The Loop Equation for Special Cubic Hodge Integrals

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Abstract
As the first step of proving the Hodge-FVH correspondence recently proposed in [17], we derive the Virasoro constraints and the Dubrovin-Zhang loop equation for special cubic Hodge integrals. We show that this loop equation has a unique solution, and provide a new algorithm for the computation of these Hodge integrals. We also prove the existence of gap phenomenon for the special cubic Hodge free energies.

Contents
1 Introduction 1
2 Virasoro constraints 6
3 Loop equation: the rational case 12
4 Loop equation: the general case 19

1 Introduction
Let $\overline{M}_{g,n}$ be the moduli space of stable algebraic curves of genus $g$ with $n$ distinct marked points, where $g$ and $n$ are non-negative integers satisfying the stability condition $2g - 2 + n > 0$. For $1 \leq k \leq n$ and $0 \leq j \leq g$, denote by $\psi_k$ the first Chern class of the $k$-th tautological line bundle $L_k$ on $\overline{M}_{g,n}$, and by $\lambda_j$ the $j$-th Chern class of the Hodge bundle $E_{g,n}$ on $\overline{M}_{g,n}$. The rational numbers defined by the formula

$$\int_{\overline{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_j^{j_1} \cdots \lambda_j^{j_m},$$

are called the Hodge integrals. These numbers take zero value unless the degree-dimension counting matches:

$$i_1 + \cdots + i_n + j_1 + \cdots + j_m = 3g - 3 + n. \quad (1.1)$$

Denote by $C_g(z) := \sum_{j=0}^{g} \lambda_j z^j$ the Chern polynomial of $E_{g,n}$. We will be particularly interested in the following class of Hodge integrals defined via cubic products of Chern polynomials, called the cubic Hodge integrals:

$$\int_{\overline{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} C_g(-p) C_g(-q) C_g(-r), \quad (1.2)$$
where \( p, q, r \) are complex parameters. These integrals are called special if \( p, q, r \) satisfy the following local Calabi-Yau condition:

\[
pq + qr + rp = 0.
\]

These Hodge integrals are important in the localization technique of computing Gromov-Witten invariants for toric three-folds \([12, 15, 20]\). Their significance was also manifested by the Gopakumar-Mariño-Vafa conjecture regarding the Chern-Simons/string duality \([11, 18]\).

Let \( \mathcal{H} = \mathcal{H}(t; p, q, r; \varepsilon) \) be the cubic Hodge free energy defined by

\[
\mathcal{H}(t; p, q, r; \varepsilon) := \sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{H}_g(t; p, q, r),
\]

\[
\mathcal{H}_g(t; p, q, r) := \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} C_g(-p)C_g(-q)C_g(-r).
\]

Here \( \mathcal{H}_g(t; p, q, r) \) is called the genus \( g \) part of the free energy \( \mathcal{H} \). Then the exponential

\[
e^{\mathcal{H}(t; p, q, r; \varepsilon)} =: \mathcal{Z}_{\text{cubic}}(t; p, q, r; \varepsilon) =: \mathcal{Z}_{\text{cubic}}
\]

is called the cubic Hodge partition function. Clearly, \( \mathcal{H}_g(t; p, q, r) \in \mathbb{C}[p, q, r][[t]] \). The genus zero free energy \( \mathcal{H}_0(t) \) is actually independent of \( p, q, r \) and has the explicit expression

\[
\mathcal{H}_0(t) = \sum_{n \geq 3} \frac{1}{n(n-1)(n-2)} \sum_{i_1 + \cdots + i_n = n-3} \frac{t_{i_1} \cdots t_{i_n}}{i_1! \cdots i_n!}.
\]

Define

\[
v(t) := \partial^2_{t_0} \mathcal{H}_0(t) = \sum_{n \geq 1} \frac{1}{n} \sum_{i_1 + \cdots + i_n = n-1} \frac{t_{i_1} \cdots t_{i_n}}{i_1! \cdots i_n!}.
\]

then it satisfies the following Riemann hierarchy:

\[
\frac{\partial v}{\partial t_i} = \frac{v^i}{i!} \frac{\partial v}{\partial t_0}, \quad i \geq 0.
\]

More generally, if one defines \( w = e^2 \partial^2_{t_0} \mathcal{H}(t; p, q, r; \varepsilon) \), then \( w \) satisfies an integrable hierarchy of Hamiltonian evolutionary PDEs \([5]\), called the (cubic) Hodge hierarchy, which is a deformation of the Riemann hierarchy. The first member of this integrable hierarchy reads

\[
w_{t_1} = w w_{t_0} + \frac{e^2}{12} (w_{t_0} t_0 - (p + q + r) w_{t_0} w_{t_0} + O(\varepsilon^4)).
\]

The Hodge-FVH correspondence is given by the following conjecture \([17]\):

**Conjecture 1.1** The Hodge hierarchy for the special cubic Hodge integrals is equivalent, under a certain Miura type transformation, to the fractional Volterra hierarchy (FVH). Furthermore, the corresponding cubic Hodge partition function gives a tau function of the FVH.

For the case with \( p = q \), the validity of the Hodge-FVH correspondence is implied by the Hodge-GUE correspondence established in \([6, 7]\). The goal of this and the subsequent papers \([16]\) is to prove the Hodge-FVH correspondence. In the present paper, we derive the
Dubrovin-Zhang loop equation by studying the Virasoro constraints for the special cubic Hodge partition function. We show that this loop equation together with the genus zero free energy uniquely determines the partition function. In the subsequent paper, we will show that the fractional Volterra hierarchy admits the same Virasoro constraints and that there exists a particular tau function of the integrable hierarchy which is uniquely determined by the same loop equation, and we prove in this way the Hodge-FVH correspondence.

From now on, we assume that $p, q, r$ satisfy the local Calabi-Yau condition \[1.3\]. The case with $p, q, r \in \mathbb{Q}$ is called rational. Note that the Virasoro constraints for the special cubic Hodge partition function in general case are quite complicated. However, in the rational case, we find explicit expressions of the Virasoro constraints which lead to the Dubrovin-Zhang loop equation. It turns out that the loop equation for the general case can be deduced from the one for the rational case.

Let us first consider the rational case. Due to the symmetry property and the homogeneity property of the cubic Hodge integrals with respect to $p, q, r$, we can assume that

$$p = \frac{1}{K_1}, \quad q = \frac{1}{K_2}, \quad r = -\frac{1}{h},$$

(1.6)

where $K_1, K_2 \in \mathbb{N}$, $(K_1, K_2) = 1$ and $h := K_1 + K_2$. We denote, for $\ell \geq 0$,

$$b_{\alpha+h\ell} := \frac{\alpha}{K_1} + \ell, \quad c_{\alpha+h\ell} := \left( \frac{b_{\alpha+h\ell}}{b_{\alpha+h\ell K_1}} \right), \quad \alpha = 0, \ldots, K_1 - 1,$$

(1.7)

$$b_{\alpha+h\ell} := -\frac{\alpha}{K_2} + \ell, \quad c_{\alpha+h\ell} := \left( \frac{b_{\alpha+h\ell}}{b_{\alpha+h\ell K_2}} \right), \quad \alpha = -(K_2 - 1), \ldots, -1,$$

(1.8)

and

$$\mathbb{N}_* = (\mathbb{N} - K_2) \setminus \{0\} \cup (h\mathbb{N} - K_2),$$

where $a\mathbb{N} - K_2 := \{ak - K_2 | k \in \mathbb{N}\}$.

Define

$$Z(x, s; \epsilon) := \exp \left( \frac{A(x, \tilde{s})}{e^2} \right) Z_{\text{cubic}} \left( t(x, s); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h}, \epsilon \right),$$

(1.9)

where $s := (s_k)_{k \in \mathbb{N}_*}$ is an infinite vector of indeterminates, $\tilde{s}_k = s_k - c_{\ell}^{-1} \delta_{k,h}$ ($k \in \mathbb{N}_*$),

$$t_i = t_i(x, s) = \sum_{k \in \mathbb{N}_*} b_k^{i+1} c_k \tilde{s}_k + \delta_{i,1} + x \delta_{i,0}, \quad i \geq 0,$$

(1.10)

and $A$ is the quadratic series

$$A := A(x, s) = \frac{1}{2} \sum_{k_1, k_2 \in \mathbb{N}_*} b_{k_1} b_{k_2} c_{k_1} c_{k_2} s_{k_1} s_{k_2} + x \sum_{k \in \mathbb{N}_*} c_k s_k.$$

(1.11)

Note that for $g \geq 0$, $\mathcal{H}_g \left( t(x, s); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h} \right)$ is a well-defined formal power series in $\mathbb{C}[[x-1, s]]$.

Indeed, for each monomial $(x - 1)^{k_0} s_{k_1} \cdots s_{k_m}$ in $\mathcal{H}_g \left( t(x, s); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h} \right)$ only finitely many monomials $t_{i_1} \cdots t_{i_n}$ contribute to its coefficient for the dimension reason \[1.1\].
Denote \( I = \{-(K_2-1),\ldots,K_1-1 \} \) and \( I_* = I \setminus \{ 0 \} \), and define a family of linear operators \( L_m = L_m(e^{-1}x,e^{-1}s,e\partial/\partial s), \ m \geq 0 \) by

\[
L_0 = \sum_{k \in \mathbb{N}_*} b_k s_k \frac{\partial}{\partial s_k} + \frac{x^2}{2\epsilon^2} + \frac{1}{24} \left( \frac{1}{\epsilon} - \frac{1}{K_1} - \frac{1}{K_2} \right),
\]

\[
L_m = \sum_{k \in \mathbb{N}_*} b_k s_k \frac{\partial}{\partial s_k + x} + \frac{\epsilon^2 m^{-1}}{2} \frac{\partial^2}{\partial s_k \partial s_{(m-\ell)}} + \frac{\epsilon^2}{2} \sum_{\alpha,\beta \in I_*} \sum_{\ell=0}^{m-1} G^{\alpha\beta}_{\alpha+\ell \partial s_{\beta+\ell(m-\ell)}},
\]

where \( (G^{\alpha\beta})_{\alpha,\beta \in I} \) is a symmetric non-degenerate constant matrix defined by

\[
G^{\alpha\beta} = \begin{cases} 
\frac{K_1}{h} \delta^{\alpha+\beta,-K_2} & \alpha,\beta < 0; \\
1 & \alpha = \beta = 0; \\
\frac{K_2}{h} \delta^{\alpha+\beta,K_1} & \alpha,\beta > 0; \\
0 & \text{elsewhere.}
\end{cases}
\]

It is easy to check that the operators \( L_m \) satisfy the following Virasoro commutation relations:

\[
[L_m, L_n] = (m - n) L_{m+n}, \quad \forall m, n \geq 0.
\]

**Theorem 1.2** The series \( Z(x,s;\epsilon) \) defined by (1.9) satisfies the Virasoro constraints

\[
L_m(e^{-1}x,e^{-1}s,e\partial/\partial s) Z(x,s;\epsilon) = 0, \quad m \geq 0.
\]

Using Theorem 1.2 and the technique developed in [8], we derive in Section 3 the Dubrovin-Zhang loop equation for the special cubic Hodge free energies in the rational case; see Theorem 3.9.

We proceed to the general case. Denote

\[
\sigma_1 = -(p + q + r), \quad \sigma_3 = -2(p^3 + q^3 + r^3).
\]

From Mumford’s relations [19], the local Calabi-Yau condition [13], the fact that the integral (1.2) is symmetric in \( p, q, r \), and the well-known relationship between the Chern polynomial and the Chern character of \( E_{g,n} \), we know that

\[
\mathcal{H}_g := \mathcal{H}_g(t;p,q,r) \in \mathbb{C}[\sigma_1,\sigma_3][[t]], \quad g \geq 0.
\]

The following theorem is the main result of the present paper.

**Theorem 1.3** The equation

\[
\sum_{i \geq 0} \left( \partial^i \Theta + \sum_{j=1}^i P_{j-1,i-j+1} \right) \frac{\partial \Delta H}{\partial z_i} = \Theta - \frac{\epsilon^2}{16} \sum_{i \geq 0} \partial^{i+2} \left( \frac{\Theta^2}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta \right) \frac{\partial \Delta H}{\partial z_i}
\]

\[
+ \frac{\epsilon^2}{2} \sum_{i,j \geq 0} P_{i+1,j+1} \left( \frac{\partial^2 \Delta H}{\partial z_i \partial z_j} + \frac{\partial \Delta H}{\partial z_i} \frac{\partial \Delta H}{\partial z_j} \right),
\]

(1.18)
which is called the Dubrovin-Zhang loop equation for the special cubic Hodge integrals, has a unique solution of the form

\[ \Delta H := \sum_{g \geq 1} e^{2g-2} H_g, \quad H_g := H_g(z_0, \ldots, z_{3g-2}; \sigma_1, \sigma_3) \]

up to the addition of a constant to each \( H_g, g \geq 1 \). These constants can be uniquely determined by the following conditions:

\[ H_1 = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0, \quad \sum_{j=1}^{3g-2} j z_j \frac{\partial H_g}{\partial z_j} = (2g - 2) H_g, \quad g \geq 2. \quad (1.19) \]

Here

\[ \partial := \sum_{k \geq 0} z_{k+1} \frac{\partial}{\partial z_k}, \quad \Theta := \frac{1}{1 - e^{z_0}/\mu}, \]

\( R_{i,j} \) are certain polynomials in \( \Theta, z_1, z_2, \ldots, \sigma_1, \sigma_3 \) which are given in Section 4 and \( \mu \) is an arbitrary parameter. Moreover, let \( v(t) \) be defined in (1.4), then the genus \( g \) (\( g \geq 1 \)) special cubic Hodge free energy has the expression

\[ \mathcal{H}_g = H_g \left( v(t), \frac{\partial v(t)}{\partial t_0}, \ldots, \frac{\partial^{2g-2} v(t)}{\partial t_0^{2g-2}}; \sigma_1, \sigma_3 \right). \quad (1.20) \]

We can recursively solve the loop equation to obtain the free energies \( H_g, g \geq 1 \). For example, the first two \( H_g \) are given by

\[ H_1 = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0, \quad (1.21) \]

\[ H_2 = \frac{1}{1152} \frac{z_4}{z_1^2} - \frac{7}{1920} \frac{z_2 z_3}{z_1^3} + \frac{1}{360} \frac{z_2}{z_1} + \frac{\sigma_1}{480} \frac{z_3}{z_1} - \frac{11 \sigma_1}{5760} \frac{z_2}{z_1^2} + \frac{7 \sigma_1^2}{5760} z_2 + \left( \frac{\sigma_1^3}{17280} - \frac{\sigma_2}{34560} \right) z_1^2. \quad (1.22) \]

These expressions agree with the results of [5].

Besides the above theorems, we also prove the following gap phenomenon for the special cubic Hodge integrals in the rational case. This type of phenomenon was used by M.-X. Huang, A. Klemm and S. Quackenbush to compute the Gromov-Witten invariants of the quintic Calabi-Yau three-fold up to genus 51 [13].

**Corollary 1.4** For the rational case, the free energies \( \mathcal{F}_g(x, s) := \mathcal{H}_g \left( t(x, s); \frac{1}{K_1}, \frac{1}{K_2}, \frac{1}{h} \right) \), \( g \geq 1 \) with \( t(x, s) \) defined in (1.10) have the following gap phenomenon:

\[ \mathcal{F}_1(x, s) = \frac{\sigma_1 - 1}{24} \log x \in \mathbb{C}[x][[s]], \quad (1.23) \]

\[ \mathcal{F}_g(x, s) - \frac{R_g(\sigma_1, \sigma_3)}{x^{2g-2}} \in \mathbb{C}[x][[s]], \quad g \geq 2. \quad (1.24) \]

Here \( R_g(\sigma_1, \sigma_3), g \geq 2 \) are certain polynomials of \( \sigma_1, \sigma_3 \) satisfying the condition

\[ \deg R_g \leq 3g - 3, \quad \text{with} \quad \deg \sigma_1 = 1, \deg \sigma_3 = 3. \quad (1.25) \]
whose degree \((3g - 3)\)-parts are given by
\[
\frac{(-1)^g}{2(2g-2)!} \frac{|B_{2g}| |B_{2g-2}|}{2g(2g-2)} \left( \frac{\sigma_3^3}{3} - \frac{\sigma_3}{6} \right)^{g-1},
\]
where \(B_j\) denote the Bernoulli numbers.

Note that in (1.23) and (1.24), the indeterminates of \(R_g(\sigma_1, \sigma_3)\) take the values
\[
\sigma_1 = \frac{1}{K_1 + K_2} - \frac{1}{K_1}, \quad \sigma_3 = \frac{2}{(K_1 + K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3}.
\]
The first two \(R_g, g \geq 2\) have the expressions
\[
R_2 = -\frac{1}{1440} + \frac{13\sigma_1}{5760} - \frac{7\sigma_1^2}{5760} + \frac{\sigma_3^3}{17280} - \frac{\sigma_3}{34560},
\]
\[
R_3 = \frac{1}{181440} - \frac{107\sigma_1}{362880} + \frac{145\sigma_1^2}{290304} + \frac{961\sigma_1^3}{4354560} + \frac{31\sigma_3}{2177280} + \frac{113\sigma_1^4}{4354560} - \frac{113\sigma_1\sigma_3}{8709120}
\]
\[-\frac{\sigma_1^6}{13063680} + \frac{\sigma_1^3\sigma_3}{13063680} - \frac{\sigma_3^2}{52254720}.
\]
We hope that the above results will be useful in the study of Gromov-Witten invariants for toric Calabi-Yau varieties.

**Organization of the paper** In Section 2, we derive the explicit expressions of Virasoro constraints for \(Z(x, s; \epsilon)\). In Section 3, we obtain the Dubrovin-Zhang loop equation for the special cubic Hodge free energies in the rational case. In Section 4, we prove Theorem 1.3.

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## 2 Virasoro constraints

In this section we first give two versions of Virasoro constraints for \(Z_{\text{cubic}}\), and then we prove Theorem 1.2. Denote by \(Z_{\text{WK}}\) the Witten-Kontsevich partition function
\[
Z_{\text{WK}}(t; \epsilon) = \exp \left( \sum_{g \geq 0} \epsilon^{2g-2} \sum_n \frac{t_1 \cdots t_n}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \right).
\]
It is well known that \(Z_{\text{WK}}(t; \epsilon)\) satisfies the following Virasoro constraints [2] [21] [14]:
\[
L_m^{\text{KdV}} \left( \epsilon^{-1} \frac{\partial}{\partial \epsilon} \right) Z_{\text{WK}}(t; \epsilon) = 0,
\]
(2.1)
where \( \tilde{t}_i = t_i - \delta_{i,1} \), and \( L^\text{KdV}_m := L^\text{KdV}_m (\epsilon^{-1}t, \epsilon \partial / \partial t) \) are linear operators given by

\[
L^\text{KdV}_{-1} := \sum_{i \geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{\epsilon^2}{2} t_1^2,
L^\text{KdV}_0 := \sum_{i \geq 0} \frac{2i + 1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16},
L^\text{KdV}_m := \sum_{i \geq 0} \frac{(2i + 2m + 1)!}{2^{m+1}(2i - 1)!!} t_i \frac{\partial}{\partial t_{i+m}} + \frac{\epsilon^2}{2} \sum_{i+j=m-1} (2i + 1)!! (2j + 1)!! \frac{\partial^2}{\partial t_i \partial t_j}, \quad m \geq 1.
\]

These operators satisfy the Virasoro commutation relations

\[
\left[ L^\text{KdV}_m, L^\text{KdV}_n \right] = (m - n) L^\text{KdV}_{m+n}, \quad \forall m, n \geq -1.
\]

Lemma 2.1 (9) The partition function \( Z_{\text{cubic}}(t; p, q, r; \epsilon) \) has the expression

\[
Z_{\text{cubic}}(t; p, q, r; \epsilon) = e^{G(\epsilon^{-1}t, \epsilon \partial / \partial t)} Z_{\text{WK}}(t; \epsilon), \tag{2.2}
\]

where the operator \( G = G(\epsilon^{-1}t, \epsilon \partial / \partial t) \) is defined by

\[
G(\epsilon^{-1}t, \epsilon \partial / \partial t) = - \sum_{i \geq 1} \frac{B_{2i}}{2i(2i - 1)} (p^{2i-1} + q^{2i-1} + r^{2i-1}) D_i,
\]

\( B_{2i} \) are Bernoulli numbers, and \( D_i = D_i (\epsilon^{-1}t, \epsilon \partial / \partial t) \) are given by

\[
D_i := - \sum_{j \geq 0} t_j \frac{\partial}{\partial t_{j+2i-1}} + \frac{\epsilon^2}{2} \sum_{j=0}^{2i-2} \frac{\partial^2}{\partial t_j \partial t_{2i-2-j}}, \quad i \geq 1.
\]

The lemma below follows from equation (2.1) and Lemma 2.1.

Lemma 2.2 (22) Define a sequence of operators \( L_{\text{cubic}}^m = L_{\text{cubic}}^m (\epsilon^{-1}t, \epsilon \partial / \partial t) \) by

\[
L_{\text{cubic}}^m (\epsilon^{-1}t, \epsilon \partial / \partial t) := e^G \circ L^\text{KdV}_m (\epsilon^{-1}t, \epsilon \partial / \partial t) \circ e^{-G}, \quad m \geq -1.
\]

Then \( L_{\text{cubic}}^m \) satisfy the Virasoro commutation relations

\[
\left[ L_{\text{cubic}}^m, L_{\text{cubic}}^n \right] = (m - n) L_{\text{cubic}}^{m+n}, \quad m, n \geq -1.
\]

Moreover, \( Z_{\text{cubic}}(t; p, q, r; \epsilon) \) satisfies the equations

\[
L_{\text{cubic}}^m (\epsilon^{-1}t, \epsilon \partial / \partial t) Z_{\text{cubic}}(t; p, q, r; \epsilon) = 0, \quad m \geq -1. \tag{2.3}
\]

We call (2.3) the Virasoro constraints for \( Z_{\text{cubic}}(t; p, q, r; \epsilon) \). Note that these Virasoro operators \( L_{\text{cubic}}^m \) are in general quite complicated, and it is difficult to use (2.3) directly for the computation of the cubic Hodge integrals. For example, the operator \( L_0^\text{cubic} \) has the explicit expression

\[
L_0^\text{cubic} = L_0^\text{KdV} - \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k - 2)!} D_k,
\]

where \( \sigma_n := \sum_{k=1}^n \binom{n+1}{k+1} t_k \).
where \( \sigma_{2k-1} = -(2k-2)! \left(p^{2k-1} + q^{2k-1} + r^{2k-1}\right) \), and the corresponding constraint given in (2.3) has the expression

\[
\begin{aligned}
&\left(3 \frac{\partial}{\partial t_1} + \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k-2)!} \frac{\partial}{\partial t_{2k}}\right) Z_{\text{cubic}} \\
= &\left(\sum_{i \geq 0} \frac{2i + 1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16} + \sum_{k \geq 1} \frac{B_{2k}}{2k} \frac{\sigma_{2k-1}}{(2k-2)!} \left(\sum_{j \geq 0} t_j \frac{\partial}{\partial t_j + 2k-1} - \sum_{j=0}^{2k-2} \frac{\partial^2}{\partial t_j \partial t_{2k-2-j}}\right)\right) Z_{\text{cubic}}.
\end{aligned}
\]

Following [6] we consider the following linear combination of Virasoro operators:

\[
\tilde{\ell}_m := \sum_{k \geq -1} \frac{m^{k+1}}{(k+1)!} L_k^{\text{cubic}}, \quad m \geq 0.
\]  

(2.4)

As it is shown in [6], the operators \( \tilde{\ell}_m^{\text{cubic}} \) also satisfy the Virasoro commutation relations

\[
\left[ \tilde{\ell}_m^{\text{cubic}}, \tilde{\ell}_n^{\text{cubic}} \right] = (m - n) \tilde{\ell}_{m+n}^{\text{cubic}}, \quad \forall m, n \geq 0.
\]

From (2.3) it follows that

\[
\tilde{\ell}_m^{\text{cubic}} \left(\epsilon^{-1} \frac{t}{\epsilon \partial/\partial t}\right) Z_{\text{cubic}}(t; p, q, r; \epsilon) = 0, \quad m \geq 0.
\]  

(2.5)

We call (2.5) the second version of Virasoro constraints for \( Z_{\text{cubic}}(t; p, q, r; \epsilon) \).

Similarly, define a family of infinitesimal symplectic transformations \( d_i, i \geq 1 \) on \( \mathcal{V} \) by

\[
d_i := z^{1-2i},
\]

then we have

\[
D_i \left(\epsilon^{-1} t, \epsilon \partial/\partial t\right) = \hat{d}_i|_{q_j \to t_j, \partial q_j \to \partial t_j, j \geq 0} + \frac{\delta_{k,0}}{16}.
\]  

(2.8)

Similarly, define a family of infinitesimal symplectic transformations \( l_k, k \geq -1 \) on \( \mathcal{V} \) by

\[
l_k := (-1)^{k+1} z^{3/2} \partial z^{k+1} z^{-1/2}, \quad k \geq -1.
\]  

(2.7)

Then we have the following formulae [10]:

\[
L_k^{\text{KdV}} = \hat{l}_k|_{q_j \to t_j, \partial q_j \to \partial t_j, j \geq 0} + \frac{\delta_{k,0}}{16}.
\]  

(2.8)

Note that the above results of this section hold true for general \( p, q, r \). The local Calabi-Yau condition (1.3) for \( p, q, r \) will be needed in what follows.

Similarly as in [6] we have the following lemma.
Lemma 2.3 Assume that \( p, q, r \) satisfy the local Calabi-Yau condition (1.3). Define
\[
\Psi(z) = \left( p^{1/p} q^{1/q} r^{1/r} \right)^{-z} \sqrt{\frac{\Gamma(1-z/p)\Gamma(1-z/q)\Gamma(1-z/r)}{\Gamma(1+z/p)\Gamma(1+z/q)\Gamma(1+z/r)}}. \tag{2.10}
\]
Then the asymptotic expansion of \( \log \Psi(z) \) as \( z \to \infty \) within a properly chosen sector is given by (up to an inessential constant \( \pm \pi i/4 \))
\[
\log \Psi(z) \sim -\infty \sum_{i=1}^{\infty} B_{2i} (p^{2i-1} + q^{2i-1} + r^{2i-1}) z^{1-2i} =: \log \Phi(z). \tag{2.11}
\]

Lemma 2.4 The operators \( \tilde{L}_m^{\text{cubic}}, m \geq 0 \) have the expressions
\[
\tilde{L}_m^{\text{cubic}} \left( \epsilon^{-1} t, \epsilon \partial / \partial t \right) = \left( z V_m(z) e^{-m \partial z} \right)^{\wedge} \mid_{q_i \mapsto t_i, \partial q_i \mapsto \partial t_i, i \geq 0} + \frac{m}{16} \frac{p + q + r}{24}, \tag{2.12}
\]
where \( V_m, m \geq 0 \) are given by
\[
V_m(z) = \frac{\Phi(z)}{\Phi(z-m)} \sqrt{\frac{z}{z-m}}. \tag{2.13}
\]
Proof Denote \( \tilde{\Phi} := \exp \left( (\log \Phi(z))^{\wedge} \mid_{q_i \mapsto t_i, \partial q_i \mapsto \partial t_i, i \geq 0} \right) \).
It follows from (2.9) and Lemma 2.2 that
\[
L_m^{\text{cubic}} \left( \epsilon^{-1} t, \epsilon \partial / \partial t \right) = \tilde{\Phi} \circ L_m^{\text{KdV}} \left( \epsilon^{-1} t, \epsilon \partial / \partial t \right) \circ \tilde{\Phi}^{-1}, \quad m \geq 0.
\]
Then by using (2.8) we obtain
\[
L_m^{\text{cubic}} \left( \epsilon^{-1} t, \epsilon \partial / \partial t \right) = \left( \Phi \circ l_m \circ \Phi^{-1} \right)^{\wedge} \mid_{q_i \mapsto t_i, \partial q_i \mapsto \partial t_i} + \frac{1}{16} \delta_{m,0} - \frac{p + q + r}{24} \delta_{m,-1}.
\]
Note that the term \(-\frac{p + q + r}{24} \delta_{m,-1}\) comes from the cocycle [10] in the quantization of \( \Phi \circ l_m \circ \Phi^{-1} \).
The lemma follows from (2.4) as well as the identity
\[
z^{3/2} \Phi(z) \circ e^{-m \partial z} \circ \frac{1}{\sqrt{z \Phi(z)}} = z \frac{\Phi(z)}{\Phi(z-m)} \sqrt{\frac{z}{z-m}} e^{-m \partial z}.
\]

We are now ready to prove Theorem 1.2.
Proof of Theorem 1.2 In order to prove the validity of (1.15), it suffices to show that the following identities hold true:
\[
L_m \left( \epsilon^{-1} x, \epsilon^{-1} s, \epsilon \partial / \partial s \right) = K^m e^{A(x,s)} \circ \tilde{L}_m^{\text{cubic}} \left( \epsilon^{-1} t, \epsilon \partial / \partial t \right) \circ e^{-A(x,s)}, \quad m \geq 0, \tag{2.14}
\]
where
\[
K = \hbar^h K_1^{-K_1} K_2^{-K_2}, \tag{2.15}
\]
Lemma 2.5 The operator $\tilde{L}_0^{\text{cubic}}$ satisfies the following relation:
\[
e^{-A(x,\tilde{k})/e^2} \circ \tilde{L}_0^{\text{cubic}} (e^{-1} \tilde{t}, \epsilon \partial / \partial t) \circ e^{-A(x,\tilde{k})/e^2} = \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k \frac{\partial}{\partial \tilde{s}_k} + \frac{x^2}{2e^2} + \frac{\sigma_1}{24},
\]

Proof By using Lemma 2.4 we have
\[
\tilde{L}_0^{\text{cubic}} (e^{-1} \tilde{t}, \epsilon \partial / \partial t) = \tilde{z} \mid_{q_i \to t_i, \partial q_i \to \partial t_i} + \frac{\sigma_1}{24} = \sum_{i \geq 1} t_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2e^2} + \frac{\sigma_1}{24}.
\]
Under the substitution (2.17), we arrive at
\[
\tilde{L}_0^{\text{cubic}} (e^{-1} \tilde{t}, \epsilon \partial / \partial t) = \sum_{i \geq 1} \left( \sum_{k \in \mathbb{N}_*} b_k^{i+1} \tilde{s}_k \right) \frac{\partial}{\partial t_{i-1}} + \frac{1}{2e^2} \left( \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{\sigma_1}{24}
\]
\[
= \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k \frac{\partial}{\partial \tilde{s}_k} + \frac{1}{2e^2} \left( \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{\sigma_1}{24}.
\]
The lemma is proved by applying the conjugation by $e^{-A(x,\tilde{k})/e^2}$ to this equality.  

Lemma 2.6 The operator $\tilde{L}_1^{\text{cubic}}$ satisfies the following relation:
\[
e^{-A(x,\tilde{k})/e^2} \circ \tilde{L}_1^{\text{cubic}} (e^{-1} \tilde{t}, \epsilon \partial / \partial t) \circ e^{-A(x,\tilde{k})/e^2} = \sum_{k \in \mathbb{N}_*} b_k V_1(-b_k) \frac{\partial}{\partial \tilde{s}_k + h} + x V_1(0) \frac{\partial}{\partial \tilde{s}_h} + \frac{e^2}{2} \sum_{\alpha = 1}^{K_1-1} \frac{\text{res}_{z=b_0} V_1(z)}{b_0} \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{K_1-\alpha}}
\]
\[
+ \frac{e^2}{2} \sum_{\alpha = -(K_2-1)}^{-1} \frac{\text{res}_{z=b_0} V_1(z)}{b_0} \frac{\partial^2}{\partial \tilde{s}_\alpha \partial \tilde{s}_{-\alpha-K_2}},
\]

where $V_1(z)$ is defined in (2.13).
Proof Since \( p = 1/K_1, q = 1/K_2, r = -1/h \), the function \( V_1(z) \) becomes rational. By employing Lemma 2.3 we find the following explicit expression of this rational function:

\[
V_1(z) = \frac{\prod_{i=1}^{K_1+K_2} \left( z - \frac{i}{K_1+K_2} \right)}{\prod_{i=1}^{K_1} \left( z - \frac{i}{K_1} \right) \prod_{i=1}^{K_2} \left( z - \frac{i}{K_2} \right)}.
\]

Note that \( V_1(z) \) has simple poles at \( b_{-(K_2-1)}, \ldots, b_{-1}, b_1, \ldots, b_{K_1-1}, \) and 1, therefore,

\[
V_1(z) = 1 + \frac{\text{res}_{y=1} V_1(y)}{z - 1} + \sum_{\alpha \in I_*} \frac{\text{res}_{y=b_{\alpha}} V_1(y)}{z - b_{\alpha}}.
\]

As \( z \) goes to infinity, the function \( V_1(z) \) has the full asymptotic expansion

\[
V_1(z) \sim 1 + \sum_{n=1}^{\infty} z^{-n} \left( \text{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} b_{\alpha}^{n-1} \text{res}_{y=b_{\alpha}} V_1(y) \right).
\]

Denote \( a_\alpha := b_{\alpha}^{-1} \text{res}_{y=b_{\alpha}} V_1(z) \), \( \alpha \in I_* \). By taking \( m = 1 \) in (2.12) we obtain

\[
\tilde{L}_{1}^{\text{cubic}}(z \! V_1(z) e^{-\partial_z}) + \frac{1}{16} + \sigma_1 = 24
\]

\[
= \sum_{i \geq 0} \left( \sum_{j=0}^{i+1} t_j \binom{i+1}{j} + \sum_{j=0}^{i+1} \sum_{n=1}^{i+1-n} (-1)^n \binom{i+1}{j+n} \left( \text{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} a_\alpha b_{\alpha}^n \right) \right) \frac{\partial}{\partial t_i}
\]

\[
+ \frac{\sigma_1}{2} \sum_{i,j \geq 0} \sum_{n=0}^{i+1} (-1)^n \binom{i+1}{n} \left( \text{res}_{y=1} V_1(y) + \sum_{\alpha \in I_*} a_\alpha b_{\alpha}^{n+1} \right) \frac{\partial^2}{\partial t_i \partial t_j} + \frac{t_0^2}{2\epsilon^2} + \frac{1}{16} + \frac{\sigma_1}{24}.
\]

Now by performing the variables substitution \( (2.17) \) we arrive at

\[
\tilde{L}_{1}^{\text{cubic}}(\epsilon^{-1} \! t, \epsilon \partial / \partial t) = \sum_{k \in \mathbb{N}_*} b_k \! V_1(-b_k) \tilde{s}_k \frac{\partial}{\partial \tilde{s}_{k+h}} + x V_1(0) \frac{\partial}{\partial \tilde{s}_h}
\]

\[
+ \frac{\sigma_1}{2} \sum_{\alpha=1}^{K_1-1} a_\alpha \frac{\partial^2}{\partial \tilde{s}_a \partial \tilde{s}_{K_1-\alpha}} + \frac{\sigma_1}{2} \sum_{\alpha=-(K_2-1)}^{K_1-1} a_\alpha \frac{\partial^2}{\partial \tilde{s}_a \partial \tilde{s}_{K_1-\alpha}}
\]

\[
+ \sum_{\alpha=1}^{K_1-1} a_\alpha \frac{\partial A(x, \tilde{s})}{\partial \tilde{s}_a} \frac{\partial}{\partial \tilde{s}_{K_1-\alpha}} + \sum_{\alpha=-(K_2-1)}^{K_1-1} a_\alpha \frac{\partial A(x, \tilde{s})}{\partial \tilde{s}_a} \frac{\partial}{\partial \tilde{s}_{K_1-\alpha}}
\]

\[
+ \frac{1}{2\epsilon^2} \left( \sum_{k \in \mathbb{N}_*} b_k \tilde{s}_k + x \right)^2 + \frac{1}{16} + \frac{\sigma_1}{24}.
\]

Here we have used the identity

\[
\sum_{j=0}^{i} \sum_{k=1}^{i+1-j} \binom{i+1}{j+k} \chi^j \mu^k = \chi^{i} \mu^{i+1} - (\chi + \mu)^{i+1} \frac{\chi^i}{\chi - \mu}.
\]

Dressing the operator \( \tilde{L}_{1}^{\text{cubic}}(\epsilon^{-1} \! t, \epsilon \partial / \partial t) \) by \( e^{\frac{A(x, \tilde{s})}{\epsilon^2}} \) we obtain (2.19) after a long but straightforward computation. The lemma is proved.

We can prove in a similar way the following lemma.
Lemma 2.7 The operator \( \tilde{L}_2^{\text{cubic}} \) satisfies the relation

\[
e^{-\frac{A(s,s)}{c^2}} \circ \tilde{L}_2^{\text{cubic}} \left( e^{-1/4 \partial} \right) \circ e^{-\frac{A(s,s)}{c^2}} = \sum_{k \in \mathbb{N}_*} b_k V_2(-b_k) \frac{\partial}{\partial s_{k+2h}} + x V_2(0) \frac{\partial}{\partial s_{2h}} + \frac{c^2}{2} b_{h+1} \text{res}_{z=0} V_2(z) \frac{\partial^2}{\partial s_{h} \partial s_{h}} + \frac{c^2}{2} \sum_{\alpha=1}^{K_1-1} \sum_{\ell=0}^{1} b_{\alpha}^{-1} \text{res}_{z=0} V_2(z) \frac{\partial^2}{\partial s_{\alpha+h \ell} \partial s_{\alpha K_1-h+1-\ell}}.
\]

Lemma 2.8 The numbers \( c_k, k \in \mathbb{N}_* \) defined in (1.7)–(1.8) have the following properties:

(i) For \( k \in \mathbb{N}_*, \) \( \ell \geq 1, \) \( c_{k+h \ell} / c_k = K^\ell V_{\ell}(-b_k). \)

(ii) For \( \ell \geq 1, \) \( c_{h \ell} = K^\ell V_{\ell}(0). \)

(iii) Let \( m, n \geq 0 \) be integers. Then

\[
c_{a+h m} c_{K_1-a+h n} = \frac{h}{K_2} K_{a+h m}^{m+n+1} \text{res}_{z=0} V_{m+n+1}(z), \quad \alpha = 1, \ldots, K_1 - 1, \]

\[
c_{a+h m} c_{-a-K_2+h n} = \frac{h}{K_1} K_{a+h m}^{m+n+1} \text{res}_{z=0} V_{m+n+1}(z), \quad \alpha = -(K_2 - 1), \ldots, -1, \]

\[
c_{h(m+1)} c_{h(n+1)} = \frac{K_{h(m+1)}^{m+n+2}}{b_{h(m+1)}} \text{res}_{z=0} V_{m+n+2}(z). \]

The proof of Lemma 2.8 is elementary with the help of Lemma 2.3, so we omit it here.

By comparing (2.19)–(2.20) with the definition of the left hand side of (2.14) and by using Lemma 2.8, we find that (2.14) is true for \( m = 1, 2. \) Thus Theorem 1.2 is proved. \( \square \)

3 Loop equation: the rational case

In this section, we derive the Dubrovin-Zhang loop equation for the special cubic Hodge integrals in the rational case, namely, we take

\[
p = 1/K_1, \quad q = 1/K_2, \quad r = -1/h,
\]

where \( K_1 \) and \( K_2 \) are coprime positive integers, and \( h = K_1 + K_2. \)

Introduce a generating series of the operators \( L_m, m \geq 0 \) defined in (1.12), (1.13) as follows:

\[
L(\lambda) = \sum_{m \geq 0} \frac{L_m}{\lambda^{m+1}}.
\]

12
Lemma 3.1 The generating series \( L(\lambda) \) can be represented as

\[
L(\lambda) = \left[ J_1(\lambda)J_2(\lambda) + \frac{1}{2} J_2(\lambda)^2 \right]_{\text{reg.}} + \frac{1}{\lambda} \left( \frac{x}{2\epsilon^2} + \sigma_1 \right), \tag{3.2}
\]

where the operators \( J_1(\lambda) \) and \( J_2(\lambda) \) are defined by

\[
J_1(\lambda) := \epsilon^{-1} \frac{x}{\sqrt{\lambda}} + \epsilon^{-1} \sum_{k \in \mathbb{N}_*} b_k \lambda^{b_k - \frac{1}{2}} s_k,
\]

\[
J_2(\lambda) := \sum_{\ell \geq 1} \frac{\epsilon}{\lambda^{b_{h\ell} + 1/2}} \partial s_{h\ell} + \sqrt{\frac{K_2}{h}} \sum_{\alpha = 1}^{K_1-1} \sum_{\ell \geq 0} \epsilon \lambda^{b_{\alpha+h\ell} + \frac{1}{2}} \partial s_{\alpha+h\ell}
\]

\[
+ \sqrt{\frac{K_1}{h}} \sum_{\alpha = -(K_2-1)}^{-1} \sum_{\ell \geq 0} \epsilon \lambda^{b_{\alpha+h\ell} + \frac{1}{2}} \partial s_{\alpha+h\ell},
\]

and \([\cdot]_{\text{reg.}}\) means keeping terms of the series with negative integer powers of \( \lambda \).

Theorem 1.2 now implies that \( L(\lambda)Z(x,s;\epsilon) = 0 \) holds true identically in \( \lambda \). From the definition of \( Z(x,s;\epsilon) \) given by (1.9) we know that its logarithm admits the genus expansion

\[
\log Z(x,s;\epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} F_g(x,s). \tag{3.3}
\]

Here we denote, as we do in Corollary 1.4,

\[
F_g(x,s) := H_g(t(x,s); \frac{1}{K_1}, \frac{1}{K_2}, -\frac{1}{h}) + A(x,\tilde{s}) \delta_{g,0}, \quad \text{with} \ \tilde{s}_k = s_k - c_k^{-1} \delta_{k,h}. \tag{3.3}
\]

In what follows we will also use the notations

\[
u(x,s) := \partial_x^2 F_0(x,s)
\]

and

\[
\Delta F = \sum_{g \geq 1} \epsilon^{2g-2} F_g(x,s). \tag{3.4}
\]

Lemma 3.2 The function \( u(x,s) \) satisfies the following equations

\[
\partial_t^i u(x,s) = \partial_t^i v(t(x,s)), \quad i \geq 0. \tag{3.5}
\]

\[
\frac{\partial u}{\partial s_k} = c_k \partial_x e^{b_k u}, \quad \frac{\partial^2 F_0}{\partial x \partial s_k} = c_k e^{b_k u}, \quad k \in \mathbb{N}_*. \tag{3.6}
\]

Proof By using (1.10) we obtain

\[
\partial_x = \partial_t_0, \quad \frac{\partial}{\partial s_k} = c_k \sum_{i \geq 0} b_k^{i+1} \frac{\partial}{\partial t_i}, \quad k \in \mathbb{N}_*.
\]

The lemma then follows from the Riemann hierarchy (1.5). \hfill \Box
Lemma 3.3 ([5]) For any given $g \geq 1$, there exists a polynomial $H_g(z_0, \ldots, z_{3g-2}; p, q, r)$ of $p, q, r, z_2, z_3, \ldots$ with coefficients depending smoothly on $z_0$ and rationally on $z_1$, such that

$$\mathcal{H}_g(t; p, q, r) = H_g \left( v(t), \frac{\partial v(t)}{\partial t_0}, \ldots, \frac{\partial^{3g-2} v(t)}{\partial t_0^{3g-2}}; p, q, r \right).$$  \hspace{1cm} (3.7)

It should be noted that Lemma 3.3 does not require $p, q, r$ satisfy the local Calabi-Yau condition.

Let us denote

$$u^{(i)} = \partial_s^i u(x, s), \quad i \geq 0.$$  \hspace{1cm} (3.8)

Then from Lemmas 3.2–3.3 we know that

$$\mathcal{F}_g(x, s) = H_g \left( u^{(0)}, u^{(1)}, \ldots, u^{(3g-2)}; p, q, r \right), \quad g \geq 1,$n

where $p = 1/K_1$, $q = 1/K_2$, $r = -1/(K_1 + K_2)$. Introduce a derivation $D(\lambda)$ on $\mathbb{C}[[x - 1, s]]$ by

$$D(\lambda) = -[J_1(\lambda) J_2(\lambda)]_{\text{reg}, -} - \epsilon^{-2} [J_2(\lambda)(\Delta \mathcal{F}) J_2(\lambda)]_{\text{reg}, -},$$

where $J_1(\lambda), J_2(\lambda)$ are defined in Lemma 3.1.

Lemma 3.4 The series $D(\lambda)(\Delta \mathcal{F})$ has the following expression:

$$D(\lambda)(\Delta \mathcal{F}) = \sum_{i \geq 0} D(\lambda) \left( u^{(i)} \right) \frac{\partial \Delta \mathcal{F}}{\partial u^{(i)}} = \frac{\sigma_1}{24\lambda} + \frac{1}{2} \sum_{i \geq 0} J_2(\lambda)^2 \left( u^{(i)} \right) \frac{\partial \Delta \mathcal{F}}{\partial u^{(i)}}$$

$$+ \frac{1}{2} \sum_{i,j \geq 0} J_2(\lambda) \left( u^{(i)} \right) J_2(\lambda) \left( u^{(j)} \right) \left( \frac{\partial^2 \Delta \mathcal{F}}{\partial u^{(i)} \partial u^{(j)}} + \frac{\partial \Delta \mathcal{F}}{\partial u^{(i)}} \frac{\partial \Delta \mathcal{F}}{\partial u^{(j)}} \right) + \frac{1}{2\epsilon^2} J_2(\lambda)^2 (\mathcal{F}_0) \right]_{\text{reg}, -}. \hspace{1cm} (3.9)$$

Proof From Lemma 3.1 we know that the Virasoro constraints $L(\lambda)Z(x, s; \epsilon) = 0$ can be represented as

$$\left[ J_1(\lambda) J_2(\lambda) (\mathcal{F}_0) + \frac{1}{2\epsilon^2} J_2(\lambda) (\mathcal{F}_0)^2 \right]_{\text{reg}, -} + \frac{x^2}{2\lambda} = 0, \quad (3.10)$$

$$\left[ J_1(\lambda) J_2(\lambda) (\Delta \mathcal{F}) + \frac{1}{\epsilon^2} J_2(\lambda) (\mathcal{F}_0) J_2(\lambda) (\Delta \mathcal{F}) \right]$$

$$+ \frac{1}{2} \left( J_2(\lambda)^2 (\Delta \mathcal{F}) + J_2(\lambda)(\Delta \mathcal{F})^2 \right) + \frac{1}{2\epsilon^2} J_2(\lambda)^2 (\mathcal{F}_0) \right]_{\text{reg}, -} + \frac{\sigma_1}{24\lambda} = 0. \quad (3.11)$$

It is easy to see that (3.11) can be rewritten in the form

$$D(\lambda)(\Delta \mathcal{F}) = \frac{\sigma_1}{24\lambda} + \left[ \frac{1}{2} \left( J_2(\lambda)^2 (\Delta \mathcal{F}) + J_2(\lambda)(\Delta \mathcal{F})^2 \right) + \frac{1}{2\epsilon^2} J_2(\lambda)^2 (\mathcal{F}_0) \right]_{\text{reg}, -}. \hspace{1cm} (3.12)$$

Since $J_2(\lambda)$ is a derivation, we obtain (3.9). \hspace{1cm} \Box

Let us proceed to simplify equation (3.9). Introduce the following Puiseux series:

$$E_\alpha = \left\{ \begin{array}{ll}
\sum_{n \geq 0} c_{\alpha + n} s^{\frac{n+\alpha}{K_1}}, & \alpha = 0, \ldots, K_1 - 1, \\
\sum_{n \geq 0} c_{\alpha + n} s^{-\frac{\alpha}{K_2}}, & \alpha = -(K_2 - 1), \ldots, -1,
\end{array} \right. \hspace{1cm} (3.13)$$
where $\zeta := e^{u(x,s)}/\lambda$. Define

$$B_{i,j} := \sum_{\alpha,\beta\in\mathcal{I}} \partial_x^i (E_\alpha) G^{\alpha\beta} \partial_x^j (E_\beta),$$

(3.14)

where $G^{\alpha\beta}$ are the constants defined in (1.14). Note that $B_{i,j} \in \mathbb{C}[u^{(1)}, u^{(2)}, \ldots][[\zeta]]$ for all $i, j \geq 0$.

**Lemma 3.5** The following formulae hold true:

$$D(\lambda) \left( u^{(i)} \right) = \frac{1}{\lambda} \left( \partial_x^i B_{0,0} + \sum_{j=1}^{i} \binom{i}{j} B_{j-1,i-j+1} \right), \quad \forall i \geq 0,$$

(3.15)

$$\epsilon^{-2} \left[ J_2(\lambda) \left( u^{(i)} \right) J_2(\lambda) \left( u^{(j)} \right) \right]_{\text{reg},-} = \frac{1}{\lambda} B_{i+1,j+1}, \quad \forall i, j \geq 0.$$

(3.16)

**Proof** Using Lemma 3.2 and the definition of $J_2(\lambda)$ given in Lemma 3.1, we have

$$\epsilon^{-1} J_2(\lambda) (\partial_x F_0) + \frac{1}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} \left[ E_0 + \sqrt{\frac{K_2}{h}} \sum_{\alpha=1}^{K_1-1} E_\alpha + \sqrt{\frac{K_1}{h}} \sum_{\alpha=(K_2-1)}^{-1} E_\alpha \right].$$

Acting $\partial_x^2$ on the both sides of equation (3.10) yields

$$D(\lambda)(u) = \left[ (\epsilon^{-1} J_2(\lambda)(\partial_x F_0) + \frac{1}{\sqrt{\lambda}})^2 \right]_{\text{reg},-} = \frac{1}{\lambda} B_{0,0},$$

which gives the validity of (3.15) for $i = 0$. Assume that formula (3.15) is true for $i = k$; then for $i = k + 1$ we have

$$D(\lambda) \left( u^{(k+1)} \right) = \partial_x D(\lambda) \left( u^{(k)} \right) + [D(\lambda), \partial_x] \left( u^{(k)} \right)$$

$$= \frac{1}{\lambda} \left( \partial_x^{k+1} B_{0,0} + \sum_{j=1}^{k} \binom{k}{j} (B_{j,k-j+1} + B_{j-1,k-j+2}) \right) + \epsilon^{-1} \left[ J_2(\lambda)(\partial_x F_0) + \frac{1}{\sqrt{\lambda}} J_2(\lambda) \left( u^{(k)} \right) \right]_{\text{reg},-}$$

$$= \frac{1}{\lambda} \left( \partial_x^{k+1} B_{0,0} + \sum_{j=1}^{k+1} \binom{k+1}{j} B_{j-1,k-j+2} \right).$$

Hence by using mathematical induction we arrive at the formula (3.15). Formula (3.16) can be verified directly. The lemma is proved.

Let us now introduce a family of polynomials $f_{i,j} \in \mathbb{C}[u^{(1)}, u^{(2)}, \ldots], i, j \geq 0$ by requiring the validity of the identity

$$\partial_x^i h(\zeta) = \sum_{j=0}^{i} f_{i,j} (\zeta \partial_\zeta)^j h(\zeta)$$

(3.17)

for any smooth function $h(\zeta)$ of $\zeta = e^{u(x,s)}/\lambda$. Clearly, $f_{i,j}$ vanishes if $i < j$. For $i \geq j$, the polynomials $f_{i,j}$ can be uniquely determined by the following recursive relations:

$$f_{i,0} = \delta_{i,0},$$

$$f_{i+1,j+1} = \partial_x f_{i,j+1} + u^{(1)} f_{i,j}.$$
Explicit expressions for $f_{i,j}$ will be given in Section 4. The functions $B_{i,j}$ defined in (3.14) can now be written as

$$B_{i,j} = \sum_{k=0}^{i} \sum_{l=0}^{j} f_{i,k} f_{j,l} \tilde{B}_{k,l},$$

(3.18)

where

$$\tilde{B}_{k,l} := \sum_{\alpha,\beta \in I} (\zeta \partial_\zeta)^k (E_\alpha) G^{\alpha\beta} (\zeta \partial_\zeta)^l (E_\beta).$$

(3.19)

**Lemma 3.6** The functions $\tilde{B}_{i,j}$, $i, j \geq 0$ defined in (3.19) satisfy the relations

$$\sum_{k \geq 0} z^{-k} \tilde{B}_{0,k} = \sqrt{z} \Phi(z) \frac{1}{1 - K\zeta e^{-\partial_z}} \left( \frac{1}{\sqrt{z} \Phi(z)} \right),$$

(3.20)

$$\tilde{B}_{i,j} = \tilde{B}_{j,i}, \quad \zeta \partial_\zeta \tilde{B}_{i,j} = \tilde{B}_{i+1,j} + \tilde{B}_{i,j+1}.$$  

(3.21)

Moreover, $\tilde{B}_{i,j}$ are polynomials of $(1 - K\zeta)^{-1}$ with degrees less than or equal to $i + j + 1$.

**Proof** By substituting (3.13) into (3.19) we obtain

$$\tilde{B}_{0,k} = \sum_{n \geq 0} A_{k,n} \zeta^n,$$

(3.22)

where

$$A_{k,n} := \sum_{\ell=1}^{n-1} b_{k+\ell}^n c_{h(\ell+h(n-\ell))} + c_{n-k,0}$$

$$+ \frac{K_2}{h} \sum_{\alpha=1}^{K_1-1} \sum_{\ell=0}^{n-1} b_{\alpha+\ell}^n c_{h(\ell+h(n-\ell))} + \frac{K_1}{h} \sum_{\alpha=-(K_2-1)}^{n-1} \sum_{\ell=0}^{n-1} b_{\alpha+\ell}^n c_{h(\ell+h(n-\ell))}.$$  

(3.23)

Using Lemma 2.8 we find that

$$K^{-n} A_{k,n} = \sum_{\alpha \in I} \sum_{\ell=0}^{n-1} b_{\alpha+\ell}^{\alpha-1} \text{res}_{z=b_{\alpha+\ell}} V_n(z) + \frac{\text{res}_{z=n} V_n(z)}{n} \delta_{k,0}.$$  

(3.24)

In particular, we have

$$A_{0,n} = K^n \left( 1 - V_n(0) + \frac{1}{n} \text{res}_{z=n} V_n(z) \right) = K^n.$$  

Hence

$$\sum_{k \geq 0} z^{-k} \tilde{B}_{0,k} = \sum_{k,n \geq 0} z^{-k} A_{k,n} \zeta^n = \sum_{n \geq 0} V_n(z) (K\zeta)^n$$

$$= \sum_{n \geq 0} \frac{\Phi(z)}{\Phi(z-n)} \sqrt{\frac{z}{z-n}} (K\zeta)^n = \sqrt{z} \Phi(z) \frac{1}{1 - K\zeta e^{-\partial_z}} \left( \frac{1}{\sqrt{z} \Phi(z)} \right).$$

(3.24)
Note that
\[
\frac{1}{1 - K\zeta e^{-\partial_z}} = \sum_{n \geq 0} \sum_{k=1}^{n+1} \frac{Q(n,k)}{(1 - K\zeta)^k} \partial_z^n, \quad Q(n,k) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} i^{n+1}. \tag{3.25}
\]
So from (3.21), (3.24) it follows that \(\tilde{B}_{i,j} \in \mathbb{C}[(1 - K\zeta)^{-1}]\). The lemma is proved. \(\square\)

**Remark 3.7** Introduce a trinomial curve
\[
XY^{K_1+K_2} - Y^{K_2} + 1 = 0. \tag{3.26}
\]
Near \(X = 0\) we have two Puiseux series solutions:
\[
Y_- = \sum_{m \geq 0} c_{-,m} X^m, \quad c_{-,0} = 1,
\]
\[
Y_+ = X^{-\frac{1}{K_1}} \sum_{m \geq 0} c_{+,m} X^{m\frac{K_2}{K_1}}, \quad c_{+,0} = 1.
\]
By using the Lagrange inversion we obtain the explicit formulae
\[
(Y_-)^{K_2} = \sum_{m \geq 0} \left( \frac{m^{K_1+K_2}}{K_2^m} \right) \frac{X^m}{1 + m\frac{K_1}{K_2}}, \quad (Y_+)^{-K_1} = \sum_{m \geq 0} \left( \frac{m^{K_1+K_2}}{K_1^m} \right) \frac{X^{1+m\frac{K_2}{K_1}}}{1 + m\frac{K_2}{K_1}}.
\]
We also observe the validity of the following relations
\[
\sum_{\alpha=0}^{K_1-1} E_{\alpha} = \frac{d}{dX} ((Y_+)^{-K_1}), \quad \sum_{\alpha=-(K_2-1)}^{0} E_{\alpha} = \frac{K_1}{K_2} X \frac{d}{dX} ((Y_-)^{K_2}) + (Y_-)^{K_2}. \tag{3.27}
\]
with \(X = \zeta^{1/K_2}\).

**Corollary 3.8** The following formula holds true:
\[
\epsilon^{-2} \left[ J_2(\lambda)^2 (\mathcal{F}_0) \right]_{\text{reg,-}} = \frac{1}{\lambda} \left( \frac{1}{8(1 - K\zeta)^2} - \left( \frac{1}{8} - \frac{\sigma_1}{12} \right) \frac{1}{1 - K\zeta} - \frac{\sigma_1}{12} \right). \tag{3.28}
\]

**Proof** By using the definition of \(J_2\) given in Lemma 3.1 and by using Lemma 3.2 we obtain
\[
\epsilon^{-2} \left[ J_2(\lambda)^2 (\mathcal{F}_0) \right]_{\text{reg,-}} = \int \frac{1}{\lambda} \frac{B_{1,1}}{u(1)} dx = \frac{1}{\lambda} \int \frac{\tilde{B}_{1,1}}{K\zeta} d(K\zeta). \tag{3.29}
\]
Here we recall that \(B_{1,1}\) is defined in (3.14), and the integration constant is chosen such that the left hand side goes to zero as \(\lambda \to \infty\). From Lemma 3.6 it follows that
\[
\tilde{B}_{1,1} = \frac{1}{4} \frac{1}{(1 - K\zeta)^2} - \left( \frac{3}{8} - \frac{\sigma_1}{12} \right) \frac{1}{(1 - K\zeta)^2} + \left( \frac{1}{8} - \frac{\sigma_1}{12} \right) \frac{1}{1 - K\zeta},
\]
which, together with (3.29), leads to the formula (3.28). The corollary is proved. \(\square\)

By using Lemmas 3.4–3.5 and Corollary 3.8 we arrive at the following theorem.
Theorem 3.9 The series $\Delta F$ defined in (3.4), (3.8) satisfies the Dubrovin-Zhang loop equation

\[
\sum_{i \geq 0} \frac{\partial \Delta F}{\partial u^{(i)}} \frac{\partial^i \Theta}{x} + \sum_{i \geq 1} \sum_{j=1}^{i} \sum_{a,\beta \in I} \frac{\partial \Delta F}{\partial u^{(i)}} \left( \frac{\partial^i - 1}{x} (E_{\alpha}) G^{\alpha \beta} \frac{\partial^j - 1}{x} (E_{\beta}) \right) = \frac{\Theta^2}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta
\]

\[
+ \sum_{i,j \geq 0} \sum_{a,\beta \in I} \left( \frac{\partial^2 \Delta F}{\partial u^{(i)} \partial u^{(j)}} + \frac{\partial \Delta F}{\partial u^{(i)}} \frac{\partial \Delta F}{\partial u^{(j)}} \right) \frac{\partial^{i+1}}{x} (E_{\alpha}) G^{\alpha \beta} \frac{\partial^{j+1}}{x} (E_{\beta}), \quad (3.30)
\]

where $\Theta = \frac{1}{1-K\zeta}$, $K = h^h K_1^{-K_1} K_2^{-K_2}$, and $\sigma_1 = -(p + q + r) = \frac{1}{k} - \frac{1}{k_1} - \frac{1}{k_2}$.

Note that each side of the loop equation (3.30) is a power series of $\Theta$. It is understood that this equation for $\Delta F$ holds identically in $\Theta$. It is easy to check that $F_g$ also satisfy the equations

\[
F_1 = \frac{1}{24} \log u^{(1)} + \frac{\sigma_1}{24} u^{(0)} - \sum_{j=1}^{3g-2} j u^{(j)} \frac{\partial F_{g-2}}{\partial u^{(j)}} = (2g-2) F_g, \quad g \geq 2. \quad (3.31)
\]

Proposition 3.10 The solution to equations (3.30)-(3.31) is unique.

Proof For $g \geq 1$, by comparing the coefficients of $e^{2g-2}$ in the both sides of (3.30), we obtain

\[
\sum_{i \geq 0} \left( \frac{\partial^i B_{0,0}}{x} + \sum_{j=1}^{i} \frac{\partial^i B_{j-i,j+1}}{x} \right) \frac{\partial F_1}{\partial u^{(i)}} = \frac{\Theta^2}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta,
\]

\[
\sum_{i \geq 0} \left( \frac{\partial^i B_{0,0}}{x} + \sum_{j=1}^{i} \frac{\partial^i B_{j-i,j+1}}{x} \right) \frac{\partial F_g}{\partial u^{(i)}} = \frac{\Theta^2}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta \frac{\partial F_{g-1}}{\partial u^{(i)}}
\]

\[
+ \sum_{i,j \geq 0} B_{i+1,j+1} \left( \frac{\partial^2 F_{g-1}}{\partial u^{(i)} \partial u^{(j)}} + \sum_{k=1}^{g-1} \frac{\partial F_k}{\partial u^{(i)}} \frac{\partial F_{g-k}}{\partial u^{(j)}} \right), \quad g \geq 2.
\]

By using the fact that $\partial F_g/\partial u^{(i)} = 0$ for $i \geq 3g-1$, and that $B_{i,j}$ are polynomials in $\Theta$ of degrees $i + j + 1$, we arrive at the following system of equations:

\[
(\Theta, \cdots, \Theta^{3g-1}) M_g \left( \frac{\partial F_g}{\partial u^{(0)}}, \cdots, \frac{\partial F_g}{\partial u^{(3g-2)}} \right) = (\Theta, \cdots, \Theta^{3g-1}) N_g, \quad g \geq 1,
\]

where $M_g$ is an invertible upper triangular $(3g-1) \times (3g-1)$ matrix, and $N_g$ is a column vector. All the elements of the matrix and the vector are differential polynomials in $u$. So the gradient of $F_g$ can be uniquely determined. The proposition is proved. □
4 Loop equation: the general case

In this section we drop the rational condition \([16]\) and consider the general case when \(p, q, r\) are arbitrary complex numbers satisfying the local Calabi-Yau condition \([13]\). We will give and prove a refined version of the loop equation for the corresponding Hodge free energies.

As it is pointed out in Lemma \([6,3]\), there are polynomials \(H_g(z_0, z_1, \ldots, z_{3g-2}; p, q, r)\) in \(p, q, r\) which satisfy the relations \((3.7)\) and the relations \((1.19)\) (see \([5]\)), here \(g \geq 1\). It is then clear that \(H_g(z_0, \ldots, z_{3g-2}; p, q, r)\) are also polynomials of \(\sigma_1, \sigma_3\) for \(g \geq 1\). For simplicity, we will denote them by \(H_g(z_0, \ldots, z_{3g-2}; \sigma_1, \sigma_3)\).

We will need the following lemma.

**Lemma 4.1** Let \(G(\sigma_1, \sigma_3)\) be a polynomial in \(\mathbb{C}[\sigma_1, \sigma_3]\). If \(G\) vanishes for the values \((\sigma_1, \sigma_3) = \left(\frac{1}{K_1+K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1+K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3}\right)\), where \(K_1, K_2 \in \mathbb{N}, (K_1, K_2) = 1\), then \(G \equiv 0\).

**Proof** Suppose \(G \not\equiv 0\). Fix \(K_1\) to be any positive integer. We observe that the points \(\left(\frac{1}{K_1+K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1+K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3}\right), K_2 \in \mathbb{N}\) belong to the irreducible algebraic curve

\[
2K_1^3\sigma_1^3 - 6K_1\sigma_1 - 6 - K_1^3\sigma_3 = 0
\]

on the \((\sigma_1, \sigma_3)\) plane. It is easy to see that there are infinitely many of such points with \((K_2, K_1) = 1\). Hence the polynomial

\[
2K_1^3\sigma_1^3 - 6K_1\sigma_1 - 6 - K_1^3\sigma_3
\]

must divide \(G(\sigma_1, \sigma_3)\). This contradicts with the fact that a polynomial must have a finite degree. The lemma is proved. \(\square\)

It follows from Lemma \([6,1]\) that if a polynomial \(P\) in \(\mathbb{C}[\sigma_1, \sigma_3][z_2, z_3, \ldots][z_1, z_1^{-1}]\) is equal to \(H_g(z_0, \ldots, z_{3g-2}; \sigma_1, \sigma_3)\) for

\[
(\sigma_1, \sigma_3) = \left(\frac{1}{K_1+K_2} - \frac{1}{K_1} - \frac{1}{K_2}, \frac{2}{(K_1+K_2)^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3}\right)
\]

with \(K_1, K_2\) being arbitrary coprime positive integers, then \(P \equiv H_g(z_0, \ldots, z_{3g-2}; \sigma_1, \sigma_3)\). Here \(g \geq 2\). For \(g = 1\), we have the explicit expression

\[
H_1(z_0, z_1; \sigma_1, \sigma_3) = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24} z_0.
\]

In order to give an efficient algorithm for the direct computation of \(H_g(v, \ldots, v_{3g-2}; \sigma_1, \sigma_3)\), \(g \geq 2\), we proceed to derive the loop equation for these functions, which is a refinement of \((3.30)\).

To this end, we first introduce some notations. Let \(B_{n,k}(X_1, \ldots, X_{n-k+1})\) be the exponential Bell polynomials. They can be defined via the generating function

\[
\sum_{n,k \geq 0} B_{n,k}(X_1, \ldots, X_{n-k+1}) \frac{y^n}{n!} z^k = \exp \left(z \sum_{j \geq 1} X_j \frac{y^j}{j!} \right).
\]

Let \(B_n\) be the complete Bell polynomials, i.e.

\[
B_0 := 1, \quad B_n(X_1, \ldots, X_n) := \sum_{k=1}^n B_{n,k}(X_1, \ldots, X_{n-k+1}), \quad n \geq 1.
\]
Now we define \( \tilde{P}_{0,k}(\xi;\sigma_1,\sigma_3) \in \mathbb{C}[\sigma_1,\sigma_3][(1 - \xi)^{-1}], k \geq 0 \) by the equation

\[
\sum_{k \geq 0} z^{-k} \tilde{P}_{0,k}(\xi;\sigma_1,\sigma_3) = \sqrt{z} \Phi(z;\sigma_1,\sigma_3) \frac{1}{1 - \xi e^{-\partial_z}} \left( \frac{1}{\sqrt{z} \Phi(z;\sigma_1,\sigma_3)} \right),
\]

where

\[
\Phi := \Phi(z;\sigma_1,\sigma_3) = \exp \left( -\sum_{i \geq 1} \frac{B_{2i}}{2i(2i - 1)} (p^{2i-1} + q^{2i-1} + r^{2i-1})z^{1-2i} \right).
\]

Equation (4.1) can be written more explicitly as follows:

\[
\sum_{n \geq 0} z^{-n}\tilde{P}_{0,n}(\xi;\sigma_1,\sigma_3) = \sum_{n \geq 0} \sum_{k=1}^{n+1} \frac{Q(n,k)}{(1 - \xi)^k} \sum_{m=0}^{n} \frac{(-1)^{n-m}(2n - 2m - 1)!!}{2^{n-m}m!(n-m)!} z^{-n+m} \Phi \partial_z^m \left( \frac{1}{\Phi} \right),
\]

where \( Q(n,k) \) are the numbers defined in (3.25), and

\[
\Phi \partial_z^m \left( \frac{1}{\Phi} \right) = B_m (-\partial_z \log \Phi(z), \ldots, -\partial_z^m \log \Phi(z)), \quad m \geq 0.
\]

We define, for \( i, j \geq 0 \), \( \tilde{P}_{i,j} = \tilde{P}_{i,j}(\xi;\sigma_1,\sigma_3) \) by the following recursion:

\[
\tilde{P}_{i,j} = \tilde{P}_{j,i},
\]

\[
\xi \partial_{\tilde{\kappa}} \tilde{P}_{i,j} = \tilde{P}_{i+1,j} + \tilde{P}_{i,j+1}.
\]

It is easy to see that the functions \( f_{i,k} \) defined in (3.17) are just the Bell polynomials \( B_{i,k}(z_1, \ldots, z_{i-k+1}) \). By using these functions we introduce, for \( i, j \geq 0 \), the following notations:

\[
P_{i,j} = \sum_{k=0}^{i} \sum_{l=0}^{j} f_{i,k}(z_1, \ldots, z_{i-k+1}) f_{j,l}(z_1, \ldots, z_{j-l+1}) \tilde{P}_{k,l}(\xi;\sigma_1,\sigma_3).
\]

**Proof of Theorem 1.3** It is easy to verify that the functions \( P_{i,j} \) defined above coincide with the functions \( B_{i,j} \) that are given in (3.18) when \( \xi = K\xi \) and

\[
(\sigma_1,\sigma_3) = \left( \frac{1}{h} - \frac{1}{K_1 - K_2}, \frac{2}{h^3} - \frac{2}{K_1^3} - \frac{2}{K_2^3} \right),
\]

\[
(z_1, z_2, \ldots) = \left( u^{(1)}, u^{(2)}, \ldots \right),
\]

where \( K_1, K_2 \) are arbitrary coprime positive integers. It then follows from Theorem 3.3 that \( \Delta H := \sum_{g \geq 1} \xi^{2g-2} H_g(z_0, \ldots, z_{3g-2};\sigma_1,\sigma_3) \) satisfies equation (1.18) for the particular values of \( (\sigma_1,\sigma_3) \). By definition we also see that \( P_{i,j} \in \mathbb{C}[\sigma_1,\sigma_3][(1 - \xi)^{-1}; z_1, z_2, \ldots] \). The existence of solution of the loop equation (1.3) is then further ensured by Lemma 4.1. As we do in the proof of Proposition 3.11 we deduce from (1.3) the following equation for each \( g \geq 1 \):

\[
(\Theta, \ldots, \Theta^{3g-1}) \overline{M}_g(\sigma_1,\sigma_3) \left( \frac{\partial H_g}{\partial z_0}, \ldots, \frac{\partial H_g}{\partial z_{3g-2}} \right) = (\Theta, \ldots, \Theta^{3g-1}) \overline{N}_g(\sigma_1,\sigma_3),
\]

20
where $\nabla g(\sigma_1,\sigma_3)$ is an invertible $(3g-1)$ by $(3g-1)$ matrix, $\nabla g(\sigma_1,\sigma_3)$ is a column vector, and their entries are polynomials of $\sigma_1,\sigma_3$. Therefore we have verified that the gradient of $H_g$ is uniquely determined by (1.3). The theorem is proved.

Example 4.2 By taking the coefficient of $e^0$ in (1.18), we obtain the following equation:

$$P_{0,0} \frac{\partial H_1}{\partial z_0} + (\partial P_{0,0} + P_{0,1}) \frac{\partial H_1}{\partial z_1} = \Theta^2 \frac{1}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta$$

for $H_1$. The coefficients read

$$P_{0,0} = \Theta, \quad P_{0,1} = \frac{z_1}{2} \left( \Theta^2 - \Theta \right).$$

So we have

$$\frac{3z_1}{2} \frac{\partial H_1}{\partial z_1} \Theta^2 + \left( \frac{\partial H_1}{\partial z_0} - \frac{3z_1}{2} \frac{\partial H_1}{\partial z_1} \right) \Theta = \Theta^2 \frac{1}{16} - \left( \frac{1}{16} - \frac{\sigma_1}{24} \right) \Theta.$$

which gives

$$\frac{\partial H_1}{\partial z_0} = \frac{\sigma_1}{24}, \quad \frac{\partial H_1}{\partial z_1} = \frac{1}{24z_1}.$$

Hence

$$H_1 = \frac{1}{24} \log z_1 + \frac{\sigma_1}{24z_0}$$

which gives (1.21). In a similar way, we obtain formula (1.22) and the following expression of $H_3$:

$$H_3 = \sum_{n=0}^{4} \frac{z^n}{5^n} \left( \frac{z_2 z_4}{z_1} - \frac{7}{46080} \frac{z_2 z_6}{z_1} - \frac{53}{161280} \frac{z_3 z_5}{z_1} + \frac{353}{322560} \frac{z_3 z_5}{z_1} - \frac{103}{483840} \frac{z_4}{z_1} \right)$$

where $\frac{z^n}{5^n}$ is the $n$th Taylor expansion of $\frac{1}{5} \log(1 + \frac{z}{5})$.

Proof of Corollary 1.4 The properties (1.23)–(1.24) follow from (1.19)–(1.20) and the simple fact that $v(t(x,s)) \in \log x + C[x][[s]]$, where it should be noted that due to Lemma 1.1
polynomials $R_g(\sigma_1, \sigma_3)$ must be unique. The degree estimate $\deg R_g(\sigma_1, \sigma_3) \leq 3g-3$ with $g \geq 2$ follows from the theorem 1.3 of [5]. To show (1.26) we first note that for $g \geq 2$,

\[ \mathcal{H}_g(t;p,q,r) = (-1)^{g-1} \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g \lambda_{g-1} \lambda_{g-2} (p^g q^{g-1} r^{g-2} + (p \leftrightarrow q \leftrightarrow r)) + (-1)^{g-1} \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g^3 p^{g-1} q^{g-1} r^{g-1} + \ldots, \tag{4.5} \]

where

\[ (p \leftrightarrow q \leftrightarrow r) := p^g r^{g-1} q^{g-2} + q^g p^{g-1} r^{g-2} + q^g r^{g-1} p^{g-2} + q^g p^{g-1} q^{g-2} + r^g q^{g-1} p^{g-2}, \]

and the dots “\ldots” denotes the terms in $\mathcal{H}_g(t;p,q,r)$, as polynomials of $p, q, r$, having degree strictly less than $3g-3$. Recalling (1.16), (1.17) and (1.25), it is clear that the degree defined for polynomials in $p, q, r$ and the one for polynomials in $\sigma_1, \sigma_3$ are consistent. We further note that the following formula holds true for $g \geq 2$:

\[ \sum_{n \geq 0} \sum_{i_1, \ldots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{1}{2(2g-2)!} \frac{|B_{2g}|}{2g} \frac{\partial^{2g-2} v(t)}{\partial t^{2g-2}}, \]

as it was proved in [5]. The corollary is then proved by using Mumford’s relation

\[ \lambda_{g-1}^3 = 2 \lambda_g \lambda_{g-1} \lambda_{g-2} \]

(see [19]), and again the fact that $v(t(x,s)) \in \log x + \mathbb{C}[x][[s]]$. \qed

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22
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