 1 \frac{1}{K\nu})}^{(1)} \leq \|V^1\|_{L^\infty} \|((\Delta (\phi - \phi_{app,1}))^{j-1})\|_{L^2(0, 1 \frac{1}{K\nu})} + \|\partial Y V^1\|_{L^\infty} \|Y(\Delta \phi_{app,1})^{j-1}\|_{L^2(0, 1 \frac{1}{K\nu})}.
$$

By using (1) with $l = 1$, we have

$$
\frac{1}{j!2^{\frac{j}{2} \nu} \frac{\nu}{3}} \sum_{l \leq j} \left( \begin{array}{c} j \cr l \end{array} \right) \|V^1_{\omega^j}1\|_{L^2(0, 1 \frac{1}{K\nu})}^{(1)} \leq \sum_{l \leq j} \frac{l!(j-l)!}{j!} \left( \frac{M_{2,j-l,1}[\Delta (\phi - \phi_{app,1})] + M_{2,j-l,1,1}[\Delta \phi_{app,1}]}{(j-l)!2^{\frac{j-l}{2} \nu} \frac{\nu}{3}} \right) \frac{1}{l!2^{\frac{l}{2} \nu} \frac{\nu}{3}} \max_{|l| = l} \left( \|V^1\|_{L^\infty} + \|\partial Y V^1\|_{L^\infty} \right).
$$

Next we observe that for all $l \in \mathbb{N} \cup \{0\}$,

$$
\#\{l | |l| = l, 1 \leq j\} = \#\{l, 2, \max(0, l - j + j_2) \leq l \leq \min(j_2, l) \} \leq \min(l + 1, l - j + l + 1),
$$

which gives the bound of the form $\sum_{l \leq j} \leq \sum_{l = 0}^{j} \min(l + 1, j - l + 1)$. Hence we have

$$
\frac{1}{j!2^{\frac{j}{2} \nu} \frac{\nu}{3}} \sum_{l \leq j} \left( \begin{array}{c} j \cr l \end{array} \right) \|V^1_{\omega^j}1\|_{L^2(0, 1 \frac{1}{K\nu})}^{(1)} \leq \sum_{l = 0}^{j} \min(l + 1, j - l + 1) \frac{l!(j-l)!}{j!} \left( \frac{M_{2,j-l,1}[\Delta (\phi - \phi_{app,1})] + M_{2,j-l,1,1}[\Delta \phi_{app,1}]}{(j-l)!2^{\frac{j-l}{2} \nu} \frac{\nu}{3}} \right) \frac{1}{l!2^{\frac{l}{2} \nu} \frac{\nu}{3}} \max_{|l| = l} \left( \|V^1\|_{L^\infty} + \|\partial Y V^1\|_{L^\infty} \right).$$

Since $\min(l + 1, j - l + 1) \frac{l!(j-l)!}{j!} \frac{1}{2^{\frac{j-l}{2} \nu} \frac{\nu}{3}}$ is uniformly bounded about $0 \leq l \leq j$, the Young inequality for $l^1$ convolution gives the inequality

$$
\sum_{j = 0}^{\nu^{-\frac{1}{2}}} \frac{1}{j!2^{\frac{j}{2} \nu} \frac{\nu}{3}} \max_{|l| = j} \sum_{l \leq j} \left( \begin{array}{c} j \cr l \end{array} \right) \|V^1_{\omega^j}1\|_{L^2(0, 1 \frac{1}{K\nu})}^{(1)} \leq C \sum_{j = 0}^{\nu^{-\frac{1}{2}}} \frac{1}{j!2^{\frac{j}{2} \nu} \frac{\nu}{3}} \max_{|l| = j} \left( \|V^1\|_{L^\infty} + \|\partial Y V^1\|_{L^\infty} \right)
$$

$$
\times \sum_{j = 0}^{\nu^{-\frac{1}{2}}} \frac{1}{j!2^{\frac{j}{2} \nu} \frac{\nu}{3}} \max_{|l| = l} \left( M_{2,j,1}[\Delta (\phi - \phi_{app,1})] + M_{2,j,1,1}[\Delta \phi_{app,1}] \right).$$

Then the desired estimate follows by noticing $\partial Y V^1 = (\partial Y V^1)^1 + 2^{\frac{1}{2}}j_2 \chi_1(\partial Y V^{(j_1,j_2)-1})$ and the bound of the form $\|f\|_2 \leq \nu^{-\frac{1}{2}} \|f\|_{2,1}$. The proof is complete.

Proposition 6.1 follows from Corollary B.1 and Lemma B.3.
Lemma C.1. The following statement holds if $\kappa$ is sufficiently small. Assume that $f \in C((0, 1/R); H^1(T \times \mathbb{R}_+))$ satisfies $\text{div} \ f = 0$ for $y > 0$ and $f_2|_{y=0} = 0$. Then

$$\|\nabla f\|_p \leq C \|\text{rot} \ f\|_p, \quad p \in [1, \infty].$$

Here $C$ is a universal constant.

Proof. We observe that $\partial_y f_1 = \text{rot} \ f + \partial_x f_2$ and $\partial_y f_2 = -\partial_x f_1$. Hence it suffices to show $\|\partial_x f\|_p \leq C \|\text{rot} \ f\|_p$. Since $f = \nabla^\perp \phi$ with the streamfunction $\phi$ and $-\Delta \phi = \omega$ with $\omega = \text{rot} \ g$ and $\phi|_{y=0} = 0$, we have

$$-(\Delta \partial_x \phi)^j = \partial_x \omega^j, \quad \omega^j = e^{-K(j+1)} \chi^{j_2} \partial_y^{j_2} \partial_x^{j_1} \omega, \quad j_1 + j_2 = j.$$ 

In virtue of the identity $-(\Delta \partial_x \phi)^j = -\nabla \cdot (\partial_x \nabla \phi)^j + j_2 \chi'(\partial_y \partial_x \phi)^j$ the integration by parts gives

$$\|\nabla \partial_x \phi\|^2 + 2 j_2 \chi'(\partial_y \partial_x \phi)^j, \partial_x \phi^j = -(\omega^j, \partial_x \phi^j)$$

Since $\partial_x \phi^j = e^{-K(j+1)} \chi^{j_2} \partial_y^{j_2} \partial_x^{j_1} \omega$ we thus have

$$\|\nabla \partial_x \phi\|^2 \leq C \|\omega\|^2 + \kappa j \|\partial_y \partial_x \phi\|^{j_1, j_2-1},$$

where $C$ is a universal constant. This estimate implies $\|\partial_x \nabla \phi\|_p \leq C \|\omega\|_p + \kappa \|\partial_y \partial_x \phi\|_p$, and thus, by taking $\kappa$ small enough, we obtain $\|\partial_x \nabla \phi\|_p \leq C \|\omega\|_p$. The proof is complete. \qed

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