Periodic vortex lattices for the Lawrence–Doniach model of layered superconductors in a parallel field

S. Alama*, A.J. Berlinsky† & L. Bronsard*

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Abstract

We consider the Lawrence–Doniach model for layered superconductors, in which stacks of parallel superconducting planes are coupled via the Josephson effect. We assume that the superconductor is placed in an external magnetic field oriented parallel to the superconducting planes and study periodic lattice configurations in the limit as the Josephson coupling parameter $r \to 0$. This limit leads to the “transparent state” discussed in the physics literature, which is observed in very anisotropic high-$T_c$ superconductors at sufficiently high applied fields and below a critical temperature. We use a Lyapunov–Schmidt reduction to prove that energy minimization uniquely determines the geometry of the optimal vortex lattice: a period-2 (in the layers) array proposed by Bulaevskiĭ & Clem. Finally, we discuss the apparent conflict with previous results for finite-width samples, in which the minimizer in the small coupling regime takes the form of “vortex planes” (introduced by Theodorakis and Kuplevakhsky.)

*Dept. of Mathematics and Statistics, McMaster Univ., Hamilton, Ontario, Canada L8S 4K1. Supported by an NSERC Research Grant.
†Dept. of Physics and Astronomy, McMaster Univ., Hamilton, Ontario, Canada L8S 4K1. Supported by an NSERC Research Grant.
1 Introduction

In this paper we continue the analysis of layered superconductors in a parallel external magnetic field started in our previous paper [AlBeBr 00]. In the previous paper we considered the case of a superconducting sample of finite width; here we treat periodic solutions in an infinitely wide superconductor.

The Lawrence–Doniach model [LaDo 71] is a mesoscopic Ginzburg–Landau type model for superconducting materials with a planar layered structure, and was originally applied to study organic superconductors and other superconducting composites manufactured by depositing successive thin layers of superconducting material with interposing insulating films. Interest in this model has been spectacularly revived by the discovery of high temperature superconductors, since nearly all of these materials exhibit a distinctly layered structure. Indeed, a pure monocrystalline sample of cuprate high-$T_c$ material (such as Bi$_2$Sr$_2$CaCu$_2$O$_8$ (BSCCO), Tl$_2$Ba$_2$CaCu$_2$O$_8$ (TBCCO), or to a lesser extent YBa$_2$Cu$_3$O$_7$ (YBCO)) consists of copper oxide superconducting planes stacked with intervening insulating (or weakly superconducting) planes.

The Lawrence–Doniach model. We model the layered superconductor as an infinite stack of superconducting planes, each parallel to the $xy$-plane and with uniform separation distance $p$. The planes are thus described by those $(x,y,z)$ with $(x,y) \in \mathbb{R}^2$ and $z = z_n := np$, $n \in \mathbb{Z}$.

We impose an applied “external” magnetic field, of constant magnitude $H$ and lying parallel to the planes, along the $y$-direction, $\vec{H} = H\hat{y}$. We make the ansatz that the local magnetic field inside the sample will be everywhere independent of $y$ and point in the $y$-direction,

$$\vec{h}(x,y,z) = h(x,z) \hat{y}.$$ 

The vector potential $\vec{A}$ may then be chosen to lie in the $xz$-plane,

$$\vec{A}(x,y,z) = A_x(x,z) \hat{x} + A_z(x,z) \hat{z}, \quad \vec{h} = \text{curl} \vec{A} = \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y}.$$ 

We study configurations which are bi-periodic, in the sense that the physically observable quantities (the magnetic field, the density of superconducting electrons, and their currents) are doubly periodic functions in the $xz$-plane. More precisely, given $q > 0$, $N \in \mathbb{N}$, and $s \in \mathbb{R}$ we define the fundamental domain of periodicity to be the parallelogram $\Pi = \Pi_{N,s,q}$ spanned by the vectors $\vec{e}_1 = (2q,0)$ and $\vec{e}_2 = (s,Np)$:

$$\Pi := \{(x,z) = t_1 e_1 + t_2 e_2, \ 0 \leq t_1, t_2 \leq 1\}.$$ 

In other words, our fields and currents will be $2q$-periodic in $x$ (along each superconducting plane), and will repeat themselves after each $N$ planes but with a horizontal translation of
We denote $z_n = np$ and $x_n = ns/N$, $n \in \mathbb{Z}$, so the part of the $n$th SC plane which lies within the basic period module $\Pi$ is described by the points $(x, z_n)$ with $x_n \leq x \leq x_n + 2q$.

In each plane we define a (complex-valued) superconducting order parameter $\psi_n(x)$, $n \in \mathbb{Z}$. We choose units in which $|\psi_n| = 1$ represents a purely superconducting state. With the assumption that the physically observable quantities are $\Pi$-periodic we may measure the free energy over a single period to obtain the following functional,

$$G_{BP}^r(\psi_n, \bar{A}) = \frac{H^2}{4\pi} \left\{ \sum_{n=1}^{N} \int_{x_n}^{x_n+2q} \left[ \frac{1}{\kappa^2} \left| \frac{d\psi_n}{dx} - iA_x \right|^2 + \frac{1}{2} (|\psi_n|^2 - 1)^2 \right] dx + \frac{r^2}{2} \sum_{n=1}^{N} \int_{x_n}^{x_n+2q} \left| \psi_n - \psi_{n-1} \exp \left( i \int_{z_{n-1}}^{z_n} A_z(x, s) ds \right) \right|^2 dx ight. \right. \left. + \frac{1}{\kappa^2} \int_{\Pi} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - H \right)^2 dxdz \right\}. \tag{1}$$

Note that $G_{BP}^r$ is expressed in non-dimensional units, chosen such that the in-plane penetration depth $\lambda_{ab} = 1$, $\kappa = \lambda_{ab}/\xi_{ab}$ is the Ginzburg Landau parameter, $r$ is the interlayer coupling parameter (or Josephson coupling parameter) and the magnetic fields are measured in units of $H_c/\kappa$, where $H_c$ is the thermodynamic critical field. (See [1096].)

The coupling between the superconducting planes given by the second sum in $G_{BP}^r$ simulates the Josephson effect, by which superconducting electrons travel from one superconducting region to another by quantum mechanical tunnelling. We will see this explicitly in the Euler–Lagrange equations, where the currents in the gaps between planes will be determined by the sine of the gauge-invariant phase difference. The interlayer coupling parameter $r$ gives the strength of the Josephson coupling.

As in our previous study of the boundary-value problem for the Lawrence–Doniach system [1096], we will study the minimizers (and low-energy solutions) for $r$ near zero. Indeed, for the high-$T_c$ cuprates at a temperature sufficiently below $T_c$ we expect $r$ is a small parameter in the Lawrence–Doniach model. In particular, for BSCCO or TBCCO we expect $r(0) \approx 10^{-4}–10^{-3}$. (See “Physical background” below.)

**Results.** We find that for $r \sim 0$ there is an unique periodic solution of the Lawrence–Doniach system which attains the minimum energy (per unit area) among any other periodic configuration, with any period geometry. This absolute minimizing solution has fundamental domain $\Pi_{N, s, q}$ with $N = 1$, $s = q = \pi/Hp$: in other words, the currents and field are $2\pi/Hp$-periodic in $x$, and shifting from one plane to the next results only in a horizontal translation.
by a half-period $s = \pi/H_p$. For example, the magnetic field satisfies
\[ h \left( x + \frac{2\pi}{H_p}, z \right) = h(x, z) = h \left( x + \frac{\pi}{H_p}, z + p \right). \tag{2} \]
Clearly this condition implies that the magnetic field, superconducting electron density and currents are $2p$-periodic in $z$. In addition, the flux per period for this solution is exactly one quantum fluxoid, $2\pi$. This configuration is optimal in the following sense: each choice of period geometry and quantized flux determines a function space in which to minimize $G^{BP}_r$. The configuration described above attains the minimum of free energy per unit cross-sectional area among all possible choices of $\Pi_{N,s,q}$ and flux quantization. In fact our result says more: for any given geometry and flux, then for all $r$ sufficiently small the minimum energy per unit cross-sectional area in that class is either always strictly larger than the $\Pi_1 \frac{\pi}{H_p}, \frac{\pi}{H_p}$-bi-periodic solution above, or exactly equal. The energies coincide if and only if the given lattice $\Pi_{N,s,q}$ is commensurate with $\Pi_1 \frac{\pi}{H_p}, \frac{\pi}{H_p}$, and its minimizer coincides with the (unique) $\Pi_1 \frac{\pi}{H_p}, \frac{\pi}{H_p}$-minimizer described above. (See Theorem 4.1.)

The minimization among configurations and among geometries and fluxes is entirely the result of direct rigorous analysis of the Lawrence–Doniach system, without recourse to numerical approximation or further restrictions other than the periodic ansatz. This is not like the case for the Abrikosov lattice in the Ginzburg–Landau model near $H_{c2}$, where the energy-minimizing geometry and flux quantization were determined numerically by comparisons within a finite collection of configurations. Our result confirms the prediction of Bulaevski˘ı & Clem [BuCm 91], who claimed that for sufficiently large applied fields $H$ the energy minimizers should form a period-two vortex lattice satisfying an ansatz of the form (2).

We also consider the case of finitely many superconducting planes, with periodic observable quantities. The result we obtain is much the same as in the bi-periodic case: minimization of the free energy per period strip dictates the choice of period $2q = \frac{2\pi}{H_p}$, and the minimizing solutions are (to order $r$) $2p$-periodic in $z$ in the interior of the sample. (See Theorem 7.1.)

In both settings we obtain a “transparent state” as observed in experiments on BSCCO by Kes, Aarts, Vinokur, & ver der Beek [KAVB 90] (see “Physical background” below.) Superconductivity is essentially unaffected by the presence of strong magnetic fields, $|\psi_n| = 1 - O(r)$, while the magnetic field is virtually unscreened by the presence of the superconducting planes, $h = H + O(r)$. In particular, the order parameters are never zero, and the “vortices”, which correspond to the local maxima of the local magnetic field, fit entirely in the gaps between the superconducting layers. In the physics literature these are referred to as Josephson vortices as opposed to the Abrikosov vortices characterized by the vanishing order parameter in the Ginzburg–Landau model.
The juxtaposition of the results obtained for the finite-width samples in our previous paper [AlBeBr 00] with the result stated above may seem contradictory: in [AlBeBr 00] we prove that for any sample of finite width \(-L \leq x \leq L\) (and for \(r \sim 0\)) the unique absolute energy minimizer occurs when the Josephson vortices are aligned vertically in a “vortex planes” geometry. These solutions were described by the physicists Theodorakis [Th 90] and Kuplevakhsky [K 99], the latter claiming that they were the only solutions to the Lawrence–Doniach system. Our explanation for this duality rests on the dependence of the interval of validity of the perturbative regime \(r \sim 0\) on the length of the sample size: the larger the sample, the smaller the interval in \(r\) for which the results of [AlBeBr 00] are valid. We will provide a more complete comparison of the two solutions in section 8.

**Physical background.** A first attempt to model layered superconductors is by an *anisotropic Ginzburg–Landau* model, which treats the sample as a three-dimensional solid with anisotropic material parameters. In particular, the coherence length along the perpendicular to the planes \(\xi_c\) is assumed to be different from the value of the coherence length \(\xi_{ab}\) within the planes (see [Γ 90] or [CDG 95], for example) and the anisotropy is measured by an “effective mass ratio” \(\Gamma := \xi_{ab}^2/\xi_c^2\). For example Iye [I 92] cites values of \(\Gamma_{YBCO} \simeq 49\), \(\Gamma_{BSCCO} \simeq 3025\), and \(\Gamma_{TBCCO} > 90000\). For certain materials (such as YBCO) and temperatures close to the critical temperature \(T_c\) this approximation seems valid, but for more anisotropic superconductors the anisotropic Ginzburg–Landau model does not give a good qualitative or quantitative description of experimental observations.

Experimental evidence of the failure of the anisotropic Ginzburg–Landau model was observed by Kes, Aarts, Vinokur, and van der Beek [KAVB 90], in the situation where the sample is placed in a strong magnetic field oriented parallel to the superconducting planes. Experiments reveal a crossover between “three-dimensional” behavior (governed by the anisotropic Ginzburg–Landau model) and “two-dimensional” behavior when the temperature is lowered beyond a critical value \(T_{c0} < T_c\). The “two-dimensional” regime is characterized by a “magnetically transparent” state, in which the applied magnetic field penetrates completely between the planes, virtually unscreened by the superconductor. These observations are not in agreement with the anisotropic Ginzburg–Landau model, where the magnetic field is largely expelled from the bulk and isolated vortices appear in a triangular “Abrikosov lattice.” For highly anisotropic materials (such as BSCCO and TBCCO) the two-dimensional regime occurs within one degree Kelvin of \(T_c\), and an analysis of the transparent state requires a model which addresses the discrete nature of the material: the Lawrence–Doniach model.

To motivate the treatment of \(r\) as a small parameter, we re-write \(r\) in the original dimen-
sional coordinates,

\[ r = r(T) = \frac{2\xi_{ab}^2\lambda_{ab}^2}{\lambda_J^2\bar{\rho}^2} = \frac{2}{\lambda_J^2\kappa^2\bar{p}^2}, \]

where \( \lambda_J = \lambda_J \lambda_{ab} \), \( \bar{\rho} = p \lambda_{ab} \) are the actual physical Josephson penetration depth and separation distance. The length scale \( \lambda_J \) in the Lawrence–Doniach model is directly related to the effective mass ratio of the anisotropic Ginzburg–Landau model via \( \Gamma = \lambda_J^2 \). The temperature dependence of \( r(T) \) is determined by the conventional dependences of \( \xi_{ab}^2 \approx \xi_{ab}^2(0)\frac{T}{T_c} \) and \( \lambda_{ab}^2 \approx \lambda_{ab}^2(0)\frac{T}{T_c} \). For the high-\( T_c \) cuprates, \( \xi_{ab}(0) \) and \( \bar{p} \) are of the same order of magnitude, and hence for highly anisotropic superconductors at temperatures sufficiently below \( T_c \), \( r \sim \lambda_J^{-2} = \Gamma^{-1} \) is small. In particular, for BSCCO or TBCCO we expect \( r(0) \approx 10^{-4}, 10^{-3} \).

We note that Chapman, Du, & Gunzburger \[CDG 95\] have proven that solutions of the Lawrence–Doniach model converge to solutions of the anisotropic Ginzburg–Landau model (and in particular the convergence of energy minimizers) under the limit \( p \to 0 \) with \( \kappa, \Gamma \) fixed. This limit does not correspond to our “two-dimensional” regime, since it would send \( r \to \infty \) and is therefore far from the “weak coupling” of superconducting planes observed in \[KAVB 90\]. Indeed, condition (1) in \[KAVB 90\], which defines the dimensional cross-over point \( T = T_{c0} \), is equivalent to \( r(T_{c0}) \leq 1 \) in the Lawrence–Doniach model. On the other hand, by fixing \( \lambda_{ab} = 1 \) in our units the limit \( T \to T_c \) (from below) sends \( p = \bar{p}/\lambda_{ab}(T) \to 0 \). Therefore it is not surprising to recover “three-dimensional” behavior for \( p \to 0 \).

**Methods.** As in our previous study of the boundary-value problem for the Lawrence–Doniach system \[AlBeBr 00\] we will use a degenerate perturbation approach to study the minimizers (and low-energy solutions) \( r \) near zero. However, the periodic and bi-periodic settings are more subtle and present some additional complications in applying the method.

First, periodic magnetic fields and currents are generally represented by non-periodic order parameters and potentials \( (\phi_n, \vec{A}) \). In order to define a variational setting for the bi-periodic problem we adopt the idea of ‘t Hooft \[H 79\] to define spaces of functions which are periodic up to gauge transformation. (See section 2.) These spaces are rather complicated from the functional analytic point of view, and therefore it is important to find an equivalent formulation. We do this by making a gauge change (in section 3) to arrive at a family of affine Hilbert spaces representing the periodic configurations.

For the finite-width case considered in \[AlBeBr 00\] the crucial observation was that when \( r = 0 \) the planes decoupled, and the energy could be minimized explicitly. Even after gauge symmetries were removed the \( r = 0 \) problem exhibited an additional symmetry, corresponding to an \( (N-1) \)-dimensional torus action (where \( N \) denotes the number of superconducting planes,) and the minimization problem at \( r = 0 \) degenerated on a finite dimensional hyperplane in function space. When \( r \neq 0 \) this symmetry was broken and by a Lyapunov–Schmidt
decomposition we reduced the problem of finding solutions with \( r \approx 0 \) to a finite dimensional variational problem on this hyperplane. In this finite-width setting, the minimum value of energy was \( O(r) \).

In the bi-periodic case the \( r = 0 \) problem already dictates a choice of period and flux: the minimum energy for the configuration \( \Pi_{N,s,q} \) will be \( O(r) \) if and only if \( q = \frac{m \pi}{H_p} \), \( m \in \mathbb{N} \), and the flux through \( \Pi_{N,s,q} \) is exactly \( 2\pi m N \). (See Lemma 4.2.) Within this more restricted class of geometries the \( r = 0 \) problem can again be solved explicitly, and the same degenerate perturbation method as in [AlBeBr 00] applies to calculate the minimum energy as an expansion in \( r \) for \( 0 < r \ll 1 \). Unlike the finite-width case, this expansion yields a constant term at order \( r \), and the computation must be carried out to order \( r^2 \) in order to select a point on the degenerate hyperplane which minimizes energy with \( r \sim 0 \). The resulting finite dimensional minimization problem may be completely resolved to give a clear choice among the remaining parameters. (See Theorem 6.1.)

The case of finitely many layers is similar (and simpler) than the bi-periodic case; we present a sketch in section 7 of the modifications necessary to treat that setting.

As in [AlBeBr 00] we recognize that in a real superconductor the parameter \( r \), while small, is not infinitesimal, which raises the question of the validity of asymptotic results in the regime \( r \sim 0 \) for physically relevant parameter choices. In section 5 of [AlBeBr 00] we derived a lower bound on the radius of validity of the \( r \)-expansion in terms of the other parameters in the problem. The same analysis applies in the periodic problem, and we conclude that the radius of validity is enhanced with smaller period \( q \) and \( \kappa \), and larger external field \( H \). Since our optimal configuration ties the period \( q \) to the reciprocal of the field \( H \), we conclude that the result is most applicable in strong external fields. In particular, the expansion of the periodic problem is more likely to be applicable than the expansion obtained in [AlBeBr 00], since the validity of finite-width expansions deteriorates for large samples, whereas treating the sample as infinitely large shifts the dependence to the period, which can be small with large \( H \).

We briefly outline the organization of the paper: in the second section we introduce a functional analytic framework for the doubly periodic problem, based on the elegant formulation of gauge periodicity due to 't Hooft [H 79]. The third section defines an equivalent space, eliminating bothersome symmetries due to gauge and continuous translation invariance. In section 4 we show how minimization at \( r = 0 \) forces the correct choice of period and flux, and the fifth section reviews the perturbation method used in [AlBeBr 00]. For the periodic problems it is necessary to expand the energy to order \( r^2 \), and hence the solutions to the projected equations (coming from the Lyapunov–Schmidt reduction) must be calculated to order \( r \): these computations occupy section 6. The seventh section sketches the procedure
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2 Variational formulation.

We begin with the doubly periodic case: we adopt the notation of the introduction, and assume that the gauge invariant quantities (fields and currents) are periodic in each SC plane, and allow the pattern of fields and currents could repeat themselves with a horizontal shift in $x$ after each period in $n$. As before, for given $N \in \mathbb{N}$, $s \in \mathbb{R}$ and $q > 0$ we define the fundamental domain $\Pi = \Pi_{N,s,q}$ as in the introduction. We seek solutions to the LD system which are periodic with respect to the lattice generated by $\vec{e}_1, \vec{e}_2$,

$$h(x + 2q, z) = h(x, z) = h(x + s, z + Np),$$

and similarly for the other gauge-invariant quantities: the density of superconducting electrons $|\psi_n|$, the in-plane current density $j_{x}^{(n)}$, and the Josephson current density in the gaps, $j_{z}^{(n)}$, defined by

$$j_{x}^{(n)} := \text{Im} \left[ \psi^{*}_n \left( \frac{d}{dx} - iA(x, z_n) \right) \psi_n \right], \quad j_{z}^{(n)} := \frac{r\kappa^2 p}{2} \text{Im} \left[ \psi^{*}_n \left( \psi_n - \psi_{n-1} e^{i \int_{z_{n-1}}^{z_n} A(x, dz)} \right) \right].$$

Of course when $s = 0$ we would have simple periodicity in $z$, with period $Np$.

The subtlety of the periodic problems is that periodic magnetic fields and currents are generally represented by non-periodic potentials $\vec{A}$ and order parameter $\psi_n$. One setting for such periodic problems is via t’Hooft boundary conditions [tH 79], for which one demands that $\vec{A}$ and $\psi_n$ be periodic up to a family of gauge transformations from one period cell to the next. Let $N, q, s$ be fixed constants which determine the period lattice $\vec{e}_1, \vec{e}_2$ as above. We say that

$$(\psi_n, \vec{A}) \in \overline{\mathcal{H}} = \overline{\mathcal{H}}(N, s, q)$$

if $\psi_n \in H^1_{\text{loc}}(\mathbb{R}), \vec{A} \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and there exist two functions $\omega_x$ and $\omega_z$ in $H^2_{\text{loc}}(\mathbb{R}^2)$ such that:

$$\psi_n(x + 2q) = \psi_n(x) \exp[i\omega_x(x, z_n)], \quad \psi_{n+N}(x + s) = \psi_n(x) \exp[i\omega_z(x, z_n)],$$

$$\vec{A}(x + 2q, z) = \vec{A}(x, z) + \nabla \omega_x(x, z), \quad \vec{A}(x + s, z + Np) = \vec{A}(x, z) + \nabla \omega_z(x, z).$$

Since $\psi_{n+N}(x + 2q)$ can be related back to $\psi_n(x)$ in two different ways (by commuting the order in which the two rules in (3) are applied,) this imposes a constraint on the functions
\(\omega_x, \omega_z\) which are permissible in this definition. For any configuration \((\psi_n, \vec{A}) \in \overline{tH}\) there exists an integer \(K \in \mathbb{Z}\) such that

\[
\omega_x(x, z_n) - \omega_x(x + s, z_n + N) + \omega_z(x + 2q, z_n) - \omega_z(x, z_n) = 2\pi K.
\]  

(7)

Clearly, by continuity \(K\) may be chosen independent of \(x\); we will see below that it is also independent of \(n\). The important role played by \(K\) is revealed by integrating the magnetic field strength over the basic unit period cell, applying Stokes' Theorem, and substituting the identity (7):

\[
\iint_{\Pi} \text{curl} \vec{A} \, dx \, dz = \oint_{\partial \Pi} \vec{A} \cdot d\vec{s} = \omega_x(0, 0) - \omega_x(s, Np) + \omega_z(2q, 0) - \omega_z(0, 0) = 2\pi K.
\]  

(8)

Hence the total flux per period cell is quantized for any element of \(\overline{tH}\). Note also that the periodicity of \(h = \text{curl} \vec{A}\) implies that the constant \(K\) in (7) may indeed be chosen independent of \(n\).

Since every \((\psi, \vec{A}) \in \overline{tH}\) is associated to an integer \(K\) via a continuous map,

\[(\psi, \vec{A}) \rightarrow \frac{1}{2\pi} \iint_{\Pi} \text{curl} \vec{A} \, dx \, dz,
\]

this discrete choice separates the space \(\overline{tH}\) into disjoint connected components,

\[\overline{tH}(N, s, q) = \bigcup_{K \in \mathbb{Z}} tH(N, s, q, K)\].

We may of course minimize the Lawrence–Doniach energy in each component \(tH(N, s, q, K)\) separately, and indeed Lemma 4.2 below will indicate the optimal choice of \(K\), for appropriate \((N, s, q)\).

**Remark 2.1** These periodic configurations may also be described within the context of complex line bundles over the 2-torus \(T\), defined as \(\mathbb{R}^2\) modulo the discrete lattice generated by \(\vec{e}_1, \vec{e}_2\). A configuration \((\psi_n, \vec{A})\) defines a connection on \(T\) by interpreting \(\vec{A}\) as a one-form, while the \(\psi_n(x)\) are restrictions to the \(N\) planes in \(\Pi\) of a section of a complex line bundle over \(T\), with structure group \(U(1)\). The 't Hooft conditions incorporate the \(U(1)\) group action as we pass from one coordinate patch to another on the manifold \(T\). In this setting, the magnetic field \(h = \text{curl} \vec{A}\) is the curvature of the connection, and (8) is a Gauss-Bonnet relation between the total curvature and the Euler number \(K\), reflecting the fact that the bundle is nontrivial.
While this is a very elegant formulation of the periodic problem it poses some practical problems for analysis of the variational problem because of the introduction of the auxiliary functions \( \omega_x, \omega_z \) as well as the usual degeneracies associated with gauge symmetry. Fortunately we will be able to simplify the setting of the problem in two ways. First, by fixing a gauge we find a much simpler Hilbert manifold setting. This will be done in section 3. Second, we will show (see Lemma 4.2 below), that the least-energy solutions will have \( |\psi_n(x)| \approx 1 \), and therefore it will be convenient to use polar coordinates for \( \psi_n \) in order to treat the phases more directly. In the remainder of this section we introduce subspaces of \( \mathcal{H} \) for \( \psi_n \) in polar form and describe their properties and the Euler–Lagrange equations for critical points in these spaces.

To define the space \( \mathcal{H} \) in terms of \( f_n, \phi_n \), with \( \psi_n(x) = f_n(x)e^{i\phi_n(x)} \) we must take into account that the periodicity conditions on \( \phi_n \) can only hold modulo \( 2\pi \). For the second relation in (5) (translating in \( z \) from \( np \rightarrow (n+N)p \)) this is not important, since the constant factor of \( 2\pi \) can be absorbed into \( \omega_z \). The degree of winding per period in the \( x \)-direction plays a particularly important role, and hence we assume that there exist integers \( k_n \) such that

\[
\phi_n(x + 2q) = \phi_n(x) + \omega_z(x, z_n) + 2\pi k_n, \quad \phi_{n+N}(x + s) = \phi_n(x) + \omega_z(x, z_n).
\]

(9)

This definition does not uniquely determine the winding numbers \( k_n \), since adding the same constant multiple of \( 2\pi \) to each merely reduces the value of the function \( \omega_x(x, z) \) by that same quantity. To remedy this problem we may (without loss of generality) fix the value

\[
k_0 = 0.
\]

Returning to the calculation (7), we see that

\[
K = k_{n+N} - k_n = k_N - k_0 = k_N,
\]

and hence the index \( K \) associated to the space \( \mathcal{H}(N, s, q, K) \) measures the net change in the winding number of \( \phi_n \) over one period \( Np \) in \( z \). Of course the moduli of the order parameters is gauge-invariant, and is \( \Pi \)-periodic,

\[
f_n(x + 2q) = f_n(x) = f_{n+N}(x + s).
\]

(10)

We may now play the same game with the winding numbers \( k_n \) (\( n \in \mathbb{Z} \)) in the polar representation \( (f_n, \phi_n, \vec{A}) \) as we did with the single index \( K \) for the configurations \( (\psi_n, \vec{A}) \in \mathcal{H} \). Since \( (f_n, \phi_n, \vec{A}) \) determines \( (k_n)_{n \in \mathbb{Z}} \) via a continuous selection, the space of admissible configurations splits into disconnected classes. Fixing \( \vec{k} := (k_n)_{n \in \mathbb{Z}} \) (with \( k_0 = 0 \) and \( k_{n+N} - \)

\[
(5)
\]

\[
(6)
\]

\[
(7)
\]

\[
(8)
\]
\( k_n = K \) we define our basic space of bi-periodic configurations \( \mathcal{BP}(N, s, q, \vec{k}) \) by choosing the corresponding connected component of \( \overline{\mathcal{H}} \):

\[
\mathcal{BP}(N, s, q, \vec{k}) := \{ ([f_n, \phi_n], \vec{A}) : f_n, \phi_n \in H^1_{\text{loc}}(\mathbb{R}), \vec{A} \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2), \text{ and there exist } \omega_x, \omega_z \in H^2_{\text{loc}}(\mathbb{R}^2) \text{ such that } (\text{[1]}), (\text{[2]}), \text{ and } (\text{[3]} \text{ hold}). \}
\]

Each component \( \mathcal{BP}(N, s, q, \vec{k}) \) describes a subclass of \( \mathcal{H}(N, s, q, K) \) with \( K = k_N \), and therefore the flux quantization

\[
\int_{\Pi} \text{curl} \vec{A} \, dx \, dz = 2\pi K = 2\pi k_N
\]

holds for every configuration \( (f_n, \phi_n, \vec{A}) \in \mathcal{BP} \). As for \( \mathcal{H}(N, s, q, K) \), it will turn out that there are preferred classes \( \mathcal{BP}(N, s, q, \vec{k}) \) for which the free energy will be smallest possible. (See Lemma 4.2 below.)

It is a simple calculation to verify that for any configuration \( (\psi_n, \vec{A}) \in \mathcal{H}(N, s, q, K) \) (and for \( (f_n, \phi_n, \vec{A}) \in \mathcal{BP}(N, s, q, \vec{k}) \)) we recover the desired periodicity conditions (see (3)) for the gauge-invariant quantities. Since only the gauge-invariant quantities appear in an expression of the free energy density, the energy density will be \( \Pi \)-periodic and hence we are justified in measuring the energy of the configuration over only one period cell. We may therefore define the Gibbs free energy \( \mathcal{G}^{BP}_r \) for the configuration \( (\psi_n, \vec{A}) \in \mathcal{H}(N, s, q, K) \) as in the introduction. When \( \psi_n \) can be represented by polar coordinates \( f_n, \phi_n \) we use the equivalent form:

\[
\Omega^{BP}_r(f_n, \phi_n, \vec{A}) = p \sum_{n=1}^{N} \int_{x_n}^{x_{n+2q}} \left[ \frac{1}{2}(f_n^2 - 1)^2 + \frac{1}{\kappa^2}(f'_n)^2 + \frac{1}{\kappa^2}(\phi'_n - A_z(x_n, y))^2 f_n^2 \right] \, dx
\]

\[
+ \frac{r}{2}p \sum_{n=1}^{N} \int_{x_n}^{x_{n+2q}} \left( f_n^2 + f_{n-1}^2 - 2 f_n f_{n-1} \cos(\Phi_{n,n-1}) \right) \, dx
\]

\[
+ \frac{1}{\kappa^2} \int_{\Pi} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - H \right)^2 \, dx \, dz,
\]

where \( \Phi_{n,n-1}(x) \) is the gauge invariant phase difference,

\[
\Phi_{n,n-1}(x) := \phi_n(x) - \phi_{n-1}(x) - \int_{x_{n-1}}^{x_n} A_z(x, z) \, dz.
\]

We observe that \( \Phi_{n,n-1}(x) \) is periodic modulo \( 2\pi \),

\[
\Phi_{n,n-1}(x + 2q) = \Phi_{n,n-1}(x) + 2\pi(k_n - k_{n-1}).
\]
Remark 2.2 Note that although we have defined \((f_n, \phi_n, \vec{A}) \in \mathcal{BP}(N, s, q, \vec{k})\) globally in \(\mathbb{R}^2\), in fact the gauge-invariant quantities \(f_n, \Phi_{n,n-1}, V_n := (\phi'_n - A(x, z_n))\), and \(h(x, z)\) are globally determined by the values of \(f_n(x), \phi_n(x)\), and \(\vec{A}(x, z)\) for \(n = 1, \ldots, N\) and \((x, z) \in \Pi\), and the values of \(\omega_z(x, 0)\). This is very clear for \(f_n, V_n, \) and \(h;\) for \(\Phi_{n,n-1}\), we require \(\phi_0(x)\) to determine \(\Phi_{1,0}(x)\), and it is here that the 't Hooft condition comes into play via the function \(\omega_z(x, 0)\). In retrospect, we could have defined \((f_n, \phi_n, \vec{A})\) only on the period module \(\Pi\) and \(\omega_x, \omega_z\) only in a neighborhood of the left and bottom edges of \(\Pi\). This would be enough to make sense of the free energy, and it allows interpretation of the 't Hooft conditions as true boundary conditions,

\[
\psi_n(x_n + 2q) = \psi_n(x_n) \exp[i\omega_x(x_n, z_n)], \quad n = 1, \ldots, N;
\]
\[
\psi_N(x + s) = \psi_0(x) \exp[i\omega_z(x, 0)], \quad 0 < x < 2q;
\]
\[
\vec{A}(x + 2q, z) = \vec{A}(x, z) + \nabla \omega_x(x, z), \quad 0 < x < s,
\]
\[
\vec{A}(x + s, Np) = \vec{A}(x, 0) + \nabla \omega_z(x, 0), \quad 0 < x < 2q.
\]

This is the interpretation of the 't Hooft conditions taken by Tarantello \[\text{Ta 96}\] for example. ♦

The Euler–Lagrange equations under 't Hooft conditions are virtually identical to those derived in \[\text{AlBeBr 00}\] except that we must take into account the periodic boundary conditions both in \(x\) and \(z\). To obtain the equations we may choose smooth genuinely \(\Pi\)-periodic test functions to vary each unknown in turn. We obtain:

\[
-\frac{1}{\kappa^2} f_n'' + (f_n^2 - 1)f_n + \frac{1}{\kappa^2}(\phi'_n - A_x(x, z_n))^2 f_n
\]
\[
= \frac{r}{2}(f_{n-1} \cos \Phi_{n,n-1} + f_{n+1} \cos \Phi_{n+1,n} - 2f_n);
\]
\[
\frac{\partial h}{\partial z}(x, z) = 0 \quad z \neq z_n;
\]
\[
h(x, z_n+) - h(x, z_n-) = -pf_n^2(x)(\phi'_n - A_x(x, z_n));
\]
\[
\frac{\partial h}{\partial x} = \frac{r\kappa^2p}{2}f_n(x)f_{n-1}(x) \sin \Phi_{n,n-1}(x), \quad \text{for} \quad z_{n-1} < z < z_n.
\]

These equations should be solved together with \(\Pi_{N,s,q}\)-periodic boundary conditions for the observable quantities, \(f_n(x), h(x, z), V_n(x) := (\phi'_n(x) - A_x(x, z_n))\), and \(\sin \Phi_{n,n-1}(x)\). For later purposes, we also record the current conservation equation

\[
\frac{1}{\kappa^2} \frac{d}{dx} \left(f_n^2(\phi'_n - A_x(x, z_n))\right) = \frac{r}{2}(f_nf_{n-1} \sin \Phi_{n,n-1} - f_{n+1}f_n \sin \Phi_{n+1,n}).
\]

This equation is not independent of the others: it can be derived by differentiating \[\text{[13]}\] and substituting from \[\text{[14]}\]. This is not surprising since \(\phi_n\) and \(\vec{A}\) are related through gauge
invariance. We note that
\[ j^{(n)}_x(x) = f^2_n(x)(\phi'_n(x) - A(x, z_n)) \]
measures the current density within the \( n^{th} \) superconducting plane, while
\[ j^{(n)}_z(x) = \frac{r\kappa^2 p}{2} f_n(x) f_{n-1}(x) \sin \Phi_{n,n-1}(x) \]
gives the Josephson current density in the gap between the \((n-1)^{st}\) and \( n^{th} \) planes. In this way we may view (15) as a semi-discrete version of the classical equation of continuity \( \text{div} \vec{j} = 0 \). This is the conservation law corresponding to the \( U(1) \) gauge invariance as guaranteed by Noether’s Theorem.

We also record a useful formula for \( \Phi_{n,n-1} \) which follows from Stokes’ Theorem:
\[ \Phi_{n,n-1}(x) = \int_0^x (V_n - V_{n-1}) \, dx + p \int_0^x h^{(n)}(\vec{x}) \, d\vec{x} + \Phi_{n,n-1}(0). \] (16)

3 Fixing a gauge

While the spaces \( t\mathcal{H}(N, s, q, K) \) and \( \mathcal{BP}(N, s, q, \vec{k}) \) are indeed unwieldy for the purposes of analysis, we will show that by fixing an appropriate gauge we can, without loss of generality, work in a much simpler setting. We begin by noting that the periodic problem has a larger symmetry group than the fixed interval problem studied in the previous paper [AlBeBr 00]. In addition to electromagnetic gauge invariance,
\[ f_n \rightarrow f_n, \quad \phi_n(x) \rightarrow \phi_n(x) - \lambda(x, z_n), \quad \vec{A} \rightarrow \vec{A} - \nabla \lambda(x, z), \]
there is also translation invariance,
\[ (f_n, \phi_n, \vec{A}) \rightarrow (f_n(\cdot - x_0), \phi_n(\cdot - x_0), \vec{A}(\cdot - x_0, \cdot)), \quad x_0 \in \mathbb{R}. \]

While this last symmetry is only one-dimensional, for our degenerate perturbation theory it is convenient to eliminate all but the essential symmetries of the \( r = 0 \) problem which are broken when \( r \neq 0 \), and so our choice of “gauge” will fix a translation as well. Note that there is also a discrete translation invariance in \( z, z \rightarrow z - kp, k \in \mathbb{Z} \), but this symmetry (being discrete) will not create analytical difficulties for our method and hence plays a less important role.

First consider the spaces \( \mathcal{BP}(N, s, q, \vec{k}) \) for which \( \vec{k} \neq \vec{0} \). (Recall that we have fixed \( k_0 = 0 \) in the definition of \( \mathcal{BP}(N, s, q, \vec{k}) \).) By the discrete translation invariance in \( z \) we may relabel the \( z \)-axis if necessary in order to obtain
\[ k_1 \neq 0 = k_0. \] (17)
We observe that the average value of the local magnetic field is fixed by the choice of the space $\mathcal{BP}(N, s, q, \vec{k})$: indeed by (8),

$$\langle h \rangle := \frac{1}{2qNp} \iiint h(x, z) \, dx \, dz = \frac{\pi K}{pq N},$$

where $h = \text{curl} \vec{A}$.

Next we define an appropriate space which eliminates gauge and translation symmetries. Assume again that $\vec{k} \neq \vec{0}$. We say that $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}_*(N, s, q, \vec{k})$ provided there are constants $\omega, d \in \mathbb{R}$ such that

$$\vec{A} \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2), \ f_n \in H^1_{\text{per}}(\mathbb{R}), \ \phi_n \in H^1_{\text{loc}}(\mathbb{R});$$

$$\phi_n(x + 2q) = \phi_n(x) + \omega + 2\pi k_n;$$

$$\phi_{n+N}(x + s) = \phi_n(x) + \frac{k_N \pi}{q} x + d;$$

$$f_n(x + 2q) = f_n(x) = f_{n+N}(x + s);$$

$$\vec{A}(x, z) = \langle h \rangle (z, 0) + (\partial_z \xi, -\partial_x \xi), \ \text{where}$$

$$\xi \in H^2_{\text{loc}}(\mathbb{R}^2), \ \xi(x + 2q, z) = \xi(x + s, z + Np) = \xi(x, z);$$

$$\int_{-q}^{q} \phi_0(x) \, dx = 0 = \int_{-q}^{q} \phi_1(x) \, dx.$$  

As usual, we may choose a definition of the phases so that

$$0 \leq \phi_n(0) < 2\pi \text{ for each } n \in \mathbb{Z}. \ (25)$$

In the case $\vec{k} = \vec{0}$, we define $\mathcal{BP}_*(N, s, q, \vec{0})$ via (19)–(23), and we demand only the vanishing of the mean value of $\phi_0(x)$ in (24) (with no restriction on the mean of $\phi_1(x)$).

It is easy to see that $\mathcal{BP}_*(N, s, q, \vec{k}) \subset \mathcal{BP}(N, s, q, \vec{k})$. We must show that the two spaces are equivalent, in the sense that for any configuration in $\mathcal{BP}(N, s, q, \vec{k})$ there is a gauge transformation which sends it to an element of $\mathcal{BP}_*(N, s, q, \vec{k})$. In particular, this implies that minimization of $\Omega_r^{BP}$ in $\mathcal{BP}_*(N, s, q, \vec{k})$ is equivalent to minimization of $\Omega_r^{BP}$ in $\mathcal{BP}(N, s, q, \vec{k})$. In addition, we will show that the choice of gauge in $\mathcal{BP}_*(N, s, q, \vec{k})$ is a “Coulomb” gauge, in the sense that the $H^1(\Pi)$ norm of $\vec{A}$ is controlled by its curl in $L^2(\Pi)$.

**Theorem 3.1** Let $N, s, q, \vec{k}$ be fixed.
(a) There exists a constant $C_0 > 0$ such that for any $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}_*(N, s, q, \vec{k})$, 
\[ \| \vec{A} \|^2_{H^1(\Pi)} \leq C_0 \| h \|^2_{L^2(\Pi)}, \]
where $\Pi$ is the period module.

(b) For any $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}(N, s, q, \vec{k})$ there exists $\lambda \in H^2_{loc}(\mathbb{R}^2)$ and $x_0 \in \mathbb{R}$ such that 
\[ \left( f_n(x - x_0), \phi_n(x - x_0) - \lambda(x - x_0, z_n), \vec{A}(x - x_0, z) - \nabla \lambda(x - x_0, z) \right) \in \mathcal{BP}_*(N, s, q, \vec{k}). \]

When $\vec{k} = \vec{0}$ the choice of $x_0$ in (b) is immaterial: we cannot remove translation invariance in that case.

**Proof of Theorem 3.1:**

Suppose $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}_*$, with $\vec{A}$ defined as in (23). Then $\Delta \xi = h - \langle h \rangle$ in $\mathbb{R}^2$, and therefore the estimate follows from standard elliptic regularity theory.

To prove (b), assume $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}(N, s, q, \vec{k})$ and let $\xi$ be a solution to the periodic problem
\[ \left\{ \begin{array}{l} \Delta \xi = \text{curl } \vec{A} - \langle h \rangle, \\
\xi(x + 2q, z) = \xi(x, z) = \xi(x + s, z + Np), \end{array} \right. \]  
(26)
and $\vec{A} = (\partial_z \xi, -\partial_x \xi)$. By standard elliptic regularity theory $\xi \in H^2_{loc}(\mathbb{R}^2)$. Then curl $(\vec{A} - \vec{A}) = 0$ so that there exists $\hat{\lambda} \in H^2_{loc}(\mathbb{R}^2)$ with $\vec{A} = \vec{A} - \nabla \hat{\lambda}$. In fact $\hat{\lambda}$ is only determined up to a constant; set $\lambda = \hat{\lambda} - c$, with $c$ as yet undetermined.

Define 
\[ \tilde{\phi}_n := \phi_n - \lambda(x, z_n). \]
It follows that 
\[ \tilde{\phi}_n(x + 2q) - \tilde{\phi}_n(x) = \omega_x(x, z_n) - \lambda(x + 2q, z_n) + \lambda(x, z_n) + 2\pi k_n. \]  
(27)
Since $\vec{A}$ is periodic in $x$, we have (for all $z$) 
\[ \nabla \omega_x(x, z) - \nabla \lambda(x + 2q, z) + \nabla \lambda(x, z) = 0, \]
and hence there exists a constant of integration $\omega$ so that 
\[ \omega_x(x, z) - \lambda(x + 2q, z) + \lambda(x, z) = \omega \]
for all $(x, z) \in \mathbb{R}^2$. Substituting this identity in (27) we verify (20) for $\tilde{\phi}_n$. Similarly 
\[ \tilde{\phi}_{n+N}(x + s) - \tilde{\phi}_n(x) = \omega_z(x, z_n) - \lambda(x + s, z_n + Np) + \lambda(x, z_n), \]
and since 
\[ \vec{A}(x + s, z + Np) - \vec{A}(x, z) = (Np, 0) \langle h \rangle \]
it follows that there is another constant of integration $d$ such that

$$\omega_z(x, z) - \lambda(x + s, z + Np) + \lambda(x, z) = Np \langle h \rangle x + d$$

for all $(x, z) \in \mathbb{R}^2$. In particular using the quantization formula (18), it follows

$$\tilde{\phi}_{n+N}(x + s) - \tilde{\phi}_n(x) = \frac{K \pi}{q} x + d,$$

and (21) is satisfied.

To complete the argument in case $\vec{k} \neq \vec{0}$, we define

$$g(t) := \int_{t-q}^{t+q} \left( \tilde{\phi}_1(x) - \tilde{\phi}_0(x) \right) dx.$$

Recalling (17), $k_0 = 0 \neq k_1$, so (20) implies that $\tilde{\phi}_1(x) - \tilde{\phi}_0(x) \sim \frac{k_1 \pi}{q} x$ for $|x|$ large. Therefore $g(t) \to \pm \infty$ as $t \to \pm \infty$, or vice-versa. Since $g(t)$ is continuous there exists $x_0 \in \mathbb{R}$ so that $g(x_0) = 0$. We now choose the constant term in $\lambda$ to be

$$c = \frac{1}{2q} \int_{x_0-q}^{x_0+q} \tilde{\phi}_1(x) dx = \frac{1}{2q} \int_{x_0-q}^{x_0+q} \tilde{\phi}_0(x) dx,$$

and the conclusion of (b) follows with $\lambda$ the desired gauge change and $x_0$ the translation.

In case $\vec{k} = \vec{0}$, we cannot assert the existence of such a translation $x_0$, but we can still choose $c = \frac{1}{2q} \int_{-q}^{q} \phi_0(x) dx$ to obtain $\tilde{\phi}_0$ with zero mean in $[-q, q]$.

The spaces $\mathcal{BP}_*(N, s, q, \vec{k})$ are quite reasonable from the analytical point of view. First, note that each $\phi_n(x)$ is the superposition of a linear function (with slope $\omega + 2\pi k_n$) with a $2q$-periodic function, and hence $\phi'_n(x)$ is $2q$-periodic. Since

$$\omega = \phi_1(x_1 + 2q) - \phi_1(x_1) - 2\pi k_1 \quad \text{and} \quad d = \frac{1}{2q} \int_{-q}^{+q} \phi_N(x) dx,$$

by the conditions (20), (21) the quantity

$$\sum_{n=1}^{N} \left\| \phi_n \right\|_{H^1[x_n, x_n + 2q]}^2$$

controls the $H^1_{loc}(\mathbb{R})$ norms of any finite collection of $\phi_n$, $n \in \mathbb{Z}$.

Indeed, for any choice of parameters $\mathcal{BP}_*(N, s, q, \vec{k})$ forms an affine Hilbert space. The tangent space (of admissible variations to $(f_n, \phi_n, \vec{A}) \in \mathcal{BP}_*$) is the space $E = E(N, s, q)$ consisting of all $(u_n, v_n, \vec{a})$ such that
(1) \( u_n, v_n \in H^1_{\text{loc}}(\mathbb{R}) \quad (n \in \mathbb{Z}) \) and there exist constants \( \omega, d \in \mathbb{R} \) such that

\[
\begin{align*}
    u_n(x + 2q) &= u_n(x) = u_{n+N}(x + s), \\
    v_n(x + 2q) &= v_n(x) + \omega, \quad v_{n+N}(x + s) = v_n(x) + d.
\end{align*}
\]

(2) \( \vec{a} = (\partial_x \xi, -\partial_z \xi) \) for some \( \xi \in H^2_{\text{loc}}(\mathbb{R}^2) \) with

\[
\xi(x + 2q, z) = \xi(x, z) = \xi(x + s, z + Np).
\]

In view of the above remarks on controlling the extension of elements of \( \mathcal{BP}^*_r(N, s, q, \vec{k}) \) beyond the basic period \( \Pi \), we may choose a norm on \( E \) of the form

\[
\| (u_n, v_n, \vec{a}) \|^2_E = \sum_{n=1}^{N} \left[ \| u_n \|^2_{H^1([x_n, x_n+2q])} + \| v_n \|^2_{H^1([x_n, x_n+2q])} \right] + \iint_\Pi [\text{curl} \vec{a}]^2 \, dx \, dz.
\]

It is clear that \( \Omega^{BP}_r \) is a smooth \((C^\infty)\) functional on \( \mathcal{BP}^*_r(N, s, q, \vec{k}) \). Since any configuration in \( \mathcal{BP}(N, s, q, \vec{k}) \) can be associated to an element in \( \mathcal{BP}^*_r(N, s, q, \vec{k}) \) (via gauge transformation and translation) with the same energy, the minimizer of \( \Omega^{BP}_r \) in \( \mathcal{BP}^*_r(N, s, q, \vec{k}) \) will also minimize \( \Omega^{BP}_r \) in \( \mathcal{BP}(N, s, q, \vec{k}) \), and hence satisfy the Lawrence–Doniach system of equations with periodic conditions.

Finally, we note that the same procedure may be used to fix a gauge in the space \( t\mathcal{H}(N, s, q, K) \) of doubly gauge-periodic configurations \((\psi_n, \vec{A})\), to obtain a nice space \( t\mathcal{H}^*_r(N, s, q, K) \) which eliminates gauge invariance. Indeed, we say \((\psi_n, \vec{A}) \in t\mathcal{H}^*_r(N, s, q, K)\) if:

(a) \( \psi_n \in H^1_{\text{loc}}(\mathbb{R}) \) and \( \vec{A} \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \);

(b) \( \vec{A} \) satisfies (23);

(c) there exist constants \( \omega, d \in \mathbb{R} \) such that

\[
\begin{align*}
    \psi_n(x + 2q) &= \psi_n(x) \exp[i \omega], \\
    \psi_{n+N}(x + s) &= \psi_n(x) \exp \left[ i \left( \frac{K \pi}{q} x + d \right) \right].
\end{align*}
\]

A version of Lemma 3.1 holds true for spaces \( t\mathcal{H}^*_r(N, s, q, K) \): for any \((\psi_n, \vec{A}) \in t\mathcal{H}(N, s, q, K)\) there exists \( \lambda \in H^2_{\text{loc}}(\mathbb{R}^2) \) such that

\[
(\psi_n \exp(i \lambda(\cdot, z_n)), \vec{A} - \nabla \lambda) \in t\mathcal{H}^*_r(N, s, q, K).
\]

And for any \((\psi_n, \vec{A}) \in t\mathcal{H}^*_r(N, s, q, K)\) the estimate of Lemma 3.1 (a) also holds.
4 Minimization at $r = 0$

We are ready to begin the process of solving for the lowest-energy periodic solution to the Lawrence–Doniach system. As we have already seen the periodic problems are indexed by several parameters: the number of layers in the period module $N$; the geometry of the period parallelogram $\Pi$ represented by the horizontal period along the SC layers $2q$ and the horizontal shift $s$ when advancing $N$ planes in $z$; and the winding numbers $k_n$, which we will denote collectively by $\vec{k}$. It will turn out that the geometry of the vortex lattice will be completely determined by energy minimization!

Denote the minimum free energy per unit cross-sectional area over all admissible lattice configurations in $\mathcal{BP}_*(N, s, q, \vec{k})$ by

$$\epsilon_r(N, s, q, \vec{k}) := \inf \left\{ \frac{\Omega^{BP}_r(f_n, \phi_n, \vec{A})}{2qNp} : (f_n, \phi_n, \vec{A}) \in \mathcal{BP}_*(N, s, q, \vec{k}) \right\}.$$ 

Our goal is to minimize twice: first for each fixed period geometry $\Pi_{N,s,q}$ and choice of winding numbers $\vec{k}$, and then to find the parameters $N, s, q, \vec{k}$ which give the least energy per unit cross sectional area. The main result we will prove is that, for $0 < r << 1$, the periodic solution with the least free energy among all possible lattice geometries is a period-2p in $z$ lattice, with period $2q = \frac{2\pi}{Hp}$ in $x$. We denote by $\mathbf{Z}$ the integer sequence, $k_n = n$ in the following statement of our theorem:

**Theorem 4.1** For any choice of $(N, s, q, \vec{k})$ there exists $\hat{r} = \hat{r}(N, s, q, \vec{k})$ such that either:

$$\epsilon_r(N, s, q, \vec{k}) > \epsilon_r(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbf{Z}) \text{ for all } r < \hat{r};$$

or

$$\epsilon_r(N, s, q, \vec{k}) = \epsilon_r(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbf{Z}) \text{ for all } r < \hat{r},$$

in which case the minimizers are achieved in $\mathcal{BP}_*(N, s, q, \vec{k})$, and they coincide with the period-2p in $z$, period $2q = \frac{2\pi}{Hp}$ in $x$ minimizers in $\mathcal{BP}_*(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbf{Z})$.

To completely clarify the above statement, we note that the magnetic field and the supercurrents $j^{(x)}_z, j^{(n)}_z$ associated to elements of the space $\mathcal{BP}_*(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbf{Z})$ make a horizontal translation of a half-period $q_1 := \frac{\pi}{Hp}$ when we increase $z$ by its period $p$:

$$f_n(x + 2q_1) = f_n(x) = f_{n+1}(x + q_1), \quad V_n(x + 2q_1) = V_n(x) = V_{n+1}(x + q_1),$$

$$h(x + 2q_1, z) = h(x, z) = h(x + q_1, z + p),$$

$$\Phi_{n,n-1}(x + 2q_1) - 2\pi = \Phi_{n,n-1}(x) = \Phi_{n+1,n}(x + q_1),$$

$$\Phi_{n,n-1}(x + 2q_1) = \Phi_{n,n-1}(x) = \Phi_{n+1,n}(x + q_1).$$

(28)
where we recall $V_n(x) = (\phi'_n(x) - A_x(x, z_n))$. It is clear that these solutions are also $2p$-periodic in $z$. These solutions have the period structure proposed by Bulaevskii & Clem [BuCm 91]. In section 6 we will show that the associated local magnetic field $h(x, z)$, in-plane and Josephson currents $j^{(n)}_x, j^{(n)}_z$ and density of superconducting electrons $f_n$ admit an expansion near $r = 0$ of the form:

$$
\begin{aligned}
  h(x, z) &= H + r(-1)^{n+1} \frac{k^2}{2H} \cos(Hpx) + O(r^2), \quad z_{n-1} < z < z_n, \\
  j^{(n)}_x(x) &= r(-1)^{n+1} \frac{k^2}{H} \cos(Hpx) + O(r^2), \\
  j^{(n)}_z(x) &= \frac{1}{2} r(-1)^n k^2 p \sin(Hpx) + O(r^2), \\
  f_n(x) &= 1 - \frac{r}{2} + O(r^2).
\end{aligned}
$$

The proof of Theorem 4.1 will be accomplished in many steps, concluding at the end of section 6. The remainder of this section is devoted to the study of the $r = 0$ problem. Section 5 introduces a degenerate perturbation method based on a Lyapunov–Schmidt decomposition at $r = 0$, and section 6 solves the reduced problems arising from the decomposition.

**Choosing the period and winding numbers.** The first step in proving Theorem 4.1 is to consider the case $r = 0$, in which case the superconducting planes decouple. We know from the previous treatment of the finite-width problem in [AlBeBr 00] that the case $r = 0$ is analogous to the self-dual point $\kappa = 1/\sqrt{2}$ in the Ginzburg–Landau model in the sense that at that point minimizers satisfy a first-order Bogomolnyi system, in addition to the (second-order) Euler–Lagrange equations. Indeed, we will see that the reduction to a first-order system of equations is only possible when the period $q$ and the winding numbers $\vec{k}$ are chosen appropriately. For a generic choice of $q, \vec{k}$ the minimum of energy will be much larger.

**Lemma 4.2** For $r = 0$,

(a) $\inf\{G_{0BP}^B(\psi_n, \vec{A}) : (\psi_n, \vec{A}) \in t\mathcal{H}_+(N, s, q, K)\} = 0$ if and only if there exists $m_0 \in \mathbb{N}$ such that $Hpq = m_0\pi$ and $K = m_0N$.

(b) Assume $Hpq = m_0\pi$ and $K = m_0N$, $m_0 \in \mathbb{N}$. For any $r \geq 0$ we have

$$
\inf\{G_{rBP}^B(\psi_n, \vec{A}) : (\psi_n, \vec{A}) \in t\mathcal{H}_+(N, s, q, K)\} \leq \frac{H^2}{2\pi} N q p r.
$$

Moreover, there exist constants $r_0, C > 0$ such that whenever $G_{rBP}^B(\psi_n, \vec{A}) \leq \frac{H^2}{2\pi} N q p r$, and $0 \leq r < r_0$, then $|\psi_n(x)| \geq 1 - Cr^{1/2} > 0$ for all $x, n$. 

Proof: We begin with (a), (c). Assume \( q = \frac{m_0\pi}{Hp} \), \( k_n = m_0n \), \( K = k_N = m_0N \), with \( m_0 \in \mathbb{N} \), and choose
\[
  f_{n0} \equiv 1, \quad \phi_{n0} = \alpha_n + nHpx, \quad \psi_{n0} = e^{i\phi_{n0}}, \quad \vec{A}_0(x, z) = (Hz, 0),
\]
for any \( \alpha_n \in \mathbb{R} \). It is easy to see that \( (f_{n0}, \phi_{n0}, \vec{A}_0) \in \mathcal{BP}_*(N, s, q, \vec{k}) \), \( (\psi_{n0}, \vec{A}_0) \in i\mathcal{H}_*(N, s, q, K) \), with \( \mathcal{G}_0^{\mathcal{BP}}(\psi_{n0}, \vec{A}_0) = 0 \) and \( \Omega_0^{\mathcal{BP}}(f_{n0}, \phi_{n0}, \vec{A}_0) = 0 \).

To see the opposite implications, assume first that \( \inf_{\mathcal{BP}_*(N, s, q, \vec{k})} \Omega_0^{\mathcal{BP}} = 0 \). By a simple estimate (see Proposition 2.1 of [AlBeBr 00]) any minimizing sequence must have \( f_n \to 1 \) uniformly. It is then easy to see that the energy controls the norm of the minimizing sequence in \( \mathcal{BP}_* \), and the minimizing sequence converges strongly to \( (\hat{f}_n, \hat{\phi}_n, \vec{A}) \in \mathcal{BP}_*(N, s, q, \vec{k}) \), and attains the minimum of zero energy. Being a sum of positive terms, \( \Omega_0^{\mathcal{BP}}(f_{n0}, \phi_{n0}, \vec{A}_0) = 0 \) implies that each term is individually zero,
\[
  f_n = 0, \quad \phi'_n - A_x(x, z_n) = 0, \quad \text{curl} \vec{A} = H.
\]

The vector potential \( \vec{A} = (Hz, 0) \) is uniquely determined by the gauge given by \( \mathcal{BP}_*(N, s, q, \vec{k}) \), and integration of the second relation leads to the solution space described by (30) above, with \( \alpha_n \in \mathbb{R} \) undetermined constants of integration. Therefore the question reduces to determine when the family (30) belongs to \( \mathcal{BP}_*(N, s, q, \vec{k}) \). Applying condition (24) we obtain
\[
  2Hpqn - \omega = 2\pi k_n, \quad \text{for all } n \in \mathbb{Z}.
\]

Since we have defined \( k_0 = 0 \), it follows that \( \omega = 0 \) and therefore \( Hpq = m\pi \) and \( k_n = mn \) for some \( m \in \mathbb{Z} \). The exact same argument applies to prove the analogous statement in (a).

To prove (b), first observe that the upper bound on the infimum follows from inserting the test configuration \( \psi_n, \vec{A} \) above into \( \mathcal{G}_r^{\mathcal{BP}} \). Then we use the simple inequalities,
\[
  \left| \frac{d}{dx} \psi_n \right|^2 \leq \left( \frac{d}{dx} - iA_x(x, z_n) \right) \psi_n^2 \leq (1 - |\psi_n(x)|^2)^2 \leq (1 - |\psi_n(x)|^2)^2,
\]
to obtain
\[
  p \sum_{n=1}^N \|1 - |\psi_n|\|^2_{H^1[x_n, x_{n+2q}]} \leq \kappa^2 p \sum_{n=1}^N \int_{x_n}^{x_{n+2q}} \left\{ (1 - |\psi_n|^2)^2 + \frac{1}{\kappa^2} \left( \frac{d}{dx} - iA_x(\cdot, z_n) \right) \psi_n^2 \right\} dx \leq \frac{4\pi}{H^2} \mathcal{G}_r^{\mathcal{BP}}(\psi_n, \vec{A}) \leq 2Nqpr.
\]
By the Sobolev embedding theorem in one dimension we conclude that for \( r \) small enough there exists a constant \( C \) with \(|\psi_n(x)| \geq 1 - Cr^{1/2}\) and (b) is proven.

Lemma 4.2 justifies certain restrictions on the family of spaces \( \mathcal{BP}_*(N, s, q, \vec{k}) \) when seeking the absolute minimizers of energy. In particular, in the remainder of the proof we fix
\[
q = q_m := m \frac{\pi}{H p}, \quad m \in \mathbb{N}, \quad \vec{k} = m \mathbb{Z}.
\]
Furthermore, statement (b) justifies the treatment of the problem in terms of the polar coordinates \( \psi_n(x) = f_n(x) \exp(i \phi_n(x)) \), which in turn allows us to restrict the winding numbers \( k_n \) as above. In the space \( \mathcal{BP}_*(N, s, q_m, m \mathbb{Z}) \) the applied flux per period per plane is \( 2\pi m \), and from (8) the total flux through a period cell is
\[
\int \int_{\Pi} \text{curl} \, \vec{A} \, dz \, dx = 2\pi N m.
\]
Therefore, for any configuration in \( \mathcal{BP}_*(N, s, q_m, m \mathbb{Z}) \) the mean magnetic field in a period cell coincides with the applied magnetic field:
\[
\langle h \rangle = \frac{1}{2q_m N p} \int \int_{\Pi} \text{curl} \, \vec{A} \, dz \, dx = H.
\]
This suggests that we are indeed in the “transparent” state referred to in the Introduction.

Note in addition that the choice \( m = 1 \) gives the smallest \( q \) for which the minimum energy is \( O(r) \) for \( r \) small, and \( 2q_1 \) corresponds to the minimal period of the explicit minimizer (30) of the \( r = 0 \) problem. Furthermore, the condition \( \vec{k} = \mathbb{Z} \) will imply that there is one Josephson vortex in between each adjacent pair of SC planes, per period in \( x \). We will see that the minimizing configuration in \( \mathcal{BP}_*(N, s, q_m, m \mathbb{Z}) \) will in fact have minimal period \( 2q_1 \), and hence reside in the space \( \mathcal{BP}_*(N, s, q_1, \mathbb{Z}) \).

In summary, with the above restrictions on the period and winding numbers, the spaces of interest are
\[
\mathcal{BP}_* \left( N, s, \frac{m\pi}{H p}, m \mathbb{Z} \right).
\]
We recall that this space is defined as all \((f_n, \phi_n, \vec{A})\) satisfying (19), (22), (23), and (24) with \( q = q_m \) and \( \langle h \rangle = H \), and
\[
\phi_n(x + 2q_m) = \phi_n(x) + 2\pi mn; \quad \phi_{n+N}(x + s) = \phi_n(x) + NpH x + d.
\]

**Identifying the solution set at \( r = 0 \).** Having restricted our choice of period and winding numbers, thanks to Lemma 4.2 we may now identify the manifold of all minimizers to the
$r=0$ problem, and verify that the setting is appropriate to apply the degenerate perturbation theory as developed in [AmCzE 87], [AmBa 98], [AlBeBr 00].

**Proposition 4.3** Assume $q=q_m$, $\vec{k}=m \in \mathbb{Z}$ for some $m \in \mathbb{N}$, and $r=0$.

(a) $\inf \{ \Omega_{0}^{BP}(f_n, \phi_n, \vec{A}) : (f_n, \phi_n, \vec{A}) \in BP_{s}(N, s, q_m, m\mathbb{Z}) \} = 0$. The minimum value is attained, and the set of all minimizers is the $(N-1)$-dimensional hyperplane

$$S := \{ (f_n, \phi_n, \vec{A}) \in BP_{s}(N, s, q_m, m\mathbb{Z}) : f_n \equiv 1, \phi_n(x) = \alpha_n + n H p x, A_x = H z, A_z = 0, \text{ where } \alpha_0 = 0 = \alpha_1, (\alpha_2, \ldots, \alpha_N) \in \mathbb{R}^{N-1}. \}$$

(b) For any element $\sigma = (f^0_n, \phi^0_n, \vec{A}^0) \in S$, the linearized operator $D^2\Omega_{0}^{BP}(\sigma) : E \rightarrow E$ defines a Fredholm operator with index zero. Moreover,

$$T_\sigma S = \ker D^2\Omega_{0}^{BP}(\sigma) \simeq \mathbb{R}^{N-1}. \quad (34)$$

Recall that the space $E = E(N, s, \frac{m \pi}{pH})$ is the tangent space to $BP_{s}(N, s, q_m, m\mathbb{Z})$, and was defined at the end of section 3. We define the second variation of energy as a quadratic form on $E$: in particular, for $(f_n^0, \phi_n^0, \vec{A}^0) \in S$ and $(u_n, v_n, \vec{a}) \in E$,

$$D^2\Omega_{0}(f_n^0, \phi_n^0, \vec{A}^0)[u_n, v_n, \vec{a}] = p \sum_{n=1}^{N} \int_{x_n-q_m}^{x_n+q_m} \left\{ 2u_n^2 + \frac{1}{\kappa^2} [u'_n]^2 + \frac{1}{\kappa^2} [v'_n - a_n(x, z_n)]^2 \right\} dx$$

$$+ \frac{1}{\kappa^2} \int_{\Pi} |\text{curl } \vec{a}|^2 dx dz.$$

The proof of Proposition 4.3 (a) follows easily from the proof of Lemma 4.2 (c), and part (b) is clear from the form of $D^2\Omega_{0}(f_n^0, \phi_n^0, \vec{A}^0)$ above (see Proposition 2.1 in [AlBeBr 00] for details.)

Note that $\mathcal{T} = T_\sigma S$ is independent of $\sigma \in S$, and that we may treat $S$ as a (compact) $(N-1)$-torus $\mathbb{T}^{N-1}$, since the energy is $2\pi$-periodic in each $\alpha_n$. Applying the condition (21) to an element of $S$ we obtain the periodicity conditions for $\alpha_n$ and $\delta_n := \alpha_n - \alpha_{n-1}$:

$$\alpha_{N+n} - \alpha_n = d - H p s (n + N), \quad \delta_{N+n} - \delta_n = -H p s \pmod{2\pi}, \quad (35)$$

where $d$ is the constant which appears in the ’t Hooft-type periodicity condition (22). Since we are given $\alpha_0 = 0 = \alpha_1$ in the definition of the space $BP_{s}(N, s, q_m, m\mathbb{Z})$, (35) provides the relation $\alpha_N = d - H N p s$. This reveals how a choice of $(\alpha_2, \ldots, \alpha_N)$ determines all $\alpha_n$: $\alpha_N$ determines the constant $d$, and the first equation of (35) then generates all the others in the sequence.

We also note that the $\alpha_n$ ($n=2, \ldots, N$) may be used as parameters for the manifold $S$. However, it will be more convenient in the end to parametrize $S$ by the phase differences,

$$\delta_n := \alpha_n - \alpha_{n-1}.$$
Since the values of \((\delta_2, \ldots, \delta_N)\) determine \((\alpha_2, \ldots, \alpha_N)\) uniquely, this is an equivalent choice of parametrization. We abuse notation and denote
\[
\sigma = \sigma(\delta_2, \ldots, \delta_N) \in \mathcal{S},
\]
\((\delta_2, \ldots, \delta_N) \in \mathbb{R}^{N-1}.
\]

5 Degenerate perturbation theory.

We now perturb away from the degenerate minima of \(\Omega^{BP}_0\), using a variational Lyapunov–Schmidt procedure, just as in [AlBeBr 00]. This method has been used by Ambrosetti, Coti-Zelati, & Ekeland [AmCzE 87], Abrosetti & Badiale [AmBa 98], Li & Nirenberg [LN 98] (and many others) in a variety of situations involving heteroclinic solutions of Hamiltonian systems and in the semiclassical limit of the nonlinear Schrödinger equation.

Since \(\mathcal{S}\) is a hyperplane, \(\mathcal{T} = T_\sigma \mathcal{S}\) is independent of \(\sigma \in \mathcal{S}\). Let \(W = \mathcal{T}^\perp\), so any \((f_n, \phi_n, \vec{A}) \in \mathcal{BP}_s(N, s, q_m, m\mathbb{Z})\) admits the unique decomposition \((f_n, \phi_n, \vec{A}) = \sigma + w\) with \(\sigma \in \mathcal{S}\), \(w \in W\), and any \(U := (u_n, v_n, \vec{a}) \in E\) decomposes uniquely as \(U = t + w\) with \(t \in \mathcal{T}\), \(w \in W\). We denote the orthogonal projection maps \(P : E \to \mathcal{T}\), \(P^\perp : E \to W\) so that \(PU = t\), \(P^\perp U = w\) when \(U = t + w\). We exploit the Hilbert space setting and interpret the first variation \(\nabla \Omega^{BP}_r(f_n, \phi_n, \vec{A})\) as an element of \(E\) itself, and project the equation \(\nabla \Omega^{BP}_r(f_n, \phi_n, \vec{A}) = 0\) into the two linear subspaces \(\mathcal{T}\) and \(W\),
\[
F_1(r, \sigma, w) := P\left[\nabla \Omega^{BP}_r(\sigma + w)\right] = 0; \quad (36)
\]
\[
F_2(r, \sigma, w) := P^\perp \left[\nabla \Omega^{BP}_r(\sigma + w)\right] = 0. \quad (37)
\]
The second equation can be solved uniquely for \(w = w(r, \sigma)\) in a neighborhood of \(\mathcal{S}\) for \(r\) small, using the Implicit Function Theorem. Because our functional \(\Omega^{BP}_r\) is smooth we can expand \(w(r, \sigma)\) in powers of \(r\), and since \(\Omega^{BP}_r(\sigma + w)\) is periodic in \(\sigma\) the expansion is uniform \(\sigma\). The result below is based on Lemma 2 of [AmBa 98], and follows directly from Lemma 3.1, [AlBeBr 00].

**Lemma 5.1** Assume \(q = q_m = \frac{m\pi}{2\kappa}, \vec{k} = m\mathbb{Z}\) for some \(m \in \mathbb{N}\).

There exist constants \(r_0 > 0\) and \(\delta > 0\), depending on \(N, m, \kappa, H, s\) and a smooth function \(w = w(r, \sigma) : (-r_0, r_0) \times \mathcal{S} \to W \subset E\) such that:

(i) There exists smooth functions \(w_1, w_2\) such that
\[
w(r, \sigma) = rw_1(\sigma) + r^2 w_2(r, \sigma)
\]
for all \(|r| < r_0\) and for all \(\sigma \in \mathcal{S}\);
(ii) \( P^\perp [\nabla \Omega^\text{BP}_r(\sigma + w(r, \sigma))] = 0 \).

(iii) Conversely, if \( P^\perp [\nabla \Omega^\text{BP}_r(\sigma + w)] = 0 \) for some \( r \in (-r_0, r_0) \) and \( w \in W \) with \( \|w\|_E < \delta \), then \( w = w(r, \sigma) \).

(iv) For any choice of \( m_0, \kappa_0, s_0, H_0 > 0 \) the constant \( r_0 \) may be chosen uniformly for all \( N \geq 1, 1 \leq m \leq m_0, 1 \leq \kappa \leq \kappa_0, H \geq H_0 \) and \( |s| \leq s_0 \).

Parts (i)–(iii) follow easily from the Implicit Function Theorem. The dependence on the various parameters in (iv) is more delicate, and was the subject of section 5 of [AlBeBr 00]. In that section we proved a lower bound on \( r_0 \) via \textit{a priori} estimates on the solutions of (37).

We note that the interval of validity of the expansion in [AlBeBr 00] was strongly affected by the sample width \( L \), with improved convergence with smaller \( L \). In the periodic problem the same method as in section 5 of [AlBeBr 00] can be used to obtain the same lower bound on the radius of validity, but with the period \( q = q_m \) replacing \( L \) in the expression of the lower bound. Since \( 2q_m \) decreases with increasing field strength \( H \) we can expect our solutions to have a large range of validity in high fields.

We define \( S_r := \{ \sigma + w(r, \sigma) : \sigma \in S \} \).

\( S_r \) is a smooth manifold parametrized by the hyperplane \( S \). The important role played by \( S_r \) is that it is a natural constraint for \( \Omega^\text{BP}_r \) (see Lemma 4 of [AmBa 98]), and hence the equation (36) may be solved variationally:

**Lemma 5.2** Assume \( q = q_m = \frac{m\pi}{Hp}, \tilde{k} = mZ \) for some \( m \in \mathbb{N} \).

(a) If \( (f_n, \phi_n, \vec{A}) \in S_r \) satisfies \( D(\Omega^\text{BP}_r|_{S_r})(f_n, \phi_n, \vec{A}) = 0 \), then \( \nabla \Omega^\text{BP}_r(f_n, \phi_n, \vec{A}) = 0 \) in \( E \).

(b) There exists a constant \( \tilde{r}_0 = \tilde{r}_0(N, m, \kappa, s, H) \) with \( 0 < \tilde{r}_0 < r_0 \) such that if \( (f_n, \phi_n, \vec{A}) \in B\mathcal{P}_s(N, s, q_m, mZ) \) is a critical point of \( \Omega^\text{BP}_r \) with

\[
\Omega^\text{BP}_r(f_n, \phi_n, \vec{A}) \leq 2q_mNpr = \frac{2\pi mN}{H} r \quad \text{and} \quad |r| < \tilde{r}_0,
\]

then \( (f_n, \phi_n, \vec{A}) \in S_r \).

The consequence of this lemma is very important: for \( r \) sufficiently small the absolute minimizer will be found by minimization on the finite dimensional manifold \( S_r \). We note that the parameter dependences for the interval of validity \( r_0 \) in Lemma 5.1 do not carry through to Lemma 5.2. This is because we have no estimate on the neighborhood \( \delta \) of Lemma 5.1 in terms of other parameters, and so we cannot conclude that configurations with very small energy must be “close enough” to \( S \) to be in the range of Lemma 5.1 (iii).

The proof of Lemma 5.2 is identical to Lemma 3.2 in [AlBeBr 00] and is omitted.
6 Vortex lattice solutions.

We are now ready to treat the finite dimensional problem \((36)\), via minimization of the constrained functional \(\Omega_r^{BP}|_{S_r}\). As opposed to the finite-width case studied in \cite{AlBeBr00} the functional \(\Omega_r^{BP}|_{S_r}\) will degenerate at order \(r\), and therefore it will be necessary to carry out the expansion to higher order. To accomplish this we must calculate an expansion of the solutions of the regular projected equation \((37)\) to order \(r\). Fortunately, the regularity of \(\Omega_r^{BP}\) and the Implicit Function Theorem allow us to do the computation explicitly, and we will obtain a straight-forward expansion of \(\Omega_r^{BP}|_{S_r}\), which permits direct minimization.

We summarize our conclusion in the following Theorem:

**Theorem 6.1** Let \(q_m = \frac{m\pi}{tp}, \vec{k} = m\mathbf{Z}, m \in \mathbb{N}\).

(i) For every \(s \in \mathbb{R}\) and \(m = 1, 2, \ldots\), there exists \(r_0 = r_0(N, s, \kappa, m, H) > 0\) such that for all \(r \in (0, r_0)\) the minimizer of \(\Omega_r^{BP}\) in \(BP_s(N, s, q_m, m\mathbf{Z})\) is a \(\Pi_{N,s,q_m}\)-periodic solution given asymptotically by

\[
\begin{align*}
 f_n = 1 + r \left[ -\frac{1}{2} + \frac{\pi^2}{2(\pi^2 + 2\kappa^2)} \left( \cos(\delta_n + Hpx) + \cos(\delta_{n+1} + Hpx) \right) \right] + o(r), \\
 h(x, z) = H - r \frac{\kappa^2}{\pi^2} \cos(\delta_n + Hpx) + o(r) \\
 j_x^{(n)}(x) = r \frac{\kappa^2}{\pi^2} \left( \cos(\delta_{n+1} + Hpx) - \cos(\delta_n + Hpx) \right) + o(r) \\
 j_z^{(n)}(x) = \frac{r}{2} \kappa^2 p \sin(\delta_n + Hpx) + o(r),
\end{align*}
\]

where \((\delta_2, \ldots, \delta_N)\) is a minimizer of the finite dimensional problem

\[
F(N, s) := \inf \left\{ \frac{1}{N} \sum_{n=1}^{N} \cos(\delta_n - \delta_{n+1}) : (\delta_2, \ldots, \delta_N) \in \mathbb{R}^{N-1}, \delta_1 = 0, \delta_{N+1} = -Hps \right\}.
\]

Moreover, the minimum energy is given by:

\[
\inf_{BP_s(N, s, q_m, m\mathbf{Z})} \Omega_r^{BP} = 2m q_1 N p \left( r + r^2 (C_0 + C_1 F(N, s)) \right) + O(r^3),
\]

where \(q_1 = \frac{\pi}{tp}\) and \(C_0, C_1 > 0\) are constants independent of \(N, s, q, m\).

(ii) \(\inf \{ F(N, s) : s \in \mathbb{R}, N = 1, 2, \ldots \} = -1\), and the minimum is attained at \((N, s)\) if and only if either:

- \(N\) is even and \(s = 2\ell q_1\) for \(\ell \in \mathbb{Z}\);
- or if

\(N\) is odd and \(s = (2\ell + 1)q_1\) for \(\ell \in \mathbb{Z}\).

In either case, the minimizer of \(\Omega_r^{BP}\) in \(BP_s(N, s, q_m, m\mathbf{Z})\) is unique and coincides with the period-2p in \(z\), period 2\(q_1\) in \(x\) lattice which minimizes \(\Omega_r^{BP}\) in \(BP_s(1, q_1, q_1, \mathbf{Z})\), satisfying \cite{28} and given asymptotically by \cite{23}.\]
We observe that the independence of the constants \( C_0, C_1 \) with respect to \( N, s, q, m \) means that the dependence of the geometry of \( \Pi \) on the energy per cross-sectional unit area is entirely encoded in the function \( F(N, s) \). In the generic case (i), \( F(N, s) \) could have several absolute minimizers \( (\delta_2, \ldots, \delta_N) \), and choosing different sequences of \( r \to 0 \) could lead to minimizers of \( \Omega_{rBP}|_{S_r} \) which are determined by different minimizers of \( F(N, s) \). In the special case (ii) the absolute minimizer of \( F(N, s) \) is unique (and non-degenerate) and we obtain the smallest possible energy for the Lawrence–Doniach energy, corresponding to the second alternative in Theorem 4.1.

Proof of Theorem 6.1: By Lemma 5.1, we may decompose an element of \( \mathcal{BP}_s(N, s, q_m, m\mathbb{Z}) \) as \( (f_n, \phi_n, \vec{A}) = \sigma + w(r, \sigma) \), with \( \sigma \in S \) and \( w \in W = [TS]^\perp \subset E \) such that the Euler-Lagrange equations hold when projected into the space \( W \). Furthermore, we may write \( w(r, \sigma) = rw_1(\sigma) + r^2w_2(r, \sigma) \), and in \( (f_n, \phi_n, A_x, A_z) \) coordinates, we denote

\[
\begin{align*}
    w_1 &= (u_{n,1}, v_{n,1}, a_{x,1}, a_{z,1}).
    \end{align*}
\]

In other words, recalling (33),

\[
\begin{align*}
    f_n &= 1 + ru_{n,1} + O(r^2), \\
    A_x &= Hz + ra_{x,1} + O(r^2), \\
    \phi_n &= \alpha_n + nHpx + v_{n,1} + O(r^2), \\
    A_z &= ra_{z,1} + O(r^2).
\end{align*}
\]

Note also that \( w_1(\sigma) = \partial_r w(0, \sigma) \).

Step 1: Expansion of the energy.

We recognize that the energy can be written in two parts, \( \Omega_{rBP}(U) = \Omega_{0BP}(U) + rG(U) \).

Since \( \Omega_{rBP}|_{S_r} = \Omega_{rBP}(\sigma + w(r, \sigma)) \) is a smooth function of \( r \) and \( \sigma \in S \), it admits a Taylor expansion at \( r = 0 \) of the form,

\[
\Omega_{rBP}(\sigma + w(r, \sigma)) = \Omega^{(0)} + r\Omega^{(1)} + \frac{r^2}{2}\Omega^{(2)} + O(r^3),
\]

with \( O(r^3) \) remainder uniform in \( \sigma \in S \), where:

\[
\begin{align*}
    \Omega^{(0)} &= \Omega_{0BP}(\sigma) = 0, \\
    \Omega^{(1)} &= \left. \frac{d}{dr} \Omega_{rBP}(\sigma + w(r, \sigma)) \right|_{r=0} \\
    &= G(\sigma) + \nabla \Omega_{0BP}(\sigma)[w_1(\sigma)] = G(\sigma) \\
    &= p \sum_{n=1}^{N} \int_{x_n}^{x_{n+2m}} (1 - \cos(\delta_n + Hpx)) \, dx \\
    &= 2Npq_{lm}, \\
    \Omega^{(2)} &= \left. \frac{d^2}{dr^2} \Omega_{rBP}(\sigma + w(r, \sigma)) \right|_{r=0} \\
    &= \frac{p^2}{2} \sum_{n=1}^{N} \int_{x_n}^{x_{n+2m}} (1 - \cos(\delta_n + Hpx)) \, dx \]
\]
here we denote
\[
\varphi_{n-1,1} = v_{n,1} - v_{n-1,1} - \int_{z=1}^{z=n} a_{z,1}(x, z) \, dz.
\] (42)

Note that the term of order \( r \) degenerates— unlike the boundary-value problem treated in [AlBeBr 00] the constant phase differences \( \delta_n \) are not determined at this point, but only at the order \( r^2 \! \).  

**Step 2:** Expansion of the solutions.

To evaluate the next order term \( \Omega(2) \) we require the first-order correction to the solution \( w_1(\sigma) \). Implicit differentiation of the equation (37) with respect to \( r \) yields:

\[
P^\perp \left[ \nabla G(\sigma) + \nabla^2 \Omega_0^{BP}(\sigma)w_1 \right] = 0.
\] (43)

Since \( \nabla^2 \Omega_0^{BP}(\sigma) \) is invertible on \( W \) (43) determines \( w_1 = w_1(\sigma) \) uniquely.

Using the expansion (11) we write the projected equation (43) in terms of the above coordinates:

\[
-\frac{1}{\kappa^2} u''_{n,1} + 2u_{n,1} = \frac{1}{2} \left( \cos \Phi_{n,n-1,0} + \cos \Phi_{n+1,n,0} - 2 \right),
\] (44)

\[
\frac{1}{\kappa^2} \frac{d}{dx} \left( v'_{n,1} - a_{x,1}(x, z_n) \right) = \frac{1}{2} \left[ \sin(\delta_n + Hpx) - \sin(\delta_{n+1} + Hpx) \right],
\] (45)

and

\[
\bar{a}_1(x, z) = \left( \frac{\partial \xi}{\partial z}, -\frac{\partial \xi}{\partial x} \right), \quad \Delta \xi = b_1(x, z), \quad \xi|_{\partial B} = 0,
\] (46)

where

\[
b_1(x, z) = b_{1}^{(n)}(x), \quad z_{n-1} < z < z_n, n = 1, \ldots, N,
\]

\[
b_{1}^{(n)}(x) - b_{1}^{(n+1)}(x) = p(v'_{n,1} - a_{x,1}(x, z_n)), n = 1, \ldots, N - 1,
\]

\[
\partial_x b_{1}^{(n)} = \frac{1}{2} \kappa^2 \sin(\delta_n + Hpx),
\]

\[
b_{1}^{(n)}(\pm L) = 0.
\] (47)
This system also coincides with the Euler–Lagrange equations for minimizing $\Omega^{(2)}$ above in the space $W$. An equivalent way to arrive at these equations is to begin with the equations for $w(r, \sigma)$ satisfying (17) as derived in section 2.5 of [AlBeBr 00], then take the order $r$ terms appearing in each equation.

These equations may be integrated explicitly: from the current conservation equation (45) we obtain:

$$v'_{n,1} - a_{x,1}(x, z_n) = C_n - \frac{\kappa^2}{2H_p} \cos(\delta_n + Hpx) + \frac{\kappa^2}{2H_p} \cos(\delta_{n+1} + Hpx), \quad n = 1, \ldots N. \tag{48}$$

where $C_n$ are as-yet undetermined constants. Next we use the equations (47) for the magnetic field inside each gap,

$$b_{1}^{(n)}(x) = b_1(x, z), \quad z_{n-1} < z < z_n, \tag{49}$$

with $D_n$ another set of undetermined constants. Using the jump condition of (47) for the magnetic field across each superconducting plane and the periodicity of $D_n$, we see that

$$-pC_n = (D_{n+1} - D_n), \quad n = 1, \ldots N. \tag{50}$$

We now solve for the order $r$ term in the gauge-invariant phase $\varphi_{n,n-1,1}$ (defined in (42)) using formula (16). We find for $n = 1, \ldots, N$,

$$\frac{d}{dx} \varphi_{n,n-1,1}(x) = (v'_{n,1} - a_{x,1}(x, z_n)) - (v'_{n-1,1} - a_{x,1}(x, z_{n-1})) + pb_{1}^{(n)}(x)$$

$$= pD_n + C_n - C_{n-1}$$

$$+ \frac{\kappa^2}{2H_p} \left( \cos(\delta_{n+1} + Hpx) - (2 + p^2) \cos(\delta_n + Hpx) + \cos(\delta_{n-1} + Hpx) \right).$$

Since each $\varphi_{n,n-1,1}(x)$ must be $2q$-periodic in $x$ we must satisfy the integrability conditions $pD_n + C_n - C_{n-1} = 0$. Together with (50), we derive a second order difference equation for $D_n$,

$$D_{n+1} - 2D_n + D_{n-1} = p^2 D_n, \quad n = 1, \ldots, N. \tag{51}$$

and note that the $\bar{e}_2$-periodicity condition (3) of $h$ together with the corresponding condition (33) for $\delta_n$ implies $D_n = D_{n+N}$. The maximum principle for second order difference equations then ensures that $D_n = 0$, and hence $C_n = 0$ by (50), and all constants are uniquely determined. We may then integrate to obtain:

$$\varphi_{n,n-1,1}(x) = \frac{\kappa^2}{2H^2p^2} \left( \sin(\delta_{n+1} + Hpx) - (2 + p^2) \sin(\delta_n + Hpx) + \sin(\delta_{n-1} + Hpx) \right).$$

We note that the arbitrary constant of integration which should normally come with $\varphi_{n,n-1,1}$ is zero here, since we are solving for each $\phi_n$ in the orthogonal to $T\mathcal{S}$. (That constant would
result from the order-\( r \) correction to our choice of \( \sigma \in \mathcal{S} \) in the minimization problem on \( \mathcal{S}_r \).

In conclusion, the gauge-invariant quantities associated with points \((\sigma + w(r, \sigma))\) on the natural constraint \( \mathcal{S}_r \) are:

\[
\begin{align*}
f_n &= 1 + r \left[ -\frac{1}{2} + \frac{\kappa^2}{2(H^2p^2 + 2\kappa^2)} \cos(\delta_n + Hpx) + \cos(\delta_{n+1} + Hpx) \right] + O(r^2), \\
h(x, z) &= H - r\frac{\kappa^2}{2H} \cos(\delta_n + Hpx) + O(r^2) \\
j^{(n)}_x(x) &= r\frac{\kappa^2}{2Hp} (\cos(\delta_{n+1} + Hpx) - \cos(\delta_n + Hpx)) + O(r^2) \\
j^{(n)}_z(x) &= \frac{\kappa^2}{2} \sin(\delta_n + Hpx) + O(r^2),
\end{align*}
\]

where \((\text{as usual}) \quad \sigma = \sigma(\delta_2, \ldots, \delta_N)\). We emphasize that Lemma 5.1 (i) ensures that all remainder terms are uniform in \( \sigma \in \mathcal{S} \).

**Step 3:** Expansion of \( \Omega^{BP}_r |_{\mathcal{S}_r} \).

We now resolve the degeneracy at order \( r^2 \) to determine which choice of \( \sigma = \sigma(\delta_2, \ldots, \delta_N) \) in \( \mathcal{S} \) gives rise to stationary solutions of the Lawrence–Doniach system. Substituting and computing the integrals,

\[
\Omega^{(2)} = \left\{ Npq_m \frac{\kappa^2}{2H^2p^2} \left[ \frac{3}{2}p^2 - 1 - \frac{H^2p^2}{H^2p^2 + 2\kappa^2} \right] - Npq_m \right\} + q_m \frac{\kappa^2}{2H^2p^2} \left( 1 - \frac{H^2p^2}{H^2p^2 + 2\kappa^2} \right) p \sum_{n=1}^{N} \cos(\delta_n - \delta_{n+1}),
\]

where we recall \( q_m = m\pi/Hp \).

In conclusion, we obtain the following expansion of \( \Omega^{BP}_r |_{\mathcal{S}_r} \), with \( \mathcal{S}_r \) parametrized by \( \sigma = \sigma(\delta_2, \ldots, \delta_N) \):

\[
\Omega^{BP}_r(\sigma + w(r, \sigma)) = 2Npq_m \left\{ r + r^2 \left( C_0 + C_1 \frac{1}{N} \sum_{n=1}^{N} \cos(\delta_n - \delta_{n+1}) \right) \right\} + O(r^3),
\]

where \( C_0 \in \mathbb{R}, \ C_1 > 0 \) are constants independent of \( N, s, q, m \). The periodicity conditions in \( BP_{\ast}(N, s, q_m, m\mathbb{Z}) \) carry over to an inhomogeneous boundary condition on \( \delta_n \) (see (33)),

\[
\delta_1 = 0, \quad \delta_{N+1} = -Hps \ (\text{mod} \ 2\pi).
\]

We recall once again that the remainder term is *uniform* in \( \sigma = \sigma(\delta_2, \ldots, \delta_N) \in \mathcal{S} \).

**Step 4:** Verifying equation (40), and the conclusion of (i).

First we observe that by Lemma 4.2 and Lemma 5.2 (b) for \( 0 < r < \tilde{r}_0 \) the infimum of \( \Omega^{BP}_r \) in \( BP_{\ast}(N, s, q_m, m\mathbb{Z}) \) will be attained on \( \mathcal{S}_r \). Since \( \mathcal{S}_r \) is finite dimensional and \( \Omega^{BP}_r |_{\mathcal{S}_r} \)
is periodic in the local coordinates \((\delta_2, \ldots, \delta_N)\), for every \(0 < r < \tilde{r}_0\) there exists (at least one) minimizer,

\[
\Omega_{r}^{BP}(\sigma + w(r, \sigma_r)) = \inf \Omega_{r}^{BP}|_{\mathcal{S}_r}, \quad \sigma_r = \sigma(\delta_2(r), \ldots, \delta_N(r)) \in \mathcal{S}.
\]

By the expansion (53) we have

\[
\Omega_{r}^{BP}(\sigma + w(r, \sigma_r)) = 2Npq_m \left\{ r + r^2 \left( C_0 + C_1 \frac{1}{N} \sum_{n=1}^{N} \cos(\delta_n(r) - \delta_{n+1}(r)) \right) \right\} + O(r^3)
\]

\[
\geq 2Npq_m \left\{ r + r^2 (C_0 + C_1 F(N, s)) \right\} + O(r^3),
\]

since \(F(N, s)\) is the infimum of the sum of cosines over all possible configurations. To obtain a complementary inequality, let \((\delta^*_2, \ldots, \delta^*_N)\) be any minimizer of \(F(N, s)\), that is

\[
F(N, s) = \frac{1}{N} \sum_{n=1}^{N} \cos(\delta^*_n - \delta^*_{n+1}).
\]

(Under the hypotheses of (i) there could be many such minimizers.) Then, applying (53) to this configuration, we obtain:

\[
\inf \Omega_{r}^{BP}|_{\mathcal{S}_r} \leq \Omega_{r}^{BP}(\sigma^* + w(r, \sigma^*)) \leq 2Npq_m \left\{ r + r^2 \left( C_0 + C_1 \frac{1}{N} \sum_{n=1}^{N} \cos(\delta^*_n - \delta^*_{n+1}) \right) \right\} + O(r^3)
\]

\[
= 2Npq_m \left\{ r + r^2 (C_0 + C_1 F(N, s)) \right\} + O(r^3).
\]

Putting together (53) and (56) we deduce the energy expansion (40) stated in Theorem 6.1 (i). Moreover, (56) implies

\[
F(N, s) \leq \frac{1}{N} \sum_{n=1}^{N} \cos(\delta_n(r) - \delta_{n+1}(r)) \leq F(N, s) + O(r).
\]

Therefore for any sequence of \(r \to 0\), the minimizers \(\sigma_r\) of \(\Omega_{r}^{BP}|_{\mathcal{S}_r}\) form a minimizing sequence for the variational problem \(F(N, s)\). Hence the \(\sigma_r\) accumulate as \(r \to 0\) at minimizers of \(F(N, s)\). To be more precise, for any sequence of \(r \to 0\) there exist subsequences and minimizers \(\sigma^* = \sigma(\delta^*_2, \ldots, \delta^*_N)\) of \(F(N, s)\) such that (along the subsequence) \(\sigma_r \to \sigma^*\). Inserting this information into (52) we obtain (38). This completes the proof of part (i) of Theorem 6.1.

**Step 5:** Proof of (ii).

By the expansion (40) the problem reduces to determining for which lattice parameters \(N, s\) does \(F(N, s)\) attain its lower bound of \(-1\). This lower bound is achieved if and only if the boundary condition (54) admits a choice of \(\delta_n\) with \(\delta_{n+1} - \delta_n = \pi \pmod{2\pi}\).
When \( s \notin \frac{\pi}{H_p} \mathbb{Z} \) the lattice is frustrated since the space \( \mathcal{BP}_s(N, s, q_m, m \mathbb{Z}) \) does not admit the configuration with \( \delta_{n+1} - \delta_n = \pi \mod 2\pi \). In that case we obtain \( F(N, s) > -1 \), and the energy per unit area of the minimizer in \( \mathcal{BP}_s(N, s, q_m, m \mathbb{Z}) \) will be strictly larger than this absolute minimum value, for all sufficiently small \( r > 0 \).

If
\[
N \text{ is even and } s = 2\ell \frac{\pi}{H_p} \text{ for } \ell \in \mathbb{Z}
\]
or if
\[
N \text{ is odd and } s = (2\ell + 1) \frac{\pi}{H_p} \text{ for } \ell \in \mathbb{Z},
\]
the choice \( \delta_n = (n - 1)\pi \mod 2\pi \) is allowed by (54) and the infimum \( F(N, s) = -1 \) is attained. Define \( g : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R} \) with
\[
g(r, \delta_2, \ldots, \delta_N) := \frac{1}{r^2} \left( \frac{\Omega^B_P(\sigma + w(r, \sigma))}{2mq_1Np} - r \right)
= C_0 + C_1 \frac{1}{N} \sum_{n=1}^{N} \cos(\delta_n - \delta_{n+1}) + O(r),
\]
by (58). In particular, \( g \) is smooth and \( g(0, \delta_2, \ldots, \delta_N) \) is minimized if and only if \( \delta_n = \delta^*_n = (n - 1)\pi \mod 2\pi \). Now, \( (\delta^*_2, \ldots, \delta^*_N) \) is a non-degenerate minimizer of \( g(0, \delta_2, \ldots, \delta_N) \): its Hessian is the familiar tridiagonal matrix with 2 on the diagonal and \(-1\) on each off-diagonal, associated with the (positive-definite) second-order difference operator. By the Implicit Function Theorem we conclude that for all sufficiently small \( r > 0 \) the function \( g(r, \delta_2, \ldots, \delta_N) \) has a unique minimum at \( (\delta_2(r), \ldots, \delta_N(r)) \), with \( \delta_n(r) = \delta^*_n + O(r) \). In other words, setting \( \sigma_r := \sigma(\delta_2(r), \ldots, \delta_N(r)) \),
\[
\inf \Omega^B_P |_{s_r} = \Omega^B_P(\sigma_r + w(r, \sigma_r)).
\]
By Lemma 5.2 when \( 0 < r < \tilde{r}_0 \) this gives the global minimizer of \( \Omega^B_P \) in \( \mathcal{BP}_s(N, s, q_m, m \mathbb{Z}) \), and inserting these optimal values for \( \delta_n \) into the asymptotic formulae (52) we obtain (29). This proves that the minimizers for any period geometry satisfying (57) or (58) coincide up to order \( r \). We will do better, and show that they are actually identical.

To gain a complete understanding of the absolute lowest energy solution we first consider the special case \( N = 1, s = q_1 = \pi/H_p \) and \( m = 1 \). In this case, there is no degenerate manifold: the \( r = 0 \) problem has a unique, non-degenerate solution and the perturbation is regular. In particular the second-order term in the energy expansion (10) is completely determined by (54): since \( \delta_0 = 0 \) and (for \( N = 1 \)) \( \delta_{N+1} = \delta_2 = -Hsp = -\pi \) we have \( F(1, q_1) = -1 \). We obtain for all sufficiently small \( r > 0 \) a unique solution \( (f_n^1, \phi_n^1, A_1^1) \) which minimizes \( \Omega^B_P \) in \( \mathcal{BP}_s(1, s, q_1, \mathbb{Z}) \). The gauge-invariant quantities associated to \( (f_n^1, \phi_n^1, A_1^1) \) will be \( 2q_1 \)-periodic in \( x \) and shifting \( z \) by \( p \) results in a translation by a half-period \( q_1 \) in \( x \) (see (28).)
Now return to the cases \((57)\) or \((58)\) where \(N \geq 2\) and \(F(N,s) = -1\). We denote by 
\[ \delta_n^* = (n - 1)\pi \mod 2\pi, \]
the unique absolute minimizer of \(F(N,s)\), and by \((\delta_2(r), \ldots, \delta_N(r))\) the coordinates of the absolute minimizer \((f_n, \phi_n, \tilde{A}) \in \mathcal{BP}_s(N, s, q_n, m\mathbb{Z})\) of \(\Omega_r^{BP}\), which we know lies on \(\mathcal{S}_r\) for all small \(r\), and for which \(\delta_n(r) = \delta_n^* + O(r)\).

It is easy to verify that the \(\mathcal{BP}_s(1, q_1, q_1, \mathbb{Z})\)-minimizer \((f_n^1, \phi_n^1, \tilde{A}^1)\) is also \(\Pi_{N,s,q_m}\) -periodic when \(N, s\) satisfy \((57)\) or \((58)\), and in fact \((f_n^1, \phi_n^1, \tilde{A}^1)\) solves the Euler–Lagrange equations for \(\Omega_r^{BP}\) in \(\mathcal{BP}_s(N, s, q_m, m\mathbb{Z})\). Moreover, \((f_n^1, \phi_n^1, \tilde{A}^1)\) also satisfies the expansion of energy given by \((\mathbf{II})\) on the period parallelogram \(\Pi_{N,s,q_m}\), and therefore by Lemma 5.2 it lies on the constraint manifold \(\mathcal{S}_r \subset \mathcal{BP}_s(N, s, q_m, m\mathbb{Z})\) in cases \((57)\) and \((58)\). Finally, the coordinates of \((f_n^1, \phi_n^1, \tilde{A}^1)\) on \(\mathcal{S}_r \subset \mathcal{BP}_s(N, s, q_m, m\mathbb{Z})\), which we denote by \((\delta_n^{(1)}(r), \ldots, \delta_N^{(1)}(r))\), satisfy

\[ \delta_n^{(1)} = \delta_n^* + O(1), \]

Since \((\delta_2^*, \ldots, \delta_N^*)\) is a non-degenerate minimizer of \(F(N,s)\) we must have \(\delta_n(r) = \delta_n^{(1)}(r)\) for all sufficiently small \(r\), that is \((f_n^1, \phi_n^1, \tilde{A}^1) = (f_n^1, \phi_n^1, \tilde{A}^1)\) for all small \(r\), that is the minimizers in \(\mathcal{BP}_s(N, s, q_m, m\mathbb{Z})\) with \((57)\) or \((58)\) coincide exactly with the minimizers in \(\mathcal{BP}_s(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbb{Z})\).

This concludes the proof of Theorem 6.1.

\(\diamondsuit\)

Finally we prove Theorem 4.1. In case \(q \not\in \frac{\pi}{Hp}\mathbb{Z}\) or \(k_n \neq mn\) (for \(m \in \mathbb{N}\) constant) we have by Lemma 4.2

\[ \epsilon_r(N, s, q, \tilde{k}) := \inf_{\mathcal{BP}_s(N,s,q,k)} \Omega_r^{BP} \geq \inf_{\mathcal{BP}_s(N,s,q,k)} \Omega_0^{BP} =: \omega_0(N, s, q, \tilde{k}) > 0, \]

with \(\omega_0(N, s, q, \tilde{k})\) a constant independent of \(r\). Using the expansion \((\mathbf{II})\) of the minimizing solution in \(\mathcal{BP}_s(1, \frac{\pi}{Hp}, \frac{\pi}{Hp}, \mathbb{Z})\) we obtain the first alternative in the statement of the Theorem. When \(q \in \frac{\pi}{Hp}\mathbb{Z}\) the remaining statements follow as a corollary to Theorem 6.1.

\(\diamondsuit\)

**Remark 6.2** The result contained in Theorems 4.1 and 6.1, namely that the smallest possible energy per unit cross-sectional area is obtained with \(N = 1, s = q_1, m = 1\), confirms the prediction of a period-2\(p\) in \(z\) staggered lattice solution made by Bulaevskii–Clem [BuCm 91]. In that paper the authors indeed claim that it minimizes energy among competing configurations, but only one other vortex lattice is treated in their paper (a period-4 lattice), and no indication is provided as to how they deduced the geometry of their solution. Here we have shown much more: we know that it is the minimizer among all periodic solutions, in the regime \(r << 1\).

The “vortex plane solution” of Theodorakis–Kuplevaksky ([Th 90], [K 99]) is obtained by taking \(\delta_n = 0\) for all \(n\) (when the value of \(s\) permits such a choice.) Note that such a choice...
maximizes $\Omega^{(2)}$ on the constraint set $S_r$. Since these would also constitute non-degenerate critical points of the finite-dimensional function $g(r, \delta_2, \ldots, \delta_N)$, by the above arguments they describe bona fida solutions to the Lawrence–Doniach system with periodic boundary conditions, but they are unstable.

**Remark 6.3** When $s \neq jq_1$, $j \in \mathbb{Z}$ the trivial choice of minima $\delta_n - \delta_{n-1} = \pm \pi$ is not admissible, and the lattice is “frustrated”. As previously mentioned, in this generic case the minimizer of $F(N, s)$ might be non-unique, leading to different asymptotics for the minimizing solutions along different subsequences $r \to 0$. Nevertheless, the dependence of the energy minimizers on subsequential limits $r \to 0$ could be eliminated when it is known that the absolute minimizer $(\delta_2^*, \ldots, \delta_N^*)$ of $F(N, s)$ is unique and non-degenerate. An example is when the minimizer of the finite dimensional problem (39) $(\delta_2^*, \ldots, \delta_N^*)$ satisfies

$$C_n := \cos(\delta_n^* - \delta_{n+1}^*) < 0 \quad \text{for all } n = 1, \ldots, N. \quad (59)$$

This will be the case when the parallelogram $\Pi_{N,s,q}$ is very close to the optimal ones described by (57) and (58). Assuming (59) holds for the minimizer, a simple calculation shows that the Hessian is the $(N-1) \times (N-1)$ tridiagonal, symmetric matrix $D^2g(0, \delta_2^*, \ldots, \delta_N^*) =: \left[ M_{m,n} \right]_{m,n=2,\ldots,N}$ with

$$M_{n,n+1} = C_n \quad n = 2, \ldots, N - 1; \quad M_{n,n} = -(C_{n-1} + C_n), \quad n = 2, \ldots, N.$$

A null vector $\vec{v} = (v_2, \ldots, v_N)$ of $M$ satisfies:

$$C_n(v_{n+1} - v_n) - C_{n-1}(v_n - v_{n-1}) = 0, \quad n = 3, \ldots, N - 1;$$

$$C_2(v_3 - v_2) - C_1 v_2 = 0, \quad C_N v_N - C_{N-1}(v_N - v_{N-1}) = 0.$$

If $v_2 = 0$ then clearly $\vec{v} = 0$, so we may assume $v_2 > 0$, in which case equations $n = 2, \ldots, N - 1$ imply $0 < v_2 < \ldots < v_N$. But this contradicts the $n = N$ equation, and therefore the only solution is the trivial one. In conclusion the minimizer is non-degenerate, and we can repeat the same arguments as in the cases (57) and (58) to conclude uniqueness for energy minimizers.

## 7 The periodic finite layer case

Finally, we consider the case of a finite number of planes $N$, each of infinite extent in $x$ and $y$, assuming that the currents and field strength are periodic functions in $x$. Since this case is very similar to the doubly periodic case treated in the previous sections we give an outline of
how to modify the formulation of the problem and its solution to fit this somewhat simpler case.

In the finite-layer case the ‘t Hooft condition is greatly simplified, since (by the argument of Theorem 3.1) we may take $\vec{A}$ to be periodic in $x$. Let $\vec{k} = (k_n)_{n \in \mathbb{Z}}$ with $k_0 = 0$, $\vec{k} \neq \vec{0}$. We say that $(f_n, \phi_n, \vec{A})$ belongs to the periodic class $P = P(q, \vec{k})$ if there exists a constant $\omega \in \mathbb{R}$ such that:

\begin{align*}
\vec{A} \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2), & \quad f_n \in H^1_{\text{per}}(\mathbb{R}), \quad \phi_n \in H^1_{\text{loc}}(\mathbb{R}); \\
\phi_n(x + 2q) = \phi_n(x) + \omega + 2\pi k_n, & \quad n = 0, \ldots, N; \\
\int_{-q}^{q} \phi_0(x) \, dx = 0 = \int_{-q}^{q} \phi_1(x) \, dx; & \\
\vec{A}(x, z) = (Hz, 0) + (\xi_z, -\xi_x), & \quad \xi \in H^2_{\text{loc}}(\mathbb{R}^2), \\
\xi(x + 2q, z) = \xi(x, z), & \quad \xi(x, 0) = \xi(x, Np) = 0. 
\end{align*}

(60)

If $\vec{k} = \vec{0}$ we define $P(q, \vec{0})$ by omitting (61). Following the proof of Theorem 3.1, any configuration satisfying a single ‘t Hooft condition (in $x$ with $\omega_x$) as in (5), (6) is gauge-equivalent to an element of $P(q, \vec{k})$. Furthermore, in this class the $L^2$ norm of curl $\vec{A}$ controls $\vec{A}$ in $H^1$.

As in the doubly periodic case, the observables are determined entirely by the values of $(f_n, \phi_n, \vec{A})$ in a single period $x \in [-q, q]$, but now the number of planes is finite (indexed by $n = 0, \ldots, N$) and each is independent. Hence we define the Gibbs free energy by integration over a single period and summation over the $N + 1$ planes,

\begin{align*}
\Omega^\text{per}_r(f_n, \phi_n, \vec{A}) = & \int_{-q}^{q} \sum_{n=0}^{N} \left[ \frac{1}{2} (f_n^2 - 1)^2 + \frac{1}{\kappa^2} (f'_n)^2 + \frac{1}{\kappa^2} (\phi'_n - A_x(x, z_n))^2 f_n^2 \right] \, dx \\
& + \frac{r}{2} \int_{-q}^{q} \sum_{n=1}^{N} \left( f_n^2 + f_{n-1}^2 - 2 f_n f_{n-1} \cos(\Phi_{n,n-1}) \right) \, dx \\
& + \frac{1}{\kappa^2} \int_0^{Np} \int_{-q}^{q} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - H \right)^2 \, dx \, dz.
\end{align*}

By taking variation of $\Omega^\text{per}_r(f_n, \phi_n, \vec{A})$ in the subclass of periodic functions, we see that the Euler–Lagrange equations are exactly the same as for the finite width sample case, see (2)–(5), (7) of [AlBeBr 00]. That is, they coincide with (11)–(15) for $n = 1, \ldots, N - 1$, and the equations involving the top and bottom surfaces $n = 0, N$ must be modified to reflect the fact that these two planes have only one “nearest neighbor”.

We may now continue as in the doubly periodic case. By Lemma 4.2 the minimum energy will be of order $r$ if and only if we choose $q = m \frac{\pi}{Hp}$ and $\vec{k} = m \mathbb{Z}$, with $m = 1, 2, \ldots$. Furthermore, with these choices the minimizers will have $f_n = |\psi_n| \sim 1$, and the use of polar
coordinated for the order parameter is well justified. We choose \( q = \frac{\pi}{Hp} \), \( \vec{k} = \mathbf{Z} \), and write \( \mathcal{P}_* := \mathcal{P}(\frac{\pi}{Hp}, \mathbf{Z}) \) in the following.

For the number \( N \) of planes, \( \kappa \), and \( H \) fixed, define the free energy per unit area in a period strip of width 2\( q \) with winding numbers \( \vec{k} \) by:

\[
\epsilon_r(q, \vec{k}) := \frac{1}{2qNp} \inf \left\{ \Omega^\text{per}_r(f_n, \phi_n, \vec{A}) : (f_n, \phi_n, \vec{A}) \in \mathcal{P}(q, \vec{k}) \right\}.
\]

We obtain the following result:

**Theorem 7.1** Let \( N, H, \kappa \) be fixed.

(a) If there exists \( m \in \mathbb{N} \) such that

\[
q = q_m := \frac{m\pi}{Hp} \quad \text{and} \quad \vec{k} = m\mathbf{Z},
\]

then there exists \( r_0 = r_0(N, H, \kappa, m) > 0 \) such that for all \( 0 < r < r_0 \), \( \epsilon_r(q, \vec{k}) = \epsilon_r(\frac{\pi}{Hp}, \mathbf{Z}) \), and the minimizers of \( \Omega^\text{per}_r \) in \( \mathcal{P}(q_m, m\mathbf{Z}) \) coincide with the minimizers of \( \Omega^\text{per}_r \) in \( \mathcal{P}_* \).

(b) For any other choice of \( q, \vec{k} \) there exist constants \( r_1, \omega_1 > 0 \) (depending on \( N, H, \kappa, q, \vec{k} \)) such that \( \epsilon_r(q, \vec{k}) \geq \omega_1 \) for all \( 0 < r < r_1 \).

We note that, as in the bi-periodic case, we may obtain an expansion of the minimizing solution in powers of \( r \). The minimizer in the special space \( \mathcal{P}_* \) (which gives the absolute minimum of energy per unit area) coincides with the period-2 lattice found in the bi-periodic case at order \( r \), except for an edge effect in the order parameter in the top and bottom planes. More precisely, the fields and currents \( h(x, z), j^{(n)}_x, j^{(n)}_z \) in a finite stack still satisfy (29) for each \( n = 0, \ldots, N \), and \( f_n \) coincides with the expression given in (29) for \( n = 1, \ldots, N - 1 \) but is modified (see (68) and (69) below) at order \( r \) for \( n = 0 \) and \( n = N \).

The proof of Theorem 7.1 follows almost line-for-line the degenerate perturbation procedure of the previous sections. In particular, the minimum value of \( \Omega^\text{per}_r \) at \( r = 0 \) is attained by elements in the \((N - 1)\)-dimensional hyperplane

\[
\mathcal{S} := \{(f_n, \phi_n, \vec{A}) \in \mathcal{P}_*: f_n \equiv 1, \ \phi_n(x) = \alpha_n + nHp x, \ A_x = Hz, \ A_z = 0, \ \text{where} \ \alpha_0 = \alpha_1 = 0, \ \text{and} \ \alpha_2, \ldots, \alpha_N \in \mathbb{R}\}.
\]
be treated differently. For example the equation of current conservation (45) now gives
\[
v'_{n,1} - a_{x,1}(x, z_n) = \begin{cases}
C_n - \frac{\kappa^2}{2p} \cos(\delta_n + Hpx) + \frac{\kappa^2}{2p} \cos(\delta_{n+1} + Hpx), & n = 1, \ldots, N - 1; \\
C_0 + \frac{\kappa^2}{2p} \cos(\delta_1 + Hpx), & n = 0; \\
C_N - \frac{\kappa^2}{2p} \cos(\delta_N + Hpx), & n = N;
\end{cases}
\]

The order \( r \) term in magnetic field \( b_1 \) is exactly as in the doubly periodic case and is given by (49) (with as yet undetermined constants \( D_1, \ldots, D_N \)). Therefore, using equation (49) and the jump conditions (47), we obtain the following conditions on the constants \( C_n \) and \( D_n \):
\[
\begin{align*}
D_{n+1} - 2D_n + D_{n-1} &= p^2 D_n, \quad n = 1, \ldots, N - 1, \\
D_2 - 2D_1 &= p^2 D_1, \quad D_{N-1} - 2D_N = p^2 D_N, \\
-pC_n &= (D_{n+1} - D_n), \quad n = 1, \ldots, N - 1, \\
-pC_0 &= D_1, \quad pC_N = D_N.
\end{align*}
\]

The maximum principle for the second-order difference equation (66) with boundary condition (67) implies that the unique solution is \( D_n = 0, C_n = 0 \) for all \( n \). In consequence the perturbed manifold \( S_r \) consists of the same configurations (52) as for the bi–periodic case, except for the superconducting order parameter which coincides with the expression given (52) for \( n = 1, \ldots, N - 1 \), but for \( n = 0 \) or \( n = N \) we obtain
\[
\begin{align*}
f_N &= 1 + \frac{r}{2} \left( -\frac{1}{2} + \frac{\kappa^2}{2(H^2p^2 + 2\kappa^2)} \cos(\delta_N + Hpx) \right) + O(r^2), \\
f_0 &= 1 + \frac{r}{2} \left( -\frac{1}{2} + \frac{\kappa^2}{2(H^2p^2 + 2\kappa^2)} \cos(Hpx) \right) + O(r^2).
\end{align*}
\]

Substitution of (52), (68), (69) into \( \Omega_{\text{per}}^r \) leads to an expansion of the energy in the same form as (53),
\[
\Omega_{\text{per}}^r(\sigma + w(r, \sigma)) = 2Npq \left\{ r + r^2 \left( \tilde{C}_0 + \tilde{C}_1 \frac{1}{N} \sum_{n=1}^{N-1} \cos(\delta_n - \delta_{n+1}) \right) \right\} + O(r^3),
\]
except for the constants \( \tilde{C}_0 \in \mathbb{R} \) and \( \tilde{C}_1 > 0 \) which differ from the doubly periodic case due to the slightly different form of solutions for the top and bottom planes. The significant difference from the previous case is that when there are finitely many planes there is only the single constraint \( \delta_1 = 0 \) (which comes from removing the translation invariance in the definition of the space \( \mathcal{P}_r \)). By the same arguments as in the previous section the minimizer of \( \Omega_{\text{per}}^r \) will be determined by minimizing the leading term in the energy expansion,
\[
G(\delta_2, \ldots, \delta_N) = \sum_{n=1}^{N-1} \cos(\delta_n - \delta_{n+1}), \quad \delta_1 = 0.
\]
By inspection, the minimizer is obtained by choosing $\delta_{n+1} - \delta_n = \pm \pi$; for example $\delta_1 = \pi$, $\delta_2 = 0$, $\delta_3 = \pi$, … Unlike the doubly periodic case we are always free to make this choice, and it is easy to verify that this configuration gives a non-degenerate minimizer of the finite-dimensional function $g(0, \sigma)$, $\sigma = (\delta_2, \ldots, \delta_N)$. In particular the period-$2p$ in $z$, period-$2q_1$ in $x$ lattice appears naturally (at order $r$) in the solutions. (See figure 1.) As in the finite-width case the solutions feel the top and bottom edges only in the first plane and first gap at order $r$; expansion to higher orders will reveal the effect of the finite stack at order $r^k$ in the $k$th stack from the top or bottom.

8 Conclusions

Finally, we summarize our results on the periodic problem, and compare with the (quite different) conclusions obtained for finite-width samples in our previous paper [AlBeBr 00].

As we have seen, in the limit $r \to 0$ energy minimization selects a preferred period geometry and quantized flux from all possible periodic configurations represented by the spaces $\mathcal{BP}_s(N, s, q, \vec{k})$. The optimal solution is $\frac{2\pi}{H_p}$-periodic in $x$, and repeats itself with a horizontal shift of a half-period in $x$ when we climb from one superconducting plane to the next. Each period parallelogram contains exactly one quantum $2\pi$ of flux, and one Josephson vortex per period in each gap between the superconducting planes. The Josephson lattice geometry is the same as was predicted by Bulaevskii & Clem [BuCm 91], but with our rigorous analytical approach we may now assert that it is the unique energy minimizing periodic configuration (among all possible geometries) for all sufficiently small $r$.

In the finite-width case [AlBeBr 00] the conclusions were surprisingly very different. For any finite width sample, $-L \leq x \leq L$, when $\sin(HpL) \neq 0$ the unique energy minimizer for $r \sim 0$ is a vortex plane configuration, with Josephson vortices vertically aligned, and the magnetic field approximately uniform in $z$ (except for edge effects at the top and bottom of a sample of finitely many planes.) The exceptional values of the applied field $H$ for which $\sin(HpL) = 0$ correspond to first-order phase transitions occurring when a new vortex plane is nucleated into the sample from the lateral edges.

We can explain this apparent conflict by examination of the expansion of the energy in powers of $r$ near the degenerate manifold $\mathcal{S}$. In (23) of [AlBeBr 00] vortex planes are preferred at order $r$ in the energy because of a surface term, a quantity which scales like the length $Np$ of the lateral edges of the sample cross-section. In the periodic case this term does not appear, and in (53) the distinction between lattice geometries appears at order $r^2$, in a term which scales as the cross-sectional area $2qNp$ of the sample. For any size sample, by making $r$ small enough the order $r$ surface term will dominate, but the value of $r$ must decrease if the surface term is to continue to prevail with increasing sample width $L$. If we
Figure 1: Period-2p (in z) vortex lattice, for a sample with a finite number of superconducting planes (Indicated by horizontal dotted lines.) Horizontal arrows indicate the in-plane currents $j^{(n)}_x$ and the vertical arrows depict the Josephson currents $j^{(n)}_z$ between adjacent planes. The magnetic field $h(x, z)$ and supercurrents $j_x, j_z$ are periodic with period 2p in the z-direction (and period $\frac{2\pi}{Hp}$ in x). The vortices (local maxima of $h$) lie along the starred segments, and form a staggered lattice. If we choose the midpoint of each segment to label each vortex the resulting lattice is diamond-shaped.

try to keep $r$ fixed while increasing $L$ then inevitably the order $r^2$ term will compete with the order $r$ term. At that point the perturbation expansion will surely have lost its validity.

This general argument is supported by the analysis of the range of validity of the expansion in Lemma 5.1 presented in section 5 of [AlBeBr 00]. Indeed, it seems clear that the radius of validity of the expansions in $r$ deteriorates with increasing sample width $L$. (See Remark 5.3 of [AlBeBr 00].)

If we approximate a macroscopic sample by an infinite one and seek periodic solutions, the period plays the role of the sample width in the estimates of the interval of validity of section 5 in [AlBeBr 00]. Since the period of the absolute minimizer is given by $\frac{2\pi}{Hp}$ we can expect the interval of validity to extend to physically appropriate values of $r$ when the applied field $H$ is large enough. Therefore we may apply our analysis to the transparent state of the high-$T_c$ superconductors in high external fields.
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