Quantum resources play crucial roles for displaying superiority in many quantum communication and computation tasks. To reveal the intrinsic relations hidden in these quantum resources, many efforts have been made in recent years. In this work, the correlations of the tripartite W-type states based on bipartite quantum resources are investigated. The inter-relations among the degree of coherence, concurrence, Bell nonlocality, and purity are presented. Considering Bell nonlocal and Bell local (satisfied the Clauser–Horne–Shimony–Holt inequality) states for the two-qubit subsystems derived from the tripartite W-type states, exact lower and upper boundaries of the degree of coherence versus concurrence are obtained. Interestingly, exact relation among the degree of coherence, concurrence, and purity is obtained. Moreover, coherence is also closely related to entanglement in two specific scenarios: the tripartite W-type state under decoherence and a practical system for a renormalized spin-1/2 Heisenberg model.

1. Introduction

Multiparticle quantum systems can show unique forms of quantum nonlocal correlations (QNCs).[1–3] QNCs and coherence are fundamental quantum resources (QRs) in many novel quantum-information tasks that cannot be achieved by classical resources. For bipartite systems shared by two parties, Alice and Bob, many efforts have been devoted to a deeper insight of nonlocal correlations, mainly taking into account three types of QNCs: entanglement,[2,3] Bell nonlocality,[4,5] and Einstein–Podolsky–Rosen (EPR) steering.[6–9] Entanglement, that serve as one of the most generally used QRs, is defined as the inseparability of quantum states and can be considered as an algebraic concept.[3] For entanglement measures, there are many mathematical measures methods, such as relative entropy of entanglement (REE),[10,11] entanglement of formation (EOF),[12] concurrence,[10,13] and negativity.[14,15] Here, the negativity is an entanglement cost measure under the positivity of partial transpose for quantum operations preserving.[14] It is well-known that the concurrence of a quantum state can always exceed its negativity, and the REE is less than its EOF.[12] For bipartite pure states, the concurrence is exactly equivalent to its negativity. Besides, another one quantum resource can be discovered by violating some Bell-type inequalities, and is termed as Bell nonlocality.[16–23] One of these Bell-type inequalities, the Clauser–Horne–Shimony–Holt (CHSH) inequality[3–5] has been often considered as a measure for the QNCs presented between spatially separated two parties that are entangled, and thereby it can violate the bound originated by the inequality.[22] It is noteworthy that, Bell-type inequalities can be violated only if their states are entangled. However, there are entangled states that can still exhibit QNCs which cannot violate any Bell-type inequality for any possible local measurements proposed by Werner[24] in 1989. Then, the sufficient and necessary conditions for arbitrary bipartite states to be Bell nonlocal states were derived[25] in 1995. The last one is called as EPR steering[6,7] which was introduced by Schrödinger[26] to analyze the EPR-paradox in 1935. Conceptually, EPR steering is able to describe that an observer can instantaneously affect a remote system by utilizing local measurements. In addition, EPR steering has been viewed as an intermediate type of quantum resource[27–29] between entanglement and Bell nonlocality in modern quantum-information theory. Except for their fundamental importance, these QRs have had many practical applications in quantum-information-science ranging from quantum key-distribution,[16,28] quantum random-number generation,[17,18] and communication complexity.[30]

On the other hand, coherence, as one of the most widely applied QRs, plays a key role in the range of biological, chemical, and physical phenomena,[31] such as quantum metrology,[32] low-temperature thermodynamics,[33–35] solid-state physics[36] and also explains the violation of the CHSH inequalities.
Meanwhile, coherence is also an important concept used to depict the interference capability of interacting fields in quantum optics research.\textsuperscript{[37–39]} Additionally, the knowledge of the internal distribution of coherence between subsystems and their correlations becomes essential for predicting the coherent evolution in the researched quantum system.\textsuperscript{[40,41]}

Even though these QRs are playing different roles in different quantum-information tasks, the intrinsic relations behind the QRs might be the same one. Recently, many researchers have made progress for this target.\textsuperscript{[41–48]} For example, Kalaga et al.\textsuperscript{[44]} have studied the relations between EPR steering and coherence in the families of the tripartite entangled states, and they disclose some mutual relations among EPR steering, entanglement, and coherence. Besides, another result is generalized to quantum discord (QD) and promoted to multipartite systems in ref. [45], in which it is indicated that QD generated by multipartite incoherent operations is restricted by coherence expended in its subsystems. All the works above are trying to connect part of these QRs. Notwithstanding, coherence, entanglement, and QD are qualitatively unified in interferometric framework,\textsuperscript{[48]} quantitative relations of these QRs is still an open question.

In this work, we will concentrate on finding mutual relations of these QRs of bipartite states. Particularly, we are interested in the quantitative relations among Bell nonlocality, entanglement, coherence, and purity based on bipartite subsystems in a general multiparticle system (a tripartite $W$-type states). Here, we severally consider Bell nonlocal and Bell local (satisfied the CHSH inequality) bipartite mixed states in the tripartite $W$-type states, and analyze the mutual relations between the degree of coherence and concurrence. Interestingly, quantitative relation among the degree of coherence, concurrence, and purity is revealed. Additionally, we consider two specific scenarios, one scenario is a tripartite $W$-type state under decoherence channel. We will observe how the phase flip channel influences the quantitative relations among these QRs. And the other one is for a practical system of a renormalized spin-1/2 Heisenberg XXZ model; in this spin system, some mutual relations between QRs and purity are attained.

The remainder of this article is organized as follows. Preliminary definitions and notations are reviewed in Section 2. Then, we investigate the intrinsic relations among these QRs in the case of the qubit–qubit subsystem, and derive some exact boundary conditions with respect to Bell nonlocal and Bell local states in Section 3. In Section 4, we discuss two specific scenarios for the tripartite $W$-type states (under a decoherence channel and a renormalized spin-1/2 Heisenberg XXZ model). Finally, we end up this paper with a brief conclusion.

\section{Preliminaries}

We consider a $2 \times 2$ dimensional quantum state $\rho_{AB}$, composed of subsystems $A$ and $B$. Commonly, the state $\rho_{AB}$ can be expressed (spectral decomposition) as $\rho_{AB} = \text{VEV}$, where $E$ is a diagonal matrix. Each subsystem is characterized by the corresponding density matrix, $\rho_A$ and $\rho_B$. Coherence can be traced to Young’s report of light field interference. And coherence measures are implemented using classical and quantum methods.\textsuperscript{[49–51]} We here utilize the degree of coherence (DC) to measure coherence. DC can be interpreted as the degree of polarization coherence in the conventional coherence optics theory. It is noteworthy that the degree of polarization coherence and coherence constraints have been investigated by Qian et al.\textsuperscript{[52]} They have exposed of the third component of the fundamental coherence triad removing the hidden coherences’ mystery in optical coherence. Firstly, we introduce that the degree of first-order coherence of each subsystem $A$ and $B$ can be given by\textsuperscript{[49,50]}

\begin{equation}
D_k = \sqrt{2\text{Tr}(\rho_k^2)} - 1, \ k = A, B
\end{equation}

Then, one can define a coherence measure named the DC for both subsystems $A$ and $B$ when they are considered independently\textsuperscript{[41,44]}

\begin{equation}
D_{AB}^2 = \frac{D_{A}^2 + D_{B}^2}{2}
\end{equation}

when both subsystems are coherent, there is $D_{AB}^2 = 1$, while only if both subsystems show no coherence, $D_{AB}^2 = 0$. Besides, we here use a popular entanglement measure, Wootters’ concurrence.\textsuperscript{[53,54]} The Wootters’ concurrence can be defined by

\begin{equation}
C_{AB} = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\}
\end{equation}

where $\lambda_i$ are the decreasing-order eigenvalues of the matrix $R = \rho_{AB}(\sigma_x \otimes \sigma_x)\rho_{AB}^\dagger(\sigma_y \otimes \sigma_y)$. This measure is a monotonic and convex function of the entanglement of formation.\textsuperscript{[55]}

Next, it is well known that the violation of Bell’s inequality in the form given by Clauser, Horne, Shimony, and Holt in two-qubit systems means that the entanglement for the bipartite appears, but not vice versa, that is, there are entangled states which do not violate the CHSH inequality. The CHSH inequality can be expressed as\textsuperscript{[3,56]}

\begin{equation}
|\langle B_{\text{CHSH}} \rangle| = |\text{Tr}(\rho_{AB} B_{\text{CHSH}})| \leq 2
\end{equation}

depending upon the CHSH operator $B_{\text{CHSH}} = a \cdot \sigma \otimes (b + b') \cdot \sigma + a' \cdot \sigma \otimes (b - b') \cdot \sigma$, where $a, a'$ and $b, b'$ are unit vectors describing the measurements on sides $A$ and $B$, respectively. In terms of the Horodecki theorem,\textsuperscript{[25]} the maximum expected value of the CHSH operator for the quantum state $\rho_{AB}$ can be given by\textsuperscript{[57,58]}

\begin{equation}
\max_{B_{\text{CHSH}}} |\text{Tr}(\rho_{AB} B_{\text{CHSH}})| = 2\sqrt{M(\rho_{AB})}
\end{equation}

where $M(\rho_{AB}) = \max_{ui}(u_i + u_i) \leq 2$ and $u_i (i = 1, 2, 3)$ are the eigenvalues of the symmetric matrix $U = T^T T$ constructed from the correlation matrix $T$ and its transpose $T^T$. The CHSH inequality can be violated if and only if (iff) $M(\rho_{AB}) > 1$\textsuperscript{[25]} In order to quantify the maximal violation of the CHSH inequality, we can use $M(\rho_{AB})$ or, equivalently\textsuperscript{[59]}

\begin{equation}
N(\rho_{AB}) = \max \left\{ 0, 2\sqrt{M(\rho_{AB})} - 2 \right\}
\end{equation}

which yields $N(\rho_{AB}) = 0$ if the CHSH inequality is not violated and $N(\rho_{AB}) = 2\sqrt{2} - 2$ for its maximal violation. In addition,
with respect to a two-qubit X-state, the three eigenvalues \( \mu_i \) of the real symmetric matrix \( U = T^T T \) are\(^{[3,20]} \)

\[
\begin{align*}
\mu_1 &= 4(|\rho_{14}| + |\rho_{23}|)^2 \\
\mu_2 &= 4(|\rho_{14}| - |\rho_{23}|)^2 \\
\mu_3 &= (\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})^2
\end{align*}
\]

respectively, \( \rho_{ij} (i, j = 1, 2, 3, 4) \) are the matrix elements for two-qubit X-state. It is easy to see that \( \mu_1 \) is always larger than \( \mu_2 \), and thus the quantified CHSH inequality named Bell nonlocality can be expressed as

\[
N(\rho_{AB}) = \max \{0, B_1, B_2\}
\]

here, \( B_1 = 2\sqrt{\mu_1 + \mu_2} - 2 \) and \( B_2 = 2\sqrt{\mu_1 + \mu_3} - 2 \).

3. Intrinsic Relations among These QRs for Different Bipartite Subsystems of The Three-Qubit Systems

In this section, we mainly investigate the internal relations among these QRs in the case of a qubit–qubit subsystem for the multiparticle systems. Here, the multiparticle system is a three-qubit W-type state. Labeling the qubits with 1, 2, and 3, the condition \( \hat{n} = (\hat{n}_1 + (\hat{n}_2) + (\hat{n}_3) = 1 \) identifies the states whose wave function can be written in the form

\[
|\psi\rangle_{123} = \alpha |001\rangle + \beta |010\rangle + \gamma |100\rangle
\]

where \( \alpha^2 + \beta^2 + \gamma^2 = 1 \). The corresponding density matrix can be expressed as

\[
\rho_{123}^{W-C} = \gamma \alpha^* |100\rangle \langle 100| + \gamma |100\rangle \langle 100| + 
\alpha \beta^* |010\rangle \langle 010| + \alpha |010\rangle \langle 010| + 
\alpha \gamma^* |001\rangle \langle 001| + \alpha |001\rangle \langle 001| + 
\beta \gamma^* |010\rangle \langle 010| + \beta |010\rangle \langle 010| + 
\alpha^2 |001\rangle \langle 001| + \beta^2 |010\rangle \langle 010| + 
\alpha^2 |001\rangle \langle 001| + \beta^2 |010\rangle \langle 010| + 
\gamma^2 |100\rangle \langle 100| + \gamma |100\rangle \langle 100|
\]

Considering different pairs of qubits described by the partially reduced density matrix from \( \rho_{123}^{W-C} \) in Equation (10), by utilizing Equations (2), (3), and (8), the corresponding DC, concurrence, and Bell nonlocality can be expressed as follows

\[
\begin{align*}
D_{11} &= 2 (\alpha^2 - \alpha^2 + \gamma^2 - \gamma^2) + 1 \\
D_{12} &= 2 (\beta^2 - \beta^2 + \gamma^2 - \gamma^2) + 1 \\
D_{21} &= 2 (\alpha^2 - \alpha^2 + \beta^2 - \beta^2) + 1 \\
C_{15} &= 2\sqrt{\alpha^2 \gamma^2}, \quad C_{12} = 2\sqrt{\beta^2 \gamma^2}, \quad C_{23} = 2\sqrt{\alpha^2 \beta^2}
\end{align*}
\]

\[
C_{ij} = D_{ij} + \frac{3}{2} C_{ij}^2 = 1
\]

respectively. The conditions of determining the generation of Bell nonlocality can be deduced from Equation (13), which do not yield Bell nonlocality if \( N(\rho_{AB}) = 0 \), and \( N(\rho_{AB}) = 2\sqrt{2} - 2 \) for the maximal values of Bell nonlocality.

In our investigations, we firstly pay attention to the relation between the DC \( D_{ij} \) and the concurrence \( C_{ij} \) for Bell nonlocal bipartite states (BNBSs) described by the density matrix \( \rho_{123}^{W-C} \) given in Equation (10). The diagram depicting mutual relations between the DC and the concurrence is drawn in Figure 1. The BNBSs are revealed in the green area. To discover the boundary conditions for the BNBSs plotted by blue dashed-dotted and red dashed curves, we should start with the expression of the DC \( D_{ij} \) in Equation (11), then, by applying Equations (12) and (13), we can obtain the boundary formed by BNBSs with the lowest possible DC assuming a fixed concurrence. Through numerical analysis, one can disclose that Bell nonlocality \( N(\rho_{AB}) \) equals to zero for these bipartite states derived from the tripartite W-type states. Via substitution of Equations (11) and (12) into Equation (13), we can derive a formula as

\[
D_{ij}^2 + \frac{3}{2} C_{ij}^2 = 1
\]
Besides, another boundary giving the maximal attainable DC for the fixed concurrence of BNBSs contains all bipartite pure states, where the DC can be written as follows

$$\begin{align*}
D_{ij}^{12} &= 2 (\alpha^2 + \gamma^2) - 1 \\
D_{ij}^{13} &= 2 (\beta^2 + \gamma^2) - 1 \\
D_{ij}^{23} &= 2 (\alpha^2 + \beta^2) - 1
\end{align*}$$

utilizing Equations (11) and (15), we can discover the upper boundary in Figure 1 for the BNBSs obtaining following formula

$$D_{ij}^2 + C_{ij}^2 = 1$$

Here, Equation (16) is similar to the coherence constraint $C_{ij}^2 + P_{ij}^2 = 1$ in ref. [52]. However, Equation (16) only applies to all bipartite pure states. For the bipartite mixed states, we will attain the different results. Then, by combining Equations (14) and (16), we can find that the DC for the BNBSs is located in an interval $D_{ij}^2 \in (1 - \frac{1}{2} C_{ij}^2, 1 - C_{ij}^2)$, which means that the DC and the concurrence have a trade-off relation: with the increasing of the concurrence, the DC will decrease, that is, the concurrence enhances at the expense of DC. Note that this condition is similar to that proposed by Svozil et al. [31] for the parameter $S_{ij}$ (accessible coherence) describing the maximal violation of the CHSH inequality $B_{ij}^{\max}$ and the DC, that is, $S_{ij} = D_{ij}^2/2 + (B_{ij}^{\max}/2\sqrt{2})^2$. In addition, for the bipartite pure states, we can derive the relation between Bell nonlocality and the concurrence

$$N_{ij}^{\max} = 2\sqrt{1 + C_{ij}^2} - 2$$

due to $D_{ij}^2 + C_{ij}^2 = 1$, we can attain the relation between Bell nonlocality and the DC as follows

$$D_{ij}^2 + 2 \left( \frac{N_{ij}^{\max} + 2}{2\sqrt{2}} \right)^2 = 2$$

namely,

$$N_{ij}^{\max} = 2\sqrt{2 - D_{ij}^2} - 2$$

Additionally, Bell nonlocality can be revealed only for bipartite entangled states with sufficiently strong nonlocal correlations. Thus, one can see that increasing the value of DC should be accompanied by decreasing the value of Bell nonlocality. Enhancing the concurrence will synchronously increase Bell nonlocality. These results can really be disclosed for BNBSs in the randomly generated ensemble of states, as demonstrated in the green (Figure 2a) and purple (Figure 2b) areas, respectively. Furthermore, for a pure N-party system, the entanglement of one party with the remaining $N - 1$ parties confirms the purity of that party’s quantum state when the rest of the quantum system is traced out. Actually, when a quantum system is mixed, it is so because of its entanglement with parties not sufficiently taken into account. [60–62] The purity of each party following the tracing out of the remaining ones can then be regarded as the basis for characterizing entanglement in multiparty systems. We know that the standard definition of purity for the bipartite states is

$$P_{AB} = \text{Tr}(\rho_{AB}^2)$$

Thus, one can obtain the purity of the above bipartite states in the tripartite W-type states

$$\begin{align*}
P_{13} &= \alpha^2 + \beta^4 + 2\alpha^2 \gamma^2 + \gamma^4 \\
P_{12} &= \alpha^2 + \beta^4 + 2\beta^2 \gamma^2 + \gamma^4 \\
P_{13} &= \alpha^2 + \beta^4 + 2\alpha^2 \beta^2 + \gamma^4
\end{align*}$$

Because of $\alpha^2 + \beta^2 + \gamma^2 = 1$, by means of combining Equations (11), (12), and (21), an interesting result can be derived

$$D_{ij}^2 + C_{ij}^2 = P_{ij}$$

Because the purity is equal to one iff the bipartite state is a pure state, we can draw a conclusion

$$D_{ij}^2 + C_{ij}^2 \leq 1$$

The inequality (23) also illustrates that the concurrence increases at the expense of the degree of coherence. In addition, Equation (22) is appropriate for all bipartite pure states and the bipartite mixed states derived from the tripartite W-type states. If
Equation (25) is a unsteerable state eventhoughthestatecannot completerelessprecise,onecannotsaythatthestateformulatedby Bell local bipartite states (BLBSs) drawn in handside of Equation (22) is equal to 1. Meanwhile, Equation (22) and onlyifthestateislimitedtobipartitepurestates,thelleftsideofEquation (22) isequalto 1. Hence, we can drawthesameconclusion: the DC of Equation (11) and the DC of Equation (14), which is already obtained for BNBSs. Besides, cointstantaneous discussion for the DC of Equation (11) and the concurrence of Equation (12), affords ustheminimal accessible DC in the scope of $C_{ij} \geq 1/2$

$$D_{ij} + C_{ij} = 1$$

(24)

Figure 3. The degree of coherence $D_{ij}$ versus the concurrence $C_{ij}$ for BLBSs derived from the tripartite W-type states $\rho_{123}^{W_\perp}$. BLBSs are discoveredin cyan areas. Red dashed, black solid, and green dashed curves are plotted in accordance with the corresponding boundary formulas given in Equations (14), (24), and (27), respectively.

and only if the state is limited to the bipartite pure states, the left-hand side of Equation (22) is equal to 1. Meanwhile, Equation (22) reduces to Equation (16).

Subsequently, we are engaged in the achieved boundaries of Bell local bipartite states (BLBSs) drawn in Figure 3. For $C_{ij} > 4/5$, the maximal attainable DC of BLBSs is given by Equation (14), which is already obtained for BNBSs. Besides, cointstantaneous discussion for the DC of Equation (11) and the concurrence of Equation (12), affords ustheminimal accessible DC in the scope of $C_{ij} \geq 1/2$

$$D_{ij} + C_{ij} = \frac{1}{2}$$

(27)

and the underlying states have the following density matrix

$$\rho_{ij} = \frac{1}{2} |00\rangle\langle 00| + (1/2 - a) |01\rangle\langle 01| + a |10\rangle\langle 10|$$

$$+ \sqrt{a/2 - a^2} |01\rangle\langle 10| + |10\rangle\langle 01|$$

(28)

The expression of DC, concurrence, purity, and Bell nonlocality can be obtained, and $D_{ij} + C_{ij} = P_{ij} \leq 1$ can also be unveiled. The discussion of the attainable DC and concurrence is showed in the diagrams of Figures 1 and 3, then, we can identify different areas in the plane ($D_{ij}, C_{ij}$) from the perspective of Bell nonlocality or not. The analysis results are shown in Table 1. In accordance with the results, BNBSs and BLBSs can be revealed in the certain areas.

### 4. Mutual Relations between QRs and Purity in Two Specific Scenarios

In this section, we will consider two specific scenarios for a tripartite W-type state. One case is a tripartite W-type state under decoherence channel; we mainly consider how the phase flip (PF) channel affects the mutual relations among these QRs. And the other one is a renormalized spin-1/2 Heisenberg XXZ model; some exact internal relations between QRs and purity are revealed in this practical system.

#### 4.1. The Tripartite W-Type State under PF Channel

We will discuss the mutual relations of the DC, the concurrence, purity, and Bell nonlocality for final states $\rho_{123}^{W_{\perp}}$ (the tripartite W-type states suffered from PF channel) described by a trace-preserving quantum operation $\vartheta(\rho)$, which is given by

$$\vartheta(\rho) = \sum_{i=1}^{3} E_i \otimes E_i \otimes E_i \cdot \rho \cdot E_i \otimes E_i \otimes E_i$$

where $\{E_i\}$ is the set of Kraus operators associated to a decohering process of a single qubit, with the trace-preserving condition reading

| Concurrence $C_{ij}$ | The degree of coherence $D_{ij}$ | Revealed states |
|----------------------|---------------------------------|-----------------|
| $0 < C_{ij} \leq \frac{3}{4}$ | $1 - \frac{1}{2} C_{ij}^2 < D_{ij} \leq 1 - C_{ij}^2$ | Only BNBSs |
| $0 < C_{ij} \leq \frac{3}{4}$ | $\frac{1}{2} C_{ij} + D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BLBSs |
| $\frac{3}{4} < C_{ij} \leq \frac{4}{5}$ | $D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BNBSs |
| $\frac{4}{5} < C_{ij} \leq 1$ | $1 - \frac{1}{2} C_{ij}^2 < D_{ij} \leq 1 - C_{ij}^2$ | Only BLBSs |
| $\frac{4}{5} < C_{ij} \leq 1$ | $D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BNBSs |

違反特定的EPR steering不等式。[64] 最後的邊界對應到綠色虛線曲線在圖3中給出的最小DC對於對應的concurrence $C_{ij} < 1/2$ 是由

$$D_{ij} + C_{ij} = \frac{1}{2}$$

(27)

和下面的邊界等式有下列密度矩陣

$$\rho_{ij} = \frac{1}{2} |00\rangle\langle 00| + (1/2 - a) |01\rangle\langle 01| + a |10\rangle\langle 10|$$

$$+ \sqrt{a/2 - a^2} |01\rangle\langle 10| + |10\rangle\langle 01|$$

(28)

表1. 昭示BNBSs和BLBSs在該區域 ($D_{ij}^2, C_{ij}^2$) 超過在的均勻性及concurrence。

| Concurrence $C_{ij}$ | The degree of coherence $D_{ij}$ | Revealed states |
|----------------------|---------------------------------|-----------------|
| $0 < C_{ij} \leq \frac{3}{4}$ | $1 - \frac{1}{2} C_{ij}^2 < D_{ij} \leq 1 - C_{ij}^2$ | Only BNBSs |
| $0 < C_{ij} \leq \frac{3}{4}$ | $\frac{1}{2} C_{ij} + D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BLBSs |
| $\frac{3}{4} < C_{ij} \leq \frac{4}{5}$ | $D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BNBSs |
| $\frac{4}{5} < C_{ij} \leq 1$ | $1 - \frac{1}{2} C_{ij}^2 < D_{ij} \leq 1 - C_{ij}^2$ | Only BLBSs |
| $\frac{4}{5} < C_{ij} \leq 1$ | $D_{ij} \leq 1 - \frac{1}{2} C_{ij}^2$ | Only BNBSs |

和下面的邊界等式有下列密度矩陣

$$\rho_{ij} = \frac{1}{2} |00\rangle\langle 00| + (1/2 - a) |01\rangle\langle 01| + a |10\rangle\langle 10|$$

$$+ \sqrt{a/2 - a^2} |01\rangle\langle 10| + |10\rangle\langle 01|$$

(28)
The Kraus operators of PF channel can be expressed as:

\[ E_1 = \sqrt{p} I_2, \quad E_2 = \sqrt{1-p} \sigma_z \]  

(29)

where \( I_2 \) is a 2 \( \times \) 2 identity matrix, \( \sigma_z \) is a Pauli matrix at site \( z \), and \( 0 \leq p = 1 - e^{-\gamma^2} \leq 1 \) is the PF channel decoherence strength and \( \eta \) is the decay rate. Then, we can obtain the final states

\[
\rho_{123}^{PF} = \gamma Y\alpha^* \begin{pmatrix} 100 \end{pmatrix} + \gamma Y\beta^* \begin{pmatrix} 100 \end{pmatrix} + \begin{pmatrix} 001 \end{pmatrix} \begin{pmatrix} 010 \end{pmatrix} + \begin{pmatrix} 010 \end{pmatrix} \begin{pmatrix} 001 \end{pmatrix} + \begin{pmatrix} 100 \end{pmatrix} \begin{pmatrix} 100 \end{pmatrix} \\
+ \alpha Y\beta^* \begin{pmatrix} 100 \end{pmatrix} + \alpha Y\gamma^* \begin{pmatrix} 100 \end{pmatrix} + \begin{pmatrix} 001 \end{pmatrix} \begin{pmatrix} 010 \end{pmatrix} + \begin{pmatrix} 010 \end{pmatrix} \begin{pmatrix} 001 \end{pmatrix} + \begin{pmatrix} 100 \end{pmatrix} \begin{pmatrix} 100 \end{pmatrix} \\
+ \beta Y\alpha^* \begin{pmatrix} 100 \end{pmatrix} + \beta Y\gamma^* \begin{pmatrix} 100 \end{pmatrix} + \begin{pmatrix} 001 \end{pmatrix} \begin{pmatrix} 010 \end{pmatrix} + \begin{pmatrix} 010 \end{pmatrix} \begin{pmatrix} 001 \end{pmatrix} + \begin{pmatrix} 100 \end{pmatrix} \begin{pmatrix} 100 \end{pmatrix} \\
+ \gamma^2 \begin{pmatrix} 100 \end{pmatrix} + \gamma^2 \begin{pmatrix} 100 \end{pmatrix} + \gamma^2 \begin{pmatrix} 100 \end{pmatrix} + \gamma^2 \begin{pmatrix} 100 \end{pmatrix} + \gamma^2 \begin{pmatrix} 100 \end{pmatrix} + \gamma^2 \begin{pmatrix} 100 \end{pmatrix} \\
\]

(30)

where \( Y = (1 - 2p^2) \). By separately tracing over the qubit 3, 2, and 1, we can obtain that the reduced density matrices are

\[
\rho_{12}^{PF} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & \beta^2 & 0 & 0 \\ 0 & 0 & Y\beta^* \gamma^* & 0 \\ 0 & 0 & 0 & \gamma^2 \end{pmatrix} 
\]

(31a)

\[
\rho_{13}^{PF} = \begin{pmatrix} \beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & Y\alpha^* \gamma^* & 0 \\ 0 & 0 & 0 & \gamma^2 \end{pmatrix} 
\]

(31b)

\[
\rho_{23}^{PF} = \begin{pmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & Y\alpha^* \beta^* & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix} 
\]

(31c)

respectively. Via employing Equations (2), (3), (8), and (20), the corresponding purity, concurrence, DC, and Bell nonlocality can be expressed as follows

\[
P_{13}(PF) = \alpha^4 + \beta^4 + 2Y^2\alpha^2\gamma^2 + \gamma^4 \\
P_{12}(PF) = \alpha^4 + \beta^4 + 2Y^2\beta^2\gamma^2 + \gamma^4 \\
P_{21}(PF) = \alpha^4 + \beta^4 + 2Y^2\alpha^2\beta^2 + \gamma^4 \\
C_{13}(PF) = 2\sqrt{Y^2\alpha^2\gamma^2} \\
C_{12}(PF) = 2\sqrt{Y^2\beta^2\gamma^2} \\
C_{21}(PF) = 2\sqrt{Y^2\alpha^2\beta^2} \\
D_{11}^2(PF) = 2(\alpha^4 - \alpha^2 + \gamma^4 - \gamma^2) + 1 \\
D_{12}^2(PF) = 2(\beta^4 - \beta^2 + \gamma^4 - \gamma^2) + 1 \\
D_{21}^2(PF) = 2(\alpha^4 - \alpha^2 + \beta^4 - \beta^2) + 1 \\
N_{13}^{PF} = \max \left\{ 0, 4\sqrt{2Y^2\alpha^2\gamma^2} - 2 \right\} \\
N_{12}^{PF} = \max \left\{ 0, 4\sqrt{2Y^2\beta^2\gamma^2} - 2 \right\} \\
N_{21}^{PF} = \max \left\{ 0, 4\sqrt{2Y^2\alpha^2\beta^2} - 2 \right\} \\
\square

(32)

(33)

(34)

(35)

(36)

(37)

(38)

respectively.

Now, we focus on the relation between the DC and the concurrence for BNBSs. The diagram depicting mutual relations between the DC and the concurrence is plotted in Figure 4. The BNBSs can be discovered in the different rainbow color areas corresponding to the different decoherence strengths in Figure 4a. The lower boundary conditions for the BNBSs in Figure 4a are use the expression of Bell nonlocality in Equation (35), then, from Equations (33) and (34), we analyze the lower boundary formed by BNBSs with the lowest possible the DC assuming a fixed concurrence. Through analysis, Bell nonlocality should be equal to zero for these bipartite states in the tripartite states \( \rho_{123}^{PF} \). Then, we can obtain a wishful lower boundary formula

\[
D_{ij}^2 + \frac{C_{ij}^2}{Y^2} = 1 
\]

(36)

The upper boundary condition for the BNBSs in Figure 4a giving the maximal attainable DC for the fixed concurrence of BNBSs for the bipartite pure states undergone the PF channel, which the corresponding DC can be obtained. By utilizing the prior method, we can reveal the upper boundary condition for the BNBSs as follows

\[
D_{ij}^2 + \frac{C_{ij}^2}{Y^2} = 1 
\]

(37)

Next, we start with the achieved boundaries of BLBSs in Figure 4b. The maximal attainable DC of BLBSs is given by the upper boundary Equation (36) already obtained for BNBSs. In addition, constantaneous discussion of the concurrence Equation (33) and the DC Equation (34), respectively, affords us the minimal accessible DC in the scope of \( C_{ij} \geq Y/2 \); we can disclose a lower boundary for BLBSs

\[
D_{ij}^2 = \left( 1 - \frac{C_{ij}}{Y} \right)^2 
\]

(38)
Here, the boundary condition (38) corresponds to the lower boundary for $C_{ij} \geq Y/2$ in Figure 4b, and the underlying states have all qubits for the Horodecki states formulated (25) undergone the PF channel. Then, another lower boundary giving the minimal DC for the corresponding concurrence $C_{ij} < Y/2$ is written as

$$D^2_{ij} + C^2_{ij} = \frac{1}{2}$$

and the underlying states have the following density matrix

$$\rho^{PF}_{ij} = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} (1/2 - a) |01\rangle \langle 01| + a |10\rangle \langle 10| + Y \sqrt{a/2 - a^2} (|01\rangle \langle 10| + |10\rangle \langle 01|)$$

As shown in Figure 4, we can see that the proportion of the areas of BNBSs relative to BLBSs will decrease with the increase of decoherence strength. Besides, the concurrence decreases with the growing intensity of decoherence, whereas, the DC is not influenced by the decoherence, that is, the DC is immune to the PF noise. Additionally, due to $\alpha^2 + \beta^2 + \gamma^2 = 1$, we still get hold of

$$D^2_{ij} + C^2_{ij} = P_{ij} \leq 1$$

which is a significant conclusion.

4.2. A Practical Physical System for Spin-1/2 Heisenberg XXZ Model

Now, we simply introduce a renormalized spin-1/2 Heisenberg XXZ model. The Hamiltonian of spin-1/2 Heisenberg XXZ model on a periodic chain of $N$ sites is

$$H = \frac{J}{4} \sum_{k=1}^{N} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \delta \sigma_k^z \sigma_{k+1}^z)$$

where $\delta$ is the anisotropy parameter, $J$ is the exchange constant, and $J, \delta > 0$. $\sigma_k^\beta (= x, y, z)$ is standard Pauli matrices at site $k$. By employing the Kadanoff’s block method, it is necessary to divide the initial system Hamiltonian shown in Equation (42) into two parts\cite{68,69}

$$H = H^D + H^{DD}$$

where $H^D$ is the block Hamiltonian and $H^{DD}$ is the interblock Hamiltonian. The perturbative implementation of this method has been discussed in refs. [67,70]. One can present this approach in the first-order correction. The effective Hamiltonian is given by\cite{71}

$$H^{eff} = P_0 H^D P_0 + P_0 H^{DD} P_0$$

where $P_0$ is a projection operator. To get a renormalized form of the Hamiltonian, one can use a three-site block procedure. Note that, choosing the three-site block is essential here to get a self-similar Hamiltonian after each quantum-renormalization-group step. Subsequently, according to Equation (44), the effective Hamiltonian of the renormalized chain can be given by\cite{68,71}

$$H^{eff} = \frac{1}{4} \sum_{k=1}^{N/3} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \delta' \sigma_k^z \sigma_{k+1}^z)$$

Hence, the iterative relations are

$$J' = J (2q/2 + q^2)^2, \quad \delta' = \delta q^2/4$$

Herein, the degenerate ground states are given by\cite{71,72}

$$|\psi_0\rangle = (2 + q^2)^{-1/2} |\uparrow\uparrow\downarrow\rangle + q |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle$$

$$|\psi_0\rangle = (2 + q^2)^{-1/2} |\uparrow\downarrow\downarrow\rangle + q |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $\sigma_z$, and $q = -(\delta + \sqrt{8 + \delta^2})/2$. The corresponding energy is given by $E_0 = -J (\delta + \sqrt{8 + \delta^2})/4$. Then, the density matrix of the ground state can be read as $\rho_{23}(XXZ) = |\psi_0\rangle \langle \psi_0|$, where $|\psi_0\rangle$ is described as Equation (47). Certainly, if we use $|\psi_0\rangle$ to construct the density matrix, through calculations, the results will be the same ones. Then, we trace over sites 1, 2, and 3, respectively, and obtain three different reduced density matrices. In the same way, via using...
Equations (2), (3), (8), and (20), one can give the corresponding concurrence, DC, purity, and Bell nonlocality as follows

\[
C_{13}(XXZ) = \frac{2}{2 + q^2}
\]

\[
C_{12}(XXZ) = C_{21}(XXZ) = \frac{2\sqrt{q^2}}{2 + q^2}
\]

\[
D_{13}^2(XXZ) = \frac{q^4}{(2 + q^2)^2}
\]

\[
D_{12}^2(XXZ) = D_{21}^2(XXZ) = \frac{2 - 2q^2 + q^4}{(2 + q^2)^2}
\]

\[
P_{13}(XXZ) = \frac{4 + q^4}{(2 + q^2)^2}
\]

\[
P_{12}(XXZ) = P_{21}(XXZ) = \frac{2q^2 + q^4}{(2 + q^2)^2}
\]

and

\[
N_{13}^{\text{max}}(XXZ) = \max \left\{ 0, \frac{2\sqrt{q^2} - 4q^2 + 8}{2 + q^2} - 2, \frac{4\sqrt{2} - 2}{2 + q^2} \right\}
\]

\[
N_{12}^{\text{max}}(XXZ) = N_{23}^{\text{max}}(XXZ)
\]

\[
= \max \left\{ 0, \frac{4\sqrt{2q^2} - 2}{2 + q^2} - 2, \frac{2\sqrt{q^2} + 4q^2}{2 + q^2} - 2 \right\}
\]

respectively.

The diagram depicting inter-relations between the DC and the concurrence is plotted in Figure 5. The BNBSs can be given in the green and red areas for the different bipartite states in the renormalized spin-1/2 Heisenberg XXZ model in Figure 5. The lower boundary conditions for the BNBSs of the state \( \rho_{13}(XXZ) \) in Figure 5a, through analysis, Bell nonlocality should equal to zero for the bipartite state, one can obtain the lower boundary formula as

\[
D_{13}^2 = \left( \frac{C_{11}^2 + \sqrt{C_{13}^2 - C_{11}^2}}{2C_{11}^2 + \sqrt{C_{11}^2 - C_{11}^2}} \right)^2
\]

(53)

For the bipartite pure states, the upper boundary for the blue solid curve is unchanging and is always \( D_{13}^2 + C_{13}^2 = 1 \). Therefore, the green areas of BNBSs can be obtained when the value of concurrence is less than 0.76. In the same way, for the other reduced states \( \rho_{12}(XXZ) \) and \( \rho_{21}(XXZ) \), the lower boundary for the black dashed curve in Figure 5b can be disclosed as follows

\[
D_{12}^2 = \frac{13C_{12}^2 - 20C_{12}^2 + 8}{2(C_{12}^2 - 2)^2}
\]

(54)

Subsequently, the red areas of BNBSs can be revealed when the value of concurrence is less than \( \sqrt{41 - 3}/2 \). From Equations (49)–(51), we can still unveil \( D_{ij}^2 + C_{ij}^2 = P_{ij} \leq 1 \) in the renormalized spin-1/2 Heisenberg XXZ model.

5. Conclusions

In this work, we have investigated the bipartite QRs in the tripartite W-type state. It is indicated that the mutual relations among concurrence, the degree of coherence, purity, and Bell nonlocality can be revealed. Surprisingly, exact quantitative relation among the degree of coherence, concurrence, and purity is obtained in all two-qubit states derived from the tripartite W-type states. Additionally, both the concurrence and Bell nonlocality increase at the expense of the degree of coherence. Then, we have derived exact lower and upper boundary conditions for Bell nonlocal and Bell local states, and identify the bipartite states revealed at these wishful boundaries. On this basis, we have illustrated the exact relation between Bell nonlocality and the degree of coherence for the considered bipartite pure states. Furthermore, we have investigated two specific scenarios: one scenario is a tripartite W-type state under decoherence channel; we exploit how the PF channel impacts the mutual relations among these QRs. It
turns out that the proportion of the areas of BNBSs relative to BLBSs will decrease with the increase of decoherence strength. Besides, the concurrence decreases with the growing intensity of decoherence, whereas, the degree of coherence is not affected by decoherence, that is, the degree of coherence is immune to the PF channel. And the other one is a practical system for a renormalized spin-1/2 Heisenberg XXZ model; some exact quantitative relations between QRs and purity have also been exposed. These results will greatly rich one understand the intrinsic relations of QRs and make one better manipulate them to implement quantum-information processing.

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

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