Normal forms of Hopf-zero singularity

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Abstract
The Lie algebra generated by Hopf-zero classical normal forms is decomposed into two versal Lie subalgebras. Some dynamical properties for each subalgebra are described; one is the set of all volume-preserving conservative systems while the other is the maximal Lie algebra of nonconservative systems. This introduces a unique conservative–nonconservative decomposition for the normal form systems. There exists a Lie-subalgebra that is Lie-isomorphic to a large family of vector fields with Bogdanov–Takens singularity. This gives rise to a conclusion that the local dynamics of formal Hopf-zero singularities is well-understood by the study of Bogdanov–Takens singularities. Despite this, the normal form computations of Bogdanov–Takens and Hopf-zero singularities are independent. Thus, by assuming a quadratic nonzero condition, complete results on the simplest Hopf-zero normal forms are obtained in terms of the conservative–nonconservative decomposition. Some practical formulas are derived and the results implemented using Maple. The method has been applied on the Rössler and Kuramoto–Sivashinsky equations to demonstrate the applicability of our results.

Keywords: Normal form, Hopf-zero singularity, $\mathfrak{sl}_2$-representation, Conservative and nonconservative decomposition.
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1. Introduction
In this paper we are concerned with computing the simplest normal form of the system
\begin{align}
\dot{x} & := \text{h.o.t.}, \\
\dot{y} & := \dot{z} + \text{h.o.t.}, \\
\dot{z} & := -y + \text{h.o.t.},
\end{align}
(1.1)
where h.o.t. denotes formal nonlinear terms (higher order terms) with respect to $(x, y, z) \in \mathbb{R}^3$. Our normal form computation is a local tool and can be used for local dynamics analysis of
formal flows; this is because the convergence of transformations are not discussed here. The system (1.1) can be transformed into the first level (classical) normal form

\[
\begin{align*}
\frac{dx}{dt} &= \sum_{i+j=2} a_{ij}x^i(y^2+z^2)^j, \\
\frac{dy}{dt} &= z + \sum_{i+j=1} \infty x^i(y^2+z^2)^j(b_{ij}y + c_{ij}z), \\
\frac{dz}{dt} &= -y + \sum_{i+j=1} \infty x^i(y^2+z^2)^j(b_{ij}z - c_{ij}y),
\end{align*}
\]

(1.2)

where \(a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}\); see also [1, equation 2.15]. All the existing results on the simplest normal forms of this singularity have only dealt with the cases where \(a_{20}b_{10} \neq 0\); see [1, 8, 9, 25, 34]. The main reason for this is that terms corresponding to \(a_{01}, b_{10}, c_{10}\), and \(a_{20}\) all have the same grades in the usual graded structures; i.e. the gradings may not distinguish these four monomial vector fields. Therefore, one needs to deal with all four terms in the computations which is a tremendously difficult job. A novelty in the results of Algbab et al [1] was to notice that the first two nonzero terms (associated with \(a_{20}\) and \(b_{10}\)) play the key role in the computations and thus, generic conditions with respect to \(a_{20}\) and \(b_{10}\) are assumed. Then, they obtained the simplest normal forms and orbital normal forms. Chen et al [8, 9] approached this family of systems using an essentially different method and provided an independent solid proof for normal form uniqueness. Yu and Yuan [34] made an efficient computer program to compute the simplest normal form of this family of systems.

Some discussions on convergence and divergence of normal forms have been made in [12]. For instance, we computed the numerically suggested radius of convergence of the second level normal forms associated with volume-preserving Hopf-zero singularity. We do not address the convergence problem in this paper; see [18, 30–32] for some related results. Therefore, all our claims with regard to dynamics analysis are limited to the formal flows of formal systems.

A motivation of this paper is to use the monomial term corresponding to \(a_{01}\) as the main player in calculations; this is to complete the existing results on this problem with a quadratic nonzero term. (Throughout this paper the only assumption is \(a_{01} \neq 0\).) The grading in our approach distinguishes this vector field (grade 1) from the other three (grade 2). This is achieved through a \(\mathfrak{sl}_2\)-representation for the classical normal forms. This greatly simplifies the computations; we obtain complete results on the simplest normal form of these systems without any extra generic conditions. Our approach can be applied to many well-known models such as Rössler and Kuramoto–Sivashinsky equations.

The computational burden has been the main obstacles of most classification problems in the normal form literature. A systematic approach is required for such computations; a Lie-graded structure is an important instrument where grade-homogeneous parts of vector fields are simplified inductively. In this process, a basis for the space of grade-homogenous vector fields is applied. The mostly used choice for the basis has been the monomial vector fields oblivious of their dynamics. This usually involves large matrices and it is hard to find the patterns of computations. The conservative and nonconservative polynomial vector fields play this role in this paper instead of the monomials. Then, computations yield more solid patterns compared to when monomial vector fields are used. Indeed both transformation generators and normal forms are presented through conservative and nonconservative grade-homogeneous vector fields. This is a new feature of this paper that distinguishes our results from the existing results on the simplest normal forms of this singularity. This is accomplished via a \(\mathfrak{sl}_2\)-representation for the classical normal form vector fields. This technique has been
mainly applied to nilpotent singularities. Thus, it may be surprising to see that the theory of $\mathfrak{sl}_2$-representation is applied to a non-nilpotent system. Another novelty of our results is to use $\mathfrak{sl}_2$-style for the second level normal forms; $\mathfrak{sl}_2$-styles have only been used for the first level normal forms in the existing literature; see also [6, 29].

Any Lie (sub)algebra structure may have interesting dynamics interpretations and normal form theory provides a powerful tool for such descriptions. For our first instance, recall that a Lie subalgebra generates a group of transformations and the group establishes an equivalence relation. Hence, it gives us a classification within the space through infinite level normal forms. These usually introduce important families of vector fields. Here, we denote $\mathcal{L}$ for the Lie algebra generated by the first level Hopf-zero normal forms. Two transversal Lie subalgebras for $\mathcal{L}$ are presented. These represent the quasi-Eulerian vector fields and volume-preserving vector fields with a first integral. The second and more interesting example is described as follows. There exists a subalgebra from $\mathcal{L}$ that is Lie-isomorphic to a subalgebra $\mathcal{L}_b$ from the Lie algebra generated by all two dimensional vector fields with Bogdanov–Takens singularity; see theorem 2.5. This gives rise to the fact that local dynamics of any planar reduced (by ignoring the phase coordinate) system from Hopf-zero singularity can be embedded into the flow of a Bogdanov–Takens singularity; see theorem 2.7. Therefore, the local planar flow associated with Hopf-zero is well-understood by studying that of Bogdanov–Takens; see our further detailed discussion following remark 2.6. In other words, the local reduced system of Hopf-zero holds less complexity than Bogdanov–Takens. The Lie isomorphism further provides an explanation for why the $\mathfrak{sl}_2$-representation works fine for non-nilpotent singularities and also suggests that our techniques may be applicable to some other non-nilpotent singularities.

The rest of this paper is organized as follows. The conservative–nonconservative decomposition for normal forms are introduced in section 2. This is achieved by a $\mathfrak{sl}_2$-representation for $\mathcal{L}$ and presenting two transversal Lie subalgebras. Some properties for each family are described. The second level normal form is computed in section 3. In section 4 we obtain the simplest normal forms. The procedure is divided into three cases and accordingly their simplest normal forms are computed in three subsections. The results are applied on Rössler and Kuramoto–Sivashinsky equations in section 5.

2. Lie algebra $\mathcal{L}$ and its $\mathfrak{sl}_2$-representation

The space of all classical normal forms governed by equation (1.2) is denoted by $\mathcal{L}$. This section provides a $\mathfrak{sl}_2$-representation for $\mathcal{L}$. Any column vector $[f_1, f_2, f_3]^T$ is associated with the vector field $v := f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}$ and vice versa. Hence, $v$ generates a system given by $[\dot{x}, \dot{y}, \dot{z}]^T := [f_1, f_2, f_3]^T$. Besides, $v$ acts as a differential operator on scalar functions, say $g(x, y, z)$, defined by $v(g) := f_1 \frac{\partial g}{\partial x} + f_2 \frac{\partial g}{\partial y} + f_3 \frac{\partial g}{\partial z}$.

Thereby, terminologies of ‘system’, ‘vector field’ and ‘differential operator’ are interchangeably used. For any vector field $w$, we define $wv := w(f_1) \frac{\partial}{\partial x} + w(f_2) \frac{\partial}{\partial y} + w(f_3) \frac{\partial}{\partial z}$.

Now $\mathcal{L}$ is a Lie algebra by $\text{ad}_v(w) := [v, w] = vw - wv$ for any $v, w \in \mathcal{L}$. Any $\mathfrak{sl}_2$ Lie algebra is represented by a triad $\{N, M, H\}$. In this paper they are introduced by $N := (y^2 + z^2) \frac{\partial}{\partial x}$.
Lemma 2.3. The structure constants for the Lie algebra $\mathcal{L}$.

A straightforward calculation proves the following lemma. The proof readily follows an induction on $n$. 

Proof. The proof readily follows an induction on $n$. 

Definition 2.1. Define

\[
F_0^{-1} := 2(y^2 + z^2) \frac{\partial}{\partial x}, \quad E_0^0 = \frac{x \partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{2} z \frac{\partial}{\partial z}, \quad \Theta_0^0 := z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z},
\]

and

\[
F_l^l := \frac{(-1)^{l+1}(k-l+1)!}{2^{l+1}(k+2)!} \text{ad}_{M^l}^{M^l}(y^2 + z^2)^k F_0^{-1}, \quad \text{for} \quad -1 \leq l \leq k, 
\]

\[
E_l^l := \frac{(-1)^{(k-l)!}}{2^{l+1}} \text{ad}_{M^l}^{M^l}(y^2 + z^2)^k E_0^0, \quad \text{for} \quad 0 \leq l \leq k, 
\]

\[
\Theta_l^l := \frac{(-1)^{(k-l)!}}{2^{l+1}} \text{ad}_{M^l}^{M^l}(y^2 + z^2)^k \Theta_0^0, \quad \text{for} \quad 0 \leq l \leq k,
\]

where $\text{ad}_M v := [M, v]$ and $\text{ad}_M^{-1} v := [M, \text{ad}_M^{-1} v]$ for any natural number $n$.

These give rise to the following theorem.

Lemma 2.2. The formulas for $F_l^l$, $E_l^l$ and $\Theta_l^l$ in terms of $(x, y, z)$-coordinates are given by

\[
F_l^l = x^l(y^2 + z^2)^{k-l} \left((k-l+1)x \frac{\partial}{\partial x} - \frac{(l+1)}{2} y \frac{\partial}{\partial y} - \frac{(l+1)}{2} z \frac{\partial}{\partial z}\right),
\]

\[
E_l^l = x^l(y^2 + z^2)^{k-l} \left(\frac{x \partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y} + \frac{1}{2} z \frac{\partial}{\partial z}\right),
\]

\[
\Theta_l^l = x^l(y^2 + z^2)^{k-l} \left(\frac{z \partial}{\partial y} - y \frac{\partial}{\partial z}\right).
\]

Proof. The proof readily follows an induction on $l$. 

A straightforward calculation proves the following lemma.

Lemma 2.3. The structure constants for the Lie algebra $\mathcal{L}$ is governed by

\[
[F_l^l, F_m^m] = (m+1)(k+2) - (l+1)(n+2) F_{k+n}^{l+m},
\]

\[
[F_l^l, E_m^m] = \frac{(n+2)(m(k+2) - n(l+1))}{(k+n+2)} E_{k+n}^{l+m} - \frac{k(k+2)}{k+n+2} F_{k+n}^{l+m},
\]

\[
[F_l^l, \Theta_m^m] = (m(k+2) - n(l+1)) \Theta_{k+n}^{l+m},
\]

\[
[E_l^l, F_m^m] = (n-k) E_{k+n}^{l+m},
\]

\[
[E_l^l, \Theta_m^m] = n \Theta_{k+n}^{l+m},
\]

\[
[\Theta_l^l, \Theta_m^m] = 0.
\]
Let
\[ F = \text{span} \left\{ a_0 F_0^{-1} + \sum_{-1 \leq l \leq k, 1 \leq k} a_k^l F_k^l \right\}, \]
and
\[ \mathcal{F} = \text{span} \left\{ \Theta^0_0 + \sum_{0 \leq l \leq k, 1 \leq k} c_k^l \Theta_k^l \right\}. \]
Hence, any \( v \in \mathcal{F} \) has only phase components in cylindrical coordinates. The set of all formal first integrals for \( v \in \mathcal{F} \) is the algebra generated by \( x \) and \( y^2 + z^2 \); see [12, lemma 2.1]. Further, denote
\[ \mathcal{E} := \left\{ \sum_{l + k \geq 1, 1 \leq k \leq l} b_k^l E_k^l \right\}. \]

The following theorem indicates that the space of all volume-preserving and conservative normal forms is given by \( \mathcal{F} \oplus \mathcal{F} \); see [24] for relevant results on three-dimensional volume-preserving vector fields and their normal forms. Furthermore, it recalls that nonzero vector fields from \( \mathcal{E} \) are conservative.

\textbf{Theorem 2.4.} The following holds.

1. Any differential system governed by equation (1.2) is associated with a vector field \( v \in \mathcal{L} \) and vice versa.
2. The vector spaces \( \mathcal{F}, \mathcal{E}, \) and \( \mathcal{F} \) are transversal Lie subalgebras in \( \mathcal{L} \), i.e.
\[ \mathcal{L} = \mathcal{F} \oplus \mathcal{E} \oplus \mathcal{F}. \]
3. The Lie subalgebra \( \mathcal{F} \) is a Lie ideal for \( \mathcal{L} \). Thus, \( \mathcal{F} \oplus \mathcal{F} \) is also a Lie subalgebra for \( \mathcal{L} \).
4. The algebra of first integrals for any nonzero element from \( \mathcal{F} \) is \( \langle x, y^2 + z^2 \rangle \).
5. Let
\[ v = \Theta^0_0 + a_0 F_0^{-1} + \sum_{1 \leq k \leq l} (a_k^l F_k^l + c_k^l \Theta_k^l) \in \mathcal{L}, \text{ where } a_0 \neq 0. \]
Then, there exists a unique formal first integral \( f = \sum f_k^l (\text{modulo scalar multiplications}) \) such that the algebra of first integrals for \( v \) is \( \langle f \rangle \), where \( f_k^l := x^{l+1}(y^2 + z^2)^{k-l+1} \).
6. There is no nonzero first integral for any nonzero vector field from \( \mathcal{E} \). Furthermore, nonzero vector fields from \( \mathcal{E} \) are not volume preserving.
7. The space \( \mathcal{F} \oplus \mathcal{F} \) is the maximal vector space of volume-preserving vector fields. The algebra of first integrals for any \( v \in \mathcal{F} \oplus \mathcal{F} \) is nontrivial.
8. The flow generated by any \( v \in \mathcal{F} \) is static in the amplitude and \( x \)-coordinates. In other words, it can only be dynamic in the phase coordinate.
9. For any \( v \in \mathcal{F} \oplus \mathcal{F} \), the generated flow is static in the phase coordinate.

\textbf{Proof.} The space \( \mathcal{L} \) is defined such that claim 1 holds. The proofs for 2 and 3 follow lemma 2.3. The claims 4, 5 and 6 are proved by [12, lemma 2.1], [12, proposition 2.2] and [13, theorem 2.3], respectively. Since \( \mathcal{L} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{F} \), the proof of 7 is straightforward. Claim 8 is true because \( \mathcal{L} \) is generated by all vector fields from \( \mathcal{L} \) with zero \( x \) and amplitude components. Since \( \mathcal{F} \oplus \mathcal{F} \) is transversal to the Lie algebra \( \mathcal{F} \), the flow generated by \( v \in \mathcal{F} \oplus \mathcal{F} \) is static in phase coordinate. \( \square \)

\textbf{Theorem 2.5.} The space \( \mathcal{F} \oplus \mathcal{E} \) is a Lie algebra and there exists a Lie-isomorphism \( \psi \) to a proper Lie subalgebra \( \mathcal{L}_b \) of the Lie algebra generated by two dimensional Bogdanov–Takens singularities.
Proof. Denote $\mathcal{L}_{BT}$ for the Lie algebra generated by all vector fields of the form
\[ w := -\frac{\partial}{\partial y} + \text{h.o.t.} \] (2.6)
The negative sign is chosen such that it matches with the notation of [5]. Define a map
\[ \psi : \mathcal{F} \oplus \mathcal{E} \to \mathcal{L}_{BT}, \] (2.7)
governed by
\[ F^i \mapsto (k + 2)A^{k-1}_l, \quad E^i \mapsto B^{k-1}_l, \]
where $A^i_k$ and $B^i_k$ are defined by Baider and Sanders [5, equations 3.6a and 3.6b]. Comparing the two sets of structure constants, $\psi$ is a Lie-monomorphism and the claim follows by $\mathcal{L}_b := \psi(\mathcal{F} \oplus \mathcal{E})$.

Remark 2.6. The claim 5 from theorem 2.4 implies that $f_l^k(x, y, z) := x^{l+1}(y^2 + z^2)^{k-1}$ refers to $F^i_k$ in $\mathcal{L}$ and from [5] the monomial $h^k_{l-1}(\overline{x}, \overline{y}) := \overline{x}^{k-1-1}\overline{y}^{l+1}$ is associated with $A^{k-1}_l$ in $\mathcal{L}_b$. The monomial $f^k_{l+1} = x^{k+2}$ does not produce a permissible vector field in $\mathcal{L}$. However, $h^k_{l+1} = \overline{x}^{k+2}$ leads to $A^{-1}_k$. Here the $\mathfrak{sl}_2$-orbits given by (2.1) are terminated when they lead to non-permissible vector fields (i.e. $l$ is increased to $k + 1$), while the $\mathfrak{sl}_2$-orbits given by [5, equations 3.6a and 3.6b] are terminated by zero (i.e. $A^2_k = 0$).

Now we conclude that the local dynamics of a Hopf-zero normal form is well-understood by the study of Bogdanov–Takens singularities. The basis of our claim is as follows. There is a common practice to ignore the phase component of the Hopf-zero normal form and obtain a planar reduced system. Then, study the dynamics of the planar reduced system and extract the full three-dimensional dynamics from the planar reduced system; see [2, 10, 17, 20–22]. Given this, we prove that for any planar reduced system obtained from a Hopf-zero normal form, there exists a Bogdanov–Takens singularity from $\mathcal{L}_b$ such that the flow of the planar reduced system is embedded into the flow of Bogdanov–Takens singularity; see theorem 2.7. It is interesting to note that the dynamics of Bogdanov–Takens singularities are expected to be more rich than that of the planar reduced systems obtained from Hopf-zero. This is because of two reasons. First, the embedding between the two flows assigns the square of the amplitude variable from the reduced systems into a state variable from Bogdanov–Takens systems that is not necessarily nonnegative. This excludes certain dynamical complexities. Second (and more important than the first), a complement space to $\mathcal{L}_b$ is expected to absorb most of the dynamics of Bogdanov–Takens singularities. This is because all nonzero nonlinear terms from $\mathcal{L}_b$ appearing in a Bogdanov–Takens singularity are simplified in the $\mathfrak{sl}_2$-classical normal forms; see remark 3.2. This contributes to the possible exclusion of many complex dynamical behaviours.

Theorem 2.7. For any Hopf-zero classical normal form, there exists a Bogdanov–Takens singularity such that the flow generated by the planar reduced system (associated with Hopf-zero) is embedded into the Bogdanov–Takens’ flow.

Proof. Consider the vector fields from $\mathcal{L}$ in cylindrical coordinates. Now for any $v \in \mathcal{L}$, let
\[ v := \tilde{v} + \hat{v}, \quad \text{where } \tilde{v} \in \mathcal{T} \quad \text{and} \quad \hat{v} \in \mathcal{F} \oplus \mathcal{E}. \]
Define $w := \psi(\hat{v}) \in \mathcal{L}_b$ and $\hat{v}$ as the two dimensional vector field obtained from $\hat{v}$. Denote nonnegative real numbers by $\mathbb{R}^+$. The changes of variables
\[ \hat{v}_b : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R} \subset \mathbb{R}^2, \]
\[ \hat{v}_b(x, \rho) = (\overline{x}, \overline{y}), \quad \overline{x}(x, \rho) := \rho^2 \quad \text{and} \quad \overline{y}(x, \rho) := x, \] (2.8)
is a homeomorphism transforming \( \tilde{v} \) into \( w \). Hence, \( \tilde{\psi} \) embeds the flow of \( \tilde{v} \) into the flow of \( w \). This completes the proof. □

**Notation 2.8.** Throughout this paper we use Pochhammer \( j \)-symbol notation, that is,

\[
(a)_j^b := \prod_{i=0}^{j-1} (a + ib),
\]

for any natural number \( j \) and real number \( b \). Further, for integer numbers \( p, q, r \), we denote

\[
p \equiv q \mod r \quad (2.10)
\]

when there exists an integer \( k \) such that \( p - r = kq \).

Assume that \( v = v_0 + v_1 + v_2 + \cdots \), where \( v_k \in \mathcal{L}_k \) and \( \mathcal{L}_k \) denotes homogenous vector fields with grade \( k \) for any \( k \).

Let

\[
L^{n,1} : \mathcal{L}_n \to \mathcal{L}_n,
\]

\[
S_n \mapsto [S_n, v_0].
\]

Next we inductively define

\[
L^{n,N} : \mathcal{L}_n \times \ker L^{n-1,N-1} \to \mathcal{L}_n,
\]

\[
(Y_n, Y_{n-1}, \ldots, Y_{n-N+1}) \mapsto \sum_{i=0}^{N-1} [Y_{n-i}, v_i].
\]

There exists a complement space \( \mathcal{C}^{n,N} \) such that \( \text{im} L^{n,N} \oplus \mathcal{C}^{n,N} = \mathcal{L}_n \). The vector field

\[
w = w_0 + w_1 + w_2 + \cdots
\]

is called an \( N \)th (infinite) level normal form when \( w_n \in \mathcal{C}^{n,N} (w_n \in \mathcal{C}^{n,n}) \) for any \( n \). Then, there exist invertible transformations such that \( v \) can be transformed into the \( n \)th and infinite level normal form; see [14].

### 3. Hypernormalization

In this section we provide a hypernormalization for the system (1.1). Any such system is transformed into equation (1.2) and then by the item 1 of theorem 2.4, it can be represented by

\[
v^{(1)} = \Theta_0^0 + a_0 F_0^{-1} + \sum_{k=1}^{\infty} (a_k F_k^l + b_k E_k^l + c_k \Theta_k^l) \in \mathcal{L}.
\]

**Lemma 3.1.** For any vector field \( v^{(1)} \) given by equation (3.1) where \( a_0 \neq 0 \), there exist invertible changes of state variables transforming \( v^{(1)} \) into

\[
v^{(2)} = \Theta_0^0 + a_0 F_0^{-1} + \sum_{k=1}^{\infty} (a_k F_k^k + b_k E_k^k + c_k \Theta_k^k).
\]

**Proof.** Define the grading function by \( \delta(F_0^l) = \delta(E_0^l) = \delta(\Theta_0^l) = l \), then proof readily follows [\( E_0^l, F_0^{-1} \) = \( -2l E_k^{l-1} \), \( [F_k^l, F_0^{-1}] = -2(l + 1) F_k^{l-1} \) and \( [\Theta_k^l, F_0^{-1}] = -2l \Theta_l^{l-1} \). □

A bounded and periodic transformation \( \varphi \) is given by

\[
[x(t), \rho(t), \Theta(t)] = \varphi(x(t), \rho(t), \theta(t)) := [x, \rho, \theta - t],
\]

where \( \Theta \) and \( \theta \) are denoted for the new and old phase variables, respectively. The change of variable in \( \varphi \) omits \( \Theta_0^0 \) from the vector field \( v^{(2)} \). Once the simplest normal form computation
is exhausted, the map \( \varphi^{-1} \) adds \( \Theta_0^n \) back into the system; also see [13, theorem 4.1]. This technique has been used in perturbation theory; see [26, lemma 5.3.6]. Since the linear part of the system \( \Theta_0^n \) commutes with the classical normal form, its elimination (using (3.3)) does not change the formal normal form. However, if one uses a truncation at a grade (say \( k \)) plus a remainder, then the transformed system is non-autonomous beyond grade \( k \). Another common way of looking at normal forms is to consider a formal normal form as smooth modulo a flat vector field; e.g. see [28]. This gives rise to an autonomous formal normal form plus a non-autonomous flat system. This was brought to our attention by James Murdock. This is new in the literature and substantially reduces the computational burden.

After eliminating \( \Theta_0^n \), we may change \( a_0 \) to any arbitrary number \( \tilde{a}_0 \) via the time rescaling \( t := \frac{a_0}{\tilde{a}_0} t \) where \( t \) and \( \tau \) denote the old and new time variables, respectively. Therefore, without the loss of generality we may assume that

\[
v^{(2)} = F_0^{-1} + \sum_{k=1}^{\infty} \left( a_k F_k^k + b_k E_k^k + c_k \Theta_k^k \right).
\]

**Remark 3.2.** All nonlinear Hamiltonian and Eulerian terms that appear in the first level normal form of Bogdanov–Takens singularity in [5] are of the form \( A_k^{-1} \) and \( B_0^n \) while these terms do not belong to \( \mathcal{L}_b \); see theorem 2.5. Therefore, the computations in hypernormalization steps in this paper are essentially independent from [5] despite similarity of the procedures and the existence of a Lie–isomorphism between \( \mathcal{L}_b \) and \( \mathcal{F} \oplus \mathcal{F} \).

For hypernormalization we need to use a normal form style. A style is a rule on how to choose complement spaces in the normal form computation. We use the \( \mathfrak{sl}_2 \)-style in a hypernormalization step. The \( \mathfrak{sl}_2 \)-style in [5] states that the only nonlinear terms from \( \text{ker } \text{ad} \Theta_0^n \) can stay in the classical normal forms. Those are \( A_k^{-1} \) and \( B_0^n \) for any \( k \); see [5]. Lemma 3.1 proves that the terms \( F_k^k \), \( E_k^k \), and \( \Theta_k^k \) (see equations (2.1)–(2.3)) may stay in our second level normal form system. These are described by \( \text{ker } \text{ad} \Theta_0^n \), if we would define \( F_0^n \). Indeed, we embed \( \mathcal{L} \) into a bigger algebra (i.e. as a proper Lie subalgebra), where the bigger algebra contains \( F_0^n \). As far as algebra is concerned, this is simply performed via a simple generalization of lemma 2.3. However, this extension may not have a justification in nonlinear dynamics; i.e. assigning any term in an ODE system or a near-identity transformation to those extra generated vector fields (beyond \( \mathcal{L}_b \)) is not permissible. Indeed, from the bigger algebra, \( F_0^n \) is only used in definitions of \( \mathcal{F} \) given by equations (4.6), (4.17) and (4.29). The map \( \mathcal{F} \) is merely an instrument that substantially facilitates the computation. The original ideas behind definition of \( \mathcal{F} \) and our gradings come from [5].

Once the second level normal form is calculated, the \( \mathfrak{sl}_2 \)-style is extended to a formal basis style. Formal basis style uses an order on normal form terms of (3.2) to distinguish between alternative terms for elimination. Here, we give priority of elimination of conservative terms over nonconservative terms of the same grade in style I and vice versa in style II; see [14–16, 27, 28] for further information on formal basis style. Let

\[
\mathcal{B}_N = \left\{ F_k^l, E_m^m, \Theta_n^i | 1 \leq l \leq k, 0 \leq m \leq n, \delta(F_k^l) = \delta(E_m^m) = \delta(\Theta_n^i) = N \right\}
\]

be a basis for the vector space \( \mathcal{L}_N \) and \( \delta \) denote for a grading function. Two formal basis styles are defined through the following orderings on \( \mathcal{B}_N \) and are used in this paper.

- **Style I:** \( E_k^l < F_m^m \prec \Theta_n^i \) for any \( l, k, m, n, i, j \). Furthermore, \( F_k^l < E_k^l \prec E_k^m \) and \( \Theta_k^l \prec \Theta_n^i \) if \( k < n \). This gives a priority for elimination of \( E_k^l \)-terms rather than \( F_m^m \)-terms.
- **Style II:** \( F_m^m \prec E_k^l \prec \Theta_n^i \) for any \( l, k, m, n, i, j \). Besides, \( F_k^l < F_m^m \prec E_k^l \prec E_k^m \) and \( \Theta_k^l \prec \Theta_n^i \) if \( k < n \). The priority of elimination is with \( F_m^m \)-terms rather than \( E_k^l \)-terms.
4. The simplest normal forms

Throughout this paper we assume that there exist \( a_l \neq 0 \) and \( b_k \neq 0 \) for some \( l \) and \( k \). Define \( r := \min\{i \mid a_i \neq 0, \ i \geq 1\} \), \( s := \min\{j \mid b_j \neq 0, \ j \geq 1\} \), and \( p := \min\{j \mid c_j \neq 0, \ j \geq 1\} \).

\[
(4.1)
\]

Then, in order to obtain complete results for the simplest normal forms we divide the problem into the following three cases:

\[
(i) \ r < s, \quad (ii) \ r > s, \quad (iii) \ r = s. \quad (4.2)
\]

In this section we compute the simplest normal forms for the three cases \((4.2)\) in the following three subsections. We hereby acknowledge N. Sadri’s help for an independent verification, and detecting a few errors, of the formulas.

4.1. Case \( i: r < s \)

Assume that

\[
v^{(2)} := F^{-1}_0 + a_r F^r_x + \sum_{k=r+1}^{\infty} a_k F^k_x + \sum_{k=s}^{\infty} b_k E^k_x + \sum_{k=1}^{\infty} c_k \Theta^k_x,
\]

and define

\[
\delta(F^l_k) = \delta(E^l_k) = r(k - l) + \frac{k}{r} \quad \text{and} \quad \delta(\Theta^l_k) = r(k - l + 1) + \frac{k}{r} + 1. \quad (4.3)
\]

The vector field

\[
F_r := F^{-1}_0 + a_r F^r_x \quad (4.4)
\]

plays an important role in further normalization of \( v^{(2)} \). For any arbitrary \( \alpha \in \mathbb{R}, \alpha > 0 \), through changes of variables

\[
t := \left( \frac{\alpha \text{sign}(a_r)}{a_r} \right)^{r+1} r, \quad x := \left( \frac{\alpha \text{sign}(a_r)}{a_r} \right)^{r+1} X, \quad y := Y, \quad \text{and} \quad z := Z,
\]

we can change \( a_r \) into \( \alpha \text{sign}(a_r) \). Thereby, without the loss of generality we assume that \( a_r \) in equation \((4.4)\) is a non-algebraic number. When it comes to practical normal form computation by using a computer, this assumption is not valid; computers do not recognize irrational numbers.

**Lemma 4.1.** The \((r + 1)\)th level normal form of \((1.1)\) is

\[
v^{(r+1)} := F^{-1}_0 + a_r F^r_x + b_k^{(r+1)} E^k_x + \sum_{k>r} a_k^{(r+1)} F^k_x + \sum_{k>s} b_k^{(r+1)} E^k_x + \sum_{k \geq 1} c_k^{(r+1)} \Theta^k_x, \quad (4.5)
\]

where in

- style I, \( a_k = 0 \) for \( k \equiv 2(r+1) \cdot 2r \), and \( k \equiv 2(r+1) r \), and \( b_k = 0 \) for \( k \equiv 2(r+1) - 1 \), while \( c_k = 0 \) for \( k \equiv 2(r+1) - 1 \).
- style II, \( a_k = 0 \) for \( k \equiv 2(r+1) \cdot 2r \), \( k \equiv 2(r+1) r \), and \( k \equiv 2(r+1) - 1 \), where \( c_k = 0 \) for \( k \equiv 2(r+1) - 1 \).

**Proof.** It is easy to see that \( v^{(2)} \in \ker \text{ad}_F \). Thereby, in order to compute the higher level normal forms we follow \([4, 11–13]\) and define

\[
\mathcal{G} := \text{ad}_F \circ \text{ad}_F. \quad (4.6)
\]
Then,

$$\mathcal{A}(E^0_k) = -4(l + 1)(k - l + 2)F^0_k + 2a_r(k - l + 1)((k - l)(r + 1) + l)F^r_{k+r},$$

$$\mathcal{A}(E^1_k) = -4(l + 1)E^1_k + \frac{2a_r(r + 2)(k - l + 1)((k - l)(r + 1) + l)}{r + k + 2}F^r_{k+r},$$

$$\mathcal{A}(e^r_{k,m}) = -4(l + 1)\theta^r_{k,m} + 2a_r(l - k)(l(r + 2) - k(r + 1))\theta^r_{k+r+1}.$$

Given the ordering for $\mathcal{B}_N$, the matrix representation of $\mathcal{A}$ is lower triangular. Thus,

$$\ker(\mathcal{A}) = \left\{F^r_{k,r}, E^0_k, X^r_{k,r}, T^0_{k,r} \mid k \in \mathbb{N}\right\},$$

where $F^r_{k,r}, X^r_{k,r}, T^0_{k,r}$ are defined in [12, equation 3.4]. In addition

$$E^0_{k,r} := \sum_{m=0}^{k} a_m^r(mr + k + 2)E^m_{mr+k} - \sum_{m=0}^{k} a_m^r e^r_{k,m}F^m_{mr+k},$$

where $\delta(e^0_{k,r}) \equiv (r+1) 0$, $e^0_{k,0} = 0$, and

$$e^r_{k,m+1} = \left(\frac{r + 2(k - 2m)(r + 1)}{2(m + 1)(r + 1)}\right) e^r_{k,m} - \frac{r(r + 2)(k - 2m)}{m!^2(m + 1)(r + 1)(mr + k + 2)^2}.$$

On the other hand

$$[E^0_{2k+1}, F_r] = a_r^{2k+1}e_{2k+2}(2k(r + 1) - r)F_{2k(r+1)+r},$$

$$[E^0_{2k+1}, F_r] = -a_r^{2k+2}(2k + 3)(2k + 1)F_{2k(r+1)+r} - 2a_r^{2k+2}e_{2k+2}(2k + 1)(r + 1) + 1 F_{2k(r+1)+r}.$$  

On the other hand [12, lemma 3.4] implies that

$$\theta^{2k+1}(r+1)+r \quad \text{and} \quad F^{2k+1}(r+1)+r$$

are simplified in the $(r + 1)$th level of hypernormalization. For any sufficiently large number $m$, the sequence $e_{2k+1,m}$ alternatively changes its sign and thus, $e_{2k,1,2k+2}$ is nonzero. Thereby, either $e_{2k,1,m}$ or $e_{2k,1,m}$ is eliminated from the $(r + 1)$th level normal form, depending on whether style I or style II is applied. Now we show that $e_{2k,2k}$ is nonzero by an induction on $m$ through which the sign of $e_{2k,m}$ is discussed. Therefore by equation (4.10), $F^{2k(r+1)+r}$ can be removed from system and the proof is complete. Since $e_{2k,1} = \frac{r}{2(r + 2k + 2)} < 0$, we have $e_{2k,m} < 0$ for $0 \leq m \leq k + 1. Then,

$$e_{2k,k+1} = \frac{-(r + 2) e_{2k,k+1}}{2(k + 2)(r + 1) + 1}, \quad \text{and} \quad e_{2k,k+2} > 0.$$

Next, $e_{2k,k+3} < 0$, $e_{2k,k+4} > 0$, and so forth. This concludes that $e_{2k,2k}$ is nonzero.  

**Lemma 4.2.** For any natural numbers $m$ and $n$, there exist $e^m_n, e^m_n$ such that

$$[e^m_n, F_r] + F^m_n = \frac{(-a_r)^{m-n} (r(n - m) + n - 2r)(m - n)(r + 1) + 2n - m - 1}{2n - (m + 2)(m + r + 3)} F^{n-m+1}_{r+n+m}.$$
There exist invertible transformations sending (1.1) into the Theorem 4.3.

Nonlinearity 28 (2015) 311
M Gazor and F Mokhtari

\[ [\mathcal{E}_n^m, \mathcal{F}_r] + E_n^m = a_r n^{-m} \left( \frac{(r)^2\phi_{n-m-1}}{n + r + 2 + r(n - m - 1)} + \frac{(r(m - n + 2) - n)\phi_{n-m-1}}{m + 1}\right) E_{r-n-m+n}^{s+1} + \frac{a_r n^{-m}(n+1)((n - m - 1)(r + 1) - m - 1)}{(m + 1)2^{n-m}(r(n - m) + n + 2)(m + r + 2)_{r+1}^{s+1}} E_{r-n-m+n}^{s+1}, \]  

(4.11)

where the sequence \( \phi_l \) follows \( \phi_0 = \phi_{-1} = 0 \),

\[ \phi_{l+1} = \frac{(r)^2}{2(n + 2 + (l + 1)(r + 1))} \phi_l, \]

and

\[ \phi_l = \frac{(n + 2)((n - m - 1)(r + 1) - m - 1)}{2^{l+1}(m + 1)(n + l + 2)(m + r + 2)_{r+1}^{s+1}}. \]  

(4.12)

**Proof.** Let

\[ \mathcal{E}_n^m := \sum_{l=0}^{n-m-1} a_r l^l \phi_l E_{n+1}^{s+1} + \sum_{l=0}^{n-m-1} a_r l^l \phi_l E_{n+1}^{s+1}, \]

\[ \delta_n^m := \sum_{l=0}^{n-m-1} \left( -a_r l^l \right) \left( m - n \right)(r + 1) + m + 2 \right)_{2(r+1)}^l \frac{1}{2^{l+1}(m + 1 + l + 2)(m + r + 2)_{r+1}^{s+1}} E_{n+1}^{s+1}. \]

Then, the proof is a straightforward calculation. \( \square \)

Define

\[ r_1 := r, \quad r_2 := \min\{ \min(i \mid a_i \neq 0, \quad i > r), \quad p_1 := \min\{ j \mid c_j \neq 0, \quad j \geq p \} \]  

(4.13)

where \( a_i, c_j \) are coefficients of the \( (r+1) \)th level. When \( s < r_2 \), we use the generator \( \lambda_{s+1}^k \) to remove \( F_{2(s+1)}^{r+s+1} \) from the \( (r+1) \)th level normal form. However, this merely represented a Lie symmetry for the system in [12].

**Theorem 4.3.** There exist invertible transformations sending (1.1) into the \((s+1)\)th level normal form

\[
\begin{align*}
\frac{dx}{dr} &= 2(y^2 + z^2) + a_{s+r_1}x^{r_1} + \sum_{k=0}^{\infty}(a_{k+r_1}x^r + \beta_{k+s}x^s)x^{k+1} + \sum_{k=0}^{\infty}\gamma_{k+p_1}x^k, \\
\frac{dy}{dr} &= \frac{a_r(r+1)}{2}x^r y - \frac{z}{2} \sum_{k=0}^{\infty}(a_{k+r_1}(k + r_1 + 1)x^r - \beta_{k+s}x^s)x^k + z \sum_{k=0}^{\infty}\gamma_{k+p_1}x^k, \\
\frac{dz}{dr} &= -y - \frac{a_r(r+1)}{2}x^r z -\frac{z}{2} \sum_{k=0}^{\infty}(a_{k+r_1}(k + r_1 + 1)x^r - \beta_{k+s}x^s)x^k - y \sum_{k=0}^{\infty}\gamma_{k+p_1}x^k, 
\end{align*}
\]

(4.14)

where \( s < r_2 \), we have

- for the case of style I, \( a_{k+r_1} = 0 \) for \( k \equiv 2r+2 \) and \( k \equiv 2r+2 \), \( k \equiv 2r+2 \) and \( k \equiv 2r+2 \), and \( \beta_{k+s} = 0 \) for \( k \equiv 2r+2 \), \( -(s+1) \), and \( \gamma_{k+p_1} = 0 \) for \( k \equiv 2r+2 \) and \( -p_1 \).
- for style II, \( a_{k+r_1} = 0 \) for \( k \equiv 2r+2 \), \( k \equiv 2r+2 \), \( k \equiv 2r+2 \), \( k \equiv 2r+2 \), \( k \equiv 2r+2 \) and \( -(s+1) \) and \( \gamma_{k+p_1} = 0 \) for \( k \equiv 2r+2 \) and \( -p_1 \).

When \( s \geq r_2 \) the following holds:

- style I: \( a_{k+r_1} = 0 \) for \( k \equiv 2r+2 \) and \( k \equiv 2r+2 \), and \( \beta_{k+s} = 0 \) for \( k \equiv 2r+2 \), \( -(s+1) \), where \( \gamma_{k+p_1} = 0 \) for \( k \equiv 2r+2 \) and \( -p_1 \).
28
4.2. Case ii: Nonlinearity

Assume that $s < r_2$ and the grading function follows equation (4.3). By theorem 4.2 there exist state solutions $E^m_n$, $S^m_n$ for $(m, n) = (j(r + 1) + s - 1, 2k + rj + s)$ such that

$$\left[ E^m_n + S^m_n, F_{r_1} \right] + [\lambda^{k+1}_{r_1}, E^r_{s_1}]$$

$$= \left( \sum_{j=0}^{k+1} \binom{k+1}{j} a_{j2k+j} + \frac{(-1)^{j+1}(2k + rj)\frac{1}{2}(2j - k)(r + 1) + s - r}{2^{2k-j}(2k + rj + s + 2)} \right) F_{r_1}^{2k(r+j)+r+1} + \frac{\phi_{2k-j}(2k(r + 1) + s - r)}{(2k + rj + s + 2)} F_{r_1}^{2k(r+j)+r+1}.$$ 

So, $F_{2k(r+j)+r+1}$ can be eliminated from the $(s + 1)$th level of normalization for any $k \in \mathbb{N}$. For $s \geq r_2$, the transformation generators $\lambda_{r_1}^{k+1}$ and $T_{2k+rj}$ are extended to a symmetry of the system and therefore, they cannot simplify the system any further; see [12, theorem 3.5]. □

4.2. Case ii: $s < r$

For this case we assume that

$$v^{(2)} := F_0^{-1} + b_j E^s_j + a_{k} F^k + \sum_{k=r}^{\infty} b_k E^k + \sum_{k=1}^{\infty} c_k \Theta^k.$$ 

The grading function is defined by

$$\delta(F^k) = \delta(E^s) = s(k - l) + k \quad \text{and} \quad \delta(\Theta^k) = s(k - l) + k + s + 1. \quad (4.15)$$

Denote $E_\delta := F_0^{-1} + b_j E^s_j$.

**Lemma 4.4.** The $(s + 1)$th level normal form of (1.1) is

$$v^{(s+1)} = F_0^{-1} + b_j E^s_j + a_{k} F^k + \sum_{k=r}^{\infty} c_k^{(s+1)} F^k + \sum_{k=s}^{\infty} b_k^{(s+1)} E^k + \sum_{k=1}^{\infty} c_k^{(s+1)} \Theta^k, \quad (4.16)$$

where in

- style I, $b_k^{(s+1)} = 0$ for $k \equiv s+2$ and $k \equiv s+1$ and $k \neq s^2 + 2s$, while $c_k^{(s+1)} = 0$ for $k \equiv s$. 
- style II, $a_k^{(s+1)} = 0$ when $k \equiv s+1$ 2$s$, $b_k^{(s+1)} = 0$ for $k \equiv s+1$ and $k \neq s^2 + 2s$, while $c_k^{(s+1)} = 0$ if $k \equiv s+1$.

**Proof.** Let

$$\mathcal{G} := \text{ad}(F_{s_1}) \circ \text{ad}(E_\delta). \quad (4.17)$$

Then,

$$\mathcal{G}(F^l_k) = -4l(k - l + 2)F^l_k - \frac{2b_k(k + 2)(k - l + 1)}{k + s + 2} F^{r+s+1}_{k+s} + 2b_s(s + 2)(k - l)(k - l + 1) E^{r+s+1}_{k+s},$$

$$\mathcal{G}(E^l_k) = -4l(k - l + 1)E^l_k - 2b_s(k - s)(k - l) E^{r+s+1}_{k+s},$$

$$\mathcal{G}(\Theta^l_k) = -4l(k - l + 1)\Theta^l_k - 2b_s(k - l) \Theta^{r+s+1}_{k+s}.$$
Then, \( \ker(\mathcal{G}) = \text{span}\{\mathcal{F}^{-1}_k, \mathcal{T}_k^0, \mathcal{E}_k^0\} \) where

\[
\mathcal{F}^{-1}_k := \sum_{m=0}^{k+1} \frac{(-b_j)^m(k+2)(k)^m}{(2s+2)^m(m!)(ms+k+2)} F_{m+s+k}^{m(s+1)-1} - \sum_{m=0}^{k+1} b_j^m h_m E_{m+s+k}^{m(s+1)-1},
\]

\[
\mathcal{E}_k^0 := \sum_{m=0}^{k} \frac{(-b_j)^m(k-s)^m}{(2s+2)^m m!} E_{m+s+k}^{m(s+1)},
\]

\[
\mathcal{T}_k^0 := \sum_{m=0}^{k} \frac{(-b_j)^m k}{(2s+2)^m m!} \mathcal{G}_{m+s+k}^{m(s+1)},
\]

for \( h_{m+1} = \frac{(-1)^{m+1}(s+2)^2(k+1)(k+2)(k-m+2)}{2m+1(s+1)^m m!(ms+k+2)^2((m+1)(s+1)-1)} - \frac{s(m-1)+k}{2(m+1)(s+1)-1} h_m. \)

Note that we have \( \delta(\mathcal{F}^{-1}_k) \equiv s+1 - 1 \), \( \delta(\mathcal{E}_k^0) \equiv s+1 0 \), and \( \mathcal{E}_s^0 = E_s^0 \). On the other hand

\[
[\mathcal{F}^{-1}_k, \mathcal{E}_k] = \frac{(-b_j)^{k+2}(k(s+1)+s)^2(k)^{k+1}}{2s+1(s+1)^k+2^2(k+1)!((k+1)(s+1)+1)} F_{k+2}^{k+2} - 2(b_j)^{k+2}h_{k+2}(k+2)(s+1)-1 E_{k+2}^{k+2},
\]

\[
[\mathcal{E}_k^0, \mathcal{E}_k] = \frac{(-b_j)^{k+1}(k-s)^{k+1}}{2(k+1)^k} E_{k+2}^{k+2},
\]

\[
[\mathcal{T}_k^0, \mathcal{E}_k] = \frac{(-b_j)^{k+1}(k)^{k+1}}{2(k+1)^k} \mathcal{G}_{k+2}^{k+2}.
\]

Notice that \( h_1 = \frac{-s(s+2)(k+2)}{2(s+1)^2} < 0 \). By an induction argument we may conclude that for any odd number \( k \), the number \( h_k \) is negative and \( h_k \) is positive for any even number \( k \). Therefore, \( h_k \neq 0 \) for any \( k \) and by equations (4.21–4.23), the proof is complete.

**Lemma 4.5.** For any natural numbers \( m \) and \( n \), there exist \( \mathfrak{A}_m^m \) and \( \mathfrak{B}_m^m \) such that

\[
F_m^m + [\mathfrak{A}_m^m, \mathcal{E}_m] = \frac{b_j \xi_{m-m-1}^n (s+2)}{n + sn - sm + 2} + b_j \xi_{m-m-1}^n (s - m + n - 2) - n \right] E_{s-n-m}^{s-n-m},
\]

\[
E_m^m + [\mathfrak{B}_m^m, \mathcal{E}_m] = \frac{b_j m (2s - n + sn + sm) (s - n)^{s-m-1}}{2^n - m (m+1) (m + s + 2)^{s-m-1}} E_{s-n-m}^{s-n-m},
\]

where

\[
\xi_i := \frac{-(s+2) b_j^i (s-n)^{i-1} ((n+ls+2)(m+2)i_{s+1} - (n+2)(m+s+2)i_{s+1})}{2i+1 (s-n) (n+ls+2) (m+s+2)i_{s+1}},
\]

\[
\zeta_i := \frac{(-b_j)^i (n+2)^{i}_{s+1}}{2i+1 (n+ls+2)(m+2)i_{s+1}}.
\]
Proof. The proof is straightforward by
\[
\mathfrak{B}_n^m := \sum_{l=0}^{m-1} b_s^l (s-n)^{-l} E_{n+l}^{m+l+s+1},
\]
\[
\mathfrak{B}_n^m := \sum_{l=0}^{m-1} \xi_l E_{m+l}^{m+l+s+1} + \sum_{l=0}^{m-1} \zeta_l E_{n+m+l}^{n+m+l+s+1}.
\]

We only use style II in the following theorem. Assume that the \((s+1)\)th level coefficients \(a_i\) and \(b_j\) are nonzero for some \(i, j \geq 1\). Define
\[s_1 := s, \quad s_2 := \min \{ j \mid b_j \neq 0 \text{ for } j > s \}, \quad r := \min \{ i \mid a_i \neq 0, i \geq 1 \}.
\]

Theorem 4.6. There exist invertible transformations sending (1.1) into the \((r+s+2)\)th level normal form
\[
\frac{dx}{dt} = 2(y^2 + z^2) + a_r x^{r+1} + b_s x^{s+1} + x \sum_{k=0}^{\infty} (\alpha_k x^r + \beta_k x^s) x^k,
\]
\[
\frac{dy}{dt} = z - \frac{a_r (r+1)}{2} x^r y + \frac{b_s}{2} b_s x^s y - \frac{y}{2} \sum_{k=1}^{\infty} (\alpha_k x^r + \beta_k x^s) x^k + z \sum_{k=1}^{\infty} \gamma_k x^k,
\]
\[
\frac{dz}{dt} = -y - \frac{a_r (r+1)}{2} x^r z + \frac{b_s}{2} b_s x^s z - \frac{z}{2} \sum_{k=1}^{\infty} (\alpha_k x^r + \beta_k x^s) x^k - y \sum_{k=1}^{\infty} \gamma_k x^k.
\]

Here \(\alpha_k = 0\) for \(k \equiv s+1 \) and \(b_k = 0\) for \(k \equiv s+1 \) where \(k \neq s^2 + 2s\), \(\gamma_k = 0\) for \(k \equiv s+1\), and if \(s_2 < r\), we have \(b_k = 0\) for \(k = s^2 + s_2 + s\),
\[\text{when } s_2 = r, \quad \alpha_k = 0 \text{ for } k = s^2 + s_2 + s,\]

for \(s_2 > r, \quad \alpha_k = 0 \text{ for } k = s^2 + r + s\).

Furthermore, the differential system (4.24) is indeed the infinite level normal form.

Proof. For \(s_2 < r\), by lemma 4.5 there exists \(\mathfrak{B}_{s_2}^s\) such that
\[
[ \mathfrak{B}_{s_2}^s, E_{s_2}^s ] + [ E_s^0, E_{s_2}^s ] = \frac{\alpha_s (-b_s)^y (s_2 + s)^{s+1}}{2(s_2 + 1)(s_2 + s + 2)^{s+1}} E_{s_2}^{s_2+s+1}.
\]

Hence, in both styles we can remove \(E_{s_2}^{s_2+s+1}\) from the system. For \(s_2 = r\), by lemma 4.5 there exist \(\mathfrak{B}_{s_2}^s\) and \(\mathfrak{B}_{s_2}^s\) such that
\[
[ \mathfrak{B}_{s_2}^s, E_{s_2}^s ] + [ E_s^0, E_{s_2}^s ] + [ E_s^0, E_{s_2}^s ] = \frac{\alpha_s (-b_s)^y (s_2 + s)^{s+1}}{2(s_2 + 1)(s_2 + s + 2)^{s+1}} E_{s_2}^{s_2+s+1} + \frac{b_s (-b_s)^y (s_2 + s)^{r+1}}{2(s_2 + 1)(s_2 + s + 2)^{r+1}} E_{s_2}^{s_2+r+1} + \frac{a_s (-b_s)^y (s_2 + s)^{s+1}}{2(s_2 + 1)(s_2 + s + 2)^{s+1}} E_{s_2}^{s_2+s+1} + \frac{b_s a_s (-b_s)^y (s_2 + s)^{s+1}}{2(s_2 + 1)(s_2 + s + 2)^{s+1}} E_{s_2}^{s_2+r+1}.
\]

Therefore, \(E_{s_2}^{s_2+r+1}\) can be eliminated in style II. When \(s_2 > r\), by lemma 4.5 there exist \(\mathfrak{B}_{s_2}^s\) and \(\mathfrak{B}_{s_2}^s\) such that
\[
[ \mathfrak{B}_{s_2}^s, E_{s_2}^s ] + [ E_s^0, E_{s_2}^s ] + [ E_s^0, E_{s_2}^s ] = \left( \frac{b_s (-b_s)^y (s_2 + s)^{r+1}}{2(s_2 + 1)(s_2 + s + 2)^{r+1}} E_{s_2}^{s_2+r+1} + \frac{(-b_s)^y (s_2 + s)^{r+1}}{2(s_2 + 1)(s_2 + s + 2)^{r+1}} E_{s_2}^{s_2+s+1} \right),
\]

Since the coefficient of \(E_{s_2}^{s_2+r+1}\) is nonzero, the proof is complete. □
4.3. Case iii: \( r = s \).

Let

\[
v^{(2)} := F_0^{-1} + a_s F_s^2 + b_s E_s^2 + \sum_{k=1}^{\infty} (a_k F_k^2 + b_k E_k^2) + \sum_{k=1}^{\infty} c_k \Theta_k^2,
\]

and assume that \( \frac{a}{b} \) is a non-algebraic number. This section is similar to the case III of Baider and Sanders [4] which is the most difficult case of the three. We essentially use the idea of Wang et al [19, 23, 33]; they assume that a certain ratio be non-algebraic. This prevents that the ratio be a root for any polynomial that appears in the computations and the results readily follow. When the corresponding fraction is an algebraic number, both problems (the case in this subsection and the third case of [4]) remain unsolved. Denote

\[
\delta(\bar{\lambda}) := s, \quad \delta(F_0) := s(k - l) + k \quad \text{and} \quad \delta(\Theta_l) := s(k - l + 1) + k + 1.
\]

**Theorem 4.7.** Let \( \frac{a}{b} \) be a non-algebraic number. Then, there exist invertible transformations that send \( v^{(1)} \) given by equation (1.1) into the infinite level normal form system

\[
\begin{align*}
\frac{dx}{dt} &= 2(y^2 + z^2) + (b_s + a_s)x^{s+1} + x \sum_{k=1}^{\infty} (\beta_k + a_k)x^k, \\
\frac{dy}{dt} &= z + (b_s - a_s(s + 1)) \frac{xy}{2} + \frac{y}{2} \sum_{k=1}^{\infty} (\beta_k - a_k(k + 1))x^k + z \sum_{k=1}^{\infty} \gamma_k x^k, \\
\frac{dz}{dt} &= -y + (b_s - a_s(s + 1)) \frac{x^2 z}{2} + \frac{z}{2} \sum_{k=1}^{\infty} (\beta_k - a_k(k + 1))x^k - y \sum_{k=1}^{\infty} \gamma_k x^k,
\end{align*}
\]

where \( \delta(\bar{\lambda}_s) = s \) and define a grading function by

\[
\delta(F_l) := \delta(E_l) := s(k - l) + k \quad \text{and} \quad \delta(\Theta_l) := s(k - l + 1) + k + 1,
\]

- **style I.** \( \beta_k = 0 \) for \( k \equiv s+1 \) and \( k \equiv s+1, 2s \), and for \( k \equiv s+1 \) we have \( \gamma_k = 0 \).
- **style II.** \( \alpha_k = 0 \) for \( k \equiv s+1 \) and \( k \equiv s+1, 2s \), while \( \gamma_k = 0 \) for \( k \equiv s+1 \).

**Proof.** Define

\[
\mathcal{G} := \text{ad}_{F_0} \circ \text{ad}_{\bar{\lambda}_s},
\]

and

\[
\begin{align*}
\mathcal{G}(F_l) &= -4l(k - l + 1)F_l + 2b_s(s + 2)(k - l)(k - l + 1)E_{k+s+1}^{l+s+1} \\
&\quad - 2(k - l + 1) \left( a_s(l - k)(s + 1) + b_s(k + 2) \right) E_{k+s+1}^{l+s+1}, \\
\mathcal{G}(E_l) &= -4l(k - l + 1)E_l + 2a_s(s + 2)(k - l + 1)F_{k+s+1}^{l+s+1} \\
&\quad - 2(k - l) \left( a_s(k + 2)(l - k)(s + 1) + b_s(k + s) \right) E_{k+s+1}^{l+s+1}, \\
\mathcal{G}(\Theta_l) &= -4l(k - l + 1)\Theta_l - 2(k - l) \left( b_s(s + 2) - k(s + 1) \right) E_{k+s+1}^{l+s+1}.
\end{align*}
\]

Hence,

\[
\ker(\mathcal{G}) = \text{span}\{F_k^{-1}, T_k^{l}, E_k^0, \Theta_k^l \mid k \in \mathbb{N}\},
\]

325
On the other hand, the coefficient of $w_{n+1}^{-1}$ is given by $(-1)^{n-k}s^m(k+2)^m$, for $k \neq s$, the constant term of $w_{n+1}^{-1}$ is $\frac{a_s(k+2)}{2m!}$.

On the other hand, the coefficient of $\left(\frac{a_s}{b_s}\right)^{m+1}$ in $u_{m+1}^0$ is governed by
\[
\frac{(-1)^m m! k^m (k+2)^m}{2^m (k+2)^m k_{k+1}}.
\]
Since \( \frac{\beta}{\alpha} \) is assumed a non-algebraic number, \( w_{k+1}^0, w_{k+1}^0, u_{k+1}^{1}, u_{k+2}^{1} \) and \( u_{k+2}^{2} \) in equation (4.35) are nonzero. Hence, the proof is complete by the rules established in styles I and II. \( \square \)

Remark 4.8. The condition \( \frac{\beta}{\alpha} \) being non-algebraic was an essential assumption in the proof of theorem 4.7. However, we usually truncate the normal form system up to certain degree, say \( N \). Then, we may only need that \( \frac{\beta}{\alpha} \) be distanced from the zeros of the polynomials generated by \( u_{k+2}^{1}, u_{k+2}^{1}, u_{k+1}^{0}, \) and \( w_{k+2}^{1} \) for \( k = 0, 1, \ldots, \left\lfloor \frac{N-3}{s+1} \right\rfloor \) and \( l = 0, 1, \ldots, \left\lfloor \frac{N}{s+1} \right\rfloor - 1 \); for an instance of this see example 5.2.

5. Examples

In this section we provide the formulas for the first few coefficients of the simplest normal forms in terms of the coefficients of the original system. These are very useful for practical applications. Next, our results are applied on the Rössler and Kuramoto-Sivashinsky equations to demonstrate the applicability of our results. Derivations of formulas follow exact (no numerical approximation) symbolic implementation of the results in Maple. The derived formulas up to the classical normal form coefficients are consistent with the results of Algaba et al [1]. We thank E. Gamero for sending us their classical normal form program and formulas of up to degree five for this comparison.

Consider a differential system

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} +
\sum_{2 \leq i+j+k \leq 3} \begin{pmatrix}
a_{ijk} \\
b_{ijk} \\
c_{ijk}
\end{pmatrix} x^i y^j z^k,
\]

(5.1)

where \( a_{0,0,0} + a_{0,0,2} \neq 0 \). Denote its cubic-truncated simplest normal form of equation (5.1) by

\[
\begin{align*}
\frac{dx}{dt} &= 2a_0 \rho^2 + (\alpha_1 + \beta_1)x^2 + (\alpha_2 + \beta_2)x^3, \\
\frac{d\rho}{dt} &= \frac{1}{2}(\beta_1 - \alpha_1)x\rho + \frac{1}{2}(\beta_2 - 3\alpha_2)x^2\rho, \\
\frac{d\theta}{dt} &= 1 + \gamma_1 x + \gamma_2 x^4.
\end{align*}

(5.2)

where

\[
\begin{align*}
\alpha_0 &:= \frac{1}{4}(c_{0,2,0} + c_{2,0,0}), \\
\alpha_1 &:= \frac{1}{3}(c_{0,0,2} - a_{1,0,1} - b_{0,1,1}), \\
\beta_1 &:= \frac{1}{3}(2c_{0,0,2} + a_{1,0,1} + b_{0,1,1}), \\
\gamma_1 &:= \frac{1}{2}(\beta_{1,0,1} - a_{0,1,1}).
\end{align*}
\]

\( \alpha_2, \beta_2 \) and \( \gamma_2 \) are given in the appendix.
Example 5.1. Consider the Rössler equation
\[ \begin{align*}
\dot{x} &= bz - cx + xz, \\
\dot{y} &= z + ay, \\
\dot{z} &= -y - x.
\end{align*} \tag{5.3}\]
This system has Hopf-zero singularity for two sets of parameters. One is for parameter values
\[ a = c, \quad b = 1, \quad 0 < a^2 < 2, \tag{5.4}\]
while the system associated with the other set of parameter values has a simple dynamics; see [1]. We denote the vector field corresponding to (5.3) by \( R_a \). Therefore, the classical normal form of equations (5.3) is given by
\[ R_a^{(i)} = \Theta_0^0 - \frac{a}{2(2 - a^2)^2} \frac{a^2 + 11}{12(2 - a^2)^2} F_0^{-1} - \frac{a^2 + 1}{3(2 - a^2)^2} F_0^1 - \frac{5a^2 + 2a}{4(2 - a^2)} F_0^2 - \frac{a}{3(2 - a^2)^2} E_1^0 - \frac{a}{4(2 - a^2)^2} E_1^1 - \frac{(a^2 + 1)}{2(2 - a^2)} E_1^2. \]
By computing the second level normal form, we notice that this system is among the case iii (see section 4.3). Then,
\[ R_a^{(s)} = \Theta_0^0 \pm \frac{1}{2} F_0^{-1} \pm \frac{2}{3} F_0^1 \pm \frac{55}{32} F_0^2 \pm \frac{1}{3} E_1^0 \pm \frac{9}{32} E_1^2 - \Theta_1, \]
and for \( a \neq \pm 1 \) the infinite level normal form is given by
\[ \begin{align*}
\frac{dx}{dt} &= -a \rho^2 - \frac{a x^2}{(2 - a^2)^3} - \frac{a(a^2 + 1)x^3}{(2 - a^2)^5}, \\
\frac{d\rho}{dt} &= \frac{a^3 x \rho}{2(2 - a^2)^3} + \frac{a \rho}{16(2 - a^2)^2}, \\
\frac{d\theta}{dt} &= 1 - \frac{(a^2 + 1)}{2(2 - a^2)} x.
\end{align*} \tag{5.5}\]
Example 5.2. The standing waves of the Kuramoto–Sivashinsky equation gives rise to
\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -\mu x - 2x^2 - y,
\end{align*} \]
which has a Hopf-zero singularity at the origin; see Chang [7]. Our formulas for second level normal form of up to degree six is given by
\[ K^{(2)} = \Theta_0^0 - \frac{1}{2} F_0^{-1} - 2F_0^1 + \frac{19}{3} \Theta_2^3 + \frac{343}{9} F_3^3 - \frac{675297}{3240} F_3^5 - \frac{57385}{108} \Theta_4^1. \]
This falls within the case i (see section 4.1). Hence, the infinite level normal form truncated up to degree six is governed by
\[ K^{(s)} = \Theta_0^0 - \frac{1}{2} F_0^{-1} - 2F_0^1 + \frac{686}{3} F_3^3 + \frac{19}{3} \Theta_2^3 - \frac{57385}{108} \Theta_4^1, \tag{5.6}\]
or equivalently,
\[ \begin{align*}
\dot{x} &= -\rho^2 - 2x^2 + \frac{686}{3} x^3, \\
\dot{\rho} &= 2x \rho + \frac{343}{3} x^3 \rho, \\
\dot{\theta} &= 1 + \frac{19}{3} x^2 - \frac{57385}{108} x^4.
\end{align*} \tag{5.7}\]
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Appendix

In this appendix we present the formulas derived by symbolic implementation of the results in Maple. Denote \( \alpha_+ := \alpha_2 \) and \( \alpha_- := \beta_2 \). Then,

\[
\alpha_+ = \frac{-1}{4(a_{0,2,0} + a_{0,0,2})} \left( (a_{0,2,0} + a_{0,0,2}) \left( c_{2,0,1} + b_{2,1,0} - a_{3,0,0} - 3 c_{2,0,0} a_{1,1,0} + 3 b_{2,0,0} a_{1,0,1} \right) \\
+ c_{2,0,0} c_{0,1,1} + 2 c_{2,0,0} b_{0,2,0} \right) \pm \frac{1}{48} \left( 2a_{2,0,0} + c_{1,0,1} + b_{1,1,0} \right) \left( 4b_{0,1,1} b_{0,0,2} + 8a_{1,2,0} + 8a_{1,0,2} \right) \\
- 12 c_{0,0,3} + 6c_{1,1,0} c_{0,2,0} - 6c_{1,1,0} d_{0,0,2} - 4c_{0,2,0} c_{0,1,1} - 8c_{0,2,0} b_{0,2,0} + 6c_{1,0,1} a_{0,1,1} - 4b_{0,1,2} \\
+ 8 c_{0,0,2} b_{0,0,2} - 6 b_{1,1,0} a_{0,1,1} + 4 b_{0,2,0} b_{0,1,1} + 6 b_{1,0,1} a_{0,2,0} - 6 b_{1,0,1} a_{0,0,2} - 4 c_{1,0,1} c_{0,0,2} \right) \\
- 12 b_{0,3,0} + 8 c_{0,2,0} d_{1,1,0} + 8 c_{0,0,2} d_{1,0,1} - 8 b_{0,2,0} a_{1,0,1} - 8 b_{0,0,2} a_{1,0,1} \right) \pm \frac{1}{3} \left( a_{2,0,0} - c_{1,0,1} \right)^2 \\
- b_{1,1,0} \left( 3c_{0,0,3} - b_{0,1,1} b_{0,0,2} + c_{0,2,0} c_{0,1,1} + 2 c_{0,2,0} b_{0,2,0} + c_{0,1,1} c_{0,0,2} - 2 c_{0,0,3} b_{0,0,2} + b_{0,1,2} \\
- b_{0,2,0} b_{0,0,1} + a_{1,0,2} + b_{0,3,0} + c_{0,2,0} d_{1,1,0} + c_{0,0,2} d_{1,0,1} - b_{0,2,0} a_{1,1,0} - b_{0,0,2} a_{1,0,1} + a_{1,2,0} \right) \right),
\]

and

\[
\gamma_2 := -\frac{1}{2} c_{2,1,0} - c_{2,0,0} c_{0,2,0} + \frac{1}{2} c_{2,0,0} b_{0,1,1} - c_{2,0,0} a_{1,0,1} - \frac{1}{8} c_{1,1,0}^2 - \frac{1}{4} c_{1,0,1} b_{0,1,1} - \frac{1}{8} c_{1,0,1}^2 \\
+ \frac{1}{4} c_{1,0,1} b_{1,1,0} + \frac{1}{2} c_{0,1,1} b_{2,0,0} - b_{2,0,0} b_{0,0,2} - b_{2,0,0} a_{0,1,1} - \frac{1}{8} b_{1,1,0}^2 - \frac{1}{8} b_{1,0,1}^2 + \frac{1}{2} b_{2,0,1}.
\]

References

[1] Algaba A, Freire E and Gamero E 1998 Hypernormal form for the Hopf-zero bifurcation Int. J. Bifur. Chaos 8 1857–87
[2] Algaba A, Freire E, Gamero E and Rodríguez-Luis A J 1999 A three-parameter study of a degenerate case of the Hopf-pitchfork bifurcation Nonlinearity 12 1177–206
[3] Baider A and Churchill R C 1988 Unique normal forms for planar vector fields Math. Z. 199 303–10
[4] Baider A and Sanders J A 1992 Further reductions of the Takens–Bogdanov normal form J. Diff. Eqns 99 205–44
[5] Baider A and Sanders J A 1991 Unique normal forms: the nilpotent Hamiltonian case J. Diff. Eqns 92 282–304
[6] Barrio R and Palacián J F 1997 Lie transforms for ordinary differential equations: taking advantage of the Hamiltonian form of terms of the perturbation Int. J. Numer. Methods Eng. 40 2289–300
[7] Chang H 1986 Traveling waves on fluid interfaces: normal form analysis of the Kuramoto–Sivashinsky equation Phys. Fluids 29 3142–7
[8] Chen G, Wang D and Yang J 2003 Unique normal forms for Hopf-zero vector fields C. R. Math. Acad. Sci. Paris 336 345–8
[9] Chen G, Wang D and Yang J 2005 Unique orbital normal form for vector fields of Hopf-zero singularity J. Dyn. Diff. Eqns 17 3–20
[10] Dumortier F, Ibáñez S, Kokubu H and Simó C 2013 About the unfolding of a Hopf-zero singularity Discrete Contin. Dyn. Syst. 33 4435–71
[11] Gazor M and Moazeni M 2015 Parametric normal forms for Bogdanov–Takens singularity; the generalized saddle-node case Discrete Contin. Dyn. Syst. 35 205–24
[12] Gazor M and Mokhtari F 2013 Volume-preserving normal forms of Hopf-zero singularity Nonlinearity 26 2809–32
[13] Gazor M, Mokhtari F and Sanders J A 2013 Normal forms for Hopf-zero singularities with nonconservative nonlinear part J. Diff. Eqns 254 1571–81
[14] Gazor M and Yu P 2012 Spectral sequences and parametric normal forms *J. Diff. Eqns* 252 1003–31
[15] Gazor M and Yu P 2010 Formal decomposition method, parametric normal forms *Int. J. Bifur. Chaos* 20 3487–515
[16] Gazor M and Yu P 2008 Infinite order parametric normal form of Hopf singularity *Int. J. Bifur. Chaos* 18 3393–408
[17] Harlim J and Langford W F 2007 The cusp–Hopf bifurcation *Int. J. Bifur. Chaos* 17 2547–70
[18] Iooss G and Lombardi E 2005 Polynomial normal forms with exponentially small remainder for analytic vector fields *J. Diff. Eqns* 212 1–61
[19] Kokubu H, Oka H and Wang D 1996 Linear grading function, further reduction of normal forms *J. Diff. Eqns* 132 293–318
[20] Langford W F 1979 Periodic and steady-state mode interactions lead to tori *SIAM J. Appl. Math.* 37 649–86
[21] Langford W F 1983 A review of interactions of Hopf and steady-state bifurcations *Nonlinear Dynamics, Turbulence (Interaction of Mechanics and Mathematics Series)* (Boston, MA: Pitman) pp 215–37
[22] Langford W F 1984 Hopf bifurcation at a hysteresis point *Differential Equations: Qualitative Theory (Colloq. Math. Soc. Janos Bolyai vol 47)* (Amsterdam: North Holland) 649–86
[23] Li J, Zhang L and Wang D 2014 Unique normal form of a class of 3 dimensional vector fields with symmetries *J. Diff. Eqns* 257 2341–59
[24] Mezić I and Wiggins S 1994 On the integrability and perturbation of three-dimensional fluid flows with symmetry *J. Nonlinear Sci.* 4 157–94
[25] Mokhtari F 2011 The simplest normal forms of Hopf-zero singularity *Master Thesis* Isfahan University of Technology (in Persian)
[26] Murdock J 2003 *Normal Forms and Unfoldings for Local Dynamical Systems* (New York: Springer)
[27] Murdock J 2004 Hypernormal form theory: foundations and algorithms *J. Diff. Eqns* 205 424–65
[28] Murdock J and Malonza D 2009 An improved theory of asymptotic unfoldings *J. Diff. Eqns* 247 685–709
[29] Palacián J 2005 Dissipative-Hamiltonian decomposition of smooth vector fields based on symmetries *Chaos* 15 033111
[30] Strózyna E 2002 The analytic and formal normal form for the nilpotent singularity. The case of generalized saddle-node *Bull. Sci. Math.* 126 555–79
[31] Strózyna E and Zoladek H 2011 Divergence of the reduction to the multidimensional nilpotent Takens normal form *Nonlinearity* 24 3129–41
[32] Strózyna E and Zoladek H 2002 The analytic and formal normal form for the nilpotent singularity *J. Diff. Eqns* 179 479–537
[33] Wang D, Li J, Huang M and Jiang Y 2000 Unique normal form of Bogdanov–Takens singularities *J. Diff. Eqns* 163 225–38
[34] Yu P and Yuan Y 2001 The simplest normal form for the singularity of a pure imaginary pair, a zero eigenvalue *Dyn. Contin. Discrete Impuls. Syst. Ser. B* 8 219–40