Lie groups as 4-dimensional Riemannian or pseudo-Riemannian almost product manifolds with nonintegrable structure

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July 13, 2021

Abstract. A Lie group as a 4-dimensional pseudo-Riemannian manifold is considered. This manifold is equipped with an almost product structure and a Killing metric in two ways. In the first case Riemannian almost product manifold with nonintegrable structure is obtained, and in the second case – a pseudo-Riemannian one. Each belongs to a 4-parametric family of manifolds, which are characterized geometrically.

Mathematics Subject Classification (2000): 53C15, 53C50

Key words: almost product manifold, Lie group, Riemannian metric, pseudo-Riemannian metric, nonintegrable structure, Killing metric

1 Preliminaries

Let $M$ be a differentiable manifold with a tensor field $P$ of type $(1,1)$ and a Riemannian metric $g$ such that

$$P^2 = id, \quad g(Px, Py) = g(x, y)$$

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on $M$. The tensor field $P$ is called an almost product structure. The manifold $(M, P, g)$ is called a Riemannian (pseudo-Riemannian, resp.) almost product manifold, if $g$ is a Riemannian (pseudo-Riemannian, resp.) metric. If $\text{tr} P = 0$, then $(M, P, g)$ is an even-dimensional manifold. The classification from [3] of Riemannian almost product manifolds is made with respect to the tensor field $F$ of type $(0,3)$, defined by

$$F(x, y, z) = g((\nabla_x P)y, z),$$

where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the following properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z).$$

In the case when $g$ is a pseudo-Riemannian metric, the same classification is valid for pseudo-Riemannian almost product manifolds, too. In these classifications
the condition
\[ F(x, y, z) + F(y, z, x) + F(z, x, y) = 0 \] (3)
defines a class \( \mathcal{W}_3 \), which is only the class of the three basic classes \( \mathcal{W}_1, \mathcal{W}_2 \) and \( \mathcal{W}_3 \) with nonintegrable structure \( P \).

The class \( \mathcal{W}_0 \), defined by the condition \( F(x, y, z) = 0 \), is contained in the other classes. For this class \( \nabla P = 0 \) and therefore it is an analogue of the class of Kählerian manifolds in the almost Hermitian geometry.

The curvature tensor field \( R \) is defined by
\[ R(x, y, z) = \nabla_{x \nabla y} z - \nabla_{y \nabla x} z - \nabla_{[x, y]} z \] and the corresponding tensor field of type \((0, 4)\) is determined by \( R(x, y, z, w) = g(R(x, y) z, w) \).

Let \( \{e_i\} \) be a basis of the tangent space \( T_p M \) at a point \( p \in M \) and \( g^{ij} \) be the components of the inverse matrix of \( g \) with respect to \( \{e_i\} \). Then the Ricci tensor \( \rho \) and the scalar curvature \( \tau \) are defined as follows
\[ \rho(y, z) = g^{ij} R(e_i, y, z, e_j), \] (4)
\[ \tau = g^{ij} \rho(e_i, e_j). \] (5)

The square norm of \( \nabla P \) is defined by
\[ \|\nabla P\|^2 = g^{ij} g^{ks} g \left( (\nabla_{e_i} P) e_k, (\nabla_{e_j} P) e_s \right). \] (6)

It is clear that \( \nabla P = 0 \) implies \( \|\nabla P\|^2 = 0 \) but the inverse implication for the pseudo-Riemannian case is not always true. We shall call a pseudo-Riemannian almost product manifold isotropic \( P \)-manifold if \( \nabla P = 0 \).

The Weyl tensor on a \( 2n \)-dimensional pseudo-Riemannian manifold \( (n \geq 2) \) is
\[ W = R - \frac{1}{2n-2} \left( \psi_1(\rho) - \frac{\tau}{2n-1} \pi_1 \right), \] (7)
where
\[ \psi_1(\rho)(x, y, z, w) = g(y, z) \rho(x, w) - g(x, z) \rho(y, w) + \rho(y, z) g(x, w) - \rho(x, z) g(y, w); \]
\[ \pi_1(x, y, z, w) = g(y, z) g(x, w) - g(x, z) g(y, w). \]

Moreover, for \( n \geq 2 \) the Weyl tensor \( W \) is zero if and only if the manifold is conformally flat.

If \( \alpha \) is a non-degenerate 2-plane spanned by vectors \( x, y \in T_p M, p \in M \), then its sectional curvature is
\[ k(\alpha) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}. \] (8)
2 A Lie group as a 4-dimensional pseudo-Riemannian manifold with Killing metric

Let $V$ be a real 4-dimensional vector space with a basis $\{E_i\}$. Let us consider a structure of a Lie algebra determined by commutators $[E_i, E_j] = C^k_{ij}E_k$, where $C^k_{ij}$ are structure constants satisfying the anti-commutativity condition $C^k_{ij} = -C^k_{ji}$ and the Jacobi identity $C^k_{ij}C^l_{ks} + C^k_{js}C^l_{ki} + C^k_{si}C^l_{kj} = 0$.

Let $G$ be the associated connected Lie group and $\{X_i\}$ be a global basis of left invariant vector fields which is induced by the basis $\{E_i\}$ of $V$. Then we have the decomposition

$$[X_i, X_j] = C^k_{ij}X_k.$$  \hspace{1cm} (9)

Let us consider the manifold $(G, g)$, where $g$ is a metric determined by the conditions

$$g(X_1, X_1) = g(X_2, X_2) = g(X_3, X_3) = g(X_4, X_4) = 1,$$
$$g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j$$ \hspace{1cm} (10)

or by the conditions

$$g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,$$
$$g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j.$$ \hspace{1cm} (11)

Obviously, $g$ is a Riemannian metric if it is determined by (10) and $g$ is a pseudo-Riemannian metric of signature $(2,2)$ if it is determined by (11).

It is known that the metric $g$ on the group $G$ is called a Killing metric \cite{1} if the following condition is valid

$$g([X, Y], Z) + g([X, Z], Y) = 0.$$ \hspace{1cm} (12)

where $X, Y, Z$ are arbitrary vector fields.

If $g$ is a Killing metric, then according to the proof of Theorem 2.1 in \cite{2} the manifold $(G, g)$ is locally symmetric, i.e. $\nabla R = 0$. Moreover, the components of $\nabla$ and $R$ are respectively

$$\nabla_{ij} = \nabla_{X_i}X_j = \frac{1}{2}[X_i, X_j],$$ \hspace{1cm} (13)

$$R_{ijks} = R(X_i, X_j, X_k, X_s) = -\frac{1}{4}g([X_i, X_j], [X_k, X_s]).$$ \hspace{1cm} (14)
3 A Lie group as a Riemannian almost product manifold with Killing metric and nonintegrable structure

In this section we consider a Riemannian manifold \((G, P, g)\) with a metric \(g\) determined by (10) and a structure \(P\) defined as follows

\[
PX_1 = X_3, \quad PX_2 = X_4, \quad PX_3 = X_1, \quad PX_4 = X_2.
\] (15)

Obviously, \(P^2 = \text{id}\). Moreover, (10) and (15) imply

\[
g(PX_i, PX_j) = g(X_i, X_j).
\] (16)

Therefore, \((G, P, g)\) is a Riemannian almost product manifold.

For the manifold \((G, P, g)\) we propose that \(g\) be a Killing metric. Then \((G, P, g)\) is locally symmetric.

From (13) we obtain

\[
(\nabla X_i P) X_j = \frac{1}{2}([X_i, PX_j] - P[X_i, X_j]).
\] (17)

Then, according to (2), for the components of \(F\) we have

\[
F_{ijk} = \frac{1}{2}g([X_i, PX_j] - P[X_i, X_j], X_k).
\] (18)

Hence, having in mind (10), (15) and (16), we get

\[
F_{ijk} + F_{jki} + F_{kij} = 0,
\] (19)

i.e. \((G, P, g)\) belong to the class \(\mathcal{W}_3\).

According to (12), we have

\[
g([X_i, X_j], X_i) = g([X_i, X_j], X_j) = 0.
\] (20)

Then the following decomposition is valid

\[
\begin{align*}
[X_1, X_2] &= C^3_{12}X_3 + C^4_{12}X_4, & [X_2, X_3] &= C^1_{23}X_1 + C^2_{23}X_4, \\
[X_1, X_3] &= C^2_{13}X_2 + C^4_{13}X_4, & [X_2, X_4] &= C^1_{24}X_1 + C^3_{24}X_3, \\
[X_1, X_4] &= C^3_{14}X_2 + C^4_{14}X_3, & [X_3, X_4] &= C^1_{34}X_1 + C^2_{34}X_2.
\end{align*}
\] (21)

Now we apply again (12) using (21). So we obtain

\[
\begin{align*}
[X_1, X_2] &= \lambda_1 X_3 + \lambda_2 X_4, & [X_2, X_3] &= \lambda_1 X_1 + \lambda_3 X_4, \\
[X_1, X_3] &= -\lambda_1 X_2 + \lambda_4 X_4, & [X_2, X_4] &= \lambda_2 X_1 - \lambda_3 X_3, \\
[X_1, X_4] &= -\lambda_2 X_2 - \lambda_4 X_3, & [X_3, X_4] &= \lambda_4 X_1 + \lambda_3 X_2.
\end{align*}
\] (22)
where $\lambda_1 = C_{12}^3$, $\lambda_2 = C_{12}^4$, $\lambda_3 = C_{23}^4$, $\lambda_4 = C_{13}^4$. We verify immediately that the Jacobi identity is satisfied in this case.

Let the conditions (22) be satisfied for a Riemannian almost product manifold $(G, P, g)$ with structure $P$ and metric $g$, determined by (15) and (10), respectively. Then we verify directly that $g$ is a Killing metric.

Therefore, the following theorem is valid.

**Theorem 3.1.** Let $(G, P, g)$ be a 4-dimensional Riemannian almost product manifold, where $G$ is the connected Lie group with an associated Lie algebra, determined by a global basis $\{X_i\}$ of left invariant vector fields, and $P$ and $g$ are the almost product structure and the Riemannian metric, determined by (15) and (10), respectively. Then $(G, P, g)$ is a $W_3$-manifold with a Killing metric $g$ iff $G$ belongs to the 4-parametric family of Lie groups, determined by (22).

From this point on, until the end of this section we shall consider the Riemannian almost product manifold $(G, P, g)$ determined by the conditions of Theorem 3.1.

Using (18), (22), (15) and (16), we obtain the following nonzero components of the tensor $F$:

\[
F_{211} = -F_{233} = 2F_{134} = 2F_{323} = -2F_{112} = -2F_{314} = \lambda_1, \\
F_{144} = -F_{122} = 2F_{212} = 2F_{423} = -2F_{234} = -2F_{414} = \lambda_2, \\
F_{322} = -F_{344} = 2F_{214} = 2F_{434} = -2F_{223} = -2F_{412} = \lambda_3, \\
F_{433} = -F_{411} = 2F_{141} = 2F_{321} = -2F_{132} = -2F_{334} = \lambda_4. 
\]

The other nonzero components of $F$ are obtained from the property $F_{ijk} = F_{ikj}$.

Let $F$ be the Nijenhuis tensor on $(G, P, g)$, i.e.

\[N_{ij} = [X_i, X_j] + P[PX_i, X_j] + P[X_i, PX_j] - [PX_i, PX_j].\]

According to (15) and (22), for the square norm $\|N\|^2 = N_{ik}N_{js}g^{ij}g^{ks}$ of $N$ we get

\[\|N\|^2 = 32\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2\right). \quad (24)\]

For the square norm of $\nabla P$, using (10), (10) and (17), we obtain

\[\|\nabla P\|^2 = 4\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2\right). \quad (25)\]

From (14), having in mind (10) and (22), we receive the following nonzero
components of the curvature tensor $R$:

$$\begin{align*}
R_{1221} &= \frac{1}{3} (\lambda_1^2 + \lambda_2^2) , \\
R_{1441} &= \frac{1}{3} (\lambda_2^2 + \lambda_3^2) , \\
R_{2442} &= \frac{1}{3} (\lambda_2^2 + \lambda_3^2) , \\
R_{1341} &= R_{2342} = \frac{1}{3} \lambda_1 \lambda_2 , \\
R_{1231} &= R_{4234} = \frac{1}{3} \lambda_2 \lambda_4 , \\
R_{1241} &= R_{3243} = -\frac{1}{3} \lambda_1 \lambda_4 , \\
R_{1342} &= R_{2132} = R_{4134} = -\frac{1}{3} \lambda_2 \lambda_3 .
\end{align*}$$

(26)

The other nonzero components of $R$ are obtained from the properties $R_{ijks} = R_{kisj}$ and $R_{ijks} = -R_{jiks} = -R_{ijsk}$.

From (1), having in mind (10), we receive the components $\rho_{ij} = \rho(X_i, X_j)$ of the Ricci tensor $\rho$. The nonzero components of $\rho$ are:

$$\begin{align*}
\rho_{11} &= \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) , \\
\rho_{22} &= \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) , \\
\rho_{33} &= \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) , \\
\rho_{12} &= \frac{1}{3} \lambda_1 \lambda_4 , \\
\rho_{13} &= -\frac{1}{3} \lambda_2 \lambda_3 , \\
\rho_{14} &= \frac{1}{3} \lambda_1 \lambda_3 , \\
\rho_{23} &= \frac{1}{3} \lambda_2 \lambda_4 , \\
\rho_{24} &= -\frac{1}{3} \lambda_1 \lambda_4 , \\
\rho_{34} &= \frac{1}{3} \lambda_1 \lambda_2 .
\end{align*}$$

(27)

The other nonzero components of $\rho$ are obtained from the property $\rho_{ij} = \rho_{ji}$.

For the scalar curvature $\tau$, using (6), we obtain

$$\tau = \frac{3}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) .$$

(28)

From (7), having in mind (10), (26), (27) and (28), we get for the Weyl tensor $W = 0$. Then $(G, P, g)$ is a conformally flat manifold.

For the sectional curvatures $k_{ij} = k(\alpha_{ij})$ of basic 2-planes $\alpha_{ij} = (X_i, X_j)$, according to (8), (26) and (10), we have:

$$\begin{align*}
k_{12} &= \frac{1}{3} (\lambda_1^2 + \lambda_2^2) , \\
k_{13} &= \frac{1}{3} (\lambda_1^2 + \lambda_4^2) , \\
k_{14} &= \frac{1}{3} (\lambda_2^2 + \lambda_4^2) , \\
k_{23} &= \frac{1}{3} (\lambda_1^2 + \lambda_3^2) , \\
k_{24} &= \frac{1}{3} (\lambda_2^2 + \lambda_3^2) , \\
k_{34} &= \frac{1}{3} (\lambda_3^2 + \lambda_4^2) .
\end{align*}$$

(29)

The obtained geometric characteristics of the considered manifold are generalized in the following

**Theorem 3.2.** Let $(G, P, g)$ be the 4-dimensional Riemannian almost product manifold where $G$ is the Lie group determined by (22), and the structure $P$ and the metric $g$ are determined by (13) and (10), respectively. Then
(i) \((G,P,g)\) is a locally symmetric \(W_3\)-manifold with Killing metric \(g\) and zero Weyl tensor;

(ii) The nonzero components of the basic tensor \(F\), the curvature tensor \(R\) and the Ricci tensor \(\rho\) are \((23)\), \((26)\) and \((27)\), respectively;

(iii) The square norms of the Nijenhuis tensor \(N\) and \(\nabla P\) are \((24)\) and \((25)\), respectively;

(iv) The scalar curvature \(\tau\) and the sectional curvatures \(k_{ij}\) of the basic 2-planes are \((28)\) and \((29)\), respectively.

Let us remark that the 2-planes \(\alpha_{13}\) and \(\alpha_{24}\) are \(P\)-invariant 2-planes, i.e. \(P\alpha_{13} = \alpha_{13}, P\alpha_{24} = \alpha_{24}\). The 2-planes \(\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}\) are totally real 2-planes, i.e. \(\alpha_{12} \perp P\alpha_{12}, \alpha_{14} \perp P\alpha_{14}, \alpha_{23} \perp P\alpha_{23}, \alpha_{34} \perp P\alpha_{34}\). Then the equalities \((29)\) imply the following

**Theorem 3.3.** Let \((G,P,g)\) be the 4-dimensional Riemannian almost product manifold where \(G\) is the Lie group determined by \((22)\), and the structure \(P\) and the metric \(g\) are determined by \((15)\) and \((10)\), respectively. Then

(i) \((G,P,g)\) is of constant \(P\)-invariant sectional curvatures iff

\[
\lambda_1^2 + \lambda_2^2 = \lambda_3^2 + \lambda_4^2;
\]

(ii) \((G,P,g)\) is of constant totally real sectional curvatures iff

\[
\lambda_1^2 = \lambda_2^2, \quad \lambda_3^2 = \lambda_4^2.
\]

4 A Lie group as a pseudo-Riemannian almost product manifold with Killing metric and non-integrable structure

In this section we consider a pseudo-Riemannian manifold \((G,P,g)\) with a metric \(g\) determined by \((11)\) and a structure \(P\) defined as follows

\[
P X_1 = X_1, \quad P X_2 = X_1, \quad P X_3 = -X_3, \quad P X_4 = -X_4.
\]

Obviously, \(P^2 = \text{id}\). Moreover, \((11)\) and \((30)\) imply

\[
g(PX_i, PX_j) = g(X_i, X_j).
\]
Therefore, \((G, P, g)\) is a pseudo-Riemannian almost product manifold. For the manifold \((G, P, g)\) we propose that \(g\) be a Killing metric. Then \((G, P, g)\) is locally symmetric and the equalities (13), (14), (17) and (18) are valid. From (11), (30) and (31) we obtain (19), i.e. \((G, P, g)\) is a \(W_3\)-manifold.

Now, the equalities (20) and (21) are also satisfied. According to (12), from (21) we obtain

\[
\begin{align*}
[X_1, X_2] &= \lambda_2 X_3 - \lambda_1 X_4, \\
[X_1, X_3] &= \lambda_2 X_2 + \lambda_4 X_4, \\
[X_1, X_4] &= -\lambda_1 X_2 - \lambda_4 X_3, \\
[X_2, X_3] &= -\lambda_2 X_1 - \lambda_3 X_4, \\
[X_2, X_4] &= \lambda_1 X_1 + \lambda_3 X_3, \\
[X_3, X_4] &= -\lambda_4 X_1 + \lambda_3 X_2,
\end{align*}
\]

(32)

where \(\lambda_1 = C_{24}^1, \lambda_2 = C_{12}^3, \lambda_3 = C_{24}^3, \lambda_4 = C_{13}^4\). We verify immediately that the Jacobi identity is satisfied in this case.

Let the conditions (32) be satisfied for a pseudo-Riemannian almost product manifold \((G, P, g)\) with structure \(P\) and metric \(g\) determined by (30) and (11), respectively. Then we verify directly that \(g\) is a Killing metric. Therefore, the following theorem is valid.

**Theorem 4.1.** Let \((G, P, g)\) be a 4-dimensional pseudo-Riemannian almost product manifold, where \(G\) is the connected Lie group with an associated Lie algebra, determined by a global basis \(\{X_i\}\) of left invariant vector fields, and \(P\) and \(g\) are the almost product structure and the pseudo-Riemannian metric, determined by (30) and (11), respectively. Then \((G, P, g)\) is a \(W_3\)-manifold with a Killing metric \(g\) iff \(G\) belongs to the 4-parametric family of Lie groups, determined by (32).

From this point on, until the end of this section we shall consider the pseudo-Riemannian almost product manifold \((G, P, g)\) determined by the conditions of Theorem 4.1.

In an analogous way of the previous section, we get some geometric characteristics of \((G, P, g)\).

We obtain the following nonzero components of the tensor \(F\):

\[
\begin{align*}
F_{124} &= -F_{214} = \lambda_1, \\
F_{213} &= -F_{123} = \lambda_2, \\
F_{423} &= -F_{324} = \lambda_3, \\
F_{314} &= -F_{413} = \lambda_4.
\end{align*}
\]

(33)

The other nonzero components of \(F\) are obtained from the properties \(F_{ijk} = F_{ikj}\).

The square norms of the Nijenhuis tensor \(N\) and \(\nabla P\) are respectively:

\[
\|N\|^2 = 24 \left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right),
\]

(34)
\[ ||\nabla P||^2 = -4 \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right). \] (35)

The nonzero components of the curvature tensor \( R \) and the Ricci tensor \( \rho \) are respectively:

\[
\begin{align*}
R_{1221} &= -\frac{1}{4} \left( \lambda_1^2 + \lambda_2^2 \right), & R_{1331} &= \frac{1}{4} \left( \lambda_2^2 - \lambda_1^2 \right), \\
R_{1441} &= -\frac{1}{4} \left( \lambda_1^2 - \lambda_2^2 \right), & R_{2332} &= \frac{1}{4} \left( \lambda_3^2 - \lambda_4^2 \right), \\
R_{2442} &= \frac{1}{4} \left( \lambda_2^2 - \lambda_3^2 \right), & R_{3443} &= \frac{1}{4} \left( \lambda_4^2 - \lambda_3^2 \right), \\
R_{1341} &= R_{2342} = -\frac{1}{4} \lambda_1 \lambda_2, & R_{2132} &= -R_{4134} = \frac{1}{4} \lambda_1 \lambda_3, \\
R_{1231} &= -R_{4234} = \frac{1}{4} \lambda_1 \lambda_4, & R_{2142} &= -R_{3143} = \frac{1}{4} \lambda_2 \lambda_3, \\
R_{1241} &= -R_{3243} = \frac{1}{4} \lambda_2 \lambda_4, & R_{3123} &= R_{4124} = \frac{1}{4} \lambda_3 \lambda_4;
\end{align*}
\] (36)

\[
\begin{align*}
\rho_{11} &= -\frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \right), & \rho_{22} &= -\frac{1}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_4^2 \right), \\
\rho_{33} &= \frac{1}{2} \left( \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right), & \rho_{44} &= \frac{1}{2} \left( \lambda_2^2 + \lambda_3^2 - \lambda_4^2 \right), \\
\rho_{12} &= -\frac{1}{2} \lambda_3 \lambda_4, & \rho_{13} &= \frac{1}{2} \lambda_1 \lambda_3, & \rho_{14} &= \frac{1}{2} \lambda_1 \lambda_4, \\
\rho_{23} &= \frac{1}{2} \lambda_1 \lambda_4, & \rho_{24} &= \frac{1}{2} \lambda_2 \lambda_4, & \rho_{34} &= -\frac{1}{2} \lambda_1 \lambda_2.
\end{align*}
\] (37)

The other nonzero components of \( R \) and \( \rho \) are obtained from the properties \( R_{ijk}s = R_{kisj}, R_{ijks} = -R_{jiks} \) and \( \rho_{ij} = \rho_{ji} \).

The scalar curvature is

\[ \tau = -\frac{3}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right). \] (38)

We get for the Weyl tensor that \( W = 0 \). Then \( (G, P, g) \) is a conformally flat manifold.

The sectional curvatures \( k_{ij} = k(\alpha_{ij}) \) of basic 2-planes \( \alpha_{ij} = (X_i, X_j) \) are:

\[
\begin{align*}
k(\alpha_{13}) &= -\frac{1}{4} \left( \lambda_2^2 - \lambda_3^2 \right), & k(\alpha_{24}) &= -\frac{1}{4} \left( \lambda_2^2 - \lambda_4^2 \right), \\
k(\alpha_{12}) &= -\frac{1}{4} \left( \lambda_1^2 + \lambda_2^2 \right), & k(\alpha_{14}) &= -\frac{1}{4} \left( \lambda_1^2 - \lambda_4^2 \right), \\
k(\alpha_{23}) &= -\frac{1}{4} \left( \lambda_2^2 - \lambda_3^2 \right), & k(\alpha_{34}) &= \frac{1}{4} \left( \lambda_3^2 + \lambda_4^2 \right).
\end{align*}
\] (39)

Since \( \alpha_{ij} = P \alpha_{ij} \) then all basic 2-planes are \( P \)-invariant. It is used to check that now \( (G, P, g) \) does not accept constant \( P \)-invariant sectional curvatures.

The obtained geometric characteristics of the considered manifold are generalized in the following

**Theorem 4.2.** Let \( (G, P, g) \) be the 4-dimensional pseudo-Riemannian almost product manifold where \( G \) is the Lie group determined by (32), and the structure \( P \) and the metric \( g \) are determined by (30) and (11), respectively. Then
(i) \((G, P, g)\) is a locally symmetric conformally flat \(W_3\)-manifold with Killing metric \(g\);

(ii) The nonzero components of the basic tensor \(F\), the curvature tensor \(R\) and the Ricci tensor \(\rho\) are \((33), (36)\) and \((37)\), respectively;

(iii) The square norms of the Nijenhuis tensor \(N\) and \(\nabla P\) are \((34)\) and \((35)\), respectively;

(iv) The scalar curvature \(\tau\) and the sectional curvatures \(k_{ij}\) of the basic 2-planes are \((38)\) and \((39)\), respectively.

The last theorem implies immediately the following

**Corollary 4.3.** Let \((G, P, g)\) be the 4-dimensional pseudo-Riemannian almost product manifold where \(G\) is the Lie group determined by \((32)\), and the structure \(P\) and the metric \(g\) are determined by \((30)\) and \((11)\), respectively. Then the following propositions are equivalent:

(i) \((G, P, g)\) is an isotropic \(P\)-manifold;

(ii) \((G, P, g)\) is a scalar flat manifold;

(iii) The Nijenhuis tensor is isotopic;

(iv) The condition \(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0\) is valid.

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