COUNTING GENERALIZED JENKINS–STREBEL DIFFERENTIALS

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Abstract. We study the combinatorial geometry of “lattice” Jenkins–Strebel differentials with simple zeroes and simple poles on CP^1 and of the corresponding counting functions. Developing the results of M. Kontsevich [K92] we evaluate the leading term of the symmetric polynomial counting the number of such “lattice” Jenkins–Strebel differentials having all zeroes on a single singular layer. This allows us to express the number of general “lattice” Jenkins–Strebel differentials as an appropriate weighted sum over decorated trees.

The problem of counting Jenkins–Strebel differentials is equivalent to the problem of counting pillowcase covers, which serve as integer points in appropriate local coordinates on strata of moduli spaces of meromorphic quadratic differentials. This allows us to relate our counting problem to calculations of volumes of these strata. A very explicit expression for the volume of any stratum of meromorphic quadratic differentials recently obtained by the authors [AEZ] leads to an interesting combinatorial identity for our sums over trees.

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1. Introduction

1.1. Counting pillowcase covers. A geometric approach to volume computation for the strata in the moduli spaces of Abelian or quadratic differentials consists in counting square-tiled surfaces or pillowcase covers, see [EO01], [EO03], [EOP], [Z00]. A pillowcase cover $\hat{\mathcal{P}} \to \mathbb{C}P^1$ is a ramified cover over $\mathbb{C}P^1$ branched over four points. Define a flat metric on $\mathbb{C}P^1$ such that the resulting pillowcase orbifold as in Figure 1 is glued from two squares of size $1/2 \times 1/2$. Choosing the four corners of the pillowcase $\mathcal{P}$ as the four ramification points, we get an induced square tiling of the pillowcase cover, see Figure 2. The flat structure on the pillowcase $\hat{\mathcal{P}}$, corresponds to the meromorphic quadratic differential $\psi_0 = (dz)^2$ on $\mathcal{P} = T/\pm$, where $T = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. The quadratic differential $\psi_0$ has four simple poles at the corners of the pillow and no other singularities. We shall see that the induced quadratic differential $\psi = \pi^* \psi_0$ on $\hat{\mathcal{P}}$ defines an integer point (in appropriate local coordinates) in the ambient stratum of meromorphic quadratic differentials.

In this paper we want to count the number of nonisomorphic connected pillowcase covers $\hat{\mathcal{P}}$ of degree at most $N$ having the following ramification pattern. All ramification points are located over the corners of the pillowcase. All preimages of the corners are ramification points of degree two with exception for $K$ ramification points of degree three and for $K+4$ unramified points. For example, the pillowcase cover in Figure 2 has $K = 3$ ramification points of degree three and $K + 4 = 7$ unramified points. We do not specify how the projections of $K + (K+4)$ distinguished points are distributed between the four corners of the pillowcase $\mathcal{P}$.

Our restriction on the ramification data implies that the quadratic differential $\psi$ has exactly $K$ simple zeroes (located at the ramification points of degree three); it has exactly $K + 4$ simple poles (located at $K + 4$ nonramified preimages of the corners), and it has no other zeroes or poles. In particular, the pillowcase cover $\hat{\mathcal{P}}$ has genus zero.

In order to count pillowcase covers we note that if $\hat{\mathcal{P}}$ is a pillowcase cover, it can be decomposed into horizontal cylinders with integer widths, with zeros and poles lying on the boundaries of these cylinders, see Figure 2. We call these boundaries singular layers. Each singular layer defines a connected graph with a certain number
of trivalent vertices, a certain number \( n \) of univalent vertices, and with no vertices of any other valence. Actually, it is more convenient to consider the singular layer as a ribbon graph by taking a small tubular neighborhood of the singular layer inside the surface. The graph is metric: all edges have certain lengths measured by means of the flat structure. Since the length of the sides of each square of the tiling of the pillowcase cover is \( 1/2 \), and the vertices of each singular layer are located at the vertices of the squares, the lengths of all edges of our graph are half-integer.

Note that the number \( l \) of cylinders adjacent to a layer is expressed in terms of the number \( m \) of zeroes and the number \( n \) of poles on the corresponding layer as

\[
   l = (m - n) + 2.
\]

Thus, by topological reasons \( m - n + 2 \) is necessarily a nonnegative even number.

\[
   \Gamma_{1,1} \quad \Gamma_{1,2} \quad \Gamma_{2,1} \quad \Gamma_{2,2} \quad \Gamma_{3}
\]

Figure 3. The list of all connected ribbon graphs with labelled boundary components having \( m \) zeros (i.e. \( m = 2 \) trivalent vertices) and \( n \) poles (i.e. \( n = 2 \) univalent vertices).

Developing the techniques of M. Kontsevich from [K92], we find a formula for the following counting function. Given \( l \) positive integer numbers \( w_1, \ldots, w_l \) we count the number of ways to join \( l \) cylinders of widths \( w_1, \ldots, w_l \) together by means of a connected half-integer ribbon graph having \( m \) trivalent and \( n \) univalent vertices; see Figure 3.

**Theorem 1.1.** Let \( m \) and \( n \) be nonnegative integer numbers not equal simultaneously to zero such that \( m - n + 2 \) is a nonnegative even number. Let \( F_{m,n}(w_1, \ldots, w_l) \), where \( l = \frac{(m-n)}{2} + 2 \), be the number of ways to attach \( l \) cylinders of integer widths \( w_1, \ldots, w_l \) to all possible layers containing \( m \) zeroes and \( n \) poles, in such way that all edges of the resulting graph are half-integer. Up to the lower order terms one has

\[
   (1.1) \quad F_{m,n}(w_1, \ldots, w_l) = \frac{m!}{(\frac{m+n}{2} - 1)!} \sum_{b_1, \ldots, b_l} \left( \frac{\binom{m+n}{2} - 1}{b_1, \ldots, b_l} \right)^2 \prod_{i=1}^{l} w_i^{2b_i}.
\]

Theorem 1.1 is of independent interest; it is is proved in §3.5. To elaborate certain geometric intuition helpful in manipulating geometric counting functions we compute in §3.2 by hands the function \( F_{2,2}(w_1, w_2) \) corresponding to ribbon graphs from Figure 3.

Having studied the enumerative geometry of singular layers let us return to global pillowcase covers. Suppose there are \( k \) cylinders of width \( w_i \) and height \( h_i \) respectively. Since the flat surface is a topological sphere, there are \( k + 1 \) singular
layers in the decomposition of $\hat{\mathcal{P}}$. The total number of pillowcase covers of degree $N$ with this type of decomposition can be written as

\begin{equation}
2^k \sum_{w \cdot h \leq N} w_1 \cdot w_2 \cdots w_k \cdot \prod_{i=1}^{k+1} F_{m_i, n_i}(w_1, \ldots, w_k),
\end{equation}

where $F_{m_i, n_i}$ is a function counting the number of ways the cylinders of width $w_i$ can be glued at the layer $i$, and $w \cdot h := \sum_{i=1}^{k} w_i h_i$. The factor $(2w_1)(2w_2)\cdots(2w_k)$ arises from the possibility of twisting each cylinder around the waist curve; see §3.2 for more details.

Representing each singular layer by a vertex of an associated graph $T$ as in Figure 2, and every cylinder by an edge of such graph, we encode the decomposition of $\hat{\mathcal{P}}$ into cylinders by a global graph $T$. We also record the information on the number $m_i$ of zeroes and the number $n_i$ of simple poles located at each layer $i$. This extra structure is referred to as a decoration. Since $\hat{\mathcal{P}}$ is a topological sphere, the graph $T$ is a tree. Taking an appropriate sum of expressions (1.2) over all decorated trees we get the leading term of the asymptotics for the number of pillowcase covers (see §3.6 and Theorem 3.10 for exact statements).

On the other hand, we have the following recent result from [AEZ]:

**Theorem.**

\[ \text{Vol} \mathcal{Q}_1(1^K, -1^{K+4}) = \frac{\pi^{2K+2}}{2K-1}. \]

It implies the main combinatorial identity stated in Theorem 3.10.

The formula for the volume $\text{Vol} \mathcal{Q}_1(1^K, -1^{K+4})$ (and, actually, a much more general formula for the volume of any stratum of meromorphic quadratic differentials with at most simple poles) is obtained in [AEZ] in a very indirect way through the analytic Riemann-Roch theorem, asymptotics of the determinant of the Laplacian of the singular flat metric, principal boundary of the moduli spaces, Siegel–Veech constants, and Lyapunov exponents of the Hodge bundle. The current paper develops a transparent geometric approach. We have to admit that from purely pragmatic point of view this natural geometric approach is, however, less efficient.

The situation with counting volumes of strata of Abelian differentials is somehow similar: the problem was solved by A. Eskin and A. Okounkov in [EO01] using methods of representation theory of the symmetric group, and developed further in [EO03] and in [EOP] using techniques of quasimodular forms. A straightforward counting of square-tiled surfaces works only for strata of small dimension, and becomes disastrously complicated when the dimension grows, see [Z00]. However, the technique elaborated in this naive geometric approach to the study of square-tiled surfaces, and the ties to various related subjects proved to be extremely helpful. For example, the separatrix diagrams (analogs of ribbon graphs representing singular layers) were used as one of the main instruments in classification [KZ00] of connected components of the strata. Multiple zeta values which appear in counting square-tiled surfaces represented by certain groups of separatrix diagrams, seem to have interesting applications to representation theory.
We believe that an ample description of the enumerative geometry of pillowcase covers combining direct geometric approach elaborated in the current paper, and the implicit analytic approach from [AEZ] could be helpful for various applications.

1.2. Reader's guide. In §2 we present the basic background material on the natural volume element the moduli spaces of quadratic differentials. Namely, in §2.1 we introduce the canonical cohomological coordinates in each stratum $Q(d_1, \ldots, d_k)$ of meromorphic quadratic differentials with at most simple poles. In §2.2 we define a canonical lattice in these coordinates which determines the natural linear volume element in the stratum. In §2.3 we show how volume calculation is related to counting of lattice points.

The original part of the paper is presented in §3. In §3.1 we show why lattice points in the stratum are represented by pillowcase covers which, in view of §2.3, explains why the volume calculation is equivalent to counting the pillowcase covers. In §3.2 we discuss in more details the functions $F_{n,m}$ from Theorem 1.1, study their elementary properties and prove formula (1.2). We consider in §3.2 a particular case $F_{2,2}$ corresponding to Figure 3 as an example, for which we perform an explicit by hand computation. In §3.4 we obtain a general expression for $F_{m,0}$ as a corollary from Kontsevich’s Theorem [K92]. We use this expression as a base of recurrence developed in §3.5, where we express $F_{m+1, n+1}$ in terms of $F_{m,n}$. This recurrence allows us to prove in §3.5 Theorem 1.1. Finally, in §3.6 we compute the sum over all decorated trees and prove the main identity stated in Theorem 3.10. We illustrate this Theorem performing a detailed computation for the strata $Q(1, -1^5)$ and $Q(1^2, -1^6)$ in §3.7 and in §3.8 respectively.

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2. Canonical volume element in the moduli space of quadratic differentials

2.1. Coordinates in a stratum of quadratic differentials. Consider a meromorphic quadratic differential $\psi$ having zeroes of arbitrary multiplicities but only simple poles on $\mathbb{CP}^1$. Let $P_1, \ldots, P_n$ be its singular points (zeros and simple poles). Consider the minimal branched double covering $p : \hat{S} \rightarrow \mathbb{CP}^1$ such that the induced quadratic differential $p^* \psi$ on the hyperelliptic surface $\hat{S}$ is already a square of an Abelian differential $p^* \psi = \omega^2$.

The zeros $\hat{P}_1, \ldots, \hat{P}_N$ of the resulting Abelian differential $\omega$ correspond to the zeros of $\psi$ in the following way: every zero $P \in \mathbb{CP}^1$ of $\psi$ of odd order is a ramification point of the covering, so it produces a single zero $\hat{P} \in \hat{S}$ of $\omega$; every zero $P \in \mathbb{CP}^1$ of $\psi$ of even order is a regular point of the covering, so it produces two zeros $\hat{P}^+, \hat{P}^- \in \hat{S}$ of $\omega$. Every simple pole of $\psi$ defines a branching point of the covering; this point is a regular point of $\omega$.

Consider the subspace $H_1^+(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{Z})$ of the relative homology of the cover with respect to the collection of zeroes $\{\hat{P}_1, \ldots, \hat{P}_N\}$ of $\omega$ which is antiinvariant with respect to the induced action of the hyperelliptic involution. We are going to construct a basis in this subspace (in complete analogy with a usual basis of absolute cycles for a hyperelliptic surface).
We can always enumerate the singular points $P_1, \ldots, P_n$ of $\psi$ in such a way that $P_n$ is a simple pole. Choose now a simple oriented broken line $P_1, \ldots, P_{n-1}$ on $\mathbb{CP}^1$ joining consecutively all the singular points of $\psi$ except the last one. For every arc $[P_i, P_{i+1}]$ of this broken line, $i = 1, \ldots, n - 2$, the difference of their two preimages defines a relative cycle in $H^{-1}(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{Z})$. By construction such a cycle is antiinvariant with respect to the hyperelliptic involution. It is immediate to see that the resulting collection of cycles forms a basis in $H^{-1}(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{Z})$.

Note that, a preimage of a simple pole does not belong to the set $\hat{P}_1, \ldots, \hat{P}_N$. Thus, a preimage of an arc $[P_i, P_{i+1}]$ having a simple pole as one of the endpoints does not define a cycle in $H_1(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{Z})$. However, since a simple pole is always a branching point, the difference of the preimages of such arc is already a well-defined relative cycle in $H_1(S, \{P_1, \ldots, P_N\}; \mathbb{Z})$.

Let $Q(d_1, \ldots, d_n)$ be the ambient stratum for the meromorphic quadratic differential $(\mathbb{CP}^1, \psi)$. The subspace $H^1(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{C})$ in the relative cohomology antiinvariant with respect to the natural involution defines local coordinates in the stratum.

2.2. **Normalization of the volume element.** For any flat surface $S$ in any stratum $Q(d_1, \ldots, d_k)$ we have a canonical ramified double cover $\hat{S} \to S$ such that the induced quadratic differential on the Riemann surface $\hat{S}$ is a global square of a holomorphic Abelian differential. We have seen in [21] that the subspace $H^1(\hat{S}, \{\hat{P}_1, \ldots, \hat{P}_N\}; \mathbb{C})$ antiinvariant with respect to the induced action of the hyperelliptic involution on relative cohomology provides local coordinates in the corresponding stratum $Q(d_1, \ldots, d_n)$ of quadratic differentials. We define a lattice
Another advantage of our choice is that the volumes of the strata \(Q\) is still always integer.)

Indeed, if a flat surface \(S\) defines a lattice point for our choice of the lattice, then the holonomy vector along a saddle connection joining distinct singularities might be half-integer. (However, the holonomy vector along any \textit{closed} saddle connection is still always integer.)

The choice of one or another lattice is a matter of convention. Our choice makes formulae relating enumeration of pillowcase covers to volumes simpler; see § 4. Another advantage of our choice is that the volumes of the strata \(Q(d_1,\ldots,d_k)\) and of the hyperelliptic components of the corresponding strata of Abelian differentials are the same (up to the factors responsible for the numbering of zeroes and of simple poles).

**Convention 2.1.** Similar to the case of Abelian differentials we choose a real hypersurface \(Q_1(d_1,\ldots,d_k)\) in the stratum \(Q_1(d_1,\ldots,d_k)\) of flat surfaces of fixed area. We abuse notation by denoting by \(Q_1(d_1,\ldots,d_k)\) the space of flat surfaces of area \(1/2\) (so that the canonical double cover has area \(1\)).

The volume element \(d\mu\) in the embodying space \(Q(d_1,\ldots,d_k)\) induces naturally a volume element \(d\mu_1\) on the hypersurface \(Q_1(d_1,\ldots,d_k)\) in the following way. There is a natural \(\mathbb{C}^*\)-action on \(Q(d_1,\ldots,d_k)\): having \(\lambda \in \mathbb{C}^*\) we associate to the flat surface \(S = (\mathbb{CP}^1,q)\) the flat surface
\[
\lambda \cdot S := (\mathbb{CP}^1, \lambda^2 \cdot q).
\]

In particular, we can represent any \(S \in Q(d_1,\ldots,d_k)\) as \(S = rS_{(1)}\), where \(r \in \mathbb{R}_+\), and where \(S_{(1)}\) belongs to the “hyperboloid”: \(S_{(1)} \in Q_1(d_1,\ldots,d_k)\). Geometrically this means that the metric on \(S\) is obtained from the metric on \(S_{(1)}\) by rescaling with linear coefficient \(r\). In particular, vectors associated to saddle connections on \(S_{(1)}\) are multiplied by \(r\) to give vectors associated to corresponding saddle connections on \(S\). It means also that \(\text{area}(S) = r^2 \cdot \text{area}(S_{(1)}) = r^2/2\), since \(\text{area}(S_{(1)}) = 1/2\). We define the \textit{volume element} \(d\mu_1\) on the “hyperboloid” \(Q_1(d_1,\ldots,d_k)\) by disintegration of the volume element \(d\mu\) on \(Q(d_1,\ldots,d_k)\):
\[
d\mu = r^{2n-1} \, dr \, d\mu_1,
\]
where
\[
2n = \dim Q(d_1,\ldots,d_k) = 2 \dim C \, Q(d_1,\ldots,d_k) = 2(k-2).
\]

Using this volume element we define the total \textit{volume of the stratum} \(Q_1(d_1,\ldots,d_k)\):
\[
\text{Vol} \, Q_1(d_1,\ldots,d_k) := \int_{Q_1(d_1,\ldots,d_k)} d\mu_1.
\]

For a subset \(E \subset Q_1(d_1,\ldots,d_k)\) we let \(C(E) \subset Q_1(d_1,\ldots,d_k)\) denote the “cone” based on \(E\):
\[
C(E) := \{ S = rS_{(1)} \mid S_{(1)} \in E, \ 0 < r \leq 1 \}.
\]
Our definition of the volume element on $Q_1(d_1, \ldots, d_k)$ is consistent with the following normalization:

\begin{equation}
\text{Vol}(Q_1(d_1, \ldots, d_k)) = \dim_{\mathbb{R}} Q(d_1, \ldots, d_k) \cdot \mu(C(Q_1(d_1, \ldots, d_k)),
\end{equation}

where $\mu(C(Q_1(d_1, \ldots, d_k))$ is the total volume of the “cone” $C(Q_1(d_1, \ldots, d_k)) \subseteq Q(d_1, \ldots, d_k)$ measured by means of the volume element $d\mu$ on $Q(d_1, \ldots, d_k)$ defined above.

### 2.3. Reduction of volume calculation to counting lattice points.

The volume of a stratum $Q_1(d_1, \ldots, d_k)$ is defined by \((\ref{2.5})\) as

\[
\text{Vol}_Q(d_1, \ldots, d_k) = \dim_{\mathbb{R}} Q(d_1, \ldots, d_k) \cdot \mu(C(Q_1(d_1, \ldots, d_k)),
\]

where $\mu(C(Q_1(d_1, \ldots, d_k))$ is the total volume of the “cone” $C(Q_1(d_1, \ldots, d_k)) \subseteq Q(d_1, \ldots, d_k)$ measured by means of the volume element $d\mu$ on $Q(d_1, \ldots, d_k)$ defined in \((\ref{2.2})\). The total volume of the cone $C(Q_1(d_1, \ldots, d_k))$ is the limit of the appropriately normalized Riemann sums.

The volume element $d\mu$ is defined as a linear volume element in cohomological coordinates, normalized by certain specific lattice. Choose a positive $\varepsilon$ such that $1/\varepsilon$ is integer, and consider a sublattice of the initial lattice of index $(1/\varepsilon)^\dim_{\mathbb{R}} Q(d_1, \ldots, d_k)$ partitioning every side of the initial lattice into $1/\varepsilon$ pieces. The corresponding Riemann sums count the number of points of the sublattices which get inside the cone. Thus, by definition of the measure $\mu$ we get

\[
\mu(C(Q_1(d_1, \ldots, d_k)) = \lim_{\varepsilon \to 0} \varepsilon^{\dim_{\mathbb{R}} Q(d_1, \ldots, d_k)}.
\]

(Number of points of the $\varepsilon$-sublattice inside the cone $C(Q_1(d_1, \ldots, d_k))$).

We assume that $1/\varepsilon$ is integer. Note that a flat surface $S$ represents a point of the $\varepsilon$-lattice, if and only if the surface $(1/\varepsilon) \cdot S$ (in the sense of definition \((\ref{2.1})\)) represents a point of the integer lattice. Denoting by $C(Q_{N}(d_1, \ldots, d_k))$ the set of flat surfaces in the stratum $Q(d_1, \ldots, d_k)$ of area at most $N/2$, and taking into consideration that

\[
\text{area}((1/\varepsilon) \cdot S) = 1/\varepsilon^2 \cdot \text{area}(S)
\]

we can rewrite the above relation as

\begin{equation}
\mu(C(Q_1(d_1, \ldots, d_k)) = \lim_{N \to +\infty} N^{-\dim_{\mathbb{R}} Q(d_1, \ldots, d_k)}.
\end{equation}

(Number of lattice points inside the cone $C(Q_{N}(d_1, \ldots, d_k))$).

### 3. Counting generalized Jenkins–Strebel differentials

In this section we pass to counting the pillowcase covers. We have seen in \((\ref{2.3})\) that volume calculation is equivalent to counting the lattice points. In \((\ref{3.1})\) we discuss in more details the pillowcase covers and show that counting of lattice points is equivalent to the counting problem for pillowcase covers. Starting from section \((\ref{3.2})\) we work exclusively with the strata $Q(1^R, -1^{K+1})$.

In \((\ref{3.2})\) we discuss in more details the functions $F_{n,m}$ from Theorem \(1.1\) study their elementary properties and prove formula \(1.2\). We consider in \((\ref{3.2})\) a particular case $F_{2,2}$ corresponding to Figure \(3\) as an example, for which we perform an explicit (by hand) computation. In \((\ref{3.3})\) we obtain a general expression for $F_{m,0}$ as a corollary from a theorem of Kontsevich \(\cite{Ko2}\). We use this expression as a base of recursion developed in \((\ref{3.4})\) where we express $F_{m+1,n+1}$ in terms of $F_{m,n}$. This
recurrence relation allows us to prove in §3.5 Theorem 1.1. Finally, in §3.6 we compute the sum over all decorated trees and prove the main identity stated in Theorem 3.10. In §3.7 and §3.8 we illustrate our formula for concrete examples of the strata $Q(1,-1^5)$ and $Q(1^2,-1^6)$ correspondingly.

3.1. Lattice points, square-tiled surfaces, and pillowcase covers. Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $T^2 = \mathbb{C}/\Lambda$ be the associated torus. The quotient $P := T^2/\pm$ by the map $z \rightarrow -z$ is known as the pillowcase orbifold. It is a sphere with four $(\mathbb{Z}/2)$-orbifold points (the corners of the pillowcase). The quadratic differential $(dz)^2$ on $T^2$ descends to a quadratic differential on $P$. Viewed as a quadratic differential on the Riemann sphere, $(dz)^2$ has simple poles at corner points. When the lattice $\Lambda$ is the standard integer lattice $\mathbb{Z} \oplus i\mathbb{Z}$, the flat torus $T^2$ is obtained by isometrically identifying the opposite sides of a unit square, and the pillowcase $P$ is obtained by isometrically identifying two squares with the side $1/2$ by the boundary, see Figure 1.

Consider a connected ramified cover $\hat{P}$ over $P$ of degree $N$ having ramification points only over the corners of the pillowcase. Clearly, $\hat{P}$ is tiled by $2N$ squares of the size $(1/2) \times (1/2)$ in such way that the squares do not superpose and the vertices are glued to the vertices. Coloring the two squares of the pillowcase $P$ one in black and the other in white, we get a chessboard coloring of the square tiling of the the cover $\hat{P}$: the white squares are always glued to the black ones and vice versa.

Lemma 3.1. Let $S$ be a flat surface in the stratum $Q(d_1,\ldots,d_k)$. The following properties are equivalent:

1. The surface $S$ represents a lattice point in $Q(d_1,\ldots,d_k)$;
2. $S$ is a cover over $P$ ramified only over the corners of the pillow;
3. $S$ is tiled by black and white $(1/2) \times (1/2)$ squares respecting the chessboard coloring.

Proof. We have just proved that (2) implies (3). To prove that (1) implies (2) we define the following map from $S$ to $P$. Fix a zero or a pole $P_0$ on $S$. For any $P \in S$ consider a path $\gamma(P)$ joining $P_0$ to $P$ having no self-intersections and having no zeroes or poles inside. The restriction of the quadratic differential $q$ to such $\gamma(P)$ admits a well-defined square root $\omega = \pm \sqrt{q}$, which is a holomorphic form on the interior of $\gamma$. Define

$P \mapsto \left( \int_{\gamma(P)} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) / \pm$.

Of course, the path $\gamma(P)$ is not uniquely defined. However, since the flat surface $S$ represents a lattice point (see the definition in §2.2), the difference of the integrals of $\omega$ over any two such paths $\gamma_1(P)$ and $\gamma_2(P)$ belongs to $\mathbb{Z} \oplus i\mathbb{Z}$, so taking the quotient over the integer lattice and over $\pm$ we get a well-defined map. By definition of the pillowcase $P$ we have, $P = (\mathbb{C} \mod \mathbb{Z} \oplus i\mathbb{Z}) / \pm$. Thus, we have defined a map $S \rightarrow P$. It follows from the definition of the map, that it is a ramified cover, and that all regular points of the flat surface $S$ are regular points of the cover. Thus, all ramification points are located over the corners of the pillowcase.

A similar consideration shows that (3) implies (1).
Let $\text{Sq}_N(d_1, \ldots, d_k)$ be the number of surfaces in the stratum $Q(d_1, \ldots, d_k)$ tiled with at most $N$ black and $N$ white squares respecting the chessboard coloring. Lemma 3.1 allows to rewrite formula (2.6) as follows:

$$\mu(C(Q_1(d_1, \ldots, d_k))) = \lim_{N \to +\infty} N^{-\dim C(Q_1(d_1, \ldots, d_k))} \cdot \text{Sq}_N(d_1, \ldots, d_k).$$

Taking into consideration (2.5) we get

$$\text{Vol} Q_1(d_1, \ldots, d_k) = 2 \dim C(Q_1(d_1, \ldots, d_k)) \cdot$$

$$\lim_{N \to +\infty} N^{-\dim C(Q_1(d_1, \ldots, d_k))} \cdot \text{Sq}_N(d_1, \ldots, d_k).$$

3.2. Local Polynomials. In order to count pillowcase covers we note that if $\tilde{\mathcal{P}}$ is a square-tiled pillowcase cover, it can be decomposed into cylinders with integer widths, with zeros and poles lying on the boundaries of these cylinders. We call these boundaries singular layers. We can form an associated graph whose vertices are singular layers and edges are cylinders. For a pillowcase cover $\tilde{\mathcal{P}}$ in $Q(1^K, (-1)^{K+1})$ the associated graph will be a tree, since $\tilde{\mathcal{P}}$ is a sphere. Figure 2 gives an example of such a tree.

Suppose there are $k$ cylinders of width $w_1, \ldots, w_k$ and height $h_1, \ldots, h_k$ respectively. Since $\tilde{\mathcal{P}}$ is a sphere, there are $k + 1$ singular layers in the decomposition of $\tilde{\mathcal{P}}$. Fix the way in which our labelled (named) zeroes and poles are distributed through singular layers (vertices of the global tree $T$).

**Lemma 3.2.** The total number of pillowcase covers of degree at most $N$ with a decomposition of a fixed type can be written as

$$2^k \sum_{w_i \cdot h_i \leq N} w_1 \cdot w_2 \cdots w_k \cdot \prod_{i=1}^{k+1} F_i(w_1, \ldots, w_k),$$

where $F_j$ is a function counting the number of ways the cylinders of width $w_i$ can be glued at vertex $j$.

Here $F_j$ depends only on the widths $w_{ij}$ associated to edges adjacent to vertex $j$.

**Proof.** Every cylinder is determined by the following parameters: by an integer perimeter (length of the waist curve) $w_i$; by a half-integer height $h_i$; and by a half-integer twist $t_i$, where $0 \leq t_i \leq w_i$. Thus, there are $2w_i$ choices for the value of the twist $t_i$, which explains the factor $(2w_1)(2w_2)\ldots(2w_k) = 2^k \prod_{i=1}^{k+1} w_i$.

The restriction on the area $\sum h_i \cdot w_i \leq N/2$ with integer $w_i$ and half-integer $h_i$ is equivalent to the restriction $\sum h_i \cdot w_i \leq N$ with integer $w_i$ and integer $h_i$. □

Our current goal is to show that up to terms of lower order the counting function $F_{m,n}$ associated to a layer with $m$ simple zeros and $n$ first order poles, is the explicit symmetric polynomial (1.1). We emphasize that the zeros and poles are labelled.

The neighborhood of a singular layer with $m$ zeros and $n$ poles is a metric ribbon graph with $m$ trivalent vertices (representing zeroes), $n$ univalent vertices (representing simple poles) and with $l$ boundary components, see Figure 2. The width $w_i$ of each boundary component is given by the sum of the lengths of the edges lying on the boundary. Thus, given a collection of integer widths of cylinders,
the counting problem can be restated as finding the number of graphs with half-
integer edge lengths yielding these widths. This is a system of linear equations, and
the number of half-integral solutions is equal to the volume of the space of all real
solutions for the edge lengths.

Note that the neighborhood of a singular layer with \( m \) zeros and \( n \) poles can
be also viewed as is a topological sphere with \( n + m \) marked points and with \( l \)
punctures. This sphere is endowed with a complex structure; the corresponding \( \bb C P^1 \)
carries a meromorphic Jenkins–Strebel differential having \( m \) simple zeros, \( n \)
simple poles, and \( l \) double poles (which are not poles of \( \mathcal P \)) corresponding to \( l \)
cylinders of widths \( w_i, i = 1, \ldots l \). The number of cylinders \( l \) is specified by the
relation \( m - n - 2l = -4 \), that is,

\[
l = \frac{(m - n)}{2} + 2.
\]

By [Str83], there is a bijective correspondence between such Jenkins–Strebel differentials and metric ribbon graphs on the sphere with \( m \) trivalent and \( n \) univalent vertices. To count these differentials, we follow an approach of Kontsevich [K92].

Given a ribbon graph on the sphere with \( m \) trivalent and \( n \) univalent vertices, we have \( v = m + n \), \( e = (3m + n)/2 \), where \( e \) and \( v \) are the number of edges and vertices respectively. Letting \( f \) denote the number of faces (i.e, complementary regions), we have \( v - e + f = 2 \), so \((n - m)/2 + f = 2\), i.e., \( 2f = 4 + m - n \). This imposes the restriction that \( m - n \in 2\bb Z \) and that \( m - n > -4 \). Also, we have \( e - f = v - 2 \), which suggests our polynomial should be a degree \( v - 2 \) polynomial in \( f \) variables.

### 3.3. Example: direct computation of \( F_{2,2} \).

Let us explicitly compute the local polynomial \( F_{2,2}(w_1, w_2) \). The list of connected ribbon graphs having two vertices of valence 3 and two vertices of valence 1 with labelled boundary components is presented at Figure 3. Note that interchanging the labelling of the boundary components for the ribbon graphs \( \Gamma_{1,1} \) and \( \Gamma_{2,1} \) we get different ribbon graphs \( \Gamma_{1,2} \) and \( \Gamma_{2,2} \) correspondingly, while changing the labelling of the boundary components of the ribbon graph \( \Gamma_3 \) we get an isomorphic ribbon graph.

Note, that since our ribbon graphs represent singular layers on a topological sphere, they are always planar, i.e., they can be embedded into a plane.

Consider, for example, the graph \( \Gamma_{1,1} \) on top on the left. The widths of the cylinders are given by \( w_1 = l_1 + l_2 \), and by \( w_2 = l_1 + l_2 + 2l_3 + 2l_4 \), so \( \Gamma_1 \) is realizable if and only if \( w_1 < w_2 \). Given \( w_1 < w_2 \) there are \( 2w_1 \) half-integer positive solutions \( l_1, l_2 \) of equation \( w_1 = l_1 + l_2 \), and for each such solution there are \( w_1 \) half-integer solutions of the equation \( w_2 = l_1 + l_2 + 2l_3 + 2l_4 \). Thus, the impact \( F_{\Gamma_{1,1}}(w_1, w_2) \) of \( \Gamma_{1,1} \) to the local polynomial \( F_{2,2}(w_1, w_2) \) has the form

\[
F_{\Gamma_{1,1}}(w_1, w_2) := \begin{cases} 
0 & \text{when } w_1 \geq w_2 \\
2w_1(w_2 - w_1) & \text{when } w_1 < w_2
\end{cases}
\]

Note that the number of quadruples of positive half-integers \( l_1, l_2, l_3, l_4 \) satisfying the above equations, can be viewed as the volume of the associated region of solutions in the positive cone \( \bb R^4_{>0} \). Consider the Laplace transform \( \hat F_{\Gamma_{1,1}}(\lambda_1, \lambda_2) = \int_{\bb R^2} e^{-\lambda w} F_{\Gamma_{1,1}}(w_1, w_2) dw \). Since \( F_{\Gamma_{1,1}}(w_1, w_2) = 0 \) for \( w_1, w_2 < 0 \) and for \( w_1 \geq w_2 \),
and since \( w_1 = l_1 + l_2, w_2 = l_1 + l_2 + 2l_3 + 2l_4 \), we obtain

\[
(3.3) \quad \tilde{F}_{1,1}(\lambda_1, \lambda_2) = 2^3 \cdot \int_{l_1, l_2, l_3, l_4 > 0} e^{-\lambda_1(l_1+l_2)} e^{-\lambda_2(l_1+l_2+2l_3+2l_4)} dl_1 dl_2 dl_3 dl_4 = \frac{1}{2} \cdot \frac{2}{\lambda_1 + \lambda_2} \cdot \frac{2}{\lambda_1 + \lambda_2} \cdot \frac{2}{2\lambda_2} \cdot \frac{2}{2\lambda_2},
\]

where the factor

\[
2^{m+n-1} = 2^3
\]

in front of the integral comes from the normalization of the volume element in cohomological coordinates. This coefficient can also be seen, in general, as follows: we have \( e = \frac{3m+n}{2} \) edges and \( f = \frac{(m+n+2)}{2} \) faces (adjacent cylinders). The latter give relations between edge lengths; the difference is our dimension

\[
\frac{3m+n}{2} - \left( \frac{m-n}{2} + 2 \right) = m + n - 2.
\]

However, in the parity count the relations are not independent. If all edge lengths are half-integer, and all perimeters of cylinders but one are integer, the last perimeter is automatically an integer. To see this compute the sum of the lengths of perimeters with natural signs. If all edge lengths are half-integer, all edges which separate different cylinders get cancelled in this sum and all other edges are counted with a factor of 2. Thus, the sum of the residues is integer. This implies that if all perimeters but one are integer, the last one is automatically an integer, and we go from \((m + n - 2)\) to \((m + n - 1)\).

The expressions for \( F_{1,1,2} \) and for \( \tilde{F}_{1,1,2} \) are symmetric to those for \( F_{1,1,1} \) and \( \tilde{F}_{1,1,1} \) respectively. Similar calculations provide the following answers for the remaining graphs:

\[
F_{1,2,1}(w_1, w_2) := \begin{cases} 
0 & \text{when } w_1 \geq w_2 \\
\frac{(w_2 - w_1)^2}{2} & \text{when } w_1 < w_2
\end{cases} \quad \tilde{F}_{2,1}(\lambda_1, \lambda_2) = \frac{1}{2} \cdot \frac{2}{\lambda_1 + \lambda_2} \left( \frac{1}{\lambda_2} \right)^3.
\]

The expressions for \( F_{2,2,1} \) and for \( \tilde{F}_{2,2,1} \) are symmetric to those for \( F_{2,1,1} \) and \( \tilde{F}_{2,1,1} \) respectively. Finally,

\[
F_{2,1,3}(w_1, w_2) := \begin{cases} 
\frac{w_2^2}{w_1} & \text{when } w_1 \geq w_2 \\
\frac{w_1^2}{w_2} & \text{when } w_1 < w_2
\end{cases} \quad \tilde{F}_{1,3}(\lambda_1, \lambda_2) = \frac{1}{2} \cdot \left( \frac{2}{\lambda_1 + \lambda_2} \right)^2 \cdot \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2}.
\]

There are \( 2! \cdot 2! \) ways to give names to 2 zeroes (i.e. to 2 trivalent vertices) and to 2 poles (i.e. to 2 univalent vertices) of the graphs \( \Gamma_{2,1,1}, \Gamma_{2,2,2} \) and \( \Gamma_{3} \), and there is \( \frac{1}{2} \cdot 2! \cdot 2! \) ways to give names to zeroes and poles of the graphs \( \Gamma_{1,1,1}, \Gamma_{1,2,2} \). Thus, the contribution of all graphs to \( F_{2,2} \) is

\[
F_{2,2}(w_1, w_2) = 2! \cdot 2! \left( \frac{1}{2} \cdot F_{1,1,1} + \frac{1}{2} \cdot F_{1,1,2} + F_{1,2,1} + F_{2,1,1} + F_{1,3} \right) = 4 \left( F_{1,1,1} + F_{1,2,1} + F_{2,1,1} \right) \quad \text{when } w_1 < w_2 =
\]

\[
4 \left( w_1 (w_2 - w_1) + \frac{(w_2 - w_1)^2}{2} + w_1^2 \right) = 2(w_1^2 + w_2^2) \quad \text{when } w_1 < w_2.
\]
Similarly,
\[ F_{2,2}(w_1, w_2) = 4 \left( F_{1,2} + F_{2,2} + F_{3,1} \right) \quad \text{when } w_1 > w_2 =
4 \left( w_2 (w_1 - w_2) + \frac{(w_1 - w_2)^2}{2} + w_2^2 \right) = 2(w_1^2 + w_2^2) \quad \text{when } w_1 > w_2. \]

We observe that, though for individual graphs \( \Gamma \) the expression \( F_{1}(w_1, w_2) \) is not symmetric in \( w_1, w_2 \), the total sum \( F_{2,2}(w_1, w_2) \) is a symmetric polynomial in \( w_1, w_2 \).

Note that formally speaking, we have calculated only the leading term of the local polynomial neglecting a small correction arising from degenerate solutions when one or several \( l_i \) vanish. A. Okounkov and R. Pandharipande prove in [OP06] that counting the degenerate solutions in an appropriate way we get a true symmetric polynomial \( F_{m,n} \) not only in the leading term, but exactly. Since for the purposes of counting the volume we are interested only in the leading term of the asymptotics, we neglect this subtlety.

We could also compute \( \hat{F}_{2,2}(\lambda_1, \lambda_2) \) directly. The advantage of this calculation is that we do not need to follow the system of inequalities, which becomes quite involved for complicated graphs. In our case we get

\[
\hat{F}_{2,2}(\lambda_1, \lambda_2) = 2! \cdot 2! \left( \frac{2}{\lambda_1 + \lambda_2} \right)^2 \left( \frac{1}{\lambda_1} \right)^2 + \left( \frac{2}{\lambda_1 + \lambda_2} \right)^2 \left( \frac{1}{\lambda_2} \right)^2 + 2 \cdot \frac{2}{\lambda_1 + \lambda_2} \left( \frac{1}{\lambda_1} \right)^3 + \frac{2 \cdot \frac{2}{\lambda_1 + \lambda_2} \left( \frac{1}{\lambda_2} \right)^3 + 2 \left( \frac{2}{\lambda_1 + \lambda_2} \right)^2 \frac{1}{\lambda_1} \frac{1}{\lambda_2} = 4 \left( \frac{1}{\lambda_1 \lambda_2^2} + \frac{1}{\lambda_2 \lambda_1^2} \right). \]

3.4. Kontsevich’s Theorem. Consider now the general setting. As above, let \( \Gamma \) be a ribbon graph on the sphere, and let \( w_1, \ldots, w_l \) be the widths of the complementary regions. Let \( F_{\Gamma}(w_1, \ldots, w_l) \) be the volume of the region in \( \mathbb{R}^{1,0}_{>0} \) corresponding to lengths of edges so that the sum of the edges adjacent to the region \( i \) is \( w_i \). Here, \( |e| \) is the number of edges of \( \Gamma \). Taking the Laplace transform, we define

\[
\hat{F}_{\Gamma}(\lambda) = 2^{m+n-1} \cdot \prod_{e \in \Gamma} \frac{1}{\lambda(e)},
\]

where the product is taken over all the edges \( e \) of the graph \( \Gamma \), and \( \lambda(e) \) denotes the sum of the \( \lambda \)'s corresponding to width variables associated to regions bordering \( e \). Our normalization is as in [33]. If \( e \) is an edge adjacent to a univalent vertex, then it is only bordered by one region. Let \( F_{m,n}(w_1, \ldots, w_l) \) denote the total volume (that is, the sum over all possible ribbon graphs \( \Gamma \) with \( m \) trivalent and \( n \) univalent vertices), and let \( \hat{F}_{m,n} \) denote the Laplace transform of \( F_{m,n} \).

\[
\hat{F}_{m,n}(\lambda) = \sum_{\Gamma} \hat{F}_{\Gamma}(\lambda) = 2^{m+n-1} \cdot \sum_{\Gamma} \prod_{e \in \Gamma} \frac{1}{\lambda(e)},
\]

where the sums are taken over all connected ribbon graphs \( \Gamma \) having \( m \) trivalent vertices, \( n \) univalent vertices and no other vertices. We have:
Theorem 3.3. [K92 §3.1, page 10] Let $m = 2k, l = k + 2$. Then

$$\hat{F}_{m,0}(\lambda_1, \ldots, \lambda_l) = 2^{k-1}m! \sum_{k_1, \ldots, k_l = k-1} \left( \frac{k - 1}{k_1, \ldots, k_l} \right) \prod_{i=1}^{l} \frac{(2k_i - 1)!!}{\lambda_i^{2k_i + 1}},$$

where $(2k_i - 1)!! = (2k_i - 1)(2k_i - 3)(2k_i - 5) \ldots 1$.

Recall that by convention $(-1)!! = 0! = 1$.

Remark: Kontsevich states this result in terms of certain intersection numbers $\langle \tau_{k_1} \ldots \tau_{k_l} \rangle$ in place of the binomial coefficients $\binom{k-1}{k_1, \ldots, k_l}$. However, the equality of the two quantities was known to Witten [W91], see, also e.g., [OP05, equation 2.3]. There are also two differences in normalization—first, we are working with labelled zeros (and later also poles), and edges, which eliminates any symmetry group factors and adds the factor of $m!$, and we normalize the half-integer lattice to have volume 1, which accounts for the difference in the factor of a power of 2.

Corollary 3.4. Theorem [77] and formula [114] are valid for $n = 0$:

$$F_{m,0}(w_1, w_2, \ldots, w_l) = m! \sum_{k_1, \ldots, k_l = k-1} \left( \frac{k - 1}{k_1, \ldots, k_l} \right) \prod_{i=1}^{l} \frac{w_i^{2k_i}}{k_i!},$$

Proof. Taking Laplace transforms, and noting that if $F(x) = x^k$, then $\hat{F}(x) = k!/\lambda^{k+1}$, we obtain

$$F_{m,0} = m! 2^{k-1} \sum_{k_1, \ldots, k_l = k-1} \left( \frac{k - 1}{k_1, \ldots, k_l} \right) \prod_{i=1}^{l} \frac{(2k_i - 1)!!}{(2k_i)!} w_i^{2k_i}.$$

The product inside the summation can be simplified using

$$\frac{(2k_i - 1)!!}{(2k_i)!} = \frac{(2k_i - 1)(2k_i - 3) \ldots 1}{(2k_i)(2k_i - 1)(2k_i - 2)(2k_i - 3) \ldots 1} = \frac{1}{(2k_i)(2k_i - 2)(2k_i - 4) \ldots 2} = \frac{1}{2^{k-1}k_i!}. \quad (3.8)$$

Noting that $\sum k_i = k - 1$ allows us to cancel the $2^{k-1}$ factor, yielding (3.6), which corresponds to the case $n = 0$ of our main theorem.

3.5. Recurrence relations and evaluation of local polynomials. To prove our general formula (1.1) for arbitrary local polynomials $F_{m,n}$, we require another lemma which gives an induction relation $F_{m+1,n+1}$ to $F_{m,n}$.

Lemma 3.5. Fix notation as in Theorem [77] Then for any nonnegative $m, n \in \mathbb{Z}$, not simultaneously equal to zero, the function $F_{m,n}$ is a polynomial in $w_i$ satisfying the relation

$$F_{m+1,n+1} = 2(m + 1) \cdot D(F_{m,n}),$$

where $D = \sum_{i=1}^{k} D_{w_i}$; operators $D_{w_i}$ are defined on monomials by $D_{w_i}(w^n) = \frac{w^{n+2}}{n+2}$ and are extended to arbitrary polynomials by linearity.

Remark 3.6. Note that the number of variables $l$ does not change, as it only depends on the difference $m - n$. 

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Proof. In terms of Laplace transforms, the statement of the lemma becomes

\[ \hat{F}_{m+1,n+1} = 2(m+1) \cdot \sum_{i=1}^{l} \frac{-1}{\lambda_i} \frac{\partial}{\partial \lambda_i} \hat{F}_{m,n}. \]  

To prove this, we proceed at a graph-by-graph level. Fix a graph \( \Gamma \) with \( m \) trivalent and \( n \) univalent labelled vertices. Define \( p_{ij}(\Gamma) \) as the number of edges of \( \Gamma \) separating regions \( i \) and \( j \). By formula (3.4) (see also the Example in §3.2)

\[ \hat{F}_\Gamma = 2m + n - 1 \cdot \prod_{i \leq j} (1 + \lambda_i \lambda_j) p_{ij}. \]

Let \( \Gamma_{i,j} \) be the graph with \( m + 1 \) trivalent and \( n + 1 \) univalent vertices formed by adding a new edge in the region \( j \) (corresponding to \( \lambda_j \)), so that the new trivalent vertex lies on an edge adjacent to the regions \( i \) and \( j \) (corresponding to \( \lambda_i \) and \( \lambda_j \); possibly \( i = j \)). By formula (3.4) (see also the Example in §3.2)

\[ \hat{F}_{i,j} = 2 \cdot \frac{1}{2\lambda_j} \cdot \frac{1}{\lambda_i + \lambda_j} \cdot \hat{F}_{\Gamma} = \frac{2}{\lambda_j} \cdot \frac{1}{\lambda_i + \lambda_j} \cdot \hat{F}_{\Gamma}. \]

We may assume that the “new” pole (univalent vertex) is located at the end of the new edge. However, there are \( m + 1 \) choices of the simple zero (trivalent vertex) at the other extremity of the new edge. From now on we will fix the labeling of the vertices of the new graph, and we multiply the final result by this factor \((m + 1)\).

Summing the above formula over all edges adjacent to the region \( j \), we obtain the contribution \( \hat{F}_{i,j} \) associated to attaching a new univalent vertex in region \( j \). Note, that the edges having region \( j \) on both sides should be counted twice, since we can attach the new edge on both sides of the original edge, producing two different graphs. Thus,

\[ \hat{F}_{i,j} = 2p_{ij}(\Gamma) \cdot \frac{1}{\lambda_j} \cdot \hat{F}_{\Gamma} + \sum_{i \neq j} p_{ij}(\Gamma) \cdot \frac{1}{\lambda_i + \lambda_j} \cdot \frac{2}{\lambda_j} \cdot \hat{F}_{\Gamma}. \]

Applying the operator \( -\frac{2}{\lambda_j} \cdot \frac{\partial}{\partial \lambda_j} \) to (3.10) we obtain exactly the same expression. Taking into consideration the factor \((m + 1)\) responsible for the numbering we prove relation (3.9). Inverting the Laplace transform and applying Corollary 3.4 and explicit evaluation \( F_{0,2} = F_{1,1} = 1 \) as the base of the recurrence, we complete the proof of the statement of the Lemma. \( \square \)

Proof of Theorem 1.1. We first consider the case \( m > n \). We know, by Corollary 3.4 that

\[ F_{m-n,0} = (m-n)! \cdot \sum_{\sum_{i=1}^{l} k_i = k-1} \binom{k-1}{k_1, \ldots, k_l} \prod_{i=1}^{l} \frac{w^{2k_i}}{k_i!}, \]  

see (3.6), where \( k \) and \( l \) are as in the statement of Theorem 3.3. Our result follows by applying Lemma 3.5 \( n \) times to (3.11), and by observing that the operator \( \prod_{\sum n_i = n} D_i^{n_i} \)
transforms the term \(\frac{w_i^{2n_i}}{k_i!}\) into

\[
\frac{w_i^{2(k_i+n_i)}}{(2k_i+2n_i)(2k_i+2n_i-2)\cdots(2k_i+2)k_i!} = \frac{1}{2^n} \frac{w_i^{2(k_i+n_i)}}{(k_i+n_i)!}.
\]

Combining the factors of 2, we obtain a \(\frac{1}{2^n}\). On the outside, we obtain the factors \((2m)(2(m-1))\cdots(2(m-n+1))\), which, combined with the \((m-n)!\), yields \(2^n(m-n)!\), so cancelling the \(2^n\) factors, we obtain

\[
(3.12) \quad F_{m,n} = m! \sum_{\sum_{i=1}^{l} k_i = k-1} \left( \frac{n}{n_1,\ldots,n_l} \right) \left( \frac{k-1}{k_1,\ldots,k_l} \right) \prod_{i=1}^{l} \frac{(w_i)^{2(k_i+n_i)}}{(k_i+n_i)!}.
\]

Rewriting (3.12) by multiplying and dividing by the factor \(a_1!a_2!\cdots a_l!\), where \(a_i = k_i+n_i\), and using the resulting factors \(a_i!\) to rearrange multinomial coefficients as products of binomial coefficients we get

\[
(3.13) \quad F_{m,n} = m! \sum_{\sum_{i=1}^{l} a_i = n+k-1} \left( \frac{(k-1)!n!}{(a_1!\ldots a_l!)^2} \right) \sum_{k_i = k-1}^{l} \prod_{i=1}^{l} \frac{(a_i)}{k_i} \prod_{i=1}^{l} \frac{y_i^{2a_i}}{a_i!}.
\]

For notational convenience, we define

\[
(3.14) \quad f(a_1,\ldots,a_l) = \frac{(l-3)!n!}{(a_1!a_2!\cdots a_l!)^2} \sum_{k_i = k-1}^{l} \prod_{i=1}^{l} \frac{(a_i)}{k_i} ;
\]

where \(l = k+2\), so that

\[
F_{m,n} = m! \sum_{\sum_{i=1}^{l} a_i = n+k-1} f(a_1,\ldots,a_l) \prod_{i=1}^{l} \frac{y_i^{2a_i}}{a_i!}.
\]

Multiplying and dividing (3.14) by the factor \((n+l-3)!\), and taking into consideration that \(n+l-3 = n+k-1 = \sum a_i\) we rewrite (3.14) as

\[
(3.15) \quad f(a_1,\ldots,a_l) = \frac{1}{a_1!\cdots a_k!} \frac{1}{(n+l-3)!} \sum_{k_i = k-1}^{l} \prod_{i=1}^{l} \frac{(a_i)}{k_i}.
\]

We have

\[
(3.16) \quad \sum_{k_i = k-1}^{l} \prod_{i=1}^{l} \frac{(a_i)}{k_i} = \binom{n+l-3}{n}
\]

by a classical combinatorial argument. Indeed, \(\binom{n+l-3}{n}\) represents the ways to select a subset of \(l-3\) elements from a set of size \(n+l-3\). On the other hand, suppose the set of size \(n+l-3\) contained elements of \(l\) distinct types. To pick \(l-3\) elements, one can choose \(k_1\) of the first kind, up to \(k_l\) of the \(l^{th}\) kind, with \(\sum k_i = l-3\). There are \(\prod (k_i)\) ways of doing this. Summing over all possible \(k_1,\ldots,k_l\) with \(\sum k_i = l-3\), we obtain \(\binom{n+l-3}{n}\). Simplifying (3.15) using (3.16), we obtain

\[
(3.17) \quad f(a_1,\ldots,a_l) = \frac{\binom{n+l-3}{n}}{a_1!\cdots a_k!} = \frac{1}{(n+l-3)!} \binom{n+l-3}{a_1,\ldots,a_l}^2.
\]
Noting that \( n + l - 3 = (m + n)/2 - 1 \), we obtain (1.1).

In the cases \( m = n \) and \( m = n - 2 \) a similar argument applied to the base polynomials \( F_{1,1}(w_1, w_2) = 1 \) and \( F_{0,2}(w) = 1 \) yields:

\[
F_{m,n} = \begin{cases} 
\sum_{i=0}^{m-1} \binom{m-1}{i}^2 w_1^i w_2^{2(m-1-i)} & m = n \\
w_1^{2m} & m = n - 2,
\end{cases}
\]  

(3.18)

**Values of** \( F_{m,n} \) **for small** \( m,n \). To illustrate the above theorem, we compute the values of \( F_{m,n} \) which are involved in volume calculations of \( Q(1^K, -1^{K+2}) \) for \( K = 1, 2 \) performed in §3.7 and §3.8.

| Valence 1 | Valence 2 | Valence 3 |
|-----------|-----------|-----------|
| \( m,n \) | \( F_{m,n} \) | \( m,n \) | \( F_{m,n} \) | \( m,n \) | \( F_{m,n} \) |
| 0, 2 | 1 | 1, 1 | 1 | 2, 0 | 2 |
| 1, 3 | \( w^2 \) | 2, 2 | \( 2(w_1^2 + w_2^2) \) | 3, 1 | \( 6(w_1^2 + w_2^2 + w_3^2) \) |
| 2, 4 | \( w^4 \) | 3, 3 | \( 3(w_1^4 + 4w_1^2w_2^2 + w_3^4) \) |
| 3, 5 | \( w^6 \) |

**3.6. Total Sums.** We first recall the following standard fact:

**Lemma 3.7.** As \( N \to \infty \),

\[
\sum_{\substack{h \in \mathbb{N}^k, w \in \mathbb{N}^k \\
h \cdot w \leq N}} w_1^{a_1+1} \ldots w_k^{a_k+1} \sim \frac{N^{a+2k}}{(a+2k)!} \prod_{i=1}^{k} (a_i + 1)! \zeta(a_i + 2),
\]

where \( a_i \in \mathbb{N} \) for \( i = 1, \ldots, k \) and \( a = a_1 + \cdots + a_k \).

**Proof.** Denote by \( \Delta^k \) the simplex \( x_1 + \cdots + x_k \leq 1 \) in \( \mathbb{R}^k_+ \). Introducing the variables \( x_i := \frac{w_i}{N} \), we can approximate the initial sum by the following sum of integrals:

\[
\sum_{\substack{h \in \mathbb{N}^k, w \in \mathbb{N}^k \\
h \cdot w \leq N}} w_1^{a_1+1} \ldots w_k^{a_k+1} \sim \sum_{h \in \mathbb{N}^k} \frac{1}{h_1^{a_1+2} \cdots h_k^{a_k+2}} .
\]

\[
\sum_{h \in \mathbb{N}^k} \int_{\Delta^k} \left( \frac{x_1 N}{h_1} \right)^{a_1+1} \ldots \left( \frac{x_k N}{h_k} \right)^{a_k+1} \left( \frac{N}{h_1} \right) dx_1 \ldots \left( \frac{N}{h_k} \right) dx_k = N^{a+2k} \int_{\Delta^k} x_1^{a_1+1} \ldots x_k^{a_k+1} dx_1 \ldots dx_k \cdot \sum_{h \in \mathbb{N}^k} \frac{1}{h_1^{a_1+2} \cdots h_k^{a_k+2}} .
\]
It remains to note that
\[
\int_{\Delta_k} x_1^{a_1+1} \cdots x_k^{a_k+1} \, dx_1 \cdots dx_k = \frac{(a_1 + 1)! \cdots (a_k + 1)!}{(a + 2k)!}.
\]

The calculations of the local polynomials allow us to obtain an expression for the number of connected pillowcase covers of degree at most \(N\) having the ramification points only over the corners of the pillowcase and having the following ramification profile (indicating the total number of ramification points over the four corners of the pillowcase together). The cover has exactly \(K\) ramification points of degree 3, \(K + 4\) nonramified points; all remaining points over the corners of the pillowcase have degree 2. Imposing this ramification profile and connectedness of \(\hat{P}\) is equivalent to requiring that \(\hat{P} \in \mathbb{Q}(1^K, -1^{K+4})\).

As explained in §3.2, see also Figure 2, every such pillowcase cover defines a “global tree” \(T\) which edges correspond to cylinders filled with horizontal periodic trajectories, and whose vertices correspond to “singular layers”. We stress that a global tree represents only the adjacency of the cylinders to the same singular layers, and have almost nothing in common with the ribbon graphs considered in §3.2 numerous ribbon graphs might be hidden behind a vertex of the global tree.

Let some horizontal singular layer \(v\) contain \(m_v\) zeroes and \(n_v\) simple poles. The valence \(l_v\) of the vertex of the global tree \(T\) represents the number of cylinders adjacent to the corresponding layer. In other words, it stands for the number of boundary components of the ribbon graph corresponding to the layer (“faces” in terminology of §3.2). We have seen in §3.2 that the valence \(l_v\) and the degree \(2a_v := \deg F_{m_v, n_v}\) of the corresponding local polynomial are related to \(m_v\) and \(n_v\) as

\[
\begin{align*}
  a_v &= \frac{m_v + n_v}{2} - 1 \\
  l_v &= \frac{m_v - n_v}{2} + 2
\end{align*}
\]

Since the number \(n_v\) is nonnegative, the degree \(2a_v\) and the valence \(l_v\) satisfy the relation

\[a_v \geq l_v - 3 \quad \text{for any vertex } v \in T.\]

Also, since the total number of zeroes and poles is \(2K + 4\), summing up the expression for \(a_v\) over all vertices of the tree \(T\), we get

\[
\sum_{v \in T} a_v = K + 2 - |V(T)|,
\]

where \(|V(T)|\) denotes the number of vertices in \(T\).

Reciprocally, given any connected tree \(T\) with at least two vertices, and any integer \(K\) satisfying

\[|V(T)| \leq K + 2,\]

consider any partition of the number \(K + 2 - |V|\) into nonnegative integers

\[a = a_{v_1} + \cdots + a_{v_{|V|}},\]

where elements \(a_v\) of the partition are enumerated by the vertices of the tree \(T\). If for every \(v\) in \(T\) the inequality (3.20) holds, equations (3.19) uniquely determine for
every vertex \( v \) a couple of nonnegative integers \( n_v, m_v \) which are not simultaneously equal to zero. By construction,

\[
\sum_{v \in T} m_v = K \quad \text{and} \quad \sum_{v \in T} n_v = K + 4.
\]

**Definition 3.8.** Given an integer \( K \in \mathbb{N} \) and a tree \( T \) with at least two and at most \( K + 2 \) vertices, by *decoration* of the tree \( T \) we call a partition \( a = a_{v_1} + \cdots + a_{v_{|V|}} \), enumerated by the vertices of the tree and satisfying relations (3.20) and (3.21).

We have just proved the following Lemma.

**Lemma 3.9.** A global tree of any pillowcase cover in \( Q(1^K, -1^{K+4}) \) is naturally decorated in the sense of Definition 3.8. Any decorated tree satisfying (3.21) corresponds to some actual pillowcase cover in \( Q(1^K, -1^{K+4}) \).

Now we are ready to count the number of pillowcase covers in \( Q(1^K, -1^{K+4}) \) of degree at most \( N \) represented by a given decorated tree \((T, a)\). Let \( V \) be the set of vertices of the tree \( T \), let \( E \) be the set of edges of \( T \). Since \( T \) is a tree we have \( |E| = |V| - 1 \). We always assume that the labellings of vertices and edges, that is, the bijections \( V \to \{1, \ldots, |V|\} \) and \( E \to \{1, \ldots, |E|\} \) are fixed.

Recall that to each edge \( e_j \) of \( T \) we associate a pair of variables \( h_j \) and \( w_j \) which represent the height of the corresponding cylinder and its width (length of the waist curve). The decoration associates a pair of nonnegative integers \( m_i, n_i \) to each vertex \( v_i \) of the tree; \( m_i, n_i \) are not simultaneously equal to zero. We associate to every vertex \( v \) the local polynomial \( F_{m_i, n_i}(w_{j_1}, \ldots, w_{j_{l(v)}}) \) where \( l(v) \) is the valence of the vertex \( v = v_i \), and indices \( \{j_1, \ldots, j_{l(v)}\} \) enumerate the edges \( e_{j_1}, \ldots, e_{j_{l(v)}} \) adjacent to \( v_i \).

Let \( |\text{Aut}(T, a)| \) be the cardinality of the automorphism group of the decorated tree \((T, a)\), and let \( k := |E| \) be the number of the edges of the tree \( T \). The number of ways to give names to \( m_i \) zeroes and to \( n_i \) poles at the layer \( v_i \), where \( i = 1, \ldots, k + 1 \), equals

\[
\frac{1}{|\text{Aut}(T, a)|} \binom{m}{m_1, \ldots, m_{k+1}} \binom{n}{n_1, \ldots, n_{k+1}}, \quad k = |E|.
\]

Hence, by Lemma 3.2, the number of pillowcase covers of degree at most \( N \) corresponding to the decorated tree \((T, a)\) is equal to the following sum (3.23)

\[
\sum_{h \in \mathbb{N}^k, \ w \in \mathbb{N}^k \atop h \cdot w \leq N} \frac{1}{|\text{Aut}(T, a)|} \binom{m}{m_1, \ldots, m_{k+1}} \binom{n}{n_1, \ldots, n_{k+1}} (2w_1) \cdots (2w_k) \prod_{i=1}^k F_{m_i, n_i},
\]

where the arguments of \( F_{m_i, n_i}(w_{j_1}, \ldots, w_{j_{l(v)}}) \) correspond to edges \( e_{j_1}, \ldots, e_{j_{l(v)}} \) adjacent to the vertex \( v_i \). Note that the definition of the decoration, and the construction of the local polynomials \( F_{m_i, n_i} \) implies that any monomial in \( w_1, \ldots, w_k \) of the above sum has total degree equal to \( \dim_{\mathbb{C}} Q(1^K, -1^{K+4}) = 2k + 2 \).

Define the formal operation

\[
\mathcal{Z} : \prod_{i=1}^k w_i^{b_i+1} \mapsto \frac{2}{(b + 2k - 1)!} \prod_{i=1}^k ((b_i + 1)! \cdot \zeta(b_i + 2))\],
\]

where \( b = \sum b_i \). For \( b + 2k = \dim_{\mathbb{C}} Q(1^K, -1^{K+4}) \) this operation corresponds to the following sequence of operations. We first apply Lemma 3.7 to the sum
\[ \sum_{h,w \leq N} \prod_{i=1}^{k} w_i^{b_i+1} \] to obtain \[ \frac{N^{b+2k}}{(b+2k)!} \prod_{i=1}^{k} (b_i + 1)! \zeta(b_i + 2). \] Then, following Lemma 3.2 we divide the resulting sum by \( N^{\dim C Q} \) and multiply the result by \( 2^{\dim C Q} \).

Summing up the contributions (3.23) of individual decorated trees, applying Lemmas 3.2 and 3.7, and using the notation \( Z \) we obtain \( \text{Vol} \ Q(1^K, -1^{K-4}) \). On the other hand, by Theorem for the volumes \( \text{Vol} \ Q(1^K, -1^{K+4}) \) stated at the end of \( \S 1.1 \)

\[ \text{Vol} \ Q_1 (1^K, -1^{K+4}) = \frac{\pi^{2K+2}}{2K-1}. \]

Comparing the two expressions for the volume, we obtain the following identity

**Theorem 3.10.** For any \( K \in \mathbb{N} \) the following identity holds:

\[ (3.24) \quad \frac{\pi^{2K+2}}{2K-1} = \sum_{k=1}^{K+1} \sum_{\text{Connected trees with } k} \sum_{\text{Admissible decorations } a} \frac{2^k}{|\text{Aut}(T,a)|} \cdot \binom{m}{m_1, \ldots, m_{k+1}} \cdot \binom{n}{n_1, \ldots, n_{k+1}} \cdot Z \left( w_1 \cdots w_k \prod_{i=1}^{k} F_{m_i, n_i} \right). \]

Below, we illustrate the volume calculations for the strata \( Q_1 (1^1, -1^5) \) and \( Q_1 (1^2, -1^6) \).

### 3.7. Stratum \( Q(1^1, -1^5) \). Let

\[ c(T, a) := \frac{1}{|\text{Aut}(T,a)|} \cdot \binom{m}{m_1, \ldots, m_{k+1}} \cdot \binom{n}{n_1, \ldots, n_{k+1}}. \]

For \( K = 1 \) there are only two trees with at least 2 and at most \( K + 2 \) vertices. Each of these trees admits a unique decoration. Thus, the sum in the right-hand side of (3.24) contains only two summands described in the table below.
Adding the two terms we get the following value for the volume (recall all zeroes and poles are numbered):

$$\text{Vol} \, Q_1(1, -1^5) = \pi^4$$

Up to the factor $120 = 5!$ coming from enumeration of simple poles it matches the value

$$\text{Vol} \, H_1(2) = \frac{\pi^4}{120}$$

from [EMZ03], where $H_1(2)$ denotes the stratum of unit-area Abelian differentials with one double zero on a genus 2 surface. The two values agree since we have the isomorphism $H(2) \cong Q(1, -1^5)$ by taking the quotient of each surface in $H(2)$ by the hyperelliptic involution.

### 3.8. Stratum $Q(1^2, -1^6)$

For $K = 2$ the tree can contain from one to three edges; corresponding decorated trees and their contributions to the right-hand side of (3.24) are presented in the table below.
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Tree} & \prod_{i=1}^{k} F_{m_i, n_i} & c(T, a) & \text{Contribution} \\
\hline
\hline
\hline
k = 1 cylinder & & & \\
\hline
\hline
\circ 2,4 \hspace{1cm} 0,2 & F_{2,4}(w_1) \cdot F_{0,2}(w_1) = \frac{1}{2} \cdot \binom{2}{2} \binom{2}{2} & \frac{60}{63} \cdot \zeta(6) = & \frac{4}{63} \cdot \pi^6 \\
\hline
\circ 1,3 \hspace{1cm} 0,2 \\
\hline
\circ 1,3 \hspace{1cm} 0,3 & F_{1,3}(w_1) \cdot F_{1,3}(w_1) = \frac{1}{2} \cdot \binom{2}{1} \binom{2}{3} & 80 \cdot \zeta(6) = & \frac{16}{189} \cdot \pi^6 \\
\hline
\hline
\text{Subtotal:} & & & \frac{4}{27} \cdot \pi^6 \\
\hline
\hline
k = 2 cylinders & & & \\
\hline
\hline
\circ 0,2 \hspace{1cm} 0,2 \\
\hline
\circ 2,2 \hspace{1cm} 0,2 & F_{0,2}(w_1) \cdot F_{2,2}(w_1, w_2) \cdot F_{0,2}(w_2) = \frac{1}{2} \cdot \binom{2}{0} \binom{2}{2} \binom{2}{2} & 72 \cdot \zeta(2) \cdot \zeta(4) = & \frac{2}{15} \cdot \pi^6 \\
\hline
\circ 1,3 \hspace{1cm} 0,2 \\
\hline
\circ 1,3 \hspace{1cm} 0,2 & F_{1,3}(w_1) \cdot F_{1,1}(w_1, w_2) \cdot F_{0,2}(w_2) = \frac{1}{2} \cdot \binom{2}{1} \binom{2}{1} \binom{2}{2} & 48 \cdot \zeta(2) \cdot \zeta(4) = & \frac{4}{45} \cdot \pi^6 \\
\hline
\hline
\text{Subtotal:} & & & \frac{2}{9} \cdot \pi^6 \\
\hline
\end{array}
\]
### Tree

| $k$ cylinders | $\prod_{i=1}^{k} F_{m_i,n_i} \cdot c(T,a)$ | Contribution |
|--------------|---------------------------------|--------------|
| $\frac{1}{2} \cdot \left( \frac{2}{6} \right) (2,0,1,1,0)$ | $24 \cdot \zeta^3(2) = \frac{1}{9} \cdot \pi^6$ |
| $\frac{1}{4} \cdot \left( \frac{6}{6} \right) (2,2,0,0,2)$ | $4 \cdot \zeta^3(2) = \frac{1}{54} \cdot \pi^6$ |

Subtotal: $\frac{7}{54} \cdot \pi^6$

Taking the total sum we get $\text{Vol} \mathcal{Q}(1^2, -1^6) = \left( \frac{4}{27} + \frac{2}{9} + \frac{7}{54} \right) \pi^6 = \frac{\pi^6}{2}$.

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