On an implicit triangular decomposition of nonlinear control systems that are 1-flat - a constructive approach

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Abstract

We study the problem to provide a triangular form based on implicit differential equations for non-linear multi-input systems with respect to the flatness property. Furthermore, we suggest a constructive method for the transformation of a given system into that special triangular shape, if possible. The well known Brunovsky form, which is applicable with regard to the exact linearization problem, can be seen as special case of this implicit triangular form. A key tool in our investigation will be the construction of Cauchy characteristic vector fields that additionally annihilate certain codistributions. In adapted coordinates this construction allows to single out variables whose time-evolution can be derived without any integration.

Key words: Differential Flatness, Differential geometry, Pfaffian systems, Nonlinear control systems, Normal-forms

1 Introduction

The concept of flatness introduced in \cite{6,7} has greatly influenced the control and systems theory community. The property of a system to be flat allows for an elegant solution for many feed-forward and/or feedback problems and is applicable for a big class of systems including the linear and the nonlinear as well as the lumped- and the distributed-parameter case. Within this paper we are interested in the system class of nonlinear multi-input systems described by ordinary differential equations. For this system class necessary and sufficient

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conditions for flatness have been proposed in [8,9] based on a polynomial matrix approach. Furthermore, for nonlinear multi-input systems with special structure further results exist, see e.g. [10,11,12,13].

Triangular forms are of special interest for nonlinear systems in the context of exact linearization or flatness. Systems that are exactly linearizable by static feedback can be converted to Brunovsky normal form (a special case of the extended Goursat form, adapted to control systems), see [5]. It is well known that systems that are flat but not exactly linearizable by static feedback can be transformed into Brunovsky normal form only after a dynamic system extension (dynamic compensator), see [4]. For systems that are 0-flat in [2] a nonlinear explicit triangular form has been proposed. We consider systems that are 1-flat, the flat output may depend on the state (0-flat) and the control (1-flat) but not on the derivatives of the control. We will consider a triangular decomposition based on implicit differential equations, and we will propose a constructive scheme how to transform a 1-flat system into that special form, if possible (this gives rise to a sufficient condition for a control system to be 1-flat). As in [5], where the extended Goursat form is discussed, we also make use of the Pfaffian system representation such that 1-forms (covector-fields) are used for the description of the (implicit) differential equations. Furthermore, we will also make use of a filtration which is connected to a triangular representation of implicit differential equations in the case of flat systems. This is in contrast to the filtration which is based on derived flags used in the exact linearization problem leading to a representation based on explicit differential equations like the Brunovsky form, see again [5]. It should be noted that the implicit triangular decomposition contains the Brunovsky form as a special case. A different approach based on differential forms, but associated with the tangent linear system, can be found in [1], where the so-called infinitesimal Brunovsky form is considered.

This contribution can be seen as further developing the ideas presented in [14,15], where a reduction and elimination procedure is considered to derive flat outputs which (in contrast to this contribution) is not based on a Pfaffian representation and the adequate tools from exterior algebra. Preliminary results have been presented already in [16], where also an extended example, the VTOL, can be found, which has been analyzed using different tools e.g. in [7,8].

Notation: Let X be an $n_x$ dimensional manifold, equipped with local coordinates $x^\alpha$, $\alpha = 1, \ldots , n_x$, i.e., $\dim(X) = \dim(x) = n_x$. We denote the partial derivatives by $\partial_{x^\alpha}$ and $\partial_x = \partial_{x^1}, \ldots, \partial_{x^n}, \ldots, \partial_{x^{n_x}}$. We will make use of the Einstein convention on sums, namely $a_i b^i = \sum_{i=1}^{n} a_i b^i$ when the index range is clear from the context. Frequently, we will use tensors in matrix representation together with index notation. Given a matrix $m(x)$ the components are given as $m^j_\alpha (x)$ where the index $j$ corresponds to the rows and $\alpha$ to the
columns. We will need to numerate matrices such that we have for example that \( m_{2,j}^{3,\alpha}(x) \) are the components of a matrix \( m_{2}^{3}(x) \). If we multiply \( m_{2}^{3}(x) \) with a vector \( w = (w^{1}, \ldots, w^{n \omega}) \) we have in components \( m_{2}^{3}(x)w = m_{3,\alpha}^{2j}(x)w^{\alpha} \) (where the components of \( w \) are represented in a list, but in matrix notation \( w \) is interpreted as a column vector) and when a vector \( v \) is partitioned into blocks e.g. \( v = (v^{1}, v^{2}) = (v^{1,1}, \ldots, v^{1,n \omega}, v^{2,1}, \ldots, v^{2,n \omega}) \) then we can compute e.g. \( m_{2}^{3}(x)v^{1} = m_{3,\alpha}^{2j}(x)v^{1,\alpha} \) (assuming appropriate dimensions of \( m_{2}^{3} \) and \( v^{1} \)). We will use the numeration of matrices and vectors to indicate to which block they belong according to the implicit triangular form to be defined.

2 The triangular form

Let us consider a nonlinear control system

\[
\dot{x} = f(x, u)
\]

with \( n_{x} \) states and \( n_{u} \) independent inputs on a manifold \( X \times U \). Roughly speaking, the system (1) is flat ((\( \kappa + 1 \))-flat), if there exist \( n_{u} \) differentially independent functions \( y(x, u, \dot{u}, \ldots, u^{(\kappa)}) \), such that the state \( x \) and the control \( u \) can be parameterized by \( y \) and its successive time derivatives. Hence, flat systems enjoy the characteristic feature that the (time) evolution of the state and input (control) variables can be recovered from that of the flat output without integration. The system is called 0-flat if \( y \) depends solely on \( x \) and 1-flat if we have \( y(x, u) \). For a rigorous definition of differential flatness, see [6,8].

The Brunovsky-form, consisting of \( n_{u} \) integrator chains, is the most simple triangular structure that can be achieved for systems (1) that are exactly linearizable by static feedback, and hence 0-flat. A different (explicit) triangular form for 0-flat systems has been proposed in [2] which is more general than the Brunovsky form. To treat the case of 1-flat systems we will present an implicit triangular form for a special subclass of systems of the form (1), that is useful regarding the property to be 1-flat.

The main idea in this contribution is to look for a diffeomorphism \((x, u) = \varphi(z)\) such that in the new coordinates \( z \) (corresponding to all the system variables including the inputs) the system (1) can be represented in a special triangular shape consisting of implicit differential equations, in a form such that the flat outputs can be read off. We first will describe some properties of the coordinates \( z \), then we introduce the implicit triangular form (definition 2), and finally we discuss how to construct the map \( \varphi \) in sections 3 and 4.

Therefore, let us consider a manifold \( Z \) where \( \dim(Z) = n_{z} \) with coordinates
z which are partitioned in \( m \) blocks of the following form

\[
z = (z^1, z^2, \ldots, z^m) = ((z^1, \ldots, z^{1,n_1}), \ldots, (z^{m,1}, \ldots, z^{m,n_m}))
\]  

(2)

where each \( z^i \) consist of \( n_{z^i} \) coordinates, i.e., \( n_z = \sum_{i=1}^{m} n_{z^i} \). In the following we will present a representation of the system (1) that among these \( z \) coordinates we will find the flat outputs \( y \) in the following form:

\[
(z^1, z^2, \ldots, z^m) = (y^1, (y^2, z^2), \ldots, (y^{m-1}, z^{m-1}), z^m)
\]  

(3)

with \( n_{z^i} = n_y + n_{\dot{z}^i} \) and \( n_y, j = 2, \ldots, m - 1. \)

**Remark 1** The \( n_{z^i} \) variables in the \( i \)-th block \( z^i = (z^{i,1}, \ldots, z^{i,n_{z^i}}) \) are decomposed into \( n_{z^i} = n_y + n_{\dot{z}^i} \) variables according to \( z^i = (y^i, \dot{z}^i) \). In the forthcoming we will show that \( y^i \) (possibly empty for \( i > 1 \)) will be part of the flat output and the \( \dot{z}^i \) will be called non-derivative variables (since they appear non-differentiated in certain blocks of the triangular form).

A desirable structure to study 1-flat system is defined by the following implicit differential equations which are decomposed into \( n_b \) blocks.

**Definition 2** The implicit differential equations \( \Xi_i^z = 0, i = 1, \ldots, n_b \) and \( m = n_b + 1 \) based on the partition (2) and (3) given as

\[
\begin{align*}
\Xi_1^z & : a_{1,1}^{1,j_1} \dot{z}^{1,a_1} - b^{1,j_1} \\
\Xi_2^z & : a_{1,2}^{2,j_2} \dot{z}^{1,a_1} + a_{2,2}^{2,2} \dot{z}^{2,a_2} - b^{2,j_2} \\
& \vdots \\
\Xi_{n_b}^z & : a_{n_b,1}^{n_b,j_{n_b}} \dot{z}^{1,a_1} + \ldots + a_{n_b,n_b}^{n_b,j_{n_b}} \dot{z}^{n_b,a_{n_b}} - b^{n_b,j_{n_b}}
\end{align*}
\]  

(4)

are termed the implicit triangular form with \( j_i = 1, \ldots, \dim(\Xi_i^z) \) and \( a_i = 1, \ldots, n_{z^i} \) for \( i = 1, \ldots, n_b \), which possesses the following properties

(a) the matrices \( a_k^i \) and (the vectors) \( b_i \) meet

\[
\begin{align*}
a^{i,j_i}_{k,a_k} & = a^{i,j_i}_{k,a_k}(z^1, \ldots, \dot{z}^i, \dot{z}^{i+1}) \\
b^{j_i} & = b^{j_i}(z^1, \ldots, \dot{z}^i, \dot{z}^{i+1})
\end{align*}
\]  

(5)

(b) \( \dim(\Xi_i^z) = \dim(\dot{z}^{i+1}) \), and the Jacobian matrices \([\partial_{\dot{z}^{i+1}} \Xi_i^z] \) are regular for

all \( i = 1, \ldots, n_b = m - 1. \)

It should be noted that \( j_i, a_k \) in (4) are indices corresponding to the rows and the columns of the matrices \( a_k^i \) and vectors \( b_i \) respectively (summation over the \( a_i \)), where each subsystem can be represented as \( \Xi_i^z : a^{i,j_i}_{k,a_k} \dot{z}^{k,a_k} - b^{j_i} \).
(summation over \( k \) and \( \alpha_k \) with \( k \leq i \)) and that due to (5) the dependence on the \( z \) coordinates is arranged in a triangular manner, as demonstrated in the following example.

**Example 3** A system in triangular form with \( n_b = 3 \) (3 blocks for the equations) and thus \( m = n_b + 1 = 4 \) (4 blocks in \( z \)) in matrix notation reads as

\[
\begin{align*}
\Xi^1_e &: a^1_1(z^1, \hat{z}^2) \dot{z}^1 - b^1(z^1, \hat{z}^2) \\
\Xi^2_e &: a^2_1(z^1, z^2, \hat{z}^3) \dot{z}^1 + a^2_2(z^1, z^2, \hat{z}^3) \dot{z}^2 - b^2(z^1, z^2, \hat{z}^3) \\
\Xi^3_e &: a^3_1(z^1, z^2, z^3, \hat{z}^4) \dot{z}^1 + \ldots + a^3_{32}(z^1, z^2, z^3, \hat{z}^4) \dot{z}^3 - b^3
\end{align*}
\]

where \( b^3 = b^3(z^1, z^2, z^3, \hat{z}^4) \) and

\[
z = (z^1, z^2, z^3, z^4) = (y^1, (y^2, \hat{z}^2), (y^3, \hat{z}^3), \hat{z}^4)
\]

such that \( y^2 \) and/or \( y^3 \) are possibly empty (but they need not, \( n_{y^2}, n_{y^3} \geq 0 \)). We require \( \dim(\Xi^i_e) = \dim(\hat{z}^{i+1}) \), and the Jacobian matrices \([\partial \hat{z}^{i+1}/\partial \Xi^i_e]\) are regular for all \( i = 1, \ldots, 3 \) such that \( \hat{z}^{i+1} \) can be computed by means of the implicit function theorem from \( \Xi^i_e \).

**Lemma 4** \( y \) is a flat output for the system (4).

To prove this Lemma, we consider the implicit equations \( \Xi^1_e = 0 \). Then \( z^1(t) = y^1(t) \) can be assigned freely, and \( \dot{z}^2(t) \) can be computed, where we make use of the implicit function theorem. We continue with the equations \( \Xi^2_e = 0 \). Two scenarios are possible: \( z^2 = \hat{z}^2 \), then it can be easily checked that \( \ddot{z}^3(t) \) can be computed using the same argument as for \( \dot{z}^2(t) \), since \( z^1(t) \) and \( z^2(t) \) are already given. If \( z^2 = (y^2, \hat{z}^2) \) then \( y^2(t) \) can be chosen freely, since the rank criteria is met for \( \dot{z}^3(t) \), which again can be computed. By continuing this procedure, we end up with the equations \( \Xi^{n_b}_e = 0 \) from which \( \dot{z}^m(t) \) can be computed since at this stage \( z^1(t), \ldots, z^{m-1}(t) \) are already known. This clearly shows that \( (y^1, \ldots, y^{m-1}) \) is a flat output for (4).

**Proposition 5** (a sufficient condition) The control system (1) is 1-flat if we can find locally a diffeomorphism \((x, u) = \varphi(z^1, \ldots, z^m)\) with \( \sum_{i=1}^m n_{z^i} = n_u + n_x \), such that it can be represented in the form (4) and \( \sum_{i=1}^{m-1} n_{y^i} = n_u \).

The fact that this proposition is sufficient for 1-flat systems, comes from the observation that the flat outputs are among the \( z \) coordinates, and therefore clearly a function of \( x \) and \( u \) as a special case also 0-flat systems are included. Furthermore, it should be noted that we consider a diffeomorphism which implies that we do not increase the dimension of the system variables.

The goal is now to provide a constructive algorithm that transforms a nonlinear multi-input system (1), if possible, into the form (4). Before we will
analyze this in detail, let us consider an example.

2.1 A motivating example

We consider a system with three state variables \((x^1, x^2, x^3)\) and two control inputs \((u^1, u^2)\) of the form

\[
\dot{x}^1 = u^1, \quad \dot{x}^2 = u^2, \quad \dot{x}^3 = \sin \left(\frac{u^1}{u^2}\right)
\]

also analyzed in [8,14] using a different approach. Let us introduce the local coordinate transformation \((x^1, x^2, x^3, u^1, u^2) = \varphi(y^1, \hat{z}^2, y^2, \hat{z}^3, \hat{z}^4)\) together with its inverse

\[
\begin{align*}
x^1 &= \hat{z}^2 \hat{z}^3, & y^1 &= x^3 \\
x^2 &= \hat{z}^3 + y^2, & \hat{z}^2 &= \frac{u^1}{u^2} \\
x^3 &= y^1, & y^2 &= x^2 - x^1 \frac{u^2}{u^1} \\
u^1 &= e^{\hat{z}^4} \hat{z}^2, & \hat{z}^3 &= x^1 \frac{u^2}{u^1} \\
u^2 &= e^{\hat{z}^4}, & \hat{z}^4 &= \ln(u^2).
\end{align*}
\]

Then the system (6) in the new coordinates can be represented as

\[
\begin{align*}
\dot{y}^1 - \sin(\hat{z}^2) &= 0 \\
-\dot{y}^2 \hat{z}^2 + \hat{z}^2 \hat{z}^3 &= 0 \\
\dot{y}^2 + \hat{z}^3 - e^{\hat{z}^4} &= 0
\end{align*}
\]

(by a suitable combination of the equations) which is an implicit system of differential equations. Following the proof of Lemma 4 it can be seen that the flat outputs are obviously \(y^1\) and \(y^2\) and in \((x, u)\) coordinates they read as \(y^1 = x^3, y^2 = x^2 - x^1 \frac{u^2}{u^1}\) based on (7).

**Example 6** The system (8) possesses the structure (4) as in example 3 with \(n_b = 3, m = 4\) and \(\dim(z^1) = \dim(z^2) = \dim(z^3) = 1, \dim(z^4) = 2\) and the matrices

\[
\begin{align*}
a^1_1 &= 1, & b^1 &= \sin(\hat{z}^2) \\
a^2_1 &= 0, & a^2_2 &= [-\hat{z}^2, \hat{z}^3], & b^2 &= 0 \\
a^3_1 &= 0, & a^3_2 &= [1, 0], & a^3_3 &= 1, & b^3 &= e^{\hat{z}^4}.
\end{align*}
\]

The key question is now, how to derive the coordinate transformation (7) and how must the equations be combined, such that the form (8) can be obtained.
These questions will both be answered at once by using a Pfaffian system representation.

3 Pfaffian representation

We will use tools from exterior algebra and Pfaffian systems in the sequel where we refer for detailed information to [3] and references therein. For a representation of nonlinear control systems in a Pfaffian form (with the focus on exact linearization with static feedback), see e.g. [5] and references therein. It should be noted that we do not base our considerations on the tangent linear system, as it is used for instance in [1].

3.1 Exterior Algebra and Properties of Pfaffian systems

We denote by $d\omega$ the exterior derivative of the $k$-form $\omega$ and by $v\lfloor\omega$ the contraction (interior product) of $\omega$ by the vector field $v$. The exterior product (wedge product) is denoted by $\wedge$.

A Pfaffian system $P$ on an $n_\zeta$-dimensional manifold $\mathcal{Z}$ with coordinates $(\zeta^\alpha)$ $\alpha = 1, \ldots, n_\zeta$ can be identified with a codistribution $P = \{\omega^1, \ldots, \omega^n\}$, with $\omega^i = m^i_\alpha(\zeta)d\zeta^\alpha$. The annihilator of a Pfaffian system $P$ is a distribution on $\mathcal{Z}$ denoted by $P^\perp := \{w \in T(\mathcal{Z}), w\lfloor\omega = 0, \forall \omega \in P\}$.

The derived flag of the Pfaffian System $P$ is the descending chain of Pfaffian systems $P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \ldots$ with $P^{(0)} = P$ and $P^{(k+1)} := \{\omega \in P^{(k)}, d\omega = 0 \mod P^{(k)}\}$.

Cauchy characteristic vector fields $v$ of $P$ meet

$$v \lfloor P = 0, \quad v \lfloor dP \subset P. \quad (9)$$

The importance of Cauchy characteristic vector fields lies in the fact that the Pfaffian system $P$ can be written using $n_\zeta - c$ coordinates (after a suitable coordinate transformation), where $c$ denotes the number of all independent Cauchy characteristic vector fields. The distribution formed by the Cauchy characteristic vector fields is denoted by $\mathcal{C}(P)$ which is involutive by construction, see [3]. The desired coordinates can be constructed by means of the Straightening out theorem (Frobenius theorem) such that coordinates are introduced that are adapted to the involutive distribution $\mathcal{C}(P)$. Indeed, by choosing adapted coordinates $(\bar{\zeta}, \tilde{\zeta}) = (\bar{\zeta}^1, \ldots, \bar{\zeta}^c, \tilde{\zeta}^{c+1}, \ldots, \tilde{\zeta}^{n_\zeta})$ such that $\mathcal{C}(P) = \{\partial_{\bar{\zeta}}\}$ the Pfaffian system can be represented solely using the $\tilde{\zeta}$ coordinates, see again [3].
In the following we consider \textit{time-invariant} dynamical systems represented as \textit{Pfaffian systems} on bundles. These systems possess a fibration with respect to the time-manifold, i.e. let $\mathcal{X}$ be an $n_\xi$-dimensional manifold, the corresponding fibration is $\mathcal{X} \times \mathbb{R} \to \mathbb{R}$, then a time-invariant Pfaffian system $P$ is identified with a codistribution on the $(n_\xi + 1)$-dimensional manifold $\mathcal{X} \times \mathbb{R}$ and is locally spanned by 1-forms $\omega$ of the form

$$\omega^i = m^i_a(\xi)d\xi^a - n^i(\xi)dt, \quad P = \{\omega^1, \ldots, \omega^{n_P}\} \quad (10)$$

with $n_P = \dim(P)$. To $\omega^i$ there correspond the implicit differential equations $\omega^i_\epsilon = 0$ with $\omega^i_\epsilon = m^i_a(\xi)\dot{\xi}^a - n^i(\xi)$, where $\dot{\xi}$ denotes the time-derivative.

Due to the fibration with respect to the time manifold we can introduce a special kind of annihilator.

**Definition 7** The vertical annihilator of (10) denoted by $\mathcal{V}(P)\perp$ is defined to be the annihilator of the extended Pfaffian system $\{P, dt\}$.

It is clear that $\mathcal{V}(P)\perp \subset P\perp$, i.e., one picks only those vector fields in $P\perp$ which are tangential to the fibration (those that do not include a $\partial_t$ component).

**Example 8** Let us consider the explicit control system $\dot{x} = f(x,u)$ written as a Pfaffian system $P = \{\omega^{\alpha x}\}$ on a manifold with coordinates $(t, x, u)$ which is fibred over the time with

$$\omega^{\alpha x} = dx^{\alpha x} - f^{\alpha x}(x,u)dt, \quad \alpha_x = 1, \ldots, n_x$$

i.e., $\xi = (x,u)$. Then we have $P\perp = \{\partial_t + f^{\alpha x}(x,u)\partial_{x^{\alpha x}}, \partial_u\}$ as well as $\mathcal{V}(P)\perp = \{\partial_u\}$.

**Definition 9** We call a Pfaffian system $P$ as in (10) parameterizable with respect to $\hat{\xi}$ when we can find appropriate coordinates $(\bar{\xi}, \hat{\xi})$ as well as a diffeomorphism $\xi = \psi(\bar{\xi}, \hat{\xi})$ with $n_\xi = n_{\bar{\xi}} + n_{\hat{\xi}}$ where $n_{\hat{\xi}} = n_P = \dim(P)$ such that $P$ is represented as

$$\bar{\omega}^i = \psi^*(\omega^i) = q^i_{a}(\bar{\xi}, \hat{\xi})d\bar{\xi}^a - r^i(\bar{\xi}, \hat{\xi})dt \quad (11)$$

$\alpha = 1, \ldots, n_\xi$ and such that the differential equations corresponding to (11), i.e. $\bar{\omega}^i_\epsilon = 0$ with $\bar{\omega}^i_\epsilon = q^i_{a}(\bar{\xi}, \hat{\xi})\dot{\bar{\xi}}^a - r^i(\bar{\xi}, \hat{\xi})$, fulfill that the Jacobian matrix $[\partial_{\xi} \bar{\omega}^i_\epsilon]$ is regular (and quadratic since $n_{\hat{\xi}} = n_P$).

The variables $\hat{\xi}$ are termed \textit{non-derivative} variables and by the implicit function theorem we locally have $\hat{\xi} = g(\bar{\xi}, \dot{\bar{\xi}})$. Furthermore, it should be noted that $\{\partial_{\xi}\} \subset \mathcal{V}(P)\perp$ holds (since no $d\xi$ appears).
3.2 The implicit triangular form in Pfaffian representation

Let us consider the implicit differential equations as in (4) written using differential forms on the bundle \( Z \times \mathbb{R} \to \mathbb{R} \) with the same properties as described in definition 2 where \( \Xi^i \) corresponds to the Pfaffian representation of \( \Xi^i _e \)

\[
\begin{align*}
\Xi^1 : & \ a_{1,i_1}^{1,j_1} dz^{1,\alpha_1} - b_{1,j_1}^{1} dt \\
\Xi^2 : & \ a_{1,i_1}^{2,j_2} dz^{1,\alpha_1} + a_{2,i_2}^{2,j_2} dz^{2,\alpha_2} - b_{2,j_2}^{2} dt \\
& \vdots \\
\Xi^n_b : & \ a_{1,i_1}^{n_b,j_{n_b}} dz^{1,\alpha_1} + \ldots + a_{n_b,i_{n_b}}^{n_b,j_{n_b}} dz^{n_b,\alpha_{n_b}} - b_{n_b,j_{n_b}}^{n_b} dt
\end{align*}
\]

(12)

Let us denote by \([1] S_{d,0}\) the system (12) and by \( S_{d,k} = \{\Xi^1, \ldots, \Xi^{n_b-k}\}\).

**Proposition 10** The system (12) with \( m = n_b + 1 \) enjoys the following properties

(a) \( \{\partial_{\hat{z}^{m-k}}\} \subset V(S_{d,k})^\perp \) are involutive distributions and \( \partial_{\hat{z}^{m-k}} \in C(S_{d,k+1}) \) for \( k = 0, \ldots, n_b - 1 \).

(b) Each subsystem \( \Xi^k \) is parameterizable with respect to the non-derivative variable \( \hat{z}^{k+1} \), i.e.

\[
\hat{z}^{k+1} = g^{k+1}(\hat{z}^1, \ldots, \hat{z}^k, \dot{z}^1, \ldots, \dot{z}^k)
\]

for \( k = 1, \ldots, n_b \).

(c) If in \( \Xi^k \), \( \hat{z}^k = (y^k, \hat{z}^k) \) such that \( n_{\hat{z}^k} > n_{\hat{z}^k} \), i.e., variables \( y^k \) are present, then \( \partial_{y^k} \in C(S_{d,m-k}) \).

The proof of this proposition is straightforward and follows from the structure of (12) together with the special structure of the \( a_{k,\alpha_k}^{i,j_1} \) and \( b^{j_1} \) according to (5) as in definition 2.

**Corollary 11** The implicit triangular decomposition (12) gives rise to the decomposition of \( S_{d,0} \) into a sequence of Pfaffian systems

\[
\ldots \subset S_{d,2} \subset S_{d,1} \subset S_{d,0}
\]

(13)

as well as to splittings of the form \( S_{d,i} = S_{d,i+1} \oplus S_{d,i+1,c} \), where all the \( S_{d,i+1,c} \) are parameterizable with respect to the corresponding non-derivative variables \( \hat{z} \).

The subscript \( d \) will always refer to a representation based on the desired triangular decomposition (12).
Example 12 (Example 3 cont.) Following the notations in proposition 10 we have 

\[ S_{d,0} = \{ \Xi_{1}, \Xi_{2}, \Xi_{3} \}, \quad S_{d,1} = \{ \Xi_{1}, \Xi_{2} \} \quad \text{and} \quad S_{d,2} = \{ \Xi_{1} \} \]

since \( n_{b} = 3 \). We observe that \( \{ \partial \hat{z}_{4} \} \subset V(S_{d,0})^\perp \) as well as \( \{ \partial \hat{z}_{4} \} \subset C(S_{d,1}) \) which is obvious since \( \hat{z}_{4} \) are non-derivative variables which only appear in \( \Xi_{3} \). The same holds true regarding \( \hat{z}_{3} \) where now \( \{ \partial \hat{z}_{3} \} \subset V(S_{d,1})^\perp \) and \( \{ \partial \hat{z}_{3} \} \subset C(S_{d,2}) \).

Proposition 10 (c) means for instance that if \( z^{2} = (y^{2}, \hat{z}^{2}) \) then \( \partial y^{2} \subset C(S_{d,2}) \) since in \( \Xi_{1} \) only \( \hat{z}^{2} \) appears.

4 A constructive algorithm

The goal is now to develop a constructive scheme that subsequently creates this sequence (13) as well as appropriate coordinate transformations based on a given control system of the form

\[ S_{0} = \{ \omega_{0}^{\alpha} \} \]

\[ \omega_{0}^{\alpha} = dx^{\alpha} - f_{\alpha}^{\alpha}(x, u)dt. \] (14)

The starting point of the scheme is the explicit system \( S_{0} \) but since linear combinations of the \( \omega_{0}^{\alpha} \) lead to implicit equations in general we demonstrate the constructive method with the system \( S_{k} = \{ \omega_{k} \} \) (here the index \( k \) refers to the \( k \)-th iteration of the reduction process) with

\[ \omega_{k}^{i} = m_{i}^{\alpha}(\xi)d\xi^{\alpha} - n_{i}(\xi)dt \] (15)

with \( i = 1, \ldots, n_{e} \) and \( n_{k} > n_{e} \), where we denote by \( \xi \) all the system variables. (14) is a special case of (15), i.e. \( \xi = (x, u) \) in \( S_{0} \). The following steps need to be performed

(a) Computation of \( V(S_{k})^\perp \), since these elements correspond to non-derivative variables. Choosing of an involutive \( F_{k} \subset V(S_{k})^\perp \) corresponds to a selection of non-derivative variables called \( \hat{w}_{k} \). (This correspondence becomes obvious in an adapted coordinate chart to be constructed by means of the Straightening out theorem.)

(b) Construction of a splitting \( S_{k} = S_{k+1} \oplus S_{k+1,c} \) such that \( F_{k} \subset C(S_{k+1}) \), since this guarantees that \( S_{k+1} \) is independent of \( \hat{w}_{k} \).

(c) Check, if \( S_{k+1,c} \) is parameterizable with respect to the \( \hat{w}_{k} \), which is possible only if \( \dim(S_{k}) = \dim(S_{k+1}) + \dim(F_{k}) \) holds.

The whole procedure will then be continued with \( S_{k+1} \).

4.1 The \( k \)-th step of the system decomposition

The constructive scheme rests on the following proposition.
Proposition 13 Let us consider the system \( S_k = \{ \omega_k^i \} \) with \( \omega_k^i \) as in (15). If we find an involutive distribution \( \mathcal{F}_k \) with \( \mathcal{F}_k \subset \mathcal{V}(S_k)^\perp \) and a sub-codistribution \( S_{k+1} \subset S_k \) such that \( \mathcal{F}_k \subset \mathcal{C}(S_{k+1}) \) is met, then we obtain a splitting \( S_k = S_{k+1} \oplus S_{k+1,c} \) with

\[
S_{k+1} : \omega_{k+1}^i = a^i_{\alpha}(w_k)dw_k^\alpha - b^i(w_k)dt \\
S_{k+1,c} : \omega_{k+1,c}^j = a^j_{\alpha,c}(w_k, \hat{w}_k)dw_k^\alpha - b^j(w_k, \hat{w}_k)dt
\]

in adapted coordinates \((w_k, \hat{w}_k)\) by using a diffeomorphism \( \xi = \varphi_k(w_k, \hat{w}_k) \) with \( n_\xi = n_{w_k} + n_{\hat{w}_k} \).

The adapted coordinates can be constructed by means of the Straightening out theorem since \( \mathcal{F}_k \) is involutive, such that in new coordinates \( \mathcal{F}_k = \{ \partial \hat{w}_k \} \). In these adapted coordinates \( \partial \hat{w}_k \subset \mathcal{V}(S_k)^\perp \) as well as \( \partial \hat{w}_k \subset \mathcal{C}(S_{k+1}) \) is met, therefore no \( d\hat{w}_k \) can appear and a basis of \( S_{k+1} \) must exist which is independent of the \( \hat{w}_k \) coordinates, since \( \partial \hat{w}_k \subset \mathcal{C}(S_{k+1}) \). Furthermore, if the system \( S_{k+1,c} \) is parameterizable with respect to \( \hat{w}_k \) and if the system \( S_{k+1} \) that can be expressed in the \( w_k \) coordinates possesses a non-trivial Cauchy characteristic, then these redundant variables are candidates for possible flat outputs. Based on these considerations we state the following corollary which additionally includes the parameterization criteria, such that proposition (13) is connected with the triangular form (4), respectively (12).

Corollary 14 The system \( S_0 \) (14) can be transformed into the form (12) if we find a sequence of codistributions

\[
\ldots \subset S_2 \subset S_1 \subset S_0
\]

as well as involutive distributions \( \mathcal{F}_l \) that meet \( \mathcal{F}_l \subset \mathcal{V}(S_l)^\perp \) as well as \( \mathcal{F}_l \subset \mathcal{C}(S_{l+1}) \) for \( l \geq 0 \) such that the systems \( S_{l+1,c} \) according to \( S_l = S_{l+1} \oplus S_{l+1,c} \) are parameterizable with respect to \( \mathcal{F}_l \). Then also \( \dim(S_{l+1,c}) = \dim(\mathcal{F}_l) \) holds where we assume that each \( S_l \) is represented by a minimal number of variables.

This sequence ends when we have a decomposition of the form \( S_{k^*} = S_{k^*+1} \oplus S_{k^*+1,c} \) with \( S_{k^*+1} \) the empty system, which means that \( S_{k^*} \) is a parameterizable system. This iterative scheme has therefore to be continued until a parameterizable system is obtained. It should be noted that in practice the effective computation of \( \mathcal{F}_l \) and \( S_{l+1} \) such that additionally parametrization is guaranteed for all elements of the sequence is a difficult task. We will comment on computational issues in section 4.3 and demonstrate on an example a possible strategy.

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4.2 The connection with the derived flag

In this short paragraph we want to discuss how the derived flag, see [3] and its application to the exact linearizability problem as described e.g. in [5] is connected to our filtration, as in corollary 14.

Let us introduce an adapted basis for $S_k$, $\dim(S_k) = h$ (with respect to the derived flag) which is $S_k = \{\bar{\Omega}_1^j, \ldots, \bar{\Omega}_i^j, \Omega_{i+1}^j, \ldots, \Omega_h^j\}$. The first derived flag of $S_k$, denoted by $S_k^{(1)}$, meets $\bar{\Omega}_i^j \in S_k^{(1)} \subset S_k$, $j = 1, \ldots, \dim(S_k^{(1)})$ such that

$$d\Omega_i^j = \alpha_i^j \wedge \bar{\Omega}_k^i + \beta_i^j \wedge \Omega_k^r$$

(17)

holds for suitable 1-forms $\alpha_i^j, \beta_i^r$.

If $\{S_k^{(1)}, dt\}$ is integrable (Frobenius theorem, [3]), we have furthermore that

$$d\bar{\Omega}_k^i = \gamma_i^j \wedge \bar{\Omega}_k^i + \rho_i^j \wedge dt$$

(18)

is met, for suitable 1-forms $\gamma_i^j, \rho_i^j$.

**Proposition 15** Let us consider any sequence of Pfaffian systems $\ldots \subset S_2 \subset S_1 \subset S_0$ with $S_0$ as in (14) with elements $S_k$ together with their first derived systems $S_k^{(1)}$. Then for all $v \in \mathcal{V}(S_k)^\perp$ we have

(a) $v|dS_k^{(1)} \subset S_k^{(1)}$ is met if $\{S_k^{(1)}, dt\}$ is integrable.

(b) $v|dS_k^{(1)} \subset S_k$

The proof of the first claim (a) follows by evaluating $v|d\bar{\Omega}_k^i$ using (17) and (18) for $v \in \mathcal{V}(S_k)^\perp$ and the second (b) can be shown by using (17) in a straightforward manner.

Systems that are exactly linearizable by static feedback meet $\{S_k^{(1)}, dt\}$ is integrable for every $k$, see [5].

**Corollary 16** From Proposition 15 it follows that for systems $S_0$ that are exactly linearizable by static feedback the sequence of the derived flags corresponds to the sequence as in Corollary 14 and $F_k$ corresponds to $\mathcal{V}(S_k)^\perp$ which is integrable by construction.

**Remark 17** The interesting case are of course examples that are not exactly linearizable by static feedback, i.e. $\{S_k^{(1)}, dt\}$ are not integrable, since then a different filtration has to be considered that may lead to an implicit triangular form.
4.3 A constructive method to derive $\mathcal{F}_k$ and $S_{k+1}$

If one is able to construct the sequence as in Corollary 14, then one eventually ends up with the form (12) (by relabeling the coordinates) where in each step the involutive distribution $\mathcal{F}_i$ has to be integrated, in order to derive the coordinate transformation. Thus, in principle a constructive method that generates the implicit triangular decomposition is stated. If then the rank and dimension condition as in corollary 14 hold the system is 0-flat/1-flat, but it should be stressed that this method is only sufficient for flatness, and a failure does not in general prove that a system is not flat.

However, the construction of $S_{k+1} \subset S_k$ such that an involutive distribution $\mathcal{F}_k$ can be found that meets $\mathcal{F}_k \subset \mathcal{V}(S_k)^\perp$ as well as $\mathcal{F}_k \subset \mathcal{C}(S_{k+1})$ is a difficult task and leads in general to partial differential equations. Furthermore, since the choice $\mathcal{F}_k \subset \mathcal{V}(S_k)^\perp$ as well as of $S_{k+1}$ with $\mathcal{F}_k \subset \mathcal{C}(S_{k+1})$ is not unique in general (branching points may appear) it might be necessary to iterate the construction of $\mathcal{F}_k$ and $S_{k+1}$ (see section 5.2) - it should be noted that based on a simple necessary condition candidates for $\mathcal{F}_k$ and $S_{k+1}$ are singled out as shown next.

We have to construct $\mathcal{F}_k \subset \mathcal{V}(S_k)^\perp$ and $S_{k+1} \subset S_k$ such that $\mathcal{F}_k | dS_{k+1} \subset S_{k+1}$ is met. Then also the necessary condition

$$\mathcal{F}_k | dS_{k+1} \subset S_k$$  \hspace{1cm} (19)

holds, since $S_{k+1} \subset S_k$. For $\mathcal{V}(S_k)^\perp = \{v_i\}$ and $S_k = \{\omega^j\}$ with $r = \text{dim}(S_k)$ we derive the purely algebraic conditions (necessary conditions)

$$c_i v_i \lrcorner d(a_j \omega^j) \wedge (\omega^1 \wedge \ldots \wedge \omega^r) = 0$$  \hspace{1cm} (20)

where $c_i$ and $a_j$ depend on all the system variables.

It should be noted that due to the requirement $\text{dim}(S_k) = \text{dim}(S_{k+1}) + \text{dim}(\mathcal{F}_k)$ one has to find $\text{dim}(S_k) - \text{dim}(\mathcal{F}_k)$ independent solutions $a_j \omega^j$ for $S_{k+1}$. From (b) in Proposition 15 we see that $S_k^{(1)}$ fulfills this necessary condition independently of $\mathcal{F}_k$. If we furthermore assume that $S_k^{(1)} \subset S_{k+1}$ then the construction of $S_{k+1}$ and $\mathcal{F}_k$ can be simplified further as demonstrated in the next section in great detail. The strategy is now to solve the necessary conditions (19) or which is the same (20) and to generate solutions for which then finally the criteria $\mathcal{F}_k | dS_{k+1} \subset S_{k+1}$ has to be checked as well as the parametrization as in corollary 14.
5 Examples

We now present two examples, in the first one we show how one algorithmically can compute the sequence of codistributions using the necessary condition (20) and the second example demonstrates a case where the algorithm stops in a dead end, and another iteration is at need.

5.1 The motivating example revisited

Let us write the equations (6) as a Pfaffian system of the form $S_0 = \{\omega_1^0, \omega_2^0, \omega_3^0\}$ with

\[
\begin{align*}
\omega_1^0 &= dx^1 - u^1 dt \\
\omega_2^0 &= dx^2 - u^2 dt \\
\omega_3^0 &= dx^3 - \sin\left(\frac{u^1}{u^2}\right) dt,
\end{align*}
\]

then we obtain the following proposition regarding the first reduction step.

**Proposition 18** Given the system $S_0$ as in (21) we derive a splitting of the form $S_0 = S_1 \oplus S_{1,c}$ as well as $v_0$ that meets $v_0 \in V(S_0) \perp$ and $v_0 \in C(S_1)$. Indeed,

\[
v_0 = u^1 \partial_{u^1} + u^2 \partial_{u^2}
\]

and $S_1 = \{\omega_1^1 = \omega_3^0, \omega_2^1 = u^2 \omega_0^1 - u^1 \omega_0^2\}$ with

\[
\begin{align*}
\omega_1^1 &= dx^3 - \sin\left(\frac{u^1}{u^2}\right) dt, \\
\omega_2^1 &= u^2 dx^1 - u^1 dx^2
\end{align*}
\]

as well as the complement $S_{1,c} = \{\omega_{1,c}^3\}$ with $\omega_{1,c}^3 = \omega_0^3 = dx^2 - u^2 dt$ possess the required properties.

The proof of this proposition follows from the observation that $V(S_0) \perp = \{\partial_{u^1}, \partial_{u^2}\}$ and that $v_0, dS_1 \subset S_1$ as desired. We will now show how one can derive $S_1$ and $v_0$.

**Calculation 19** The first derived system $S_0^{(1)}$ is given by the single form

\[
\Phi = \cos\left(\frac{u^1}{u^2}\right) \left(\frac{u^1}{u^2} dx^2 - dx^1\right) + u^2 (dx^3 - \sin\left(\frac{u^1}{u^2}\right) dt)
\]

and a basis for $S_0$ can be alternatively given as $S_0 = \{\omega_0^1, \omega_0^2, \Phi\}$. To construct
we assume that $S_0^{(1)} \subset S_1$ and consider the relation (according to (20))
\[
(c_1^1 \partial_{u^1} + c_1^2 \partial_{u^2}) \left[ d(a_1^1 \omega_0^1 + a_2^1 \omega_0^2 + a_3^1 \Phi) \lor \omega_0^1 \lor \omega_0^2 \lor \Phi = 0 \right.
\]
where $c_1^i$ and $a_1^i$ are functions of all system variables that have to be computed.

From (23) we are left with the equation $c_1^1 a_1^1 + c_1^2 a_1^2 = 0$ or $a_1^i = -a_1^2 \frac{c_1^1}{c_1^2}$. This means that the forms $\Phi$ and $\omega_0^2 - \frac{c_1^2}{c_1^1} \omega_0^1$ fulfill the necessary conditions for the vector field $v_0 = c_1^1 \partial_{u^1} + c_1^2 \partial_{u^2}$. To determine $c_1^1$ and $c_1^2$ we consider the criteria
\[
(c_1^1 \partial_{u^1} + c_1^2 \partial_{u^2}) \left( \Phi \lor (\omega_0^2 - \frac{c_1^2}{c_1^1} \omega_0^1) \lor \Phi = 0. \right.
\]

For the solution of (24) of the form $c_1^1 = c_2^2 \omega^1$ we have that $\Phi$ and $\omega_0^2 - \frac{c_1^2}{c_1^1} \omega_0^1$ clearly correspond to $S_1$ as in (22) as can be checked easily (by linear combinations) and that
\[
(v_0 \lor \omega_0^2) \lor \omega_0^1 \lor \omega_0^2 = 0
\]
is fulfilled, such that $v_0 \lor \omega_1 \subset S_1$ is met, because of (24,25).

To straighten out $v_0$ we consider the coordinate transformation $(x^1, x^2, x^3, u^1, u^2) = \varphi_0(w^1, w^2, w^3, w^4, \dot{w})$ with $x^i = w^i$ for $i = 1, 2, 3$ and
\[
\begin{align*}
  u^1 &= e^{\dot{w}} w^4 \\
  u^2 &= e^{\dot{w}}
\end{align*}
\]
which is based on the flow of $v_0$. In new coordinates we obtain a basis for $S_1$ as
\[
\begin{align*}
  \omega_1^1 &= d\omega^3 - \sin(w^4)dt \\
  \omega_1^2 &= d\omega^1 - w^4dw^2
\end{align*}
\]
and for the complement $S_{1,c} = \{\omega_1^c\}$ with $\omega_1^c = d\omega^2 - e^{\dot{w}}dt$ and it can be checked easily that $\dim(F_0) = 1$, $\dim(S_1) + 1 = \dim(S_0)$ and that the Jacobian $\partial_{\dot{w}}(\dot{w}_2 - e^{\dot{w}})$ has maximal rank.

Remark 20 We want to point out again, that $v_0$ is a Cauchy characteristic vector field for $S_1$, i.e. $v_0 \in C(S_1)$ and this guarantees that there is a basis for $S_1$ which does not depend on the coordinate $\dot{w}$, since in new coordinates $\partial_{\dot{w}} \in C(S_1)$ is met.

Then we continue our considerations with $S_1$ and the following proposition states the second reduction step.
Proposition 21. Given the system \( S_1 \) as in (26) we derive a splitting of the form \( S_1 = S_2 \oplus S_{2,c} \) as well as \( v_1 \) that meets \( v_1 \in \mathcal{V}(S_1)^\perp \) and \( v_1 \in \mathcal{C}(S_2) \) with

\[
v_1 = w^4 \partial_{w^1} + \partial_{w^2}
\]

and

\[
S_2 = \{ \omega_2^1 \} , \quad \omega_2^1 = dw^3 - \sin \left( w^4 \right) dt
\]

and \( S_{2,c} = \{ \omega_{2,c}^2 \} \) with \( \omega_{2,c}^2 = dw^1 - w^4 dw^2 \).

The proof follows again from the fact that \( \mathcal{V}(S_1)^\perp = \{ w^4 \partial_{w^1} + \partial_{w^2} \} \) and \( v_1 \) \( dS_2 \subset S_2 \). The construction of \( S_2 \) can be performed in the same manner as above. (Observe however that \( S_1^{(1)} \) is empty, but from

\[
(c_2^1(w^4 \partial_{w^1} + \partial_{w^2}) + c_2^2 \partial_{w^4}) \lhd (a_1^2 \omega_1^1 + a_2^2 \omega_1^2) \wedge \omega_1^1 \wedge \omega_1^2 = 0
\]

that result follows at once).

Based on the flow of \( v_1 \) we derive the map \( w = \varphi_1(q, \hat{q}) \) in the form

\[
\begin{align*}
  w^1 &= \hat{q}q^4, \quad w^3 = q^3 \\
  w^2 &= \hat{q} + q^2, \quad w^4 = q^4.
\end{align*}
\]

With

\[
(y^1 = q^3, y^2 = q^2, \hat{z}^2 = q^4, \hat{z}^3 = \hat{q}, \hat{z}^4 = \hat{w})
\]

it is easily seen that the composition of \((x, u) = \varphi_0(w, \hat{w}) \) and \( w = \varphi_1(q, \hat{q}) \) together with (29) gives the desired transformation \((x, u) = \varphi(z) \) as in (7). Furthermore, the sequence of systems \( S_2 \subset S_1 \subset S_0 \) leads at once to the desired normal-form

\[
\begin{align*}
  \omega_{d,0}^1 &= dy^1 - \sin \left( \hat{z}^2 \right) dt \\
  \omega_{d,0}^2 &= \hat{z}^3 d\hat{z}^2 - \hat{z}^2 dy^2 \\
  \omega_{d,0}^3 &= d\hat{z}^3 + dy^2 - e^{z_4} dt.
\end{align*}
\]

as in (8). The flow parameters \( \hat{w} \) and \( \hat{q} \) correspond to the non-derivative variables \( \hat{z}^4 \) and \( \hat{z}^3 \), respectively. Furthermore, \( y^2 \) is a flat output since \( \omega_{d,0}^2 \) is parameterizable with respect to \( \hat{z}^3 \) and \( \partial_{q^2} \subset \mathcal{C}(S_2) \) with \( q^2 = y^2 \).

5.2 A further example

Let us consider the system \( S_0 = \{ \omega_0^1, \omega_0^2, \omega_0^3, \omega_0^4 \} \) also treated in [4] in a different context.
where we again have $V(S_0)^\perp = \{\partial_{u^1}, \partial_{u^2}\}$. The triangular form is based on the decompositions $S_0 = S_1 \oplus S_{1,c}$ with $S_1 = \{\omega_1 = \omega_0^1, \omega_2 = \omega_0^2, \omega_3 = \omega_0^3\}$

\[
\begin{align*}
\omega_0^1 &= dx^1 - (x^2 + x^3 u^2) dt \\
\omega_0^2 &= dx^2 - (x^3 + x^1 u^2) dt \\
\omega_0^3 &= dx^3 - (u^1 + x^2 u^2) dt \\
\omega_0^4 &= dx^4 - u^2 dt \\
\end{align*}
\]

and $S_{1,c} = \{\omega_{1,c}^1 = dx^3 - (u^1 + x^2 u^2) dt\}$ as well as on $S_1 = S_2 \oplus S_{2,c}$ with $S_2 = \{\omega_2^1 = \omega_1^1 - u^2 \omega_1^3, \omega_2^2 = \omega_3^1\}$ with

\[
\begin{align*}
\omega_2^1 &= dx^1 - u^2 dx^2 - (x^2 - x^1(u^2)^2) dt \\
\omega_2^2 &= dx^4 - u^2 dt \\
\end{align*}
\]

and $S_{2,c} = \{\omega_{2,c}^3 = dx^2 - (x^3 + x^1 u^2) dt\}$. The distributions $\mathcal{F}_0 = \{\partial_{u^1}\} \subset V(S_0)^\perp, \mathcal{F}_1 = \{\partial_{x^3}\} \subset V(S_1)^\perp$ and $\mathcal{F}_2 = \{\partial_{x^2} + u^2 \partial_{x^1}\} \subset V(S_2)^\perp$ were used and the flat outputs $y^1 = x^1 - u^2 x^2$ and $y^2 = x^4$ follow at once by applying a coordinate transformation based on the flow of $\mathcal{F}_2$ and regarding $\mathcal{F}_0$ and $\mathcal{F}_1$ no coordinate transformation is at need, since $u^1$ and $x^3$ are already non-derivative variables.

**Remark 22** Also in this example we have that $S_0^{(1)} \subset S_1$ and $S_1^{(1)} \subset S_2$ which enables one to construct the solutions based on the necessary condition (20) very easily.

However, a different possible solution for $S_1 \oplus S_{1,c}$ (branching point) can be based on choosing the distribution $\mathcal{F}_0 = \{\partial_{u^1}, \partial_{u^3}\}$ together with $S_1 = S_0^{(1)} = \{\omega_1^1, \omega_2^1\}$

\[
\begin{align*}
\omega_1^1 &= dx^1 - x^3 dx^4 - x^2 dt \\
\omega_1^2 &= dx^2 - x^1 dx^4 - x^3 dt \\
\end{align*}
\]

and $S_{1,c} = \{dx^3 - (u^1 + x^2 u^2) dt, dx^4 - u^2 dt\}$, where obviously $S_{1,c}$ is parameterizable with respect to $u^1$ and $u^2$. This choice for $\mathcal{F}_0$ and $S_1$ however leads to a 'dead end' since for $S_1$ the necessary condition (20) does not lead to a splitting $S_1 = S_2 \oplus S_{2,c}$. 
6 Discussion

We have characterized a suitable normal form for 1-flat systems, which is in implicit triangular shape, see (12), that possesses the properties as in proposition 10 based on exterior algebra. Furthermore, we have discussed a constructive calculation scheme to transform 1-flat systems into that desired form. It should be mentioned again that we only provide sufficient conditions for a system to be 1-flat and that the constructive algorithm is in general not unique, and iterations might be necessary. Nevertheless, we believe that the presented normal-form is of interest in the analysis of the flatness problem, and our examples show that this implicit triangular form can be achieved by successive coordinate transformations in a rather straightforward manner. Additionally, the well known Brunovsky form for systems that are linearizable by static feedback is naturally included in our approach, based on proposition 15.

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