Breather wave, resonant multi-soliton and M-breather wave solutions for a (3+1)-dimensional nonlinear evolution equation

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Abstract: In this paper, a (3+1)-dimensional nonlinear evolution equation is considered. First, its bilinear formalism is derived by introducing dependent variable transformation. Then, its breather wave solutions are obtained by employing the extend homoclinic test method and related figures are presented to illustrate the dynamical features of these obtained solutions. Next, its resonant multi-soliton solutions are obtained by using the linear superposition principle. Meanwhile, 3D profiles and contour plots are presented to exhibit the process of wave motion. Finally, M-breather wave solutions such as one-breather, two-breather, three-breather and hybrid solutions between breathers and solitons are constructed by applying the complex conjugate method to multi-soliton solutions. Furthermore, their evolutions are shown graphically by choosing suitable parameters.

Keywords: breather wave solutions; rogue waves; resonant multi-soliton solutions; M-breather solutions; hybrid solutions

Mathematics Subject Classification: 35A25, 35G50, 35Q35, 37K10

1. Introduction

It is known that Hirota bilinear method [1] is a direct and effective method to solve a large number of nonlinear evolution equations, which can depict physical phenomenon in nonlinear science. Recently, Professor Ma considered the Hirota conditions of a (2+1)-dimensional combined equation and derived its the N-soliton solution in [2]. Furthermore, Ma presented N-soliton solutions of the combined pKP-BKP equation in [3] and soliton solutions to the B-type Kadomtsev-Petviashvili equation under general dispersion relations in [4] by using Hirota bilinear method. In recent literature, many types of exact solutions have been obtained by employing the Hirota bilinear method, such as soliton solutions [5–7], breather solutions [8, 9], lump wave solutions [10], interaction solutions [11, 12], rogue wave solutions [13, 14], resonant multi-soliton solutions [15–17], bifurcation solitons [18, 19], bright and dark soliton solutions [20, 21] and so on.
In 2009, Dai proposed the homoclinic test approach and extended homoclinic test approach to search solitary-wave solution of high dimensional nonlinear equation in [22]. Then Xu proposed the homoclinic breather limit method for searching rogue wave solution to nonlinear evolution equation in [23]. In 2011, Ma constructed the resonant multi-soliton solutions of nonlinear equations by introducing introducing the linear superposition principle [24]. Afterwards the linear superposition principle was used to establish the resonant multiple wave solutions of a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation by Lin in [25]. By comparing the linear superposition principle and the velocity resonant condition, Kuo [26, 27] concluded that the linear superposition principle is more effective, although they could result in the same results. In recent years, the complex conjugate method was applied to N-soliton solutions to construct M-breather solutions and corresponding hybrid solutions in [28–30]. As early as 1977, lump wave solutions were first proposed by Manakov et al. [31] to indicate that the type of wave does not decrease in the direction of \((x, y)\)-plane. Later, Satsuma and Ablowitz [32] proposed the long wave limit method for seeking M-lump solutions based on the collision effects of lump and N-soliton. And it is proved to be the most effective method to construct M-lump solutions of nonlinear evolution equations.

In the past few years, a lot of literature on M-lump solutions and hybrid solutions between lump and solitons of nonlinear evolution equations has been appeared such as the (2+1)-dimensional combined pKP-BKP equation [33], the (3+1)-dimensional potential-YTSF equation [34], a (2+1)-dimensional generalized KDKK equation [35], the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation [36], (2+1)-dimensional HSI equation [37], (3+1)-dimensional Kadomtsev-Petviashvili equation [38], (3+1)-dimensional gCH-KP equation [39], the (4+1)-dimensional Boiti-LeonManna-Pempinelli equation [40] and so on.

Recently, Zhang etc [41] considered a (3+1)-dimensional nonlinear evolution equation which was written by

\[-4u_{\xi t} + u_{xxxxx} + 3\alpha u_{yy} + 4u_{x}u_{xz} + 2u_{xx}u_{z} = 0.\]  

(1.1)

Its M-lump and interactive solutions of Eq (1.1) were given in [41] by using long wave limit method. when \(\alpha = 1\), Eq (1.1) can be reduced to the potential-YTSF equation [34], whose nontravelling wave solutions were discussed by Yan in [42]. Based on the above literature, we aim to consider the breather wave solutions and resonant multi-soliton solutions, M-breather solutions of Eq (1.1).

The rest of this paper is organized as follows. In Section 2, the bilinear formalism of the (3+1)-dimensional equation (1.1) is derived via introducing variable transformation; In Section 3, breather wave solutions are derived by using the extend homoclinic test approach. Meanwhile, a rogue wave solution is derived by using the homoclinic breather limit method; In Section 4, resonant multi-soliton solutions are obtained by using linear superposition principle; In Section 5, multi-soliton solutions are derived by Hirota bilinear method, then its M-breather solutions and hybrid solutions are constructed based upon the obtained multi-soliton solutions. Finally, related remarks are given.

### 2. Bilinear formalism for the (3+1)-dimensional equation (1.1)

Through simple calculation, we can see that it is impossible to establish the Hirota bilinear formalism of the (3+1)-dimensional equation (1.1) directly. By introducing a dependent variable transformation \(\xi = x + kz\) in (1.1), the (3+1)-dimensional equation (1.1) can be transformed into the following form

\[-4u_{\xi t} + k u_{\xi\xi\xi\xi} + 3\alpha u_{yy} + 6k u_{\xi}u_{\xi\xi} = 0,\]  

(2.1)

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where $k$ is a real constant. In what follows, we introduce another potential transformation

$$u = cq_\xi. \quad (2.2)$$

By substituting (2.2) into (2.1), we can get

$$E(q) = -4cqq_\xi + c\alpha_\xi q_{\xi\xi} + 3\alpha_\xi q_{\xi\xi\xi} + 6k\alpha_\xi q_{\xi\xi\xi\xi} = 0. \quad (2.3)$$

Integrating the Eq (2.3) with respect to $\xi$ once, we have

$$-4cqq_\xi + c\alpha_\xi q_{\xi\xi} + 3\alpha_\xi q_{\xi\xi\xi} + 3c^2kq_{\xi\xi\xi\xi} = 0. \quad (2.4)$$

According to the results between Bell polynomials and bilinear formalism in Ref. [43] and taking $c = 1$, the above expression (2.4) leads to the following bilinear form of (2.1)

$$[-4D_\xi D_\xi + kD_{\xi\xi} + 3\alpha D_{\xi\xi\xi}] \varphi \cdot \varphi = 0, \quad (2.5)$$

with the help of the following transformation relationship

$$q = 2\ln(\varphi) \Leftrightarrow u = q_\xi = 2(\ln \varphi)_\xi. \quad (2.6)$$

3. Breather wave and rogue wave solutions

In the beginning, we aim to investigate the breather wave solutions of Eq (1.1) by using the extend homoclinic test method [22]. We choose the test function as the following form

$$\varphi = \exp(-p_1(\xi + m_1y + c_1t)) + \delta_1\cos(p_2(\xi + m_2y + c_2t)) + \delta_2\exp(p_1(\xi + m_1y + c_1t)), \quad (3.1)$$

where $p_1, p_2, m_1, m_2, c_1, c_2, \delta_1, \delta_2$ are undetermined real constants. By substituting Eq (3.1) into Eq (2.5), an algebraic equation can be obtained. Setting all coefficients for the powers of $\exp(\pm p_1(\xi + m_1y + c_1t)), \sin(p_2(\xi + m_2y + c_2t)), \cos(p_2(\xi + m_2y + c_2t))$ to be zero, some algebraic equations can be obtained. Taking $p_1 = p_2 = p$, we can get

$$\begin{aligned}
2p^2(-6\alpha_1m_1m_2 + 4c_1\delta_1 + 4c_2\delta_1) = 0, \\
2p^2(3\alpha_1m_1^2 - 3\alpha_1m_2^2 - 4p^2k\delta_1 - 4c_1\delta_1 + 4c_2\delta_1) = 0, \\
2p^2(3\alpha_1\delta_1^2m_1^2 - 3\alpha_1\delta_1\delta_2m_2^2 - 4p^2k\delta_1\delta_2 - 4c_1\delta_1\delta_2 + 4c_2\delta_1\delta_2) = 0, \\
2p^2(6\alpha_1\delta_1\delta_2m_1m_2 - 4c_1\delta_1\delta_2 - 4c_2\delta_1\delta_2) = 0, \\
2p^2(-3\alpha_1\delta_1^2m_2^2 + 4p^2k\delta_1^2 + 12\alpha_1\delta_2m_1^2 + 16p^2k\delta_2 + 4c_2\delta_1^2 - 16c_2\delta_2) = 0.
\end{aligned} \quad (3.2)$$

Solving the obtained equations (3.2) by means of symbolic computation, two cases are obtained:

Case 1.

$$\delta_1^2 = \frac{4((m_1 - m_2)^2\alpha + 4p^2k)\delta_1}{(m_1 - m_2)^2\alpha - 4p^2k}, \quad c_1 = \frac{(3m_1^2 + 6m_1m_2 - 3m_2^2)\alpha}{8} - \frac{p^2k}{2}. \quad (3.3)$$

$$c_2 = \frac{(-3m_1^2 + 6m_1m_2 + 3m_2^2)\alpha}{8} + \frac{p^2k}{2}.$$
in which $p, \delta_2, m_1, m_2$ are arbitrary real numbers with $\frac{4((m_1-m_2)^2x^2+4p^2k)\delta_2}{(m_1-m_2)^2x^2-4p^2k} > 0$.

**Case 2.**

$$\delta_1^2 = -4\delta_2, \ c_1 = \frac{3m_2\alpha}{4} - \frac{p^2k}{2}, \ c_2 = \frac{3m_2\alpha}{4} + \frac{p^2k}{2}, \ (3.4)$$

in which $\delta_2, p, m_1, m_2$ are arbitrary real numbers with $\delta_2 < 0$.

For Case 1, substituting (3.3) along with $\xi = x + kz$ into (3.1), we obtain the following results

$$\varphi = \exp(-p(x + m_1y + kz + \Omega_1t)) \pm \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t)) + \frac{\delta_2}{2}\exp(p(x + m_1y + kz + \Omega_1t)), \ (3.5)$$

where $\Omega_1 = \frac{(3m_1^2t+6m_1m_2-3m_2^2\alpha)}{8} - \frac{p^2k}{2}, \ \Omega_2 = \frac{4((m_1-m_2)^2x^2+4p^2k)\delta_2}{(m_1-m_2)^2x^2-4p^2k}, \ \Omega_3 = \frac{(-3m_1^2t+6m_1m_2+3m_2^2\alpha)}{8} + \frac{p^2k}{2}$.

When $\delta_2 > 0$, (3.5) can be reduced to the following equation

$$\varphi = 2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t)) \pm \frac{1}{2}\ln(\delta_2)) \pm \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t)). \ (3.6)$$

According to the variable transformation $u = \sqrt{\varphi}\xi$, we obtain the exact wave solutions for (1.1) as

$$u_1 = \frac{4p\sqrt{\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t)) \pm \frac{1}{2}\ln(\delta_2)) - 2p\sqrt{\Omega_2}\sin(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t)) \pm \frac{1}{2}\ln(\delta_2)) + \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))}, \ (3.7)$$

$$u_2 = \frac{4p\sqrt{\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t)) \pm \frac{1}{2}\ln(\delta_2)) + 2p\sqrt{\Omega_2}\sin(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t)) \pm \frac{1}{2}\ln(\delta_2)) - \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))}, \ (3.8)$$

where $u_1$ and $u_2$ are two homoclinic breather wave solutions of (1.1). Their dynamical properties are much similar, so we only discuss $u_2$ for an example. By selecting appropriate parameters, the profile of $u_2$ is shown in Figure 1, which is of great value in understanding the dynamical behaviors of the breathers (3.7) and (3.8).

For $u_2$ in Eq (3.8) at $p = 0$, taking $\delta_2 = 1$, we can obtain the following Taylor expansions

$$\sinh(p(x + m_1y + kz + \Omega_1t)) = p(x + m_1y + kz + \Omega_1t) + O(p^2),$$

$$\sin(p(x + m_2y + kz + \Omega_3t)) = p(x + m_2y + kz + \Omega_3t) + O(p^2),$$

$$\cosh(p(x + m_1y + kz + \Omega_1t)) = 1 + \frac{1}{2}(p(x + m_1y + kz + \Omega_1t))^2 + O(p^3),$$

$$\cos(p(x + m_1y + kz + \Omega_1t)) = 1 - \frac{1}{2}(p(x + m_2y + kz + \Omega_3t))^2 + O(p^3). \ (3.9)$$

Supposing an arbitrary constant $u_0$ is the equilibrium solution of Eq (1.1), substituting above Taylor expansions into Eq (3.8), then the following rouge wave solution for Eq (1.1) can be obtained

$$u_3 = u_0 + \frac{8x + 4(m_1 + m_2)y + 8kz + 6m_1m_2t}{\left(x + m_1y + kz + \frac{(3m_1^2t+6m_1m_2-3m_2^2\alpha)}{8}\right)^2 + \left(x + m_2y + kz + \frac{(-3m_1^2t+6m_1m_2+3m_2^2\alpha)}{8}\right)^2}. \ (3.10)$$

By choosing the same parameters as in Figure 1, we exhibit the profile of rouge wave solution (3.10) in Figure 2.
Figure 1. (Color online) Plots of breather solution (3.8) with $\alpha = 2, p = 1, \delta_2 = 1, m_1 = 3, m_2 = 1, k = -1, z = 1$ at time $t = 0$. (a) 3D plot; (b) Overhead view of the wave.

Figure 2. (Color online) Rouge wave for Eq (1.1) described by (3.10) with $u_0 = 1, \alpha = 2, k = 3, m_1 = 2, m_2 = -1, z = 1$ at $t = 2$. (a) 3D profile; (b) Vertical view of the wave.

We can see that there is one peak and one trough in Figure 2(a). For any given time $t$, when $x^2 + y^2 + z^2 \to +\infty$, $u_3$ tends to the equilibrium solution $u_0$. If taking $u_0 = 0$, when $p$ tends to 0, $u_3$ is exactly the limit of $u_2$. From the mathematical point of view, the rogue wave solution (3.10) is a limit behavior of breather wave solution (3.8). From the physical point of view, we may think perhaps the energy collection and superposition of breather waves in many periods generate a rogue wave.
For Case 2, when $\delta_2 < 0$, (3.5) can be reduced to the following equation

$$\varphi = -2\sqrt{-\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(-\delta_2)) \pm \sqrt{\Omega_2}\cos(p_2(x + m_2y + kz + \Omega_3t)).$$

(3.11)

According to the variable transformation $u = 2\ln(\varphi)\xi$, we obtain the exact wave solutions for (1.1) as

$$u_4 = \frac{4p\sqrt{-\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(-\delta_2)) + 2p\sqrt{\Omega_2}\sinh(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{-\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(-\delta_2)) - \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))},$$

(3.12)

$$u_5 = \frac{4p\sqrt{-\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(-\delta_2)) - 2p\sqrt{\Omega_2}\sinh(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{-\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(-\delta_2)) + \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))},$$

(3.13)

where $u_4$ and $u_5$ are also two homoclinic breather wave solutions of (1.1). They also have similar dynamical properties, here we only choose $u_4$ for an example. By selecting proper parameters, we present the profile of $u_5$ in Figure 3, which will help us better understand the dynamical behaviors of the breather wave solutions (3.12) and (3.13).

![Figure 3](image-url)

**Figure 3.** (Color online) Breather wave for Eq.(1.1) described by (3.13) with $\alpha = -2, p = -1, \delta_2 = -1, m_1 = 2, m_2 = 1, k = -1, z = 2$ at time $t = 1$. (a) Front view of the wave. (b) Overhead view of the wave.

Substituting (3.4) along with $\xi = x + kz$ into (3.1), we have the following expression

$$\varphi = \exp(-p(x + m_1y + kz + \Phi_1t)) \pm 2\sqrt{-\delta_2}\cos(p(x + m_2y + kz + \Phi_2t)) + \frac{1}{2}\exp(p_1(\xi + m_1y + kz + \Phi_1t)),$$

(3.14)

where $\Phi_1 = \frac{3m_2^2\alpha}{4} - \frac{\nu^2k}{2}, \Phi_2 = \frac{3m_2^2\alpha}{4} + \frac{\nu^2k}{2}$. When $\delta_2 < 0$, (3.14) can be reduced to the following equation

$$\varphi = -2\sqrt{-\delta_2}\sinh(p(x + m_1y + kz + \Phi_1t)) \pm 2\sqrt{-\delta_2}\cos(p(x + m_2y + kz + \Phi_2t)).$$

(3.15)
According to the variable transformation \( u = 2\ln(\phi_{\xi}) \), we obtain the breather wave solution for (1.1) as

\[
\begin{align*}
  u_6 &= \frac{2p \cosh(p(x + m_1y + k z + \Phi_1t)) + 2p \sin(p(x + m_2y + k z + \Phi_2t))}{\sinh(p(x + m_1y + k z + \Phi_1t)) - \cos(p(x + m_2y + k z + \Phi_2t))}, \\
  u_7 &= \frac{2p \sinh(p(x + m_1y + k z + \Phi_1t)) - 2p \sin(p(x + m_2y + k z + \Phi_2t))}{\sinh(p(x + m_1y + k z + \Phi_1t)) + \cos(p(x + m_2y + k z + \Phi_2t))},
\end{align*}
\]

where \( u_6 \) and \( u_7 \) are another two homoclinic breather wave solutions of (1.1). They also have similar dynamical properties, here we only choose \( u_7 \) for an example. By selecting appropriate parameters, the profile of \( u_7 \) is presented in Figure 4. By comparing the other two pairs of breather wave solutions, we can better comprehending the dynamical behaviors of the breather wave solutions (3.16) and (3.17).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{(Color online) Breather depicted by (3.17) with \( \alpha = 2, p = 1, \delta_2 = -1, m_1 = 1, m_2 = \frac{1}{2}, k = 1, z = 1 \) at \( t = 1 \). (a) Perspective view of the wave. (b) Overhead view of the wave.}
\end{figure}

\section{4. Resonant multi-soliton solutions of Eq (1.1)}

In what follows, we would like to construct the resonant multi-soliton solution of Eq (1.1). For the bilinear formula Eq (2.5), we choose \( N \)-exponential wave function as the following form

\[
\phi = \sum_{j=1}^{N} \mu_j \phi_j = \sum_{j=1}^{N} \mu_j \exp(\zeta_j) = \mu_1 \exp(\zeta_1) + \mu_2 \exp(\zeta_2) + \cdots + \mu_N \exp(\zeta_N).
\]

(4.1)

in which the resonant multi-soliton variables \( \zeta_j = h_j \xi + l_j y + r_j t \) and \( \mu_j \) are nonzero (1 \( \leq j \leq N \)). On the basis of the properties of differential operator to exponential functions, substituting the expression (4.1) into the bilinear equation (2.5), we have

\[
\left[ -4D_{\xi}D_{t} + kD_{\xi}^4 + 3\alpha D_{t}^2 \right] \phi \cdot \phi = \sum_{j,l=1}^{N} \mu_j \mu_l P(h_j - h_l, l_j - l, r_j - r_l) \exp(\zeta_j + \zeta_l) = 0.
\]

(4.2)
therefore the Eq (4.2) holds if and only if $P(h_j - h_i, l_j - l_i, r_j - r_i) = 0$. Hence the resonant multi-soliton condition can be written as

$$kh_j^4 - 4kh_j^3h_j + 6kh_j^2h_j^2 - 4kh_jh_j^3 + kh_j^4 + 3\alpha l_j^2 - 6\alpha hl_j + 3\alpha l_j^2 - 4h_jr_i + 4r_jh_i + 4h_jr_i - 4r_jh_j = 0. \quad (4.3)$$

By balancing the power of $h_i, l_j, r_j$, we conjecture the following wave related numbers

$$h_j = h_j, l_j = c_1h_j^2, r_j = c_2h_j^3. \quad (4.4)$$

Substituting (4.4) into (4.2), we can obtain the following solution for bilinear equation (2.5)

$$\varphi = \mu_1\exp(h_1\xi + c_1h_1^2y + c_2h_1^3t) + \mu_2\exp(h_2\xi + c_1h_2^2y + c_2h_2^3t) + \cdots + \mu_N\exp(h_N\xi + c_1h_N^2y + c_2h_N^3t), \quad (4.5)$$

with the condition $c_1 = \sqrt{\frac{k}{2}}, c_2 = k$. Through the transformation $u = 2\ln(\varphi)_c$ and $\xi = x + kz$, we can derive the resonant multi-soliton solution of the equation (1.1), which can be written as

$$u = 2\left(\frac{\partial}{\partial x}\ln\left(\sum_{j=1}^{N} \mu_j\exp(h_jx + \sqrt{\frac{k}{2}h_j^2y - kh_jz + kh_j^3t})\right)\right). \quad (4.6)$$

Especially, the 2-kink solution is written as

$$u_{kink} = \frac{2h_1\mu_1\exp(h_1x + \sqrt{\frac{k}{2}h_1^2y + kh_1z + kh_1^3t}) + 2h_2\mu_2\exp(h_2x + \sqrt{\frac{k}{2}h_2^2y + kh_2z + kh_2^3t})}{\mu_1\exp(h_1x + \sqrt{\frac{k}{2}h_1^2y + kh_1z + kh_1^3t}) + \mu_2\exp(h_2x + \sqrt{\frac{k}{2}h_2^2y + kh_2z + kh_2^3t})}. \quad (4.7)$$

In Figure 5, the physical behaviors at $t = 2$ are vividly presented by selecting proper parameters. For (a) and (d) with the following parameters: $N = 2, \mu_1 = 1, \mu_2 = 3, h_1 = 0.5, h_2 = 1, k = 4, \alpha = 3, z = 1$; For (b) and (e) with the following parameters $N = 4, \mu_1 = 1, \mu_2 = 3, \mu_3 = 0.5, \mu_4 = 2, h_1 = 0.5, h_2 = 1, h_3 = 0.2, h_4 = 3, k = 4, \alpha = 3, z = 1$; For(c) and (f) with the following parameters: $N = 6, \mu_1 = 0.5, \mu_2 = 1.5, \mu_3 = 2, \mu_4 = 0.6, \mu_5 = 0.7, \mu_6 = 1.4, h_1 = 0.8, h_2 = 3, h_3 = 1.2, h_4 = 0.5, h_5 = 1.4, h_6 = 1.8, k = 2, \alpha = 4, z = 0$. As a matter of fact, the waves shown in Figure 5 are inelastic. It is clear that two convex waves appeared in (a) and (d), three convex waves appeared in (b) and (e), while four convex waves appeared in (c) and (f). Their fashions and motion trend in $(x, y)$-plane are vividly exhibited separately. Consequently, the resonant solutions expressed by (4.6) emerge a novel type of interactions.
5. M-breather and hybrid solutions

In the end, we aim to derive M-breather and hybrid solutions of Eq (1.1) by using complex conjugate method. To achieve this goal, we consider its multi-soliton solutions. Based upon the bilinear formalism (2.5), we expand function $\varphi(\xi, y, t)$ into a series of infinitesimal $\varepsilon$,

$$\varphi = 1 + \varphi^{(1)}\varepsilon + \varphi^{(2)}\varepsilon^2 + \varphi^{(3)}\varepsilon^3 + \cdots.$$  (5.1)

Substituting Eq (5.1) into Eq (2.5), we obtain the following bilinear equations by comparing the coefficients of the same power with respect to $\varepsilon$:

$$-4\varphi^{(1)}_{\xi t} + k\varphi^{(1)}_{\xi\xi\xi\xi} + 3\alpha\varphi^{(1)}_{yy} = 0.$$  (5.2a)

$$\left[-4D_\xi D_t + kD_\xi^4 + 3\alpha D_y^2\right] \varphi^{(1)} \cdot \varphi^{(1)} = 2 \left(4\varphi^{(2)}_{\xi t} - k\varphi^{(2)}_{\xi\xi\xi\xi} - 3\alpha\varphi^{(2)}_{yy}\right).$$  (5.2b)

$$-4\varphi^{(3)}_{\xi t} + k\varphi^{(3)}_{\xi\xi\xi\xi\xi} + 3\alpha\varphi^{(3)}_{yy} = \left[4D_\xi D_t - kD_\xi^4 - 3\alpha D_y^2\right] \varphi^{(1)} \cdot \varphi^{(2)}.$$  (5.2c)

From the (5.2a), we can get a solution of $\varphi^{(1)}$ in this form $\varphi^{(1)} = \exp(\eta_1), \eta_1 = l_1(\xi + m_1y + n_1t) + \eta_{01}, n_1 = \frac{kl_1^2 + 3am_1^2}{4}$. Substituting $\varphi^{(1)}$ into (5.2b), we have $4\varphi^{(2)}_{\xi t} - k\varphi^{(2)}_{\xi\xi\xi\xi} - 3\alpha\varphi^{(2)}_{yy} = 0$. If we take $\varphi^{(2)} = 0$, we can derive $-4\varphi^{(3)}_{\xi t} + k\varphi^{(3)}_{\xi\xi\xi\xi\xi} + 3\alpha\varphi^{(3)}_{yy} = 0$ from (5.2c) such that $\varphi^{(3)} = 0$. Continuing to study like this, we
can further derive $\varphi^{(4)} = \varphi^{(5)} = \cdots = 0$. So a finite form of series (5.1) can be derived by truncation. If taking $\varepsilon = 1$, we have $\varphi = 1 + \exp(\eta_1)$. By means of transformation $u = 2\ln(\varphi_\xi)$, we derive the one-soliton solution $u = 2\frac{\partial}{\partial x}\ln(1 + \exp(\eta_1))$ of equation (1.1), and $\eta_1 = l_1(x + m_1y + kz + n_1t) + \eta_01$. On account of the linear property for Eq (5.2a), $\varphi^{(1)} = \exp(\eta_1) + \exp(\eta_2)$ is also its solution. By substituting $\varphi^{(1)} = \exp(\eta_1) + \exp(\eta_2)$ into (5.2b), according to the identity of Hirota bilinear operators, it is clear that $\varphi^{(2)} = \exp(\eta_1 + \eta_2 + A_{12})$ is a solution of (5.2b), where $A_{12} = -\frac{4(l_1-l_2)(l_1-m_1-m_2)\varepsilon}{l_1-l_2} - \frac{3\varepsilon}{2}(l_1-m_1-m_2)^2$, $\eta_0 = \frac{l_1(\xi + m_1y + n_1t) + \eta_{00}}{2}$ and $n_1$ is defined as before. From (5.2b), we can further obtain $\varphi^{(4)} = \varphi^{(5)} = \cdots = 0$. Thus, the truncation solution at $\varepsilon = 1$ with $\varphi = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12})$ can be derived. In addition the two-soliton solution $u = 2\frac{\partial}{\partial x}(\ln(1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}))$ can also be obtained. According to the existence theorem of $N$-soliton solution, with the help of the mathematical induction, $N$-soliton solution for the (3+1)-dimensional equation (1.1) can be derived. Therefore, we have the following conclusions

$$\varphi_N = \sum_{i=0,1}^{\infty} \exp\left( \sum_{i=1}^{N} \lambda_i \eta_i + \sum_{i<j}^{N} \lambda_i \lambda_j A_{ij} \right).$$  (5.3)

through transformation $u = 2\ln(\varphi_\xi)$ and $\xi = x + kz$, $N$-soliton solutions of equation (1.1) can be written as

$$u = 2\frac{\partial}{\partial x}\ln\left( 1 + \sum_{i=0,1}^{\infty} \exp\left( \sum_{i=1}^{N} \lambda_i \eta_i + \sum_{i<j}^{N} \lambda_i \lambda_j A_{ij} \right) \right).$$  (5.4)

in which $\exp(A_{ij}) = \sum_{i=0,1}^{\infty} \frac{\lambda_i \lambda_j A_{ij}}{1} + \exp(\eta_1 + \eta_2 + A_{12})$ is all possible sums with regard to $\lambda_i = 0$ or 1 under the condition $1 \leq i < j \leq N$. Furthermore, $l_i, m_i$ are non-zero constants.

In what follows, we would like to construct M-breather based on multi-soliton solutions. By taking $N = 2$ in (5.4), we can obtain the following result

$$u = 2\frac{\partial}{\partial x}(\ln(1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}))).$$  (5.5)

By selecting $l_1 = l_2 = a_1 + a_2i, m_1 = m_2$ in solution (5.5), we can derive the one-breather solution. Hence the function $\varphi$ can be expressed as

$$\varphi = 1 + 2\cosh(\theta)\cos(\rho) + 2\sinh(\theta)\cos(\rho) + \exp(A_{12})(\cosh(2\theta) + \cosh(2\theta)).$$  (5.6)

where

$$\theta = a_1 \left( x + m_1y + kz + \frac{k l_1^2 + 3a m_1^2}{4} t \right),$$  (5.7)

$$\rho = b_1 \left( x + m_1y + kz + \frac{k l_1^2 + 3a m_1^2}{4} t \right).$$

Therefore, we obtain the 1-breather solution of Eq (1.1) as the following form

$$u = \frac{2(\cosh(\theta) + \sinh(\theta))(a_1\cos(\rho) - b_1\sin(\rho)) + 2a_1\exp(A_{12})(\cosh(2\theta) + \cosh(2\theta))}{1 + 2\cosh(\theta)\cos(\rho) + 2\sinh(\theta)\cos(\rho) + \exp(A_{12})(\cosh(2\theta) + \cosh(2\theta))}.$$  (5.8)
If taking $N = 4$ in (5.4), the function $\varphi$ can be written as the following form

$$\varphi = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \exp(\eta_4) + \exp(\eta_1 + \eta_2 + A_{12}) + \exp(\eta_2 + \eta_3 + A_{23})$$
$$+ \exp(\eta_3 + \eta_4 + A_{34}) + \exp(\eta_1 + \eta_3 + A_{13}) + \exp(\eta_1 + \eta_4 + A_{14}) + \exp(\eta_2 + \eta_4 + A_{24})$$
$$+ \exp(\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}) + \exp(\eta_2 + \eta_3 + \eta_4 + A_{23} + A_{24} + A_{34})$$
$$+ \exp(\eta_1 + \eta_2 + \eta_4 + A_{12} + A_{14} + A_{24}) + \exp(\eta_1 + \eta_3 + \eta_4 + A_{13} + A_{14} + A_{34})$$
$$+ \exp(\eta_1 + \eta_2 + \eta_3 + \eta_4 + A_{12} + A_{13} + A_{14} + A_{23} + A_{24} + A_{34}).$$

(5.9)

which is exactly the four-soliton solution for bilinear equation (2.5). According to the complex conjugate method, taking special selections $l_1 = l_2 = a_1 + b_1i, l_2 = l_3 = a_2 + b_2i, m_1 = m_2 = c_1 + d_1i, m_2 = m_3 = c_2 + d_2i$ in (5.4), then the two-breather solution can be derived. By taking $N = 6$ and selecting $l_1 = l_1^* = a_1 + b_1i, l_2 = l_3 = a_2 + b_2i, l_3 = l_5 = a_3 + b_3i, m_1 = m_2 = c_1 + d_1i, m_2 = m_3 = 34, m_3 = m_4 = c_3 + d_3i$ in (5.4), hence the three-breather solution will be obtained. As the expressions of two-breather solution and three-breather solution are too complex, so we omit their expressions here.

In order to describe their evolution process, appropriate parameters for two-breather are selected with $N = 4, k = 1, \alpha = 2, a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 3, c_1 = 2, d_1 = 3, c_2 = 2, d_2 = 5$ at $t = -5, 0, 5$. The motion pattern of the wave is shown in Figure 6. By the similar way, we select suitable parameters for three-breather solution with $N = 6, k = 2, \alpha = 3, a_1 = 0, a_2 = 0, a_3 = 0, b_1 = 4, b_2 = 3, b_3 = 2, c_1 = 1, c_2 = 2, c_3 = 2, d_1 = 2, d_2 = 3, d_3 = 1$ at $t = -30, 0, 30$. Its fluctuation form is shown in Figure 7.

By analyzing these pictures, several column waves emerge for two-breather while many column waves emerge for three-breather. In Figure 6, when $t = -5$, it appears three peaks, two of them are taller than the third one. Then their heights are different from each other at $t = 0$. At $t = 5$, it appears four or more peaks. As can be seen from Figure 7, when $t = -30$, it appears many peaks and troughs with different heights. Then their numbers decrease at $t = 0$. At $t = 30$, it appears many peaks and troughs, but their height differences are not obvious. They maintain a forward posture for both the two-breather and three-breather waves.

![Figure 6](image-url)

**Figure 6.** (Color online) Profiles of two-breather solution at different times with parameters described above: (a) At $t = -5$ with $z = 1$; (b) At $t = 0$ with $z = 1$; (c) At $t = 5$ with $z = 1$. 

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In the next, we would like to construct hybrid solutions based on multi-soliton solutions. Taking $N = 3$ in (5.5), the function $\varphi$ can be written as the following form

$$\varphi = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \exp(\eta_1 + \eta_2 + A_{12}) + \exp(\eta_1 + \eta_3 + A_{13}) + \exp(\eta_2 + \eta_3 + A_{23}) + \exp(\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}),$$

which is exactly the three-soliton solution for bilinear equation (2.5). By selecting $l_1 = l_3^* = a_1 + b_1 i, m_1 = m_3^* = c_1 + d_1 i$ in (5.10), we can derive the hybrid solution between one-soliton and one-breather. Similarly, by selecting $N = 5$ and $k_1 = k_3^* = a_1 + b_1 i, k_2 = k_4^* = a_2 + b_2 i, l_1 = l_3^* = c_1 + d_1 i, l_2 = l_5^* = c_2 + d_2 i$ in (5.4), the hybrid solution between one-soliton and two-breather can also be obtained. As their expressions are very long, we omit them here in order to save space.

In order to depict the evolution progress of hybrid solutions between breather and soliton, proper parameters are selected with $N = 3, k = 2, \alpha = 4, a_1 = \frac{1}{2}, b_1 = \frac{1}{3}, c_1 = \frac{1}{4}, d_1 = \frac{1}{5}$ at $t = -20, 0, 20$ to exhibit its wave motions. Meanwhile, proper parameters are selected with $N = 5, k = 1, \alpha = 3, a_1 = \frac{1}{10}, a_2 = \frac{1}{20}, b_1 = \frac{1}{2}, b_2 = \frac{1}{4}, c_1 = \frac{3}{5}, c_2 = \frac{3}{10}, d_1 = \frac{4}{5}, d_2 = \frac{9}{10}, k_5 = \frac{3}{7}, l_3 = 3$ at $t = -5, 0, 10$ to depict its fluctuations. As is shown in Figure 8, it appear many breathers at $t = -20$. When colliding occurs at $t=0$ between the breather and soliton, we can see that the breather changes a lot at $t = 20$. It keeps moving forward over time. It can be seen from Figure 9, two parts constitute the hybrid wave solution. It is clear that one part is soliton wave and the other part is two parallel breathers. Both the two parallel breathers maintain their original propagation path. They gradually separate from each other after colliding with solitons.
Figure 8. (Color online) Evolution progress of hybrid solution between one-soliton and one-breather: (a) Profile of hybrid solution at $t = -20$ with $z = 3$; (b) Profile of hybrid solution at $t = 0$ with $z = 3$; (c) Profile of hybrid solution at $t = 20$ with $z = 3$.

Figure 9. (Color online) Evolution progress of hybrid solution between one-soliton and two-breather: (a) Profile of hybrid solution at $t = -5$ with $z = 2$; (b) Profile of hybrid solution at $t = 0$ with $z = 2$; (c) Profile of hybrid solution at $t = 10$ with $z = 2$.

6. Conclusions

In order to establish the bilinear formalism of a (3+1)-dimensional equation, a dependent variable transformation was introduced. This bilinear expression plays a key role in this paper. Then we derived three pairs of breather solutions by employing the homoclinic test method. Among these breather solutions, the rouge wave solution can only be derived from a from $u_2$. Next we obtained resonant multi-soliton solutions by employing the linear superposition principle. Obviously, the resonant multi-soliton solutions which we obtained don’t depend on dispersion relation. By choosing $N = 2, 4, 6$, the physical shape of the waves were shown graphically. In what follows, M-breather solutions including one-breather, two-breather and three-breather were obtained by applying the complex conjugate...
method to the multi-soliton solutions, and their evolution progress were exhibited by choosing appropriate parameters. Compared their dynamical behaviors, we can see that breather solutions which were derived by using the homoclinic test method are different from the ones which were derived by using the complex conjugate method. Finally, the hybrid solutions between breathers and solitons were constructed and their dynamic properties were exhibited in the form of plotting. Hence, these solutions would amend the existing literature on the exact solutions of nonlinear evolution equation. Furthermore, the method in this paper can be more effectively used in other nonlinear evolution equations.

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Conflict of interest

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

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