Gauge Singularities in the SU(2) Vacuum on the Lattice

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Abstract

I summarize and extend the qualitative results, obtained previously by inspection of SU(2) lattice gauge field configurations. These configurations were generated by the Wilson action, then transformed to a Landau gauge and smoothed by Fourier filtering. This leads to sharp peaks in field strengths and related quantities, the characteristics of which are very well separated from a background. These spikes are caused by gauge singularities. Their density is determined as \( 1.5/\text{fm}^4 \), with very good scaling properties as a function of the bare coupling constant. The number of spikes within a configuration vanishes when approaching the deconfinement region. Furthermore, the Landau-gauging procedure becomes unique, if the probability to find a spike is much smaller than unity. The relation of the spikes to the instantons obtained by cooling is investigated. Finally, a correlation between the presence of spikes and the infrared behaviour of the gluon propagator is demonstrated.

1 Introduction

If one transforms a single configuration of SU(2) lattice gauge fields to a Landau gauge and then removes the high momentum Fourier components of the gauge fields by some exponential cut-off ("Fourier filtering", F.f., or "smoothing"), one observes the following phenomenon \[1\]: For sufficiently large lattices and/or sufficiently large bare coupling constants, at a couple of lattice positions the gauge fields show a rapid variation in all space-time directions and for all colour components. These variations lead to narrow spikes in the Wilson action density \( S(x) \) and in the topological charge density \( \hat{q}(x) \). As an example, in fig. 1 the charge density \( \hat{q}(x) \) is shown for a plane of a large lattice. The cones have been generated by the VRML-language\[2\] and visualized by the browser GLVIEW\[3\]. The positions of almost all the spikes do not vary in the different gauges which are due to different Gribov copies \[4\]. For sufficiently modest filtering, the spatial extension of these spikes amounts to one or two lattice units, which roughly corresponds to 0.05 fm.

Due to Fourier smoothing, the quantities \( S(x) \) and \( \hat{q}(x) \) are gauge dependent, and their physical interpretation is not straightforward. The observation that, within the spikes, one observes self-duality, i.e. \( \hat{q}(x) \approx \pm S(x) \) within a 10% accuracy, might suggest that we observe narrow instantons. Two facts contradict

\[1\]The normalization of these two quantities will be given in eqs. 13 and 14.
Figure 1: The topological charge density on a lattice plane running through a typical spike. The direction of the cones indicates the orientation of the charge in three-dimensional colour space. The lattice size is $48^3 \times 64$, and $\beta = 2.85$. The cut-off for the Fourier transformed gauge fields is $\lambda^2 = 0.5a^2$. 
this interpretation. First, one observes that the distribution in terms of \( S(x) \) summed over a few lattice sites around the spikes is strongly peaked for large action values. Those correspond to small-sized instanton-like objects. For details see section 3. On the contrary, several lattice studies in SU(2) predict a size distribution with a maximum for sizes in the range from 0.2 \( \text{fm} \) \([5]\) to 0.4 \( \text{fm} \) \([6]\) and around 0.3 \( \text{fm} \) or higher in SU(3) \([7]\). Secondly, if the configuration is smoothed by the standard cooling technique \([8]\), then one indeed observes instantons at the position of the spikes, but their size is much larger than that of the spikes.

If one transforms to a Landau gauge prior to cooling, one can directly observe that the gauge fields are singular (regulated by the lattice) at the positions of the spikes. All observations point towards the interpretation that the Landau gauging leads to singularities in the gauge fields, if the lattice parameters are within the confinement region. A gauge singularity at the origin can be created by imposing the gauge transformation\(^2\)

\[
g(x) = \left( x_0 + i\vec{x}\tau \right) / |x| \tag{1}
\]
on a relatively smooth gauge field configuration, e.g. on a large-size instanton in the regular gauge\(^3\). Then, close to the origin, there occurs a cancellation between the derivative terms in the action and the cubic and higher gauge field terms.

This cancellation will be mutilated by filtering, which acts differently on the two contributions. Accordingly, a singular gauge field distribution will develop, under F.f., a sharp peak in the action and in \( \hat{q}(x) \), even if both quantities are smooth in \( x \) without filtering\(^4\). Thus, the plaquette action and \( \hat{q}(x) \) -after smoothing- no longer possess a definite physical meaning. They have to be understood as a measure for unexpected things that happen while gauging.

In this paper, I will concentrate on the phenomenology of the spikes, i.e. on their density and on their correlation with other nonperturbative phenomena. The following topics will be treated:

1. For large \( \beta = 4/g_0^2 \), the spikes are very well defined objects, if the action (or the topological charge density) -summed over a few lattice units around the action maximum- is taken as a probe. This is demonstrated by the histogram in fig. 2 which is based on \( O(30) \) configurations on large lattices and large \( \beta \). The histogram displays a well defined valley between a peak containing the spikes (all satisfying self-duality \( \hat{q}(x) \approx \pm S(x) \)), and a background which contains distinctly different parts of the lattice configuration. This allows to determine the density of spikes as

\[
\rho_{\text{spikes}} = (1.5 \pm 0.20) / \text{fm}^4. \tag{2}
\]

Since the average number of spikes per configuration is well determined, i.e. without a significant systematic error, the density of spikes in physical units can be used for a scaling study as a function of \( \beta \). Very good scaling with the string tension results \([12]\) is observed in the interval \( 2.70 \leq \beta \leq 2.85 \). For details see section 3. The relation of the number of spikes and the deconfinement transition will also be discussed in that section.

\(^2\)The origin \( x = 0 \) has to be chosen apart from a lattice site. \( \tau \) are the Pauli matrices.

\(^3\)The generalization to several singularities at different positions is obvious.

\(^4\)This phenomenon has been checked by analytic calculations in the continuum by H. Joos.
2. The density of spikes is correlated with the number of Gribov copies which are found during the Landau gauging process. If the lattice is chosen so small that the probability for finding one or more spikes in the configuration is much less than unity, then, empirically, the gauging procedure is practically unique. This means that the Landau gauging algorithm ends up -with high probability- preferentially in the same gauge, i.e. one rarely observes the appearance of Gribov copies [4] if only a finite number of gauging trials is performed (a trial is defined by applying a random gauge prior to the Landau gauging). The observation is that if copies are found, the probability to find spikes is enhanced too. This positive correlation between the appearance of spikes and copies will be demonstrated in section 4.

3. It is instructive to study alternatives to the Fourier filtering technique, notably the cooling technique [8]. In section 5 I will show that there is an interesting connection between the outcome of cooling and of F.f. If one starts from a configuration in the Landau gauge, finds the position of its spikes and then cools the unfiltered lattice configuration, one observes the following: The absolute values of the gauge fields, $|\vec{u}_\mu(x)|$ (see eq. 4), decrease under cooling almost everywhere, except at the positions of the spikes. There, gauge singularities show up. The local action maxima, which always accompany the singularities, are broad in most cases.

4. The fact that the spikes manifest themselves as zeros in the gauge fields -in all directions and for all colours- suggests that the gluon propagator and other correlators of the gauge fields behave differently if spikes are present in a configuration or if not. This is the case, as it will be shown in section 6.

2 Landau Gauge and Filtering

Here a couple of topics relevant for gauging and filtering will be discussed. The Landau gauging algorithm on the lattice maximizes the sum of the "large" SU(2)-components $u_{0,\mu}(x)$ in the link representation

$$U_\mu(x) = u_{0,\mu}(x) + i\vec{\tau}\vec{u}_\mu(x) \quad (3)$$

with

$$u_{0,\mu}^2(x) + \vec{u}_\mu^2(x) = 1. \quad (4)$$

Thus, one searches numerically for maxima of

$$F(U) = \sum_{x,\mu} u_{0,\mu}(x). \quad (5)$$

under local gauge transformations $U$. If the maximum condition is fulfilled, one has

$$\sum_\mu \vec{u}_\mu(x) - \vec{u}_\mu(x - e_\mu) = 0, \quad (6)$$

which corresponds to

$$\partial_\mu \vec{A}_\mu = 0 \quad (7)$$

in the continuum. At the center of an instanton in the singular gauge, eq. 4 makes no sense. This fact requires to describe an instanton around the center by
gauge fields in the regular gauge and at infinity by the singular gauge. On the lattice, however, the singular gauge is tolerable, if the center is not located at a lattice site, and lattice configurations in the Landau gauge seem to choose this option.

The Gribov ambiguity [4] implies, that there exist -for sufficiently large lattices in physical units\(^5\)- many different maxima where eq. 6 is satisfied. A priori, it is not clear whether gauge dependent average values of link variables, e.g. the gauge field propagator, depend strongly on the special Landau gauges, which are characterized by the different values of the gauge function eq. 5 for a given configuration. Empirically, the gauge field propagator does not depend strongly on the special gauge. This can be demonstrated by studying the correlation between numerical results for the gluon propagator and the values of the gauge function \(F(U)\) for different gauges. The latter can be generated either by different gauging algorithms (e.g. overrelaxation or conjugate gradient) or by different random gauges, applied before the iterative gauging starts. The following numerical results have been obtained:

For a lattice of size \(24^3 \times 32\) at \(\beta=2.5\), corresponding to a physical size \((1.9)^3 \times 2.5 \text{ fm}^4\), 20 random gauges were used for each configuration, and the values for the gluon propagator at the smallest momenta were ordered with respect to the values of the gauge function, i.e. whether \(F(U)\) was smaller or larger than the average value (which is determined for the individual configuration). No significant difference between the two data sets could be observed, with an overall relative accuracy of 5% for the gluon propagator at \(q^2 = 0\), and with 1.5% at \(q^2 > 0\). The upper limit for the difference between one of the ordered data sets and the average over the random gauges is 2% for \(q^2 > 0\). The stochastic noise, due to measuring on different configurations, is comparable to that due to measuring at different gauges, with no significant correlation between the value of the gauge function and the value of the propagator.

The Landau gauging drives the configuration to the region of small vector components \(\vec{u}_\mu(x)\) in the sense that link variables with \(u_{0,\mu}(x)<0\) become very rare. For presently available coupling constants, \(\beta \leq 2.85\), this goal has almost been achieved. This allows for a linearization and subsequent Fourier transformation of the gauge field variables. Besides the trivial linearization,

\[
\vec{u}_\mu(x) \Rightarrow \vec{A}_\mu(x),
\]

I use a stereographic transformation in order to minimize errors due to residual negative \(u_{0,\mu}(x)\). There I define gauge fields by

\[
\vec{u}_\mu \sqrt{\frac{2}{1+u_{0,\mu}}} = \vec{w}_\mu(x) \Rightarrow \vec{A}_\mu.
\]

This stereographic transformation brings the ”south-pole” \(u_0 = -1\) not to infinity, but to \(\vec{w}_\mu^2 = 4\). The Fourier filtering is applied to the gauge fields \(\vec{A}_\mu(x)\), (see eq. 10) and after suppression of high momentum Fourier amplitudes, the gauge fields are transformed back to \(\text{SU}(2)\)-variables. After an eventual smoothing of the fields, \(\vec{A}_\mu \Rightarrow \vec{\tilde{A}}_\mu\) by F.f., normalized link variables will be reconstructed by

\(^5\text{In section 4 we will see, that the critical size is about } (0.9 \text{ fm})^4.\)
the obvious inversion:

\[
\vec{w}_\mu = \vec{A}_\mu',
\]

(10)

\[
\vec{u}_\mu' = \sqrt{1 - \vec{w}_\mu^2/4 \vec{w}_\mu},
\]

(11)

\[
u_0,\mu' = \pm \sqrt{1 - \vec{u}'_\mu^2}, \text{ sign for } \vec{w}_\mu^2 > 2.
\]

(12)

The differences between the two methods of linearization amount to 5% for the field strengths, with no impact on the existence and properties of the spikes.

With these variables, the action and the topological charge density will be defined in the following way: First of all, the field strength tensor \(F^{a}_{\mu \nu}(x)\) is calculated by averaging the link products around the 4 plaquettes \(\rho, \sigma\) with \(\rho, \sigma \neq \mu, \nu\), which are connected to \(x\). Given \(F^{(a)}_{\mu \nu}(x)\), electric fields \(\vec{E}^{(a)}(x)\) and magnetic fields \(\vec{B}^{(a)}(x)\) are defined, and the action is

\[
S(x) = \frac{1}{2} \sum_a (\vec{E}^{(a)}(x)\vec{E}^{(a)}(x) + \vec{B}^{(a)}(x)\vec{B}^{(a)}(x)).
\]

(13)

For convenience, I define

\[
\hat{q}(x) = \sum_a \vec{E}^{(a)}(x)\vec{B}^{(a)}(x)
\]

(14)

and call it topological charge density, in spite of the nonstandard normalization. For self-dual objects, one has \(S(x) = \pm \hat{q}(x)\).

Due to the slight non-locality of the operator \(\hat{q}(x)\), it should have a negative correlator

\[
< \hat{q}(x)\hat{q}(0) > \leq 0
\]

(15)

only for \(x^2 > 4a^2\). This property has been successfully checked for 10 configurations on large lattices. The negativity breaks down for filtered and for cooled configurations. This can be read off, for instance, from fig. 16 of ref. [6], and it also has been verified for the cooled configurations used in section 5.

### 3 Density of Spikes

Here it will be shown that the occurrence of spikes is a well defined phenomenon in the sense that the spikes are separated from a background of action maxima by a deep valley, such that their density can be determined without a significant systematic error. This is true, especially, for large values of \(\beta\). For smaller values, the depth of the valley diminishes somewhat, without a severe loss of significance. Thus, it can be tested quite well whether the density of spikes scales under a variation of \(\beta\) in accordance with the string tension, in spite of the limited number of spikes identified so far.

An interesting question is how the average number of spikes varies with the lattice volume, if \(\beta\) is kept fixed. The preliminary evidence is that the density

\[6\] Obviously, the negativity excludes the dominance of four-dimensional, ”sign coherent structures” in the true vacuum, as has been emphasized by [10].

\[7\] This is true because the change of scale enters the density with the 4th power.
is roughly independent of the volume, and that the parameters of the critical volume, where one expects just one spike per configuration, are close to the deconfinement transition. This will be discussed at the end of this section.

The following quantitative analysis is based on A) 34 configurations on a large lattice with size $48^3 \times 64$ at $\beta=2.85$, with twisted boundary conditions [11], and B) 25 configurations on a smaller lattice with size $32^3 \times 48$ at $\beta=2.70$, also with twisted boundary conditions.

Each of these configurations has been gauged to a Landau gauge and then Fourier filtered, with the following substitution for the momentum space amplitudes

$$\vec{A}_\mu(q) \rightarrow \vec{A}_\mu'(q) = \vec{A}_\mu(q) \exp(-\lambda^2 q^2).$$

(16)

For identifying spikes, I use $\lambda^2 = 0.5a^2$. This modest filtering already leads to a strong suppression of the perturbative noise of the $\vec{A}_\mu$. The filtered gauge fields were transformed back to $x$-space and subsequently reunitarized (see eq. 10).

For each of these filtered configurations, I have identified $O(500)$ local maxima of the action density $S(x)$. Then, $S(x)$ has been summed within a radius of $R = 2a$ around the positions $x_m$ of these maxima,

$$\Sigma_m(\lambda^2, R) = \sum_x S(x)$$

(17)

For case A, the distribution of the values of $\Sigma_m(\lambda^2, R)$ is shown as the solid line histogram of fig. 2. The distribution is normalized by the physical volume $V(\beta)$ of the lattice (see below). It is remarkable how clearly the peak at large $\Sigma_m(\lambda^2, R)$ is separated from the background at smaller values, with a dip around $\Sigma_m(\lambda^2, R) = 8$. Furthermore, the data for case B are shown in fig. 2 as a broken line. In order to compare these data with those of case A, one has to fix the ratios of the physical volumes. For this, I use

$$V(\beta=2.7)/V(\beta=2.85) = a(\beta=2.70)^4 \times 32^3 \times 48$$

$$a(\beta=2.85)^4 \times 48^3 \times 64$$

(18)

with [12] $a(\beta=2.70)/a(\beta=2.85) = 1.6$.

The fact that the dip is less pronounced at $\beta = 2.7$ than at the higher value of $\beta$, is, most likely, not due to a statistical fluctuation. Data at $\beta = 2.5$ show that the peak for high values of $\Sigma_m(\lambda^2, R)$ changes in shape towards a plateau. At low $\beta$, the spikes are probably not as stable as at larger $\beta$, and a snapshot of a single configuration will find both well developed spikes and decaying/growing ones.

It is natural to define the average density of spikes per configuration by the number of objects with $\Sigma_m(\lambda^2, R) \geq 8$. In case A, this gives

$$N_{spikes}(\beta = 2.85) = 6.6 \pm 0.5 \text{ spikes per configuration},$$

(19)

and, using the string tension scale [12] $a(\beta = 2.85) = 0.028\text{ fm}$,

$$\rho_{spikes}(\beta = 2.85) = 1.52 \pm 0.2 \text{ spikes / fm}^4.$$  

(20)
The corresponding results for case B are:

\[ N_{\text{spikes}}(\beta = 2.70) = 9.6 \pm 0.6 \text{ spikes per configuration} \quad (21) \]

and

\[ \rho_{\text{spikes}}(\beta = 2.70) = 1.51 \pm 0.2 \text{ spikes/fm}^4. \quad (22) \]

Thus, eqs. 20 and 22 show with good accuracy, that the density of spikes obeys scaling according to the non-asymptotic variation of the string tension, and that the density itself is a rather dilute one. The spikes themselves are better defined for large values of \( \beta \) than for small ones, and this implies, together with the scaling of the density, that they are not a remnant of the strong coupling regime, which would become irrelevant in the continuum limit.

It remains to show how the number of spikes varies when -at fixed \( \beta \)- the lattice size is chosen so small that one approaches the deconfinement region. I take the variation of certain glueball masses as a signal for (de-)confinement, especially the mass ratio of \( A_1^+ \) and \( E^+ \), which shows a step as a function of the lattice size \( L \) around

\[ z_{A_1^+} = m_{A_1^+} L \approx 7. \quad (23) \]

The details of the transition to this region depend on the lattice geometry, and the present geometry is not optimally chosen for this purpose.
Combining this with the ratio of the root of the string tension, $\sqrt{K}$, to the mass $m_{A_1^+}$, $\sqrt{K}/m_{A_1^+} \approx 0.25$, and with the value of the string tension $a^2 K(\beta = 2.85) = 0.0040$, I find for the deconfining lattice size

$$L_{\text{step}} = 27.7a.$$  \hfill (24)

Now, the important question is how many spikes have to be expected on a lattice of size $27.7^3 \times 37$. As will be shown in the next section, the density of spikes is (within an accuracy of 10\%) independent of the volume. Thus, from eq. (19) we have to expect $0.73 \pm 0.15$ spikes per configuration. This value allows to conclude that at the deconfinement transition the average number of spikes falls below unity.

### 4 Gribov Copies and Spikes

The existence of Gribov copies is made evident by applying random gauge transformations prior to the Landau gauging. Then, on large lattices, the final value of the gauge function $F(U)$, eq. (5) will differ for each set of random numbers. It remains to be seen how small the lattices have to be such that the gauging procedure is practically unique in the sense that for a large number of random trials one always ends up in the same gauge. Here I will show that the naively estimated number of spikes provides a good measure for the transition from non-unique to unique gauging.

In detail, it will be studied how the appearance of Gribov copies is correlated with the existence of spikes. The study is restricted to a lattice size close to the critical one for which the expected number of spikes per configuration is just one, in the following sense: Given the number of spikes, $N_{\text{spikes}}$, on a physically large lattice, $V_{\text{large}}$, I define a critical volume by

$$V_{\text{crit}} = V_{\text{large}}/N_{\text{spikes}}.$$  \hfill (25)

Thus, the number of spikes on $V_{\text{crit}}$ were one if the density of spikes were independent of the volume, which, of course, need not be the case a priori.

A lattice slightly smaller than the critical one seems to be best suited to study the correlation between the two phenomena. The reason is that for lattices which are much larger than the critical one, the number of Gribov copies cannot be determined reliably, since for every random gauge one will end up in a different copy. On the other hand, for lattices much smaller than the critical one, copies are very rare, and it will take too many sweeps to find any.

At $\beta = 2.85$, I found the average number of spikes on the large lattice, eq. (19), and accordingly we have

$$V_{\text{crit}} = (0.90 \text{fm})^4.$$  \hfill (26)

Thus, at $\beta = 2.85$ and on a lattice of size $24^3 \times 32$, which corresonds to a volume $V = (0.72 \text{fm})^4$, we would expect 0.41 spikes by naive geometrical scaling. The simulation, described in the following, shows that the average number of spikes per configuration is

$$N_{\text{spikes}}(\beta = 2.85, 24^3 \times 32) = 0.37 \pm 0.03 \text{ spikes per configuration.}$$  \hfill (27)
This is well compatible with the number 0.41 expected from naive geometrical scaling. I conclude that -within reasonable accuracy- the density of spikes is independent of the lattice volume.

For the above lattice, the number of Gribov copies for a given lattice configuration turns out to be strongly correlated with the average number of spikes observed on this configuration. The observation is based on two long sequences with about 200,000 updates for each sequence. The approximate number of Gribov copies has been determined after every 1,000 sweeps. For this purpose, 12 random Landau gauges were performed, and the 12 values of the gauge function were compared and the numbers of different gauge functions, found in this way, were recorded. Simultaneously, the number of spikes was found by filtering the configuration with a cut-off $\lambda^2 = 0.5a^2$ and by searching for local action maxima, which exceeded the background by a factor three or more. For such maxima, $|\hat{q}(x)|$ agrees with the action within an accuracy of 10 %. The results for copies and spikes are the following:

- Configurations without Gribov copies follow each other in long sequences, i.e. for $O(10,000)$ lattice updates. When spikes occur, they preferentially show up in all random gauges, and very often more than one spike is found. Thus, spikes and Gribov copies occur, during the process of updating, in lumps.

- The appearance of Gribov copies is not exactly correlated with the number of spikes, i.e. not on a 1:1 basis for each configuration. There are configurations without copies, but with spikes, and vice versa.

- If both the number of copies and the number of spikes per configuration are smeared over a few adjacent measured configurations (just to improve the presentation), a striking correlation shows up. This is demonstrated in fig. 3. On the left-hand side, the first 100,000 sweeps are shown, with the slim curve giving the number of copies, and the fat curve giving the number of spikes (averaged over the 12 random gauges). The results from the next 100,000 sweeps are plotted on the right hand side.

It is evident, that the appearance of Gribov copies and of gauge singularities both do not stop abruptly when the lattice becomes small. Presumably, both phenomena are connected with the tunneling of configurations from one ”smooth” state to another one.

## 5 Cooling a Gauged Configuration

The nature of the spikes can be studied further if smoothing by F.f. is confronted with the results of the cooling technique. By the latter, the quantum noise in gauge invariant observables is reduced during the approach to metastable minima of the action. In this way, non-perturbative aspects of the configuration, like instantons, may emerge. Since instanton configurations are close to local minima of the action, the distinction between genuine instantons and artifacts of the cooling procedure, is subjective. A comparison of the gauge fields etc. of a cooled...
configuration with the same configuration modified by F.f., may be helpful to understand in what sense instantons exist on the lattice.

For this purpose it is necessary to quantify the impact of F.f. on instantons. This depends on the gauge. For standard instantons in the regular gauge, the effect of filtering is quite modest: The peak action of an artificial instanton of radius $7a$ (this corresponds to a radius $\rho = 0.20 \text{fm}$ at $\beta = 2.85$) will be reduced by F.f. not too strongly, if the filtering is done with a strong cut-off, i.e. $\lambda^2 = 4.0a^2$. The reduction amounts to a factor

$$S(x = 0, \lambda^2 = 4.0a^2)/S(x = 0, \lambda^2 = 0.50a^2) \approx 0.49.$$  

Thus, F.f. will not seriously affect those instantons, which are the most numerous ones in SU(2), according to ref. [5]. Smaller ones will feel a stronger reduction in peak height, such that they essentially look like broader instantons.

On the other hand, as has been stated already, instantons in the singular gauge will show up, under F.f., as spikes and, therefore, cannot be missed. I assume, more generally, that F.f. will not reduce the number of visible instantons, if the creation and observation of spikes is properly taken into account. Further-
more, F.f. will not create quasi-instantons nor improve the self-dual properties of maxima, in obvious contrast to the cooling technique. I have studied correlations between spikes and cooled objects in two ways. First of all, around the location of spikes in a given filtered configuration (denoted by $\mathcal{F}$), the cooled configuration is studied visually. Also, action maxima are studied, which are not associated with spikes. Secondly, the inverse correlation will be investigated, i.e. maxima in the cooled configuration, called $\mathcal{C}$, are selected and the filtered configuration $\mathcal{F}$ is visualized around the corresponding locations. In the final subsection, the probability for obtaining spikes by gauging cooled configurations is investigated. The number of spikes and -to a large degree- their positions seem to be independent on the amount of cooling.

5.1 Spikes $\Rightarrow$ Cooled Configuration

The cooling sequence starts from the same Landau gauged configurations as the filtered ones. Four configurations have been cooled both with 30 and 100 steps, using "strong" cooling. There, each step rotates each link variable to the local action minimum. A total of 37 spikes has been found, out of an ensemble of 128 action maxima.

For the spikes, the resulting correlation between the filtered configuration and the cooled one is quite simple: At the location of the 36 spikes one finds, first of all, always gauge singularities in $\mathcal{C}$, and, secondly, one observes that the spikes are, apart from 7 cases\(^\text{10}\), associated with instantons.

The first observation means that the cooled gauge fields $A_\mu^a(x)$ show clean peaks around the position of the spikes, including sign flips in all colours and directions, with a diameter of 3 or 4 lattice unit. Outside of the peaks, the gauge fields are noisy and not particularly small. This is due to the fact that cooling does not minimize the gauge fields, and Landau gauging still preserves their perturbative fluctuations. The positions of the gauge singularity and the nature of the gauge field peaks are independent of the number of cooling steps, as has been checked by comparing the fields for 30 and 100 cooling steps.

The second observation means that at the position of a spike, pronounced maxima of the action and of the topological charge density show up in $\mathcal{C}$. The topological charge density has the opposite sign in the two cases cooling and filtering\(^\text{11}\). After cooling, the heights of the action-peaks vary strongly from peak to peak. The explanation is that for instantons, the height of the action-maxima is a strongly decreasing function of the instanton size, such that large instantons do not show up spectacularly under visualization techniques. Obviously, selecting positions by spikes does not select instantons of a particular radius.

For action-maxima observed after F.f. which are not spikes, the situation at the same position in $\mathcal{C}$ is not very clear-cut. At some maxima one may find an instanton-like object, at others there is one close by, and in many cases one cannot observe any activity in a neighbourhood of reasonable size.

\(^{10}\)An inspection of these exceptions reveals that either the cooled maximum of $\hat{q}(x)$ is quite low, or that cooling leads to an annihilation of a instanton-anti-instanton pair.

\(^{11}\)This is in agreement with the interpretation of spikes as a mismatch of the quadratic terms and the higher order terms, caused by suppression of high frequency terms via filtering.
5.2 Maxima of Cooled Configuration \Rightarrow Filtered Configuration.

The inverse correlation, i.e. between cooled action \((|\hat{q}(x)|)\) maxima selected in \(C\) on the one hand, and between structures in \(F\) (obtained by filtering) on the other hand, is more complicated to investigate than the previous one. This holds because there is much freedom in the selection of maxima. It is not the purpose of this paper to follow the elaborate filtering techniques of ref. [8] for extracting the best instanton candidates among the many maxima which show up during the cooling process. I simply start from the 4 configurations which have been "strongly" cooled by 100 sweeps, select for each configuration 32 positions associated with the 32 highest maxima of \(|\hat{q}(x)|\), and investigate the properties of \(F\) around the positions of these maxima. The filtering is done both with parameters \(\lambda^2 = 4a^2\) and \(\lambda^2 = 0.5a^2\).

The first observation is that one recovers \(\approx 80\%\) of the spikes among the first \(O(30)\) maxima. This is not trivial. The sizes of the instantons which are associated with spikes, vary considerably, such that a selection via the \(|\hat{q}(x)|\)-peak height might lead to a failure in identifying these maxima within the first few dozens of peaks. I conclude that the spikes are tightly correlated with instantons which are produced by cooling and selected among the spatially less extended ones. Obviously, the spikes have a significance beyond the gauge dependent filtering technique.

Next, it is highly interesting to investigate those locations in the filtered configuration, \(F\), which on the one hand are associated with peaks in \(C\) but, on the other hand, have no singularity in the gauge fields at this location. In a fraction of about 2/3 of those maxima, one encounters also a peak in \(F\) both in the action and in \(|\hat{q}(x)|\), with identical signs of \(\hat{q}(x)\) in both configurations, i.e. (anti-)instantons in non-singular gauge in \(C\) may be associated with candidates for (anti-)instantons in \(F\). In the other 1/3 cases, there is no significant activity in \(F\).

Now, the crucial question is whether these candidates have the correct properties to be unambiguously identified with instantons. This has to be doubted for the following reasons:

- In \(F\), the orientation of the colour components in colour space is not along the diagonal. This fact is best recognized by visual inspection, and it will not be specified in detail here.

- The electric and magnetic fields in \(F\) are not perfectly self-dual. In order to be quantitative, I consider the measure of self-dual quality, \(Q^{(a)}(y)\), defined by

\[
Q^{(a)}(y) = \frac{2E^*(a)(y)B^*(a)(y)}{E^2(a)(y) + B^2(a)(y)}, \quad a = 1, 2, 3. \tag{29}
\]

where \(y\) is taken at the lattice positions around the maxima position \(x\) in \(C\), with \(|x - y| < 4a\). This distance amounts to half the radius of a typical SU(2)-instanton. A histogram of this quantity, taken between \(-1 \geq Q^{(a)}(y) \geq 1\), is peaked beautifully at the limits \(\pm 1\) for the cooled configuration. For the filtered configuration, however, \(Q^{(a)}(y)\) shows a
broad histogram for all $a$, extending down to $Q^{(a)}(y) \approx 0$. Thus, self-duality is not realized well in the case of F.f.

• The spatial sizes of the peaks in $\mathcal{F}$ are much smaller than in $\mathcal{C}$. Since the shape of the topological objects seems to be rather irregular, their radii are difficult to determine directly. A simpler way is to observe the reduction of the peak height as a function of the filtering parameter, when it is increased from $\lambda^2 = 0.5a^2$ to $\lambda^2 = 4a^2$. This reduction amounts approximately to a factor $0.03$, in sharp contrast to the reduction by a factor $0.49$ for an artificial lattice instanton (see eq. 28).

It has to be concluded that a close correspondence between cooled configurations and filtered ones exists mainly at the position of the spikes. In the cooled configuration, the gauge singularity -the origin of the spike- is preserved, and an instanton-like maximum of the action etc. has developed. Action-maxima in $\mathcal{C}$, which are not associated with spikes, and which may therefore be displayed in the filtered configuration, do not reveal the typical properties of instantons. Of course, by a more refined search among the many maxima in $\mathcal{C}$ one may eventually find better instanton candidates.

5.3 Gauge and Cooling (In-)Dependence of Spike Positions

In the previous subsection, it has been stated that almost all spikes in the filtered configurations are found at a position close to maxima in the cooled configurations. The relevant maxima are those with the largest values of the action density. This is rather important, since the positions of these cooled maxima are gauge independent. Thus, also the position of spikes has a gauge invariant meaning, at least in some probabilistic sense.

Furthermore, in the process of cooling, most small scale fluctuations are eliminated which, in principle, could induce the gauge singularities. The presence of such ”dislocations” which can contribute to the topological charge is a drawback of using the Wilson action both for updating configurations and for cooling.

In the following, the effect of cooling will be studied once more in a different way. In determining the correlation between the cooled maxima and the spikes, the latter were defined by Landau gauging a configuration which had all short range fluctuations undamped. Here, the order of cooling and gauging will be reversed, i.e. a non-gauged configuration will be cooled with up to 10 strong cooling steps. This eliminates all plaquette values smaller than 0.9. If one then finds a Landau gauge and applies filtering to this cooled configuration, one finds approximately the same number of spikes as compared to the case without cooling. Most of the spikes show up at the same positions in the two cases.

In detail, on a lattice of size $24^3 \times 32$ at $\beta = 2.65$, 20 configurations have been cooled with 3 steps (case (a)), and 20 configurations with 10 steps (case (b)). After a random gauge had been applied, first the Landau gauging and then filtering were performed. In case (a) one finds 93 spikes, whereas for the corresponding uncooled configurations one finds 87 spikes. In case (b), the corresponding numbers are 99 and 91. When the spike positions are compared between the cooled
and the uncooled configurations, it turns out that in 15 configurations out of the 20 cooled ones (in case (b)), more than 50% of the spikes show up at the same positions as in the uncooled case (within 1 or 2 lattice units).

Because of the strong suppression of plaquettes with trace values \(< 0.9\), this correlation of spike positions makes it rather unlikely that the spikes are induced by dislocations, in so far as these are characterized by large negative plaquette values.

6 The Gauge Field Propagator and Spikes

It is evident that close to a spike, the gauge fields show a rapid flip of sign, for all colours, for all space time orientations and along all directions. Thus, it is natural to expect that the gauge field correlators behave differently for configurations with spikes as compared to configurations without spikes. A convenient tool to study this effect is a measurement of the gluon propagator, evaluated separately for the two specimens of configurations. It is to be expected that the propagator in x-space falls off more rapidly for increasing spatial separation, when spikes are present than if none are around. In momentum space, this effect is reproduced if the zero momentum propagator is reduced and the small -but non-zero- momentum propagator is enhanced when spikes are present, as compared to the spike-free configurations.

This can be tested on lattices which are so small that -simply by geometrical considerations- the probability to find a spike is considerably smaller than one. For our standard value of \(\beta = 2.85\), a lattice of size \(24^3 \times 32\) has a density of 0.4 spikes per configuration (see eq. [27]). Since often there are more than one spike per configuration, the probability to find at least one spike is around 0.2, i.e. considerably smaller than 0.4. Thus for a total of 600 configurations which have been measured\(^{12}\), one finds 480 configurations without spikes and 120 configurations with one or more spikes. The gauge field propagator has been measured separately for the two classes of configurations.

The difference between the two values of the propagator, -spikes present or not- depends on the momentum \(q^2\). For \(q^2 = 0\), the propagator with spikes is slightly smaller than without spikes:

\[
G(q^2 = 0)_{\text{no spikes}} \approx (1.19 \pm 0.05) G(q^2 = 0)_{\text{spikes}}.
\]

This difference is just significant. For \(q^2 > 0\), the error is significantly smaller, and we have

\[
G(q^2 \approx 0.1a^{-2})_{\text{no spikes}} \approx (0.78 \pm 0.03) G(q^2 \approx 0.1a^{-2})_{\text{spikes}}.
\]

For \(q^2 > 1/a^2\), no significant difference between the propagators of the two classes could be observed. It is straightforward to transform the different behaviour of the propagators to gauge field correlators in x-space. If one sums the gauge fields over 3 spatial dimensions and considers the correlator of this average along the t-direction,

\[
C(T) = \langle \sum_{\overrightarrow{x}} A(t, \overrightarrow{x}) \sum_{\overrightarrow{x}} A(t + T, \overrightarrow{x}) \rangle / \langle (\sum_{\overrightarrow{x}} A(t, \overrightarrow{x}))^2 \rangle,
\]

\(^{12}\)Out of 300,000 sweeps, after 500 updates Landau gauging and searching for spikes was performed.
then the inequalities $30$ and $31$ imply that the correlator with spikes decreases faster with $T$ than the correlator without spikes. The effect is small but significant.

7 Conclusions

Configurations of SU(2) lattice theory, when transformed to a Landau gauge, reveal special points with a scale invariant $^{13}$ density of $1.5$ points per $fm^4$, where the gauge fields show a singularity, regulated by the lattice. This singularity clearly shows up either if the gauged configuration is cooled -with $O(50)$ cooling steps- or if the high momentum Fourier components are suppressed by some exponential cut-off. In detail, the phenomena are:

- After cooling a gauged configuration, the gauge fields resemble those of a singular gauge (see eq. 1), i.e. they shoot up and change sign at the special points. The gauge invariant action and the topological charge density signal the appearance of instantons of various sizes around those singularities, with the gauge fields in the singular gauge. Other -nonsingular- instanton-like objects show up visually with a density which is higher than that of the singular ones. This density drops quickly under prolonged cooling. The instantons associated with singularities are stable under cooling.

- After removing the high momentum amplitudes by Fourier filtering, the singularities show up as spikes in the Wilson action and in the topological charge density. This is so because the removal of short range Fourier amplitudes destroys the cancellation between linear and quadratic terms in the field strengths. The gauge fields show zeroes, as a function of lattice positions, in all colours and directions, but the peaks are smoothed out relative to the cooling procedure. When more and more Fourier amplitudes are removed, the spikes get, of course, broader. However, the positions do not vary essentially.

- The positions of the spikes are not completely independent of the special Landau gauge, but almost so. This means that for different Gribov copies of a given configuration, almost all spikes appear at the same position, with a mismatch in the order of $10\%$ (see [1]). Furthermore, the spikes are strongly correlated with the -gauge invariant- positions of the leading action maxima which are generated by cooling.

- The density of spikes is correlated with the appearance of of Gribov copies. This holds in the sense that on physically small lattices, where Landau gauging is almost unique and where the probability for the occurrence of spikes is smaller than unity, Gribov copies preferentially show up in the same configurations as spikes do.

- The gauge field propagator is different for configurations with spikes as compared to the case without spikes. This shows up as a faster temporal decrease of zero momentum gauge field correlators.

$^{13}$This holds if the scale as a function of $\beta$ is taken from a measurement of the string tension.
In summary, it is evident that the presence of spikes is strongly correlated with other nonperturbative phenomena on the lattice. In particular, the correlation with the gluon propagator implies that the presence of spikes is connected with the decrease of the propagator in x-space. The sign flip of gauge fields, which are associated with gauge singularities, intuitively provides a mechanism for a fast decorrelation of the fields. According to ref. [16], such a decorrelation can be responsible for deconfinement. A study of the correlation of the string tension with the spike density on larger lattices seems to be topic for future investigations.

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