Marsden-Weinstein reduction
for
symplectic connections

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Abstract. We propose a reduction procedure for symplectic connections with symmetry. This is applied to coadjoint orbits whose isotropy is reductive.

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The aim of this paper is to show that under very mild conditions, Marsden-Weinstein reduction is “compatible” with a symplectic connection. This means that if a symplectic manifold \((M, \omega)\) is endowed with a strongly Hamiltonian action of a connected Lie group \(G\) and with a \(G\)-invariant symplectic connection \(\nabla\), there is a natural way to construct a symplectic connection \(\nabla^r\) on a reduced manifold \((M^r, \omega^r)\). The construction always works when \(G\) is compact, and in many non-compact cases as well.

The interest of the construction if two-fold. First it leads to interesting examples of symplectic connections when \((M, \omega)\) is a very simple symplectic manifold and \(G\) is, for example, one-dimensional or multidimensional but abelian (see [2]). Secondly, it may be a useful tool in dealing with the general problem of commutation of quantization and reduction in the framework of deformation quantization.

The paper is organized as follows. We first recall some classical results about strongly Hamiltonian actions. In the second paragraph we show how to construct a reduced connection with a technical assumption and we prove that this is always possible in the compact case. The third paragraph collects several examples where this construction gives interesting results. We finally indicate some possible further developments.

Let \((M, \omega)\) be a symplectic manifold and let \(\sigma: G \times M \to M\) be a strongly Hamiltonian action of a connected Lie group \(G\), \((g, x) \mapsto g \cdot x\), which we will assume to be effective. If \(\mathfrak{g}\) is the Lie algebra of \(G\), we denote by \(J: M \to \mathfrak{g}^*\) the corresponding \(G\)-equivariant momentum map:

\[
i(X^*)\omega = d(J^*X), \ \forall X \in \mathfrak{g}
\] (1)

where \(X^*\) is the infinitesimal generator of the action corresponding to \(X\):

\[
X^*_x = \frac{d}{dt}\exp(-tX) \cdot x \bigg|_{t=0}
\] (2)

and \(J^*: \mathfrak{g} \subset C^\infty(\mathfrak{g}^*) \to C^\infty(M)\) the map defined by

\[
(J^*X)(x) = \langle J(x), X \rangle, \ \forall x \in M.
\] (3)
Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$ and let $\Sigma_\mu = J^{-1}(\mu)$ be the constraint manifold; it is a closed embedded submanifold of $M$.

The following two lemmas are classical [1] and presented for sake of completeness.

1 Lemma. In the neighborhood of $\Sigma_\mu$, the action of $G$ is locally free, i.e. for any $x \in \Sigma_\mu$, there exists a neighborhood $\Omega_x$ of the identity element $e$ of $G$ and a neighborhood $U_x$ of $x$ in $M$ such that for any $g \in \Omega_x$, $y \in U_x$, the equation $g \cdot y = y$ implies $g = e$.

Proof. Let $x \in \Sigma_\mu$. The map $J_*: T_xM \to T_\mu \mathfrak{g}^* \cong \mathfrak{g}^*$ is surjective; hence the map $(J_*)^*: (\mathfrak{g}^*)^* \cong \mathfrak{g} \to T_x^*M$ is injective, i.e. $\forall X \in \mathfrak{g}$, $X \neq 0$, one has:

$$(J_*)^*(X) = (dJ^*X)_x = i(X^*_x)\omega_x \neq 0;$$

hence $X_x^* \neq 0$. This means that the stabilizer $G_x$ of $x$ is discrete. Let $\chi: G \times M \to M \times M$ be the map $(g, y) \mapsto (g \cdot y, y)$. By the above $\chi_*|_{\Omega_x} \times U_x$ is injective. □

Let $G_\mu$ be the stabilizer of $\mu$ under the coadjoint action.

2 Lemma.

(i) Let $x \in \Sigma_\mu$ and denote by $\mathcal{O}_x$ the orbit of $x$ under the action of $G$. Then $(T_x\Sigma_\mu)^\perp = T_x\mathcal{O}_x$ (where $\perp$ means orthogonal with respect to $\omega_x$).

(ii) Let $\Delta_x = (T_x\Sigma_\mu)^\perp \cap T_x\Sigma_\mu$; then $\Delta_x$ has constant dimension (independent of $x$) and the orbit of $x$ under the action of $G_\mu$ is an integral manifold of $\Delta$.

Proof.

(i) For $Z \in T_xM$, we have:

$$J_*Z = 0 \Leftrightarrow Z \in T_x\Sigma_\mu \Leftrightarrow \omega_x(X^*_x, Z) = \langle J_*Z, X \rangle = 0, \forall X \in \mathfrak{g}.$$ 

Consequently $T_x\Sigma_\mu \subset (T_x\mathcal{O}_x)^\perp$. But $\dim \Sigma_\mu = \dim M - \dim G = \dim \mathcal{O}_x$ (by Lemma 1). Hence $T_x\Sigma_\mu = (T_x\mathcal{O}_x)^\perp$.

(ii) If $Z \in T_x\Sigma_\mu$, $\omega_x(Z, X^*) = -\langle J_*Z, X \rangle = 0, \forall X \in \mathfrak{g}$; if $Z \in (T_x\Sigma_\mu)^\perp$, there exists $Y \in \mathfrak{g}$ such that $Z = Y^*$; then $Y \in \mathfrak{g}_\mu$, by equivariance of $J$, where $\mathfrak{g}_\mu$ is the
Lie algebra of $G_{\mu}$. Hence, $\dim \Delta_x \leq \dim g_{\mu}$. But (i) implies that $\dim g_{\mu} \leq \dim \Delta_x$. From the above $\Delta_x$ is both the radical of $\omega|_{T_x \Sigma_{\mu} \times T_x \Sigma_{\mu}}$ and the tangent space to the orbit of $G_{\mu}$ passing through $x$. ■

**Assumption 1.** The constraint manifold $\Sigma_{\mu}$ is a $G_{\mu}$-principal bundle over the reduced manifold $M^r = G_{\mu} \backslash \Sigma_{\mu}$.

**Remark.** If the action of $G$ on $M$ is free and proper, Assumption 1 is satisfied; in particular this is true if the action is free and the group $G$ is compact.

The restriction to the constraint submanifold $\Sigma_{\mu}$ of the tangent bundle $TM$, denoted $TM|_{\Sigma_{\mu}}$ is a vector bundle over $\Sigma_{\mu}$; the group $G_{\mu}$ acts by automorphisms on this bundle. It contains four $G_{\mu}$-stable vector subbundles, $T\Sigma_{\mu}$, $(T\Sigma_{\mu})^\perp$, $T\Sigma_{\mu} + (T\Sigma_{\mu})^\perp$ and $T\Sigma_{\mu} \cap (T\Sigma_{\mu})^\perp$.

**Assumption 2.** There exists a $G_{\mu}$-stable vector subbundle $\tilde{S}$ of $TM|_{\Sigma_{\mu}}$ such that:

$$TM|_{\Sigma_{\mu}} = (T\Sigma_{\mu} + (T\Sigma_{\mu})^\perp) \oplus \tilde{S}. $$

**Remark.** If the group $G$ is compact, such a vector subbundle always exists. Indeed, we can build a $G_{\mu}$-invariant metric on $TM|_{\Sigma_{\mu}}$ and choose $\tilde{S}$ to be the orthogonal complement, relative to this metric, of $T\Sigma_{\mu} + (T\Sigma_{\mu})^\perp$.

**3 Lemma.** One may assume that $\tilde{S}$ is isotropic (relative to $\omega$).

**Proof.** By dimension argument, $\dim \tilde{S} = \dim (T\Sigma_{\mu} \cap (T\Sigma_{\mu})^\perp)$ and $\omega$ induces a non-singular pairing between these two $G_{\mu}$-invariant subbundles. Let $x \in \Sigma_{\mu}$ and let $V_x$ be the symplectic subspace of $T_x M$ defined by:

$$V_x = \tilde{S}_x \oplus \Delta_x.$$

There is a unique linear map $L_x: \tilde{S}_x \rightarrow \Delta_x$ such that, $\forall u, v \in \tilde{S}_x$,

$$\omega_x(L_x u, v) = \omega_x(u, L_x v),$$

$$\omega_x(L_x u, v) + \omega_x(u, L_x v) = -\omega_x(u, v).$$
The graph of \( L_x \) in \( V_x \) is an isotropic subspace \( S_x \) of \( V_x \) such that

\[
V_x = S_x \oplus \Delta_x.
\]

Let \( g \in G \); then

\[
0 = \omega_x(L_x u, v) - \omega_x(u, L_x v) = (g^* \omega)_x(L_x u, v) - (g^* \omega)_x(u, L_x v)
\]

\[
= \omega_{g \cdot x}(g_* L_x u, g_* v) - \omega_{g \cdot x}(g_* u, g_* L_x v)
\]

\[
-\omega_x(u, v) = -(g^* \omega)_x(u, v) = -\omega_{g \cdot x}(g_* u, g_* v)
\]

\[
= \omega_x(L_x u, v) + \omega_x(u, L_x v) = (g^* \omega)_x(L_x u, v) + (g^* \omega)_x(u, L_x v)
\]

\[
= \omega_{g \cdot x}(g_* L_x u, g_* v) + \omega_{g \cdot x}(g_* u, g_* L_x v).
\]

By unicity, \( L_{g \cdot x} = g_* \circ L_x \circ g^{-1} \) and hence the subbundle \( S \) is \( G_\mu \)-stable. ■

**Remark.** By dimension argument:

\[
(S \oplus \Delta)^\perp = ((S \oplus \Delta)^\perp \cap T\Sigma) \oplus ((S \oplus \Delta)^\perp \cap T\Sigma^\perp)
\]

\[
\not= W_1 \oplus W_2
\]

and the two subbundles \( W_1 \) and \( W_2 \) are \( G_\mu \)-stable.

2. We consider the situation where one has a symplectic manifold \((M, \omega)\), a Hamiltonian action \( \sigma: G \times M \to M \) of a connected Lie group \( G \) and a symplectic connection \( \nabla \) which is \( G \)-invariant.

**4 Lemma.** If the group \( G \) is compact such a connection always exist.

**Proof.** Let \( \nabla \) be any symplectic connection and let \( X, Y \) be smooth vector fields on \( M \). Define:

\[
(\nabla_X Y)_x = \int_G \left[(g \cdot \nabla)X \right]_x dg = \int_G \left(g_*^{-1} \nabla g_* X g_* Y\right)(x)dg.
\]

One checks that \( \nabla \) is a torsion free linear connection. Furthermore:

\[
\omega_x(\nabla_X Y, Z) + \omega_x(Y, \nabla_X Z) =
\]
\[
\begin{align*}
&= \int_G \left[ \omega_x \left( g_x^{-1} \tilde{\nabla}_x^* g_* Y, Z \right) + \omega_x \left( Y, g_x^{-1} \tilde{\nabla}_x^* g_* Z \right) \right] \, dg \\
&= \int_G \left[ \omega_{g^*x} \left( \tilde{\nabla}_x^* g_* Y, g_* Z \right) + \omega_{g^*x} \left( g_* Y, \tilde{\nabla}_x^* g_* Z \right) \right] \, dg \\
&= \int_G (g_* X)_{g^*x} \omega(g_* Y, g_* Z) \, dg = \int_G X_x \omega(Y, Z) \, dg \\
&= X_x \omega(Y, Z),
\end{align*}
\]

if the Haar measure \( dg \) is properly normalized. \( \blacksquare \)

If Assumptions 1 and 2 are satisfied, \( \Sigma_\mu \) (the constraint manifold) is a \( G_\mu \)-principal bundle over the reduced manifold \( M^r \):

\[
\pi: \Sigma_\mu \to M^r.
\]

Furthermore, at a point \( x \in \Sigma_\mu \), the tangent space \( T_x \Sigma_\mu \) is the direct sum of two \( G_\mu \)-invariant distributions:

\[
T_x \Sigma_\mu = \Delta_x \oplus (W_1)_x
\]

where \( \Delta_x = \ker \pi_x = \text{rad}^\omega (T_x \Sigma_\mu) \). The distribution \( W_1 \) will be called the horizontal distribution. To \( W_1 \) is canonically associated a connection 1-form \( \alpha \) on \( \Sigma_\mu \) (with values in \( \mathfrak{g}_\mu \)):

\[
\alpha(U) = X,
\]

if \( U = \delta + w_1 \) with \( \delta_x = (d/dt) \exp(-tX) \cdot x \bigg|_{t=0} = X_x^* \). Observe that in this framework

\[
T_x M = \Delta_x \oplus (W_1)_x \oplus (W_2)_x \oplus S_x.
\]

Hence we have a projection operator \( P_x: T_x M \to T_x \Sigma_\mu \).

5 Definition. If \( X, Y \) are smooth vector fields, along \( \Sigma_\mu \), tangent at each point to \( \Sigma_\mu \), we define a linear connection \( \nabla \) along \( \Sigma_\mu \), by:

\[
\nabla_X Y = P(\tilde{\nabla}_X Y). \tag{4}
\]

6 Lemma. \( \nabla \) is a torsion free linear connection on \( \Sigma_\mu \). Furthermore, \( G_\mu \) is a group of affine transformations of \( \nabla \).
Proof. One has for \( f \in C^\infty(\Sigma_\mu)\):

\[
[\nabla_X (fY)]_x = P(\tilde{\nabla}_X fY)_x = P((Xf)Y + f\tilde{\nabla}_X Y)_x = (X_x f)_x Y_x + f(x)(\nabla_X Y)_x
\]

\[
\nabla_X Y - \nabla_Y X - [X,Y] = P(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]) = 0.
\]

Also, if \( Z \in \mathfrak{g}_\mu \):

\[
(\mathcal{L}_{Z^*} \nabla)_X Y = [Z^*, \nabla_X Y] - \nabla_{[Z^*, X]} Y - \nabla_X [Z^*, Y]
\]

\[
= [Z^*, P\tilde{\nabla}_X Y] - P\tilde{\nabla}_{[Z^*, X]} Y - P\tilde{\nabla}_X [Z^*, Y]
\]

\[
= P \left( [Z^*, \tilde{\nabla}_X Y] - \tilde{\nabla}_{[Z^*, X]} Y - \tilde{\nabla}_X [Z^*, Y] \right)
\]

using the \( G_\mu \)-invariance of \( P \). Hence the conclusion as \( \tilde{\nabla} \) is \( G_\mu \)-invariant.

7 Lemma. The orbits of \( G_\mu \) in \( \Sigma_\mu \) are totally geodesic with respect to \( \nabla \) if and only if for all \( X,Y \in \mathfrak{g}_\mu \) and for all vector fields \( Z \) on \( M \), one has:

\[
\omega(P\tilde{\nabla}_X Y^*, PZ) = 0.
\]

Proof. The totally geodesic condition means that \( (\nabla_X Y^*)(x) \) belongs to \( \Delta_x \) which is the radical of \( T_x \Sigma_\mu \).

8 Definition. The reduced connection \( \nabla^r \) on \( M^r \) is defined as follows. Let \( X,Y \) be smooth vector fields on \( M^r \); denote by \( \tilde{X}, \tilde{Y} \) their horizontal lifts to \( \Sigma_\mu \). Then:

\[
(\nabla^r_X Y)(x) = (\nabla_X \tilde{Y})(x) - [\alpha_x(\nabla_X \tilde{Y})]^*.
\]

9 Proposition. Formula (5) defines a torsion free linear connection on \( M^r \). Furthermore, if \( \omega^r \) is the 2-form on \( M^r \) such that

\[
\omega^r_{\pi(x)}(X,Y) = \omega_x(\tilde{X}, \tilde{Y}),
\]

then \( \omega^r \) is symplectic and parallel relative to \( \nabla^r \).
Proof. Formula (5) defines a linear connection on $\mathcal{M}$. Indeed, one has, if $g \in G$:

$$\nabla_{\bar{X}}\bar{Y}|_{x\cdot g} - \left[ \alpha_{x\cdot g}(\nabla_{\bar{X}}\bar{Y}) \right] = \nabla_{\tilde{R}_gX}\tilde{R}_g\bar{Y}|_{x\cdot g} - \tilde{R}_g*(\text{Ad}(g)\alpha_x(\nabla_{\bar{X}}\bar{Y}))^*$$

$$= \tilde{R}_g*[\nabla_{\bar{X}}\bar{Y}|_x - \alpha_x(\nabla_{\bar{X}}\bar{Y})]^*,$$

where $\tilde{R}_g$ is the right action which corresponds to $\sigma$: $\tilde{R}_g(x) = \sigma(g^{-1}, x)$. Thus formula (5) is independent of the choice of $x$ in the fibre over $\pi(x)$. Also:

$$\nabla_{\bar{X}}\bar{Y} - \nabla_{\bar{Y}}X - [X, Y] = \nabla_{\bar{X}}\bar{Y} - \alpha_x(\nabla_{\bar{X}}\bar{Y})^* - \nabla_{\bar{Y}}X + \alpha_x(\nabla_{\bar{Y}}X)^* - [X, Y]$$

$$= [\bar{X}, \bar{Y}] - \alpha_x([\bar{X}, \bar{Y}])^* - [\bar{X}, \bar{Y}] = 0$$

and $\nabla^r$ is torsion free.

The 2-form $\omega^r$ has constant rank; furthermore, if $S$ denotes the cyclic sum, we have:

$$\left( d\omega^r \right)_{\pi(x)}(X, Y, Z) = S_{X, Y, Z} \left[ X_{\pi(x)}\omega^r(Y, Z) - \omega^r_{\pi(x)}([X, Y], Z) \right]$$

$$= S_{X, Y, Z} \left[ X_x\omega(Y, Z) - \omega_x([X, Y] - \alpha_x([X, Y])^*, Z) \right]$$

$$= (d\omega)_x(\bar{X}, \bar{Y}, \bar{Z}),$$

hence $\omega^r$ is closed. Finally:

$$X_{\pi(x)}\omega^r(Y, Z) = \tilde{X}_x\omega(Y, Z) = \omega_x(\tilde{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}) + \omega_x(\bar{Y}, \tilde{\nabla}_{\bar{X}}\bar{Z})$$

$$= \omega_x(\nabla_{\bar{X}}\bar{Y}, \bar{Z}) + \omega_x(\bar{Y}, \nabla_{\bar{X}}\bar{Z})$$

$$= \omega^r_{\pi(x)}(\nabla_{\bar{X}}\bar{Y}, Z) + \omega^r(Y, \nabla_{\bar{X}}\bar{Z}),$$

which proves that $\nabla^r$ is symplectic.

Formula for the curvature of the reduced connection. Let $X, Y, Z$ be
vector fields on $M^r$. Then:

\[
R^r(X,Y)Z = \left( \nabla^r_X \nabla^r_Y - \nabla^r_Y \nabla^r_X - \nabla^r_{[X,Y]} \right) Z
\]

\[
= \nabla_X (\nabla_Y Z) - \alpha (\nabla_X (\nabla_Y Z))^* - \nabla_Y (\nabla_X Z) + \alpha (\nabla_Y (\nabla_X Z))^*
\]

\[
- \nabla_{[X,Y]} Z + \alpha \left( \nabla_{[X,Y]} Z \right)^*
\]

\[
= \nabla_X (\nabla_Y Z - \alpha (\nabla_Y Z)^*) - \alpha \left( \nabla_X (\nabla_Y Z - \alpha (\nabla_Y Z)^*) \right)^*
\]

\[
- \nabla_Y (\nabla_X Z - \alpha (\nabla_X Z)^*) + \alpha \left( \nabla_Y (\nabla_X Z - \alpha (\nabla_X Z)^*) \right)^*
\]

\[
- \nabla_{[X,Y]} - \alpha ([\alpha, [X,Y]] \cdot Z) + \alpha \left( \nabla_{[X,Y]} - \alpha ([\alpha, [X,Y]] \cdot Z) \right)^*
\]

\[
= R(\tilde{X}, \tilde{Y})Z - \alpha (R(\tilde{X}, \tilde{Y}) \tilde{Z})^* - \nabla_X \alpha (\nabla_Y \tilde{Z})^* + \alpha \left( \nabla_X \alpha (\nabla_Y \tilde{Z})^* \right)^*
\]

\[
+ \nabla_Y \alpha (\nabla_X \tilde{Z})^* - \alpha \left( \nabla_Y \alpha (\nabla_X \tilde{Z})^* \right)^* + \nabla_{\alpha ([\alpha, [X,Y]])} \tilde{Z}
\]

\[
- \alpha \left( \nabla_{\alpha ([\alpha, [X,Y]])} \right)^* \tilde{Z}
\]

In the special case where $\Sigma_\mu$ is autoparallel with respect to the connection $\hat{\nabla}$, we have $\nabla_X Y = \hat{\nabla}_X Y$ for all vector fields $X, Y$ tangent to $\Sigma_\mu$ and the vertical subbundle in $\Sigma_\mu$ (which coincides with the radical of $\omega |_{\Sigma_\mu}$) is preserved by the connection $\nabla$. Furthermore, the reduced connection $\nabla^r$ does not depend on the choice of $S$. Indeed, for another subbundle $\hat{S}$ with the same properties as $S$, we have another horizontal distribution $\hat{W}_1$; if $X$ is a vector field on $M^r$, $\hat{X}$ and $\tilde{X}$ its horizontal lifts with respect to $W_1$ and $\hat{W}_1$, and $\hat{\alpha}$ the connection 1-form defining $\hat{W}_1$, then $\hat{X} = \tilde{X} + \alpha(\tilde{X}) = \tilde{X} - \hat{\alpha}(\tilde{X})$. If $\nabla^\hat{r}$ is the reduced connection defined by (5) for the connection $\hat{\alpha}$, then one easily sees that $\nabla^\hat{r}_X Y = \nabla_X^\hat{r} Y - \hat{\alpha} (\nabla^\hat{r}_X Y) = \nabla_X Y$, which simply means that $\nabla^r$ and $\nabla^\hat{r}$ coincide. The reduction of the symplectic connection when $\Sigma_\mu$ is autoparallel is natural and can be performed without the machinery we introduce here (see [5] for more details).

3. Coadjoint orbits are standard examples of reduced symplectic manifolds [1] [4]. Let $p : T^*G \to G$ be the cotangent bundle to a connected Lie group $G$; it can be identified, as manifold, to the direct product $G \times \mathfrak{g}^*$ by:

$$\phi : T^*G \to G \times \mathfrak{g}^*, \quad a \mapsto (g, L^*_g a), \quad g = p(a),$$
where $\mathfrak{g}$ is the Lie algebra of $G$. The left translation by $g_1$ of $G$, lifts to $T^*G$ and can be read by the above identification, as:

$$L(g_1): G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*, \ (g, \xi) \mapsto (g_1 g, \xi).$$

Similarly, the right translation by $g_1$ reads:

$$R(g_1): G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*, \ (g, \xi) \mapsto (g g_1, \text{Coad}(g_1^{-1}) \xi).$$

The Liouville 1-form $\theta$ on $T^*G$, reads on $G \times \mathfrak{g}^*$:

$$\left((\phi^{-1})^* \theta\right)_{(g, \xi)} (L_{g*} X + \eta) = (\not{\bar{\theta}}(g, \xi))(L_{g*} X + \eta) = \xi(X),$$

for $X \in \mathfrak{g}$, $\eta \in \mathfrak{g}^*$. This gives the symplectic form

$$\omega_{(g, \xi)}(L_{g*} X + \eta, L_{g*} X' + \eta') = \langle \eta, X' \rangle - \langle \eta', X \rangle - \langle \xi, [X, X'] \rangle.$$

The fundamental vector field corresponding to the left action is

$$X^l(g, \xi) = -R_{g*} X.$$

Similarly, the fundamental vector field corresponding to the right action is

$$X^r(g, \xi) = L_{g*} X + \xi \circ \text{ad}(X).$$

From this one deduces the expression of the left (resp. right) momentum maps:

$$J^l(g, \xi) = \text{Coad}(g) \xi$$

$$J^r(g, \xi) = \xi.$$

If $\mu \in \mathfrak{g}^*$ one constructs a constraint submanifold $\Sigma^l_\mu$ (resp. $\Sigma^r_\mu$) corresponding to the left (resp. right) action:

$$\Sigma^l_\mu = \{(g, \text{Coad}(g^{-1}) \mu) \mid g \in G\}$$

$$\Sigma^r_\mu = \{(g, \mu) \mid g \in G\}.$$
Let us consider the constraint manifold corresponding to the right action:

\[ T_{(g,\mu)}\Sigma^r = \{ L_g X \mid X \in \mathfrak{g} \} \]

\[ (T_{(g,\mu)}\Sigma^r)^\perp = \{ X^r(g, \xi) \mid X \in \mathfrak{g} \} \]

\[ (T\Sigma^r \cap (T\Sigma^r)^\perp)_{(g,\mu)} = \{ Y \in \mathfrak{g} \mid \mu \circ \text{ad}(Y) = 0 \} \cong \mathfrak{g}_\mu, \]

where \( \mathfrak{g}_\mu \) is the Lie algebra of the stabilizer \( G_\mu \) of \( \mu \) in the coadjoint action.

\[ (T\Sigma^r + (T\Sigma^r)^\perp)_{(g,\mu)} = \{ L_{g*} X + \mu \circ \text{ad}(X) \mid X \in \mathfrak{g} \}. \]

10 Lemma. On \( (T^*G \cong G \times \mathfrak{g}^*, \omega) \) there exists a symplectic connection \( \nabla \) invariant by the right action of \( G \).

Proof. Let \( \tilde{\nabla} \) be the linear connection on \( G \times \mathfrak{g}^* \) defined by:

\[ \tilde{\nabla}_{\hat{X} + \hat{\eta}}(\hat{X}' + \hat{\eta}') = \frac{1}{2}[\hat{X}, \hat{X}'], \]

where the \( \hat{\ } \) means the corresponding left invariant vector field. This connection is right and left invariant but not symplectic; indeed, one has:

\[ (\tilde{\nabla}_{\hat{X} + \hat{\eta}}(\hat{Y} + \hat{\zeta}, \hat{Y}' + \hat{\zeta}')) = (\hat{X} + \hat{\eta})[(\langle \hat{\zeta}, Y' \rangle - \langle \hat{\zeta}', Y \rangle - \langle \hat{\zeta}, [Y, Y'] \rangle] \]

\[ - \frac{1}{2} (- \langle \hat{\zeta}', [X, Y] \rangle - \langle \hat{\zeta}, [X, Y], Y' \rangle) \]

\[ - \frac{1}{2} (\langle \hat{\zeta}, [X, Y'] \rangle - \langle \hat{\zeta}, [Y, [X, Y']] \rangle) \]

\[ = - \langle \hat{\eta}, [Y, Y'] \rangle + \frac{1}{2} \langle \hat{\zeta}', [X, Y] \rangle - \frac{1}{2} \langle \hat{\zeta}, [X, Y'] \rangle \]

\[ + \frac{1}{2} \langle \hat{\zeta}, [X, [Y, Y']] \rangle. \]

This can be projected on the space of symplectic connections as follows. Write

\[ \nabla_U V = \tilde{\nabla}_U V + A(U)V \]

where \( A(U) \) is an endomorphism such that

\[ A(U)V = A(V)U \quad \text{(torsion free condition)} \].
Then choose:

\[ \omega(A(U)V, W) = \frac{1}{3}[(\tilde{\nabla}_U \omega)(V, W) + (\tilde{\nabla}_V \omega)(U, W)]. \]

This gives a symplectic connection which is \( G \)-invariant. 

11 Proposition. If the group \( G_\mu \) is reductive, there exists on the reduced symplectic manifold a symplectic connection.

Proof. The action of \( G \) on \( T^*G \) is free; hence Assumption 1 is satisfied. The reductiveness hypothesis ensures Assumption 2.

Curvature properties of these reduced connections are worth investigating. We recall in particular the examples given in [2]. It seems also worthwhile to read the nice Gotay-Tuynman paper [3] thinking of connections.

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References

[1] R. Abraham, J. E. Marsden: Foundations of Mechanics, Addison-Wesley Publishing Company, Inc., pp. 299, 302 (1978)

[2] P. Baguis, M. Cahen: A construction of symplectic connections through reduction, Lett. Math. Phys. (under press)

[3] M. J. Gotay, G. M. Tuynman: $\mathbb{R}^{2n}$ is a universal symplectic manifold for reduction, Lett. Math. Phys. Vol. 18: 55–59 (1989)

[4] J. Rawnsley: Private communication

[5] Vaisman, I.: Connections under Symplectic Reduction, math.SG/0006023 v1, 4 Jun 2000