Thermoelastic Response of Fractional-Order Nonlocal and Geometrically Nonlinear Beams

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This study presents both the theoretical framework and the finite element solution to the fractional-order nonlocal thermoelastic response of beams. The constitutive relations of the fractional-order medium are developed based on thermodynamic principles. Remarkably, it is shown that the fractional model allows the rigorous application of thermodynamic balance principles at every point within the domain. The theory is applied to the analysis of the nonlocal response of Euler-Bernoulli beams under combined thermo-mechanical loads. The governing fractional-differential equations and the associated boundary conditions for the elastic field variables are derived using variational principles. The fractional finite element method (f-FEM) is used to numerically solve the linear and nonlinear fractional-order system of equations. Further, the numerical model is used to study the thermoelastic response of the nonlocal beam subject to various thermo-mechanical loads and boundary conditions. The fractional thermoelasticity framework is expected to provide consistent models for nonlocal interactions in complex nonlocal structures exposed to a thermo-mechanical environment.

Keywords: Fractional calculus, Nonlocal Elasticity, Thermoelasticity, Variational Calculus, Finite Element Method

I. INTRODUCTION

Several theoretical and experimental studies have shown that size-dependent effects, also referred to as nonlocal effects, are prominent in the response of complex structures of great relevance for many real-world applications. These size-dependent effects can be traced back to medium heterogeneity, existence of surface stresses, presence of thermal loads, and even medium geometry. More specifically, in the case of micro- and nano-structures, size-dependent effects have been traced back to the existence of surface and interface stresses due to nonlocal atomic interactions and Van der Waals forces [1–3]. In the case of macroscale structures, nonlocal effects can result from an ensemble of factors including material heterogeneity, interactions between layers (e.g. in FGMs or composite media) or unit cells (e.g. in periodic media), and geometric inhomogeneity [4–7]. In other terms, nonlocal governing equations for macrostructures often result from a process of homogenization of the initial inhomogeneous system. Further, geometric effects such as changes in curvature have also been shown to induce nonlocal size-dependent effects in nano-, micro-, and macro-structures [3,8].

The above mentioned complex slender structures have important applications in engineering and biotechnology. As an example, macroscale structures made from functionally graded materials (FGM) or sandwiched designs have been largely used in weight-critical applications such as aerospace, naval, and automotive systems [9,10]. Similarly, thin films, carbon nanotubes, monolayer graphene sheets and micro tubules have far-reaching applications in atomic devices, micro/nano-electromechanical devices, as well as sensors and biological implants [8,11,12]. Independently on the spatial scale, the key design constraints in the above applications include restrictions on space and weight. As a result, structural assemblies for lightweight applications are typically made of a combination of slender components like beams, plates, and shells. There are several applications where these structures are subject to large and rapidly varying mechanical and thermal loads that drive the system into a geometrically nonlinear regime. A practical example includes the analysis of supersonic or hypersonic aerospace systems where the combination of large and quickly varying aero-thermo-mechanical loads induces highly a nonlinear response [9,10,13]. Similarly, the ability to account for coupled thermomechanical nonlinear effects is also critical in applications involving nano- and micro-structures such as, for example, in the design of biological implants, measurement devices, and sensors [11,12,14,15]. Despite the undeniable need for proper theoretical frameworks and computational tools capable of simulating the thermomechanical response of nonlinear and nonlocal structures, only a limited amount of studies focusing on geometrically nonlinear thermomechanical response of nonlocal slender structures are available in the literature. In the following, we briefly review the main characteristics of these studies and discuss key limitations.

Seminal works from Kro{"e}ner [16] and Eringen [17] have explored the role of nonlocality in elasticity and laid its
theoretical foundation. The key principle behind nonlocal theories relies on the idea that all the particles located within a prescribed area, typically indicated as the horizon of nonlocality, influence one another by means of long range cohesive forces. This interaction between particles is accounted for by using gradient or integral relations for the strain field within the constitutive equations. These approaches lead to so-called weak gradient methods or strong integral methods, respectively. Integral methods [18, 19] define constitutive laws in the form of convolution integrals between the strain and the spatially dependent elastic properties over the horizon of nonlocality, whereas gradient elasticity theories [20, 22] account for the nonlocal behavior by introducing strain gradient dependent terms in the stress-strain constitutive law. As emphasized earlier, several applications involving nonlocal slender structures also experience thermal loads and geometric nonlinearities. Although several studies are available on the topics of geometrically nonlinear response of nonlocal slender structures [11, 12, 14, 23, 24] and on the thermomechanical response of nonlocal structures [15, 18, 25, 26], it appears to be a shortage of theoretical and numerical methods capable of addressing the geometrically nonlinear thermomechanical response of nonlocal structures.

As mentioned previously, the classical studies on nonlocal (linear or nonlinear) elasticity and nonlocal (linear) thermoelasticity encounter some key shortcomings. As an example, gradient theories experience serious difficulties when enforcing the boundary conditions associated with the strain gradient-dependent terms [20, 21]. On the other side, the integral methods are better suited to deal with boundary conditions but they lead to mathematically ill-posed governing equations. This mathematical ill-posedness leads to erroneous predictions such as the absence of nonlocal effects or the occurrence of hardening behavior for certain combinations of boundary conditions [27, 28]. In this class of problems, the ill-posedness stems from the fact that the constitutive relation between the bending moment field and the curvature is a Fredholm integral of the first kind, whose solution does not generally exist and, if it exists, it is not necessarily unique [27, 28]. Additionally, in both these classes of methods, there are no available explicit relations to estimate the stress at a given point given the strain at that particular point. This latter aspect prevents the application of variational principles [29, 30] and has critical implications on the development of a thermodynamic framework for the classical nonlocal approaches. More specifically, the modeling of nonlocality through nonlocal stress-strain constitutive relations allows only a weak application (in a domain integral sense) and prevents the localized (pointwise) application of the thermodynamic balance laws. As discussed in [18], the weak application of thermodynamic balance laws, particularly the second law, leads to inconsistencies in the nonlocal continuum framework.

In recent years, fractional calculus has emerged as a powerful mathematical tool to model a variety of nonlocal and multiscale phenomena. Fractional derivatives, which are a differ-integral class of operators, are intrinsically multiscale and provide a natural way to account for nonlocal effects [31]. Given the multiscale nature of fractional operators, fractional calculus has found widespread applications in nonlocal elasticity [6, 7, 32, 37]. In a series of papers, Patnaik et al. [7, 38, 41] have shown that a nonlocal continuum approach based on fractional-order kinematic relations provides an effective way to address the previously mentioned shortcomings of classical approaches to nonlocal elasticity. Note that, unlike gradient elasticity methods, additional essential boundary conditions are not required when using Caputo fractional derivatives [6, 7]. Further, the nonlocal model based on fractional-order kinematic relations allows the application of variational principles and leads to well-posed governing equations that admit unique solutions [38, 40].

In this study, we build upon the fractional-order nonlocal continuum theory and develop a thermodynamic framework for the geometrically nonlinear response of nonlocal beams. The overall goal of this study is two-fold. First, we develop a thermodynamic framework for the fractional-order continuum formulation. We will show that the fractional-order continuum formulation allows for a rigorous application of all thermodynamic principles. More specifically, the use of fractional-order kinematic relations prevents the requirement of additional integral constitutive stress-strain relations as seen in classical nonlocal approaches (see, for example, [18]). As a result, the thermodynamic balance laws in the fractional-order theory are free from nonlocal residual terms which greatly simplifies the constitutive modeling of the nonlocal continuum theory while providing a rigorous implementation of the thermodynamic principles at each point in the solid. The latter observation highlights an important benefit and a key motivation to pursue a fractional-order formulation to nonlocal thermoelasticity. Second, we develop a fractional-order finite element method (f-FEM) capable of accurately solving the nonlinear fractional-order thermomechanical equations. This numerical model builds on the f-FEM developed in [39] for the analysis of geometrically nonlinear response of nonlocal beams using fractional calculus.

We note that several fractional-order thermomechanical models have been developed and presented in previous literature [12, 45]. However, all these studies have focused primarily on the use of fractional-order operators to model complex thermal exchanges. More specif-
ically, time-fractional derivatives have been used within the heat conduction equation in \cite{42,43} in order to model dissipation within the thermal processes, while space-fractional derivatives have been used in \cite{43,46} to model spatial diffusion in the thermal processes. We emphasize that the stress-strain constitutive relations in all these studies are local whereas, on the contrary, we have considered a nonlocal solid with fractional-order-continuum relations. The heat conduction equation used in our study matches the classical integer-order heat conduction equation. In this regard, we merely note that the use of a space-fractional heat conduction equation leads to inconsistencies in the application of the second law of thermodynamics \cite{46}. This latter observation motivates the use of the classical (integer-order) heat conduction equation in our study.

The remainder of the paper is structured as follows: we begin with the development of a thermodynamic framework for the fractional-order approach to nonlocal elasticity. We use this framework to derive the governing equations of the nonlinear thermoelastic response of a nonlocal beam using variational principles. Next, we develop a f-FEM to obtain the numerical solution to the nonlinear fractional-order governing equations. Finally, we use the f-FEM to analyze the effect of the fractional-order nonlocality on the static response of the beam subject to different thermomechanical loads and boundary conditions.

II. CONSTITUTIVE MODEL FOR FRACTIONAL-ORDER THERMOELASTICITY

In this section, we develop the constitutive model for fractional-order nonlocal thermoelasticity. As discussed earlier, the nonlocal beam theory presented here below builds on the formulation for a fractional-order nonlocal continuum presented in \cite{7}. In the following, we briefly review the key highlights of the continuum theory.

A. Fundamentals of the fractional-order nonlocal continuum formulation

Analogous to the classical approach to continuum mechanics, the response of a nonlocal solid can be analysed by introducing two configurations, namely, the reference (undeformed) and the current (deformed) configurations. The motion of the body from the reference configuration (denoted as \(X\)) to the current configuration (denoted as \(x\)) is assumed as:

\[ x = \Phi(X, t) \]  \hspace{1cm} (1)

such that \(\Phi(X, t)\) is a bijective mapping operation. The above mapping operation is used to model the differential line elements \(d\mathbf{x}\) and \(d\mathbf{x}\) in the undeformed and deformed configurations of the nonlocal solid using fractional-order operators. Such result is achieved by defining fractional-order deformation gradient tensors \cite{7}.

In analogy with classical strain measures, the nonlocal strain can be defined using the fractional-order differential line elements as \(d\mathbf{x}d\mathbf{x} - d\mathbf{x}d\mathbf{x}\). The above formalism leads to the following expression for the Lagrangian strain tensor in the nonlocal medium \cite{7,39}:

\[ \hat{E} = \frac{1}{2} \left( \nabla^\alpha \mathbf{U}_X + \nabla^\alpha \mathbf{U}_X^T + \nabla^\alpha \mathbf{U}_X^T \nabla^\alpha \mathbf{U}_X \right) \]  \hspace{1cm} (2)

where \(\mathbf{U}(X) = \mathbf{x}(X) - \mathbf{X}\) denotes the displacement field. The fractional gradient denoted by \(\nabla^\alpha \mathbf{U}_X\) is given as \(\nabla^\alpha \mathbf{U}_X_{ij} = D^\alpha_{ij} \mathbf{U}_X\) and consists of space-fractional derivatives. The space-fractional derivative \(D^\alpha_{ij} \mathbf{U}(X, t)\) is taken according to a Riesz-Caputo (RC) definition with order \(\alpha \in (0, 1)\) and it is defined on the interval \(\mathbf{X} \in (\mathbf{X}_A, \mathbf{X}_B) \subseteq \mathbb{R}^3\) in the following manner:

\[ D^\alpha_{ij} \mathbf{U} = \frac{1}{2} \Gamma(2-\alpha) [ \mathbf{L}_{ij}^{-1} \mathbf{C}_{\mathbf{X}_A} D^\alpha \mathbf{U} - \mathbf{I}^{\alpha-1} \mathbf{C}_{\mathbf{X}_B} D^\alpha \mathbf{U} ] \]  \hspace{1cm} (3)

where \(\Gamma(\cdot)\) is the Gamma function, and \(\mathbf{C}_{\mathbf{X}_A} D^\alpha \mathbf{U}\) and \(\mathbf{C}_{\mathbf{X}_B} D^\alpha \mathbf{U}\) are the left- and right-handed Caputo derivatives of \(\mathbf{U}\), respectively. Before proceeding, we highlight certain implications of this definition of the fractional-order derivative. The interval of the fractional derivative \((\mathbf{X}_A, \mathbf{X}_B)\) defines the horizon of nonlocality (also called attenuation range in classical nonlocal elasticity). The length scale parameters \(\mathbf{L}_{ij}^{-1}\) and \(\mathbf{L}_{ij}^{\alpha-1}\) ensure the dimensional consistency of the deformation gradient tensor, and along with the term \(\frac{1}{2} \Gamma(2-\alpha)\) ensure the frame invariance of the constitutive relations \cite{7}.

In this formulation, nonlocality was introduced by using fractional-order kinematic relations. The differential line elements in the undeformed and deformed nonlocal configurations were modeled using fractional-order deformation gradients which, in turn, were used to obtain the strain in the nonlocal medium. This definition of the strain has critical implications on the thermodynamic framework for the fractional-order model of nonlocal continuum. In the following section, we will show that the above approach to modeling nonlocality allows enforcing in a strong (localized) sense the first and second law of thermodynamics. In other terms, the fundamental laws of thermodynamics can be applied in a strict sense at each point in the nonlocal continuum as opposed to what happens in classical nonlocal approaches.
B. Thermodynamic framework for fractional-order thermoelasticity

In order to cast the fractional-order nonlocal model presented above within a thermodynamic framework, consider a nonlocal solid occupying a domain $\Omega$ and assume that it possesses an internal energy density $\epsilon$. The first step is to enforce the first law of thermodynamics (i.e. the conservation of energy) based on the fractional-order continuum model. For the nonlocal solid, in addition to the local strain energy, the statement of the conservation of energy should also include the energy exchange due to the long range cohesive forces between different material points. Given the fractional-order kinematic relations described in (1), the contribution of the energy contained in these long range cohesive forces are fully captured in the fractional-order nonlocal strain $\tilde{\epsilon}$. It is immediate to see that the internal energy density of the solid can now be expressed as a function of the fractional-order strain $\tilde{\epsilon}$. Further, the strain energy density is also a function of the entropy $\tilde{\eta}$. Consequently, we have $\epsilon = \epsilon(\tilde{\epsilon}, \tilde{\eta})$. This functional relationship is in sharp contrast with the classical nonlocal thermodynamic framework where the internal energy is only a function of the strain field. More specifically, the thermodynamic framework for classical nonlocal approaches leads to $\epsilon = \epsilon(\epsilon, \mathcal{R}(\epsilon), \tilde{\eta})$ where $\epsilon$ is the classical (local) strain field and $\mathcal{R}(\epsilon)$ denotes a linear integral operator which models nonlocality in the solid [17, 21].

Note that the fractional-order strain $\tilde{\epsilon}$ allows to combine the local integer-order strain $\epsilon$ and its integral $\mathcal{R}(\epsilon)$ into a single term (see Eq. 3) [2]. Indeed, this is precisely the reason that allows expressing the internal energy density as $\epsilon = \epsilon(\tilde{\epsilon}, \tilde{\eta})$. The latter observation is significant as the first law of thermodynamics can be applied in a strict sense at every point in the domain in the following manner:

$$\dot{\epsilon} = \tilde{\sigma}_{ij} \dot{\epsilon}_{ij} + h - q_{i,i} \quad \forall X \in \Omega$$  \hspace{1cm} (4)

where $\tilde{\sigma}$ denotes the fractional-order stress tensor in the nonlocal solid, $h$ is the heat generated internally per unit volume and $q$ is the vector of the heat flux density. The comma in the subscript $(q_{i,i})$ denotes the integer-order spatial derivative and the accent $(\dot{\cdot})$ denotes the first integer-order derivative with respect to time. We emphasize again that this results is not in net contrast with classical nonlocal approaches where the conservation of the first law can only be applied in a weak sense (see, for example, [18]).

Next, we apply the second law of thermodynamics to the fractional-order continuum model. The internal entropy production rate, $\dot{\hat{\eta}}_{\text{int}}$, is defined as:

$$\dot{\hat{\eta}}_{\text{int}} = \dot{\hat{\eta}} - \left[ \frac{h}{T} - \nabla \cdot \left( \frac{q}{T} \right) \right]$$  \hspace{1cm} (5)

where $\nabla(\cdot)$ denotes the integer-order divergence and $T$ denotes the temperature of the solid. Recall that the second law of thermodynamics states that the internal entropy production rate is non-negative for all points inside the solid, that is $\dot{\hat{\eta}}_{\text{int}} \geq 0 \ \forall \ X \in \Omega$. Classical approaches to nonlocal thermoelasticity allow satisfying this inequality only in a weak sense, that is $\int_{\Omega} \dot{\hat{\eta}}_{\text{int}} \geq 0$ [17, 18, 47, 48]. A detailed discussion of the corresponding physical inconsistencies can be found in [18].

In analogy with the classical approach, we introduce the Legendre transformation $\psi = e - T\tilde{\eta}$, where $\psi$ denotes the Helmholtz free energy. It follows that $\psi = \psi(\tilde{\epsilon}, T)$ which is contrary to classical nonlocal approaches wherein $\psi = \psi(\epsilon, \mathcal{R}(\epsilon), T)$ [18]. By using the Legendre transform along with Eq. (5) in the second law of thermodynamics, we obtain:

$$T \dot{\hat{\eta}}_{\text{int}} = \tilde{\sigma}_{ij} \dot{\epsilon}_{ij} - \psi - \eta T - T \dot{q}_{i} \frac{q_{i}}{T} \geq 0$$  \hspace{1cm} (6)

Remarkably, the above fractional-order inequality matches, in its functional form, the classical Clausius-Duhem inequality: a clear difference with classical nonlocal formulations where additional terms appear within the inequality as a result of the functional dependence of $\psi$ on $\mathcal{R}(\epsilon)$. As discussed in [18], these additional terms within the inequality disappear only when a weak form is considered. However, as mentioned previously, the satisfaction of the second law of thermodynamics in a weak sense is nonphysical.

The inequality in Eq. (6) is used to derive the thermodynamically consistent constitutive equations for the fractional-order nonlocal elasticity. By substituting the expression for the time derivative of the Helmholtz free energy, the inequality in Eq. (6) is expressed as:

$$T \dot{\hat{\eta}}_{\text{int}} = \left( \tilde{\sigma}_{ij} - \frac{\partial \psi}{\partial \tilde{\epsilon}_{ij}} \right) \dot{\epsilon}_{ij} - \left( \eta + \frac{\partial \psi}{\partial T} \right) T - T \dot{q}_{i} \frac{q_{i}}{T} \geq 0$$  \hspace{1cm} (7)

Since the above inequality must hold for all thermoelastic processes as well as for arbitrary choices of the independent fields $\tilde{\epsilon}$ and $T$, we obtain the following constitutive laws:

$$\tilde{\sigma}_{ij} = \frac{\partial \psi}{\partial \tilde{\epsilon}_{ij}} \quad \forall X \in \Omega$$ \hspace{1cm} (8a)

$$\tilde{\eta} = \frac{\partial \psi}{\partial T} \quad \forall X \in \Omega$$ \hspace{1cm} (8b)
Further, by using the above constitutive relations within Eq. (6), the inequality reduces to:
\[ T\dot{\delta}_{\text{int}} = -T\alpha \frac{q_i}{T} \geq 0 \quad \forall \mathbf{X} \in \Omega \] (9)
which establishes the second law of thermodynamics. The relations in Eqs. (8a) can be expressed as:

**Theorem:** The constitutive relations for fractional-order nonlocal thermoelasticity do not violate the Clausius-Duhem inequality if they are of the form given in Eq. (8a) and subject to Eq. (9).

A few additional comments on this thermodynamic framework are needed. At first glance, the form of the stress-strain constitutive relation in Eq. (8a) might be deceiving as it appears to lead to a classical constitutive relation. Although this is, formally, a correct statement it is misleading as it appears to lead to a classical constitutive relation. Stress defined through the Eq. (8a) is also nonlocal in order kinematic relations given in Eq. (2). Therefore, the relation. Although this is, formally, a correct statement it is deceptively so as it appears to lead to a classical constitutive relation. Stress defined through the Eq. (8a) is also nonlocal in order kinematic relations given in Eq. (2). Therefore, the relation.

The relations in Eqs. (8,9) can be expressed as:

\[
\begin{align*}
\psi &= a_0 + a_1 \dot{\varepsilon}_1 + a_2 \dot{\varepsilon}_2 + a_3 \dot{\varepsilon}_3 + a_4 \theta + a_5 \dot{\theta}^2 + a_6 \dot{\theta}^2 + a_7 \dot{\theta}^3 + a_8 \dot{\varepsilon}_1 \dot{\varepsilon}_2 + a_9 \dot{\varepsilon}_1 \dot{\varepsilon}_3 + a_{10} \dot{\varepsilon}_2 \dot{\varepsilon}_3 + (10a) \\
&= a_{11} \theta^2 + a_{12} \dot{\varepsilon}_2 \dot{\varepsilon}_1 + a_{13} \dot{\varepsilon}_2 \dot{\varepsilon}_3 + a_{14} \dot{\varepsilon}_3 \dot{\varepsilon}_2 + \text{h.o.t}
\end{align*}
\]

where \(a_k\) are material constants and

\[
\dot{\varepsilon}_1 = \dot{\varepsilon}_1; \quad \dot{\varepsilon}_2 = \frac{1}{2} (\varepsilon_{ii} \varepsilon_{jj} - \varepsilon_{ij} \varepsilon_{ij}); \quad \dot{\varepsilon}_3 = \text{det}(\dot{\varepsilon}_{ij})
\]

are the invariants of the nonlinear strain tensor \(\dot{\varepsilon}\). Assuming that the solid is stress free in the undeformed state and the free energy is restricted to linear isotropic thermoelasticity (i.e. ignoring the higher order terms in Eq. (10a)), we obtain the following expression for \(\psi\):

\[
\psi = a_2 \dot{\varepsilon}_2 + a_5 \dot{\theta}^2 + a_{11} \theta^2 + a_{12} \dot{\varepsilon}_2 \dot{\varepsilon}_1
\]

where the material constants are given as [50]:

\[
a_2 = -2\mu; \quad a_5 = \frac{\lambda + 2\mu}{2}; \quad a_{11} = -\frac{C_v^0}{2T_0}; \quad a_{12} = (3\lambda + 2\mu) \alpha_0
\]

(12)

The material constants \(\lambda\) and \(\mu\) are the isothermal Lamé parameters for the isotropic solid, \(\alpha_0\) is the coefficient of volumetric thermal expansion and \(C_v^0\) is the specific heat at constant strain. The parameters \(a_0\) and \(C_v^0\) are all defined in the reference state at \(T_0\). Thus, the Helmholtz free energy density for the thermoelastic response of a nonlocal isotropic solid is given by:

\[
\psi = \frac{1}{2} \lambda \varepsilon_{kk} \varepsilon_{ii} + \mu \varepsilon_{ij} \varepsilon_{ij} - (3\lambda + 2\mu) \alpha_0 \varepsilon_{kk} \theta - \frac{C_v^0}{2T_0} \theta^2
\]

(13)

By using the above expression for \(\psi\) together with Eqs. (8a,8b), the thermoelastic constitutive relations relating the different physical quantities for the isotropic solid are obtained as:

\[
\dot{\sigma}_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} - (3\lambda + 2\mu) \alpha_0 \delta_{ij} \theta
\]

(14a)

\[
\dot{\eta} = (3\lambda + 2\mu) \alpha_0 \varepsilon_{kk} + \frac{C_v^0}{T_0} \theta
\]

(14b)

C. Thermoelastic fractional-order continuum mechanics

Remembering that the fractional-order nonlocal formulation allows a localized implementation of the thermodynamic principles, we write the Helmholtz free energy density for the thermoelastic response following the typical approach for local elasticity (albeit by using the fractional-order strain). Further, we use the Helmholtz free energy density to construct the material constitutive relations for nonlocal thermoelasticity. Using the above arguments, the free energy for an isotropic material is expressed as a series expansion of the fractional-order strain \(\dot{\varepsilon}_{ij}\) and the temperature difference \(\theta\) as [49]:
Using the above results, the Helmholtz free energy in Eq. [13] is recast in the following manner:

\[ \psi = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} - \frac{1}{2} \tilde{\eta} \theta \]  \hfill (15)

III. THERMOELASTIC EULER-BERNOULLI NONLOCAL BEAM MODEL

In this section, we use the thermoelastic constitutive relations developed for the nonlocal solid to analyze the thermoelastic response of a fractional-order Euler-Bernoulli beam. Building on [38, 39] we derive the geometrically nonlinear governing equations and the corresponding boundary conditions for the thermoelastic boundary value problem (BVP) using variational principles.

A. Geometrically nonlinear constitutive relations

Consider a nonlocal beam subject to distributed transverse mechanical and thermal loads as illustrated in Fig. [1]. As indicated in the schematic, the Cartesian coordinates for the current study are chosen such that \( x_3 = \pm h/2 \) coincide with the top and bottom surfaces of the beam, and \( x_1 = 0 \) and \( x_1 = L \) are the ends of the beam along the longitudinal direction. The surface \( x_3 = 0 \) coincides with the mid-plane of the beam and the origin of the reference frame is chosen at the intersection of the mid-plane with the left-end of the beam.

![Schematic of an elastic beam subject to distributed transverse mechanical load \( F_l(x_1) \) and thermal load \( T(x_1) \).](image)

The axial and transverse components of the displacement field \( u(x_1, x_3) \) are denoted by \( u_1(x_1, x_3) \) and \( u_3(x_1, x_3) \), respectively. These displacement fields are given by the Euler-Bernoulli theory as:

\[ u_1(x_1, x_3) = u_0(x_1) - x_3 \left[ \frac{dw_0(x_1)}{dx_1} \right] \]  \hfill (16a)

\[ u_3(x_1, x_3) = w_0(x_1) \]  \hfill (16b)

where \( u_0(x_1) \) and \( w_0(x_1) \) are the mid-plane axial and transverse displacements, respectively. For a geometrically nonlinear analysis, assuming moderate rotations \((10^\circ - 15^\circ)\) but small strains, the fractional-order Lagrangian strain tensor in Eq. [2] can be further simplified using von-Kármán relations. The resulting fractional-order von-Kármán strain-displacement relations are given as [39]:

\[ \varepsilon_{11}(x_1, x_3) = D_{x_1}^\alpha u_1(x_1) + \frac{1}{2} \left[ D_{x_1}^\alpha u_3(x_1) \right]^2 \]  \hfill (17)

where \( D_{x_1}^\alpha (\cdot) \) denotes the fractional-order RC derivative defined in Eq. [3]. As discussed previously, for the RC derivative used above, the 1D domain \((x_A, x_B)\) along the mid-plane of the beam is the horizon of nonlocal interaction at \( x(x_1, 0) \). The end-points of the nonlocal horizon \( x_A(x_A, 1) \) and \( x_B(x_B, 1) \) are the terminals of the left- and right-handed Caputo derivatives within the RC derivative. It follows from Eq. [3] that \( l_A = x_1 - x_A \) and \( l_B = x_B - x_1 \) are the length scales along \( x_1 \) to the left and right hand sides of the point \( x(x_1, 0) \), respectively.

By combining the fractional-order nonlinear axial strain in the above equation along with the Euler-Bernoulli displacement field given in Eq. [16], the axial strain can be recast as:

\[ \varepsilon_{11}(x_1, x_3) = \varepsilon_0(x_1) + x_3 \kappa(x_1) \]  \hfill (18)

In the above equation, \( \varepsilon_0(x_1) \) and \( \kappa(x_1) \) denote the fractional-order axial and bending strains, respectively. They are expressed in terms of the mid-plane field variables as:

\[ \varepsilon_0(x_1) = D_{x_1}^\alpha u_0(x_1) + \frac{1}{2} \left[ D_{x_1}^\alpha w_0(x_1) \right]^2 \]  \hfill (19a)

\[ \kappa(x_1) = -D_{x_1}^\alpha \left[ \frac{dw_0(x_1)}{dx_1} \right] \]  \hfill (19b)

The axial stress in the nonlocal isotropic solid subject to thermoelastic loads is obtained from Eq. [14a] as:

\[ \tilde{\sigma}_{11}(x_1, x_3) = E (\varepsilon_{11}(x_1, x_3) - \alpha_0 \theta(x_1, x_3)) \]  \hfill (20)

where \( E \) is the Young’s modulus of the isotropic solid and \( \alpha_0 \) is the coefficient of thermal expansion for the isotropic solid, as defined earlier.

Using the above defined fractional-order strain and stress fields, the deformation energy density \( U \) of the non-
local beam is obtained as:

\[ U = \frac{1}{2} \int_\Omega \tilde{\sigma}_{11}(x_1, x_3) \tilde{\epsilon}_{11}(x_1, x_3) \, d\Omega \]  
\tag{21}

where \( \Omega \) denotes the volume occupied by the beam. The total potential energy functional of the beam subject to distributed axial \( (F_a(x_1)) \) and transverse forces \( (F_t(x_1)) \) acting on the mid-plane, assuming that no body forces, is given by:

\[
\Pi[u(x)] = U - \int_0^L F_a(x_1) u_0(x_1) \, dx_1 \\
- \int_0^L F_t(x_1) u_0(x_1) \, dx_1
\]
\tag{22}

We now derive the governing equations and the associated boundary conditions for the thermoelastic response of the nonlocal beam in the strong form by imposing optimality conditions on the above functional. Before presenting the governing equations, we highlight that the objective of this study is to evaluate the elastic response of a 1D beam when subject to combined thermal and mechanical loads. The thermal load consists of a steady-state temperature distribution applied along the length of the beam on the face at \( x_3 = \pm h/2 \). Thus, the independent variation of temperature field \( \delta T \), and thereby \( \delta \theta \), is identically zero.

### B. Governing equations

The fractional-order governing equations for the thermoelastic response of geometrically nonlinear and nonlocal beams is obtained using variational principles (i.e. by minimizing the total potential energy given in Eq. (22)). The resulting form is given as follows:

\[ \mathcal{D}^\alpha_{x_1} \mathcal{N}(x_1) + F_a(x_1) = 0 \]  
\tag{23a}

\[ D^1_{x_1} \left[ \mathcal{D}^\alpha_{x_1} \mathcal{M}(x_1) \right] + \mathcal{D}^\alpha_{x_1} \left[ \mathcal{N}(x_1) D^\alpha_{x_1} [w_0(x_1)] \right] + F_t(x_1) = 0 \]  
\tag{23b}

The corresponding essential and natural boundary conditions \( (\forall \ x_1 \in \{0, L\}) \) are obtained as:

\[ \mathcal{N}(x_1) = 0 \text{ or } \delta u_0(x_1) = 0 \]  
\tag{24a}

\[ \mathcal{M}(x_1) = 0 \text{ or } \delta \left[ D^1_{x_1} w_0(x_1) \right] = 0 \]  
\tag{24b}

\[ D^1_{x_1} \mathcal{M}(x_1) + \mathcal{N}(x_1) D^1_{x_1} [w_0(x_1)] = 0 \text{ or } \delta u_0(x_1) = 0 \]  
\tag{24c}

Note that the detailed steps leading to the above fractional-order nonlinear governing equations extend directly from the geometrically nonlinear analysis of fractional-order beams presented in [39], hence they are not provided here. In the above Eqs. [23], \( D^1_{x_1} \) denotes the first integer-order derivative with respect to the axial variable \( x_1 \). \( \mathcal{D}^\alpha_{x_1} \) is the Riesz Riemann-Liouville derivative of order \( \alpha \) which is defined as:

\[ \mathcal{D}^\alpha_{x_1} \phi = \frac{\Gamma(2 - \alpha)}{2} \left[ \int_{x_1-l_B}^{x_1+l_A} - \left( x_1-x_0 \right)^{\frac{1}{\alpha}} \phi \right] \]  
\tag{25}

where \( \phi \) is an arbitrary function and \( x_1-l_B \) \( D^\alpha_{x_1} \phi \) and \( x_1+l_A \alpha \) \( D^\alpha_{x_1} \phi \) are the left- and right-handed Riemann Liouville derivatives of \( \phi \) to the order \( \alpha \), respectively. Note that the fractional derivative \( \mathcal{D}^\alpha_{x_1} \) is defined over the interval \( (x_1-l_B, x_1+l_A) \) unlike the fractional derivative \( D^\alpha_{x_1} \) which is defined over the interval \( (x_1-l_B, x_1+l_B) \). This change in the terminals of the interval of the Riesz Riemann-Liouville fractional derivative follows from the standard integration by parts technique used to simplify the variational integrals (see [38]). Further, \( \mathcal{N}(x_1) \) and \( \mathcal{M}(x_1) \) are axial and bending stress resultants defined in the following manner:

\[ \mathcal{N}(x_1) = \int_{-h/2}^{h/2} \tilde{\sigma}_{11}(x_1, x_3) \, dx_3 \, dx_2 \]  
\tag{26a}

\[ \mathcal{M}(x_1) = \int_{-h/2}^{h/2} x_3 \tilde{\sigma}_{11}(x_1, x_3) \, dx_3 \, dx_2 \]  
\tag{26b}

By using the constitutive relations for a homogeneous isotropic solid given in Eq. (20) along with the above definitions, the stress resultants are obtained as:

\[ \mathcal{N}(x_1) = A_{11} \tilde{\epsilon}_0(x_1) - N_0(x_1) \]  
\tag{27a}

\[ \mathcal{M}(x_1) = -D_{11} \tilde{\gamma}_0(x_1) - M_0(x_1) \]  
\tag{27b}

where \( A_{11} = Ebh \) and \( D_{11} = Ebh^3/12 \) are the axial and bending stiffness coefficients of the beam, respectively. The thermal resultants \( N_0(x_1) \) and \( M_0(x_1) \) for the isotropic beam are given as:

\[ \{N_0(x_1), M_0(x_1)\} = Eba_0 \int_{-h/2}^{h/2} \{1, x_3\} \theta(x_1, x_3) \, dx_3 \]  
\tag{28}
Note that for a general distribution of material properties across the thickness of the beam, additional terms due to the bending-extension coupling would be noted in the Eqs.\footnote{20,28}.

In the following, we discuss a few characteristics of the thermomechanical governing equations given in Eq.\footnote{23}. First, observe that the stress resultants given in Eq.\footnote{27} introduce the thermoelastic variables into the governing equations in Eq.\footnote{23}. In the absence of thermal loads \((\theta(x_1,x_3) = 0)\), the constitutive models reduce to the expressions derived in \cite{39} for the geometrically nonlinear fractional-order nonlocal beams. Owing to the non-linear nature of the structural response, the axial and transverse displacement fields are coupled unlike what seen in the linear elastic case \cite{38}. Second, we emphasize that the fractional-order model for the linear thermomechanical fractional-order BVP builds upon the constitutive relations developed above. The linear thermomechanical model of the fractional-order nonlocal beam will be discussed further in \cite{41}. Finally, the classical thermomechanical models are recovered for \(\alpha = 1\).

\section{IV. Nonlinear Fractional Finite Element Method (f-FEM)}

Given the nonlinear and integro-differential nature of the governing equations, it is not possible to obtain closed form solutions for the most general loading and boundary conditions. Therefore, we develop a fractional-order finite element method (f-FEM) to obtain the numerical solution of the nonlinear fractional-order governing differential equations. The f-FEM developed for the thermomechanical fractional-order BVP builds upon the numerical models developed for fractional-order models of nonlocal elasticity \cite{38,39}. Although, the f-FEM is developed here for thermoeastic response of fractional-order beams, it can be easily extended for higher dimensional structures like plates and shells. Analogously to traditional FEM, the f-FEM is formulated starting from a discretized form of the total potential energy functional \(\Pi[u(x)]\) given in Eq.\footnote{22}. For this purpose, the 1D domain \(\Omega = [0,L]\) of the beam indicated in Fig.\footnote{1} is uniformly discretized into disjoint two noded elements \(\Omega_i = (x_1,x_1^{+1})\) of length \(l_e\) such that \(\cup_{i=1}^{N_e} \Omega_i = \Omega\), \(N_e\) being the total number of discretized elements. It is immediate that \(\Omega_i^j \cap \Omega_k^l = \emptyset \ \forall \ j \neq k\). The unknown field variables \(u_0(x_1)\) and \(w_0(x_1)\) in Eq.\footnote{23} can now be evaluated at any point \(x_1 \in \Omega_e^i\) by interpolating the corresponding nodal values for \(\Omega_e^i\) as:

\[
\{u_0(x_1)\} = [S_u(x_1)] \{U_e(x_1)\} \quad \text{(29a)}
\]

\[
\{w_0(x_1)\} = [S_w(x_1)] \{W_e(x_1)\} \quad \text{(29b)}
\]

where \(\{U_e(x_1)\}\) and \(\{W_e(x_1)\}\) are the axial and transverse displacement degrees of freedom of the element \(\Omega_e^i\). For the two-noded element used in this study, the displacements are given by:

\[
\{U_e(x_1)\}^T = \begin{bmatrix} u_0^{(1)} & u_0^{(2)} \end{bmatrix} \quad \text{(29c)}
\]

\[
\{W_e(x_1)\}^T = \begin{bmatrix} w_0^{(1)}(1) & w_0^{(2)}(1) & w_0^{(2)}(2) \end{bmatrix} \quad \text{(29d)}
\]

The superscripts (\(\cdot\))\(^{(1)}\) and (\(\cdot\))\(^{(2)}\) in the above equations denote the local node numbers of the two-noded element. \([S_u(x_1)]\) and \([S_w(x_1)]\) are shape function matrices given as:

\[
[S_u(x_1)] = [L_1^u(x_1) \ L_2^u(x_1)] \quad \text{(29e)}
\]

\[
[S_w(x_1)] = [H_1^u(x_1) \ H_2^u(x_1) \ H_3^u(x_1) \ H_4^u(x_1)] \quad \text{(29f)}
\]

where \(L_i\ (i = 1,2)\) and \(H_j\ (j = 1,2,3,4)\) are the Lagrangian and Hermitian interpolation functions, respectively. These functions are chosen to enforce the continuity of the axial and transverse displacement fields for the Euler-Bernoulli beam theory. The RC fractional derivative \(D_1^\alpha\ [u_0(x_1)]\) defined in Eq.\footnote{7} is expressed as:

\[
D_1^\alpha\ [u_0(x_1)] = \frac{1}{2}(1-\alpha) \left[ \int_{x_1-l_A}^{x_1} \frac{D_1^1[u_0(s_1)]}{(x_1-s_1)^\alpha} \ dx_1 + \int_{x_1+l_B}^{x_1+l_A} \frac{D_1^1[u_0(s_1)]}{(s_1-x_1)^\alpha} \ dx_1 \right] \quad \text{(30)}
\]

where \(s_1\) is a dummy variable in the axial direction, used within the definition of the fractional-order derivative. The above expression can be recast as:

\[
D_1^\alpha\ [u_0(x_1)] = \int_{x_1-l_A}^{x_1+l_B} A(x_1,s_1,l_A,l_B,\alpha) D_1^1[u_0(s_1)] \ dx_1 \quad \text{(31a)}
\]

where the kernel \(A(x_1,s_1,l_A,l_B,\alpha)\) is interpreted as the \(\alpha\)-order power-law function for the convolution integral of integer-order derivative \(D_1^1[u_0(x_1)]\) over the nonlocal interaction zone \((x_1-l_A,x_1+l_B)\). It follows from Eq.\footnote{30}
that the attenuation function is:

\[
A = \begin{cases} 
\frac{1}{2}(1-\alpha)I_A^{\alpha-1}(x_1-s_1)^{-\alpha} & s_1 \in (x_1-l_A, x_1) \\
\frac{1}{2}(1-\alpha)I_B^{\alpha-1}(s_1-x_1)^{-\alpha} & s_1 \in (x_1, x_1+l_B) 
\end{cases} 
\]

Note that the kernel is a function of the relative distance \(|x_1-s_1|\) and physically, serves as the attenuation function corresponding to the fractional-order model for nonlocal elasticity. The above result for the RC fractional derivative of the axial displacement \(u_0(x_1)\) (Eq. (33)) can be extended directly to derive the expressions for \(D^1_{s_1}[w_0(x_1)]\) and \(D^2_{s_1}[D^1_{s_1}[w_0(x_1)]]\). These expressions can also be found in [39].

The integer-order derivatives \(D^1_{s_1}[u_0(s_1)], D^1_{s_1}[w_0(s_1)]\) and \(D^2_{s_1}[w_0(s_1)]\) that occur within the fractional-order derivatives \(D^\alpha_{s_1}[u_0(x_1)], D^\alpha_{s_1}[w_0(x_1)]\) and \(D^\alpha_{s_1}[D^1_{s_1}[w_0(x_1)]]\), are expressed in terms of the nodal values of the element containing the point \(s_1\).

More specifically, from Eq. [29] we have:

\[
\begin{align*}
D^1_{s_1}[u_0(s_1)] &= [B_u(s_1)]\{U_e(s_1)\} \\
D^1_{s_1}[w_0(s_1)] &= [B_w(s_1)]\{W_e(s_1)\} \\
D^2_{s_1}[w_0(s_1)] &= [B_\vartheta(s_1)]\{W_e(s_1)\}
\end{align*}
\]

where the different matrices \([B_{\square}(s_1)\)] are given as:

\[
\begin{align*}
[B_u(s_1)] &= \frac{d[S_u(s_1)]}{ds_1} \\
[B_w(s_1)] &= \frac{d[S_w(s_1)]}{ds_1} \\
[B_\vartheta(s_1)] &= \frac{d^2[S_w(s_1)]}{ds_1^2}
\end{align*}
\]

Finally, by using the above expressions in Eq. (31a) we have:

\[
\begin{align*}
D^\alpha_{s_1}[u_0(x_1)] &= \int_{x_1-l_A}^{x_1+l_B} A(x_1, s_1, l_A, l_B, \alpha)[B_u(s_1)]\{U_e(s_1)\}ds_1 \\
D^\alpha_{s_1}[w_0(x_1)] &= \int_{x_1-l_A}^{x_1+l_B} A(x_1, s_1, l_A, l_B, \alpha)[B_w(s_1)]\{W_e(s_1)\}ds_1 \\
D^\alpha_{s_1}[D^1_{s_1}[w_0(x_1)]] &= \int_{x_1-l_A}^{x_1+l_B} A(x_1, s_1, l_A, l_B, \alpha)[B_\vartheta(s_1)]\{W_e(s_1)\}ds_1
\end{align*}
\]

It is immediate that the evaluation of the above fractional derivatives requires a convolution of the integer-order derivatives across the horizon of nonlocality \((x_1-l_A, x_1+l_B)\). Thus, while obtaining the FE approximation in Eq. (33), the nonlocal contributions from the different finite elements in the horizon \((x_1-l_A, x_1+l_B)\) must be properly attributed to the corresponding nodes. In order perform this mapping of the nonlocal contributions from elements in the horizon, the nodal values \([U_i(s_1)\] and \([W_i(s_1)\] must be transformed into the respective global variable vectors \([U_g\)] and \([W_g\)] with the help of the connectivity matrices \([C_u(x_1, s_1)\)] and \([C_w(x_1, s_1)\]), respectively. A detailed discussion on these connectivity matrices can be found in [38, 39]. Following this transformation, the fractional derivatives can be expressed as:

\[
\begin{align*}
D^\alpha_{s_1}[u_0(x_1)] &= [\tilde{B}_u(x_1)]\{U_g\} \\
D^\alpha_{s_1}[w_0(x_1)] &= [\tilde{B}_w(x_1)]\{W_g\} \\
D^\alpha_{s_1}[D^1_{s_1}[w_0(x_1)]] &= [\tilde{B}_\vartheta(x_1)]\{W_g\}
\end{align*}
\]

where the different matrices \([\tilde{B}_{\square}(x_1)\)] are given as:

\[
\tilde{B}_u(x_1) = \int_{x_1-l_A}^{x_1+l_B} A(x_1, s_1, l_A, l_B, \alpha)[B_u(s_1)]\{C_u(x_1, s_1)\}ds_1
\]
We use the above derived expressions for FE approximations of the different fractional-order derivatives to obtain the algebraic governing equations corresponding to the geometrically nonlinear thermoelastic response of the fractional-order beam. In the interest of a more compact notation, the functional dependence of the different physical quantities on the spatial variables will be implied, unless stated to be constant. By using the expressions in Eqs. (34), the first variation of the virtual strain energy $\delta U$ defined in Eq. (22) is obtained as:

$$
\delta \Pi = b \int_0^L \int_{-h/2}^{h/2} \delta \tilde{\epsilon}_{11} \tilde{\sigma}_{11} dx_3 dx_1 - \int_0^L F_0 \delta w_0 dx_1 - \int_0^L F_1 \delta \vartheta_0 dx_1
$$

By using the strain-displacement relations in Eq. (18) and the stress-resultants in Eq. (26) we obtain:

$$
\delta \Pi = \int_0^L \left\{ N \left[ D^\alpha_{x_1} (\delta u_0) \right] + N \left[ D^\alpha_{x_1} w_0 \right] [D^\alpha_{x_1} (\delta w_0)] - M \left[ D^\alpha_{x_1} \left[ D^1_{x_1} (\delta w_0) \right] \right] F_0 \delta u_0 - F_1 \delta \vartheta_0 \right\} dx_1
$$

The minimum potential energy principle, $\delta \Pi = 0$, is enforced to obtain the algebraic equations of equilibrium. Collecting the terms containing the independent variations $\delta u_0$ and $\delta w_0$, and using Eq. (27) gives the following pair of equations:

$$
\begin{align*}
\int_0^L \left\{ [D^\alpha_{x_1} (\delta u_0)] \left[ A_{11} \left( D^\alpha_{x_1} u_0 + \frac{1}{2} \left( D^\alpha_{x_1} w_0 \right)^2 \right) - N_0 \right] - F_0 \delta u_0 \right\} dx_1 &= 0 \\
\int_0^L \left\{ [D^\alpha_{x_1} w_0] [D^\alpha_{x_1} (\delta w_0)] \left[ A_{11} \left( D^\alpha_{x_1} u_0 + \frac{1}{2} \left( D^\alpha_{x_1} w_0 \right)^2 \right) - N_0 \right] - F_1 \delta \vartheta_0 - [D^\alpha_{x_1} \left[ D^1_{x_1} (\delta w_0) \right] \left[ -D_{11} \left( D^\alpha_{x_1} \left[ D^1_{x_1} (\delta w_0) \right] \right) - M_0 \right] \right\} dx_1 &= 0
\end{align*}
$$

From the above equation, note that the geometric nonlinearity introduces additional nonlinear thermomechanical coupled terms. This nonlinear behavior is dependent on the thermal properties of the beam as evident from the expressions of the thermal stress resultants given in Eq. (28). In fact, these nonlinear effects are expected to be significant at high temperature. These additional nonlinear thermomechanical terms can be accounted for in two ways: approach #1: the terms are treated as an external nonlinear thermal force; and approach #2: the contribution of these terms is accounted for through the stiffness matrix of the system. The equivalence of the results obtained through both these approaches and a comparison of their accuracy and stability is presented in [51]. In this study, we follow the approach #1 so that the linear analysis of the nonlinear system, for small displacements, be-
comes straightforward without requiring changes to the
stiffness matrix.

The nonlinear governing Eq. (37) in the weak form
is converted into algebraic equations in the nodal dis-
placement degrees of freedom. More specifically, by using
the numerical approximations developed for the different
fractional derivatives (see Eq. (34)) and then enforcing
the minimization of the total potential energy, we obtain
the following system of nonlinear algebraic equations in
\{U_0\} and \{W_0\}:
\[
\begin{bmatrix}
[K_{11}] & [K_{12}] \\
[K_{21}] & [K_{22}]
\end{bmatrix}
\begin{bmatrix}
[U_0] \\
[W_0]
\end{bmatrix} = 
\begin{bmatrix}
\{F_A + F_{A0}\} \\
\{F_T + F_{T0}\}
\end{bmatrix}
\tag{38}
\]
where the different stiffness matrices are given by:
\[
[K_{11}] = \int_0^L A_{11}[\tilde{B}_u(x_1)]^T[\tilde{B}_u(x_1)]dx_1 
\tag{39a}
\]
\[
[K_{12}] = \frac{1}{2} \int_0^L A_{11} \left(D_{x_1}^\alpha [w_0(x_1)]\right) [\tilde{B}_u(x_1)]^T[\tilde{B}_w(x_1)]dx_1 
\tag{39b}
\]
\[
[K_{21}] = \int_0^L A_{11} \left(D_{x_1}^\alpha [w_0(x_1)]\right) [\tilde{B}_w(x_1)]^T[\tilde{B}_u(x_1)]dx_1 
\tag{39c}
\]
\[
[K_{22}] = \int_0^L D_{11} [\tilde{B}_\theta(x_1)]^T[\tilde{B}_\theta(x_1)] dx_1 + 
\frac{1}{2} \int_0^L \left[A_{11} \left(D_{x_1}^\alpha [w_0(x_1)]\right)^2\right] [\tilde{B}_w(x_1)]^T[\tilde{B}_w(x_1)]dx_1 
\tag{39d}
\]

The algebraic equations (38) are solved for the nodal val-
ues of the generalized displacement coordinates for an
isotropic beam subject to distributed thermal and me-
chanical loads. The solution to these equations along
with Eq. (10) gives the displacement field at any point
within the beam. The geometric nonlinearity in the sys-
tem is clearly evident from the expressions of the stiffness
matrices. Further, as previously discussed, the addi-
tional nonlinear thermomechanical terms are introduced
into the model as a nonlinear transverse force as evident
from Eq. (40d). Given the nonlinear nature of the FE
algebraic equations, a Newton-Raphson (NR) iterative
numerical scheme was adopted to obtain the solution of
the Eq. (38). Similar to classical nonlinear models, the
NR procedure for the fractional-order nonlinear equa-
tions also requires the evaluation of the tangent stiffness
matrix. The procedure to evaluate the tangent stiffness
matrix as well as the NR scheme can be found in [39].

Note that the nonlinear f-FEM also involves the nu-
merical evaluation of the stiffness matrix, the tangent
stiffness matrix, and the force vector. The procedure
to numerically evaluate the force vector follows directly
from classical FE formulations. The stiffness matrix of
the nonlocal system given in Eq. (39) requires the evalua-
tion of the different nonlocal matrices \[\tilde{B}_\square\], \[\square \in \{u, w, \theta\}\] given in Eq. (34). As evident from Eq. (34), this in-
volves a convolution of the integer-order derivatives with
the fractional-order attenuation function over the hori-
zon of nonlocality. Clearly, the FE approximation for
fractional-order derivative involves additional integra-
tions over the horizon of nonlocality to account for the
nonlocal interactions. A numerical procedure to account
for these nonlocal interactions has been presented in se-
mral classical numerical approaches to nonlocal elastic-
ity [18 52]. However, an additional complexity in the fra-
tonial-order theory stems from the singularity of the
kernel at \[x_1 = s_1\] (see Eq. (31)). The fractional-order
nonlocal interactions as well as the end-point singularity
are addressed in detail in [38 39]. We emphasize that
the numerical integration procedure presented in [38 39]
directly extends to the stiffness and tangent stiffness ma-
trices of the nonlinear FE governing equations derived in
this study.

The linear f-FEM for the thermomechanical response of the
nonlocal isotropic beam can be obtained from the above
model by ignoring the contribution of the nonlinear coupling term, that is \((D^\alpha_{x_1}[w_0(x)])^2\) in the system matrices as well as in the force vectors. Note that the axial and transverse displacement fields for the linear elastic response due to thermo-mechanical loads are decoupled. Finally, the f-FEM reduces to a local thermoelastic study of beams when the fractional-order is set to \(\alpha = 1\).

V. NUMERICAL RESULTS AND DISCUSSION

We use the numerical model developed in [44] to analyze both the linear and geometrically nonlinear thermoelastic response of fractional-order nonlocal isotropic beams. In order to satisfy the underlying assumptions of the Euler-Bernoulli beam theory, the beam is assumed to be slender with an aspect ratio of \(L/h = 100\). In the following studies, the length of the beam is maintained at \(L = 1\text{m}\) and the width of the beam is considered to be unity. The beam is assumed made out of aluminum that is \(E = 70\text{GPa}\) and \(\alpha_0 = 23 \times 10^{-6}\text{K}^{-1}\). The constitutive parameters of the fractional-order continuum model, order \(\alpha\) and length scales \(l_A\) and \(l_B\) are provided wherever necessary. We assumed that the length scales \(l_A\) and \(l_B\) at a point within the domain of the isotropic beam are equal, that is \(l_A = l_B = l_f\). However, following the discussion in [44], these length scales are truncated for points close to geometric boundaries of the beam. We analyzed numerically the effect of the fractional parameters \(\alpha\) and \(l_f\) on the response of the beam subject to different loading and boundary conditions. Both the linear and nonlinear cases are considered.

Before presenting the results, we make a few remarks concerning the validation and convergence of the f-FEM. In this regard, the f-FEM procedure has already been validated for linear BVPs in [7] and for nonlinear BVPs in [39]. Further, as discussed in [7, 39], the convergence of the f-FEM with finer element discretization is controlled by the dynamic rate of convergence defined as: \(N_{inf}(= l_f/l_e)\), where \(l_e\) is the length of the discretized element. This parameter was shown to be dependent on the fractional model parameters \(\alpha\) and \(l_f\) that determine the strength of the nonlocal interactions between distant elements [7, 39]. Following these convergence studies, the mesh discretization \(N_{inf} = 10\) was chosen. This choice allows for sufficient number of elements to be included in the horizon of nonlocality at any point, in order to accurately capture the nonlocal interactions [7, 39].

**Linear Response**: we considered a simply supported beam subject to a uniformly distributed transverse load (UDTL) of magnitude \(q_0\) (in N/m) and to the following thermal load:

\[
\theta(x_1, x_3) = \theta_1 \left(1 + \frac{2x_3}{h}\right)
\]  

(41)

The bottom surface of the beam was maintained at the ambient temperature \(T_0\). It follows, from Eq. (41), that the temperature of the top surface is \(T_1(= T_0 + 2\theta_1)\). The nonlocal elastic response to these thermo-mechanical loads for different values of fractional constitutive parameters \(\alpha\) and \(l_f\) are compared in Fig. (2). The transverse displacement along the length of the simply supported beam was explored for different values of \(\alpha\) while maintaining \(l_f\) constant (Fig. (2a)). Similarly, the transverse displacement was evaluated for different values of \(l_f\) while maintaining \(\alpha\) constant (Fig. (2b)). The increase in transverse displacement with the increasing degree of nonlocality, achieved either by reducing \(\alpha\) (see Fig. (2a)) or by increasing \(l_f\) (see Fig. (2b)), points towards the reduction of the stiffness of the fractional-order beam. We emphasize that the consistent softening of the structure with increasing degree of nonlocality was also observed for beams subject to different loading and boundary conditions. In order to facilitate the analysis of the thermoelastic response of the nonlocal beam, we have provided, as a reference, the transverse displacement of the local beam (\(\alpha = 1\)) for two different cases: (a) absence of thermal load, i.e. \(\theta_1 = 0\); and (b) linear thermal load given in Eq. (41). From Fig. (2), note that the nonlocal results converge to the local results for \(\alpha\) approaching 1 and \(l_f/L << 1\).

**Nonlinear response**: in this case, the beam was subjected to a UDTL of magnitude \(q_0\) (in N/m) and a uniform thermal field \(\theta(x_1, x_3) = \theta_0\) (in K). First, we considered a beam that is clamped at both ends and subjected to the thermo-mechanical loads described above. The transverse displacement of the beam for a fixed UDTL and varying magnitude of the thermal load was obtained and compared for different values of \(\alpha\) and \(l_f\). The magnitude of the UDTL was fixed at \(q_0 = 5 \times 10^4\text{N/m}\) and the value of the uniform thermal field \(\theta_0\) was varied in order to analyze the effect of the thermal load on the response of the beam. Additionally, in order to analyze the effect of the fractional model parameters on the response of the beam, the transverse displacement of the beam was compared for different values of \(\alpha\) and \(l_f\). The results of this study are presented in Fig. (3) in terms of the thermal load versus displacement. The displacement values presented in Fig. (3) correspond to the maximum displacement of the mid-plane of the beams, obtained at \(x_1 = L/2\). The effect of the fractional-order \(\alpha\), with \(l_f\) being held constant, is compared in Fig. (3a), while the
FIG. 2: Transverse displacement corresponding to the linear response of a simply supported beam for $q_0 = 10^4$ N/m and $\theta_1 = 10$K. The plot is parameterized and compared for different values of (a) the fractional-order and (b) the length scale.

effect of $l_f$ for fixed $\alpha$ are presented in Fig. (3b). The result for the local case ($\alpha = 1$) is provided for reference in both cases. As evident from Fig. (3), the nonlocal beam exhibits consistent softening with increasing thermal loads and increasing degree of nonlocality. As observed earlier, the thermoelastic response of the nonlocal beam converges to the corresponding local elastic response for $\alpha$ approaching 1 and $l_f/L << 1$.

Finally, the study was repeated for a beam pinned at both ends. The effect of the fractional model parameters over the geometrically nonlinear response of the pinned-pinned beam subject to thermo-mechanical loads is presented in Fig. (4). Observations analogous to those drawn for the clamped-clamped beam can be extended to this case. Remarkably, the fractional-order approach to the modeling of nonlocal elasticity exhibits a coherence across boundary conditions as well as loading conditions for both the linear and geometrically nonlinear responses. This behavior differs sharply from the paradoxical results reported in the literature for either gradient or integral based approaches to nonlocal elasticity [27, 28, 53].

VI. CONCLUSIONS

This study established the theoretical and computational framework for the analysis of the thermoelastic response of nonlocal solids according to the fractional-order continuum theory. Starting from fractional-order strain-displacement relations, the presented approach allowed achieving simplified material constitutive relations resulting in a rigorous point-wise enforcement of the thermodynamic balance laws. This result highlights a critical difference with respect to traditional integer-order methods that can satisfy the thermodynamic laws only in a (weak) integral sense. In fact, the fractional-order framework greatly simplifies the determination of the free energy density and the subsequent constitutive modeling of nonlocal thermoelasticity. The resulting thermodynamically-consistent fractional-order continuum theory is well suited to develop accurate models to capture nonlocal interactions, heterogeneity, and scale effects in complex elastic solids operating in a thermo-mechanical environment. The efficacy of the modeling approach was illustrated by applying the methodology to the analysis of the static response of a nonlocal Euler-Bernoulli beam subject to combined thermo-mechanical loads. An accurate numerical finite element method (referred to as f-FEM) for the solution of the nonlocal beam under different boundary conditions was also developed. The f-FEM is capable of solving the nonlinear integro-differential governing equations despite the presence of a singular kernel characteristic of the fractional operators. A variety of numerical results highlighted the extremely robust nature of the fractional models by illustrating the consistency of the predicted nonlocal response across different thermo-mechanical loads and boundary conditions; a property not achievable in classical integer-order methods. The current study may be extended further to include the fractional-order heat conduction laws so to
FIG. 3: Transverse displacement of a clamped-clamped beam subject to $q_0 = 5 \times 10^4$N/m and thermal load $\theta_0$. The curves are parameterized and compared for different values of (a) the fractional-order and (b) the length scale.

FIG. 4: Transverse displacement of a pinned-pinned beam subject to $q_0 = 5 \times 10^4$N/m and thermal load $\theta_0$. The curves are parameterized and compared for different values of (a) the fractional-order and (b) the length scale.

develop coupled fractional-order governing equations for a thermoelastic solid.

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[1] N. A. Fleck, G. M. Muller, M. F. Ashby, and J. W. Hutchinson. Strain gradient plasticity: theory and experiment. *Acta Metallurgica et materialia*, 42(2):475–487,
[2] W. D. Nix and H. Gao. Indentation size effects in crystalline materials: a law for strain gradient plasticity. *Journal of the Mechanics and Physics of Solids*, 46(3):411–425, 1998.

[3] CY Wang, T Murmu, and S Adhikari. Mechanisms of nonlocal effect on the vibration of nanoplates. *Applied Physics Letters*, 98(15):153101, 2011.

[4] N. A. Fleck and J. W. Hutchinson. A reformulation of strain gradient plasticity. *Journal of the Mechanics and Physics of Solids*, 49(10):2245–2271, 2001.

[5] Jani Romanoff, JN Reddy, and Jasmin Jelovica. Using non-local timoshenko beam theories for prediction of micro- and macro-structural responses. *Composite Structures*, 156:410–420, 2016.

[6] John P Hollkamp, Mihir Sen, and Fabio Semperlotti. Analysis of dispersion and propagation properties in a periodic rod using a space-fractional wave equation. *Journal of Sound and Vibration*, 441:204–220, 2019.

[7] S. Pataiaik and F. Semperlotti. A generalized fractional-order elastodynamic theory for nonlocal attenuating media. *Under Review*, 2019.

[8] LJ Sudak. Column buckling of multiwalled carbon nanotubes using nonlocal continuum mechanics. *Journal of applied physics*, 94(11):7281–7287, 2003.

[9] MA Kouchakzadeh, M Rasekh, and H Haddadpour. Panel flutter analysis of general laminated composite plates. *Composite Structures*, 92(12):2906–2915, 2010.

[10] P Marzozza, SA Fazelzadeh, and M Hosseini. A review of nonlinear aero-thermo-elasticity of functionally graded panels. *Journal of Thermal Stresses*, 34(5-6):536–568, 2011.

[11] S. A. Emam. A general nonlocal nonlinear model for buckling of nanobeams. *Applied Mathematical Modelling*, 37(10-11):6929–6939, 2013.

[12] F. Najar, S. El-Borgi, J. N. Reddy, and Jasmin Jelovica. Using non-local timoshenko beam theories for prediction of micro- and macro-structural responses. *Composite Structures*, 156:410–420, 2016.

[13] F. Ebrahimi and E. Salari. Nonlocal thermo-mechanical principle. *International Journal of Solids and Structures*, 38(42-43):7359–7380, 2001.

[14] A. C. Eringen and D. G. B. Edelen. On nonlocal elasticity. *International Journal of Engineering Science*, 10(3):233–248, 1972.

[15] C. Polizzotto. Nonlocal elasticity and related variational principles. *International Journal of Solids and Structures*, 38(42-43):7359–7380, 2001.

[16] Z. P. Bažant and M. Jirásek. Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics*, 128(11):1119–1149, 2002.

[17] R. H. J. Peerlings, M. G. D. Geers, R. De Borst, and W. A. M. Brekelmans. A critical comparison of nonlocal and gradient-enhanced softening continua. *International Journal of Solids and Structures*, 38(44-45):7723–7746, 2001.

[18] E. C. Aifantis. Update on a class of gradient theories. *Mechanics of materials*, 35(3-6):259–280, 2003.

[19] S Sidhardh and MC Ray. Element-free galerkin model of nano-beams considering strain gradient elasticity. *Acta Mechanica*, 229(7):2765–2786, 2018.

[20] J. Yang, L. L. Ke, and S. Kitipornchai. Nonlinear free vibration of single-walled carbon nanotubes using nonlocal timoshenko beam theory. *Physica E: Low-dimensional Systems and Nanostructures*, 42(5):1727–1735, 2010.

[21] A. R. Srinivasa and J. N. Reddy. A model for a constrained, infinitely deforming, elastic solid with rotation gradient dependent strain energy, and its specialization to von kármán plates and beams. *Journal of the Mechanics and Physics of Solids*, 61(3):873–885, 2013.

[22] L. E. Shen, H-S. Shen, and C-L. Zhang. Nonlocal plate model for nonlinear vibration of single layer graphene sheets in thermal environments. *Computational Materials Science*, 48(3):680–685, 2010.

[23] A. Tounsi, S. Bengueldiab, A. Sennah, M. Zidou, et al. Nonlocal effects on thermal buckling properties of double-walled carbon nanotubes. *Advances in nano research*, 1(1):1, 2013.

[24] Giovanni Romano, Raffaele Barretta, Marina Diaco, and Francesco Marotti de Sciarra. Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. *International Journal of Mechanical Sciences*, 121:151–156, 2017.

[25] Giovanni Romano, Raimondo Luciano, Raffaele Barretta, and Marina Diaco. Nonlocal integral elasticity in nanostructures, mixtures, boundary effects and limit behaviours. *Continuum Mechanics and Thermodynamics*, 30(3):641–655, 2018.

[26] JK Phadikar and SC Pradhan. Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates. *Computational materials science*, 49(3):492–499, 2010.

[27] Amin Anjomshoa. Application of ritz functions in buckling analysis of embedded orthotropic circular and elliptical micro/nano-plates based on nonlocal elasticity theory. *Mechanica*, 48(6):1337–1353, 2013.

[28] I. Podlubny. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, volume 198. Elsevier, 1998.

[29] G. Cottone, M. Di Paola, and M. Zingales. Elastic waves propagation in 1d fractional non-local continuum. *Physica E: Low-dimensional Systems and Nanostructures*, 42(2):95–103, 2009.

[30] M. Di Paola and M. Zingales. Long-range cohesive interactions of non-local continuum faced by fractional cal-
culus. *International Journal of Solids and Structures*, 45(21):5642–5659, 2008.

[34] A. Carpinteri, P. Cornetti, and A. Sapora. Nonlocal elasticity: an approach based on fractional calculus. *Meccanica*, 49(11):2551–2569, 2014.

[35] W. Sumelka and T. Blaszczyk. Fractional continua for linear elasticity. *Archives of Mechanics*, 66(3):147–172, 2014.

[36] W. Sumelka, T. Blaszczyk, and C. Liebold. Fractional euler–bernoulli beams: Theory, numerical study and experimental validation. *European Journal of Mechanics-A/Solids*, 54:243–251, 2015.

[37] John P Hollkamp and Fabio Semperlotti. Application of fractional order operators to the simulation of ducts with acoustic black hole terminations. *Journal of Sound and Vibration*, 465:115035, 2020.

[38] Sansit Patnaik, Sai Sidhardh, and Fabio Semperlotti. A ritz-based finite element method for a fractional-order boundary value problem of nonlocal elasticity. *arXiv preprint arXiv:2001.06885*, 2020.

[39] Sai Sidhardh, Sansit Patnaik, and Fabio Semperlotti. Geometrically nonlinear response of a fractional-order nonlocal model of elasticity. *arXiv preprint arXiv:2002.07148*, 2020.

[40] Sansit Patnaik, Sai Sidhardh, and Fabio Semperlotti. Fractional-order models for the static and dynamic analysis of nonlocal plates. *arXiv preprint arXiv:2002.10244*, 2020.

[41] Sansit Patnaik, Sai Sidhardh, and Fabio Semperlotti. Geometrically nonlinear analysis of nonlocal plates using fractional calculus. *Under Review*.

[42] Yu Z Povstenko. Fractional heat conduction equation and associated thermal stress. *Journal of Thermal Stresses*, 28(1):83–102, 2004.

[43] YZ Povstenko. Thermoelasticity that uses fractional heat conduction equation. *Journal of Mathematical Sciences*, 162(2):296–305, 2009.

[44] Yurii Povstenko. *Fractional thermoelasticity*, volume 219. Springer, 2015.

[45] Monika Žecová and Ján Terpák. Heat conduction modeling by using fractional-order derivatives. *Applied Mathematics and Computation*, 257:365–373, 2015.

[46] L. Vázquez, J. Trujillo, and M. P. Velasco. Fractional heat equation and the second law of thermodynamics. *Fractional Calculus and Applied Analysis*, 14(3):334–342, 2011.

[47] A. C. Eringen. Theory of nonlocal thermoelasticity. *International Journal of Engineering Science*, 12(12):1063–1077, 1974.

[48] F. Balta and E. S. Şuhubi. Theory of nonlocal generalised thermoelasticity. *International Journal of Engineering Science*, 15(9-10):579–588, 1977.

[49] J. T. Oden. Finite element analysis of nonlinear problems in the dynamical theory of coupled thermoelasticity. *Nuclear Engineering and Design*, 10(4):465–475, 1969.

[50] V. A. Lubarda. On thermodynamic potentials in linear thermoelasticity. *International Journal of Solids and Structures*, 41(26):7377–7398, 2004.

[51] GN Praveen and JN Reddy. Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates. *International journal of solids and structures*, 35(33):4457–4476, 1998.

[52] A. A. Pisano, A. Sofi, and P. Fuschi. Nonlocal integral elasticity: 2d finite element based solutions. *International Journal of Solids and Structures*, 46(21):3836–3849, 2009.

[53] N. Challamel and C. M. Wang. The small length scale effect for a non-local cantilever beam: a paradox solved. *Nanotechnology*, 19(34):345703, 2008.