One-instanton calculations in $N = 2$ $SU(N_c)$ Supersymmetric QCD

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Abstract

We study the low-energy effective theory in $N = 2$ $SU(N_c)$ supersymmetric QCD with $N_f \leq 2N_c$ fundamental hypermultiplets in the Coulomb branch by microscopic and exact approaches. We calculate the one-instanton correction to the modulus $u \equiv \langle \frac{1}{2} \text{Tr} A^2 \rangle$ from microscopic instanton calculation. We also study the one-instanton corrections from the exact solutions for $N_c = 3$ with massless hypermultiplets. They agree with each other except for $N_f = 2N_c - 2$ and $2N_c$ cases. These differences come from possible ambiguities in the constructions of the exact solutions or the definitions of the operators in the microscopic theories.

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The prepotential of the low energy effective theories of the $N = 2$ supersymmetric gauge theories in the Coulomb phase have been obtained exactly by using the holomorphy and the duality arguments [1]. The prepotential receives a non-perturbative instanton correction in the semi-classical region [2]. On the other hand, the microscopic instanton calculations in supersymmetric gauge theories [3, 4, 5] give reliable results when the semi-classical approximation is valid. Thus the comparison between these approaches provides a non-trivial check on the method of the microscopic instanton calculus as well as the assumptions in the derivation of the exact solutions.

Such comparisons have been studied in the case of $N = 2$ supersymmetric Yang-Mills theories and $SU(2)$ supersymmetric QCDs (SQCDs) [6]-[10]. The results have been consistent with the exact solutions so far, while some discrepancies have been reported in the $SU(2)$ SQCDs with $N_f = 3, 4$ [10]. These discrepancies do not seem to contradict with the assumptions in the derivations of the exact solutions in the sense that they could come from the possible ambiguities in the exact solutions or the definitions of the quantum observables in the microscopic theories.

In this letter we study the one-instanton correction to the prepotential in the $N = 2$ $SU(N_c)$ SQCD with $N_f \leq 2N_c$ fundamental hypermultiplets both from the microscopic and the exact viewpoints. In the case of $N = 2$ $SU(2)$ SQCD, the contributions from the odd numbers of instantons vanish due to the anomalous $Z_2$ symmetry since the fundamental representation of $SU(2)$ is pseudoreal [1]. But, for $N_c \geq 3$, one can expect that all the instanton corrections appear in general.

We introduce an $N = 1$ chiral multiplet $\phi = (A, \psi)$ in the adjoint representation and an $N = 1$ vector multiplet $W_\alpha = (v_\mu, \lambda)$, which form an $N = 2$ vector multiplet. The $N = 1$ chiral multiplets $Q_i = (q_i, \psi_{mi})$ and $\tilde{Q}_i = (\tilde{q}_i, \tilde{\psi}_{mi})$ $(i = 1, \cdots N_f)$ form the $N_f$ $N = 2$ matter hypermultiplets in the fundamental representation. The microscopic $N = 2$ Lagrangian for $N = 2$ SQCD is given by

$$
\mathcal{L} = 2 \int d^4\theta Tr \left( \phi^\dagger e^{-2gV} \phi e^{2gV} \right) + \frac{1}{2g^2} \left( \int d^2\theta W^\alpha W_\alpha + \text{h.c.} \right) \\
+ \int d^4\theta \sum_{k=1}^{N_f} \left( Q_k^\dagger e^{-2gV} Q_k + \tilde{Q}_k e^{2gV} \tilde{Q}_k^\dagger \right) + \left( i\sqrt{2g} \int d^2\theta \sum_{k=1}^{N_f} \tilde{Q}_k \phi Q_k + \text{h.c.} \right)
$$
\[ + \left( \int d^2 \theta \sum_{k=1}^{N_f} m_k \tilde{Q}_k Q_k + h.c. \right), \]  

where \( g \) is the gauge coupling constant and the trace is taken in the fundamental representation. Here the color indices are suppressed. We will examine the euclidean lagrangian of (1) in terms of the component fields in the Wess-Zumino gauge.

The Coulomb branch of this theory is parameterized by the expectation values of the adjoint scalar vacuum expectation values \( A_0^i = a_i \delta_i^j \) and \( A_0^i = \bar{a}_i \delta_i^j \). For generic values of \( a_i \) and \( \bar{a}_i \) the non-abelian gauge symmetry completely breaks down to the \( U(1)^{N_c-1} \), and the system is in the Coulomb phase.

Since the holomorphy argument \[5, 11\] for the gauge coupling \( g \) shows that the calculation in the region \( g \ll 1 \) is enough to obtain reliable results, we will perform microscopic instanton calculation in the lowest order of \( g \). In this approximation, the equation of motion of the gauge field \( D_\mu G^{\mu\nu} = 0 \) has the instanton solutions \[12\]. Their bosonic zero modes of the one-instanton solutions are the instanton location \( x_0 \) in the euclidean space, the instanton size \( \rho \) and the location in the color space. The integration over the location in the color space is defined by the integration over the minimal embedding of the subgroup \( SU(2) \), where the one-instanton configuration resides, into the gauge group \( SU(N_c) \) \[13\]. The generators of the minimally embedded \( SU(2) \) subgroup can be characterized by \( \Omega^\dagger J^a \Omega \), where \( \Omega \in SU(N_c) \), and \( J^a \) are the generators of the \( SU(2) \) subgroup obtained by the upper-left-hand corner embedding of the two-dimensional representation of \( SU(2) \) into the \( N_c \)-dimensional representation of \( SU(N_c) \) \[13\]. Hence the integration in the color space is performed by sweeping \( \Omega \) in \( V(N_c) \equiv SU(N_c)/SU(2) \times U(1) \times SU(N_c-2) \), where the \( U(1) \times SU(N_c-2) \) is the stability group of the embedding and the additional \( SU(2) \) \[14\] comes from the fact that we are interested only in the observables symmetric under the space rotation. By the global gauge transformation, the group integration can be performed by rotating the scalar vacuum expectation values \( \langle A \rangle = \Omega \Lambda_0 \Omega^\dagger, \langle A^\dagger \rangle = \Omega \Lambda_0^\dagger \Omega^\dagger \), while the instanton configuration is fixed at the upper-left-hand corner \[3\].

The equations of motion of the adjoint fermions are given by \( \tau_\mu D_\mu \psi = \tau_\mu D_\mu \lambda = 0 \) in the lowest order of \( g \). From the index theorem, each of these equations has \( 2N_c \) zero modes. Similarly, each of the lowest order equations of motions of the matter fermions \( \tau_\mu D_\mu \psi_{m\bar{i}} = \tau_\mu D_\mu \bar{\psi}_{m\bar{i}} = 0 \) has one zero mode. We use \( \xi, \zeta, \eta \) and \( \bar{\eta} \) to label the zero-modes...
of \( \lambda, \psi, \bar{\psi}_m \) and \( \bar{\psi}_m \), respectively. Then, under an appropriate choice of the normalization of these fermionic zero modes\(^1\), the integration measure of the \( N = 2 \) one-instanton zero modes is given by [13, 5, 15]

\[
2^{10} \pi^{2N_c+2} \Lambda_d^{b_1} g^{-4N_c} \int d^4x_0 \int_0^\infty d\rho \rho^{4N_c-5} \int_{V(N_c)} d\Omega \int d^2N_c \xi d^2N_c \zeta dN_f \eta dN_f \bar{\eta},
\]

where \( \mu \) is the Pauli-Villars regulator and \( b_1 = 2N_c - N_f \) is the one-loop coefficient of the beta function.

Out of the \( 4N_c \) zero-modes of the adjoint fermions, four are the supersymmetric zero-modes \( \xi_{SS}^a, \xi_{SS} \) obtained by the supersymmetry transformations of the one-instanton configuration of the gauge field. Another four are the superconformal zero-modes \( \xi_{SC}^a, \zeta_{SC} \) obtained by the superconformal transformations. The fermionic zero-modes other than the supersymmetric ones, say \( \xi' \equiv (\xi_{SS}^a, \xi, \bar{\xi}_a) \) and \( \zeta' \equiv (\xi_{SC}^a, \zeta, \bar{\zeta}_a) \) \((a = 3, \ldots, N_c)\), cease to be zero-modes by taking into account the mass terms and the Yukawa terms [3, 4].

One of the lowest order contributions may be obtained by substituting the solution of the scalar field equation of motion in the lowest order \( D_\mu A^\dagger(x) = 0 \) with the asymptotic value \( \langle A^\dagger \rangle \), and the fermionic zero-modes into the Yukawa coupling term \( g \int d^4x \text{Tr}(\bar{\psi}[A^\dagger, \lambda]) \).

Thus we obtain a bilinear term \( \xi' g M(\langle A^\dagger \rangle) \xi' \) with

\[
g M(\langle A^\dagger \rangle) = ig \begin{pmatrix}
\sqrt{2} \varepsilon \langle A^\dagger \rangle_{d1}^1 \\
\langle A^\dagger \rangle_{d1}^3 \\
(\varepsilon \langle A^\dagger \rangle_{d2}^2)^t \\
(\varepsilon \langle A^\dagger \rangle_{d2}^3)^t \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
(\langle A^\dagger \rangle_{d1}^3)^t \\
0 \\
-\frac{\text{Tr}(A^\dagger)^{(1)}}{\sqrt{2}} I_{N_c-2} + \sqrt{2} \langle (A^\dagger)^{(4)} \rangle^t \\
0
\end{pmatrix},
\]

where \( \varepsilon \) and \( I_{N_c-2} \) are a two by two antisymmetric tensor with \( \varepsilon^{12} = 1 \) and an \( N_c - 2 \) by \( N_c - 2 \) identity matrix, respectively. Here we have divided the row and column of the scalar field into the following blocks;

\[
A = \begin{pmatrix}
A^{(1)} & A^{(2)} \\
A^{(3)} & A^{(4)}
\end{pmatrix},
\]

where \( A^{(1)}, A^{(2)}, A^{(3)} \) and \( A^{(4)} \) are \( 2 \times 2, 2 \times (N_c - 2), (N_c - 2) \times 2 \) and \( (N_c - 2) \times (N_c - 2) \) matrices, respectively, and \( \langle A^\dagger \rangle_{d1}^{(1)} \) is the traceless part of \( \langle A^\dagger \rangle^{(1)} \). Another contribution of

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\(^1\)The normalizations of the zero-modes are taken to be unity under the norm \( \int d^4x 2 \text{Tr}(\phi^\dagger \phi) \) and \( \int d^4x \phi^\dagger \phi \) for adjoint and fundamental fields, respectively.
order $g$ is the term
\[ -\sum_{k=1}^{N_f} \left( \frac{g i}{\sqrt{2}} \text{Tr}(\langle A \rangle^{(1)}) + m_k \right) \bar{\eta}_k \eta_k, \]
which comes from the Yukawa term $g \bar{\psi}_m A \psi_m$. Here we have added also the contributions from the mass terms of the matter fermions $\bar{\psi}_m \psi_m$.

The mass terms among the zero-modes (3), (5) have a weight $\sqrt{g}$ per one fermionic zero-mode. Therefore there exist other contributions with the same order in the case of SQCD. In fact, one needs to introduce the terms with four fermionic zero-modes of order $g^2$. One of such contributions comes from the Yukawa terms $g \bar{\psi}_m \lambda \bar{q}^\dagger$ and $g \bar{q} \psi_m$ mediated by the propagator of $\bar{q}$. Using the scalar propagators in the instanton backgrounds [10], we obtain
\[ \Delta_{\bar{q}}S = -\frac{g^2}{12\pi^2 \rho^2} \sum_{k=1}^{N_f} \bar{\eta}_k \eta_k \sum_{a=3}^{N_c} \xi_a \bar{\xi}_a - \frac{g^2}{32\pi^2 \rho^2} \sum_{k=1}^{N_f} \bar{\eta}_k \eta_k \xi_{SC} \bar{\xi}_{SC}. \]

The contribution $\Delta_{q}S$ mediated by the propagator of $q$ can be obtained in a similar manner. Due to $SU(2)_R$ symmetry, the sum $\Delta_{q}S + \Delta_{\bar{q}}S$ is simply given by replacing $\xi_a \bar{\xi}_a$ by $\xi_a \bar{\xi}_a + \bar{\xi}_a \xi_a$ in the first term in (6), while the second term vanishes.

The other contribution of order $g^2$ comes from the Yukawa terms $g \text{Tr}(\lambda \psi \bar{A}^\dagger)$ and $g \bar{\psi}_m A \psi_m$ mediated by the propagator of $A$, and the result is
\[ \Delta_{A}S = -\frac{g^2}{24\pi^2 \rho^2} \sum_{k=1}^{N_f} \bar{\eta}_k \eta_k \sum_{a=3}^{N_c} (\xi_a \bar{\xi}_a + \bar{\xi}_a \xi_a). \]

Hence, including $8\pi^2 \rho^2 f$ from the contribution of the kinetic term of $A$, we obtain the classical action of the instanton configuration as
\[ S = 8\pi^2 \rho^2 f - g \xi' M \xi' - \sum_{k=1}^{N_f} \left( \frac{g i}{\sqrt{2}} \text{Tr}(\langle A \rangle^{(1)}) + m_k \right) \bar{\eta}_k \eta_k - \frac{g^2}{24\pi^2 \rho^2} \sum_{k=1}^{N_f} \bar{\eta}_k \eta_k \sum_{a=3}^{N_c} (\xi_a \bar{\xi}_a + \bar{\xi}_a \xi_a), \]
where
\[ f(\langle A \rangle, \langle A^\dagger \rangle) = \text{Tr} \left( \langle A^\dagger \rangle^{(1)} \langle A \rangle^{(1)} + \frac{1}{2} \langle A \rangle^{(2)} \langle A^\dagger \rangle^{(2)} + \frac{1}{2} \langle A^\dagger \rangle^{(3)} \langle A \rangle^{(3)} \right). \]

The supersymmetric zero-modes must be canceled by the insertion of appropriate operators. An approach to do this is to consider the four fermion correlation function
\[ f(\langle A \rangle, \langle A^\dagger \rangle) = \text{Tr} \left( \langle A^\dagger \rangle^{(1)} \langle A \rangle^{(1)} + \frac{1}{2} \langle A \rangle^{(2)} \langle A^\dagger \rangle^{(2)} + \frac{1}{2} \langle A^\dagger \rangle^{(3)} \langle A \rangle^{(3)} \right). \]

\[ \text{We treat the mass terms perturbatively. See [11] for another treatment.} \]
of the classically massless modes of $\psi$ and $\lambda$. But it turns out that this approach is not so convenient for the cases $2N_c - 2 \leq N_f \leq 2N_c$, because it detects only a certain combination of the fourth derivatives of the prepotential of the effective theory and this is always zero for the massless $2N_c - 2 \leq N_f \leq 2N_c$ cases from the dimensional analysis. Thus we insert the modulus $u \equiv \langle \frac{1}{2} \text{Tr} A^2 \rangle$ [9, 10], which has the following direct relation to the prepotential of the effective field theory for the massless cases [17]:

$$ib_1 u = -\Lambda \frac{\partial F}{\partial \Lambda} = \sum_{i=1}^{N_c} a_i \frac{\partial F}{\partial a_i} - 2F.$$  \hspace{1cm} (10)

Here the prepotential has the following expansion in the weak coupling region for $N_f < 2N_c$:

$$F(a) = \frac{\tau_0}{2} \sum_{i=1}^{N_c} a_i^2 + i \frac{1}{4\pi} \left( \sum_{i<j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\Lambda^2} - \frac{N_f}{2} \sum_{i=1}^{N_c} a_i^2 \log \frac{a_i^2}{\Lambda^2} \right) - i \frac{\infty}{\pi} \sum_{n=1}^{\infty} F_n(a) \Lambda^{b_1 n},$$  \hspace{1cm} (11)

where the first and the second terms are the classical and the one-loop parts, respectively, and the last ones are the instanton corrections. The logarithmic parts contribute to the classical part of $u$. Further corrections to $u$ come purely from the instanton effects: $u = \frac{1}{2} \sum_{i=1}^{N_c} a_i^2 + \sum_{k=1}^{\infty} u_k \Lambda^{b_1 k}$, where $u_k = k F_k$. On the other hand, one can obtain the modulus $u = \frac{1}{2} \sum_{i=1}^{N_c} a_i^2 + \sum_{k=1}^{\infty} u_{k}\Lambda^{b_1 k}$ from the instanton calculation. In the massive case, $u_{k}^{\text{inst.}}$ depends on $a_i$ and $m_k$. In the following, we will determine the one-instanton contributions $u_1$ and $u_{1}^{\text{inst.}}$ from the exact solutions and the microscopic calculation, respectively.

The contribution of the supersymmetric zero-modes to the field $A$ is obtained by solving the classical equation of the motion $D_0^2 A_{SS} + \sqrt{2} g i[\lambda_{SS}, \psi_{SS}] = 0$. The solution is given by $A_{SS} = \frac{ib_1}{4\pi} \xi_{SS} \psi_{SS}$, so we obtain [4]

$$\int d^4x_0 u = -\frac{g^2}{2^5 \pi^2} \left( \frac{1}{2} \xi_{SS}^2 \sum_{k=1}^{\infty} \right) \left( \frac{1}{2} \xi_{SS}^2 \right)$$  \hspace{1cm} (12)

for the part with the supersymmetric zero-modes.

For the massless case, after the integration over the bosonic and fermionic zero-modes, we obtain

$$\Lambda_{d,N_c,N_f}^{b_1} u_{1}^{\text{inst.}}(N_c, N_f) = i^{N_f} 2^{-b_1/2+1} \Lambda_{d,N_c,N_f}^{b_1} U_{1}^{\text{inst.}}(N_c, N_f),$$  \hspace{1cm} (13)
where we have rescaled the field $\phi \rightarrow g\phi$ to match with the standard convention used in the exact solutions, and

$$U_{1}^{\text{inst.}}(N_c, N_f) \equiv 2^{-5N_c+9-N_f} \pi^{-2N_c+4} \int_{V(\Omega)} d\Omega \sum_{k=0}^{\min[N_f, 2N_c-4]} N_f C_k \Gamma(2N_c - 2 - k) \times \frac{\langle \text{Tr}(\langle A \rangle^{(1)}) \rangle N_f - k}{f(\langle A \rangle, \langle A^\dagger \rangle)^{2N_c-2-k}} \langle M(\langle A^\dagger \rangle) \rangle$$

$$\det_k(M(\langle A^\dagger \rangle)) \equiv \int d^{2N_c-2} \zeta d^{2N_c-2} \xi' \left( \sum_{a=3}^{N_c} (\xi_a \bar{\zeta}_a + \bar{\xi}_a \zeta_a) \right)^k \exp \left( (\xi' M(\langle A^\dagger \rangle)) \xi' \right). \quad (14)$$

Firstly we will enumerate $U_{1}^{\text{inst.}}(N_c, N_f)$ by estimating the structures of the poles [4]. Although the integrand depends both on $a_i$ and $\bar{a}_i$, the holomorphy argument tells that $U_{1}^{\text{inst.}}(N_c, N_f)$ should be a function only of the holomorphic variables $a_i$. A pole may exist when the denominator of the integrand has some zeros in the integration region. This condition turns out that two of the $a_i$ coincides, because in this case $\langle A \rangle^{(1)}_{ll} = \langle A \rangle^{(2)} = \langle A \rangle^{(3)} = 0$ is realized. Let us study the structures of the poles with the highest order. This comes from the $k = 0$ term in the sum (14) [4]. To estimate the structure of the pole at $a_1 = a_2$, let us introduce an infinitesimally small parameter $\epsilon$ by $a_1 - a_2 = \epsilon$. Since $f = O(\epsilon)$ at $\Omega = 1$, we restrict the integration region to the infinitesimally small region $\Omega = \exp(-i\sqrt{\epsilon} \omega) \in V(N_c)$ to keep $f$ to be $O(\epsilon)$. Then the nonlinear integration region of $\Omega$ is linearized, and one obtains easily the following behavior of $U_{1}^{\text{inst.}}(N_c, N_f)$:

$$U_{1}^{\text{inst.}}(N_c, N_f) \sim \frac{(a_1 + a_2)^{N_f}}{2^{N_f} (a_1 - a_2)^2 \prod_{j=2}^{N_c} (a_1 - a_j)^2}. \quad (15)$$

The full expression should have the similar poles at $a_i = a_j (i \neq j)$, and so we obtain the following result up to possible gauge invariant regular terms:

$$U_{1}^{\text{inst.}}(N_c, N_f) = \frac{\sum_{i<j}^{N_c} (a_i + a_j)^{N_f} \prod_{k<l:j \neq i,j}^{N_c} (a_k - a_l)^2 \prod_{k \neq i,j}^{N_c} (a_i - a_k)(a_j - a_k)}{2^{N_f} \prod_{i<j}^{N_c} (a_i - a_j)^2}. \quad (16)$$

The gauge invariant regular term may exist only for the cases $N_f = 2N_c - 2$, $2N_c$ from the dimensional analysis. For $N_f = 2N_c - 2$, it is a constant term, while it is a term of the form $\text{const.} \sum_{i=1}^{N_c} a_i^2$ for the case $N_f = 2N_c$. These terms can not be fixed by the present method.

\[3\text{The explicit integration discussed below shows that the } k = 0 \text{ term results in the expected holomorphic terms as well as some unwanted terms such as terms with logarithmic poles and non-holomorphic terms. These terms cancel exactly with the terms from } k > 0.\]
of estimating the structures of the poles, but the explicit integrations for the cases \( N_c = 3 \) with \( N_f \leq 2N_c \) show that the regular terms in fact vanish\(^4\). The explicit integration is rather cumbersome even for the case \( N_c = 3 \). This is simplified enormously by putting the antiholomorphic variables \( \bar{a}_i \) to special values so that the integrand takes simple forms \(^7\), because the function \( U_1^{\text{inst}}(N_c, N_f) \) should be independent of the antiholomorphic variables \( \bar{a}_i \). Putting \( \bar{a}_1 = \bar{a}_2 = 1, \bar{a}_3 = -2 \) and taking into account the delta functional contribution at \( \Omega = 1 \) \(^7\), we obtain (16) for the cases \( N_f \leq 2N_c \) with \( N_c = 3 \).

We may also consider the massive cases. We expand with the mass terms in the instanton action (8). Since these terms cancel part of the matter fermionic zero-modes in the instanton measure, we obtain

\[
T_k(m) = \sum_{i_1 < \cdots < i_k} m_{i_1} \cdots m_{i_k}, \tag{17}
\]

Now we will check the consistency of the above microscopic results (13), (16) and (17) with the physical matching condition of the dynamical scales. The physical matching condition of the dynamical scales in the Pauli-Villars regularization scheme is given by \(^3\)

\[
\prod_i m_{Q,i}^\Lambda_d, \prod_i m_{W,i}^\Lambda_d = \Lambda_d^\Lambda_{d,i}', \tag{18}
\]

where \( \Lambda_d \) denotes the dynamical scale of the original system, and \( \Lambda_d' \) denotes that of the induced system after integrating out the heavy modes of the vector multiplets with masses \( m_{W,i} \)'s and the matter multiplets with masses \( m_{Q,i} \)'s. First consider the case that some of the masses of the matters, say \( m_{Q,i} (i = 1, \cdots, k) \), are very large compared to the others. One can show easily that, taking the limit \( \Lambda_d \to 0 \) of (17) with fixing \( \prod_{i=1}^k m_{Q,i} \Lambda_d^{b_i} = \Lambda_d^{b_i}' \), will give a similar expression of (17) with the substitution \( N_f \to N_f - k \). Another check is given by the Higgs breaking \( (N_c, N_f) \to (N_c - 1, 0) \) by taking the limit \( b \to \infty \) in \( a_i = a_i' - b(i = 1, \cdots, N_c - 1), a_{N_c} = (N_c - 1)b \). The heavy masses due to the large vacuum expectation value \( b \) are given by \( m_{W} = \sqrt{2N_c}b \) and \( m_{Q} = \sqrt{2ib} \). Hence the physical matching condition of the scales is given by

\[
\Lambda_{d,N_c-2}^{2N_c-2} = 4^{N_f/2} 2^{N_f/2-1}b^{N_f-2}N_c^{-2} \Lambda_{d,N_c,N_f}^{2N_c-N_f}. \tag{19}
\]

\(^4\)Eq. (16) is correct also for \( N_c = 2 \).
This is consistent with (13) and (16) because of
\[ U_{1}(N_{c}, N_{f}) \sim b^{N_{f}-2}N_{c}^{-2}U_{1}(N_{c} - 1, 0) \]
in the \( b \rightarrow \infty \) limit.

The exact solutions are determined by the hyperelliptic curve and the meromorphic one-form on it [1]. There are some proposals with non-perturbative differences consistent with the symmetries of the system [18, 19, 20]. Firstly, we shall use the hyperelliptic curves in [18]. The hyperelliptic curve and the meromorphic one-form \( \lambda \) for the SU\( (N_{c}) \) QCD with \( N_{f}(< 2N_{c}) \) flavors are given as follows:

\begin{align*}
F(x) &= \prod_{i=1}^{N_{c}}(x - e_{i}) + \left\{ 2^{-2}b_{1}^{N_{f}-N_{c}} \sum_{i=0}^{N_{f}-N_{c}} x^{N_{f}-N_{c}-i}t_{i}(m) \right\} \quad \text{for} \ N_{f} < N_{c}, \\
G(x) &= b_{1}^{N_{f}} \prod_{i=1}^{N_{f}}(x + m_{i}), \\
\lambda &= \frac{xdx}{2\pi iy} \left( \frac{FG'}{2G} - F' \right). \quad (20)
\end{align*}

The curve for the case \( N_{f} = 2N_{c} \) is given by the following substitution in the above definitions:

\begin{align*}
F(x) &= x^{N_{c}} + l(q) \prod_{i=0}^{N_{c} - 2} s_{N_{c} - i}x^{i} + 2^{-2}L(q) \sum_{i=0}^{N_{c}} x^{i}t_{N_{c} - i}(m), \\
s_{k} &= (-1)^{k} \sum_{i_{1} < \cdots < i_{k}} e_{i_{1}} \cdots e_{i_{k}}, \\
G(x) &= L(q) \prod_{i=1}^{N_{c}}(x + l(q)m_{i}), \\
\lambda &= \frac{1}{l(q)} \frac{xdx}{2\pi iy} \left( \frac{FG'}{2G} - F' \right). \quad (21)
\end{align*}

Here \( q \equiv \exp(2\pi i\tau) = \exp(-8\pi^{2}/g_{\text{ex}}^{2}) \) and the \( L(q) \) and \( l(q) \) are defined by

\begin{align*}
L(q) &= \frac{4\theta[1/2]}{\theta[0]^{4}}, \quad l(q) = \frac{\theta[0 \ 1]}{\theta[0]^{4}}, \\
\theta[m_{1} \ m_{2}] &= \sum_{n \in \mathbb{Z}^{N_{c}-1}} \exp\left\{ 2\pi i \left[ \frac{1}{2}(n + m_{1})^{t}\tau(n + m_{1}) + (n + m_{1})^{t}m_{2} \right] \right\}, \quad (22)
\end{align*}

where \( \tau_{ij} = \tau(\delta_{ij} + 1) \ (i, j = 1, \cdots, N_{c} - 1) \), and 0 and \( \frac{1}{2} \) denote the zero vector and a vector with one of its entries being \( \frac{1}{2} \) and the others are zeros, respectively.
The vacuum expectation values of the scalar field $a_i$ can be written as periods of the one form $\lambda$ on the curve $[1]$: \[
(\lambda_i, a) = \oint_{A_i} \lambda, \tag{23}\]
where $\lambda_i$ are the fundamental weights and $A_i$ are the appropriate homology cycles on the curve. The equation (23) gives $(\lambda_i, a)$'s as functions of $e_i$'s. Inverting them, one obtains the modulus $u \equiv \frac{1}{2} \sum_{i=1}^{N_c} \epsilon_i^2$ in terms of $a_i$'s. For $N_c = 3$, the explicit form of the contour integral (23) may be obtained by solving the Picard-Fuchs equations [21] with respect to $u$ and $v = e_1 e_2 e_3$. In the semi-classical region, the power series type solutions for $N_f$ flavors ($1 \leq N_f \leq 5$) takes the form \[
(\lambda_1, a) = w(\alpha_{N_f}, \beta_{N_f}; x_1, x_2) + \frac{1}{2} w(\gamma_{N_f}, \delta_{N_f}; x_1, x_2),
(\lambda_2, a) = w(\alpha_{N_f}, \beta_{N_f}; x_1, x_2) - \frac{1}{2} w(\gamma_{N_f}, \delta_{N_f}; x_1, x_2), \tag{24}\]
where \[
x_1 = \frac{v^2}{u^3}, \quad x_2 = \Lambda^{6-N_f} u^{N_f/2-3+(-1)^{N_f}} / \Lambda^{-(1-(-1)^{N_f})/2}, \tag{25}\]
and $\alpha_{N_f} = -\frac{1}{4(6-N_f)}, \beta_{N_f} = \delta_{N_f} = -\frac{1}{6-N_f}, \gamma_{N_f} = \frac{9+(-1)^{N_f}(3-2N_f)}{4(6-N_f)}$. $w(\alpha, \beta; x_1, x_2)$ denotes a power series of the form $\sum_{m,n \geq 0} d_{m,n} x_1^m x_2^n$ with $d_{0,0} = 1$. The coefficients $d_{m,n}$ are determined recursively. For $N_f = 6$ the explicit evaluation of the contour integral would be effective instead of using the Picard-Fuchs equations. For massless cases we may derive the one-instanton correction to $u$ for $N_c = 3$ by the explicit enumeration of the integral, which is expanded in powers of $\epsilon = \Lambda^{b_i/2}$. The calculation goes as follows. Let $e'_i$ and $e''_i$ denote the two branch points of the curves [20], [21] approaching to $e_i$ in the limit $\epsilon \to 0$. When the homology cycle in the right hand side of (23) is taken around the $e'_i$ and $e''_i$, one obtains directly $a_i$. The $e'_i$ and $e''_i$ can be expanded in the integral powers of $\epsilon$, and we take the terms up to order $\epsilon^4$. We expand the one-form $\lambda$ up to order $\epsilon^3$, after the change of variable $x = e_i + \epsilon z$. Performing the contour integral explicitly, we obtain $a_i$ in terms of $e_j$'s. Inverting the results, we obtain $u = \frac{1}{2} \sum_{i=1}^{N_c} \epsilon_i^2$ in terms of $a_i$'s. Taking the term with $\epsilon^2$, we obtain the one-instanton correction as \[
\Lambda_{N_c,N_f}^{b_i} u_1 = 2^{-1} \Lambda_{N_c,N_f}^{b_i} U_1(N_c, N_f), \quad U_1(N_c, N_f) = \sum_{i=1}^{N_c} a_i N_f \Delta_{N_c-1}(a_1, \ldots, \delta_2, \ldots, a_{N_c}) + A_{N_c} \delta_{N_f,2N_c-2} + B_{N_c} \delta_{N_f,2N_c} \sum_{i=1}^{N_c} a_i^2, \tag{26}\]
\[ \Delta^m(a_1, \ldots, a_m) \equiv \prod_{k<l}^m (a_k - a_l)^2 \]  

(26)

for the cases \( N_f \leq 2N_c \) with \( N_c = 3 \), where \( A_3 = 0 \). For the case \( N_f = 2N_c \) with \( N_c = 3 \), the scale parameter \( \Lambda_{N_c,N_f}^{b_1} \) should be replaced by the \( L(q) \sim 64q \), and we obtain \( B_3 = -\frac{7}{8} \) by using \( l(q) \sim 1 - 40q \).

We have also calculated the modulus \( u \) in other curves. For the curve presented in [19], we obtain \( \Lambda_{N_c,2N_c}^{b_1} \sim -64q \), \( A_3 = -\frac{1}{2} \) and \( B_3 = -\frac{1}{2} \). For the curve [20], we get \( \Lambda_{N_c,2N_c}^{b_1} \sim -108q \), \( A_3 = -\frac{1}{2} \) and \( B_3 = 0 \).

In order to obtain the relation between the dynamical scales and the scale parameters in the curves, let us consider the substitution \( e_i = e_i' - b (i = 1, \ldots, N_c - 1), e_{N_c} = (N_c - 1)b \) and take the \( b \to \infty \) limit in the curves (20), (21). This corresponds to the Higgs breaking we considered in the check of the microscopic instanton calculation, and in fact the curve with \( (N_c, N_f) \) reduces to that with \( (N_c - 1, 0) \) after the rescaling and shift of \( x \) and \( y \) and the substitution

\[ \Lambda_{N_c-1,0}^{2N_c-2} = N_c^{-2}b^{N_f-2}\Lambda_{N_c,N_f}^{2N_c-N_f}. \]  

(27)

Comparing (27) with the physical matching condition (19) and using the known relation \( \Lambda_{d,2,0} = \Lambda_{2,0} \) [6, 7], we obtain the relation between the dynamical scales and the scale parameters of the curves as:

\[ \Lambda_{N_c,N_f}^{b_1} = i^{N_f-2}b_1^{N_f-2}\Lambda_{d,N_c,N_f}^{b_1}. \]  

(28)

Now let us discuss the differences between the microscopic instanton calculation and the exact solution. Comparing (13) and (26) with using (28), the \( u_{1}^{inst.} \) and the \( u_1 \) is the same except \( U_{1}^{inst.} \) and \( U_1 \). One can show easily that the structures of the poles are the same between \( U_{1}^{inst.} \) and \( U_1 \), and hence the possible difference is a regular term. From the gauge invariance and the dimensional counting, this regular term is restricted in the following form:

\[ U_1(N_c, N_f) = U_{1}^{inst.}(N_c, N_f) + C_{N_c}\delta_{N_f,2N_c-2} + D_{N_c}\delta_{N_f,2N_c} \sum_{i=1}^{N_c} a_i^2. \]  

(29)

\(^5\)For \( N_f = 2N_c \) case, one notices that the square inverses of the gauge coupling constant \( 1/g^2 \) in the microscopic theory and \( 1/g_{ex}^2 \) appearing in the exact solution are different by a constant shift in general. This is not an inconsistency even for the scale invariant case, because finite renormalizations might exist in general.
In fact, $C_3, D_3 \neq 0$ for all the above proposed curves.

These differences do not seem to lead to any inconsistencies because, for the $N_f = 2N_c - 2, 2N_c$ cases, the anomaly free symmetries certainly allow the above ambiguities in the construction of the curves [18, 19, 20]. There would also be some non-perturbative ambiguities in the definitions of the quantum operator $\text{Tr}A^2$ in the microscopic theory. For $N_c = 2$ case, this type of resolution has been discussed in [22]. We point out that these differences cause some qualitative differences in the non-perturbative renormalization procedures under reductions of the systems. Consider for example the Higgs breaking $a_i = a_i' - b (i = 1, \cdots, N_c - 1), a_{N_c} = (N_c - 1)b (b \to \infty)$ while $K$ matters are kept massless; $m_i = i\sqrt{2}b (i = 1, \cdots, K)$. Then the system with $(N_c, N_f)$ reduces to that with $(N_c - 1, K)$. When it is applied to the microscopic result (16), there appear divergences which cannot be absorbed into the matching condition (18). But it can be easily shown that the divergences are regular terms and (16) becomes consistent if the operator $u$ is subtracted by divergent regular terms $u^{\text{inst.}}_{N_c-1,K} = u^{\text{inst.}}_{N_c,N_f} - (\text{regular terms})$ under the reduction. On the other hand, we do not need such a subtraction for the modulus $u$ under the similar reduction in the curves proposed in refs. [19, 20]. Another example is the reduction in the mixed branch. If one sets $a_{N_c} = 0$ in the curve of [19] with $(N_c, N_f)$ for the massless case, the curve reduces to the one with $(N_c - 1, N_f - 2)$. This can be justified physically by assuming that the mixed branch $a_{N_c} = 0$ with $q^i = \langle q \rangle \delta_{i,j} \delta^N_{a}$, $\tilde{q}^a_i = \langle q \rangle \delta_i^N \delta^N_{a}$ touches the Coulomb branch at $\langle q \rangle \sim 0$ [19]. On the other hand, the microscopic one-instanton result (14) is not consistent with naively setting $a_{N_c} = 0$, and $u$ needs again regular term shifts. A further analysis would be necessary in order to clarify the physical meaning of this difference.

After completion of this work, we noticed the preprints [23], in which the explicit evaluations of the prepotentials of the exact solutions are discussed for $N = 2$ supersymmetric gauge theories.  

\footnote{One would have $u^{\text{inst.}} = c(q)u \ (c(0) = 1)$ and $u^{\text{inst.}} = u + c'\Lambda^2$ for massless $N_f = 2N_c$ and $N_f = 2N_c - 2$ cases, respectively, which could explain the discrepancies in (29). For $N_c = 2$ case, see refs. [22].}
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