An Efficient PTAS for Two-Strategy Anonymous Games

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Abstract. We present a novel polynomial time approximation scheme for two-strategy anonymous games, in which the players’ utility functions, although potentially different, do not differentiate among the identities of the other players. Our algorithm computes an $\epsilon$-approximate Nash equilibrium of an $n$-player 2-strategy anonymous game in time $\text{poly}(n) \cdot (1/\epsilon)^{O(1/\epsilon^2)}$, which significantly improves upon the running time $n^{O(1/\epsilon^2)}$ required by the algorithm of Daskalakis & Papadimitriou, 2007. The improved running time is based on a new structural understanding of approximate Nash equilibria: We show that, for any $\epsilon$, there exists an $\epsilon$-approximate Nash equilibrium in which either only $O(1/\epsilon^3)$ players randomize, or all players who randomize use the same mixed strategy. To show this result we employ tools from the literature on Stein’s Method.

1 Introduction

It has been recently established that computing a Nash equilibrium is an intractable problem [19, 11, 6, 14], even in the case of two-player games [7]. In view of this hardness result, research has been directed towards the computation of approximate Nash equilibria, which are states of the game in which no player has more than some small $\epsilon$ incentive to change her strategy. But, despite much research in this direction [23, 22, 12, 18, 13, 5, 28], only constant $\epsilon$’s can be achieved in polynomial time. Yet, an approximate Nash equilibrium in which the players have regret equal to a significant fraction of their payoffs is not an attractive solution concept; after all, there is no reason to expect a player to keep her strategy if she can significantly improve by changing to a different one. On the contrary, if $\epsilon$ were arbitrarily small, it could be that the cost of switching one’s strategy is larger than the regret $\epsilon$ that she suffers. Hence, approximate equilibria with arbitrarily close approximation could be credible solutions concepts. The following question then emerges: Is there a Polynomial Time Approximation Scheme for approximate Nash equilibria?

The question remains open for general games, but there are special classes known to be tractable. It is well-known, for example, that zero-sum games are solvable exactly in polynomial time by Linear Programming [25, 9]. This tractability result has been extended to a generalization of zero-sum games,

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called two-player low-rank games, in which the sum of the players’ payoff tables has fixed rank; in this case there is a PTAS for approximate Nash equilibria. It has also been shown that symmetric multi-player games with (about logarithmically) few strategies per player can be solved exactly in polynomial time by a reduction to the theory of real closed fields [26]. In congestion games, we can compute in polynomial time a pure Nash equilibrium, if the game is a symmetric network congestion game [17], and an approximate pure Nash equilibrium, if the congestion game is symmetric (but not necessarily network) and the utilities are somehow “continuous” [8].

In this paper, we consider another important class of games, called anonymous. These are games in which each player’s utility function does not differentiate among the identities of the other players. That is, the payoff of a player depends on the strategy that she chooses and only the number of other players choosing each strategy. Anonymous games comprise a broad and well studied class of games (see, e.g., [3, 4, 20, 24] for recent work on this subject by economists) which are of special interest to the Algorithmic Game Theory community, as they capture important aspects of auctions and markets, as well as of Internet congestion.

But, what do we know about computing Nash equilibria in anonymous games? It was recently established that there is a PTAS for the case of a constant number of strategies per player [15, 16]. The running time of the algorithm given in [16] is $n^{O(f(s, 1/\epsilon))}$, where $\epsilon$ is the desired approximation, $s$ the number of strategies available to the players, and $f$ some function which is polynomial in $1/\epsilon$, but superpolynomial in $s$. Hence, although theoretically efficient for any fixed $\epsilon$ and $s$, the algorithm is highly non-practical. Even for the simpler case of two-strategy anonymous games the running time achieved by [15] is $n^{O(1/\epsilon^2)}$.

In this paper, we present a more efficient algorithm for 2-strategy anonymous games, which runs in time $\text{poly}(n) \cdot (1/\epsilon)^{O(1/\epsilon^2)}$. The improved running time is due to a novel understanding of certain structural properties of approximate Nash equilibria. In particular, we show that, for any integer $k$, there exists an $\epsilon$-approximate Nash equilibrium, with $\epsilon = O(1/k)$, in which

(a) either at most $k^3 = O((1/\epsilon)^3)$ players use randomized strategies, and their strategies are integer multiples of $1/k^2$; \footnote{Note that, since every player has 2 strategies, a mixed strategy is a number in [0, 1].}

(b) or all players who randomize choose the same mixed strategy which is also an integer multiple of $1/k$.

To derive the above characterization, we study mixed strategy profiles in the proximity of a Nash equilibrium. We establish that there always exists a nearby mixed strategy profile which is of one of the types (a) or (b) described above and satisfies the Nash equilibrium conditions to within an additive $\epsilon$, thus corresponding to an $\epsilon$-approximate equilibrium. Given this structural result (see Theorem 1), an $\epsilon$-approximate equilibrium can be found by dynamic programming (see Theorem 2).