Effects of a magnetic field on the one-dimensional spin-orbital model

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Abstract

We study the effects of a uniform magnetic field on the one-dimensional spin-orbital model in terms of effective field theories. Two regions are examined: one around the SU(4) point \( J=K/4 \) and the other with \( K \ll J \) [see Eq. (1)]. We found that when \( J \leq K/4 \), the spin and orbital correlation functions exhibit power-law decay with non-universal exponents. In the region with \( J > K/4 \), the excitation spectrum has a gap. When the magnetic field is beyond some critical value, a quantum phase transition occurs. However, the correlation functions around the SU(4) point and the region with \( K \ll J \) exhibit distinct behavior. This results from different structures of excitation spectra in both regime.
I. INTRODUCTION

The interest in studying the role of orbital degrees of freedom stems from the understanding of the magnetic structures of transition metal compounds [1,2]. In these systems, the low-lying electron states have orbital degeneracy as well as the usual spin degeneracy. This may result in interesting magnetic properties of the Mott insulating phase. For example, the magnetic ordering is influenced by the orbital structure which may change under the pressure or the magnetization is a nonlinear function of the magnetic field even in the case of an isotropic exchange interaction of the type $\vec{S}_i \cdot \vec{S}_j$ [1]. These distinguish the spin-orbital models from the ordinary Heisenberg model with the spin degrees of freedom only. Thus, to understand the magnetic properties of these compounds, an investigation of the interplay between spin and orbital fluctuations is necessary.

As a prototypical model in which the quantum fluctuations of the orbital degrees of freedom are important, we would like to consider the following Hamiltonian:

$$H = \sum_i [K(\vec{S}_i \cdot \vec{S}_{i+1})(\vec{T}_i \cdot \vec{T}_{i+1}) + J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{T}_i \cdot \vec{T}_{i+1} - hS_z^i],$$

where $\vec{T}_i$ is the pseudospin to represent the orbital degrees of freedom. We assume that $K$ and $h$ are positive. The Hamiltonian [4] is related to the recently discovered spin-gapped materials, Na$_2$Ti$_2$Sb$_2$O [3] and Na$_2$V$_2$O$_5$ [4]. These materials have a quasi-1D structure and are modeled by the quater-filled two-band Hubbard model which is equivalent to Eq. (1) ($h = 0$) in the strong Coulomb repulsion limit. Instead of a magnetic field, we can also impose a uniaxial pressure $P$ which gives rise to a term of the type $APT_i^z$ in the Hamiltonian. In other words, a uniaxial pressure affects the orbital sector in much the same way that a magnetic field affects the spin sector. Therefore, our results also apply to this case where the roles of spin and orbital operators interchange.

The Hamiltonian [4] with $h = 0$ is invariant under independent SU(2) rotations in the spin and orbital spaces. When $J_1 = J_2 = J$, it is also invariant under the exchange between $\vec{S}$ and $\vec{T}$. We restrict our consideration to this more symmetric case and comment on the...
asymmetric case in the conclusion. The phase diagram has been studied in Ref. [7,12] and the properties of the ground state depend on the ratio $J/K$. When $J > K/4$, the ground state is doubly degenerate with alternating spin and orbital singlets. The excitation spectrum has a finite gap. In the region with $-K/4 < J \leq K/4$, it is a critical theory. The point with $J = K/4$ is special because the symmetry is enlarged to SU(4) and it is Bethe-ansatz solvable [10,14]. There are three gapless bosonic modes with the same velocity. This implies that the central charge $c = 3$ and as shown by Affleck [15], the fixed point Hamiltonian is described by the SU(4) Wess-Zumino-Novikov-Witten (WZNW) model. The predictions of the conformal field theory have been confirmed by numerical work [6,8]. Away from the SU(4) point, the low energy excitations of the Hamiltonian [1] are described by different effective theories at different range of $J/K$. As shown in Ref. [9], an appropriate starting point at weak coupling regime ($J \gg K$) is two decoupled Heisenberg chains [16] while near the SU(4) point, we have to begin with the O(6) Gross-Neveu (GN) model.

In the present paper, we study Eq. (1) on the symmetric line with $h \neq 0$. The case with $J = K/4$ has been done in Ref. [11] with Bethe ansatz and numerical methods. Although the magnetic field is not directly coupled to the orbital degrees of freedom, it still affects the orbital structure. In one dimension, this is reflected on the change of the orbital correlation functions. For $J \leq K/4$, the excitations are gapless at any finite $h$ and the leading behavior of the correlation functions is the same as that at $J = K/4$. The orbital correlators is hardly influenced by the magnetic field except the $2k_F$ component. Not only the characteristic momentum becomes incommensurate but the corresponding exponent is nonuniversal and dependent of the magnetization now. Especially, it shows a discontinuous jump compared with that at vanishing magnetic field. This is because a marginally irrelevant operator at $h = 0$ becomes a marginal one under the magnetic field.

For a gapped spin liquid, there is a quantum phase transition induced by the magnetic field. When $h > h_c$ (the critical field), the magnetization $M \neq 0$ whereas $M = 0$ for $h < h_c$. The corresponding quantum critical point (QCP) is determined by the gap of the $S^z = \pm 1$ components. When the magnetic field exceeds this gap, the corresponding
excitations become gapless due to the condensation of these \( S^z = \pm 1 \) bosons. For \( J > K/4 \) but still near the \( SU(4) \) point, the excitations with \( S^z = 0 \) are still gapped and we conjecture that they are described by the \( O(4) \) GN model. This results in a change on the spectrum when \( M \neq 0 \). For \( h > h_c \), the low energy excitations with \( S^z = 0 \) are kinks while for \( h < h_c \), they become massive fermions which can be considered as the bound states of kinks. In this case, the orbital correlators are exponentially decaying functions with or without algebraically decaying prefactors while the spin correlators show algebraic decay. However, when it comes to the region with \( K \ll J \), the behavior changes though the ground states in both regions have similar properties. Now the quantum phase transition occurs at the value of twice the gap instead of the gap. In addition, both types of correlation functions exhibit power-law decay with universal exponents. Indeed, as suggested in Ref.\(^{[9,12]}\), the excitation spectrum at \( K/4 < J < K/2 \) is different from that at \( J \geq K/2 \). In the former case, the structure factor has an incoherent background with the top at \( q = \pi \) and a coherent magnon peak at \( q = \pi/2 \). The relative amplitude of the peak at \( q = \pi/2 \) with respect to the incoherent background at \( q = \pi \) decreases with increasing \( J/K \) and vanishes at \( J = K/2 \). As for the latter, there is only an incoherent background at \( q = \pi \).\(^{[17]}\) Our analysis indicates that both regions have distinct responses to the magnetic field (or uniaxial pressure).

II. THE EFFECTIVE HAMILTONIAN AROUND THE \( SU(4) \) POINT

We start by a brief review of the derivation of the low energy effective Hamiltonian to fix our notation. Following Ref.\(^{[3]}\), the low energy effective Hamiltonian of Eq. (1) \((h = 0)\) at \( J = K/4 \) can be derived by considering the following repulsive \( SU(4) \) Hubbard model \((U > 0)\) at quarter-filling:

\[
H_U = -t \sum_{i\sigma} (c_{i+1\sigma}^+ c_{i\sigma} + \text{H.c.}) \\
+ \frac{U}{2} \sum_{iab\sigma\sigma'} n_{i\sigma\sigma} n_{i\sigma\sigma'} (1 - \delta_{ab}\delta_{\sigma\sigma'}).
\]

(2)
Here $c_{ia\sigma}^+$ creates an electron with the orbital index $a = 1, 2$ and spin $\sigma = \uparrow, \downarrow$ and $n_{ia\sigma} = c_{ia\sigma}^+ c_{ia\sigma}$. The quarter-filling electron band implies that the Fermi momentum $k_F = \frac{\pi}{4a_0}$ where $a_0$ is the lattice spacing.

The low energy physics can be described by the right-moving ($R_{a\sigma}$) and left-moving ($L_{a\sigma}$) fermions, which is related to the original lattice electron operator as: $c_{ia\sigma}/\sqrt{a_0} = R_{a\sigma}(x) \exp (ik_F x) + L_{a\sigma}(x) \exp (-ik_F x)$ where $x = ia_0$. We introduce bosonic fields $\phi_{R(L)\alpha\sigma}$, which satisfy the commutation relation $[\phi_{L\alpha\sigma}, \phi_{R\alpha\sigma'}] = -\frac{i}{4} \delta_{\alpha\alpha} \delta_{\sigma\sigma'}$, to bosonize the right and left movers as: $R(L)_{a\sigma} = (2\pi a_0)^{-1/2} \eta_{a\sigma} \exp (\pm i\sqrt{4\pi} \phi_{R(L)\alpha\sigma})$ where $\eta_{a\sigma}$ are Klein factors. It is more convenient to employ the following basis:

$$
\begin{align*}
\Phi_c &= (\phi_{1\uparrow} + \phi_{1\downarrow} + \phi_{2\uparrow} + \phi_{2\downarrow})/2, \\
\Phi_s &= (\phi_{1\uparrow} - \phi_{1\downarrow} + \phi_{2\uparrow} - \phi_{2\downarrow})/2, \\
\Phi_f &= (\phi_{1\uparrow} + \phi_{1\downarrow} - \phi_{2\uparrow} - \phi_{2\downarrow})/2, \\
\Phi_{sf} &= (\phi_{1\uparrow} - \phi_{1\downarrow} - \phi_{2\uparrow} + \phi_{2\downarrow})/2.
\end{align*}
$$

In this new basis, $\Phi_c$ represents the total charge degree of freedom while other bosonic fields $\Phi_a (a = s, f, sf)$ correspond to the spin-orbital degrees of freedom.

Substitution of the above representations into Eq. (2) gives a Hamiltonian which consists of decoupled charge and spin-orbital degrees of freedom. The charge sector is described by a Gaussian model of $\Phi_c$ perturbed by an Umklapp process: $\cos (\sqrt{16\pi/K_c \Phi_c})$ where $K_c$ is an increasing function of $U$ at weak coupling. As shown in Ref. [18], there exists a critical value $U_c (~2.8)$ such that $K_c = 2$ where a Mott transition occurs and the system becomes an insulator when $U > U_c$. In the following, we assume that we stay in the insulating phase and focus on the spin-orbital sector. By introducing six Majorana fermions:

$$
\begin{align*}
(\xi^1 + i\xi^2)_{R(L)} &= \frac{\eta_1}{\sqrt{\pi a_0}} \exp (\pm i\sqrt{4\pi} \Phi_{sR(L)}), \\
(\xi^3 + i\xi^4)_{R(L)} &= \frac{\eta_2}{\sqrt{\pi a_0}} \exp (\pm i\sqrt{4\pi} \Phi_{fR(L)}), \\
(\xi^5 + i\xi^6)_{R(L)} &= \frac{\eta_3}{\sqrt{\pi a_0}} \exp (\pm i\sqrt{4\pi} \Phi_{sfR(L)}),
\end{align*}
$$

(4)
where $\eta_a$ ($a = 1, 2, 3$) are Klein factors \[19\]. The corresponding Hamiltonian becomes

$$
H = -\frac{i}{2} v_f \sum_{a=1}^{6} (\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L) + G_3 (\sum_{a=1}^{6} \xi^a_R \xi^a_L)^2,
$$

where $G_3 = -Ua_0/2$. This is an O(6) GN model with marginally irrelevant 4-fermion interaction due to $G_3 < 0$. Therefore, the low energy behavior of the SU(4) spin-orbital model is described by the SO(6)$_1$ (SU(4)$_1$) WZNW model.

Away from the SU(4) point, i.e. $J \neq K/4$, there is an additional term in the Hamiltonian:

$$
\Delta H = \frac{G}{\pi} (\partial_x \Phi_s)^2 + G (\sum_{a=3}^{5} \xi^a_R \xi^a_L)^2 - \frac{2iG}{\pi a_0} \xi^6_R \xi^6_L \cos (\sqrt{4\pi} \Phi_s),
$$

where $G = c(J - K/4)$ and $c$ is a positive constant. With the above, the low energy effective Hamiltonian of Eq. (1) with $J_1 = J_2$ becomes

$$
H = H_s + H_{so} + H_{int},
$$

where

$$
H_s = \frac{v^0_s}{2} [\left( \partial_x \Theta_s \right)^2 + (\partial_x \Phi_s)^2] + \frac{g_1}{\pi} (\partial_x \Phi_s)^2 - \frac{h}{\sqrt{\pi}} \partial_x \Phi_s,
$$

$$
H_{so} = -\frac{i}{2} v_f \sum_{a=3}^{6} (\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L) + g_1 (\sum_{a=3}^{5} \xi^a_R \xi^a_L)^2 + g_2 \xi^6_R \xi^6_L (\sum_{a=3}^{5} \xi^a_R \xi^a_L),
$$

$$
H_{int} = -\frac{i}{\pi a_0} \cos (\sqrt{4\pi} \Phi_s) (g_2 \sum_{a=3}^{5} \xi^a_R \xi^a_L + 2g_1 \xi^6_R \xi^6_L),
$$

with $g_1 = G_3 + G$ and $g_2 = 2G_3$. Here, the velocity of $\Phi_s$ is represented by a different notation. Because under the magnetic field, the spin SU(2) rotational symmetry and the exchange symmetry are broken. We do not expect that all fields have the same velocity. Note that only $\Phi_s$ is directly coupled to $h$. Physically, this reflects the fact that $\Phi_s$ represents the $S^z = \pm 1$ components of magnons while other fields have $S^z = 0$. Eq. (7) is our working Hamiltonian.

Let us consider $H_{so}$ first. The one-loop RG equations of coupling constants are

$$
\frac{dg_\pm}{dl} = \frac{g_\pm^2}{\pi},
$$

where \( g_\pm = g_3 \).
where \( g_{\pm} = g_1 \pm g_2/2 \) and \( l \) is the logarithm of the length scale. The parameter space is divided into three regions where the boundaries are the same as those in the case without the magnetic field [3]: (i) Region I \((G \leq 0)\). Both \( g_{\pm} \) flow to zero couplings. (ii) Region II \((0 < G < -2G_3)\). \( g_+ \) flows to zero coupling while \( g_- \) flows to strong coupling. (iii) Region III \((G > -2G_3)\). Both flow to strong couplings. Based on the RG flow, we discuss the properties of the Hamiltonian (7) in the following sections.

**III. CORRELATION FUNCTIONS AROUND THE SU(4) POINT**

We now discuss the spectrum of the Hamiltonian (7) and the behavior of correlation functions around the SU(4) point.

**A. \( J \leq K/4 \)**

Eq. (8) tells us that \( H_{so} \) describes four free massless Majorana fermions at low energy when \( J \leq K/4 \). Integrating out the high energy modes of these fermions generates two kinds of interactions through \( H_{int} \): \((\partial_x \Phi_s)^2 \) and \( \cos (\sqrt{16\pi l} \Phi_s) \) where \( l \) is an integer. (It also renormalizes \( v_s^0 \).) The former can be absorbed into the \( g_1 \) term of \( H_s \). In addition, it consists of higher powers of \( g_1 \) and \( g_2 \) and we do not expect this will change the sign of \( g_1 \). The latter (the cosine term) is associated with the lattice translation symmetry of the original lattice model: \( \Phi_s \rightarrow \Phi_s + \sqrt{\pi}/2 \) and \( \xi_R^a \xi_L^a \rightarrow -\xi_R^a \xi_L^a \ (a = 3, \cdots, 6) \). They are irrelevant operators. Therefore, to calculate the long distance behavior of correlation functions, we can treat \( H_s \) and \( H_{so} \) as independent sectors and throw away \( H_{int} \).

Keeping the above approximation in mind, \( H_s \) can be diagonalized as the following:

\[
H_s = \frac{v_s}{2} \left[ \frac{1}{\alpha} (\partial_x \Theta_s)^2 + \alpha (\partial_x \Phi_s)^2 \right],
\]

where \( \alpha = \sqrt{1 + \frac{2g_1}{\pi v_s^0}} \) and \( v_s = \alpha v_s^0 \). We also translate \( \Phi_s \) as: \( \Phi_s \rightarrow \Phi_s + \frac{\hbar}{\sqrt{\pi v_s \alpha}} x \). With the definition of the magnetization \( M = 2S_{tot}^z/N \ (0 \leq M \leq 1) \) where \( N \) is the number of lattice points, we obtain \( M = \frac{2\hbar v_s^0}{\pi v_s \alpha} \). (This relation is valid when \( M \ll 1 \).) In general, we expect \( \alpha \)
and $v_s$ depend on the magnetization or magnetic field. Moreover, $\alpha < 1$ because $g_1 < 0$ in this case.

Now we can calculate the spin and orbital correlation functions. The spin and orbital pseudospin operators are, respectively, defined as: $\vec{S}_i = \frac{1}{2} \sum_a c^+_ia \vec{\sigma} c_i$ and $\vec{T}_i = \frac{1}{2} \sum_\sigma c^+_ia \vec{\tau} c_\sigma$, where $\vec{\sigma}$ and $\vec{\tau}$ are Pauli matrices in the spin and orbital spaces, respectively. Near the SU(4) point, they can be expressed by the WZNW fields as: $\vec{S}_i/a_0 = \vec{J}_s(x) + (e^{2ikFx} \vec{N}_s(x) + \text{H.c.}) + (-1)^{\vec{m}} \vec{r}_s(x)$ and $\vec{T}_i/a_0 = \vec{J}_t(x) + (e^{2ikFx} \vec{N}_t(x) + \text{H.c.}) + (-1)^{\vec{m}} \vec{r}_t(x)$ where $x = ia_0$.

We leave the detail expressions of these WZNW fields in Appendix A. With the help of Eq. ([A1]) and shifting $\Phi_+ \rightarrow \Phi_+ + \frac{\sqrt{\pi} M}{2a_0} x$, the correlation functions in the presence of a weak magnetic field can be shown as follows:

$$\langle T^x_i(\tau) T^y_j(0) \rangle = \langle T^y_i(\tau) T^y_j(0) \rangle = \langle T^z_i(\tau) T^z_j(0) \rangle$$

$$= \frac{a_0^2}{4\pi^2} \left[ \frac{1}{(v_s \tau + i x)^2} + \frac{1}{(v_s \tau - i x)^2} \right] + A_1 \frac{\cos \left[ \frac{\pi}{2a_0} (1 + M) x \right] + \cos \left[ \frac{\pi}{2a_0} (1 - M) x \right]}{(v_s^2 \tau^2 + x^2)^{1/4\alpha} (v_s^2 \tau^2 + x^2)^{1/2}}$$

$$+ (-1)^{\vec{m}} \frac{A_2}{v_s \tau^2 + x^2},$$

$$\langle S^y_i(\tau) S^y_j(0) \rangle = \frac{M^2}{4} + \frac{a_0^2}{4\pi^2 \alpha} \left[ \frac{1}{(v_s \tau + i x)^2} + \frac{1}{(v_s \tau - i x)^2} \right] + A_1 \frac{\cos \left[ \frac{\pi}{2a_0} (1 + M) x \right] + \cos \left[ \frac{\pi}{2a_0} (1 - M) x \right]}{(v_s^2 \tau^2 + x^2)^{1/4\alpha} (v_s^2 \tau^2 + x^2)^{1/2}}$$

$$+ A_3 \cos \left[ \frac{\pi}{2a_0} (1 - M) x \right] (v_s^2 \tau^2 + x^2)^{1/\alpha},$$

$$\langle S^y_i(\tau) S^y_j(0) \rangle = \frac{B_1}{(v_s^2 \tau^2 + x^2)^{\gamma_1/2}} \left[ e^{i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau - i x}{v_s \tau + i x} \right) + e^{-i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau + i x}{v_s \tau - i x} \right) \right]$$

$$+ B_2 \cos \left( \frac{\pi}{2a_0} \right) \left[ e^{i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau - i x}{v_s \tau + i x} \right) + e^{-i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau + i x}{v_s \tau - i x} \right) \right] + (-1)^{\vec{m}} \frac{B_3}{(v_s^2 \tau^2 + x^2)^{\gamma_1/2}} \left[ e^{i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau - i x}{v_s \tau + i x} \right) + e^{-i \frac{\pi M}{2a_0} x} \left( \frac{v_s \tau + i x}{v_s \tau - i x} \right) \right],$$

where $\tau$ is the imaginary time and $x = a_0 |i - j|$. $\gamma_1 = 1 + \frac{\alpha}{2} + \frac{1}{2\gamma} > 2$. $A_i$ and $B_i$ ($i = 1, 2, 3$) are nonuniversal constants.

The various characteristic momenta in Eq. (10) can be understood as the combination of
the shifted Fermi momenta under the magnetic field: $k_{F\uparrow} = \frac{\pi}{4a_0}(1+M)$ and $k_{F\downarrow} = \frac{\pi}{4a_0}(1-M)$. The structures of orbital (spin) correlation functions are not affected by the magnetic field (uniaxial pressure) except the $2k_F$ component. The original degenerate $2k_F$-excitations now split into two incommensurate soft modes at $2k_{F\uparrow} = \frac{\pi}{2a_0}(1+M)$ and $2k_{F\downarrow} = \frac{\pi}{2a_0}(1-M)$. Moreover, the corresponding exponent $\gamma_{2k_F} = 1 + \frac{1}{\sqrt{2}} > 1.5$ even at $M = 0^+$ and $J = K/4$. ($\gamma_{2k_F} = 1.5$ when $M = 0$ and $J = K/4$.)

This is because the marginally irrelevant coupling $G_3$ in Eq. (5) now turns into a marginal one and thus changes the compactification radius of $\Phi_s$ as shown in Eq. (9). The $4k_F$ mode does not appear in $\langle T^\alpha_i(\tau)T^\beta_j(0) \rangle$. (Note that $4k_F\downarrow \equiv 4k_F\uparrow$ (mod. $2\pi$) and thus the $4k_F\uparrow$ mode is not independent.) This means that it does not carry nontrivial orbital quantum numbers. The corresponding exponent $\gamma_{4k_F} = 2/\alpha = 4(\gamma_{2k_F} - 1)$, which is consistent with the prediction of conformal field theory [11].

The only enhanced fluctuation is the $2k_F$ component of $\langle S^+_i(\tau)S^-_j(0) \rangle$ (the $B_2$ term) of which the exponent becomes $1 + \alpha/2 < 1.5$. Another interesting observation is that the position of the cusp at $q = \pi/2$ in the static transverse spin structure factor does not shift under the magnetic field.

### B. $J > K/4$

When $G > 0$, there are two massive regions. As suggested in Ref. [9], the Hamiltonian (4) with $J > K/4$ falls into the region with $0 < G < -2G_3$ in the absence of the magnetic field. We expect that under a weak field it is still the case which corresponds to the region II. In this region, because $g_+$ is marginally irrelevant and the O(4) symmetry is restored as $g_+ = 0$ ($G = -2G_3$), we conjecture that up to logarithmic corrections, $H_{so}$ is equivalent to the O(4) GN model at strong coupling regime. Next, we examine the effect of $H_{int}$ on $H_s$. Integrating out $\xi^a$ ($a = 3, \cdots, 6$) gives the interaction term $-\frac{m}{\pi a_0} \cos (\sqrt{4\pi}\Phi_s)$ where $m = i(g_2\langle \sum_{a=3}^5 \xi^a_R \xi^a_L \rangle + 2g_1\langle \xi^6_R \xi^6_L \rangle)$. (The term $(\partial_x \Phi_s)^2$ is possibly generated. But its effect can be absorbed into the $g_1$ term in $H_{so}$.) By replacing $H_s$ with the following one:

$$\tilde{H}_s = \int dx \left\{ \frac{v_s}{2} [(\partial_x \Theta_s)^2 + (\partial_x \Phi_s)^2] - \frac{m}{2\pi a_0} \cos (\sqrt{4\pi}\Phi_s) + \frac{g_1}{\pi} (\partial_x \Phi_s)^2 - \frac{h}{\sqrt{\pi}} \partial_x \Phi_s \right\}, \quad (11)$$
where $v_s$ has been substituted into $v_s^0$ to incorporate its renormalization effect, we assume that the low energy and long distance behavior of the Hamiltonian (11) with $J > K/4$ and $h > h_c$ is approximately described by $H_{so}$ and $\tilde{H}_s$.

By introducing a Dirac fermion $\psi_{R(L)}$, $\tilde{H}_s$ can be fermionized as:

$$\tilde{H}_s = \int dx \left[ -i v_s (\psi_R^+ \partial_x \psi_R - \psi_L^+ \partial_x \psi_L) - im (\psi_R^+ \psi_L - \psi_L^+ \psi_R) - h (\psi_R^+ \psi_R + \psi_L^+ \psi_L) \right] + \frac{g_1}{\pi} (\psi_R^+ \psi_R + \psi_L^+ \psi_L)^2. \tag{12}$$

We see that $h$ is equivalent to the chemical potential of fermions and $h = h_c = |m|$ is a quantum critical point. We are concerned with the case where $h > h_c$. In this case, the low energy excitations of the $\Phi_s$ sector are described by the following effective Hamiltonian [21]:

$$\tilde{H} = \int dx \frac{\tilde{v}_s}{2} \left[ \frac{1}{g} (\partial_x \tilde{\phi})^2 + g (\partial_x \tilde{\phi})^2 \right], \tag{13}$$

where $\tilde{\phi}$ is obtained via shifting $\Phi_s$ by an amount $\sqrt{\pi M} a_0 x$. $g$ is a parameter determined by $g_1$ and $g < 1$ ($g > 1$) when $g_1 < 0$ ($g_1 > 0$). The scattering ($g_1$) term has only negligible effects in the limit $M \to 0$ where we will get free fermions with $g \to 1$.

Based on our assumption that $H_{so}$ is equivalent to an O(4) GN model in the strong coupling regime, we replace $H_{so}$ with the following one:

$$H_{so} = -\frac{i}{2} \frac{\tilde{v}_s}{v_f} \sum_{a=3}^{6} (\xi_R^a \partial_x \xi_L^a - \xi_L^a \partial_x \xi_R^a) + \tilde{g} (\sum_{a=3}^{6} \xi_R^a \xi_L^a)^2, \tag{14}$$

where $\tilde{g} > 0$. We have performed the duality transformation: $\xi_R^6 \to \xi_R^6$, $\xi_L^6 \to -\xi_L^6$, and $\sigma_6 \leftrightarrow \mu_6$. The same transformation must be applied to Eq. (A1) before calculating correlation functions. The spectrum of O(4) GN model is distinguished from that of the O(N) GN model with $N \geq 6$ [22]. The latter consists of the elementary fermions, the bound states of them, and kinks and can be qualitatively captured by the large N approximation. The existence of kinks is intimately related to the spontaneous breaking of discrete chiral symmetry ($Z_2$ symmetry), which corresponds to the breaking of lattice translation symmetry in our case. On the contrary, in the O(4) case, the elementary fermions are unstable against decay into kinks which become the only stable excitations. As a consequence, the magnetic field
dramatically changes the spectrum of the $S^z = 0$ sector. As shown in Ref. [4], the low energy excitations in the absence of the magnetic field are massive fermions which belong to the representations $(S_{tot}, T_{tot}) = (1, 0), (0, 1)$. Under a magnetic field, the symmetry becomes $U(1) \times SU(2)$ and the good quantum numbers are $S^z$ and $T_{tot}$. As we expect, the excitations of $\Phi_s$ carry $(S^z, T_{tot}) = (\pm 1, 0)$. Nevertheless, those massive fermions carrying $S^z = 0$ are no longer stable when $h > h_c$. The elementary excitations turns out to be kinks which carry $(S^z, T_{tot}) = (0, 1/2)$.

A convenient way to deal with Eq. (14) is to transform it into two decoupled sine-Gordon models [23]:

\[
H_{so} = \sum_{a = \pm} \left\{ \tilde{v}_f \left[ \frac{1}{l} (\partial_x \Theta_a)^2 + l(\partial_x \Phi_a)^2 \right] - \tilde{g} \pi a_0 \cos \sqrt{8\pi} \Phi_a \right\},
\]

where $l = \sqrt{1 + \frac{2\tilde{g}}{\pi v_f}}$ and $\tilde{v}_f = l v_f$. Since $l = 1^+$, the coupling constant of the sine-Gordon model $\beta^2 = 8\pi^-$. The kinks of the $O(4)$ GN model are nothing but the solitons of the two sine-Gordon models. Within our approximation Eqs. (13) and (14), we are able to discuss the dynamical structure factors of spin and orbital sectors, which are defined as:

\[
S^{\alpha\beta}(\omega, q) = 2 \text{Im} \lim_{\omega \to \omega + i0^+} \int_{-\infty}^{\infty} dx d\tau \langle S_i^\alpha(\tau) S_j^\beta(0) \rangle e^{i\omega \tau - iqx},
\]

\[
T^{\alpha\beta}(\omega, q) = 2 \text{Im} \lim_{\omega \to \omega + i0^+} \int_{-\infty}^{\infty} dx d\tau \langle T_i^\alpha(\tau) T_j^\beta(0) \rangle e^{i\omega \tau - iqx},
\]

where $x = a_0|i - j|$. For $T^{\alpha\beta}$, it suffices to compute $T^{zz}$ because the orbital $SU(2)$ symmetry remains intact. We shall see that the $2k_F$ components depend on $x + y$ as well as $x - y$. This is a manifestation of the spontaneous $Z_2$ symmetry breaking of the ground state. In that case, we just list the correlators in the coordinate space. The details of computations are left in Appendix B and we show the results in the following. First, we consider the spin correlators. They are

\[
S^{zz}(\omega, q) = \frac{a_0^2}{g \tilde{v}_s} \omega \delta(\omega - \tilde{v}_s q) + \delta(\omega + \tilde{v}_s q)] + C_1 \left( \frac{4\tilde{v}_s^2/a_0^2}{(\omega^2 - \tilde{v}_s^2(q - 4k_F))} \right)^{1 - \frac{1}{2}} \times [\Theta(\omega - \tilde{v}_s(q - 4k_F))\Theta(\omega + \tilde{v}_s(q - 4k_F)) - (\omega \to -\omega)],
\]

\[
\langle S_i^+(\tau) S_j^-(0) \rangle \sim \cos \left( \frac{\pi}{2a_0} x \right) \cos \left( \frac{\pi}{2a_0} y \right) \left( \frac{a_0}{|z|} \right)^{\frac{5}{2}},
\]

\[
(17)
\]
where $x = ia_0$, $y = ja_0$, and $z = \tilde{v}_s \tau + i(x - y)$. $C_1$ is a nonuniversal constant. For $S^{zz}$, the $C_1$ term is correct only near its low energy threshold and $q \approx 4 k_{F_1}$. In addition, it diverges at the low energy threshold as $[\omega \pm \tilde{v}_s(q - 4 k_{F_1})]^{-1 + \frac{1}{2}}$ for $g > 1$ while it approaches zero with the same functional form for $g < 1$. As $M \to 0$, the exponent becomes zero. For $\langle S_i^+(\tau) S_j^-(0) \rangle$, as the case with $J \leq K/4$, the characteristic momentum $q = \pi/2$ does not shift under the magnetic field. Moreover, the exponent approaches $\frac{1}{2}$ as $M \to 0$.

Next, we consider the orbital correlators. The results are as follows:

$$T^{zz}(\omega, q \approx 0) = 8a_0^2 \frac{s^3 \sqrt{s^2 - 4m^2}}{s^3 \sqrt{s^2 - 4m^2}} \text{sgn}(\omega),$$

$$\langle T_i^z(\tau) T_j^z(0) \rangle_{2k_F} \sim 2 \cos \left[ \frac{\pi}{2a_0} (x + y) \right] \cos \left[ \frac{\pi M}{2a_0} (x - y) \right] \left( \frac{a_0}{|z|} \right)^{\frac{1}{2}}$$

$$\times [W_1(\tau, x - y) - W_2(\tau, x - y)]$$

$$+ \{ \cos [2k_F^+(x - y)] + \cos [2k_F^-(x - y)] \} \left( \frac{a_0}{|z|} \right)^{\frac{1}{2}}$$

$$\times [W_1(\tau, x - y) + W_2(\tau, x - y)],$$

$$T^{zz}(\omega, q \approx \frac{\pi}{a_0}) = C_2 \frac{m^2}{u \sqrt{u^2 - 4m^2}} \times \left[ \frac{|F_0(2\chi(u))|^2}{\cos \left( \frac{\pi u}{\zeta} \right) \cosh \left( \frac{2\pi u}{\zeta} \chi(u) \right)} \right] \text{sgn}(\omega),$$

where $x = ia_0$, $y = ja_0$, $z = \tilde{v}_s \tau + i(x - y)$ and $m$ is the mass of kinks. $s^2 = \omega^2 - v_f^2 q^2$, $u^2 = \omega^2 - v_f^2 (q - \pi/a_0)^2$ and $\zeta = \pi \beta^2/(8\pi - \beta^2)$ with $\beta^2 = 8\pi^2$. $\chi(s) = \cosh^{-1}(s/2m)$. The functions $f(x)$ and $F_0(x)$ are defined in Eqs. (B7) and (B11), respectively. $C_2$ is a mere constant. The result for $T^{zz}(\omega, q \approx 0)$ is exact when $4m^2 < s^2 < 16m^2$ while that for $T^{zz}(\omega, q \approx \frac{\pi}{a_0})$ is valid when $4m^2 < u^2 < 16m^2$. For higher energy, there are small contributions from four, six, eight, etc. particles states. The functions $W_{1,2}$, which are defined in the following:

$$W_1(\tau, x) = F(\sin \left[ \sqrt{\pi/2}(\Phi_+ + \Phi_-) \right] \sin \left[ \sqrt{\pi/2}(\Theta_+ - \Theta_-) \right]),$$

$$W_2(\tau, x) = F(\cos \left[ \sqrt{\pi/2}(\Phi_+ + \Phi_-) \right] \cos \left[ \sqrt{\pi/2}(\Theta_+ - \Theta_-) \right]),$$

with $F(\hat{O}) \equiv \langle \hat{O}(\tau, x) \hat{O}(0, 0) \rangle$, involve the computation of form factors containing $\Phi_\pm$ and their dual fields $\Theta_\pm$. At present, no one knows how to compute them. However, we can still discuss some features of these functions. The operators we are considering are like $\hat{A}_+ \cdot \hat{A}_-$. 

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where \( \hat{A}_\pm = \cos[\text{or } \sin](\sqrt{\pi/2} \Phi_\pm) \times \cos[\text{or } \sin](\sqrt{\pi/2} \Theta_\pm) \). According to Ref. [27], the operators \( \cos[\text{or } \sin](\sqrt{\pi/2} \Theta_\pm) \) are fermionic while \( \cos[\text{or } \sin](\sqrt{\pi/2} \Phi_\pm) \) are bosonic for \( \beta^2 = 8\pi^- \). Thus, \( \hat{A}_\pm \) must be fermonic. This implies that the first nontrivial form factors of \( \hat{A}_\pm \) start from the one-soliton states. As a result, the leading contributions to \( W_i \) come from two-particle states. We expect that in the momentum space, the \( 2k_F \) component exhibits incoherent background with two-particle thresholds.

In summary, when \( h > h_c \), the spin correlators exhibit algebraic decay while orbital correlators remain massive behavior. Low energy modes appear close to \( q = 0 \) and \( 4k_{F_\perp} \) for \( S^{zz} \) or \( q = \frac{\pi}{2a_0} \) for \( S^{+-} \). The \( 2k_F \) component of \( S^{zz} \) and the uniform and \( 4k_F \) components of \( S^{+-} \) involve massive excitations [see Eq. (A1)] and thus exhibit similar behavior to that of \( \langle T_i^z(\tau)T_j^z(0)\rangle_{2k_F} \). The leading contributions to the uniform and \( 4k_F \) components of \( \langle T_i^\alpha(\tau)T_j^\beta(0)\rangle \) come from the two-particle states, which is similar to that in the absence of the magnetic field. The main distinction between the cases with \( h > h_c \) and \( h < h_c \) is the \( 2k_F \) component. For the former case, it also exhibits two-particle thresholds in contrast to the latter case where it shows a coherent ”magnon” peak. This can be detected experimentally by examining the dynamical spin structure factors under a uniaxial pressure.

### IV. WEAK COUPLING REGIME: \( K \ll J \)

When \( K \ll J \), a proper starting point is to consider the \( K \) term in Eq. (1) as a perturbation, which was discussed by Nersesyan and Tsvelik [16] in the context of two-leg spin ladders. The unperturbed Hamiltonian is composed of two antiferromagnetic (AF) Heisenberg chains which can be treated by using the standard relation between the spin (pseudospin) operators and WZNW fields [20]: \( \tilde{S}_n/a_0 = \tilde{j}_s + (-1)^{\frac{x}{a_0}} \tilde{m}_s \) and \( \tilde{T}_n/a_0 = \tilde{j}_t + (-1)^{\frac{x}{a_0}} \tilde{m}_t \) where \( x = na_0 \). By defining \( \phi_\pm = (\phi_s \pm \phi_t)/\sqrt{2} \) where \( \phi_s \) and \( \phi_t \) are, respectively, the bosonic fields describing the spin and orbital degrees of freedom, we obtain

\[
H = \sum_{a=\pm} \int dx \left\{ \frac{v}{2} \left( (\partial_x \theta_a)^2 + (\partial_x \phi_a)^2 \right) + \frac{m}{\pi a_0} \cos \left( \sqrt{4\pi} \phi_a \right) - \frac{h}{\sqrt{4\pi}} \partial_x \phi_a \right\},
\]  

(20)
where \( v \sim J a_0 \) and \( m = \lambda^2 K/(2\pi^3) \). The Hamiltonian (21) can be further fermionized as:

\[
H = \sum_{a=\pm} \int dx \left[ -iv(\psi_{aR}^+ \partial_x \psi_{aR} - \psi_{aL}^+ \partial_x \psi_{aL}) + im(\psi_{aR}^+ \psi_{aL} - \psi_{aL}^+ \psi_{aR}) - \frac{h}{2}(\psi_{aR}^+ \psi_{aR} + \psi_{aL}^+ \psi_{aL}) \right].
\] (21)

From Eq. (21), we see that in the region with \( K \ll J \), the low energy excitations in the absence of the magnetic field are described by two free massive Dirac fermions or four massive Majorana fermions, which are kinks connecting two degenerate ground states [16]. In our case, these Majorana fermions belong to the representation \((S_{tot}, T_{tot}) = (1/2, 1/2)\) and are coupled to the magnetic field simultaneously. This leads to the result that the QCP moves to \( h_c = 2m \) instead of \( m \). Beyond that value, all excitations become gapless and the correlation functions are algebraic decay. The situation is quite different from the two-leg spin ladder model by considering \( \vec{T}_i \) in Eq. (1) as the spin operator on the other chain. In that case, the magnetic field is only coupled to \( \psi_+ \) and the QCP is located at \( h_c = m \). Beyond this value, \( \psi_- \) is still massive and we expect that the behavior of correlation functions is similar to that of the usual spin ladders.

What we are concerned is the case where \( h > h_c \). The effective Hamiltonian describing the low energy excitations around the Fermi point is as the following:

\[
H = \sum_{a=\pm} \int dx \frac{v}{2} \left[ (\partial_x \bar{\phi}_a)^2 + (\partial_x \bar{\theta}_a)^2 \right].
\] (22)

Note that Eq. (22) is in fact a theory describing two free massless fermions. We are now in a position to calculate the spin and orbital correlation functions. In terms of \( \bar{\phi}_a \) and \( \bar{\theta}_a \), the spin and orbital operators are expressed as follows [21]:

\[
S^z_n = \frac{M}{2} + \frac{a_0}{2\sqrt{\pi}} \partial_x (\bar{\phi}_+ + \bar{\phi}_-) + \lambda_1(-1)^n \sin \left[ \frac{\pi M}{a_0} x + \sqrt{\pi}(\bar{\phi}_+ + \bar{\phi}_-) \right],
\]

\[
S^+_n = \frac{1}{2\pi} \left[ e^{\frac{\pi M}{a_0} x} e^{i\sqrt{\pi}(\bar{\phi}_+ + \bar{\phi}_-)} e^{i\sqrt{\pi}(\bar{\phi}_+ - \bar{\phi}_-)} + e^{-i\frac{\pi M}{a_0} x} e^{-i\sqrt{\pi}(\bar{\phi}_+ - \bar{\phi}_-)} e^{-i\sqrt{\pi}(\bar{\phi}_+ - \bar{\phi}_-)} \right] + \lambda_2(-1)^n e^{i\sqrt{\pi}(\bar{\phi}_+ + \bar{\phi}_-)},
\]

\[
T^z_n = \frac{a_0}{2\sqrt{\pi}} \partial_x (\bar{\phi}_+ - \bar{\phi}_-) + \lambda_3(-1)^n \sin \left[ \sqrt{\pi}(\bar{\phi}_+ - \bar{\phi}_-) \right],
\]

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\[ T_n^+ = \frac{1}{2\pi} \left[ e^{i\sqrt{\pi}(\phi_+ + \theta_+)} e^{-i\sqrt{\pi}(\phi_- - \theta_-)} + e^{-i\sqrt{\pi}(\phi_+ - \theta_+)} e^{i\sqrt{\pi}(\phi_- - \theta_-)} \right] + \lambda_3 (-1)^n e^{i\sqrt{\pi}(\theta_+ - \theta_-)}, \] 

where \( x = na_0 \) and \( \lambda_i \ (i = 1, 2, 3) \) are constants. Note that \( S_n^z \) contains the \( Q = \frac{\pi}{a_0}(1 - M) \) term in contrast to the two-leg spin ladder where the term next to the uniform component is the \( 2Q \) one \([21]\). The reason is that in the spin ladder, the antisymmetrical mode \( \phi_- \) is still massive even under the magnetic field and leads to the exponential decay of the \( Q \) term. However, in our case, \( \phi_- \) becomes massless when \( h > h_c \). With the help of Eq. (23), we get the correlators as the following:

\[
\begin{align*}
\langle S_i^z(\tau)S_j^z(0) \rangle &= \frac{M^2}{4} + \frac{a_0^2}{8\pi^2} \left( \frac{1}{z^2} + \frac{1}{\bar{z}^2} \right) + D_1 \cos(Qx) \frac{a_0}{|z|}, \\
\langle S_i^+(\tau)S_j^-(0) \rangle &= \frac{a_0^2}{4\pi^2|z|^2} \left( e^{i\frac{\pi}{a_0}x\frac{z}{z}} e^{-i\frac{\pi}{a_0}x\frac{\bar{z}}{\bar{z}}} + e^{-i\frac{\pi}{a_0}x\frac{z}{z}} e^{i\frac{\pi}{a_0}x\frac{\bar{z}}{\bar{z}}} \right) + D_2(-1)^\frac{a_0}{|z|}, \\
\langle T_i^z(\tau)T_j^z(0) \rangle &= \langle T_i^y(\tau)T_j^y(0) \rangle = \langle T_i^z(\tau)T_j^z(0) \rangle = \frac{a_0^2}{8\pi^2} \left( \frac{1}{z^2} + \frac{1}{\bar{z}^2} \right) + (-1)^\frac{a_0}{|z|} D_3 \frac{a_0}{|z|},
\end{align*}
\] 

(24)

where \( x = a_0|i - j|, z = v\tau + ix \) and \( D_i \ (i = 1, 2, 3) \) are constants. Note that all correlators in Eq. (24) behave like the spin correlation functions of an AF Heisenberg chain when \( h \to h_c^+ \) \((M \to 0^+)\). Low energy modes appear close to \( q = 0 \) and \( Q \) for \( S^{zz} \) or \( q = \frac{\pi M}{a_0} \) and \( \frac{\pi}{a_0} \) for \( S^{+-} \) while for spin ladders they are close to \( q = 0 \) and \( \frac{\pi M}{a_0} \) for \( S^{zz} \) or \( q = Q \) and \( \frac{\pi}{a_0} \) for \( S^{+-} \) \([21]\). The orbital correlators are identical to the spin correlation functions of an AF Heisenberg spin chain up to some numerical prefactors for any finite \( h \) \((> h_c)\).

V. CONCLUSION

In this paper, we discuss the effects of a magnetic field on the one dimensional spin-orbital model with \( J_1 = J_2 = J \). In the gapless phase, i.e. \( J \leq K/4 \), the spin and orbital correlation functions still exhibit power-law decay. The magnetic field manifests itself on the emergence of incommensurate soft modes with characteristic momenta depending on the magnetization. Furthermore, the exponents are also dependent of the magnetization.
though they have universal values at zero field. A distinctive feature in the case with orbital degeneracy is that the structures of orbital correlators are hardly influenced by the magnetic field. This is associated with the unbroken orbital SU(2) symmetry.

In the massive phase, i.e. $J > K/4$, we expect that there is a quantum phase transition induced by increasing the magnetic field. However, because of the different excitation spectrum at different range of $J/K$, the correlation functions exhibit distinct behavior between the region near the SU(4) point and the one with $K \ll J$. In the former case, the magnetic field exerts a great influence on the $S^z = 0$ sector such that the spectrum is completely different from that in the absence of the magnetic field. We propose that this change can be experimentally examined by studying the dynamical spin structure factors under a uniaxial pressure. In the latter case, the spin and orbital correlation functions become power-law decay when the magnetic field is greater than the critical value. Furthermore, the exponents take universal values which are identical to those of an AF Heisenberg chain. This is due to the fact that the underlying system is described by a free fermion theory.

Finally, we would like to mention the applicability of our results to the asymmetric case ($J_1 \neq J_2$) around the SU(4) point. The massless and massive phases on the symmetric line now extend to large anisotropic regions [13]. The phase boundary passes through the SU(4) point. The anisotropy affects not only the coupling constants in the Hamiltonian (7) but also the velocities of the spin and orbital sectors such that they become different. For the massless phase with $G_2 = c(J_2 - K/4) < 0$ ($c > 0$), the correlators under the magnetic field are similar to Eq. (10) except that the velocity of $\xi^6$ is different from the one of the orbital sector. The symmetry of the fixed point Hamiltonian is in general $U(1) \times SO(3) \times \mathbb{Z}_2$ instead of $U(1) \times SO(4)$. In the massive phase with $G_2 > 0$, we still have an approximate $U(1) \times SO(4)$ symmetry at low energy for $h > h_c$ and the correlators are the same as Eqs. (17) and (18). On the other hand, in the rest part of the phase diagram around the SU(4) point, it is possible that another phase transition of the KT type occurs on the $S^z = 0$ sector when $h$ further increases [13]. However, it is just a transition from the massive (our region II) to massless (our region I) phases. Therefore, the correlators are still of the forms
we obtained in the corresponding regions.

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APPENDIX A: THE SPIN AND ORBITAL OPERATORS

In this Appendix, we list the expressions of the WZNW fields $\vec{J}_s(t)$, $\vec{N}_s(t)$, and $\vec{n}_s(t)$ in the following:

$\vec{J}_\nu = \frac{i}{2} \vec{\xi}_\nu \times \vec{\xi}_\nu, J_s^z = \frac{1}{\sqrt{\pi}} \partial_x \Phi_s,$

$J_{s+} = -\frac{i\eta_1}{\sqrt{\pi a_0}} \left\{ \xi_R^6 \exp[-i\sqrt{\pi}(\Phi_s - \Theta_s)] + \xi_L^6 \exp[i\sqrt{\pi}(\Phi_s + \Theta_s)] \right\},$

$N_s^z = A [\sin(\sqrt{\pi}\Phi_s)\mu_3\mu_4\mu_5\mu_6 + i \cos(\sqrt{\pi}\Phi_s)\sigma_3\sigma_4\sigma_5\sigma_6],$

$N_{s\pm} = A \eta_1 \eta_3 (\mu_3\mu_4\mu_5\sigma_6 \pm i \sigma_3\sigma_4\sigma_5\mu_6) \exp(\pm i\sqrt{\pi}\Theta_s),$

$N_l^z = A [\cos(\sqrt{\pi}\Phi_s)\sigma_3\sigma_4\mu_5\mu_6 + i \sin(\sqrt{\pi}\Phi_s)\mu_3\mu_4\sigma_5\sigma_6],$

$N_{l\pm} = -i A \eta_2 \eta_3 (\mu_3\sigma_4 \pm i \mu_3\sigma_4) [\sin(\sqrt{\pi}\Phi_s)\mu_5\sigma_6 \mp \cos(\sqrt{\pi}\Phi_s)\sigma_5\mu_6],$

$\vec{n}_l = -i B \vec{\xi}_l \times \vec{\xi}_l, n_s^z = \frac{B}{\pi a_0} \sin(\sqrt{4\pi}\Phi_s),$

$n_{s\pm} = \frac{B\eta_1}{\sqrt{\pi a_0}} \left\{ \xi_R^6 \exp[i\sqrt{\pi}(\Phi_s + \Theta_s)] + \xi_L^6 \exp[-i\sqrt{\pi}(\Phi_s - \Theta_s)] \right\}, \quad (A1)$

where $\nu = R, L$ and $\hat{O}_\pm = \hat{O}^x \pm i \hat{O}^y$ ($\hat{O} = J_s(t), N_s(t), n_s(t)$). $A$ and $B$ are nonuniversal constants. $\vec{\xi}_\nu = (i\xi_3^\nu, -i\xi_4^\nu, i\xi_5^\nu)$ is a vector under orbital SO(3) rotations. $\sigma_a$ and $\mu_a$ ($a = 3, \ldots, 6$) are order and disorder parameters of the corresponding Ising models.

APPENDIX B: CORRELATION FUNCTIONS IN THE REGION WITH $J > K/4$

Following the appendix of Ref. [21], we found that in this case it is not necessary to include higher harmonics. When $h > h_c$, the spin operators become
\[ S_i^z = \frac{M}{2} + \frac{a_0}{\sqrt{\pi}} \partial_x \tilde{\phi} + \lambda(-1)^\frac{n}{2} \sin \left( \frac{M}{a_0} x + \sqrt{4\pi} \tilde{\phi} \right), \]
\[ S_i^\pm \sim e^{i\pi a_0 x} e^{\pm i\sqrt{\pi} \tilde{\phi}} \cos \left( \sqrt{2\pi} \Phi_\pm \right) + \text{H.c.}, \tag{B1} \]

where \( \lambda \) is a constant. As for the orbital operators, the corresponding WZNW fields become

\[ J_i^z = \frac{1}{\sqrt{2\pi}} \partial_x (\Phi_+ + \Phi_-), \]
\[ N_i^z \sim \cos \left( \frac{\pi M}{2a_0} x + \sqrt{\pi} \tilde{\phi} \right) \sin \left[ \sqrt{\frac{\pi}{2}} (\Phi_+ + \Phi_-) \right] \sin \left[ \sqrt{\frac{\pi}{2}} (\Theta_+ - \Theta_-) \right] \]
\[ + i \sin \left( \frac{\pi M}{2a_0} x + \sqrt{\pi} \tilde{\phi} \right) \cos \left[ \sqrt{\frac{\pi}{2}} (\Phi_+ + \Phi_-) \right] \cos \left[ \sqrt{\frac{\pi}{2}} (\Theta_+ - \Theta_-) \right], \]
\[ n_i^z \sim \sin \left[ \sqrt{2\pi} (\Phi_+ + \Phi_-) \right]. \tag{B2} \]

The computations involving the \( \tilde{\phi} \) sector are straightforward. To compute the longitudinal dynamical spin structure factor, we need the following integral:

\[ \int_{-\infty}^{\infty} dx dt \frac{e^{i\omega t - iqx}}{(x + vt - i0^+)^\alpha (x - vt + i0^+)^\beta} = \Theta(\omega - vq) \Theta(\omega + vq) \frac{2\pi^2 e^{i\pi (\alpha - \beta)/2}}{v \Gamma(\alpha) \Gamma(\beta)} \times \left( \frac{2v}{\omega - vq} \right)^{1-\alpha} \left( \frac{2v}{\omega + vq} \right)^{1-\beta}. \tag{B3} \]

The correlators involving \( \Phi_\pm \) only can be calculated in terms of the exact results of form factors on the sine-Gordon model \([24 \sim 27]\). The calculations can be partially simplified by noting that the Hamiltonian \((13)\) is invariant under the exchange \( \Phi_+ \leftrightarrow \Phi_- \) and \( \Phi_+ \) and \( \Phi_- \) are decoupled. We now compute \( T^{zz}(\omega, q \approx 0) \). (A similar calculation has been done in the context of a spin ladder model \([29]\).) With the help of Eq. \((B2)\), it is

\[ T^{zz}(\omega, q \approx 0) = \frac{2a_0^2}{\pi} \text{Im} \lim_{i\omega \to +i0^+} \int_{-\infty}^{\infty} dx d\tau \langle \partial_x \Phi_+(\tau, x) \partial_x \Phi_+(0, 0) \rangle e^{i\omega \tau - iqx} \]
\[ = 2a_0^2 \text{Im} i \int_{-\infty}^{\infty} dx \int_0^{\infty} dt \ e^{i(\omega + i0^+)t - iqx} \langle [j_0(t, x), j_0(0, 0)] \rangle, \tag{B4} \]

where \( j_0 = \frac{1}{\sqrt{\pi}} \partial_x \Phi_+ \) is the temporal component of the current operator in the sine-Gordon model and we set \( v_f = 1 \). In one dimension, energy and momentum can be parametrized by rapidity \( \theta \) as \( \epsilon = m \cosh \theta \) and \( p = m \sinh \theta \) where \( m \) is the mass of the sine-Gordon solitons. The resolution of the identity is given by

\[ 1 = \sum_{n=0}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |\theta_n, \cdots, \theta_1, \theta|_{\alpha_n \cdots \alpha_1} \langle \theta_1, \cdots, \theta_n |, \tag{B5} \]

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where \( n \) is the number of particles. \( \alpha_i = 1, -\frac{1}{2} \) for solitons and antisolitons, respectively. Inserting Eq. (B5) into Eq. (B4) and performing the integrations over space and time yields

\[
T_{zz}(\omega, q \approx 0) = -4\pi a_0^2 \text{Im} \sum_{n=0}^{\infty} \sum_{\alpha_i} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |F^{j_0}_{\alpha_1 \cdots \alpha_n}(\theta_1, \ldots, \theta_n)|^2 \\
\times \left[ \frac{\delta(q - m \sum_j \sinh \theta_j)}{\omega - m \sum_j \cosh \theta_j + i0^+} - \frac{\delta(q + m \sum_j \sinh \theta_j)}{\omega + m \sum_j \cosh \theta_j + i0^+} \right],
\]

(B6)

where

\[
F^{j_0}_{\alpha_1 \cdots \alpha_n}(\theta_1, \ldots, \theta_n) \equiv \langle 0|j_0(0,0)|\theta_n, \ldots, \theta_1 \rangle_{\alpha_n \cdots \alpha_1}
\]

is the sine-Gordon current form factor. (For an operator \( \hat{O}(t,x) \), the form factor \( F^{\hat{O}}_{\alpha_1 \cdots \alpha_n}(\theta_1, \ldots, \theta_n) \equiv \langle 0|\hat{O}(0,0)|\theta_n, \ldots, \theta_1 \rangle_{\alpha_n \cdots \alpha_1} \). Note that an \( n \)-particle state only contributes to Eq. (B6) when \( s^2 = \omega^2 - q^2 \geq n^2m^2 \) and \( n \) must be an even integer. Thus, at low energy \( s^2 < 16m^2 \), only two-particle states contribute. The corresponding form factor [24] is

\[
F^{j_0}_{\frac{1}{2},-\frac{1}{2}}(\theta_1, \theta_2) = -2m \sinh \left( \frac{\theta_1 + \theta_2}{2} \right) f(\theta_{12}),
\]

where \( \theta_{12} = \theta_1 - \theta_2 \) and

\[
f(\theta) = \frac{i}{2\pi} \sinh (\theta/2) \times \exp \left\{ \int_0^\infty dx \frac{x}{x} \frac{\sin^2 [(\theta - i\pi)x/2]}{\sinh (\pi x)} [\tanh (\pi x/2) - 1] \right\}.
\]

(B7)

After performing the integrations over \( \theta_i \), we obtain

\[
T_{zz}(\omega, q \approx 0) = 8a_0^2 m^2 q^2 |f(2\chi(s))|^2 \frac{s^3}{s^2 - 4m^2} \text{sgn}(\omega),
\]

(B8)

where \( \chi(s) = \cosh^{-1}(s/2m) \) and \( 4m^2 < s^2 < 16m^2 \).

Next, we compute \( T_{zz}(\omega, q \approx \frac{s}{a_0}) \). According to Ref. [28], \( \langle \cos (\frac{q}{a_0}\phi) \rangle = \mathcal{G}_{\beta/2} \) where

\[
\mathcal{G}_a \equiv \langle e^{ia\phi} \rangle = \left[ \frac{m\sqrt{\pi} \Gamma(4\pi/(8\pi - \beta^2))}{2\Gamma(\beta^2/(16\pi - 2\beta^2))} \right]^{\frac{2}{\pi}}
\times \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh^2 (2a\beta t)}{2\sinh (\beta^2t) \sinh (8\pi t) \cosh ((8\pi - \beta^2)t)} - \frac{a^2}{4\pi} e^{-16\pi t} \right] \right\},
\]

(B9)

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and $\beta < \sqrt{8\pi}$ is the coupling constant of the sine-Gordon model. Therefore, the leading term of $\langle n_z(\tau, x)n_z(0, 0) \rangle \propto G^2_{\beta/2} F(\sin(\frac{\beta}{2} \Phi_+))$ where $\beta^2 = 8\pi$. Then,

$$T^{zz}(\omega, q \approx \frac{\pi}{a_0}) \propto \text{Im} \left[ \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dt e^{i(\omega+i0^+)t-i(q-\frac{\pi}{a_0})x} \langle [\sin(\frac{\beta}{2} \Phi_+)(t, x), \sin(\frac{\beta}{2} \Phi_+)(0, 0)] \rangle \right] \propto G^2 \frac{\beta}{2} F(\sin(\frac{\beta}{2} \Phi_+)),$$

\[ \text{Im} \left[ \sum_{n=0}^{\infty} \sum_{\alpha_i} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} F_{\alpha_1 \cdots \alpha_n}^{\sin(\frac{\beta}{2} \Phi_+)}(\theta_1, \cdots, \theta_n) \right]^2 \times \left[ \frac{\delta(q-m \sum_j \sin \theta_j)}{\omega - m \sum_j \cosh \theta_j + i0^+} - \frac{\delta(q+m \sum_j \sin \theta_j)}{\omega + m \sum_j \cosh \theta_j + i0^+} \right]. \tag{B10} \]

The leading contributions to Eq. (B10) come from the two-particle states and the corresponding form factor [27] is

$$F_{12}^{\sin(\frac{\beta}{2} \Phi_+)}(\theta_{12}) = -F_{21}^{\sin(\frac{\beta}{2} \Phi_+)}(\theta_{12}) \propto \frac{F_0(\theta_{12})}{\cosh(\frac{\theta_{12}}{2} - i\pi)},$$

where $\zeta = \pi \beta^2/(8\pi - \beta^2)$ and

$$F_0(\theta) = -i \sinh(\theta/2) \times \exp \left\{ \int_{0}^{\infty} dx \sinh(\frac{\pi - \zeta}{2} x) \sin^2(\frac{\pi}{2} \theta - i\pi) \right\}.$$ \tag{B11}

Here the indices of the form factor $\alpha_i = 1, 2$ correspond to the neutral fermions which are related to the sine-Gordon solitons through $Z_\pm = Z_1 \pm iZ_2$ where $Z_\pm$ and $Z_{1,2}$ are, respectively, annihilation operators of solitons (anti-solitons) and neutral fermions [27]. Inserting Eq. (B11) into Eq. (B10) and performing the integrations over $\theta_i$, we get

$$T^{zz}(\omega, q \approx \frac{\pi}{a_0}) \propto \frac{m^2}{u \sqrt{u^2 - 4m^2}} \times \frac{|F_0(2\chi(u))|^2}{\cos(\frac{\pi^2}{\zeta}) + \cosh(\frac{2\pi}{\zeta} \chi(u))} \text{sgn}(\omega), \tag{B12}$$

where $4m^2 < u^2 = \omega^2 - (q - \pi/a_0)^2 < 16m^2$. 

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\[ \eta_{2\uparrow} \eta_{2\downarrow} \eta_{1\uparrow} \eta_{3} = 1 = -\eta_{1\uparrow} \eta_{1\downarrow} \eta_{1\uparrow} \eta_{3} = \eta_{1\uparrow} \eta_{2\uparrow} \eta_{2\downarrow} \eta_{3} = \eta_{1\downarrow} \eta_{2\downarrow} \eta_{2\downarrow} \eta_{3}. \]

These are all what we need.

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