Nonequilibrium Dynamic Phase Transition in the Ferromagnetic Potts Model

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Abstract. The nonequilibrium dynamic phase transition arises in various magnetic systems such as ferromagnetic Ising model when system is driven by both an external field and temperature simultaneously. In this paper, the nonequilibrium dynamic phase transition in the q-state Potts model in the presence of a time dependent oscillating magnetic field on a simple cubic lattice is investigated by mean field theory and time dependent Ginzburg-Landau equation. For the case of \( q = 2 \), with decrease of the temperature the system undergoes a dynamically ordered phase transition, which is characterized by the period averaged magnetization \( Q \), from a dynamically disordered state \( Q = 0 \) to the dynamically ordered state \( Q \neq 0 \). On the other hand, for the case of \( q = 3 \) the system has indicated more complicated behavior than that of \( q = 2 \). We investigated those characteristic features of the dynamical phase state from the behavior of the dynamic magnetization and the Liapunov exponent.

1. Introduction

The nonequilibrium phase transition characterized by the period time averaged magnetization is called as the dynamic phase transition (DPT) [1]. The DPT was first observed in a study of mean-field calculation of the time dependent magnetization for an Ising ferromagnetic system in an oscillating field [2]. Since then, many investigations have been performed by using various methods including mean-field type and Monte Carlo calculation [3, 4, 5], for various systems such as Blume-Capel model [6] and Ising ferrimagnets [7] as well as standard Ising ferromagnetic systems. The DPT has been also experimentally observed in Co films on a Cu (001) ultrathin magnetic films [8] and [Co/Pt]₃ multilayers [9]. From these studies it is now well established that there appears a genuine continuous phase transition between the symmetric and asymmetric dynamic phase for some region in the parameter space spanned by temperature and magnetic field amplitude and frequency.

In this paper, we have investigated the nonequilibrium DPT by use of a mean field theory (MFT) and the time dependent Ginzburg-Landau equation for the kinetic Potts model in an oscillating field on a simple cubic lattice.
2. Theory

We consider the $q$-state ferromagnetic Potts system under an oscillating magnetic field and the Hamiltonian is given by

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij}(q\delta_{\sigma_i,\sigma_j} - 1) - h(t) \sum_i (q\delta_{\sigma_i,1} - 1).$$  \hspace{1cm} (1)

In equation (1), $J_{ij}$ is the ferromagnetic interaction between the nearest neighboring site $i$ and $j$, and $h(t) = h_0 \cos \omega t$ where $h_0$ and $\omega$ are the amplitude and the frequency of the oscillating field, respectively. The $\sigma_i$ represents discrete spin variable for site $i$ which can take $q$-state ($\sigma_i = 1, 2, \ldots, q$). In the case of $q = 2$, this system reduces to the simple $s = \pm \frac{1}{2}$ Ising model.

Here, we define a set of the unit vector $e_i(i = 1, 2, \ldots, q)$ pointing to $q$ directions in the $(q - 1)$-dimensional space with the restriction that any two of them make the same angle with each other and the sum of all unit vector is zero, i.e. $\sum_{i=1}^{q} e_i = 0$. Then we introduce the vector magnetization: $m(\sigma_i) = \sum_{j=1}^{q} \delta_{\sigma_i,j} e_j$, so that the interaction energy $\delta_{\sigma_i,\sigma_j}$ is given by

$$\delta_{\sigma_i,\sigma_j} = \frac{1}{q} \{ 1 + (q - 1)m(\sigma_i) \cdot m(\sigma_j) \}. \hspace{1cm} (2)$$

By substituting equation (2) into (1), we get

$$\mathcal{H} = -(q - 1) \sum_{\langle ij \rangle} J_{ij} m(\sigma_i) \cdot m(\sigma_j) - (q - 1) \sum_i h(t) \cdot m(\sigma_i), \hspace{1cm} (3)$$

with $h(t) = h(t)e_1$. We denote the projection of $m(\sigma_i)$ on $e_1$ by $m(\sigma_i)$, that is $m(\sigma_i) \cdot e_1 = m(\sigma_i)$, and it can be given by

$$m(\sigma_i) = \frac{q\delta_{\sigma_i,1} - 1}{q - 1}. \hspace{1cm} (4)$$

In the present study, for simplicity we consider only the case of $J_{ij} = J$, and applied a mean field theory (MFT) to the Hamiltonian (3) and derived it’s free energy $f$ as a function of spin average $\langle m(\sigma_i) \rangle \equiv m$.

The evolution equation for $m$ is directly obtained by use of the time dependent Ginzburg-Landau equation, $\frac{d}{dt}m = -\gamma \frac{d}{dm}f$, where $\gamma$ is a diffusion constant. Finally, we get the following equation for $m$,

$$\Omega \frac{dm}{d\xi} = -m + \frac{1 - \exp\{-\frac{q}{q - 1}(m + h \cos \xi)\}}{1 + (q - 1) \exp\{-\frac{q}{q - 1}(m + h \cos \xi)\}}. \hspace{1cm} (5)$$

Hereafter we use the dimensionless parameters which are defined by $\xi = \omega t$, $T = (z\beta J)^{-1}$, $h = h_0/zJ$, and $\Omega = \omega \tau$, where $z$ is the coordination number and $\tau$ is a relaxation time proportional to $\gamma^{-1}$. The equation (5) can be also derived by applying the Glauber-type stochastic dynamics to the Hamiltonian (1). In the next section, by use of equation (5), we will study the stationary solutions of this system and its dependence on the parameters, $T$, $h$ and $\Omega$.

3. Results and Discussion

The stationary solutions of equation (5) will be a periodic function of $\xi$ with period $2\pi$, that is $m(\xi + 2\pi) = m(\xi)$, and for the $q = 2$ state Potts model those solutions are also distinguished by whether they satisfy $m(\xi) = -m(\xi + \pi)$ or not. Here we introduce the dynamical order parameter $Q$ which is defined by the average of the total magnetization in a period, $Q = \frac{1}{2\pi} \int_{0}^{2\pi} m(\xi) d\xi$. Then, for the case of $q = 2$ we can classify the solutions of the equation (5) in two types: a symmetric one where $m(\xi)$ completely follows the field and oscillates around zero leading $Q = 0$, and
and an asymmetric one where \( m(\xi) \) does not follow the field and oscillates around a finite value leading \( Q \neq 0 \). On the other hand, in the case of \( q = 3 \), \( m(\xi) \) does not satisfy the relation \( m(\xi) = -m(\xi + \pi) \) as shown later, and \( Q \) will not play the role of the order parameter adequately.

For several fixed values of \( h = h_0/zJ \) and \( Q = \omega T \), the equation (5) is solved by using fourth order Runge-Kutta method and the quantity \( Q \) is calculated as a function of \( T = (\beta zJ)^{-1} \).

In the calculation, we also checked and verified the results and stability of the solutions by estimating the Liapunov exponent adapted for the present periodic system. If we write equation (5) as \( \Omega \frac{d}{d\xi} F(m, \xi) \), then the Liapunov exponent \( \lambda \) is given by \( \Omega \lambda = \frac{1}{2\pi} \int_0^{2\pi} \frac{dF}{dm} d\xi \). When the Liapunov exponent \( \lambda \) is negative, the solution is stable.

The temperature dependences of \( Q \) and \( \lambda \) for the \( q = 2 \) state Potts model at \( h = 0.4 \) and \( \Omega/2\pi = 1.0 \) are shown in figure 1. Here we denote \( Q \) and \( \lambda \) corresponding to the state in which most of the sites are in the spin state \( \sigma_i = 1(\sigma_i = 2) \) by \( Q_{\sigma_i=1}(Q_{\sigma_i=2}) \) and \( \lambda_{\sigma_i=1}(\lambda_{\sigma_i=2}) \), respectively. From the calculation for various fixed values of \( h \), we can construct the phase diagram for the \( q = 2 \) state Potts model in \((T, h)\) plane as shown in figure 2. We confirmed that the present MFT gives DPT in the region of sufficiently small \( h \) and small \( T \). In addition to the ferromagnetic (F) and paramagnetic (P) phase there appears the coexistence phase F+P. The DPT between the F+P phase and F (P) phase is of first order, while the DPT between F and P phase is of second order. Now it is believed and confirmed by various Monte Carlo simulation that for the present system the P and F phase do not coexist and the tricritical point (TCP) does not appear [5]. Then, it is considered that the coexistence of P and F phase is an artifact of MFT.

![Figure 1](image1.png)  
**Figure 1.** The temperature dependences of the dynamical order parameter \( Q \) and the Liapunov exponent \( \lambda \) for the \( q = 2 \) state Potts system at \( h = 0.4 \) and \( \Omega/2\pi = 1.0 \). \( \lambda_p \) indicates the Liapunov exponent for the paramagnetic state.

![Figure 2](image2.png)  
**Figure 2.** The phase diagram projected in \((T, h)\) plane for the \( q = 2 \) state Potts system \((\Omega/2\pi = 1.0)\). The P, F and P+F are explained in the text. TCP is denoted by cross mark \((\times)\). The phase boundaries are shown by solid lines.

The similar calculation has been performed for the \( q = 3 \) state Potts model. In figure 3, we present the result of the temperature dependence of \( Q \) and \( \lambda \) for the \( q = 3 \) state Potts model at \( h = 0.2 \) and \( \Omega/2\pi = 0.2 \). In the low temperature region of \( T \leq 0.7 \), both \( \lambda_{\sigma_i=1} \) and \( \lambda_{\sigma_i\neq1} \) are negative until the latter reaches to zero at \( T = T_1 \) with increase of the temperature. On the other hand, the former preserves negative value with steep cusp at \( T = T_2 \). These results indicate that the two stable states exist below \( T_1 \) and only the one corresponding to the spin state having \( \sigma_i = 1 \) for most of the sites becomes stable above \( T_1 \) while the quantity \( Q_{\sigma_i=1} \) does not show any singularity even in the vicinity of \( T \sim T_2 \). Figure 4 shows the result for the case of stronger field of \( h = 0.7 \). In this case only one state is stable at all temperatures and we
cannot see any clear phase change. Those behaviors different from that of the $q = 2$ case will be attributed to that the applied oscillating magnetic field is not symmetric between $\sigma_i = 1$ state and $\sigma_i \neq 1$ state in the present formalism. The cusp of $\lambda_{\sigma_i=1}$ found in figure 3 and 4 may be suggesting an increase of dynamic spin state instability at around $T \sim T_1$. In order to have a deep insight into this problem it will need more sophisticated analysis than that of MFT.

**Figure 3.** The temperature dependence of the time-averaged magnetization $Q$ and the Liapunov exponent $\lambda$ for the $q = 3$ state Potts system at $h = 0.2$ and $\Omega/2\pi = 0.2$.

**Figure 4.** The temperature dependence of the time-averaged magnetization $Q$ and the Liapunov exponent $\lambda$ for the $q = 3$ state Potts system at $h = 0.7$ and $\Omega/2\pi = 0.2$.

4. **Concluding Remarks**

By use of the mean field theory and time dependent Ginzburg-Landau equation, we have analyzed the dynamical phase transition (DPT) for the $q$-state ferromagnetic Potts model in an oscillating magnetic field. While the system undergoes DPT in the case of $q = 2$, the dynamic behavior in the $q = 3$ system is different from that because of an asymmetry of the effect of the oscillating field. In order to elucidate the complicated dynamic properties of the present system, it is interesting to perform the Monte Carlo simulation and such calculation is now in progress.

**References**

[1] Chakrabarti B K 1999 *Rev. Mod. Phys.* **71** No.3 847
[2] Tóme T and de Oliveira M J 1990 *Phys. Rev. A* **41** 4251
[3] Acharyya M Electronic preprint: http://xxx.lanl.gov/abs/cond-mat/0403530
[4] Sides S W, Rikvold P A and Novotny N A 1998 *Phys. Rev. Lett.* **81** 834
[5] Korniss G, Rikvold, P A, Novotny M A 2002 *Phys. Rev. E* **66** 056127
[6] Keskin M, Canko O, Deviren B 2007 *J. Magn. Magn. Mater.* **313** L1-L5
[7] Buendía G M and Machado E 1998 *Phys. Rev. E* **58** 1260
[8] Jiang Q, Yang H N and Wang G C *Phys. Rev. B* **52** 14911
[9] Robb D T, Xu Y H, Hellwig O, McCord J, Berger A, Novotny M A and Rikvold P A 2008 *Phys. Rev. B* **78** 134422