Sequence Design to Minimize the Weighted Integrated and Peak Sidelobe Levels

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Abstract—Sequences with low aperiodic autocorrelation sidelobes are well known to have extensive applications in active sensing and communication systems. In this paper, we consider the problem of minimizing the weighted integrated sidelobe level (WISL), which can be used to design sequences with impulse-like autocorrelation and zero (or low) correlation zone. Two algorithms based on the general majorization-minimization method are developed to tackle the WISL minimization problem and the convergence to a stationary point is guaranteed. In addition, the proposed algorithms can be implemented via fast Fourier transform (FFT) operations and thus are computationally efficient, and an acceleration scheme has been considered to further accelerate the algorithms. Moreover, the proposed methods are extended to optimize the $\ell_2$-norm of the autocorrelation sidelobes, which lead to a way to minimize the peak sidelobe level (PSL) criterion. Numerical experiments show that the proposed algorithms can efficiently generate sequences with virtually zero autocorrelation sidelobes in a specified lag interval and can also produce very long sequences with much smaller PSL compared with some well known analytical sequences.

Index Terms—Autocorrelation, majorization-minimization, peak sidelobe level, weighted integrated sidelobe level, unit-modulus sequences.

I. INTRODUCTION

SEQUENCES with good autocorrelation properties lie at the heart of many active sensing and communication systems. Important applications include synchronization of digital communication systems (e.g., GPS receivers or CDMA cellular systems), pilot sequences for channel estimation, coded sonar and radar systems and even cryptography for secure systems [11–15]. In practice, due to the limitations of sequence generation hardware components (such as the maximum signal amplitude clip of analog-to-digital converters and power amplifiers), unit-modulus sequences (also known as polyphase sequences) are of special interest because of their maximum energy efficiency [3].

Let $\{x_n\}_{n=1}^N$ denote a complex unit-modulus sequence of length $N$, then the aperiodic autocorrelations of $\{x_n\}_{n=1}^N$ are defined as

$$r_k = \sum_{n=1}^{N-k} x_n^* x_{n+k}, \quad k = 0, \ldots, N-1. \quad (1)$$

The problem of sequence design for good autocorrelation properties usually arises when small autocorrelation sidelobes (i.e., $k \neq 0$) are required. To measure the goodness of the autocorrelation property of a sequence, two commonly used metrics are the integrated sidelobe level (ISL)

$$\text{ISL} = \sum_{k=1}^{N-1} |r_k|^2, \quad (2)$$

and the peak sidelobe level (PSL)

$$\text{PSL} = \max_{k} |r_k|^{N-1}. \quad (3)$$

Owing to the practical importance of sequences with low autocorrelation sidelobes, a lot of effort has been devoted to identifying such sequences. Binary Barker sequences, with their peak sidelobe level (PSL) no greater than 1, are perhaps the most well-known such sequences [6]. However, it is generally accepted that they do not exist for lengths greater than 13. In 1965, Golomb and Scholtz [7] started to investigate more general sequences called generalized Barker sequences, which obey the same PSL maximum, but may have complex (polyphase) elements. Since then, a lot of work has been done to extend the list of polyphase Barker sequences [8]–[12], and the longest one ever found is of length 77. It is still unknown whether there exist longer polyphase Barker sequences. Apart from searching for longer polyphase Barker sequences, some families of polyphase sequences with good autocorrelation properties that can be constructed in closed-form have also been proposed in the literature, such as the Frank sequences [13], the Chu sequences [14], and the Golomb sequences [15]. It has been shown that the PSL’s of these sequences grow almost linearly with the square root of the length $N$ of the sequences [16], [17].

In recent years, several optimization based approaches have been proposed to tackle sequence design problems, see [18]–[22]. Among them, [22] and [18] proposed to design unit-modulus sequences with low autocorrelation by directly minimizing the true ISL metric or a simpler criterion that is “almost equivalent” to the ISL metric. Efficient algorithms based on fast Fourier transform (FFT) operations were developed and shown to be capable of producing very long sequences (of length $10^4$ or even larger) with much lower autocorrelation compared with the Frank sequences and Golomb sequences.

Why the ISL metric was chosen as the objective in the optimization approaches? It is probably because the ISL metric is more tractable compared with the PSL metric from an optimization point of view. But as in the definition of Barker sequences, PSL seems to be the preferred metric in many cases. So it is of particular interest to also develop efficient optimization algorithms that can minimize the PSL metric. Additionally, in some applications, instead of sequences with...
impulse-like autocorrelation, zero correlation zone (ZCZ) sequences (with zero correlations over a smaller range) would suffice [23, 24]. In [18], an algorithm named WeCAN (weighted cyclic algorithm new) was proposed to design sequences with zero or low correlation zone.

In this paper, we consider the problem of minimizing the weighted ISL metric, which includes the ISL minimization problem as a special case and can be used to design zero or low correlation zone sequences by properly choosing the weights. Two efficient algorithms are developed based on the general majorization-minimization (MM) method by constructing two different majorization functions. The proposed algorithms can be implemented by means of FFT operations and are thus very efficient in practice. The convergence of the algorithms to a stationary point is proved. An acceleration scheme is also introduced to further accelerate the proposed algorithms. We also extend the proposed algorithms to minimize the $\ell_p$-norm of the autocorrelation sidelobes. The resulting algorithm can be adopted to minimize the PSL metric of unit-modulus sequences.

The remaining sections of the paper are organized as follows. In Section II, the problem formulation is presented. In Sections III and IV, we first give a brief review of the MM method and then two MM algorithms are derived, followed by the convergence analysis and an acceleration scheme in Section V. In Section VI, the algorithms are extended to minimize the $\ell_p$-norm of the autocorrelation sidelobes. Finally, Section VII presents some numerical results and the conclusions are given in Section VII.

Notation: Boldface upper case letters denote matrices, boldface lower case letters denote column vectors, and italics denote scalars. $\mathbf{R}$ and $\mathbf{C}$ denote the real and complex field, respectively. $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote the real and imaginary part, respectively. $\arg(\cdot)$ denotes the phase of a complex number. The superscripts $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote transpose, complex conjugate, and conjugate transpose, respectively. $\circ$ denotes the Hadamard product. $X_{i,j}$ denotes the $(i, j)$-th element of matrix $X$ and $x_i$ denotes the $i$-th element of vector $x$. $X_{i,:}$ denotes the $i$-th row of matrix $X$. $X_{:,j}$ denotes the $j$-th column of matrix $X$, and $X_{i:j,k:l}$ denotes the submatrix of $X$ from $X_{i,k}$ to $X_{j,l}$.

\text{diag}(\cdot)$ is a column vector consisting of all the diagonal elements of $X$. $\text{vec}(X)$ is a diagonal matrix formed with $X$ as its principal diagonal. $I_n$ denotes an $n \times n$ identity matrix.

II. PROBLEM FORMULATION

Let $\{x_n\}_{n=1}^N$ denote the complex unit-modulus sequence to be designed and $\{r_k\}_{k=1}^{N-1}$ be the aperiodic autocorrelations of $\{x_n\}_{n=1}^N$ as defined in (1), then we define the weighted integrated sidelobe level (WISL) as

$$\text{WISL} = \sum_{k=1}^{N-1} w_k |r_k|^2,$$

(4)

where $w_k \geq 0$, $k = 1, \ldots, N - 1$. It is easy to see that the WISL metric includes the ISL metric as a special case by simply taking $w_k = 1$, $k = 1, \ldots, N - 1$.

The problem of interest in this paper is the following WISL minimization problem:

\begin{align*}
\text{minimize} & \quad \text{WISL} \\
\text{subject to} & \quad |x_n| = 1, \quad n = 1, \ldots, N, \quad (5)
\end{align*}

which includes the ISL minimization problem considered in [23] as a special case and can be used to design zero (or low) correlation zone sequences by assigning larger weights for the sidelobes that we want to minimize.

An algorithm named WeCAN was proposed in [18] to tackle this problem. However, instead of directly minimizing the WISL metric as in (5), WeCAN tries to minimize an “almost equivalent” criterion. Moreover, the WeCAN algorithm requires computing the square-root of an $N \times N$ matrix at the beginning and $N$ FFT’s at each iteration, which could be costly for large $N$. In the next section, we will develop algorithms that directly minimize the original WISL metric, and at the same time are computationally much more efficient than the WeCAN algorithm.

III. SEQUENCE DESIGN VIA
MAJORIZATION-MINIMIZATION

In this section, we first introduce the general majorization-minimization (MM) method briefly and then apply it to derive simple algorithms to solve the problem (5).

A. The MM Method

The MM method refers to the majorization-minimization method, which is an approach to solve optimization problems that are too difficult to solve directly. The principle behind the MM method is to transform a difficult problem into a series of simpler problems. Interested readers may refer to [23, 26] and references therein for more details (recent generalizations include [27, 28]).

Suppose we want to minimize $f(x)$ over $\mathcal{X} \subseteq \mathbb{C}^n$. Instead of minimizing the cost function $f(x)$ directly, the MM approach optimizes a sequence of approximate objective functions that majorize $f(x)$. More specifically, starting from a feasible point $x^{(0)}$, the algorithm produces a sequence $\{x^{(k)}\}$ according to the following update rule:

$$x^{(k+1)} \in \arg \min_{x \in \mathcal{X}} u(x, x^{(k)}),$$

(6)

where $x^{(k)}$ is the point generated by the algorithm at iteration $k$, and $u(x, x^{(k)})$ is the majorization function of $f(x)$ at $x^{(k)}$. Formally, the function $u(x, x^{(k)})$ is said to majorize the function $f(x)$ at the point $x^{(k)}$ if

$$u(x, x^{(k)}) \geq f(x), \quad \forall x \in \mathcal{X},$$

(7)

$$u(x^{(k)}, x^{(k)}) = f(x^{(k)}).$$

(8)

In other words, function $u(x, x^{(k)})$ is an upper bound of $f(x)$ over $\mathcal{X}$ and coincides with $f(x)$ at $x^{(k)}$.

To summarize, to minimize $f(x)$ over $\mathcal{X} \subseteq \mathbb{C}^n$, the main steps of the majorization-minimization scheme are

1) Find a feasible point $x^{(0)}$ and set $k = 0$.
2) Construct a function $u(x, x^{(k)})$ that majorizes $f(x)$ at $x^{(k)}$. 

"Almost equivalent" criterion. Moreover, the WeCAN algorithm requires computing the square-root of an $N \times N$ matrix at the beginning and $N$ FFT’s at each iteration, which could be costly for large $N$. In the next section, we will develop algorithms that directly minimize the original WISL metric, and at the same time are computationally much more efficient than the WeCAN algorithm.
Let $\mathbf{x}^{(k+1)} \in \arg\min_{\mathbf{x} \in \mathcal{X}} u(\mathbf{x}, \mathbf{x}^{(k)})$.

4) If some convergence criterion is met, exit; otherwise, set $k = k + 1$ and go to step (2).

It is easy to show that with this scheme, the objective value is monotonically decreasing (nonincreasing) at every iteration, i.e.,

$$f(x(k+1)) \leq u(x(k+1), x(k)) \leq u(x(k), x(k)) = f(x(k)).$$

(9)

The first inequality and the third equality follow from the the properties of the majorization function, namely (7) and (8) respectively and the second inequality follows from (2). The monotonicity makes MM algorithms very stable in practice.

B. WISL Minimization via MM

To solve the problem (5) via majorization-minimization, the key step is to find a majorization function of the objective such that the majorized problem is easy to solve. For that purpose we first present a simple result that will be useful later.

**Lemma 1** (22). Let $\mathbf{L}$ be an $n \times n$ Hermitian matrix and $\mathbf{M}$ be another $n \times n$ Hermitian matrix such that $\mathbf{M} \succeq \mathbf{L}$. Then for any point $x_0 \in \mathbb{C}^n$, the quadratic function $\mathbf{x}^H \mathbf{M} \mathbf{x}$ is majorized by $\mathbf{x}^H \mathbf{L} \mathbf{x} + 2 \mathbf{Re} (\mathbf{x}^H (\mathbf{L} - \mathbf{M}) x_0) + x_0^H (\mathbf{M} - \mathbf{L}) x_0$ at $x_0$.

Let us define $\mathbf{U}_k, k = 0, \ldots, N - 1$ to be $N \times N$ Toeplitz matrices with the $k$th diagonal elements being 1 and 0 elsewhere. Noting that

$$r_k = \text{Tr}(\mathbf{U}_k \mathbf{x} \mathbf{x}^H), k = 0, \ldots, N - 1,$$

we can rewrite the problem (5) as

$$\begin{align*}
\text{minimize} & \quad \sum_{k=1}^{N-1} w_k |\text{Tr}(\mathbf{U}_k \mathbf{x})|^2 \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N.
\end{align*}$$

(10)

Let $\mathbf{U}_{-k} = \mathbf{U}_k^T, k = 1, \ldots, N - 1$, it is easy to see that $\text{Tr}(\mathbf{U}_{-k} \mathbf{x}) = r_{-k} = r_k$, so we can rewrite the problem (11) in a more symmetric way:

$$\begin{align*}
\text{minimize} & \quad \frac{1}{2} \sum_{k=1-N}^{N-1} w_k |\text{Tr}(\mathbf{U}_k \mathbf{x})|^2 \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N,
\end{align*}$$

(12)

where $w_{-k} = w_k$ and $w_0 = 0$. Since $\text{Tr}(\mathbf{U}_k \mathbf{x}) = \text{vec}(\mathbf{x})^H \text{vec}(\mathbf{U}_k)$, the problem (12) can be further rewritten using vec notation as (the constant factor $\frac{1}{2}$ is ignored):

$$\begin{align*}
\text{minimize} & \quad \sum_{k=1-N}^{N-1} w_k \text{vec}(\mathbf{x})^H \text{vec}(\mathbf{U}_k) \text{vec}(\mathbf{U}_k)^H \text{vec}(\mathbf{x}) \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N.
\end{align*}$$

Let us define

$$\mathbf{L} = \sum_{k=1-N}^{N-1} w_k \text{vec}(\mathbf{U}_k) \text{vec}(\mathbf{U}_k)^H$$

(14)

and denote the objective function of (13) by $f(\mathbf{X})$, then $f(\mathbf{X}) = \text{vec}(\mathbf{X})^H \mathbf{L} \text{vec}(\mathbf{X})$, which is clearly a quadratic function in $\mathbf{X}$. According to Lemma 1 we can construct a majorization function of $f(\mathbf{X})$ by simply choosing a matrix $\mathbf{M}$ such that $\mathbf{M} \succeq \mathbf{L}$. A simple choice of $\mathbf{M}$ can be $\mathbf{M} = \lambda_{\text{max}}(\mathbf{L}) \mathbf{I}$, where $\lambda_{\text{max}}(\mathbf{L})$ is the maximum eigenvalue of $\mathbf{L}$. Owing to the special structure of $\mathbf{L}$, it can be shown that $\lambda_{\text{max}}(\mathbf{L})$ can be computed efficiently in closed form.

**Lemma 2.** Let $\mathbf{L}$ be the matrix defined in (14). Then the maximum eigenvalue of $\mathbf{L}$ is given by $\lambda_{\text{max}}(\mathbf{L}) = \max_k \{ w_k (N - k)|k = 1, \ldots, N - 1 \}$.

**Proof:** It is easy to see that the set of vectors $\{ \text{vec}(\mathbf{U}_k) \}_{k=1-N}$ are mutually orthogonal. For $k = 1 - N, \ldots, N - 1$, we have

$$\begin{align*}
\text{vec}(\mathbf{U}_k) &= \sum_{j=1-N}^{N-1} w_j \text{vec}(\mathbf{U}_j) \text{vec}(\mathbf{U}_j)^H \text{vec}(\mathbf{U}_k) \\
&= w_k \text{vec}(\mathbf{U}_k)^H \text{vec}(\mathbf{U}_k) \\
&= w_k (N - k) \text{vec}(\mathbf{U}_k).
\end{align*}$$

Thus $w_k (N - |k|), k = 1 - N, \ldots, N - 1$ are the nonzero eigenvalues of $\mathbf{L}$ with corresponding eigenvectors $\text{vec}(\mathbf{U}_k), k = 1 - N, \ldots, N - 1$. Since $w_{-k} = w_k$ and $w_0 = 0$, the maximum eigenvalue of $\mathbf{L}$ is given by $\max_k \{ w_k (N - k)|k = 1, \ldots, N - 1 \}$.

Then given $\mathbf{X}^{(l)} = \mathbf{X}^{(l)} (\mathbf{X}^{(l)})^H$ at iteration $l$, by choosing $\mathbf{M} = \lambda_{\text{max}}(\mathbf{L}) \mathbf{I}$ in Lemma 1 we know that the objective of (13) is majorized by the following function at $\mathbf{X}^{(l)}$:

$$\begin{align*}
\text{minimize} & \quad \text{Re}(\text{vec}(\mathbf{X})^H (\mathbf{L} - \lambda_{\text{max}}(\mathbf{L}) \mathbf{I}) \text{vec}(\mathbf{X})) \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N.
\end{align*}$$

(17)

Substituting $\mathbf{L}$ in (14) back into (17), the problems becomes

$$\begin{align*}
\text{minimize} & \quad \sum_{k=1-N}^{N-1} w_k \text{Re}(\text{Tr}(\mathbf{U}_{-k} \mathbf{X}^{(l)}) \text{Tr}(\mathbf{U}_k \mathbf{X})) \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N.
\end{align*}$$

(18)

Since $\text{Tr}(\mathbf{U}_{-k} \mathbf{X}^{(l)}) = r_{-k}^{(l)}$, the problem (18) can be rewritten as

$$\begin{align*}
\text{minimize} & \quad \text{Re} \left( \text{Tr} \left( \sum_{k=1-N}^{N-1} w_k r_{-k}^{(l)} \mathbf{U}_k \mathbf{X} \right) \right) \\
\text{subject to} & \quad \mathbf{x} = \mathbf{x} \mathbf{x}^H \\
& \quad |x_n| = 1, n = 1, \ldots, N.
\end{align*}$$

(19)
which can be further simplified as

\[
\begin{align*}
\text{minimize} & \quad x^H (R - \lambda_{\text{max}}(L)x^{(l)})(x^{(l)})^H x \\
\text{subject to} & \quad |x_n| = 1, \, n = 1, \ldots, N,
\end{align*}
\]

where

\[
R = \sum_{k=1}^{N-1} w_k r_{-k}^T U_k
\]

(20)

\[
= \begin{bmatrix}
0 & w_1 r_{-1}^T & \cdots & w_{N-1} r_{-N+1}^T \\
w_1 r_1^T & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
w_{N-1} r_{N-1}^T & \cdots & w_1 r_{-1}^T & 0
\end{bmatrix}
\]

(21)

is a Hermitian Toeplitz matrix, and \(\{r_k^{(l)}\}_{k=1}^{N-1}\) are the autocorrelations of the sequence \(\{x_n^{(l)}\}_{n=1}^N\).

It is clear that the objective function in (20) is quadratic in \(x\), but the problem (20) is still hard to solve directly. So we propose to majorize the objective function of problem (20) at \(x^{(l)}\) again to further simplify the problem that we need to solve at each iteration. Similarly, to construct a majorization function of the objective, we need to find a matrix \(M\) such that \(M \succeq R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H\). As one choice, one may choose \(M = \lambda_{\text{max}}(R) - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H\) as in the first majorization step. But in this case, to compute the maximum eigenvalue of the matrix \(R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H\), some iterative algorithms are needed, in contrast to the simple closed form expression in the first majorization step. To maintain the simplicity and the computational efficiency of the algorithm, here we propose to use some upper bound of \(\lambda_{\text{max}}(R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H)\) that can be easily computed instead. To derive the upper bound, we first introduce a useful result regarding the bounds of the extreme eigenvalues of Hermitian Toeplitz matrices [29].

**Lemma 3.** Let \(T\) be an \(N \times N\) Hermitian Toeplitz matrix defined by \(\{t_k\}_{k=0}^{N-1}\) as follows

\[
T = \begin{bmatrix}
t_0 & t_1^* & \cdots & t_{N-1}^* \\
t_1 & t_0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
t_{N-1} & \cdots & t_1 & t_0
\end{bmatrix}
\]

and \(F\) be a \(2N \times 2N\) FFT matrix with \(F_{m,n} = e^{-j \frac{2\pi}{2N} mn}, 0 \leq m, n < 2N\). Let \(c = [t_0, t_1, \ldots, t_{N-1}, 0, t_{N-1}^*, \ldots, t_1^*, 0]^T\) and \(\mu = F c\) be the discrete Fourier transform of \(c\). Then

\[
\lambda_{\text{max}}(T) \leq \frac{1}{2} \left( \max_{1 \leq i \leq N} \mu_{2i} + \max_{1 \leq i \leq N} \mu_{2i-1} \right), \quad \lambda_{\text{min}}(T) \geq \frac{1}{2} \left( \min_{1 \leq i \leq N} \mu_{2i} + \min_{1 \leq i \leq N} \mu_{2i-1} \right).
\]

(22)

(23)

**Proof:** See [29].

Since the matrix \(R\) is Hermitian Toeplitz, according to Lemma 3 we know that

\[
\lambda_{\text{max}}(R) \leq \frac{1}{2} \left( \max_{1 \leq i \leq N} \mu_{2i} + \max_{1 \leq i \leq N} \mu_{2i-1} \right), \quad \lambda_{\text{min}}(R) \geq \frac{1}{2} \left( \min_{1 \leq i \leq N} \mu_{2i} + \min_{1 \leq i \leq N} \mu_{2i-1} \right).
\]

(24)

where \(\mu = Fc\) and \(c = [0, w_1 r_1^T, \ldots, w_{N-1} r_{N-1}^T, 0, w_{N-1} r_{1-N}^T, \ldots, w_1 r_{-1}^T]^T\).

Let us denote the upper bound of \(\lambda_{\text{max}}(R)\) at the right hand side of (24) by \(\lambda_u\), i.e.,

\[
\lambda_u = \frac{1}{2} \left( \max_{1 \leq i \leq N} \mu_{2i} + \max_{1 \leq i \leq N} \mu_{2i-1} \right).
\]

(25)

Since \(\lambda_{\text{max}}(L) \geq 0\), it is easy to see that \(\lambda_u \geq \lambda_{\text{max}}(R) \geq \lambda_{\text{max}}(R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H)\).

Thus, we may choose \(M = \lambda_u I\) in Lemma 1 and the objective of (20) is majorized by

\[
u_2(x, x^{(l)}) = \lambda_u x^H x + 2\text{Re} \left( x^H (R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H - \lambda_u I)x^{(l)} \right) + \left( x^{(l)} \right)^H \left( \lambda_u I - R + \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H \right)x^{(l)}.
\]

(26)

(27)

Since \(x^H x = N\), the first term of (28) is a constant. Again by ignoring the constant terms, we have the majorized problem of (20):

\[
\begin{align*}
\text{minimize} & \quad \text{Re} \left( x^H (R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H - \lambda_u I)x^{(l)} \right) \\
\text{subject to} & \quad |x_n| = 1, \, n = 1, \ldots, N,
\end{align*}
\]

(29)

which can be rewritten as

\[
\begin{align*}
\text{minimize} & \quad \| x - y \|_2 \\
\text{subject to} & \quad |x_n| = 1, \, n = 1, \ldots, N,
\end{align*}
\]

(30)

where

\[
y = -(R - \lambda_{\text{max}}(L)x^{(l)}(x^{(l)})^H - \lambda_u I)x^{(l)} = (\lambda_{\text{max}}(L)N + \lambda_u) x^{(l)} - Rx^{(l)}.
\]

(31)

It is easy to see that the problem (30) has a closed form solution, which is given by

\[
x_n = e^{j \text{arg}(y_n)}, \, n = 1, \ldots, N.
\]

(32)

Note that although we have applied the majorization-minimization scheme twice at the point \(x^{(l)}\), it can be viewed as directly majorizing the objective function of (5) at \(x^{(l)}\) by the following function:

\[
u_2(x, x^{(l)}) = \nu_2(x, x^{(l)}) + \lambda_{\text{max}}(L)N^2 - \sum_{k=1}^{N-1} w_k |r_k^{(l)}|^2
\]

(33)

and the minimizer of \(u(x, x^{(l)})\) over the constraint set is given by (32).

According to the steps of the majorization-minimization scheme described in section III-A, we can now readily have a straightforward implementation of the algorithm, which at each iteration computes \(y\) according to (31) and update \(x\) via (32). It is easy to see that the main cost is the computation of \(y\).
in (31). To obtain an efficient implementation of the algorithm, here we further explore the special structures of the matrices involved in the computation of $y$.

We first notice that to compute $\lambda_u$, we need to compute the FFT of the vector $c$ in (25) and the autocorrelations \{$r_k\}_{k=1-N}$ of \{$x_n\}_n$ are needed to form the vector $c$. It is well known that the autocorrelations can be computed efficiently via FFT (IFFT) operations, i.e.,

$$
\begin{align*}
[r^{(l)}_0, r^{(l)}_1, \ldots, r^{(l)}_{N-1}, 0, r^{(l)}_{1-N}, \ldots, r^{(l)}_{-1}]^T
= \frac{1}{2N} F^H F[x^{(l)}T]0_{1 \times N}^T \bigg|_2,
\end{align*}
$$

where $F$ is the $2N \times 2N$ FFT matrix and $|\cdot|^2$ denotes the element-wise absolute-squared value. Next we present another simple result regarding Hermitian Toeplitz matrices that can be used to compute the matrix vector multiplication $R \mathbf{x}^{(l)}$ efficiently via FFT (IFFT).

**Lemma 4.** Let $T$ be an $N \times N$ Hermitian Toeplitz matrix defined as follows

$$
\begin{align*}
T = \begin{bmatrix}
t_0 & t_1^* & \cdots & t_{N-1}^* \\
t_1 & t_0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
t_{N-1} & t_1 & \cdots & t_0
\end{bmatrix}
\end{align*}
$$

and $F$ be a $2N \times 2N$ FFT matrix with $F_{m,n} = e^{-j \frac{2\pi mn}{2N}}, 0 \leq m, n < 2N$. Then $T$ can be decomposed as $T = \frac{1}{2N} F^H_{1 \times N} \text{Diag}(Fc) F_{1 \times N}$, where $c = [t_0, t_1, \ldots, t_{N-1}, 0, t_{N-1}^*, \ldots, t_1^*]^T$.

**Proof:** The $N \times N$ Hermitian Toeplitz matrix $T$ can be embedded in a circulant matrix $C$ of dimension $2N \times 2N$ as follows:

$$
C = \begin{bmatrix}
T & W \\
W & T
\end{bmatrix},
$$

where

$$
W = \begin{bmatrix}
0 & t_{N-1} & \cdots & t_1 \\
t_{N-1}^* & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
t_1^* & \cdots & t_{N-1}^* & 0
\end{bmatrix}
$$

The circulant matrix $C$ can be diagonalized by the FFT matrix $F$, i.e.,

$$
C = \frac{1}{2N} F^H \text{Diag}(Fc) F,
$$

where $c$ is the first column of $C$, i.e., $c = [t_0, t_1, \ldots, t_{N-1}, 0, t_{N-1}^*, \ldots, t_1^*]^T$. Since the matrix $T$ is just the upper left $N \times N$ block of $C$, we can easily obtain $T = \frac{1}{2N} F^H_{1 \times N} \text{Diag}(Fc) F_{1 \times N}$.

Since the matrix $R$ is Hermitian Toeplitz, from Lemma 4 we easily have

$$
R = \frac{1}{2N} F^H_{1 \times N} \text{Diag}(Fc) F_{1 \times N},
$$

which is the same as the one defined in (25), which can be reused here. With the decomposition of $R$ given in (38), it is easy to see that the matrix vector multiplication $R \mathbf{x}^{(l)}$ can be performed by means of FFT (IFFT) operations.

Now we are ready to summarize the overall algorithm and it is given in Algorithm 1. The algorithm falls into the general framework of MM algorithms and thus preserves the monotonicity of such algorithms. It is also worth noting that the per iteration computation of the algorithm is dominated by four FFT (IFFT) operations and thus computationally very efficient.

**Algorithm 1** MWISL - Monotonic minimizer for Weighted ISL.

**Require:** sequence length $N$, weights $\{w_k \geq 0\}_{k=1}^{N-1}$

1. Set $l = 0$, initialize $\mathbf{x}^{(0)}$.
2. $\lambda_L = \max_k \{w_k (N - k) | k = 1, \ldots, N - 1\}$
3. repeat
   4. $f = F[x^{(l)}T, 0_{1 \times N}]^T$
   5. $r = \frac{1}{2N} F^H [f]^2$
   6. $c = r \circ [0, w_1, \ldots, w_{N-1}, 0, w_{N-1}, \ldots, w_1]^T$
   7. $\mu = Fc$
   8. $\lambda_u = \frac{1}{2} \left( \max_{1 \leq i \leq N} \mu_{2i} + \max_{1 \leq i \leq N} \mu_{2i-1} \right)$
   9. $y = x^{(l)} - \frac{r^H_{\lambda_u}}{2N(\lambda_u N + \lambda_L)}$
10. $x^{(l+1)} = e^{\text{arg}(ym)}$, $n = 1, \ldots, N$
11. $l \leftarrow l + 1$
12. until convergence

**IV. WISL MINIMIZATION WITH AN IMPROVED MAJORIZATION FUNCTION**

As described in the previous section, the proposed algorithm is based on the majorization-minimization principle, and the nature of the majorization functions usually dictate the performance of the algorithm. In this section, we further explore the special structure of the problem and construct a different majorization function.

Notice that to obtain the simple algorithm in the previous section, a key point is that the first term of the majorization-minimization problem \(u_1 \mathbf{X}, \mathbf{X}^{(l)}\) in (16) is just a constant and can be ignored, which removes the higher order term in the objective of the majorized problem (17). In the previous section, we have chosen $\mathbf{M} = \lambda_{\max}(\mathbf{L}) \mathbf{I} \geq \mathbf{L}$ such that $\text{vec}(\mathbf{X})^H \mathbf{M} \text{vec}(\mathbf{X}) = \lambda_{\max}(\mathbf{L}) \text{vec}(\mathbf{X})^H \text{vec}(\mathbf{X})$ is a constant over the constraint set. But it is easy to see that, to ensure the term $\text{vec}(\mathbf{X})^H \mathbf{M} \text{vec}(\mathbf{X})$ being constant, it is sufficient to require $\mathbf{M}$ being diagonal, i.e., choosing $\mathbf{M} = \text{Diag}(\mathbf{b}) \geq \mathbf{L}$. To construct a tight majorization function, here we consider the following problem

$$
\begin{align*}
\min_{\mathbf{b}} & \quad \text{Tr}(\text{Diag}(\mathbf{b}) - \mathbf{L}) \\
\text{subject to} & \quad \text{Diag}(\mathbf{b}) \geq \mathbf{L},
\end{align*}
$$

i.e., we choose the diagonal matrix $\text{Diag}(\mathbf{b})$ that minimizes the sum of eigenvalues of the difference $\text{Diag}(\mathbf{b}) - \mathbf{L}$. Since $\mathbf{L}$ is a constant matrix, the problem (39) can be written as

$$
\begin{align*}
\min_{\mathbf{b}} & \quad \mathbf{b}^T \mathbf{1}_n \\
\text{subject to} & \quad \text{Diag}(\mathbf{b}) \geq \mathbf{L}.
\end{align*}
$$

The problem (40) is an SDP (semidefinite programming), and there is no closed form solution in general. However, due to the special properties (i.e., symmetry and nonnegativity) of the
Since \( b \) is majorized by the following function at the equality achieved by \( b = L1_n \).

The objective in (47) is quadratic in \( x \) and similar as before we would like to find a matrix \( M \) such that \( M \succeq R - B \circ (x^{(l)}(x^{(l)})^H) \). For the same reason as in the previous section, here we use some upper bound of \( \lambda_{max}(R - B \circ (x^{(l)}(x^{(l)})^H)) \) to construct the matrix \( M \).

Then given \( \lambda \) is an eigenvalue of \( B \) and \( z \) is the corresponding eigenvector, i.e., \( Bz = \lambda z \), then

\[
\lambda = \text{diag}(\text{vec}(x^{(l)}))(x \circ z),
\]

which means \( \lambda \) is also an eigenvalue of the matrix \( B \circ (xx^H) \), with the corresponding eigenvector given by \( (x \circ z) \).

With Lemma 7, we have

\[
\lambda_{max}(R - B \circ (x^{(l)}(x^{(l)})^H)) \\
\leq \lambda_{max}(R) - \lambda_{min}\left(B \circ (x^{(l)}(x^{(l)})^H)\right) \\
= \lambda_{max}(R) - \lambda_{min}(B),
\]

Noticing that the matrix \( B \) is symmetric Toeplitz, according to Lemma 3 we know that

\[
\lambda_{min}(B) \geq \frac{1}{2} \left( \min_{1 \leq i \leq N} \nu_{2i} + \min_{1 \leq i \leq N} \nu_{2i-1} \right),
\]
where $\nu = Fw$ and 
$$\tilde{w} = [0, w_1(N-1), \ldots, w_{N-1}, 0, w_{N-1}, \ldots, w_1(N-1)]^T.$$  
By defining 
$$\lambda_B = \frac{1}{2} \left( \min_{1 \leq i \leq N} \nu_{2i} + \min_{1 \leq i \leq N} \nu_{2i-1} \right),$$  
we now have 
$$\lambda_{\max} \left( \begin{bmatrix} R - B & (x^{(l)})(x^{(l)})(H) \end{bmatrix} \right) \leq \lambda_u - \lambda_B,$$  
where $\lambda_u$ is defined in (20) and by choosing $M = (\lambda_u - \lambda_B)I$ in Lemma 1 we know that the objective of (47) is majorized by 
$$\tilde{u}_{2}(x, x^{(l)}) = (\lambda_u - \lambda_B) x^H x + 2Re \left( x^H (R - B \circ (x^{(l)})(x^{(l)})(H)) - (\lambda_u - \lambda_B)I \right) x^{(l)} + \left( (x^{(l)})(H) \right) (\lambda_u - \lambda_B) - R + B \circ (x^{(l)})(x^{(l)})(H)) x^{(l)}).$$

By ignoring the constant terms, the majorized problem of (47) is given by 
$$\begin{array}{ll}
\text{minimize} & \Re x^H (R - B \circ (x^{(l)})(x^{(l)})(H)) x^{(l)} \\
\text{subject to} & |x_n| = 1, n = 1, \ldots, N.
\end{array}$$  
In section 5.2, we also admits a closed form solution given by 
$$x_n = e^{j\arg(\tilde{y}_n)}, n = 1, \ldots, N,$$  
where 
$$\tilde{y} = (\lambda_u - \lambda_B) x^{(l)} + (\circ (x^{(l)})(x^{(l)})(H)) x^{(l)} - R x^{(l)} + \left( (x^{(l)})(H) \right) (\lambda_u - \lambda_B) - R + B \circ (x^{(l)})(x^{(l)})(H)) x^{(l)}.$$  
It is worth noting that the $\tilde{y}$ in (57) can be computed efficiently via FFT operations. To see that, we first note that $R x^{(l)}$ can be computed by means of FFT as described in the previous section. According to Lemma 6 we have 
$$\left( B \circ (x^{(l)})(x^{(l)})(H) \right) x^{(l)} = \text{diag}(B \text{Diag}(x^{(l)})(x^{(l)})(H)) x^{(l)} = \text{diag}(B(x^{(l)})(x^{(l)})(H)) x^{(l)} = (B1)(x^{(l)})(H) x^{(l)} = (B1) x^{(l)}.$$  
Since $B$ is symmetric Toeplitz, by Lemma 4, we can decompose $B$ as 
$$B = \frac{1}{2N} F^H_{1,1:N} \text{Diag}(Fw) F_{1,1:N},$$  
and thus $B1$ can also be computed efficiently via FFT operations. We further note that we only need to compute $B1$ once. The overall algorithm is then summarized in Algorithm 2 for which the main computation of each iteration is just four FFT (IFFT) operations and thus of order $O(N \log N)$.

### V. CONVERGENCE ANALYSIS AND ACCELERATION SCHEME

#### A. CONVERGENCE ANALYSIS

The MWISL and MWISL-Diag algorithms given in Algorithm 1 and 2 are based on the general majorization-minimization framework, thus according to subsection III-A we know that the sequence of objective values (i.e., weighted ISL) evaluated at $\{x^{(l)}\}$ generated by the algorithms is non-increasing. And it is easy to see that the weighted ISL metric in (4) is bounded below by 0, thus the sequence of objective values is guaranteed to converge to a finite value.

Now we further analyze the convergence property of the sequence $\{x^{(l)}\}$ itself. In the following, we will focus on the sequence generated by the MWISL algorithm (i.e., Algorithm 1), and prove the convergence to a stationary point. The same result can be proved for the MWISL-Diag algorithm (i.e, Algorithm 2) similarly.

To make it clear what is a stationary point in our case, we first introduce a first-order optimality condition for minimizing a smooth function over an arbitrary constraint set, which follows from [32].

**Proposition 8.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $x^* \in \mathbb{X}$ be a local minimum of $f$ over a subset $\mathbb{X}$ of $\mathbb{R}^n$. Then 
$$\nabla f(x^*)^T z \geq 0, \forall z \in T_x(\mathbb{X}),$$  
where $T_x(\mathbb{X})$ denotes the tangent cone of $\mathbb{X}$ at $x^*$.

A point $x \in \mathbb{X}$ is said to be a stationary point of the problem 
$$\begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & x \in \mathbb{X}
\end{array}$$
if it satisfies the first-order optimality condition (60).

To facilitate the analysis, we further note that upon defining 
$$\bar{x} = [\Re(x)^T, \Im(x)^T]^T,$$  
it follows from (54) that 
$$\lambda_{\max} \left( \begin{bmatrix} R - B & (x^{(l)})(x^{(l)})(H) \end{bmatrix} \right) \leq \lambda_u - \lambda_B,$$  
and thus the sequence of objective values is guaranteed to converge to a finite value.

### Algorithm 2 MWISL-Diag - Monotonic minimizer for Weighted ISL.

**Require:** sequence length $N$, weights $\{w_k \geq 0\}_{k=1}^{N}$

1: Set $l = 0$, initialize $x^{(0)}$.
2: $\bar{w} = [0, w_1(N-1), \ldots, w_{N-1}, 0, w_{N-1}, \ldots, w_1(N-1)]^T$
3: $\nu = F\bar{w}$
4: $p = F^H_{1,1:N}(\nu \circ (F_{1,1:N})$)
5: $\lambda_B = \frac{1}{2} (\min_{1 \leq i \leq N} \nu_{2i} + \min_{1 \leq i \leq N} \nu_{2i-1})$
6: **repeat**
7: $\mathbf{f} = F[x^{(l)}]_T, 0_{1 \times N}]^T$
8: $\mathbf{r} = \frac{1}{\nu} F^H_{1,1:N} [\mathbf{f}]$
9: $c = r \circ [0, w_1, \ldots, w_{N-1}, 0, w_{N-1}, \ldots, w_1]^T$
10: $\mu = Fc$
11: $\lambda_u = \frac{1}{2} (\max_{1 \leq i \leq N} \mu_{2i} + \max_{1 \leq i \leq N} \mu_{2i-1})$
12: $\tilde{y} = x^{(l)} + \frac{\mu x^{(l)} - \nu x^{(l)}}{2\nu (\lambda_u - \lambda_B)}$
13: $x_n^{(l+1)} = e^{j\arg(\tilde{y}_n)}, n = 1, \ldots, N$
14: $l \leftarrow l + 1$
15: **until** convergence
and based on the expression of $r_k$ in (10), it is straightforward to show that the complex WISL minimization problem (5) is equivalent to the following real one:

$$
\begin{align*}
&\minimize_{\tilde x} \\
&\quad \sum_{k=1}^{N-1} w_k \left( (\tilde x^T \tilde U_k \tilde x)^2 + (\tilde x^T \tilde U_k \tilde x)^2 \right) \tag{64} \\
&\text{subject to } \tilde x_n^2 + \tilde x_{n+N}^2 = 1, n = 1, \ldots, N.
\end{align*}
$$

We are now ready to state the convergence properties of MWISL.

**Theorem 9.** Let \{\tilde x^{(l)}\} be the sequence generated by the MWISL algorithm in Algorithm 7. Then every limit point of the sequence \{\tilde x^{(l)}\} is a stationary point of the problem (5).

**Proof:** Denote the objective functions of the problem (5) and its real equivalent (64) by $f(\tilde x)$ and $\tilde f(\tilde x)$, respectively. Denote the constraint sets of the problem (5) and (64) by $C$ and $\tilde C$, respectively, i.e., $C = \{x \in \mathbb{C}^N | x_n = 1, n = 1, \ldots, N\}$ and $\tilde C = \{x \in \mathbb{R}^{2N} | \tilde x_n^2 + \tilde x_{n+N}^2 = 1, n = 1, \ldots, N\}$. From the derivation of MWISL in subsection III-B, we know that, at iteration $l$, $f(x)$ is majorized by the function $u(x, \tilde x^{(l)})$ in (63) at $\tilde x^{(l)}$ over $\tilde C$. Then according to the general MM scheme described in subsection III-A, we have

$$
\begin{align*}
f(x^{(l+1)}) &\leq u(x^{(l+1)}, \tilde x^{(l)}) \leq u(x^{(l)}, \tilde x^{(l)}) = f(x^{(l)}),
\end{align*}
$$

which means \{f(\tilde x^{(l)})\} is a nonincreasing sequence.

Since the sequence \{\tilde x^{(l)}\} is bounded, we know that it has at least one limit point. Consider a limit point $\tilde x^{(\infty)}$ and a subsequence \{\tilde x^{(l_j)}\} that converges to $\tilde x^{(\infty)}$, we have

$$
\begin{align*}
u(x^{(l_j+1)}, \tilde x^{(l_j)}) = f(x^{(l_j+1)}) \leq f(x^{(l_j+1)}) \\
\leq u(x^{(l_j+1)}, \tilde x^{(l_j)}) \leq u(x, \tilde x^{(l_j)}), \forall x \in C.
\end{align*}
$$

Letting $j \to +\infty$, we obtain

$$
u(x^{(\infty)}, \tilde x^{(\infty)}) \leq u(x, \tilde x^{(\infty)}), \forall x \in C, \tag{65}$$

i.e., $\tilde x^{(\infty)}$ is a global minimizer of $u(x, \tilde x^{(\infty)})$ over $\tilde C$. With the definitions of $\tilde x$, $\tilde U_k$ and $\tilde U_k$ given in (61), (62) and (63), and by ignoring the constant terms in $u(x, \tilde x^{(\infty)})$, it is easy to show that minimizing $u(x, \tilde x^{(\infty)})$ over $\tilde C$ is equivalent to the following real problem:

$$
\begin{align*}
\minimize_{\tilde x} &\quad 4\tilde x^T d - (\lambda_{\max}(L)N + \lambda_u) \tilde x^T \tilde x \tag{66} \\
n\text{subject to } &\quad \tilde x \in \tilde C,
\end{align*}
$$

where $\tilde x^{(\infty)} = \{\Re(x^{(\infty)})^T, \Im(x^{(\infty)})^T\}^T$ and

$$
d = \sum_{k=1}^{N-1} w_k \left( \tilde x^{(\infty)}^T \tilde U_k \tilde x^{(\infty)} \tilde U_k + \tilde x^{(\infty)}^T \tilde U_k \tilde x^{(\infty)} \tilde U_k \right) \tilde x^{(\infty)}. \tag{67}
$$

Since $\tilde x^{(\infty)}$ minimizes $u(x, \tilde x^{(\infty)})$ over $\tilde C$, $\tilde x^{(\infty)}$ is a global minimizer of (66) and since $\tilde x^T \tilde x$ is just a constant over $\tilde C$, $\tilde x^{(\infty)}$ is also a global minimizer of

$$
\begin{align*}
\minimize_{\tilde x} &\quad 4\tilde x^T d - (\lambda_{\max}(L)N + \lambda_u) (2\tilde x^T \tilde x^{(\infty)} - \tilde x^T \tilde x) \\
n\text{subject to } &\quad \tilde x \in \tilde C. \tag{68}
\end{align*}
$$

Then as a necessary condition, we have

$$
\nabla \tilde u(\tilde x^{(\infty)})^T z \geq 0, \forall z \in T_{\tilde C}(\tilde x^{(\infty)}), \tag{69}
$$

where $\tilde u(\tilde x)$ denotes the objective function of (68). It is easy to check that

$$
\nabla \tilde f(\tilde x^{(\infty)}) = \nabla \tilde u(\tilde x^{(\infty)}) = 4d.
$$

Thus we have

$$
\nabla \tilde f(\tilde x^{(\infty)})^T z \geq 0, \forall z \in T_{\tilde C}(\tilde x^{(\infty)}), \tag{70}
$$

implying that $\tilde x^{(\infty)}$ is a stationary point of the problem (64). Due to the equivalence of problem (64) and (5), the proof is complete.

### B. Acceleration Scheme

In MM algorithms, the convergence speed is usually dictated by the nature of the majorization functions. Due to the successive majorization steps that we have carried out in the previous sections to construct the majorization functions, the convergence of MWISL and MWISL-Diaq seems to be slow. In this subsection, we briefly introduce an acceleration scheme that can be applied to accelerate the proposed MM algorithms. It is the so called squared iterative method (SQUAREM), which was originally proposed in [33] to accelerate any Expectation–Maximization (EM) algorithms. SQUAREM adapts the idea of the Cauchy-Barzilai-Borwein (CBB) method [34], which combines the classical steepest descent method and the two-point step size gradient method [35], to solve the nonlinear fixed-point problem of EM. It only requires the EM updating scheme and can be readily implemented as an off-the-shelf accelerator. Since MM is a generalization of EM and the update rule is also just a fixed-point iteration, SQUAREM can be easily applied to MM algorithms after some minor modifications.

Suppose we have derived an MM algorithm to minimize $f(x)$ over $X \subseteq \mathbb{C}^n$ and let $F_{MM}(\cdot)$ denote the nonlinear fixed-point iteration map of the MM algorithm:

$$
x^{(k+1)} = F_{MM}(x^{(k)}). \tag{71}
$$

For example, the iteration map of the MWISL algorithm is given by (62). Then the steps of the accelerated MM algorithm based on SQUAREM are given in Algorithm 3. A problem of the general SQUAREM is that it may violate the nonlinear constraints, so in Algorithm 3 we need to project wayward points back to the feasible region by $P_X(\cdot)$. For the unit-modulus constraints in the problem under consideration, the projection can be done by simply applying the function $e^{j\arg(\cdot)}$ element-wise to the solution vectors. A second problem of SQUAREM is that it can violate the descent property of the original MM algorithm. To ensure the descent property, a strategy based on backtracking has been adopted in Algorithm 3, which repeatedly halves the distance between $\alpha$ and $-1, \alpha \leftarrow (\alpha - 1)/2$ until the descent property is maintained. To see why this works, we first note that $P_X(x^{(k)} - 2\alpha x + \alpha^2 x) = x_2$ if $\alpha = -1$. In addition, since $f(x_2) \leq f(x^{(k)})$ due to the descent property of original MM steps, $f(x) \leq f(x^{(k)})$ is guaranteed to hold as $\alpha \to -1$. 

$$
\begin{align*}
\tilde U_k &= \frac{1}{2} \begin{bmatrix} U_k + U_k^T & 0 \\ 0 & U_k + U_k^T \end{bmatrix}, \tag{62} \\
\tilde U_k &= \frac{1}{2} \begin{bmatrix} 0 & U_k - U_k^T \\ U_k - U_k^T & 0 \end{bmatrix}, \tag{63}
\end{align*}
$$
It is worth mentioning that, in practice, usually only a few back-tracking steps are needed to maintain the monotonicity of the algorithm.

Algorithm 3 The acceleration scheme for MM algorithms.

Require: parameters
1: Set $k = 0$, initialize $x^{(0)}$.
2: repeat
3: $x_1 = F_{MM}(x^{(k)})$
4: $x_2 = F_{MM}(x_1)$
5: $r = x_1 - x^{(k)}$
6: $v = x_2 - x_1 - r$
7: Compute the step-length $\alpha = \frac{-\|r\|}{\|v\|}$
8: $x = \mathcal{P}_X(x^{(k)} - 2\alpha r + \alpha^2 v)$
9: while $f(x) > f(x^{(k)})$ do
10: $\alpha \leftarrow (\alpha - 1)/2$
11: $x = \mathcal{P}_X(x^{(k)} - 2\alpha r + \alpha^2 v)$
12: end while
13: $x^{(k+1)} = x$
14: $k \leftarrow k + 1$
15: until convergence

VI. MINIMIZING THE $\ell_p$-NORM OF AUTOCORRELATION SIDELOBE

In previous sections, we have developed algorithms to minimize the weighted ISL metric of a unit-modulus sequence. It is clear that the (unweighted) ISL metric is just the squared $\ell_2$-norm of the autocorrelation sidelobes and in this section we would like to consider the more general $\ell_p$-norm metric of the autocorrelation sidelobes defined as

$$
\left( \sum_{k=1}^{N-1} |r_k|^p \right)^{1/p}
$$

(72)

with $2 \leq p < \infty$. The motivation is that by choosing different $p$ values, we may get different metrics of particular interest. For instance, by choosing $p \rightarrow +\infty$, the $\ell_p$-norm metric tends to the $\ell_\infty$-norm of the autocorrelation sidelobes, which is known as the peak sidelobe level (PSL). So it is well motivated to consider the more general $\ell_p$-norm ($2 \leq p < \infty$) metric minimization problem

minimize

$$
\left( \sum_{k=1}^{N-1} |r_k|^p \right)^{1/p}
$$

subject to

$$
|x_n| = 1, \; n = 1, \ldots, N,
$$

which is equivalent to

minimize

$$
\sum_{k=1}^{N-1} |r_k|^p
$$

subject to

$$
|x_n| = 1, \; n = 1, \ldots, N.
$$

(74)

To tackle the problem (72) via majorization-minimization, we need to construct a majorization function of the objective and the idea is to majorize each $|r_k|^p$ by a quadratic function of $|r_k|$. It is clear that when $p > 2$, it is impossible to construct a global quadratic majorization function of $|r_k|^p$. However, we can still majorize it by a quadratic function locally based on the following lemma.

Lemma 10. Let $f(x) = x^p$ with $p \geq 2$ and $x \in [0, t]$. Then for any given $x_0 \in [0, t]$, $f(x)$ is majorized at $x_0$ over the interval $[0, t]$ by the following quadratic function

$$
a x^2 + (p x_0^{p-1} - 2 a x_0) x + a x_0^p - (p - 1) x_0^p,
$$

where

$$
a = \frac{t^p - x_0^p - p x_0^{p-1} (t - x_0)}{(t - x_0)^2}. $$

(76)

Proof: See Appendix A

Given $\|r_k\|$ at iteration $l$, according to Lemma 10 we know that $|r_k|^p$ ($p \geq 2$) is majorized at $|r_k^l|$ over $[0, t]$ by

$$
|a_k| |r_k|^2 + b_k |r_k| + a_k \left( |r_k^l|^2 - (p - 1) |r_k^l|^p \right),
$$

where

$$
a_k = \frac{t^p - |r_k^l|^p - p |r_k^l|^{p-1} (t - |r_k^l|)}{(t - |r_k^l|)^2},
$$

$$
b_k = p |r_k^l|^{p-1} - 2 a_k |r_k^l|.
$$

(79)

Since the objective decreases at every iteration in the MM framework, at the current iteration $l$, it is sufficient to majorize $|r_k|^p$ over the set on which the objective is smaller, i.e.,

$$
\sum_{k=1}^{N-1} |r_k|^p \leq \sum_{k=1}^{N-1} |r_k^l|^p,
$$

which implies

$$
|r_k| \leq \left( \sum_{k=1}^{N-1} |r_k^l|^p \right)^{1/p} / p.\] Hence we can choose $t = \left( \sum_{k=1}^{N-1} |r_k^l|^p \right)^{1/p}$ in (78). Then the majorized problem of (72) in this case is given by (ignoring the constant terms)

$$
\begin{align*}
\min_{x_n} & \sum_{k=1}^{N-1} (a_k |r_k|^2 + b_k |r_k|) \\
\text{subject to} & \; |x_n| = 1, \; n = 1, \ldots, N.
\end{align*}
$$

(80)

We can see that the first term $\sum_{k=1}^{N-1} a_k |r_k|^2$ in the objective is just the weighted ISL metric with weights $w_k = a_k$, and thus can be further majorized as in Section III-B. Following the steps in Section III-B until (20) (i.e., just the first majorization), we know that it is majorized at $x^{(l)}$ by (with constant terms ignored)

$$
x^H \left( R - \lambda_{\max}(L)x^{(l)}(x^{(l)})^H \right) x,
$$

(81)

where $R$ and $L$ are defined in (21) and (14) with $w_k = a_k$. For the second term, since it can be shown that $b_k \leq 0$, we
have
\[ \sum_{k=1}^{N-1} b_k |r_k| \leq \sum_{k=1}^{N-1} b_k \text{Re} \left\{ \frac{r_k^{(l)}}{r_k} \right\} \]
\[ = \sum_{k=1}^{N-1} b_k \text{Re} \left\{ \text{Tr}(U_k xx^H) \frac{r_k^{(l)}}{r_k} \right\} \]
\[ = \text{Re} \left\{ x^H \left( \sum_{k=1}^{N-1} b_k \frac{r_k^{(l)}}{r_k} U_k \right) x \right\} \]
\[ = \frac{1}{2} x^H \left( \sum_{k=1}^{N-1} b_k \frac{r_k^{(l)}}{r_k} U_k \right) x, \tag{85} \]
where \( b_{-k} = b_k, k = 1, \ldots, N - 1, b_0 = 0 \). By adding the two majorization functions, i.e., (81) and (85), and defining
\[ \tilde{w}_{-k} = \tilde{w}_k = a_k + \frac{b_k}{2} \frac{|r_k|}{|r_k^{(l)}|} = \frac{p}{2} \frac{|r_k|}{|r_k^{(l)}|} \tilde{r}_k, k = 1, \ldots, N - 1, \]
we have the majorized problem of (80) given by
\[ \begin{array}{ll}
\text{minimize} & x^H \left( \tilde{R} - \lambda_{\text{max}}(L) x^{(l)} (x^{(l)})^H \right) x \\
\text{subject to} & |x_n| = 1, n = 1, \ldots, N,
\end{array} \tag{86} \]
where
\[ \tilde{R} = \sum_{k=1}^{N} \tilde{w}_k r_k^{(l)} U_k. \tag{87} \]

We can see that the problem (87) has the same form as (20), then by following similar steps as in section II-B we can perform one more majorization step and get the majorized problem
\[ \begin{array}{ll}
\text{minimize} & \|x - y\|_2 \\
\text{subject to} & |x_n| = 1, n = 1, \ldots, N,
\end{array} \tag{88} \]
where
\[ y = (\lambda_{\text{max}}(L) N + \lambda_u) x^{(l)} - \tilde{R} x^{(l)}, \tag{89} \]
and this time \( \lambda_{\text{max}}(L) \) should be computed based on weights \( a_k \) in (75) and \( \lambda_u \) is based on the weights \( \tilde{w}_k \) in (80). As in previous cases, we only need to solve (89) in closed form at every iteration and the overall algorithm is summarized in Algorithm 4. Note that, to avoid numerical issues, we have used the normalized \( a_k \) and \( \tilde{w}_k \) (i.e., divided by \( p^l \)) in Algorithm 4 which is equivalent to divide the objective in (80) by \( p^l \) during the derivation. It is also worth noting that the algorithm can be accelerated by the scheme described in subsection III-B.

VII. NUMERICAL EXPERIMENTS
To compare the performance of the proposed MWISL algorithm and its variants with existing algorithms and to show the potential of proposed algorithms in designing sequences for various scenarios, we present some experimental results in this section. All experiments were performed on a PC with a 3.20GHz i5-3470 CPU and 8GB RAM.
which means the majorization function proposed in Section \[IV\] is somehow better than the one in Section \[III-B\]. From Fig. 2, we can see that in terms of the computational time the superiority of the proposed algorithms is more significant, more specially the accelerated MWISL and MWISL-Diag algorithms take only 0.07 and 0.06 seconds respectively, while the WeCAN algorithm takes more than 1000 seconds. It is because the proposed MWISL (and MWISL-Diag) algorithms require only four FFT operations per iteration, while each iteration of WeCAN requires \(N\) computations of \(2N\)-point FFTs. The correlation level of the output sequence of the accelerated MWISL-Diag algorithm is shown in Fig. 3, where the correlation level is defined as

\[
\text{correlation level} = 20 \log_{10} \left| \frac{r_k}{p_0} \right|, \quad k = 1 - N, \ldots, N - 1.
\]

We can see in Fig. 3 that the autocorrelation sidelobes are suppressed to almost zero (about -160dB) at the required lags.

![Figure 1. Evolution of the weighted ISL with respect to the number of iterations.](image1)

![Figure 2. Evolution of the weighted ISL with respect to time (in seconds). The plot within the time interval [0, 10] second is zoomed in and shown in the upper right corner.](image2)

![Figure 3. Correlation level of the sequence of length \(N = 100\) designed by accelerated MWISL-Diag algorithm with weights in \([41]\).](image3)

### B. PSL Minimization

In this subsection, we test the performance of the proposed Algorithm \([4]\) in Section \[VII\] in minimizing the peak sidelobe level (PSL) of the autocorrelation sidelobes, which is of particular interest. To apply the algorithm, we need to choose the parameter \(p\). To examine the effect of the parameter \(p\), we first apply the accelerated version of Algorithm \([4]\) (denoted as MM-PSL) with four different \(p\) values, i.e., \(p = 10, 100, 1000\) and \(10000\), to design a sequence of length \(N = 400\). Frank sequences \([13]\) are used to initialize the algorithm, which are known to be sequences with good autocorrelation. More specifically, Frank sequences are defined for lengths that are perfect squares and the Frank sequence of length \(N = M^2\) is given by

\[
x_n M^k = e^{j2\pi nk/M}, \quad n, k = 0, 1, \ldots, M - 1.
\]

For all \(p\) values, we stop the algorithm after \(5 \times 10^4\) iterations and the evolution curves of the PSL are shown in Fig. 4. From the figure, we can see that smaller \(p\) values lead to faster convergence. However, if \(p\) is too small, it may not decrease the PSL at a later stage, as we can see that \(p = 100\) finally gives smaller PSL compared with \(p = 10\). It may be explained by the fact that \(\ell_p\)-norm with larger \(p\) values approximates the \(\ell_{\infty}\)-norm better. So in practice, gradually increasing the \(p\) value is probably a better approach.

In the second experiment, we consider both an increasing scheme of \(p\) (denoted as MM-PSL-adaptive) and the fixed \(p\) scheme with \(p = 100\). For the increasing scheme, we apply the MM-PSL algorithm with increasing \(p\) values \(2, 2^2, \ldots, 2^{13}\). For each \(p\) value, the stopping criterion was chosen to be

\[
\|\text{obj}(x^{(k+1)}) - \text{obj}(x^{(k)})\| / \text{obj}(x^{(k)}) \leq 10^{-5} / p, \text{ with } \text{obj}(x) \text{ being the objective in } (73), \text{ and the maximum allowed number of iterations was set to be } 5 \times 10^3.\]

For \(p = 2\), the algorithm is initialized by the Frank sequence and for larger \(p\) values, it is initialized by the solution obtained at the previous \(p\). For the fixed \(p\) scheme, the stopping criterion was chosen to be

\[
\|\text{obj}(x^{(k+1)}) - \text{obj}(x^{(k)})\| / \text{obj}(x^{(k)}) \leq 10^{-10}, \text{ and the maximum allowed number of iterations was } 2 \times 10^5.\]

In this...
case, in addition to the Frank sequence, the Golomb sequence [15] was also used as the initial sequence, which is also known for its good autocorrelation properties. In contrast to Frank sequences, Golomb sequences are defined for any positive integer and a Golomb sequence \( \{x_n\}_{n=1}^{N} \) of length \( N \) is given by

\[
x_n = e^{j\pi(n-1)n/N}, \quad n = 1, \ldots, N.
\]

The two schemes are applied to design sequences of the following lengths: \( N = 5^2, 7^2, 10^2, 20^2, 30^2, 50^2, 70^2, 100^2 \), and the PSL’s of the resulting sequences are shown in Fig. 5. From the figure, we can see that for all lengths, the MM-PSL(G) and MM-PSL(F) sequences give nearly the same PSL; both are much smaller than the PSL of Golomb and Frank sequences, while a bit larger than the PSL of MM-PSL-adaptive sequences. For example, when \( N = 10^4 \), the PSL values of the MM-PSL(F) and MM-PSL-adaptive sequences are 4.36 and 3.48, while the PSL values of Golomb and Frank sequences are 48.03 and 31.84, respectively. The correlation level of the Golomb, Frank and the MM-PSL-adaptive sequences are shown in Fig. 6. We can notice that the autocorrelation sidelobes of the Golomb and Frank sequences are relatively large for \( k \) close to 0 and \( N - 1 \), while the MM-PSL-adaptive sequence has much more uniform autocorrelation sidelobes across all lags.

VIII. CONCLUSION

We have developed two efficient algorithms for the minimization of the weighted integrated sidelobe level (WISL) metric of unit-modulus sequences. The proposed algorithms are derived based on applying two successive majorization steps and we have proved that they will converge to a stationary point of the original WISL minimization problem. By performing one more majorization step in the derivations, we have extended the proposed algorithms to tackle the problem of minimizing the \( \ell_p \)-norm of the autocorrelation sidelobes. All the algorithms can be implemented by means of FFT operations and thus are computationally very efficient in practice. An acceleration scheme that can be used to further speed up the proposed algorithms has also been considered. By some numerical examples, we have shown that the proposed WISL minimization algorithms can generate sequences with virtually zero autocorrelation sidelobes in some specified lag intervals with much lower computational cost compared with the state-of-the-art. It has also been observed that the proposed \( \ell_p \)-metric minimization algorithm can produce long sequences with much more uniform autocorrelation sidelobes and much smaller PSL compared with Frank and Golomb sequences, which are known for their good autocorrelation properties.
APPENDIX A

PROOF OF LEMMA 10
Proof: For any given $x_0 \in [0, t]$, let us consider a quadratic function of the following form
\[ g(x|x_0) = f(x_0) + f'(x_0)(x - x_0) + a(x - x_0)^2, \]  
where $a > 0$. It is easy to check that $f(x_0) = g(x_0|x_0)$. So to make $g(x|x_0)$ be a majorization function of $f(x)$ at $x_0$ over the interval $[0, t]$, we need to further have $f(x) \leq g(x|x_0)$ for all $x \in [0, t]$, $x \neq x_0$. Equivalently, we must have
\[ a \geq \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x-x_0)^2} \]  
for all $x \in [0, t]$, $x \neq x_0$. Let us define the function
\[ A(x|x_0) = \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x-x_0)^2} \]  
for all $x \neq x_0$. The derivative of $A(x|x_0)$ is given by
\[ A'(x|x_0) = \frac{f'(x) + f'(x_0) - 2(f(x) - f(x_0))/((x-x_0)^2).} \]  
Since $f'(x) = px^{p-1}$ is convex on $[0, t]$ when $p \geq 2$, we have
\[ \frac{f(x) - f(x_0)}{x - x_0} = \int_0^1 f'(x_0 + \tau(x - x_0))d\tau \leq \int_0^1 (f'(x_0) + \tau(f'(x) - f'(x_0)))d\tau = \frac{1}{2}(f'(x) + f'(x_0)), \]  
which implies $A'(x|x_0) \geq 0$ for all $x \in [0, t]$, $x \neq x_0$. Thus, $A(x|x_0)$ is increasing on the interval $[0, t]$ and the maximum is achieved at $x = t$. Then the smallest $a$ we may choose is
\[ a = \max_{x \in [0,t], x \neq x_0} A(x|x_0) = \frac{t^p - x_0^p - px_0^{p-1}(t-x_0)}{(t-x_0)^2}. \]  
By substituting $a$ into $g(x|x_0)$ in (94) and appropriately rearranging terms, we can obtain the function in (75). \[ \square \]

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