INFINITELY MANY SHIMURA VARIETIES IN THE JACOBIAN LOCUS FOR \( g \leq 4 \)

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Abstract. We study families of Galois covers of curves of positive genus. It is known that under a numerical condition these families yield Shimura subvarieties generically contained in the Jacobian locus. We prove that there are only 6 families satisfying this condition, all of them in genus 2,3 or 4. We also show that these families admit two fibrations in totally geodesic subvarieties, generalizing a result of Grushevsky and Möller. Countably many of these fibres are Shimura. Thus the Jacobian locus contains infinitely many Shimura subvarieties of positive dimension of any \( g \leq 4 \).

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1. Introduction

Denote by \( \mathcal{M}_g \) the moduli space of curves, by \( \mathcal{A}_g \) the moduli space of principally polarized abelian varieties and by \( j : \mathcal{M}_g \to \mathcal{A}_g \) the period map. The Torelli locus \( \mathcal{T}_g \) is the closure of \( j(\mathcal{M}_g) \) in \( \mathcal{A}_g \). A special or Shimura subvariety of \( \mathcal{A}_g \) is by definition a Hodge locus for the tautological family of principally polarized abelian varieties on \( \mathcal{A}_g \). A subvariety \( Z \subset \mathcal{A}_g \) is generically contained in \( j(\mathcal{M}_g) \) if \( Z \subset \mathcal{T}_g \) and and \( Z \cap j(\mathcal{M}_g) \neq \emptyset \).

Expectation 1.1 (Coleman-Oort). For large \( g \) there are no special subvarieties of positive dimension generically contained in \( j(\mathcal{M}_g) \).

(See [4, 32] and [29] for a thorough survey. See [5, 6, 7, 9, 10, 11, 12, 14, 17, 15, 19, 23, 24, 28] for related results.)

Shimura subvarieties are totally geodesic, i.e they are images of totally geodesic submanifolds of the Siegel space, that we denote by \( \mathcal{G}_g \). More precisely, by results of Mumford and Moonen an algebraic subvariety of \( \mathcal{A}_g \)
which is totally geodesic is a Shimura subvariety if and only if it contains a CM point, see [30, 27]. While the notion of CM point is arithmetic, the condition of being totally geodesic relates to the locally symmetric geometry of $A_g$ coming from the Siegel space. One expects the Torelli embedding to be very curved with respect to this locally symmetric ambient geometry. In particular $T_g$ should contain very few totally geodesic subvarieties of $A_g$.

Yet there are at least some Shimura (hence totally geodesic) subvarieties contained in $T_g$. They are all in genus $g \leq 7$. To describe them consider the following construction. Take a family of Galois covers $C \to C' = C/G$, where the genera $g(C') = g'$, $g(C) = g$, the number of ramification points and the monodromy are fixed. Let $Z$ denote the closure in $A_g$ of the locus described by $[JC]$ for $C$ varying in the family. The simple numerical condition

\[(*) \quad \dim(S^2(H^0(K_C))^G = \dim H^0(2K_C)^G)\]

is sufficient to ensure that $Z$ is Shimura [9, 10]. Moonen [28] proved that when $g' = 0$ and the group $G$ is cyclic (*) is also necessary for $Z$ to be Shimura. Mohajer and Zuo [26] extended this to the case where $g' = 0$, $G$ is abelian and the family is one–dimensional. In both cases the authors also showed that condition (*) holds only in the known examples. These results are proved using methods from positive characteristic. A completely different Hodge theoretic argument was given in [5, Prop. 5.2], but only works for some of the families of cyclic covers of $\mathbb{P}^1$. It is unknown whether (*) is necessary in general for a family of covers to yield a Shimura subvariety or whether other families exist which satisfy (*).

Shimura subvarieties via families of Galois covers of elliptic curves satisfying (*) were constructed in [10]. There are 6 such families, but 4 yield Shimura subvarieties already gotten using coverings of $\mathbb{P}^1$. In [10] also families of Galois covers over curves of genus $g' > 1$ were considered. No example was found, but it was shown that when (*) holds then $g \leq 6g' + 1$.

Our first result in this paper is the following:

**Theorem 1.2.** The only positive dimensional families of Galois covers $C \to C' = C/G$ with $g' \geq 1$ and satisfying (*) are the 6 families found in [10]. In particular all of them have $g' = 1$.

See Theorem 2.5. This shows that condition (*) is very strong when $g' > 0$. Of course the moduli image of some family could be a Shimura subvariety even if (*) does not hold.

To prove Theorem 1.2 we first prove that $g' \leq 3$ (Theorem 2.3). This reduces the problem to the analysis of a finite number of cases. The étale covers are ruled out using some elementary representation theory (Lemma 2.4). The ramified cases are checked by a computer program as in [10].

One of the 2 families of coverings over elliptic curves found in [10] had been studied previously by Pirola [33] in order to disprove a conjecture of Xiao. This family was also studied by Grushevsky and Möller [18], who got the following remarkable result: the Prym map for this family is a fibration in curves, which are totally geodesic. As consequence they obtained uncountably many totally geodesic curves generically contained in $T_4$, countably many of which are Shimura.
In this paper we show that this phenomenon appears for all the Shimura subvarieties found in [10]:

**Theorem 1.3.** Consider a family of Galois covers $C \to C' = C/G$ with $g' \geq 1$ and satisfying $(\star)$ (i.e. one of the 6 families in [10]). If $F$ is an irreducible component of a fibre of the Prym map, then $W := \mathcal{J}(F)$ is a totally geodesic subvariety of $\mathbb{A}_g$ of dimension $g'(g' + 1)/2$.

(See Theorem 3.9 for the precise statement.) In particular also $T_2$ and $T_3$ contain uncountably many totally geodesic curves and countably many Shimura curves.

Next we study the map $\varphi$ that maps the covering $[C \to C']$ to $[J\mathcal{C}]$. Then $\varphi$ is a fibration of these families onto $\mathbb{A}_{g'}$.

**Theorem 1.4.** Consider a family of Galois covers $C \to C' = C/G$ with $g' \geq 1$ and satisfying $(\star)$ (i.e. one of the 6 families in [10]). If $F$ is an irreducible component of a fibre of $\varphi$ then $X := \mathcal{J}(F)$ is a totally geodesic subvariety of $\mathbb{A}_g$ of codimension 1.

Also in this case countably many of the irreducible components of the fibres of $\varphi$ are Shimura subvarieties.

In light of Theorem 1.2, Theorems 1.3 and 1.4 concern just 6 explicit families of coverings. Nevertheless it seems hard to prove them by a direct analysis of the families. Our approach is different, we prove our results using some general arguments on isogenies and totally geodesic subvarieties that are explained in Section 3. These are of independent interest.

In the last part of the paper we analyse several features of the examples in relation with the two fibrations $\mathcal{P}$ and $\varphi$. We describe all the inclusions among the families of Galois covers of $\mathbb{P}^1$ or of elliptic curves yielding Shimura subvarieties of $\mathbb{A}_g$ known so far. We show that some of them occur as irreducible components of fibres of the Prym map or are contained in fibres of the map $\varphi$ of one of the 6 families in [10].

2. FAMILIES OF COVERINGS WITH $g' > 0$

For $r \geq 0$ and $g' \geq 1$ set $\Gamma_{g',r} := \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_r | \prod_{j=1}^r \gamma_i \cdot \prod_{j=1}^{g'} [\alpha_j, \beta_j] \rangle$. A **datum** is a pair $\Delta := (G, \theta)$, where $G$ is a finite group and $\theta : \Gamma_{g',r} \to G$ is an epimorphism. We denote by $m_j$ the order of $\theta(\gamma_j)$. After making some choices, one can associate to a datum $(G, \theta)$ a monomorphism from $G$ to the mapping class group $\text{Map}_{g'}$, where $g'$ depends on $g'$ and $m_1, \ldots, m_r$ via Riemann-Hurwitz formula. Denote by $\mathcal{T}_g$ the Teichmüller space and by $\mathcal{T}_g^G$ the set of fixed points of $G$, which is a non-empty connected complex submanifold of $\mathcal{T}_g$ of dimension $3g' - 3 + r$. The image of $\mathcal{T}_g^G$ in $M_g$, which we denote by $M_\Delta$, depends only on the datum, not on the choices made. It is an algebraic subvariety of dimension $3g' - 3 + r$. The equivalence class of the representation of $G$ on $H^0(C, K_C)$ does not change for $[C]$ varying in $M_\Delta$. Therefore the number

$$N(\Delta) := \dim \left( S^2 H^0(C, K_C) \right)^G$$

depends only on the datum. See [10 §2] and references therein for more details on this construction.
Theorem 2.1. Fix a datum \( \Delta = (G, \theta) \) and assume that
\[(*) \quad N(\Delta) = 3g' - 3 + r.\]
Then \( \overline{j(M_\Delta)} \) (closure in \( \mathbb{A}_g \)) is a special subvariety of PEL type of \( \mathbb{A}_g \) that is generically contained in the Torelli locus.

(See [9, Thm. 3.9] and [10, Thm. 3.7].) In principle condition \((*)\) is only sufficient for \( \overline{j(M_\Delta)} \) to be special. It is known to be necessary if \( G \) is cyclic [28] or if \( G \) is abelian, \( g' = 0 \) and \( r = 4 \) [26, Thm. 6.2]. It is not known whether it is necessary in the general case.

In the following we concentrate on the case where \( g' \geq 1 \). A first result was proved in [10, Thm. 4.11].

Theorem 2.2. If \( \Delta = (G, \theta) \) is a datum with \( g' \geq 1, 3g' + r > 3 \) (i.e. \( \dim M_\Delta > 0 \)) and which satisfies condition \((\star)\), then \( g \leq 6g' + 1 \). In particular, for \( g \geq 8 \), (resp. \( g \geq 14 \)), there is no positive-dimensional datum with \( g' = 1 \), (resp. \( g' = 2 \)) and which satisfy condition \((*)\).

Fix a datum \( \Delta = (G, \theta) \) and a point of \( M_\Delta \). This point represents the isomorphism class of a curve \( C \) of genus \( g \), which admits an effective holomorphic action of \( G \). Denote by \( C' := C/G \) the quotient, which has genus \( g' \) and by \( f : C \to C' \) the quotient map. Then \( f \) is a Galois covering with branch locus \( B \). The unramified covering \( f|_{f^{-1}(C' - B)} \) is the \( G \)-cover associated to the epimorphism \( \theta : \Gamma_{g',r} \cong \pi_1(C' - B) \to G \). If \( g' \geq 1 \), we can consider the norm map \( Nm : JC \to JC' \) and the Prym variety \( P(f) := \ker Nm \) of the covering \( f : C \to C' \). Call \( \delta \) the type of the polarization obtained by restricting the theta divisor of \( JC \) to \( P(f) \). Denote by \( \mathbb{A}_{g-g'}^\delta \) the moduli space of abelian varieties of dimension \( g - g' \) with a polarization of type \( \delta \).

Since \( f \) and \( Nm \) are \( G \)-invariant, \( P(f) \) is \( G \)-invariant, thus \( G \) is a group of automorphisms of \( P(f) \) as a polarized abelian variety. Denote by
\[ P_\Delta \subset \mathbb{A}_{g-g'}^\delta \]
the Shimura variety parametrizing abelian varieties with an action of \( G \) of the same type as \( P(f) \). This variety is constructed as in [9, §3]. Next denote by
\[ \mathcal{P} : M_\Delta \to P_\Delta \]
the Prym map that associates to \( [C] \in M_\Delta \) the isomorphism class of \( (P(f), \Theta_{|P(f)}) \). If \( g' = 0 \), then \( \mathcal{P} \) is just the Torelli morphism, so we are just in the setting of Theorem 2.1, which in fact asserts that under condition \((\star)\) we have \( \overline{j(M_\Delta)} = P_\Delta \). Instead, when \( g' \geq 1 \), the Prym map gives rise to some additional geometry.

Theorem 2.3. Consider a datum \( \Delta = (G, \theta) \) with \( g' \geq 1, 3g' + r > 3 \) (i.e. \( \dim M_\Delta > 0 \)), and which satisfies condition \((\star)\). Then \( g' \leq 3 \) and \( \mathcal{P} \) is dominant.

Proof. Both \( M_\Delta \) and \( P_\Delta \) are complex orbifolds and \( \mathcal{P} \) is an orbifold map. We wish to show that \( \mathcal{P} : M_\Delta \to P_\Delta \) is generically submersive.

Fix a point \( x = [C] \in M_\Delta \), denote by \( f : C \to C' := C/G \) the covering and set \((A, \Theta) = P(f)\), so that \( \mathcal{P}(x) = [A, \Theta] \). The orbifold tangent space
of $M_{\Delta}$ at $x$ is $H^1(C, T_C)^G$, while the orbifold tangent space of $P_{\Delta}$ at $\mathcal{P}(x)$ is $(S^2 H^0(A, \Omega^1_A))^G$. Observe that

\[(2.1) \quad H^0(K_C) = H^0(K_C)^G \oplus H^0(K_C)^-\]

where $H^0(K_C)^G \cong H^0(G', K_{C'})$ and $H^0(K_C)^- \cong H^0(A, \Omega^1_A)$ denotes the sum of the non-trivial isotypic components. Moreover $T^*_\Delta M_{\Delta} \cong H^0(C, 2K_C)^G$ and $T^*_\mathcal{P}(x) P_{\Delta} \cong (S^2 H^0(C, K_C^-))^G$. So the codifferential (i.e. the dual of the differential) of $\mathcal{P}$ at $x$ is a map

\[d\mathcal{P}^*_x : (S^2 H^0(K_C^-))^G \to H^0(C, 2K_C)^G.\]

This map is just the restriction of the multiplication map

\[(2.2) \quad m : S^2 H^0(K_C) \to H^0(C, 2K_C).\]

Denote by $\text{HE}_g \subset M_g$ the hyperelliptic locus. Assume first that $M_{\Delta}$ is not contained in $\text{HE}_g$ and that $x \notin \text{HE}_g$. Then the multiplication map $(2.2)$ is surjective by Noether theorem. By Schur lemma $m((S^2 H^0(K_C))^G) = H^0(C, 2K_C)^G$. Since $\dim M_{\Delta} = 3g' - 3 + r$ and $\dim P_{\Delta} = N(\Delta)$, condition \(\dagger\) implies that

\[(2.3) \quad m((S^2 H^0(K_C))^G) : (S^2 H^0(K_C))^G \to H^0(C, 2K_C)^G\]

is in fact an isomorphism. But $(S^2 H^0(K_C^-))^G \subset (S^2 H^0(K_C))^G$, so we conclude that $d\mathcal{P}^*_x$ is injective. Hence $d\mathcal{P}_x$ is surjective. This shows that $\mathcal{P}$ is surjective at $x$.

Assume now that $M_{\Delta} \subset \text{HE}_g$ and denote by $\sigma : C \to C$ the hyperelliptic involution. Then $m$ maps $S^2 H^0(K_C)$ onto $H^0(C, 2K_C)^{\langle \sigma \rangle}$. If $\sigma \in G$, then $H^0(C, 2K_C)^G \subset H^0(C, 2K_C)^{\langle \sigma \rangle}$. Just as before Schur lemma shows that $(2.3)$ is onto and \(\dagger\) yields that $(2.3)$ is an isomorphism. It follows that $\mathcal{P}$ is submersive at $x$. If instead $\sigma \notin G$, denote by $\hat{G}$ the subgroup of $\text{Aut}(C)$ generated by $G$ and $\sigma$. Arguing as above we conclude that the multiplication map $(S^2 H^0(K_C))^\hat{G} \to H^0(C, 2K_C)^\hat{G}$ is surjective. Since $M_{\Delta} \subset \text{HE}_g$, we have $H^0(2K_C)^G = H^0(2K_C)^{\langle \sigma \rangle}$ and since $\sigma$ acts by multiplication by $-1$ on $H^0(K_C)$, it acts trivially on $S^2 H^0(K_C)$, therefore $(S^2 H^0(K_C))^G = (S^2 H^0(K_C))^\hat{G}$. So also in this case the multiplication map $(2.3)$ is surjective, by \(\dagger\) it is an isomorphism and $d\mathcal{P}_x$ is surjective.

We have proved that in case $M_{\Delta} \subset \text{HE}_g$, $\mathcal{P}$ is submersive on $M_{\Delta}$ in the orbifold sense, while in case $M_{\Delta}$ is not contained in $\text{HE}_g$, $\mathcal{P}$ is submersive on $M_{\Delta} - \text{HE}_g$. At any case $\mathcal{P}$ is generically submersive (in the orbifold sense) hence it is dominant.

From $(2.1)$ we get

\[(2.4) \quad (S^2 H^0(K_C))^G \cong S^2 H^0(K_{C'}) \oplus (S^2 H^0(K_C^-))^G.\]

Since the multiplication map $(2.3)$ at a generic point is an isomorphism, its restriction to $S^2 H^0(K_{C'})$ is injective. Moreover it maps $S^2 H^0(K_{C'})$ to $H^0(2K_{C'}) \subset H^0(2K_C)^G$. Hence $\dim(S^2 H^0(K_{C'})) \leq \dim H^0(2K_{C'})$, which yields $g' \leq 3$. \(\square\)

**Lemma 2.4.** There do not exist positive dimensional families of étale coverings $f : C \to C' = C/G$ with $g' = g(C') \geq 2$ satisfying condition $(*)$.  

Theorem 2.5. □

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From Theorem (2.3) we know that
Proof. (condition
More precisely, consider a differentiable manifold \( M \) with an affine connection \( \nabla \), i.e. a linear connection on \( TM \). With the aid of \( \nabla \) one can define geodesics, the exponential map, completeness and curvature. A diffeomorphism \( f : M \to M \) is an affine transformation if it preserves the connection

\[ \dim(S^2 H^0(K_C))^G = 3g' - 3 = \dim S^2 H^0(K_C) + \dim(S^2(V^-))^G \geq \dim S^2 H^0(K_{C'}) = \frac{g'(g' + 1)}{2} = 3g' - 3, \]

since \( g' = 2, 3 \). This implies \( (S^2(V^-))^G = 0 \). By Chevalley-Weil formula \([3]\) (see also \([31]\) or \([8]\) Thm. 2.8)) we have \( \nu_\chi = (\dim V_\chi)(g' - 1) > 0 \), for all non-trivial irreducible character \( \chi \). So for any \( \chi \in I \) we have \( (S^2 V_\chi)^G = 0 \) and for any \( \chi, \chi' \in I \), we have \( (V_\chi \otimes V_{\chi'})^G = 0 \) if \( \chi \neq \chi' \). If there is a non-trivial 1-dimensional representation \( V_\chi \), this is impossible. In fact let \( \chi' \) be the character of \( V_\chi^* \). If \( \chi \neq \chi' \), then \( (V_\chi \otimes V_{\chi'})^G \neq 0 \). If \( \chi = \chi' \), then \( 0 \neq (V_\chi \otimes V_{\chi'})^G \cong (S^2(V_\chi))^G \), since \( \Lambda^2 V_\chi = 0 \) because \( \dim V_\chi = 1 \).

By Theorem (2.3) we know that \( g' \leq 3 \) and from \([10]\) Thm. 1.2] that \( g \leq 6g' + 1 \). Denote by \( d := |G| \). If \( g' = 2 \), we have \( d = g - 1 \leq 6g' = 12 \) by Riemann-Hurwitz, while in case \( g' = 3 \), we have \( g - 1 = 2d \), hence \( d = \frac{g - 1}{2} \leq 3g' = 9 \). So at any case \( d \leq 12 \) and all groups with \( |G| \leq 12 \) admit non-trivial 1-dimensional irreducible representations. This concludes the proof. □

Theorem 2.5. The only positive dimensional families of Galois coverings \( f : C \to C' = C/G \) with \( g' = g(C') \geq 1 \) and \( g = g(C) \) which satisfy condition (\( * \)) have \( g' = 1 \) and are the 6 families found in \([10]\).

Proof. From Theorem (2.3) we know that \( g' \leq 3 \). From Theorem 1.2 of \([10]\) we know that \( g \leq 6g' + 1 \). From Lemma (2.4) we know that the covering has to be ramified and by computer calculations as in \([10]\) we find exactly the 6 families of \([10]\) with \( g' = 1 \). □

3. Families of isogenous Abelian varieties

Riemannian symmetric spaces appear in Hodge theory as parameter spaces of Abelian varieties. In the sequel we will compare totally geodesic submanifolds of two Siegel spaces with respect to different polarizations. To do that we will embed both Siegel spaces in a larger parameter space for complex tori that is a non-Riemannian symmetric space. Therefore we start by recalling the definition and the main facts regarding this more general class of symmetric spaces, that are less known than the Riemannian ones. We will follow \([22]\) vol. II, Ch. XI]

The definition of a general (i.e. not necessarily Riemannian) symmetric space is obtained from definition of Riemann symmetric space by dropping the request that the connection is compatible with a Riemannian metric. More precisely, consider a differentiable manifold \( M \) with an affine connection \( \nabla \), i.e. a linear connection on \( TM \). With the aid of \( \nabla \) one can define geodesics, the exponential map, completeness and curvature. A diffeomorphism \( f : M \to M \) is an affine transformation if it preserves the connection
The group \( \text{Aff}(M) \) of all affine transformations is a Lie group acting smoothly on \( M \) \cite{22} vol. I, p. 225). We say that \((M, \nabla)\) is a symmetric space if (1) \( \nabla \) is symmetric i.e. \( T(\nabla) = 0 \), (2) \( M \) is connected, (3) \( M \) is complete with respect to \( \nabla \) and (4) for each point \( x \in M \) there is a symmetry at \( x \), i.e. an affine transformation \( s_x : M \to M \) such that \( s_x(x) = x \) and \( (ds_x)_x = -\text{id}_{TM} \). As in the Riemannian case there is a local version of condition (4) which is equivalent to the fact that \( \nabla R = 0 \).

Assume that \((M, \nabla)\) is a symmetric space and let \( G \) denote the connected component of the identity in \( \text{Aff}(M) \). Then \( G \) acts transitively on \( M \). Fix a point \( o \in M \). If \( g \in G \), then \( \sigma(g) := s_o \circ g \circ s_o \in G \) and \( \sigma : G \to G \) is an involutive automorphism of \( G \). If we set \( H := G_o \), then \( M = G/H \) and \( H \) lies between \( G^\sigma = \{ g \in G : \sigma(g) = g \} \) and the identity component of \( G^\sigma \).

Conversely assume that \((G, H, \sigma)\) is a symmetric triple, i.e. \( G \) is a connected Lie group, \( \sigma \) is an involutive automorphism of \( G \) and \( H \) is a closed subgroup lying between \( G^\sigma \) and its component of the identity. Then \( M := G/H \) is a reductive homogeneous space (see \cite{22} vol. II, ch. X). As such it admits the so-called canonical connection \( \nabla \). With this connection \( M \) is a symmetric space. \cite{22} vol. II, pp. 230-231]. If \( m := \{ X \in g : d\sigma(X) = -X \} \) then \( g = h \oplus m \) and there is an isomorphism

\[
(3.1) \quad m \cong T_o M, \quad X \mapsto \frac{d}{dt} \bigg|_{t=0} \exp(tX) \cdot o.
\]

If \( M \) is a manifold with an affine connection \( \nabla \) and \( M' \subset M \) is a submanifold, denote by \( p \) the canonical projection \( TM|_{M'} \to N := TM|_{M'}/TM' \). The second fundamental form is defined as \( B(X, Y) := p(\nabla_X Y) \). If \( \nabla \) is symmetric, then \( B \in \Gamma(S^2T' M' \otimes N) \). A totally geodesic submanifold of \( M \) is a submanifold \( M' \) such that any geodesic \( \gamma : \mathbb{R} \to M \) passing through \( x \in M' \) at time \( t = 0 \) with \( \dot{\gamma}(0) \in T_x M' \) remains in \( M' \) for \( |t| \) small enough. It can be proved in the usual way that \( M' \) is totally geodesic iff its second fundamental form vanishes identically.

Let \((G, H, \sigma)\) be a symmetric triple. A subtriple is a triple \((G', H', \sigma')\) with \( G' \) a connected Lie subgroup of \( G \) invariant by \( \sigma \), \( H' = G' \cap H \) and \( \sigma' := \sigma|_{G'} \). A subtriple is automatically a symmetric triple.

**Theorem 3.1.** (1) Let \((G, H, \sigma)\) be a symmetric triple and let \((G', H', \sigma')\) be a subtriple. Then the inclusion \( G' \subset G \) induces an embedding \( G'/H' \cong M' := G' \cdot o \subset M = G/H \) and \( M' \) is a totally geodesic submanifold of \( M \). The canonical connection of \( M \) restricts to the canonical connection of \( M' \).

(2) Conversely set \( o := H \in M = G/H \) and let \( M' \subset M \) be a complete and connected totally geodesic submanifold of \( M \). Set \( G'' = \{ g \in G : g(M') = M' \} \). Then \( G'' \) is a Lie subgroup of \( G \). Let \( G' \) denote the identity component of \( G'' \). Then \( G' \) is invariant by \( \sigma \), so \((G', H' := H \cap G', \sigma' := \sigma|_{G'} \) is a subtriple and \( M' = G'/H' \).

(3) If \( M' \subset M \) is a totally geodesic submanifold through \( o \), then via \((3.1)\) we have \( T_o M' \cong m' \) for a Lie triple system \( m' \), i.e. for a vector subspace \( m' \subset m \) satisfying \( [m', m'], m' \subset m' \).

See \cite{22} vol. II, p. 234-237] for a proof.
Set $V := \mathbb{R}^{2g}$ and define
\begin{equation}
\mathcal{C}(V) := \{ J \in \text{End} V : J^2 = - \text{id}_V \}
\end{equation}
This set is invariant by the adjoint action of $\text{GL}(V)$ on $\text{End} V$. Let $V_J$ denote the complex vector space with underlying real space $V$ and complex multiplication defined by $i \cdot v := Jv$. Fix $J \in \mathcal{C}(V)$, and let $J'$ be another point of $\mathcal{C}(V)$. Fix bases $\{ e_i \}$ and $\{ e'_i \}$ of $V_J$ and $V_{J'}$ respectively. Thus $\{ e_1, \ldots, e_g, Je_1, \ldots, Je_g \}$ is a basis of $V$ and the same for $\{ e'_1, J'e'_1 \}$. Hence there is a unique map $a \in \text{GL}(V)$ such that $a(e_i) = e'_i$ and $a(Je_i) = J'e'_i$. It follows that $aJ = J'a$, i.e. $\text{Ad}(J) = J'$. This shows that $\mathcal{C}(V) \cong \text{GL}(V)/ \text{GL}(V_J)$. Thus $\mathcal{C}(V)$ is a manifold with two connected components. The connected component containing $J$ is the orbit $\text{GL}^+(V) : J \cong \text{GL}^+(V)/ \text{GL}(V_J)$.

Fix $J \in \mathcal{C}(V)$ and consider the automorphism
\[
\sigma_J : \text{GL}(V) \rightarrow \text{GL}(V), \quad \sigma_J(a) := \text{Ad}(J)(a) = -JaJ.
\]
Since $J^2 = -\text{id}_V$, $\sigma_J$ is involutive and $\text{GL}(V_J)$ is its fixed point set. Thus we have proved that the connected component of $\mathcal{C}(V)$ containing $J$ is the symmetric space associated with the symmetric triple $(\text{GL}^+(V), \text{GL}(V_J), \sigma_J)$. Since the stabilizer of $J$ is non-compact, this symmetric space is not Riemannian. On the other hand, we can replace $\text{GL}^+(V)$ with $\text{SL}(V)$ and $\text{GL}(V_J)$ with $\text{GL}(V_J) \cap \text{SL}(V)$. Since $\text{SL}(V)$ is simple, Theorem 3.4 of [22, vol. II, p. 232] implies that this space admits a symmetric pseudo-Riemannian metric. In the following we will often refer to $\mathcal{C}(V)$ as a symmetric space, even tough this is true only of its connected components.

Let $\omega$ be a symplectic form on $V$. If $J \in \mathcal{C}(V)$, we have $J^*\omega = \omega$ if and only if the bilinear form $g_J := \omega(\cdot, J \cdot)$ is symmetric. Set
\[
\mathcal{S}(V, \omega) := \{ J \in \mathcal{C}(V) : J^*\omega = \omega, g_J \text{ is positive definite} \}.
\]
This is the Siegel space of $(V, \omega)$.

**Proposition 3.2.** The Siegel space $\mathcal{S}(V, \omega)$ is a totally geodesic submanifold of $\mathcal{C}(V)$. It is itself a symmetric space and in fact a Riemannian one.

**Proof.** It is well-known that $\mathcal{S}(V, \omega) \neq \emptyset$ and that it is connected. Fix $J \in \mathcal{S}(V, \omega)$. Then $\text{Sp}(V, \omega)$ is invariant by $\sigma_J$, since $\sigma_J = \text{Ad}(J)$ and $J \in \text{Sp}(V, \omega)$. Set $G' := \text{Sp}(V, \omega)$, $H' := G' \cap \text{GL}(V_J)$ and $\sigma'_J := \sigma_J|_{G'}$. Then $(G', H', \sigma'_J)$ is a subtriple of $(\text{GL}^+(V), \text{GL}(V_J) \cap \text{GL}^+(V), \sigma_J)$, so by Theorem 3.3 $\mathcal{S}(V, \omega) = G'/H'$ is totally geodesic submanifold and also a symmetric space itself. On $V_J$ consider the Hermitian product $H_J(x, y) := g_J(x, y) - i\omega(x, y)$. Then $\text{GL}(V_J) \cap \text{GL}^+(V)$ is the unitary group $U(V_J, H_J)$. Since this is a compact group the symmetric space $\mathcal{S}(V, \omega)$ is Riemannian, see [20, p. 209].

Set $\Lambda := \mathbb{Z}^{2g}$. As usual if $F \subset \mathbb{C}$ is a field we set $\Lambda_F := \Lambda \otimes \mathbb{Z} F$. Then $V = \Lambda_F$ and $T := V/\Lambda$ is a real torus of dimension $2g$. Since the tangent bundle to $T$ is trivial, any $J \in \mathcal{C}(V)$ yields a complex structure on $T$. We denote $T_J$ the complex torus obtained in this way. Any complex torus of dimension $g$ is isomorphic to $T_J$ for some $J$. In this sense $\mathcal{C}(V)$ is a parameter space for $g$-dimensional complex tori. If $f : T_J \rightarrow T_{J'}$ is an isomorphism, then $f$ lifts to an isomorphism $a : V \rightarrow V$ such that $a(\Lambda) = \Lambda$. 

\[\Box\]
and \( \text{Ad} a(J) = J' \). Thus \( T_J \) and \( T_{J'} \) are isomorphic if and only if \( a \in \text{GL}(\Lambda) \) such that \( \text{Ad} a(J) = J' \).

More generally an isogeny of \( T_J \) onto \( T_{J'} \) is a surjective morphism \( f : T_J \to T_{J'} \) with finite kernel. This lifts to a linear map \( a : V \to V \) of maximal rank, hence invertible, such that \( J' a = a J \) (i.e. \( f \) is holomorphic) and \( a(\Lambda) \subset \Lambda \). It follows that \( a \in \text{GL}(\Lambda_Q) \). Conversely, given \( a \in \text{GL}(\Lambda_Q) \) such that \( J' a = a J \), multiplying \( a \) by an appropriate positive integer \( m \), we get a linear map \( m \cdot a : V \to V \) such that \( m \cdot a(\Lambda) \subset \Lambda \). This induces a surjective holomorphic morphism \( f : T_J \to T_{J'} \), which is an isogeny. Thus \( T_J \) is isogenous to \( T_{J'} \) if and only if there is \( a \in \text{GL}(\Lambda_Q) \) such that \( \text{Ad} a(J) = J' \).

The map \( J \mapsto V_J^{0,1} \) is a diffeomorphism of \( \mathcal{C}(V) \) onto the set \( \Omega := \{ W \in \mathcal{G}(g, V_C) : W \cap W = \{0\} \} \), which is an open subset of the Grassmannian in the analytic topology. So \( \mathcal{C}(V) \) is a complex manifold.

**Lemma 3.3.** Let \( Z_1, Z_2 \subset \mathcal{C}(V) \) be irreducible analytic subsets. Let \( \Omega \) be a non-empty open subset of \( Z_1 \). Assume that for any \( J_1 \in \Omega \) there is some \( J_2 \in Z_2 \) such that \( T_{J_1} \) is isogenous to \( T_{J_2} \). Then there is \( a \in \text{GL}(\Lambda_Q) \) such that \( \text{Ad} a(Z_1) \subset Z_2 \).

**Proof.** Given \( a \in \text{GL}(\Lambda_Q) \), let \( \Gamma_a \subset \mathcal{C}(V) \times \mathcal{C}(V) \) denote the graph of \( \text{Ad} a \).

If \( \pi_j : \mathcal{C}(V) \times \mathcal{C}(V) \to \mathcal{C}(V) \) denotes projection on the \( j \)-th factor, the assumption is equivalent to saying that

\[
\Omega \subset \bigcup_{a \in \text{GL}(\Lambda_Q)} \pi_1(\Gamma_a \cap \pi_2^{-1}(Z_2)).
\]

Indeed, if \( J_1 \in \Omega \), there is \( J_2 \in Z_2 \) such that \( T_{J_1} \) and \( T_{J_2} \) are isogenous, i.e. there is \( a \in \text{GL}(\Lambda_Q) \) such that \( (J_1, J_2) \in \Gamma_a \).

For each \( a \in \text{GL}(\Lambda_Q) \) the intersection \( \Gamma_a \cap \pi_2^{-1}(Z_2) \) is an analytic subset, so we can find a sequence \( \{ K_{a,i} \}_{i \in \mathbb{N}} \) of compact subsets of \( \mathcal{C}(V) \times \mathcal{C}(V) \) such that

\[
\Gamma_a \cap \pi_2^{-1}(Z_2) = \bigcup_{i=1}^{\infty} K_{a,i}.
\]

Then

\[
\Omega = \bigcup_{a, i} \Omega \cap \pi_1(K_{a,i}).
\]

Since \( K_{a,i} \) is compact, the set \( \Omega \cap \pi_1(K_{a,i}) \) is closed in \( \Omega \). As \( i \) and \( a \) vary in countable sets, Baire theorem [L p. 57] implies that there are \( i \) and \( a \) such that \( \pi_1(K_{a,i}) \) contains an open subset \( U \) of \( \Omega \). The set \( U \) is clearly open also in \( Z_1 \) and satisfies \( U \subset \pi_1(\Gamma_a \cap \pi_2^{-1}(Z_2)) \). This means that if \( J \in U \), there is \( J' \in Z_2 \) such that \( (J, J') \in \Gamma_a \). In other words \( \text{Ad} a(J) \in Z_2 \) for any \( J \in U \). So, setting \( f = \text{Ad} a : \mathcal{C}(V) \to \mathcal{C}(V) \), we have \( f(U) \subset Z_2 \). Therefore \( f^{-1}(Z_2) \cap Z_1 \) is an analytic subset of \( Z_1 \), which contains the open subset \( U \subset Z_1 \). By the Identity Lemma [L p. 167] this implies that \( f^{-1}(Z_2) \cap Z_1 = Z_1 \) i.e. \( f(Z_1) \subset Z_2 \). \(\square\)

**Proposition 3.4.** Let \( \omega_1, \omega_2 \) be symplectic forms on \( V \). Assume that \( Z_1 \) is an irreducible analytic subset of \( \mathcal{G}(V, \omega_1) \) and that \( Z_2 \) is a totally geodesic submanifold of \( \mathcal{G}(V, \omega_2) \). Let \( \Omega \) be a non-empty open subset of \( Z_1 \) with the
property that for any $J_1 \in \Omega$ there is some $J_2 \in Z_2$ such that $T_{J_1}$ is isogenous to $T_{J_2}$. Assume moreover that $\dim Z_1 = \dim Z_2$. Then there is $a \in \text{GL}(\Lambda_1)$ such that $\text{Ad}_a(Z_1) = Z_2$. Moreover $Z_1$ is a totally geodesic submanifold of $\mathcal{S}(V, \omega_1)$.

**Proof.** By Lemma 3.3 there is $a \in \text{GL}(\Lambda_1)$ such that $\text{Ad}_a(Z_1) \subset Z_2$. Since $Z_2$ is irreducible, a proper analytic subset of $Z_2$ is nowhere dense in $Z_2$, see e.g. [16, p. 168]. Since $\dim Z_1 = \dim Z_2 = n$ we conclude that $\text{Ad}_a(Z_1) = Z_2$. This proves the first assertion. By assumption $Z_2$ is totally geodesic in $\mathcal{S}(V, \omega_2)$. By Proposition 3.2 $\mathcal{S}(V, \omega_2)$ is itself totally geodesic in $\mathcal{C}(V)$. Thus $Z_2$ is totally geodesic in $\mathcal{C}(V)$. The same is true for $Z_1$ since $\text{Ad}_a$ is an affine transformation of $\mathcal{C}(V)$. As $Z_1 \subset \mathcal{S}(V, \omega_2)$ by assumption, $Z_1$ is in fact a totally geodesic submanifold of $\mathcal{S}(V, \omega_1)$ as desired. \qed

The following definition goes back to Moonen [21].

**Definition 3.5.** Let $\omega$ be a polarization of type $D$. Denote by $\pi : \mathcal{S}(V, \omega) \to \mathcal{A}_D$ the canonical projection. A totally geodesic subvariety of $\mathcal{A}_D$ is a closed algebraic subvariety $W \subset \mathcal{A}_D$, such that $W = \pi(Z)$ for some totally geodesic submanifold $Z \subset \mathcal{S}(V, \omega)$.

We wish to prove an analogue of Proposition 3.4 for subvarieties of $\mathcal{A}_g$ instead of $\mathcal{S}_g$. A difficulty in passing from $\mathcal{S}_g$ to $\mathcal{A}_g$ comes from the fact that the map $\pi : \mathcal{S}_g \to \mathcal{A}_g$ is of infinite degree and ramified. It helps to factor $\pi$ as an unramified covering of infinite degree followed by a finite map. From this one easily gets Proposition 3.4 below, which is enough to “descend” Proposition 3.4 to $\mathcal{A}_g$.

**Lemma 3.6.** Let $X, Y, Z$ be reduced complex analytic spaces. Let $p : X \to Y$ be an unramified covering and let $q : Y \to Z$ be a finite Galois covering. If $Z' \subset Z$ is an irreducible analytic subset and $X'$ is an irreducible component of $(qp)^{-1}(Z')$, then $qp(X') = Z'$.

**Proof.** Let $\{Y_i\}$ be the irreducible components of $q^{-1}(Z')$. We claim that $q(Y_i) = Z'$ for each $i$. Since $q$ is a finite map each $q(Y_i)$ is an analytic subset of $Z'$. Obviously $Z' = \bigcup q(Y_i)$. By Baire theorem there is some $i_0$ such that $q^{-1}(Y_{i_0})$ has non-empty interior, therefore $Z' = q(Y_{i_0})$. Since $q : Y \to Z$ is a Galois cover with finite Galois group $G$, it follows that $q^{-1}(Z') = G \cdot Y_{i_0}$, so for each $i$ we have $Y_i = g \cdot Y_{i_0}$ for some $g \in G$ and hence $q(Y_i) = q(Y_{i_0}) = Z'$. This proves the claim. Next fix a point $x$ of $X'$ that does not lie in any other irreducible component of $(qp)^{-1}(Z')$. Since $p(X') \subset q^{-1}(Z')$ and $X'$ is irreducible, $p(X')$ is contained in a unique irreducible component $Y'$ of $q^{-1}(Z')$. By the above $q(Y') = Z'$. To conclude it is enough to show that $p(X') = Y'$. Set $y := p(x) \in Y'$. Let $u : \check{Y}' \to Y'$ be the universal cover. Fix $\check{y} \in p^{-1}(y)$. By the lifting theorem there is a holomorphic map $\check{f} : (\check{Y}', \check{y}) \to (X, x)$ such that $p\check{f} = f := iu$, where $i : Y' \to Y$ denotes the inclusion. Since $Y'$ is irreducible, also $\check{Y}'$ is irreducible, hence $\check{f}(\check{Y}')$ is contained in a unique irreducible component, which is necessarily $X'$ since $\check{f}(\check{y}) = x$. So $\check{f}(\check{Y}') \subset X'$. It follows that $Y' = iu(\check{Y}') = f(\check{Y}') = p\check{f}(\check{Y}') \subset p(X')$. On the other hand we have $p(X') \subset Y'$ by construction. Hence $p(X') = Y'$ as desired. \qed
Proposition 3.7. Let $\omega$ be a symplectic form of type $D$. Denote by $\pi : \mathcal{G}(V, \omega) \rightarrow A_D$ the canonical projection. If $W \subset A_D$ is an irreducible analytic subset, then for any irreducible component $Z$ of $\pi^{-1}(W)$ we have $\pi(Z) = W$.

Proof. The polarisation $\omega$ is a non-degenerate alternating form $\omega : \Lambda^2(\Lambda) \rightarrow \mathbb{Z}$ of type $D = (d_1, \ldots, d_g)$, where $d_1 | d_2 | \cdots | d_g$. Let $n$ be a natural number such that $(d_g, n) = 1$. Consider a symplectic level $n$ structure, i.e. a symplectic isomorphism of the set of $n$-torsion points $A[n]$ of $A = V/\Lambda$ with $(\mathbb{Z}/n\mathbb{Z})^2g$.

Denote by $\Gamma_D(n)$ the subgroup of the automorphisms of the pair $(\Lambda, \omega)$ which induce the trivial action on $\Lambda/\Lambda$. If $n$ is large enough with respect to the polarisation $D = (d_1, \ldots, d_n)$, then the quotient $\mathcal{G}(V, \omega)/\Gamma_D(n) =: A_D^{(n)}$ is smooth and the map $p : \mathcal{G}(V, \omega) \rightarrow A_D^{(n)}$ is a topological covering (cf. e.g. [13]). The map $\pi$ factors through the topological covering $p : \mathcal{G}(V, \omega) \rightarrow A_D^{(n)}$ and a finite Galois covering $q : A_D^{(n)} \rightarrow A_D$. The result follows from Lemma 3.6.

Theorem 3.8. Let $D_1$ and $D_2$ be types of $g$-dimensional abelian varieties. Let $W_1 \subset A_{D_1}$ and $W_2 \subset A_{D_2}$ be irreducible analytic subsets. Assume that there is a non-empty subset $U$ of $W_1$ such that (1) $U$ is open in the complex topology, (2) any $[A_1] \in U$ is isogenous to some $[A_2] \in W_2$. Then $\dim W_1 \leq \dim W_2$. Moreover if $\dim W_1 = \dim W_2$ and $W_2$ is a totally geodesic subvariety, then $W_1$ also is totally geodesic.

Proof. Denote by $\pi_i : \mathcal{G}(V, \omega_i) \rightarrow A_{D_i}$ the canonical projections. Let $Z_i$ be an irreducible component of $\pi_i^{-1}(W_i)$. By Proposition 3.7 $\pi_i(Z_i) = W_i$. Set $\Omega := Z_1 \cap \pi_1^{-1}(U)$. Clearly for any $J_1 \in \Omega$ there is some $J_2 \in Z_2$ such that $T_{J_1}$ and $T_{J_2}$ are isogenous. By Lemma 3.3 there is $a \in \text{GL}(\Lambda_\mathbb{Q})$ such that $\text{Ad}_a(Z_1) \subset Z_2$. Hence $\dim W_1 = \dim Z_1 \leq \dim Z_2 = \dim W_2$. This proves the first assertion. To prove the second, recall that by definition 3.3 we can assume that $Z_2 \subset \mathcal{G}(V, \omega_2)$ is a totally geodesic submanifold. By Proposition 3.4 $Z_1$ is also totally geodesic, hence the result.

Theorem 3.9. Consider a datum $\Delta = (G, \theta)$ with $g' \geq 1$, $3g' + r > 3$ (i.e. $\dim M_{\Delta} > 0$), and which satisfies condition (a). Then for every $y \in \text{Im} \mathcal{P}$ and for every irreducible component $F$ of $\mathcal{P}^{-1}(y)$, the closure $W := j(F)$ is a totally geodesic subvariety of $A_g$ of dimension $g'(g' + 1)/2$.

Proof. We start by proving the dimension statement. First we compute the dimension of generic fibers of $\mathcal{P}$.

By (a) $\dim M_{\Delta} = \dim (S^2 H^0(K_C))^G$. Fix $x$ such that $\mathcal{P}$ is submersive at $x$. Set $y = \mathcal{P}(x)$. Then, using (2.4) we get

$$
\dim_x \mathcal{P}^{-1}(y) = \dim M_{\Delta} - \dim P_{\Delta} = \dim (S^2 H^0(K_C))^G - \dim (S^2 H^0(K_C^-))^G = \dim S^2 H^0(K_C^-) = \frac{g'(g' + 1)}{2} \geq 1.
$$

Hence the generic fiber of $\mathcal{P}$ has dimension $g'(g' + 1)/2$, therefore $\dim W = \dim F \geq g'(g' + 1)/2$.

If $y = [A, \Theta] \in \text{Im} \mathcal{P}$, denote by $W' \subset A_D$ the closure of the variety parametrizing abelian varieties isomorphic to products $A \times j(C')$, where
$[C'] \in M_g$ (with $D$ denoting the appropriate product polarization). Observe that $W$ is an irreducible analytic subset of $A_g$ and $W'$ is an irreducible analytic subset of $A_D$. Clearly the generic point of $W$ is isogenous to some point of $W'$.

By Theorem 3.11, $g' \leq 3$, so $W'$ parametrizes abelian varieties of the form $A \times B$ with $|B| \in A_{g'}$. Hence $W'$ is a totally geodesic subvariety of $A_D$ of dimension $g'(g' + 1)/2$. Theorem 3.8 implies that $\dim W \leq \dim W'$, so $\dim W = g'(g' + 1)/2$. This proves the dimension statement. Now we can apply the second part of Theorem 3.8 to conclude that $W$ is totally geodesic.

**Remark 3.10.** Since we know that the data satisfying the assumptions of the Theorem have $g' = 1$, the fibres are 1-dimensional.

Fix a datum $\Delta = (G, \theta)$ with $g' \geq 1$, $3g' + r > 3$, and consider the map

$$\varphi : M_\Delta \to A_{g'}$$

which associates to $[C \to C'] \in M_\Delta$ the multiplication map

$$m : (S^2 H^0(K_C)^G) \cong S^2 H^0(K_{C'}) \oplus (S^2 H^0(K_C)^{-})^G \to H^0(2K_C)^G$$

is an isomorphism. This implies that the restriction of $m$ to $S^2 H^0(K_{C'})$ is injective. Since this is the codifferential of $\varphi$ at $x$, we have proved that

$$d\varphi_x : H^1(T_C)^G \to S^2 H^0(K_{C'})^*$$

is surjective. Hence the dimension of the generic fiber $\varphi^{-1}(y)$ is $N(\Delta) - \frac{g'(g' + 1)}{2}$, so $\dim(X) = \dim(Y) \geq N(\Delta) - \frac{g'(g' + 1)}{2}$.

Let $y = [J(C'), \Theta] \in j_M(g')$, denote by $X' \subset A_D$ the closure of the variety parametrizing abelian varieties isomorphic to products $j(C') \times B$, where $B \in P_\Delta$ (with $D$ denoting the appropriate product polarization). Since $P_\Delta$ is a Shimura subvariety of $A_g$, $X'$ is a totally geodesic subvariety of $A_D$ of dimension $N(\Delta) - \frac{g'(g' + 1)}{2}$. The generic point of $X$ is isogenous to some point of $X'$, so by Theorem 3.8 $\dim X \leq \dim X'$, therefore $\dim X = N(\Delta) - \frac{g'(g' + 1)}{2}$. This proves the dimension statement. Now we can apply the second part of Theorem 3.8 to conclude that $X$ is totally geodesic.

**Remark 3.12.** Since we know that the data satisfying the assumptions of the Theorem have $g' = 1$, we have in fact $d = N(\Delta) - 1 = r - 1$.

**Remark 3.13.** Theorem 3.8 says roughly that at the level of Siegel space there is a global isogeny between (the liftings of) $Z := j(M_\Delta)$ and $A_1 \times P_\Delta$. In particular this shows again that $Z$ is Shimura. It is natural to ask whether this global isogeny can be chosen so that it also realizes the isogenies between the fibres of $\mathcal{P}$ (resp. the fibres of $\varphi$) and the subsets $A_1 \times \{pt\}$ (resp. $\{pt\} \times P_\Delta$). It seems unlikely that such a global isogeny can be gotten by
methods similar to those of Theorem 3.8. Since we are dealing with only 6 families an explicit analysis could in principle answer this question. Yet this is probably non-trivial.

4. Analysis of the families fibered in totally geodesic subvarieties

There are exactly 6 families with $g' = 1$ that satisfy $(\ast)$ (i.e. $N(\Delta) = r$). The purpose of this section is to give a detailed analysis of these families from the point of view of the fibrations $P$ and $\varphi$. The 6 families are the following:

(1e) $g = 2$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 2$.
(2e) $g = 3$, $G = \mathbb{Z}/2\mathbb{Z}$, $N = r = 4$.
(3e) $g = 3$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 2$.
(4e) $g = 3$, $G = \mathbb{Z}/4\mathbb{Z}$, $N = r = 2$.
(5e) $g = 3$, $G = Q_8$, $N = r = 1$.
(6e) $g = 4$, $G = \mathbb{Z}/3\mathbb{Z}$, $N = r = 3$.

All these families except, (2e) and (6e), yield Shimura subvarieties which can be obtained also using coverings of $\mathbb{P}^1$. In fact in [10] it was shown that:

- (1e) gives the same subvariety as (26) of Table 2 in [9] (this was already found in [29]).
- (3e) gives the same subvariety as (31) of Table 2 in [9].
- (4e) gives the same subvariety as (32) of Table 2 in [9].
- (5e) gives the same subvariety as (34)=(23)=(7) of Table 2 in [9] (see also Table 1 in [9] to see that these families are the same).
- Apart from (1e), none of these families is contained in the hyperelliptic locus.

Corollary 4.1. Families (1e), (2e), (3e), (4e), (6e) are fibered in totally geodesic curves via their Prym maps and are fibered in totally geodesic subvarieties of codimension 1 via the map $\varphi$. Therefore they contain infinitely many totally geodesic subvarieties and countably many Shimura subvarieties. The Prym map of family (5e) is constant.

Proof. This follows immediately from Theorems 3.9 and 3.11. Since all these families yield Shimura subvarieties of $A_g$, they contain countably many CM points, hence the fibers of the two maps $P$ and $\varphi$ passing through these points are Shimura subvarieties. Since family (5e) is one dimensional, it is itself a fibre of its Prym map $P$, which therefore is constant.

Now we will give all the possible inclusions between the families of Galois covers of $\mathbb{P}^1$ or of genus 1 curves known so far and listed in Tables 1,2 of [9] and in [10], yielding Shimura varieties. In this way we will also check which of these families are contained in a fiber of the Prym map $P$ or of the map $\varphi$ of one of the 6 families above.

In genus 2 we have the following diagram, where the arrows denote the inclusions. In the lowest line we have the one dimensional families, family
(1e) = (26) has dimension 2, while family (2) has dimension 3 and is $M_2$. 

\[
\begin{array}{c}
(2) \\
\downarrow \\
(1e) = (26) \\
\end{array}
\]

(3) = (5) = (28) = (30) 
(4) = (29)

Proposition 4.2. In genus 2 families (3) = (5) = (28) = (30) and (4) = (29) are not contained in a fibre of $\mathcal{P}$, nor in a fibre of $\varphi$ of the family (1e).

Proof. Since both the curves in family (30) and the ones in family (29) have jacobians that are isogenous to the self product of an elliptic curve (see e.g. [6] p. 16), none of these two families are fibres of the Prym map of (1e), nor of the map $\varphi$ of (1e). 

Note that the bielliptic locus in genus 2 has codimension 1 in $M_2$ and it is totally geodesic in $A_2$. This only happens in genus 2. In fact in [9] it is shown that if $g \geq 3$ and $Y \subset M_g$ is an irreducible divisor, then there is no proper totally geodesic subvariety of $A_g$ containing $j(Y)$.

In genus 3 we have the following diagram. In the lowest line we have the one dimensional families, in the second one the two dimensional ones, family (27) has dimension 3 and (2e) has dimension 4.

\[
\begin{array}{c}
(2e) \\
\bigtriangleup \\
(27) \\
\end{array}
\]

(6) (31) = (3e) 
(8) (32) = (4e) 
(9) (22) 
(33) = (35) 
(7) = (23) = (34) = (5e)

Proposition 4.3.

i) Family (34) is a fibre of the Prym map of the bielliptic locus (2e) and also a fibre of the Prym map of (4e).

ii) Families (9) and (22) are both contained in fibers of the map $\varphi$ of (2e) and are not contained in any fiber of the map $\mathcal{P}$ of (2e).

iii) Family (33) = (35) is not contained in any fibre of $\varphi$ nor in any fibre of $\mathcal{P}$ of both (2e) and (3e).

iv) Families (31) = (3e) and (32) = (4e) are not contained in fibres of the map $\varphi$ of (2e).

v) Family (27) is not contained in a fibre of the map $\varphi$ of (2e).
Proof. In the following, for the description of the families of [33], we will denote by \( \theta : \Gamma_{0,r} \to G \) the monodromy, we set \( x_i = \theta(\gamma_i), i = 1, \ldots, r \) and \( m = (m_1, \ldots, m_r) \), where \( m_j = o(x_j) \). For simplicity we will just write \( x = (x_1, \ldots, x_r) \) to describe the monodromy. We will also use the notation of MAGMA for the irreducible representations of the group \( G \). We will analyse the different families one at a time.

\((5e) = (7) = (23) = (34)\). The description of (34) in [33] is as follows:

\[
G = \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g_1, g_2, g_3 : g_1^2 = g_2^2 = g_3^4 = 1, g_2g_3 = g_3g_2, g_1^{-1}g_2g_1 = g_2g_3g_1^{-1}g_3g_1 = g_3 \rangle,
\]

\( m = (2, 2, 2, 4) \), \( x = (g_1, g_1g_2g_3, g_2g_3^2, g_3^3) \).

Using the notation of MAGMA, we have \( H^0(C, K_C) \cong V_6 \oplus V_{10} \), where \( V_i \) are irreducible representations of \( G \) and \( \dim(V_6) = 1 \), \( \dim(V_{10}) = 2 \). Moreover \( (S^2H^0(C, K_C))^G \cong S^2V_6 \), which has dimension 1, hence condition (\( \ast \)) holds. The group algebra decomposition for the curves \( C \) in the family (see [2], [34]), gives us a decomposition of the Jacobian \( JC \) up to isogeny: \( JC \sim B_6 \times B_{10}' \), where both the \( B_i \)'s have dimension one.

Choose \( H = \langle g_3^2 \rangle \) and consider the map \( f : C \to C/H \). Call \( z_1, \ldots, z_4 \) the critical values of the map \( \psi : C \to C/G \). One immediately verifies that there are only four critical points of index two for \( f \), all placed in the fiber \( \psi^{-1}(z_4) \). Applying Riemann-Hurwitz formula, we obtain \( g(C/H) = 1 \). Notice that this gives us the inclusion of (34) in (2e).

Denote by \( E := C/H \), we get \( B_6 \sim E \), and \( H^0(E, K_E) = H^0(C, K_C)^H = V_6 \). So the curve \( E \) moves, and the Prym variety \( P(C, E) \sim B_{10}' \). So the Prym variety \( P(C, E) \) doesn’t move, hence family (34) is a fiber of the Prym map of the family (2e).

Now we consider the normal subgroup \( Q_8 = \langle g_3^2, g_2g_3, g_1g_3, g_1g_2g_3^2 \rangle \) which is isomorphic to the quaternion group. The degree 8 map \( C \to C/Q_8 \) has a single branch point and the quotient \( C/Q_8 \) has genus 1. So (34) = (5e).

Now we show that this family is also a fiber of the Prym map of family (4e). In fact, consider the subgroup \( H' = \langle g_1g_3 \rangle \) of order 4, which contains \( H \), and the map \( f' : C \to C/H' \). One can check that \( f' \) has four critical points of order two and two critical values and \( C/H' \) has genus 1. This provides the inclusion (34) \( \subset \) (4e). Moreover \( H^0(C, K_C)^{H'} = V_6 \), so the curve \( E' \) = \( C/H' \) moves and it is isogenous to \( E = C/H \) while \( P(C, E') \) is fixed and it is isogenous to \( P(C, E) \). Hence (34) is also a fiber of the Prym map of (4e). Finally one can see that the elliptic curve \( B_{10} \) is obtained as the quotient \( C/(g_1) \).

\((33) = (35)\).

\( G = S_4 \), set \( g_1 = (12), g_2 = (123), g_3 = (13)(24) \) and \( g_4 = (14)(23) \). Monodromy: \( (x_1, x_2, x_3, x_4) = (g_1g_2^3, g_3g_4, g_1, g_2g_4) \), \( m = (2, 2, 2, 3) \), \( H^0(C, K_C) \cong V_4 \) (\( V_4 \) is an irreducible representation of dimension 3), so \( (S^2H^0(C, K_C))^G \cong (S^2V_4)^G \).

Considering the group algebra decomposition of the Jacobians of these curves we obtain \( JC_4 \sim B_3^5 \), where \( B_3 \) is an elliptic curve.

Let’s describe it. Take the subgroup \( H = \langle g_2 \rangle \), which has order three, and
consider the quotient map \( f \). We have two critical points of \( f \) in \( \psi^{-1}(z_1) \) of multiplicity equal to three. Applying Riemann-Hurwitz formula we get an elliptic curve \( E := C/H \). We can see that \( H^0(E, K_E) \) is a one dimensional subspace of \( V_4 \), hence \( E \) is isogenous to \( B_4 \). The family (35) is one dimensional so \( E \) moves, hence family (35) is not a fibre of the Prym map of (2e) and it is not contained in a fibre of the map \( \varphi \) of (2e). Finally, observe that \( f \) has exactly two critical values. This shows the inclusion (35) \( \subset \) (3e).

(9) \[ G := \mathbb{Z}/6 = \langle g_1, g_2 \mid g_1^2 = g_2^3 = 1 \rangle, \text{ monodromy: } \langle g_1, g_2^2, g_2^3 g_1, g_2^3 g_1^2 \rangle, \ m = (2, 3, 3, 6), H^0(C, K_C) \cong V_4 \oplus V_5 \oplus V_6 \text{ (the } V_i \text{'s are irreducible representations of } G) \text{ and } (S^2 H^0(C, K_C))^G \cong V_4 \oplus V_6. \]

Let \( C \) be a curve in the family and denote as usual the quotient map by \( \psi : C \to C/G \). Now take the subgroup \( H := \langle g_1 \rangle \). The corresponding quotient map \( C \to C/H \) has three critical points, of index two, in \( \psi^{-1}(z_1) \) and a single one, of the same type, in \( \psi^{-1}(z_4) \). By Riemann-Hurwitz formula we see that \( E := C/H \) is an elliptic curve with \( H^0(E, K_E) = V_5 \). First observe that this curve doesn’t move. Moreover we get (9) \( \subset \) (2e).

The group algebra decomposition gives \( JC \sim B_4 \times B_5 \), where \( B_5 \) is isogenous to the elliptic curve \( E \), that is fixed, \( B_4 \sim P(C, E) \) has dimension 2 and it moves. This shows that family (9) is contained in a fibre of the map \( \varphi \) of (2e). Hence this fibre of \( \varphi \) has an irreducible component which is a Shimura subvariety of \( \mathbb{A}_3 \) of dimension 3.

(22) \[ G := \mathbb{Z}/2 \times \mathbb{Z}/4, \text{ the monodromy is generated by } \langle g_3, g_2 g_3, g_1 g_2, g_1 g_3 \rangle, \text{ where } g_1 = (0, 1), g_2 = (1, 0) \text{ and } g_3 = (0, 2), \ m = (2, 2, 4, 4). \ We have \( H^0(C, K_C) \cong V_4 \oplus V_7 \oplus V_8 \text{ and } (S^2 H^0(C, K_C))^G \cong V_3 \oplus V_7. \ The group algebra decomposition gives } JC \sim B_3 \times B_8, \text{ with } \dim(B_3) = 2 \text{ and } \dim(B_8) = 1. \text{ Consider the subgroup } H = \langle g_2 g_3 \rangle. \text{ One easily checks that the quotient curve } E = C/H \text{ has genus one, with } (H^0(C, K_C))^H \cong V_8. \text{ Therefore } E \text{ is isogenous to } B_8 \text{ and it remains fixed. Now take another subgroup: } H' = \langle g_2 \rangle. \text{ The quotient map } C \to C/H' \text{ is étale so, applying Riemann-Hurwitz, we have } g(C/H') = 2. \text{ Being } \dim(V_8^{H'}) = 1 = s_8 V_3, \text{ where } s_8 V_3 \text{ is the Schur index of } V_3, \text{ and since } \dim(V_8^{H'}) = 0 \text{ we apply [21] Lemma 1} \text{ and we obtain } B_3 \sim JC'. \]

Now we call \( C/H' = C' \) and we study the degree 4 map \( C' \to C'/\langle \gamma \rangle \cong \mathbb{P}^1 \), where \( \langle \gamma \rangle \cong G/H' \cong \mathbb{Z}/2 \mathbb{Z}. \) One easily sees that the family \( C' \to C'/\langle \gamma \rangle \) is family (4) = (29). The group algebra decomposition of the Jacobians for family (29) is \( JC' \sim F^2 \), where \( F \) is an elliptic curve. Thus we conclude \( JC \sim E \times F^2 \). Since the curve \( E \) is fixed we see that family (22) is also contained in a fibre of the map \( \varphi \) of (2e), which then has an irreducible component that is a Shimura subvariety of \( \mathbb{A}_3 \) of dimension 3.

Now we analyze the families of dimension two \( (N = 2). \)

(31) \[ G := S_3 = \langle g_1, g_2 : g_1^2 = g_2^3 = 1, g_1^{-1} g_2 g_1 = g_2^2 \rangle, \text{ with monodromy given by} \]
where

\[(g_1g_2^2, g_1g_2, g_1, g_1g_2^2, g_2^2, m = (2, 2, 2, 3, 2)).\]

We have \(H^0(C, K_C) = V_2 \oplus V_3\), where \(V_2\) has dimension 1, \(V_3\) has dimension 2, and \((S^2H^0(C, K_C))^G = S^2V_2 \oplus (S^2V_3)^G\). The group algebra decomposition gives \(JC \sim B_2 \times B_3^2\), where both terms have dimension one. Consider a curve \(C\) of the family and denote as usual by \(\psi : C \to C/G \cong \mathbb{P}^1\) the quotient map. Denote 

\[H = \langle g_2 \rangle \subset G\]

and by \(\alpha : C \to C/H\) the quotient map. We have two critical points of index three in \(\psi^{-1}(z_3)\), hence \(E := C/\langle g_2 \rangle\) has genus one and one can show that \(H^0(E, K_E) = V_2\). Thus \(B_2 \sim E\) and we also have shown that \((31) = (3e)\). Finally one can easily see that \(C/\langle g_1 \rangle\) has genus 1 and \(B_3 \sim J(C/\langle g_1 \rangle)\). This gives the inclusion \((31) \subset (2e)\). Notice that both \(B_2\) and \(B_3\) move, so \((31)\) is not contained in a fibre of the map \(\varphi\).

\[(32)\]

\[G = D_4 = \langle x, y : x^4 = y^2 = 1, y^{-1}xy = x^3 \rangle,\]

monodromy given by \((x^3y, y, x^2, y, xy), m = (2, 2, 2, 2, 2)\). Moreover \(H^0(C, K_C) \cong V_4 \oplus V_5\), where \(V_4\) has dimension 1, \(V_5\) has dimension 2 and \((S^2H^0(C, K_C))^G \cong S^2V_4 \oplus (S^2V_5)^G\). The group algebra decomposition yields \(JC \sim B_4 \times B_5^2\).

Take the subgroup \(H = \langle x \rangle\) and consider the quotient map \(\alpha : C \to C/H\).

We get four critical points in \(\psi^{-1}(z_3)\) of index two, hence \(g(C/H) = 1\). This shows the inclusion \((32) \subset (4e)\). Consider the subgroup \(H' = \langle x^2 \rangle\) of \(H\). We can factor the degree four map \(\alpha\) into two maps of degree two. The map \(\alpha' : C \to C/H'\) has four critical points of index two in \(\psi^{-1}(z_3)\), hence \(C/H'\) has genus 1 and is isogenous to \(C/H\). This gives the inclusion \((32) \subset (2e)\).

We have \(H^1(C/H) \cong V_4\), therefore \(C/H \sim C/H' \sim B_4\).

Consider the subgroup \(K = \langle y \rangle\). One immediately checks that \(C/K\) has genus 1 and \(H^1(C/K) \cong V_5\), hence \(B_5 \sim E\). So both \(B_4\) and \(B_5\) move, hence \((32)\) is not contained in a fibre of the map \(\varphi\).

Let’s now describe the only family of dimension \(N = 3\).

\[(27)\]

\[G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g_1, g_2 : g_1^2 = g_2^2 = 1 \rangle,\]

monodromy: \((g_2, g_1g_2, g_1, g_1g_2, g_1, g_2)\), \(H^0(C, K_C) \cong V_2 \oplus V_3 \oplus V_4\), \((S^2H^0(C, K_C))^G \cong S^2V_2 \oplus S^2V_3 \oplus S^2V_4\). The Jacobian decomposes up to isogeny as \(JC \sim B_2 \times B_3 \times B_4\), where \(B_i\)’s are three different elliptic curves.

One easily checks that \(B_2 \sim C/H\), \(B_3 \sim C/H'\) and finally \(B_4 \sim C/H''\), where \(H = \langle g_2 \rangle\), \(H' = \langle g_1 \rangle\) and \(H'' = \langle g_1g_2 \rangle\). This gives the inclusion \((27) \subset (2e)\). All the three elliptic curves move, so \((27)\) is not a fibre of the map \(\varphi\).
This ends up the discussion of inclusions in genus 3. In genus 4 we have the following diagram of inclusions.

\[
\begin{array}{c}
\text{(10)} \\
\text{(11)} \\
\text{(12)} \\
\text{(13) = (24)} \\
\text{(25) = (38) (6e) (14)} \\
\text{(37) (36)}
\end{array}
\]

Families in the lowest line are one-dimensional, (14) has dimension 2, while (10) and (6e) have dimension 3.

**Proposition 4.4.** Family (12) is contained in a fibre of the Prym map of (6e), while families (25) = (38) and (37) are contained in fibers neither of the Prym map nor of the map \( \varphi \) of (6e). Family (37) is not contained in a fibre of \( \mathcal{P} \), while it is contained in a fibre of the map \( \varphi \) of (6e).

**Proof.** Let us start by considering family (12):

\[ G = \mathbb{Z}/6 = \langle g_1g_2 : g_1^2 = g_2^3 = 1 \rangle, x = (g_1, g_1g_2, g_1g_2, g_1g_2), \]

\[ m = (2, 6, 6, 6), \quad H^0(C, K_C) \cong V_2 \oplus V_3 \oplus 2V_6, \quad (S^2 H^0(C, K_C))^G \cong S^2 V_2. \]

The group algebra decomposition gives \( JC \sim B_2 \times B_3 \times B_6 \), where the first two terms have dimension one while the third one has dimension equal to two. Consider the subgroup \( H := \langle g_2 \rangle \cong \mathbb{Z}/3 \). Call \( z_1, z_2, z_3, z_4 \) the branch points for the map \( \psi : C \to C/G \) and consider the quotient map \( f : C \to C/H \). There are three critical points for \( f \), of order three, in the preimage \( \psi^{-1}(z_i), i = 2, 3, 4 \). Applying Riemann-Hurwitz formula we get that the genus of \( E := C/H \) is one. This gives us the inclusion \( (12) \subset (6e) \).

Moreover, since \( H^0(C, K_C)^H = V_2 \), we get \( B_2 = E \). Consider \( H' := \langle g_1 \rangle \cong \mathbb{Z}/2 \). Each critical point for \( \psi \) is also critical of order two for the quotient map \( C \to C/H' \). Riemann-Hurwitz formula implies that \( C/H' \) is an elliptic curve. We have \( H^0(C/H', K_{C/H'}) = V_3 \) and thus \( B_3 \sim C/H' \).

Notice that the terms \( B_3 \) and \( B_6 \) don’t move and their product is isogenous to the Prym variety \( P(C, E) \). Thus, as already observed in [IS], (12) is contained in a fiber of the Prym map of (6e).

\[(25) = (38)\]

\[ G = \mathbb{Z}/3 \times S_3 \] with generators \( g_1 = ([0]_3, (12)), g_2 = (1, (1)) \) and \( g_3 = ([0]_3, (123)) \) and monodromy \( (g_1g_3^2, g_1g_3, g_2g_3, g_3^2) \). We know that \( H^0(C, K_C) \cong V_3 \oplus V_4 \oplus V_6 \) and that \( (S^2 H^0(C, K_C))^G \cong V_4 \oplus V_6 \). The first two \( V_i \)’s have dimension one while \( V_6 \) has dimension equal to two. The Jacobian decomposes as \( JC \sim B_3 \times B_6^2 \), the first term is 2-dimensional while the second is 1-dimensional.

Set \( H := \langle g_2g_3 \rangle \cong \mathbb{Z}/3 \) and consider the quotient map \( f : C \to C/H \). We get three critical points of \( f \) of order three all contained in \( \psi^{-1}(z_3) \). Hence \( g(C/H) = 1 \) and we also see the inclusion \( (38) \subset (6e) \). Moreover \( H \) fixes a one-dimensional subspace of \( V_8 \), thus we get \( C/H \sim B_8 \). Note that this
term of the decomposition doesn’t move. Now take the quotient for $H' = \langle g_3 \rangle \cong \mathbb{Z}/3$. The correspondent quotient map is étale. Therefore we obtain $g(C/H') = 2$. Due to the fact that $\dim(V_3^{H'}) = 1 = s_{V_3}$, where $s_{V_3}$ is the Schur index of $V_3$, and since $\dim(V_3^{H'}) = 0$, there exists an isogeny between $J(C/H')$ and $B_3$ (see [21, Lemma 1]). Since $H'$ is normal in $G$ we can now look at the map $C/H' \to \mathbb{P}^1$. This is a Galois covering with Galois group $G/H' \cong \mathbb{Z}/6$ and $m = (2, 2, 3, 3)$. Actually we have only one one-dimensional family with this datum and it corresponds to family (5) of [9]. Using the group algebra decomposition on the Jacobian of family (5) of \cite{9}. We get $B_3 \sim J(C/H') \sim E^2$, where $E$ is an elliptic curve. Since $E$ moves, family (38) isn’t a fiber of the Prym map of (6e) and it is not contained in a fibre of $\varphi$ of (6e).

(37) $G = A_4$, the generators for the monodromy are $(g_3,g_1g_3,g_1, g_1g_2g_3)$, where $g_1 = (123)$, $g_2 = (12)(34)$, and $g_3 = (13)(24)$, $m = (2, 3, 3, 3)$. Moreover we have $H^0(C, K_C) \cong V_2 \oplus V_4$, where $V_2$ has dimension one and $V_4$ has dimension equal to three and $(S^2H^0(C, K_C))^G \cong (S^2V_4)^G$. The Jacobian decomposes completely as $JC \sim B_2 \times B_4^3$.

Take the quotient $\psi : C \to C/G$ and call, as usual, the branch points $z_i, \ i=1, 2, 3, 4$. Now we consider the subgroup generated by $(g_1)$. It is a cyclic group of order three. Studying the map $C \to C/(g_1) := E$ we get three critical points, of order three, respectively in the fibers $\psi(z_i)^{-1}, \ i = 2, 3, 4$ and so we have $g(E) = 1$ (and also the inclusion in the family (6e)). Moreover $(g_1)$ fixes a one-dimensional subspace of $V_4$ and thus $E \sim B_4$.

If we consider the subgroup $H := \langle g_2, g_3 \rangle$ and its associated quotient map $f : C \to C/H$, all the critical points in the fiber of $z_1$ are critical points of order two for $f$. Thanks to Riemann-Hurwitz formula we see that the curve $E := C/H$ is elliptic. Since $\text{Fix}(H) = V_2$ we obtain $E \sim B_2$. Although this curve remains constant for the family, we have that $P(C, E) \sim B_2 \times B_4^2$, hence it moves. Thus (37) isn’t a fiber of the Prym map of (6e), but it is contained in a fiber of the map $\varphi$ of (6e). Hence this fibre of $\varphi$ has an irreducible component which is a Shimura subvariety of $A_4$ of dimension 2.

\[\square\]

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