Added Masses of an Elliptical Cylinder with Unsteady Motion or in Unsteady Flow

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1. Introduction

Okamoto et al\(^\text{1)}\) proposed a new method for wind tunnel testing of a heaving wing. In the method, the wing remains stationary and the flow has an unsteady velocity due to the heaving motion of the wind tunnel. This method is different from the conventional method where a wing has a heaving motion in a steady flow.\(^\text{2)}\)

On the other hand, a study by Hino\(^\text{3)}\) indicated that the added mass of a circular cylinder in an unsteady flow is twice that of a circular cylinder subjected to unsteady motion. In the present research, the hydrodynamic force acting on an elliptical cylinder in an unsteady flow is compared to that of an elliptical cylinder with unsteady motion. The parameters are the ratio between the lengths of the two axes and the angle between the axes and the direction of unsteady flow or unsteady motion.

2. Analysis

An elliptical cylinder in a \(x–y\) plane as shown in Fig. 1 is considered. The center of the elliptical cylinder moves with the following velocity,

\[
\dot{z}_0(t) = -(H_0 + H_1 \cos \omega t) - i(V_0 + V_1 \cos \omega t)
\]

Equation (B.52) presented by Imai\(^\text{4)}\) indicates the hydrodynamic forces on a moving elliptical cylinder in an unsteady flow of \(U\) in \(x\) direction as

\[
X + iY = -i\rho i \Gamma (\dot{z}_0 - U) + \rho (\pi ab) \dot{z}_0
\]

\[= \frac{\pi}{2} \rho \frac{d}{dt} \left[ (a + b)^2 (\dot{z}_0 - U) \right]
\]

\[+ \frac{\pi}{2} \rho \frac{d}{dt} \left[ (a^2 - b^2) (\dot{z}_0 - U) e^{-2i\alpha} \right]
\]

Here, \(\Gamma\) is a constant, and \(U, a, b,\) and \(\alpha\) are varied over time. Note that \(\Gamma\) and \(\alpha\) in Eq. (2) correspond to \(-\Gamma\) and \(-\chi\) in the study by Imai.\(^\text{3)}\)

The equation for \(U = 0\) is given by Eq. (B.54) in the study by Imai.\(^\text{4)}\) When the elliptical cylinder is in an unsteady flow with \(U_*(t) = (H_0 + H_1 \cos \omega t) + i(V_0 + V_1 \cos \omega t)\), the last term in Eq. (2) is replaced with

\[
\frac{\pi}{2} \rho \frac{d}{dt} \left[ (a^2 - b^2) (\dot{z}_0 - \dot{U}_*) e^{-2i\alpha} \right]
\]

Hereafter, the two cases, Case 1 and Case 2, are considered. In Case 1, the elliptical cylinder has no motion in an unsteady flow \(U_*(t)\). In Case 1, \(\dot{z}_0(t) = 0\). In Case 2, the elliptical cylinder has the motion of

\[
\dot{z}_0(t) = -(H_0 + H_1 \cos \omega t) - i(V_0 + V_1 \cos \omega t)
\]

under \(U_* = 0\). The hydrodynamic forces acting on the ellip-

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Fig. 1. An elliptical cylinder.
Table 1. Hydrodynamic forces on an elliptical cylinder.

| Case 1 | \[ \frac{\pi}{2} \rho \omega \sin \omega t \left[ H_1 \left[ a^2 (\cos 2\alpha - 1) - 2ab - b^2 (1 + \cos 2\alpha) \right] - V_1 (a^2 - b^2) \sin 2\alpha \right] + \frac{\pi}{2} \rho \omega \sin \omega t \left[ -V_1 a^2 (\cos 2\alpha + 1) + 2ab + b^2 (1 - \cos 2\alpha) \right] - H_1 (a^2 - b^2) \sin 2\alpha \right] - \rho \Gamma V_0 + \rho \Gamma \sin \omega t \] | Case 2 | \[ \frac{\pi}{2} \rho \omega \sin \omega t \left[ H_1 \left[ a^2 (\cos 2\alpha - 1) - 2ab + b^2 (1 + \cos 2\alpha) \right] - V_1 (a^2 - b^2) \sin 2\alpha \right] + \frac{\pi}{2} \rho \omega \sin \omega t \left[ -V_1 a^2 (\cos 2\alpha + 1) + 2ab - b^2 (1 - \cos 2\alpha) \right] - H_1 (a^2 - b^2) \sin 2\alpha \right] - \rho \Gamma V_0 + \rho \Gamma \sin \omega t \] |
| --- | --- | --- | --- |

Table 2. Added masses of the elliptical cylinder.

| Case 1 | Case 2 |
| --- | --- |
| \( m_{xx} \) | \( \frac{\rho \pi \alpha^2}{2} \left[ \left( 1 + \frac{b}{a} \right)^2 - \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha \) | \( \frac{\rho \pi \alpha^2}{2} \left[ \left( 1 + \frac{b}{a} \right)^2 - \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha \) |
| \( m_{yy} \) | \( \frac{\rho \pi \alpha^2}{2} \left[ \left( 1 + \frac{b}{a} \right)^2 + \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha \) | \( \frac{\rho \pi \alpha^2}{2} \left[ \left( 1 + \frac{b}{a} \right)^2 + \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha \) |
| \( m_{xy} \) | \( \frac{\rho \pi \alpha^2}{2} \left[ 1 - \left( \frac{b}{a} \right)^2 \right] \sin 2\alpha \) | \( \frac{\rho \pi \alpha^2}{2} \left[ 1 - \left( \frac{b}{a} \right)^2 \right] \sin 2\alpha \) |

Fig. 2. Relation between \( b/a \) and \( R_x, R_y \).

\[ R_x = \frac{1 + \left( \frac{b}{a} \right)^2 + \left[ 1 - \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha}{\left( 1 + \frac{b}{a} \right)^2 + \left[ 1 - \left( \frac{b}{a} \right)^2 \right] \cos 2\alpha} \]  

(5)

Relations between \( b/a \) and \( R_x, R_y \) are shown in Fig. 2. The following conditions hold:

1. When \( b/a = 0 \), the \( m_{xx} \) for \( \alpha = 0 \) and the \( m_{yy} \) for \( \alpha = 90 \text{deg} \) are zero. Therefore, the value of \( R_x \) for \( \alpha = 0 \) can be defined.

2. When \( b/a = 0 \), the value of \( R_x \) for \( \alpha = 90 \text{deg} \) cannot be defined.

3. Both \( R_x \) for \( \alpha = 0 \) and \( R_x \) for \( \alpha = 90 \text{deg} \) are equal to 1.

4. When \( b/a = 1 \), \( R_x = R_y = 0.5 \) independently of \( \alpha \). This corresponds to the fact that, in case of a circular cylinder, \( m_{xx} \) and \( m_{yy} \) for Case 1 are twice as much as those for Case 2.

5. If \( R_x \leq 1 \) and \( R_y \leq 1 \), this means that \( m_{xx} \) and \( m_{yy} \) in Case 1 are not smaller than those in Case 2.

6. As \( \alpha \) increases, the difference in \( m_{xx} \) between Case 1 and Case 2 is reduced, while the difference in \( m_{yy} \) between Case 1 and Case 2 is increased.
and Case 2 also increases. Conversely, as $\alpha$ increases, the difference in $m_{xy}$ between Case 1 and Case 2 decreases.

(7) When $b/a$ is small, the difference in $m_{xx}$ between Case 1 and Case 2, and that in $m_{yy}$, are not small when $\alpha \cong 0$ and $\alpha \cong 90$ deg, respectively.

3. Conclusion

The added masses $m_{xy}$ and $m_{yx}$ of an elliptical cylinder are identical when the cylinder remains stationary in an unsteady flow (Case 1) as well as when it is accelerating in a still fluid (Case 2). The added masses $m_{xx}$ and $m_{yy}$ are not identical between these cases. The difference in the added masses between the two cases corresponds to the mass of fluid and the volume of the elliptical cylinder. When an elliptical cylinder is thick, the difference in the added masses, $m_{xx}$ and $m_{yy}$, between these cases is large. Furthermore, even if an elliptical cylinder is thin, the difference in the added masses, $m_{xx}$ and $m_{yy}$, between these cases is large when the angle between the long axis of the circular cylinder and the direction of acceleration is small.

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