A DISTRIBUTIVE LAW FOR COMPACT SYMMETRIC MULTICATEGORIES

SOPHIE RAYNOR

Abstract. Compact symmetric multicategories (CSMs) generalise a number of operad-like devices, such as wheeled properads and modular operads, that admit a contraction operation as well as an operadic multiplication. They were introduced in 2011 by Joyal and Kock, who constructed them as the algebras for an endofunctor on a certain presheaf category. CSMs are known to exhibit strange behaviour, related to the contraction of units, that presents some difficulty for describing their combinatorics, and hence for proving a nerve theorem. Here, non-unital CSMs are constructed as algebras for a monad defined in terms of connected graphs, and this provides the foundation for a new construction of CSMs based on a distributive law. This enables a proof, in the style of Joyal and Kock, of a corresponding nerve theorem characterising CSMs in terms of a Segal condition.

Introduction

Compact symmetric multicategories (CSMs) were introduced by André Joyal and Joachim Kock in an extended abstract [19]. CSMs are operad-like devices that, as well as an operadic multiplication, admit a unary contraction operation. They generalise (in an appropriate sense) such structures as modular operads [12, 13], wheeled properads [14, 36], and small compact closed categories [20]. In fact, they may legitimately be described as ‘the connected parts of compact closed categories’ (see Section 5).

The project outlined in [19], like the works [14, 15, 16, 17], fits into two significant research strands of recent years: that of defining and classifying operad-like structures [23, 27], and that of describing their ∞-analogues (for example [28, 8, 14, 15, 17].) Whilst the concepts explored in [19] are firmly grounded in this context, Joyal and Kock’s construction of CSMs involved the introduction of a particularly innovative and elegant graphical toolkit.

However, there is a problem, related to contraction of multiplicative units, with one part of the construction outlined in [19]. The present paper fills this gap. Using a modification of the CSM endofunctor of [19], the category of CSMs is constructed as the Eilenberg-Moore (EM) category of algebras for a composite monad, and a nerve theorem in the style of [4] is proved, characterising CSMs in terms of a Segal condition.

In the spirit of [19], the construction given in this paper relies only on basic categorical arguments. As such, it lays bare the combinatorics of ‘loops’, and, in particular, highlights the complications that may arise when working with contracted units. This work is complementary to [16], which also presents a monadic construction and nerve theorem for CSMs (there called modular operads). There, however, the constructions proceed ‘by hand’ so that, in [17], the authors are able to construct a suitable model category whose fibrant objects are Segal CSMs (Segal modular operads), providing a model for (∞, 1)-CSMs.
0.1. Definitions and context. Graphical species provide the suitable notion of ‘coloured collection’ on which the structure of a CSM may be defined [19, Section 4].

An (involutive) palette is a pair \((C, \omega)\) consisting of a set \(C\) and involution \(\omega: C \rightarrow C\), and denoted by \((C, \omega)\). Let \(\text{Set}\) be the category of sets and set maps, and \(\text{FinSet}_{iso} \subseteq \text{Set}\) be the groupoid of finite sets and bijections.

Given a palette \((C, \omega)\), a \((C, \omega)\)-coloured graphical species is a presheaf \(S: \text{FinSet}_{iso}^{op} \rightarrow \text{Set}, \ X \mapsto S_X\), together with projections \(p_x: S_X \rightarrow C\), defined for all \(X\) and all \(x \in X\), that are equivariant with respect to the action of \(\text{FinSet}_{iso}\). An element \(\phi \in S_X\) may be thought of as a corolla or star graph whose unique vertex is decorated by \(\phi\) and whose ‘legs’ are indexed by \(X\) so that, for each \(x \in X\), the corresponding leg is ‘coloured’ by \(p_x(\phi)\) (as in Figure 0.1 below). Graphical species are the objects of a category \(\mathcal{GS}\) whose morphisms are given by pairs of palette maps and presheaf morphisms that respect all structure.

A \((C, \omega)\)-coloured CSM is a \((C, \omega)\)-coloured graphical species \(S\) together with two (partial) operations: the multiplication \(\phi \circ_{x,y} \psi \in S_{X \amalg Y}\) along \((x, y)\) is defined for \(\phi \in S_{X \amalg \{x\}}\) and \(\psi \in S_{Y \amalg \{y\}}\) whenever \(p_x(\phi) = \omega p_y(\psi)\), and, for any \(\phi \in S_{X \amalg \{x\}}\) such that \(p_x(\phi) = \omega p_y(\phi)\), there is a contraction \(\zeta_{x,y}(\phi) \in S_X\) of \(\phi\) along \((x, y)\). Multiplication and contraction are equivariant with respect to the \(\text{FinSet}_{iso}\) action, and satisfy some axioms governing their composition.

![Figure 1. CSM multiplication and contraction.](image)

Crucially for the present work, CSM multiplication is unital: if a CSM \(S\) has palette \((C, \omega)\), then, for each \(c \in C\) of \(S\), there is a unique element \(\iota_c\) of \(S_2\) that acts a \(c\)-coloured unit for the partial multiplication. In particular, \(p_1(\iota_c) = c\) and \(p_2(\iota_c) = \omega c\) and so \(\zeta(\iota_c) \in S_0\) exists for all \(c\). (It is these contracted units \(\zeta(\iota_c)\) that present the challenge for describing the combinatorics of CSMs.)

CSMs are the objects in a category \(\text{CSM}\) whose morphisms are morphisms of graphical species that respect the multiplication, contraction and units. They appear in a variety of contexts:

- **Wheeled properads.** For any palette \((C, \omega)\), the terminal \((C, \omega)\)-coloured graphical species \(Z^{(C, \omega)}\) given by

\[
Z_X^{(C, \omega)} = C^X, \quad \text{and} \quad p_x(\underline{x})(x) = \underline{x}(x) \quad \text{for} \quad x \in X \text{ and } \underline{x} \in C^X,
\]

\[\underline{x}\]

I will use the single-word ‘palette’ in place of the more standard ‘colour set’.

trivially has the structure of a CSM. In particular, if $\sigma_2$ is the unique non-identity permutation on two elements, objects of the slice category $\text{CSM} \downarrow Z^{(2, \sigma_2)}$ are wheeled properads. (See Examples 1.43, 1.46, 3.60.)

- **Compact closed categories.** A (small) compact closed category $(\mathcal{C}, \otimes, 1, *)$ describes an $(\text{ob}(\mathcal{C}), *)$-coloured CSM $S^\mathcal{C}$ by

$$S_n^\mathcal{C} = \prod_{(c_1, \ldots, c_n) \in \text{ob}(\mathcal{C})^n} C(c_1 \otimes \cdots \otimes c_n, 1),$$

where, for $n \in \mathbb{N}_{\geq 1}$, $n$ denotes the set $\{1, \ldots, n\}$. Multiplication in $S^\mathcal{C}$ is induced by categorical composition in $\mathcal{C}$ and contraction is obtained from the obvious compositions with ‘cup’ and ‘cap’. (Details are given in [30].)

- **Modular operads.** The ‘connected part’ of the 2-dimensional cobordism category $\text{Sur}$ is described by a one-coloured CSM $\text{Sur}$. Elements of $\text{Sur}_n$ are, up to homeomorphism, compact oriented 2-manifolds with $n$ closed boundary components.

Since the topological type of a compact oriented surface with boundary is determined only by its genus and number of boundary components, $\text{Sur}_n = \{S(g, n)\}_{g \in \mathbb{N}} \cong \mathbb{N}$ for all $n$. The multiplication $\circ$ in $\text{Sur}$, which corresponds to gluing two surfaces along a pair of boundary components, is additive on genus, whereas the contraction $\zeta$, corresponding to gluing two distinct boundary components of a single surface, raises the genus by one.

$$S(g_1, m) \circ S(g_2, n) = S(g_1 + g_2, m + n - 2), \text{ and } \zeta(S(g, n')) = S(g + 1, n' - 2).$$

In particular, the cylinder $S(0, 2)$ is the unit for $\circ$. It is clear how this may be generalised to more interesting geometric structures on surfaces.

This example illustrates the relation of CSMs, described as ‘coloured modular operads’ in the abstract of [19], to (single-coloured) modular operads as originally defined by Getzler and Kapranov [12]. The latter are bigraded (by ‘genus’ and ‘boundary components’) objects $M = \{M(g, n)\}_{g, n}$ equipped with an additive-on-genus multiplication and a contraction that increases the genus by one. A further ‘stability condition’ — that says that $M(g, n)$ is only defined when $2g + n - 2 \leq 0$ — implies that modular operads in the sense of [12], unlike the CSMs in this paper, are non-unital.

[19] consider related 2-coloured stable modular operads arising from gluing of surfaces along open and closed boundary components. A sequel paper [9] gives many of the same constructions, but this time in terms of open-closed 2 cobordism categories.

- In recent years, a number of authors have considered (generalised) operadic methods in the context of applications involving networks (for example [32] [1]). Inspired by ideas from neuroscience, (specifically brain plasticity), [18] describe topological categories of weighted directed graphs where connections may be broken and created. Another motivation for studying such structures comes from understanding varying effects of noise in a network of signal connections. Using the methods described in this paper and [30], the categories of [18] may be obtained as free objects in an EM category of algebras for a composite monad on topological graphical species.

As demonstrated by the examples above, [19] represents a break from the standard of studying directed (e.g. wheeled properads) and undirected (e.g. modular operads) families of operads separately. The definition of CSMs in terms of an involutive palette means that these are specific instances of one and the same construction, and is a direct consequence of Joyal and Kock’s simple but innovative definition

---

6See Section 5 and [30].

7 Compare, for example, [14] and [16, 17], but also [15] and [11] mentioned below. In each case, the earlier work involved palettes with trivial involution.
of graphs, also adopted in [16, 17]. (See [22] and 2 Section 15 for a comparison of Joyal-Kock graphs and, for example, the graphs used in [14, 6].)

Moreover, where similar approaches to constructing operad families involve restrictions to a given palette, Joyal and Kock’s definition of graphs enables the construction of the category CSM of all CSMs of all palettes, as the EM category of algebras for a single monad on GS. This makes CSMs accessible to a nerve theorem in the style of [4, 35]. And, unlike palette-specific constructions – that do have the advantage of being easily generalisable to other enriching categories, but require more data and often proceed ‘by hand’ – the method of [19] is based entirely on simple diagrams of finite sets and some basic categorical constructions. (See also [17, Remark 2.15].)

This approach has implications beyond generality, and beyond CSMs. For example, the category of simply connected graphs used in [15] does not imbed fully in the category of cyclic operads (see [15, ex. 5.7, rmk. 5.10]). This is not the case in [11] where cyclic operads are defined in terms of an involutive palette. (The implications of an involutive palette in such constructions are also discussed in some detail in [17 p. 2].)

0.2. Outline of the problem, and summary of the construction. As usual, let $\Delta$ denote the simplex category of finite ordinals $[n]$ ($n \in \mathbb{N}_{\geq 0}$) and order preserving maps, and $\text{sSet} \overset{\text{def}}{=} PSh(\Delta)$ is the category of presheaves on $\Delta$, or simplicial sets. Objects of $\Delta$ may be viewed as directed linear graphs $[n] \rightarrow 1 \rightarrow \cdots \rightarrow n \rightarrow$.

The classical nerve theorem for categories (see e.g. [31]) states that there is a fully faithful functor $\text{Cat} \rightarrow \text{sSet}$ whose essential image consists of precisely those $P \in \text{sSet}$ the satisfy the Segal condition: for $n \geq 1$, each $n$-cell in $P_n$ is an $n$-fold fibred product of 1-cells.

\[ P_n \cong P_1 \times_{P_0} \cdots \times_{P_0} P_1, \]

In general, if $M$ is monad on a category $C$ with dense subcategory $D \subset C$, then the canonical identity on objects/fully faithful factorisation

\[ D \overset{i_{\alpha}}{\rightarrow} D^M \overset{f.f.}{\rightarrow} C^M \]

induces a nerve functor

\[ N_M : C^M \rightarrow PSh(D^M), \quad d \mapsto C^M(-, d) \]

from the EM category $C^M$ of $M$, to the presheaf category $PSh(D^M)$ on $D^M$.

In [4 Sections 1,2], it is shown that if $M$ has certain properties – specifically, if it preserves certain colimits and therefore is said to ‘have arities $D$’ – then this nerve $N_M$ is fully faithful and its essential image is characterised by a Segal condition.

[19 Section 2] gives a concise outline of the situation for categories and the role of the free category functor $\text{PSh}([0] \Rightarrow [1]) \rightarrow \text{Cat}$. It is shown that, in the context of [4], the classical nerve theorem corresponds to the statement that the category of linear graphs and ‘successor-preserving’ maps provides arities for the free category monad on $\text{PSh}([0] \Rightarrow [1])$.

The combinatorics of CSMs are related to connected graphs (admitting cycles) as categories are related to directed linear graphs. Hence, to obtain a suitable ‘generalisation’ of the nerve theorem to CSMs, [19] defines a category of elGr of elementary graphs whose presheaf category $PSh(\text{elGr})$ is precisely $GS$, and whose role is analogous to that of the full subcategory $([0] \Rightarrow [1]) \subset \Delta$ in the classical nerve theorem. Moreover, elGr embeds densely in a larger category Gr of connected graphs, and, there is a canonical chain of fully faithful embeddings

\[ \text{elGr} \overset{\text{dense}}{\rightarrow} \text{Gr} \overset{\text{dense}}{\rightarrow} GS \overset{\text{dense}}{\rightarrow} \text{PSh(Gr)}, \]
where the last of these describes how to ‘decorate’ a graph $G$ by elements of a graphical species $S$.

In particular, if $M$ is a monad on $G_S$ whose EM category is of algebras is isomorphic to $CSM$, the Segal condition says that a presheaf $S \in PSh(G^M_S)$ is a CSM precisely when

$$S(G) = \lim_{(C,b) \in \mathcal{G} \downarrow \mathcal{S}} S(C), \text{ for all } G \in \text{ob}(G).$$

Unfortunately, though there is a monad on $G_S$ whose algebras are CSMs (see Section 3.5), it does not have arities $Gr$. So, without modification, the theory of [4] does not apply to CSMs.

A brief sketch of the obstruction is worthwhile. As with related constructions, such as [27] and [15] (and also the free category monad [19, Section 2]), monadic multiplication for the CSM monad on $G_S$ is described in terms of a rule for substituting the vertices of a given graph by other graphs.

---

**Figure 2.** Graph substitution.

[19] obtain multiplicative units of CSMs by degenerate substitution of bivalent vertices by the exceptional ‘stick’ graph (i) with no vertices. However, together with CSM contraction, this implies the existence of a special case arising from a degenerate substitution into the unique vertex of the wheel with one vertex $W$.

In other formalisms (for example [14, 26, 36]) – where graph substitution is defined via orderings on the graph ‘legs’ and the edges incident at each vertex – this substitution results in an exceptional loop with no vertices, that, in order to define the monadic multiplication, must be included as a graph in the formalism.

---

**Figure 3.** Visualising the exceptional loop $\bigcirc$.

The situation in [19], where graph substitution is defined in terms of colimits of certain small diagrams, is somewhat more subtle: the degenerate substitution into the unique vertex of the wheel actually results

---

8 This is generally unproblematic, though the loop must still be treated with care. For example, it is an obstruction to the wheeled graphical category $\Gamma_{\odot}$ (which generalises the dendroidal category $\Omega$ of [28]) of [14] being Reedy (though this is simply remedied in [13]) and it has an ambiguous role in the proof of the corresponding nerve theorem [14, 15].
in the stick graph (\cite{17}). The loop arises because, in this one special case, the construction does not respect the notion of graph equivalence required to define the monad multiplication. Moreover, it is not possible to resolve this issue by adding the appropriate (loop) equivalence class to the category \( \text{Gr} \), since the resulting category does not embed in \( \text{GS} \). (See Subsection 3.1.)

Nonetheless, by forbidding degenerate substitutions, the single problematic case of graph substitution is eliminated, and a well-defined monad \( T \) on \( \text{GS} \) is obtained. Algebras for \( T \) are almost CSMs – except that they do not admit units for the multiplication.

Fortunately, just as the free category monad on directed graphs admits a decomposition, via a distributive law, into a monad that governs (semi-)categorical composition and one that adjoins distinguished ‘unit morphisms’, the same is true for the CSM monad on \( \text{GS} \). It is quite straightforward to construct a monad \( D \) on \( \text{GS} \) that adjoins distinguished elements to a graphical species \( S \) according to the defining properties of multiplicative units, and also their contractions, and to define a distributive law \( TD \to DT \) with the desired features.

**Theorem A.** The EM category of algebras for the composite monad \( DT \) on \( \text{GS} \) is precisely the category \( \text{CSM} \) of CSMs.

Since \( DT \) does not have arities \( \text{Gr} \), it is not immediately obvious how to prove a nerve theorem for this composite monad \( DT \) in the style of \cite{17}. However, \cite[Section 3]{3} implies the following corollary.

**Corollary A.** There is a monad \( T_\ast \) on \( \text{GS}_\ast \) \( \overset{\text{def}}{=} \text{GS}^D \) whose EM category of algebras \( \text{GS}_\ast T_\ast \) is canonically isomorphic to \( \text{CSM} \).

In particular, there are categories \( \text{elGr}_\ast, \text{Gr}_\ast \) and \( \widetilde{\text{Gr}}_\ast \), obtained via canonical factorisations, such that the following diagram of functors (where all the horizontal inclusions are full) commutes.

\[
\begin{array}{cccccc}
\text{elGr}_\ast & \hookrightarrow & \text{Gr}_\ast & \hookrightarrow & \text{GS}_\ast & \to & \text{PSh}((\text{Gr}_\ast)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{elGr} & \hookrightarrow & \text{Gr} & \hookrightarrow & \text{GS} & \to & \text{PSh}((\text{Gr}).
\end{array}
\]

This leads to the desired nerve theorem and Segal condition (Theorem 4.3 in the text).

**Theorem B.** The monad \( T_\ast \) has arities \( \text{Gr}_\ast \) and hence the nerve functor \( N : \text{CSM} \to \text{PSh}((\text{Gr}_\ast)) \) is fully faithful. Its essential image consists of precisely those presheaves on \( \widetilde{\text{Gr}}_\ast \) that satisfy the Segal condition. In other words, a presheaf \( S \) on \( \widetilde{\text{Gr}}_\ast \) is in the essential image of \( N \) if and only if \( S(G) = \lim_{\text{elGr}_\ast G} S \) for all graphs \( G \).\[8\]

---

9In \cite{17}, the stick and the loop have the same description as diagrams of finite sets and are distinguished from each other only by specifying their ‘boundary’. Though more data is required to define the graphs, this is a clever approach to solving the issue of loops, at least in order to obtain a well-defined monad, though not sufficient to obtain a Weber-style nerve theorem \cite{17} \cite{35}.

10Corollary \[3\] and Theorem \[1\] first appeared in \cite{29}, though without explicit reference to the distributive law. See Remark \[3.3\] for how these results relate to \cite[Section 6]{19} and \cite[Theorem 4.1]{17}.
0.3. Structure of the paper. The first section of the paper introduces the categories \( \text{elGr} \subset \text{Gr} \subset \mathcal{G}S \). Most of this section stays very close to the original construction of [19]. Since this was just a short note, it contained very few proofs, and so most of the results underlying the graph constructions are proved in full here.

Rather than immediately describing the construction in [19] and the problem of loops, Section 2 is devoted to a description of the monad \( T = (T, \mu, \eta) \) on \( \mathcal{G}S \) – obtained by a slight modification of the CSM endofunctor in [19] – whose algebras are non-unital CSMs.

In Section 3, the focus turns to the construction of (unital) CSMs. It is from this point that the distinct approach of this work – as compared with [19, 16, 17] and other related work – becomes clear. Subsection 3.1 explains the ‘problem of loops’ in [19] in more detail, and it is shown why the ‘obvious’ fix will not work. In Subsection 3.2, important properties of multiplicative units and their contractions are encoded via the addition of morphisms to the category of elementary graphs \( \text{elGr} \). The resulting category \( \text{elGr}_\ast \supset \text{elGr} \) of pointed elementary graphs embeds in a new category \( \text{Gr}_\ast \supset \text{Gr} \) of pointed graphs, obtained via a factorisation. This category \( \text{Gr}_\ast \) and, therein some surprising morphisms that appear to encode the original problem of loops (Subsection 3.1), are explained in detail in Subsections 3.3 and 3.4.

The category \( \mathcal{G}S \ast \) is, itself, the EM category of algebras for a monad \( D \) on \( \mathcal{G}S \) that adjoins (contracted) units. In Subsection 3.5 it is shown that there is a distributive law \( \lambda : TD \to DT \) and, moreover, that the composite \( DT \) is the desired CSM-monad on \( \mathcal{G}S \), and hence prove Theorem A and Corollary A.

In Section 4, the proof of the nerve theorem B (Theorem 4.3 in the main text), by showing that the monad \( T \ast \) from Corollary A has arities \( \text{Gr}_\ast \), proceeds without problems, and relies heavily on the more unusual features of the category \( \text{Gr}_\ast \). The desired nerve theorem and Segal condition in terms of \( \text{Gr} \) and \( \mathcal{G}S \) then follows using Diagram 0.2.

0.4. Acknowledgements. This work owes its existence to the ideas of A. Joyal and J. Kock and I thank Joachim for taking time to speak with me about it. P. Hackney, M. Robertson and D. Yau’s work has been an invaluable resource, and conversations with Marcy have been particularly helpful. The article builds on my PhD research at the University of Aberdeen, UK and funded by the EPFL, Switzerland, and I thank my supervisors R. Levi and K. Hess. Thanks to the members of the Centre for Australian Category Theory at Macquarie University for providing the ideal mathematical home for these results to mature.

1. Graphs and graphical species

1.1. Feynman graphs. Following Joyal and Kock [19], a (Feynman) graph \( \mathcal{G} \) is a diagram of finite sets

\[
\tau \subseteq E \xleftarrow{s} H \xrightarrow{t} V
\]

such that \( s : H \to E \) is injective and \( \tau : E \to E \) is an involution without fixed points.

The set \( V \) is called the vertex set of \( \mathcal{G} \) and \( E \) is called the edge set of \( \mathcal{G} \). The elements of \( H \) are called half edges of \( \mathcal{G} \) and \( h \in H \) will often be written as an ordered pair \( h = (s(h), t(h)) \). The set of \( \tau \)-orbits in \( E \) is denoted \( E/\tau \) and \( \bar{e} \in E/\tau \) denotes the image of \( e \in E \) under the quotient \( q : E \to E/\tau \).

Let \( n \overset{\text{def}}{=} \{1, 2, \ldots, n\} \) for all \( n \in \mathbb{N}_{\geq 1} \). The empty set \( \emptyset \) will also be denoted \( 0 \).

Example 1.1. The stick graph \((\bar{)}\) has no vertices and edge set \( 2 = \{1, 2\} \).

\[
(\bar{)} \overset{\text{def}}{=} \bigcup 2 \xleftarrow{0} 0 \xrightarrow{0} 0.
\]

\[\text{This definition contrasts with, for example [6], where leaves of the graph are defined as fixed points for the involution on } E. \text{ A good comparison of various graph formalisms, including that of [19], can be found in [2] Section 15}.\]
Example 1.2. Let \( X \) be a finite set. The (Feynman) \( X \)-corolla \( \mathcal{C}_X \) has the form

\[
\mathcal{C}_X : \quad \tau_X \quad \bigcirc_X \quad X^H \quad \xrightarrow{s_X} \quad X^H \quad \to \quad \{*\}
\]

where \( X^H \) is a copy of \( X \) and \( s_X : X^H \to X^H \) is the inclusion. The involution \( \tau_X \) takes an element in \( X \) to the same element in \( X^H \).

Remark 1.3. A graph \( \mathcal{G} \) can be realised geometrically by a one-dimensional space \(|\mathcal{G}|\) where the set \( \{s_v \mid v \in V\} \) is the set of 0-cells of \( |\mathcal{G}| \) and, for each \( e \in E \), we take a copy \([0, \frac{1}{2}]_e\) of the interval \([0, \frac{1}{2}]\) and identify

- \( 0_{s(h)} \sim s(t(h)) \) for \( h \in H \),
- \( (\frac{1}{2})_e \sim (\frac{1}{2})_{\tau e} \) for all \( e \in E \).

Example 1.4. Let \( \mathcal{W} = W_1 \) be the wheel graph with one vertex

\[
\bigcirc_{(e, \tau e)} \quad \xrightarrow{\tau} \quad \{h^{\prime}, h^{\tau e}\} \quad \xrightarrow{\{\}} \quad \{*\}.
\]

All edges of \( \mathcal{W} \) are inner edges. It has the realisation

\[
|\mathcal{W}| : \quad \bigcirc
\]

Definition 1.5. An inner edge of \( \mathcal{G} \) is an element \( e \in E \) such that the \( \tau \)-orbit \( \{e, \tau e\} \) of \( e \) is contained in the image of \( s \). The set of inner edges of \( \mathcal{G} \) is denoted \( E_I \).

The set \( E_0 = E - \text{im}(s) \) is the set of ports of \( \mathcal{G} \).

Example 1.6. The stick graph \((i)\) has no internal edges and all edges are ports. That is \( E(i) = E_0(i) \). For a finite set \( X \), \( E_0(\mathcal{C}_X) = X \) and \( E_I(\mathcal{C}_X) = \emptyset \). All edges of the wheel graph \( \mathcal{W} \) are internal \( E_0(\mathcal{W}) = \emptyset \) and \( E_I(\mathcal{W}) = E(\mathcal{W}) \).

Let \( \mathcal{G} \) be a graph. For each \( v \in V \), let \( H\{v\} = t^{-1}(v) \subset H \), and \( E\{v\} = s(H\{v\}) \). There is a natural grading \( V = \bigsqcup_{n \in \mathbb{N}} V_n \) of \( V \), where \( V_n = \{v \in V \mid |H\{v\}| = n\} \). So \( V_0 \) is the set of isolated vertices of \( \mathcal{G} \). The pullback along \( t \) induces gradings \( H = \bigsqcup_{n \in \mathbb{N}} H_n \) and \( E = \bigsqcup_{n \in \mathbb{N}} E_n \) where \( H_n = t^{-1}(V_n) \) and \( E_n = s(H_n) \) for \( n \geq 1 \).

The category \( \text{Gr}^f \) has Feynman graphs as objects. Morphisms are triples of finite set maps

\[
f = (f_E, f_H, f_V) \in \text{Gr}^f(\mathcal{G}, \mathcal{G}')
\]

such that diagrams of the form

(1.7) \[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\
E & \xrightarrow{\tau f} & E' \\
H & \xrightarrow{s f} & H' \\
V & \xrightarrow{t f} & V'
\end{array}
\]

commute. In particular, \( \tau' f(e) = f(\tau e) \) for all \( e \in E \).

Definition 1.8. If \( \mathcal{G} \) is a graph with edge set \( E \), then the morphism \( 1 \mapsto e \in E \) in \( \text{Gr}^f(\mathcal{G}, \mathcal{G}) \) that ‘chooses’ \( e \in E \) is denoted \( ch_e \), or sometimes \( ch_e^f \).

The map \( E \to \text{Gr}(i, \mathcal{G}), e \mapsto ch_e \) is a bijection for all graphs \( \mathcal{G} \).

Lemma 1.9. (Corresponds to [21] Proposition 1.1.11.) For any morphism \( f = (f_E, f_H, f_V) \in \text{Gr}^f(\mathcal{G}, \mathcal{G}') \), the map \( f_H \) is completely determined by \( f_E \). Moreover if \( V_0 = \emptyset \), then \( f_E \) also determines \( f_V \).
Proof. By injectivity of $s$, $f_H(h) = s^{-1} f_E s(h)$ is well-defined for $h \in H$. If $V_0 = \emptyset$, then for each $v \in V$, $H\{v\}$ is non-empty and $f_V(v) = t'(s^{-1} f_E s(h))$ does not depend on the choice of $h \in H\{v\}$. □

**Definition 1.10.** A weak injection $\mathcal{G} \to \mathcal{G}'$ is a morphism $w \in \text{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ for which $w_V$ and $w_H$, but not necessarily $w_E$, are injective.

**Remark 1.11.** A morphism in $\text{Gr}^\sharp$ is a weak injection precisely if it is a monomorphism.

**Definition 1.12.** A morphism $f \in \text{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is called étale if the righthand square in the defining diagram

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\
\downarrow f & & \downarrow \phi \\
\mathcal{G} & \xrightarrow{f} & \mathcal{G}'
\end{array}
\]

is a pullback in the category $\text{FinSet}$ of finite sets and set maps.

The category $\text{Gr}^\sharp_{\text{et}} \subseteq \text{Gr}^\sharp$ is the wide subcategory of Feynman graphs and étale morphisms.

The following proposition is immediate from the definition and tells us that the étale morphisms are precisely those that preserve (vertex) velancy.

**Proposition 1.14.** A morphism $f \in \text{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is étale if and only if, for all $v \in V$, $f_H$ induces a bijection $H\{v\} \xrightarrow{\cong} H\{f(v)\}$, that is

\[
H\{v\} \cong f_H(H\{v\}) \quad \text{and} \quad f_H(H\{v\}) = H\{f(v)\}.
\]

In particular, for composable morphisms $f, g \in \text{Gr}^\sharp$, if any two of $f, g$ and $g \circ f$ are étale, then so is the third.

Let $\text{diag}$ be the small category with objects and non-identity morphisms given by $\bigcup \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$. The functor category $\text{GrShape} \overset{\text{def}}{=} \text{Fun}(\text{diag}, \text{FinSet})$ is the category of graph-like diagrams in $\text{FinSet}$.

Clearly, there are inclusions of categories $\text{Gr}^\sharp_{\text{et}} \subseteq \text{Gr}^\sharp \subseteq \text{GrShape}$ and $\text{Gr}^\sharp$ and $\text{Gr}^\sharp_{\text{et}}$ inherit a cocartesian monoidal structure from $\text{GrShape}$ where the coproduct $+$ is induced pointwise by disjoint union of sets, and the initial object is the empty graph $\emptyset \overset{\text{def}}{=} (\emptyset \leftarrow \emptyset \rightarrow \emptyset)$.

Since $\text{FinSet}$ admits finite (co)limits, so does $\text{GrShape}$, and these (co)limits can be computed pointwise (see, for example [5 Proposition 2.15.1]). A (co)limit in $\text{Gr}^\sharp$ is a (co)limit in $\text{GrShape}$ since the inclusion $\text{Gr}^\sharp \subseteq \text{GrShape}$ is full. By the two out of three property for étale morphisms (Proposition 1.14) (co)limits in $\text{Gr}^\sharp_{\text{et}}$, when they exist, correspond under inclusion to (co)limits in $\text{GrShape}$.

**Definition 1.15.** A graph $\mathcal{G}$ is connected if, for every finite sum of graphs $\sum^n_{i=1} \mathcal{H}_i$,

\[
\text{Gr}^\sharp(\mathcal{G}, \sum^n_{i=1} \mathcal{H}_i) \cong \coprod^n_{i=1} \text{Gr}^\sharp(\mathcal{G}, \mathcal{H}_i).
\]

**Definition 1.17.** A (connected) component of a graph $\mathcal{G}$ is a maximal connected subgraph of $\mathcal{G}$.

A second characterisation of connectedness is sometimes useful.

**Lemma 1.18.** A graph $\mathcal{G} \in \text{ob}(\text{Gr})$ is connected if and only if for each $f \in \text{GrShape}(\mathcal{G}, \star + \star)$, the pullback $\mathcal{P}$ in $\text{GrShape}$, of the diagram
proof. For any $\sum_{i=1}^{k} \mathcal{H}_i \in ob(Gr^i)$, and $1 \leq j \leq k$, let $p_j \in GrShape(\sum_{i=1}^{k} \mathcal{H}_i, \star + \star)$ be the morphism that projects $\mathcal{H}_j$ onto the first summand, and $\sum_{i\neq j} \mathcal{H}_i$ onto the second summand. Then, for any graph $\mathcal{G}$ and any $f \in Gr^i(\mathcal{G}, \sum_{i=1}^{k} \mathcal{H}_i)$, the diagram

\[
\begin{array}{c}
P_j \\
\downarrow \ \\ \mathcal{H}_j \\
\downarrow \ \\ \star \\
\end{array}
\begin{array}{c}
P \\
\downarrow f \\
\mathcal{G} \\
\downarrow j \\
\star + \star \\
\end{array}
\]

where the top square is a pullback, commutes in $GrShape$. Since the lower square is a pullback by construction, so is the outer rectangle.

In particular, if $\mathcal{G}$ satisfies the condition of the lemma, then $P_j$ is either empty or isomorphic to $\mathcal{G}$ itself. But this implies that there is some unique $1 \leq j \leq k$ such that $f$ factors through the inclusion $inc_j \in Gr^i(\mathcal{H}_j, \sum_{i=1}^{k} \mathcal{H}_k)$. In other words, $Gr^i(\mathcal{G}, \sum_{i=1}^{k} \mathcal{H}_i) = \prod_{i=1}^{k} Gr^i(\mathcal{G}, \mathcal{H}_i)$.

Conversely, observe that any morphism $f \in GrShape(\mathcal{G}, \star + \star)$ factors as a morphism $\tilde{f} \in Gr^i(\mathcal{G}, W + W)$ followed by the componentwise projection $W + W \rightarrow \star + \star$ in $GrShape$ (where $\tilde{f}$ is unique if and only if $E(\mathcal{G}) = \emptyset$). So, let $\sum_{i=1}^{k} \mathcal{H}_k = W + W$ in Diagram (1.19). Then, if $\mathcal{G}$ is connected, $P_j = \emptyset$ or $P_j \cong \mathcal{G}$ for $j = 1, 2$ so $\mathcal{G}$ satisfies the condition of the lemma.

\[\square\]

Definition 1.20. The category $Gr$ is the full subcategory of $Gr^i_{et}$ whose objects are connected.

Remark 1.21. Clearly $Gr$ does not inherit the monoidal structure from $Gr^i_{et}$. In contrast to $Gr^i$ and $Gr^i_{et}$, it is also not closed under pullbacks.

Example 1.22. For $k \geq 0$, let $L^k$ be the connected line graph with $k$ vertices such that $V(L^k) = V_2(L^k)$, and $E_0(L^k) \cong 2$. 

\[L^k = \bigcirc 2 + 2k \xrightarrow{2k} k\]

In particular, there is an ordering $(v_i)_{i=1}^{k}$ on $V(L^k)$ such that $E(L^k) = \bigsqcup_{i=1}^{k} \{e_i, e_{i+1}\}$ with $e_1 = 1 \in E_0$ and $\tau e_k = 2 \in E_0$, and $E\{v_i\} = \{\tau e_i, e_{i+1}\}$ for $1 \leq i \leq k$.

For $k > k'$, $Gr(L^k, L^{k'}) = \emptyset$ and, if $k \leq k'$, then $Gr(L^k, L^{k'}) \cong 2(k' - k)$. Informally, a morphism in $Gr(L^k, L^{k'})$ corresponds to a choice of ‘oriented’ inclusion of $L^k$ in $L^{k'}$.

Example 1.23. For $l \geq 1$, $W^l$ is the connected wheel graph with $l$ vertices, and such that $V(W^l) = V_2(W^l)$ and $E_0(W^l) = \emptyset$.

\[W^l = \bigcirc 2l \xrightarrow{2l} l\]

So, with the labelling of edges described in Example 1.22, $W^l \cong L^l / (e_1 \sim \tau e_l)$.

For $l, m \geq 1$, $Gr(W^l, W^m) = \emptyset$ if $m$ does not divide $l$. If $m$ does divide $l$ then $Gr(W^l, W^m) \cong 2m$. For all $k \geq 0, m \geq 1$, $Gr(L^k, W^m) \cong 2m$. 
1.2. Elementary graphs and essential categories.

**Definition 1.24.** An elementary graph is a connected, non-empty Feynman graph without inner edges. The category \( \text{elGr} \) is the full subcategory of \( \text{Gr} \) whose objects are elementary graphs.

Elementary graphs with one vertex are corollas. Up to isomorphism, the stick graph \((i)\) is the only elementary graph with no vertices.

Since there are no connected graphs with multiple vertices and no internal edges, and étale morphisms preserve vertex valency, \( \text{elGr} \subset \text{Gr} \) is completely described by

\[
\text{elGr}(i, \mathcal{C}) = \begin{cases} \{id, \tau\}, & \mathcal{C} = (i), \\ \{ch_x\}_{x \in X} \amalg \{ch_x \circ \tau\}_{x \in X}, & \mathcal{C} = \mathcal{C}_X, \text{ } X \text{ a finite set}, \end{cases}
\]

\[
\text{elGr}(\mathcal{C}_X, \mathcal{C}_Y) = \text{FinSet}_{\text{iso}}(X, Y), \text{ for finite sets } X, Y.
\]

Elementary graphs form the basic building blocks from which all other graphs are built. In the graphical construction of CSMs, the role of \( \text{elGr} \) is analogous to that of \( ([0] \to [1]) \subset \Delta \) for categories. In particular, the appropriate Segal condition for CSMs will be defined in terms of \( \text{elGr} \).

**Remark 1.25.** The constructions in this paper are given in terms of the category \( \text{elGr} \supset \text{FinSet}_{\text{iso}} \) rather than its skeleton \( \Sigma^+ \), the full subcategory on the objects \((i)\) and \( \{\mathcal{C}_n\}_{n \in \mathbb{N}} \). It greatly simplifies the description of the multiplication and contraction operations to work with arbitrary finite sets rather than finite cardinals. Moreover, many of the constructions used here involve elementary graphs whose ports have a canonical labelling by an unordered finite set.

For any graph \( \mathcal{G} \), let \( \text{el}(\mathcal{G}) \) be the restriction to \( \text{elGr} \) of the slice above \( \mathcal{G} \) in \( \text{Gr}_{\text{et}} \). We may define the category \( \text{es}(\mathcal{G}) \subset \text{el}(\mathcal{G}) \) of canonical or essential elements of \( \mathcal{G} \). Objects of \( \text{es}(\mathcal{G}) \) are indexed by \( \mathcal{G} \amalg V \) and non-trivial morphisms are indexed by \( H \).

Precisely, for \( v \in V \), denote the formal involution of \( E\{v\} \) by \( P\{v\} \xymatrix{ \tau_{(v)} \ar[rr] & & E\{v\} } \), and let \( \mathcal{C}_{P\{v\}} \) be the corresponding corolla. The inclusion \( E\{v\} \subset E \) induces a weak subgraph \( \mathcal{V} : \mathcal{C}_{P\{v\}} \to \mathcal{G} \). For each \( e \in E \), \( \tilde{\mathcal{V}} : (i) \to \mathcal{G} \) is the canonical inclusion of the stick graph \((i)\) with edge set \( \{e, \tau e\} \subset E \).

If \( h = (e, v) \in H \), then \( s(h) = e \) is the unique element in the intersection \( E\{e\} \cap E(\mathcal{C}_{P\{v\}}) \cap E \). So, \( h \) describes the unique morphism \( \mathcal{H} \in \text{elGr}(i, \mathcal{C}_{P\{v\}}) \) above \( \mathcal{G} \) that fixes \( e \).

**Definition 1.26.** For a graph \( \mathcal{G} \), the category \( \text{es}(\mathcal{G}) \) of essential elements of \( \mathcal{G} \) is the full subcategory of \( \text{el}(\mathcal{G}) \) whose objects are of the form \( x \) (or \( x^{\mathcal{G}} \)) with \( x \in \mathcal{G} \amalg V \), and whose non-identity morphisms are of the form \( \mathcal{H} : \tilde{\mathcal{V}} \to \mathcal{V} \) for \( h = (e, v) \in H \).

**Remark 1.27.** Since the restriction of the domain functor \( \text{el} \to \text{elGr} \) is faithful on \( \text{es}(\mathcal{G}) \), \( \text{es}(\mathcal{G}) \) may be regarded interchangeably as a subcategory of \( \text{elGr} \).

**Lemma 1.28.** Let \( \mathcal{G} \) be a graph. Then \( \text{es}(\mathcal{G}) \) is a skeleton for \( \text{el}(\mathcal{G}) \).

**Proof.** The essential category \( \text{es}(\mathcal{G}) \) is skeletal by definition. Each \((\mathcal{C}, b) \in \text{ob}(\text{elGr} \downarrow \mathcal{G}) \) factors uniquely as an isomorphism followed by some \( x \) in \( \text{es}(\mathcal{G}) \) so the inclusion is essentially surjective on objects.

To see that the inclusion is full, let \( \tilde{e} \in E \) and \( v \in V \). A morphism \( g \in \text{elGr} \downarrow \mathcal{G}(\tilde{e}, v) \) is given by a commuting diagram of the form

\[
\begin{array}{ccc}
(i) & \xymatrix{ \tilde{e} \ar[rr]^g & & \mathcal{C}_{P\{v\}} } \\
& H \ar[ru]_v & \\
& \mathcal{G} & \\
\end{array}
\]
where $g$ fixes an element of $\{e, re\}$. Therefore $g = h$ for $h = (e, v)$ (or $h = (re, v)$) in $H(\mathcal{G})$. Hence the inclusion $\text{es}(\mathcal{G}) \hookrightarrow \text{elGr} \downarrow \mathcal{G}$ is also full and therefore an equivalence.

It is also useful to describe the inverse of the inclusion $\text{es}(\mathcal{G}) \to \text{elGr} \downarrow \mathcal{G}$.

**Definition 1.29.** The essential retract of $\text{elGr} \downarrow \mathcal{G}$ is the functor $\lfloor \cdot \rfloor^\mathcal{G} : \text{elGr} \downarrow \mathcal{G} \to \text{es}(\mathcal{G})$ described by $[C, b]^\mathcal{G} = x$ where $b = x \circ b_{\text{iso}} \in \text{Gr}_{et}^b(C, \mathcal{G})$ and $b_{\text{iso}}$ is an isomorphism in $\text{elGr}$.

So, a morphism $f \in \text{Gr}_{et}^b(\mathcal{G}, \mathcal{G}')$ induces a functor $\text{es}(\mathcal{G}) \to \text{es}(\mathcal{G}')$ by $x \mapsto [f \circ x]^\mathcal{G}'$.

**Proposition 1.30.** The inclusion $\text{elGr} \hookrightarrow \text{Gr}_{et}$ is dense. That is, for all graphs $\mathcal{G}$

$$\mathcal{G} = \text{colim}_{(C, b) \in \text{el}(\mathcal{G})} C$$

canonically.

**Proof.** By Lemma [28] it suffices to prove that

$$\mathcal{G} = \text{colim}_{x \in \text{es}(\mathcal{G})} \text{dom}(x).$$

By definition, $\text{es}(\mathcal{G})$ forms a cocone in $\text{Gr}_{et}^b$ above $\mathcal{G}$ and it follows immediately from the definitions that this is universal. \qed

1.3. Graphical Species. The inclusion $\Phi : \text{elGr} \hookrightarrow \text{Gr}$ induces a geometric morphism (e.g. by [25], Theorem 2, page 359),

$$\Phi_* : \text{PSh(Gr)} \quad\quad\quad\downarrow\quad\quad\quad\downarrow\quad\quad\quad\text{PSh(Gr)} : \Phi^*$$

where

$$\Phi^* : \text{PSh(Gr)} \to \text{PSh(Gr)}, \quad P \mapsto (C \mapsto P(\Phi C)), \quad C \in \text{ob(Gr)}$$

is the pullback, and $\Phi_*$ is the right Kan extension along $\Phi$ of a presheaf $S$ on $\text{elGr}$ to a presheaf on $\text{Gr}$

$$(1.31) \quad \Phi_* : \text{PSh(Gr)} \to \text{PSh(Gr)}, \quad S \mapsto (G \mapsto \text{lim}_{(C, b) \in \text{el}(G)} S(C)).$$

By [25], $\Phi^*$ also has a left adjoint $\Phi_! \dashv \Phi^*$

$$(1.32) \quad \Phi_! : \text{PSh(Gr)} \to \text{PSh(Gr)}, \quad S \mapsto \left( \mathcal{G} \mapsto \begin{cases} S(\mathcal{G}) & \mathcal{G}' \in \text{ob(Gr)} \\ \emptyset & \text{otherwise} \end{cases} \right)$$

and $\Phi_*, \Phi_! : \text{PSh(Gr)} \to \text{PSh(Gr)}$ are fully faithful since $\Phi : \text{elGr} \hookrightarrow \text{Gr}$ is [25] pages 377 and 378].

Since $\Phi$ is dense, the pullback $Y = \Phi^* y : \text{Gr} \to \text{PSh}(\text{elGr})$ of the Yoneda embedding $y : \text{Gr} \hookrightarrow \text{PSh(Gr)}$ along $\Phi$

$$Y : \mathcal{G} \mapsto (C \mapsto \text{Gr}(C, G)), \quad C \in \text{ob(Gr)}, \quad \mathcal{G} \in \text{ob(Gr)}$$

is fully faithful (see e.g. [24], Chapter X., Section 6]).

The full inclusion $\Phi : \text{elGr} \hookrightarrow \text{Gr}$ canonically induces a Grothendieck topology $J$ on $\text{Gr}$ where, for each graph $\mathcal{G}$, a subfunctor $U \subset y\mathcal{G}$ is a cover at $\mathcal{G}$ for $J$ if and only if $\Phi_! Y U \subset U$. Equivalently $U$ is a cover for $J$ if precisely if

$$U(C) = \text{Gr}(C, \mathcal{G}) \text{ for all } C \in \text{ob(Gr)}.$$

**Definition 1.33.** Sheaves for $(\text{Gr}, J)$ are called graphical species, and the category $\text{GS}$ is the category of sheaves $Sh(Gr, J) \subset \text{PSh(Gr)}$.

**Proposition 1.34.** The categories $\text{GS}$ and $\text{PSh(Gr)}$ are canonically isomorphic.
Proof. A presheaf \( S \in \mathbf{PSh}(\mathbf{Gr}) \) is a sheaf for \((\mathbf{Gr}, J)\), if and only if for all graphs \( \mathcal{G} \),

\[
S(\mathcal{G}) \cong \text{PSh}(\mathbf{Gr})(y\mathcal{G}, S) \quad \text{by Yoneda,}
\]

\[
= \text{PSh}(\mathbf{Gr})(\Phi_\ast Y\mathcal{G}, S) \quad \text{by definition of a sheaf for } J,
\]

\[
= \text{PSh}(\mathbf{elGr})(Y\mathcal{G}, \Phi_\ast S) \quad \text{since } \Phi_\ast \dashv \Phi_!,
\]

from which it follows that \( \Phi_\ast \) induces an isomorphism \( \text{PSh}(\mathbf{elGr}) \cong \mathbf{GS} \).

\[\square\]

So, a graphical species \( S \in \text{sh}(\mathbf{Gr}, J) \) with \( S(\iota) = C \) and \( S(\tau(\iota)) = \omega \) is described by \((C, \omega)\) together with a set \( S_X \), for every finite set \( X \), equipped with projections \( S(ch_x) : S_X \to C \), for each \( x \in X \), and a right action of \( \mathbf{FinSet}_{iso} \): for each bijection \( f : Y \to X \) of finite sets, there is an isomorphism \( S_f : S_X \iso S_Y \) that preserves the projections \( S(ch_x) \).

\textbf{Remark 1.35 (Remark on notation).} It is convenient to abuse notation slightly and write \( S(\iota) = (C, \omega) \). The pair \((C, \omega)\) is called the \textit{(involutive) palette of \( S \) and \( S \) a \((C, \omega)\)-coloured graphical species}.

\textbf{Definition 1.36.} For a graphical species \( S \), objects of the category \( \mathbf{el}(S) \) of \( \mathbf{elGr}\)-elements of \( S \) are pairs \((C, \phi)\) with \( C \in \text{ob}(\mathbf{elGr}) \) and \( \phi \in S(C) \), and morphisms \((C, \phi) \to (C, \phi')\) in \( \mathbf{el}(S) \) are morphisms \( g \in \mathbf{elGr}(C, C') \) such that \( S(g)(\phi') = \phi \).

\textbf{Remark 1.37 (Remark on notation).} By an application of the Yoneda Lemma, the category \( \mathbf{el}(Y\mathcal{G}) \) of \( \mathbf{elGr}\)-elements, is canonically isomorphic to the slice category \( \mathbf{el}(\mathcal{G}) \) \(\text{def} = \mathbf{elGr} \downarrow \mathcal{G} \), and these will not be distinguished notionally. In general, since \( \mathbf{Gr} \) is a full subcategory of \( \mathbf{GS} \) under \( Y \), where there is no risk of confusion, I will henceforth write \( \mathcal{G} \in \text{ob}(\mathbf{GS}) \) rather than \( Y\mathcal{G} \in \text{ob}(\mathbf{GS}) \).

\textbf{Example 1.38.} If \( Z \in \text{ob}(\mathbf{GS}) \) is the terminal graphical species \( Z \in \text{ob}(\mathbf{GS}) \), then \( Z(C) = \{ * \} \) for all \( C \in \text{ob}(\mathbf{elGr}) \).

\textbf{Definition 1.39.} For each element \( \mathcal{L} = (c_x)_{x \in X} \in C^X \), the \( \mathcal{L} \)-(coloured) arity \( S_\mathcal{L} \) is the fibre above \( \mathcal{L} \in C^X \) of the map \( (S(ch_x))_{x \in X} : S_X \to C^X \).

\textbf{Definition 1.40.} A morphism \( \gamma \in \mathbf{GS}(S, S') \) is called \textit{palette-preserving} if the component \( \gamma(\iota) : S(\iota) \to S'(\iota) \) of \( \gamma \) at \( \iota \) is the identity map.

If \( \gamma \in \mathbf{GS}(S, S') \) is a palette-preserving morphism of \((C, \omega)\)-coloured graphical species, then \( \gamma_X S_\mathcal{L} \subset S'_{\mathcal{L}} \) for all finite sets \( X \) and all \( \mathcal{L} \in C^X \).

\textbf{Definition 1.41.} For a given palette \((C, \omega)\), \( \mathbf{GS}^{(C, \omega)} \subset \mathbf{GS} \) is the category of \((C, \omega)\)-coloured graphical species and palette-preserving morphisms.

\textbf{Example 1.42.} Let \((C, \omega)\) be an involutive palette. If \( Z^{(C, \omega)} \) is the terminal \((C, \omega)\)-coloured graphical species in \( \mathbf{GS}^{(C, \omega)} \), then

\[
Z_X^{(C, \omega)} = C^X \quad \text{for all finite sets } X.
\]

\textbf{Example 1.43.} Let \( D_\iota \text{def} = Z^{(C, \omega)} \) where \( C = \{ \text{in, out} \} \iso 2 \) and \( \omega = \sigma \) the unique non-trivial involution on \( C \).

Then \( \mathbf{el}(D_\iota) \) is the category of \textit{directed elementary graphs} whose objects, up to isomorphism, are the \textit{directed exceptional edge} \((\uparrow)\) and the \textit{directed corollas} \( C_{X,Y} \) where \( X \) and \( Y \) are finite sets. Morphisms in \( \mathbf{el}(D_\iota) \) are morphisms of elementary graphs that preserve the direction.

A morphism \( \xi \in \mathbf{GS}(S, D_\iota) \) equips a graphical species \( S \) with the structure of a presheaf on \( \mathbf{el}(D_\iota) \). Hence \( S \) is described by a set \( D \) (if \( S \) has palette \((C, \omega)\) then \( D \iso C/\omega \) canonically) and, for each pair of finite sets \( X,Y \), and each \( \xi \in D^X, d \in D^Y \), sets \( S[\xi;d] \), together with an induced action of \( \mathbf{FinSet}_{iso} \times \mathbf{FinSet}_{op}^{op} \).
**Definition 1.44.** Let $S$ be a graphical species. The category $\text{Gr}(S)$ of $S$-structured graphs is the category of $\text{Gr}$-elements of $S$ whose objects, called $S$-structured graphs are pairs $(\mathcal{G}, \alpha)$ with $\mathcal{G} \in \text{ob}(\text{Gr})$ and $\alpha \in S(\mathcal{G})$, and whose morphisms $(\mathcal{G}, \alpha) \to (\mathcal{G}', \alpha')$ are morphisms $g \in \text{Gr}(\mathcal{G}, \mathcal{G}')$ such that $S(g)(\alpha') = \alpha$.

Given a graph $\mathcal{G} \in \text{ob}(\text{Gr})$, $S(\mathcal{G})$ is called the set of $S$-structures on $\mathcal{G}$.

**Remark 1.45 (Remark on notation).** By Yoneda, $S(\mathcal{G}) \cong S(\mathcal{G}, S)$ canonically the same notation $\alpha$ (or $(\mathcal{G}, \alpha)$) will be used for both an $S$-structure $\alpha \in S(\mathcal{G})$, and the corresponding morphism $\alpha \in S(\mathcal{G}, S)$.

**Example 1.46.** Let $Di$ be the graphical species defined in Example 1.43. Then $\text{Gr}(Di)$ is the category of connected directed graphs whose objects are pairs $(\mathcal{G}, \xi)$, $\mathcal{G} \in \text{ob}(\text{Gr})$, $\xi \in Di(\mathcal{G})$. In particular, a $Di$-structure $\xi \in Di(\mathcal{G})$ induces a partition $E = E_{\text{in}} \sqcup E_{\text{out}}$ and $H = H_{\text{in}} \sqcup H_{\text{out}}$ according to $e \in E$ if $\xi(ch_e) = X$, and $h \in H$ if $s(h) \in E_X$ for $X \in \{\text{in}, \text{out}\}$. So, an object $(\mathcal{G}, \xi) \in \text{ob}(\text{Gr}(Di))$ is given by a diagram of finite sets.

![Diagram of finite sets](image)

where the maps $s_{\text{in}}, s_{\text{out}}$, respectively $t_{\text{in}}, t_{\text{out}}$ denote the appropriate restrictions of $s : H \to E$, respectively $t : H \to V$. Since $E_{\text{in}} \cong E_{\text{out}} \cong E/\tau$ canonically, this is equivalently a diagram

$$E/\tau \xleftarrow{s_{\text{in}}/\tau} H_{\text{in}} \xrightarrow{t_{\text{in}}} V \xleftarrow{t_{\text{out}}} H_{\text{out}} \xrightarrow{s_{\text{out}}/\tau} E/\tau,$$

where $s/\tau$ is the obvious composition $H \xrightarrow{\sim} E \xrightarrow{E/\tau}$ of $s$ with the quotient $q : E \to E/\tau$.

Morphisms in $\text{Gr}(Di)$ are quadruples of finite set maps making the obvious diagrams commute, and such that the outer left and right squares are pullbacks. In this case $\text{Gr}(Di)$ is the full subcategory of connected directed graphs in the category of directed graphs and étale morphisms described in [21, Section 1.5].

### 1.4. Gluing constructions

The categories $\text{Gr}^x$ and $\text{Gr}^x_{\text{et}}$ are not closed under finite (co)limits.

**Example 1.48.** Since a singleton set does not admit a non-trivial involution, the terminal object $\star$ in $\text{GrShape}$

$$\star = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.8]
    \node (i) at (0,0) {$\star$};
    \node (0) at (2,0) {0};
    \node (1) at (2,1.5) {1};
    \node (2) at (4,0) {2};
    \draw (i) -- (0);
    \draw (i) -- (1);
    \draw (1) -- (2);
\end{tikzpicture}$$

is not a graph, so $\text{Gr}^x$ and $\text{Gr}^x_{\text{et}}$ are not closed under finite limits.

**Example 1.49.** Recall from Example 1.4 that $W$ is the wheel graph given by

$$\tau_{(W)} \xleftarrow{\{e, \tau e\}} \{h^e, h^{\tau e}\} \xrightarrow{\{\star\}}.$$

The coequaliser of the two distinct morphisms $id_W, \tau_W \in \text{Gr}^x(W, W)$ is $\star$. So $\text{Gr}^x$ and $\text{Gr}^x_{\text{et}}$ are not closed under finite colimits.

**Example 1.50.** Of particular significance to this work is the pair of parallel morphisms $id, \tau : (i) \cong (i)$. In $\text{GrShape}$, the coequaliser of these morphisms is the diagram

$$\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.8]
    \node (i) at (0,0) {$\star$};
    \node (0) at (2,0) {0};
    \draw (i) -- (0);
\end{tikzpicture}$$

Clearly $\circ \notin \text{ob}(\text{Gr}^x)$ since a singleton set does not admit a non-trivial involution.

One important class of diagrams always admit colimits in $\text{Gr}^x_{\text{et}}$.

---

12 See [4] and [17] Section 1 for details concerning comparisons of the various different notions of graph.

13 A construction enlarging $\text{Gr}^x$ to include $\circ$ is discussed in Section 3.1.
Definition 1.51. Let $G$ be a Feynman graph. A $G$-shaped graph of graphs is a functor $\Gamma : \text{el}(G) \to \text{Gr}_{et}^G$ such that

$$\Gamma(a) = (i), \quad \text{for all } (i, a) \in \text{ob}(\text{el}(G)),$$

and

$$E_0(\Gamma(b)) = X \quad \text{for all } (\mathcal{C}_X, b) \in \text{ob}(\text{el}(G)).$$

A $G$-shaped graph of graphs $\Gamma : \text{el}(G) \to \text{Gr}_{et}^G$ is called non-degenerate if, for all $v \in V$, $\Gamma(v)$ has no stick components. Otherwise, $\Gamma$ is called degenerate.

Example 1.52. For any graph $G$, the (non-degenerate) identity $G$-shaped graph of graphs is defined by the functor $\text{dom} : \text{el}(G) \to \text{Gr}, (C, b) \mapsto C$ and has colimit $G$.

Proposition 1.53. A non-degenerate $G$-shaped graph of graphs $\Gamma : \text{el}(G) \to \text{Gr}_{et}^G$ admits a colimit in $\text{Gr}_{et}^G$.

Some auxiliary notions will be used to prove the proposition.

Lemma 1.54. Let $L$ be a finite set, $G$ a graph, and, in, out $\in \text{Gr}_{et}^G(\sum_{l \in L}(\{i\}), G)$ be injective morphisms from an $L$-indexed disjoint union of stick graphs such that the images of in and out are disjoint in $G$, and

$$\text{in}(1_l) \in E_0, \quad \text{out}(2_l) \in E_0, \quad \text{for all } l \in L.$$

Then the diagram

$$(1.55)\quad \sum_{l \in L}(\{i\}) \xrightarrow{\text{in, out}} G$$

admits a coequaliser $\vartheta \in \text{Gr}_{et}^G(G, \overline{\text{Gr}}_{G_{i,j}})$.

Proof. The diagram $1.55$ has a coequaliser $\vartheta : G \to \overline{G} = (E, \overline{H}, \overline{V}, \overline{s}, \overline{t}, \tau)$ in the category $\text{GrShape}$, and so, for $X = E, H, V$, elements of $\overline{X}$ are equivalence classes of elements of $X$.

Since in and out are injective, for each $\tau \in \overline{E}$, either $\vartheta^{-1}(\tau) = \{e\}$ in which case $e \in E - (\text{im}(\text{in}) \cup \text{im}(\text{out}))$, or $\vartheta^{-1}(\tau) = \{\text{in}(j_l), \text{out}(j_l)\} \subset E$, for some $l \in L, j = 1, 2$, in which case $\text{im}(\text{in}(j_l)) \neq \tau \text{out}(j_l) \in E(G)$ since in and out have disjoint images.

If there exist distinct elements $\overline{h}, \overline{h} \in \overline{H}$ such that $\overline{s}(\overline{h}) = \overline{\text{in}(s)}$. In which case, there are $h, h' \in H$ such that $s(h) = \text{in}(j_l)$, $s(h') = \text{out}(j_l)$ for some $l \in L, j = 1, 2$. Since in, out have disjoint images, this contradicts the condition that $\text{in}(1_l) \in E_0, \text{out}(2_l) \in E_0$ for all $l \in L$.

If $\vartheta^{-1}(\tau) = \{e\}$ for some $e \in E - (\text{im}(\text{in}) \cup \text{im}(\text{out}))$, then also $\tau e \in E - (\text{im}(\text{in}) \cup \text{im}(\text{out}))$ so the restriction of $\vartheta$ to points of this form is just the corresponding restriction of $\tau$. Since in and out have disjoint images, if $\tau \in \overline{E}$ is such that $\vartheta^{-1}(\tau) = \{\text{in}(j_l), \text{out}(j_l)\}$ for some $l \in L, j = 1, 2$, then

$$\vartheta^{-1}(\tau e) = \{\tau G \text{in}(j_l), \tau G \text{out}(j_l)\}.$$ 

So $\tau$ is fixed point free since $\tau G$ and $\tau \sum_{l \in L}(\{i\})$ are. Hence $G \in \text{ob}(\text{Gr}_{et}^G)$ and, since $\text{Gr}_{et}^G$ is a full subcategory of $\text{GrShape}$, $\vartheta \in \text{Gr}_{et}^G(G, \overline{G})$ is a coequaliser for in, out in $\text{Gr}_{et}^G$. By the two out of three property for étale maps $\vartheta \in \text{Gr}_{et}^G(G, \overline{G})$ is étale.

Definition 1.56. A shrub $S$ is a graph that is isomorphic to a disjoint union of stick graphs. Following [21] Section 1.5, a pair of disjoint injective morphisms in, out : $S \to G$ is called a gluing datum if there exists a graph isomorphism $\xi : S \xrightarrow{\text{cong}} \sum_{l \in L}$ such that the induced maps $\sum_{l \in L}(\{i\}) \to G$ satisfy the conditions of the lemma above.
Definition 1.57. (See [21] page 8) The core \( G^* = (E^*, H^*, V^*, s^*, t^*, \tau^*) \subset G \) of a graph \( G \) is the maximal closed subgraph of \( G \) in \( Gr^d \). So \( V^* = V \), \( E^* \overset{\text{def}}{=} E_I \) and the corestriction \( s^* : H \supset H^* \to E^* \) of \( s : H \to E \) is an isomorphism. The inclusion \( i_{G}^* : Gr^d(G^*, G) \) is called a core inclusion.

A core inclusion \( i_{G}^* : G^* \hookrightarrow G \) canonically induces a fully faithful functor

\[
i_{G}^* : \text{el}(G^*) \to \text{es}(G) \hookrightarrow \text{el}(G).
\]

Recall [24] Section 3, Chapter IX that a functor \( \Theta : C \to D \) is final if the slice category \( \Theta \downarrow d \) is non-empty and connected for all \( d \in \text{ob}(D) \).

Lemma 1.58. For any graph \( G \) without stick components, the functor

\[
i_{G}^* : \text{el}(G^*) \to \text{es}(G) \hookrightarrow \text{el}(G)
\]

induced by the core inclusion \( i_{G}^* : G^* \hookrightarrow G \) is final.

Proof. It suffices to check that, given a graph \( G \), \( I_{G}^* \downarrow x \) is non-empty and connected for \( x \in \text{ob}(\text{es}(G)) \). An object \( x \) of \( \text{es}(G) \) is either in the image of \( i_{G}^* \), or of the form \( \bar{e}\bar{v} \) where \( \bar{e} = q(e) \in E/\tau \) for some port \( e \in E_0 \).

If \( G \) has no stick components, \( \tau e = s(h) \) for some \( h \in H \) so \( h^G \in \text{es}(G)(\bar{e}, v) \) for \( v = t(h) \in V = V^* \), so \( I_{G}^* \downarrow x \) is non-empty for all \( x \in \text{ob}(\text{es}(G)) \). Connectivity is immediate. \( \square \)

By e.g. [24] page 217, Theorem 1 \( \Theta : C \to D \) is final if and only if for any functor \( \Phi : D \to E \) such that \( \text{colim}_C(\Phi \circ \Theta) \) exists in \( E \), then \( \text{colim}_D\Phi \) exists in \( E \), and the universal morphism

\[
\text{colim}_C\Phi \circ \Theta \to \text{colim}_D\Phi
\]

induced by the inclusion \( \text{im}(\Theta) \subset D \) is an isomorphism. In particular, by the lemma, for any category \( C \) the colimit of any functor \( F : \text{el}(G) \to C \), if it exists, may be computed as the colimit of \( F \circ I_{G}^* : \text{el}(G^*) \to C \).

Proof of Proposition 1.53. To prove that, for all \( G \), a non-degenerate \( G \)-shaped graph of graphs \( \Gamma \) has a colimit in \( Gr \), assume first that \( G \) has no stick components.

Let \( S(E_I) \overset{\text{def}}{=} \sum_{e \in E_I/\tau}(e) \) be the shrub on the internal edges of \( G \), and \( \xi : S(E_I) \overset{\text{cong}}{\to} \sum_{e \in E_I/\tau}((e)) \) be an isomorphism. Then \( \xi \) induces a partition \( H^* = H_1 \sqcup H_2 \) on \( H^* = s^{-1}(E_I) \) by \( h = (e, v) \in H^j \) whenever \( e = \bar{x}^{-1}(j_l) \) for some \( l \in L \).

Therefore, for \( j \in \{1, 2\} \), the morphism

\[
H_j^* \overset{\text{def}}{=} \left( \sum_{h \in H_j} h \right) \in \text{Gr}_{el}^d \left( \sum_{e \in E_I/\tau} (e), \sum_{v \in V} C_{P(v)} \right)
\]

is well defined, and \( H_1^*(e) \in E_0 \) whenever \( h \in H_2 \), \( H_2^*(e) \in E_0 \) whenever \( h \in H_1 \), and, since \( s \) is injective, the images of \( H_1^*, H_2^* \) are disjoint, and they define a gluing datum.

Now, given a non-degenerate \( G \)-shaped graph of graphs \( \Gamma \) and a half-edge \( h = (e, v) \in H^* = s^{-1}(E_I) \) there is a canonical morphism \( \Gamma(h) \in \text{Gr}_{el}^d((e), (\bar{e}, \bar{v})) \) given by \( \tau e \mapsto \tau_{e, v} \in E_0(\Gamma(v)) = E_0(C_{P(v)}) \).

Since, \( \Gamma \) is non-degenerate, the pair of induced morphisms \( H_1^*, H_2^* : \text{Gr}_{el}^d(\sum_{e \in E_I/\tau} (e), \sum_{v \in V} \Gamma(v)) \), with

\[
H_j^* \overset{\text{def}}{=} \sum_{h \in H_j} \Gamma(h) \in \text{Gr}_{el}^d(\sum_{e \in E_I/\tau} (e), \sum_{v \in V} \Gamma(v)) \cong \sum_{h \in H_j} \text{Gr}_{el}^d((e), \Gamma(v)),
\]

for \( j = 1, 2 \), also defines a gluing datum in \( \text{Gr}_{el}^d \).
Let $\mathcal{G}$ be the colimit of this gluing datum in $\text{Gr}_{et}^x$. For $j = 1, 2$ and each $h = (e, v) \in H^j$, the diagram

\[
\begin{array}{ccc}
\text{colim} (\Gamma \circ I^*_G) & \xrightarrow{\Gamma \circ I^*_G(h^*)} & \Gamma \circ I^*_G(C_{P(v^*)}) \\
\downarrow{i_\varepsilon} & & \downarrow{\Gamma(h)} \\
S(E_I) & \xrightarrow{\Gamma(H')} & \sum_{v \in V} \Gamma(v) \\
\downarrow{\sum_{v \in V} \Gamma(v)} & & \sum_{v \in V} \Gamma(v) \\
\mathcal{G}, & & \\
\end{array}
\]

whose unlabelled arrows are the canonical weak inclusions, commutes in $\text{Gr}_{et}^x$, and hence, by Lemma 1.58, $\mathcal{G} = \overline{\mathcal{G}}$.

By density of $\text{elGr} \subset \text{Gr}_{et}^x$, $\text{el}(\mathcal{G})$ is a connected category if and only if $\mathcal{G}$ is a connected graph. In particular, for general $\mathcal{G} \in \text{ob}(\text{Gr}^x)$, the colimit of a $\mathcal{G}$-shaped graph of graphs $\Gamma : \text{el}(\mathcal{G}) \to \text{Gr}_{et}^x$, if it exists, may be constructed component-wise on $\mathcal{G}$. To complete the proof, it therefore remains to observe that any $(i)$-shaped graph of graphs is isomorphic the ‘identity’ $(i) \mapsto (i)$ (see Example 1.52). $\square$

**Remark 1.60.** In fact, all graphs of graphs admit a colimit in $\text{Gr}$. However, the non-degeneracy condition significantly simplifies the proof of the proposition, and is included here since a stronger result is not needed in what follows.

Informally, a non-degenerate $\mathcal{G}$-shaped graph of graphs is a rule for ‘inserting graphs into vertices of $\mathcal{G}$’ by replacing the essential corolla at each vertex of $\mathcal{G}$ with a graph with matching ports. However, this intuitive description of a graph of graphs in terms of graph insertion does not always apply to degenerate graphs of graphs (see also Subsection 3.1).

**Corollary 1.61.** If $\mathcal{G}$ is a graph, and $\Gamma$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\overline{\Gamma}$, then $\Gamma$ induces an identity $E_0(\mathcal{G}) \xrightarrow{\sim} E_0(\overline{\Gamma})$, and, for each $(\mathcal{C}, b) \in \text{ob}(\text{el}(\mathcal{G}))$, the universal map $\Gamma(b) \to \overline{\Gamma}$ is a weak inclusion.

In particular,

$$E(\overline{\Gamma}) \cong E(\mathcal{G}) \amalg \prod_{v \in V} E_I(\Gamma(v)).$$

**Proof.** The final statement follows directly from the first two.

By the proof of Proposition 1.53 only the internal edges of $\mathcal{G}$, and, for each $(\mathcal{C}, b) \in \text{ob}(\text{el}(\mathcal{G}))$, the edges $(E - E_I)(\Gamma(b)) \cong E(\mathcal{C})$ are involved in forming the colimit $\overline{\Gamma}$ of $\Gamma$. Hence the induced map

$$\sum_{\varepsilon \in E/\tau} \varepsilon \xrightarrow{\sim} \sum_{\varepsilon \in E} \Gamma(\varepsilon) \mapsto \overline{\Gamma}$$

is a strict inclusion and restricts to an identity $E_0(\mathcal{G}) \xrightarrow{\sim} E_0(\overline{\Gamma})$. The second statement is immediate. $\square$

**Lemma 1.62.** (Corresponds to [21] Lemma 1.5.12.) If $\Gamma$ is non-degenerate and $\Gamma(\mathcal{C}, b) \in \text{ob}(\text{Gr})$ is connected for all $(\mathcal{C}, b) \in \text{el}(\mathcal{G})$, then $\overline{\Gamma} = \text{colim} \Gamma(\mathcal{G}) \in \text{ob}(\text{Gr}_{et}^x)$ is connected if and only if $\mathcal{G}$ is connected.

**Proof.** We may assume that $\mathcal{G}$ has no stick components since a $(i)$-shaped graph of graphs is, up to isomorphism the identity functor $(i) \mapsto (i)$ with colimit $(i)$. 


So, let $\Gamma : \text{el}(\mathcal{G}) \to \text{Gr}$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\overline{\Gamma}$.

A morphism $\gamma \in \text{GrShape}(\overline{\Gamma}, \star + \star)$ is equivalently a coequaliser in $\text{GrShape}$ of a gluing datum
\begin{equation}
S(E_{\ell}) \xrightarrow{a} \sum_{v \in V(\mathcal{G})} \Gamma(v) \xrightarrow{b} \star + \star.
\end{equation}

If $\Gamma(v)$ is connected for each $v \in V$, then all maps $\Gamma(v) \to \star + \star$ are constant and hence morphisms $\sum_{v \in V} \Gamma(v) \to \star + \star$ are in one-to-one correspondence with morphisms $\sum_{v \in V} C_{p(v)} \to \star + \star$.

So (by the proof of Proposition 1.65 below), there is a bijection between morphisms $\Gamma \to \star + \star$ in $\text{GrShape}$ and morphisms $\mathcal{G} \to \star + \star$ in $\text{GrShape}$. It then follows from Proposition 1.18 that $\overline{\Gamma}$ is connected if and only if $\mathcal{G}$ is connected. $\square$

1.5. Port preserving morphisms.

Remark 1.64 (Notation in this subsection). For any graph $\mathcal{G}$, $x \in \text{ob(\text{es}(\mathcal{G}))}$ usually denotes the essential element corresponding to $x \in E/\tau \Pi V$. For this section only, given $x \in E \Pi V$, $x$ will denote the essential element corresponding to $x$. This differs from the convention elsewhere in that $x \in \text{ob(\text{es}(\mathcal{G}))}$ may correspond to some $x \in E$ rather than $x \in E/\tau$. For each $x \in E \Pi V$, the domain of $x$ will be denoted $D_x \in \text{ob(\text{elGr})}$.

In general, morphisms $f \in \text{Gr}^\ell_{\text{et}}(\mathcal{G}, \mathcal{G}')$ do not satisfy $f(E_0) \subset E_0'$. By Proposition 1.65 below, those that do behave like 'graphical covering maps'.

Let $f \in \text{Gr}^\ell_{\text{et}}(\mathcal{G}, \mathcal{G}')$. Define $k = k(f) : E' \Pi V' \to \mathbb{N}$ to be the function $x' \mapsto |f^{-1}(x')|$. In particular, $k(e') = k(\tau e')$ for all $e' \in E'$.

Proposition 1.65. For any morphism $f \in \text{Gr}^\ell_{\text{et}}(\mathcal{G}, \mathcal{G}')$, $f(E_0) \subset E_0'$ if and only if $k : E' \Pi V' \to \mathbb{N}$ is constant on each connected component of $\mathcal{G}'$.

Proof. If $f(E_0) \not\subset E_0'$ then there is an $e \in E_0$ and $v' \in V'$ such that $f(e) = e' \in E\{v'\}$. Since $f$ is étale, this implies that $k(e') > k(v')$.

For each $x' \in E' \Pi V'$, let $w \in \text{Gr}^\ell_{\text{et}}(\sum_{x' \in f^{-1}(x')} D_x, \mathcal{G})$, be the canonical (weak) inclusion and let $p \in \text{Gr}^\ell_{\text{et}}(\sum_{x \in f^{-1}(x')} D_x, D_{x'})$ be the canonical projection.

To prove that, if $f(E_0) \subset E_0'$, then $k$ is constant on connected components of $\mathcal{G}'$, the first step is to show that $f(E_0) \subset E_0'$ implies that, for each $x' \in E' \Pi V'$, the lower square in the diagram 1.66 is a pullback.

\begin{equation}
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{p} & \mathcal{D}_{x'} \\
\downarrow{a} & & \downarrow{f} \\
\sum_{x \in f^{-1}(x')} D_x & \xrightarrow{w} & \mathcal{G} \\
\end{array}
\end{equation}

This is straightforward for $x' = e'$ since $\text{Gr}^\ell_{\text{et}}(\mathcal{H}, D_{e'})$ is nonempty if and only if $\mathcal{H}$ is a disjoint union of stick graphs, and the map $b : \mathcal{H} \to \mathcal{G}$ factors uniquely through $\sum_{e \in f^{-1}(e')} \tilde{e}$. 

\begin{itemize}
\item For any graph $\mathcal{G}$, $x \in \text{ob(\text{es}(\mathcal{G}))}$ usually denotes the essential element corresponding to $x \in E/\tau \Pi V$. For this section only, given $x \in E \Pi V$, $x$ will denote the essential element corresponding to $x$. This differs from the convention elsewhere in that $x \in \text{ob(\text{es}(\mathcal{G}))}$ may correspond to some $x \in E$ rather than $x \in E/\tau$. For each $x \in E \Pi V$, the domain of $x$ will be denoted $D_x \in \text{ob(\text{elGr})}$.
\end{itemize}
Now let \( x' = v' \in V' \). Given a commuting diagram of the form \ref{eq:1.60}, we may assume, without loss of generality, that
\[
\mathcal{H} = \left( \sum_{i=1}^{m} (i) \right) + \left( \sum_{j=1}^{n} (C_{P(v')})_j \right), \text{ for some } m, n \in \mathbb{N},
\]
and that the restriction
\[
a^{v'} \overset{\text{def}}{=} a|_{\sum_{i=1}^{m_i} (C_{P(v')}), j} : \sum_{j=1}^{n} (C_{P(v')})_j \to C_{P(v')}
\]
is the canonical fold morphism. Then \( a^{v'} \) factors uniquely through \( w \) since \( f \circ b = v' \circ a : \mathcal{H} \to G \) and so \( \text{im}(b_{V(\mathcal{H})}) \subset f^{-1}(v') \).

Now, if \( e' \in E\{v'\} \), then for each \( v \in f^{-1}(v') \) there is a unique element \( e_v \in E\{v\} \cap f^{-1}(e') \). Since \( f(E_0) \subset E'_0 \), all elements of \( p^{-1}(e') \) are of this form. Hence the restriction, \( b_{\sum_{i=1}^{m}(i)} \in \mathcal{G}_c'(\mathcal{G}) \) factors uniquely through \( \sum_{e \in V} v \). Therefore, the lower square of \ref{eq:1.65} is a pullback.

It is now straightforward to show that \( k \) is constant on connected components of \( \mathcal{G} \). Namely, if \( h' = (e', v') \in H' \) and \( h \in f^{-1}(h') \subset H \) then \( h \) has the form \((e, v)\) where \( e \in f^{-1}(e') \subset E \) and \( v \in f^{-1}(v') \subset V \). Let \( x' = e' = s'(h') \), then the lower square of Diagram \ref{eq:1.65} factors as
\[
\sum_{e \in f^{-1}(e')} h \xrightarrow{\sum_{h \in f^{-1}(h')}} \sum_{v \in f^{-1}(v')} C_{P(v)} \xrightarrow{\sum_{h \sum_{v}}} \mathcal{G} \xrightarrow{f} \mathcal{G}'
\]
In particular, since this is a pullback, \( k(e') = k(v') \) for all \( h' = (e', v') \in H' \). Therefore \( k \) defines a functor from \( \text{es}(\mathcal{G}') \) to the discrete category \( \mathbb{N} \). So, \( k \) is constant on connected components of \( \mathcal{G}' \).

\[ \square \]

**Definition 1.67.** A morphism \( f \in \mathcal{G}_{et}'(\mathcal{G}, \mathcal{G}') \) is port preserving if \( f_{E_0} \) is an isomorphism \( E_0 \cong E'_0 \). If \( S \) is a graphical species, a morphism in \( \mathcal{G}(S) \) is port preserving if its underlying morphism of graphs is port preserving.

**Corollary 1.68.** If \( \mathcal{G} \) and \( \mathcal{G}' \) are connected graphs and \( E_0(\mathcal{G}) \neq \emptyset \), a morphism \( f \in \mathcal{G}(\mathcal{G}, \mathcal{G}') \) is port preserving if and only if it is an isomorphism.

**Proof.** Let \( f \in \mathcal{G}(\mathcal{G}, \mathcal{G}') \). Since \( \mathcal{G}, \mathcal{G}' \) are connected, \( k : E' \Pi V' \to N \) has constant value \( K \) by Proposition \ref{eq:1.65}. In particular, if \( E_0 \neq \emptyset \) then \( f \) is port preserving if and only if \( K = 1 \). In other words, \( f \) is an étale morphism of graphs that is bijective on \( E \) and \( V \) (and hence on \( H \)), and therefore an isomorphism.

\[ \square \]

**Remark 1.69.** The condition that \( E_0 \neq \emptyset \) is necessary in the statement of Proposition \ref{eq:1.68}. For example, for any \( l > 1 \), each of the two morphisms in \( \mathcal{G}(W^l, W) \) is trivially port preserving, but certainly not an isomorphism. (However, the method of the proof of Lemma \ref{eq:1.65} can be applied to show that a port preserving morphism from a non-empty graph to a connected graph is always surjective.)

**Corollary 1.70.** Let \( \mathcal{G} \) be a connected graph with only bivalent vertices, i.e. \( V = V_2 \). Then \( \mathcal{G} = \mathcal{L}^k \) or \( \mathcal{G} = W^l \) (see Examples \ref{eq:1.22}, \ref{eq:1.23}) for some \( k \geq 0 \) or \( l \geq 1 \).

**Proof.** As usual, let \( \mathcal{L}^k \) be the connected line graph with \( k \) vertices. Observe that, for any \( f \in \mathcal{G}_{et}'(\mathcal{L}^k, \mathcal{G}) \), the image of \( f \) in \( \mathcal{G} \) is either isomorphic to \( \mathcal{L}^k \) (if \( f \) is injective) or a connected component of \( \mathcal{G} \) of the form \( W^l \) (the wheel graph with \( l \) vertices) for \( 1 \leq l \leq k \). (This can be proved directly using the ordering \( (v_i)_{i=1}^k \) of the vertex set of \( \mathcal{L}^k \) described in Example \ref{eq:1.22})
Now, let $\mathcal{G}$ be any connected graph with $V = V_2$. Recall that $E_0(\mathcal{L}^X) = 2$. If $E_0 \neq \emptyset$, let $e \in E_0$ and let $K$ be the largest number such that there is a morphism $f \in \text{Gr}(\mathcal{L}^X, \mathcal{G})$ satisfying $f(1) = e$. We may assume that $K > 0$ since otherwise $V = V_2 = \emptyset$, in which case $\mathcal{G} = (i) = \mathcal{L}^0$. Now $e' \xrightarrow{\text{def}} f(2) \in E_0$. If not, then $e' \in E\{v'\}$ for some $v' \in V'$ so we can define an etale morphism $\mathcal{L}^{K+1} \to \mathcal{G}$ such that $1 \mapsto e$ and $v_{K+1} \mapsto v'$ contradicting maximality of $K$. Furthermore $e \neq e'$, since otherwise $f$ is not injective, in which case $\text{im}(f) \cong V^l$ for some $1 \leq l \leq k$. So, $E_0(\mathcal{L}^k) \subset E_0$ and therefore, by Corollary 1.68, $\mathcal{G} \cong \mathcal{L}^K$.

On the other hand, if $E_0 = \emptyset$, let $M$ be the largest number such that there is a weak injection $g \in \text{Gr}(\mathcal{L}^M, \mathcal{G})$. In particular, $M > 0$ since $\mathcal{G} \neq (i)$. Set $e = g(1)$. Then $g(2) = \tau e$, since otherwise, either $g(2) \in E_0$ – contradicting the assumption that $E_0 = \emptyset$ – or there is a weak injection $\mathcal{L}^{M+1} \to \mathcal{G}$ with $1 \mapsto e$ and $v_{K+1} \mapsto v' \in V' - \text{im}(g)$, contradicting maximality of $M$. It follows that $g$ factors as a weak injection $\mathcal{L}^M \to \mathcal{W}^M \xrightarrow{\tilde{g}} \mathcal{G}$. In particular since $\tilde{g}(E_0(V^l)) \subset E_0$ trivially and $g$ is a weak injection, $\mathcal{G} \cong \mathcal{W}^M$ by Proposition 1.65.

\[\square\]

2. Non-unital CSMs

The goal of the current section is the construction of a monad $T = (T, \mu, \eta)$ on $\text{GS}$ whose algebras satisfy the definition of CSMs up to admitting a unit for the operadic multiplication.

2.1. $X$-graphs.

**Definition 2.1.** Let $X$ be a finite set. An $X$-graph is a pair $\mathcal{X} = (G, \rho)$, where $G \in \text{ob}(\text{Gr})$ is a connected graph such that $V \neq \emptyset$ and $\rho : E_0 \xrightarrow{\cong} X$ is a bijection of finite sets called an $X$-labelling for $\mathcal{G}$.

Given $X$-graphs $\mathcal{X} = (G, \rho), \mathcal{X'} = (G', \rho')$, an $X$-isomorphism $\mathcal{X} \to \mathcal{X'}$ is an isomorphism $g \in \text{Gr}(G, G')$ that preserves the $X$-labelling. That is, $\rho' \circ g_{E_0} = \rho : E_0 \to X$.

The groupoid $X\text{Gr}_{\text{iso}}$ is the groupoid of $X$-graphs and $X$-isomorphisms.

**Remark 2.2 (Remark on notation).** It is often convenient to use the same notation for labelled and unlabelled graphs. In particular, an $X$-graph $\mathcal{X} = (G, \rho)$ is denoted simply by $G$ when the labelling $\rho$ is trivial or completely canonical. For example, for all finite sets $X$, the corolla $\mathcal{C}_X$ canonically defines an $X$-graph $\mathcal{C}_X = (\mathcal{C}_X, \text{id})$.

Though the notation $\mathcal{X}$ always denotes a labelled graph $(G, \rho)$, where there is no risk of confusion, $\mathcal{X}$ may sometimes be used in place of $G$, even for constructions defined in terms of unlabelled graphs. For example, if $S$ is a graphical species, $S(\mathcal{X}) \xrightarrow{\text{def}} S(G) \times \{\rho\}$, and $\text{el}(\mathcal{X})$ may be used to denote $\text{el}(G)$.

**Example 2.3.** The line graph $\mathcal{L}^k$, with $E_0(\mathcal{L}^k) = 2$, $k \geq 0$ is canonically labelled by $id_2$ and therefore has the structure of a $2$-graph when $k \geq 1$. The other labelling on $\mathcal{L}^k$ is the unique non-identity permutation $\sigma_2 \in \Sigma_2$.

Notice, however, that $\mathcal{L}^0 = (i)$ is not a $2$-graph since its vertex set is empty.

**Example 2.4.** Let $X$ be a finite set and $\mathcal{X}$ an $X$-graph. If $\Gamma : \text{el}(\mathcal{X}) \to \text{Gr}$ is a non-degenerate $\mathcal{X}$-shaped graph of graphs, then the colimit $\mathbf{T} = \text{colim}_{\text{el}(\mathcal{X})}\Gamma$ exists by Proposition 1.53 and, by Corollary 1.61 $\mathbf{T}$ inherits the labelling $\rho$ from $\mathcal{X}$.

**Definition 2.5.** Let $S$ be a graphical species and $X$ a finite set. Objects of the groupoid $X\text{Gr}_{\text{iso}}(S)$ of $S$-structured $X$-graphs are elements $\alpha \in S(\mathcal{X})$ for $\mathcal{X} \in \text{ob}(X\text{Gr}_{\text{iso}})$ and morphisms in $X\text{Gr}_{\text{iso}}(S)(\alpha, \alpha')$ are isomorphisms $g \in X\text{Gr}_{\text{iso}}(G, G')$ such that $S(g)(\alpha') = \alpha \in S(G)$.

**Remark 2.6.** For $\mathcal{X} \in \text{ob}(X\text{Gr}_{\text{iso}})$, let $\text{Aut}_X(\mathcal{X}) \xrightarrow{\text{def}} X\text{Gr}_{\text{iso}}(\mathcal{X}, \mathcal{X})$ denote the automorphism group of $\mathcal{X}$ in the groupoid $X\text{Gr}_{\text{iso}}$. 

If \( g, g' \in X \text{Gr}_{iso}(\mathcal{X}, \mathcal{X}') \) are \( X \)-isomorphisms, then there is some \( \sigma \in \text{Aut}_X(\mathcal{X}) \) and \( \sigma' \in \text{Aut}_X(\mathcal{X}') \) such that \( g' = \sigma' g \sigma \). In particular,

\[
\frac{S(\mathcal{X})}{\text{Aut}_X(\mathcal{X})} = \frac{S(\mathcal{X}')}{\text{Aut}_X(\mathcal{X}')},
\]

insofar as there is a completely canonical (independent of \( g \in X \text{Gr}_{iso}(\mathcal{X}, \mathcal{X}') \)) choice of natural (in \( \mathcal{X} \)) isomorphism

\[
[\alpha] \mapsto [g(\alpha)], \text{ for } \alpha \in S(\mathcal{X}).
\]

**Definition 2.9.** For a graph \( \mathcal{G} \), and graphical species \( S \), a (non-degenerate) \( \mathcal{G} \)-shaped graph of \( S \)-structured graphs is a functor \( \Gamma : \text{el}(\mathcal{G}) \to \text{Gr}(S) \) such that \( \Gamma_e(ch_e) \in S(e) \) for all \( e \in E \), and, for all \( (C_X, b) \in \text{ob}(\text{el}(\mathcal{G})) \),

\[
\Gamma(b) \in \text{ob}(X \text{Gr}_{iso}(S)).
\]

The category of non-degenerate \( \mathcal{G} \)-shaped graphs of \( S \)-structured graphs \( \text{Gr}^{(\mathcal{G})}(S) \) is the subcategory of the functor category \( \text{Fun}(\text{el}(\mathcal{G}), \text{Gr}(S)) \) whose objects are non-degenerate \( \mathcal{G} \)-shaped graphs of \( S \)-structured graphs \( \Gamma : \text{el}(\mathcal{G}) \to \text{Gr}(S) \) and whose morphisms are natural transformations.

The following Corollary of Proposition 1.63 follows immediately from the universal property of colimits.

**Corollary 2.10** (Corollary to Proposition 1.63). For any graphical species \( S \) and any graph \( \mathcal{G} \), a \( \mathcal{G} \)-shaped graph of \( S \)-graphs \( \Gamma \) has a colimit \( \text{colim}_{\text{el}(\mathcal{G})} \Gamma \) in \( \text{Gr}(S) \) given by \( \alpha \in S(\mathcal{T}_Z) \), where \( \mathcal{T}_Z \) is the colimit of the \( \mathcal{G} \)-shaped graph of graphs \( \Gamma : \text{el}(\mathcal{G}) \to \text{Gr} \) obtained by composition of \( \Gamma \) with the forgetful functor \( \text{dom} : \text{Gr}(S) \to \text{Gr} \). In particular, if \( i_b \in \text{Gr}(\mathcal{T}_Z(b), \mathcal{T}_Z) \) is the universal map to the colimit for \( (C, b) \in \text{ob}(\text{el}(\mathcal{G})) \), then \( \alpha \in S(\mathcal{T}_Z) \) is the unique \( S \)-structure satisfying

\[
S(i_b \circ v^{\Gamma_z(b)})(\alpha) = S(v)(\Gamma(b))
\]

for all \( (C, b) \) and all \( v \in V(\mathcal{T}_Z(b)) \).

**Lemma 2.11.** For \( \mathcal{G} \neq \mathcal{C}_0 \), two \( \mathcal{G} \)-shaped graphs of \( S \)-structured graphs \( \Gamma^1, \Gamma^2 \in \text{ob}(\text{Gr}^{(\mathcal{G})}) \) are in the same connected component if and only if \( \Gamma^1(C, b) \cong \Gamma^2(C, b) \) for all \( (C, b) \in \text{ob}(\text{el}(\mathcal{G})) \).

In particular, if \( \Gamma^1 \) and \( \Gamma^2 \) are in the same connected component of \( \text{Gr}^{(\mathcal{G})}(S) \), then \( \Gamma^1 \) and \( \Gamma^2 \) have isomorphic colimits in \( \text{Gr}(S) \).

**Proof.** Let \( \phi : \Gamma^1 \Rightarrow \Gamma^2 \) be a morphism in \( \text{Gr}^{(\mathcal{G})}(S) \). For each \( (C_X, b) \in \text{ob}(\text{el}(\mathcal{G})) \), \( \phi(c_X, b) \in \text{Gr}(S)(\Gamma^1(b), \Gamma^2(b)) \) is, by definition, a port-preserving morphism in \( \text{Gr}(S) \). Since \( \mathcal{G} \neq \mathcal{C}_0 \), \( E_0(\Gamma^j(b)) \neq \emptyset \) for \( j = 1, 2 \), and so, by Corollary 1.68, \( \phi(c_X, b) \) is an isomorphism of \( S \)-structured graphs.

The converse is immediate, as is the final statement.

\[\square\]

### 2.2. A monad for non-unital CSMs.

Recall that a monad \((M, \mu^M, \eta^M)\) on a category \( \mathcal{C} \) is an endofunctor \( M : \mathcal{C} \to \mathcal{C} \) together with natural transformations \( \mu^M : M^2 \Rightarrow M \) (the monad multiplication) and \( \eta^M : \text{id} \Rightarrow M \) (the monad unit), such that, for all \( c \in \text{ob}(\mathcal{C}) \), the diagrams

\[
\begin{align*}
\begin{array}{ccc}
M^3c & \xrightarrow{\mu^MMc} & M^2c \\
\mu^MMc & \searrow & \mu^Mc \\
M^2c & \downarrow & Mc,
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
Mc & \xrightarrow{M\eta^Mc} & M^2c \\
\mu^Mc & \searrow & \mu^Mc \\
Mc & \downarrow & Mc
\end{array}
\end{align*}
\]

commute in \( \mathcal{C} \).
Definition 2.14. The free non-unital CSM functor \( T : GS \to GS \) is the endofunctor defined by

\[
TS(i) = S(i),
\]

and

\[
(2.15) \quad TS_X = \text{colim}_{X \in \text{Gr}_{iso}} S(X)
\]

for all \( S \in \text{ob}(GS) \) and finite sets \( X \).

For a finite set \( X \) and graphical species \( S \), it follows from a direct but lengthy verification (see [19, Section 5], and details in [29]) that

\[
TS_X = \prod_{[X] \in \pi_0(X\text{Gr}_{iso})} \frac{S(X)}{\text{Aut}_X([X])}
\]

(2.16)

where, for \( X = (G, \rho) \), \([X] \in \pi_0(X\text{Gr}_{iso})\) is the connected component of \( X \) in \( X\text{Gr}_{iso} \). The second expression is well defined by 2.6.

So, elements of \( TS_X \) may be viewed as isomorphism classes of \( S \)-structured \( X \)-graphs and two \( X \)-labelled \( S \)-structured graphs \((X, \alpha)\) and \((X', \alpha')\) represent the same class \([X, \alpha] \in TS_X\) precisely when there is an isomorphism \( g \in X\text{Gr}_{iso}(X, X') \) such that \( S(g)(\alpha') = \alpha \).

Isomorphisms of corollas induce relabelings of graph ports in \( TS \).

To describe the projections \( TS(ch_x) \to S(i) \), let \( X \) be an \( X \)-graph and

\[
ch_X^X \overset{\text{def}}{=} ch_{\rho^{-1}(x)} \in \text{Gr}(i, G), \quad 1 \mapsto \rho^{-1}(x) \in E_0(G).
\]

Then \( TS(ch_x) : TS_X \to TS(i) = S(i) \) is the morphism

\[
[X, \alpha] \mapsto S(ch_X^X)(\alpha).
\]

This is well defined since, if \((X, \alpha)\) and \((X', \alpha')\) represent the same element of \( TS_X \), then there is a \( g \in X\text{Gr}_{iso}(X, X') \) such that \( S(g)(\alpha') = \alpha \in S_X \). Therefore \( ch_X^{X'} = g \circ ch_X^X \in \text{Gr}(i, G') \), and

\[
S(ch_X^{X'})(\alpha') = S(ch_X^X) \circ S(g)(\alpha') = S(ch_X^X)(\alpha) \in S(i).
\]

Let us define a natural transformation \( \mu^T : T^2 \Rightarrow T \) in terms of graphs of graphs.

Let \( S \) be a graphical species and \( X \) a finite set. An element of \( T^2S_X \) is represented by an \( X \)-graph \( \mathcal{X} \) together with an element \( \beta \in TS(\mathcal{X}) \), that can be thought of as a functor

\[
e_{\text{el}}(\beta) : \text{el}(\mathcal{X}) \to \text{el}(TS), \quad (C_{X_a}, b) \mapsto (C_{X_b}, S(b)(\beta)), \quad \text{where } S(b)(\beta) \in TS_{X_b}.
\]

In other words, \( \beta \) is represented by \( \mathcal{X} \)-shaped graphs of \( S \)-structured graphs. If, for \( j = 1, 2 \), \((\mathcal{X}^j, \Gamma^j) : \text{el}(\mathcal{X}^j) \to \text{Gr}(S)\) both represent \([X, \beta] \in T^2S_X\) then, \( X^1 \cong X^2 \) and,

\[
(2.17) \quad \text{colim}_{\text{el}(\mathcal{X}^1)} \Gamma^1 = \text{colim}_{\text{el}(\mathcal{X}^2)} \Gamma^2 \in X\text{Gr}_{iso}(S)
\]

by Lemma 2.11, and the assignment

\[
\mu^T S : T^2S \to TS, \quad [X, \beta] \mapsto [\text{colim}_{\text{el}(\mathcal{X})} \Gamma]
\]

is well-defined.

\[\text{This functor is very similar to the CSM endofunctor defined in [19, Section 5], but avoids the problematic case. Kock applies a similar construction, though not restricted to connected graphs, in [22]. (See also Section 5.)}\]
To see that $\mu^T S$ defines a morphism $T^2 S \to TS$ in $\mathbf{GS}$, let $[\mathcal{X}, \beta] \in T^2 S X$ be represented by a graph of graphs $\Gamma : \text{el}(\mathcal{X}) \to \text{Gr}(S)$ with colimit $(\bar{\Gamma}, \alpha) \in X \text{Gr}_{\text{iso}}(S)$. By Corollary 1.61, there is a canonical inclusion $E(\mathcal{X}) \hookrightarrow E(\bar{\Gamma})$ and, for each $e \in E(\mathcal{X})$, 

$$S(ch_{\bar{\Gamma}}(\alpha)) = S(ch_{\mathcal{X}}(\beta)) \in S(i).$$

Hence, for all $x \in X$, there is a commutative diagram of sets

$$
\begin{array}{ccc}
T^2 S X & \xrightarrow{\mu^T S X} & TS X \\
\downarrow & & \downarrow \\
T^2 S(ch_X) & \rightarrow & TS(ch_X)
\end{array}
$$

And so $\mu^T S$ defines a morphism in $\mathbf{GS}(T^2 S, TS)$.

Naturality of $\mu^T S$ in $S$ is immediate and a straightforward modification of [21] section 2.2.3-4] then shows that $\mu^T : T^2 \Rightarrow T$ satisfies the monad axiom 2.12 for $T$.

The unit $\eta^T : id_{\mathbf{GS}} \Rightarrow T$ for $T$ is induced by inclusion of the corollas $C X \in \text{ob}(X \text{Gr}_{\text{iso}})$. Given a graphical species $S$, $\eta^T S(i) \overset{\text{def}}{=} S(i)$, and

$$\eta^T S X \overset{\text{def}}{=} \{[C X, \alpha]\}_{\alpha \in S X} \subset TS X.$$

[21] section 2.2.3-4] may be easily modified to obtain a proof that $\mathbb{T} = (T, \mu^T, \eta^T)$ satisfies the axioms 2.12 and 2.13 for a monad.

Recall that an algebra for a monad $M = (M, \mu^M, \eta^M)$ on a category $C$ is a pair $(c, \theta)$ with $c \in \text{ob}(C)$ and $\theta \in C(Mc, c)$ such that the diagrams

$$
\begin{align*}
\begin{array}{ccc}
M^2 c & \xrightarrow{\mu^M c} & Mc \\
\downarrow & & \downarrow \theta \\
Mc & \xrightarrow{\theta} & c,
\end{array} & \quad
\begin{array}{ccc}
c & \xrightarrow{\eta^M c} & Mc \\
\downarrow & & \downarrow \theta \\
c & \xrightarrow{\theta} & c.
\end{array}
\end{align*}
$$

commute in $C$.

The \textit{Eilenberg-Moore (EM) category} $C^M$ of \textit{algebras for the monad} $M = (M, \mu^M, \eta^M)$ has $M$-algebras as objects and morphisms in $C^M((c, \theta), (c', \theta'))$ are morphisms $f \in C(c, c')$ such that the diagram

$$
\begin{array}{ccc}
Mc & \xrightarrow{Mf} & Mc' \\
\downarrow \theta & & \downarrow \theta' \\
c & \xrightarrow{f} & c'
\end{array}
$$

commutes in $C$.

\textbf{Remark 2.20.} Since $\mu^T S$ and $\eta^T S$ are palette-preserving morphisms in $\mathbf{GS}$, then for all palettes $(C, \omega)$ and all graphical species $S$, $\mathbb{T}$ restricts to a monad $\mathbb{T}^{(C, \omega)}$ on $\mathbf{GS}^{(C, \omega)}$. Then $A \in \text{ob}(\mathbf{GS}^{(C, \omega)})$, together with a morphism $h \in \mathbf{GS}(TA, A)$ is a $\mathbb{T}$-algebra if and only if $(A, h)$ is a $\mathbb{T}^{(C, \omega)}$-algebra.

\textbf{Example 2.21.} Recall from Example 1.38 that $Z$ is the terminal graphical species, so $\text{Gr}(Z) \cong \text{Gr}$ and elements of $TZ$ are port-preserving isomorphism classes of graphs in $\text{Gr}$. The unique morphism $r \in \mathbf{GS}(TZ, Z)$ makes $Z$ into an algebra for $\mathbb{T}$ by assigning to $G \in \text{ob}(X \text{Gr}_{\text{iso}})$ the corolla $C X$.
Example 2.22. Recall Examples 1.41, 1.43 and 1.46. The unique palette-preserving morphism $r^{(C,\omega)} \in GS^{(C,\omega)}(TZ^{(C,\omega)}, Z^{(C,\omega)})$ makes $Z^{(C,\omega)}$ into an algebra for $T$. In particular, $(D_i, r^{D_i})$ is an algebra for $T$, where a directed graph is assigned to the directed coralla with the same oriented ports.

Definition 2.23. Let $S$ be a graphical species. A multiplication $\circ$ on $S$ is given by a family of partial maps

$$- \circ_{x,y}^{X,Y} : S_{X(x)} \times S_{Y(y)} \to S_{X(Y)},$$

defined for all $\phi \in S_{X(x)}$, $\psi \in S_{Y(y)}$ satisfying $S(ch_x)(\phi) = S(ch_y \circ \tau(\tilde{e})))(\psi)$.

The multiplication $\circ$ is commutative if

$$\psi \circ_{y,x}^{Y,X} \phi = \phi \circ_{x,y}^{X,Y} \psi$$

for all finite sets $X, Y$ and all pairs $\phi \in S_{X(x)}$, $\psi \in S_{Y(y)}$ for which the map is defined.

It is invariant if it commutes with the action of $\text{FinSet}_{\text{iso}}$ on $S$.

A contraction $\zeta$ on $S$ is given by a family of partial maps

$$\zeta_{x,y}^X : S_{X(x)} \to S_X$$

defined for all finite sets $X$ and all $\phi \in S_{X(x)}$, satisfying $S(ch_x)(\phi) = \omega S(ch_y \circ \tau(\tilde{e})))(\phi)$.

The contraction $\zeta$ is equivariant if it commutes with the action of $\text{FinSet}_{\text{iso}}$ on $S$.

In particular, if $\zeta$ is an equivariant contraction on $S$, then it is commutative:

$$\zeta_{x,y}^X \phi = \zeta_{y,x}^X \phi$$

for all $X$ and for all $\phi \in S_{X(x)}$ for which the map is defined.

Remark 2.24 (Remark on notation). Let $S$ be a $(C,\omega)$-coloured graphical species, and let $c \in C$, and $c \in C^X, \bar{d} \in C^Y$ for finite sets $X, Y$.

The notation $\circ_{x,y}^{c,\bar{d}} : S_{(c,c)} \times S_{(d,\omega c)} \to S_{cd}$

for the corresponding restriction of $\circ_{x,y}$ is useful. In particular, if $\circ$ is commutative, then

$$\phi \circ_{x,y}^{c,\bar{d}} \psi = \psi \circ_{x,y}^{d,c} \phi$$

whenever either is defined.

Similarly, the notation

$$\zeta_{x,y}^{c} : S_{(c,c,\omega c)} \to S_c$$

is used for the corresponding restriction of $\zeta_{x,y}^X$. If $\zeta$ is commutative, then

$$\zeta_{x,y}^{c} \phi = \zeta_{x,y}^{c} \phi$$

whenever either is defined.

Where the context is clear, it is convenient to drop superscripts and write simply $\circ_{x,y}$ (or $\circ_c$), and $\zeta_{x,y}$ (or $\zeta_c$).

Proposition 2.27. A $(C,\omega)$-coloured graphical species $S$ underlies a $(C,\omega)$-coloured algebra for $T$ if and only if $S$ is a non-unital CSM. That is, $S$ admits an equivariant commutative multiplication $\circ$, and an equivariant contraction $\zeta$ that together satisfy the following four coherence conditions:

The first two conditions concern the multiplication and contraction separately:
(C1.) The multiplication $\circ$ in $S$ is associative. Precisely, for finite sets $X_1, X_2$ and $X_3$, $b \in C^{X_1}, c \in C^{X_2}, d \in C^{X_3}$, and $c, d \in C$, the following diagram commutes.

\[
\begin{array}{ccc}
S_{(c, c)} \times S_{(c, c, d)} \times S_{(d, c, d)} & \xrightarrow{\circ_c \times \text{id}} & S_{(b, b, d)} \times S_{(d, c, d)} \\
\text{id} \times \circ_d & \downarrow & \circ_d \\
S_{(b, c, c)} \times S_{(c, c, d)} & \xrightarrow{\circ_c} & S_{(b, c, d)}
\end{array}
\]

(C2.) For a finite set $X$, $e \in C^X$, and $c, d \in C$, the following diagram commutes.

\[
\begin{array}{ccc}
S_{(e, c, c, d, d)} & \xrightarrow{\zeta_e} & S_{(e, c, d, d)} \\
\zeta_d & \downarrow & \zeta_d \\
S_{(e, c, c)} & \xrightarrow{\zeta_e} & S_e
\end{array}
\]

The second pair of coherence conditions relate the multiplication $\circ$ and the contraction $\zeta$ to each other.

(C3.) For finite sets $X_1$ and $X_2$, $e \in C^{X_1}, d \in C^{X_2}$, and $c, d \in C$, the following diagram commutes.

\[
\begin{array}{ccc}
S_{(e, e, c, c, d, d)} \times S_{(d, c, d)} & \xrightarrow{\zeta_e \times \text{id}} & S_{(e, e, c, d, d)} \times S_{(d, c, d)} \\
\circ_d & \downarrow & \circ_d \\
S_{(e, e, c, c, c)} \times S_{(c, c, c, d)} & \xrightarrow{\zeta_e} & S_{(e, e, c, c, d)}
\end{array}
\]

(C4.) For finite sets $X_1, X_2$, $e \in C^{X_1}, d \in C^{X_2}$, and $c, d \in C$, the diagram

\[
\begin{array}{ccc}
S_{(e, c, d, d)} \times S_{(d, c, d, d)} & \xrightarrow{\circ_c} & S_{(e, c, d, d)} \times S_{(d, c, d, d)} \\
\circ_d & \downarrow & \circ_d \\
S_{(e, c, c, c, d)} \times S_{(c, c, c, c, d)} & \xrightarrow{\zeta_e} & S_{(e, c, c, d)}
\end{array}
\]

commutes.

This correspondence extends to an isomorphism between the EM category $\mathbf{GS}^T$ of $T$-algebras and the category of non-unital CSMs whose morphisms $(S, \circ, \zeta) \to (S', \circ', \zeta')$ are morphisms $\gamma \in \mathbf{GS}(S, S')$ that respect the multiplication and contraction.

Proof. Let $X$ and $Y$ be finite sets and let $\mathcal{M}^{X,Y}_{x, y}$ be the $X \amalg Y$-graph obtained by gluing the corollas $C_{X \amalg \{x\}}$ and $C_{Y \amalg \{y\}}$ along the ports labelled by $x$ and $y$, and applying the induced labelling. Precisely, $\mathcal{M}^{X,Y}$ is the coequaliser of the gluing datum

\[
ch_x, (ch_y \circ \tau) : (i) \Rightarrow (C_{X \amalg \{x\}} + C_{Y \amalg \{y\}}).
\]
For a graphical species $S$, elements of $S(M^{X,Y}_{x,y})$ are determined by ordered pairs $(\phi, \psi) \in S(\underline{c}) \times S(\underline{d};\omega_c)$, where $\underline{c} \in C^X$, $\underline{d} \in C^Y$, and $c \in C$. Let $\mathcal{T}_{x,y}^S(\phi, \psi) \in TS_{\underline{d};\omega_c}$ be the image of $(\phi, \psi) \in S(M^{X,Y}_{x,y})$ under the quotient map $S(M^{X,Y}_{x,y}) \rightarrow TS_{X \coprod Y}$. This map is an inclusion unless $X$ and $Y$ are empty, in which case two distinct ordered pairs $(\phi_1, \psi_1), (\phi_2, \psi_2) \in S(\underline{c}) \times S(\omega_c)$ represent the same element of $TS_0$ if and only if $(\phi_1, \psi_1) = (\psi_1, \phi_1)$.

If $(A, h)$ is a $(C, \omega)$-coloured $T$-algebra, the family of maps defined by the composition

$$h \circ: S(\underline{c}) \times S(\underline{d};\omega_c) \xrightarrow{\mathcal{T}_{X \coprod Y}^S} TS_{\underline{d};\omega_c} \xrightarrow{h} S_{\underline{d};\omega_c},$$

(where $X, Y$ are finite sets, $\underline{c} \in C^X$, $\underline{d} \in C^Y$, and $c \in C$) defines a multiplication on $A$. This multiplication is commutative and equivariant by construction.

Similarly, for a finite set $X$, let $N^X_{x,y}$ be the $X$-graph obtained by gluing the ports of $C_{X \coprod \{x,y\}}$ labelled by $x$ and $y$. Precisely, $N^X_{x,y}$ is the coequaliser of the gluing datum

$$ch_x, ch_y \circ \tau: (i) \xrightarrow{i} C_{X \coprod \{x,y\}}.$$

If $(A, h)$ is a $(C, \omega)$-coloured algebra for $T$, the family of maps defined by the composition

$$h \xi: A_{\underline{c};\omega_c} \xrightarrow{\mathcal{T}_{X}^S} TA_{\underline{c}} \xrightarrow{h} A_{\underline{c}} \ (\text{where } X \text{ is a finite set and } \underline{c} \in C^X, c \in C),$$

defines an equivariant contraction on $A$. 

---

**Figure 4.** Construction of the $(X \coprod Y)$-graph $M^{X,Y}_{x,y}$.

**Figure 5.** Construction of the $X$-graph $N^X_{x,y}$. 
For condition (C1.) (see Figure 6),
\[(\phi_1 \circ_{w,x} \phi_2) \circ_{y,z} \phi_3 = h((\tau^A \phi_1 \circ_{w,x} \phi_2, \phi_3)) = h \mu^T A ((\tau^A \phi_1 \circ_{w,x} \phi_2, \eta^A \phi_3)) = h \mu^T A ((\tau^A \phi_1, \tau^A \phi_2, \phi_3)) = h((\psi^A \phi_1, \phi_2 \circ_{y,z} \phi_3)) = \phi_1 \circ_{w,x} (\phi_2 \circ_{y,z} \phi_3)\]

using the defining properties of monad algebras, and the definition of \( \circ = h \circ \tau \).

\[\text{Figure 6. Coherence condition (C1.) Applying } \mu^T A : T^2 A \to A \text{ amounts to erasing inner nesting.}\]

The coherence conditions (C2.)-(C4.) all follow in the same way from the defining properties 2.18 and 2.19 of monad algebras. Figures 7 - 9 illustrate each condition.

The proof of the converse closely resembles [12, Theorem 3.7].

Namely, let \((S, \circ, \zeta)\) satisfy the conditions in the statement of the proposition. The idea is to construct a structure morphism \(h \in GS(TS, S)\) by successively ‘collapsing’ internal edge orbits of \(S\)-structured graphs \((\mathcal{G}, \alpha)\) to obtain a finite sequence of \(S\)-structured graphs terminating in an \(S\)-structured corolla.

As usual, let \(X\) be a finite set and \([X, \alpha] \in TS_X\).

If \(\mathcal{G}\) has no internal edges, then \(X = C_X\), and so \([X, \alpha] = \eta^T S(\phi)\) for some \(\phi \in S_X\). In this case, define (2.28)

\[h[X, \alpha] \overset{\text{def}}{=} \phi \in S_X.\]

Otherwise, let \(\{e, \tau e\}\) be an internal edge \(\tau\)-orbit of \(\mathcal{G}\). Let \(v = t(e)\) and \(v' = t(\tau e)\), and let \(\mathcal{G}_e\) be the graph obtained from \(\mathcal{G}\) by ‘collapsing’ the \(\tau\)-orbit \(\{e, \tau e\}\) and identifying the endpoints. If \(v \neq v'\), then

\[\mathcal{G}_e \overset{\text{def}}{=} \tau \bigcup (E - \{e, \tau e\}) s^{-1} \tau (H - s^{-1} \{e, \tau e\}) \tau \to V/(v \sim v'),\]

where \(\tau\) is the composition of \(t = t : H \to V\) with the quotient \(V \to V/(v \sim v')\). The class of \(v, v'\) in \(V/(v \sim v')\) is given by \(\tau\), and \(\nabla_{k\alpha}\) is the corresponding essential element of \(\text{es}(\mathcal{G}_e)\). Then, the edge collapse
induces an $S$-decoration $\alpha_\varepsilon$ on $G_\varepsilon$ by

\[
S(\mathbf{w}^\varepsilon)(\alpha_\varepsilon) = \phi_{\varepsilon,\tau\varepsilon} \psi, \quad \text{and} \quad S(\mathbf{w}^\varepsilon)(\alpha_\varepsilon) = S(\mathbf{w}^\varepsilon)(\alpha) \quad w \in V - \{v, v'\}.
\]

Otherwise, if $v = v'$, the graph $G_\varepsilon$ obtained from $G$ by ‘collapsing’ $\{e, \tau e\}$ has the form

\[
G_\varepsilon \overset{\text{def}}{=} \tau \left( E - \{e, \tau e\} \right)^s \quad (H - s^{-1} \{e, \tau e\}) \quad \tau \rightarrow V.
\]
If $S(v)(\alpha) = \phi \in S_{E_{\tau}}$, then we may define an $S$-decoration $\alpha_{\varepsilon}$ on $G_\varepsilon$ by

$$S(v^{\phi})(\alpha_{\varepsilon}) = \zeta_{\varepsilon, \tau} \phi,$$

and

$$S(w^{\phi})(\alpha_{\varepsilon}) = S(w^\phi)(\alpha) \text{ for } w \in V - \{v\}.$$

Hence, an ordering $(\varepsilon_1, \ldots, \varepsilon_N)$ of the set $E_I/\tau$ of internal $\tau$-orbits of $G$, defines a terminating sequence of $S$-structured $X$-graphs

$$(X, \alpha) \mapsto (X_{\varepsilon_1}, \alpha_{\varepsilon_1}) \mapsto (X_{\varepsilon_1}, \alpha_{\varepsilon_2}) \mapsto \cdots \mapsto ((X_{\varepsilon_1})_{\varepsilon_N}, (\alpha_{\varepsilon_1}, \ldots, \varepsilon_N)).$$

Since $((X_{\varepsilon_1})_{\varepsilon_N})$ has no internal edges, and is therefore an $X$-corolla, there exists an element $\phi(X, \alpha) \in S_X$ such that

$$(\alpha_{\varepsilon_1}, \ldots, \varepsilon_N) = \eta^\tau S(\phi(X, \alpha)) \in T S_X.$$

The coherence conditions (C1.)-(C4.) are equivalent to the statement that $\phi(X, \alpha)$ is independent of the choice of ordering of $E_I/\tau$.

If $(\mathcal{X}, \alpha)$ and $(\mathcal{X}', \alpha')$ represent $[\mathcal{X}, \alpha] \in T S_X$, then there is an isomorphism $g \in X \mathcal{G}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha$ and so $g$ uniquely defines isomorphisms $((\alpha_{\varepsilon_1}), \ldots, \varepsilon_k) \cong ((\alpha_{\varepsilon_1'}, \ldots, \varepsilon_k')$ at each successive edge collapse. In particular since $g$ preserves ports, $\phi(X, \alpha) = \phi(X', \alpha') \in S_X$ so the assignment

$$h[\mathcal{X}, \alpha] \overset{\text{def}}{=} \phi(X, \alpha)$$

is well-defined and it is clear from the construction that it extends to a morphism $h \in \mathcal{G}(T S, S)$.

To complete the proof of the proposition, it remains to establish that $h$ satisfies the algebra axioms (2.18) (2.19) for the monad $T$. Compatibility of $h$ with $\eta^\tau$ is immediate from Equation (2.28). Compatibility of $h$ with $\mu^T$ follows since the coherence conditions (C1.) - (C4.) ensure that $h[\mathcal{X}, \alpha]$ is independent of the order of collapse of the internal edges of $\mathcal{X}$. Let $\mathbf{T} : \mathcal{G}(S) \rightarrow \mathcal{G}(S)$ represent $[\mathcal{X}, \beta] \in T^2 S_X$ where $(\mathcal{X}, \beta)$ has colimit $(\mathbf{T}, \alpha) \in \mathcal{G}(S)$, with $\alpha \in S(\mathbf{T})$. By Corollary (1.61)

$$E(\mathbf{T}) = E(\mathcal{X}) \amalg \bigsqcup_{v \in V(\mathcal{X})} E_I(dom \mathbf{T}(v))$$

and, by the coherence conditions (C1.) - (C4.), it doesn’t matter
if we first collapse the internal edges of each $\Gamma(b)$, and then the internal edges of $X$, or if we first take the colimit $\Gamma$ and collapse the edges in any order.

So $(S,\circ,\zeta)$ defines a $T$-algebra $(S, h)$. It is straightforward to observe that these correspondences are inverse to one another and extend to the desired isomorphism of categories. Hence the proposition is proved.

The following definition will be used in the proof of Theorem 3.46.

**Definition 2.29.** Let $S$ be a $(C,\omega)$-coloured graphical species. For all finite sets $X,Y$ and pairs $(\phi,\psi) \in S(M_{X,Y}^X)$ with $\phi \in S(C,c)$ and $\psi \in S(\underline{d},\omega_C)$, as in the proof above, the image

$$\overline{\phi,\psi} \in TS_{\underline{d}}$$

under the quotient map $S(M_{X,Y}^X) \to TS_{X,Y}$ is called the $(S)$-premultiplication of $\phi$ and $\phi$. Likewise, for all finite sets $X$, and $\phi \in S(N_{X}^X)$ with $\phi \in S(C,c,\omega_C)$, the image

$$\overline{\phi} \in TS_{C}$$

under the quotient map $S(N_{X}^X) \to TS_X$ is called the $(S)$-precontraction of $\phi$.

### 3. Monads associated to CSMs

Proposition 2.27 identifies the category of non-unital CSMs with the EM category of algebras for the monad $T$ on $G_S$. The question now is how to modify this in order to obtain (unital) CSMs.

**Definition 3.1.** A CSM $(S,\circ,\zeta,\iota)$ is a graphical species $S$ equipped with a commutative, equivariant multiplication $\circ$ and an equivariant contraction $\zeta$ that together satisfy conditions (C1.) – (C4.) in the statement of Proposition 2.27 and such that, if $(C,\omega)$ is the palette of $S$, then, for each $c \in C$, there is an element $\iota_c \in S_2$ such that, for all finite sets $X$ and all $\phi \in S_{X,l}(z)$ with $S(\chi_X)(\phi) = c \in C$,

$$\phi \circ_{X,\phi}^{(1)} \iota_c = \phi.$$

#### 3.1. The ‘problem of loops’

Before giving the main construction and results of the present paper, it is worthwhile to say a little more about the obstruction to the construction outlined in [19]. This also provides insight into why, for example, the graphical categories used to define the (wheeled properad and modular operad) monads in [14, 18, 17] do not embed into the presheaf categories on which those monads are defined.

The method for constructing multiplicative units in [19] will be familiar to anyone who has seen other constructions of operad families as EM categories for monads ([15, 27, 28]), and relies on a small adjustment to the Definition 2.11 of $X$-graphs to allow degenerate substitutions.

**Definition 3.3.** Let $X$ be a finite set. Objects of the groupoid $XGr_{iso}^{JK}$ are pairs $X = (G,\rho)$ where $G \in ob(Gr)$ is a connected graph with possibly empty vertex set, and $\rho: E_G \xrightarrow{\cong} X$ is a bijection of finite sets. Morphisms in $XGr_{iso}^{JK}$ are graph isomorphisms that preserve the labelling of the ports.

In other words, for all $X \neq 2$, $XGr_{iso}^{JK} = XGr_{iso}$, and $2Gr_{iso} = 2Gr_{iso} \amalg \{(i, id), (i, \sigma_2)\}$. (Note that $(\iota, \sigma_2) = (\tau, id) \in 2Gr_{iso}^{JK}$.)

If it were possible to replace $XGr_{iso}$ by $XGr_{iso}^{JK}$ everywhere in the definition of the monad $T$ (Section 1) and thereby obtain a well-defined monad $T^{JK}$ on $G_S$, then the EM category of $T^{JK}$-algebras would be isomorphic to CSM. In this case, given a $T^{JK}$-algebra $(A, h)$, for each $c \in A(i)$, the element

$$\iota_c \overset{\text{def}}{=} h(i, c) \in A_2$
would define the $c$-coloured multiplicative unit for the corresponding CSM.

However, the definition of a CSM $(S, \circ, \zeta, \iota)$ requires, not only that $S$ admits multiplicative units, but also that these units may be contracted. In other words as well as elements $\iota_c \in S_2$, there are also distinguished elements $\zeta(\iota_c) \in S_0$, that satisfy certain conditions. It is precisely these contracted units that form an obstruction to defining a multiplication for the desired monad $T^{JK}$.

Namely, let $T^{JK} : GS \to GS$ be the endofunctor (defined in [19, Section 5]) defined on objects by

$$T^{JK}S(i) \overset{\text{def}}{=} S(i), \quad T^{JK}S_X \overset{\text{def}}{=} \text{colim}_X \in X \text{Gr}^{JK}_S(X),$$

for all finite sets $X$.

For all graphical species $S$, $TS \subset T^{JK}S$, and hence also $T^2S \subset (T^{JK})^2S$. And, by Proposition 2.27 if $T^{JK}$ admits a multiplication $\mu_{JK} : (T^{JK})^2 \Rightarrow T^{JK}$, it must restrict to $\mu^T_S$ on $T^2S$.

Given an $X$-graph $X$, a functor $\Gamma : \text{el}(X) \to \text{Gr}(S)$ satisfying $\Gamma(a) \in S(i)$ for $a \in \text{Gr}(1, G)$ and

$$\Gamma(b) = (X b, \alpha_b) \in \text{ob}(X \text{Gr}^{JK}_S(S)),$$

for $(C_Y, b) \in \text{ob}(\text{el}(X))$ is called an $X$-shaped graph of $S$-structured $JK$-graphs, and represents an element $[X, \beta] \in (T^{JK})^2S_X$.

It could be hoped that there is a natural transformation $\mu_{JK} : (T^{JK})^2 \Rightarrow T^{JK}$ such that the restriction of $\mu_{JK}S$ to $T^2S$ is just

$$\mu^T_S : T^2S \Rightarrow S, \ [X, \beta] \mapsto \text{colim}_{\text{el}(X)} \Gamma, \rho,$$

where $\Gamma$ is an $X$-shaped graph of $S$-structured graphs that represents $\beta$. However, this is not the case.

As usual, let $W$ be the wheel graph with edge set $\{e, \tau e\}$ (Example 1.4). The essential category $\text{es}(W)$ is isomorphic to the small category with objects $(i, ch_e)$ and $(C_2, b : 1_{C_2} \Rightarrow \tau e)$ and non-identity morphisms

$$ch^C_2 : (i, ch_e) \to (C_2, b) \text{ induced by } 1 \mapsto 2_{C_2}, \quad \text{and } ch^C_1 \circ \tau(\bar{e}) : (i, ch_e) \to (C_2, b) \text{ induced by } 2 \mapsto 1_{C_2}.$$  

Now, for each $c \in C$, there is an element $[W, \beta_c] \in (T^{JK})^2S_0$ represented by the $W$-shaped graph of $S$-structured $JK$-graphs $O^c : \text{el}(W) \to \text{Gr}(S)$ defined by

$$O^c(b) = (i, c) \text{ and } O^c(ch_e) = (i, c)$$

(so the $O^c(b)$-induced isomorphism $E_0(C_2) \cong E_0(i)$ is the canonical one).

Then

$$O^c(ch^C_2)O^c(b) = S(ch^C_2^{(i)})O^c(b) = (i, \omega c),$$

and likewise

$$O^c(ch^C_2 \circ \tau(\bar{e}))O^c(b) = S(ch^{C_1}_1 \circ \tau(\bar{e}))O^c(b) = (i, \omega c).$$

In other words, $O^c(ch^C_2) = O^c(ch^C_1 \circ \tau(\bar{e}))$, whence it follows that the colimit $\text{colim}_{\text{el}(W)}O^c$ exists in $\text{Gr}(S)$ and is given by $(i, c)$.

In the first place this is surprising since $E_0(i) \neq E_0(W)$ so Corollary 1.61 does not hold for $O^c$. Moreover, if $O^{wc} : \text{el}(W) \to \text{Gr}(S)$ is the functor defined by $(C_2, b) \mapsto (i, \omega c)$, then

$$\text{colim}_{\text{el}(W)}O^{wc} = (i, c) \neq (\tau(\bar{e})(i), c) = (i, \omega c) = \text{colim}_{\text{el}(W)}O^c$$

whenever $c \neq \omega c$.

However, $W$ admits a unique non-trivial but trivially port preserving automorphism $\tau(W) : W \to W$, such that $O^c$ and $O^{wc}$ are related by

$$O^{wc}(C, b) = O^c(C, \tau(W) \circ b) \text{ for all } (C, b) \in \text{ob}(\text{el}(W)).$$

In particular, $O^c$ and $O^{wc}$ represent the same element of $(T^{JK})^2S_0$. 

So, to extend $\mu^T S$ to a well-defined natural transformation $\mu_{JK} S : (T^{JK})^2 S \Rightarrow T^{JK} S$, it is necessary to identify $[i, c]$ and $[\tau(i), c]$ in $T^{JK} S$. But, $\tau \in \text{Gr}(i, i)$ is emphatically not port-preserving, and the coequaliser $id_{(i, \tau)} : (i) \rightrightarrows (i)$ in $\text{GrShape}$ is precisely the exceptional loop $\emptyset \not\in \text{ob} (\text{Gr})$ described in Example 1.50. So, whenever $c \neq \omega c$ there is no element of $T^{JK} S$ represented by both $(i, c)$ and $(i, \omega c)$.

An obvious first attempt at a resolution of this problem would be to enlarge $\text{Gr}^t$, and $\text{Gr}$, to include the exceptional loop $\emptyset$. To this end, let us define the category $\text{Gr}^t_{\emptyset}$ of fully generalised Feynman graphs and étale morphisms whose objects are diagrams of the form

$$\tau \circ s \quad H \xrightarrow{t} V$$

where $s : H \rightarrow E$ is injective and $\tau : E \rightarrow E$ is an involution whose restriction to $\text{im}(s) \subset E$ has no fixed points. The involution $\tau$ may, however, have fixed points in $E - \text{im}(s)$.

All other definitions remain unchanged. So, $\text{Gr}^\emptyset \subset \text{Gr}^t_{\emptyset}$ is the full subcategory of fully generalised connected graphs obtained from $\text{Gr}$ by adding the object $\emptyset$ and a unique morphism $(i) \rightarrow \emptyset$.

Define $\text{el}(\emptyset) \overset{\text{def}}{=} \text{elGr} \downarrow \text{Gr}^\emptyset$. Any graphical species $S$, viewed as a presheaf on $\text{elGr}$, may be extended to a presheaf on $\text{Gr}^\emptyset$ by $S(G) \overset{\text{def}}{=} \text{lim}_{\text{elGr}(G)} S(G)$. Since the loop graph $\emptyset$ is the coequaliser in $\text{Gr}^\emptyset$ of the maps $id, \tau : (i) \rightrightarrows (i)$, $\text{el}(\emptyset) \cong \text{el}(G)(i)$ canonically (and both are isomorphic to the connected groupoid of singleton sets) and therefore $S(\emptyset) = S(i)$. It follows that the inclusion $\text{elGr} \rightarrow \text{Gr}^\emptyset$ is not dense and $\text{Gr}^\emptyset$ does not embed in $\text{GS}$.

In particular, if $T^\emptyset : \text{GS} \rightarrow \text{GS}$ is the endofunctor

$$T^\emptyset S(i) \overset{\text{def}}{=} S(i),$$

$$T^\emptyset S_X \overset{\text{def}}{=} \text{colim}_{(\emptyset, \rho) \in X \text{Gr}^\emptyset} S(G),$$

and $(C_2, b) \in \text{ob}(\text{el}(W))$ as before, then the pair $O^e, O^{\omega c}$ of $W$-shaped graphs of $S$-structured JK-graphs

$$O^e : (C_2, b) \mapsto (i, c), \quad \text{with colimit } (i, c) \in \text{Gr}^\emptyset(S)$$

$$O^{\omega c} : (C_2, b) \mapsto (i, \omega c), \quad \text{with colimit } (i, \omega c) \in \text{Gr}^\emptyset(S)$$

represent the same element $[W, \beta] \in (T^\emptyset)^2 S_0$. But since $S(\emptyset) \cong S(i) = C$, the elements $[\emptyset, c], [\emptyset, \omega c] \in T^\emptyset S_0$ are distinct whenever $c \neq \omega c$. It follows that $\mu^T$ does not extend to a multiplication $\mu^\emptyset : (T^\emptyset)^2 \Rightarrow T^\emptyset$.

Remark 3.4 (Some details on [19] and [17]). Let $\eta_{JK} : 1_{\text{GS}} \Rightarrow T^{JK}$ be the natural transformation $\eta_{JK} : 1_{\text{GS}} \Rightarrow T^{JK}$ of endofunctors on $\text{GS}$ obtained as the composition of $\eta^T : 1_{\text{GS}} \Rightarrow T$ with the inclusion $T \subset T^{JK}$. Then $(T^{JK}, \eta_{JK})$ is a pointed endofunctor on $\text{GS}$ and algebras for $(T^{JK}, \eta_{JK})$ have a canonical CSM structure.

It is shown in [17] Section 4) that, if $\text{Gr} \supset \text{Gr}$ is the category described in [19] Section 6) whose objects are connected graphs and whose morphisms factor as ‘refinements’ followed by étale maps [4] then the induced nerve $N_{JK} : \text{CSM} \rightarrow \text{PSh} \text{Gr}$ is fully faithful and is obtained as a right Kan extension of the fully faithful nerve $N_U : \text{CSM} \rightarrow \text{PSh} \text{U}$ described in [17] Theorem 3.6 [see [17]] Section 4) for the details). Here $U \subset \text{Gr}$ is the Reedy wide subcategory that is used in [16] to define Segal modular operads. In each case, the essential image of the induced nerve consists precisely of those presheaves on $\text{Gr}$, respectively $U$.

\footnote{In [19], (a version of) $\text{Gr}$ is called $\text{Gr}$. This is described both as a category of graphs whose morphisms admit a particular factorisation, and as a category of graphs obtained in an identity on objects/faithful factorisation of functors. Though the first is a subcategory of the second, they are not identical. Here $\text{Gr}$ refers to the smaller category described in terms of morphisms admitting a particular factorisation.}
that satisfy the Segal condition. In other words for this description of the category \( \Gr \) the statement of the main theorem in [19, Section 6] and restated as [17, Theorem 4.1] holds.

However, since \( T^{JK} \) does not underly a monad, there is no canonical CSM-structure on objects in its image, so the theorem can’t be proved directly by the method outlined in [19, Section 5]. In particular, \( \Gr \) is a proper wide subcategory of the category \( \widetilde{Gr}_\ast \), obtained in the identity on objects/fully faithful factorisation of \( \Gr \to GS \to \text{free CSM} \to \text{CSM} \) (see Section 3). \( \Gr \) does not include the morphisms in \( \widetilde{Gr}_\ast \) that describe the contracted units. In fact, the functor \( \Gr \to \text{CSM} \) is not fully faithful (see [17]) as is easily observed by considering, for example, the free CSMs on the graphs \( \mathcal{C}_0 \) and (i) (see also Remark 4.5).

3.2. Pointed graphical species. Let us pursue a different approach and examine the definition 3.1 of CSMs more closely.

**Lemma 3.5.** If \((S, \circ, \zeta, \iota)\) is a \((C, \omega)-\text{coloured CSM}\), then, for each \(c \in C\), the \(c\)-coloured unit \(\iota_c\) is unique, and commutes with the involutions \(\omega : C \to C\) and the non-identity automorphism \(\sigma_2 \in \text{Gr}(C_2, C_2)\).

**Proof.** For all finite sets \(X, Y\), and all \(\phi \in S_{X \cup \{x\}}\) with \(S(ch_x) = c\), \(\psi \in S_{Y \cup \{y\}}\) with \(S(ch_y)(\psi) = \omega c\),

\[
\phi \circ_{x, y}^{X, Y} \psi = (\phi \circ_{x, 2}^{X, \{1\}} \iota_c) \circ_{1, y}^{\{2\}, Y} \psi \quad \text{by definition of units},
\]

\[
(3.6) \quad \phi \circ_{x, 2}^{X, \{1\}} (\iota_c \circ_{1, y}^{\{2\}, Y} \psi) \quad \text{by (C.1.)},
\]

\[
\phi = \phi \circ_{x, 2}^{X, \{1\}} (\sigma_2 \iota_c \circ_{1, y}^{\{2\}, Y} \psi) \quad \text{by equivariance of } \circ. \]

In particular, if \(X = \{1\}, x = 2 \) and \(\phi = \iota_{\omega c} \in S_2\), then (3.6) gives

\[
(3.7) \quad \psi = \sigma_2 \iota_c \circ_{2, y}^{\{1\}, Y} \psi
\]

so \(\sigma_2 \iota_{\omega c}\) is a \(\omega c\)-coloured unit for \(\circ\).

Now, let \(\lambda_c\) be another \(c\)-coloured unit for \(\circ\). Then, \(\sigma_2 \lambda_c\) is an \(\omega c\) coloured unit and so

\[
\iota_c = \iota_c \circ_{2, \{1\}}^{\{1\}, \{1\}} \sigma_2 \lambda_c \quad \text{by (3.7)}
\]

\[
(3.8) \quad \iota_c = \iota_c \circ_{2, \{1\}}^{\{1\}, \{2\}} \lambda_c \quad \text{by equivariance of } \circ,
\]

\[
\lambda_c = \lambda_c \quad \text{by commutativity of } \circ. \]

Moreover, for all \(c \in C\), \(\zeta_c(\iota_c) = \zeta_{\omega c}(\iota_{\omega c}) \in S_0\) by the lemma and equivariance of \(\zeta\).

It follows that

1. if a \((C, \omega)\)-coloured graphical species \(S\) admits a unital multiplication \(\circ\) then, for each \(c \in C\), there is a distinguished element \(\iota_c \in S_{(c, \omega c)} \subset S_2\) satisfying \(S(\sigma_2)(\iota_c) = \iota_{\omega c}\). In other words, \(S\) is equipped with an injective map \(\iota : C \to S_2\) that is compatible with the involutions.

2. if a \((C, \omega)\)-coloured graphical species \(S\) admits an equivariant contraction \(\zeta\), and, for each \(c \in C\), a distinguished element \(\iota_c \in S_2\) such that \(S(\sigma_2)(\iota_c) = \iota_{\omega c}\), then \(S\) is equipped with a map \(\zeta^i : \iota(C) \to S_0\) such that \(\zeta^i(\iota_c) = \zeta^i(\iota_{\omega c})\) for all \(c \in C\) and hence a map \(\iota : C \to S_0\) factoring through \(C/\omega\).

The idea now is to add morphisms to \(\text{elGr}\) that encode this additional structure on \(GS\) and then modify the constructions in Sections 1 and 2 for this new category \(\text{elGr}_\ast\).

**Definition 3.9.** The category \(\text{elGr}_\ast \supset \text{elGr}\) of pointed elementary graphs is obtained from \(\text{elGr}\) by formally adjoining morphisms \(u : C_2 \to (i)\) and \(z : C_0 \to (i)\) subject to the relations

- \(u \circ ch_1^{C_2} = id \in \text{elGr}((i), i)\) and \(u \circ ch_2^{C_2} = \tau \in \text{elGr}((i), i)\),
- \(\tau \circ u = u \circ \sigma_2 \in \text{elGr}_\ast((C_2), i)\), and
- \(z = \tau \circ z \in \text{elGr}_\ast((C_0), i)\).
The morphism sets of \( \text{elGr}_* \) are easily described. A direct verification reveals that \( \text{elGr}_*(C,C') = \text{elGr}(C,C') \) for all \( C' \in \text{ob}(\text{elGr}) \), and all \( C \in \text{ob}(\text{elGr}) \) such that \( C \not\cong C_0, C \not\cong C_2 \). In particular
\[
\text{elGr}_*(i, i) = \text{elGr}(i, i) = \{id, \tau\}.
\]

Since \( z = \tau \circ z \),
\[
\text{elGr}_*(C_0, i) = \{z\},
\]
and it follows that
\[
\text{elGr}_*(C_0, C_X) = \begin{cases} 
\{ch_x \circ z \}_{x \in X} \cong X, & \text{for } X \neq 0 \\
\text{elGr}(C_0, C_0) = \{\ast\}, & \text{for } X = 0.
\end{cases}
\]

Again,
\[
\text{elGr}_*(C_2, i) = \{u, \tau \circ u\} = \{u, u \circ \sigma_2\}
\]
follows from a straightforward verification, and therefore
\[
\text{elGr}_*(C_2, C_X) = \text{elGr}(C_2, C_X) \amalg \{ch_x \circ u\}_{x \in \text{elGr}(i, C_X)}.
\]

**Definition 3.10.** The category \( \text{GS}_* \) of pointed graphical species is the category \( \text{PSh}(\text{elGr}_*) \) of presheaves on \( \text{elGr}_* \).

Thus, a pointed graphical species is a graphical species \( S \in \text{ob}(\text{GS}) \) equipped with

1. an injection \( \iota \overset{\text{def}}{=} S(u) : S(i) \rightarrow S_2 \), such that
   - \( S(ch_1^{C_2}) \circ \iota = S(u \circ ch_1^{C_2}) = id_{S(i)} \), and \( S(ch_2^{C_2}) \circ \iota = S(u \circ ch_2^{C_2}) = S(\tau) \),
   - \( S(\sigma_2) \circ \iota = \iota \circ S(\tau) \),
2. and a map
   \[
o \overset{\text{def}}{=} S(z) : S(i) \rightarrow S_0,
\]
that factors through \( S(i)/S(\tau) \).

A morphism \( \gamma : (S, \iota, o) \rightarrow (S', \iota', o') \) of pointed graphical species is a morphism \( \gamma \in \text{GS}(S, S') \) of the underlying graphical species that preserves (contracted) units.

**Remark 3.11** (Remark on notation and terminology). For a \((C, \omega)\)-pointed graphical species, let
\[
\iota_c \overset{\text{def}}{=} \iota(c) \in S((c, \omega_c)) \subset S_2,
\]
and
\[
o_c \overset{\text{def}}{=} o(c) = o(\omega c) \in S_0.
\]
To emphasise the underlying graphical species \( S \), the notation \( \iota^S = \iota \), and \( o^S = o \) will sometimes be used.

Although a pointed graphical species \((S, \iota, o)\) is not, in general, equipped with a multiplication, it will often be convenient to slightly abuse terminology and call the map \( \iota : C \rightarrow S_2 \), and elements \( \iota_c, o_c \), units for \( S \). Similarly, \( o : C \rightarrow S_0 \), and \( o_c \), will be called (contracted) units for \( S \). By contrast, when \( S \) is viewed as a (unpointed) graphical species under the forgetful functor \( \text{GS}_* \rightarrow \text{GS} \), the elements \( \iota_c, o_c \) are simply called distinguished (or specified) elements.

**Definition 3.12.** A morphism \( \gamma \in \text{GS}_*((S, \iota, o), (S', \iota', o')) \) is palette-preserving if the underlying morphism of graphical species is palette-preserving (see Definition 1.40). That is,
\[
\gamma_{(i)} = id : S(i) \rightarrow S'(i).
\]

Given a palette \((C, \omega)\), \( \text{GS}_*(C, \omega) \subset \text{GS}_* \) is the category of \((C, \omega)\)-coloured pointed graphical species and palette-preserving morphisms of graphical species.

**Example 3.13.** The terminal species \( Z \in \text{ob}(\text{GS}) \) trivially has the structure of a pointed graphical species, and is therefore terminal in \( \text{GS}_* \). For any palette \((C, \omega)\), the terminal \((C, \omega)\)-coloured graphical species \( Z^{(C, \omega)} \) is also trivially a pointed species and so is terminal in \( \text{GS}_*(C, \omega) \).
3.3. The category $\text{Gr}_*$.  

**Lemma 3.14.** The category $\text{GS}_*$ is the EM category of algebras for a monad $\mathbb{D}$ on $\text{GS}$.  

**Proof.** Let $(\cdot)^+ : \text{GS} \to \text{GS}_*$ be given by left Kan extension along the inclusion $\text{elGr}^{op} \to \text{elGr}_*^{op}$. Its right adjoint $U^\mathbb{D} : \text{GS}_* \to \text{GS}$ is just the functor that forgets (contracted) units. The monad $\mathbb{D}$ is induced by $U^\mathbb{D}(\cdot \cdot \cdot)^+$. □

If $S$ is a $(C, \omega)$-coloured graphical species, then $S^+$ is described explicitly by

$$S^+(i) = S(i) = C,$$

and

$$S^+_X = \begin{cases} S_X^+, & X \neq 0, 2, \\ S_2 \sqcup \{x^+_c \}_{c \in C} \cong S_2 \sqcup C, & X = 2, \\ S_0 \sqcup \{o^+_c \}_{c \in C/\omega} \cong S_0 \sqcup C/\omega, & X = 0, \end{cases}$$

(3.15)

together with the obvious maps

$$i^+ : C \to S^+_2, \quad c \mapsto i^+_c,$$

$$o^+ : C \to S^+_0, \quad c \mapsto o^+_c,$$

$$S^+(ch^C_x) : S^+_X \to C,$$

$$S^+(ch^C_x)(\phi) = S(ch^C_x)(\phi), \quad \phi \in S_X;$$

$$S^+(ch^C_1)(i^+_x) = c,$$

$$S^+(\sigma^C_2)(\nu^+_x) = c,$$

The right action of $\text{FinSet}_{iso}$ on $S$ is extended to $S^+_X$ by $S^+(\sigma^2)(\nu^+_c) = \nu^+_c$.

Since $(\cdot)^+$ does nothing more than adjoin (contracted) units to $S$, the monadic unit $\eta^\mathbb{D}$ is provided by the inclusion $S \hookrightarrow DS$, and the multiplication $\mu^\mathbb{D}$ is induced by the identity on $S \to DS \hookrightarrow D^2S$ together with the canonical projections $D^2S_2 \to DS_2 : t^D_2 \to \ell^D_2, \ell^D_2 \to t^D_2, D^2S_0 \to DS_0 : o^D_0 \to o^D_0$.

**Definition 3.16.** The category $\text{Gr}_*$ of pointed graphs is the category obtained in the identity on objects/fully-faithful factorisation of $(Y^--)^+ : \text{Gr} \to \text{GS} \to \text{GS}_*$.  

**Proposition 3.17.** The category $\text{elGr}_*$ of pointed elementary graphs is dense in $\text{Gr}_*$, and canonically induces a Grothendieck topology $J_*$ on $\text{Gr}_*$ such that $\text{Sh}(\text{Gr}_*, J_*) \cong \text{GS}_*$.  

**Proof.** (Recall Subsection 1.3)

By definition, the diagram

![Diagram](3.18)

commutes, and, since $Y_* : \text{Gr}_* \to \text{GS}_*$ is fully faithful by construction, it follows immediately by factoring $(Y \circ \Phi)(\cdot)^+ : \text{elGr} \to \text{GS}_*$ that $Y_*|_{\text{elGr}_*} = Y_* \circ \Psi$ is the Yoneda embedding, and the inclusion $\Psi : \text{elGr}_* \hookrightarrow \text{Gr}_*$ is dense.

The pullback $\Psi_* : \text{PSh}(\text{elGr}_*) \to \text{PSh}(\text{Gr}_*)$ has fully faithful right and left adjoints

$$\Psi_* : \text{PSh}(\text{elGr}_*) \rightleftarrows \text{PSh}(\text{Gr}_*) : \Psi^* , \quad \Psi_* : S \mapsto (G \mapsto \lim_{(\mathcal{C}, b) \in \text{elGr}_* \downarrow b} S(\mathcal{C})).$$
and
\[
\Psi^*: \text{PSh}(\text{Gr}_*) \xrightarrow{\perp} \text{PSh}(\text{elGr}_*) : \Psi_1, \Psi_1^* : S \longrightarrow \left( G \mapsto \begin{cases} S(G) & G \in \text{ob}(\text{elGr}_*) \\ \emptyset & \text{otherwise} \end{cases} \right)
\]
and there is a canonical isomorphism \(sh(\text{Gr}_*, J_*) \cong \text{Gs}_*\) where \(J_*\) is the induced topology on \(\text{Gr}_*\) given by
\[
yG \supset UG \in J_*(G) \text{ if and only if } \Psi_1Y_\ast G \subset U.
\]
\[\square\]

For any graph \(G\),
\[
Y_\ast G = \text{Gr}_*(-, G) \overset{\text{def}}{=} (Y_\ast G)^+.
\]
It follows that, if \(C\) is elementary, morphisms \(b \in \text{Gr}_*(C, G)\) have a unique decomposition \(b = x \circ a\) where \(a\) is a morphism in \(\text{elGr}_*\) and \(x \in \text{es}(G)\).

**Definition 3.19.** Objects of the category \(\text{el}_*(S)\) of pointed elements of a pointed graphical species \(S = (S, 1, 0)\) are pairs, \((C, \phi)\) with \(C \in \text{ob}(\text{elGr}_*)\) and \(\phi \in S(C)\). Morphisms \(g \in \text{el}_*(S)(\phi, \phi')\) are morphisms \(g \in \text{elGr}_*(C, C')\) such that \(S(g)(\phi') = \phi\). For all graphs \(G\), \(\text{el}_*(G) \overset{\text{def}}{=} \text{el}_*(Y_\ast G) \cong \text{elGr}_* \downarrow G\) is the category of pointed elements of \(G\).

**Remark 3.20 (Remark on notation).** Since \(Y_\ast : \text{Gr}_* \hookrightarrow \text{Gs}_*\) is a fully faithful embedding, in general, where there is no danger of confusion, \(Y_\ast G \in \text{ob}(\text{Gs}_*)\) will not be distinguished notationally from \(G \in \text{ob}(\text{Gr})\) and both will be denoted simply \(G\).

It is an immediate consequence of the definitions that, for any graph \(G\), the inclusion \(\text{el}(G) \hookrightarrow \text{el}_*(G)\) is final, and, for all graphs \(G\), there is a canonical functor \(\lfloor \cdot \rfloor^G_\ast : \text{el}_*(G) \to \text{es}(G) \hookrightarrow \text{el}(G)\) that extends \(\lfloor \cdot \rfloor^G : \text{el}(G) \to \text{es}(G)\) and, for all \(e \in E\),

- \(\lfloor C_2, ch_c \circ u\rfloor^G_\ast = \lfloor 1, ch_c \rfloor^G\), \(= \bar{e} \in \text{ob}(\text{es}(G))\), and
  \(\lfloor u\rfloor^G = \text{id}_{\bar{e}} : \lfloor C_2, ch_c \circ u\rfloor^G_\ast \to \lfloor 1, ch_c \rfloor^G_\ast\),
- \(\lfloor C_0, ch_c \circ z\rfloor^G_\ast = 1, ch_c \rfloor^G_\ast = \bar{e}\) and
  \(\lfloor z\rfloor^G = \text{id}_{\bar{e}} : \lfloor C_0, ch_c \circ z\rfloor^G_\ast \to \lfloor 1, ch_c \rfloor^G_\ast\).

For a graph \(G\), and \(j = 0, 2\), let \(E^j = E\) and \(q^j : E \to E^j / \tau\) denote the quotient. Recall (Subsection 1.1) that, for \(n \in \mathbb{N}\),
\[
V_n \overset{\text{def}}{=} \{ v \in E \mid |t^{-1}(v)| = n \}, \text{ and } H_n \overset{\text{def}}{=} t^{-1}(V_n) \subset H.
\]

**Lemma 3.21.** Morphisms \(f \in \text{Gr}_*(G, G')\) are characterised by commuting diagrams \(\bar{f}\) of the form
\[
(3.22) \begin{array}{cccccccc}
G & \xrightarrow{\tau} & E & \xleftarrow{s} & H & \xrightarrow{t} & V \\
\downarrow f & & \downarrow f_E & \downarrow f_E & \downarrow f_H & \downarrow f_V \\
G' & \xleftarrow{\tau'} & E' & \xrightarrow{s'} & H' & \xrightarrow{t'} & V'
\end{array}
\]
such that \(\bar{f}_{V_j}^{-1}(E' / \tau')^j \subset V_j\), for \(j = 0, 2\), and the righthand square is a pullback.

**Proof.** Observe first that composition of diagrams of the form \((3.22)\) is well-defined and associative and that, for the defining diagrams of the form \((1.7)\) for morphisms in \(\text{Gr} \hookrightarrow \text{Gr}_*\) satisfy \((3.22)\).
The diagrams

\[(3.23)\]

\[
\begin{array}{ccc}
\tau \circ \{1, 2\} & \overset{id}{\longrightarrow} & \{1, 2\} \\
\downarrow & & \downarrow q_i \\
\{1\} & \longrightarrow & \{v\}
\end{array}
\]

and

\[(3.24)\]

\[
\begin{array}{ccc}
\tau \circ \{1, 2\} & \overset{e_1, e_2}{\longrightarrow} & \{(e_1, v), (e_2, v)\} \\
\downarrow & & \downarrow q_i \\
\{1, 2\} & \longrightarrow & \{v\}
\end{array}
\]

have the form \[3.22\] and satisfy the axioms for \(z : C_0 \to (\cdot)\) and \(u : C_2 \to (\cdot)\) in \(\mathfrak{elGr}_s\).

Therefore, since morphisms in \(\mathfrak{Gr}\) have the form \[3.22\] so do morphisms in \(\mathfrak{elGr}_s\). Moreover, for all \(G\), and all pointed elements \((C, b) \in \mathfrak{el}_s(G)\), \(b\) factors as \(b = x \circ a\) with \(a \in \text{mor}(\mathfrak{Gr})\) and \(x = [b]^\sharp \in \text{es}(\mathfrak{Gr})\). So, there is a diagram \(b\) satisfying \[3.22\] for \(b\), obtained by a composition of diagrams \[3.22, 3.23\] and diagrams in \(\mathfrak{Gr}\).

Now, let \(f : G \to G'\) be a diagram as in \[3.22\]. Then, for all pointed elements, \((C, b) \in \mathfrak{el}_s(G)\), the composition of diagrams \(C \overset{b}{\longrightarrow} G \overset{f}{\longrightarrow} G'\) is easily seen to describe a morphism \(f_b \in \mathfrak{Gr}_s(C, G')\). Hence, \(f\) uniquely determines a natural transformation \(Y_s G \Rightarrow Y_s G'\) satisfying \((C, b) \mapsto (C, f_b)\) for all \((C, b) \in \mathfrak{el}_s(G)\).

By definition, this is precisely a morphism \(f \in \mathfrak{Gr}_s(G, G')\).

For the converse, a morphism \(f \in \mathfrak{Gr}_s(G, G')\) is a natural transformation \(Y_s G \Rightarrow Y_s G : \mathfrak{el}_s(G)^{op} \to \text{Set}\). Let \(f_C\) be the component of \(f\) at \(C \in \text{ob}(\mathfrak{elGr}_s)\). By the above, we may assume that \(G \not\in \text{ob}(\mathfrak{elGr}_s)\) and therefore that \(V_0 = \emptyset\).

Define a map \(f_E : E \to E'\) by

\[f_E(e) \overset{\text{def}}{=} f_u(\hat{e})(e).\]

(Or, equivalently \(f_E(e) \overset{\text{def}}{=} f_c(ch_e)(1)\).)

And define \(f_V : V \to V' \amalg E'/\tau'\) by \(f_V(v) \overset{\text{def}}{=} f_{c_{p(v)}}(v)(v)\) so that

\[f_V(v) \in \begin{cases} E'/\tau' & \text{if } [G']^\sharp f_{c_{p(v)}}(v) = \hat{e}', \text{ for some } e' \in E', \\ V' & \text{otherwise.} \end{cases}\]

It follows that \(f_H : H \to H' \amalg E'\) satisfies

\[f_H(e, v) = \begin{cases} f_E(e) \in E' & \text{if } f_V(v) \in E'/\tau', \\ (f_E(e), f_V(v)) \in H' & \text{otherwise.} \end{cases}\]

By construction, the diagram so formed is commutative and satisfies the conditions for \[3.22\]

\[\square\]

**Remark 3.25.** The image of a morphism \(f \in \mathfrak{Gr}_s^e(G, G')\) in \(G'\) is the strict subgraph of \(G'\) induced by \(f_E(E) \subseteq E', f_H(H) \subseteq H' \amalg E'^2\) and \(f_V(V) \subseteq V' \amalg E'/\tau'^2 \amalg E'/\tau'^0\) together with the appropriate restrictions of \(s', t', \tau'\), and corestriction of \(q' \amalg q'\).
Explicitly, \( \text{im}(f) \hookrightarrow \mathcal{G}' \) is given by the commuting diagram of finite sets

\[
\begin{array}{ccc}
E'(f) \cup (q'^{-1}(f(V) \cap E'/\tau')) & \xrightarrow{\tau'} & f(E) \cup (q^{-1}(f(V) \cap E'/\tau')) \\
\uparrow & & \uparrow \\
E' & \xrightarrow{\tau'} & E' \\
\end{array}
\]

Observe, in particular, that, since \( \mathcal{G} \) is connected, the two copies \( E'^0, E'^2 \) of \( E' \) (and the corresponding quotients) in the description \([3.22]\) are only necessary to define the étale condition on diagrams \([3.22]\) in terms of a pullback. For any given morphism \( f \in \text{Gr}_*(\mathcal{G}, \mathcal{G}') \), it suffices to adjoin one copy of \( E' / \tau' \) to \( V' \).

**Example 3.26.** A surprising consequence of the definitions is that the morphism set \( \text{Gr}_*(\mathcal{W}, i) \) is non-empty.

Recall from Subsection \([3.1]\) that the essential category \( \text{es}(\mathcal{W}) \) of the wheel graph \( \mathcal{W} \) with \( E(\mathcal{W}) = \{ e, \tau e \} \) is isomorphic to the small category with objects \((i, ch_e)\) and \((\mathcal{C}_2, b : 1_{\mathcal{C}_2} \rightarrow \tau e)\), and non-identity morphisms

\[
\begin{align*}
ch_{\mathcal{C}_2}^2 : (i) & \rightarrow \mathcal{C}_2 \quad \text{given by } 1, \mapsto 2_{\mathcal{C}_2}, \text{ and } ch_{\mathcal{C}_2}^2 \circ \tau(e) : (i) \rightarrow \mathcal{C}_2 \quad \text{given by } 2, \mapsto 1_{\mathcal{C}_2}.
\end{align*}
\]

So, there are two morphisms \( e, \tau \circ e \in \text{Gr}_*(\mathcal{W}, i) \) corresponding to the diagrams

\[
\begin{align*}
\text{(3.27)} \\
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{1_{\mathcal{C}_2} \rightarrow \tau e} & \mathcal{C}_2 \\
\downarrow & & \downarrow \\
(i) & \xrightarrow{ch_{\mathcal{C}_2}} & (i) \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{(3.28)} \\
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{1_{\mathcal{C}_2} \rightarrow \tau e} & \mathcal{C}_2 \\
\downarrow & & \downarrow \\
(i) & \xrightarrow{ch_{\mathcal{C}_2}} & (i) \\
\end{array}
\end{align*}
\]

It follows that, for all graphs \( \mathcal{G} \not\ni \mathcal{W} \),

\[
\text{(3.29) } \quad \text{Gr}_*(\mathcal{W}, \mathcal{G}) \cong E(\mathcal{G}) \quad \text{by } ch_e \circ e \mapsto e.
\]

These morphisms play a crucial role in the proof of the nerve theorem, Theorem \([1.3]\).

The following, slightly weakened version of Lemma \([1.9]\) holds in \( \text{Gr}_* \).

**Lemma 3.30.** If \( f \in \text{Gr}_*(\mathcal{G}, \mathcal{G}') \) is a morphism, then \( f \) is completely determined by \( f_E \) and \( f_V \). If \( \mathcal{G} \neq \mathcal{C}_0 \) and \( \mathcal{G}' \not\ni \mathcal{W} \), then \( f_E \) is sufficient to define \( f \).

**Proof.** By the proof of Lemma \([1.9]\), it is immediate that \( f_H \) is determined by \( f_E \) and \( f_V \).

So, assume that \( \mathcal{G} \neq \mathcal{C}_0 \) (hence \( E \neq \emptyset \)) and \( \mathcal{G}' \not\ni \mathcal{W} \). By Proposition \([1.9]\) it suffices to check that \( f_E \) determines \( f_V(v) \) for \( v \in V_2 \).

Let \( v' \in V_2 \) with \( E\{v'\} = \{ e_1', e_2' \} \). Then \( e_1' \neq \tau' e_2' \), since \( \mathcal{G}' \not\ni \mathcal{W} \).

So, if \( v \in V_2 \) with \( E\{v\} = \{ e_1, e_2 \} \subset E_2 \) and

\[
f_E(e_1) = f_E(\tau e_2) \in E',
\]

then it must be the case that

\[
f_V(v) = q'(f_E(e_1)) = q'(f_E(\tau e_2)) \in E'/\tau'.
\]
Otherwise, if \( f_E(e_1) \neq f_E(\tau e_2) \), then \( f_V(v) = t's^{-1}(f_E(e_1)) \in V' \).

\[ \square \]

Example 3.31. The lemma does not hold if \( G' = W \), since in this case, for all graphs \( G \), all \( f \in \Gr_*(G, W) \), and all \( v \in V_2 \) with \( E\{v\} = \{e_1, e_2\} \), \( f_E(e_1) = f_E(\tau e_2) \) by definition of \( W \).

For example, let \( G \) be the wheel graph \( W^2 \) with two vertices \( V(W^2) = \{w^1, w^2\} \) such that, for \( j = 1, 2 \), \( E\{w^1\} = \{\tau e_1, e_2\} \) and therefore \( E\{w^2\} = \{e_1, \tau e_2\} \), and \( W \) is the wheel graph with one vertex \( v \) and edge set \( \{e', \tau W e'\} \).

The morphism \( f \) in \( \Gr_*(W^2, W) \) given by
\[
\begin{align*}
\begin{aligned}
w^1 &\mapsto v, \\
e^1 &\mapsto e', \\
e^2 &\mapsto \tau W e
\end{aligned}
\end{align*}
\]

is distinct from the unique morphism \( g \) in \( \Gr(W^2, W) \cong 2 \) such that \( e_1 \mapsto e' \). Nonetheless, \( f_E = g_E : E(W^2) \to E(W) \).

3.4. A factorisation system on \( \Gr_* \). Let \( G \in \ob(\Gr_*) \) be any connected graph and \( W \subset V_2 \) a subset of bivalent vertices.

Definition 3.32. A vertex deletion functor (for \( W \)) is a \( G \)-shaped (degenerate) graph of graphs \( \Gamma^G_{/W} : \el(G) \to \elGr_* \subset \Gr_* \) such that for \( v \in V \) and \( v \in \ob(\es(G)) \), the corresponding essential element,
\[
\Gamma^G_{/W} : v \mapsto \begin{cases} (i) & \text{if } v \in W, \\ C_p(v) & \text{otherwise.} \end{cases}
\]

If \( \Gamma \) is a vertex deletion functor for \( W \), then there is a canonical natural transformation \( \gamma : \dom \Rightarrow \Gamma^G_{/W} : \el(G) \to \elGr_* \) whose components \( \gamma_b \) on objects \( (C, b) \in \ob(\el(G)) \) are morphisms in \( \elGr_* \),
\[
\gamma_b = \begin{cases} u \in \elGr_*(C_2, i) & \text{when } |b|^G = v \text{ for } v \in W \text{ hence } \dom(b) \cong C_2, \\ id_{\dom(b)} & \text{otherwise.} \end{cases}
\]

In particular, if \( \Gamma^G_{/W} \) has a colimit \( \bar{G}_{/W} \) in \( \Gr_* \), then this corresponds to a cocone of \( \el(G) \) above \( G_{/W} \) and hence, by finality of \( \el(G) \to \el_*(G) \), a morphism \( \del_{/W} \in \Gr_*(G, G_{/W}) \).

Definition 3.33. If the vertex deletion functor \( \Gamma^G_{/W} : \el(G) \to \elGr_* \subset \Gr_* \) admits a colimit \( \bar{G}_{/W} \) in \( \Gr_* \), then the induced morphism \( \del_{/W} \in \Gr_*(G, G_{/W}) \) is called the vertex deletion morphism corresponding to \( W \).

Example 3.34. For \( G = C_2 \) and \( W = V = \{*, \} \), \( \Gamma^G_{/W} \) is determined by
\[
\begin{align*}
\begin{aligned}
(C_2, id) &\mapsto (i), \\
(i, ch_1) &\mapsto (i), \\
(i, ch_2 \circ \tau) &\mapsto (i), \\
\end{aligned}
\end{align*}
\]

and
\[
f \mapsto id_{(i)} \text{ for all } f \in \mor(\el(G)).
\]

So, trivially, \( \Gamma^G_{/W} \) has colimit \( (i) \) in \( \Gr_* \) and \( \del_{/W} = u \in \Gr_*(C_2, i) \).

More generally, let \( G = L^k \) and let the vertex set \( V = (v_i)_{i=1}^k \), and edge set \( E = (e_i, \tau e_i)_{i=1}^k \) have the labelling described in Example 3.22. So \( e_1 = 1 \in E_0, \tau e_k = 2 \in E_0 \) and \( E\{v_i\} = \{e_{i+1}, \tau e_i\}, 1 \leq i \leq k \).

In particular this labelling is fixed by morphisms
\[
b_i \in \Gr(C_2, G), \quad 1 \mapsto e_i, \quad v \mapsto v_i, \quad 1 \leq i \leq k
\]

Then the collection of objects \( b_i \in \Gr(C_2, L^k) \), together with
\[
a_i \in \Gr(i, G), \quad 1 \mapsto e_i,
\]
and morphisms

\[ f_i \overset{\text{def}}{=} ch_1 \in \text{el}(G)(a_i, b_i) \quad \text{and} \quad g_i \overset{\text{def}}{=} ch_2 \circ \tau \in \text{el}(G)(a_{i+1}, b_i) \]

for \(1 \leq i \leq k\), describe a skeleton for \(\text{el}(L^k)\).

If \(W = V\), then, for all \(1 \leq i \leq k\),

\[ \Gamma^g_{/W}(C_2, b_i) = \Gamma^g_{/W}(i, a_i) = (i), \]

and

\[ \Gamma^g_{/W}(f_i) : (i) \to (i) \quad \text{is determined by } 1 \mapsto 1, \]

\[ \Gamma^g_{/W}(g_i) : (i) \to (i) \quad \text{is determined by } 2 \mapsto 2. \]

In other words, \(\Gamma^g_{/W}(f) = \text{id}_{(i)}\) for all \(f \in \text{mor}(\text{el}(G))\).

Hence \(L^k_{/V} = \text{colim}_{\text{el}(L^k)} \Gamma^g_{/V}\) exists in \(\text{Gr}_a\) and is isomorphic to \((i)\).

Example 3.35. By Example 3.26 \(W_{/(v)} = \text{colim}_{\text{el}(W)} \Gamma^W_{/V}\) exists and is isomorphic to \((i)\) in \(\text{Gr}\). (See also Section 3.1) The induced morphism \(\text{del}_{/(v)}\) is precisely \(\epsilon : W \to (i)\).

More generally, let \(G = W^l\) be the wheel graph with \(l\) cyclically ordered vertices \((v_i)_{i=1}^l\). As usual, the edges of \(W^l\) are labelled so that, for each \(v_i\), \(E\{v_i\} = \{\tau e, e_{i+1}\}\), with \(i, i+1 \in \mathbb{Z}/l\mathbb{Z}\). Then the elements

\[ b_i \in \text{Gr}(C_2, G), \quad 1 \mapsto e_i \]

and

\[ a_i \in \text{Gr}(i, G), \quad 1 \mapsto e_i, \]

in \(\text{el}(W^l)\), together with the morphisms

\[ f_i \overset{\text{def}}{=} ch_1 \in \text{el}(G)(a_i, b_i) \quad \text{and} \quad g_i \overset{\text{def}}{=} ch_2 \circ \tau \in \text{el}(G)(a_{i+1}, b_i) \]

for all \(1 \leq i \leq l\), form a skeleton for \(\text{el}(W^l)\).

If \(W = V\), then, for all \(1 \leq i \leq l\), \(\Gamma^g_{/W}(C_2, b_i) = \Gamma^g_{/W}(i, a_i) = (i)\), and

\[ \Gamma^g_{/W}(f_i) : (i) \to (i) \quad \text{is determined by } 1 \mapsto 1, \]

\[ \Gamma^g_{/W}(g_i) : (i) \to (i) \quad \text{is determined by } 2 \mapsto 2. \]

So, all morphisms in \(\text{el}(G)\) are mapped by \(\Gamma^g_{/W}\) to the identity on \((i)\). Hence \(W^l_{/V} = \text{colim}_{\text{el}(W^l)} \Gamma^W_{/V}\) exists in \(\text{Gr}_a\) and is isomorphic to \((i)\).

In particular, for all \(l \geq 1\), there are two distinct morphisms in \(\text{Gr}_a(W^l, i)\) and, for all graphs \(G\), and all \(l \geq 1\)

\[ \text{Gr}_a(W^l, G) \cong \text{Gr}(W^l, G) \amalg E. \]

Now, let \(G, G'\) be graphs such that \(G \neq C_0\) and \(f \in \text{Gr}_a(G, G')\) any morphism. Let \(W_f \subset V_2\) be the set of bivalent vertices such that \(f(v) \in E'/\tau'\). So \(v \in W_f\) if and only if

\[ [f \circ v]^{G'}_a = \tilde{e}' \in \text{es}(G') \quad \text{for some } e' \in E'. \]

In other words, if \([b]^{G} = v\) for \((C_2, b) \in \text{el}(G)\) and \(v \in W\), then,

\[ f \circ b = ch_{e'} \circ u, \text{ or } f \circ b = ch_{e'} \circ \tau \circ u. \]

By construction, if \(G_{/W_f}\) and \(\text{del}_{/W_f} \in \text{Gr}(G, G_{/W_f})\) exist, then they are universal for all graphs \(H\) and all morphisms \(g \in \text{Gr}_a(G, H)\) such that \(W_f \subset W_g\).

Proposition 3.36. For all \(G \in \text{ob}(\text{Gr}_a)\) and all \(W \subset V_2\), \(G_{/W}\) exists in \(\text{Gr}_a\).

Moreover, \(E_0(G) = E_0(G_{/W})\) unless \(G = W^l\) and \(W = V\) for some \(l \geq 1\).
defines a fixed-point free involution.

Clearly if \( W \) is empty, then \( \mathcal{G}/W = \mathcal{G} \) and \( \text{del}/W = \text{id}_\mathcal{G} \in \text{Gr}_s(\mathcal{G}, \mathcal{G}) \). So, let us assume that \( W \neq V \) and \( W \neq \emptyset \).

The graph \( \mathcal{G}/W \) may be constructed as follows. Let \( \mathcal{G}' \subset \mathcal{G} \) be the subgraph with \( V(\mathcal{G}') = W \), \( H(\mathcal{G}') = \prod_{w \in W} H\{w\} \). The edge set \( E(\mathcal{G}') \) is the closure under \( \tau \) of \( \prod_{w \in W} E\{w\} \). By connectedness of \( \mathcal{G} \) and Corollary 3.3.7, \( \mathcal{G}' = \prod_{i=1}^m \mathcal{L}_i \) is a disjoint union of line graphs such that \( k_i \geq 1 \) for all \( i \). In particular \( E_0(\mathcal{G}') = \prod_{i=1}^m \{1, 2\} \).

Set

\[
\begin{align*}
V/W & \overset{\text{def}}{=} V - W, \\
H/W & \overset{\text{def}}{=} H - H(\mathcal{G}') \\
E/W & \overset{\text{def}}{=} E - (E(\mathcal{G}') - E_0(\mathcal{G}'))
\end{align*}
\]

Since \( E(\mathcal{G}') \cap E/W = \prod_{i=1}^m \{1, 2\} \), the map \( \tau/W : E/W \to E/W \) given by

\[
\begin{align*}
\tau/W(e) &= \tau e \quad \text{for } e \in E - E(\mathcal{G}'), \\
\tau/W(1) &= 2_i \quad \text{for } 1 \leq i \leq m
\end{align*}
\]
defines a fixed-point free involution.

Then,

\[
\tau/W \bigcirc E/W \leftarrow s/W \rightarrow H/W \rightarrow t/W \rightarrow V/W,
\]

(where \( s/W, t/W \) are just the restrictions of \( s \) and \( t \)) is a well-defined graph and canonically isomorphic to \( \mathcal{G}/W \).

Moreover, when \( W \neq V \), the morphism \( \text{del}/W \in \text{Gr}_s(\mathcal{G}, \mathcal{G}/W) \) restricts to the identity on ports. Hence the final statement, that \( E_0(\mathcal{G}) = E_0(\mathcal{G}/W) \) except when \( \mathcal{G} = \mathcal{W} \) and \( W = V \) for some \( l \geq 1 \) follows immediately\[^6\]

The following corollary is immediate from the universal property of \( \text{del}/W \in \text{Gr}_s(\mathcal{G}, \mathcal{G}/W) \) and Proposition 1.6.8.

**Corollary 3.37.** Let \( \mathcal{G} \in \text{ob}(\text{Gr}_s) \) with \( \mathcal{G} \neq \mathcal{C}_0 \), and let \( f \in \text{Gr}_s(\mathcal{G}, \mathcal{G}') \). Then \( f \) factors uniquely as a vertex deletion morphism \( \text{del}/W_f \in \text{Gr}_s(\mathcal{G}, \mathcal{G}/W_f) \) followed by a morphism \( f/W_f \in \text{Gr}(\mathcal{G}/W_f, \mathcal{G}') \).

In particular, if \( E_0(\mathcal{G}) \neq \emptyset \) and \( f \in \text{Gr}_s(\mathcal{G}, \mathcal{G}') \) induces an isomorphism \( E_0 \overset{\cong}{\rightarrow} E_0' \) (i.e. \( f \) is port preserving), then \( f/W_f \in \text{Gr}(\mathcal{G}/W_f, \mathcal{G}') \) is an isomorphism.

In particular, if \( \mathcal{V} \mathcal{D} \) is the class of morphisms in \( \text{Gr}_s \) consisting of \( z : \mathcal{C}_0 \to (i) \), the vertex deletion morphisms, and the isomorphisms, then \( (\mathcal{V} \mathcal{D}, \text{mor}(\text{Gr})) \) forms an orthogonal factorisation system on \( \text{Gr}_s \).

**Example 3.38.** For \( \mathcal{G} \neq \mathcal{C}_0 \), let \( (\mathcal{G})^- \) be the set of graphs in \( \text{ob}(\text{Gr}) \) obtained from \( \mathcal{G} \in \text{ob}(\text{Gr}) \) by deleting a (possibly empty) subset of \( V_2 \). Then

\[
\text{Gr}^*(\mathcal{G}, \mathcal{G}') = \prod_{\mathcal{G}^- \in (\mathcal{G})^-} \text{Gr}(\mathcal{G}^-, \mathcal{G}').
\]

\[^6\]In [17], the extra specification of ‘boundary’ as a subset of \( E_0 \) means that morphisms corresponding to \( \text{del}/W \) always induce isomorphisms on boundaries, even when \( W = V \) and \( \mathcal{G} = \mathcal{W} \).
In particular, for all \( k \in \mathbb{N} \),
\[
\text{Gr}_*(\mathcal{L}^k, G) \cong \prod_{j=0}^{k} \binom{k}{j} \text{Gr}(\mathcal{L}^j, G).
\]

**Definition 3.40.** (Compare Definition 3.44.) Let \((S, \iota, o)\) be a pointed graphical species. Objects of the category \( \text{Gr}_*(S) = \text{Gr}_*(S, \iota, o) \) of \( S \) structured graphs are pairs \((G, \alpha)\) with \( G \in \text{ob}(\text{Gr}_*) \) and \( \alpha \in S(G) = \lim_{e \in G} S_e \), and morphisms in \( \text{Gr}_*(S)((G, \alpha), (G', \alpha')) \) are morphisms \( g \in \text{Gr}_*(G, G') \) such that \( S(g)(\alpha') = \alpha \).

**Example 3.41.** Let \( G \) be a graph and \( W \subset V_2 \), \( W \neq V \). If \( \alpha \in S(G) \) is such that the vertex deletion morphism \( \text{del}_W \) induces a morphism of pointed \( S \)-structured graphs in \( \text{Gr}_*(S)((G, \alpha), (G/W, \alpha')) \), then
\[
S(v^G)(\alpha) = S(v^{G/W})(\alpha') \quad \text{for all } v \in V - W,
\]
and, by definition of \( \text{del}_W \), there is a unique \( e \in E/W \) such that
\[
S(b)(\alpha) = S(b)S(\text{del}_W)(\alpha') = S(\text{ch}_\nu \circ u)(\alpha') = \iota \circ S(\text{ch}_\nu)(\alpha').
\]
So,
\[
S(b)(\alpha) \in \text{im}(\iota)
\]
for each \((C, b) \in \text{ob}(\text{el}(G))\) such that \( |b|^G = v \), with \( v \in W \).

**Definition 3.42.** Given \((G, \alpha) \in \text{ob}(\text{Gr}_*(S))\), \( G \neq C_0 \), the set
\[
W_\alpha = \{ v \mid \text{there exists } (C, b) \in \text{ob}(\text{el}(G)) \text{ such that } |b|^G = v \text{ and } S(b)(\alpha) \in \text{im}(\iota) \} \subset V_2
\]
is the set of vertices decorated by units (for \( \alpha \)).

For \( \alpha \in S(G) \) such that \( W \subset W_\alpha \), the unique element \( \alpha' \in S(G/W) \) satisfying \( \text{del}_W \in \text{Gr}_*(S)((G, \alpha), (G/W, \alpha')) \) is denoted by \( \alpha/W \). If \( W = W_\alpha \), then \( \alpha/W \in S(G/W_\alpha) \) will be denoted by \( \alpha_0 \).

Informally, \( \text{del}_W \in \text{Gr}_*(S)((G, \alpha), (G/W, \alpha/W)) \) deletes a subset of vertices ‘\( \alpha \)-decorated’ by units. Clearly, for any \( \alpha \in S(G) \), then \( \alpha_0 \in S(G/W_\alpha) \) is obtained by deleting all vertices \( \alpha \)-decorated by units and hence, if \( \text{del}_W \in \text{Gr}_*(S)((G/W_\alpha, \alpha_0), (G/W_\alpha/W_\alpha, \alpha_0/W_\alpha)) \) then \( W' = \emptyset \).

**Corollary 3.43.** (Corollary to Proposition 3.30) Let \((S, \iota, o)\) be a pointed graphical species and \((G, \alpha) \in \text{ob}(\text{Gr}_*(S))\). Further, let \( W_1, W_2 \) be disjoint subsets of \( W_\alpha \). Then \((G/W_1)/W_2 = G/W_1 W_2\) and the induced vertex deletion morphism
\[
\text{del}_{W_1 W_2}^G \in \text{Gr}_*(G/W_1, G/W_1 W_2)
\]
defines a morphism in \( \text{Gr}_*(S)((G/W_1, \alpha/W_1), (G/W_1 W_2, \alpha/W_1 W_2)) \).

**Proof.** This is straightforward. For \( W_1 \cap W_2 = \emptyset \) then \((G/W_1)/W_2 = G/W_1 W_2\) by Examples 3.34 and 3.35. Otherwise, if \( W_1 \cap W_2 \neq \emptyset \), then the graphs \((G/W_1)/W_2 \) and \( G/W_1 W_2 \) are constructed via the method of the proof of Proposition 3.30. In particular, they have the same vertex, and half edge sets, and \( \text{del}_{W_1}^G \) induces a port-preserving morphism \( \text{del}_{W_1 \cap W_2}^G \in \text{Gr}_*(G/W_1, G/W_1 W_2) \). Therefore \( E_0(G/W_2)/W_1) \cong E_0(G/W_2) \), and hence \( E(G/W_1)/W_2) \cong E(G/W_1 W_2) \) canonically, such that all the induced structure maps agree.

**Definition 3.44.** Let \((S, \iota, o) \in \text{ob}(\text{Gr}_*)\) be a \((C, \omega)\)-coloured pointed graphical species and let \( \mathcal{L}^k \) be the line graph with labelled edges \( \{e_i, \tau e_i\}_{i=1}^k \) and vertices \( \{v_i\}_{i=1}^k \) together with, for \( 1 \leq i \leq k \), morphisms \( b_i \in \text{Gr}_*(C, \mathcal{L}^k) \), \( 1 \to e_i, \quad v \mapsto v_i \) as in Example 3.32. For each \( c \in C \), the \( c \)-unit \( S \)-structure \( \mathcal{L}^k(\iota_c) \in S(\mathcal{L}^k) \) is defined by
\[
S(b_i)(\mathcal{L}^k(\iota_c)) = \iota_c \quad \text{for all } 1 \leq i \leq k.
\]
and morphisms $b_i \in \text{Gr}(C_2, W^l)$, $1 \mapsto e_i$, $v \mapsto v_i$ as in Example 3.35. For each $c \in C$, the $c$-unit $S$-structure $W^l(\iota_c) \in S(W^l)$ is defined by $S(b_i)(W^l(\iota_c)) = \iota_c$ for all $1 \leq i \leq l$.

For all graphs $G$, all $e \in E$, and all $S$-structures $\alpha \in S(G)$, if $c = S(ch_e)(\alpha)$, then $ch_e \circ \text{def}_{/W} \in \text{Gr}_*(S)(\mathcal{H}(\iota_c), \alpha)$ for $\mathcal{H} = L^k, W^l, k \geq 0, l \geq 1$.

3.5. A distributive law for CSMs.

**Lemma 3.45.** There is a distributive law $\lambda : TD \Rightarrow DT$ describing a composite monad $\mathbb{D} \mathbb{T}$ on $\text{GS}$. 

**Proof.** Let $S$ be a $(C, \omega)$-coloured graphical species.

For a finite set $X$, let $[X, \alpha] \in TDS_X$ be represented by $(X, \alpha) \in DS(X)$. The set $W = W_\alpha$ of vertices of $X$ ‘decorated by adjoined distinguished elements’ is defined as $W = W_\alpha \overset{\text{def}}{=} \{v \mid DS(v)(\alpha) \notin S(C_{P(v)})\} \subset V$.

By definition of $D$, $TS \subset DTS$. There is also a canonical inclusion $TS \to TDS$ where, for each finite set $X$, $TS_X \subset TDS_X$ corresponds to the elements $[X, \alpha]$ with $W_\alpha = \emptyset$. The natural transformation $\lambda : TD \Rightarrow DT$ will restrict to the identity on $TS$.

Assume first that $W = V$. Then, $X = 0$ or $X \cong 2$ by Corollary 1.70. If $X = 0$, then either $(X, \alpha) = (G_0, o_c^{DS})$, or $(X, \alpha) = W^l(\iota_c^{DS})$ for some $c \in C$. In each case, set $\lambda[X, \alpha] = o_c^{DTS}$.

If $X = 2$, then $(X, \alpha) = L^k(\iota_c^{DS})$ for some $c \in C$ and $k \geq 1$, and we can set $\lambda[L^k(\iota_c^{DS})] = \iota_c^{DTS}$.

Otherwise, if $W \neq V$ and $W \neq \emptyset$, set $\lambda[X, \alpha] = [X_{W_\alpha}, \alpha_0] \in TS_X$ (see Definition 3.42).

The verification that $\lambda$ so defined extends to a natural transformation $TD \Rightarrow DT$, and that it moreover satisfies the four axioms [3] Section 1 for a distributive law is straightforward, and is omitted here, except to observe the role of Corollary 3.43 in the proof that the diagram\[ \begin{array}{cccc}
T(D^2) & \xrightarrow{\lambda_D} & DT(D) & \xrightarrow{D\lambda} & D^2T \\
\downarrow{T(D^2)} & & \downarrow{D\lambda} & & \downarrow{(\mu^D)T} \\
TD & \xrightarrow{\lambda} & DT & & DT
\end{array} \]

commutes. Namely, for any $[X, \alpha] \in TD^2S_X$ denote by $[X, \alpha']$ the element $T\mu^D D[X, \alpha] \in TDS_X$, and define the sets $W^2 \overset{\text{def}}{=} \{v \mid D^2S(v)(\alpha) \notin DS(C_{P(v)})\} \subset V$,
and
\[ W^1 \equiv \{ v \mid D^2 S(v)(\alpha) \not\in S(C_{P(v)}) \} - W^2 \subset V. \]

So, for \( j = 1, 2 \), \( W^j \) is the set of vertices decorated by distinguished elements adjoined in the \( j \)th application of \( D \).

Then, if \( W^1 \cup W^2 \not= V \), the diagram gives
\[ [X, \alpha] \xrightarrow{\lambda D} [X/W^2, \alpha/W^2] \xrightarrow{D\lambda} [(X/W^2)/W^1, (\alpha/W^2)/W^1] \in TS_X \]
and
\[ [X, \alpha'] \xrightarrow{T(\mu^\ast)} [X, \alpha'] \xrightarrow{\lambda} [X/((W^1W^2), \alpha/(W^1W^2))], \]
and the result follows from an application of Lemma 3.48.

The remainder of this section is concerned with the proof of the first main theorem.

**Theorem 3.46.** The EM category \( GS^{\text{DT}} \) of algebras for the composite monad \( \text{DT} \) is canonically isomorphic to \( CSM \).

The extra structure of the (contracted) units in \( GS_* \) enables an enlargement of the categories \( XGr_{iso}(S) \) [2.1]. Namely, for all pointed graphical species \((S, t, o)\), there exist port-preserving morphisms \( g \) in \( Gr_*(S) \) from \( S \)-structured graphs \((G, \alpha)\) that are not isomorphisms, even when \( E_0(G) \not= \emptyset \).

**Definition 3.47.** Let \((S, t, o)\in ob(GS_*)\) be a \((C, \omega)\)-coloured pointed graphical species. The category \( XGr_*^{sim}(S, t, o) \) or \( XGr_*(S) \), of similar \( S \)-structured \( X \)-graphs is the category whose objects are elements \( \alpha \in S(\mathcal{X}) \) with \( \mathcal{X} = (G, \rho) \in XGr_{iso} \) and whose morphisms in \( XGr_{iso}(\alpha, \alpha') \) are induced by port-preserving morphisms \( g \in Gr_*(S)((G, \alpha), (G', \alpha')) \) such that \( g = g_{/W^1} \circ \text{del}_{/W^2} \) where \( g_{/W^1} \in Gr(G_{/W^1}, G') \) is an isomorphism, and \( \text{del}_{/W^2} : G \to G_{/W^2} \) is a vertex deletion morphism.

Objects \((\lambda^1, \alpha^1), (\lambda^2, \alpha^2) \in ob(XGr_*(S))\) are called similar, written \((\lambda^1, \alpha^1) \sim (\lambda^2, \alpha^2)\), or just \( \alpha^1 \sim \alpha^2 \) if they are in the same connected component of \( XGr_*(S) \).

So, \( XGr_*(S) \) is obtained from \( XGr_{iso}(S) \) by adjoining vertex deletion morphisms.

**Example 3.48.** For all \( c \in C \) and all \( k, k', l, l' \geq 1 \),
\[ L^k(t_c) \sim L^{k'}(t_c) \in 2Gr_*(S) \text{ and } W^l(t_c) \sim W^{l'}(t_c) \in 0Gr_*^{sim}(S). \]
However \( L^k(t_c) \not\sim L^l(t_{\omega c}) \) unless \( c = \omega c \).

**Lemma 3.49.** For any finite set \( X \), and \((\lambda, \alpha) \in ob(XGr_*^{sim}(S))\), the connected component \( (XGr_*^{sim}(S))_{(\lambda, \alpha)} \) containing \((\lambda, \alpha)\) has a terminal object. In particular, objects \((\lambda^1, \alpha^1), (\lambda^2, \alpha^2) \in XGr_*^{sim}(S)\) are similar if and only if there is a \((\lambda, \alpha) \in XGr_*^{sim}(S)\) and a cospan
\[ (\lambda^1, \alpha^1) \longrightarrow (\lambda, \alpha) \longleftarrow (\lambda^2, \alpha^2) \]
in \( XGr_*^{sim}(S) \).

**Proof.** If \( \mathcal{X} = C_0 \), then the component of \((\lambda, \alpha)\) is trivial.

Assume therefore that \( \mathcal{X} \not= C_0 \). For \( \alpha \in S(\mathcal{X}) \), \( W_\alpha \subset V_2 \) is, as usual, the set of vertices decorated by units (Definition 3.42). If \( W_\alpha \not= V \), then clearly \((\lambda^1, \alpha_0) \) is terminal in the component of \((\lambda, \alpha)\) in \( XGr_*^{sim}(S) \) by definition.

When \( W = V \), then \((\lambda, \alpha) = L^k(t_c) \) or \((\lambda, \alpha) = W^l(t_c) \) for \( 2Gr_*^{sim}(S) \), and that \( W(t_c) \cong W(t_{\omega c}) \) and it is also immediate that \((C_2, t_c) \) is the terminal object in the component of \( t_c \) in \( 0Gr_*^{sim}(S) \).
The converse is true by definition. □

Definition 3.50. For a graph \( \mathcal{G} \) and pointed graphical species \((S, \iota, o)\), a (non-degenerate) \( \mathcal{G} \)-shaped pointed graph of \( S \)-structured graphs is a functor \( \Gamma_* : \text{el}_*(\mathcal{G}) \to \text{Gr}_* (S) \) whose restriction to \( \text{el}(\mathcal{G}) \) is a (non-degenerate) \( \mathcal{G} \)-shaped graph of \( S \)-structured graphs (Definition 2.54). Precisely, \( \Gamma_*(ch_e) \in S(1) \) for all \( e \in E \), and for all \((C, b) \in \text{ob}(\text{el}(\mathcal{G}))\),

\[
\Gamma_*(b) = (X_b, \alpha_b) \text{ where } X_b \text{ is an } X \text{-graph and } \alpha_b \in S(X_b)
\]

By Corollary 2.10 and finality of the inclusion \( \text{el}(\mathcal{G}) \to \text{el}_*(\mathcal{G}) \), the colimit \( \text{colim}_{\text{el}_*(\mathcal{G})} \Gamma_* = \text{colim}_{\text{el}(\mathcal{G})} \Gamma_* \) of any non-degenerate \( \mathcal{G} \)-shaped pointed graph of \( S \)-structured graphs \( \Gamma_* \) exists in \( \text{Gr}_* (S) \).

Remark 3.51. If \( S \) is the terminal graphical species \( Z \), then \( \Gamma_* : \text{el}_*(\mathcal{G}) \to \text{Gr}_* (S) = \text{Gr}_* \) is called a (non-degenerate) \( \mathcal{G} \)-shaped pointed graph of graphs and its restriction to \( \text{el}(\mathcal{G}) \) is then precisely a non-degenerate \( \mathcal{G} \)-shaped graph of graphs (Definition 1.51).

Remark 3.52. Given a \( \mathcal{G} \)-shaped graph of \( S \)-structured graphs \( \Gamma \) with colimit \( \Gamma_* \), \( \Gamma \) may always be extended to a \( \mathcal{G} \)-shaped pointed graph of \( S \)-structured graphs \( \Gamma_* \), with the same colimit, by

\[
\Gamma_*(C, b) = \Gamma(C, b), \text{ for all } (C, b) \in \text{ob}(\text{el}(\mathcal{G}))
\]

and, for all \( e \in E \),

\[
\Gamma_*(C_2, ch_e \circ u) = \iota_2(\Gamma_*(ch_e)) \in S_2, \text{ and } \Gamma_*(C_0, ch_e \circ z) = o(\Gamma_*(ch_e)) \in S(C_0).
\]

Definition 3.53. For \( \mathcal{G} \in \text{ob}(\text{Gr}_*) \) and \((S, \iota, o) \in \text{ob}(\text{GS}_*)\), the category \( \text{Gr}^{(\mathcal{G})}_*(S) \) of \( \mathcal{G} \)-shaped pointed graph of \( S \)-structured graphs is the subcategory of the functor category \( \text{Fun}(\text{el}_*(\mathcal{G}), \text{Gr}_* (S)) \) whose objects are \( \mathcal{G} \)-shaped pointed graphs of \( S \)-structured graphs \( \Gamma_* : \text{el}_*(\mathcal{G}) \to \text{Gr}_* (S) \) and whose morphisms are natural transformations.

If \( S \) is the terminal pointed species \( Z \), then \( \text{Gr}^{(Z)}_*(S) \) is denoted \( \text{Gr}^{(\mathcal{G})}_*(S) \).

Lemma 3.54. Given a pointed species \((S, \iota, o)\) and a graph \( \mathcal{G} \neq \mathcal{C}_0 \), pointed graphs of \( S \)-structured graphs \( \Gamma_1^*, \Gamma_2^* \in \text{ob}(\text{Gr}^{(\mathcal{G})}_*(S)) \) are in the same connected component of \( \text{Gr}^{(\mathcal{G})}_*(S) \) if and only if there is a \( \Gamma_* \in \text{ob}(\text{Gr}^{(\mathcal{G})}_*(S)) \), and a cospan

\[
\begin{array}{ccc}
\Gamma_1^* & \longrightarrow & \Gamma_* \\
\Gamma_* & \leftarrow & \Gamma_2^*
\end{array}
\]

in \( \text{Gr}^{(\mathcal{G})}_*(S) \).

In particular, if, for \( j = 1, 2 \), \( \Gamma_* \overset{\text{def}}{=} \text{colim}_{\text{el}_*(\mathcal{G})} \Gamma_j^* \), and \( \Gamma_* = \text{colim}_{\text{el}(\mathcal{G})} \Gamma_* \), then there is a cospan

\[
\begin{array}{ccc}
\Gamma_1^* & \longrightarrow & \Gamma_* \\
\Gamma_* & \leftarrow & \Gamma_2^*
\end{array}
\]

of port preserving morphisms in \( \text{Gr}_* (S) \).

Proof. Let \( \mathcal{G} \in \text{ob}(\text{Gr}) \), \( \mathcal{G} \neq \mathcal{C}_0 \). Given \( \Gamma_*, \Gamma_* \in \text{ob}(\text{Gr}^{(\mathcal{G})}_*(S)) \), a morphism in \( \text{Gr}^{(\mathcal{G})}_*(S)(\Gamma_*, \Gamma_*) \) is determined by port preserving morphisms

\[
\gamma : \Gamma_*(C_X, b) \to \Gamma'_*(C_X, b)
\]

in \( \text{Gr}_* (S) \) for all \((C_X, b) \in \text{ob}(\text{el}_*(\mathcal{G}))\). Since \( \mathcal{G} \neq \mathcal{C}_0 \), by Corollaries 3.37 and 1.70 these are morphisms in \( X \text{Gr}^{(\mathcal{G})}_*(S)(\Gamma_*(b), \Gamma'_*(b)) \).

For each \( e \in E \), \( \Gamma_*(u) \) is a morphism in \( \text{Gr}_* (S)(\Gamma_*(ch_e \circ u), \Gamma_*(ch_e)) \). So, since \( \Gamma_*(ch_e) \in S(1) \),

\[
\Gamma_*(C_2, ch_e \circ u) = \mathcal{L}^k(\iota_{\omega_e}) \text{ for some } k \geq 0, e \in S(1).
\]

(In this case, \( \Gamma_*(C_2, ch_e \circ u) = \mathcal{L}^k(\iota_{\omega_e}) \))
Likewise, for each $e \in E$, $\Gamma_e(z)$ describes a morphism in $\text{Gr}_* (S)(\Gamma_e(ch_e \circ z), \Gamma_e(ch_e))$, so

$$\Gamma_e(C_0, ch_e \circ z) = (C_0, o_e) \text{ or } W^l(\iota_e)$$

for some $l \geq 1$ and $e \in S(i)$, but since $ch_e \circ z = ch_{te} \circ z$, and, in general, $\iota_e \neq \iota_{te}$, it must be the case that

$$\Gamma_e(C_0, ch_e \circ z) = (C_0, o_e).$$

Therefore, by Lemma 3.49 there is a span $(3.55)$ in $\text{Gr}_*^G(S)$.

The converse is immediate, as is the final statement. \qed

**Proof of Theorem 3.49.** By Proposition 3.2, a DT algebra $(A, h : DTA \to A)$ admits structure morphisms $h \circ D\eta^\tau A : DA \to A$ and $h \circ D\eta^\sigma A : h|TA : TA \to A$ in $\text{Gr}_*^G(S)$. In particular, by Lemma 3.14, $A$ is equipped with the structure of a pointed graphical species $(A, \iota, o)$, with $\iota = h(l^{DTA}) : A(i) \to A_2$ and $o = h(o^{DTA}) : A(i) \to A_0$. As a $\Gamma$-algebra, $A$ admits a multiplication $\circ$ and contraction $\zeta$ induced by $h$ by Proposition 2.27.

Since $(A, h)$ is an algebra for DT, there are commuting diagrams

$$\begin{align*}
(3.56) & \quad A \xrightarrow{\eta^\tau \eta^\sigma A} DTA \\
& \quad A, \quad h \\
& \quad A \\
& \quad DT \quad DTA \quad DT \quad DTA \\
& \quad h \\
& \quad A.
\end{align*}$$

Let $A$ have palette $(C, \omega)$. To show that $\iota$ gives the units for $\circ$ and $o$ the contracted units, let $(C, \omega)$ be the palette of $A$, and let $\phi \in A_{\text{XII}(x)}$ with $A(ch_x)(\phi) = c$, and $\psi \in A_{\text{YII}(y)}$ with $A(ch_y)(\psi) = \omega c$. Then, by the proof of Proposition 2.27, $\phi \circ_{x,y} \psi = h \llcorner_{x,y}(\phi, \psi)$ (Definition 2.29).

In particular, since $S \subset DS$,

$$\llcorner^{(DTA)}_{x,y} \left( (\eta^\iota^{DTA}(\phi), (\iota^ DT A) \right) \in ((DT)^2 A)_{\text{XII}(x)}$$

and, since Diagram 3.57 commutes

$$\phi \circ X, \{1\}_{x,2} \iota_c = h \circ \llcorner_{x,2}^{A}(\phi, \iota_c) = \phi$$

and so $\iota_c$ is the $c$-coloured unit for the multiplication $\circ$ induced by $h$. In other words $(A, h)$ naturally has the structure of a CSM.

For the converse, let $(S, o, \zeta, \iota)$ be a $(C, \omega)$-coloured CSM. Then $S$ has the structure of a pointed graphical species $(S, \iota, o)$ with $o = \zeta \iota : C \to C/\omega \to S_0$, and by Proposition 2.27 $(S, o, \zeta)$ corresponds to a $\Gamma$-algebra $(S, h_T)$ where $h_T : TS \to S$ satisfies

$$\circ = h_T \circ \llcorner, \quad \zeta = h_T \circ \zeta.$$

In particular, since $\iota$ is a unit for $o$,

$$o_c = h_T \left( \zeta(L^k(\iota_c)) \right) = h_T(W^k(\iota_c)), \quad \text{for all } c \in C, \quad \text{and all } k \geq 1.$$

So, $h : DTS \to S$ may be defined by

$$h|TS = h_T : TS \to S,$$

and

$$h(x^{DTS}) = \iota : C \to S_2, \quad \text{and } h(o^{DTS}) = o : C \to S_0.$$
Then, clearly \( h\eta \gamma S = \text{id}_S \), and it remains to check that the diagram

\[
\begin{array}{c}
\xymatrix{
(DT)^2 S \ar[r]^{DT} & D^2T^2 S \ar[r]^{\mu DT^2 S} & DTS \\
DT S \ar[d]_h & & \ar[d]^{h}
}
\end{array}
\]

commutes.

On distinguished elements \( \ell (DT)^2 S, \ell (DT)^2 S \), this is clear. So, let \( [X, \beta] \in TDT S \) be represented by \( (X, \beta) \in DTS (X) \). If \( X = C_0 \) then either \( \beta \in im (\ell \ell T S) \), or \( \beta \in S_0 \), and, in each case, the result is immediate.

Therefore, assume that \( X \neq C_0 \) (and hence that \( V_0 = \emptyset \)), and define the subset \( W = W (\beta) \subset V_2 \)

\[
W = W (\beta) \overset{\text{def}}{=} \{ v \mid \text{there is } (C_2, b) \in \text{el}(G) \text{ such that } [b]^G = v \text{ and } DTS (b) (\beta) \in im (\ell T S) \}.
\]

In other words, \( v \in W \) if and only if \( DTS (v) (\beta) \) is not in \( TSP (v) \).

In case \( W \neq V \), then \( \lambda \text{T} S (X, \beta) = [X/W, \beta/W] \). Now, if \( \alpha \in S (X) \) is given by \( Th [X, \beta] = [X, \alpha] \in TS X \), then \( S (b) (\alpha) \in im (\lambda) \) for all \( v \in W \), and \( (C_2, b) \in \text{el}(X) \), such that \( [b]^X = v \).

It follows that the vertex deletion morphism \( \text{del}/W \) induces a similarity morphism in \( X \text{Gr}^s (S) (X, \alpha), (X/W, \alpha/W) \), and therefore the images of \( [X, \beta] \) under \( \lambda T S \circ Th \) and \( Th \circ \text{del}/W \) correspond in \( TS X \).

\[
\xymatrix{
[X, \beta] \ar[r]^{Th} \ar[d]_{\lambda T S} & [X, \alpha] \ar[d]^{[\text{del}/W]} \\
[X/W, \beta/W] \ar[r]_{Th} & [X/W, \alpha/W].
}
\]

Now, since \( h_T \) is induced by the CSM structure on \( S \), it is invariant under similarity morphisms in \( X \text{Gr}^s (S) \). In particular, \( h_T [X, \alpha] = h_T [X/W, \alpha/W] \in S X \), and therefore, since \( (S, h_T) \) is an algebra for \( T \), chasing \( [X, \beta] \) around Diagram 3.58 gives commuting paths

\[
\xymatrix{
[X, \beta] \ar[r]^{\lambda T S} \ar[d]_{Th} & [X/W, \beta/W] \ar[r]^{\mu DT^2 S} & \mu DT^2 S [X/W, \beta/W] \ar[d]_{h_T} \\
[X, \alpha] \ar[r]_{[\text{del}/W]} & [X/W, \alpha/W] \ar[r]_{h_T} & h_T [X/W, \alpha/W].
}
\]

Finally, when \( W = V \), then, by Corollary 1.70, either \( [X, \beta] = L^k (\ell T S) \) for some \( c \in C \) and \( k \geq 1 \), or \( [X, \beta] = W^l (\ell T S) \) for some \( c \in C, l \geq 1 \), and it is immediately verified that Diagram 3.58 commutes for these cases.

So, \( (S, h) \) naturally admits the structure of a \( DT \)-algebra. It is straightforward to verify that the functors \( CSM \Rightarrow GS DT \) so defined are each others’ inverses.

\[\square\]

**Remark 3.59.** There is also a distributive law \( DT \Rightarrow TD \). Algebras for the composite monad \( TD \) are just the cofibred coproducts of algebras for \( D \) and \( T \). There is no further relationship between the two structures.\[\uparrow\]

\[\uparrow\]This corresponds precisely to the categorical case: the monad that adjoins distinguished units to directed graphs distributes over the free semi-category monad, but the two structures do not interact in the composite.
Example 3.60. The terminal graphical species $Z^{(C,\omega)}$ trivially admits the structure of a CSM for all palettes $(C,\omega)$. In particular, the slice category $\text{CSM} \downarrow Di$ (see Examples 1.43 and 1.46) is canonically equivalent to the category of (coloured) wheeled properads [36, 14].

3.6. A structured-graph construction of $T_*$. By [3 Section 3], $\lambda : TD \to DT$ induces a commuting square of monadic adjunctions,

\[
\begin{array}{c}
\text{CSM} \\
\downarrow
\end{array}
\begin{array}{c}
\text{GS}^T
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{GS}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{GS}_*
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{GS}
\end{array}
\end{array}
\]

where for each pair, the right adjoint is the corresponding forgetful (monadic) functor.

In particular, the following corollary of Theorem 3.46 is immediate.

Corollary 3.62. There is a monad $T_*$ on $\text{GS}_*$ whose EM category $\text{GS}^{T_*}_*$ is canonically isomorphic to CSM.

The remainder of this work is devoted to giving an explicit description, in the style of [19] and the construction of $T$ in Subsection 2.2, of the monad $T_*$ on $\text{GS}_*$, and to showing that it has arities $\text{Gr}_*$ (see [4 Definition 1.8]). The desired nerve theorem for CSMs will then follow from [4, Theorem 1.10].

Let $(S,\iota,o)$ be a $(C,\omega)$-coloured pointed graphical species and let $\pi : DS \to S$ denote the canonical structure map that restricts to the identity on $S \subset DS$ and $\iota_{DS} \mapsto \iota$, $o_{DS} \mapsto o$. By [3, Section 3], the endofunctor $T_*$ for $T_*$ is described by the coequaliser

\[
\begin{array}{c}
(DT)S \\
\downarrow
\end{array}
\begin{array}{c}
(DT)^2S
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
T_*S
\end{array}
\]

So, elements of $T_*S$ are equivalence classes of elements of $DTS$.

In fact, each element of $T_*S$ has a representative in $TS$, and is therefore represented in turn by an $S$-structured graph with non-empty vertex set. Namely, for all $c \in C$ and $k \geq 1$,

\[
\text{L}^k(\iota_{DS}) \overset{T\pi}{\longrightarrow} \text{L}^k(\iota_c),
\]

and

\[
\text{L}^k(\iota_{DS}) \overset{T\eta_{DS}}{\longrightarrow} \text{L}^k(\iota^{DTS}_c) \overset{D\pi}{\longrightarrow} i_c^DTS \overset{\mu^{\text{DTS}}}{\longrightarrow} i_c^{DTS},
\]

so, by Diagram (3.63) $\text{L}^k(\iota_c) \in TS_2 \subset DTTS_2$ and $\iota^{DTS}_c \in DTTS_2$ represent the same element $\iota^{T_*S}_c$ of $T_*S_2$.

Similarly, for all $l \geq 1$,

\[
\text{W}^l(\iota_{DS}) \overset{T\pi}{\longrightarrow} \text{W}^l(\iota_c),
\]

and

\[
\text{W}^l(\iota_{DS}) \overset{T\eta_{DS}}{\longrightarrow} \text{W}^l(\iota^{DTS}_c) \overset{D\pi}{\longrightarrow} o_c^{DTS} \overset{\mu^{\text{DTS}}}{\longrightarrow} o_c^{DTS}.
\]

Moreover,

\[
[\mathcal{O}_0, o^{'DS}_c] \overset{T\pi}{\longrightarrow} [\mathcal{O}_0, o_c],
\]
and
\[ C_0, o_0^{DS} \xrightarrow{\text{TD}_c S} C_0, o_0^{DTS} \xrightarrow{\text{D}_{\lambda}^{TS}} o_0^{D^2\eta^S} = \mu^\ast S \xrightarrow{\text{TD}_c S} o_0^{DTS}, \]
so \( W^l(t_c) \) and \( (C_0, o_c) \) represent the same element as \( o_0^{DTS} \) in \( T_\ast S_0 \). In particular, these correspond to the (contracted) units for \( T_\ast S \).

To describe the remaining elements of \( T_\ast S \), let \( X \) be a finite set and let \((\mathcal{X}, \alpha^D), \mathcal{X} \neq C_0 \) represent an element of \( TDS_X \). As usual
\[(3.64) \quad W_{\alpha^D} \overset{\text{def}}{=} \{ v \} \text{ there exists } (C_2, b) \in \text{el}(\mathcal{G}) \text{ such that } [b]^G = v \text{ and } DS(b)(\alpha^D) \in \text{im}(\nu^D S) \]
is the set of bivalent vertices of \( \mathcal{X}, \alpha^D \)-decorated by adjoined distinguished elements.

Assume, moreover, that \( W_{\alpha^D} \neq V \). Then, let \( \alpha \in S(\mathcal{X}) \) be the element defined by the \([\mathcal{X}, \alpha] \overset{\text{def}}{=} T\pi[\mathcal{X}, \alpha^D], \]
\[(3.65) \quad S(b)(\alpha) = \begin{cases} S(b)(\alpha^D) & \text{for } b = v, v \in V - W, \\
\iota_c & \text{for } b \in \text{el}(\mathcal{G}). \end{cases}
\]
So, \([\mathcal{X}, \alpha] = T\pi[\mathcal{X}, \alpha^D] \in TS_X \).

Further let \( \alpha^D_{\eta^S} \in DTS_X[\mathcal{X}] \) be the element defined by \([\mathcal{X}, \alpha^D_{\eta^S}] \overset{\text{def}}{=} DTD_{\eta^S}S[\mathcal{X}, \alpha^D] \) that replaces vertex decorations by \( S \) in \( \alpha \) with decorations by \( \eta^S S \).
\[
DT S(\alpha^D_{\eta^S})(b) = \begin{cases} \eta^S(S(\alpha^D)(b)) & \text{for } b = v, v \in V - W, \\
S(b)(\alpha^D) & \text{whenever } S(\alpha^D)(b) = \iota_c^{DS}. \end{cases}
\]
Then, \([\mathcal{X}, \alpha^D] \xrightarrow{\text{DT}_{\lambda} S} [\mathcal{X}, \alpha^D_{\eta^S}], \)
\[
[\mathcal{X}, \alpha^D_{\eta^S}] \xrightarrow{\text{D}_{\lambda}^{TS}} [\mathcal{X}/W_{\alpha^D}, (\alpha^D_{\eta^S})/W_{\alpha^D}] \xrightarrow{\mu^\ast S} [\mathcal{X}/W_{\alpha^D}, \alpha^D_{/W_{\alpha^D}}],
\]
and, by definition
\[
\alpha^D_{/W_{\alpha^D}} = \alpha / W_{\alpha^D} \in TS(\mathcal{X}/W_{\alpha^D}).
\]
Note, in particular, that for all \( \alpha \in S(\mathcal{X}) \) and all \( W \subset W_{\alpha} \), such that \( W \neq V \), then there is a unique \( (\mathcal{X}, \alpha^D) \in DTS(\mathcal{X}) \) with \( W_{\alpha^D} = W \) (Equation 3.64) and satisfying 3.65. Therefore, by the defining Diagram 3.63 for \( T_\ast \) and Lemma 3.49 if \((\mathcal{X}, \alpha),(\mathcal{X}', \alpha')\) are similar in \( XGr_{\ast}^{\sim}(S) \), then they represent the same element of \( T_\ast S_X \).

With this in mind, let \( T_P : GS_\ast \rightarrow GS_\ast \) be the endofunctor
\[(3.66) \quad T_P S(\cdot) = S(\cdot), \text{ and, for all finite sets } X,
\]
\[
T_P S_X = \text{colim}_{\mathcal{X} \in XGr_{\ast}^{\sim}(S)} S(\mathcal{X}).
\]

The \( \mathcal{D} \)-algebra structure on \( T_P S \) is given by
\[
\begin{align*}
\iota_{T_P S} & \overset{\text{def}}{=} T_P S(u) : S(\cdot) \rightarrow T_P S_X, \quad c \mapsto [\mathcal{L}^k(t_c)]_P, \\
o_{T_P S} & \overset{\text{def}}{=} T_P S(z) : S(\cdot) \rightarrow T_P S_X, \quad c \mapsto [\mathcal{W}^l(t_c)]_P,
\end{align*}
\]
where \([\mathcal{X}, \alpha]_P \) denotes the class in \( T_P S_X \) of \([\mathcal{X}, \alpha] \in TS_X \) under the quotient \( TS_X \rightarrow T_P S_X \).

Now, for all pointed graphical species \((S, l, o), \)
\[
T_P S_X = \begin{cases} T_\ast S_X & X \neq \emptyset \\
T_\ast S_X / ([\mathcal{W}^l(t_c)]_P \sim [C_0, o_c]_P)_{c \in C} & X = \emptyset.
\end{cases}
\]

For any palette \((C, \omega)\), the category \( GS_\ast(C, \omega) \subset GS_\ast \) of \((C, \omega)\)-coloured pointed graphical species has an initial object \((I, l', o') = (I^{(C, \omega)}, l^{(C, \omega)}, o^{(C, \omega)}) \) that trivially admits the structure of a CSM.
For finite sets $X$

$$I_X = \begin{cases} 
\emptyset, & X \notin \{0, 2\}, \\
\{i^I_c\}_{c \in C} \cong C & X = 2, \\
\{0^I_c\}_{c \in C/\omega} \cong C/\omega & X = 0,
\end{cases}$$

and therefore

$$TP I_X = \begin{cases} 
\emptyset, & X \notin \{0, 2\}, \\
\{[L^I_1(i^I_c)]_{c \in C} \cong C, & X = 2, \\
\{[C_0, 0^I_c]\}_{c \in C/\omega} \{[W^I(i^I_c)]_{c \in C} \cong C/\omega II C/\omega, & X = 0.
\end{cases}$$

and there is a canonical palette-preserving morphism in $GS_*(TP I, I)$ such that, for all $cinC$, $[L^k(i^I_c)]_{p} \mapsto i^I_c$ and $[W^I(i^I_c)]_{p} \mapsto 0^I_c$.

**Lemma 3.67.** For a pointed graphical species $(S, i, o)$ with palette $(C, \omega)$, let $j = j^S \in GS_*(I, S)$ be the unique morphism. Then $T_\star S$ corresponds naturally (in $GS_\star$) to the pushout

$$TP I \xrightarrow{TP j} TP S$$

in $GS_\star$.

**Proof.** Observe first that, since the image of $I$ in $T_{p,o} S$ consists precisely of the (contracted) units of $T_{p,o} S$, the universal map $TP S_X \to T_{p,o} S_X$ is a surjection for all $X$. Indeed, since the canonical maps $TP I_2 \to TP S_2$ and $TP I_2 \to I_2$ are injective, the projection $TP S_X \to T_{p,o} S_X$ is an isomorphism whenever $X \neq 0$. The units in $T_{p,o} S_2$ are represented by the classes $[L^k(i^I_c)]_p$.

Since $[W^I(i^I_c)]_p$ and $[C_0, 0^I_c]$ in $TP (I(C, \omega))_0$ both map to $0^I_c$ in $I_0$, their images $[W^I(i^I_c)]_p$ and $[C_0, 0^I_c]_p$ in $TP S_0$ are identified in $T_{p,o} S_0$. These represent the contracted unit $0^I_{p,o} S$ and are the only extra identifications formed by the pushout. In particular, the identifications are compatible with all the projections to $C$. Hence, $T_\star S$ is described by the pushout. Naturality is immediate.

$\square$

By Diagram 3.68 if $\eta_\star : 1_{GS_\star} \Rightarrow T_\star$ and $\mu_\star : T_\star \Rightarrow T^2_\star$ are the unit and multiplication for $T_\star$, then, for all pointed species $(S, i, o)$, and all finite sets $X$, there are commuting diagrams of sets

$$\begin{align*}
S_X \xrightarrow{\eta^S X} TS_X & \quad (3.69) & T^2 S_X \xrightarrow{\mu^S X} TS_X & \quad (3.70) \\
S_X \xrightarrow{\eta^2 S_X} T^2 S_X & \quad \downarrow \\
S_X \xrightarrow{\eta S_X} T S_X, & \quad \downarrow \\
T^2 S_X \xrightarrow{\mu S_X} T S_X, & \quad \downarrow \\
T S_X \xrightarrow{\mu^2 S_X} T S_X, & \quad \downarrow \\
T^2 S_X \xrightarrow{\mu^2 S_X} T^2 S_X.
\end{align*}$$

in which the vertical arrows are the canonical projections. So, if $[X, \alpha]_\star$ denotes the image of $[X, \alpha] \in TS_X$ under the projection defined in Lemma 3.67 then $\eta_\star : 1_{GS_\star} \Rightarrow T_\star$, $\phi \mapsto [\eta^2 \phi]_\star$, by 3.69.

To describe the multiplication $\mu_\star$ for $T_\star$ explicitly, observe that, for all finite sets $X$, the canonical diagram of projections

$$T^2 S_X \xrightarrow{\mu^2 S_X} T^2 S_X$$

$$TT S_X \xrightarrow{TT^2 S_X} T^2 S_X$$

$\square$
commutes. So, an element \([X, \beta] \in T^2S_X\), represents a well-defined element \([X, \beta]_{**} \in T^3S_X\), and, by (3.70) \(\mu_\ast[X, \beta]_{**} \overset{\text{def}}{=} [\mu[X, \beta]]_\ast\).

Let \((X^1, \beta^1) \in \text{ob}(X\text{Gr}^{\text{simp}}(T, S))\) be \(T, S\)-structured \(X\)-graphs representing the same element \([X, \beta]_{**}\) of \(T^3S_X\). If \((X^1, \beta^1)\) is not of the form \((C_0, o_{C}^T, S)\), then by Lemma (3.49) there is a \(T, S\)-structured \(X\)-graph \((X, \beta)\) and a cospan
\[
(X^1, \beta^1) \xrightarrow{g^1} (X, \beta) \xleftarrow{g^2} (X^2, \beta^2)
\]
in \(X\text{Gr}^{\text{simp}}(T, S)\) and therefore, for \(j = 1, 2\), and all \((C, b) \in \text{ob}(el_\ast(X^j))\),
\[
S(b)(\beta^j) = S(g^j \circ b)(\beta) \in TS(C).
\]
In particular, if \((C_2, b) \in \text{ob}(el_\ast(GX^j))\) is such that \(g^j \circ b = ch_e \circ u\) for some \(e \in E = E(X)\), then
\[
S(b)(\beta^j) = S(u)S(ch_e)(\beta) = \iota^T e \in T_pS_2
\]
for \(c = S(ch_e)(\beta) \in S(i)\).

For all \((C_Y, b) \in \text{ob}(el_\ast(X^j))\), if pointed graphs of \(S\)-structured graphs (Definition (3.50)) \(\Gamma_\ast^T : el_\ast(X^j) \rightarrow \text{Gr}_\ast(S)\) represent \([X^j, \beta^j]\) and \(\Gamma_\ast^T : el_\ast(X^j) \rightarrow \text{Gr}_\ast(S)\) represents \([X, \beta]\), then, if \((C_Y, g^j \circ b) \in \text{ob}(el_\ast(X^j))\),
\[
\Gamma_\ast^T(C_Y, b) \sim \Gamma_\ast^T(C_Y, g^j \circ b) \in Y\text{Gr}^{\text{simp}}_\ast(S).
\]

On the other hand, if \((C_2, b) \in \text{ob}(el_\ast(GX^j))\) is such that \((C_2, g^j \circ b) = (C_2, ch_e \circ u) \in \text{ob}(el_\ast(X^j))\) for some \(e \in E\),
\[
\Gamma_\ast^T(C_2, b) \in \text{ob}(Gr_\ast(S)) \text{ represents } \iota^T e \in T_pS_2, \text{ where } c = \Gamma_\ast^T(i, ch_e) \in S(i).
\]
So,
\[
\Gamma_\ast^T(C_2, b) = \mathcal{L}_j^k(\iota_e)
\]
for some \(k \geq 1\).

Then, by Lemma (3.55)
\[
\text{colim}_{(C, b) \in el_\ast(X^j)} \Gamma_\ast^T(C, b) \sim \text{colim}_{(C, b) \in el_\ast(X^j)} \Gamma_\ast^T(C, g^j \circ b) \in X\text{Gr}^{\text{simp}}_\ast(S)
\]
as expected.

If \((X^1, \beta^1) = (C_0, o_{C}^T, S)\) for some \(C \in C\), we may assume that \((X^2, \beta^2) \in \mathcal{W}^l(\iota^T e, S)\) for some \(l \geq 1\).

In particular, if \(V(\mathcal{W}^l) = (v_i)_{i=1}^l\), a \(\mathcal{W}^l\)-shaped pointed graph of \(S\)-structured graphs \(\Gamma_\ast\) representing \(\mathcal{W}^l(\iota^T e, S)\) is defined by a collection of morphisms \(b_i \in Gr(C_2, \mathcal{W}^l)\) (as in Examples (3.28) and (3.35), with \([b_i] \sim v_i\) for each \(1 \leq i \leq l\) and
\[
\Gamma_\ast(b_i) = \mathcal{L}_i^k(\iota_e).
\]
Hence, if \(L \overset{\text{def}}{=} \sum_{i=1}^l k_i\), then
\[
\text{colim}_{el_\ast(\mathcal{W}^l)} \Gamma_\ast = \mathcal{W}^l(\iota_e) \in TS_0,
\]
and therefore represents \(o_{C}^T, S \in T, S_0\).

So, the multiplication \(\mu_\ast S\) is described by the assignment \(\mu_\ast S : [X, \beta]_{**} \mapsto \text{colim}_{el_\ast(X)} \Gamma_\ast \in T, S_X \)
where \(\Gamma_\ast : el_\ast(X) \rightarrow Gr_\ast(S)\).

**Example 3.71.** Given a set \(C\) with involution \(\omega\), the terminal \((C, \omega)\)-species \(Z(C, \omega)\), together with the unique morphism \(v(C, \omega) : T, Z(C, \omega) \rightarrow Z(C, \omega)\) is an algebra for \(T\).

**Example 3.72.** The full subcategory \(\overline{Gr}_\ast \subset CSM\), with objects the free CSMs on \(Gr \subset GS\), plays an important role (analogous to that of \(\Delta\) in the classical case) in the nerve theorem for CSMs (Theorem (4.39)).

Let \(H = (E, H, V, s, t, \tau) \in \text{ob}(Gr_\ast)\) be a graph. Of course, \(T_\ast H(i) = Gr_\ast(i, H) \cong E\) with involution \(\tau: ch_e \mapsto ch_{\tau e}\).
For each $e \in E$,

$$\iota_e \defeq ch_e \circ u \in Gr_*(\mathcal{C}_2, \mathcal{H})$$

is the corresponding unit element of $\mathcal{H} = Y\mathcal{H} \in ob(\mathcal{GS}_*)$, and

$$o_e \defeq ch_e \circ z \in Gr_*(\mathcal{G}_0, \mathcal{H})$$

is the corresponding contracted unit of $\mathcal{H} \in ob(\mathcal{GS}_*)$.

So, by Lemma 3.67 the (contracted) units in $T_*\mathcal{H}$ are given by

$$\iota^*_e \mathcal{H} = [\mathcal{L}^k(\iota_e)]_*$, \quad \text{and} \quad o^*_e \mathcal{H} \defeq [\mathcal{W}^l(\iota_e)]_* = [\mathcal{G}_0, o_e]_*$$

for each $e \in E$.

In other words, $\iota^*_e \mathcal{H}$ is represented by morphisms of the form $ch_e \circ \text{del}_{/V(\mathcal{L}^k)} \in Gr_*(\mathcal{L}^k, \mathcal{H})$ and $ch_e \circ \text{del}_{/V(\mathcal{W}^l)} \in Gr_*(\mathcal{W}^l, \mathcal{H})$. The contracted units $o^*_e \mathcal{H}$ are represented by $ch_e \circ z \in Gr_*(\mathcal{G}_0, \mathcal{H})$.

For $X$ a finite set, an element $[\mathcal{X}, f]_* \in T_*\mathcal{H}_X$ is represented by a pair $(\mathcal{X}, f)$ of an $X$-graph $\mathcal{X} = (\mathcal{G}, \rho)$, together with a morphism $f \in Gr_*(\mathcal{G}, \mathcal{H})$ and, by Lemma 3.49, pairs $(\mathcal{X}^1, f^1)$ and $(\mathcal{X}^1, f^2)$ such that for $j = 1, 2$, $(\mathcal{X}^j, f^j)$ is not of the form $(\mathcal{G}_0, ch_e \circ z)$, represent the same element $[\mathcal{X}, f]_* \in \mathcal{GS}_*(\mathcal{C}_X, T_*\mathcal{H})$ if and only if there is a commuting diagram

$$
\begin{array}{ccc}
G^1 & \xrightarrow{g^1} & G \\
\downarrow{f^1} & & \downarrow{f} \\
G^2 & \xrightarrow{g^2} & G
\end{array}
$$

in $Gr_*$ such that $g^j \in Gr_*(G^j, G)$ are port-preserving vertex-deletion morphisms for $j = 1, 2$. So, for all finite sets $X$ and all $\phi \in T_*\mathcal{H}[X]$, $\phi \notin \text{im}(o^*_e \mathcal{H})$, the subcategory of representatives of $\phi$ in $Gr_* \downarrow \mathcal{H}$ is connected.

Surprisingly, this is also the case for the contracted units $o^*_e \mathcal{H} \in T_*\mathcal{H}_0$ since, although $(\mathcal{G}_0, o_e)$ and $\mathcal{W}^l(\iota_e)$ lie in distinct components of $0\text{Gr}_*^{\text{sim}}(\mathcal{H})$, for any edges $e \in E(\mathcal{H})$ and $e_W \in E(\mathcal{W}^l)$, the diagram

$$
\begin{array}{ccc}
\mathcal{G}_0 & \xrightarrow{ch_{e_W} \circ o_e} & \mathcal{W}^l \\
\downarrow{ch_e \circ o_e} & & \downarrow{ch_e \circ \text{del}_{/V(\mathcal{W}^l)}} \\
\mathcal{H} & \xrightarrow{ch_e \circ z} & \mathcal{H}
\end{array}
$$

commutes in $Gr_*$. This observation is essential in the proof of Theorem 4.3.

### 4. A Nerve Theorem

Let $F_* : \mathcal{GS}_* \Rightarrow \text{CSM} : U_*$ be the free-faithful adjunction arising from the monad $T_*$ on $\mathcal{GS}_*$.

**Definition 4.1.** The (CSM) graphical category $\widetilde{\text{Gr}}_*$ is the category obtained in the factorisation of $F_*Y_* : \text{Gr}_* \Rightarrow \text{CSM}$ as an identity on objects functor $j : \text{Gr}_* \Rightarrow \widetilde{\text{Gr}}_*$ followed by a fully faithful functor $i : \text{Gr}_* \Rightarrow \text{CSM}$.

In the following diagram of functors, the left hand square commutes by construction.
The right hand vertical arrow is the pullback along \( j \) and the right hand horizontal arrows are the associated nerve functors. In particular \( \text{sh} : \text{GS}_* \to \text{PSh}(\text{Gr}_*) \) is the fully faithful inclusion of sheaves functor. It is straightforward to observe that the right hand square commutes up to natural isomorphism. (See [4, Section 1] for details.)

The goal of this section is to prove the following.

**Theorem 4.3.** The functor \( N : \text{CSM} \to \text{PrSh}(\text{Gr}_*) \) is full and faithful. Its essential image consists of precisely those presheaves on \( \text{Gr}_* \) whose restriction to \( \text{PrSh}(\text{Gr}) \) is a graphical species.

The monad \( T_* \) is said to have *arities* \( \text{Gr}_* \) if, for each pointed graphical species \( (S, \iota, \omicron) \), the image under \( NF_* \) of the ‘representable’ colimit cocone \( \text{Gr}_* \downarrow S \) is a colimit cocone in \( \text{PSh}(\text{Gr}_*) \) (see [4, Definition 1.8]).

By [4, Propositions 1.5 and 1.9, Theorem 1.10], if \( T_* \) has arities \( \text{Gr}_* \) then \( N \) is fully faithful and its essential image consists of those presheaves \( S \) on \( \text{Gr}_* \) whose restriction to \( \text{Gr}_* \) is a sheaf with respect to \( J_* \). It then follows from Proposition [4, 4.18] and Diagram [4.18] specifically, that this is the case if and only if the restriction of \( S \) to \( \text{PSh}(\text{Gr}) \) is a sheaf, hence a graphical species.

So, the remainder of this section is devoted to proving that \( T_* \) has arities \( \text{Gr}_* \). The observation, at the end of Example [4.72] that [4.74] commutes in \( \text{Gr}_* \) will be significant. Indeed, it is precisely because \( \mathcal{W}^d \) and \( \mathcal{C}_0 \) are in different connected components of \( \text{Gr}_* \) that the CSM monad \( \mathbb{D}_T \) on \( \text{GS} \) cannot have arities \( \text{Gr}_* \).

**4.1. The category \( \text{Gr}_* \).** By definition, \( \text{Gr}_* \) is the restriction to \( \text{Gr}_* \) of the Kleisli category of \( T_* \). Hence, \( \text{Gr}_*(\mathcal{G}, \mathcal{H}) = \text{GS}_*(\mathcal{G}, T_* \mathcal{H}) \cong T_* \mathcal{H}(\mathcal{G}) \) for all pairs \( (\mathcal{G}, \mathcal{H}) \) of graphs. If \( \mathcal{G} \in \text{ob}(\text{elGr}) \) is elementary, then \( \text{Gr}_*(\mathcal{G}, \mathcal{H}) \cong T_* \mathcal{H}(\mathcal{G}) \) has been described in Example [4.72].

So, \( \mathcal{G} \) and \( \mathcal{H} = (E, H, V, s, t, \tau) \) be any graphs, \( \mathcal{G} \neq \mathcal{C}_0 \). Then,

\[
\text{Gr}_*(\mathcal{G}, \mathcal{H}) \cong T_* \mathcal{H}(\mathcal{G}) = \lim_{\mathcal{C}, b \in \text{el}(\mathcal{G})} T_* (\mathcal{H}(\mathcal{C})) = \lim_{\mathcal{C}, b \in \text{el}(\mathcal{G})} T_* (\mathcal{H}(\mathcal{C})),
\]

so \( \alpha \in T_* \mathcal{H}(\mathcal{G}) \) is represented by a \( \mathcal{G} \)-shaped pointed graph of graphs \( \Gamma_* \), and, by the universal property of the colimit \( \Gamma_* = \text{colim}_{\text{el}(\mathcal{G})} \Gamma_* \), a morphism \( f \in \text{Gr}_*(\Gamma_*, \mathcal{H}) \).

By Lemma [4.52] since \( \mathcal{G} \neq \mathcal{C}_0 \), pairs \( (\Gamma_*^1, f^1), (\Gamma_*^2, f^2) \) represent the same element \( \text{Gr}_*(\mathcal{G}, \mathcal{H}) \) whenever there is a cospan

\[
\begin{array}{ccc}
\Gamma_*^1 & \longrightarrow & \Gamma_*^2 \\
\downarrow & & \downarrow \\
\Gamma_* & \leftarrow & \Gamma_*^2
\end{array}
\]

in \( \text{Gr}_*(\mathcal{G}) \), and a morphism \( f \in \text{Gr}_*(\Gamma_*, \mathcal{H}) \) (where \( \Gamma_* = \text{colim}_{\text{el}(\mathcal{G})} \Gamma_* \)) such that

\[
\begin{array}{ccc}
\Gamma_*^1 & \longrightarrow & \Gamma_*^2 \\
\downarrow f^1 \downarrow & & \downarrow f^2 \\
\Gamma_* & \leftarrow & \Gamma_*^2
\end{array}
\]

commutes in \( \text{Gr}_* \).

The assignment \( f \mapsto (\text{dom}, f) \) where \( \text{dom} : \text{el}(\mathcal{G}) \to \text{Gr}_* \) is the identity \( \mathcal{G} \)-shaped graph of graphs with colimit \( \mathcal{G} \) (Example [1.52]) induces the inclusion \( \text{Gr}_* \hookrightarrow \text{Gr}_* \).

The following terminology is from [21].
Definition 4.4. A (pointed) free map in $\widetilde{\text{Gr}}_*$ is a morphism in $\widetilde{\text{Gr}}_*(\mathcal{G}, \mathcal{H})$ represented by $(\text{dom}, f)$, for $f \in \text{Gr}_*(\mathcal{G}, \mathcal{H})$, and a refinement in $\widetilde{\text{Gr}}_*$ is a morphism in $\widetilde{\text{Gr}}_*(\mathcal{G}, \mathcal{H})$ with a representative of the form $(\Gamma_*, id_\mathcal{H})$.

So, a free map ‘is’ a morphism in $\text{Gr}_*$. A refinement in $\widetilde{\text{Gr}}_*(\mathcal{G}, \mathcal{H})$ corresponds to a connected component $[\Gamma_*]$ in the category $\text{Gr}_*(\mathcal{G})$ of $\mathcal{G}$-shaped pointed graphs of graphs such that each object $\Gamma_* \in [\Gamma_*]$ has colimit $\mathcal{H}$.

From the discussion above it follows that, for all $\mathcal{G}, \mathcal{H} \in \text{ob}(\text{Gr}_*)$ and for all morphisms $\alpha \in \widetilde{\text{Gr}}_*(\mathcal{G}, \mathcal{H})$, $\alpha$ factors as a refinement followed by a free map.

Remark 4.5. The graphical category $\overline{\text{Gr}}$ whose morphisms are described in [19] Section 6 (and discussed in Remark 3.3 and the footnote on page 52) is the wide subcategory of $\text{Gr}_*$ that does not contain the morphisms $z : \mathcal{C}_0 \to (i)$ or any morphisms in $\text{Gr}_*(\mathcal{W}^l, \mathcal{G})$ that factor through some $\text{ch}_\epsilon \in \text{Gr}_*(\epsilon, \mathcal{G})$. Therefore, $\overline{\text{Gr}}$ does not embed fully faithfully in $\text{CSM}$. It is worthwhile to note that, though the morphisms in $\text{Gr}_*$ that are missing from $\overline{\text{Gr}}$ are somewhat mysterious when viewed as maps of graphs, they are easy to understand as morphisms of CSMs in $\text{Gr}_*$. For example, the morphism $z \in \text{CSM}(T, \mathcal{C}_0, T, i)$ maps the unique arity 0 element of the CSM $T, \mathcal{C}_0$ with empty palette, to the unique contracted unit in $T, (i)_0$.

In [16, 17], the wide subcategory $U \subset \overline{\text{Gr}} \subset \text{Gr}_*$ is constructed precisely so that it is Reedy, and, in order to obtain the desired structure, covering morphisms in $\text{Gr} \to \text{Gr}_*$ are excluded from $U$. These are those – somewhat mysterious – morphisms $g \in \text{Gr}(\mathcal{G}, \mathcal{G}')$ such that $E_0(\mathcal{G}) = E_0(\mathcal{G}')$ but $\mathcal{G}$ is not an isomorphism (so $E_0(\mathcal{G}) = \emptyset$ by Corollary 1.68).

In [16], it is shown that, both $U$ and $\overline{\text{Gr}}$ yield a fully faithful nerve functor on $\text{CSM}$ whose essential image consists of those presheaves that satisfy the Segal condition.

Remark 4.6. The factorisation used to obtain the Reedy structure on $U$ [16, 17], does not correspond with the restriction to $U$ of the (pointed) free/refinement factorisation on $\text{Gr}_*$ described above. For example, $u \in \text{Gr}_*(\mathcal{C}_2, i)$ is (pointed) free in $\text{Gr}_*$ by definition 4.4 but the corresponding map in $U$ is a refinement in [16].

However, there is also a canonical factorisation system on $\text{Gr}_*$ that does restrict to the factorisation on $U$ in [16]. By Corollary 3.3 and there is a ternary factorisation system on $\text{Gr}_*$ factoring morphisms uniquely as refinement followed by vertex deletion followed by morphism in $\text{Gr}$. The refinements described above (Definition 4.4), taken together with the vertex deletion morphisms restrict to ‘refinements’ in $U$. Indeed, this is not surprising since, by 3.18 $\text{Gr}_*$ is also the category obtained in the identity on objects/fully faithful factorisation of $DTY : \text{Gr} \to \text{CSM}$.

4.2. Factorisation categories. More generally, let $\text{GS}_{S,T}$ be the Kleisli category of $T$, with morphism sets $\text{GS}_{S,T}(S, S') = \text{GS}_{S}(S, T_*S')$. For any graph $\mathcal{G}$ and pointed graphical species $(S, \iota, o)$, a morphism $\beta \in \text{GS}_{S,T}(\mathcal{G}, S)$ is represented by a refinement $\Gamma_*$ in $\text{Gr}_*$ with colimit $\Gamma_*$, followed by a morphism $\alpha \in \text{GS}_{S}(\Gamma_*, S)$.

Namely, for all finite sets $X$,

$$\text{GS}_{S}(\mathcal{G}, T_*S) \cong \lim_{(C,b) \in \mathcal{G}} T_*S(C),$$

and elements of $T_*S(C_X)$ are represented by $S$-structured $X$-graphs $(X, \alpha)$. So an element $\beta \in \text{GS}_{S}(\mathcal{G}, T_*S)$ is represented by an $X$-shaped pointed graph of $S$-structured graphs $\Gamma_*$ that admits a colimit $\Gamma_*$ in $\text{Gr}_*(S)$.

Let $\Gamma_*^Z : \mathcal{E}_{\iota}(X) \to \text{Gr}_*$ be the $X$-shaped pointed graph of graphs obtained by composition of $\Gamma_*$ with the forgetful morphism $\text{dom} : \text{Gr}_*(S) \to \text{Gr}_*(Z) = \text{Gr}_*$, so $\Gamma_*^Z : (\mathcal{C}_X, b) \mapsto (X_0, \alpha_0) \mapsto (X_0)$, and let $\Gamma_*^Z$ be the colimit of $\Gamma_*^Z$ in $\text{Gr}_*$. Then $\Gamma_*^Z$ is a refinement in $\text{Gr}_*(\mathcal{G}, \Gamma_*^Z)$, and $\Gamma_*^Z$ is given by an
\begin{align*}
\alpha \in S(\Gamma_s, \mathbb{Z}) & \cong GS_\ast(\Gamma_s, \mathbb{Z}, S). \\
\end{align*}

**Definition 4.7.** Let \( G \in \text{ob}(Gr_\ast) \), \((S, \iota, o) \in \text{ob}(GS_\ast)\), and \( \beta \in GS_\ast(G, T, S) \). The factorisation category \( \text{fact}(\beta) \) of \( \beta \) is the category whose objects are pairs \((\Gamma_s, \alpha)\) where \( \Gamma_s \) is a non-degenerate \( G \)-shaped graph of pointed graphs with colimit \( \Gamma_s \) and \( \alpha \in S(\Gamma_s) \) is such that the composition

\[
\begin{array}{ccc}
G & \xrightarrow{[\Gamma_s]} & \Gamma_s \\
& \alpha \downarrow & S \\
& \Gamma^2_s & \\
\end{array}
\]

is equal to \( \beta \).

Morphisms in \( \text{fact}(\beta)([\Gamma^1_s, \alpha^1], [\Gamma^2_s, \alpha^2]) \) are commuting diagrams

\[
\begin{array}{ccc}
\Gamma_s & \xrightarrow{\alpha^1} & S \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Gamma^2_s & \xrightarrow{\alpha^2} & S \\
\end{array}
\]

in \( GS_\ast T_\ast \), where \( g \) is a morphism in \( Gr_\ast((\Gamma^1_s, \Gamma^2_s)) \).

**Lemma 4.8.** For all \( G \in \text{ob}(Gr_\ast) \) and all \( \beta \in GS_\ast(G, T, S) \), the category \( \text{fact}(\beta) \) is connected.

**Proof.** This follows directly from the discussion above, and in particular Example 3.72.

Let \((S, \iota, o)\) be \((C, \omega)\)-coloured pointed graphical species.

An element \( c \in T_s S(l) = C \) may be represented by a stick graph \( (l) \cong (l) \), together with an element \( c_l \in T_s S(l) \cong C \), and a choice of isomorphism \( g \in Gr_\ast(l, l) \) such that \( S(g)(c_l) = c \).

For \( X \) a finite set, \( S \)-structured \( X \)-graphs \((X^1, \alpha^1), (X^2, \alpha^2)\) represent the same element of \( T_s S_X \) if and only if they are similar in \( XGr^\ast \mathbb{Z}(S) \), or if \((X^1, \alpha^1) = (C_0, o^2_\mathbb{Z}) \), and \((X^2, \alpha^2) = W^l(\iota c) \) for some \( c \in C \), and \( l \geq 1 \).

In the first case, \( \text{fact}(\beta) \) is connected by Lemma 3.49 and in the second case, since \( o^2_\mathbb{Z} = S(x)(c) \) and \( S(ch_x)(W^l(\iota c)) = c \), if \( e \in E(W^l) \) is any edge of \( W^l \), there is a commuting diagram

\[
\begin{array}{ccc}
C_0 & \xrightarrow{e} & W^l \\
\downarrow & \quad & \downarrow \\
S & \xrightarrow{o^2_\mathbb{Z}} & W^l(\iota c) \\
\end{array}
\]

in \( GS_\ast \).

For general \( G \in \text{ob}(Gr_\ast) \), if \( G \neq C_0 \), elements of \( GS_\ast(G, T, S) \cong T_s S(G) \) are represented by \( G \)-shaped pointed graphs of \( S \)-structured graphs. Since there is no object of the form \((C_0, b)\) in \( el(G) \), two such graphs of graphs, \( \Gamma^1_s, \Gamma^2_s \) represent the same element of \( T_s S(G) \) if and only if

\[
\Gamma^1_s(C_X, b) \sim \Gamma^2_s(C_X, b) \in XGr^\ast \mathbb{Z}(S)
\]

for all \((C_X, b) \in el(G) \). In other words, \( \Gamma^1_s, \Gamma^2_s \) represent the same element of \( T_s S(G) \) if and only if they are in the same connected component of \( Gr^\ast \mathbb{Z}(S) \) (see Definition 3.39). The result follows by an application of Lemma 3.54.

\[\square\]

Theorem 4.3 now follows from [4, Sections 1.2].
Proof of Theorem 4.3. The category $\text{Gr}_*$ is dense in $\text{GS}_*$. By \[4\] Proposition 2.5, $T_*$ has arities $\text{Gr}_*$ if and only if, for all $G \in \text{ob}(\text{Gr}_*)$ and all $\beta \in T_*, S(\beta)$, the category $\text{fact}(\beta)$ is connected. So $T_*$ has arities $\text{Gr}_*$ by Lemma \[1, 8\].

In particular, since $\text{Gr}_* \subset \text{GS}_*$ is dense and $T_*$ has arities $\text{Gr}_*$, it follows from \[4\] Propositions 1.5 and 1.9, that the the functor $N : \text{CSM} \to \text{PrSh}(\tilde{\text{GS}}_*)$ is fully faithful. Moreover, by \[4\] Theorem 1.10 its essential image is the subcategory of those presheaves on $\tilde{\text{Gr}}_*$ whose restriction to $\text{Gr}_*$ are in the image of the fully faithful embedding $\text{GS}_* \hookrightarrow \text{PrSh}(\text{Gr}_*)$. In other words, the essential image of the inclusion $\text{CSM} \hookrightarrow \text{PrSh}(\tilde{\text{Gr}}_*)$ consists of precisely those presheaves whose restriction to $\text{Gr}_*$ is a sheaf on $(\text{Gr}_*, J_*)$.

So $S \in \text{ob}(\text{PSh}(\tilde{\text{Gr}}_*))$ is in the image of $N$ if and only if

$$S(G) = \lim_{(C, b) \in \text{el}(\tilde{G})} S(C)$$

and, by finality of $\text{el}(\tilde{G}) \subset \text{el}_*(\tilde{G})$, this is the case precisely if

$$S(G) = \lim_{(C, b) \in \text{el}(G)} S(C).$$

\[\square\]

5. Relationship to other work

In Remark 3.4, I have mentioned \[16, 17\] and their relationship to \[19\]. Relative advantages and disadvantages of the method of \[16, 17\] compared with a Joyal-Kock style construction are discussed in detail in \[17\] Remark 2.15]. The graph substitution underlying the monad multiplication in \[16, 17\] proceeds largely ‘by hand’ and involves a lot of data. In contrast to the method presented here, this approach does not rely so heavily on the combinatorial power of the involutive stick graph $(\cdot)$. In particular, the ‘modular operad monads’ are defined in terms of coloured graphs and are therefore palette-specific. And, in this case, it is obvious how to construct monads for CSMs enriched in categories other than $\text{Set}$. Indeed, the modular operads of \[17\] are introduced there as an ‘enriched version of the compact symmetric multicategories introduced in \[17, 18\]. By contrast, CSMs as defined here are, in particular, presheaves on $\text{Gr}$. So, a generalisation of the definition to apply to functors $P : \text{Gr}^{\text{op}} \to \mathcal{E}$ (for some $\mathcal{E}$ other than $\text{Set}$) would still require that $P(\cdot)$ is an object of $\mathcal{E}$. As a result, the framework presented here is not the correct one for studying enriched CSMs in the sense of \[16, 17\]. However, it does provide a natural context for considering CSMs internal to a category $\mathcal{E}$ with finite limits.

In particular, it is possible to define Segal CSMs (Segal modular operads) in the sense of \[16\] Section 3.2], in terms of $\text{sSet}$-valued presheaves on $\tilde{\text{Gr}}_*^{\text{op}}$. The goal of \[16\] is to establish that these are legitimately ‘up to homotopy’ CSMs. To this end, graphical category $U$ of \[16\] is precisely constructed to admit a Reedy structure, enabling the authors to define a model structure on $\text{sSet}^{\text{op}}$ – as a localisation of the Reedy model structure – whose fibrant objects are the Segal CSMs (\[16\] Theorem 3.8]). This approach to the nerve is very different from the one presented here. Nonetheless, they do admit comparison. In \[17\] Section 4 it has already been shown that the nerve theorem \[17\] Theorem 3.6], implies the nerve theorem of \[19\] Section 6] (under the correct interpretation of the latter) and these comments can be extended to presheaves on $\tilde{\text{Gr}}_*$ and Theorem \[13\]. It is not yet known whether it is possible to obtain a (suitable) model structure on $\text{sSet}^{\tilde{\text{Gr}}_*^{\text{op}}}$ without first restricting to $U$.

This paper’s sequel \[30\] gives a precise sense to the claim made in the introduction that ‘CSMs are the connected part of compact closed categories’, by defining CSMs with product. These stand in the same relation to CSMs as props do to properads. Vallette introduced properads in \[34\] as the connected part of props and they contain “all the information to code [relevant] algebraic structure”. In particular, he used properads to prove Koszul duality for quadratic props.
In relation to ‘prop-like structures’, \cite{22} defines a monad $T^\times$ on $\mathbf{GS}$ whose algebras are non-unital CSMs with product. For any palette $(C, \omega)$, the terminal $(C, \omega)$-coloured graphical species $Z^{(C, \omega)}$ trivially admits a $T^\times$-algebra structure. In particular, objects of the slice category $\mathbf{GS}^{T^\times} \downarrow D$ have a wheeled semi-prop, (or ‘non-unital wheeled prop’) structure (see Examples \cite[1.43, 1.46 and 3.60]{22}). The monad $T^\times$, though similar to $T$, does not distribute over $D$. However (as noted, though not explicitly described in \cite[Sections 1, 2, 7]{22}), $T^\times$ is itself a composite monad. In fact, there are iterated distributive laws (see \cite[7]{22}) that enable the construction of the desired monad for CSMs with product as a composite of three monads on $\mathbf{GS}$. Moreover, there is a canonical adjunction between the EM-category of algebras for this composite monad and the category of small compact closed categories. (Details to appear in \cite[30]{22}.)

### References

1. John C. Baez, John Foley, Joseph Moeller, and Blake S. Pollard. Network Models. arXiv e-prints, page arXiv:1711.00037, Oct 2017.
2. M. A. Batanin and C. Berger. Homotopy theory for algebras over polynomial monads. Theory Appl. Categ., 32:Paper No. 6, 148–253, 2017.
3. Jon Beck. Distributive laws. In Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), pages 119–140. Springer, Berlin, 1969.
4. Clemens Berger, Paul-André Melliès, and Mark Weber. Monads with arities and their associated theories. J. Pure Appl. Algebra, 216(8-9):2029–2048, 2012.
5. Francis Borceux. Handbook of categorical algebra. 1, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994. Basic category theory.
6. Dennis V. Borisov and Yuri I. Manin. Generalized operads and their inner cohomomorphisms. In Geometry and dynamics of groups and spaces, volume 265 of Progr. Math., pages 247–308. Birkhäuser, Basel, 2008.
7. Eugenia Cheng. Iterated distributive laws. Math. Proc. Cambridge Philos. Soc., 150(3):459–487, 2011.
8. Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal sets and simplicial operads. J. Topol., 6(3):705–756, 2013.
9. Martin Doubek, Branislav Jurčo, and Lada Peksova. Properads and homotopy algebras related to surfaces. 2017.
10. Martin Doubek, Branislav Jurčo, and Korbinian Münster. Modular operads and the quantum open-closed homotopy algebra. J. High Energy Phys., (12):158, front matter+54pp, 2015.
11. Gabriel C. Drummond-Cole and Philip Hackney. Dwyer–Kan homotopy theory for cyclic operads. 2018.
12. E. Getzler and M. M. Kapranov. Modular operads. Compositio Math., 110(1):65–126, 1998.
13. Jeffrey Giansiracusa. Moduli spaces and modular operads. Morfismos, 17(2):101–125, 2013.
14. Philip Hackney, Marcy Robertson, and Donald Yau. Infinity properads and infinity wheeled properads, volume 2147 of Lecture Notes in Mathematics. Springer, Cham, 2015.
15. Philip Hackney, Marcy Robertson, and Donald Yau. On factorizations of graphical maps. Homology Homotopy Appl., 20(2):217–238, 2018.
16. Philip Hackney, Marcy Robertson, and Donald Yau. A graphical category for higher modular operads. arXiv e-prints, page arXiv:1906.01143, Jun 2019.
17. Philip Hackney, Marcy Robertson, and Donald Yau. Modular operads and the nerve theorem. arXiv e-prints, page arXiv:1906.01144, Jun 2019.
18. K. Hess, R. Levi, and S. Raynor. Enriched categories of plastic networks. In preparation, 2019.
19. A. Joyal and J. Kock. Feynman graphs, and nerve theorem for compact symmetric multicategories (extended abstract). Electronic Note in Theoretical Computer Science, 270(2):105 – 113, 2011. Proceedings of the 6th International Workshop on Quantum Physics and Logic (QPL 2009).
20. G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. J. Pure Appl. Algebra, 19:193–213, 1980.
21. Joachim Kock. Graphs, hypergraphs, and properads. Collect. Math., 67(2):155–190, 2016.
22. Joachim Kock. Cospan construction of the graph category of Borisov and Manin. Publ. Mat., 62(2):331–353, 2018.
23. Tom Leinster. Higher operads, higher categories, volume 298 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2004.
24. Saunders Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.
25. Saunders Mac Lane and Ieke Moerdijk. Sheaves in geometry and logic. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.
26. M. Markl, S. Merkulov, and S. Shadrin. Wheeled PROPs, graph complexes and the master equation. J. Pure Appl. Algebra, 213(4):496–535, 2009.
[27] Martin Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, volume 5 of *Handb. Algebr.*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008.

[28] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebr. Geom. Topol.*, 7:1441–1470, 2007.

[29] S. Raynor. *Compact Symmetric Multicategories and the problem of loops*. PhD thesis, University of Aberdeen, 2018.

[30] S. Raynor. CSMs with product: an iterated distributive law for compact closed categories. *In preparation, expected submission December 2019*, 2019.

[31] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[32] D. Spivak. The operad of wiring diagrams. *Preprint*, 2013. arXiv:1305.0297.

[33] Ulrike Tillmann. The classifying space of the (1+1)-dimensional cobordism category. *J. Reine Angew. Math.*, 479:67–75, 1996.

[34] Bruno Vallette. A Koszul duality for PROPs. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007.

[35] Mark Weber. Familial 2-functors and parametric right adjoints. *Theory Appl. Categ.*, 18:No. 22, 665–732, 2007.

[36] Donald Yau and Mark W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.