Effect of Diffraction on Wigner Distributions of Optical Fields and how to Use It in Optical Resonator Theory. 
I – Stable Resonators and Gaussian Beams

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Abstract. The first part of the paper is devoted to diffraction phenomena that can be expressed by fractional Fourier transforms whose orders are real numbers. According to a scalar theory, diffraction acts on the amplitude of the electric field as well as on its spherical angular spectrum, and Wigner distributions can be defined on a space-frequency phase-space. The phase space is equipped with an Euclidean structure, so that the effects of diffraction are rotations of Wigner distributions associated with optical fields. Such a rotation is shown to split into two specific elliptical rotations. Wigner distributions associated with transverse modes of a resonator are invariant in these rotations, and a complete theory of stable optical resonators and Gaussian beams is developed on the basis of this property, including waist existence and related formulae, and naturally introducing the Gouy phase.

Keywords: Diffraction, Fourier optics, fractional order Fourier transformation, Gaussian beams, optical resonators, spherical angular spectrum, Wigner distribution.

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1 Introduction

A lot of works have been devoted to Wigner distributions in many areas, such as Quantum Mechanics or Signal Processing [1]. In Optics, since the works of Walther [2] and Bastiaans [3], Wigner distributions have been used for representing optical fields, for dealing with radiometry and coherence theories [2], with applications to tomography [1], or more recently for simulating wave effects in graphics [4] or in ray tracing [5].
If we restrict our attention to light propagation, it has been shown that the effect of a GRIN medium is a rotation of the Wigner distribution associated with the optical field [6,7]. The link has been made with real-order fractional Fourier transformations, whose effects are also rotations in an appropriate phase-space [1,6]. Nevertheless, the effect of Fresnel diffraction or of propagation in free space (through Fresnel transforms), considered between two transverse planes, is generally seen like a “horizontal” shear of the Wigner distribution [1,8,10], not a rotation. We notice that Lohmann expresses Fraunhofer diffraction as a $\pi/2$–rotation, but does not generalize to Fresnel diffraction [8], so that the previous descriptions introduce a breaking between the effects of Fresnel or Fraunhofer phenomena, a shearing or a rotation.

In the following, we consider diffraction in a broad meaning, including both Fresnel and Fraunhofer phenomena. The difference between them can be made as follows: the integral expressing the field transfer by diffraction—see Eq. (2)—includes a quadratic phase factor when expressing a Fresnel phenomenon, and no quadratic phase factor for a Fraunhofer phenomenon. (The quadratic phase factor in front of the integral in Eq. (2) does not matter, since it has no effect on the irradiance of the diffraction pattern.)

The effect of diffraction (scalar theory) on Wigner distributions associated with optical fields can be obtained by linking two results:

1. A diffraction phenomenon between a spherical emitter and a spherical receiver is expressed by a fractional order Fourier transform [11–13].

2. Once chosen appropriate scaled variables, fractional order Fourier transformations operate as rotations in the phase space on which Wigner distributions are defined [1,6,9,14].

As far as we know, no work has been dealing with establishing such a link, which constitutes the main subject of the present paper, and clearly, the result will be that the effect of diffraction is a rotation of the Wigner distribution. We do not obtain a shearing, as proposed by several authors [1,8,10] because these authors consider diffraction between two planes, and not between spherical caps as we do. Our approach will make the above mentioned breaking between Fresnel and Fraunhofer phenomena disappear: generally, Fraunhofer diffraction is physically obtained from Fresnel diffraction by continuously increasing the distance at which the diffracted irradiance is observed; its effect on the Wigner representation will be deduced from the effect of Fresnel diffraction by continuously varying the rotation angle up to $-\pi/2$.

The optical field is described by a function of two real spatial variables, that is a function of a 2–dimensional vector variable, so that the corresponding phase-space is 4–dimensional and the Wigner distribution is a function of four real variables. Almost every paper on the Wigner function deals with functions of time (1–dimensional variable); even when dealing with optical fields, most authors (excepting Lohmann [8]) restrict themselves to functions of one spatial variable, whose associated Wigner functions are easier to produce [15] and draw [1,10]. In the present paper we will manage with Wigner distributions in four variables, so that the effect of diffraction will be a 4–rotation. Doing so, we will notice that the 4–rotation cannot be an arbitrary rotation: it is such that it can split into two 2–rotations (as seen by Lohmann [8]).

The (one-dimensional) fractional Fourier transformation of order $\alpha$ is sometimes defined as a rotation of angle $-\alpha$ in phase space, generally the time–frequency space [14]. Speaking of rotation in such a space makes no sense, indeed, as far as appropriate scaled variables are not used.

In the time–frequency space, a rotation of angle $-\alpha$ would transform the $t$–$v$ axes into $u$–$v$ axes, say, and if we use matrix notation, we should write something like

$$
\begin{pmatrix}
u \\
u
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
t \\
\nu
\end{pmatrix},
$$

(1)

and then $u = t \cos \alpha + \nu \sin \alpha$, which makes no sense since $t$ is a time and $\nu$ a frequency, unless appropriate scaled dimensionless (or homogeneous) variables have been defined [7,8].

We conclude that the effect of diffraction on the Wigner distribution can be properly described only after choosing appropriate scaling factors and thus, by defining an appropriate phase-space.
on which rotations make sense. We propose a solution in Sects. 2–5 and present a synthetic approach, linking together diffraction phenomena, fractional order Fourier transformations and Wigner distributions.

Finally, we apply our theory to optical resonators and Gaussian beams. The main result is that the Wigner distribution associated with the optical field inside a resonator must be invariant by a 4-rotation of a particular kind. The whole theory of stable optical resonators can be deduced from that one property: we deduce the existence of transverse modes, represented by Hermite-Gauss functions; we prove the existence of the waist and provide usual related formulae; we also introduce the Gouy phase.

2 Field transfer by diffraction: real-order transfer

We use and adapt the representation of a diffraction phenomenon by a fractional order Fourier transformation as developed in various papers [11–13], in the framework of a scalar theory. We consider a spherical emitter $A_1$ (Fig. 1), that is, a spherical cap emitting monochromatic light (wavelength $\lambda$ in the considered homogeneous and isotropic propagation medium): if its vertex is $\Omega_1$ and its center of curvature $C_1$, the radius of curvature of $A_1$ is $R_1 = \Omega_1 C_1$, where $\Omega_1 C_1$ is an algebraic measure. Algebraic measures are positive if taken in the sense of light propagation. More generally, $A_1$ can be an immaterial spherical cap, illuminated by a light wave. We also consider a spherical receiver $A_2$ (radius of curvature $R_2$) at a distance $D$ from $A_1$ (we write “distance”, but $D$ is an algebraic measure —we also use “algebraic distance”: $D = \Omega_2 \Omega_1$, where $\Omega_2$ is the vertex of $A_2$; distance $D$ is negative if $A_2$ is a virtual receiver). A point $M$ on $A_1$ is represented by the coordinates $(x, y)$ of its projection $m$ on the plane tangent to $A_1$ at its vertex $\Omega_1$ (Fig. 1).

Let $U_1$ denote the electric field amplitude on $A_1$, and $U_2$ the amplitude on $A_2$. With Cartesian coordinates and vectorial notations $r = (x, y)$ on $A_1$ and $r' = (x', y')$ on $A_2$, the field transfer from $A_1$ to $A_2$ is expressed by

$$U_2(r') = \frac{1}{\lambda D} \exp\left[-\frac{i\pi}{\lambda} \left(\frac{1}{R_2} + \frac{1}{D}\right) r'^2\right] \int_{\mathbb{R}^2} \exp\left[-\frac{i\pi}{\lambda} \left(\frac{1}{D} - \frac{1}{R_1}\right) r^2\right] \exp\left(\frac{2i\pi}{\lambda D} r \cdot r'\right) U_1(r) \, dr,$$

where $dr = dx \, dy$ and $r = (x^2 + y^2)^{1/2}$, and where $r \cdot r'$ denotes the Euclidean scalar product of $r$ and $r'$. A phase factor $\exp(-2i\pi D/\lambda)$ has been omitted in Eq. (2) (this factor will be reintroduced later on, when necessary).

Figure 1: Diffraction from a spherical emitter $A_1$ to a spherical receiver $A_2$ at a distance $D$. The point $M$ is represented by the coordinates $(x, y)$ of its projection $m$ on the plane $P_1$ tangent to $A_1$ at its vertex $\Omega_1$. 
We now write Eq. (2) by using a fractional order Fourier transformation. For a function $f$ of two variables, such a transformation is defined by

$$\mathcal{F}_\alpha[f](\rho') = \frac{ie^{-ia}}{\sin \alpha} \exp(-i\rho^2 \cot \alpha) \int_{\mathbb{R}^2} \exp(-i\rho \rho') \exp\left(\frac{2\pi \rho \cdot \rho'}{\sin \alpha}\right) f(\rho) \, d\rho, \quad (3)$$

where $\alpha$ is the order of the transformation, and $\rho$ and $\rho'$ are 2-dimensional real vectors without physical dimension. The standard Fourier transformation is $\mathcal{F}_{\pi/2}$.

In the first part of the paper we will restrict ourselves to real values of $\alpha$. The second part of the paper is devoted to complex $\alpha$.

We define $J = (R_1 - D)(D + R_2) / D(D - R_1 + R_2)$, (4)

and we assume $J \geq 0$ ($J < 0$ is analyzed in the second part of the paper).

Then we define the order $\alpha$ of the transformation associated with diffraction from $A_1$ to $A_2$, as expressed by Eq. (2), by

$$\cot^2 \alpha = J, \quad (5)$$

with $-\pi < \alpha < \pi$, and $\alpha D \geq 0$.

We remark that Eq. (4) is equivalent to

$$\cos^2 \alpha = \left(1 - \frac{D}{R_1}\right) \left(1 + \frac{D}{R_2}\right), \quad (6)$$

which can also be used in defining $\alpha$.

So far, the sign of $\cot \alpha$ is not determined. We then introduce the auxiliary parameter $\varepsilon_1$ such that

$$\varepsilon_1 = \frac{D}{R_1 - D} \cot \alpha, \quad \varepsilon_1 R_1 > 0, \quad (7)$$

and which determines the sign of $\cot \alpha$, so that $\alpha$ is also totally determined, after taking into account the previous definitions.

At last, we define the auxiliary parameter $\varepsilon_2$ such that

$$\varepsilon_2 = \frac{D}{R_2 + D} \cot \alpha. \quad (8)$$

In Appendix A, we show that $\varepsilon_2 R_2 > 0$.

One of the main points in associating a fractional order Fourier transformation with a diffraction phenomenon is the choice of scaled vectorial variables on the emitter and on the receiver. We choose

$$\rho = \frac{r}{\sqrt{\lambda \varepsilon_1 R_1}}, \quad \text{and} \quad \rho' = \frac{r'}{\sqrt{\lambda \varepsilon_2 R_2}}, \quad (9)$$

on $A_1$ and $A_2$ respectively. We remark that both $\rho$ and $\rho'$ are physically dimensionless.

We also use the following scaled field amplitudes on $A_1$ and $A_2$

$$V_1(\rho) = \sqrt{\frac{\varepsilon_1 R_1}{\lambda}} U_1 \left(\sqrt{\lambda \varepsilon_1 R_1} \rho\right), \quad (10)$$

$$V_2(\rho') = \sqrt{\frac{\varepsilon_2 R_2}{\lambda}} U_2 \left(\sqrt{\lambda \varepsilon_2 R_2} \rho'\right). \quad (11)$$

Then Eq. (2) becomes

$$V_2 = e^{ia} \mathcal{F}_\alpha[V_1], \quad (12)$$

which expresses the field transfer from $A_1$ to $A_2$ through a fractional order Fourier transformation.

Generally Eq. (12) corresponds to a Fresnel phenomenon. Fraunhofer diffraction is a limit case, obtained for $\alpha = \pi/2$. 

4
3 Transfer of the spherical angular spectrum

The spherical angular spectrum \( S \) of the field amplitude \( U \) on the spherical emitter (or receiver) \( \mathcal{A} \) is defined by\(^{17,18}\)

\[
S(\Phi) = \frac{1}{\lambda^2} \hat{U} \left( \frac{\Phi}{\lambda} \right),
\]

where \( \hat{U} \) denotes the Fourier transform of \( U \), and the 2-dimensional vectorial variable \( \Phi \) is the angular spatial frequency, related to the spatial frequency \( F \) by \( \Phi = \lambda F \).

The propagation of the spherical angular spectrum from the emitter \( \mathcal{A}_1 \) to the receiver \( \mathcal{A}_2 \) at a distance \( D \) is given by\(^{17,18}\)

\[
S_2(\Phi') = i R_1 R_2 \lambda(D - R_1 + R_2) \exp \left( -\frac{i \pi R_1 (D + R_2)}{\lambda(D - R_1 + R_2)} \phi^2 \right)
\times \int_{\mathbb{R}^2} \exp \left( -\frac{i \pi R_1 (D + R_2)}{\lambda(D - R_1 + R_2)} \phi \right) \exp \left( \frac{2i \pi R_1 R_2}{\lambda(D - R_1 + R_2)} \phi \cdot \phi' \right) S_1(\Phi) d\Phi.
\]

We remark that Eq. (14) is similar to Eq. (2), where \( r \) and \( r' \) are replaced by \( \Phi \) and \( \Phi' \) and where \( D, R_1 \) and \( R_2 \) are changed according to

\[
D \mapsto D - R_1 + R_2, \tag{15}
\]

\[
R_1 \mapsto -\frac{D - R_1 + R_2}{R_1 D}, \tag{16}
\]

\[
R_2 \mapsto -\frac{D - R_1 + R_2}{R_2 D}. \tag{17}
\]

We choose \( \alpha, \epsilon_1 \) and \( \epsilon_2 \) as in Eqs. (5), (7) and (8). By introducing the following scaled angular variables on \( \mathcal{A}_1 \) and on \( \mathcal{A}_2 \) respectively

\[
\phi = \sqrt{\frac{\epsilon_1 R_1}{\lambda}} \Phi, \quad \text{and} \quad \phi' = \sqrt{\frac{\epsilon_2 R_2}{\lambda}} \Phi', \tag{18}
\]

and the scaled spherical angular spectra

\[
T_1(\phi) = \sqrt{\frac{\lambda}{\epsilon_1 R_1}} S_1 \left( \sqrt{\frac{\lambda}{\epsilon_1 R_1}} \phi \right), \quad \text{and} \quad T_2(\phi') = \sqrt{\frac{\lambda}{\epsilon_2 R_2}} S_2 \left( \sqrt{\frac{\lambda}{\epsilon_2 R_2}} \phi' \right), \tag{19}
\]

it can be proved\(^{17,18}\) that Eq. (14) becomes

\[
T_2 = e^{i \alpha} \mathcal{F}_\alpha[T_1]. \tag{20}
\]

Eq. (20) is identical to Eq. (12): the same fractional order Fourier transformation expresses the scaled spherical angular spectrum propagation as well as the scaled field amplitude propagation.

Both \( \phi \) and \( \phi' \) are 2-dimensional vectors without physical dimensions. We have

\[
r \cdot F = \frac{1}{\lambda} r \cdot \Phi = \rho \cdot \phi, \tag{21}
\]

\[
r' \cdot F' = \frac{1}{\lambda} r' \cdot \Phi' = \rho' \cdot \phi', \tag{22}
\]

so that \( \phi \) (resp. \( \phi' \)) is the conjugate variable of \( \rho \) (resp. \( \rho' \)). Eqs. (21) and (22) make sense if rational units are used both for lengths and spatial frequencies (for example mm and mm\(^{-1}\)).
We point out that $T_1$ is no more than the (2–dimensional) Fourier transform of $V_1$: from Eq. (10), indeed, we deduce

$$\hat{V}_1(\phi) = \frac{1}{\lambda \sqrt{\lambda \varepsilon_1 R_1}} \hat{U}_1 \left( \frac{\phi}{\sqrt{\lambda \varepsilon_1 R_1}} \right) = \sqrt{\lambda} \varepsilon_1 R_1 \left( \sqrt{\lambda} \varepsilon_1 R_1 \phi \right) = T_1(\phi).$$

(23)

Of course $T_2$ is the Fourier transform of $V_2$, and finally, Eq. (20) can be written

$$\hat{V}_2 = e^{i \alpha F_\alpha} \hat{V}_1.$$

(24)

Eq. (24) can also be deduced from Eq. (12) by using the commutativity of the product of fractional Fourier transformations ($F_\alpha \circ F_\beta = F_\beta \circ F_\alpha$).

The propagation of the optical field is summarized in the following diagram

$$V_1 \xrightarrow{e^{i \alpha F_\alpha}} V_2 \quad \xrightarrow{F_{\pi/2}} \quad \xrightarrow{F_{\pi/2}} \quad \hat{V}_1 \xrightarrow{e^{i \alpha F_\alpha}} \hat{V}_2,$$

(25)

where the symmetry between propagations of the field amplitude and of the spherical angular spectrum is conspicuous.

4 Wigner distribution associated with the field

The Wigner distribution of a $x$–function $f$, is a function $W(x, y)$, where $y$ is the conjugate variable of $x$, and $W(x, y)$ represents the localization of $f$ in the phase space $x$–$y$.

To obtain the Wigner distribution corresponding to an optical field—whose amplitude is referred to spatial variables—it is natural to use also the angular spectrum, since it represents the field in the domain of spatial frequencies, which are the conjugates of spatial variables.

There is, indeed, a basic reason for using the spherical angular spectrum instead of the planar angular spectrum as usually defined in Fourier Optics [19]: the spherical angular spectrum, unlike the planar angular spectrum, propagates in the same way as the field amplitude, as shown in Sect. 3 Eqs. (12) and (20) and diagram (25). In Fourier optics [19], and considering diffraction between two planes, propagation of the optical field generally corresponds to the following diagram

$$U_1 \xrightarrow{\ast h} U_2 \quad \xrightarrow{F_{\pi/2}} \quad \xrightarrow{F_{\pi/2}} \quad \hat{U}_1 \xrightarrow{\times H} \hat{U}_2,$$

(26)

where $\times H$ denotes the multiplication by a transfer function of the form $H(F) = \exp(i\pi\lambda DF^2)$, and $\ast h$ denotes the convolution product by $h(r) = (i/\lambda D) \exp(-i\pi r^2/\lambda D)$. The symmetry of diagram (25) is broken.

The previous difference between the two angular spectra is an important feature to be taken into account in defining the Wigner representation. The effect of diffraction in the phase space will be homogeneous only if the Wigner distribution is related to the spherical angular spectrum.

We then define the Wigner distribution of the optical field on a spherical cap $A$ as the Wigner distribution of the scaled field amplitude $V$ on $A$, that is,

$$W(\rho, \phi) = \int_{\mathbb{R}^2} V(\rho + \frac{\tau}{2}) \bar{V}(\rho - \frac{\tau}{2}) e^{2i\pi \tau \cdot \phi} d\tau,$$

(27)

where $\bar{V}$ denotes the complex conjugate of $V$.

The Wigner distribution is defined in the 4–dimensional phase space $\rho \cdot \phi$, which has no physical dimension and which will be called the “scaled phase-space”.

6
It can be proved that

\[
W(\rho, \phi) = \int_{\mathbb{R}^2} \hat{V} \left( \phi + \frac{\eta}{2} \right) \hat{V} \left( \phi - \frac{\eta}{2} \right) e^{-2i\rho \cdot \eta} \, d\eta,
\]

(28)

so that the above defined Wigner distribution has the usual properties assigned to Wigner distributions. We also have

\[
\int_{\mathbb{R}^2} W(\rho, \phi) \, d\phi = |V(\rho)|^2, \quad \text{and} \quad \int_{\mathbb{R}^2} W(\rho, \phi) \, d\rho = |\hat{V}(\phi)|^2.
\]

(29)

5 Effect of diffraction on the Wigner distribution: real-order transfer

Let \( \alpha \) be the order of the fractional Fourier transformation associated with the field transfer from the spherical emitter \( A_1 \) to the spherical receiver \( A_2 \) (see Sect. 2). Let \( W_j \) denote the Wigner distribution associated with the field amplitude \( U_j \) on \( A_j \) \((j = 1, 2)\). The effect of diffraction on the Wigner distribution is expressed by

\[
W_2(\rho, \phi) = W_1(\rho \cos \alpha - \phi \sin \alpha, \rho \sin \alpha + \phi \cos \alpha).
\]

(30)

A proof is as follows. For sake of conciseness we define \( E(x) = \exp(i\pi x) \). We use Eqs. \[12\] and \[3\] and write

\[
W_2(\rho, \phi) = \frac{1}{\sin^2 \alpha} \int_{\mathbb{R}^2} E \left( \left\| \rho + \frac{\tau}{2} \right\|^2 \cot \alpha \right) \times \int_{\mathbb{R}^2} E(-\rho'^2 \cot \alpha) \left[ \frac{2\rho'}{\sin \alpha} \cdot \left( \rho - \frac{\tau}{2} \right) \right] E(\rho'^2 \cot \alpha) V_1(\rho') \, d\rho' \\
\times \int_{\mathbb{R}^2} E(\rho'^2 \cot \alpha) E\left( \frac{2\rho \cdot \rho'}{\sin \alpha} \right) V_1(\rho') \, d\rho' \\
\times \int_{\mathbb{R}^2} E(-2\rho \cdot \tau \cot \alpha) E\left( \frac{\rho'^2 + \rho''^2}{\sin \alpha} \cdot \tau \right) E(2\tau \cdot \phi) \, d\tau.
\]

(31)

If \( \delta \) denotes the Dirac generalized function, the last integral in Eq. \[31\] is equal to

\[
\delta \left( \phi - \rho \cot \alpha + \frac{\rho' + \rho''}{2\sin \alpha} \right) = 4 \sin^2 \alpha \delta \left( 2\phi \sin \alpha - 2\rho \cos \alpha + \rho' + \rho'' \right),
\]

(32)

so that Eq. \[31\] becomes

\[
W_2(\rho, \phi) = 4 \int_{\mathbb{R}^2} E(-\rho'^2 \cot \alpha) \left[ \frac{2\rho \cdot \rho'}{\sin \alpha} \right] E\left( \|2\rho \cos \alpha - 2\phi \sin \alpha - \rho'\|^2 \cot \alpha \right) \times \int E\left[ -\frac{2\rho}{\sin \alpha} \cdot (2\rho \cos \alpha - 2\phi \sin \alpha - \rho') \right] V_1(\rho') V_1(2\rho \cos \alpha - 2\phi \sin \alpha - \rho') \, d\rho'.
\]

(33)

We change \( \rho' \) into \( \tau = 2\rho' - 2\rho \cos \alpha + 2\phi \sin \alpha \), so that Eq. \[33\] becomes

\[
W_2(\rho, \phi) = \int_{\mathbb{R}^2} V_1 \left( \rho \cos \alpha - \phi \sin \alpha + \frac{\tau}{2} \right) V_1 \left( \rho \cos \alpha - \phi \sin \alpha - \frac{\tau}{2} \right) \times E\left[ 2(\rho \sin \alpha + \phi \cos \alpha) \cdot \tau \right] \, d\tau,
\]

(34)

which is eq. \[30\] once more. The proof is complete.
Equation (30) shows that the effect of a diffraction phenomenon is a rotation of the Wigner distribution associated with the scaled field amplitude. The rotation operates in the 4-dimensional scaled phase-space \( \rho \cdot \phi \). The value of the function \( W_2 \) at point \((\rho, \phi)\) is equal to the value of the function \( W_1 \) at point \( (\rho \cos \alpha - \phi \sin \alpha, \rho \sin \alpha + \phi \cos \alpha) \), so that the “angle” of the rotation is equal to \(-\alpha\), and opposite to the order of the fractional Fourier transformation associated with the field amplitude transfer (see Appendix B).

We remark that the 4-dimensional scaled phase-space is homogeneous to \( \mathbb{R}^4 \). It is equipped with an Euclidean norm defined by

\[
|| (\rho, \phi) ||^2 = \rho_x^2 + \rho_y^2 + \phi_x^2 + \phi_y^2,
\]

where \( \rho = (\rho_x, \rho_y) \) and \( \phi = (\phi_x, \phi_y) \). Then the above mentioned rotation makes sense.

A matrix representation of the effect of diffraction can be seen as a coordinate transformation in the scaled phase-space and is the following. We use \( \rho = (\rho_x, \rho_y) \) and \( \phi = (\phi_x, \phi_y) \) and a 1 or 2 index for the emitter \( A_1 \) or the receiver \( A_2 \). Then

\[
\begin{pmatrix}
\rho_{x2} \\
\rho_{y2} \\
\phi_{x2} \\
\phi_{y2}
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & 0 & \sin \alpha & 0 \\
0 & \cos \alpha & 0 & \sin \alpha \\
-\sin \alpha & 0 & \cos \alpha & 0 \\
0 & -\sin \alpha & 0 & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\rho_{x1} \\
\rho_{y1} \\
\phi_{x1} \\
\phi_{y1}
\end{pmatrix}.
\]

An equivalent matrix form is obtained by reordering the variables, that is,

\[
\begin{pmatrix}
\rho_{x2} \\
\rho_{y2} \\
\phi_{x2} \\
\phi_{y2}
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \alpha & \sin \alpha \\
0 & 0 & -\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\rho_{x1} \\
\rho_{y1} \\
\phi_{x1} \\
\phi_{y1}
\end{pmatrix}.
\]

In the following, we call “Wigner rotation of parameter (or angle) \(-\alpha\)” a 4-dimensional rotation whose matrix is given by Eq. (37). We denote it by \( R_{-\alpha} \).

The 4-rotation matrix can be written as the (commutative) product of two matrices

\[
\begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \alpha & \sin \alpha \\
0 & 0 & -\sin \alpha & \cos \alpha
\end{pmatrix},
\]

and each matrix represents a rotation on a 2-dimensional subspace of \( \mathbb{R}^4 \), that is, a plane. In these planes, the effect of diffraction is expressed through a 2-rotation of angle \(-\alpha\), whose matrix is

\[
\begin{pmatrix}
\rho_{x2} \\
\rho_{y2} \\
\phi_{x2} \\
\phi_{y2}
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \alpha & \sin \alpha \\
0 & 0 & -\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\rho_{k3} \\
\rho_{k4} \\
\phi_{k3} \\
\phi_{k4}
\end{pmatrix} \text{ where } k = x, y.
\]

This will help in concretely representing the effect of diffraction on the Wigner distribution.

We conclude that the matrix representation of the effect of diffraction on the Wigner distribution in the scaled phase-space is not an arbitrary rotation matrix, but a matrix that can split according to (38).

### 6 Composition of two transformations

According to the Huygens–Fresnel principle, the field transfer from an emitter \( A_1 \) to a receiver \( A_2 \) can be thought of as the composition of two transfers: from \( A_1 \) to \( A_3 \) and from \( A_3 \) to \( A_2 \), where \( A_3 \) is an intermediate spherical cap located between \( A_1 \) and \( A_2 \).

More generally, \( A_3 \) can be every (immaterial) spherical cap, not necessarily located between \( A_1 \) and \( A_2 \). For example if light encounters first \( A_1 \), then \( A_2 \) and \( A_3 \) at last, the transfer from \( A_3 \) to \( A_2 \) is virtual.
We denote \( D_1 \) the (algebraic) distance from \( A_1 \) to \( A_3 \) and \( D_2 \) from \( A_3 \) to \( A_2 \); the (algebraic) distance from \( A_1 \) to \( A_2 \) is \( D = D_1 + D_2 \) (this relation holds true even if \( A_3 \) is not between \( A_1 \) and \( A_2 \); for example \( D_2 \) is negative when the transfer from \( A_3 \) to \( A_2 \) is virtual).

The transfer from \( A_1 \) to \( A_3 \) is described by a fractional Fourier transformation whose order is \( \alpha_1 \) and the transfer from \( A_3 \) to \( A_2 \) by a transformation whose order is \( \alpha_2 \). According to the Huygens–Fresnel principle the field transfer from \( A_1 \) to \( A_2 \) is expressed by a fractional Fourier transformation whose order is \( \alpha = \alpha_1 + \alpha_2 \). Interpreted in terms of Wigner representation, the Huygens–Fresnel principle is equivalent to the composition of two Wigner rotations whose respective parameters are \( -\alpha_1 \) and \( -\alpha_2 \), which results in a Wigner rotation whose parameter is \( -\alpha = -\alpha_1 - \alpha_2 \).

The previous composition makes sense only if scaled variables on \( A_3 \) associated with the first rotation are also scaled variables for the second rotation. We introduce parameters \( \varepsilon_{11} \) and \( \varepsilon_{12} \) (transfer from \( A_1 \) to \( A_3 \)) and \( \varepsilon_{21} \) and \( \varepsilon_{22} \) (transfer from \( A_3 \) to \( A_2 \)). Scaled variables on \( A_3 \) must be the same for both transfers, which implies \( \varepsilon_{12} = \varepsilon_{21} \), that is,

\[
\frac{D_1(R_1 - D_1)}{(R_3 + D_1)(D_1 - R_1 + R_3)} = \frac{D_2(R_2 + D_2)}{(R_3 - D_2)(D_2 - R_3 + R_2)}. \tag{40}
\]

Eq. (40) reduces, indeed, to a first degree equation in \( R_3 \), whose solution is \( R_3 = \frac{D_1(R_2 + D_2)(R_1 - D) + D_2(R_2 + D)(R_1 - D_1)}{D_1(R_1 - D) + D_2(R_2 + D)}. \tag{41}
\]

Finally, if \( V_j \) \((j = 1, 2, 3)\) denotes the scaled field amplitude on \( A_j \), the Huygens–Fresnel principle is expressed by

\[
V_2 = e^{i\alpha_2}F_{\alpha_2}[V_3] = e^{i\alpha_2}e^{i\alpha_1}F_{\alpha_2} \circ F_{\alpha_1}[V_1] = e^{i\alpha}F_{\alpha}[V_1], \tag{42}
\]

where \( \alpha = \alpha_1 + \alpha_2 \). Since fractional order Fourier transformations commute, we also have

\[
\hat{V}_2 = F_{\pi/2}[V_2] = e^{i\alpha_2}F_{\alpha_2} \circ F_{\pi/2}[V_3] = e^{i\alpha_2}e^{i\alpha_1}F_{\alpha_2} \circ F_{\alpha_1}[\hat{V}_1] = e^{i\alpha}F_{\alpha}[\hat{V}_1]. \tag{43}
\]

If \( W_j \) denotes the Wigner distribution on \( A_j \), we draw the following diagrams

\[
\begin{align*}
V_1 & \xrightarrow{e^{i\alpha_1}F_{\alpha_1}} V_2 \\
V_3 & \xrightarrow{e^{i\alpha_2}F_{\alpha_2}} \hat{V}_3
\end{align*}
\]

\[
\begin{align*}
\hat{V}_1 & \xrightarrow{e^{i\alpha_2}F_{\alpha_2}} \hat{V}_2 \\
\hat{V}_3 & \xrightarrow{e^{i\alpha_2}F_{\alpha_2}} \hat{V}_3
\end{align*}
\]

\[
\begin{align*}
W_1 & \xrightarrow{R_{\alpha_1}} W_2 \\
W_3 & \xrightarrow{R_{\alpha_2}} W_3
\end{align*}
\]

which express the Huygens-Fresnel principle. For every spherical cap \((A_1, A_2 \) or \( A_3 \)) two fractional Fourier transformations are involved that apply to a same scaled amplitude (and then the same scaled variable). For example \( V_1 \) is the input scaled amplitude on \( A_1 \) for both \( F_{\alpha} \) and \( F_{\alpha_1} \).

7 Application to optical resonator theory

7.1 Sign conventions

Since we are going to consider mirrors, we will adopt the following sign conventions. We maintain the rule that an algebraic measure is positive if taken in the sense of light propagation. If a mirror has a vertex \( \Omega \) and a center of curvature \( C \), its radius is \( R = \overline{OC} \). For a concave mirror, \( \overline{OC} \) is negative if we consider light propagation before reflection on the mirror (since light travels to the mirror, that is from \( C \) to \( \Omega \)); it is positive after reflection (since light goes away from the mirror, that is, from \( \Omega \) to \( C \)). For that reason we associate two radii with a mirror: the objet radius, denoted \( R \), is related to light propagation before reflection on the mirror; the image radius, denoted \( R' \), is related to light propagation after reflection. We have \( R' = -R \).
7.2 Round trip

We consider an optical resonator made up of two spherical mirrors \( M_1 \) and \( M_2 \). The object radius of \( M_1 \) is \( R_1 \) and its image radius is \( R'_1 \). They are \( R_2 \) and \( R'_2 \) for \( M_2 \).

For diffraction from \( M_1 \) to \( M_2 \), the distance to be considered is the distance from \( M_1 \) to \( M_2 \), say \( D \), after reflection on \( M_1 \). For diffraction from \( M_2 \) to \( M_1 \), the distance to be considered, say \( D' \), is taken from \( M_2 \) to \( M_1 \) and is related to light propagation after reflection on \( M_2 \). According to our sign convention, we have \( D = D' \). We then introduce the “length” of the resonator, which is \( L = D = D' \), and which is independent of the sense of light propagation (\( L \) can be negative for a virtual resonator; \( L \) should be called “algebraic length”).

We consider first the field transfer from \( M_1 \) to \( M_2 \) and will apply the result of Sect. 2. Light propagates from \( M_1 \) to \( M_2 \) after reflection on \( M_1 \) and before reflection on \( M_2 \); according to our convention, the radii to be considered are \( R'_1 \) and \( R_2 \). Then the field transfer from \( M_1 \) to \( M_2 \) is described with the help of \( \alpha_0 \) such that

\[
\cot^2 \alpha_0 = \frac{(R'_1 - L)(L + R_2)}{L(L - R'_1 + R_2)},
\]

with \( \alpha_0 L \geq 0 \) (we assume that \( \alpha_0 \) is a real number). We also have

\[
\varepsilon_1 = \frac{L}{R'_1 - L} \cot \alpha_0, \quad \varepsilon_1 R'_1 > 0,
\]

and

\[
\varepsilon_2 = \frac{L}{R_2 + L} \cot \alpha_0, \quad \varepsilon_2 R_2 > 0.
\]

With appropriate scaled variables on \( M_1 \) and \( M_2 \) we have

\[
V_2 = e^{i \alpha_0} \mathcal{F}_{\alpha_0}[V_1].
\]

For the field transfer from \( M_2 \) to \( M_1 \), the emitter is \( M_2 \) and the receiver is \( M_1 \); light propagates after reflection on \( M_2 \) and before reflection on \( M_1 \). The radii to be considered are \( R'_2 \) for the emitter and \( R_1 \) for the receiver. Then the field transfer from \( M_2 \) to \( M_1 \) is described with the help of \( \alpha'_0 \) such that

\[
\cot^2 \alpha'_0 = \frac{(R'_2 - L)(L + R_1)}{L(L - R'_2 + R_1)}, \quad \alpha'_0 L \geq 0,
\]

\[
\varepsilon'_1 = \frac{L}{R'_2 - L} \cot \alpha'_0, \quad \varepsilon'_1 R'_2 > 0,
\]

and

\[
\varepsilon'_2 = \frac{L}{R_1 + L} \cot \alpha'_0, \quad \varepsilon'_2 R_1 > 0.
\]

Since \( R'_1 = -R_1 \) and \( R'_2 = -R_2 \), then \( \alpha_0 = \alpha'_0 \). We also have \( \varepsilon_1 R'_1 = \varepsilon'_1 R_1 \) and \( \varepsilon_2 R_2 = \varepsilon'_2 R'_2 \), so that scaled variables on \( M_1 \) and \( M_2 \) are identical for both transfers: their composition makes sense. Finally, the transfer from \( M_1 \) to \( M_1 \) in a round trip (that is, after a reflection on \( M_2 \)) is expressed by

\[
V_1 = e^{2i \alpha_0} \mathcal{F}_{2 \alpha_0}[V_1],
\]

and the transfer from \( M_2 \) to \( M_2 \) by

\[
V_2 = e^{2i \alpha_0} \mathcal{F}_{2 \alpha_0}[V_2].
\]
7.3 Eigenmodes

We consider an intermediate spherical cap $S_\alpha$ at a distance $D_1$ from $M_1$. The distance from $S_\alpha$ to $M_2$ is $D_2$ and the length of the resonator is such that $L = D_1 + D_2$. Let $\alpha_1$ be the order of the fractional Fourier transformation associated with the field transfer from $M_1$ to $S_\alpha$.

We choose the radius $R_\alpha$ of $S_\alpha$ such that composition of transfers from $M_1$ to $S_\alpha$ and from $S_\alpha$ to $M_2$ makes sense, that is, according to Eq. (41)

$$R_\alpha = \frac{D_1(R_2 + D_2)(R_1' - L) + D_2(R_2 + L)(R_1' - D_1)}{D_1(R_1' - L) + D_2(R_2 + L)}.$$ (54)

Then scaled variables on $M_1$ are identical for both transfers from $M_1$ to $S_\alpha$ and from $M_1$ to $M_2$, so that the scaled amplitude on $S_\alpha$ is $V_{\alpha_1}$ such that

$$V_{\alpha_1} = e^{i\alpha_1} F_{\alpha_1}[V_1],$$ (55)

where $V_1$ is the $V_1$ of Eq. (52).

We imagine now that $S_\alpha$ is a mirror, without changing its radius. A round trip from $M_1$ to $M_1$ after reflection on $S_\alpha$ is expressed by

$$V_1 = e^{2i\alpha_1} F_{2\alpha_1}[V_1].$$ (56)

Since $S_\alpha$ is an arbitrary spherical cap (not necessarily located between $M_1$ and $M_2$), with appropriate radius, the order $\alpha_1$ can be an arbitrary real number. We conclude that the scaled field amplitude $V_1$ on $M_1$ is invariant in every fractional order Fourier transformation, whatever its order. Since Hermite–Gauss functions are eigenfunctions of all fractional order Fourier transformations [16], then $V_1$ is represented by a Hermite–Gauss function, or by a linear combination of such functions (explicit expressions of Hermite–Gauss functions will be given in Sect. 8.2).

According to Eq. (55), the field amplitude on $S_\alpha$ is obtained from the amplitude on $M_1$ by applying a fractional order Fourier transformation, so that the scaled field amplitude on $S_\alpha$ is expressed by the same Hermite–Gauss function as the amplitude on $M_1$. Since $S_\alpha$ is an arbitrary spherical cap (with appropriate radius), we conclude that for every spherical surface of the family $\{S_\alpha\}$ the amplitude is expressed by a Hermite–Gauss function, which remains the same (up to a scaling factor) along the whole resonator. Hence the notions of transverse Hermite–Gauss modes and of Gaussian beams. (Gaussian beams also propagate outside the resonator.)

7.4 Stability

So far, we assumed that orders of fractional Fourier transformations associated with diffraction phenomena were real numbers. By considering Eq. (52), we conclude that after a round trip, the field amplitude is multiplied by a factor $\exp(2i\alpha_0)$ (the analysis of the phase factor will be done in Sect. 8.3). The resonator is said to be stable [20].

According to Eq. (55), if

$$\frac{(R_1' - L)(L + R_2)}{L (L - R_1' + R_2)} < 0,$$ (57)

the order $\alpha_0$ is a complex number so that the factor $\exp(2i\alpha_0)$ represents an attenuation. There are losses by diffraction and the resonator is said to be unstable (see part II of the paper).

We conclude that a resonator is stable if, and only if, the order of the fractional Fourier transformation associated with the field transfer from a mirror to the other is a real number. If we change $R_2$ into $R_2' = -R_2$ in (57), then a resonator of length $L$ is stable if, and only if,

$$\frac{(R_1' - L)(L - R_2')}{L (L - R_1' - R_2')} \geq 0,$$ (58)

where $R_1'$ and $R_2'$ are the image radii of the mirrors. Inequality (58) is equivalent to

$$0 \leq \left(1 - \frac{L}{R_1'}\right) \left(1 - \frac{L}{R_2'}\right) \leq 1,$$ (59)
which is a usual stability condition as given in most textbooks on optical resonators.

In the following, as previously stated, we consider only stable resonators.

### 7.5 Consequence for the Wigner distribution of the field amplitude inside a resonator

First, we deduce from Eq. (52) that the Wigner distribution associated with the scaled amplitude on $M_1$ is invariant in a Wigner rotation of angle $-\alpha_0$. More general, the analysis of Sect. 7.3—Eq. (50)—shows that it is invariant in every Wigner rotation. The result holds true for the scaled field amplitude on every appropriate wave surface inside or outside the resonator (by appropriate wave surface, we mean a spherical cap whose radius is taken according to Eq. (41)). We conclude that the Wigner distribution associated with the scaled field amplitude inside a resonator is invariant in a Wigner rotation, whatever the angle. The result also holds outside the resonator, that is, for Gaussian beams.

An equivalent statement is:

**Theorem.** The Wigner distribution associated with the scaled field-amplitude of a transverse mode of a resonator is invariant in a Wigner rotation, whatever the rotation angle.

In the following, usual properties and usual relations that hold for stable optical resonators and Gaussian beams are proved as a consequence of the previous theorem.

### 7.6 Wigner distribution of a Gaussian field

We consider a resonator as before. We assume the field amplitude on $M_1$ to be of the form

$$U_1(r) = U_0 \exp \left( -\frac{r^2}{w_1^2} \right), \quad (60)$$

where $U_0$ is a dimensionless constant, and $w_1 (w_1 > 0)$ is called the transverse radius of the field on $M_1$, and is such that the emittance at point $r$ is higher than or equal to $|U_0|^2 e^{-2}$ if, and only if, $r \leq w_1$.

The Wigner distribution of the scaled field amplitude on $M_1$ is

$$W_1(\rho, \phi) = \frac{\lambda}{\lambda_1} |U_0|^2 \int_{\mathbb{R}^2} \exp \left( -\frac{\lambda \rho_1' R_1'}{2 w_1^2} \right) \frac{\rho}{\rho_1} \left( \rho + \frac{\tau}{2} \right)^{2} \cdot \exp \left( -\frac{\lambda \rho_1' R_1'}{2 w_1^2} \right) \frac{\rho - \frac{\tau}{2}}{\rho_1'}^{2} \cdot e^{2i\pi \rho \cdot \phi} d\rho d\phi \quad (61)$$

We use (2-dimensional Fourier pair)

$$\exp \left( -\frac{\lambda \rho_1' R_1'}{2 w_1^2} \right) = \frac{2\pi w_1^2}{\lambda \lambda_1 R_1^2} \exp \left( \frac{-2\pi w_1^2}{\lambda \lambda_1 R_1^2} \phi^2 \right), \quad (62)$$

and we obtain

$$W_1(\rho, \phi) = \frac{2\pi w_1^2}{\lambda^2} |U_0|^2 \exp \left( -\frac{2\lambda \rho_1' R_1'}{w_1^2} - \rho^2 \right) \cdot \exp \left( -\frac{2\pi w_1^2}{\lambda \lambda_1 R_1^2} \phi^2 \right) \quad (63)$$

### 7.7 Transverse radius of the field amplitude on a mirror

We consider hyper-surfaces of equal amplitudes for the Wigner distribution $W_1$ of the previous section. In the $(\rho, \phi)$-space, their equations are written

$$\frac{\lambda \rho_1' R_1'}{w_1^2} \rho^2 + \frac{\pi^2 w_1^2}{\lambda \lambda_1 R_1^2} \phi^2 = \frac{\lambda \rho_1' R_1'}{w_1^2} (\rho_x^2 + \rho_y^2) + \frac{\pi^2 w_1^2}{\lambda \lambda_1 R_1^2} (\phi_x^2 + \phi_y^2) = C, \quad (64)$$
where $C$ is a constant.

The effect of diffraction on the Wigner distribution is a Wigner rotation that splits into two 2rotations operating on planes $(\rho_x, \phi_x)$ and $(\rho_y, \phi_y)$ respectively. Then we consider equal amplitude sections of the hyper-surface. These sections are curves whose equations are written

$$\frac{\lambda \varepsilon_1 R_1'}{w_1^2} \rho_x^2 + \pi^2 w_1^2 \frac{1}{\lambda \varepsilon_1 R_1'} \phi_x^2 = C_x,$$

in the $(\rho_x, \phi_x)$ plane, and

$$\frac{\lambda \varepsilon_1 R_1'}{w_1^2} \rho_y^2 + \pi^2 w_1^2 \frac{1}{\lambda \varepsilon_1 R_1'} \phi_y^2 = C_y,$$

in the $(\rho_y, \phi_y)$ plane, where $C_x$ and $C_y$ are constants.

Generally, Eqs. (65) and (66) are equations of ellipses. But for eigenmodes of optical resonators, the curves of equal amplitudes of the associated Wigner distribution are circles, because they must be invariant by every Wigner rotation, according to the previous theorem.

Eq. (65) corresponds to a circle if

$$\frac{w_1^4}{\varepsilon_1 R_1^2} = \frac{\lambda \varepsilon_1 R_1'}{\pi^2 w_1^2} \cos 2 \alpha + \frac{\lambda \varepsilon_1 R_1'}{\pi^2 w_1^2} \sin 2 \alpha.$$

Then

$$w_1^4 = \frac{1}{\pi^2} \lambda^2 \varepsilon_1^2 R_1'^2.$$

From Eqs. (5) and (7), and using $R_2 = -R_2'$, we obtain

$$w_1^4 = \frac{\lambda^2 R_1'^2 (L - R_2') \pi^2 (R_1' - L)(L - R_1' - R_2')}{R_2' - L(R_1' - R_2')^2}.$$

Since propagation from $M_1$ to $M_2$ corresponds to a rotation in the $(\rho_x, \phi_x)$–plane, the above mentioned circle is unchanged in the rotation, so that if $w_2$ is the field transverse radius on $M_2$, we have

$$\frac{w_2^2}{\varepsilon_2 R_2} = \frac{w_1^2}{\varepsilon_1 R_1'},$$

that is,

$$w_2^4 = \frac{\lambda^2 R_2'^2 (L - R_1') \pi^2 (R_2' - L)(L - R_1' - R_2')}{R_2' - L(R_1' - R_2')^2}.$$

Eqs. (69) and (71) are classical relations for Gaussian beams in optical resonators [13].

### 7.8 Waist

We consider the mirror $M_1$ once more. We are looking for the field transverse radius $w$ on the spherical cap $S$ (curvature radius $R$) at distance $D_1$ from $M_1$. The field transfer from $M_1$ to $S$ is expressed by a fractional Fourier transformation whose order is $\alpha$, and we denote $\varepsilon$ the parameter that corresponds to $\varepsilon_2$ in the general transfer of Sect. 2. We know that equal amplitude curves of sections of the Wigner distribution are circles. Then we adapt Eqs. (67) and (70), and deduce

$$\frac{w^2}{\lambda \varepsilon R} = \frac{w_1^2}{\lambda \varepsilon_1 R_1'} \cos^2 \alpha + \frac{\lambda \varepsilon_1 R_1'}{\pi^2 w_1^2} \sin^2 \alpha.$$

From Eqs. (7) and (8), we obtain

$$w^2 = w_1^2 \frac{R_1' - D_1}{R + D_1} - \frac{R}{R_1'} \cos^2 \alpha + \frac{\lambda^2}{\pi^2 w_1^2} \cdot \frac{D_1^2 R_1' \cot^2 \alpha}{(R_1' - D_1)(R + D_1)} \sin^2 \alpha.$$

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Finally, Eq. (5) leads to
\[
\cos^2 \alpha = \frac{(D_1 + R_1)(R_1' - D_1)}{R_1'R},
\]
so that Eq. (75) becomes
\[
w^2 = w_1^2 + \frac{\lambda^2 D_1^2}{\pi^2 w_1'^2} + w_1^2 \frac{D_1}{R_1'} \left( \frac{D_1}{R_1'} - 2 \right).
\]
Eq. (75) provides the transverse radius of the field amplitude on an arbitrary wave surface \(S\) at a distance \(D_1\) from mirror \(M_1\).

From Eq. (75) we obtain \(dw^2/dD_1 = 0\), if
\[
D_1 = \frac{R_1'}{1 + \frac{\lambda^2 R_1'^2}{\pi^2 w_1'^2}}.
\]
(76)
If we report the value of \(D_1\), as given by Eq. (76), in Eq. (75) we obtain an extremum value of \(w^2\), which is \(w_0^2\), given by
\[
w_0^2 = \frac{w_1^2}{1 + \frac{\lambda^2 R_1'^2}{\pi^2 w_1'^2}}.
\]
(77)
According to Eq. (75), \(w^2\) tends to +\(\infty\) when \(D_1\) tends to \(\pm\infty\). Then \(w_0^2\) is a minimum for \(w^2\).

We now prove that the minimum \(w_0\) (\(w_0 > 0\)) is obtained on a plane. First, we use Eqs. (68) and (76) and obtain
\[
D_1 = \frac{R_1'}{1 + \varepsilon_1^2},
\]
(78)
where \(\varepsilon_1\) corresponds to the field transfer from \(M_1\) to \(M_2\). We use Eqs. (7) and (45) so that
\[
D_1 = \frac{L(L + R_2)}{2L - R_1' + R_2}.
\]
(79)
We introduce then \(D_2\), such that \(L = D_1 + D_2\), and obtain from Eq. (79)
\[
D_1(R_1' - L) + D_2(R_2 + L) = 0,
\]
(80)
which means that the radius of \(S\) is infinite, according to Eq. (54).

We conclude that among the wave surfaces of the \(\{S_\alpha\}\) family, a surface exists, on which the transverse radius of the field is minimum. This surface is a plane and its distance from \(M_1\) is given by Eq. (79). The disk of points \(r\) such that \(r \leq w_0\) is the resonator waist. The waist is on a plane and its transverse radius is \(w_0\), as given by Eq. (77).

### 7.9 The waist according to the resonator geometry

We consider the previous resonator, with mirrors \(M_1\) and \(M_2\), and whose waist plane is denoted \(W_0\). We then consider an hypothetic resonator whose mirrors are \(M_1\) and a plane mirror located at \(W_0\), that is, a resonator whose length is \(D_1\), given by Eq. (79). For infinite \(R_2'\) (plane mirror), Eq. (71) gives
\[
w_0^4 = \frac{\lambda^2}{\pi^2} D_1(R_1' - D_1),
\]
(81)
and Eq. (80) leads to (we use $R'_2 = -R_2$)

$$w_0^4 = \frac{\lambda^2 L(R'_1 - L)(L - R'_2)(L - R'_1 - R'_2)}{(2L - R'_1 - R'_2)^2},$$

(82)

which gives the waist radius of the initial resonator (mirrors $M_1$ and $M_2$) as a function of its geometry. (Since $\alpha_0$ is a real number, the numerator on the right side of Eq. (82) is positive according to Eq. (5).)

7.10 Rayleigh parameter

The transverse radius of the field on a given wave surface $S$ of a Gaussian beam is denoted $w$, and we introduce the Rayleigh parameter on $S$, that is, $\zeta$ such that

$$\zeta = \frac{\pi w^2}{\lambda}.$$

(83)

The Rayleigh parameter is $\zeta_0 = \pi w_0^2/\lambda$ on the waist plane, and is known then as the Rayleigh distance.

We consider two spherical caps of the family $\{S_\alpha\}$, say $S_{\alpha_1}$ (radius $R_1$) and $S_{\alpha_2}$, with corresponding Rayleigh parameters $\zeta_1$ and $\zeta_2$. From Eq. (75) we obtain

$$\zeta_2 = \zeta_1 + \frac{D^2}{\zeta_1} + \zeta_1 \frac{D}{R_1} \left( \frac{D}{R_1} - 2 \right),$$

(84)

where $D$ is the distance (algebraic measure) from $S_{\alpha_1}$ to $S_{\alpha_2}$.

7.11 Formulae related to the waist

The waist is located on $S_{\alpha_1}$, if this surface is a plane ($R_1$ is infinite). Then $\zeta_1 = \zeta_0$. According to Eq. (84), the Rayleigh parameter $\zeta$ on the surface $S$ at distance $d$ from the waist is

$$\zeta = \zeta_0 + \frac{d^2}{\zeta_0}.$$

(85)

If we set $D_1 = -d$ and $R'_1 = R$ in Eq. (84), we obtain the radius of curvature of the wave surface $S$ at a distance $d$ from the waist as

$$R = -d - \frac{\zeta_0^2}{d}.$$

(86)

Equations (85) and (86) are also classical in Gaussian beam theory.

8 The field amplitude on a wave surface. Longitudinal modes

8.1 Fundamental mode. Gouy phase

The results in the present section hold true for stable optical resonators as well as for Gaussian beams.

The fundamental mode amplitude of an optical resonator is proportional to a Gaussian function. The amplitude on the waist plane, chosen as reference for the phase, is

$$U(r) = U_0 \exp \left( -\frac{r^2}{w_0^2} \right).$$

(87)

The field transfer from the waist to a wave surface $S$ at a distance $d$ is obtained by applying a fractional Fourier transform whose order is $\alpha$, and according to Eq. (12) diffraction introduces a phase factor $\exp (i\alpha)$. 

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For infinite $R_1$, Eq. (5) gives
\[ \cot^2 \alpha = -1 - \frac{R_2}{d}, \] (88)
and then Eq. (86)
\[ \tan^2 \alpha = \frac{d^2}{\zeta_0^2}. \] (89)
Since $\alpha d > 0$, we have
\[ \tan \alpha = \frac{d}{\zeta_0} = \frac{\lambda d}{\pi w_0^2}. \] (90)
At last, we introduce the phase factor $\exp(-2\pi d/\lambda)$, which was not written in Eq. (2). If $w$ is the transverse radius on $S$, the field amplitude on $S$ is written
\[ U_d(r) = \frac{w_0}{w} U_0 \exp \left( -\frac{r^2}{w^2} - \frac{2\pi d}{\lambda} + i\alpha \right), \] (91)
where $\alpha$ is given by Eq. (90).

The factor $w_0/w$ in Eq. (91) can be explained if we consider that the power that passes through the waist (transverse radius $w_0$) is also the power that passes through the circle of radius $w$ on the wave surface $S$. The power density is proportional to the square of the amplitude modulus and to the inverse of the area ($\pi w_0^2$ and $\pi w^2$). Another explanation is based on the property of the Wigner distribution, that is, on Eq. (70). If $S_1$ and $S_2$ are two wave surfaces, then according to Eq. (70), which is a consequence of the rotation invariance of the Wigner distribution of a Gaussian field, we obtain
\[ \frac{w_1^2}{w_2^2} = \varepsilon_1 R_1 \varepsilon_2 R_2. \] (92)
From Eq. (29) and from the invariance of the Wigner distribution in a Wigner rotation, we deduce
\[ \int_{R^2} |V_2|^2 = \int_{R^2} |V_1|^2, \] (93)
and then, from Eqs. (10) and (11),
\[ \int_{R^2} |U_1|^2 = \varepsilon_2 R_2 \int_{R^2} |V_1|^2 = \frac{w_2^2}{w_1^2} \int_{R^2} |V_2|^2, \] (94)
The factor $w_0/w$ in Eq. (91) is obtained by applying Eq. (94) to $S$ and the waist plane.

Finally, we remark that the order $\alpha$ in Eq. (91) is usually called the Gouy phase [24] and is such that $\tan \alpha = d/\zeta_0$, according to Eq. (90) [22]. We conclude that the effect of diffraction on Wigner distributions associated with eigenmodes of a resonator is a rotation whose angle is opposite to the Gouy phase. (The origin of the Gouy phase is taken on the waist plane.)

### 8.2 Higher modes
If $m$ and $n$ are two positive integers, the $(m,n)$-mode corresponds to the Hermite–Gauss function $\varphi_{m,n}$ defined by
\[ \varphi_{m,n}(\xi,\eta) = H_m(\sqrt{2\pi} \xi)H_n(\sqrt{2\pi} \eta)e^{-\pi(\xi^2+\eta^2)}, \] (95)
where $H_m$ is the Hermite polynomial of order $m$, defined by
\[ H_m(\xi) = (-1)^m \exp(\xi^2) \frac{d^m}{d\xi^m} \exp(-\xi^2). \] (96)
Hermite–Gauss functions are eigenfunctions of fractional order Fourier transforms, that is, for every $\alpha$

$$\mathcal{F}_\alpha[\varphi_{m,n}] = \exp[i(m + n)\alpha] \varphi_{m,n}. \quad (97)$$

The field amplitude of the $(m, n)$–mode on the wave surface $S$ at a distance $d$ from the waist is then

$$U_d(x, y) = \frac{w_0}{w} U_0 H_m \left( \sqrt{2} \frac{x}{w} \right) H_n \left( \sqrt{2} \frac{y}{w} \right) \exp \left[ -\frac{x^2 + y^2}{w^2} - \frac{2i\pi d}{\lambda} + i(1 + m + n)\alpha \right]. \quad (98)$$

### 8.3 Resonator longitudinal modes

A factor $\exp(-2i\pi D/\lambda)$ has been omitted in Eq. (2). We consider the resonator of Sect. 7.2 once more; its length is $L$, so that by taking into account the previous factor, Eq. (52) becomes

$$V_1 = \exp \left( 2i\alpha_0 \left( 1 + m + n \right) - \frac{4i\pi L}{\lambda} \right) V_1. \quad (99)$$

We assume that $V_1$ corresponds to the $(m, n)$–eigenmode, that is, $V_1 = \varphi_{m,n}$. We set $\alpha = \alpha_0$ in Eq. (97); then Eq. (99) becomes

$$V_1 = \exp \left[ 2i\alpha_0(1 + m + n) - \frac{4i\pi L}{\lambda} \right] V_1. \quad (100)$$

We conclude that $L$, $\alpha_0$ and $\lambda$ are such that

$$\frac{2\pi L}{\lambda} - \alpha_0(1 + m + n) = q\pi, \quad (101)$$

where $q$ is an integer.

Since $\alpha_0$ depends only on the resonator geometry (resonator length and mirror radii), then only waves whose wavelengths satisfy Eq. (101) can propagate in the resonator. Hence the notion of longitudinal modes.

### 9 Concluding graphical analysis

We conclude with an elementary graphical analysis, which will be useful in discriminating stable and unstable resonators (second part of the paper). We first change the order of variables: So far an arbitrary point in the phase space has been denoted $(\rho, \phi) = (\rho_x, \rho_y, \phi_x, \phi_y)$; this point is now denoted $p = (\rho_x, \phi_x, \rho_y, \phi_y) = (P, Q)$ where $P$ and $Q$ are the projections of $p$ in the two-dimensional subspaces $(\rho_x, \phi_x)$ and $(\rho_y, \phi_y)$. The Wigner distribution $W$ is changed into the function $W_s$ defined by

$$W_s(p) = W_s(\rho_x, \phi_x, \rho_y, \phi_y) = W(\rho_x, \rho_y, \phi_x, \phi_y). \quad (102)$$

The function $W_s$, as well as $W$, represents the field amplitude, and its value $W_s(p)$ represents the state of the field at the scaled point $(\rho_x, \rho_y)$ and scaled angular variables $(\phi_x, \phi_y)$, that is, after using back scaled factors, the state of the field at a point in the physical space and at a spatial frequency.

For the sake of simplicity we still call “Wigner distribution” the function $W_s$, and from now on, denote it by $W$.

We consider a resonator made up of two mirrors $M_1$ and $M_2$ and denote $W_j$ the Wigner distribution associated with the field on $M_j$. Let $\alpha_0$ be the order of the fractional Fourier transform associated with the field transfer from $M_1$ to $M_2$. Let $p_0$ be a point in the phase space. Then $W_1(p_0)$ is also the value of the Wigner distribution $W_2$ associated with the optical field on mirror $M_2$, taken at point $p_1$ that is deduced from $p_0$ in a 4–Wigner rotation of angle $-\alpha_0$, that is
$W_2(p_1) = W_1(p_0)$. We have $p_1 = (P_1, Q_1)$, where $P_1$ (resp. $Q_1$) is deduced from $P_0$ (resp. $Q_0$) in a 2-dimensional rotation of angle $-\alpha_0$ (see Fig. 2).

We build a sequence of points $(p_j)$ as follows: point $P_{j+1}$ (resp. $Q_{j+1}$) is deduced from $P_j$ (resp. $Q_j$) in the 2-rotation of angle $-\alpha_0$ and $p_{j+1} = (P_{j+1}, Q_{j+1})$. Let $d_j$ be the Euclidean distance from $p_j$ to the origin $O$. Then the sequence $(d_j)$ is a constant sequence. This is also true for the distance from $P_j$ to $O$ and from $Q_j$ to $O$ as shown in Fig. 2. This is equivalent to saying that curves of constant amplitude of the Wigner distribution in the 2-dimensional planes $(\rho_x, \phi_x)$ and $(\rho_y, \phi_y)$ are circles; or that the Wigner distribution associated with a transverse mode of the field is invariant under Wigner rotations, as seen before.

In the second part of the paper, we will consider diffraction transfers that are represented by complex order fractional Fourier transforms. We will prove that the effect on the Wigner distribution reduces to two 2-dimensional hyperbolic rotations. We will also build sequences of representative points, and will see that sequences $(d_j)$ are diverging and correspond to unstable resonators.

**Appendix A. Proof of $\varepsilon_2 R_2 > 0$**

We assume $\alpha$ real and will prove that $\varepsilon_1 R_1$ and $\varepsilon_2 R_2$ have the same sign. We start from the identity $D(D - R_1 + R_2) = (D - R_1)(D + R_2) + R_1 R_2$, and deduce from Eq. (5) and $\cot^2 \alpha \geq 0$ ($\alpha$ real) that

$$\frac{R_1 R_2}{(R_1 - D)(D + R_2)} \geq 1. \quad (103)$$

We conclude that

$$\frac{R_1 R_2 D^2}{(R_1 - D)(D + R_2)} \geq 0, \quad (104)$$

which means that $R_1 D(R_1 - D)$ and $R_2 D(D + R_2)$ have the same sign. Then from Eqs. (7) and (8), we conclude that $\varepsilon_1 R_1$ and $\varepsilon_2 R_2$ have the same sign.
Appendix B. Angle of rotation

Eq. (30) corresponds to a rotation of the Wigner distribution, and we briefly explain why the angle of rotation is equal to $-\alpha$.

It will be enough to consider a function of two variables, say $f$. Let $g$ be defined by

$$
g(x, y) = f(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha).$$

(105)

If $\alpha = \pi/2$, we have $g(x, y) = f(-y, x)$, which means that the value of $g$ at point $P = (x, y)$ is the value of $f$ at point $Q = (-y, x)$. We notice that $P$ is deduced from $Q$ in the rotation of angle $-\pi/2$. We conclude that the graph of $g$ is deduced from the graph of $f$ in a rotation of angle $-\pi/2 = -\alpha$. The result holds true for every $\alpha$.

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