On cosmic acceleration in four dimensional Einstein–Gauss–Bonnet gravity

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Abstract

We study the possibility that the Gauss-Bonnet term proposed in [1] gives rise to cosmic acceleration in four-dimensional ($D = 4$) FLRW space-time. Inserting a Gauss-Bonnet term multiplied by $1/(D - 4)$ in the action produces non-trivial terms in the dynamical equations and changes the cosmic evolution. We consider a model consisting of this term (playing the role of dark energy) and a barotropic fluid (e.g. dark matter) to describe a Universe dominated by dark matter and dark energy, then the evolution of this Universe is investigated.

1 Introduction

It has been more than two decades that we have found that the expansion of the Universe is positively accelerating [2,3]. Many models have been proposed to describe this acceleration in line with astrophysical data that has become much more accurate and complete in recent years. The negative pressure required for this expansion cannot be described in the framework of the standard cosmology with the known particles in the standard model of particle physics. Therefore some authors introduced exotic matter such as scalar field [4–13] and so on, in the framework of Einstein standard model of gravity, while some others modified the usual gravitational model [14–19]. Meanwhile, many authors used both methods simultaneously to correct each model’s defects according to observational data [20–26]. So in the literature, we encounter actions comprising many complicated scalar terms made by combinations of matter fields and (modified) geometrical functions of Riemann curvature, Einstein tensor, Ricci curvature, torsion, etc. One of these geometrical terms is the Gauss-Bonnet term. As the variation of this term

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with respect to the metric vanishes in four dimensions, it does alter the Einstein equation and does not alter the dynamics of the system. That is why, instead of using the pure Gauss-Bonnet term, which is a total derivative, modified Gauss-Bonnet model \cite{27-30} or generalized models in which the Gauss-Bonnet term is coupled to other fields have been used to study the Universe acceleration \cite{31-33}.

Recently a new model has been proposed in which the Gauss-Bonnet term appears with a factor $1/(D - 4)$ in the Lagrangian, where $D$ is the dimension of the space-time \cite{1}. As the contribution of Gauss-Bonnet in the equations of motion is proportional to $D - 4$, inserting this factor produces a nontrivial contribution in the Einstein equations in four dimensions. In other words, the infinity is eliminated by a zero of the same order in the classical equation of motion. This elimination looks somehow similar to the dimensional renormalization technique used in quantum field theory, where quantum infinite are canceled out by adding appropriate counterfeiters to the Lagrangian. However, in dimensional regularization, the dimension is treated as a continuous parameter. Despite the singular coefficients in the Lagrangian, it is the equations of motions which describes the classical behaviour of the system. In the last few months, many subjects have been studied in this framework, such as black hole solutions and their stability \cite{34-36}, Quasinormale modes \cite{37}, wormhole solutions \cite{38}, cosmological evolution \cite{39-41}, etc.

In this paper, we will consider the possible acceleration of the Universe in the context of this novel model. In this study, we employ only the Gauss-Bonnet term and do not involve other dark energy candidates to see how this term alone can lead to an acceleration in the cosmic evolution. The scheme of the paper is as follows: In the second section, we first introduce the model and present a detailed computation to derive the Friedmann equations in Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time. Then using these equations we will investigate the dynamics of the Universe.

We use units $\hbar = c = 1$ through the paper.

### 2 Friedmann equations in four dimensional EGB model

We consider the Einstein-Gauss-Bonnet (EGB) action

$$S = \int d^N x \sqrt{-g} \left( \frac{\mathcal{M}_P^2 R}{2} \right) + S_{GB} + S_m$$

\hspace{1cm} (1)

in an N-dimensional spatially flat FLRW space-time

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2 + \ldots + dx_{N-1}^2),$$

\hspace{1cm} (2)
where $a(t)$ is the scale factor. $M_P$ is the reduced Planck mass $M_P = 2.436 \times 10^{18} GeV$, $S_{GB}$ is the Gauss-Bonnet action

$$S_{GB} = \int d^N x \sqrt{-g} \alpha \mathcal{G}$$  \hspace{1cm} (3)$$

in which $\alpha$ is a constant and $\mathcal{G}$ is the Gauss-Bonnet term

$$\mathcal{G} = R^\rho\sigma R_{\mu\nu}^\rho - 4 R^\mu_{\nu} R^\nu_{\mu} + R^2.$$  \hspace{1cm} (4)$$

We have placed all matter components (baryonic and dark) in $S_m$. Variation of the Gauss-Bonnet action $S_{GB}$ with respect to the metric gives

$$\frac{g_{\mu\rho}}{\sqrt{-g}} \frac{\delta S_{GB}}{\delta g_{\mu\rho}} = A^\mu_{\nu} + \frac{1}{2} G \delta^\mu_{\nu}$$  \hspace{1cm} (5)$$

where $A^\mu_{\nu}$ is

$$A_{\mu\nu} = -2 R^\rho\alpha R_{\rho\alpha \nu} + 4 R^\mu\alpha R^\nu_\alpha + 4 R^\mu_\alpha R^\nu_\alpha - 2 RR_{\mu\nu}$$  \hspace{1cm} (6)$$

To determine (5) we need to compute Riemann curvature tensor, the Ricci tensor, the Ricci scalar, and the Gauss-Bonnet term for the space-time (2).

By computing the Riemann curvature tensor components we find

$$R_{\dot{i}\dot{0}0} = -a\ddot{a}, \hspace{0.5cm} R_{\dot{0}\dot{0}0} = a^2 \dddot{a}$$  \hspace{1cm} (7)$$

Leading to following components for the Ricci curvature

$$R_{00} = -(N-1)\frac{\dddot{a}}{a}, \hspace{0.5cm} R_{i\dot{i}} = (N-2)\dot{a}^2 + \dddot{a}$$  \hspace{1cm} (8)$$

So the scalar curvature is obtained as

$$R = (N-1)(N-2)(\dot{a}^2 + 2(N-1)\frac{\dddot{a}}{a})$$  \hspace{1cm} (9)$$

The Gauss Bonnet scalar, is derived as

$$\mathcal{G} = R^\rho\sigma R_{\rho\sigma\nu} - 4 R^\mu_{\nu} R^\nu_{\mu} + R^2$$

$$= (N-3)(N-2)(N-1) \left( (N-4)\frac{\dddot{a}^2}{a^3} + 4 \frac{\dot{a}^2 \dddot{a}}{a^3} \right)$$  \hspace{1cm} (10)$$

Using (7), (8), and (9), the tensor $A^\mu_{\nu}$, is computed as

$$A^i_{\dot{i}} = -2(N-2)(N-3) \left( (N-4)\frac{\dddot{a}^4}{a^4} + 3 \frac{\dot{a}^2 \dddot{a}}{a^3} \right)$$

$$A_{0\dot{0}} = -2(N-1)(N-2)(N-3)\frac{\dddot{a}^2}{a^2}$$  \hspace{1cm} (11)$$
So by using (10) and (11) the contribution of the Gauss-Bonne term to the equations of motion is given by

\[
\frac{g_{\mu \rho}}{\sqrt{-g}} \frac{\delta S_{GB}}{\delta g_{\mu \rho}} = \frac{1}{2} (N-2)(N-3)(N-4) \left( (N-5) \frac{\dot{a}^4}{a^4} + \frac{4a^2 \ddot{a}}{a^3} \right)
\]

\[
\frac{g_{0 \rho}}{\sqrt{-g}} \frac{\delta S_{GB}}{\delta g_{0 \rho}} = \frac{1}{2} (N-1)(N-2)(N-3)(N-4) \frac{\dot{a}^4}{a^4}
\]

which identically vanishes for \( N = 4 \). The Einstein tensor \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \) has the following non-zero components

\[
G_{00} = \frac{(N-1)(N-2)}{2} \frac{\dot{a}^2}{a^2}
\]

\[
G_{ii} = \frac{(N-2)(3-N)}{2} \dot{a}^2 + (2-N) \ddot{a}
\]

So collecting all together, by variation of the action (11) with respect to the metric, for the time (00) component, we obtain

\[
\frac{(N-1)(N-2)}{2} M_P^2 H^2 = -\alpha (N-4)(N-3)(N-2)(N-1) H^4 + \rho_m \quad (14)
\]

and for \( ii \) components we derive

\[
M_P^2 \left( \frac{(N-2)(3-N)}{2} H^2 + (2-N) \frac{\dot{a}}{a} \right) =
\]

\[
P_m + \alpha (N-4)(N-3)(N-2) \left( (N-5) H^2 + \frac{\dot{a}}{a} \right) H^2
\]

(15)

where \( H := \frac{\dot{a}}{a} \) is the Hubble parameter and the matter ingredient in \( S_m \) is assumed to be a perfect fluid with energy density \( \rho_m \) and pressure \( P_m \). Using the identity \( \frac{\dot{a}}{a} = H + H^2 \), (15) can be rewritten as

\[
\dot{H} = -\frac{P_m + \rho_m}{(N-2) (4\alpha(N-3)(N-4) H^2 + M_P^2)}
\]

(16)

(14) and (16) are Friedmann equations in \( N \) dimensions. In four dimensions they reduces to familiar Friedmann equations, unless \( (N-4)\alpha \) gains a finite nonzero value. This is only possible for \( \alpha \propto \frac{1}{N-4} \). By setting \( \alpha \to \frac{\alpha}{N-4} \), in four dimensions (14) and (16) reduce to

\[
3M_P^2 H^2 = \rho_m - 6\alpha H^4
\]

\[
\dot{H} = -\frac{P_m + \rho_m}{2 (4\alpha H^2 + M_P^2)}
\]

(17)

note that the equations (17), imply that the matter satisfies the continuity equation

\[
\dot{\rho}_m + 3H(P_m + \rho_m) = 0
\]

(18)
3 Universe acceleration in four dimensional Einstein–Gauss–Bonnet cosmology

Based on these modified Friedmann equations, we will study the possible acceleration of FLRW Universe whose dominant component is a barotropic matter (e.g. dark matter) $\rho_m$, with pressure $P_m = w_m \rho_m$. In the following, unless we explicitly mention, we assume that the equation of state (EoS) parameter satisfies $w_m \geq -\frac{1}{3}$ (so that it does not act as dark energy).

In order for the Hubble parameter to be real, the following condition must hold
\[ \alpha \rho_m > -\frac{3}{8} M_P^4 \]  
(19)

Also the positivity of $\rho_m$ requires
\[ \alpha H^2 \geq -\frac{1}{2} M_P^2 \]  
(20)

By substituting $\rho_m$ from the first equation of (17) in the second one, the EoS parameter of the Universe $w$, and the deceleration parameter $q = -1 - \frac{\dot{H}}{H^2}$ are obtained as

\[
q = -1 + \frac{3}{2} \gamma_m \left( \frac{1 + 2\alpha \frac{H^2}{M_P^2}}{1 + 4\alpha \frac{H^2}{M_P^2}} \right)
\]  
(21)

\[
w = -1 + \gamma_m \frac{1 + 2\alpha \frac{H^2}{M_P^2}}{1 + 4\alpha \frac{H^2}{M_P^2}}
\]  
(22)

respectively. Where $\gamma_m = w_m + 1$. If we took $\gamma_m = 0$ (e.g. a cosmological constant), the Gauss-Bonnet contribution would have no effect on the deceleration parameter, $q = -1$. We have acceleration ($q < 0$), provided that

\[
\gamma_m \left( \frac{1 + 2\alpha \frac{H^2}{M_P^2}}{1 + 4\alpha \frac{H^2}{M_P^2}} \right) < \frac{2}{3}
\]  
(23)

and for

\[
\left( \frac{1 + 2\alpha \frac{H^2}{M_P^2}}{1 + 4\alpha \frac{H^2}{M_P^2}} \right) < 0
\]  
(24)

the super-acceleration ($q < -1$ or equivalently $\dot{H} > 0$) occurs.

3.1 $\{ \alpha > 0, \gamma_m > \frac{2}{3} \}$

For $\alpha > 0$, the acceleration condition (23) reduces to

\[
2\alpha (3\gamma_m - 4) \frac{H^2}{M_P^2} < 2 - 3\gamma_m.
\]  
(25)
As $\gamma_m > 2/3$, acceleration requires $\gamma_m < 4/3$. Taking the matter as cold dark matter $\gamma_m = 1$, (25) becomes

$$2\alpha \frac{H^2}{M_p^2} > 1 \tag{26}$$

Hence if one intends to study the present acceleration of the Universe in this context, as $\frac{H_0^2}{M_p^2} \ll 1$, where $H_0 = H(a = 1)$ is the present Hubble parameter, he must choose a huge value of order $\frac{M_p^2}{H_0^2}$ for $\alpha$. Indeed $\frac{M_p^2}{H_0^2} \sim 10^{120}$ is of the same order as the ratio of the theoretical vacuum energy density to the observed cosmological density, encountered in the cosmological constant problem. For $H_0 = 67.4 \text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$ [42], $\alpha$ is obtained as $\alpha = 0.4 \times 10^{120}$.

Note that by considering $\gamma_m > 2/3$ and $\alpha > 0$, the minimum of the EoS parameter of this Universe is $w_{\text{min.}} = -2/3$. For $\gamma = 1$ (cold dark matter), $w$ cannot be less than $-1/2$. If based on [42], cosmological results, one takes the ratio density of dark energy (or cosmological constant) as $\Omega_d = 0.68$, and its EoS parameter as $w_d = -1.03 \pm 0.03$, he finds the equation of state parameter of the universe as $w = w_d \Omega_d + w_m \Omega_m \approx -0.7$ which is smaller than our minimum.

Although it seems that there exist acceleration for this model, but what about the transition from deceleration to acceleration? For an expanding Universe in order that these transitions occur, we require to have $\frac{dq}{dH^2} > 0$ at the transition point. From (21) we have

$$\frac{dq}{dH^2} = -\frac{3\alpha \gamma_m M_p^2}{(M_p^2 + 4\alpha H^2)^2} \tag{27}$$

which is negative for $\alpha > 0$. So in the $\alpha > 0$ case, although an acceleration solution may exist, but the transition cannot be described by this special model alone. As an illustrative example, By using (17) and (18), we have depicted the deceleration parameter in the case of $\alpha > 0$ in fig. (1), in terms of dimensionless time $\tau = tH_0$, where $H_0 = H(a = 1)$, showing an accelerated expanding Universe which ends to a deceleration state. At $a = 1$, we have taken $\frac{\rho_m(0)}{M_p H_0^2} = 0.9$ [42].

3.2 \{ $\gamma_m > \frac{2}{3}, \alpha < 0$\} 

Getting $\alpha < 0$, we find a singularity at $4\alpha \frac{H^2}{M_p^2} = -1$ diving the problem into two branches $4\alpha \frac{H^2}{M_p^2} > -1$ and $4\alpha \frac{H^2}{M_p^2} < -1$.

For $4\alpha \frac{H^2}{M_p^2} < -1$ and by considering (20), from (25) we find that the Universe is in a supper-accelerated regime.
Figure 1: Deceleration parameter in terms of dimensionless time $\tau = tH_0$, with $\alpha \frac{H^2}{M_P^2} = 20$ and initial conditions $\frac{\rho_m(0)}{M_P^2H_0^2} = 0.9$

For $4\alpha \frac{H^2}{M_P^2} > -1$ and by considering (20), we obtain

$$\left(1 + 2\alpha \frac{H^2}{M_P^2}\right) \left(1 + 4\alpha \frac{H^2}{M_P^2}\right) > 1 \quad (28)$$

So by comparing with (23), we find out that the Universe is in decelerated regime.

For $\alpha < 0$, we have

$$\frac{dq}{dH^2} = -\frac{3\alpha\gamma_m M_P^2}{(M_P^2 + 4\alpha H^2)^2} > 0 \quad (29)$$

Therefore $q$ increases in the super-acceleration regime, and tends to $q = -1$ eventually, while decreases in the deceleration regime and tends to $q = 0$ without crossing it. So like the $\alpha > 0$ case, we are unable to describe transitions in this context. As we will see later, $q = -1$ is a table fixed point.

By using (17) and (18), we have depicted the deceleration parameter in the case of $\alpha < 0$ for $4\alpha \frac{H^2}{M_P^2} < -1$ and $4\alpha \frac{H^2}{M_P^2} > -1$, in terms of dimensionless time $\tau = tH_0$, where $H_0 = H(a = 1)$, showing a super-acceleration expanding Universe which tends asymptotically to the fixed point $q = -1$ in the former fig.(2) and a deceleration Universe tending to $q = 1/2$ eventually fig.(3).
Figure 2: Deceleration parameter in terms of dimensionless time $\tau = tH_0$, with $\alpha \frac{H^2}{M_P^2} = -0.4$ and initial conditions $\frac{\rho_m(0)}{M_P^2H_0^2} = 0.9$

Figure 3: Deceleration parameter in terms of dimensionless time $\tau = tH_0$, with $\alpha \frac{H^2}{M_P^2} = -0.2$ and initial conditions $\frac{\rho_m(0)}{M_P^2H_0^2} = 0.9$

In the absence of any matter $\rho_m = 0$, and for $\alpha < 0$, we have de Sitter solution characterized by the constant Hubble parameter $H^2 = -\frac{M_P^2}{2\alpha}$. This is the critical point, and is stable. (17) and (18) form a system of autonomous equations with the critical point $\{\bar{H}^2 = -\frac{M_P^2}{2\alpha}, \bar{\rho}_m = 0\}$. Perturbing the system around its critical point $\bar{H} = \bar{H} + \delta H$, $\rho_m = \bar{\rho}_m + \delta \rho$, we find

$$\delta \dot{\rho}_m + 3\bar{H}\gamma_m \delta \rho_m = 0$$

(30)

and

$$-M_P^2 \left(1 + \frac{4\alpha}{M_P^2} \bar{H}^2\right) \delta \dot{H} = \frac{1}{2} \gamma_m \delta \rho_m,$$

(31)

and after some computation, we derive

$$\delta \rho_m = -6M_P^2 \bar{H} \delta H$$

(32)
Therefore from (31), (32), we obtain
\[ \delta \dot{H} = -3\gamma_m H \delta H \] (33)

Hence
\[ \delta \rho_m \propto \delta H \propto e^{-3\gamma_m H t} \] (34)

So for an expanding Universe \( \dot{H} > 0 \), the critical point is stable.

### 3.3 Gauss-Bonnet term as a dark energy component

For \( \alpha < 0 \), one may attribute a dark energy density \( \rho_d = -6\alpha H^4 \), and a pressure \( P_d \) to the Gauss-Bonnet contribution, and rewrites the Friedmann equations as
\[
H^2 = \frac{1}{3M_P^2}(\rho_m + \rho_d) \\
\dot{H} = -\frac{1}{2M_P^2}(\gamma_m \rho_m + P_d + \rho_d) \] (35)

By comparing (35) with (17), after some calculation we derive
\[ P_d = \frac{2\alpha H^2(12\alpha H^4 + 3M_P^2 H^2 - 2\gamma_m \rho_m)}{4\alpha H^2 + M_P^2} \] (36)

The EoS parameter is then
\[ w_d = -1 + 2\gamma_m \frac{\Omega_m}{1 + 4\alpha \frac{H^2}{M_P^2}} \] (37)

The dark energy EoS parameter depends completely on the matter’s EoS parameter and its ratio density \( \Omega_m := \frac{\rho_m}{3M_P^2 H^2} \). In the absence of matter \( \Omega_m = 0 \) or for \( \gamma_m = -1 \) (which is not of our interest in this paper), we obtain \( w_d = -1 \) and the Gauss-Bonnet term behaves as a cosmological constant. The ratio density of the dark sector is \( \Omega_d = \frac{\rho_d}{3M_P^2 H^2} \), but \( \rho_d = -6\alpha H^4 \), therefore
\[ \Omega_d = -2\alpha \frac{H^2}{M_P^2} \] (38)

This equation may be employed to obtain \( \alpha \). Putting \( \alpha = -\frac{1}{2}\Omega_d \frac{M_P^2}{H^2} \), back into (37), gives the EoS parameter as
\[ w_d = -1 + 2\gamma_m \frac{1 - \Omega_d}{1 - 2\Omega_d} \] (39)

From \( w = \omega_m \Omega_m + w_d \Omega_d \), the EoS of the Universe is
\[ w = \frac{(\gamma_m - 2)\Omega_d + 1 - \gamma_m}{2\Omega_d - 1} \] (40)
For $\Omega_d < \frac{1}{2}$, we have $w_d > 1/3$ and $w > -1/3$. As a result, in our study where $w_m \geq -1/3$, $\Omega_d < 1/2$ corresponds to a decelerating phase. In this situation as $H$ is decreasing the system eventually tends to $\{H = 0, \rho_m = 0, q = 0.5\}$.

For $\Omega_d > 1/2$, we have $w < -1$ and $w_d < -1$, which describes a super-acceleration phase. If the matter is considered as pressureless dark matter then $w_d = \frac{1}{1-2\Omega_d}$ and $w = \frac{\Omega_d}{1-2\Omega_d}$. In this phase $\frac{dq}{dH^2} > 0$ so by increasing of $H$, $q$ increases too, and the system tends to $w = -1$ which is a stable fixed point as proved before. In this situation too, we have only a super-acceleration regime, and the model is unable to describe the possible transition to or from (normal)acceleration regime.

So for $\alpha < 0$, we have two distinct branches $\Omega_d > 1/2$ and $\Omega_d < 1/2$. $\Omega_d < 1/2$ corresponds to a decelerating phase. In this case, $w_d$ is positive, and the Gauss-Bonnet term plays the role of a matter with positive pressure, while for $\Omega_d < 1/2$ the Gauss-Bonnet term plays the role of phantom-like dark energy with $w_d < -1$ causing supera-acceleration evolution.

4 Conclusion

We considered the new model introduced in [1], where the Gauss-Bonnet term appears with a singular coefficient in the action. This coefficient is eliminated in the equations of motion in four dimensions, and new contributions from the Gauss-Bonnet term emerge, giving rise to physical results. In an FLRW space-time filled nearly with a barotropic matter, we precisely derived the modified Friedmann equations and studied generally the cosmological consequences of the new terms in the acceleration of the Universe. To avoid the influence of other dark energies in our results, we restricted our model to contain only matter whose equation of state parameter satisfies $w_m > -1/3$. Based on Friedmann equations, the conditions required to have accelerating solutions were derived. It was shown that when the coefficient sign is positive we may obtain a solution with positive acceleration but the deceleration to acceleration transition cannot be explained by this model and eventually, the Universe decelerates. Besides, the EoS parameter of the Universe cannot be near $-1$. In this situation the regularized coefficient of the Gauss Bonnet term is the same as the discrepancy of vacuum and observable dark energy densities encountered in the cosmological constant problem.

For the negative coefficient, the Gauss-Bonnet term imitates the role of a (dark energy) component whose equation of state parameter is a function of its ratio density $\Omega_d$, as well as the equation of state parameter of the other ingredient: $w_m$. The solutions were classified into two distinct sets $\Omega_d < 1/2$ and $\Omega_d > 1/2$ separated by a singularity at $\Omega_d = 1/2$. For $\Omega_d > 1/2$, the Universe is in a super-accelerated phase and eventually, tends to a de Sitter
stable fixed point. For $\Omega_d < 1/2$, the Gauss-Bonnet term acts as a matter with positive pressure and the Universe is in the deceleration phase.

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