A Fast Approximation Scheme for Low-Dimensional $k$-Means

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Abstract

We consider the popular $k$-means problem in $d$-dimensional Euclidean space. Recently Friggstad, Rezapour, Salavatipour [FOCS’16] and Cohen-Addad, Klein, Mathieu [FOCS’16] showed that the standard local search algorithm yields a $(1+\varepsilon)$-approximation in time $(n,k)^{1/\varepsilon^{O(d)}}$, giving the first polynomial-time approximation scheme for the problem in low-dimensional Euclidean space. While local search achieves optimal approximation guarantees, it is not competitive with the state-of-the-art heuristics such as the famous $k$-means++ and $D^2$-sampling algorithms.

In this paper, we aim at bridging the gap between theory and practice by giving a $(1+\varepsilon)$-approximation algorithm for low-dimensional $k$-means running in time $n^{1/\varepsilon^{O(d)}}k\log n$, and so matching the running time of the $k$-means++ and $D^2$-sampling heuristics up to polylogarithmic factors. We speed-up the local search approach by making a non-standard use of randomized dissections that allows to find the best local move efficiently using a quite simple dynamic program. We hope that our techniques could help design better local search heuristics for geometric problems.

1 Introduction

The $k$-means objective is arguably the most popular clustering objective among practitioners. While originally motivated by applications in image compression, the $k$-means problem has proven to be a successful objective to optimize in order to pre-process and extract information from datasets. Its most successful applications are now stemming from machine learning problems like for example learning mixture of Gaussians, Bregman clustering, or DP-means [33, 51, 16]. Thus, it has become a classic problem in both machine learning and theoretical computer science.

Given a set of points in a metric space, the $k$-means problem asks for a set of $k$ points, called centers, that minimizes the sum of the squares of the distances of the points to their closest center.

The most famous algorithm for $k$-means is arguably the Lloyd heuristic introduced in the 80s [50] and sometimes referred to as “the $k$-means algorithm”. While this algorithm is very competitive in practice and yields empirically good approximate solutions to real-world inputs, it is known that its running time can be exponential in the input size and that it can return arbitrarily bad solutions in the worst-case (see [8]). This induces a gap between theory and practice.

Thus, to fix this unsatisfactory situation, Arthur and Vassilvitskii [9] have designed a variant of the Lloyd heuristic, called $k$-means++, and proved that it achieves an $O(\log k)$ approximation. The $k$-means++ algorithm has now become a standard routine that is part of several machine learning libraries (e.g., [59]) and is widely-used in practice. While this has been a major step for reducing the gap between theory and practice, it has remained an important problem as to conceive algorithms with nearly-optimal approximation guarantee.

Unfortunately, the $k$-means problem is known to be APX-Hard even for (high dimensional) Euclidean inputs [14]. Hence, to design competitive approximation schemes it is needed to restrict our attention to classes of “more structured” inputs that are important in practice. The low-dimensional Euclidean inputs form a class of inputs that naturally arise in image processing and machine learning (see examples of [59] or in [43]). Thus, finding a polynomial time approximation scheme (PTAS) for $O(1)$-dimensional Euclidean inputs of $k$-means has been an important research problem for the last 20 years since the seminal work of [42]. Recently, Friggstad et al. [35] and Cohen-Addad et al. [26] both showed that the classic local search

\footnote{Also referred to as Lloyd-Forgy}
heuristic with neighborhood of magnitude \((d/\varepsilon)^{O(d)}\) achieves a \((1+\varepsilon)\)-approximation. While this has been an important result for the theory community, it has a much weaker impact for practitioners since the running time of the algorithm is \(n^{(d/\varepsilon)^{O(d)}}\). Therefore, to reduce the gap between theory and practice, it is natural to ask for near-optimal approximation algorithms with competitive running time. This is the goal of this paper.

**Fast local search techniques are important.** The result of Friggstad et al. and Cohen-Addad et al. has been preceded by several results showing that local search achieves good approximation bound or even exact algorithms for various problems (see e.g., [50, 15, 19, 51, 25]). Furthermore, there is a close relationship between local search and clustering showing that the standard local search heuristics achieve very good approximation guarantees in various settings (in addition to the two aforementioned papers, see [12, 43, 27, 19, 53]). Moreover, local search approaches are extremely popular in practice because they are easy to tune, easy to implement, and easy to run in parallel.

Thus, it has become part of the research agenda of the theory community to develop fast local search approaches while preserving the guarantees on the quality of the output (see e.g., [30, 57]).

### 1.1 Our Results

We show that our fast local search algorithm (Algorithm 1) yields a PTAS for the slightly more general variants of the \(k\)-means problem where centers can have an opening cost (a.k.a. weight).

**Theorem 1.1.** There exists a randomized algorithm (Algorithm 7) that returns a \((1+\varepsilon)\) approximation to the center-weighted \(d\)-dimensional Euclidean \(k\)-means problem in time \(n \cdot k \cdot (\log n)^{(d-1)O(d)}\), with probability at least \(1/2\).

We would like to remark that the doubly exponential dependency in \(d\) is needed: Awasthi et al. [14] showed that the Euclidean \(k\)-means is APX-Hard when \(d = \Omega(\log n)\). Note that it is possible to obtain an arbitrarily small probability of failure \(p > 0\) by repeating the algorithm \(\log 1/p\) times.

As far as we know, this is the first occurrence of a local search algorithm whose neighborhood size only impacts the running time by polylogarithmic factors.

### 1.2 Other Related Work

The \(k\)-means problem is known to be NP-Hard, even when restricted to inputs lying in the Euclidean plane (Mahajan et al. [52], and Dasgupta and Freud [31]) and was recently shown to be APX-hard in Euclidean space of dimension \(\Omega(\log n)\) (14).

There has been a large body of work on approximation algorithms for the Euclidean \(k\)-means problem (see e.g., [43]), very recently Ahmadian et al. [2] gave a 6.357-approximation improving over the 9-approximation of Kanungo et al. [43].

Given the hardness results, researchers have focused on different scenarios. There have been various \((1+\varepsilon)\)-approximation algorithms when \(k\) is considered a fixed-parameter (see e.g., [34, 18]). Another successful approach has been through the definition of “stable instances” to characterize the real-world instances stemming from machine learning and data analysis (see for example [15, 13, 18, 17, 21, 46, 58, 28, 3]). In the case of low-dimensional inputs, Bandyapadhyay and Varadarajan showed that local search with neighborhood of size \(\varepsilon^{-O(d)}\) achieves a \((1+\varepsilon)\)-approximation [19] when allowed to open \(O(\varepsilon k)\) extra centers. As mentioned before, this results has been improved by Friggstad et al. [35] and Cohen-Addad et al. [26] who showed that even when constrained to open exactly \(k\) centers, local search achieves a \((1+\varepsilon)\)-approximation.

**Related work on local search** Local Search heuristics belong to the toolbox of all practitioners, (see Aarts and Lenstra [1] for a general introduction). As mentioned before, there is a tight connection between local search and clustering: Arya et al. [12] proved that local search with a neighborhood size of \(1/\varepsilon\) yields a \(3 + 2\varepsilon\) approximation to \(k\)-median. For the \(k\)-means problem, Kanungo et al. [43] showed a similar result by proving that the approximation guarantee of local search for Euclidean \(k\)-means is \(9 + \varepsilon\). For more applied
examples of local search an clustering see [20, 22, 38, 60, 37, 4, 41]. For other theoretical example of local search for clustering, we refer to [29, 32, 36, 40, 61].

Related work on k-median. The k-median problem has been widely studied. For the best known results in terms of approximation for general metric spaces inputs we refer to Li and Svensson [49] and Byrka et al. [24]. More related to our results are the approximation schemes for k-median in low-dimensional Euclidean space given by Arora et al. [6] who gave a \((1 + \varepsilon)\)-approximation algorithm running in time \(n^{\varepsilon/\delta} \). This was later improved by Kolliopoulos and Rao [44] who obtained a running time of \(2^\varepsilon \cdot n \cdot \text{polylog } n \). Quite surprisingly, it is pretty unclear whether the techniques used by Arora et al. and Kolliopoulos and Rao could be used to obtain a \((1 + \varepsilon)\)-approximation for the k-means problem; this has induced a 20-year gap between the first PTAS for k-median and the first PTAS for k-means in low-dimensional Euclidean space. See Section 1.3 for more details.

1.3 Overview of the Algorithm and the Techniques

Our proof is rather simple. Given a solution \(L\), our goal is to identify – in near-linear time – a minimum cost solution \(L'\) such that \(|L - L'| + |L' - L| \leq \delta\) for some (constant) parameter \(\delta\). If \(\text{cost}(L) - \text{cost}(L') = O(\varepsilon \cdot \text{cost}(\text{OPT})/k)\), then we can immediately apply the result of Friggstad et al. [35] or Cohen-Addad et al. [20]: the solution \(L\) is locally optimal and its cost is at most \((1 + O(\varepsilon))\cdot \text{cost}(\text{OPT})\). Finding \(L\) has to be done in near-linear time since we could repeat this process up to \(\Theta(k)\) times until reaching a local optimum. Hence, the crux of the algorithm is to efficiently identify \(L'\); it proceeds as follows (see Algorithm 1 for a full description):

1. Compute a random recursive decomposition of \(L\) (see Section 2);
2. Apply dynamic programming on the recursive decomposition; we show that there exists a near-optimal solution whose interface between different regions has small complexity.

To obtain our recursive decomposition we make a quite non-standard use of the classic quadtree dissection techniques (see Section 2). Indeed, the k-means problem is famous for being “resilient” to random quadtree approaches – this is partly why a PTAS for the low-dimensional k-median problem was obtained 20 years ago but the first PTAS for the k-means was only found last year. More precisely, the classic quadtree approach (which works well for k-median) defines portals on the boundary of the regions of the dissection and forces the clients of a given region that are served by a center that is in a different region (in the optimal solution) to make a detour through the closest portal. This is a key property as it allows to bound the complexity of OPT between regions. Unfortunately, when dealing with squared distances, making a detour could result in a dramatic cost increase and it is not clear that it could be compensated by the fact that the event of separating a client form its center happens with small probability (applying the analysis of Arora et al. [6] or Kolliopoulos et al. [44]). This problem comes from the fact that some facilities of OPT and \(L\) might be too close from the boundary of the dissection (and so, their clients might have to make too important detour (relative to their cost in OPT) through the portals). We call these facility the “moat” facilities (as they fall in a bounded-size “moat” around the boundaries).

We overcome this barrier by defining a more structured near-optimal solution as follows: (1) when a facility of OPT is too close from the boundary of our dissection we simply remove it and (2) if a facility of the current solution \(L\) is too close from a boundary of our dissection we add it to OPT.

Of course, this induces two problems: first we have to bound the cost of removing the facilities of OPT and second we have to show that adding the facilities of \(L\) does not result in a solution containing more than \(k\) centers. This is done through some technical lemmas and using the concept of isolated facilities inherited from Cohen-Addad et al. [20]. Section 3 shows the existence of near-optimal solution \(S^*\) whose set of moat facilities (i.e., facilities that are too close from the boundaries) is exactly the set of moat facilities of \(L\) (and so, we already know their location and so the exact cost of assigning a given client to such a facility).

We now aim at using the result of Friggstad et al. [35] and Cohen-Addad et al. [20]: if for any sets \(\Delta_1 \subseteq L\) and \(\Delta_2 \subseteq S^*\), we have \(\text{cost}(L) - \text{cost}(L - \Delta_1 \cup \Delta_2) = O(\varepsilon \cdot \text{cost}(\text{OPT})/k)\), then we have \(\text{cost}(L) \leq (1 + O(\varepsilon))\cdot \text{cost}(S^*)\). Thus, we provide a dynamic program (in Section 5) that allows to find the best solution \(S\) that is such that (1) its set of moat facilities is exactly the set of moat facilities of \(L\) and (2) \(|S - L| + |L - S| \leq \delta\).
for some fixed constant δ. The dynamic program simply “guess” the approximate location of the centers of \((L - S) \cup (S - L)\), we show that since each such center is far from the boundary, its location can be approximated.

1.4 Preliminaries

In this article, we consider the k-means problem in a \(d\)-dimensional Euclidean space: Given a set \(A\) of points (also referred to as clients) and candidate centers \(C\) in \(\mathbb{R}^d\), the goal is to output a set \(S \subseteq C\) of size \(k\) that minimizes: \(\sum_{a \in A} \text{dist}(a, C)^2\), where \(\text{dist}(a, C) = \min_{c \in C} \text{dist}(a, c)\). We refer to \(S\) as a set of centers or facilities. Our results naturally extends to any objective function of the form \(\sum_{a \in A} \text{dist}(a, C)^p\) for constant \(p\). For ease of exposition, we focus on the \(k\)-means problem. As Friggstad et al [3], we also consider the more general version called weighted \(k\)-means for which, in addition of the sets \(A\) and \(C\), we are given a weight function \(w: C \to \mathbb{R}_+\) and the goal is to minimize \(\sum_{a \in A} w(c) + \sum_{a \in A} \text{dist}(a, C)^2\).

A classic result of Matousek [55] shows that, if \(C = \mathbb{R}^d\), it is possible to compute in linear time a set \(C'\) of linear size (and polynomial dependency in \(d\) and \(\varepsilon\)) such that the optimal solution using the centers in \(C'\) cost at most \((1 + \varepsilon)\) times the cost of the optimal solution using \(C\). Hence, we assume without loss of generality that \(|C|\) has size linear in \(|A|\) and we let \(n = |A| + |C|\).

Isolated Facilities. We make use of the notion of isolated facilities introduced by Cohen-Addad et al. [26] defined as follows. Let \(\varepsilon_0 < 1/2\) be a positive constant and \(L\) and \(G\) be two solutions for the Euclidean \(k\)-means problem.

**Definition 1.2.** Let \(\varepsilon_0 < 1/2\) be a positive number and let \(L\) and \(G\) be two solutions for the \(k\)-clustering problem with parameter \(p\). Given a facility \(f_0 \in G\) and a facility \(f \in L\), we say that the pair \((f_0, f)\) is \(1-1\) \(\varepsilon_0\)-isolated if most of the clients served by \(f\) in \(L\) are served by \(f_0\) in \(G\), and most of the clients served by \(f_0\) in \(G\) are served by \(f\) in \(L\); formally, if

\[
|V_L(f) \cap V_{OPT}(f_0)| \geq \max \left\{ \frac{1 - \varepsilon_0}{|V_L(f)|}, \frac{1 - \varepsilon_0}{|V_{OPT}(f_0)|} \right\}
\]

When \(\varepsilon_0\) is clear from the context we refer to \(1-1\) \(\varepsilon_0\)-isolated pairs as \(1-1\) isolated pairs.

**Theorem 1.3** (Theorem III.7 in [26]). Let \(\varepsilon_0 < 1/2\) be a positive number and let \(L\) and \(G\) be two solutions for the \(k\)-clustering problem with exponent \(p\). Let \(k\) denote the number of facilities \(f\) of \(G\) that are not in a \(1-1\) \(\varepsilon_0\)-isolated region. There exists a constant \(c\) and a set \(S_0\) of facilities of \(G\) of size at least \(\varepsilon_0^3 k / 6\) that can be removed from \(G\) at low cost: \(\text{cost}(G - S_0) \leq (1 + c \cdot \varepsilon_0) \cdot \text{cost}(G) + c \cdot \varepsilon_0 \cdot \text{cost}(L)\).

Observe that since \(\varepsilon_0 < 1/2\), each facility of \(L\) belongs to at most one isolated region. Let \(\tilde{G}\) denote the facilities of \(G\) that are not in an isolated region. In the rest of the paper, we will use it with \(\varepsilon_0 = \varepsilon^3\).

1.5 Fast Local Search

This section is dedicated to our description of the fast local search algorithm for the \(k\)-means problem. It relies on a dynamic program called \textsc{FindImprovement} that allows to ﬁnd the best improvement of the current solution in time \(n \cdot \text{poly}_{\varepsilon, d}(\log n)\). We then show that total number of iterations of the do-while loop is \(O(k)\).

2 Dissection Procedure

In this section, we recall the classic deﬁnition of quadtree dissection. For simplicity we give the deﬁnition for \(\mathbb{R}^2\), the deﬁnition directly generalizes to any ﬁxed dimension \(d\), see Arora [5] and Arora et al. [6] for a complete description. Our deﬁnition of quadtree is standard and follows the deﬁnition of [5] [6], our contribution in the structural properties we extract from the dissection and is summarized by Lemma 2.3.

Let \(L\) be the length of the bounding box of the client set \(A\) (i.e., the smallest square containing all the points in \(A\)). Applying standard preprocessing techniques, see in [44] and [6], it is possible to assume that
Algorithm 1 Fast Local Search for $k$-Means

1: Input: An $n$-element client set $A$, an $m$-element candidate center set $C$, a positive integer parameter $k$, an opening cost function $w : C \mapsto \mathbb{R}_+$, and an error parameter $0 < \varepsilon < 1/2$.
2: $L \leftarrow O(1)$-approximation.
3: Round up the weights of the candidate centers to the closest $(1 + \varepsilon)^i \cdot \varepsilon \cdot \text{cost}(L)/n$ for some integer $i$.
4: do
5: Improv $\leftarrow 0$
6: $L^* \leftarrow L$
7: repeat log $k$ times (to boost the success probability)
8: Compute a random decomposition $D$ of $L$ (as in Sec. 2). Let $\mathcal{M}$ be the moat centers of $L$.
9: $L' \leftarrow$ output of FindImprovement($L$, $D$, $\mathcal{M}$, $d^{O(d)}\varepsilon^{-O(d)}$)
10: if Improv $\leq \text{cost}(L) - \text{cost}(L')$ then
11: Improv $\leftarrow \text{cost}(L) - \text{cost}(L')$
12: $L^* \leftarrow L'$
13: end if
14: $L \leftarrow L^*$
15: while Improv $> \varepsilon \text{cost}(L)/k$
16: Output: $L$

the points lie on a unit grid of size polynomial in the number of input points. This incurs an additive error of $O(\text{OPT}/n^c)$ for some constant $c$ and yields $L = n \cdot \text{poly}(\varepsilon^{-d})$.

We define a quadtree dissection $D$ of a set of points $P$ as follows. A dissection of (the bounding box of) $L$ is a recursive partitioning into smaller squares. We view it as a 4-ary tree whose root is the bounding box of the input points. Each square in the tree is partitioned into 4 equal squares, which are its children. We stop partitioning a square if it has size $< 1$ (and therefore at most one input point). It follows that the depth of the tree is $\log L = O(\log n)$. Standard techniques show that such a quadtree can be computed in $n \cdot \log n \cdot \text{poly}(\varepsilon^{-d})$, see [44] for more details. The total number of nodes of the quadtree is $n \cdot \log n \cdot \text{poly}(\varepsilon^{-d})$.

Given two integers $a, b \in [0, L]$, the $(a, b)$-shifted dissection consists in shifting the $x$- and $y$-coordinates of all the vertical and horizontal lines by $a \mod L$ and $b \mod L$ respectively. For a shifted dissection, we naturally define the level of a bounding box to be its depth in the quadtree. From this, we define the level of a line to be the level of the square it bounds.

For a given square of the decomposition, each boundary of the square defines a subline of one of the $2L$ lines of the grid. It follows that each line at level $i$ consists of $2^i$ sublines of length $L/2^i$.

Given an $(a, b)$-shifted quadtree dissection of a set of $n$ points, and given a set of points $U$, we say that a point $p$ of $U$ is an $i, \gamma$-moat point if it is at distance less than $\gamma \cdot L/2^i$ of a line of the dissection that is at level $i$. We say that a point $p$ of $U$ is a $\gamma$-moat point if there exists an $i$ such that $p$ is at level $i$. When $\gamma$ is clear from the context, we simply call such a point a moat point. We have:

**Lemma 2.1.** For any $p \in U$, the probability that $p$ is a $\gamma$-moat point is at most $\gamma \log L = O(\gamma \log n)$.

**Proof.** Let $i$ be an integer in $[0, \ldots, \log L]$ and consider the horizontal lines at level $i$ (an analogous reasoning applies to the vertical lines). By definition, the number of dissection lines that are at distance at most $\gamma L/2^i$ from $p$ is $\gamma L/2^i$. We now bound the probability that one of them is at level $i$ (and so the probability of $p$ being at distance less than $\gamma L/2^i$ of a horizontal line of length $L/2^i$). For any such line $l$, we have: $Pr_{a}[l$ is at level $i] = 2^i/L$. Hence, $Pr_{a}[p$ is a $i, \gamma$-moat point] $\leq \sum_{d : \text{dist}(l, p) \leq \gamma L/2^i} Pr_{a}[l$ is at level $i] \leq \gamma$. The lemma follows by taking a union bound over all $i$. ☐

We now consider an optimal solution OPT and any solution $L$. In the following, we will focus on $\gamma$-moat centers of $L$ and OPT for $\gamma = \varepsilon^{13}/\log n$. In the rest of the paper, $\gamma$ is fixed to that value and so $\gamma$-moat centers are simply called moat centers.

Let $\ell$ the facilities of OPT that are not 1-1 isolated. We define a weigh function $\tilde{w} : L \cup \text{OPT} \cup \text{OPT} \mapsto \mathbb{R}_+$ as follows. For each facility $s \in \ell$ we define $\tilde{w}(s)$ as the sum of $w(s)$ and the cost of serving all the clients served by $s$ in OPT by the closest facility $\ell$ in $L$ plus $w(\ell)$. For each facility $s \in L$ we define $\tilde{w}(s) = w(s) + \sum_{c \text{ served by } s}$ in $L \cdot \text{dist}(c, s)$. Similarly, for each $s \in \text{OPT} - \ell$, we let $\tilde{w}(s) = w(s) + \sum_{c \text{ served by } s}$ in $\text{OPT} \cdot \text{dist}(c, s)$. 5
We show:

**Lemma 2.2.** There exists a constant $c_0$ such that $\sum_{s \in \iota} \bar{w}(s) \leq c_0(\text{cost}(\text{OPT}) + \text{cost}(L))$.

*Proof.* Consider a facility $s \in \iota$ and the closest facility $l$ in $L$ that (1) serves in $L$ at least one client that is served by $s$ in OPT and that (2) minimizes the following quantity: $\eta = \min_{l \in S} \text{dist}(c, l)^2 + \text{dist}(c, s)^2$. Let $c^*$ be a client that minimizes the quantity $\text{dist}(c, l)^2 + \text{dist}(c, s)^2$. Let $N(s)$ be the set of all clients served by $s$ in OPT. We have: $|N(s)|\eta \leq \sum_{c \in N(s)} \text{dist}(c, L)^2 + \text{dist}(c, \text{OPT})^2$.

We have that the total cost of sending all the clients in $N(s)$ is at most (by triangle inequality): $\sum_{c \in N(s)} (\text{dist}(c, s) + \text{dist}(s, l))^2 \leq \sum_{c \in N(s)} (\text{dist}(c, s) + \text{dist}(s, c) + \text{dist}(c, l))^2$. Note that there exists a constant $c_0$ such that the above sum is at most $c_0 \sum_{c \in N(s)} \text{dist}(c, s)^2 + \text{dist}(s, c^*)^2 + \text{dist}(c^*, l)^2$ and so at most $c_0(|N(s)|\eta + \sum_{c \in N(s)} \text{dist}(c, \text{OPT})^2)$. The lemma follows by combining with the above bound on $|N(s)|\eta$. \qed

In the following we denote by $\phi : \iota \rightarrow L$ the mapping from each non-isolated facility of OPT to its closest facility in $L$. We define $\iota_L$ to be the set of non-isolated facilities of $L$.

We define Event $\mathcal{E}(L \cup \text{OPT}, \bar{w})$ as follows:

1. The set of moat centers $S_1$ of $L \cup \iota$ is such that $\bar{w}(S_1) = \sum_{c \in S_1} \bar{w}(c) \leq \varepsilon^9 \sum_{c \in L\cup \iota} \bar{w}(c) = \varepsilon^9 \bar{w}(L \cup \iota)$, and

2. The set of moat centers $S_1$ of $\iota_L \cup \iota$ is such that $|S_0| \leq \varepsilon^9 |\iota_L \cup \iota| = \varepsilon^9 \bar{k}$ (recall that $\bar{k}$ is the number of non 1-1-isolated facilities of OPT (and so of $L$ as well)).

The following lemma follows from Lemma 2.1, applying Markov’s inequality and taking a union bound over the probability of failures of property (1) and (2).

**Lemma 2.3.** The probability that Event $\mathcal{E}(L \cup \text{OPT}, \bar{w})$ happens is at least $1/2$.

*Proof.* We apply Lemma 2.1 and obtain that any element of $L \cup \text{OPT}$ is a moat center with probability $O(\rho^{-1} \log n)$. Since $\rho^{-1} = (\varepsilon^9 \log n)^{-1}$, we obtain that this probability is at most $\varepsilon^{12}$. Thus, taking linearity of expectation we have that the expected size of $S_0$ is at most $\varepsilon^{12}|S \cup \iota|$ and that the expected value of $\bar{w}(S_0)$ is at most $\varepsilon^{12} \bar{w}(S \cup \iota)$. Applying Markov’s inequality to obtain concentration bounds on both quantities and then taking a union bound over the probabilities of failure yields the lemma. \qed

We finally conclude this section with some additional definitions that are used in the following sections. We define a basic region of a decomposition of a to be a region of the dissection that contains exactly 1 points of $L$. The other squares of the decomposition are simply called regions.

## 3 A Structured Near-Optimal Solution

This section is dedicated to the following proposition.

**Proposition 3.1.** Let $L$ be any solution. Let $D_L$ be a random quadtree dissection of $L$ as per Sec. 2. Suppose Event $\mathcal{E}(L \cup \text{OPT}, \bar{w})$ happens. Then there exists a constant $c$ and a solution $S^*$ of cost at most $(1 + c \cdot \varepsilon)\text{cost}(\text{OPT}) + c \cdot \varepsilon \text{cost}(L)$ and such that the set of moat centers of $S^*$ is equal to the set of moat centers of $L$.

We prove Proposition 3.1 by explicitly constructing $S^*$. We iteratively modify OPT in four main steps:

1. Modify OPT by replacing $f_0$ by $\ell_0$ for each 1-1 isolated pair $(f_0, \ell_0)$ where $\ell_0$ or $f_0$ is a moat center. This yields a near-optimal solution $S_0$ (Lemma 3.2).

2. Replace in OPT each moat center $s$ that is in $\iota$ by $\phi(s)$ (as per Section 2). This yields a near-optimal solution $S_1$ (Lemma 3.3).

3. Apply Theorem 1.3 (i.e., Theorem III.7 in [24]) to obtain a near-optimal solution $S_2$ that has at most $k - c_2 \varepsilon^9 \cdot \bar{k}$ where $\bar{k}$ is the number of facilities of OPT that are not 1-1 isolated (Lemma 3.4).

4. Add the moat centers of $L$ that are non 1-1 isolated to $S_2$. This yields a near-optimal solution $S_3$ that has at most $k$ centers.

See Section 3.4 for a detailed proof.
3.1 1-1 Isolated Pairs

We start from OPT and for each 1-1 isolated facility \((f_0, \ell_0), f_0 \in \text{OPT}, \ell_0 \in L\), where \(\ell_0\) or \(f_0\) is a moat center, we replace \(f_0\) by \(\ell_0\) in OPT.

This results in a solution \(S_0\) whose structural properties are captured by the following lemma. For any \(f_0 \in \text{OPT}\) (resp. \(\ell_0 \in \text{OPT}\)), let \(V_{\text{OPT}}(f_0)\) (resp. \(V_{L}(\ell_0)\)) be the set of clients served by \(f_0\) in OPT (resp. the set of clients served by \(\ell_0\) in \(L\)).

Lemma 3.2. Assuming Event \(\mathcal{E}(L \cup \text{OPT}, \tilde{w})\) happens, there exists a constant \(c_0\) such that \(\text{cost}(S_0) \leq (1 + c_0 \cdot \varepsilon) \cdot \text{cost}(\text{OPT}) + c_0 \cdot \varepsilon \cdot \text{cost}(L)\).

Proof. Since Event \(\mathcal{E}(L \cup \text{OPT}, \tilde{w})\) happens, we have by Lemma 2.3 that the total opening cost of the moat centers plus the total service cost induced by the clients served by the moat centers is bounded by \(c_1 \cdot \varepsilon \cdot (\text{cost}(L) + \text{cost}(\text{OPT}))\) for some constant \(c_1\). More formally, we can write:

\[
(w(\ell_0) + \sum_{a \in V_L(\ell_0)} \text{dist}(a, \ell_0)) \leq c_1 \cdot \varepsilon \cdot (\text{cost}(L) + \text{cost}(\text{OPT}))
\]

Thus, we need to bound the cost for the clients that are in \(V_{\text{OPT}}(f_0) - V_{L}(\ell_0)\), for each 1-1 isolated pair \((f_0, \ell_0)\) where \(\ell_0\) or \(f_0\) is a moat center. Consider such a pair \((f_0, \ell_0)\). We bound the cost of the clients served by \(f_0\) in OPT by the cost of rerouting them toward \(\ell_0\).

We can thus write for each such client \(c\):

\[
dist(c, \ell_0, f_0)^2 \leq (\text{dist}(c, f_0) + \text{dist}(\ell_0, f_0))^2.
\]

Also, \(\text{dist}(c, \ell_0, f_0)^2 \leq (1 + \varepsilon)^2 \text{dist}(c, f_0)^2 + (1 + \varepsilon^{-1})^2 \text{dist}(\ell_0, f_0)^2\). Note that \(\text{dist}(c, f_0)^2\) is the cost paid by \(c\) in OPT. Thus we aim at bounding \(\text{dist}(\ell_0, f_0)\).

Applying the triangle inequality, we obtain \(\text{dist}(\ell_0, f_0)^2 \leq (\text{dist}(\ell_0, c_1) + \text{dist}(c_1, f_0))^2\), for any \(c_1\) in \(V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)\). Hence,

\[
\text{dist}(\ell_0, f_0)^2 \leq \frac{1}{|V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)|} \sum_{c_1 \in V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)} (\text{dist}(\ell_0, c_1) + \text{dist}(c_1, f_0))^2.
\]

Now,

\[
\sum_{c_1 \in V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)} (\text{dist}(\ell_0, c_1) + \text{dist}(c_1, f_0))^2 \leq 3 \sum_{c_1 \in V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)} \text{dist}(c_1, \ell_0)^2 + \text{dist}(c_1, f_0)^2.
\]

Combining, we obtain that the total cost for the clients in \(V_{\text{OPT}}(f_0) - V_{L}(\ell_0)\) is at most

\[
\sum_{c \in V_{\text{OPT}}(f_0) - V_{L}(\ell_0)} \text{dist}(c, \ell_0)^2 \leq (1 + \varepsilon)^2 \left( \sum_{c \in V_{\text{OPT}}(f_0) - V_{L}(\ell_0)} \text{dist}(c, f_0)^2 \right) + |V_{\text{OPT}}(f_0) - V_{L}(\ell_0)| (1 + \varepsilon^{-1})^2 \text{dist}(\ell_0, f_0)^2
\]

\[
\leq (1 + \varepsilon)^2 \left( \sum_{c \in N(f_0) - N(\ell_0)} \text{dist}(c, f_0)^2 \right) + (1 + \varepsilon^{-1})^2 \frac{|V_{\text{OPT}}(f_0) - V_{L}(\ell_0)|}{|V_{\text{OPT}}(f_0) \cup V_{L}(\ell_0)|} \cdot 3 \sum_{c_1 \in V_{\text{OPT}}(f_0) \cap V_{L}(\ell_0)} \text{dist}(c_1, f_0)^2 + \text{dist}(c_1, \ell_0)^2.
\]

The lemma follows from applying the definition of 1-1 isolation: \(\frac{|V_{\text{OPT}}(f_0) - V_{L}(\ell_0)|}{|V_{\text{OPT}}(f_0) \cup V_{L}(\ell_0)|} \leq \varepsilon^4\), and summing over all 1-1 isolated pair.

\[\square\]

3.2 Replacing the Moat Centers of OPT

In this section, we consider the solution \(S_0\) and define a solution \(S_1\) whose set of moat centers is a subset of the moat centers of \(L\). Namely:

Lemma 3.3. There exists a constant \(c_1\) and a solution \(S_1\) such that the moat centers of \(S_1\) are a subset of the moat centers of \(L\) and \(\text{cost}(S_1) \leq (1 + c_1 \varepsilon) \cdot \text{cost}(S_0) + c_1 \varepsilon \cdot \text{cost}(L)\).
Proof. Note that by Lemma 3.2 all the 1-1 isolated facilities of $S_0$ that are moat centers are also in $L$. Thus, we focus on the moat centers of $S_0$ that are not 1-1-isolated (and so, by definition, in $\iota$).

We replace each center $s \in \iota$ by the center $\phi(s) \in L$ (as per Section 2); the bound on the cost follows immediately from the definition of $\hat{w}$ and by combining Lemma 2.3 and Lemma 2.2.

3.3 Making Room for Non-Isolated Moat Facilities

We now consider the solution $S_1$ described in the previous section that satisfies the condition of Lemma 3.2. The following lemma is a direct corollary of Theorem 1.3 (from [26]).

Lemma 3.4. Given $S_1$ and $L$, there exists a solution $S_2 \subseteq S_1$ and constants $c_2, c_3$ such that

1. $|S_2| \leq k - c_2 \cdot \varepsilon^2 k$, where $k$ is the number of facilities that are not 1-1 isolated in $OPT$ (or in $L$ it is the same number), and

2. $cost(S_2) \leq (1 + c_3 \cdot \varepsilon) cost(OPT) + c_3 \cdot \varepsilon \cdot cost(L)$.

3.4 Adding the Non-Isolated Moat Facilities and Proof of Proposition 3.1

We now consider the solution $S_3$ consisting of the centers in $S_2$ and the non-1-1-isolated moat centers of $L$.

Proof of Proposition 3.1. Combining Lemmas 3.2, 3.3, and 3.4 yields the bound on the cost of $S_3$.

By definition of $S_3$ and applying Lemmas 3.2 and 3.3, we have that the set of moat centers of $S_3$ is exactly the set of moat centers of $L$.

By Lemma 3.4 (and because Event $E(L \cup OPT, \hat{w})$ happens), we have that the total number of centers in $S_3$ is at most $k$.

4 Proof of Theorem 1.1

We summarize: By Proposition 3.1, we have that there exists a near-optimal solution $S^*$ whose set of moat centers is the set of moat centers of $L$. By Proposition 5.3, we have that, FINDIMPROVEMENT identifies a solution $S'$ that is $\delta$-close w.r.t. $L$, whose set of moat centers is the set of moat centers of $L$, and such that $cost(L) - cost(S') \geq (1 - \varepsilon)(cost(L) - cost(OPT_3))$, where OPT_3 is the minimum cost solution $S$ such that $|S - L| + |L - S| \leq \delta$. We now argue that: If FINDIMPROVEMENT outputs a solution $S_4$ such that $cost(L) - cost(S_4) \leq \varepsilon cost(OPT)/k$ then there exists a constant $c^*$ such that $cost(L) \leq (1 + c^* \cdot \varepsilon) cost(OPT)$.

Assuming cost($L$) - cost($S_4$) \leq \varepsilon cost(OPT)/k implies by Proposition 5.3 that cost($L$) - cost(OPT_3) \leq 2\varepsilon cost(OPT)/k, since $\varepsilon < 1/2$. Now, consider the solution $S^*$ defined in Section 3.

By Theorem 1 in [35] (see also [26] for a slightly better dependency in $\varepsilon$ in the unweighted case), if for any pair of sets $\Delta_1 \subseteq L, \Delta_2 \subseteq S_2$ such that $|\Delta_2| \leq |\Delta_1| = (d\varepsilon^{-1})O(d)$, we have cost($L$) - cost($L - \Delta_1 \cup \Delta_2$) \leq \varepsilon cost($S_2$), then there exists a constant $c^*$ such that cost($L$) \leq (1 + c^* \cdot \varepsilon) cost($S_2$). To obtain such a bound we want to apply Proposition 5.3 and so we need to show that any such solution $M = L - \Delta_1 \cup \Delta_2$ is such that its moat centers are the moat centers of $L$. This follows immediately from Proposition 3.1 the moat centers of $S^*$ are the moat centers of $L$.

Therefore, we can apply Proposition 5.3 and we have cost($L$) \leq (1 + c^* \cdot \varepsilon) cost(OPT), for some constant $c^*$.

We now bound the running time of Algorithm 1.

Lemma 4.1. The running time of Algorithm 1 is at most $n \cdot k \cdot (\log n)^{(d\varepsilon^{-1})O(d)}$.

Proof. By Proposition 5.3 we only need to bound the number of iterations of the do-while loop (lines 4 to 55) of Algorithm 1. Let cost($S_0$) denotes the cost of the initial solution. The number of iterations of the do-while loop is

$$\frac{\log(cost(S_0)/cost(OPT))}{\log(\frac{1}{1-1/k})}.$$
Assuming $\text{cost}(S_0) \leq O(\text{OPT})$, we have that the total number of iterations is at most $O(k)$. To obtain $\text{cost}(S_0) \leq O(\text{OPT})$ it is possible to use the algorithm of Guha et al. which outputs an $O(1)$-approximation in time $n \cdot k \cdot \text{polylog}(n)$, as a preprocessing step (i.e., for the computations at line 2), without increasing the overall running time.

We conclude by bounding the probability of failure: By lemma 2.3 Event $\mathcal{E}$ happens with probability at least $1/2$. Since the random dissection is repeated independently $c \cdot \log(k)$ times, the probability of failure of Event $\mathcal{E}$ for a given iteration of the while loop is at most $2^{-c \cdot \log k} = k^{-c}$. Now, by Lemma 1.1 the do-while loop is repeated a total of at most $O(k)$ times, thus the probability of failure is at most $O(k^{-c+1})$.

## 5 A Dynamic Program to Find the Best Improvement

For a given solution $L$, we define a solution $L'$ to be $\delta$-close from $L$ if $|L - L'| + |L' - L| \leq \delta$. Let $\text{OPT}_\delta$ denote the cost of the best solution that is $\delta$-close from $L$ and whose set of moat centers is the set of moat centers of $L$. In the following we refer to this solution by the **best $\delta$-close solution**.

As a preprocessing step, we round the weights of the centers to the closest integer $i$. It is easy to see that this only modify the total value by a factor $(1 + \varepsilon)$.

For each region $R$, we define the center of the region $c_R$ to be the center of the square $R$. For each point $p$ that is outside of $R$ and at distance at least $\varepsilon \delta R/\log n$ of $R$, we define the coordinates of $p$ w.r.t. $c_R$ as follows. Consider the coordinates of the vector $c_{R'} p$, rounded to the closest $(1 + \varepsilon/\log n)^{14} \delta R/\log n$, for some integer $i$. Let $\tilde{c}_{R'}$ be the resulting list of coordinates. Let $s$ be the point such that the coordinates of the vector $\hat{c}_{R'} s$ are equal to the coordinates of $\tilde{c}_{R'} p$. We define the coordinates of $p$ rounded w.r.t. $R$ to be the coordinates of $s$.

For each region $R$ we also define the grid $G_R$ of $R$ as the $d$-dimensional grid of size $2 \log n/\varepsilon^{14} \times \ldots \times 2 \log n/\varepsilon^{14}$ on $R$. Note that the distance between two consecutive points of the grid is $\varepsilon^{14} \delta R/2 \log n$. For each point $p$ that is inside of $R$, we define the coordinates of $p$ rounded w.r.t. $R$ to be the coordinates of the closest grid point.

The following fact follows from the definition and recalling that the region sizes are in the interval $[1, \text{poly}(n)]$.

**Fact 1.** For any region $R$, the number of different coordinates rounded w.r.t. to $R$ is at most $O((\log n/\varepsilon^{14})^{2d})$.

We now describe the dynamic program. Each entry of the table is defined by the following parameters:

- a region $R$,
- a list of the rounded coordinates w.r.t. $R$ of the centers of $L - \text{OPT}_\delta$ and $\text{OPT}_\delta - L$,
- a list of the (rounded) weights of the centers of $L - \text{OPT}_\delta$ and $\text{OPT}_\delta - L$.
- a boolean vector of length $\delta$ indicating whether $i$th center in the above lists is in $L - \text{OPT}_\delta$ (value 0) or in $\text{OPT}_\delta - L$ (value 1).

The following fact follows from the definition and Fact 1.

**Fact 2.** The total number of entries that are parameterized by region $R$ is at most $(\log n/\varepsilon^{14})^{O(d \cdot \delta)}$.

We now explain how to fill-up the table. We maintain the following constraint when we compute a (possibly partial) solution $L'$: there is no center of $(L' - L) \cup (L - L')$ that is moat. Under this constraint, we proceed as follows, starting with the basic regions which define the base-case of our DP. The base-case regions contains only one single candidate center. Hence, the algorithm proceeds as follows: it fills up table entries that are parameterized by:

- any boolean vector of length $\delta$.
- the rounded coordinates of the unique candidate center inside $R$ and a set of $\delta - 1$ rounded coordinates for the centers of $(L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$ outside $R$, or
- a set of $\delta$ rounded coordinates for the centers of $(L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$ outside $R$.

Additionally, we require that the boolean vector is consistent with the rounded coordinates: the candidate center inside $R$ is already in $L$ if and only if its corresponding boolean entry is 0.

It iterates over all possible rounded coordinates for the at most $\delta$ centers of $(L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$ outside $R$ and for each of possibility, it computes the cost. Note that this can be done in time $n \cdot \delta$.

We now consider the general case which consists in merging table entries of child regions. Fix a table entry parameterized by a region $R$, and the rounded coordinates of the centers of $(L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$. We define which tables entries of the child regions are compatible given the rounded coordinates. For a table entry of a child region $R_i$, with the rounded coordinates of the centers of $(L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$, we require the following for all center $c_0 \in (L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$. Denote by $c^1_0$ the coordinates of $c_0 \in (L - \text{OPT}_\delta) \cup (\text{OPT}_\delta - L)$ rounded w.r.t. $c_{R_i}$, namely its values in the table entry for $R_i$. Let $\tilde{c}^R_0$ denote its rounded coordinate w.r.t. $R$, namely its values in the table entry for $R$. We require:

- If $\tilde{c}^R_0$ is outside $R$, we say that the table entries are compatible for $c_0$ if the coordinates of the vector $c_{R_i}/c_0 \text{ and } c_{R_i}/\tilde{c}^R_0$ are all within a $(1 + \varepsilon/\log n)$ factor.

- If $c_0$ is inside $R$, we say that the table entries are compatible for $c_0$ if the point of the grid $G_R$ that is the closest to $\tilde{c}^R_0$ is $\tilde{c}^R_0$.

- the entries corresponding to $c_0$ in the boolean vectors are the same.

The following lemma follows immediately from the above facts and the definition.

**Lemma 5.1.** The running time of the dynamic program is $n(\log n/\varepsilon)^{O(\delta \delta)}$.

We now turn to the proof of correctness. For a given region $R$, and a $\delta$-close solution $S$ we define the table entry of $R$ induced by $S$ to be the table entry parameterized by $R$ and the coordinates of the centers of $(L - S) \cup (S - L)$ rounded w.r.t. $R$.

**Lemma 5.2.** Consider the best $\delta$-close solution $\text{OPT}_\delta$. For any level $i$ of the quadtree dissection, for any region $R$ at level $i$, we have that the table entry induced by $\text{OPT}_\delta$ has cost at most $\sum_{c \in R}((1 + \varepsilon/\log n)^{\delta} \text{dist}(c, \text{OPT}_\delta))^2$.

**Proof.** Observe that we consider a $\delta$-close solution that has the same set of moat centers than $L$. Hence, if a client is served by a moat center in $\text{OPT}_\delta$, we know exactly the position of this center (as it is also in $L$ and so cannot be removed). Thus, for any region $R$, the set of clients in $R$ is served by either a center in $R$ or a center at distance at least $\varepsilon \delta R/\log n$ from the boundary of $R$ or a center of $L$ that is a moat center.

We now proceed by induction. We consider the base case: let $R$ be a region at the maximum level. Consider the table entry induced by $\text{OPT}_\delta$. We claim that for each client $c$ in $R$, the cost induced by the solution for this table entry is at most $(1 + \varepsilon/\log n) \text{dist}(c, \text{OPT}_\delta)$. Indeed, since $\text{OPT}_\delta$ has the same set of moat centers each client that is served by a center outside of $R$ that is at distance at most $\varepsilon \delta R/\log n$ from the boundary is served by a moat center of $L$ and so there is no approximation in its service cost. Each client that is served by a center of $\text{OPT}_\delta - L$ is at distance at least $\varepsilon \delta R/\log n$ and so, the error induced by the rounding is at most $\varepsilon \text{dist}(c, \text{OPT}_\delta)/\log n$. Finally, the cost for the clients in $R$ served by the unique center of $R$ (if there is one) is exact.

Thus, assume that this holds up to level $i - 1$. Consider a region $R$ at level $i$ and the table entry induced by $\text{OPT}_\delta$. The inductive hypothesis implies that for each of the table entries of the child regions that are induced by $\text{OPT}_\delta$, the cost for the clients in each subregion is at most $(1 + \varepsilon/\log n)^{\delta - 1} \sum_{c \in R} \text{dist}(c, \text{OPT}_\delta)^2$.

By definition, we have that each client of $R$ that is served in $\text{OPT}_\delta$ by a center that is outside $R$ is at distance at least $\varepsilon \delta R/\log n$ or a moat center of $L$ (and so the distance is known exactly). Thus the rounding error incurred for the cost of the clients of $R$ served by a center outside $R$ is at most $(1 + \varepsilon/\log n)^{\delta} \text{dist}(c, \text{OPT}_\delta)$.

We now turn to the rounding error introduced for the centers that are inside $R$. Let $c$ be such a center. We have that the error introduced is at most the distance between two consecutive grid points and so at most $\varepsilon \delta R/2\log n$. Now, observe that again because $\text{OPT}_\delta$ shares the same moat centers than $L$, each client $a$ of a child region $R_i$ that suffers some rounding error and that is served by $c$, is at distance at least $\varepsilon \delta R_i/\log n$ from $c$ and so, combining with the inductive hypothesis the error incurred is at most $(1 + \varepsilon/\log n)^{\delta} \text{dist}(a, c)$.

\[\square\]
Proposition 5.3 follows from combining Lemmas 5.1 and 5.2.

**Proposition 5.3.** Let $\varepsilon > 0$ be a small enough constant. Let $L$ be a solution and $D$ be a decomposition and $M$ be the moat centers of $L$. The dynamic program FindImprovement output a solution $\text{OPT}_\delta$ such that 
\[
\text{cost}(L) - \text{cost}(\text{OPT}_\delta) \geq (1-\varepsilon)(\text{cost}(L) - \text{cost}(\text{OPT}_\delta)),
\]
where $\text{OPT}_\delta$ is a minimum-cost solution that is $\delta$-close from $L$ and whose set of moat centers is the set of moat centers of $L$. Its running time is $n \cdot (\log n/\varepsilon^{14})^{O(d^2)}$.

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