Tilting complexes for group graded self-injective algebras

Andrei Marcus\textsuperscript{a} and Shengyong Pan\textsuperscript{b,}\textsuperscript{*}

Faculty of Mathematics and Computer Science, Babes\textendash;Bolyai University, Cluj-Napoca, Romania
E-mail: marcus@math.ubbcluj.ro

\textsuperscript{b} Department of Mathematics, Beijing Jiaotong University, Beijing 100044, People’s Republic of China
E-mail: shypan@bjtu.edu.cn

Abstract

We construct derived equivalences between group graded self-injective algebras, starting from equivalences between their 1-components, obtained via a construction of J. Rickard and S. Al-Nofayee.

1 Introduction

Construction of tilting complexes for group graded algebras was primarily motivated by the problem of finding reduction methods for Broué’s Abelian Defect Group Conjecture. In [6] two-sided tilting complexes are discussed, while in [7], Okuyama’s work prompted the need for one-sided group graded tilting complexes. The paper [8] starts from a method, due to Rickard [11], to lift stable equivalences to derived equivalences by characterizing objects that correspond to simple modules. The context in [11] is that of symmetric algebras. This result has been generalized to self-injective algebras by Al-Nofayee [2], and then further extended by Rickard and Rouquier [12].

In this paper we obtain group graded derived equivalences between self-injective algebras starting from the main results of [2] and [12]. Thus we generalize here the main result of [8], and for this, we rely on the properties of the Nakayama functor in the group graded setting. In Section 2 we recall the definition and characterization of group graded tilting complexes, and we point out the it is no need to assume the finiteness of the group $G$. Our main results in Section 3 are group graded versions of [2, Theorem 4] and of [12, Theorem 3.9]. One of the applications in Section 4 is the combination of these results with Okuyama’s strategy to lift stable equivalences. Another application is related to a construction of tilting complexes by Abe and Hoshino [1].

\textsuperscript{*} Corresponding author. Email: shypan@bjtu.edu.cn
2000 Mathematics Subject Classification: 18E30,16G10;16S10,18G15.
Keywords: self-injective algebra, graded derived equivalence, strongly graded algebra.
In this paper, rings are associative with identity, and modules are left, unless otherwise specified. We denote by $A$-Mod the category of left $A$-modules, and by $A$-mod its full category consisting of finitely generated $A$-modules. If $X$ is an object of an additive category $\mathcal{A}$, $\text{add}(X)$ denotes the full subcategory of $\mathcal{A}$ whose objects are direct summands of finite direct sums of copies of $X$. The notations $\mathcal{H}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ stand for the (unbounded) homotopy, respectively derived category of an abelian category $\mathcal{A}$. We freely use basic facts from [6], [7], [8].

2 Preliminaries

Let $k$ be a commutative ring and $G$ be a group (not necessarily finite). Suppose that $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ are $G$-graded $k$-algebras such that $R$ is $k$-flat. Throughout we denote by $A = R_1$ and $B = S_1$ the identity components of $R$ and $S$, respectively. We denote by $R$-Gr the category of $G$-graded $R$-modules, and by $R$-gr the category of finitely generated $G$-graded $R$-modules.

2.1. The group $G$ acts on $G$-graded $R$-modules $M \in R$-Gr by letting $M(g) = \bigoplus_{h \in G} M(g)_h$ be the $g$-suspension of $M$, where $M(g)_h = M_{hg}$ for all $g, h \in G$. If $R$ is strongly graded, then $G$ acts on $A$-modules $X \in A$-Mod by conjugation $X \mapsto R_1 \otimes_A X$. Note that $(R \otimes_A X)(g)$ is naturally isomorphic to $R \otimes_{A}(R_g \otimes_A X)$ in $R$-Gr.

2.2. A $G$-graded $(R, S)$-bimodule $M$ can be regarded as an $R \otimes S^{op}$-module graded by the $G \times G$-set $G \times G/\delta(G)$, where $\delta(G)$ is the diagonal subgroup of $G \times G$, with 1-component $M_1$ a module over the diagonal subalgebra

$$\Delta(R \otimes S^{op}) := (R \otimes S^{op})_{\delta(G)} = \bigoplus_{g \in G} R_g \otimes S_{g^{-1}}.$$ 

If $R$ and $S$ are strongly graded, then $M$ and $(R \otimes S^{op}) \otimes_{\Delta(R \otimes S^{op})} M_1$ are naturally isomorphic $G$-graded $(R, S)$-bimodules.

2.3. Recall that an object $\tilde{T}$ of $\mathcal{D}(R$-Gr) is called a $G$-graded tilting complex if it satisfies the following conditions:

(i) $\tilde{T} \in R$-grperf; this means that, regarded as a complex of $R$-modules, $\tilde{T} \in R$-perf, that is $\tilde{T}$ is bounded, and its terms are finitely generated projective $R$-modules.

(ii) $\bigoplus_{g \in G} \text{Hom}_{\mathcal{D}(R$-Gr)}(\tilde{T}, \tilde{T}(g)[n]) = 0$ for $n \neq 0$.

(iii) $\text{add}\{\tilde{T}(g) \mid g \in G\}$ generates $R$-grperf as a triangulated category.

The following result was proved in [7] Theorem 2.4] and [6] Theorem 4.7], based on Keller’s approach [4], but note that the assumption that $G$ is finite is not needed.

Theorem 2.4. The following statements are equivalent:

(1) There is a $G$-graded tilting complex $\tilde{T} \in \mathcal{D}(R$-Gr) and an isomorphism $S \simeq \text{End}_{\mathcal{D}(R)}(\tilde{T})^{op}$ of $G$-graded algebras.
(2) There is a complex $\tilde{U}$ of $G$-graded $(R, S)$-bimodules such that the functor

$$\tilde{U} \otimes^L S^{-} : \mathcal{D}(S) \to \mathcal{D}(R)$$

is an equivalence.

(3) There are equivalences

$$F : \mathcal{D}(R) \to \mathcal{D}(S) \quad \text{and} \quad F^\text{gr} : \mathcal{D}(R-\text{Gr}) \to \mathcal{D}(S-\text{Gr})$$

of triangulated categories such that $F^\text{gr}$ is $G$-graded functor and the diagram

\[
\begin{array}{ccc}
\mathcal{D}(R-\text{Gr}) & \xrightarrow{F^\text{gr}} & \mathcal{D}(S-\text{Gr}) \\
\downarrow U & & \downarrow U \\
\mathcal{D}(R) & \xrightarrow{F} & \mathcal{D}(S),
\end{array}
\]

is commutative.

(4) There are equivalences

$$F^\text{perf} : \mathcal{D}(R-\text{perf}) \to \mathcal{D}(S-\text{perf}) \quad \text{and} \quad F^\text{gr,perf} : \mathcal{D}(R-\text{grperf}) \to \mathcal{D}(S-\text{grperf})$$

of triangulated categories such that $F^\text{gr,perf}$ is $G$-graded functor and $U \circ F^\text{gr,perf} = F^\text{perf} \circ U$.

(5) (provided that $R$ and $S$ are strongly graded) There are (bounded) complexes $U$ of $\Delta(R \otimes S^\text{op})$ modules and $V$ of $\Delta(S \otimes R^\text{op})$-modules, and isomorphisms $U \otimes^L_R V \simeq A$ in $\mathcal{D}(\Delta(R \otimes R^\text{op}))$ and $V \otimes^L_A U \simeq B$ in $\mathcal{D}(\Delta(S \otimes S^\text{op}))$.

**Proof.** We have an isomorphism $S \simeq \text{End}_{\mathcal{D}(R)}(\tilde{T})^\text{op}$ of $G$-graded algebras for any group $G$, because $\tilde{T}$ is bounded, and each component of $\tilde{T}$ is finitely generated. It follows that the proofs of [7, Theorem 2.4] and [6, Theorem 4.7] carry over to the general situation. \qed

### 2.5.

A complex $\tilde{T} \in \mathcal{H}(R-\text{Gr})$ is called $G$-invariant if $\tilde{T}(g) \simeq \tilde{T}$ in $\mathcal{H}(R-\text{Gr})$ for all $g \in G$. More generally, $\tilde{T}$ is called weakly $G$-invariant if $\tilde{T}(g) \in \text{add}(\tilde{T})$ in $\mathcal{H}(R-\text{Gr})$ for all $g \in G$.

If $R$ is strongly $G$-graded, then a complex $T \in \mathcal{H}(A)$ is called $G$-invariant if $R_g \otimes_A T \simeq T$ in $\mathcal{H}(A)$ for all $g \in G$, and $T$ is called weakly $G$-invariant if $R_g \otimes_A T \in \text{add}(T)$ in $\mathcal{H}(A)$ for all $g \in G$.

Note that if $R$ is strongly $G$-graded, then the functor $R \otimes_A - : A-\text{Mod} \to R-\text{Gr}$ is an equivalence, hence a complex $T \in \mathcal{H}(R-\text{Gr})$ is $G$-invariant (weakly $G$-invariant) if and only if its identity component $T \in \mathcal{H}(A)$ is $G$-invariant (weakly $G$-invariant).

The following statement is also true for arbitrary $G$. It is essentially proved in [8, Proposition 2.1 and Remark 2.2], but for convenience, we include a proof here.

**Proposition 2.6.** Assume that $R$ is strongly $G$-graded. Let $\tilde{T}$ be a weakly $G$-invariant object in $\mathcal{H}^b(R-\text{Gr})$. Denote by $T$ the identity component of $\tilde{T}$, and let $S = \text{End}_{\mathcal{D}(R)}(\tilde{T})^\text{op}$. Then, $T$ is a tilting complex for $A$ if and only if $\tilde{T}$ is a $G$-graded tilting complex for $R$. Moreover, in this case, $S$ is strongly $G$-graded, and it is a crossed product if and only if $\tilde{T}$ is $G$-invariant.
Proof. Since \( R \) is strongly graded, the functor \( R \otimes_A - : A\text{-Mod} \to R\text{-Gr} \) is an equivalence, and a \( G \)-graded \( R \)-module is projective in \( R\text{-Gr} \) if and only if it is projective in \( A\text{-Mod} \). It follows that \( \tilde{T} \) is a bounded complex of finitely generated projective \( R \)-modules if and only if \( T \) is a bounded complex of finitely generated projective \( A \)-modules. For each \( m \in \mathbb{Z} \), we have

\[
\text{Hom}_{\mathcal{H}(R)}(\tilde{T}, \tilde{T}[m]) \cong \bigoplus_{g \in G} \text{Hom}_{\mathcal{H}(R\text{-Gr})}(\tilde{T}, \tilde{T}[m](g))
\]

and

\[
\text{Hom}_{\mathcal{H}(R\text{-Gr})}(\tilde{T}, \tilde{T}[m](g)) \cong \text{Hom}_{\mathcal{H}(A)}(T, R_g \otimes T[m]).
\]

Since \( \tilde{T} \) is weakly \( G \)-invariant, we have that for \( m \neq 0 \), \( \text{Hom}_{\mathcal{H}(A)}(T, R_g \otimes T[m]) = 0 \) if and only if \( \text{Hom}_{\mathcal{H}(R\text{-Gr})}(\tilde{T}, \tilde{T}[m](g)) = 0 \). If \( A \) is in the triangulated subcategory generated by \( \text{add} (T) \) in \( \mathcal{D}^b(A) \), then \( R \) is in the triangulated subcategory generated by \( \text{add} (\tilde{T}) \) in \( \mathcal{D}^b(R\text{-Gr}) \). Conversely, if \( R \) belongs to the triangulated subcategory generated by \( \text{add} (\tilde{T}) \) in \( \mathcal{D}^b(R\text{-Gr}) \), then \( A R \) is in the triangulated subcategory generated by \( \text{add} (A \tilde{T}) \) in \( \mathcal{D}^b(A) \). Since \( A R \) is a finite direct sum of copies of \( A \), and \( A \tilde{T} \) is a finite direct sum of copies of \( T \), \( A \) is in the triangulated subcategory generated by \( \text{add} (T) \) in \( \mathcal{D}^b(A) \). The last statement is clear, and also note that for the identity component of \( S \) the have the isomorphism

\[
S_1 = \text{End}_{\mathcal{H}(R\text{-Gr})}(\tilde{T}) \cong \text{End}_{\mathcal{H}(A)}(T)\text{op}
\]
of \( k \)-algebras.

\[\square\]

3 \quad \textbf{G-graded self-injective algebras}

In this section we assume that \( R \) is a strongly \( G \)-graded algebra over the field \( k \), where \( G \) is a finite group, and the identity component \( A := R_1 \) is a finite dimensional algebra. For simplicity, we also assume that the field \( k \) is algebraically closed, but the results below easily generalize to arbitrary fields (see [11, Section 8]).

\textbf{Proposition 3.1.} Let \( T \) be a \( G \)-invariant object in \( \mathcal{H}(A) \), and denote \( \tilde{T} = R \otimes_A T \) and \( S = \text{End}_{\mathcal{D}(R)}(\tilde{T})\text{op} \). If \( T \) is a tilting complex for \( A \) and \( A \) is self-injective, then \( S \) is a strongly \( G \)-graded self-injective algebra.

\textbf{Proof.} We know by Proposition 2.6 that \( \tilde{T} \) is a \( G \)-graded tilting complex for \( R \) and that \( S \) is strongly \( G \)-graded. It is easy to see that \( R \) is self-injective if and only if \( A \) is self-injective (see, for instance, [6, 5.1]). Finally, self-injectivity is preserved by derived equivalences by [1] Lemmas 1.7 and 1.8 (see also [12, Corollary 3.12]).

Next we extend S. Al-Nofayee’s construction [2] to the case of strongly \( G \)-graded algebras.

3.2. If there is a derived equivalence between the self-injective \( k \)-algebras \( A \) and \( B \), then the set \( S = \{ X_i \mid i \in I \} \) of objects corresponding to the simple \( B \)-modules, must satisfy the following conditions.

(a) \( \text{Hom} (X_i, X_j[m]) = 0 \) for \( m < 0 \).

(b) \( \text{Hom} (X_i, X_j) = k \) if \( i = j \) and \( 0 \) otherwise.
(c) The objects $X_i$, $i \in I$, generate $\mathcal{D}^b(A\text{-mod})$ as a triangulated category.

(d) The Nakayama functor $\nu$ permutes the set $S$, that is, there is a permutation $\sigma$ on $I$ such that $\nu(X_i) = X_{\sigma(i)}$.

In order to obtain a graded derived equivalence, we need to consider the conjugation action of $G$ on $A$-modules. Assume that $I$ is a finite $G$-set, and that the objects $X_i$ also satisfy the condition:

(e) $R_g \otimes_A X_i \simeq X_{gi}$ for all $g \in G$ and $i \in I$.

Lemma 3.3. Let $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, be objects satisfying conditions 3.2 (a) to (e). There exist bounded complexes $T_i = I_S(X_i)$ of finitely generated injective modules, and bounded complexes $T'_i = P_S(X_i)$ of finitely generated projective modules, such that

$$\text{Hom}(T_i, X_j[m]) = \begin{cases} k, & \text{if } i = j \text{ and } m = 0 \\ 0, & \text{otherwise.} \end{cases},$$

$$\text{Hom}(X_j[m], T'_i) = \begin{cases} k, & \text{if } i = j \text{ and } m = 0 \\ 0, & \text{otherwise.} \end{cases},$$

and moreover,

$$R_g \otimes_A T_i \simeq T_{gi}, \quad R_g \otimes_A T'_i \simeq T'_{gi},$$

for all $g \in G$ and $i, j \in I$.

Proof. The proof given in [7, Theorem 2.4] is based on [11, Section 5], and it works for self-injective algebras. For convenience, we give a brief proof. Let $g \in G$ and $i \in I$. The construction of the complexes $T_i$ go by induction as follows.

Set $X_i^{(0)} := X_i$, then $R_g \otimes_A X_i^{(0)} = X_{gi}^{(0)}$. By induction on $n$, we shall construct a sequence

$$X_i^{(0)} \rightarrow X_i^{(1)} \rightarrow X_i^{(2)} \rightarrow \ldots \rightarrow X_i^{(n)} \rightarrow \ldots$$

of objects and maps in $\mathcal{D}^b(A)$. Assuming that $X_i^{(n-1)}$ and $X_{gi}^{(n-1)}$ are constructed such that $R_g \otimes_A X_i^{(n-1)} = X_{gi}^{(n-1)}$, we may construct that $X_i^{(n)}$ and $X_{gi}^{(n)}$ such that $R_g \otimes_A X_i^{(n)} = X_{gi}^{(n)}$ and we have the commutative diagram

$$\begin{array}{ccc}
R_g \otimes_A X_i^{(n-1)} & \rightarrow & X_{gi}^{(n-1)} \\
\downarrow & & \downarrow x \\
R_g \otimes_A X_i^{(n-1)} & \rightarrow & X_{gi}^{(n-1)}
\end{array}$$

Finally, let $T_i = \text{hocolim}(X_i^{(n)})$, so it follows that $R_g \otimes_A T_i \simeq T_{gi}$. \qed

Lemma 3.4. The permutation induced by the Nakayama functor $\nu$ commutes with the conjugation action of $G$ on $A$-modules.
**Proof.** Recall that \( \nu = \text{DHom}_A(\cdot, A) \), where \( \text{D} = \text{Hom}_k(\cdot, k) \). Since for any \( g \in G \), the bimodule \( R_g \) induces a Morita auto-equivalence of \( A \)-mod with quasi-inverse \( R_{g^{-1}} \), for any \( g \in G \) and \( i \in I \) we have

\[

\nu(R_g \otimes_A X_i) \simeq \text{DHom}_A(R_g \otimes_A X_i, A) \simeq \text{DHom}_A(X_i, \text{Hom}_A(R_g, A)) \\
\simeq \text{DHom}_A(X_i, R_{g^{-1}}) \simeq \text{D}(\text{Hom}_A(X_i, A) \otimes_A R_{g^{-1}}) \\
\simeq \text{Hom}_A(R_{g^{-1}}, \text{DHom}_A(X_i, A)) \simeq \text{Hom}_A(R_{g^{-1}}, \nu(X_i)) \\
\simeq \text{Hom}_A(R_{g^{-1}}, A) \otimes \nu(X_i) \simeq R_g \otimes_A \nu(X_i) \\
\simeq X_{\sigma(g)i},
\]

On the other hand,

\[

\nu(R_g \otimes_A X_i) \simeq \nu(X_{gi}) \simeq X_{\sigma(gi)},
\]

so \( \sigma(gi) = g\sigma(i) \) for all \( g \in G \) and \( i \in I \). \( \square \)

**Theorem 3.5.** Let \( R \) be a strongly \( G \)-graded self-injective algebra with \( R_1 = A \), let \( I \) be a finite \( G \)-set, and let \( X_i \in \mathcal{D}^b(A \text{-mod}) \), \( i \in I \), be objects satisfying conditions \( 3.2 \) (a) to (e).

Then there is another self-injective crossed product \( G \)-algebra \( S \), and a \( G \)-graded derived equivalence between \( R \) and \( S \), whose restriction to \( A \) sends \( X_i, i \in I \), to the simple \( S_1 \)-modules.

**Proof.** By \([2, \text{Lemma 5}]\), there is a tilting complex \( T = \bigoplus_{i \in I} T_i \) for \( A \) such that

\[
\text{Hom}(T_i, X_j[m]) = \begin{cases} 
  k, & \text{if } \sigma(i) = j \text{ and } m = 0, \\
  0, & \text{otherwise}.
\end{cases}
\]

It follows by Lemma \( 3.4 \) and by the definition of the homotopy colimit that \( \nu(T_i) \simeq T_{\sigma(i)} \) for all \( i \in I \) (see \([2, \text{Lemma 9}]\)). By Lemma \( 3.3 \) the summands \( T_i \) can be constructed to satisfy the additional condition

\[
R_g \otimes_A T_i \simeq T_{gi},
\]

for all \( i \in I \) and \( g \in G \). Consequently, \( T \) is \( G \)-invariant, and Proposition \( 2.6 \) applies \( \square \)

**3.6.** Let \( T = \mathcal{D}^b(A) \), and let \((T^<, T^>)\) be the bounded \( t \)-structure on \( T \) as in \([12, \text{Proposition 3.4}]\). Denote by \( A \) the heart of this \( t \)-structure, and by \( H^0 \) the \( H^0 \)-functor associated to this \( t \)-structure. Then the set of the simple objects of \( A \) is \( S \). Let \( T_i = I \mathcal{S}(X_i) \) and \( T'_i = P_S(X_i) \), \( i \in I \), be the complexes defined in Lemma \( 3.3 \), and let \( T' = \bigoplus_{i \in I} T'_i \).

Consider the finite dimensional \( G \)-graded DG algebra (see \([7, 2.3]\))

\[
S = \text{End}_A^\ast(R \otimes_A \bigoplus_{i \in I} P_S(X_i))
\]

with 1-component

\[
B = \text{End}_A^\ast(\bigoplus_{i \in I} P_S(X_i)) = \bigoplus_{i \in I} \text{Hom}_A(\bigoplus_{i \in I} P_S(X_i), \bigoplus_{m \in I} P_S(X_i)[m]).
\]
We may now extend [12, Theorem 3.9] to strongly $G$-graded algebras.

**Theorem 3.7.** We have:
1) $H^n(B) = 0$ for $m > 0$ and for $m \ll 0$.
2) There is a $G$-graded derived equivalence $\mathcal{D}^b(S\text{-mod}) \simeq \mathcal{D}^b(R\text{-mod})$.
3) There is a $G$-equivalence $H^0(B\text{-mod}) \simeq A$.

**Proof.** 1) By [12, Theorem 3.9], we have that $H^n(B) = 0$ for $m > 0$ and for $m \ll 0$. Note that
\[ \text{Hom}_R(R \otimes_A P_S(X_j), R \otimes_A P_S(X_i)[m]) \simeq \text{Hom}_A(P_S(X_j), \bigoplus_{g \in G} R_g \otimes_A P_S(X_i)[m]), \]
and, by Lemma 3.3, $\text{Hom}(P_S(X_j), R_g \otimes_A P_S(X_i)[m]) = 0$ for all $i, j \in I$ if and only if $\text{Hom}(P_S(X_j), P_S(X_i)[m]) = 0$ for all $i, j \in I$. Consequently, $H^i(S) = 0$ for $m > 0$ and for $m \ll 0$.

2) We also know that the functor $\text{Hom}_A^*(\bigoplus_{i \in I} P_S(X_i), -) : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$ is an equivalence. Since $R \otimes_A - : A\text{-Mod} \rightarrow R\text{-Gr}$ is an equivalence, and a $G$-graded $R$-module is projective in $R\text{-Gr}$ if and only if it is projective in $A\text{-Mod}$, it is clear that $R \otimes_A (\bigoplus_{i \in I} P_S(X_i))$ is perfect object in $\mathcal{D}(R\text{-Gr})$ if and only if $\bigoplus_{i \in I} P_S(X_i)$ is a perfect complex of $A$-modules. Therefore, we get the $G$-graded derived equivalence
\[ \text{Hom}_R^*(R \otimes_A \bigoplus_{i \in I} P_S(X_i), -) : \mathcal{D}^b(R\text{-mod}) \rightarrow \mathcal{D}^b(S\text{-mod}). \]

3) By [12, Theorem 3.9], $\mathcal{T}'(H^0(T'))$ is a progenitor for $A$ with endomorphism algebra $H^0(B)$. As in Lemma 3.3, $T'$ is $G$-invariant, hence $\mathcal{T}'(H^0(T'))$ is also $G$-invariant, and the statement follows. \hfill \Box

## 4 Applications and examples

Okuyama’s strategy to lift a stable equivalence to a derived equivalence also generalizes to strongly $G$-graded self-injective algebras. We assume that $k$ is a field, and here we need to assume in addition that the order of $G$ is invertible in $k$, that is, the characteristic of $k$ does not divide $|G|$.

**Corollary 4.1.** Let $R$ and $S$ be strongly $G$-graded self-injective algebras. Assume that $|G|$ is invertible in $k$, and let $M$ be a $G$-graded $R\text{-S-bimodule}$ inducing a stable equivalence of Morita type between $R$ and $S$.

If there are objects $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, be objects satisfying conditions 3.2 (a) to (e), and such that $X_i$ is stably isomorphic to $M_i \otimes_B S_i$, for all $i \in I$, then there is a $G$-graded derived equivalence between $R$ and $S$.

**Proof.** By Theorem 3.5 there is a self-injective crossed product $R'$ and a $G$-graded derived equivalence between $R$ and $R'$ By [7, Remark 3.4], we obtain a $G$-graded stable equivalence of Morita type between $R$ and $R'$. Consequently, we have a stable Morita equivalence between $R'$ and $S$ induced by a $G$-graded $R' \otimes S^{op}$-bimodule $M'$. Since simple $A'$-modules are sent to simple $B$-modules, a theorem...
of Linckelmann [5] Theorem 2.1] says that a direct \( A \otimes B^{\text{op}} \)-summand \( M \) of \( M' \) induces a Morita equivalence between \( A' \) and \( B \). Since \( [G] \) is invertible in \( k \), we have that \( M \) is a \( \Delta(R' \otimes S^{\text{op}}) \)-summand of \( M' \), so by [6] Theorem 3.4], \( (R' \otimes S^{\text{op}}) \otimes_{\Delta(R' \otimes S^{\text{op}})} M \) induces a \( G \)-graded Morita equivalence between \( R' \) and \( S \). By composing this equivalence with the \( G \)-graded derived equivalence between \( R \) and \( R' \), we obtain a \( G \)-graded derived equivalence between \( R \) and \( S \).

\[ \square \]

**Remark 4.2.** By [2] Section 4] and [11] Section 6.3], we must have

\[ X_i \simeq \Omega^i (M_i \otimes_B S)[n_i]. \]

Here we only have to verify condition [3.2] (e). But this follows immediately since \( R_g \otimes_A M \otimes_B S \) is a \( (A, B) \)-bimodule, and the syzygy functor \( \Omega \) also commutes with the \( G \)-conjugation functor \( R \). The case of representation-finite algebras. Here we adapt [1] Th eorem 3.6] in order to obtain a \( G \)-invariant tilting complex, so that Proposition 2.6 can be applied. Here \( G \) is not assumed to be finite.

**Proposition 4.3.** Assume that \( A \) is representation-finite. Let \( P \) be a bounded complex of finitely generated projective \( A \)-modules such that \( \text{Hom}_{\mathcal{D}(A)}(P, R_g \otimes_A P[m]) = 0 \) for all \( m \neq 0 \) and \( g \in G \), and add \( P = \text{add} \nu P \). Then there exists a bounded complex of finitely generated projective \( A \)-modules \( Q \) such that \( Q \otimes P \) is a \( G \)-invariant tilting complex.

**Proof.** By adding \( G \)-conjugates of \( P \), we may assume that \( P \) is \( G \)-invariant. The complex \( P \) defines a torsion theory \( (\mathcal{T}, \mathcal{F}) \), where \( \mathcal{T} = \nu H^0(P) \) and \( \mathcal{F} = \mathcal{T}^\perp \) are invariant under the Nakayama functor \( \nu \). Since \( P \) is \( G \)-invariant, we have that \( \mathcal{T} \) and \( \mathcal{F} \) are also closed under \( G \)-conjugation. Let \( \{ e_j \mid j \in J \} \) be a basic set of orthogonal local idempotents in \( A \). The \( G \)-conjugation action of \( G \) on \( A \)-modules induces a \( G \)-set structure on \( J \), such that then the subsets \( J_1 = \{ j \in J \mid Ae_j \in \mathcal{T} \} \) and \( J_2 = \{ j \in J \mid Ae_j \in \mathcal{F} \} \) are \( G \)-stable. On can easily deduce from these observations and [3.4] that the tilting complex \( T \) constructed in [11] Lemma 3.4] and the complex \( Q \) from the proof of [11] Theorem 3.6] are \( G \)-invariant. \[ \square \]

**Example 4.4.** This example is related to [3] Example 9.5]. Let \( A \) be a finite dimensional \( k \)-algebra given by the quiver

\[
1 \swarrow_{\alpha_1} \searrow_{\alpha_3} \downarrow_{\alpha_2} 2 \swarrow_{\beta_1} \searrow_{\beta_2} 3
\]

with relations

\[
\alpha_1 \alpha_2 \alpha_3 \alpha_1 = \alpha_2 \alpha_3 \alpha_1 \alpha_2 = \alpha_3 \alpha_1 \alpha_2 \alpha_3 = 0.
\]

Let \( P = P_2 \oplus P_3 \). Then add \( P = \text{add} \nu(P) \). Then there is a complex

\[ Q := 0 \to P_2 \to P_1 \to 0 \]

with \( P_2 \) in degree 0 such that \( P \oplus Q \) is a tilting complex for \( A \), and the endomorphism algebra \( B \) of \( P \oplus Q \) is the algebra given by the quiver

\[
1 \swarrow_{\alpha_1} \searrow_{\alpha_2} 2 \swarrow_{\beta_1} \searrow_{\beta_2} 3
\]
with relations
\[ \alpha_1 \beta_1 = \beta_2 \alpha_2 = 0, \quad \alpha_1 \alpha_2 \alpha_1 = \beta_2 \beta_1 \beta_2 = \alpha_2 \alpha_1 - \beta_1 \beta_2 = 0. \]

Moreover, consider the infinite cyclic group \( G = \langle g, g^{-1} \mid gg^{-1} = g^{-1}g = 1 \rangle \) acting on \( A \) and \( B \) as follows: The element \( g \) fixes all the vertices and the edge \( \alpha_1 \) in \( A \), and \( g(\alpha_i) = \alpha_i + \alpha_i \alpha_{i+1} \alpha_{i+2} \alpha_i \) (mod 3) for \( i \neq 1 \), while \( g \) fixes all the vertices and all \( \alpha_i \) in \( B \), and \( g(\beta_i) = \beta_i + \beta_i \beta_{i+1} \beta_i \) (mod 2) for \( i \neq 1 \). Then the complexes \( P \) and \( Q \) are \( G \)-invariant, and \( R \otimes_A (P \oplus Q) \) induces a \( G \)-graded derived equivalence between the skew group algebras \( R = A \ast G \) and \( S = B \ast G \).

**Example 4.5.** Let \( A \) be a finite dimensional \( k \)-algebra given by the quiver
\[
\begin{array}{c}
1 \rightarrow \alpha_1 \\
\alpha_2 \\
2 \rightarrow
\end{array}
\]
(see [10, 1.4] and [9, Example (2)]). Let \( G = \{1, g\} \), with \( g \) acting on \( A \) by interchanging vertices 1 and 2, and consider the skew group algebra \( R = A \ast G \). Let \( T = P_1 \oplus P_2 \oplus I_3 \). Then \( R \otimes_A T \) is a \( G \)-graded tilting complex for \( R \).

**Acknowledgements.** Shengyong Pan is funded by China Scholarship Council.

The authors thank the referee for his/her observations which improved the presentation of the paper.

**References**

[1] H. Abe and M. Hoshino, On derived equivalences for selfinjective algebras, *Comm. Algebra* 34 (2006), 4441–4452.

[2] S. Al-Nofayee, Equivalences of derived categories for self-injective algebras, *J. Algebra* 313 (2007), 897–904.

[3] H. Asashiba, A generalization of Gabriel’s Galois covering functors and derived equivalences, *J. Algebra* 334 (2011), 109–149.

[4] B. Keller, On the construction of triangle equivalences. In: Derived equivalences for group rings (Springer-Verlag, 1998), pp. 155-176.

[5] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and \( p \)-groups, Math. Z. 223 (1996) 87–100.

[6] A. Marcus, Equivalences induced by graded bimodules, *Comm. Algebra* 26 (1998), 713–731.

[7] A. Marcus, Tilting complexes for group graded algebras, *Journal of Group Theory* 6 (2003), 175–193.

[8] A. Marcus, Tilting complexes for group graded algebras II, *Osaka J. Math.* 42 (2005), no.2, 453-462.

[9] J. i. Miyachi, Extensions of rings and tilting complexes, *J. Pure Appl. Algebra* 105 (1995), 183–194.

[10] I. Reiten and C. Riedtmann, Skew group algebras in the representation theory of Artin algebras, *J. Algebra* 92 (1985), 224–282.

[11] J. Rickard, Equivalences of derived categories for symmetric algebras, *J. Algebra* 257 (2002), 460–481.

[12] J. Rickard and R. Rouquier, Stable categories and reconstruction, *J. Algebra* 475 (2017), 287–307.