A Bernstein-type inequality for stochastic processes of quadratic forms of Gaussian variables

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Abstract: We introduce a Bernstein-type inequality which serves to uniformly control quadratic forms of gaussian variables. The latter can for example be used to derive sharp model selection criteria for linear estimation in linear regression and linear inverse problems via penalization, and we do not exclude that its scope of application can be made even broader.

A Bernstein-type inequality for quadratic forms of gaussian variables

The concentration phenomenon of stochastic processes around their mean is of key importance in statistical estimation by model selection for getting non-asymptotic bounds for some statistics. For example in model selection via penalization, for devising sharp penalties and proving useful upper bounds for the risk of an estimator, one needs generally to control uniformly the statistic of the risk of an estimator by means of a sharp concentration inequality. This topic has received since recently (late nineties) a considerable interest among the statistical community above all further to the amazing series of works of Michel Talagrand which can be seen as the infinite dimensional analogue of the Bernstein’s inequality (see in particular (7) for an overview and (8) for later advances). Their application in non-asymptotic model selection has first been discovered by Birgé and Massart (e.g. (2)), then refined and popularized by the same authors (e.g. (3; 4)). For beautiful lectures on the topic, we refer the dear reader to (5; 6).

In this small body of work, we establish a new Bernstein-type inequality which serves to control (e.g. uniformly) quadratic forms of Gaussian variables and which happens to be useful for controlling, for example, uniformly the quadratic risk of a finite (or a countable) set of linear estimators in linear regression and linear inverse problems (see (1) for an application). In the remainder, we will give both the uncorrelated form and the correlated form of such an inequality.

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Lemma 0.1. Let \( a = (a_k)_{k=1,p} \) and \( b = (b_k)_{k=1,p} \) be two \( p \)-dimensional real vectors, and consider the following random expression: \( T = \sum_{k=1}^p a_k z_k^2 + b_k z_k \), where \( z_k, k = 1, \ldots, p \) are i.i.d. \( N(0,1) \), and let’s put: \( a^+ = \sup\{\sup_{k=1,\ldots,p}(a_k), 0\} \), \( a^- = \sup\{\sup_{k=1,\ldots,p}(-a_k), 0\} \). Then the following two concentration results hold true for all \( x > 0 \):

\[
\mathbb{P} \left[ T \geq \sum_{k=1}^p a_k + 2 \sqrt{\sum_{k=1}^p a_k^2 + \frac{b_k^2}{2} x + 2a^+x} \right] \leq \exp[-x] \tag{1.1}
\]

\[
\mathbb{P} \left[ T \leq \sum_{k=1}^p a_k - 2 \sqrt{\sum_{k=1}^p a_k^2 + \frac{b_k^2}{2} x - 2a^- x} \right] \leq \exp[-x] \tag{1.2}
\]

The proof of the lemma is rather technical, so it is deferred to the appendix section.

The following lemma uses the concentration results of lemma (1.1) to control more general quadratic forms of Gaussian variables involving a matrix.

Lemma 0.2. Consider the random expression \( T = z^T A z + b^T z \), where \( A \) is \( p \) by \( p \) real square matrix, \( b \) is a \( p \)-dimensional real vector, and \( z = (z_k)_{k=1,p} \) is a \( p \)-dimensional standard gaussian vector, i.e. \( z_k, k = 1, p \) are i.i.d. zero-mean gaussian variables with standard deviation 1. Let’s denote by \( s_k, k = 1, p \) the eigen values of the symmetric matrix \( \frac{1}{2}(A + A^T) \), and let’s put \( s^+ = \sup\{\sup_{k=1,\ldots,p}(s_k), 0\} \), and \( s^- = \sup\{\sup_{k=1,\ldots,p}(-s_k), 0\} \). Then, the following two concentration results hold true for all \( x > 0 \):

\[
\mathbb{P} \left[ T \geq \text{tr}(A) + 2 \sqrt{\frac{1}{4} ||A + A^T||^2 + \frac{1}{2} ||b||^2} \sqrt{x} + 2s^+ x \right] \leq \exp[-x] \tag{1.3}
\]

\[
\mathbb{P} \left[ T \leq \text{tr}(A) - 2 \sqrt{\frac{1}{4} ||A + A^T||^2 + \frac{1}{2} ||b||^2} \sqrt{x} - 2s^- x \right] \leq \exp[-x] \tag{1.4}
\]

Proof. One can rewrite \( T \) as follows: \( T = z^T A z + b^T z = z^T A z + b^T z = \frac{1}{2} z^T (A + A^T) z + b^T z \), and by using the eigen value decomposition of the symmetric matrix \( \frac{1}{2}(A + A^T) \) one derives \( T = \sum_{k=1}^p \frac{1}{2} s_k z_k'^2 + b' z' \), where \( s_k, k = 1, p \) are the respective eigen values of \( \frac{1}{2}(A + A^T) \), \( z' = U^T z \) with \( U \) standing for the (orthonormal) eigen matrix of \( \frac{1}{2}(A + A^T) \), and \( b' = U^T b \). Then, by noticing that \( z' \) stands for a \( p \)-dimensional standard gaussian vector, \( ||b'||^2 = ||b||^2 \), \( \sum_{k=1}^p s_k = \text{tr}(A) \), and \( \sum_{k=1}^p s_k^2 = \frac{1}{2} ||A + A^T||^2 \), so by applying lemma (1.1), the proof of lemma (1.2) follows immediately.

References

[1] Bechar, I. (2009). Non-asymptotic model selection for linear non least-squares estimation in regression models and inverse problems. submitted to Elec. J. Stat.
Appendix A: Proof of Lemma (0.1)

Proof. We make use of the following lemma for proving lemma (0.1)

**Lemma A.1** (Birgé & Massart 1998). If a random variable $\xi$ satisfies for some two real positive numbers $u$ and $v$ the following inequality :

$$\log \left( \mathbb{E} \left[ \exp(y\xi) \right] \right) \leq \frac{(uy)^2}{1-vy}, \text{ for all } 0 < y < \frac{1}{v} \quad (A.1)$$

then

$$\mathbb{P} \left[ \xi \geq 2u\sqrt{x} + vx \right] \leq \exp[-x], \text{ for all } x > 0 \quad (A.2)$$

We refer the reader to (2) for a proof of this lemma.

Now, to prove lemma (0.1), one can notice first that concentration inequality (0.2) can be obtained from (0.1) by considering the random quantity

$$T' = -T = \sum_{k=1}^{p} (-a_k)z_k^2 + (-b_k)z_k$$

and by applying (0.3) on $T'$ instead of $T$. So, we need to prove only (0.3). To do this, let us rewrite $T$ as follows: $T = \sum_{k=1}^{p} T_k$, where $T_k = a_kz_k^2 + b_kz_k$, and let us compute $\log \left[ \mathbb{E} \left( \exp(y(T - \bar{T})) \right) \right]$, where $\bar{T} = \sum_{k=1}^{p} a_k$. We have

$$\mathbb{E} \left[ \exp(yT_k) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2} ((1-2a_ky)t^2 - 2yb_kt) \right] dt$$
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\[ E[\exp[yT_k]] = \exp \left[ \frac{b_k^2 y^2}{1 - 2a_k y} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2} \left( \sqrt{1 - 2a_k y} - \frac{b_k y}{\sqrt{1 - 2a_k y}} \right)^2 \right] dt \right) \right] \]

\[ E[\exp[yT_k]] = \exp \left[ \frac{b_k^2 y^2}{1 - 2a_k y} \right] \]

\[ E[\exp[y(T_k - a_k)]] = \exp \left[ \frac{b_k^2 y^2}{1 - 2a_k y} \exp[-ya_k] \right] \]

\[ \log \left( E[\exp[y(T_k - a_k)]] \right) = \frac{b_k^2 y^2}{1 - 2a_k y} - \frac{1}{2} \log \left( 1 - 2a_k y \right) - a_k y \]

Then, by putting \( a^+ = \sup \left\{ \sup_{k=1,\ldots,p} \{a_k\}, 0 \right\} \), one derives (see the technical details below) that for all \( 0 < y < \frac{1}{2a^+} \):

\[ \log \left( E[\exp[y(T_k - a_k)]] \right) \leq \frac{(a_k^2 + \frac{b_k^2}{2}) y^2}{1 - 2a^+ y} \leq \frac{(a_k^2 + \frac{b_k^2}{2}) y^2}{1 - 2a^+ y} \]

which implies by independence that for all \( 0 < y < \frac{1}{2a^+} \):

\[ \log \left( \mathbb{E}[\exp[y(T - \bar{T})]] \right) \leq \sum_{k=1}^{p} \left( \frac{(a_k^2 + \frac{b_k^2}{2}) y^2}{1 - 2a^+ y} \right) \]

\[ \log \left( \mathbb{E}[\exp[y(T - \bar{T})]] \right) \leq \frac{\left( \sum_{k=1}^{p} (a_k^2 + \frac{b_k^2}{2}) \right) y^2}{1 - 2a^+ y} \]

Finally, by applying lemma (A.2) below with \( u = \sqrt{\sum_{k=1}^{p} (a_k^2 + \frac{b_k^2}{2})} \), and \( v = 2a^+ \), one derives that for all \( x > 0 \):

\[ \mathbb{P} \left[ T \geq \left( \sum_{k=1}^{p} a_k \right) + 2 \sqrt{\sum_{k=1}^{p} (a_k^2 + \frac{b_k^2}{2}) \sqrt{x} + 2a^+ x} \right] \leq \exp[-x] \]

This terminates the proof of lemma (0.1) \( \square \)

**Some additional technical details about the proof**

We will show here that for all \( r > 0, a \geq r \), and \( 0 < y < \frac{1}{2a} \), one has

\[ \frac{-1}{2} \log(1 - 2ry) - ry \leq \frac{r^2 y^2}{1 - 2ay} \quad (A.3) \]
and that for all $r \leq 0$, for all $a > 0$, and for all $0 < y < \frac{1}{2a}$, one has
\[
-\frac{1}{2} \log(1 - 2ry) - ry \leq \frac{r^2y^2}{1 - 2ay}
\]  
(A.4)

Proof. let us start by showing inequality (A.3). To do this, let us consider the following function
\[
f_{r,a}(y) = -\frac{1}{2} \log(1 - 2ry) - ry - \frac{r^2y^2}{1 - 2ay}
\]
One first notices that $f_{r,a}(0) = 0$, then a sufficient condition for inequality (A.3) to hold true is that $f_{r,a}(y) \leq 0$, for all $0 < y < \frac{1}{2a}$. We have
\[
f_{r,a}(y) = -\frac{1}{2} \log(1 - 2ry) - ry + \frac{r^2y^2}{2a} + \frac{r^2}{(2a)^2} - \frac{r^2}{1 - 2ay}
\]
one then derives that
\[
f_{r,a}(y)' = \frac{r}{1 - 2ry} - r + \frac{r^2}{2a} - \frac{r^2}{(1 - 2ay)^2}
\]
\[
f_{r,a}(y)' = \frac{2r^2y}{1 - 2ry} - \frac{r^2y}{(1 - 2ay)} - \frac{r^2y}{(1 - 2ay)^2}
\]
\[
f_{r,a}(y)' \leq \frac{2r^2y}{1 - 2ry} - \frac{2r^2y}{(1 - 2ay)}
\]
and finally since \(\frac{1}{1 - 2ry} \leq \frac{1}{1 - 2ay}\), one deduces that
\[
f_{r,a}(y)' \leq \frac{2r^2y}{1 - 2ay} - \frac{2r^2y}{(1 - 2ay)} = 0
\]
then we have shown (A.3).
We proceed in the same way as for showing inequality (A.3) to show inequality (A.4). So let us consider the following function
\[
g_{r,a}(y) = -\frac{1}{2} \log(1 - 2ry) - ry - \frac{r^2y^2}{1 - 2ay}
\]
One first notices that $g_{r,a}(0) = 0$, then a sufficient condition for inequality (A.4) to hold true is that $g_{r,a}(y)' \leq 0$ for all $0 < y < \frac{1}{2a}$. One derives that
\[
g_{r,a}(y)' = \frac{r}{1 - 2ry} - r + \frac{r^2}{2a} - \frac{r^2}{(1 - 2ay)^2}
\]
\[
g_{r,a}(y)' = \frac{2r^2y}{1 - 2ry} - \frac{r^2y}{(1 - 2ay)} - \frac{r^2y}{(1 - 2ay)^2}
\]
and finally, since \( \frac{1}{1-2y} \leq \frac{1}{1-2ay} \), one finds that

\[
gr_{r,a}(y)' \leq \frac{2r^2y}{1-2ay} - \frac{2r^2y}{(1-2ay)} = 0
\]

\[\square\]

**Birge's & Massart concentration inequality**

**Lemma A.2.** If a random variable \( \xi \) satisfies for some two real positive numbers \( u \) and \( v \) the following inequality:

\[
\log \left( \mathbb{E} \left[ \exp(y \xi) \right] \right) \leq \frac{(uy)^2}{1-vy}, \text{ for all } 0 < y < \frac{1}{v} \quad (A.5)
\]

then

\[
P \left[ \xi \geq 2u\sqrt{x} + vx \right] \leq \exp[-x], \text{ for all } x > 0 \quad (A.6)
\]

The proof of this lemma can be found in (2).