QUASITRIANGULAR AND DIFFERENTIAL STRUCTURES ON
BICROSSPRODUCT HOPF ALGEBRAS

Edwin Beggs
Department of Mathematics
University of Wales, Swansea
Singleton Park, Swansea SA2 8PP, UK

Shahn Majid
Department of Mathematics, Harvard University
Science Center, Cambridge MA 02138, USA
Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, UK

ABSTRACT
Let $X = GM$ be a finite group factorisation. It is shown that the quantum
double $D(H)$ of the associated bicrossproduct Hopf algebra $H = kM \bowtie k(G)$ is itself a
bicrossproduct $kX \bowtie k(Y)$ associated to a group $YX$, where $Y = G \times M^{op}$. This
provides a class of bicrossproduct Hopf algebras which are quasi-
triangular. We also construct a subgroup $Y^{\theta}X^{\theta}$ associated to every order-reversing
automorphism $\theta$ of $X$. The corresponding Hopf algebra $kX^{\theta} \bowtie k(Y^{\theta})$ has the same
coalgebra as $H$. Using related results, we classify the first order bicovariant differ-
ential calculi on $H$ in terms of orbits in a certain quotient space of $X$.

1 INTRODUCTION

The quantum double[1] of the bicrossproduct Hopf algebra $H = kM \bowtie k(G)$ associated to a finite
group factorisation $X = GM$ has been studied recently in [2]. Here we continue this study with
further results in the same topic, including a concrete application to the classification of the
bicovariant differential calculi on a bicrossproduct.

The bicrossproduct Hopf algebras have been introduced in [3] and [4], and extensively studied
since then. Factorisations of groups abound in mathematics, so these Hopf algebras, which are

1 Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge
2 During 1995 & 1996
non-commutative and non-cocommutative, are quite common. In the context of [4] they are viewed as systems which combine quantum mechanical ideas with geometry in a unified way. To develop this idea a natural step is to compute and study the algebra of differential forms (the so-called differential calculus) over them, which we do here. Another open problem is the close connection which one may expect between these Hopf algebras and the method of inverse scattering in soliton theory. We make a connection of this type also in the paper. All our results, however, will be in a strictly algebraic setting with finite groups and finite-dimensional Hopf algebras.

We recall two principal results from [2]. One result was that $D(H)$ is isomorphic to a twist (i.e. up to a categorical equivalence of representations) to the much simpler double $D(X)$ of the group algebra $kX$. We provide in Section 2 a variant of this result, namely that $D(H)$ is also isomorphic to a bicrossproduct $kX\bowtie k(Y)$ where $Y$ is the ‘dressing group’ $G \times M_{\text{op}}$ (the terminology comes from soliton theory [5]). The bicrossproduct is associated to a certain double cross product group $Y\bowtie \triangleleft X$ which factorises into $X, Y$. Our new result answers affirmatively the question ‘can a bicrossproduct Hopf algebra be quasitriangular?’ Among non-commutative and non-cocommutative Hopf algebras, the quasitriangular [1] ones have a special place with special properties. We compute the quasitriangular structure in the present case, and some of its consequences.

The second principal result in [2] was that associated to every order reversing isomorphism $\theta$ of $X$ is a Hopf algebra isomorphism $\Theta$ of the quantum double. In Section 3 we provide more results about $\Theta$. We then construct two new groups $Y^\theta, X^\theta$ forming a subgroup $Y^\theta\bowtie X^\theta$ of $Y\bowtie X$. We study the associated Hopf algebra $kX^\theta\bowtie k(Y^\theta)$. It has an isomorphic coalgebra to that of our original bicrossproduct $H$ associated to the factorisation $GM$. However we give a finite group example where these are not isomorphic as algebras. Another example involving upper and lower triangular matrices is given. In this case we use the order reversing isomorphism given by the operation $g \mapsto (g^*)^{-1}$.

In Section 4 we study a certain Hilbert space representation of $H$ and find that our various constructions over $\mathbb{C}$ respect the $\ast$-structures. This is also one of the motivations behind the main isomorphism in Section 2.

In Section 5 we turn to a specific application of the quantum double, namely to the construc-
tion of first order bicovariant differential calculi. By definition a first order differential calculus over an algebra $A$ means an $A$-bimodule $\Omega^1$ over $A$ and a map $d : A \to \Omega^1$ obeying the Leibniz rule (which makes sense using the bimodule structure). When $A$ is a Hopf algebra the calculus is said to be left-, right- or bi-covariant when $d$ interwines the left regular coaction of $A$, the right regular coaction, or both, with a coaction given on $\Omega^1$. It is known that first order bicovariant calculi are related to the representation theory of the quantum double\cite{6}, allowing us to apply our previous results\cite{2} when $A = k(M)\bowtie kG$. We find that the irreducible bicovariant calculi correspond to the choice of an orbit in a certain quotient space of the group $X$ along with an irreducible subrepresentation of the isotropy subgroup associated to the orbit. We actually classify the bicovariant quantum tangent spaces using the techniques developed in \cite{7}, and obtain the corresponding 1-forms later by dualisation. The result is a constructive method which provides the entire moduli space of bicovariant calculi on a bicrossproduct, as we demonstrate on some nontrivial examples.

We note that in the physics literature an important bicrossproduct Hopf algebra is the $\kappa$-deformed Poincaré algebra\cite{8}, and for this Hopf algebra some examples of bicovariant calculi have been obtained by other means\cite{9}. Here the group $G$ is the Lorentz group and $M$ is $\mathbb{R}^3 \rtimes \mathbb{R}$. Another important (and very similar) example is with $G = SO(3)$ and $M = \mathbb{R}^2 \rtimes \mathbb{R}$ in \cite{4}\cite{10}, where the bicrossproduct is the algebra of observables (or quantum phase space) of a particular quantum mechanical system. It seems likely that a more geometrical version of the present results should include such examples as well. Armed with a choice of differential calculus, one may proceed to gauge theory (i.e. to bundles and connections) using the formalism of \cite{11}.

**Preliminaries**

We use the notation and conventions of \cite{2}, which are also the notations and conventions in the text \cite{10}. Briefly, let $X = GM$ be a group which factorises into two subgroups. Then each group acts on the other through left and right actions $\triangleright, \triangleleft$ defined by $su = (s\triangleright u)(s\triangleleft u)$ for all $s \in M$ and $u \in G$. Conversely, given two actions $\triangleright, \triangleleft$ obeying certain matching conditions

$$s\triangleleft e = s, \quad (s\triangleleft u)v = s\triangleleft(uv); \quad e\triangleleft u = e, \quad (st)\triangleleft u = (s\triangleleft t\triangleright u)(t\triangleleft u)$$

$$e\triangleright u = u, \quad s\triangleright(t\triangleright u) = (st)\triangleright u; \quad s\triangleright e = e, \quad s\triangleright(uv) = (s\triangleright u)((s\triangleleft u)\triangleright v).$$

\begin{equation}
\end{equation}
we can build a double cross product group on $G \times M$ with

$$(u, s)(v, t) = (u(s \triangleright v), (s \triangleleft v)t), \quad e = (e, e), \quad (u, s)^{-1} = (s^{-1}\triangleright u^{-1}, s^{-1}\triangleleft u^{-1}). \quad (2)$$

The associated bicrossproduct Hopf algebra $H = kM \triangleright k(G)$ has the smash product algebra structure by the induced action of $M$ and the smash coproduct coalgebra structure by the induced coaction of $G$. Explicitly,

$$(s \otimes \delta_u)(t \otimes \delta_v) = \delta_{u,t \triangleright v}(st \otimes \delta_v), \quad \Delta(s \otimes \delta_u) = \sum_{xy=uv} s \otimes \delta_x \otimes s \triangleleft x \otimes \delta_y$$

$$(1 = \sum_u e \otimes \delta_u, \quad \epsilon(s \otimes \delta_u) = \delta_{u,e}, \quad S(s \otimes \delta_u) = (s \triangleleft u)^{-1} \otimes (s \triangleright u)^{-1}. \quad (3)$$

We work over a general ground field $k$. There is also a natural $*$-algebra structure $(s \otimes \delta_u)^* = s^{-1} \otimes \delta_{s\triangleleft u}$ when the ground field has an involution. This happens over $\mathbb{C}$, but can also be supposed for any field with $\bar{\lambda} = \lambda$ for all $\lambda \in k$. The dual $H^*$ has a similar structure $k(M)\otimes kG$ on the dual basis,

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{s,u,t\triangleright v}(\delta_s \otimes uv), \quad \Delta(\delta_s \otimes u) = \sum_{ab=uv} \delta_a \otimes b\triangleright u \otimes \delta_b \otimes u$$

$$1 = \sum_s \delta_s \otimes e, \quad \epsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = (s\triangleright u)^{-1} \otimes (s\triangleleft u)^{-1},$$

and $(\delta_s \otimes u)^* = \delta_{s\triangleleft u} \otimes u^{-1}$ when the ground field has an involution. The quantum double is a general construction $D(H) = H^{\text{op}} \triangleright \triangleleft H$ built on $H^* \otimes H$ with a double cross product algebra structure and tensor product coalgebra structure. In our case the cross relations between $H, H^{\text{op}}$ are

$$(1 \otimes t \otimes \delta_v)(\delta_s \otimes u \otimes 1) = \delta_{t'(s\triangleleft vu)^{-1}}(t\triangleright vu^{-1}) \otimes (t\triangleleft vu^{-1}) \triangleright u \otimes t' \otimes \delta_{(s\triangleleft u)vu^{-1}}, \quad (5)$$

where $t' = t\triangleleft(s\triangleleft u)^{-1}$.

2 More about $D(H)$

Here we extend results about the quantum double associated to a bicrossproduct in [2]. For our first observation, it is known that to every factorisation $X = GM$ there is a ‘double factorisation’ $YX$ where $Y$ is also $G \times M$ as a set and the action of $X$ is the adjoint action viewed as an action on $Y$ [4]. Here we give a similar but different ‘double factorisation’ more suitable for our needs.

**Proposition 2.1** Let $Y = G \times M^{\text{op}}$ with group law $(us)(vt) = uvts$. Then there is a double cross product group $Y \triangleright \triangleleft X$ (factorising into $Y, X$) defined by actions

$$us\triangleleft vt = ((s \triangleleft v)ts^{-1} \triangleright u^{-1})^{-1} (s\triangleleft v)$$
\[us\tilde{\circ} vt = us(vt)(us)^{-1} = u(s\triangleright v)((s\triangleleft v)ts^{-1}u^{-1})((s\triangleleft v)ts^{-1}u^{-1}).\]

The second line is the adjoint action on \(X\) which we view as an action on the set \(Y\).

**Proof** We show that these actions are matched in the required sense (see [10]). First note that the following results are immediate from the definitions:

\[us\tilde{\circ} e = us, \ e\tilde{\circ} vt = e, \ us\tilde{\circ} e = e, \ e\tilde{\circ} vt = vt.\]

For the other results we must work rather harder. To derive the equation

\[us\tilde{\circ} ((vt).(wr)) = (us\tilde{\circ} vt).((us\tilde{\circ} vt)\tilde{\circ} wr),\]

begin by the formula given for \(\tilde{\circ}\) as \(us\tilde{\circ} vt = u(s\triangleright v)yp\) where \(y = (s\triangleleft v)ts^{-1}u^{-1}\) and \(p = (s\triangleleft v)ts^{-1}au^{-1}\). Then starting from \(us\tilde{\circ} vt = y^{-1}(s\triangleleft v)\) we calculate

\[(us\tilde{\circ} vt)\tilde{\circ} wr = y^{-1}((s\triangleleft v)\triangleright w)((s\triangleleft ww)r(s\triangleleft v)^{-1}\triangleright y)((s\triangleleft ww)r(s\triangleleft v)^{-1}y),\]

and on applying the rules for multiplication in \(Y\) we find the required formula,

\[(us\tilde{\circ} vt).((us\tilde{\circ} vt)\tilde{\circ} wr) = u(s\triangleright v)((s\triangleleft v)\triangleright w)((s\triangleleft ww)r(s\triangleleft v)^{-1}\triangleright y)((s\triangleleft ww)r(s\triangleleft v)^{-1}y)p\]

\[= u(s\triangleleft vw)((s\triangleleft vw)rts^{-1}u^{-1})((s\triangleleft vw)rts^{-1}u^{-1})\]

\[= us\tilde{\circ} ((vt).(wr)).\]

Now we must prove that

\[((wr).(us))\tilde{\circ} vt = (wr\tilde{\circ} (us\tilde{\circ} vt)).(us\tilde{\circ} vt).\]

Begin by calculating

\[wr\tilde{\circ} (us\tilde{\circ} vt) = wr\tilde{\circ} (u(s\triangleright v)yp) = ((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y),\]

and then use the definition of multiplication in \(X\) to find

\[(wr\tilde{\circ} (us\tilde{\circ} vt)).(us\tilde{\circ} vt) = ((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y)\]

\[= ((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y)(s\triangleleft v)\]

\[= ((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}((r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y)(s\triangleleft v)\]

\[= ((r\triangleleft u(s\triangleright v)y)(r\triangleleft u(s\triangleright v)y)pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y)\]

\[= ((r\triangleleft u(s\triangleright v)y)(pr^{-1}\triangleright w^{-1})^{-1}(r\triangleleft u(s\triangleright v)y)\].
Now consider a part of this equation

\[ y(pr^{-1}w^{-1}) = ((sv)ts^{-1}u^{-1})((sv)ts^{-1}u^{-1})^{-1}(r^{-1}w^{-1}) \]

\[ = (sv)ts^{-1}u^{-1}(r^{-1}w^{-1}) , \]

and putting this into the previous equation we get

\[ (wr\tilde{\alpha}(us\tilde{\alpha}vt))(us\tilde{\alpha}vt) = (((r\triangleleft u)(sv)ts^{-1}u^{-1}(r^{-1}w^{-1}))^{-1}((r\triangleleft u)s\triangleleft v) \]

\[ = (((r\triangleleft u)s\triangleleft w)ts^{-1}u^{-1}(r^{-1}w^{-1}))^{-1}((r\triangleleft u)s\triangleleft v) . \]

On comparing this with

\[ ((wr)(us))\tilde{\alpha}vt = (w(r\triangleright u)(r\triangleright u)s)\tilde{\alpha}vt = (((r\triangleright u)s\triangleleft w)ts^{-1}u^{-1}(r\triangleright u)w^{-1})^{-1}((r\triangleright u)s\triangleleft v) \]

we get identical results since

\[ (r\triangleleft u)^{-1}(r\triangleright u)^{-1}w^{-1} = (((r\triangleleft u)^{-1}(r\triangleright u)^{-1}w^{-1})^{-1}((r\triangleleft u)s\triangleleft v) \]

\[ \square \]

**Theorem 2.2** \( D(H) \cong kX \circledast k(Y) \) as Hopf algebras, by

\[ \psi : D(H) \to kX \circledast k(Y), \quad \psi(\delta_s \otimes u \otimes t \otimes \delta_v) = (sv)^{-1}t \otimes \delta_{v(t\omega)^{-1}s^{-1}t} . \]

*Over a field with involution, the map preserves the star operation.*

**Proof** The structure of \( D(H) \) in the basis used is in [3]. We check that the linear map \( \psi \) is an algebra isomorphism to the smash product induced by \( \tilde{\alpha} \). Start with \( \alpha = \delta_x \otimes q \otimes t \otimes \delta_v \), and \( \beta = \delta_s \otimes u \otimes r \otimes \delta_w \) in \( D(H) \), and multiply them together to get

\[ \alpha \beta = \delta_{s'q'u'} \otimes \delta_{v'r'} \otimes \delta_{w'} \]

where

\[ t' = t \triangleleft (sv)^{-1} , \quad v' = (sv)vu^{-1} , \quad s' = t's(t\omega)^{-1}u^{-1} , \quad u' = (t\omega)^{-1}vu \]
Now we can calculate

\[ \psi(\alpha \beta) = \delta_{s' < u', x} \delta_{t', r \bowtie w} \left( (s' > u')^{-1} t' r \otimes \delta_{u'(t'^{r \bowtie w})^{-1} s' t'} \right) \]
\[ = \delta_{s' < u', x} \delta_{t', r \bowtie w} \left( \left( (s' > u') t' r \right) \otimes \delta_{u'(r \bowtie w)^{-1} t' s'} \right) \]
\[ = \delta_{t(s > u)(t \bowtie w)^{-1} , x} \delta_{r \bowtie w \bowtie u^{-1} , r \bowtie u} \left( (x > q)^{-1} \left( t' s \bowtie u \right) \otimes \delta_{w \bowtie w^{-1} t' s^{-1} t'} \right) \]
\[ = \delta_{t(s > u)(t \bowtie w)^{-1} , x} \delta_{r \bowtie w \bowtie u^{-1} , r \bowtie u} \left( (x > q)^{-1} \left( t' \psi \left( \alpha \beta \right) \right) \otimes \delta_{w \bowtie w^{-1} t' s^{-1} t'} \right) \].

Here we have used

\[ (s' < u') = (t < (s > u)^{-1}) s(t < w u^{-1})^{-1} \triangleq (t < w u^{-1} > u) \]
\[ = (t < w^{-1} > u) \]
\[ = (t < (s > u)^{-1} s > u) (t < w u^{-1} > u) \]
\[ = t(s > u)(t < w u^{-1} > u) \]
\[ = t(s > u)(t < w u^{-1} > u) \]
\[ = t(s > u)(t \bowtie w)^{-1} \].

Conversely we can calculate the product in \( kX \bowtie k(Y) \) as

\[ \psi(\alpha) \psi(\beta) = \left( (x > q)^{-1} t \otimes \delta_{u(t \bowtie w)^{-1} x^{-1} t} \right) \left( (s > u)^{-1} t \otimes \delta_{w(r \bowtie w)^{-1} s^{-1} t} \right) \]
\[ = \delta_{u(t \bowtie w)^{-1} x^{-1} t} \left( s > u \right)^{-1} t \left( t > w \right)^{-1} \delta_{w(r \bowtie w)^{-1} s t} \]
\[ = \delta_{u(t \bowtie w)^{-1} x^{-1} t} \left( s > u \right)^{-1} t \left( t > w \right)^{-1} \delta_{w(r \bowtie w)^{-1} s t} \]
\[ = \delta_{u(t \bowtie w)^{-1} r \bowtie w} \left( t > w \right)^{-1} \delta_{w(r \bowtie w)^{-1} s t} \].

It is now apparent that the expressions for \( \psi(\alpha \beta) \) and \( \psi(\alpha) \psi(\beta) \) are the same.

To compare the coproducts, we use the coproduct of \( D(H) \), which is the tensor product one

\[ \Delta_{D(H)}(\delta_x \otimes u \otimes t \otimes \delta_x) = \sum_{ab=s, \; xy=v} (\delta_x \otimes b \otimes u \otimes t \otimes \delta_x) (\delta_x \otimes u \otimes t \otimes \delta_x) \]

Applying \( \psi \otimes \psi \) to this, we find the following expression for \( (\psi \otimes \psi) \Delta_{D(H)}(\delta_x \otimes u \otimes t \otimes \delta_x) \):

\[ \sum_{ab=s, \; xy=v} \left( (s > u)^{-1} t \otimes \delta_x(t \bowtie x)^{-1} a^{-1} t \right) \otimes \left( (b > u)^{-1} (t \bowtie x) \otimes \delta_w(t \bowtie w)^{-1} b^{-1} (t \bowtie x) \right) \]

Alternatively we can calculate

\[ \Delta_{kX \bowtie k(Y)}(\psi(\delta_x \otimes u \otimes t \otimes \delta_x)) = \Delta_{kX \bowtie k(Y)}((s > u)^{-1} t \otimes \delta_x(t \bowtie w)^{-1} s^{-1} t) \]
\[ = \sum_{yz=n(t \bowtie w)^{-1} s^{-1} t} (s > u)^{-1} t \otimes \delta_x \otimes (s > u)^{-1} t \otimes y \otimes \delta_z \].
where \( y, z \in Y \), which is the smash coproduct for the coaction induced by the back-reaction \( \bar{s} \).

We begin with the calculation
\[
(x(t \bowtie x)^{-1} a^{-1} t) \cdot (w(t \bowtie v)^{-1} b^{-1} (t \bowtie x)) = x w(t \bowtie v)^{-1} b^{-1} a^{-1} t ,
\]
which shows that if we replace \( y \) by \( x(t \bowtie x)^{-1} a^{-1} t \) and \( z \) by \( w(t \bowtie v)^{-1} b^{-1} (t \bowtie x) \), that the conditions of the summations are the same. It now remains to calculate
\[
(s \triangleright u)^{-1} t \bar{e} y = (s \triangleright u)^{-1} t \bar{e} x (t \bowtie x)^{-1} a^{-1} t = (a^{-1} s \triangleright u)^{-1} (t \bowtie x) = (b \triangleright u)^{-1} (t \bowtie x) ,
\]
as required.

If we apply \( \psi \) to the unit of \( D(H) \), we get
\[
\psi \sum_{s,v} \delta_s \otimes e \otimes e \otimes \delta_v = \sum_{s,v} e \otimes \delta_{s^{-1}} ,
\]
which is the unit in \( kX \bowtie k(Y) \).

Next we consider the counit,
\[
\epsilon_{kX \bowtie k(Y)}(\psi(\delta_s \otimes u \otimes t \otimes \delta_v)) = \epsilon_{kX \bowtie k(Y)}((s \triangleright u)^{-1} t \otimes \delta_v(t \bowtie v)^{-1} s^{-1} t) = \delta_v(t \bowtie v)^{-1} s^{-1} t ,
\]
The last \( \delta \)-function splits into \( \delta_{v,e} \delta_{(t \bowtie v)^{-1} s^{-1} t} \), which is equal to \( \delta_{v,e} \delta_{s,e} \), which is in turn equal to \( \epsilon_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v) \).

Finally we consider the antipode,
\[
S_{kX \bowtie k(Y)}(\psi(\delta_s \otimes u \otimes t \otimes \delta_v)) = (s \triangleright u)^{-1} t \triangleright v(t \bowtie v)^{-1} s^{-1} t \otimes \delta_{(s \triangleright u)^{-1} t \triangleright v(t \bowtie v)^{-1} s^{-1} t})^{-1}
\]
\[
= (u^{-1}(t \bowtie u))^{-1} t \triangleright v(t \bowtie v)^{-1} s^{-1} t \otimes \delta_{(s \triangleright u)^{-1} t \triangleright v(t \bowtie v)^{-1} s^{-1} t})^{-1}
\]
\[
= (t \bowtie v)^{-1} u \otimes \delta_{(s \triangleright u)^{-1} t \bowtie v} \otimes \delta_{s \bowtie u} \otimes \delta_{s \bowtie u} \otimes \delta_{s \bowtie u} \otimes \delta_{s \bowtie u}
\]
where we remember in the last line to take the inverse for the \( Y \) group operation, and compare this with
\[
\psi S_{D(H)}(\delta_s \otimes u \otimes t \otimes \delta_v) = \psi((1 \otimes S(t \otimes \delta_v))(S^{-1}(\delta_s \otimes u) \otimes 1))
\]
\[
= \psi((1 \otimes (t \bowtie u)^{-1} \otimes \delta_{(t \bowtie v)^{-1}}(\delta_{s \bowtie u})^{-1} \otimes (s \triangleright u)^{-1} \otimes 1))
\]
\[
= \psi(\delta_{t \bowtie u}^{-1} \otimes (t \bowtie u)^{-1} \otimes \delta_{(s \bowtie u)^{-1}} \otimes (s \triangleright u)^{-1} \otimes 1))
\]

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where \( t = (t \triangleright u)^{-1} \), \( \bar{v} = (t \triangleright v)^{-1} \), \( s = (s \triangleright u)^{-1} \), \( \bar{u} = (s \triangleright v)^{-1} \) and \( t' = \bar{t} \triangleright (s \triangleright \bar{u})^{-1} \). Applying the definition of \( \psi \), and using the fact that \( s \triangleright \bar{u} = u^{-1} \), we find

\[
\psi_{S_{D(H)}}(s \otimes u \otimes t \otimes \bar{u}) = (t' \bar{s} \triangleright \bar{u})^{-1} t' \otimes \delta((s \triangleright u) \triangleright \bar{u})^{-1} (t' \triangleright (s \triangleright \bar{u})^{-1} \bar{u})^{-1} (t \triangleright \bar{u})^{-1} t' t' t'
\]

\[
= (t' \triangleright u^{-1})^{-1} t' \otimes \delta_u^{-1} \bar{u}^{-1} \bar{s}^{-1}
\]

\[
= (t' \triangleright u^{-1}) u \otimes \delta_u^{-1} \psi(s \triangleright u) (s \triangleright u)
\]

\[
= (t \triangleright u)^{-1} u \otimes \delta_u^{-1} (t \triangleright v)^{-1} s \triangleright u
\]

again as required. This concludes the proof of the Hopf algebra isomorphism. Now we show that the star operation is preserved.

\[
\psi * (s \otimes u \otimes t \otimes \bar{u}) = \psi((I \otimes t^{-1} \otimes \delta_{t \triangleright v}) (s \triangleright u \otimes u^{-1} \otimes I))
\]

\[
= \psi(\delta_{t'/((s \triangleright u) \triangleright t \triangleright v)^{-1)} \otimes (t^{-1} \psi(t \triangleright v) u) \triangleright u^{-1} \otimes t' \otimes \delta_{((s \triangleright u) \triangleright u^{-1}) \triangleright (t \triangleright v) \triangleright u})
\]

where \( t' = t^{-1} \psi(s \triangleright u) \). Applying the definition of \( \psi \), we get

\[
\psi * (s \otimes u \otimes t \otimes \bar{u}) = (t' \triangleright (s \triangleright u) \triangleright u)^{-1} t' t' \otimes \delta((s \triangleright u) \triangleright (t \triangleright v) \triangleright u) (s \triangleright u)^{-1} (t^{-1} \psi(t \triangleright v) u) \triangleright u^{-1} \otimes t' \otimes \delta_{((s \triangleright u) \triangleright u^{-1}) \triangleright (t \triangleright v) \triangleright u}
\]

\[
= (t^{-1} \psi) \otimes \delta_{((s \triangleright u) \triangleright (t \triangleright v) \triangleright u) (s \triangleright u)^{-1}}
\]

\[
* \psi(s \otimes u \otimes t \otimes \bar{u}) = (s \triangleright u)^{-1} \otimes \delta_{(t \triangleright v)} (s \triangleright u)^{-1}
\]

\[
= (t^{-1} \psi) \otimes \delta_{((s \triangleright u) \triangleright (t \triangleright v) \triangleright u) (s \triangleright u)^{-1}}
\]

again as required. □

This means that \( kX \triangleright k(Y) \) inherits many of the nice properties of \( D(H) \). In particular, it has a quasitriangular structure \( R \) and associated elements \( Q = R_{21} R \) (the ‘quantum inverse Killing form’) and \( u = \sum (SR_{(2)}) R_{(1)} \) (the element which implements the square of the antipode) in Drinfeld’s general theory of quasitriangular Hopf algebras\[I\].

**Corollary 2.3** The bicharacteristic \( kX \triangleright k(Y) \) is quasitriangular, with

\[
R = \sum_{u,s,t} u^{-1} \otimes \delta_{us} \otimes s^{-1} \otimes \delta_{sv} t.
\]
The ‘quantum inverse Killing form’ is

\[ \mathcal{R}_{21} = \sum_{u,v\in G, s,p\in M} s^{-1} u^{-1} \otimes \delta_{u(s\triangleright v)(p\triangleleft u^{-1})} v^{-1} p^{-1} \otimes \delta_{p\triangleleft u^{-1}}. \]

and is non-degenerate as a bilinear functional \( k(X) \triangleright k(Y) \). The element \( u \) is the canonical element \( u = \sum_{x\in X} x \otimes \delta_x \) in \( kX \triangleright k(Y) \), and is central.

**Proof**  The computation is straightforward. Thus

\[
(\psi \otimes \psi)(\mathcal{R}) = \sum (s\triangleright u)^{-1} \otimes \delta_{us} - 1 \otimes \delta_{us} (s\triangleleft u^{-1}) t^{-1} s
\]

which yields the formula shown on a change of variables \( v \rightarrow s\triangleright v \). We then compute \( \mathcal{R}_{21} \mathcal{R} \) using the product in \( kX \triangleright k(Y) \).

\[
\mathcal{R}_{21} = \left( \sum_{u,s,v,t} s^{-1} \otimes \delta_{(s\triangleright v)t} \otimes v^{-1} \otimes \delta_{us} \right) \left( \sum_{u',s',v',t'} \psi^{-1} \otimes \delta_{u's'} \otimes s'^{-1} \otimes \delta_{(s'^{-1}v')t'} \right)
\]

From the \( \delta \)-functions here we can read off \( s\triangleright v = v^{-1} u'(s'^{-1}v') \), \( t = s\triangleleft v' \), \( u = v' \), and \( s = (s'^{-1}v')^{-1} t'^{-1} s' \). If we rewire these as \( v' = u, s' = t\triangleleft u^{-1}, u' = u(s\triangleright v)(t\triangleright u^{-1}) \) and \( t' = ts(t\triangleleft u^{-1})^{-1} \), then on substituting \( p = t\triangleleft u^{-1} \):

\[
\mathcal{R}_{21} = \sum_{u,v\in G, s,p\in M} s^{-1} u^{-1} \otimes \delta_{u(s\triangleright v)(p\triangleleft u^{-1})} v^{-1} p^{-1} \otimes \delta_{p\triangleleft u^{-1}}
\]

Nondegeneracy of \( \mathcal{R}_{21} \mathcal{R} \) as a linear map \( D(H^*) \rightarrow D(H) \) is the so-called factorisability property holding for any quantum double\[10\]. Hence it carries over in our case to a linear isomorphism \( k(X) \triangleright k(Y) \rightarrow kX \triangleright k(Y) \) or, equivalently, to a nondegenerate bilinear functional on \( k(X) \triangleright k(Y) \).

Finally,

\[
\psi(u) = \sum (s\triangleleft u)^{-1} \otimes \delta_{(s\triangleleft u)^{-1}} = \sum (s\triangleleft u)^{-1} \otimes \delta_{s\triangleleft u}^{-1} = \sum s\otimes \delta_{us}
\]
on a change of summation variables in which \( s \triangleright u \) is replaced by \( s^{-1} \) and \( s \triangleright u \) by \( u^{-1} \). Finally, the element \( u \) in any quasitriangular Hopf algebra implements the square of the antipode. But for any bicrossproduct, the antipode is involutive, hence \( u \) here is central. □

We also consider the \(*\)-structure in the case where the ground field is equipped with an involution.

**Corollary 2.4** \( kX \triangleright k(Y) \) is antireal-quasitriangular in the sense \((* \otimes *)(R) = R^{-1}\).

**Proof** This is known for the quantum double \( D(H) \) of any Hopf \(*\)-algebra[12], and hence follows from Theorem 2.2: here we provide a direct proof for our particular case. Computing using the bicrossproduct \(*\)-structure, we have

\[
(* \otimes *)(R) = \sum_{u,s,v,t} v \otimes \delta_{u^{-1}(us)} \otimes s \otimes \delta_{v^{-1}((s \triangleright v)t)} = \sum_{u,s,v,t} v \otimes \delta_{u^{-1}u(s \triangleright v)(sv) \otimes s \otimes \delta_{v(s \triangleright v)^{-1}ts}}
\]

after a change of \( u, t \) variables in the last step. Meanwhile,

\[
R^{-1} = (S \otimes \text{id})(R) = \sum_{u,s,v,t} (v^{-1} \delta us)^{-1} \otimes \delta_{v^{-1} \delta us}^{-1} \otimes s^{-1} \otimes \delta_{s \triangleright v} = \sum_{u,s,v,t} v \otimes \delta_{u(s \triangleright v)} \otimes s \otimes \delta_{vt}
\]

using the actions in Proposition 2.1. Note that the inversion in \( \delta_{( \_ )^{-1}} \) is the inverse in \( Y = G \times M^\text{op} \). This gives the same as \((* \otimes *)(R)\) after changing variables to \( v' = s \triangleright v \), \( s' = s^{-1}, u' = (s \triangleright v)^{-1} u^{-1} v \). Here \((s \triangleright v)^{-1} = s^{-1} \triangleright (s \triangleright v) = s' \triangleright v'. □

As an application, the finite-dimensional modules of any quasitriangular Hopf algebra have a natural ‘quantum dimension’ \( \dim \) defined as the trace of \( u \) in the representation. The modules of \( kX \triangleright k(Y) \), as a cross product algebra, are just the \( Y \)-graded \( X \)-modules \( V \) such that \(|x \triangleright v| = x\triangleright |v|\) for all \( v \in V \) homogeneous of degree \(|\_|\).

**Proposition 2.5** The quantum dimension of a general \( kX \triangleright k(Y) \)-module \( V \) is

\[
\dim(V) = \sum_{y \in Y} \text{trace}_{V_y} \pi(y)
\]

where \( V_y \) is the subspace of degree \( y \) and \( \pi(y) : V_y \rightarrow V_y \) is the restriction to \( V_y \) of the action of \( y \) viewed as an element of \( X \).
Proof We write $V = \oplus_y V_y$ for our $Y$-graded $X$-module. The action of $f \in k(Y)$ is $f(y)$ on $V_y$. A general element $x \in X$ acting on $V$ sends $V_y \to V_{x \triangleright y}$. Hence, in particular, $y$ viewed in $X$ sends $V_y \to V_y$ as $y \triangleleft y = y$ from Proposition 2.1. This is the operator on $V_y$ denoted $\pi(y)$. Let $\{e_a^{(y)}\}$ be a basis of $V_y$, with dual basis $\{f_a^{(y)}\}$. Then
\[
\text{Tr} (u) = \sum_{y \in Y, x \in X} (f_a^{(y)}, (x \otimes \delta_x) e_a^{(y)}) = \sum_{y \in Y} \sum_a (f_a^{(y)}, y e_a^{(y)}) = \sum_{y \in Y} \text{Tr} V_y \pi(y).
\]
\[\square\]

For example, we may take the natural representation in $k(Y)$ by left multiplication of $k(Y)$ and the left action of $X$ induced by its action on $Y$. This is the so-called Schrödinger representation of any cross product algebra. The spaces $V_y$ are 1-dimensional with basis $\{\delta_y\}$ and $\pi(y)\delta_y = \delta_y(y^{-1}y) = \delta_y$ is the identity. So $\dim(k(Y)) = |Y| = \dim k(Y)$, where $|Y|$ is the order of group $Y$. So for this representation the quantum dimension is the usual dimension.

Example 2.6 We consider the factorisation of the group $S_3$ into a subgroup of order 3 and a subgroup of order 2.

Proof Consider a factorisation of the group $S_3$ of permutations of 3 objects, which we label 1, 2 and 3. Let $G$ be the subgroup consisting of the 3-cycles and the identity, and let the subgroup $M$ consist of the transposition $(1, 2)$ and the identity. Then, in the notation of this section, $X = S_3$, and $Y = GM^{op}$ is a cyclic group of order 6. The left action of $X$ on $Y$ is the adjoint action of the group $S_3$ on the set $S_3$, and the right action of $Y$ on $X$ is given by
\[
u \triangleright u = u, \quad u \triangleright (1, 2) = u^{-1}, \quad u(1, 2) \triangleright v = u(1, 2), \quad u(1, 2) \triangleright v(1, 2) = u^{-1}(1, 2),
\]
where $u$ and $v$ are any 3-cycles or the identity. This leads to a quasitriangular structure $R$ on $kX \triangleright k(Y)$, given by
\[
R = \sum_{u, v \in G, t \in M} v^{-1} \otimes \delta_u \otimes e \otimes \delta_v + \sum_{u, v \in G, t \in M} v^{-1} \otimes \delta_{u(1, 2)} \otimes (1, 2) \otimes \delta_{v^{-1}t}.
\]
What actually is the group $Y \triangleright X$ in this case? It is of order 36, and a short calculation will show that it has no center. The possibilities, read off from a table of groups, are $S_3 \times S_3$ and $C_4 \ltimes (C_3 \times C_3)$, where the $C_4 = \{0, 1, 2, 3\}$ action is given by $\triangleright (x, y) = (y, -x)$. The group $S_3 \times S_3$ has 15 elements of order 2, and $C_4 \ltimes (C_3 \times C_3)$ has 9 elements of order 2. A brief check
of the group $Y \triangleright \triangleleft X$ shows that it has 15 elements of order 2, so it is isomorphic to $S_3 \times S_3$. However the isomorphism does not seem to be obvious! □

3 Subfactorisation from an order-reversing isomorphism

Let $\theta$ be an automorphism of $X$ which reverses its factors $GM$ (i.e. $\theta(G) = M$ and $\theta(M) = G$). It is shown in [2] that $\theta$ induces an semi-skew automorphism of $D(H)$ (i.e. an algebra antiautomorphism and coalgebra automorphism), which we denote $\Theta$:

$$\Theta(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{\theta(t \triangleright v)} \otimes \theta(t \triangleleft v) \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s \triangleleft u)}.$$ (6)

Via Theorem 2.2, we may view this as a semi-skew automorphism of $kX \triangleright \triangleleft k(Y)$. When the ground field is equipped with an involution, we may follow $\Theta$ by the star operation and obtain an antilinear Hopf algebra automorphism $\ast \Theta$.

Lemma 3.1 If $\theta$ is a factor-reversing automorphism of $X$ then the induced antilinear automorphism of $kX \triangleright \triangleleft k(Y)$ is given by

$$\ast \Theta(x \otimes \delta_y) = \theta(x) \otimes \delta_{\theta(y)^{-1}}$$

when $y$ is viewed in $X$ (and the inverse is also in $X$).

Proof We define $\ast \Theta$ via $\psi$ and (6). Thus,

$$\ast \psi(\delta_{\theta(t \triangleright v)} \otimes \theta(t \triangleleft v) \otimes \theta(s \triangleright u) \otimes \delta_{\theta(s \triangleleft u)})$$

$$= \ast \left( (\theta(t \triangleright v) \otimes \theta(t \triangleleft v))^{-1} \otimes \delta_{\theta(s \triangleright u)} (\theta(s \triangleright u) \otimes \theta(s \triangleleft u))^{-1} \otimes \delta_{\theta(s \triangleright u)} \right)$$

$$= \ast \left( (\theta(s \triangleright u)^{-1} \otimes \delta_{\theta(t \triangleright v)^{-1}}) \otimes \theta(s \triangleright u)^{-1} \otimes \theta(s \triangleright u)^{-1} \otimes \delta_{\theta(s \triangleright u)^{-1}} \right)$$

$$= \ast \Theta \psi(\delta_s \otimes u \otimes t \otimes \delta_v) = \ast \Theta((s \triangleright u)^{-1} t \otimes \delta_v(t \triangleright v)^{-1} s^{-1} t).$$

Comparing these expressions gives the result for $\ast \Theta$ after changing variables to general elements of $X, Y$. □

We observe that $\ast \Theta$-invariant basis elements $x \otimes \delta_y$ are characterised by the property that $\theta x = x$ and $\theta(y) = y^{-1}$ (computed in $X$).
Proposition 3.2 There is a subgroup $X^\theta$ of $X$ consisting of those elements $x$ for which $\theta x = x$, and a subset $Y^\theta$ of $Y$ consisting of those elements $y$ for which $\theta y = y^{-1}$ (inverse in $X$). The actions $\tilde{\cdot}, \tilde{\cdot}$ restrict to $X^\theta,Y^\theta$, forming a double cross product group $Y^\theta \bowtie X^\theta$ factorising into $Y^\theta, X^\theta$. The corresponding bicrossproduct $kX^\theta \bowtie k(Y^\theta)$ Hopf algebra has an isomorphic coalgebra to that of $kM \bowtie k(G)$.

Proof The proof that the $\tilde{\cdot}$ action restricts is immediate. If we take $x \in X^\theta$ and $y \in Y^\theta$, then $x \tilde{\cdot} y = xyx^{-1}$ (adjoint action in the $X$ multiplication). If we apply $\theta$ to this, then

$$\theta(x \tilde{\cdot} y) = \theta(xyx^{-1}) = \theta(x)\theta(y)\theta(x^{-1}) = xy^{-1}x^{-1},$$

which in the inverse (in $X$) of $x \tilde{\cdot} y = xyx^{-1}$, so $x \tilde{\cdot} y \in Y^\theta$.

The proof for the other action is rather more difficult. It will be most convenient to find formulae for the elements of $X^\theta$ and $Y^\theta$ first.

If $y = vt \in Y^\theta$, then $\theta y = y^{-1}$ (inverse in $X$), so we can substitute $\theta(y) = \theta(v)\theta(t)$ and $y^{-1} = t^{-1}v^{-1}$ and use uniqueness of factorisation to say that $\theta(v) = t^{-1}$. Then we can write $y = Y(v) = v\theta(v)^{-1}$. Now we can write out a simple formula for the multiplication in $Y^\theta$ as

$$y.y' = Y(v).Y(v') = (v\theta(v)^{-1})(v'\theta(v')^{-1}) = vv'\theta(vv')^{-1} = Y(vv').$$

This shows that $Y^\theta$ is actually isomorphic to $G$.

If $x = us \in X^\theta$, then $\theta x = x$, so we can substitute

$$\theta(x) = \theta(u)\theta(s) = x = us = (s^{-1}u^{-1})^{-1}(s^{-1}u^{-1})^{-1},$$

and use uniqueness of factorisation to say that $\theta(s) = (s^{-1}u^{-1})^{-1}$. Then $u^{-1} = s\bowtie\theta(s)^{-1}$, so we can write $x = X(s) = (s\bowtie\theta(s)^{-1})^{-1}s$. This shows that $X^\theta$ is bijective as a set with $M$.

Now consider the right action,

$$X(s)\tilde{\cdot}Y(v) = (s\bowtie\theta(s)^{-1})^{-1}s\tilde{\cdot}\theta(v)^{-1} = ((s\bowtie\theta(v)^{-1}s^{-1}\bowtie\theta(s)^{-1}))^{-1}(s\bowtie\theta(v))$$

$$= ((s\bowtie\theta(v)^{-1}\bowtie\theta(s)^{-1})^{-1}(s\bowtie\theta(v))$$

$$= ((s\bowtie\theta(s)^{-1})^{-1}(s\bowtie\theta(v)) = X(s\bowtie\theta(v)).$$

In particular this shows that the result is in $X^\theta$. 

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We therefore have a bicrossproduct Hopf algebra \(kX^\theta \bowtie k(Y^\theta)\). Its coalgebra is determined by the action of \(Y^\theta\) on \(X^\theta\) and the group structure of \(Y^\theta\), hence we see that it is isomorphic via the maps \(X, Y\) to the coalgebra of \(kM \bowtie k(G)\). \(\Box\)

Also, by construction, we see that if we equip \(k\) with the trivial involution then

\[
kX^\theta \bowtie k(Y^\theta) \subseteq (kX \bowtie k(Y))^*\Theta
\]

as algebras. The right hand side denotes the fixed point subalgebra under the algebra automorphism \(*\Theta\). The inclusion is clear from Lemma 5.1 (????) and the inclusion \(X^\theta \subseteq X\) as subgroups and \(k(Y^\theta) \subseteq k(Y)\) (extension by zero) as \(X^\theta\)-module algebras. That \(*\Theta\) is also a coalgebra automorphism tells us further that the coproduct of \(kX \bowtie k(Y)\) applied to elements of the fixed subalgebra yields elements invariant under \(*\Theta \otimes *\Theta\). It is natural to ask to what extent the quasitriangular structure of \(kX \bowtie k(Y)\) is likewise invariant.

**Proposition 3.3** The quasitriangular structure of \(kX \bowtie k(Y)\) obeys

\[
(\Theta \otimes \Theta)(R) = R_{21}
\]

When the field has an involution, we have \((*\Theta \otimes *\Theta)(R) = R_{21}^{-1}\). Moreover, if \(\theta^2 = \text{id}\) then

\[
\Theta^2 = \text{id} \quad \text{and} \quad (*\Theta)^2 = \text{id}.
\]

**Proof** It is easier to do the first computations in \(D(H)\). There, we have

\[
(\Theta \otimes \Theta)(R) = \sum_{u,v,s,t} \Theta(\delta_s \otimes u \otimes e \otimes \delta_v) \otimes \Theta(\delta_t \otimes e \otimes s \otimes \delta_u)
\]

\[
= \sum_{u,v,s,t} (\delta_{\theta(e \bowtie v)} \otimes \theta(e \bowtie v) \otimes \theta(s \bowtie u) \otimes \delta_{\theta(s \bowtie u)}) \otimes (\delta_{\theta(s \bowtie u)} \otimes \theta(s \bowtie u) \otimes \theta(t \bowtie e) \otimes \delta_{\theta(t \bowtie e)})
\]

\[
= \sum_{u,s} (1 \otimes \theta(s \bowtie u) \otimes \delta_{\theta(s \bowtie u)}) \otimes (\delta_{\theta(s \bowtie u)} \otimes \theta(s \bowtie u) \otimes 1)
\]

where the sum over \(v, t\) are replaced by sums over \(v' = \theta(v), t' = \theta(t)\) and give the unit elements of \(k(M)\) and \(k(G)\) respectively. Then we change variables from \(u, s\) to \(u' = \theta(s \bowtie u), u' = \theta(s \bowtie u)\), to recognise \(R_{21}\) in \(D(H)\). Hence the same result applies for \(kX \bowtie k(Y)\). This combines with Corollary 2.4 to obtain the corresponding property for \(*\Theta\). Also, the automorphism \(\Theta\) in (???) clearly obeys

\[
\Theta^2(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{\theta^2(s)} \otimes \theta^2(u) \otimes \theta^2(t) \otimes \delta_{\theta^2(v)}
\]

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and hence $\Theta^2 = \text{id}$ when $\theta^2 = \text{id}$. The same feature for $\ast \Theta$ is immediate from Lemma 3.1. $\square$

Hence $\mathcal{R}$ is not invariant in the usual sense (unless $G, M$ are trivial), due to the nondegeneracy of $\mathcal{R}_{21} \mathcal{R}$ in Corollary 2.3. Rather, one should note that for any quasitriangular Hopf algebra, $\mathcal{R}_{21}^{-1}$ defines a second ‘conjugate’ quasitriangular structure. It corresponds in topological applications to reversing braid crossings; we see that our quasitriangular structure is invariant up to conjugation in this sense. Although $kX^\theta \triangleright k(Y^\theta)$ does not in general inherit a quasitriangular structure from $\mathcal{R}$, its inclusion as fixed point subalgebra in a quasitriangular Hopf algebra equipped with such an automorphism might be a useful substitute. Of course, it may still happen that $kX^\theta \triangleright k(Y^\theta)$ is quasitriangular for some other reason, which is the case in the first example below.

Another natural question, in view of Proposition 3.2, is whether $kX^\theta \triangleright k(Y^\theta)$ is in fact isomorphic as a Hopf algebra to our original bicrossproduct $kM \triangleright k(G)$. The following example also demonstrates that it is not necessarily isomorphic to it.

**Example 3.4** We consider the example in [BGM] of the doublecrossproduct of two cyclic groups of order 6 ($C_6$) which gives the product of two symmetric groups $S_3 \times S_3$. In this case $kX^\theta \triangleright k(Y^\theta)$ is isomorphic to $kS_3 \triangleright k(S_3)$ in Example 2.6 and hence quasitriangular.

**Proof** Consider the group $X = S_3 \times S_3$ as the permutations of 6 objects labelled 1 to 6, where the first factor leaves the last 3 objects unchanged, and the second factor leaves the first 3 objects unchanged. We take $G$ to be the cyclic group of order 6 generated by the permutation $1_G = (123)(45)$, and $M$ to be the cyclic group of order 6 generated by the permutation $1_M = (12)(456)$. Our convention is that permutations act on objects on their right, for example $1_G$ applied to 1 gives 2. The intersection of $G$ and $M$ is just the identity permutation, and counting elements shows that $GM = MG = S_3 \times S_3$. We write each cyclic group additively, for example $G = \{0_G, 1_G, 2_G, 3_G, 4_G, 5_G\}$. The action of the element $1_M$ on $G$ is seen to be given by the permutation $(1_G, 5_G)(2_G, 4_G)$, and that of $1_G$ on $M$ is given by the permutation $(1_M, 5_M)(2_M, 4_M)$.

The factor reversing automorphism $\theta$ can be taken to be conjugation by the permutation $(1, 4)(2, 5)(3, 6)$. Then if we split the elements of $X$ into $S_3 \times S_3$, we see that the elements of $X^\theta$ are of the form $\sigma \times \sigma$, for $\sigma \in S_3$, and that the elements of $Y^\theta$ are of the form $\sigma \times \sigma^{-1}$. Then
$X^\theta$ is isomorphic to $S_3$, and the action of $X^\theta$ on $Y^\theta$ is the adjoint action of the group $S_3$ on the set $S_3$.

This is enough information to show that, despite having the same dimension, that $kX^\theta \bowtie k(Y^\theta)$ is not isomorphic to $kG \bowtie k(M)$ or $k(G) \rhd kM$. All these can, as algebras over $\mathbb{C}$, be decomposed into a direct sum of matrix algebras. According to [4], the algebras $kG \bowtie k(M)$ or $k(G) \rhd kM$ have at most $2 \times 2$ matrices in their decompositions. However the existence of an orbit of size 3 in the adjoint action of the group $S_3$ on the set $S_3$ shows that at least one $3 \times 3$ matrix occurs in the decomposition of $kX^\theta \bowtie k(Y^\theta)$.

So what is $kX^\theta \bowtie k(Y^\theta)$? In fact we have already met it, it is the bicrossproduct given by the product of $S_3$ and a cyclic group of order 6 in example 2.6. The explicit correspondence is given by deleting the points 4, 5 and 6 from the example here. Then we have the maps $\sigma \times \sigma \mapsto \sigma$ for $X^\theta$, and $\sigma \times \sigma^{-1} \mapsto \sigma$ for $Y^\theta$, corresponding to the subsets of $S_3$ used in in example 2.6. But from the previous example we see that $kX^\theta \bowtie k(Y^\theta)$ is actually the double of a bicrossproduct obtained from a group of order 2 and a group of order 3. We deduce that as a result $kX^\theta \bowtie k(Y^\theta)$ is actually quasitriangular. However there is no obvious relation between this quasitriangular structure and the standard structure on $kX \bowtie k(Y)$. □

**Example 3.5** The upper-lower triangular example, in which $kX^\theta \bowtie k(Y^\theta)$ is isomorphic to $kM \bowtie k(G)$.

**Proof** Let $G$ be the group $T_+$ of upper triangular $n \times n$ matrices with ones on the diagonal, with the usual matrix multiplication. Also let $M$ be the group $T_-$ of lower triangular $n \times n$ matrices with ones on the diagonal. As in [10] we define actions, using $\theta(a) = (a^T)^{-1}$,

\[ s \triangleright u = 1 + \theta(s)(u - 1), \quad s \triangleleft u = 1 + (s - 1)\theta(u). \]

Using these actions we find

\[ X(t).X(s) = X((t \triangleright (s \triangleleft \theta(s)^{-1}))s) = X(2ts - s - t + 1). \]

Hence there is a group isomorphism $X^\theta \cong T_-$ defined by $X(t) \mapsto 2t - I$. We call the group isomorphism $\alpha : T_- \rightarrow X^\theta$, where $\alpha(s) = X((s + 1)/2)$. Then we calculate

\[ \alpha(s) \triangleright Y(v) = X((s+1)/2) \triangleright Y(v) = X(s + 1/2 \triangleright v) = X(1 + (s + 1/2 - 1)\theta(v)) = X(1 + (s - 1/2)\theta(v)) = \alpha(s \triangleleft v). \]
As the map \( v \rightarrow Y(v) \) gives a group isomorphism from \( T_+ \) to \( Y^\theta \) we might think that we are on our way to showing that the \( X^\theta Y^\theta \) doublecross product is isomorphic to the original \( T_+ T_- \) doublecross product. To check this, we must perform the calculation of the left action:

\[
X(s) \triangleright Y(v) = Y(1 + (2 - \theta(s))^{-1} \theta(s)(v - 1)) = Y(1 + (2\theta(s)^{-1} - 1)^{-1}(v - 1))
\]

Now we use the fact that \( \theta(s)^{-1} = s^T \), and calculate

\[
\alpha(t) \triangleright Y(v) = X((t+1)/2) \triangleright Y(v) = Y(1 + (t^T)^{-1}(v - 1)) = Y(t \triangleright v)
\]

as required to prove the isomorphism of the doublecross products. □

4 A *-representation of \( D(H) \) on a Hilbert space

In this section we provide a Hilbert space representation of \( D(H) \) which is one of the motivations behind Theorem 2.2. Recall that it was shown in \[ \] that representations of \( D(H) \) are \( G - M \)-bigraded bicrossed \( G - M \)-bimodules. We shall use \( |w| \) for the \( G \)-grading and \( \langle w \rangle \) for the \( M \)-grading of a homogeneous element \( w \) of the representation.

**Proposition 4.1** There is a representation of \( D(H) \) on a vector space \( E \) with basis \( \{ \eta_{s,u} | s \in M, u \in G \} \), with gradings

\[
|\eta_{s,u}| = u, \quad \langle \eta_{s,u} \rangle = s
\]

and the group actions

\[
t \triangleright \eta_{s,u} = \eta_{ts(t \triangleright u)^{-1}, t \triangleright u}, \quad \eta_{s,u} \triangleleft v = \eta_{s^{-1}v(t \triangleright u)^{-1}, uv}
\]

The corresponding action of \( D(H) \) is

\[
(\delta_s \otimes u \otimes t \otimes \delta_v) \triangleright \eta_{r,w} = \delta_{v,w} \delta_{t^{-1}s(t \triangleright v), r} \eta_{s^{-1}v, (s \triangleright v)^{-1}uv}
\]

**Proof** The definition of the group actions is made precisely so that the matching conditions in \[ \] are true. The corresponding actions of the Hopf algebras \( H^* \) and \( H \) are

\[
(t \otimes \delta_v) \triangleright \eta_{s,u} = \delta_{v,u} t \triangleright \eta_{s,u}, \quad \eta_{s,u} \triangleleft (\delta_t \otimes v) = \delta_{s,t} \eta_{s,u} \triangleleft v
\]

and the formula \( (a \otimes h) \triangleright w = (h \triangleright w) \triangleleft a \) gives the action of \( D(H) \) as

\[
(I \otimes t \otimes \delta_v) \triangleright \eta_{s,u} = \delta_{v,u} \eta_{ts(t \triangleright u)^{-1}, t \triangleright u}, \quad (\delta_t \otimes v \otimes I) \triangleright \eta_{s,u} = \delta_{t,s} \eta_{s^{-1}v, (s \triangleright v)^{-1}uv}
\]
which gives the formula stated. □

As far as the original group doublecross product is concerned, the $E$ representation is more symmetric than the standard ‘Schroedinger’ representation of $D(H)$ in $H$, as we do not have to decide to take the group algebra of one factor subgroup, and the function algebra of the other. The $E$ representation motivates the isomorphism in Theorem 2.2 in the following manner: If we rewrite $\rho_{wr^{-1}} = \eta_{r,w}$, then the action above gives

$$(\delta_s \otimes u \otimes t \otimes \delta_v) \triangleright \rho_{wr^{-1}} = \delta_v(t\omega v)^{-1}s^{-1}t_{wr^{-1}} \rho(s\omega u)_{r^{-1}t_{(wr^{-1})}t^{-1}(s\omega u)}.$$ 

Compare this to the action of a single bicrossproduct $kX \triangleright k(Y)$, with $Y$-grading $\parallel \parallel$ on homogeneous elements:

$$(x \otimes \delta_y) \triangleright (t) = \delta_y, \parallel x \parallel \triangleright (t).$$

We see that the formulae agree if we set $x = (s\omega u)^{-1}t$, $y = v(t\omega v)^{-1}s^{-1}t$, $\parallel \rho_{wr^{-1}} \parallel = wr^{-1}$, and use the adjoint action of $X$, $x \triangleright \rho_{wr^{-1}} = \rho_x(x^{-1}w^{-1})$. This suggests trying a formula of the type used for $\psi$ in Section 2.

**Proposition 4.2** Over $\mathbb{C}$, there is an inner product $(\eta, \zeta)$ on $E$ (conjugate-linear in the first variable and linear on the second), defined by

$$(\eta_{s,u}, \eta_{r,w}) = \delta_{s,r} \delta_{u,w}.$$ 

With respect to this, $D(H)$ with its natural $*$-structure is represented as a $*$-algebra.

**Proof** The inner product given is non-degenerate (in fact the $\eta_{s,u}$ form an orthonormal basis). We can then check the condition that

$$(\alpha \triangleright \eta, \zeta) = (\eta, \alpha^* \triangleright \zeta)$$

for any $\alpha \in D(H)$. We shall only prove this in the case $\alpha = I \otimes t \otimes \delta_v$, and leave the other case to the reader. First we calculate

$$(\alpha \triangleright \eta_{su}, \eta_{s'u'}) = \delta_{v,u} \delta_{s(t\omega u)^{-1}s', \delta_{u,u'}}.$$ 

Now $\alpha^* = I \otimes t^{-1} \otimes \delta_{v'u'}$, and

$$(\eta_{su}, \alpha^* \triangleright \eta_{s'u'}) = \delta_{v'u', u} \delta_{s,t^{-1}s'(t^{-1}u')^{-1}} \delta_{u,t^{-1}u'}.$$ 

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which is the same condition. □

**Proposition 4.3** $E$ has a coalgebra structure

$$\Delta_E(\eta_{s,u}) = \sum_{ab=s, wz=u} \eta_{a,w} \otimes \eta_{b,z} = \sum \eta_{(1)} \otimes \eta_{(2)} , \quad \epsilon(\eta_{s,u}) = \delta_{s,e} \delta_{u,e} ,$$

and becomes a left module coalgebra under the action of $D(H)$.

**Proof** To show that $E$ is a coalgebra, we first show that it is coassociative, which is

$$(I \otimes \Delta_E) \Delta_E(\eta_{s,u}) = \sum_{abc=s, wzu=v} \eta_{a,w} \otimes \eta_{b,z} \otimes \eta_{c,v} = (\Delta_E \otimes I) \Delta_E(\eta_{s,u}) .$$

Next we show that $\epsilon$ is a counit;

$$(I \otimes \epsilon) \Delta_E(\eta_{s,u}) = \sum_{ab=s, wz=u} \eta_{a,w} \delta_{c,z} \delta_{b,e} = \eta_{s,u} = (\epsilon \otimes I) \Delta_E(\eta_{s,u}) .$$

For the module coalgebra condition, first we have to prove

$$\Delta_E(\alpha \triangleright \eta) = \sum \alpha_{(1)} \triangleright \eta_{(1)} \otimes \alpha_{(2)} \triangleright \eta_{(2)} = \Delta(\alpha) \triangleright \Delta_E(\eta) .$$

Again we shall only prove this in the case $\alpha = I \otimes t \otimes \delta_v$, and leave the other case to the reader.

We set $\eta = \eta_{s,u}$. The coproduct of $D(H)$ is

$$\Delta(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)} = \sum_{xy=v} (I \otimes t \otimes \delta_x) \otimes (I \otimes t \langle x \rangle \otimes \delta_y) ,$$

and so we obtain

$$\sum \alpha_{(1)} \triangleright \eta_{(1)} \otimes \alpha_{(2)} \triangleright \eta_{(2)} = \delta_{v,u} \sum_{xy=u} \eta_{a(t \langle x \rangle)^{-1},t \langle x \rangle \otimes \eta_{(t \langle x \rangle)b(t \langle u \rangle)^{-1},(t \langle x \rangle)\triangleright y} .$$

Now we use $\alpha \triangleright \eta = \delta_{v,u} \eta_{ls(t \langle u \rangle)^{-1},t \triangleright u}$ and calculate

$$\Delta_E(\alpha \triangleright \eta) = \delta_{v,u} \sum_{x'y'=t \triangleright u, a'b'=ts(t \langle u \rangle)^{-1}} \eta_{a',x'} \otimes \eta_{b',y'} .$$

If we now use the correspondences $a' = ta(t \langle x \rangle)^{-1}$, $b' = (t \langle x \rangle)b(t \langle u \rangle)^{-1}$, $x' = t \triangleright x$ and $y' = (t \langle x \rangle)\triangleright y$ we see that the formulae for $\alpha_{(1)} \triangleright \eta_{(1)} \otimes \alpha_{(2)} \triangleright \eta_{(2)}$ and $(\alpha \triangleright \eta)_{(1)} \otimes (\alpha \triangleright \eta)_{(2)}$ agree.
Lastly we must prove that $\epsilon(\alpha \triangleright \eta) = \epsilon(\alpha)\epsilon(\eta)$. Using $\alpha = \delta_s \otimes u \otimes t \otimes \delta_v$ and $\eta = \eta_{r,w}$, we have

$$
\epsilon(\alpha \triangleright \eta) = \delta_{v,w} \delta_{I,s(t\triangleright v),r} \epsilon(\eta_{s\triangleright u,(s\triangleright u)^{-1}(t\triangleright w)u,e}) = \delta_{v,w} \delta_{I,s(t\triangleright v),r} \delta_{s\triangleright u,e} \delta_{(s\triangleright u)^{-1}(t\triangleright v)u,e} = \\
\delta_{v,w} \delta_{I,t\triangleright v,r} \delta_{s,e} \delta_{(s\triangleright u)\triangleright v,u,e} = \delta_{v,w} \delta_{I,t\triangleright v,r} \delta_{s,e} \delta_{w,e} = \\
\delta_{v,w} \delta_{I,t\triangleright v,r} \delta_{s,e} \delta_{w,e} = \delta_{v,w} \delta_{I,t\triangleright v,r} \delta_{s,e} \delta_{w,e} = \\
\epsilon(\delta_s \otimes u \otimes t \otimes \delta_v) \epsilon(\eta_{t\triangleright w}).
$$

□

Now suppose that $\theta$ is an order reversing isomorphism of the doublecross product group $X = GM$. As previously mentioned, from [2] we have an antilinear Hopf algebra automorphism $\hat{\Theta} : D(H) \rightarrow D(H)$.

**Proposition 4.4** There is an antilinear map $\hat{\Theta} : E \rightarrow E$ defined by

$$
\hat{\Theta}(\eta_{s,u}) = \eta_{\theta(s),\theta(u)}.
$$

which obeys

$$
\hat{\Theta}(\alpha \triangleright \eta) = (\hat{\Theta}\alpha \triangleright \eta) \forall \alpha \in D(H).
$$

**Proof** As usual we shall only prove this in the case $\alpha = I \otimes t \otimes \delta_v$, and leave the other case to the reader. We begin with

$$
*\Theta(I \otimes t \otimes \delta_v) = *(\delta_{\theta(t\triangleright v)} \otimes \theta(t\triangleright v)) \otimes I = \delta_{\theta(v)} \otimes \theta(t\triangleright v)^{-1} \otimes I,
$$

where we have used the equation $\theta(t\triangleright v)\triangleright \theta(t\triangleright v) = \theta(v)$. Now

$$
(\delta_{\theta(v)} \otimes \theta(t\triangleright v)^{-1} \otimes I) \triangleright \eta_{\theta(u),\theta(s)} = \delta_{\theta(v)} \otimes \theta(t\triangleright u) \eta_{\theta(u),\theta(s)} \eta_{\theta(u),\theta(t\triangleright v)^{-1}} \eta_{\theta(u),\theta(t\triangleright v)^{-1}} \eta_{\theta(u),\theta(t\triangleright v)^{-1}}.
$$

Since $\theta$ is 1-1, we can replace $\delta_{\theta(v)} \otimes \theta(t\triangleright v)^{-1}$ by $\delta_{v,u}$. Also we calculate

$$
\theta(u)\triangleright \theta(t\triangleright v)^{-1} = \theta((t\triangleright u)\triangleright v)^{-1} \text{ and } \theta(u)\triangleright \theta(t\triangleright v)^{-1} = \theta((t\triangleright u)\triangleright v)^{-1} \text{,}
$$

so the equation above becomes

$$
(\delta_{\theta(v)} \otimes \theta(t\triangleright v)^{-1} \otimes I) \triangleright \eta_{\theta(u),\theta(s)} = \delta_{v,u} \eta_{\theta(t\triangleright u)\triangleright v)^{-1}} \eta_{\theta(t\triangleright u)\triangleright v)^{-1}} \eta_{\theta(t\triangleright v)^{-1}} \eta_{\theta(t\triangleright v)^{-1}} \eta_{\theta(t\triangleright v)^{-1}}.
$$
Now using the $\delta_{v,u}$ to put $v = u$ in the equation;

$$(\delta_{\theta(v)} \otimes \theta(t^*v)^{-1} \otimes I) \eta_{\theta(tv),\theta(s)} = \delta_{v,u} \eta_{\theta(tv),u} \cdot \theta(t)\theta(s)\theta(t^*u)^{-1} = \delta_{v,u} \eta_{\theta(tv),u} \cdot \theta(ts(t^*u)^{-1})$$

which is the formula for $\hat{\theta}(\alpha \triangleright \eta_{s,u})$ as required. □

**Proposition 4.5** The coproduct on $E$ and the inner product are related by the formula

$$(\Delta_E \eta, \Delta_E \zeta) = |X| (\eta, \zeta),$$

where we use the tensor product inner product on $E \otimes E$.

**Proof** It is sufficient to prove this for basis elements,

$$(\Delta_E \eta_{s,\alpha}, \Delta_E \eta_{s',\alpha'}) = \sum_{ab = s, a'b' = s', \omega = u, \omega' = u'} (\eta_{a,w}, \eta_{a',w'}) (\eta_{b,z}, \eta_{b',z'})$$

$$= \sum_{ab = s, a'b' = s', \omega = u, \omega' = u'} \delta_{a,a'} \delta_{b,b'} \delta_{z,z'}$$

$$= \sum_{ab = s = s', \omega = u = u'} 1 = \delta_{s,s'} \delta_{u,u'} |G| |M| = |X| (\eta_{s,u}, \eta_{s',u'}).$$

□

5 First order bicovariant differential calculi on $H$

In this section, we regard the Hopf algebra $A = H^* = k(M) \triangleright kG$ associated to a group factorisation $GM$ as a 'coordinate ring' of some non-commutative geometric phase space. This is the point of view introduced in [4], where $H^*$ is an algebraic model for the quantization of a particle on $M$ moving along orbits under the action of $G$. Here we develop some of the ‘noncommutative geometry’ associated to this point of view.

First of all, on any algebra $A$, one may define a first-order differential calculus or ‘cotangent space’ $\Omega^1$ in a standard way cf[13]

1. $\Omega^1$ is an $A$-bimodule.
2. $d : A \rightarrow \Omega^1$ is a linear map obeying the Leibniz rule $\text{d}(ab) = (\text{d}a)b + abd$.
3. $A \otimes A \rightarrow \Omega^1$ given by $a \otimes b \mapsto abd$ is surjective.

When $A$ is a Hopf algebra, it is natural to add to this left and right covariance (bicovariance) under $A$. Thus[6]
4. $\Omega^1$ is an $A$-bicomodule and the the bimodule structures and $d$ are bicomodule maps. Here $\Omega^1 \otimes A$ and $A \otimes \Omega^1$ have the induced tensor product bicomodule structures, where $A$ is a bicomodule under its coproduct.

Cf. [14] one knows that compatible bimodules and bicomodules (Hopf bimodules) are of the form (say) $\Omega^1 = V \otimes A$ for some left crossed $A$-module $V$. The latter in our finite-dimensional setting means nothing more than left modules of the Drinfeld quantum double $D(A)$. A particular module is $\ker \epsilon \subset A$, a restriction of the canonical ‘Schroedinger’ representation whereby $D(A)$ acts on $A$ (by left multiplication and the coadjoint action, see [14]). As observed in [3], the further conditions for $\Omega^1,d$ amount to requiring $V$ to be a quotient of $\ker \epsilon$ as a quantum double module. Then

$$da = (\pi \otimes \text{id})(a_{(1)} \otimes a_{(2)} - 1 \otimes a), \quad \forall a \in A,$$

(7)

where $\pi : \ker \epsilon \rightarrow V$ is the quotient map. The right (co)module structure on $V \otimes A$ is by right (co)multiplication in $A$, the left module structure is the tensor product of $V$ and left (co)multiplication in $A$.

In the finite-dimensional setting which concerns us, one may equally well work in the dual picture in terms of $H = kM \bowtie k(G)$ and $L = V^*$. By definition cf [8], a bicovariant quantum tangent space $L$ for $H$ is a submodule of $\ker \epsilon \subset H$ under the quantum double $D(H)$. Here $D(H)$ acts on $\ker \epsilon$ by

$$h \triangleright g = h_{(1)} g S h_{(2)}, \quad a \triangleright g = \langle a, g_{(1)} \rangle g_{(2)} - \langle a, g \rangle 1, \quad \forall h \in H, \quad g \in \ker \epsilon \subset H, \quad a \in H^*$$

as a projection to $\ker \epsilon$ of the Schroedinger representation. We say that a quantum tangent space is irreducible if $L$ is irreducible as a quantum double module. It corresponds to $\Omega^1$ having no bicovariant quotients. This dual point of view has been used recently in [7], where the irreducible bicovariant quantum tangent spaces over a general quasitriangular Hopf algebra have been classified under the assumptions of $R_{21}R$ non-degenerate and a Peter-Weyl decomposition for the left regular representation. In the same manner, we now classify the irreducible bicovariant quantum tangent spaces $L$ when $H$ is a bicrossproduct. The corresponding $\Omega^1$ will be given as well.

The Schroedinger representation of $D(H)$ in $H$ has already been computed for $H = kM \bowtie k(G)$ in [2]. Using the description of $D(H)$ modules as $M - G$ bicrossed bimodules (as recalled in
Section 4), it has

\[ \left| t \otimes \delta_v \right| = (t \triangleright v)v^{-1}, \quad \langle t \otimes \delta_v \rangle = t \]

\[ s \rhd (t \otimes \delta_v) = st s^{-1} \otimes \delta_{s \triangleright v}, \quad (t \otimes \delta_v) \triangleleft u = t \triangleleft u \otimes \delta_u v^{-1}, \]

where \( s' = s \triangleleft (t \triangleright v)v^{-1} \). We now use the isomorphism \( D(H) \cong D(X) \) (the quantum double of the group algebra of \( X \)) also in 2 to transfer to an action of \( D(X) \). Since \( D(X) = k(X) \triangleright \triangleleft kX \) as an algebra, this should make it easier to decompose representations into irreducibles. One can also use our new isomorphism in Theorem 2.2 to transfer to an action of \( kX \triangleright \triangleleft k(Y) \), but this appears to be less natural for the present purpose.

Lemma 5.1 The action (8) defines an action of \( k(X) \triangleright \triangleleft kX \) on \( H \) given by

\[ us \rhd (t \otimes \delta_v) = (s''ts^{-1}) \triangleleft u^{-1} \otimes \delta_{u(s \triangleright v)}, \quad \delta_{us \rhd (t \otimes \delta_v)} = \delta_{u,v((t \triangleright v)^{-1} \triangleright v^{-1})} \delta_{s \triangleright v, (t \triangleright v)^{-1}} t \otimes \delta_v, \]

where \( s'' = s \triangleleft (t \triangleright v)v^{-1} \).

Proof According to the general results in 2, the corresponding \( X \)-grading \( || \) and \( X \)-action can be computed from the \( M - G \)-bicrossed bimodule structure as \( ||w|| = \langle w \rangle^{-1} |w| \) and \( us \triangleright w = ((s \triangleleft |w|^{-1}) \triangleright w) \triangleleft u^{-1} \) for all \( w \) in the module. Hence,

\[ ||t \otimes \delta_v|| = t^{-1}(t \triangleright v)v^{-1} = v((t^{-1} \triangleleft (t \triangleright v)v^{-1})(t^{-1} \triangleleft ((t \triangleright v)v^{-1})) \]

\[ = v((t \triangleleft v)^{-1} \triangleright v^{-1})((t \triangleleft v)^{-1} \triangleleft v^{-1}) = v(t \triangleleft v)^{-1} v^{-1} \]

and

\[ us \triangleright (t \otimes \delta_v) = (s''ts^{-1}) \triangleleft u^{-1} = (s''ts^{-1} \otimes \delta_{s \triangleright v}) \triangleleft u^{-1} = (s''ts^{-1}) \triangleleft u^{-1} \otimes \delta_{u(s \triangleright v)}. \]

\( \square \)

Motivated by the form of \( ||t \otimes \delta_v|| \) in the proof of the preceding lemma, we chose new bases for the vector space on which \( D(X) \) acts.

Lemma 5.2 Let \( \phi_{vt} = t^{-1} \triangleleft v^{-1} \otimes \delta_v \). Here \( \{ \phi_{vt} \} \) is a basis of \( H \) labelled by \( vt \in X \). Then the action in Lemma 5.1 is

\[ us \triangleright \phi_{vt} = \phi_{usvt(s \triangleright v)^{-1}}, \quad \delta_{us \triangleright \phi_{vt}} = \delta_{us, vt v^{-1}} \phi_{vt} \]
Proof  Here $||\phi_{vt}|| = vtv^{-1} = v(t\triangleright v^{-1})(t\triangleright v^{-1})$ gives the action of $\delta_{us}$ by evaluation against the degree. This can be written more explicitly as $\delta_{us} \triangleright \phi_{vt} = \delta_{u,v(t\triangleright v^{-1})}\delta_{s,t\triangleright v^{-1}}\phi_{vt}$ and is thereby equivalent to the action in Lemma 5.1. Moreover, 

$$s''(t^{-1}\triangleright v^{-1})s^{-1} = (s\triangleright ((t^{-1}\triangleright v^{-1})t\triangleright v^{-1}))(t^{-1}\triangleright v^{-1})s^{-1} = (s\triangleright v(t^{-1}\triangleright v^{-1}))(t^{-1}v^{-1})s^{-1}$$

$$= (((s\triangleright v)t^{-1}\triangleright v^{-1})s^{-1} = ((s\triangleright v)t^{-1}(s\triangleright v)^{-1}v(s\triangleright v)^{-1} = ((s\triangleright v)(s\triangleright v)^{-1}v^{-1} = \phi_{usvt}(s\triangleright v)^{-1}.$$ 

Note that $(s\triangleright v)^{-1}(s\triangleright v)^{-1} = v^{-1}$ and $(s\triangleright v)^{-1}(s\triangleright v)^{-1} = s^{-1}$ for any matched pair of groups. Then

$$us\triangleright \phi_{vt} = us\triangleright (t^{-1}\triangleright v^{-1} \otimes \delta_{v}) = (s''(t^{-1}\triangleright v^{-1})s^{-1})u^{-1} \otimes \delta_{u(s\triangleright v)}$$

$$= ((s\triangleright v)t(s\triangleright v)^{-1})^{-1}(t\triangleright u(s\triangleright v)^{-1})^{-1} \otimes \delta_{u(s\triangleright v)} = \phi_{u(s\triangleright v)}(s\triangleright v)^{-1} = \phi_{usvt(s\triangleright v)^{-1}}.$$ 

$\square$

Our task is to decompose $\ker \epsilon \subset H$ into irreducibles under this action of $k(X)\rtimes kX$. We begin by decomposing the action in Lemma 5.2 into irreducibles and afterwards projecting to $\ker \epsilon$. Note that Lemma 5.2 tells us that when we identify $H \cong kX$ as linear spaces by the above basis, the action of $X$ is the linear extension of a certain group action of $X$ on itself.

Proposition 5.3 Let $X$ act on itself by the action $us\triangleright vt = usvt(s\triangleright v)^{-1}$ as in Lemma 5.2. Let $||vt|| \equiv ||\phi_{vt}|| = vtv^{-1}$ as an $X$-valued function on $X$.

(i) Let $\sim$ denote the equivalence relation on $X$ defined by $vt \sim us$ if and only if $||us|| = ||vt||$. Then $\triangleright$ descends to an action of $X$ on the quotient space $X/\sim$.

(ii) Let $\Xi_{[vt]} \subseteq X$ denote the isotropy subgroup of an equivalence class $[vt] \in X/\sim$. Then 

$$\Xi_{[vt]} = \{us \in X \mid us||vt|| = ||vt||us\},$$

the centraliser of $||vt||$ in $X$.

Proof (i) may be verified directly. However, it follows from Lemma 5.2 since an action of $k(X)\rtimes kX$ (where $X$ acts on $X$ by the adjoint action in the semidirect product) requires $||us\triangleright \phi_{vt}|| = us||\phi_{vt}||(us)^{-1}$. In terms of the group $X$, this is $||us\triangleright vt|| = us||vt||(us)^{-1}$. This also implies (ii) since the group $\Xi_{[vt]}$ consists of $us \in X$ such that $us\triangleright vt \sim vt$, i.e. such that $||us\triangleright vt|| = ||vt||$. $\square$
We denote by $O_{[vt]}$ the orbit containing the point $[vt]$ in $X/\sim$.

**Example 5.4** We may restrict attention to orbits of the form $O_{[s]}$, where $s \in M$. Then the elements of the equivalence class $[s]$ may be identified with the subset of $M$ fixed under the action of $s$, $[s] = \{sv \mid s \triangleright v = v\}$. The stabiliser $\Xi_{[s]}$ consists of all elements of $X$ which commute with $s$. The action of $\Xi_{[s]}$ on elements of the equivalence class $[s]$ is given by $ut\triangleright sv = su(t\triangleright v)$. In the particular case where $s = e$, we get $[e] = G$ and $\Xi_{[e]} = X$.

**Proof** This may seem to be a rather specialised example, but in fact any orbit $O_{[us]}$ in $X/\sim$ contains a point of the given form, since $[s] \in O_{[us]}$. We compute,

$$[s] = \{vt \mid vtv^{-1} = e\} = \{vt \mid v(tv^{-1}) = e, tv^{-1} = s\} = \{v(sv) \mid sv = v\}.$$ 

This can be simplified if we note that if $sv = v$, then $v(sv) = (sv)(sv) = sv$, giving the result stated. The action of $\Xi_{[s]}$ is computed as

$$ut\triangleright sv = ut\triangleright (sv) = utv(sv)(tv^{-1})^{-1} = utsv(tv^{-1})^{-1} = sutv(tv^{-1})^{-1} = su(tv^{-1})$$

as stated. $\square$

For $\chi \in X/\sim$, define $S_{\chi} = kp^{-1}(\chi) \subset kX$, where $p$ is the canonical projection to $X/\sim$. Here $S_{\chi}$ is the linear span of the elements of $\chi$ viewed as a leaf in $X$, and is a linear $\Xi_{\chi}$ representation since, by definition, the action of $\Xi_{\chi}$ sends $p^{-1}(\chi)$ to itself.

**Proposition 5.5** Let $O$ be an orbit in $X/\sim$ under the action of $X$. Then $M_{O} = \bigoplus_{\chi \in O} S_{\chi} \subset kX$ is a subrepresentation under the action of $k(X) \triangleright kX$ in Lemma 5.2. Moreover, $kX = \bigoplus_{O} M_{O}$ is a decomposition of $kX$ into subrepresentations.

**Proof** The action of $\delta_{usu^{-1}} \in k(X)$ on $kX$ in Lemma 5.2 is the projection operator

$$\delta_{usu^{-1}} \triangleright \phi_{vt} = \delta_{usu^{-1},vtv^{-1}} \phi_{vt}.$$ 

This is evident since the action of $k(X)$ is evaluation against $||\phi_{vt}||$ (or, explicitly, put $\delta_{usu^{-1}} = \delta_{u(su^{-1})(su^{-1})}$ into Lemma 5.2.) We see that $\pi_{[us]} = \delta_{usu^{-1}} \triangleright$ projects $kX$ onto the subspace $S_{[us]}$ and

$$kX = \bigoplus_{\chi} S_{\chi} = \bigoplus_{O \in O} S_{\chi} = \bigoplus_{O} M_{O}$$

(9)

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as vector spaces. Then the operator

$$\pi_O = \sum_{\chi \in O} \pi_{\chi}$$

commutes with the action of $k(X)\rtimes kX$, and is a projection to $M_O$. To see that $\pi_O$ does commute with the action of the algebra, we can calculate

$$\pi_{\chi}(x \otimes \delta_y) = (x \otimes \delta_y) \cdot \pi_{x^{-1} \tilde{\chi}}$$

and note that the operation $x^{-1} \tilde{\chi}$ is a 1-1 correspondence on the set $O$. □

In what follows, we fix an orbit $O \subset X/\sim$ and a base point $\chi_0$ on it. We denote by $\Xi$ the isotropy subgroup at $\chi_0$, and $S = S_{\chi_0}$.

**Proposition 5.6** Let $S = S_1 \oplus \cdots \oplus S_n$ be a decomposition into irreducibles under the action of $\Xi$. Let $\chi, S_i = us \triangleright S_i$ when $\chi = us \tilde{\chi}_0$ (this is independent of the choice of $us$). Then $M_i = \oplus_{\chi \in O \chi, S_i \subset M_O}$ are irreducible subrepresentations under $k(X)\rtimes kX$. Moreover, $M_O = \oplus_i M_i$ is a decomposition of $M_O$ into irreducibles.

**Proof** Let $x_\chi \in X$ be a choice of $us$ such that $us \tilde{\chi}_0 = \chi$. Define $\chi, S_i = x_\chi \triangleright S_i$. Now if we take any $x$ so that $x \tilde{\chi}_0 = \chi$, then $x^{-1} x \in \Xi$, so

$$x \triangleright S_i = x_\chi \triangleright (x^{-1} x \triangleright S_i) = x_\chi \triangleright S_i = \chi, S_i,$$

as $S_i$ is a representation of $\Xi$. Moreover, $\chi, S_i \cup \eta, S_i = \{0\}$ for $\chi, \eta$ distinct, so $M_i$ spanned as shown is a direct sum.

Next we show that $k(X)\rtimes kX$ acts on $M_i$. Clearly, $x \triangleright M_i \subset M_i$ for all $x \in X$ since $x \triangleright x_\chi \triangleright S_i = x x_\chi \triangleright S_i = \eta, S_i$, where $\eta = x \tilde{\chi} = x x_\chi \tilde{\chi}_0$ is another point in $O$. Meanwhile, The action of the element $\delta_{us}$ in $k(X)$ is either either zero or one of the projections associated to (8), and these are all zero or the identity on each $S_\chi$. Therefore the whole action of $k(X)\rtimes kX$ preserves the decompositions of the $S_\chi$, with the result that $M_i$ are subrepresentations and $M_O = \oplus_i M_i$.

We now prove irreducibility. Let $m \in M_i$ be nonzero. Then there is some $\chi$ such that the projection $m_\chi$ to $\chi, S_i$ is nonzero, and then $x^{-1}_\chi \triangleright m_\chi$ is a nonzero element of $S_i$. Since $S_i$ is irreducible under $\Xi \subseteq X$, we know that vectors of the form $\xi x^{-1}_\chi \triangleright m_\chi$, for $\xi \in \Xi$, span all of $S_i$. 27
Since the projection is itself the action of an element of $k(X) \bowtie kX$, we see that $S_i$ is contained in the space spanned by the action of this algebra on $m$. Moreover by using $x \triangleright S_i = \eta_i S_i$ we see that every $\eta_i S_i$ is contained in the image of $m$ under $k(X) \bowtie kX$. Hence $M_i = (k(X) \bowtie kX).m$, i.e. $M_i$ is irreducible. \qed

These two propositions give a total decomposition of $kX$ into irreducibles. In particular, we obtain irreducible subrepresentations for every choice of orbit and every irreducible subrepresentation of the associated isotropy group. The converse also holds by similar arguments to those in the preceding proposition.

**Proposition 5.7** Let $M \subset kX$ be an irreducible subrepresentation under $k(X) \bowtie kX$ in Lemma 5.2. Then as a vector space, $M$ is of the form

$$M = \bigoplus_{\chi \in \mathcal{O}} \chi.M_0$$

for some orbit $\mathcal{O}$ (with base point $\chi_0$) and some irreducible subrepresentation $M_0 \subset S$ under $\Xi$. (Here $\chi.M_0 = \text{us} \triangleright M_0$ when $\chi = \text{us} \triangleright \chi_0$.)

**Proof** Consider $M \subset kX$ and let $M_\chi = \pi_\chi(M)$ for any $\chi \in X/\sim$. Choose a $\chi_0$ so that $M_{\chi_0}$ is nonzero, write $M_0 = M_{\chi_0}$, and let $\Xi$ be the stabiliser of $\chi_0$. Now $M_0$ must be a representation of $\Xi$. If $S_1$ is an irreducible subrepresentation of $M_0$ under $\Xi$, then the previous proposition shows that

$$\bigoplus_{\chi \in \mathcal{O}} \chi.S_1 \subset M$$

is an (irreducible) representation of the algebra, where $\mathcal{O}$ is the orbit containing $\chi_0$. Since $M$ is irreducible, this representation must be equal to $M$, and in particular $M_0$ is an irreducible representation of $\Xi$. \qed

It is now a short step to obtain the classification for subrepresentations of $\ker \epsilon \subset H$. Note that every Hopf algebra is a direct sum $k1 \oplus \ker \epsilon$ as vector spaces, with associated projection $\Pi(h) = h - 1\epsilon(h)$. In our case, remembering that

$$\epsilon(\phi_{vit}) = \epsilon(t^{-1} \delta_v^{-1} \otimes \delta_v) = \delta_{v,e},$$
we see that \( \ker \epsilon \) is spanned by the projected basis elements

\[
\tilde{\phi}_{vt} \equiv \Pi(\phi_{vt}) = \phi_{vt} - \delta_{v,e} \sum_{u \in G} \phi_{ue}.
\]

**Lemma 5.8** The action in Lemma 5.2 and the projected action

\[
u s \triangleright \tilde{\phi}_{vt} = \tilde{\phi}_{us \triangleright vt}, \quad \delta_{us} \triangleright \tilde{\phi}_{vt} = \delta_{us,vtv^{-1}} \tilde{\phi}_{vt}
\]
on \( \ker \epsilon \) are intertwined by \( \Pi : kX \to \ker \epsilon \).

**Proof** First we check the \( kX \) actions, using \( \Pi \) and then acting in \( \ker \epsilon \) by \( us \), to get

\[
\Pi us \triangleright \Pi \phi_{vt} = \Pi(us \triangleright \phi_{vt}) - \delta_{v,e} \Pi(1) = \Pi(us \triangleright \phi_{vt}),
\]
using the facts that \( us \triangleright 1 = 1 \) and \( \Pi(1) = 0 \). Now we check the \( k(X) \) actions, using \( \Pi \) and then acting in \( \ker \epsilon \) by \( \delta_{us} \), to get

\[
\Pi \delta_{us} \triangleright \Pi \phi_{vt} = \Pi(\delta_{us} \triangleright \phi_{vt}) - \delta_{v,e} \Pi(\delta_{us} \triangleright 1) = \Pi(\delta_{us} \triangleright \phi_{vt}) - \delta_{v,e} \Pi(1) = \Pi(\delta_{us} \triangleright \phi_{vt}).
\]

\( \square \)

**Theorem 5.9** The irreducible quantum tangent spaces \( L \subset \ker \epsilon \) are all given by the following two cases:

(a) For an orbit \( O \neq \{[e]\} \) in \( X/\sim \), choose a base point \( \chi_0 \in O \). For each irreducible subrepresentation \( M_0 \subset S \) of \( \Xi \) we have an irreducible quantum double subrepresentation \( M = \bigoplus_{\chi \in O \chi} M_0 \subset kX \), and an isomorphic quantum double subrepresentation \( L = \Pi(M) \subset \ker \epsilon \).

(b) For the orbit \( O = \{[e]\} \) in \( X/\sim \), choose the base point \( \chi_0 = [e] \in O \). For any irreducible subrepresentation \( M_0 \subset S \) of \( \Xi \) other than the trivial one (multiples of 1), we have an irreducible quantum double subrepresentation \( M = \bigoplus_{\chi \in O \chi} M_0 \subset kX \), and an isomorphic quantum double subrepresentation \( L = \Pi(M) \subset \ker \epsilon \). Here \( S = kG \) and \( \Xi = X \), as in Example 5.4.
Proof In the cases above, $\mathcal{M} = \bigoplus_{\chi \in \mathcal{O}} \mathcal{M}_0$ is an irreducible representation of the unprojected action. By the previous lemma $\Pi : \mathcal{M} \to L$ is a map of representations, where $L \subset \ker \epsilon$ uses the projected representation. The map is onto, and if it is 1-1 then the two representations are isomorphic, and hence $L$ is also irreducible.

The only case where the map $\Pi$ is not 1-1 is where $1 \in \mathcal{M}$. Since $k1$ is a representation and $\mathcal{M}$ is assumed irreducible, this is just the nontriviality exclusion stated in the theorem. We have shown that the cases described lead to irreducible representations of $\ker \epsilon$.

Now suppose that we have $L$, an irreducible representation of $\ker \epsilon$. Then its inverse image $\Pi^{-1}L \subset H$ is a representation of $k(X) \rtimes kX$, and it contains the subrepresentation $k1$. If $L \neq \{0\}$, then there is at least one other irreducible subrepresentation $\mathcal{M} \subset H$ so that $k1 \oplus \mathcal{M} \subset \Pi^{-1}L$. But now $\mathcal{M}$ must be of the form described earlier, and by irreducibility $\Pi \mathcal{M} = L$. \qed

We are now left with the problem of finding the irreducible subrepresentations of $\mathcal{S}$ under the group $\Xi$. We can decompose the representation $\mathcal{S}$ into a sum of irreducibles $\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_r$, so in this case there would be at least $r$ irreducible subrepresentations. However if there are any equivalent representations in this list, there are many more possible irreducible subrepresentations. This is a standard situation in representation theory and we briefly recall its resolution, as follows.

**Lemma 5.10** Let $\mathcal{N}$ be an irreducible representation of the group $\Xi$, and let $\mathcal{M} \subset \mathcal{N} \oplus U$ be an irreducible representation of $\Xi$, for another representation $U$. Suppose that there is a vector in $\mathcal{M}$ with a nonzero $\mathcal{N}$ component. Then there is a $\Xi$-map $T : \mathcal{N} \to U$ so that $\mathcal{M} = \{w \oplus Tw \mid w \in \mathcal{N}\}$.

**Proof** First we show that any $w \in \mathcal{N}$ occurs as the first coordinate of a vector in $\mathcal{M}$. The projection $\pi : \mathcal{N} \oplus U \to \mathcal{N}$ is a $\Xi$-map, so $\pi \mathcal{M} \subset \mathcal{N}$ is a subrepresentation of $\mathcal{N}$. As $\mathcal{N}$ is irreducible, we see that $\pi \mathcal{M} = \mathcal{N}$.

Now we show that for any $w \in \mathcal{N}$ there is at most one $u \in U$ for which $w \oplus u \in \mathcal{M}$. If we had $w \oplus u \in \mathcal{M}$ and $w \oplus u' \in \mathcal{M}$, then on subtraction we would also have $0 \oplus (u - u') \in \mathcal{M}$. But now using $0 \oplus (u - u')$ as a cyclic vector we could construct a representation contained in $\mathcal{M}$ which consisted purely of vectors with zero $\mathcal{N}$ component. This would contradict the irreducibility of $\mathcal{M}$ unless $u = u'$. 

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Now simply label this unique u as T(w). As ξ(w ⊕ Tw) = ξw ⊕ ξTw ∈ M for ξ ∈ Ξ, we must have T(ξw) = ξT(w). □

**Proposition 5.11** Suppose that M, an irreducible representation of the group Ξ, is contained in \( N_1^{\oplus n_1} \oplus \ldots \oplus N_k^{\oplus n_k} \), where the \( N_i \) are inequivalent irreducible representations, and \( n_i \) gives the corresponding multiplicities in the sum. Then there is an index \( j \), and \( (\lambda_1, \ldots, \lambda_{n_j}) \in k^{n_j} \) (not all zero) so that

\[
M = \left\{ 0 \oplus \ldots \oplus 0 \oplus (\lambda_1 w \oplus \ldots \oplus \lambda_{n_j} w) \oplus 0 \oplus \ldots \oplus 0 \in N_1^{\oplus n_1} \oplus \ldots \oplus N_k^{\oplus n_k} \mid w \in N_j \right\}.
\]

**Proof** A nonzero vector in \( M \) must have a nonzero component in at least one of the irreducible summands. To reduce confusion, we shall assume that the first \( N_1 \) summand has a such a nonzero component. Then we can write \( M \subset N_1 \oplus U \), where U is the sum of the rest of the \( N_i \)s. By the previous lemma, there is a \( \Xi \)-map \( T : N_1 \to U \) so that \( M = \{ w \oplus Tw \mid w \in N_1 \} \).

Now we can follow the map \( T \) with a projection to one of the \( N_i \) summands in U and examine the resulting map. If we project to another of the \( N_1 \) summands, the result is a \( \Xi \)-map from an irreducible representation to itself, and by Schur’s lemma this must be scalar multiplication. If we project to a \( N_i \) summand for \( i \neq 1 \), the result is a \( \Xi \)-map from an irreducible representation to an inequivalent irreducible representation, and by Schur’s lemma this must be zero. □

Thus the irreducible subrepresentations of \( S \) in Theorem 5.9 are themselves classified as follows. For a given orbit \( O \) with base point \( \chi_0 \), decompose \( S \) into irreducible representations under the action of \( \Xi \). Write this as \( S = M_1^{\oplus n_1} \oplus \ldots \oplus M_k^{\oplus n_k} \), where the \( M_i \) are inequivalent irreducible representations, and \( n_i \) gives the corresponding multiplicities in the sum. The data is then \( (O, M_0, \lambda) \), where \( M_0 \) is an irreducible representation of \( \Xi \) occurring in \( S \) with multiplicity \( n > 0 \), and \( \lambda \in \mathbb{P}(k^n) \). (Note that we take the projective space as only the ratios of the scalars in the last proposition are required to specify the subspace.)

Bicrossproducts interpolate between group algebras and group function algebras. As a check, we recover the seemingly quite disparate results known separately for these two cases.

**Example 5.12** Suppose that \( G \) is trivial. Then \( H = kM \). In this case \( \sim \) is the same as equality and \( X/\sim = X \). In this case the equivalence classes are singletons corresponding to points in \( M \).
and $s \triangleright t = sts^{-1}$ is conjugation. Hence the orbits $O$ are conjugacy classes in $M$. The action of the isotropy group is trivial and hence this is the only data. We recover the result [7,7] that the irreducible quantum tangent spaces are the projected spans of the conjugacy classes.

**Example 5.13** Suppose that $M$ is trivial. Then $H = k(G)$. In this case $X$ is one entire equivalence class and $X/\sim = \{e\}$. Then there is only one orbit $O = [e]$ and Theorem 5.9 reduces to the classification of quantum tangent spaces in $\ker \epsilon \subset k(G)$ (so $O = kG$) which have been classified by the second author in [3] as irreducible subspaces under the left regular representation induced by $u\triangleright v = uv$.

We can also keep $M,G$ non trivial but let $\triangleright$ or $\triangleleft$ be trivial in the matched pair. In these cases $X$ is a semidirect product. Note that the two cases are quite different. In one, $H$ is a tensor product algebra. In the other it is a cross product $kM\triangleleft k(G)$ where $M$ acts by group automorphisms of $G$.

**Example 5.14** Suppose that the action $\triangleright$ is trivial. If we start with an orbit $O$ containing $[s]$, then $[s] = sG$ and $\Xi_{[s]} = \{ut| tst^{-1} = s\triangleleft u\}$. The canonical projection $ut \mapsto u$ is a group homomorphism $\Xi \rightarrow G$ and, identifying $\mathcal{S} = kG$, the action of $\Xi$ is the pull back along this of the left regular representation.

**Example 5.15** Suppose that the action $\triangleleft$ is trivial. If we start with an orbit $O$ containing $[s]$, then $[s] = \{sv| sv = vs\}$ may be identified with $G_s$, the $\sqcup G$ of the centraliser of $s \in X$. The isotropy group is $\Xi_{[s]} = \{ut| su = us \text{ and } st = ts\} = G_s \times M_s$, where $M_s$ is the centraliser of $s \in M$. Identifying $\mathcal{S} = kG_s$, the action of $\Xi$ is with $M_s$ acting by $\triangleright$ and $G_s$ acting by the left regular representation.

**Example 5.16** We completely solve the problem by hand for Example 2.6 where $X = S_3$. We just decompose $k(X)$ into irreducible subspaces $L$ under (5.8) and project to $\ker \epsilon$. More precisely, we follow the above theory in this special case and then convince ourselves by direct means that it is the right answer.
The orbit decomposition  
First we find the allowed orbits:

Orbit 1 : $\mathcal{O} = \{ [e] \}$, choosing base point $[e]$. Then $[e] = \{ e, (123), (321) \}$ and $\Xi = S_3$. The action of $\Xi$ on $[e]$ is given by the formula $ut\bar{\phi}v = utv^{-1}$. We have the eigenspaces of the $G$ action $M_0 = \langle e + (123) + (321) \rangle$, $M_1 = \langle e + \omega(123) + \omega^2(321) \rangle$, and $M_2 = \langle e + \omega^2(123) + \omega(321) \rangle$. Now $M_0$ forms a 1-dimensional representation of $\Xi$, but this is annihilated by $\Pi$. The action of $(12) \in \Xi$ is to swap $M_1$ and $M_2$, so we get a 2 dimensional irreducible representation $M_1 \oplus M_2$ of $\Xi$, giving a 2 dimensional irreducible subrepresentation in $\ker \epsilon$.

Orbit 2 : $\mathcal{O} = \{ [(12)], [(13)], [(23)] \}$, choosing base point $[(12)]$. Then $[(12)] = \{ (12) \}$ and $\Xi = M$. The action of $\Xi$ on $[(12)]$ is the trivial 1 dimensional representation, giving a 3 dimensional irreducible representation in $\ker \epsilon$.

The direct approach  
The space $\ker \epsilon$ is spanned by the vectors $\{ \bar{\phi}_x | x \in S_3 \}$, where $\bar{\phi}_x = \Pi \phi_x$, and there is the linear relation $\bar{\phi}_e + \bar{\phi}_{(123)} + \bar{\phi}_{(321)} = 0$. We use the relation to rewrite $\bar{\phi}_{(321)} = -\bar{\phi}_e - \bar{\phi}_{(123)}$, giving a basis consisting of 5 elements. This is then split into two parts by the action of $X$:

1. The space spanned by the elements $\{ \bar{\phi}_e, \bar{\phi}_{(123)} \}$. This has the $X$ action $us\bar{\phi}_e = \bar{\phi}_u$ and $us\bar{\phi}_{(123)} = \bar{\phi}_{us(123)s^{-1}}$, where we remember to rewrite $\bar{\phi}_{(321)} = -\bar{\phi}_e - \bar{\phi}_{(123)}$. This gives a 2 dimensional irreducible representation.

2. The space spanned by the elements $\{ \bar{\phi}_{(12)}, \bar{\phi}_{(13)}, \bar{\phi}_{(23)} \}$. This has the $X$ action $us\bar{\phi}_{v(12)} = \bar{\phi}_{usv^{-1}(12)}$ for all $v \in G$. This gives a 3 dimensional irreducible representation.

Example 5.17  
We apply the theory above to Example 3.4 where $X = C_6 \rtimes C_6$ and spell out the final result. This is rather complicated to do directly (hence justifying our methods). It is one of the simplest examples of a true bicrossproduct Hopf algebra.

First we identify the possible values of $\| . \|$, and the orbits of these values under $X = S_3 \times S_3$. Since $\| vt \| = vt v^{-1}$, the possible values and the orbits are simply given by looking at the conjugacy classes of the elements $t \in M$ in $X$. These are:

(Orbit 0)  
$t = [0]_M$ gives the conjugacy class consisting only of the identity. We choose $[0]_M$ as the base point for this orbit. Then $\Xi[0] = X$, and the equivalence class is $[0]_M = G$ (as previously noted). The action of $\Xi[0] = X$ on $[0]_M = G$ is given by the formula $ut\bar{\phi}v = u(t\phi v)$.

Let us now look at the decomposition of $S = kG$ into irreducibles under the action of
The element $1_G \in \Xi$ acts on any irreducible $M \subset S$, and its action diagonalises, that is $M$ is a sum of $M_r$ ($r \in \mathbb{Z}_6$), where each $M_r$ is zero or $\langle f_r \rangle$, where

$$f_r = 0_G + \omega^r 1_G + \omega^{2r} 2_G + \omega^{3r} 3_G + \omega^{4r} 4_G + \omega^{5r} 5_G,$$

for $\omega$ a primitive 6th root of unity. The action of $1_M$ is to send $f_r$ to $f_{-r}$, so we get the 4 irreducible representations $S_1 = \langle f_0 \rangle$, $S_2 = \langle f_1, f_5 \rangle$, $S_3 = \langle f_2, f_4 \rangle$, and $S_4 = \langle f_3 \rangle$. The two 1-dimensional representations $S_1$ and $S_4$ are not equivalent, as $S_1$ is the trivial representation and $S_4$ is not trivial. To show that $S_2$ and $S_3$ are not equivalent we use the trace of $1_G$ on the representations, which is $\omega + \omega^{-1}$ on $S_2$ and $\omega^2 + \omega^{-2}$ on $S_3$.

There are four inequivalent irreducible representations for this orbit, but one is annihilated by $\Pi$, leaving three irreducible representations of ker $\epsilon$ on application of $\Pi$.

(Orbit 1) $t = 1_M$ and $t = 5_M$ give the conjugacy class consisting of elements of the form (any 2-cycle in 1,2,3)(any 3-cycle in 4,5,6). We choose $[1_M]$ as the base point for this orbit. Then $\Xi_{[1]} = M$, and the equivalence class is $[1_M] = \{1_M v | 1_M \triangleright v = v\}$. Since $1_M$ acts on $G$ by the permutation $(1_G, 5_G)(2_G, 4_G)$, we find that $[1_M] = \{1_M 0_G, 1_M 3_G\}$. The action of $\Xi_{[1]} = M$ on $[1_M]$ is given by the formula $t \triangleright 1_M v = 1_M (t \triangleright v)$, which is the trivial action since both $0_G$ and $3_G$ are fixed by the left action of $M$.

The decomposition of $S = k\{1_M 0_G, 1_M 3_G\}$ into irreducibles under the action of $\Xi_{[1]} = M$ gives two copies of the trivial one-dimensional representation.

(Orbit 2) $t = 2_M$ and $t = 4_M$ give the conjugacy class consisting of elements of the form (any 3-cycle in 4,5,6). We choose $[2_M]$ as the base point for this orbit. Then $\Xi_{[2]} = S_3 \times C_3$, where $C_3$ is the group of permutations of $\{4, 5, 6\}$ consisting of the identity and the two 3-cycles. In terms of the factorisation, $\Xi_{[2]}$ consists of all elements of the form $u t$ for $u \in \{0_G, 2_G, 4_G\}$ and $t \in M$. The equivalence class is $[2_M] = \{2_M v | 2_M \triangleright v = v\}$. Since $2_M$ has the trivial action on $G$, we find that $[2_M] = \{2_M 0_G, 2_M 1_G, 2_M 2_G, 2_M 3_G, 2_M 4_G, 2_M 5_G\}$. Under the $\Xi_{[2]}$ action $u t \triangleright v = su(t \triangleright v)$, $[2_M]$ splits into two orbits, $\{2_M 0_G, 2_M 2_G, 2_M 4_G\}$ and $\{2_M 1_G, 2_M 3_G, 2_M 5_G\}$.

First we decompose $k\{2_M 0_G, 2_M 2_G, 2_M 4_G\}$ into irreducibles under the action of $\Xi_{[2]}$. The action of $2_G$ on this vector space diagonalises, that is $k\{2_M 0_G, 2_M 2_G, 2_M 4_G\} = M_0 \oplus M_1 \oplus M_2$, where $M_0 = \langle 2_M 0_G + 2_M 2_G + 2_M 4_G \rangle$, $M_1 = \langle 2_M 0_G + \omega 2_M 2_G + \omega^2 2_M 4_G \rangle$, and $M_2 = \langle 2_M 0_G + \omega^2 2_M 2_G + \omega 2_M 4_G \rangle$ ($\omega$ being a primitive 3rd root of unity). The effect of $1_M$ on these
eigenspaces is to swap \( M_1 \) and \( M_2 \). The decomposition into irreducibles gives \( S_1 = M_0 \) (trivial 1-dimensional) and \( S_2 = M_1 \oplus M_2 \).

Next we decompose \( k\{2_M 1_G, 2_M 3_G, 2_M 5_G\} \) into irreducibles under the action of \( \Xi_{[2]} \). The action of \( 2_G \) on this vector space diagonalises, that is \( k\{2_M 1_G, 2_M 3_G, 2_M 5_G\} = N_0 \oplus N_1 \oplus N_2 \), where \( N_0 = \langle 2_M 1_G + 2_M 3_G + 2_M 5_G \rangle_{k} \), \( N_1 = \langle 2_M 1_G + \omega 2_M 3_G + \omega^2 2_M 5_G \rangle_{k} \), and \( N_2 = \langle 2_M 1_G + \omega^2 2_M 3_G + \omega 2_M 5_G \rangle_{k} \). The effect of \( 1_M \) on these eigenspaces is to swap \( N_1 \) and \( N_2 \). The decomposition into irreducibles gives \( S_3 = N_0 \) (trivial 1-dimensional) and \( S_4 = N_1 \oplus N_2 \).

In fact the two 2-dimensional representations \( S_2 \) and \( S_4 \) are isomorphic, using the map \( 2_M 0_G + \omega 2_M 2_G + \omega^2 2_M 4_G \mapsto 2_M 1_G + \omega 2_M 3_G + \omega^2 2_M 5_G \) and \( 2_M 0_G + \omega^2 2_M 2_G + \omega 2_M 4_G \mapsto \omega^2 (2_M 1_G + \omega^2 2_M 3_G + \omega 2_M 5_G) \).

(\textbf{Orbit 3}) \( t = 3_M \) gives the conjugacy class consisting of elements of the form (any 2-cycle in 1,2,3). We choose \( [3_M] \) as the base point for this orbit. Then \( \Xi_{[3]} = C_2 \times S_3 \), where \( C_2 \) is the group of permutations of \( \{1, 2, 3\} \) consisting of the identity and \( (1, 2) \). In terms of the factorisation, \( \Xi_{[3]} \) consists of all elements of the form \( ut \) for \( u \in \{0_G, 3_G\} \) and \( t \in M \).

The equivalence class is \( [3_M] = \{3_M v \mid 3_M \triangleright v = v\} \). Since \( 3_M \) acts on \( G \) by the permutation \( (1_G, 5_G)(2_G, 4_G) \), we find that \( [3_M] = \{3_M 0_G, 3_M 3_G\} \). The \( \Xi_{[3]} \) action on \( [3_M] \) is given by \( ut \triangleright sv = su(t \triangleright v) = suv \).

Now decompose \( S = k\{3_M 0_G, 3_M 3_G\} \) into irreducibles under the action of \( \Xi_{[3]} \). The action of \( 3_G \) on this vector space diagonalises, that is \( S = M_1 \oplus M_2 \), where \( M_1 = \langle 3_M 0_G + 3_M 3_G \rangle_{k} \) and \( M_1 = \langle 3_M 0_G - 3_M 3_G \rangle_{k} \). The decomposition into irreducibles gives \( S_1 = M_1 \) and \( S_2 = M_2 \), which are not equivalent.

This completes our classification of the bicovariant quantum tangent spaces on bicrossproducts. It remains to dualise the results to obtain the corresponding first order differential calculi, as outlined at the start of the section. Explicitly, the dual of the inclusion \( i_L : L \to \ker \epsilon_H \) is a surjection \( i_L^* : (\ker \epsilon_H)^* \to V \), where \( V = L^* \). On the other hand, the inclusion \( j : \ker \epsilon_H \to H \) dualises to a map \( j^* : A \to (\ker \epsilon_H)^* \) where \( A = H^* \). Since \( \ker j^* = (\text{image} \ j)^\perp = (\ker \epsilon_H)^\perp = k 1_A \), the restriction \( j^*|_{\ker \epsilon_A} : \ker \epsilon_A \to (\ker \epsilon_H)^* \) is an isomorphism. Putting these together, we get a quotient map \( \pi_V : \ker \epsilon_A \to V \). Recall also that we can describe a representation \( L \) of \( D(H) \) as
a left $H$ and right $H^*$-module, with actions obeying the compatibility condition

$$h \triangleright (x \triangleleft a) = \sum ((h(1) \triangleleft a(1)) \triangleright x) \triangleleft (h(2) \triangleleft a(2)),$$

for all $x \in L$, which can be further computed in terms of the mutual coadjoint actions. (We freely identify a left $H^{*\text{op}}$ module as a right $H^*$-module.) For a given $a \in H^*$, the action $\triangleleft a : L \to L$ dualises to $(\triangleleft a)^* : V \to V$ where $V = L^*$. Similarly for the operators $h \triangleright$. We obtain in this way a left action of $A = H^*$ and a right action of $H$ on $V$, by $a \triangleright v = (\triangleleft a)^*(v)$ and $v \triangleleft h = (h \triangleright)^*(v)$, which make $V$ into a $D(A)$ representation. This is a general observation about the dualisation of quantum double modules. Combined with the projection $\pi_V$ and $d$ in (7), we obtain the corresponding first order differential calculus $\Omega^1 = V \otimes A$.

To fully specify the resulting exterior differential $d$ it is equivalent and more convenient to give its evaluations $\partial_v = (\langle v, \cdot \rangle \otimes \text{id}) \circ d$ against all $v \in L$. These operators $\partial_v : A \to A$ are the\textit{ braided vector fields} associated to elements $v \in L$. The term is justified because they obey a braided version of the Leibniz rule\cite{[7]}. Since we obtain $L$ from the map $\Pi : \mathcal{M} \to L$, we specify equivalently the operators

$$\partial_m = (\langle m, \cdot \rangle \otimes \text{id})(\Pi^* \otimes \text{id}) \circ d, \quad \forall m \in \mathcal{M}.$$\textbf{Example 5.18} The $\partial$ operators in a few cases from Example 5.17.

We take orbit 0 in Example 5.17, and write

$$f_r = 0_M \otimes \delta_0G + \omega^r 0_M \otimes \delta_1G + \omega^{2r} 0_M \otimes \delta_2G + \omega^{3r} 0_M \otimes \delta_3G + \omega^{4r} 0_M \otimes \delta_4G + \omega^{5r} 0_M \otimes \delta_5G,$$

where $\omega$ is a primitive 6th root of unity. From its definition,

$$\partial_{f_r}(\delta_t \otimes v) = \sum_{ab=t} \langle \delta_a \otimes b \triangleright v, f_r \rangle \delta_b \otimes v - \sum_{a} \langle \delta_a \otimes 0_G, f_r \rangle \delta_a \otimes v,$$

After a little calculation this reduces to

$$\partial_{f_r}(\delta_t \otimes v) = (\omega^{r(t \triangleright v)} - 1) \delta_t \otimes v.$$\textbf{Example 5.18} The $\partial$ operators in a few cases from Example 5.17.

We use the allowed spaces for $\mathcal{M}$ for this orbit, which are $\langle f_3 \rangle_k, \langle f_1, f_5 \rangle_k,$ and $\langle f_2, f_4 \rangle_k$. 

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Next we take a case from orbit 2, the irreducible representation given by $\mathcal{M}_0 = \langle g_1, g_2 \rangle_k$, where

$$g_r = \phi_{2M}G + \omega^r \phi_{2M}G + \omega^{2r} \phi_{2M}G = 4_M \otimes \delta_0G + \omega^r 4_M \otimes \delta_2G + \omega^{2r} 4_M \otimes \delta_4G,$$

and $\omega$ is a primitive 3rd root of unity. In this case the orbit consists of more than one point, in fact $\mathcal{O} = \{[2M], [4M]\}$. We choose $x_{[4M]} \in X$ so that $x_{[4M]}[2M] = [4M]$, for example $x_{[4M]} = 1_G$. Now we can add $1_G \tilde{\triangleright} g_1$ and $1_G \tilde{\triangleright} g_2$ to $g_1$ and $g_2$ to get a basis of the 4-dimensional representation specified by $\mathcal{O}$ and $\mathcal{M}_0$, that is $\mathcal{M} = \langle g_1, g_1, 1_G \tilde{\triangleright} g_1, 1_G \tilde{\triangleright} g_2 \rangle_k$, where

$$1_G \tilde{\triangleright} g_r = \phi_{1G}2M + \omega^r \phi_{3G}2M + \omega^{2r} \phi_{5G}2M = 2_M \otimes \delta_1G + \omega^r 2_M \otimes \delta_3G + \omega^{2r} 2_M \otimes \delta_5G.$$

Then we may calculate the evaluations

$$\partial_{g_r}(\delta_t \otimes v) = \left( \delta_{(t-4)\otimes 0} + \omega^r \delta_{(t-4)\otimes 2} + \omega^{2r} \delta_{(t-4)\otimes 4} \right) \delta_{t-4} \otimes v - \delta_t \otimes v,$$

$$\partial_{1_G \tilde{\triangleright} g_r}(\delta_t \otimes v) = \left( \delta_{(t-2)\otimes 1} + \omega^r \delta_{(t-2)\otimes 3} + \omega^{2r} \delta_{(t-2)\otimes 5} \right) \delta_{t-2} \otimes v.$$

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