Bohr–Sommerfeld–Heisenberg theory in geometric quantization

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Abstract. In the framework of geometric quantization we extend the Bohr–Sommerfeld rules to a full quantum theory which resembles the Heisenberg matrix theory. This extension is possible because Bohr–Sommerfeld rules not only provide an orthogonal basis in the space of quantum states, but also give a lattice structure to this basis. This permits the definition of appropriate shifting operators. As examples, we discuss the 1-dimensional harmonic oscillator and the coadjoint orbits of the rotation group.

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1. Introduction

The desire to understand the energy spectrum of completely integrable Hamiltonian systems leads to Bohr–Sommerfeld theory, also called old quantum theory. Bohr [2] explained Planck’s hypothesis by the spectrum of the harmonic oscillator, which he obtained using his quantum conditions. Sommerfeld [21] extended Bohr’s quantization rules to a system with Hamiltonian

\[ H = \sqrt{p^2 + m^2} + \frac{k}{r}. \]  

(1)

The first term in (1) is the relativistic expression for the kinetic energy of a particle with mass \( m \) and momentum \( p \) and the second term is the potential energy of a charged particle in the electric field produced by a stationary charged particle at the origin. With an appropriate choice of the parameters \( m \) and \( k \), we can use the Hamiltonian \( H \) as an approximation to the energy of an electron in the hydrogen atom in the limit of infinite mass of its nucleus. The energy spectrum of the hydrogen atom obtained by Sommerfeld
agreed exactly with the observed spectrum.\footnote{It is remarkable that the energy spectrum obtained by Sommerfeld agrees exactly with the energy spectrum obtained by solving the Dirac equation for an electron in the same electric field \cite{9}. Even more puzzling is the fact that a modification of Bohr–Sommerfeld conditions by a term $\frac{1}{2}\hbar$ gives rise to the energy spectrum for a $\pi$ meson in the same electric field, which can be obtained by solving the Klein–Gordon equation \cite{19}.} The Bohr–Sommerfeld theory was applied with varying success to other systems.

The problem with Bohr–Sommerfeld theory is that it gives only the joint spectrum of energy and angular momentum. It does not provide a way to discuss the probability of transition between states. The next stage in the development of understanding of the nature of quantum physics was provided by the matrix theory of Heisenberg \cite{14} and the wave theory of Schrödinger \cite{18}. Heisenberg postulated that dynamical variables were not functions on the phase space of the system but are matrices in some vector space, possibly infinite dimensional. One can infer that Heisenberg’s matrices are linear transformations in the space of physical states relative to a basis provided by Bohr–Sommerfeld joint eigenstates of energy and angular momentum. Heisenberg’s approach was further developed by Born and Jordan \cite{3}, who used it to study various physical systems. For Schrödinger, physical states were described by complex-valued functions on the configuration space of the system; while dynamical variables were represented by differential operators obtained from classical dynamical variables by replacing the momentum by $i/\hbar$ times the operator of differentiation with respect to the position variable. Here $\hbar$ is Planck’s constant divided by $2\pi$. Dirac \cite{10} showed that the theories of Heisenberg and Schrödinger are equivalent. Since then, the Schrödinger equation has become the computational basis of quantum mechanics. Heisenberg’s theory is discussed in works mainly of historical interest \cite{17}.

At present, quantization of a completely integrable Hamiltonian system is discussed in the Schrödinger framework. The energy spectra obtained there tend to their Bohr–Sommerfeld counterparts as $\hbar \to 0$ \cite{23}.

The aim of this paper is to find a place for the Heisenberg matrix formalism within the framework of geometric quantization. A completely integrable Hamiltonian system defines a singular real polarization $F = D \otimes \mathbb{C}$ of the phase space of the underlying classical system, which is a symplectic manifold $(P, \omega)$. We denote by $S_F^\infty (L)$ the space of sections of the prequantization line bundle $L$ over $P$ that are covariantly constant along $F$. The Bohr–Sommerfeld conditions identify those leaves of $F$ that admit lifts to covariantly constant sections of $L$ \cite{20}. The space $S_F^\infty (L)$ consists of generalized sections (distribution sections) of $L$ that are supported on unions of Bohr–Sommerfeld leaves. For each Bohr–Sommerfeld leaf, we can choose a section supported on that leaf. In this way, we obtain a basis of complex vector space $S_F^\infty (L)$. We may choose a scalar product on $S_F^\infty (L)$ so that this basis is orthogonal. Let $\mathcal{H}$ denotes the space of sections of $S_F^\infty (L)$ that are normalizable with respect to this scalar product.