Quantum Sheaves
An outline of results

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Abstract

In this paper we start with the development of a theory of presheaves on a lattice, in particular on the quantum lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of a complex Hilbert space $\mathcal{H}$, and their associated etale spaces. Even in this early state the theory has interesting applications to the theory of operator algebras and the foundations of quantum mechanics. Among other things we can show that classical observables (continuous functions on a topological space) and quantum observables (selfadjoint linear operators on a Hilbert space) are on the same structural footing.
1 Introduction

Classically, an observable is a real valued (measurable, continuous or differentiable) function on the phase space $M$ of a physical system. In quantum mechanics, however, an observable is a selfadjoint linear operator defined on a dense subspace of a complex Hilbert space $\mathcal{H}$.

Apparently, these are quite different concepts. One of the aims of our work is to show that both concepts are on the same structural footing. This insight is an outcome of a theory of presheaves on an arbitrary lattice which we begin to study here.

The theory of presheaves and sheaves, and in particular the cohomology theory of sheaves, is an indispensable tool in Penrose’s twistor theory (8, 12). Butterfield, Isham and Hamilton have used presheaves in the sense of topos theory for a new interpretation of the Kochen-Specker theorem in quantum mechanics (3, 4). There are also attempts to use sheaf theory in the development of a theory of quantum gravity (7, 9). So sheaf theory begins to establish in mathematical physics.

A presheaf $S$ on a topological space $M$ assigns to every open subset $U$ of $M$ a set $S(U)$ (or a set $S(U)$ with some algebraic structure: e.g. an $R$-module, a vectorspace or an algebra) and to each pair $(U, V)$ of open sets so that $U \subseteq V$ a “restriction map” $\rho_{UV} : S(V) \rightarrow S(U)$ that respects the algebraic structure of the sets $S(U), S(V)$. A complete presheaf (usually called a sheaf) on $M$ is a presheaf that has the property that one can glue together compatible local data to global ones in a unique manner (definition 2.2). The definition of presheaves and sheaves can be translated literally from the lattice $\mathcal{T}(M)$ of open subsets of $M$ to an arbitrary lattice $L$. The most important lattice that we have in mind is the orthocomplemented lattice $L(\mathcal{H})$ of closed subspaces of a complex Hilbert space $\mathcal{H}$. This is the “quantum lattice”: it is isomorphic to the lattice of orthogonal projections of the Hilbert space $\mathcal{H}$.

Each serious mathematical theory has to present some interesting and convincing examples. So our first result is a disappointment: there are no non-trivial complete presheaves on the quantum lattice $L(\mathcal{H})$. But there are non-trivial presheaves on $L(\mathcal{H})$. There are canonical ones, namely presheaves of spectral families in $L(\mathcal{H})$, that we shall discuss in section 6.

In ordinary sheaf theory (over topological spaces) there is a natural construction that assigns to each presheaf a sheaf, namely the sheaf of local sections.

\footnote{For the definition of a lattice see definition 2.1. A lattice in our sense has nothing to do with the following notion in use: a group isomorphic to a subgroup of the abelian group $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. Here we use “lattice” in the sense of the german “Verband”, whereas the other meaning is called in german “Gitter”.}
of the etale space of the presheaf. The etale space of a presheaf is the space of
germs of elements \( f \in S(U) \) \( (U \in \mathcal{T}(M)) \) at points of \( U \). We can generalize
the notion of a “point” so that it makes sense in an abstract (complete) lattice (definition 3.1). Unfortunately, in important lattices like \( L(\mathcal{H}) \) there are
no points at all. For the definition of germs, however, we do not need points
but only “filter bases”. For maximal filter bases (these are not ultrafilters
in general!) we have coined the name \textit{quasipoints}, for they are substitutes
for the non-existing points in a general lattice. It turns out that quasipoints
are already known in lattice theory: they are the maximal dual ideals of the
lattice. The set of quasipoints of a lattice carries a natural topology. This
topology has been introduced for the case of Boolean algebras by M.H. Stone
in the thirties (of the bygone bestial century). Therefore we call the set of
quasipoints of a general lattice with its natural topology the \textit{Stonean space}
of the lattice.

We study Stonean spaces in section 4 and show how to associate a sheaf of
local sections over the Stonean space \( Q(L) \) to a presheaf on a lattice \( L \).
In section 5 we consider the maximal distributive sublattices of the quantum
lattice \( L(\mathcal{H}) \) (called \textit{Boolean sectors}) and their Stonean spaces. The Boolean
sectors \( B \subseteq L(\mathcal{H}) \) are in one to one correspondence to the maximal abelian
von Neumann subalgebras of the algebra \( L(\mathcal{H}) \) of bounded linear operators
on \( \mathcal{H} \) (theorem 5.1). Moreover, if \( C^*(B) \) denotes the \( C^* \)-algebra generated
by the projections \( P_U \) \( (U \in B) \), then the Gelfand spectrum \( \Omega_B \) of \( C^*(B) \) can
be identified with the Stonean space \( Q(B) \) of the Boolean sector \( B \) (theorem
5.2).

In section 6 we consider a canonical example of a presheaf on \( L(\mathcal{H}) \). This
presheaf consists of spectral families of selfadjoint operators on \( \mathcal{H} \), i.e. of
quantum mechanical observables. We show that one can define restriction
maps in close analogy to the lattice formulation of the restriction of func-
tions. The restriction of spectral families to one dimensional subspaces of \( \mathcal{H}
\) define functions on the projective Hilbert space \( \mathbb{P}(\mathcal{H}) \). These functions can
be characterized abstractly without any reference to linear operators on \( \mathcal{H}
\) (theorem 6.2). We therefore call these functions \textit{observable functions}. They
induce upper semicontinuous functions on the Stonean space of the quant-
num lattice \( L(\mathcal{H}) \) and on the Stonean spaces \( Q(B) \) of each Boolean sector
\( B \subseteq L(\mathcal{H}) \). The induced observable functions on \( Q(B) \) are precisely the
Gelfand transforms of selfadjoint operators in \( C^*(B) \) (theorem 6.3). Using
these concepts, we show (theorem 6.4) that in a precise measure theoretical
sense the number \( < \rho; A > := tr(\rho A) \), where \( A \) is a bounded selfadjoint op-
terator (an observable) and \( \rho \) a positive operator of trace 1 (a state), is an
expectation value.

In section 7 we show how continuous real valued functions on a topological
space $M$ (classical observables) can be characterized by spectral families with values in the lattice $\mathcal{T}(M)$ (theorem 7.1). This shows that classical and quantum mechanical observables are on the same structural footing: either as functions or as spectral families.

The results presented here show that the theory of quantum sheaves, i.e. the theory of presheaves on a lattice and their etale spaces, deserves to be developed further.

In this paper only few of the results are proved in detail. Rather long proofs of the main results are only sketched. A detailed version will appear in appropriate form.

Thanks are due especially to Andreas Döring for several discussions on the foundations of quantum mechanics.
2 Preliminaries

Definition 2.1 A lattice is a partially ordered set \((L, \leq)\) with a zero element 0 (i.e. \(\forall a \in L : 0 \leq a\)), a unit element 1 (i.e. \(\forall a \in L : a \leq 1\)), such that any two elements \(a, b \in L\) possess a maximum \(a \lor b \in L\) and a minimum \(a \land b \in L\).

Let \(m\) be an infinite cardinal number.

The lattice \(L\) is called \(m\)-complete, if every family \((a_i)_{i \in I}\) has a maximum \(\bigvee_{i \in I} a_i\) and a minimum \(\bigwedge_{i \in I} a_i\) in \(L\), provided that \(#I \leq m\) holds. A lattice \(L\) is simply called complete, if every family \((a_i)_{i \in I}\) in \(L\) (without any restriction of the cardinality of \(I\)) has a maximum and a minimum in \(L\).

A lattice \(L\) is called distributive if the two distributive laws

\[
a \land (b \lor c) = (a \land b) \lor (a \land c)
\]

\[
a \lor (b \land c) = (a \lor b) \land (a \lor c)
\]

hold for all elements \(a, b, c \in L\).

\(\bigvee_{i \in I} a_i\) is characterized by the following universal property:

1. \(\forall j \in I : a_j \leq \bigvee_{i \in I} a_i\)
2. \(\forall c \in L : ((\forall i \in I : a_i \leq c) \Rightarrow \bigvee_{i \in I} a_i \leq c)\).

An analogous universal property characterizes the minimum \(\bigwedge_{i} a_i\).

Note that if \(L\) is a distributive complete lattice, then in general

\[
a \land (\bigvee_{i \in I} b_i) \neq \bigvee_{i \in I} (a \land b_i),
\]

so completeness and distributivity do not imply complete distributivity!

Let us give some important examples.

Example 2.1 Let \(M\) be a topological space and \(\mathcal{T}(M)\) the topology of \(M\), i.e. the set of all open subsets of \(M\). \(\mathcal{T}(M)\) is a distributive complete lattice.

The maximum of a family \((U_i)_{i \in I}\) of open subsets \(U_i\) of \(M\) is given by

\[
\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i,
\]

the minimum, however, is given by

\[
\bigwedge_{i \in I} U_i = \text{int}(\bigcap_{i \in I} U_i),
\]

where \(\text{int}N\) denotes the interior of a subset \(N\) of \(M\).
Example 2.2 If $U \in \mathcal{T}(M)$, then always

$$U \subseteq \text{int} \bar{U},$$

but $U \neq \text{int} \bar{U}$ in general. $U$ fails to be the interior of its adherence $\bar{U}$, if for example $U$ has a “crack” or is obtained from an open set $V$ by deleting some points of $V$.

We call $U$ a regular open set, if $U = \text{int} \bar{U}$. Each $U \in \mathcal{T}(M)$ has a pseudocomplement, defined by

$$U^c := M \setminus \bar{U},$$

and together with the operation of pseudocomplementation $\mathcal{T}(M)$ is a Heyting algebra:

$$\forall U \in \mathcal{T}(M) : U^{ccc} = U^c.$$

$U \in \mathcal{T}(M)$ is regular if and only if $U = U^{cc}$. Let $\mathcal{T}_r(M)$ be the set of regular open subsets of $M$. If $U, V \in \mathcal{T}_r(M)$, then also $U \cap V \in \mathcal{T}_r(M)$. The union of two regular open sets, however, is not regular in general. Therefore one is forced to define the maximum of two elements $U, V \in \mathcal{T}_r(M)$ as

$$U \lor V := (U \cup V)^c.$$

It is then easy to see that $\mathcal{T}_r(M)$ is a distributive complete lattice with the lattice operations

$$U \land V := U \cap V, \quad U \lor V := (U \cup V)^c.$$

The pseudocomplement on $\mathcal{T}(M)$, restricted to $\mathcal{T}_r(M)$, gives an orthocomplement $U \mapsto U^c$ on $\mathcal{T}_r(M)$:

$$U^{cc} = U, \quad U^c \lor U = M, \quad U^c \land U = \emptyset, \quad (U \land V)^c = U^c \lor V^c$$

for all $U, V \in \mathcal{T}_r(M)$. Thus $\mathcal{T}_r(M)$ is a complete Boolean lattice.

Example 2.3 Let $M$ be a topological space and $\mathcal{B}(M)$ the set of Borel subsets of $M$. $\mathcal{B}(M)$ together with the usual set theoretic operations is a distributive $\aleph_0$-complete Boolean lattice, usually called the $\sigma$-algebra of Borel subsets of $M$. 

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Example 2.4 Let $\mathcal{H}$ be a (complex) Hilbert space and $\mathbb{L}(\mathcal{H})$ the set of all closed subspaces of $\mathcal{H}$. $\mathbb{L}(\mathcal{H})$ is a complete lattice with lattice operations defined by

$$U \wedge V := U \cap V$$
$$U \vee V := (U + V)$$
$$U^\perp := \text{orthogonal complement of } U \text{ in } \mathcal{H}.$$ 

Contrary to the foregoing examples $\mathbb{L}(\mathcal{H})$ is highly non-distributive!

Of course $\mathbb{L}(\mathcal{H})$ is isomorphic to the lattice $P(\mathcal{L}(\mathcal{H})) := \{P_U \mid U \in \mathbb{L}(\mathcal{H})\}$ of all orthogonal projections in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators of $\mathcal{H}$. The non-distributivity of $\mathbb{L}(\mathcal{H})$ is equivalent to the fact that two projections $P_U, P_V \in P(\mathcal{L}(\mathcal{H}))$ do not commute in general.

$\mathbb{L}(\mathcal{H})$ is the basic lattice of quantum mechanics ([6]). It represents the “quantum logic” in contrast to the classical “Boolean logic”.

Traditionally, the notions of a presheaf and a complete presheaf (complete presheaves are usually called “sheaves”) are defined for the lattice $\mathcal{T}(M)$ of a topological space $M$. The very definition of presheaves and sheaves, however, can be formulated also for an arbitrary lattice:

**Definition 2.2** A presheaf of sets (R-modules) on a lattice $\mathbb{L}$ assigns to every element $a \in \mathbb{L}$ a set (R-module) $S(a)$ and to every pair $(a, b) \in \mathbb{L} \times \mathbb{L}$ with $a \leq b$ a mapping (R-module homomorphism)

$$\rho^b_a : S(b) \to S(a)$$

such that the following two properties hold:

1. $\rho^a_a = \text{id}_{S(a)}$ for all $a \in \mathbb{L}$,
2. $\rho^c_b \circ \rho^a_b = \rho^c_a$ for all $a, b, c \in \mathbb{L}$ such that $a \leq b \leq c$.

The presheaf $(S(a), \rho^b_a)_{a \leq b}$ is called a complete presheaf (or a sheaf for short) if it has the additional property

3. If $a = \bigvee_{i \in I} a_i$ in $\mathbb{L}$ and if $f_i \in S(a_i)$ $(i \in I)$ are given such that

$$\forall i, j \in I : (a_i \wedge a_j \neq 0 \implies \rho^a_{a_i \wedge a_j}(f_i) = \rho^a_{a_i \wedge a_j}(f_j),$$

then there is exactly one $f \in S(a)$ such that

$$\forall i \in I : \rho^a_a(f) = f_i.$$
The mappings \( \rho_b^a : S(b) \to S(a) \) are called **restriction maps**.

One of the most elementary and at the same time instructive examples is the sheaf of locally defined continuous complex valued functions on a topological space \( M \): \( S(U) \) is the space of continuous functions on the open set \( U \subseteq M \) and for \( U, V \in \mathcal{T}(M) \) with \( U \subseteq V \)

\[
\rho^V_U : S(V) \to S(U)
\]

is the restriction map \( f \mapsto f|_U \). Property (3) in definition 2.2 expresses the elementary fact that one can glue together a family of locally defined continuous functions \( f_i : U_i \to \mathbb{C} \) which agree on the non-empty overlaps \( U_i \cap U_j \) to a continuous function \( f \) on \( \bigcup_{i \in I} U_i \) which coincides with \( f_i \) on \( U_i \) for each \( i \in I \).

Are there interesting new examples for sheaves on a lattice other than \( \mathcal{T}(M) \), in particular on the quantum lattice \( \mathbb{L}(\mathcal{H}) \)?

The story begins with a disappointing answer:

**Proposition 2.1** Let \((S(U), \rho^V_U)_{U \subseteq V}\) be a complete presheaf of sets on the quantum lattice \( \mathbb{L}(\mathcal{H}) \). Then

\[
\#S(U) = 1
\]

for all \( U \in \mathbb{L}(\mathcal{H}) \setminus \{0\} \).
Thus complete presheaves on \( \mathbb{L}(\mathcal{H}) \) are completely trivial!

The proof of this result is based on the following observation: For each \( U \in \mathbb{L}(\mathcal{H}) \setminus \{0\} \) we have

\[
U = \bigvee_{C_x \subseteq U} \mathbb{C}x,
\]

and if \( \mathbb{C}x, \mathbb{C}y \subseteq U \) are different one dimensional subspaces, then \( \mathbb{C}x \cap \mathbb{C}y = 0 \). Hence the compatibility conditions in (3) of definition 2.2 are void and therefore

\[
S(U) \cong \prod_{C_x \subseteq U} S(C_x).
\]

Also on the lattice \( \mathcal{T}_r(M) \) (although a distributive complete lattice of open sets) there are only trivial sheaves.

There are, however, non-trivial presheaves on \( \mathbb{L}(\mathcal{H}) \) and one of them, which we shall study in section 6, turns out to be quite fruitful for quantum mechanics and the theory of operator algebras.
Moreover, there is also another perspective of sheaves: the etale space of a presheaf. Classically, for a topological space $M$, a presheaf $S$ on $\mathcal{T}(M)$ induces a sheaf of local sections of the etale space of $S$. This sheaf on $\mathcal{T}(M)$ is called the “sheafification of the presheaf $S$”. We can imitate this natural construction in the general case of a lattice $L$ and shall obtain a sheaf - not on $L$ because of the foregoing result - on the lattice $\mathcal{T}(Q(L))$ where $Q(L)$ is the Stonean space of the lattice $L$. 
3 Points and Quasipoints in a lattice

Let $M$ and $N$ be topological spaces. The elements of $N$ are in one-to-one correspondence to the constant mappings $f : M \to N$. These constant mappings correspond via the inverse image morphisms

$$V \mapsto f^{-1}(V) \quad (V \in \mathcal{T}(N))$$

to the continuous lattice morphisms

$$\Phi : \mathcal{T}(N) \to \mathcal{T}(M)$$

with the property

$$\forall V \in \mathcal{T}(N) : \Phi(V) \in \{\emptyset, M\}.$$

It is immediate that the set

$$p := \{V \in \mathcal{T}(N) \mid \Phi(V) = M\}$$

has the following properties:

1. $\emptyset \notin p$.
2. If $V, W \in p$, then $V \cap W \in p$.
3. If $V \in p$ and $W \supseteq V$ in $\mathcal{T}(N)$, then $W \in p$.
4. If $(V_i)_{i \in I}$ is a family in $\mathcal{T}(N)$ and $\bigcup_{i \in I} V_i \in p$, then there is at least one $i_0 \in I$ such that $V_{i_0} \in p$.

Now these properties make perfectly sense in an arbitrary $m$-complete lattice, so we can use them to define points in a lattice:

**Definition 3.1** Let $\mathbb{L}$ be an $m$-complete lattice. A non-empty subset $\mathfrak{p} \subseteq \mathbb{L}$ is called a point in $\mathbb{L}$ if the following properties hold:

1. $0 \notin \mathfrak{p}$.
2. $a, b \in \mathfrak{p} \Rightarrow a \land b \in \mathfrak{p}$.
3. $a \in \mathfrak{p}, b \in \mathbb{L}, a \leq b \Rightarrow b \in \mathfrak{p}$.
4. Let $(a_i)_{i \in I}$ be a family in $\mathbb{L}$ such that $\#I \leq m$ and $\bigvee_{i \in I} a_i \in \mathfrak{p}$ then $a_i \in \mathfrak{p}$ for at least one $i \in I$. 
Example 3.1 Let $M$ be a non-empty set and $L \subseteq \text{pot}(M)$ an $m$-complete lattice such that

\[
0_L = \emptyset, \\
1_L = M, \\
\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i \quad (#I \leq m).
\]

Then for each $x \in M$

\[
p_x := \{U \in L \mid x \in U\}
\]

is a point in $L$.

Conversely, if $L$ is the lattice $\mathcal{T}(M)$ of open sets of a regular topological space $M$ we have

**Proposition 3.1** Let $M$ be a regular topological space. A non-empty subset $p \subseteq \mathcal{T}(M)$ is a point in the lattice $\mathcal{T}(M)$ if and only if $p$ is the set of open neighbourhoods of an element $x \in M$. $x$ is uniquely determined by $p$.

Unfortunately there are important lattices that do not possess any points! There are plenty of points in $\mathcal{T}(M)$ and $\mathcal{B}(M)$; $\mathcal{T}_{r}(M)$ and $\mathbb{L}(\mathcal{H})$ possess no points at all. We will show this here only for the lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of the Hilbert space $\mathcal{H}$.

**Proposition 3.2** If $\dim \mathcal{H} > 1$, then there are no points in $\mathbb{L}(\mathcal{H})$.

**Proof:** Let $p \subseteq \mathbb{L}(\mathcal{H})$ be a point. If $(e_\alpha)_{\alpha \in A}$ is an orthonormal basis of $\mathcal{H}$ then

\[
\bigvee_{\alpha \in A} \mathbb{C}e_\alpha = \mathcal{H} \in p,
\]

so $\mathbb{C}e_{\alpha_0} \in p$ for some $\alpha_0 \in A$. It follows that each $U \in p$ must contain the line $\mathbb{C}e_{\alpha_0}$. Now choose $U \in \mathbb{L}(\mathcal{H})$ such that neither $U$ nor $U^\perp$ contains $\mathbb{C}e_{\alpha_0}$. Then $U, U^\perp \notin p$ but $U \vee U^\perp = \mathcal{H} \in p$ which is a contradiction to property (4) in the definition of a point in a lattice. Therefore there are no points in $\mathbb{L}(\mathcal{H})$. □

Let $\mathcal{P} = (\mathcal{P}(U), \rho^U_V)_{V \subseteq U}$ be a presheaf on the topological space $M$. The **stalk** of $\mathcal{P}$ at $x \in M$ is the direct limit

\[
\mathcal{P}_x := \lim_{U \in \mathbb{M}(x)} \mathcal{P}(U)
\]

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where \( \mathcal{U}(x) \) denotes the set of open neighbourhoods of \( x \) in \( M \), i.e. the point in \( \mathcal{T}(M) \) corresponding to \( x \).

For the definition of the direct limit, however, we do not need the point \( \mathcal{U}(x) \), but only a partially ordered set \( I \) with the property

\[
\forall \alpha, \beta \in I \quad \exists \gamma \in I : \gamma \leq \alpha \quad \text{and} \quad \gamma \leq \beta.
\]

In other words: a filter basis \( B \) in a lattice \( \mathbb{L} \) is sufficient. It is obvious how to define a filter basis in an arbitrary lattice \( \mathbb{L} \):

**Definition 3.2** A filter basis \( B \) in a lattice \( \mathbb{L} \) is a non-empty subset \( B \subseteq \mathbb{L} \) such that

(1) \( 0 \notin B \).

(2) \( \forall a, b \in B \exists c \in B : c \leq a \land b \).

The set of all filter bases in a lattice \( \mathbb{L} \) is of course a vast object. So it is reasonable to consider maximal filter bases in \( \mathbb{L} \). (By Zorn’s lemma, every filter basis is contained in a maximal filter basis in \( \mathbb{L} \).) This leads to the following

**Definition 3.3** A nonempty subset \( \mathcal{B} \) of a lattice \( \mathbb{L} \) is called a quasipoint in \( \mathbb{L} \) iff

(1) \( 0 \notin \mathcal{B} \)

(2) \( \forall a, b \in \mathcal{B} \exists c \in \mathcal{B} : c \leq a \land b \)

(3) \( \mathcal{B} \) is a maximal subset having the properties (1) and (2).

**Proposition 3.3** Let \( \mathcal{B} \) be a quasipoint in the lattice \( \mathbb{L} \). Then

\[
\forall a \in \mathcal{B} \forall b \in \mathbb{L} : (a \leq b \implies b \in \mathcal{B}).
\]

In particular

\[
\forall a, b \in \mathcal{B} : a \land b \in \mathcal{B}
\]
Proof: Let \( c \in \mathcal{B} \). Then \( a \land c \leq b \land c \) and from \( a, c \in \mathcal{B} \) we obtain a \( d \in \mathcal{B} \) such that
\[
d \leq a \land c \leq b \land c.
\]
Therefore \( \mathcal{B} \cup \{ b \} \) is a filter basis in \( \mathcal{L} \) containing \( \mathcal{B} \). Hence \( \mathcal{B} = \mathcal{B} \cup \{ b \} \) by the maximality of \( \mathcal{B} \), i.e. \( b \in \mathcal{B} \). \( \square \)

This proposition shows that a quasipoint in \( \mathcal{L} \) is nothing else but a maximal dual ideal in the lattice \( \mathcal{L} \) (\( \mathbb{I} \)).

We shall now determine the quasipoints in some important lattices.

Let \( M \) be a locally compact Hausdorff space and \( \mathcal{L} := \mathcal{T}(M) \). Let \( \mathcal{B} \) be a quasipoint in \( \mathcal{L} \). We distinguish two cases. In the first case we assume that \( \mathcal{B} \) has an element that is a relatively compact open subset of \( M \). Let \( U_0 \in \mathcal{B} \) be such an element. Then
\[
\bigcap_{U \in \mathcal{B}} \bar{U} \neq \emptyset,
\]
for otherwise \( \bigcap_{U \in \mathcal{B}} U \cap U_0 = \emptyset \) and from the compactness of \( \bar{U}_0 \) we see that there are \( U_1, \ldots, U_n \in \mathcal{B} \) such that \( \bigcap_{i=1}^n U_i \cap U_0 = \emptyset \).

But then \( U_0 \cap U_1 \cap \ldots \cap U_n = \emptyset \), contrary to the defining properties of a filter basis. The maximality of \( \mathcal{B} \) implies that every open neighbourhood of \( x \in \bigcap_{U \in \mathcal{B}} \bar{U} \) belongs to \( \mathcal{B} \). Therefore, as \( M \) is a Hausdorff space, \( \bigcap_{U \in \mathcal{B}} \bar{U} \) consists of precisely one element of \( M \). We will denote this element by \( \text{pt}(\mathcal{B}) \) and call \( \mathcal{B} \) a quasipoint over \( x = \text{pt}(\mathcal{B}) \).

Now consider the other case in which no element of the quasipoint \( \mathcal{B} \) is relatively compact. It can be easily shown, using the maximality of \( \mathcal{B} \) again, that in this case \( M \setminus K \in \mathcal{B} \) for every compact subset \( K \) of \( M \). We summarize these facts in the following

**Proposition 3.4** Let \( M \) be a locally compact Hausdorff space and \( \mathcal{B} \) a quasipoint in the lattice \( \mathcal{T}(M) \) of open subsets of \( M \). Then either \( M \setminus K \in \mathcal{B} \) for all compact subsets \( K \) of \( M \) or there is a unique element \( x \in M \) such that \( \bigcap_{U \in \mathcal{B}} \bar{U} = \{ x \} \).

In the first case \( \mathcal{B} \) is called an unbounded quasipoint, in the second case a bounded quasipoint over \( x \).

For a non-compact space \( M \) let \( M_\infty := M \cup \{ \infty \} \) be the one-point compactification of \( M \). Then the unbounded quasipoints in \( \mathcal{T}(M) \) can be considered as quasipoints over \( \infty \) in \( \mathcal{T}(M_\infty) \).

Next we consider a Boolean \( \sigma \)-algebra \( \mathcal{B} \), i.e. a \( \sigma \)-complete complemented distributive lattice. As a consequence of distributivity we have the following
Lemma 3.1 Let \( A \mapsto A^c \) be the complement operation in the \( \sigma \)-algebra \( \mathcal{B} \). Then a filter basis \( \mathcal{B} \subseteq \mathcal{B} \) is a quasipoint in \( \mathcal{B} \) iff
\[
\forall A \in \mathcal{B} : A \in \mathcal{B} \text{ or } A^c \in \mathcal{B}.
\]
Consequently every point in \( \mathcal{B} \) is a quasipoint in \( \mathcal{B} \).

Moreover, we can show:

Proposition 3.5 Let \( \mathcal{I} \) be a \( \sigma \)-ideal in the \( \sigma \)-algebra \( \mathcal{B} \) and let \( \pi : \mathcal{B} \to \mathcal{B}/\mathcal{I} \) be the canonical projection onto the quotient \( \sigma \)-algebra \( \mathcal{B}/\mathcal{I} \). Then \( \mathcal{B} \subseteq \mathcal{B}/\mathcal{I} \) is a quasipoint if and only if \( \pi^{-1}(\mathcal{B}) \) is a quasipoint in \( \mathcal{B} \) such that
\[
\pi^{-1}(\mathcal{B}) \cap \mathcal{I} = \emptyset.
\]

By a theorem of Loomis and Sikorski ([10]) every \( \sigma \)-algebra is the quotient of a \( \sigma \)-algebra of Borel sets of a compact space modulo a \( \sigma \)-ideal. It is not difficult to determine the quasipoints in the \( \sigma \)-algebra of all Borel subsets of a topological space (satisfying some mild topological conditions). Thus the quasipoints in a \( \sigma \)-algebra are known in principal.

Our third example is the complete lattice \( \mathbb{L}(\mathcal{H}) \) of closed subspaces of a Hilbert space \( \mathcal{H} \). As in the topological situation the quasipoints in \( \mathbb{L}(\mathcal{H}) \) fall into two different classes:

Proposition 3.6 Let \( \mathcal{B} \) be a quasipoint in \( \mathbb{L}(\mathcal{H}) \). \( \mathcal{B} \) contains an element of finite dimension if and only if there is a unique line \( \mathbb{C}x_0 \) in \( \mathcal{H} \) such that
\[
\mathcal{B} = \{ U \in \mathbb{L}(\mathcal{H}) \mid \mathbb{C}x_0 \subseteq U \}.
\]
\( \mathcal{B} \) does not contain an element of finite dimension if and only if \( W \in \mathcal{B} \) for all \( W \in \mathbb{L}(\mathcal{H}) \) of finite codimension.

Proof: Let \( U_0 \in \mathcal{B} \) be finite dimensional. Then \( U \cap U_0 \neq 0 \) for all \( U \in \mathcal{B} \) and therefore \( \{ U \cap U_0 \mid U \in \mathcal{B} \} \) contains an element \( V_0 \) of minimal dimension. Hence \( V_0 \subseteq U \) for all \( U \in \mathcal{B} \) and by the maximality of \( \mathcal{B} \) \( V_0 \) must have dimension one.

Assume that a quasipoint \( \mathcal{B} \) in \( \mathbb{L}(\mathcal{H}) \) contains every \( W \in \mathbb{L}(\mathcal{H}) \) of finite codimension. Let \( U \) be a finite dimensional subspace of \( \mathcal{H} \). Then \( U^\perp \in \mathcal{B} \) and therefore \( U \not\in \mathcal{B} \) because of \( U \cap U^\perp = 0 \).
Let $V \in \mathbb{L}(\mathcal{H})$ be of finite codimension and $V \notin \mathfrak{B}$. Then there is some $U \in \mathfrak{B}$ such that $U \cap V = 0$. Consider the orthogonal projection

$$P_{V^\perp} : \mathcal{H} \to V^\perp$$

onto $V^\perp$. $U \cap V = 0$ means that the restriction of $P_{V^\perp}$ to $U$ is injective. As $V^\perp$ is finite dimensional, $U$ must be finite dimensional too. □

Quasipoints in $\mathbb{L}(\mathcal{H})$ that contain a line are called atomic, otherwise they are called continuous.

Whereas the structure of atomic quasipoints is trivial, the set of continuous quasipoints mirrors the whole complexity of spectral theory of linear operators in $\mathcal{H}$. 
4 Stonean spaces and the etale space of a presheaf on a lattice

In 1936 M.H.Stone ([11]) showed that the set $Q(B)$ of quasipoints in a Boolean algebra $B$ can be given a topology such that $Q(B)$ is a compact zero dimensional Hausdorff space and that the Boolean algebra $B$ is isomorphic to the Boolean algebra of all closed open subsets of $Q(B)$. A basis for this topology is simply given by the sets

$$Q_U(B) := \{ B \in Q(B) \mid U \in B \}$$

where $U$ is an arbitrary element of $B$.

Of course we can generalize this construction to an arbitrary lattice $L$: Let $Q(L)$ be the set of quasipoints in $L$ and for $U \in L$ let

$$Q_U(L) := \{ B \in Q(L) \mid U \in B \}.$$  

It is quite obvious from the definition of a quasipoint that

$$Q_{UV}(L) = Q_U(L) \cap Q_V(L)$$

holds. Hence $\{ Q_U(L) \mid U \in L \}$ is a basis for a topology on $Q(L)$. Moreover it is easy to see, using the maximality of quasipoints, that in this topology the sets $Q_U(L)$ are open and closed. Therefore the topology defined by the basic sets $Q_U(L)$ is zero dimensional. $Q(L)$ together with this topology is called the Stonean space of the lattice $L$.

**Remark 4.1** The Stonean space $Q(L)$ of the lattice $L$ is a completely regular Hausdorff space.

This follows immediately from the fact that the sets $Q_U(L)$ are open and closed, so their characteristic functions are continuous.

In contrast to the case of Boolean algebras, Stonean spaces are not compact in general. The situation can be even worse, as the following important example shows:

**Remark 4.2** Let $H$ be a Hilbert space of dimension greater than one. Then the Stonean space $Q(H) := Q(L(H))$ is not locally compact.

This is an easy consequence of Baire’s category theorem and the general fact that the Stonean space $Q(L_U)$ of the principal ideal $L_U := \{ V \in L \mid V \leq U \}$ of an arbitrary lattice $L$ and $U \in L \setminus \{0\}$ is homeomorphic to $Q_U(L)$.
If $\mathcal{M}$ is a complete lattice isomorphic to $\mathcal{L}$ via a lattice isomorphism $\Phi : \mathcal{L} \to \mathcal{M}$, then it is easy to see that $\Phi$ induces a homeomorphism $\Phi_* : Q(\mathcal{L}) \to Q(\mathcal{M})$ of the corresponding Stonean spaces:

$$\Phi_*(\mathfrak{B}) := \{ \Phi(a) \mid a \in \mathfrak{B} \}. $$

The opposite conclusion, however, is not true. In fact we can show that the Stonean spaces $Q(\mathcal{T}(M))$ and $Q(\mathcal{T}_r(M))$ are homeomorphic for every topological space $M$. But in general the lattice $\mathcal{T}(M)$ of open subsets of $M$ is not isomorphic to the lattice $\mathcal{T}_r(M)$ of regular open subsets of $M$, because $\mathcal{T}(M)$ possesses points whereas in general $\mathcal{T}_r(M)$ does not.

In section 2 we have seen that $\mathcal{T}_r(M)$ is a Boolean algebra with complement operation

$$U \mapsto U^c$$

where $U^c := M \setminus \bar{U}$. Now it is easy to see that

$$U \cap V = \emptyset \implies U^{cc} \cap V^{cc} = \emptyset$$

holds for all open sets $U, V \subseteq M$. From this fact we get

**Lemma 4.1** Let $M$ be a topological space and let $\mathfrak{B}$ be a quasipoint in $\mathcal{T}(M)$. Then

$$\mathfrak{B}^r := \{ U^{cc} \mid U \in \mathfrak{B} \}$$

is a quasipoint in $\mathcal{T}_r(M)$.

**Proposition 4.1** The mapping

$$\rho : Q(\mathcal{T}(M)) \to Q(\mathcal{T}_r(M))$$

$$\mathfrak{B} \mapsto \mathfrak{B}^r$$

is a homeomorphism of Stonean spaces.

**Sketch of proof:** The first thing to show is that every quasipoint $\mathfrak{R}$ in $\mathcal{T}_r(M)$ is contained in exactly one quasipoint in $\mathcal{T}(M)$. Thus $\rho$ is a bijection. Moreover

$$U \in \mathfrak{B} \iff U^{cc} \in \mathfrak{B}^r$$
for every quasipoint $\mathcal{B}$ in $\mathcal{T}(M)$. This implies

$$\rho(\mathcal{Q}_U(\mathcal{T}(M))) = \mathcal{Q}_{U^c}(\mathcal{T}_r(M))$$

and

$$\rho^{-1}(\mathcal{Q}_W(\mathcal{T}_r(M))) = \mathcal{Q}_W(\mathcal{T}(M)),$$

i.e. $\rho$ is a homeomorphism. $\square$

**Corollary 4.1** The Stonean space $\mathcal{Q}(\mathcal{T}(M))$ is compact.

**Corollary 4.2** Let $M$ be a compact Hausdorff space and let

$$pt : \mathcal{Q}(\mathcal{T}(M)) \rightarrow M$$

be the map that assigns to $\mathcal{B} \in \mathcal{Q}(\mathcal{T}(M))$ the element $pt(\mathcal{B}) \in M$ determined by $\bigcap_{U \in \mathcal{B}} \overline{U}$. Then the quotient topology of $M$ induced by $pt$ coincides with the given topology of $M$.

This follows from the fact that $pt$ is a continuous mapping and therefore the quotient topology is finer than the given topology. It cannot be strictly finer because both topologies are compact and Hausdorff.

This result gives an extreme example for the fact that the projection onto the quotient by an equivalence relation need not be an open mapping: let $M$ be a connected compact Hausdorff space. The compactness of the Stonean space $\mathcal{Q}(\mathcal{T}(M))$ implies that $pt$ is a closed mapping. If it was also an open mapping the total disconnectedness of $\mathcal{Q}(\mathcal{T}(M))$ would imply that the image $M$ of $pt$ is totally disconnected, too. As $M$ is connected, this is only possible for the trivial case that $M$ consists of a single element.

In what follows we shall show that to each presheaf on a (complete) lattice $\mathbb{L}$ one can assign a sheaf on the Stonean space $\mathcal{Q}(\mathbb{L})$. The construction is quite similar to the well-known construction called “sheafification of a presheaf”. If $\mathcal{P}$ is a presheaf, say, of modules on a topological space $M$, i.e. on the lattice $\mathcal{T}(M)$, then the corresponding etale space $\mathcal{E}(\mathcal{P})$ of $\mathcal{P}$ is the disjoint union of the stalks of $\mathcal{P}$ at points in $\mathcal{T}(M)$:

$$\mathcal{E}(\mathcal{P}) = \coprod_{x \in M} \mathcal{P}_x$$
where
\[ P_x = \lim_{U \to U} P(U). \]

Now in a general lattice we need not have points. Our most important example for this situation is the quantum lattice \( \mathbb{L}(\mathcal{H}) \) of closed subspaces of the Hilbert space \( \mathcal{H} \). However, we always have plenty of quasipoints, and we can define the stalk of a presheaf \( P \) on a lattice \( \mathbb{L} \) over a quasipoint \( B \in \mathcal{Q}(\mathbb{L}) \) in the very same manner as in the topological situation.

Let \( P = (P(U), \rho^U_V)_{V \subseteq U} \) be a presheaf on the (complete) lattice \( \mathbb{L} \).

**Definition 4.1** \( f \in P(U) \) is called equivalent to \( g \in P(V) \) at the quasipoint \( B \in \mathcal{Q}(\mathcal{L}) \) if and only if
\[ \exists W \in \mathcal{B} : W \leq U \wedge V \text{ and } \rho^U_W(f) = \rho^V_W(g). \]

If \( f \) and \( g \) are equivalent at the quasipoint \( \mathcal{B} \) we write \( f \sim_{\mathcal{B}} g \).

It is easy to see that \( \sim_{\mathcal{B}} \) is an equivalence relation. The equivalence class of \( f \in P(U) \) at the quasipoint \( \mathcal{B} \in \mathcal{Q}(\mathbb{L}) \) is denoted by \([f]_{\mathcal{B}}\). It is called the germ of \( f \) at \( \mathcal{B} \). Note that this only makes sense if \( \mathcal{B} \in \mathcal{Q}_U(\mathbb{L}) \). Let \( \mathcal{B} \in \mathcal{Q}_U(\mathbb{L}) \). Then we obtain a canonical mapping
\[ \rho^U_{\mathcal{B}} : P(U) \to P_{\mathcal{B}} \]
of \( P(U) \) onto the set \( P_{\mathcal{B}} \) of germs at the quasipoint \( \mathcal{B} \), defined by the composition
\[ P(U) \xrightarrow{i_U} \prod_{V \subseteq \mathcal{B}} P(V) \xrightarrow{\pi_{\mathcal{B}}} \left( \prod_{V \subseteq \mathcal{B}} P(V) \right) / \sim_{\mathcal{B}} \]
where \( i_U \) is the canonical injection and \( \pi_{\mathcal{B}} \) the canonical projection of the equivalence relation \( \sim_{\mathcal{B}} \). \( (P_{\mathcal{B}} := (\prod_{V \subseteq \mathcal{B}} P(V)) / \sim_{\mathcal{B}} \) is nothing else but the direct limit \( \lim_{V \to \mathcal{B} \subseteq V} P(V) \) (\([4]\)) and \( \rho^U_{\mathcal{B}}(f) \) is just another notation for the germ \([f]_{\mathcal{B}} \) of \( f \in P(U) \).

Let \( P \) be a presheaf on the lattice \( \mathbb{L} \) and
\[ E(\mathcal{P}) := \prod_{\mathcal{B} \in \mathcal{Q}(\mathbb{L})} P_{\mathcal{B}}. \]

Moreover, let
\[ \pi_P : E(\mathcal{P}) \to \mathcal{Q}(\mathbb{L}) \]
be the projection defined by
\[ \pi_P(\mathcal{P}_B) := \{ B \}. \]
We will define a topology on \( \mathcal{E}(\mathcal{P}) \) such that \( \pi_P \) is a local homeomorphism. For \( U \in \mathcal{L} \) and \( f \in \mathcal{P}(U) \) let
\[ O_{f,U} := \{ \rho_U^f(f) \mid B \in \mathcal{Q}(U) \}. \]
It is quite easy to see that \( \{ O_{f,U} \mid f \in \mathcal{P}(U), U \in \mathcal{L} \} \) is a basis for a topology on \( \mathcal{E}(\mathcal{P}) \). Together with this topology, \( \mathcal{E}(\mathcal{P}) \) is called the etale space of \( \mathcal{P} \) over \( \mathcal{Q}(\mathcal{L}) \). By the very definition of this topology the projection \( \pi_P \) is a local homeomorphism, for \( O_{f,U} \) is mapped bijectively onto \( \mathcal{Q}(U) \).

If \( \mathcal{P} \) is a presheaf of modules or algebras, the algebraic operations can be transferred fibrewise to the etale space \( \mathcal{E}(\mathcal{P}) \). Addition, for example, gives a mapping from
\[ \mathcal{E}(\mathcal{P}) \circ \mathcal{E}(\mathcal{P}) := \{ (a, b) \in \mathcal{E}(\mathcal{P}) \times \mathcal{E}(\mathcal{P}) \mid \pi_P(a) = \pi_P(b) \} \]
to \( \mathcal{E}(\mathcal{P}) \) defined as follows:
Let \( f \in \mathcal{P}(U), g \in \mathcal{P}(V) \) be such that
\[ a = \rho^U_{\pi_P(a)}(f), \quad b = \rho^V_{\pi_P(b)}(g) \]
and let \( W \in \pi_P(a) \) be some element such that \( W \leq U \land V \). Then
\[ a + b := \rho^W_{\pi_P(a)}(\rho^U_W(f) + \rho^V_W(g)) \]
is a well defined element of \( \mathcal{E}(\mathcal{P}) \).
By standard techniques one can prove that the algebraic operations
\[ \mathcal{E}(\mathcal{P}) \circ \mathcal{E}(\mathcal{P}) \to \mathcal{E}(\mathcal{P}) \]
\((a, b) \mapsto a - b \)
(and \((a, b) \mapsto ab\) if \( \mathcal{P} \) is a presheaf of algebras) and
\[ \mathcal{E}(\mathcal{P}) \to \mathcal{E}(\mathcal{P}) \]
\(a \mapsto \alpha a \)
(scalar multiplication with \( \alpha \)) are continuous.

From the etale space \( \mathcal{E}(\mathcal{P}) \) over \( \mathcal{Q}(\mathcal{L}) \) we obtain - as in ordinary sheaf theory - a complete presheaf \( \mathcal{P}^\mathcal{Q} \) on the topological space \( \mathcal{Q}(\mathcal{L}) \) by
\[ \mathcal{P}^\mathcal{Q}(\mathcal{V}) := \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{P})) \]
where \( \mathcal{V} \subseteq Q(\mathbb{L}) \) is an open set and \( \Gamma(\mathcal{V}, E(\mathcal{P})) \) is the set of continuous sections of \( \pi_{\mathcal{P}} \) over \( \mathcal{V} \), i.e. of all continuous mappings \( s_{\mathcal{V}} : \mathcal{V} \to E(\mathcal{P}) \) such that \( \pi_{\mathcal{P}} \circ s_{\mathcal{V}} = id_{\mathcal{V}} \). If \( \mathcal{P} \) is a presheaf of modules, then \( \Gamma(\mathcal{V}, E(\mathcal{P})) \) is a module, too.

**Definition 4.2** The complete presheaf \( \mathcal{P}^Q \) on the Stonean space \( Q(\mathbb{L}) \) is called the sheaf associated to the presheaf \( \mathcal{P} \) on \( \mathbb{L} \).

We will postpone the study of the general situation to later work. Instead we will consider a concrete presheaf on the quantum lattice \( \mathbb{L}(\mathcal{H}) \) and some of its connections to quantum mechanics and the theory of operator algebras.
5 Boolean sectors and Boolean quasipoints in the quantum lattice

In the following $\mathcal{H}$ is a fixed complex Hilbert space and $\mathcal{L}(\mathcal{H})$ denotes the algebra of bounded linear operators of $\mathcal{H}$.

Let $A$ be a selfadjoint (not necessarily bounded) operator of $\mathcal{H}$. The spectral theorem states that $A$ determines a unique family $(P_\lambda)_{\lambda \in \mathbb{R}}$ of orthogonal projections $P_\lambda \in \mathcal{L}(\mathcal{H})$ such that

(a) $P_\lambda \leq P_\mu$ for $\lambda \leq \mu$,
(b) $P_\lambda = \lim_{\mu \searrow \lambda} P_\mu$ for all $\lambda \in \mathbb{R}$, and
(c) $\lim_{\lambda \to -\infty} P_\lambda = 0$, $\lim_{\lambda \to \infty} P_\lambda = I$,

where we understand the limits with respect to the strong operator topology, from which $A$ can be recovered as a Riemann-Stieltjes integral

$$A = \int_{-\infty}^{\infty} \lambda dP_\lambda.$$ 

Moreover, if $A$ is bounded, then $A$ commutes with an operator $B \in \mathcal{L}(\mathcal{H})$ if and only if $B$ commutes with every spectral projection $P_\lambda$ ($\lambda \in \mathbb{R}$).

Now the properties (a), (b), (c) of the family $(P_\lambda)_{\lambda \in \mathbb{R}}$ can be reformulated in terms of the closed subspaces

$$\sigma_A(\lambda) := P_\lambda \mathcal{H} \in \mathbb{L}(\mathcal{H})$$

as follows:

(1) $\sigma_A(\lambda) \subseteq \sigma_A(\mu)$ for $\lambda \leq \mu$,
(2) $\sigma_A(\lambda) = \bigcap_{\mu > \lambda} \sigma_A(\mu)$ for all $\lambda \in \mathbb{R}$, and
(3) $\bigcap_{\lambda \in \mathbb{R}} \sigma_A(\lambda) = 0$, $\bigvee_{\lambda \in \mathbb{R}} \sigma_A(\lambda) = \mathcal{H}$.

Definition 5.1 A mapping $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ with the properties (1), (2), (3) above is called a spectral family in the quantum lattice $\mathbb{L}(\mathcal{H})$.

Now let $\mathcal{D}$ be a sublattice of $\mathbb{L}(\mathcal{H})$. The following fact is well known:
Remark 5.1 A sublattice $\mathbb{D}$ of $\mathbb{L}(\mathcal{H})$ is distributive if and only if for all $U, V \in \mathbb{D}$ the orthogonal projections $P_U, P_V$ onto $U$ and $V$ respectively, commute.

It is obvious from Zorn’s lemma that each distributive sublattice of $\mathbb{L}(\mathcal{H})$ is contained in a maximal distributive one.

Definition 5.2 A maximal distributive sublattice $\mathbb{B}$ of $\mathbb{L}(\mathcal{H})$ is called a Boolean sector of $\mathbb{L}(\mathcal{H})$.

In what sense this is a “sector” will become clear soon.

Boolean sectors have an important interpretation in the theory of operator algebras.

Theorem 5.1 Boolean sectors of $\mathbb{L}(\mathcal{H})$ are in one-to-one correspondence with maximal abelian von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$.

The proof relies on the facts that

- a von Neumann subalgebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ is maximal abelian iff $\mathcal{M} = \mathcal{M}'$ where

$$\mathcal{M}' := \{ T \in \mathcal{L}(\mathcal{H}) \mid \forall S \in \mathcal{M} : ST = TS \}$$

- a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ is generated by the lattice $P(\mathcal{A})$ of the projections contained in $\mathcal{A}$, i.e.

$$\mathcal{A} = P(\mathcal{A})''.$$

If $\mathbb{B} \subseteq \mathbb{L}(\mathcal{H})$ is a Boolean sector and

$$P(\mathbb{B}) := \{ P_U \mid U \in \mathbb{B} \}$$

is the corresponding Boolean algebra of projections then

$$W^*(\mathbb{B}) := P(\mathbb{B})''$$

is a maximal abelian subalgebra of $\mathcal{L}(\mathcal{H})$. Conversely, if $\mathcal{M}$ is a maximal abelian subalgebra of $\mathcal{L}(\mathcal{H})$, then the lattice $P(\mathcal{M})$ of its projections is contained in $P(\mathbb{B})$ for some Boolean sector $\mathbb{B}$. Using the maximality of $\mathcal{M}$, one shows that $\mathcal{M} = P(\mathbb{B})''$ holds.

It is easy to see that Boolean sectors of $\mathbb{L}(\mathcal{H})$ are complete Boolean algebras.
Definition 5.3 A subset $\beta \subseteq \mathbb{L}(\mathcal{H})$ is called a Boolean quasipoint in $\mathbb{L}(\mathcal{H})$ iff

1. $0 \neq \beta$
2. $\forall U, V \in \beta \exists W \in \beta : W \subseteq U \cap V$
3. $\forall U, V \in \beta : P_U P_V = P_V P_U$
4. $\beta$ is a maximal set fulfilling the properties (1), (2), (3).

As in the case of ordinary quasipoints in $\mathbb{L}(\mathcal{H})$ the defining properties of Boolean quasipoints imply

5. Let $\beta$ be a Boolean quasipoint in $\mathbb{L}(\mathcal{H})$ and let $V \in \mathbb{L}(\mathcal{H})$ be such that $P_U P_V = P_V P_U$ for all $U \in \beta$ and that $W \subseteq V$ for some $W \in \beta$. Then $V \in \beta$.

Remark 5.2 Let $\mathbb{B} \subseteq \mathbb{L}(\mathcal{H})$ be a Boolean sector. A subset $\beta \subseteq \mathbb{B}$ is a Boolean quasipoint in $\mathbb{L}(\mathcal{H})$ if and only if $\beta$ is a quasipoint in the Boolean algebra $\mathbb{B}$.

Obviously every Boolean quasipoint is contained in some Boolean sector. The term “sector” is motivated by the following

Proposition 5.1 Each Boolean quasipoint in $\mathbb{L}(\mathcal{H})$ is contained in exactly one Boolean sector.

Definition 5.4 We call a Boolean quasipoint (or a Boolean sector) atomic if it possesses a finite dimensional element. Otherwise we speak of a continuous Boolean quasipoint or sector.

Remark 5.3 An atomic Boolean quasipoint (or Boolean sector) possesses an element that is a line in $\mathcal{H}$. If a Boolean quasipoint $\beta$ is contained in an atomic quasipoint in $\mathbb{L}(\mathcal{H})$, then $\beta$ is itself atomic and hence is contained in exactly one quasipoint in $\mathbb{L}(\mathcal{H})$.

There are continuous sectors in $\mathbb{L}(\mathcal{H})$. To see this, consider a hermitean operator $T$ of $\mathcal{H}$ that has no eigenvalues. Let $\mathbb{B}$ be a Boolean sector that contains the spectral family of $T$. Each element of $\mathbb{B}$ is a $T$-invariant subspace. Hence a one dimensional element of $\mathbb{B}$ would give eigenvectors of $T$. Thus each Boolean sector that contains the spectral family of $T$ is continuous.

The simplest Boolean sectors correspond to diagonalizable operators.
Remark 5.4 Let $b = (e_\alpha)_{\alpha \in A}$ be an orthonormal basis of $\mathcal{H}$. Then there is exactly one Boolean sector $\mathbb{B}_b$ that includes $\{Ce_\alpha \mid \alpha \in A\}$. The elements of $\mathbb{B}_b$ are the $b$-adapted elements of $L(\mathcal{H})$, i.e. the closed subspaces $U$ of $\mathcal{H}$ such that

$$\forall \alpha \in A : e_\alpha \in U \text{ or } e_\alpha \in U^\perp.$$  

Each orthonormal basis of $\mathcal{H}$ is contained in exactly one Boolean sector, and two orthonormal bases of $\mathcal{H}$ that are included in the same Boolean sector differ only by a permutation of their members.

We have seen that the Boolean sectors of $L(\mathcal{H})$ correspond to the maximal abelian von Neumann algebras in $L(\mathcal{H})$. We will now show that also the Stonean space $Q(\mathbb{B})$ of a Boolean sector $\mathbb{B}$ has an interpretation in the context of operator algebras.

Let $\mathbb{B}$ be a Boolean sector, $P(\mathbb{B})$ the corresponding Boolean algebra of projections and $C^*(\mathbb{B})$ the $C^*$-algebra generated by $P(\mathbb{B})$, i.e. the closure of $\text{span}P(\mathbb{B})$ in the norm-toplogy of $L(\mathcal{H})$. $C^*(\mathbb{B})$ is an abelian $C^*$-algebra with unity and is therefore, by the Gelfand representation theorem, isometrically *-isomorphic to the $C^*$-algebra $C(\Omega_\mathbb{B})$ of continuous functions $\Omega_\mathbb{B} \to \mathbb{C}$ on some compact Hausdorff space $\Omega_\mathbb{B}$. $\Omega_\mathbb{B}$ is the set of all multiplicative linear functionals $\tau : C^*(\mathbb{B}) \to \mathbb{C}$, equipped with the weak*-topology.

$\Omega_\mathbb{B}$ is called the Gelfand spectrum or the space of characters of the $C^*$- algebra $C^*(\mathbb{B})$.

**Theorem 5.2** The Gelfand spectrum $\Omega_\mathbb{B}$ of the $C^*$-algebra $C^*(\mathbb{B})$ is homeomorphic to the Stonean space $Q(\mathbb{B})$ of all quasipoints in the Boolean algebra $\mathbb{B}$.

With respect to this homeomorphism the strongly continuous characters correspond to the atomic quasipoints in $\mathbb{B}$.

**Sketch of proof:** Let $\tau \in \Omega_\mathbb{B}$. Then it is easy to see that

$$\beta_\tau := \{ U \in \mathbb{B} \mid \tau(P_U) = 1 \}$$

is a quasipoint in $\mathbb{B}$.

Consider $\sigma, \tau \in \Omega_\mathbb{B}$. Then $\beta_\tau = \beta_\sigma$ is equivalent to

$$\forall U \in \mathbb{B} : \tau(P_U) = 1 \iff \sigma(P_U) = 1.$$  

Because of

$$\text{im } \tau|_{P(\mathbb{B})} = \text{im } \sigma|_{P(\mathbb{B})} = \{0, 1\}$$
this implies
\[ \tau|_{\text{span}(P(B))} = \sigma|_{\text{span}(P(B))}. \]

As \(\sigma\) and \(\tau\) are continuous, it follows that \(\sigma = \tau\). Hence the mapping
\[
\Omega_B \rightarrow Q(B) \\
\tau \mapsto \beta_\tau
\]
is injective.

Conversely, if \(\beta \in Q(B)\) is given, we define a mapping
\[ \tau_\beta : P(B) \rightarrow \{0, 1\} \]
by
\[
\tau_\beta(P_U) := \begin{cases} 
1 & \text{if } U \in \beta \\
0 & \text{otherwise.}
\end{cases}
\]
The defining properties of quasipoints show that \(\tau_\beta\) is a multiplicative mapping. The technical difficulty is to prove that \(\tau_\beta\) can be extended to a continuous linear mapping \(\text{span}(P(B)) \rightarrow \mathbb{C}\).

Observe that each linear combination
\[ T = \sum_{k=1}^{n} a_k P_{U_k} \in \text{span}(P(B)) \]
can be written as \(T = \sum_{j=1}^{m} b_j P_{V_j}\) with subspaces \(V_j \in B\) that are orthogonal in pairs. We call this an orthogonal representation of \(T\). There is a canonical orthogonal representation of \(T = \sum_{k=1}^{n} a_k P_{U_k}\), namely
\[
\sum_{k=1}^{n} a_k P_{U_k} = (a_1 + \ldots + a_n)P_{U_1 \cap \ldots \cap U_n}
\]
\[ + \sum_{i=1}^{n} (a_1 + \ldots + \hat{a}_i + \ldots + a_n)P_{U_1 \cap \ldots \cap U_i^\perp \cap \ldots \cap U_n} \]
\[ + \sum_{1 \leq i < j \leq n} (a_1 + \ldots + \hat{a}_i + \ldots + a_j + \ldots + a_n)P_{U_1 \cap \ldots \cap U_i^\perp \cap \ldots \cap U_j^\perp \cap \ldots \cap U_n} \]
\[ + \ldots + \sum_{i=1}^{n} a_i P_{U_i^\perp \cap \ldots \cap U_i^\perp \cap \ldots \cap U_n}. \]
If $V_1, \ldots, V_m \in B$ are orthogonal in pairs, then $V_i \in \beta$ for at most one $i \leq m$. Moreover, there exists a (unique) $j_\beta \leq m$ such that $V_{j_\beta} \in \beta$ if and only if $V_1 \lor \ldots \lor V_m \in \beta$.

Hence the following definition is reasonable:

$$\tilde{\tau}_\beta \left( \sum_{j=1}^m b_j P_{V_j} \right) := \begin{cases} b_{j_\beta} & \text{if } V_1 \lor \ldots \lor V_m \in \beta \\ 0 & \text{otherwise.} \end{cases}$$

One proves that this definition is independent of the orthogonal representation of $T \in \text{span}(P(B))$. Using the canonical orthogonal representation of $T = \sum_{k=1}^n a_k P_{U_k}$ we obtain

$$\tilde{\tau}_\beta(T) = a_{j_1} + \ldots + a_{j_s},$$

where $j_1, \ldots, j_s$ are those indices, for which $U_{j_1}, \ldots, U_{j_s}$ are elements of $\beta$. This shows the linearity of $\tilde{\tau}_\beta$.

Continuity of $\tilde{\tau}_\beta$ is obvious from

$$\left| \sum_{j=1}^m b_j P_{V_j} \right| = \max_{j \leq m} |b_j|,$$

where $\sum_{j=1}^m b_j P_{V_j}$ is an orthogonal representation of $T$. Hence $\tau_\beta$ has a unique extension to a character $\tilde{\tau}_\beta \in \Omega_B$. By construction

$$\beta \tilde{\tau}_\beta = \beta.$$

The continuity of $\tau \mapsto \beta_\tau$ follows from

$$\tau \in N_{U,\epsilon}(\tau_0) \iff \beta_\tau \in Q_U(B)$$

where $\epsilon \in ]0,1[ \setminus \tau_0 \in \Omega_B$, $U \in B$ such that $\tau_0(U) = 1$ and

$$N_{U,\epsilon}(\tau_0) := \{ \tau \in \Omega_B \mid |\tau(P_U) - \tau_0(P_U)| < \epsilon \}.$$

Since $\Omega_B$ and $Q(B)$ are compact, $\tau \mapsto \beta_\tau$ is a homeomorphism.

Finally, the strong limit of the monotonous net $(P_U)_{U \in \beta}$ is

$$\lim_{U \in \beta} P_U = \begin{cases} P_{C_X} & \text{if } \beta \text{ is atomic} \\ 0 & \text{otherwise,} \end{cases}$$

and $\tau_\beta(P_U) = 1$ for all $U \in \beta$. Hence $\tau_\beta$ is strongly continuous if and only if $\beta$ is atomic.

□

In the next section we will study a canonical presheaf on the quantum lattice $\mathbb{L}({\mathcal{H}})$ and show how it determines the Gelfand transform

$$C^*(B) \rightarrow C(\Omega_B)$$

of the abelian $C^*$- algebra $C^*(B)$. 

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6 Observable functions

We know that there is only the trivial sheaf on the quantum lattice \( \mathbb{L}(\mathcal{H}) \) of closed subspaces of the Hilbert space \( \mathcal{H} \). But what about presheaves?

An obvious example is the following one: For \( U \in \mathbb{L}(\mathcal{H}) \) let \( \mathcal{P}(U) := \mathcal{L}(U) \) be the space of bounded linear operators \( U \to U \) and for \( V \in \mathbb{L}(\mathcal{H}) \), \( V \subseteq U \), we define a “restriction map”

\[
\rho^U_V : \mathcal{L}(U) \to \mathcal{L}(V)
\]

by

\[
\rho^U_V(A) := P_V A |_V.
\]

Clearly these data give a presheaf on \( \mathbb{L}(\mathcal{H}) \).

This example looks somewhat artificial because the restriction maps defined above do not coincide with the usual idea of restricting a mapping from its domain to a smaller set. The elements of the stalks of this presheaf, however, have a quantum mechanical interpretation.

Remark 6.1 Let \( A \in \mathcal{L}(U) \) and let \( \mathcal{B}_{\mathbb{C}x} \in \mathcal{Q}_U(\mathcal{H}) \) be an atomic quasipoint (in \( \mathbb{L}(U) \)). Then the germ of \( A \) in \( \mathcal{B}_{\mathbb{C}x} \) is given by \( \langle Ax, x \rangle \), where \( x \in S^1(\mathcal{H}) \cap \mathbb{C}x \).

Namely, if \( A, B \in \mathcal{L}(U) \), then \( A \sim_{\mathbb{B}_{\mathbb{C}x}} B \) if and only if \( P_{\mathbb{C}x} AP_{\mathbb{C}x} = P_{\mathbb{C}x} BP_{\mathbb{C}x} \).

Now if \( x \in S^1(\mathcal{H}) \) then

\[
\forall z \in \mathcal{H} : P_{\mathbb{C}x} AP_{\mathbb{C}x} z = \langle Ax, x \rangle z, x > x.
\]

Hence \( P_{\mathbb{C}x} AP_{\mathbb{C}x} = P_{\mathbb{C}x} BP_{\mathbb{C}x} \) if and only if \( \langle Ax, x \rangle = \langle Bx, x \rangle \).

If \( A \) is a hermitian operator and \( x \in S^1(\mathcal{H}) \), then \( \langle Ax, x \rangle \) is interpreted as the expectation value of the observable \( A \) when the quantum mechanical system is in the pure state \( \mathbb{C}x \).

In order to obtain a more natural example of a presheaf on \( \mathbb{L}(\mathcal{H}) \), we shall reformulate the operation of restricting a continuous function \( f : U \to \mathbb{R} \) to an open subset \( V \subseteq U \) in the language of lattice theory.

Let \( M \) and \( N \) be regular Hausdorff spaces. A continuous mapping \( f : M \to N \) induces a lattice homomorphism

\[
\Phi_f : \mathcal{T}(N) \to \mathcal{T}(M)
\]

\[
W \mapsto \frac{1}{f(W)}
\]

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that is continuous in the following sense:

\[ \Phi_f \left( \bigcup_{i \in I} W_i \right) = \bigcup_{i \in I} \Phi_f(W_i) \]

for each family \((W_i)_{i \in I}\) in \(\mathcal{T}(N)\). Conversely:

**Theorem 6.1** Each continuous lattice homomorphism \(\Phi : \mathcal{T}(N) \to \mathcal{T}(M)\) induces a unique continuous mapping \(f : M \to N\) such that \(\Phi = \Phi_f\).

The proof is based on the observation that for any point \(p\) in \(\mathcal{T}(M)\) the inverse image \(\Phi^{-1}(p)\) is a point in \(\mathcal{T}(N)\). Because the points in \(\mathcal{T}(M)\) correspond to the elements of \(M\), this gives a mapping \(f : M \to N\). It is then easy to show that \(f\) has the required properties.

Now we can describe the restriction of a continuous mapping \(f : M \to N\) to an open subset \(U\) of \(M\) in the following way:

**Proposition 6.1** Let \(f : M \to N\) be a continuous mapping between regular Hausdorff spaces, \(\Phi_f : \mathcal{T}(N) \to \mathcal{T}(M)\) the continuous lattice homomorphism induced by \(f\), and \(U\) an open subset of \(M\). Then

\[
\Phi_f^U : \mathcal{T}(N) \to \mathcal{T}(U)
\]

\[
W \mapsto \Phi_f(W) \cap U
\]

is a continuous lattice homomorphism and the corresponding continuous mapping \(U \to N\) is the restriction of \(f\) to \(U\).

Let \(\mathcal{H}\) be a Hilbert space. The observables of a quantum mechanical system are selfadjoint operators of \(\mathcal{H}\). Equivalently we can think of observables as spectral families in \(\mathcal{L}(\mathcal{H})\). To begin with, we restrict our attention to those spectral families \(\sigma : \mathbb{R} \to \mathcal{L}(\mathcal{H})\) that are bounded from above:

\[ \exists \lambda_1 \in \mathbb{R} : \sigma(\lambda_1) = \mathcal{H}. \]

Let \(U \in \mathcal{L}(\mathcal{H})\) and \(\sigma : \mathbb{R} \to \mathcal{L}(\mathcal{H})\) be a spectral family that is bounded from above. Then it easy to see that

\[ \sigma^U : \lambda \mapsto \sigma(\lambda) \cap U \]

is a spectral family in \(\mathcal{L}(U)\) that is bounded from above, too. \(\sigma^U\) is called the **restriction of \(\sigma\) to \(U\)**.
Let $\Sigma_{ba}(\mathcal{H})$ be the set of spectral families in $\mathbb{L}(\mathcal{H})$ that are bounded from above. We define for $V \subseteq U$ in $\mathbb{L}(\mathcal{H})$ the restriction map

$$
\rho_V^U : \Sigma_{ba}(U) \to \Sigma_{ba}(V) \quad \sigma \mapsto \sigma^V.
$$

Clearly, $(\Sigma_{ba}(U), \rho_V^U)_{V \subseteq U}$ is a presheaf on $\mathbb{L}(\mathcal{H})$. Let $\Sigma_b(U)$ be the set of bounded spectral families in $\mathbb{L}(U)$. These correspond to the bounded hermitian operators of $U$. Obviously $(\Sigma_b(U), \rho_V^U)_{V \subseteq U}$ is a sub-presheaf of $(\Sigma_{ba}(U), \rho_V^U)_{V \subseteq U}$.

If the spectral family $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ is not bounded from above then $\sigma^U$ may fail to be a spectral family in $\mathbb{L}(U)$. Of course the properties

1. $\sigma^U(\lambda) \subseteq \sigma^U(\mu)$ for $\lambda \leq \mu$
2. $\sigma^U(\lambda) = \bigcap_{\mu > \lambda} \sigma^U(\mu)$ for all $\lambda \in \mathbb{R}$
3. $\bigcap_{\lambda \in \mathbb{R}} \sigma^U(\lambda) = 0$

hold, but

$$
\bigvee_{\lambda \in \mathbb{R}} \sigma^U(\lambda) \neq U
$$

in general.

**Example 6.1** Let $\mathcal{H}$ be a separable Hilbert space and $(e_n)_{n \in \mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. Then

$$
\sigma(\lambda) := \bigvee_{n \leq \lambda} \mathbb{C}e_n \quad (\lambda \in \mathbb{R})
$$

defines a spectral family in $\mathbb{L}(\mathcal{H})$. One can show that this spectral family corresponds (up to some scaling) to the Hamilton operator of the harmonic oscillator. Take $x \in S^1(\mathcal{H})$ such that

$$
\forall \ n \in \mathbb{N} : \langle x, e_n \rangle \neq 0.
$$

This means that $x \notin \sigma(\lambda)$ for all $\lambda \in \mathbb{R}$ and hence

$$
\sigma^{Cx}(\lambda) = \sigma(\lambda) \cap \mathbb{C}x = 0
$$

for all $\lambda \in \mathbb{R}$. Therefore

$$
\bigvee_{\lambda \in \mathbb{R}} \sigma^{Cx}(\lambda) = 0 \neq \mathbb{C}x.
$$
Remark 6.2 Of course we can drop the requirement
\[ \bigvee_{\lambda \in \mathbb{R}} \sigma(\lambda) = \mathcal{H} \]
in the definition of spectral families. Then we obtain the notion of a generalized spectral family. Operators that are given by generalized spectral families are not necessarily densely defined, but their domain of definition is only dense in the closed subspace \( \bigvee_{\lambda \in \mathbb{R}} \sigma(\lambda) \) of \( \mathcal{H} \).

Let us consider the restriction of a spectral family \( \sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H}) \) to a one dimensional subspace \( \mathbb{C}x \) more closely. If \( \mathbb{C}x \subseteq \sigma(\lambda) \) for some \( \lambda \in \mathbb{R} \), then the hermitian operator corresponding to the spectral family \( \sigma^{\mathbb{C}x} : \mathbb{R} \to \mathbb{L}(\mathbb{C}x) \)
is a (real) scalar multiple \( cI_1 \) of the identity \( I_1 : \mathbb{C}x \to \mathbb{C}x \). Now \( \mathbb{L}(\mathbb{C}x) = \{0, \mathbb{C}x\} \), hence

\[ \sigma^{\mathbb{C}x}(\lambda) = \begin{cases} 0 & \text{for } \lambda < c \\ \mathbb{C}x & \text{for } \lambda \geq c \end{cases} \]
and

\[ c = \inf\{\lambda \in \mathbb{R} \mid \mathbb{C}x \subseteq \sigma(\lambda)\} \]

Using the convention
\[ \inf \emptyset = \infty \]
we obtain in this way a function on the projective Hilbert space \( \mathbb{P}\mathcal{H} \) with values in \( \mathbb{R} \cup \{\infty\} \),

\[ f_\sigma : \mathbb{P}\mathcal{H} \to \mathbb{R} \cup \{\infty\}, \]
defined by

\[ f_\sigma := \inf\{\lambda \in \mathbb{R} \mid \mathbb{C}x \subseteq \sigma(\lambda)\} \]

Clearly, if \( \sigma \) is bounded from above then \( f_\sigma \) is bounded from above, too. Moreover \( f_\sigma \) is a bounded function if and only if \( \sigma \) is a bounded spectral family, i.e. the corresponding selfadjoint operator \( A_\sigma \) is bounded.
The canonical topology on projective Hilbert space $\mathbb{P} \mathcal{H}$ is the quotient topology defined by the projection

$$pr : \mathcal{H} \setminus \{0\} \to \mathbb{P} \mathcal{H}$$

$$x \mapsto Cx.$$ 

This means that a subset $W \subseteq \mathbb{P} \mathcal{H}$ is open if and only if $\overline{pr(W)}$ is an open subset of $\mathcal{H} \setminus \{0\}$.

The function $f_\sigma$ has some remarkable properties:

**Proposition 6.2** Let $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ be a spectral family and let

$$f_\sigma : \mathbb{P} \mathcal{H} \to \mathbb{R} \cup \{\infty\}$$

be the function defined by

$$f_\sigma(Cx) := \inf\{\lambda \in \mathbb{R} \mid Cx \subseteq \sigma(\lambda)\}.$$

Then

1. $f_\sigma$ is lower semicontinuous on $\mathbb{P} \mathcal{H}$;
2. if $Cx, Cy, Cz$ are elements of $\mathbb{P} \mathcal{H}$ such that $Cz \subseteq Cx + Cy$, then

$$f_\sigma(Cz) \leq \max(f_\sigma(Cx), f_\sigma(Cy));$$
3. $f_\sigma(\mathbb{R})$ is dense in $\mathbb{P} \mathcal{H}$.

Lower semicontinuity follows from

$$\overline{pr(f_\sigma(]-\infty, \lambda[)) \cup \{0\}} = \sigma(\lambda);$$

for then $f_\sigma(]-\infty, \lambda[)$ is closed in $\mathbb{P} \mathcal{H}$ for all $\lambda \in \mathbb{R}$ and therefore $f_\sigma$ is lower semicontinuous. The two other properties are obvious from the definitions.

**Definition 6.1** A function $f : \mathbb{P} \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is called an observable function if it has the following properties:

1. $f$ is lower semicontinuous;
2. if $Cx, Cy, Cz$ are elements of $\mathbb{P} \mathcal{H}$ such that $Cz \subseteq Cx + Cy$ then

$$f(Cz) \leq \max(f(Cx), f(Cy));$$
\( (3) \ f^{-1}(\mathbb{R}) \) is dense in \( \mathbb{P}\mathcal{H} \).

The point is that we can reconstruct spectral families in \( L(\mathcal{H}) \) from observable functions on \( \mathbb{P}\mathcal{H} \):

**Theorem 6.2** The mapping \( \sigma \mapsto f_\sigma \) is a bijection from the set of spectral families in \( L(\mathcal{H}) \) onto the set of observable functions on \( \mathbb{P}\mathcal{H} \). This mapping is compatible with restrictions:

\[
f_{\sigma ^U} = f_\sigma \mid_{\mathbb{P}\mathcal{U}}.
\]

Moreover, \( \sigma \in \Sigma_{ba}(\mathcal{H}) \) if and only if \( f_\sigma(\mathbb{R}) = \mathbb{P}\mathcal{H} \), and \( \sigma \in \Sigma_b(\mathcal{H}) \) if and only if \( f_\sigma \) is bounded.

**Sketch of proof:** The construction of a spectral family from an observable function \( f \) is roughly as follows: for \( \lambda \in \mathbb{R} \) let

\[
\sigma(\lambda) := -1 \text{pr}(\lambda^{-1} - \infty, \lambda]) \cup \{0\}.
\]

Property (1) assures that \( \sigma(\lambda) \) is closed in \( \mathcal{H} \) and property (2) implies that \( \sigma(\lambda) \) is a subspace of \( \mathcal{H} \). It is not difficult to show that \( \sigma : \lambda \mapsto \sigma(\lambda) \) is a spectral family in \( L(\mathcal{H}) \) and that

\[
f_\sigma = f
\]

holds. It follows from Baire’s category theorem that \( \sigma \in \Sigma_{ba}(\mathcal{H}) \) if and only if \( f_\sigma(\mathbb{R}) = \mathbb{P}\mathcal{H} \). \( \square \)

If \( A \) is the selfadjoint operator corresponding to the spectral family \( \sigma \), then we also write \( f_A \) instead of \( f_\sigma \).

The **spectrum** \( \text{spec}(A) \) of a selfadjoint operator on \( \mathcal{H} \) is given by the corresponding observable function \( f_A \) in a surprisingly simple manner:

**Proposition 6.3** Let \( A \) be a selfadjoint operator on \( \mathcal{H} \). Then

\[
\text{spec}(A) = f_A(f_A^{-1}(\mathbb{R})),
\]

which simplifies to

\[
\text{spec}(A) = f_A(\mathbb{P}\mathcal{H})
\]

if \( A \) is bounded from above.
In what follows we investigate the rôle of observable functions for the étale space corresponding to the presheaf \((\Sigma_{ba}(U), \rho_{U}^V)_{V \subseteq U}\) and for the Gelfand representation of the \(C^\ast\)-algebra \(C^\ast(\mathcal{B})\) of a Boolean sector \(\mathcal{B} \subseteq L(\mathcal{H})\).

We begin with a reformulation of the definition of observable functions. Let \(O(\mathcal{H})\) be the set of observable functions on \(\mathcal{P}(\mathcal{H}), O_{ba}(\mathcal{H})\) the set of observable functions that are bounded from above, and \(O_b(\mathcal{H})\) the set of bounded observable functions.

Let \(f \in O(\mathcal{H}), \mathcal{C}x \in f^{-1}(\mathbb{R})\) and \(\mathcal{B}_{\mathcal{C}x} \subseteq Q(\mathcal{L}(\mathcal{H}))\) the atomic quasipoint defined by \(\mathcal{C}x\). Let further \(\sigma\) be the spectral family corresponding to \(f\). Then

\[
\hat{f}(\mathcal{B}) := \inf\{\lambda \in \mathbb{R} | \mathcal{B} \subseteq \sigma(\lambda)\}.
\]

Using this formulation, we can extend the definition of observable functions to arbitrary quasipoints in \(L(\mathcal{H})\):

**Definition 6.2** Let \(f \in O(\mathcal{H})\) and let \(\sigma_f : \mathbb{R} \to \mathcal{L}(\mathcal{H})\) be the spectral family corresponding to \(f\). The function

\[
\hat{f} : Q(\mathcal{L}(\mathcal{H})) \to \mathbb{R} \cup \{\infty\},
\]

defined by

\[
\hat{f}(\mathcal{B}) := \inf\{\lambda \in \mathbb{R} | \sigma(\lambda) \in \mathcal{B}\},
\]

is called the **observable function on** \(Q(\mathcal{L}(\mathcal{H}))\) **induced by** \(f\).

The observable function \(\hat{f}\) induced by \(f \in O_b(\mathcal{H})\) can also be expressed directly in terms of \(f\):

**Proposition 6.4** Let \(f\) be a bounded observable function. Then the observable function \(\hat{f}\) induced by \(f\) is given by

\[
\forall \mathcal{B} \in Q(\mathcal{L}(\mathcal{H})) : \hat{f}(\mathcal{B}) = \inf_{U \in \mathcal{B}} \sup_{\mathcal{C}x \subseteq U} f(\mathcal{C}x).
\]

**Proposition 6.5** Let \(f \in O_{ba}(\mathcal{H})\). Then the induced observable function \(\hat{f} : Q(\mathcal{L}(\mathcal{H})) \to \mathbb{R}\) is bounded from above and **upper semicontinuous**.

From now on we will denote the observable function \(Q(\mathcal{L}(\mathcal{H})) \to \mathbb{R} \cup \{\infty\}\) induced by \(f \in O(\mathcal{H})\) also with the letter \(f\).

Next we will show how observable functions \(f : Q(\mathcal{L}(\mathcal{H})) \to \mathbb{R}\) can be used to assign a value to the germ \([\sigma]\) of a spectral family \(\sigma\) in the quasipoint \(\mathcal{B} \in Q(\mathcal{L}(\mathcal{H}))\). We recall that spectral families \(\sigma \in \Sigma_{ba}(U)\) and \(\tau \in \Sigma_{ba}(V)\) are equivalent at the quasipoint \(\mathcal{B} \in Q_{U \cap V}(\mathcal{L}(\mathcal{H}))\) if and only if there is an element \(W \in \mathcal{B}\) such that \(W \subseteq U \cap V\) and \(\sigma^W = \tau^W\) holds.
Proposition 6.6 Let $\sigma \in \Sigma_{ba}(U)$, $\tau \in \Sigma_{ba}(V)$ be spectral families with corresponding observable functions $f_\sigma$ and $f_\tau$ respectively. If $\sigma$ and $\tau$ are equivalent at $B \in Q_{U \cap V}(\mathbb{L}(\mathcal{H}))$, then

$$f_\sigma(B) = f_\tau(B)$$

holds.

This follows directly from the observation that the definition of equivalence at $B$ implies

$$\{\lambda \in \mathbb{R} \mid \sigma(\lambda) \in B\} = \{\lambda \in \mathbb{R} \mid \tau(\lambda) \in B\}.$$ 

The proposition shows that we obtain a mapping

$$v : \mathcal{E}(\Sigma_{ba}) \to \mathbb{R}$$

defined by

$$v([\sigma]_B) = f_\sigma(B)$$

on the etale space $\mathcal{E}(\Sigma_{ba})$. $v([\sigma]_B)$ is called the value of the germ $[\sigma]_B$.

Let us consider a simple example. Let $U$ be a non-zero element of $\mathbb{L}(\mathcal{H})$. Then the spectral family of the orthogonal projection $P_U$ onto $U$ is given by

$$\sigma(\lambda) = \begin{cases} 
0 & \text{for } \lambda < 0 \\
U^\perp & \text{for } 0 \leq \lambda < 1 \\
\mathcal{H} & \text{for } 1 \leq \lambda 
\end{cases}$$

and therefore the corresponding observable function $f_\sigma$ on $Q(\mathbb{L}(\mathcal{H}))$ is given by

$$f_\sigma(B) = \begin{cases} 
0 & \text{if } U^\perp \in B \\
1 & \text{if } U^\perp \not\in B.
\end{cases}$$

Of course $U \in B$ implies $U^\perp \not\in B$. The converse, however, is not true. But for a Boolean quasipoint $\beta$ we have

$$U^\perp \notin \beta \iff U \in \beta.$$ 

So the situation becomes much simpler for Boolean quasipoints.
Definition 6.3 Let $f : \mathcal{P}\mathcal{H} \to \mathbb{R}$ be an observable function, $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ a Boolean sector and $\beta \in \mathcal{Q}(\mathcal{B})$. Then the function $$f^\mathcal{B} : \mathcal{Q}(\mathcal{B}) \to \mathbb{R}$$ defined by $$f^\mathcal{B}(\beta) := \inf\{ \lambda \in \mathbb{R} \mid \sigma_f(\lambda) \in \beta \}$$ is called the $\mathcal{B}$-observable function induced by $f$.

We therefore obtain for $\sigma = \sigma_{\mathcal{P}\mathcal{U}}$: $$f^\mathcal{B} = \chi_{\mathcal{Q}\mathcal{U}(\mathcal{B})}$$ where $\chi_{\mathcal{Q}\mathcal{U}(\mathcal{B})}$ denotes the characteristic function of the subset $\mathcal{Q}\mathcal{U}(\mathcal{B}) \subseteq \mathcal{Q}(\mathcal{B})$. Now $\mathcal{Q}\mathcal{U}(\mathcal{B})$ is open and closed in the Stonean topology of $\mathcal{Q}(\mathcal{B})$, hence $f^\mathcal{B}_\sigma$ is a continuous function.

This is no accident. If $\mathcal{B}$ is a Boolean sector and $C^*(\mathcal{B})$ the $C^*$-algebra generated by $\{P_U \mid U \in \mathcal{B}\}$, we have seen in theorem 5.2 that $\mathcal{Q}(\mathcal{B})$ is homeomorphic to the Gelfand spectrum of $C^*(\mathcal{B})$. Let $C^*(\mathcal{B})_{sa}$ be the real subalgebra of hermitian elements of $C^*(\mathcal{B})$.

We can show that $f^\mathcal{B}_A$ $(A \in C^*(\mathcal{B})_{sa})$ is a continuous function on $\mathcal{Q}(\mathcal{B})$ and that the mapping $$C^*(\mathcal{B})_{sa} \to C(\mathcal{Q}(\mathcal{B}))$$ $$A \mapsto f^\mathcal{B}_A$$ is the restriction of the Gelfand transform $$C^*(\mathcal{B}) \to C(\mathcal{Q}(\mathcal{B}))$$ $$A \mapsto (\hat{A} : \beta \mapsto \tau_\beta(A))$$ to the real subalgebra $C^*(\mathcal{B})_{sa}$.

The proof proceeds in several steps. We denote by $\mathcal{A}(\mathcal{B})$ the complex algebra generated by $\{P_U \mid U \in \mathcal{B}\}$. In our special situation this is just $\text{span}_\mathbb{C}\{P_U \mid U \in \mathcal{B}\}$. $\mathcal{A}(\mathcal{B})$ is dense in $C^*(\mathcal{B})$. Let $A \in \mathcal{A}(\mathcal{B})$. We have called $$A = \sum_{j=1}^m b_j P_{V_j}$$ an orthogonal representation of $A$ if $V_1, \ldots, V_m \in \mathcal{B}$ are orthogonal in pairs. In the previous section we have seen that each element of $\mathcal{A}(\mathcal{B})$ possesses
at least one orthogonal representation. Let \( \sum_{j=1}^{m} b_j P_{V_j} \) and \( \sum_{k=1}^{n} c_k P_{W_k} \) be orthogonal representations of \( A \in \mathcal{A}(\mathcal{B}) \). One can show that
\[
\sum_{j=1}^{m} b_j \chi_{Q_{V_j}}(\mathcal{B}) = \sum_{k=1}^{n} c_k \chi_{Q_{W_k}}(\mathcal{B}).
\]
Therefore we obtain a mapping
\[
\mathcal{F}_B : \mathcal{A}(\mathcal{B}) \to C(\mathcal{Q}(\mathcal{B}))
\]
defined by
\[
\mathcal{F}_B(A) := \sum_{j} b_j \chi_{Q_{V_j}}(\mathcal{B})
\]
where \( \sum_j b_j P_{V_j} \) is any orthogonal representation of \( A \). One shows that \( \mathcal{F}_B \) is an isometric homomorphism of complex algebras. By the Stone-Weierstrass theorem the subalgebra \( \text{spanc}\{ \chi_{Q_U}\mathcal{B} \mid U \in \mathcal{B} \} \) is uniformly dense in \( C(\mathcal{Q}(\mathcal{B})) \). Therefore \( \mathcal{F}_B \) can be extended uniquely to an isometric isomorphism \( C^*(\mathcal{B}) \to C(\mathcal{Q}(\mathcal{B})) \) of \( C^* \)-algebras. We denote this isomorphism by \( \mathcal{F}_B \) again.

In the next step one shows that \( \mathcal{F}_B : C^*(\mathcal{B}) \to C(\mathcal{Q}(\mathcal{B})) \) coincides with the Gelfand transform of \( C^*(\mathcal{B}) \) (where we have identified \( \mathcal{Q}(\mathcal{B}) \) with the Gelfand spectrum \( \Omega_B \) of \( C^*(\mathcal{B}) \) according to theorem 5.2).

The second major part of the proof is to show that for \( A \in \mathcal{A}(\mathcal{B})_{sa} \), i.e. \( A \in \text{span}_R \{ P_U \mid U \in \mathcal{B} \} \), the induced observable function
\[
f_A : \mathcal{Q}(\mathcal{B}) \to \mathbb{R}
\]
equals \( \mathcal{F}_B(A) \).

Finally, in the third part of the proof, one shows that the equality \( f_A = \mathcal{F}_B(A) \) holds on \( C^*(\mathcal{B})_{sa} \).

Summarizing we obtain

**Theorem 6.3** Let \( \mathcal{B} \) be a Boolean sector of \( \mathbb{L}(\mathcal{H}) \). Then the induced observable functions \( f_A^\mathcal{B} : \mathcal{Q}(\mathcal{B}) \to \mathbb{R} \) are continuous for all \( A \in C^*(\mathcal{B})_{sa} \) and the mapping
\[
C^*(\mathcal{B})_{sa} \to C(\mathcal{Q}(\mathcal{B})),
\]
\[
A \mapsto f_A^\mathcal{B}
\]
is the restriction of the Gelfand transform \( \mathcal{F}_B : C^*(\mathcal{B}) \to C(\mathcal{Q}(\mathcal{B})) \) to \( C^*(\mathcal{B})_{sa} \).
The induced observable functions $f_A : \mathcal{Q}(\mathcal{L}(\mathcal{H})) \to \mathbb{R}$ and $f_A^B : \mathcal{Q}(\mathcal{B}) \to \mathbb{R}$ of an hermitian operator $A$ are closely related:

**Remark 6.3** Let $\beta \in \mathcal{Q}(\mathcal{B})$ and let $\mathcal{B} \in \mathcal{Q}(\mathcal{L}(\mathcal{H}))$ be a quasipoint that contains $\beta$. Then

$$f_A^B(\beta) = f_A(\mathcal{B})$$

for all $A \in W^*(\mathcal{B})_{sa}$.

If $X$ is a topological space, we denote by $\mathfrak{N}(X)$ the set of upper semicontinuous functions $X \to \mathbb{R}$.

**Proposition 6.7** The mapping

$$\mathcal{L}(\mathcal{H})_{sa} \to \mathfrak{N}(\mathcal{Q}(\mathcal{L}(\mathcal{H})))$$

$$A \mapsto f_A$$

is injective. Moreover, if $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ is a Boolean sector, the mapping

$$W^*(\mathcal{B})_{sa} \to \mathfrak{N}(\mathcal{Q}(\mathcal{B}))$$

$$A \mapsto f_A^B$$

is injective, too.

**Corollary 6.1** The hermitian operators $A \in W^*(\mathcal{B})$ whose induced observable functions $f_A^B$ are continuous are precisely the hermitian elements of $C^*(\mathcal{B})$.

In the following we will show that a positive operator of finite trace induces a bounded positive Radon measure $\mu_\rho^B$ on the compact Stonean space $\mathcal{Q}(\mathcal{B})$ of each Boolean sector $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$. $\mu_\rho^B$ will be a probability measure on $\mathcal{Q}(\mathcal{B})$ if and only if $tr\rho = 1$.

Let $\varphi \in C(\mathcal{Q}(\mathcal{B}))$ and let

$$\varphi = \varphi_1 + i\varphi_2$$

be the decomposition of $\varphi$ into real and imaginary part. There are uniquely determined hermitian operators $A_1, A_2 \in C^*(\mathcal{B})_{sa}$ such that

$$\varphi_k = f_{A_k}^B \quad (k = 1, 2).$$

Then

$$A_\varphi := A_1 + iA_2 \in C^*(\mathcal{B})$$
and we define
\[ f^B_{A\varphi} := f^B_{A1} + if^B_{A2}. \]

\( f^B_{A\varphi} \) is called the observable function corresponding to \( A\varphi \). Obviously
\[ f^B_{A\varphi} = \varphi. \]

**Proposition 6.8** Let \( \rho \in \mathcal{L}(\mathcal{H}) \) be a positive operator of finite trace. Then
\[ \mu^B_\rho : C(\mathcal{Q}(\mathbb{B})) \to \mathbb{C} \]
\[ \varphi \mapsto tr(\rho A\varphi) \]
is a bounded positive Radon measure on the Stonean space \( \mathcal{Q}(\mathbb{B}) \) of the Boolean sector \( \mathbb{B} \subseteq \mathcal{L}(\mathcal{H}) \). Moreover, \( \mu^B_\rho \) has norm \( tr\rho \), so \( \mu^B_\rho \) is a probability measure on \( \mathcal{Q}(\mathbb{B}) \) if and only if \( tr\rho = 1 \).

The following result shows that the value \( <Ax, x> \in \mathbb{R} \) for a hermitian operator \( A \) (an observable) and an \( x \in S^1(\mathcal{H}) \) (a pure state) is really an “expectation value” unless \( x \) is an eigenvector for \( A \). In that case the system in state \( x \) answers to the observable \( A \) with the eigenvalue \( \lambda \) corresponding to the eigenvector \( x \).

This result supports the conventional wisdom in quantum mechanics.

**Theorem 6.4** Let \( \mathcal{H} \) be a separable Hilbert space, \( \mathbb{B} \subseteq \mathcal{L}(\mathcal{H}) \) a Boolean sector and \( \rho \) a state, i.e. a positive operator on \( \mathcal{H} \) of trace 1. Then the Radon measure \( \mu^B_\rho \) on \( \mathcal{Q}(\mathbb{B}) \) is the point measure \( \varepsilon_{\beta_0} \) for some \( \beta_0 \in \mathcal{Q}(\mathbb{B}) \), if and only if there is an \( x \in S^1(\mathcal{H}) \) such that \( \mathbb{C}x \in \mathbb{B} \), \( \beta_0 = \beta_{\mathbb{C}x} \) and \( \rho = P_{\mathbb{C}x} \).

**Proof:** Let \( x \in S^1(\mathcal{H}) \) such that \( \mathbb{C}x \in \mathbb{B} \) and let \( \rho := P_{\mathbb{C}x} \). Let \( \varphi \) be a real valued function on \( \mathcal{Q}(\mathbb{B}) \) and let \( A\varphi \in \mathcal{L}(\mathcal{H}) \) be the corresponding hermitian operator:
\[ \varphi = f^B_{A\varphi}. \]

\( P_{\mathbb{C}x} \) commutes with \( A\varphi \), so \( x \) is an eigenvector of \( A\varphi \). Let \( \lambda \) be the corresponding eigenvalue. Then
\[ tr(\rho A\varphi) = <A\varphi x, x> = \lambda <x, x> = \lambda. \]

On the other hand
\[ \lambda = f_{A\varphi}(\mathbb{C}x) = f^B_{A\varphi}(\beta_{\mathbb{C}x}) = \varepsilon_{\beta_{\mathbb{C}x}}(f^B_{A\varphi}) = \varepsilon_{\beta_{\mathbb{C}x}}(\varphi). \]
Therefore

\[ \mu_{\rho}^B = \varepsilon_{\beta_0} \cdot \]

Conversely, let \( \rho \) be a state and \( B \) a Boolean sector such that \( \mu_{\rho}^B \) is the point measure \( \varepsilon_{\beta_0} \) for some \( \beta_0 \in \mathcal{Q}(B) \). Then for all \( U \in B \)

\[ \text{tr}(\rho P_U) = \mu_{\rho}^B(\chi_Q(U)) = \chi_Q(U)(\beta_0) \]

and therefore

\[ \forall U \in B : (U \in \beta_0 \iff \text{tr}(\rho P_U) = 1) . \]

Let \( U \in \beta_0 \) and let \( (e_k)_{k \in \mathbb{N}} \) be an \( U \)-adapted orthonormal basis of \( \mathcal{H} \), i.e. \( e_k \in U \cup U^\perp \) for all \( k \in \mathbb{N} \). Then

\[
1 = \text{tr}(\rho P_U) \\
= \text{tr}(P_U \rho) \\
= \sum_k < P_U \rho e_k, e_k > \\
= \sum_k < \rho e_k, P_U e_k > \\
= \sum_{e_k \in U} < \rho e_k, e_k > .
\]

Because of \( < \rho e_k, e_k > \geq 0 \) for all \( k \in \mathbb{N} \) and \( \text{tr} \rho = 1 \) we conclude that

\[ \forall e_k \in U^\perp : < \rho e_k, e_k > = 0 . \]

Hence \( \rho e_k = 0 \) for all \( e_k \in U^\perp \) and therefore

\[ \rho(I - P_U) = 0 \]
i.e.

\[ \rho = \rho P_U . \]

In particular

\[ \rho P_U = \rho = \rho^* = P_U \rho . \]

Now for all \( y \in \mathcal{H} \)

\[ \rho y = \rho P_U y = P_U \rho y \]
and therefore
\[ \forall U \in \beta_0 \forall y \in \mathcal{H} : \rho y \in U. \]
This implies
\[ \forall U \in \beta_0 : \text{im}\rho \subseteq U, \]
and from the maximality of $\beta_0$ we conclude that $\text{im}\rho \in \beta_0$. Hence $\text{im}\rho = \text{im}\rho = \mathbb{C}x$ for a unique $\mathbb{C}x \in \mathbb{L}(\mathcal{H})$. $\beta_0$ is contained in exactly one Boolean sector and therefore $\mathbb{C}x \in \mathbb{B}$.
There is a unique $\lambda_0 \in \mathbb{C}$ such that
\[ \rho x = \lambda_0 x. \]
$\rho \geq 0$ and $\text{tr}\rho = 1$ imply $\lambda_0 = 1$. Hence for all $y \in \mathcal{H}$
\[ \rho^2 y = \rho(\rho y) = \rho(\lambda x) = \lambda \rho x = \lambda x = \rho y \]
and therefore
\[ \rho = P_{\mathbb{C}x}. \quad \square \]
7 Classical observables and spectral families

In the previous section we have seen that a bounded hermitian operator $A$ on a Hilbert space $\mathcal{H}$ induces bounded upper semicontinuous real valued functions on the Stonean spaces $\mathcal{Q}(\mathcal{B})$ of those Boolean sectors $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ that contain the spectral family of $A$. These functions are continuous if $A$ belongs to the adherence of the linear span of its spectral projections with respect to the norm topology of $\mathcal{L}(\mathcal{H})$.

In this section we will show that a continuous real valued function on a topological space $M$ can be described by a spectral family with values in the complete lattice $\mathcal{T}(M)$ of open subsets of $M$. These spectral families $\sigma : \mathbb{R} \to \mathcal{T}(M)$ can be characterized abstractly by a certain property of the mapping $\sigma$. Thus also a classical observable has a “quantum mechanical” description. This shows that classical and quantum mechanical observables are on the same structural footing: either as functions or as spectral families.

It is quite natural to generalize the definition of a spectral family in the quantum lattice $\mathcal{L}(\mathcal{H})$ to a general complete lattice $\mathcal{L}$:

**Definition 7.1** Let $\mathcal{L}$ be a complete lattice. A spectral family in $\mathcal{L}$ is a mapping $\sigma : \mathbb{R} \to \mathcal{L}$ with the following properties:

1. $\sigma(\lambda) \leq \sigma(\mu)$ for $\lambda \leq \mu$,
2. $\sigma(\lambda) = \bigwedge_{\mu > \lambda} \sigma(\mu)$ for all $\lambda \in \mathbb{R}$, and
3. $\bigwedge_{\lambda \in \mathbb{R}} \sigma(\lambda) = 0$, $\bigvee_{\lambda \in \mathbb{R}} \sigma(\lambda) = 1$.

We postpone the investigation of spectral families in a general complete lattice to later work and concentrate here mainly on spectral families in the lattice $\mathcal{T}(M)$ for a topological space $M$.

Recall that in the lattice $\mathcal{T}(M)$ the (infinite) lattice operations are given by

$$\bigvee_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} U_\alpha$$

and

$$\bigwedge_{\alpha \in A} U_\alpha = \text{int}(\bigcap_{\alpha \in A} U_\alpha),$$

where $\text{int}N$ denotes the interior of the subset $N$ of $M$. We would like to begin with some simple examples:
Example 7.1 The following settings define spectral families \( \sigma_{id}, \sigma_{abs}, \sigma_{ln}, \sigma_{step} \) in \( \mathcal{T}(\mathbb{R}) \):

\[
\begin{align*}
\sigma_{id}(\lambda) &:= ]-\infty, \lambda[, \quad (1) \\
\sigma_{abs}(\lambda) &:= ]-\lambda, \lambda[, \quad (2) \\
\sigma_{ln}(\lambda) &:= ]-\exp(\lambda), \exp(\lambda)[, \quad (3) \\
\sigma_{step}(\lambda) &:= ]-\infty, [\lfloor \lambda \rfloor [. \quad (4)
\end{align*}
\]

where \( \lfloor \lambda \rfloor \) denotes the “floor of \( \lambda \in \mathbb{R} \)”: 
\[
\lfloor \lambda \rfloor = \max\{ n \in \mathbb{Z} | n \leq \lambda \}.
\]

The names of these spectral families sound somewhat crazy at the moment, but we will justify them soon.

In close analogy to the case of spectral families in the lattice \( \mathbb{L}(\mathcal{H}) \), each spectral family in \( \mathcal{T}(M) \) induces a function on a subset of \( M \).

Definition 7.2 Let \( \sigma : \mathbb{R} \to \mathcal{T}(M) \) be a spectral family in \( \mathcal{T}(M) \). Then
\[
\mathcal{D}(\sigma) := \{ x \in M | \exists \lambda \in \mathbb{R} : x \notin \sigma(\lambda) \}
\]
is called the admissible domain of \( \sigma \).

Remark 7.1 The admissible domain \( \mathcal{D}(\sigma) \) of a spectral family \( \sigma : \mathbb{R} \to \mathcal{T}(M) \) is dense in \( M \).

This follows directly from the observation that \( U \cap \mathcal{D}(\sigma) = \emptyset \) for some \( U \in \mathcal{T}(M) \) implies \( U \subseteq \bigwedge_{\lambda \in \mathbb{R}} \sigma(\lambda) = \emptyset \).

On the other hand it may happen that \( \mathcal{D}(\sigma) \neq M \). The spectral family \( \sigma_{ln} \) is a simple example:
\[
\forall \lambda \in \mathbb{R} : 0 \in \sigma_{ln}(\lambda).
\]

Each spectral family \( \sigma : \mathbb{R} \to \mathcal{T}(M) \) induces a function \( f_\sigma : \mathcal{D}(\sigma) \to \mathbb{R} \):

Definition 7.3 Let \( \sigma : \mathbb{R} \to \mathcal{T}(M) \) be a spectral family with admissible domain \( \mathcal{D}(\sigma) \). Then the function \( f_\sigma : \mathcal{D}(\sigma) \to \mathbb{R} \), defined by 
\[
\forall x \in \mathcal{D}(\sigma) : f_\sigma(x) := \inf\{ \lambda \in \mathbb{R} | x \in \sigma(\lambda) \},
\]
is called the function induced by \( \sigma \).

In complete analogy to the operator case we define the spectrum of a spectral family \( \sigma \):
Definition 7.4 Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family. Then

$$R(\sigma) := \{ \lambda \in \mathbb{R} \mid \sigma \text{ is constant on a neighborhood of } \lambda \}$$

is called the **resolvent set** of $\sigma$, and

$$\text{Spec}(\sigma) := \mathbb{R} \setminus R(\sigma)$$

is called the **spectrum** of $\sigma$.

Obviously $\text{Spec}(\sigma)$ is a closed subset of $\mathbb{R}$.

**Proposition 7.1** Let $f_{\sigma} : D(\sigma) \to \mathbb{R}$ be the function induced by the spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$. Then

$$\text{Spec}(\sigma) = \overline{\text{im} f_{\sigma}}.$$

The functions induced by our foregoing examples are

- $f_{\sigma_{\text{id}}}(x) = x \quad \text{(5)}$
- $f_{\sigma_{\text{abs}}}(x) = |x| \quad \text{(6)}$
- $f_{\sigma_{\text{ln}}}(x) = \ln |x| \quad \text{and } D(\sigma_{\text{ln}}) = \mathbb{R} \setminus \{0\} \quad \text{(7)}$
- $f_{\sigma_{\text{step}}} = \sum_{n \in \mathbb{Z}} n \chi_{[n,n+1]} \quad \text{(8)}$

There is a fundamental difference between the spectral families $\sigma_{\text{id}}, \sigma_{\text{abs}}, \sigma_{\text{ln}}$ on the one side and $\sigma_{\text{step}}$ on the other. The function induced by $\sigma_{\text{step}}$ is not continuous. This fact is mirrored in the spectral families: the first three spectral families have the property

$$\forall \lambda < \mu: \overline{\sigma(\lambda)} \subseteq \sigma(\mu).$$

Obviously $\sigma_{\text{step}}$ fails to have this property.

**Definition 7.5** A spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is called **continuous** if

$$\forall \lambda < \mu: \overline{\sigma(\lambda)} \subseteq \sigma(\mu)$$

holds.

Using the pseudocomplement $U^c := M \setminus \bar{U} \quad (U \in \mathcal{T}(M))$ we can express the condition of continuity in purely lattice theoretic terms as

$$\forall \lambda < \mu: \sigma(\lambda)^c \cup \sigma(\mu) = M.$$
Remark 7.2 The admissible domain $D(\sigma)$ of a continuous spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is an open (and dense) subset of $M$.

Remark 7.3 If $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is a continuous spectral family, then for all $\lambda \in \mathbb{R}$, $\sigma(\lambda)$ is a regular open set, i.e.

$$\sigma(\lambda)^{\text{ce}} = \sigma(\lambda).$$

The importance of continuous spectral families becomes manifest in the following

Theorem 7.1 Let $M$ be a topological space. Then every continuous function $f : M \to \mathbb{R}$ induces a continuous spectral family $\sigma_f : \mathbb{R} \to \mathcal{T}(M)$ by

$$\forall \lambda \in \mathbb{R} : \sigma_f(\lambda) := \text{int} f([-\infty, \lambda]).$$

The admissible domain $D(\sigma_f)$ equals $M$ and the function $f_{\sigma_f} : M \to \mathbb{R}$ induced by $\sigma_f$ is $f$. Conversely, if $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is a continuous spectral family, then the function

$$f_\sigma : D(\sigma) \to \mathbb{R}$$

induced by $\sigma$ is continuous and the induced spectral family $\sigma_{f_\sigma}$ in $\mathcal{T}(D(\sigma))$ is the restriction of $\sigma$ to the admissible domain $D(\sigma)$:

$$\forall \lambda \in \mathbb{R} : \sigma_{f_\sigma}(\lambda) = \sigma(\lambda) \cap D(\sigma).$$

The proof is, although not trivial, an exercise in general topology.

Note that any function $f : M \to \mathbb{R}$ induces a spectral family $\sigma_f : \mathbb{R} \to \mathcal{T}(M)$ by

$$\sigma_f(\lambda) := \text{int} f([-\infty, \lambda]).$$

There is a close connection between bounded spectral families $\mathbb{R} \to \mathbb{L}(\mathcal{H})$ with values in a given Boolean sector $\mathbb{B} \subseteq \mathbb{L}(\mathcal{H})$ and spectral families $\mathbb{R} \to \mathcal{T}(\mathbb{Q}(\mathbb{B}))$.

Let $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ be a bounded spectral family such that $\sigma(\lambda) \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$.

For $\lambda \in \mathbb{R}$ define

$$\hat{\sigma}(\lambda) := \bigwedge_{\mu > \lambda} \mathbb{Q}_{\sigma(\mu)}(\mathbb{B}) \in \mathcal{T}(\mathbb{Q}(\mathbb{B})).$$
Proposition 7.2 $\hat{\sigma} : \mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ is a spectral family with admissible domain $Q(\mathbb{B})$.

The connection between bounded spectral families $\mathbb{R} \to \mathbb{L}(\mathcal{H})$ with values in $\mathbb{B}$ and spectral families $\mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ rests on the following

Proposition 7.3 Let $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ be a bounded spectral family with values in the Boolean sector $\mathbb{B} \subseteq \mathbb{L}(\mathcal{H})$, $f_{\sigma} : Q(\mathbb{B}) \to \mathbb{R}$ the function induced by $\sigma$, $\hat{\sigma} : \mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ the spectral family induced by $\sigma$ and $f_{\hat{\sigma}} : Q(\mathbb{B}) \to \mathbb{R}$ the function induced by $\hat{\sigma}$. Then $f_{\hat{\sigma}} = f_{\sigma}$.

From this and from theorem 7.1 we obtain the following interesting result:

Corollary 7.1 Let $\sigma : \mathbb{R} \to \mathbb{L}(\mathcal{H})$ be a bounded spectral family with values in the Boolean sector $\mathbb{B}$. Then $f_{\sigma} : Q(\mathbb{B}) \to \mathbb{R}$ is continuous if and only if the spectral family $\hat{\sigma} : \mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ is continuous.

By the way this shows that in general the simpler “Ansatz”

$$\tilde{\sigma}(\lambda) := Q_{\sigma(\lambda)}(\mathbb{B})$$

cannot give a spectral family $\tilde{\sigma}$.

Now let $\tau : \mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ be a continuous spectral family and let $f_{\tau}$ be the continuous function induced by $\tau$. There is exactly one $A \in C^*(\mathbb{B})_{sa}$ such that $f_{\sigma_A} = f_{\tau}$ where $\sigma_A$ denotes the spectral family of $A$. Hence

Corollary 7.2 Let $\tau : \mathbb{R} \to \mathcal{T}(Q(\mathbb{B}))$ be a continuous spectral family. Then there is a unique hermitian operator $A \in C^*(\mathbb{B})$ such that $\hat{\sigma}_A = \tau$

where $\sigma_A$ is the spectral family of $A$.

In the following $\mathfrak{A}$ is an abstract $\sigma$-algebra. Thus $\mathfrak{A}$ is not necessarily a sub-$\sigma$-algebra of the power set of some set $M$.

Definition 7.6 We define a spectral family in the $\sigma$-algebra $\mathfrak{A}$ as a mapping

$$\sigma : \mathbb{R} \to \mathfrak{A}$$

with the following properties:
(1) $\sigma(\lambda) \leq \sigma(\mu)$ for $\lambda < \mu$.

(2) $\sigma(\lambda) = \bigwedge_{n \in \mathbb{N}} \sigma(\lambda_n)$ for every sequence $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lambda_n \searrow \lambda$.

(3) $\bigwedge_{n \in \mathbb{N}} \sigma(\lambda_n) = 0$ for every unbounded monotonously decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$.

(4) $\bigvee_{n \in \mathbb{N}} \sigma(\lambda_n) = 1$ for every unbounded monotonously increasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$.

If $\mathfrak{A}$ is a sub-$\sigma$-algebra of the power set $\text{pot}(M)$ of some non-empty set $M$ (that means that $\bigwedge_n U_n = \bigcap_n U_n$, $\bigvee_n U_n = \bigcup_n U_n$ etc.), we define as in the topological case

**Definition 7.7** Let $\mathfrak{A} \subseteq \text{pot}(M)$ a sub-$\sigma$-algebra and $\sigma : \mathbb{R} \to \mathfrak{A}$ a spectral family. Then the function $f_\sigma : M \to \mathbb{R}$, defined by

$$f_\sigma(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\},$$

is called the function induced by $\sigma$.

Obviously we have

**Remark 7.4** Let $\sigma : \mathbb{R} \to \mathfrak{A}$ be a spectral family in the sub-$\sigma$-algebra $\mathfrak{A} \subseteq \text{pot}(M)$. Then

$$\forall \lambda \in \mathbb{R} : \sigma(\lambda) = f_\sigma(\lambda),$$

**Corollary 7.3** Let $\mathfrak{A}$ be as above and $\sigma$ a spectral family in $\mathfrak{A}$. Then the function $f_\sigma : M \to \mathbb{R}$ is $\mathfrak{A}$-measurable. (We always assume that $\mathbb{R}$ is equipped with the $\sigma$-algebra of Borel sets.)

Conversely, every $\mathfrak{A}$-measurable function $f : M \to \mathbb{R}$ defines a spectral family $\sigma_f : \mathbb{R} \to \mathfrak{A}$ by

$$\sigma_f(\lambda) := f_{\sigma_f}[\lambda],$$

and it can be easily seen that these constructions are inverse to each other:

**Proposition 7.4** Let $\mathfrak{A} \subseteq \text{pot}(M)$ be a sub-$\sigma$-algebra. Then the spectral families $\mathbb{R} \to \mathfrak{A}$ are in bijective correspondence to the $\mathfrak{A}$-measurable functions $M \to \mathbb{R}$:

$$\sigma_{f_\sigma} = \sigma \quad \text{and} \quad f_{\sigma_f} = f.$$
This result shows that the following definition of a real valued random variable for an arbitrary $\sigma$-algebra $\mathcal{A}$ is adequate:

**Definition 7.8** Let $\mathcal{A}$ be a $\sigma$-algebra. An $\mathcal{A}$-random variable is a spectral family $X : \mathbb{R} \to \mathcal{A}$.

The consequences of this definition will be investigated in a forthcoming paper.

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