ZERO VOLUME BOUNDARY FOR EXTENSION DOMAINS
FROM SOBOLEV TO BV

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Abstract. In this note, we prove that the boundary of a \((W^{1,p}, BV)\)-extension domain is of volume zero under the assumption that the domain \(\Omega\) is 1-fat at almost every \(x \in \partial \Omega\). Especially, the boundary of any planar \((W^{1,p}, BV)\)-extension domain is of volume zero.

1. Introduction

Given \(1 \leq q \leq p \leq \infty\), a bounded domain \(\Omega \subset \mathbb{R}^n, n \geq 2\), is said to be a \((W^{1,p}, W^{1,q})\)-extension domain if there exists a bounded extension operator

\[ E : W^{1,p}(\Omega) \rightarrow W^{1,q}(\mathbb{R}^n) \]

and is said to be a \((W^{1,p}, BV)\)-extension domain if there exists a bounded extension operator

\[ E : W^{1,p}(\Omega) \rightarrow BV(\mathbb{R}^n). \]

The theory of Sobolev extensions is of interest in several fields in analysis. Partial motivations for the study of Sobolev extensions comes from the theory of PDEs, for example, see [Maz11]. It was proved in [Cal61, Ste70] that for every Lipschitz domain in \(\mathbb{R}^n\), there exists a bounded linear extension operator \(E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)\) for each \(k \in \mathbb{N}\) and \(1 \leq p \leq \infty\). Here \(W^{k,p}(\Omega)\) is the Banach space of all \(L^p\)-integrable functions whose distributional derivatives up to order \(k\) are \(L^p\)-integrable. Later, the notion of \((\epsilon, \delta)\)-domains was introduced by Jones in [Jon81], and it was proved that for every \((\epsilon, \delta)\)-domain, there exists a bounded linear extension operator \(E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)\) for every \(k \in \mathbb{N}\) and \(1 \leq p \leq \infty\).

In [VGL79], a geometric characterization of planar \((W^{1,2}, W^{1,2})\)-extension domain was given. By later results in [Kos98, Shv10, KRZ15, KRZ17], we now have geometric characterizations of planar simply connected \((W^{1,p}, W^{1,p})\)-extension domains for all \(1 \leq p \leq \infty\). A geometric characterization is also known for planar simply connected \((L^{k,p}, L^{k,p})\)-extension domains with \(2 < p \leq \infty\), see [SZ16, Whi34, Zob99]. Here \(L^{k,p}(\Omega)\) denotes the homogeneous Sobolev space which contains locally integrable functions whose \(k\)-th order distributional derivative is \(L^p\)-integrable. Beyond the planar simply connected case, geometric characterizations of Sobolev extension domains are still missing. However, several necessary properties have been obtained for general Sobolev extension domains.

For a measurable subset \(F \subset \mathbb{R}^n\), we use \(|F|\) to denote its Lebesgue measure. In [HKT08a, HKT08b], Hajlasz, Koskela and Tuominen proved for \(1 \leq p < \infty\) that a \((W^{1,p}, W^{1,p})\)-extension domain \(\Omega \subset \mathbb{R}^n\) must be Ahlfors regular which means that there exists a positive
constant $C > 1$ such that for every $x \in \overline{\Omega}$ and $0 < r < \min \{1, \frac{1}{4} \text{diam } \Omega\}$, we have

$$|B(x, r)| \leq C|B(x, r) \cap \Omega|.$$  \hfill (1.1)

From the results in [KMS10, GBR21], we know that also $(BV, BV)$-extension domains are Ahlfors regular. For Ahlfors regular domains, the Lebesgue differentiation theorem then easily implies $|\partial \Omega| = 0$.

In the case where $\Omega$ is a planar Jordan $(W^{1,p}, W^{1,p})$-extension domain, $\Omega$ has to be a so-called John domain when $1 \leq p \leq 2$ and the complementary domain has to be John when $2 \leq p < \infty$. The John condition implies that the Hausdorff dimension of $\partial \Omega$ must be strictly less than 2, see [KR97]. Recently, Lučić, Takanen and the first named author gave a sharp estimate on the Hausdorff dimension of $\partial \Omega$, see [LRT21]. In general, the Hausdorff dimension of a $(W^{1,p}, W^{1,p})$-extension domain can well be $n$.

The outward cusp domain with a polynomial type singularity is a typical example which is not a $(W^{1,p}, W^{1,p})$-extension domain for $1 \leq p < \infty$. However, it is a $(W^{1,p}, W^{1,q})$-extension domain, for some $1 \leq q < p \leq \infty$, see the monograph [MP97] and the references therein. Hence, for $1 \leq q < p \leq \infty$, it is not necessary for a $(W^{1,p}, W^{1,q})$-extension domain to be Ahlfors regular. In the absence of Ahlfors regularity, one has to find alternative approaches for proving $|\partial \Omega| = 0$. The first approach in [Ukh20, Ukh99] was to generalize the Ahlfors regularity (1.1) to a Ahlfors-type estimate

$$|B(x, r)|^q \leq C \Phi^{p-q}(B(x, r))|B(x, r) \cap \Omega|^q$$ \hfill (1.2)

for $(W^{1,p}, W^{1,p})$-extension domains with $n < q < p < \infty$. Here $\Phi$ is a bounded and quasiadditive set function generated by the $(W^{1,p}, W^{1,q})$-extension property and defined on open sets $U \subset \mathbb{R}^n$, see Section 3. By differentiating $\Phi$ with respect to the Lebesgue measure, one concludes that $|\partial \Omega| = 0$ if $\Omega$ is a $(W^{1,p}, W^{1,q})$-extension domain for $n < q < p < \infty$.

Recently, Koskela, Ukhlov and the second named author [KUZ21] generalized this result and proved that the boundary of a $(W^{1,p}, W^{1,q})$-extension domain must be of volume zero for $n - 1 < q < p < \infty$ (and for $1 \leq q < p < \infty$ on the plane). For $1 \leq q < n - 1$ and $(n - 1)q/(n - 1 - q) < p < \infty$, they constructed a $(W^{1,p}, W^{1,q})$-extension domain $\Omega \subset \mathbb{R}^n$ with $|\partial \Omega| > 0$. For the remaining range of exponents where $1 \leq q \leq n - 1$ and $q < p \leq (n - 1)q/(n - 1 - q)$, it is still not clear whether the boundary of every $(W^{1,p}, W^{1,q})$-extension domain must be of volume zero.

As is well-known, for every domain $\Omega \subset \mathbb{R}^n$, the space of functions of bounded variation $BV(\Omega)$ strictly contains every Sobolev space $W^{1,q}(\Omega)$ for $1 \leq q \leq \infty$. Hence, the class of $(W^{1,p}, BV)$-extension domains contains the class of $(W^{1,p}, W^{1,q})$-extension domains for every $1 \leq q \leq p < \infty$. As a basic example to indicate that the containment is strict when $n \geq 2$, we can take the slit disk (the unit disk minus a radial segment) in the plane. The slit disk is a $(W^{1,p}, BV)$-extension domain for every $1 \leq p < \infty$, and even a $(BV, BV)$-extension domain; however it is not a $(W^{1,p}, W^{1,q})$-extension domain for any $1 \leq q \leq p < \infty$. This basic example also shows that it is natural to consider the geometric properties of $(W^{1,p}, BV)$-extension domains. In this paper, we focus on the question whether the boundary of a $(W^{1,p}, BV)$-extension domain is of volume zero. Our first theorem tells us that the $(BV, BV)$-extension property is equivalent to the $(W^{1,1}, BV)$-extension property. Hence, a $(W^{1,1}, BV)$-extension domain is Ahlfors regular and so its boundary is of volume zero.
Theorem 1.1. A domain $\Omega \subset \mathbb{R}^n$ is a $(BV, BV)$-extension domain if and only if it is a $(W^{1,1}, BV)$-extension domain.

Since, $W^{1,1}(\Omega)$ is a proper subspace of $BV(\Omega)$ with $\|u\|_{W^{1,1}(\Omega)} = \|u\|_{BV(\Omega)}$ for every $u \in W^{1,1}(\Omega)$, $(BV, BV)$-extension property implies $(W^{1,1}, BV)$-extension property immediately. The other direction from $(W^{1,1}, BV)$-extension property to $(BV, BV)$-extension property is not as straightforward, as $W^{1,1}(\Omega)$ is only a proper subspace of $BV(\Omega)$. The essential tool here is the Whitney smoothing operator constructed by García-Bravo and the first named author in [GBR21]. This Whitney smoothing operator maps every function in $BV(\Omega)$ to a function in $W^{1,1}(\Omega)$ with the same trace on $\partial \Omega$, so that the norm of the image in $W^{1,1}(\Omega)$ is uniformly controlled from above by the norm of the corresponding preimage in $BV(\Omega)$.

With an extra assumption that $\Omega$ is $(q, 1)$-fat at almost every point on the boundary $\partial \Omega$, in [KUZ21] it was shown that the boundary of a $(W^{1,p}, W^{1,q})$-extension domain is of volume zero when $1 \leq q < p < \infty$. The essential point there was that the $q$-fatness of the domain on the boundary guarantees the continuity of a $W^{1,q}$-function on the boundary. Maybe a bit surprisingly, the assumption that the domain is $1$-fat at almost every point on the boundary also guarantees that the boundary of a $(W^{1,p}, BV)$-extension domain is of volume zero. In particular, every planar domain is $1$-fat at every point of the boundary. Hence, we have the following theorem.

Theorem 1.2. Let $\Omega \subset \mathbb{R}^n$ be a $(W^{1,p}, BV)$-extension domain for $1 \leq p < \infty$, which is $1$-fat at almost every $x \in \partial \Omega$. Then $|\partial \Omega| = 0$. In particular, for every planar $(W^{1,p}, BV)$-extension domain $\Omega$ with $1 \leq p < \infty$, we have $|\partial \Omega| = 0$.

In light of the results and example given in [KUZ21], the most interesting open question is what happens in the range $1 < p \leq (n-1)/(n-2)$ of exponents. For this range, we do not know whether the boundary of a $(W^{1,p}, BV)$-extension domain must be of volume zero. If a counterexample exists in this range, it might be easier to construct it in the $(W^{1,p}, BV)$-case rather than the $(W^{1,p}, W^{1,1})$-case. Hence we leave it as a question here.

Question 1.3. For $1 < p \leq (n-1)/(n-2)$, is the boundary of a $(W^{1,p}, BV)$-extension domain of volume zero?

2. Preliminaries

For a locally integrable function $u \in L^1_{\text{loc}}(\Omega)$ and a measurable subset $A \subset \Omega$ with $0 < |A| < \infty$, we define

$$u_A := \int_E u(y) \, dy = \frac{1}{|A|} \int_A u(y) \, dy.$$ 

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. For every $1 \leq p \leq \infty$, we define the Sobolev space $W^{1,p}(\Omega)$ to be

$$W^{1,p}(\Omega) := \{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega; \mathbb{R}^n) \},$$

where $\nabla u$ denotes the distributional gradient of $u$. It is equipped with the nonhomogeneous norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$ 

Now, let us give the definition of functions of bounded variation.
Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $u \in L^1(\Omega)$ is said to have bounded variation and denoted $u \in BV(\Omega)$ if
\[ \|Du\|(\Omega) := \sup \left\{ \int_{\Omega} f(\nabla \phi) \, dx : \phi \in C_0^1(\Omega; \mathbb{R}^n), |\phi| \leq 1 \right\} < \infty. \]
The space $BV(\Omega)$ is equipped with the norm
\[ \|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega). \]

Definition 2.3. We say that a domain $\Omega \subset \mathbb{R}^n$ is a $(W^{1,p}, BV)$-extension domain for $1 \leq p < \infty$, if there exists a bounded extension operator $E : W^{1,p}(\Omega) \to BV(\mathbb{R}^n)$ such that for every $u \in W^{1,p}(\Omega)$, we have $E(u) \in BV(\mathbb{R}^n)$ with $E(u)|_\Omega = u$ and
\[ \|E(u)\|_{BV(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)} \]
for a constant $C > 1$ independent of $u$.

Let $U \subset \mathbb{R}^n$ be an open set and $A \subset U$ be a measurable subset with $\overline{A} \subset U$. The $p$-admissible set $W_p(A; U)$ is defined by setting
\[ W_p(A; U) := \left\{ u \in W_0^{1,p}(U) : u|_A \geq 1 \right\}. \]

Definition 2.4. Let $U \subset \mathbb{R}^n$ be an open set and $A \subset U$ with $\overline{A} \subset U$. The relative $p$-capacity $Cap_p(A; U)$ is defined by setting
\[ Cap_p(A; U) := \inf_{u \in W_p(A; U)} \int_U |\nabla u(x)|^p \, dx. \]

Following Lahti [Lah17], we define 1-fatness below.

Definition 2.5. Let $A \subset \mathbb{R}^n$ be a measurable subset. We say that $A$ is 1-thin at the point $x \in \mathbb{R}^n$, if
\[ \lim_{r \to 0^+} \frac{Cap_1(A \cap B(x, r); B(x, 2r))}{|B(x, r)|} = 0. \]
If $A$ is not 1-thin at $x$, we say that $A$ is 1-fat at $x$. Furthermore, we say that a set $U$ is 1-finely open, if $\mathbb{R}^n \setminus U$ is 1-thin at every $x \in U$.

By [Lah17, Lemma 4.2], the collection of 1-finely open sets is a topology on $\mathbb{R}^n$. For a function $u \in BV(\mathbb{R}^n)$, we define the lower approximate limit $u^*$ by setting
\[ u^*_n(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{|B(x, r) \cap \{u < t\}|}{|B(x, r)|} = 0 \right\} \]
and the upper approximate limit $u^*_u$ by setting
\[ u^*_u(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{|B(x, r) \cap \{u > t\}|}{|B(x, r)|} = 0 \right\}. \]
The set
\[ S_u := \{ x \in \mathbb{R}^n : u^*_u(x) < u^*_u(x) \} \]
is called the jump set of $u$. By the Lebesgue differentiation theorem, $|S_u| = 0$. Using the lower and upper approximate limits, we define the precise representative $\tilde{u} := (u^* + u^*_u)/2$. The following lemma was proved in [Lah17, Corollary 5.1].

Lemma 2.6. Let $u \in BV(\mathbb{R}^n)$. Then $\tilde{u}$ is 1-finely continuous at $\mathcal{H}^{n-1}$-almost every $x \in \mathbb{R}^n \setminus S_u$. 
The following lemma for \( u \in W^{1,1}(\mathbb{R}^n) \) was proved in [KUZ21, Lemma 2.6], which is also a corollary of a result in [HKM93]. We generalize it to \( BV(\mathbb{R}^n) \) here.

**Lemma 2.7.** Let \( \Omega \subset \mathbb{R}^n \) be a domain which is 1-fat at almost every point \( x \in \partial \Omega \). If \( u \in BV(\mathbb{R}^n) \) with \( u|_{B(x,r) \cap \Omega} = c \) for some \( x \in \partial \Omega \), \( 0 < r < 1 \) and \( c \in \mathbb{R} \). Then \( u(y) = c \) for almost every \( y \in B(x,r) \cap \partial \Omega \).

**Proof.** Let \( u \in BV(\mathbb{R}^n) \) satisfy the assumptions. Then the precise representative \( \tilde{u}|_{B(x,r) \cap \Omega} = c \). Since \( |S_u| = 0 \), by Lemma 2.6, there exists a subset \( N_1 \subset \mathbb{R}^n \) with \( |N_1| = 0 \) such that \( \tilde{u} \) is \( 1 \)-finely continuous on \( \mathbb{R}^n \setminus N_1 \). By the assumption, there exists a measure zero set \( N_2 \subset \partial \Omega \) such that \( \Omega \) is 1-fat on \( \partial \Omega \setminus N_2 \). By Definition 2.5, one can see that \( B(x,r) \cap \Omega \) is also 1-fat on \( (B(x,r) \cap \partial \Omega) \setminus N_2 \). For every \( y \in (B(x,r) \cap \partial \Omega) \setminus (N_1 \cup N_2) \), since \( \tilde{u} \) is \( 1 \)-finely continuous on it and any 1-fine neighborhood of \( y \) must intersect \( B(x,r) \cap \Omega \), we have \( \tilde{u}(y) = c \). Hence \( u(y) = c \) for almost every \( y \in B(x,r) \cap \partial \Omega \). □

The following coarea formula for \( BV \) functions can be found in [EG15, Section 5.5]. See also [GBR21, Theorem 2.2].

**Proposition 2.8.** Given a function \( u \in BV(\Omega) \), the superlevel sets \( u_t = \{ x \in \Omega : u(x) > t \} \) have finite perimeter in \( \Omega \) for almost every \( t \in \mathbb{R} \) and

\[
\| Du \|(F) = \int_{-\infty}^{\infty} P(u_t, F) \, dt
\]

for every Borel set \( F \subset \Omega \). Conversely, if \( u \in L^1(\Omega) \) and

\[
\int_{-\infty}^{\infty} P(u_t, \Omega) \, dt < \infty
\]

then \( u \in BV(\Omega) \).

See [AFP00, Theorem 3.44] for the proof of the following \((1,1)\)-Poincaré inequality for \( BV \) functions.

**Proposition 2.9.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. Then there exists a constant \( C > 0 \) depending on \( n \) and \( \Omega \) such that for every \( u \in BV(\Omega) \), we have

\[
\int_{\Omega} |u(y) - u_\Omega| \, dy \leq C \| Du \|(\Omega).
\]

In particular, there exists a constant \( C > 0 \) only depending on \( n \) so that if \( Q, Q' \subset \mathbb{R}^n \) are two closed dyadic cubes with \( \frac{1}{4}l(Q') \leq l(Q) \leq 4l(Q') \) and \( \Omega := \text{int}(Q \cup Q') \) connected, then for every \( u \in BV(\Omega) \),

\[
\int_{\Omega} |u(y) - u_\Omega| \, dy \leq Cl(Q) \| Du \|(\Omega).
\]  

3. A set function arising from the extension

In this subsection, we introduce a set function defined on the class of open sets in \( \mathbb{R}^n \) and taking nonnegative values. Our set function here is a modification of the one originally introduced by Ukhlov [Ukh20, Ukh99]. See also [VU04, VU05] for related set functions. The modified version of the set function we use is from [KUZ21], where it was used by Koskela, Ukhlov and the second named author to study the size of the boundary of a \((W^{1,p}, W^{1,q})\)-extension domains. Let us recall that a set function \( \Phi \) defined on the class of open subsets
of \( \mathbb{R}^n \) and taking nonnegative values is called quasiadditive (see for example [VU04]), if for all open sets \( U_1 \subset U_2 \subset \mathbb{R}^n \), we have

\[
\Phi(U_1) \leq \Phi(U_2),
\]

and there exists a positive constant \( C \) such that for arbitrary pairwise disjoint open sets \( \{U_i\}_{i=1}^\infty \), we have

\[
\sum_{i=1}^\infty \Phi(U_i) \leq C \Phi \left( \bigcup_{i=1}^\infty U_i \right). \tag{3.1}
\]

Let \( \Omega \subset \mathbb{R}^n \) be a \((W^{1,p},BV)\)-extension domain for some \( 1 < p < \infty \). For every open set \( U \subset \mathbb{R}^n \) with \( U \cap \Omega \neq \emptyset \), we define

\[
W_0^p(U,\Omega) := \left\{ u \in W^{1,p}(\Omega) \cap C(\Omega) : u \equiv 0 \text{ on } \Omega \setminus U \right\}.
\]

For every \( u \in W_0^p(U,\Omega) \), we define

\[
\Gamma(u) := \inf \left\{ \|Dv\|_L(U) : v \in BV(\mathbb{R}^n), v|_{\Omega} \equiv u \right\}.
\]

Then we define the set function \( \Phi \) by setting

\[
\Phi(U) := \begin{cases} 
\sup_{u \in W_0^p(U,\Omega)} \left( \frac{\Gamma(u)}{\|u\|_{W^{1,p}(\Omega)}} \right)^k, & \text{with } \frac{1}{k} = 1 - \frac{1}{p}, \text{ if } U \cap \Omega \neq \emptyset, \\
0, & \text{otherwise}.
\end{cases} \tag{3.2}
\]

The proof of the following lemma is almost the same as the proof of [KUZ21, Theorem 3.1]. One needs to simply replace \( \|Dv\|_L(U) \) by \( \|Dv\|_L(U) \) in the proof of [KUZ21, Theorem 3.1] and repeat the argument.

**Lemma 3.1.** Let \( 1 < p < \infty \) and let \( \Omega \subset \mathbb{R}^n \) be a bounded \((W^{1,p},BV)\)-extension domain. Then the set function defined in (3.2) for all open subsets of \( \mathbb{R}^n \) is bounded and quasiadditive.

The upper and lower derivatives of a quasiadditive set function \( \Phi \) are defined by setting

\[
\overline{D}\Phi(x) := \limsup_{r \to 0^+} \frac{\Phi(B(x,r))}{|B(x,r)|} \quad \text{and} \quad \underline{D}\Phi(x) := \liminf_{r \to 0^+} \frac{\Phi(B(x,r))}{|B(x,r)|}.
\]

By [RR55, VU04], we have the following lemma. See also [KUZ21, Lemma 3.1].

**Lemma 3.2.** Let \( \Phi \) be a bounded and quasiadditive set function defined on open sets \( U \subset \mathbb{R}^n \). Then \( \overline{D}\Phi(x) < \infty \) for almost every \( x \in \mathbb{R}^n \).

The following lemma immediately comes from the definition (3.2) for the set function \( \Phi \).

**Lemma 3.3.** Let \( 1 < p < \infty \) and let \( \Omega \subset \mathbb{R}^n \) be a bounded \((W^{1,p},BV)\)-extension domain. Then, for a ball \( B(x,r) \) with \( x \in \partial \Omega \) and every function \( u \in W_0^p(B(x,r),\Omega) \), there exists a function \( v \in BV(B(x,r)) \) with \( v|_{B(x,r) \cap \Omega} \equiv u \) and

\[
\|Dv\|(B(x,r)) \leq 2\Phi^{\frac{1}{k}}(B(x,r))\|u\|_{W^{1,p}(B(x,r) \cap \Omega)}, \quad \text{where } \frac{1}{k} = 1 - \frac{1}{p}. \tag{3.3}
\]
4. Proofs of the results

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let us first assume that $\Omega \subset \mathbb{R}^n$ is a $(BV, BV)$-extension domain with the extension operator $E$. Since $W^{1,1}(\Omega) \subset BV(\Omega)$ with $\|u\|_{BV(\Omega)} = \|\nabla u\|_{L^1(\Omega)}$ for every $u \in W^{1,1}(\Omega)$, we have

$$\|E(u)\|_{BV(\Omega)} \leq C\|u\|_{BV(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}.$$

This implies that $\Omega$ is a $(W^{1,1}, BV)$-extension domain with the same operator $E$ restricted to $W^{1,1}(\Omega)$.

Let us then prove the converse and assume that $\Omega \subset \mathbb{R}^n$ is a $(W^{1,1}, BV)$-extension domain with an extension operator $E$. Let $S_{\Omega, \Omega}$ be the Whitney smoothing operator defined in [GBR21]. Then by [GBR21, Theorem 3.1], for every $u \in BV(\Omega)$, we have $S_{\Omega, \Omega}(u) \in W^{1,1}(\Omega)$ with

$$\|S_{\Omega, \Omega}(u)\|_{W^{1,1}(\Omega)} \leq C\|u\|_{BV(\Omega)}$$

for a positive constant $C$ independent of $u$, and

$$\|D(u - S_{\Omega, \Omega}(u))\| (\partial \Omega) = 0,$$ \hfill (4.1)

where $u - S_{\Omega, \Omega}(u)$ is understood to be defined on the whole space $\mathbb{R}^n$ via a zero-extension. Then $E(S_{\Omega, \Omega}(u)) \in BV(\mathbb{R}^n)$ with

$$\|E(S_{\Omega, \Omega}(u))\|_{BV(\mathbb{R}^n)} \leq C\|S_{\Omega, \Omega}(u)\|_{W^{1,1}(\Omega)} \leq C\|u\|_{BV(\Omega)}.$$

Now, define $T: BV(\Omega) \rightarrow BV(\mathbb{R}^n)$ by setting for every $u \in BV(\Omega)$

$$T(u)(x) := \begin{cases} u(x), & \text{if } x \in \Omega \\ E(S_{\Omega, \Omega}(u))(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

By (4.1), we have $T(u) \in BV(\mathbb{R}^n)$ with

$$\|T(u)\|_{BV(\mathbb{R}^n)} \leq \|E(S_{\Omega, \Omega}(u))\|_{BV(\mathbb{R}^n)} + \|u\|_{BV(\Omega)} \leq C\|u\|_{BV(\Omega)}.$$

Hence, $\Omega$ is a $BV$-extension domain. \hfill \Box

Proof of Theorem 1.2. Assume towards a contradiction that $|\partial \Omega| > 0$. By the Lebesgue density point theorem and Lemma 3.2, there exists a measurable subset $U$ of $\partial \Omega$ with $|U| = |\partial \Omega|$ such that every $x \in U$ is a Lebesgue point of $\partial \Omega$ and $\|\nabla \Phi(x)\| < \infty$. Fix $x \in U$. Since $x$ is a Lebesgue point, there exists a sufficiently small $r_x > 0$, such that for every $0 < r < r_x$, we have

$$|B(x, r) \cap \overline{\Omega}| \geq \frac{1}{2n-1} |B(x, r)|.$$

Let $r \in (0, r_x)$ be fixed. Since $|\partial B(x, s)| = 0$ for every $s \in (0, r)$, we have

$$\left|B \left( x, \frac{r}{4} \right) \cap \overline{\Omega} \right| \geq \frac{1}{2n-1} \left|B \left( x, \frac{r}{4} \right) \right| \geq \frac{1}{2n-1} |B(x, r)| \geq \frac{1}{2n} |B(x, r)| \geq \frac{1}{2n} |B(x, r)|,$$ \hfill (4.2)

and

$$\left|B(\overline{x}, r) \setminus B \left( x, \frac{r}{2} \right) \right| \cap \overline{\Omega} \geq |B(x, r) \setminus \Omega| - \left|B \left( \frac{r}{2} \right) \right| \geq \frac{1}{2n} |B(x, r)|.$$ \hfill (4.3)
Define a test function \( u \in W^{1,p}(\Omega) \cap C(\Omega) \) by setting
\[
\begin{cases}
1, & \text{if } y \in B(x, \frac{r}{4}) \cap \Omega, \\
\frac{1}{r} |y - x| + 2, & \text{if } y \in (B(x, \frac{r}{2}) \setminus B(x, \frac{r}{4})) \cap \Omega, \\
0, & \text{if } y \in \Omega \setminus B(x, \frac{r}{2}).
\end{cases}
\]  
(4.4)

We have
\[
\left( \int_{B(x,r) \cap \Omega} |u(y)|^p + |\nabla u(y)|^p \, dx \right)^{\frac{1}{p}} \leq \frac{C}{r} |B(x,r) \cap \Omega|^{\frac{1}{p}}. 
\]  
(4.5)

Since \( u \equiv 0 \) on \( \Omega \setminus B(x,r/2) \), we have \( u \in W_0^p(B(x,r), \Omega) \). Then, by the definition (3.2) of the set function \( \Phi \) and by Lemma 3.3, there exists a function \( v \in BV(B(x,r)) \) with \( v|_{B(x,r) \cap \Omega} \equiv u \) and
\[
\|Dv\|(B(x,r)) \leq 2\Phi^{\frac{1}{p}}(B(x,r))\|u\|_{W^{1,p}(B(x,r) \cap \Omega)}. 
\]  
(4.6)

By the Poincaré inequality of \( BV \) functions stated in Proposition 2.9, we have
\[
\int_{B(x,r)} |v(y) - v_{B(x,r)}| \, dy \leq C r \|Dv\|(B(x,r)). 
\]  
(4.7)

Since \( \Omega \) is 1-fat on almost every \( z \in \partial \Omega \), by Lemma 2.7, \( v(z) = 1 \) for almost every \( z \in B(x, \frac{r}{2}) \cap \partial \Omega \) and \( v(z) = 0 \) for almost every \( z \in (B(x,r) \setminus B(x, \frac{r}{2})) \cap \partial \Omega \). Hence, on one hand, if \( v_{B(x,r)} \leq \frac{1}{2} \), we have
\[
\int_{B(x,r)} |v(y) - v_{B(x,r)}| \, dy \geq 1 \left| B(x, \frac{r}{4}) \cap \Omega \right| \geq c|B(x,r)|.
\]

On the other hand, if \( v_{B(x,r)} > \frac{1}{2} \), we have
\[
\int_{B(x,r)} |v(y) - v_{B(x,r)}| \, dy \geq 1 \left| \left( B(x,r) \setminus B(x, \frac{r}{2}) \right) \cap \Omega \right| > c|B(x,r)|.
\]

All in all, we always have
\[
\int_{B(x,r)} |v(y) - v_{B(x,r)}| \, dy \geq c|B(x,r)| 
\]  
(4.8)

for a sufficiently small constant \( c > 0 \). Thus, by combining inequalities (4.5)-(4.8), we obtain
\[
\Phi(B(x,r))^{p-1}|B(x,r) \cap \Omega| \geq c|B(x,r)|^p 
\]

for a sufficiently small constant \( c > 0 \). This gives
\[
|B(x,r) \cap \partial \Omega| \leq |B(x,r)| - |B(x,r) \cap \Omega| \leq |B(x,r)| - C \frac{|B(x,r)|^p}{\Phi(B(x,r))^{p-1}}.
\]

Since \( D\Phi(x) < \infty \), we have
\[
\limsup_{r \to 0^+} \frac{|B(x,r) \cap \partial \Omega|}{|B(x,r)|} \leq \limsup_{r \to 0^+} \left( 1 - \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} \right) 
\]
\[
\leq \limsup_{r \to 0^+} \left( 1 - \frac{|B(x,r)|^{p-1}}{\Phi(B(x,r))^{p-1}} \right) \leq 1 - cD\Phi(x)^{1-p} < 1.
\]

This contradicts the assumption that \( x \) is a Lebesgue point of \( \partial \Omega \). Hence, we conclude that \( |\partial \Omega| = 0 \).
Let us then consider the case $\Omega \subset \mathbb{R}^2$. By [HK14, Theorem A.29], for every $x \in \partial \Omega$ and every $0 < r < \min \{1, \frac{1}{4} \text{diam}(\Omega)\}$, we have
\[ \text{Cap}_1(\Omega \cap B(x,r); B(x,2r)) \geq cr \]
for a constant $0 < c < 1$. This implies that $\Omega$ is 1-fat at every $x \in \partial \Omega$. Hence, by combining this with the first part of the theorem, we have that the boundary of any planar $(W^{1,p}, BV)$-extension domain is of volume zero. □

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