Schrödinger operators with $\delta$-interactions supported on conical surfaces

J Behrndt$^1$, P Exner$^2$ and V Lotoreichik$^1$

$^1$Department of Computational Mathematics, Graz University of Technology, Steyrergasse 30, 8010, Graz, Austria
$^2$Doppler Institute for Mathematical Physics and Applied Mathematics, Czech Technical University in Prague, Břehová 7, 11519 Prague, and Nuclear Physics Institute ASCR, 25068 Rež near Prague, Czech Republic

E-mail: behrndt@tugraz.at, exner@ujf.cas.cz and lotoreichik@math.tugraz.at

Received 8 April 2014, revised 17 June 2014
Accepted for publication 30 June 2014
Published 18 August 2014

Abstract
We investigate the spectral properties of self-adjoint Schrödinger operators with attractive $\delta$-interactions of constant strength $\alpha > 0$ supported on conical surfaces in $\mathbb{R}^3$. It is shown that the essential spectrum is given by $[\alpha^{-2}/4, \infty)$ and that the discrete spectrum is infinite and accumulates to $-\alpha^{-2}/4$. Furthermore, an asymptotic estimate of these eigenvalues is obtained.

Keywords: Schrödinger operator, delta potential, infinite discrete spectrum

1. Introduction
The purpose of this paper is to analyse the spectrum of the three-dimensional Schrödinger operator $-\Delta_{a,C_\theta}$ with an attractive $\delta$-interaction of constant strength $\alpha > 0$ supported on the conical surface

$$C_\theta := \{(x, y, z) \in \mathbb{R}^3 : z := \cot(\theta) \sqrt{x^2 + y^2} \}, \quad \theta \in (0, \pi/2).$$

The Schrödinger operator $-\Delta_{a,C_\theta}$ is defined via the first representation theorem [17, Theorem VL2.1] as the unique self-adjoint operator in $L^2(\mathbb{R}^3)$ which is associated with the closed, densely defined, symmetric and semibounded quadratic form...
\[ a_{\Delta,C_\theta}[\psi] = \| \nabla \psi \|^2_{L^2(\mathbb{R}^3)} - \alpha \int_{C_\theta} |\psi|^2 \, d\sigma, \quad \text{dom } a_{\Delta,C_\theta} = H^1(\mathbb{R}^3); \quad (1) \]

cf [1, 4]. In a short form the main result of this note is the following theorem.

**Theorem.** For any \( \theta \in (0, \pi/2) \) and \( \alpha > 0 \) the essential spectrum of the operator \( -\Delta_{a,C_\theta} \) is \( [\alpha^{-2}/4, +\infty) \), the discrete spectrum is infinite and accumulates to \( -\alpha^2/4 \).

In addition, we obtain an asymptotic estimate of the eigenvalues of \( -\Delta_{a,C_\theta} \) lying below \( -\alpha^2/4 \), see theorem 3.2. Roughly speaking, the infiniteness of the discrete spectrum is induced by global geometrical properties of the conical surface \( C_\theta \) and is not related to the singularity at the tip or other local geometrical properties. In fact, the same effect remains present after a local deformation of \( C_\theta \); cf theorem 3.3.

Various relations between the geometry and the bound states of quantum systems have been studied intensively in recent decades (see, e.g. [19]) after it had been realized in [14] that curvature can give rise to an effective attractive interaction. In addition to systems with a hard-wall confinement the so-called ‘leaky’ structures attracted attention, see the review paper [11]. Their advantage is that they make it possible to take quantum tunnelling into account. The model discussed in this paper can describe, for instance, a structure composed of two semiconductors: a conical substrate of one material on the top of which we have a thin layer of the second one, covered by a bulk mass of the former.

The proof of our main result is based on standard techniques in spectral theory of self-adjoint operators: we construct singular sequences and use Neumann bracketing in the spirit of [13] to show the assertion on the essential spectrum; for the infiniteness of the discrete spectrum we employ variational principles. The same approach was applied in [25] in the context of Schrödinger operators with slowly decaying negative regular potentials, see also [23, §XIII.3]. Similar arguments were also used in [10, 15] for the closely related question of infiniteness of the discrete spectrum for the Dirichlet Laplacian in a conical layer, see also [7, 18, 19, 21, 23] for further progress in this problem. We also point out [6, 9, 12] for related spectral problems for Schrödinger operators with \( \delta \)-potentials.

### 2. Essential spectrum of \( -\Delta_{a,C_\theta} \)

In this section we show that the essential spectrum of the operator \( -\Delta_{a,C_\theta} \) is given by \( [\alpha^{-2}/4, +\infty) \). The proof of the inclusion \( \sigma_{\text{ess}}(\Delta_{a,C_\theta}) \supseteq [-\alpha^2/4, +\infty) \) makes use of singular sequences and for the other inclusion a specially chosen Neumann bracketing is used. A similar type of argument was also employed in [1, 13] for \( \delta \) and \( \delta' \)-interactions on broken lines in the two-dimensional setting.

**Theorem 2.1.** Let \( -\Delta_{a,C_\theta} \) be the self-adjoint operator in \( L^2(\mathbb{R}^3) \) associated to the form (1) and let \( \alpha > 0 \) and \( \theta \in (0, \pi/2) \). Then

\[ \sigma_{\text{ess}}(-\Delta_{a,C_\theta}) = [-\alpha^2/4, +\infty) \]

**Remark 2.2.** For completeness we mention that the above theorem is also valid in the case \( \theta = \pi/2 \), that is, the conical surface is a plane, and the statement can be shown directly via separation of variables.
Proof of theorem 2.1 Step 1. We verify the inclusion \( \alpha_{\text{ess}}(-\Delta_{\text{ext}}, C_{\beta}) \supseteq \mathbb{R} (\alpha^2/4, +\infty) \) by constructing singular sequences for the operator \(-\Delta_{\text{ext}}\) at every point of the interval \([-\alpha^2/4, +\infty)\). Let us start by fixing a function \( \chi_1 \in C_0^\infty(1, 2) \) such that
\[
\| \chi_1 \|_{L^1(1,2)} = 1, \tag{2}
\]
and a function \( \chi_2 \in C_0^\infty(-\varepsilon, \varepsilon) \) with some fixed \( \varepsilon \in (0, \tan \theta) \), which satisfies
\[
0 \leqslant \chi_2 \leqslant 1 \quad \text{and} \quad \chi_2(t) = 1 \text{ for } |t| < \varepsilon/2. \tag{3}
\]
Define for all \( p \in \mathbb{R} \) and \( n \in \mathbb{N} \) the functions \( \omega_{n,p} : \mathbb{R}^2_+ \to \mathbb{C} \) as
\[
\omega_{n,p}(s, t) := \frac{1}{\sqrt{n}} \left( \chi_1 \left( \frac{s}{n} \right) e^{i p s} \right) \left( \chi_2 \left( \frac{t}{n} \right) e^{-\alpha \frac{|t|}{2}} \right) \in C(\mathbb{R}^2_+) \tag{4}
\]
in the coordinate system \((s, t)\) in figure 1. Here \( \mathbb{R}^2_+ \) denotes open right half-plane \( \{(r, z) \in \mathbb{R}^2 : r > 0\} \).

Note that because of the choice of \( \varepsilon \in (0, \tan \theta) \) we have \( \text{supp} \omega_{n,p} \subset \mathbb{R}^2_+ \) for all \( n \in \mathbb{N} \) and, moreover, the distances between the axis and the supports of \( \omega_{n,p} \) satisfy
\[
\rho_n := \inf \{ r \colon (r, z) \in \text{supp} \omega_{n,p} \} \to +\infty, \quad n \to \infty. \tag{5}
\]

By dominated convergence, using (2) and (3), we get
\[
\| \omega_{n,p} \|^2_{L^2(\mathbb{R}^2_+)} = \int_{-\infty}^{\infty} \left( \frac{1}{n} \int_{n}^{2n} \left( \chi_1 \left( \frac{s}{n} \right) e^{i p s} \right) \left( \chi_2 \left( \frac{t}{n} \right) e^{-\alpha \frac{|t|}{2}} \right) ds \right)^2 dt \to \int_{-\infty}^{\infty} e^{-\alpha \frac{|t|}{2}} dt = \frac{2}{\alpha}, \quad n \to \infty. \tag{6}
\]

We denote by \( \omega_{n,p, z} \) the restrictions of \( \omega_{n,p} \) onto the open subsets
\[
S_+ = \left\{ (r, z) \in \mathbb{R}^2_+ : z > r \cot \theta \right\} \quad \text{and} \quad S_- = \left\{ (r, z) \in \mathbb{R}^2_+ : z < r \cot \theta \right\}
\]
of \( \mathbb{R}^2_+ \). The partial derivatives of \( \omega_{n,p, z} \) with respect to \( s \) and \( t \) are given by
\[
\partial_s \omega_{n,p, z} = \frac{1}{\sqrt{n}} \left( \chi_1 \left( \frac{s}{n} \right) e^{i p s} + i p \chi_1 \left( \frac{s}{n} \right) e^{i p s} \right) \left( \chi_2 \left( \frac{t}{n} \right) e^{-\alpha \frac{|t|}{2}} \right),
\]
\[
\partial_t \omega_{n,p, z} = \frac{1}{\sqrt{n}} \left( \chi_1 \left( \frac{s}{n} \right) e^{i p s} \right) \left( \chi_2 \left( \frac{t}{n} \right) e^{-\alpha \frac{|t|}{2}} \pm \alpha \frac{|t|}{n} \right) \left( \chi_2 \left( \frac{t}{n} \right) e^{-\alpha \frac{|t|}{2}} \right).
\]

Similarly as in (5), using dominated convergence, we get
\[
\| \nabla \omega_{n,p} \|^2_{L^2(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} \left( \partial_s \omega_{n,p} \right)^2 + \left( \partial_t \omega_{n,p} \right)^2 ds dt \to \left( \frac{p^2 + \alpha^2}{4 \alpha} \right) \frac{2}{\alpha}, \quad n \to \infty. \tag{7}
\]

Let us define the sequence of functions \( \psi_{n,p} : \mathbb{R}^3 \to \mathbb{C} \) as
\[
\psi_{n,p}(r, \varphi, z) := \frac{\omega_{n,p}(r, z)}{\sqrt{2\pi r}}, \quad n \in \mathbb{N}, \tag{8}
\]
where the functions \( \omega_{n,p} : \mathbb{R}^3 \to \mathbb{C} \) are interpreted as rotationally invariant functions on \( \mathbb{R}^3 \) in the cylindrical coordinate system \((r, \varphi, z)\). The hypersurface \( C_0 \) separates the Euclidean space \( \mathbb{R}^3 \) into two unbounded Lipschitz domains \( \Omega_+ \) and \( \Omega_- \), where
We use the notation \( \psi_{n,p} := \psi_{n,p} |_{\Omega_n} \). Then \( \psi_{n,p} \in C^\infty(\Omega_\pm) \) and from (5) we obtain
\[
\| \psi_{n,p} \|^2_{L^2(\mathbb{R}^2)} = \| \alpha_{n,p} \|^2_{L^2(\mathbb{R}^2)} \to \frac{2}{\alpha^2}, \quad n \to \infty.
\] (8)

We claim that \( \psi_{n,p} \in \text{dom} (-\Delta_{x,C} \mp) \). For this we still need to check that the boundary conditions
\[
\psi_{n,p,+} \big| \!\big| \!\big|_{\partial \Omega_n} = \psi_{n,p,-} \big| \!\big| \!\big|_{\partial \Omega_n} \quad \text{and} \quad \partial_{z_+} \psi_{n,p,+} \big| \!\big| \!\big|_{\partial \Omega_n} + \partial_{z_-} \psi_{n,p,-} \big| \!\big| \!\big|_{\partial \Omega_n} = \alpha \omega_{n,p} \big| \!\big| \!\big|_{\partial \Omega_n}
\] (9)
are satisfied; cf [1, Theorem 3.3(i)]. In fact, by the definition of \( \omega_{n,p} \) we have \( \omega_{n,p} \big| \!\big| \!\big|_{\partial \Omega_n} = \omega_{n,p} \big| \!\big| \!\big|_{\partial \Omega_n} \), where \( \omega_{n,p} \) are interpreted as rotationally invariant functions on \( \Omega_n \).

This implies that the first condition (9) holds. Furthermore, one computes
\[
\partial_{z_+} \omega_{n,p,+} \big| \!\big| \!\big|_{\partial \Omega_n} + \partial_{z_-} \omega_{n,p,-} \big| \!\big| \!\big|_{\partial \Omega_n} = \alpha \frac{1}{\sqrt{H}} \left( \chi \left( \frac{1}{n} \right) \exp (ip\theta) \right) = \alpha \omega_{n,p} \big| \!\big| \!\big|_{\partial \Omega_n}.
\] (10)

The gradient of \( \psi_{n,p,\pm} \) can be expressed as
\[
\nabla \psi_{n,p,\pm} = \frac{1}{\sqrt{2\pi T}} \nabla \omega_{n,p,\pm} + \omega_{n,p,\pm} \nabla \left( \frac{1}{\sqrt{2\pi T}} \right),
\]
where \( \nabla \) acts on the functions \( (r, \varphi, z) \mapsto \omega_{n,p,\pm}(r, z) \) and \( (r, \varphi, z) \mapsto 1/\sqrt{2\pi T} \). Hence, we obtain

\[\text{Figure 1.} \] The right half-plane \( \mathbb{R}^2_+ \) with the coordinate system \((r, z)\). The ray \( \Gamma_\theta \) emerges from the origin and constitutes the angle \( \theta \in (0, \pi/2) \) with the \( z \)-axis. The coordinate system \((s, t)\) is associated with \( \Gamma_\theta \).
\[ \partial_r \psi_{n,p,+}^+ \Big| \big|_{r_o} + \partial_r \psi_{n,p,-}^- \Big| \big|_{r_o} = \left( \frac{1}{\sqrt{2\pi r}} \right) \left( \partial_r, \omega_{n,p,+}^+ \Big| \big|_{r_o} + \partial_r, \omega_{n,p,-}^- \Big| \big|_{r_o} \right) + \left( \omega_{n,p} \Big| \big|_{r_o} \right) \left( \frac{1}{\sqrt{2\pi r}} \right) \partial_r \left( \frac{1}{\sqrt{2\pi r}} \right) \Big| \big|_{r_o} \right) + \partial_r \left( \frac{1}{\sqrt{2\pi r}} \right) \Big| \big|_{r_o} \right) \]

\[ = \left( \frac{1}{\sqrt{2\pi r}} \right) \partial_r \left( \omega_{n,p} \Big| \big|_{r_o} \right) = \omega \psi_{n,p} \Big| \big|_{r_o}, \]

where (10) was used in the second equality. Thus we have verified (9) and therefore \( \psi_{n,p} \in \text{dom}( -\Delta_{n,C_0} ) \). Moreover, according to [1, Theorem 3.3(i)] we also have

\[ -\Delta_{n,C_0} \psi_{n,p} = \left( \begin{array}{c} -\Delta \psi_{n,p,+} \\ -\Delta \psi_{n,p,-} \end{array} \right). \]

Using the expression for the three-dimensional Laplacian in cylindrical coordinates we find

\[ -\Delta \psi_{n,p,+} = -\frac{1}{r} \partial_r \left( r \partial_r \psi_{n,p,z} \right) - \partial_z^2 \psi_{n,p,z}, \]

where the angular term is absent since the functions \( \psi_{n,p,z} \) do not depend on \( \varphi \). The above expression can be rewritten as

\[ -\Delta \psi_{n,p,z} = -\partial_r^2 \psi_{n,p,z} - \partial_z^2 \psi_{n,p,z} - \frac{1}{r} \left( \partial_r \psi_{n,p,z} \right). \]

Next we compute the first and second order partial derivatives of \( \psi_{n,p,z} \) with respect to \( r \):

\[ \partial_r \psi_{n,p,z} = \frac{\partial_r \omega_{n,p,z}}{\sqrt{2\pi r}} - \frac{\omega_{n,p,z}}{2\sqrt{2\pi r}^3/2}. \]

\[ \partial_r^2 \psi_{n,p,z} = \frac{\partial_r^2 \omega_{n,p,z}}{\sqrt{2\pi r}} - \frac{\partial_r \omega_{n,p,z}}{\sqrt{2\pi r}^3/2} + \frac{3}{4} \frac{\omega_{n,p,z}}{\sqrt{2\pi r}^5/2}. \]

The last two summands in the expression for \( \partial_r^2 \psi_{n,p,z} \) can be estimated in \( L^2 \)-norm as

\[ \frac{9}{16} \left\| \frac{\omega_{n,p,z}}{\sqrt{2\pi r}^5/2} \right\|_{L^2(\mathbb{R})}^2 \leq \frac{9}{16} \beta_n \left\| \omega_{n,p} \right\|_{L^2(\mathbb{R}; c)}^2 \to 0, \quad n \to \infty, \]

\[ \left\| \partial_r \omega_{n,p,z} \right\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{\rho_n^2} \left\| \omega_{n,p} \right\|_{L^2(\mathbb{R}; c)}^2 \to 0, \quad n \to \infty, \]

where we have used (4), (5) and (6). The second order partial derivatives of \( \psi_{n,p,z} \) with respect to \( z \) are

\[ \partial_z^2 \psi_{n,p,z} = \frac{\partial_z^2 \omega_{n,p,z}}{\sqrt{2\pi r}}. \]

Using (13), (14), (15) and the invariance of the Laplacian under rotation of the coordinate system we obtain that

\[ -\partial_z^2 \psi_{n,p,z} - \partial_z^2 \psi_{n,p,z} = -\frac{1}{\sqrt{2\pi r}} \left( \partial_z^2 \omega_{n,p,z} + \partial_z^2 \omega_{n,p,z} \right) + o(1), \quad n \to \infty; \]
here and in the following we understand \( o(1) \) in the strong sense with respect to the corresponding \( L^2 \)-norm. With the help of (13) the norm of the last summand on the right hand side in (12) can be estimated as
\[
\left\| \frac{\partial \omega_{n,p,\pm}}{r} \right\|_{L^2(\mathbb{R})} \leq \left\| \frac{\partial^2 \omega_{n,p,\pm}}{\sqrt{2\pi} r^{3/2}} \right\|_{L^2(\mathbb{R})} + \left\| \frac{\omega_{n,p,\pm}}{2\sqrt{2\pi} r^{3/2}} \right\|_{L^2(\mathbb{R})},
\]
and from (14) we conclude
\[
\left\| \frac{\partial \psi_{n,p,\pm}}{r} \right\|_{L^2(\mathbb{R})} = o(1), \quad n \to \infty.
\]
From (12), the latter result and (16) we obtain
\[
-\Delta \psi_{n,p,\pm} = -\frac{1}{\sqrt{2\pi} r} \left( \frac{\partial^2 \omega_{n,p,\pm}}{\sqrt{2\pi} r^{3/2}} + \omega_{n,p,\pm} \right) + o(1), \quad n \to \infty. \tag{17}
\]
Again using dominated convergence we compute
\[
\frac{\partial^2 \omega_{n,p,\pm}}{r^{3/2}} = \frac{1}{\sqrt{n}} \left( X_n \left( \frac{\sqrt{3}}{\sqrt{n}} e^{ip} \right) \left( \frac{\sqrt{3}}{\sqrt{n}} e^{ip} + \frac{2i}{n} \right) \left( \frac{s}{n} \right) - p^2 n \left( \frac{s}{n} \right) e^{ip} \right)
\]
\[
= -p^2 \omega_{n,p,\pm} + o(1), \quad n \to \infty, \tag{18}
\]
and
\[
\frac{\partial^2 \omega_{n,p,\pm}}{4r} = \frac{1}{\sqrt{n}} \left( X_n \left( \frac{\sqrt{3}}{\sqrt{n}} e^{ip} \right) \left( \frac{\sqrt{3}}{\sqrt{n}} e^{ip} + \frac{\alpha}{n} \right) \left( \frac{t}{n} \right) e^{it} + \frac{\alpha^2}{4} \right) \left( \frac{t}{n} \right) e^{it}
\]
\[
= \frac{\alpha^2}{4} \omega_{n,p,\pm} + o(1), \quad n \to \infty. \tag{19}
\]
Finally, employing (11), (17), the definition of \( \psi_{n,p} \) in (7) and (18), (19) we arrive at
\[
-\Delta \psi_{n,p} = \left( -\frac{\alpha^2}{4} + p^2 \right) \psi_{n,p} + o(1), \quad n \to \infty. \tag{20}
\]
Since the supports of \( \psi_{n,p} \) and \( \psi_{n,p^*}, \ k \neq k^* \), are disjoint, the sequence \( \{ \psi_{n,p} \} \) converges weakly to zero. Moreover, by (8) we have liminf \( \| \psi_{n,p} \|_{L^2(\mathbb{R})} > 0 \) and hence (20) implies that \( \{ \psi_{n,p} \} \) is a singular sequence for the operator \( -\Delta_{\sigma_\alpha} \) corresponding to the point \(-\alpha^2/4 + p^2\). Therefore, \(-\alpha^2/4 + p^2 \in \sigma_{\text{ess}}(-\Delta_{\sigma_\alpha}) \) for all \( p \in \mathbb{R} \) (see, e.g. [3, Theorem 9.1.2] or [24, Proposition 8.11]) and it follows that \( [-\alpha^2/4, +\infty) \subseteq \sigma_{\text{ess}}(-\Delta_{\sigma_\alpha}) \).

Step 2. In this step we show the inclusion \( \sigma_{\text{ess}}(-\Delta_{\sigma_\alpha}) \subseteq [-\alpha^2/4, +\infty) \) using form decomposition methods. For sufficiently large \( n \in \mathbb{N} \) we define three subsets of the closed half-plane \( \mathbb{R}_+^2 := \{ (r, z) \in \mathbb{R}^2 : r \geq 0, \ z \in \mathbb{R} \} \)
\[
\pi_{n,1} := \left\{ (r(s, t), z(s, t)) \in \mathbb{R}_+^2 : s > n, \ |r| < \sqrt{n} \right\} \subset \mathbb{R}_+^2,
\]
\[
\pi_{n,2} := \left\{ (r(s, t), z(s, t)) \in \mathbb{R}_+^2 : s < n, \ |r| < \sqrt{n} \right\} \subset \mathbb{R}_+^2,
\]
\[
\pi_{n,3} := \left\{ (r(s, t), z(s, t)) \in \mathbb{R}_+^2 : |r| > \sqrt{n} \right\} \subset \mathbb{R}_+^2,
\]
as shown in figure 2.
The ray \( I_\theta \), which emerges from the origin and constitutes the angle \( \theta \) with \( z \)-axis, is decomposed into

\[ \Gamma_\theta = \{ (r(s, t), z(s, t)) \in I_\theta : s > n \} \]

\[ \Gamma_\theta^2 = \{ (r(s, t), z(s, t)) \in I_\theta : s < n \} \]

The splitting \( \{ \pi_n^k \}_{k=1}^3 \) of \( \mathbb{R}^3 \) induces the splitting of \( \mathbb{R} \) into three domains

\[ \Omega_n^k = \{ (r, \varphi, z) : (r, z) \in \pi_n^k, \varphi \in [0, 2\pi) \} \subset \mathbb{R} \]

and the splitting of the conical surface \( \mathbb{C}_0 \) into two parts

\[ C_{\theta, n}^1 = \{ (r, \varphi, z) : (r, z) \in I_{\theta, n}^1, \varphi \in [0, 2\pi) \} \subset \mathbb{C}_0 \]

\[ C_{\theta, n}^2 = \{ (r, \varphi, z) : (r, z) \in I_{\theta, n}^2, \varphi \in [0, 2\pi) \} \subset \mathbb{C}_0 \]

We agree to denote the restriction of \( \psi \in L^2(\mathbb{R}^3) \) onto \( \Omega_n^k \) with \( k = 1, 2, 3 \) by \( \psi_k \).

Consider the quadratic form

\[ a_{n, C_n}(\psi) := \sum_{k=1}^3 \| \nabla \psi_k \|_{L^2(\Omega_n^k, \varepsilon^0)}^2 - \alpha \| \psi_1 |_{C_{\alpha, n}^1} \|_{L^2(c_{\alpha, n}^1)}^2 - \alpha \| \psi_2 |_{C_{\alpha, n}^2} \|_{L^2(c_{\alpha, n}^2)}^2 \cdot \]

\[ \text{dom } a_{n, C_n} := \mathfrak{B} H^1(\Omega_n^k) \]

As in the proof of [1, Proposition 3.1] one verifies that the form \( a_{n, C_n} \) is closed, densely defined, symmetric and semibounded from below. Hence \( a_{n, C_n} \) induces a self-adjoint operator \(-\Delta_{n, C_n}\) in \( L^2(\mathbb{R}^3) \) via the first representation theorem [17, Theorem VI.2.1]. The operator \(-\Delta_{n, C_n}\) can be decomposed into an orthogonal sum \( \bigoplus_{k=1}^3 H_{n,k} \) of self-adjoint operators \( H_{n,k} \) in \( L^2(\Omega_n^k) \) with respect to the orthogonal decomposition \( L^2(\mathbb{R}^3) = \bigoplus_{k=1}^3 L^2(\Omega_n^k) \), where \( H_{n,1} \) and \( H_{n,2} \) correspond to the quadratic forms.
respectively, and $H_{n,3}$ corresponds to the quadratic form

$$a_{n,3}[\psi_3] = \| \nabla \psi_3 \|_{L^2(\Omega_n^3)}^2, \quad \text{dom } a_{n,3} = H^1(\Omega_n^3),$$

Let us first estimate the spectrum of $H_{n,1}$. For this, note that $C^\infty(\Omega_n^1) \cap H^1(\Omega_n^1)$ is a core of $a_{n,1}$ and thus it suffices to use functions from this set in the estimates below (see, e.g. [8, Theorem 4.5.3]). For any $\psi_1 \in C^\infty(\Omega_n^1) \cap H^1(\Omega_n^1)$ normalized as $\| \psi_1 \|_{L^2(\Omega_n^1)} = 1$ we obtain

$$a_{n,1}[\psi_1] \geq \int_0^{2\pi} \left( \int_0^{+\infty} \int_{-\pi}^{\pi} r(s, t) |\partial_t \psi_1(s, t, \varphi)|^2 \, dr \, ds \right) \, d\varphi,$$

where we have used the form of the gradient in cylindrical coordinates and the invariance of the gradient with respect to rotations of the coordinate system, and the non-negative terms corresponding to the partial derivatives of $\psi_1$ with respect to $\varphi$ and $s$ were estimated from below by zero. Note that for simple geometric reasons we have $r(s, t) \geq r(s, -\sqrt{n})$ for all $(s, t) \in \Omega_n^1$. Using this observation we get

$$a_{n,1}[\psi_1] \geq \int_0^{2\pi} \int_0^{+\infty} r(s, -\sqrt{n}) \left( \int_{-\pi}^{\pi} |\partial_t \psi_1(s, t, \varphi)|^2 \, d\varphi \right) \, dr,$$

Consider the closed, densely defined, symmetric and semibounded form

$$b[h] = \int_{-\pi}^{+\pi} |h(t)|^2 \, dt - \beta |h(0)|^2, \quad \text{dom } b = H^1(( -\sqrt{n}, \sqrt{n})),$$

and denote by $\mu(\beta, 2\sqrt{n}) < 0$ the lower bound of the spectrum of the associated 1-D Schrödinger operator on the interval $(-\sqrt{n}, \sqrt{n})$ with Neumann boundary conditions at the endpoints and attractive $\delta$-interaction of strength $\beta > 0$ located at 0. Then

$$b[h] \geq \mu(\beta, 2\sqrt{n}) \int_{-\pi}^{+\pi} |h(t)|^2 \, dt$$

holds for all $h \in H^1((-\sqrt{n}, \sqrt{n}))$ and hence (21) can be further estimated as

$$a_{n,1}[\psi_1] \geq \int_0^{2\pi} \int_{-\pi}^{+\pi} \mu \left( \frac{ar(s, 0)}{r(s, -\sqrt{n})} \right) \int_{-\sqrt{n}}^{\sqrt{n}} r(s, -\sqrt{n}) |\psi_1(s, t, \varphi)|^2 \, dr \, d\varphi. \quad (22)$$

By the definition of $\pi_n^1$ one has

$$r(s, -\sqrt{n}) = r(s, t) \left( 1 + O\left( \frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty,$$
for \((s, t) \in \pi_n\), where the remainder is uniform in \(s\). Hence, we obtain from (22) and (23)
\[
a_{n,1}[\psi] \geq \mu \left( \alpha \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right), 2, \sqrt{n} \right) \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty, \tag{24}
\]
where we used that
\[
\int_0^{2\pi} \int_{-\infty}^{\infty} r(s, t) \left| \psi_1(s, t, \varphi) \right|^2 \, dr \, ds \, d\varphi = \| \psi_1 \|_{L^2(\Omega_n^1)}^2 = 1.
\]
According to [16, Proposition 2.5] the following estimate
\[
\mu(\beta, 2, \sqrt{n}) \geq - \frac{\beta^2}{4} - C \beta^2 \exp \left( - \frac{1}{2} \beta \sqrt{n} \right)
\]
holds with some constant \(C > 0\) and \(n\) sufficiently large. Hence,
\[
\mu \left( \alpha \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right), 2, \sqrt{n} \right) \geq - \frac{\alpha^2}{4} + O \left( \frac{1}{\sqrt{n}} \right), \quad n \to \infty.
\]
Plugging the above estimate into (24) we arrive at
\[
a_{n,1}[\psi] \geq - \frac{\alpha^2}{4} + O \left( \frac{1}{\sqrt{n}} \right), \quad n \to \infty.
\]
Hence, for any \(\varepsilon > 0\) there exists a sufficiently large \(n\) for which
\[
\inf \sigma(H_{n,1}) \geq - \frac{\alpha^2}{4} - \varepsilon.
\tag{25}
\]
As \(H^1(\Omega_n^2)\) is compactly embedded into \(L^2(\Omega_n^2)\) the essential spectrum of \(H_{n,2}\) is empty. The operator \(H_{n,3}\) is non-negative and hence \(\sigma(H_{n,3}) \subseteq [0, +\infty)\). Due to the orthogonal decomposition \(-\Delta_{\alpha,C_{\alpha,n}} = \bigoplus_{k=1}^3 H_{n,k}\) the property (25) implies that for any \(\varepsilon > 0\) there exists a sufficiently large \(n\) for which
\[
\inf \sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha,n}} \right) \geq - \frac{\alpha^2}{4} - \varepsilon.
\tag{26}
\]
Finally, we apply a Neumann bracketing argument. Notice that the ordering \(a_{\alpha,C_{\alpha,n}} \leq a_{\alpha,C_{\alpha}}\) holds in the sense of quadratic forms; cf [17, §VI.5]. Hence by [3, Theorem 10.2.4]
\[
\inf \sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha,n}} \right) \leq \inf \sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha}} \right).
\tag{27}
\]
In view of (27) the estimate (26) implies that for any \(\varepsilon > 0\)
\[
\inf \sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha}} \right) \geq - \frac{\alpha^2}{4} - \varepsilon
\]
and thus passing to the limit \(\varepsilon \to 0^+\) we arrive at
\[
\inf \sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha}} \right) \geq - \frac{\alpha^2}{4},
\]
which shows the inclusion \(\sigma_{\text{ess}} \left( -\Delta_{\alpha,C_{\alpha}} \right) \subseteq \left[ -\frac{\alpha^2}{4}, +\infty \right)\) and finishes the proof of theorem 2.1. \(\square\)
3. Discrete spectrum of $-\Delta_{\alpha, C_0}$

In this section we show that the discrete spectrum of the self-adjoint operator $-\Delta_{\alpha, C_0}$ below the bottom $-\alpha^2/4$ of the essential spectrum is finite for all angles $\theta \in (0, \pi/2)$ and we estimate the rate of the convergence of these eigenvalues to $-\alpha^2/4$ with the help of variational principles. The following lemma will be useful.

**Lemma 3.1.** Let $a_{\alpha, C_0}$ be the form in (1). For $\omega \in H^1(\mathbb{R}_+^2)$ with compact support $\operatorname{supp} \omega \subset \mathbb{R}_+^2$, define the function $\psi(r, \varphi, z) := \frac{\omega(r, \varphi)}{2\pi r}$. Then $\psi \in H^1(\mathbb{R}^3)$ and

$$ a_{\alpha, C_0}[^\psi] = \| \nabla \omega \|^2_{L^2(\mathbb{R}_+^2; \mathbb{R}^3)} - \int_{\mathbb{R}_+^2} \frac{1}{4\pi^2} |\omega(r, z)|^2 \, d\varphi \, dz - \alpha \| \omega \|^2_{L^2(I_{\theta}(\mathbb{R}_+^2))}, $$

where $I_\theta$ is the ray in figure 1.

**Proof.** First of all observe that

$$ \| \psi \|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{2\pi} \frac{|\omega(r, \varphi)|^2}{2\pi r} \, r \, d\varphi \, dz = \| \omega \|^2_{L^2(\mathbb{R}_+^2)} < \infty. \tag{29} $$

Moreover, we compute

$$ \partial_r \psi = \frac{\partial_r \omega}{\sqrt{2\pi r}} - \frac{\omega}{2r \sqrt{2\pi r}} \quad \text{and} \quad \partial_z \psi = \frac{\partial_z \omega}{\sqrt{2\pi r}}, \tag{30} $$

and setting $\rho := \inf \{ r : (r, \varphi, z) \in \operatorname{supp} \omega \} > 0$ we obtain

$$ \| \nabla \psi \|^2_{L^2(\mathbb{R}^3)} = \| \partial_r \psi \|^2_{L^2(\mathbb{R}^3)} + \| \partial_z \psi \|^2_{L^2(\mathbb{R}^3)} \leq 2 \left( \left\| \frac{\partial_r \omega}{\sqrt{2\pi r}} \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \frac{\partial_z \omega}{\sqrt{2\pi r}} \right\|^2_{L^2(\mathbb{R}^3)} \right) \leq 2 \| \partial_r \omega \|^2_{L^2(\mathbb{R}_+^2)} + \frac{1}{2\rho^2} \| \omega \|^2_{L^2(\mathbb{R}_+^2)} + \| \partial_z \omega \|^2_{L^2(\mathbb{R}_+^2)} < \infty. \tag{31} $$

Hence (29) and (31) imply $\psi \in H^1(\mathbb{R}^3)$. Next we substitute $\psi$ in the form $a_{\alpha, C_0}$ in (1). It follows from the form of $\partial_r \psi$ in (30) and $\| \psi \|_{L^2(I_{\theta}(\mathbb{R}_+^2))} = \| \omega \|_{L^2(I_{\theta}(\mathbb{R}_+^2))}$ that

$$ a_{\alpha, C_0}[\psi] = \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_r \psi|^2 2\pi r \, dr \, dz + \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_z \psi|^2 2\pi r \, dr \, dz - \alpha \| \psi \|_{L^2(I_{\theta}(\mathbb{R}_+^2))}^2 $$

$$ = \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_r \omega|^2 2\pi r \, dr \, dz + \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_z \omega|^2 2\pi r \, dr \, dz - \alpha \| \omega \|^2_{L^2(I_{\theta}(\mathbb{R}_+^2))}. \tag{32} $$

Denote the first integral by $I_{\psi}$. Making use of $\partial_r \psi$ in (30) we rewrite $I_{\psi}$ as

$$ I_{\psi} = \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_r \omega|^2 \, dr \, dz + \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{4\pi^2} |\omega|^2 \, dr \, dz - \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{r} \operatorname{Re}(\partial_r \omega \bar{\omega}) \, dr \, dz \tag{33} $$

and the last term can be further rewritten as

$$ \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{r} \operatorname{Re}(\partial_r \omega \bar{\omega}) \, dr \, dz = \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{2r^2} |\omega|^2 \, dr \, dz = \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{2r^2} |\omega|^2 \, dr \, dz, \tag{34} $$

where we integrated by parts and used the fact that $\operatorname{supp} \omega$ is contained in the open half-plane $\mathbb{R}_+^2$. Hence, (33) and (34) imply...
Substituting this expression for the first integral in (32) we obtain (28).

Now we are ready to formulate and prove our main result on the finiteness of the discrete spectrum of $-\Delta_{\alpha,C_0}$ below the bottom of the essential spectrum for all $\alpha > 0$ and $\theta \in (0, \pi/2)$. Recall that $-\Delta_{\alpha,C_0}$ is bounded from below, and hence it also follows that the discrete spectrum has a single accumulation point, namely $-\alpha^2/4$. This result illustrates the typical phenomenon that curvature induces bound states. The peculiarity in this three-dimensional system is that the global geometry of the interaction support plays an important role. We point out that in the case $\theta = \pi/2$ the conical surface $C_0$ coincides with a plane, in which case it follows by separation of variables that the discrete spectrum is empty.

**Theorem 3.2.** Let $-\Delta_{\alpha,C_0}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to the form (1) and let $\alpha > 0$ and $\theta \pi \in (0, 2)$. Recall that $-\Delta_{\alpha,C_0}$ is bounded from below, and hence it also follows that the discrete spectrum has a single accumulation point, namely $-\alpha^2/4$. This result illustrates the typical phenomenon that curvature induces bound states. The peculiarity in this three-dimensional system is that the global geometry of the interaction support plays an important role. We point out that in the case $\theta = \pi/2$ the conical surface $C_0$ coincides with a plane, in which case it follows by separation of variables that the discrete spectrum is empty.

**Proof.** Let us pick a function $\chi_1 \in H^1_0(0, 1)$ with $\| \chi_1 \|_{H^1_0(0,1)} = 1$ such that

$$\| \chi_1' \|_{L^2(0,1)}^2 < \frac{1}{4 \sin^2 \theta} \int_0^1 \left| \chi_1(t) \right|^2 dt$$

holds; [5, Lemma in §1]. Let us fix $\epsilon > 0$ and choose $\chi_2 \in C^\infty_0(-\epsilon, \epsilon)$ such that $0 \leq \chi_2 \leq 1$ and $\chi_2(t) = 1$ for $|t| \leq \epsilon/2$. In the coordinate system $(s, t)$ in figure 1 we define the sequence of functions

$$\omega_n(s, t) := \frac{1}{n} \chi_1 \left( \frac{s - n t}{n^2} \right) \chi_2 \left( \frac{t}{\sqrt{n}} \right) \exp \left( - \frac{\alpha}{2} |t| \right) \in H^1_0(\mathbb{R}^2_+)$$

where the support of $\omega_n$ satisfies

$$\text{supp } \omega_n \subset [n, n + n^2] \times [-\epsilon \sqrt{n}, \epsilon \sqrt{n}], \quad n \in \mathbb{N}.$$  

in the coordinate system $(s, t)$.

For sufficiently large $n \in \mathbb{N}$ the functions $\omega_n$ satisfy the conditions of lemma 3.1. The function $\omega_n$ can also be viewed as a function in $r$ and $z$; cf figure 1. Then we define

$$\psi_n(r, \varphi, z) := \frac{\omega_n(r, z)}{\sqrt{2\pi r}}, \quad n \in \mathbb{N}.$$  

Using lemma 3.1 we compute the values

$$S_n := a_{\alpha,C_0} \| \psi_n \|_{L^2(\mathbb{R}^3)}^2 + \frac{\alpha^2}{4} \| \omega_n \|_{L^2(\mathbb{R}^3)}^2$$

$$= \| \nabla \omega_n \|_{L^2(\mathbb{R}^3)}^2 - \int_{\mathbb{R}^3} \frac{1}{4r^2} |\omega_n|^2 dr dz - \alpha \| \omega_n \|_{L^2(\mathbb{R}^3)}^2 + \frac{\alpha^2}{4} \| \omega_n \|_{L^2(\mathbb{R}^3)}^2.$$  

11
The choice of $\chi_1$ in (36) together with a subtle treatment of the second term in (40) will finally lead to $S_n < 0$ for sufficiently large $n \in \mathbb{N}$. First of all it is not difficult to check the asymptotics

\[
\int_{-\epsilon}^{\epsilon} \left| x_2 \left( \frac{t}{\sqrt{n}} \right) \right|^2 e^{-\alpha t} dt = \frac{\alpha}{2} + O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

(41)

\[
\int_{-\epsilon}^{\epsilon} \left| x_2' \left( \frac{t}{\sqrt{n}} \right) \right|^2 e^{-\alpha t} dt = O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

(42)

\[
\int_{-\epsilon}^{\epsilon} x_2 \left( \frac{t}{\sqrt{n}} \right) x_2' \left( \frac{t}{\sqrt{n}} \right) e^{-\alpha t} dt = O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

(43)

with some constant $c > 0$. Using (41) we get

\[
\frac{\alpha^2}{4} \left\| \omega_n \right\|^2_{L^2(\mathbb{R}^*_+)} = \frac{\alpha^2}{4} \left( \frac{1}{n^2} \int_n^{n^2} \left| x_1 \left( \frac{s-n}{n^2} \right) \right|^2 \frac{ds}{n^2} \left( \int_{-\epsilon}^{\epsilon} \left| x_2 \left( \frac{t}{\sqrt{n}} \right) \right|^2 e^{-\alpha t} dt \right) \right.
\]

\[
= \frac{\alpha}{2} + O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

(44)

and

\[
\left\| \partial_n \omega_n \right\|^2_{L^2(\mathbb{R}^*_+)} = \left( \int_{-\epsilon}^{\epsilon} \left| x_2 \left( \frac{t}{\sqrt{n}} \right) \right|^2 e^{-\alpha t} dt \right) \left( \frac{1}{n^2} \int_n^{n^2} \left| x_1' \left( \frac{s-n}{n^2} \right) \right|^2 \frac{ds}{n^2} \right)
\]

\[
= \frac{2}{\alpha n^2} \left\| x_1' \right\|^2_{L^2(0, 1)} + O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

and from (42) and (43) we obtain

\[
\left\| \partial_n \omega_n \right\|^2_{L^2(\mathbb{R}^*_+)} = \left( \int_{-\epsilon}^{\epsilon} \left| x_2 \left( \frac{t}{\sqrt{n}} \right) \right|^2 e^{-\alpha t} dt \right) \left( \frac{1}{n^2} \int_n^{n^2} \left| x_1' \left( \frac{s-n}{n^2} \right) \right|^2 \frac{ds}{n^2} \right)
\]

\[
= \frac{\alpha}{2} + O\left( e^{-\epsilon^2} \right), \quad n \to \infty,
\]

(45)

that is,

\[
\left\| V\omega_n \right\|^2_{L^2(\mathbb{R}^*_+; c^\alpha)} = \frac{2}{\alpha n^2} \left\| x_1' \right\|^2_{L^2(0, 1)} + \frac{\alpha}{2} + O\left( e^{-\epsilon^2} \right), \quad n \to \infty.
\]

(46)

It is straightforward to see that

\[
\alpha \left\| \omega_n \right\|^2_{L^2(\mathbb{R}^*_+)} = \frac{\alpha}{n^2} \int_n^{n^2} \left| x_1 \left( \frac{s-n}{n^2} \right) \right|^2 \frac{ds}{n^2} = \alpha \left\| x_1 \right\|^2_{L^2(0, 1)} = \alpha,
\]

(47)

and hence it remains to estimate the term $\int_{\mathbb{R}^*_+} \left| \frac{1}{4n^2} \omega_n \right|^2$ in (40). For that we make the following splitting.
\[
\int_{\mathbb{R}^2_+} \frac{1}{4r^2} |\omega_n(r, z)|^2 \, dr\, dz = \int_{-\pi}^{\pi} \int_n^{n + n^2} \frac{1}{4r} \omega_n(s, t)^2 \, ds \, dt = I_n + J_n, \tag{48}
\]

where
\[
I_n := \int_{-\pi}^{\pi} \int_n^{n + n^2} \frac{1}{4r} \omega_n(s, t)^2 \, ds \, dt
\tag{49}
\]

and
\[
J_n := \int_{-\pi}^{\pi} \int_n^{n + n^2} \left( \frac{1}{4r} - \frac{1}{4r^3} \right) \omega_n(s, t)^2 \, ds \, dt.
\]

The term \(J_n\) can be further rewritten as
\[
J_n = \int_{-\pi}^{\pi} \int_n^{n + n^2} \frac{(r(s, 0) - r(s, t))(r(s, 0) + r(s, t))}{4r(s, t)^2r(s, 0)^2} |\omega_n(s, t)|^2 \, ds \, dt. \tag{50}
\]

For geometric reasons we have \(|r(s, 0) - r(s, t)| \leq a \sqrt{n}\) with some \(0 < a \leq \varepsilon\) and \(r(s, t) > b n\) with some \(b > 0\) for all \((s, t) \in \omega_n\). We first conclude from (50) that
\[
|J_n| \leq a \sqrt{n} \int_{-\pi}^{\pi} \int_n^{n + n^2} \frac{2}{4r(s, t)^2r(s, 0)^2} + \frac{r(s, 0) - r(s, t)}{4r(s, t)^2r(s, 0)^2} |\omega_n(s, t)|^2 \, ds \, dt
\]

and hence
\[
|J_n| \leq \left( \frac{2a}{b \sqrt{n}} + \frac{a^2}{b^2 n} \right) I_n \tag{51}
\]

follows together with (49). For \(I_n\) we have
\[
I_n = \left( \frac{1}{n^2} \int_n^{n + n^2} \frac{1}{4 s^2 \sin^2 \theta} \left| \chi_1 \left( s - \frac{n}{n^2} \right) \right|^2 \, ds \right) \left( \int_{-\pi}^{\pi} \frac{1}{4 \sin^2 \theta} \, d\theta \right) e^{-\alpha \Pi} dt
\]

and the choice of \(\chi_1\) (see (36)) together with monotone convergence yields
\[
\int_0^1 \frac{|\chi_1(u)|^2}{(u + 1/n)^2} \, du = \int_0^1 \frac{|\chi_1(u)|^2}{u^2} \, du + o(1), \quad n \to \infty.
\]

Hence we conclude from (41) and (52) that
\[
I_n = \frac{2}{\alpha \, n^4} \frac{1}{4 \sin^2 \theta} \int_0^1 \frac{|\chi_1(u)|^2}{u^2} \, du + o \left( \frac{1}{n^2} \right), \quad n \to \infty,
\]

and from (51) we find
\[
J_n = o \left( \frac{1}{n^2} \right), \quad n \to \infty.
\]
It follows that (48) becomes
\[
\int_{\mathbb{R}^3} \frac{1}{4\pi^2} |\omega_n(r, z)|^2 \, dr \, dz = 2 \frac{1}{\alpha n^4} \frac{1}{4 \sin^2(\theta)} \int_0^1 \frac{|\chi(\omega)|^2}{u^2} \, du + o \left( \frac{1}{n^4} \right), \quad (53)
\]
as \( n \to \infty \). Finally, (44), (46), (47) and (53) yield
\[
S_n = 2 \frac{1}{\alpha n^4} \left( \| \chi' \|_{L^2(0, 1)}^2 - \int_0^1 \frac{|\chi(\omega)|^2}{4 \sin^2(\theta) u^2} \, du \right) + o \left( \frac{1}{n^4} \right), \quad n \to \infty, \quad (54)
\]
for \( S_n \) in (40). In view of the above asymptotics and according to (36) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have
\[
S_n \leq - \frac{2\gamma(\theta)}{an^4}, \quad (55)
\]
for some constant \( \gamma(\theta) > 0 \). Let us consider a sequence \( \{n_k\} \), where \( n_1 := N \) and \( n_{k+1} := n_k^2 + n_k \) for \( k \in \mathbb{N} \). Then by (38) the measure of \( \text{supp} \, \omega_{n_k} \cap \text{supp} \, \omega_{n_l} \) is zero for all \( k, l \in \mathbb{N} \), \( k \neq l \), and hence it follows from the definition (39) that the measure of \( \text{supp} \, \psi_{n_k} \cap \text{supp} \, \psi_{n_l} \) is zero for all \( k, l, n_k, n_l \neq N \), and, in particular, the functions \( \psi_{n_k} \) are orthogonal in \( L^2(\mathbb{R}^3) \). The space
\[
F_k := \text{span} \{ \psi_{n_1}, \psi_{n_2}, \ldots, \psi_{n_k} \} \subset H^1(\mathbb{R}^3),
\]
has dimension \( k \) and for an arbitrary \( \psi = \sum_{i=1}^k a_i \psi_{n_i} \in F_k \), \( a_i \in \mathbb{C} \), we get
\[
\| \psi \|_{L^2(\mathbb{R}^3)}^2 = \sum_{i=1}^k |a_i|^2 \| \psi_{n_i} \|_{L^2(\mathbb{R}^3)}^2 = \sum_{i=1}^k |a_i|^2 \| \omega_{n_i} \|_{L^2(\mathbb{R}^3)}^2 \leq 2 \sum_{i=1}^k |a_i|^2, \quad (56)
\]
where we have also used the estimate \( \| \omega_{n_i} \|_{L^2(\mathbb{R}^3)}^2 \leq \frac{2}{a} \). Employing (55) we obtain
\[
\alpha_{a,C_0}[\psi] + \frac{\alpha^2}{4} \| \psi \|_{L^2(\mathbb{R}^3)}^2 = \sum_{i=1}^k |a_i|^2 S_{n_i} \leq - \frac{2\gamma(\theta)}{an_k} \sum_{i=1}^k |a_i|^2,
\]
where we have again used that the mutual intersections of the supports of \( \{ \psi_{n_i} \}_{i=1}^k \) are of measure zero.

Combining the above estimate with (56) we get
\[
\frac{\alpha_{a,C_0}[\psi]}{\| \psi \|_{L^2(\mathbb{R}^3)}^2} = \frac{\alpha^2}{4} + \frac{\alpha_{a,C_0}[\psi] + \left( \frac{\alpha^2}{4} \right) \| \psi \|_{L^2(\mathbb{R}^3)}^2}{\| \psi \|_{L^2(\mathbb{R}^3)}^2} \leq \frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^2} < - \frac{\alpha^2}{4}. \quad (57)
\]
Hence, according to [3, Theorem 10.2.3] the operator \( -\Delta_{a,C_0} \) has at least \( k \) eigenvalues below the bottom of the essential spectrum \( -\alpha^2/4 \). The above construction works for any \( \theta \in (0, \pi/2) \) and \( C_0 \) be the conical surface as above. A hypersurface \( \Sigma \subset \mathbb{R}^3 \), which for some compact set \( K \subset \mathbb{R}^3 \) satisfies the condition \( \Sigma K = C_0 \setminus K \) and which splits the space \( \mathbb{R}^3 \) into two unbounded Lipschitz domains, is called a local deformation of \( C_0 \); cf [1, Section 4.2]. Below we consider the self-adjoint Schrödinger operator \( -\Delta_{a,\Sigma} \) with an attractive
\( \delta \)-interaction of constant strength \( \alpha > 0 \) supported on the Lipschitz hypersurface \( \Sigma \). This Schrödinger operator is defined via the closed, densely defined, symmetric and semibounded quadratic form

\[
\alpha_{\alpha, \Sigma} \psi = \| \nabla \psi \|_{L^2(\mathbb{R}^3)}^2 - \alpha \int_{\Sigma} |\psi|^2 d\sigma, \quad \text{dom } \alpha_{\alpha, \Sigma} = H^1(\mathbb{R}^3). \tag{58}
\]

The assertion on the essential spectrum in the next theorem is a consequence of \([1, \text{Theorem 4.7}]\); the infiniteness of the discrete spectrum can be shown as in the proof of theorem 3.2 using the same functions \( \psi_n \) in (39) and \( n \in \mathbb{N} \) sufficiently large.

**Theorem 3.3.** Let \( \theta \in (0, \pi/2) \) and \( \alpha > 0 \). Let \( \Sigma \) be a local deformation of the cone \( \mathcal{C}_0 \) and let \( -\Delta_{\alpha, \Sigma} \) be the self-adjoint operator in \( L^2(\mathbb{R}^3) \) associated to (58). Then

\[
\sigma_{\text{ess}}(-\Delta_{\alpha, \Sigma}) = [ -\alpha^2/4, +\infty),
\]

the discrete spectrum below \(-\alpha^2/4\) is infinite, accumulates at \(-\alpha^2/4\), and the eigenvalues \( \lambda_k < -\alpha^2/4 \) (enumerated in non-decreasing order with multiplicities taken into account) satisfy the estimate

\[
\lambda_k \leq -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^2}, \quad k \in \mathbb{N},
\]

where \( \gamma(\theta) > 0 \), \( n_{k+1} := n_k^2 + n_k \) for \( k \in \mathbb{N} \), and \( n_1 = N \) with \( N \in \mathbb{N} \) sufficiently large.

**Acknowledgments**

The authors gratefully acknowledge financial support by the Austrian Science Fund (FWF), project P 25162-N26, Czech Science Foundation (GAČR), project 14–06818S, and the Austrian Agency for International Cooperation in Education and Research (OeAD), Austria-Czech Republic cooperation grant CZ01/2013.

**References**

[1] Behrndt J, Exner P and Lotoreichik V 2014 Rev. Math. Phys. doi:10.1142/S0129055X14500159
[2] Behrndt J, Langer M and Lotoreichik V 2013 Ann. Henri Poincaré 14 385–423
[3] Birman M Sh and Solomjak M Z 1987 Spectral Theory of Self-adjoint Operators in Hilbert Spaces (Holland: Dordrecht)
[4] Brasche J F, Exner P, Kuperin Yu A and Šeba P 1994 J. Math. Anal. Appl. 184 112–39
[5] Brezis H and Marcus M 1997 Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 217–37
[6] Brown B M, Eastham M S P and Wood I G 2009 J. Math. Anal. Appl. 354 24–30
[7] Carron G, Exner P and Krejčířik D 2004 J. Math. Phys. 45 774–84
[8] Davies E B 1995 Spectral Theory and Differential Operators (Cambridge: Cambridge University Press)
[9] Duchêne V and Raymond N 2014 J. Phys. A 47 155203
[10] Duclos P, Exner P and Krejčířik D 2001 Comm. Math. Phys. 223 13–28
[11] Exner P 2008 (Proc. Symp. Pure Math. vol 77) ed P Exner, J P Keating, P Kuchment, T Sunada and A Teplayaev (Providence, R.I: Amer. Math. Soc.) pp 523–64
[12] Exner P and Kondej S 2002 Ann. Henri Poincaré 3 967–81
[13] Exner P and Němcová K 2003 J. Phys. A 36 10173–93
[14] Exner P and Šeba P 1989 J. Math. Phys. 30 2574–80
[15] Exner P and Tater M 2010 J. Phys. A 43 474023
[16] Exner P and Yoshitomi K 2002 J. Geom. Phys. 41 344–58
[17] Kato T 1995 Perturbation Theory for Linear Operators. Reprint of the 1980 Edition (Berlin: Springer-Verlag)
[18] Kovářík H and Vugalter S 2008 J. Math. Anal. Appl. 345 566–72
[19] Lin C and Lu Z 2007 J. Funct. Anal 244 1–25
[20] Londergan J T, Carini J P and Murdock D P 1999 Binding and Scattering in Two-Dimensional Systems. Applications to Quantum Wires, Waveguides and Photonic Crystals (Berlin: Springer)
[21] Lu Z and Rowlett J 2012 J. Math. Phys. 53 073519
[22] Mao J 2013 J. Math. Anal. Appl. 397 791–9
[23] Reed M and Simon B 1978 Methods of Modern Mathematical Physics, IV. Analysis of Operators (New York: Academic)
[24] Schmüdgen K 2012 Unbounded Self-adjoint Operators on Hilbert Space (Dordrecht: Springer)
[25] Simon B 1970 Helv. Phys. Acta 43 607–30