THE COUNTERFACTUAL MEANING OF THE ABL RULE

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The Aharonov-Bergmann-Lebowitz rule assigns probabilities to quantum measurement results at time $t$ on the condition that the system is prepared in a given way at $t_1 < t$ and found in a given state at $t_2 > t$. The question whether the rule can also be applied counterfactually to the case where no measurement is performed at the intermediate time $t$ has recently been the subject of controversy. I argue that the counterfactual meaning may be understood in terms of the true value of an observable at $t$. Such a value cannot be empirically determined for, by stipulation, the measurement that would yield it is not performed. Nevertheless, it may or may not be well-defined depending on one’s proposed interpretation of quantum mechanics. Various examples are discussed illustrating what can be asserted at the intermediate time without running into contradictions.

1 Introduction

The Aharonov-Bergmann-Lebowitz (ABL) rule was proposed in 1964 to compute measurement result probabilities of systems that are both preselected and postselected. It was meant to provide a time-reversal invariant formulation of nonrelativistic quantum mechanics.

As long as a measurement is actually carried out between pre- and postselection, the ABL rule is a straightforward consequence of the Born probability rule and usual assumptions on the quantum state following a measurement. Attempts have been made, however, to interpret the ABL rule counterfactually, that is, to cases where no intermediate measurement is made. The resulting controversy is, I believe, partly due to the fact that proposed definitions of counterfactuals in the present quantum-mechanical context do not adequately capture the intuitive meaning of the notion.

In this paper I first review the derivation of the ABL rule and some of its properties, in particular contextuality. Next I analyze alleged proofs of the impossibility of interpreting the rule counterfactually, as well as various arguments supporting or opposing them. I attempt to clarify the meaning of a counterfactual interpretation, and point out circumstances in which the counterfactual assertion of the ABL rule is or is not correct.
2 The ABL rule

Consider a quantum system $S$ and three observables $A$, $C$ and $B$ pertaining to it. For simplicity, we shall first assume that these observables are discrete and nondegenerate.

Suppose that at time $t_1$, $S$ is prepared in an eigenstate $|a\rangle$ of $A$. At $t > t_1$, the observable $C$ is measured. According to the Born rule, the probability of obtaining result $c_i$ (where $\{c_j\}$ is the set of eigenvalues of $C$) is given by $|\langle c_i|a\rangle|^2$.\(^a\)

Suppose the result $c_i$ is indeed obtained upon measurement at $t$. We assume the measurement interaction is such that immediately after $t$, the system is in state $|c_i\rangle$. At a later time $t_2$, the observable $B$ is measured. The probability of obtaining result $b$ is then given by $|\langle b|c_i\rangle|^2$. Hence the probability of result $c_i$ and result $b$, conditional on preparation $|a\rangle$, is equal to

$$P(c_i \land b|a) = |\langle b|c_i\rangle \langle c_i|a\rangle|^2.$$  \(^1\)

We are interested here in the probability of obtaining $c_i$ at $t$, on the condition that $S$ is prepared in $|a\rangle$ at $t_1 < t$ and found in $|b\rangle$ at $t_2 > t$. This we shall denote by $P(c_i|a,b)$. From the definition of conditional probability, we have (for $P(b|a) \neq 0$)

$$P(c_i|a,b) = \frac{P(c_i \land b|a)}{P(b|a)}.$$  \(^2\)

Here $P(b|a)$ is the total probability of $b$ (given preparation $|a\rangle$), equal to the sum of $P(c_j \land b|a)$ over all possible results $c_j$. From \(^1\) we obtain

$$P(c_i|a,b) = \frac{|\langle b|c_i\rangle \langle c_i|a\rangle|^2}{\sum_j |\langle b|c_j\rangle \langle c_j|a\rangle|^2} = \frac{\text{Tr}(P_bP_{c_i}P_aP_{c_i})}{\sum_j \text{Tr}(P_bP_{c_j}P_aP_{c_j})}.$$  \(^3\)

This is the ABL rule \(^1\). In the right-hand side, $P_a$ and $P_b$ are projectors on states $|a\rangle$ and $|b\rangle$. The operator $P_{c_i}$ projects on $|c_i\rangle$ or, if the eigenvalue $c_i$ is degenerate, on the associated subspace.\(^b\) The ABL rule (expressed in terms of projection operators) also holds in that case, provided the intermediate measurement obeys the L"uders rule $|a\rangle \rightarrow NP_{c_i}|a\rangle$ (where $N$ is a normalization constant). The ABL rule can also be written for multiple intermediate measurements or for selection by means of density matrices rather than pure states, but we won’t need these generalizations here.

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\(^a\)All kets are normalized. We assume that the system’s Hamiltonian vanishes in between measurements. If it does not, evolution operators must be appropriately inserted, or kets expressed in the Heisenberg picture.

\(^b\)The sum over $j$ is then carried out on these subspaces.
A much discussed property of the ABL rule is its contextuality. Let \( |u_1 \rangle, |u_2 \rangle \) and \( |u_3 \rangle \) be an orthonormal basis in a three-dimensional Hilbert space. Each of these kets can represent a state wherein a particle is in one of three disjoint boxes. Take the initial and final states as

\[
\begin{align*}
|a\rangle &= \frac{1}{\sqrt{3}} \left( |u_1 \rangle + |u_2 \rangle + |u_3 \rangle \right), \\
|b\rangle &= \frac{1}{\sqrt{3}} \left( |u_1 \rangle + |u_2 \rangle - |u_3 \rangle \right).
\end{align*}
\]

Let the intermediate observable \( C \) be so chosen as to have \( |u_1 \rangle \) as one of its eigenvectors, corresponding to a nondegenerate eigenvalue \( c_1 \). The ABL rule then yields

\[
P(c_1|a,b) = \frac{\text{Tr}(P_a P_c P_a P_c)}{\sum_j \text{Tr}(P_b P_c P_a P_c)}.
\]

It turns out that the right-hand side of this equation is not well-defined unless \( C \) itself is well-defined. Let \( c_1, c_2 \) and \( c_3 \) be three different real numbers and define

\[
C = c_1 |u_1\rangle \langle u_1| + c_2 |u_2\rangle \langle u_2| + c_3 |u_3\rangle \langle u_3|.
\]

Elementary algebra shows that \( P(c_1|a,b) = 1/3 \). But if

\[
C' = c_1 |u_1\rangle \langle u_1| + c_2 \{|u_2\rangle \langle u_2| + |u_3\rangle \langle u_3|\},
\]

then \( P(c_1|a,b) = 1 \). Hence the ABL probability of \( c_1 \) depends not only on the eigenspace associated with that eigenvalue, but also on the structure of the observable in the orthogonal eigenspace. This is contextuality. In terms of boxes, this means the following. If an observable \( (C') \) distinguishes the first box from the other two taken together, then the ABL probability of being in the first box is 1. If another observable \( (C) \) distinguishes the three boxes one from another, the ABL probability of being in the first box is 1/3.

It is easy to check that if \( C = A \),

\[
P(a|a,b) = 1.
\]

Likewise if \( C = B \),

\[
P(b|a,b) = 1.
\]

From \( \text{(9)} \) and \( \text{(10)} \), Albert, Aharonov and D’Amato have argued that non-commuting observables \( A \) and \( B \) must be simultaneously well-defined at \( t \). This conclusion hinges on a counterfactual use of the ABL rule. It was indeed intended to apply specifically to the case where neither \( A \) nor \( B \) are measured between pre- and postselection.
3 Counterfactual interpretation

Can the ABL rule be interpreted counterfactually? That question was answered in the negative by Sharp and Shanks, Cohen and Miller, and much debated afterwards.

The crux of the Sharp and Shanks argument (as well as others) can be stated rather succinctly. Let \( |b\rangle \) and \( |b'\rangle \) be the possible final states of a two-state system. Suppose \( C \) is not measured, and assume that the ABL rule correctly gives the probability of nondegenerate result \( c_1 \), conditional on pre- and postselection, had \( C \) been measured. The total probability of \( c_1 \) should then be given as a weighted sum on the possible final states, that is,

\[
P(c_1|a) = |\langle b | a \rangle|^2 P(c_1|a, b) + |\langle b' | a \rangle|^2 P(c_1|a, b').
\] (11)

Here \( |\langle b | a \rangle|^2 \) is the probability of final state \( |b\rangle \) when no intermediate measurement is made, and \( P(c_1|a, b) \) is the conditional probability of \( c_1 \) based on the counterfactual interpretation of the ABL rule. But according to standard quantum mechanics, \( P(c_1|a) \) is given by \( |\langle c_1 | a \rangle|^2 \). Hence we should have

\[
|\langle c_1 | a \rangle|^2 = |\langle b | a \rangle|^2 \sum_j \text{Tr}(P_b P_{c_j} P_a P_{c_j}) + |\langle b' | a \rangle|^2 \sum_j \text{Tr}(P_{b'} P_{c_j} P_a P_{c_j}).
\] (12)

Since this is not true in general (counterexamples are easily found), Sharp and Shanks conclude that the counterfactual use is invalid.

The validity of the counterfactual use and the relevance of the Sharp and Shanks argument were debated between Vaidman and Kastner. Vaidman’s objection to the proof consists in pointing out that the weight \( |\langle b | a \rangle|^2 \) in (11) is the probability of \( b \) if no intermediate measurement occurs. Since we are asking for the total probability of result \( c_1 \), we must use the expression for the probability of \( b \) if \( C \) is measured at \( t \). That probability is given by \( \sum_j \text{Tr}(P_b P_{c_j} P_a P_{c_j}) \). But then Eq. (11) becomes

\[
P(c_1|a) = \sum_j \text{Tr}(P_b P_{c_j} P_a P_{c_j}) P(c_1|a, b) + \sum_j \text{Tr}(P_{b'} P_{c_j} P_a P_{c_j}) P(c_1|a, b')
\]

\[
= \text{Tr}(P_b P_{c_1} P_a P_{c_1}) + \text{Tr}(P_{b'} P_{c_1} P_a P_{c_1})
\]

\[
= |\langle c_1 | a \rangle|^2,
\] (13)

in accordance with standard quantum mechanics.

The significance of this calculation of \( P(c_1|a) \) is best brought out by quoting Vaidman’s definition of the counterfactual meaning of the ABL rule.
If a measurement of an observable $C$ were performed at time $t$, then the probability for $C = c_j$ would equal $P(c_j)$, provided that the results of measurements performed on the system at times $t_1$ and $t_2$ are fixed.

This is a statement about the statistical distribution of results of unperformed experiments, made on the basis of a law derived from a large number of performed experiments identical to the former in all relevant respects. It is like saying, on the basis of numerous tosses of a die yielding essentially uniform outcomes, that had the die been tossed one additional time, the probability of obtaining “3” would have been $1/6$. Both this statement and Vaidman’s definition express the regularity of Nature. Both assert, on the basis of a general rule drawn from numerous experimental trials, that if the experiment were done over again in the same relevant conditions, the results would fall under the same general rule. Hence Vaidman’s statement is (presumably) true, genuinely counterfactual, but not particularly illuminating, at least as far as specificities of quantum mechanics are concerned.

In contrast with Vaidman, Kastner defended the Sharp and Shanks proof and argued that a nontrivial counterfactual assertion of the ABL rule (i.e. one that provides information about specific systems) is false. Kastner’s analysis draws from both Goodman’s and Lewis’s theories of counterfactuals. Her argument can be summarized as follows.

A counterfactual $p \square \rightarrow q$ (read “If it were that $p$, it would be that $q$.”) is true if $p$ is not true and the conjunction of $p$ with laws of nature $L$ and suitable background conditions $S$ implies $q$, that is,

\[(p \land L \land S) \rightarrow q. \quad (14)\]

Now clearly, there must be restrictions on $S$ for, if $S$ includes the statement $\neg p$ (“not $p$”), implication (14) will hold trivially. These restrictions usually state that $S$ should be “cotenable” with $p$, a notion not so easy to define but essentially meaning that $S$ is independent of the truth of $p$.

Suppose that in the real world, preparation $|a\rangle$ is followed by postselection $|b\rangle$, with no intermediate measurement. Then had there been an intermediate measurement, the intermediate state would have changed, and the final measurement result $b$ could no longer be expected to obtain. In other words, result $b$ is not cotenable with measuring $C$.

\(^c\)Admittedly, our belief in the noncounterfactual validity of the ABL rule comes from more indirect evidence, but this does not affect the present discussion.

\(^d\)The relation between weak values and measurement results with unit ABL probability does not affect that remark, since it can be established solely on the basis of the noncounterfactual use of the ABL rule.
Does it follow from this argument that the nontrivial counterfactual assertion of the ABL rule is false? The answer is affirmative if the background conditions must include the result $b$. But this is not necessarily the case. The background conditions can be construed as encompassing everything that characterizes the state of the system at $t$, including whatever might induce it to yield result $b$ upon measurement of $B$ at $t_2$. With such a definition, the result $b$ itself is not part of the background conditions. And although cotenability might be analyzed in that context, I shall use a different approach.

I submit that the proper way of investigating the counterfactual validity of the ABL rule is to enquire about the “true value” of $C$ at $t$. Strict empiricists will no doubt quit reading right here, since it is absolutely impossible, in a situation where $C$ has not been measured between $t_1$ and $t_2$, to empirically ascertain what the true value of $C$ was at the intermediate time $t$. But that doesn’t prevent different theories or interpretations to make claims on what the true value is, as we will presently see.

4 What can be said at $t$?

It is instructive to analyze the Sharp and Shanks argument in terms of true values. If, following standard quantum mechanics, $P(c_1|a)$ is equal to $|\langle c_1|a \rangle|^2$, then the first equality in (13) holds identically. But Sharp and Shanks claim that the counterfactual meaning of the ABL rule is encapsulated in Eq. (11). It is easy to see that (11) and (13) will coincide if

$$|\langle b|a \rangle|^2 = \sum_j \text{Tr}(P_b P_{c_j} P_a P_{c_j}),$$

(15)

together with a similar equation with $b$ replaced by $b'$. In quantum mechanics, Eq. (15) does not hold in general.

The left-hand side of (15) is the uncontroversial probability of $b$ when no intermediate measurement is performed, while the right-hand side is the uncontroversial probability of $b$ when $C$ is measured at the intermediate time. That the equality is false means that whichever true value $C$ does or does not have at the intermediate time $t$, it cannot be one that is simply revealed in and unaffected by an eventual measurement.

It was pointed out in 4 that (15) holds if $(P_a, \{P_{c_j}\}, P_b)$ makes up a consistent family of histories. Indeed in that case one can maintain that $C$ has a well-defined value (equal to one of its eigenvalues) without running into contradictions. But of course one is not compelled to do so. Associating or

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*For relevant definitions see 18, Sect. 12.6.*
not associating true values to observables belonging to a consistent family of histories contributes in defining the interpretation of the theory.

It is instructive to recall the example of Sect. 2, with \( |a \rangle \) and \( |b \rangle \) defined as in Eqs. (4) and (5). Let \( C \) and \( C' \) be defined as in (7) and (8) and let

\[
C'' = c_1 |u_1\rangle \langle u_1| + |u_3\rangle \langle u_3| + c_2 |u_2\rangle \langle u_2|.
\]  

(16)

It is easy to check that

\[
(P_a, \{P_{u_1}, I - P_{u_1}\}, P_b)
\]  

(17)

and

\[
(P_a, \{P_{u_2}, I - P_{u_2}\}, P_b)
\]  

(18)

make up distinct consistent families of histories. The ABL probability that \( C' = c_1 \) is one, and so is the ABL probability that \( C'' = c_2 \). We won’t run into contradictions if we maintain that the particle was surely in box 1 at \( t \). Likewise we won’t run into contradictions if we maintain that it was in box 2. Of course both statements cannot be held at once, since they are logically contradictory. Note that the family of histories

\[
(P_a, \{P_{u_1}, P_{u_2}, P_{u_3}\}, P_b),
\]  

(19)

more refined than either (17) or (18), is not consistent.

The families \( (P_a, \{P_{a_k}\}, P_b) \) and \( (P_b, \{P_{b_l}\}, P_b) \), where \( \{a_k\} \) and \( \{b_l\} \) stand for the set of eigenvalues of \( A \) and \( B \), are always consistent. Hence it can be maintained that \( A \) has the true value \( a \) at intermediate times, or that \( B \) has the true value \( b \). We have assumed that \( P(b|a) \neq 0 \). Therefore \( |b\rangle \) and \( |a\rangle \) are not orthogonal. So it is not a priori logically inconsistent to assume that \( A \) and \( B \) both have true values at intermediate times.\(^4\) It is not clear whether a full-fledged interpretation can be implemented along these lines, for arbitrary observables \( A \) and \( B \).

The ABL rule is symmetric under permutation of preselection and postselection. This means that for the purpose of making probabilistic statements about intermediate measurement outcomes, the initial and final states \( |a\rangle \) and \( |b\rangle \) have exactly the same utility. This does not necessarily entail that they are equally useful for the purpose of making ontological statements. In von Neumann’s measurement theory,\(^5\) for instance, a measurement is an interaction between a quantum system and an apparatus, followed by a collapse. From the time \( t_1 \) when the system is prepared in state \( |a\rangle \) to the time \( t \) when an

\(^4\)Note that if \( P_{ab} \) denotes the projector on the subspace spanned by \( |a\rangle \) and \( |b\rangle \), then the family of histories \( (P_a, \{P_{ab}, I - P_{ab}\}, P_b) \) is consistent.
ontological statement is to be made, no physical action occurs on the system. Such is not the case, however, between $t$ and $t_2$. Indeed a physical interaction of the quantum system with an apparatus has to occur sometime before $t_2$, for the system to collapse to $|b\rangle$ at $t_2$. In that context, it may be more natural to hold that $|a\rangle$, rather than $|b\rangle$, is the correct state at $t$, and that $A = a$, rather than $B = b$, expresses a true value.

In some interpretations of quantum mechanics, true values of observables can be assigned outside the context of consistent families of histories. Their statistical distributions, however, will not obey the ABL rule. An example is Bohmian mechanics, where the true value of position is defined at any intermediate time $t$. But in general, $(P_a, \{P_x\}, P_b)$ does not make up a consistent family. In this context, the meaning of background conditions proposed at the end of the last section is particularly clear. Suppose that $|a\rangle$ and $|b\rangle$ correspond to one-dimensional wave functions $\psi_a(x)$ and $\psi_b(x)$, with $\psi_a$ a Gaussian and $\psi_b$ a function uniform over some interval $\Delta$ and zero elsewhere (i.e. $|b\rangle$ postselects through a slit of width $\Delta$). All Bohmian trajectories going through $\Delta$ at $t_2$ have gone through some other interval $\Delta_t$ at $t$. Hence the background conditions associated with preselection $|a\rangle$ and postselection $|b\rangle$ are the wave function at $t$ together with the interval $\Delta_t$ of true positions. Of course if position is measured at $t$, the wave function will change accordingly, and so will the measurement result probabilities at $t_2$.

It was pointed out by Vaidman that no hidden variable theory can reproduce the ABL rule in all situations. Indeed take an ensemble of spin $\frac{1}{2}$ particles prepared in the state $|a\rangle = |\pm\rangle$. If there is no intermediate measurement, postselection in the state $|b\rangle = |a\rangle$ will in fact introduce no additional selection. Hence if no backward causality is assumed, true values at $t$ must be the same whether postselection does or does not occur, irrespective of any hidden variables. But then the probability that a measurement at $t$ of the spin along $\hat{n}$ yields $+$ must be equal to $\cos^2\{\frac{1}{2} \cos^{-1}(\hat{n} \cdot \hat{z})\}$, which differs from the ABL value.

In the Copenhagen interpretation of quantum mechanics (or at least in the most instrumentalistic versions of it), an observable $C$ has a value only when a measurement of $C$ indicates that value. This is also the case in Mohrhoff’s more recent interpretation, which incorporates the ABL rule explicitly. The fact that the ABL rule predicts a statistical distribution of values of $C$ implies, according to Mohrhoff, an objective fuzziness of $C$ in the case where the experiment is not performed. I believe that statement is genuinely and

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8 Vaidman is well aware that in hidden-variable theories, counterfactual statements involve fixing the values of hidden variables. See for instance his discussion in Sect. 5.
nontrivially counterfactual. It asserts that no unmeasured observable of no individual system whatsoever has a true value in the interval between pre- and postselection.

5 Conclusion

The noncounterfactual meaning of the ABL rule is not controversial. The validity of the rule is then a straightforward consequence of standard quantum mechanics and usual hypotheses on the state of a quantum system immediately after measurement. The rule is also true counterfactually if it simply expresses the reproducibility of experiments.

A counterfactual meaning more relevant to the specificities of quantum mechanics refers to the true value of an observable at an intermediate time, when no observable is actually measured between pre- and postselection. In general the ABL rule is then counterfactually false. It can be true, however, when associated with a consistent family of histories or when asserting objective fuzziness or nonvaluedness of observables not being measured.

Although statements about true values between pre- and postselection may not have definite empirical meaning, they can fruitfully be viewed as contributing to define the interpretation of the quantum-mechanical theory.

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