DOUBLE EXPONENTIAL GROWTH OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER EQUATION

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Abstract. For two-dimensional Euler equation on the torus, we prove that the $L^\infty$-norm of the vorticity gradient can grow as double exponential over arbitrary long but finite time provided that at time zero it is already sufficiently large. The method is based on the perturbative analysis around the singular stationary solution studied by Bahouri and Chemin in [1]. Our result on the growth of the vorticity gradient is equivalent to the statement that the operator of Euler evolution is linearly unbounded in Lipschitz norm for any time $t > 0$.

1. Introduction and some upper bounds

Consider the two-dimensional Euler equation for the vorticity
\[ \dot{\theta} = \nabla \theta \cdot \psi, \quad \psi = \nabla \perp \Delta^{-1} \theta, \quad \theta(x, y, 0) = \theta_0(x, y) \] (1)
and $\theta$ is $2\pi$-periodic in both $x$ and $y$ (that is, the equation is considered on the torus $T^2$). We assume that $\theta_0$ has zero average over $T^2$ and then $\Delta^{-1}$ is well-defined since the Euler flow is area-preserving and the average of $\theta(\cdot, t)$ is zero as well. Denote the operator of Euler evolution by $E_t$, i.e.,
\[ \theta(t) = E_t \theta_0 \]
The global existence of the smooth solution for smooth initial data is well-known and is due to Wolibner [13] (see also [10]). The estimate on the possible growth of the Sobolev norms, however, is double exponential. We sketch the proof of this bound for $H^2$-norm. The estimates for $H^s$, $s > 2$ can be obtained similarly. More general results on regularity can be found in [4]. Let
\[ j_k(t) = \|\theta(t)\|_{H^k} \]

Lemma 1.1. If $\theta$ is the smooth solution of (1), then
\[ j_2(t) \leq \exp\left(\frac{1 + 2 \log^+ j_2(0)}{2} \exp(C\|\theta_0\|_{L^\infty} t) - 1\right) \] (2)
Proof. Acting on (1) with Laplacian we get
\[ \Delta \dot{\theta} = \Delta \theta_x \psi_y + 2 \nabla \theta_x \cdot \nabla \psi_y - \Delta \theta_y \psi_x - 2 \nabla \theta_y \cdot \nabla \psi_x \]
Multiply by $\Delta \theta$ and integrate over $T^2$ to get
\[ \partial_t \|\theta\|^2_{H^2} \lesssim \|H(\psi)\|_{L^\infty} \|\theta\|^2_{H^2} \] (3)

Keywords: Two-dimensional Euler equation, growth of the vorticity gradient
2000 AMS Subject classification: primary 76B99, secondary 76F99.
where $H(\psi)$ denotes the Hessian of $\psi$. The next inequality follows from the Littlewood-Paley decomposition (see [4], proposition 1.4 for more general result)

$$\|H(\psi)\|_\infty < C(\sigma)\|\theta\|_\infty (1 + \log^+ \|\theta\|_{H^\sigma})$$ (4)

for any $\sigma > 1$. Notice that $\|\theta\|_\infty$ is invariant under the motion so combine (3) and (4) to get (2).

\[ \tag{2} \]

**Remark 1.** In the same way one can prove bounds for higher Sobolev norms, e.g.,

$$ \log j_2(t) \lesssim (1 + \log^+ j_4(0)) \exp(C\|\theta_0\|_\infty t) - 1 \quad (5) $$

The natural questions one can ask then are the following: first, how fast can the Sobolev norms grow in time and what is the mechanism that leads to their growth? Secondly, for fixed $t$, how does $\|E(t)\|_{H^s}$ depend on $\|\theta_0\|_{H^s}$, when the last expression grows to infinity? For example, given $\|\theta_0\|_s \sim 1$, the right hand side in (2) grows as a power function in $j_2(0)$, the degree grows exponentially in $t$ and is more than one for any $t > 0$.

Instead of working with Sobolev norms, we will be studying the uniform norm of the vorticity gradient (or Lipschitz norm) as this norm is more natural for the method used in the proof. It allows the similar upper bound. We again give the sketch of the proof for completeness.

**Lemma 1.2.** If $\theta_0$ is smooth and $\|\theta_0\|_\infty \sim 1$, then

$$ \|\nabla E_1\theta_0\|_\infty \lesssim \exp(C(1 + \log^+ \|\nabla \theta_0\|_\infty) e^{Ct}) \quad (6) $$

**Proof.** If $\Psi(z,t)$ is area-preserving Euler diffeomorphism, then

$$(E_1\theta_0)(z) = \theta_0(\Psi^{-1}(z,t))$$

On the other hand, $\Psi(z,t)$ solves

$$ \dot{\Psi} = -u(\Psi, t), \quad \Psi(z, 0) = z $$

where $u(z,t) = \nabla^1 \Delta^{-1} \theta(t, z)$. For the Riesz transform we have a trivial estimate

$$ \|H(\Delta^{-1} \theta)\|_\infty \lesssim 1 + \|\theta\|_\infty (1 + \log^+ \|\nabla \theta\|_\infty) \quad (7) $$

Indeed, without loss of generality we can evaluate the integral at zero and assume $\theta(0) = 0$. Then, e.g.,

$$ \int_{B_{1}(0)} \frac{\xi_1 \xi_2 \theta(\xi)}{\|\xi\|^4} d\xi \lesssim \int_{B_{3}(0)} \frac{1}{\|\xi\|^2} |\theta(\xi)| d\xi + \int_{\delta < |z| < 1} \frac{1}{\|\xi\|^2} |\theta(\xi)| d\xi $$

where $\delta^{-1} = \max\{\|\nabla \theta\|_\infty, 2\}$. Apply now the Lagrange formula to the first term to get (7).

So

$$ |u(w_1, t) - u(w_2, t)| \lesssim |w_1 - w_2| b, \quad b = 1 + \log^+ \|\nabla \theta(t)\|_\infty $$

Therefore, we have

$$ |\dot{f}| \lesssim fb, \quad f(t) = |\Psi(z_2, t) - \Psi(z_1, t)|^2 $$

After integration

$$ |z_2 - z_1| \exp\left(-C \int_0^t b(\tau) d\tau \right) \leq |\Psi(z_2, t) - \Psi(z_1, t)| \leq |z_2 - z_1| \exp\left(C \int_0^t b(\tau) d\tau \right) $$
Since
\[ \| \nabla \theta(z,t) \|_\infty = \sup_{z_1, z_2} \left| \frac{\theta_0(\Psi^{-1}(z_2,t)) - \theta_0(\Psi^{-1}(z_1,t))}{|z_2 - z_1|} \right| \]
we get inequality
\[ \| \nabla \theta(z,t) \|_\infty \lesssim , \| \nabla \theta_0 \|_\infty \exp \left( C \int_0^t b(\tau) d\tau \right) \]
Taking log of the both parts and applying the Gronwall-Bellman, we get (6).

In this paper, we will work only with large \( \| \nabla \theta_0 \|_\infty \). For that case, we will show that, given arbitrarily large \( \lambda \), the estimate \( \max_{t \in [0,T]} \| \nabla \theta(\cdot,t) \|_\infty > \lambda e^{T-1} \| \nabla \theta_0 \|_\infty \) can hold for some infinitely smooth initial data. This is far from showing that (2) or (3) are sharp however it already is equivalent to the statement that \( \mathcal{E}_t \) is linearly unbounded. The question of whether \( \| \nabla \theta_0 \|_\infty \) can be taken \( \sim 1 \) is left wide open, see discussion in the last section.

Our results rigorously confirm the following observation: if the 2D incompressible inviscid fluid dynamics gets into a certain “instability mode” then the Sobolev norms can grow very fast in local time (i.e. counting from the time the “instability regime” was reached). Can the Sobolev norms grow at all infinitely in time assuming that initially they are small? The answer to this question is yes, see [5] and [9, 12, 8, 11]. The important questions of linear and nonlinear instabilities were addressed before (see, e.g., [6] and references there). In the recent paper [7], it was proved that \( \mathcal{E}_t \) is not uniformly continuous on the unit ball in Sobolev spaces.

**Remark 2.** It must be mentioned here that 2D Euler allows rescaling which provides the tradeoff between the size of \( \theta_0 \) and the speed of the process, i.e. if \( \theta(x,y,t) \) is a solution then \( \mu \theta(x,y,\mu t) \) is also a solution for any \( \mu > 0 \). However, in our construction we will always have \( \| \theta \|_p \sim 1, \forall p \in [1, \infty] \).

**Remark 3.** If one replaces \( \Delta^{-1} \) in (1) by \( \Delta^{-\alpha} \) with \( \alpha > 1 \), then the growth of the vorticity gradient is at most exponential, e.g.
\[ \| \nabla \mathcal{E}_t^{(\alpha)} \theta_0 \|_\infty \lesssim \| \nabla \theta_0 \|_\infty \exp(C \| \theta_0 \|_\infty t) \]
Moreover, the lower exponential bound can hold for all times as long as \( \theta_0 \) is properly chosen (see [5]).

2. The singular stationary solution and dynamics on the torus

The following singular stationary solutions was studied before (see, e.g., [1, 4] in the context of \( \mathbb{R}^2 \)). We consider the following function
\[ \theta_0(x,y) = sgn(x) \cdot sgn(y), \quad |x| \leq \pi, |y| \leq \pi \]
This is a steady state. Indeed, the function \( \psi_0 = \Delta^{-1} \theta_0 \) is odd with respect to each variable as can be verified on the Fourier side. That, in particular, implies that \( \psi_0 \) is zero on the coordinate axes so its gradient is orthogonal to them. This steady state, of course, is a weak solution, a vortex-patch steady state. Another consequence of \( \psi_0 \) being odd is that the origin is a stationary point of the dynamics.

By the Poisson summation formula, we have
\[ \sum_{n \in \mathbb{Z}^2, n \neq (0,0)} |n|^{-2} e^{in \cdot z} = C \ln |z| + \phi(z), \quad z \sim 0 \]
where $\phi(z)$ is smooth and even.

Therefore, around the origin we have

$$\nabla \psi_0(x, y) \sim \int \int_{B_0(0)} \frac{(x - \xi_1, y - \xi_2)}{(x - \xi_1)^2 + (y - \xi_2)^2} \text{sgn}(\xi_1) \text{sgn}(\xi_2) d\xi_1 d\xi_2 + (O(y), O(x))$$

Due to symmetry, it is sufficient to consider the domain $D = \{0 < x < y < 0.001\}$. Then, taking the integrals, we see that

$$\mu(x, y) = (\mu_1, \mu_2) = (\nabla^\perp \psi_0)(x, y) = (8)$$

The correction terms $r_{1(2)}$ are smooth. Without loss of generality we will later assume that $c_2 = 1$ in the last formula (so $c_1 = 0.5$). That can always be achieved by time-rescaling. Notice also that the flow given by the vector-field $\mu$ is area-preserving.

Thus, the dynamics of the point $(\alpha, \beta) \in D, \alpha \ll \beta$ is

$$(C_1 \beta)^{\varepsilon} \lesssim y(t) \lesssim (C_2 \beta)^{\varepsilon}, \quad \alpha(C_1 \beta)^{-\varepsilon} \lesssim x(t) \lesssim \alpha(C_2 \beta)^{-\varepsilon}$$

where $t_0$ is the time the trajectory leaves the domain $D$. These estimates therefore give a bound on $t_0$. The attraction to the origin, the stationary point, is double exponential along the vertical axis and the repulsion along the horizontal axis is also double exponential.

3. The idea

The idea of constructing the smooth initial data for a double exponential scenario is quite simple and roughly can be summarized as follows: given any $T > 0$, we will smooth out the singular steady state such that the dynamics is double exponential over $[0, T]$ in a certain domain away from the coordinate axes. Then we will place a small but steep bump in the area of double exponential behavior and will let it evolve hoping that the vector field generated by this bump itself is not going to ruin the double exponential contraction in $OY$ direction. The rest of the paper verifies that this indeed is the case.

4. The Model Equation

Consider the following system of ODE’s

$$\begin{cases}
\dot{x} = \mu_1(x, y) + \nu_1(x, y, t), \quad x(\alpha, \beta, 0) = \alpha \\
\dot{y} = \mu_2(x, y) + \nu_2(x, y, t), \quad y(\alpha, \beta, 0) = \beta
\end{cases}$$

(10)

Here we assume the following

$$|\nu_{1(2)}| < 0.0001 v r, \quad r = \sqrt{x^2 + y^2}$$

(11)

and

$$|\nabla \nu_{1(2)}| < 0.0001 v$$

(12)

with small $v$ (to be specified later) and these estimates are valid in the area of interest

$$R = \{y > \sqrt{x}\} \cap \{y < \epsilon_2\} \cap \{x > \epsilon_1\}$$
where
\[ v \ll \epsilon_1 \ll \epsilon_2 \]
The functions \( \nu_1(2) \) are infinitely smooth in all variables in \( \mathbb{R} \) but we have no control over higher derivatives. We also assume that the flow given by (10) is area preserving. Our goal is to study the behavior of trajectories within the time interval \([0, T]\). In this section, the parameters will eventually be chosen in the following order
\[ T \rightarrow \epsilon_2 \rightarrow \epsilon_1 \rightarrow v \]
Here are some obvious observations:
1. If \( \epsilon_1(2) \) are small and
\[ \alpha \gg v \left| \frac{\beta}{\log \beta} \right| \]
then \( x(t) \) increases and \( y(t) \) decreases. This monotonicity persists as long as the trajectory stays within \( \mathbb{R} \). Assuming that \( \epsilon_1(2) \) are fixed, (13) can always be satisfied by taking \( v \) small enough, i.e.,
\[ v \lesssim \frac{\epsilon_1 |\log \epsilon_2|}{\epsilon_2} \] (14)
2. We have estimates
\[ x(\log y + C) + vy > \dot{x} > x(\log y - C) - vy, \quad -y(|\log y| + C) < \dot{y} < -y(|\log y| - C) \] (15)
The second estimate yields
\[ e^{e^T (\log \beta + C)} > y(t) > e^{e^T (\log \beta - C)} \] (16)
Let us introduce
\[ \kappa(T, \beta) = e^{e^T (\log \beta - C)} \]
For \( x(t) \), we have
\[ x(t) \leq \alpha \exp \left( Ct - \int_0^t \log y(\tau) d\tau \right) + v \int_0^t y(\tau) \exp \left( C(t - \tau) - \int_\tau^t \log y(s) ds \right) d\tau \]
\[ x(T) < (\alpha + v \beta T) \exp \left( T(C + |\log \kappa(T, \beta)|) \right) \]
Thus, the trajectory will stay inside \( \mathbb{R} \) for any \( t \in [0, T] \) as long as
\[ \alpha < \kappa^{3+T} - v \epsilon_2 T \]
and if we have
\[ v < \kappa^{4+T}(T, \beta) \] (17)
then the condition
\[ \alpha < \beta^{8e^{2T}} \] (18)
is sufficient for the trajectory to stay inside \( \mathbb{R} \) for \( t \in [0, T] \). Thus, we are taking
\[ \epsilon_1 < \epsilon_2^{8e^{2T}} \]
and focus on the nonempty domain
\[ \Omega_0 = \{ (\alpha, \beta) : \epsilon_1 < \alpha < \beta^{8e^{2T}}, \beta < \epsilon_2 \} \]
The condition on \( v \) is (17), so taking the smallest possible \( \kappa(T, \beta) \) within \( \Omega_0 \) we get, e.g.,
\[ v < \epsilon_1^{10} \] (19)
Then, any point from $\Omega_0$ stays inside $\mathcal{K}$ over $[0, T]$, $x(t)$ grows monotonically and $y(t)$ monotonically decays with the double-exponential rate given in (16).

Now, we will prove that the derivative in $\alpha$ of $x(\alpha, \beta, t)$ grows with the double-exponential rate and this will be the key calculation. For any $t \in [0, T]$, (18) yields

\begin{align*}
\begin{cases}
\dot{x}_\alpha = -0.5x_\alpha \log(x^2 + y^2) + x_\alpha r_1 + \\
\dot{y}_\alpha = 0.5y_\alpha \log(x^2 + y^2) + y_\alpha r_2 + yx_\alpha r_2 + \\
+ y_\alpha r_2 y + \nu_2 x_\alpha + \nu_2 y_\alpha - y_\alpha \arctan(xy^{-1})
\end{cases}
\end{align*}

(20)

and $x_\alpha(\alpha, \beta, 0) = 1$, $y_\alpha(\alpha, \beta, 0) = 0$. Let

\begin{align*}
&f_{11}(t) = \nu_1 x - 0.5 \log(x^2 + y^2) + r_1 + xr_1 x \\
&f_{12}(t) = xr_1 y + \nu_1 y - \arctan(xy^{-1}) \\
&f_{21}(t) = yr_2 x + \nu_2 x + \arctan(xy^{-1}) \\
&f_{22}(t) = 0.5 \log(x^2 + y^2) + r_2 + yr_2 y + \nu_2 y
\end{align*}

Then

\begin{align*}
x_\alpha &= \exp\left(\int_0^t f_{11}(\tau) d\tau \right) \hat{x}, \\
y_\alpha &= \exp\left(\int_0^t f_{22}(\tau) d\tau \right) \hat{y}
\end{align*}

If

$$g = f_{11} - f_{22}$$

then

$$\hat{x}(t) = 1 + \int_0^t \hat{x}(s) f_{22}(s) \int_s^t f_{21}(\tau) \exp\left(-\int_s^\tau g(\xi) d\xi \right) d\tau ds$$

Since the trajectory is inside $\mathcal{K}$, we have $y > \sqrt{x}$ and so

$$|f_{12}| \lesssim y + v, \quad |f_{21}| \lesssim 1, \quad f_{11} > e^t (-\log \beta + C), \quad g(t) > 1$$

From (16), we get

$$|\hat{x}(t) - 1| \lesssim v \int_0^t |\hat{x}(\tau)| d\tau + \int_0^t |\hat{x}(s)| \left(\int_s^t e^{\tau(\log \beta + C)} e^{-(r-s)} d\tau \right) ds$$

The following estimate is obvious

$$\int_s^t e^{\tau(\log \beta + C)} e^{-(r-s)} d\tau \ll e^{-s}$$

as $\beta$ is small. Assuming that

$$v \ll (T + 1)^{-1}$$

and $\epsilon_2$ is small, we have

$$\hat{x}(t) \sim 1$$

and

$$x_\alpha(\alpha, \beta, T) > \left(\frac{1}{\beta}\right)^{(e^T - 1)/2}$$

(22)

The estimate (22) is the key estimate that will guarantee the necessary growth.

Now, let us place a circle $S_\gamma(\hat{x}, \hat{y})$ with radius $\gamma$ and center at $(\hat{x}, \hat{y})$ into the zone $\Omega_0$. Consider also the line segment $l = [A_1, A_2]$, $A_1 = (\hat{x} - \gamma/2, \hat{y})$, $A_2 = (\hat{x} + \gamma/2, \hat{y})$ in the center, parallel to $OX$. We will track the evolution of this disc and this line segment under the flow. We have by the Lagrange formula

$$x(A_2, T) - x(A_1, T) > \beta^{-(e^T - 1)/2}|A_2 - A_1|$$
From the positivity of $x_\alpha(\alpha, \beta, T)$ it follows that the image of $l$ under the flow is a curve given by the graph of a smooth function $\Gamma(x)$. Thus, the image of $l$ (call it $l'$) has length at least $\beta^{-(\varepsilon^T-1)/2}|A_2 - A_1|$. Denote the distance from $l'$ to $S'_\gamma(\tilde{x}, \tilde{y})$, the image of the circle, by $d$. Then, the domain $\{\Gamma(x) - d < y(x) < \Gamma(x) + d, x \in (x(A_1, T), x(A_2, T))\}$ is inside $B'_1(\tilde{x}, \tilde{y})$. The area of this domain is at least

$$d \cdot \beta^{-(\varepsilon^T-1)/2}|A_2 - A_1|$$

Thus, assuming that the flow preserves the area, we have

$$d \leq \beta^{(\varepsilon^T-1)/2}$$

Consequently, if we place a bump in $\Omega_0$ such that the $l$ and $S_\gamma(\tilde{x}, \tilde{y})$ correspond to level sets, say, $h_2$ and $h_1$ (and, what is crucial, $h_{1/2}$ are essentially arbitrary $0 < h_1 < h_2 < 0, 0001$), then the original slope of at least $\sim |h_2 - h_1|/\gamma$ will become not less than

$$\beta^{-(\varepsilon^T-1)/2} \cdot |h_2 - h_1|/\gamma$$

thus leading to double-exponential growth of arbitrarily large gradients.

**Remark 1.** If $\beta$ is a fixed small number, we have growth in $T$. If $T$ is any positive fixed moment of time, we have the growth if $\beta \to 0$.

**Remark 2.** Let us reiterate the order in which the parameters are chosen: we first fix any $T$, then small $\varepsilon_2$, then $\varepsilon_1 < \varepsilon_2^{\varepsilon_2^{\varepsilon_2}}$. How small $\varepsilon_2$ must be taken will be determined by how large the parameter $\lambda$ is chosen in the theorem 7.1 below. This defines the set $\Omega_0$. For the whole argument to work we need to collect all conditions on $\nu$: (14), (19), (21) which leads to

$$\nu < \varepsilon_1^{10}$$

(24)

5. **Small perturbations of a singular cross can also generate double exponential contraction in $\mathbb{R}$**

Assume that the function $\theta_1$ at any given time $t \in [0, T]$ is such that

$$\theta_1(x, y, t) = \theta_0^s(x, y)$$

outside the “cross”-domain $A = \{|x - \pi k| < \tau\} \cup \{|y - \pi l| < \tau\}$ where $\tau$ is small and $k, l \in \mathbb{Z}$. Inside the domain $A$ we only assume that $\theta_1$ is bounded by one in absolute value, is even, and has zero average. Notice here that the Euler flow preserves property of the function to be even. Given fixed $\varepsilon_{1(2)}$ and the domain $\mathbb{R}$ defined by these constants, we are going to show that the flow generated by $\theta_1$ can be represented in $\mathbb{R}$ in the form (10) with $\nu(\tau) \to 0$ as $\tau \to 0$. We assume of course that $\tau \ll \varepsilon_1$.

For that, we only need to study

$$F_1 = \nabla \Delta^{-1} p, \quad p = \theta_1 - \theta_0^s$$

Here are some obvious properties of $F_1$

1. $F_1(0) = 0$ as $\theta_1$ and $\theta_0^s$ are both even.
2. We have

$$F_1(z) \sim \int_A \left( \frac{\xi - z}{|\xi-z|^2} - \frac{\xi}{|\xi|^2} \right) p(\xi)d\xi$$

Using the formula

$$\left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| = \left| \frac{x-y}{|x|\cdot|y|} \right|$$
we get

$$|F_1(z)| \lesssim |z| \tau |\log \tau| \frac{1}{\epsilon_1}$$

Thus, by taking $\tau$ small, we can satisfy (11). How about (12)? For the Hessian, we have

$$|H \Delta^{-1} p| \lesssim \epsilon_1^{-2} \tau$$

and after combining we must have

$$\epsilon_1^{-2} \tau |\log \tau| \lesssim \epsilon_1^{10}$$

by (24). Thus, this condition on the size of the cross guarantees that the arguments in the previous section work.

6. The flow generated by a small steep bump in $\mathbb{R}$

In this section, we assume that at a given moment $t \in [0, T]$, we have a smooth even function $b(x, y, t)$ with support in $\mathbb{R} \cup -\mathbb{R}$, with zero average, and

$$\|b\|_2 < \omega, \quad \|\nabla b\|_\infty < M$$

(here one should think about small $\omega$ and large $M$). We will study the flow generated by this function. Let

$$F_2 = \nabla \Delta^{-1} b$$

Here are some properties of $F_2$

1. $F_2(0) = 0$.
2. To estimate the Hessian of $\Delta^{-1} b$, consider the second order derivatives. For example,

$$(\Delta^{-1} b)_{\alpha\beta}(\alpha, \beta) \sim \int \frac{(\alpha - \xi)(\beta - \eta)}{((\alpha - \xi)^2 + (\beta - \eta)^2)^2} b(\xi, \eta, t) d\xi d\eta =$$

$$= \int_{(\alpha - \xi)^2 + (\beta - \eta)^2 > \rho^2} \frac{(\alpha - \xi)(\beta - \eta)}{((\alpha - \xi)^2 + (\beta - \eta)^2)^2} b(\xi, \eta, t) d\xi d\eta +$$

$$\int_{(\alpha - \xi)^2 + (\beta - \eta)^2 < \rho^2} \frac{(\alpha - \xi)(\beta - \eta)}{((\alpha - \xi)^2 + (\beta - \eta)^2)^2} [b(\alpha, \beta, t) + \nabla b(\xi', \eta', t) \cdot (\xi - \alpha, \eta - \beta)] d\xi d\eta$$

The first term is controlled by $\omega \rho^{-1}$. By our assumption, the second term is dominated by $M \rho$. Optimizing in $\rho$ we have

$$\|H \Delta^{-1} b\|_\infty \lesssim \sqrt{M \omega}$$

To guarantee the conditions that lead to double exponential growth with arbitrary a priori given $M$, we want to make $\omega$ so small that conditions (11) and (12) are satisfied with $\nu$ as small as we need (i.e., (24)). The condition (12) is immediate and (11) follows from $F_2(0) = 0$, Lagrange formula and the estimate on the Hessian.
7. ONE STABILITY RESULT AND THE PROOF OF THE MAIN THEOREM

It is well known that given \( \theta_0 \in L^\infty(\mathbb{T}^2) \), the weak solution exists and the flow can be defined by the homeomorphic maps \( \Psi_\theta_0(x,y,t) \) for all \( t \) so that \( \theta(x,y,t) = \theta_0(\Psi_\theta_0^{-1}(x,y,t)) \) where \( \Psi_\theta_0 \) itself depends on \( \theta_0 \). The continuity of this map though is rather poor \([3\text{, theorem 2.3, p.99}\]) in this section, we will need to take smooth \( \theta_0 \) such that

\[
\max_{t \in [0,T]} \max_{z \in \mathbb{T}^2} |\Psi_{\theta_0}(z,t) - \Psi_{\theta_0}(z,t)| \to 0 \tag{26}
\]

To this end, we will consider \( \theta_0 = \theta_0^s \) outside the domain \( \mathcal{D} \) of small area. Inside this domain we assume \( \theta_0 \) to be bounded by some universal constant. The proof of Yudovich theorem (see, e.g., the argument on pp. 313–318, proof of Proposition 8.2, \([2]\)) implies

\[
\max_{t \in [0,T]} \max_{z \in \mathbb{T}^2} |\Psi_{\theta_0^s}(z,t) - \Psi_{\theta_0}(z,t)| \to 0 \tag{26}
\]

as \( |\mathcal{D}| \to 0 \).

This is the only stability result with respect to initial data that we are going to need in the argument below.

**Theorem 7.1.** For any large \( \lambda \) and any \( T > 0 \), we can find smooth initial data \( \theta_0 \) so that 

\[
\|\theta_0\|_\infty < 2 \quad \text{and} \quad \max_{t \in [0,T]} \|\nabla \theta(t,\cdot)\|_\infty > \lambda^{e^{-1}} \|\nabla \theta_0\|_\infty
\]

**Proof.** Fix any \( T > 0 \) and find \( \epsilon_1(2) \). For larger \( \lambda \), we have to take smaller \( \epsilon_2 \) (see remark 1 in the fourth section). Identify the domain \( \Omega_0 \) and place a bump (call it \( b(z) \)) in \( \Omega_0 \cup -\Omega_0 \) so that the resulting function is even. Make sure that this bump has zero average, height \( h_2 \) and diameter of support \( h_1 \) so that the gradient initially is of the size \( \sim h_2/h_1 \). Here \( h_1 \ll h_2 \ll 1 \) will be adjusted later.

Take a smooth even function \( \omega(x,y) \) supported on \( B_1(0) \) such that

\[
\int_{\mathbb{T}^2} \omega(x,y) \, dx \, dy = 1
\]

For positive small \( \sigma \), consider

\[
\tilde{\theta}_\sigma(x,y) = \theta_0^s \ast \omega_\sigma \in C^\infty, \quad \omega_\sigma = \sigma^{-2} \omega(x/\sigma, y/\sigma)
\]

We take \( \sigma \ll \epsilon_1 \) so \( \tilde{\theta}_\sigma(x,y) \) and \( \theta_0^s(x,y) \) coincide in \( \mathbb{N} \).

As the initial data for Euler dynamics we take a sum

\[
\tilde{\theta}_\sigma(z) + b(z)
\]

Then, since \( \theta_0^s \) is stationary under the flow, the stability result \([26]\) guarantees that given any \( \tau \) and keeping the same value of \( h_2/h_1 \), we can find \( \sigma \) and \( h_1 \) so small that over the time interval \([0,T]\) we satisfy

1. The “evolved bump” \( b(z,t) \) stays in the domain \( \mathbb{N} \) (e.g., \( \Psi_{\theta_0}(t)\left(\text{supp } b(z)\right) \subset \mathbb{N} \)).
2. Outside the cross of size \( \tau \) (the one considered in section five) and the support of the evolved bump \( b \), the solution is identical to \( \theta_0^s \).

Fix \( \sigma \) and \( h_1^s \) so small that for any \( h_1 < h_1^s \) we have the size of \( A \) being small, i.e. \( \tau \) is small as we wish. The value of \( \tau \) must be small enough to ensure the double exponential scenario, the conditions \([11]\) and \([12]\). For that, we need \([26]\).
Next, we proceed by contradiction. Assume that for all \( t \in [0, T] \) we have
\[
\|\nabla \theta(z, t)\|_\infty < M = (h_2/h_1) \lambda^{\epsilon-1}.
\]
Then, because \( \|b(z, t)\|_2 \) is constant in time as the flow is area-preserving and \( \|b(z, t)\|_2 \lesssim h_1 h_2 \), we only need to take \( h_2 \) so small that \( \sqrt{M h_1 h_2} \) is small enough to guarantee the double exponential scenario and the estimate \((23)\). This gives us a contradiction as the double exponential scenario makes the gradient’s norm more than \( M \) (provided that \( \epsilon_2 \ll \lambda^{-2} \)). For the initial value,
\[
\|\nabla \theta_0\|_\infty \sim \sigma^{-1} + h_2/h_1 \sim \sigma h_2/h_1
\]
by arranging \( h_1(2) \) (and keeping \( h_1 < h_1' \)).

Here is an order in which parameters are chosen in this construction:
\[
\{T, \lambda\} \rightarrow \epsilon_2 \rightarrow \epsilon_1 \rightarrow \{\sigma, h_1(2)\}
\]
\[\square\]

8. The operator \( \mathcal{E}_t \) is linearly unbounded.

The theorem \([7,1]\) is equivalent to the following

**Proposition 1.** The operator \( \mathcal{E}_t \) is linearly unbounded for any \( t > 0 \), i.e.
\[
\sup_{\theta_0 \in C^\infty(T^2), 0 < \|\theta_0\|_\infty \leq 1, \theta_0 \perp 1} \frac{\|\nabla \mathcal{E}_t \theta_0\|_\infty}{\|\nabla \theta_0\|_\infty} = +\infty
\]

**Proof.** The proof is immediate. Indeed, given any fixed \( t \), we have
\[
\sup_{\tau \in [0, t]} \left( \sup_{\theta_0 \in C^\infty(T^2), \|\theta_0\|_\infty = 1, \theta_0 \perp 1} \frac{\|\nabla \mathcal{E}_\tau \theta_0\|_\infty}{\|\nabla \theta_0\|_\infty} \right) = +\infty
\]
by taking \( \lambda \to \infty \) in the theorem \([7,1]\). Then, to have the statement at time \( t \), we only need to multiply \( \theta_0 \) by a suitable number and use remark 2 from the first section.

Vice versa, in the theorem \([7,1]\) the combination \( \lambda^{\epsilon t} \) can be replaced by arbitrary large number. In this formulation, the statement follows from the proposition. \[\square\]

As the statement of the theorem \([7,1]\) holds with any \( \lambda \), the double exponential function is not relevant at all in the formulation itself. However, it is very special hyperbolic scenario with double exponential rate of contraction that ultimately provided the superlinear dependence on the initial data.

The interesting and important question is whether the vorticity gradient can grow in the same double exponential rate starting with initial value \( \sim 1 \)? We do not know the answer to this question yet and the best known bound is (see, e.g., \([5]\))

\[
\max_{t \in [0, T]} \|\nabla \theta(\cdot, t)\|_\infty > e^{0.01 T}
\]
for arbitrary \( T \) and for \( T \)-dependent \( \theta_0 \) with \( \|\theta_0\|_\infty \sim \|\nabla \theta_0\|_\infty \sim 1 \).

9. Acknowledgment

This research was supported by NSF grants DMS-1067413 and DMS-0758239. The hospitality of the Institute for Advanced Study, Princeton, NJ is gratefully acknowledged. We are grateful to T. Tao for pointing out that the theorem \([7,1]\) is equivalent to \( \mathcal{E}_t \) being linearly unbounded and to R. Killip and A. Kiselev for interesting comments.
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