Abstract

A Monte Carlo program is presented for the computation of the most general cross section for two-photon production in $e^+e^-$ collisions at fixed two-photon invariant mass $W$. Functions implemented for the five $\gamma^*\gamma^*$ structure functions include three models of the total hadronic cross section and the lepton-pair production cross section. Prospects of a structure-function determination through a study of the azimuthal dependence between the two scattering planes are outlined. All dependences on the electron mass and the photon virtualities $Q_i^2$ are fully kept. Special emphasis is put on a numerically stable evaluation of all variables over the full $Q_i^2$ range from $Q_{i\min}^2 \sim m_e^2(W/\sqrt{s})^4 \ll m_e^2$ up to $Q_{i\max}^2 \sim s$. A comparison is made with an existing Monte Carlo program for lepton-pair production and an equivalent-photon approximation for hadronic cross sections.
Program Summary

Title of program:        GALUGA
Program obtainable from: G.A. Schuler, CERN–TH,
                        CH-1211 Geneva 23, Switzerland;
                        Gerhard.Schuler@cern.ch
Licensing provisions:   none
Computer for which the program is designed and
others on which it is operable:  all computers
Operating system under which the program has
been tested:             UNIX
Programming language used: FORTRAN 77
Number of lines:         1445
Keywords:               Monte Carlo, two-photon, e\(^+\)\(^-\), azimuthal dependence
Subprograms used:       VEGAS [1] (included, 229 lines)
                        RANLUX [2] (included, 305 lines)
                        HBOOK [3] and DATIME [4] for
                        the test program (367 lines)

Nature of physical problem:
Hadronic two-photon reactions in a new energy domain are becoming accessible with
LEP2. Unlike purely electroweak processes, hadronic processes contain dominant non-
perturbative components parametrized by suitable structure functions, which are functions
of the two-photon invariant mass \(W\) and the photon virtualities \(Q_1\) and \(Q_2\). It is
hence advantageous to have a Monte Carlo program that can generate events at fixed,
user-defined values of \(W\) and, optionally, at fixed values of \(Q_i\). Moreover, at least one
program with an exact treatment of both the kinematics and the dynamics over the whole
range \(m^2 \gg m(W/\sqrt{s})^4 \lesssim Q_i^2 \lesssim s\) (\(m\) is the electron mass and \(\sqrt{s}\) the \(e^+e^-\) c.m. energy)
is needed, (i) to check the various approximations used in other programs, and (ii) to be
able to explore additional information on the hadronic physics, e.g. coded in azimuthal
dependences.

Method of solution:
The differential cross section for \(e^+e^- \rightarrow e^+e^-X\) at fixed two-photon invariant mass \(W\) is
rewritten in terms of four invariants with the photon virtualities \(Q_i\) as the two outermost
integration variables in order to simultaneously cope with antitag and tagged electron
modes. Due care is taken of numerically stable expressions while keeping all electron-mass
and \(Q_i\) dependences. Special care is devoted to the azimuthal dependences of the cross
section. Cuts on the scattered electrons are to a large extent incorporated analytically
and suitable mappings introduced to deal with the peaking structure of the differential
cross section. The event generation yields either weighted events or unweighted ones (i.e.
equally weighted events with weight 1), the latter based on the hit-or-miss technique.
Optionally, VEGAS can be invoked to (i) obtain an accurate estimate of the integrated
cross section and (ii) improve the event generation efficiency through additional variable
mappings provided by the grid information of VEGAS. The program is set up so that
additional hadronic (or leptonic) reactions can easily be added.

Typical running time:
The integration time depends on the required cross-section accuracy and the applied cuts.
For instance, 13 seconds on an IBM RS/6000 yields an accuracy of the VEGAS integration of about 0.1% for the antitag mode or of about 0.2% for a typical single-tag mode; within the same time the error of the simple Monte Carlo integration is about 0.5% for either mode. Event generation with or without VEGAS improvement and for either tag mode takes about $4 \times 10^{-4}$ ($2 \times 10^{-3}$) seconds per event for weighted (unweighted) events.
1 Introduction

Two-photon physics is facing a revival with the advent of LEP2. Measurements of two-photon processes in a new domain of γγ c.m. energies $W$ are ahead of us \[5\]. Any two-photon process is, in general, described \[6\] by five non-trivial structure functions (two more for polarized initial electrons). Purely QED (or electroweak) processes are fully calculable within perturbation theory. Several sophisticated Monte Carlo event generators exist \[7,8,9\] to simulate 4-fermion production in $e^+e^-$ collisions. Indeed, the differential cross section is not explicitly decomposed as an expansion in the five $\gamma^*\gamma^*$ structure functions. Rather, the full matrix element for the reaction $e^+e^- \rightarrow e^+e^-\ell^+\ell^-$ is calculated as a whole, partly even including QED radiative corrections. Such a procedure is, however, not possible for hadronic two-photon reactions since the hadronic behaviour of the photon is of non-perturbative origin. The decomposition into the above-mentioned five structure functions (and their specification, of course) is hence mandatory for a full description of hadronic reactions.

Monte Carlo event generators for hadronic two-photon processes can be divided into two classes. Programs of the first kind \[10,11,12,13,14,15\] put the emphasis on the QCD part but are (so far) restricted to the scattering of two real photons. The two-photon sub-processes are then embedded in an approximate way in the overall reaction of $e^+e^-$ collisions. A recent discussion of the so-called equivalent-photon approximation can be found in \[16\].

The other type of programs \[17,18\] treat the kinematics of the vertex $e^+e^- \rightarrow e^+e^-\gamma\gamma$ more exactly, but they contain only simple models of the hadronic physics. Moreover, the event generation is done in the variables that are tailored for $ee \rightarrow ee\gamma\gamma$, namely the energies and angles (or virtualities) of the photons and the azimuthal angle $\phi$ between the two lepton-scattering planes in the laboratory system. Hence, both the hadronic energy $W$ and the azimuthal angle $\phi$ in the photon c.m.s. (which enters the decomposition of the $e^+e^-$ cross section into the five hadronic structure functions) are highly non-trivial functions of these variables\[1\].

In the study of hadronic physics one prefers to study events at fixed values of $W$. Not only is $W$ the crucial variable that determines the nature of the hadronic physics, but through studies of events at fixed $W$ can $\gamma\gamma$ collisions be compared with $\gamma p$ and $p\bar{p}$ ones \[19\]. Next to $W$, the virtualities $Q_1$ and $Q_2$ of the two photons determine the hadronic physics. At fixed values of $W$ and one of the $Q$’s, $Q_1$ say, one obtains the cross section of deep-inelastic electron–photon scattering. Varying $Q_2$ one can investigate the so-called target-mass effects, i.e. the influence of non-zero values of $Q_2$ on the extraction of the photon structure function $F_2$. Hence it is desirable to have an event generator that keeps $W$ fixed and in which $Q_1$ and $Q_2$ are the outermost integration variables so that these can be held constant.

The remaining two non-trivial integration variables, which complete the phase space of $e^+e^- \rightarrow e^+e^-X$, should be chosen such that three conditions are fulfilled. First, cuts on the scattered electrons are usually imposed in experimental analyses. Hence, the efficiency and accuracy of the program is improved if these can be treated explicitly rather than incorporated by a simple rejection of those events that fall outside the allowed region. Second, the peaking structure of the differential cross section should be reproduced as

\footnote{The only program that contains the $\tilde{\phi}$-dependences is TWOGAM \[17\]. However, the expressions taken from \[13\] are numerically very unstable at small $Q_i$; see the discussion following \[10\]. Moreover, $\phi$ itself is not calculated.}
well as possible in order to reduce the estimated Monte Carlo error and to improve the efficiency of the event generation. And third, it should be possible to achieve a numerically stable evaluation of all variables needed for a complete event description. These three conditions are met to a large extent by the choice of subsystem squared invariant masses $s_1$ and $s_2$ as integration variables besides $Q_1^2$ and $Q_2^2$. In the laboratory frame, $s_i$ are related to the photon energies $\omega_i$ by $s_{1/2} - m^2 = 2\omega_{1/2}/\sqrt{s}$, where $m$ denotes the electron mass.

In the interest of those readers not interested in calculational details, the paper starts with a presentation of a few results in section 2. The differential cross section for the reaction $e^+e^- \rightarrow e^+e^-X$ is rewritten in terms of the four invariants $Q_i^2$ and $s_i$ ($i = 1, 2$) in section 3 where also models for the cross section $\sigma(\gamma^*\gamma^* \rightarrow X)$ are described. The integration boundaries with $Q_1^2$ and $Q_2^2$ as the two outermost integration variables are specified in section 4. The derivation of the integration limits is standard \[20\] but tedious. Here the emphasis is put on numerically stable expressions. To our knowledge, numerical stable forms of $\phi$ and $\tilde{\phi}$ are presented here for the first time. All dependences on the electron mass and the virtualities of the two photons are kept. The formulas are stable over the whole range from $Q_{i \text{min}}^2 \sim m^2(W/\sqrt{s})^4 \ll m^2$ up to $Q_{i \text{max}}^2 \sim s$, i.e. the program covers smoothly the antitag and tag regions. An equivalent-photon approximation is also implemented (section 5). The complete representation of the four-momenta of the produced particles in terms of the integration variables is given in section 6. Section 7 describes the incorporation of cuts on the scattered electrons. Details of the Monte Carlo program GALUGA are given in section 8.

2 A few results

In order to check GALUGA, we include the production of lepton pairs, for which several well-established Monte Carlo generators \[7,8,9\] exist. The five structure functions for $\gamma^*\gamma^* \rightarrow \ell^+\ell^-$ as quoted in \[6\] have been implemented. For the comparison we have modified the two-photon part of the four-fermion program DIAG36 \[8\] (i.e. DIAG36 restricted to the multiperipheral diagrams) in such a way that it can produce events at fixed values of $W$. The agreement is excellent. Two examples are shown in Fig. 1, the first corresponding to a no-tag setup and the second to a single tagging mode.

Next we study the (integrated) total hadronic cross section. Figures 2 and 3–5 compare different ansätze for the $Q_i^2$ behaviour of the various cross sections for transverse and longitudinal photons. The results of the two models of generalized-vector-meson-dominance type (GVMD \[4\] and VMDc \[5\], dash-dotted and dotted histograms, respectively) are hardly distinguishable in the no-tag case, but may deviate by more than 20% in a single-tag case. In the contrast, the different $Q_i^2$ behaviour of a simple $\rho$-pole (dashed histograms) shows up already in the no-tag mode. Note that this model includes scalar photon contributions, but does not possess an $1/Q^2$ “continuum” term for transverse photons. These differences imply that effects of non-zero $Q_i^2$ values must not be neglected for a precision measurement of $\sigma_{\gamma\gamma}(W^2)$.

During the course of the LEP2 workshop, sophisticated programs to generate the full (differential) hadronic final state in two-photon collisions have been developed \[21\]. The description of hadronic physics with one (or both) photons off-shell by virtualities

\footnote{A similar phase-space decomposition with $s_1$ replaced by $\Delta = [(s - 2m^2)(W^2 + Q_1^2 + Q_2^2) - (s_1 + Q_2^2 - m^2)(s_2 + Q_1^2 - m^2)]/4$ is presented in \[6\].}
$Q_i^2 \ll W^2$ is still premature. Indeed, existing programs are thus far for real photons and hence use, in one way or another, the equivalent-photon approximation (EPA) to embed the two-photon reactions in the $e^+e^-$ environment. It is hence indispensable to check the uncertainties associated with the EPA. Hadronic physics is under much better theoretical control for deep-inelastic scattering, i.e. the setup of one almost real photon probed by the other that is off-shell by an amount $Q^2$ of the order of $W^2$. Corresponding event generators exist \cite{21} but also in this case it is desirable to check the equivalent-photon treatment of the probed photon.

An improved EPA has recently been suggested in \cite{16}. In essence, the prescription consists in neglecting $Q_i^2$ w.r.t. $W^2$ in the kinematics but to keep the full $Q_i^2$ dependence in the $\gamma^*\gamma^*$ structure functions. In addition, non-logarithmic terms proportional to $m^2/Q_i^2$ in the luminosity functions are kept as well. The study \cite{16} shows that this improved EPA works rather well for the integrated $e^+e^-$ cross section. In Fig. 2 we show that this EPA (solid compared to dash-dotted histograms) works well also for differential distributions, with the exception of the polar-angle distribution of the hadronic system at large angles, where it can, in fact, fail by more than an order of magnitude! (There, of course, the cross section is down by several orders.)

The EPA describes also rather well the dynamics of the scattered electrons in the single-tag mode except in the tails of the distributions (Fig. 3). The same holds for the distributions in the photon virtualities, see Fig. 4. Sizeable differences do, however, show up (Fig. 4) in the distributions of the subsystem invariant masses $\sqrt{s_i}$. These then lead to the wrong shapes for the energy and momentum distributions of the hadronic system shown in Fig. 5. The EPA should, therefore, not be used for single-tag studies.

Finally we study the prospects of a determination of additional structure functions besides $F_2$. One such possibility was outlined in \cite{5}, namely the study of the azimuthal dependence in the $\gamma\gamma$ c.m.s. between the plane of the scattered (tagged) electron and the plane spanned by the beam axis and the outgoing muon or jet. Here we propose to study the azimuthal angle $\tilde{\phi}$ between the two electron scattering planes, again in the $\gamma\gamma$ c.m.s. Although such a study requires a double-tag setup, the event rates need not be small, since one can fully integrate out the hadronic system but for its invariant mass $W$. In order to demonstrate the sensitivity of such a measurement we show, as a preparatory exercise, the $\tilde{\phi}$ distribution for muon-pair production in Fig. 6. Fitting to the functional form

$$\frac{d\sigma}{d\tilde{\phi}} \propto 1 + A_1 \cos \tilde{\phi} + A_2 \cos 2 \tilde{\phi}, \quad (1)$$

we find

$$A_1 = 0.098, \quad A_2 = -0.028. \quad (2)$$

Let us emphasize that the selected tagging ranges have in no way been optimized for such a study. Nonetheless, given the magnitudes of $A_i$, a measurement appears feasible.

All but one \cite{1} event generators for two-photon physics use the azimuthal angle $\phi$ between the two scattering planes in the laboratory frame as one of the integration variables. In fact, $\phi$ appears as a trivial variable in these programs. None of these up to now provides the calculation of $\tilde{\phi}$. An expression for $\tilde{\phi}$ in terms of $t_i, \phi$, and two other invariants is given in \cite{6} (see \cite{19} below) and, in principle, is available in TWOGAM \cite{17}. However, the factor $\sqrt{t_1t_2}$ appears explicitly in the denominator of $\cos \tilde{\phi}$ but not in its numerator. Hence, at small values of $-t_i$ this factor will be the result of the cancellation of several much larger terms, rendering this expression for $\cos \tilde{\phi}$ numerically very unstable. (Recall
that $|t_i|_{\text{min}} \sim m^2(W/\sqrt{s})^4 \ll m^2$, while the numerator contains terms of order $s$.) In contrast, we use the numerically stable expression given in \((50)\):

$$
\cos \tilde{\phi}_{\text{approx}} = \cos \phi + \sin^2 \phi \frac{Q_1 Q_2 (2s - s_1 - s_2)}{(W^2 - t_1 - t_2) \sqrt{(s - s_1)(s - s_2)}}.
$$

An approximation for $\tilde{\phi}$ in terms of $\phi$ is proposed in \((50)\):

Indeed, the correlation between $\tilde{\phi}$ and its approximation is very high in the no-tag case, where, however, the dependence on $\tilde{\phi}$ is almost trivial (i.e. flat). Figure 6 exhibits that there is still a correlation for a double-tag mode, but formula \((3)\) fails to reproduce the correct $\tilde{\phi}$ dependence: a fit to \((3)\) yields $A_1 = 0.084$ and $A_2 = 0.017$, quite different from \((3)\).

3 Notation and cross sections

Consider the reaction

$$
e^+(p_a) + e^-(p_b) \to e^+(p_1) + X(p_X) + e^-(p_2)
$$

proceeding through the two-photon process

$$
\gamma(q_1) + \gamma(q_2) \to X(p_X).
$$

The cross section for \((4)\) depends on six invariants, which we choose to be the $e^+e^-$ c.m. energy $\sqrt{s}$, the $\gamma\gamma$ c.m. (or hadronic) energy $W$, the photon virtualities $Q_i$, and the subsystem invariant masses $\sqrt{s_i}$:

$$
s = (p_a + p_b)^2, \quad W^2 = p_X^2,
$$

$$
s_1 = (p_1 + p_X)^2 = (p_a + q_2)^2, \quad -Q_1^2 = t_1 = q_1^2 \equiv (p_a - p_1)^2,
$$

$$
s_2 = (p_2 + p_X)^2 = (p_b + q_1)^2, \quad -Q_2^2 = t_2 = q_2^2 \equiv (p_b - p_2)^2.
$$

We find it convenient to introduce also the dependent variables:

$$
u = \frac{1}{2} (W^2 - t_1 - t_2),
$$

$$
u_1 = s_2 - m^2 - t_1, \quad K = \frac{1}{2W} \sqrt{\lambda(W^2, t_1, t_2) = \frac{1}{W} \sqrt{\nu^2 - t_1 t_2}},
$$

$$
\beta = \sqrt{1 - \frac{4m^2}{s}}, \quad y_i = \sqrt{1 - \frac{4m^2}{t_i}}.
$$

where $\lambda(x, y, z) = (x - y - z)^2 - 4yz$ and $m$ denotes the electron mass. Note that $K$ is the photon three-momentum in the $\gamma\gamma$ c.m.s. In terms of these variables the $e^+e^-$ cross section at fixed values of $\sqrt{s}$ and $\tau = W^2/s$ is given by:

$$
\frac{d\sigma[e^+e^- \to e^+e^- + X]}{d\tau} = \frac{\alpha^2 K W}{2 \pi^4 Q_1^2 Q_2^2} dR_3 \Sigma(W^2, Q_1^2, Q_2^2, s_1, s_2, \tilde{\phi}; s, m^2),
$$

where $R_3$ is the phase space for \((4)\).

\footnote{This form of $\tilde{\phi}$ could, with only minor modifications, be implemented \([7]\).}
The hadronic physics is fully encoded in five structure functions. Three of these can be expressed through the cross sections $\sigma_{ab}$ for scalar $(a, b = S)$ and transverse photons $(a, b = T)$ ($\sigma_{ST} = \sigma_{TS}(q_1 \leftrightarrow q_2)$). The other two structure functions $\tau_{TT}$ and $\tau_{TS}$ correspond to transitions with spin-flip for each of the photons with total helicity conservation. Introducing $\tilde{\phi}$, the angle between the scattering planes of the colliding $e^+$ and $e^-$ in the photon c.m.s., these structure functions enter the cross sections as:

$$
\Sigma = 2 \rho_1^{++} 2 \rho_2^{++} \sigma_{TT} + 2 \rho_1^{+0} \rho_2^{00} \sigma_{TS} + \rho_1^{00} 2 \rho_2^{+0} \sigma_{ST} + \rho_1^{00} \rho_2^{00} \sigma_{SS}
+ 2 |\rho_1^{+-}| \tau_{TT} \cos 2 \tilde{\phi} - 8 |\rho_1^{+0} \rho_2^{00}| \tau_{TS} \cos \tilde{\phi} .
$$

(9)

The density matrices of the virtual photons in the $\gamma\gamma$-helicity basis are given by

$$
2 \rho_1^{++} = \frac{(u_2 - \nu)^2}{K^2 W^2} + 1 + \frac{4 m^2}{t_1}
$$
$$
\rho_1^{00} = \frac{(u_2 - \nu)^2}{K^2 W^2} - 1
$$
$$
|\rho_1^{+-}| = \rho_1^{++} - 1
$$
$$
|\rho_1^{+0}| = \sqrt{(\rho_1^{00} + 1) |\rho_1^{+-}|} = \frac{u_2 - \nu}{K W} \sqrt{\rho_1^{++} - 1}
$$

(10)

with analogous formulas for photon 2.

A few remarks about the numerical stability of the $\tilde{\phi}$-dependent terms are in order. Thus far, these terms are implemented solely in the TWOGAM [17] event generator, using the formulas quoted in [8]. Given in [8] and coded in [17] are the products $X_2 = 2 |\rho_1^{+-}| \rho_2^{+-} \cos 2 \tilde{\phi}$ and $X_1 = 8 |\rho_1^{+0} \rho_2^{00}| \cos \tilde{\phi}$ in terms of invariants. Now, the expressions for $X_i$ contain explicit factors of $t_1 t_2$ ($X_2$) and $\sqrt{t_1 t_2}$ ($X_1$) in the denominators but not in the numerators. Clearly, the evaluation of $X_i$ becomes unstable for small values of $|t_1|$. On the other hand, the factors multiplying $\cos \tilde{\phi}$ and $\cos 2 \tilde{\phi}$ in $X_i$ approach perfectly stable expressions in the limit $m^2/W^2 \to 0$ and $t_i/W^2 \to 0$:

$$
|\rho_1^{+-}| \to \frac{2}{x_1^2} (1 - x_1) + \frac{2 m^2}{t_1}
$$
$$
|\rho_1^{+0}| \to \frac{2 - x_1}{x_1} \sqrt{|\rho_1^{+-}|}
$$

(11)

where $x_i = W^2/s_i \approx s_k/s$ ($i \neq k$). Hence a numerically stable evaluation of $\tilde{\phi}$ guarantees a correct evaluation of the $\tilde{\phi}$-dependent terms.

The structure functions $\sigma_{ab}$ and $\tau_{ab}$ for lepton-pair production are well quoted in the literature; the formulas of [1] are implemented in the program. Much less is known about the structure functions for hadronic processes. Since we are not aware of a model for $\tau_{ab}$, the current version of the program assumes

$$
\tau_{TT} = 0 = \tau_{TS} .
$$

(12)

The cross sections $\sigma_{ab}$ are uncertain at small values of $Q_i$. Three models for $\sigma_{ab}$ are provided, all based upon the assumption

$$
\sigma_{ab}(W^2, Q_i^2) = h_a(Q_i^2) h_b(Q_i^2) \sigma_{\gamma\gamma}(W^2) ,
$$

(13)

which is valid for $Q_i^2 \ll W^2$, which is justified in most applications. Note the cross section for the scattering of two real photons $\sigma_{\gamma\gamma}(W^2)$ that enters as a multiplicative factor in (13).
The three models are defined as follows. The first one is based upon a parametrization [23] of the $\gamma^* p$ cross section calculated in a model of generalized vector-meson dominance (GVMD):

$$h_T(Q^2) = r P_1^{-2}(Q^2) + (1-r) P_2^{-1}(Q^2)$$

$$h_S(Q^2) = \xi \left\{ r \frac{Q^2}{m_1^2} P_1^{-2}(Q^2) + (1-r) \left[ \frac{m_2^2}{Q^2} \ln P_2(Q^2) - P_2^{-1}(Q^2) \right] \right\}$$

$$P_i(Q^2) = 1 + \frac{Q^2}{m_i^2},$$ \hspace{1cm} (14)

where we take $\xi = 1/4$, $r = 3/4$, $m_1^2 = 0.54$ GeV$^2$ and $m_2^2 = 1.8$ GeV$^2$.

The second model [24] adds a continuum contribution to simple (diagonal, three-mesons only) vector–meson dominance (VMDc):

$$h_T(Q^2) = \sum_{V=\rho,\omega,\phi} r_V \left( \frac{m_V^2}{m_V^2 + Q^2} \right)^2 + r_c \frac{m_0^2}{m_0^2 + Q^2}$$

$$h_S(Q^2) = \sum_{V=\rho,\omega,\phi} \frac{\xi Q^2}{m_V^2} r_V \left( \frac{m_V^2}{m_V^2 + Q^2} \right)^2,$$ \hspace{1cm} (15)

where $r_\rho = 0.65$, $r_\omega = 0.08$, $r_\phi = 0.05$, and $r_c = 1 - \sum_V r_V$.

Since photon-virtuality effects are often estimated by using a simple $\rho$-pole only, we include also the model defined by ($\rho$-pole):

$$h_T(Q^2) = \left( \frac{m_\rho^2}{m_\rho^2 + Q^2} \right)^2,$$ $h_S(Q^2) = \frac{\xi Q^2}{m_\rho^2} \left( \frac{m_\rho^2}{m_\rho^2 + Q^2} \right)^2.$$ \hspace{1cm} (16)

Since the program is meant to be used at fixed $W$, we take

$$\sigma_{\gamma\gamma}(W) = 1.$$ \hspace{1cm} (17)

Finally we give the relation between the cross section at fixed values of $\tau$ and $Q_2^2$ and the usual form used in deep-inelastic scattering:

$$\frac{d\sigma}{d\tau dt_2} = \frac{x^2 s}{Q_2^2} \frac{d\sigma}{dx dQ_2^2},$$ \hspace{1cm} (18)

where $x$ is the Bjorken-$x$ variable defined by

$$x = \frac{Q_2^2}{2 q_1 \cdot q_2} = \frac{Q_2^2}{W^2 + Q_3^2 + Q_4^2}.$$ \hspace{1cm} (19)

### 4 Phase space

The phase space can be expressed in terms of four invariants:

$$dR_3 \equiv \prod_{i=1,2,3} \int \frac{d^3p_i}{2 E_i} \delta^4 \left( p_a + p_b - \sum p_i \right)$$

$$= \frac{1}{16 \beta s} \int dt_1 dt_2 ds_1 ds_2 \frac{\pi}{\sqrt{-\Delta_4}},$$ \hspace{1cm} (20)
where $\Delta_4$ is the $4 \times 4$ symmetric Gram determinant of any four independent vectors formed out of $p_a, p_b, p_1, p_X, p_2$. The physical region in $t_2, t_1, s_1, s_2$ for fixed $s$ satisfies $\Delta_4 \leq 0$. Since $\Delta_4$ is a quadratic polynomial in any of its arguments, the boundary of the physical region, $\Delta_4 = 0$, is a quadratic equation and has two solutions. Picking $s_2$ as the innermost integration variable, the explicit evaluation of $\Delta_4$ yields

$$16 \Delta_4 = a s_2^2 + b s_2 + c = a (s_2 - s_{2+}) (s_2 - s_{2-}) ,$$

where

$$a = \lambda(s_1, t_2, m^2)$$
$$b = -2 s m^2 t_1 - 2 m^2 s_1^2 + 8 t_2 m^4 - 2 m^2 t_2^2 - 2 s s_1 W^2 + 2 m^2 s W^2 + 2 t_1 s s_1 + 2 s t_2 s_1 + 4 m^2 s_1 W^2 + 4 m^4 s_1 + 2 t_1 t_2 s - 2 t_2 m^2 t_1 - 2 t_2^2 s - 2 m^2 t_2 s + 2 t_1 t_2 s_1 - 4 m^4 W^2 + 2 t_1 s_1^2 + 2 s t_2 W^2 - 2 m^6 + 2 m^4 t_1$$
$$c = -2 s m^4 W^2 - 2 t_2^2 m^2 s_1 - 2 t_1 t_2 s^2 + 2 s t_1 t_2 s_1 - 2 s t_2 s_1 + t_1^2 s_1^2 + t_1^2 s_1^2 + t_2^2 s_1^2 + t_2^2 s_2^2 + m^4 s_1^2 + m^4 t_1^2 - 6 m^6 t_1 - 2 m^6 s_1 - 4 m^4 s_1 W^2 + 2 m^4 t_2 s + 2 m^4 t_2 s_1 + 8 m^4 t_1 s_1 - 2 s t_2 W^2 - 2 t_1 s^2 W^2 - 2 m^2 t_1 s_1^2 + m^8 - 2 m^2 s t_2 s_1 + 4 m^6 W^2 + m^4 t_2^2 + 4 m^2 t_1 t_2 s - 2 m^2 t_1 t_2 s_1 - 6 m^6 t_1 + s^2 W^4 + 6 m^2 s t_2 W^2 - 4 s m^2 W^4 - 2 s m^2 t_2 + 2 s t_1 m^4 - 2 s m^2 t_2^2 + 2 t_1 t_2 m^4 + 2 s m^2 s_1 W^2 - 4 s t_1 t_2 W^2 - 2 s t_1 m^2 s_1 + 6 s t_1 m^2 W^2 + 2 s t_1 s_1 W^2 .$$

A numerical stable form for the $s_2$ limits is

$$s_{2+} = \frac{-b + \sqrt{\Delta}}{2a}$$
$$s_{2-} = \frac{c}{a s_{2+}} ,$$

where $\Delta = b^2 - 4 a c$ is given below in a numerically stable form, in (24).

In order to remove the singularity due to $(-\Delta_4)^{-1/2}$ (in the limit $|t_1|, m^2 \ll s_i, W^2$, the $s_2$ integration degenerates to an integration over the $\delta$-function $\delta(s_2 - s W^2/s_1)$), it is advisable to change variable from $s_2$ to $x_4$, $0 \leq x_4 \leq 1$:

$$s_2 = \frac{1}{2a} \int_{s_{2-}}^{s_{2+}} \frac{ds_2}{\sqrt{-\Delta_4}} = \frac{4 \pi}{\sqrt{a}} \int_0^1 dx_4 .$$

For later use we also need a numerically stable form of the Gram determinant, which reads

$$16 \Delta_4 = -\frac{\Delta \sin^2(x_4 \pi)}{4a} .$$

The $s_1$-integration limits follow from the requirement $\Delta > 0$. They are most easily derived when realizing that the discriminant $\Delta$ is given as the product of two $3 \times 3$ symmetric Gram determinants or, equivalently, the product of two kinematic $G$ functions

$$\frac{1}{4} \Delta = 4 G_3 G_4 = 64 D_3 D_4 .$$

9
where

\[-4D_3 \equiv -4\Delta_3(p_a,p_b,q_2) = G(s,t_2,s_1,m^2,m^2,m^2) \equiv G_3\]
\[-4D_4 \equiv -4\Delta_3(p_a,q_1,q_2) = G(t_1,s_1,t_2,m^2,m^2,W^2) \equiv G_4.\quad (27)\]

Since any $3 \times 3$ Gram determinant $\Delta_3$ satisfies $\Delta_3 \geq 0$, the physical region is that where both $G_3$ and $G_4$ are simultaneously negative. Solving $G_i$ for $s_1$

\[
G_3 = m^2 (s_1 - s_{11+}) (s_1 - s_{11-}) \\
= m^2 s_2^2 - 2 m^2 s_1 - s t_2 s_1 - 3 m^2 t_2 s + m^6 + t_2 s^2 + t_2^2 s \\
G_4 = t_1 (s_1 - s_{12+}) (s_1 - s_{12-}) \\
= -2 t_1 m^2 s_1 - t_2 m^2 t_1 + m^4 t_1 - m^2 W^2 t_1 + m^2 t_2^2 + t_2 W^2 t_1 - t_1 s_1 W^2 \\
- 2 m^2 t_2 W^2 + m^2 W^4 + t_1 s_1^2 + t_1^2 s_1 - t_1 t_2 s_1
\]

we find

\[
s_{11\pm} = \frac{t_2 S + 2 m^4 \pm \sqrt{\lambda(S,m^2,m^2)\lambda(t_2,m^2,m^2)}}{2 m^2} \\
s_{12\pm} = \frac{t_2}{2} + m^2 + \frac{W^2}{2} - \frac{t_1}{2} \pm \frac{\sqrt{\lambda(t_1,t_2,W^2)\lambda(t_1,m^2,m^2)}}{2 t_1} \\
s_{11+} s_{11-} = \frac{t_2 S (-3 m^2 + S + t_2)}{m^2} + m^4 \\
s_{12+} s_{12-} = (W^2 - m^2) (-m^2 + t_2) + \frac{m^2 (W^2 - t_2)^2}{t_1}.\quad (29)
\]

Note that $s_{12+} \leq s_{12-}$. Since $G_3$ is always negative between its two roots, the range of integration over $s_1$ is $s_{12-} \leq s_1 \leq s_{11+}$. Numerically it is more advantageous to calculate the limits as

\[
s_{1\text{min}} = s_{12-} = m^2 + \frac{1}{2} \left( W^2 + t_2 + y_1 \sqrt{\lambda(W^2,t_1,t_2)} \right) \\
s_{1\text{max}} = s_{11+} = m^2 + \frac{2 (s + t_2 - 4 m^2)}{1 + \beta y_2}.\quad (30)
\]

The dominant behaviour of the $s_1$ integration is given by the factor $\lambda^{-1/2}(s_1,t_2,m^2)$, see (24). (In the limit $t_2, m^2 \ll s_1$, this becomes $ds_1/s_1$ integration.) This factor can be transformed away by the variable transformation from $s_1$ to $x_3$, $0 \leq x_3 \leq 1$,

\[
s_1 = X_1/2 + m^2 + t_2 + 2 m^2 t_2/X_1 \\
X_1 = (\nu + K W) (1 + y_1) \exp(\delta_1 x_3) \\
\delta_1 = \ln \frac{s (1 + \beta)^2}{(\nu + K W) (1 + y_1) (1 + y_2)},\quad (31)
\]

such that

\[
\int_{s_{1\text{min}}}^{s_{1\text{max}}} ds_1 \frac{4\pi}{\sqrt{a}} = 4\pi \delta_1 \int_0^1 dx_3.\quad (32)
\]

The physical region in the $t_1-t_2$ plane is defined by the requirement $G_i < 0$ for all $s_1$ values between the limits $(m + W)^2 \leq s_1 \leq (\sqrt{s} - m)^2$. Since for the reaction considered
here the masses of the particles involved are such that the values \( t_1 = (m_a - m_1)^2 \), \( t_2 = (m_b - m_2)^2 \) cannot be reached and \( t_2 \) is never larger than zero, the boundary curve in the \( t_1-t_2 \) plane is simply given by \( s_{12-} = s_{11+} \). Equivalently, the \( t_1 \) limits can be found by solving \( G_4 = 0 \) with \( s_1 = s_{11+} \) for \( t_1 \):

\[
t_{1\text{min}} = -\frac{1}{2} \left( \frac{b_1}{a_1} + \Delta t_1 \right) , \quad t_{1\text{max}} = \frac{c_1}{a_1 t_{1\text{min}}},
\]

where

\[
\Delta t_1 = \frac{\sqrt{\Delta_1}}{a_1} , \quad a_1 = 2 (Q + t_2 + 2 m^2 + W^2) , \quad b_1 = Q^2 - W^4 + 2 W^2 t_2 - t_2^2 - 8 m^2 t_2 - 8 m^2 W^2 , \quad c_1 = 4 m^2 (W^2 - t_2)^2 ,
\]

\[
\Delta_1 \equiv b_1^2 - 4 a_1 c_1 = (Q + t_2 - W^2 + 4 m W) (Q + t_2 - W^2 - 4 m W) (Q^2 - 2 Q t_2 + 2 Q W^2 + t_2^2 + W^4 - 16 m^2 t_2 - 2 W^2 t_2) .
\]

\[
Q = \frac{1}{m^2} \left\{ t_2 s - m^2 t_2 - m^2 W^2 + \sqrt{\lambda(s, m^2, m^2)\lambda(t_2, m^2, m^2)} \right\}
\]

\[
= \frac{4 (s + t_2 - 4 m^2)}{1 + \beta y_2} - t_2 - W^2 .
\]

Finally, the \( t_2 \)-integration limits follow from requiring \( \Delta_1 \geq 0 \):

\[
\Delta_1 = \frac{(t_2 - t_{21+})(t_2 - t_{21-})(t_2 - t_{22+})(t_2 - t_{22-})}{F_1} \frac{t_2(t_2 - t_{23})^2}{F_2} = \frac{t_2(t_2 - t_{23})^2}{F_3} ,
\]

where

\[
F_1 = \frac{4 s}{t_2 s \beta y_2 + t_2 s - 2 W^2 m^2 + 4 m^3 W} , \quad F_2 = \frac{4 s}{t_2 s \beta y_2 + t_2 s - 2 W^2 m^2 - 4 m^3 W} , \quad F_3 = -16 m^4 / \left\{ -2 t_2 s^2 \beta y_2 + 4 t_2 s \beta y_2 m^2 - t_2 s^2 - t_2 s^2 \beta^2 + 4 t_2 s m^2 + 4 s^2 \beta^2 m^2 - 4 t_2 m^4 + 16 m^6 \right\} ,
\]

\[
t_{21\pm} = -\frac{2 m W - W^2 + s - 4 m^2 \pm \beta \sqrt{(s-W^2)(s-W^2)}}{2} ,
\]

\[
t_{22\pm} = -\frac{2 m W - W^2 + s - 4 m^2 \pm \beta \sqrt{(s-W^2)(s-W^2)}}{2} ,
\]

\[
t_{23} = \frac{(s - 2 m^2)^2}{m^2} ,
\]

\[
W_\pm = W \pm 2 m .
\]

Equivalently, they are arrived at by solving \( G_3 = 0 \) with \( s_1 = (m + W)^2 \) for \( t_2 \):

\[
t_{2\text{min}} = t_{22+} = -\frac{1}{2} \left( s - W^2 - 2 m W - 4 m^2 + \Delta t_2 \right) , \quad t_{2\text{max}} = t_{22-} = \frac{m^2 W^2 W^2}{s t_{2\text{min}}} ,
\]

(37)
\[
\Delta t_2 = \beta \sqrt{(s - W^2)(s - W^2)}.
\]  

(38)

The phase space finally becomes
\[
dR_3 = \frac{\pi^2}{4 \beta s} \int_{t_{2\min}}^{t_{2\max}} dt_2 \int_{t_{1\min}}^{t_{1\max}} dt_1 \delta_1(t_1, t_2) \int_0^1 dx_3 \int_0^1 dx_4.
\]  

(39)

The dominant \( t_i \) behaviour is taken into account through a logarithmic mapping, so that we end up with a cross section of the form
\[
d\sigma = \prod_{i=1}^{4} dx_i \int dt_1 \frac{t_1^{t_{1\max}}}{t_1^{t_{1\min}}} \ln \frac{s}{s_2} \ln \frac{s}{s_1} \delta_2(s_1 - s) \frac{(1 + \beta)^2}{(\nu + K W) (1 + y_1) (1 + y_2)} \frac{\alpha^2 K W}{8 \pi^2 \beta^2 s}\Sigma. \]

(40)

5 Equivalent-photon approximation

An approximation is arrived at by neglecting as much as possible the electron-mass and \( t_i \) dependences in the kinematics, but keeping the full dependence on \( W \) and \( Q_i \) in the hadronic cross sections \( \sigma_{ab}(W^2, Q^2_1, Q^2_2) \): 
\[
\frac{d\sigma}{d\tau} = \prod_{i=1}^{4} dx_i \int dt_1 \frac{t_1^{t_{1\max}}}{t_1^{t_{1\min}}} \ln \frac{s}{s_2} \ln \frac{s}{s_1} \delta_2(s_1 - s) \frac{(1 + \beta)^2}{(\nu + K W) (1 + y_1) (1 + y_2)} \frac{\alpha^2 K W}{8 \pi^2 \beta^2 s}\Sigma.
\]

(41)

The integration limits are given by:
\[
t_{ia} = -\frac{m^2 x_i^2}{1 - x_i} - (1 - x_i) \sin^2 \frac{\theta_i}{2},
\]
\[
t_{ib} = -\frac{m^2 x_i^2}{1 - x_i} - (1 - x_i) \sin^2 \frac{\theta_i}{2},
\]

(42)

where \( x_1 = s_2/s \) and \( x_2 = s_1/s \).

The approximate forms of the photon density matrices read:
\[
2 \rho_{1\text{approx}}^{++} = 2 \frac{x_1}{x_1^2} \left\{ 1 + (1 - x_1)^2 - \frac{2 m^2 x_1^2}{Q_i^2} \right\},
\]
\[
\rho_{1\text{approx}}^{00} = \frac{4}{x_1^2} (1 - x_1).
\]

(43)

6 Momenta

Here we present the particle momenta in the laboratory frame. The particle energies follow simply from \( E_i = (p_a + p_b) \cdot p_i/\sqrt{s} \):
\[
E_1 = \frac{s + m^2 - s_2}{2 \sqrt{s}}.
\]

12
\[ E_2 = \frac{s + m^2 - s_1}{2\sqrt{s}} \]
\[ E_X = \frac{s_1 + s_2 - 2m^2}{2\sqrt{s}} \]  \hspace{1cm} (44)

and the moduli of the three-momenta from \( P_i^2 = E_i^2 - m_i^2 \). The polar angles \( \theta_i \) with respect to the beam axis could be calculated from \( p_b \cdot p_i = E_b E_i - P_b P_i \cos \theta_i \)

\[ \cos \theta_1 = \frac{s - s_2 + 2t_1 - 3m^2}{2\beta \sqrt{s} P_1} \]
\[ \cos \theta_2 = \frac{s - s_1 + 2t_2 - 3m^2}{2\beta \sqrt{s} P_2} \]
\[ \cos \theta_X = \frac{s_2 - s_1 + 2(t_2 - t_1)}{2\beta \sqrt{s} P_X}. \]  \hspace{1cm} (45)

Typically, the polar angles are very small and it is better to calculate them in a numerically stable form from

\[ \sin \theta_1 = \frac{2\sqrt{D_1}}{s \beta P_1} \]
\[ \sin \theta_2 = \frac{2\sqrt{D_3}}{s \beta P_2} \]
\[ \sin \theta_X = \frac{2\sqrt{D_5}}{s \beta P_X}. \]  \hspace{1cm} (46)

Equations (45) are then only used to resolve the ambiguity \( \theta_i \leftrightarrow \pi - \theta_i \). The quantity \( D_3 \) is defined in (27–29); \( D_1 \) is obtained from \( D_3 \) by the interchange \( t_1 \leftrightarrow t_2 \) and \( s_1 \leftrightarrow s_2 \). The same interchange relates \( D_2 \), needed below, with \( D_4 \), given in (27–29). Furthermore, we have:

\[ D_5 = D_1 + D_3 + 2D_6 \]
\[ D_6 = \frac{s}{8} \left[ -(s - 4m^2)(W^2 - t_1 - t_2) + (s_1 - t_2 - m^2)(s_2 - t_1 - m^2) + t_1 t_2 \right] \]
\[ - \frac{m^2}{4}(s_1 - m^2)(s_2 - m^2). \]  \hspace{1cm} (47)

The polar angles \( \phi_1 (\phi_2) \) between the \( e^+ (e^-) \) plane and the hadronic plane and the polar angle \( \phi \) between the two lepton planes in the \( e^+e^- \) c.m.s. are again best calculated using the numerically more stable form for the sinus function

\[ \cos \phi = \frac{D_6}{\sqrt{D_1 D_3}} \]
\[ \sin \phi = \frac{s \beta \sqrt{-\Delta_4}}{2\sqrt{D_1 D_3}} \]
\[ \sin \phi_1 = \frac{2\sqrt{-\Delta_4}}{s \beta P_X \sin \theta_X P_1 \sin \theta_1} \]
\[ \sin \phi_2 = \frac{2\sqrt{-\Delta_4}}{s \beta P_X \sin \theta_X P_2 \sin \theta_2} \]
\[
\cos \phi_1 = \frac{D_1 + D_6}{\sqrt{D_1 D_5}} \\
\cos \phi_2 = \frac{D_3 + D_6}{\sqrt{D_3 D_5}} = \frac{\sqrt{D_3} + \sqrt{D_1} \cos \phi}{\sqrt{D_3 + D_1 + 2 \sqrt{D_1 D_3} \cos \phi}}.
\] (48)

An expression for the azimuthal angle between the lepton planes in the $\gamma \gamma$ c.m.s. can be deduced from the formulas given in [3]:
\[
\cos \tilde{\phi} = \frac{-2 s + u_1 + u_2 - \nu + 4 m^2 + \nu(u_2 - \nu)(u_1 - \nu)/(K^2 W^2)}{\sqrt{t_1 t_2 (2 \rho_1^{++} - 2)(2 \rho_2^{++} - 2)}}.
\] (49)

Numerically more stable is the following form
\[
\sin \tilde{\phi} = \frac{K W \sqrt{-\Delta_4}}{\sqrt{D_2 D_4}} \quad \text{and} \quad \cos \tilde{\phi} = \frac{D_7}{\sqrt{D_2 D_4}},
\] (50)

where
\[
16 D_7 = 2 W^2 \left( s_1 s_2 - s W^2 \right) - 2 t_1 \left( -t_1 s_1 + st_1 + s_1 s_2 + W^2 s_1 - 2 s W^2 \right) \\
- 2 t_2 \left( -t_2 s_2 + t_2 s + s_1 s_2 - 2 s W^2 + s_2 W^2 \right) + 2 t_1 t_2 \left( -s_1 + 2 s + 2 W^2 - s_2 \right) \\
- 2 m^2 \left( m^2 t_2 - t_2^2 - m^2 W^2 + m^2 t_1 - 2 W^4 - t_1^2 + W^2 s_1 + 2 t_1 t_2 \right) \\
+ 3 t_1 W^2 - t_1 s_1 + 3 t_2 W^2 - t_2 s_2 - t_2 s_1 + s_2 W^2 - t_1 s_2 \right) \right) .
\] (51)

A numerically stable relation between $\phi$ and $\tilde{\phi}$ at $-t_i$, $m^2 \ll W^2$ is provided by
\[
4 s^2 D_2 = 4 s^2 D_3 + 2 t_2 s^2 r_2 \cos \phi s_2 - 2 t_1 t_2 s s_2 \left( s - s_1 \right) \\
- st_1 t_2 \left( -2 t_1 t_2 - st_1 + t_1 s_1 + 3 t_2 s_2 - t_2 s + 2 r_2 \cos \phi s \right) \\
- 4 m^2 s r_2 \cos \phi s_1 s_2 + O \left( m^4 s^2 s_2^2/s, m^2 t_1 t_2 s s_2 \right),
\] (52)

an analogous expression for $D_4$, and
\[
16 s \sqrt{D_2 D_4} \cos \tilde{\phi} = 16 W^2 \sqrt{D_1 D_3} \cos \phi - 4 t_2 s^2 r_2 \cos \phi - 4 t_1 s^2 r_2 \cos \phi \\
+ 4 t_1 t_2 s \left( -s_1 + 2 s - s_2 + t_1 + t_2 \right) + 8 m^2 \cos \phi r_2 s_1 s_2 \\
+ \frac{2 m^2 t_2}{s} \left( -t_2 s^2 + 4 s t_2 s_2 - 2 t_2 s^2 - 4 s s_1 s_2 + 2 s_1 s_2^2 \\
- 8 r_2 \cos \phi s s_2 + s^2 s_1 + s^2 s_2 + 8 r_2 \cos \phi s \right) \\
+ \frac{2 m^2 t_1}{s} \left( -t_1 s^2 + 4 t_1 s s_1 - 2 t_1 s_1^2 - 4 s s_1 s_2 + 2 s_1 s_2^2 \\
+ 8 r_2 \cos \phi s^2 - 8 r_2 \cos \phi s s_1 + s^2 s_1 + s^2 s_2 \right) \\
+ O \left( m^4 s^2 s_2/s, m^2 t_1 t_2 s \right),
\] (53)

where
\[
r_2^2 = \left[ m^2 \left( \frac{s_1}{s} \right)^2 + t_2 \left( 1 + \frac{t_2}{s} - \frac{s_1}{s} \right) \right] \left[ m^2 \left( \frac{s_2}{s} \right)^2 + t_1 \left( 1 + \frac{t_1}{s} - \frac{s_2}{s} \right) \right].
\] (54)
For $m \to 0$ and $t_i/W^2 \to 0$, (52) and (53) lead to the approximate relation (3).

The four-momenta are now given by

\[ p_a = \frac{1}{2} \sqrt{s} (1, 0, 0, -\beta) \]
\[ p_b = \frac{1}{2} \sqrt{s} (1, 0, 0, \beta) \]
\[ p_1 = (E_1, -P_1 \sin \theta_1 \cos \phi_1, -P_1 \sin \theta_1 \sin \phi_1, -P_1 \cos \theta_1) \]
\[ p_2 = (E_2, -P_2 \sin \theta_2 \cos \phi_2, P_2 \sin \theta_2 \sin \phi_2, P_2 \cos \theta_2) \]
\[ p_X = (E_X, P_X \sin \theta_X, 0, P_X \cos \theta_X) \, . \quad (55) \]

7 Experimental cuts

If cuts on the angle $\theta_2$ and the energy $E_2$ of the scattered electron are applied, the $(s_1, t_2)$-integration region shrinks as follows (see Fig. 7):

\[ s_{1\text{low}} = \min \left\{ (m + W)^2, m^2 + s \left(1 - \frac{2E_{2\text{max}}}{\sqrt{s}}\right) \right\} \]
\[ s_{1\text{upp}} = \max \left\{ (\sqrt{s} - m)^2, m^2 + s \left(1 - \frac{2E_{2\text{min}}}{\sqrt{s}}\right) \right\} \quad (56) \]

and

\[ T_2(s_1, \theta_2_{\text{max}}) < t_2 < T_2(s_1, \theta_2_{\text{min}}) \, , \]

where

\[ T_2(s_1, \theta_2) = \frac{1}{2} \left(3 m^2 - s + s_1 + \beta \cos \theta_2 \sqrt{\lambda(s_1, m^2)}\right) \]
\[ = -\frac{2 m^2 (s_1 - m^2)^2}{s \left[\beta \lambda^{1/2}(s, s_1, m^2) + s - s_1 - 3 m^2\right]} - \beta \lambda^{1/2}(s, s_1, m^2) \sin^2 \frac{\theta_2}{2} \]
\[ \to -\left\{ \frac{m^2 x_2^2}{1 - x_2} + s \left(1 - x_2\right) \sin^2 \frac{\theta_2}{2} \right\} \, . \quad (58) \]

The approximate form holds for $m^2 \ll s_1$ and a small angle $\theta_2$ and is used in (12).

If, as in our case, $t_2$ is the outer integration, then its lower limit becomes

\[ t_{2\text{min}} = \min \left\{ T_2(s_{1\text{upp}}, \theta_2_{\text{max}}), T_2(s_{1\text{low}}, \theta_2_{\text{min}}) \right\} \, , \quad (59) \]

while the upper limit is more complicated

\[ t_{2\text{max}} = T_2(s_{1\text{upp}}, \theta_2_{\text{min}}) \quad \hat{s}_1 > s_{1\text{upp}} \]
\[ = T_2(s_{1\text{low}}, \theta_2_{\text{min}}) \quad \hat{s}_1 < s_{1\text{low}} \]
\[ = \hat{t}_2 \quad s_{1\text{low}} < \hat{s}_1 < s_{1\text{upp}} \, , \quad (60) \]

where

\[ \hat{s}_1 = s + m^2 - \frac{2m \sqrt{s}}{\sqrt{X}} \]
\[ = m^2 + \frac{s \beta^2 \sin^2 \theta_2}{X \left(1 + 2m / \sqrt{s X}\right)} \]
\[ \hat{t}_2 = 2m^2 - m\sqrt{s}X \quad (\theta_2 < \pi/2) \]
\[ = 2m^2 - m\sqrt{s}\frac{1 + \beta^2 \cos^2 \theta_2}{\sqrt{X}} \quad (\theta_2 > \pi/2) \]
\[ X = \frac{4m^2}{s} + \beta^2 \sin^2 \theta_2. \] (61)

The \( s_1 \)-integration range is a rather complicated function of \( t_2 \) and may even consist of two separated ranges (Fig. 7). Moreover, the \( s_1 \)-integration range is affected by \( t_1 \) and cuts on \( E_1 \) and \( \theta_1 \). Then it is better to use the Monte Carlo method. In any case, since the \( t_1 \) integration are the most singular ones, the most important constraints are taken into account through (59) and (60) and the analogous formulas for \( t_1 \).

8 Details of the program

8.1 Common blocks

The user can decide whether to keep \( t_2 \) at a fixed, user-defined value or to integrate over \( t_2 \), i.e. to calculate (8) or (18). In the case of integration over all variables, the user can choose between the exact or an approximate treatment (41) of the kinematics.

\begin{verbatim}
Common /ggLapp/ t2user,iapprx,ivegas,iwaght

  t2user: Fixed value of \( t_2 \) chosen by user for \( iapprx = 1 \).
  iapprx: = 1: \( t_2 \) is kept fixed at the user value;
            = 2: approximate kinematics is used, \( t_2 \) is integrated over;
            = 0: all variables are integrated using exact kinematics.
  ivegas: = 1: VEGAS integration;
            = 0: Simple integration.

Cuts on the scattered leptons are set in

Common /ggLcut/th1min,th1max,E1min,E1max,th2min,th2max,E2min,E2max

  th1min,th1max: Minimum and maximum scattering angles of scattered \( e^+ \)
                 w.r.t. direction of incident \( e^+ \).
  th2min,th2max: Minimum and maximum scattering angles of scattered \( e^- \)
                 w.r.t. direction of incident \( e^- \).
                 Tighter cuts should be applied to the \( e^- \).
  E1min,E1max: Minimum and maximum energies of scattered \( e^+ \).
  E2min,E2max: Minimum and maximum energies of scattered \( e^- \).

Models for the \( \gamma^*\gamma^* \) cross sections and their parameters are chosen in

Common /ggLmod/ imodel

  imodel = 1 \quad GVMD model [14]
  imodel = 2 \quad VMDc model [15]
  imodel = 20 \quad \rho\text{-pole model [16]}
  imodel = 0 \quad \rho\text{-pole model [16]} with \( h_S(Q^2) = 0 \)
  imodel = 3 \quad Exact cross section for lepton-pair production.
\end{verbatim}
Parameters for (14–16): $r$, $\xi$, $m_1^2$, $m_2^2$, $r_\rho$, $r_\omega$, $r_\phi$, $r_c$, $m_\rho^2$, $m_\omega^2$, $m_\phi^2$, $m_0^2$.

The integration variables and the particle momenta are stored in

Common /ggLvar/
&yar(4), t2, t1, s1, s2, E1, E2, EX, P1, P2, PX, th1, th2, thX, phi1, phi2, phi, pht

*yar(i)* Integration variables for VEGAS.
*t2, t1, s1, s2* Invariants $t_2$, $t_1$, $s_1$, $s_2$.
*E1, E2, EX* Energies $E_1$, $E_2$, $E_X$.
*P1, P2, PX* Three-momenta $P_1$, $P_2$, $P_X$.
*th1, th2, thX* Polar angles $\theta_1$, $\theta_2$, $\theta_X$.
*phi1, phi2, phi, pht* Azimuthal angles $\phi_1$, $\phi_2$, $\phi$, $\tilde{\phi}$.

Common /ggLvec/ mntum(7,5)

Particle four-momenta *mntum(i,k)*: $k = 1 \ldots 5$ for $p_x$, $p_y$, $p_z$, $E$, $\text{sign}(p^2) \times \sqrt{|p^2|}$; $i = 1 \ldots 7$ for incident e$, \text{incident e}^-$, photon from e$, \text{photon from e}^-$, scattered e$, \text{scattered e}^-$, hadronic system $X$.

Parameter for the simple integration and results of the integration and event generation are stored in

Common /ggLuno/ cross, error, Fmax, Fmin, Weight, npts, nzero, ntrial

cross Estimate of luminosity.
error Estimate of error on luminosity.
Fmax Maximum function value, calculated in *ggLcrs*; checked in *ggLgen*.
Fmin Minimum function value, calculated in *ggLuF*.
Weight Weight if weighted events requested.
npts Number of function evaluations for simple integration.
nzero Number of cases where function was put to zero in *ggLuF* because it failed the cuts; initialized to zero in *ggLcrs*, *ggLgen*.
ntrial Number of trials necessary in *ggLgen* to generate an event; incremented by each call.

Parameters for the VEGAS integration are set in

Common /ggLvg1/ xl(10), xu(10), acc, ndim, nfcall, itmx, nprn

*acc* VEGAS accuracy (Default (D): $10^{-4}$).
*ndim* Number of integration variables (D: 4).
*nfcall* Maximum number of function calls per iteration for VEGAS (D: $10^5$).
*itmx* Number of iterations for VEGAS (D: 4).
*nprn* Print flag for VEGAS (D: 2).

Additional common blocks
Common /ggLprm/ s,roots,Whad,m,Pi,alem

s Overall c.m. energy square s.
roots Overall c.m. energy $\sqrt{s}$ (twice the beam energy),
set by user through call to ggLcrs.
Whad Hadronic mass $W$, set by user through call to ggLcrs.
m Electron mass (D: 511 keV).
Pi $\pi$
alem $\alpha_{em}$ (D: 1/137).

Common/ggLvg2/XI(50,10),SI,SI2,SWGT,SCHI,NDO,IT

Common /ggLerr/
& it1,iD1,iD3,iD5,itX,iph,ip1,ip2,ia1,ia2,ia3,ia4,ie1,ie2,ipt,is

Block Data ggLblk

8.2 Subroutines

ggLcrs(rs,W) Integrates $d \sigma/d\tau$ and finds $F_{max}$; $rs = \sqrt{s}$, $W = W$
ggLmom Builds up four-momenta.
ggLprt Prints four-momenta and checks momentum sum.
ggLgen(Flag) Generates one event;
Flag=F if a new maximum is found; then it is advisable to restart event generation with adjusted maximum.

8.3 Double-precision functions

ggLInt(W2,m2,Q1s,Q2s,s1,s2,phi,s) $\Sigma$ as defined in (4).
ggLxF(xar,wgt) $F(x_i)$ as defined in (40).
ggLhTT(W2,Q1s,Q2s) $\sigma_{TT}(W^2,Q^2_1,Q^2_2)$
ggLhTS(W2,Q1s,Q2s) $\sigma_{TS}(W^2,Q^2_1,Q^2_2)$
ggLhSS(W2,Q1s,Q2s) $\sigma_{SS}(W^2,Q^2_1,Q^2_2)$
ggLrTS(W2,Q1s,Q2s) $\tau_{TS}(W^2,Q^2_1,Q^2_2)$
ggLrTT(W2,Q1s,Q2s) $\tau_{TT}(W^2,Q^2_1,Q^2_2)$
ggLhT(Qs) $h_T(Q^2)$
ggLhS(Qs) $h_S(Q^2)$
ggLggg(W2) $\sigma_{\gamma\gamma}(W^2)$
ggLxG(z) Makes the variable transformation from $x_i$ in (40) to
those used by the simple or VEGAS integration.
8.4 Excerpt from the demonstration program

* Initialize the random number generator RanLux
  Call rLuxGo(3,314159265,0,0)
*
* Initialize GALUGA; get luminosity within cuts
  Call ggLcrs(rs,W)
*
* Initialize plotting
  Call User(0)
*
* Timing:
  Call Timex(time1)
  Call rLuxGo(3,314159265,0,0)
*
* Event loop
  Do 10 i=1,Nev
    Call ggLgen(Flag)
    If(.not.Flag) Write(6,*) 'Caution: new maximum'
*
* Calculate 4-momenta
  Call ggLmom
*
* Display first 3 events
  If(i.le.3) call ggLprt
*
* Fill histograms
  Call User(1)
  10 Continue
*
  Call Timex(Time2)
  Write(6,300) Nev,Time2-Time1,(Time2-Time1)/real(Nev),
  & iwaght,ntrial,nzero,Fmax
*
* Finalize plotting
  Call User(-Nev)
*
300 Format(/,3x,'time to generate ',I8,' events is ',E12.5,/, &
  &3x,'resulting in an average time per event of ',E12.5,/, &
  &3x,'unweighted events requested if 1: ',I8,/, &
  &3x,'the number of trials was:    ',I8,/, &
  &3x,'the number of zero f was:    ',I8,/, &
  &3x,'the (new) maximum f value was: ',E12.5)
*
  Stop
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Figure 1: Comparison of muon-pair production in GALUGA (dashed histograms) and DIAG36 (solid histograms) at $\sqrt{s} = 130$ GeV and $W = 10$ GeV. Top: distribution in the logarithm of the polar angle of the $\mu^+\mu^-$ system; no cuts are applied on the scattered electrons. Bottom: distribution in $t_1$ under the cuts: $1.55 < \theta_1 < 3.67^\circ$ and $30$ GeV $< E_1$. 
Figure 2: Distributions in $E_1$, $\ln \theta_1$, $E_X$, and $\theta_X$ for the integrated total hadronic cross section at $\sqrt{s} = 130\text{ GeV}$ and $W = 10\text{ GeV}$. No cuts on the scattered electrons are applied. Histogram line-styles correspond to GVMD model in the EPA (solid), $\rho$-pole model (dashed), GVMD model (dash-dotted), VMDc model (dotted).
Figure 3: Distributions in $E_1$, $E_2$, $\ln \theta_1$, and $\ln \theta_2$ for the integrated total hadronic cross section at $\sqrt{s} = 130$ GeV and $W = 10$ GeV. The cuts $\theta_1 < 1.43^\circ$, $1.55 < \theta_2 < 3.67^\circ$, and $30$ GeV $< E_2$ have been applied. Histogram line-styles correspond to GVMD model in the EPA (solid), $\rho$-pole model (dashed), GVMD model (dash-dotted), VMDc model (dotted).
Figure 4: Same as Fig. 3 but for the distributions in $\sqrt{s_1}$, $\ln(-s/t_1)$, and $t_2$. 
Figure 5: Same as Fig. 3, but for the distributions in $E_X$ and $\theta_X$. 
Figure 6: At the top, the correlation between $\tilde{\phi}_{\text{approx}}$ (3), proposed in [22], and $\tilde{\phi}$; at the bottom, the distribution in $\tilde{\phi}$ (solid histogram) and its approximation (dashed histogram) for the integrated muon-pair cross section at $\sqrt{s} = 130$ GeV and $W = 10$ GeV. The cuts $1.55^\circ < \theta_i$, $5$ GeV < $E_1$, and $30$ GeV < $E_2$ have been applied. Also shown is a fit to $d\sigma/d\phi$ of the form $1 + A_1 \cos \phi + A_2 \cos 2\phi$. 
Figure 7: Phase space in the variables \((t_2, s_1)\) for \(\sqrt{s} = 4\) and \(m = 1\). The solid lines correspond to \(\theta_2 = 0, \pi/4, \pi/2, 3\pi/4,\) and \(\pi\) (from \(t_2 = 0\) to \(t_2 = -12\) at \(s_1 = 1\)). The dashed lines are \(s_1 = \hat{s}_1\) and \(t_2 = \hat{t}_2\) at \(\theta_2 = \pi/4\).