DEFORMATION QUANTIZATION OF $A_\infty$-MORITA EQUivalences

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We show that Deformation Quantization of quadratic Poisson structures preserves the $A_\infty$-Morita equivalence of a given pair of Koszul dual $A_\infty$-algebras.

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1. Introduction

In this paper we consider a finite dimensional vector space $X$ over a field $\mathbb{K}$ of characteristic 0 and the associative algebras with zero differentials $A = S(X^*)$ resp. $B = \wedge(X)$ i.e. the symmetric algebra over $X^*$, resp. the exterior algebra over $X$. For simplicity we choose $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In [3] it is shown that it is possible to endow $\mathbb{K}$ with an $A_\infty$-$A_\infty$-bimodule given by a codifferential $d_\mathbb{K}$ whose Taylor components are defined by certain perturbative expansions in Feynman diagrams. The expansions are written by considering configuration spaces of points on the complex upper half plane and differential 1-forms called the 4-colors propagators. This construction and those in [5], [6] are the first partial example of multi-brane generalization of the results by M. Kontsevich on Deformation Quantization of Poisson manifolds; see [13]. In [3] it is shown that the $A_\infty$-$A_\infty$-bimodule $(K,d_\mathbb{K})$ is s.t. the classical Koszul duality between $A$ and $B$ holds, i.e. there exists isomorphisms

$$A \simeq \text{Ext}_B(K, \mathbb{K}), \quad B \simeq \text{Ext}_A(\mathbb{K}, \mathbb{K})^{\text{op}},$$

of algebras: as left $A_\infty$-$A$-module and right $A_\infty$-$B$-module $K$ is in fact the classical augmentation module.

Our first goal is to prove an $A_\infty$-derived Morita equivalence for the pair $(A,B)$ explicitly, i.e. the equivalence of certain triangulated subcategories of the derived categories of strictly unital $A_\infty$-right-modules over $A$ and $B$ by using the $A_\infty$-bimodule $(K,d_K)$: $A$ and $B$ are just associative algebras with zero differential but we consider categories of $A_\infty$-modules over them.

It is natural to introduce a bigrading on the triple $(A,K,B)$: the first grading is cohomological; the second grading is called internal; consequently we consider only bigraded $A_\infty$-structures, i.e. bigraded $A_\infty$ modules, bimodules, morphisms between them etc. By definition, the internal grading is preserved by the $A_\infty$-structures and morphisms between them.

The $A_\infty$-Morita equivalence for the pair $(A,B)$ has been already proved in [27], where a more general result is shown. In [27], (see prop. 1.14, 3.1. and thm. 5.7, 5.8 loc. cit.) the authors prove the aforementioned equivalence by “returning” to the differential bigraded level by considering the derived categories of differential bigraded modules over the enveloping algebras $UA$, resp. $UB$ of $A$ resp. $B$. The enveloping algebra $UA'$ of any bigraded $A_\infty$-algebra $A'$ is a differential bigraded algebra. It is introduced in [27] as the theory of differential bigraded algebras is, in general, simpler than the theory of bigraded $A_\infty$-algebras. Such an approach has the advantage of using the already
well-known results on the enveloping algebras and (bar) resolutions of differential bigraded algebras. On the other way, using this approach one introduces the iterated use of the Koszul dual functor \(E(\cdot)\), which associates to any augmented \(A\)-algebra \(A'\) its \(A\)-Koszul dual \(E(A') = \text{Hom}(U(A'), \mathbb{K})\). Moreover the enveloping algebra \(U(A')\) is a rather “big” bigraded object, as by definition it is the cobar construction of the bar construction over \(A'\).

Our approach is alternative to the one presented in [27]; we use the \(A\_\infty\)-bimodule \(K\) to prove the Morita equivalence at the \(A\_\infty\)-level, without using the enveloping algebras \(U(A, UB)\) and returning to the differential bigraded level.

The key observation in our construction is that the left derived derived actions ([10], [3])

\[
L_A : A \rightarrow \text{End}_\mathbb{K}(K), \quad R_B : B \rightarrow \text{End}_\mathbb{K}(K)^{op},
\]

are quasi-isomorphisms of strictly unital \(A\_\infty\-A\-A\)-bimodules and strictly unital \(A\_\infty\-B\-B\)-bimodules; this is done in subsection 5.0.3. We use this fact to prove the equivalences of categories before and after deformation quantization.

The pair of functors inducing the equivalence is studied in subsection 6.0.10. We define them by using the tensor products \(\otimes\) of \(A\_\infty\)-modules described in subsection 4.0.6. The main advantage of such “pure” \(A\_\infty\)-approach, aside from the explicit use of the bimodule \(K\), is represented by the possibility of quantizing the equivalences: this is the content of section 8. Let \(\hbar\pi\) be an \(h\)-formal quadratic Maurer-Cartan-element of cohomological degree 1 in \(T_{poly}(X)[[\hbar]]\), the ring of formal power series in \(\hbar\) with coefficients in \(T_{poly}(X) = S(X^\ast) \otimes \wedge^{\ast+1}(X)\).

\(T_{poly}(X)[[\hbar]]\) is a differential graded Lie algebra with zero differential and graded Lie bracket \([\cdot, \cdot]_\hbar\) obtained by extending \(\mathbb{K}[[\hbar]]\)-linearly the Schouten-Nijenhuis bracket \([\cdot, \cdot]\) on \(T_{poly}(X)\). With such a choice of Poisson bivector the internal grading on the triple on \((A, K, B)\) is provided by the \(\hbar\)-module \(\mathbb{K}[[\hbar]]\)-level, without using the enveloping algebras

\[
\pi \in \text{End}_\mathbb{K}(K), \quad \text{with } \pi : K \rightarrow K.
\]

Our approach is quite “down-to-earth”: we adapt the definitions and results in [20] to our topological \(A\_\infty\)-setting. It follows that the quantizations \(A_\hbar\), resp. \(B_\hbar\) of \(A\), resp. \(B\) are associative bigraded algebras with zero differentials. The quantized bimodule \(K_\hbar = (K[[\hbar]], d_K)\) satisfies the quantized version of the Keller condition, and it is a left \(A\_\hbar\)-module and a right \(B\_\hbar\)-module with zero differential. Furthermore, one is able to construct a bar resolution \(M^\bullet \otimes_{\mathbb{K}[[\hbar]]} K\) for \(K\) in the natural way, using this approach one introduces the iterated use of the Koszul dual functor

\[
\text{End}_\mathbb{K}(K)^{op},
\]

Let \(\pi \in \text{End}_\mathbb{K}(K)\) be any \(K\)-bimodule, and \(\pi, \pi'\) be their thickenings.

The main result of these notes is then the Deformation Quantization of \((A, K, B)\). The triangulated functor

\[
\mathcal{F}_A : D^b_{\text{Der}}(A) \rightarrow D^b_{\text{Der}}(K), \quad \mathcal{F}_A(\pi) = \pi \otimes_{A_\hbar} K_\hbar
\]

induces the equivalence of triangulated categories

\[
\triang_{\text{Der}}(A_\hbar) \simeq \triang_{\text{Der}}(K_\hbar), \quad \text{thick}_{\text{Der}}(A_\hbar) \simeq \text{thick}_{\text{Der}}(K_\hbar).
\]

The main result of these notes is then

\[
\mathcal{F}_A : D^b_{\text{Der}}(A) \rightarrow D^b_{\text{Der}}(K), \quad \mathcal{F}_A(\pi) = \pi \otimes_{A_\hbar} K_\hbar
\]

Let \((K, d_K)\) be the \(A\_\infty\-B\-B\)-bimodule with \(K = K\) and \(d_K\) obtained from \(d_K\) exchanging \(A\) and \(B\); the triangulated functor

\[
\mathcal{F}_B : D^b_{\text{Der}}(B) \rightarrow D^b_{\text{Der}}(A), \quad \mathcal{F}_B(\pi) = \pi \otimes_{B_\hbar} K_\hbar
\]

is a \(K\)-bimodule, and \(\pi, \pi'\) be their thickenings.

The main result of these notes is then

\[
\mathcal{F}_B : D^b_{\text{Der}}(B) \rightarrow D^b_{\text{Der}}(A), \quad \mathcal{F}_B(\pi) = \pi \otimes_{B_\hbar} K_\hbar
\]
induces the equivalence of triangulated categories

\[
\text{triang}^\infty_{A_b}(B_h) \simeq \text{triang}^\infty_{B_b}(B_h), \quad \text{thick}^\infty_{A_b}(B_h) \simeq \text{thick}^\infty_{B_b}(B_h).
\]

In other words, Deformation Quantization of \(h\)-formal quadratic Poisson bivectors preserves the \(A\infty\)-Morita equivalence of the Koszul dual \(A\infty\)-algebras \(A\) and \(B\).

In Appendix A we show the proof of prop. \ref{prop:2} while in Appendix B-C we prove thm. \ref{thm:3} and thm. \ref{thm:4} in some detail. Such proof are conceptually quite easy; using the very definition of the triangulated subcategories \(\text{triang}^{\infty}_{A_b}(A_h)\) … \(\text{thick}^{\infty}_{B_b}(K_h)\) we just need to check the commutativity of diagrams in which the quasi-isomorphisms of \(A\infty\)-bimodules of section \ref{section:6} appear. Moreover, the proof of thm. \ref{thm:4} is analogous to the one of thm. \ref{thm:3} with mild changes.

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3. Notation and Conventions

Let \(K\) be a field of characteristic 0. Throughout this work we fix \(K = \mathbb{R}\) or \(\mathbb{C}\). Let \(bG_{\infty}\) be the category of \(Z\)-bigraded vector spaces, i.e. collections \(\{M^i\}_{i \in \mathbb{Z}}\) of vector spaces over \(K\). The upper grading is also called the “cohomological grading”. The lower index denotes the “internal grading”. The space of morphisms \(\text{Hom}_{bG_{\infty}}(M, N)\) is the \(Z\)-bigraded vector space with \((r, s)\) component

\[
\text{Hom}_{bG_{\infty}}^{r,s}(M, N) = \prod_{n,m \in \mathbb{Z}} \text{Hom}(M^n_m, N^{n+r}_{m+s}),
\]

for every \(r, s \in \mathbb{Z}^2\).

Any \(f \in \text{Hom}_{bG_{\infty}}^{r,s}(M, N)\) is said to be a bigraphed morphism of bidegree \((r, s)\). The identity morphisms in \(bG_{\infty}\) are denoted simply by \(1\). For any object \(M\) in \(bG_{\infty}\), we denote by \(M[n]\) the object in \(bG_{\infty}\) such that \((M[n])^i_j := M^{i+n}_j\); the degree 1-isomorphism \(s: M \to M[1]\), \(s(m) := m(\) called the suspension map; its inverse of degree 1 \(s^{-1}: M[1] \to M\) is the desuspension. Both are endofunctors of \(bG_{\infty}\) with \((s^{-1})^2 \circ s = (s^{-1})^3 \circ 1\). We use the short notation \(s|\) for \(s(m) \in M[1]\). The cohomological degree of bihomogeneous elements of \(M\) is denoted by \(|\cdot|\); in particular \(|m| = |m| - 1\), for every \(m \in M[1]\).

Similarly, the object \(M[j]\) in \(bG_{\infty}\) is s.t. \(M[j]^n_m := M_{m+j}^n\), for any \(j \in \mathbb{Z}\). It follows that \(\text{Hom}_{bG_{\infty}}^{r,s}(M, N) = \text{Hom}_{bG_{\infty}}^{r,0,0}(M, N[r]|\langle s\rangle)\).

The tensor product \(M \otimes N\) of any two objects in \(bG_{\infty}\) is the object in \(bG_{\infty}\) with bihomogeneous components

\[
(M \otimes N)^n_m = \bigoplus_{p+q=n} M^p_p \otimes N^q_q,
\]

for every \(n, m \in \mathbb{Z}\) with \(\otimes = \otimes_{\infty}\). Throughout this work we will use the shorthand conventions \(m_1, \ldots, m_n \equiv m_1 \otimes \cdots \otimes m_n\), and \((m_1|\cdots|n) \equiv s(m_1) \cdots \otimes s(m_n)\), for any \(m_1, \ldots, m_n \in M \in bG_{\infty}\). So, in particular, \((m_1, m_2|m_3) = m_1 \otimes s(m_2) \otimes s(m_3)\) and \((m_1|m_2, m_3) = s(m_1) \otimes s(m_2) \otimes m_3\). In what follows we assume that the Koszul sign rule holds.

4. \(A\infty\)-Structures

In this section we introduce \(A\infty\)-structures from a purely algebraic point of view. We recall the concept of \(A\infty\)-algebra, \(A\infty\)-module, \(A\infty\)-bimodule and their morphisms. We focus our attention on unital \(A\infty\)-structures, augmented \(A\infty\)-algebras. The tensor product of \(A\infty\)-modules is also considered; it contains the bar resolution of a module over a given unital algebra as special case. \(A\infty\)-algebras have been introduced by Stasheff [25] in the sixties in algebraic topology; in the nineties they have been further popularized by Kontsevich’s [14] in his Homological Mirror Symmetry conjecture. The material here presented is standard; we refer to [12, 9, 19, 26] for all details, in particular the definitions of coalgebras, coderivations, comodules etc. For the interested reader, we just note that such definitions can be deduced by taking the “limit” \(h = 0\) in the formulæ appearing in section 8. Tensoring of \(A\infty\)-bimodules has been introduced explicitly in [18], extending the case of right \(A\infty\)-modules contained in [12]. In what follows we will consider only bigraded \(A\infty\)-structures; the rule of thumb is that the maps defining the \(A\infty\)-structures themselves preserve the internal grading. In this sense, there is not substantial difference between the graded and bigraded case.
4.0.1. $A_\infty$-algebras. Let $A$ be an object of $bG_K$. The coassociative counital tensor coalgebra on $A$ is the triple
\[
B(A) := \left(T^c(A[1]), \Delta, \epsilon\right),
\]
where $T^c(A[1]) = \sum_{k\geq 0} A[1]^k \otimes_k A[1]^k$. The coassociative coproduct $\Delta(a_1 \ldots a_n) = 1 \otimes (a_1 \ldots a_n) + (a_1 \ldots a_n) \otimes 1 + \sum_{n' \geq 1} a_{n'1} \otimes (a_{n'2} \ldots a_n)$ and the counit $\epsilon$ denotes the projection onto $K$; by definition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta = 1$.

**Definition 1.** (J. Stasheff, [23].) An $A_\infty$-algebra is a pair $(A, d_A)$, where $A$ is an object of $bG_K$ and $d_A$ is a bidgree $(1, 0)$ coderivation on $B(A)$ s.t.
\[
d_A \circ d_A = 0.
\]

By the lifting property of coderivations on $B(A)$, such $d_A$ is uniquely determined by its Taylor components, i.e. the family of morphisms $d_A^n := pr_{A[1]} c d_A | A[1]^{\otimes n}$, $n \geq 0$, denoting by $pr_{A[1]}$ the projection $pr_{A[1]} : T^c(A[1]) \rightarrow A[1]$.

Then $d_A \circ d_A = 0$ is equivalent to
\[
\sum_{k=0}^n \sum_{j=1}^{k-s_i+1} (-1)^j d_A^{k-s_i+1} \left(a_i \ldots \langle a_{j-1}, d_A^s (a_{j} \ldots a_{s_{i+1}}) \rangle a_{s_i+j} \ldots a_k\right) = 0,
\]
for every $k \geq 0$ and $(a_1, \ldots, a_k) \in B(A)$. The Koszul sign is simply $\epsilon = \sum_{i=1}^j (a_i - 1)$. Equivalently, we can consider the bidgree $(2, n)$ maps $m_n$ defined through
\[
d_A^0 = - s \circ m_0,
\]
\[
d_A^n = - s \circ m_n \circ (s^{-1})^\otimes n, \quad n \geq 1.
\]

An $A_\infty$-algebra $(A, d_A)$ is said to be flat if $d_A^0 = 0$. In this case $m_1$ is a differential and $m_2$ is associative up to homotopy. It reduces to an associative product on the cohomology $H(A)$ with respect to $m_1$. If a flat $A_\infty$-algebra is s.t. $m_3 = m_4 = \cdots = 0$, then it is a differential bigraded algebra. If $(A, d_A)$ is not flat, then it is called curved, with curvature $d_A^1$ (or $d_A^1(1)$; we use both notations). In presence of non trivial curvature, $d_A^1$ is not a differential. Any graded associative algebra $A$ s.t. $d_A^1(1)$ is a degree 2 element in the center of $A$ is a curved $A_\infty$-algebra. Curvature appears naturally in Deformation Quantization: see for example [2]. Curved $A_\infty$-algebras are also related to models in theoretical physics [3]. With a little abuse of notation we introduce the following

**Definition 2.** Let $(A, d_A)$ and $(B, d_B)$ be $A_\infty$-algebras. A morphism $F : A \rightarrow B$ of $A_\infty$-algebras is a morphism $F \in \operatorname{Hom}_{bG_K}^0 (B(A), B(B))$ of coassociative counital coalgebras s.t.
\[
F \circ d_A = d_B \circ F.
\]

$F : T^c(A[1]) \rightarrow T^c(B[1])$ is uniquely determined by the family of morphisms $F_n : A[1]^{\otimes n} \rightarrow B[1]^{\otimes n}$ s.t. $pr_{B[1]} c F | A[1]^{\otimes n} = F_n$ and $F(1) = 1$. The morphisms $F_n$ are called the Taylor components of $F$. $F \circ d_A = d_B \circ F$ is equivalent to a tower of quadratic relations involving the Taylor components $F_n$, $d_A^n$ and $d_B^n$ of $F$, $A$ and $B$, respectively. If $(A, d_A)$, resp. $(B, d_B)$, are curved $A_\infty$-algebras with curvature $d_A^1$, resp. $d_B^1$, then, by definition of $F$: $F_1(d_A^1(1)) = d_B^1(1)$.

It is useful to introduce the degree $1 - n$ unsuspended morphisms $f_n : A^{\otimes n} \rightarrow B$ in $bG_K$, through
\[
f_n = s \circ f_n \circ (s^{-1})^\otimes n,
\]
for every $n \geq 0$. A morphism $F : A \rightarrow B$ of $A_\infty$-algebras is said to be strict if $f_n = 0$ for $n \geq 2$. If $A$ and $B$ are flat, $F$ is a quasi-isomorphism if $F_1$ is a quasi-isomorphism in $bG_K$.

4.0.2. Units and augmentations in flat $A_\infty$-algebras. Let $(A, d_A)$ be an $A_\infty$-algebra; the maps $m_n$, $n \geq 0$ and $f_m$, $m \geq 1$, have been defined in [2], (3).

**Definition 3.** An $A_\infty$-algebra $(A, d_A)$ is said to be strictly unital if it contains an element $1_A \in A^0_0$ s.t.
\[
m_2(a, 1_A) = m_2(1_A, a) = a,
\]
for any $a \in A$ and $m_{n}(a_1, \ldots, a_n) = 0$ for $n \geq 3$ if $a_i = 1$ for some $i = 1, \ldots, n$.

We note that, if $A$ is strictly unital, then $d_A^1(sA) = 0$, also in presence of curvature on $A$.

A morphism $F : A_1 \rightarrow A_2$ of strictly unital $A_\infty$-algebras is said to be strictly unital if $f_1(1_{A_1}) = 1_{A_2}$, and $f_m(a_1, \ldots, a_m) = 0$ for $m \geq 2$ if $a_i = 1_{A_i}$, for some $i = 1, \ldots, m$. In particular, it follows that $d_B^1(F_1(1_{A_1})) = 0$.

**Lemma 1.** Any strictly unital flat $A_\infty$-algebra $A$ with unit $1_A$ comes equipped with a strict strictly unital morphism $\eta : K \rightarrow A$, sending the unity $1$ of the ground field $K$ to $1_A$. 

This allows us to introduce the following

**Definition 4.** A strictly unital flat $A_{\infty}$-algebra $(A,d_A)$ with unit $1_A$ is augmented if there exists a strictly unital $A_{\infty}$-algebra morphism $\epsilon : A \to K$, s.t. $\epsilon \circ \eta = 1$.\footnote{For any $A_{\infty}$-algebra $B$, the identity morphism $1 : B \to B$ is the strict $A_{\infty}$-morphism with non trivial Taylor component $\bar{1}^1(b) = b$, for every $b \in B$.}

We note that the morphism $\epsilon \circ \eta$ is strict as $\epsilon$ is strictly unital. If $A$ is an augmented $A_{\infty}$-algebra with augmentation $\epsilon$, then we call ker $\epsilon$ the augmentation ideal of $A$.

**4.0.3. $A_{\infty}$-modules and $A_{\infty}$-bimodules.** In this subsection $(A,d_A)$ and $(B,d_B)$ are $A_{\infty}$-algebras.

**Definition 5.** A left $A_{\infty}$-$A$-module is pair $(M,d_M)$, where $M$ is an object in $\mathbf{bG}_K$ and $d_M \in \text{Hom}^{1,0}_{\mathbf{bG}_K}(\mathcal{L}(M),\mathcal{L}(M))$ is a codifferential on $\mathcal{L}(M) := T(A[1]) \otimes M[1]$ s.t.

$$d_M \circ d_M = 0.$$ 

As in the case of morphisms and coderivatives on the tensor coalgebra $T^*(V)$, the codifferential $d_M$ is uniquely determined by its Taylor components $\bar{d}_M^s : A[1]^{\otimes s} \otimes M[1] \to M[1]$, $s \geq 0$, via

$$d_M^s = \sum_{s_1=0}^{k} \sum_{j=1}^{k-s_1+1} 1^{\otimes j-1} \otimes \bar{d}_A^{s_1} \otimes 1^{\otimes k-s_1-j+1} + \sum_{s_2=0}^{k} 1^{\otimes k-s} \otimes \bar{d}_M^s,$$

where the $\bar{d}_A^s$ denote the Taylor components of the coderivation $d_A$ defining the $A_{\infty}$-algebra structure on $A$.

Let $(M,d_M)$ be a left $A_{\infty}$-$A$-module. $d_M \circ d_M = 0$ is equivalent to

$$\sum_{s_1=0}^{k} \sum_{j=1}^{k-s_1+1} (-1)^{i_1} \bar{d}_A^{s_1} (a_1|...|a_{j-1}, \bar{d}_M^s (a_j|...|a_{s_1+j-1})|a_{s_1+j}|...|a_k|m) +$$

$$\sum_{s_2=0}^{k} (-1)^{s_2} \bar{d}_M^{s_2} (a_1|...|a_{k-s_2}, \bar{d}_M^s (a_{k-s_2+1}|...|a_k|m)) = 0,$$

with $\epsilon_1 = \sum_{i=1}^{s_1} (|a_i| - 1)$, $\epsilon_2 = \sum_{i=1}^{s_2} (|a_i| - 1)$.

**Remark 2.** With obvious changes it is possible to define right $A_{\infty}$-$A$-modules on the right $B(A)$-counital comodule $R(M) = M[1] \otimes T(A[1])$.

If $A$ is curved then $\bar{d}_M^s (d_B^1(1), sm) + \bar{d}_M^s (d_M(sm)) = 0$, i.e. in presence of non trivial curvature $d_B^1(1)$, $\bar{d}_M^s$ is not a differential on $M[1]$.

**Definition 6.** A morphism $F : M \to N$ of left $A_{\infty}$-modules $(M,d_M), (N,d_N)$ is a morphism $F \in \text{Hom}^{0,0}_{\mathbf{bG}_K}(\mathcal{L}(M),\mathcal{L}(N))$ of left-$B(A)$-counital-comodules s.t.

$$F \circ d_M = d_N \circ F.$$ 

Any morphism $F : M \to N$ of left $A_{\infty}$-modules is uniquely determined by its Taylor components $F_n : A[1]^{\otimes n} \otimes M[1] \to N[1]$.

**Definition 7.** A morphism $F : M \to N$ of left-$A_{\infty}$-modules is said to be strict if $F_n = 0$ for $n \geq 1$. If $A$ is flat, $F$ is a quasi-isomorphism if $F_0$ is a quasi-isomorphism.

**Definition 8.** An $A_{\infty}$-$A$-$B$-bimodule is a pair $(M,d_M)$, where $M$ is an object in $\mathbf{bG}_K$ and $d_M \in \text{Hom}^{1,0}_{\mathbf{bG}_K}(\mathcal{B}(M),\mathcal{B}(M))$ is a codifferential on $\mathcal{B}(M) = T(A[1]) \otimes M[1] \otimes T(B[1])$ s.t.

$$d_M \circ d_M = 0.$$
Once again, it is possible to show that the codifferential $d_M$ is uniquely determined by the Taylor components $d^k_M := A[1] \otimes k \otimes M[1] \otimes B[1]^{[\infty]} \to M[1]$, $k, l \geq 0$, with
\[
d^{k,l}_M = \sum_{s_1=0}^k \sum_{j=1}^{l-s_1+1} 1^{\otimes j-1} \otimes \tilde{d}^1_A \otimes 1^{\otimes k-j-s_1+1+l+1} + \sum_{s_2=0}^l \sum_{j=1}^{l-s_2+1} 1^{\otimes k+l+1} \otimes 1^{\otimes j-1} \otimes \tilde{d}^2_B \otimes 1^{\otimes l-j-s_2+1} + \sum_{s_3=0}^k \sum_{s_4=0}^l 1^{\otimes k-s_3} \otimes \tilde{d}^{s_3,s_4}_M \otimes 1^{\otimes l-s_4}.
\]

Then $d_M \circ d_M = 0$ is equivalent to a tower of quadratic relations similar to $[1]$, with due differences. In presence of non trivial curvatures on $A$ and/or $B$, then $d^0_M$ is not a differential on $M[1]$.

**Lemma 2** ([15]). Let $(A, d_A)$, $(B, d_B)$ be $A_\infty$-algebras and $(M, d_M)$ be an $A_\infty$-$A$-$B$-bimodule.

- If $B$ is flat, then the family $d^{k,0}_M : A[1] \otimes k \otimes M[1] \to M[1]$ defines a left-$A_\infty$-$A$-module structure on $M$.
- If $A$ is flat, then the family $d^{0,l}_M : M[1] \otimes B[1]^{[\infty]} \to M[1]$, $l \geq 0$, defines a right-$A_\infty$-$B$-module structure on $M$.

**Remark 3.** Every $A_\infty$-algebra $(A, d_A)$ is an $A_\infty$-$A$-$A$-bimodule with $A_\infty$-bimodule structure given by the Taylor components $d_A^{k,l} : A[1] \otimes k \otimes A[1] \otimes A[1]^{[\infty]} \to A[1]$, with $d_A^{k,l} = \tilde{d}_A^{k+l+1}$.

**Definition 9.** Let $(M, d_M)$ and $(N, d_N)$ be two $A_\infty$-$A$-$B$-bimodules, with $B(M) = T(A[1]) \otimes M[1] \otimes T(B[1])$, and similarly for $B(N)$. A morphism of $A_\infty$-$A$-$B$-bimodules is a morphism $F \in \text{Hom}_{bG_\infty}(B(M), B(N))$ of $B(A)$-$B(B)$-codifferential- counital bimodules s.t.
\[
F \circ d_M = d_N \circ F.
\]

Any $A_\infty$-$A$-$B$-bimodule morphism $F$ is uniquely determined by its Taylor components $F^{k,l} : A[1] \otimes k \otimes M[1] \otimes B[1]^{[\infty]} \to N[1]$, $k, l \geq 0$. Explicitly
\[
F^{k,l} = \sum_{s_1=0}^k \sum_{s_2=0}^l 1^{\otimes k-s_1} \otimes F^{s_1,s_2} \otimes 1^{\otimes l-s_2},
\]
where $F^{k,l} := F|_{A[1]^{[s]} \otimes M[1]^{[s]} \otimes B[1]^{[s]}}$. If $A$, resp. $B$ are curved with curvature $\tilde{d}_A^{0}$, resp. $\tilde{d}_B^{0}$, then $F^{0,0}$ does not commute with $d_M^{0,0}$ and $d_N^{0,0}$ (which are not differentials).

### 4.0.4. Units in $A_\infty$-modules

Let $(A, d_A)$ be a strictly unital $A_\infty$-algebra with unit $1_A$ and $(M, d_M)$ a left $A_\infty$-$A$-module. We introduce the desuspended maps
\[
d^1_M = -s \circ d^1_M \circ (s^{-1})^{[\infty]}, \quad l \geq 0.
\]

**Definition 10.** The module $(M, d_M)$ is strictly unital if
\[
d^1_M(1_A, m) = m,
\]
for every $m \in M$ and $d^a_M(a_1, \ldots, a_n, m) = 0$ for $n \geq 2$ with $a_i = 1_A$ for some $i = 1, \ldots, n$.

Similar considerations hold for right $A_\infty$-modules. A strictly unital morphism of strictly unital $A_\infty$-modules is an $A_\infty$-morphism $F$ s.t.
\[
F^a(a_1| \ldots | a_n|m) = 0, \quad n \geq 2
\]
with $a_i = 1_A$ for some $i = 1, \ldots, n$ and $F^1(1_A|m) = -sm$.

Similar definitions hold for unital $A_\infty$-$B$-bimodules over strictly unital $A_\infty$-algebras.

### 4.0.5. Homotopies of strictly unital $A_\infty$-modules

Let $A$ be a strictly unital $A_\infty$-algebra and $(M, d_M)$, $(N, d_N)$ be strictly unital $A_\infty$-$A$-modules. Let $f, g : M \to N$ be morphisms of $A_\infty$-$A$-modules; we say that $M$ and $N$ are $A_\infty$-homotopy equivalent (alternatively: $A_\infty$-homotopic) if there exists an $A_\infty$-homotopy between them, i.e. a bidegree $(-1,0)$ morphism $H \in \text{Hom}_{bG_\infty}(M[1] \otimes T(A[1]), N[1] \otimes T(A[1]))$ of counital $T(A[1])$-comodules, s.t.
\[
f_h - g_h = d_{N_h} \circ H_h + H_h \circ d_{M_h}.
\]
4.0.6. The tensor product of $A_{\infty}$-bimodules. We consider now three $A_{\infty}$-algebras $(A, d_A)$, $(B, d_B)$ and $(C, d_C)$. Furthermore, we introduce an $A_{\infty}$-$A$-$B$-bimodule $(K_1, d_{K_1})$ and an $A_{\infty}$-$B$-$C$-bimodule $(K_2, d_{K_2})$.

Definition 11. The tensor product $K_1 \otimes_B K_2$ of $K_1$ and $K_2$ over $B$ is the object

$$K_1 \otimes_B K_2 = K_1 \otimes T(B[1]) \otimes K_2$$

in $bG$. 

Proposition 1 ([13]). $K_1 \otimes_B K_2$ is endowed with an $A_{\infty}$-$A$-$C$-bimodule structure given by the codifferential $d_{K_1 \otimes_B K_2}$ with Taylor components $d_{K_1 \otimes_B K_2}^{m,n}$ given by

$$d_{K_1 \otimes_B K_2}^{m,n} a_1 \cdots a_m | b_1 \cdots b_n = 0, \quad m, n > 0$$

and

$$d_{K_1 \otimes_B K_2}^{m,n} a_1 \cdots a_m | b_1 \cdots b_n = 0$$

Furthermore, we introduce an $A_{\infty}$-$A$-$B$-bimodule structure given by the codifferential $d_{K_1 \otimes_B K_2}$ with Taylor components $d_{K_1 \otimes_B K_2}^{m,n}$ given by

$$d_{K_1 \otimes_B K_2}^{m,n} a_1 \cdots a_m | b_1 \cdots b_n = 0, \quad m, n > 0$$

and

$$d_{K_1 \otimes_B K_2}^{m,n} a_1 \cdots a_m | b_1 \cdots b_n = 0$$

which induces an isomorphism of objects in $bG$.

Proposition 2 ([13]). Let $K_1$ be an $A_{\infty}$-$A$-$B$-bimodule, $K_2$ an $A_{\infty}$-$B$-$C$-bimodule and $K_3$ an $A_{\infty}$-$C$-$D$-bimodule. The tensor product of $A_{\infty}$-bimodules is associative, i.e. there exists a strict $A_{\infty}$-$A$-$D$-bimodule morphism

$$\Theta : (K_1 \otimes_B K_2) \otimes_C K_3 \to K_1 \otimes_B (K_2 \otimes_C K_3)$$

which induces an isomorphism of objects in $bG$.

4.0.7. The $A_{\infty}$-bar constructions of an $A_{\infty}$-bimodule. We consider two $A_{\infty}$-algebras $(A, d_A)$, $(B, d_B)$ and an $A_{\infty}$-$A$-$B$-bimodule $(M, d_M)$. We recall that $A$ can be canonically endowed with an $A_{\infty}$-$A$-$A$-bimodule structure; see Remark 11. Same holds for $B$ with due changes.

Definition 12. The $A_{\infty}$-$A$-$B$-bimodule $(A \otimes M, d_{A \otimes M})$ is called the $A_{\infty}$-bar construction of $(M, d_M)$ as left $A_{\infty}$-$A$-module. Similarly, the $A_{\infty}$-$A$-$B$-bimodule $(M \otimes_B B, d_{M \otimes_B B})$ is called the $A_{\infty}$-bar construction of $(M, d_M)$ as right $A_{\infty}$-$B$-module.

By definition, both $A \otimes M$ and $M \otimes_B B$ are $A_{\infty}$-$A$-$B$-bimodules. Let $A$ and $B$ be unital algebras and $M$ an $A$-$B$-bimodule. Then $A \otimes M$ is the bar resolution of $M$ as left $A$-module. Similarly, $M \otimes_B B$ is the bar resolution of $M$ as right $B$-module.

Proposition 3 ([13]). Let $(A, d_A)$, $(B, d_B)$ be $A_{\infty}$-algebras and $(M, d_M)$ be an $A_{\infty}$-$A$-$B$-bimodule. There exists a natural morphism

$$\mu : A \otimes_A M \to M,$$

of $A_{\infty}$-$A$-$B$-bimodules. If $A$, $B$ are both flat, and $A$, $M$ are left unital as $A_{\infty}$-$A$-module, then the morphism $\mu$ is a quasi-isomorphism.

4.0.8. On the $A_{\infty}$-bar construction: a remark. We continue our analysis of the $A_{\infty}$-bar constructions and the morphisms

$$\mu_A : A \otimes_A K \to K, \quad \mu_B : K \otimes_B B \to K$$

of strictly unital $A_{\infty}$-$A$-$B$-bimodules introduced in the above subsection. In the following lemma we restrict to the case of augmented associative algebras with zero differentials as they will appear later on.

Lemma 3. Let $(A, d_A)$ and $(B, d_B)$ be augmented associative algebras with zero differential and $(K, d_K)$ be a strictly unital $A_{\infty}$-$A$-$B$-bimodule.
• There exists strictly unital quasi-isomorphisms
\[ K \rightarrow A \otimes A K, \quad K \rightarrow K \otimes B, \]
of $A_\infty$-$A$-$B$-bimodules.

**Proof.** We denote by
\[ A_+ := \ker \epsilon_A, \quad B_+ := \ker \epsilon_B \]
the augmentation ideals in $A$, resp. $B$, denoting by $\epsilon_A$ resp. $\epsilon_B$ the augmentation maps on $A$, resp. $B$. We recall that the augmentation maps are morphisms of algebras. So the augmentation ideals are subalgebras.

We prove the first statement. The second is analogous. We want to show that there exists a strictly unital quasi-isomorphism $\Phi$ of $A_\infty$-$A$-$B$-bimodules; it is the natural inclusion. The quasi-isomorphism $K \rightarrow A \otimes A K$ is the composition
\[ K \xrightarrow{\Phi} A \otimes A K \xrightarrow{\tilde{\Phi}} A \otimes_+ A K \]
where the (bidegree $(0,0)$) morphism $\Phi$ is given as follows. Its $(n,m)$-th Taylor component $\Phi_{n,m} : A[1]^\otimes n \otimes K[1] \otimes B[1]^\otimes m \rightarrow (A \otimes A_+) K[1]$ is simply
\[ \Phi_{n,m} = s \circ \Phi_{n,m} \circ (s^{-1})^{n+m+1} \]
with
\[ \Phi_{n,m}(a_1, \ldots, a_n, k, b_1, \ldots, b_m) = 0 \text{ if } m \geq 1, \]
and
\[ \Phi_{n,0}(a_1, \ldots, a_n, k) = \left\{ \begin{array}{ll}
(1)^{n-1} \sum_{i=1}^{n-1} [a_i]^{-1}(1, a_1, \ldots, a_n, k) & \text{ if } a_i \in A_+, \text{ for all } i = 1, \ldots, n, \\
0 & \text{ otherwise}
\end{array} \right. \]
Note that $\Phi_{n,0}$ is of bidegree $(-n,0)$; $\Phi$ is strictly unital by construction. To check that
\[ \Phi \circ d_K = d_{A \otimes A_+} K \circ \Phi, \]
is straightforward. We need to consider (7) on all the possible strings of elements $(a_1 \ldots a_n | k | b_1 \ldots | b_m) \in T(A[1]) \otimes K[1] \otimes T(B[1]), n, m \geq 0$ paying attention whether $(a_1 \ldots a_n) \in A_+[1]^\otimes n$ or $a_i \in K[1]$, for some $i$. As $\Phi_{0,0}(1) = 1 \otimes 1$, then $\Phi$ is a quasi-isomorphism.

**Corollary 2.** Let $A$, $B$ and $K$ be as above.

• $K$ and $A \otimes A K$ are homotopy equivalent as strictly unital $A_\infty$-$A$-$B$-bimodules.

• $K$ and $K \otimes B$ are homotopy equivalent as strictly unital $A_\infty$-$A$-$B$-bimodules.

**Proof.** We prove the first statement; the second is analogous. We want to show that there exists a strictly unital $A_\infty$-homotopy $\tilde{H} : K \rightarrow A \otimes A K$ of $A_\infty$-$A$-$B$-bimodules, s.t.
\[ \tilde{H} \circ \mu_A = 1 + d_{A \otimes A K} \circ \tilde{H} + \tilde{H} \circ d_{A \otimes A K}, \]
\[ \mu_A \circ \tilde{H} = 1, \]
denoting by $\mu_A$ the $A_\infty$-morphism appearing in prop.[2] and by $\tilde{H}$ the one appearing in lem.[3]. The bidegree $(-1,0)$ Taylor components $\hat{H}_{m,n} : A[1]^\otimes m \otimes (A \otimes A K)[1] \otimes B[1]^\otimes n \rightarrow (A \otimes A K)[1]$ are given by $\hat{H}_{n,m} = 0$ if $m \geq 1$, and
\[ \hat{H}_{n,0}(a_1 | \ldots | a_n (a_a a'_{a_1} \ldots | a'_q, k)) = \left\{ \begin{array}{ll}
s(1, a_1 \ldots | a_n a | a'_1 \ldots | a'_q, k) & \text{ if } a_i \in A_+, \text{ for all } i = 1, \ldots, n, \\
0 & \text{ otherwise}
\end{array} \right. \]
$\mu_A \circ \tilde{H} = 1$ easily follows as $K$ is strictly unital. The equality involving $H$ is long to prove, but straightforward. By definition, the identity 1 is a strict and strictly unital $A_\infty$-morphism. \[ \square \]
5. The triple \((A,K,B)\)

Let \(X\) be a finite dimensional vector space over the field \(\mathbb{K} = \mathbb{R}, \mathbb{C}\). In [3] it is shown that, choosing a pair \((U,V)\) of subspaces in \(X\), then it is possible to introduce a pair \((A,B)\) of \(A_\infty\)-algebras associated to the subspaces themselves and an \(A_\infty\)-bimodule \(K\) associated to the intersection \(U \cap V\). Choosing \((U,V) = (X, \{0\})\) we arrive at the pair of \(A_\infty\)-algebras
\[
A = S(X^*), \quad B = \wedge(X).
\]

\(A\) and \(B\) are objects in \(bG_\mathbb{K}\); let us discuss their bigrading. We put

\[
A = \bigoplus_{i \geq 0} A_i, \quad A_i = A_i^0,
\]

where \(A_i\) denotes the vector space of homogeneous polynomials of degree \(i\). It follows that \(A_0 = A_0^0 = \mathbb{K}\). \(A\) is concentrated in cohomological degree 0. The \(A_\infty\)-structure on \(A\) is encoded in a codifferential \(d_A\) whose only non trivial Taylor component is \(d_A^1 : A[1]\to A[1]\). For the exterior algebra \(B\) we put

\[
B = \bigoplus_{i \geq 0} B^i, \quad B^i = B^i_{-i},
\]

with \(B^i : = \wedge^i X\). A bihomogeneous element \(b \in B^i\) has bidegree \((i,-i)\). Also in this case \(B_0 = B_0^0 = \mathbb{K}\). The \(A_\infty\)-structure on \(B\) is encoded in a codifferential \(d_B\) whose only non trivial Taylor component is \(d_B^1 : B[1]\to B[1]\). In summary, the generators of \(B\) are bihomogeneous of bidegree \((1,-1)\); the dual generators in \(A\) are bihomogeneous with bidegree \((0,1)\). Both \(A\) and \(B\) are augmented \(A_\infty\)-algebras with augmentation ideals \(A_\infty = \bigoplus_{i \geq 1} A_i^0\) and \(B_\infty = \bigoplus_{i \geq 1} B_i^\infty\). Moreover

**Proposition 3 ([3]).** Let \(X\) be a finite dimensional vector field over \(\mathbb{K}\), \(A = S(X^*)\), and \(B = \wedge(X)\). There exists a one-dimensional strictly unital \(A_\infty\)-\(A\)-\(B\)-bimodule \(K\) which, as a left \(A\)-module and as a right \(B\)-module, is the augmentation module.

The \(A_\infty\)-\(A\)-\(B\)-bimodule structure on \(K\) is specified by a codifferential \(d_K\), with Taylor components \(d_K^{0,i} : A[1]\otimes K[1] \otimes B[1] \to K[1]\). We remind that, by definition, \(d_K\) (and so \(d_K^k\) for every \(k,l \geq 0\)) is of cohomological degree 1. The explicit construction in terms of Feynman diagrams implies that \(d_K(k|a_1|\ldots|a_k|b_1|\ldots|b_l)\) is non vanishing iff

\[
\sum_{i=1}^k \deg a_i = \sum_{i=1}^l |b_i| = k + l - 1,
\]

where \(\deg a_i\) denotes the internal degree of the homogeneous polynomial \(a_i \in A\) and \(|b_i|\) the cohomological grading of \(b_i \in B[0]\). But (3) implies that \(d_K^{0,i}\) is of degree 0 w.r.t. the internal grading on \(A\), \(B\) and \(K\), for every \(k,l \geq 0\): we recall that suspension and desuspension do not shift the internal degree.

The explicit construction of the codifferential \(d_K\) implies that \(K\) is a strictly unital \(A_\infty\)-\(A\)-\(B\)-bimodule.

5.0.9. **On the Keller condition for \((A,K,B)\).** We return to a more general setting.

**Definition 13.** Let \((A,d_A)\) and \((B,d_B)\) be flat \(A_\infty\)-algebras and \((K,d_K)\) be a right \(A_\infty\)-\(B\)-module. We set \(\mathcal{R}(K) := K[1] \otimes \mathbb{T}(B[1]).\) \((\text{End}_0(K),d_{\text{End}_0(K)})\) is the flat \(A_\infty\)-algebra defined as follows. As bigraded object

\[
\text{End}_0(K) := \text{Hom}_{bG_\mathbb{K}}(\mathcal{R}(K),K[1]);
\]

the codifferential \(d_{\text{End}_0(K)}\) has non trivial Taylor components

\[
\begin{align*}
\bar{d}_1^A & : (\text{End}_0(K),d_{\text{End}_0(K)}) \to \text{End}_0(K), \\
\bar{d}_2^A & : (\text{End}_0(K),d_{\text{End}_0(K)}) \to \text{End}_0(K).
\end{align*}
\]

We can define \((\text{End}_0(K),d_{\text{End}_0(K)})\) almost verbatim.

**Proposition 4 ([13]).** Let \((A,d_A)\) and \((B,d_B)\) be flat \(A_\infty\)-algebras and \((K,d_K)\) be a right \(A_\infty\)-\(B\)-module. \(K\) is an \(A_\infty\)-\(A\)-\(B\)-bimodule\(^3\) if and only if there exists a morphism

\[
L_A : A \to \text{End}_0(K)
\]

of \(A_\infty\)-algebras.

\(^3\)We define a codifferential \(D_K\) s.t. \(D_K^0 = \bar{d}_k^A\), for every \(l \geq 0\).
Proof. A detailed proof can be found in [3]; here we sketch it. Let \((K, d_K)\) be endowed with an \(A_{\infty}-A-B\)-bimodule structure \(D_K\) s.t. \(\text{d}_K^{0} = \text{d}_K^{1}\). The maps

\[ L_A(a_1 \ldots a_k) \in \text{End}_{B}(K)[1], \quad L_A(a_1 \ldots a_k) := s \circ \mathcal{L}_A(a_1 \ldots | a_k) \]

with \(\mathcal{L}_A(a_1 \ldots | a_k)\) of bidegree \((1, 0)\) given by

\[ \mathcal{L}_A(a_1 \ldots | a_k)(1 | b_1 \ldots | b_q) := D_K^{i,j}(a_1 \ldots | a_k | b_1 \ldots | b_q), \]

are the Taylor components of an \(A_{\infty}\)-algebra morphism \(A \to \text{End}_{B}(K)\), for every \((a_1 \ldots | a_k) \in A[1]^{\otimes k}\), \((1 | b_1 \ldots | b_q) \in K[1] \otimes B[1]^{\otimes q}\) and \(k \geq 1, q \geq 0\). Viceversa, let \(L_A : A \to \text{End}_{B}(K)\) be an \(A_{\infty}\)-algebra morphism with Taylor components as in (10). Then the maps \(\bar{D}_K^{i,j}\) in (11) are the Taylor components of a codifferential \(\bar{D}_K\) on \(T(A[1]) \otimes K[1] \otimes T(B[1])\), extending the given right \(A_{\infty}-B\)-module structure on \(K\). \(\square\)

We call \(L_A\) in prop. [3] the derived left \(A\)-action. A similar statement can be proved in the case of the derived right \(B\)-action, i.e. the \(A_{\infty}\)-algebra morphism \(R_B : B^{op} \to \text{End}_{A}(K)\) with obvious Taylor components. The \(A_{\infty}\)-algebra \(B^{op}\) has \(A_{\infty}\)-structure canonically induced by the one on \(B\), but the signs are not trivialized. We refer to [27] for all details.

Definition 14 ([10]). Let \((A, d_A)\) and \((B, d_B)\) be flat \(A_{\infty}\)-algebras and \((K, d_K)\) be an \(A_{\infty}-A-B\)-bimodule. The triple \((A, K, B)\) satisfies the Keller condition if the derived actions

\[ L_A : A \to \text{End}_{B}(K), \]

and

\[ R_B : B^{op} \to \text{End}_{A}(K), \]

are quasi-isomorphism of \(A_{\infty}\)-algebras.

5.0.10. The Keller condition for the triple \((A, K, B)\). Let \((A, K, B)\) be the triple of bigraded \(A_{\infty}\)-structures given in section [3]. The bigrading on the triple \((A, K, B)\) is such that

\[ \text{End}_{B}(K) = \left\{ \begin{array}{ll}
\text{Hom}_{B_{\text{Gr}}}(K[1] \otimes B[1]^{\otimes i+j}, K[1][i](j)) & i + j < 0 \\
0 & i + j \geq 0
\end{array} \right. \]

Note that \(\text{End}_{B}^{0,0}(K) \cong \mathbb{K}\) and \(\text{d}_K^{0,1} \in \text{End}_{B}^{0,1}(K)\). Similar considerations hold for \(\text{End}_{A}(K)\). The derived left action \(L_A\) preserves the internal grading, by definition. Moreover, for every \(k \geq 1\) and \((a_1 \ldots | a_k) \in A[1]^{\otimes k}\), then \(L_A(a_1 \ldots | a_k)\) is an element of \(\text{End}_{B}^{0,m}(K)\), with \((n, m) := (-k + 1, \sum_{i=1}^{k} \deg a_i)\).

For any \(l \geq 0\) and \((1 | b_1 \ldots | b_l) \in (K[1] \otimes B[1]^{\otimes l})\), with \((a, b) = (-1 + \sum_{i=1}^{l} | b_i| - l, -\sum_{i=1}^{l} | b_i|)\), we have

\[ L_A(a_1 | a_k)(1 | b_1 \ldots | b_l) := D_K^{n,a}(a_1 \ldots | a_k | b_1 \ldots | b_l) \in (K[1])^{a+n \ b}. \]

This implies that

\[ n + a + 1 = 0 \Rightarrow \sum_{i=1}^{l} | b_i| = k + l - 1, \quad m + b = 0 \Rightarrow \sum_{i=1}^{k} \deg a_i = \sum_{i=1}^{l} | b_i|. \]

In other words, the wordlength \(l\) is uniquely determined by the constraint \(l = 1 - k + \sum_{i=1}^{k} \deg a_i\), for any choice of \((b_1 \ldots | b_l) \in (B[1])^{\otimes l}\) as above. This analysis applies to \(R_B\), with due changes. In [3] it is shown the important

Proposition 5. The triple \((A, K, B)\) given is section [3] is s.t. the derived left \(A\)-action \(L_A\) and the derived right \(B\)-action \(R_B\) are quasi-isomorphism of strictly unital \(A_{\infty}\)-algebras.

As in the proof of proposition [3] we introduce the notation

\[ \mathcal{L}_A(a_1 \ldots | a_n) \in \text{End}_{B}(K), \quad \mathcal{L}_A(a_1 \ldots | a_n)(1 | b_1 \ldots | b_q) = d_K^{n,q}(a_1 \ldots | a_n | b_1 \ldots | b_q), \]

i.e. \(\mathcal{L}_A(a_1 \ldots | a_n) = s \circ \mathcal{L}_A(a_1 \ldots | a_n)\) and \(r = \sum_{i=1}^{n} (|a_i| - 1) + 1, \quad m = \sum_{i=1}^{n} \deg a_i\). We note that \(\mathcal{L}_A(a_1 \ldots | a_n)\) is of cohomological degree \(+1\). \(A = S(X^*)\) is canonically endowed with a strictly unital \(A_{\infty}-A\)-bimodule structure \(\text{d}_A\) whose non trivial Taylor components are \(d_A^{1,0} = d_A^{0,1} = \text{d}_A^2\).
Proposition 6. There exists a strictly unital $A_\infty$-$A$-$A$-bimodule structure $d_{\text{End}_A(K)}^\infty$ on $\text{End}_A(K)$ such that the derived action $L_A$ descends to a quasi-isomorphisms of strictly unital $A_\infty$-$A$-$A$-bimodules. $d_{\text{End}_A(K)}^\infty$ has Taylor components

\[
\begin{align*}
\tilde{d}_{\text{End}_A(K)}^{0,0} &= -s \circ \partial_{\text{End}_A(K)} \circ s^{-1}, \\
\tilde{d}_{\text{End}_A(K)}^{n,m}(a_1, \ldots, a_n) &= s \circ D_{\text{End}_A(K)}^{n,m}(a_1, \ldots, a_n), \quad (n \geq 1)
\end{align*}
\]

with

\[
\begin{align*}
\partial_{\text{End}_A(K)}(\varphi) &= (-1)^{|\varphi|} \varphi \circ d_K + d_K \circ \varphi, \\
D_{\text{End}_A(K)}^{n,m}(a_1, \ldots, a_n) &= (-1)^{\sum |a_i| - 1} M(A_1, \ldots, a_n) \circ \varphi, \\
D_{\text{End}_A(K)}^{n,m}(\varphi, a_1, \ldots, a_m) &= (-1)^{|\varphi|} \varphi \circ L(A_1, \ldots, a_m),
\end{align*}
\]

and $\tilde{d}_{\text{End}_A(K)}^{n,m} = 0$, otherwise.

Proof. See Appendix A. \(\square\)

It can also be verified that the derived right-$B$ action $R_B$ descends to a quasi-isomorphism of $A_\infty$-$B^{op}$-$B^{op}$-bimodules.

6. $A_\infty$-Morita theory

6.0.11. On thm. 5.7. in [27]. In this section we study the $A_\infty$-Morita theory for the triple $(A, K, B)$. Our approach to the Morita equivalence is purely $A_\infty$; all we need is the $A_\infty$-$A$-$B$-bimodule structure on $K$ we described in the previous section to prove the equivalence of certain triangulated subcategories of $A_\infty$-modules in the derived categories $D^\infty(A)$ and $D^\infty(B)$ of $A$ and $B$. The functors giving the equivalences are defined through the $A_\infty$-tensor product of $A_\infty$-modules and bimodules. The formalism is quite simple, using the associativity of the $A_\infty$-tensor product. The main advantage in using such "pure" $A_\infty$-approach is represented by the fact that the computations which follow are all explicit: the quasi-isomorphisms of $A_\infty$-bimodules which are the core of the equivalences are induced by the Keller condition on $(A, K, B)$.

6.0.12. On some bigraded $A_\infty$-modules. Let $M$ be an $A_\infty$-$A$-$B$-bimodule and $N$ be an $A_\infty$-$B$-$C$-bimodule, where $A$, $B$, and $C$ are $A_\infty$-algebras. We have already introduced the $A_\infty$-$A$-$B$-bimodule $(B_B(M, d_B(M))$, where $B_B(M) := M \otimes_B B$, calling it the $A_\infty$-bar construction of $M$ as right $A_\infty$-$B$-module. It is an $A_\infty$-right-$B$-module. If $B$ is a differential bigraded algebra, then $B_B(M)$ is a right-$B$-module. Note that $A$ and $B$ are not necessarily augmented. Similarly, $(B_B(N, d_B(N)))$, with $B_B(N) := B \otimes_B N$, is the $A_\infty$-bar construction of $N$ as left $A_\infty$-$B$-module. It is an $A_\infty$-left-$B$-module. If $B$ is a differential bigraded algebra, then $B_B(M)$ is a left-$B$-module. The following lemma is almost tautological, but it is helpful to fix notation.

Lemma 4. Let $A$, $C$ be flat $A_\infty$-algebras, $B$ be a unital associative algebra and $(N, d_N)$ be an $A_\infty$-$B$-$C$-bimodule. If $(M, d_M)$ is an $A_\infty$-$A$-$B$-bimodule such that $d_M^{k,l} = 0$ if $(k, l) \neq (0, 0), (0, 1), (k, 0)$ and it is unital as right $B$-module, then there exists a strict $A_\infty$-$A$-$C$-bimodule isomorphism

\[
(M \otimes_B N) \cong M \otimes_B B(B(N)).
\]

The $A_\infty$-$A$-$C$-bimodule $M \otimes_B B(B(N))$ in lem. [3] is given as follows. As bigraded object we have

\[
(M \otimes_B B(B(N)))_j^i := \bigoplus_{i_1 + i_2 = i, j_1 + j_2 = j} M_{j_1}^{i_1} \otimes_B B(B(N))_{j_2}^{i_2}/Q_j^i,
\]

where $Q^i_j = \bigoplus_{i_1 + i_2 = i, j_1 + j_2 = j} Q \cap (M_{j_1}^{i_1} \otimes_B B(B(N))_{j_2}^{i_2})$ and $Q$ denotes the submodule in $M \otimes_B B(B(N))$ generated by elements of the form $m \cdot b \otimes B - m \otimes b \cdot B$, with $m \in M$, $b \in B$ and $B \in B(B(N))$. $M \otimes_B B(B(N))$ is endowed with an $A_\infty$-$A$-$C$-bimodule structure given by a codifferential $d_{|M \otimes_B B(B(N))}$ with Taylor components

\[
\begin{align*}
\tilde{d}_{M \otimes_B B(B(N))}^{0,0} &= -s \circ D^{0,0} \circ s, \\
\tilde{d}_{M \otimes_B B(B(N))}^{n,m} &= -s \circ D^{n,m} \circ s, \quad \tilde{d}_{M \otimes_B B(B(N))}^{n,m}.
\end{align*}
\]
We have also the isomorphism
\[ \tilde{d}^{0,0}_M(m \otimes_B \tilde{B}) = s^{-1}(\tilde{d}^{0,0}_M(sm)) \otimes_B \tilde{B} + (-1)^{|m|}m \otimes_B s^{-1}(\tilde{d}^{0,0}_{nB(N)}(s\tilde{B})), \]
\[ \tilde{d}^{r,0}_{M \otimes nB(N)}(a_1|\ldots|a_r|(m \otimes_B \tilde{B})) = s(s^{-1}(\tilde{d}^{r,0}_M(a_1|\ldots|a_r|m)) \otimes_B \tilde{B}), \]
\[ \tilde{d}^{0,m}_{M \otimes nB(N)}((m \otimes_B (b \otimes b_1|\ldots|b_q \otimes n))|c_1|\ldots|c_m) = (-1)^{|m|+|b|+\sum_{i=1}^{q}\{|b_i|-1\}} \sum_{q'=0}^q s(m \otimes_B (b \otimes b_1|\ldots|b_{q'} \otimes n) \otimes s^{-1}(\tilde{d}^{0,n}_N(b_{q'+1}|\ldots|b_q|n)|c_1|\ldots|c_m))), \]
and zero otherwise.

**Remark 4.** Exchanging the role of \( M \) and \( N \) in lemma 4 we can describe the strict \( A_\infty-B \)-bimodule isomorphism \( M \otimes_B N \cong B_B(M) \otimes_B N \).

**Remark 5.** In what follows we only consider the triple \((A, K, B)\) of bigraded \( A_\infty \)-objects with \( A_\infty \)-algebras \((A, d_A)\) and \((B, d_B)\) s.t. \( A = S(X^*) \), \( B = \wedge(X) \) and \( A_\infty \)-bimodule \((K, d_K)\), \( K = \mathbb{K} \).

6.0.13. On the right derived \( A_\infty \)-module \( K \). Let \( B_B(K) := K \otimes_B B \) denote the bar construction of \( K \) as right \( B \)-module. By definition,
\[ B_B(K)_j := \bigoplus_{q \geq 0} (K \otimes B[1]^{\otimes q} \otimes B)_j, \]
and
\[ B_B(K)_j = \begin{cases} 0 & \text{if } i + j > 0, \\ K \otimes (B[1]^{\otimes -(i+j)} \otimes B)_j & \text{if } i + j \leq 0. \end{cases} \]
We have also the isomorphism \( B_B(K) \cong M \otimes_B B \) in \( bG_\mathbb{K} \), where \( M^j = \bigoplus_{q \geq 0} K \otimes (B[1]^{\otimes q})_j = K \otimes (B[1]^{\otimes -(i+j)})_j \).

**Definition 15.** The right derived dual module \( K \) of \( K \) is the object
\[ \hat{K} = \text{Hom}_B(B_B(K), B) \]
in \( bG_\mathbb{K} \).

We recall that, for every pair \( M, N \) of right \( B \)-modules, then \( \text{Hom}_B(M, N) \) is the object in \( bG_\mathbb{K} \) with bihomogeneous components \( \text{Hom}_B^{ij}(M, N) = \{ \varphi \in \text{Hom}_{bG_\mathbb{K}}(M, N), \varphi \text{ right } B \text{-linear} \} \).

**Lemma 5.** \( \hat{K} \) can be endowed with a strictly unital \( A_\infty-B \)-bimodule structure \( d_{\hat{K}} \) with Taylor components given by
\[ \tilde{d}^{0,0}_{\hat{K}} = -s \circ \partial_{\hat{K}} \circ s^{-1}, \]
\[ \tilde{d}^{0,0}_{\hat{K}}(b|\varphi) = s \circ D^0_\mathbb{K}(b|\varphi), \]
\[ \tilde{d}^{0,m}_{\hat{K}}(a_1|\ldots|a_m) = s \circ D^0_\mathbb{K}(\varphi|a_1|\ldots|a_m), \]
with
\[ \partial_{\hat{K}}(\varphi) = (-1)^{|\varphi|}\varphi \circ \partial_{B_B(K)}^{0,0}, \]
\[ D^1_\hat{K}(b|\varphi)(1, b_1|\ldots|b_q, b') = (-1)^{|b|}b \cdot \varphi(1, b_1|\ldots|b_q, b'), \]
\[ D^m_\hat{K}(\varphi|a_1|\ldots|a_m)(1, b_1|\ldots|b_q, b') = (-1)^{|\varphi|-1+\sum_{i=1}^{m}\{|a_i|-1\}} \sum_{q'=0}^q \varphi(s^{-1}\tilde{d}^{0,q'}_{\hat{K}}(a_1|\ldots|a_m|1|b_1|\ldots|b_q), b'_{q'+1}|\ldots|b_q, b), \]
and \( \tilde{d}^{0,0}_{B_B(K)} = -s \circ \partial_{B_B(K)}^{0,0} \circ s^{-1}, \tilde{d}^{0,m}_{B_B(K)} = 0, \) otherwise.

**Corollary 3.** \((\hat{K}, d_{\hat{K}})\) is a strictly unital differential bigraded left \( B \)-module; we have a strict isomorphism
\[ \hat{K} \otimes_B \hat{K} = B_B(K) \otimes_B \hat{K} \]
of strictly unital \( A_\infty-A \)-bimodules.
6.0.14. On the quasi-isomorphism $A \to \bigotimes_{G} K$.

Definition 6.16. $\text{End}_{B}(B_{B}(K))$ is the object in $b\mathcal{G}$ with bihomogeneous components

$$(15) \quad \text{End}_{B}^{0}(B_{B}(K)) = \text{Hom}^{0,0}_{B_{B}}\left(K \otimes_{B} B, (K \otimes_{B} B)[i][j]\right).$$

Lemma 6. $\text{End}_{B}(B_{B}(K))$ can be endowed with a strictly unital $A_{\infty}$-$A$-$A$-bimodule structure $d_{\text{End}_{B}(B_{B}(K))}$ with Taylor components

$$
d_{0,\text{End}_{B}(B_{B}(K))}(s) = -s \circ d_{\text{End}_{B}(B_{B}(K))} \circ s^{-1},$$

$$
d_{1,\text{End}_{B}(B_{B}(K))}(a) = s \circ d_{\text{End}_{B}(B_{B}(K))}(a),$$

$$
d_{m,\text{End}_{B}(B_{B}(K))}(\varphi_{1} \ldots \varphi_{m}) = s \circ d_{\text{End}_{B}(B_{B}(K))}(\varphi_{1} \ldots \varphi_{m}),$$

with

$$
d_{0,\text{End}_{B}(B_{B}(K))} = -s \circ d_{\text{End}_{B}(B_{B}(K))} \circ s^{-1},$$

$$
d_{1,\text{End}_{B}(B_{B}(K))} = s \circ d_{\text{End}_{B}(B_{B}(K))},$$

$$
d_{m,\text{End}_{B}(B_{B}(K))} = s \circ d_{\text{End}_{B}(B_{B}(K))},$$

and $d_{0,\text{End}_{B}(B_{B}(K))}$ is given in proposition $\Box$.

Let $(\text{End}_{B}(K), d_{\text{End}_{B}(B_{B}(K))})$ be the strictly unital $A_{\infty}$-$A$-$A$-bimodule described in proposition $\Box$. We recall that the bar resolution $B_{B}(K) = K \otimes_{B} B$ is homotopy equivalent to $K$ in $b\mathcal{G}$ (but not as right bigraded $B$-modules); the maps giving such homotopy equivalence are the projection $p : K \otimes_{B} B \to K$ and the inclusion $i : K \to K \otimes_{B} B$, with $p(1, b) = b(0)$ and $p(1, b_{1} \ldots b_{q}, b) = 0$ for $q \geq 1$.

Proposition 7. $(\text{End}_{B}(B_{B}(K)), d_{\text{End}_{B}(B_{B}(K))})$ and $(\text{End}_{B}(K), d_{\text{End}_{B}(B_{B}(K))})$ are homotopy equivalent as strictly unital $A_{\infty}$-$A$-$A$-bimodules.

Proof. We define the strict (and strictly unital) morphism $\mathcal{H} : \text{End}_{B}(B_{B}(K)) \to \text{End}_{B}(K)$ of strictly unital $A_{\infty}$-$A$-$A$-bimodules, via $\mathcal{H} = s \circ H \circ s^{-1}$, where, for any $(i, j) \in \mathbb{Z}^{2}$, $H : \text{End}_{B}^{0}(B_{B}(K)) \to \text{End}_{B}^{0}(K)$ is the composition $H := (s \circ 1 \circ s^{-1}) \circ P \circ \mathcal{T}$, denoting by $\mathcal{I}$ and $\mathcal{P}$ the morphisms

$$
\text{Hom}^{0,0}_{B_{B}}(B_{B}(K), B_{B}(K)[i][j]) \xrightarrow{\mathcal{H}} \text{Hom}^{0,0}_{B_{B}}(K \otimes T(B[1]), B_{B}(K)[i][j]) \xrightarrow{\text{End}_{B}^{0}} \text{Hom}^{0,0}_{B_{B}}(K[1] \otimes T(B[1]), (K[1])[i][j]),
$$

with $\mathcal{I}(\varphi)(1, b_{1} \ldots b_{q}) := \varphi(1, b_{1} \ldots b_{q}, 1)$, $P(\psi) := p \circ \psi$. More explicitly, if $\varphi \in \text{End}_{B}^{0}(B_{B}(K))$, then

$$(16) \quad (H \varphi)(1, b_{1} \ldots b_{q}) = s((\varphi(0, 1, b_{1} \ldots b_{q}, 1)(0)),$$

denoting by $\varphi(0, 1, b_{1} \ldots b_{q}, 1)$ the projection of $\varphi(1, b_{1} \ldots b_{q}, 1)$ onto $K \otimes B$.

To prove

$$\mathcal{H} \circ d_{\text{End}_{B}(K)} = d_{\text{End}_{B}(K)} \circ \mathcal{H}$$

is straightforward; the only issue is represented by the signs; all details are contained in $[7]$. $\square$

Proposition 8. $(K \otimes_{B} K, d_{K \otimes_{B} K})$ and $(\text{End}_{B}(B_{B}(K)), d_{\text{End}_{B}(B_{B}(K))})$ are strictly isomorphic as strictly unital $A_{\infty}$-$A$-$A$-bimodules.

Proof. We recall that $B_{B}(K) = M \otimes B$ in $b\mathcal{G}$. The strict isomorphism of $A_{\infty}$-$A$-$A$-bimodules

$${G} : B_{B}(K) \otimes_{B} K \to \text{End}_{B}(B_{B}(K)),$$

with $G = s \circ G \circ s^{-1}$ is given as follows. The morphism $G$ is defined by the commutative diagram
Proposition 9. A module $bG$ with $G$ the obvious isomorphisms. Note the sign in $T_2((m \otimes b) \otimes B \varphi) = (-1)^{|m|+|b|+|\varphi|}m \otimes b \varphi$.

More explicitly

$G((m \otimes b) \otimes B \varphi)(m' \otimes b') := (-1)^{|m|+|b|+|\varphi|}m \otimes b \varphi(m' \otimes b')$.

By definition, $G(Q^i_j) = 0$ for every $(i,j) \in \mathbb{Z}^2$, where $Q^i_j$ is the submodule in $(B_B(K) \otimes \text{Hom}_B(B_B(K), B))_j$ introduced in the proof of lemma [1]. So $G$ is well defined, as morphism in $bG_K$. Note that $T_2(Q^i_j) = 0$, as well. $G$ is an isomorphism in $bG_K$; so $G$ is an isomorphism in $bG_K$ as well; in fact $M$ is an object in $bG_K$ with finite dimensional bihomogeneous components $M^i_j = K \otimes (B[1] \otimes [i])_j$, for every $i,j \in \mathbb{Z}^2$. We finish the proof of proposition [5] by checking that $G$ is a chain map and commutes with the left and right $A_{\infty}$-actions on $B_B(K) \otimes B_K$ and $\text{End}_B(B_B(K))$. The only issue is represented by the signs appearing in $G$ and in the Taylor components of the coderivatives on $B_B(K) \otimes B_K$ and $\text{End}_B(B_B(K))$. In particular, the non trivial sign in [13] is necessary to prove compatibility between $G$ and the right $A_{\infty}$-module structures, i.e.

$\text{d}_{G}^0 \text{End}_B(K)[G((m \otimes b) \otimes B \varphi), a_1|\ldots|a_m] = G(\text{d}_{G}^0 \text{End}_B(K) \otimes B_K)((m \otimes b) \otimes B \varphi)[a_1|\ldots|a_m],$

with l.h.s. equal to (applying it on $m' \otimes b'$ with $m' \otimes b' = 1, b_1|\ldots|b_q \otimes b'$)

$$\sum_{q'=0}^{q} m \otimes b \varphi(s^{-1}d^{m,q'}_K(a_1|\ldots|a_m|1|b_1|\ldots|b_{q'})), b_{q'}|\ldots|b_{q'}, b'),$$

and r.h.s. equal to

$$(-1)^{|m|+|b|}G((m \otimes b) \otimes B s^{-1}(d^0_m a_1|\ldots|a_m))) = (-1)^{|q|+\sum_{i=1}^{m}(a_{i}-1)m \otimes b \cdot s^{-1}(d^0_m (a_1|\ldots|a_m)(1, b_1|\ldots|b_q, b'))).$$

The Taylor components $d^0_m$ generate the sign $(-1)^{|q|+\sum_{i=1}^{m}(a_{i}-1)-1}$; so we are done.

We summarize the results so far into

Corollary 4. $(K \otimes_{bG_K} K, d_{K \otimes_{bG_K} K})$ and $(\text{End}_B(K), d_{\text{End}_B(K)})$ are homotopy equivalent as strictly unital $A_{\infty}$-$A_{\infty}$-bimodules.

Proposition 9. There exists a strictly unital quasi-isomorphism

$$A \to K \otimes_{bG_K} K$$

of strictly unital $A_{\infty}$-$A_{\infty}$-bimodules.

Proof. Just compose the homotopy equivalence in the above corollary with the left derived action $L_A$. \qed

6.0.15. On the quasi-isomorphism $B \to K \otimes_{bG_K} K$. Following the example of $K$, we can introduce the left derived bimodule

$$K = \text{Hom}_A(A \otimes_{bG_K} K, A).$$

As $A[1]$ is concentrated in cohomological degree $-1$, then

$$A \otimes_{bG_K} K = A \otimes N$$

in $bG_K$, with $N_j = 0$ if $i > 0$, $N_0 = K$ and

$$N_j = \bigoplus_{j_1 + \ldots + j_i = j} (A[1])^{-1}_{j_1} \otimes \cdots \otimes (A[1])^{-1}_{j_i}$$

for any $i < 0$. Every bihomogeneous component of $N$ is finite dimensional. In what follows $A_B(K) := A \otimes_{bG_K} K$. 

}\[\text{bimodules.}]

\[\text{and checking that}\]

\[\text{By definition,}\]

\[\text{applying it on}\]

\[\text{More explicitly}\]

\[\text{with l.h.s. equal to}\]

\[\text{and}\]

\[\text{and the right}\]

\[\text{A}_{\infty}-\text{A}-\text{A}\]-bimodules.

\[\text{of strictly unital}\]

\[\text{Proof. Just compose the homotopy equivalence in the above corollary with the left derived action} L_A.\] \qed

\[\text{On the quasi-isomorphism} B \to K \otimes_{bG_K} K.\]

\[\text{As} A[1] \text{is concentrated in cohomological degree} -1, \text{then}\]

\[A \otimes_{bG_K} K = A \otimes N\]

\[\text{in} bG_K, \text{with} N_j = 0 \text{if} i > 0, N_0 = K \text{and}\]

\[N_j = \bigoplus_{j_1 + \ldots + j_i = j} (A[1])^{-1}_{j_1} \otimes \cdots \otimes (A[1])^{-1}_{j_i}\]

\[\text{for any} i < 0. \text{Every bihomogeneous component of} N \text{is finite dimensional. In what follows} A_B(K) := A \otimes_{bG_K} K.\]
By definition, $\overline{K}$ is a strictly unital $A_\infty$-$B$-$A$-bimodule with codifferential $\overline{d}_{\overline{K}}$ whose Taylor components are given by

$$
\overline{d}^{0,0}_{\overline{K}} = - s \circ \partial \overline{K} \circ s^{-1}, \quad \overline{d}^{k,0}_{\overline{K}}(b_1|\ldots|b_k|\varphi) = s \circ D^k_{\overline{K}}(b_1|\ldots|b_k|\varphi), \quad \overline{d}^{0,1}_{\overline{K}}(\varphi|a) = s \circ D^1_{\overline{K}}(\varphi|a),
$$

with

$$
\partial \overline{K}(\varphi) = (-1)^{|\varphi|} \varphi \circ \overline{d}^{0,0}_{\overline{K}}, \quad D^k_{\overline{K}}(b_1|\ldots|b_k|\varphi)(a,a_1|\ldots|a_q,1) = (-1)^{(|\varphi|+|a|+\sum_i(|a_i|-1)+1)\sum_j(|b_i|-1)+1} \sum_{q'=0}^q \varphi(a,a_1|\ldots|a_{q'-1}|a_{q'+1}|\ldots|a_q|1|b_1|\ldots|b_k), \quad D^1_{\overline{K}}(\varphi|a')(m) = (-1)^{|\varphi|+|a||m|} \varphi(m) \cdot a,
$$

where $\overline{d}^{0,0}_{\overline{B}(K)} = - s \circ \partial \overline{B}(K) \circ s^{-1}$ and $\overline{d}^{m,n}_{\overline{K}} = 0$, otherwise.

To check that $(\overline{K}, \overline{d}_{\overline{K}})$ is a strictly unital $A_\infty$-$B$-$A$-bimodule is long but straightforward.

**Definition 17.** $\text{End}_{A}(A_\infty B(K))^{op}$ is the object in $bG_K$ with bihomogeneous components

$$
\text{End}_{A}(A_\infty B(K))^{op} = \text{Hom}_{A_\infty A}(A_\otimes A K, (A_\otimes A K)[i,j]).
$$

**Lemma 7.** $\text{End}_{A}(A_\infty B(K))^{op}$ can be endowed with a strictly unital $A_\infty$-$B$-$A$-bimodule structure $\overline{d}_{\text{End}_{A}(A_\infty B(K))^{op}}$ with Taylor components

$$
\overline{d}^{0,0}_{\text{End}_{A}(A_\infty B(K))^{op}} = - s \circ \partial \text{End}_{A}(A_\infty B(K))^{op} \circ s^{-1}, \quad \overline{d}^{i,j}_{\text{End}_{A}(A_\infty B(K))^{op}}(b_1|\ldots|b_i|\varphi)(a,a_1|\ldots|a_q,1) = (-1)^{|\varphi|+|a|+\sum_i(|a_i|-1)+1)\sum_j(|b_i|-1)+1} \sum_{q'=0}^q \varphi(a,a_1|\ldots|a_{q'-1}|a_{q'+1}|\ldots|a_q|1|b_1|\ldots|b_i), \quad (i \geq 1)
$$

$$
\overline{d}^{0,m}_{\text{End}_{A}(A_\infty B(K))^{op}}(\varphi|b_1|\ldots|b_m)(a,a_1|\ldots|a_q,1) = (-1)^{|a|+\sum_i(|a_i|-1)+1)\sum_j(|b_i|-1)+1} \overline{d}^{0,m}_{A_\infty B(K)}(\varphi(a,a_1|\ldots|a_q,1|b_1|\ldots|b_m), \quad (m \geq 1)
$$

and $\overline{d}^{0,0}_{A_\infty B(K)} = - s \circ \partial \overline{B}(K) \circ s^{-1}$, where $\overline{d}^{0,0}_{A_\infty B(K)}$ is given in proposition 2 and $\overline{d}^{m,n}_{\text{End}_{A}(A_\infty B(K))^{op}} = 0$, otherwise.

**Proposition 10.** $(\overline{K}_{\otimes A K}, \overline{d}_{\overline{K}_{\otimes A K}})$ and $(\text{End}_{A}(A_\infty B(K))^{op}, \overline{d}_{\text{End}_{A}(A_\infty B(K))^{op}})$ are strictly isomorphic as strictly unital $A_\infty$-$B$-$A$-bimodules.

**Proof.** The proof is similar to the one of prop. 3 with due changes. \qed

**Proposition 11.** There exists a strictly unital quasi-isomorphism

$$
B \to \overline{K}_{\otimes A K}
$$

of strictly unital $A_\infty$-$B$-$B$-bimodules.

**Proof.** Just compose the homotopy equivalence in the above prop. with the right derived action $R_B$. \qed

6.0.16. $A_\infty$-Morita theory for the triple $(A, K, B)$. 

6.0.17. On the functors. Let us consider the functors
\[ F' : \text{Mod}_\infty(A) \to \text{Mod}^{\text{strict}}_\infty(B), \quad G' : \text{Mod}_\infty(B) \to \text{Mod}^{\text{strict}}_\infty(A), \]
given by
\[ F'(M) := M \otimes_k K, \quad G'(N) := N \otimes_k K, \]
on objects \( M \in \text{Mod}_\infty(A) \) and \( N \in \text{Mod}_\infty(B) \), while on morphisms \( f : M_1 \to M_2 \) in \( \text{Mod}_\infty(A) \) and \( g : N_1 \to N_2 \) in \( \text{Mod}_\infty(B) \) we set
\[ F'(f) := M_1 \otimes_k K \to M_2 \otimes_k K, \quad G'(g) := N_1 \otimes_k K \to N_2 \otimes_k K, \]
with
\[ F'(f) := (s^{-1} \circ F \circ s) \otimes 1, \quad G'(g) := (s^{-1} \circ G \circ s) \otimes 1. \]
We have denoted by \( F : \mathcal{R}(M_1) \to \mathcal{R}(M_2) \), respectively \( G : \mathcal{R}(N_1) \to \mathcal{R}(N_2) \), the unique lifting of \( f \) (resp. \( g \)) to a \( T(A[1]) \)-counital-comodule morphism, respectively a \( T(B[1]) \)-counital-comodule morphism. In this notation, \( \mathcal{R}(M_1) := M_1[1] \otimes T(A[1]) \) and similarly for \( \mathcal{R}(N_1) \), with due changes. Let \( \mathcal{F} \) and \( \mathcal{G} \) be the functors given by the compositions
\[ \mathcal{F} : \text{Mod}_\infty(A) \to \text{Mod}^{\text{strict}}_\infty(B) \cong \text{Mod}_\infty(B), \]
and
\[ \mathcal{G} : \text{Mod}_\infty(B) \to \text{Mod}^{\text{strict}}_\infty(A) \cong \text{Mod}_\infty(A), \]
denoting by \( i \) the inclusion of the subcategories \( \text{Mod}^{\text{strict}}_\infty(A) \) (resp. \( \text{Mod}^{\text{strict}}_\infty(B) \)) in \( \text{Mod}_\infty(A) \) (resp. \( \text{Mod}_\infty(B) \)). We remark that \( \text{Mod}^{\text{strict}}_\infty(A) \) and \( \text{Mod}^{\text{strict}}_\infty(B) \) are not full subcategories.

If two morphisms \( f \) and \( g \) in \( \text{Mod}_\infty(A) \) are \( (A_\infty) \)-homotopic, then we write \( f \sim g \). An analogous notation holds true in \( \text{Mod}_\infty(B) \). If the homotopy between \( f \) and \( g \) is strict, then we write \( f \sim_{\text{strict}} g \).

**Lemma 8. a) Let** \( f \sim g \) **in** \( \text{Mod}_\infty(A) \), **resp. in** \( \text{Mod}_\infty(B) \). **Then**
\[ F'(f) \sim_{\text{strict}} F'(g), \]
in \( \text{Mod}^{\text{strict}}_\infty(B) \), resp.
\[ G'(f) \sim_{\text{strict}} G'(g), \]
in \( \text{Mod}^{\text{strict}}_\infty(A) \).

**b) The functors** \( F' \) **and** \( G' \) **send strictly unital homotopy equivalences to strict and strictly unital homotopy equivalences.**

**Proof.** Part a). Let \( f, g : M \to N \) with \( f \sim g \) in \( \text{Mod}_\infty(A) \), i.e. \( f - g = d_N h + h d_M \), where \( h : M \to N \) is a strictly unital \( A_\infty \)-homotopy. By definition, \( h \) is a degree \(-1\) map with components \( h_n : M[1] \otimes A[1] \otimes n \to N[1] \), \( n \geq 0 \). We claim that \( H : M \otimes_k K \to N \otimes_k K \), where
\[ H_0 := (s^{-1} \circ h \circ s) \otimes 1 \]
and \( H_n = 0 \) for \( n > 0 \), is a strict \( A_\infty \)-homotopy between \( F'(f) \) and \( F'(g) \), i.e.
\[ F'(f) - F'(g) = d_N \otimes_k K \circ H + H \circ d_M \otimes_k K. \]

Eq. (19) is equivalent to
\[ (F'(f) - F'(g))(s(m, a_1, \ldots | a_q, 1)) = (d_N \otimes_k K \circ H + H \circ d_M \otimes_k K)(s(m, a_1, \ldots | a_q, 1)), \]
and
\[ 0 = (d_N \otimes_k K \circ H + H \circ d_M \otimes_k K)((m, a_1, \ldots | a_q, 1)|b_1, \ldots | b_l), \]
for every \( q, l \geq 0 \). Let us consider at first eq. (20); on the l.h.s. we have terms involving the Taylor components of the codifferential \( d_M \) on \( M \) and the \( A_\infty \)-homotopy \( h \) by the homotopy hypothesis \( f \sim g \); all we need to prove is that on the r.h.s. the terms involving the Taylor components of the codifferential \( d_K \) on \( K \) cancel. This is true because these terms appear in
\[ \sum_{q_1=0}^q \sum_{q_2=0}^{q+1} (-1)^{1+|m|+\sum_{i=2}^{|a_1|-1}}((h_{q_1}(m, a_1, \ldots | a_{q_1})a_{q_1+1} \ldots | a_{q-q_2}, d^{q_2}_{K}(a_{q-q_2+1} | a_q | l))) + \]
\[ \sum_{q_1=0}^q \sum_{q_2=0}^{q+1} (-1)^{|m|+|a_1|-1}(h_{q_1}(m, a_1, \ldots | a_{q_1})a_{q_1+1} \ldots | a_{q-q_2}, d^{q_2}_{K}(a_{q-q_2+1} | a_q | l)) = 0, \]
as the $A_\infty$-homotopy $h$ has degree $-1$. Eq. (21) is equivalent to

$$0 = \sum_{q_i=0}^q d_N \otimes_K (s^{-1}(h_{\overline{q_i}}(m|a_1| \ldots |a_{q_i}))|a_{q_i+1}| \ldots |a_q|1) +$$

$$\sum_{q_i=0}^q (-1)^{q+|m|+\sum_{i=1}^q |a_i|-1} H(m|a_1| \ldots |a_{q-q_2}| d_{\overline{K}}^q(a_{q-q_2+1}| \ldots |a_q|1|b_1| \ldots |b_q)),$$

which is verified by the same argument we used for eq. (20) and (21). The case $f \sim g$ in $\text{Mod}_\infty(B)$ is similar.

Part b). The morphism $f : M \to N$ is a homotopy equivalence in $\text{Mod}_\infty A$ if there exists a morphism $g : N \to M$ in $\text{Mod}_\infty(A)$ s.t. $f \circ g \sim 1$ and $g \circ f \sim 1$. We denote by $h_1 : N \to N$, resp. $h_2 : M \to M$ the $A_\infty$-homotopies between $f \circ g$ and $1_N$, resp. $g \circ f$ and $1_M$. We want to prove that

$$F'(f) \circ F'(g) = 1 + d_N \otimes_K h_1 + h_1 \circ d_N \otimes_K,$$

$$F'(g) \circ F'(f) = 1 + d_M \otimes_K h_2 + h_2 \circ d_M \otimes_K,$$

with strict $A_\infty$-homotopies

$$H_i := (s^{-1} \circ h_i \circ s) \otimes 1,$$

for $i = 1, 2$. Using the proof of a) we get the statement. The case for $G'$ is similar. \hfill \Box

6.0.18. On the derived categories. In this section we introduce the derived categories $D^\infty(A)$, respectively $D^\infty(B)$, of right unital $A_\infty$-modules over $A$, respectively $B$, with strictly unital $A_\infty$-morphisms. Using the theory of closed model categories it is possible to prove

**Theorem 6** (K. Lefèvre-Hasegawa, [12]). Let $A$ be an augmented $A_\infty$-algebra\(^3\) quasi-isomorphisms in $\text{Mod}_\infty(A)$ are homotopy equivalences of strictly unital $A_\infty$-A-modules.

This results implies that

$$D^\infty(A) = \text{Mod}_\infty(A)/\sim,$$

and similarly for $D^\infty(B)$. In this setting quasi-isomorphisms of strictly unital $A_\infty$-modules are already isomorphisms in the homotopy categories; no localization is needed. The main advantage is represented by the explicit structure of the morphisms in the derived categories themselves; no “roofs” manipulation is needed.

We discuss now the triangulated structures on the derived categories. The direct sum of two objects in $\text{Mod}_\infty(A)$ is again a strictly unital $A_\infty$-module; the cohomological grading shift functor $\Sigma(M) := M[1]$ in actually an endofunctor on $D^\infty(A)$ and $D^\infty(B)$. It follows that $\Sigma(\cdot) := [1]$ is an autoequivalence of $D^\infty(A)$ and $D^\infty(B)$. More precisely, let $(M, d_M)$ be an object of $D^\infty(A)$. The bigraded object $M[1]$ can be endowed with a strictly unital $A_\infty$-A module structure as follows. The codifferential $d_{M[1]}$ has Taylor components $d_{M[1]}^d : (M[1])[1] \otimes B[1]^\otimes \to (M[1])[1]$ given by

$$d_{M[1]}^d = -s \circ d_M^d \circ (s^{-1} \otimes 1).$$

Proving that $d_{M[1]}^d = 0$ is a straightforward sign-check. Given any morphism $F : M[1] \to N[1]$ in $D^\infty(A)$ with Taylor components (of bidegree $(0,0)$) $F^d : M[1] \otimes B[1]^\otimes \to N[1]$, we get the induced morphism $\tilde{F} : (M[1])[1] \to (N[1])[1]$ in $\text{Mod}_\infty(A)$ with Taylor components

$$\tilde{F}^d = s \circ F^d \circ (s^{-1} \otimes 1).$$

Once again, the proof of $\tilde{F} \circ d_{M[1]} = d_{N[1]} \circ \tilde{F}$ is a straightforward sign check. Same considerations hold in $D^\infty(B)$. The inverse functor $\Sigma^{-1}$ is given by $\Sigma^{-1}(\cdot) = [-1]$.

**Definition 18** ([12]). The triangulated structure on the derived category $D^\infty(A)$ is given as follows. The autoequivalence $\Sigma$ is simply the (cohomological) grading shift functor $\Sigma = [1]$. The distinguished triangles are isomorphic to those induced by semi-split sequences of strict $A_\infty$-morphisms

$$M \xrightarrow{f} M' \xrightarrow{\rho} M''$$

in $\text{Mod}_\infty A$, i.e. sequences such that

$$0 \to M \xrightarrow{f} M' \xrightarrow{\rho} M'' \to 0$$

is an exact sequence in $bG_k$, and such that there exists a splitting $\rho \in \text{Hom}_{bG_k}(M', M)$ of $f$ with

$$\rho \circ d_M^d = d_{M'}^d \circ (\rho \otimes 1^\otimes 1), \quad i \geq 2.$$

\(^3\)Augmentation w.r.t. a ground field $K$ of characteristic 0.
For the derived category of $B$ the definition is analogous. The splitting $\rho$ in the exact sequence (22) does not commute with the differentials $\partial_i^{A\otimes B}$ and $\partial_j^{A\otimes B}$, in general. The above exact triangles endow $\mathcal{D}^\infty(A)$ with a triangulated category structure; the proof is contained in thm. 2.4.3.1 in [12]: the idea is induce the triangulated category structure on $\mathcal{D}^\infty(A)$ by using the one on $\mathcal{D}(UA)$, denoting by $UA$ the enveloping algebra of $A$; by definition $UA$ is a differential (bi)graded algebra we refer to [12], [21] for all details; its derived category $\mathcal{D}(UA)$ is a well-known object. The equivalence of categories $\mathcal{D}(UA) \to \mathcal{D}^\infty(A)$ becomes then an equivalence of triangulated categories.

Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{D}^\infty(A)$; it is isomorphic to a triangle of the form $M \xrightarrow{f} M' \xrightarrow{g} M'' \xrightarrow{0}$.

be a semi-split exact sequence with $f$, $g$ strict, and splitting $\rho : M' \to M$, $\rho \circ f = 1$. This implies that

$M'_{ij} \cong M'_i \otimes M''_j$

as vector spaces over $\mathbb{K}$, for any $(i, j) \in \mathbb{Z}$; in virtue of this we assume that $M' = (M \oplus M'', d_{M \oplus M''})$, where $d_{M \oplus M''} = (d_M - h, d_{M''})$. It follows that $d_{M \oplus M''} \cdot d_{M \oplus M''} = 0$ if and only if $h : M'' \to M[1]$ defines an $A_\infty$-morphism of strictly unital $A_\infty$-bimodules. Thanks to this, we will consider the semi-split exact sequence $0 \to M \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$ with $i$ and $p$ the natural inclusion and projection (which are strict morphisms in $\mathcal{D}^\infty(A)$), and complete it to the exact triangle

$M \xrightarrow{f} M' \xrightarrow{g} M'' \xrightarrow{h} M[1].$

A small momento: in section 8.0.19 we will discuss the triangulated structure on some “deformed” derived categories of topologically free modules; some examples will be given: taking there the “limit” $h = 0$ we obtain further examples of exact triangles in $\mathcal{D}^\infty(A)$ and $\mathcal{D}^\infty(B)$.

6.0.19. On the functors $\mathcal{F}$ and $\mathcal{G}$. Collecting the results on the derived categories of $A$ and $B$ and the definitions of the functors $\mathcal{F}$ and $\mathcal{G}$ we arrive at the pair of functors

$\mathcal{F} : \mathcal{D}^\infty(A) \xrightarrow{E_A} \mathcal{Mod}^\text{strict}(B) / \sim_{\text{strict}} \xrightarrow{\lambda} \mathcal{D}^\infty(B),$

and

$\mathcal{G} : \mathcal{D}^\infty(B) \xrightarrow{E_B} \mathcal{Mod}^\text{strict}(A) / \sim_{\text{strict}} \xrightarrow{\lambda} \mathcal{D}^\infty(A),$

with a little abuse of notation.

**Proposition 12.** Let $(\mathcal{F}, \mathcal{G})$ be the pair of functors introduced above. Then $\mathcal{F}(A) \simeq K$, $\mathcal{F}(K) \simeq B$, in $\mathcal{D}^\infty(B)$, and $\mathcal{G}(B) \simeq K$, $\mathcal{G}(K) \simeq A$ in $\mathcal{D}^\infty(A)$.

**Proof.** The quasi-isomorphisms of strictly unital $A_\infty$-bimodules

$K \to A \otimes_A K \to (K \otimes_B K) \otimes_A K = \mathcal{F}(G(K))$

and

$A \to K \otimes_B K \to A \otimes_A (K \otimes_B K) \equiv (A \otimes_A K) \otimes_B K = \mathcal{G}(\mathcal{F}(A))$

give both the statements. We used lem. [3] prop. [9] and prop. [11].

**Lemma 9.** $(\mathcal{F}, \varphi_1)$ and $(\mathcal{G}, \varphi_2)$ are exact functors w.r.t the triangulated category structures on $\mathcal{D}^\infty(A)$ and $\mathcal{D}^\infty(B)$; for any $M \in \mathcal{D}^\infty(A)$:

$\varphi_1(\mathcal{F}(M)) : (M \otimes_A K)[1] \to M[1] \otimes_A K$, $\varphi_1(\mathcal{F}(M))(s(m, a_1 \ldots a_i, k)) := (m|a_1 \ldots a_i, k)$,

and similarly for $\varphi_2$.

**Proof.** $\mathcal{F}$ and $\mathcal{G}$ send quasi-isomorphisms into quasi-isomorphisms as quasi-isomorphisms in the derived categories $\mathcal{D}^\infty(A)$ and $\mathcal{D}^\infty(B)$ are homotopy equivalences. To prove that $\mathcal{F}$ (and $\mathcal{G}$) are exact w.r.t. the triangulated structures on the derived categories it is sufficient to consider triangles of the form (20), i.e. $M \xrightarrow{f} M' \xrightarrow{g} M'' \xrightarrow{h} M[1]$. Applying $\mathcal{F}$ to such a triangle, and using the above lemmata we get the sequence

$M \xrightarrow{\mathcal{F}(f)} (M \oplus M') \otimes_A K \xrightarrow{\mathcal{F}(g)} M' \otimes_A K \mathcal{F}(\mathcal{F}(M)) \mathcal{G}(M \otimes_A K)[1]$
in $D^\infty(B)$; the short exact sequence ($\mathcal{F}$ is additive)

$$(24) \quad 0 \to M \otimes_A K \xrightarrow{F(i)} (M \oplus M') \otimes_A K \xrightarrow{F(p)} M'' \otimes_A K \to 0$$

is semi-split w.r.t. the splitting

$$F(p) := \rho \otimes 1,$$

denoting by $\rho : M' \to M$ the splitting of the short exact sequence $0 \to M' \xrightarrow{\rho} M \oplus M' \xrightarrow{h} M' \to 0$. In fact

$$F(\rho) \circ F(\alpha) = 1,$$

and $F(\rho) \circ (s^{-1} \circ d_{M \otimes_A K}^0) = (s^{-1} \circ d_{M' \otimes_A K}^0) \circ (F(\rho) \otimes 1^\otimes i)$,

for $i \geq 1$. Then (24) can be completed to the distinguished triangle

$$M \otimes_A K \xrightarrow{F(i)} (M \oplus M') \otimes A K \xrightarrow{F(p)} M'' \otimes_A K \xrightarrow{h'} (M \otimes_A K)[1],$$

with $h' := F(h)$. In summary $\mathcal{F}$ sends exact triangles into exact triangles. Same considerations holds true for $\mathcal{G}$.  

With $\text{triang}_A^\infty(M)$ we denote the full triangulated subcategory in $D^\infty(A)$ generated by $\{M[i](j), i \in \mathbb{Z}\}$. $\text{thick}_A^\infty(M_A)$, resp. $\text{thick}_A^\infty(N_B)$ are the thick subcategories of direct summands of objects in $\text{triang}_A^\infty(M_A)$, resp. $\text{triang}_B^\infty(N_B)$. We refer to Appendix C for all definitions. Finally, we can state the main theorem of this section.

**Theorem 7.** Let $X$ be a finite dimensional vector space on $\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$. Let $(A, K, B)$ be the triple of $A_\infty$-structures with $A = S(X^*)$ and $B = \wedge(X)$ Koszul dual augmented differential bigraded algebras with zero differential and $K = \mathbb{K}$ endowed with the bigraded $A_\infty$-$A$-$B$-bimodule structure $\mathcal{d}_K$ given in (3). The triangulated functor

$$\mathcal{F} : D^\infty(A) \to D^\infty(B), \quad \mathcal{F}(\bullet) = \bullet \otimes_A K$$

induces the equivalence of triangulated categories

$$\text{triang}_A^\infty(A) \simeq \text{triang}_B^\infty(K), \quad \text{thick}_A^\infty(A) \simeq \text{thick}_B^\infty(K).$$

Let $(\tilde{K}, \mathcal{d}_{\tilde{K}})$ be the $A_\infty$-$B$-$A$-bimodule with $\tilde{K} = K$ and $\mathcal{d}_{\tilde{K}}$ obtained from $\mathcal{d}_K$ exchanging $A$ and $B$; then the triangulated functor

$$\mathcal{F}'' : D^\infty(B) \to D^\infty(A), \quad \mathcal{F}''(\bullet) = \bullet \otimes_B \tilde{K}$$

induces the equivalence of triangulated categories

$$\text{triang}_B^\infty(B) \simeq \text{triang}_A^\infty(\tilde{K}), \quad \text{thick}_B^\infty(B) \simeq \text{thick}_A^\infty(\tilde{K}).$$

**Proof.** Appendix B.  

7. Deformation Quantization of $A_\infty$-structures

In this section we study the quantizations $(A_k, K_k, B_k)$ of the $A_\infty$-structures on the triple $(A, K, B)$. In this contest, the term “quantization”, or more properly, “Deformation Quantization” refers to a technique that produces new $A_\infty$-structures from already given $A_\infty$-data: the latter are recovered from the former through a “limiting” procedure. For the original idea we refer to [1]. $A_\infty$-structures on bigraded topologically free $\mathbb{K}[[\hbar]]$-modules are said to be topological. The deformations are obtained through certain Feynman diagrams expansions, a “two branes” Formality theorem and an explicit choice of an $h$-formal quadratic Poisson bivector $\pi_h = h\pi$ on $X$, the finite dimensional vector space underlying $A$ and $B$. For the full construction and the 2-branes formality theorem we refer to [3]; the diagrammatic techniques there described generalize those introduced in [13]. The choice of a quadratic Poisson bivector field is motivated by the necessity of preserving the internal grading on the Deformation Quantization of triple $(A, K, B)$; its main consequences are

- The Deformation Quantizations $(A_k, B_k)$ of $(A, B)$ are flat bigraded $A_\infty$-algebras.
- The Deformation Quantization $K_k$ of $K$ is a left $A_k$-module and a right $B_k$-module with zero differential.
- It is possible to quantize the bimodules $A \otimes_A K, K \otimes_B B, K, \text{End}_A(K)$ and $\text{End}_B(K)$ straightforwardly by using the “classical” $A_\infty$-bimodule structures with due changes.
- The quantized left and right derived actions are quasi-isomorphisms of topological $A_\infty$-algebras and topological $A_\infty$-bimodules.
7.0.20. On modules over $\mathbb{K}[[h]]$. We consider the local ring $\mathbb{K}[[h]]$ of formal power series with coefficients in $\mathbb{K}$. Topological free $\mathbb{K}[[h]]$ modules are modules over $\mathbb{K}[[h]]$ isomorphic to $\mathbb{K}[[h]]$-modules of the form $M[[h]]$, with $M$ a $\mathbb{K}$ vector space. Let $M$ and $N$ be $\mathbb{K}[[h]]$-modules. The $\mathbb{K}[[h]]$-module $M \otimes_{\mathbb{K}[[h]]} N$ is the quotient of the tensor product $M \otimes \mathbb{K}[[h]] N$ by the subspace generated by all elements of the form $km \otimes n - m \otimes kn$, with $k \in \mathbb{K}[[h]]$ and $m, n \in N$. We denote by $\langle \cdot \rangle$ the completed tensor product $M \otimes \mathbb{K}[[h]] N$. If $M$ and $N$ are topologically free, i.e. $M = M_1[[h]]$ and $N = N_1[[h]]$, then $M[[h]] \otimes \mathbb{K}[[h]] N[[h]]$ is topologically free as well; in fact $M \otimes \mathbb{K}[[h]] N[[h]] = (M_1 \otimes N_1) [[h]]$.

Let $\text{Hom}_{\mathbb{K}[[h]]}(M[[h]], N[[h]])$ be the space of $\mathbb{K}[[h]]$-linear morphisms from $M[[h]]$ to $N[[h]]$; there exists an isomorphism $\mathcal{I} : \text{Hom}(M, N) [[h]] \rightarrow \text{Hom}_{\mathbb{K}[[h]]}(M[[h]], N[[h]])$ of $\mathbb{K}[[h]]$-modules. Any $\varphi \in \text{Hom}_{\mathbb{K}[[h]]}(M[[h]], N[[h]])$ is uniquely determined by a formal power series

$$\sum_{i \geq 0} \varphi_i h^i \in \text{Hom}(M, N) [[h]].$$

We observe that any $\varphi \in \text{Hom}_{\mathbb{K}[[h]]}(M[[h]], N[[h]])$ is continuous w.r.t. the $h$-adic topology on $M[[h]]$ and $N[[h]]$. In the sequel we will use the formal power series description of morphisms extensively.

7.0.21. On the category $\mathbf{bG}_{\mathbb{K}[[h]]}$. Let $\mathbf{bG}_{\mathbb{K}[[h]]}$ be the category of bigraded $\mathbb{K}[[h]]$-modules; an object in $\mathbf{bG}_{\mathbb{K}[[h]]}$ is a collection $\{M^j_i\}_{i,j \in \mathbb{Z}}$ of $\mathbb{K}[[h]]$-modules; the space of morphisms $\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(M, N)$ is the object in $\mathbf{bG}_{\mathbb{K}[[h]]}$ with bihomogeneous components

$$\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(M, N) = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathbb{K}[[h]]}(M^j_i, N^{j+i}).$$

7.0.22. Topologically free modules in $\mathbf{bG}_{\mathbb{K}[[h]]}$. We say that an object $M_h$ in $\mathbf{bG}_{\mathbb{K}[[h]]}$ is topologically free if

$$M_h = \{ (M_h)^j_i \}_{(i,j) \in \mathbb{Z}^2}, \text{ with } (M_h)^j_i = M^j_i [[h]].$$

Let $M_h$ and $N_h$ be topologically free objects in $\mathbf{bG}_{\mathbb{K}[[h]]}$; with $M_h = M[[h]]$ and $N_h = N[[h]]$, for $M, N$ objects in $\mathbf{bG}_{\mathbb{K}[[h]]}$; then $\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(M_h, N_h)$ is the topologically free object in $\mathbf{bG}_{\mathbb{K}[[h]]}$ with bihomogeneous components

$$\text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(M_h, N_h) = \text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(M, N) [[h]].$$

For any topologically free $M_h$ in $\mathbf{bG}_{\mathbb{K}[[h]]}$, the objects $M_h[k]$ and $M_h[l]$ in $\mathbf{bG}_{\mathbb{K}[[h]]}$ are defined via

$$M_h[k] = \{ (M_h[k])^j_i \}_{(i,j) \in \mathbb{Z}^2}, \text{ with } (M_h[k])^j_i := M^{j+k}_j [[h]];$$

and

$$M_h[l] = \{ (M_h[l])^j_i \}_{(i,j) \in \mathbb{Z}^2}, \text{ with } (M_h[l])^j_i := M^{i+l}_j [[h]];$$

for any $(k, l) \in \mathbb{Z}^2$. Topologically free objects in $\mathbf{bG}_{\mathbb{K}[[h]]}$ form a full subcategory in $\mathbf{bG}_{\mathbb{K}[[h]]}$ which is not abelian; we endow it with a monoidal structure induced by the completion $\langle \cdot \rangle$, w.r.t. the $h$-adic topology, of the tensor product of topologically free $\mathbb{K}[[h]]$-modules. More precisely, for any $M_h$ and $N_h$ topologically free in $\mathbf{bG}_{\mathbb{K}[[h]]}$ and $(i,j) \in \mathbb{Z}^2$, we write (with a little abuse of notation)

$$(M_h \hat{\otimes} N_h)^j_i = \bigoplus_{i_1 + i_2 = i} M^j_{i_1} [[h]] \hat{\otimes} N^{i_2}_j [[h]],$$

where $\hat{\otimes}$ on the right hand side is the completed tensor product of topologically free $\mathbb{K}[[h]]$-modules introduced above.

8. Topological $A_{\infty}$-structures.

8.0.23. Topological $A_{\infty}$-algebras.

Definition 19. Let $A_h$ be a topologically free object in $\mathbf{bG}_{\mathbb{K}[[h]]}$. The topological tensor coalgebra over $A_h$ is the triple $(T(A_h[[1]]), \Delta_h, \epsilon_h)$ where

$$T(A_h[[1]]) := \bigoplus_{q \geq 0} A_h[[1]] \hat{\otimes} q = T(A[[1]]) [[h]],$$

in $\mathbf{bG}_{\mathbb{K}[[h]]}$, and

$$\Delta_h \in \text{Hom}_{\mathbf{bG}_{\mathbb{K}[[h]]}}(T(A_h[[1]]), T(A_h[[1]]) \hat{\otimes} T(A_h[[1]]))$$

given by $\Delta_h = \sum_{i \geq 0} \Delta^{(i)} h^i = \Delta^0 = \Delta$, where $\Delta$ denotes the coproduct on $T(A[[1]])$ and $\epsilon_h = \epsilon$, where $\epsilon$ is the counit in $T(A[[1]])$.

By definition $(1 \hat{\otimes} \Delta_h) \circ \Delta_h = (\Delta_h \hat{\otimes} 1) \circ \Delta_h$ and $(\epsilon_h \hat{\otimes} 1) \circ \Delta_h = (1 \hat{\otimes} \epsilon_h) \circ \Delta_h = 1$. 


8.0.24. On codifferentials: definitions.

Definition 20. A coderivation on $T(A_h[1])$ is a morphism $d_{A_h} : \text{Hom}_{bG_K[[\hbar]]}^1 (T(A_h[1]), T(A_h[1]))$ s.t. $(1 \otimes d_{A_h} + d_{A_h} \otimes 1) \circ \Delta_h = \Delta_h \circ d_{A_h}$ and
\[
d_{A_h}^2 = 0
\]

Let $d_{A_h}$ be the coderivation on $T(A_h[1])$ uniquely determined by the formal power series
\[
d_{A_h} = \sum_{i \geq 0} d_{A_h}^{(i)} \hbar^i, \quad d_{A_h}^{(i)} \in \text{Hom}_{bG_K}^1 (T(A[1]), T(A[1])).
\]

Then, by definition of $d_{A_h}$, each $d_{A_h}^{(i)}$ is uniquely determined by the family of Taylor components $d_{A_h}^{(i,k)} = p_{A_h[1]} \circ d_{A_h}^{(i)}|_{A_h[1]^{\otimes k}}$. The quadratic relations \([25]\) are equivalent to a tower of quadratic relations with the Taylor components $d_{A_h}^{(i,k)} : k \geq 0, i \geq 0$.

Definition 21. Let $A_h$ be a topologically free object in $bG_K[[\hbar]]$. A topological $A_{\infty}$-algebra structure on $A_h$ is the datum of a coderivation on the topological tensor coalgebra over $A_h$.

Lemma 10. Let $(A_h, d_{A_h})$ be a topological $A_{\infty}$-algebra. Then $(A, d_A)$, $d_A := d_{A_h}^{(0)}$, is an $A_{\infty}$-algebra (on $K$).

8.0.25. Topologically free $A_{\infty}$-modules.

Definition 22. Let $M_h$ be topologically free module in $bG_K[[\hbar]]$; $R_h(M_h)$ is the object
\[
R_h(M_h) := M_h \otimes T(A_h[1]) = (M[1] \otimes T(A[1]))[[\hbar]]
\]
in $bG_K[[\hbar]]$. A right $(T(A_h[1]), \Delta_h, \varepsilon_h)$-counital-comodule structure on $R_h(M_h)$ is the morphism
\[
\Delta_h^R \in \text{Hom}_{bG_K[[\hbar]]}^1 (R_h(M_h), R_h(M_h) \otimes T(A_h[1])), \quad \Delta_h^R = \Delta_h^R(0) = \Delta_h^R,
\]
satisfying $(1 \otimes \Delta_h) \circ \Delta_h^R = (\Delta_h^R \otimes 1) \circ \Delta_h^R$ and $(1 \otimes \varepsilon_h) \circ \Delta_h^R = 1$, denoting by $\Delta_h^R$ the usual counital-$T(A[1])$-comodule structure on $M[1] \otimes T(A[1])$.

Definition 23. A codifferential on the right $T(A_h[1])$-comodule $R_h(M_h)$ is a morphism $d_{M_h} : \text{Hom}_{bG_K[[\hbar]]}^1 (R_h(M_h), R_h(M_h))$ s.t.
\[
\Delta_h \circ d_{M_h} = (1 \otimes d_{M_h} + d_{M_h} \otimes 1) \circ \Delta_h^R \quad \text{and}
\]
\[
d_{M_h}^2 = 0.
\]

By definition, if $d_{M_h} = \sum_{i \geq 0} d_{M_h}^{(i)} \hbar^i$, then each $d_{M_h}^{(i)} \in \text{Hom}_{bG_K}^1 (M[1] \otimes T(A[1]), M[1] \otimes T(A[1]))$ is uniquely determined by its Taylor components $d_{M_h}^{(i,n)} = p_{M[1]} \circ d_{M_h}^{(i)}|_{M[1]^{\otimes n}}$, for any $i, n \geq 0$. The quadratic relations \([26]\) are equivalent to a tower of quadratic relations involving the aforementioned maps $d_{M_h}^{(i,n)}$.

Definition 24. Let $M_h$ be an object in $bG_K[[\hbar]]$. A topological right $A_{\infty}$-$A_h$-module structure on $M_h$ is the datum of a codifferential $d_{M_h}$ on $R_h(M_h)$.

Lemma 11. Let $M_h$ be a topological right $A_{\infty}$-$A_h$-module. Then $M$ is a right $A_{\infty}$-$A$-module.

In the same spirit, one can define topological left $A_{\infty}$-modules and topological $A_{\infty}$-bimodules, with due changes.

8.0.26. On morphisms, quasi-isomorphisms and homotopy equivalences.

Definition 25. Let $(M_h, d_{M_h})$ and $(N_h, d_{N_h})$ be topological $A_{\infty}$-$A_h$-modules, with $(A_h, d_{A_h})$ topological $A_{\infty}$-algebra.

A morphism $f_h : M_h \to N_h$ of topological $A_{\infty}$-$A_h$-modules is a map $f_h \in \text{Hom}_{bG_K[[\hbar]]}^1 (R_h(M_h), R_h(N_h))$ which is a morphism of $T(A_h[1])$-counital-comodules s.t.
\[
f_{N_h} \circ f_h = f_h \circ d_{M_h}.
\]

Such a morphism is uniquely determined by a formal power series $f_h = \sum_{i \geq 0} f_h^{(i)} \hbar^i$, with $f_h^{(i)} \in \text{Hom}_{bG_K}^1 (M[1] \otimes T(A[1]), N[1] \otimes T(A[1]))$, for any $i \geq 0$. Each component $f_h^{(i)}$ is a morphism of counital-$T(A[1])$-comodules, and so it admits an explicit description by Taylor components $f_h^{(i,n)} : M[1] \otimes A[1]^{\otimes n} \to N[1]$, for any $i, n \geq 0$.

Lemma 12. Let $f_h : M_h \to N_h$, $f_h = \sum_{i \geq 0} f_h^{(i)} \hbar^i$ be a morphism of topological $A_{\infty}$-$A_h$-modules. Then $f_h^{(0)} : M \to N$ is a morphism of $A_{\infty}$-$A$-modules.
Definition 26. Let $f_h, g_h : M_h \to N_h$ be morphisms of topological $A_{\infty}$-$A_h$-modules; we say that they are topological $A_{\infty}$-homotopy equivalent (alternatively: top. $A_{\infty}$-homotopic) if there exists a topological $A_{\infty}$-homotopy between them, i.e. a map $H_h : M_h \to N_h$ of $T(A_h[1])$-counital-comodules with

$$H_h = \sum_{i \geq 0} H_h^{(i)} h^i,$$

$$H_h^{(i),n} \in \text{Hom}_{\mathcal{D}G}(M[1] \otimes A[1] \otimes_n A[1], M[1]), \ n \geq 0,$$

such that

$$f_h - g_h = d_{M_h} \circ H_h + H_h \circ d_{M_h}$$

holds true, order by order in $h$.

8.0.27. On units. Let $A_h$ in $bG\mathcal{E}_{\mathbb{K}}[[h]]$ be a topological $A_{\infty}$-algebra with codifferential $d_{A_h}$. We say that the right $A_{\infty}$-$A_h$-module structure $d_{M_h}$ on $M_h$ is strictly unital if

$$d_{M_h}^{(i),n}(m[a_1] \ldots |\eta| \ldots |a_n]) = 0,$$

for any $n \geq 2$ and $i \geq 0$. We have denoted by $\eta$ the unit in $A$ and by $d_{M_h}^{(i),n}$ the $n$-th Taylor component of $d_{M_h}^{(i)}$.

A morphism $f_h : M_h \to N_h$ of topological $A_{\infty}$-$A_h$-modules is strictly unital if

$$f_h^{(i),n}(m[a_1] \ldots |\eta| \ldots |a_n]) = 0,$$

for any $n \geq 1$, $i \geq 0$, where $f_h^{(i),n}$ is the $n$-th Taylor component of $f_h$. Strictly unital homotopies are defined similarly.

8.0.28. Quantizing $(A, K, B)$ via quadratic Poisson structures. By $X$ we denote a finite dimensional vector space of dimension $n$ on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(T_{\text{poly}}(X)[[h]], [-, -], \eta)$ be the trivial deformation of $(T_{\text{poly}}(X), [-, -])$, with Schouten-Nijenhuis bracket $[-, -]$, obtained by extending $[-, -] \mathbb{K}[[h]]$-linearly. Let $\{x_i\}_{i \in I}$ be a set of global coordinates on $X$, with $I = n$. We say that the Poisson bivector $\pi \in T_{\text{poly}}(X)$ is quadratic if it can be written as

$$\pi = \sum_{i,j=1}^{n} \pi^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \pi^{ij} = \sum_{k,l=1}^{n} \pi^{ij}_{kl} x_k x_l,$$

for some constant coefficients $\pi^{ij}_{kl} \in \mathbb{K}$ such that $\pi^{ij}_{kl} = -\pi^{ij}_{lk}$, for any $k, l \in I$. In $\mathbb{K}$ a 2-branes Formality theorem is proved; we refer to $\mathbb{K}$ for all details; here we sketch the construction in the special case in which the triple $(A, K, B)$ appears.

To the $A_{\infty}$-triple $(A, K, B)$ it is possible to associate a unital $A_{\infty}$-category $\text{Cat}_{A_{\infty}}(A, K, B)$; its objects are the branes $U = X$ and $V = \{0\}$ in the vector space $X$ and the spaces of morphisms are given by $\text{Hom}(U, U) = A$, $\text{Hom}(V, V) = B$, $\text{Hom}(U, V) = K$ and $\text{Hom}(V, U) = 0$. In this “local” setting the branes are linear subspaces on the ambient space $X$. The unital $A_{\infty}$-category structure on $\text{Cat}_{A_{\infty}}(A, K, B)$ is induced by the associative algebra structures on $A, B$ and the $A_{\infty}$-$A-B$-bimodule structure on $K$. The 2-branes Formality theorem states the existence of a quasi-isomorphism of $L_\infty$-algebras

$$U : (T_{\text{poly}}^{\star+1}(X), [-, -], 0) \to (C^{+1}(\text{Cat}_{A_{\infty}}(A, K, B)), [-, -]_{C}, \partial)$$

between the differential graded Lie algebra (shortly, DGLA) of polynomial polyvector fields on $X$ and the DGLA of Hochschild cochains on the $A_{\infty}$-category $\text{Cat}_{A_{\infty}}(A, K, B)$ endowed with the Gerstenhaber bracket $[-, -]_{C}$ and Hochschild differential $\partial$. As a graded object $C^{+1}(\text{Cat}_{A_{\infty}}(A, K, B))$ decomposes in the direct sum of three components: $C^{+1}(A, A)$, $C^{+1}(B, B)$ and $C^{+1}(A, K, B)$. $C^{+1}(A, A)$ and $C^{+1}(B, B)$ are the DGLAs of Hochschild cochains of $A$ and $B$; they are sub complexes of $C^{+1}(\text{Cat}_{A_{\infty}}(A, K, B))$. $C^{+1}(A, K, B)$ is given by

$$C^n(A, K, B) = \bigoplus_{p+q+r=n-1} \text{Hom}^q(A^p \otimes K \otimes B^{+r}, K).$$

The proof of the 2-branes Formality theorem is based on Stokes’ theorem on manifolds with corners and the properties of the 4-color propagators $[\mathbb{S}, \mathbb{S}, \mathbb{M}, \mathbb{M}]$ at the boundary components. In the general case, we have to consider short loops in the Feynman diagrams describing the $L_\infty$-quasi-isomorphism $U$.

The $L_\infty$-quasi-isomorphism $U$ induces an isomorphism between the sets of Maurer-Cartan elements (MCEs) on the DGLAs $(T_{\text{poly}}^{\star+1}(X), [-, -], 0)$ and $(C^{+1}(\text{Cat}_{A_{\infty}}(A, K, B)), [-, -]_{C}, \partial)$. MCEs in $T_{\text{poly}}(X)$ are Poisson structures on $X$; they are mapped to MCEs on $C^{+1}(A, A)$ and $C^{+1}(B, B)$ which are $A_{\infty}$-deformations of the graded associative algebra structures on $A$ and $B$ and to an $A_{\infty}$-deformation of the $A_{\infty}$-$A-B$-bimodule structure on $K$.

Let $h\pi$ be a MCE in $T_{\text{poly}}(X)[[h]]$; it satisfies

$$[h\pi, h\pi]_h = 0.$$
order by order in \( h \), denoting by \([\cdot, \cdot]_h\) the Lie bracket on \( T_{\text{poly}}(X) [[h]]\) obtained by extending \( \mathbb{K} [[h]]\)-linearly the Lie bracket \([\cdot, \cdot]\) on \( T_{\text{poly}}(X) \).

Let \( U(h) = U_A(h\pi) + U_B(h\pi) + U_K(h\pi) \) be the MCE in \( C^{*+1}(\text{Cat}_\infty(A, K, B)) [[h]]\) where \( U_A(h\pi) \) is the component of \( U \) on \( C^{*+1}(A, A) [[h]]\), \( U_B(h\pi) \) is the one on \( C^{*+1}(B, B) [[h]]\) and \( U_K(h\pi) \) on \( C^{*+1}(A, K, B) [[h]]\). The three components are defined through an expansion in Feynman graphs in which "areal vertices" appear.

**Proposition 13** ([3], section 8.1). Let \( h\pi \) be an \( h\)-formal quadratic MCE in \( T_{\text{poly}}(X) [[h]]\).

- The \( A_\infty\)-deformations of \( A \), resp. \( B \) are given by \( (A [[h]], \cdot, + U_A(h\pi)), (B [[h]], \wedge + U_B(h\pi)) \).
- In other words, \( A \) and \( B \) are deformed into bigraded associative algebras with free differentials. The deformed products preserve the internal grading.
- The \( A_\infty\)-\( A [[h]] \)-\( B [[h]] \)-bimodule deforming \( K \) is given by \( (K [[h]], d, K_a, dK_a = dK + U_K(h\pi)). \)
- The codifferential \( d_{K_a} \) is such that
  \[
d^{(i), n, 0}_{K_a} = d^{(i), 0, m}_{K_a} = 0
\]
  if either \( m = n = 0 \) or \( m, n \geq 2 \), for any \( i \geq 0 \).

Choosing a general Poisson structure \( \pi \) on \( X \) we obtain different quantizations \( A_h \) resp. \( B_h \) of \( A \), resp. \( B \), in general curved as \( A_\infty \)-algebras. For a specific example we refer to [2].

8.0.29. Quantizing bimodules. In section 2 we have defined left and right bar resolutions of \( A_\infty \)-bimodules.

The \( A_\infty \)-components of the codifferentials on such resolutions are given by the formulae ([3]). We quantize the resolutions considering the triple \((A_h, B_h, K_h)\).

**Definition-lemma 1.** Let \((M_h, dM_h)\) be a topological \( A_\infty \)-\( A_h \)-\( B_h \) bimodule. The left topological bar resolution of \( M_h \) is the object \((A_h \hat{\otimes} A_h M_h = (A_h \otimes T(A_h[1]) \otimes M_h)) \) in \( bG_K [[h]]\). It is a topological \( A_\infty \)-\( A_h \)-\( B_h \) bimodule with codifferential \( d_{A_h \hat{\otimes} A_h M_h} = \sum_{i \geq 0} d^{(i)}_{A_h \hat{\otimes} A_h M_h} h^i \). For any \( i \geq 0 \), the \((k, l)\)-th Taylor component \( d^{(i), k, l}_{A_h \hat{\otimes} A_h M_h} \) is given by the formulae ([3]) with the insertion of the operators \( d^{(i), 2}_{A_h}, d^{(i), 1}_{M_h} \) and \( d^{(i), 2}_{B_h} \).

**Proof.** It is easy but quite long to check that \( d_{A_h \hat{\otimes} A_h M_h} \circ d_{A_h \hat{\otimes} A_h M_h} = 0 \) follows from associativity of the products on \( A_h, B_h \) and the quadratic relations \( d^{2}_{M_h} = 0 \).

We can define right topological bar resolutions, or bar resolutions of topological \( A_\infty \)-bimodules, with due changes. In the sequel we will consider the bimodule \( K_h \) and the topological bar resolutions \( A_h \hat{\otimes} A_h K_h \) and \( K_h \hat{\otimes} B_h B_h \).

Let \( K \) and \( K \) be the \( A_\infty \)-\( B \)-\( A \) bimodules introduced in section 2.

**Definition-lemma 2.** The quantization \( K_h \) of the \( A_\infty \)-\( B \)-\( A \) bimodule \( K \) is the object

\[
K_h = \text{Hom}_{bG}(K_h \hat{\otimes} B_h B_h, B_h)
\]

in \( bG_K [[h]]\). It is a strictly unital topological \( A_\infty \)-\( B_h \)-\( A_h \) bimodule with codifferential \( d_{K_h} = \sum_{i \geq 0} d^{(i)}_{K_h} h^i \). For any \( i \geq 0 \), the \((k, l)\)-th Taylor component

\[
d^{(i), k, l}_{K_h} \in \text{Hom}_{bG}^0((A[1] \otimes (A \hat{\otimes} A(M)) \otimes B[1] \otimes, K[1])
\]

of \( d^{(i)}_{K_h} \) is given by the formulae in lemma 3, with the insertion of the operators \( d^{(i), 2}_{A_h}, d^{(i), 1}_{K_h} \) and \( d^{(i), 2}_{B_h} \).

**Proof.** The proof of the topological \( A_\infty \)-bimodule structure is similar to the one for \( K_h \hat{\otimes} A_h M_h \): we use the associativity of the products on \( A_h, B_h \) and the quadratic \( A_\infty \)-bimodule structure on \( K_h \).

The definition of \( K_h \) is analogous, with due changes. Similarly, we can introduce the quantizations \( \text{End}_{A_h}(K_h) \), resp. \( \text{End}_{A_h}(K_h) \) of \( \text{End}_{A_h}(K_h) \), resp. \( \text{End}_{A_h}(K_h) \); their topological \( A_\infty \)-algebra structures are induced by the topological \( A_\infty \)-structures on \((A_h, K_h, B_h)\). We use the same classical formula introduced in section 6, with due changes. Such quantizations are topologically free objects in \( bG_K [[h]]\). Let \( B_h(K_h) = A_h \hat{\otimes} A_h K_h, B_h(K_h) = K_h \hat{\otimes} B_h B_h \) and \( \text{End}_{A_h}(B_h(K_h)) \), \( \text{End}_{A_h}(B_h(K_h)) \) be the topologically free objects in \( bG_K [[h]]\):

\[
\text{End}_{A_h}(B_h(K_h)) = \varphi \in \text{End}_{bG}^0(B_h(K_h), B_h \text{ - linear}),
\]

\[
\text{End}_{A_h}(B_h(K_h)) = \varphi \in \text{End}_{bG}^0(B_h(K_h), A_h \text{ - linear})
\]
They are canonically endowed with topological $A_\infty$-bimodule structures; the formulae are induced by the classical constructions presented in the previous sections. The constructions used to quantize $\text{End}_A(K)$ and $\text{End}_B(K)$ are replied here, with due changes.

**Proposition 14.** The quantized derived actions

\[ L_{A_h} : A_h \to \text{End}_{B_h}(K_h), \]
\[ R_{B_h} : B_h \to \text{End}_{A_h}(K_h)^{op}, \]

are quasi-isomorphisms of topological $A_\infty$-algebras.

**Proof.** The proposition is proved in [3], section 8.1. □

**Corollary 5.** $L_{A_h}$ and $R_{B_h}$ descend to quasi-isomorphisms of topological $A_\infty$-bimodules.

**Proof.** The bimodule structures on $A_h$, $B_h$, $\text{End}_{B_h}(K_h)$ and $\text{End}_{A_h}(K_h)$ are those described in section 6 for $A$, $B$, $\text{End}_A(K)$ and $\text{End}_B(K)$ with due changes. □

8.0.30. Some quasi-isomorphisms of quantized bimodules.

**Proposition 15.**

- There exist quasi-isomorphisms

  \[ \mu_{K_h} : K_h \otimes_{B_h} B_h \to K_h, \quad \mu_{K_h} : A_h \otimes_{A_h} K_h \to K_h, \]
  \[ \Phi_{K_h} : K_h \to K_h \otimes_{B_h} B_h, \quad \Phi_{K_h} : K_h \to A_h \otimes_{A_h} K_h. \]

  of strictly unital topological $A_\infty-A_h-B_h$-bimodules.

- There exists an isomorphism

  \[ \Theta_h^1 : K_h \otimes_{A_h} A_h \to \text{End}_{B_h}(B_h(K_h)), \]

  of strictly unital topological $A_\infty-A_h-A_h$-bimodules.

- There exists an isomorphism

  \[ \Theta_h^2 : K_h \otimes_{A_h} A_h \to \text{End}_{A_h}(B_h(K_h))^{op} \]

  of strictly unital topological $A_\infty-B_h-B_h$-bimodules.

**Proof.** On $\mu_{K_h}$. The morphism is defined using the formula for the morphism $\mu$ in proposition 2, section 2, with due changes. So the compatibility with the topological $A_\infty$-bimodule structures follows. In other words,

\[ \mu_{K_h}(\sum a_i h^i) = \sum \sum n \geq 0 \geq j=n \mu^{(i)}_{h}(a_i) h^n, \]

with $\mu^{(i)}_{h} \in \text{Hom}_{hG}(T(A[1]) \otimes (A_{i}\otimes K) \otimes T(B[1]), K[1])$ uniquely determined by the Taylor components $\mu^{(i),k,l}_{h}$, with

\[ \mu^{(i),k,l}_{h}(a_1|...|a_k, s(1)b_1|...|b_q|b'_1|...|b'_l) = \pm d_{K_h}^{(i),k,q+1+l}(a_1|...|a_k|1|b_1|...|b_q|b'_1|...|b'_l). \]

The relations $d_{K_h} \circ \mu_{K_h} = \mu_{K_h} \circ d_{K_h} \otimes_{B_h} B_h$ are equivalent to $\sum_{i+j=n} d_{h}^{(i)} \circ d_{h}^{(j)} = 0$, for any $n \geq 0$. We recall that in the classical case the relations

\[ d_{K} \circ \mu = \mu \circ d_{K} \otimes_{B} B \]

are equivalent to $d_{K}^{2} = 0$.

It is also clear that $\mu_{K_h}|_{b=0} = \mu_{K_h}|_{b=0} = \mu$; as $\mu$ is a quasi-isomorphism, then $\mu_{K_h}$ is a quasi-isomorphism as well. Similar considerations hold for $\mu'_{K_h}$. For $\Phi_{K_h}$ and $\Phi'_{K_h}$ the conclusions are similar, with due changes: all we need is to consider the trivially “quantized” version of the formula introduced in lemma 3. $\Theta_{h}^1$ is given by the formal power series $\Theta_{h}^1 = \sum_{i \geq 0} \Theta_{h}^{(i)} h^i$, with $\Theta_{h}^{(i)} = 0$ for $i \geq 1$ and $\Theta_{h}^{(0)} = I \circ G$, where $I : K \otimes_{A} K \to B(A) \otimes K$ is an isomorphism of $A_\infty-A-A$-bimodules and $G : B(A) \otimes K \to \text{End}_{A}(B(A))$ is the isomorphism $A_\infty-A-A$-bimodules given in prop. 3.

On any element $\sum_{i \geq 0} a_i h^i$ in $(T(A[1]) \otimes (K \otimes_{B} K)[1] \otimes T(A[1]))[[h]]$ the relations

\[ d_{\text{End}_{B_h}(B_h(K_h))} \circ \Theta_h^1 = \Theta_h^1 \circ d_{\text{End}_{B_h}(B_h(K_h))} \]

are equivalent to

\[ \sum_{i+j=n} d_{\text{End}_{B_h}(B_h(K_h))}^{(i)}(\Theta_h^{(i)}(a_i)) = \sum_{i+j=n} \Theta_h^{(i)}(d_{\text{End}_{B_h}(B_h(K_h))}^{(j)}(a_i)) \]
for any \( n \geq 0 \); the above relations are verified (projecting both sides onto \( \text{End}_{A_h}(B_{B_h}(K_h)) \), as usual ) as \( \Theta_h^{(0)} \) commutes with \( d_{\text{End}_{B_h}(B_{B_h}(K_h))}^{(j)} \cdot d_{K_h \otimes_{A_h} K_h}^{(j)} \), for any \( j \geq 0 \).

We recall that the Taylor components \( d_{\text{End}_{B_h}(B_{B_h}(K_h))}^{(j)} \cdot d_{K_h \otimes_{A_h} K_h}^{(j)} \) are given by the Taylor components \( d_{\text{End}_{B_h}(B_{B_h}(K_h))}^{(j)} \cdot d_{K_h \otimes_{A_h} K_h}^{(j)} \) via the substitutions \( d_A^{(j)} \mapsto d_A^{(j),2} \), \( d_B^{(j)} \mapsto d_B^{(j),2} \) and \( d_{K_h}^{(j)} \mapsto d_{K_h}^{(j),2} \).

\( \Theta_h^{(0)} \) is an isomorphism in \( bG_k \); it follows that \( \Theta_h^2 \) is an isomorphism in \( bG_k[[\hbar]] \). For \( \Theta_h^3 \) similar considerations hold, with due changes.

\[ \square \]

**Corollary 6.**

- There exists a homotopy equivalence
  \[ K_h \otimes_{A_h} K_h \to \text{End}_{B_h}(K_h), \]
  of strictly unital topological \( A_{\infty} \)-\( A_h \)-\( A_h \)-modules.

- There exists a homotopy equivalence
  \[ K_h \otimes_{A_h} K_h \to \text{End}_{A_h}(K_h)^{op} \]
  of strictly unital topological \( A_{\infty} \)-\( B_h \)-\( B_h \)-modules.

**Proof.** The classical homotopy equivalence \( H : \text{End}_{B}(bG(K)) \to \text{End}_{B}(K) \) defined in prop. I induces the homotopy equivalence \( H_h : \text{End}_{B_h}(bG_h(K_h)) \to \text{End}_{B_h}(K_h) \), \( H_h = H \). The check is immediate; in fact the codifferentials on \( \text{End}_{B_h}(bG_h(K_h)) \) and \( \text{End}_{B_h}(K_h) \) are constructed by using \( d_{K_h}^{(j),k} \), \( d_{K_h}^{(j),l} \) and \( d_{B_h}^{(j)} \). It is necessary to prove the compatibility of the quantized homotopy equivalence with these operators. But this goes on like in the classical case. Similar considerations hold for the second statement, with due changes.

\[ \square \]

Composing with the quantized derived action we arrive at

**Corollary 7.**

- There exists a quasi-isomorphism
  \[ A \to K_h \otimes_{A_h} K_h, \]
  of strictly unital topological \( A_{\infty} \)-\( A_h \)-\( A_h \)-modules.

- There exists a quasi-isomorphism
  \[ B_h \to K_h \otimes_{A_h} K_h, \]
  of strictly unital topological \( A_{\infty} \)-\( B_h \)-\( B_h \)-modules.

8.9.31. On the categories \( \text{Mod}^{\gamma}_f(A_h) \) and \( \text{Mod}^{\gamma}_f(B_h) \).

**Definition 27.** Let \( (A_h, K_h, B_h) \) be the triple quantizing \((A, K, B)\) w.r.t. an \( \hbar \)-formal quadratic Poisson bivector \( \pi_h \).

- \( \text{Mod}^{\gamma}_f(A_h) \) is the category of strictly unital topological \( A_{\infty} \)-\( A_h \)-modules.

- \( \text{Mod}^{\gamma}_f(B_h) \) is the category of strictly unital topological \( A_{\infty} \)-\( B_h \)-modules.

\( \text{Mod}^{\gamma}_f(A_h) \) and \( \text{Mod}^{\gamma}_f(B_h) \) are additive categories. The direct sum \( M_h \oplus N_h \) of objects \((M_h,d_{M_h})\) and \((N_h,d_{N_h})\) in \( \text{Mod}^{\gamma}_f(A_h) \) (or \( \text{Mod}^{\gamma}_f(B_h) \)) is the topologically free module

\[ M_h \oplus N_h := (M \oplus N) [[\hbar]] \]

if \( M_h = M [[\hbar]] \) and \( N_h = N [[\hbar]] \) in \( bG_k[[\hbar]] \), endowed with the strictly unital topological \( A_{\infty} \)-module structure given by the codifferential \( d_{M_h} \oplus d_{N_h} \). The natural inclusion and projection

\[ i_h : M_h \to M_h \oplus N_h, \quad p_h : M_h \oplus N_h \to N_h, \]

are the strict topological \( A_h \)-module morphisms defined via

\[ i_h = i_h^{(0)} = i_h^{(0),0} = i : M \to M \oplus N, \quad m \mapsto m \oplus 0, \]

and

\[ p_h = p_h^{(0)} = p_h^{(0),0} = p : M \oplus N \to N, \quad m \oplus n \mapsto n. \]
8.0.32. On quasi-isomorphisms in $\text{Mod}^\infty_f(A_h)$ and $\text{Mod}^\infty_f(B_h)$. The category $bG_{K[[\hbar]]}$ of all bigraded $K[[\hbar]]$-modules is abelian; clearly $\text{Mod}^\infty_f(A_h)$ and $\text{Mod}^\infty_f(B_h)$ are (not full) subcategories of $bG_{K[[\hbar]]}$.

In general, the cohomology of a topologically free differential bigraded $K[[\hbar]]$-module is not topologically free; so we introduce the following definition.

**Definition 28.** A quasi-isomorphism $f_h: N_h \to M_h$ of objects in $\text{Mod}^\infty_f(A_h)$ (or $\text{Mod}^\infty_f(B_h)$) is a morphism of topological $A_\infty$-modules s.t. $H(f_h): H(N_h) \to H(M_h)$ is an isomorphism in the abelian category $bG_{K[[\hbar]]}$. Quasi-isomorphisms of strictly unital top. $A_\infty$-modules are not, in general, homotopy equivalences. A counterexample is given by

**Example 1** (B. Keller, [8]). Let $A_h = K[[\hbar]]$ and $(M_h,d_{M_h})$ be the object in $\text{Mod}^\infty_f(A_h)$ given by

$$M_h = \{ M_0^0 \[ [\hbar] \], M_1^0 \[ [\hbar] \], M_2^0 \[ [\hbar] \] \}, \quad M_0^0 = M_1^0 = K, \quad M_2^0 = 0,$$

with codifferential $d_{M_h}$ s.t. $d_{M_h}^{(0),0}: M_0^0 \[ [\hbar] \] \to M_1^0 \[ [\hbar] \]$ is the multiplication by $h$, and $d_{M_h}^{(0),1}: M_2^0 \otimes A_h \to M_1^0$ is the multiplication in $K[[\hbar]]$. All the other components are set to be zero. The strict quasi-isomorphism ($K$ is concentrated in bideg $(0,0)$) of strictly unital top. $A_\infty$-modules

$$f_h: M_h \to K$$

admits no inverse $g_h$ (up to homotopy); in fact such an inverse would have a $K[[\hbar]]$-linear $(0,0)$-th component $\partial^{(0),k}=0: K \to K[[\hbar]]$. But this implies that $g_h$ is the zero map.

8.0.33. On $\text{H}^\infty_f(A_h)$, $\text{H}^\infty_f(B_h)$ and their triangulated structures. As $\text{Mod}^\infty_f(A_h)$ and $\text{Mod}^\infty_f(B_h)$ are additive categories, we can introduce the homotopy categories

$$\text{H}^\infty_f(A_h) := \text{H}(\text{Mod}^\infty_f(A_h)), \quad \text{H}^\infty_f(B_h) := \text{H}(\text{Mod}^\infty_f(B_h)).$$

The objects in $\text{H}^\infty_f(A_h)$, resp. $\text{H}^\infty_f(B_h)$ are the same objects of $\text{Mod}^\infty_f(A_h)$, resp. $\text{Mod}^\infty_f(B_h)$. The morphisms are equivalence classes w.r.t. the equivalence relation defined as follows; two morphisms $f_h,g_h: X_h \to Y_h$ in $\text{Mod}^\infty_f(A_h)$, resp. $\text{Mod}^\infty_f(B_h)$ are equivalent, i.e. $f_h \sim g_h$, if there exists a strictly unital topological $A_\infty$-homotopy $H_h$ (see 8.3.1., subsection “On morphisms, quasi-isomorphisms and homotopy equivalences ”) s.t. $f_h-g_h = d_{X_h} \circ H_h + H_h \circ d_{X_h}$, denoting by $d_{X_h}$, resp. $d_{Y_h}$ the codifferentials on $X_h$, resp. $Y_h$. $\sim$ is an equivalence relation on morphisms in $\text{Mod}^\infty_f(A_h)$, resp. $\text{Mod}^\infty_f(B_h)$. We want to prove that $\text{H}^\infty_f(A_h)$ and $\text{H}^\infty_f(B_h)$ are triangulated categories.

8.0.34. Triangulated structure on $\text{H}^\infty_f(A_h)$, $\text{H}^\infty_f(B_h)$. We endow the categories $\text{H}^\infty_f(A_h)$ and $\text{H}^\infty_f(B_h)$ with a triangulated structure such that, for $h=0$ it reduces to the triangulated structure on $\text{H}_\infty(A)$ and $\text{H}_\infty(B)$. We refer to Appendix A for the notation on triangulated categories. As usual we give the definition for $\text{H}^\infty_f(A_h)$; it applies to $\text{H}^\infty_f(B_h)$ as well, with due changes.

Let $0 \to M_h \overset{f_h}{\to} M_h' \overset{g_h}{\to} M_h'' \to 0$ be a short exact sequence of objects in $\text{H}^\infty_f(A_h)$ with $f_h$ and $g_h$ strict. This means that, for any $(i,j) \in \mathbb{Z}^2,$ then

$$0 \to (M_h)_i^j \overset{f_h}{\to} (M_h')_i^j \overset{g_h}{\to} (M_h'')_i^j \to 0$$

is short exact as sequence of $K[[\hbar]]$-modules.

**Definition 29.** The triangulated structure on the additive category $\text{H}^\infty_f(A_h)$ is given as follows. The endofunctor $\Sigma$ is simply the (cohomological) grading shift functor $\Sigma = [1]$. The distinguished triangles are isomorphic to those induced by semi-split sequences of strict morphisms

$$M_h \overset{f_h}{\to} M_h' \overset{g_h}{\to} M_h'' \to 0$$

in $\text{Mod}^\infty_f(A_h)$, i.e. sequences such that $0 \to M_h \overset{f_h}{\to} M_h' \overset{g_h}{\to} M_h'' \to 0$ is an exact sequence in $bG_{K[[\hbar]]}$, and such that there exists a strict splitting

$$\rho_h = \sum_{k \geq 0} \rho^{(k)} h^k, \quad \rho^{(k)} = \rho^{(k),0} \in \text{Hom}^{0,0}_{bG_{K}} (M'[1], M[1])$$

of $f_h$, i.e.

$$(27) \quad \rho_h \circ f_h = 1_h,$$

with

$$\rho_h \circ d_{M_h} = d_{M_h'} \circ (\rho_h \circ \text{Id}_{M'}_i^{i-1}), \quad i \geq 2.$$

By the very definition if the triangulated structure on $\text{H}^\infty_f(A_h)$ we have
Corollary 8. The “evaluation at \( \hbar = 0 \)” functor \( (E_\hbar,1), E_\hbar : \mathcal{H}(A_b) \to \mathrm{Mod}_WH(A) \) with \( E_\hbar(M_h) := M_h/hM_h \), is exact w.r.t. the triangulated category structures on exact triangles. What follows is a suitable topological \( A_\infty \)-version of the analysis contained in [20] on the triangulated structure of the homotopy category \( \mathcal{K}(A) \) of any additive category \( A \). The goal is to show that it is possible to lift those computations to the aforementioned topological \( A_\infty \)-case.

8.0.35. Characterization of exact triangles in \( \mathcal{H}(A_b) \). Before proving that the endofunctor \( [1] \) and the class of exact triangles in the above definition endow \( \mathcal{H}(A_b) \) with a triangulated category structure, let us better characterize the exact triangles. What follows is a suitable topological \( A_\infty \)-version of the analysis contained in [20] on the triangulated structure of the homotopy category \( \mathcal{K}(A) \) of any additive category \( A \).

8.0.36. Cones and cylinders. Exact sequences of topologically free modules. We recall that, given a topological \( A_\infty \)-module \( M_h \), the bigraded object \( M_h[\pm1] \) can be endowed with a topological \( A_\infty \)-module structure via

\[
\delta^{(i),t}_{M_h[\pm1]} = -s \circ \delta^{(i),t}_{M_h} \circ (s^{-1} \otimes 1).
\]

Let \( f_h : (M_h,dM_h) \to (N_h,dN_h) \) be a morphism in \( \mathrm{Mod}_WH(A_b) \); \( f_h \), \( dM_h \) and \( dN_h \) are uniquely determined by formal power series whose \( i \)-th components are \( f_h(i), dM_h(i) \) and \( dN_h(i) \).

Definition 30. A cone \( C(f_h) \) of \( f_h \) is the object

\[
C(f_h) := M_h[1] \overset{\tilde{g}}{\to} N_h
\]

with topological \( A_\infty \)-structure given by the differential \( dC(f_h) \), such that

\[
D_{C(f_h)} = \begin{pmatrix}
        dM_h[1] & 0 \\
        s^{-1} \circ f_h & s^{-1} \circ dN_h \circ s
\end{pmatrix}
\]

Definition 31. The \( A_\infty \)-cylinder \( \text{Cyl}(f_h) \) is the object

\[
\text{Cyl}(f_h) = M_h \oplus M_h[1] \overset{\tilde{g}}{\to} N_h
\]

with codifferential \( D_{\text{Cyl}(f_h)} \) given by

\[
D_{\text{Cyl}(f_h)} = \begin{pmatrix}
        s^{-1} \circ dM_h \circ s & -i_h \circ s^{-1} & 0 \\
        0 & dM_h[1] & 0 \\
        0 & s^{-1} \circ f_h & s^{-1} \circ dN_h \circ s
\end{pmatrix}.
\]

The natural inclusion

\[
i_h : M_h \to \text{Cyl}(f_h), \quad i_h = \sum_{i \geq 0} \iota_h^{(i)h_i}
\]

with

\[
i_h^{(i),n} = 0, \quad \text{for } i,n \geq 1.
\]

and \( \iota_h^{(0),0} = i : M \to M \oplus M[1] \oplus N, \ m \mapsto (m,0,0) \), is a strict morphism of topological \( A_\infty \)-\( A_h \)-modules.

Proposition 16. For any morphism \( f_h \) in \( \mathrm{Mod}_WH(A_b) \)

\[
dC(f_h) \circ dC(f_h) = d_{\text{Cyl}(f_h)} \circ d_{\text{Cyl}(f_h)} = 0.
\]

Proof. Using (29) and (30), the proof is immediate.

For any morphism \( f_h : (M_h,dN_h) \to (N_h,dM_h) \) we consider the sequence

\[
0 \to M_h \overset{h}{\to} \text{Cyl}(f_h) \overset{c}{\to} C(f_h) \to 0
\]

The natural projection \( \pi_h \) is the strict morphism of topological \( A_\infty \)-modules \( \pi_h = \sum_{i \geq 0} \pi_{h,0} h_i^{(i)} \), with \( \pi_{h,0} = 0 \), for \( i \geq 1 \) and \( \pi_{h,0}^{(0),0} = \pi : M \oplus M[1] \oplus N \to M[1] \oplus N \). The sequence (31) is exact in \( bG_{[A],[B]} \); actually more can be said: as ker \( \pi_h = M_h = \text{im } i_h \) then (31) is exact in \( \mathrm{Mod}_WH(A_b) \).

Proposition 17. Let \( (M_h,dM_h), (N_h,dN_h) \) and \( (L_h,dL_h) \) be objects in \( \mathrm{Mod}_WH(A_b) \) and \( f_h : M_h \to N_h, g_h : N_h \to L_h \) be strict morphisms in \( \mathrm{Mod}_WH(A_b) \). Any short exact sequence

\[
0 \to M_h \overset{f_h}{\to} N_h \overset{g_h}{\to} L_h \to 0
\]

in \( bG_{[A],[B]} \) is quasi-isomorphic in \( \mathrm{Mod}_WH(A_b) \) to the short exact sequence \( 0 \to M_h \overset{h}{\to} \text{Cyl}(f_h) \overset{c}{\to} C(f_h) \to 0 \).

Proof. Like in [20], prop. 5, section III, with due changes.
For any morphism \( f_h : M_h \to N_h \) in \( \text{Mod}^{f}_{\mathbb{L}}(A_h) \) the sequence
\[
0 \to M_h \xrightarrow{i_h} \text{Cyl}(f_h) \xrightarrow{\rho_h} C(f_h) \to 0
\]
is exact. Something more can be said; in fact the sequence is semi-split with strict splitting \( \rho_h : \text{Cyl}(f_h) \to M_h \) given by
\[
\rho_h^0(m, sm', l) = m, \quad \rho_h^i = 0 \quad \text{for} \quad i \geq 1.
\]
It is important to note that \( \rho_h \) does not commute with the components \( d_{\text{Cyl}(f_h)}^{(i,0)} \) and \( d_{M_h}^{(i,0)} \) of the codifferentials on \( \text{Cyl}(f_h) \) and \( M_h \) for any \( i \geq 0 \), but
\[
\rho_h^0(s^{-1}(d_{\text{Cyl}(f_h)}^{(i,0)}(m, sm', l|a_{\infty}^n))) = d_{M_h}^{(i,0)}(\rho_h^0(m, sm', l|a_{\infty}^n)), \quad \text{for any} \quad n \geq 1.
\]
Like in the classical case, the presence of the inclusion \( i_h \) in the definition of the codifferential \( d_{\text{Cyl}(f_h)} \) plays a major role. In summary,
\[
M_h \xrightarrow{i_h} \text{Cyl}(f_h) \xrightarrow{\rho_h} C(f_h) \to M_h[1]
\]
is an exact triangle in \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \) for any morphism \( f_h : M_h \to L_h \) in \( \text{Mod}^{f}_{\mathbb{L}}(A_h) \).

Let
\[
M_h \xrightarrow{i_h} \text{Cyl}(f_h) \xrightarrow{\rho_h} C(f_h) \to M_h[1];
\]
be a sequence in \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \); it is isomorphic in \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \) to the exact triangle \( M_h \xrightarrow{i_h} \text{Cyl}(f_h) \xrightarrow{\rho_h} C(f_h) \xrightarrow{r_h} M_h[1] \) via
\[
M_h \xrightarrow{i_h} \text{Cyl}(f_h) \xrightarrow{\rho_h} C(f_h) \xrightarrow{r_h} M_h[1]
\]
with strict \( A_\infty \)-morphism
\[
\alpha_h : L_h \to \text{Cyl}(f_h), \quad \alpha_h^{(0,0)}(l) = (0, 0, 1)
\]
and \( \alpha_h^i = 0 \), for \( i \geq 1 \). In summary, \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \) is an exact triangle in \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \), as well.

8.0.37. Other exact triangles: using the splitting. Let
\[
0 \to M_h \xrightarrow{i_h} N_h \xrightarrow{s_h} Q_h \to 0
\]
be a semi-split exact sequence like in def. 25, with splitting \( \rho_h \) and \( N_h = N[[h]] \), \( M_h = M[[h]] \), \( Q_h = Q[[h]] \) in \( \mathbf{bG}_{\mathbb{K}}[[h]] \).

At the order \( k^0 \) eq. \( \mathcal{H}_{\mathbb{L}} \) is equivalent to \( \rho_h^0 \circ f_h^0 = 1 \). This implies that \( N_j^i \cong M_j^i \oplus Q_j^i \) as \( \mathbb{K} \)-modules, for any \( (i, j) \in \mathbb{Z}^2 \); in virtue of this we assume that \( N_h = ((M \oplus Q)[[h]], d_{M_h \oplus Q_h}) \), where
\[
d_{M_h \oplus Q_h} = \begin{pmatrix}
0 & -f_h \\
\text{id}_{M_h} & d_{Q_h}
\end{pmatrix}.
\]
and \( M_h \oplus Q_h \equiv (M \oplus Q)[[h]] \) in \( \mathbf{bG}_{\mathbb{K}}[[h]] \). It follows that \( d_{M_h \oplus Q_h} \circ d_{M_h \oplus Q_h} = 0 \) if and only if
\[
f_h : Q_h \to M_h[1]
\]
defines an \( A_\infty \)-morphism of topological \( A_\infty \)-\( A_h \)-modules. By definition of the triangulated structure on \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \), the sequence \( M_h \xrightarrow{i_h} M_h \oplus Q_h \xrightarrow{p_h} Q_h \xrightarrow{i_h} M_h[1] \) is an exact triangle.

**Theorem 8.** The homotopy categories \( \mathcal{H}^{f}_{\mathbb{L}}(A_h) \) and \( \mathcal{H}^{f}_{\mathbb{L}}(B_h) \) are triangulated; the triangulated structure is the one given in def. 25.

**Proof.** [20] pag. 246, with due changes; we sketch the proof for sake of clarity. On the axiom (T1) (see the Appendix); the sequence
\[
X_h \xrightarrow{i_h} X_h \to 0 \to X_h[1]
\]
is isomorphic to \( X_h \xrightarrow{i_h} X_h \to C(1_h) \to X_h[1] \) as the zero morphism \( 0 \to C(1_h) \) is homotopic to the identity morphism \( 1'_h : C(1_h) \to C(1_h) \) on \( C(1_h) \); in fact
\[
1'_h = H_h \circ d_{C(1_h)} + d_{C(1_h)} \circ H_h,
\]
with strict topological $A_{\infty}$-homotopy $H_h = H^{(0),0} = H^{(0),0}(sx \oplus x') = (x', 0)$. Compatibility with the codifferentials follows easily. Axiom (T2) is proved similarly. Let

$$X_h \xrightarrow{u} Y_h \xrightarrow{v} C(u_h) \xrightarrow{p_h} X_h[1]$$

be an exact triangle. We want to prove that the sequence

$$X_h \xrightarrow{u} Y_h \xrightarrow{v} C(u_h) \xrightarrow{p_h} X_h[1]$$

is isomorphic to the exact triangle

$$X_h \xrightarrow{u_h} C(v_h) \xrightarrow{s_h} C(v_h) \xrightarrow{p_h} X_h[1].$$

All we need is to introduce the topological $A_{\infty}$-homotopy equivalence

$$\theta_h : X_h[1] \rightarrow C(v_h),$$

with

$$\theta_h^{(0),0}(sx) = (-su_h^{(0),0}(sx), sx, 0), \theta_h^{(1),n}(x|a_1| \ldots |a_n) = (-su_h^{(1),n}(x|a_1| \ldots |a_n)), 0, 0), n \geq 1$$

and to check that $s_h \circ 1_h - \theta_h \circ p_h = d_{C(v_h)}H_h + H_h$ with strict $A_{\infty}$-homotopy $H_h : C(u_h) \rightarrow C(v_h)$. $H^{(0),0}(sx, y) = (y, 0, 0)$, $H^{(1),n} = 0$ otherwise.

The computations are long but straightforward; $\theta_h$ is a homotopy equivalence because it admits the strict inverse

$$\psi_h : C(v_h) \rightarrow X_h[1], \psi_h^{(0),0}(sy, sx, y') = sx,$$

and $\psi_h^{(1),n} = 0$ otherwise. Clearly $\psi_h \circ \theta_h = 1_h$, but $\theta_h \circ \psi_h = 1_h + d_{C(v_h)}H_h + H_h$ with $H_h^{(0),0}(sy, sx, y') = (y', 0, 0)$ and zero otherwise.

Axiom (T3) is proved by using cones and (T4) follows by using semi split exact sequences.

8.0.38. Localizing w.r.t. topological $A_{\infty}$-quasi-isomorphisms: on the derived categories $D^r_{A_{\infty}}(A_h)$ and $D^r_{B_{\infty}}(B_h)$. In [20], def.6, section III, localizing classes of morphisms are defined. In our setting we have

**Proposition 18.** The class $Qis$ of quasi-isomorphisms in the homotopic categories $\mathcal{H}^r_{A_{\infty}}(A_h)$ and $\mathcal{H}^r_{B_{\infty}}(B_h)$ is localizing.

**Proof.** We prove the statement for $\mathcal{H}^r_{A_{\infty}}(A_h)$. We refer to the proof of thm. 4, pag 160 in [23]. We “translate” it in our topological $A_{\infty}$-case, with due changes.

Thanks to the above proposition the following definition makes sense.

**Definition 32.** The localizations

$$D^r_{A_{\infty}}(A_h) := H^r_{A_{\infty}}(A_h)[Qis^{-1}], \text{ resp. } D^r_{B_{\infty}}(B_h) := H^r_{B_{\infty}}(B_h)[Qis^{-1}]$$

are said to be the derived categories of $Mod^r_{A_{\infty}}(A_h)$, resp. $Mod^r_{B_{\infty}}(B_h)$.

The objects in $D^r_{A_{\infty}}(A_h)$, resp. $D^r_{B_{\infty}}(B_h)$ are the same objects of $H^r_{A_{\infty}}(A_h)$, resp. $H^r_{B_{\infty}}(B_h)$ while the morphisms are described through the equivalence classes of “roofs”, as in [20]. We use the notation $D = D^r_{A_{\infty}}(A_h), D^r_{B_{\infty}}(B_h)$. Any morphism $\phi_h : X_h \rightarrow Y_h$ in $D$ is represented by an equivalence class of roofs; if two roofs $(s_h, \phi_h)$ and $(t_h, \psi_h)$ representing the same morphism in $D$ are equivalent, we will simply write $(s_h, \phi_h) = (t_h, \psi_h)$.

In what follows we will state that the morphism $\phi_h : X_h \rightarrow Y_h$ in $D$ is represented by the roof $(s_h, \phi_h)$, for simplicity. The identity morphism $1_h : X_h \rightarrow X_h$ in $D$ is represented by

$$\begin{align*}
X_h & \xrightarrow{1_h} X_h \\
X_h & \xrightarrow{1_h} X_h
\end{align*}$$

for any $X_h$ in $D$. The composition

$$(34) \psi_h \circ \phi_h$$

of morphisms $\phi_h : X_h \rightarrow Y_h, \psi_h : Y_h \rightarrow Z_h$ in $D$ represented by the roofs $(s_h, \phi_h)$ and $(t_h, \psi_h)$ will be denoted also by $(t_h, \bar{\psi}_h) \circ (s_h, \bar{\phi}_h)$. 

Corollary 9. The class of quasi-isomorphisms in $\mathcal{H}_\infty^f(A_h)$ and $\mathcal{H}_\infty^f(B_h)$ is compatible with triangulation; the derived categories

\[ \text{D}_1^\infty(A_h), \text{D}_2^\infty(B_h). \]

are triangulated.

Proof. See [20]; the proofs there apply here with straightforward changes. □

8.0.40. On the quantized Functors. Let us define the functors

\[ F_h : \text{Mod}_{\infty}^f(A_h) \to \text{Mod}_{\infty}^{f,\text{strict}}(B_h), \quad G_h : \text{Mod}_{\infty}^f(B_h) \to \text{Mod}_{\infty}^{f,\text{strict}}(A_h), \]

via

\[ F_h(M_h) := M_h \widehat{\otimes}_{A_h} K_h, \quad G_h(N_h) := N_h \widehat{\otimes}_{A_h} K_h, \]

on objects $M_h \in \text{Mod}_{\infty}^f(A_h)$ and $N_h \in \text{Mod}_{\infty}^f(B_h)$. Let $f_h : M_h \to N_h$ be a morphism in $\text{Mod}_{\infty}^f(A)$. Then $F_h(f_h)$ is the strict morphism in $\text{Mod}_{\infty}^f(B)$ given by

\[ F_h(f_h) = \sum_{i \geq 0} F_h^{(i)}(f_h) h^i, \quad F_h^{(0)}(f_h) = \sum_{k \geq 0} f_h^{(i),k} \otimes 1. \]

and zero otherwise. Similar definition holds true for $G_h$. Here $\text{Mod}_{\infty}^{f,\text{strict}}(A_h)$ denotes the subcategory of $\text{Mod}_{\infty}^{f}(A_h)$ with same objects and strict topological $A_{\infty}$-morphisms. Same convention holds true for $\text{Mod}_{\infty}^{f,\text{strict}}(B_h)$.

Lemma 13. Let $f_h : M_h \to N_h$ be a quasi-isomorphism in $\text{Mod}_{\infty}^{f}(A_h)$; then $F_h(f_h) : M_h \widehat{\otimes}_{A_h} K_h \to N_h \widehat{\otimes}_{A_h} K_h$ is a quasi-isomorphism in $\text{Mod}_{\infty}^{f,\text{strict}}(A_h)$.

Similar considerations hold for the functor $G_h$.

8.0.40. The quantized functors on the derived categories; compatibility with the triangulated structures. We discuss now the above quantized functors lifting them on the derived categories $\text{D}_1^\infty(A_h)$ and $\text{D}_2^\infty(B_h)$.

Definition 33. $\mathcal{F}_h$ is the unique functor $\mathcal{F}_h : \text{D}_1^\infty(A_h) \to \text{D}_2^\infty(B_h)$ s.t.

\[ \mathcal{F}_h \circ Q_{A_h} = \mathcal{T}_h, \]

denoting by $Q_{A_h} : \mathcal{H}_{\infty}^f(A_h) \to \text{D}_1^\infty(A_h)$ the canonical functor

\[ Q_{A_h}(X) = X, \quad Q_{A_h}(f_h) = (1, f_h), \]

and by $\mathcal{T}_h : \mathcal{H}_{\infty}^f(A_h) \to \text{D}_2^\infty(B_h)$ the composition

\[ \mathcal{T}_h = Q_{B_h} \circ \mathcal{F}_h, \]

where $\mathcal{F}_h : \mathcal{H}_{\infty}^f(A_h) \to \mathcal{H}_{\infty}^f(B_h)$ is the functor induced by $F_h$ on the homotopy categories of $\text{Mod}_{\infty}^f(A_h)$ and $\text{Mod}_{\infty}^f(B_h)$.

The functor $\mathcal{G}_h : \text{D}_2^\infty(B_h) \to \text{D}_1^\infty(A_h)$ is defined similarly. By definition

\[ \mathcal{F}_h(X_h) = \mathcal{F}_h(X_h) = F_h(X_h), \]

on every object $X_h \in \text{D}_2^\infty(B_h)$ and on any morphism $(s_h, f_h)$ in $\text{D}_2^\infty(A_h)$:

\[ \mathcal{F}_h(s_h, f_h) = (\mathcal{F}_h(s_h), \mathcal{F}_h(f_h)). \]

Both $(\mathcal{F}_h, \phi_1^h)$ and $(\mathcal{G}_h, \phi_2^h)$ are exact functors w.r.t. the triangulated structure on $\text{D}_1^\infty(A_h)$ and $\text{D}_2^\infty(B_h)$. Here $\phi_1^h : \mathcal{F}_h \circ [1] \to [1] \circ \mathcal{F}_h$ denotes the obvious morphism of functors, and similarly for $\phi_2^h$.

Proposition 19. Let $(\mathcal{F}_h, \mathcal{G}_h)$ be the pair of functors defined above.

- $A_h$ is isomorphic to $\mathcal{G}_h(\mathcal{F}_h(A_h))$ in $\text{D}_1^\infty(A_h)$.
- $K_h$ is isomorphic to $\mathcal{F}_h(\mathcal{G}_h(K_h))$ in $\text{D}_2^\infty(B_h)$.

Proof. The first isomorphism is represented by

\[ \xymatrix{ A_h \ar[r]_{\mathcal{G}_h(\mathcal{F}_h(A_h))} & \mathcal{G}_h(\mathcal{F}_h(A_h)) } \]
with
\[ \tilde{\psi}_A : A_h \to K_h \otimes_{D_k} K_h \to (A_h \otimes_{A_h} K_h) \otimes_{D_k} K_h = G_h(F_h(A_h)) \]
the second isomorphism is represented by
\[
\begin{array}{c}
\tilde{\psi}_A \\
\downarrow \\
F_h(G_h(K_h))
\end{array}
\quad \begin{array}{c}
\tilde{\psi}_K \\
\downarrow \\
K_h
\end{array}
\]
with
\[ \tilde{\psi}_K : K_h \to A_h \otimes_{A_h} K_h \to (K_h \otimes_{D_k} K_h) \otimes_{A_h} K_h = F_h(G_h(K_h)) \]
\(\tilde{\psi}_A\) and \(\tilde{\psi}_K\) are defined in cor. \ref{cor}

\section{Main result}

Denoting by \(\text{triang}^\infty_{A_h}(A_h)\) the triangulated subcategory of \(D^\infty_{ij}(A_h)\) generated by \(A_h[i,j]\) and by \(\text{triang}^\infty_{K_h}(K_h)\) the triangulated subcategory of \(D^\infty_{ij}(B_h)\) generated by \(K_h[i,j]\), \(i,j \in \mathbb{Z}\), we arrive at the main result of these notes.

\begin{theorem}
Let \(X\) be a finite dimensional vector space over \(\mathbb{K} = \mathbb{R}\), or \(C\) and \((A,K,B)\) be the triple of bigraded \(A\)-structures introduced in section 6. By \(\hbar \pi \in (T_{pol}(X))([h],0,[,\cdot])\) we denote an \(h\)-formal quadratic Poisson bivector on \(X\) such that \((A_h,K_h,B_h)\) is the Deformation Quantization on \((A,K,B)\) w.r.t. \(h\pi\). The functor
\[ F_h : D^\infty_{ij}(A_h) \to D^\infty_{ij}(B_h), \quad F_h(*) = \bullet \otimes_{A_h} K_h \]
induces equivalences of triangulated categories
\[ \text{triang}^\infty_{A_h}(A_h) \simeq \text{triang}^\infty_{K_h}(K_h), \quad \text{thick}^\infty_{A_h}(A_h) \simeq \text{thick}^\infty_{K_h}(K_h). \]
Let \((\tilde{K},d_{K})\) be the \(A\)-\(B\)-bimodule with \(\tilde{K} = K\) and \(d_{K}\) obtained from \(d_{K}\) exchanging \(A\) and \(B\) and \((\tilde{K},d_{\tilde{K}})\) be its quantization w.r.t. \(\pi_h\); the functor
\[ F_h^*: D^\infty_{ij}(B_h) \to D^\infty_{ij}(A_h), \quad F_h^*(*) = \bullet \otimes_{D_k} \tilde{K}_h \]
induces the equivalence of triangulated categories
\[ \text{triang}^\infty_{K_h}(\tilde{K}_h) \simeq \text{triang}^\infty_{B_h}(B_h), \quad \text{thick}^\infty_{K_h}(\tilde{K}_h) \simeq \text{thick}^\infty_{B_h}(B_h). \]
\end{theorem}

\section*{Appendix A. Proof of prop. \ref{prop}}

\begin{itemize}
\item On the quadratic relations \(d^2_{\text{End}_{n}(K)} = 0\).
\end{itemize}

First of all we noted that the maps \(D^0_{\text{End}_{n}(K)}(a_1|\ldots|a_n|\phi)\) and \(D^0_{\text{End}_{n}(K)}(\phi|a_1|\ldots|a_m)\) have cohomological degree 2; we have already remarked that \(\mathcal{L}(a_1|\ldots|a_n)\) has cohomological degree 1, instead. The relations \(d^0_{\text{End}_{n}(K)}(\partial_{\text{End}_{n}(K)}(\phi)) = 0\) are immediate to prove. We prove the case \(n \geq 2, m = 0\), i.e.
\[
\sum_{j=1}^{n} (-1)^{\sum_{i=1}^{j-1}(|a_i| + 1)} d^1_{\text{End}_{n}(K)}(a_1|\ldots|a_{j-1},\tilde{d}^2_{\text{End}_{n}(K)}(a_j|a_{j+1})|a_{j+2}|\ldots|a_n)|\phi) + \]
\[
\sum_{i=1}^{n-1} (-1)^{\sum_{i=1}^{n-1}(|a_i| + 1)} d^0_{\text{End}_{n}(K)}(a_1|\ldots|a_{n-i},\tilde{d}^0_{\text{End}_{n}(K)}(a_{n-i+1}|\ldots|a_n)|\phi) + \]
\[
\sum_{i=1}^{n-1} d^0_{\text{End}_{n}(K)}(\partial_{\text{End}_{n}(K)}(a_1|\ldots|a_n)|\phi)) = 0;
\]
such quadratic relations are equivalent to
\[
\sum_{j=1}^{n} (-1)^{\sum_{i=1}^{j-1}(|a_i| + 1)} \mathcal{L}(a_1|\ldots|a_{j-1},\tilde{d}^2_{\text{End}_{n}(K)}(a_j|a_{j+1})|a_{j+2}|\ldots|a_n) \circ \phi + \]
\[
\sum_{i=1}^{n-1} (-1)^{\sum_{i=1}^{n-i}(|a_i| + 1)} \mathcal{L}(a_1|\ldots|a_{n-i},\tilde{d}^0_{\text{End}_{n}(K)}(a_{n-i+1}|\ldots|a_n)) \circ \phi + \]
\[
(-1)^{\sum_{i=1}^{n-1}(|a_i| + 1)} \partial_{\text{End}_{n}(K)}(\mathcal{L}(a_1|\ldots|a_n) \circ \phi) + \mathcal{L}(a_1|\ldots|a_n) \circ \partial_{\text{End}_{n}(K)}(\phi) = 0.
\]
The last contributions on the l.h.s. of (36) can be written as

$$(-1)^{a_1-1}\partial_{\End(K)}\left(\mathcal{L}_A(a_1)\ldots|a_n|\circ \varphi + \mathcal{L}_A(a_1)\ldots|a_n|\circ \partial_{\End(K)}(\varphi)\right) =$$

$$(-1)^{a_1-1}\partial_{\End(K)}\left(\mathcal{L}_A(a_1)\ldots|a_n|\circ \varphi =$$

$$(\mathcal{L}_A(a_1)\ldots|a_n|\circ \partial_K + (-1)^{a_1-1}d_K\circ \mathcal{L}_A(a_1)\ldots|a_n|)\circ \varphi,$$

because $|\mathcal{L}_A(a_1)\ldots|a_n|) = \sum_{i=1}^n(a_i - 1) + 1$.

At the end, multiplying both sides of (36) by $(-1)^{\sum_{i=1}^n(a_i-1)}$ and using the associativity of the product $\circ$ we get that (36) is equivalent to a finite sum of equations of the type $d_K^n\left(\mathcal{L}_A(a_1)\ldots|a_n|\varphi(\ldots)\right) = 0$.

We continue with the case $n = 0, m \geq 2$, i.e.

$$\sum_{j=1}^m (-1)^{|a_j|+\sum_{i=1}^{j-1}(|a_i|-1)}d_{\End(K)}^0(a_1|a_j-1, \partial_{\End(K)}(a_1|a_{j+1})|a_{j+2})\ldots|a_m) +$$

$$\sum_{m' = 1}^{m-1} \sum_{j=1}^{m'-1} (-1)^{|a_j|+\sum_{i=1}^{j-1}(|a_i|-1)}d_{\End(K)}^0(a_1|a_m|\circ \varphi)|a_1|\ldots|a_{m-1})\circ \mathcal{L}_A(a_1|a_m) = 0,$$

which are easily verified, as

$$(-1)^{|a_j|+1}\partial_{\End(K)}(\varphi \circ \mathcal{L}_A(a_1)\ldots|a_m)) + (-1)^{|a_j|}\partial_{\End(K)}(\varphi \circ \mathcal{L}_A(a_1)\ldots|a_m)) =$$

$$\varphi \circ \left((-1)^{\sum_{i=1}^{j-1}(a_i-1)}\mathcal{L}_A(a_1)\ldots|a_m) \circ d_K - d_K \circ \mathcal{L}_A(a_1)\ldots|m)\right).$$

Note the overall $-1$ sign, which plays no role.

The equations expressing compatibility between the left and right actions on $\End(K)$ (for $n, m \geq 1$) are

$$(-1)^{\sum_{i=1}^{n}(a_i-1)}d_{\End(K)}^0(a_1|\ldots|a_{n+1})\circ \partial_{\End(K)}(\varphi|\ldots|a_{n+1}) +$$

$$\partial_{\End(K)}^0(a_1|\ldots|a_{n+1})\circ \partial_{\End(K)}(\varphi|\ldots|a_{n+1}) = 0;$$

they are equivalent to

$$\mathcal{L}_A(a_1)\ldots|a_n|\circ D_{\End(K)}^0(\varphi|\ldots|a_{n+1}) +$$

$$(-1)^{|a_j|+\sum_{i=1}^{j-1}(a_i-1)}D_{\End(K)}^0(\varphi|\ldots|a_{n+1})\circ \mathcal{L}_A(a_1)\ldots|a_{n+1}) = 0,$$

or

$$(1-)^{|a_j|+1}\mathcal{L}_A(a_1)\ldots|a_n|\circ (\varphi \circ \mathcal{L}_A(\ldots|a_{n+1}) +$$

$$(-1)^{|a_j|}(\mathcal{L}_A(a_1)\ldots|a_n)\circ \varphi \circ \mathcal{L}_A(\ldots|a_{n+1}) = 0.$$

We finish by checking the compatibility of the actions with the differential, i.e.

(37) $\left(-1)^{a_1-1}\partial_{\End(K)}(sa\partial_{\End(K)}(\varphi) + \partial_{\End(K)}(\partial_{\End(K)}(a_1)\varphi)\right) = 0,$

and

(38) $\partial_{\End(K)}^0(\varphi |a) + \partial_{\End(K)}^0(\partial_{\End(K)}(\varphi |a)) = 0;$

(37) is equivalent to

$$\mathcal{L}_A(sa)\circ \partial_{\End(K)}(\varphi) + (-1)^{|a_1-1}\partial_{\End(K)}(\mathcal{L}_A(sa) \circ \varphi) = 0;$$

(38) gives

$$\partial_{\End(K)}(\varphi \circ \mathcal{L}_A(sa)) - \partial_{\End(K)}(\varphi \circ \mathcal{L}_A(sa) = 0.$$
Both relations are satisfied by checking that

\[ \partial_{\text{End}_{\mathbb{A}}(K)}(\mathcal{L}_A(sa)) = 0. \]

- \( L_A \) descends to a morphism of \( A_{\infty} \)-\( A \)-bimodules.

We prove that (39)

\[ L_A \circ \partial_{n+1}(d) = d_{\text{End}_{\mathbb{A}}(K)} \circ L_A. \]

We check (39) on strings \((a_1|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l) \in A[1]^{\oplus k+l+1}\) and \((b_1|...|b_q) \in K[1] \otimes B[1]^{\oplus q}, \) for any \( k, l, q \geq 0. \)

If \( k, l \geq 1, \) the l.h.s. of (39) is

\[
\sum_{j=1}^{k} (-1)^{\sum_{i=1}^{j} \{0, \bar{a}, a\}} d_{K}^{k+l,q}(a_1|...|a_{j-1}, d_{A}(a_{j}a_{j+1}))a_{j+2}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_q) + \\
\sum_{j=1}^{k} (-1)^{\sum_{i=1}^{j} \{0, \bar{a}, a\}} d_{K}^{k+l,q}(a_1|...|a_{j-1}, d_{A}(a_{j}a_{j+1}))a_{j+2}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_q) + \\
\sum_{j=1}^{k} (-1)^{\sum_{i=1}^{j} \{0, \bar{a}, a\}} d_{K}^{k+l,q}(a_1|...|a_{j-1}, d_{A}(a_{j}a_{j+1}))a_{j+2}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_q) + \\
\sum_{j=1}^{k} (-1)^{\sum_{i=1}^{j} \{0, \bar{a}, a\}} d_{K}^{k+l,q}(a_1|...|a_{j-1}, d_{A}(a_{j}a_{j+1}))a_{j+2}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_q).
\]

As \( L_A(a_1|...|a_n) = s \circ \mathcal{L}_A(a_1|...|a_n) \) then the r.h.s. of (39) is given by the terms

\[
\sum_{j=1}^{q} (-1)^{\sum_{i=1}^{j} \{0, \bar{a}, a\}} d_{K}^{k+l,q+1}(a_1|...|a_{j-1}, d_{A}(a_{j}a_{j+1}))a_{j+2}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_{j-1}, d_{B}^{q+1}(b_{j+1})|b_1|...|b_q),
\]

the sum over \( k' \in \{1, \ldots, k\} \) of terms

\[
\sum_{q' + 1}^{q} (-1)^{\sum_{i=1}^{q'} \{0, \bar{a}, a\}} d_{K}^{k'-l+q,q'}(a_{k'+1}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_{q'+1}b_q),
\]

and the sum over \( l' \in \{0, \ldots, l - 1\} \) of terms

\[
\sum_{q' + 1}^{q} (-1)^{\sum_{i=1}^{q'} \{0, \bar{a}, a\}} d_{K}^{k'+l'+q,q'}(a_{k'+1}|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_{q'+1}b_q),
\]

i.e. those contributions in the r.h.s. of (39) corresponding to the right actions on elements in \( \text{End}_{\mathbb{B}}(K) \).

Moving the terms on the r.h.s. of (39) (note the overall \(-1\) sign) to the l.h.s., we get that (39) equivalent to

\[
\partial_{n+1}(d)_{K}(a_1|...|a_k|\bar{a}|\hat{a}_1|...|\hat{a}_l|b_1|...|b_q) = 0.
\]

The other cases, i.e. \( k = 0, l \geq 1, l \geq 1, l = 0 \) and \( k = l = 0 \) are trivial sign check. We are done.

**Appendix B. Proof of thm. \( \Box \)**

The proof of thm. 7 is shown in detail. We note that all the proof is based on checking the commutativity of diagrams in which objects belonging the classes \( \mathcal{S}_1, \mathcal{S}_2 \) appear (see below). Commutativity of the other diagrams follows from these two special cases. Moreover, we do not need to perform any explicit computation; we just need to apply the definition of the \( A_{\infty} \)-morphisms we introduced in section \( \Box \).
Proof. On objects in $S_1$, $S'_1$. We introduce $S_1 = \{ A(i)[n], i, n \in \mathbb{Z} \}$, and $S'_1 = \{ K(i)[n], i, n \in \mathbb{Z} \}$. By definition of the functors $(F, G)$ and by proposition 7 and 8 we get $G(F(X)) \simeq X$, $F(G(Y)) \simeq Y$, for every $X \in S_1$, $Y \in S'_1$, with $F(X) \in S'_1$ for every $X \in S_1$, and $G(Y) \in S_1$ for every $Y \in S'_1$. 

On morphisms of objects in $S_1$. 

We want to prove that 

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
G(F(X)) & \xrightarrow{G(F(f))} & G(F(Y))
\end{array}
\]

commutes, for every $X$ and $Y$ in $S_1$ and natural quasi-isomorphisms $\varphi_X, \varphi_Y$ in $D^\infty(A)$.

Let $f : A(k_1)[i] \to A(k_2)[j]$ be a morphism in $D^\infty(A)$, for any $k_1, k_2, i, j \in \mathbb{Z}$. As usual $f_n : (A(k_1)[i])[1] \otimes A[1]^{\otimes n} \to (A(k_2)[j])[1]$ denotes its $n$-th Taylor component, for $n \geq 0$. As $A[1]$ is concentrated in cohomological degree $-1$ and the morphism $f$ is of bidegree $(0, 0)$, then $f_n \neq 0$ if and only if $n = j - i$, i.e. there exists one and only one non trivial Taylor component, if $j - i \geq 0$. We denote by $f_n = s \circ f_n \circ s^{-1}$ its desuspension. If $n = 0$, then $f_0$ is a right $A$-linear map.

If $X = A(k_1)[i]$ and $Y = A(k_2)[j]$ we need to check the commutativity of the diagram 

\[
\begin{array}{cccc}
A(k_1)[i] & \xrightarrow{f_{j-i}} & A(k_2)[j] \\
\downarrow \varphi_1 & & \downarrow \varphi_2 \\
(A \otimes_A A)(k_1)[i] & \xrightarrow{f_{j-i}} & (A \otimes_A A)(k_2)[j] \\
\downarrow \varphi_3 & & \downarrow \varphi_4 \\
(A \otimes_A (K \otimes_B K))(k_1)[i] & \xrightarrow{G(F(f))} & (A \otimes_A (K \otimes_B K))(k_2)[j] \\
\end{array}
\]  

(40) 

Let us describe it in some detail. We give the definitions of the maps $V_1$, up to suspensions and desuspensions w.r.t. the cohomological and internal degree. The strict quasi-isomorphism $V_1$ is induced by the $A_1$-quasi-isomorphism $\Phi : A \to A \otimes_A A$, described in lem. 3. A similar formula holds true for $V_2$. The quasi-isomorphism $V_3$ is given by $V_3 = 1 \otimes L_A$ and the morphism $V_4$ is defined similarly. $V_5$ (and similarly $V_6$) is 

\[ V_5 = 1 \otimes T \]

where $T$ is the homotopy equivalence of $A_\infty$-$A$-bimodules $T : \text{End}_B(K) \to \text{End}_B(B_B(K)) \to K \otimes_B K$ given in prop. 7 and prop. 8.

Let us prove commutativity of (40). The morphism $f$ has a unique non trivial Taylor component $f_n$, for $n = j - i \geq 0$. For any 

\[ (a, a_1, \ldots, a_n) \in A \otimes A^{\otimes n} \]

we distinguish the following cases.

- $n' < n$. Going east and then south in (40) we get 0; going south-east, instead, we arrive at 

\[ G(F(f))(a, a_1, \ldots, a_n') \in (L_A(1)) = 0, \]

because $L_A$ is strictly unital (here we write $L_A(1) = L_A(1)$); all Taylor components $L_A^{m' + m'' + \ldots + 1} \in (L_A(1))$ such that with $1 \leq m' + m''$ identically vanish.

- $n' = n$. Going east and then south in (40) we arrive at 

\[
f_n(a, a_1, \ldots, a_n) \otimes T(L_A(1)),
\]

as 

\[
V_2(f_n(a, a_1, \ldots, a_n)) = f_n(a, a_1, \ldots, a_n) \in (A \otimes A)^0 \subset (A \otimes_A A)^0.
\]

\]
denoting by \( r \geq 0 \) the internal degree of the string \((a, a_1, \ldots, a_n)\). Going south-east in (40) we arrive at (41) as well. In fact
\[
V_1(a, a_1, \ldots, a_n) = (a, a_1, \ldots, a_n, 1) + \sum_{n'=0}^{n-1} (V_1^{n'}(a, a_1, \ldots, a_{n'}, \otimes a_{n'+1}, \ldots, a_n) \in \mathcal{A} \otimes \mathcal{A},
\]
and
\[
\mathcal{G}(\mathcal{F}(f))(V_2(V_1(a, a_1, \ldots, a_n, 1))) = \mathcal{G}(\mathcal{F}(f))(a, a_1, \ldots, a_n) \otimes T(L_A(1));
\]
this is because \( L_A \) is strictly unital and \( \mathcal{G}(\mathcal{F}(f)) \) strict. The diagram (40) commutes.

- \( n' > n \). Going east and then south in (40) we arrive at
\[
\mathcal{G}(\mathcal{F}(f))(a, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n'}, T(L_A(1))) = f_n(a, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n'}, T(L_A(1)),
\]
as \( \mathcal{G}(\mathcal{F}(f)) \) is strictly unital.

In summary, (40) commutes.

\section*{Induction: \( \mathcal{S}_r \)}
We denote by
\[
\mathcal{S}_r = \mathcal{S}_1 \circ \cdots \circ \mathcal{S}_1,
\]
the \( r \)-th extensions of \( \mathcal{S}_1 \), for every \( r \geq 1 \). We want to prove the commutativity of any diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
\mathcal{G}(\mathcal{F}(X)) & \xrightarrow{\mathcal{G}(\mathcal{F}(f))} & \mathcal{G}(\mathcal{F}(Y))
\end{array}
\]
(42)
with \( X, Y \in \text{triang}^\infty(A) \) and \( \varphi_X, \varphi_Y \) isomorphisms in \( \mathcal{D}^\infty(A) \). If \( X, Y \in \text{triang}^\infty(A) \), then by definition there exist \( r, r' \geq 1 \) such \( X \in \mathcal{S}_r \), \( Y \in \mathcal{S}_{r'} \) and exact triangles
\[
X_1 \to X \to X'_{r-1} \xrightarrow{f_1} X_1[1]
\]
and
\[
Y_1 \to Y \to Y'_{r'-1} \xrightarrow{g_1} Y_1[1]
\]
in \( \mathcal{D}^\infty(A) \) for some morphisms \( f \) and \( g \) and \( X_1, Y_1 \in \mathcal{S}_r \), \( X'_{r-1} \in \mathcal{S}_{r-1} \) and \( Y'_{r'-1} \in \mathcal{S}_{r'-1} \). They are isomorphic to the exact triangles \( X_1 \xrightarrow{f_1} X_1 \oplus X'_{r-1} \xrightarrow{F_{X_1 \oplus X'_{r-1}}} X'_{r-1} \xrightarrow{f_1} X_1[1] \) and \( Y_1 \xrightarrow{g_1} Y_1 \oplus Y'_{r'-1} \xrightarrow{F_{Y_1 \oplus Y'_{r'-1}}} Y'_{r'-1} \xrightarrow{g_1} Y_1[1] \) for some isomorphisms \( \rho_X : X \to X_1 \oplus X'_{r-1} \) in \( \text{triang}^\infty(A) \) and \( \rho_Y : Y \to Y_1 \oplus Y'_{r'-1} \) in \( \text{triang}^\infty(A) \). Commutativity of (42) is equivalent to the commutativity of
\[
\begin{array}{ccc}
X_1 \oplus X'_{r-1} & \xrightarrow{f_1} & Y_1 \oplus Y'_{r'-1} \\
\downarrow \varphi_{X_1 \oplus X'_{r-1}} & & \downarrow \varphi_{Y_1 \oplus Y'_{r'-1}} \\
\mathcal{G}(\mathcal{F}(X_1 \oplus X'_{r-1})) & \xrightarrow{\mathcal{G}(\mathcal{F}(f_1))} & \mathcal{G}(\mathcal{F}(Y_1 \oplus Y'_{r'-1}))
\end{array}
\]
(43)
where
\[
\tilde{f}_1 = \rho_Y \circ f_1 \circ \rho_X^{-1}, \quad \varphi_X = \mathcal{G}(\mathcal{F}(\rho_X))^{-1} \circ \varphi_{X_1 \oplus X'_{r-1}} \circ \rho_X
\]
and similarly for \( \varphi_Y \). Let us discuss (43) and the isomorphisms
\[
\varphi_{X_1 \oplus X'_{r-1}} : X_1 \oplus X'_{r-1} \to \mathcal{G}(\mathcal{F}(X_1 \oplus X'_{r-1})) \equiv \mathcal{G}(\mathcal{F}(X_1)) \oplus \mathcal{G}(\mathcal{F}(X'_{r-1})),
\]
\[
\varphi_{Y_1 \oplus Y'_{r'-1}} : Y_1 \oplus Y'_{r'-1} \to \mathcal{G}(\mathcal{F}(Y_1 \oplus Y'_{r'-1})) \equiv \mathcal{G}(\mathcal{F}(Y_1)) \oplus \mathcal{G}(\mathcal{F}(Y'_{r'-1}));
\]
we distinguish two cases.

- If \( r = 2 \), and \( r' = 2 \), then such isomorphisms are simply
\[
\varphi_{X_1 \oplus X'_{1}} := \varphi_{X_1} \oplus \varphi_{X'_{1}}, \quad \varphi_{Y_1 \oplus Y'_{1}} := \varphi_{Y_1} \oplus \varphi_{Y'_{1}};
\]
we have \( \varphi_{X_1 \oplus X'_{1}} \circ d_{X_1 \oplus X'_{1}} = \mathcal{G}(\mathcal{F}(X_1)) \circ \varphi_{X_1 \oplus X'_{1}} \), i.e.
\[
\varphi_{X_1 \oplus X'_{1}} \circ \begin{pmatrix} d_{X_1} & -f \\ 0 & d_{X'_{1}} \end{pmatrix} = \begin{pmatrix} \mathcal{G}(\mathcal{F}(X_1)) & -\mathcal{G}(\mathcal{F}(f)) \\ 0 & \mathcal{G}(\mathcal{F}(X'_{1})) \end{pmatrix} \circ \varphi_{X_1 \oplus X'_{1}}
\]
(44)
\[ \varphi_X : X \to \mathcal{G}(\mathcal{F}(X)) \]

is explicitly given (up to suspensions and desuspensions) by

\[ \varphi_X(x|a_1| \ldots |a_n) = \sum_{n'=0}^{n} ((x,a_1| \ldots |a_{n'})|T(L(1))|a_{n'+1}| | a_n), \]

for any \( X \in S_1 \). We are left to prove

\[ \varphi_{Y,1 \oplus Y'} \circ \tilde{f}_1 = \mathcal{G}(\mathcal{F}(\tilde{f}_1)) \circ \varphi_{X,1 \oplus X'}; \]

The strategy is clear: we use the same techniques introduced in the previous subsection, for the \( S_1 \) case. We just need to consider any string

\[ ((x_1 \oplus x_1')|a_1| \ldots |a_n) \in (X_1 \oplus X_1')|A[1]|^\oplus; \]

(46) is equivalent to

\[ \sum_{n'\geq 0} \varphi_{Y,1 \oplus Y'}(\tilde{f}_{1,n'}((x_1 \oplus x_1')|a_1| \ldots |a_{n'})|a_{n'+1}| |a_n)) = \]

\[ \sum_{n'\geq 0} \left[ \varphi_{Y,1 \oplus Y'}(\tilde{f}_{1,n'}((x_1 \oplus x_1')|a_1| \ldots |a_{n'})|a_{n'+1}| |a_n)) \right] = \]

\[ \mathcal{G}((\mathcal{F}(\tilde{f}_1))((x_1 \oplus x_1')|a_1| \ldots |a_n)) | \mathcal{G}(\mathcal{F}(\tilde{f}_1))((x_1 \oplus x_1')|a_1| \ldots |a_n)) = \]

(47) unchanged: we proved this last equality in (47). We check

\[ \exists \tilde{f}_{1,n'} \] such that \( \mathcal{G}(\tilde{f}_{1,n'}) = \mathcal{F}(\tilde{f}_1)|t_1 \oplus 1 \).

By definition (45), \( \varphi_{Y,1 \oplus Y'} \) leave the contributions \( \tilde{f}_{1,n'}((x_1 \oplus x_1')|a_1| \ldots |a_{n'}) \in Y_1 \) and \( \tilde{f}_{1,n'}((x_1 \oplus x_1')|a_1| \ldots |a_{n'}) \in Y_1' \) unchanged: then (46) follows and commutativity of (42) is proven.

- If \( r \geq 3 \) or \( r' \geq 3 \) in (43), one needs to further decompose

the objects \( X_{r-1}' \) and \( Y_{r'-1}' \) using the above techniques, i.e. introducing suitable exact triangles, arriving at the isomorphisms

(48) \( \rho_{X} : X \to X_1 \oplus X_1' \oplus \ldots X_{r-1}' \), \( \rho_{Y} : Y \to Y_1 \oplus Y_1' \oplus \ldots Y_{r'-1}' \);

with \( X_1, X_1', \ldots, X_{r-1}', Y_1, Y_1', \ldots, Y_{r'-1}' \in S_1 \). We have reduced our problem to a finite direct sum of the \( S_1 \) case.

There is no substantial difference with the \( r = 2 \), \( r' = 2 \) case, both conceptually and computationally. We conclude that \( \mathcal{G} \circ \mathcal{F} \simeq 1 \) on \( \text{triang}_{\infty}(A) \).

On morphisms of objects in \( S_1' \)

Let us consider \( S_1' \) and a strictly unital \( A_\infty \)-morphism \( g : K(i)[l] \to K(j)[r] \) with Taylor components \( \tilde{g}_n : K(i)[l+1] \otimes B[1]|^\oplus \to K(j)[r+1] \), for \( n \geq 0 \). Once again, as \( K(j)[r+1] \) is concentrated in bidegree \((r+1-1,-j)\) and \( g \) is of bidegree \((0,0)\), then \( \tilde{g}_n = 0 \) for \( n \neq r-l+j-0 \): there exists one and only one non trivial component \( \tilde{g}_n \).

We check the commutativity of any diagram

\[ X \quad \xrightarrow{g} \quad Y \]

(49) \[ \xymatrix{ X \ar[d]_{\psi_X} & Y \ar[d]_{\psi_Y} \\
\mathcal{G}(\mathcal{F}(X)) & \mathcal{G}(\mathcal{F}(Y)) \ar[l]_(\mathcal{F}(\mathcal{G}(g))} \]

with \( X \) and \( Y \) in \( S_1' \) and \( \psi_X, \psi_Y \) natural quasi-isomorphisms in \( \mathcal{D}^\infty(B) \). We need some preliminary results to prove this statement.

We recall that the \( A_\infty-B-A \)-bimodules \( K\otimes A K \) are such that \( A_\infty \)-quasi-isomorphisms of strictly unital \( A_\infty \)-bimodules \( \nu_A : A \to K\otimes A K \) and \( \nu_B : B \to K\otimes A K \) exist. Introducing the functor

(50) \[ \tilde{G} : \mathcal{D}^\infty(B) \to \mathcal{D}^\infty(A), M \mapsto \tilde{G}(M) := M\otimes_A K, \]

we prove that
Lemma 14.  \( \mathcal{G}(M) \simeq \mathcal{G}(M) \) in \( \mathcal{D}^\infty(A) \)\(^4\) for any \( M \in \mathcal{D}^\infty(B) \).

- The functor \( \mathcal{G} \) is exact w.r.t. the triangulated structures on \( \mathcal{D}^\infty(B) \) and \( \mathcal{D}^\infty(A) \);
- \( \mathcal{G}(M) \in \text{triang}_A^\infty(A) \), for any object \( M \) in \( \text{triang}_B^\infty(K) \).

Proof. Let us consider the following diagram:

\[
\begin{array}{ccc}
(M \otimes_B K) \otimes_A (K \otimes_B K) & \xrightarrow{\Phi_M \otimes \Phi_K} & M \otimes_B (B \otimes_B K) \\
1 \otimes \nu_A & & 1 \otimes \nu_B \otimes 1 \\
M \otimes_B K = \mathcal{G}(M) & & M \otimes_B K
\end{array}
\]

(51)

where

\[ \Phi_M \otimes \Phi_K : M \otimes_B K \to (M \otimes_B K) \otimes_A K \]

and similarly for \( \Phi_K \) are the maps described in lemma \( [3] \). All arrows are quasi-isomorphisms of strictly unital \( A_{\infty} \)-\( A \)-\( A \)-bimodules, i.e. homotopy equivalences. We denote by

\[ \varphi_M : \mathcal{G}(M) \to \mathcal{G}(M) \]

the above quasi-isomorphism in \( \mathcal{D}^\infty(A) \), obtained by inverting the quasi-isomorphisms

\[ \eta_{M,B} := 1 \otimes \nu_B \otimes 1, \]

and \( 1 \otimes \Phi_K \); their inverses exist as quasi-isomorphisms in \( \mathcal{D}^\infty(A) \) and \( \mathcal{D}^\infty(B) \) are homotopy equivalences. In other words,

\[ \varphi_M = (1 \otimes \Phi_K)^{-1} \circ \eta_{M,B}^{-1} \circ (1 \otimes \nu_A) \circ (\Phi_M \otimes \Phi_K) := (\varphi_M^2)^{-1} \circ \varphi_M^1. \]

I.e. \( \varphi_M^1 \) is the composition on the l.h.s. of (51), while \( \varphi_M^2 \) is the one on the r.h.s. We also introduce the notation

\[ T(M) = (M \otimes_B K) \otimes_A (K \otimes_B K) \cong M \otimes_B (K \otimes_B K) \otimes_B K. \]

By definition \( \varphi_M \) is not strict.

The first statement of the lemma follows by the very definition of \( \varphi_M \).

The second statement is proved easily using the same techniques showing that \( \mathcal{G} \) is an exact functor w.r.t. the triangulated structures on \( \mathcal{D}^\infty(B) \) and \( \mathcal{D}^\infty(A) \). One the third statement; choosing \( M = K \), then (51) implies \( \mathcal{G}(K) \simeq A \), or \( \mathcal{G}(X) \in \mathcal{S}_i = \{ A[i] \langle j \rangle, i, j \in \mathbb{Z} \} \), for any \( X \in \mathcal{S}_i' = \{ K[i] \langle j \rangle, i, j \in \mathbb{Z} \} \). By definition of \( \mathcal{S}_i \) and \( \mathcal{S}_i' \) we have \( \mathcal{G}(X) \in \mathcal{S}_i \), for any \( X \in \mathcal{S}_i' \).

\[ \square \]

Corollary 10. \( \mathcal{F} \circ \mathcal{G} \simeq 1 \) on \( \text{triang}_B^\infty(K) \).

Proof. Thanks to lemma \([14] \) if \( M \in \text{triang}_B^\infty(K) \), then \( \mathcal{G}(M) \simeq \mathcal{G}(M) \) and \( \mathcal{F}(\mathcal{G}(M)) \simeq \mathcal{F}(\mathcal{G}(M)) \), as \( \mathcal{F} \) sends quasi-isomorphisms to quasi-isomorphisms. It is easy to prove that \( \mathcal{F} \circ \mathcal{G} \simeq 1 \) on \( \text{triang}_B^\infty(K) \) by following the subsection “Induction: \( \mathcal{S}_r \)” above. Checking the commutativity of

\[ X \xrightarrow{\varphi_X} Y \]

\[ \psi_X \]

\[ \mathcal{F}(\mathcal{G}(X)) \xrightarrow{\mathcal{F}(\mathcal{G}(X))} \mathcal{F}(\mathcal{G}(Y)) \]

for any \( X, Y \in \mathcal{S}_r' \) is immediate; note that \( \psi_X \) and \( \psi_Y \) are explicit; in fact

\[ \psi_X : X \to \mathcal{F}(\mathcal{G}(X)), \ X \in \mathcal{S}_r' \]

is given by (up to suspensions and desuspensions)

\[ \psi_X(x|b_1| \ldots |b_n) = \sum_{n \geq 0} (x, b_1 | \ldots | b_n) \nu(R_B(1))|b_{n+1} \ldots |b_n), \]

\[ ^{\text{More precisely, the quasi-isomorphisms are all of strictly unital } A_{\infty} \text{-} A \text{-} A \text{-bimodules.}} \]
where $N: \text{End}_4(K) \to \text{End}_4K$ is s.t. $N(R_B(1)) = \varphi \otimes 1$, with $\varphi: (A\otimes_4 K)_0 \to K$, and $\varphi(1 \otimes 1) = 1$. This follows from the very definition of $R_B(1)$; in fact $R_B(1)(a_1)\ldots[a_1] = 0$ for $l \geq 1$ and $R_B(1)(1) = 1 \cdot 1 = 1$. The relations $\mathcal{F}(\mathcal{G}(g)) \circ \psi_X = \psi_Y \circ g$.

i.e.

$$(g \otimes 1) \circ \psi_X = \psi_Y \circ g$$

follow, using (54). Then one moves to diagrams in which $X, Y \in \text{triang}^\infty_B(K)$ and any morphism $g: X \to Y$ in $\text{D}^\infty(B)$ appear; the proof of commutativity is done as in subsection “Induction: $S_r$”. We decompose objects in $S'_r$, $r \geq 2$ into direct sums of objects in $S'_1$; as the case for $r = 1$ is explicit, thanks to (51) and (55), then we can repeat verbatim the considerations in “Induction: $S_r$”, ending the proof of the equivalence $\mathcal{F} \circ \mathcal{G} \simeq 1$ on $\text{triang}^\infty_B(K)$.

We are left to prove

$$\mathcal{F} \circ \mathcal{G} \simeq 1$$

on $\text{triang}^\infty_B(K)$ by induction on $S'_r$, $r \geq 1$; we begin with the case $r = 1$. Let $X$ and $Y$ be in $S'_1$; any diagram

$$\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow \rho_X & & \downarrow \rho_Y \\
\mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y))
\end{array}$$

can be decomposed into the subdiagrams

$$\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow \psi_X & & \downarrow \psi_Y \\
\mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y))
\end{array} \quad \begin{array}{ccc}
\mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y)) \\
\downarrow \mathcal{F}(\psi_X) & & \downarrow \mathcal{F}(\psi_Y) \\
\mathcal{F}(\mathcal{G}(X)) & \xrightarrow{\mathcal{F}(\mathcal{G}(g))} & \mathcal{F}(\mathcal{G}(Y))
\end{array}$$

where $\psi_X, \psi_Y$ are given by (51), $\varphi_X$ and $\varphi_Y$ by (52) and $\rho_X = \mathcal{F}(\varphi_X) \circ \psi_X, \rho_Y = \mathcal{F}(\varphi_Y) \circ \psi_Y$ are quasi-isomorphisms.

We have already proved that the upper subdiagram in (56) commutes; the lower one commutes if we prove the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{G}(X) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(Y) \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
\mathcal{G}(X) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(Y)
\end{array}$$

as $\mathcal{F}$ is a functor. Once again, using the definition (52) of $\varphi_X$ and $\varphi_Y$ we decompose (57) into

$$\begin{array}{ccc}
\mathcal{G}(X) & \xrightarrow{\mathcal{G}(g)} & \mathcal{G}(Y) \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
\bar{\mathcal{G}}(X) & \xrightarrow{\bar{\mathcal{G}}(g)} & \bar{\mathcal{G}}(Y)
\end{array} \quad \begin{array}{ccc}
\bar{\mathcal{G}}(X) & \xrightarrow{\bar{\mathcal{G}}(g)} & \bar{\mathcal{G}}(Y) \\
\downarrow (1 \otimes \nu_A) \circ (\Phi_B) & & \downarrow (1 \otimes \nu_A) \circ (\Phi_B) \\
T(X) & \xrightarrow{T(g)} & T(Y)
\end{array}$$

where the morphisms appear in the definition (52) and $T(X)$ (similarly for $T(Y)$) is defined in (53).

The map $T(g): T(X) \to T(Y)$ is simply $T(g) = g \otimes 1$.

But

$$(1 \otimes \nu_A) \circ (\Phi_B) \circ \mathcal{F}(g) = T(g) \circ (1 \otimes \nu_A) \circ (\Phi_B),$$

as one can easily check just using the definitions of the morphisms; in fact the identity has to be verified on any string, say

$$(x, b_1|\ldots|b_m)\varphi(a_1|\ldots|a_n) \in (X\otimes_B K)[1] \otimes T(A[1]);$$
The l.h.s. of (58) is equal to (up to signs)
\[ g((x, b_1 | ... | b_q), b_{q'+1} | ... | b_{q'+\nu}, a_1 | ... | a_n, \nu A(1), \]
as the morphism \( g : X \rightarrow Y \) (\( X \) and \( Y \) are in \( \mathcal{S}_r \)) has only one non trivial Taylor component \( g_{q', \bar{q}} \geq 0, \) due to the bigrading on \( X \) and \( Y \); we already used this fact in the proof of thm. 7. The r.h.s. of (58) gives the same result, by definition of \( T(g) \), which is clearly a strict \( A_\infty \)-morphism.

On the other hand
\[ G(g) \circ (1 \otimes \Phi_K)^{-1} \circ \eta_{X,B}^{-1} = (1 \otimes \Phi_K)^{-1} \circ \eta_{Y,B}^{-1} \circ T(g) \]
holds true if and only if
\[ (59) \]
is easily verified, as we did for (58); so (59) commutes. Let us verify (59) explicitly, on any string
\[ (x, b_1 | ... | b_q, a_1 | ... | a_n) \in (X \otimes_{\mathcal{B}} K)[1] \otimes T(A)[1], n \geq 0. \]
If \( n \geq 1 \), as all morphisms in (58) are strict, then (59) is trivially verified. Note that, by definition, \( \Phi_K \) “sees” the left \( \mathcal{B} \)-module structure on \( K \), i.e. \( \Phi_K : \mathcal{K} \rightarrow \mathcal{B} \otimes_{\mathcal{A}} K \). If \( n = 0 \), recalling that \( g \) has only one non trivial Taylor component, say \( g_q, \bar{q} \geq 0, \) due to the bigradings on \( X \) and \( Y \), we arrive at
\[ \sum_{q' = q+1}^g g((x, b_1 | ... | b_q), b_{q'+1} | ... | b_{q'+\nu}, a_1 | ... | a_n), \]
for the l.h.s. of (59) (up to suspensions and desuspensions). We recall that \( \nu_{\mathcal{B}} : \mathcal{B} \rightarrow K \otimes_{\mathcal{A}} K \) is strictly unital, so \( \nu_{\mathcal{B}}(b_1 | ... | b_q, a_1 | ... | a_n) = 0 \) if \( q' \geq 1 \). The r.h.s. of (59) gives the same result, as \( T(g) = g \otimes 1 \).

The first step of the induction is proven. If \( X \in \mathcal{S}_r \) and \( Y \in \mathcal{S}_r \), then we prove the commutativity of
\[ \begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow \psi_X & & \downarrow \psi_Y \\
\mathcal{F}(G(X)) & \xrightarrow{\mathcal{F}(G(g))} & \mathcal{F}(G(Y)) \\
\downarrow \mathcal{F}(\psi_X) & & \downarrow \mathcal{F}(\psi_Y) \\
\mathcal{F}(G(X)) & \xrightarrow{\mathcal{F}(G(g))} & \mathcal{F}(G(Y)) \\
\end{array} \]
introducing the isomorphisms
\[ \rho_X : X \rightarrow \bigoplus_{i=1}^r X_i, \rho_Y : Y \rightarrow \bigoplus_{j=1}^{r'} Y_j \]
in \( D^{\infty}(\mathcal{B}) \), for some \( X_1, \ldots, X_r, Y_1, \ldots, Y_{r'} \in \mathcal{S}_1 \). The considerations that lead us to prove (44) hold here, with due changes; we are just considering finite direct sums of \( A_\infty \)-modules and homotopy equivalences.

Exchanging \( A \) and \( B \); i.e. using the \( A_\infty \)-\( B \)-\( A \)-bimodule \( (\mathcal{K}, d_\mathcal{K}) \) and the new functors \( \mathcal{F}' = \cdot \otimes_{\mathcal{B}} \mathcal{K} \) and \( \mathcal{G}' = \cdot \otimes_{\mathcal{A}} \mathcal{K} \), we can prove the equivalence of the triangulated categories \( \text{triang}_{\mathcal{B}}^{\infty}(\mathcal{B}) \) and \( \text{triang}_{\mathcal{A}}^{\infty}(\mathcal{K}) \) with the same techniques introduced above.

B.0.41. On thick subcategories. The statement on the thick subcategories follows by additivity of \( \mathcal{F} \) and \( \mathcal{G} \) (\( \mathcal{F}' \) and \( \mathcal{G}' \) as well), w.r.t. the coproduct in \( D^{\infty}(A) \) and \( D^{\infty}(\mathcal{B}) \), i.e. the direct sum of strictly unital \( A_\infty \)-modules.

More precisely, let \( X \in \text{thick}_{\mathcal{B}}^{\infty}(A) \); there exists a \( Z \in \text{triang}_{\mathcal{B}}^{\infty}(K) \) s.t.
\[ Z \simeq X \oplus Y, \]
for some \( Y \in D^{\infty}(A) \). Let us call such isomorphism \( \varphi_X \), i.e. \( \varphi_X : Z \rightarrow X \oplus Y \). It follows that \( (X) \in \text{thick}_{\mathcal{B}}^{\infty}(K) \), as \( \mathcal{F} \) is additive and preserves quasi-isomorphisms. For any morphism \( f : X_1 \rightarrow X_2 \), with \( X_1, X_2 \in \text{thick}_{\mathcal{B}}^{\infty}(A) \), we want to prove that the diagram
\[ \begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
\mathcal{G}(\mathcal{F}(X_1)) & \xrightarrow{\eta_1} & \mathcal{G}(\mathcal{F}(X_2)) \\
\end{array} \]
commutes, for some isomorphisms $\psi_i : X_i \to G(F(X_i))$. All we need is to check the commutativity of the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow{i_1} & & \downarrow{i_2} \\
Z_1 & \xrightarrow{\psi_1} & X_1 \oplus Y_1 \\
\downarrow{\psi_{Z_1}} & & \downarrow{\psi_{Z_2}} \\
& & X_2 \oplus Y_2 \\
& & \xrightarrow{\psi_{Z_2}^{-1}} \\
& & Z_2 \\
\end{array}
$$

with

$$\rho_1 := G(F(\varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1}$$

and similarly for $\rho_2$. The isomorphisms $\psi_{Z_i}$ do exist as $Z_i \in \operatorname{triang}_{\infty}^P(A)$. The maps $\pi_2 : X_2 \oplus Y_2 \to X_j$ and $i_j : X_j \to X_j \oplus Y_j$ are morphisms in $D^\infty(A)$. We want to prove that the central square in (61) commutes, i.e.,

$$\rho_2 \circ (i_2 \circ f \circ \pi_1) = G(F(i_2 \circ f \circ \pi_1)) \circ \rho_1;$$

clearly

$$\begin{array}{ccc}
Z_1 & \xrightarrow{\psi_{Z_1}^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \pi_2} & Z_2 \\
\downarrow{\psi_{Z_1}} & & \downarrow{\psi_{Z_2}} \\
& & G(F(Z_2)) \\
\end{array}
$$

commutes, as $Z_i \in \operatorname{triang}_{\infty}^P(A)$; in other words

$$\begin{aligned}
\psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1 &= G(F(\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \Leftrightarrow \\
\psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) &= G(F(\varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1} \Leftrightarrow \\
G(F(\varphi_2)) \circ \psi_{Z_2} \circ \varphi_2^{-1} \circ (i_2 \circ f \circ \pi_1) &= G(F((i_2 \circ f \circ \pi_1) \circ \varphi_1)) \circ \psi_{Z_1} \circ \varphi_1^{-1}.
\end{aligned}$$

i.e. (62). The upper central and the lower central squares in (61) clearly commute; the morphisms

$$\psi_j : X_j \to G(F(X_j)), \quad \psi_j = G(F(\pi_j)) \circ \rho_j \circ i_j$$

are actually isomorphisms with inverses given by

$$\psi_j^{-1} : G(F(X_j)) \to X_j, \quad \psi_j^{-1} = \pi_j \circ \rho_j^{-1} \circ G(F(i_j));$$

this last statement is proved by writing explicitly $\rho_j$ and recalling the decompositions (53), for any object in $\operatorname{triang}_{\infty}^P(A)$. As $G(F(\cdot))$ is of the form $G(F(g)) = g \oplus 1$, for any morphism $g$ in $\operatorname{triang}_{\infty}^P(A)$, the claim follows. The thick subcategory $\operatorname{thick}_{\infty}^P(K)$ is studied analogously.

\appendix
\section{Proof of thm. \ref{thm:main}.}

We show the proof of thm. \ref{thm:main} in some detail. Such proof is analogous to the one of thm. \ref{thm:main} modulo technical issue due to the presence of “roofs”. Once again, all we need is to prove the commutativity of “easier” diagrams in which objects of the form $A_k[i][j]$ and $K_n[n][m]$ appear.

\begin{proof}
We study the exact functors $F_h$ and $G_h$ on $D^\infty_f(A_k)$ and $D^\infty_f(B_h)$ to prove that they induce an equivalence of triangulated categories between $\operatorname{triang}_{\infty}^P(A_k)$ and $\operatorname{triang}_{\infty}^P(B_h)$.

On objects in $S_1, S_1'$.

We introduce $S_1 = \{A_k[i][n], i, n \in \mathbb{Z}\}$, and $S_1' = \{K_n[n], i, n \in \mathbb{Z}\}$. By definition, objects of $S_1$ are (all isomorphism classes of the) objects $A_k[i][n]$ in $D^\infty_f(A_k)$ and similarly for $S_1'$. By definition of the functors $(F_h, G_h)$ and proposition \ref{prop:functors} we have

$$G_h(F_h(A_k[i][n])) \simeq A_k(i)[n]$$

\end{proof}
in $D_{\ell}^\infty(A_h)$ and
\[ F_h(G_h(K_h(j)[m]) \cong K_h(j)[m] \]
in $D_{\ell}^\infty(B_h)$, with $F_h(X) \in S_0$ and $G_h(Y) \in S_1$, for every $X \in S_1$, $Y \in S'_1$.

C.0.42. On commutative diagrams in the derived categories $D_{\ell}^\infty(A_h)$ and $D_{\ell}^\infty(B_h)$. To prove the theorem, we need to consider the following general setting. Let $X_h$, $Y_h$ be objects in $\text{triang}_{D_{\ell}^\infty}(A_h)$ and let $f_h : X_h \to Y_h$ be a morphism in $D_{\ell}^\infty(A_h)$. We want to prove that there exist isomorphisms
\[ \varphi_{X_h} : X_h \to G_h(F_h(X_h)), \varphi_{Y_h} : Y_h \to G_h(F_h(Y_h)) \]
in $D_{\ell}^\infty(A_h)$, such that
\[ \begin{array}{c}
X_h \\
\downarrow \varphi_{X_h} \\
G_h(F_h(X_h))
\end{array} \quad \begin{array}{c}
f_h \\
\downarrow \varphi_{Y_h} \\
G_h(F_h(Y_h))
\end{array} \]
commutes in $D_{\ell}^\infty(A_h)$.

Let us represent the morphism $f_h$ by the roof $(s_h, \bar{f}_h)$, i.e.
\[ \begin{array}{c}
X_h \\
\downarrow \varphi_{X_h} \\
G_h(F_h(X_h))
\end{array} \quad \begin{array}{c}
\bar{f}_h \\
\downarrow \varphi_{Y_h} \\
G_h(F_h(Y_h))
\end{array} \]
for some $X'_h$ in $D_{\ell}^\infty(A_h)$; by definition of $\text{triang}_{D_{\ell}^\infty}(A_h)$ (property (SO)) we can infer that $X'_h$ is an object in $\text{triang}_{D_{\ell}^\infty}(A_h)$: in fact $s_h$ is an isomorphism in $D_{\ell}^\infty(A_h)$. Then the morphism $G_h(F_h(X_h)) \to G_h(F_h(Y_h))$ is represented by the roof $(G_h(F_h(s_h)), G_h(F_h(\bar{f}_h)))$, i.e.
\[ \begin{array}{c}
G_h(F_h(X_h)) \\
\downarrow G_h(F_h(s_h)) \\
G_h(F_h(Y_h))
\end{array} \quad \begin{array}{c}
\downarrow G_h(F_h(\bar{f}_h)) \\
G_h(F_h(Y_h))
\end{array} \]

We are interested in proving also the commutativity of diagrams
\[ \begin{array}{c}
W_h \\
\downarrow \varphi_{W_h} \\
F_h(G_h(W_h))
\end{array} \quad \begin{array}{c}
\bar{f}_h \\
\downarrow \varphi_{Z_h} \\
F_h(G_h(Z_h))
\end{array} \]
in $D_{\ell}^\infty(B_h)$, with $W_h$ and $Z_h$ in $\text{triang}_{D_{\ell}^\infty}(K_h)$, $\varphi_{W_h}$, $\varphi_{Z_h}$ isomorphisms in $D_{\ell}^\infty(B_h)$ and morphisms $g_h$ represented by some roof, like the morphisms $f_h$ introduced above.

We need the following lemma, which reduces the problem of commutativity in the derived categories to the check of certain relations involving morphisms in the corresponding homotopy categories. We state the lemma in the case of diagrams of the form (63); the other case is analogous.

**Lemma 15.** Let $X_h$, $Y_h$ be objects in $\text{triang}_{D_{\ell}^\infty}(A_h)$ and let us consider a diagram of the form (63), with $f_h$, $\varphi_{X_h}$, $\varphi_{Y_h}$ and $G_h(F_h(\bar{f}_h))$ as above. If there exists a quasi-isomorphism
\[ \bar{\varphi}_{X'_h} : X'_h \to G_h(F_h(X'_h)) \]
in $\mathcal{H}_{D_{\ell}^\infty}(A_h)$ s.t. for any morphism $g_h : X'_h \to Y_h$ in $\mathcal{H}_{D_{\ell}^\infty}(A_h)$ the relation
\[ \bar{\varphi}_{Y_h} \circ g_h = G_h(F_h(g_h)) \circ \bar{\varphi}_{X'_h} \]
holds true, then, representing the isomorphism
\[ \varphi_{X'_h} : X'_h \to G_h(F_h(X'_h)), \]

in $\mathbf{D}^\mathcal{G}_\mathcal{Y}(A_h)$ by the roof

\[
\begin{array}{c}
X'_h \\
\downarrow t_h \quad \downarrow \varphi_{X'_h}
\end{array}
\rightarrow
\begin{array}{c}
\varphi_{X'_h} \\
\uparrow \varphi_{X'_h}
\end{array}
\rightarrow
\begin{array}{c}
G_h(F_h(X'_h)) \\
G_h(F_h(X'_h))
\end{array}
\]

the diagrams of the form \((63)\) commute.

\textbf{Proof.} Commutativity of \((63)\) is equivalent to

\[
(1, \varphi_{Y_h}) \circ (s_h, f_h) = (G_h(F_h(s_h)), G_h(F_h(f_h))) \circ (1, \varphi_{X'_h})
\]

in $\mathbf{D}^\mathcal{G}_\mathcal{Y}(A_h)$; the l.h.s. reads

\[
(1, \varphi_{Y_h}) \circ (s_h, f_h) = (s_h \alpha_h, \varphi_{Y_h} \beta_h),
\]

with $\beta = f_h \alpha_h$, for some roof

\[
\begin{array}{c}
Z_h \\
\downarrow \alpha_h \quad \downarrow \beta_h
\end{array}
\rightarrow
\begin{array}{c}
X'_h \\
\downarrow t_h
\end{array}
\rightarrow
\begin{array}{c}
Y_h \\
\uparrow \varphi_{X'_h}
\end{array}
\]

Then

\[
(1, \varphi_{Y_h}) \circ (s_h, f_h) = (s_h \alpha_h, \varphi_{Y_h} \beta_h) = (s_h, \varphi_{Y_h} f_h) = (s_h, G_h(F_h(f_h)) \varphi_{X'_h})
\]

where in the last equality we used \((65)\) and the second equality holds true as $\alpha_h$ is a quasi-isomorphism\(^{\text{6}}\), but

\[
(s_h, G_h(F_h(f_h)) \varphi_{X'_h}),
\]

i.e. the roof

\[
\begin{array}{c}
X'_h \\
\downarrow t_h \quad \downarrow \varphi_{X'_h}
\end{array}
\rightarrow
\begin{array}{c}
G_h(F_h(X'_h)) \\
G_h(F_h(X'_h))
\end{array}
\]

is equal to

\[
\begin{array}{c}
X'_h \\
\downarrow t_h \quad \downarrow \varphi_{X'_h}
\end{array}
\rightarrow
\begin{array}{c}
G_h(F_h(X'_h)) \\
G_h(F_h(X'_h))
\end{array}
\]

i.e. the composition

\[
(G_h(F_h(s_h)), G_h(F_h(f_h))) \circ (1, \varphi_{X'_h}),
\]

which is the r.h.s. of \((66)\), if and only if

\[
\varphi_{X_h} \circ s_h = G_h(F_h(s_h)) \circ \varphi_{X'_h}.
\]

This latter is nothing but \((65)\) applied to the quasi-isomorphism $s_h$. \hfill \Box

In virtue of the above lemma, we prove that diagrams of the form \((63)\) commute, for any $X_h, Y_h$ in $\text{triang}^\mathcal{Y}_A(A_h)$, by choosing a representative for the morphisms and checking the relations \((66)\).

---

\(^{\text{6}}\)We recall that equality "$=\$" between roofs is, by definition, the equivalence relation between them.
C.0.43. On $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\text{triang}^\infty_{A^\times_h}(A_h)$. We begin by proving $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\text{triang}^\infty_{A^\times_h}(A_h)$ on any pair

$X_h := A_h(i')[n'], \ Y_h := A_h(j)[m]$ of objects in $\mathcal{S}_1$ and any morphism $f_h : A_h(i')[n'] \rightarrow A_h(j)[m]$ in $\text{D}^\infty_{\mathcal{F}}(A_h)$ represented by the roof

$$A_h(i)[n] \xrightarrow{f_h} A_h(j)[m]$$

(67)

where $i, i', n, n', j, m \in \mathbb{Z}$.

Let $f_h^{(l)} : (A(i)[n])[1] \otimes A[1] \otimes r \rightarrow (A(j)[m])[1]$ be the $r$-th Taylor component of $f_h^{(l)}$ for any $l, r \geq 0$. A quick degree analysis (we recall that $A_h$ is concentrated in cohomological degree 0) implies that $f_h^{(l),r} \neq 0$ if and only if $r = m - n$, i.e. there exists one and only one non trivial Taylor component of $f_h^{(l)}$, if $m - n \geq 0$, for any $l \geq 0$. To prove the commutativity of the diagram

$$A_h(i)[n] \xrightarrow{\varphi_{A_h(i)[n]}} A_h(j)[m]$$

(68)

is sufficient, thanks to lemma 19 to prove

$$\varphi_{A_h(j)[m]} \circ f_h = \mathcal{G}_h(\mathcal{F}_h(f_h)) \circ \varphi_{A_h(i)[n]}$$

representing the isomorphisms

$$\varphi_{A_h(k)[l]} : A_h(k)[l] \rightarrow \mathcal{G}_h(\mathcal{F}_h(A_h(k)[l]))$$

(69)

for any $k, l \in \mathbb{Z}$. The quasi-isomorphisms $\varphi_{A_h(j)[m]}$ and $\varphi_{A_h(i)[n]}$ can be deduced by the diagram 10, with due changes. Up to suspensions and desuspensions w.r.t. both the cohomological and internal degree, we have

$$\varphi_{A_h(i)[n]} = (1 \otimes \mathcal{T}_h) \circ (1 \otimes \mathcal{L}_{A_h}) \circ \Phi_{A_h},$$

where $\Phi_{A_h} : A_h \rightarrow A_h \otimes A_h$ and $\mathcal{T}_h : \text{End}_{\mathcal{O}_h}(K_h) \rightarrow K_h \otimes_{\mathcal{O}_h} K_h$ is described in Corollary 5. To check 68 is immediate, once we recall that $\mathcal{G}_h(\mathcal{F}_h(f_h)) = f_h^{(l)} \otimes 1$.

We finish the proof of the equivalence $\mathcal{G}_h \circ \mathcal{F}_h \simeq 1$ on $\text{triang}^\infty_{A^\times_h}(A_h)$ considering the general case. Denoting by

$$S_r = \mathcal{S}_1 \cdots \mathcal{S}_1,$$

the $r$-th extension of $\mathcal{S}_1$ for every $r \geq 1$, we note that $\mathcal{F}_h(X_h) \in \mathcal{S}_r$ for every $X_h \in \mathcal{S}_r$ and $\mathcal{G}_h(Y_h) \in \mathcal{S}_r$ for every $Y_h \in \mathcal{S}_r$, and $r, r' \geq 1$.

Let $X_h$ and $Y_h$ be objects in $\text{triang}^\infty_{A^\times_h}(A_h)$ and $f_h : X_h \rightarrow Y_h$ be a morphism in $\text{D}^\infty_{\mathcal{F}}(A_h)$ represented by the roof

$$X_h \xrightarrow{f_h} Y_h$$

(71)

It follows that $X_h'$ is an object in $\text{triang}^\infty_{A^\times_h}(A_h)$ as well. we show that the diagram

$$X_h \xrightarrow{f_h} Y_h$$

(72)

$$\varphi_{X_h} \downarrow \quad \varphi_{Y_h} \downarrow$$

$$\mathcal{G}_h(\mathcal{F}_h(X_h)) \xrightarrow{\mathcal{G}_h(\mathcal{F}_h(f_h))} \mathcal{G}_h(\mathcal{F}_h(Y_h))$$

is commutative.
commutes in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$. To do so, we introduce the roofs

\[
\begin{array}{ccc}
X_h & \xrightarrow{\phi} & Y_h \\
\downarrow{\varphi_{X_h}} & & \downarrow{\varphi_{Y_h}} \\
X_{\mathcal{J}} & & Y_{\mathcal{J}} \\
\end{array}
\]

representing $\varphi_{X_h}$, $\varphi_{Y_h}$ and $\varphi_{\mathcal{J}}(\mathcal{J}(f_h))$, for some $\varphi_{X_h}$ and $\varphi_{Y_h}$ still to define (see below) On the other hand, $\varphi_{\mathcal{J}}(\mathcal{J}(f_h)) = f_h \circ 1$. By definition of the triangulated subcategory $\text{triang}_{\mathcal{J}}(A_h)$, there exist $r', r \geq 0$ s.t. $X_h^r \in \mathcal{S}_{r'}$, and $Y_h \in \mathcal{S}_r$, i.e. there exist exact triangles

\[
(72) \quad X_h^r \rightarrow X_h^r \rightarrow X_h^{r-1} \xrightarrow{\gamma_h} X_h^1[1]. \quad Y_h^r \rightarrow Y_h^r \rightarrow Y_h^{r-1} \xrightarrow{\beta_h} Y_h^1[1].
\]

in $\mathcal{D}_{\mathcal{J}}^\infty(A)$ for some morphisms $\gamma_h$ and $\beta_h$, with $X_h^r, Y_h^r \in \mathcal{S}_1$ and $X_h^{r-1}, Y_h^{r-1} \in \mathcal{S}_{r-1}$. Let us focus on the first exact triangle in (72); for the second one the analysis is analogous. By definition of exact triangles in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$ (and $\text{triang}_{\mathcal{J}}^\infty(A_h)$), such exact triangle is isomorphic in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$ to a sequence of the form

\[
(73) \quad W_h \xrightarrow{\alpha_h} Z_h \xrightarrow{\beta_h} R_h \xrightarrow{\gamma_h} W_h[1],
\]

where (73) is the image under the canonical functor $\mathcal{Q}_{A_h} : H_{\mathcal{J}}^1(A_h) \rightarrow \mathcal{D}_{\mathcal{J}}^\infty(A_h)$ of the exact triangle $W_h \xrightarrow{\alpha_h} Z_h \xrightarrow{\beta_h} R_h \xrightarrow{\gamma_h} W_h[1]$. In other words, the morphism $\alpha_h$ is represented by the roof

\[
\begin{array}{ccc}
W_h & \xrightarrow{\alpha_h} & Z_h \\
\downarrow{\gamma_h} & & \downarrow{\beta_h} \\
R_h & & W_h
\end{array}
\]

and similarly for $\beta_h$, $\gamma_h$.

But (73) is isomorphic in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$ to the exact triangle

\[
(74) \quad W_h \xrightarrow{i_h} W_h \oplus R_h \xrightarrow{p_h} R_h \xrightarrow{\gamma_h} W_h[1], \quad d_{W_h \oplus R_h} = \begin{pmatrix} d_{W_h} & -\gamma_h \\ 0 & d_{R_h} \end{pmatrix},
\]

where $i_h$ and $p_h$ are represented by

\[
\begin{array}{ccc}
W_h & \xrightarrow{i_h} & W_h \oplus R_h \\
\downarrow{\varphi_{W_h}} & & \downarrow{\varphi_{W_h \oplus R_h}} \\
W_h & & W_h \oplus R_h \\
\end{array}
\]

with canonical topological inclusion $\varphi_{W_h}$ and topological projection $\varphi_{W_h \oplus R_h}$. In summary, collecting the isomorphisms of the exact triangles so far, we arrive at the isomorphisms $X_h^r \simeq W_h$ and $X_h^{r-1} \simeq R_h$ in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$, implying that $W_h \in \mathcal{S}_1$ and $R_h \in \mathcal{S}_{r-1}$; the isomorphism

\[
\rho_{X_h} : (X_h^r, d_{X_h^r}) \rightarrow (W_h \oplus R_h, d_{W_h \oplus R_h}),
\]

in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$ follows, as well. Repeating the same analysis for the exact triangle in which $Y_h$ appears-see (72)-we get the isomorphism

\[
\rho_{Y_h} : (Y_h, d_{Y_h}) \rightarrow (M_h \oplus N_h, d_{M_h \oplus N_h}),
\]

in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$ for some $M_h \in \mathcal{S}_1$ and $N_h \in \mathcal{S}_{r-1}$.

In virtue of the above isomorphisms in $\mathcal{D}_{\mathcal{J}}^\infty(A_h)$, (71) commutes if and only if

\[
\begin{array}{ccc}
W_h \oplus R_h & \xrightarrow{f_h} & M_h \oplus N_h \\
\downarrow{\varphi_{W_h \oplus R_h}} & & \downarrow{\varphi_{M_h \oplus N_h}} \\
\mathcal{G}_h(\mathcal{J}(W_h \oplus R_h)) & \xrightarrow{\mathcal{G}_h(\mathcal{J}(f_h))} & \mathcal{G}_h(\mathcal{J}(M_h \oplus N_h))
\end{array}
\]

does, where

\[
\begin{align*}
\tilde{f}_h &= \rho_{Y_h} \circ f_h \circ \rho_{X_h}^{-1}, \\
\varphi_{W_h \oplus R_h} &= \rho_{\mathcal{G}_h(\mathcal{J}(X_h))} \circ \varphi_{X_h} \circ \rho_{X_h}^{-1}, \\
\varphi_{M_h \oplus N_h} &= \rho_{\mathcal{G}_h(\mathcal{J}(Y_h))} \circ \varphi_{Y_h} \circ \rho_{Y_h}^{-1}.
\end{align*}
\]
In the sequel we will explicitly define \( \varphi_{W_h \oplus R_h} \) and \( \varphi_{M_A \oplus N_h} \); thanks to (76) \( \varphi_{X_h} \) and \( \varphi_{Y_h} \) will be explicit, as well. As we did in the proof of thm. 7 in the subsection “Induction: \( \mathcal{S}_r \)”, we need to distinguish two cases: if \( r' = r = 2 \), i.e. \( W_h, R_h, M_h, N_h \in \mathcal{S}_1 \), we represent \( \varphi_{W_h \oplus R_h} \) and \( \varphi_{M_A \oplus N_h} \) by the roofs

\[
\begin{align*}
W_h \oplus R_h & \rightarrow \varphi_{W_h \oplus R_h} \rightarrow \varphi_{W_h \oplus R_h}, \\
M_h \oplus N_h & \rightarrow \varphi_{M_A \oplus N_h} \rightarrow \varphi_{M_A \oplus N_h},
\end{align*}
\]

where \( \varphi_{W_h}, \ldots, \varphi_{N_h} \) are given by (76). Note that \( \varphi_{W_h} \oplus \varphi_{R_h} \) and \( \varphi_{M_A} \oplus \varphi_{N_h} \) are quasi-isomorphisms in \( \mathcal{H}^{\text{df}}(A_h) \) as, by definition, \( \varphi_{W_0}, \ldots, \varphi_{N_0} \) and are homotopy equivalences, i.e. isomorphisms, in \( \mathcal{D}^{\infty}(A) \), as noted in the subsection “Induction: \( \mathcal{S}_r \)” in the proof of thm. 7. Commutativity of (76) is easily proved: we are just following the lines of the proof in thm. 7 with due changes.

If \( r' \geq 3 \) or \( r \geq 3 \), we need to further decompose \( \bar{X}_{r'} \) and \( \bar{Y}_{r'} \), repeating the above considerations, a finite number of times. The proof of commutativity of (76) is conceptually analogous to the one for \( r' = r = 2 \); we are just considering a “trivially” quantized version of the computations which appear at the very end of the proof of thm. 7.

In summary, we have proven the equivalence \( \mathcal{G}_h \circ \mathcal{F}_h \simeq 1 \) on \( \mathcal{D}_{ij}^{\infty}(A_h) \).

C.0.44. On \( \mathcal{F}_h \circ \mathcal{G}_h \simeq 1 \) on \( \mathcal{D}_{ij}^{\infty}(K_h) \). To prove \( \mathcal{F}_h \circ \mathcal{G}_h \simeq 1 \) on \( \mathcal{D}_{ij}^{\infty}(K_h) \) we begin by considering pair of objects in \( \mathcal{S}_1 \), following (once again!) the lines in the proof of thm. 7. We define the derived functor

\[
\mathcal{G}_h : \mathcal{D}_{ij}^{\infty}(B_h) \rightarrow \mathcal{D}_{ij}^{\infty}(A_h),
\]

with \( M_h \mapsto \mathcal{G}_h(M_h) := M_h \bar{\otimes}_{B_h} K_h \). We consider the following diagram:

\[
\begin{align*}
M_h \bar{\otimes}_{B_h} K_h & \rightarrow \mathcal{G}_h(M_h) = M_h \bar{\otimes}_{B_h} K_h, \\
M_h \bar{\otimes}_{B_h} K_h & \rightarrow \mathcal{G}_h(M_h) = M_h \bar{\otimes}_{B_h} K_h, \\
M_h \bar{\otimes}_{B_h} K_h & \rightarrow \mathcal{G}_h(M_h) = M_h \bar{\otimes}_{B_h} K_h,
\end{align*}
\]

where the quasi-isomorphisms \( \Phi_{M_h \bar{\otimes}_{B_h} K_h} \) and \( \Phi_{K_h} \) have been described in prop. 15 while the quasi-isomorphisms \( \nu_{A_h} \) and \( \nu_{B_h} \) appear in Cor. 7. All arrows in the above diagram are quasi-isomorphisms of strictly unital topological \( A_{\infty} \)-\( A_{\infty} \)-\( A_{\infty} \)-bimodules. Introducing the notation

\[
(77) \quad T_h(M_h) := M_h \bar{\otimes}_{B_h} (K_h \bar{\otimes}_{A_h} K_h) \bar{\otimes}_{B_h} K_h,
\]

the quasi-isomorphism

\[
(78) \quad \varphi_{M_h} : \mathcal{G}_h(M_h) \rightarrow \mathcal{G}_h(M_h)
\]

in \( \mathcal{D}_{ij}^{\infty}(A_h) \) is the composition

\[
\varphi_{M_h} = \alpha_{M_h} \circ \beta_{M_h},
\]

choosing the roofs \( (1_h, \tilde{\alpha}_{M_h}) \) and \( (\tilde{\beta}_{M_h}, 1_h) \) for \( \alpha_{M_h} \) and \( \beta_{M_h} \), where

\[
\tilde{\alpha}_{M_h} = (1 \hat{\otimes} \nu_{A_h}) \circ \Phi_{M_h \bar{\otimes}_{B_h} K_h}, \quad \tilde{\beta}_{M_h} = (1 \hat{\otimes} \nu_{B_h} \hat{\otimes} 1) \circ (1 \hat{\otimes} \Phi_{K_h}).
\]

\( \mathcal{F}_h \) and \( \mathcal{G}_h \) are exact w.r.t. the triangulated structures on \( \mathcal{D}_{ij}^{\infty}(B_h) \) and \( \mathcal{D}_{ij}^{\infty}(A_h) \); the statement for \( \mathcal{G}_h \) follows as well: the analysis is similar. Moreover \( \mathcal{G}_h(M_h) \in \mathcal{D}_{ij}^{\infty}(A_h) \), for any object \( M_h \) in \( \mathcal{D}_{ij}^{\infty}(B_h) \); in fact

\[
(79) \quad \mathcal{G}_h(K_h) = K_h \bar{\otimes}_{B_h} K_h \simeq A_h
\]

in \( \mathcal{D}_{ij}^{\infty}(A_h) \) as it follows by considering the quasi-isomorphism \( \varphi_{K_h} \) and using \( K_h \bar{\otimes}_{B_h} K_h \simeq A_h \); the statement for any object in \( \mathcal{S}_1 \) easily follows; for \( r \geq 2 \) we just need to look at exact triangles. We continue with the equivalence \( \mathcal{F}_h \circ \mathcal{G}_h \simeq 1 \) on \( \mathcal{D}_{ij}^{\infty}(K_h) \); its proof is done by decomposing objects in \( \mathcal{S}_r \), \( r \geq 2 \), into finite direct sums of objects
in $S_1'$, as we did in the preceding subsection. We need to prove the step $r = 1$ explicitly. Let $K_h(i')[n'], K_h(j)[r]$ be two objects in $S_1'$ and let $g_h : K_h(i')[n'] \to K_h(j)[r]$ be a morphism in $D^\infty(B_h)$ represented by the roof
\[
\begin{tikzpicture}
  \node (A) {$K_h(i)[l]$};
  \node (B) [below right of=A] {$K_h(i')[n']$};
  \node (C) [below right of=B] {$K_h(j)[r]$};
  \node (D) [below left of=A] {$K_h(j)[r]$};
  \node (E) [below left of=A] {$K_h(i)[l]$};
  \draw[->] (A) -- (B) node[above] {$g_h$};
  \draw[->] (B) -- (C) node[above] {$g_h$};
  \draw[->] (A) -- (D) node[above] {$K_h(i)[l]$};
  \draw[->] (A) -- (E) node[above] {$K_h(j)[r]$};
\end{tikzpicture}
\]
(80)

where $i, i', l, n', j, r \in \mathbb{Z}$. By degree reasons $g_h$ has a unique non trivial Taylor component $g_h^{(r)}$, for some $r \geq 0$. The commutativity of the diagram
\[
K_h(i')[n'] \xrightarrow{g_h} K_h(j)[r] \quad \xrightarrow{\psi_{K_h(i')[n']}} \quad \xrightarrow{\psi_{K_h(j)[r]}}
\]
\[
F_h(G_h(K_h(i')[n']))) \xrightarrow{F_h(G_h(g_h)))} F_h(G_h(K_h(j)[r]))
\]
is proven once we show that
\[
K_h(i)[l] \xrightarrow{\psi_{K_h(i)[l]}} K_h(j)[r] \quad \xrightarrow{\psi_{K_h(j)[r]}}
\]
(81)
\[
(\overline{K_h} \otimes_{B_h} B_h)(i)[l] \quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)^{op}(i)[l] \quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)^{op}(j)[r]
\]
\[
\quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)(j)[r] \quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)^{op}(j)[r]
\]
\[
\quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)^{op}(i)[l] \quad \xrightarrow{\overline{K_h} \otimes_{B_h} B_h}(K_h)(j)[r]
\]
commutes, where the morphisms $Z_h^0$ are those appearing in the proof of thm. 7, section 7, with due changes. Considering (81), given any $X_h \in S_1'$, we denote by $\overline{\psi}_{X_h} : X_h \to F_h(G_h(X_h))$ the composition
\[
\overline{\psi}_{X_h} = Z_h^0 \circ Z_h^1 \circ Z_h^2 = (1 \otimes R_h) \circ (1 \otimes R_h) \circ (\Phi_{K_h}).
\]
(82)

where $\Phi_{K_h}$ is described in prop 15. $R_h$ is the quantized derived right action and $R_h$ appears in cor 6. $\overline{\psi}_{X_h}$ is a quasi-isomorphism in $H^*_h(A_h)$ as $\overline{\psi}_{X_h}^{(0)}$ is a quasi-isomorphism, i.e. a homotopy equivalence in $D^\infty(A)$. Using the decomposition of objects in $S_1'$ into finite direct sums of objects in $S_1$ we finish the proof of the equivalence $F_h \circ G_h \simeq 1$ on $\text{triang}^{\infty}_h(K_h)$. We can repeat the analysis of the previous subsection almost verbatim.

The equivalence $F_h \circ G_h \simeq 1$ on $\text{triang}^{\infty}_h(K_h)$. is proved following the same strategy that lead us to $F_h \circ G_h \simeq 1$ on $\text{triang}^{\infty}_h(K_h)$: all we need is to consider the case $r = 1$ using $F_h \circ G_h \simeq 1$ on $\text{triang}^{\infty}_h(K_h)$ . All computations and decompositions that appear in the proof of thm. 7 can be repeated here, with due changes.

C.0.45. Last part of the proof. Exchanging $A_h$ and $B_h$; i.e. using the topological $A_{\infty}$-$B_h$-$A_{\infty}$-bimodule $(\tilde{K}_h, d_{K_h})$ and the new functors $F_h'' = \tilde{\otimes}_{B_h} \tilde{K}_h$ and $G_h'' = \tilde{\otimes}_{A_h} \tilde{K}_h$ we can prove the equivalence of the triangulated categories $\text{triang}^{\infty}_h(B_h)$ and $\text{triang}^{\infty}_h(K_h)$ with the same techniques introduced above.

The statement on the thick subcategories follows by additivity of $F_h$ and $G_h$ ($F_h''$ and $G_h''$ as well), w.r.t. the coproduct in $D^\gamma_h(A_h)$ and $D^\gamma_h(B_h)$, i.e. the direct sum of strictly unital topological $A_{\infty}$-modules.

\[\square\]

**APPENDIX D. ON TRIANGULATED CATEGORIES**

In this section we collect some known facts on triangulated categories and thickness. We follow the expositions in [20] and [16]. Let us consider the pair $(\mathcal{T}, \Sigma)$, where $\mathcal{T}$ is an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$, $\Sigma(X) := 2X$ an additive autoequivalence.

**Definition 34.** A triangle in $\mathcal{T}$ is a triple $(\alpha, \beta, \gamma)$ of morphisms in $\mathcal{T}$
\[
X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma Z,
\]

and a morphism between two triangles \((\alpha, \beta, \gamma), (\alpha', \beta', \gamma')\) is a triple \((\varphi_1, \varphi_2, \varphi_3)\) of morphisms in \(T\) s.t. the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} \\
X' & \xrightarrow{\alpha'} & Y' \\
\end{array}
\quad \begin{array}{ccc}
Y & \xrightarrow{\beta} & Z \\
\downarrow{\varphi_2} & & \downarrow{\varphi_3} \\
Y' & \xrightarrow{\beta'} & Z' \\
\end{array}
\quad \begin{array}{ccc}
\Sigma X & \xrightarrow{\gamma} & \\
\downarrow{\Sigma(\varphi_1)} & & \\
\Sigma X' & \xrightarrow{\gamma'} & \\
\end{array}
\]

commutes.

**Definition 35.** The category \(T\) is said to be triangulated if it is equipped with a class of distinguished triangles, called the exact triangles, satisfying the following axioms.

- **(T1)** A triangle isomorphic to an exact triangle is exact. For any object \(X\), the triangle \(0 \to X \xrightarrow{1} X \to 0\) is exact. Any morphism \(\alpha : X \to Y\) in \(T\) can be completed to an exact triangle \(\alpha \to Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X\).
- **(T2)** A triangle \((\alpha, \beta, \gamma)\) is exact if and only if \((\beta, \gamma, -\Sigma \alpha)\) is exact.
- **(T3)** Given two exact triangles \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\), each pair of morphisms \(\varphi_1\) and \(\varphi_2\) satisfying \(\varphi_2 \circ \alpha = \alpha' \circ \varphi_1\) can be completed (not necessarily uniquely) to a morphism of triangles \((\varphi_1, \varphi_2, \varphi_3)\):

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} \\
X' & \xrightarrow{\alpha'} & Y' \\
\end{array}
\quad \begin{array}{ccc}
Y & \xrightarrow{\beta} & Z \\
\downarrow{\varphi_2} & & \downarrow{\varphi_3} \\
Y' & \xrightarrow{\beta'} & Z' \\
\end{array}
\quad \begin{array}{ccc}
\Sigma X & \xrightarrow{\gamma} & \\
\downarrow{\Sigma(\varphi_1)} & & \\
\Sigma X' & \xrightarrow{\gamma'} & \\
\end{array}
\]

- **(T4)** Given exact triangles \((\alpha_1, \alpha_2, \alpha_3)\), \((\beta_1, \beta_2, \beta_3)\) and \((\gamma_1, \gamma_2, \gamma_3)\) with \(\gamma_1 = \beta_1 \circ \alpha_1\), there exists an exact triangle \((\delta_1, \delta_2, \delta_3)\) making the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha_1} & Y \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} \\
Z & \xrightarrow{\gamma_3} & V \\
\downarrow{\beta_2} & & \downarrow{\beta_3} \\
W & \xrightarrow{\beta_1} & \Sigma Y \\
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\alpha_2} & U \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} \\
Z & \xrightarrow{\gamma_3} & V \\
\downarrow{\beta_2} & & \downarrow{\beta_3} \\
W & \xrightarrow{\beta_1} & \Sigma Y \\
\end{array}
\quad \begin{array}{ccc}
\Sigma X & \xrightarrow{\Sigma(\alpha_2)} & \\
\downarrow{\Sigma(\gamma_1)} & & \\
\Sigma U & & \\
\end{array}
\]

commutative.

If the category \(T\) satisfies only the axioms (T1)-(T2)-(T3), then it is said to be a pre-triangulated category.

**Definition 36.** Let \(T\) be a pre-triangulated category and \(\text{Ab}\) be the category of abelian groups. A functor \(F : T \to \text{Ab}\), with \(\text{Ab}\) abelian, is said to be cohomological if it sends each exact triangle in \(T\) to an exact sequence in \(\text{Ab}\).

**Lemma 16.** For each \(\epsilon \in T\), the representable functors

\[
\text{Hom}_T(\epsilon, -) : T \to \text{Ab}, \quad \text{Hom}_T(-, X) : T^{\text{op}} \to \text{Ab},
\]

are cohomological.

From the above lemma it follows

**Lemma 17.** Let \((\varphi_1, \varphi_2, \varphi_3)\) be a morphism between exact triangles in \(T\). If two maps in \(\{\varphi_1, \varphi_2, \varphi_3\}\) are isomorphisms, then also the third.

**Definition 37.** Let \((T, \Sigma_1)\) and \((U, \Sigma_2)\) be triangulated categories. An exact functor \(T \to U\) is a pair \((F, \eta)\) consisting of a functor \(F : T \to U\) and a natural isomorphism \(\eta : F \circ \Sigma_1 \to \Sigma_2 \circ F\) s.t., for every exact triangle \(X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma_1 X\) in \(T\), the triangle

\[
F(X) \xrightarrow{F(\alpha)} F(Y) \xrightarrow{F(\beta)} F(Z) \xrightarrow{\eta F(\gamma)} \Sigma_2 F(X)
\]

is exact in \(U\).

\footnote{This is the celebrated octahedral axiom.}
The following basic example is well studied in [11].

**Example 2.** Let \((T, Σ)\) be a triangulated category. The autoequivalence \((Σ, -1)\) is an exact functor w.r.t. the triangulated structure on \(T\).

D.0.46. On triangulated subcategories. Let \((T, Σ)\) be a triangulated category.

**Definition 38.** A non-empty full additive subcategory \(C\) is a triangulated subcategory if

- \((S0)\) \(C\) is strict; any object isomorphic to an object in \(S\) belongs to \(S\).
- \((S1)\) \(Σ^nX ∈ C\) for all \(X ∈ C\) and \(n ∈ \mathbb{Z}\).
- \((S2)\) Let \(X → Y → Z → ΣX\) be any exact triangle in \(T\). If any two objects from \(\{X, Y, Z\}\) belong to \(C\), so also the third.

A triangulated subcategory \(C\) inherits a canonical triangulated structure from \(T\). Let \(U\) and \(V\) be classes whose objects are (isomorphism classes of) objects in \(T\), where \(T\) is triangulated. The class \(U * V\) is defined as follows:

\[ U * V := \{ X ∈ T : U → X → V → ΣU \} \]

The composition \(*\) is associative with the octahedral axiom \((T4)\). The following notation

\[ S_r = S_1 * S_1 * ... * S_1 \]

\(r\)-times

is unambiguous for \(r ≥ 1\), for any class \(S_r\) of objects in \(T\). The objects in \(S_r\) are called the extensions of length \(r\) of objects of \(S_1\). If \(T\) is a triangulated category and \(M\) is an object in \(T\), the full triangulated subcategory generated by \(M\) consists of all objects belonging to \(S_r\) \((r ≥ 1\), as above\) with \(S_1 = \{ M[i], i ∈ \mathbb{Z} \}\) (in \(S_1\) we consider equivalence classes of isomorphic objects). Such triangulated subcategory is the smallest full triangulated subcategory in \(T\) containing \(M\).

Its thickening is the full triangulated subcategory of \(T\) consisting of all objects \(X\) in \(T\), s.t. there exist an object \(Z\) in the triangulated subcategory generated by \(M\) with \(Z ≃ X ⊕ V\). The thickening is closed under direct summands; actually it is the smallest full triangulated subcategory in \(T\) containing the triangulated subcategory generated by \(M\) and closed under direct summands.

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