Radiation Reaction by Massive Particles and Its Non-Analytic Behavior

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Abstract

We derive a massive analog of the ALD (Abraham, Lorentz and Dirac) equation, i.e., the equation of motion of a relativistic charged particle with a radiation reaction term induced by emissions of massive fields. We show that the radiation reaction term has a non-analytic behavior as a function of the mass \( M \) of the radiation field and both expansions with respect of \( M \) and \( 1/M \) are generally invalid. Hence the massive ALD equation cannot be written as a local equation with derivative expansions. We especially investigate the radiation reaction in three specific motions, uniform acceleration, a circular motion and a scattering process.

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1 Introduction

A charged particle emits radiation when it is accelerated and loses the kinetic energy. Effectively, the emission induces a friction term in the equation of motion of the charged particle. The modified equation with the radiation reaction term becomes

\[ m\ddot{z}^\mu = F_{\text{ext}}^\mu + \alpha (\dot{z}^\mu + \dot{z}^\mu \ddot{z}^\nu \ddot{z}_\nu ) \]  

(1.1)

and it is called the ALD equation [1]. The first term is the external force while the second one corresponds to the back-reaction of the radiation of electromagnetic fields. The coefficient \( \alpha \) of the radiation reaction force is \( \alpha = e^2 / 6\pi \) when the massless photon is radiated. The ALD equation itself is well established but it has an infamous problem of the runaway solutions. The ALD equation contains the third derivative term and the point particle is accelerated to the speed of light by the radiation force. One can remove such pathological solutions by imposing a regular boundary condition at the infinite future. But then, the solutions must be accelerated before the external force is applied. It is against the causality and called the problem of the preacceleration. A pragmatic resolution is to treat the backreaction term as a perturbation [2], and replace the ALD equation by the so called Landau Lifshitz equation. The runaway solution and the problem of preacceleration can then be removed but the validity of the Landau-Lifshitz equation is restricted to a situation when the acceleration is sufficiently small. It is also important to investigate how the correction to the ALD equation is obtained from the quantum field theory calculation in which there is no such pathological problems (see, e.g. [3], [4], [5], [6]).

An important aspect of the ALD equation can also be inferred from the study of the radiation reaction in other situations such as in other space-time dimensions [7], in curved backgrounds or the backreaction due to the gravitational radiation [8], [10], [9]. In all of these situations the radiation reaction term becomes nonlocal, and a local differential equation such as the ALD equation can be obtained only in special limits. Hence it will be important to see whether such a nonlocal effect also arises when the radiation field becomes massive, and to see how such a nonlocal effect is related to the problem of runaway solutions.

In this paper, we consider a relativistic point particle coupled with a massive scalar field, and investigate properties of radiation reaction by emissions of the massive scalar fields. This system is considered as a toy model of the ALD equation with massive photons in plasma. We derive an analog of the ALD equation, but the radiation reaction term becomes generally nonlocal unlike the massless ALD equation, and the derivative expansion (1/M expansion) can not be applied. Furthermore, we show that the massless limit \( M \to 0 \) is nonanalytic: the coefficient of the radiation reaction term contains a logarithmic term like \( M^2 \log M \) or \( M^4 \log M \).

The paper is organized as follows. In section 2, we derive the massive analog of the ALD
equation. We first develop several tools for the calculation and compare the radiation reaction in massless and massive cases. The backreaction (radiation reaction) term in the massive case behaves non-analytically as a function of the mass $M$ and contains a logarithmic term like $M^2 \log M$. We also show that the derivative expansion (namely an expansion with respect to $1/M$) is generally invalid. Since the term itself is not singular as a function of $M$, it shows that the radiation reaction term is essentially nonlocal. In section 3, in order to see the explicit behavior of the backreaction, we evaluate the radiation reaction term numerically for specific motions, uniform acceleration and a circular motion. We show that, in the uniform acceleration, the radiation reaction term behaves like $M^2 \log M$ near $M = 0$. We also discuss a scattering process (a motion when the external force is applied during a finite time interval) to see that the nonanalytic behavior is not specific to the uniformly accelerated motion. In appendix A we discuss the non-analytic behavior of the backreaction based on the properties of the Bessel function which appears in the propagator of the massive scalar fields.

2 Massive ALD Equation and Radiation Reaction

2.1 Derivation of the Radiation Reaction Force

The action of a relativistic charged particle interacting with a scalar field is given by

$$ S = -m_0 \int d\tau \sqrt{\dot{z}^\mu \dot{z}_\mu} + \int d^4x \, j(x; z) \phi(x) + \int d^4x \frac{1}{2} (\partial_\mu \phi(x) \partial^\mu \phi(x) - M^2 \phi^2(x)) $$

(2.1)

where the scalar current $j(x; z)$ is defined as

$$ j(x; z) = e \int d\tau \sqrt{\dot{z}^\mu \dot{z}_\mu} \delta^4(x - z(\tau)). $$

(2.2)

In the paper, we set the speed of light $c = 1$. This model is considered as a toy model of a charged point particle in massive photon fields in plasma. The equations of motion of the coupled system of the position of the particle $z(\tau)$ and the scalar field $\phi(x)$ are given by

$$ m_0 \ddot{z}^\mu = -e(\partial^\mu - \dot{z}^\mu \partial_\nu \partial^\nu - \ddot{z}^\mu)\phi(z), $$

$$ (\partial^\mu \partial_\mu + m^2)\phi(x) = j(x; z). $$

(2.3)  (2.4)

The parameter $\tau$ is chosen to satisfy the gauge condition $\dot{z}^\mu \dot{z}_\mu = 1$. The above equation of motion is consistent with this gauge condition. In order to derive the massive ALD equation, we write the scalar field $\phi$ as a sum of an external part and a self-interaction part, $\phi(x) = \phi_{ext}(x) + \phi_{self}(x)$. Then the equation of motion (2.3) for $z(\tau)$ becomes

$$ m_0 \ddot{z}^\mu = F^\mu_{ext} + F^\mu_{self}. $$

(2.5)
where the external force $F_{\text{ext}}^\mu$ is given by the external field $\phi_{\text{ext}}(x)$ as

$$F_{\text{ext}}^\mu(z) = -e(\partial^\mu - \dot{z}^\nu \partial_\nu - \ddot{z}^\mu)\phi_{\text{ext}}(z). \quad (2.6)$$

On the contrary, the self-force (radiation reaction force) $F_{\text{self}}^\mu$ is generated by the charged current of the particle itself, and written in terms of the induced field $\phi_{\text{self}}(x)$ as

$$F_{\text{self}}^\mu(z) = -e(\partial^\mu - \dot{z}^\nu \partial_\nu - \ddot{z}^\mu)\phi_{\text{self}}(z). \quad (2.7)$$

Here the induced field $\phi_{\text{self}}(x)$ is solved in terms of the current $j(x';z)$ by using the retarded Green function

$$\phi_{\text{self}}(x) = \int d^4x' G_R(x,x') j(x';z) \quad (2.8)$$

where $G_R(x,x')$ satisfies the equation

$$(\partial^\mu \partial_\mu + M^2)G_R(x,x') = \delta^{(4)}(x - x'). \quad (2.9)$$

Let us first discuss some general properties of the radiation reaction force without using the explicit form of $G_R(x,x')$.

Due to the Lorentz symmetry, the Green function can be written as a function of the distance $\sigma = (x - x')^2$,

$$G_R(x,x') = \theta(x^0 - x'^0)G(\sigma). \quad (2.10)$$

Substituting this expression in $\phi_{\text{self}}(x)$, one can rewrite the self-force as

$$F_{\text{self}}^\mu(z) = -e^2(\partial^\mu - \dot{z}^\nu \partial_\nu - \ddot{z}^\mu) \int_{-\infty}^{\tau(x^0)} d\tau' G_R((x - z(\tau'))^2)|_{x=z(\tau)}, \quad (2.11)$$

where $\tau(x^0)$ is defined by the proper time $\tau$ which satisfies $z^0(\tau) = x^0$. In taking derivatives with respect to $x$, one first need to evaluate the integral at general $x$, and then take the limit of $x$ to a point $z(\tau)$ on the trajectory of the particle. But noting that $G(\sigma)$ depends on $x^\mu$ and $x'^\mu$ only through their distance $\sigma$, one can express the right hand side of (2.11) in terms of the quantities on the trajectory,

$$\partial_\mu \int_{-\infty}^{\tau(x^0)} d\tau' G(\sigma)|_{x=z(\tau)} = \int_{-\infty}^{\tau} d\tau' \frac{s}{\sigma}G(\sigma). \quad (2.12)$$

The integrand is a function of $\sigma = (z(\tau) - z(\tau'))^2$ and its derivatives while the integral is performed over the proper time $\tau'$ of the trajectory. Since $\sigma$ and $s = \tau - \tau'$ are related to each
other, we can either express the integral \( \text{(2.12)} \) in terms of \( \sigma \) or \( s \). Either expression has its own advantage, so we will explain both expressions in the following.

We first express the integral \( \text{(2.12)} \) and the self-force term (2.11) in terms of \( s = \tau - \tau' \). First \( z(\tau) - z(\tau') \) can be expanded in a power series of \( s \) as

\[
y^{\mu}(s; \tau) \equiv z(\tau) - z(\tau') = - \sum_{n=1}^{\infty} \frac{(-s)^n z^{(n)\mu}}{n!} = s \ddot{z}^\mu(\tau) - \frac{s^2}{2} \dddot{z}^\mu(\tau) + \frac{s^3}{6} \ddddot{z}^\mu(\tau) + \cdots, \tag{2.13}
\]

where \( z^{(n)\mu}(\tau) = d^n z^\mu(\tau)/d\tau^n \). We fix \( \tau \) and change the variable from \( \tau' \) to \( s = \tau - \tau' \). By using the gauge condition \( \dddot{z}^\mu = 1, \dddot{z}^{\nu} = 0 \) and \( \dddot{z}^{\nu} \dddot{z}_\mu + \dddot{z}^{\mu} \dddot{z}_\mu = 0 \), it is straightforward to check the following relations

\[
\begin{align*}
\sigma(s; \tau) &= y^{\mu} y_\mu = s^2(1 - \frac{s^2}{12} \dddot{z}^\mu \dddot{z}_\mu + \cdots), \\
\frac{d\sigma(s; \tau)}{ds} &= 2y^{\mu} \dot{y}_\mu = 2s(1 - \frac{s^2}{6} \dddot{z}^\mu \dddot{z}_\mu + \cdots). \tag{2.14}
\end{align*}
\]

Using the relation \( d/d\sigma = (ds/d\sigma) d/ds \), one obtains

\[
F_{\mu \text{self}}^\mu = -e^2 \int_0^\infty ds \left( P^\mu_\nu \frac{y^\nu}{y^\mu y_\rho} \frac{d}{ds} - \dddot{z}^\mu(\tau) \right) G(\sigma(s; \tau)), \tag{2.15}
\]

where \( P^\mu_\nu = \delta^\mu_\nu - \dddot{z}^\mu(\tau) \dddot{z}_\nu(\tau) \) is a projection operator which satisfies \( P^\mu_\nu \dddot{z}^\nu(\tau) = 0 \). The surface term does not contribute because \( P^\mu_\nu (y^\nu/y^\mu y_\rho) \rightarrow P^\mu_\nu \dddot{z}^\nu = 0 \) near \( s = 0 \). So we can perform an integration by parts and get the expression of the self-force term in terms of \( s \)

\[
F_{\mu \text{self}}^\mu = e^2 \int_0^\infty ds \left[ P^\mu_\nu \frac{d}{ds} \left( \frac{y^\nu}{y^\mu y_\rho} \right) + \dddot{z}^\mu(\tau) \right] G(\sigma). \tag{2.16}
\]

In order to evaluate it, let us first expand the integrand in powers of \( s \),

\[
F_{\mu \text{self}}^\mu = e^2 \int_0^\infty ds \left[ P^\mu_\nu \left\{ \frac{d}{ds} \left( -\frac{s}{2} \dddot{z}^\mu + \frac{s^2}{6} (\dddot{z}^\mu + \dddot{z}^{\mu} \dddot{z}_\rho + \cdots) \right) \right\} + \dddot{z}^\mu \right] G(\sigma)
\]

\[
= e^2 \int_0^\infty ds \left[ \frac{1}{2} \dddot{z}^\mu + \frac{s}{3} (\dddot{z}^\mu + \dddot{z}^{\mu} \dddot{z}_\rho + \cdots) \right] G(\sigma). \tag{2.17}
\]

As we will see later, for a massless scalar field where \( G(\sigma) \sim \delta(\sigma) \), higher order terms of \( s \) vanish except for the mass renormalization and the radiation reaction term. Thus one obtains the ALD equation, which is written as a local equation. On the other hand, for a massive scalar field, \( G(\sigma) \) only damps as an inverse-power of \( \sigma \) at \( s \rightarrow \infty \). Then integrals of \( s \) for sufficiently higher orders diverge. Since the integral \( \text{(2.16)} \) is shown to be finite after the mass renormalization, the divergence of the coefficients for higher orders indicates that the derivative expansion of the radiation reaction term is not valid. Hence the massive analog of the ALD
equation cannot be written as a sum of local terms. We therefore need to evaluate the integral (2.16) directly without the derivative expansion, and for this purpose, it is more convenient to express the integral in terms of $\sigma$ instead of $s$.

The Green function $G(\sigma)$ is generally a complicated function of $\sigma$ (e.g. see (2.26) for a massive scalar field. Therefore it is sometimes more appropriate to express the integral in terms of $\sigma$ instead of $s$, especially in evaluating the radiation reaction term numerically. The integral (2.16) is rewritten by changing the integration variable from $s$ to $\sigma$ as

$$F_{\text{self}}^\mu = e^2 \int_0^\infty d\sigma \left[ P^\mu_\nu \left( \frac{d}{d\sigma} 2y^\nu \frac{ds}{d\sigma} \right) + \ddot{z}^\mu(\tau) \frac{ds}{d\sigma} \right] G(\sigma).$$

(2.18)

In the region $0 < s < \infty$, the function $\sigma(s)$ is single valued and $d\sigma/ds > 0$ is satisfied. Hence we can solve $s$ as a function of $\sigma$. By using the variable $l = \sqrt{\sigma}$, the self-force can be expressed as

$$F_{\text{self}}^\mu = e^2 \int_0^{\infty} dl \left[ P^\mu_\nu \left( \frac{d}{dl} y^\nu 1 \frac{ds}{dl} \right) + \ddot{z}^\mu(\tau) \frac{ds}{dl} \right] G(l^2)$$

$$= e^2 \int_0^{\infty} dl \left[ \ddot{z}^\mu 2 + (\ddot{z}^\mu + \dddot{z}^\nu \dddot{z}_\nu) \frac{l}{3} + \cdots \right] G(l^2).$$

(2.19)

In the second equality, we have used the following relations

$$\frac{ds}{dl} = 1 + \ddot{z}^\nu \dddot{z}_\nu \frac{l^2}{2} + o(l^3),$$

$$y_\mu = \dot{z}_\mu l - \ddot{z}_\mu \frac{l}{2} + \left( \dddot{z}_\mu + \dddot{z}^\nu \dddot{z}_\nu \frac{4}{4} \right) \frac{l^3}{6} + o(l^4).$$

(2.20)

The usefulness of the expression (2.19) becomes apparent when we consider, e.g. the renormalization of the mass term $m \ddot{z}$. The radiation reaction term, (2.16) or (2.18), contains a divergence which is proportional to $\ddot{z}^\mu$, and should be absorbed by the mass renormalization. For a massless scalar field with $G(\sigma) \sim \delta(\sigma) = \delta(s^2)$, it makes no difference whether we subtract the divergent term in (2.16) or (2.18). However, for a massive case, $G(\sigma)$ is a nontrivial function of $\sigma(z, z') = (z^\mu(\tau) - z^\mu(\tau'))^2$, and since $\sigma$ depends on $z^\mu$ and $\tau'$ in a complicated way, an integral of $G(\sigma)$ such as $\int ds f(s) \dddot{z}^\mu(\tau) G(\sigma)$ depends not only on $\dddot{z}^\mu(\tau)$ but also on the details of the trajectory $z(\tau')$. On the other hand, an integral of $G(\sigma)$ such as $\int ds f(s) \dddot{z}^\mu(\tau) G(\sigma)$ is a constant multiplied by $\ddot{z}^\mu(\tau)$. Therefore the mass renormalization of the massive ALD equation should be evaluated not in the expression (2.16) but in (2.18).

### 2.2 Massless Radiation

Before considering a massive case, we briefly review the evaluation of the radiation reaction by a massless scalar field. For a massless scalar field, the retarded Green function is simply given
by

\[ G_R(\mathbf{x} - \mathbf{x}') = \theta(x^0 - x'^0) \frac{\delta(\sigma)}{2\pi}. \quad (2.21) \]

By substituting it in (2.17) or (2.19), we can obtain the ALD equation

\[ m\ddot{z}^\mu = F^\mu_{\text{ext}} + \frac{e^2}{12\pi} (\dot{\tau}^\mu + \dot{z}^\mu \dot{\tau}^\nu \dot{z}_{\nu}). \quad (2.22) \]

Here the divergent term

\[ e^2 \int_0^\infty dl \frac{\delta(l)}{4l} \quad (2.23) \]

is absorbed by the mass renormalization.

We would like to mention the derivation of the ALD equation from the quantum field theory calculations [5]. The authors evaluated two types of diagrams which contribute to the backreaction. One type of diagrams corresponds to the radiation processes. The contribution of this type to the backreaction turns out to be written in the form of (2.11) but with \( G_R \) replaced by \( G_- \)

\[ G_-(\mathbf{x}, \mathbf{x}') = G_R(\mathbf{x}, \mathbf{x}') - G_A(\mathbf{x}, \mathbf{x}') = \frac{\theta(x^0 - x'^0) - \theta(x'^0 - x^0)}{2} G(\sigma), \quad (2.24) \]

where \( G_A(\mathbf{x}, \mathbf{x}') \) is the advanced Green function. With this replacement, the terms with even powers of \( s \) in (2.17) vanish and the remaining terms become finite. Another type of diagrams corresponds to the self-energy (including the mass renormalization) of the point particle. It has a form of (2.11) but with \( G_R \) replaced by \( G_+ \)

\[ G_+(\mathbf{x}, \mathbf{x}') = G_R(\mathbf{x}, \mathbf{x}') + G_A(\mathbf{x}, \mathbf{x}'). \quad (2.25) \]

It gives a divergent term which can be absorbed by the mass counter term. Summing these two types of diagrams, one can reproduce the ALD equation.

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1 The integration seems subtle since the delta function is just at the edge of the integration range. But the subtlety can be removed by going back to the original equation (2.11). Here the integral is performed over the region \( \tau(x^0) > \tau' > -\infty \) and the delta function is always inside the region.

2 One might expect to subtract a divergence by replacing \( G_R \) by \( G_- \) or \( G_F \) also for the massive case discussed in the next subsection. But the Green function in the massless case depends on \( \sigma \) only through \( \delta(\sigma) \), and the acausal time dependence of \( G_A \) inside \( G_- \) and \( G_F \) does not appear in the final results. However, in a massive case, the Green function has a tail and a simple replacing of \( G_R \) by another type of Green functions such as \( G_{\pm} \) violates the causality explicitly.
2.3 Massive Radiation

The retarded Green function of a massive scalar field is given by

\[ G_R(x, x') = \frac{\theta(x^0 - x'^0)}{4\pi} \left( 2\delta(\sigma) - \theta(\sigma) \frac{MJ_1(M\sqrt{\sigma})}{\sqrt{\sigma}} \right). \tag{2.26} \]

The first term in the parentheses is the same as the massless case while the second term represents a modification of the Green function by the mass of the radiation field. Since the second term disappears in the massless limit, one might expect that the massive ALD equation can be written as a perturbation of the massless ALD equation by higher derivative terms. The situation is not so simple, however. By the dimensional analysis, the coefficient of the derivative expansion must contain an inverse power of \( M \), but such an expansion is shown to be invalid because of the nonanalyticity at \( M = 0 \).

To see such nonanalytic behaviors of the derivative expansion, let us try to evaluate each coefficient of the derivative expansion by substituting the Green function (2.26) in (2.19). The leading order term is proportional to \( \dddot{z}^\mu \) which corresponds to the mass renormalization. The next order term gives the coefficient of the ALD term. The coefficient, however, vanishes due to the cancellation between the massless propagator \( \delta(\sigma) \) and the massive modification of the Green function,

\[ e^2 (\dddot{z}^\mu + \dot{z}^\mu \ddot{z}_\nu) \int_0^\infty dl \frac{l}{12\pi} \left( 2\delta(l^2) - \frac{MJ_1(Ml)}{l} \right) = 0. \tag{2.27} \]

This cancellation is universal and does not depend on the value of \( M \). We are tempted to conclude that the ALD term vanishes for a massive scalar field, but it is not correct since the derivative expansion is shown to be invalid. It suggests that something strange happens in the massless limit. Let us evaluate the coefficients of higher order terms. They are written as

\[ e^2 \int_0^\infty dl \ l^{n+1} \frac{MJ_1(Ml)}{l}. \tag{2.28} \]

Since the Bessel function behaves as \( J_1(x) \sim 1/\sqrt{x} \) for large \( x \) (see Appendix A), the integral diverges for \( n > 1/2 \). Therefore the derivative expansion (or equivalently a power-series expansion with respect to \( l = \sqrt{\sigma} \)) is not justified. One can show that the expansion in terms of \( s \) in (2.17) is also divergent. Such singular behavior of the derivative expansion in the massive case shows a peculiarity of the massless ALD equation where the radiation reaction force is written as a local derivative term.

Though one can not write the radiation reaction term in terms of local derivative terms, the backreaction itself is finite even in the massive case. To confirm the finiteness of the backreaction, we go back to (2.18) or (2.16). The integration can be divided into two parts, one
involving the integration of $\delta(\sigma)$ and the other involving the integration of $J_1(M\sqrt{\sigma})$. Since the first term is the same as the massless case, it is sufficient to show the finiteness of the integrals of the second part,
\[
\int_0^\infty d\sigma \ P^\mu_\nu \frac{d}{d\sigma} \left( y^\nu \frac{ds}{d\sigma} \right) \frac{MJ_1(M\sqrt{\sigma})}{\sqrt{\sigma}},
\] (2.29)
and
\[
\int_0^\infty d\sigma \ frac{ds}{d\sigma} MJ_1(M\sqrt{\sigma}) \sqrt{\sigma}.
\] (2.30)

Note that for general trajectories $P^\mu_\nu(d/d\sigma)(y^\nu ds/d\sigma)$ or $ds/d\sigma$ is regular at finite $\sigma$ and falls off faster than $\sigma^{-1/2}$ at $\sigma \to \infty$ (or $s \to \infty$). On the other hand, $MJ_1(M\sqrt{\sigma})/\sqrt{\sigma}$ is regular at finite $\sigma$ and behaves like
\[
\frac{1}{\sigma} \cos \sqrt{\sigma}
\] (2.31)
at infinity. So the integration (2.18) is finite for a fixed $M$, and we get a finite amount of backreaction by emissions of a massive scalar field as expected. We see this finiteness explicitly in the next section for special types of trajectories.

We have seen that, if we perform the derivative expansion, the coefficient of the ALD term vanishes for a finite $M$ while it becomes $(e^2/12\pi)$ at $M = 0$. One may also be interested in the mass expansion near $M = 0$ of the backreaction term and how the local equation can be derived from the massive ALD equation with a nonlocal radiation reaction term, (2.18) or (2.16). But it is not so simple as one might expect. For example, consider the convergent integral (2.29). If we naively expand it as a power series of $M$,
\[
M^2 \int_0^\infty P^\mu_\nu \frac{d}{d\sigma} \left( y^\nu \frac{ds}{d\sigma} \right) \sum_{n=0}^{\infty} \frac{(-1)^{n}(M\sqrt{\sigma}/2)^{2n}}{N!\Gamma(n+2)}.
\] (2.32)
But the higher order terms of $M$ are divergent, and such an expansion is not valid. The backreaction must have a non-analytic behavior at $M = 0$. We will investigate this kind of non-analyticity in the next section.

## 2.4 Mass Renormalization

In this subsection, we discuss the mass renormalization. Because of the non-locality of the backreaction term, the mass renormalization becomes more subtle than the massless case. Since the non-local term is convergent, one may naively renormalize the mass only by the divergent integral. But then the renormalization of mass becomes independent of $M$ and survives even
in the infinite $M$ limit where the emission of such a massive field is highly suppressed. More explicitly, because of the identity of the Bessel function,

$$\lim_{M \to \infty} M \theta(\sigma) \frac{J_1(M \sqrt{\sigma})}{\sqrt{\sigma}} = 2\delta(\sigma),$$  \hspace{1cm} (2.33)

the Green function vanishes in the large $M$ limit

$$\lim_{M \to \infty} G_R(x, x') = \lim_{M \to \infty} \frac{\theta(x^0 - x'^0)}{4\pi} \left( 2\delta(\sigma) - \theta(\sigma) \frac{MJ_1(M \sqrt{\sigma})}{\sqrt{\sigma}} \right) = 0. \hspace{1cm} (2.34)$$

The backreaction therefore must vanish in the limit $M \to \infty$. Thus we also need to take into account the non-local but convergent integrals, (2.29) and (2.30), in discussing the mass renormalization at finite $M$.

In order to make a consistent mass renormalization, we first introduce a function $f(l)$ and define the renormalized mass term by

$$\delta m \ddot{z}^\mu = -e^2 \int_0^\infty dl \, f(l)G(l^2). \hspace{1cm} (2.35)$$

Then the self-force term can be divided into the mass renormalization $\delta m \ddot{z}^\mu$ and the radiation reaction force $F_{\mu,0,self}$ as

$$F_{\mu, self}^\mu = F_{\mu,0,self}^\mu + \delta m \ddot{z}^\mu. \hspace{1cm} (2.36)$$

The function $f(l)$ must satisfy the condition $f(0) = 1/2$ so that the divergent term is correctly subtracted in (2.16). It is consistent with the mass renormalization in the massless case. For the massive scalar field, the above subtraction (2.35) has a contribution not only from the term $\delta(\sigma)$ in the Green function but also from the non-local term $\sim J_1(M \sqrt{\sigma})/\sqrt{\sigma}$. Another condition for $f(l)$ is that the mass renormalization should vanish in the large $M$ limit. The simplest choice is $f(l) = 1/2$, but the choice is not unique. The ambiguous part of the mass renormalization is written as

$$\delta m_n = -e^2 \int_0^\infty dl f(l) \frac{MJ(Ml)}{l}. \hspace{1cm} (2.37)$$

It is independent of the trajectory of the particle. Hence a different choice of $f(l)$ does not affect how the backreaction term depends on the details of the trajectory such as the acceleration or the velocity of the particle’s motion. In this paper, we take the simplest choice $f(l) = 1/2$. In the next section, we will show, if we renormalize the mass by $f(l) = 1/2$, the radiation reaction term becomes $o(a^3)$ when a particle is uniformly accelerated with an acceleration $a$. For other choices, it is generally proportional to $o(a)$. This seems to justify the simplest choice $f(l) = 1/2$. 

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3 Radiation Reaction for Specific Trajectories

In order to investigate the non-analytic properties of the backreaction, we evaluate (2.16) and (2.18) explicitly for some specific trajectories. We consider three cases, a uniform acceleration, a circular motion and a scattering process.

3.1 Uniform Acceleration

A uniformly accelerated point particle is described by the trajectory

\[ z^\mu = \left( \frac{1}{a} \sinh a\tau, \frac{1}{a} \cosh a\tau, 0, 0 \right), \tag{3.1} \]

where \( a \) is the acceleration and \( \ddot{z}^\nu \dot{z}_\nu = -a^2 \). \( \ddot{z}^\mu(\tau) \) and \( \dddot{z}^\mu \) form a two dimensional vector space and other derivatives are written in terms of them as follows

\[ z^{(2n)\mu} = a^{2(n-1)} \dot{z}^\mu, \quad z^{(2n+1)\mu} = a^{2n} \ddot{z}^\mu, \quad \dddot{z}^\mu \dot{z}_\mu = 0. \tag{3.2} \]

For the trajectory (3.1), the radiation reaction term vanishes

\[ \dddot{z}^\mu + \dddot{z}^\nu \dot{z}_\nu = 0. \tag{3.3} \]

\( l(s) \) or \( \sigma(s) \) has the following simple form

\[ l = \sqrt{\sigma} = \sqrt{(z^\mu(\tau) - z^\mu(\tau'))(z_\mu(\tau) - z_\mu(\tau'))} = \frac{2}{a} \sinh \frac{a(\tau - \tau')}{2}. \tag{3.4} \]

The backreaction becomes proportional to \( \dddot{z}^\mu \) since \( P^\nu_\nu \dddot{z}^\nu = 0 \) and is given by

\[ F^\mu_{\text{self}} = \frac{\ddot{z}^\mu}{a} F(a, M) = e^2 a (\sinh a\tau, \cosh a\tau, 0, 0) \int_0^\infty dl \left( \frac{1 + a^2 l^2/2}{2(\sqrt{1 + a^2 l^2/4})^3} - \frac{1}{2} \right) G_R(l). \tag{3.5} \]

Because of (3.3) the delta function term in (2.26) vanishes in \( F^\mu_{\text{self}} \). Then \( F(a, M) \) becomes

\[ F(a, M) = -e^2 a \int_0^\infty dl \left( \frac{1 + a^2 l^2/2}{2(\sqrt{1 + a^2 l^2/4})^3} - \frac{1}{2} \right) \frac{M J_1(Ml)}{4\pi l} \]

\[ = -e^2 a M \int_0^\infty dt \left( \frac{1 + 2 gt^2}{2(\sqrt{1 + gt^2})^3} - \frac{1}{2} \right) \frac{J_1(t)}{4\pi t} \]

\[ = e^2 a M \bar{F}(g), \tag{3.6} \]

where \( g = a^2/4M^2 \). In the second equality, we changed the integration variable from \( l \) to \( t = Ml \). All information of the backreaction to the uniformly accelerated particle is contained in the function \( F(a, M) \) or equivalently in \( \bar{F}(g) \).
Figure 1: The function $\bar{F}(g)$ is plotted as a function of $g = a^2/4M^2$.

The function $\bar{F}(g)$ can be expressed by using the Meijer’s G-function (see for example [11])

$$\bar{F}(g) = \frac{1}{8\pi} \left( 1 - \int_0^\infty dt \frac{1 + 2gt^2}{(1 + gt^2)^3} J_1(t) \right)$$

$$= \frac{1}{8\pi} - \frac{I_1(\frac{1}{\sqrt{g}})}{8\pi \sqrt{g}} K_1(\frac{1}{\sqrt{g}}) + \frac{g}{\pi^{3/2}} G_{21}^{13} \left( \frac{1}{4g^{3/2}3/2,1/2} \right). \quad (3.7)$$

The behavior of the function $\bar{F}(g)$ is plotted in figure. In the two limiting cases of $g = 0$ and $g \to \infty$, it can be approximated as

$$\lim_{g \to 0} \bar{F}(g) = -\frac{1}{16\pi} g + \cdots,$$

$$\lim_{g \to \infty} \bar{F}(g) = \frac{1}{8\pi} \left( 1 - \frac{\log \sqrt{g}}{\sqrt{g}} \right) + \cdots. \quad (3.8)$$

Substituting $g = a^2/4M^2$ in $\bar{F}(g)$, one obtains the dependence of the backreaction on the acceleration $a$ of the point particle and the mass $M$ of the radiation field.

Let us first look at the behavior of the backreaction $F(a, M)$ as a function of $M$ with the acceleration $a$ fixed. It is plotted in Figure. From (3.8), one can see that near $M = 0$ the backreaction $F(a, M)$ becomes

$$\lim_{M \to 0} F(a, M) = \lim_{M \to 0} e^2aM \bar{F}(\frac{a^2}{4M^2}) = \frac{e^2aM}{8\pi} - \frac{e^2M^2}{4\pi} \log M + \cdots, \quad (3.9)$$

Note that $e^2aM \bar{F}(g)$ vanishes at $M = 0$. However it is not analytic at $M = 0$ and contains a logarithmic term proportional to $M^2 \log M$. This is the reason why one could not expand the backreaction term with respect to $M$. One can also obtain the large mass limit, $M \to \infty$, as

$$\lim_{M \to \infty} F(a, M) = -\frac{1}{16\pi} \frac{e^2a^3}{4M} + \cdots. \quad (3.10)$$
The backreaction vanishes in the limit as expected.

We then fix $M$ and look at the dependence of the backreaction on the acceleration $a$. The behavior of $F(a, M) = aM\tilde{F}(g)$ as a function of $a$ is plotted in Figure. In the limit $a \to \infty$, the back reaction term $F(a, M)$ becomes

$$
\lim_{a \to \infty} F(a, M) = \lim_{a \to \infty} \left( \frac{e^2 a M}{8\pi} + e^2 M^2 \log a \right).
$$

It is proportional to $a$ in the leading order, and has a correction of $\log a$. The logarithmic factor $\log a$ has the same origin as the logarithmic factor $\log M$ in (3.9). In the limit $a \to 0$ which corresponds to the limit of the point charge staying at the origin, the backreaction is suppressed as $a^3$

$$
\lim_{a \to 0} e^2 a M F(g) = -\lim_{a \to 0} \frac{e^2}{16\pi} \frac{a^3}{4M} \to 0
$$

If we took a different choice of $f(l)$, the coefficient of the term proportional to $a$ did not vanish. This implies the plausibility of the simplest choice $f(l) = 1/2$. Indeed, if we choose a different

\footnote{It is written by an inverse power of $M$, but it does not mean that a derivative expansion is valid since an explicit trajectory was used to derive the result.}
$f(l)$, the change of the backreaction term is given by

$$e^2 a M \int_0^\infty dt \left( f(t/M) - \frac{1}{2} \right) \frac{J_1(t)}{4\pi t} = e^2 a \delta F(M). \quad (3.13)$$

It is proportional to $a$ multiplied by a function of $M$. Though a different choice of $f(l)$ changes the mass renormalization and the leading behavior of the radiation reaction in the $a \to 0$ or $a \to \infty$ limit, it does not change the logarithmic behavior ($\log a$ or $\log M$).

Another important feature is the sign of the backreaction $F(a, M)$. As shown in figure 2 $F(a = 1, M)$ becomes either negative or positive, and such a behavior of $F(a, M)$ is not special to $f(l) = 1/2$, but also holds for a general choice of $f(l)$ (see the appendix B). Since the backreaction is given by $F^{\mu}_{\text{self}} = (\ddot{z}^\mu /a)F(a, M)$, a negative value of $F(a, M)$ is physically natural since it means that the backreaction term suppresses a change of the particle motion.

On the contrary, if $F(a, M)$ is positive, it enhances the acceleration when the particle is accelerated. Hence it is an indication of an instability of particle motion or a breakdown of the framework which we are using. The positiveness $F(a, M) > 0$ for a specific region of the parameters is generally satisfied for $a \gg M$. It is interesting to investigate a relation with the problem of runaway solutions in the massless ALD equation.

### 3.2 A Scattering Process

In the previous section, we have seen a non-analytic behavior of the backreaction for a uniform acceleration. In this case, the point charge is accelerated for an infinitely long time, and the total radiation becomes infinite. Hence one may suspect that such a non-analytic behavior is caused by the infinite acceleration. In this section, we see that a similar non-analytic behavior does arise even when a particle is accelerated for a finite time interval.

Now we consider a charged particle with a constant four velocity $v^\mu$ at $\tau < 0$. During a finite time interval $0 < \tau < L$, the particle is scattered (or feels an external force) and then after $\tau > L$ it goes away with a velocity $v'^\mu$. The trajectory is given by

$$z^\mu = \begin{cases} 
  v^\mu \tau, & \text{for } \tau < 0 \\
  q^\mu(\tau), & \text{for } 0 < \tau < L \\
  v'^\mu \tau + c^\mu, & \text{for } \tau > L
\end{cases} \quad (3.14)$$

where $q^\mu(\tau)$ is a function satisfying the boundary conditions $q^\mu(0) = 0$ and $q^\mu(L) = c^\mu + v'^\mu L$.

At $\tau < 0$, the backreaction is 0 since no radiation has been emitted yet. It is confirmed from the equation (2.16). Since $y^\mu = sv^\mu$, the backreaction trivially vanishes by the projection
operator $P^\mu_\nu = \delta^\mu_\nu - \ddot{z}^\mu \dot{z}_\nu$. During the time interval $0 < \tau < L$, the point particle is scattered by the target (or source of the external field). Let us divide the integral (2.16) into two parts, one with $\int_0^{\tau} ds$ and the other with $\int_{\tau}^\infty ds$. The first part does not cause any non-analytic behavior since the integration range $\int_0^{\tau} ds$ is finite. So we are interested in the second part,

$$F^\mu_\infty = -e^2 \int_\tau^\infty ds \left( P^\mu_\nu \frac{dy^\nu}{ds} + \ddot{z}(\tau) \right) \frac{MJ_1(Ml)}{4\pi l}. \tag{3.15}$$

Using the following relations

$$y^\mu = z^\mu(\tau) - z^\mu(\tau') = q^\mu(\tau) - v^\mu\tau', \quad \dot{y}^\mu = \frac{dy^\mu}{ds} = v^\mu, \quad \sigma = y^\mu y_\mu = q^\mu q_\mu + \tau'^2 - 2q^\mu v_\mu \tau', \tag{3.16}$$

one can rewrite the integral as

$$F^\mu_\infty = -e^2 \int_\tau^\infty ds \left( \frac{\delta^\mu_\nu - \ddot{q}^\mu \dot{q}_\nu}{(v^\rho q_\rho - \tau')^2} \left[ v^\rho q^\alpha - q^\rho v^\alpha \right] v_\alpha \right) + \ddot{q}^\mu \frac{MJ_1(Ml)}{4\pi l}. \tag{3.17}$$

Using the property of $l \sim \tau' \sim s$ as $s \to \infty$, the integration becomes

$$F^\mu_\infty \to -e^2 \int_\tau^\infty dl \left( \ddot{q}^\mu + \left( \delta^\mu_\nu - \ddot{q}^\mu \dot{q}_\nu \right) \left( v^\rho q^\alpha - q^\rho v^\alpha \right) v_\alpha \frac{1}{l^2} \left( 1 - 2\frac{v^\rho q_\rho}{l} + \cdots \right) \right) \frac{MJ_1(Ml)}{4\pi l}. \tag{3.18}$$

The leading power of $l$ at $l \to \infty$ in the parenthesis is given by the term proportional to $l^{-3}$. Due to the property of an integral of the Bessel function (see Appendix A), the integration gives a term proportional to $M^4 \log M$. This term vanishes in the massless limit, but it is non-analytic and we cannot expand the backreaction term with respect to the mass $M$. Hence, even for a scattering process where the point particle is accelerated only during a finite time interval, the backreaction behaves non-analytically at $M = 0$.

### 3.3 A Circular Motion

The third example we consider is a circular motion. When the particle is moving on a circle, the trajectory is given by

$$z^\mu = (\gamma \tau, \rho \cos \gamma \omega \tau, \rho \sin \gamma \omega \tau, 0). \tag{3.19}$$

The parameter $\rho$ is the radius of the circular motion and $\rho \omega (< 1)$ is the velocity of the point charge. These parameters $(\rho, \omega, \gamma)$ must satisfy

$$\gamma = \frac{1}{\sqrt{1 - \rho^2 \omega^2}}. \tag{3.20}$$

\[\] Since the integral is always convergent, we can obtain a power series of $M^n$ by expanding $MJ_1(Ml)$ with respect to $M$. 14
in order to hold the gauge condition \( \dot{z}^\mu \dot{z}_\mu = 1 \). Since \( \dot{z}^\mu \), \( \ddot{z}^\mu \) and \( \dddot{z}^\mu \) are all independent of each other, the backreaction can be generally written in the form

\[
F^\mu_{self}(z) = e^2 [F_m(0, \cos \gamma \omega_T, \sin \gamma \omega_T, 0) + F_{ALD}(\rho \omega, -\sin \gamma \omega_T, \cos \gamma \omega_T, 0)], \tag{3.21}
\]

where the first term is proportional to \(-\dddot{z}^\mu\) and the second term is proportional to \(-\dot{z}^\mu - (\dddot{z}^\nu \dddot{z}_\nu)\). A term proportional to \(\dot{z}^\mu\) vanishes due to the projection operator \(P^\mu_{\nu}\). The following relations are useful

\[
\dddot{z}^\mu + \dot{z}^\mu \dddot{z}_\nu = \rho \gamma^3 \omega^3 (\rho \omega, \sin \gamma \omega_T, -\cos \gamma \omega_T, 0), \quad \dddot{z}_\nu = -\rho^2 \gamma^4 \omega^4,
\]

\[
\sigma = (z(\tau)^\mu - z(\tau')^\mu)(z(\tau)_\mu - z(\tau')_\mu) = \gamma^2 s^2 - 2 \rho^2 (1 - \cos \gamma \omega s). \tag{3.22}
\]

The coefficient \(F_m\) is given by

\[
F_m = -\int_0^\infty ds \rho \left( \frac{(1 + \rho^2 \omega^2)(\cos \gamma \omega s - 1) + \gamma \omega s \sin \gamma \omega s}{(\gamma s - \rho^2 \omega \sin \gamma \omega s)^2} + \gamma^2 \omega^2 \left( \frac{1}{2} \frac{dl}{ds} - 1 \right) \right) \frac{M J_1(Ml)}{4\pi l}, \tag{3.23}
\]

and determines the mass renormalization. The other term \(F_{ALD}\) gives a coefficient of the ALD term

\[
F_{ALD} = -\int_0^\infty ds \rho \gamma \omega s \cos \gamma \omega s - \sin \gamma \omega s \frac{M J_1(Ml)}{4\pi l} - \frac{\rho \gamma^3 \omega^3}{12\pi}. \tag{3.24}
\]

The second term \(-\rho \gamma^3 \omega^3 / 12\pi\) is equal to the ALD force when the radiation field is massless while the first term gives a correction in the massive case and comes from the non-local part of the Green function.

In order to see the dependence of the backreaction on \(M\), we first note that \(l \sim \gamma s\) at \(s \to \infty\). Then both of the integrands of \(F_{ALD}\) and \(F_m\) are proportional to \(\sin \omega l\) or \(\cos \omega l\) at \(l \to \infty\). Thus there are no non-analytic terms such as \(\log M\) (see appendix A for details) in the case of the circular motion.

Finally let us see how the radiation reaction changes as a function of \(M\). In Figure. 4 the ratio of \(F_{ALD}\) to the radiation reaction in the massless case is plotted. It becomes 1 at \(M \to 0\) and decreases as the mass of the radiation field \(M\) increases. Eventually it vanishes at \(M \to \infty\). This result is consistent with our expectation that the radiation reaction is suppressed by the effect of the mass of the radiation field. In Figure. 5 we plot the same ratio as a function of \(\omega\). The ratio becomes 1 in the relativistic limit \(\rho \omega \to 1\). When the frequency \(\omega\) decreases and the particle’s motion becomes nonrelativistic, the radiation reaction is more suppressed in the massive case. This can be naturally expected since in the nonrelativistic region with a small \(\omega\) the emission is highly suppressed by the mass of the radiation field.

\[\ast\ast\] There is also a non-oscillating term \(\sim l^{-2}\) in \(F_m\). But as we see in the appendix A, such a term with an even power of \(l\) does not produce a logarithmic behavior at \(M = 0\).
Figure 4: $F_{\text{ALD}}/F_{\text{ALD massless}} = F_{\text{ALD}}/(−\rho \gamma^5(\omega^3/12\pi)$ as a function of $M$, with $\rho = 1$ and $\omega = 0.8$

Figure 5: $F_{\text{ALD}}/F_{\text{ALD massless}}$ as a function of $\omega$ with $\rho = 1, M = 2$

4 Conclusions

In this paper, we investigated the radiation reaction of a charged particle interacting with a massive scalar field. We first obtained a massive analog of the ALD equation. The most important observation is that the massive ALD equation cannot be written as a local equation with higher derivative terms. The coefficients of higher derivative terms are divergent and the derivative expansion of the radiation reaction term is invalid. Only in the massless limit the ALD equation becomes local. This sounds strange since the massless limit is more sensitive to the infrared effect and the nonlocal effect seem to become more important than a massive case. Technically speaking the locality arises in the massless ALD equation because the retarded Green function of a massless field has a support only on the light cone and is proportional to $δ(σ)$. On the contrary, the Green function of a massive field is distributed around the light cone and thus the radiation reaction term becomes nonlocal which is written only as an integral form.

We also studied the nonanalytic behavior of the radiation reaction term in various situations. First we showed that it has a non-analytic behavior as $M^p \log M^2$ at $M \to 0$, where $p$ is a positive integer depending on the details of the trajectory of the particle. Such non-analytic behavior generally appears at $M \to 0$ even when the particle is accelerated during a finite
time interval. In appendix A we studied how such a non-analytic behavior with log $M$ appears by looking at various integrals of the Bessel function. We also evaluated the backreaction in specific motions, a uniform acceleration and a circular motion. In both cases, we evaluated the backreaction (radiation reaction) as a function of the mass $M$ and showed that it is suppressed when the mass becomes large. It is consistent with our physical intuition that the emission is suppressed in the massive case and accordingly the backreaction is reduced.

Finally we would like to comment on a possible resolution to the runaway solutions in the massless ALD equation. As we saw at the end of Section 3.1 the positive region of the backreaction $F(a, M)$ suggests that an instability will occur in the region $a \gg M$. It may be related to the pathological behavior of the massless ALD equation. In the massive case, when a particle is accelerated from a nonrelativistic region, the backreaction term $F(a, M)$ is negative and suppresses the acceleration. Hence it tends to avoid the runaway type solution. It is interesting to investigate that if one can avoid the pathology of the runaway solution by introducing a small mass of the radiation field and then taking a massless limit.

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**A Some properties of $J_1(x)$**

The Bessel function $J_1(x)$ is given by the series

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{n! \Gamma(n + 2)},$$

which is an odd function of $x$. It is regular on the whole complex plane except at infinity. The asymptotic behavior of $J_1(x)$ is given by

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{3\pi}{4}) + o(x^{-3/2}).$$

Though it falls off very slowly as $1/\sqrt{x}$ at infinity, the oscillating factor makes the following integral convergent

$$\int_0^{\infty} dx \ J_1(s) = 1.$$

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In this appendix, we focus on the behavior of the function
\[ I(M, l) = \frac{MJ_1(Ml)}{l}. \] (A.4)
and its integrals in the two limiting situations, \( M \to \infty \) and \( M \to 0 \).

**A.1 \( M \to \infty \) limit**

We first show that the function \( I(M, l) \) has the following property
\[ \lim_{M \to \infty} I(M, l) = 2\delta(l^2). \] (A.5)

In order to show this, we introduce a continuous and regular function \( g(l) \) in the region \( 0 \leq l < \infty \) with the boundary condition \( \lim_{l \to \infty} g(l) = 0 \). Then the following integral becomes
\[
\lim_{M \to \infty} \int_0^\infty d(l^2) I(M, l)g(l^2) = \lim_{M \to \infty} \int_0^\infty dl \ 2MJ_1(Ml)g(l^2)
= \lim_{M \to \infty} 2 \int_0^\infty ds J_1(s)g(s^2/M^2)
= 2 \int_0^\infty ds J_1(s)g(0) = 2g(0). \] (A.6)

In the last equality we changed the ordering of \( \lim_{M \to \infty} \) and \( \int_0^\infty ds \). If it is justified, the equality (A.5) is proved. To justify it, one needs to show that
\[ \lim_{M \to \infty} J_1(s)g(s^2/M^2) = J_1(s)g(0) \]
is uniformly convergent at \( 0 \leq s < \infty \). Namely we need to show that for any \( \epsilon \) there exists \( N_\epsilon \) such that the inequality
\[ |J_1(s)(g(s^2/N_\epsilon^2) - g(0))| < \epsilon, \] (A.7)
is satisfied for all \( 0 \leq s < \infty \).

Such \( N_\epsilon \) can be found as follows. Let us denote the maximum value of \( |g(s) - g(0)| \) for \( 0 < s < \infty \) by \( \delta g_{\text{max}} \). Then, since \( J_1(s) \) behaves like \( 1/\sqrt{s} \) at large \( s \), there exists \( S_\epsilon \) such that \( |J_1(s)\delta g_{\text{max}}| < \epsilon \) is satisfied for \( s \geq S_\epsilon \). On the other hand, for the region \( s \leq S_\epsilon \), we can always find \( N_\epsilon \) such that \( |g(s^2/N_\epsilon^2) - g(0)| < \epsilon \) since \( g(l) \) is continuous at \( l = 0 \). To summarize, for the region \( s \geq S_\epsilon, |J_1(s)(g(s^2/N_\epsilon^2) - g(0))| < |J_1(s)\delta g_{\text{max}}| < \epsilon \) is satisfied and for the region \( s \leq S_\epsilon, |J_1(s)(g(s^2/N_\epsilon^2) - g(0))| < |g(s^2/N_\epsilon^2) - g(0)| < \epsilon \). Thus the uniform convergence (A.7) is proved.
A.2 Massless limit: $M \to 0$

We then investigate the behavior of the integration

$$Q(M) = \int_0^\infty dl \ I(M, l)g(l),$$

in the massless limit $M \to 0$. One can similarly prove that

$$\lim_{M \to 0} I(M, l)g(l) = 0$$

is uniformly convergent at $0 \leq l < \infty$. We can exchange the ordering of the limit and the integral and show that

$$Q(0) = \lim_{M \to 0} Q(M) = 0.$$  

Hence the non-local term like (2.29) vanishes in the massless limit and it is reduced to the massless ALD equation.

However, the analytic behavior of $Q(M)$ is not so simple since logarithmic terms like $M^2 \log M$ are expected to appear around $M = 0$. In the following, we show how to derive such a non-analytic term of $Q(M)$. First note that the integrand $I(M, l)$ is analytic for both parameters, $l$ and $M$. Then an integration over a finite range

$$Q(M; L) = \int_0^L dl \ I(M, l)g(l),$$

is also an analytic function of $M$. Non-analytic behavior like $\log M$ appears only when we take the integral region infinite $\Delta \to \infty$.

Let $g(l)$ be an analytic function at finite $l$ which falls off as $l^{-n}$ at infinity. Here $n$ is a non-negative integer. We encountered two examples in the analysis of the radiation reaction. In the case of the uniform acceleration (3.6), an integral with $g(l) \sim l^{-1}$ appears. In a scattering process (3.18), $g(l)$ contains a term proportional to $l^{-3}$. In order to see the non-analytic behavior of these integrals, we divide the integral $Q(M)$ into a regular part $Q(M; L)$ and a (possibly) singular part $Q(M; \infty)$ as

$$Q(M) = Q(M; L) + Q(M; \infty) = \int_0^L dl \ I(M, l)g(l) + \int_\infty^L dl \ I(M, l)g(l),$$

where $L$ is chosen to be large enough so that one can make the approximation $g(l) = \alpha l^{-n}$. By changing the integration variable to $s = ML$, the second part becomes

$$Q(M, \infty) = M \int_{ML}^\infty ds \ \frac{J_1(s)}{s} g(s/M) \approx \alpha M^{n+1} \int_{ML}^\infty ds \ \frac{J_1(s)}{s^{n+1}}.$$ 

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The approximation in the second equality is valid even in the massless limit because $s/M$ is always larger than $L$ and $g(s/M) \sim \alpha(s/M)^{-n}$ is a good approximation. The $M$ dependence only comes from the integration around $s = ML$. For a small $M$, one can expand $J(s) \approx s/2 - (s)^3/16 + \cdots$ near $s = ML$, and $Q(M, \infty)$ becomes

$$Q(M, \infty) \approx \alpha M^{s+1} \int_{ML}^{\infty} ds \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k-n}}{2^{2k+1} k!(k+1)!}.$$  

for $s_{\infty} = s/2 - (s)^3/16 + \cdots$. It is a regular function for an even integer $n$. For an odd integer $n$, the logarithmic term $\log M$ arises. The coefficient of the logarithmic term $\log M$ is given by

$$\alpha M^{n+1} \int_{ML}^{\infty} \frac{J_1(s) \sin (ks/M)}{s^{n+1}}.$$  

(B.16)

it is determined by the behavior of $g(l) \sim \alpha l^{-n}$ at infinity and the behavior of the Bessel function $J_1(s)$ near $s = 0$. As an example, due to the asymptotic behavior (3.10), the integral (3.6) at $a = 1$ is reduced to the above integral $Q(M)$ with $n = 1$ and $\alpha = -e^2/2\pi$, and the logarithmic dependence in (3.11) is reproduced.

When $g(l)$ is an oscillating function, the behavior of the integral becomes different. It is the case corresponding to the circular motion in (3.23) and (3.24). In this case, $g(l)$ contains oscillating factors like $\sin kl$ at $l \to \infty$. Suppose that $g(l)$ behaves like $g(l) \sim l^{-n} \sin kl$ for $l \to \infty$. If we divide the integral $Q(M)$ into $Q(M, L)$ and $Q(M, \infty)$ as above, a non-analytic behavior, if exists, must come from the second part $Q(M, \infty)$. By changing the integration variable to $s = ML$, it becomes

$$Q(M, \infty) \approx \alpha M^{n+1} \int_{ML}^{\infty} ds \frac{J_1(s) \sin (ks/M)}{s^{n+1}}.$$  

(B.17)

Since $ks/M > kL \gg 1$, the term $\sin (ks/M)$ is oscillating rapidly. Since $J_1(s) \sim s$ for small $s$, $M$-dependence of $Q(M, \infty)$ is determined by the integration,

$$\int_{ML}^{\infty} \frac{\sin ks/M}{s^n} ds = M^{n+1} \int_{L}^{\infty} \frac{\sin kt}{t^n} dt,$$  

which is a power function of $M$. Thus no non-analytic behavior appears when $g(l)$ is an oscillating function at infinity.

**B  Existence of a positive region of $F(a, M)$**

In this appendix we prove that, in the uniform acceleration, whatever $f(l)$ we choose the backreaction must take a positive value for a sufficiently large $g = a^2/4M^2$. This can be
shown as follows. If we take a different choice of $f(l)$ from the simplest choice $f(l) = 1/2$, the backreaction changes as in (3.13). This change of the backreaction must satisfy the condition

$$
\lim_{M \to \infty} \delta F(M) = 0
$$

(B.18)
since it should vanish in the large $M$ limit. Hence, for any small $\epsilon$, there exists $M_\epsilon$ that satisfies $|\delta F(M)| < \epsilon$ for $M > M_\epsilon$. On the other hand, as shown in Figure, the function $\tilde{F}(g)$ is positive for $g > 1$ and grows as a function of $g$ and there exists $g_\epsilon$ such that $\tilde{F}(g) > \epsilon$ for $g > g_\epsilon$. For a fixed $M > M_\epsilon$, we can always take $a$ sufficiently large so that $g = a^2/4M^2 > g_\epsilon$. Then in such values of $a$ and $M$, the backreaction becomes positive;

$$
\frac{F(a_l, M_l)}{e^2a_lM_l} = \delta F(M_l) + \tilde{F}(\frac{a_l^2}{4M_l^2}) > 0.
$$

(B.19)

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