DIFFERENCE EQUATIONS IN THE COMPLEX PLANE:
QUASICLASSICAL ASYMPTOTICS AND BERRY PHASE

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Abstract. We study solutions to the difference equation \( \Psi(z + h) = M(z)\Psi(z) \)
where \( z \) is a complex variable, \( h > 0 \) is a parameter, and \( M : \mathbb{C} \to SL(2, \mathbb{C}) \)
is a given analytic function. We describe the asymptotics of its analytic solutions as \( h \to 0 \). The asymptotic formulas contain an analog of the geometric (Berry) phase well-known in the quasiclassical analysis of differential equations.

1. Introduction

For \( M : \mathbb{C} \to SL(2, \mathbb{C}) \) being a given analytic function, we consider the equation
\[
\Psi(z + h) = M(z)\Psi(z) \tag{1.1}
\]
where \( z \) is a complex variable, and \( h > 0 \) is a parameter. We describe asymptotics of analytic vector solutions \( \Psi \) to (1.1) as \( h \to 0 \).

Formally, \( \Psi(z + h) = e^{h\pi i} \psi(z) \), and being a small parameter in front of the derivative, \( h \) can be regarded as a quasiclassical asymptotic parameter.

The quasiclassical asymptotics of solutions to the ordinary differential equation
\[
ih\frac{d\Psi}{dx}(x) = M(x)\Psi(x) \tag{1.2}
\]
as \( h \to 0 \) are described by means of the famous WKB (Wentzel, Kramers and Brillouin) method. There is a huge literature devoted to this method and its applications. If \( M \) is analytic, one uses a method often called the complex WKB method, see, e.g., chapters 3 and 5 in [8] and chapter 7 in [25]. This method allows to study solutions to (1.2) on the complex plane. Even when the input problem does not require to go into the complex plane, one uses this method to simplify the analysis: it allows to go around, say, turning points or singularities of solutions located on the real line, and to compute the asymptotics of their Wronskians in the domains where they are easy to be computed. The latter makes the complex WKB method very efficient for computing exponentially small quantities. Actually, in [8] one can find various interesting examples of problems solved using this method.

To study difference equations on the real axis in the quasiclassical approximation, one uses methods similar to the classical WKB methods (e.g., [15]), pseudodifferential operator theory (e.g., [20]), and Maslov’s canonical operator method (e.g., [9]). For difference equations on the complex plane, a complex WKB method can play the same role as for differential ones, and an analog of the complex WKB method for difference equations is being developed in [3, 15, 17, 13, 14] and in the present paper. In [3, 15, 17, 13, 14] the authors develop an analog of the complex WKB method for the one-dimensional difference Schrödinger equation
\[
\psi(z + h) + \psi(z - h) + v(z)\psi(z) = 0, \tag{1.3}
\]

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where \( v \) is an analytic function. In this paper we extend this method to the matrix difference equation (1.1), get asymptotic formulas for its solutions, and, in particular, find an analog of the geometric phase (Berry phase) well-known in the case of differential equations, see [2, 23].

Our work is motivated by the analysis of the spectrum of the Harper operator acting in \( L_2(\mathbb{R}) \) by the formula
\[
H\psi(z) = \psi(z + h) + \psi(z - h) + 2\lambda \cos z \psi(z).
\]
This operator arises in the solid state physics when studying an electron in a crystal submitted to a magnetic field, see, e.g., the introduction sections in [20, 19] and references therein. For irrational \( h \) the spectrum coincides with one of the famous almost Mathieu operator, and is a Cantor set, see, e.g. [1]. In [20] heuristically, and in [26] rigorously, the authors obtain in the quasiclassical approximation a description of the spectrum similar to one of the classical Cantor set: they discovered step by step sequences of smaller and smaller spectral gaps. Note that the gaps of each sequence appear to be exponentially small with respect to the gaps of the previous one.

To study the geometrical properties of the spectrum, Buslaev and Fedotov have suggested a renormalization approach based on ideas of the Floquet theory, see [9]. A crucial role in their analysis is played by the minimal entire solutions to the Harper equation
\[
\psi(z + h) + \psi(z - h) + 2\lambda \cos z \psi(z) = E\psi(z),
\]
where \( E \) is a spectral parameter. To study them in the quasiclassical approximation, a version of the complex WKB for difference equations was developed, see [3, 5, 9]. The analysis of geometrical properties of the spectrum also requires to analyze solutions to matrix difference equations of the form (1.1) with complex coefficients (see, section 2.3.2 in [9]). Similar problems arise when, instead of the Harper operator, one studies more general difference and differential one-dimensional quasiperiodic Schrödinger equations with two frequencies, see [9].

In this paper we study two cases: the case when \( M \) is analytic in a bounded domain, and the case when \( M \) is a trigonometric polynomial.

In the next section we describe the main objects of the complex WKB method for equation (1.1): the complex momentum, geometric phase, canonical curves and canonical domains. Then, we formulate and discuss two our theorems on the existence of analytic solutions to (1.1) having a simple quasiclassical behavior in certain complex domains. In section 3 we turn to the geometric phase appearing in the asymptotic formulas. It is very natural to consider it as an integral of a meromorphic differential (meromorphic differential 1-form) on a Riemann surface, and we study this differential in details. In section 4 we prove the existence of analytic solutions having simple asymptotic behavior in bounded domains, and in section 5 we turn to the case where \( M \) is a trigonometric polynomial.

Our results were announced in a short note [16] (conference proceedings).

2. The main construction of the complex WKB method

We begin with formulating our assumptions on the matrix \( M \).

2.1. Our assumptions. We assume that either the domain of analyticity of \( M \) is bounded or \( M \) is a trigonometric polynomial, i.e.,
\[
M(z) = \sum_{j=-k}^{l} M_j e^{2\pi i jz}, \quad z \in \mathbb{C},
\]
(2.1)
where \( M_j \) are Fourier coefficients. We do not consider the degenerate case where \( M_{12}M_{21} \equiv 0 \) (in this case equation (1.1) can be solved explicitly).
In the case of (2.1), we assume also that \( k, l > 0 \), that \( \text{tr} M_{-k} \text{tr} M_l \not\equiv 0 \), and that
\[
M_{22}(z)/M_{11}(z) \text{ stays bounded as } |\text{Im} z| \to \infty. \tag{2.2}
\]
The last hypothesis can be removed and is made just for the sake of simplicity.

2.2. The complex momentum. The complex momentum \( p \) is the multivalued analytic function defined in the domain of analyticity of \( M \) by the formula
\[
2 \cos p(z) = \text{tr} M(z), \quad z \in D. \tag{2.3}
\]
The branch points of \( p \) satisfy the equations \( \text{tr} M(z) = \pm 2 \). We call points where \( \text{tr} M(z) \in \{ \pm 2 \} \) turning points. We say that a subset of the domain of analyticity of \( M \) regular if it contains no turning points.

As \( \det M(z) \equiv 1 \), the eigenvalues of \( M(z) \) are equal to \( e^{\pm i p(z)} \). If \( z \) is regular, one has \( e^{ip(z)} \not= e^{-ip(z)} \).

2.3. The geometric phase. Let \( R \subset \mathbb{C} \) be a regular simply connected domain. We fix in \( R \) an analytic branch \( p \) of the complex momentum.

Let \( r^\pm : R \to \mathbb{C}^2 \) be two nontrivial analytic functions satisfying the equations
\[
M(z)r^\pm(z) = e^{\pm i p(z)} r^\pm(z), \quad z \in R. \tag{2.4}
\]
We set
\[
l^\pm(z) = (r^\mp)^T(z) \sigma, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.5}
\]
where \( ^T \) denotes transposition. By Lemma 3.1, we have
\[
l^\pm(z)M(z) = e^{\pm i p(z)} l^\pm(z), \quad z \in R. \tag{2.6}
\]
If \( r^+(z) \not= 0 \) ( \( r^-(z) \not= 0 \) ), then \( r^+(z) \) ( resp., \( r^-(z) \) ) is a right eigenvector of \( M(z) \), and \( l^+(z) \) ( resp., \( l^-(z) \) ) is its left eigenvector.
The analytic functions \( z \mapsto l^\pm(z)r^\pm(z) \) are not identically zero (in view of (2.1)), and we define in \( R \) two meromorphic differentials \( \Omega^\pm \) by the formulas
\[
\Omega^\pm(z) = \frac{i}{2} dp(z) - \frac{l^\pm(z)}{l^\pm(z)r^\pm(z)} dr^\pm(z). \tag{2.7}
\]
Let us note that the poles of \( \Omega^\pm \) are located at points where \( r^\pm(z) = 0 \) (in view of Theorem 2.1). Let \( z_0 \in R \), and \( r^\pm(z_0) \not= 0 \). The integrals \( \int_{z_0}^z \Omega^\pm \) are called geometric phases. Very close objects are well-known in the WKB analysis of differential equations, see section 2.6.3. But, it looks like their properties has not been systematically studied as properties of functions of the complex variable.

We study \( \Omega^\pm \) in section 3. For two column vectors \( u, v \) in \( \mathbb{C}^2 \), we denote by \( (u \ v) \) the \( 2 \times 2 \)-matrix with the columns \( u \) and \( v \). In section 3.3 we check

**Theorem 2.1.** Let \( z_0 \in R \) and \( r^\pm(z_0) \not= 0 \). In the domain \( R \) each of the functions
\[
V^\pm : z \mapsto \exp \left( \int_{z_0}^z \Omega^\pm(z) \right) r^\pm(z) \tag{2.8}
\]
is analytic, does not vanish and is independent of the choice of \( r^\pm \) up to a constant factor. Moreover, one has
\[
\det(V^+(z) V^-(z)) = \det(r^+(z_0) r^-(z_0)) \not= 0. \tag{2.9}
\]

We call \( V^\pm \) analytic eigenvectors of \( M \) normalized at \( z_0 \). The facts that \( V^\pm \) are independent of the choice of \( r^\pm \) and satisfy (2.9), are proved by means of the ideas used to check similar facts in the case of differential equations on \( \mathbb{R} \), see, e.g., section 3 of chapter 5 in [8].
2.4. The canonical curves. For \( z \in \mathbb{C} \), we let \( x = \text{Re} \ z, \ y = \text{Im} \ z \).

A curve \( \gamma \subset \mathbb{C} \) is called *vertical* if, along \( \gamma \), \( x \) is a piecewise continuously differentiable function of \( y \). We say that \( \gamma \) is infinite if along it \( y \) increases from \( -\infty \) to \( \infty \).

Let \( R \) be a regular simply connected domain, and \( z_0 \in R \). We fix in \( R \) an analytic branch of the complex momentum \( p \). Let \( \gamma \subset R \) be a vertical curve, and let \( z(\gamma) \) be the point of \( \gamma \) with the imaginary part equal to \( y \). This curve is called *canonical* with respect to the branch \( p \) if, at all the points of \( \gamma \) where \( z' \) exists, one has

\[
\frac{d}{dy} \text{Im} \int_{z_0}^{z(y)} p(z) \, dz > 0, \quad \text{and} \quad \frac{d}{dy} \text{Im} \int_{z_0}^{z(y)} (p(z) - \pi) \, dz < 0, \quad (2.10)
\]

and, at the points where \( dz/dy \) is discontinuous, these inequalities hold for the left and right derivatives.

2.5. The canonical domains. The definitions of the bounded and unbounded canonical domains are slightly different.

2.5.1. Bounded canonical domains. We call a domain horizontally connected if, for any its two points having one and the same imaginary part, the straight line segment that connects them is contained in this domain.

Let \( K \) be a bounded regular horizontally connected domain, \( p \) be a branch of the complex momentum analytic in \( K \), and \( z_1, z_2 \) be two regular points of the boundary of \( K \). We call \( K \) *canonical* with respect to \( p \) if, \( \forall z \in K \), there is a curve \( \gamma \) connecting \( z_1 \) and \( z_2 \) in \( K \), containing \( z \) and canonical with respect to \( p \).

2.5.2. Unbounded canonical domains. If \( M \) is a trigonometric polynomial, we consider the unbounded canonical domains that contain infinite vertical curves.

We call a domain horizontally connected if \( |\text{Re} \ z| \) stays bounded for all \( z \) in it.

Let \( K \subset \mathbb{C} \) be an unbounded regular, horizontally connected and horizontally bounded domain, let \( p \) be a branch of the complex momentum analytic in it. We call the domain \( K \) *canonical* with respect to \( p \) if for any \( z \in K \) there is an infinite curve \( \gamma \subset K \) canonical with respect to \( p \) and containing \( z \).

2.6. Main theorems. Below \( K \subset \mathbb{C} \) is a domain canonical with respect to a branch \( p \), and \( V^\pm \) are analytic eigenvectors of \( M \) normalized at \( z_0 \in K \) and corresponding to the eigenvalues \( e^{\pm i\pi(z)} \).

2.6.1. Locally uniform asymptotics. Let us recall that an asymptotic representation is locally uniform in a domain \( D \) if it is uniform in any fixed compact subset of \( D \).

First, we describe locally uniform asymptotics of solutions to \( (1.1) \). One has

**Theorem 2.2.** For sufficiently small \( h \), in \( K \) there exist \( \Psi^\pm \), two analytic solutions to \( (1.1) \), admitting the following locally uniform asymptotic representations :

\[
\Psi^\pm(z) = e^{\pm \frac{\pi}{2} \int_{z_0}^{z} p(z) \, dz} \left( V^\pm(z) + O(h) \right), \quad h \to 0. \quad (2.11)
\]

2.6.2. Asymptotics in unbounded domains. Here, we concentrate on the case where \( K \) is an unbounded canonical domain, and describe the behavior of the solutions \( \Psi^\pm(z) \) from Theorem 2.2 for large \( |\text{Im} \ z| \).

For a fixed \( \delta > 0 \), we call the domain \( K \) without the \( \delta \)-neighborhood of its boundary an admissible subdomain of \( K \).

**Theorem 2.3.** Let, in the case of the previous theorem, the domain \( K \) be unbounded, and \( A \) be its admissible subdomain. For sufficiently large \( Y \), for \( |\text{Im} \ z| \geq Y \) the solutions \( \Psi^\pm \) admit in \( A \) the following uniform asymptotic representations :

\[
\Psi^\pm(z) = e^{\pm \frac{\pi}{2} \int_{z_0}^{z} p(z) \, dz + g} \left( \frac{V^\pm_1(z)(1 + O(h))}{V^\pm_2(z)(1 + O(h))} \right), \quad h \to 0. \quad (2.12)
\]
Here \(|g| \leq C (1 + |z|) h\) with a constant \(C > 0\) independent of \(h\). If \(\frac{M_{22}(z)}{M_{11}(z)} \to 0\) as \(|\ln z| \to \infty\), then \(|g| \leq C h\).

In (2.12) the \(O(h)\) decay exponentially as \(|y| \to \infty\) (for \(\Psi^+\) see 5.41 and 5.53).

2.6.3. Known results for differential equations. For equation (1.2), for sufficiently small \(h\) one constructs vector solutions \(\Psi_j, \ j = 1, 2,\) such that

\[
\Psi_j(x) = \frac{1}{\sqrt{2\pi i}} \int_{\gamma_j} e^{\frac{1}{2} \int_{z_0}^x pj_{ij} dx - \frac{1}{2} \int_{z_0}^x \sqrt{\frac{\rho_{ij}(x)}{\rho_{ij}(z_0)}} dx} r_j, \quad h \to 0.
\] (2.13)

where \(p_j\) are eigenvalues of \(M\), and \(l_j\) and \(r_j\) are the corresponding left and right eigenvectors, see section 4 of chapter 5 in [8], and we have written only the leading terms of the asymptotics.

The expressions \(-\int_{x_0}^x \frac{j_{ij}(x)}{\rho_{ij}(x)} dx, \ j = 1, 2\) are often called geometric phases or Berry phases (see [2]) and have a well-known geometric interpretation (see [23]).

2.6.4. An example. Let us consider the scalar difference equation (1.3) with an analytic function \(v\). A vector function \(\Psi\) satisfies (1.1) with the matrix \(M(z) = \begin{pmatrix} -v(z) & -1 \\ 1 & 0 \end{pmatrix}\) if and only if \(\Psi(z) = \begin{pmatrix} \psi(z) \\ \psi(z-h) \end{pmatrix}\) where \(\psi\) is a solution to (1.3).

Let us deduce the quasiclassical asymptotics of solutions to (1.3) from Theorem 222.

For the above \(M(z)\), the complex momentum is defined by the relation \(2 \cos p(z) + v(z) = 0\), and as eigenvectors of \(M(z)\) one can choose

\[
r^\pm = \begin{pmatrix} 1 \\ e^{\mp ip(z)} \end{pmatrix} \quad \text{and} \quad l^\pm = \begin{pmatrix} e^{\pm ip(z)} \\ -1 \end{pmatrix}.
\]

Then

\[
\int_{z_0}^z \Omega_{\pm} = \frac{i}{2} \int_{z_0}^z \left( p'(s) + \frac{2 p'(s) e^{\mp ip(s)}}{e^{\pm ip(s)} - e^{\mp ip(s)}} \right) ds = -\frac{i}{2} \int_{z_0}^z p'(s) \left( e^{ip(s)} + e^{-ip(s)} \right) ds = -\ln \sqrt{\sin p(z)}_{z_0}.
\]

This leads to the following formulas for two analytic solutions to (1.3) :

\[
\psi_{\pm}(z) = \frac{1}{\sqrt{\sin p(z)}} e^{\pm \int_{z_0}^z p(z) dz + O(h)}.
\] (2.14)

These formulas were obtained in, e.g., [17].

3. The meromorphic differentials \(\Omega_{\pm}\)

3.1. Preliminaries.

3.1.1. The left and right eigenvectors of unimodular 2\times2-matrices. Here, we assume only that \(M \in SL(2, \mathbb{C})\). Then the eigenvalues of \(M\) are of the form \(e^{\pm ip}\), where \(p\) is a complex number. We assume that \(e^{ip} \neq e^{-ip}\). Let \(r^\pm\) be right eigenvectors of \(M\) corresponding to the eigenvalues \(e^{\pm ip}\). One has

**Lemma 3.1.** The row vectors defined by formula (2.5) are left eigenvectors corresponding to the eigenvalues \(e^{\pm ip}\).

**Proof.** As \(M \in SL(2, \mathbb{C})\), we have

\[
\sigma M = (M^{-1})^T \sigma.
\]

Hence

\[
l^T M = r^\pm^T \sigma M = r^\pm^T (M^{-1})^T \sigma = (M^{-1} r^\pm)^T \sigma = e^{\mp ip} r^\pm^T \sigma = e^{\mp ip} l^\pm.
\]

\(\square\)
Proof. One has
\[ l^\pm r^\pm = \pm \det (r^+ - r^-), \quad \text{and} \quad l^\pm r^0 = 0, \] (3.1)
where \((r^+ - r^-)\) is the matrix with the columns \(r^+\) and \(r^-\).

Lemma 3.2. One has
\[ l^\pm r^\pm = r^\pm T \sigma r^\pm = -(\sigma r^\pm)T r^\pm = - (\sigma r^\pm, r^\pm)_{\mathbb{R}^2} = - \det (r^+ r^-) = \pm \det (r^+ r^-). \]
This proves the first two equalities. The remaining two are proved similarly. □

Lemma 3.2 can be equivalently formulated in the following form. Let us denote by \((l^+, l^-)\) the matrix with the rows \(l^+\) and \(-l^-.\) One has

\[
\begin{pmatrix}
  l^+ \\
  -l^-
\end{pmatrix} (r^+ r^-) = \det (r^+ r^-) \cdot I.
\] (3.2)

3.1.2. Analytic solutions to equation (2.4) and differentials \(\Omega_\pm\). Let us come back to (2.4). Let \(R \subset D\) be a simply connected regular domain, and let \(p\) be a branch of the complex momentum analytic in \(R\). By Theorem 2.1, up to constant factors, the vectors \(V_\pm (z)\) are independent of the choice of \(r_\pm\), analytic solutions to (2.4), used to construct them. Throughout this paper \(r^\pm\) are vectors given by the formulas:

\[
r^\pm (z) = \begin{pmatrix} M_{12} (z) \\ e^{\mp ip(z)} - M_{11} (z) \end{pmatrix}.
\] (3.3)

One has
\[
\det (r^+ (z) r^- (z)) = - 2i M_{12} (z) \sin p (z).
\] (3.4)

As \(R\) is regular, \(\sin p (z) \neq 0\). So, the determinant vanishes only at zeros of \(M_{12}\). Actually, one has

Lemma 3.3. If \(M_{12} (z) = 0\) at \(z \in R\), then one and only one of the vectors \(r^\pm (z)\) equals zero.

Proof. As \(M_{12} (z) = 0\), the numbers \(M_{11} (z)\) and \(M_{22} (z)\) are eigenvalues of \(M (z)\). As \(z\) is regular, \(M_{11} (z) \neq M_{22} (z)\). So, either \(M_{11} (z) = e^{ip (z)}\) and \(M_{11} (z) \neq e^{-ip (z)}\) or \(M_{11} (z) = e^{-ip (z)}\) and \(M_{11} (z) \neq e^{ip (z)}\). This and (3.3) imply the statement. □

Now, we define two row vectors \(l^\pm (z)\) by the formula (2.4). One has
\[
l^\pm (z) = \begin{pmatrix} e^{\mp ip(z)} - M_{11} (z) \\ - M_{12} (z) \end{pmatrix}.
\] (3.5)

Let us compute the differentials \(\Omega_\pm\) corresponding to the chosen \(r^\pm\).

For our choice of \(r^\pm\), formulas (3.4) and (3.3) imply that
\[
l^\pm (z) r^\pm (z) = \mp 2i M_{12} (z) \sin p (z).
\] (3.6)

Using (3.3), (3.4), and the definitions of \(\Omega_\pm\), we prove that
\[
\Omega_\pm (z) = \mp \frac{idp (z)}{2} \pm \frac{e^{\mp ip(z)} - M_{11} (z)}{2i \sin p (z)} \cdot d \ln M_{12} (z) - d (e^{\pm ip(z)} - M_{11} (z)).
\] (3.7)

In the rest of this paper, \(\Omega_\pm\) are the differentials given by these formulas.
3.2. Differentials $\Omega_{\pm}$ in regular domains. Let, again, $R$ be a regular domain. By (3.7), $\Omega_{\pm}$ can have poles in $R$ only at the points where $M_{12}(z) = 0$ (as $e^{\pm ip(z)}$ differ in $R$). Let $z_0 \in R$, and let $M_{12}(z_0) \neq 0$. One has

**Proposition 3.1.** Pick $s \in \{\pm\}$. Let $z_* \in R$ and $M_{12}(z_*) = 0$. If $r^s(z_*) \neq 0$, then $\Omega_s$ is holomorphic at $z_*$. If $r^s(z_*) = 0$, then the function $z \mapsto \exp \left( \int_{z_0}^{z} \Omega_s(z) \right) r^s(z)$ is analytic and does not vanish at $z_*$. In this case, in a neighborhood of $z_*$, one has

$$\Omega_s(z) = -d \ln M_{12}(z) + a \text{ holomorphic differential}. \quad (3.8)$$

This proposition and Lemma 3.3 give quite a complete description of the poles of $\Omega_{\pm}$ in a regular domain.

**Proof.** For the sake of definiteness, we assume that $s = +$. By (3.7), one has

$$\Omega_+(z) = \frac{e^{-ip(z)} - M_{11}(z)}{2i \sin p(z)} d \ln M_{12}(z) + (a \text{ holomorphic differential}), \quad z \sim z_. \quad (3.9)$$

First, let us assume that $r^+(z_*) \neq 0$. Then, by Lemma 3.3, $r^-(z_*) = 0$, and, therefore, by (3.7), one has $M_{11}(z_*) = e^{-ip(z_*)} \neq e^{ip(z_*)}$. But

$$\frac{e^{-ip(z)} - M_{11}(z)}{M_{12}(z)} = -\frac{M_{21}(z)}{e^{ip(z)} - M_{11}(z)}. \quad (3.10)$$

Indeed, as $2 \cos p(z) = M_{11}(z) + M_{22}(z)$, and $M_{11}(z)M_{22}(z) - M_{12}(z)M_{21}(z) = 1$, $(e^{-ip(z)} - M_{11}(z))(e^{ip(z)} - M_{11}(z)) = 1 - 2 \cos p(z)M_{11}(z) + M_{11}(z)^2 = -M_{12}(z)M_{21}(z)$.

Formulas (3.8) and (3.10) imply that $\Omega_+$ is holomorphic at $z_*$. Let us assume that $r^+(z_*) = 0$. By (3.7), one has $M_{11}(z_*) = e^{ip(z_*)} \neq e^{-ip(z_*)}$. So,

$$\frac{e^{-ip(z)} - M_{11}(z)}{2i \sin p(z)} = \frac{e^{-ip(z)} - e^{ip(z)}}{2i \sin p(z)} + o(1) = -1 + o(1) \quad \text{as} \quad z \to z_*.$$

Therefore (3.7) implies that, in a neighborhood of $z_*$, one has (3.8). This and (3.3) imply that up to a non-vanishing analytic factor

$$\exp \left( \int_{z_0}^{z} \Omega_+(z) \right) r^+(z) \sim \left( \frac{1}{e^{ip(z)} - M_{11}(z)} \right), \quad z \sim z_* \quad (3.11)$$

Now, the analyticity of $z \mapsto \exp \left( \int_{z_0}^{z} \Omega_+(z) \right) r^+(z)$ follows from the equality

$$\frac{e^{ip(z)} - M_{11}(z)}{M_{12}(z)} = -\frac{M_{21}(z)}{e^{-ip(z)} - M_{11}(z)}$$

that is equivalent to (3.10). The fact that the right hand side in (3.11) does not vanish at $z_*$ is obvious. This completes the proof of Proposition 3.1. \hfill \Box

Finally, we check

**Lemma 3.4.** In $R$ one has

$$\Omega_+ + \Omega_- = -d \ln \det(r^+ r^-). \quad (3.12)$$

**Proof.** The statement follows from (3.7) and (3.4). \hfill \Box
3.3. Proof of Theorem 2.1 Let \( z_0 \in R \) and \( r_\pm(z_0) \neq 0. \) Then, \( M_{12}(z_0) \neq 0, \) and \( \det(r^+(z_0) r^{-}(z_0)) \neq 0. \)

Let us construct \( V^\pm \) in terms of \( r^\pm \) defined by (3.2). As in \( R \) the differentials \( \Omega^\pm \) have poles only at zeros of \( M_{12} \) (see (3.4.1)), then, in view of Proposition 3.1, \( V^\pm \) are analytic in \( R. \) As, outside the set of zeros of \( M_{12}, \) \( r^\pm \) do not vanish and \( \Omega^\pm \) are holomorphic, the same proposition implies that \( V^\pm \) do not vanish in \( R. \)

Let us check (2.3). Near \( z_0 \) the \( \Omega^\pm \) are holomorphic, and, using Lemma 2.3, we get

\[
\det(V^+(z) V^-(z)) = \exp \left( \int_{z_0}^{z} \Omega^+ + \Omega^- \right) \det(r^+(z) r^-(z)) = \det(r^+(z_0) r^-(z_0)).
\]

This is formula (2.4). It is valid in the whole domain \( R \) as \( \det(V^+ V^-) \) is analytic. Now, let us check that in \( R \) any analytic eigenvectors \( \tilde{V}^\pm \) normalized at \( z_0 \) coincide with \( V^\pm \) up to constant factors. For this, we consider \( \tilde{r}^\pm, \) two nontrivial solutions to (2.3) analytic in \( R \) and such that \( \det(\tilde{r}^+(z_0) \tilde{r}^-(z_0)) \neq 0. \) In terms of \( \tilde{r}^\pm, \) we define \( \tilde{l}^\pm, \Omega^\pm \) and \( \tilde{V}^\pm \) as we defined \( l^\pm, \Omega^\pm \) and \( V^\pm \) in terms of \( r^\pm. \) One has

\[
\tilde{r}^\pm(z) = c^\pm(z) r^\pm(z), \quad z \in R,
\]

where \( c^\pm \) are nontrivial functions meromorphic in \( R. \) Clearly, \( c^\pm \) can vanish only at points where \( \tilde{r}^\pm \) vanish, and \( c^\pm \) can have poles only at points where \( r^\pm \) vanish.

So, near \( z_0 \) the functions \( c^\pm \) are analytic, do not vanish, and one has

\[
\int_{z_0}^{z} \tilde{l}^\pm d\tilde{r}^\pm = \int_{z_0}^{z} l^\pm dr^\pm + \ln c^\pm|_{z_0}^z,
\]

\[
\tilde{c}^\pm(z) = c^\pm(z_0) c^\pm(z).\]

This implies that in \( R, \) one has \( \tilde{V}^\pm = c^\pm(z_0) V^\pm, i.e., \) any analytic eigenvectors normalized at \( z_0 \) coincide with \( V^\pm \) up to constant factors.

Finally, as \( \tilde{V}^\pm(z) = c^\pm(z_0) V^\pm(z) \) and \( \tilde{r}^\pm(z) = c^\pm(z) r^\pm(z), \) formula (2.3) valid for \( V^\pm \) is valid also for \( \tilde{V}^\pm. \) This completes the proof of Theorem 2.1. \( \square \)

Corollary 3.2 (from the proof of Theorem 2.1). Let \( \tilde{r}^\pm \) be nontrivial analytic solutions to equation (2.3) and \( \tilde{l}^\pm \) be constructed by formula (2.3). Then \( z \mapsto \tilde{l}^\pm(z) \tilde{r}^\pm(z) \) are nontrivial analytic functions.

Proof. The statement follows from (3.2) as \( \tilde{r}^\pm(z) = c^\pm r^\pm, \) where \( c^\pm \) are nontrivial meromorphic functions. \( \square \)

3.4. \( \Omega^\pm \) near turning points.

3.4.1. The complex momentum near a turning point. Let \( z_0 \in D \) be a turning point for equation (1.1). We call it simple if \( (\text{Tr } M)'(z_0) \neq 0. \)

One can easily see that, near a simple turning point \( z_0, \) the complex momentum is an analytic function of \( \tau = \sqrt{z - z_0}, \) and one has

\[
p(z) = p(z_0) + p_1 \tau + o(\tau), \quad \tau \to 0,
\]

(3.14) where \( p_1 \) is a non-zero constant. Below, near a simple branch point \( z_0, \) we choose \( \tau = \sqrt{z - z_0} \) as the local coordinate.

3.4.2. \( \Omega^\pm \) near a turning point. Let \( z_0 \) be a simple turning point. One has

Lemma 3.5. The point \( z_0 \) is a simple pole of \( \Omega^\pm. \) If \( M_{12}(z_0) \neq 0, \) then \( \text{res } z_0 \Omega^\pm = -\frac{1}{2}. \) Otherwise, \( \text{res } z_0 \Omega^\pm = -\frac{1}{2}. \)

Proof. For the sake of definiteness, we prove this lemma only for \( \Omega_+. \) First, we consider the case where \( M_{12}(z_0) \neq 0. \) Let us consider the terms in the right hand side of (3.15). In a neighborhood of \( \tau = \sqrt{z - z_0} = 0, \) one has:

- as \( p \) is analytic in \( \tau, \) and thus, \( dp \) is a holomorphic differential;
• as $z = z_0 + \tau^2$, $e^{-ip} - M_{11}$ is analytic in $\tau$;
• as $dz = 2\tau d\tau$, one has $dM_{11} = \tau g(\tau) d\tau$, where $g$ is analytic;
• as $M_{12}(z_0) \neq 0$, one has $d\ln M_{12} = \tau f(\tau) d\tau$, where $f$ is analytic;
• as $p(z_0) \in \mathbb{Z}$, and in view of (3.14), $\sin \tau$ is analytic in $\tau$ and has a simple zero at $\tau = 0$.

These observations and formula (3.7) imply that, in a neighborhood of $\tau = 0$,

$$
\Omega_+ = -\frac{de^ip}{2i \sin p} + \text{a holomorphic differential}.
$$

As $\frac{de^ip}{2i \sin p} = \frac{1}{2} d\ln(e^{2ip} - 1)$, and as $e^{2ip} - 1$ has a simple zero at $z_0$, see formula (3.14), this implies that $z_0$ is a simple pole of $\Omega_+$, and that res$_{z_0} \Omega_+ = -1/2$.

Now, we assume that $M_{12}(z_0) = 0$. Then, in a neighborhood of $\tau = 0$, the differential

$$
\Omega_+ - \left(\frac{(e^{-ip} - M_{11})}{2i \sin p} \frac{d \ln M_{12}}{\tau} - \frac{de^ip}{2i \sin p}\right)
$$

is holomorphic. Let us consider the first term in the brackets.

We have $e^{ip(z_0)} = e^{-ip(z_0)}$. On the other hand, as $M_{12}(z_0) = 0$, and as det $M \equiv 1$, $M_{11}(z_0)$ and $M_{22}(z_0)$ are eigenvalues of $M(z)$. So, we have $M_{11}(z_0) = M_{22}(z_0)$.

Using the definition of the complex momentum, we get

$$
e^{-ip} - M_{11} = \frac{e^{-ip} - M_{11}}{2} + \frac{M_{22} - e^{ip}}{2} = -i \sin p + \frac{M_{22} - M_{11}}{2}.
$$

Therefore, near $\tau = 0$, one has

$$
\frac{e^{-ip} - M_{11}}{2i \sin p} = -\frac{1}{2} + O(\tau). \quad (3.16)
$$

Now, to complete the proof, it suffices to check that near $\tau = 0$

$$
d\ln M_{12} = \frac{2d\tau}{\tau} + \text{a holomorphic differential.} \quad (3.17)
$$

Indeed, this and (3.16) imply that the first term in the brackets in (3.15) has a simple pole with the residue equal to $-1$. On the other hand, we have already seen that at $\tau = 0$ the second term in the brackets has a simple pole with the residue equal to $-\frac{1}{2}$.

These observations lead to the second statement of the lemma.

As $\frac{d\tau}{\tau} = \frac{dz}{z}$, to prove representation (3.17), we need only to check that the zero of $M_{12}$ at $z_0$ is simple. As det $M \equiv 1$, and as $M_{11}(z_0) = M_{22}(z_0) \neq 0$, we have

$$M_{12}(z_0)|_{z=0} = M_{11}'M_{22} + M_{12}'M_{21} |_{z=0} = M_{11}(z_0)(\text{Tr} M)'(z_0). \quad (3.18)
$$

So, as $z_0$ is a simple turning point, one has $M_{12}(z_0) \neq 0$. The proof is completed. \[ \square \]

3.5. The behavior of $p$ and $\Omega_\pm$ as $\text{Im } z \to \infty$. Below, we assume that $M$ is a trigonometric polynomial satisfying the assumptions formulated in section 2.1.

We assume that $Y > 0$ is so large that the half-planes $\mathbb{C}_+(Y) = \{\text{Im } z \geq Y\}$ and $\mathbb{C}_d(Y) = \{\text{Im } z \leq -Y\}$ are regular, and $M_{12}$ does not vanish in them.

Here, we study the complex momentum and $\Omega_\pm$ in $\mathbb{C}_u(Y)$ and $\mathbb{C}_d(Y)$. In particular, we get their asymptotic representations as $|y| \to \infty$, $y = \text{Im } z$.

Below $C$ denotes different positive constants, and $O(f(z))$ denotes an expression bounded by $C|f(z)|$ in the domain we consider.

For a trigonometric polynomial $P$, $P(z) = \sum_{j=-l}^k P_j e^{2\pi jz}$, where $P_j$ are Fourier coefficients, and $P_l \neq 0$, we let $P_0 = P_{-l}$, $P_0 = P_k$, $n_u(P) = l$ and $n_d(P) = k$.

Let $t = \text{tr } M$. In view our assumptions made in section 2.1, one has

$$n_u(M_{12}), \ n_u(M_{21}), \ n_u(M_{22}) \leq n_u(M_{11}) = n_u(t) > 0, \quad s \in \{u, d\}. \quad (3.19)$$
We also note that this and the equality det $M \equiv 1$ imply that
\[ n_u(M_{22}) \leq n_u(M_{12}). \quad (3.20) \]

### 3.5.1. The behavior of the complex momentum.

Let us fix in $\mathbb{C}_u(Y)$ an analytic branch $p$ of the complex momentum. In view of (2.23), one has
\[ p(z) = s_u \left( 2\pi n_u(t)z + i \ln t_u + O(e^{-2\pi y}) \right), \quad y = \text{Im } z, \quad z \in \mathbb{C}_u(Y), \quad (3.21) \]
where $s_u \in \{\pm 1\}$ and the branch of $\ln$ is determined by the choice of the branch $p$. We note that by our assumptions $n_u(t) > 0$, see section 2.1

By means of the Cauchy estimates for the derivatives of analytic functions, we deduce from (3.21) the estimates:
\[ p'(z) = 2s_u\pi n_u(t) + O(e^{-2\pi y}), \quad p''(z) = O(e^{-2\pi y}), \quad y = \text{Im } z, \quad z \in \mathbb{C}_u(Y). \quad (3.22) \]

We also note that
\[ p(z+1) = p(z) + 2s_u\pi n_u(t), \quad z \in \mathbb{C}_u(Y). \quad (3.23) \]

Indeed, it follows from (2.23), that $p(\cdot + 1)$ is a branch of the complex momentum analytic in $\mathbb{C}_u(Y)$. This and (2.23) imply that $p(\cdot + 1) = sp(\cdot) \bmod 2\pi$, $s \in \{\pm 1\}$. This and (3.21) imply (3.23).

Let us fix in $\mathbb{C}_d(Y)$ an analytic branch $p$ of the complex momentum. Reasoning as for $\text{Im } z \geq Y$, we now prove that
\[ p(z) = s_d \left( -2\pi n_d(t)z + i \ln t_d + O(e^{-2\pi y}) \right), \quad z \in \mathbb{C}_d(Y), \quad (3.24) \]
where $s_d \in \{\pm 1\}$. Note that $n_d(t) > 0$. Furthermore, we have
\[ p'(z) = -2s_d\pi n_d(t) + O(e^{-2\pi y}), \quad p''(z) = O(e^{-2\pi y}), \quad z \in \mathbb{C}_d(Y), \quad (3.25) \]
and
\[ p(z+1) = p(z) - 2s_d\pi n_d(t), \quad z \in \mathbb{C}_d(Y). \quad (3.26) \]

### 3.5.2. The behavior of the complex momentum.

Let $p$ be a branch of the complex momentum analytic in $\mathbb{C}_u(Y)$ satisfying (3.21) with $s_u = 1$. Here we study in $\mathbb{C}_u(Y)$ the differentials $\Omega_Y$ defined in terms of this branch $p$ by (3.24).

The half-plane $\mathbb{C}_u(Y)$ being regular, we can represent there $\Omega_{\pm}(z) = \omega_{\pm}(z)dz$. For our choice of $Y$, the functions $\omega_{\pm}$ are analytic in $\mathbb{C}_u(Y)$.

Thanks to (3.23), one has
\[ \omega_{\pm}(z+1) = \omega_{\pm}(z), \quad z \in \mathbb{C}_u(Y). \quad (3.27) \]

Let us check

**Proposition 3.2.** For sufficiently large $Y$ and $z \in \mathbb{C}_u(Y)$, one has
\[ \omega_+(z) = \pi in_u(t) + O(e^{-2\pi y}), \]
\[ \omega_-(z) = \pi in_u(t) + 2\pi in_u(M_{12}) + O(e^{-2\pi y}). \quad (3.28) \]

**Proof.** Let us begin with $\omega_+$. Using (3.7) and (2.23), we get
\[ \omega_+(z) = -\frac{i p'(z)}{2} - \frac{M_{22}(z) M'_{12}(z)}{M_{12}(z)} + M'_{11}(z) - e^{ip(z)} \left( \frac{M'_{12}(z)}{M_{12}(z)} + ip'(z) \right). \]

In view of (3.22) and as $s_u = 1$, one has
\[ e^{ip(z)} = O(e^{-2\pi y}), \quad e^{-ip(z)} / \text{tr } M(z) \to 1, \quad y \to \infty. \quad (3.29) \]

This and (3.22) lead to the formula
\[ \omega_+(z) = -\pi in_u(t) - \frac{M_{22}(z) M'_{12}(z)}{M_{12}(z) + M_{22}(z)} + O(e^{-2\pi y}). \quad (3.30) \]
Now, we consider the case where \( n_u(M_{12}) = n_u(M_{11}) \). Then as \( z \to \infty \) one has
\[
\frac{M_{12}(z)}{M_{12}(z)} = -2\pi in_u(M_{11}) + O(e^{-2\pi y}), \quad \frac{M_{11}(z)}{M_{12}(z)} = -2\pi in_u(M_{11}) + O(e^{-2\pi y}),
\]
and, in view of (3.19), we get
\[
\frac{M_{22}(z)\frac{M_{12}(z)}{M_{12}(z)} + M_{11}(z)}{M_{11}(z) + M_{22}(z)} = -2\pi in_u(t) + O(e^{-2\pi y}). \tag{3.31}
\]
This and (3.30) leads to the first formula in (3.28).

To complete the proof, we have to analyze the case where \( n_u(M_{12}) < n_u(M_{11}) \).
Then, in view of (3.20), one has \( M_{22}(z)/M_{11}(z) = O(e^{-2\pi y}) \), and we again come to (3.31), and thus to the first formula in (3.28). This complete its proof.

Now, let us turn to \( \omega_- \). Instead of (3.30), we now get
\[
\omega_-(z) = \pi in_u(t) \frac{M_{12}(z)\frac{M_{12}(z)}{M_{12}(z)} + M_{11}(z)}{M_{11}(z) + M_{22}(z)} + O(e^{-2\pi y}), \tag{3.32}
\]
and considering consequently the case where \( n_u(M_{12}) = n_u(M_{22}) \) and then the case where \( n_u(M_{12}) > n_u(M_{22}) \) (and, therefore, \( n_u(M_{11}) > n_u(M_{22}) \)) we prove that
\[
\frac{M_{11}(z)\frac{M_{12}(z)}{M_{12}(z)} + M_{22}(z)}{M_{11}(z) + M_{22}(z)} = -2\pi in_u(M_{12}) + O(e^{-2\pi y}).
\]
This leads to the second formula in (3.28). The proof is complete. \( \Box \)

Let \( p \) be a branch of the complex momentum analytic in \( C_d(Y) \) and satisfying (3.24) with \( s_d = 1 \). Now, we study in \( C_d(Y) \) the \( \Omega_\pm \) defined in terms of this \( p \) by (3.27). One has \( \Omega_\pm(z) = \omega_\pm(z)dz \), where \( \omega_\pm \) are analytic in \( C_d(Y) \) functions.

We get the formula
\[
\omega_\pm(z + 1) = \omega_\pm(z), \quad z \in C_d(Y), \tag{3.33}
\]
and

**Proposition 3.3.** Let \( Y \) be sufficiently large. Then in \( C_d(Y) \)
\[
\omega_+(z) = -\pi in_d(M_{11}) + O(e^{-2\pi |y|}), \quad \omega_-(z) = -\pi in_d(M_{11}) - 2\pi in_d(M_{12}) + O(e^{-2\pi |y|}). \tag{3.34}
\]
The proof of this proposition being similar to one of Proposition 3.2, we omit it.

3.6. **Remarks on the Riemann surface of \( \Omega_\pm \).** The differentials \( \Omega_\pm \) are two branches of a meromorphic differential \( \Omega \) defined on the Riemann surface of the analytic function \( w : z \mapsto e^{\Omega(z)} \) (this Riemann surface has two sheets).

As \( trM \) is a trigonometric polynomial, it is natural to consider \( w \) as a function of the variable \( u = e^{2\pi iz} \). Then the Riemann surface \( \Gamma \) of \( w \) appears to be a hyperelliptic curve. In particular, in the case where \( trM \) is a first order trigonometric polynomial, relation (2.28) implies that
\[
w + 1/w = t_1u + t_0 + t_{-1}/u, \quad u \in \mathbb{C}, \tag{3.35}
\]
where \( t_1, t_0 \) and \( t_{-1} \) are constants, and \( |t_1|^2 + |t_{-1}|^2 \neq 0 \). Therefore \( w \) is single-valued on the Riemann surface of the function \( u \mapsto \sqrt{(t_1u^2 + t_0u + t_{-1})^2 - 4u^2} \), which is a hyperelliptic curve of genus one, see [21].

The analysis done in the previous sections shows that on \( \Gamma \) the differential \( \Omega \) has simple poles at zeros of \( M_{12} \) (on the sheets where \( w(z) - M_{11}(z) \) vanishes), at all the branch points of \( p \), at zero and at infinity.

The fact that \( \Omega \) is meromorphic on a hyperelliptic curve \( \Gamma \) is important for applications of the complex WKB method. In particular, it implies that the differential
We prove with

We define the matrix $\Psi_0$ equation (1.1) in the form $\Psi(\pm \infty)$ leading terms in (2.11), i.e., the vectors $V_\pm \in R$. Also, for a matrix-function $A$, $A^{-1}(z)$ is the matrix inverse to $A(z)$.

In this section all the estimates and asymptotics are locally uniform in $z$. We pick $z_0 \in R$ so that $\det(r^+(z_0) - r^-(z_0)) \neq 0$, and define in terms of $\Omega_\pm$ and $r^\pm$ the analytic eigenvectors $V^\pm$ of $M$ normalized at $z_0$.

4. THE PROOF OF THEOREM 2.2 FOR BOUNDED CANONICAL DOMAINS

We prove Theorem 2.2 by reducing the analysis of equation (1.1) to analyzing a finite difference equation of precisely the same form as the one studied in [15]. Below $R$ is a regular horizontally connected domain, and $p$ is a branch of the complex momentum analytic in $R$. We always assume that $z,z+h \in R$.

Also, for a matrix-function $A$, $A^{-1}(z)$ is the matrix inverse to $A(z)$.

In this section all the estimates and asymptotics are locally uniform in $z$. We pick $z_0 \in R$ so that $\det(r^+(z_0) - r^-(z_0)) \neq 0$, and define in terms of $\Omega_\pm$ and $r^\pm$ the analytic eigenvectors $V^\pm$ of $M$ normalized at $z_0$.

4.1. ASYMPTOTIC TRANSFORMATION OF THE MATRIX IN (1.1)

Let us note that the leading terms in (2.11), i.e., the vectors

$$\Psi_0(z) = e^{\pm \int_{z_0}^{z_0} p(z) dz} V^\pm(z),$$

are eigenvectors of $M(z)$, corresponding to its eigenvalues $e^{\pm ip(z)}$. In view of (2.9),

$$\det(\Psi_0(z)) = \det(r^+(z_0) - r^-(z_0)).$$

We define the matrix $\Psi_0(z) = (\Psi_0^+(z) \Psi_0^-(z))$ and represent a vector solution $\Psi$ to equation (1.1) in the form $\Psi(z) = \Psi_0(z) X(z)$. Then $X$ satisfies the equation

$$X(z+h) = T(z) X(z)$$

with

$$T(z) = \Psi_0^{-1}(z+h) M(z) \Psi_0(z).$$

We prove

Proposition 4.1. As $h \to 0$, one has

$$T(z) = \mathcal{I} + \left( \begin{array}{cc} O(h^2) & O(h) e^{-2 \theta(z)/h} \\ O(h) e^{2 \theta(z)/h} & O(h^2) \end{array} \right), \quad \theta(z) = \int_{z_0}^{z} p(z) dz.$$

Proof. As $\Psi_0^\pm(z)$ are eigenvectors of $M(z)$ corresponding to its eigenvalues $e^{\pm ip(z)}$,

$$T(z) = \Psi_0^{-1}(z+h) \Psi_0(z) \left( \begin{array}{cc} e^{ip(z)} & 0 \\ 0 & e^{-ip(z)} \end{array} \right).$$

In view of (4.1), we have

$$\Psi_0(z) = V(z) \left( \begin{array}{cc} e^{\theta(z)/h} & 0 \\ 0 & e^{-\theta(z)/h} \end{array} \right), \quad V(z) = (V^+(z) V^-(z)).$$

Formulas (4.6) and (4.7) imply that

$$T(z) = \left( \begin{array}{cc} e^{-2 \theta(z)/h} & 0 \\ 0 & e^{2 \theta(z)/h} \end{array} \right) W(z) \left( \begin{array}{cc} e^{\theta(z)+ip(z)/h} & 0 \\ 0 & e^{-\theta(z)-ip(z)/h} \end{array} \right),$$

where $W(z) = V^{-1}(z+h) V(z)$. To continue, we need

Lemma 4.1. As $h \to 0$, one has

$$W(z) = \left( \begin{array}{cc} e^{ibp'/2+O(h^2)} & O(h) \\ O(h) & e^{-ibp'/2+O(h^2)} \end{array} \right).$$
Proof. Using the Taylor’s theorem, we get
\[
W(z) = V^{-1}(z + h)V(z) = I + h(V^{-1})'(z)V(z) + O(h^2).
\]
As \((V^{-1})'V = 0\), one has \((V^{-1})'^2V = -V^{-1}V'\), and
\[
W(z) = I - hw(z) + O(h^2), \quad w(z) = V^{-1}(z)V'(z).
\]

It suffices to check that
\[
w_{11}(z) = -ip'(z)/2, \quad w_{22}(z) = ip'(z)/2.
\]

Let us prove the first formula.

Let \(e^\pm(z) = e^{\int z \Omega z}\). One has
\[
V(z) = (r^+(z)e^+(z) \quad r^-(z)e^-(z)) = (r^+(z) \quad r^-(z)) \begin{pmatrix} e^+(z) & 0 \\ 0 & e^-(z) \end{pmatrix},
\]

(4.11)

Therefore, in view of Corollary 3.1
\[
V^{-1}(z) = \frac{1}{\det(r^+(z) r^-(z))} \begin{pmatrix} 1/e^+(z) & 0 \\ 0 & 1/e^-(z) \end{pmatrix} \begin{pmatrix} l^+(z) & l^-(z) \\ l^-(-z) & -l^+(z) \end{pmatrix},
\]

(4.12)

and using (3.1), we get finally
\[
w_{11}(z) = \frac{1}{l^+(z)r^+(z)} \frac{l^+(z)}{l^+(z)r^+(z)} (r^+ e^+)'(z),
\]

and using the definition of \(\Omega_\pm\), see (2.7), we get
\[
w_{11}(z) = \frac{e^+(z)}{e^+(z)} + \frac{l^+(z)(r^+)'(z)}{l^+(z)r^+(z)} = -ip'(z)/2.
\]

This proves the first formula in (4.10). The second one is checked similarly. □

As \(\theta(z+h) = \theta(z) + p(z+h)h + p'(z)h^2/2 + O(h^3)\), substituting representation (4.9) into formula (4.8), we come to (4.11). This completes the proof of Proposition 4.1. □

4.2. Solutions to equations (4.3) and (1.1). Equation (1.3) with a matrix \(T\) of the form (1.5) is precisely the equation we study in [13], see the beginning of section 4 and Lemma 4.1 in [15]. Most of [15] (sections 4–6) is devoted to the analysis of this equation. The results of this analysis are described as properties of a vector-function \(X\) defined in terms of \(X\) by formulas (5.1) and (5.2) in [15]. Below, we describe these results as properties of \(X\).

In [15], in the formula analogous to (1.5), \(p\) is a function analytic in a regular domain \(R\), and, in terms of this function \(p\), one defines the canonical domains exactly as in Section 2.3. Then one proves that, given a bounded canonical domain \(K \subset R\), for sufficiently small \(h\), there exist two solutions to equation (4.3) that are analytic in \(K\) and admit there as \(h \to 0\) the asymptotic representations

\[
X^+(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(h), \quad X^-(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left( e^{-2i\phi(z)} + O(h) \right).
\]

The representation for \(X^+\) follows from Lemma 5.1 in [15], the representation for \(X^-\) is obtained as described in section 6.3 in [15]. We omit further details and note only that \(X^\pm\) satisfy singular integral equations on a vertical curve \(\gamma\), and that the crucial observation is that if \(\gamma\) is a canonical curve, then the norms of the integral operators are small.

Having constructed \(X^\pm\), one constructs the solutions \(\Psi^\pm\) from Theorem 2.2 by
the formulas $\Psi^\pm(z) = \Psi_0(z)X^\pm(z)$. This completes the proof of Theorem 2.2 for bounded canonical domains.

5. THE PROOFS OF THEOREMS 2.2 AND 2.3 FOR UNBOUNDED CANONICAL DOMAINS

In [17] we studied the one-dimensional difference Schrödinger equations with the potentials being trigonometric polynomials. Now, we consider equation (1.1) with $M$ being a trigonometric polynomial and prove Theorems 2.2 and 2.3 for unbounded canonical domains by means of the method developed in [17].

Again, $R$ is a regular horizontally connected domain, and $p$ is a branch of the complex momentum analytic in $R$. As before $V^\pm$ are normalized at $z_0 \in R$.

Now we assume that the domain $R$ contains an infinite vertical curve and is horizontally bounded. Also, almost up to the end of this section, we assume that

$$e^{ip(z)} \to 0 \quad \text{as} \quad |\text{Im} \, z| \to \infty,$$

i.e., that the coefficients $s_u$ and $s_d$ in (5.7) and (5.5) are equal to +1.

Finally, $C$ denotes different positive constants independent of $h$, and, for $z$ being in the domains we consider, $O(f(z, h))$ is bounded by $C[f(z, h)]$.

5.1. Asymptotic transformation of the matrix in (1.1). We begin with transforming equation (1.1) as in the previous section.

5.1.1. The matrix $W(z)$. The statements of Lemma 4.1 remain valid in any compact subset of $R$. Now, we assume that $Y$ is so large that $\mathbb{C}_u(Y) \cup \mathbb{C}_d(Y)$ is regular and $M_{12}(z) \neq 0$ in $\mathbb{C}_u(Y) \cup \mathbb{C}_d(Y)$. Then we continue analytically the functions $p$, $r^\pm$, $V^\pm$ and $W$ in $\mathbb{C}_u(Y) \cup \mathbb{C}_d(Y)$ from $R$, and prove

Lemma 5.1. Let $s \in \{d, u\}$. If $Y$ is sufficiently large, then, for $z \in \mathbb{C}_s(Y)$ one has

$$W_{11}(z) = e^{i b^+/2 + h^2 g_1(z)}, \quad W_{22}(z) = e^{-i b^+/2 + h^2 g_2(z)},$$

$$g_1(z) = c_s + O(e^{-2\pi|y|}), \quad g_2(z) = -c_s + O(e^{-2\pi|y|}),$$

$$W_{12}(z) = O(h e^{-2\pi n_s(M_{12})|y|}), \quad W_{21}(z) = O(h e^{-2\pi n_u(M_{12})|y|}).$$

Here $c_s$ is a constant analytic in $h$. If $M_{12}(z)W_{11}(z) \to 0$ as $|y| \to \infty$, then $c_s = 0$.

Proof. Below, we assume that $z \in \mathbb{C}_u(Y)$; the case of $z \in \mathbb{C}_d(Y)$ is treated similarly. We use notations from the proof of Lemma 4.1. The analysis is broken into several steps. We begin with studying $W_{11}$.

1. Let $d_0 = \text{det}(r^+(z_0) r^-(z_0))$. By (4.11), (4.12) and (2.9),

$$W_{11}(z) = \frac{1}{d_0} e^{-z} e^{+} (z + h) r^+(z) r^+(z).$$

2. Using (3.14), we get

$$e^{-z} e^{+} (z + h) e^{+}(z) = e^{2\pi i (n_u(M_{11}) + n_u(M_{12})) + a_0 + a_1 h + O(e^{-2\pi y})},$$

where $a_0$ and $a_1$ are constants independent of $h$. Therefore,

$$e^{-z} e^{+} (z + h) e^{+}(z) = O(e^{-2\pi i (n_u(M_{11}) + n_u(M_{12})) y}).$$

We also note that in, view of (3.14), (5.3) and (5.4), $e^{-} (\cdot + h) e^{+} (\cdot)$ is 1-periodic.

3. By means of (3.14), (5.3) and (5.4), we check that

$$l^+(z + h) r^+(z) = (M_{22}(z + h) - e^{i p(z+h)}) M_{12}(z) + (M_{11}(z) - e^{i p(z)}) M_{12}(z + h).$$

Estimates (2.2) and (3.20), and formula (5.3) imply that

$$l^+(z + h) r^+(z) = O(e^{2\pi i (n_u(M_{11}) + n_u(M_{12})) y}).$$

4. We have chosen $Y$ so that $W_{11}$ is analytic in $\mathbb{C}_u(Y)$. Moreover, in view of (5.9), (5.7), and (5.5), it is bounded there.
5. The element $W_{11}$ is 1-periodic in $z \in \mathbb{C}_u(Y)$. Indeed, by the first step the product $e^+(z+h)e^-(z)$ is 1-periodic in $z$, and \((5.29)\) and the 1-periodicity of $M$ imply the 1-periodicity of $l^+$ and $r_2^-$. This and \((5.5)\) imply the needed.

6. Now we prove the representation for $W_{11}$ from \((5.2)\). As $\mathbb{C}_u(Y)$ is regular, $p$ is analytic there. By \((3.23)\) it is 1-periodic in $C_s$. Now we prove the representation for $W_{11}$ in \((5.2)\). Combining this with \((5.6)\), we see that

$$W_{11}(z) = e^{ihp(z)/2} + h^2c + O(h^2e^{-2\pi(\gamma-Y)}), \quad c = f(0)/h^2, \quad z \in \mathbb{C}_u(Y).$$

As $f(0) = O(h^2)$, $p$ satisfies \((8.22)\), and $Y$ is a fixed positive number, this implies the representation for $W_{11}$ from \((5.2)\).

7. Let us assume that $M_{21}(z)/M_{11}(z) \to 0$ as $|y| \to \infty$, and prove that $c_u \equiv 0$. Now, instead of \((5.9)\), for sufficiently large $y$, for $z \in C_u(Y)$, we get

$$l^+(z+h)r^-(z) = (M_{11}u)(M_{12}u) e^{-2\pi i(n_u(M_{11})+n_u(M_{12}))z} - f_1(z) + O(e^{-2\pi y}).$$

Combining this with \((5.6)\), we see that $W_{11}(z) = e^{i\theta_0+\phi_h+O(e^{-2\pi r})}$ with some constants $\theta_0$ and $\phi_h$ independent of $h$. In view of \((8.22)\), this representation implies that the constant $c_u$ in the formula for $W_{11}$ in \((5.7)\) is zero. The proof of the statements of Lemma \((5.7)\) concerning $W_{11}$ is completed.

8. Let us turn to $W_{12}$ and $W_{21}$. Instead of \((5.6)\), \((5.7)\) and \((5.8)\) we get

$$W_{12}(z) = 1/d_0 e^-(z+h)e^-(z) l^+(z+h)r^-(z),$$

$$e^-(z+h)e^-(z) = O(e^{-2\pi y}), \quad f_1(z) = O(e^{-2\pi y}),$$

$$l^+(z+h)r^-(z) = (M_{22}u)(M_{12}u) e^{-2\pi i(n_u(M_{22})+n_u(M_{12}))z} + f_2(z), \quad f_2(z) = O(e^{-2\pi y}).$$

Let us assume that $e^{ip(z)} = o(M_{22}(z))$ as $y \to \infty$. If $Y$ is sufficiently large, then

$$M_{12}(z) = (M_{12}u)e^{-2\pi i n_u(M_{12})z+f_1(z)}, \quad f_1(z) = O(e^{-2\pi y}),$$

$$M_{22}(z) = e^{ip(z)} = (M_{22}u)e^{-2\pi i n_u(M_{22})z+f_2(z)}, \quad f_2(z) = O(e^{-2\pi y}),$$

where $f_1$ and $f_2$ are analytic in $z$. These formulas and \((5.12)\) imply that

$$l^+(z+h)r^-(z) = (M_{22}u)(M_{12}u) e^{-2\pi i(n_u(M_{22})+n_u(M_{12}))z} + f_2(z) \times$$

$$e^{-2\pi i n_u(M_{21})z+f_2(z)} - f_2(z) = O(e^{-2\pi y}),$$

Using the Cauchy estimates for the derivatives of analytic functions, in $\mathbb{C}_u(Y)$ (possibly with a larger $Y$) we get

$$f_j(z+h) - f_j(z) = O(h e^{-2\pi y}), \quad j \in \{1, 2\}. \quad (5.16)$$

Therefore,

$$l^+(z+h)r^-(z) = O(h e^{-2\pi y}).$$

This and \((5.11)\) imply that

$$W_{12}(z) = O(h e^{-2\pi i n_u(M_{12})-n_u(M_{22})+n_u(M_{12}))y}) \text{ if } e^{ip(z)} = o(M_{22}(z)) \text{ as } y \to \infty.$$
and, therefore,

\[ W_{12}(z) = O(h e^{-2\pi n_u(M_{12})y - 2\pi y}) \text{ if } n_u(M_{22}) = n_u(M_{11}). \]

Finally, if \( M_{22}(z) = O(e^{ip(z)}) \), then, using (5.21), we get

\[ W_{12}(z) = O(h e^{-2\pi(2n_u(M_{11}) + n_u(M_{12}))y}). \] (5.17)

The obtained estimates lead to the estimate for \( W_{12} \) from (5.3). Similarly one proves the estimate for \( W_{21} \).

The proof of Lemma 5.1 is complete. \( \square \)

5.1.3. Completing the asymptotic transformation. To use the method developed in Proposition 4.1, we now get

Proposition 5.1. Let \( s \) be either \( u \) or \( d \). For sufficiently large \( Y \), in \( C_s(Y) \),

\[ T_{11}(z) = e^{h^2 c_s + O(h^2 e^{-2\pi |z|})}, \quad T_{22}(z) = e^{-h^2 c_s + O(h^2 e^{-2\pi |z|})}, \] (5.18)

\[ T_{12}(z) = e^{-\frac{2i\theta(z)}{h}} O(h e^{+2\pi(2n_u(M_{12}) - n_u(M_{11}))|y| - 2\pi |y|}), \]

\[ T_{21}(z) = e^{\frac{2i\theta(z)}{h}} O(h e^{-2\pi(2n_u(M_{12}) - n_u(M_{11}))|y|}). \] (5.19)

Proof. In view of (5.22) and (5.24), in \( C_s(Y) \cap R \), one has

\[ \theta(z + h) = \theta(z) + p(z)h + p'(z)h^2/2 + O(h^3 e^{-2\pi |y|}). \] (5.20)

This and (5.22) lead to (5.18). Moreover, using (5.21), we get the estimates

\[ T_{12}(z) = e^{-\frac{2i\theta(z)}{h}} O(e^{-2ip(z)}W_{12}(z)), \quad T_{21}(z) = e^{\frac{2i\theta(z)}{h}} O(e^{+2ip(z)}W_{21}(z)). \]

This and representations (5.21) and (5.24) with \( s_u = s_d = 1 \) lead to (5.19). \( \square \)

5.1.2. The matrix \( T \). Let us recall that \( T(z) \) is described by (4.8). In addition to Proposition 4.1, we now get

Definition 5.1. We call a vertical curve \( \gamma \) strictly vertical if the angles between \( \gamma \) and \( \mathbb{R} \) at all the points \( z \in \gamma \) are uniformly bounded away from zero.

First, we assume that \( R \) contains a strictly vertical curve \( \gamma \) with some its \( \delta \)-neighborhood \( V_\delta \) and its boundary \( \partial V_\delta \).

Next, for each \( j \in \{1, 2\} \), we fix a branch of \( \ln T_{jj} \) in the corresponding equation in (5.21). For this, we choose \( Y \) as in Proposition 5.1. In view of Propositions 4.1 and 5.1 for sufficiently small \( h \), we can choose and choose the branch of \( \ln T_{jj} \) that is analytic and equals \( O(h^2) \) in \( V_\delta \cup C_u \cup C_d \). Then, we prove
Lemma 5.2. For sufficiently small $h$, there exist functions $\phi_1$ and $\phi_2$ analytic in $V_3 \cup C_u \cup C_d$ and satisfying there the corresponding equations in (5.23). Moreover, in $V_3$ one has
\[
|\phi_j(z)| \leq C h(1 + |y|). \tag{5.24}
\]
In (5.24) the right hand side can be replaced by $Ch$ if $c_u = c_d = 0$.

Proof. Below we assume that $h$ is sufficiently small. We fix $j \in \{1, 2\}$. To construct $\phi_j$, a solution to the corresponding equation in (5.23), we use a known construction for a solution to a first order difference equation, see, e.g., section 3.5 in [3].

For $z \in V_3$, we denote by $\gamma(z)$ the curve containing $z$ and obtained from $\gamma$ by translation. Clearly, $\gamma(z)$ is a strictly vertical, and $\gamma(z) \subset V_3$.

Let $l_j(t) = \ln T_{jj}(\cdot - h/2)$, where $\ln T_{jj}$ is the branch we have chosen just before formulating the lemma. If $h$ is sufficiently small, $l_j$ is analytic in $V_3 \cup C_d(Y) \cup C_u(Y)$ and equals $O(h^2)$ there. We fix $z_0 \in V_3$ and let
\[
\phi_j(z) = \frac{\pi}{2h^2} \int_{\gamma(z)} \int_{z_0}^{\zeta} l_j(t) \, dt \, d\zeta, \quad z \in V_3. \tag{5.25}
\]

The fact that the integral in (5.25) converges and defines an analytic function follows from the estimate
\[
\int_{\gamma(z)} l_j(t) \, dt = O((1 + |y|)h^2), \quad z \in V_3. \tag{5.26}
\]

We note that if $c_u = c_d = 0$, then the right hand side in (5.26) can be replaced by $O(h^3)$ (see (5.15)).

The fact that, for $z, z + h \in V_3$, $\phi_j$ given by (5.25) satisfies (5.23) follows from the residue theorem.

Having constructed $\phi_j$ in $V_3$, one continues it analytically in $C_d \cup C_u$ just by means of the corresponding equation from (5.23).

Finally, (5.24) follows from the estimate
\[
|\phi_j(z)| \leq C \int_{-\infty}^{\infty} \frac{1 + |\eta + y|}{\cosh^2 \frac{\pi \eta}{h}} \, d\eta. \tag{5.27}
\]

If $z \in \gamma$, (5.27) follows from (5.26), (5.25) and the fact that $\gamma$ is strictly vertical. If $z \not\in \gamma$, then one also uses the fact that $\gamma(z)$ is obtained from $\gamma$ by a translation.

If $c_u = c_d = 0$, then, instead of (5.27), one obtains the estimate $|\phi_j(z)| \leq C \int_{-\infty}^{\infty} \frac{d\eta}{\cosh \frac{\pi \eta}{h}}$, and it implies (5.24) with the right hand side replaced by $Ch$. □

Let $\tilde{\phi}_1$ and $\tilde{\phi}_2$ in (5.22) be the functions from Lemma 5.2. Then, for sufficiently small $h$, Lemma 5.2 and Propositions 4.1 and 5.1 imply that, in $V_3 \cup C_d \cup C_u$, the coefficients $S_{12}$ and $S_{21}$ from (5.24) are analytic and admit the representations
\[
S_{12}(z) = h e^{-\frac{2i\delta(z)}{h}} \cdot g_{12} \quad \text{and} \quad S_{21}(z) = h e^{\frac{2i\delta(z)}{h}} \cdot g_{21}, \tag{5.28}
\]
\[
g_{12}(z) = O(e^{-2\pi(2n_s(t) - n_s(M_{12}) - 1 + c h)|y|}), \quad g_{21}(z) = O(e^{-2\pi(2n_s(t) - n_s(M_{12}) - h)|y|}), \tag{5.29}
\]
for $z \in C_u(0) \cap V_3, \ s \in \{u, d\}$. Here $c = 0$ if $c_u = c_d = 0$.

5.1.4. Completing the proofs Theorems 2.2 and 2.3 for unbounded canonical domains. Equation (5.21) with $S_{12}$ and $S_{21}$ of the form (5.28) was studied in [17], compare (5.21) and (5.28) with (2.12) and (2.15) from [17]. Actually, one can use the method of [17] if there are positive constants $C, C_1$ and $C_2$ (independent of $h$) such that
\[
|g_{12}(z)| \leq C e^{C_1|y|}, \quad |g_{21}(z)| \leq C e^{C_1|y|}, \quad |g_{12}(z)g_{21}(z)| \leq C e^{-C_2|y|},
\]
and this occurs in our case. So, we construct analytic solutions to (5.21) as in [17], focusing only on the modifications.

In view of Lemmas 3.3 from [17], in the case when tr $M$ is a trigonometric polynomial, any unbounded canonical domain $D$ (containing an infinite vertical curve) can be extended to a canonical domain $\tilde{D}$ such that $D \subset \tilde{D}$, and that, $\forall z \in \tilde{D}$, there is a strictly canonical curve containing $z$ and contained in $\tilde{D}$ with some its $\delta$-neighborhood. Here, all the canonical curves and domains are canonical with respect to one and the same branch of the complex momentum.

Clearly, it suffices to prove Theorem 2.3 only for the extended canonical domains. We assume that $K = \mathbb{R}$ is such an extended canonical domain and that it is canonical with respect to the branch $p$ fixed above in $\mathbb{R}$.

Below all the canonical curves are in $K$ and are canonical with respect to $p$.

1. Let $\gamma$ be a strictly vertical curve contained in $K$ together with some its $\delta$-neighborhood $V_\delta$ and its boundary $\partial V_\delta$. Let us consider the matrix $S$ constructed in $V_\delta$ as in section 5.1.3. We shall study the integral equation

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{L}(S X)$$

(5.30)

where $\mathcal{L}$ is the singular integral operator acting by the formula

$$\mathcal{L}f(z) = \frac{1}{2i} \int_\gamma \left( \cot \left( \frac{\pi(\zeta - z - 0)}{h} \right) - i \right) f(\zeta) d\zeta,$$

(5.31)

on a suitable space of functions defined on $\gamma$. We note that, formally, equation (5.30) can be obtained from (5.21) by inverting the difference operator in the left hand side of (5.21), see section 7.1 in [17].

The matrix $S$ being anti-diagonal, we readily deduce from (5.30) an equation for the first element of the vector $X$. In view of (5.28), it can be written in the form

$$X_1 = 1 + \hbar^2 \mathcal{L} \left( g_{12} K_+ \left( g_{21} X_1 \right) \right),$$

(5.32)

$$K_+ f(z) = e^{-\frac{2\pi i (z)}{h}} \mathcal{L} \left( e^{\frac{2\pi i}{h}} f \right)(z).$$

(5.33)

2. To study (5.32), we fix $a \in (0,1)$ and define

$$\Pi_{\gamma,a} = \{ z \in \mathbb{C} : \exists \zeta \in \gamma : \text{Im} \zeta = \text{Im} z \text{ and } |\text{Re} \zeta - \text{Re} z| < ah \}.$$  

(5.34)

Furthermore, let $b,c > 0$ and

$$\rho(z,b,c) = \begin{cases} e^{+2\pi b y} & \text{if } y \geq 0, \\ e^{-2\pi c y} & \text{if } y \leq 0. \end{cases}$$

(5.35)

Let $H_{\gamma,a,b,c}$ be the space of functions $f$ analytic in $\Pi_{\gamma,a}$ and such that the numbers

$$\| f \|_{\gamma,a,b,c} = \sup_{z \in \Pi_{\gamma,a}} |\rho(z,b,c) f(z)|$$

(5.36)

are finite. This is a Banach space with the norm $\| \cdot \|_{\gamma,a,b,c}$. In addition to Definition 5.1 we need

**Definition 5.2.** We call a canonical curve $\gamma$ strictly canonical if it is strictly vertical, and the derivatives in (2.10) are bounded away from zero uniformly in $y \in \mathbb{R}$.

We recall that $C$ denote different positive constants independent of $h$. One has

**Proposition 5.2.** Let $\alpha$ be an infinite strictly vertical curve, and let $b,c > 0$. Then

$$\| \mathcal{L}_+ \|_{H_{\alpha,a,b,c} \rightarrow H_{\alpha,a,b,c}} \leq C/h.$$
Let \( \alpha \) be an infinite strictly canonical curve located in \( K \) with some its \( \delta \)-neighborhood, and let \( b, c \in \mathbb{R} \). Then, for sufficiently small \( h \),
\[
\| \mathcal{L}_+ \|_{H^{n,a,b,c}} \leq C.
\]

Mutandis mutatis, the first and the second statements are proved respectively as Propositions 5.1 and 5.2 from [17]. Below we assume that that the curve \( \gamma \) is strictly canonical. Let us fix \( \alpha, \beta, \gamma \), such that \( \| \mathcal{L}_+ \|_{H^{\alpha,\beta,\gamma}} \leq \epsilon \), for sufficiently small \( \epsilon \) and \( \gamma < \gamma_{0,0} \), i.e., that \( \forall z \in \Pi_{\gamma,\alpha} \)
\[
|X(x) - 1| \leq C_h.
\]

3. We define the function \( X_2 \) in \( \Pi_{\gamma,\alpha} \) by the formula
\[
X_2 = h \mathcal{L}_+(e^{\frac{2i\theta}{\epsilon}} g_{21}(X_1)).
\]

As \( \mathcal{L}_+(e^{\frac{2i\theta}{\epsilon}} g_{21}(X_1)) \), Proposition 5.2 and estimates (5.29) and (5.37) imply that, for sufficiently small \( h \), the function \( X_2 \) satisfies the estimate
\[
\| e^{-\frac{2i\theta}{\epsilon}} \mathcal{L}_+(X_2) \|_{\gamma,\alpha,2(1-\infty) - n_s(t)} - ch.\end{equation}

Therefore, as \( 2an_s(t) < 1, s \in \{d, u\}, \) for sufficiently small \( h \), for all \( z \in \Pi_{\gamma,\alpha} \)
\[
|e^{-\frac{2i\theta}{\epsilon}} X_2(z)| \leq C_\epsilon e^{2\pi (n_s(t) - n_s(M_{1)} - ch)}\|y\|.
\]

Furthermore, as \( n_s(t) \geq n_s(M_{12}), \) for all \( z \in \Pi_{\gamma,\alpha} \), we get
\[
|e^{-\frac{2i\theta}{\epsilon}} X_2(z)| \leq C_\epsilon.
\]

4. It follows from (5.28) and (5.30) that \( \mathcal{X} \), the vector with the elements \( X_1 \) and \( X_2 \), satisfies equation (5.30) in \( \Pi_{\gamma,\alpha} \).

5. Let \( A \) be an admissible subdomain of \( K \) containing \( \gamma \) with some its \( \delta \)-neighborhood. Let us assume that \( h \) is sufficiently small, and prove that the function \( X \) is analytic in \( A \) and satisfies equation (5.21) if \( z, z + h \in A \). The function \( X \) is analytic between \( \gamma - ah \) and \( \gamma + (a + 1)h \). Indeed, \( X \) is defined and analytic between \( \gamma - ah \) and \( \gamma + ah \). This and the definition of \( \mathcal{L}_+ \) imply that the function \( \mathcal{L}_+(S\mathcal{X}) \) is analytic between \( \gamma - ah \) and \( \gamma + (a + 1)h \). As \( X \) satisfies equation (5.30), this implies that it is also analytic there. Let us assume that \( z + h, z \) are located between \( \gamma - ah \) and \( \gamma + (a + 1)h \). Computing the difference \( \mathcal{L}_+(S\mathcal{X})(z + h) - \mathcal{L}_+(S\mathcal{X})(z) \) by means of the residue theorem, we check that \( \mathcal{X} \) satisfies equation (5.21), and so \( \mathcal{X}(z + h) = (I + S(z))\mathcal{X}(z) \). This allows to continue \( \mathcal{X} \) analytically from the strip bounded by \( \gamma - ah \) and \( \gamma + (a + 1)h \) into the part of \( K \) located on the right of \( \gamma \). In view of (5.28) and (5.29), we have \( \det(I + S(z)) = 1 + O(h^2 e^{-2\pi(1-2ch)}|y|) \) in \( A \). So, for sufficiently small \( h \), for \( z \in A \) the matrix \( I + S(z) \) is invertible, and \( \mathcal{X}(z) = (I + S(z))^{-1}\mathcal{X}(z + h) \). This allows to continue \( \mathcal{X} \) analytically from the strip bounded by \( \gamma - ah \) and \( \gamma + (a + 1)h \) in the part of \( A \) located on the left of \( \gamma \). By construction \( \mathcal{X} \) satisfies equation (5.21) if \( z, z + h \in A \).

6. Let us show that, for sufficiently small \( h \), estimates (5.37) and (5.40) are valid and uniform in any compact subset of \( A \).

Let \( z^0 \in A \), and let \( \gamma^0 \) be a strictly canonical curve containing \( z^0 \) and contained in \( K \) with some its \( \delta \)-neighborhood. Clearly, a strictly canonical curve remains strictly canonical if we deform it only in a \( \delta \)-neighborhood of its point and if this deformation is sufficiently small in \( C^1 \)-topology. Therefore, \( z^0 \) is an internal point of a simply connected domain \( D^0 \subset A \) bounded by two strictly canonical curves \( \gamma^1 \subset K \) and \( \gamma^2 \subset K \) that coincide with \( \gamma^0 \) outside a neighborhood of the point \( z^0 \). Let \( j \in \{1, 2\} \). Deforming in equation (5.22) the integration path \( \gamma \) inside \( K \) to \( \gamma^j \),
one shows that $X_1$ is a solution to this equation in $H_{\gamma',a,0,0}$, and, therefore, satisfies estimate (5.37) uniformly in $z \in \Pi_{\gamma',a}$. Deforming the integration path in (5.39) to $\gamma$, we come to estimate (5.40) uniform in $z \in \Pi_{\gamma',a}$. 

As estimates (5.37) and (5.40) hold on the boundary of $D^0$, by the Maximum modulus principle for analytic functions, they hold in $D^0$. This implies the needed. 

7. For sufficiently large $Y > 0$, and for sufficiently small $h$, the vector $X$ satisfies estimates (5.37) and (5.39) in the domain $A(Y) = \{z \in A : \left| \text{Im} \right| z > Y \}$. Indeed, when proving Lemma 8.1 from [17] (steps 2 and 3 of the proof), we checked that if $Y$ is sufficiently large, and $h$ is sufficiently small, then there is a constant $C_0 > 0$ independent of $h$ and such that for any point $z_0 \in A(Y)$, there is a canonical curve $\gamma \subset A$ containing $z_0$ and such that the estimates of Proposition 5.2 with $C = C_0$ hold. This implies the needed. 

8. As $X$ satisfies (5.21), one constructs a vector solution to (1.1) by the formulas 

$$
\Psi^+(z) = \Psi_0(z) \begin{pmatrix} e^{\phi_1(z)} & 0 \\ 0 & e^{\phi_2(z)} \end{pmatrix} X(z)
$$

(5.41)

Being defined and analytic in a strip between $\gamma - ah$ and $\gamma + ah + h$, the solution $\Psi^+$ can be analytically continued up to an entire function just by means of equation (1.1).

9. Using (5.41), (5.37), (5.40) and (5.21), we get 

$$
\Psi^+(z) = e^{i\phi_1(z)} V^+(z) + O(he^{\infty})
$$

(5.42)

In view of the definition of $\theta$, see (4.3), (5.42) implies representation (2.11) locally uniform in $z$. This completes the proof of Theorem 2.2 for the solution $\Psi^+$ in the case we study (when $e^{\phi(z)} \to 0$ as $\left| \text{Im} \right| z \to \infty$).

10. Now, let us prove Theorem 2.3 for the solution $\Psi^+$. We fix $Y > 0$ sufficiently large. In view of (2.8), (3.3), (3.28) and (3.34), for $s \in (u, d)$ and $z \in C_0(Y)$, we get the estimates:

$$
V_1^+(z) \asymp e^{-\pi n_2(t)} |y|, \quad V_2^+(z) \asymp e^{\pi n_2(t)} |y|,
$$

$$
V_1^-(z) = O(e^{-\pi n_2(t)} |y|), \quad V_2^-(z) = O(e^{\pi n_2(t)} |y|).
$$

This, (5.37) and (5.39) imply that, for $j = 1, 2$, for sufficiently small $h$,

$$
e^{-2i\phi_1(z) + \phi_2(z) - \phi_j(z)} \frac{V_j^+(z)}{X_1(z)} = O(h e^{2\pi n_2(t)} |y|) = O(h)
$$

(5.43)

uniformly in $z \in A(Y)$. This and (5.41) imply the statement of Theorem 2.3 on $\Psi^+$ in the case we study. To construct the solution $\Psi^-$, one proceeds as suggested in section 9 of [17]. We have studied the case of (5.21). The complementary cases are analyzed similarly; in section 10 of [17], we indicated the way to do this. We omit further details.

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