Empirical Evaluation of Project Scheduling Algorithms for Maximization of the Net Present Value

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Abstract

This paper presents an empirical performance analysis of three project scheduling algorithms dealing with maximizing projects’ net present value with unrestricted resources. The selected algorithms, being the most recently cited in the literature, are: Recursive Search (RS), Steepest Ascent Approach (SAA) and Hybrid Search (HS). The main motivation for this research is the lack of knowledge about the computational complexities of the RS, SAA, and HS algorithms, since all studies to date show some gaps in the analysis. Furthermore, the empirical analysis performed to date does not consider the fact that one algorithm (HS) uses a dual search strategy, which markedly improved the algorithm’s performance, while the others don’t. In order to obtain a fair performance comparison, we implemented the dual search strategy into the other two algorithms (RS and SAA), and the new algorithms were called Recursive Search Forward-Backward (RSFB) and Steepest Ascent Approach Forward-Backward (SAAFB). The algorithms RSFB, SAAFB, and HS were submitted to a factorial experiment with three different project network sampling characteristics. The results were analyzed using the Generalized Linear Models (GLM) statistical modeling technique that showed: a) the general computational costs of RSFB, SAAFB, and HS; b) the costs of restarting the search in the spanning tree as part of the total cost of the algorithms; c) and statistically significant differences between the distributions of the algorithms’ results.

Keywords: Empirical evaluation, max-npv, project scheduling algorithms, unrestricted resources.

1. Introduction

Net Present Value (NPV), probably the most used method for the financial evaluation of projects, consists in calculating the sum of all discounted cash flows generated by the project activities. The premise behind the method is that the higher its NPV, the more financially attractive this project is. However, the application of this method may involve taking into account several project implementation constraints, such as: a) precedence between activities; b) dates imposed for the start or end of activities; c) lag between activities; e) amount and type of resources mobilized; f) and resource renewal capacity.

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Early discussions on how to maximize project NPV, also known as \textit{max-npv} problems, originate from the pioneering works of Battersby (1964) and Russell (1970). Such contributions paved the way for the research on different classes \textit{max-npv} of problems, as shown in several works published later. The literature review of Herroelen et al. (1997) registered thirty-four works related to the subject and grouped them into six categories.

The category referred to as \textit{Deterministic Unconstrained max-npv} assumes (a) deterministic activity durations, (b) finish-start precedence relations with zero time-lag, (c) deterministic cash flows and (d) deterministic discount rate. This type of \textit{max-npv} problem consists of the calculation of the set of start times of all activities that maximize a project’s NPV. This will be the category discussed in this paper. The literature review of on the \textit{Deterministic Unconstrained max-npv} problem by Herroelen et al. (1997) presented seven works cataloged. However, two more recent known algorithms (useful in this category) were published after their work. These algorithms are \textit{Steepest Ascent Approach} (SAA) from Schwindt & Zimmermann (2001), and \textit{Hybrid Search} (HS) from Vanhoucke (2006).

The main motivation for this research is the lack of knowledge about the computational complexities of the RS, SAA, and HS algorithms since they all show some gaps in the analysis. An additional motivation was that the empirical analysis performed to date does not consider that one algorithm (HS) uses a dual search strategy, which markedly improved the algorithm’s performance, while the others don’t. Therefore, in order to be able to obtain a fair performance comparison, we implemented the dual search strategy into the other two algorithms (RS and SAA), and the new algorithms were called \textit{Recursive Search Forward-Backward} (RSFB) and \textit{Steepest Ascent Approach Forward-Backward} (SAAFB).

Thus, this paper shows the results of an empirical study of the performance of the algorithms RSFB, SAAFB, and HS. They were submitted to a factorial experiment with three different project network sampling characteristics using two different performance metrics. The first metric that deals with the total cost is a proxy to the big-O complexity measure, while the second metric deals with the number of tree searches used by the algorithms. The remaining of this paper contains the following sections: \textit{The max-npv scheduling problem; Related works; Algorithms chosen for evaluation; The experiment; Results; Analysis and discussion; and Conclusions.}

2. The max-npv scheduling problem

The \textit{max-npv} class problem presented in this paper concerns the maximization of the Net Present Value (NPV) of projects, with precedence constraints between activities and unrestricted resources. In this sense, the information about the variables and restrictions of the problem can be displayed as a graph $G(V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. Two special vertices are considered and are called initial \textit{dummy} and final \textit{dummy}, respectively demarcating the beginning and end of the project. Each vertex (representing an activity) is associated with the following attribute set: a) undiscounted cash flow (in the case of \textit{dummies} with a value of zero); b) a duration; c) a start date; d) and an end date. The vertices are arranged in ascending order, starting at the initial \textit{dummy} and ending at the final \textit{dummy}. Edges indicate a precedence relation between activities of the type \textit{finish-start} with zero \textit{lag}. Also, the end date of the last \textit{dummy} must
be less than or equal to the project’s deadline.

Using of the three-field notation, originally presented by [Graham et al. (1979)], expanded by [Blazewicz et al. (1983)], and specialized for the context of project scheduling by [Demeulemeester et al. (1997)], the problem of interest can be expressed as: \( c | cpm, \delta_n, c_j | \text{max-npv} \). With this notation, \( c \) indicates unrestricted resources, \( cpm \) indicates precedence between finish-start activities with zero lag; \( \delta_n \) indicates that there is a deadline for the project; \( c_j \) denotes a cash flow for each activity; and \( \text{max-npv} \) means that the objective function is to maximize the net present value.

The objective function can be given as:

\[
\max \sum_{i=2}^{n-1} c_i e^{-\alpha (s_i + d_i)}
\]

in which, \( c_i \) indicates the cash flow of each activity \( i \) (at its finish time), \( s_i \) means the start of each activity, \( d_i \) indicates the duration of each activity and \( e^{-\alpha} = 1/(1 + r) \) means the discount factor, where \( r \) refers to the discount rate.

Subject to the following restrictions:

\[
s_i + d_i \leq s_j; \forall (i,j) \in E
\]

(2)

\[
s_1 = 0
\]

(3)

\[
s_n \leq \delta_n
\]

(4)

\[
s_i \in \mathbb{N}; i = 2, 3, 4..., n
\]

(5)

Expression (1) indicates that the objective function is to maximize the NPV. The expression (2) limits the end of any activity to be less than or equal to the start of any of its successors (\( s_j \)). The expression (3) indicates that the start time of the initial dummy must be equal to zero. The expression (4) indicates that the start of the final dummy (\( s_n \)) is less or equal than to the project’s deadline date (\( \delta_n \)). The last expression (5) indicates that starting dates must belong to the set of natural numbers.

It is worth mentioning that some authors have already shown that, for this class of problem, it is possible to obtain a solution in polynomial time ([Cavalcante et al., 1998; Chang & Edmonds, 1985; Gröflin et al., 1982; Juang, 1994; Margot et al., 1990; Rouny et al., 1991; Sankaran et al., 1999]). Furthermore, [Grinold (1972)] demonstrates that it is possible to transform the nonlinear programming problem into a linear programming problem using an event-driven approach.

3. Related works

The literature review on the max-npv problem by [Herroelen et al. (1997)] describes some of the works related to this article. This review contains seven works cataloged in the category Deterministic Unconstrained max-npv, which is directly related to the scheduling problem of interest. However, three new works directly related to our problem were published after that. The main characteristics of these ten works are described in the sequence.
The first of these works is that of Russell (1970), one of the pioneers in promoting and formulating the maximization of net present value in projects. The author showed that, although the problem involves maximizing a nonlinear objective function (subject to linear constraints), it is possible to obtain a solution with successive applications of linear programming. The author proposed an algorithm composed of several steps but did not present any discussion either about its complexity or an empirical evaluation.

The second work, Grinold (1972), demonstrates that it is possible to have a solution for the nonlinear objective function (with linear constraints) using linear programming. The author also shows that the best solution to the problem can be obtained in a (viable) tree extracted from the project network and that the search can be restricted to tree structures. The author proposes two algorithms: the first solves the problem through a single deadline, and the second solves the problem through several possible deadlines. These algorithms are related to the so-called special procedure of Markowitz for the weighted distribution problem using triangular systems of equations.

The third work, published by Smith-Daniels (1986), proposes a forecasting model for max-npv with unrestricted resources, considering a single cash flow at project completion, characteristic of projects under lump-sum contracts. In addition, summary measures have been proposed to predict the value of max-npv. However, the article did not propose algorithms or computational experiments.

In the fourth work, Elmaghraby & Herroelen (1990) demonstrated that the approaches of the papers by Russell (1970) and Grinold (1972) can produce inconclusive results, as they do not explicitly specify start and end restrictions for scheduling. Under these arguments, Elmaghraby & Herroelen (1990) proposed an algorithm that builds tree structures iteratively and with the determination of displacement intervals. However, the paper does not present any discussion about complexity neither about computational experiments.

The fifth work by Herroelen & Gallens (1993) presents a simplified version of the algorithm of Elmaghraby & Herroelen (1990). The authors submitted this algorithm to a computational experiment that allowed empirical comparisons with the solutions obtained via linear programming using the software Super LINDO. The paper does not present any discussion about complexity.

In the sixth work, Kazaz & Sepil (1996) argued that in real projects, it is more common for cash flows to be associated with regular periods, such as months, rather than events such as the completion of activities. Thus, they proposed costs of activities divided by their duration and the formulation of the problem through integer programming. They also presented rules of a random generator of networks with results of an experiment through the software LINDO. However, there were no discussions about complexity.

In the seventh work, Demeulemeester et al. (1996) describe an algorithm called Recursive Search (RS) composed of three steps, which performs recursive searches in tree structures, assuming positive cash flow is anticipated as much as possible, and the negative ones are delayed as much as possible. This work presented a computational experiment but did not discuss any aspects of complexity.
In the eighth work, although the main focus was resource-constrained max-npv, Vanhoucke et al. (1999) also proposed a refinement for the RS algorithm by adding an extra edge in the spanning tree between the last and first vertices. This extra edge favored the recursion process of the algorithm proposed initially in Demeulemeester et al. (1996). The work of Vanhoucke et al. (1999) also presented a computational experiment but did not discuss the algorithm's complexity. However, on the version of RS with the extra edge, Demeulemeester & Herroelen (2002) and Vanhoucke et al. (2000) stated that the first two steps could be implemented with cost $O(n^2)$, but the third has unknown complexity.

In the ninth work, Schwindt & Zimmermann (2001) presented an algorithm called Steepest Ascent Approach (SAA) with a generalized approach to the precedence of activities, admitting minimum and maximum intervals. The paper presents a computational experiment and highlights that two of its three component algorithms can be implemented respectively with cost $O(n)$ and $O(m \log m)$. However, the complexity of the third component algorithm was considered an open question.

In the tenth and last work, Vanhoucke (2006) proposed an algorithm called Hybrid Search (HS) that combines RS and SAA strategies, as well as includes the ability to reverse the search and displacement direction when the design has more than half of the activities with negative cash flow. In this work, the author presented the results of a computational experiment but did not discuss any aspects of complexity.

Table 1 presents a summary of all relevant related works, where Algorithm means that the work describes an algorithm, Complexity that it presents any complexity analysis, and Experiment if the paper details any form of empirical analysis.

| Authors                     | Algorithm | Complexity | Experiment |
|-----------------------------|-----------|------------|------------|
| 1) Russell (1970)           | yes       | no         | yes        |
| 2) Grinold (1972)           | yes       | no         | no         |
| 3) Smith-Daniels (1986)     | no        | no         | no         |
| 4) Elmaghraby & Herroelen (1990) | yes     | no         | no         |
| 5) Herroelen & Gallens (1993) | yes     | no         | yes        |
| 6) Kazaz & Sepil (1996)     | yes       | no         | yes        |
| 7) Demeulemeester et al. (1996) | yes     | no         | yes        |
| 8) Vanhoucke et al. (1999)  | yes       | partial    | yes        |
| 9) Schwindt & Zimmermann (2001) | yes     | partial    | yes        |
| 10) Vanhoucke (2006)        | yes       | no         | yes        |

4. Algorithms chosen for evaluation

The algorithms selected and implemented for the empirical evaluation were Recursive Search Forward-Backward (RSFB), Steepest Ascent Approach Forward-Backward (SAAFB), and Hybrid Search (HS). As relates to these algorithms, it is important to point out that RSFB and SAAFB are
proposed variations of the Recursive Search (RS) and Steepest Ascent Approach (SAA) algorithms. The following sections present the main features of each.

4.1. Recursive Search Forward-Backward

Recursive Search (RS) is one of the fundamental algorithms in this work. It was originally proposed by Demeulemeester et al. (1996) and refined by Vanhoucke et al. (1999). Its composition includes three steps. In the first step, RS creates a tree called Early Tree (ET), with anticipation of the dates of the vertices (activities) as much as possible. Moreover, RS assumes the existence of an extra edge between the first and last vertices (dummies) in the ET, useful for the recursion process. In the second step, RS makes a copy of ET called Current Tree (CT). According to the precedence constraints, the second step also delays as much as possible the vertices of CT without successors with negative cash flow. Finally, in the third step, RS recursively searches for subtrees with negative cash flow in the CT, and each of them is shifted according to the constraints.

Because the first two steps are meant for data preprocessing, all the interesting RS algorithmic work is done in the third step. Such a step comprises of two-component algorithms called Step_3 and Recursion. The first component algorithm Step_3 (with pseudocode in Algorithm 1) is used to initiate a recursive depth-first search in the spanning tree. For this, Step_3 invokes the second component algorithm Recursion (with pseudocode in Algorithm 2). Thus, Recursion can identify candidate subtrees for displacement. Each subtree identified by Recursion is immediately displaced, and the search is restarted on the spanning tree with a new invocation to Step_3.

Algorithm 1: Step_3 - Forward.

1. procedure Step_3() {CA is a global structure}
2. \[ \text{total\_Step}_3 = \text{total\_Step}_3 + 1 \]
3. \[ \text{CA} \leftarrow \emptyset \]
4. \[ \text{SA}', \text{DC}' \leftarrow \text{Recursion}(1) \]
5. Report the optimal solution DC'

With respect to the complexity of RS, the authors state that the first two steps are $O(n^2)$ and the third step has unknown complexity [Vanhoucke et al., 2000]. According to the authors, one point that makes this complexity an open question refers to the unknown number of times that searches are restarted in the spanning tree with invocations to the component Step_3. As a consequence, the overall complexity of RS is an open question.

Although RS is a fundamental algorithm, the version implemented and used in the experiment refers to a variation called Recursive Search Forward-Backward (RSFB). This variation incorporates the same inversion strategy for search and displacement proposed by Vanhoucke (2006) in the Hybrid Search (HS) algorithm. Thus, RSFB reverses the search and shifts reference when the project has more than half of the activities with negative cash flow. In that case, RSFB creates a Late Tree delaying the activities as much as possible and takes the deadline as a reference. Then, with the inverted search, the start occurs by the final dummy, and the subtrees identified with positive cash flow are shifted toward the initial dummy.

In this section, the pseudocodes for the RSFB components (under Algorithms 1 and 2) refer only to the forward approach, i.e., when the project has up to half of the activities with negative
cash flow. However, the pseudocodes corresponding to the backward approach follow as part of Appendix A. In both approaches (forward and backward), the algorithms have their differences in underlined parts of their lines. As relates to the main terms of Step 3 (Algorithm 1) the following stand out: a) CA that refers to the Considered Activities in the last search; b) SA′ that refers to the Set of Activities to shift; c) and DC′ that refers to the Discounted Cash flow. In addition, the variable total Step 3 was used to count the times the search is restarted in the spanning tree. This count refers to one of the metrics discussed in the experiment section.

As relates to the main terms of Recursion (Algorithm 2) the following stand out: a) the function Compute $v_{k\ast l\ast}$ that identifies the smallest distance between a vertex $\in SA$ and a vertex $\notin SA$; b) $f_l$, $d_k$ and $f_k$ which respectively indicate the end of vertex $l$, the duration of vertex $k$ and the end of vertex $k$; c) and $G$ which indicates the graph with all project constraints. In addition, the variable total Recursion (line 2 in Algorithm 2) is used to count the number of recursive calls to the algorithm component Recursion. This global variable starts with a zero value only once immediately before the first call to the Step 3 component. This count also refers to one of the metrics discussed in the experiment section.

Algorithm 2: Recursion - Forward.

```plaintext
1 function Recursion(newnode) {CA, CT are global structures}
2 total_Recursion = total_Recursion + 1
3 SA ← {newnode}; DC ← DC_{newnode}; CA ← CA + newnode
4 for each (i | i $\notin$ CA and i succeeds newnode $\in$ CT) do
5   SA′, DC′ ← Recursion(i)
6   if DC′ $\geq$ 0 then
7     SA ← SA + SA′; DC ← DC + DC′
8   else
9     CT ← CT − (newnode, i)
10    Compute $v_{k\ast l\ast} = \min \{ f_l - d_k - f_k \}_{(k\ast,l\ast)\in G}$; CT ← CT + (k\ast, l\ast)
11   end
12 for each (i | i $\notin$ CA and i precedes newnode $\in$ CT) do
13   SA′, DC′ ← Recursion(i)
14   SA ← SA + SA′; DC ← DC + DC′
15 end
16 return (SA, DC)
```

4.2. Steepest Ascent Approach Forward-Backward

Proposed by Schwindt & Zimmermann [2001], Steepest Ascent Approach (SAA) is the second fundamental algorithm in this work. Although his approach generalizes the precedence relationship between activities, an adaptation is easily performed, considering only the type finish-start and zero lag, as has been done in similar work [Vanhoucke et al., 2000]. Its organization includes the three-component algorithms Steepest Ascent Direction (SAD), Vertex Ascent (VA), and Steepest-Ascent Procedure (SAP). Thus, the SAD component algorithm iteratively searches for subtrees with
negative cash flow in the spanning tree. The VA component algorithm identifies destinations for the subtrees found with SAD, considering the closest constraints. The third SAP component algorithm synthesizes the presented strategy, including calls to the SAD and VA components. With this, SAP calls SAD, and when subtrees are identified, VA is executed to perform the displacements. This way, SAP iterates as long as subtrees are identified with SAD.

With respect to complexity, the authors claim that SAD can be implemented in $O(n)$ and VA in $O(m \log m)$ which is equivalent to $O(m \log n^2) = O(2m \log n) = O(m \log n)$. However, the lack of knowledge of the number of times that searches in the spanning tree are restarted is an open question, as it also occurs in RS. Hence, the overall complexity of SAA remains an open question.

Although SAA is another fundamental algorithm, the version implemented and used in the experiment refers to a variation called *Steepest Ascent Approach Forward-Backward* (SAAFB). Similar to what was proposed in RSFB, SAAFB also incorporates the inversion strategy for searching and shifting of the HS [Vanhoucke et al., 2000]. The pseudocodes of the SAAFB components (Algorithms 3, 4, and 5) are presented in the forward approach. However, versions in the backward follow as part of Appendix A. The differences between these approaches are highlighted in underlined passages in the pseudocodes.

Algorithm 3: Steepest Ascent Direction (SAD) - Forward.

1. function SAD() {ST(Vst, Est) is a global structure}
2. Z ← $\emptyset$; V ← Vst
3. $\forall i \in V$ do $C(i) ← i; φ_i ← -C_i e^{-α(s_i+d_i)}$
4. while $V ≠ \{1\}$ do
5.   if (V has a node source $i ≠ 1$) & (at most one successor $j$) then
6.     iteration_SAD = iteration_SAD + 1
7.     $C(j) ← C(j) + C(i); φ_j ← φ_j + φ_i; V ← V - i$
8.   else
9.     if (V has a node sink $j ≠ 1$) & (only one predecessor $i$) then
10.    iteration_SAD = iteration_SAD + 1
11.    if $φ_j > 0$ then $Z ← Z + C(j)$
12.    else $φ_i ← φ_i + φ_j; C(i) ← C(i) + C(j); V ← V - j$
13.  end
14.  end
15. end
16. return(Z)

The main terms of the SAD component are: a) ST as the spanning tree, with $Vst$ being the vertices and $E_{st}$ the edges; b) $Z$ is the set of candidate vertices for displacement; c) $C(i)$ a vector used to group vertices without successors or predecessors; d) $φ_i$ refers to the partial derivative of the discounted cash flow of each activity $i$. 

Algorithm 4: Vertex Ascent (VA) - Forward.

function VA(S, Z) {ST, G are global structures}

∀ (i,j) ∈ ST | (j ∉ C(i)) & (i ∉ C(j)) : ST ← ST - (i,j)

while Z ≠ ∅ do

Store _k*_l* = min {S_l - S_k - d_k}

∀ k ∈ Z : ST ← ST - (k,l)

∀ i ∈ C(j) : S_i ← S_i + v_k*l

Z ← Z - C(j); ST ← ST + (k,l)

end

return (S)

The main terms of the VA component (Algorithm 4) are: a) _S_ as the vector that contains the schedule for the start of the activities of the spanning tree; b) and _Compute v_k*l_ as the function that calculates the shortest distance between _k_ ∈ Z and _l_ ∉ Z. The main terms contained in the SAP component (Algorithm 14) have already been highlighted.

Algorithm 5: Steepest Ascent Procedure (SAP) - Forward.

procedure SAP() {ST is a global structure}

S, ST ← Determine the Early Schedule (S) as a vector and a corresponding initial Spanning Tree (ST) through the original graph G.

total_SAD = 0

while Z ≠ ∅ do

total_SAD = total_SAD + 1

S ← VA(S, Z)

Z ← SAD()

end

Report the optimal solution _S_

Finally, the variables _iteration_SAD_ (Algorithm 3) and _total_SAD_ (Algorithm 5) refer respectively to the total number of iterations performed in the search for vertices that meet the conditionals of lines 5 and 9 (SAD), and the total number of searches initiated in the spanning tree with the invocation of SAD. Such counts are metrics discussed in the experiment section.

4.3. Hybrid Search

Proposed by Vanhoucke (2006), Hybrid Search (HS) is the last algorithm considered in this work. Its approach combines RS and SAA strategies and is the pioneer algorithm in the inversion of search and displacement when more than half of the activities have negative cash flow. HS includes three-component algorithms called _Recursion_, _Shift_activities_ and _Hybrid Recursive Search_ (HRS). Thus, _Recursion_ uses recursive depth-first search to identify candidate subtrees for displacement (with pseudocode in Algorithm 6). The _Shift_activities_ component (with pseudocode in Algorithm 7) finds destinations and performs the displacements of the subtrees identified in the last depth-first search. The _HRS_ component (with pseudocode in Algorithm 8) summarizes the entire scheduling approach, invoking the _Recursion_ and _Shift_activities_ components.
Algorithm 6: Recursion de HS - Forward.

```plaintext
1 function Recursion(newnode) { CA, ST, SS are global structures}
2 total_Recursion = total_Recursion + 1
3 SA ← {newnode}; DC ← DC_{newnode}; CA ← CA + newnode
4 for each (i| i ∉ CA and i succeeds newnode ∈ ST) do
5     SA', DC' ← Recursion(i)
6     if DC' ≥ 0 then
7         SA ← SA + SA'; DC ← DC + DC'
8     else
9         ST ← ST -(newnode, i); SS ← SS + SA'
10    end
11 end
12 for each (i| i ∉ CA and i precedes newnode ∈ ST) do
13     SA', DC' ← Recursion(i)
14     SA ← SA + SA'; DC ← DC + DC'
15 end
16 return (SA, DC)
```

Although the subtree search strategy is also recursive like RS, the HS algorithm can identify several subtrees before performing displacements, just like the SAA algorithm (but the latter in an iterative way). Therefore, several subtrees may have been identified in the last search performed when HS starts to perform displacements, similar to SAA.

Algorithm 7: Shift activities - Forward.

```plaintext
1 procedure Shift activities() { SS, ST, and G are global structures}
2 Z ← ∅; ∀ i ∈ SA | SA ∈ SS : Z ← Z + i
3 while Z ≠ ∅ do
4     Compute v_{k^*, l^*} = min \{ s_i - s_k - d_k \}
5     \( (k^*, l^*) \in G \), \( i \in Z \)
6     ∀ i ∈ SA|k* ∈ SA : s_i ← s_i + v_{k^*, l^*} and Z ← Z - i
7 end
```

There are no complexity considerations for HS, making this aspect an open question. The main terms of the Recursion are: a) SA as the candidate set of activities; b) DC as discounted cash flow; c) CA as activities considered in the search; d) ST as spanning tree; e) and SS as a set of set of activities. The main terms of the Shift activities component are: a) Z as the vertices that must be shifted; b) and the Compute \( v_{k^*, l^*} \) as a function to calculate the shortest distance.

Algorithm 8: Hybrid Recursive Search (HRS) - Forward.

```plaintext
1 procedure HRS() { CA and SS are global structures}
2 total_HRS = total_HRS + 1
3 CA ← SS ← ∅
4 SA, DC' ← Recursion(1)
5 if SS ≠ ∅ then
6     Shift activities()
7     HRS()
8 else Report the optimal solution DC'
```

10
Finally, the variables \( \text{total\_Recursion} \) (Algorithm 6) and \( \text{total\_HRS} \) (Algorithm 8) refer respectively to the total number of recursive calls in searches by displacement candidate subtrees and the total number of searches initiated in the spanning tree. These counts are metrics discussed in the experiment section.

### 4.4. Compute \( v_{k+l} \)

Compute \( v_{k+l} \) refers to a function contained in the fundamental algorithms and in all the algorithms (variations) implemented for the experiment of this research. Its purpose is to identify the shortest distance between a vertex that must be moved to a vertex that must not. Although this purpose is at the heart of the logic for scheduling, the fundamental algorithms (RS, SAA, and HS) treat Compute \( v_{k+l} \) as a black box. In other words, the original works referring to fundamental algorithms did not present open pseudocode for this function. For this reason, the \textit{forward} version of Compute \( v_{k+l} \) implemented together with the RSFB, SAAFB and HS algorithms remains explicit in Algorithm 9. The pseudocode corresponding to the \textit{backward} approach is contained in Appendix A. Underlined points highlight the differences between the \textit{forward} and \textit{backward} approaches (Algorithm 9).

**Algorithm 9:** Compute \( v_{k+l} \) - Forward.

```plaintext
1 function Compute \( v_{k+l}(Z) \)
2 \( v_{k+l} \leftarrow \delta \)
3 \( k \leftarrow \emptyset; l \leftarrow \emptyset \)
4 for node \( \in Z \) do
5   if \( k = \emptyset \) then \( k \leftarrow \text{node} \)
6     for suc \( \in \text{successors of} \text{ node} \) do
7       \( \text{edge\_checked} \leftarrow \text{edge\_checked} + 1 \)
8       if suc \( \notin Z \) do
9         if \( s_l - s_k < 0 \) then
10            \( \text{current\_min} = s_l - s_k - \delta \)
11         else
12            \( \text{current\_min} = s_l - s_k - d_k \)
13         end
14         if current\_min < \( v_{k+l} \) then
15           \( v_{k+l} \leftarrow \text{current\_min} \)
16           if node is the last node then \( l \leftarrow 1 \)
17           else
18             \( l \leftarrow \text{suc} \)
19           end
20         end
21       else
22         if \( l = \emptyset \) then \( l \leftarrow \text{suc} \)
23     end
24 return \( (k, l, v_{k+l}) \)
```

As all the main terms contained in Compute \( v_{k+l} \) have already been explained in the descriptions of the algorithms, only one remark about the variable \( \text{edge\_checked} \) should be made. This variable refers to the count of edges checked in moving into the nearest constraints. It is treated as a global variable started only once (with a zero value) at the instant when any of the algorithms are also started. Thus, the section dealing with the experiment discusses the count of checked edges as one of the metrics.
5. The experiment

5.1. Independent variables: the experimental factors

The experiment aimed to investigate the absolute and relative computational performance of the RSFB, SAAFB, and HS algorithms under different characteristics of the project networks. These characteristics or experimental factors constitute independent variables of the experiment. Table 2 shows the experimental factors and their respective meanings.

| Factor    | Code | Description                                           |
|-----------|------|-------------------------------------------------------|
| vertices  | f_1  | Number of vertices of the network graph.              |
| layers    | f_2  | Number of layers of the network graph.                |
| maxDegree | f_3  | Maximum degree (in and out) of the vertices of the network graph. |
| discRate(%)| f_4  | The discount rate used in the project.                |
| percNeg(%)| f_5  | Percentage of activities with negative cash flow.     |
| cpMult    | f_6  | Project deadline as a multiple of the critical path duration. |
| edges     | f_7  | Number of edges of the network graph.                 |

5.2. Dependent variables: direct metrics

The dependent variables of the experiment were selected in such a way as to provide a measure of the computational cost of the selected algorithms. The dependent variables were defined by two metrics: the first metric, called \textit{computational cost}, represents the total computational cost incurred by the algorithms to evaluate the optimal schedules. It is obtained by adding the number of iterations (in iterative algorithms), or the number of recursive calls (in recursive algorithms) to the number of edges checked in the search for the shortest displacement distance of subtrees. The formula of the \textit{computational cost} metric is displayed below:

\[
\text{computational cost} = \text{total\_Recursion} + \text{edge\_checked}
\]

In the RSFB and HS algorithms, the composition of \textit{computational cost} refers to the one indicated in equation (6). In this case, the variable \textit{total\_Recursion} (line 2 of the pseudocodes in Algorithms 2 and 6) counts the number of recursive calls and the variable \textit{edge\_checked} (line 7 of the pseudocode in Algorithm 9) counts the edges checked. In the SAAFB algorithm, the composition of \textit{computational cost} refers to the one indicated in the equation (7). In this case, the variable \textit{iteration\_SAD} (lines 6 and 10 of the pseudocode in Algorithm 3) counts the iterations performed and the variable \textit{edge\_checked} (also in line 7 of the pseudocode in Algorithm 9) counts the edges checked.

The second metric, called \textit{restarted search}, represents the number of times a new search is restarted in the spanning tree. In the case of the RSFB algorithm, \textit{restarted search} is represented by the variable \textit{total\_Step\_3}, indicated in line 2 of the pseudocode in Algorithm 1. This variable accounts for all calls to the \textit{Step\_3} component of RSFB. In the case of the SAAFB algorithm, \textit{restarted search} is represented by the variable \textit{total\_SAD}, indicated in line 3 of the pseudocode in
Algorithm 5. This variable accounts for all calls to the SAD component of SAAFB. In the case of the HS algorithm, restarted search is represented by the variable total_HRS, indicated in line 2 of the pseudocode in Algorithm 8. This variable accounts all calls to the HRS component of HS.

5.3. Dependent variables: the upper-bound metric

The most common computational measures of algorithm cost are a) asymptotic upper bound ($O$), b) asymptotic lower bound ($\Omega$), c) and asymptotic upper and lower bound ($\Theta$). In this sense, this experiment was concerned with finding a statistical asymptotic upper bound ($O$) in the form of an empirical maximum cost function as an approximate return of the maximum computational cost as a function of an experimental factor, i.e., $\maxCost(factor = value_{factor})$. In this case, the function returns the maximum computational cost obtained in the experiment when the specific factor assumes value_{factor}.

Since each one of the experimental factors $f_i$ assumes values in a discrete set $Df_i$, the whole sample space of the experiment is given by:

$$S = \prod_{i=1}^{i=6} Df_i$$

So, the sample space of the experiment is the relation containing all tuples that can be formed with all possible combinations of experimental factor values, as follows:

$$S = \{(v_{f_1}, v_{f_2}, \ldots, v_{f_n})|(v_{f_1} \in Df_1, v_{f_2} \in Df_2, \ldots, v_{f_n} \in Df_n)\}$$

Thus, on the sample space, the computational cost function is $S \rightarrow R^+$ and a sample sub-space is $S_{f_i,v_{f_i}} = \{s \in S|s(\ldots,f_i = v_{f_i},\ldots)\}$, which is subset of $S$, containing all tuples where the factor $f_i$ assumes the value $v_{f_i}$. In the same way, the empirical computational cost function is $Cost(v_{f_1}, v_{f_2}, \ldots, v_{f_n}) = Cost(f_1 = v_{f_1}, f_2 = v_{f_2}, \ldots, f_n = v_{f_n})$, which the experimental value obtained for each element of the sample space $S$. Finally, the conditional empirical function $\maxCost(f_i, v_{f_i})$ can be defined as: $\maxCost(f_i, v_{f_i}) = \max(Cost(s_i|s_i \in S_{f_i,v_{f_i}}))$.

5.4. Characteristics of the experimental samples

The experiment ran the three algorithms in three batches comprising 14,000 project network instances (5,000 in Sample 1, 5,000 in Sample 2, and 4,000 in Sample 3). The instances were created from a random network generator, developed exclusively for the experiment, inspired by Erdős & Rényi (1960). The random instance generator obtains random instances from sampling values in the ranges of values defined for the independent variable parameters, as shown in Table 3.
Table 3: Random Sampling.

| Parameter      | Sample 1       | Sample 2       | Sample 3       |
|----------------|----------------|----------------|----------------|
| vertices       | 16..80         | 16..320        | 16..320        |
| layers         | 2..vertices − 1| 2..vertices − 1| 2              |
| maxDegree      | 2 or 3         | 2 or 3         | (vertices − 2)/2|
| discRate(%)    | 1..20          | 1..20          | 1..20          |
| percNeg(%)     | 0,10,20,...100 | 0,10,20,...100 | 0,10,20,...50  |
| cpMult         | 1..2           | 1..2           | 1..2           |
| cashFlow       | -100..100      | -100..100      | -100..100      |
| activityDur    | 5..10          | 5..10          | 5..10          |

Samples 1 and 2 differ only in the number of vertices of the networks, the purpose being to evaluate the behavior of the algorithms on two sets of project networks: small/medium networks (Sample 1) and large networks (Sample 2).

Sample 3, on the other hand, was created to evaluate the behavior of algorithms in graphs with only two layers (disregarding dummies), in which all vertices of the first layer are connected to all vertices of the second layer. Such a configuration generated graphs with the maximum number of edges, i.e., complete bipartite digraphs (disregarding dummies). It provided a sample of data designed to stress the algorithms with a higher number of edges in the search subtrees to be displaced.

The algorithms were coded in Python 3 and run on a personal computer, with a 2.50 GHz core i5 processor, with 32 GB of RAM, running under Windows 10. It is worth noting that although there are instances available in the literature, it was decided to create and use a specific generator for this experiment to establish ranges of parameter values with a gradual increase and flexibility.

5.5. Statistical tools: GLM

The classical linear regression was the first data analysis technique considered. However, since the assumptions of the classical linear regression approach, such as normality of the residuals of the distributions and homoscedasticity, were not satisfied, even with Box-Cox and logarithmic transformations, a different tool had to be used - the generalized regression. This approach is part of the group called Generalized Linear Models (GLM), introduced by Nelder & Wedderburn (1972). In this case, the generalized regression approach admits linear and non-linear models, besides dependent variables with distributions such as Bernoulli, Poisson, Poisson-Gamma, and Gaussian (normal).

According to Fávero & Belfiore (2019), the proper definition of a generalized linear model must consider the dependent variable’s characteristic. The characteristics of two types of dependent variables presented by Fávero & Belfiore (2019) are worth noting. In the first one, the generalized model is called linear type when the dependent variable is quantitative and adherent to the normal distribution. In the second one, when the dependent variable is quantitative, being a count data (integer and non-negative values), not adherent to the Gaussian, it is considered Poisson or Poisson-Gamma. It is worth mentioning that the difference between a Poisson distribution and a Poisson-
Gamma distribution is the long tail to the right of the second distribution (Poisson-Gamma), characterizing overdispersion of the data. In other words, when the variance is statistically greater than the mean (with count data), the Poisson-Gamma distribution should be chosen. In this case, the generalized model is called a negative binomial.

In this work, all results were analyzed using generalized models with negative binomial distribution (Poisson-Gamma).

6. Results: preliminary data analysis

The preliminary analysis of the experimental results took the following steps: (1) generate a summary analysis of the data, (2) check for the similarity between the empirical distribution, and (3) verify the degree of correlation between the factors and the dependent variables.

6.1. Dependent variable summary measures

Tables 4, 5 and 6 present the minimum, first quartile, median, mean, third quartile and maximum, by metric and sample type.

Table 4: Summary Sample 1.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 18   | 63     | 132  | 301  | 375    | 4208 |
| RSFB | 18   | 77     | 261  | 614  | 685    | 2588 |
| SAAFB| 17   | 91     | 180  | 298  | 371    | 4298 |

(b) restarted search metric.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 1    | 2      | 3    | 4    | 4      | 15   |
| RSFB | 1    | 3      | 9    | 15   | 20     | 339  |
| SAAFB| 1    | 2      | 3    | 4    | 4      | 15   |

Table 5: Summary Sample 2.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 18   | 310    | 978  | 2795 | 3008   | 58409|
| SAAFB| 17   | 307    | 973  | 2791 | 3003   | 58391|

(b) restarted search metric.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 1    | 2      | 3    | 4    | 5      | 26   |
| RSFB | 1    | 2      | 3    | 4    | 5      | 26   |

Table 6: Summary Sample 3.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 18   | 2626   | 31749| 351981| 271013 | 10180880|
| RSFB | 17   | 2620   | 31725| 351940| 270980 | 10180639|

(b) restarted search metric.

| Algo | Min  | 1st Q. | Med. | Mean | 3rd Q. | Max  |
|------|------|--------|------|------|--------|------|
| HS   | 1    | 8      | 21   | 32   | 48     | 241  |
| RSFB | 1    | 8      | 21   | 32   | 48     | 241  |

6.2. Empirical result distribution types

Figures 1 and 2 show that the distributions have a greater concentration of results on the left and present an exponential drop with a long tail on the right. This pattern suggests that the distributions can be Poisson or Poisson-Gamma, as pointed by Fávero & Belfiore (2019), considering that these variables are counting data.
Figures 1 and 2 show a concentration of results on the left and a long tail on the right.

Figures 3 and 4 also show a concentration of results on the left and a long tail on the right.
6.2.1. Empirical frequency distribution similarity tests

The distributions were compared using the Kolmogorov-Smirnov (KS) test, which allowed the statistical evaluation of the similarity between pairs of the empirical frequency distributions as shown in Table (a) and (b). The SAAFB and HS distributions are statistically similar in all cases, while the RSFB algorithm distribution has no statistically significant similarity with any others.
Table 7: Distribution comparison with KS

(a) computational cost metric.

| Sample | Comparison  | Statistic D | p value | Similar |
|--------|-------------|-------------|---------|---------|
| 1      | RSFB vs SAAF | 0.17428     | 2.2e-16 | No      |
| 1      | RSFB vs HS   | 0.17428     | 2.2e-16 | No      |
| 2      | SAAF vs HS   | 0.01049     | 0.9518  | Yes     |
| 3      | SAAF vs HS   | 0.00228     | 1.0000  | Yes     |

(b) restarted search metric.

| Sample | Comparison  | Statistic D | p value | Similar |
|--------|-------------|-------------|---------|---------|
| 1      | RSFB vs SAAF | 0.57613     | 2.2e-16 | No      |
| 1      | RSFB vs HS   | 0.57613     | 2.2e-16 | No      |
| 2      | SAAF vs HS   | 0.00000     | 1.0000  | Yes     |
| 3      | SAAF vs HS   | 0.00000     | 1.0000  | Yes     |

6.3. Correlation between factors and performance metrics

Spearman’s Coefficient correlation method was used due to the non-parametric nature of the data. The correlation took into account only the maximum result of each algorithm per factor of the experiment (in both metrics). In the case of the factor \( f_5 \) (percNeg), in all algorithms, the correlation was evaluated only for values with up to 50% of negative activities, as the algorithms reverse the search direction when this value is greater.

Table 8: Spearman’s Coefficient (computational cost vs factors) - Sample 1.

| vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|----------|-------|-----------|----------|---------|--------|-------|
| RSFB     | 0.96  | -0.20     | NA       | 0.67    | 0.94   | NA    |
| SAAF     | 0.96  | -0.18     | NA       | 0.70    | 1.00   | NA    |
| HS       | 0.96  | -0.18     | NA       | 0.70    | 1.00   | NA    |

Table 9: Spearman’s Coefficient (restarted search vs factors) - Sample 1.

| vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|----------|-------|-----------|----------|---------|--------|-------|
| RSFB     | 0.78  | -0.40     | NA       | 0.79    | 0.94   | NA    |
| SAAF     | 0.78  | -0.40     | NA       | 0.44    | 0.71   | NA    |
| HS       | 0.78  | -0.40     | NA       | 0.44    | 0.71   | NA    |

Table 10: Spearman’s Coefficient (computational cost vs factors) - Sample 2.

| vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|----------|-------|-----------|----------|---------|--------|-------|
| SAAF     | 0.93  | -0.13     | NA       | 0.37    | 1.00   | NA    |
| HS       | 0.93  | -0.13     | NA       | 0.37    | 1.00   | NA    |

Table 11: Spearman’s Coefficient (restarted search cost vs factors) - Sample 2.

| vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|----------|-------|-----------|----------|---------|--------|-------|
| SAAF     | 0.70  | -0.32     | NA       | 0.58    | 0.39   | NA    |
| HS       | 0.70  | -0.32     | NA       | 0.58    | 0.39   | NA    |

Table 12: Spearman’s Coefficient (computational cost vs factors) - Sample 3.

| vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|----------|-------|-----------|----------|---------|--------|-------|
| SAAF     | 0.93  | NA        | 0.96     | 0.75    | 1.00   | NA    |
| HS       | 0.93  | NA        | 0.96     | 0.75    | 1.00   | NA    |
Table 13: Spearman’s Coefficient (restarted search vs factors) - Sample 3.

|       | vertices | layer | maxDegree | discRate | percNeg | cpMult | edges |
|-------|----------|-------|-----------|----------|---------|--------|-------|
| SAAFB | 0.85     | NA    | 0.88      | 0.88     | 0.99    | NA     | 0.36  |
| HS    | 0.85     | NA    | 0.88      | 0.88     | 0.99    | NA     | 0.36  |

Table 14: Spearman’s Coefficient (computational cost vs runtime).

|       | Sample 1 | Sample 2 | Sample 3 |
|-------|----------|----------|----------|
| RSFB  | 0.90     | NA       | NA       |
| SAAFB | 0.79     | 0.79     | 0.96     |
| HS    | 0.86     | 0.92     | 0.96     |

6.4. Discussion on the preliminary data analysis

6.4.1. Computational cost and restarted search

Table 4(a), referring to computational cost in Sample 1, shows that the values for the three algorithms in the first quartile and also in the second quartile (median) are very close. However, in the third quartile, the value for the RSFB algorithm is almost double the respective values of SAAFB and HS. It is also worth noting that the maximum value of RSFB is about five times greater than the respective values of SAAFB and HS.

In Table 4(b), referring to restarted search in Sample 1, the median is three times greater than the value of SAAFB and HS as the mean. In the third quartile, the value of RS is five times greater than the others. The maximum value of RS is about twenty times higher than others. It is also possible to notice that the maximum value of RSFB is about five times greater than the respective values of SAAFB and HS.

At the same time, for the metric computational cost, it can be seen that the highest results in Sample 3 are on the order of 10,000,000, while in Sample 2 the highest results are on the order of 60,000. It is important to remember that Samples 2 and 3 have graphs that range from 16 to 320 vertices but that the highest results of Sample 3 are about 166 times greater than those of Sample 2 with the same algorithms (SAAFB and HS). This big difference is related to the high number of edges in the graphs of Sample 3, which are complete bipartite digraphs (disregarding dummies).

6.4.2. Empirical distribution similarity tests

Results show that SAAFB and HS empirical distributions are statistically similar while the distribution of the RSFB algorithm is statistically different from the other two.

6.4.3. Metrics correlation with experimental factors

According to Table 8 in Sample 1, the highest correlation coefficients of the metric computational cost are with the factors $f_1$ (vertices), $f_5$ (percNeg), and $f_7$ (edges). For the metric restarted search the highest correlation coefficients were with $f_1$ (vertices), $f_5$ (percNeg), $f_7$ (edges), and $f_4$ (discRate). The other factors did not present relevant coefficients. Since the factors $f_3$ (maxDegree) and $f_6$ (cpMult), have sampling intervals with two values, they were not included in the analysis.

In Sample 2, as shown in Table 10, the two factors with the highest correlation coefficients with the metric computational cost were also $f_1$ (vertices) and $f_5$ (percNeg), both with values above 90%. The other factors did not present relevant correlation coefficients. Table 11 shows that the
highest correlation of restarted search was with $f_1$ (vertices).
In Sample 3, according to Tables 12 and 13, four factors showed strong correlation coefficients: $f_1$ (vertices), $f_3$ (maxDegree), $f_5$ (percNeg), and $f_4$ (discRate). In this sample, unlike the first two, the factor $f_5$ (maxDegree) includes a range of values between $(16 - 2)/2$ and $(320 - 2)/2$ (according to Table 3). Therefore, the correlation coefficient, in this case, presented a relevant value. The other factors did not present relevant values.

Table 14 shows the correlation between the metric computational cost with the execution time of the algorithms (time expressed in milliseconds, runtime). As the algorithms can reverse the search, the values considered for the factor $f_5$ (percNeg) were up to 50%. Since the RSFB algorithm was not used in Samples 2 and 3, NA refers to not applicable. In the other cases, there is a strong ($\geq 70\%$) or very strong ($\geq 90\%$) correlation between the computational cost and the time measure (which was not considered a dependent variable). No correlation study was made between restarted search and execution time, as this metric does not refer to the total cost, as is the case with computational cost.

7. Results: statistical models

The experimental results were analyzed with the Generalized Linear Model (GLM) statistical modeling technique. Twenty-eight statistical models were obtained from results of the two metrics (computational cost and restarted search). All the models assumed a Poisson-Gamma distribution (negative binomial), which agrees with the results from the preliminary analysis. Two model types were developed, with one factor and two factors, considering the maxCost formulation of the subsection 5.3.

7.1. Models of Sample 1

The results from Sample 1 (networks between 16 and 80 vertices) enabled the creation of eleven models: nine referring to the computational cost metric and three referring to the restarted search metric.

7.1.1. Models for computational cost (Sample 1)

Model 1 refers to maxCost(vertices) for the RSFB algorithm as a function of factor vertices. Figure 5(a) highlights the curve, polynomial, and the proportion of deviation explained with $D^2$ (GLM equivalent to $R^2$). Table 15 shows the respective estimated value, standard error, value of Wald’s z statistic and the value $Pr(|z|)$. In Model 1, the polynomial is: $maxCost(x) = 8 + 7x - 1x^2 + 1x^3$, where $x$ is vertices.

Table 15: RSFB maxCost(vertices) for computational cost - Sample 1.

| (Model 1) | Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|----------|------------|---------|----------|
| Intercept | 8.1181   | 0.0425     | 191.19  | 0.0000   |
| poly(x, 3)1 | 7.2567 | 0.3427     | 21.18   | 0.0000   |
| poly(x, 3)2 | -1.1065 | 0.3426     | -3.23   | 0.0012   |
| poly(x, 3)3 | 0.7793  | 0.3426     | 2.27    | 0.0229   |

Models 2 and 3 refer to the maxCost(vertices) of the SAAFB and HS algorithms. Figure 5(b) and Figure 6(a) highlight the respective curves, polynomials and $D^2$ of the models. Tables 16(a)
and show the results of the models. Models 2 and 3 have the same polynomial: \( \text{maxCost}(x) = 7 + 6x - 1x^2 \), where \( x \) refers to the vertices.

Table 16: SAAF and HS maxCost(vertices) for computational cost - Sample 1.

| (a) Model 2 - SAAF | (b) Model 3 - HS |
|--------------------|-----------------|
| (Intercept) | 6.9091 | 6.9174 |
| poly(x, 2)1 | 6.4801 | 6.4379 |
| poly(x, 2)2 | -0.9662 | -0.9455 |
| Std. Error | 0.2115 | 0.2109 |
| z value | 30.64 | 30.53 |
| Pr(>|z|) | 0.0000 | 0.0000 |

Figure 5: RSFB and SAAF maxCost(vertices) for computational cost - Sample 1.

Figure 6: HS maxCost(vertices), RSFB maxCost(percNeg) for computational cost - Sample 1.

Models 4, 5, and 6 refer to the maxCost(percNeg) of the RSFB, SAAF, and HS algorithms for the computational cost. Figures 6(b), 7(a), and 7(b) highlight the respective curves, polynomials, and \( D^2 \). Tables 17 and 18 display the model results. The polynomial of Model 4 is: \( \text{maxCost}(x) = 8 - 6x^2 - 1x^4 \), where \( x \) is percNeg. Models 5 and 6 have the same polynomial: \( \text{maxCost}(x) = 7 - 4x^2 + 1x^3 - 1x^4 \), where \( x \) is percNeg.
Table 17: RSFB maxCost(percNeg) for computational cost - Sample 1.

| Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|------------|---------|----------|
| (Intercept) | 7.7174 | 0.1495 | 52.08 | 0.0000 |
| poly(x, 4)1 | -0.7447 | 0.4037 | -1.84 | 0.0652 |
| poly(x, 4)2 | -6.1344 | 0.4047 | -15.16 | 0.0000 |
| poly(x, 4)3 | 0.7539 | 0.4029 | 1.87 | 0.0614 |
| poly(x, 4)4 | -1.0992 | 0.4008 | -2.74 | 0.0061 |

Table 18: SAAFB and HS maxCost(percNeg) for computational cost - Sample 1.

(a) Model 5 - SAAFB

| Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|------------|---------|----------|
| (Intercept) | 7.0701 | 0.0799 | 88.45 | 0.0000 |
| poly(x, 4)1 | -0.5058 | 0.2691 | -1.88 | 0.0602 |
| poly(x, 4)2 | -3.7575 | 0.2696 | -13.94 | 0.0000 |
| poly(x, 4)3 | 0.9398 | 0.2681 | 3.51 | 0.0005 |
| poly(x, 4)4 | -0.8128 | 0.2662 | -3.05 | 0.0023 |

(b) Model 6 - HS

| Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|------------|---------|----------|
| (Intercept) | 7.0762 | 0.0799 | 88.53 | 0.0000 |
| poly(x, 4)1 | -0.5035 | 0.2691 | -1.87 | 0.0613 |
| poly(x, 4)2 | -3.7462 | 0.2696 | -13.90 | 0.0000 |
| poly(x, 4)3 | 0.9379 | 0.2681 | 3.50 | 0.0005 |
| poly(x, 4)4 | -0.8118 | 0.2662 | -3.05 | 0.0023 |

Figure 7: SAAFB and HS maxCost(percNeg) for computational cost - Sample 1.

In addition to the one-factor models, three two-factor models were developed for computational cost. Figures 8(a), 8(b), and 8(c) highlight each of these models with their original data. The axes of the models refer to x(vertices), z(percNeg) and y(maxCost). Table 19 refers to the first model with two factors, i.e., maxCost(vertices,percNeg) for the RSFB. In Model 15, the polynomial is: maxCost(x, y) = 6 + 19x - 3y - 3x^2 - 35y^2 + 4y^3 - 3y^4 - 3y^6, where x refers to the vertices and y the percNeg. D^2 of Model 15 was 88%.

Table 19: RSFB maxCost(vertices,percNeg) for computational cost - Sample 1.

| Estimate | Std. Error | z value | Pr(>|z|) |
|-----------|------------|---------|----------|
| (Intercept) | 6.2855 | 0.0180 | 349.40 | 0.0000 |
| poly(y, 6)1 | -0.5058 | 0.2691 | -1.88 | 0.0602 |
| poly(y, 6)2 | -3.7575 | 0.2696 | -13.94 | 0.0000 |
| poly(y, 6)3 | 0.9398 | 0.2681 | 3.51 | 0.0005 |
| poly(y, 6)4 | -0.8128 | 0.2662 | -3.05 | 0.0023 |

Tables 20(a) and 20(b) refer to Models 16 and 17. These models have the same polynomial: maxCost(x) = 6 + 19x - 1y - 3x^2 - 22y^2 + 6y^3 - 3y^4 + 2y^5 + 2y^6, where x is vertices and y is percNeg. D^2 of Models 16 and 17 were 89%.
Table 20: HS and SAAFB \( \maxCost(\text{vertices}, \text{percNeg}) \) for computational cost - Sample 1.

(a) Model 16 - HS

| Model 16 | Estimate | Std. Error | \( z \) value | \( P(>|z|) \) |
|----------|----------|------------|----------------|----------------|
| (Intercept) | 5.7844 | 0.0127 | 455.76 | 0.0000 |
| poly(x, 2)1 | 16.4096 | 0.3407 | 48.17 | 0.0000 |
| poly(x, 2)2 | -2.7184 | 0.3400 | -7.99 | 0.0000 |
| poly(y, 6)1 | -0.6613 | 0.3463 | -1.91 | 0.0562 |
| poly(y, 6)2 | -22.2871 | 0.3467 | -64.29 | 0.0000 |
| poly(y, 6)3 | 5.8396 | 0.3444 | 16.95 | 0.0000 |
| poly(y, 6)4 | -3.1664 | 0.3413 | -9.28 | 0.0000 |
| poly(y, 6)5 | 1.8862 | 0.3386 | 5.57 | 0.0000 |
| poly(y, 6)6 | -2.4509 | 0.3368 | -7.28 | 0.0000 |

(b) Model 17 - SAAFB

| Model 17 | Estimate | Std. Error | \( z \) value | \( P(>|z|) \) |
|----------|----------|------------|----------------|----------------|
| (Intercept) | 5.7697 | 0.0127 | 453.51 | 0.0000 |
| poly(x, 2)1 | 16.3963 | 0.3415 | 48.60 | 0.0000 |
| poly(x, 2)2 | -2.7960 | 0.3409 | -8.20 | 0.0000 |
| poly(y, 6)1 | -0.6749 | 0.3473 | -1.94 | 0.0520 |
| poly(y, 6)2 | -22.4429 | 0.3477 | -64.55 | 0.0000 |
| poly(y, 6)3 | 5.8525 | 0.3454 | 16.95 | 0.0000 |
| poly(y, 6)4 | -3.1730 | 0.3421 | -9.27 | 0.0000 |
| poly(y, 6)5 | 1.8941 | 0.3394 | 5.58 | 0.0000 |
| poly(y, 6)6 | -2.4553 | 0.3375 | -7.27 | 0.0000 |

7.1.2. Models for restarted search (Sample 1)

For the restarted search metric, in Sample 1, three models were built. Models 1b, 2b, and 3b refer to the \( \maxCost(\text{vertices}) \) of the RSFB, SAAFB, and HS algorithms, as shown in Figures 9 and 10. Table 21 shows that the polynomial of Model 1b is: \( \maxCost(x) = 3 + 0x \), where \( x \) is \( \text{vertices} \). Tables 22(a) and 22(b) show that the Models 2b and 3b have the same polynomial: \( \maxCost(x) = 1 + 0x \), where \( x \) is \( \text{vertices} \).
Table 22: SAAFB and HS maxCost(vertices) for restarted search - Sample 1.

(a) Model 2b - SAAFB

|        | Estimate | Std. Error | z value | Pr(>|z|) |
|--------|----------|------------|---------|----------|
| (Intercept) | 1.4832   | 0.1364     | 10.87   | 0.0000   |
| a       | 0.0110   | 0.0024     | 4.61    | 0.0000   |

(b) Model 3b - HS

|        | Estimate | Std. Error | z value | Pr(>|z|) |
|--------|----------|------------|---------|----------|
| (Intercept) | 1.4832   | 0.1364     | 10.87   | 0.0000   |
| a       | 0.0110   | 0.0024     | 4.61    | 0.0000   |

Figure 9: RSFB maxCost(vertices) for restarted search - Sample 1.

Figure 10: SAAFB and HS maxCost(vertices) for restarted search - Sample 1.

It is worth noting that models for the factor edges were also obtained for RSFB, SAAFB, and HS. However, their results were similar to models with the factor vertices due to the proportionality between vertices and edges in data from Sample 1.

7.2. Models of Sample 2

The results from Sample 2 (networks between 16 and 320 vertices) enabled the creation of eight models: six referring to the computational cost and two referring to the restarted search.

7.2.1. Models for computational cost (Sample 2)

Models 7 and 8 refer to the maxCost(vertices) of the SAAFB and HS algorithms for the computational cost metric, as shown in Figures 11(a) and 11(b). Tables 23(a) and 23(b) show that the models have the same polynomial: \( \text{maxCost}(x) = 9 + 2.2x - 6x^2 + 2x^3 \), where \( x \) is vertices.
Table 23: SAAFB, HS maxCost(vert)ices for computational cost - Sample 2.

(a) Model 7 - SAAFB

|         | Estimate | Std. Error | z value | Pr(>|z|) |
|---------|----------|------------|---------|----------|
| (Intercept) | 8.7534   | 0.0228     | 383.93  | 0.0000   |
| poly(x, 3)1 | 22.4161  | 0.3987     | 56.23   | 0.0000   |
| poly(x, 3)2 | -6.0673  | 0.3987     | -15.22  | 0.0000   |
| poly(x, 3)3 | 2.2616   | 0.3987     | 5.67    | 0.0000   |

(b) Model 8 - HS

|         | Estimate | Std. Error | z value | Pr(>|z|) |
|---------|----------|------------|---------|----------|
| (Intercept) | 8.7560   | 0.0228     | 384.63  | 0.0000   |
| poly(x, 3)1 | 22.3701  | 0.3981     | 56.20   | 0.0000   |
| poly(x, 3)2 | -6.0328  | 0.3981     | -15.15  | 0.0000   |
| poly(x, 3)3 | 2.2387   | 0.3981     | 5.62    | 0.0000   |

Figure 11: SAAFB, HS maxCost(vert)ices for computational cost - Sample 2.

Models 9 and 10 refer to the maxCost(percNeg) of the SAAFB and HS algorithms, as shown in Figures 12(a) and 12(b). Tables 24(a) and 24(b) highlight that the models have the same polynomial: maxCost(x) = 9 - 1x - 5x^2 + x^3 - 1x^5 + 1x^5 - 1x^6, where x is percNeg.

Table 24: SAAFB, HS maxCost(percNeg) for computational cost - Sample 2.

(a) Model 9 - SAAFB

|         | Estimate | Std. Error | z value | Pr(>|z|) |
|---------|----------|------------|---------|----------|
| (Intercept) | 9.1172   | 0.0349     | 260.94  | 0.0000   |
| poly(x, 6)1 | -0.8530  | 0.1187     | -7.19   | 0.0000   |
| poly(x, 6)2 | -5.1860  | 0.1191     | -43.54  | 0.0000   |
| poly(x, 6)3 | 1.3668   | 0.1181     | 11.57   | 0.0000   |
| poly(x, 6)4 | -0.6875  | 0.1166     | -5.89   | 0.0000   |
| poly(x, 6)5 | 0.5134   | 0.1154     | 4.45    | 0.0000   |
| poly(x, 6)6 | -0.7549  | 0.1146     | -6.58   | 0.0000   |

(b) Model 10 - HS

|         | Estimate | Std. Error | z value | Pr(>|z|) |
|---------|----------|------------|---------|----------|
| (Intercept) | 9.1186   | 0.0349     | 261.04  | 0.0000   |
| poly(x, 6)1 | -0.8529  | 0.1186     | -7.19   | 0.0000   |
| poly(x, 6)2 | -5.1829  | 0.1191     | -43.53  | 0.0000   |
| poly(x, 6)3 | 1.3665   | 0.1181     | 11.57   | 0.0000   |
| poly(x, 6)4 | -0.6871  | 0.1166     | -5.89   | 0.0000   |
| poly(x, 6)5 | 0.5138   | 0.1154     | 4.45    | 0.0000   |
| poly(x, 6)6 | -0.7546  | 0.1146     | -6.58   | 0.0000   |

Figure 12: SAAFB, HS maxCost(percNeg) for computational cost - Sample 2.
With two factors, Models 18 and 19 refer to the \( \text{maxCost}(\text{vertices,percNeg}) \) of the HS and SAAFB algorithms, as shown in Figures 13(a) and 13(b). Tables 25(a) and 25(b) show that the models have the same polynomial: \( \text{maxCost}(x) = 7 + 58x - 6y - 15x^2 - 61y^2 + 5x^3 + 19y^3 - 1y^4 + 1y^5 - 6y^6 \), where \( x \) is \( \text{vertices} \) and \( y \) is \( \text{percNeg} \). Models 18 and 19 present the same \( D^2 \), 42%.

### Table 25: HS and SAAFB \( \text{maxCost}(\text{vertices,percNeg}) \) for computational - Sample 2.

(a) Model 18 - HS

| Model (18) | Estimate | Std. Error | z value | Pr(>|z|) |
|------------|----------|------------|---------|----------|
| (Intercept) | 6.9367   | 0.0262     | 265.07  | 0.0000   |
| poly(x, 3)1 | 57.9835  | 1.5165     | 38.24   | 0.0000   |
| poly(x, 3)2 | -14.8337 | 1.5165     | -9.78   | 0.0000   |
| poly(y, 6)1 | -5.7769  | 1.5168     | -3.81   | 0.0001   |
| poly(y, 6)2 | -60.6295 | 1.5168     | -39.97  | 0.0000   |
| poly(y, 6)3 | 18.8305  | 1.5168     | 12.42   | 0.0000   |
| poly(y, 6)4 | 1.0439   | 1.5157     | 0.69    | 0.4910   |
| poly(y, 6)5 | -6.1972  | 1.5154     | -4.09   | 0.0000   |

(b) Model 19 - SAAFB

| Model (19) | Estimate | Std. Error | z value | Pr(>|z|) |
|------------|----------|------------|---------|----------|
| (Intercept) | 6.9308   | 0.0262     | 265.00  | 0.0000   |
| poly(x, 3)1 | 58.2700  | 1.5156     | 38.45   | 0.0000   |
| poly(x, 3)2 | -15.0443 | 1.5156     | -9.93   | 0.0000   |
| poly(x, 3)3 | 5.3331   | 1.5156     | 3.52    | 0.0004   |
| poly(y, 6)1 | -5.7890  | 1.5160     | -3.82   | 0.0001   |
| poly(y, 6)2 | -60.7725 | 1.5160     | -40.09  | 0.0000   |
| poly(y, 6)3 | 18.8451  | 1.5157     | 12.43   | 0.0000   |
| poly(y, 6)4 | -0.8444  | 1.5148     | -0.56   | 0.5774   |
| poly(y, 6)5 | 1.0551   | 1.5145     | 0.70    | 0.4861   |
| poly(y, 6)6 | -6.1917  | 1.5145     | -4.09   | 0.0000   |

Models for the factor \( \text{edges} \) were also obtained for SAAFB and HS. However, as in Sample 1, their results were similar to models with the factor \( \text{vertices} \) due to the proportionality between \( \text{vertices} \) and \( \text{edges} \).

#### 7.2.2. Models for restarted search (Sample 2)

For the \( \text{restarted search} \) metric, two models were built, in Sample 2. Models 7b and 8b refer to \( \text{maxCost}(\text{vertices}) \) of SAAFB and HS, as shown in the Figures 14(a) and 14(b). Tables 26(a) and 26(b) display that the models have the same polynomial: \( \text{maxCost}(x) = 2 + 0x \), where \( x \) is \( \text{vertices} \).

### Table 26: SAAFB and HS \( \text{maxCost}(\text{vertices}) \) for restarted search - Sample 2.

(a) Model 7b - SAAFB

| Model (7b) | Estimate | Std. Error | z value | Pr(>|z|) |
|------------|----------|------------|---------|----------|
| (Intercept) | 1.7439   | 0.0457     | 38.46   | 0.0000   |
| x           | 0.0034   | 0.0002     | 15.81   | 0.0000   |

(b) Model 8b - HS

| Model (8b) | Estimate | Std. Error | z value | Pr(>|z|) |
|------------|----------|------------|---------|----------|
| (Intercept) | 1.7439   | 0.0457     | 38.46   | 0.0000   |
| x           | 0.0034   | 0.0002     | 15.81   | 0.0000   |
7.3. Models of Sample 3

In Sample 3 (networks between 16 and 80 vertices, but as complete bipartite digraphs without dummies), eight models were elaborated: six referring to the computational cost metric and two referring to the restarted search metric.

7.3.1. Models for computational cost (Sample 3)

Models 11 and 12 refer to maxCost(vertices) of the SAAFB and HS algorithms, as shown in Figures 18(a) and 18(b). Tables 27(a) and 27(b) show that the models have the same polynomial: maxCost(x) = 13 + 31x − 11x^2 + 6x^3, where x is vertices.

| Model 11 - SAAFB |  |  |  |
|------------------|--|--|--|
| (Intercept)      | 13.2331 | 0.0272 | 486.10 | 0.0000 |
| poly(x, 3)1      | 31.5093 | 0.4755 | 66.27 | 0.0000 |
| poly(x, 3)2      | -10.8626 | 0.4755 | -22.85 | 0.0000 |
| poly(x, 3)3      | 5.7247  | 0.4755  | 12.04 | 0.0000 |

| Model 12 - HS  |  |  |  |
|----------------|--|--|--|
| (Intercept)    | 13.2335 | 0.0272 | 486.37 | 0.0000 |
| poly(x, 3)1    | 31.4995 | 0.4752 | 66.29 | 0.0000 |
| poly(x, 3)2    | -10.8533 | 0.4752 | -22.84 | 0.0000 |
| poly(x, 3)3    | 5.7168  | 0.4752  | 12.03 | 0.0000 |

Models 13 and 14 refer to the maxCost(percNeg) of SAAFB and HS, as shown in Figures 16(a) and 16(b). Tables 28(a) and 28(b) show that the models have the same polynomial: maxCost(x) = 13 + 9 − 4^2 + 2x^3 − 1x^4 + 1x^5 − 1x^6, where x is percNeg.

| Model 13 - SAAFB |  |  |  |
|------------------|--|--|--|
| (Intercept)      | 13.2331 | 0.0272 | 486.10 | 0.0000 |
| poly(x, 3)1      | 31.5093 | 0.4755 | 66.27 | 0.0000 |
| poly(x, 3)2      | -10.8626 | 0.4755 | -22.85 | 0.0000 |
| poly(x, 3)3      | 5.7247  | 0.4755  | 12.04 | 0.0000 |

| Model 14 - HS  |  |  |  |
|----------------|--|--|--|
| (Intercept)    | 13.2335 | 0.0272 | 486.37 | 0.0000 |
| poly(x, 3)1    | 31.4995 | 0.4752 | 66.29 | 0.0000 |
| poly(x, 3)2    | -10.8533 | 0.4752 | -22.84 | 0.0000 |
| poly(x, 3)3    | 5.7168  | 0.4752  | 12.03 | 0.0000 |
Table 28: SAAFB and HS maxCost(percNeg) for computational - Sample 3.

(a) Model 13 - SAAFB

| Estimate    | Std. Error | z value | Pr(>|z|) |
|-------------|------------|---------|----------|
| Intercept   | 13.3799    | 0.0341  | 392.41   | 0.0000   |
| poly(x, 6)2 | -4.0915    | 0.1147  | -35.47   | 0.0000   |
| poly(x, 6)3 | 1.7654     | 0.1147  | 15.40    | 0.0000   |
| poly(x, 6)4 | 1.7251     | 0.1121  | 15.40    | 0.0000   |
| poly(x, 6)5 | 0.7251     | 0.1121  | 6.44     | 0.0000   |

(b) Model 14 - HS

| Estimate    | Std. Error | z value | Pr(>|z|) |
|-------------|------------|---------|----------|
| Intercept   | 13.3803    | 0.0341  | 392.54   | 0.0000   |
| poly(x, 6)2 | -4.0899    | 0.1146  | -35.47   | 0.0000   |
| poly(x, 6)3 | 1.7642     | 0.1146  | 15.39    | 0.0000   |
| poly(x, 6)4 | 1.7246     | 0.1126  | 6.43     | 0.0000   |

Figure 16: SAAFB and HS maxCost(percNeg) for computational - Sample 3.

Models 20 and 21 refer to maxCost(vertices, percNeg) of HS and SAAFB, in Sample 3, as shown in Figures 17(a) and 17(b). Tables 29(a) and 29(b) demonstrate that the models have the same polynomial: maxCost(x, y) = 11 + 86x + 120y − 26x² − 48y² + 12x³ + 17y³ − 7y⁴ + 5y⁵, where x is vertices and y is percNeg.

Table 29: HS and SAAFB maxCost(vertices, percNeg) for computational - Sample 3.

(a) Model 20 - HS

| Estimate    | Std. Error | z value | Pr(>|z|) |
|-------------|------------|---------|----------|
| Intercept   | 10.5202    | 0.0386  | 272.89   | 0.0000   |
| poly(x, 3)2 | -25.1565   | 2.2337  | -11.26   | 0.0000   |
| poly(x, 3)3 | 11.9616    | 2.2336  | 5.36     | 0.0000   |
| poly(y, 5)2 | -47.7613   | 2.2338  | -21.38   | 0.0000   |
| poly(y, 5)3 | 17.2933    | 2.2333  | 7.74     | 0.0000   |
| poly(y, 5)4 | -7.4775    | 2.2329  | -3.35    | 0.0008   |
| poly(y, 5)5 | 5.3003     | 2.2328  | 2.38     | 0.0175   |

(b) Model 21 - SAAFB

| Estimate    | Std. Error | z value | Pr(>|z|) |
|-------------|------------|---------|----------|
| Intercept   | 10.5172    | 0.0385  | 272.84   | 0.0000   |
| poly(x, 3)2 | -25.3112   | 2.2335  | -11.33   | 0.0000   |
| poly(x, 3)3 | 12.0930    | 2.2334  | 5.41     | 0.0000   |
| poly(y, 5)2 | -47.8990   | 2.2336  | -21.44   | 0.0000   |
| poly(y, 5)3 | 17.2933    | 2.2333  | 7.74     | 0.0000   |
| poly(y, 5)4 | -7.5425    | 2.2329  | -3.38    | 0.0007   |
| poly(y, 5)5 | 5.3410     | 2.2326  | 2.39     | 0.0167   |
Models for the edges factor were also tested. However, their results showed low explanatory power with $D^2$.

### 7.3.2. Models for restarted search (Sample 3)

For the restarted search metric, in Sample 3, two more models were built. Models 11b and 12b refer to $\text{maxCost} (\text{vertices})$ of SAAFB and HS, as shown in Figures 18(a) and 18(b). Tables 30(a) and 30(b) show that the models have the same polynomial: $\text{maxCost}(x) = 3 + 0x$, where $x$ is vertices.

| (a) Model 11b - SAAFB | (b) Model 12b - HS |
|------------------------|-------------------|
| (Intercept) 3.3821     | (Intercept) 3.3821|
| x 0.0055               | x 0.0055          |
| 0.0404                 | 0.0404            |
| 83.65                  | 83.65             |
| 0.0000                 | 0.0000            |

8. Analysis and discussion

The results of models are summarized by sample in Tables 31, 32, and 33. These results are organized by metric, factors of the $\text{maxCost}()$ function (vertices; percNeg; and vertices with
percNeg), and algorithms (RSFB, SAAFB and HS). The results are reduced to terms with the highest degree of the respective polynomials, highlighting only the order of magnitude in each case. It is important to remember that the computational cost metric refers to the total cost of each algorithm and the restarted search metric refers to the number of times the search is restarted in the spanning tree by each algorithm (open questions for the three algorithms).

8.1. Discussion about Sample 1

Table [31] summarize all models of Sample 1, where the first three rows show the models for the computational cost metric (as comp. cost) and the last row shows the models for the restarted search metric (as rest. search). So, in first row, for computational cost with maxCost(vertices) the performance of RSFB was $\text{vertices}^3$ and performance of SAAFB and HS were $\text{vertices}^2$. In the second row for computational cost with maxCost(percNeg) all algorithms showed the same performance, i.e., $\text{percNeg}^4$. In the third row for the computational cost with maxCost(vertices, percNeg) all results were $\text{vertices}^2 + \text{percNeg}^6$. In the last row, for the restarted search with maxCost(vertices) all algorithms showed the same results, i.e., $\text{vertices}^1$.

It should be noted that the KS test, described in Table [7], pointed out that, in Sample 1 for the computational cost: RSFB $\not\sim$ SAAFB; RSFB $\not\sim$ SAAFB; and HS $\sim$ SAAFB. Such test reinforces the results of Table [31]. In other words, the polynomial degrees when the metric is computational cost and the maxCost() functions of RSFB diverge from the degrees of the respective functions of SAAFB and HS. However, it is noticed that between SAAFB and HS, comparing by function maxCost(), the polynomial degrees are always the same.

Table [31]: Highest degree of polynomials by metric - Sample 1.

| Metric          | maxCost Factors | RSFB degree | SAAFB degree | HS degree |
|-----------------|-----------------|-------------|--------------|-----------|
| comp. cost      | vertices        | $\text{vertices}^3$ | $\text{vertices}^2$ | $\text{vertices}^2$ |
| comp. cost      | percNeg         | $\text{percNeg}^4$ | $\text{percNeg}^4$ | $\text{percNeg}^4$ |
| comp. cost      | vertices, percNeg | $\text{vertices}^2 + \text{percNeg}^6$ | $\text{vertices}^2 + \text{percNeg}^6$ | $\text{vertices}^2 + \text{percNeg}^6$ |
| rest. search    | vertices        | $\text{vertices}^1$ | $\text{vertices}^1$ | $\text{vertices}^1$ |

8.2. Discussion about Sample 2

Similar to the previous section, the results of the models in Sample 2 are summarized in Table [32]. In the first row, for computational cost with maxCost(vertices), the results of SAAFB and HS were $\text{vertices}^3$. In the second row for computational cost with maxCost(percNeg) the results of SAAFB and HS were $\text{percNeg}^6$. In the third row for the computational cost with maxCost(vertices, percNeg) the results were $\text{vertices}^2 + \text{percNeg}^6$. In the last row, for the restarted search with maxCost(vertices), the results of SAAFB and HS were $\text{vertices}^1$, as well as in Sample 1.

With respect to the KS test, according to Table [7] (in Sample 2) for the computational cost: SAAFB $\sim$ HS. This result is in harmony with the results of Table [32]. In other words, the maximum degrees of the polynomials are the same in all maxCost() functions between SAAFB and HS.
8.3. Discussion about Sample 3

In Sample 3, for computational cost with maxCost(vertices), through Table 33, it is noted that the polynomials of SAAFB and HS do not exceed degree 3. In the third row, for the computational cost with maxCost(vertices, percNeg), the results were vertices$^2 + percNeg^5$. The other results were the same as in Sample 2.

Table 33: Highest degree of polynomials by metric - Sample 3.

| Metric                         | maxCost Factors | SAAFB degree     | HS degree     |
|--------------------------------|-----------------|------------------|---------------|
| computational cost             | vertices        | vertices$^4$     | vertices$^4$  |
| computational cost             | percNeg         | percNeg$^6$     | percNeg$^6$   |
| computational cost             | vertices, percNeg | vertices$^3 + percNeg^5$ | vertices$^3 + percNeg^5$ |
| restarted search               | vertices        | vertices$^1$     | vertices$^1$  |

9. Conclusions

In this article, the most recent algorithms (RSFB, SAAFB, and HS) for the problem described in Section 2 were presented and submitted to a factorial experiment. The experiment structure employed a function called maxCost, whose parameters were network factors such as vertices and percNeg (percentage of negative activities). The results were expressed through two metrics: 1) computational cost and 2) restarted search. While the first metric referred to the total cost for each of the algorithms, the second metric referred to the number of search restarts in the spanning tree for each algorithm.

The results were obtained using three samples from different network configurations. In the first sample, graphs with 16 to 80 vertices were used, randomized in terms of vertices, layers, and percentage of negative activities, among other parameters. The second sample used graphs with 16 to 320 vertices and randomization similar to the first sample. Finally, the third sample referred to a set of convenience graphs to stress the algorithms with a high number of edges. In this case, complete bipartite digraphs were used (disregarding dummies) with vertices between 16 and 320.

9.1. Main Conclusions
9.1.1. SAAFB and HS outperform RSFB for the first metric

As indicated, the RSFB algorithm was used only in the networks of Sample 1. In this case, it is worth noting that the highest degree of the polynomial identified for the metric computational cost with maxCost(vertices) was vertices$^3$, while the algorithms SAAFB and HS were expressed in polynomials of degree two, i.e., vertices$^2$. The empirical results show that by using maxCost(vertices)
as a proxy for $O(\text{vertices})$, we can estimate that RSFB has a time cost of $O(\text{vertices}^3)$, while SAAFB and HS presented a time cost of $O(\text{vertices}^2)$, with the networks in Sample 1. In addition, the results of the KS test, according to Table 7, showed a statistically significant difference between the distributions of RSFB and SAAFB; and RSFB and HS.

9.1.2. SAAFB and HS have the same performance for both metrics

In the case of SAAFB and HS, the results of the KS test (in the three samples and both metrics) did not present any statistically significant difference, as shown by the factor comparisons of the maxCost function (Table 7). Furthermore, they presented equal values for the maximum degree of polynomials for all factors and metrics analyzed. So we can conclude, within the limits of statistical precision, that both SAAFB and HS present the same order of cost performance.

9.1.3. RSFB, SAAFB and HS have similar performance for the second metric

The results of the restarted search metric in all samples by function maxCost of the three algorithms were always the same. In this case, the highest degree of the polynomials of the elaborated models was always $\text{vertices}^1$. In other words, RSFB, SAAFB, and HS presented a time cost of $O(\text{vertices}^1)$. It is an important finding, as it refers to one of the open questions in the three algorithms.

9.1.4. Performance most influential factor

The models that include the percNeg factor presented polynomials with higher degrees than those that do not have such a factor. However, the percNeg factor considers a range with values from 0% to 100%. In this case, it was possible to verify that the influence of percNeg on the scheduling cost increases until the value is around 50%, after which the influence decreases. On the other hand, the $\text{vertices}$ factor influences the cost of scheduling without a defined limit. Overall, this indicates that as long as the $\text{vertices}$ factor can grow, the scheduling cost will also grow without limit. Therefore, the factor $\text{vertices}$ exclusively is more appropriate to express the order of cost growth of the algorithms in the three samples with the two metrics. In this sense, where $n$ is $\text{vertices}$, the cost of scheduling was: a) $O(n^3)$ for RSFB, with the computational cost metric in Sample 1; b) $O(n^2)$, with the computational cost metric for SAAFB and HS, in Sample 1; c) $O(n^3)$, with the computational cost metric for SAAFB and HS, in Samples 2 and 3; d) and $O(n)$, with the restarted search metric for RSFB, SAAFB and HS, in the three samples.

9.2. Future Research

Among the possibilities for future research are: 1) consider samples with larger graphs than those treated in this experiment (above 320 vertices) and graphs with redundant edges by transitivity. Such cases can subject the algorithms to higher stress levels; 2) implement the RSFB algorithm in some language that supports double recursive stacking (characteristic of the algorithm) in networks with 320 (or more) vertices. Then perform a new experiment. With networks in the order of 320 (or more) vertices, it is conjectured that the order of cost of RSFB is greater than $O(n^3)$, which was verified in this experiment for the algorithms SAAFB and HS.
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Appendix A. Experiment algorithms in backward

Appendix A.1. Recursive Search Forward-Backward (RSFB)

Algorithm 10: Step_3 - Backward.

1. procedure Step_3() {CA is a global structure}
2. total_Step_3 = total_Step_3 + 1
3. CA ← ∅
4. SA', DC' ← Recursion(n)
5. Report the optimal solution DC'

Algorithm 11: Recursion - Backward.

1. function Recursion(newnode) {CA, CT are global structures}
2. total_Recursion = total_Recursion + 1
3. SA ← {newnode}; DC ← DC_{newnode}; CA ← CA + newnode
4. for each (i| i ∈ CA and i precedes newnode ∈ CT) do
   5. SA', DC' ← Recursion(i)
   6. if DC' ≥ 0 then
      7. SA ← SA + SA'; DC ← DC + DC'
   8. else
      9. \[ \text{Compute} v_{l,k} = \min_{\begin{subarray}{c} \{s_k - f_j\} \\text{for} \ l \in \text{SA} \\ k \notin \text{SA} \\end{subarray}} \]
     10. \[ \text{CT} ← \text{CT} - (i, \text{newnode}) \]
     11. \[ \text{for each} j \in \text{SA}' : f_j ← f_j - v_{l,k} \]
     12. Step_3()
   13. end
5. for each (i| i ∉ CA and i succeeds newnode ∈ CT) do
6. SA', DC' ← Recursion(i)
7. SA ← SA + SA'; DC ← DC + DC'
8. end
9. return (SA, DC)
Appendix A.2. Steepest Ascent Approach Forward-Backward (SAAFB)

Algorithm 12: Steepest Ascent Direction (SAD) - Backward.

1 function SAD() \{ST(V_{st}, E_{st}) is a global structure\}
2 \[ Z \leftarrow \emptyset; V \leftarrow V_{st} \]
3 \[ \forall i \in V \text{ do } C(i) \leftarrow i; \phi_i \leftarrow -\alpha c_i e^{-\alpha(s_i + d_i)} \]
4 \[ \text{while } V \neq \{1\} \text{ do} \]
5 \[ \text{if (V has a node sink } i \neq 1 \text{) \& (at most one predecessor } j \text{) then} \]
6 \[ \text{iteration}_{\text{SAD}} = \text{iteration}_{\text{SAD}} + 1 \]
7 \[ C(j) \leftarrow C(j) + C(i); \phi_j \leftarrow \phi_j + \phi_i; V \leftarrow V - i \]
8 \[ \text{else} \]
9 \[ \text{if (V has a node source } j \neq 1 \text{) \& (only one successor } i \text{) then} \]
10 \[ \text{iteration}_{\text{SAD}} = \text{iteration}_{\text{SAD}} + 1 \]
11 \[ \text{if } \phi_j \leq 0 \text{ then } Z \leftarrow Z + C(j) \]
12 \[ \text{else } \phi_i \leftarrow \phi_i + \phi_j; C(i) \leftarrow C(i) + C(j); V \leftarrow V - j \]
13 \[ \text{end} \]
14 \[ \text{end} \]
15 \[ \text{return}(Z) \]

Algorithm 13: Vertex Ascent (VA) - Backward.

1 function VA(S, Z) \{ST, G are global structures\}
2 \[ \forall (i,j) \in ST \mid (j \notin C(i)) \& (i \notin C(j)) : ST \leftarrow ST - (i,j) \]
3 \[ \text{while } Z \neq \emptyset \text{ do} \]
4 \[ \text{Compute } V_{l^*k^*} = \min \{ S_k - f_{l^*} \}_{(l^*,k^*) \in G} \]
5 \[ \text{Take the set } C(j) \in Z \text{ where } k^* \text{ is contained} \]
6 \[ \forall i \in C(j) : S_i \leftarrow S_i - V_{l^*k^*} \]
7 \[ Z \leftarrow Z - C(j); ST \leftarrow ST + (l^*,k^*) \]
8 \[ \text{return}(S) \]

Algorithm 14: Steepest Ascent Procedure (SAP) - Backward.

1 procedure SAP() \{ST is a global structure\}
2 \[ S, ST \leftarrow \text{Determine the Late Schedule } (S) \text{ as a vector and a corresponding initial Spanning}\]
3 \[ \text{Tree } (ST) \text{ through the original graph } G. \]
4 \[ \text{total}_{\text{SAD}} = 0 \]
5 \[ Z \leftarrow \text{SAD}() \]
6 \[ \text{while } Z \neq \emptyset \text{ do} \]
7 \[ \text{total}_{\text{SAD}} = \text{total}_{\text{SAD}} + 1 \]
8 \[ S \leftarrow VA(S, Z) \]
9 \[ Z \leftarrow \text{SAD}() \]
10 \[ \text{end} \]
11 \[ \text{Report the optimal solution } S \]
Appendix A.3. Hybrid Search (HS)

Algorithm 15: Recursion de HS - Backward.

1 function Recursion(newnode) \{CA, ST, SS are global structures\}
2 \[total\_Recursion = total\_Recursion + 1\]
3 \[SA ← \{newnode\}; DC ← DC_{newnode}; CA ← CA + newnode\]
4 for each (i|i \∉ CA and i precedes newnode ∈ ST) do
5 \[SA', DC' ← Recursion(i)\]
6 if \[DC' < 0\] then
7 \[SA ← SA + SA'; DC ← DC + DC'\]
8 else
9 \[ST ← ST - (i, newnode); SS ← SS + SA'\]
10 end
11 end
12 for each (i|i \∉ CA and i succeeds newnode ∈ ST) do
13 \[SA', DC' ← Recursion(i)\]
14 \[SA ← SA + SA'; DC ← DC + DC'\]
15 end
16 return \((SA, DC)\)

Algorithm 16: Shift activities - Backward.

1 procedure Shift activities() \{SS, ST, and G are global structures\}
2 \[Z ← \emptyset; ∀ i \in SA \mid SA ∈ SS : Z ← Z + i\]
3 while \[Z \neq \emptyset\] do
4 \[\text{Compute} v_{l+k} = \min_{(l*, k*) \in \mathcal{G}} \left\{ s_{k*} - f_{l*}\right\}\]
5 \[∀ i \in SA | k* ∈ SA : s_i ← s_i - v_{k*+l*} \text{ and } Z ← Z - i\]
6 \[ST ← ST + (l*, k*)\]
7 end

Algorithm 17: Hybrid Recursive Search (HRS) - Backward.

1 procedure HRS() \{CA and SS are global structures\}
2 \[total\_HRS = total\_HRS + 1\]
3 \[CA ← SS ← \emptyset\]
4 \[SA, DC' ← Recursion(n)\]
5 if \[SS \neq \emptyset\] then
6 \[Shift\_activities()\]
7 \[HRS()\]
8 else Report the optimal solution \[DC'\]
Algorithm 18: Compute $v_{lsk^*}$ - Backward.

1 function Compute $v_{lsk^*}(Z)$
2     $v_{lsk^*} \leftarrow \delta$
3     $k \leftarrow \emptyset$; $l \leftarrow \emptyset$
4     for node $\in Z$ do
5         if $k = \emptyset$ then $k \leftarrow node$
6             for pred $\in$ predecessorsof node do
7                 edge checked $\leftarrow$ edge checked + 1
8                 if pred $\notin Z$ do
9                     if $s_k - s_l < 0$ then
10                        current min $= s_k - f_l - (-\delta)$
11                     else
12                        current min $= s_k - f_l$
13                     end
14                 if current min $< v_{lsk^*}$ then
15                     $v_{lsk^*} \leftarrow$ current min
16                 $k \leftarrow$ node
17                 $l \leftarrow$ pred
18                 if $v_{lsk^*} \mid 0$ then $v_{lsk^*} = v_{lsk^*} - 1$
19             end
20             if $l = \emptyset$ then $l \leftarrow suc$
21         end
22     return $(k, l, v_{lsk^*})$