Neutrino Propagation in a Fluctuating Sun

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Abstract

We adapt to neutrino physics a general formulation for particle propagation in fluctuating media, initially developed for applications to electromagnetism and neutron optics. In leading approximation this formalism leads to the usual MSW effective Hamiltonian governing neutrino propagation through a medium. Next-to-leading contributions describe deviations from this description, which arise due to neutrino interactions with fluctuations in the medium. We compute these corrections for two types of fluctuations: (i) microscopic thermal fluctuations, and (ii) macroscopic fluctuations in the medium’s density. While the first of these reproduces standard estimates, which are negligible for applications to solar neutrinos, we find the second can be quite large, since it grows in size with the correlation length of the fluctuation. We consider two models in some detail. For fluctuations whose correlations are extend only over a local region in space of length $\ell$, appreciable effects for MSW oscillations arise if $(\delta n/n)^2 \ell \gtrsim 100$ m or so. Alternatively, a crude model of helioseismic $p$-waves gives appreciable effects only when $(\delta n/n) \gtrsim 1\%$. In general the dominant effect is to diminish the quality of the resonance, making the suppression of the $^7$Be neutrinos a good experimental probe of fluctuations deep within the sun. Fluctuations can also provide a new mechanism for reducing the solar neutrino flux, giving an energy-independent suppression factor of $\frac{1}{2}$ away from the resonant region, even for small vacuum mixing angles.
1. Introduction and Summary

We do not understand our sun as well as we should. Although the sun shines brightly in neutrinos, it is not so bright as it ought to be according to our present understanding of its workings and of neutrino properties.

In recent years experimental and theoretical lines of research have converged to bring this Solar Neutrino Problem (SNP) to a head. On the one hand, confidence in the experimentally-measured neutrino fluxes has grown with the observation of the solar-neutrino shortfall in four independent experiments [1], including those which are capable of detecting neutrinos from the principal $p - p$ cycle of nuclear reactions. Moreover, it has recently become possible to calibrate these detectors by exposing them to very intense radioactive sources here on earth. On the theoretical side, confidence in the neutrino-flux predictions of solar models has also improved, for two reasons [2]. First, the redundancy of the experiments permits the discrepancy to be mainly based on the predictions for the $p - p$ neutrinos. Since the $p - p$ reactions are largely responsible for generating the sun’s energy, the theoretical uncertainty in their reaction rates is minimal. Second, the rise of the field of helioseismology has made available an abundance of experimental data about the solar interior, thereby significantly improving the constraints on the assumptions which must be made in constructing solar models.

If the problem is not the sun, then the measured neutrino shortfall must arise while the neutrinos are *en route* to the earth. Besides gaining support from the improvements in understanding of solar models, the credibility of such a neutrino solution to the SNP is also boosted by the existence of a very plausible and elegant mechanism for depleting the observed solar neutrino flux. The mechanism consists of resonant (MSW) oscillations of the neutrinos as they pass through the sun [3]. In this picture the small influence of the solar medium on neutrino propagation plays an important role by resonating with the equally small vacuum oscillations which generically arise once neutrinos are endowed with masses. Considerable effort has been invested in understanding the nature of these material-dependent oscillations.

A common feature of the majority of these studies has been the approximation in which the influence of the solar medium is described in terms of an effective hamiltonian, depending on the mean values of the quantities to which the neutrinos couple. Less has been done to study scattering from the deviations away from this mean. Some researchers have investigated the effects of neutrino scattering from position-dependent densities [4], although usually ignoring the potentially decohering effects [5] — more about which later — of such scattering. Incoherent scattering due to interactions with the particles which
make up the medium has also been studied within the context of supernovae, and the early universe, [6][7][8], in which case neutrinos themselves can be part of the underlying medium. This type of scattering is entirely negligible within the sun [9]. Until quite recently, [10][11][12], less attention has been devoted to the effects for solar neutrinos of more macroscopic fluctuations.

The purpose of the present work is to develop a framework for describing the influence of all such fluctuations on neutrino propagation, with the goal of identifying when each can be important. To this end we adapt to neutrino physics a formalism which has been extensively used to describe the interaction of electromagnetic waves and neutrons with fluctuations in matter [13]. As we describe in detail herein, our results agree with earlier approaches when applied to the fluctuations they consider.

In order of magnitude, fluctuation effects contribute to neutrino evolution with strength \( G_F^2 \langle \delta n \delta n \rangle \ell_\parallel \). Here the average is over different quantities in different situations, and \( \delta n = n - \langle n \rangle \) denotes the deviation of the particle density from its mean. \( \ell_\parallel \) is the correlation length along the direction of neutrino motion. The relative size of this term as compared to the usual MSW evolution term, \( G_F \langle n \rangle \ell_\parallel \epsilon^2 \), where \( \langle \delta n \delta n \rangle = \epsilon^2 \langle n \rangle^2 \), is therefore of order \( G_F \langle n \rangle \ell_\parallel \epsilon^2 \), where \( \langle n \rangle \sim 10^{26}/\text{cm}^3 \), we see that sizable effects can be expected only for large-scale fluctuations: \( \ell_\parallel \epsilon^2 \lesssim (G_F \langle n \rangle)^{-1} \sim 100 \) km. Our more detailed analysis shows that for resonant oscillations this estimate is too large, and sizable effects can arise starting from \( \epsilon^2 \ell_\parallel \gtrsim 100 \) m. These scales are of potential interest for solar neutrinos, since they are typical of scales which can arise from physics within the sun.

Some work on this kind of macroscopic-fluctuation-driven effects for neutrino propagation has appeared recently in the literature, starting with the pioneering work of ref. [5]. This reference considered a fixed density profile which varied in space, and computed the time evolution of the reduced density matrix which governs the flavour degrees of freedom. Decohering fluctuation effects were found when the neutrino momenta were integrated out to obtain the flavour evolution. We argue in Section 5 that this type of fluctuation is unlikely to be important for solar-neutrino physics, although it could well play a role in other applications. We reach this conclusion because we find that the only fluctuations which can decohere neutrinos as they pass through, are those whose size, \( \ell_\perp \), transverse to the direction of neutrino propagation, is smaller than the transverse size of the detector. But since fluctuations in the sun can in any case only affect neutrino evolution for \( \ell_\parallel \gtrsim 100 \) m, correlation lengths as small as typical neutrino detector sizes can play no significant role, so long as \( \ell_\perp \sim \ell_\parallel \). The same need not be true for other applications, such as to supernovae, however.
More recently, the main approach to fluctuations that has been pursued to date, \textit{e.g.} by refs. [10] and [12], is to model the density as a Gaussian random variable subject to the assumption that all correlations exist only over distances that are negligibly small compared to the neutrino oscillation lengths which are of interest. Our results extend these analyses in several ways. First, since we work from first principles, we can give explicit expressions which are applicable to any kind of density ensemble. In particular, we do \textit{not} assume the correlation length of the fluctuations to be small compared to neutrino oscillation lengths, and so can apply our results to density profiles which vary on scales that are comparable to the size of the sun. We can also incorporate arbitrary variation in space and time of the fluctuation’s mean and variance. This permits us to consider such real density variations as helioseismic waves, which are known to exist in the sun. Our equations reduce to those of refs. [10] and [12] in the limit that our assumptions overlap, but some of our most interesting applications are to situations for which previous analyses do not apply.

Our results are presented in the following way:

\begin{itemize}
  \item \textit{1:} We present a general formalism in Section 2 for describing the interactions of any particle with arbitrary matter fluctuations. One of the main features of such fluctuations is that they generically destroy the coherence of neutrino propagation by evolving pure states into mixed states. As a result it is typically \textit{impossible} to describe them in terms of a matter-dependent effective hamiltonian, since any such hamiltonian would necessarily take pure states to pure states. This section culminates in a master formula for the rate of change of the density matrix describing particle propagation in an arbitrary medium, which naturally divides into a term which defines a mean effective matter-dependent hamiltonian, plus a fluctuation-dependent term. A nice feature of the formalism is its recursive nature, which permits fluctuations to be successively integrated out on larger and larger distance scales.
  
  \item \textit{2:} Section 3 applies the general results of Section 2 to neutrinos moving in the presence of microscopic fluctuations, which are those for which the correlation length is negligible compared to the interesting scales for neutrino propagation. We consider in some detail the special case of thermal fluctuations, and rederive the usual result that these are small for neutrinos in the sun.

  \item \textit{3:} Section 4 then considers matter fluctuations on larger scales. These fluctuations arise because, although any one neutrino sees a fixed density profile, successive neutrinos see different ones. We describe this by considering neutrinos to pass through an ensemble of density profiles. The ensemble is characterized by expanding the density in terms of a
complete set of modes whose amplitudes are taken to be uncorrelated random variables. The nature of the underlying physics governs the basis of modes which are uncorrelated in any given application. We consider two types of bases for illustrative purposes: (i) fluctuations which are localized in position, within cells of slowly-varying length, \( \ell \); and (ii) fluctuations in the amplitude of the normal modes which describe acoustic density waves. In the limit of small, constant, \( \ell \) the first of these bases reduces to the case studied in refs. [10] and [12]. The second is new, and is meant as a crude model of a helioseismic \( p \)-wave of fluctuating amplitude. We evaluate, for both examples, the mean Hamiltonian and the contributions of fluctuations to neutrino evolution.

- **4**: Section 5 takes the previous results and integrates out the neutrino momentum degrees of freedom, to obtain the reduced evolution equation which governs the reduced density matrix describing neutrino flavour and spin. We argue that previous derivations [5], which trace over neutrino momenta without taking into account that neutrino positions are ultimately measured in real experiments, give mistakenly large estimates of the size of the decoherence which this trace introduces for solar neutrinos. A general expression is found for the fluctuation contributions to the neutrino-flavour evolution, which are found to be characterized by a single parameter, \( A_{ab} \). This parameter is evaluated for the two density ensembles introduced in Section 4.

- **5**: Section 6 specializes the general results to the two flavour case, and integrates the time evolution to obtain the electron-neutrino survival probability. An approximate analytic form for this integration is obtained, which is a generalization of Parke’s formula for standard MSW mixing. The decoherence due to fluctuations appears in the evolution as a damping term, similar (but not identical) to what would happen if the neutrinos were decaying. Numerical integration is also performed, and found to agree well with the analytical results.

We perform the MSW analysis for both models of density fluctuations that are described in Section 4. It is found that appreciable changes to the usual MSW scenario arise for surprisingly small amplitude fluctuations. In terms of \( \epsilon^2 = \langle \delta n \delta n \rangle / \langle n \rangle^2 \), and the correlation length \( \ell || \), we find deviations for \( \epsilon^2 \ell || \gtrsim 10 \) m. Startlingly large changes from MSW behavior arise for \( \epsilon^2 \ell || \gtrsim 1 \) km. A generic new feature of fluctuations is the introduction of a universal energy-independent reduction of the survival probability to \( \frac{1}{2} \) for small \( E/\delta m^2 \).

The model of a fluctuating helioseismic \( p \)-wave gives discernible effects which are comparatively small. For a wave with a 30 minute period, discernible effects require \( \epsilon \gtrsim 1\% \). We understand the size of this effect to be due to the small wavelength the
wave typically has near the resonance region, due to the increase of the speed of sound with depth in the medium. More realistic simulations are currently under way to see if the same is true for neutrinos propagating through both solar p and g waves.

• Finally, in Section 7 the general formalism is applied to derive the effective evolution equation governing the reduced neutrino flavour/spin density matrix in the presence of a magnetic moment interaction. Our conclusions are briefly summarized in Section 8.

2. The General Formalism

In order to keep all approximations explicit it is instructive to first formulate our problem within its most general context. Suppose, therefore, that our system consists of two sectors, A and B, of which we wish to follow the evolution of degrees of freedom in sector A while ignoring (or partially ignoring – see below) those in sector B. For example, when examining the influence of matter fluctuations on neutrino propagation we will take A to describe the neutrino states of the system while B consists of the states which are available to the electrons and/or nucleons which make up the medium through which the neutrinos move.

It is sometimes necessary to consider the more general case where a partial measurement is made on sector B, in addition to the measurements which are performed in sector A. We do this in order to set up the treatment of resonant oscillations, for which A consists only of the flavour (and spin) sectors of the single-particle neutrino sector, while B contains both the neutrino position/momentum information as well as all medium-related effects. (We argue in Section 5 that an improper treatment of this case has in the past led to a mistaken estimate of the size of incoherent effects purely due to this removal of momentum degrees of freedom.) The slightly more general formulation is required to analyze this situation since neutrino position information is in practice never completely ignored (i.e. neutrinos are all detected on Earth).

At an initial time, \( t' \), we suppose these two sectors to be completely uncorrelated. That is, suppose the initial density matrix for the entire system factorizes:

\[
\rho(t') = \rho_A \otimes \rho_B. \tag{1}
\]

We imagine here that \( \rho_A \) acts only in the A sector of the Hilbert space and \( \rho_B \) acts only in the B sector, and we take \( \rho_A \) and \( \rho_B \) to be separately normalized within their own sectors: \( \text{Tr}_A \rho_A = \text{Tr}_B \rho_B = 1 \). Here \( \text{Tr}_A \) (or \( \text{Tr}_B \)) denotes a trace taken only over the A (or B) sector of the Hilbert space.
Next suppose the Hamiltonian for the system takes the form:

\[ H = H_0 + \hat{V}, \]  

(2)

in which \( H_0 = H_A + H_B \) describes the separate evolution of the A and B sectors, while \( \hat{V} \) is the interaction which couples these sectors together. Using this Hamiltonian we may evolve \( \rho \) to later times, \( t > t' \). We assume for this purpose that \( \varrho_A \) and \( \varrho_B \) respectively commute with \( H_A \) and \( H_B \). Within the interaction representation the time evolution of \( \rho(t) \) is then described by:

\[ \frac{\partial \rho}{\partial t} = -i \left[ V(t), \rho \right], \]  

(3)

where \( V(t) \equiv e^{iH_0t} \hat{V} e^{-iH_0t} \). Alternatively:

\[ \rho(t) = U(t,t') \rho(t') U^*(t,t') \]

with \( \frac{\partial U(t,t')}{\partial t} = -iV(t) U(t,t'). \)  

(4)

In general this time evolution will introduce correlations between sectors A and B and so won’t preserve the factorized form of eq. (1). The remainder of this section is devoted to explicitly displaying these, and other, effects as sector A evolves in the presence of sector B.

2.1) Coherent and Diffuse Scattering

Suppose, now, that only observables associated with sector A are to be measured at some time \( t > t' \). The probability of the results of any such measurement are completely described by the reduced density matrix, \( \varrho_A(t) \), defined by tracing the full density matrix over only the B sector of states:

\[ \varrho_A(t) \equiv \text{Tr}_B \left[ \rho(t) \right]. \]  

(5)

Notice that eq. (1) implies \( \varrho_A(t) \) satisfies the initial condition \( \varrho_A(t = t') = \varrho_A \).
We now wish to split the time evolution for $\rho_A(t)$ into a piece which describes the mean features of sector $B$ — ‘coherent’ scattering\(^1\) — plus a piece which describes the fluctuations about this mean — ‘diffuse’ scattering. Our guiding principle in so doing is to ensure that final time-evolved probabilities may be written as the non-interfering sum of a coherent part plus a diffuse part.

Define, then, the mean (or coherent) evolution operator, $\mathcal{U}(t,t')$, as the average of $U(t,t')$ over the $B$ sector, as follows:

$$\mathcal{U}(t,t') \equiv \langle U(t,t') \rangle_B,$$

where the $B$-average of any quantity is defined by: $\langle \cdots \rangle_B \equiv \text{Tr}_B[\rho_B(\cdots)]$. The difference between $U(t,t')$ and $\mathcal{U}(t,t')$ we denote:

$$\Delta U(t,t') \equiv U(t,t') - \mathcal{U}(t,t'),$$

and so satisfies the defining identity $\langle \Delta U \rangle_B = 0$.

With these definitions all probabilities calculated at times $t > t'$ are the sum of a coherent piece and a diffuse piece. That is, for any hermitian observable acting only in the $A$ sector, $O_A$, eqs. (1), (4), (6) and (7) imply:

$$\langle O_A \rangle(t) \equiv \text{Tr}[\rho(t)O_A]$$

$$= \text{Tr}\left[U(t,t')\rho(t')U^*(t,t')O_A\right]$$

$$= \text{Tr}_A\left[\mathcal{U}(t,t')\rho_A\mathcal{U}^*(t,t')O_A\right] + \text{Tr}\left[\Delta U(t,t')\rho(t')\Delta U^*(t,t')O_A\right]$$

$$\equiv \langle O_A \rangle_c(t) + \langle O_A \rangle_d(t).$$

The cross terms involving both $\mathcal{U}(t,t')$ and $\Delta U(t,t')$ vanish by virtue of the identity $\langle \Delta U \rangle_B = 0$. This last equality defines the mean (or coherent) and fluctuation (or diffuse) parts of $\langle O_A \rangle(t)$.

\(^1\) We borrow the descriptions ‘coherent’ and ‘diffuse’ from the analogous applications of this formalism to the propagation of X-rays and neutrons through matter. This split, as made precise in eq. (8), is our definition of coherent scattering for the present purposes. Notice that the ‘coherent’ part, as defined here, is coherent only in a weaker — though more useful, for present purposes — sense than is sometimes used in electromagnetic applications. For instance, coherence in the present context need not imply phase coherence between incident and scattered waves.
The distinction between diffuse and coherent evolution can also be made directly for the reduced density matrix itself. That is, \( \rho_A(t) = \rho^c_A(t) + \rho^d_A(t) \), where

\[
\rho^c_A(t) \equiv U(t, t') \rho_A U^*(t, t'), \\
\rho^d_A(t) \equiv \text{Tr}_B \left[ \Delta U(t, t') \rho(t') \Delta U^*(t, t') \right].
\]

For many applications — including the description of neutrino oscillations — it is preferable to formulate the diffuse-coherent split for \( \partial \rho / \partial t \) rather than for the integrated evolution operator, \( U(t, t') \). This is because it is often possible to use perturbation theory for \( \partial \rho / \partial t \) but not for the long-time evolution of \( \rho(t) \). This leads us to formulate the main result of this section. The differential evolution equation for \( \rho^c_A(t) \) may be written:

\[
\frac{\partial \rho^c_A}{\partial t} = -i \left[ \mathcal{V}(t) \rho^c_A(t) - \rho^c_A(t) \mathcal{V}^*(t) \right],
\]

where \( \mathcal{V}(t) \) is the effective interaction hamiltonian which is defined in such a way as to ensure that \( \mathcal{V} \) is related to \( U(t, t') \) in the same way that \( V \) is related to \( U \). That is:

\[
\mathcal{V}(t) \equiv i \frac{\partial U}{\partial t} U^{-1},
\]

which need not be hermitian (since \( U \) need not be unitary).

Similarly, the differential evolution equation for \( \rho^d_A(t) \) is

\[
\frac{\partial \rho^d_A}{\partial t} = \frac{\partial}{\partial t} \text{Tr}_B \left[ \Delta U(t, t') \rho(t') \Delta U^*(t, t') \right].
\]

2.2) Incorporating a Partial Measurement in Sector B

For some applications it is true that measurements do not completely ignore what is going on in sector \( B \). For instance, in applications to solar-neutrino oscillations we will follow common practice and take \( A \) to describe only the neutrino flavour and spin degrees of freedom. This involves banishing all neutrino position and momentum information into sector \( B \), even though any realistic measurements do include some information concerning neutrino position, such as that they are detected on earth. This section describes the slight generalization of the formalism which is required to handle such cases.
We therefore relax the assumption that all of the observables of interest, \( \mathcal{O} \), need act only in sector \( A \). Instead we assume them to involve a specific observation in sector \( B \), which is uncorrelated with all measurements in sector \( A \). That is, consider the class of observables having the form:

\[
\mathcal{O} = \mathcal{O}_A \otimes \mathcal{O}_B,
\]

with \( \mathcal{O}_A \) (or \( \mathcal{O}_B \)) acting only in sector \( A \) (or \( B \)). In this case expressions similar to those found above may be derived, in which all averages over sector \( B \) are weighted by the observable \( \mathcal{O}_B \).

Specifically, define once more the evolution operators \( \overline{U}(t,t') \) and \( \Delta U(t,t') \) as in eqs. (6) and (7), but with the \( B \)-average now defined by:

\[
\langle \cdots \rangle_B \equiv \frac{\text{Tr}_B \left[ (\cdots) \rho_B \mathcal{O}_B \right]}{\text{Tr}_B \left[ \rho_B \mathcal{O}_B \right]}.
\]

This choice preserves the property that \( \langle \mathcal{O} \rangle(t) \) may be written as the non-interfering sum of a diffuse and coherent contribution, although eq. (8) is slightly modified to become:

\[
\langle \mathcal{O} \rangle(t) \equiv \text{Tr} \left[ \rho(t) \mathcal{O} \right] = \text{Tr}_A \left[ \overline{U}(t,t') \rho_A \overline{U}'(t,t') \mathcal{O}_A \right] \text{Tr}_B \left[ \rho_B \mathcal{O}_B \right] + \text{Tr} \left[ \Delta U(t,t') \rho(t') \Delta U^*(t,t') \mathcal{O} \right] = \langle \mathcal{O} \rangle_c(t) + \langle \mathcal{O} \rangle_d(t).
\]

As before, the time evolution of any such observable may be completely described in terms of a reduced density matrix, \( \rho_A(t) \), for which the definition, eq. (5), is now replaced by:

\[
\rho_A(t) \equiv \text{Tr}_B \left[ \rho(t) \mathcal{O}_B \right],
\]

satisfying the initial condition: \( \rho_A(t = t') = \mathcal{Q}_A \text{Tr}_B[\rho_B \mathcal{O}_B] \). Notice that \( \rho_A(t) \) defined this way is not normalized. The differential time evolution of its coherent part is now given by the analog of eq. (10):

\[
\frac{\partial \rho_A^c}{\partial t} = -i \left[ \mathcal{V}(t) \rho_A^c(t) - \rho_A^c(t) \mathcal{V}^*(t) \right] \text{Tr}_B \left[ \rho_B \mathcal{O}_B \right].
\]

\(^2\) Beware: the operator ordering in this definition has the counterintuitive implication that \( \langle X \rangle_B \) need not equal \( \langle X^* \rangle_B \).
Notice that this equation lacks a term proportional to \( \frac{\partial O_B}{\partial t} \), even though \( O_B(t) \) generally depends on time within the interaction picture with which we are working. Its omission from eq. (17) is justified since there \( O_B \) should be evaluated at the time, \( t_m \), when the measurement is performed, rather than at the time, \( t \), of the evolution. \( \bar{V}(t) \) is once again defined by eq. (11).

The diffuse evolution now becomes:

\[
\frac{\partial \rho^d_A}{\partial t} \equiv \frac{\partial}{\partial t} \text{Tr}_B \left[ \Delta U(t,t') \rho(t') \Delta U^*(t,t') O_B \right].
\]  

(18)

2.3) Perturbative Expressions

It is instructive to evaluate eqs. (11) and (18) perturbatively in the interaction \( V \). To this end we use the familiar series solution to eq. (4):

\[
U(t,t') = \sum_{n=0}^{\infty} (-i)^n \int_{t'}^t d\tau_1 \cdots \int_{t'}^{\tau_{n-1}} d\tau_n V(\tau_1) \cdots V(\tau_n).
\]  

(19)

Using this expression in the previous results leads to the following formula for effective Hamiltonian:

\[
\bar{V}(t) = \langle V(t) \rangle_B - i \int_{t'}^t d\tau \langle \delta V(t) \delta V(\tau) \rangle_B + O(V^3),
\]  

(20)

with \( \delta V(t) \equiv V(t) - \langle V(t) \rangle_B \).

Notice that the antihermitian part of \( \bar{V} \) first arises at second order in \( V \). For instance, if \([\varrho_B, O_B] = 0\):

\[
\frac{1}{2} \langle \bar{V} + \bar{V}^* \rangle = \langle V \rangle_B - \frac{i}{2} \int_{t'}^t d\tau \left\langle \left[ \delta V(t), \delta V(\tau) \right] \right\rangle_B + O(V^3),
\]

\[
-\frac{i}{2} \langle \bar{V} - \bar{V}^* \rangle = -\frac{1}{2} \int_{t'}^t d\tau \left\langle \left\{ \delta V(t), \delta V(\tau) \right\} \right\rangle_B + O(V^3)
\]

(21)

Similarly, the rate of change of \( \rho^d_A \) is:

\[
\frac{\partial \rho^d_A}{\partial t} = \int_{t'}^t d\tau \text{Tr}_B \left[ \left( \delta V(t) \rho(t') \delta V(\tau) + \delta V(\tau) \rho(t') \delta V(t) \right) O_B \right] + O(V^3).
\]  

(22)

Eqs. (20) and (22) are our starting point for applications of this formalism to neutrino propagation through matter.
2.4) Long-Time Evolution and Master Equations

Regardless of how small the interaction hamiltonian should be, perturbation theory eventually fails if one follows the system’s evolution for sufficiently long times. Worse, it is often precisely the long-time behaviour which is of interest in particular applications. In this section we outline how to use the above perturbative expressions for time scales which are sufficiently large compared to the correlation times which govern the fluctuations in the medium.

Suppose, then, that the correlation, \( \langle \delta V(t) \delta V(t') \rangle \), is negligible for \( |t - t'| \) greater than some correlation time, \( \tau \). Suppose also that the system’s time evolution is required over timescales, \( T \), for which \( T \gg \tau \). In this case perturbative expressions for \( \frac{\partial \rho_A}{\partial t} \) may be useful provided that \( \tau \) is small enough to justify perturbation theory, even if the same would not be true for timescales as large as \( T \).

In this limit a coarse-grained time derivative of the density matrix may be defined for times which are large compared to \( \tau \) [13]. It is given by neglecting the difference between \( \rho_A \) and \( \rho_A^\circ \) in the expressions for \( \frac{\partial \rho_A^\circ}{\partial t} \), as well as neglecting the difference between \( \rho(t) \) and \( \rho(t') \) in \( \frac{\partial \rho_A^d}{\partial t} \). Since these differences are higher order in the perturbation, \( V \), this neglect is justified over time scales which are short enough to lay within the domain of perturbation theory. With these approximations, the perturbative expression for the sum of eqs. (10) and (12) becomes:

\[
\frac{\partial \rho_A}{\partial t} = \frac{\partial \rho_A^\circ}{\partial t} + \frac{\partial \rho_A^d}{\partial t} = -i \left[ V_2(t) \rho_A(t) - \rho_A(t) V_2(t) \right] + \int_{t'}^t d\tau \text{Tr}_B \left[ (\delta V(t) \rho(t) \delta V(\tau) + \delta V(\tau) \rho(t) \delta V(t)) O_B \right] + O(V^3),
\]

where \( V_2(t) \) denotes the second-order expression, eq. (20).

If the correlation scale, \( \tau \), is now assumed to be small compared to the time scale over which the coarse graining is taken, then we may neglect correlations on the right-hand-side of eq. (23), by writing \( \rho(t) \approx \rho_A(t) \otimes \rho_B \). With this choice eq. (23) describes a Markov-like process, for which \( \frac{\partial \rho_A}{\partial t} \) depends only on \( \rho_A(t) \), and not on the behaviour of \( \rho_A \) for times previous to \( t \). This represents a great simplification once eq. (23) is integrated to obtain the evolution of \( \rho_A \) for very long times. It is this form of the time-evolution equations which is used in Section 6 for describing neutrino evolution within the sun.
2.5) Unitarity and Decoherence

There are two general features of the above expressions which bear special emphasis. Notice first that the condition $\text{Tr } \rho = 1$ implies the same is true for the reduced density matrix (when $O_B = I$): $\text{Tr}_A \rho_A = 1$. This is easily seen to follow from eqs. (10), (20) and (22) by virtue of the following identity, which expresses the optical theorem in the present example:

$$\frac{\partial}{\partial t} \text{Tr}_A \rho_A = -i \text{Tr}_A \left\{ \rho_A^c(t) \left[ \mathbf{V}(t) - \mathbf{V}^*(t) \right] \right\} + \text{Tr}_A \frac{\partial \rho_A^d}{\partial t} = 0. \quad (24)$$

Second, $\partial \rho_A^d/\partial t$ and $\mathbf{V} - \mathbf{V}^*$ both cause a loss of coherence within sector $A$. That is, if the system is initially prepared in a pure state, for which $\rho_A^2 = \rho_A$, then it need not remain so after interacting with sector $B$. The endpoint of evolution is therefore generally a mixed state. Quantitatively, starting from an initially pure state eqs. (10) and (12) imply the following rate of coherence loss:

$$\frac{\partial}{\partial t} \left( \rho_A^2 - \rho_A \right) \big|_{\rho_A^2 = \rho_A} = -i \rho_A \left( \mathbf{V} - \mathbf{V}^* \right) \rho_A + \left\{ \rho_A, \frac{\partial \rho_A^d}{\partial t} \right\} - \frac{\partial \rho_A^d}{\partial t}, \quad (25)$$

where we have neglected the difference between $\rho_A$ and $\rho_A^c$ in the first term on the right-hand-side. When this is nonzero it clearly makes no sense to define the time evolution in terms of the Schrödinger evolution, $i \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$, for a pure state: $\rho_A = |\psi\rangle \langle \psi|.$

2.6) Recursiveness

Notice that the definitions of the effective hamiltonian, $\mathbf{V}$, and of the diffuse scattering term, $\partial \rho_A^d/\partial t$, are recursive, in the following sense. Suppose that sector $A$ in the previous discussion were itself to be divided into independent subsectors, $A'$ and $B'$, and only observables acting in subsector $A'$ were measured. Suppose also that the initial state did not involve any correlations between these two subsectors: $\rho_A = \rho_A' \otimes \rho_B'$. Then we may further reduce the density matrix to act only within this subsector:

$$\rho_{A'}(t) \equiv \text{Tr}_{B'} \left[ \rho_A(t) \right] = \text{Tr}_{B' \cup B} \left[ \rho(t) \right]. \quad (26)$$

Then we define the coherent and diffuse part of the evolution in sector $A'$ in such a way as to ensure that the coherent part takes the same form regardless of whether the
trace over sectors $B$ and $B'$ are performed separately, or all at once. That is, with the definitions

$$
\overline{U}_B \equiv \text{Tr}_B[\rho_B U] \quad \text{and} \quad \overline{U}_{B'} \equiv \text{Tr}_{B'}[\rho_{B'} \overline{U}_B] = \text{Tr}_{B \cup B'}[(\rho_B \otimes \rho_{B'})U],
$$

and

$$
\Delta U_B \equiv U - \overline{U}_B \quad \text{and} \quad \Delta U_{B'} \equiv \overline{U}_B - \overline{U}_{B'},
$$

we have $\text{Tr}_B[\rho_B \Delta U_B] = \text{Tr}_{B'}[\rho_{B'} \Delta U_{B'}] = 0$, and so the expectation of any observable in only sector $A'$ may be written:

$$
\langle O_{A'} \rangle (t) \equiv \text{Tr}_A[\rho(t)O_{A'}] = \text{Tr}_A[\overline{U}_B(t,t')\rho_{A'} \overline{U}_B^*(t,t')O_{A'}] + \text{Tr}_A[\Delta U_B(t,t')\rho(t')\Delta U_B^*(t,t')O_{A'}]
$$

$$
= \text{Tr}_A[\overline{U}_{B'}(t,t')\rho_{A'} \overline{U}_{B'}^*(t,t')O_{A'}] + \text{Tr}_A[\Delta U_{B'}(t,t')\rho_{B'} \Delta U_{B'}^*(t,t')O_{A'}] + \text{Tr}[\Delta U_B(t,t')\rho(t')\Delta U_B^*(t,t')O_{A'}].
$$

A similar expression holds for $\frac{\partial \rho_{A'}}{\partial t}$:

$$
\frac{\partial \rho_{A'}}{\partial t} = -i [\nabla_{B'}(t)\rho_{A'}(t) - \rho_{A'}(t)\nabla_{B'}^*(t)] + \frac{\partial \rho_{B'}^d}{\partial t} + \frac{\partial \rho_B^d}{\partial t},
$$

where:

$$
\rho_{B'}^d \equiv \text{Tr}_{B'}[\Delta U_{B'}(t,t')\rho(t')\Delta U_{B'}^*(t,t')]
$$

$$
\rho_B^d \equiv \text{Tr}_{B' \cup B}[\Delta U_B(t,t')\rho(t')\Delta U_B^*(t,t')].
$$

The effective hamiltonians, $\nabla_B(t)$ and $\nabla_{B'}(t)$, are respectively defined, as usual, in terms of $\overline{U}_B(t,t')$ and $\overline{U}_{B'}(t,t')$, using eq. (11). In particular, using the notation $\langle \cdots \rangle_{B'} = \text{Tr}_{B'}[\cdots \rho_{B'}]$, we have the following very useful perturbative expression:

$$
\nabla_{B'}(t) = \langle \langle V(t) \rangle_{B'} \rangle_{B'} - i \int_{t'}^t dt' \left\{ \left[ \langle \langle V(t) \rangle_{B'} \langle V(\tau) \rangle_B \rangle_{B'} - \langle \langle V(t) \rangle_B \langle V(\tau) \rangle_{B'} \rangle_{B'} \right] + \left[ \langle \langle V(t) \rangle_B \langle V(\tau) \rangle_{B'} \rangle_{B'} - \langle \langle V(t) \rangle_{B'} \langle V(\tau) \rangle_B \rangle_{B'} \right] \right\} ,
$$

$$
= \langle \langle V(t) \rangle_{B'} \rangle_{B'} - i \int_{t'}^t dt' \left\{ \langle \langle V(t) \rangle_{B'} \rangle_{B'} - \langle \langle V(t) \rangle_{B'} \rangle_{B'} \right\} .
$$
The recursive nature of these definitions is a very attractive feature. This is because it lends itself to a renormalization-group-like analysis of the effects of a medium on particle propagation, in which the effects of fluctuations on successively larger distance scales are separately integrated out.

2.7) Neutrino Interactions

Our later applications use the formalism just presented for the special case of neutrinos interacting with matter through the weak interactions. (Magnetic moment interactions are briefly considered in Section 7.) We therefore pause here to gather the relevant expressions for the interaction hamiltonian. None of the details of the nature of the medium are needed at this point, although we suppose for simplicity that it does not include a significant component of neutrinos themselves. This permits a clean separation between the neutrino states and the states which are available to the medium.

The neutrinos may be described by $N_\nu$ majorana neutrino fields, $\nu_i$, $i = 1, \ldots, N_\nu$, without loss of generality. In the absence of light sterile neutrinos this consists of the usual ($N_\nu = 3$) neutrino eigenstates. The coupling between neutrinos and the medium is mediated by the weak interactions. To keep as broad as possible the applications of this section we choose:

$$\mathcal{L} = i \bar{\nu}_i \gamma_\mu \left( \gamma_L g^a_{ij} + \gamma_R h^a_{ij} \right) \nu_j J^\mu_a, \quad (33)$$

where $\gamma_L$ and $\gamma_R$ project onto left- and right-handed spinors; $J^\mu_a$ are a set of hermitian operators involving the degrees of freedom of the medium, and $g^a_{ij}$ and $h^a_{ij}$ are corresponding $N_\nu \times N_\nu$ matrices of couplings. The reality of $\mathcal{L}$ implies $g^a_{ij}$ and $h^a_{ij}$ must all be hermitian.

Since the most important applications are to the Standard Model (SM), possibly supplemented by various sterile neutrinos and/or neutrino masses, we record explicit expressions for the quantities, $J^\mu_a$, $g^a_{ij}$, and $h^a_{ij}$ in this case. The couplings to charged leptons, $\ell_m$, are given by:

$$\begin{align*}
(J^\mu_a)_{mn} &= i \bar{\ell}_m \gamma^\mu \left( 1 \pm \gamma_5 \right) \ell_n, \\
\text{with } (g^m_{ij})_{ij} &= \sqrt{2} G_F \left[ V_{mj} V_{ni}^* + N_{ij} \left( -\frac{1}{2} + s_w^2 \right) \right], \\
(g^m_{ij})_{ij} &= \sqrt{2} G_F N_{ij} s_w^2, \\
\text{and } (h^{m\pm}_{ij})_{ij} &= 0,
\end{align*} \quad (34)$$

---

3 Although more than adequate for applications to the sun, this assumption can break down for supernovae or the early universe.

4 Our conventions are $\gamma_L = \frac{1}{4}(1 + \gamma_5)$, $\gamma_R = \frac{1}{4}(1 - \gamma_5)$ and $\nu = i \nu^\dagger \gamma^0 = -i \nu^\dagger \gamma_0$. 

15
where $V_{mi}$ are the leptonic CKM matrix elements which arise in the charged-current couplings once neutrinos acquire masses, and $N_{ij}$ are the analogous matrices which can arise in the neutrino neutral-current couplings. $s_w$ denotes, as usual, the weak mixing angle. The corresponding couplings to hadrons arise through their quark content:

$$(J^\mu_\pm)_a = i\bar{q}_a\gamma^\mu(1 \pm \gamma_5)q_a,$$

with

$$(g^a_+)(ij) = \sqrt{2}G_FN_{ij}(T^a_3 - Q_as^2_w),$$

$$(g^a_-)(ij) = \sqrt{2}G_FN_{ij}(-Q_as^2_w),$$

and

$$(h^a_\pm)(ij) = 0,$$

where $T^a_3$ and $Q_a$ are the third component of weak isospin and electric charge of the corresponding quark.

For many practical applications the energies involved are sufficiently low that the relevant hadronic degrees of freedom are just protons and neutrons. In this case we may approximate eqs. (35) by the following effective macroscopic currents:

$$(g^a_+)(ij)(J^\mu_+)_a + (g^a_-)(ij)(J^\mu_-)_a \approx \frac{G_F}{\sqrt{2}}N_{ij}[(1 - 4s^2_w)i\bar{p}\gamma^\mu p - i\bar{n}\gamma^\mu n] + (\text{axial-current and higher-derivative terms}),$$

We now use these expressions to compute the quantities $V$ and $\partial\rho\partial t$ in two different regimes. We first consider the case for which the fluctuations of interest occur on scales which are microscopic in comparison to those relevant to neutrino propagation. This is followed, in Section 4, by a consideration of macroscopic fluctuations, for which correlation lengths are much larger.

3. Microscopic Fluctuations

Consider first the case of fluctuations having microscopic characteristic correlation lengths. For neutrinos in the sun this includes the thermal fluctuations among the particles making up the solar interior, and so the expressions obtained in this section may in this case be tested against standard results.

Our goal is to compute the quantities $\overline{V}$ and $\frac{\partial\rho^d}{\partial t}$ of Section 2. In order to apply the formalism we must first split the system into sectors, A and B. We choose $A = \nu$ to consist of all of the states in the neutrino sector, while sector $B = E$ (‘environment’) represents
the sector describing the other particles — electrons, nucleons *etc.* — through which the neutrinos propagate.\(^5\)

At first order in the interaction, eq. (33), we have \(\Delta U = 0\) and:

\[
\mathbf{V}_1 = \langle \mathbf{V} \rangle_E = - \int d^3 x \, i \mathcal{V}_i \gamma_\mu \left( \gamma_\mu g_{ij}^a \gamma^a_h \right) \nu_j \langle \mathbf{J}_a \rangle_E. \tag{37}
\]

To this order neutrino evolution is simply described by replacing the interaction current, \(J_a^\mu\), with its mean, \(\langle J_a^\mu \rangle_E = \text{Tr}_E \left[ \varrho_E J_a^\mu \right]\), evaluated using \(\varrho_E(= \varrho_B)\), which describes the initial state of the medium through which the neutrinos pass.

More can be said about the mean currents, \(j_a^\mu(x) = \langle J_a^\mu(x) \rangle_E\), given more information about the state \(\varrho_E\). With the sun in mind we take this to describe a mixture of nonrelativistic electrons, protons and neutrons which are mutually interacting dominantly through the electromagnetic and strong interactions. This permits the use of parity invariance to limit the form taken by the mean currents. Moreover, we also work in an electroweak basis, for which \(N_{ij} = \delta_{ij}\) and \(V_{mi} = \delta_{mi}\) for the three usual neutrinos, and \(N_{ij} = V_{mi} = 0\) for any light sterile neutrinos. (The influence of neutrino masses is described in more detail once the trace over neutrino momentum states is performed in Section 5.) With these choices we obtain the usual estimate for the mean matter currents. We find \(\langle h_{ij}^a J_a^\mu \rangle_E = 0\), and:

\[
\langle g_{ij}^a J_a^\mu \rangle_E \approx \frac{G_F}{\sqrt{2}} \left\{ 2\delta_{ie} \delta_{je} j_e^\mu(x) - \delta_{ij} j_n^\mu(x) + \delta_{ij} (1 - 4s_w^2) \left[ j_p^\mu(x) - j_e^\mu(x) \right] \right\}, \tag{38}
\]

where \(j_e^\mu = i \bar{e} \gamma^\mu e\), \(j_p^\mu = i \bar{p} \gamma^\mu p\) and \(j_n^\mu = i \bar{n} \gamma^\mu n\) are respectively the local electron, proton and neutron currents. This expression is obtained by averaging eqs. (34) and (35), using eq. (36). The axial-vector parts of the weak currents drop out of eq. (38) by virtue of parity invariance of the solar medium. Notice that, although the neutron current distribution is independent of the others, for slowly moving particles local electric neutrality implies \(j_n^\mu(x) = j_p^\mu(x)\), and so the terms in the last equation which are proportional to \((1 - 4s_w^2)\) cancel. For nonrelativistic particles there is also a further simplification, since we may neglect of all of the spatial components of the mean currents: \(j_a^\mu \approx n_a \delta^\mu_0\).

At second order in the weak interactions two new things happen. First, \(\mathbf{V}\) acquires a second-order correction, typically introducing to it an antihermitean part. Second, \(\frac{\partial \rho}{\partial t}\)

\(^5\) Our use of subscripts ‘\(\nu\)’ and ‘\(E\)’, in place of ‘\(A\)’ and ‘\(B\)’, is meant to avoid confusion with the different choice for \(A\) and \(B\) which is made in the subsequent sections.
becomes nonzero, introducing decoherence into any propagating neutrino state. We find:

\[
V_2 = i \int_{t'}^t d\tau d^3 x d^3 x' \overline{\nu}_i \gamma_\mu (\gamma_L g_{ij}^a + \gamma_R h_{ij}^a) \nu_j \overline{\nu}_k \gamma_\lambda (\gamma_L g_{kl}^b + \gamma_R h_{kl}^b) \nu'_l \left\langle \delta J_\mu^a \delta J_\lambda^b \right\rangle_E,
\]

and

\[
\left( \frac{\partial \rho_\nu^d}{\partial t} \right)_2 = -2 \int_{t'}^t d\tau d^3 x d^3 x' \left\{ \overline{\nu}_i \gamma_\mu (\gamma_L g_{ij}^a + \gamma_R h_{ij}^a) \nu_j \rho_\nu \overline{\nu}_k \gamma_\lambda (\gamma_L g_{kl}^b + \gamma_R h_{kl}^b) \nu'_l \left\langle \delta J_\mu^b \delta J_\lambda^a \right\rangle_E + (t \leftrightarrow \tau) \right\}.
\]

In these expressions \(\rho_\nu(= \rho_\lambda)\) denotes the neutrino density matrix at time \(t\). A prime on any field denotes that it is evaluated at the spacetime point \((x', \tau)\) — e.g. \(\nu' = \nu(x', \tau)\) — while all unprimed fields are evaluated at \((x, t)\).

At this point we use the information that the scale of the fluctuations, \(\ell\), are microscopic in comparison with the distances of interest for neutrino propagation. This means the correlations, \(\left\langle \delta J_\mu^a \delta J_\lambda^b \right\rangle_E\), can be written in the approximate form

\[
\left\langle \delta J_\mu^a \delta J_\lambda^b \right\rangle_E \approx C_{\mu\lambda}^{ab} \delta^3(x - x'),
\]

where the coefficient functions, \(C_{\mu\lambda}^{ab}\), are explicitly calculable given the state \(\rho_E\) — a point to which we return below.

Eq. (41) permits eqs. (39) and (40), for \(V_2\) and \(\left( \frac{\partial \rho_\nu^d}{\partial t} \right)_2\), to be written as follows:

\[
V_2 = i \int_{t'}^t d\tau d^3 x C_{ab}^{\mu\lambda} \overline{\nu}_i \gamma_\mu (\gamma_L g_{ij}^a + \gamma_R h_{ij}^a) \nu_j \overline{\nu}_k \gamma_\lambda (\gamma_L g_{kl}^b + \gamma_R h_{kl}^b) \nu'_l,
\]

and

\[
\left( \frac{\partial \rho_\nu^d}{\partial t} \right)_2 = -2 \int_{t'}^t d\tau d^3 x C_{ba}^{\lambda\mu} \overline{\nu}_i \gamma_\mu (\gamma_L g_{ij}^a + \gamma_R h_{ij}^a) \nu_j \rho_\nu \overline{\nu}_k \gamma_\lambda (\gamma_L g_{kl}^b + \gamma_R h_{kl}^b) \nu'_l.
\]

Notice that these interactions describe processes, such as \(\nu\nu \rightarrow \nu\nu\) or \(\nu \rightarrow \nu\nu\nu\), in which neutrinos scatter from medium-dependent fluctuations. Similar interactions are familiar for electromagnetic propagation through matter, where the analogs of eqs. (42) and (43) are quadratic in the electromagnetic field and so describe the scattering of electromagnetic waves by microscopic fluctuations [14]. (Similar quadratic terms also arise for
neutrinos at second order in their charged-current interactions with the particles in the bath.) Since the matter-dependent effects depend differently on neutrino energies than do the same processes in vacuo they can, in principle, be separated from one another.

The potential size of the matter-dependent correlation coefficients, $C^\mu\lambda_{ab}$, may be estimated by computing them for thermal fluctuations in a system of nonrelativistic particles in local thermal equilibrium\(^6\) and for which, for simplicity, we imagine there is a single conserved particle number, $\mathcal{N} = \int d^3x \, J^0$, to whose current the neutrinos couple.

For thermal fluctuations we compute all averages over $E$ using the density matrix, $\varrho_E$, of the grand canonical ensemble:

$$\varrho_E = Z^{-1} e^{-\frac{(H_E - \mu \mathcal{N})}{T}}. \quad (44)$$

$Z$ here is the standard normalization constant: $Z = \text{Tr}_E e^{-\frac{(H_E - \mu \mathcal{N})}{T}}$. More generally, we consider media which are in local thermal equilibrium, and so for which $\varrho_E$ has a similar form, but with mean thermodynamic properties which vary (over macroscopic distances) from place to place. The grand canonical ensemble is the one which is locally appropriate for this case [15], since the number of particles in any local region of the medium is not fixed.

With these choices we may compute the local fluctuations of $\mathcal{N}$. Using the assumption that the constituents of the medium are nonrelativistic, we neglect all but the time component of the current: $\langle J^\mu(x) \rangle_E = n(x) \delta^\mu_0$. We find [16]:

$$C^\mu\lambda = \delta^\mu_0 \delta^\lambda_0 \left[ n^2 \kappa_T T + f_\ell(x) \right], \quad (45)$$

where $T(x)$ is the local temperature, and $\kappa_T(x)$ is the system’s specific isothermal compressibility: $\kappa_T = \frac{1}{n} \left( \frac{\partial n}{\partial p} \right)_T$, where $p$ is the pressure.

In eq. (45), $f_\ell(x)$ is a function whose scale of variation is the microscopic fluctuation length, $\ell$, and which satisfies the defining condition: $\int d^3x \, f_\ell(x) = 0$. Because of these conditions, $f_\ell(x)$ can be neglected for macroscopic applications, such as when eq. (45) is used in eqs. (42) and (43) to describe neutrino evolution over scales which are much larger than $\ell$.

Substituting eq. (45) into expressions (42) and (43), and using the results in eq. (10), reproduces the usual expressions [6][7][8] for neutrino scattering from a thermal ensemble

---

\(^6\) This application presumes not being near a critical point for which thermal fluctuations need not be microscopic in size.
of particles. This can be seen by using the ideal-gas equation of state, \( p = nT \), in which case \( \kappa_T = \frac{1}{nT} \). It follows that the combination \( n^2 \kappa_T T \) — which through eq. (45) governs neutrino scattering — simply reduces to the particle density, \( n \). This leads to a neutrino scattering rate, \( \Gamma \sim \sigma n \sim G_F^2 m E n \), which for solar neutrinos in the sun’s centre (\( E \sim 1 \) MeV and \( n_c \sim 10^{26} / \text{cm}^3 \) ) scattering from nucleons (\( m \sim 1 \) GeV) is negligibly small: \( \Gamma^{-1} \sim 10^{10} \) km.

4. Macroscopic Density Fluctuations

Our second application of the formalism of section 2 is to macroscopic variations in the mean currents, \( \langle J^\mu_a \rangle_E \), that arise in eq. (37). We do so partly because this source of fluctuations has until recently been ignored in the literature. More importantly, this type of fluctuation can produce effects which are much larger than those which arise microscopically. There are many situations in electromagnetism for which macroscopic fluctuations can furnish the dominant medium-dependent effects. A familiar example is furnished by the case of light propagating through a cloud. In this case the cloud is opaque because of density fluctuations on the scale of the water droplets which make up the cloud, rather than fluctuations on more microscopic scales.\(^7\) Our purpose here, and in Section 4, is to analyze the analogue of such fluctuations for neutrinos.

Eq. (45) implies that thermal fluctuations have negligible effects for neutrinos passing through the sun, so our starting point is the mean hamiltonian, eq. (37), which describes neutrino propagation after averaging over microscopic matter fluctuations:

\[
V_E = -\int d^3x \, j^\mu_a(x) \, i \bar{\nu}_i \gamma_\mu (\gamma_L g^a_{ij} + \gamma_R h^a_{ij}) \nu_j, \tag{46}
\]

where we write \( j^\mu_a(x) = \langle J^\mu_a(x) \rangle_E \) for the mean current.

Now comes the main point. For any fixed current profile, \( j^\mu_a(x) \), the propagation of any particular neutrino through the sun is perfectly well described by pure-state evolution using the mean hamiltonian given in eq. (37)(or, equivalently, eq. (46)). (This point is demonstrated in detail in Section 5.) It is, however, not in general true that successive neutrinos see the same profile, \( j^\mu_a(x) \). On the contrary, successive neutrinos arriving at a detector may have been produced at different places within the sun and so can pass

\(^7\) Of course, this analogy can be misleading if applied too literally to neutrino physics, since the absence of multiple scattering precludes neutrinos from ‘refracting’ from a large scale density fluctuation in the sun in the same way that light refracts through a water droplet.
through entirely different density profiles while *en route* to the earth. Alternatively, the density profile itself can change in the interval between the passage through the same region of different neutrinos. As a result, the neutrino flux to which a detector is exposed can be thought to have been processed through a constantly changing kaleidoscope of density profiles.

We wish to adapt the formalism of Section 2 to describe the influence on neutrinos of this eternally varying current profile. We do so by modelling these density variations as being random in character. We therefore consider passing neutrinos through an *ensemble* of density profiles — whose properties are elaborated below — over which we must average to obtain the neutrino signal as seen by a detector on earth. Taking advantage of the recursive nature of the formalism of Section 2, we may simply take these formulae over in whole cloth, but with the mean hamiltonian of eq. (37) now interpreted as the microscopic hamiltonian, and with the averages over sector $B = \mathcal{E}$ (‘\(\mathcal{E}\)nssemble’) now interpreted as ensemble averages.

The results are immediate. To first order in $V_E$, the mean hamiltonian after the ensemble average now becomes:

$$
\overline{V}_1 = -\int d^3 x \langle j^\mu_a(x) \rangle_\mathcal{E} i \overline{\nu_i} \gamma_\mu (\gamma_L g^a_{ij} + \gamma_R h^a_{ij}) \nu_j.
$$

(47)

Similarly, at second order we find:

$$
\overline{V}_2 = i \int_{t'}^t d\tau d^3 x d^3 x' \overline{\nu}_i \gamma_\mu (\gamma_L g^a_{ij} + \gamma_R h^a_{ij}) \nu_j \\
\overline{\nu}'_k \gamma_\lambda (\gamma_L g^b_{kl} + \gamma_R h^b_{kl}) \nu'_l \langle \delta j^\mu_a \delta j^\nu_b \rangle_\mathcal{E},
$$

(48)

and

$$
\left( \frac{\partial \rho'_\nu}{\partial t} \right)_2 = -\int_{t'}^t d\tau d^3 x d^3 x' \left\{ \overline{\nu}_i \gamma_\mu (\gamma_L g^a_{ij} + \gamma_R h^a_{ij}) \nu_j \\
\rho_v \overline{\nu}'_k \gamma_\lambda (\gamma_L g^b_{kl} + \gamma_R h^b_{kl}) \nu'_l \langle \delta j^\nu_a \delta j^\lambda_b \rangle_\mathcal{E} + (t \leftrightarrow \tau) \right\},
$$

(49)

where $\delta j^\nu_a(x) \equiv j^\nu_a(x) - \langle j^\nu_a(x) \rangle_\mathcal{E}$.

What remains is to estimate the ensemble averages, $\langle j^\nu_a(x) \rangle_\mathcal{E}$ and $\langle \delta j^\lambda_a(x) \delta j^\nu_b(x') \rangle_\mathcal{E}$, which appear in these expressions. A key difference between these averages and those considered previously is that we may no longer assume the currents to be delta-correlated, as in eq. (41).
4.1) The Ensemble Properties

The precise nature of these ensemble averages depends on the kinds of physics which is responsible for the varying currents that successive neutrinos see. It is useful to have a systematic framework within which to couch our later models of these fluctuations. This section outlines such a framework.

Suppose, then, that the currents $j^{\mu}_{a}(x)$ are expanded in terms of a complete set of orthonormal functions, $\phi_{N}(x)$, as follows:

$$ j^{\mu}_{a}(x) = \overline{j^{\mu}_{a}(x)} \left[ 1 + \sum_{N} C_{aN} \phi_{N}(x) \right]. \tag{50} $$

We take the coefficients, $C_{aN}$, to be random variables having vanishing mean, which are uncorrelated for different modes:

$$ \langle C_{aN} \rangle_{E} = 0 \quad \text{and} \quad \langle C_{aN} C_{bM} \rangle_{E} = C_{abN}^{2} \delta_{NM}. \tag{51} $$

This implies the following for the density distributions themselves:

$$ \langle j^{\mu}_{a}(x) \rangle_{E} = \overline{j^{\mu}_{a}(x)}, \quad \langle \delta j^{\mu}_{a}(x) \delta j^{\nu}_{b}(x') \rangle_{E} = \overline{j^{\mu}_{a}(x)} \overline{j^{\nu}_{b}(x')} \sum_{N} C_{abN}^{2} \phi_{N}(x) \phi_{N}(x'). \tag{52} $$

Notice that the completeness of the basis functions implies that the currents, $j^{\mu}_{a}(x)$, become microscopically correlated, $\langle \delta j^{\mu}_{a}(x) \delta j^{\nu}_{b}(x') \rangle_{E} \propto \delta(x - x')$, if $C^{2}_{abN}$ should be the same for all $N$.

Clearly it cannot be true that the variables, $C_{aN}$, are uncorrelated for all choices of basis functions. Different physical origins for the underlying randomness can lead to a different choice for the preferred, uncorrelated, basis. In what follows we use the following two models for the density fluctuations in the sun.

- **Locally-Varying Density Fluctuations:**

  As our first model of solar fluctuations we picture $j^{\mu}_{a}(x)$ to be varying randomly from place to place. Motivated by the picture of the solar medium consisting of turbulent regions of fluid we imagine dividing the sun into cells, labelled by the index $N$, whose volume we
denote by \( V_N \). We permit this volume to vary slowly as one moves around the sun. We then choose the basis functions to be:

\[
\phi_N(x) = \begin{cases} 
V_N^{-\frac{1}{2}} & \text{if } x \text{ lies within cell } N, \\
0 & \text{otherwise.}
\end{cases}
\]  

(53)

With this choice we find the correlations:

\[
\langle \delta j^\mu_a(x) \delta j^\nu_b(x') \rangle_E = \begin{cases} 
\epsilon_{ab}^2 V_N & \text{if } x \text{ and } x' \text{ both lie within cell } N, \\
0 & \text{otherwise.}
\end{cases}
\]  

(54)

We introduce here the dimensionless quantity \( \epsilon_{abN} \) by:

\[C_{abN}^2 \equiv \epsilon_{abN}^2 V_N,\]

to remove the dependence on the cell volume, \( V_N \), which enters due to the normalization condition for the basis functions, \( \phi_N(x) \).

How big might \( \epsilon_{abN} \) and \( V_N \) reasonably be expected to be? For solar applications the convective zone is known to contain density variations on many scales [17]. Granules on the solar surface are \( \sim 100 \) km across. Giant convection cells are believed to have dimensions which are comparable to the depth of the convective zone itself: \( \sim 2 \times 10^5 \) km. Of more interest for neutrino propagation are the scales at the depths where neutrinos are produced, and where they resonate. Unfortunately, both of these regions lie within the radiative zone, where intuition based on the convective zone is unlikely to apply. Our analysis in subsequent sections of how these fluctuations modify MSW oscillations indicates that \( \epsilon^2 \ell > 100 \) m is the range which is likely to have phenomenologically interesting implications for neutrino oscillations, where \( \ell \) is the length of a typical cell in the direction of neutrino motion.

This type of random model is very much in the spirit of refs. [10] and [12], for which the electron density is modelled as a random variable that is delta-correlated in space. In fact, eq. (54) directly reduces to the ensemble used in ref. [10] in the limit of negligible correlation length (taken in ref. [10] to be \( \ell = 10 \) km), and when \( \epsilon^2 \ell \) is taken to be constant. Ref. [12] makes a slightly different choice, ensuring a small correlation length by continually adjusting \( \ell \) to be a tenth of the neutrino matter oscillation length, as this varies throughout the sun. Differences between our results and those of ref. [12] do arise for some regimes, which we believe to be due to this difference in treatment of the correlation length.

Besides not assuming negligibly small correlation lengths, a more important difference between eq. (54) and refs. [10] and [12] is that we may take the fluctuations to vary differently as a function of position and time, as may be appropriate for some kinds of solar physics. Our next example presents an illustration of such a case.
Helioseismic Waves: Oscillatory Normal Modes:

Our second simple model of fluctuations in the sun is meant to model helioseismic $p$-waves. As a source of fluctuations through which neutrinos propagate, helioseismic waves have the great advantage of actually being known to exist. Furthermore, a fair amount is known about the spectrum and amplitude of these waves [18].

In order to strip away as much extraneous detail as possible, we start here with a simple wave within a rectangular geometry. (We report elsewhere on the results of more detailed modelling of neutrino interactions with helioseismic waves [19].) That is, we choose our neutrinos to be moving up the positive $z$ axis, through a medium whose length in the $z$ direction is $L = 2R_\odot$. A basis of modes which vanishes at the boundaries of this volume is given by:

$$
\phi_+ = N_+ \cos \left( \frac{2\pi z}{\ell_+} \right) \cos \left( \frac{2\pi t}{\tau_+} \right),
$$

$$
\phi_- = N_- \sin \left( \frac{2\pi z}{\ell_-} \right) \cos \left( \frac{2\pi t}{\tau_-} \right),
$$

where we take the period and wavelength to be related in terms of the speed of sound, $c_s$. To start with we take $c_s$ to be a constant, but we also present some results with $c_s = c_s(z)$ chosen to more accurately mimic the properties of the sun. For constant $c_s$, momentum is conserved in the $z$ direction, and $\ell_\pm$ are determined by the boundary conditions to be a positive integer: $\ell_+ = L/(n - \frac{1}{2})$ and $\ell_- = L/n$.

Finally, $N_\pm = N_\pm(r_\perp)$, denotes the dependence of the modes on the two other directions, $r_\perp = (x,y)$, transverse to the direction of neutrino motion. For a $z$-dependent speed of sound, $c_s(z)$, we take $N_\pm$ to also be plane waves, as in eq. (55), labelled by the conserved transverse momentum, $k_\perp = k_x e_x + k_y e_y$. For slowly varying wave amplitudes we then take the wave number in the $z$-direction to be given by

$$
k_z^2 = \frac{\omega^2}{c_s^2(z)} - k_\perp^2,
$$

with $\omega = 2\pi/\tau_\pm$. In our subsequent numerical applications we take the wave amplitude to vanish for those $z$ for which the resulting $k_z$ is imaginary.

There are several types of physics which might be expected to produce a normal modes with a randomly varying amplitude. First, even if the sun were to be oscillating with a single mode, the amplitude of this mode as seen by successive neutrinos would differ. This is because, although any one neutrino sees an essentially static density profile, (since the time scale for neutrinos to entirely escape the sun is quite short — $\ell \leq L \sim$ several seconds...
compared to typical wave periods — \( \tau \sim \) several minutes) successive neutrinos can catch the wave at differing points in its cycle. Neutrinos passing through at random times would therefore see a wave with a randomly varying amplitude.

Of course, the real sun does not simply ring with constant amplitude because various (poorly understood) processes permit energy to be transferred into and out of the various normal modes. This leads to additional randomness to the mode amplitudes, as seen from the neutrino’s perspective.

5. Tracing Over the Neutrino Momenta

We now fill in the neutrino part of the picture, and compute how the fluctuations considered in the preceding Sections can enter into single-particle neutrino evolution. To this end we again apply the results of Section 2 to trace over the momentum/position part of the single-particle neutrino Hilbert space, with the goal of deriving the explicit form for eq. (10) acting in neutrino flavour space, with which we can analyze resonant neutrino oscillations.

Two issues must be borne in mind when applying the results of Section 2 in this way. First, since we wish to keep track of the second-order effects, described above, due to matter fluctuations, we must use the recursive form for the effective description which was given in section 2.5. We therefore divide the neutrino sector itself into two subsectors, \( A' \) and \( B' \), with \( B' \) consisting of the span of all of the momentum states of the neutrino sector, whilst \( A' \) comprises the sector labelled by neutrino spins and flavours. To avoid confusion with our earlier choices for \( A \) and \( B \), we introduce the new notation \( A' = F \) (‘flavour’) and \( B' = P \) (‘position’) for this part of the analysis.

Second, since all practical measurements of neutrino flavour also involve a position measurement — i.e. neutrino \( x \) is measured to be of flavour \( y \) when it arrived at point \( z \) — we must also remember to adopt the formulation of Section 2.2, in which we perform a partial measurement in sector \( B' = P \). This innocuous point has important implications for the form of the fluctuation terms in the evolution equations for the reduced density matrix in flavour/spin space.

To proceed we must choose the density matrix which describes the initial neutrino state. Assuming that the neutrino flavour/spin sector is initially uncorrelated with the neutrino momentum, \( \varrho_\nu = \varrho_\nu \otimes \varrho_p \), we must choose an explicit form for \( \varrho_p \), in order to evolve the spin/flavour state, \( \rho_\nu \), forward in time. For a single neutrino, we would take this to be a pure, single-particle state, \( \varrho_p = |\psi\rangle\langle\psi| \), describing an outgoing spherical wave packet which starts at \( t = t' = 0 \) at the nucleus whose fusion produced the neutrino. The
spatial width, $\xi$, of this packet we imagine to be of negligible, microscopic, dimensions. Since applications to neutrino oscillations involve observations of this wave a very long way away from its centre, it suffices for our purposes to dispense with the spherical geometry and treat $\psi_{k,z_0}(p)$ as a plane wave packet, starting at $z = z_0$ at $t = 0$, and travelling along the $z$ axis with average momentum, $k \sim \text{MeV}$:

$$\langle p|\psi_{k,z_0}\rangle = \psi_k(p) = \left[\frac{2\xi}{\sqrt{2\pi}}\right]^\frac{1}{2} e^{-\xi^2(p_x-k)^2-ipy z_0} \delta(p_x) \delta(p_y). \quad (57)$$

$|\psi_{k,z_0}\rangle$ so defined is continuum normalized in the $x$ and $y$ directions. For solar neutrinos this pure state must be averaged over the initial distribution for producing such a neutrino within the sun.

Next we must define the partial neutrino position measurement, which is meant to express the fact that we know where neutrino measurements are performed: the Earth. We therefore choose observables of the form (13) ($O = O_F \otimes O_P$) with

$$O_P(t_m) \equiv |r; t_m\rangle\langle r; t_m|, \quad (58)$$

corresponding to neutrino detection at the point, $r$, at a measurement time $t = t_m$. For a long-term exposure to a constant flux (such as for solar neutrinos) we integrate over the appropriate range for $t_m$.

5.1) First-Order Effects

With these choices we may now evaluate the quantity $\nabla_1$. (Second order effects due to the averaging over neutrino momenta are explored in the next section.) This will reproduce the usual MSW hamiltonian. The proper description of neutrino scattering, including the effects of matter fluctuations, is therefore found by tracing eqs. (47), (48) and (49) (or, for microscopic fluctuations, eqs. (37), (39) and (40)) over the neutrino momentum sector.

This trace is straightforward to perform, subject to two important approximations.

• Negligible Neutrino Masses: The first of these is the assumption, previously encountered in Section 3, that all neutrino masses may be neglected when evaluating $\nabla_1$. This is a good approximation for the masses and mixings which are relevant for solar neutrino oscillations. (Of course, neutrino masses do play an important role once $\nabla_1$ is used to evolve neutrino states forward in time.)
**Slowly-Varying Density Profile**: Secondly, \( j_\mu^a(x) \) is assumed to vary negligibly over distances comparable to the packet width, \( \xi \), and to the neutrino wavelength, \( \lambda_\nu = 2\pi/k \). This approximation implies the only significant scattering from macroscopic fluctuations is in the forward direction. For applications to solar neutrinos \( j_\mu^a \) varies macroscopically while \( \xi \) and \( \lambda_\nu \) are of atomic dimensions, so this last approximation also holds extremely well.

We find the following effective hamiltonian, to first order in \( V \):

\[
[V_1(r, t, t_m)]_{i\lambda; j\sigma} = -i \int d^3x \, d^3q \, \psi^*_k(z_0) \psi_k(z_0) \, j_\mu^a(x, t) \langle r, \lambda, i; t_m | \bar{\nu} \gamma_\mu (\gamma_L g^a + \gamma_R h^a) \nu | q, \sigma, j \rangle \approx \pi_a(r, t) \left[ M^a_{ij} \theta(-\sigma) - M^a_{ij}^* \theta(\sigma) \right] \delta_{\lambda \sigma}. \tag{59}
\]

This result acts trivially on the spin labels, \( \lambda \) and \( \sigma \), involving only the step function, \( \theta(\sigma) = \frac{1}{2}[1 + \text{sign} \sigma] \), which projects onto left-handed (LH: \( \sigma = -\frac{1}{2} \)) and right handed (RH: \( \sigma = +\frac{1}{2} \)) states. We label the spin space using the projection of the spin in the \( z \) (or propagation) direction. For massive neutrinos this choice is made in the neutrino rest frame, while for massless neutrinos it applies in any Lorentz frame. Since \( \lambda = \frac{1}{2} \) corresponds to a left-handed state, we see that the within the Standard Model the states for which \( \lambda = +\frac{1}{2} \) are antineutrinos. For \( N_\nu \) neutrino species the \( N_\nu \)-by-\( N_\nu \) matrices \( M^a_{ij} \) represent the action of \( V_1 \) on the flavour indices, \( i \) and \( j \). They are given explicitly in terms of the coupling matrices by \( M^a_{ij} = g^a_{ij} - h^a_{ji} = g^a_{ij} - h^a_{ij}^* \).

Finally, the quantity \( \pi_a(r, t) \) in this equation denotes the following:

\[
\pi_a(r, t) = \overrightarrow{\pi}_a[r_\perp, z_0 + vt, t] - \overleftarrow{\pi}_a[r_\perp, z_0 + vt, t], \tag{60}
\]

where \( r_\perp = (r_x, r_y) \) is the measurement position transverse to the neutrino propagation direction, and \( v \) denotes the speed, \( v = k/E_k \approx 1 \), associated with the central momentum, \( k \), of the wave packet. Recall that \( z_0 \) denotes the point of origin of the neutrino, which is to be averaged at the end of the calculation.

Using eq. (38) for the mean currents, \( \overrightarrow{J}_a^\mu \), together with the nonrelativistic approximation, which permits the neglect of \( \overrightarrow{J}_a^\mu \) — certainly good for the electrons and nucleons within the sun — we see that eq. (59), is recognizable as the standard MSW starting point for analyzing resonant neutrino mixing in matter. With this encouragement, we now proceed to compute the second-order contributions to neutrino evolution.
5.2) Second-Order Contributions

The second-order contributions come in two kinds, as can be seen from eqs. (30) and (32) of Section 2.5. First and foremost, there are the matter fluctuations — i.e. eqs. (48) and (49) due to macroscopic fluctuations in the ensemble. But, a priori, there could also be diffuse contributions due to the neutrino momentum average itself, as was first discussed in ref. [5].

In this section we apply the general treatment of Section 2 to both types of fluctuations. For matter fluctuations we do so using the two ensemble models which were introduced in Section 4.1. The effective hamiltonian we obtain in this way will turn out to have interesting implications for MSW oscillations in the next section. By contrast, we find no phenomenologically interesting effects for solar neutrinos due to fluctuations which arise due to integrating out the neutrino momenta. Since this conclusions differs somewhat from that of ref. [5], we reproduce his results in our formalism, and show why our conclusions differ.

We start with the formalism of Section 2, with the following three sectors: (i) $A' = F$ for neutrino spins and flavour; (ii) $B' = P$ for neutrino momenta and position; and (iii) $B = \mathcal{E}$ (or $E$) for the matter degrees of freedom. Using the same approximations as were used to obtain eq. (59) for $\mathbf{V}_1$, we find the second-order contribution to $\frac{\partial \rho_F}{\partial t}$ to have the following form, regardless of the source of fluctuations:

$$\begin{align*}
(\mathbf{V}_2)_{i\lambda;j\sigma} &\approx -iA_{ab}(r, t, t_m) \left[ (\mathcal{M}^a \mathcal{M}^b)_{ij} \theta(-\sigma) + (\mathcal{M}^{a*} \mathcal{M}^{b*})_{ij} \theta(\sigma) \right] \delta_{\lambda\sigma}, \\
\left(\frac{\partial \rho_F^d}{\partial t}\right)_{i\lambda;j\sigma} &\approx 2A_{ab}(r, t, t_m) \left[ (\mathcal{M}^a \rho_F \mathcal{M}^b)_{ij} \theta(-\sigma) + (\mathcal{M}^{a*} \rho_F \mathcal{M}^{b*})_{ij} \theta(\sigma) \right] \delta_{\lambda\sigma}.
\end{align*}$$

(61)

(Recall here $r$ represents the position where the neutrino is detected at time $t = t_m$.) The two kinds of fluctuations discussed above differ only in their predictions for the key coefficient, $A_{ab}(r, t, t_m)$. We now give expressions for this quantity for each of the two cases.

- **Fluctuations due to Tracing out Neutrino Momenta:**

  Before computing $A_{ab}$ for matter fluctuations, we briefly pause to discuss the fluctuations which arise on integrating out the neutrino momentum sector. We do so partly to make explicit the contact with ref. [5]. We also do so partly because such fluctuations can arise and may be important in some circumstances. We argue here why solar neutrinos are unlikely to be one such case.
A direct application of the formulae of Section 2 to this type of second-order contributions shows them to vanish, within the approximations outlined above. This is because we have chosen to measure the neutrino position arbitrarily accurately, at precisely one point, \((r, t_m)\). It is therefore instructive to integrate the observable, \(O_P\) of eq. (58), over a finite detector volume, \(D\). That is, we now replace eq. (58) by:

\[
O_P = \int_D d^3r \, |r, t_m\rangle \langle r, t_m|.
\]

With these choices, formulae (30) and (32), when applied to the neutrino-sector trace, give eqs. (61), with

\[
A_{ab}(r, t, t_m) \approx \int_t^{t'} d\tau \, \langle \delta \pi_a[r_{\perp}, z_0 + vt, t] \delta \pi_b[r_{\perp}, z_0 + v\tau, \tau] \rangle_P.
\]

Here \(\pi_a\) is as defined in eq. (60), and the average is over the detector volume transverse to the neutrino momentum. For any quantity, \(A(r, t)\), this average is defined by:

\[
\langle A(r, t) \rangle_P \equiv \frac{1}{D_\perp} \int_{D_\perp} d^2r_\perp \, A(r_\perp, r_z, t),
\]

where \(D_\perp\) denotes the area which the detector presents transverse to the neutrino beam. As usual, \(\delta \pi_a\), denotes the deviation of \(\pi_a\) from this transverse mean: \(\delta \pi_a[r, t] \equiv \pi_a[r, t] - \langle \pi_a[r, t] \rangle_P\). The key point here is that this deviation vanishes, \(\delta \pi_a = 0\), in the limit that the transverse detector size, \(D_\perp\), is much smaller than the scales over which \(\delta n\) varies appreciably.

This result makes sense physically. In the absence of matter fluctuations, neutrino evolution in the presence of a fixed density profile can be computed as an exercise in scattering from a fixed potential. Scattering only arises from variations, \(\delta n\), in the density from its spatial average. The main point is that the interference term between the scattered and initial waves in this problem is proportional to \(\int_{D_\perp} d^2r_\perp \, \delta n(r_\perp)\), and so vanishes only if the transverse area of the detector is sufficiently large on the scales over which \(\delta n\) varies. It follows that the scattering is incoherent only for such large detectors.

The relation of this result with that of ref. [5] is now clear. In this reference, the reduced density matrix for neutrino flavours is defined by completely tracing over all neutrino momenta, without taking into account the position measurement, eq. (58). This is equivalent to taking the detector volume to fill all space, and our eqs. (63) and (64) indeed
reduce to ref. [5]'s in this limit. But working in the limit of an extremely large detector misses the important suppression of these effects by the detector size.

Since we find in later chapters that significant neutrino effects require fluctuations on scales of hundreds of metres and up, we are led to conclude that this kind of neutrino incoherence likely plays no role for solar neutrinos.

**Matter Fluctuations:**

For matter fluctuations, the second-order contribution to the neutrino evolution equation is given by eq. (61), with

\[
A_{ab}(r, t, t_m) = \int_{t'}^t d\tau \langle \delta n_a[r_\perp, z_0 + vt, t] \delta n_b[r_\perp, z_0 + v\tau, \tau] \rangle_E. \tag{65}
\]

As before \(n_a\) denotes the difference \(j^0_a - j^z_a\) (or simply the density, \(j^0_a\), in the nonrelativistic limit), while \(\delta n_a\) is the difference between \(n_a\), and its ensemble average, \(\overline{n}_a = \langle n_a \rangle_E\).

Eq. (65) may be explicitly computed within the two models of fluctuations which were introduced in Section 4. For the case of locally-varying density fluctuations, eq. (53), we have:

\[
A_{ab}(r, t, t_m) = \epsilon_{abN}^2 \overline{n}_a[r_\perp, z_0 + vt, t] \int_{\text{cell} N} \overline{n}_b[r_\perp, z_0 + v\tau, \tau] d\tau
\approx \epsilon_{abN}^2 \ell_N \overline{n}_a[r_\perp, z_0 + vt, t] \overline{n}_b[r_\perp, z_0 + vt, t] \tag{66}
\]

(locally-varying density fluctuations).

Here \(\ell_N\) is the length of cell \(N\) along the neutrino line of flight, and \(N\) labels the specific cell which contains the point \((r_\perp, z_0 + vt, t)\). The approximate equality in the second line is derived under the assumption that the mean current, \(\overline{n}_a\), does not vary appreciably over the size of this cell. For the approximate exponential density profile [2]:

\[
\overline{n}_a(z) = (n_a)_c e^{-z/h}, \tag{67}
\]

which we use for electrons in the sun, the neglect of the variation of \(\overline{n}_a\) over a cell requires \(\ell \ll h = R_\odot/10.5 = 6.6 \times 10^4\) km. (Notice this is a much weaker condition than requiring \(\ell\) to be much smaller than neutrino propagation scales.) For electrons, the central density we use is \((n_a)_c = 1.5 \times 10^{26}/\text{cm}^3\).

Similarly, \(A_{ab}(r, t, t_m)\) may also be evaluated for the case of an oscillatory density profile. For solar applications, since \(\tau_{\pm}\) is of order several minutes and (the light-travel...
time across) $\ell_\pm \lesssim R_\odot$ is of order a few seconds, it is sufficient to neglect powers of $\ell_\pm/\tau_\pm$ and $R_\odot/\tau_\pm$. Again using the exponential profile, eq. (67), for $n_z(x)$, we find:

$$A_{ab}^{\pm}(r_\pm, t, t_m) = (n_a)_c (n_b)_c \varepsilon^2_\pm(r_\pm) F_\pm[z_0 + vt, t, t'],$$

(68)

(oscillatory density fluctuations)

where $\varepsilon^2_\pm \equiv C^2_\pm N^2_\pm$ defines the dimensionless size of the fluctuation, and the function $F$ is defined by

$$F_\pm(z_0 + vt, t, t') \equiv e^{-(z_0+vt)/\hbar} f_\pm(z_0 + vt) \int_{z_0+t'}^{z_0+t} dx \ e^{-x/\hbar} f_\pm(x)$$

(69)

with $f_-(z) = \sin\left(\frac{2\pi z}{\ell_\pm}\right)$ and $f_+(z) = \cos\left(\frac{2\pi z}{\ell_\pm}\right)$. The integral is elementary and is given (writing $a = 1/\hbar$ and $b = 2\pi/\ell$) by:

$$F_- (z_0, t - t') = \frac{e^{-a(z_0+t)} \sin b(z_0 + t)}{a^2 + b^2} \left\{ e^{-a(z_0+t')} \left[ b \cos b(z_0 + t') + a \sin b(z_0 + t') \right] - (t' \rightarrow t) \right\},$$

$$F_+ (z_0 + t, t, t') = \frac{e^{-a(z_0+t)} \cos b(z_0 + t)}{a^2 + b^2} \left\{ e^{-a(z_0+t')} \left[ a \cos b(z_0 + t') + b \sin b(z_0 + t') \right] - (t' \rightarrow t) \right\}.$$  \hspace{1cm} (70)

To summarize, fluctuations can indeed influence the neutrino evolution. Their effects are quantified by equations (61) and (65), which are the main results of this section. Their implications for resonant MSW oscillations can be sizable, as is now explored in more detail.

6. Applications to MSW Oscillations

In this section we evolve the neutrino density matrix to second order in $G_F$. With solar neutrinos in mind we follow the usual practice and suppose the initial neutrino spin to be purely left-handed, and focus on the evolution in flavour space. The plan is to use eqs. (59) and (61) to evaluate the right-hand-side of eqs. (10) and (12), and then to integrate the result to determine the electron-neutrino survival probability, $P_e(t) = \rho_{ee}(t)$. 

31
6.1) The Evolution Equations

For simplicity we specialize also to the case of two neutrino flavours, whose electroweak eigenstates we denote \( e \) and \( \mu \), although we might equally well imagine mixing the \( e \)-type and \( \tau \)-type neutrinos. Eq. (23) then takes the following form:

\[
\frac{\partial \rho}{\partial t} = -i [V_0 + V_1, \rho] - 2 G_F^2 \sum_{a,b = e,n} A_{ab} (g^a g^b \rho + \rho g^a g^b - 2 g^a \rho g^b),
\]

where \( \rho = \rho_F \) is the neutrino-sector density matrix in flavour space, and

\[
V_0 \equiv \left[ k^2 + m^\dagger m \right]^{1/2} \approx k + \frac{m^\dagger m}{2k} + \cdots,
\]

\[
V_1(t) \equiv \sqrt{2} G_F \left[ g^e \pi_e(t) + g^n \pi_n(t) \right].
\]

Here \( m, g^e \) and \( g^n \) are \( 2 \times 2 \) matrices which represent the left-handed-neutrino mass matrix, and the neutrino charged- and neutral-current coupling matrices. In an electroweak basis these are given explicitly by:

\[
m = \begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{e\mu} & m_{\mu\mu} \end{pmatrix}, \quad g^e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad g^n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.
\]

In general \( m \) may be a generic symmetric complex matrix, although in the absence of CP violation it may be chosen to be real. \( \pi_e(t) \) and \( \pi_n(t) \) are the spatially-averaged electron and neutron currents, as defined in eq. (60).

The second-order contribution to eq. (71) is given by the three terms proportional to \( \mathcal{A}_{ab}(t) \), which is defined for \( a, b = e,n \) by eq. (65). The terms of the form \( g^2 \rho \) and \( \rho g^2 \) are the contributions due to \( \mathcal{V}_2 \), while the \( g \rho g \) term comes from \( \frac{\partial \rho^2}{\partial t} \). Notice that because \( g^n \) is proportional to the unit matrix, all but the term involving \( \mathcal{A}_{ee} \) — which we henceforth denote simply by \( \mathcal{A} \) — give zero in the sum in the first of eqs. (71). As a result, it is only fluctuations in the electron density profile which are relevant for neutrino evolution in the sun.

In order to integrate this equation it is useful to expand all matrices in terms of the unit and Pauli matrices, \( \{ I, \tau \} \). We have

\[
\rho = \rho_0 + \vec{\rho} \cdot \tau \quad \text{and} \quad V_0 = M_0 + \vec{M} \cdot \tau,
\]

\[
g^e = \frac{1}{2} (1 + \tau_3), \quad \text{and} \quad g^n = -\frac{1}{2}.
\]

\[
(74)
\]
with
\[ \rho_{ee} = \rho_0 + \rho_3, \quad \rho_{\mu\mu} = \rho_0 - \rho_3, \quad \rho_{e\mu} = \rho_1 - i\rho_2, \] (75)
and
\[
\begin{align*}
M_0 &= k + \frac{|m_{e\mu}|^2 + \frac{1}{2} (|m_{ee}|^2 + |m_{\mu\mu}|^2)}{2k} \quad M_1 = \frac{\text{Re} (m_{ee}^* m_{e\mu} + m_{e\mu}^* m_{\mu\mu})}{2k}, \\
M_2 &= -\frac{\text{Im} (m_{ee}^* m_{e\mu} + m_{e\mu}^* m_{\mu\mu})}{2k}, \quad M_3 = \frac{|m_{ee}|^2 - |m_{\mu\mu}|^2}{4k}.
\end{align*}
\] (76)

Of these components, \(M_0\) plays no role in the evolution of \(\rho\) since it drops out of the right-hand-side of eq. (71). Similarly, since the trace of the right-hand side of eq. (71) vanishes identically, it follows that the coefficient \(\rho_0\) is independent of time: \(\frac{\partial \rho_0}{\partial t} = 0\).

With these definitions the evolution equation, (71), for the remaining three components of \(\rho\) may be written:
\[
\begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial t}
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix} = \begin{pmatrix}
-2a & -2(M_3 + b) & 2M_2 \\
2(M_3 + b) & -2a & -2M_1 \\
-2M_2 & 2M_1 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix} \equiv \mathcal{H}
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix},
\] (77)
where
\[
a(t) \equiv G_f^2 A(t) \quad \text{and} \quad b(t) \equiv \frac{G_f \bar{n}_e(t)}{\sqrt{2}}.
\] (78)

This is the form for the evolution which are integrated in subsequent sections.

For the purposes of exploring the implications of electron density fluctuations it suffices to restrict our attention to CP-conserving neutrino physics, for which the components of \(V_0\) simplify somewhat because the neutrino mass matrix, \(m_{ij}\), may be chosen to be real. For this case \(M_1\) through \(M_3\) can be expressed in terms of the heavy and light neutrino mass eigenvalues, \(m_h\) and \(m_l\), by:
\[
\begin{align*}
M_1 &= \frac{\delta m^2 \sin 2\theta_V}{4k}, \quad M_2 = 0, \quad M_3 = -\frac{\delta m^2 \cos 2\theta_V}{4k},
\end{align*}
\] (79)
where \(\delta m^2 = m_h^2 - m_l^2\) and \(\theta_V\) is the vacuum mixing angle, for which
\[
\nu_\ell = \nu_e \cos \theta_V - \nu_\mu \sin \theta_V, \quad \nu_\ell = \nu_e \sin \theta_V + \nu_\mu \cos \theta_V.
\] (80)

---

8 If \(A(t)\) is assumed to be a constant times \(\bar{n}_e(t)\), and in the absence of CP violation, then this equation agrees with that used in ref. [10]. It also agrees with ref. [12] if the correlation length is adjusted as explained in Section 4.1.
We now turn to integrating eq. (77). Before turning to a numerical solution using an exponentially falling electron density profile [2], we first pause for the instructive exercise of solving this equation in the adiabatic and Parke limits.

6.2) Adiabatic Evolution and the Generalized Parke Formula

If \(a\) and \(b\) are slowly varying, then it is straightforward to analytically integrate eq. (77) simply by performing a \((t\text{-dependent})\) rotation which diagonalizes the matrix \(\mathcal{H}\). Once these adiabatic solutions are obtained, then Parke’s more general expression may then be derived by starting with these states as bases and computing the transition probability as the neutrinos pass through the resonance.

The adiabatic result is:

\[
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix}(t) = R(t) \begin{pmatrix}
e^{\int_{t'}^t \lambda_1(x) dx} & e^{\int_{t'}^t \lambda_2(x) dx} & e^{\int_{t'}^t \lambda_3(x) dx}
\end{pmatrix} \begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix}_{t=t'},
\]

where \(\lambda_i(t)\) are the three time-dependent eigenvalues of \(\mathcal{H}\), and \(R(t)\) is the matrix for which \(R^\dagger(t)\mathcal{H}(t)R(t) = \text{diag}(\lambda_1(t), \lambda_2(t), \lambda_3(t))\).

Keeping in mind that \(a\) arises at second order in \(G_F\) and so is smaller than all of the other elements, we may solve for the \(\lambda_i\) and \(R\) perturbatively in \(a\). The resulting eigenvalues are, to linear order in \(a\):

\[
\lambda_0 = -2\gamma_0, \quad \lambda_\pm = \pm 2i\kappa - 2\gamma,
\]

with

\[
\kappa = \sqrt{M_1^2 + (M_3 + b)^2}, \quad \gamma_0 = \frac{aM_1^2}{M_1^2 + (M_3 + b)^2}, \quad \gamma = \frac{a[M_1^2 + 2(M_3 + b)^2]}{2[M_1^2 + (M_3 + b)^2]}.
\]

To the lowest (zeroeth) order in \(a\) the corresponding matrix, \(R\), is:

\[
R = \begin{pmatrix}
\frac{M_1}{\kappa} & -\frac{M_3+b}{\sqrt{2}\kappa} & -\frac{M_3+b}{\sqrt{2}\kappa} \\
0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\
\frac{M_3+b}{\kappa} & \frac{M_1}{\sqrt{2}\kappa} & \frac{M_1}{\sqrt{2}\kappa}
\end{pmatrix} = \begin{pmatrix}
sin 2\theta_m & \frac{1}{\sqrt{2}} \cos 2\theta_m & \frac{1}{\sqrt{2}} \cos 2\theta_m \\
0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\
-\cos 2\theta_m & \frac{1}{\sqrt{2}} \sin 2\theta_m & \frac{1}{\sqrt{2}} \sin 2\theta_m
\end{pmatrix},
\]

where the last equality defines the usual matter mixing angle, \(\theta_m\).
Using eqs. (82), (83) and (84) in eq. (81) then gives the adiabatic prediction for the electron-neutrino survival probability, given the initial condition \( \rho(t') = \text{diag} (1, 0) \):

\[
P_e(t) \equiv \rho_{ee}(t) = \frac{1}{2} \left\{ 1 + e^{-2 \int_{t'}^{t} \gamma_0(x)dx} \cos 2\theta_{m}(t') \cos 2\theta_{m}(t) \right. \\
\left. + e^{-2 \int_{t'}^{t} \gamma(x)dx} \sin 2\theta_{m}(t) \sin 2\theta_{m}(t') \cos \left[ 2 \int_{t'}^{t} \kappa(x)dx \right] \right\}.
\]  

(85)

The oscillatory term, on the second line of eq. (85), averages to zero once we sum over a long enough measurement interval.

For nonadiabatic evolution it is straightforward to use these adiabatic results to derive a generalization of Parke’s formula. After averaging over the production and detection times we find:

\[
P_e(t) = \frac{1}{2} + \left( \frac{1}{2} - P_J \right) e^{-2 \int_{t'}^{t} \gamma_0(x)dx} \cos 2\theta_{m}(t') \cos 2\theta_{m}(t),
\]  

(86)

in which \( P_J = \exp \left[ -\frac{\pi}{2} \left( \frac{\sin^2 2\theta_{V}}{\cos 2\theta_{V}} \right) \left( \frac{\delta m^2 h}{2k} \right) \right] \) is the ‘jump’ probability as one passes through the resonance point.

There are many reasons to become emotionally involved with eq. (86):

- **1:** In the absence of fluctuations, \( a \to 0 \), eqs. (85) and (86) reduce to the standard results for matter oscillations.

- **2:** When \( a \) is small, but not zero, its dominant influence is to damp the neutrino oscillations, by introducing an imaginary part to the masses of the mass eigenstates in matter. Such an imaginary contribution might have been expected given the antihermitian form found in eq. (61) for \( \nabla_2 \).

The resulting damped oscillations are similar to what arises when neutrinos decay, but with an important difference. The difference is the appearance of the term \( \frac{\partial \rho_{\ell}^\ell}{\partial t} \) in the evolution equation, (10). As a result, there is no net loss of probability in the neutrino sector: \( \text{Tr} \rho_{\ell}(t) \equiv 1 \) for all \( t \). For neutrinos the damping is a reflection of the conversion of the incoming neutrino from a pure to a mixed state, due to the decoherence introduced by the fluctuations in the matter through which it passes.

- **3:** The relative size of the fluctuation parameter, \( a = G_F^2 A \), in comparison with the usual MSW effective hamiltonian, \( b = G_F \bar{m}/\sqrt{2} \), is given (for locally-varying fluctuations) by

\[
\frac{a}{b} \sim G_F \bar{m} \epsilon^2 \ell \sim \left( \frac{\epsilon^2 \ell}{100 \text{ km}} \right) \left( \frac{\pi}{10^{26} \text{ cm}^{-3}} \right),
\]  

(87)
which gives a rough indication how large fluctuations must be in order to contribute significantly to neutrino evolution.

Important effects arise even for smaller $\epsilon^2 \ell$, however, in the case of resonant oscillations. This is because even a small damping term can ruin the quality of the resonance. This is borne out by our numerical integrations, which show that the first modifications arise precisely at the resonant point. It can also be seen analytically from eq. (86), as we now argue.

* 4: The strength of the decohering effect is completely parameterized by the integral over $\gamma_0$ which appears in eq. (86), after the oscillatory contributions are averaged out. This integral is elementary (for an exponentially falling electron density profile) when $A(x)$ is a constant times $n_e^2(x)$, as is the case for the locally-varying density-fluctuation scenario, discussed earlier. In this case we have $A = \epsilon^2 \ell n_e^2$ and so

$$R \equiv \exp \left\{ -2 \int_{t_e}^{t_{ex}} \gamma_0(x) \, dx \right\} = \exp \left\{ -2h \int_0^{n'} \gamma_0(n) \frac{dn}{n} \right\},$$

$$= \exp \left\{ - \left( \frac{\epsilon^2 \ell h (\delta m^2)^2 \sin^2 2\theta_V}{4k^2} \right) \left[ \ln \left( \frac{\sin 2\theta_V}{\sin 2\theta'_m} \right) + \cot 2\theta_V \left( \arctan \cot 2\theta_V - \arctan \cot 2\theta'_m \right) \right] \right\},$$

$$\approx \exp \left\{ - \left( \frac{\pi \epsilon^2 \ell h (\delta m^2)^2 \sin 2\theta_V}{4k^2} \right) \left( 1 + O(\sin 2\theta_V) \right) \right\},$$

$$\approx \exp \left\{ - \left( \frac{\epsilon^2 \ell}{7 \text{ km}} \left( \frac{\delta m^2 / k}{1 \text{ eV}^2 / \text{ MeV}} \right)^2 \left( \frac{\sin 2\theta_V}{0.1} \right) \right) \right\}.$$  

Here $t_{ex}$ is the time the neutrino exits the sun, and so after which $n_e(t)$ and $\gamma_0(t)$ both vanish. $\theta'_m = \theta_m(t')$ and $n' = n_e(t')$ are the matter mixing angle and electron density at the production point, deep within the solar interior.

Eq. (88) shows that significant damping is possible for solar neutrinos when $\epsilon^2 \ell \gtrsim$ a few kilometers.

* 5: For the smallest amplitude fluctuations, the first place where the damping becomes noticeable is for resonant, adiabatic oscillations (see Figures 1 and 2). That is, for $P_J \approx 0$ and $\cos 2\theta'_m \approx -1$, and taking $\theta_m(t) = \theta_V$, eq. (86) becomes:

$$P_e \approx \frac{1}{2} \left[ 1 - e^{-2 \int_{t'}^{t} \gamma_0(x) dx} \cos 2\theta_V \right].$$

36
For \( a = 0 \) (no fluctuations) this gives the usual suppression: \( P_e(t) \approx \sin^2 \theta_V \). For small damping this suppression is weakened.

Since the success of MSW oscillations in explaining the solar-neutrino data uses this suppression to virtually remove the \(^7\)Be neutrino line, a good measurement of the strength of this flux promises to give information about the strength of electron density fluctuations deep within the solar interior.

- **6:** For sufficiently large times, \( \int_t^{t'} \gamma_0(x) dx \gg 1 \), eq. (86) has the universal prediction: \( P_e \to \frac{1}{2} \). (This limit is also seen in our numerical integrations, as well as in those of ref. [10]. We believe its absence in ref. [12] is due to the use there of a correlation length which follows the neutrino matter oscillation length.) This suggests a new solution for the solar neutrino problem: an approximately energy-independent suppression of the solar neutrino flux by a factor of 2 due to solar fluctuations, even for small neutrino mixing angles in vacuum. This type of solution will be disfavored if improvements in the data should confirm the present indications for an energy-dependent neutrino suppression (i.e. for which \( p - p \) neutrinos are untouched while \(^7\)Be neutrinos are essentially removed).

- **7:** Resonance is defined by the condition \( M_3 + b = 0 \). Interestingly, although this resonance condition minimizes \( \gamma \), it actually maximizes \( \gamma_0 \). This is clearest when eq. (83) is written: \( \gamma_0(t) = a(t) \sin^2 2\theta_m(t) \). The sharper the resonance, the sharper the peak there in \( \gamma_0 \). In the limit of a sudden resonance we therefore expect \( R, \) of eq. (88), to be controlled by the value of \( A \) evaluated at the resonance point. In this limit we expect the fluctuation size at the resonance point to be what determines the size of the damping contribution.

### 6.3) Numerical Results

We have numerically integrated eq. (77) to determine the electron-neutrino survival probability, \( P_e(t) = \rho_{ee}(t) \), using \( A(t) \) as predicted by both the locally-varying density-fluctuation and the oscillating-mode model of density fluctuations. For the oscillating-mode case we have examined both the case of constant sound speed, and a speed which varies as a function of \( z \).\(^9\) We use the exponential density profile, eq. (67).

The present section is devoted to presenting the results of these calculations. We have compared these numerical results with the analytical result, eq. (86), which we find works extremely well for all of the parameters of interest.

\(^9\) We use the form \( c(z)=c_1+c_2(\hat{\tau})+c_3(\hat{\tau})^2+c_4(\hat{\tau})^3+c_5(\hat{\tau})^4 \), in units of the speed of light, with coefficients \( c_1=0.00170, c_2=0.000581, c_3=-0.0106, c_4=0.0177 \) and \( c_5=-0.09929 \), as found by fitting to the profile given in ref. [18].
We present our results through Figures 1 and 2. These present plots of the survival probability, $P_e(t)$, as a function of the neutrino energy normalized to its squared mass: $E/\delta m^2$. We use $\sin 2\theta_V = 0.1$. In Figure 1 we use the locally-varying density fluctuation model, with successive curves representing different values for the fluctuation strength, $\epsilon^2\ell$. As expected based on eq. (86), sizable effects first appear by deteriorating the quality of the suppression at resonance. The first deviations appear when $\epsilon^2\ell = 0.01$ km, until the resonance is completely destroyed for $\epsilon^2\ell = 100$ km. For small $E/\delta m^2$ and $\epsilon^2\ell < 10$ km, the survival probability approaches its asymptotic MSW form $P_e \approx \cos^2 2\theta_V \approx 1$. For $\epsilon^2\ell$ larger than this value, $P_e$ instead tends to the limit $\frac{1}{2}$ for small $E/\delta m^2$.

Figure 2 presents a similar plot for an oscillatory fluctuation, using the position-dependent speed of sound. The wave period is chosen to be 30 minutes, and its transverse momentum, $k_\perp$, is chosen to be as large as is possible: $k_\perp \sim 2\pi/L$, for $L \sim R_\odot$. The wavenumber, $k_z$, in the direction of neutrino motion is fixed as a function of $z$ using the dispersion relation, eq. (56), with a realistic $z$-dependent speed of sound. As was described in section (4.1), the wave’s amplitude is set to zero when $k_z$ so determined is imaginary, corresponding to a damped wave. Our plots are made using an ‘odd’ mode, $\phi_-(x)$.

The figure displays the resulting survival probability as a function of $E/\delta m^2$ for various amplitudes, $\epsilon$. Sizable deviations start for $\epsilon \gtrsim 1\%$. The size of this deviation roughly agrees with what would be estimated using the locally-varying density fluctuation results of Fig. 1, with the correlation length, $\ell$ taken as the wavelength at the resonant point. This leads us to expect a larger effect in a more accurate simulation using $g$ waves than for $p$-waves, since these have larger amplitudes and wavelengths in the resonance region. More realistic simulations to investigate these issues are under way [19].

7. Magnetic-Moment Couplings

This section briefly applies the formalism of Section 2 to neutrinos which interact with magnetic fields through a magnetic-moment interaction:

$$\mathcal{L} = \frac{\mu_{ij}}{4} \left[ \bar{\nu}_i \gamma^{\mu\nu} \gamma_L \nu_j \right] F_{\mu\nu} + \text{c.c.}$$

$$= \frac{1}{4} \left[ \bar{\nu}_i \gamma^{\mu\nu} (\text{Re} \, \mu_{ij} + i\gamma_5 \text{Im} \, \mu_{ij} \nu_j) \right] F_{\mu\nu}. \quad (90)$$

Since the $\nu_i$ are majorana, the matrix $\mu_{ij}$ is (complex) antisymmetric. Fluctuating magnetic fields have been considered in ref. [10], in the limit of negligibly short correlation length.
With this choice, the first-order contribution to the effective Hamiltonian governing neutrino evolution in spin and flavour space as it moves along the $z$ axis is:

$$(\mathcal{V}_1)_{i\lambda;j\sigma} = \mathcal{F}_{ij} \text{sign}(\sigma) \delta_{-\lambda,\sigma},$$

with $\mathcal{F}_{ij}$ representing the following combination:

$$\mathcal{F}_{ij} \equiv \text{Re} \mu_{ij} (\mathcal{E}_x + \mathcal{B}_y) + \text{Im} \mu_{ij} (\mathcal{E}_y - \mathcal{B}_x)
= \text{Re} \left[ \mu_{ij} (\mathcal{E}_x - i\mathcal{E}_y) \right] - \text{Im} \left[ \mu_{ij} (\mathcal{B}_x - i\mathcal{B}_y) \right].$$

Here $\mathcal{E}_k$ and $\mathcal{B}_l$ denote the (microscopic or ensemble) mean electric and magnetic fields, averaged over the matter sector:

$$\mathcal{E}_k = \langle F_{0k} \rangle_{\mathcal{E}} \quad \text{and} \quad \mathcal{B}_k = \frac{1}{2} \varepsilon_{klm} \langle F_{lm} \rangle_{\mathcal{E}},$$

and $\mathcal{E}_k \equiv \mathcal{E}_k + i\mathcal{B}_k$.

Similarly, the second-order contributions to the evolution equation become:

$$(\mathcal{V}_2)_{i\lambda;j\sigma} = \delta_{\lambda\sigma} \int_{t'}^t d\tau \left( \langle \delta \mathcal{F} \delta \mathcal{F} \rangle_{\mathcal{E}} \right)_{ij}
= \text{sign}(\lambda) \text{sign}(\sigma) \int_{t'}^t d\tau \left( \langle \delta \mathcal{F} \rho \delta \mathcal{F} \rangle_{\mathcal{E}} \right)_{i\lambda;j\sigma},$$

where $\delta \mathcal{F}_{ij} \equiv \mathcal{F}_{ij} - \mathcal{F}_{ij}$.

Clearly these expressions can be used to extend the analysis of fluctuations in magnetic fields to systems for which the fluctuations are uncorrelated in momentum space. Based on our experience with the MSW oscillations, for reasonably small fluctuations we expect appreciable consequences dominantly in the presence of resonant oscillations.

8. Conclusions

In this paper we have obtained the following results:

- **1:** We set up a general formalism for describing neutrino propagation through fluctuating media. This formalism has the virtue that it is derived from first principles, and so there
are no hidden assumptions which limit its applications. As a result it can be applied to any source of fluctuations which may be of interest.

- 2: We have applied this formalism to many of the sources of fluctuations for neutrinos propagating through the sun. When applied to microscopic, thermal fluctuations we reproduce standard estimates, which give a negligible impact on neutrino propagation. When applied to larger-scale macroscopic density fluctuations the effects can be larger, typically becoming important once $\epsilon^2 \ell > 100$ m or so. Here $\epsilon \sim \delta n/n$ is the relative amplitude of the density fluctuation, and $\ell$ is a measure of its correlation length in the direction of neutrino motion. If this varies from place to place, it is its value at the point where resonant oscillations occur that is most important.

- 3: The neutrino evolution equation which we obtain — eq. (77) — agrees, when restricted to the domain of common validity, with those of previous workers [6][7][8][5][10][12]. We find the fluctuations found in ref. [5] to only significantly decohere the neutrinos if their size is small compared to the size of the detector. Since the strength of their influence on neutrino propagation is itself proportional to the fluctuation size, such small fluctuations are likely to be negligible for solar neutrinos.

- 4: Two models of macroscopic solar fluctuations were developed. One of these, having cells of constant, fluctuating density, reduces to ref. [10], and is similar to ref. [12], in the limit of small cell size. The other is completely different, and models the oscillatory density variations which occur in solar acoustic $p$-waves. (A more realistic version of this model is currently under study.) For these oscillatory waves, taking a period of 30 minutes, we find appreciable neutrino effects for density fluctuations which are at least a percent in size. We trace this comparatively small effect to the relatively small wavelength at the neutrino resonance point, which comes about because, for $p$ waves, the wavelength decreases with depth due to the increase of the speed of sound. We do not expect the same suppression to apply to $g$ waves.

- 5: We integrate the resulting neutrino evolution equations for the case of two neutrino flavours (with no CP violation) to obtain a generalized Parke’s formula — eq. (86) — for the electron-neutrino survival probability. This formula reproduces well our numerical integrations. The decoherence due to the fluctuations enters the neutrino evolution like a friction term, causing the oscillations to damp.

Two features emerge from the result. First, for small fluctuations deviations from the usual MSW survival probability first arise for adiabatic resonant transitions, for which the MSW suppression deteriorates because the fluctuations partially ruin the resonance.
Because MSW oscillations use this suppression to agree with the solar neutrino data by eliminating the $^7$Be line, measurements of this line in neutrino detectors promises to shed light on the nature of density fluctuations deep within the sun.

Second, for small $E/\delta m^2$ and for sufficiently large fluctuations, the survival probability falls to 0.5, independent of energy. This introduces a new fluctuation-driven mechanism for solving the solar neutrino puzzle, although its energy dependence is not favoured by current measurements.

• 6: Finally, our formalism is used to derive the effective neutrino hamiltonian which is relevant to magnetic moment couplings to magnetic fields.

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10. Figure Captions

(1) The electron neutrino survival probability as a function of $E/\delta m^2$ assuming the locally-varying density fluctuations model, as described in the text. Each curve represents a different value for the parameter combination $\epsilon^2\ell$, where $\epsilon = \delta n/n$ is the fractional amplitude of the fluctuation, and $\ell$ is the fluctuation’s correlation length in the direction of neutrino motion.

(2) The electron neutrino survival probability as a function of $E/\delta m^2$ assuming the oscillatory density fluctuations model, which is a crude model of a helioseismic $p$-wave. Each curve represents a different value for the fractional amplitude. $\epsilon = \delta n/n$. The figure assumes a wave period of 30 minutes, and uses the speed of sound as a function of depth given in ref. [18]. The transverse wavenumber, $k_\perp$, is chosen as small as possible, corresponding to a wave which penetrates as deeply as possible into the solar interior.
\[ \sin^2 2\theta = 0.01 \]

**Figure 1**

\[ P(\nu_e - \nu_e) \]

\[ \frac{E}{\Delta m^2} \text{ [MeV/eV}^2\] \]

- \(\epsilon^2 l = 1 \text{ m} \)
- \(\epsilon^2 l = 10 \text{ m} \)
- \(\epsilon^2 l = 10^2 \text{ m} \)
- \(\epsilon^2 l = 10^3 \text{ m} \)
- \(\epsilon^2 l = 10^4 \text{ m} \)
- \(\epsilon^2 l = 10^5 \text{ m} \)
- \(\epsilon^2 l = 10^6 \text{ m} \)

**Figure 2**

\[ P(\nu_e - \nu_e) \]

\[ \frac{E}{\Delta m^2} \text{ [MeV/eV}^2\] \]

- \(\epsilon = 0.001 \)
- \(\epsilon = 0.010 \)
- \(\epsilon = 0.024 \)
- \(\epsilon = 0.032 \)
- \(\epsilon = 0.057 \)
- \(\epsilon = 0.075 \)

\(\sin^2 2\theta = 0.01\)