Optimizing the RVB state on a triangular lattice:

Presence of the long-range order

Yong-Cong Chen
Department of Mathematics, Rutgers University
New Brunswick, NJ 08903

and

†Department of Physics
University of Science & Technology of China
Hefei, Anhui 230026, China

Abstract

We present a Schwinger-boson approach for the RVB state of the spin-1/2 Heisenberg antiferromagnet on a triangular lattice. It is shown that a Gutzwiller projection of the mean-field state that includes both antiferromagnetic and ferromagnetic decouplings leads to optimizing the RVB pair amplitudes within a self-consistent approximation. The resulting state yields, by Monte Carlo simulations, energies and spin-spin correlations in excellent agreement with the exact diagonalization result on finite lattices (up to 36 sites). We conclude that the optimized RVB wavefunction possesses a long-range three-sublattice order.

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†Permanent address.
The spin-1/2 antiferromagnetic Heisenberg model on a triangular lattice has been a subject of great interest in recent years. Unlike the square lattice case, the corresponding Ising model is disordered at all temperature with finite entropy [1]. This was also the original model for Anderson’s proposed resonant valence bond (RVB) state [2]. The system was thereafter often referred to be in quantum liquid state for the lack of long-range magnetic order (LRMO). However, this simple picture was not so obvious: The quantum fluctuations may as well lift the classical degeneracies and lead to ordered ground-state. Many studies were further inspired by the magnetic properties of doped CuO$_2$ planes in the copper-oxide superconductors where the frustration again plays a central role. Almost every possible scheme has been employed in the literature. These include the earlier variational approaches [3, 4, 5], the spinwave and large-$N$ theories [5, 6], the spontaneous symmetry breaking consideration [7], the numerical series expansion methods [8, 9], and the exact diagonalizations on finite clusters [10, 11, 12]. Most of recent analytic approaches tend to conclude with LRMO. But they all seem to suffer from uncontrolled approximations or inaccuracies. On the other hand, the numerical groups are divided into with and without LRMO (or just right on the critical point as suggested by Singh and Huse [8]), mainly due to insufficient knowledge on larger clusters and different methods of extrapolation. The main question, whether this frustrated system possesses LRMO remains controversial.

Even RVB states could have LMRO if the bonds decayed sufficiently slowly. This was first found on a square lattice by Liang, Doucot and Anderson [13]. Does this also take place on the triangular lattice? In this Letter, we shall address to this question based on a new use of the Schwinger-boson formalism [14, 15]. The central idea is to work with the RVB states resulting from Gutzwiller projection of mean-field states onto the right Hilbert space. This method has been working extremely well on the square lattice [15]. Note only the ground-state energy and the staggered magnetization obtained are virtually identical to the exact ones (cited by [10]), the low lying spin-flip spectrum also agrees with the best numerical result found by supercomputer (cf. [17]). On the triangular lattice, one needs to face the frustration. we shall show that it is important to have, at the mean-field level, both the antiferromagnetic and ferromagnetic fields in order to account for this frustration. With a proper balance between the two, the Gutzwiller projection then leads to optimizing
the RVB state. The latter is also explicitly derived within a self-consistent approximation in the “loop-gas” statistics\cite{R8, R13}. Our approach provides a simple but powerful way of founding the best RVB state and the right mean-field solution \textit{regardless of the frustration}. The optimized parameter-free RVB state is found to have energies much closer to the exact ones than those previously found\cite{R3, R4}. The system is predicted to be long-range ordered with a staggered sublattice magnetization $\sim 0.275$.

The nearest neighbor antiferromagnetic Heisenberg model may be written in terms of Schwinger bosons as\cite{R15}

$$
\hat{H} = \sum_{<ij>} \left[ -\frac{(1+\nu)}{4} \hat{A}_{ij}^\dagger \hat{A}_{ij} + \frac{(1-\nu)}{4} \hat{F}_{ij}^\dagger \hat{F}_{ij} + \nu S^2 - \frac{(1-\nu)}{2} S \right],
$$

(1)

where

$$
\hat{A}_{ij} = \hat{b}_{i\uparrow} \hat{b}_{j\downarrow} - \hat{b}_{i\downarrow} \hat{b}_{j\uparrow}, \quad \hat{F}_{ij} = \sum_{\sigma=\uparrow,\downarrow} \hat{b}_{i\sigma} \hat{b}_{j\sigma}, \quad \sum_{\sigma=\uparrow,\downarrow} \hat{b}_{i\sigma}^\dagger \hat{b}_{i\sigma} = 2S.
$$

(2)

We have deliberately broken up in (1) the Hamiltonian into two pieces (the first two terms): The first (second) is suitable for an antiferromagnetic (ferromagnetic) mean-field decoupling. The separation ($\nu$ is a free parameter to be fixed) is motivated by the fact that the pairing of nearest neighbor spins is frustrated on the triangle lattice. A ferromagnetic field will therefore be essential for an accurate description of the original system. This will become clear below when we minimize the energy of the RVB state. The partition function of the system can be calculated via

$$
Z(\beta) = \text{Tr}[\hat{P}_G \hat{\rho}] = \text{Tr} \left\{ \prod_{i=1}^N \hat{P}_i \hat{\rho} \right\},
$$

where $\hat{\rho} = \exp(-\beta \hat{H})$, and $\hat{P}_G$ ($\hat{P}_i$) stands for the Gutzwiller projection operator for the whole lattice (the $i$th site) which enforces the constraint in (2). Introduce the so-called coherent state, $|\alpha_i> = \exp \left( \sum_{\sigma=\uparrow,\downarrow} \hat{b}_{i\sigma}^\dagger b_{i\sigma} \right) |0>$, where $\alpha_i = (b_{i\uparrow}, b_{i\downarrow})$ are ordinary complex numbers. We can further write\cite{R14}

$$
Z(\beta) = \left[ \prod_{i=1}^N P_i \right] <\{\alpha_i^\dagger\}|\hat{\rho}|\{\alpha_i\}>_{\{\alpha_i^*, \alpha_i = 0\}},
$$

(3)

$$
P_i = \frac{1}{(2S)!} \left[ \sum_{\sigma=\uparrow,\downarrow} \frac{\partial}{\partial b_{i\sigma}} \frac{\partial}{\partial b_{i\sigma}^\dagger} \right]^{2S}
$$

(4)
provided that the matrix elements of \( \hat{\rho} \) are known.

Eq. (3) would be rigorous if the exact matrix elements were used. This is of course rarely possible for interacting systems. One therefore needs a trial \( \hat{\rho} \). We shall consider in this Letter a mean-field type of approximation in which the Hamiltonian is replaced by

\[
\hat{H}_{\text{mf}} = E_0 + \lambda \sum_{i,\sigma} \hat{b}_i^\dagger \hat{b}_{i\sigma} + \sum_{<ij>} \left\{ -D_{ij} \hat{A}_{ij}^\dagger + Q_{ij} \hat{F}_{ij} + \text{h.c.} \right\}.
\]

The Lagrangian multiplier \( \lambda \) and the mean fields \( D_{ij} \) and \( F_{ij} \) are to be determined via the usual self-consistency requirements,

\[
\sum_{\sigma = \uparrow, \downarrow} <\hat{b}_{i\sigma}^\dagger \hat{b}_{i\sigma}> = 1, \quad D_{ij} = \frac{1 + \nu}{4} <\hat{A}_{ij}> , \quad Q_{ij} = \frac{1 - \nu}{4} <\hat{F}_{ij}^\dagger> . \tag{5}
\]

\( \hat{H}_{\text{mf}} \) after Fourier transforms can be readily diagonalized by a Bogoliubov transformation. This results in a mean-field spectrum \( \omega_k + M_k \) with

\[
\omega_k = \sqrt{\lambda_k^2 - |D_k|^2}, \quad M_k = (Q_k - Q_k)/2, \quad \lambda_k = \lambda + (Q_k - Q_k)/2.
\]

The matrix elements of \( \hat{\rho} \) can be calculated accordingly. We find

\[
<\{\alpha_i^\dagger\}|\hat{\rho}_{\text{mf}}|\{\alpha_i\}> = Z_0(\beta) \exp \left( \sum_k \left[ \sum_{\sigma = \uparrow, \downarrow} W_k^{(1)} b_{k\sigma}^\dagger b_{k\sigma} + (W_k^{(2)} b_{k\uparrow}^\dagger b_{-k\downarrow}^\dagger + \text{c.c.}) \right] \right)
\]

where

\[
W_k^{(1)} = B_k \exp(-\beta M_k), \quad W_k^{(2)} = (D_k/\omega_k) B_k \sinh(\beta \omega_k), \quad B_k = ([\lambda_k/\omega_k] \sinh(\beta \omega_k) + \cosh(\beta \omega_k))^{-1}. \tag{6}
\]

Note that \( Q_k = Q_k^*, \quad D_k = -D_{-k} \).

Now return to (3) and consider in what follows \( S = 1/2 \). Write \( Z(\beta) = Z_0(\beta) \times Y_N \) and take a closer look at the structure of \( Y_N \). The differentiations amount to dividing the whole lattice into subsets of self-avoiding loops. One then sums over all possible configurations of the loops. This kind of loop gas statistics has been known for the RVB states. The more general case is that the bond, say, from sites \( i \) to \( j \) is replaced by a \( 4 \times 4 \) transfer matrix \( T_{ij} \) of the form (\( \sigma_z \) below is the third Pauli matrix)

\[
T_{ij} = \begin{pmatrix} G_{ij} & 0 \\ 0 & \sigma_z G_{ij} \sigma_z \end{pmatrix}, \quad G_{ij} = \begin{pmatrix} W_{ij}^{(1)} & -W_{ij}^{(2)*} \\ W_{ij}^{(2)} & W_{ij}^{(1)*} \end{pmatrix}. \tag{7}
\]
It brings, upon taking $P_i$, see (3), the prefactor $(\chi_{i1}b_{i\uparrow} + \chi_{i2}b_{i\downarrow} + \chi_{i3}b_{i\uparrow} + \chi_{i4}b_{i\downarrow})$ to a new one at site $j$ via multiplying $T_{ij}$ to the column of the $\chi$’s. $W_{ij}^{(1)}$, $W_{ij}^{(2)}$ are, respectively, the (inverse) Fourier transforms of $W_k^{(1)}$, $W_k^{(2)}$. Tracing over the matrix (divided by two) after completing a loop gives its contribution.

To illustrate this prescription, one can decompose $Y_N$ at a given $j_0$ into (all $j_k$’s below are self-avoiding)

$$Y_N = \sum_{n=0}^{N-1} \left\{ \sum_{\{j_k|i,k\neq 0\}} Y_{N-n-1}(\{j_k\}) \times \text{Tr}(G_{j_0j_1} \cdots G_{j_{n}j_{0}}) \right\}. \quad (8)$$

The arguments of $Y_{N-n-1}$ represent the sites excluded. Various physical quantities can be calculated by multiplying the operators to $\hat{\rho}$ and then compute the matrix elements as in (3). In the coherent states one has the simple substitutions, $\hat{\rho}$ calculated by multiplying the operators to $\hat{\rho}$ and then compute the matrix elements as in (3). In the coherent states one has the simple substitutions, $\hat{b}_{i\sigma}^\dagger \rightarrow \hat{b}_{i\sigma}$, $\hat{b}_{i\sigma} \rightarrow (\partial/\partial b_{i\sigma}^\dagger)$, which may modify the transfer matrix at site $i$. In general, operators that conserve boson numbers can be presented by some $4 \times 4$ matrices inserted at the relevant sites. One then can proceed to evaluate the modified $Y_N$ as usual. For the spin-spin correlation $\hat{S}_i \cdot \hat{S}_j$ we have the matrices

$$\frac{1}{4} \left( \begin{array}{cc} \sigma_z & 0 \\ 0 & -\sigma_z \end{array} \right)_i \left( \begin{array}{cc} \sigma_z & 0 \\ 0 & -\sigma_z \end{array} \right)_j + \frac{1}{2} \left[ \left( \begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right)_i \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)_j \right].$$

This leads us to ($i \neq j$)

$$< \hat{S}_i \cdot \hat{S}_j > = \frac{1}{Y_N} \sum_{n=1}^{N-1} \left\{ \sum_{\{j_k\}} Y_{N-n-1}(\{j_k\}) \times \frac{3}{4} \sum_{m=1}^{n} \text{Tr}[\sigma_z G_{j_0j_m}^{(1)} \sigma_z G_{j_{m-1}j_0}^{(2)}] \right\} \bigg|_{j_0=i,j_m=j}, \quad (9)$$

$$G_{j_0j_m}^{(1)} \equiv G_{j_0j_1} \cdots G_{j_{m-1}j_m}, \quad G_{j_{m-1}j_0}^{(2)} \equiv G_{j_{m}j_{m+1}} \cdots G_{j_{n}j_{0}}. \quad (10)$$

We shall take here a specific mean-field solution Consider the simplest symmetry $Q_{ij} = Q$, $D_{ij} = iD \exp(i3\phi_{ij})$ with $\phi_{ij}$ being the polar angle of the bond. Let $\gamma_k = (1/3) \sum_{l=1}^{3} \cos(k \cdot e_l)$, $\Gamma_k = (1/3) \sum_{l=1}^{3} \sin(k \cdot e_l)$ where $e_1 = (1/2, -\sqrt{3}/2)$, $e_2 = (1/2, +\sqrt{3}/2)$, and $e_3 = (-1, 0)$. At zero temperature, the mean-field dispersion $\omega_k/\lambda = \tilde{\omega}_k = \sqrt{(1 + d_1\gamma_k)^2 - (d_2\Gamma_k)^2}$ becomes gapless. Namely we have ($z = 6$ is the coordination number)

$$1 + d_1\gamma_{\pm k_0} = \pm d_2\Gamma_{\pm k_0}, \quad d_1 = zQ/\lambda, \quad d_2 = zD/\lambda. \quad (11)$$
where \( k_0 = 2\pi/3(1, \sqrt{3}) \). Condensation, denoted by \( \alpha \) which has to be inserted in (\( \mathbb{F} \)), occurs at \( \pm k_0 \) [or their equivalences on the reciprocal lattice spanned by \( G_1 = \pi(2, \sqrt{3}) \) and \( G_2 = (0, 4\pi/\sqrt{3}) \)]. Consequently, the mean-field equations of (\( \mathbb{F} \)) read (“BZ” means the first Brillouin zone)

\[
1 = \alpha + \frac{1}{2} \int_{BZ} \frac{\sqrt{3}d^2k}{8\pi^2} \left[ 1 + d_1\gamma_k \right],
\]

\[
\frac{D}{1 + \nu} = \alpha \Gamma_{k_0} + \frac{1}{4} \int_{BZ} \frac{\sqrt{3}d^2k}{8\pi^2} \left[ d_2\Gamma_k^2 \right],
\]

\[
\frac{Q}{1 - \nu} = \alpha \gamma_{k_0} + \frac{1}{4} \int_{BZ} \frac{\sqrt{3}d^2k}{8\pi^2} \left[ \gamma_k(1 + d_1\gamma_k) \right].
\]

The matrix elements \( W_{ij} \)'s can be found upon substituting the solution back to (\( \mathbb{F} \)), see (\( \mathbb{I} \)) below.

We next show that this set of \( W_{ij} \)'s minimizes the RVB ground-state energy at \( \nu = 0 \) within a self-consistent approximation for the loop gas. Going back to the transfer matrix (\( \mathbb{I} \)), \( W^{(1)}_k \) vanishes at zero temperature. The physical picture of (\( \mathbb{I} \))–(\( \mathbb{V} \)) become apparent: Here \( W^{(2)}_{ij} \) stands for the amplitude of a RVB pair connecting site \( i \) to \( j \), more explicitly, \( W_{ij}(\hat{b}^+_i \hat{d}^+_j - \hat{b}^+_i \hat{d}^+_j) \) [the superscript “(2)” is hereafter dropped]. \( Y_N \) is nothing but \( < \Phi_{RVB} | \Phi_{RVB} > \) with \( |\Phi_{RVB} > \) containing all possible superpositions of the pair products. Eq. (\( \mathbb{E} \)) then gives the explicit way of computing the spin-spin correlations. The question is reduced to finding the set of best amplitudes that minimizes the energy.

Rigorous evaluation of the RVB expectation values appears problematic. We thus introduce below a self-consistency approach. Let us first approximate \( Y_N / Y_{N-1} \rightarrow y^{n+1} \). This assigns a uniform weight \( 1/y^{n+1} \) for a loop of \( (n+1) \) sites. We are still left to deal with the self-avoiding restriction. Let us try to ignore this in the first place, denoting the corresponding \( y \) by \( y_0 \). The correct \( y \) is then obtained by properly identifying the over-counting. This allows a full analytic summation over the loops using Fourier transforms. The problem is reduced to finding the set of best amplitudes that minimizes the energy.

The matrix after summing over the paths connecting \( i \) to \( j \) reads \([i \neq j, \text{see } (\mathbb{I})]\),

\[
R_{ij} = \frac{1}{N} \sum_k \exp(-ik \cdot r_{ij}) \left( \begin{array}{cc} 1 & W_k^*/y_0 \\ W_k/y_0 & 1 \end{array} \right).
\]

The parameter \( y_0 \) can be determined by \( \text{Tr}(R_{ii} - 1) = 1 \) [cf. (\( \mathbb{L} \)) below]. Now \( y \) can be approximately recovered by grouping the extra winding at a given site (say \( i \)) into \( 1/y \),
which gives
\[ y^{-1} = y_0^{-1} \times \text{Tr} R_{ii}/2 = (3/2) \times y_0^{-1}. \] (14)

More detail on this over-counting problem has been discussed in [15].

The same procedure can be applied to the spin-spin correlations. We now have one path starting from \( i \) to \( j \) and the other from \( j \) to \( i \). Ingoring again the self-avoiding restriction and the overlaps between the two paths, the result of the summation is simply \( R_{ij} \) and \( R_{ji} \). However, taking two independent paths results in an additional over-counting of a factor 3/2 [see (14)] which has to be deducted. Substituting (13) back to (1) and assuming, as in the mean-field case, the three-fold symmetry for \( W_k \), we end up with the expression (drop \( y_0 \) hereafter as \( W_k \) is variational)

\[ E_{\text{bond}} = \left| \frac{1}{N} \sum_k \frac{\gamma_k}{1 - |W_k|^2} \right|^2 - \left| \frac{1}{N} \sum_k \frac{\Gamma_k W_k}{1 - |W_k|^2} \right|^2 \]

subject to \( W_k = W_k^* = -W_{-k} \) and the constraint
\[ \frac{1}{N} \sum_k \frac{1}{1 - |W_k|^2} = \frac{3}{2}. \] (15)

Eq. (13) can be treated via the method of Lagrangian multiplier. For simplicity, one can assume here real \( W_k \)'s. It is now a straightforward exercise to obtain (picking up the right solution satisfying \( |W_k| \leq 1 \))

\[ W_k = \frac{d_2 \Gamma_k}{1 + d_1 \gamma_k + \sqrt{(1 + d_1 \gamma_k)^2 - (d_2 \Gamma_k)^2}} \] (16)

with
\[ \frac{d_1}{d_2} = \frac{(1/N) \sum_k [\gamma_k/(1 - W_k^2)]}{(1/N) \sum_k [\Gamma_k W_k/(1 - W_k^2)]}. \] (17)

Note that (16) is nothing but the zero-temperature limit of (8) when the mean-field solution discussed above is used. Moreover, it can be easily checked that (17) and (14), after some manipulations using the specific form of (14), reduce to (11) and (12) of the mean-field result with \( \nu = 0 \). Note that, in passing to the continuous limit, the same condensation \( \alpha \) is also required in (13) and (14). This proves our claim that two approaches give the same RVB state.
Solving (15) and (17), or equivalently (11) and (12), we find
\[ \alpha = 0.275, \quad d_1 = -0.7223, \quad d_2 = 1.572 \] and \[ E_{\text{bond}} = -0.1899. \] Because of the condensation at the particular modes \( \pm k_0 \), the system possesses a long-range three-sublattice order with
\[
\lim_{|r_{ij}| \to \infty} < \hat{S}_i \cdot \hat{S}_j > = \begin{cases} 
\alpha^2 & i, j \in \text{same sublattices} \\
-\alpha^2/2 & i, j \in \text{different sublattices}
\end{cases}.
\]

The structure of (8) and (9) allows also a direct evaluation via Monte Carlo simulation. This is important for justifying our theory. It amounts to sampling over the distribution of the loops. At zero temperature, the rule to calculate the spin-spin correlations is particularly simple: For a given configuration,
\[
< \psi_L | \hat{S}_i \cdot \hat{S}_j | \psi_R > = \begin{cases} 
-3/4 < \psi_L | \psi_R > & i, j \in \text{same loop} \quad n(i) - n(j) = \text{odd} \\
+3/4 < \psi_L | \psi_R > & i, j \in \text{same loop} \quad n(i) - n(j) = \text{even} \\
0 & i, j \in \text{different loops}
\end{cases}
\]

where \( n(k) \) stands for the order of the site \( k \) in its own loop. Each loop contains even number of sites. In the simulation, the loop configurations are updated by randomly choosing a pair of nearest or next nearest neighbor sites and exchanging their loop connections with a probability satisfying detailed balance condition \[13, 15\]. For the triangular lattice, the sign problem shows up in that some of the loops contribute negative weights. We use their absolute values to regulate this problem. This limits the practical sizes of the simulation. The periodic boundary condition for the RVB amplitudes also presents a subtle problem. Normally, we should use those obtained by summing over a discrete set of \( k \)'s in accordance with the finite size of the lattice. If we did so, there would be no sign problem for sizes up to \( 6 \times 6 \). But the energy would also get higher. We therefore use the amplitudes calculated on a much larger lattice \( 200 \times 200 \). We then go to the smaller size and cut them into a full hexgon (i.e. with inversion symmetry). Bonds outside are moved into the hexgon via the periodic boundary condition except for those that are required to vanish by symmetry [e.g. for \( i = (0, 0) \) and \( j = \pm(n, -n) \) when the two axes are chosen to be \( (1,0) \) and \( (-1, \sqrt{3})/2 \). The correlations as a function of distance can be obtained by averaging over the six-fold equivalences.

We wish to compare our results simulated on finite lattices with those based on exact diagonalizations. On a \( 4 \times 4 \) lattice, the energy per bond is \( E_{\text{RVB}} = -0.1754(3) \) vs.
\( E_{\text{exact}} = -0.1782 \) (cited from [11]). The difference is probably due to that the symmetry of the bonds are not very suited on this cluster. The result on a \( 6 \times 6 \) lattice is presented in Table 1. One sees that they are in excellent agreement. Perhaps, the only notable difference between the two sets is at \( n = 3 \). The result also shows a nice approach to (18), i.e., a three-sublattice magnetization. We stress that our wavefunction is parameter-free. These results themselves should be of interest for variational approaches in the presence of frustration. Finally, the small difference in the approximate and the true values of \( E_{\text{bond}} \) should justify our analytic work.

Let us summarize the main results of this Letter. Starting with a Schwinger-boson mean-field state that includes both the antiferromagnetic and ferromagnetic fields, we have shown that a Gutzwiller projection leads to optimizing the RVB state on the triangle lattice. This in turn justifies the use of the two fields. The ground-state energies of the optimized state were calculated rigorously on finite lattices and were found to approach virtually the exact values. It is apparently much better than all previous variational states [3, 4]. we have demonstrated, though our prediction for larger lattices is still based on analytic approximation, that by properly optimizing the RVB pair amplitudes one is most likely to get long-range correlations. Namely, the system is predicted to possess a long-range order within the RVB context.

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References

[1] G. H. Wannier, Phys. Rev. 79, 357 (1950).

[2] P. W. Anderson, Mater. Res. Bull. 8, 153 (1973); P. Fazekas and P. W. Anderson, Philos. Mag. 30, 423 (1974).

[3] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987).
[4] D. A. Huse and V. Elser, Phys. Rev. Lett. 60, 2531 (1988).

[5] Th. Jolicoeur and J. C. Le Guillou, Phys. Rev. B. 40 2727 (1989); S. J. Miyake, J. Phys. Soc. Jpn. 61, 983 (1992).

[6] S. Sachdev, Phys. Rev. B 45, 12377 (1992).

[7] P. Azaria, B. Delamotte and D. Mouhanna, Phys. Rev. Lett. 70, 2483 (1993).

[8] R. R. P. Singh, D. A. Huse, Phys. Rev. Lett. 68, 1766 (1992).

[9] N. Elstner, R. R. P. Singh, and A. P. Young, Phys. Rev. Lett. 71, 1629 (1993).

[10] B. Bernu, C. Lhuillier, and L. Pierre, Phys. Rev. Lett. 69, 2590 (1992).

[11] K. Yang, L. K. Warman, and S. M. Girvin, Phys. Rev. Lett. 70, 2641 (1993).

[12] P. W. Leung, K. J. Runge, Phys. Rev. B 47, 5861 (1993).

[13] S. Liang, B. Doucot, and P. W. Anderson, Phys. Rev. Lett. 61, 365 (1988).

[14] Y.-C. Chen, Physica C 202, 345 (1992); ibid 204, 88 (1992).

[15] Y.-C. Chen, Phys. Lett. A 174, 329 (1993); Y.-C. Chen and K. Xiu, ibid 181, 373 (1993).

[16] E. Manousakis, Rev. Mod. Phys. 63, 1 (1991).

[17] G. Chen, H.-Q. Ding, and W. A. Goddard III, Phys. Rev. B 46, 2933 (1992).

[18] B. Sutherland, Phys. Rev. B 37, 3786 (1988); ibid 38, 6855 (1988).

[19] The two decouplings were first introduced by D. P. Arovas and A. Auerbach, Phys. Rev. B 38, 316 (1988).
Table Caption

Table 1 The spin-spin correlation $C(n)$ as a function of distance on the $6 \times 6$ lattice. The simulation takes up to $2 \times 10^9$ runs and the statistical errors are about $\pm 0.002$. The exact result is quoted from Leung and Runge[12].
| $n = $ | 1   | 2   | 3   | 4   | 5   | 6   |
|-------|-----|-----|-----|-----|-----|-----|
| RVB   | $-0.186$ | 0.155 | $-0.062$ | $-0.065$ | 0.118 | 0.117 |
| exact | $-0.1868$ | 0.1535 | $-0.0548$ | $-0.0664$ | 0.1136 | 0.1174 |

Table 1