ON RESTRICTION PROPERTIES OF EQUIVARIANT K-THEORY RINGS.

ABDELOUAHAB AROUCHE

Abstract. An important ingredient in the completion theorem of equivariant K-theory given by S. Jackowski is that the representation ring $R(Γ)$ of a compact Lie group satisfies two restriction properties called ($N$) and ($RF$).

We give in this note sufficient conditions on a (compact) $Γ$-space $Z$ such that these properties hold with $K^∗_Γ(Z)$ instead of $R(Γ)$. As an example, we consider the space $Z(Γ; G)$ of the so called “elementary cocycles with coefficients in $G$” invented by H. Ibisch in his construction of a universal $(Γ; G)$-bundle.

1. Introduction

In what follows, let $Γ$ be a compact Lie group and $Z$ a compact $Γ$-space. Consider, for every subgroup $Λ ≤ Γ$, the ring $M_Λ = K^*_Λ(Z)$. We have a restriction morphism $res : M_Γ → M_Λ$ whose kernel is denoted by $I^Γ_Λ$ or $I_Λ$ for short. If $F$ is a family of (closed) subgroups of $Γ$ (stable under subconjugation), then the $F$-adic topology on $M_Γ$ is defined by the set of ideals

$$I(F) = \{I_{Λ_1}, I_{Λ_2}, ..., I_{Λ_n} : Λ_i ∈ F, i = 1, ..., n\}.$$ 

If $Λ ≤ Γ$, then $F ∩ Λ = \{Ω ∈ F : Ω ≤ Λ\}$ is a family of subgroups of $Λ$.

Let $X$ be a compact $Γ$-space with a $Γ$-map $σ : X → Z$. Define the two pro-rings

$$K^*_Γ [F](X) = \{K^*_Γ (X × K) : K compact ⊆ EF\},$$

and

$$(K^*_Γ / I(F))(X) = \{K^*_Γ (X) / I.K^*_Γ (X) : I ∈ I(F)\}.$$

Here $EF$ denotes the classifying space of T. tom Dieck [7]. The completion theorem asserts that the pro-homomorphism

$$p_X(F) : K^*_Γ [F](X) → (K^*_Γ / I(F))(X)$$

induced by the projection $X × EF → X$, is actually an isomorphism, provided $M_Λ$ satisfies the conditions $(N)$ and $(RF)$, and $K_Λ(X)$ is finitely generated over $M_Λ$ (via $σ$), for all $Λ ≤ Γ$.

In order to prove his completion theorem [4], S. Jackowski required the following two conditions, and showed they are satisfied when $Z = *$ is a point :

$(N) : M_Γ$ is noetherian and $M_Λ$ is a finitely generated $M_Γ$-module, $∀ Λ ≤ Γ$.

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For every family $\mathcal{F}$ of subgroups of $\Gamma$, and every subgroup $\Lambda \leq \Gamma$, the $\mathcal{F}$-adic topology on $M_{\Lambda}$ defined by the restriction $\text{res} : M_{\Gamma} \to M_{\Lambda}$ coincides with the $(\mathcal{F} \cap \Lambda)$-adic topology.

2. Some nilpotent elements in $K^+_1(Z)$.

Let $z \in Z$, and $\Gamma_z$ be its isotropy subgroup. Then an element $\zeta \in M_{\Gamma}$ belongs to $\ker \left\{ K^+_1(Z) \to \prod_{y \in Z} R(\Gamma_y) \right\}$ if and only if its restriction to the orbit $\Gamma.z$ is zero. It is well known that the elements of $\ker \left\{ K^+_1(Z) \to \prod_{y \in Z} R(\Gamma_y) \right\}$ are nilpotent [6](5.1).

We have the following:

Lemma 1. Let $z \in Z$ and $\zeta \in \ker \left\{ K^+_1(Z) \to R(\Gamma_z) \right\}$. Then there exists a relatively compact open $\Gamma$-neighbourhood $V(\zeta)$ of $z$ such that the restriction of $\zeta$ to $V(\zeta)$ is nilpotent.

Proof. If $\zeta = E - T \in \ker \left\{ K^+_1(Z) \to R(\Gamma_z) \right\}$, then $E_z - T = 0 \in R(\Gamma_z)$, where $T$ is seen as a $\Gamma_z$-module. Therefore, there exists a $\Gamma_z$-module $R$ such that $E_z \oplus R \cong T \oplus R$. Consider the isomorphism between $\Gamma_z$-vector bundles:

$$E \oplus R \big|_{\{z\}} \cong T \oplus R \big|_{\{z\}}.$$  

According to [6](1.2), there is a $\Gamma_z$-neighbourhood $U$ of $z$ and a $\Gamma_z$-isomorphism

$$E \oplus R \big|_{U} \cong T \oplus R \big|_{U}.$$  

Let $S$ be a slice at $z$ ([2] II.4.1). Then $\Gamma.(S \cap U)$ is an open $\Gamma$-neighbourhood of $z$ such that $\zeta \in \ker \left\{ K^+_1(Z) \to R(\Gamma_y) \right\}$, $\forall y \in \Gamma.(S \cap U)$. Now let $V(\zeta)$ be any open $\Gamma$-neighbourhood of $z$ satisfying $z \in V(\zeta) \subseteq \overline{V(\zeta)} \subseteq \Gamma.(S \cap U)$.

□

Proposition 1. Assume $K^+_1(Z)$ is noetherian. Then there exists a finite subset $Y \subseteq Z$ such that the ideal $\ker \left\{ K^+_1(Z) \to \prod_{y \in Y} R(\Gamma_y) \right\}$ consists of nilpotent elements.

Proof. For each $z \in Z$, let $\ker \left\{ K^+_1(Z) \to R(\Gamma_z) \right\}$ be generated by $\zeta_1, ..., \zeta_{n_z}$, and for $j = 1, ..., n_z$, let $V(\zeta_j)$ be as in lemma (2.1). Put

$$V_z = \bigcap_{j=1, ..., n_z} V(\zeta_j).$$

Since $Z$ is compact, it can be covered by a finite number of such open subsets:

$$Z = \bigcup_{i=1, ..., n} V_{z_i}.$$  

It is easy to show that the ideal $\ker \left\{ K^+_1(Z) \to \prod_{i=1, ..., n} R(\Gamma_{z_i}) \right\}$ consists of nilpotent elements. □
3. Supports of primes

In [4], G. Segal defined the support \( \text{supp} \) of a prime \( y \in \text{Spec} R(\Gamma) \) to be a minimal subgroup from which \( y \) comes. We have shown in [1] that such a notion does exist for primes in \( M_\Gamma \). Indeed, if \( p : Z \to * \) is the projection of a \( \Gamma \)-space \( Z \) on a fixed point and \( p^* : R(\Gamma) \to M_\Gamma \) is the induced homomorphism, we have the following:

**Proposition 2.** Let \( \Lambda \leq \Gamma \) be a (closed) subgroup. Assume \( M_\Gamma \) is noetherian and let \( x \in \text{Spec} M_\Gamma \). Then the following properties are equivalent:

1. \( x \) come from \( M_\Lambda \), i.e. \( \exists y \in \text{Spec} M_\Lambda : x = \text{res}^{-1}(y) \).
2. \( x \) contains \( I_\Lambda \).
3. \( \text{supp}(p^* - 1(x)) \) is subconjugate to \( \Lambda \).

In order to prove proposition (3.1), we need the following lemma:

**Lemma 2.** Let \( x \in \text{Spec} M_\Gamma \) and \( z \in Z \) be such that there exists a prime \( t \in \text{Spec} R(\Gamma \cdot z) \), with support \( \Sigma \), satisfying \( x = \varphi^{-1}(t) \), \( \varphi_z : M_\Gamma \to R(\Gamma \cdot z) \).

Let \( \gamma \in \Gamma \) and \( z' = \gamma \cdot z \). Then there exists a prime \( t' \in \text{Spec} R(\Gamma \cdot z') \), with support \( \Sigma' = \gamma \Sigma \gamma^{-1} \), satisfying \( x = \varphi_z^{-1}(t') \), \( \varphi_{z'} : M_\Gamma \to R(\Gamma \cdot z') \).

**Proof.** Let \( \Gamma \cdot z \hookrightarrow Z \) be the inclusion of the orbit of \( z \). We have the following commutative diagram:

\[
\begin{array}{ccc}
K_\Gamma^*(Z) & \xrightarrow{i^*} & K_\Gamma^*(\Gamma \cdot z) \\
\varphi_z & & \varphi_{\gamma \cdot z} \\
R(\Gamma \cdot z) & \xleftarrow{\chi_{\gamma \cdot z}} & R(\Gamma \cdot z)
\end{array}
\]

It is clear that both \( \chi_z \) and \( \chi_{\gamma \cdot z} \) are isomorphisms. Put \( h = \chi_z^{-1} \circ \chi_{\gamma \cdot z} \). Then \( h \) admits a restriction \( \tilde{h} : R(\Sigma) \to R(\gamma \Sigma \gamma^{-1}) \), which makes the following diagram commute and ends the proof:

\[
\begin{array}{ccc}
R(\Gamma \cdot z) & \xrightarrow{\text{res}} & R(\Sigma) \\
\chi_z & & \chi_{\gamma \cdot z} \\
K_\Gamma^*(Z) & \xrightarrow{i^*} & K_\Gamma^*(\Gamma \cdot z) \\
\xleftarrow{\gamma \cdot z} & & \xleftarrow{\gamma \cdot z} \\
R(\Gamma \cdot z) & \xrightarrow{\text{res}} & R(\gamma \Sigma \gamma^{-1})
\end{array}
\]

**Proof.** Now we proceed to prove proposition (3.1) (see [1]). 1) \( \implies \) 2) and 2) \( \implies \) 3) are obvious. Let us prove 3) \( \implies \) 1). To this end, let \( z_1, \ldots, z_n \) be elements of \( Z \) such that \( \ker \left\{ \varphi : K_\Gamma^*(Z) \to \prod_{i=1}^n R(\Gamma \cdot z_i) \right\} \) consists of nilpotent elements. Since every \( R(\Gamma \cdot z_i) \) is finitely generated over \( M_\Gamma \) (it is even finitely generated over \( R(\Gamma) \) [5]), the induced map on spectra \( \varphi_* : \prod_{i=1}^n \text{Spec} R(\Gamma \cdot z_i) \to \text{Spec} M_\Gamma \) is onto. So there
is an element $z \in Z$ and a prime $t \in \text{Spec}R(\Gamma_z)$ such that $x = \varphi^{-1}_z(t)$. Following [5] (3.7), we have $\text{supp}(p^{*^{-1}}(x)) = \text{supp}(t)$. Let $\Sigma = \text{supp}(p^{*^{-1}}(x))$. By assumption, the exists $\gamma \in \Gamma$ such that $\gamma \Sigma \gamma^{-1} \leq \Lambda$. Put $z' = \gamma \cdot z$. According to lemma (3.2), there exists a prime $t' \in \text{Spec}R(\Gamma_{z'})$, with support $\Sigma' = \gamma \Sigma \gamma^{-1}$, and satisfying $x = \varphi^{-1}_{z'}(t')$, $\varphi_{z'} : M_\Gamma \to R(\Gamma_{\gamma z})$.

Therefore, $x \in \text{Spec}M_\Gamma$ comes from $\Lambda$, as shown by the commutative diagram:

\[
\begin{array}{ccc}
M_\Gamma & \to & M_\Lambda \\
\downarrow & & \downarrow \\
R(\Gamma_{\gamma z}) & \to & R(\Lambda_{\gamma z}) \\
\gamma & & \gamma \\
& \text{supp}(\Sigma') & \\
\end{array}
\]

□

Hence, $\text{supp}(p^{*^{-1}}(x))$ is, up to conjugation, the minimal subgroup of $\Gamma$ from which $x$ comes. We denote it by $\text{supp}(x)$. Consequently, if $(x, y) \in \text{Spec}M_\Gamma \times \text{Spec}M_\Lambda$ is such that $x = \text{res}^{-1}(y)$, then $\text{supp}(x) = \text{supp}(y)$.

4. The restriction properties

Let us first consider the condition $(R_\mathcal{F})$. We have the following:

**Theorem 1.** Assume $M_\Lambda$ is noetherian $\forall \Lambda \leq \Gamma$. Then, for every family $\mathcal{F}$ of subgroups of $\Gamma$, the condition $(R_\mathcal{F})$ is satisfied.

**Proof.** For each $\Omega \in \mathcal{F}$ we have $I^\mathcal{F}_\Omega \supseteq I^\Gamma_{\Omega \Lambda}$. So it remains only to show that for any ideal $I.M_\Lambda$, where $I \in I(\mathcal{F} \cap \Lambda)$ such that $J \subseteq r(I.M_\Lambda)$, since $M_\Lambda$ is noetherian. It is enough to do that for $I = I^\mathcal{F}_\Omega$. So let $p_1, ..., p_n$ be minimal prime ideals in $M_\Lambda$ containing $I.M_\Lambda$, and let $\Sigma_1, ..., \Sigma_n$ be their supports. Put $J = I_{\Sigma_1} \cap ... \cap I_{\Sigma_n}$. Since $p_i$ comes from $\Sigma_i$, we have $J \subseteq I_{\Sigma_1} \cap ... \cap I_{\Sigma_n} \subseteq p_1 \cap ... \cap p_n = r(I)$. Now, if we put $q_i = \text{res}^{-1}(p_i)$, then $\Sigma_i = \text{supp}(q_i)$. But $q_i$ comes from $\Omega$ because it contains $I^\mathcal{F}_{\Omega i}$. Hence, $\Sigma_i$ is subconjugate to $\Omega$. Since $\Omega \in \mathcal{F}$, so doe $\Sigma_i$, for $i = 1, ..., n$, that is, $J \in I(\mathcal{F} \cap \Lambda)$.

**Corollary 1.** If the condition $(N)$ is satisfied, then so is the condition $(R_\mathcal{F})$.

□

It is easy to see that the condition $(N)$ is fulfilled if $Z$ is a (compact) differentiable $\Gamma$-manifold, and more generally, for any finite $\Gamma$-CW-complex, for $M_\Lambda$ is a finitely generated module over $R(\Lambda)$, $\forall \Lambda \leq \Gamma$. This can be shown by induction and use of Mayer-Vietoris sequence.

**Corollary 2.** Assume $M_\Lambda$ is finitely generated over $R(\Gamma)$, $\forall \Lambda \leq \Gamma$. Then the conditions $(N)$ and $(R_\mathcal{F})$ are satisfied for every family $\mathcal{F}$ of subgroups of $\Gamma$. 
5. AN EXAMPLE

For a topological group \( G \) and a family \( \mathcal{F} \) of subgroups of a compact Lie group \( \Gamma \), H.Ibisch has constructed in [2] a universal \((\Gamma; G)\)-bundle \( E(\mathcal{F}; G) \) whose “components” are the spaces

\[
Z(\Lambda; G) = \{ f : \Lambda \times \Lambda \to G : f(\lambda_1, \lambda_2, \lambda_3) = f(\lambda_1, \lambda_2, \lambda_3), \forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda \}, \Lambda \leq \Gamma.
\]

The crucial fact is that for a \( \Gamma \)-space \( X \), trivial \((\Gamma; G)\)-bundles over \( X \) correspond to \( \Gamma \)-maps \( \sigma : X \to Z(\Gamma; G) \). We then take \( Z = Z(\Gamma; G) \). Actually, the \( \Gamma \)-space \( Z(\Gamma; G) \) turns to be \( \Gamma \)-homeomorphic to the space of maps \( \Gamma \to G \) sending \( 1_\Gamma \) to \( 1_G \). Moreover, \( Z(\Gamma; G)^\Gamma \) is the space of (continuous) homomorphisms \( \Gamma \to G \). When \( \Gamma \) is finite, the space \( Z(\Gamma; G) \) has a further description. For instance, if \( \Gamma = \mathbb{Z}_n \) is the cyclic group of order \( n \), then \( Z(\Gamma; G) \) is the product \( G^{n-1} \times \{ 1_G \} \) with the action \( \gamma^i(g_1, \ldots, g_n = 1_G) = (g_1+ig_1^{-1}, \ldots, g_n+ig_n^{-1}) \), where \( \gamma \) is a generator of \( \mathbb{Z}_n \) and the indexation is mod \([n]\), hence \( Z(\mathbb{Z}_2; G) \) is no but \( G \) with involution \( g \mapsto g^{-1} \). If \( G \) is moreover a compact Lie group, then \( Z(\Gamma; G) \) is a compact \( \Gamma \)-manifold. Accordingly, the conditions \((N)\) and \( R(\mathcal{F}) \) are satisfied.

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