We have calculated the shear viscosity coefficient \( \eta \) of the strongly interacting matter in the relaxation time approximation, where a quasi particle description of quarks with its dynamical mass is considered from NJL model. Due to the thermodynamic scattering of quarks with pseudo scalar type condensate (i.e. pion), a non zero Landau damping will be acquired by the propagating quarks. This Landau damping may be obtained from the Landau cut contribution of the in-medium self-energy of quark-pion loop, which is evaluated in the framework of real-time thermal field theory.

From the basic idea of the QCD asymptotic freedom at high temperatures and densities, a weakly interacting quark gluon plasma (QGP) is naturally expected to be produced in the experiments of heavy ion collision (HIC). However, the experimental data from RHIC, especially the measured elliptic flow indicates that non-zero shear viscosity. The shear viscosity due to the Landau damping from quark-pion interaction (HIC). However, the experimental data from RHIC, must have very small shear viscosity. The relaxation time approximation, where a quasi particle description of quarks with its dynamical mass is considered from NJL model. Due to the thermodynamic scattering of quarks with pseudo scalar type condensate (i.e. pion), a non zero Landau damping will be acquired by the propagating quarks. This Landau damping may be obtained from the Landau cut contribution of the in-medium self-energy of quark-pion loop, which is evaluated in the framework of real-time thermal field theory.

\[
\eta = \frac{8\beta}{5} \int \frac{d^3 \bar{k}}{(2\pi)^3} \frac{\bar{k}^4}{\omega_{\bar{k}}} \frac{n_{\bar{Q}}(1-n_{\bar{Q}})}{\Gamma_{\bar{Q}}}
\]

FIG. 1: The diagram of quark (A) and pion (B) self-energy for quark-pion and quark-anti quark loops respectively.

\[
\eta = 8\beta \int \frac{d^3 \bar{k}}{(2\pi)^3} \frac{\bar{k}^4}{\omega_{\bar{k}}} \frac{n_{\bar{Q}}(1-n_{\bar{Q}})}{\Gamma_{\bar{Q}}}
\]

where \( n_{\bar{Q}} = \frac{1}{e^{\beta \omega_{\bar{Q}}}} + 1 \) and \( n_{\pi} = \frac{1}{e^{\beta \omega_{\pi}}} + 1 \) are respectively Fermi-Dirac distribution of quark and Bose-Einstein distribution of pion with \( \omega_{\bar{Q}} = \sqrt{k^2 + M_{\bar{Q}}^2} \) and \( \omega_{\pi} = \sqrt{k^2 + m_{\pi}^2} \). The \( \Gamma_{\bar{Q}} \) and \( \Gamma_{\pi} \) are Landau damping of quark and pion respectively. Following the quasi particle description of Nambu-Jona-Lasinio (NJL) model [22], the dynamical quark mass \( M_{\bar{Q}} \) is considered and it is generated due to quark condensate

\[
\langle \bar{\psi}_f \psi_f \rangle = -\frac{M_{\bar{Q}} - m_{\bar{Q}}}{2G}
\]

where \( m_{\bar{Q}} \) is the current quark mass. In the medium, above relation become (for \( \mu = 0 \))

\[
M_{\bar{Q}} = m_{\bar{Q}} + 4 N_f N_c G \int \frac{d^3 \bar{k}}{(2\pi)^3} \frac{M_{\bar{Q}}}{\omega_{\bar{Q}}(1 - 2n_{\bar{Q}})}
\]

This relation shows that the constituent quark mass tends to be the current quark mass at very high temperature where the non-zero quark condensate becomes small.

This Landau damping \( \Gamma_{\bar{Q}} \) and \( \Gamma_{\pi} \) may be estimated from the self-energy graphs of quark and pion at finite temperature for quark-pion and quark-anti quark loops respectively. These are respectively expressed as

\[
\Gamma_{\bar{Q}} = -\text{Im}\Sigma^R(k_0 = \sqrt{\bar{k}^2 + M_{\bar{Q}}^2}, \bar{k})
\]

and

\[
\Gamma_{\pi} = -\frac{1}{m_{\pi}} \text{Im}\Pi^R(k_0 = \sqrt{\bar{k}^2 + m_{\pi}^2}, \bar{k})
\]
where $\Sigma^R$ and $\Pi^R$ are respectively retarded part of quark and pion self-energy at finite temperature. Their diagrammatic representations are shown in Fig.1(A) and (B) respectively. Following the real-time formalism of thermal field theory, the retarded part of in-medium quark self energy for quark-pion loop is given by

$$
\Sigma^R(k_0, \vec{k}) = \int \frac{d^4l}{(2\pi)^4} \frac{1}{4\omega^2\omega^\prime} \left[ (1-n^0_L)\omega^2_L + n^0_L\omega^2_Q \right] \left[ \frac{1}{k_0 - \omega_Q - \omega^\prime + i\eta} + \frac{n^0_L\omega^2_Q}{k_0 - \omega_Q - \omega^\prime + i\eta} + \frac{-n^0_L\omega^2_Q}{k_0 + \omega_Q - \omega^\prime + i\eta} + \frac{n^0_L\omega^2_Q}{k_0 + \omega_Q + \omega^\prime + i\eta} \right].
$$

(6)

where $L^0_Q, i = 1, \ldots, 4$ denote the values of $L^Q(l_0, \vec{l})$ for $l_0 = \omega_Q, \omega^\prime_Q, k_0 + \omega^\prime_Q, k_0 + \omega_Q$ respectively with $\omega^\prime_Q = \sqrt{k^2 + M_Q^2}$ and $\omega_Q = \sqrt{(-\vec{k} - \vec{l})^2 + m^2}$. Here $n^0_L(\omega^\prime_Q)$ is Fermi-Dirac distribution function of quark whereas $n^0_Q(\omega_Q)$ denotes Bose-Einstein distribution function of $\pi$ meson.

During extracting the imaginary part of $\Sigma^R(k_0, \vec{k})$ we will get four delta functions associated with the four individual terms of Eq. (6), which generate four different region in $k_0$ axis where the $\text{Im} \Sigma^R(k_0, \vec{k})$ will be non-zero. From the non-zero values of $\text{Im} \Sigma^R(k_0, \vec{k})$ the region of discontinuities or branch cuts of $\Sigma^R(k_0, \vec{k})$ can be identified. The regions coming from the 1st and 4th terms of (6) are respectively $(k_0 = -\infty$ to $-\sqrt{k^2 + (m_\pi + M_Q^2)^2})$ and $(k_0 = \sqrt{k^2 + (m_\pi + M_Q^2)^2}$ to $\infty$). These are known as unitary cuts and different kind of forward and inverse decay processes are associated with these cut contributions [29, 30]. Similarly the regions $(k_0 = -\sqrt{k^2 + (m_\pi - M_Q^2)^2}$ to $0)$ and $(k_0 = 0$ to $\sqrt{k^2 + (m_\pi - M_Q^2)^2})$ are coming from 2nd and 3rd terms respectively. These purely medium dependent cuts are known as Landau cuts and different kind of forward and inverse scattering processes are physically interpreted by these cut contributions [29, 30]. So the 3rd term of $\text{Im} \Sigma^R(k_0, \vec{k})$ at the on-shell mass $(k_0 = \sqrt{k^2 + M_Q^2}, \vec{k})$ of quark is responsible for the Landau damping $\Gamma_Q$ and it is given by [30]

$$
\Gamma_Q = -\text{Im} \Sigma^R(k_0 = \sqrt{k^2 + M_Q^2}, \vec{k}) = \int \frac{d^4l}{(2\pi)^4} \frac{L^Q}{4\omega^2\omega^\prime} \delta(k_0 - \omega_Q - \omega^\prime) \left[ n^0_L(\omega_Q) + n^0_Q(\omega_Q) \right]_{k_0 = \sqrt{k^2 + M_Q^2}}.
$$

(7)

Rearranging the statistical weight factor by

$$
(n^0_L + n^0_Q)(1 + n^0_L) + n^0_Q(1 - n^0_Q),
$$

we can find thermalized $\pi$ and $\bar{\pi}$ with Bose enhanced probability $(1 + n^0_L)$ and Pauli blocked probability $(1 - n^0_Q)$ respectively. With the help of Eq. (5), the physical significance of the Landau contribution may expressed as follows. During the propagation of $u$ quark, it may absorb the thermalized $\bar{\pi}$ from the heat bath and create a thermalized $\pi$ in the bath (indicated by the second part of Eq. (5)). Again the thermalized $\pi$ may be absorbed by the medium and create the thermalized $\bar{\pi}$ along with a propagating $u$, which is slightly off-equilibrium with the medium (indicated by the first part of Eq. (5)).

To calculate $L^Q$ from quark-pion interaction, let us start with free Lagrangian of quarks and demanding the invariance properties of Lagrangian under chiral transformation,

$$
\psi^\prime = \exp(i\vec{\pi} \cdot \vec{\gamma}_5 \frac{\vec{r}}{2F_\pi}) \psi_f
$$

(9)

where the chiral angle is associated with the pion field $\vec{\pi}$ and $F_\pi$ is pion decay constant. Expanding up to first order of pion field, we obtain the quark-pion interaction term [33, 34].

$$
L_{\pi QQ} = -i\frac{M_Q}{F_\pi} \vec{\pi} \cdot \vec{\gamma}_5 \psi_f
$$

As we are interested to calculate one-loop self-energy ($\Sigma^R$) of any quark flavor, $u$ (say) hence we have to consider two possible loops - $u\pi^0$ and $d\pi^+$. Due to isospin symmetry consideration in Lagrangian, we can evaluate anyone of the loops, say $u\pi^0$ loop and then we have to multiply it by a isospin factor

$$
I_F = (1)^2 + (\sqrt{2})^2 = 3.
$$

(11)

From the interaction part,

$$
L_{\pi \pi uu} = -i g_{\pi QQ} \vec{\pi} \gamma_5 \pi^0 u, \quad \text{with} \quad g_{\pi QQ} = \frac{M_Q}{F_\pi}
$$

(12)

we can calculate $L^Q_{\pi}(l_0, \vec{l}) = -iF_\pi g^2_{\pi QQ}(l - m_l)_{ab}$, where $a, b$ are Dirac indices. For simplification we have taken the scalar part only i.e. $L^0_{\pi}(l_0, \vec{l}) = iF_\pi g^2_{\pi QQ}m_l$. We have taken the parameters $m = 0.0056$ GeV, $M_Q = 0.4$ GeV (for $T = 0$), three momentum cut-off $A = 0.588$ GeV and corresponding $T_c = 0.222$ GeV for $\mu = 0$ [28].

Similar to Eq. (4), the pion self-energy $\Pi^R$ for quark-antiquark quark loop is also received similar kind of form only the quantities $n^0_L$, $\omega^\prime_Q = \sqrt{(\vec{l} - \vec{k})^2 + m^2}$, $U = k - l$ and $L^Q_{\pi}$’s are changed to $-n^0_L$, $\omega^\prime_Q = \sqrt{(\vec{l} - \vec{k})^2 + m^2}$, $U = -k + l$ and $L^Q_{\pi}$’s respectively [31, 32]. As pion on-shell mass point $(k_0 = \sqrt{k^2 + M_Q^2}, \vec{k})$ will be inside the unitary cut region $(k_0 = \sqrt{k^2 + (M_Q + M_Q)^2}$ to $\infty$) of $\Pi^R$, therefore

$$
\Gamma_\pi = -\text{Im} \Pi^R(k_0 = \sqrt{k^2 + M_Q^2}, \vec{k}) = -\frac{1}{m_\pi} \int \frac{d^4l}{(2\pi)^4} \frac{L^0_{\pi}}{4\omega^2\omega^\prime}
$$

$$
= -\frac{1}{m_\pi} \int \frac{d^4l}{(2\pi)^4} \frac{L^0_{\pi}}{4\omega^2\omega^\prime}
$$
where $L^T = 4I_F g_2^2 Q Q [M_Q^2 - l^2 - k \cdot ℓ]$ can be obtained from (12).

In Fig. (2), we can see the temperature dependency of Landau damping $\Gamma$ (upper panel) and collision time $\tau = \frac{1}{\Gamma}$ (lower panel) of quark (dotted line) and pion (dashed line) for their momentum $\vec{k} = 0$. Owing to the on-shell condition, the $\Gamma_Q$ and $\Gamma_\pi$ are received the non-zero values only in the temperature range where $m_\pi > 2M_Q$, which are clearly seen from dotted and dashed lines respectively. A corresponding non-divergent collisional times are also achieved by them in the same temperature domain. Due to decay width of $\pi \rightarrow QQ$, it will more realistic to consider the pion resonance of finite width in Eq. (4). The pion spectral function due to $QQ$ width may be defined as

$$A_\pi(M) = \frac{1}{\pi} \Im \left[ \frac{1}{M^2 - m_\pi^2 + i\Im \Pi_{vac}(k_0, \vec{k})} \right]$$

where $\Im \Pi_{vac}(k_0, \vec{k})$ is vacuum part of $\Im \Pi^R(k_0, \vec{k})$ and $M = \sqrt{k_0^2 - \vec{k}^2}$. The variation of $\Im \Pi^R(M)/m_\pi$ and $A_\pi$ with $M$ for two different temperatures are respectively shown in lower and upper panel of Fig. (3). Replacing $m_\pi$ of $\Gamma_Q$ in (7) by $M$ and then convoluting or folding it by $A_\pi(M)$, we have

$$\Gamma_Q(m_\pi) = \frac{1}{N_\pi} \int \Gamma_Q(M) A_\pi(M) dM^2$$

where $N_\pi = \int A_\pi(M) dM^2$. One should notice that in the narrow width approximation i.e. for $\Im \Pi_{vac}^R \rightarrow 0$, Eq. (15) is merged to (7). The $T$ dependency of $\Gamma_Q$ and its corresponding $\tau$ after folding are shown by solid line in the upper and lower panel of Fig. (2) respectively. Due to folding, $\Gamma_Q$ at low $T$ domain (where $m_\pi < 2M_Q$) has acquired some non-zero values from its vanishing contributions and at the same time corresponding $\tau$ recover from its divergence up to the approximate freeze out temperature ($T \sim 120 - 150$ GeV) of the strongly interacting matter.

By using $\Gamma_Q(T, \vec{k})$ from Eq. (7) and (15) in the quark component (first term) of Eq. (1), we get the results of shear viscosity as function of $T$, which are respectively described by dotted and solid line of Fig. (4). Being proportional to collisional time, the divergence of $\eta$ is removed after folding in those temperature region, where $m_\pi < 2M_Q$. The contribution of $\eta$ due to $\Gamma_\pi(T, \vec{k})$ from Eq. (15) is shown by dashed line in Fig. (4). After similar kind of folding as done in Eq. (15), an almost negligible ($\sim 10^{-5}$ GeV$^3$) contribution of $\eta$ for pion component can be obtained which is not included in final results.

In low temperature region, $\eta$ is decreasing with increasing of $T$ which is analogous to the behavior of liquid (From our daily life experience, we see that the cooking oil behaves like a less viscous medium when it is heated).
Whereas in high temperature domain, $\eta$ become an increasing function of $T$ just like a system of gas.

The magnitude of $\eta$ in our approach is very close to the results of Sasaki and Redlich (indicated by triangles) but underestimated with respect to the earlier estimation in NJL model by Zhuang et al. (indicated by solid line in the lower panel of Fig. 2), we see that $\eta$ below the $T \sim 160$ MeV exceeds the typical value of time period ($\sim 30 - 50$ fm) during which a strongly interacting matter survive in the labs of heavy ion collisions. Therefore the estimation of $\eta$ in low temperature domain is quite higher than the standard calculations of $\eta$ of hadronic matter. The earlier calculations of NJL model also displayed these discrepancy in the hadronic temperature domain.

In summary we have investigated the shear viscosity of strongly interacting matter in the relaxation time approximation, where quarks with its dynamical mass may have some non zero Landau damping because of its various forward and inverse scattering with pions. This Landau damping can be obtained from the thermal field theoretical calculation of quark self-energy for quark-pion loop. The temperature dependency of shear viscosity is coming from the thermal distribution functions, the temperature dependence of Landau damping as well as the constituent quark mass, supplied by the temperature dependent gap equation in the NJL model. Due to this gap equation, this constituent quark mass drops rapidly towards its current mass near $T_c$ to restore the chiral symmetry. A non-trivial influence of all these temperature dependency on $\eta(T)$ is displayed in our results.

Acknowledgment: S. G. thanks to Saurav Sarkar, Tamal K. Mukherjee, Sounitma Maity, Ramaprasad Adak, Kinkar Saha, Sudipa Upadhaya for some pieces of discussions which have some direct and indirect influence on our present work.