Measuring the general relativistic curvature of wave-fronts

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Einstein’s general theory of relativity predicts that an initially plane wave-front will curve because of gravity. This effect can now be measured using Very Long Baseline Interferometry (VLBI). A wave-front from a distant point source will curve as it passes the gravitational field of the Sun. We describe an idealised experiment to directly measure this curvature, using four VLBI stations on earth, separated by intercontinental distances. Expressed as a time delay, the size of the effect is a few hundred picoseconds and may be measureable with present technology.

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It is now possible to do interferometry with intercontinental baselines (VLBI) and clocks of picosecond accuracy [1,2]. This technological advance of the last few years suggests a new test of general relativity. The test consists of directly measuring the curvature of a wave-front coming from a distant radio source. In the absence of general relativistic (GR) effects, the wave-front from the distant source would appear plane. General relativity predicts that the wave-front will curve because of gravitational effects. By means of four VLBI stations located on the earth, one can directly measure this curvature. The effect, whose size we estimate below, may be measurable with currently available VLBI techniques. We describe an idealised version of the experiment in the hope of motivating VLBI astronomers to design more realistic experiments to detect the curvature of wave fronts.

That gravity curves wave-fronts is already established from astronomical observations of gravitational lensing. Multiple images, caustics and “luminous arcs” prove that initially plane wave-fronts curve under the influence of gravity. However, all these gravitational lenses are outside the solar system and we can only guess at and imperfectly model their structure. Tests of general relativity within the solar system are far more under our control, since we are on familiar ground. We can model the lenses in detail, work out the predictions of the theory and quantitatively confront theory with experiment.

Two general relativistic effects in the propagation of light have already been accurately measured—the Shapiro time delay [3] and the bending of light [4,5]. It is worth noting that analogous effects exist even in special relativity. The bending of light manifests itself as a change in the apparent direction of the source. In special relativity, uniform motion of the observer can result in aberration—an apparent change in the position of the source. Time delays too can be induced by uniform motion of the observer as in the Doppler effect. However curving of plane wave-fronts is a purely general relativistic effect that has no special relativistic analog. A plane wave in one inertial frame appears plane in all inertial frames. The experiment proposed in this paper consists of directly measuring the curvature of a wave-front, a purely GR effect with no special relativistic analog.

This letter is organised as follows: we first describe the theory behind the proposed experimental test. We then describe the experiment in an idealised form and show how to extract the general relativistic effect from measured quantities.

Consider a plane wave incident on an isolated static spherically symmetric body (the sun) of mass $m$ and radius $R$. We note that that $\epsilon := 2Gm/(c^2 R) = 2.10^{-8} << 1$ and neglect terms of order $\epsilon^2$. The effect of gravity can be calculated using the Schwarzschild metric. Neglecting terms of second and higher order in $\epsilon$, the Schwarzschild metric is given in standard co-ordinates by (we henceforth set $c = 1$).

$$ds^2 = (1 - \rho/r) dt^2 - (1 + \rho/r) dr^2 - r^2 d\Omega^2,$$

where $\rho = 2Gm$ is the Schwarzschild radius of the body. We can write the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(\epsilon^2)$, where $h_{\mu\nu}$ is the General Relativistic perturbation of the metric and our notation $O(\epsilon^2)$ means that we neglect quantities which are second order and higher in $\epsilon$. The propagation of electromagnetic waves [6] can be described in the geometrical optics approximation by the eikonal equation [7]

$$g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi = 0$$

and the solution $\psi(x)$ is the phase of the wave, a scalar function of the general co-ordinate $x^\mu, \mu = 0, 1, 2, 3$. Let us choose Minkowskian coordinates $x^\mu = (t, \vec{r})$ adapted to the flat metric $\eta_{\mu\nu}$. Let $\omega$ be the frequency at infinity of the incident plane wave and let $k^M_\mu$ be wave vector at infinity: $k^M_\mu = (\omega, 0, 0, -\omega)$, where we have chosen the direction of the incident plane wave along the positive $z$ axis. In Minkowski space the function $\psi^M(x) = k^M_\mu x^\mu$ with $\eta^{\mu\nu} k^M_\mu k^M_\nu = 0$ solves the eikonal equation. To first Post-Minkowskian order, the solution is $\psi = \psi^M + \phi$, where $\phi$ satisfies the differential equation

$$\eta^{\mu\nu} k^M_\mu \partial_\nu \phi = 1/2 h^{\mu\nu} k^M_\mu k^M_\nu.$$

Then (2) leads to
\[
\phi(x, y, z) = -\frac{\rho \omega}{2} \int_{z_-}^{z_+} dz \left[ \frac{1}{f^{1/2}} + \frac{u^2}{f^{3/2}} \right],
\]
where \( f = u^2 + x^2 + y^2 \) and \( z_- \to -\infty \). Integration yields:
\[\phi = \rho \omega \ln((r - z) + z/2r) \bigg|_{z_-}^{z_+}\]
where \( r = \sqrt{x^2 + y^2 + z^2} \). For lines of sight close to the sun \( (x, y << z) \) it is more illuminating to use the simple approximate form
\[\phi = \frac{(1 + \gamma) \rho \omega}{2} \ln((x^2 + y^2)/(2z))\]
which we arrive at using a binominal expansion and dropping a constant independent of \( (x, y, z) \). In (5), we have introduced the PPN (parametrised post-Newtonian [5]) parameter \( \gamma \), which is equal to 1 in general relativity.

The total phase of the wave is given by
\[\psi = k_\mu^M x^\mu + \phi,\]
where \( \phi \) is the small general relativistic correction, the Shapiro delay. The arrival of a wave-front at \( (x, y, z) \) is delayed by a time \( \phi(x, y, z)/\omega \) due to gravitational effects. Thus the gravitational field acts as a retarder, just as a glass slab does with light. There is a close mathematical relation between the Shapiro delay [3], the bending of light [4] and the curvature of wave-fronts which is the subject of this paper. Let us expand the eikonal \( \psi \) in a Taylor expansion about an event \( x_0 \):
\[
\psi(x^\mu) = \psi(x_0) + (x^\mu - x_0^\mu) \partial_{\mu} \psi|_{x_0} + 1/2(x^\mu - x_0^\mu)(x^\nu - x_0^\nu) \partial_{\mu} \partial_{\nu} \psi|_{x_0} + ... \quad (7)
\]
The Shapiro delay measures \( \psi(x_0) \), the eikonal. The bending of light is a measurement of its first derivative \( k_\mu = \partial_{\mu} \psi|_{x_0} \) and the curvature effect described here measures its second derivative. While these effects are related mathematically, they are distinct physical effects and should therefore be separately measured and checked against theoretical predictions. In these three effects the gravitational field acts respectively as a retarder, a prism and a lens, which act by delaying, tilting and curving a wave-front.

In the experiment to be described below, all observation points are on the earth (see figure 1). As a result the numerical value of \( h_{\mu \nu} \), the perturbation of the metric tensor is around \( 10^{-8} \). We will therefore neglect \( h_{\mu \nu} \) at the earth’s location and use the Minkowskian metric \( g_{\mu \nu} \) to raise and lower indices. Henceforth all dot products are formed using the flat Minkowskian metric.

A wave-front is a three dimensional surface in four dimensional space-time. For easy visualisation let us pick a slice \( t = 0 \). The wave-front is now a two dimensional surface described by
\[\mathbf{k} \cdot \mathbf{x} + \phi(x) = \text{constant}.\]
In order to detect the curvature [8] of this wave-front one needs to sample at least four points on it. For, given three points (or less), one can always find a plane passing through all of them. The deviation of the fourth point from the plane passing through the first three gives a measure of the curvature of the wave-front. More symmetrically, one can compute the volume of the tetrahedron with these four points as vertices. A non-zero volume would imply that these four points do not lie in a plane. This simple three dimensional argument provides the intuition we will use in describing the experiment. The principle of the experiment is that given the arrival events at three VLBI stations, one can model the incident wave as a plane wave and absorb any deviation in a redefinition of the apparent direction of the source. However, with the fourth station this freedom does not exist and one can measure genuine curvature effects [9]. Our method does not require an absolute determination of the direction of the source.

Consider four VLBI stations \( T_a, a = 0, 1, 2, 3 \) with intercontinental separations viewing a radio source along a line of sight passing near the sun. The VLBI experiment involves simultaneously observing a structureless point radio source from four telescopes. The signal voltage received at each antenna is beat against an ultra-stable local oscillator, low pass filtered, digitized and recorded. The recorded data from two antennae are cross-correlated with a delay to find the ‘fringe’—the delay at which the cross-correlation peaks. Thus one can accurately locate the events at which a given wave-front arrives at the four antennae.

Such VLBI techniques accurately measure the arrival times \( t_0 \) of a wave-front at each of these stations located at \( X_a \). Here \( x_a^\mu = (t_a, X_a) \) are the four four-vectors used to locate the four events, the arrival of a wave-front at each of the four stations. We will now describe how the measured quantities \( x_a^\mu, a = 0, 1, 2, 3 \) can be used to detect the curvature of the wave-front. Let us define \( X_a^\mu = x_a^\mu - x_0^\mu \), the three baselines [10] connecting the reference station \( T_0 \) to the other three stations. We write the components of \( X_a^\mu \) as \( (X_a, Y_a, Z_a, T_a) \). Note that the baselines have spatial as well as temporal components. We now construct the following determinant from these twelve numbers:
\[S = \det \left| \begin{array}{ccc}
X_1 & X_2 & X_3 \\
Y_1 & Y_2 & Y_3 \\
(Z_1 - T_1) & (Z_2 - T_2) & (Z_3 - T_3)
\end{array} \right| \quad (9)
\]
If the wave-front were plane with wave-vector \( k_\mu^M = (\omega, 0, 0, -\omega) \), the last row in the determinant \( S \) would vanish and so would \( S \). If the incident wave is plane, and has wave-vector \( k_\mu = (k_0, k_1, k_2, k_3) \), where \( k_1, k_2 \) are of order \( \epsilon \), we find again that \( S \) vanishes: From \( k_\mu k^\mu = \).
and the naive expectation \( t_3 \). From (9), we find that to first order in \( \epsilon \), \( k_0 = k_3 \). It then follows from

\[
k \cdot X_a = k_0(T_a - Z_a) - k_1X_a - k_2Y_a = 0
\]

that the last row of (9) is a linear combination of the first two and so \( S \) vanishes. A non-vanishing \( S \) implies that the wave-front is not plane and therefore is a diagnostic for the curvature of the wave-front.

Theoretically the measured arrival time difference \( T_a \) between stations \( a \) and \( 0 \) can be written (to first order in \( \epsilon \)) as

\[
T_a = Z_a - \tau_a
\]

where \( Z_a \) is the expected arrival time difference in the absence of gravitational effects and \( \tau_a = (\phi(x_a) - \phi(x_0))/\omega \). Plugging this into the expression for \( S \) we find that

\[
S = (\tau_1 a_1 + \tau_2 a_2 + \tau_3 a_3
\]

where \( a_1 = X_2 Y_3 - X_3 Y_2 \) (and cyclic) are the areas of the parallelograms formed by the spatial baselines projected on the \( x - y \) plane.

Let us express \( \tau_a \) using a Taylor expansion for \( \phi(x) \) keeping terms up to second order in \( x - x_0 \), as in (11). It is convenient to choose co-ordinates so that the reference event \( x_0 \) is in the \( x - z \) plane, \( x_0^y = 0 \). Setting \( x_0^y = b \), the impact parameter, we find that

\[
\tau_a = (1 + \gamma_0)\rho [X_a/b + (Y_a^2 - X_a^2)/b^2] + ...
\]

As expected, the first term, being linear in \( X_a \), drops out of the expression for \( S \) and \( S \) is given by

\[
\frac{(1 + \gamma_0)\rho}{b^2} [a_1(Y_1^2 - X_1^2) + a_2(Y_2^2 - X_2^2) + a_3(Y_3^2 - X_3^2)]
\]

A non-vanishing \( S \) signals curvature of the wave-front. We can express the effect in time units. Given all the measured quantities \( x_0^y \) except \( t_3 \), if the wave-front were plane we would expect that the arrival time of the wave-front at station three would be \( t_3^a \), where \( t_3^a \) satisfies the equation

\[
(Z_1 - T_1)a_1 + (Z_2 - T_2)a_2 + (Z_3 - t_3 + t_0)a_3 = 0,
\]

This expectation would be false, because \( S \) does not vanish and we have

\[
(Z_1 - T_1)a_1 + (Z_2 - T_2)a_2 + (Z_3 - t_3 + t_0)a_3 = S
\]

Subtracting (15) from (16) we find that \( t_3 \) would differ from \( t_3^a \) by an amount \( \Delta t_3 = t_3 - t_3^a \) given by

\[
\Delta t_3 = S/a_3
\]

The difference \( \Delta t_3 \) between the measured arrival time \( t_3 \) and the naive expectation \( t_3^a \) would detect the curvature of the wave-front. The size of the effect is easily estimated from (12,17). The main purpose of this letter is to draw attention to the fact that this effect may be measurable with present VLBI technology. Expressed in time units, it is of order

\[
10^{-5}|X|^2/b^2 s,
\]

where \( |X| \) is the typical size of the baseline (the intercontinental separation of the telescopes) and \( b \) is the impact parameter. If one chooses the impact parameter to be three solar radii, \( (b = 3R_\odot) \), the effect is 100 ps, which may be measurable. At grazing incidence, \( (b = R_\odot) \), the effect is as large as a nanosecond. However, lines of sight close to the Sun suffer from the problem of noise due to the Solar ionosphere. As one chooses larger impact parameters, this noise is reduced, but so also is the effect of interest (which decreases as the square of the impact parameter \( b \)). The optimal choice of impact parameter may be best determined by trial and error.

The ideal source for this experiment would be a strong, distant point source. A number of such sources have been already identified in the ICRF (International Celestial Reference Frame) catalog [11]. These sources are densely distributed over the sky with an average separation of a few degrees and can be detected using integration times of just a few minutes. Apart from the non-dispersive gravitational effects of interest there are some dispersive non-gravitational effects such as due to ionospheric fluctuations. Such phases can be removed by dual or multi-frequency observations, which is a standard technique in VLBI. There are also non-dispersive, non-gravitational effects such as due to the troposphere. These can be removed by the techniques of phase referencing [1] between sources, provided the sources are within a few degrees of each other.

As we mentioned earlier, the effect described here is contained in the Shapiro effect, just as the bending of light is. Since the Shapiro time delays are readily accounted for in VLBI observations, the proposed effect has already been implicitly measured. The purpose of this letter is to motivate observers to explicitly measure this effect that general relativity predicts. Although the Shapiro effect has been measured with high precision under varied conditions [12], this does not constitute a direct measurement of the curvature of a wave front, since one does not sample four points on the same wavefront. We suggest here that a a the curvature of the wave front is worth measuring directly. Indeed, the mean curvature of the wave-front is related via Raychaudhuri’s equation [13] to the integral of the Ricci curvature along the line of sight. In Einstein’s theory, we expect the wavefront to be a minimal surface (zero mean curvature). This prediction could be explicitly checked. To conclude, we have proposed an idealised experiment to directly measure the curvature of the wave-front from a distant radio source.
We hope to motivate VLBI astronomers to design a more realistic version of the experiment to measure the relativistic curvature of wave fronts.

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[8] Theoretically the curved wave-front has principal curvatures which are equal and opposite. Thus, the wave-front is a minimal surface with negative Gaussian curvature.
[9] We are referring here to the curvature of an incident plane wave-front due to the Sun’s gravitational field. We are not talking about the curvature of the wave-fronts [2] of radiation emitted by bodies at a finite distance.
[10] We suppose the locations of the stations so chosen that of the four three dimensional vectors $X_1, X_2, X_3, k$ every triple has a triple product well away from zero. This will prevent one station “shadowing” another and linearly dependent baselines.
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FIG. 1. Figure 1 shows disposition of the radio telescopes, all on the same hemisphere as the source S1, which is in common view of all the telescopes. The source S2 about three degrees away from S1 is used for phase referencing. The figure is not to scale.