String clouds and radiation flows as sources of gravity in static or rotating cylinders

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Static and stationary cylindrically symmetric space-times in general relativity are considered, supported by distributions of cosmic strings stretched in the azimuthal ($\varphi$), longitudinal ($z$) or radial ($x$) directions or and by pairs of mutually opposite radiation flows in any of these directions. For such systems, exact solutions are obtained and briefly discussed, except for radial strings (a stationary solution for them is not found); it is shown that static solutions with $z$- and $\varphi$-directed radiation flows do not exist while for $z$-directed strings a solution is only possible with negative energy density. Almost all solutions under discussion contain singularities, and all stationary solutions have regions with closed timelike curves, hence only their well-behaved regions admit application to real physical situations.

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1. Introduction

Cylindrically symmetric space-times have attracted the researchers’ attention soon after the advent of general relativity (GR) as examples of the simplest geometries describing extreme deviations from spherical symmetry and sources like jets and flows, rods and (as we know now) cosmic strings. The story began with the Levi-Civita static vacuum solution (1919) and its stationary counterparts with rotation and new solutions are still being found and studied. The relative simplicity of cylindrical symmetry as compared to more general axial symmetry is favorable for studying rotation effects in GR. Among sources of gravity in known stationary exact solutions of GR one can mention the cosmological constant $\Lambda$, scalar fields with or without self-interaction potentials, rigidly or differentially rotating dust, electrically charged dust, dust combined with a scalar field, perfect and anisotropic fluids etc, see also the reviews and references therein.

We adhere to GR as a perfectly successful theory on scales, say, from centimeters to parsecs, while its diverse extensions are mostly designed for much smaller or much larger scales. In addition, solutions of many alternative theories can be obtained.
from those of GR by various solution generation methods like conformal mapping between the Einstein and Jordan frames in $f(R)$ and scalar-tensor theories.

In our previous papers, some static and stationary cylindrical wormhole solutions were found and discussed, and a construction of asymptotically flat wormhole configuration was suggested that does not contain exotic matter violating the Weak Energy Condition. The latter required, as a source of gravity, either a special kind of anisotropic fluid or a stiff perfect fluid. This paper is devoted to a search for other sources of gravity possibly providing wormhole solutions for stationary cylindrically symmetric space-times. Specifically, we consider distributions of cosmic strings stretched in the azimuthal, longitudinal or radial directions, and mutually opposite radiation flows in any of these directions. By the structure of their stress-energy tensors (SETs) $T^\mu_\nu$, they represent special examples of anisotropic fluids. We follow the formalism described in Ref. 21 which includes splitting of the Ricci ($R^\mu_\nu$) and Einstein ($G^\mu_\nu$) tensors into static and rotational parts and using the harmonic radial coordinate for cylindrical metrics. We enumerate possible exact solutions with such sources and briefly discuss their basic properties.

2. Basic relations

Consider a stationary cylindrically symmetric metric

$$ds^2 = e^{2\gamma(x)} [dt - E(x) e^{-2\gamma(x)} d\varphi]^2 - e^{2\alpha(x)} dx^2 - e^{2\beta(x)} dz^2 - e^{2\delta(x)} d\varphi^2,$$

(1)

where $x$, $z \in \mathbb{R}$, $\varphi \in [0, 2\pi)$ are the radial, longitudinal and angular coordinates, respectively. The coordinate $x$ is specified up to a change $x \rightarrow f(x)$, and its range depends on both its choice and on the geometry itself. The only off-diagonal metric component $E$ describes rotation, with the vorticity $\omega(x)$ (in terms of an arbitrary coordinate $x$) given by $\omega(x) = \frac{1}{2} (E e^{-2\gamma})' e^{-\beta - \alpha}$ (the prime denotes $d/dx$). Next, choosing the reference frame comoving to matter (as it moves along $\varphi$), we require $T^3_3 = 0$, then the Einstein equations lead to $R^3_3 \sim (\omega e^{2\gamma + \mu})' = 0$, so that

$$\omega = \omega_0 e^{-\mu - 2\gamma} \quad \text{and} \quad E(x) = 2\omega_0 e^{2\gamma(x)} \int e^{\alpha + \beta - \mu - 3\gamma} dx,$$

(2)

with $\omega_0 = \text{const}$. The nontrivial components of the Einstein equations $R^\nu_\mu = -\kappa T^\nu_\mu$ \(\equiv -\kappa (T^\nu_\mu - \frac{1}{2} \delta^\nu_\mu T^\alpha_\alpha)\) and the constraint equation $G^1_1 = -\kappa \tilde{T}^1_1$ may be written as

$$R^0_0 = -e^{-2\alpha} [\gamma'' + \gamma'(\sigma' - \alpha')] - 2\omega^2 = -\kappa \tilde{T}^0_0,$$

$$R^2_2 = -e^{-2\alpha} [\mu'' + \mu'(\sigma' - \alpha')] = -\kappa \tilde{T}^2_2,$$

$$R^3_3 = -e^{-2\alpha} [\beta'' + \beta'(\sigma' - \alpha')] + 2\omega^2 = -\kappa \tilde{T}^3_3,$$

$$R^0_3 = G^0_3 = E e^{-2\gamma} (R^3_3 - R^0_0) = -\kappa E e^{-2\gamma} (T^3_3 - T^0_0),$$

$$G^1_1 = e^{-2\alpha} (\beta' \gamma' + \beta' \mu' + \gamma' \mu') + \omega^2 = -\kappa \tilde{T}^1_1.$$

(3)

where $\sigma = \beta + \gamma + \mu$ and $\kappa = 8\pi G$ is the gravitational constant. As follows from 3, the diagonal components $R^\nu_\nu$ and $G^\nu_\nu$ split into static parts $\delta R^\nu_\nu$ and $\delta G^\nu_\nu$ (those
for the metric (1) with \( E = 0 \) plus contributions from \( \omega \) in particular,
\[
G_\mu^\nu = \omega G_\mu^\nu + \omega G_\mu^\nu, \quad \omega G_\mu^\nu = \omega^2 \text{diag}(-3,1,-1,1),
\]
(4)

Each of the tensors \( \omega G_\mu^\nu \) and \( \omega G_\mu^\nu \) separately obeys the conservation law \( \nabla_\nu G_\mu^\alpha = 0 \) according to this static metric. Thus \( \omega G_\mu^\nu / \omega \) acts as an additional SET of a “vortex gravitational field” with a negative effective energy density \( T_{\mu}^\nu = -3\omega^2 / \omega \), favorable for wormhole existence, as is confirmed by a few examples. \(^{[6,7,25,27]}\)

What is important, it is sufficient to solve the diagonal components of the Einstein equations, their single off-diagonal component then holds as well. \(^{[9]}\)

**Anisotropic fluids.** Consider \(^{[15,16,21]}\) nondissipative matter with energy density \( \rho \) and three principal pressures \( p_i \) in mutually orthogonal directions in a comoving reference frame, so that the SET has the form
\[
T^{(\alpha \beta)} = \text{diag}(\rho, p_1, p_2, p_3)
\]
in an orthonormal tetrad \( \{ e^\mu_\alpha \} = \{ u^\mu, \phi^\mu, \chi^\mu, \psi^\mu \} \), where \( u^\mu \) is the fluid 4-velocity \( (u_\mu u^\mu = 1) \) while the vectors \( \phi^\mu, \chi^\mu, \psi^\mu \) are spacelike. Choosing (arbitrarily) one of them as \( \phi^\mu = e^\mu_1 \), we have \( \phi^{\mu
u} = u^{\mu}u^{\nu} - \phi^{\mu} \phi^{\nu} - \chi^{\mu} \chi^{\nu} - \psi^{\mu} \psi^{\nu} \)
\[
T^{\mu\nu} = (p + p_1)u^{\mu}u^{\nu} - p_1 g^{\mu\nu} + (p_2 - p_1)\chi^{\mu} \chi^{\nu} + (p_3 - p_1)\psi^{\mu} \psi^{\nu}.
\]
(5)

Consider now this kind of matter in the metric (1), assuming that \( p_1 = p_x \) is the radial pressure, \( p_2 = p_z \) and \( p_3 = p_\varphi \) being pressures in the \( z \) and \( \varphi \) directions. Then one can verify that the conservation law \( \nabla_\nu T^{\nu}_{\mu} = 0 \) does not contain \( E \) and takes the same form for rotating and nonrotating fluids:
\[
p'_x + (p + p_x)\gamma' + (p_x - p_z)\mu' + (p_x - p_\varphi)\beta' = 0.
\]
(6)
Let us now assume in (5) the equations of state
\[
p_x = w_1 \rho, \quad p_z = w_2 \rho, \quad p_\varphi = w_3 \rho, \quad w_i = \text{const},
\]
(7)
and choose the harmonic radial coordinate defined by the condition \(^{[22]}\) \( \alpha = \beta + \gamma + \mu \). Then three diagonal equations (3) and the constraint read
\[
e^{-2\alpha} \gamma'' = -2\omega^2 + \frac{1}{2} \kappa \rho (1 + w_1 + w_2 + w_3),
\]
(8)
\[
e^{-2\alpha} \mu'' = \frac{1}{2} \kappa \rho (-1 + w_1 - w_2 + w_3),
\]
(9)
\[
e^{-2\alpha} \beta'' = 2\omega^2 + \frac{1}{2} \kappa \rho (-1 + w_1 + w_2 - w_3),
\]
(10)
\[
e^{-2\alpha} (\beta' \gamma' + \beta' \mu' + \gamma' \mu') = -\omega^2 + w_1 \kappa \rho.
\]
(11)

3. **String clouds**

Let us now discuss special cases of Eqs. (8)–(11) corresponding to distributions of cosmic strings stretched in one of the directions \( \phi^\mu, \chi^\mu, \psi^\mu \), so that the factors \( (w_1, w_2, w_3) \) in the expressions (7) take the forms \((-1,0,0), (0,-1,0) \) or \((0,0,-1) \).

In all cases there are five unknowns: the metric functions \( \beta(x), \gamma(x), \mu(x), E(x) \) and the density \( \rho(x) \).

\(^{a}\) Tetrad indices are written in parentheses.
Azimuthal strings: \((w_1, w_2, w_3) = (0, 0, -1)\). For such distributions of coaxial circular string loops, the stationary solution for the metric reads
\[
ds^2 = 2|\omega_0| r_0 x \left[ dt + \frac{1 - E_0 x}{2 \omega_0 x} d\varphi \right]^2 - r_0^2 e^{2\mu} dx^2 - e^{2\mu} dz^2 - \frac{r_0}{2|\omega_0|x} d\varphi^2. \tag{12}\]
where \(r_0 = \text{const}\) is an arbitrary length scale, and \(E_0\) is an integration constant from (2). The unknowns \(\rho(x)\) and \(\mu(x)\) are connected by the relation due to Eq. (9),
\[
r_0^2 \kappa \rho = -\mu'' e^{-2\mu}, \tag{13}\]
thus the density distribution in the string cloud remains arbitrary. The behavior of \(g_{00} = e^{2\gamma}\) indicates singularities on both ends, \(x \to 0\) and \(x \to \infty\). Furthermore, the metric component \(g_{33}\) is
\[
g_{33} = -e^{2\beta} + e^{-2\gamma} E^2 = \frac{r_0 E_0 (E_0 x - 2)}{2|\omega_0|}. \tag{14}\]
It means that if \(E_0 > 0\), \(g_{33} < 0\), and the spatial section \(t = \text{const}\) of the manifold (12) has a normal signature only in the range \(0 < x < 2/E_0\), while at \(x > 2/E_0\) we have \(g_{33} > 0\), the closed coordinate lines of \(\varphi\) are timelike, violating the causality principle. If \(E_0 < 0\), \(g_{33} > 0\) in the whole range \(x \in \mathbb{R}_+\). If \(E_0 = 0\), then \(g_{33} = 0\), making the spatial sections \(t = \text{const}\) degenerate. The metric as a whole is certainly not degenerate, and other spatial sections can have a normal signature.

The corresponding static solution is obtained from the same equations with \(\omega_0 = 0\). They lead to \(\beta' = \gamma' = 0\) and the same relation (13) for \(\mu\) and \(\rho\). Without loss of generality, the metric can be written as
\[
ds^2 = dt^2 - e^{2\mu(x)}(r_0^2 dx^2 + dz^2) - r_0^2 d\varphi^2. \tag{15}\]
It describes a congruence of cylinders of equal radii but an \(x\)-dependent scale along the \(z\) axis, with a distribution of closed strings as \(\varphi\)-circles with an \(x\)-dependent density profile.

Longitudinal strings: \((w_1, w_2, w_3) = (0, -1, 0)\). For such strings stretched along the \(z\) axis, the conservation law (6) leads to \(\gamma' = -\mu'\), hence \(\mu'' = -\gamma''\). Comparing the expressions for \(\mu''\) from (9) and for \(\gamma''\) from (5), we see that
\[
\mu'' = -\kappa \rho e^{2\alpha} = -\gamma'' = 2\omega_0^2 e^{2\beta - 2\gamma}, \tag{16}\]
we see that \(\rho < 0\). Thus we inevitably obtain a negative energy density and do not consider this case any more.

For static systems we again obtain (16) but now with \(\omega_0 = 0\), hence \(\rho \equiv 0\), vacuum. Thus static cylindrically symmetric distributions of longitudinal cosmic strings do not exist, while stationary ones require negative energy density.

Radial strings: \((w_1, w_2, w_3) = (-1, 0, 0)\). It is a cloud of radial strings which, imagined around a symmetry axis in flat space, would resemble a barbed wire. In curved space there can be a geometry without a symmetry axis, e.g., that of a wormhole. Let us consider the corresponding equations.
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The conservation law (13) leads to $\rho = \rho_0 e^{-\mu - \beta}$. Using it in Eqs. (8)–(11) leads to $\mu'' = \beta'' + \gamma''$ (hence we can put $\mu = \beta + \gamma + mx$, $m = \text{const}$) and

\begin{align}
\gamma'' &= -2\omega_0^2 e^{2\beta - 2\gamma}, \\
\beta'' &= 2\omega_0^2 e^{2\beta - 2\gamma} - \kappa\rho_0 e^{2\gamma + \mu + \beta}, \\
\beta'\gamma' + (\beta' + \gamma')(\beta' + \gamma' + m) &= -\omega_0^2 e^{2\beta - 2\gamma} - \kappa\rho_0 e^{2\gamma + \mu + \beta},
\end{align}
(17) (18) (19)

It is hard to solve these equations analytically, but some qualitative observations can be made. In particular, wormhole solutions (implying a regular minimum of $\beta(x)$) are not forbidden in general, but if $m = 0$, any regular extremum of $\beta(x)$ or $\gamma(x)$ is incompatible with $\rho > 0$ due to negativity of the r.h.s. in (19). Wormholes, if any, require $m \neq 0$ and are asymmetric under the change $x \to -x$.

Static solutions ($\omega_0 = 0$) are easily found from (17)–(19). We have $\gamma'' = 0 \Rightarrow \gamma = hx$ ($h = \text{const}$, a $t$-scale is chosen), and $\beta$ is derived from (19). The resulting solutions consist of two branches. In the first one, the metric and the density are

\begin{align}
ds^2 &= e^{2hx}dt^2 - \frac{k^2}{\kappa\rho_0^2 \cosh^2 kx} e^{-2hx} dx^2 - \frac{k^2}{\cosh^2 kx} \left( \frac{e^{2mx}}{r_0} dz^2 + r_0 e^{-2mx} d\varphi^2 \right), \\
\rho &= \kappa\rho_0^2 e^{2hx} \cosh^2 kx, \quad k = \sqrt{h(h + 2m)} \geq 0, \quad h(h + 2m) \geq 0.
\end{align}
(20)

In the second branch,

\begin{align}
ds^2 &= e^{2hx}dt^2 - \rho_0^2 e^{-2hx} dx^2 - e^{2(m-h)x} dz^2 - \rho_0^2 e^{-2(m+h)x} d\varphi^2, \\
\rho &= (\rho_0/r_0) e^{2hx}, \quad \kappa\rho_0 r_0 = h(h + 2m).
\end{align}
(21)

In both branches the range of $x$ is $x \in \mathbb{R}$. The metrics (20) and (21) do not describe wormholes at any values of the parameters; the absence of a minimum of $\beta(x)$ is already clear from Eq. (18) since $\beta'' < 0$ if $\omega_0 = 0$. A regular axis corresponding to $e^\beta = 0$ is also impossible, which is natural since, in non-wormhole space-time, radial strings must begin from a source on a symmetry axis.

4. Radiation flows

A null radiation flow with intensity $\Phi$ has the SET $T_{\mu\nu}^\nu = \Phi k_\mu k_\nu$, where $k_\mu$ is a null vector in a given spatial direction. If there are two such flows of equal intensity in mutually opposite directions, say, $\pm x^1$, their SETs add into $T_{\mu\nu}^\nu = \rho \text{diag}(1, -1, 0, 0)$ ($\rho = 2\Phi$), describing a special case of anisotropic fluid (21) (22). Let us consider such flows in different directions as sources of gravity in Eqs. (8)–(11).

Asimutual flows: $(w_1, w_2, w_3) = (0, 0, 1)$. The conservation law leads to $\beta' = \gamma'$, while Eqs. (8)–(11) yield $\beta'' + \gamma'' = \mu'' = 0$, and without loss of generality we have

\begin{align}
\beta &= bx + \ln r_0, \quad \gamma = bx, \quad \mu = mx, \quad m = -\frac{b^2 + \omega_0^2 r_0^2}{2b},
\end{align}
(22)
where \( b = \text{const.} \), and the last equality follows from (11). The integral in (2) is easily found, while \( \rho \) is obtained from (3), so that finally
\[
\begin{align*}
\kappa \rho &= 2\omega_0^2 e^{-2ax}, \\
\kappa &= \frac{3b^2 - \omega_0^2 r_0^2}{2b}, \\
E_0 &= \text{const.}
\end{align*}
\]
(23)
The solution is defined for \( x \in \mathbb{R} \). At \( a = 0 \) (that is, \( 3b^2 = \omega_0^2 r_0^2 \)), we obtain \( \rho = \text{const.} \), a homogeneous density. In other cases \( \rho \to \infty \) as either \( x \to \infty \) or \( x \to -\infty \). If (say) \( b > 0 \), \( x \to \infty \) is spatial infinity and \( x \to -\infty \) the axis. A feature of interest is that if \( 3b^2 > \omega_0^2 r_0^2 \) (that is, \( a > 0 \)), then \( \rho \) infinitely grows at spatial infinity. Lastly, a calculation of \( g_{33} \) shows that \( \varphi \)-circles are inevitably timelike at large \( x \), e.g., at \( |x| > 1/(2|\omega_0|r_0) \) if \( E_0 = 0 \).

In the static case \( \omega_0 = 0 \), we obtain \( \rho = 0 \), hence the vacuum solution (11)

**Longitudinal flows:** \((w_1, w_2, w_3) = (0, 1, 0).\) For this case, the solution is already known (22) and consists of two branches, one with a constant \( b > 0 \), the other with \( b = 0 \). The metric reads
\[
\begin{align*}
ds^2 &= (e^{2bx} - 1)^{2/3} \left[ dt - 2\omega_0 (E_0 - b \coth bx) d\varphi \right]^2 - r_0^2 b^{4/3} e^{2bx} dx^2 \\
&\quad - \left( e^{2bx} - 1 \right)^{2/3} dz^2 - r_0^2 \left( \frac{b^2 e^{bx}}{\sinh bx} \right)^{2/3} d\varphi^2
\end{align*}
\]
(24)
for \( b > 0 \), while for \( b = 0 \) we have
\[
\begin{align*}
ds^2 &= x^{2/3} \left[ dt - \frac{2\omega_0}{x} (E_0 x - 1) d\varphi \right]^2 - r_0^2 dx^2 - x^{2/3} dz^2 - r_0^2 x^{4/3} d\varphi^2
\end{align*}
\]
(25)
where \( E_0 = \text{const.} \). For the density we obtain
\[
\kappa \rho_0^2 = \frac{8b^2}{3(e^{2bx} - 1)^2} \quad \text{for} \ b > 0, \quad \kappa \rho_0^2 = 2/(3x^2) \quad \text{for} \ b = 0.
\]
(26)
The solutions are defined for \( x \in \mathbb{R}_+ \). In all cases the value \( x = 0 \) corresponds to radii \( r = e^\beta \to \infty \) which is a singularity due to \( \rho \to \infty \). Other features depend on \( b \) and \( E_0 \). Closed timelike curves are observed at large \( x \) if \( b = 0 \), \( E_0 \neq 0 \).

In the static case, the equations lead again to \( \rho = 0 \), the vacuum solution.

**Radial flows:** \((w_1, w_2, w_3) = (1, 0, 0).\) The conservation law (6) leads to \( \rho = \rho_0 e^{-\beta - 2\gamma - m}, \rho_0 = \text{const.} \). Equation (6) reads \( \mu'' = 0 \Rightarrow \mu = mx, \ m = \text{const.} \). For \( \gamma(x) \) and \( \beta(x) \) we then obtain the equations
\[
\begin{align*}
\gamma'' &= -2\omega_0^2 e^{2\beta - 2\gamma} + \kappa \rho_0 e^{\beta + mx}, \\
\beta'' &= 2\omega_0^2 e^{2\beta - 2\gamma}.
\end{align*}
\]
(27)
It is hard to find their general solution, but a special one can be found by assuming \( e^{\beta - 2\gamma} = e^{mx}/r_0, \ r_0 = \text{const.} \). With this ansatz Eqs. (27) lead to
\[
\kappa \rho_0 = 3b^2 \omega_0^2, \quad \text{and} \quad \beta'' - \gamma'' = \omega_0^2 e^{2\beta - 2\gamma}
\]
(28)
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which is easily solved. Substituting the solution to (2) to find \( E(x) \) and taking into account the constraint (11), we finally obtain for \( m \neq 0 \)

\[
ds^2 = \frac{k^2 e^{-2mx}}{r_0^2 \omega^2 \sinh^2 kx} \left[ dt + \frac{2}{\omega}(k \coth kx - E_0) d\varphi \right]^2 - \frac{k^6 e^{-2mx}}{r_0^4 \omega^6 \sinh^6 kx} dx^2 - e^{2mx} dz^2 - \frac{k^4 e^{-2mx}}{r_0^2 \omega^4 \sinh^4 kx} d\varphi^2, \tag{29}
\]

where \( k = |m|/\sqrt{2} \) and \( E_0 \) are constants. For \( m = 0 \) we arrive at

\[
ds^2 = \frac{1}{r_0^2 \omega^2 x^2} \left[ dt + \frac{2}{\omega_0 x}(1 - E_0 x) d\varphi \right]^2 - \frac{dx^2}{r_0^2 \omega^2 x^6} - dz^2 - \frac{d\varphi^2}{r_0^2 \omega^4 x^4}; \tag{30}
\]

the range of \( x \) in both (29) and (30) is \( x > 0 \). For the density we have

\[
\rho = \frac{\rho_0}{k^4 \omega^4 x^3} e^{2mx} \sinh^4 kx \quad \text{for} \quad m \neq 0, \quad \rho = \rho_0 \omega^4 r_0^4 x^4 \quad \text{for} \quad m = 0. \tag{31}
\]

In all cases there exist timelike \( \varphi \)-circles at sufficiently large \( x \), and \( x \to \infty \) is a singularity where \( e^\beta \to 0 \) and \( \rho \to \infty \).

For static configurations \( (\omega = 0) \) we have \( \rho = \rho_0 e^{-\beta - 2\gamma - \mu} \) and \( \mu'' = \beta'' = 0 \Rightarrow \mu = mx, \beta = bx + \ln r_0, \quad m, b, r_0 = \text{const} \). From (11) we obtain an expression for \( \gamma(x) \), and the solution takes the form

\[
ds^2 = e^{2\gamma} dt^2 - \frac{r_0^2}{\omega^2} e^{2\gamma + 2(m+b)x} dx^2 - e^{2mx} dz^2 - \frac{r_0^2}{\omega^4} e^{2bx} d\varphi^2, \]

\[
\rho = \frac{\rho_0}{r_0} e^{-2\gamma -(b+m)x}, \quad e^\gamma = \exp \left[ \frac{-\rho_0}{(m+b)^2} e^{(m+b)x} - \frac{mbx}{m+b} \right], \tag{32}
\]

where \( b + m \neq 0 \). The properties of this simple solution depend on the values of \( b \) and \( m \) and are evident from the expressions (32).

5. Concluding remarks

1. Among new exact solutions obtained here, with such sources of gravity as clouds of cosmic strings and pairs of mutually opposite radiation flows, there are no nonsingular ones, and stationary solutions contain regions with closed timelike curves. It evidently means that not these global solutions but only their physically acceptable regions can be used in modeling some real situations in the nature.

2. It has been shown that stationary solutions with \( z \)-directed strings require \( \rho < 0 \), while static solutions with \( z \)- or \( \varphi \)-directed radiation flows do not exist. On the contrary, \( \varphi \)-directed cosmic strings admit arbitrary density distributions.

3. String distributions and radiation flows in \( z \) and \( \varphi \) directions possess zero radial pressure \( p_r \), making it possible to match the corresponding solutions to external vacuum ones, with or without rotation, on any hypersurface \( x = \text{const} \), without need for surface densities and tensions.

4. Among the exact solutions under consideration, there are no wormhole ones. Throats as minima of \( r(x) = e^\beta(x) \) are possible in the cases of Eqs. (17), (19) and (27) (to be solved numerically), and only solutions to (27) can be symmetric with respect to the throat. In the author’s view, these cases deserve a separate study.
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