Tarski’s Undefinability Theorem and Diagonal Lemma

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Abstract

We prove the equivalence of the semantic version of Tarski’s theorem on the undefinability of truth with a semantic version of the Diagonal Lemma, and also show the equivalence of syntactic Tarski’s Undefinability Theorem with a weak syntactic diagonal lemma. We outline two seemingly diagonal-free proofs for these theorems from the literature, and show that syntactic Tarski’s theorem can deliver Gödel-Rosser’s Incompleteness Theorem.

Keywords: Diagonal Lemma, Diagonal-Free Proofs, Gödel’s Incompleteness Theorem, Rosser’s Theorem, Self-Reference, Tarski’s Undefinability Theorem.

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1 Introduction

ONE OF THE CORNERSTONES of modern logic (and theory of incompleteness after Gödel) is the Diagonal Lemma (aka Self-Reference, or Fixed-Point Lemma) due to Gödel and Carnap (see [10] and the references therein). The lemma states that (when \( \alpha \mapsto \lceil \alpha \rceil \) is a suitable Gödel coding which assigns the closed term \( \lceil \alpha \rceil \) to a syntactic expression or object \( \alpha \)) for a given formula \( \Psi(x) \) with the only free variable \( x \), there exists some sentence \( \theta \) such that the equivalence \( \Psi(\lceil \theta \rceil) \leftrightarrow \theta \) holds; “holding” could mean either being true in the standard model of natural numbers \( \mathbb{N} \) or being provable in a suitable theory \( T \) (which is usually taken to be a consistent extension of Robinson’s arithmetic). When the equivalence \( \Psi(\lceil \theta \rceil) \leftrightarrow \theta \) holds in \( \mathbb{N} \) we call it the Semantic Diagonal Lemma (studied in Section 2); when \( T \) proves the equivalence, we call it the Syntactic Diagonal Lemma. The Weak Diagonal Lemma states the consistency of the sentence \( \Psi(\lceil \theta \rceil) \leftrightarrow \theta \) with \( T \), for some sentence \( \theta \) which depends on the given arbitrary formula \( \Psi(x) \) and the theory \( T \) (studied in Section 3).

The Diagonal Lemma has been used in proving many fundamental theorems of mathematical logic, such as Gödel’s First and (also) Second Incompleteness Theorems, Rosser’s (strengthening of Gödel’s Incompleteness) Theorem, and Tarski’s Theorem (on the Undefinability of Truth). One problem with the Diagonal Lemma is its standard proof which is a kind of magic (or “pulling a rabbit out of the hat”, see e.g. [15]); indeed it is not easy to remember its typical proof, even after several years of teaching it. Here, we quote some texts on the proof of this lemma from the literature:

(1998) S. Buss writes in [1] that the proof of the Diagonal Lemma is “quite simple but rather tricky and difficult to conceptualize.”

(2002) V. McGee states in [8] that by the diagonal (aka self-referential) lemma there exists a sentence \( \phi \) for a given formula \( \Psi(x) \) such that \( \Psi(\lceil \phi \rceil) \leftrightarrow \phi \) is provable in Robinson’s arithmetic. “You would hope that such a deep theorem would have an insightful proof. No such luck. I am going to write down a sentence \( \phi \) and verify that it works. What I won’t do is give you a satisfactory explanation for why I write down the particular formula I do. I write down the formula because Gödel wrote down the formula,
and Gödel wrote down the formula because, when he played the logic game he was able to see seven or eight moves ahead, whereas you and I are only able to see one or two moves ahead. I don’t know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma works. It has a rabbit-out-of-a-hat quality for everyone.”

(2004) H. Kotlarski [7] said that the diagonal lemma “being very intuitive in the natural language, is highly unintuitive in formal theories like Peano arithmetic. In fact, the usual proof of the diagonal lemma […] is short, but tricky and difficult to conceptualize. The problem was to eliminate this lemma from proofs of Gödel’s result.”

(2006) G. Serény [12] attempts to make “the proof of the lemma completely transparent by showing that it is simply a straightforward translation of the Grelling paradox into first-order arithmetic.”

(2006) H. Gaifman mentions in [3] that the proof of the Diagonal Lemma is “extremely short”. However, the “brevity of the proof does not make for transparency; it has the aura of a magician’s trick.”

In this paper, we attempt at giving some explanations and motivations for this basic lemma, in a way that we will have a satisfactory understanding for at least some weaker versions of it. For that purpose, we will first see the equivalence of the semantic form of the diagonal lemma with Tarski’s theorem on the undefinability of arithmetical truth (in Section 2). In other words, the diagonal lemma holds in the standard model of natural numbers just because the set of (the Gödel codes of) the true arithmetical sentences is not definable. As a matter of fact, different proofs for Tarski’s undefinability theorem can lead to different proofs for the semantic version of this lemma. We will review two such proofs (presented in [2, 5, 6, 7, 11]) which are supposedly diagonal-free. Having different proofs will, hopefully, shed some new light on the nature of this lemma and will increase our understanding about it. Then, secondly, we will see that a syntactic version of Tarski’s theorem is equivalent to a weak (syntactic) version of the diagonal lemma (in Section 3). This weak form of the diagonal lemma is still sufficiently strong to prove Gödel-Rosser’s incompleteness theorem. So, different proofs of the syntactic version of Tarski’s theorem will provide some seemingly diagonal-free proofs for Rosser’s theorem (cf. [14], in which Gödel’s second incompleteness theorem is derived from Tarski’s undefinability theorem by some circular-free arguments).

2 The Diagonal Lemma, Semantically

Definition 2.1 (Semantic Diagonal Lemma):
The following statement is called the Semantic Diagonal Lemma:

\[ \text{For every formula } \Psi(x) \text{ there exists a sentence } \theta \text{ such that } \mathbb{N} \models \Psi(\langle\theta\rangle) \iff \theta. \]

This (weaker) form of the Diagonal Lemma serves to prove the semantic version of Gödel’s Incompleteness Theorem (see [13] Theorem 6.3):

Theorem 2.2 (Gödel’s Incompleteness Theorem for Sound Definable Theories):
For every definable and sound theory \( T \) there exists a true sentence independent from \( T \).

Proof:
If \( T \) is definable then there exists a formula \( \Pr_T(x) \) such that for every sentence \( \eta \) we have \( T \vdash \eta \) if and only if \( \mathbb{N} \models \Pr_T(\langle\eta\rangle) \). Now, by the Semantic Diagonal Lemma we have \( \mathbb{N} \models \gamma \leftrightarrow \neg\Pr_T(\langle\gamma\rangle) \) for some sentence \( \gamma \). It can be seen that \( T \not\vdash \gamma \), since \( T \vdash \gamma \) implies on the one hand that \( \mathbb{N} \models \Pr_T(\langle\gamma\rangle) \), and on the other hand (by the soundness of \( T \)) that \( \mathbb{N} \models \gamma \) and so \( \mathbb{N} \models \neg\Pr_T(\langle\gamma\rangle) \), a contradiction. So, \( T \not\vdash \gamma \), therefore \( \mathbb{N} \models \neg\Pr_T(\langle\gamma\rangle) \), whence \( \mathbb{N} \models \gamma \), which also implies (by the soundness of \( T \)) that \( T \not\vdash \neg\gamma \).
Also, Tarski’s Theorem on the Undefinability of (arithmetical) Truth follows from the Semantic Diagonal Lemma (see [4, Exercise 3.7] and cf. [4, Chapter 9]):

**Theorem 2.3** (Tarski’s Theorem on the Undefinability of Arithmetical Truth):
The Gödel codes of the set of true sentences, i.e. \( \{ \langle \eta \rangle \mid \mathbb{N} \models \eta \} \), is not definable in \( \mathbb{N} \).

**Proof:**
If \( \{ \langle \eta \rangle \mid \mathbb{N} \models \eta \} \) is definable by some \( \Gamma(x) \), then \( \mathbb{N} \models \Gamma(\langle \theta \rangle) \iff \theta \) holds for every sentence \( \theta \). Now, by the Semantic Diagonal Lemma \( \mathbb{N} \models \neg \Gamma(\langle \lambda \rangle) \iff \lambda \) holds for a sentence \( \lambda \); so we have \( \mathbb{N} \models \lambda \iff \Gamma(\langle \lambda \rangle) \iff \neg \lambda \), which is a contradiction. \[Q.E.D.\]

The fact of the matter is that the Semantic Diagonal Lemma is equivalent to Tarski’s Theorem on the Undefinability of Truth and to Semantic Incompleteness Theorem of Gödel:

**Theorem 2.4** (Semantic Diagonal Lemma \( \iff \) Semantic Gödel’s Incompleteness Theorem \( \iff \) Tarski’s Theorem):
The following statements are equivalent:

1. Semantic Diagonal Lemma (Definition 2.1);
2. Semantic Gödel’s Incompleteness Theorem (2.2);
3. Semantic Tarski’s Undefinability Theorem (2.3).

**Proof:**
The implication (1 \( \implies \) 2) is proved in Theorem 2.2 (and 1 \( \implies \) 3 is proved in Theorem 2.3).

(2 \( \implies \) 3): If \( \text{Th}(\mathbb{N}) = \{ \eta \mid \mathbb{N} \models \eta \} \) were definable, then since it is sound, there would be some sentence independent from it (by 2); but it is a complete theory.

(3 \( \implies \) 1): Suppose that the set \( \{ \langle \theta \rangle \mid \mathbb{N} \models \theta \} \) is not definable by any formula. Then for a given formula \( \Psi(x) \) the formula \( \neg \Psi(x) \) cannot define this set, and so we cannot have \( \mathbb{N} \models \neg \Psi(\langle \beta \rangle) \iff \mathbb{N} \models \beta \), for all sentences \( \beta \); whence there should exists some sentence \( \theta \) such that \( \mathbb{N} \not\models \neg \Psi(\langle \theta \rangle) \iff \theta \). Now, by the classical propositional tautology \( \neg(p \iff q) \equiv (\neg p \iff q) \), we have \( \mathbb{N} \models \Psi(\langle \theta \rangle) \iff \theta \). So, for every \( \Psi(x) \) there exists some \( \theta \) for which the equivalence \( \Psi(\langle \theta \rangle) \iff \theta \) holds in \( \mathbb{N} \). \[Q.E.D.\]

So, after all, Tarski’s Undefinability Theorem (in its semantic form) is not very much different from the Diagonal Lemma (in the semantic form). Therefore, it may seem at the first glance that the only way to prove Tarski’s theorem is to use the Diagonal Lemma (as is done in almost all the textbooks). But as a matter of fact, there are some, supposedly, diagonal-free proofs for Tarski’s theorem in the literature (see e.g. [5]) which by Theorem 2.4 can give us some diagonal-free proofs for the Diagonal Lemma itself! We will outline two of them below.

Assume that \( \Upsilon(x) \) defines truth in \( \mathbb{N} \); i.e., \( \mathbb{N} \models \Upsilon(\langle \beta \rangle) \iff \beta \) for all sentences \( \beta \). \[\text{(C)}\]

### 2.1 The First Proof

**Convention:** Let us make the convention that all the individual variables of our syntax are \( x, x', x'', x''', \cdots \) whose lengths are 1, 2, 3, 4, \( \cdots \), respectively. By this convention, there will be at most finitely many formulas with length \( n \) for a given \( n \in \mathbb{N} \) (otherwise the formulas \( x = x, y = y, z = z, \cdots \) all would have length three).

**Definition 2.5** (length \( \text{len}(x) \), \( \pi \), definability, \( D(x) \), \( \text{Def}^z_T(y), \text{Berry}^{x''}_T(u), \ell_T, B_T(x) \)):
• Let \( \text{len}(x) \) denote the length of the formula with Gödel code \( x \).

• For \( n \in \mathbb{N} \), let \( \mathfrak{F} \) be the term that represents the number \( n \), i.e., \( 0 = 0, \mathfrak{I} = 1 \), and for every \( m \geq 1 \) we have \( m + \mathfrak{F} = 1 + (\mathfrak{m}) \).

• We say that a number \( n \in \mathbb{N} \) is definable by the formula \( \varphi(x) \), in which \( x \) is the only free variable, when \( \forall \zeta[\varphi(\zeta) \leftrightarrow \zeta = m] \) is true (in \( \mathbb{N} \)).

• Let \( D(x, y) \) be the Gödel code of the formula which states that the formula with Gödel code \( x \) defines the number \( y \); so, \( D(\text{\textsuperscript{\text{\textup{\overline{\varphi}}}}, y) = \forall \zeta[\varphi(\zeta) \leftrightarrow \zeta = y] \).

• Let \( \text{Def}^{\prec\zeta}(y) \) be the formula \( \exists \alpha (\text{Formula}(\alpha) \land \text{len}(\alpha) < z \land \mathfrak{Y}[D(\alpha, y)]) \) which states that the number \( y \) is definable by a formula with length less than \( z \) if \( \mathfrak{Y} \) is a truth predicate; needless to say, Formula(\alpha) states that \( \alpha \) is the Gödel code of a formula.

• Let \( \text{Berry}^{\prec\zeta}(u) \) be the formula \( \neg \text{Def}^{\prec\zeta}(u) \land \forall w < u \text{Def}^{\prec\zeta}(w) \), which states that \( u \) is the least number not defined by a formula with length less than \( v \).

• Let \( \ell_\mathfrak{Y} \) be the length of the formula \( \text{Berry}^{\prec\zeta}(x) \).

• Let \( B_\mathfrak{Y}(x) \) be the formula \( \exists x'[x' = \overline{\mathfrak{Y}} \land \text{Berry}^{\prec\zeta}(x)] \).

Here is an alternative proof (from [2, 7, 11]) for contradicting (C):

**Proof:**

The length of \( B_\mathfrak{Y}(x) \) is less than \( 6\ell_\mathfrak{Y} \); since it can be seen to be equal to \( 10 + 1\text{en}(5) + 1\text{en}(\ell_\mathfrak{Y}) + \ell_\mathfrak{Y} = 24 + 5\ell_\mathfrak{Y} \), as we have \( 1\text{en}(m) = 4m - 3 \) for every \( m \geq 1 \). So, the formula \( B_\mathfrak{Y}(x) \) with length less than \( 6\ell_\mathfrak{Y} \) states that \( x \) is the least number that is not definable by any formula with length less than \( 6\ell_\mathfrak{Y} \). Whence, if \( B_\mathfrak{Y}(x) \) holds, then \( x \) should not be definable by \( B_\mathfrak{Y}(x) \) itself. But this is a contradiction, since if \( B_\mathfrak{Y}(x) \) holds, then \( x \) is definable by \( B_\mathfrak{Y}(\zeta) \). That is because \( B_\mathfrak{Y}(x) \) implies \( \forall \zeta[B_\mathfrak{Y}(\zeta) \leftrightarrow \zeta = x] \) by the sentence \( \forall u, v[B_\mathfrak{Y}(u) \land B_\mathfrak{Y}(v) \rightarrow u = v] \), which follows in turn from the sentence \( \forall u, v, w[\text{Berry}^{\prec\zeta}(u) \land \text{Berry}^{\prec\zeta}(v) \rightarrow u = v] \) that can be proved from the basic laws of the order relation. So, for no \( x \) can \( B_\mathfrak{Y}(x) \) hold. Now, in reality, there exists a number \( b \in \mathbb{N} \) that is not definable by any formula of length less than \( 6\ell_\mathfrak{Y} \) (since by our convention there are only finitely many formulas with length less than \( 6\ell_\mathfrak{Y} \)). So, \( \neg \text{Def}^{\prec\zeta}(b) \) is true, and since it is the least such number then \( \forall w < \overline{b} \text{Def}^{\prec\zeta}(w) \) is true too. Thus, \( \text{Berry}^{\prec\zeta}(b) \) is true; and so is \( B_\mathfrak{Y}(b) \), which is a contradiction. \( \square \)

This proof of Tarski’s theorem \( \text{[2, 3]} \) does not use the Diagonal Lemma (and so it can be called diagonal-free in a way), though it can be debated whether the proof is genuinely circular-free or not. By incorporating the proof of Theorem \( \text{[2, 4]} \) into this proof (e.g. by taking \( \mathfrak{Y} \equiv \neg \mathfrak{W} \)), one can get a proof for the Semantic Diagonal Lemma, which is different from the standard (textbook) proofs (see [10]).

### 2.2 The Second Proof

**Definition 2.6 (Definable and Dominating Functions):**

A function \( f: \mathbb{N} \rightarrow \mathbb{N} \) is called definable whenever there exists a formula \( \varphi(u, v) \) such that for every \( m, n \in \mathbb{N} \) we have \( f(m) = n \iff \mathbb{N} \models \varphi(\overline{m}, \overline{n}) \).

A function \( F: \mathbb{N} \rightarrow \mathbb{N} \) is said to dominate a function \( f: \mathbb{N} \rightarrow \mathbb{N} \), whenever there exists some \( n \in \mathbb{N} \) such that \( F(x) > f(x) \) holds for all \( x \geq n \). \( \square \)
Indeed, for a given countably indexed family of functions \( \{f_i : \mathbb{N} \to \mathbb{N}\}_{i \in \mathbb{N}} \) one can find a function that dominates all the functions of this family: put

\[
F(x) = 1 + \max_{i \leq x} f_i(x);
\]

then for every \( k \in \mathbb{N} \) and every \( x \geq k \), \( f_k(x) \leq \max_{i \leq x} f_i(x) < [1 + \max_{i \leq x} f_i(x)] = F(x) \) holds. This idea is used in the following proof of Tarski’s theorem (2.3); cf. [6, 7]:

**Proof:**

Define the function \( F : \mathbb{N} \to \mathbb{N} \) as

\[
F(x) = \min\{y \mid \forall \alpha \leq x[\exists z \alpha(x, z) \to \exists z < y \alpha(x, z)]\}.
\]

We show that the function \( F \) dominates every definable function, but is itself definable if (\( \Box \)) holds; and this is a contradiction (since no function can dominate itself). To see that \( F \) dominates the family of all definable functions, assume that a function \( f : \mathbb{N} \to \mathbb{N} \) is definable by a formula \( \varphi(u, v) \). Now, for every \( m \geq [\varphi] \) we show that \( F(m) > f(m) \) holds: if \( F(m) \leq f(m) \), then from \( \varphi(m, f(m)) \) we have \( \exists z \varphi(m, z) \) and so \( \exists z < F(m) : \varphi(m, z) \) by the definition of \( F \), which implies \( \exists z < f(m) : \varphi(m, z) \) by the assumption \( F(m) \leq f(m) \); but for every \( k \neq f(m) \) we have \( \neg \varphi(m, k) \), and so \( \exists z < f(m) : \neg \varphi(m, z) \), a contradiction. Now, if (\( \Box \)) holds for \( \Upsilon \), then \( F \) is actually definable by \( \psi(u, v) \land \forall w < v \neg \psi(u, w) \) where \( \psi(u, v) \) is the formula \( \forall \alpha \leq u[\exists z \Upsilon(\varphi(u, z)) \to \exists z < v \Upsilon(\varphi(u, z))] \).

As a matter of fact, the function \( F \) used by Kotlarski [6, 7] is defined as

\[
F(x) = \min\{y \mid \forall \alpha, u \leq x[\exists z \alpha(u, z) \to \exists z < y \alpha(u, z)]\}
\]

which corresponds to \( F(x) = 1 + \max_{i,j \leq x} f_i(j) \) that dominates \( \{f_i : \mathbb{N} \to \mathbb{N}\}_{i \in \mathbb{N}} \).

### 3 The Diagonal Lemma, Syntactically

The Diagonal Lemma is usually stated as the provability of the equivalence \( \Psi(\vartheta) \leftrightarrow \vartheta \) in a theory like Robinson’s arithmetic, for some sentence \( \vartheta \) which depends on the given formula \( \Psi(x) \). Let us call this the *Syntactic Diagonal Lemma*. A syntactic version of Tarski’s theorem on the undefinability of truth is as follows:

**Definition 3.1 (Syntactic Tarski’s Theorem):**

For a formula \( \Psi(x) \), let \( \text{TB}^\Psi \) be the set of all truth biconditionals \( \Psi(\vartheta) \leftrightarrow \beta \), where \( \beta \) ranges over all the sentences. That is \( \text{TB}^\Psi = \{\Psi(\vartheta) \leftrightarrow \beta \mid \beta \text{ is a sentence}\} \).

The following statement is called the *Syntactic Tarski’s Theorem* on a consistent \( T \):

For every \( \Psi(x) \) we have \( T \not\vdash \text{TB}^\Psi \).

**Definition 3.2 (Weak Diagonal Lemma):**

The following statement is called the *Weak Diagonal Lemma* about a consistent theory \( T \):

For every \( \Psi(x) \) there exists a sentence \( \vartheta \) such that \( \Psi(\vartheta) \leftrightarrow \vartheta \) is consistent with \( T \).

We show that Syntactic Tarski’s Theorem is equivalent to Weak (Syntactic) Diagonal Lemma.

**Theorem 3.3 (Weak Diagonal Lemma \( \iff \) Syntactic Tarski’s Theorem):**

The Weak Diagonal Lemma is equivalent to Syntactic Tarski’s Theorem.
PROOF:
First, suppose that the Weak Diagonal Lemma holds for a consistent theory $T$. Take any formula $\Psi(x)$; we show that $T \nvdash \text{TB}^\Psi$. There exists a sentence $\theta$ such that the theory $T$ is consistent with $\neg \Psi(\neg \theta) \leftrightarrow \theta$. Thus, $T \nvdash \Psi(\neg \theta) \leftrightarrow \theta$ and so $T \nvdash \text{TB}^\Psi$.

Second, suppose that $T \nvdash \text{TB}^\Psi$ for all formulas $\Psi(x)$. Take any formula $\Psi(x)$; we show the existence of some $\theta$ such that $T$ is consistent with $\Psi(\neg \theta) \leftrightarrow \theta$. Since $T \nvdash \text{TB}^\Psi$, there should exist some sentence $\theta$ such that $T \nvdash \neg \Psi(\neg \theta) \leftrightarrow \theta$. Therefore, $T$ is consistent with the sentence $\Psi(\neg \theta) \leftrightarrow \theta$.

As a matter of fact, the Weak Diagonal Lemma cannot show the independence of the Gödel sentence (when the theory is sound even):

REMARK 3.4 (Weak Diagonal Lemma vs. Gödel’s Proof):
For a consistent and recursively enumerable theory $T$ extending Robinson’s arithmetic, the consistency of $\neg \text{Pr}_T(\neg \theta)$ implies that $\theta$ is unprovable in $T$, but does not imply that $\theta$ is independent from $T$, even if $T$ is $\omega$-consistent:

1. If $T \vdash \theta$ then $T \vdash \text{Pr}_T(\neg \theta)$, and so $T + [\neg \text{Pr}_T(\neg \theta) \leftrightarrow \theta] \vdash \neg \theta$, therefore $T + [\neg \text{Pr}_T(\neg \theta) \leftrightarrow \theta]$ cannot be consistent.

2. For a contradictory sentence like $\delta = (0 \neq 0)$, the sentence $\neg \text{Pr}_T(\neg \delta)$ is consistent with $T$ (by Gödel’s Second Incompleteness Theorem), but $\delta$ is not independent from $T$ (as $T$ proves its negation).

It is stated in [9, p. 202] that for every sentence $\sigma$, $T \vdash \sigma \iff T \vdash \neg \text{Pr}_T(\neg \sigma)$ if $T$ is sound. Unfortunately, this is not true since for e.g. $\sigma = (0 \neq 0)$ we do have that $T \vdash \sigma \iff T \vdash \neg \text{Pr}_T(\neg \sigma)$ by Gödel’s second incompleteness theorem, but trivially $T \vdash \neg \sigma$ holds. If we replace $\text{Pr}_T(x)$ with Rosser’s provability predicate $\text{RPr}_T(x)$, then it is true that for every $\varrho$ that satisfies $T \vdash \varrho \iff T \vdash \neg \text{RPr}_T(\neg \varrho)$ we have $T \nvdash \varrho, \neg \varrho$ if $T$ is (only) consistent; see the next theorem.

However, the Weak Diagonal Lemma is sufficiently strong to prove Rosser’s theorem:

THEOREM 3.5 (Weak Diagonal Lemma $\implies$ Rosser’s Theorem):
If the Weak Diagonal Lemma holds for a consistent and recursively enumerable theory that extends Robinson’s arithmetic, then there exists a sentences which is independent from that theory.

PROOF:
For such a theory $T$, suppose that $\text{prf}_T(x, y)$ is its proof predicate (stating that $x$ is the Gödel code of a proof of the sentence with Gödel code $y$ in $T$). By the Weak Diagonal Lemma there exists a sentence $\varrho$ such that the following theory is consistent:

$$U = T + \left( \forall x [\text{prf}_T(x, \neg \varrho) \rightarrow \exists y < x \text{prf}_T(y, \neg \neg \varrho)] \leftrightarrow \varrho \right).$$

The standard proof of Rosser’s theorem can show that $\varrho$ is independent from $T$:

- If $T \vdash \varrho$, then $T \vdash \text{prf}_T(\overline{k}, \neg \varrho)$ for some $k \in \mathbb{N}$ and so $U \vdash \exists y < \overline{k} \text{prf}_T(y, \neg \neg \varrho)$, by the definition of $U$, which contradicts $\forall m \in \mathbb{N} \neg \text{prf}_T(m, \neg \neg \varrho)$ (that holds by $T \nvdash \neg \varrho$).

- If $T \nvdash \neg \varrho$, then $T \vdash \text{prf}_T(\overline{k}, \neg \neg \varrho)$ for some $k \in \mathbb{N}$. Reason inside $U$:

  for some $x$ we have $\text{prf}_T(x, \neg \varrho) \land \forall y < x \neg \text{prf}_T(y, \neg \neg \varrho)$; now $\overline{k} < x$ is impossible, and so $x \leq \overline{k}$, whence $\forall i \leq k \text{prf}_T(\overline{i}, \neg \varrho)$, therefore $\forall i \leq k \text{prf}_T(\overline{i}, \neg \varrho)$. 

QED
Hence, \( U \vdash \bigwedge_{i \leq k} \text{prf}_T(i, \gamma \rho \gamma) \), but this contradicts \( \bigwedge_{m \in \mathbb{N}} U \vdash \neg \text{prf}_T(m, \gamma \rho \gamma) \) (that holds by \( T \not\models \rho \)). Therefore, \( T \not\models \rho, \neg \rho \).

So, the Weak Diagonal Lemma is worthy of studying further. Unfortunately, the second proof (Subsection 2.2) for Tarski’s Theorem cannot be carried over to the Syntactic Tarski’s Theorem (cf. [14]). However, the first proof (Subsection 2.1) can be adapted for it:

**THEOREM 3.6 (Syntactic Tarski’s Theorem):**

If \( T \) is a consistent extension of Robinson’s arithmetic, then for no formula \( \Psi(x) \) can we have \( T \supseteq TB^\Psi \).

**PROOF:**

Assume that the consistent theory \( T \) contains Robinson’s arithmetic, and it also contains the set \( TB^T \) for a formula \( \Upsilon(x) \). We work with the Convention of Subsection 2.1 (and also Definition 2.5). Let \( q \) be the term \( \overline{6} \cdot \ell_T \) which represents the number \( 6\ell_T \). Fix a number \( n \in \mathbb{N} \); and reason inside the theory \( T \):

Assume \( B_T(\pi) \); so, \( \text{Berry}^\Upsilon(\pi) \) thus (1) \( \neg \text{Def}^\Upsilon_T(\pi) \) and (2) \( \forall w < \pi \text{Def}^\Upsilon_T(w) \) hold. Fix \( \zeta \); if \( B_T(\zeta) \) holds, then (i) \( \neg \text{Def}^\Upsilon_T(\zeta) \) and (ii) \( \forall w < \zeta \text{Def}^\Upsilon_T(w) \). We also have either \( \zeta < \pi \) or \( \zeta = \pi \) or \( \zeta > \pi \) (which holds in Robinson’s arithmetic). Now, \( \zeta < \pi \) contradicts (i) and (2), and \( \zeta > \pi \) contradicts (1) and (ii). Whence, \( \zeta = \pi \); which shows that \( \forall \zeta[B_T(\zeta) \iff \zeta = \pi] \) holds. Thus, \( D(\neg B_T, \pi) \) holds and since \( \text{len}(B_T(x)) < q \), then we have \( \text{Def}^\Upsilon_T(\pi) \) by \( TB^T \); which contradicts (1). Therefore, the assumption \( B_T(\pi) \) leads to a contradiction. Whence, \( \neg B_T(\pi) \) holds, so \( \neg \text{Berry}^\Upsilon_T(\pi) \), thus we have (\( * \)) \( \bigwedge_{i<n} \text{Def}^\Upsilon_T(\bar{i}) \rightarrow \text{Def}^\Upsilon_T(\pi) \).

Therefore, \( T \vdash \text{Def}^\Upsilon_T(\pi) \) can be shown by induction on \( n \in \mathbb{N} \) from (\( * \)). Let \( p \in \mathbb{N} \) be greater than all the Gödel codes of formulas with length less than \( 6\ell_T \). Therefore, for all \( n \in \mathbb{N} \) we have \( T \vdash \exists \alpha < \overline{p} \Upsilon(\neg D(\alpha, \pi)) \).

So, inside \( T \) for any \( n \in \mathbb{N} \) there exists some formula \( \alpha_n(x) \) that defines \( n \), i.e., \( \forall \zeta[\alpha_n(\zeta) \iff \zeta = \pi] \) holds by \( TB^T \), and the Gödel codes of all \( \alpha_n \)’s are less than \( p \). This contradicts the Pigeonhole’s Principle a version of which is provable in Robinson’s arithmetic: since both of the sentences \( \forall x(x \not\equiv 0) \) and \( \forall x(x < k+1 \rightarrow x \equiv k) \) are provable in this arithmetic, then for every \( \{\alpha_k < \overline{p}\}_{k \leq p} \) there should exist some \( i < j \leq p \) such that \( \alpha_i = \alpha_j \).

Reason inside \( T \) again:

There are some \( \bar{i} < \bar{j} \leq \overline{p} \) and some formula \( \varphi(x) \) for which we have \( \Upsilon(\neg D(\varphi, \bar{i}) \rangle\) and \( \Upsilon(\neg D(\varphi, \bar{j}) \rangle\).

So, by \( TB^T \), both \( \forall \zeta[\varphi(\zeta) \iff \zeta = \bar{i}] \) and \( \forall \zeta[\varphi(\zeta) \iff \zeta = \bar{j}] \) hold, combining which implies that \( \bar{i} = \bar{i} \rightarrow \varphi(\bar{i}) \rightarrow \bar{i} = \bar{j} \), a contradiction.

So, \( T \) is inconsistent, which is a contradiction with the assumption.

**QED**

4 Conclusion

The semantic form of Tarski’s Undefinability Theorem, that the set \( \{\gamma \eta \mid \mathbb{N} \models \eta\} \) is not definable in arithmetic, is equivalent to the semantic form of the Diagonal Lemma, that for a given \( \Psi(x) \) there exists a sentence \( \theta \) such that \( \mathbb{N} \models \Psi(\gamma \theta \gamma) \iff \theta \). We outlined two seemingly diagonal-free proofs for these equivalent theorems. The syntactic form of Tarski’s Theorem, that no consistent extension of Robinson’s arithmetic contains the set of truth biconditionals \( TB^\Psi = \{\Psi(\gamma \beta \gamma) \iff \beta \mid \beta \text{ is a sentence}\} \), is equivalent to the Weak (syntactic) Diagonal Lemma, that for every \( \Psi(x) \) there exists a sentence \( \theta \) such that \( \Psi(\gamma \theta \gamma) \iff \theta \) is consistent with such a theory. Even though Gödel’s proof does not work with the Weak Diagonal Lemma, the weak lemma is sufficiently strong to prove Rosser’s theorem. So, the syntactic form of Tarski’s theorem can derive Gödel and Rosser’s incompleteness theorem (by combining the proofs of Theorems 3.6, 3.3 and 3.5).
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