Quantum Statistical Enhancement of the Collective Performance of Multiple Bosonic Engines

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Introduction.— Technical advances have allowed us to miniaturize thermal machines to the nanoscale and beyond, where quantum effects can play an important role [1]. A paradigmatic instance of a thermal machine is a quantum heat engine (QHE). First conceived in the late 50s, a QHE is a quantum system that serves as the working fuel of a thermodynamic cycle [2–7]. More recently, a synergy between technology and progress on the foundations of quantum thermodynamics [8–13] has led to a surge of activity on the study of quantum machines [14–22], consolidating it as an active area of research [23]. A prominent challenge in this context is the identification of situations where quantum effects govern and lead to an enhanced performance with no classical counterpart [19, 24–26]. One strategy to identify such situations is to consider thermal machines composed of multiple components [17, 24, 27–36] described by collective quantum states with collective unitary operations on the constituents. Alternatively, the natural process of extracting work from a QHE by outcoupling it to drive another quantum system [37, 38] or even the process of storing work in a quantum system [28, 34] can also lead to the manifestation of genuine quantum effects. In this letter, we identify a third route, in which quantum statistics leads to a genuine quantum enhancement of the performance as a result of the statistical indistinguishability of the constituent work resources. Specifically, we consider multiple work resources, each composed of a single QHE with an individual piston [20, 39], coupled to a single external system and show that the internal energy change of the external system displays quantum enhancement when the QHEs are indistinguishable. We note that such setting is fundamentally different from a single QHE with a working fluid consisting of multiple particles [24, 27, 30, 36, 40–42].

Setup.— Consider N work resources (working entities) collectively denoted by R coupled to an external quantum system S on which the work is performed; see Fig. 1. The global Hamiltonian of the whole system is the sum of that of the work resources, the external system, and the coupling C between them,

\[ H(t) = H_R(t) + H_C(t) + H_S, \]

where the external system is assumed to be time independent. If the work resources are QHEs, \( H_R(t) \) collectively represents the Hamiltonian for N engines and the two common baths. The coupling Hamiltonian \( H_C \) takes the general form

\[ H_C(t) = \sum_j g_{C,j}(t) V_{R,j} \otimes V_{S,j}, \]

where \( g_{C,j}(t) \) is a time-dependent coupling constant, and \( V_{R,j} \)

- **FIG. 1.** Schematic picture of the setup. Work \( w \) is transferred from multiple work resources collectively denoted by \( R \) to an external system \( S \) through the mutual coupling Hamiltonian \( H_C \). If the work resources are \( N \) quantum heat engines \( E_1, E_2, \cdots, E_N \), all the engines and the heat baths collectively denoted by \( E \) and \( B \), respectively, are included in \( R \).
be zero without loss of generality, the average work is given by \( \langle w_N \rangle = \sum_{i \neq 0} \lambda_i E_i \rho_i \), where the probability \( \rho_i \) for measuring the \( i \)th eigenvalue \( E_i \) of \( H_S \) as an outcome of the energy measurement at \( t = T \) reads

\[ p_i = \text{Tr}_R \left[ \langle S \{ i | U(T) (0) \rho_0 U(T)^\dagger (0) \} | i \rangle_S \right]. \]

Here, \( \text{Tr}_R[\cdots] \) is the trace over the Hilbert space of the work resources and \( | i \rangle_S \) is the \( i \)th eigenvector of \( H_S \). To gain an analytical insight, we first resort to the weak coupling regime where \( \int_0^T g_C(t) \, dt \ll 1 \). In this limit, expanding the propagator to leading order as \( U(T) (0) \approx 1 - i \int_0^T g_C(t) V_R^{(t)} (t) \otimes V_S^{(t)} (t) \) in Eq. (4), the excitation probability of the system reduces to

\[ p_i \approx \int_0^T dt \int_0^T \left[ g_C(t) V_R^{(t)} \right]_S \left[ | i \rangle_S \langle i | \right]_S \rho_0^S. \]

with \( \langle \cdots \rangle_\rho^S \equiv \text{Tr}_R[\cdots]_\rho^S \).

Quantum statistical enhancement.—To demonstrate the genuinely quantum mechanical advantage of indistinguishable bosons in comparison to indistinguishable fermions and distinguishable particles as the work resources, we consider \( N \) QHEs, each performing an Otto cycle with the two lowest internal energy levels of a bosonic atom prepared in its center of mass (COM) ground state as a working fluid. As sketched in Fig. 1, the work resources \( R \) contain these engines, together with the hot and cold heat baths.

The four strokes of the Otto cycle are performed as follows: (0) Initial state.—In the absence of the coupling to the external system \( S \), \( g_C (0) = 0 \), all two-level atoms are prepared in thermal equilibrium with the common cold bath at inverse temperature \( \beta_c \). Thus the initial reduced density matrix \( \rho_0^b \equiv \text{Tr}_B \rho_0^B \) of the engine part is \( \rho_0^b = Z_{\beta_c}^{-1} \exp[-\beta_c H_E (0)] \), with \( Z_{\beta_c} \equiv \text{Tr}_E \exp[-\beta_c H_E (0)] \), where \( \text{Tr}_E[\cdots] \) are the trace over the Hilbert space of the engines and that of the baths, respectively. The baths are assumed to be time-independent and in the canonical state of \( H_B \) throughout the cycle. (1) Isentropic compression.—From \( 0 < t < T/2 \), all the engines are decoupled from the baths, \( H_{EB} = 0 \), and the level distance of all the two-level atoms are slowly increased in the same manner. (2) Hot isochore.—At \( t = T/2 \), setting \( g_C = 0 \), all the two-level atoms are brought into weak contact with a common hot bath and thermalized at inverse temperature \( \beta_h \). At the end of this process, the state of the engine is given by \( \rho_0^E = \text{Tr}_B \rho_0^B \equiv Z_{\beta_h}^{-1} \exp[-\beta_h H_E (T/2)] \) with \( Z_{\beta_h} = \text{Tr}_E \exp[-\beta_h H_E (T/2)] \). (3) Isentropic expansion.—From \( T/2 < t < T \), all engines are decoupled from the baths, \( H_{EB} = 0 \), and the energy separation of each two-level atom is decreased slowly in the same way. (4) Cold isochore.—At \( t = T \), setting \( g_C = 0 \), and all the two-level atoms are brought into contact with the common cold bath again and quickly return to the initial state.

First we focus on the case of indistinguishable atoms. We

\[ H_C (t) = g_C (t) V_R \otimes V_S, \]

the extension to the general form (2) being straightforward.

The work done by the work resources is evaluated by energy measurements on the external system \( S \) at the beginning and the end of the cycle at \( t = 0 \) and \( T \), respectively [38]. For simplicity, we turn off the coupling \( g_C (t) \) at \( t = 0 \) and \( T \), and thus \( [H_S, H (t)] = 0 \) at these moments in time. Consequently, measurements of the system energy \( H_S \) at these two times do not affect the state of the work resources \( R \). The external system is initially prepared in its ground state \( | 0 \rangle_S \), and the initial state \( \rho_0 \) of the total system is \( \rho_0 = \rho_0^b \otimes | 0 \rangle_S \otimes | 0 \rangle_S \) with \( \rho_0^b \) being the initial state of the work resources.

Average of work by outcoupled work resources.—We consider the average of work \( \langle w_N \rangle \) done on the system \( S \) by the \( N \) work resources. In the rotating frame with respect to \( H_0 (t) \equiv H_R (t) + H_S \), the propagator in the interaction picture is \( U_R^{(t)} (0) = \mathcal{T} \exp \left[ -i \int_0^t H_C^{(t)} (t') \, dt' \right] \), where \( \mathcal{T} \) is the time-ordering operator and \( H_C^{(t)} (t) \equiv U_0^{(t)} (0) H_C (t) U_0^{(t)} (0) \) with \( U_0^{(t)} (0) \equiv \mathcal{T} \exp \left[ -i \int_0^t H_0^{(t)} (t') \, dt' \right] \).

Regarding the coupling Hamiltonian \( H_C \) in Eq. (3), \( H_C^{(t)} (t) \) reads \( H_C^{(t)} (t) = g_C (t) V_R^{(t)} (t) \otimes V_S^{(t)} (t) \) with \( V_R^{(t)} (t) \equiv U_R^{(t)} (0) V_R U_R^{(t)} (0), \ V_S^{(t)} (t) \equiv e^{i H_S t} V_S e^{-i H_S t}, \) and \( U_R^{(t)} (0), \rho_0 \equiv \mathcal{T} \exp \left[ -i \int_0^t H_0^{(t)} (t') \, dt' \right] \).

Setting the energy of the ground state \( | 0 \rangle_S \) of the system to
choose $V_R = 2S_x$ in the coupling Hamiltonian (3)

$$H_C(t) = g_C(t) 2S_x \otimes V_S,$$

with $S_x \equiv (a^b a + b^b a)/2$, where $a^b$ and $a$ are creation and annihilation operators of the ground-state atoms in the lowest COM level, and $b^b$ and $b$ are those of the excited-state atoms, respectively. While we keep the external system general in this discussion, we note that if the external system is a harmonic oscillator (HO) and $V_S = c^\dagger + c$ with $c^\dagger$ and $c$ being creation and annihilation operators of the HO, Eq. (6) reduces to the standard dipole coupling between an ensemble of atoms and a single-mode HO. In order to compute the average work using the probability in Eq. (5), we now choose the following engine Hamiltonian

$$H_E(t) = 2\Omega(t) S_z + 2\Delta S_z,$$

with $S_z \equiv (a^b a - b^b a)/2$. In our model of the Otto cycle, the baths are time-independent and couple to the engine degrees of freedom only during the hot and cold isochores. Thus the time evolution of the work resource $R$ arises purely from the engine Hamiltonian (7). In order to account for this time evolution and estimate the work done in the compression and expansion strokes, we resort to an adiabatic approximation for the engine dynamics and write the propagator as

$$U_R(t,t_0) \approx \sum_{m=-N/2}^{N/2} |m, \theta(t)\rangle_E \langle m, \theta(t_0)| e^{-i\phi(t,t_0)} ,$$

with $\phi(t,t_0) = \int_{t_0}^{t} dt' 2E_r$. Here, we denote by $|m, \theta(t)\rangle_E$ the eigenstate of the instantaneous engine Hamiltonian $H_E(t)$ with eigenvalue $2E_r m$, where $E_r = \sqrt{\Omega(t)^2 + \Delta^2}$. Further, $\theta(t)$ is defined by $tan \theta(t) = -\Omega(t)/\Delta$. The initial time is $t_0 = 0$ for the isentropic compression and $t_0 = T/2$ for the isentropic expansion strokes. With this adiabatic propagator, we can immediately obtain the autocorrelation function of the operator $V_R$, when $t_0 \leq \{t, t'\} \leq t_0 + T/2$, in Eq. (5) as [43]

$$\langle V_R^{(t)}(t')V_R^{(t')}(t) \rangle_{\rho_B^g} = 4 \cos \theta_i \cos \theta_f \sum_{m=\pm} \left[ N \left( \frac{N}{2} + 1 \right) - F_\sigma(N,\beta_0 E_0) \right] ,$$

with $\beta_0 \equiv \beta_*. \beta_{t/2} \equiv \beta_0$ and $F_\sigma(N,\beta_0 E_0) \equiv \langle m^2 \rangle \pm \langle m \rangle$, where the expectation values are defined with respect to the thermal state of $H_E(t_0)$ at $\beta_0$. On the other hand, if $\tau$ and $\tau'$ are separated by the thermalization process at $\tau = T/2$, for instance $\tau < T/2$ and $\tau' > T/2$, the autocorrelation function takes the factorized form $\langle V_R^{(t)}(t')V_R^{(t')}(t) \rangle_{\rho_B^g} = \langle V_R^{(t)}(t') \rangle_{\rho_B^g} \langle V_R^{(t')}(t) \rangle_{\rho_B^g}$ with

$$\langle V_R^{(t)}(t) \rangle_{\rho_B^g} = 2 \cos \theta_i \langle m^2 \rangle .$$

Next, we examine the case in which all the atoms are distinguishable. In this case, the coupling Hamiltonian (6) reduces to $H_C(t) = g_C(t) 2S_x \otimes V_S$ and the engine Hamiltonian (7) to $H_E(t) = 2\Omega(t) (\sum_{j=1}^{N} (\sigma_{j,x}/2) + 2\Delta \sum_{j=1}^{N} (\sigma_{j,x}/2))$, where $\sigma_{j,} \sigma_{j,z}$ are the Pauli matrices of $j$th atom. Assuming adiabatic dynamics like in the indistinguishable case, we find the following autocorrelation function

$$\langle V_R^{(t)}(t')V_R^{(t')}(t) \rangle_{\rho_B^g} = \cos \theta_i \cos \theta_f \left[ N + (N-1) \tanh^2(\beta_0 E_0) \right] + \frac{N \sin \theta_i \sin \theta_f}{2 \cosh(\beta_0 E_0)} \sum_{\sigma=\pm} e^{i(\phi(t') - \phi_0 - \beta_0 E_0)},$$

and the average

$$\langle V_R^{(t)}(t) \rangle_{\rho_B^g} = N \cos \theta_i \tanh(\beta_0 E_0),$$

allowing the calculation of the probability in Eq. (5) for the distinguishable case. We emphasize that in the distinguishable case, while taking the trace over the engine states, all the possible $2^N$ configurations of the atomic pseudo-spins have to be considered, while in the indistinguishable case the trace is taken over only $N+1$ symmetrized eigenstates of $S_z$.

To clearly evidence that Bose statistics leads to a quantum advantage, we specialize the coupling protocol to the impulse form $g_C(t) = g \delta(t - t_1)$, with $0 \leq t_1 \leq T/2$. In this case the expressions for the probability (5) are simplified greatly and the average work equals

$$\langle W_N \rangle \simeq g^2 \left\langle \left| V_R^{(t_1)}(t_1) \right|^2 \right\rangle_{\rho_B^g} \sum_{\ell \neq 0} \left| \ell \right|^2 \left| \left\langle \ell | V_R^{(t_1)}(t_1) | 0 \right\rangle \right|^2 .$$

Thus, for the impulse form of the coupling, the average work for the indistinguishable and distinguishable cases differ by the value of the variance $(\left\langle \left| V_R^{(t_1)}(t_1) \right|^2 \right\rangle_{\rho_B^g} - \left\langle \left| V_R^{(t_1)}(t_1) \right|^2 \right\rangle_{\rho_B^g})$ that can be evaluated from Eqs. (9) and (11). Remarkably, we find that, in the indistinguishable case, this variance is larger than or equal to that of the distinguishable case for any values of the parameters
$N \geq 1$, $\beta_i E_0$, and $\theta_i$ (see [44]). This fact guarantees that the work $\langle w_N \rangle^{\text{indist}}$ performed by $N$ indistinguishable bosonic engines is always larger than the work $\langle w_N \rangle^{\text{dist}}$ done by the same number of distinguishable engines. The resulting enhancement can be quantified by the ratio $\mathcal{E} = \langle w_N \rangle^{\text{indist}} / \langle w_N \rangle^{\text{dist}}$.

In Fig. 2 (a), we compare our analytical result for the enhancement $\mathcal{E}$, for different values of $\theta_i$ and $N$, with numerical simulations for a specific choice of the system as a HO with frequency $\omega$, i.e., $H_S = \omega c^\dagger c$ and $V_S = c^\dagger + c$ in the coupling Hamiltonian (6). For the engine Hamiltonian $H_E(t)$, we consider linear sweeps of $\Omega(t)$ as $\Omega(t) = -vt$ and $\Omega(t) = v(t - T)$ for the isentropic compression and expansion strokes respectively. Hereafter, let us focus on the situation with $\Delta = 0$ (i.e., $\theta_i = -\pi/2$), where the difference between the indistinguishable and distinguishable cases is most prominent [45]. When $\Delta = 0$, using Eq. (11) we see that $\langle w_N \rangle \propto N$ for the distinguishable case while in the indistinguishable case using Eq. (9) the dependence on $N, \beta_E$ is more involved (see [44]).

In general, although $\langle [V_R^{(j)}(t_1)]^2 \rangle_{\rho_0}^\sigma$ contains both terms proportional to $N^2$ and $N$, we find that $\langle w_N \rangle$ shows $N^2$ scaling for moderate values of $N$ with $N\beta_E \Omega(0) \lesssim 1$ [46]. We see this behavior in Fig. 2 (b), where we plot $\sqrt{\langle w_N \rangle / \langle w_1 \rangle}$ to bring out the quadratic scaling. We also note that for sufficiently large $N$, the $N^2$ scaling of $\langle [V_R^{(j)}(t_1)]^2 \rangle_{\rho_0}^\sigma$ for the indistinguishable case turns into a linear scaling with an enhanced slope of $\omega_N \approx i + p T$.

Considering general coupling protocols $g(t)$, when $\Delta = 0$, we find that the autocorrelation $\langle V_R^{(j)}(t'')V_R^{(j)}(t) \rangle_{\rho_0}^\sigma$ is factorized, and hence vanishes when $t$ and $t'$ are separated by a thermalization step with the hot bath. This allows us to simplify (5) and write the probability of excitation of the driven system in the indistinguishable case as

$$p_i^{\text{indist}} \simeq \sum_{\sigma = \pm 1} \frac{N(N + 2)}{4} \left(1 + \sigma \tan(\beta_i E_0)\right).$$

and in the distinguishable case as

$$p_i^{\text{dist}} \simeq \sum_{\sigma = \pm 1} \left(1 + \sigma \tan(\beta_i E_0)\right) \right),$$

where the positive, coupling-protocol-dependent terms are determined by the amplitudes $c_i^{\sigma}(t_0) = \int_{t_0}^{t_0 + T/2} dt \, g(t) \langle S|c_i^{(j)}(t)|S \rangle e^{i \phi(t_0)}$.

In order to widen the scope of our results we also consider engine strokes with $\Delta \neq 0$ as described in more detail in [44]. There, we find that the enhancement persists for small values of $N$ independent of the form of the coupling and the external system Hamiltonian. The fact that the enhancement is guaranteed for small $N$ most likely will be of particular relevance to experiments in the near future that will presumably implement with small $N$ in most platforms. Furthermore, we demonstrate for a given choice of the Hamiltonian how to identify generic parameter regimes and coupling protocols that lead to enhancement. Lastly, we also extend our results to non-perturbative continuous coupling $g_c(t)$ (see [44] for the exact functional form) using numerical simulations for a HO external system. From Fig. 3 we see that for small values of $N$ and moderate values of the coupling strength, there is enhancement and the average work for indistinguishable engines scales as $N^2$. The fact that we are able to find enhancement for a generic set of parameters as in Fig. 3 without fine tuning the parameters suggests the general applicability of our result.

Finally, we briefly discuss the case in which each engine is made of an identical non-interacting two-level fermionic atom. In this case, $\langle w_N \rangle$ displays a pronounced even-odd dependence with respect to $N$. At $T_{\text{COM}} = 0$, $\langle w_N \rangle = 0$ for even $N$, and $\langle w_N \rangle = \langle w_1 \rangle$ for odd $N$. This is because the atoms inside the Fermi sphere, in which both the ground and excited internal levels are fully occupied, cannot contribute to work because of the Pauli blocking: In the case of odd $N$, there is one fermion on the top of the Fermi surface, and this is the only atom which can change the internal states in the engine operation. By contrast, there is no such atom for even $N$. At nonzero $T_{\text{COM}}$, $\langle w_N \rangle$ takes non-trivial values due to the contribution of atoms near the Fermi surface, where the COM levels are only partially occupied. In the regime where $T_{\text{COM}}$ is much smaller than the energy difference $\omega_{\text{trap}}$ between the highest occupied and lowest unoccupied trap levels, the ratio $\lambda = \langle w_N \rangle / \langle w_1 \rangle$ for an arbitrary trap in general can be well characterized by only $\beta_{\text{COM}} \omega_{\text{trap}}$. Symbols in Fig. 4 show $\lambda$ numerically calculated for multiple fermionic engines in the Otto cycle for several values of $N$. In this example, we assume that atoms are trapped in a HO potential with frequency $\omega_{\text{trap}}$. Another HO with frequency $\omega$ is taken as an external system $S$. This numerical result is well fitted by [44]

$$\langle w_N \rangle / \langle w_1 \rangle \simeq \begin{cases} 8 \exp(-2 \beta_{\text{COM}} \omega_{\text{trap}}) & \text{(even } N), \\ 1 + 8 \exp(-2 \beta_{\text{COM}} \omega_{\text{trap}}) & \text{(odd } N). \end{cases}$$

At higher COM temperatures $T_{\text{COM}}$ for which $\beta_{\text{COM}} \omega_{\text{trap}} \lesssim 1$, the work $\langle w_N \rangle$ depends on details of the system such as the spectrum of the COM degrees of freedom determined by the shape of the trapping potential and the Fermi energy. A detailed study of this dependence is beyond the scope of the present work and will be taken up in the future.

In conclusion, we have demonstrated that the statistical indistinguishability of work resources can be exploited to gain a genuine quantum-enhancement in quantum thermodynamics. The predicted enhancement of the work outcoupled from multiple indistinguishable heat engines to a generic external system is readily testable with current or near future experimental realizations of quantum heat engines, e.g., in nitrogen-vacancy centers [17], trapped ions [49] and ultracold gases. While we have considered bosonic and fermionic statistics, exotic fractional statistics [36, 50] may lead to further interesting results.
FIG. 4. Performance of fermionic engines at nonzero temperature $T_{\text{COM}}$. The left panel shows the work outcome for even $N$, with $N = 2$ (red) and $N = 4$ (blue) obtained by numerical simulations. They are well reproduced by the analytical expression $\lambda = 8\exp(-\beta_{\text{COM}}\omega_{\text{trap}})$ (dashed line). Odd values of $N$ are considered in the right panel, with $N = 3$ (green) and $N = 5$ (orange), in good agreement with $\lambda = 1 - 8\exp(-2\beta_{\text{COM}}\omega_{\text{trap}})$ (dashed line). Parameters are $\Delta = 1$, $\nu = 0.5/\Delta^2$, $T = 20/\Delta$, $\beta E_0 = 1$, $\beta \mu E_T/2 = 1/8$, $\omega = 2\pi \times 0.05/T$, $g = 0.5$, $\delta = 0.98$, and $\alpha = 2000/T$.

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respect to the state $\rho_{\uparrow/\downarrow}$.

[44] See Supplemental Material.

[45] For $\Delta \equiv 0$, the above results are accurate even in the diabatic case since the exact time evolution operator and the adiabatic time evolution operator of the engine coincide for this case since $H_E$ is diagonal throughout.

[46] It should be noted that the $N^2$ scaling is obtained for the thermal initial state of the engines, which is strikingly different from the Dicke superradiance [47, 48].

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### Supplemental Material

#### Impulse-type coupling between the engines and the system

The average work with impulse type coupling at time $t_1$ (assuming $0 \leq t_1 \leq T/2$) is discussed in the main manuscript, see Eq. (13). This discussion carries over to the case $T/2 \leq t_1 \leq T$ up to the fact that the variance have to be taken with respect to the work resource state $\rho_{R,1}^E$ and is given by

$$\langle w_N \rangle \simeq 2 N^2 \langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_{0}^E} \sum_{i \neq 0} \epsilon_i^S \langle s(\bar{i}|V_S^{(I)}(t_1)|0)_S \rangle^2. \quad (S1)$$

An important simplification arising from the impulse coupling becomes evident in the above equation: the average work is a product of a term that depends only on the engine or work-resource operators and a term that contains the system operators alone. The engine-dependent quantity is the second moment of the coupling $\langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_0^E}$, and it can be obtained using the general expression for the two-time correlators $\langle |V_R^{(I)}(t)|V_R^{(I)}(t')\rangle_{\rho_0^E}$ provided in Eqs. (9) and (11) of the main paper as

$$\langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_0^E} = \frac{1}{2} N(N+1) - 2 f(N, \beta, E_0) \sin^2 \theta_1 + 4 f(N, \beta, E_0) \cos^2 \theta_1 \quad (S2)$$

for the indistinguishable case and as

$$\langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_0^E} = \frac{N^2 \sin^2 \theta_1 + [N+N(N-1)\tanh^2(\beta, E_0)] \cos^2 \theta_1}{(S3)}$$

for the distinguishable case. Here, we have introduced the function $f(N, \beta, E_0) = \langle m^2 \rangle$ to denote

$$f(N, \beta, E_0) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{4 m^2 e^{-\beta_0^2 E_0 m} \beta_0^2 E_0}{e^{\beta_0^2 E_0 m}} \, d\beta_0^2 E_0 = \frac{\langle H_2^2(0) \rangle_{\rho_0^E}}{4 E_0^2} = \frac{N^2 \sinh[(N+3)\beta_0 E_0] + (N+2)^2 \sinh[(N-1)\beta_0 E_0] - 2 (N^2+2N-2) \sinh[(N+1)\beta_0 E_0]}{16 \sinh[(N+1)\beta_0 E_0] \sinh^2(\beta_0 E_0)}. \quad (S4)$$

Now, using the inequalities

$$f(N, x) \leq \frac{N^2}{4}, \quad 4 f(N, x) \geq [N+N(N-1)\tanh^2(x)], \quad (S5)$$

we compare the coefficients of the $\sin^2 \theta_1$ and $\cos^2 \theta_1$ terms in Eqs. (S2) and (S3). It follows that the second moment $\langle |V_R^{(I)}(t_1)|^2 \rangle$, and thus the average work as well, are larger for the indistinguishable case, as we stated in the main text.

Let us now consider $\Delta = 0$ (i.e., $\cos \theta_1 = 0$) and examine the scaling of the average work with $N$. To this end, note from Eq. (S1) that it is sufficient to examine the scaling of the second moment with $N$. Nonetheless, to make the comparison with numerical solutions as in Fig. 2(b) of the main paper, we note that when the external system is chosen as a harmonic oscillator with frequency $\omega$, one finds $s(\bar{i}|V_S^{(I)}(t)|0)_S = e^{i\omega t} \delta_{\bar{i}1}$ and the average work (S1) becomes

$$\langle w_N \rangle \simeq \omega g^2 \langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_0^E}. \quad (S7)$$

Turning our attention to the second moment for the distinguishable case, we first see that $\langle |V_R^{(I)}(t_1)|^2 \rangle_{\rho_0^E} = N$. To examine the behavior of the variance in Eq. (S2) for the indistinguishable case, consider first Eq. (S4) in the limit of $N \to \infty$

$$f(N, \beta, E_0) \to \frac{N^2}{4} + \frac{N}{2} [1 - \coth(\beta, E_0)] + \frac{1}{2} [\coth(\beta, E_0) - 1] \coth(\beta, E_0). \quad (S8)$$
Eq. (S4), is also given by a moment of the partition function as
\[
\langle [V_R^{(f)}(t_1)]^2 \rangle_{\rho_0^G} \rightarrow \left\{ \text{coth} (\beta_i E_0^i) N - \left[ \text{coth} (\beta_i E_0^i) - 1 \right] \text{coth} (\beta_i E_0^i) \right\},
\]
and generalizes the expressions presented in the main paper. The function \( h \) here, the coupling-protocol- and engine-state-dependent amplitudes are given by
\[
N_x = \text{coth} (\beta_i E_0^i) - 1 \text{coth} (\beta_i E_0^i)
\]
and scales linearly with \( N \) with a slope \( \text{coth}(\beta_i E_0^i) \geq 1 \), as stated in the main paper. This term is thus larger in the indistinguishable case than in the distinguishable counterpart and is responsible of the quantum statistical enhancement of the average work. In the other extreme with \( N = 1 \), since \( f(N = 1, \beta_i E_0^i) = 1/4 \), the second moment in the indistinguishable and distinguishable cases are both equal to 1, as expected. For \( N \beta_i \Omega(0) \sim 1 \) or less we have, \( \langle [V_R^{(f)}(t_1)]^2 \rangle_{\rho_0^G} \sim f_1(\beta_i \Omega(0)) N^2 + f_2(\beta_i \Omega(0)) N \), with \( f_1(x) = x (x \text{coth} x - 1) \text{coth} x/ \sinh^2 x \text{ and } f_2(x) = 1 - [(x \text{coth} x - 1)/ \sinh^2 x] \). Therefore, \( \langle w_N \rangle \) shows \( N^2 \) scaling for moderate values of \( N \) with \( N \beta_i \Omega(0) \lesssim 1 \) for the indistinguishable case and is larger than the distinguishable case which scales linearly with \( N \).

**General coupling between the engines and the system**

For a general continuous coupling, the probability of exciting the \( i \)th eigenstate of the system reads
\[
p_i \simeq \int_0^T dt \int_0^T dt' g_C(t) g_C(t') s(i|V_S^{(f)}(t)|0)_S s(0|V_S^{(f)}(t')|i)_S \langle V_R^{(f)}(t')V_R^{(f)}(t) \rangle_{\rho_0^G}.
\]

Using the two-time correlators and one-time averages stated in the main manuscript (see Eqs. (9)–(12) there), the excitation probability for the case of many indistinguishable and distinguishable engines can be respectively written as
\[
p_i^{\text{indist}} = \sum_{t_0,T/2} 4|d_i(t_0)|^2 f(N, \beta_0 E_0) + |\tilde{c}_i^+(t_0)|^2 \left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - F_+(N, \beta_0 E_0) \right] + |\tilde{c}_i^-(t_0)|^2 \left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - F_-(N, \beta_0 E_0) \right] + 8 \mathcal{R}[d_i(t_0) d_i^*(T/2)] h(N, \beta_0 E_0) h(N, \beta_0 E_{T/2}),
\]
\[
p_i^{\text{dist}} = \sum_{t_0=0,T/2} |d_i(t_0)|^2 \left[ N + N(N-1) \tanh^2(\beta_0 E_0) \right] + |\tilde{c}_i^+(t_0)|^2 \left[ \frac{N}{2} \left( 1 + \tanh(\beta_0 E_0) \right) \right] + |\tilde{c}_i^-(t_0)|^2 \left[ \frac{N}{2} \left( 1 - \tanh(\beta_0 E_0) \right) \right] + 2 \mathcal{R}[d_i(t_0) d_i^*(T/2)] N^2 \tanh(\beta_0 E_0) \tanh(\beta_0 E_{T/2}).
\]

Here, the coupling-protocol- and engine-state-dependent amplitudes are given by
\[
\tilde{c}_i^±(t_0) = \int_{t_0}^{t_0+T/2} dt g_C(t) \sin \theta_S(i|V_S^{(f)}(t)|0)_S e^{±i\phi(t,t_0)},
\]
\[
d_i(t_0) = \int_{t_0}^{t_0+T/2} dt g_C(t) \cos \theta_S(i|V_S^{(f)}(t)|0)_S,
\]

and generalize the expressions presented in the main paper. The function \( h(N,x) \), similar to the function \( f(N,x) \) defined in Eq. (S4), is also given by a moment of the partition function as
\[
h(N, \beta_0 E_0) = \frac{N/2}{\beta_0 E_0} \sum_{m=-N/2}^{N/2} m e^{-\beta_0 E_0 m} = \frac{\langle H_E(0) \rangle_{\rho_0^G}}{2E_0} = \frac{1}{4} \frac{(N + 2) \sinh(N \beta_0 E_0) - N \sinh[(N + 2) \beta_0 E_0]}{\sinh(\beta_0 E_0) \sinh[(N + 1) \beta_0 E_0]}.
\]

In Eqs. (S11) and (S12), the common protocol-dependent, and \( N \), \( \beta_0 \)-independent parts of the first three terms given by \( |d_i(t_0)|^2 \) and \( |\tilde{c}_i^+(t_0)|^2 \) are positive but the third term (on the second line of the equations) proportional to \( \mathcal{R}[d_i(t_0) d_i^*(T/2)] \) is not necessarily positive. As discussed in the main paper, for \( \Delta = 0 \), \( \cos \theta = 0 \) at all \( t \) and the non-positive term vanishes. In this case, one can directly compare the probabilities for the indistinguishable and distinguishable cases in Eqs. (S11) and (S12), by examining the relative size of the \( N \)- and \( \beta_0 \)-dependent factors. To this end, we use the following relation that holds for any \( N > 1 \) and \( x > 0 \):
\[
\left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - F_±(N,x) \right] \geq \left[ \frac{N}{2} \left( 1 \mp \tanh(x) \right) \right].
\]
This leads to one of the central results given in the main paper: when $\Delta = 0$, for any form of the coupling protocol $g_C(t)$, $N > 1$, system Hamiltonian and value of $\beta_c$ and $\beta_h$, one finds that $P_t^{\text{indist}} > P_t^{\text{dist}}$, i.e., the work done in the indistinguishable case is larger than the distinguishable case.

Let us now consider $\Delta \neq 0$. While in this case the enhancement of work for indistinguishable engines is not guaranteed in general, in what follows we identify conditions under which enhancement can be expected. We first note that the inequality Eq. (S6) allows us to compare the first term of Eqs. (S11) and (S12) proportional to $|d_i(t_0)|^2$ and see it is larger in the indistinguishable case. Secondly, the $N$- and $\beta_0$-dependent positive factors multiplying $2\Re[d_i(0)d_i^*(T/2)]$ satisfy

$$4h(N, \beta_c E_0) h(N, \beta_h E_{T/2}) \geq N^2 \tanh(\beta_c E_0) \tanh(\beta_h E_{T/2}),$$

for any $N > 1$, $\beta_c E_0$, and $\beta_h E_{T/2}$, with the equality holding for $N = 1$. Thus, from the above discussion together with the previous considerations leading up to Eq. (S16), it follows that $P_t^{\text{indist}} > P_t^{\text{dist}}$ which enhances the average work, for coupling protocols and external system Hamiltonians that satisfy the constraint $\Re[d_i(0)d_i^*(T/2)] \geq 0$. Note that $h(N, \beta_c E_0) h(N, \beta_h E_{T/2})$ is an increasing function of $N$ with magnitude comparable to the functions on the left-hand side of the inequalities in Eqs. (S6) and (S16) for small $N$. Thus, for small $N$, the magnitude of the negative term when $\Re[d_i(0)d_i^*(T/2)] < 0$ is off-set by the positive terms in Eq. (S11) and enhancement persists. In other words, for small enough $N$, enhancement is generally present, regardless of the form of the coupling function, the external system Hamiltonian, and the temperatures of the baths, even when $\Delta \neq 0$.

We will now demonstrate that we can, without much fine tuning, identify parameter regimes where enhancement is possible for a given choice of the external system Hamiltonian and smooth form for the coupling function. Let us choose for the external system, a harmonic oscillator (HO) of frequency $\omega$ coupled to the engines. For the coupling, we consider a function which is constant for most of the duration of the work stroke and turns on and off smoothly. A suitable choice is the one used in our previous work [S1]

$$g_C(t) = \frac{g}{\delta_T} \sum_{n=0}^{1} \left\{ \tanh \left[ \alpha \left( t - t_{\text{on}} - \frac{nT}{2} \right) \right] - \tanh \left[ \alpha \left( t - t_{\text{off}} - \frac{nT}{2} \right) \right] \right\},$$

where $\alpha$ is the switching rate and $\delta_T (0 < \delta_T < 1)$ is the ratio of the duration in which the coupling is turned on from $t_{\text{on}} + nT/2$ to $t_{\text{off}} + nT/2 (n = 0$ and $1)$ in the duration of the half of the cycle for $T/2$: $t_{\text{on}} = (1 - \delta_T)T/4$ and $t_{\text{off}} = t_{\text{on}} + \delta_T T/2$. We note that this is the coupling form used in the numerical calculations generating the results in Figs. 3 and 4 of the main paper. Let us, for the sake of concreteness, assume that the engine compression and expansion strokes are represented by a linear protocol with

$$\omega \Delta$$

for small $\Delta$-dependent positive factors multiplying $2\Re[d_i(0)d_i^*(T/2)]$.

Thus, we have

$$d_i(t_0) = -\int_{t_0}^{t_0 + T/2} dt \frac{g}{T} e^{i\omega \Delta} \frac{\Delta}{\Omega(t)^2 + \Delta^2},$$

When the external system is an oscillator with small frequency $\omega T \ll 1$, we can ignore the oscillating phase factor in the integral and find

$$d_i(t_0) \approx -\frac{g\Delta}{\omega T} \log \left[ \sec \theta_{T/2} + \tan \theta_{T/2} \right],$$

leading to

$$\Re[d_i(0)d_i^*(T/2)] = \frac{g^2 \Delta^2}{\omega^2 T^2} \left( \log \left[ \frac{\sec \theta_{T/2} + \tan \theta_{T/2}}{\sec \theta_{0} + \tan \theta_{0}} \right] \right)^2 \geq 0.$$ 

Thus, in this limit of small oscillator frequency a quantum enhancement is always present. On the other hand when $\omega T \gg 1$,

$$d_i(t_0) \approx -\frac{g\Delta}{i\omega T} \left[ \frac{e^{i\omega(t_0 + T/2)}}{\sqrt{\Omega(t_0 + T/2)^2 + \Delta^2}} - \frac{e^{i\omega t_0}}{\sqrt{\Omega(t_0)^2 + \Delta^2}} \right],$$

and

$$\Re[d_i(0)d_i^*(T/2)] \approx \frac{g^2 \Delta^2}{\omega^2 T^2} \left[ \frac{2\cos(\omega T/2)}{\sqrt{\Omega(0)^2 + \Delta^2}(\Omega(T/2)^2 + \Delta^2)} - \frac{\cos(\omega T)}{\Omega(0)^2 + \Delta^2} - \frac{1}{\Omega(T/2)^2 + \Delta^2} \right].$$
with $\Omega(T/2) = \Omega(0) - \nu T/2$. The presence of the oscillating factors in the above expression can lead to $\Re[d_i(0)d_i^*(T/2)] < 0$. While the dependence on $\Delta$, $\Omega(0)$, and $\nu$ implied by Eq. (S21) is complicated, we can immediately see that when $\Delta \ll \{\Omega(0), \omega\}$ the magnitude of the term is suppressed. Thus even if $\Re[d_i(0)d_i^*(T/2)] < 0$, in this limit this term will not be able to suppress the other positive terms in $\rho_i^{\text{indist}}$. Thus we can anticipate that in the small $\Delta$ regime, in agreement with the idea of continuity with the $\Delta = 0$ result, enhancement is exhibited. In the other extreme, when $\Omega(0) = 0$ and $\Delta \gg \nu T$ Eq. (S21) reduces to

$$
\Re[d_i(0)d_i^*(T/2)] \approx \frac{\delta^2}{\omega^2 T^2} [2 \cos(\omega T/2) - \cos(\omega T) - 1].
$$

In this regime, varying $\omega T > \pi$ thus unveils the regions in which enhancement appears and disappears as a result of the oscillatory nature of the sign of $\Re[d_i(0)d_i^*(T/2)]$. To summarize and support the above discussion, Fig. S1 shows contour plots, as a function of $\Delta/\Omega(0)$ and $\omega T$, of the regions where enhancement is present (grey) or absent (white) for different fixed values of $2 \leq N \leq 20$. Here, the expressions (S11) and (S12) are used to calculate the average work in the indistinguishable and distinguishable case. The main features are in agreement with the analytical arguments above, namely, enhancement is present at small $N$ independent of other parameter choice (for all the parameter space region explored). Further, regions with no enhancement appear for $\omega T > \pi$ at large enough $N$. Such regions also shrink and vanish as $\Delta/\Omega(0) \to 0$.

**Multiple fermionic engines trapped in a harmonic oscillator potential**

For indistinguishable fermionic engines trapped in a HO potential, the ratio $\lambda \equiv \langle w_N \rangle / \langle w_1 \rangle$ at nonzero temperature $T_{\text{COM}}$ of the center of mass (COM) degrees of freedom is discussed in the main paper. The numerical result of the outcoupled work for the Otto cycle is well fitted by the analytical formula given by Eq. (16) in the main text. This formula can be obtained by considering isolated engines without coupling to an external system as we show below.

The Hamiltonian $H_E(t)$ of the engines consists of that of the COM degrees of freedom $H_{\text{COM}}$ and that of the internal degrees
of freedom $H_{\text{in}}(t), H_E(t) = H_{\text{COM}} + H_{\text{in}}(t)$, with

$$H_{\text{COM}} = \omega_{\text{trap}} \sum_l \left( l + \frac{1}{2} \right) \left( a_l^\dagger a_l + b_l^\dagger b_l \right),$$ \hspace{1cm} (S22)

$$H_{\text{in}}(t) = \sum_l \left[ \Omega(t) (a_l^\dagger a_l - b_l^\dagger b_l) + \Delta (a_l^\dagger b_l + b_l^\dagger a_l) \right].$$ \hspace{1cm} (S23)

Here, $\omega_{\text{trap}}$ is the frequency of the HO trapping potential whose levels are labeled by $l$. The operators $a_l^\dagger (a_l)$ and $b_l^\dagger (b_l)$ are creation (annihilation) operators of a spin-up and spin-down atom at $l$th level of the trap, respectively, satisfying the fermionic anti-commutation relations $\{a_l, a_p^\dagger\} = \{b_l, b_p^\dagger\} = \delta_{l,p}$ and $\{a_l, a_p\} = \{a_l^\dagger, a_p^\dagger\} = \{b_l, b_p\} = \{b_l^\dagger, b_p^\dagger\} = 0$ with $\{A,B\} \equiv AB + BA$ being the anti-commutator. The distribution of the population $n_l \equiv a_l^\dagger a_l + b_l^\dagger b_l$ of the trap levels is conserved, which is set by $T_{\text{COM}}$ and follows the canonical distribution $\propto \exp (-\beta_{\text{COM}} H_{\text{COM}})$ with $\beta_{\text{COM}} \equiv 1/T_{\text{COM}}$.

Using the internal degrees of freedom of the two-level fermionic atoms described by the above Hamiltonian, we consider the quantum heat engines in the Otto cycle. The average of work done by $N$ engines is

$$\langle w_N \rangle = (\epsilon_b - \epsilon_c) (\tanh \beta_b \epsilon_c - \tanh \beta_b \epsilon_b) f_N (\beta_{\text{COM}} \omega_{\text{trap}}),$$ \hspace{1cm} (S24)

with $\pm \epsilon_b \equiv \pm \sqrt{\Delta^2 + \Omega^2_b}$ and $\pm \epsilon_c \equiv \pm \sqrt{\Delta^2 + \Omega^2_c}$ being the energy eigenvalues of the internal degrees of freedom in the hot and cold isochore process, respectively. Here, $f_N$ is a function of $\beta_{\text{COM}} \omega_{\text{trap}}$ whose detailed form varies with $N$. Since $f_{N=1} = 1$, the ratio $\lambda$ is

$$\lambda = f_N (\beta_{\text{COM}} \omega_{\text{trap}}),$$ \hspace{1cm} (S25)

which is independent of the temperatures of the heat baths and the spectrum of the internal degrees of freedom. Although the exact form of $f_N$ nontrivially depends on $N$ in general, in the low COM temperature limit of $\beta_{\text{COM}} \omega_{\text{trap}} \gg 1$ the form of $f_N$ differs only by the parity of $N$ as

$$f_N (\beta_{\text{COM}} \omega_{\text{trap}}) \simeq \begin{cases} 8 e^{-\beta_{\text{COM}} \omega_{\text{trap}}} & \text{(even $N$)}, \\ 1 + 8 e^{-2 \beta_{\text{COM}} \omega_{\text{trap}}} & \text{(odd $N$)}. \end{cases}$$ \hspace{1cm} (S26)

Though Eq. (S26) is obtained for the HO trapping potential, this expression is applicable for any type of the trapping potential in the low temperature limit, $\beta_{\text{COM}} \omega_{\text{trap}} \gg 1$, by replacing $\omega_{\text{trap}}$ by the energy difference between the highest occupied and the lowest unoccupied levels of the trap at zero COM temperature. Although the analytical formula (S26) is obtained for the isolated engines without outcoupling, it agrees with the numerical results for outcoupled case as shown in Fig. 4 in the main text.

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