Abstract

This paper deals with the recoverable robust spanning tree problem under interval uncertainty representations. A polynomial time, combinatorial algorithm for the recoverable spanning tree problem is first constructed. This problem generalizes the incremental spanning tree problem, previously discussed in literature. The algorithm built is then applied to solve the recoverable robust spanning tree problem, under the traditional interval uncertainty representation, in polynomial time. Moreover, the algorithm allows to obtain, under some mild assumptions about the uncertainty intervals, several approximation results for the recoverable robust spanning tree problem under the Bertsimas and Sim interval uncertainty representation and the interval uncertainty representation with a budget constraint.

Keywords: robust optimization; interval data; recovery; spanning tree

1 Introduction

Let \( G = (V, E) \), \(|V| = n\), \(|E| = m\), be an undirected graph and let \( \Phi \) be the set of all spanning trees of \( G \). In the minimum spanning tree problem, a cost is specified for each edge, and we seek a spanning tree in \( G \) of the minimum total cost. This problem is well known and can be solved efficiently by using several polynomial time algorithms (see, e.g., \([1, 18]\)).

In this paper, we first study the recoverable spanning tree problem (Rec ST for short). Namely, for each edge \( e \in E \), we are given a first stage cost \( C_e \) and a second stage cost \( c_e \) (recovery stage cost). Given a spanning tree \( X \in \Phi \), let \( \Phi^k_X \) be the set of all spanning trees \( Y \in \Phi \) such that \( |Y \setminus X| \leq k \) (the recovery set), where \( k \) is a fixed integer in \([0, n-1]\), called the recovery parameter. Note that \( \Phi^k_X \) can be seen as a neighborhood of \( X \) that contains all spanning trees which can be obtained from \( X \) by exchanging up to \( k \) edges. The Rec ST problem can be stated formally as follows:

\[
\text{Rec ST : } \min_{X \in \Phi} \left( \sum_{e \in X} C_e + \min_{Y \in \Phi^k_X} \sum_{e \in Y} c_e \right).
\] (1)

We thus seek a first stage spanning tree \( X \in \Phi \) and a second stage spanning tree \( Y \in \Phi^k_X \), so that the total cost of \( X \) and \( Y \) for \( C_e \) and \( c_e \), respectively, is minimum. Notice that Rec ST
generalizes the following incremental spanning tree problem, investigated in [8]:

\[
\text{INC ST} : \min_{Y \in \Phi^k_{\hat{X}}} \sum_{e \in Y} c_e, \quad (2)
\]

where \(\hat{X} \in \Phi\) is a given spanning tree. So, we wish to find an improved spanning tree \(Y\) with the minimum cost, within a neighborhood of \(\hat{X}\) determined by \(\Phi^k_{\hat{X}}\). Several interesting practical applications of the incremental network optimization were presented in [8]. It is worth pointing out that INC ST can be seen as the Rec ST problem with a fixed first stage spanning tree \(\hat{X}\), whereas in Rec ST both the first and the second stage trees are unknown. It has been shown in [8] that INC ST can be solved in strongly polynomial time by applying the Lagrangian relaxation technique. On the other hand, no combinatorial algorithm for Rec ST has been known to date. Thus proposing a combinatorial algorithm for this problem is one of the main results of this paper.

The Rec ST problem, beside being an interesting problem per se, has an important connection with a more general problem. Namely, it is an inner problem in the recoverable robust model with uncertain recovery costs, discussed in [4, 5, 6, 7, 15, 17]. Indeed, the recoverable spanning tree problem can be generalized by considering its robust version. Suppose that the second stage costs \(c_e, e \in E\), are uncertain and let \(U\) contain all possible realizations of the second stage costs, called scenarios. We will denote by \(c^S_e\) the second stage cost of edge \(e \in E\) under scenario \(S \in U\), where \(S = (c^S_e)_{e \in E}\) is a cost vector. In the recoverable robust spanning tree problem (Rob Rec ST for short), we choose an initial spanning tree \(X\) in the first stage with the cost equal to \(\sum_{e \in X} C_e\). Then, after scenario \(S \in U\) reveals, \(X\) can be modified by exchanging at most \(k\) edges, obtaining a new spanning tree \(Y \in \Phi^k_{\hat{X}}\). The second stage cost of \(Y\) under scenario \(S \in U\) is equal to \(\sum_{e \in Y} c^S_e\). Our goal is to find a pair of trees \(X\) and \(Y\) such that \(|X \setminus Y| \leq k\), which minimizes the sum of the first and the second stage costs \(\sum_{e \in X} C_e + \sum_{e \in Y} c^S_e\) in the worst case. The Rob Rec ST problem formally is defined as follows:

\[
\text{Rob Rec ST} : \min_{X \in \Phi} \left( \sum_{e \in X} C_e + \max_{S \in U} \min_{Y \in \Phi^k_{\hat{X}}} \sum_{e \in Y} c^S_e \right). \quad (3)
\]

If \(C_e = 0\) for each \(e \in E\) and \(k = 0\), then Rob Rec ST is equivalent to the following min-max spanning tree problem, examined in [2, 13, 12], in which we seek a spanning tree that minimizes the largest cost over all scenarios, that is

\[
\text{Min-Max ST} : \min_{X \in \Phi} \max_{S \in U} \sum_{e \in X} c^S_e. \quad (4)
\]

If \(C_e = 0\) for each \(e \in E\) and \(k = n - 1\), then Rob Rec ST becomes the following adversarial problem [17] in which an adversary wants to find a scenario which leads to the greatest increase in the cost of the minimum spanning tree:

\[
\text{Adv ST} : \max_{S \in U} \min_{Y \in \Phi} \sum_{e \in Y} c^S_e. \quad (5)
\]

We now briefly recall the known complexity results on Rob Rec ST. It turns out that its computational complexity highly relies on a way of defining the scenario set \(U\). There are two popular methods of representing \(U\), namely the discrete and interval uncertainty representations. For the discrete uncertainty representation (see, e.g., [13]), scenario set,
denoted by $U^D$, contains $K$ explicitly listed scenarios, i.e. $U^D = \{S_1, S_2, \ldots, S_K\}$. In this case, the Rob Rec ST problem is known to be NP-hard for $K = 2$ and any constant $k$ \cite{11}. Furthermore, it becomes strongly NP-hard and not at all approximable when both $K$ and $k$ are part of input \cite{11}. It is worthwhile to mention that MIN-MAX ST is NP hard even when $K = 2$ and becomes strongly NP-hard and not approximable within $O(\log^{1-\epsilon} n)$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME}(n^{\text{polylog} n})$, when $K$ is a part of input \cite{13, 12}. It admits an FPTAS, when $K$ is a constant \cite{2} and it is approximable within $O(\log^2 n)$ \cite{12}, when $K$ is a part of input. The ADV ST problem, under scenario set $U^D$, is polynomially solvable, since it boils down to solving $K$ traditional minimum spanning tree problems.

For the interval uncertainty representation, which is considered in this paper, one assumes that the second stage cost of each edge $e \in E$ is known to belong to the closed interval $[c_e, c_e + d_e]$, where $c_e$ is a nominal cost of $e \in E$ and $d_e \geq 0$ is the maximum deviation of the cost of $e$ from its nominal value. In the traditional case $U$, denoted by $U^I$, is the Cartesian product of all these intervals \cite{13}, i.e. $U^I = \{S = (c^S_e)_{e \in E} : c^S_e \in [c_e, c_e + d_e], e \in E\}$. \cite{6}

In \cite{4} a polynomial algorithm for the recoverable robust matroid basis problem under scenario set $U^I$ was constructed, provided that the recovery parameter $k$ is constant. In consequence, ROB Rec ST under $U^I$ is also polynomially solvable for constant $k$. Unfortunately, the algorithm proposed in \cite{4} is exponential in $k$. Interestingly, the corresponding recoverable robust version of the shortest path problem ($\Phi$ is replaced with the set of all $s - t$ paths in $G$) has been proven to be strongly NP-hard and not at all approximable even if $k = 2$ \cite{5}. It has been recently shown in \cite{10} that ROB Rec ST under $U^I$ is polynomially solvable when $k$ is a part of input. In order to prove this result, a technique called the iterative relaxation of a linear programming formulation, whose framework was described in \cite{14}, has been applied. This technique, however, does not imply directly a polynomial time combinatorial algorithm for ROB Rec ST.

In \cite{3} a popular and commonly used modification of the scenario set $U^I$ has been proposed. The new scenario set, denoted as $U^I_1(\Gamma)$, is a subset of $U^I$ such that under each scenario in $U^I_1(\Gamma)$, the costs of at most $\Gamma$ edges are greater than their nominal values $c_e$, where $\Gamma$ is assumed to be a fixed integer in $[0, m]$. Scenario set $U^I_1(\Gamma)$ is formally defined as follows:

$$U^I_1(\Gamma) = \{S = (c^S_e)_{e \in E} : c^S_e \in [c_e, c_e + \delta_e d_e], \delta_e \in \{0, 1\}, e \in E, \sum_{e \in E} \delta_e \leq \Gamma\}. \quad (7)$$

The parameter $\Gamma$ allows us to model the degree of uncertainty. When $\Gamma = 0$, then we get REC ST (ROB Rec ST with one scenario $S = (c_e)_{e \in E}$). On the other hand, when $\Gamma = m$, then we get ROB Rec ST under the traditional interval uncertainty $U^I$. It turns out that the ADV ST problem under $U^I_1(\Gamma)$ is strongly NP-hard (it is equivalent to the problem of finding $\Gamma$ most vital edges) \cite{17, 16, 9}. Consequently, the more general ROB Rec ST problem is also strongly NP-hard. Interestingly, the corresponding MIN-MAX ST problem with $U^I_1(\Gamma)$ is polynomially solvable \cite{3}.

Yet another interesting way of defining scenario set, which allows us to control the amount of uncertainty, is called the scenario set with a budget constraint (see, e.g., \cite{17}). This scenario set, denoted as $U^I_2(\Gamma)$, is defined as follows:

$$U^I_2(\Gamma) = \{S = (c^S_e)_{e \in E} : c^S_e = c_e + \delta_e, \delta_e \in [0, d_e], e \in E, \sum_{e \in E} \delta_e \leq \Gamma\}. \quad (8)$$
where $\Gamma \in \mathbb{R}_+^+$ is a fixed parameter that can be seen as a budget of an adversary, and represents the maximum total increase of the edge costs from their nominal values. Obviously, if $\Gamma$ is sufficiently large, then $U^I_2(\Gamma)$ reduces to the traditional interval uncertainty representation $U^I$. The computational complexity of Rob Rec ST for scenario set $U^I_2$ is still open. We only know that its special cases, namely Min-Max ST and Adv ST, are polynomially solvable [17].

In this paper we will construct a polynomial combinatorial algorithm for Rec ST. We will apply this algorithm for solving Rob Rec ST under scenario set $U^I_2$ in polynomial time. Moreover, under some assumptions about the uncertainty intervals, which are not particularly restrictive, we will show how the algorithm for Rec ST can be used to obtain several approximation results for Rob Rec ST, under scenario sets $U^I_1(\Gamma)$ and $U^I_2(\Gamma)$. These approximation algorithms can be useful, since no exact algorithms are known for Rob Rec ST under $U^I_1(\Gamma)$ and $U^I_2(\Gamma)$. This paper is organized as follows. Section 2 contains the main result of this paper – a polynomial time combinatorial algorithm for Rec ST. Section 3 discusses Rob Rec ST under the interval uncertainty representations $U^I, U^I_1(\Gamma)$, and $U^I_2(\Gamma)$.

2 The recoverable spanning tree problem

In this section we construct a polynomial time, combinatorial algorithm for Rec ST. Since $|X| = n - 1$ for each $X \in \Phi$, Rec ST (see (1)) is equivalent to the following mathematical programming problem:

$$\min \sum_{e \in X} C_e + \sum_{e \in Y} c_e$$

s.t. $|X \cap Y| \geq L$, $X, Y \in \Phi$, $(9)$

where $L = n - 1 - k$. Problem (9) can be expressed as the following MIP model:

$$Opt = \min \sum_{e \in E} C_e x_e + \sum_{e \in E} c_e y_e$$

s.t. $x_e = n - 1$, $\forall e \in E$ $(10)$

$$\sum_{e \in E(U)} x_e \leq |U| - 1, \quad \forall U \subset V,$$ $(11)$

$$\sum_{e \in E(U)} y_e = n - 1,$$ $(12)$

$$\sum_{e \in E(U)} y_e \leq |U| - 1, \quad \forall U \subset V,$$ $(13)$

$$x_e - z_e \geq 0, \quad \forall e \in E,$$ $(14)$

$$y_e - z_e \geq 0, \quad \forall e \in E,$$ $(15)$

$$\sum_{e \in E} z_e \geq L,$$ $(16)$

$$x_e, y_e, z_e \geq 0, \text{ integer} \quad \forall e \in E,$$ $(17)$

where $E(U)$ stands for the set of edges that have both endpoints in $U \subseteq V$. We first apply the Lagrangian relaxation (see, e.g., [1]) to (10)-(18) by relaxing the cardinality constraint (17).
with a nonnegative multiplier $\theta$. We also relax the integrality constraints \((\ref{18})\). We thus get the following linear program (with the corresponding dual variables which will be used later):

$$\phi(\theta) = \min \sum_{e \in E} C_e x_e + \sum_{e \in E} c_e y_e - \theta \sum_{e \in E} z_e + \theta L \tag{19}$$

s.t. \(\sum_{e \in E} x_e = n - 1\), \([\mu]\),

\[- \sum_{e \in E(U)} x_e \geq -(|U| - 1), \quad \forall U \subset V, \quad [w_U],\]

\[\sum_{e \in E} y_e = n - 1, \quad [v],\]

\[- \sum_{e \in E(U)} y_e \geq -(|U| - 1), \quad \forall U \subset V, \quad [v_U],\]

\[x_e - z_e \geq 0, \quad \forall e \in E, \quad [\alpha_e],\]

\[y_e - z_e \geq 0, \quad \forall e \in E, \quad [\beta_e],\]

\[x_e, y_e, z_e \geq 0, \quad \forall e \in E.\]

For any $\theta \geq 0$, the Lagrangian function $\phi(\theta)$ is a lower bound on $Opt$. It is well-known that $\phi(\theta)$ is concave and piecewise linear. By the optimality test (see, e.g., \([1]\)), we obtain the following theorem:

**Theorem 1.** Let \((x_e, y_e, z_e)_{e \in E}\) be an optimal solution to \((\ref{19})\) for some $\theta \geq 0$, feasible to \((\ref{10})-(\ref{18})\) and satisfying the complementary slackness condition $\theta(\sum_{e \in E} z_e - L) = 0$. Then \((x_e, y_e, z_e)_{e \in E}\) is optimal to \((\ref{10})-(\ref{18})\).

Let \((X, Y)\), \(X, Y \in \Phi\), be a pair of spanning trees of $G$ (a pair for short). This pair corresponds to a feasible $0-1$ solution to \((\ref{19})\), defined as follows: $x_e = 1$ for $e \in X$, $y_e = 1$ for $e \in Y$, and $z_e = 1$ for $e \in X \cap Y$; the values of the remaining variables are set to 0. From now on, by a pair \((X, Y)\) we also mean a feasible solution to \((\ref{19})\) defined as above. Given a pair \((X, Y)\) with the corresponding solution \((x_e, y_e, z_e)_{e \in E}\), let us define the partition \((E_X, E_Y, E_Z, E_W)\) of the set of the edges $E$ in the following way: $E_X = \{e \in E : x_e = 1, y_e = 0\}$, $E_Y = \{e \in E : y_e = 1, x_e = 0\}$, $E_Z = \{e \in E : x_e = 1, y_e = 1\}$ and $E_W = \{e \in E : x_e = 0, y_e = 0\}$. It holds $X = E_X \cup E_Z$, $Y = E_Y \cup E_Z$ and $E_Z = E_X \cap E_Y$. Our goal is to establish sufficient optimality conditions for a given pair \((X, Y)\) in the problem \((\ref{19})\). The dual to \((\ref{19})\) has the following form:

$$\phi^D(\theta) = \max - \sum_{U \subset V} (|U| - 1) w_U + (|V| - 1) \mu - \sum_{U \subset V} (|U| - 1) v_U + (|V| - 1) \nu + \theta L \tag{20}$$

s.t. \[- \sum_{\{U : e \in E(U)\}} w_U + \mu \leq C_e - \alpha_e, \quad \forall e \in E,\]

\[- \sum_{\{U : e \in E(U)\}} v_U + \nu \leq c_e - \beta_e, \quad \forall e \in E,\]

\[\alpha_e + \beta_e \geq \theta, \quad \forall e \in E,\]

\[w_U, v_U \geq 0, \quad U \subset V,\]

\[\alpha_e, \beta_e \geq 0, \quad \forall e \in E.\]
Lemma 1. The dual problem \((20)\) can be rewritten as follows:

$$\max_{\{\alpha_e \geq 0, \beta_e \geq 0, \alpha_e + \beta_e \geq \theta, e \in E\}} \left( \min_{X \in \Phi} \sum_{e \in X} (C_e - \alpha_e) + \min_{Y \in \Phi} \sum_{e \in Y} (c_e - \beta_e) \right) + \theta L.$$  

Proof. Fix some \(\alpha_e\) and \(\beta_e\) such that \(\alpha_e + \beta_e \geq \theta\) for each \(e \in E\) in \((20)\). For these constant values of \(\alpha_e\) and \(\beta_e\), \(e \in E\), using the dual to \((20)\), we arrive to \(\min_{X \in \Phi} \sum_{e \in X} (C_e - \alpha_e) + \min_{Y \in \Phi} \sum_{e \in Y} (c_e - \beta_e) + \theta L\) and the lemma follows. \(\square\)

Lemma [1] allows us to establish the following result:

Theorem 2 (Sufficient pair optimality conditions). A pair \((X, Y)\) is optimal to \((19)\) for a fixed \(\theta \geq 0\) if there exist \(\alpha_e \geq 0, \beta_e \geq 0\) such that \(\alpha_e + \beta_e = \theta\) for each \(e \in E\) and

(i) \(X\) is a minimum spanning tree for the costs \(C_e - \alpha_e\), \(Y\) is a minimum spanning tree for the costs \(c_e - \beta_e\),

(ii) \(\alpha_e = 0\) for each \(e \in E_X\), \(\beta_e = 0\) for each \(e \in E_Y\).

Proof. By Lemma 1, \(X, Y, \alpha_e\) and \(\beta_e\), \(e \in E\), correspond to a feasible solution to the dual problem \((20)\). The pair \((X,Y)\) is feasible in \((19)\). It thus suffices to show that the objective values of \((20)\) for \(X, Y, \alpha_e, \beta_e\) and \((19)\) for \(X, Y\) are the same. Indeed

$$\sum_{e \in X} (C_e - \alpha_e) + \sum_{e \in Y} (c_e - \beta_e) + \theta L = \sum_{e \in E_X} C_e + \sum_{e \in E_Y} c_e + \sum_{e \in E_Z} (C_e + c_e - \theta) + \theta L$$

$$= \sum_{e \in E_X} C_e + \sum_{e \in E_Z} C_e + \sum_{e \in E_Y} c_e + \sum_{e \in E_Z} c_e - \theta|E_Z| + \theta L$$

$$= \sum_{e \in E_X} C_e + \sum_{e \in E_Y} c_e - \theta|E_Z| + \theta L.$$  

The Weak Duality Theorem implies the optimality of \((X,Y)\) in \((19)\) for a fixed \(\theta \geq 0\). \(\square\)

Lemma 2. A pair \((X, Y)\), which satisfies the sufficient pair optimality conditions for \(\theta = 0\), can be computed in polynomial time.

Proof. Let \(X\) be a minimum spanning tree for the costs \(C_e\) and \(Y\) be a minimum spanning tree for the costs \(c_e\), \(e \in E\). Since \(\theta = 0\), we set \(\alpha_e = 0, \beta_e = 0\) for each \(e \in E\). It is clear that \((X,Y)\) satisfies the sufficient pair optimality conditions. \(\square\)

Assume that \((X,Y)\) satisfies the sufficient pair optimality conditions for some \(\theta \geq 0\). If, for this pair, \(|E_Z| \geq L\) and \(\theta(|E_Z| - L) = 0\), then we are done, because by Theorem 1 the pair \((X,Y)\) is optimal to \((10)-(18)\). Suppose that \(|E_Z| < L\) \(((X,Y)\) is not feasible \((10)-(18)\). We will now show a polynomial time procedure for finding a new pair \((X',Y')\), which satisfies the sufficient pair optimality conditions and \(|E_Z'| = |E_Z| + 1\). This implies a polynomial time algorithm for the problem \((10)-(18)\), since it is enough to start with a pair satisfying the sufficient pair optimality conditions for \(\theta = 0\) (see Lemma 2) and repeat the procedure at most \(L\) times, i.e. until \(|E_Z'| = L\).

Given a spanning tree \(T\) in \(G = (V,E)\) and edge \(e = \{k,l\} \notin T\), let us denote by \(P_T(e)\) the unique path in \(T\) connecting nodes \(k\) and \(l\). Consider a pair \((X,Y)\) that satisfies the sufficient pair optimality conditions for some fixed \(\theta \geq 0\). Set \(C^*_e = C_e - \alpha_e\) and \(c^*_e = c_e - \beta_e\) for every

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The second step. Observe that each arc \((v, X)\) on \(X\) is easily seen that if \((v, X)\) is not reachable from any node \(v\) and \(c_e = c_f\), we say that \(v_e \in V^A\) is admissible if \(e \in E_Y\), or \(v_e\) is reachable from a node \(v_y \in V^A\), such that \(g \in E_Y\), by a directed path in \(G^A\). In the second step we remove from \(G^A\) all nodes which are not admissible, together with their incident arcs. An example of an admissible graph is shown in Figure 1. Each node of this admissible graph is reachable from some node \(v_e, e \in E_Y\). Note that the arcs \((v_{e7}, v_{e9})\) and \((v_{e7}, v_{e10})\) are not present in \(G^A\), because \(e_7\) is not reachable from any node \(v_e, e \in E_Y\). These arcs are removed from \(G^A\) in the second step. Observe that each arc \((v_e, v_j)\) in the admissible graph represents a move on \(X\) or \(Y\) (the corresponding label is shown in the admissible graph, see Figure 1b), namely \(X' = X \cup \{e_i\} \\setminus \{e_j\}\) or \(Y' = Y \cup \{e_j\} \\setminus \{e_i\}\), which creates new spanning tree \(X'\) or \(Y'\). Notice also that the cost of \(X'\) is the same as \(X\) and the cost of \(Y'\) is the same as \(Y\). It is easily seen that if \((v_{e1}, v_{e2})\) belongs to \(E^A\), then \(e_i \notin E_X\) and \(e_j \notin E_Y\). Furthermore \(e_i\) and \(e_j\) cannot both belong to \(E_W\) or \(E_Z\).

![Diagram](image_url)

Figure 1: (a) A pair \((X, Y)\) such that \(X = \{e_2, e_3, e_4, e_6, e_{10}\}\) and \(Y = \{e_1, e_3, e_5, e_9, e_{10}\}\). (b) The admissible graph \(G^A\) for \((X, Y)\).

We will consider two cases: \(E_X \cap \{e \in E : v_e \in V^A\} \neq \emptyset\) and \(E_X \cap \{e \in E : v_e \in V^A\} = \emptyset\). The first case means that there is a directed path from \(v_e, e \in E_Y\), to a node \(v_f, f \in E_X\), in the admissible graph \(G^A\) and the second case is just opposite. We will show that in the first case it is possible to find a new pair \((X', Y')\) which satisfies the sufficient pair optimality conditions and \(|E_{Z'}| = |E_Z| + 1\). The idea will be to perform the sequence of moves on \(X\) and \(Y\), indicated by the arcs on some suitably chosen path from \(v_e, e \in E_Y\), to \(v_f, f \in E_X\) in the admissible graph \(G^A\). Let us formally handle this case.
Lemma 3. If \( E_X \cap \{ e \in E : v_e \in V^A \} \neq \emptyset \), then there exists a pair \((X', Y')\) with \(|E_{Z'}| = |E_Z| + 1\), which satisfies the sufficient pair optimality conditions for \( \theta \).

Proof. We will show that if the assumption of the lemma holds, then it is possible to transform the pair \((X, Y)\) into a pair \((X', Y')\) such that \(|E_{Z'}| = |E_Z| + 1\). We begin by introducing the notion of the cycle graph \( G(T) = (V^T, A^T) \) corresponding to a given spanning tree \( T \) of graph \( G = (V, A) \). We build \( G(T) \) as follows: we associate with each edge \( e \in E \) a node \( v_e \) and include it to \( V^T \), \(|E| = |V^T|\); then we add arc \((v_e, v_f)\) to \( A^T \) if \( e \notin T \) and \( f \in P_T(e) \). An example is shown in Figure 2.

![Figure 2](image)

Figure 2: (a) A graph \( G \) with a spanning tree \( T \) (the solid lines). (b) The cycle graph \( G(T) \).

The following claim shows the usefulness of the notion of the cycle graph (for its proof see Appendix A).

Claim 1. Given a spanning tree \( T \) of \( G \), let \( \mathcal{F} = \{(v_{e_1}, v_{f_1}), (v_{e_2}, v_{f_2}), \ldots, (v_{e_\ell}, v_{f_\ell})\} \) be a subset of arcs of \( G(T) \), where all \( v_{e_i} \) and \( v_{f_i} \) (resp. \( e_i \) and \( f_i \)), \( i \in [\ell] \), are distinct. If \( T' = T \cup \{e_1, \ldots, e_\ell\} \setminus \{f_1, \ldots, f_\ell\} \) is not a spanning tree, then \( G(T) \) contains a subgraph depicted in Figure 3, where \( \{j_1, \ldots, j_k\} \subseteq [\ell] \).

![Figure 3](image)

Figure 3: A subgraph of \( G(T) \) from Claim 1

Let us illustrate Claim 1 by using the sample graph in Figure 2. Suppose that \( \mathcal{F} = \{(v_{e_1}, v_{f_1}), (v_{e_2}, v_{f_2}), (v_{e_3}, v_{f_3})\} \). Then \( T' = T \cup \{e_1, e_2, e_3\} \setminus \{f_1, f_2, f_3\} \) is not a spanning tree and \( G(T) \) contains the subgraph composed of the arcs (see Figure 2):

\[
(v_{e_1}, v_{f_2}), (v_{e_2}, v_{f_2}), (v_{e_2}, v_{f_3}), (v_{e_3}, v_{f_3}), (v_{e_3}, v_{f_2}), (v_{e_1}, v_{f_3}).
\]

After this preliminary step, we can now return to the main proof. If \( E_X \cap \{ e \in E : v_e \in V^A \} \neq \emptyset \), then, by the construction of the admissible graph, there exists a directed path in \( G^A \) from a node \( v_e, e \in E_Y \), to a node \( v_f, f \in E_X \). Let \( P \) be a shortest such a path, i.e. the path consisting of the fewest number of arcs, called augmenting path. We need to consider the following cases:
1. If the augmenting path $P$ is of the form:

$$E_Y \quad E_X$$
$$v_e \quad \rightarrow \quad v_f$$

then $X' = X \cup \{e\} \setminus \{f\}$ if $C_e^{*} = C_f^{*}$ (resp. $Y' = Y \cup \{f\} \setminus \{e\}$ if $c_e^{*} = c_f^{*}$) is an updated spanning tree of $G$. Obviously $|X' \cap Y| = |E_Z| + 1$ (resp. $|X \cap Y'| = |E_Z| + 1$), $X'$ is a minimum spanning tree with respect to $C_e^{*}$ (resp. $Y'$ is a minimum spanning tree with respect to $c_e^{*}$) and the new pair $(X', Y')$ (resp. $(X, Y')$) satisfies the sufficient pair optimality conditions. An example can be seen in Figure 5. There is a path $v_{e_1} \rightarrow v_{e_2}$ in the admissible graph. In this case, this path represents two possible moves $X' = X \cup \{e_1\} \setminus \{e_2\}$ or $Y' = Y \cup \{e_2\} \setminus \{e_1\}$. Choosing one of them gives us a desired pair $(X', Y')$ or $(Y', X)$.

2. If the augmenting path $P$ is of the form:

$$E_Y \quad E_Z \quad E_W \quad E_Z \quad E_W \quad E_Z \quad E_W \quad E_Z \quad E_X$$
$$v_{e_1} \quad \rightarrow \quad v_{e_2} \quad \rightarrow \quad v_{e_3} \quad \rightarrow \quad v_{e_4} \quad \rightarrow \quad \cdots \quad \rightarrow \quad v_{e_\ell} \quad \rightarrow \quad v_{f_1} \quad \rightarrow \quad v_{f_\ell}\quad \rightarrow \quad v_{e_\ell+1}$$

then $X' = X \cup \{e_1, \ldots, e_\ell\} \setminus \{f_1, \ldots, f_\ell\}$ and $Y' = Y \cup \{e_2, \ldots, e_{\ell+1}\} \setminus \{f_1, \ldots, f_\ell\}$ for case (a), and $X' = X \cup \{e_2, e_3, e_4\} \setminus \{f_1, f_2, f_3, f_4\}$ and $Y' = Y \cup \{e_3, e_4\} \setminus \{f_1, f_2, f_3, f_4\}$. An example of the case (a) is shown in Figure 4. Thus $X' = X \cup \{e_1, e_2, e_3, e_4\} \setminus \{f_1, f_2, f_3, f_4\}$ and $Y' = Y \cup \{e_3, e_4\} \setminus \{f_1, f_2, f_3, f_4\}$. An example of the case (b) is shown in Figure 5. In this example $X'$ is the same as in the previous case and $Y' = Y \cup \{e_2, e_3, e_4\} \setminus \{f_1, f_2, f_3\}$. In both cases $|E_{Z'}| = |E_Z| + 1$ holds. We now have to show that the resulting pair $(X', Y')$ is indeed a pair of spanning trees. Suppose that $X'$ is not a spanning tree. Observe that $(v_{e_1}, v_{f_1}), \ldots, (v_{e_\ell}, v_{f_\ell})$ are arcs of the cycle graph $G(X)$. Thus by Claim 4 the cycle graph $G(X)$ must contain a subgraph depicted in Figure 3 where $\{j_1, \ldots, j_k\} \subseteq \{\ell\}$. A trivial verification shows that all edges $e_i, f_i, i \in \{j_1, \ldots, j_k\}$ must have the same costs $C_{e_i}^{*}, C_{f_i}^{*}$. Indeed, if some costs are different, then there exists an edge exchange which decreases the cost of $X$. This contradicts our assumption that $X$ is a minimum spanning tree with respect to $C_e^{*}$. Finally, there must be an arc $(v_{e_{i'}}, v_{f_{i''}})$ in the subgraph such that $i' < i''$. Since the costs $C_{e_{i'}}^{*}$ and $C_{f_{i''}}^{*}$ are equal, $(v_{e_{i'}}, v_{f_{i''}})$ is present in the admissible graph $G^4$. This leads to a contradiction with our assumption that $P$ is a shortest path. The proof that $Y'$ is spanning tree is similar. The cost of $X'$ for $C_e^{*}$ is the same as $X$ and the cost of $Y'$ for $c_e^{*}$ is the same as $Y$. In consequence, $X'$ and $Y'$ are optimal for the costs $C_e^{*}$ and $c_e^{*}$, respectively. Furthermore, $E_{X'} \subseteq E_X$ and $E_{Y'} \subseteq E_Y$, so $(X', Y')$ satisfies the sufficient pair optimality conditions.

3. If the augmenting path $P$ is of the form:

$$E_Y \quad E_W \quad E_Z \quad E_W \quad E_Z \quad E_W \quad E_Z \quad E_W \quad E_X$$
$$v_{e_1} \quad \rightarrow \quad v_{f_1} \quad \rightarrow \quad v_{e_2} \quad \rightarrow \quad v_{f_2} \quad \rightarrow \quad v_{e_3} \quad \rightarrow \quad v_{f_3} \quad \rightarrow \quad \cdots \quad \rightarrow \quad v_{e_\ell} \quad \rightarrow \quad v_{f_\ell} \quad \rightarrow \quad v_{e_{\ell+1}}$$

The proof is similar to the previous case.
then $X' = X \cup \{f_1, \ldots, f_\ell\} \setminus \{e_2, \ldots, e_{\ell+1}\}$ for the case (a), $X' = X \cup \{f_1, \ldots, f_{\ell-1}\} \setminus \{e_2, \ldots, e_{\ell}\}$ for the case (b) and $Y' = Y \cup \{f_1, \ldots, f_\ell\} \setminus \{e_1, \ldots, e_\ell\}$ are spanning trees in $G$ such that $|E_{Z'}| = |X' \cap Y'| = |E_Z| + 1$. An example of the case (a) is shown in Figure 6. Thus $X' = X \cup \{f_1, f_2, f_3, f_4\} \setminus \{e_2, e_3, e_4, e_5\}$ and $Y' = Y \cup \{f_1, f_2, f_3, f_4\} \setminus \{e_1, e_2, e_3, e_4\}$. An example for the case (b) is shown in Figure 7. The spanning tree $Y'$ is the same as in the previous case and $X' = X \cup \{f_1, f_2, f_3\} \setminus \{e_2, e_3, e_4\}$. The proof that $X'$ and $Y'$ are spanning trees follows by the same arguments as for the symmetric case described in case 2. Again, the trees $X'$ and $Y'$ are optimal for the costs $C^*_e$ and $c^*_e$, respectively, $E_{X'} \subseteq E_X$, $E_{Y'} \subseteq E_Y$, so $(X', Y')$ satisfies the sufficient pair optimality conditions.

We now turn to the case $E_X \cap \{e \in E : v_e \in V^A\} = \emptyset$. Fix $\delta > 0$ (the precise value of $\delta$ will be specified later) and set:

\begin{align*}
C_e(\delta) &= C^*_e - \delta, & c_e(\delta) &= c^*_e, & v_e \in V^A, & (22a) \\
C_e(\delta) &= C^*_e, & c_e(\delta) &= c^*_e - \delta, & v_e \notin V^A. & (22b)
\end{align*}
Lemma 4. There exists a sufficiently small $\delta > 0$ such that the costs $C_e(\delta)$ and $c_e(\delta)$ satisfy the tree optimality conditions for $X$ and $Y$, respectively, i.e:

$$\begin{align*}
&\text{for every } e \notin X, C_e(\delta) \geq C_f(\delta) \quad \text{for every } f \in P_X(e), \quad (23a) \\
&\text{for every } e \notin Y, c_e(\delta) \geq c_f(\delta) \quad \text{for every } f \in P_Y(e). \quad (23b)
\end{align*}$$

Proof. If $C_e^* > C_f^*$ (resp. $c_e^* > c_f^*$), $e \notin X, f \in P_X(e)$ (resp. $e \notin Y, f \in P_Y(e)$), then there is $\delta > 0$, such that after setting the new costs (22) the inequality $C_e(\delta) \geq C_f(\delta)$ (resp. $c_e(\delta) \geq c_f(\delta)$) holds. Hence, one can choose a sufficiently small $\delta > 0$ such that after setting the new costs (22), all the strong inequalities are not violated. Therefore, for such a chosen $\delta$ it remains to show that all originally tight inequalities in (21) are preserved for the new costs. Consider a tight inequality of the form:

$$C_e^* = C_f^*, \ e \notin X, f \in P_X(e). \quad (24)$$

On the contrary, suppose that $C_e(\delta) < C_f(\delta)$. This is only possible when $C_e(\delta) = C_e^* - \delta$ and $C_f(\delta) = C_f^*$. Hence and from the construction of the new costs, we have $v_f \notin V^A$ (see (22b)).
and \( v_e \in V^A \) (see (22a)). By (24), we obtain \((v_e, v_f) \in E^A\). Thus \( v_f \in V^A \), a contradiction. Consider a tight inequality of the form:

\[
c_e^* = c_f^*, \ e \notin Y, \ f \in P_Y(e).
\]

On the contrary, suppose that \( c_e(\delta) < c_f(\delta) \). This is only possible when \( c_e(\delta) = c_e^* - \delta \) and \( c_f(\delta) = c_f^* \). Thus we deduce that \( v_e \notin V^A \) and \( v_f \in V^A \) (see (22)). From (25), it follows that \((v_f, v_e) \in E^A\) and so \( v_e \in V^A \), a contradiction.

We are now ready to give the precise value of \( \alpha^* \). We do this by increasing the value of \( \delta \) until some inequalities, originally not tight in (21), become tight. Namely, let \( \delta^* > 0 \) be the smallest value of \( \delta \) for which an inequality originally not tight becomes tight. Obviously, it occurs when \( C_e^* - \delta^* = C_f^* \) for \( e \notin X, \ f \in P_X(e) \) or \( C_f^* - \delta^* = C_e^* \) for \( f \notin Y, \ e \in P_Y(f) \).

By (22), \( v_e \in V^A \) and \( v_f \notin V^A \). Accordingly, if \( \delta = \delta^* \), then one arc (or more arcs) are added to \( G^A \). Observe also that no arc can be removed from \( G^A \) - the admissibility of nodes remains unchanged. It follows from the fact that each tight inequality for \( v_e \in V^A \) and \( v_f \in V^A \) is still tight. This leads to the lemma that handles the case \( E_X \cap \{e \in E : v_e \in V^A\} = \emptyset \).

Lemma 5. If \( E_X \cap \{e \in E : v_e \in V^A\} = \emptyset \), then \((X, Y)\) satisfies the sufficient pair optimality conditions for each \( \theta' \in [0, \theta + \delta^*] \).

Proof. Set \( \theta' = \theta + \delta, \ \delta \in [0, \delta^*] \). Lemma 4 implies that \( X \) is optimal for \( c_e(\delta) \) and \( Y \) is optimal for \( c_e(\delta) \). From (22) and the definition of the costs \( C_e^* \) and \( c_e^* \), it follows that \( C_e(\delta) = C_e - \alpha_e' \) and \( c_e(\delta) = c_e - \beta_e' \), where \( \alpha_e' = \alpha_e + \delta \) and \( \beta_e' = \beta_e + \delta \) for each \( v_e \in V^A \).

\( \alpha_e' = \alpha_e \) and \( \beta_e' = \beta_e + \delta \) for each \( v_e \notin V^A \). Notice that \( \alpha_e' + \beta_e' = \alpha_e + \beta_e + \delta = \theta + \delta = \theta' \) for each \( e \in E \). By (22), \( c_e(\delta) = c_e \) for each \( e \in E_Y \), and thus \( \beta_e = 0 \) for each \( e \in E_Y \). Since \( E_X \cap \{e \in E : v_e \in V^A\} = \emptyset \), \( C_e(\delta) = C_e^* \) holds for each \( e \in E_X \), and so \( \alpha_e = 0 \) for each \( e \in E_X \). We thus have shown that there exist \( \alpha_e', \beta_e' \geq 0 \) such that \( \alpha_e' + \beta_e' = \theta' \) for each \( e \in E \) satisfying the conditions (i) and (ii) in Theorem 2, which completes the proof.

We now describe a polynomial procedure that for a given pair \((X, Y)\) satisfying the sufficient pair optimality conditions for some \( \theta \geq 0 \), finds a new pair of spanning trees \((X', Y')\) which also satisfies the sufficient pair optimality conditions with \( |E_Z| = |E_Z| + 1 \). We start by building the admissible graph \( G^A = (V^A, E^A) \) for \((X, Y)\). If this graph contains an augmenting path, then by Lemma 3 we are done, i.e. one can compute a pair of spanning trees \((X', Y')\) with \( |E_Z'| = |E_Z| + 1 \) which satisfies the sufficient pair optimality conditions for \( \theta \). Otherwise, we determine \( \delta^* \) and modify the costs by using (22). Lemma 5 shows that \((X, Y)\) satisfies the sufficient pair optimality conditions for \( \theta + \delta^* \). For \( \delta^* \) some new arcs can be added to the admissible graph \( G^A \), of course, if their ending nodes are admissible (all the previous arcs must be still present in \( G^A \)). Thus \( G^A \) is updated and we set \( C_e^* := C_e(\delta^*), \ c_e^*: = c_e(\delta^*) \) for each \( e \in E \), and \( \theta := \theta + \delta^* \). We repeat this until there is an augmenting path in \( G^A = (V^A, E^A) \). Note that such a path must appear after at most \( m = |E| \) iterations.

Sample computations are shown in Figure 8. We start with the pair \((X, Y)\), where \( X = \{e_2, e_4, e_5, e_6, e_9, e_{10}\} \) and \( Y = \{e_2, e_3, e_5, e_8, e_9, e_{11}\} \), which satisfies the sufficient pair optimality conditions for \( \theta = 0 \) (see Figure 8a). Observe that in this case it is enough to check that \( X \) is optimal for the costs \( C_e \) and \( Y \) is optimal for the costs \( c_e \). For \( \theta = 0 \), the admissible graph does not contain any augmenting path. We thus have to modify the costs \( C_e^* \) and \( c_e^* \), according to (22). For \( \delta^* = 1 \), a new inequality becomes tight and one arc is
Theorem 3. The \( \text{Rec ST} \) problem is solvable in \( O(Lm^2n) \) time, where \( L = n - 1 - k \).

3 The recoverable robust spanning tree problem

In this section we are concerned with the \( \text{Rob Rec ST} \) problem under the interval uncertainty representation, i.e. for the scenario sets \( \mathcal{U}' \), \( \mathcal{U}_1^t(\Gamma) \), and \( \mathcal{U}_2^t(\Gamma) \). Using the polynomial combinatorial algorithm for \( \text{Rec ST} \) constructed in Section 2, we will provide a polynomial algorithm for \( \text{Rob Rec ST} \) under \( \mathcal{U}^t \) and some approximation algorithms for a wide class of problems.
ROB REC ST under \( \mathcal{U}_1^1(\Gamma) \) and \( \mathcal{U}_2^1(\Gamma) \) (some mild assumptions on the uncertainty intervals will be imposed). The idea will be to solve REC ST for a suitably chosen second stage costs.

Let \( F(X) = \sum_{e \in X} C_e + \max_{S \in \mathcal{U}_1^1} \min_{Y \in \Phi_X^1} \sum_{e \in Y} c_e^S \) and \( f(Y, S) = \sum_{e \in Y} c_e^S \). It is worth pointing out that under scenario sets \( \mathcal{U}_1^1 \) and \( \mathcal{U}_2^1(\Gamma) \), the value of \( F(X) \), for a given spanning tree \( X \), can be computed in polynomial time \([8, 17]\). On the other hand, computing \( F(X) \) under \( \mathcal{U}_1^1(\Gamma) \) turns out to be strongly NP-hard \([17, 9]\). Given scenario \( S = (c_e^S)_{e \in E} \), consider the following REC ST problem:

\[
\min_{X \in \Phi} \left( \sum_{e \in X} C_e + \min_{Y \in \Phi_X^e} f(X, S) \right). \tag{26}
\]

Problem \( (26) \) is equivalent to the formulation \( (11) \) for \( S = (c_e)_{e \in E} \) and it is polynomially solvable, according to the result obtained in Section 2. As in the previous section, we denote by pair \((X,Y)\) a solution to \( (26) \), where \( X \in \Phi \) and \( Y \in \Phi_X^k \). Given \( S \), we call \((X,Y)\) an optimal pair under \( S \) if \((X,Y)\) is an optimal solution to \( (26) \).

The ROB REC ST problem with scenario set \( \mathcal{U}_1^1 \) can be rewritten as follows:

\[
\min_{X \in \Phi} \left( \sum_{e \in X} C_e + \max_{S \in \mathcal{U}_1^1} \min_{Y \in \Phi_X^e} \sum_{e \in Y} c_e \right) = \min_{X \in \Phi} \left( \sum_{e \in X} C_e + \min_{Y \in \Phi_X^e} \sum_{e \in E} (c_e + d_e) \right). \tag{27}
\]

Thus \( (27) \) for \( S = (c_e + d_e)_{e \in E} \in \mathcal{U}_1^1 \). Hence and from Theorem 3, we immediately get the following theorem:

**Theorem 4.** For scenario set \( \mathcal{U}_1^1 \), the ROB REC ST problem is solvable in \( O((n - 1 - k)m^2 n) \) time.

We now address ROB REC ST under \( \mathcal{U}_1^1(\Gamma) \) and \( \mathcal{U}_2^1(\Gamma) \). Our basic assumption is that \( c_e \geq \alpha(c_e + d_e) \) for each \( e \in E \), where \( \alpha \in (0, 1] \) is a given constant. It is reasonable to assume that, for every \( e \in E \), the edge nominal cost \( c_e \) is positive (later we will show how to weaken this assumption), so the value of \( \alpha \) is well defined, and assume that \( c_e + d_e \) is at most \( 1/\alpha \) greater than \( c_e \) for each \( e \in E \), where \( 1/\alpha \) is not very large. Under this, not particularly restrictive assumption, we get the following approximation result:

**Lemma 6.** Suppose that \( c_e \geq \alpha(c_e + d_e) \) for each \( e \in E \), where \( \alpha \in (0, 1] \), and let \((\hat{X}, \hat{Y})\) be an optimal pair under \( \mathcal{S} = (c_e)_{e \in E} \). Then for the scenario sets \( \mathcal{U}_1^1(\Gamma) \) and \( \mathcal{U}_2^1(\Gamma) \) the inequality \( F(\hat{X}) \leq \frac{1}{\alpha} F(X) \) holds for any \( X \in \Phi \).

**Proof.** We give the proof only for the scenario set \( \mathcal{U}_1^1(\Gamma) \). The proof for \( \mathcal{U}_2^1(\Gamma) \) is the same. The following inequality is satisfied:

\[
F(X) = \sum_{e \in X} C_e + \max_{S \in \mathcal{U}_1^1} \min_{Y \in \Phi_X^k} f(Y, S) = \sum_{e \in X} C_e + f(Y^*, S^*) \geq \sum_{e \in X} C_e + f(Y^*, \mathcal{S}).
\]

Clearly, \((X, Y^*)\) is a feasible pair to \( (26) \) under \( \mathcal{S} \). From the definition of \((\hat{X}, \hat{Y})\) we get

\[
F(X) \geq \sum_{e \in \hat{X}} C_e + f(\hat{Y}, \mathcal{S}) = \sum_{e \in \hat{X}} C_e + \sum_{e \in \hat{Y}} c_e \geq \sum_{e \in \hat{X}} C_e + \sum_{e \in \hat{Y}} \alpha(c_e + d_e) = \sum_{e \in \hat{X}} C_e + \alpha \sum_{e \in \hat{Y}} f(\hat{Y}, \mathcal{S}), \tag{28}
\]
where $\bar{S} = (c_e + d_e)_{e \in E}$. Hence
\[
F(X) \geq \sum_{e \in \hat{X}} C_e + \alpha \max_{S \in U_2^l} f(\hat{Y}, S) \geq \sum_{e \in \hat{X}} C_e + \alpha \max_{S \in U_2^l} \min_{Y \in \Phi_X} f(Y, S) \\
\geq \alpha \left( \sum_{e \in \hat{X}} C_e + \max_{S \in U_2^l} \min_{Y \in \Phi_X} f(Y, S) \right) = \alpha F(\hat{X})
\]
and the lemma follows. \(\square\)

The assumption $c_e \geq \alpha(c_e + d_e)$ for each $e \in E$, in Lemma 6, can be weakened and, in consequence, the set of instances of the problem to which the algorithm applies can be extended. Indeed, from inequality (28) it follows that the bounds of the uncertainty intervals are only required to meet the condition $\sum_{e \in \hat{Y}} c_e \geq \alpha \sum_{e \in \hat{Y}} (c_e + d_e)$. Now $\sum_{e \in \hat{Y}} c_e$ is unlikely to be zero in practical applications.

We now focus on ROB Rec ST for $U_2^l(\Gamma)$. Define $D = \sum_{e \in E} d_e$ and suppose that $D > 0$ (if $D = 0$, then the problem is equivalent to Rec ST for the second stage costs $c_e$, $e \in E$). Consider scenario $S'$ under which $c_e^2 = \min\{c_e + d_e, c_e + \Gamma \frac{d_e}{D}\}$. Obviously, $S' \in U_2^l(\Gamma)$, since $\sum_{e \in E} \delta_e \leq \sum_{e \in E} \Gamma \frac{d_e}{D} \leq \Gamma$. The following theorem provides another approximation result for ROB Rec ST with scenario set $U_2^l(\Gamma)$ under some mild assumptions about the uncertainty intervals:

**Lemma 7.** Let $(\hat{X}, \hat{Y})$ be an optimal pair under $S'$. Then

(i) If $\Gamma \geq \beta D$, $\beta \in (0, 1]$, then $F(\hat{X}) \leq \frac{1}{\beta} F(X)$ for any $X \in \Phi$.

(ii) If $\Gamma \leq \gamma F(\hat{X})$, $\gamma \in [0, 1]$ then $F(\hat{X}) \leq \frac{1}{1-\gamma} F(X)$ for any $X \in \Phi$.

**Proof.** Since $S' \in U_2^l(\Gamma)$, we get
\[
F(X) = \sum_{e \in X} C_e + \max_{S \in U_2^l} \min_{Y \in \Phi_X} f(Y, S) \geq \sum_{e \in X} C_e + \min_{Y \in \Phi_X} f(Y, S') \tag{29}
\]
We first prove implication (i). By (29) and the definition of $(\hat{X}, \hat{Y})$, we obtain
\[
F(X) \geq \sum_{e \in \hat{X}} C_e + f(\hat{Y}, S') = \sum_{e \in \hat{X}} C_e + \sum_{e \in \hat{Y}} \min\{c_e + d_e, c_e + \Gamma \frac{d_e}{D}\} \\
\geq \sum_{e \in \hat{X}} C_e + \sum_{e \in \hat{Y}} \min\{c_e + d_e, c_e + \beta d_e\} = \sum_{e \in \hat{X}} C_e + \sum_{e \in \hat{Y}} (c_e + \beta d_e) \geq \sum_{e \in \hat{X}} C_e + \beta f(\hat{Y}, S),
\]
where $S = (c_e + d_e)_{e \in E}$. The rest of the proof is the same as in the proof of Lemma 6. We now prove implication (ii). By (29) and the definition of $(\hat{X}, \hat{Y})$, we have
\[
F(X) \geq \sum_{e \in \hat{X}} C_e + f(\hat{Y}, S') \geq \sum_{e \in \hat{X}} C_e + f(\hat{Y}, S) \geq \sum_{e \in \hat{X}} C_e + \max_{S \in U_2^l} f(\hat{Y}, S) - \Gamma \\
\geq \sum_{e \in \hat{X}} C_e + \min_{S \in U_2^l} \min_{Y \in \Phi_X} f(Y, S) - \Gamma = F(\hat{X}) - \Gamma.
\]
If $L \leq \gamma F(\hat{X})$. Then $F(X) \geq F(\hat{X}) - \gamma F(\hat{X}) = (1 - \gamma) F(\hat{X})$ and $F(\hat{X}) \leq \frac{1}{1-\gamma} F(X)$. \(\square\)
Note that the value of $F(\hat{X})$ under $U^I_2(\Gamma)$ can be computed in polynomial time [17]. In consequence, the constants $\beta$ and $\gamma$ can be efficiently determined for every particular instance of the problem. If $\alpha$, $\beta$ and $\gamma$ are the constants from Lemmas 6 and 7 then the following theorem summarizes the approximation results:

**Theorem 5.** Rob Rec ST is approximable within $\frac{1}{\alpha}$ under scenario set $U^I_1(\Gamma)$ and it is approximable within $\min\{\frac{1}{\beta}, \frac{1}{\alpha}, \frac{1}{1-\gamma}\}$ under scenario set $U^I_2(\Gamma)$.

### 4 Conclusions

In this paper we have studied the recoverable robust spanning tree problem (ROB Rec ST) under various interval uncertainty representations. The main result is the polynomial time combinatorial algorithm for the recoverable spanning tree. We have applied this algorithm for solving ROB Rec ST under the traditional uncertainty representation (see, e.g., [13]) in polynomial time. Moreover, we have used the algorithm for providing several approximation results, under some mild assumptions about the uncertainty intervals, for Rec ST with the scenario set introduced by Bertsimas and Sim [3] and the scenario set with a budget constraint (see, e.g., [17]). There is a number of open questions concerning the considered problem. Perhaps, the most interesting one is to resolve the complexity of the robust problem under the interval uncertainty representation with budget constraint. It is possible that this problem can be solved in polynomial time by some extension of the algorithm constructed in this paper. One can also try to extend the algorithm for more general recoverable matroid base problem, which has been also shown to be polynomially solvable in [10].

### Acknowledgements

The second and the third authors were supported by the National Center for Science (Narodowe Centrum Nauki), grant 2013/09/B/ST6/01525.

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### A The proof of Claim 1

**Proof.** We form $T'$ by performing a sequence of moves consisting in adding edges $e_i$ and removing edges $f_i \in P_T(e_i)$, $i \in [\ell]$. Suppose that, at some step, a cycle appears, which is formed by some edges from $\{e_1, \ldots, e_\ell\}$ and the remaining edges of $T$ (not removed from $T$). Let us relabel the edges so that $\{e_1, \ldots, e_s\}$ are on this cycle, i.e. the first $s$ moves consisting in adding $e_i$ and removing $f_i$ create this cycle, $i \in [s]$. An example of such a situation for $s = 4$ is shown in Figure 9. The cycle is formed by the edges $e_1, \ldots, e_4$ and the paths $P_{v_2v_3}$, $P_{v_4v_5}$ and $P_{v_1v_6}$ in $T$. Consider the edge $e_1 = \{v_1, v_2\}$. Because $T$ is a spanning tree, $P_T(e_1) \subseteq P_{v_2v_3} \cup P_T(e_2) \cup P_T(e_3) \cup P_{v_4v_5} \cup P_T(e_4) \cup P_{v_1v_6}$. Observe that $f_1 \in P_T(e_1)$ cannot belong to any of $P_{v_2v_3}$, $P_{v_4v_5}$ and $P_{v_1v_6}$. Hence, it must belong to $P_T(e_2) \cup P_T(e_3) \cup P_T(e_4)$. The above argument is general and using it we can show that for each $i \in [s]$, $f_i \in P_T(e_j)$ for some $j \in [s] \setminus \{i\}$.
We are now ready to build a subgraph depicted in Figure 3. We first choose arc \((v_{e_1}, v_{f_1})\), then we add arc \((v_{e_j}, v_{f_1})\), where \(f_1 \in P_T(e_j), j \neq 1\), we then add \((v_{e_j}, v_{f_j})\), etc. (see Figure 9). After at most \(s\) such steps we obtain a subgraph of \(G(T)\) that has the form shown in Figure 3. Observe that not all nodes corresponding to edges \(e_i, f_i, i \in [s]\) must belong to this subgraph, i.e. \(\kappa \leq s\).