A Necessary and Sufficient Condition for Stability of Linear Difference Equations with Two Delays

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Abstract

This note provides a necessary and sufficient condition for guaranteeing exponential stability of the linear difference equation

\[ x(t) = Ax(t - a) + Bx(t - b) \]

where \( a > 0, b > 0 \) are constants and \( A, B \) are \( n \times n \) square matrices, in terms of a linear matrix inequality of size \( 3n^2 \times 3n^2 \) and the spectral radius of the matrix \( A + B \).

**Keywords:** Linear difference equations; Exponential stability; Necessary and sufficient conditions; Linear matrix inequality.

1 Introduction

Throughout this note, we use \( A \otimes B \) to denote the Kronecker product of matrices \( A \) and \( B \). For a matrix \( A \), the symbols \( |A|, \|A\|, A^T, A^H \), and \( \rho(A) \) denote respectively its determinant, norm, transpose, conjugate transpose, and spectral radius. For a square matrix \( P, P > 0 \) denotes that it is positive definite.

The linear (continuous-time) difference equation

\[ x(t) = \sum_{i=1}^{N} A_i x(t - r_i), \quad t \in \mathbb{R}^+, \]

where \( r_i > 0, i = 1, 2, \ldots, N \), are constants, \( A_i, i = 1, 2, \ldots, N \), are square matrices, is frequently encountered in neutral-type time-delay systems [6, 10, 11, 15] and coupled differential-functional equations [5, 9]. The stability of such linear difference equation (1) is usually the necessary condition for ensuring the asymptotic stability of the above two types of time-delay systems, and thus has attracted considerable attentions in the literature [1, 2, 4, 5, 7, 12, 13].

In this note, we restrict ourself to a special case of (1) where \( N = 2 \), for which we rewrite (1) as

\[ x(t) = Ax(t - a) + Bx(t - b), \]

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where \( a, b \) are positive constants, and \( A, B \) are \( n \times n \) square matrices. The initial condition (function) for the difference equation (2) is \( \varphi(s), s \in [-\max\{a, b\}, 0] \), which is such that \( \varphi(0) = A\varphi(-a) + B\varphi(-b) \). Denote the corresponding solution by \( x(t, \varphi), t \geq 0 \). Regarding the existence of a solution, the continuity/discontinuity of the solution, and definitions for stability of the solution, readers are suggested to refer [1] and [7] for details. Here we only recall that the difference equation (2) is said to be exponentially stable if there exist two positive constants \( k \) and \( \alpha \) such that
\[
\|x(t, \varphi)\| \leq ke^{-\alpha t} \sup_{s \in [-\max\{a, b\}, 0]} \|\varphi(s)\|, \quad \forall t \geq 0.
\]
As we have mentioned in the above, stability of difference equation (2) has been extensively investigated in the literature. We first recall the following well-known result.

**Lemma 1** [7, 8] The statements below are equivalent:

1. The difference equation (2) is exponentially stable for all delay \((a, b)\) that satisfy \( a > 0 \) and \( b > 0 \).

2. There exists an \( \varepsilon > 0 \) such that the difference equation (2) is exponentially stable for all delay \((a, b)\) that satisfy
\[
|a - a^*| < \varepsilon, \quad |b - b^*| < \varepsilon,
\]
for some fixed nominal delay \((a^*, b^*)\), where \( a^* > 0 \) and \( b^* > 0 \).

3. There holds
\[
\rho(A + Be^{j\theta}) < 1, \quad \forall \theta \in \mathbb{R}.
\] (3)

It follows that the exponential stability of the difference equation (2) is guaranteed under small parameter deviation (no matter how small the deviation is) is equivalent to the exponential stability of the difference equation for all possible positive delays [3], which case sometimes is referred to as strong stability. The strong stability concept is important since in practical applications these delays are generally subject to small errors [5]. In this note, we are interested in such kind of stability. Notice that the necessary and sufficient condition (3) is not tractable in general since the spectral radius should be tested for all \( \theta \in [0, 2\pi] \). Therefore, such a necessary and sufficient condition was not directly adopted in the existing literature, while sufficient conditions that are easy to test are more popular. We next recall some existing sufficient conditions for guaranteeing the stability of the difference equation (2).

**Lemma 2** [1] The linear difference equation (2) is exponentially stable in the sense of Lemma [1] if
\[
\|A\| + \|B\| < 1.
\] (4)
Lemma 3 [10, 14] The linear difference equation (3) is exponentially stable in the sense of Lemma 1 if there exists a scalar \( \delta \in (0, 1) \) such that
\[
\rho \left( \delta A \otimes A + \frac{1}{1 - \delta} B \otimes B \right) < 1. \tag{5}
\]

Lemma 4 [1, 4] The linear difference equation (3) is exponentially stable in the sense of Lemma 1 if there exist two positive definite matrices \( X \in \mathbb{R}^{n \times n} \) and \( Y \in \mathbb{R}^{n \times n} \) such that the following linear matrix inequality (LMI) is satisfied
\[
\begin{bmatrix}
A & B \\
I_n & 0
\end{bmatrix}^T \begin{bmatrix}
X & 0 \\
0 & Y
\end{bmatrix} \begin{bmatrix}
A & B \\
I_n & 0
\end{bmatrix} - \begin{bmatrix}
X & 0 \\
0 & Y
\end{bmatrix} < 0. \tag{6}
\]

Lemma 5 [2] The linear difference equation (3) is exponentially stable in the sense of Lemma 1 if there exist four positive definite matrices \( S_1 \in \mathbb{R}^{n \times n}, S_2 \in \mathbb{R}^{n \times n} \), and
\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{12}^T & P_{22}
\end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{7}
\]
\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{12}^T & Q_{22}
\end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{8}
\]
such that the following LMI is satisfied
\[
\Pi = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & A^T Q_{12} \\
\Pi_{12}^T & \Pi_{22} & B^T Q_{12} - P_{12}^T \\
Q_{12}^T A & Q_{12}^T B - P_{12} & Q_{22} - P_{11}
\end{bmatrix} < 0, \tag{9}
\]
where
\[
\Pi_{11} = A^T (S_1 + S_2 + P_{11} + Q_{11}) A + P_{22} - Q_{11} - S_1 + P_{12}^T A + A^T P_{12},
\]
\[
\Pi_{12} = A^T (S_1 + S_2 + P_{11} + Q_{11}) B + P_{12}^T B - Q_{12},
\]
\[
\Pi_{22} = -P_{22} - Q_{22} - S_2 + B^T (S_1 + S_2 + P_{11} + Q_{11}) B.
\]

In this note, motivated by the method in [3], we will provide for the difference equation (3) a necessary and sufficient condition for guaranteeing exponential stability in the sense of Lemma 1. The obtained condition is expressed by an LMI of size \( 3n^2 \times 3n^2 \) and the spectral radius of a matrix.

2 The Main Result

The main result of this note is the following theorem.

**Theorem 1** The linear difference equation (3) is exponentially stable in the sense of Lemma 1 if and only if
\[
\rho (A + B) < 1, \tag{10}
\]
and there exist two symmetric matrices \( P_1 \in \mathbb{R}^{n^2 \times n^2}, P_2 \in \mathbb{R}^{n^2 \times n^2} \) and a matrix \( P_3 \in \mathbb{R}^{n^2 \times n^2} \) such that
\[
\begin{bmatrix}
-P_1 & 0 & -P_3 \\
0 & -P_2 & P_3^T \\
-P_3^T & P_3 & P_1 + P_2
\end{bmatrix} < E^T E,
\] (11)
where
\[
E = \begin{bmatrix}
B^T \otimes A & A^T \otimes B & A^T \otimes A + B^T \otimes B - I_n \otimes I_n
\end{bmatrix}.
\]

It is not difficult to see that if we change the order of \( A \) and \( B \), the LMI (11) keeps the same if we change the role of \( P_1 \) and \( P_2 \), which is reasonable since there is no assumption on the order of \( A \) and \( B \). To the best of our knowledge, necessary and sufficient stability conditions for the difference equation (2) other than (3) have never been reported in the literature. Therefore, we believe that the condition in Theorem 1 is the first necessary and sufficient stability condition that is easy to test.

**Remark 1** We mention that the spectral radius condition (10) is not implied by the LMI (11) and thus cannot be removed from our necessary and sufficient condition. To see this, we consider
\[
A = \begin{bmatrix}
-0.4 & -0.3 \\
-0.5 & 0.15
\end{bmatrix}, \quad B = \begin{bmatrix}
0.1 & 0.25 \\
-0.9 & 1.6
\end{bmatrix}.
\]
It follows that (11) is not satisfied since \( \rho(A + B) = 1.7836 > 1 \). However, the LMI (11) is indeed feasible for this pair of \((A, B)\).

### 3 A Numerical Example

**Table 1:** The maximal allowable \( \gamma \) for different methods

| Different Methods | Lem 2 | Lem 3 | Lem 4 | Lem 5 | Thm 1 | Exact |
|-------------------|------|------|------|------|------|------|
| The Maximal \( \gamma \) | 0.9999 | 0.9999 | 0.9999 | 1.4142 | 1.4142 | \( \sqrt{2} \) |

Consider the continuous-time linear difference equation (2) with
\[
A = A(\gamma) = \gamma \begin{bmatrix}
\frac{7}{2} & 0 \\
0 & -\frac{7}{2}
\end{bmatrix}, \quad B = B(\gamma) = \gamma \begin{bmatrix}
0 & \frac{7}{2} \\
-\frac{7}{2} & 0
\end{bmatrix},
\]
where \( \gamma > 0 \) is a constant. It is easy to see that \( \rho(A(\gamma) + B(\gamma)) = 0 \). According to the computation in [1], [3] is satisfied if and only if \( \gamma < \sqrt{2} = 1.4142 \cdots \), which is the necessary and sufficient condition for guaranteeing exponential stability of the difference equation. When \( \gamma = 1 \), it is shown in [1] that the LMI in Lemma 3 is not feasible. Our Theorem 1 can indeed provide a necessary and sufficient stability condition since the LMI in (11) is solvable for \( \gamma \leq 1.4142 < \sqrt{2} \). For the comparison purpose, the maximal allowable \( \gamma \) for different methods
as recalled in Lemmas 2, 5 are recorded in Table 1. It is interesting to notice that Lemma 5 also gives a necessary and sufficient stability condition. We have also done other thousands of numerical examples, and without exception, both Lemma 5 and Theorem 1 lead to the same conclusion. This motivates us to state the following conjecture.

**Conjecture 1** The linear difference equation (2) is exponentially stable in the sense of Lemma 5 if and only if the condition in Lemma 5 is satisfied.

### 4 Conclusion

This paper has established a necessary and sufficient condition for guaranteeing exponential stability of linear difference equations with two delays. The obtained condition is expressed by a linear matrix inequality and the spectral radius of a matrix. A numerical example demonstrates the effectiveness of the proposed method.

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