EN'TIRE SOLUTIONS OF THE ALLEN–CAHN–NAGUMO EQUATION IN A MULTI-DIMENSIONAL SPACE

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ABSTRACT. The Allen–Cahn–Nagumo equation is a reaction-diffusion equation with a bistable nonlinearity. This equation appears to be simple, however, it includes a rich behavior of solutions. The Allen–Cahn–Nagumo equation features a solution that constantly maintains a certain profile and moves with a constant speed, which is referred to as a traveling wave solution. In this paper, the entire solution of the Allen–Cahn–Nagumo equation is studied in multidimensional space. Here an entire solution is meant by the solution defined for all time including negative time, even though it satisfies a parabolic partial differential equation. Especially, this equation admits traveling wave solutions connecting two stable states. It is known that there is an entire solution which behaves as two traveling wave solutions coming from both sides in one dimensional space and annihilating in a finite time and that this one-dimensional entire solution is unique up to the shift. Namely, this entire solution is symmetric with respect to some point. There is a natural question whether entire solutions coming from all directions in the multi-dimensional space are radially symmetric or not. To answer this question, radially asymmetric entire solutions will be constructed by using super-sub solutions.

1. Introduction. Consider the reaction–diffusion equation:

\[ u_t = \Delta u + f(u) \]  \hspace{1cm} (1)

in \( x \in \mathbb{R}^N \) and \( t > 0 \) where \( N \geq 2 \) and \( f \) is a smooth function on \( \mathbb{R} \) with only three zeros \( 0, a, 1 \) satisfying

\[
\begin{align*}
& f(0) = f(a) = f(1) = 0, \quad f(u) \neq 0 \quad \text{if} \ a \neq 0, a, 1, \\
& f'(0) < 0, \quad f'(1) < 0, \quad f'(a) > 0, \quad \int_0^1 f(s) \, ds > 0.
\end{align*}
\]  \hspace{1cm} (2)

A typical example satisfying (2) is \( f(u) = u(1 - u)(u - a) \). This equation (1) is often called the Allen–Cahn equation when \( a = 1/2 \) or the Nagumo equation when \( a \neq 1/2 \). In this paper, this is referred to as the Allen–Cahn–Nagumo equation. This equation appears to be simple, however, it includes a rich behavior of solutions. A solution that constantly maintains a certain profile and moves with a constant speed.

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is called a traveling wave solution. Here it is always assumed that there is the unique one-dimensional traveling wave solution \( \Phi(x - c_0 t) \) with the speed \( c_0 \) satisfying

\[
- c_0 \Phi'(\xi) = \Phi''(\xi) + f(\Phi(\xi)), \quad \Phi'(\xi) > 0 \quad \text{for} \quad \xi \in \mathbb{R},
\]

\[
\Phi(-\infty) = 0, \quad \Phi(\infty) = 1, \quad \Phi(0) = a.
\]

Refer to [13, 8] for instance. By the condition (2), we see that \( c_0 < 0 \). That is, the region where \( u \) is close to 1 invades the state \( u \equiv 0 \) with a finite speed \( |c_0| \).

The existence of traveling wave solutions in a multi-dimensional space is also expected. If it exists, without loss of generality, one can assume that a traveling wave solution propagates in a given direction \( x_N \) and introduce the moving coordinate \( (x', z) := (x', x_N - ct) \in \mathbb{R}^N \) where \( x' = (x_1, \ldots, x_{N-1}) \). By defining

\[
\mathcal{L}[V] := -cV_z - \Delta V - f(V),
\]

this problem is reduced to finding a solution \((c, V(x', z))\) for the following nonlinear elliptic equation:

\[
\mathcal{L}[V] = 0 \quad (4)
\]

with

\[
\lim_{z \to -\infty} V(x', z) = 1, \quad \lim_{z \to \infty} V(x', z) = 0. \quad (5)
\]

The one-dimensional traveling wave solution \( \Phi(x \cdot n - c_0 t) = \Phi(x' \cdot n' + n_N z) \) satisfies (4) and (5) with

\[
c = \frac{c_0}{n_N} > 0
\]

for any \( n = (n', n_N) \in S^{N-1} \) if \( n_N < 0 \). This is often referred to as a planar traveling wave solution.

In the two dimensional case, there are two choices for \( n \in S^1 \) that satisfies (6). By combining the corresponding two planar traveling wave solution, a V-shaped traveling wave solution can be constructed. See Ninomiya and Taniguchi [18, 19] and Hamel, Monneau, and Roquejoffre [9, 10] for the details. In the multi-dimensional space, there are many choices of \( n \) that satisfies (6). By combining these planar traveling wave solutions, new traveling wave solutions can be constructed. More precisely, let \( n_1, \ldots, n_M \in S^{N-1} \) satisfy (6). Then one can define

\[
V^-(x) := \Phi\left( \max_{1 \leq j \leq M} (n_j \cdot x) \right).
\]

Taniguchi [22, 23] proved the existence of the traveling wave solution which is asymptotically close to \( V^- \). This is called a pyramidal traveling wave solution. One of the extensions of the multi-dimensional traveling wave solutions is the conical traveling wave solution that was proposed by Hamel, Monneau, and Roquejoffre [9, 10], which is radially symmetric in \( z \)-axis with an appropriate shift. In [24, 25], it has been shown that by increasing the number of lateral faces for a pyramidal traveling wave solution, it converges to a conical traveling wave solution. Taniguchi [24, 25] extended a conical traveling wave solution to a traveling wave solution whose level sets are close to equidistant hypersurfaces from any strictly convex set in \( \mathbb{R}^{N-1} \) all principal curvatures of which are positive. It can be called a convex-conical traveling wave solution.

Next, a solution that is defined for all time is considered, which is known as the entire solution. The entire solutions are important to characterize the dynamics of the solutions because these solutions include the transient dynamics. A traveling wave solution and a stationary solution are typical examples of entire solutions. A
connecting orbit between $a$ and 0 (or 1) is a homogeneous entire solution. As an inhomogeneous extension of the connecting orbits, Fukao, Morita, and Ninomiya [6] showed the existence of the entire solutions between an inhomogeneous stationary solution, which is a so-called standing wave solution and a stable constant steady state $u \equiv 0$ or 1.

On the other hand, many researchers have studied the entire solutions which behave like two traveling wave solutions as $x \to \pm \infty$ when $t \to -\infty$. Yagisita [28] first constructed it in the bistable case. Hamel and Nadirashvili, [11, 12] studied for the monostable case. Morita and Ninomiya [15] also demonstrated the existence of entire solutions such that they behave as a traveling wave solution connecting between 0 and $a$ in the left hand half space and does as a traveling wave solution connecting between $a$ and 1 in the right hand half space as $t$ tends to $-\infty$. This is referred to as the entire solution with merging fronts. Recently, Chen, Guo, Ninomiya and Yao [3] extended these studies to entire solutions originating from $k$-fronts. See [3] for the details.

Chen and Guo [2] showed that the entire solutions which behave like two facing traveling wave solutions in one dimensional space are unique up to the shift. We emphasize that the traveling wave solutions connect between two stable equilibria 0 and 1 for their uniqueness. The natural question arises: for the entire solutions that come from all directions in the multi-dimensional space, are they radially symmetric or not? Indeed, Polacik [20] proved that any entire solution satisfying
\[
\lim_{|x| \to \infty} \sup_{t<T} |1 - u(x, t)| = 0
\]
with some $T$ is radially symmetric under an appropriate shift. However, the entire solution of this type violates the above condition. Thus there is a chance to have radially asymmetric entire solutions.

As pointed out in [17], the existence of the $(N-1)$-dimensional entire solution seems to be deeply related to the existence of $N$-dimensional traveling wave solution. First, we explain heuristically how the $N$-dimensional traveling wave solutions can be expected from $(N-1)$-dimensional entire solutions. By arraying the snapshots of the $(N-1)$-dimensional entire solution appropriately in the $x_N$ direction, we can expect the $N$-dimensional traveling wave solution. Indeed, in [17], the author proved the existence of a two-dimensional traveling wave solution such that it converges to the standing wave solution as $x_2 \to -\infty$ and behaves like a traveling wave solution connecting between 0 and 1/2 in $x_1 < 0$ and does like a traveling wave solution connecting between $1/2$ and 1 in $x_1 > 0$ as $x_2 \to \infty$ when $f(u) = u(1-u)(u-1/2)$. This is known as a zipping wave solution. This traveling wave solution can be expected by the existence of entire solutions with merging fronts shown in [15].

Next we explain heuristically how we can expect $(N-1)$-dimensional entire solutions from $N$-dimensional traveling wave solutions. Stay at some point in two-dimensional space and observe a V-shaped traveling wave solution with a high speed along the $x_2$-direction. One can observe the state $u = 0$ as $t \to -\infty$ and we observe two fronts approaching from both sides of large $|x_1|$ as $t$ increases. The two fronts collide at some time and we only see the state $u = 1$ as $t \to \infty$. This suggests the existence of the one-dimensional entire solution which behaves like two facing traveling wave solutions connecting 0 and 1, which is shown in [28, 6, 7, 2]. Applying this heuristic relation between traveling wave solutions and entire solutions to the convex-conical traveling wave solutions in [24, 25], we may expect the existence of a multi-dimensional radially asymmetric entire solution whose level
sets are approximately equidistant from any convex body as \( t \) tends to \(-\infty\). To explain this rigorously, a mathematical framework will be prepared. We always assume that \( K_0 \) is a strictly convex set such that its boundary \( \Gamma_0 := \partial K_0 \) is an \((N - 1)\) dimensional closed \( C^2 \) hypersurface that can be represented by

\[
\Gamma_0 = \{ g(\omega) \omega \mid \omega \in S^{N-1} \},
\]

and all principal curvatures of \( \Gamma_0 \) are positive where \( g \) is \( C^2(S^{N-1}) \). In addition, the convex hull of a set is denoted as \( D \) by \( \text{Conv} \). From this, \( K_0 = \text{Conv} \Gamma_0 \).

For a positive constant \( \alpha \), we set

\[
\Gamma_\alpha := \{ x \in \mathbb{R}^N \mid \text{dist}(x, \text{Conv} \Gamma_0) = \alpha \}, \\
K_\alpha := \text{Conv} \Gamma_\alpha.
\]

We also define a signed distance function \( h \) from \( K_0 \) by

\[
h(x) := \begin{cases} 
\text{dist}(x, \Gamma_0) & \text{if } x \notin K_0, \\
-\text{dist}(x, \Gamma_0) & \text{if } x \in K_0
\end{cases}
\]

(see Section 2 for the details).

**Theorem 1.1.** Let \( K_0 \) be any \( N \)-dimensional strictly convex body as described above. Then there exists an entire solution \( U \) of (1) that satisfies

\[
\lim_{t \to -\infty} \inf_{p > 0} \sup_{x \in \mathbb{R}^{N-1}} \left| U(x, t) - \Phi(h(x) - p) \right| = 0.
\]

In addition, Yagisita [27] constructed the solution starting from any convex set and showed that the initial perturbation of the solution remains for \( t > 0 \); however, it is not an entire solution. Notice that the behavior of the entire solution is opposite.

Let \( U(x, t) \) be an entire solution of (1). If there are a function \( \Psi \) and a point \( x_0 \in \mathbb{R}^N \) satisfies

\[
U(x, t) = \Psi(|x - x_0|, t)
\]

for any \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^N \), then we call it a \textit{radially symmetric} entire solution. Otherwise, it is called a \textit{radially asymmetric} entire solution. By taking \( K_0 \) a convex set except for a disk, we can expect the existence of radially asymmetric entire solutions. Actually we can show the following result.

**Corollary 1.2.** There exists a radially asymmetric entire solution of (1) when \( N \geq 2 \).

This paper is organized as follows. In the successive section, several lemmas are prepared in order to construct the super-sub solutions. In particular, the auxiliary ordinary differential equation is proposed to determine the locations of the super-sub solutions. In Section 3, the proof of Theorem 1.1 is provided. In the last section, an example of the radially asymmetric entire solutions is presented.

2. Preliminaries. First, several constants need to be defined that depend on \( f \). By the condition of \( f \), set

\[
\beta_* := \frac{1}{2} \min\{-f'(0), -f'(1)\} > 0.
\]

One can choose \( \delta_* \in (0, 1/4) \) satisfying

\[
\min_{|s| < 2\delta_*} (-f'(s)) > \beta_*, \quad \min_{|s-1| < 2\delta_*} (-f'(s)) > \beta_*,
\]

(8)
Moreover, the following is set:
\[ M_f := \max_{-2\delta \leq s \leq 1+2\delta} |f'(s)|. \]

Secondary, recall the property of the traveling wave solution \( \Phi \). It is well known that there are positive constants \( A \) and \( \lambda \) such that
\[ \begin{align*}
0 < \Phi'(\xi) &< Ae^{-\lambda|\xi|} & \text{for} & \xi \in \mathbb{R}, \\
0 < \Phi(\xi) &< Ae^{\lambda \xi} & \text{for} & \xi \leq 0, \\
0 < 1 - \Phi(\xi) &< Ae^{-\lambda \xi} & \text{for} & \xi \geq 0, \\
|\Phi''(\xi)| &< Ae^{-\lambda|\xi|} & \text{for} & \xi \in \mathbb{R}.
\end{align*} \]

There is a positive constant \( R_* \) such that
\[ \Phi(\xi) < \delta_*, \quad \Phi(\xi) > 1 - \delta_* \quad \text{for any} |\xi| \geq R_. \]

One can take \( R_* \) so large that
\[ R_* > \max \left\{ 1, \frac{3}{A} \right\}, \quad R_* e^{-\lambda R_*} \leq \frac{\beta \delta_*}{8A\kappa_+^2}, \]
where \( \kappa_+ \) is the maximum of the principal curvatures of \( \Gamma_0 \). One also sets
\[ \gamma_* := \min_{|\xi| \leq R_*} \Phi'(\xi) > 0. \]

Thirdly, the property of the distance function of \( \Gamma_0 \) will be investigated. The interface \( \Gamma_0 \) is assumed to be represented by (7) after an appropriate translation. The outer normal vector and the mean curvature of \( \text{Conv} \Gamma_0 \) at \( x \in \Gamma_0 \) are denoted by \( \nu(x; a) \) and \( \kappa(x; a) \). For simplicity of the notation, \( \nu(x) = \nu(x; 0) \) and \( \kappa(x) = \kappa(x; 0) \) when \( x \in \Gamma_0 \). The distance function from the hypersurface was introduced shown by [16] and [1]. From this, the results are stated as follows.

**Lemma 2.1.** For any \( x \in \mathbb{R}^N \setminus \text{Conv} \Gamma_0 \), there is a unique point \( s(x) \in \Gamma_0 \) such that
\[ \text{dist} (x, \Gamma_0) = |x - s(x)|, \quad x = s(x) + h(x)\nu(x), \]
Moreover, \( s \) is a \( C^2 \) function from \( \mathbb{R}^N \setminus \text{Conv} \Gamma_0 \) to \( \Gamma_0 \) and
\[ \nu(x; a) = \nu(s(x)) \]
for any positive \( a \).

Even for any \( x \in \text{Conv} \Gamma_0 \), there is a point \( s(x) \in \Gamma_0 \) satisfying \( \text{dist} (x, \Gamma_0) = |x - s(x)| \), which may not be determined uniquely. Define
\[ h(x) := \begin{cases} 
\min_{\omega \in \mathbb{S}^{N-1}} (x \cdot \nu(x) - g(\omega) \omega \cdot \nu(x)) & \text{if} \ x \in \mathbb{R}^N \setminus \text{Conv} \Gamma_0, \\
\max_{\omega \in \mathbb{S}^{N-1}} (x \cdot \nu(x) - g(\omega) \omega \cdot \nu(x)) & \text{if} \ x \in \text{Conv} \Gamma_0.
\end{cases} \]

It is easily checked that \( h(x) \) is a signed distance function from \( K_0 \).

**Lemma 2.2.** The function \( h \) defined by (13) is a \( C^2 \) function and is the distance between \( \Gamma_0 \) and \( x \) if \( x \in \mathbb{R}^N \setminus \text{Conv} \Gamma_0 \). Namely, \( x \in \Gamma_h(x) \) for \( x \in \mathbb{R}^N \setminus \text{Conv} \Gamma_0 \). Moreover,
\[ \nabla h(x) = \nu(s(x)), \quad |\nabla h| = 1, \]
\[ \Delta h = \sum_{j=1}^{N-1} \frac{\kappa_j(s(x))}{1 + h(x)\kappa_j(s(x))}, \quad |\nabla h| = 1, \]
if \( x \in \Gamma_0 \) with some \( a = h(x) > 0 \), where \( \kappa_1(s), \cdots, \kappa_{N-1}(s) \) are principal curvatures at \( s \in \Gamma_0 \).
This lemma was basically shown in [16, 1, 25]. Since it seems difficult to find a proof of the second equality in the literature, the proof is provided for the readers’ convenience.

**Proof.** Since
\[ h(x + \varepsilon \nu(s)) = h(x) + \varepsilon, \quad s(x + \varepsilon \nu(s)) = s(x) \]
for any \( x \not\in K_0 = \text{Conv} \Gamma_0 \) and any positive constant \( \varepsilon \), one has
\[ \nu(s(x)) = \nabla h(x) \]
from \( x = s(x) + h(x)\nu(s(x)) \). Similarly, the directional derivative of \( x \) along \( n \) yields
\[ n = (n \cdot \nabla)s(x) + (n \cdot \nabla h(s(x))) + h(x)(n \cdot \nabla)\nu(s(x)). \]
Let \( \tau_j \) (\( j = 1, \cdots, N-1 \)) be a unit vector of the principal directions of the tangential space of \( \Gamma_0 \) at \( s(x) \). By taking the inner product with \( \tau_j \), one gets
\[ n \cdot \tau_j = (n \cdot \nabla)s(x) \cdot \tau_j + h(x)(n \cdot \nabla)\nu(s(x)) \cdot \tau_j \]
for \( j = 1, \cdots, N-1 \). In particular, substituting \( n = \tau_j \) into the above equation yields
\[ (\tau_j \cdot \nabla)s(x) \cdot \tau_j = \frac{1}{1 + h(x)\kappa_j}. \]
Noting \( (\nu \cdot \nabla)s(x) = 0 \), one obtains
\[ \nabla \cdot s(x) = (\nu \cdot \nabla)s(x) \cdot \nu + \sum_{j=1}^{N-1} (\tau_j \cdot \nabla)s(x) \cdot \tau_j = \sum_{j=1}^{N-1} \frac{1}{1 + h(x)\kappa_j}. \]
Moreover, since
\[ \nabla \cdot x = \nabla \cdot s(x) + \nabla h(x) \cdot \nu(x) + h(x)\nabla \cdot \nabla h(x), \]
one gets
\[ N = \nabla \cdot s(x) + \nabla h(x) \cdot \nu(x) + h(x)\Delta h(x), \]
which implies
\[ h(x)\Delta h(x) = N - 1 - \nabla \cdot s(x) \]
\[ = N - 1 - \sum_{j=1}^{N-1} \frac{1}{1 + h(x)\kappa_j(s(x))} \]
\[ = \sum_{j=1}^{N-1} \frac{h(x)\kappa_j(s(x))}{1 + h(x)\kappa_j(s(x))}, \]
which implies
\[ \Delta h(x) = \sum_{j=1}^{N-1} \frac{\kappa_j(s(x))}{1 + h(x)\kappa_j(s(x))}. \]
This completes the proof. \( \square \)
Let \( \kappa_- \) (resp. \( \kappa_+ \)) be the minimum (resp. maximum) of the principal curvatures of \( \Gamma_0 \). Under the assumption of \( \Gamma_0 \), the following is obtained:

\[
0 < (N - 1)\kappa_- \leq \kappa(s) \leq (N - 1)\kappa_+ \tag{14}
\]

for \( s \in \Gamma_0 \). Moreover, there is a \( C^2 \) function \( H \) from \( \mathbb{R}^N \) to \( \mathbb{R} \) that satisfies

\[
H(x) := \begin{cases} 
  h(x) & \text{if } x \in \mathbb{R}^N \setminus K_1, \\
  0 & \text{if } x \in K_0 = \text{Conv} \Gamma_0
\end{cases}
\]

by modifying the function in \( K_1 = \text{Conv} \Gamma_1 \).

Lastly, the solution of the following ordinary equation

\[
p'(t) = c_0 - \frac{(N - 1)k}{1 + kp(t)} + \frac{\sigma}{(1 + kp(t))^2}, \quad p(s) = p_0 \tag{15}
\]

is denoted by \( p(t; s, p_0, k, \sigma) \) with \( k > 0 \) and \( s, p_0, \sigma \in \mathbb{R} \). Note that \( p(t; s, p_0, k, \sigma) = p(t - s; 0, p_0, k, \sigma) \). We recall that \( c_0 < 0 \). Fix a positive constant \( \rho_1 \) that satisfies

\[
1 + \kappa_+ \rho_1 > 0, \quad \frac{2(N - 1)\kappa_+}{1 + \kappa_- \rho_1} + \frac{2\sigma}{(1 + \kappa_- \rho_1)^2} < -c_0. \tag{16}
\]

Let \( p_+(t) \) be a solution \( p(t; 0, \rho_1, \kappa_+, -\sigma) \) of (15). Then \( p_+(t) \) satisfies

\[
p_+''(t) = c_0 - \frac{(N - 1)\kappa_+}{1 + \kappa_- p_+(t)} - \frac{\sigma}{(1 + \kappa_- p_+(t))^2} < c_0 < 0 \tag{17}
\]

for \( p_+(t) \geq \rho_1 \) and \( p(t) \) is monotone decreasing in \( t \). Moreover, since

\[
p_+'(t) > \frac{3}{2} c_0,
\]

\( p_+(t) \) is bounded for any negative time \( t \). Since \( \rho_1 - p_+(t) < -c_0 t \), it is easily seen that \( p_+(t) \to \infty \) as \( t \to -\infty \). Let \( p_-(t; s) \) be a solution \( p(t; s, p_+(s), \kappa_-, \sigma) \) of (15) for \( s \leq t \leq 0 \).

**Lemma 2.3.** Under (16), let \( p_+(t) \) and \( p_-(t; s) \) be as above. Then the limit \( \lim_{s \to -\infty} p_-(t; s) \) exists for \( t \leq 0 \), which is denoted by \( p_-(t) \). Moreover,

\[
\frac{2}{9|c_0|} \left\{ \frac{\log \frac{\kappa_+(1 + \kappa_- p_+(t))}{\kappa_-(1 + \kappa_- p_-(t))}}{\kappa_+(1 + \kappa_- p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_- p_+(t))} + \frac{\sigma}{\kappa_-(1 + \kappa_- p_-(t))} \right\} \leq p_-(t) - p_+(t) \leq \frac{3}{|c_0|} \left\{ \frac{\log \frac{\kappa_+(1 + \kappa_- p_+(t))}{\kappa_-(1 + \kappa_- p_-(t))}}{\kappa_+(1 + \kappa_- p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_- p_+(t))} + \frac{\sigma}{\kappa_-(1 + \kappa_- p_-(t))} \right\} \tag{18}
\]

for any \( t \leq 0 \). In particular,

\[
\lim_{t \to -\infty} (p_-(t) - p_+(t)) = 0.
\]

**Proof.** Since \( p_-(t; s) \) satisfies

\[
p_-'(t; s) = c_0 - \frac{(N - 1)\kappa_-}{1 + \kappa_- p_-(t; s)} - \frac{\sigma}{(1 + \kappa_- p_-(t; s))^2} < 0 \tag{19}
\]

by (16), \( p_-(t; s) \geq \rho_1 \) for \( s \leq t \leq 0 \) and it is monotone decreasing in \( t \). Since

\[
-\frac{\kappa_-}{1 + \kappa_+} \leq \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} < 0 < \frac{\sigma}{(1 + \kappa_- p)^2},
\]

one has

\[
\frac{3}{2} c_0 < p_+'(t) \leq p_-'(t; s) \leq \frac{c_0}{2} < 0
\]
for \( s < t \leq 0 \). Especially,

\[ \rho_1 < p_+(t) < p_-(t; s) \quad \text{for} \quad s < t \leq 0. \]

From (17) and (19), it follows that

\[
\int_{p_+(t)}^{p_+(s)} dp \left[ \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right] = t - s,
\]

\[
\int_{p_-(t; s)}^{p_+(s)} dp \left[ \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right] = t - s.
\]

Subtracting the two equalities yields

\[
\int_{p_+(t)}^{p_-(t; s)} dp \left[ \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right] = 2\left( \int_{p_+(t)}^{p_-(t; s)} dp \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) dp.
\]

The right-hand side of (20) can be rewritten as

\[
\int_{p_+(t)}^{p_+(s)} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) - \left| c_0 \right| + \frac{\kappa_+}{1 + \kappa_+ p} + \frac{\sigma}{(1 + \kappa_+ p)^2} \right) dp = \int_{p_+(t)}^{p_+(s)} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \left| c_0 \right| + \frac{\kappa_+}{1 + \kappa_+ p} + \frac{\sigma}{(1 + \kappa_+ p)^2} \right) dp.
\]

Thus, \( p_-(t; s) \) is decreasing in \( s \). By (16), one gets

\[
\frac{2(p_-(t; s) - p_+(t))}{3\left| c_0 \right|} \leq \int_{p_+(t)}^{p_-(t; s)} \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \frac{2(p_-(t; s) - p_+(t))}{\left| c_0 \right|}.
\]

The right-hand side of (20) can be estimated as follows:

\[
\int_{p_+(t)}^{p_+(s)} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \int_{p_+(t)}^{\infty} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \int_{p_+(t)}^{\infty} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \int_{p_+(t)}^{\infty} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \int_{p_+(t)}^{\infty} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right) \leq \int_{p_+(t)}^{\infty} \left( \left| c_0 \right| + \frac{\kappa_-}{1 + \kappa_- p} - \frac{\sigma}{(1 + \kappa_+ p)^2} \right)
\]

Combining (21) with the above inequality yields

\[
\frac{3}{\left| c_0 \right|} \left\{ \log \frac{\kappa_+(1 + \kappa_+ p_+(t))}{\kappa_-(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_+ p_+(t))} - \frac{\sigma}{\kappa_-(1 + \kappa_+ p_+(t))} \right\} \leq \frac{3}{\left| c_0 \right|} \left\{ \log \frac{\kappa_+(1 + \kappa_+ p_+(t))}{\kappa_-(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_+ p_+(t))} - \frac{\sigma}{\kappa_-(1 + \kappa_+ p_+(t))} \right\}.
\]
This implies that the limit \( \lim_{s \to -\infty} p_-(t; s) \) exists for \( t \leq 0 \). This is denoted by \( p_-(t) \). In particular,

\[
p_-(t) - p_+(t) \leq \frac{3}{|c_0|} \left\{ \log \frac{\kappa_+(1 + \kappa_- p_+(t))}{\kappa_-(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_-(1 + \kappa_- p_+(t))} \right\}
\]

by (22). Similarly, the following is obtained:

\[
\int_{p_+(t)}^{\infty} \left( \frac{1}{|c_0|} + \frac{\kappa_-}{1 + \kappa_- |p|} - \frac{\sigma}{(1 + \kappa_+ |p|)^2} \right) dp - \int_{p_+(t)}^{\infty} \left( \frac{1}{|c_0|} + \frac{\kappa_+}{1 + \kappa_+ |p|} + \frac{\sigma}{(1 + \kappa_- |p|)^2} \right) dp
\]

\[
= \int_{p_+(t)}^{\infty} \left( \frac{\kappa_+ - \kappa_-}{|c_0|} + \frac{\sigma}{(1 + \kappa_+ |p|)(1 + \kappa_- |p|)} + \frac{\sigma}{(1 + \kappa_- |p|)^2} + \frac{\sigma}{(1 + \kappa_+ |p|)^2} \right) dp
\]

\[
= \frac{4}{|c_0|^2} \left\{ \log \frac{\kappa_+(1 + \kappa_- p_+(t))}{\kappa_-(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_-(1 + \kappa_- p_+(t))} \right\}.
\]

Combining (21) yields

\[
p_-(t) - p_+(t) \geq \frac{2}{|c_0|} \left\{ \log \frac{\kappa_+(1 + \kappa_- p_+(t))}{\kappa_-(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_+(1 + \kappa_+ p_+(t))} + \frac{\sigma}{\kappa_-(1 + \kappa_- p_+(t))} \right\}.
\]

Hence, (18) is obtained. The last inequality of (18) immediately implies that the difference between \( p_-(t) \) and \( p_+(t) \) converges to 0 as \( t \to -\infty \). This completes the proof.

\[\square\]

3. Proof of Theorem 1.1. This section delivers the proof of Theorem 1.1. To this end, we construct the supersolutions and the subsolutions. Set

\[
\mathcal{F}[U] := U_t - \Delta U - f(U), \quad (23)
\]

\[
U_\pm(x, t) := \Phi(H(x) - p_\pm(t)) \pm \frac{\eta}{(1 + \kappa_\pm p_\pm(t))^2}, \quad (24)
\]

where \( H(x), p_\pm(t) \) and \( \kappa_\pm \) are given in the previous section and \( \eta \) is a positive constant specified later. Plugging (24) into (23) and using Lemma 2.2 yield

\[
\mathcal{F}[U_\pm] = \Phi'(-p_\pm') + 2\eta(1 + \kappa_\pm p_\pm)^{-3}\kappa_\pm p_\pm' - \nabla(\Phi'\nabla H) - f(\Phi \pm \eta(1 + \kappa_\pm p_\pm)^{-2})
\]

\[
= \Phi'p_\pm' - \Phi'\nabla H^2 - \Phi'\Delta H - f(\Phi \pm \eta(1 + \kappa_\pm p_\pm)^{-2}) \mp 2\eta(1 + \kappa_\pm p_\pm)^{-3}\kappa_\pm p_\pm'
\]

\[
= -(p_\pm' + \Delta H)\Phi' + \left( c_0 \Phi' + f(\Phi) \right) \nabla H^2 - f(\Phi \pm \eta(1 + \kappa_\pm p_\pm)^{-2})
\]

\[
\mp 2\eta(1 + \kappa_\pm p_\pm)^{-3}\kappa_\pm p_\pm'
\]

\[
= I_{1, \pm} + I_{2, \pm} + I_{3, \pm},
\]

where

\[
I_{1, \pm} := \left( c_0 \nabla H^2 \right) \Phi'
\]

\[
\left( c_0 \nabla H^2 \right) \Phi' = \left( c_0 \nabla H^2 - c_0 + \frac{(N - 1)\kappa_\pm}{1 + p_\pm(t)\kappa_\pm} \pm \frac{\sigma}{(1 + \kappa_\pm p_\pm)^2} - \Delta H \right) \Phi',
\]

\[
I_{2, \pm} := f(\Phi)(\nabla H^2 - 1),
\]

\[
I_{3, \pm} := f(\Phi) - f(\Phi \pm \eta(1 + \kappa_\pm p_\pm)^{-2}) \mp 2\eta(1 + \kappa_\pm p_\pm)^{-3}p_\pm'.
\]
respectively. The estimate of $\mathcal{F}[U]$ is split into four cases:

(i) $H(x) < 1$ and $|H(x) - p_\pm(t)| < R_*$;
(ii) $H(x) < 1$ and $|H(x) - p_\pm(t)| \geq R_*$;
(iii) $H(x) \geq 1$ and $|H(x) - p_\pm(t)| < R_*$;
(iv) $H(x) \geq 1$ and $|H(x) - p_\pm(t)| \geq R_*$.

Set
\[
\rho_2 := \max \left\{ R_* + 1, 2R_*, \frac{4}{\lambda}, \rho_1 \right\}.
\]

There is a constant $T_1$ such that $p_\pm(t) \geq \rho_2$ for $t \leq T_1$. Then, since $p_\pm(t) \geq \rho_2$, case (i) never occurs; thus, only cases (ii) – (iv) need to be considered.

First, consider case (ii). By (25), the case where $p_\pm(t) \geq \rho_2$, case (i) never occurs; thus, only cases (ii) – (iv) need to be considered.

Next, consider case (ii). By (26), one gets
\[
|H(x)| \leq M_H, \quad |\Delta H(x)| \leq M_H \quad \text{for} \quad x \in K_1 = \text{Conv} \Gamma_1.
\]

**Lemma 3.1.** For any $\eta \in (0, \delta_*)$, $t \in (-\infty, T_1)$ and any $x \in \mathbb{R}^N$ that satisfies $H(x) < 1$ and $|H(x) - p_\pm(t)| \geq R_*$, the following inequalities hold:

\[
\pm I_{1,\pm} \geq -\left( |c_0| (M_H^2 + 1) + M_H \right) Ae^{-\lambda(p_\pm - 1)},
\]
\[
|I_{2,\pm}| \leq M_f (M_H^2 + 1) Ae^{-\lambda(p_\pm - 1)},
\]
\[
\pm I_{3,\pm} \geq \frac{\beta_* \eta}{(1 + \kappa_\pm p_\pm)^2}.
\]

**Proof.** By (25), the case where $p_\pm(t) \geq H(x) + R_*$ only takes place. For $p_\pm(t) - H(x) \geq R_*$, the inequality $\Phi(H(x) - p_\pm) \leq \delta_*$ holds. Using (9) and (26), one gets
\[
I_{1,\pm} \geq -\left( |c_0| M_H^2 + M_H \right) \Phi' \geq -\left( |c_0| M_H^2 + 1 \right) M_H Ae^{-\lambda(p_\pm - H)}.
\]

The estimate for $I_{1,\pm}$ can be obtained similarly.

Next, consider $I_{2,\pm}$. Since $H(x) \leq 1$ and $p_\pm(t) - H(x) \geq R_*$, one has
\[
|I_{2,\pm}| \leq (M_H^2 + 1) M_f \Phi \leq (M_H^2 + 1) M_f Ae^{-\lambda(p_\pm - H)}
\]
by the definition of $M_f$. 

**Figure 1.** The case when $p_\pm(t)$ is large. The dashing curve in the gray region indicates the set of $H(x) = p_\pm(t)$. The gray region indicates (iii). The region surrounded by the solid curve is $K_0$. One can observe that the curvature is getting small as $p_\pm$ goes to infinity.
Finally, consider the estimate for $I_{3, \pm}$. Note that $\Phi(H(x) - p_+) + \eta(1 + \kappa_+ p_+)^{-2} \leq 2\delta_*$ by $\eta \in (0, \delta_*)$. Since $c_0 < 0$, one gets
\[
I_{3,+} \geq f(\Phi) - f(\Phi + \eta(1 + \kappa_+ p_+)^{-2}) - 2\eta(1 + \kappa_\pm p_\pm)^{-3} \left( c_0 - \frac{(N - 1)\kappa_+}{1 + \kappa_+ p_+} - \frac{\sigma}{1 + \kappa_+ p_+} \right)
\geq - \int_0^1 f'(\Phi + \theta\eta(1 + \kappa_+ p_+)^{-2})d\theta \cdot \eta(1 + \kappa_+ p_+)^{-2}
\geq \frac{\beta_* \eta}{(1 + \kappa_+ p_+)^2}
\]
by the definition of $\beta_*$. Because the inequality for $I_{3,-}$ can be shown similarly, the proof is omitted. 

To consider case (iv), $T_2$ is introduced as follows. For any $\eta \in (0, \delta_*)$, there is a positive constant $\rho_3$ such that
\[
p(1 + p\kappa_\pm) e^{-\lambda p} \leq \frac{\beta_\eta}{(N - 1)A\kappa_\pm^2} \quad \text{for any } p \geq \rho_3.
\] (27)
Take $T_2 \in (-\infty, T_1]$ so that
\[
p_\pm(t) > \max\{\rho_2, \rho_3\} \quad \text{for any } t < T_2.
\] (28)

**Lemma 3.2.** For any $\eta \in (\delta_*/4, \delta_*)$, $t < T_2$ and $x \in \mathbb{R}^N$ that satisfies $H(x) \geq 1$, $|H(x) - p_\pm(t)| \geq R_*$, the following inequalities hold:
\[
\pm I_{1,\pm} \geq - \frac{\beta_\eta}{(1 + p_\pm \kappa_\pm)^2}
\] 
\[
I_{2,\pm} = 0,
\]
\[
\pm I_{3,\pm} \geq \frac{\beta_\eta}{(1 + \kappa_\pm p_\pm)^2}.
\]

**Proof.** Recall that $|\nabla H(x)| \equiv 1$ where $H(x) \geq 1$. This yields $I_{2,\pm} = 0$. Using (15) and Lemma 2.2, one gets
\[
I_{1,\pm} = \left( \frac{(N - 1)\kappa_\pm}{1 + p_\pm(t)\kappa_\pm} \pm \frac{\sigma}{1 + p_\pm(t)\kappa_\pm} \right) \Phi'
\]
\[
= \left( \frac{(N - 1)\kappa_\pm}{1 + p_\pm \kappa_\pm} - \frac{(N - 1)\kappa_\pm}{1 + H\kappa_\pm} \right) + \frac{(N - 1)\kappa_\pm}{1 + H\kappa_\pm} \pm \kappa_\pm \frac{\tilde{\eta}(s(x))}{1 + H\kappa_\pm(s(x))} \pm \frac{\sigma}{(1 + p_\pm \kappa_\pm)^2} \Phi'
\]
where $H = H(x)$ and $p_\pm = p_\pm(t)$ for simplicity. If $H(x) - p_\pm \geq 0$, then this results in $I_{1,\pm} \geq 0$. Only the case where $H(x) - p_\pm < 0$ needs to be considered. In this case, one gets
\[
I_{1,\pm} \geq - \frac{(N - 1)\kappa_\pm^2 (p_\pm - H(x))}{(1 + p_\pm \kappa_\pm)(1 + H(x)\kappa_\pm)} A e^{-\lambda(p_\pm - H(x))}.
\]
To estimate the right hand side of the above inequality, the auxiliary function is set as follows:
\[
g(s) := \frac{se^{-\lambda s}}{1 + (p_\pm - s)\kappa_\pm},
\]
for $s \in [R_*, p_+]$ where $p_+ > R_* > 0$. The simple calculation leads us to

$$g'(s) = \frac{[\lambda \kappa_+ s^2 - \lambda (1 + \kappa_+ p_+) s + 1 + \kappa_+ p_+] e^{-\lambda s}}{(1 + (p_+ - s) \kappa_+)^2}.$$

The roots of the numerator are as follows:

$$s_1 = \left( 1 - \sqrt{1 - \frac{4 \kappa_+}{\lambda (1 + \kappa_+ p_+)}} \right) \frac{1 + \kappa_+ p_+}{2 \kappa_+},$$

$$s_2 = \left( 1 + \sqrt{1 - \frac{4 \kappa_+}{\lambda (1 + \kappa_+ p_+)}} \right) \frac{1 + \kappa_+ p_+}{2 \kappa_+},$$

By (25) and (28),

$$1 + \kappa_+ p_+ \geq \frac{4 \kappa_+}{\lambda}.$$

This implies that

$$\frac{1}{\lambda} \leq s_1 = \frac{1}{1 + \sqrt{1 - \frac{4 \kappa_+}{\lambda (1 + \kappa_+ p_+)}}} \cdot \frac{2}{\lambda} \leq \frac{2}{\lambda} < R_*.$$

Therefore,

$$\max_{R_* \leq s \leq p_+} g_1(s) = \max\{g(R_*), g(p_+)\} = \max \left\{ \frac{R_* e^{-\lambda R_*}}{1 + (p_+ - R_*) \kappa_+} p_+ e^{-\lambda p_+} \right\}.$$

By (25), $1 + (p_+ - R_*) \kappa_+ \geq 1 + p_+ \kappa_+ / 2 \geq (1 + p_+ \kappa_+ / 2).$ Since (11) implies that

$$2AR_* \kappa_+^2 e^{-\lambda R_*} \leq \beta_* \eta$$

for $\eta \in (\delta_*/4, \delta_*)$, one gets

$$\frac{R_* e^{-\lambda R_*}}{1 + (p_+ - R_*) \kappa_+} \leq \frac{2AR_* \kappa_+^2 e^{-\lambda R_*}}{1 + \kappa_+ p_+} \leq \frac{\beta_* \eta}{2AR_* \kappa_+^2 (1 + \kappa_+ p_+)}.$$

By (27), the following holds:

$$p_+ e^{-\lambda p_+} \leq \frac{\beta_* \eta}{(N - 1) A \kappa_+^2 (1 + \kappa_+ p_+)}.$$

Gathering these two inequalities implies

$$\max_{R_* \leq s \leq p_+} g(s) \leq \frac{\beta_* \eta}{(N - 1) A \kappa_+^2 (1 + \kappa_+ p_+)}.$$

As a result,

$$I_{1,+} \geq -\frac{(N - 1) A \kappa_+^2}{1 + p_+ \kappa_+} g(p_+ - H(x)) \geq -\frac{\beta_* \eta}{(1 + \kappa_+ p_+)^2}.$$ 

In addition, the case $I_{1,-}$ can be treated similarly.

The estimate for $I_{3,\pm}$ can be obtained as in Lemma 3.1; this proof has been omitted.

Next, consider case (iii).
Lemma 3.3. For any \( \eta \in (0, \delta_\ast) \) and any \( x \in \mathbb{R}^N \) that satisfies \( H(x) \geq 1, |H(x) - p_{\pm}| < R_\ast \) and \( t < T_2 \), the following inequalities hold:
\[
\pm I_{1, \pm} \geq \frac{\sigma - (N - 1)\kappa_{\pm}^2 R_\ast (1 + \kappa_{\pm} R_\ast)}{(1 + \kappa_{\pm} p_{\pm})^2} \gamma_\ast,
\]
\[
I_{2, \pm} = 0,
\]
\[
\pm I_{3, \pm} \geq -\frac{M_\eta}{(1 + \kappa_+ p_\ast)^2}.
\]

Proof. Since \( |\nabla H(x)| \equiv 1 \), \( I_{2, \pm} = 0 \). As in Lemma 3.2, one gets
\[
I_{1, \pm} = \left( \frac{(N - 1)\kappa_{\pm}^2 (H - p_{\pm})}{(1 + \kappa_{\pm} p_{\pm})(1 + \kappa_{\pm} R_\ast)} \pm \frac{\sigma}{(1 + \kappa_{\pm} p_{\pm})^2} + \sum_{j=1}^{N-1} \frac{\kappa_{\pm} - \kappa_j(s)}{(1 + H \kappa_{\pm})(1 + H \kappa_j(s))} \right) \Phi'.
\]
From \( \kappa_\pm \leq \kappa_j(s(x)) \leq \kappa_\ast \) \((j = 1, \cdots, N - 1)\) it follows that
\[
\pm I_{1, \pm} \geq \left( \frac{\sigma}{(1 + \kappa_{\pm} p_{\pm})^2} - \frac{(N - 1)\kappa_{\pm}^2 R_\ast}{(1 + \kappa_{\pm} p_{\pm})(1 + \kappa_{\pm} R_\ast)} \right) \Phi'.
\]
Since \( p_{\pm} < H + R_\ast, p_{\pm} > 0 \) and \( H > 0 \), the following inequality holds:
\[
\frac{1 + \kappa_{\pm} p_{\pm}}{1 + \kappa_{\pm} R_\ast} \leq 1 + \kappa_{\pm} R_\ast \quad \text{for} \ |H - p_{\pm}| < R_\ast.
\]
This implies that
\[
\pm I_{1, \pm} \geq \frac{\sigma - (N - 1)\kappa_{\pm}^2 R_\ast (1 + \kappa_{\pm} R_\ast)}{(1 + \kappa_{\pm} p_{\pm})^2} \Phi' \geq \frac{\sigma - (N - 1)\kappa_{\pm}^2 R_\ast (1 + \kappa_{\pm} R_\ast)}{(1 + \kappa_{\pm} p_{\pm})^2} \gamma_\ast.
\]
For the estimate of \( I_{3, \pm} \), one has
\[
\pm I_{3, \pm} \geq \pm \left\{ f(\Phi) - f(\Phi \pm \eta(1 + \kappa_{\pm} p_{\pm})^{-2}) \right\}
\]
by \( p_{\pm}' < 0 \).
\[
\pm I_{3, \pm} \geq - \int_0^1 f'(\Phi \pm \theta \eta(1 + \kappa_{\pm} p_{\pm})^{-2}) d\theta \cdot \eta(1 + \kappa_{\pm} p_{\pm})^{-2}
\geq -M_\eta \eta(1 + \kappa_{\pm} p_{\pm})^{-2}.
\]
The proof is complete. \( \square \)

Proof of Theorem 1.1. Let \( K_0 \) be as in Theorem 1.1 and take positive constants \( \kappa_{\pm} \) and \( \delta_\ast \) that satisfy (14) and (8) respectively. The constant \( R_\ast \) is chosen so as to satisfy (11) and the constant \( \gamma_\ast \) is determined by (12). Set \( \eta = \delta_\ast/2 \) for simplicity. By the above parameters, the positive constant \( \sigma \) is chosen so as to satisfy
\[
\sigma \geq \frac{M_\eta}{\gamma_\ast} + (N - 1)\kappa_{\pm}^2 R_\ast (1 + \kappa_{\pm} R_\ast).
\]  
(29)
By taking a positive constant \( \rho_1 \) as in (16), \( p_\pm(t) \) can be defined as in Lemma 2.3. From this, one can choose positive constants \( \rho_2 \) and \( \rho_3 \) as in (25) and (27) respectively. Moreover, the constant \( \rho_4 \) is so large that
\[
(1 + \kappa_+ \rho_4)^2 e^{-\lambda \rho_4} \leq \frac{\beta_\ast \eta}{A \{(|c_0| + M_\ast)(M_{\ell_+}^2 + 1) + M_\ast \}} e^{\lambda}.
\]  
(30)
Then, there is a constant \( T_3 \) such that \( p_\pm(t) > \max\{\rho_1, \rho_2, \rho_3, \rho_4\} \) for all \( t < T_3 \). Now it will be shown that \( U_+ \) is a supersolution and \( U_- \) is a subsolution for \( t < T_3 \).
First, consider the case where \( H(x) \leq 1 \). By Lemma 3.1 and (28), one obtains
\[
\pm F[U_{\pm}] \geq \frac{\beta_s \eta}{(1 + \kappa_\pm p_\pm)^2} - \left\{ |c_0| (M_H^2 + 1) + M_H \right\} \lambda \left( p_{\pm} - 1 \right) - M_f (M_H^2 + 1) A e^{-\lambda \left( p_{\pm} - 1 \right)}
\]
\[
= \frac{1}{(1 + \kappa_\pm p_\pm)^2} \left[ \beta_s \eta - \left\{ |c_0| + M_f (M_H^2 + 1) + M_H \right\} \lambda (1 + \kappa_\pm p_\pm)^2 e^{-\lambda \left( p_{\pm} - 1 \right)} \right]
\]
\[
\geq \frac{1}{(1 + \kappa_\pm p_\pm)^2} \left[ \beta_s \eta - \left\{ |c_0| + M_f (M_H^2 + 1) + M_H \right\} A e^{\lambda \left( 1 + \kappa_\pm p_\pm \right)^2 e^{-\lambda \left( p_{\pm} - 1 \right)}} \right].
\]

The assumption (30) guarantees us that \( F[U_{\pm}] \geq 0 \) and \( F[U_{\pm}] \leq 0 \) in \( H(x) \leq 1 \).

Next, consider the case where \( H(x) \geq 1 \) and \( |p_\pm - H(x)| \geq R_* \). Lemma 3.2 implies that
\[
\pm F[U_{\pm}] \geq \frac{\beta_s \eta}{(1 + \kappa_\pm p_\pm)^2} - \frac{\beta_s \eta}{(1 + p_\pm \kappa_\pm)^2} = 0.
\]

Finally, consider the case where \( H(x) \geq 1 \) and \( |p_\pm - H(x)| < R_* \). Lemma 3.3 and (29) imply that
\[
\pm F[U_{\pm}] \geq \frac{(\sigma - (N - 1) \kappa_\pm^2 R_* (1 + \kappa_\pm R_*)) \gamma_* - M_f \eta}{(1 + \kappa_\pm p_\pm)^2} \geq 0.
\]
Thus, \( U_{\pm} \) is a supersolution and \( U_{\pm} \) is a subsolution for \( t < T_3 \).

One can see that \( U_{-} \leq U_{+} \) for any \( x \in \mathbb{R}^N \) and \( t < T_3 \). Indeed,
\[
U_{+}(x, t) - U_{-}(x, t)
= \Phi(H(x) - p_+(t)) - \Phi(H(x) - p_-(t)) + \frac{\eta}{(1 + \kappa_+ p_+(t))^2} + \frac{\eta}{(1 + \kappa_- p_-(t))^2}
\geq \int_0^1 \Phi'(H(x) - \theta p_+(t) - (1 - \theta) p_-(t)) d\theta (p_-(t) - p_+(t)) > 0
\]
by Lemma 2.3. Therefore, there is an entire solution \( U(x, t) \) between them. Namely,
\[
\Phi(H(x) - p_-(t)) - \frac{\eta}{(1 + \kappa_- p_-(t))^2} \leq U(x, t) \leq \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+ p_+(t))^2}.
\]

Indeed, let \( u(x, t; s, u_0) \) be a solution of (1) with \( u(x, s; s, u_0) = u_0(x) \). It immediately follows from (31) that
\[
\Phi(H(x) - p_-(t)) - \frac{\eta}{(1 + \kappa_- p_-(t))^2} \leq u(x, t; s, \Phi(H(\cdot) - p_-(s)) - \frac{\eta}{(1 + \kappa_- p_-(s))^2})
\]
\[
\leq \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+ p_+(t))^2}
\]
for any \( s < t < T_3 \). By letting \( s \to -\infty \), one gets the entire solution
\[
U(x, t) = \lim_{s \to -\infty} u(x, t; s, \Phi(H(\cdot) - p_-(s)) - \frac{\eta}{(1 + \kappa_- p_-(s))^2}),
\]
which satisfies (31) when \( t < T_3 \). See [21, 26, 6] for the details.

Next, show
\[
\lim_{t \to -\infty} \inf_{p > 0} \sup_{x \in \mathbb{R}^N} \left| U(x, t) - \Phi(h(x) - p) \right| = 0.
\]
Indeed, the following inequality holds:
\[
|U(x, t) - \Phi(h(x) - p)| \leq \max \left\{ \left| \Phi(H(x) - p_-(t)) - \frac{\eta}{(1 + \kappa_-p_-(t))^2} - \Phi(h(x) - p) \right|, \right. \\
\left. \left| \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+p_+(t))^2} - \Phi(h(x) - p) \right| \right\}.
\]
As a result, this yields
\[
\inf_{p > 0} \left| \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+p_+(t))^2} - \Phi(h(x) - p) \right| \\
\leq \left| \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+p_+(t))^2} - \Phi(h(x) - p_+(t)) \right| \\
\leq \sup_{\pi \in \mathcal{K}_1} \left| \Phi(H(x) - p_+(t)) - \Phi(h(x) - p_+(t)) \right| + \frac{\eta}{(1 + \kappa_+p_+(t))^2}
\]
by the definition of \( H \). By (9), one obtains
\[
\inf_{p > 0} \left| \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+p_+(t))^2} - \Phi(h(x) - p) \right| \\
\leq 2Ae^{-\lambda(p_+(t)-1)} + \frac{\eta}{(1 + \kappa_+p_+(t))^2} \to 0
\]
as \( t \to -\infty \). The estimate for \( |\Phi(H(x) - p_-(t)) - \eta/(1 + \kappa_-p_-(t))^2 - \Phi(h(x) - p)| \)
can be shown similarly. Hence, the proof of the theorem is complete. \( \square \)

4. **Radially asymmetric entire solutions.** This section provides an example of a radially asymmetric entire solution of (1) when \( N \geq 2 \). Let \( a_1, a_2 \) be positive constants. Assume that \( 0 < a_1 < a_2 \). Define \( \Gamma_0 \) by
\[
\Gamma_0 := \left\{ (a_1 \cos t, a_2 \sin x' \sin t) \in \mathbb{R}^N \mid 0 \leq t \leq \pi, x' \in \mathbb{R}^{N-1}, |x'| = 1 \right\}.
\]
which is an ellipsoid. It is easily checked that the unit normal vector at \((a_1 \cos t, a_2 \sin x' \sin t) \in \Gamma_0\) is
\[
\nu = \left( \frac{a_2 \cos t}{\sqrt{a_1^2 \sin^2 t + a_2^2 \cos^2 t}}, \frac{a_1 \sin t}{\sqrt{a_1^2 \sin^2 t + a_2^2 \cos^2 t}} \right).
\]
The mean curvature of \( \Gamma_0 \) at \((a_1 \cos t, a_2 \sin x' \sin t) \in \Gamma_0\) is calculated as follows:
\[
\kappa = \frac{1}{N-1} \left( \frac{a_1 \cos \theta}{(a_1^2 \sin^2 t + a_2^2 \cos^2 t) \cos t} + \frac{N-2}{a_2 \sin t} \sin \theta \right) \\
= \frac{1}{N-1} \left( \frac{a_1 a_2}{(a_1^2 \sin^2 t + a_2^2 \cos^2 t)^{3/2}} + \frac{(N-2)a_1}{a_2^2 \sin^2 t + a_2^2 \cos^2 t} \right),
\]
where \( \tan \theta = a_1 \sin t/(a_2 \cos t) \). For example, see [4, Section 2] for the mean curvature of a surface of revolution in \( \mathbb{R}^N \). Set
\[
\kappa_+ := \frac{1}{N-1} \left( \frac{a_2}{a_1^2} + \frac{N-2}{a_2^2} \right), \quad \kappa_- := \frac{a_1}{a_2^2}.
\]
By $0 < a_1 < a_2$, one has

$$\kappa_+ > \kappa_-.$$ 

Note that $\kappa_+$ and $\kappa_-$ are defined reversely in the case where $0 < a_2 < a_1$. Moreover,

$$\Gamma_R := \left\{ (a_1 \cos t, a_2 x' \sin t) \middle| \frac{(a_2 \cos t, a_2 x' \sin t)}{\sqrt{a_1^2 \sin^2 t + a_2^2 \cos^2 t}} R \in \mathbb{R}^N \right\}$$

It can be observed that the difference between the semi-major axes and the semi-minor axis is always

$$(a_2 + R) - (a_1 + R) = a_2 - a_1.$$ 

**Lemma 4.1.** For any positive constant $R \in [(a_2 - a_1)/3, \infty)$, the region $K_{R+(a_2-a_1)/3} \setminus K_{R-(a_2-a_1)/3}$ in $\mathbb{R}^N$ does not include any sphere surrounding $K_{R-(a_2-a_1)/3}$.

**Proof.** Assume that there is a sphere $C_0$ centered at $x_0 \in \mathbb{R}^N$ with radius $r_0$ which includes $K_{R-(a_2-a_1)/3}$ and is included in $K_{R+(a_2-a_1)/3} \setminus K_{R-(a_2-a_1)/3}$. Then the diameter $2r_0$ must be greater than $2(a_2 + R - (a_2 - a_1)/3)$ because $K_{R-(a_2-a_1)/3} \subset \text{Conv } C_0$. Similarly, due to $C_0 \subset K_{R+(a_2-a_1)/3}$, $2r_0 \leq 2(a_1 + R + (a_2 - a_1)/3)$. Therefore one gets

$$a_2 + R - \frac{a_2 - a_1}{3} < a_1 + R + \frac{a_2 - a_1}{3},$$

which implies a contradiction. 

**Proof of Corollary 1.2.** Let $U$ be the entire solution as in Theorem 1.1 with the ellipsoid $\Gamma_0$ given by (32). To show that $U$ is not radially symmetric, consider the level set of $U$. By (31), the level set $\{ x \in \mathbb{R}^N \mid U(x, t) = a \}$ satisfies

$$\Phi(H(x) - p_-(t)) - \frac{\eta}{(1 + \kappa_- p_-(t))^2} \leq a \leq \Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+ p_+(t))^2}.$$ 

By (12), one gets

$$\Phi(H(x) - p_-(t) - \frac{\gamma_+ \eta}{(1 + \kappa_+ p_+(t))^2}) \leq \Phi(H(x) - p_-(t)) - \frac{\eta}{(1 + \kappa_- p_-(t))^2}$$

and

$$\Phi(H(x) - p_+(t)) + \frac{\eta}{(1 + \kappa_+ p_+(t))^2} \leq \Phi(H(x) - p_+(t) + \frac{\gamma_+ \eta}{(1 + \kappa_- p_-(t))^2}).$$

Gathering the above inequalities yields

$$\Phi(H(x) - p_-(t) - \frac{\gamma_+ \eta}{(1 + \kappa_- p_-(t))^2}) \leq a \leq \Phi(H(x) - p_+(t) + \frac{\gamma_+ \eta}{(1 + \kappa_+ p_+(t))^2}).$$

These inequalities implies that

$$H(x) - p_-(t) - \frac{\gamma_+ \eta}{(1 + \kappa_- p_-(t))^2} \leq \Phi^{-1}(a) \leq H(x) - p_+(t) + \frac{\gamma_+ \eta}{(1 + \kappa_+ p_+(t))^2}$$

and that the level set is included in

$$K_{\Phi^{-1}(a) + p_-(t) + \gamma_- \eta(1 + \kappa_- p_-(t))^{-2}} \setminus K_{\Phi^{-1}(a) + p_+(t) - \gamma_+ \eta(1 + \kappa_- p_-(t))^{-2}}.$$

It follows from Lemma 2.3 that

$$p_-(t) + \gamma_- \eta(1 + \kappa_- p_-(t))^{-2} - \left\{ p_+(t) - \gamma_+ \eta(1 + \kappa_+ p_+(t))^{-2} \right\} \to 0$$

as $t \to -\infty$. Thus the level set must be included in $K_{R+(a_2-a_1)/3} \setminus K_{R-(a_2-a_1)/3}$ with some $R$ if $-t$ is large enough. Lemma 4.1 implies that the level set is not
radially symmetric. Therefore, $U$ is a radially asymmetric entire solution of (1) in $\mathbb{R}^N$ when $N \geq 2$.

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