Covariant description of parametrized nonrelativistic Hamiltonian systems

Mauricio Mondragón and Merced Montesinos

Departamento de Física, Centro de Investigación y de Estudios Avanzados del I.P.N.,
Av. I.P.N. No. 2508, 07000 Ciudad de México, México.
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The various phase spaces involved in the dynamics of parametrized nonrelativistic Hamiltonian systems are displayed by using Crnkovic and Witten’s covariant canonical formalism. It is also pointed out that in Dirac’s canonical formalism there exists a freedom in the choice of the symplectic structure on the extended phase space and in the choice of the equations that define the constraint surface with the only restriction that these two choices combine in such a way that any pair (of these two choices) generates the same gauge transformation. The consequence of this freedom on the algebra of observables is also discussed.

I. INTRODUCTION

There is currently a growing interest in the study of the fundamentals of both classical and quantum mechanics motivated, in part, by several theoretical approaches that try to build a quantum theory of gravity [see, for instance, Ref. \[1\]]. The various conceptual issues found in the construction of generally covariant quantum theories frequently make people to go back to the fundamentals of both classical and quantum mechanics to try to remove what is non-essential and get the generic aspects of them which could be implemented later on in realistic theories [see, for instance, Refs. \[2, 3, 4, 5, 6\] and references therein].

In this paper we focus in the covariant description of Hamiltonian mechanics. The geometrical structure underlying parametrized nonrelativistic Hamiltonian systems is obtained by using the approach of Ref. \[7\], from which the extended phase space \((Γ_{\text{ext}}, Ω_{\text{ext}})\) and the presymplectic phase space \((Σ, Ω_Σ)\) involved are obtained [see also Ref. \[8\] for more details]. Once this is done, the definition of physical observables is implemented and this fact allows us to reach the physical phase space \((Γ_{\text{phys}}, Ω_{\text{phys}})\) for the system. This is displayed in SubSecs. \[II A\] and \[II B\]. In spite of working with the covariant canonical formalism, the usual symplectic structure is used. The implications of choosing alternative symplectic structures in Dirac’s formalism are analyzed in Secs. \[III\], \[IV\], \[V\], and \[VI\] where it is shown that there are many ways of choosing the symplectic structure on the extended phase space if the equation that defines the constraint surface is, in the generic case, accordingly modified in such a way that the gauge transformation is not altered. Due to the fact that the gauge transformation is not modified the gauge-invariant functions are also not modified, however, the ‘algebra of observables’ is, in the generic case, modified because it depends on the particular symplectic structure chosen. Section \[VII\] contains a generalization of these results to generally covariant systems with first class constraints only. Our conclusions are collected in Sec. \[VIII\].

II. THE GEOMETRY AND THE PHYSICS OF PARAMETRIZED NONRELATIVISTIC HAMILTONIAN SYSTEMS

Let us begin by considering the Hamiltonian formulation associated with a system with a finite number of degrees of freedom obtained from the action principle

\[
S[q^j, p_j] = \int_{t_1}^{t_2} \left[ \frac{dq^j}{dt} p_j - H(q^j, p_j, t) \right] dt, \quad j = 1, ..., n,
\]

subject to the standard boundary conditions

\[
q^j(t_α) = q^j_α, \quad α = 1, 2,
\]

\[1\] Associate Member of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.
[Electronic address: mo@fis.cinvestav.mx]
[Electronic address: merced@fis.cinvestav.mx]
\[1\] In some cases, the equation that defines the constraint surface is not modified [see Sec. \[III\]].
where \( q^j_\alpha \) are prescribed numbers. It is assumed that the dynamical system is well-defined, i.e., that there exists a solution to the dynamical problem that matches the boundary conditions. By definition, the coordinates \( q^i \) locally label the points of the *configuration space* \( C \) for the system; the cotangent bundle of \( C \), \( \Gamma = T^* C \), is the *phase space* whose points are locally labelled by the coordinates \( x^a, x^i = q^i \) and \( x^{i+n} = p_i, a = 1, \ldots, 2n \). The equations of motion for the system have the canonical form

\[
\dot{x}^a = \omega^{ab} \frac{\partial H}{\partial x^b},
\]

where \( H \) is the Hamiltonian, \( \omega^{ab} \) are the components of the inverse of the symplectic matrix

\[
\{x^a, x^b\} := \omega^{ab}, \quad (\omega^{ab}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

with \( I \) the \( n \times n \) identity matrix and \( 0 \) the \( n \times n \) zero matrix.

The symplectic structure

\[
\omega := \frac{1}{2} \omega_{ab} dx^a \wedge dx^b,
\]

induces a Poisson structure on \( \Gamma = T^* C \) defined by

\[
\{f(x,t), g(x,t)\} := \frac{\partial f}{\partial x^a} \omega^{ab} \frac{\partial g}{\partial x^b},
\]

where, as usual, the coordinate \( t \) is treated as a parameter \([9]\).

### Parameterizing the system

If the time variable is considered as a canonical variable, the action for this Hamiltonian system becomes

\[
S[q^j, p_j, t, p_t, \lambda] = \int_{\tau_1}^{\tau_2} \left[ \dot{q}^j p_j + \dot{p}_t - \lambda \gamma \right] d\tau,
\]

\[
\gamma := p_t + H(q^j, p_j, t),
\]

subject to the standard boundary conditions

\[
q^j(\tau_\alpha) = q^j_{\alpha}, \quad t(\tau_\alpha) = t_\alpha, \quad \alpha = 1, 2,
\]

where \( p_t \) is the canonical variable conjugate to \( t \), \( \lambda \) is the Lagrange multiplier associated with the constraint \( \gamma = 0 \) that comes from the definition of \( p_t \), and the dot means total derivative with respect to the parameter \( \tau \). The new configuration space for the system is the *extended configuration space* \( C_{\text{ext}} = C \times R \) whose points are locally labelled by \((q^j, t)\) where \( R \) stands for the \( t \) coordinate. Its corresponding phase space will be analyzed below.

### A. Hamilton’s principle

Following the conventions used in Ref. [10], let \( \delta \) denote the arbitrary variation of coordinates \((q^j, t)\), momenta \((p_j, p_t)\), and Lagrange multiplier \( \lambda \) at \( \tau \) fixed

\[
\delta x^\mu(\tau) := x'^\mu(\tau) - x^\mu(\tau),
\]

\[
\delta \lambda(\tau) := \lambda'(\tau) - \lambda(\tau),
\]

where \((x^\mu) = (q^j, t, p_j, p_t), \mu = 1, \ldots, 2(n+1)\). The object \( \delta \) is, but not always, called the *total variation*, other authors call it a virtual variation.

\[2\] It is frequently asserted that the parameter \( \tau \) has no ‘physical meaning.’ However, this assertion is not completely true. For instance, in the case of the relativistic free particle the parameter \( \tau \) might be chosen to be the proper time which, of course, has a physical content; it is the reading of a clock moving together with the relativistic particle, i.e., the proper time can be measured by a device.
The variation of the action \( S \) under arbitrary transformations \( \tilde{\delta} \) keeping at most first order terms in \( \tilde{\delta} \) is

\[
\tilde{\delta} S := \int_{\tau_1}^{\tau_2} \left[ -\left( \dot{p}_i + \lambda \frac{\partial \gamma}{\partial t} \right) \tilde{\delta} t + \left( i - \lambda \frac{\partial \gamma}{\partial t} \right) \tilde{\delta} p_i - \left( \dot{p}_j + \lambda \frac{\partial \gamma}{\partial q^j} \right) \tilde{\delta} q^j 
+ \left( \dot{q}^j - \lambda \frac{\partial \gamma}{\partial p_j} \right) \tilde{\delta} p_j - \gamma \tilde{\delta} \lambda \right] \, d\tau.
\] (10)

To get the equations of motion for the system, Hamilton’s principle will be used. According to it, the evolution of the system from \( \tau_1 \) to \( \tau_2 \), keeping the end points fixed \( \tilde{\delta} q^j(\tau_\alpha) = 0 = \tilde{\delta} t(\tau_\alpha) \), is along the path such that the total variation of the action vanishes, \( \tilde{\delta} S = 0 \). On the other hand, the variations \( \tilde{\delta} t, \tilde{\delta} p_j, \tilde{\delta} q^j, \tilde{\delta} p_t, \) and \( \tilde{\delta} \lambda \) are arbitrary in the ‘bulk’ \( (\tau_1, \tau_2) \). This, together with \( \tilde{\delta} S = 0 \) and the fact that \( \tilde{\delta} q^j(\tau_\alpha) = 0 = \tilde{\delta} t(\tau_\alpha) \) imply that the coefficients of the variations in the integrand must vanish

\[
\begin{align*}
\tilde{\delta} q^j : & \quad \dot{p}_j = -\lambda \frac{\partial \gamma}{\partial q^j}, \\
\tilde{\delta} t : & \quad \dot{p}_t = -\lambda \frac{\partial \gamma}{\partial t}, \\
\tilde{\delta} p_j : & \quad \dot{q}^j = \lambda \frac{\partial \gamma}{\partial p_j}, \\
\tilde{\delta} p_t : & \quad \dot{t} = \lambda \frac{\partial \gamma}{\partial p_t}, \\
\tilde{\delta} \lambda : & \quad \gamma = 0,
\end{align*}
\] (11)

which are the equations of motion for the parametrized Hamiltonian system \( S \).

Note that the Lagrange multiplier \( \lambda \) cannot be determined from the evolution of the constraint. To see this, the evolution with respect to \( \tau \) of \( \gamma \) is computed. If \( f = f(\mu, \tau) \) then

\[
\frac{d}{d\tau} f(\mu, \tau) = \frac{\partial f}{\partial q^j} \dot{q}^j + \frac{\partial f}{\partial t} \dot{t} + \frac{\partial f}{\partial p_j} \dot{p}_j + \frac{\partial f}{\partial p_t} \dot{p}_t + \frac{\partial f}{\partial \tau},
\] (12)

where the Poisson bracket \( \{ \cdot, \cdot \} \) is defined by

\[
\{ f, g \} := \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p^j} + \frac{\partial f}{\partial t} \frac{\partial g}{\partial p_t} - \left( f \leftarrow g \right) = \frac{\partial f}{\partial x^\mu} \Omega^\mu_{\nu} \frac{\partial g}{\partial x^\nu},
\] (13)

where \( \Omega^\mu_{\nu} \) are the components of the inverse of the symplectic matrix

\[
\{ x^\mu, x^\nu \} := \Omega^\mu_{\nu}, \quad \Omega^\mu_{\nu} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\] (14)

where \( I \) is the \((n + 1) \times (n + 1)\) identity matrix and 0 the \((n + 1) \times (n + 1)\) zero matrix.

In particular, if \( f = \gamma \) then \( \{ \gamma, \gamma \} = 0 \) and \( \frac{\partial \gamma}{\partial \tau} = 0 \) and so \( \frac{d\gamma}{d\tau} = 0 \). One could alternatively say that no more constraints arise and that \( \gamma \) is the only one constraint in the formalism. In Dirac’s terminology \( \gamma \) is first class [11].

B. Extended, presymplectic, and physical phase spaces

Let us denote the integrand of Eq. \( S \) as

\[
L := \dot{q}^j p_j + \dot{t} p_t - \lambda \gamma.
\] (16)
Following Ref. [7] and using Eq. (10) it follows that the total variation of $L$ is

$$
\delta L = -\left(\dot{p}_i + \lambda \frac{\partial \gamma}{\partial p_i}\right) \delta t + \left(i - \lambda \frac{\partial \gamma}{\partial p_i}\right) \delta p_t - \left(\dot{p}_j + \lambda \frac{\partial \gamma}{\partial q^j}\right) \delta q^j
$$

$$
+ \left(\dot{q}^j - \lambda \frac{\partial \gamma}{\partial p_j}\right) \delta p_j - \gamma \delta \lambda + \frac{d}{dt} \left(p_i \delta t + p_j \delta q^j\right).
$$

Computing again $\delta$ and using $\delta^2 L = 0$

$$
\frac{d\Omega_{\text{ext}}}{d\tau} = \delta \left(\dot{p}_i + \lambda \frac{\partial \gamma}{\partial p_i}\right) \wedge \delta t - \delta \left(i - \lambda \frac{\partial \gamma}{\partial p_i}\right) \wedge \delta p_t + \delta \left(\dot{p}_j + \lambda \frac{\partial \gamma}{\partial q^j}\right) \wedge \delta q^j
$$

$$
+ \delta \gamma \wedge \delta \lambda - \delta \left(\dot{q}^j - \lambda \frac{\partial \gamma}{\partial p_j}\right) \wedge \delta p_j,
$$

where

$$
\Omega_{\text{ext}} := \delta p_i \wedge \delta q^i + \delta p_t \wedge \delta t,
$$

[cf. Refs. [4, 12, 13]].

**Definition.** The extended phase space for the system is the couple $(\Gamma_{\text{ext}}, \Omega_{\text{ext}})$, where $\Gamma_{\text{ext}}$ is equal to the cotangent bundle of the extended configuration space $C_{\text{ext}}$, $\Gamma_{\text{ext}} = T^*C_{\text{ext}} = T^*C \times R^2$. The points of $\Gamma_{\text{ext}}$ can be labelled by $(x^\mu) = (q^i, t, p_i)$. The symplectic structure on $\Gamma_{\text{ext}}$ is given in Eq. (19), which is the one already defined in Eq. (13). Note that $\Omega_{\text{ext}}$ is closed and non-degenerate, i.e., it is a symplectic structure on $\Gamma_{\text{ext}}$.

Note also that ‘on-shell’, i.e., if the equations of motion (11) hold, then their total variation vanishes too

$$
\delta \left(\dot{p}_i + \lambda \frac{\partial \gamma}{\partial p_i}\right) = 0,
$$

$$
\delta \left(i - \lambda \frac{\partial \gamma}{\partial p_i}\right) = 0,
$$

$$
\delta \left(\dot{p}_j + \lambda \frac{\partial \gamma}{\partial q^j}\right) = 0,
$$

$$
\delta \left(\dot{q}^j - \lambda \frac{\partial \gamma}{\partial p_j}\right) = 0,
$$

$$
\delta \gamma = 0.
$$

Thus, from Eqs. (19), (20) and (21)

$$
\frac{d\Omega_{\Sigma}}{d\tau} = 0,
$$

where

$$
\Omega_{\Sigma} := i^*\Omega_{\text{ext}} = \delta p_i \wedge \delta q^i - \delta H (q, p, t) \wedge \delta t,
$$

is the pull-back of $\Omega_{\text{ext}}$ to the constraint surface $\Sigma$ defined by $\gamma = 0$ through the inclusion map $i : \Sigma \rightarrow \Gamma_{\text{ext}}$ [cf. Refs. [4, 12, 13]]. To be precise, $\Sigma := \{(q^i, t, p_i) \in \Gamma_{\text{ext}} \mid p_i = -H(q, p, t)\}$. It is clear that $\Sigma$ is a $(2n + 1)$-dimensional submanifold of $\Gamma_{\text{ext}}$ and that $(y^A) = (q^i, p_i, t)$ can be taken as independent coordinates for labelling the points on $\Sigma$. The inclusion map $i$ from $\Sigma$ to $\Gamma_{\text{ext}}$ is defined by

$$
i : \Sigma \rightarrow \Gamma_{\text{ext}},
$$

$$(q, p, t) \mapsto (q, p, t, p_i = -H(q, p, t)).
$$

Moreover, Eq. (21) means that $\Omega_{\Sigma}$ is conserved in $\tau$-evolution. The two-form (22) is degenerate in the sense that

$$
(\Omega_{\Sigma})_{AB} v^B = 0,
$$

for a non-trivial vector field $v^B$ on $\Sigma$. Solving last equation one finds that there is a single null direction, as expected because there is a single first class constraint, given by

$$
v = a(q, p, t) \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^j} \frac{\partial}{\partial p_j} + \frac{\partial}{\partial t}\right),
$$
where \( a(q,p,t) \) is an arbitrary non-vanishing function on \( \Sigma \). However, \( \gamma \) has associated the Hamiltonian vector field \( X_{d\gamma} \) on \( \Gamma_{ext} \), \((\Omega_{ext})\cdot X_{d\gamma} = d\gamma \), where the dot \( \cdot \) stands for contraction, and it is given by

\[
X_{d\gamma} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}.
\]

(26)

\( X_{d\gamma} \) is globally defined on \( \Gamma_{ext} \) and, in particular, on \( \Sigma \). From the condition

\[
i_*v = X_{d\gamma},
\]

(27)

it follows that \( a(q,p,t) = 1 \). So,

\[
v = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}.
\]

(28)

Definition. The couple \((\Sigma,\Omega_\Sigma)\) is the \textit{presymplectic phase space} for the parametrized Hamiltonian system. The name presymplectic comes from the fact that the structure of Eq. (22) is closed, degenerate, and defined on the odd-dimensional manifold \( \Sigma \).

Note that the presymplectic phase space \((\Sigma,\Omega_\Sigma)\) is a well-defined structure even though \( H \) might not explicitly depend on the time variable \( t \). Moreover, Eq. (22) can be written as

\[
\Omega_\Sigma = \tilde{\delta} p_i \wedge \tilde{\delta} q^i - \tilde{\delta} H(q,p,t) \wedge \tilde{\delta} t,
\]

(29)

with

\[
\alpha_i := \tilde{\delta} p_i + \frac{\partial H}{\partial q^i} \tilde{\delta} t,
\]

\[
\beta^i := \tilde{\delta} q^i - \frac{\partial H}{\partial p_i} \tilde{\delta} t.
\]

(30)

So far, only the symplectic \((\Gamma_{ext},\Omega_{ext})\) and the presymplectic \((\Sigma,\Omega_\Sigma)\) phase spaces have been analyzed using the procedure of Ref. [7]. Now, it will be studied the so-called physical or reduced phase space. To do this, it will be convenient to analyze first the issue of Dirac observables.

Definition. The \textit{Dirac or physical observables} \( \mathcal{O} \) for the system are real functions on \( \Sigma \), \( \mathcal{O}: \Sigma \rightarrow \mathbb{R} \), killed by the null vector \( v \) of Eq. (28)

\[
v \mathcal{O} = \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \right) \mathcal{O} = 0.
\]

(31)

This means that the \( \mathcal{O} \)'s are constant along the orbits on \( \Sigma \) to which \( v \) is tangent. Equation (31) can be written as

\[
\{ \mathcal{O}, H \} + \frac{\partial \mathcal{O}}{\partial t} = 0 \iff \frac{d\mathcal{O}}{dt} = 0.
\]

(32)

Thus, we have got the following:

**result 1.** Dirac observables or physical observables \( \mathcal{O} \) are the \textit{constants of motion with respect to} \( t \) of the standard (i.e., non-parametrized) Hamiltonian system because the bracket in Eq. (32) is the standard Poisson bracket on \( \Gamma \) given by Eq. (4).

Moreover, due to the fact \( \Sigma \) is a \((2n+1)\)-dimensional manifold and one has a single linear differential equation on it for the unknowns \( \mathcal{O} \), then Eq. (31) or, equivalently, Eq. (32) has \( 2n \) independent solutions, say, \((Q^i(q,p,t), P_i(q,p,t))\). In addition, \((Q^i(q,p,t), P_i(q,p,t))\) can be chosen to be canonical (again, with respect to the bracket of Eq. (8))

\[
\{ Q^i(q,p,t), Q^j(q,p,t) \} = 0, \\
\{ Q^i(q,p,t), P_j(q,p,t) \} = \delta^i_j, \\
\{ P_i(q,p,t), P_j(q,p,t) \} = 0.
\]

(33)
The specification of the physical observables \((Q_i^i(q, p, t), P_i(q, p, t))\) to the independent concrete values \((Q_i^i, P_i)\) they can have

\[
\begin{align*}
Q_i^i &= Q_i^i(q, p, t), \\
P_i &= P_i(q, p, t),
\end{align*}
\]

(34)
gives rise to curves in \(\Sigma\) (one curve for each value of \((Q_i^i, P_i))\), which are called motions, orbits, stories, or physical states. These curves are precisely those to which \(\psi\) in Eq. (23) is tangent.

**Definition.** The reduced or physical phase space for the parametrized Hamiltonian system is the couple \((\Gamma_{phys}, \Omega_{phys})\), where the points of the \(2n\)-dimensional manifold \(\Gamma_{phys}\) represent all the possible motions for the system; each point being (locally) labelled by the independent coordinates \((Q_i^i, P_i)\), and the symplectic structure \(\Omega_{phys}\) (written in these coordinates) is defined by

\[
\Omega_{phys} := \tilde{\delta} P_i \wedge \tilde{\delta} Q^i. \tag{35}
\]

It is clear that the locus of Eq. (34) defines a projection map \(\pi\) from the presymplectic phase space \(\Sigma\) to the physical phase space \(\Gamma_{phys}\)

\[
\pi : \Sigma \rightarrow \Gamma_{phys}, \\
p \mapsto \mathbf{q} = \pi(p),
\]

(36)
that sends a point \(p\) on \(\Sigma\) to the orbit \(q\) to which it belongs. More precisely, let \((Q_i^i, P_i)\) be local coordinates around the point \(q = \pi(p)\)

\[
\begin{align*}
(Q_i^i(\pi(p)), P_i(\pi(p))) &= ((Q_i^i \circ \pi)(p), (P_i \circ \pi)(p)), \\
&= (\pi^* Q_i^i)(p), (\pi^* P_i)(p)), \\
&= (Q_i^i(p), P_i(p)),
\end{align*}
\]

(37)
i.e.,

\[
\begin{align*}
\pi^* Q_i^i &= Q_i^i, \\
\pi^* P_i &= P_i.
\end{align*}
\]

(38)
Taking the map \(\pi\) into account, the pull-back of various geometrical objects on \(\Gamma_{phys}\) to \(\Sigma\) can be computed. In particular, the pull-back of the one-forms \(\tilde{\delta} Q^i\) and \(\tilde{\delta} P_i\) on \(\Gamma_{phys}\) to \(\Sigma\) are

\[
\begin{align*}
\pi^* \tilde{\delta} Q_i^i &= \tilde{\delta} (\pi^* Q_i^i) = \tilde{\delta} Q_i^i, \\
&= \frac{\partial Q_i^i}{\partial q^i} \tilde{\delta} q^i + \frac{\partial Q_i^i}{\partial p_j} \tilde{\delta} p_j + \frac{\partial Q_i^i}{\partial t} \tilde{\delta} t, \\
&= \frac{\partial Q_i^i}{\partial q^i} \tilde{\delta} q^i + \frac{\partial Q_i^i}{\partial p_j} \tilde{\delta} p_j - \{Q_i^i, H\} \tilde{\delta} t, \\
&= \frac{\partial Q_i^i}{\partial q^i} \left( \tilde{\delta} q^i - \frac{\partial H}{\partial p_j} \tilde{\delta} p_j \right) + \frac{\partial Q_i^i}{\partial p_j} \left( \tilde{\delta} p_j + \frac{\partial H}{\partial q^i} \tilde{\delta} t \right), \\
&= \frac{\partial Q_i^i}{\partial q^i} \beta^j + \frac{\partial Q_i^i}{\partial p_j} \alpha^j, \tag{39}
\end{align*}
\]

\[
\begin{align*}
\pi^* \tilde{\delta} P_i &= \tilde{\delta} (\pi^* P_i) = \tilde{\delta} P_i, \\
&= \frac{\partial P_i}{\partial q^i} \tilde{\delta} q^i + \frac{\partial P_i}{\partial p_j} \tilde{\delta} p_j + \frac{\partial P_i}{\partial t} \tilde{\delta} t, \\
&= \frac{\partial P_i}{\partial q^i} \tilde{\delta} q^i + \frac{\partial P_i}{\partial p_j} \tilde{\delta} p_j - \{P_i, H\} \tilde{\delta} t, \\
&= \frac{\partial P_i}{\partial q^i} \left( \tilde{\delta} q^i - \frac{\partial H}{\partial p_j} \tilde{\delta} p_j \right) + \frac{\partial P_i}{\partial p_j} \left( \tilde{\delta} p_j + \frac{\partial H}{\partial q^i} \tilde{\delta} t \right), \\
&= \frac{\partial P_i}{\partial q^i} \beta^j + \frac{\partial P_i}{\partial p_j} \alpha^j. \tag{40}
\end{align*}
\]
where Eq. (32) was used. Thus, the right side of Eq. (39) (equivalently, the right side of Eq. (40)) is the pull-back of \( \delta Q^i \) to \( \Sigma \). In addition, the right side of Eq. (41) (equivalently, the right side of Eq. (42)) is the pull-back of \( \delta P_i \) to \( \Sigma \).

Using these results, the pull-back of the symplectic structure \( \Omega_{phys} \) on \( \Gamma_{phys} \) to the constraint surface \( \Sigma \) can be obtained. The resulting two-form on \( \Sigma \) is precisely the presymplectic structure \( \Omega_\Sigma \):

\[
\pi^* \Omega_{phys} = \Omega_\Sigma. \tag{43}
\]

**Proof:** using Eqs. (35), (40) and (42)

\[
\pi^* \Omega_{phys} = \left( \pi^* \delta P_i \right) \wedge \left( \pi^* \delta Q^i \right),
\]

\[
= \frac{1}{2} \left( \frac{\partial P_i}{\partial q^k} \frac{\partial Q^i}{\partial p^j} - \frac{\partial P_i}{\partial p^k} \frac{\partial Q^i}{\partial q^j} \right) \beta^k \wedge \beta^j + \frac{1}{2} \left( \frac{\partial P_i}{\partial q^k} \frac{\partial Q^i}{\partial q^j} - \frac{\partial P_i}{\partial p^k} \frac{\partial Q^i}{\partial p^j} \right) \alpha_k \wedge \alpha_j
\]

\[
+ \left( \frac{\partial P_i}{\partial p^k} \right) \delta Q^i = \alpha_i \wedge \beta^i,
\]

\[
= \Omega_\Sigma, \tag{44}
\]

because of Eq. (38) [cf. Refs. 4, 12, 13].

Note that the meaning and the range of the variables \((Q^i, P_i)\) do not correspond, in the generic case, with the ones of the variables \((q^i, p_i)\) at the initial time \(t_0\), given by \((q_{i0}, p_{i0})\). Of course, in particular \((Q^i, P_i)\) can be taken to be \((q_{i0}, p_{i0})\).

In summary, the various geometrical structures involved in the dynamics of parametrized Hamiltonian systems are: \((\Gamma_{ext}, \Omega_{ext})\), \((\Sigma, \Omega_\Sigma)\), \((\Gamma_{phys}, \Omega_{phys})\), and \((\Gamma, \omega)\). In particular, the physical observables for the system introduces \((\Gamma, \omega)\) again in the formalism of the parametrized Hamiltonian system and this allows us to get the physical phase space for the system \((\Gamma_{phys}, \Omega_{phys})\). The interplay between these structures is shown in the next diagram

\[
T^* C_{ext} \xleftarrow{i} \Sigma \xrightarrow{\pi} \Gamma_{phys}, \tag{45}
\]

or, in a more familiar notation

\[
\Gamma \times R^2 \xleftarrow{i} \Gamma \xrightarrow{\pi} \Gamma_{phys}, \tag{46}
\]

or, equivalently

\[
T^* C \times R^2 \xleftarrow{i} T^* C \xrightarrow{\pi} \Gamma_{phys}, \tag{47}
\]

[cf. Refs. 4, 12, 17].

**Relationship with the canonical formalism a la Dirac.** In the canonical formalism a la Dirac it is possible to get the physical phase space by fixing the gauge freedom. In the present case, a single gauge condition would be enough because there is a single first class constraint. In general, by choosing a gauge condition, Dirac’s method leads to canonical coordinates to label the points of \( \Gamma_{phys} \). On the other hand, in the present approach no gauge condition has been chosen to reach \( \Gamma_{phys} \). What is then the relationship of the present approach to the fact of choosing a gauge condition in Dirac’s method to reach \( \Gamma_{phys} \)? Well, the answer is as follows: in the present approach there still exists the freedom to choose the particular set of Dirac observables that form \((Q(q, p, t), P_t(q, p, t))\), i.e., there exists the freedom to choose the canonical coordinates on \( \Gamma_{phys} \). This freedom corresponds, precisely, to the freedom of picking a gauge condition in Dirac’s method.

### III. FREEDOM IN THE CHOICE OF THE SYMPLECTIC STRUCTURE \( \Omega_{ext} \) IN \( \Gamma_{ext} \) KEEPING THE SAME CONSTRAINT SURFACE \( \Sigma \)

Note that a more covariant description of \((\Gamma_{phys}, \Omega_{phys})\) (and therefore of \((\Gamma_{ext}, \Omega_{ext})\)) would be given by labelling the points of \( \Gamma_{phys} \) with arbitrary labels \(X^a\) (not necessarily canonical ones) with the only restriction that the independent coordinates \(X^a\) label physically distinct states. In these coordinates, \(\Omega_{phys}\) would have the form

\[
\Omega_{phys} = \frac{1}{2} (\Omega_{phys})_{ab} (X) dX^a \wedge dX^b,
\]

\[
\{X^a, X^b\}_{phys} := \Omega_{phys}^{ab}(X). \tag{48}
\]
To illustrate this point consider the action principle (7) with \((x^\mu) = (x, y, t, p_x, p_y, p_t)\) and
\[
G_0 := p_t + \frac{1}{2} \left( \frac{p_x^2}{m} + m \omega^2 x^2 + \frac{p_y^2}{m} + m \omega^2 y^2 \right),
\]
which yields the equations of motion
\[
\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{t} = \lambda,
\]
\[
\dot{p}_x = -\lambda m \omega^2 x, \quad \dot{p}_y = -\lambda m \omega^2 y, \quad \dot{p}_t = 0,
\]
and the constraint
\[
G_0 \approx 0. \tag{51}
\]
As it was explained in Sec. II, the standard procedure consists in taking \((x^\mu) = (x, y, t, p_x, p_y, p_t)\) as canonical coordinates with the standard symplectic structure on \(\Gamma_{ext}\) given by
\[
\Omega_0 = dp_x \wedge dx + dp_y \wedge dy + dp_t \wedge dt, \tag{52}
\]
and the corresponding symplectic structure \(\omega_0\) on \(\Gamma_{phys}\) given by
\[
\omega_0 = dp_{x0} \wedge dx_0 + dp_{y0} \wedge dy_0. \tag{53}
\]
However, it is not necessary that the coordinates \((x^\mu) = (x, y, t, p_x, p_y, p_t)\) which label the points of \(\Gamma_{ext}\) are canonical ones. It is possible, for instance, to choose\(^3\)
\[
\Omega = \Lambda m^2 \omega^2 xy dx \wedge dy - dx \wedge dp_x + \Lambda xypdx \wedge dp_y - ydp_x dy \wedge dp_x - dy \wedge dp_y + \Lambda \frac{p_x p_y}{m^2 \omega^2} dp_x \wedge dp_y + dp_t \wedge dt, \tag{54}
\]
where \(\Lambda\) is an arbitrary real constant, as the symplectic structure on \(\Gamma_{ext}\) with the corresponding symplectic structure\(^8\)
\[
\omega = \Lambda m^2 \omega^2 x_0 y_0 dx_0 \wedge dy_0 - dx_0 \wedge dp_{x0} + \Lambda x_0 p_{y0} dx_0 \wedge dp_{y0} - y_0 dp_{x0} dy_0 \wedge dp_{y0} + \Lambda \frac{p_{x0} p_{y0}}{m^2 \omega^2} dp_{x0} \wedge dp_{y0}, \tag{55}
\]
or, equivalently,
\[
\{x_0, y_0\} = \Lambda \frac{p_{x0} p_{y0}}{m^2 \omega^2}, \quad \{x_0, p_{x0}\} = 1, \quad \{x_0, p_{y0}\} = -\Lambda y_0 p_{x0}, \quad \{y_0, p_{x0}\} = \Lambda x_0 p_{y0}, \quad \{y_0, p_{y0}\} = 1, \quad \{p_{x0}, p_{y0}\} = \Lambda m^2 \omega^2 x_0 y_0, \tag{56}
\]
on \(\Gamma_{phys}\) and keeping the same constraint \((51)\). In fact, note that the equations
\[
\frac{dx^\mu}{dt} = \{x^\mu, \lambda G_0\} = \lambda \Omega^{\mu\nu} \frac{\partial G_0}{\partial x^\nu}, \tag{57}
\]
where \((\Omega^{\mu\nu})\) is the inverse matrix of the symplectic matrix given in \((54)\), exactly reproduce the equations of motion of Eq. \((58)\). Moreover, note that the presymplectic two-forms \(i^* \Omega_0\) and \(i^* \Omega\), defined on \(\Sigma\), have the same null vector given in Eq. \((60)\). Furthermore, the gauge transformation on the variables \((x^\mu) = (x, y, t, p_x, p_y, p_t)\) computed with the standard symplectic structure \((52)\) is exactly the same one computed with the symplectic structure given in Eq. \((54)\) [see also Secs. IV, V, and VI]. These two choices can be represented in the following diagram:
\[
(\Gamma_{ext}, \Omega_0) \xleftarrow{i} (\Sigma, i^* \Omega_0) \overset{\pi}{\longrightarrow} (\Gamma_{phys}, \omega_0), \tag{58}
\]
\[
(\Gamma_{ext}, \Omega) \xleftarrow{i} (\Sigma, i^* \Omega) \overset{\pi}{\longrightarrow} (\Gamma_{phys}, \omega), \tag{59}
\]
\(^3\) Note that we are not making a change of coordinates, we are working with the same set of coordinates, i.e., what changes is the symplectic structure.
respectively. Thus, we have got the following:

**Result 2.** The specification of both the constraint (surface) $\gamma \approx 0$ and the dynamical equations of motion [see Eqs. (50) and (51)] do *not* uniquely fix the symplectic structure on the extended phase space $\Gamma_{\text{ext}}$ and therefore do *not* uniquely fix the symplectic structure on the physical phase space $\Gamma_{\text{phys}}$. In particular, there is not need of choosing the coordinates that label the points of $\Gamma_{\text{ext}}$ and $\Gamma_{\text{phys}}$ to be canonical coordinates, which is always the case in Dirac’s approach, and there exists the freedom to choose non-canonical symplectic structures on $\Gamma_{\text{ext}}$ and $\Gamma_{\text{phys}}$. Of course, there are many other ways of choosing the symplectic structure on $\Gamma_{\text{ext}}$ and the corresponding presymplectic structure on $\Sigma$ and the symplectic structure on $\Gamma_{\text{phys}}$ keeping the same constraint (51). We have just listed two of these possible choices.

Finally, note that by means of Darboux’s theorem all of these non-canonical expressions for the symplectic structure on $\Omega_{\text{ext}}$ (and therefore non-canonical expressions for $\Omega_{\text{phys}}$) can (locally) acquire the canonical form. However, it would be interesting not to write them in a canonical form and to explore the consequences of these possible choices in the quantum theory. In particular, to understand how this situation is handled in the framework of Dirac’s quantization as well as in the framework of reduced phase space quantization.

### IV. Freedom in the Choice of Both the Symplectic Structure $\Omega_{\text{ext}}$ in $\Gamma_{\text{ext}}$ and the Constraint Surface $\Sigma$

It is remarkable that $H$ in Eq. (1) does not need to be the energy, even for conservative systems. To see this, consider again the equations of motion for the two-dimensional isotropic harmonic oscillator

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{p}_x = -m\omega^2 x, \quad \dot{p}_y = -m\omega^2 y, \quad (60)$$

where the dot ‘.’ stands for the total derivative with respect to $t$. It is well known that the equations of motion (60) can be obtained from the action principle

$$S[x, y, p_x, p_y] = \int_{t_1}^{t_2} [\dot{x}p_x + \dot{y}p_y - H_0] \, dt,$$

$$H_0 = \frac{1}{2} \left( \frac{p_x^2}{m} + m\omega^2 x^2 + \frac{p_y^2}{m} + m\omega^2 y^2 \right). \quad (61)$$

However, the equations of motion (60) can also, for instance, be obtained from the action principle

$$S[x, y, p_x, p_y] = \int_{t_1}^{t_2} [\dot{x}p_y + \dot{y}p_x - J_1] \, dt,$$

$$J_1 = \frac{p_xp_y}{m} + m\omega^2 xy. \quad (62)$$

Note that $H_0$ in Eq. (61) is bounded from below while $J_1$ in Eq. (62) is *not*. In addition, in Eq. (61) the canonical momenta of $(x, y)$ are their corresponding kinetic momenta while in Eq. (62) the canonical momenta of $(x, y)$ are not their kinetic momenta $^4$.

The action principles (61) and (62) can be parametrized incorporating the variable $t$ as a configuration variable. By doing this one gets

$$S[x, y, t, p_x, p_y, p_t, \lambda] = \int_{\tau_1}^{\tau_2} [\dot{x}p_x + \dot{y}p_y + \dot{t}p_t - \lambda G_0] \, d\tau,$$

$$G_0 := p_t + H_0, \quad (63)$$

and

$$S_1[x, y, t, p_x, p_y, p_t, \lambda] = \int_{\tau_1}^{\tau_2} [\dot{x}p_y + \dot{y}p_x + \dot{t}p_t - \lambda G_1] \, d\tau,$$

$$G_1 := p_t + J_1; \quad (64)$$

$^4$ In classical field theory is a common fact that the canonical momenta of fields are not their linear momenta obtained, for instance, from the energy-momentum tensor. However, Eq. (62) shows that this property is also present in systems with a finite number of degrees of freedom.
respectively.

**Extended phase space.** In the case of the action (63), the extended phase space \((\Gamma_{\text{ext}}, \Omega)\) is such that \((x, y, t, p_x, p_y, p_t)\) are independent coordinates for labelling the points of \(\Gamma_{\text{ext}}\) and the symplectic structure \(\Omega\) on \(\Gamma_{\text{ext}}\) is given by

\[
\Omega = dp_y \wedge dx + dp_x \wedge dy + dp_t \wedge dt ,
\]

or, equivalently, the nonvanishing Poisson brackets are

\[
\{x, p_y\}_{\Omega} = 1, \quad \{y, p_x\}_{\Omega} = 1, \quad \{t, p_t\}_{\Omega} = 1,
\]

instead of \(\Omega_0\), given in Eq. (52), or equivalently,

\[
\{x, p_x\}_{\Omega_0} = 1, \quad \{y, p_y\}_{\Omega_0} = 1, \quad \{t, p_t\}_{\Omega_0} = 1
\]

which corresponds to the action (68).

**Presymplectic phase space.** According to (68), the presymplectic phase space \(\Sigma\) is defined as \(\Sigma = \{(x, y, t, p_x, p_y, p_t) \in \Gamma_{\text{ext}} \mid G_1 = p_t + J_1 = 0\}\). The presymplectic two-form \(\Omega_{\Sigma}\) on \(\Sigma\), induced by (65), is given by

\[
\Omega_{\Sigma} = i^*\Omega_1 = dp_y \wedge dx + dp_x \wedge dy - dJ_1 \wedge dt = (dp_y + m\omega^2 y dt) \wedge \left(dx - \frac{p_x}{m} dt\right) + (dp_x + m\omega^2 x dt) \wedge \left(dy - \frac{p_y}{m} dt\right).
\]

The null vector \(v\) of \(\Omega_{\Sigma}\) in Eq. (68) is given by

\[
v = \frac{p_x}{m} \frac{\partial}{\partial x} + \frac{p_y}{m} \frac{\partial}{\partial y} - m\omega^2 \frac{\partial}{\partial p_x} - m\omega^2 y \frac{\partial}{\partial p_y} + \frac{\partial}{\partial t}.
\]

Note that this null vector is also the null vector of the standard symplectic structure \(\Omega_{\Sigma_0}\)

\[
\Omega_{\Sigma_0} = i^*\Omega_0 = dp_x \wedge dx + dp_y \wedge dy - dH_0 \wedge dt.
\]

Therefore, we have got the following:

**result 3.** There exists the freedom to choose both the symplectic structure \(\Omega_{\text{ext}}\) on \(\Gamma_{\text{ext}}\) and the equation that defines the constraint surface \(\Sigma\) with the only restriction that these two choices combine in such a way that the null vector \(v\) is the same for any choice of the pair \((\Omega_{\text{ext}}, \Sigma)\). Note that the integral curves to which \(v\) is tangent belong to \(\Sigma\) and also to \(\Sigma_1\) (and also to any other constraint surface \(\Sigma_2\) such that \((\Omega_2, \Sigma_2)\) generates the same null vector, see the end part of this section). Therefore, the integral curves are also integral curves of the intersection of all these surfaces \(\Sigma_0, \Sigma_1, \Sigma_2\), etc.

**result 4.** Due to the fact that the null vector \(v\) is the same for any choice and because the physical observables \(O\) for the system are those functions on \(\Sigma\) such that \(vO = 0\) then Dirac observables \(O\) are the same for any choice of the couple \((\Omega_{\text{ext}}, \Sigma)\). Moreover, in Sec. II B it has been shown that Dirac observables \(O\) are the constants of motion with respect to \(t\) of the unparametrized system. It is important to recall that to be a constant of motion \(O\) is just a property of the equations of motion, and does not depend on the choice of the Hamiltonian or of the symplectic structure. This explains why Dirac observables \(O\) are independent of the choice of the symplectic structure \(\Omega_{\text{ext}}\) on \(\Gamma_{\text{ext}}\) and of the specification of \(\Sigma\) in the sense explained above.

**Physical phase space.** The points of \(\Gamma_{\text{phys}}\) can be labelled with the independent coordinates \((x_0, y_0, p_{x0}, p_{y0})\) and the symplectic two-form on it is

\[
\omega_1 := dp_{y0} \wedge dx_0 + dp_{x0} \wedge dy_0 ,
\]

or, equivalently, the nonvanishing Poisson brackets are

\[
\{x_0, p_{y0}\} = 1, \quad \{y_0, p_{x0}\} = 1,
\]

in opposition to \(\omega_0\), given in Eq. (58), which corresponds to (68).

In summary, from (64) one has

\[
(\Gamma_{\text{ext}}, \Omega_1) \leftrightarrow (\Sigma_1, \Omega_{\Sigma_1}) \rightarrow (\Gamma_{\text{phys}}, \omega_1) ,
\]
while from \( \mathbf{[13]} \) one has

\[
(\Gamma_{\text{ext}}, \Omega_0) \xrightarrow{\varepsilon} (\Sigma_0, \Omega_{\Sigma_0}) \xrightarrow{\pi} (\Gamma_{\text{phys}}, \omega_0).
\] (74)

There are many other ways of choosing the pair \((\Omega_{\text{ext}}, \Sigma)\) or, equivalently, the pair \((\Omega_{\text{ext}}, G)\) where \(G\) is the first class constraint. We just list other two of these pairs:

1) \[
(\Gamma_{\text{ext}}, \Omega_2) \xrightarrow{\varepsilon} (\Sigma_2, \Omega_{\Sigma_2}) \xrightarrow{\pi} (\Gamma_{\text{phys}}, \omega_2),
\] (75)

with

\[
\begin{align*}
\Omega_2 & := dx \wedge dp_x - dy \wedge dp_y + dp_t \wedge dt, \\
\Omega_{\Sigma_2} & = dx \wedge dp_x - dy \wedge dp_y - dJ_2 \wedge dt \\
& = dx \wedge dp_x - dy \wedge dp_y + \omega^2 x dx \wedge dt - \omega^2 y dy \wedge dt \\
& \quad + \frac{p_x}{m} dp_x \wedge dt - \frac{p_y}{m} dp_y \wedge dt, \\
\omega_2 & = dx_0 \wedge dp_{x_0} - dy_0 \wedge dp_{y_0},
\end{align*}
\]

because of

\[
G_2 := p_t + J_2 \approx 0, \quad J_2 = \frac{\rho_x^2 - \rho_x^2}{2m} + \frac{1}{2} \omega^2 (y^2 - x^2).
\] (77)

2) \[
(\Gamma_{\text{ext}}, \Omega_3) \xrightarrow{\varepsilon} (\Sigma_3, \Omega_{\Sigma_3}) \xrightarrow{\pi} (\Gamma_{\text{phys}}, \omega_3),
\] (78)

with

\[
\begin{align*}
\Omega_3 & := m \omega^2 dx \wedge dy + \frac{1}{m \omega} dp_x \wedge dp_y + dp_t \wedge dt, \\
\Omega_{\Sigma_3} & = m \omega^2 dx \wedge dy + \frac{1}{m \omega} dp_x \wedge dp_y - dJ_3 \wedge dt \\
& = m \omega^2 dx \wedge dy + \frac{1}{m \omega} dp_x \wedge dp_y - \omega p_x dx \wedge dt \\
& \quad - \omega x dp_y \wedge dt + \omega p_x dy \wedge dt + \omega y dp_x \wedge dt, \\
\omega_3 & = m \omega^2 dx_0 \wedge dy_0 + \frac{1}{m \omega} dp_{x_0} \wedge dp_{y_0},
\end{align*}
\]

because of

\[
G_3 := p_t + J_3 \approx 0, \quad J_3 = \omega (yp_x - xp_y).
\] (80)

Note that neither \(J_2\) nor \(J_3\) are bounded from below. We have seen that the vector \(\mathbf{[69]}\) is the null vector of \(\Omega_{\Sigma_0}\) and \(\Omega_{\Sigma_1}\). In addition, the vector \(\mathbf{[69]}\) is also the null vector of \(\Omega_{\Sigma_2}\) and \(\Omega_{\Sigma_3}\). Due to the fact \((x, y, p_x, p_y, t)\) are local coordinates for \(\Sigma_\mu, \mu = 0, 1, 2, 3\), one would say that all the surfaces \(\Sigma_\mu\) are diffeomorphic.

Finally, note that even though the symplectic structures \(\Omega_1, \Omega_2, \text{ and } \Omega_3\) (after re-scaling the coordinates) have the canonical form specified by Darboux’s theorem, they are distinct to the usual canonical form \{,\} \(0\) given by Eq. \(\mathbf{[22]}\).

V. COVARIANT DESCRIPTION OF GAUGE TRANSFORMATIONS

The infinitesimal gauge transformation of the variables \((x, y, t, p_x, p_y, p_t)\) (which label the points of \(\Gamma_{\text{ext}}\)) generated by the constraint \(G_0 = p_t + H_0 \approx 0\) in \(\mathbf{[13]}\) and the symplectic structure of Eq. \(\mathbf{[22]}\) is

\[
\begin{align*}
\delta_x x & = \varepsilon \{x, G_0\}_0 = \varepsilon \frac{p_x}{m} \{x, p_x\}_0 = \varepsilon \frac{p_x}{m}, \\
\delta_x y & = \varepsilon \{y, G_0\}_0 = \varepsilon \frac{p_y}{m} \{y, p_y\}_0 = \varepsilon \frac{p_y}{m}, \\
\delta_x t & = \varepsilon \{t, G_0\}_0 = \varepsilon \{t, p_t\}_0 = \varepsilon, \\
\delta_x p_x & = \varepsilon \{p_x, G_0\}_0 = \varepsilon \omega^2 x \{p_x, x\}_0 = - \varepsilon \omega^2 x, \\
\delta_x p_y & = \varepsilon \{p_y, G_0\}_0 = \varepsilon \omega^2 y \{p_y, y\}_0 = - \varepsilon \omega^2 y, \\
\delta_x p_t & = \varepsilon \{p_t, G_0\}_0 = \varepsilon.
\end{align*}
\] (81)
On the other hand, the infinitesimal gauge transformation of the variables \((x, y, t, p_x, p_y, p_t)\) generated by the constraint \(G_1 = p_t + J_1 \approx 0\) in (51) and the symplectic structure \(\Omega_1\) of Eq. (55) is

\[
\begin{align*}
\delta \varepsilon x &= \varepsilon \{x, G_1\}_1 = \varepsilon \frac{p_x}{m} \{x, p_y\}_1 = \varepsilon \frac{p_x}{m}, \\
\delta \varepsilon y &= \varepsilon \{y, G_1\}_1 = \varepsilon \frac{p_y}{m} \{y, p_x\}_1 = \varepsilon \frac{p_y}{m}, \\
\delta \varepsilon t &= \varepsilon \{t, G_1\}_1 = \varepsilon \{t, p_t\}_1 = \varepsilon, \\
\delta \varepsilon p_x &= \varepsilon \{p_x, G_1\}_1 = \varepsilon m \omega^2 x \{p_x, y\}_1 = -\varepsilon m \omega^2 x, \\
\delta \varepsilon p_y &= \varepsilon \{p_y, G_1\}_1 = \varepsilon m \omega^2 y \{p_y, x\}_1 = -\varepsilon m \omega^2 y, \\
\delta \varepsilon p_t &= \varepsilon \{p_t, G_1\}_1 = 0. \quad (82)
\end{align*}
\]

In the same way, the infinitesimal gauge transformation generated by the constraint \(G_2 = p_t + J_2 \approx 0\) and the symplectic structure \(\Omega_2\) of Eq. (56) is

\[
\begin{align*}
\delta \varepsilon x &= \varepsilon \{x, G_2\}_2 = -\varepsilon \frac{p_x}{m} \{x, p_x\}_2 = \varepsilon \frac{p_x}{m}, \\
\delta \varepsilon y &= \varepsilon \{y, G_2\}_2 = \varepsilon \frac{p_y}{m} \{y, p_y\}_2 = \varepsilon \frac{p_y}{m}, \\
\delta \varepsilon t &= \varepsilon \{t, G_2\}_2 = \varepsilon \{t, p_t\}_2 = \varepsilon, \\
\delta \varepsilon p_x &= \varepsilon \{p_x, G_2\}_2 = -\varepsilon m \omega^2 x \{p_x, x\}_2 = -\varepsilon m \omega^2 x, \\
\delta \varepsilon p_y &= \varepsilon \{p_y, G_2\}_2 = \varepsilon m \omega^2 y \{p_y, y\}_2 = -\varepsilon m \omega^2 y, \\
\delta \varepsilon p_t &= \varepsilon \{p_t, G_2\}_2 = 0. \quad (83)
\end{align*}
\]

Finally, the infinitesimal gauge transformation generated by the constraint \(G_3 = p_t + J_3 \approx 0\) and the symplectic structure \(\Omega_3\) of Eq. (59) is

\[
\begin{align*}
\delta \varepsilon x &= \varepsilon \{x, G_3\}_3 = -\varepsilon \omega p_x \{x, y\}_3 = -\varepsilon \frac{p_x}{m}, \\
\delta \varepsilon y &= \varepsilon \{y, G_3\}_3 = \varepsilon \omega p_y \{y, x\}_3 = \varepsilon \frac{p_y}{m}, \\
\delta \varepsilon t &= \varepsilon \{t, G_3\}_3 = \varepsilon \{t, p_t\}_3 = \varepsilon, \\
\delta \varepsilon p_x &= \varepsilon \{p_x, G_3\}_3 = \varepsilon \omega x \{p_x, p_y\}_3 = -\varepsilon \omega \omega^2 x, \\
\delta \varepsilon p_y &= \varepsilon \{p_y, G_3\}_3 = -\varepsilon \omega y \{p_y, p_x\}_3 = -\varepsilon \omega \omega^2 y, \\
\delta \varepsilon p_t &= \varepsilon \{p_t, G_3\}_3 = 0. \quad (84)
\end{align*}
\]

Note that the right hand side of Eqs. (51), (52), (53), and (54) are the same. We have got the following:

**result 5.** Equations (51), (52), (53), and (54) mean that the gauge transformation of the variables that label the points of \(\Gamma_{ext}\) is independent of the choice of the symplectic structure \(\Omega_{ext}\) on \(\Gamma_{ext}\) and of the form of specifying \(\Sigma\) in the sense explained above. There is an easy way of understanding the cause of this phenomenon. Assuming, for the moment, that there exists just a single first class constraint \(G_0 \approx 0\) (which is the case considered so far in this paper) then the Hamiltonian formalism *a la Dirac* [11] says that the gauge transformation on any function \(F\) generated by \(G_0 \approx 0\) is

\[
\delta \varepsilon F = \varepsilon \{F, G_0\}_0. \quad (85)
\]

However, the right-hand side of Eq. (51) can, instead of using the pair \((\{\cdot, \cdot\}_0, G_0)\) in Eq. (55), be obtained from a new, different, pair \((\{\cdot, \cdot\}_{new}, G_{new})\)

\[
\delta \varepsilon F = \varepsilon \{F, G_{new}\}_{new}. \quad (86)
\]

i.e., there exists an ambiguity, a freedom, in the choice of the symplectic structure (or, equivalently, the Poisson brackets) and in the form that the first class constraint is specified \((G_0 \approx 0\) or \(G_{new} \approx 0\)) in such a way that any pair (of these choices) generates the same gauge transformation on \(F\) [5]. In Dirac’s approach one uses the usual canonical

---

5 Note that this ambiguity is not of the same kind than the one that it is involved in the Abelianization of constraints [21].
form of the symplectic structure (equivalently, the canonical form of the Poisson brackets, \{ , \}_0). However, we have seen that it is not mandatory to choose \{ , \}_0 on \Gamma_{\text{ext}} and that other choices are allowed, the only restriction on these choices is that they generate the same gauge transformation by choosing the appropriate form for the first class constraint.

**result 6.** Using the result 5 or, equivalently, Eqs. \[31\] and \[32\] one has that any gauge-invariant function \(O\) on \(\Gamma_{\text{ext}}\) (under the gauge transformation) has the same functional form independent of the choice of the couple \((\Omega_0, G_0)\), \((\Omega_1, G_1)\), \((\Omega_2, G_2)\), and \((\Omega_3, G_3)\) (and also of any other allowed choices).

**VI. THE ‘ALGEBRA’ OF GAUGE-INVARIANT FUNCTIONS ON \(\Gamma_{\text{ext}}\) DEPENDS ON THE CHOICE OF \(\Omega_{\text{ext}}\) AND \(\Sigma\)**

**result 7.** Even though gauge-invariant functions \(O : \Gamma_{\text{ext}} \to R\) have the same functional form independently of the choice of the couple \((\Omega_0, G_0)\) or \((\Omega_1, G_1)\) (or any other allowed choice), the Poisson brackets among the gauge-invariant functions \(O\) on \(\Gamma_{\text{ext}}\) might or might not, in the generic case, form a Lie algebra simply because the Lie algebra directly depends on the choice of the symplectic structure on \(\Gamma_{\text{ext}}\).

1) **su(2) algebra of observables.** The gauge-invariant functions on \(\Gamma_{\text{ext}}\)

\[
\begin{align*}
J_1 &= \frac{p_x p_y}{m} + m \omega^2 x y, \\
J_2 &= \frac{p_x^2 - p_y^2}{2m} + \frac{1}{2} m \omega^2 (y^2 - x^2), \\
J_3 &= \omega (xp_y - y p_x),
\end{align*}
\]

(87)

satisfy \[19\]

\[
\{J_i, J_j\}_0 = 2 \omega \epsilon_{ijk} J_k,
\]

(88)

with respect to the usual symplectic structure \(\Omega_0\) on \(\Gamma_{\text{ext}}\). It is clear that \[18\] is a Lie algebra isomorphic to the \(\text{su}(2)\) (and \(\text{so}(3)\)) algebra.

2) **su(1, 1) algebra of observables.** On the other hand, with respect to the symplectic structure \(\Omega_1\) the gauge-invariant functions \[37\] satisfy

\[
\begin{align*}
\{J_1, J_2\}_1 &= 0, \\
\{J_1, J_3\}_1 &= 0, \\
\{J_2, J_3\}_1 &= -2 \omega H_0,
\end{align*}
\]

(89)

which means that the gauge-invariant functions \[37\] do not form a Lie algebra with respect to the symplectic structure \(\Omega_1\). Nevertheless, the set of gauge-invariant functions \(\{H_0, J_2, J_3\}\) satisfy \[19\]

\[
\begin{align*}
\{H_0, J_2\}_1 &= 2 \omega J_3, \\
\{H_0, J_3\}_1 &= -2 \omega J_2, \\
\{J_2, J_3\}_1 &= -2 \omega H_0,
\end{align*}
\]

(90)

with respect to the symplectic structure \(\Omega_1\) of Eq. \[35\], which means that \(\{H_0, J_2, J_3\}\) generate an algebra isomorphic to \(\text{su}(1, 1)\).\[10\]

Note also that even though \(H_0\) and \(J_3\) are in involution with respect to the symplectic structure \[32\], they are not in involution with respect to the symplectic structure \[35\] because of the second line of Eq. \[36\].

Finally, note also that

\[
J_1^2 + J_2^2 + J_3^2 = H_0^2,
\]

(92)

holds independently of the choice of the symplectic structure, i.e., it does not depend on \[32\] or \[35\]. However, the gauge-invariant function that plays the role of Casimir directly depends on the symplectic structure chosen.
VII. SYSTEMS WITH FIRST CLASS CONSTRAINTS ONLY

The results of the previous sections for parametrized nonrelativistic Hamiltonian systems can be generalized to any generally covariant system with first class constraints only. In the Hamiltonian formalism a la Dirac one has

\[ \frac{dx^\mu}{d\tau} = \{x^\mu, \lambda^a \gamma_a\}_0, \]

\[ \gamma_a \approx 0, \]

\[ \{\gamma_a, \gamma_b\}_0 = C_{ab}^c \gamma_c, \]

where \((x^\mu) = (q^i, p_i), \{f, g\}_0 = \Omega^\mu_0 \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\mu}\), and \(\{x^\mu, x^\nu\}_0 = \Omega^\mu_0\) (which has the usual canonical form). If one does not want to use this symplectic structure then the system \((93)\) can be replaced with

\[ \frac{dx^\mu}{d\tau} = \{x^\mu, \lambda^a G_a\}, \]

\[ G_a \approx 0, \]

\[ \{G_a, G_b\} = D_{ab}^c G_c, \]

with \(\{f, g\} = \Omega^\mu \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\mu\}\) and \(\{x^\mu, x^\nu\} = \Omega^\mu\nu\) provided that the gauge transformation of any function \(F : \Gamma_{ext} \rightarrow R\) is the same for both cases

\[ \delta_x F = \varepsilon^a \{F, \gamma_a\}_0 = \varepsilon^a \{F, G_a\}. \]

(95)

Note that, by construction, the \textit{evolution} in \(\tau\) of \(x^\mu\) is \textit{not} modified by the choice of \((93)\) or \((94)\) simply because \(\tau\)-evolution is a gauge transformation, i.e., the explicit form of this transformation is the same for both cases. In fact, from \((93)\)

\[ x^\mu(\tau + d\tau) = x^\mu(\tau) + d\tau \frac{dx^\mu}{d\tau} = x^\mu(\tau) + \varepsilon^a \{x^\mu, \gamma_a\}_0 = x^\mu(\tau) + \varepsilon^a \{x^\mu, G_a\}, \]

with \(\varepsilon^a = \lambda^a dt\) and last equality follows from Eq. \((94)\).

In some particular cases, the \(\gamma\)'s in \((93)\) will form a Lie algebra. However, it might be possible that the \(G\)'s in \((94)\) form a Lie algebra, \textit{distinct} to that of the \(\gamma\)'s or even that the \(G\)'s do not form a Lie algebra. In addition, suppose that there exists a set of observables (same for both cases) which form a Lie algebra with respect to the Poisson brackets of Eq. \((93)\). They might or might not form a Lie algebra with respect to the Poisson brackets of Eq. \((94)\). Moreover, it might or not be possible to find a set of observables that form a Lie algebra with respect to the Poisson brackets of Eq. \((94)\).

VIII. CONCLUDING REMARKS

The covariant canonical formalism applied to parametrized nonrelativistic Hamiltonian systems clearly displays the various geometrical structures involved in their dynamics. In particular, the reduced phase space is reached by using Dirac’s observables which are the constants of motion with respect to \(t\) of the standard (i.e., non-parametrized) Hamiltonian system. In contrast to what happens in Dirac’s method, in the covariant canonical formalism there is not need to choose a gauge condition to get the reduced phase space. In spite of using the techniques of the covariant canonical formalism to analyze the geometry of parametrized nonrelativistic Hamiltonian systems, the usual symplectic structure was used.

To avoid the use of the usual symplectic structure in the extended phase space \(\Gamma_{ext}\), it was explored what changes in Dirac’s canonical formalism if alternative symplectic structures are chosen. It was shown that there exists the freedom to choose the symplectic structure on the extended phase space if the equation that defines the constraint surface is, in the generic case, accordingly modified in such a way that the gauge transformation is not altered. Moreover, due to the fact that the null vectors are the same for any choice of the pair \((\Omega_{ext}, G)\) where \(\Omega_{ext}\) is the symplectic structure on \(\Gamma_{ext}\) and \(G \approx 0\) defines the constraint surface \(\Sigma\) then the reduced phase space \(\Gamma_{phys}\) is also not modified, what changes is the symplectic structure \(\Omega_{phys}\) on \(\Gamma_{phys}\) which depends on the pair \((\Omega_{ext}, G)\) chosen. The generalization of these results to any generally covariant systems with a finite number of first class constraints was also discussed.

Finally, due to the fact that the canonical analysis is a first step towards canonical quantization and because it was seen how the algebra of observables directly depends on the symplectic structure chosen in the extended phase space then it would interesting to know how this phenomenon (i.e., the fact of choosing distinct symplectic structures and distinct ways of expressing the constraints surface) is handled, for instance, in Dirac’s quantization as well as in the framework of algebraic and/or refined algebraic quantizations which heavily depend on the algebra of observables.
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