ON AFFINE RIGIDITY

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Abstract. We study the properties of affine rigidity of a hypergraph and prove a variety of fundamental results. First, we show that affine rigidity is a generic property (i.e., depends only on the hypergraph, not the particular embedding). Then we prove that a graph is generically neighborhood affinely rigid in $d$-dimensional space if it is $(d+1)$-vertex-connected. We also show neighborhood affine rigidity of a graph implies universal rigidity of its squared graph. Our results, and affine rigidity more generally, have natural applications in point registration and localization, as well as connections to manifold learning.

1. Introduction

Suppose one has a number of overlapping “scans” of a set of points in some space, and that the corresponding points shared between scans have been identified. One naturally may want to register these scans and merge them together into a single configuration [23, inter alia]. Such a merging problem is called a realization problem. The study of the uniqueness of the solutions to such realization problems is known as rigidity.

We model the combinatorics of this problem using a hypergraph $\Theta$, with vertices representing the points, and hyperedges representing the sets of points in each scan. The geometry of the problem is modeled with a configuration $p$, associating each vertex with a point in space.

One natural setting is the Euclidean setting, where the scans are known to be related by Euclidean transforms. In this case it is sufficient to study just the case of a graph, where we think of each edge as its own scan with only 2 points. Unfortunately, many of the Euclidean problems are NP-HARD [18]. In this paper, we study what happens when one relaxes the problem to the affine setting, that is, one assumes that the scans are known to be related by affine transforms. Under this relaxation, much of the analysis reduces to linear algebra, and uniqueness questions reduce to rank calculations. We prove a variety of fundamental results about this type of rigidity and also place it in the context of other rigidity classes such as global rigidity and universal rigidity.

We also specifically investigate the case of hypergraphs $\Theta$ that arises by starting with an input graph $\Gamma$, and considering each one-ring in $\Gamma$ as a hyperedge in $\Theta$. We call such a hypergraph the neighborhood hypergraph of $\Gamma$. Such neighborhood hypergraphs naturally arise when studying molecules [13], when applying a divide and conquer approach to sensor network localization [20] and in machine learning [16].
1.1. **Summary of Results.** We start by describing how affine rigidity in $\mathbb{R}^d$ is fully characterized by the kernel size of one of its associated “affinity matrices”. (This result was first shown by Zha and Zhang [26].) We show how this implies a number of interesting corollaries including the fact that affine rigidity is a generic property. That is, given a hypergraph $\Theta$ and dimension $d$, either all generic embeddings of $\Theta$ are affinely rigid in $\mathbb{R}^d$ or all generic embeddings are affinely flexible in $\mathbb{R}^d$. The specific geometric positions of the vertices are irrelevant to this property, as long as they are in sufficiently general position. Thus we call such a hypergraph *generically affinely rigid* in $\mathbb{R}^d$.

Next we relate affine rigidity in $\mathbb{R}^d$ to the related notion of universal Euclidean rigidity. A framework is universally Euclideanly rigid if it is rigid (in the Euclidean sense) in any dimension. In this context, we prove that affine rigidity in $\mathbb{R}^d$ implies universal Euclidean rigidity.

We then prove the following sufficiency result: if a graph $\Gamma$ is $d + 1$ (vertex) connected, then its neighborhood hypergraph is generically affinely rigid in $\mathbb{R}^d$; alternatively, we say that the graph $\Gamma$ itself is generically neighborhood affinely rigid in $\mathbb{R}^d$. In particular we will show that almost every *non-symmetric equilibrium stress matrix* for any generic embedding of $\Gamma$ in $\mathbb{R}^d$ will have co-rank $d + 1$ (i.e., rank $v - d - 1$).

Putting these two results together, we show that if a graph is $d + 1$ connected, then any generic embedding of its square graph in $\mathbb{R}^d$ is universally rigid. This result is interesting, as very few families of graphs have been proven to be generically universally rigid.

We give examples showing that many of the implications proved in this paper do not reverse.

Finally we discuss some of the motivating applications.

The main properties of frameworks of graphs and their implications proven in this paper are summarized below.

| Property | Graph ... |
|----------|-----------|
| GGR      | is generically globally rigid in $\mathbb{R}^d$ |
| DP1C     | is $d + 1$ connected ([10]) |
| GNSESM   | generically has non-symmetric equilibrium stress matrix of rank $v - d - 1$ (Proposition 5.8) |
| GNAR     | is generically neighborhood affine rigid in $\mathbb{R}^d$ (Proposition 5.9) |
| GNUR     | is generically neighborhood universally rigid in $\mathbb{R}^d$ (Corollary 4.2) |
| GNGR     | is generically neighborhood globally rigid in $\mathbb{R}^d$ (by definition) |

2. **A rigidity zoo**

In this paper we will consider several different rigidity theories. They all fit in to a unifying framework, which we now explain.

Most generally, rigidity questions (of any type) ask if all of the geometric information about a set of points is determined by information from small subsets. In the usual Euclidean rigidity problem, we measure the distances between pairs of points. However, in other cases it is not enough to consider pairs of points for the small subsets; as such, we need to consider hypergraphs rather than just graphs.
Definition 2.1. A hypergraph $\Theta$ is a set of $v$ vertices $V(\Theta)$ and $h$ hyperedges $E(\Theta)$, where $E(\Theta)$ is a set of subsets of $V(\Theta)$. We will typically write just $V$ or $E$, dropping the hypergraph $\Theta$ from the notation.

There are natural ways to pass from a hypergraph back and forth to a graph.

Definition 2.2. Given a hypergraph $\Theta$, define its body graph $B(\Theta)$ as follows. For each vertex in $\Theta$, we have a vertex in $B(\Theta)$. For each hyperedge $h$ in $\Theta$ and each pair of vertices in $h$ we have an edge in $B(\Theta)$.

See Figure 1 for an example.

Definition 2.3. Given a graph $\Gamma$, define its neighborhood hypergraph, written as $N(\Gamma)$ as follows. For each vertex in $\Gamma$, we have an associated vertex in $N(\Gamma)$. For each vertex in $\Gamma$ we have a hyperedge in $N(\Gamma)$ consisting of that vertex and its neighbors in $\Gamma$.

Definition 2.4. Given a graph $\Gamma$, its squared graph $\Gamma^2$ is obtained by adding to $\Gamma$ an edge between two vertices $i$ and $j$ if $i$ and $j$ share some neighbor vertex $k$.

Lemma 2.5. For any graph $\Gamma$, $B(N(\Gamma)) = \Gamma^2$.

Proof. Immediate from the definitions. (See Figure 2 for an example.)

Definition 2.6. A $k$-hypergraph $\Theta$ is a hypergraph where each hyperedge has exactly $k$ vertices. For any $k \in \mathbb{N}$ and hypergraph $\Theta$, let $B_k(\Theta)$ be the $k$-hypergraph whose hyperedges are all the subsets $S$ of size $k$ of vertices that are contained together in at least one hyperedge of $\Theta$:

$$E(B_k(\Theta)) = \{ S \mid \exists h \in E(\Theta), S \subset h, |S| = k \}.$$  

For a vertex set $S$, the complete $k$-hypergraph on $S$, written $K_k(S)$, is the $k$-hypergraph whose hyperedges are all $\binom{|S|}{k}$ subsets of $S$ of size $k$.

For instance, a 2-hypergraph is a graph, and $B_2(\Theta)$ is just the body graph $B(\Theta)$. 
Definition 2.7. A configuration of the vertices $V(\Theta)$ of a hypergraph in $\mathbb{R}^d$ is a mapping $p$ from $V(\Theta)$ to $\mathbb{R}^d$. Let $C^d(\mathbb{V})$ be the space of configurations in $\mathbb{R}^d$. For $p \in C^d(\mathbb{V})$ and $u \in V(\Theta)$, we write $p(u) \in \mathbb{R}^d$ for the image of $u$ under $p$.

A framework $\rho = (p, \Theta)$ of a hypergraph is the pair of a hypergraph and a configuration of its vertices. $C^d(\Theta)$ is the space of frameworks $(p, \Theta)$ with hypergraph $\Theta$ and $p \in C^d(V(\Theta))$. We may also write $\rho(u)$ for $p(u)$ where $\rho = (p, \Theta)$ is a framework of the configuration $p$.

A framework of a hypergraph has also been called a body-and-multipin framework [13].

Definition 2.8. Let $G$ be a group acting on $\mathbb{R}^d$, for instance the group $\text{Eucl}(d)$ of all Euclidean isometries of $\mathbb{R}^d$. Two frameworks $\rho, \sigma \in C^d(\Theta)$ are $G$-equivalent if for each hyperedge $h \in E(\Theta)$, the positions of the vertices in $\rho$ can be mapped to their positions in $\sigma$ by an element of $G$ depending only on $h$. That is, for each $h \in E$, there is a $g_h \in G$ so that for each $u \in h$, we have $g_h(\rho(u)) = \sigma(u)$. The two are $G$-congruent if the positions of all the vertices in $\rho$ can be mapped to their positions in $\sigma$ by a single element $g \in G$ (not depending on $h$).

A framework $\rho \in C^d(\Theta)$ is globally $G$-rigid if for any other framework $\sigma \in C^d(\Theta)$ that is $G$-equivalent to $\rho$, we also have that $\rho$ and $\sigma$ are $G$-congruent. Otherwise we say that $\rho$ is globally $G$-flexible in $\mathbb{R}^d$.

Similarly, a framework $\rho \in C^d(\Theta)$ is locally $G$-rigid in $\mathbb{R}^d$ if there is a small neighborhood $U$ of $\rho$ in $C^d(\Theta)$ so that for any $\sigma \in U$ that is $G$-equivalent to $\rho$, we also have that $\rho$ and $\sigma$ are $G$-congruent. Otherwise we say that $\rho$ is locally $G$-flexible in $\mathbb{R}^d$.

A related notion of group based rigidity has been explored in the computer aided design literature [19].

In this paper, we are mainly concerned with the cases when $G$ is either Eucl($d$) or Aff($d$), the group of all invertible affine linear maps of $\mathbb{R}^d$, in which case we speak about Euclidean or affine rigidity, respectively. But there are other interesting possibilities, like the group of
projective transformations. Another interesting case is when \( G \) is the group of dilations and translations of \( \mathbb{R}^d \) (with no rotations); this gives the theory of parallel-line redrawings [25].

In this terminology, Euclidean rigidity is the default: if the group \( G \) is not specified, it is the Euclidean group. In much of the rigidity literature, local rigidity is the default, and the qualifier “local” is dropped. However, in this paper this distinction is important and we will write “local” or “global” when the distinction is meaningful.

**Lemma 2.9.** A framework \( (p, \Theta) \) is locally (resp. globally) Euclideanly rigid iff the body framework \( (p, B(\Theta)) \) is locally (resp. globally) Euclideanly rigid.

*Proof.* This easily follows from the fact that, for each hyperedge \( h \in \mathcal{E}(\Theta) \), the complete graph on \(|h|\) vertices is globally rigid. \(\square\)

Thus we only need to consider Euclidean rigidity for frameworks of graphs, not hypergraphs.

In the next section (Corollary 3.5) we will see that a framework is locally affinely rigid iff it is globally affinely rigid. Thus we can drop the local/global distinction for affine rigidity.

**Definition 2.10.** For \( d < d' \), we view \( C^d(V) \) as contained in \( C^{d'}(V) \) by the inclusion of \( \mathbb{R}^d \) as the first \( d \) dimensions of \( \mathbb{R}^{d'} \).

Let \( G \) be a family of groups \( G_d \) acting on \( \mathbb{R}^d \), so that for \( d < d' \), \( G_d \) is the subgroup of \( G_{d'} \) that fixes \( \mathbb{R}^d \) as a subset of \( \mathbb{R}^{d'} \). A framework \( \rho \in C^d(\Theta) \) is universally locally (resp. globally) \( G \)-rigid if it is locally (resp. globally) \( G \)-rigid as a framework in \( C^{d'}(\Theta) \) for all \( d' \geq d \).

Note that universal rigidity of any sort implies rigidity of the same sort.

**Lemma 2.11.** A framework \( p \in C^d(\Theta) \) is universally globally Euclideanly rigid iff it is universally locally Euclideanly rigid.

*Proof.* For any two equivalent frameworks \( \rho \) in \( C^d(\Theta) \) and \( \rho' \) in \( C^{d'}(\Theta) \), Bezdek and Connelly [3] exhibited an explicit flex between \( \rho \) and \( \rho' \) in \( C^{d+d'}(\Theta) \). In particular, if \( \rho \) is a \( d \)-dimensional framework with a equivalent but non-congruent framework in \( d' \) dimensions, then this flex shows that \( \rho \) is not locally rigid in \( \mathbb{R}^{d+d'} \). \(\square\)

Thus we can also drop the local/global distinction in the case of universal Euclidean rigidity.

**Definition 2.12.** A framework \( (p, \Gamma) \) of the graph \( \Gamma \) is neighborhood rigid (of any of the sorts above) if the corresponding framework \( (p, N(\Gamma)) \) of the neighborhood hypergraph is rigid (of the same sort).

For instance, Lemmas 2.9 and 2.5 tell us that neighborhood Euclidean rigidity of \( (\rho, \Gamma) \) is equivalent to the Euclidean rigidity of \( (\rho, \Gamma^2) \).

**Definition 2.13.** A configuration \( p \) in \( C^d(V) \) is generic if the coordinates do not satisfy any non-zero algebraic equation with rational coefficients. A framework is generic if its configuration is generic.

A property is generic in \( \mathbb{R}^d \) if, for every (hyper)graph, either all generic frameworks in \( C^d(\Theta) \) have the property or none do. For instance, local and global Euclidean rigidity in \( \mathbb{R}^d \) are both generic properties of graphs and therefore for hypergraphs as well [2,8]. For any property \( P \) (generic or not) of frameworks, a (hyper)graph \( \Theta \) is generically \( P \) in \( \mathbb{R}^d \) if every
generic framework in \( C^d(\Theta) \) has \( P \). (For a non-generic property like universal Euclidean rigidity, there are (hyper)graphs that are neither generically \( P \) or generically not \( P \).)

The above adjectives are to be read in the reverse of the order we have defined them. Thus, for any framework, we may talk about

\[(\text{generic}/\emptyset) \ (\text{universal}/\emptyset) \ (\text{local}/\text{global}) \ (\text{Euclidean}/\text{affine}) \text{ rigidity.}\]

where by \( \emptyset \) we mean that this term has been dropped.

3. **Affine Rigidity in \( \mathbb{R}^d \)**

We now move on the main focus of this work, affine rigidity, as defined in the previous section. Though the definitions start from a different point of view, this notion of affine rigidity is identical to the one defined by Zha and Zhang [26] and the concept is also informally mentioned by Brand [4]. Additionally, Theorem 1 below is essentially equivalent to [26, Theorem 5.2]. Our contribution here, described by the corollaries, is showing how affine rigidity fits in to the general scheme of rigidity problems.

**Lemma 3.1.** Any general position framework of a complete \((d + 2)\)-hypergraph is affinely rigid.

**Proof.** Clear. □

**Proposition 3.2.** A framework \((p, \Theta)\) in general position is affinely locally (resp. globally) rigid iff the associated framework \((p, B_{d+2}(\Theta))\) is affinely locally (resp. globally) rigid.

(Compare Lemma 2.9.)

**Proof.** First consider a hyperedge with less than \(d + 2\) vertices. Any two generic embeddings in \(\mathbb{R}^d\) of these vertices are related by a \(d\)-dimensional affine transform. Therefore this hyperedge does not affect affine equivalence of these vertices, and so this hyperedge may be dropped without affecting affine rigidity. Next consider a hyperedge with \(k\) vertices, with \(k > d + 2\). By Lemma 3.1, one can replace this hyperedge with \(\binom{k}{d+2}\) hyperedges corresponding to all subsets of \(d + 2\) vertices. The framework of the new hypergraph will be affinely rigid in \(\mathbb{R}^d\) iff the original one is. □

**Definition 3.3.** An affinity matrix of a framework \((p, \Theta)\) in \(C^d(\Theta)\) is any matrix where each row encodes some affine relation between the coordinates of the vertices in a hyperedge of \((p, \Theta)\) as a homogeneous linear equation. That is, there are \(v\) columns in the matrix, the only non-zero entries in a row correspond to vertices in some hyperedge, the sum of the entries in a row is 0, and each of the coordinates of \(p\), thought of as a vector of length \(v\), is in the kernel of the matrix.

An affinity matrix is strong if it encodes all of the affinely independent relations in every hyperedge of \((p, \Theta)\).

**Lemma 3.4.** If the framework \((q, \Theta)\) is affinely equivalent to \((p, \Theta)\) then the coordinates of \(q\) are in the kernel of any affinity matrix for \((p, \Theta)\). Additionally, the converse is true if the affinity matrix is strong.

**Proof.** Clear from the definitions. □

The kernel of an affinity matrix of a framework \((p, \Theta) \in C^d(\Theta)\) always contains the subspace of \(\mathbb{R}^v\) spanned by the coordinates of \(p\) along each axis and the vector \(\vec{1}\) of all 1’s.
This corresponds to the fact that any affine image of \( p \) is equivalent to \( p \). If \( p \) is a proper \( d \)-dimensional configuration (with full \( d \)-dimensional affine span), these vectors are independent and span a \((d + 1)\)-dimensional space. In particular, a generic framework of a hypergraph with at least \( d + 1 \) vertices in \( \mathbb{R}^d \) is proper, so for such frameworks the corank of any of its affinity matrices must be no less than \( d + 1 \).

The rank of strong affinity matrices fully characterize affine rigidity.

**Theorem 1.** Let \( \Theta \) be a hypergraph with at least \( d + 1 \) vertices. Let \((p, \Theta)\) be any proper, \( d \)-dimensional framework and let \( M \) be any strong affinity matrix for \((p, \Theta)\). Then \((p, \Theta)\) is affinely rigid in \( \mathbb{R}^d \) iff \( \dim(\ker(M)) = d + 1 \).

**Proof.** By Lemma 3.4, for any other configuration \( q \) in \( C^d(\mathcal{V}) \) such that \( (q, \Theta) \) is affinely equivalent to \((p, \Theta)\), the coordinates of \( q \) must be in the kernel of \( M \). When \( \dim(\ker(M)) = d + 1 \), the kernel of \( M \) contains only one-dimensional projections of \( p \) and the all-ones vector. Thus any \((q, \Theta)\) that is affinely equivalent to \((p, \Theta)\) must in fact be affinely congruent to \((p, \Theta)\).

Conversely, if the corank is higher, then the kernel must contain an “extra” vector that is not a one-dimensional projection of \( p \). Adding any amount of this vector to one of the coordinates of \( p \) must, by Lemma 3.4, produce a \( q \) such that \((q, \Theta)\) is affinely equivalent to \((p, \Theta)\) but not congruent to it. \( \square \)

It is easy now to prove the following corollaries.

**Corollary 3.5.** If \((p, \Theta)\) is affinely globally flexible in \( \mathbb{R}^d \) then it is affinely locally flexible.

**Proof.** From the proof of Theorem 1, when \((p, \Theta)\) is affinely globally flexible in \( \mathbb{R}^d \) there is an extra vector \( \delta \) in the kernel of a strong affinity matrix, and we can add any multiple of \( \delta \) to one of the coordinates of \( p \) to get an affinely equivalent but not congruent framework. \( \square \)

**Remark 3.6.** In fact, for any two affinely equivalent frameworks \((p, \Theta)\) and \((q, \Theta)\), there is a continuous path of affinely equivalent frameworks in \( C^d(\Theta) \) connecting them, namely \(((1 - t)p + tq, \Theta)\).

**Corollary 3.7.** A framework \((p, \Theta) \in C^d(\Theta)\) is affinely rigid in \( \mathbb{R}^d \) iff it is affinely rigid when considered as a (degenerate) framework in \( \mathbb{R}^{d'} \) for \( d' \geq d \).

**Proof.** Follows from the proof of Theorem 1. \( \square \)

Thus there is no distinct notion of “universal” affine rigidity.

**Corollary 3.8.** Affine rigidity in \( \mathbb{R}^d \) is a generic property of a hypergraph.

**Proof.** The condition that \( M \) is an affinity matrix for \((p, \Theta)\) is linear in the entries in \( M \). The corollary then follows from Proposition A.1. \( \square \)

**Remark 3.9.** Though we will not pursue the details here, one can use the concept of an affinity matrix to derive an efficient randomized discrete algorithm for testing generic affine rigidity of a hypergraph in \( \mathbb{R}^d \). To do this, one needs to use integers of sufficiently many bits, and do the arithmetic modulo a suitably large prime. The details parallel those in the global rigidity case [8, Section 5].
4. Universal Euclidean Rigidity

We now turn to universal Euclidean rigidity. To begin, we need the following technical definition:

**Definition 4.1.** We say that the edge directions of a graph framework \((p, \Gamma) \in C^d(\Gamma)\) are **on a conic at infinity** if there exists a symmetric \(d\)-by-\(d\) matrix \(Q\) such that for all edges \((u, v)\) of \(\Gamma\), we have

\[
[p(u) - p(v)]^t Q [p(u) - p(v)] = 0.
\]

The edge directions of \((p, \Gamma)\) are on a conic at infinity iff there is a continuous family of \(d\)-dimensional non-Euclidean affine transforms that preserve all of the edge lengths \([6]\). This is a very degenerate situation which is very easy to rule out. For example, if in a hypergraph framework \((p, \Theta)\) some hyperedge in \(\Theta\) has vertices whose positions in \(p\) affinely span \(\mathbb{R}^d\), then the edge directions \((p, B(\Theta))\) cannot be on a conic at infinity.

**Theorem 2.** If a framework \((p, \Theta)\) of a hypergraph \(\Theta\) with \(p \in C^d(\mathcal{V})\) is affinely rigid in \(\mathbb{R}^d\), and the edge directions of \((p, B(\Theta))\) are not on a conic at infinity, then \((p, \Theta)\) is universally Euclidean rigid.

**Proof.** Let \(q \in C^{d'}(\mathcal{V})\) be a configuration with \(d' > d\) such that \((q, \Theta)\) is Euclidean equivalent in \(\mathbb{R}^{d'}\) to \((p, \Theta)\). Then \((q, \Theta)\) is affinely equivalent in \(\mathbb{R}^{d'}\) to \((p, \Theta)\). Since \((p, \Theta)\) is affinely rigid in \(\mathbb{R}^d\), from Corollary 3.7, we have that \(p\) is affinely congruent to \(q\) in \(\mathbb{R}^{d'}\) and the affine span of \(q\) must be of the same dimension as that of \(p\). Let \(R(q)\) be a rotation of \(q\) down to \(\mathbb{R}^d\). Then \(R(q)\) is affine congruent in \(\mathbb{R}^d\) to \(p\) and \((R(q), \Theta)\) is Euclidean equivalent in \(\mathbb{R}^d\) to \((p, \Theta)\).

Since the edge directions of \((p, B(\Theta))\) are not on a conic at infinity, any \(d\)-dimensional transform that is affine but not Euclidean must change some of the inter-vertex lengths within some hyperedge of \(\Theta\). Putting this together with the fact that \((R(q), \theta)\) and \((p, \Theta)\) are affinely congruent and Euclidean equivalent, we can conclude that they are Euclidean congruent. Likewise \(q\) must be Euclidean congruent to \(p\) in \(\mathbb{R}^{d'}\). Thus we conclude that \((p, \Theta)\) is universally rigid. \(\square\)

**Corollary 4.2.** Let \(\Theta\) be a hypergraph with at least \(d + 2\) vertices. If a generic framework \((p, \Theta)\) of a hypergraph \(\Theta\) with \(p \in C^d(\mathcal{V})\) is affinely rigid in \(\mathbb{R}^d\) then \((p, \Theta)\) is universally rigid.

**Proof.** Any generic framework of a hypergraph \(\Theta\) with at least \(d + 2\) vertices that is affinely rigid in \(\mathbb{R}^d\) must have at least one hyperedge \(h\) with at least \(d + 2\) vertices, whose vertices in \(p\) have a \(d\)-dimensional affine span. Thus \((p, B(\Theta))\) must include a generic framework of a \((d + 1)\)-simplex and thus cannot have edge directions at a conic at infinity. Then Theorem 2 applies. \(\square\)

There can be frameworks that are universally rigid but not affinely rigid in \(\mathbb{R}^d\). (See Figure 3.)

5. Neighborhood affine rigidity

In this section we prove the following theorem about the generic neighborhood affine rigidity of a graph.
The framework in $\mathbb{R}^2$ of the hypergraph on the left is not affinely rigid as each hyperedge (shown as a dashed ellipse) has only 3 vertices. But this framework is universally Euclidean rigid, as its body graph (right) is a Cauchy polygon.

**Theorem 3.** Let $\Gamma$ be a graph with at least $d + 1$ vertices. If $\Gamma$ is $(d + 1)$-vertex-connected, then $\Gamma$ is generically neighborhood affinely rigid in $\mathbb{R}^d$.

We will build up the proof of Theorem 3 with a series of lemmas.

**Definition 5.1.** An equilibrium stress matrix of a framework $(p, \Gamma)$ of a graph in $C^d(\Gamma)$ is a matrix $\Omega$ indexed by $V \times V$ so that

1. for all $u, w \in V$, we have $\Omega(u, w) = \Omega(w, u)$;
2. for all $u, w \in V$ with $u \neq w$ and $\{u, w\} \not\in E$, we have $\Omega(u, w) = 0$;
3. for all $u \in V$, we have $\sum_{w \in V} \Omega(u, w) = 0$; and
4. for all $u \in V$, we have $\sum_{w \in V} \Omega(u, w)p(w) = 0$.

A non-symmetric equilibrium stress matrix of a framework $(p, \Gamma)$ is a matrix that satisfies properties (2)–(4) above.

Observe first that an equilibrium stress matrix (symmetric or not) $\Omega$ of $(p, \Gamma)$ is an affinity matrix of $(p, N(\Gamma))$. From the properties of affinity matrices, the kernel of $\Omega$ always contains the subspace spanned by the coordinates of $p$ along each axis and the vector $\mathbf{1}$ of all 1’s.

**Definition 5.2.** We say that a framework of a graph in $C^d(\Gamma)$ has the convex containment property if

1. the one-ring of each vertex has an affine span of dimension $d$, and
2. except for $d + 1$ exceptional vertices, each of the other vertices in the framework is contained in the strict $d$-dimensional convex hull of its neighbors.

**Lemma 5.3.** Let $\Gamma$ be a graph with at least $d + 1$ vertices. Suppose $\Gamma$ is $(d + 1)$-connected. Then there exists a generic framework $(q, \Gamma)$ in $C^d(\Gamma)$ with the convex containment property.

**Proof.** Pick any $d + 1$ vertices to be exceptional. Constrain the exceptional vertices to fixed generic positions in $\mathbb{R}^d$ (at the vertices of a simplex). Associate generic positive weights $\omega_{ij}$ with each undirected edge $ij$. Find the “rubber band” configuration consistent with the
constrained vertices and these weights. Namely, find a framework \((r, \Gamma)\) so that each non-exceptional vertex is the weighted linear average of its neighbors:

\[
\sum_{j \in N(i)} \omega_{ij}(r(i) - r(j)) = 0,
\]

where \(N(i)\) are the neighbors of vertex \(i\). This involves solving \(d\) systems of linear equations, one for each component of \(r\). Note that the resulting configuration \(r\) may not be generic.

From [15], we know that if \(\Gamma\) is \((d + 1)\)-connected and the constraints on the exceptional vertices and the edge weights \(\omega\) are generic, then no set of \(d + 1\) vertices in \(r\) will be contained in a \((d - 1)\)-dimensional affine plane, giving us the first condition.

By construction, any non-exceptional vertex in \((r, \Gamma)\) must be contained in the convex hull of its neighbors. Again, from [15], the convex containment must be strict.

Finally, we perturb each vertex in \(\mathbb{R}^d\) to obtain a generic configuration in \(q \in C^d(\mathcal{V})\). By the first convex containment condition, the convex hull of the neighbors of a vertex has non-empty interior, so a sufficiently small perturbation will maintain both conditions. □

**Definition 5.4.** Suppose that \((q, \Gamma)\) has the convex containment property and \(\Omega\) is a non-symmetric equilibrium stress matrix for \((q, \Gamma)\). We call a row of \(\Omega\) non-exceptional if its corresponding vertex is in the strict \(d\)-dimensional convex hull of its neighbors.

**Lemma 5.5.** Let \(\Gamma\) be a graph with at least \(d + 1\) vertices. Suppose \(\Gamma\) is a \((d + 1)\)-connected graph, and we have a framework \((q, \Gamma)\) in \(C^d(\mathcal{V})\) with the convex containment property. Then there is a non-symmetric equilibrium stress matrix \(\Omega\) of \((q, \Gamma)\), such that for every non-exceptional row \(i\), we have the following property: If there is an edge connecting vertex \(i\) and vertex \(j\), then \(\Omega_{ij}\) is positive.

**Proof.** All vertices have \(d + 1\) or more neighbors. For each vertex \(i\), we can therefore find “barycentric coordinates”: non-zero edge weights \(\omega_{ij}\) on the adjoining edges so that

\[
\sum_{j \in N(i)} \omega_{ij}(q(j) - q(i)) = 0.
\]

If \(i\) is a non-exceptional vertex, due to the convex containment property we can choose the \(\omega_{ij}\) to be positive. We then choose \(\Omega_{ij} = \omega_{ij}\) for \(i \neq j\) and \(\Omega_{ii} = -\sum_j \omega_{ij}\). □

**Remark 5.6.** This lemma is false if we require the stress matrix to be symmetric, because this prevents us from choosing \(\omega_{ij}\) and \(\omega_{ji}\) independently.

**Lemma 5.7.** Let \(\Gamma\) be a graph with at least \(d + 1\) vertices. Suppose \(\Gamma\) is a \((d + 1)\)-connected, and we have a framework \((q, \Gamma)\) in \(C^d(\mathcal{V})\) with the convex containment property. Then there is a non-symmetric equilibrium stress matrix \(\Omega\) of \((q, \Gamma)\) with co-rank \(d + 1\).

**Proof.** From Lemma 5.5 we find for \((q, \Gamma)\) a non-symmetric equilibrium stress matrix \(\Omega\) with the desired positive entries. We now show that \(\Omega\) has the stated rank.

First remove the \(d + 1\) rows and columns associated with the exceptional vertices to create a smaller matrix \(\Omega'\). Due to the sign pattern assumed in \(\Omega\), as well as property (3) of any equilibrium stress matrix, \(\Omega'\) must be weakly diagonally dominant.

Let us call a vertex \(EN\) if it has an exceptional neighbor and refer to its corresponding row in \(\Omega'\) as \(EN\). Any \(EN\) row must be strictly diagonally dominant (since at least one non-zero off-diagonal entry of \(\Omega\) have been removed from this row).

Since all entries corresponding to edges are non-zero, the reducible components of \(\Omega'\) correspond to vertex subsets that remain connected after the exceptional vertices have been
removed. Each reducible component of $\Omega'$ includes such an EN row, thus $\Omega'$ must be full rank. (See, e.g., [24, Theorem 1.21].)

Since $\Omega'$ has co-rank 0, the co-rank of $\Omega$ must be at most $d + 1$. It is no less since any equilibrium stress matrix must have a $(d+1)$-dimensional kernel spanned by the coordinates of $q$ and the all-ones vector.

\begin{proposition}
Let $\Gamma$ be a graph with at least $d+1$ vertices. Suppose $\Gamma$ is $(d+1)$-connected, and $p$ is generic in $C^d(V)$. Then there is a non-symmetric equilibrium stress matrix $\Omega$ of $(p, \Gamma)$ with co-rank $d+1$.
\end{proposition}

\begin{proof}
From Lemma 5.3, there must exist a generic framework $(q, \Gamma)$ in $C^d(\Gamma)$ that has the convex containment property. From Lemma 5.7, $(q, \Gamma)$ must have a non-symmetric equilibrium stress matrix of co-rank $d+1$. Thus from Proposition A.1, any generic framework $(p, \Gamma)$ must have such a matrix as well.
\end{proof}

See Figure 4 for an example showing that the converse of Proposition 5.8 is not true. Here we show a neighborhood affinely rigid framework in $\mathbb{R}^2$ with a non-symmetric equilibrium stress matrix of co-rank $d + 1 = 3$. This framework is not 3-connected.

Note that from the proof of Proposition A.1 it is clear that if $\Gamma$ is $(d + 1)$-connected, then almost every non-symmetric stress matrix for almost every $(p, \Gamma)$ in $C^d(\Gamma)$ will have co-rank $d + 1$. Moreover, each row of such a non-symmetric stress matrix of $p$ can be constructed independently from the other rows, and we still expect to find this minimal co-rank.

\begin{proposition}
Let $\Gamma$ be a graph with at least $d + 1$ vertices. Suppose $(p, \Gamma)$, a framework in $C^d(\Gamma)$, has a non-symmetric equilibrium stress matrix $\Omega$ that has co-rank $d + 1$. Then $(p, \Gamma)$ is neighborhood affinely rigid in $\mathbb{R}^d$.
\end{proposition}

\begin{proof}
$\Omega$ is a (not strong) affinity matrix of $(p, N(\Gamma))$ and so the proof follows that of the first direction of Theorem 1.
\end{proof}

\begin{proof}[Proof of Theorem 3]
The theorem now follows directly from Propositions 5.8 and 5.9.
\end{proof}

See Figure 5 for an example showing that the converse of Proposition 5.9 is not true. Here we show a neighborhood affinely rigid framework in $\mathbb{R}^2$ that does not have a non-symmetric equilibrium stress matrix of co-rank $d + 1 = 3$.

\begin{remark}
Generic global rigidity of a graph $\Gamma$ in $\mathbb{E}^d$ can be characterized either using the dimension of the kernel of a single symmetric stress matrix of a generic framework $(p, \Gamma)$ or using the dimension of the \textit{shared symmetric stress kernel} of a generic $p$: the intersection of the kernels of all stress matrices of $p$ [8].
\end{remark}
Figure 5. This framework in $\mathbb{R}^2$ does not have a non-symmetric equilibrium stress matrix of co-rank $d+1 = 3$, but is (trivially) neighborhood affinely rigid.

By contrast, the analogous statement is not true in the affine rigidity case. By “vertically concatenating” a sufficient number of non-symmetric equilibrium stress matrices of $(p, \Gamma)$, we can create a strong affinity matrix for $(p, N(\Gamma))$. The kernel of the vertical concatenation will be the shared non-symmetric stress kernel of $(p, \Gamma)$, and the dimension of this kernel characterizes affine rigidity. Since the converse of Proposition 5.9 is not true, we see that neighborhood affine rigidity cannot in general be characterized by the rank of one single (say, generic) non-symmetric equilibrium stress matrix for $(p, \Gamma)$.

Note that there is a different sufficiency condition for affine rigidity given by Zha and Zhang [26]. Their condition is complementary to our condition (neither strictly stronger or weaker), and (like trilateration [7]) is greedy in nature. Their condition on generic frameworks of a hypergraph requires that one can walk between any two hyperedges such that for each sequence $(i, j)$ of two hyperedges along the walk, $h_i$ and $h_j$ share at least $d+1$ vertices. When translated to a neighborhood hypergraph $N(\Gamma)$, it states that one can walk between any two vertices such that for each sequence $(i, j)$ of two vertices along the walk, the neighborhoods of these vertices share at least $d + 1$ vertices in $\Gamma$.

Figure 6 shows a graph which clearly fails Zha and Zhang’s condition, but is 3-connected, showing that their condition does not imply Theorem 3. It is not hard to construct examples in the opposite direction, as well.

From Corollary 4.2, we also have the following corollary of Theorem 3.

**Corollary 5.11.** Let $\Gamma$ be a graph with at least $d + 1$ vertices. If $\Gamma$ is $(d + 1)$-connected, then any generic framework of $\Gamma^2$ in $\mathbb{R}^d$ is universally rigid.

**Remark 5.12.** This corollary can also be proven without reference to affine rigidity and Corollary 4.2. In particular, Proposition 5.8 guarantees a non symmetric maximal-rank stress $\Omega$ for $(p, \Gamma)$, and then $\Omega'\Omega$ is a symmetric, positive semi-definite, maximal rank stress for $(p, \Gamma^2)$. Universal rigidity then follows by a theorem of Connelly [5]. (See also [9].)

We wish to highlight this corollary since the only other known (to us) class of graphs that are universally rigid for all generic configurations in $\mathbb{R}^d$ are the $d$-trilateration graphs. (A $d$-trilateration graph is one that can be obtained from a complete graph by successively adding vertices, each connected to at least $d + 1$ old vertices.)

See Figure 6 for an example of a framework whose square is universally rigid by this corollary but is not a trilateration graph.

We also mention that a theorem of a related nature, showing a relationship between the connectivity of a graph and global rigidity in the squared graph, has been described by Anderson et al. [1].
Figure 6. A drawing of a hexagonal lattice on the torus. (The vertices on the top edge should be identified with those on the bottom and similarly with the left and right, as indicated by the arrows.) This graph is 3-connected but its neighborhood hypergraph does not satisfy the sufficiency condition of Zha and Zhang [26], and its squared graph is not a 2-trilateration graph.

6. Applications

6.1. Registration. There are many applications where one has multiple views of some underlying configuration, but it is not known how these views all fit together. We assume that these views share some points in common, and this correspondence is known. (Of course in practice, establishing such a correspondence could in itself be a very challenging problem.) For example, in computer vision, one may have multiple uncalibrated laser scans of overlapping parts of some three-dimensional object.

In our setting we model all of the points as vertices, and each of the views as a hyperedge. The geometry of the vertices in each hyperedge \( h \) is given up to some unknown transform \( T_h \) from a relevant class. The goal in registration then is to realize the entire hypergraph up to the relevant congruence class.

**Affine case:** Suppose we wish to realize a framework \((p, \Theta)\) where we are given as input the geometry of each hyperedge \( h \) up to an affine transform \( A_h \). Theorem 1 tells us that if \((p, \Theta)\) is affine rigid, then we can compute the realization just using linear algebra. In particular, we can use the data for each hyperedge to build its associated rows in a strong affinity matrix. Then we can solve for its kernel, giving us our answer \( p \).

If our hypergraph \( \Theta \) happens to be the neighborhood graph of an underlying graph \( \Gamma \), then one could also construct a (smaller) non-symmetric equilibrium stress matrix \( \Omega \) for \((p, \Gamma)\). This is not guaranteed to work; even when \((p, \Gamma)\) is neighborhood affinely rigid in \( \mathbb{R}^d \), the matrix \( \Omega \) may have co-rank larger than \( d + 1 \). But Theorem 3 states that if \( \Gamma \) is \((d + 1)\)-connected, this method will indeed work for almost every \( p \) in \( \mathbb{R}^d \) (and, in fact, using almost any non-symmetric equilibrium stress matrix for \((p, \Gamma)\)).

**Euclidean case:** The Euclidean framework registration problem is perhaps more natural and common.
When \((p, \Theta)\) is globally rigid in \(\mathbb{R}^d\), this problem is well posed, but it is general hard to solve, as the graph case includes the graph realization problem which is strongly NP-HARD [18].

When \((p, \Theta)\) is, in fact, also universally rigid there is an efficient algorithm: we can solve the Euclidean registration problem using semi-definite programming. One simply sets up the program that looks for the Gram matrix of an embedding of the vertices in \(\mathbb{R}^v\) (a semi-definite constraint on a Gram matrix) subject to the length constraints (linear constraints on the Gram matrix) [14]. Due to universal rigidity, one does not need to explicitly enforce the (non-convex) constraint that the embedding have a \(d\)-dimensional affine span [22].

When \((p, \Theta)\) is, furthermore, affinely rigid then we can solve the Euclidean registration problem using linear algebra. We can simply reduce this problem to an affine registration problem above, and find \(p\) using the kernel vectors of an affinity matrix. This determines \(p\) up to some global affine transform. Moreover, for \((p, \Theta)\) that is generically globally rigid in \(\mathbb{R}^d\), we can solve a second (least squares) linear system to remove the unwanted global affine transform, leaving just the unknown global Euclidean transform (see Appendix B). This approach is morally the same affine relaxation used in the initialization step of the registration method of Krishnan et al. [12] (though in their case, they think of the inter-patch transforms as the unknown variables instead of the point positions).

6.2. Global embeddings from edge lengths. Similar approaches have been applied to the (NP-HARD) problem of solving for the framework of a graph given its edge lengths. In these approaches one first attempts to find local \(d\)-dimensional embeddings for each one-ring of the framework up to an unknown local Euclidean transform. This step alone is NP-HARD and can fail. But assuming this step is (approximately) successful one can reduce the rest of the problem to the Euclidean registration problem above.

In the As-Affine-As-Possible (AAAP) method [11, 27], this was done using what is essentially a strong affinity matrix. In the Locally-Rigid-Embedding (LRE) method [20] this was done using a non-symmetric equilibrium stress matrix. Both approaches then removed the global affine transform using the least squares linear system described in Appendix B.

6.3. Manifold learning. Many of the ideas of affine rigidity first appeared in the context of manifold learning. Suppose one has \(d\)-dimensional smooth manifold \(\mathcal{M}\) which is a topological \(d\)-ball embedded in a larger \(D\)-dimensional space \(\mathbb{R}^D\). Also suppose that one has a set \(V\) of sample vertices on the manifold. In manifold learning, one first connects nearby samples to form a proximity graph \(\Gamma\). One then looks for a framework \((p, \Gamma)\) of this graph in \(\mathbb{R}^d\) that in some way that preserves some of the geometric relations of the points in \(\mathbb{R}^D\). This is used to represent a parametrization of \(\mathcal{M}\).

To compute the coordinates \(p\), the Locally Linear Embedding (LLE) method [16] builds a matrix \(\Omega\) with structure similar to a non-symmetric equilibrium stress matrix. In particular, row \(i\) encodes one affine relation between vertex \(i\) and its neighbors in \(\mathbb{R}^D\). Then (after ignoring the all ones vector) the smallest \(d\) eigenvectors of \(\Omega^T\Omega\) are used to form the coordinates of \(p\) in \(\mathbb{R}^d\). Unfortunately, since the original embedding is in \(\mathbb{R}^D\), for a graph of high enough valence and assuming no noise, \(\Omega\) must have a kernel of size at least \(D + 1\), which is much larger than \(d + 1\). Thus it is not clear how the numerically smallest \(d + 1\) eigenvectors will behave. The paper suggests to add an additional regularization term, possibly to address this issue.
A follow up to the LLE paper [17] describes a PCA-LLE variant where a \(d\)-dimensional local PCA is computed to "flatten" each one-ring before calculating its corresponding row in the matrix \(\Omega\). Thus \(\Omega\) is designed to represent \(d\)-dimensional affine relations between the points.

The Local-Tangent-Space-Alignment (LTSA) method [28] is an interesting variant of PCA-LLE. In this method, a \(v \times v\) matrix \(N\) is formed that is the Hessian of a quadratic energy. This energy vanishes only for vectors that are affinely equivalent to each of the flattened one-rings. Thus this matrix plays the role of a strong affinity matrix. It is in this context that Zha and Zhang investigated the rank of this matrix and affine rigidity [26].

In all of these methods, an understanding of affine rigidity is important. In particular it tells us what the rank of the computed matrix would be if the original \(d\)-dimensional manifold was in fact embedded in \(\mathbb{R}^d\). For example, if such a framework was not affinely rigid in \(\mathbb{R}^d\), then the kernel would be too big, and we would not expect a manifold learning technique to succeed. However, in manifold learning the embedding has an affine span greater than \(d\) and the analysis becomes more difficult. The kernel of a strong \(D\)-dimensional affinity matrix is too large, while the kernel of a strong affinity matrix for the locally flattened configurations contains only the all-ones vector but it is hoped that the numerically next smallest \(d\) eigenvectors are somehow geometrically meaningful. For an analysis along these lines, see [21].

**Appendix A. Matrix rank**

For completeness, we recall the necessary material for determining the generic matrix rank. This material is standard. For a more detailed treatment, see, e.g., [8, Section 5].

We will consider the general setting where there is a set of linear constraints that must be satisfied by a vector \(m \in \mathbb{R}^n\). The entries of \(m\) are then arranged in some fixed manner as the entries of a matrix \(M\), whose rank we wish to understand. The linear constraints are described by a constraint matrix with \(n\) columns: \(C(p)\). Each of the entries of \(C(p)\) is defined by some polynomial function, with coefficients in \(\mathbb{Q}\), of the coordinates of an input configuration \(p\). We wish to study the behavior of the rank of \(M\) as one changes \(p\).

We apply this in the proof of Corollary 3.8 where the constraints \(C\) specify that the matrix \(M\) is an affinity matrix for \((p, \Theta)\) and in the proof of Proposition 5.8 where the constraints \(C\) specify that the matrix \(M\) is a non-symmetric equilibrium stress matrix for \((p, \Gamma)\).

**Proposition A.1.** Suppose that for some generic \(p\), there is a matrix \(M\) of rank \(s\) consistent with \(C(p)m = 0\).

Then for all generic \(p\), there is some matrix \(M\) of rank \(\geq s\) consistent with \(C(p)m = 0\).

**Proof.** To prove this proposition we first need the following lemma.

**Lemma A.2.** Let \(M(\pi)\) be a matrix whose entries are polynomial functions with rational coefficients in the variables \(\pi \in \mathbb{R}^n\). Let \(r\) be a rank achieved by some \(M(\pi_0)\). Then \(\text{rank}(M(\pi)) \geq r\) for all points \(\pi\) that are generic in \(\mathbb{R}^n\).

**Proof.** The rank of the \(M(\pi)\) is less than \(r\) iff the determinants of all of the \(r \times r\) submatrices vanish. Let \(\pi_0 \in \mathbb{R}^n\) be a choice of parameters so \(M(\pi_0)\) has rank \(r\). Then there is an \(r \times r\) submatrix \(T(\pi_0)\) of \(M(\pi_0)\) with non-zero determinant. Thus \(\det(T(\pi))\) is a non-zero polynomial of \(\pi\). For any \(\pi\) with \(\text{rank}(M(\pi)) < r\), this determinant must vanish. Thus, any such \(\pi\) cannot be generic. \(\square\)
Next we recall that for a non-singular \( n \times n \) matrix \( \hat{C} \),
\[
\text{adj}(\hat{C}) = \det(\hat{C})\hat{C}^{-1},
\]
where \( \text{adj} \) \( C \) is the adjugate matrix of \( C \), the conjugate of the cofactor matrix of \( C \). In particular, \( \text{adj} \) \( C \) is a polynomial in \( \hat{C} \).

For a given \( p \), let \( t(p) \) be the rank of \( C(p) \). Let \( t := \max_{p} t(p) \). By Lemma A.2 this maximum is obtained for generic \( p \).

For each \( p \) we add a set \( H \) of \( n - t \) additional rows to \( C(p) \) to obtain a matrix \( C(p, H) \), and determine \( m \) by solving the linear system \( C(p, H)m = b \) where \( b \in \mathbb{R}^n \) is a vector of all zeros except for a single 1 in one of the positions of a row in \( H \) (if there are any rows in \( H \)). This \( m \) is then converted to a matrix \( \hat{M}(p, H) \). \( M(p, H) \) is well-defined iff this linear system has a unique solution, i.e., iff \( C(p, H) \) has rank \( n \). Note that this happens for generic \( p \) and \( H \).

Let \( p_0 \) be generic and have a compatible matrix \( M_0 \) with rank \( s \), as in the hypotheses of the proposition. Find a set \( H_0 \) of additional rows so that \( C(p_0, H_0) \) has rank \( n \) and \( C(p_0, H_0)m_0 = b \). Let \( \hat{C}(p, H) \) be an \( n \times n \) submatrix of \( C(p, H) \) so that \( \hat{C}(p_0, H_0) \) is invertible. \( \hat{C} \) necessarily uses \( t \) rows from \( C(p) \) and all rows of \( H \). Define \( \hat{b} \) similarly, let \( \hat{m}(p, H) := \text{adj}(\hat{C})\hat{b} \), and let \( \hat{M}(p, H) \) be the associated matrix.

By Lemma A.2, the rank of \( \hat{C}(p, H) \) is equal to its maximum value, \( n \), at all points \( (p, H) \) that are not zeros of a polynomial \( P_1(p, H) := \det(\hat{C}(p, H)) \). Moreover, when \( P_1(p, H) \neq 0 \), the linear equation defining \( M(p, H) \) has a unique solution and the adjugate matrix \( \hat{M}(p, H) \) is a scalar multiple of \( M(p, H) \). In particular we have assumed \( (p_0, H_0) \) is not a zero of \( P_1 \) and thus \( \hat{M}(p_0, H_0) \) has rank \( s \). By Lemma A.2 again, the rank of \( \hat{M}(p, H) \) is less than \( s \) only at the zeros of a non-zero polynomial \( P_2(p, H) \).

For any generic \( p \), there must be some generic point \( (p, H) \). Such a generic \( (p, H) \) cannot be a zero of \( P_1 \) or \( P_2 \) and thus \( \hat{M}(p, H) \) and \( M(p, H) \) must have rank no less than \( s \). \( \square \)

**Appendix B. Removing the Affine Transform**

Suppose one has solved for \( q \) -- a configuration in \( \mathbb{R}^d \) -- up to an unknown global affine transform \( A \) of the true configuration \( p : p = A(q) \). Given a set of edge lengths for \( p \), it is possible to compute \( A \) up to an unknown global Euclidean transform. This approach was described by Singer [20].

In particular, let \( L \) be a \( d \times d \) matrix representing the linear portion of \( A \) and let \( G := L^T L \). Now consider the following set of linear equations (in the \( \frac{d(d+1)}{2} \) unknowns of \( G \)): For each pair of vertices \( i, j \), whose edge lengths are known, we require
\[
(q(i) - q(j))^T G (q(i) - q(j)) = (p(i) - p(j))^T (p(i) - p(j))
\]
(Since we have more constraints than unknowns, for numerical purposes we would solve this as a least squares linear system in the unknown \( G \).

The only remaining concern is whether this system has more than one solution. The solution to Eq. (2) will be unique as long as our set of edges with known lengths do not lie on a conic at infinity.

Fortunately, we can conclude from Proposition 4.3 of [6] that if our known lengths form a graph \( B(\Theta) \) with minimal valence at least \( d \) and \( p \) is generic, then the edges do not lie on a conic at infinity. This property holds for any hypergraph that is generically globally rigid in \( \mathbb{E}^d \).
Using Cholesky decomposition on $G$ then yields $L$ up to a global Euclidean transform.

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