AN ELEMENTARY PROOF OF THE LACK OF NULL CONTROLLABILITY FOR THE HEAT EQUATION ON THE HALF LINE

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Abstract. In this note, we give an elementary proof of the lack of null controllability for the heat equation on the half line by employing the machinery inherited by the unified transform, known also as the Fokas method. This approach also extends in a uniform way to higher dimensions and different initial-boundary value problems governed by the heat equation, suggesting a novel methodology for studying problems related to controllability.

1. Introduction

The Uniform Transform Method (UTM), also known as the Fokas method, is a powerful tool for obtaining solutions of initial - (inhomogeneous) boundary-value problems. This method was first introduced in [1] for the analysis of initial-boundary value problems for integrable nonlinear partial differential equations (PDEs). However, in later works it was proven to produce novel results for a general class of linear PDEs; see [2, 3]. Recently researchers utilized the UTM to produce rigorous wellposedness results in Sobolev and Bourgain spaces for dispersive PDEs; see for instance [4] and [8] for the local and global wellposedness analysis of nonlinear Schrödinger type PDEs and [5] for a similar analysis on the Korteweg-de Vries equation.

To date, there is no work on the boundary controllability of PDEs that utilizes the advantages of the UTM. This method has two basic elements: (i) the so-called global relation, an identity that relates the initial datum and a suitable time transform of known and unknown boundary values, and (ii) the integral representation of the solution. We illustrate a new methodology by making use of these two elements in order to provide an elementary proof of the lack of null controllability for the heat equation on the half line.

To this end, let us consider the following canonical initial-boundary value problem:

\begin{align}
(1.1) \quad & u_t = u_{xx}, \quad x \in \mathbb{R}_+, \quad t \in (0,T), \\
(1.2) \quad & u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\
(1.3) \quad & u(0,t) = g(t), \quad t \in (0,T).
\end{align}

We say (1.1)-(1.3) is null controllable in [0, T] if given $u_0 \in L^2(\mathbb{R}_+)$ there is $g \in L^2(0,T)$ such that $u(x, T) \equiv 0$.

It is well known that the above property does not hold for (1.1)-(1.3) for those solutions in $C([0,T];L^2(\mathbb{R}_+))$; see for example [6] for a proof of this result. Our goal is to provide an alternate, yet very short proof of this fact. More precisely, we prove the following theorem.

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Theorem 1.1. There exists $u_0 \in L^2(\mathbb{R}_+)$ such that $u(x, T) \neq 0$ for any $g \in L^2(0, T)$ if $u \in C([0, T]; L^2(\mathbb{R}_+))$ and it solves (1.1)-(1.3).

Orientation. In Section 2, we provide a proof of Theorem 1.1 via the global relation. In Section 3, we extend Theorem 1.1 to the $N$-dimensional half space by outlining the straightforward and simple extension of the proof presented in Section 2 to $N$ dimensions. In Section 4, we discuss alternative pathways through the Fokas method, introducing also a characterisation for the null-controllability problem on the finite interval. In Section 5, we discuss the main results of this work, as well as its future implications.

2. Proof of Theorem 1.1

By introducing the half-line Fourier transform, namely

$$\hat{f}(\lambda) = \int_0^\infty e^{-i\lambda x} f(x)dx, \quad \text{Im} \lambda \leq 0,$$

and the transform

$$\tilde{f}(\lambda, t) = \int_0^t e^{\lambda \tau} f(\tau)d\tau, \quad t > 0, \quad \lambda \in \mathbb{C},$$

the global relation for (1.1)-(1.3), given by the Fokas method (see [3]) can be written in the following form:

$$e^{\lambda T} \hat{u} = \hat{u}_0 - \tilde{r}(\lambda^2, t) - i\lambda \tilde{g}(\lambda^2, t), \quad \text{Im} \lambda \leq 0,$$

where $r(t) = u_x(0, t)$ and $g(t) = u(0, t), \quad t > 0$. Applying the condition $u(x, T) \equiv 0$, we obtain that

$$0 = \hat{u}_0 - \tilde{r}(\lambda^2, T) - i\lambda \tilde{g}(\lambda^2, T), \quad \text{Im} \lambda \leq 0.$$

Letting $\lambda \to -\lambda$ in (2.4) and subtracting the resultant expression (which is valid for $\text{Im} \lambda \geq 0$) from (2.4) we obtain the following equation:

$$2i\lambda \tilde{g}(\lambda^2, T) = \hat{u}_0 - \hat{u}_0(-\lambda), \quad \lambda \in \mathbb{R}.$$

Let $0 \neq u_0 \in L^1 \cap L^2(\mathbb{R}_+)$. It is clear that if $g \equiv 0$, then $\hat{u}_0(\lambda) = \hat{u}_0(-\lambda)$ for all $\lambda \in \mathbb{R}$, which would contradict with the assumption that $0 \neq u(0) = u_0$. Thus, if there is $g \neq 0$ for which $u(x, T) = 0$, it must satisfy (2.5).

Employing the definition of $\tilde{g}$ we obtain the uniform bound below for some $M > 0$:

$$\left| \int_0^T e^{\lambda^2 t} g(t)dt \right| = \frac{1}{2\lambda} \left| [\hat{u}_0 - \hat{u}_0(-\lambda)] \right| < M, \quad \lambda^2 > 1.$$
3. The N-dimensional half space

In this section we extend Theorem 1.1 to the higher dimensional half space \( \mathbb{R}^+_N = \mathbb{R}^{N-1} \times \mathbb{R}_+ \), \( N > 1 \) (see also [4]). The methodology we used previously for the proof of Theorem 1.1 provides a straightforward path to study the (lack of) null controllability for

\[
(3.1) \quad u_t = \Delta u, \quad x = (x', x_N) \in \mathbb{R}^+_N, \quad t \in (0, T),
\]

\[
(3.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^+_N,
\]

\[
(3.3) \quad u(x', 0, t) = g(x', t), \quad x' \in \mathbb{R}^{N-1}, \quad t \in (0, T).
\]

The relevant result can be obtained by using half space Fourier transform

\[
\hat{u}(\lambda) = \int_{\mathbb{R}^{N-1}} \int_0^\infty e^{-i\lambda \cdot x} u(x) dx_n dx', \quad \lambda = (\lambda', \lambda_N) \in \mathbb{R}^{N-1} \times \mathbb{C}, \quad \text{Im} \lambda_N \leq 0
\]

and applying Fokas’s method only to the last variable \( x_N \). Indeed, half space Fourier transform yields the global relation

\[
(3.4) \quad e^{i|\lambda|^2 t} \hat{u}(\lambda, t) = \hat{u}_0(\lambda) - \hat{h}(\lambda, t) - i\lambda_N \hat{g}(\lambda, t), \quad \text{Im} \lambda_N \leq 0,
\]

where

\[
(3.5) \quad \hat{g}(\lambda, t) = \int_0^t e^{i|\lambda|^2 s} g(x', s) ds \quad \text{and} \quad \hat{h}(\lambda, t) = \int_0^t e^{i|\lambda|^2 s} h(x', s) ds,
\]

with \( h(x', t) \equiv u_{x_N}(x', 0, t) \) and \( \hat{g}, \hat{h} \) denoting Fourier transforms of \( g \) and \( h \) with respect to \( x' \).

The proof of the lack of null controllability for solutions in the class \( C([0, T]; L^2(\mathbb{R}^+_N)) \) follows the exact same steps with the proof of Theorem 1.1. Hence, (2.6) is now replaced with

\[
(3.6) \quad \left| \int_0^T e^{\lambda_N^2 t} F(\lambda', t) dt \right| \leq \frac{1}{2\lambda_N} \left| \hat{u}_0(\lambda', \lambda_N) - \hat{u}_0(\lambda', -\lambda_N) \right| < M, \quad \lambda_N^2 > 1,
\]

where \( F(\lambda', t) := e^{i|\lambda'|^2 t} \hat{g}(\lambda', t) \). Applying Lemma 2.1 for each fixed \( \lambda' \in \mathbb{R}^{N-1} \), we conclude that \( F \equiv 0 \), which in turn implies that \( g \equiv 0 \).

4. Alternative Pathways

In this section, we provide an alternative pathway to obtain a proof of Theorem 1.1 via the integral representation of the Fokas method. Furthermore, this pathway provides a characterisation of the control for the finite interval problem given in (4.6). In this sense it suggests a more general viewpoint on studying controllability problems through this methodology.

The Half Line. The integral representation of the solution of (1.1)-(1.3) given by the Fokas method (see [3]) takes the form:

\[
(4.1) \quad u(x, t) = \frac{1}{2\pi} \int^{\infty}_{-\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) d\lambda
\]

\[
- \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[ 2i\lambda \hat{g}(\lambda^2, t) + \hat{u}_0(-\lambda) \right] d\lambda,
\]

where \( \partial D^+ \) is depicted in Figure 1.
By applying $u(x, T) \equiv 0$, deforming $\partial D^+$ to the real line and taking the inverse Fourier transform of both sides in the resultant expression, we obtain (2.5). Then, the proof of Theorem 1.1 follows by the exact same arguments of Section 2.

The Finite Interval. It is well known that the null controllability is true, for instance in $C([0, T]; L^2(\Omega))$, if one replaces the infinite domain $\mathbb{R}_+$ by the finite one $(0, L)$. Here, we wish to give a characterization of the set of suitable boundary controllers, say acting at the right Dirichlet boundary condition, using the integral representation obtained from the Fokas method. Thus, we consider the following problem:

$$
\begin{aligned}
&u_t = u_{xx}, & x \in (0, L), & t \in (0, T), \\
&u(0, t) = 0, & u(L, t) = h(t), & t \in (0, T) \\
&u(x, 0) = u_0(x), & x \in (0, L)
\end{aligned}
$$

(4.2)

and the goal is to find a sufficient condition for the boundary controller $h$ so that it steers the given initial datum $u_0$ to $u_T \equiv 0$ at $t = T$.

In analogy with the half line problem, one introduces the following Fourier transform where the integral is taken over the given spatial domain $(0, L)$:

$$
\check{u}(\lambda, t) = \int_0^L e^{-i\lambda x} u(x, t) dx, & \lambda \in \mathbb{C}.
$$

(4.3)

Then, the corresponding global relation for the above problem evaluated at $t = T$ becomes

$$
0 = \check{u}_0(\lambda) + i\lambda e^{-i\lambda L} \check{h}(\lambda^2, T) - \check{g}_1(\lambda^2, T) + e^{-i\lambda L} \check{h}_1(\lambda^2, T), & \lambda \in \mathbb{C},
$$

(4.4)

with $g_1(t) = u_x(0, t)$, $h_1(t) = u_x(L, t)$, and $h(t) = u(L, t)$.

Similarly, the integral representation of the solution evaluated at $t = T$ becomes

$$
0 = u(x, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 T} \check{u}_0(\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 T} \check{g}_1(\lambda^2, T) d\lambda
$$

$$
- \frac{1}{2\pi} \int_{\partial D^-} e^{-i\lambda (L-x) - \lambda^2 T} \left[ \check{h}_1(\lambda^2, T) + i\lambda \check{h}(\lambda^2, T) \right] d\lambda,
$$

(4.5)

for all $x \in (0, L)$, where the contours $\partial D^\pm$ are depicted in Figure 1.

We next utilise the standard approach of Fokas method: Using the invariances of the global relation under the transformation $\lambda \mapsto -\lambda$, the unknown boundary transforms ($\check{g}_1$ and $\check{h}_1$) can be eliminated from the integral representation. Through
short and straightforward calculations, and by employing the definition of \( \tilde{h} \), equation (4.5) yields the following relation:

\[
\int_{\partial D^+} R(\lambda; x, T, L) d\lambda + \int_{\partial D^-} R(\lambda; x, T, L) d\lambda = U_0(x; T), \quad \forall x \in (0, L),
\]

where the integrand \( R(\lambda; x, T, L) \) is given by

\[
R(\lambda; x, T, L) := \frac{i}{\pi} \frac{\lambda e^{i\lambda x - \lambda^2 T} - e^{-i\lambda L}}{e^{i\lambda L} - e^{-i\lambda L}} \left[ \int_0^T e^{i\lambda^2 s} h(s) ds \right]
\]

and the known \( U_0(x; T) \) is given by

\[
U_0(x; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 T} \hat{u}_0(\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 T} \left[ \frac{e^{i\lambda L} \hat{u}_0(\lambda) - e^{-i\lambda L} \hat{u}_0(-\lambda)}{e^{i\lambda L} - e^{-i\lambda L}} \right] d\lambda
\]

\[
- \frac{1}{2\pi} \int_{\partial D^-} e^{-i\lambda(L-x) - \lambda^2 T} \left[ \frac{\hat{u}_0(\lambda) - \hat{u}_0(-\lambda)}{e^{i\lambda L} - e^{-i\lambda L}} \right] d\lambda,
\]

with the contours \( \partial D^\pm \) depicted in Figure 1, and the red dots denoting the zeros of \( \exp(i\lambda L) - \exp(-i\lambda L) \) on the real axis.

Thus, we obtain the following characterization for the problem of null controllability: The problem (4.2) is null controllable at time \( t = T \) if and only if there exists \( h = h(t) \) which satisfies (4.6).

5. Discussion

In this work we analyse a family of null-controllability problems governed by the heat equation, using the machinery provided by the Fokas method. In this connection we make the following three remarks:

- It is straightforward but more technical to generalise the proof of Theorem 1.1 so that one constructs a function \( u_0 \) satisfying Theorem 1.1 with \( u_0 \in L^2(\mathbb{R}_+) \), but not necessarily \( u_0 \in L^1(\mathbb{R}_+) \).

- The methodology appearing in the current work can be applied to boundary value problems of higher dimensions such as \((\mathbb{R}_+)^N, N > 1\), where all the spatial coordinates are positive. The relevant proof, which will be presented elsewhere, is based on the analysis of the Fokas method presented in [2] for the case of \( N = 2 \), namely the quarter plane.

- If \( u_0 \in L^2(\mathbb{R}_+) \) and \( g \in L^2(0, T) \), then (1.1)-(1.3) possesses a solution \( u \in C([0, T]; L^2(\mathbb{R}_+)) \) in the transposition sense, and moreover this solution can be represented as in (4.1). Therefore, Theorem 1.1 concerns such solutions. If the condition \( u \in C([0, T]; L^2(\mathbb{R}_+)) \) is removed, then one can recover the null controllability in a larger class of solutions. This was proved in [9] for the linearized KdV, heat, and Schrödinger equations.

The possibility of applying the methodology introduced in the current work to null-controllability problems governed by other linear PDEs is currently under investigation.
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REFERENCES

1. Athanassios S. Fokas, *A unified transform method for solving linear and certain nonlinear pdes*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, vol. 453 (1997), no. 1962, 1411–1443.
2. ________, *A new transform method for evolution partial differential equations*, IMA Journal of Applied Mathematics, vol. 67 (2002), no. 6, 559–590.
3. ________, *A unified approach to boundary value problems*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 78, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. MR 2451953
4. Athanassios S. Fokas, A. Alexandrou Himonas, and Dionyssios Mantzavinos, *The nonlinear Schrödinger equation on the half-line*, Trans. Amer. Math. Soc. vol. 369 (2017), no. 1, 681–709. MR 3557790
5. A. Alexandrou Himonas, Dionyssios Mantzavinos, and Fangchi Yan, *The Korteweg–de Vries equation on an interval*, J. Math. Phys. vol. 60 (2019), no. 5, 051507, 26. MR 3947621
6. Sorin Micu and Enrique Zuazua, *On the lack of null-controllability of the heat equation on the half-line*, Trans. Amer. Math. Soc. vol. 353 (2001), no. 4, 1635–1659. MR 1806726
7. ________, *On the lack of null-controllability of the heat equation on the half-space*, Portugal. Math. vol. 58 (2001), no. 1, 1–24.
8. Türker Özşarı and Nermin Yolcu, *The initial-boundary value problem for the biharmonic Schrödinger equation on the half-line*, Commun. Pure Appl. Anal. vol. 18 (2019), no. 6, 3285–3316.
9. Lionel Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line*, SIAM J. Control Optim. vol. 39 (2000), no. 2, 331–351. MR 1788062
10. Köskku Yosida, *Functional analysis*, sixth ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 123, Springer-Verlag, Berlin-New York, 1980. MR 617913

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