ON OSCILLATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. An interrelationship is found between the accumulation points of zeros of non-trivial solutions of \( f'' + Af = 0 \) and the boundary behavior of the analytic coefficient \( A \) in the unit disc \( \mathbb{D} \) of the complex plane \( \mathbb{C} \).

It is also shown that the geometric distribution of zeros of any non-trivial solution of \( f'' + Af = 0 \) is severely restricted if
\[
|A(z)|(1 - |z|^2)^2 \leq 1 + C(1 - |z|), \quad z \in \mathbb{D},
\]
(*)

for any constant \( 0 < C < \infty \). These considerations are related to the open problem whether (*) implies finite oscillation for all non-trivial solutions.

1. Introduction

The following result plays a decisive role in the oscillation theory of solutions of linear differential equation
\[
f'' + Af = 0 \tag{1.1}
\]
in the unit disc \( \mathbb{D} \) of the complex plane \( \mathbb{C} \). If \( A \) is an analytic function in \( \mathbb{D} \) for which
\[
|A(z)|(1 - |z|^2)^2 \leq 1, \quad z \in \mathbb{D},
\]
then each non-trivial solution \( f \) of (1.1) vanishes at most once in \( \mathbb{D} \). This statement corresponds to the well-known result of Z. Nehari [11, Theorem 1], which provides a sufficient condition for injectivity of any locally univalent meromorphic function \( w \) in \( \mathbb{D} \) in terms of the size of its Schwarzian derivative
\[
S_w = \frac{w''}{w'} - \frac{3}{2} \left( \frac{w''}{w'} \right)^2 = \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2.
\]

The corresponding necessary condition was invented by W. Kraus [9], and rediscovered by Nehari [11, Theorem 1] some years later. In the setting of differential equations it states that, if \( A \) is analytic in \( \mathbb{D} \), and each solution \( f \) of (1.1) vanishes at most once in \( \mathbb{D} \), then \( |A(z)|(1 - |z|^2)^2 \leq 3 \) for all \( z \in \mathbb{D} \). An important discovery of B. Schwarz [15, Theorems 3–4] shows that the condition
\[
\sup_{z \in \mathbb{D}} |A(z)|(1 - |z|^2)^2 < \infty,
\]
which allows non-trivial solutions of (1.1) to have infinitely many zeros in \( \mathbb{D} \), is both necessary and sufficient for zeros of all non-trivial solutions to be separated with respect to the hyperbolic metric.

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Our first objective is to consider the interrelationship between the accumulation points of zeros of non-trivial solutions \( f \) of \((1.1)\) and the boundary behavior of the coefficient \( A \). The second objective is a question of more specific nature. We consider differential equations \((1.1)\) in which the growth of the coefficient barely exceeds the bound \((1.2)\) that ensures finite oscillation.

2. Results

2.1. Accumulation points of zeros of solutions. The point of departure is a result, which associates the zero-sequences of non-trivial solutions of \((1.1)\) to the boundary behavior of the coefficient. This theorem sets the stage for more profound oscillation theory.

**Theorem 1.** Let \( A \) be an analytic function in \( D \), and let \( \zeta \in \partial D \).

If there exists a sequence \( \{w_n\} \subset D \) converging to \( \zeta \), such that

\[
|A(w_n)|(1 - |w_n|^2)^2 \to c \tag{2.1}
\]

for some \( c \in (3, \infty] \), then for each \( \delta > 0 \) there exists a non-trivial solution of \((1.1)\) having two distinct zeros in \( D(\zeta, \delta) \cap D \).

Conversely, if for each \( \delta > 0 \) there exists a non-trivial solution of \((1.1)\) having two distinct zeros in \( D(\zeta, \delta) \cap D \), then there exists a sequence \( \{w_n\} \subset D \) converging to \( \zeta \) such that \((2.1)\) holds for some \( c \in [1, \infty) \).

We point out that \((2.1)\) with \( c \in (3, \infty] \) does not necessarily imply infinite oscillation for any non-trivial solution of \((1.1)\), see Example 2 below.

The proof of Theorem 1 is based on theorems by Nehari and Kraus, and on a principle of localization. One of the key factors is an application of a suitable family of conformal maps under which the image of \( D \) has a smooth boundary, that intersects \( \partial D \) precisely on an arc centered at \( \zeta \in \partial D \). The second assertion of Theorem 1 is implicit in the proof of \([11, \text{Theorem 1}]\), and follows directly from the following property: if \( z_1, z_2 \in D \) are two distinct zeros of a non-trivial solution \( f \) of \((1.1)\), then there exists a point \( w \in D \), which belongs to the hyperbolic geodesic going through \( z_1 \) and \( z_2 \), such that

\[
|A(w)|(1 - |w|^2)^2 > 1.
\]

2.2. Chuaqui-Stowe question. Schwarz \([15, \text{Theorem 1}]\) supplemented the oscillation theory by proving that, if there exists a constant \( 0 < R < 1 \) such that

\[
|A(z)|(1 - |z|^2)^2 \leq 1, \quad R < |z| < 1,
\]

then each non-trivial solution of \((1.1)\) has at most finitely many zeros. Schwarz also gave an example \([15, \text{p. 162}]\) showing that the constant one in the right-hand side of \((1.2)\) is best possible. That is, for each \( \gamma > 0 \), the functions

\[
A(z) = \frac{1 + 4\gamma^2}{(1 - z^2)^2} \quad \text{and} \quad f(z) = \sqrt{1 - z^2} \sin \left( \gamma \log \frac{1 + z}{1 - z} \right)
\]

satisfy \((1.1)\), while \( f \) has infinitely many (real) zeros in \( D \). Example 1 below shows the sharpness of Kraus’ result.

M. Chuaqui and D. Stowe \([4, \text{Theorem 5}]\) constructed an example showing that for each continuous function \( \varepsilon: [0, 1) \to [0, \infty) \) satisfying \( \varepsilon(r) \to \infty \) as \( r \to 1^- \), there exists an analytic function \( A \) such that

\[
|A(z)|(1 - |z|^2)^2 \leq 1 + \varepsilon(|z|)(1 - |z|), \quad z \in D, \tag{2.2}
\]

while \((1.1)\) admits a non-trivial solution having infinitely many zeros. In other words, if \( \varepsilon(r)(1 - r) \) in \((2.2)\) does not decay to zero as fast as linear rate as \( r \to 1^- \),
then non-trivial solutions of (1.1) may have infinitely many zeros. This is in contrast to the case of real differential equations (1.1) on the open interval \((-1, 1)\), since then \(\varepsilon(r) = (1 - r)^{-1}(-\log(1-r))^{-2}\) distinguishes finite and infinite oscillation, see [2, 4] for more details. Chuaqui and Stowe [4, p. 564] left open a question whether

\[ |A(z)|(|1 - |z|^2|) \leq 1 + C(1 - |z|), \quad z \in \mathbb{D}, \]

(2.3)

with some or any \(0 < C < \infty\), implies finite oscillation for all non-trivial solutions of (1.1). The following results do not give a complete answer to this question, however, they indicate that both the growth and the zero distribution of non-trivial solutions of (1.1) are severely restricted if (2.3) holds for some \(0 < C < \infty\).

2.2.1. Growth of solutions. An analytic function \(f\) in \(\mathbb{D}\) belongs to the growth space \(H^\infty_{\alpha}\) for \(0 \leq \alpha < \infty\), if

\[ \|f\|_{H^\infty_{\alpha}} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha < \infty. \]

It is known that the growth of \(A\) restricts the growth of solutions of (1.1). If \(A \in H^\infty_{\alpha}\), then there exists a constant \(p = p(\|A\|_{H^\infty_{\alpha}})\) with \(0 \leq p < \infty\) such that all solutions \(f\) of (1.1) satisfy \(f \in H^\infty_{p}\). This result can be deduced by using classical comparison theorems [14, Example 1], Gronwall’s lemma [8, Theorem 4.2] or successive approximations [7, Theorem I], for example. We conclude this result by means of straightforward integration. See Example 3 for sharpness discussion.

**Theorem 2.** Let \(A\) be an analytic in \(\mathbb{D}\) such that \(|A(z)|(|1 - |z|^2|) \leq K + \varepsilon(|z|)\) for all \(z \in \mathbb{D}\), where \(0 \leq K < \infty\) is a constant, and \(\varepsilon(|z|) \to 0\) as \(|z| \to 1^-\). Then all solutions of (1.1) belong to \(H^\infty_{p}\) for any \((\sqrt{1 + K} - 1)/2 < p < \infty\).

2.2.2. Separation of zeros of solutions. The following result establishes a connection between the separation of zeros of non-trivial solutions of (1.1) and the growth of the coefficient function \(A\); compare to [15, Theorems 3 and 4].

If \(z_1, z_2\) are two distinct points in \(\mathbb{D}\), then the pseudo-hyperbolic distance \(\varrho_p(z_1, z_2)\) and the hyperbolic distance \(\varrho_h(z_1, z_2)\) between \(z_1\) and \(z_2\) are given by

\[ \varrho_p(z_1, z_2) = |\varphi_{z_1}(z_2)|, \quad \varrho_h(z_1, z_2) = \frac{1}{2} \log \frac{1 + \varrho_p(z_1, z_2)}{1 - \varrho_p(z_1, z_2)}, \]

where \(\varphi_z(a) = (a - z)/(1 - \overline{z}a), a \in \mathbb{D}\). Moreover, let \(\xi_h(z_1, z_2)\) denote the hyperbolic midpoint between \(z_1\) and \(z_2\). Correspondingly,

\[ \Delta_p(a, r) = \{z \in \mathbb{D} : \varrho_p(z, a) < r\}, \quad \Delta_h(a, r) = \{z \in \mathbb{D} : \varrho_h(z, a) < r\}, \]

are the pseudo-hyperbolic and hyperbolic discs of radius \(r > 0\) centered at \(a \in \mathbb{D}\), respectively.

**Theorem 3.** Let \(A\) be an analytic function in \(\mathbb{D}\).

If the coefficient \(A\) satisfies \((2.3)\) for some \(0 < C < \infty\), then the hyperbolic distance between any distinct zeros \(z_1, z_2 \in \mathbb{D}\) of any non-trivial solution of (1.1), for which \(1 - |\xi_h(z_1, z_2)| < 1/C\), satisfies

\[ \varrho_h(z_1, z_2) \geq \log \frac{2 - C^{1/2}(1 - |\xi_h(z_1, z_2)|)^{1/2}}{C^{1/2}(1 - |\xi_h(z_1, z_2)|)^{1/2}}. \]

(2.4)

Conversely, if there exists a constant \(0 < C < \infty\) such that any two distinct zeros \(z_1, z_2 \in \mathbb{D}\) of any non-trivial solution of (1.1), for which \(1 - |\xi_h(z_1, z_2)| < 1/C\), satisfies \((2.3)\), then

\[ |A(z)|(|1 - |z|^2|) \leq 3 \left(1 + \Theta_C(|z|)(1 - |z|)^{1/3}\right), \quad 1 - |z| < (8C)^{-1}, \]

(2.5)
where \( \Psi_C \) is positive, and satisfies \( \Psi_C(|z|) \longrightarrow (2(8C)^{1/3})^+ \) as \( |z| \rightarrow 1^- \).

Concerning Theorem 3 note that, if \( 1 - |\xi_h(z_1, z_2)| < 1/C \), then (2.4) implies
\[
g_h(z_1, z_2) \geq \frac{1}{2} \log \frac{1}{C} + \frac{1}{2} \log \frac{1}{1 - |\xi_h(z_1, z_2)|},
\]
and hence \( g_h(z_1, z_2) \) is large whenever \( \xi_h(z_1, z_2) \) is close to the boundary \( \partial \mathbb{D} \).

If the coefficient \( A \) satisfies (2.3) for some \( 0 < C < \infty \), and \( f_1 \) and \( f_2 \) are linearly independent solutions of (1.1), then the quotient \( w = f_1/f_2 \) is a normal function (in the sense of Lehto and Virtanen) by [16, Corollary, p. 328]. As a direct consequence we deduce the following corollary, which states that the zero-sequences of \( f_1 \) and \( f_2 \) are hyperbolically separated from each other, see also Example 4 below.

**Corollary 4.** Let \( A \) be an analytic function in \( \mathbb{D} \), which satisfies (2.3) for some \( 0 < C < \infty \), and let \( \{z_n\} \) and \( \{\zeta_m\} \) be the zero-sequences of two linearly independent solutions \( f_1 \) and \( f_2 \) of (1.1). Then, there is a constant \( \delta = \delta(f_1, f_2) \) such that \( g_h(z_n, \zeta_m) > \delta > 0 \) for all \( n \) and \( m \).

The following result shows that, if (2.3) does not imply finite oscillation for non-trivial solutions of (1.1), then infinite zero-sequences tend to \( \partial \mathbb{D} \) tangentially. Any disc \( D(\zeta, 1 - |\zeta|) \) for \( \zeta \in \mathbb{D} \), which is internally tangent to \( \mathbb{D} \), is called a horodisc. Note that Theorem 5 remains valid in the limit case \( C = 0 \) by the classical theorems of Nehari and Kraus.

**Theorem 5.** Let \( A \) be an analytic function in \( \mathbb{D} \).

If \( A \) satisfies (2.3) for some \( 0 < C < \infty \), then any non-trivial solution of (1.1) has at most one zero in any Euclidean disc \( D(\zeta, (1 + C)^{-1}) \) for \( |\zeta| \leq C/(1 + C) \).

Conversely, if there exists \( 0 < C < \infty \) such that any non-trivial solution of (1.1) has at most one zero in any Euclidean disc \( D(\zeta, (1 + C)^{-1}) \) for \( |\zeta| \leq C/(1 + C) \), then
\[
|A(z)|(1 - |z|^2)^2 \leq 3 \left( 1 + \Psi_C(|z|) (1 - |z|) \right), \quad \frac{C}{1 + C} < |z| < 1,
\]
where \( \Psi_C \) is positive, and satisfies \( \Psi_C(|z|) \longrightarrow (2C)^+ \) as \( |z| \rightarrow 1^- \).

**2.2.3. Geometric distribution of zeros of solutions.** The set
\[
Q = Q(I) = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r < 1 \}
\]
is called a Carleson square based on the arc \( I \subset \partial \mathbb{D} \), where \( |I| = \ell(Q) \) denotes the normalized arc length of \( I \) (i.e., \( |I| \) is the Euclidean arc length of \( I \) divided by \( 2\pi \)).

**Theorem 6.** If \( A \) is an analytic function in \( \mathbb{D} \) such that (2.3) holds for some \( 0 < C < \infty \), then the zero-sequence \( \{z_n\} \) of any non-trivial solution of (1.1) satisfies
\[
\sum_{z_n \in Q} (1 - |z_n|)^{1/2} \leq K \ell(Q)^{1/2}, \tag{2.6}
\]
for any Carleson square \( Q \). Here \( K = K(C) \) with \( 0 < K < \infty \) is a constant independent of \( f \).

If \( A \) is analytic in \( \mathbb{D} \) and satisfies (2.3) for some \( 0 < C < \infty \), then the zero-sequence of any non-trivial solution of (1.1) is interpolating by Theorem 6 because the zero-sequences are separated by the classical result of Schwarz.
3. Examples

We turn to consider some non-trivial examples, the first of which shows the sharpness of Kraus’ result.

**Example 1.** Let

\[ A(z) = -\frac{3}{4(1 - z)^2}, \quad z \in \mathbb{D}. \]

A solution base \( \{f_1, f_2\} \) of (1.1) is given by the non-vanishing functions

\[ f_1(z) = (1 - z)^{-1/2}, \quad f_2(z) = (1 - z)^{3/2}, \quad z \in \mathbb{D}. \]

Let \( f \) be any non-trivial solution of (1.1). If \( f \) is linearly dependent to \( f_1 \) or \( f_2 \), then \( f \) is non-vanishing. Otherwise, there exist \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \) such that \( f(z) = \alpha f_1(z) + \beta f_2(z) \), and \( f(z) = 0 \) if and only if \( (1 - z)^2 = -\alpha/\beta \). This equation has two solutions \( z_1, z_2 \in \mathbb{C} \), and only one zero of \( f \), say \( z_1 \), satisfies \( \text{Re} \, z_1 < 1 \). This follows from the fact that \( 1 - z_1 = z_2 - 1 \). Consequently, each solution of (1.1) has at most one zero in \( \mathbb{D} \), while the coefficient function \( A \) satisfies \( |A(z)|(1 - |z|^2)^2 \to 3 \) as \( z \to 1^- \) along the positive real axis.

To conclude that (2.1) with \( c \in (3, \infty) \) does not imply infinite oscillation for any solution of (1.1), we recall Hille’s example [15, Eq. (2.12)]. The same example is also used in [3, Example 20].

**Example 2.** Let \( A(z) = a/(1 - z^2)^2 \), where \(-\infty < a < 0\) is a real parameter. If

\[ f_1(z) = \sqrt{1 - z^2} \left( \frac{1 - z}{1 + z} \right)^{1/2} \sqrt{1 - a}, \quad f_2(z) = \sqrt{1 - z^2} \left( \frac{1 - z}{1 + z} \right)^{-1/2} \sqrt{1 - a}, \]

then \( \{f_1, f_2\} \) is a solution base of (1.1) of non-vanishing functions. Let \( f \) be any non-trivial solution of (1.1). If \( f \) is linearly dependent to \( f_1 \) or \( f_2 \), then \( f \) is non-vanishing. Otherwise, there exist \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \) such that \( f = \alpha f_1 + \beta f_2 \). In this case \( f(z) = 0 \) if and only if

\[ \frac{-\beta}{\alpha} = \left( \frac{1 - z}{1 + z} \right) \sqrt{1 - a}. \quad (3.1) \]

Since \( z \mapsto (1 - z)/(1 + z) \) maps \( \mathbb{D} \) onto the right half-plane, \( \sqrt{1 - a} \leq 4 \) ensures that each solution of (1.1) has at most two zeros in \( \mathbb{D} \), see also [15, p. 174]. Further, if \( 2 < \sqrt{1 - a} \), then there exists a solution having exactly two zeros in \( \mathbb{D} \). In particular, if \(-\beta/\alpha\) is real and strictly negative, then \( \alpha f_2 + \beta f_2 \) has two zeros in \( \mathbb{D} \) by (3.1), and these zeros are complex conjugate numbers in \( \mathbb{D} \). Note that \( 2 < \sqrt{1 - a} \leq 4 \) if and only if \(-15 \leq a < -3 \), and then \( |A(x)|(1 - |x|^2)^2 = |a| > 3 \) for all \( x \in (0, 1) \).

We fix \( a = -8 \), and discuss the zeros of the solution \( f = f_1 + k f_2 \) for \( k > 0 \). By (3.1), the zeros of \( f \) in \( \mathbb{D} \) are solutions of \(-k = (1 - z^2)/(1 + z^2)\). We conclude that \( f \) has exactly two zeros in \( \mathbb{D} \) given by

\[ z_1 = \frac{1 - \sqrt[3]{k} \exp (i \pi/3)}{1 + \sqrt[3]{k} \exp (i \pi/3)}, \quad z_2 = \frac{1 - \sqrt[3]{k} \exp (-i \pi/3)}{1 + \sqrt[3]{k} \exp (-i \pi/3)}. \]

If \( k \to 0^+ \), then \( z_1 \) and \( z_2 = \bar{z}_1 \) converge to \( z = 1 \) inside the unit disc. Now, for each \( \delta > 0 \) there exists a solution of (1.1) having two distinct zeros in \( D(1, \delta) \cap \mathbb{D} \).

The following example concerns the sharpness of Theorem 2.
Example 3. Let $0 \leq K < \infty$, and let $A$ be the analytic function
\[ A(z) = -\frac{K + 4\sqrt{1 + K} \left( \log \frac{e}{1-z} \right)^{-1}}{4(1-z)^2}, \quad z \in \mathbb{D}. \]

Now
\[ |A(z)|(1 - |z|^2)^2 \leq K + \frac{4\sqrt{1 + K}}{\log \frac{e}{1-|z|}}, \quad z \in \mathbb{D}. \]

However, the analytic function
\[ f(z) = \frac{1}{(1-z)^{\sqrt{1+K}-1/2}} \log \frac{e}{1-z} \]

is a solution of (1.1) such that $f \notin H_p^\infty$ for $p = (\sqrt{1+K} - 1)/2$.

The following example shows that the number of zeros of a solution of (1.1) may be larger than any pregiven number, while the coefficient function $A$ satisfies (2.3) for some sufficiently large $0 < C < \infty$. See [1] for similar examples concerning the cases of $A \in H_0^\infty$ and $A \in H_1^\infty$. Before the example, we recall some basic properties of the Legendre polynomials $P_0, P_1, P_2, \ldots$, which can be recovered from Bonnet’s recursion formula
\[ nP_n(z) = (2n-1)zP_{n-1}(z) - (n-1)P_{n-2}(z), \quad P_0(z) = 1, \quad P_1(z) = z. \]

For every $n \in \mathbb{N}$, the Legendre polynomial $P_n$ is known to have $n$ distinct zeros in the interval $(-1, 1)$, and $P_n$ is a solution of Legendre’s differential equation
\[ (1 - z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0, \quad z \in \mathbb{D}, \quad n \in \mathbb{N} \cup \{0\}. \quad (3.2) \]

Example 4. Let $P_n$ be the Legendre polynomial for $n \in \mathbb{N} \cup \{0\}$. By (3.2),
\[ P''_n(z) + a_1(z)P'_n(z) + a_0(z)P_n(z) = 0, \quad a_1(z) = \frac{-2z}{1-z^2}, \quad a_0(z) = \frac{n(n+1)}{1-z^2}, \]

for any $z \in \mathbb{D}$. Define $b(z) = -(1/2) \log (1 - z^2)$ for $z \in \mathbb{D}$, and note that then $b$ is a primitive of $-a_1/2$. According to [10, p. 74] the analytic function
\[ f(z) = P_n(z)e^{-b(z)} = P_n(z)(1 - z^2)^{1/2}, \quad z \in \mathbb{D}, \]

which is bounded and has precisely $n$ zeros in $\mathbb{D}$, is a solution of (1.1) with
\[ A(z) = a_0(z) - \frac{1}{4} \left( a_1(z) \right)^2 - \frac{1}{2} a'_1(z) = \frac{1 + n(n+1)(1-z^2)}{(1-z^2)^2}, \quad z \in \mathbb{D}. \]

Let us consider the case $n = 0$ more closely. It is easy to verify that a solution base $\{f_1, f_2\}$ of (1.1) with $A(z) = (1-z^2)^{-2}$ is given by the non-vanishing functions
\[ f_1(z) = (1 - z^2)^{1/2}, \quad f_2(z) = (1 - z^2)^{1/2} \log \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \]

Let $0 < \alpha < \infty$. Then
\[ f(z) = f_1(z) - \frac{1}{\alpha} f_2(z) = (1 - z^2)^{1/2} \left( 1 - \frac{1}{\alpha} \log \frac{1+z}{1-z} \right) \]

is also a solution of (1.1). Evidently, the point $z = z(\alpha)$ is a zero of $f$ if and only if $z(\alpha) = (e^\alpha - 1)/(e^\alpha + 1)$. Since $z(\alpha)$ is a continuous function of $0 < \alpha < \infty$, we conclude that there exists a solution base $\{f_1 - f_2/\alpha_1, f_1 - f_2/\alpha_2\}$ of (1.1), where $0 < \alpha_1 < \alpha_2 < \infty$, such that the hyperbolic distance $g_h(z(\alpha_1), z(\alpha_2))$ is smaller than any pregiven number. In particular, the constant $\delta > 0$ in Corollary 4 depends on the choice of linearly independent solutions, even if $A$ satisfies (2.3) for $C = 0$. 
By considering similar examples one can investigate the sharpness of the second assertions of Theorems 3 and 5. Details are left for the interested reader.

We next offer two concrete examples of equations whose solutions admit infinite oscillation, but the coefficient satisfies

\[ |A(z)|(1 - |z|^2)^2 \leq 1 + \varepsilon(|z|), \quad z \in \mathbb{D}, \quad (3.3) \]

where \( \varepsilon(|z|) \) decays to zero slower than the linear rate as \( |z| \to 1^- \). The following example is similar to [3, Example 12].

**Example 5.** Let \( p \) be a locally univalent analytic function in \( \mathbb{D} \). The functions

\[ f_1(z) = \left( p'(z) \right)^{-1/2} \sin p(z), \quad f_2(z) = \left( p'(z) \right)^{-1/2} \cos p(z), \quad z \in \mathbb{D}, \]

are linearly independent solutions of \((1.1)\) with \( A = (p')^2 + S_p/2 \). We consider the equations \((1.1)\) with \( A = A_1 \) and \( A = A_2 \) induced by

\[ p_1(z) = \log \left( \log \frac{e^e}{1 - z} \right), \quad p_2(z) = \left( \log \frac{e}{1 - z} \right)^q, \]

where \( 0 < q < 1 \). In the first case

\[ A_1(z)(1 - z)^2 = \frac{1}{4} + \frac{5 + \left( \log \frac{e^e}{1 - z} \right)^2}{\left( \log \frac{e^e}{1 - z} \right)^2}, \]

and it follows that \((3.3)\) holds for \( \varepsilon_1(r) \sim 5(\log(e/(1 - r)))^{-2} \) as \( r \to 1^- \); the zeros of the solution \( f_1 \) are \( z_k = 1 - \exp(e - \exp(k\pi)) \), where \( k \in \mathbb{Z} \). In the second case

\[ A_2(z)(1 - z)^2 = \frac{1}{4} + \frac{q^2}{\left( \log \frac{e}{1 - z} \right)^{2(1-q)}} + \frac{1}{4} \frac{1 - q^2}{\left( \log \frac{e}{1 - z} \right)^2}, \]

and so \( \varepsilon_2(r) \sim 4q^2(\log(e/(1 - r)))^{2(q-1)} \) as \( r \to 1^- \); the zeros of the solution \( f_1 \) are \( z_k = 1 - \exp \left( 1 - (k\pi)^{1/q} \right) \), where \( k \in \mathbb{Z} \).

4. **Proof of Theorem 1**

The proof is grounded on an application of a suitable family of conformal maps. The following construction, including Lemma [A] below, is borrowed from [6, p. 576]. Without loss of generality, we may assume \( \zeta = 1 \).

![Figure 1](https://example.com/figure1.png)

**Figure 1.** The boundary of \( \Omega_{3/8,1/4} \) consists of the colorized bold curves.
Let $\tau, \varrho \in (0, 1)$ such that $2\tau + \varrho < 1$. Consider the circles $\partial \mathbb{D}$, $\partial D(1 + \varrho, 1)$ and $\partial D(c_{\pm}, r)$, where $c_{\pm} = (1 + \varrho)(1 \pm i \tan \tau)/2$ and $r = |e^{i\tau} - c_+|$. The discs $D(c_{\pm}, r)$ are contained in both $\mathbb{D}$ and $D(1 + \varrho, 1)$. Moreover, the circles $\partial D(c_{\pm}, r)$ intersect $\partial \mathbb{D}$ on the points $e^{\pm i\tau}$, and the common points of $\partial D(c_{\pm}, r)$ and $\partial D(1 + \varrho, 1)$ are the reflections of $e^{\pm i\tau}$ with respect to the line $\Re z = (1 + \varrho)/2$. Let us call them $\gamma_{\pm}$ according to the sign of their imaginary parts. Let $\Omega_{\tau, \varrho}$ be the Jordan domain formed by the shortest four circular arcs connecting $e^{\pm i\tau}$.

The existence of a such mapping is ensured by the Riemann mapping theorem, which also shows that under the additional conditions $\varphi_{\tau, \varrho}(0) = (1 + \varrho)/2$ and $\varphi'_{\tau, \varrho}(0) > 0$ this mapping is unique.

The following lemma produces an estimate for the growth of the Schwarzian derivative of $\varphi_{\tau, \varrho}$. An alternative approach is explained in [12, pp. 198–208]: the Schwarzian derivative of $\varphi_{\tau, \varrho}$ is explicitly determined by the boundary arcs of $\Omega_{\tau, \varrho}$, since the boundary $\partial \Omega_{\tau, \varrho}$ forms a curvilinear polygon.

**Lemma A ([6, Lemma 8]).** Let $0 < p < \infty$, and let $\tau, \varrho \in (0, 1)$ such that $2\tau + \varrho < 1$. Then the function $\varphi_{\tau, \varrho}$ satisfies $(\log \varphi'_{\tau, \varrho})' \in H^p$, $\varphi''_{\tau, \varrho} \in H^p$ and

$$
\int_{\mathbb{D}} |\varphi''_{\tau, \varrho}(z)|^p \frac{dm(z)}{|\varphi'_{\tau, \varrho}(z)|} \to 0, \quad \tau \to 0^+.
$$

Before the proof of Theorem 1 we make some observations about $\varphi_{\tau, \varrho}$. To conclude that $\varphi_{\tau, \varrho}$ maps the open interval $(-1, 1)$ into the real axis, we follow [13, p. 11]. Evidently, $\varphi_{\tau, \varrho}$ admits a Taylor expansion

$$
\varphi_{\tau, \varrho}(z) = \frac{1 + \varrho}{2} + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,
$$

where $a_1 > 0$. Define an auxiliary function $\tilde{\varphi}(z) = \varphi_{\tau, \varrho}(\varphi(z))$. Function $\tilde{\varphi}$ is analytic and univalent in $\mathbb{D}$, and it has a Taylor expansion

$$
\tilde{\varphi}(z) = \frac{1 + \varrho}{2} + a_1 z + \overline{a}_2 z^2 + \overline{a}_3 z^3 + \cdots.
$$

Note that $\tilde{\varphi}(\mathbb{D}) = \varphi_{\tau, \varrho}(\mathbb{D})$, and hence $\tilde{\varphi} \equiv \varphi_{\tau, \varrho}$ according to the uniqueness part of the Riemann mapping theorem. Since the Taylor expansion of $\varphi_{\tau, \varrho}$ is unique, we conclude that coefficients $a_j$ are real for all $j \in \mathbb{N}$. This means that $\varphi_{\tau, \varrho}$ maps the interval $(-1, 1)$ into the real axis, and hence is typically real. Furthermore, since $\varphi_{\tau, \varrho}$ is univalent, and as a real function of a real variable it is increasing at $z = 0$ by $\varphi'_{\tau, \varrho}(0) > 0$, we have shown that $z = 1$ is a fixed point of $\varphi_{\tau, \varrho}$. Recall that $\varphi_{\tau, \varrho}$ has an injective and continuous extension to the closed unit disc $\overline{\mathbb{D}}$ by the famous theorem of Carathéodory.

Moreover, $\varphi'_{\tau, \varrho}$ has a continuous extension to $\overline{\mathbb{D}}$, and

$$
\frac{|\varphi''_{\tau, \varrho}(z)|}{|\varphi'_{\tau, \varrho}(z)|} (1 - |z|^2)^{1/2} \to 0, \quad |z| \to 1^-,
$$

since both $\varphi''_{\tau, \varrho}$ and $(\log \varphi'_{\tau, \varrho})'$ belong to $H^p$ for all $0 < p < \infty$, see [5, Theorems 3.11 and 5.9]. Standard estimates yield

$$
|S_{\varphi_{\tau, \varrho}}(z)|(1 - |z|^2)^{1 + \frac{1}{p}} \to 0, \quad |z| \to 1^-,
$$

for all $1 \leq p < \infty$. 

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To prove the first assertion of Theorem 1, assume that there exists \( \delta > 0 \) such that any non-trivial solution of (1.1) has at most one zero in \( D(1, \delta) \cap \mathbb{D} \). Fix \( \tau, \varrho \in (0, 1) \) such that \( 2\tau + \varrho < 1 \) and \( \varphi_{\tau, \varrho}(\mathbb{D}) = \Omega_{\tau, \varrho} \subset D(1, \delta) \cap \mathbb{D} \). Write \( T = \varphi_{\tau, \varrho} \) for short. Then, for any given linearly independent solutions \( f_1 \) and \( f_2 \) of (1.1), the meromorphic function \( f_1 / f_2 \circ T \) is univalent in \( \mathbb{D} \). Therefore
\[
|S_{f_1 / f_2}(T(z)) (T'(z))^2 + S_T(z) (1 - |z|^2)^2| \leq 6, \quad z \in \mathbb{D},
\]
by Kraus theorem [9], see also [13, pp. 67–68] regarding the meromorphic case.

Let \( \{w_n\} \) be any sequence of points in \( \mathbb{D} \) such that \( w_n \to 1 \), and define \( z_n \) by the equation \( T(z_n) = w_n \). Then
\[
\lim_{n \to \infty} z_n = \lim_{n \to \infty} T^{-1}(w_n) = T^{-1}(\lim_{n \to \infty} w_n) = T^{-1}(1) = 1,
\]
because \( z = 1 \) is a fixed point of \( T \). Let \( L_n \) denote the straight line segment from \( z_n \in \mathbb{D} \) to \( z_n / |z_n| \in \partial \mathbb{D} \). For all \( n \) sufficiently large, we have
\[
1 - |T(z_n)| = |T(z_n / |z_n|)| - |T(z_n)| \leq \left| \int_{L_n} T'(z) \, dz \right| \leq (1 - |z_n|) \sup_{z \in L_n} |T'(z)|.
\]
We point out that \( T' \) is continuous in \( \Omega \), and \( T'(1) \neq 0 \) by the Julia-Caratheodory theorem. By the Schwarz-Pick lemma, we deduce
\[
1 \leq \frac{1 - |T(z_n)|^2}{|T'(z_n)|(1 - |z_n|^2)} \leq \frac{1 + |T(z_n)|}{1 + |z_n|} \frac{\sup_{z \in L_n} |T'(z)|}{|T'(z_n)|} \to 1, \quad n \to \infty.
\]
This, together with (4.1) and (4.2), yields
\[
\limsup_{n \to \infty} |A(w_n)|(1 - |w_n|^2)^2 = \limsup_{n \to \infty} \frac{1}{2} |S_{f_1 / f_2}(T(z_n)) (1 - |T(z_n)|^2)^2
\]
\[
= \limsup_{n \to \infty} \frac{1}{2} |S_{f_1 / f_2}(T(z_n))| |T'(z_n)|^2 (1 - |z_n|^2)^2 \left( \frac{1 - |T(z_n)|^2}{|T'(z_n)|(1 - |z_n|^2)} \right)^2 \leq 3,
\]
which is a contradiction.

5. PROOF OF THEOREM 2

Let \( f \) be a solution of (1.1), and suppose that \( 0 \leq \delta < R < 1 \). We have
\[
|f(z)| \leq \int_{\delta}^{1} \int_{\delta}^{t} \left| f'' \left( s \frac{z}{|z|^2} \right) \right| ds \, dt + M(\delta, f') + M(\delta, f), \quad \delta < |z| < 1,
\]
where \( M(\delta, \cdot) \) is the maximum modulus on \( |z| = \delta \). By means of (1.1), we obtain
\[
\sup_{\delta < |z| < R} (1 - |z|^2)^p |f(z)| \leq \left( \sup_{\delta < |z| < R} (1 - |z|^2)^p |f(\zeta)| \right) \left( \sup_{\delta < |\zeta| < R} (1 - |\zeta|^2)^2 \left| A(\zeta) \right| \right)
\]
\[
\cdot \sup_{\delta < |z| < R} \left( 1 - |z|^2 \right)^p \int_{\delta}^{1} \int_{\delta}^{t} \frac{ds \, dt}{\left( 1 - s^2 \right)^{p+2}} + M(\delta, f') + M(\delta, f).
\]

\[ (5.1) \]
If $0 < p < \infty$, then
\[
\lim_{|z| \to 1} \left( (1 - |z|^2)^p \int_{\delta}^{z} \frac{ds}{(1 - s^2)^{p+2}} \right) = \frac{1}{4p(p+1)}
\]
by the Bernoulli-l'Hôpital theorem. The estimate (5.1) implies that $\|f\|_{H_p^\infty} < \infty$ provided that $K < 4p(p+1)$. We conclude $\|f\|_{H_p^\infty} < \infty$ for any $p > (\sqrt{1+K} - 1)/2$.

6. Proof of Theorem 3

To prove the first assertion of Theorem 3, suppose that $f_1$ is any non-trivial solution of (1.1) which has two distinct zeros $z_1, z_2 \in \mathbb{D}$ such that their hyperbolic midpoint $\xi = \xi_h(z_1, z_2)$ satisfies $1 - |\xi| < 1/C$. Let $\{f_1, f_2\}$ be a solution base of (1.1). Define $h = f_1/f_2$, which implies that $S_h = 2A$.

Let $a \in \mathbb{D}$ such that $1 - 1/C < |a| < 1$. If we define $r_a = 1 - C^{1/2}(1 - |a|)^{1/2}$, then $0 < r_a < 1$. Set $g_a(z) = (h \circ \varphi_a)(r_a z)$. Then the assumption (2.3) yields
\[
|S_{g_a}(z)|(1 - |z|^2)^2 = |S_h(\varphi_a(r_a z))| |\varphi'_a(r_a z)|^2 r_a^2 (1 - |z|^2)^2 \leq 2 \left( 1 + C(1 - |\varphi_a(r_a z)|) \right) \frac{1 - |z|^2}{1 - r_a^2 |z|^2} r_a^2 \leq 2 \left( 1 + C(1 - |\varphi_a(r_a z)|) \right) r_a^2 \leq 2, \quad z \in \mathbb{D},
\]
where the last inequality follows from
\[
1 - \left( 1 + C(1 - |\varphi_a(r_a z)|) \right) r_a^2 = C \frac{(1 + r_a)(1 - |a|)}{1 - |a| r_a} \left( \frac{1 - |a| r_a}{1 - r_a^2} - r_a^2 \right) \geq 0.
\]
According to [11] Theorem 1] the function $g_a$ is univalent in $\mathbb{D}$, and hence $h = f_1/f_2$ is univalent in the pseudo-hyperbolic disc $\Delta_p(a, r_a)$.

The argument above shows that $h$ is univalent in $\Delta_p(\xi, r_\xi)$, and hence
\[
\varrho_h(z_1, z_2) = \log \frac{1 + \varrho_p(z_1, \xi)}{1 - \varrho_p(z_1, \xi)} \geq \log \frac{1 + r_\xi}{1 - r_\xi} = \log \frac{2 - C^{1/2}(1 - |\xi|)}{C^{1/2}(1 - |\xi|)^{1/2}}.
\]

To prove the second assertion of Theorem 3, suppose that the hyperbolic distance between any distinct zeros $z_1, z_2 \in \mathbb{D}$ of any non-trivial solution of (1.1), for which $1 - |\xi_h(z_1, z_2)| < 1/C$, satisfies (2.4) to some $0 < C < \infty$. In another words,
\[
\varrho_h(z_1, z_2) \geq \log \frac{1 + r_\xi}{1 - r_\xi}, \quad r_\xi = 1 - C^{1/2}(1 - |\xi|)^{1/2}, \quad \xi = \xi_h(z_1, z_2). \quad (6.1)
\]

First, we show that each non-trivial solution of (1.1) has at most one zero in
\[
\Delta_h \left( a, \frac{1}{2} \log \frac{1 + R_a}{1 - R_a} \right), \quad R_a = 1 - (8C)^{1/3}(1 - |a|)^{1/3}, \quad 1 - |a| < (8C)^{-1}.
\]
Assume on the contrary that there exists a non-trivial solution having two distinct zeros $z_1, z_2 \in \Delta_p(a, R_a)$ for some $1 - |a| < (8C)^{-1}$. By hyperbolic geometry we conclude $\xi \in \Delta_p(a, R_a)$, and hence
\[
1 - r_\xi \leq C^{1/2} \left( 1 - \frac{|a| - R_a}{1 - |a|R_a} \right)^{1/2} = \frac{C^{1/2}(1 - |a|)^{1/2}(1 + R_a)^{1/2}}{(1 - |a|R_a)^{1/2}},
\]
which implies

\[
\frac{1 + r_\xi}{1 - r_\xi} \cdot \frac{1 - R_a}{1 + R_a} \geq \frac{(1 + r_\xi)(1 - |a|R_a)^{1/2}}{C^{1/2}(1 - |a|)^{1/2}(1 + R_a)^{1/2}} \cdot \frac{(8C)^{1/3}(1 - |a|)^{1/3}}{1 + R_a} \geq \frac{(8C)^{1/3} \cdot (1 - R_a)^{1/2}}{2\sqrt{2}C^{1/2}(1 - |a|)^{1/6}} = 1.
\]

We deduce

\[
\varrho_h(z_1, z_2) \leq \varrho_h(z_1, a) + \varrho_h(a, z_2) < \log \frac{1 + R_a}{1 - R_a} \leq \log \frac{1 + r_\xi}{1 - r_\xi},
\]

which contradicts (6.1).

Second, we derive the estimate (2.5). Let \(\{f_1, f_2\}\) be a solution base of (1.1) and set \(h = f_1/f_2\) so that \(S_h = 2A\). Set \(g_a(z) = (h \circ \varphi_a)(R_az)\). Since \(h\) is univalent in each pseudo-hyperbolic disc \(\Delta(a, R_a)\) for \(1 - |a| < (8C)^{-1}\), it follows that \(g_a\) is univalent in \(D\) for those values of \(a\), and hence

\[
|S_{g_a}(z)|(1 - |z|^2)^2 = |S_h(\varphi_a(R_az))| |\varphi_a'(R_az)|^2 R_a^2 (1 - |z|^2)^2 = 2|A(\varphi_a(R_az))| \frac{(1 - |a|^2)^2}{|1 - \overline{a}R_az|^4} R_a^2 (1 - |z|^2)^2 \leq 6, \quad z \in D,
\]

by Kraus theorem [9]. By choosing \(z = 0\), we conclude

\[
|A(a)|(1 - |a|^2)^2 \leq 3 \frac{3}{R_a^2} = 3 \left(1 + (1 - R_a) \frac{1 + R_a}{R_a^2}\right), \quad 1 - |a| < (8C)^{-1}.
\]

7. Proof of Theorem 5

We begin with the first assertion of Theorem 5. Suppose that the coefficient \(A\) satisfies (2.3) for some \(0 < C < \infty\), and there exists a non-trivial solution \(f_1\) of (1.1) having two distinct zeros \(z_1, z_2 \in D(\zeta, (1 + C)^{-1})\) for some \(|\zeta| \leq C/(1 + C)\). Let \(f_2\) be a solution of (1.1) linearly independent to \(f_1\). By setting \(h = f_1/f_2\), we deduce \(S_h = 2A\). The Möbius transformation \(T(z) = \zeta + (1 + C)^{-1}z\) is a conformal map from \(D\) onto \(D(\zeta, (1 + C)^{-1})\). We proceed to prove that \(g = h \circ T\) is univalent in \(D\). Now

\[
|S_g(z)|(1 - |z|^2)^2 = |S_h(T(z))| |T'(z)|^2(1 - |z|^2)^2 \leq 2 \frac{1 + C(1 - |\zeta + (1 + C)^{-1}z|)}{(1 - |\zeta + (1 + C)^{-1}z|^2)^2} (1 + C)^{-2}(1 - |z|^2)^2, \quad z \in D.
\]

The proof of the first assertion is divided into two separate cases. By differentiation, there exists \(0 < t_C < 1/3\) such that the auxiliary function

\[
\mu(t) = (1 + C(1 - t))(1 - t^2)^{-2}, \quad 0 < t < 1,
\]

is decreasing for \(0 < t < t_C\), and increasing for \(t_C < t < 1\).

(i) Suppose that \(z \in D\), and \(|\zeta + (1 + C)^{-1}z| > t_C\). By the triangle inequality \(|\zeta + (1 + C)^{-1}z| \leq (1 + C)^{-1}(C + |z|)\), and hence

\[
|S_g(z)|(1 - |z|^2)^2 \leq 2 \frac{1 + C(1 - (1 + C)^{-1}(C + |z|))}{(1 - (1 + C)^{-2}(C + |z|)^2)^2} (1 + C)^{-2}(1 - |z|^2)^2
\]

\[
= 2 \frac{(1 + 2C - C|z|)(1 + C)(1 + |z|)^2}{(|z| + 2C + 1)^2} \leq 2. \quad (7.1)
\]
The inequality in (7.1) follows by differentiation, since the quotient is an increasing function of $|z|$ for $0 < |z| < 1$.

(ii) Suppose that $z \in \mathbb{D}$, and $|\zeta + (1 - a)z| \leq tC$. Since $\mu(0) \geq \mu(t)$ for all $0 < t \leq tC$, we deduce

$$|S_g(z)|(1 - |z|^2)^2 \leq \frac{2}{1 + C} < 2.$$ 

By means of (i), (ii) and [11, Theorem 1], we conclude that $g$ is univalent in $\mathbb{D}$. This is a contradiction, since the preimages $T^{-1}(z_1) \in \mathbb{D}$ and $T^{-1}(z_2) \in \mathbb{D}$ are distinct zeros of $g$. The first assertion of Theorem 5 follows.

We turn to consider the second assertion of Theorem 5. Suppose that there exists $0 < C < \infty$ such that any non-trivial solution of (1.1) has at most one zero in any Euclidean disc $D(\zeta, (1 + C)^{-1})$ for $|\zeta| \leq C/(1 + C)$. Let $\{f_1, f_2\}$ be a solution base of (1.1), and set $h = f_1/f_2$ which implies $S_h = 2A$. Suppose that $a \in \mathbb{D}$ and $|a| > C/(1 + C)$. Set $g_a(z) = (h \circ \varphi_a)(ra_z)$, where

$$r_a^2 = \frac{|a| - \frac{C}{1 + C}}{|a| (1 - |a|)\frac{C}{1 + C}}, \quad 0 < r_a < 1.$$ 

Then

$$\Delta_p(a, r_a) = D\left(\frac{a}{|a|} \cdot \frac{C}{1 + C}, \frac{r_a(1 - |a|^2)}{1 - r_a^2|a|^2}\right) \subset D\left(\frac{a}{|a|} \cdot \frac{C}{1 + C}, \frac{1}{1 + C}\right).$$

Since $z \mapsto \varphi_a(ra_z)$ maps $\mathbb{D}$ onto $\Delta_p(a, r_a)$, it follows that $g_a$ is univalent in $\mathbb{D}$ by the assumption. Hence

$$|S_{g_a}(z)|(1 - |z|^2)^2 = |S_h(\varphi_a(ra_z))| \frac{|\varphi'_a(ra_z)|^2 r_a^2 (1 - |z|^2)^2}{|1 - \overline{a}ra_z|^2}$$

$$= 2 |A(\varphi_a(ra_z))| \left(\frac{1 - |a|^2}{1 - r_a^2|a|^2}\right)^2 r_a^2 (1 - |z|^2) \leq 6, \quad z \in \mathbb{D},$$

by Kraus theorem [9]. By choosing $z = 0$, we obtain

$$|A(a)|(1 - |a|^2)^2 \leq \frac{3}{r_a^2} = 3 \left(1 + \frac{C}{1 + C} \frac{(1 + |a|)}{|a| - \frac{C}{1 + C}(1 - |a|)}\right), \quad \frac{C}{1 + C} < |a| < 1.$$ 

8. PROOF OF THEOREM [10]

If $A$ is analytic in $\mathbb{D}$, and satisfies (2.3) for some $0 < C < \infty$, then any non-trivial solution $f$ of (1.1) has at most one zero $z_n$ in any horodisc

$$D_\theta = D\left(e^{i\theta} \frac{C}{1 + C}, \frac{1}{1 + C}\right), \quad e^{i\theta} \in \partial \mathbb{D},$$

by Theorem [5]. Suppose that $0 < 1 - r < 2/(1 + C)$. Now $r \in \partial D_\theta$ if and only if

$$\left|e^{i\theta} \frac{C}{1 + C} - r\right|^2 = \frac{1}{(1 + C)^2}.$$ 

The positive solution $\theta = \theta(r)$ of this equation satisfies

$$\theta(r) = \arccos \frac{C - 1 + r^2(1 + C)}{2rC} \sim \sqrt{2/C (1 - r)^{1/2}}, \quad r \to 1^-.$$
This implies that the zeros \( z_n \) of \( f \), for which \( 0 < 1 - |z_n| < 2/(1 + C) \), induce pairwise disjoint zero-free tent-like domains \( \Omega_n \subset \mathbb{D} \), which intersect \( \partial \mathbb{D} \) on arcs \( I_n = \overline{\Omega_n} \cap \partial \mathbb{D} \) of normalized length

\[
\ell(I_n) \sim \frac{\sqrt{2}}{\pi \sqrt{C}} (1 - |z_n|)^{1/2}, \quad n \to \infty.
\]

Consequently, if \( Q \) is any Carleson square for which \( \ell(Q) < 2/(1 + C) \), then

\[
\sum_{z_n \in Q} (1 - |z_n|)^{1/2} \lesssim \sum_{z_n \in Q} \ell(I_n) \lesssim \ell(Q)^{1/2},
\]

where the comparison constants depend on \( 0 < C < \infty \). By a standard argument, this implies (2.6) for any Carleson square \( Q \).

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