Differential equations of order 1 in differential fields of zero characteristic\(^1\)

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Abstract

I begin from a particular field of generalised Puiseux series and investigate a class of nonlinear differential equations in the field. It is appeared that the main part of differential equation determines solvability and positions of resonance i.e. the appearance of a free constant in solutions. Secondly, for any singular differential equation in a differential field of the form \(Q(y)\dot{y} = P(y)\), \(P, Q\) polynomials. Then the greatest Picard-Vessiot extension exists. It is shown that the arising set of solutions is obtained from algebraic equations labelled by constants. Sufficient conditions for the PV extension being extended liouvillian are delivered.

2000 Mathematics Subject Classification. 16W60, 34G20, 12F99

Key words and phrases. Puiseux series, valuation, singular differential equation, liouvillian extension

1 Preliminaries

Differential Galois theory gives a complete view of algebraic extension structure of solutions for a given linear differential equation \([7, 4]\). For proceeding the research of general, nonlinear equation

\[
\dot{y} = \frac{P(y)}{Q(y)}, \tag{1}
\]

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where $P$ and $Q$, polynomials under a field $K$ have no common roots in algebraic closure of $K$, firstly I elaborate a special field of generalised Puiseux series in the section. The existence problem was partially solved in [1] by the method of Newton polygons. For other approaches see for example [3]. Section 2 refines the result. The introducing contour of differential equation defines position of solutions beginning and resonance. For a general differential field the existence of a maximal Picard-Vessiot extension (the greatest one with the isomorphism identification) via Eq. (1) is proved in section 3, theorem 12 and corollary 13. In the place of investigation of the group of symmetries of the extension [7] for the case of linear equations, here, one focuses on a rational functions associating with (1), (29). In the way a sufficient condition for realizing the extension by finite number of operations: successive adding an exponent of integral, an integral (liouvillian extension) or algebraic extension appears as theorem [8].

I begin from a definition of (generalised) Puiseux field. Everywhere in the paper a field $K$ is of zero characteristic. Therefore, $\mathbb{Q} \subset K$. I will further assume that an ordered $\mathbb{Q}$-linear space with order $(R_K, \leq)$ is given, i.e. $v_1 > v_2 \Rightarrow v_1 + v > v_2 + v$ and $\mathbb{Q} \ni q > 0 \Rightarrow R_K \ni v > 0 \Rightarrow qv > 0$.

An example is $R_K = \mathbb{Q}$ or $R_K = \mathbb{R} \cap K$. The last case is equivalent to extra property $\forall (r, \epsilon \in R_K) \exists (N \in \mathbb{N}) (r > 0 \land \epsilon > 0) \Rightarrow ne > r$.

The completing of $R_K$ to $\mathbb{Q}$-space with linear order $\tilde{R}_K$ by Peano construction enriches the space with suprema and infima of nonempty and properly bounded subsets. $\tilde{R}_K$ is also a completion with respect to the norm $|\cdot|: \tilde{R}_K \mapsto R_K$ such that $|r| = r$ for $r \geq 0$ and $|r| = -r$ for $r < 0$.

**Definition 1.** Let $S \subset 2^{\tilde{R}_K}$ be the family of all well ordered sets of $(R_K, \leq)$. The Puiseux field $K_P[[x]]$ under the field $K$ is the sum $\bigcup (K - \{0\})^s$ under all $s \in S$. The operations are defined by the identification of the nonzero elements of $K_P[[x]]$ with series of the form $c_0 x^{P_0} + c_1 x^{P_1} + \ldots$, where the set of indices $\{\mu_i\} \in S$ and the set of coefficients $c_i \in K - \{0\}$.

The multiplication is properly defined because of the property of finite decomposability of elements of $s$, $s \in S$, with respect to the additive operation in $\tilde{R}_K$. The set of non-decomposable elements of $s$ is referred to as $B(s) = \{ \mu \in s | \mu = \mu_1 + \mu_2 \Rightarrow (\mu_1 = \mu_\lor \mu_2 = \mu) \}$. If $s_1, s_2 \in S$ I write $s_1 + s_2$ for $s_1 \cup s_2$, $s_1 \cdot s_2$ in the place of $\{r_1 + r_2 | r_1 \in s_1 \land r_2 \in s_2 \}$ and $[s_i]$ for $s_i + s_i \cdot s_i + \ldots$. If $\forall (x \in s_i) x \geq 0$ and $R_K \subset \mathbb{R}$ then $[s_i] \in S$. Extending the monoid $(S, +)$ to the group $\tilde{S} \equiv \{ s_1 - s_2 | s_1, s_2 \in S \}$ one obtains a ring $(\tilde{S}, +, \cdot)$. Let $G(s)$ means the subset of $s$, $s \in S$, of elements
without the proceeding element in $s$. I put $N(s) := k$ for all $s$ in $S$ such that $G^k(s) = GG \ldots G(s)$ is a finite set or the image of a divergent to infinity sequence of elements of $s$ and $k \in \{0\} \cup \mathbb{N} \cup \{+\infty\}$ is the smallest such number and $N(-s) = N(s)$.

The set of series $K^0_P[[x]]$ is a subring of $K_P[[x]]$, where $K^0_P[[x]]$ contains by definition indices only from $N_{-1}(0)$. If, additionally, $R_K \subset \mathbb{R}$ then $K^0_P[[x]]$ is a subfield.

Finding solutions of algebraic equations with coefficients in $K^0_P[[x]]$ may be realized explicitly. In the way one may state

**Theorem 1.** Let $K$ be an algebraically closed field of zero characteristic. Then $K_P[[x]]$ is algebraically closed.

**Proof.** Any equation of degree one has a solution in $K_P[[x]]$. Let all equations of degree less than $N - 1$, $N \geq 1$, have a solution in the field. Now, I take an algebraic equation,

$$w(y) := \sum_{i=0}^{N} \alpha_i y^i = 0, \quad (2)$$

$\alpha_i \in K_P[[x]]$, $\alpha_N \alpha_0 \neq 0$ and $\alpha_i = \alpha_0(i) x^{\nu_0(i)} + \ldots$ for $\alpha_i \neq 0$. Let $y = c_0 x^{\mu_0} + c_1 x^{\mu_1} + \ldots$ fulfills the equation. The contour function $\mathfrak{f} : R_K \mapsto R_K$ is defined by $\mathfrak{f}(x) = \min\{\nu_0(i) + ix | i = 0, \ldots, N\}$. The nonempty finite set of breaking points $\{x_b\}$ of $\mathfrak{f}$ determines all starting $\mu_0$ admitted by the equation by $\mu_0 = x_b$. The associated with $x_b$ polynomial $p_0(c_0) = \sum_{i \in B} \alpha_0(i) c_0^i$, where $B \subset \{0, \ldots, N\}$ counts realizations of the minimum of $\mathfrak{f}$ in $x_b$, defines initial $c_0 \neq 0$ admitted for $\mu_0 = x_b$. One may observe that the number of available pairs $(\mu_0, c_0) \in R_K \times K^*$ including multiplicities of $c_0$ is equal to $N$. Really, if a polynomial has all roots in $K_P[[x]]$ then each $(\mu_0, c_0)$ including multiplicities begins a solution.

Let’s assume that one possesses $y_0 = c_0 x^{\mu_0} + \ldots \in K_P[[x]]$. Then $y = y_0 + \bar{y}$ defines a new variable $\bar{y}$. Eq. (2) may be rewritten to

$$\sum_{i=0}^{N} \beta_i \bar{y}^i = 0, \quad (3)$$

$$\beta_i = \sum_{l=1}^{N} \binom{l}{i} \alpha_l y_0^{l-i} = \frac{1}{i!} \frac{d^i}{dy} w|_{y=y_0}. \quad (4)$$

I will refer to $y_0 \in K_P[[x]]$ as a partial solution iff for all $\Delta y_0 = y_0 - y_0^\sigma$, where $y_0^\sigma \neq y_0$ is the restriction of $y_0$ to the indices less or less or equal to
σ, the term \( c_\sigma x^{\mu_0} \) from \( \Delta y_0 := c_\sigma x^{\mu_0} + \ldots \) is an acceptable initial term of a solution obtained from the contour of the appropriate Eq. (3). In (3) still \( \beta_N = \alpha_N \neq 0 \). If \( \beta_0 = 0 \) then \( \bar{y} = 0 \) delivers a complete solution.

Now, I will construct a solution in the following way. On each step I use the contour function for the redefined equation to choose the largest \( \mu \), \( \mu = x_b \), accompanied by appropriate \( c \in K^* \), \( p_\beta(c) = 0 \). By the move I kill the lowest index element \( c_\sigma x^\sigma \) in \( \beta_0 \). Therefore, in the next step \( \sigma' \), the lowest index element of \( \beta'_0 \), will be greater then \( \sigma \). A set \( T \) of triples \((\sigma, c, \mu)\) is arisen, where \( \{\sigma\} \) appears to be a well ordered set by the method of the construction. The minimal \( \sigma \) is equal to the lowest index of \( \alpha_0 \). I will show that in each step labelled by \( \sigma \) I always take the additional \( cx^\mu \) such that \( \mu > g_\sigma \equiv \sup\{\mu| (\sigma,c,\mu) \in T \land \sigma < \sigma\} \). Firstly, I assume that the supremum \( g_\sigma \) is the maximum. The case is reduced to the problem: having a partial solution \( c_0 x^{\mu_0} \) of (3) arising as in the construction find a continuation \( c_0 x^{\mu_0} + c_1 x^{\mu_1} \) according to the above rule. Let \( c_0 x^{\mu_0} \) cancels \( c_\sigma x^\sigma \) in \( \beta_0 \) and the new constituent to remove in \( \beta'_0 = \sum_{i=0}^{N} \beta_i (c_0 x^{\mu_0})^i \) is equal to \( c_{\sigma'} x^{\sigma'} \), where \( \sigma' > \sigma \). The lowest index expressions arising as a consequence of introducing \( c_1 x^{\mu_1} \) has the forms

\[
c_{1}p^{(1)}(c_{0})x^{M+(\mu_1-\mu_0)}, \ldots, c_{1}p^{(k)}(c_{0})x^{M+k(\mu_1-\mu_0)},
\]

(5)

where \( p \) is the polynomial from the contour for \( c_0 x^{\mu_0} \), \( p^{(i)}(y) = \frac{d^i}{dy^i}p(y) \) is its \( i \)th derivative, \( k = \deg p \) and \( M = f_{0}(\mu_{0}) \) with the appropriate for the first term contour function \( f_{0} \). At least one of them is nonzero. If \( \mu_1 \) goes to \( \mu_0^+ \) then the indices of the term goes to \( \sigma \). Therefore, \( \mu_1 > \mu_0 \) claimed to cancel \( c_{\sigma'} x^{\sigma'} \) exists. In turn, let’s assume that a partial solution \( y_0 \) exists such that \( g_{\sigma'} \) is not the maximum. Let’s consider \( y_0 + c_1 x^{\mu_1} \), where \( \mu_1 = g_{\sigma'} \) and \( c_1 \in K^* \). Again, the lowest index of \( \sum_{i=0}^{N} \beta_i (y_0 + c_1 x^{\mu_1})^i \) containing \((\mu_0, c_0)\) is equal to \( \sigma_1 \leq \sigma < \sigma_2 \), where \( c_1 \) is treated as a variable. The proper cancellation may be done.

The rational functions \( K(x) \) are a subset of \( K[[x]][x^{-1}] \subset K[[x^{\frac{1}{2}}]] \), where \( K[[x^{\frac{1}{2}}]] := \bigcup_{n \in N} K[[x^{\frac{1}{2}}]][x^{-\frac{1}{2}}] \) is the (classical) Puiseux series field. The field appears to be algebraically closed as one of consequences of theorem [4].

**Corollary 2.** \( K^0_P[[x]] \), \( K^0_Q[[x]] \) and \( K[[x^{\frac{1}{2}}]] \) are algebraically closed fields, where \( R_K \subset \mathbb{R} \) and \( K^0_Q[[x]] \) is obtained as a restriction of \( K^0_P[[x]] \) to the rational indices.
Proof. Let \( w(y) = y^N + \alpha_{N-1}y^{n-1} + \ldots + \alpha_0 = 0 \) be an algebraic equation with coefficients in \( K_P[[x]] \), \( N > 0 \). Also, let algebraic equations of degree less then \( N \) with coefficients from \( K_P[[x]] \) are solvable in \( K_P[[x]] \). Let \( y_s \in K_P[[x]] \) be a solution of \( w \) and \( \mu = \limsup \mu_n \in \bar{R}_K \), where \( \{\mu_n\}_{n=1}^\infty \) is an increasing sequence of the set of indices of \( y_s \), be the minimal number with the property. Following the theorem, in the series \( w(y_s) \) the converging indices appears initially via \( p^{(i)}(c_0)c_{\mu_n}x^{M+i(\mu_n-\mu_0)} \), where \( p^{(i)}(c_0) \neq 0 \) and \( i = 1, \ldots, N \) is the smallest such derivative. But by the rescaling \( y \to x^{-M}y \) I may assume that \( M = 0 \). Now, the expressions can not find counterparts under the assumption. Therefore, \( y_s \in K_P[[x]] \).

The corollary for the restriction of indices to \( \mathbb{Q} \) follows from the construction of solutions in theorem 1.

The final case is the coefficients in \( K[[x^\mathbb{Q}]] \). At the moment we know that a solution \( y_s \) is at least from \( K_\mathbb{Q}[[x]] \). I make the proof of the part by induction with respect to degree of algebraic equation. Let \( \lim_{n\to\infty} \mu_n = \infty \), where \( \{\mu_n\} \) are ordered by \( \mathbb{N} \) indices of \( y_s \). Otherwise, the set of the indices is finite and \( y_s \in K[[x^\mathbb{Q}]] \). Also, let \( w^{(i)}(y_s) \neq 0 \) for all \( i = 1, \ldots, N - 1 \). It implies that the contours appearing in the construction of the solution for \( n > N_0, N_0 \in \mathbb{N} \), differ only by horizontal lines defined by \( \sigma_n \). Moreover, theorem 1 guarantees that all partial solutions of the algebraic equations have a continuation to a complete solution, so all of them are subsets of a solution. The number of solutions is finite, so one finds that for \( n > N_1 \) the solution is constructed via right choice of breaking points as in the proof of the theorem and then the right breaking points of the contours, \( \mu_n, n > N_2 \), are the intersection of the horizontal lines and the line \( r \to \nu_0(1) + r \). Otherwise, an infinite number of different partial solutions beginning different solutions exists. Let \( N = \max\{N_0, N_1, N_2\} \). All coefficients and the partial solution \( y_s|_N \) have indices in \( \frac{1}{N}\mathbb{N} \) for \( s \in \mathbb{N} \). The procedure of extension behaves the pointed set indices. The proof is done.

Remark 1. Using the transformations \( y \to x^My \) or \( y \to y^{-1} \) one finds that the following situation is generic in the space of leading indices \( \{\nu_0(i)\}_{i=0}^N \) of coefficients. Namely, \( \alpha_N = 1, \nu_0(N - k) < 0 \) for a \( N - k \in \{1, \ldots, N\} \) and \( \nu_0(i) \geq 0 \) for \( i \neq N - k \). Then the explicit solution may be proposed.
according to [6, 5]:

\[ y_s = (-a_k)^{1/k} - \sum_{i=0}^{\infty} \left( \sum_{p=1}^{i+1} \frac{1}{p!} k^p \prod_{j=1}^{p-1} (i - jk) \sum_{i_1 + \ldots + i_p = i+1 \atop i_q \neq k} a_{i_1} \cdot \ldots \cdot a_{i_p} \right) (-a_k)^{-1/k}, \]

where \( a_i = \alpha_{N-i}, R_K \subset \mathbb{R} \) and I assume all conventions from the paper. The formal series on finite intervals reduce under the generic restrictions to an algebraic expressions with the \( k \)-root. Then the theorem and the corollary are an immediate consequence of (6).

For a further context it is essential that the schedule of subspaces

\[ K[[x]] \subset K_Q^0[[x]] \subset K_Q[[x]] \subset K_P[[x]] \supset K_P^0[[x]] \]

are defined only by referring to properties of indices and, additionally, they are closed for \([\mu]\) operation for nonnegative indices \( \mu \) for \( R_K \subset \mathbb{R} \).

Keeping the extra assumption and enriching the Puiseux field with the standard differentiation

\[ (c_i x^{\mu_i})^\bullet := c_i \mu_i x^{\mu_i - 1} \]

I may return to Eq. (1). The \( \frac{P(y)}{Q(y)} \in K_P[[x]](y) \) I will interpret as an element of \( K_P[[x]]Q[[y]] \). The change may leads to limitations for admitted substitution of \( y \). Nevertheless, the whole initial domain \( K_P[[x]] - \{ y | Q(y) = 0 \} \) may be realized by considering transformations \( y \mapsto y - y_0, Q(y_0) \neq 0 \), to reach all solutions of (1). In the next section I replace the problem of solving the equation by a new one:

\[ \dot{y} = f_{ij} x^{\nu_i} y^{\sigma_j} \]

with the additional condition

\[ f \in K_P[[x]]Q[[y]] \cap K_Q[[y]]P[[x]] \]

and \( f_i \neq 0, f_j \neq 0 \). I assume that a choice of branches of \( y^{\frac{1}{k}} \) is given.

**Language of valuation**

The presented analysis and the approach of the next section suggests a general view. Let \( K \subset L \) be an algebraically closed subfield of a field \( L \). Then a
minimal valuation ring $O$ exist such that $K \subset O \subsetneq L$. Then the field $O/O_+$ may be identified with $K$, where $O_+$ is the maximal ideal of $O$. Really, the maximal subfield $\tilde{K}$ in $O - O_+ \supset \tilde{K}$ is isomorphic to $O/O_+$. If $\tilde{K} \neq K$ then a valuation ring $\tilde{O}$ in $\tilde{K}$ such that $K \subset \tilde{O} \subsetneq \tilde{K}$ exists and the ring generated by $\tilde{O}, O_+$ is a valuation ring in $K$ smaller then $O$. The contradiction.

The valuation function $v : (L^*, \cdot) \mapsto (R_K, \oplus)$ is the groups homomorphism defined by $O$, where $(R_K, \oplus) \equiv (L/O^*, \cdot)$ has no nilpotent elements and has the linear order structure $\leq$ such that $k_1 \leq k_2 \iff k_1O \supset k_2O$ for $k_1, k_2 \in R_K$. Additionally, $k_1 \leq k_2 \Rightarrow k_1 + k \leq k_2 + k$ for each $k \in R_K$.

Such linearly ordered group is naturally extendible treated as a $\Z$-module to $Q$ linear space with order $(R_K, \oplus, \cdot, \leq)$. For example, if $L$ is algebraically closed then the multiplication by $q \in \Q$ in $R_K$ is already present and is defined by $kO^* \mapsto k^qO^*$.

Now, let $L$ be algebraically closed and $\{b_i > 0\}$ be a basis of $R_K$ and a choice $\{x(b_i)\} \subset L$ is given such that $v(x(b_i)) = b_i$. The $\{x(b_i)\}$ have no relations, therefore, $K(\{b_i\}) \subset L$. The referring to $x(b_i)^{q_1} \cdot \ldots \cdot x(b_n)^{q_n}$ by $x^r$, where $q_i \in \Q$ and $r = q_1b_1 + \ldots + q_nb_n \in R_K$, is assumed. Let $K_P[[x]]$ be the Puiseux series with the set of index equals to $R_K$. Then $L \subset K_P[[x]]$. For a proof, it is enough to show that any nontrivial extension of $K_P[[x]] \supset \tilde{O}$ leads necessary to set theory extension of $R_K$, where $\tilde{O}$ is the valuation ring of $v(k) = \mu_0$.

**Proposition 3.** Let $K$ be an algebraically closed field, $R_K$ a $\Q$-linear space with a linear order and $K_P[[x]](y) \supset D$ be an extension of $K_P[[x]] \supset \tilde{O}$. Additionally, let $K = D/D^*$. Then $K(y) = K$ or $R_K \subsetneq R_K \equiv K_P[[x]](y)/D^*$.

**Proof.** If $v(y - k) = 0$ for each $k \in K_P[[x]]$ then $K(y) \subset K$. Therefore, let $v(y) > 0$. The rest cases are equivalent. I assume that $v(y) \in R_K$. Let $v(x^{\lambda_0}) = v(y)$ and $\mu_0 \in R_K$. If $v(y - c_0x^{\lambda_0}) > v(y)$ for $c_0 \in K^*$ then $c_0$ is unique. If the procedure has no any halt point then $y = c_0x^{\lambda_0} + c_1x^{\lambda_1} + \ldots$ in $v$-topology (i.e. $\lim y_n = y \iff \forall(r \in R_K) \exists(N \in \N) \forall(n > N) y - y_n \in O_r$, $O_r := \{k \in L|v(K) \geq r\}$). So, for $\bar{y} := y - k, k \in K_P[[x]], v(\bar{y} - c_0x^{\lambda_0}) = \mu > 0$ for all $c \in K$. It implies that $v(y/x^{\mu} - c) = 0$ and $y/x^{\mu} \in D/D^* - \tilde{O}/\tilde{O}^*$.

If an injection of $\Q$-linear space $i : R_K \mapsto K$ exists one may identify $R_K$ with a subset of $K$. In the case a differentiation is defined by:

$$
(x^{\mu})' = \frac{\mu - x^{\mu}}{x(b)}\]$$

(11)
where $b_1$ is an element of a chosen basis of $R_K$. An identification $\mathbb{Q} \cdot b_1$ with $\mathbb{Q}$ via $b_1 = 1$, assumption that $K = \{0\}$ and the rule of independent differentiation of each ”monomial” in $k \in K_F[[x]]$ return us to formula (4).

The inequality $v(\dot{y}) - v(y) \geq 1$ becomes an equality for $v(y) \neq 0$, where $y \in K^0_Q[[x]]$. In the approach the case $v(y) = 0$ is distinguished.

2 Existing theorem and resonance

2.1 Domain of differential equation

Let’s start from a characterization of the domain of (9). All $y = c_0 x^{\mu_0} + \ldots$, where $\mu_0 > 0$, are properly substituted to the equation, compare proposition 4 below. Nonzero constants from the domain constitute the set $D_0 \equiv \{c \in K | \forall (i \in \nu) f_i(c) < \infty\}$. It is reasonable to consider also a class of fields $K^{ns}$ being also a normalized space over $\mathbb{Q}$, for example $K^{ns} = \mathbb{C}$. In the way, the condition $c \in D_0$ reads as

$$\forall_i \sum_j f_{ij} c^{\sigma_j} < \infty$$

admits infinite summations. One defines $D'_0 \subset D_0$ by

$$c_0 \in D'_0 \iff \forall_{i \in \nu} \limsup_{j \to \infty} \sqrt[|f_{ij} c_0^{\sigma_j}| < 1}.\quad (13)$$

For $y = c_0 + \ldots, c_0 \in K^{ns} - \{0\}$, the condition $c_0 \in D_0$ is necessary for staying in the domain. It is appeared that $c_0 \in D'_0$ is sufficient. We have

**Proposition 4.** Let $D_f \equiv \{y \in K^p_F[[x]] | f(y) \in K^r_p[[x]]\}$ be the domain of the differential equation. Then all $y = c_0 + \ldots, c_0 \in D'_0$ and $y = c_0 x^{\mu_0} + \ldots, \mu_0 > 0$, belong to $D_f$.

**Proof.** We start from the observation that

$$(c_0 x^{\mu_0} + c_1 x^{\mu_1} + \ldots)^{\sigma_j} = c_0^{\sigma_j} x^{\sum j \mu_0} (1 + d_1(j)x^{\mu_1 - \mu_0} + \ldots)\quad (14)$$

Therefore, the indices after substitution have the form $\nu_i + \sigma_j \mu_0 + [\mu_k - \mu_0]$. Then for $\mu_0 > 0$ the sums of coefficients before an admitted index in $f(c_0 x^{\mu_0} + c_1 x^{\mu_1} + \ldots)$ are always finite. Let $\mu_0 = 0$. Then the convergence of the lowest term coefficient in $f(y)$ is guaranteed by (13). The coefficients $d_k(j)$ of $x^{\delta \mu(k)}$, for $|\mu_i - \mu_0| \geq \delta \mu(k) > 0$, are polynomials of $c_{k(1)}/c_0, \ldots, c_{k(l)}/c_0$ spanned.
by a finite number of monomials $m_n(k) := \prod_i (c_{k(i)}/c_0)^{n_i}$ such that $\delta\mu(k) = \sum n_i \delta\mu(k(i))$, where $i = 1, 2, \ldots, I_k, n_i, I_k \in \mathbb{N}$ and $\delta\mu(k(i)) \in [\mu(i)] - \{0\}$ are elements of admitted decompositions of $\delta\mu(k)$. For $\delta\mu(l) \in [\mu_i - \mu_0] - \{\mu_i - \mu_0\}$ I put $c_{\delta\mu(l)} = 0$. The coefficients $\gamma_n(k, j) \in \mathbb{Q}$ defined by

$$d_k(j) = \sum \gamma_n(k, j)m_n(k)$$

may be calculated directly. For $\sigma_j \in \mathbb{N}$

$$\gamma_n(k, j) = \frac{\sigma_j!}{n_0! n_1! \ldots n_{\sigma_j}!},$$

where one introduces $n_0 := \max\{\sigma_j - \sum_{l=1} n_l, 1\}$. For $\sigma_j = -1$ the coefficient is a number. For $\sigma_j = \frac{1}{q}, q \in \mathbb{N}$ one need to solve a finite number of algebraic equations defined by relations arising among $x^{\delta\mu(k(i))}$. The well-defined number is a rational function of $\sigma_j$ depending on expressions as in (13). In the same way coefficient arise for general $\sigma_j \in \mathbb{Q}$. Now, the summability of $\sum f_{ij}c_{k}^{\sigma_j}d_k(j)$ keeping constant $\nu_i + \delta\mu_k$ is under consideration. Again (13) is sufficient for convergence, because the only infinite sum may appear is through $j$. \(\square\)

**Remark 2.** For considering any $K$ and any initial index one must assume that the set $\{\sigma_j\}$ is finite. Then $D_0 = K$ and $D_f = K_P[[x]]$.

For any $K$ and $\mu_0 \geq 0$ admitted as the beginning index it is enough that $\forall(i) \sum_j f_{ij} < \infty$ i.e. there are only a finite number of $f_{ij}$ for each fixed $i$. In the situation $D_0 = K$ and $D_f$ contains all Puiseux series with nonnegative indices.

### 2.2 Necessary conditions

For investigating of necessary conditions for existence of solutions of Eq. (9) one defines the contour function for the equation

$$f(x) := \min\{\nu_i + \sigma_j x, x - 1\} | x \in R_K \land f_{ij} \neq 0\}.$$  

(17)

The function’s graph for $x \geq 0$ (or for $x \in R_K$, while $\sigma$ is finite) is a broken line with finite number of breaking points. Also for each $x > 0$ (resp. $x \in R_K$) a finite set of pairs $(\nu_i, \sigma_j)$ exists such that $f(x) = \nu_i + \sigma_j x$. The following proposition is an answer for the first step of the equation solving:
Proposition 5. If \( y \in \mathbb{K}[[x]], \ y \neq 0, \) is a solution of (9) then the set of indices of \( y, i(y), \) begins from \( \mu_0 \) such that

(a) \( \mu_0 = 0 \) or

(b) \( \mu_0 \) is a breaking point of \( \hat{f} \) and \( \mu_0 \neq 0 \) or

(c) \( f(\mu_0) = \mu_0 - 1 \) and \( \mu_0 = f_{ij} \neq 0, \nu_i = -1, \sigma_j = 1. \)

Proof. Let \( \mu_0 \neq 0 \) and \( y = c_0 x^{\mu_0} + \ldots \) fulfils the differential equation. Then the algebraic equation for \( c_0, \)

\[
p_{\mu_0}(c_0) := \sum_{i,j,\nu_i+\sigma_j\mu_0=f(\mu_0)} f_{ij} c_0^{\sigma_j} - \mu_0 c_0 = 0, \tag{18}\]

admits the nonzero roots only for the cases (b) or (c). \( \square \)

Remark 3. The choice of branches of \( y_j^\sigma \) may sometimes cause Eq. (18) has no nonzero solutions. Further, taking \( c_0 \neq 0 \) I will suppose a coherent initial definitions.

The above proposition state that in the accepted area of initial indices there are only finite number of possibilities. If there are no breaking points of \( \hat{f} \) then \( y = 0 \) is a solution. Moreover, it is appeared that \( c_0 \) is not defined in item (c) and no (b) of the proposition. The same is true in (a) under

\[
\nu_0 + 1 > 0 \tag{19}\]

i.e. \( c_0 \in D_0. \) If (19) is not the case then the following condition for \( c_0 \) immediately arises \( \sum_j f_{0j} c_0^{\sigma_j} = 0. \)

The distinction is close to a general classification of solutions. The algebraic type solution of Eq. (9) is such which in first step of solving does not bind left hand side of the equation. The rest of solutions are called proper ones. In the way (c) leads only to proper solutions, (a) without (14) only to algebraic type and (b) refers to proper ones iff the breaking point lies on \( y = x - 1. \) From the properties of \( \hat{f} \) it implies that the maximal initial indices’ number of proper solutions is 4, in case (b) at most two.

It will appear that the free parameters \( c_i \) may arise only one time in a solution. For each \( (\mu_0, c_0) \) from cases (b)(c) of proposition 5 I define the resonant index \( \mu_r \) by

\[
\mu_r = \sum f_{ij} c_0^{\sigma_j - 1} \sigma_j, \tag{20}\]

10
where the summation is taken through \( \{(i, j) | \nu_i + \sigma_j \mu_0 = f(\mu_0)\} \). One finds that the case (c) begins from the resonant value \( \mu_0 = \mu_r \). For terminological consistency case (a) restricted to (19) will also be called as resonant.

In turn, I will find a well ordered set covering indices of proper solutions \( y = c_0 x^{\mu_0} + \ldots \).

**Proposition 6.** Let \( R_K \subset \mathbb{R} \) and \( c_0 x^{\mu_0}, \mu_0 \geq 0 \) and \( \neg(\mu_r > \mu_0) \), may start a proper solution. If \( y = c_0 x^{\mu_0} + \ldots \) is a solution then the indices set \( i(y) \) is a subset of \( i_{\text{max}}(c_0 x^{\mu_0}) := [\{\nu + 1\} \cdot \{\mu_0(\sigma - 1)\}] \cdot \{\mu_0\} \).

**Proof.** Let \( y = c_0 x^{\mu_0} + c_1 x^{\mu_1} + \ldots \) be a solution of (9) written in the convention such that \( \{\mu\} = [\mu] \) and \( c_i = 0 \) are admitted for \( i \neq 0 \). One defines a function \( f_0(x) := \min\{\nu_i + \mu_j x | x \in R_K \land f_{ij} \neq 0\} \). Let \( \mu_0 \neq 0 \). Then

\[
\mu_0 - 1 = f_0(\mu_0) = \nu_i + (\sigma_j - 1)\mu_0 + \mu_0
\]

for some \( \nu_i, \sigma_j \) realizing the minimum. Let \( \delta \mu_i := \mu_i - \mu_0 \). Then \( y^{\sigma_j} = c_0^{\sigma_j} x^{\mu_0 \sigma_0} (1 + \ldots + d_i(j) x^{\mu_i} + \ldots) \). A comparison of both sides of (9) pitches the condition for the indices:

\[
\delta \mu_l = \nu_i + 1 + (\sigma_j - 1)\mu_0 + \delta \mu_k
\]

with \( \nu_i + 1 + (\sigma_j - 1)\mu_0 \geq 0 \) for all \( i, j \) such that \( f_{ij} \neq 0 \). The accompanying coefficient equation is the following

\[
c_l \mu_l = \sum f_{ij} c_0^{\sigma_j} d_j(k) = \sum_{i,j,\mu_0=\nu_i+\sigma_j \mu_0} f_{ij} c_0^{\sigma_j^{-1}} \sigma_j c_l + \sum_{k < l}.
\]

Now, if in Eq. (22) \( \delta \mu_l \) has no realization for \( k < l \) then (23) implies \( c_l \mu_l = c_l \mu_r + 0 \), so \( c_l = 0 \). Therefore, starting from (21) and following by (22) for \( k \neq l \) one gains the thesis. The proof for \( \mu_0 = 0 \) follows similarly.

2.3 Existence theorems

Now, I pass to the proper solutions’ continuation problem.

**Theorem 7.** Let \( R_K \subset \mathbb{R} \) and \( c_0 x^{\mu_0}, \mu_0 \geq 0 \), is admitted as an initial term of a proper solution. Then the following statements are fulfilled:

(i) if \( \mu_0 \neq 0, \neg(\mu_r > \mu_0) \) then a unique continuation exists,
(ii) if \( \mu_0 \neq 0, \mu_r > \mu_0 \) and \( \mu_r \notin i_{\text{max}}(c_0x^{\mu_0}) \) then any \( c_r \in K \) determines a continuation, where \( c_r \) is the coefficient before \( x^{\mu_r} \).

(iii) if \( \mu_r > \mu_0 > 0 \) and \( \mu_r \in i_{\text{max}}(c_0x^{\mu_0}) \) then a finite number of steps solves the alternative: the continuation is as in case (ii) or the solution is terminated (negative resonance).

(iv) if \( \mu_0 = 0 \) then there is exactly one continuation.

Proof. Item (i) follows directly from the proof of proposition 6. Each \( c_l \in K \) for \( i_{\text{max}}(c_0x^{\mu_0}) \ni \mu_l > \mu_0 \) is uniquely countable from (23). In case (ii) \( \delta \mu_r := \mu_r - \mu_0 \) can not be of the form (22). This is the reason why appropriate (23) becomes a tautology. Any \( c_r \in K \) is admitted and the successive solving may be continued from \( \mu_r \) with additional \( c_r \cdot x^{\mu_r} \). The new set of admitted indices is equal to \( i_{\text{max}}(c_0x^{\mu_0}) + i_{\text{max}}(c_0x^{\mu_0}) \cdot [\mu_r - \mu_0] \).

The situation (iii) delivers an extra condition on the level of \( \mu_r \). The index \( \mu_r \) defined a finite set of non-decomposable elements \( B_r \subset B(i_{\text{max}}(c_0x^{\mu_0})) \) which may appear in decomposition of \( \mu_r \). According to (22) and (23) the coefficient \( c_\mu, \mu \in [B_r] \) and \( \mu < \mu_r \) are uniquely determined by a finite number of algebraic equations. They meet themselves on the level \( \mu_r \) in appropriate (23) and the relation, non-containing \( c_r \), needs to be justify with respect to its logical value. For example, if \( \mu_r \in B(i_{\text{max}}(c_0x^{\mu_0})) \) then one meets case (ii). Finally, in the last case \( \mu_0 = 0 \) the lowest Eq. (22) has the form

\[
\mu_1 = \nu_0 + 1 > \mu_0,
\]

which implies that all coefficient equations have no any obstacles in solving and \( c_0 \in D'_0 \) determines uniquely the rest coefficients.

Remark 4. For finite \( \sigma \) the beginning coefficients \( \mu_0 < 0 \) may be translated to nonnegative one via redefinition \( y \to x^M y, M \in R_K \), and then theorem 7 covers also the situation.

Remark 5. The theorem for a general \( R_K \) may be realized similarly with omitting the predefining of indices. Then the case \( \neg(\mu_r > \mu_0) \) and \( \mu_r \notin R_K \) may be treated as a resonant case after an extension of \( R_K \subset R'_K \ni \mu_r \) such that \( \mu_r > \mu_0 \).

With help of the distinction (7) a classification of Eq. (9) may be imposed. From proposition 6 and the above theorem one concludes the following fact.
Corollary 8. Let $R_K \subset \mathbb{R}$. If the sets of indices $\{\nu_i\}$ and $\{\sigma_j\}$ belong to a subfield of (4) then for the cases (i), (ii) with $c_r = 0$, (iii) and (iv) the proper solutions belong to the subfield as well.

It needs to be remarked that the proper solution with $\mu_0 = 0$ implies (24), so (19), but not inversely. For all choices of $c_0 \neq 0$ such that $\sum_j f_{0j} c_0^j = 0$ we are in algebraic type solutions. Nevertheless, for (19) being fulfilled theorem 7 is applied (item (iv)). A different treating of the other cases is needed.

Theorem 9. Let $R_K \subset \mathbb{R}$ and $c_0 x^{\mu_0}$ is admitted as an algebraic type solution and $\{\sigma\}$ is finite. If $\mu_0 = 0$ I additionally assume that $\nu_0 + 1 < 0$ and that $f_0$ has a breaking point in 0. Then the set of continuations as a solution $C(c_0, \mu_0)$ in nonempty and $\#C(c_0, \mu_0) \leq s 2^{(\sigma_{\max} - \sigma_{\min}) s - 1}$, where $\sigma_{\max} := \max \{\sigma\}$, $\sigma_{\min} := \min \{\sigma\}$ and $s \in \mathbb{N}$ is the smallest common divisor of $\{\sigma\}$.

Proof. Eq. (9) has the form $\dot{y} = \sum_j p_j(x)(y_1^\frac{1}{s})^{n_j}$, where $p_j(x) := \sum_i f_{ij} x^{\nu_i}$, a branch of $y_1^\frac{1}{s}$ is fixed and $n_j := s \sigma_j \in \mathbb{Z}$. Without lost of generality of the consideration one may put $s = 1$. The algorithm of solving begins from the term such that $f(\mu_0) < \mu_0 - 1$ for $\mu_0 \neq 0$ or $\mu_0 = 0$ is a breaking point of $f_0$ with $\sum_j f_{0j} c_0^j = 0$. The next step is to solve the algebraic equation

$$\sum_j p_j(x) y^{n_j} = 0 \quad (25)$$

for given $(\mu_0, c_0)$, see theorem 4. Let $\mu_0 \neq 0$ (the proof is analogous for $\mu_0 = 0$). The solution of (24), $y_0$, will be up to index $\mu$, $\mu - \mu_0 = \mu_0 - 1 - f(\mu_0) > 0$, the solution of the differential equation. Subsequently, I modify (24) to

$$\sum_j p_j(x) y^{n_j} - y_0 = 0. \quad (26)$$

Eq. (26) has a solution, $y_0$, being a continuation of $y_0|_{<\mu}$, see the proof of theorem 4. Now, the coincidence with the differential equation is improved up to $\mu'\,\,\text{such that}\,\,\mu' - \mu_0 = \mu - 1 - f(\mu_0) = 2(\mu - \mu_0)$. The cut series $y_0|_{<\mu}, y_0|_{<\mu'}, \ldots$ become a sequence of approximation of a solution of the differential equation. The induction follows.

Each solution in the form $y_1^\frac{1}{s}$ may bifurcate at most $(\sigma_{\max} - \sigma_{\min} - 1)$ times. The numerical coefficient is estimated. \hfill \Box
The construction offers the closeness of $K_0[[x]], K_0[[y]], K_0[[z]]$ for solving in algebraic type.

Remark 6. The first omitted case in the theorems about continuation is the following: $\mu_0 = 0, \nu_0 + 1 \leq 0$ and $f_0$ has no breaking point in 0. They have no continuation. For example, equations having no solutions in $K_0[[x]]$ at all are of the form:

$$\dot{y} = y^\sigma (f_0x^{-1} + \ldots),$$

where $\mathbb{Q} \ni \sigma < 0$ and $f_0 \neq 0$. To solve them ln $x$ function is needed.

The second and last case is $\mu_0 = 0, \nu_0 + 1 = 0$ and $f_0$ has breaking point in 0. Let $\{\sigma\}$ is finite. Then one may redefine the differential equation by $y \to x^\epsilon y$, where $\epsilon > 0$. The admitted, finite set of initial terms $c_0$ appears as $c_0x^\epsilon$ in the new equation, but, now, as a beginning of a proper solution (cases (i)-(iii) from theorem 7). Elementary contour analysis shows that for each $c_0$ the resonance is placed in $\mu_r = \sum_j f_{0j}c_{0j}^{-1}\sigma_j = 0$.

3 Picard-Vessiot extension

An algebraic approach to Eq. (1) in a differential field (of zero characteristic) $(K, \cdot)$ will be concentrated on existence of maximal set of solutions, its algebraic structure and a decomposition of the extension into liouvillian steps. It is assumed that the field of constants $C \subset K$ is algebraically closed.

I begin from an observation that the algebraic closure $\overline{K} \supset K$ is a differential field extension with the uniquely defined extension of the differentiation. It behaves the field of constants. Further, all extensions $(\tilde{K}, \cdot) \supset (K, \cdot)$ such that the field of constants is kept unchanged and that they are generated by solutions of (1) will be called Picard-Vessiot (PV) extensions. Firstly, let $K$ be algebraically closed.

Lemma 10. Let $\{y_{sol}\}$ be the set of all solutions of (1) in $K = \overline{K}$. Then the following statements are equivalent:

(i) the differential ring

$$R := \left( K[y] \left[ \frac{1}{Q} \left( \frac{1}{y - y_{sol}} \right) \right], \dot{y} = \frac{P(y)}{Q(y)} \right)$$

does not possess new invertible constants (NIC),

(ii) a nontrivial PV extension of $K$ exists.
Proof. (i) is a necessary condition of (ii). Therefore, it is enough to explain the implication (i) \( \Rightarrow \) (ii). The only nonzero, prime, proper differential ideal of \( R \) is of the form \( (y - a), a \in K \). Then \( R/I = K \), so \( y \) is a solution from \( K \). The contradiction imply that \( \{0\} \) is the maximal differential ideal. In the way \( (K(y), \dot{y} = \frac{P(y)}{Q(y)}) \) is a PV extension. \( \Box \)

Proposition 11. Let \( K = \mathcal{K} \) does not contain any solution of \( \{4\} \). Then item (i) of lemma \( \{11\} \) is fulfilled.

Proof. An invertible element of \( R \) has the form

\[
\alpha(y - y_1)^{k_1} \cdots (y - y_n)^{k_n},
\]

where \( \alpha \in K^* \) and \( y_i \in \{y_{sd}\} \subset K \vee Q(y_i) = 0 \), and \( k_i \in \mathbb{Z} - \{0\} \), and \( i \neq j \Rightarrow y_i \neq y_j \). It is a constant iff

\[
\prod_j (y - y_j) \left( \frac{\dot{\alpha}}{\alpha} Q(y) + \sum_i k_i \frac{-y_i Q(y) + P(y)}{y - y_i} \right) = 0. \tag{30}
\]

I substitute \( y = y_i \) to (30). One obtains that \( y_i \) is a solution. \( \Box \)

Let’s assume the following notations. For a given differential filed \( K I_{\infty} \subset K[I_{\infty}] \equiv R \) is the differential ideal in the differential ring \( R \) generated by the equations:

\[
eq_{i+j+k=N} w_{1,i}w_{2,j}q_k - w_{1,i}w_{2,j}q_k + (i+1)w_{1,i+1}w_{1,j}p_k - w_{1,i}(j+1)w_{2,j+1}p_k = 0. \tag{31}
\]

Now, the proper statement has the following shape.

Theorem 12. For any differential field \( (K, \dot{\cdot}) \) and Eq. \( \{3\} \) in the field the greatest PV extension exists.

Proof. A maximal differential ideal \( I_{\max} \supset I_{\infty} \mid_N \) of a differential ring

\[
K[1/(w_{1,k}w_{2,l} - w_{1,l}w_{2,k})[\{w_{1,i}, w_{2,i}, \dot{w}_{1,i}, \dot{w}_{2,i}, \ldots\}_{i=0}^N] \supset I_{\infty} \mid_N \tag{32}
\]

exists such that \( (w_{1,k}w_{2,l} - w_{1,l}w_{2,k})^n \notin I_{\infty} \mid_N \) for \( n \in \mathbb{N} \) and \( N \geq k, N \geq l \) and \( \mid_N \) means a restriction by \( w_{i,t}^{(0)} = 0 \) for \( t = 1, 2, i > N \) and \( l \in \mathbb{N} \cup \{0\} \). Really, let’s assume that \( (w_{1,k}w_{2,l} - w_{1,l}w_{2,k})^n \in I_{\infty} \mid_N, k \neq l \). Then \( (w_{1,k}w_{2,l} - w_{1,l}w_{2,k})^n = \sum e_i \mid_N r_i \), where \( r_i \in R \). It implies that \( (w_{1,k}w_{2,l} - w_{1,l}w_{2,k})^n = \)

15
\[(k - 1)w_{1,k}w_{2,l} - (l - 1)w_{1,l}w_{2,k})r, \text{ where } r \in R \text{ and } w^{(s)}_{t,i} = 0 \text{ for } s > 0, \]
\[t = 1, 2, \text{ and } w_{t,i} = 0 \text{ for } i \neq k \vee i \neq l. \text{ It is not possible. In turn,}\]
\[c : K[1/(w_{1,k}w_{2,l} - w_{1,l}w_{2,k})][\{w_{1,i}, w_{2,i}, \hat{w}_{1,i}, \hat{w}_{2,i}, \ldots\}_{i=0}^{N}] \mapsto K[1/(w_{1,k}w_{2,l} - w_{1,l}w_{2,k})][\{w_{1,i}, w_{2,i}, \hat{w}_{1,i}, \hat{w}_{2,i}, \ldots\}_{i=0}^{N}] / I_{\max} := \hat{R}\]
realizes a differential fields extension \(K \subset \hat{K} := Q(R/I_{\max}),\) where \(Q(\cdot)\) is the field of fractions of a domain. \(\overline{K}\) does not possess any PV extension via \(\hat{K}\). Really, the finite set \(\{c(w_{1,i}), c(w_{2,i})\}_{i=0}^{N}\) defines \(\hat{w}_{i}^{(y)}(y)/\hat{w}_{2}^{(y)}(y) \in \hat{K}(y) = K\) such that
\[\begin{align*}
(w_{1}^{(y)}(y)/\hat{w}_{2}^{(y)}(y))|_{\hat{y} = P(y)/Q(y)} &= 0.
\end{align*}\]
Let \(K \subset K'(\subset \overline{K})\) be a maximal PV extension of \(K\) via \(\hat{K}\) in \(\overline{K}\). Moreover, I assume that \(\overline{K}\) has a PV extension via \(\hat{K}\) to \((\overline{K}(y), \hat{y} = P(y)/Q(y))\).
Therefore, \(y \notin K\), but \(y\) fulfills \(33\) and its integral, so is algebraic over \(K\).
The contradiction ends the proof.

\[\square\]

One needs to remark that the above \(\hat{K}\) is not PV extension of \(K\), so \(K\) does not possess necessary an integral of the type \(a(y - y_{1})/(y - y_{2})\).

Additionally, it is true that:

**Corollary 13.** Let \(K_{\min} := C(P, Q, \hat{P}, \hat{Q}, \ldots) \subset K\), \(P, Q \in K[y]\). Then \(K_{\min}^{PQ} = K_{\min}\langle y_{\text{sol}} \rangle\), where \(\{y_{\text{sol}}\} \subset K_{\min}^{PQ}\) is the set of solutions of \(4\) and \(L^{PQ}\) is the greatest PV extension via the equation of any differential field \(L \supset K_{\min}\).

An inview into the structure of the arising set brings an example.

**Example 7.** Let the equation has the form
\[\dot{y} = y^{2} + by + a\]
(for \(K = \mathbb{C}\) this is Riccati equation). Then by the transformation \(y = -\frac{\dot{z}}{z}\) it becomes equivalent to
\[\ddot{z} - b\dot{z} + az = 0.\]
Let \(z_{1}, z_{2}\) be linear independent solutions of the equation from the PV extension \(K^{PQ}\) via \(30\). Then all solutions of \(33\) are defined by the following algebraic set from \(K^{PQ} \times C\):
\[\left(\left(\left(\dot{z}_{2}y - \dot{z}_{2}\right)c + \left(z_{1}y - \dot{z}_{1}\right)\right) \left(y - \frac{\dot{z}_{2}}{z_{2}}\right)\right) = 0.\]
Additionally, the greatest PV extension of $K$, $\widetilde{K} \supset K$, via (35) may be extended to $K^{PQ}$ via $\dot{z} + y_s z = 0$, where $y_s \in \widetilde{K}$ is a solution of (29). Therefore, the questions about quadratures are equivalent for the two equations.

It is appeared that the algebraicity is a general feature of the greatest sets of solutions.

**Proposition 14.** Let $L \equiv K^{PQ} \supset K$ be the greatest PV extension of $K$ via (I). Then the set of solutions is the intersection of an algebraic set in $L \times L$ and $\{ (y, z)| y \in L \land z \in C \}$ subtracted a finite set.

**Proof.** From lemma 10 one concludes that the extension of differential fields $L(y) \supset K$ contains NIC, where $\dot{y} = \frac{P(y)}{Q(y)}$. Let $\vec{k} \in (\mathcal{F}_0(\{y_{sol}\}, \mathbb{Z}), +), \vec{k} \neq 0$, be an integer value function of finite support defined on the set of all solutions. The following equation for $(y, c) \in L(\alpha_\vec{k}) \times C$ contain the solutions set of (I):

\[
\left( \alpha_\vec{k} \prod_{l, k_l > 0} (y - y_l)^{k_l} - c \prod_{l, k_l < 0} (y - y_l)^{-k_l} \right) \left( \prod_{l, k_l < 0} (y - y_l)^{-k_l} \right) = 0. \tag{38}
\]

Moreover, the characteristic of $K$ is zero, so $\mathbb{Q} \subset C$ and there are infinite number of solutions of (I), roots of (29). Therefore,

\[
\alpha_\vec{k} = c_0 \prod_l (y_0 - y_l)^{-k_l} \in L \tag{39}
\]

for a constant $c_0$ and a solution $y_0$. We may restrict to $L \times C$. (38) is equivalent to

\[
\prod_i (y - y_i) \left( \frac{\dot{\alpha_\vec{k}}}{\alpha_\vec{k}} + \sum_i k_l \frac{1}{y - y_i} (\dot{y} - \dot{y}_i) \right) = 0. \tag{40}
\]

Therefore, if $y$ belongs to solutions of (38) for a $c \in C$ then $y$ is a solution of (I) or

\[
\sum_i k_l \frac{1}{y - y_i} = 0. \tag{41}
\]

\[\square\]

The common part of all admitted sets (38) leads as a result to a set $\Sigma_{PQ}$ in $\{ (y, z) \in L \times L \}$ intersected with $\{ (y, z)| \dot{z} = 0 \}$ covering all solutions and a finite number of *ghost solutions* $y_g$ which are not solutions of the differential
equation and fulfil all additionally arising equations \( \text{(II)} \). \( L[y, z] \) is a Noether ring. It implies that a finite set of equations \( \text{(II)} \) determined by a set \( \{ \vec{k}_f \}_{f \in 1} \) may be chosen. The following equations is also valid for ghost solutions

\[
\frac{\dot{\alpha}_k}{\alpha_k} = \sum_i \frac{k_i \dot{y}_i}{y - y_i}, \tag{42}
\]

which follows from \( \text{(II)} \). In the way \( \text{(II)} \) and \( \text{(II)} \) for \( \vec{k} = \vec{k}_1, \ldots, \vec{k}_F \) contains all ghost solutions.

The ghost solutions appearance is natural and immovable. Consider a formal differentiation defined by: \( dp(y) := \dot{p}(y)dx + p_y(y)dy \) for \( p(y) \in K[y] \) and \( d(a(y)dx + b(y)dy) := (-a_y(y) + b(y))dx \wedge dy \). The notation is such that: \( \dot{p}(y) \equiv \sum \dot{p}_i y^i \) and \( p_y(y) \equiv \sum i p_i y^{i-1} \) for \( p(y) \in K[y] \). We have that \( d^2 = 0, H^0 = C \) and \( H^2 = 0 \), where \( H^i \) are \( C \)-linear spaces of cohomology. If \( \forall (k \in K) \exists k \in K \) then we come to the classical Poincare lemma: \( H^1 = 0 \). In turn, the Eq. \( \text{(II)} \) may be identify with \( \omega = P(y)dx - Q(y)dy = 0 \). Then the necessary integrability condition is \( P_y(y) + \dot{Q}(y) = 0 \) i.e. \( dw = 0 \). Ghosts appear if an integral factor is needed.

The equation for an integral factor in \( K(y) \), \( \dot{y} = P(y)/Q(y) \), has the form:

\[
\dot{f} = -\frac{P_y(y) + \dot{Q}(y)}{Q(y)}f. \tag{43}
\]

Therefore, if it exists it is unique up to a constant of \( K(y) \).

For general type of equations

\[
p(y, \dot{y}) = 0, \tag{44}
\]

\( p \in K[y, \dot{y}] \) and \( p_{\dot{y}} \neq 0 \) the situation is similar iff the roots \( z \) of \( p(y, z) = 0 \) are in \( K(y) \). Otherwise one concludes the following statement.

**Proposition 15.** Let \( p(y, z) = 0 \) has at least one root in \( K(y) - K(y) \). Then for any set \( \{ y_i \} \) the extension \( (K(\{ y_i \}), \cdot) \supset (K, \cdot) \) defined with agreement with \( p(y_i, \dot{y}_i) = 0 \) is a PV extension of \( K \) via \( p(y_i, \dot{y}_i) = 0 \).

**Proof.** The extension is defined by taking for each \( \dot{y}_i \) a nonrational root of \( p \) in \( K(y_i) - K(y) \). If new constants in \( K(\{ y_i \}) \) exist then a new constant \( w \in K(\{ y_i \}) \) must belong to \( K(y_1, \ldots, y_N) \) for \( N \in \mathbb{N} \). Then \( w_{y_i} \neq 0 \) for an \( i \in [1, N] \), so \( \dot{y}_i \in K(\ldots, y_{i-1}, y_{i+1}, \ldots)[y_i] \). The contradiction. \( \square \)
A formal series analog of the algebraic constant of lemma 10 may be found.

**Proposition 16.** For \( P, Q \in K[[y]], P(y) = p_0y^{\mu_p} + \ldots \) and \( Q(y) = q_0y^{\mu_q} + \ldots, p_0 \neq 0, q_0 = 1, \) there is a PV extension \((\tilde{K}, \cdot) \supset (K, \cdot)\) such that the equation:

\[
\dot{w}Q + w_y P = 0 \tag{45}
\]

has a nonzero solution \( w = w_0y^{\mu_0} + \ldots, w_0 \neq 0, \) in \( \tilde{K}[[y]][y^{-1}] \). Moreover

A for \( \mu_q + 1 \leq \mu_p \) \( w_k \) belongs to a liouvillian extension of the differential field \( K_{\min}\langle \{w_l\}_{l=0}^{k-1}, \{\dot{w}_l\}_{l=0}^{k-1}, \ldots \rangle, K_{\min} := C\langle p_i, q_j, \dot{p}_i, \dot{q}_j, \ldots \rangle, \) for all \( k \geq 0 \)

B for \( \mu_q + 1 > \mu_p \) \( w_k \) belongs to the differential field \( K_{\min}\langle \{w_l\}_{l=0}^{k-1}, \{\dot{w}_l\}_{l=0}^{k-1}, \ldots \rangle \)

for all \( k > 0 \).

**Proof.** The used method is a direct substitution. At the beginning, let \( \mu_0 \in \mathbb{Z} - \{0\} \) and \( w = x^{\mu_0}(w_0 + w_1y + \ldots), w_0 \neq 0. \) Then necessary \( \mu_q + 1 < \mu_p \) is kept and then

\[
\begin{align*}
q_0\dot{w}_0 &= 0 \\
\vdots \\
q_0w_{\delta+k} + \ldots + q_{\delta+k}\dot{w}_0 + (\mu_0 + k)p_0w_k + \ldots + \mu_0p_kw_0 &= 0 \\
\vdots
\end{align*}
\tag{46}
\]

where \( \delta := \mu_p - \mu_q - 1, \) or \( \mu_q + 1 = \mu_p \) and

\[
\begin{align*}
q_0\dot{w}_0 + \mu_0p_0w_0 &= 0 \\
\vdots \\
q_0\dot{w}_k + \ldots + q_kw_0 + (\mu_0 + k)p_0w_k + \ldots + \mu_0p_kw_0 &= 0 \\
\vdots
\end{align*}
\tag{47}
\]

Therefore, we are in case A. Now, let \( \mu_0 = 0. \) If \( \gamma := \mu_q + 1 - \mu_p > 0 \) then

\[
\begin{align*}
q_0\dot{w}_0 + \gamma p_0w_\gamma &= 0 \\
\vdots \\
q_0\dot{w}_k + \ldots + q_kw_0 + (\mu_0 + k + \gamma)p_0w_{k+\gamma} + \ldots + (\mu_0 + \gamma)p_kw_{0+\gamma} &= 0 \\
\vdots
\end{align*}
\tag{48}
\]
and \( w_l = 0 \) for \( l \in (0, \delta) \). It is case B. The rest conditions for \( \mu_p, \mu_q \) are of type A and are verified similarly.

I return to the rational equation (41). Taking into the place \( w = w_0 y^{\mu_0} + \ldots \) a solution of (15) in the form of (25), (41) is then (30), one obtains:

**Proposition 17.** The greatest PV extension of \( K, (K, \cdot) \subset (K^{PQ}, \cdot) \), via (4) is in an extended liouvillian extension (i.e. via a finite sequence of extensions via an affine differential equation and an extension contained in algebraic closure) if \( K \subset K(w_0, \dot{w}_0, \ldots) \) is in an extended liouvillian one, where \( w = w_0 y^{\mu_0} + \ldots \notin K \) is a rational solution of (43).

**Proof.** Assume that

\[
    w = \alpha \frac{a(y)}{b(y)} := \alpha y^{\mu_0} \frac{1 + a_1 y + \ldots + a_N y^N}{1 + \ldots + b_M y^M} \in L(y) - L
\]

is a solution of (15) and \( K \subset K(w_0, \dot{w}_0, \ldots) \subset L \) is extended liouvillian and \( a(y), b(y) \) have no common divisors, \( \alpha \neq 0 \). Its existence follows from lemma (10) and theorem (12). Then a finite set of coefficients of \( w = w_0 y^{\mu_0} + \ldots \) define \( \alpha, a_i, b_j \) by algebraic equations. Using proposition (16) one states that they are in an extended liouvillian extension \( L' \supset K(w_0, \dot{w}_0, \ldots) \). Moreover, \( \{y \mid \exists c \in C \ a(y)b(y) = cb(y)^2 \} \) consists from all solutions of (4) and a finite set of ghost solutions, proposition (13). Therefore, \( K^{PQ} \) lies in an extended liouvillian extension of \( K \).

The inverse implication is obvious.

Finally, from propositions (10), (14) and the proof of proposition (17) we reach the following fact:

**Theorem 18.** If \( \mu_q + 1 \leq \mu_p \) then \( K \subset K^{PQ} \) is an extended liouvillian extension.

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