THE DENSITY FUNCTION FOR THE VALUE-DISTRIBUTION
OF THE LERCH ZETA-FUNCTION AND ITS APPLICATIONS

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Abstract. The probabilistic study of the value-distributions of zeta-functions is one of the modern topics in analytic number theory. In this paper, we study a certain probability measure related to the value-distribution of the Lerch zeta-function. We prove that it has a density function which we can explicitly construct. Moreover, we prove an asymptotic formula for the number of zeros of the Lerch zeta-function on the right side of the critical line, whose main term is associated with the density function.

1. Introduction

1.1. Value-distribution of the Riemann zeta-function. Let \( \zeta(s) \) be the Riemann zeta-function with \( s = \sigma + it \) a complex variable. The study of the distribution of values \( \zeta(\sigma + it) \) is a classical topic in number theory. In 1914, Bohr and Courant [3] proved that the set \( \{ \zeta(\sigma + it) \mid t \in \mathbb{R} \} \) is dense in the complex plane \( \mathbb{C} \) for any fixed \( \frac{1}{2} < \sigma \leq 1 \). This result was refined by Bohr and Jessen [4,5].

Theorem 1.1 (Bohr–Jessen limit theorem). Let \( \sigma > \frac{1}{2} \) be a fixed real number and \( R \) be a fixed rectangle in \( \mathbb{C} \) whose edges are parallel to the axes. Then the limit value

\[
W_\sigma(R) = \lim_{T \to \infty} \frac{1}{T} \mu_1 \left( \{ t \in [0, T] \mid \log \zeta(\sigma + it) \in R \} \right)
\]

exists, where \( \mu_1 \) indicates the one-dimensional Lebesgue measure. Moreover, there exists a non-negative real valued continuous function \( F_\sigma(z) \) such that

\[
W_\sigma(R) = \int_R F_\sigma(z) |dz|,
\]

where \( |dz| = (2\pi)^{-1} dx dy \) for \( z = x + iy \).

Here, we determine the branch of logarithm as follows. For \( \sigma > 1 \), we recall that \( \zeta(s) \) has the Euler product representation

\[
\zeta(s) = \prod_p (1 - p^{-s})^{-1},
\]

where \( p \) runs through all prime numbers. According to this, we define \( \log \zeta(s) = -\sum_p \log (1 - p^{-s}) \) for \( \sigma > 1 \) with \( \log(z) \) the principal branch. In order to extend

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the definition of $\log \zeta(s)$, we put
\[ G = \{\sigma + it \mid \sigma > 1/2\} \setminus \bigcup_{\text{Re}(\rho)>1/2} \{\sigma + i\text{Im}(\rho) \mid 1/2 < \sigma \leq \text{Re}(\rho)\}, \]
where $\rho$ runs through all possible zeros and poles of $\zeta(s)$ with $\text{Re}(\rho) > 1/2$. Then $\log \zeta(s)$ is defined for $s \in G$ by the analytic continuation along the horizontal path from right.

Let $\mathcal{B}(\mathbb{C})$ be the class of Borel sets of $\mathbb{C}$. For $\sigma > 1/2$ and $T > 0$, we define a probability measure $P_{\sigma,T}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as
\[ P_{\sigma,T}(A) = \frac{1}{T} \mu_1 (\{t \in [0,T] \mid \log \zeta(\sigma+it) \in A\}). \]

Jessen and Wintner [20] proved that there exists an absolutely continuous probability measure $P_\sigma$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that $P_{\sigma,T}$ converges weakly to $P_\sigma$ as $T \to \infty$. Thus we have
\[ \lim_{T \to \infty} P_{\sigma,T}(A) = P_\sigma(A) \]
for any $A \in \mathcal{B}(\mathbb{C})$ whose boundary $\partial A$ satisfies $\mu_2(\partial A) = 0$ with the two-dimensional Lebesgue measure $\mu_2$. Note that the limit value $W_\sigma(R)$ of Theorem [1.1] agrees with $P_\sigma(R)$, and the function $F_\sigma(z)$ is the density of the probability measure $P_\sigma$.

Borchsenius and Jessen [6] applied the probabilistic research on values $\log \zeta(\sigma+it)$ to the study of the distribution of $a$-points of $\zeta(s)$. Here, we call $z \in \mathbb{C}$ is an $a$-point of $\zeta(s)$ if $z$ satisfies $\zeta(z) = a$. If $a \neq 0$, then there exist $a$-points of $\zeta(s)$ in the half plane $\sigma > 1/2$ while the Riemann Hypothesis asserts that there are no zeros in this region. Borchsenius and Jessen proved that there are a lot of $a$-points of $\zeta(s)$ in the half plane $\sigma > 1/2$ when $a \neq 0$.

**Theorem 1.2** (Borchsenius–Jessen). Let $1/2 < \sigma_1 < \sigma_2 < \infty$ and $a \in \mathbb{C}$. Then there exists a limit value
\[ (1.4) \quad C_a(\sigma_1, \sigma_2) = \lim_{T \to \infty} \frac{1}{T} N_a(T, \sigma_1, \sigma_2), \]
where we denote the number of $a$-points of $\zeta(s)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ by $N_a(T, \sigma_1, \sigma_2)$. Furthermore, the limit value $C_a(\sigma_1, \sigma_2)$ is positive for $\sigma_1 < 1$ if $a \neq 0$.

Especially, there are infinitely many $a$-points of $\zeta(s)$ in the strip $1/2 < \sigma < 1$ if $a \neq 0$. Theorems [1.1] and [1.2] have been developed in various ways. For example, we consider
\[ D_{\sigma,T}(R) = P_{\sigma,T}(R) - P_\sigma(R), \]
where $\sigma > 1/2$, $T > 0$, and $R$ is a fixed rectangle as above. Matsumoto [25] obtained an upper bound for $D_{\sigma,T}(R)$, and it was improved by Harman and Matsumoto [14]. Lamzouri–Lester–Radziwill [21] proved the estimate
\[ (1.5) \quad \sup_R |D_{\sigma,T}(R)| = O \left( \frac{1}{(\log T)^\sigma} \right) \]
holds as $T \to \infty$. Furthermore, they refined limit formula [1.4]. In fact, they obtained that the formula
\[ (1.6) \quad N_a(T, \sigma_1, \sigma_2) = C_a(\sigma_1, \sigma_2) T + O \left( T \frac{\log \log T}{(\log T)^{\sigma_1/2}} \right) \]
holds for \(1/2 < \sigma_1 < \sigma_2 < \infty\) and \(a \neq 0\) as \(T \to \infty\). Their methods for the proof of (1.5) and (1.6) were partly based on Guo [12,13], who studied the value-distribution of \((\zeta'/\zeta)(s)\) instead of \(\log \zeta(s)\).

In [20–28], Matsumoto generalized Theorem 1.1 in a class of \(L\)-functions possessing polynomial Euler products. Then Ihara [16] named density functions such as \(\mathcal{F}_\sigma(z)\) “\(M\)-functions” for the value-distributions of \(L\)-functions. Ihara and Matsumoto [17–19] studied \(M\)-functions for various value-distributions of Dirichlet \(L\)-functions, Hecke \(L\)-functions, and so on; see the survey of Matsumoto [29]. In this paper, we study the \(M\)-function for the value-distribution of the Lerch zeta-function \(L(\lambda, \alpha, s)\), which does not have the Euler product in general.

1.2. Hurwitz and Lerch zeta-functions. Let \(\lambda \in \mathbb{R}\) and \(0 < \alpha \leq 1\). Then we define the Hurwitz zeta-function \(\zeta(s, \alpha)\) as
\[
\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s}
\]
for \(\sigma > 1\), which is continued holomorphically to the whole complex plane except for a simple pole at \(s = 1\). Since we have \(\zeta(s, 1) = \zeta(s)\) by definition, the Hurwitz zeta-function is a generalization of the Riemann zeta-function. The Lerch zeta-function
\[
L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^s}
\]
is a further generalization of \(\zeta(s)\). Indeed, it deduces the Hurwitz zeta-function if \(\lambda \in \mathbb{Z}\). Some of the results on \(\zeta(s)\) are naturally generalized to \(\zeta(s, \alpha)\) and \(L(\lambda, \alpha, s)\). For instance, let \(N(T; \lambda, \alpha)\) be the number of non-trivial zeros \(\rho\) of \(L(\lambda, \alpha, s)\) with \(0 < \Im(\rho) < T\). Then we have
\[
N(T; \lambda, \alpha) = \frac{T}{2\pi} \log \frac{T}{2\pi e\alpha\lambda} + O(\log T)
\]
for every \(0 < \lambda \leq 1\) and \(0 < \alpha \leq 1\); see [10, Theorem 6]. It is a generalization of the classical result of von-Mangoldt on \(N(T) = N(T; 1, 1)\). On the other hand, a significant difference between \(\zeta(s)\) and \(\zeta(s, \alpha)\) or \(L(\lambda, \alpha, s)\) arises when we consider their Euler products and zeros off the critical line \(\sigma = 1/2\). By definition of \(L(\lambda, \alpha, s)\), we have

\[
L(1, 1/2, s) = (2^s - 1)\zeta(s) \quad \text{and} \quad L(1/2, 1, s) = (1 - 2^{1-s})\zeta(s).
\]
Hence \(L(1, 1, s), L(1, 1/2, s), \text{ and } L(1/2, 1, s)\) have no zeros in the half plane \(\sigma > 1\) due to the Euler product of the Riemann zeta-function (1.2). However, \(\zeta(s, \alpha)\) and \(L(\lambda, \alpha, s)\) do not have the Euler product representation in general. For example, we see that

\[
\zeta(s, 1/3) = 3^s \sum_{n=1}^{\infty} \frac{\delta_{\equiv 1(\mod 3)}(n)}{n^s},
\]
where \(\delta_{\equiv 1(\mod 3)}(n)\) is equal to 1 if \(n \equiv 1(\mod 3)\) and 0 otherwise, which is not multiplicative. Because of the lack of Euler products, one may hope that Hurwitz and Lerch zeta-functions have zeros in the half plane \(\sigma > 1\). Davenport and Heilbronn [8] proved that there are infinitely many zeros of the Hurwitz zeta-function \(\zeta(s, \alpha)\) in \(\sigma > 1\) if \(\alpha \neq 1, 1/2\) is a rational or transcendental number. Moreover, Cassels [7] proved the same result in the remaining case, i.e. \(\zeta(s, \alpha)\) has infinitely
many zeros in \( \sigma > 1 \) even if \( \alpha \) is algebraic irrational. We then proceed to considering zeros in the strip \( 1/2 < \sigma \leq 1 \). Although it is conjectured that \( \zeta(s) \) has no zeros there, \( \zeta(s, \alpha) \) has infinitely many zeros in \( 1/2 < \sigma \leq 1 \) if \( \alpha \neq 1, 1/2 \) is either rational or transcendental. This result is a consequence of a property of \( \zeta(s, \alpha) \) so-called the strong universality proved by Bagchi [1] and Gonek [11]. Unfortunately, such a property has not yet been known in the case that \( \alpha \) is an algebraic irrational number.

Garunkštis and Laurinčikas proved similar results on zeros of the Lerch zeta-function on the right side of the critical line; see [24, Chapter 8]. Let \( N(T, \sigma_1, \sigma_2; \lambda, \alpha) \) be the number of zeros of \( L(\lambda, \alpha, s) \) in the rectangle \( \sigma_1 < \sigma < \sigma_2, 0 < t < T \). Then, combining the results of Garunkštis and Laurinčikas, we obtain the following result.

**Theorem 1.3** (Garunkštis–Laurinčikas). Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Assume that \( \alpha \) is a transcendental number. Then there exists a constant \( \delta(\alpha) > 0 \) depending only on \( \alpha \) such that, for any fixed real numbers \( \sigma_1, \sigma_2 \) with \( 1/2 < \sigma_1 < \sigma_2 < 1 + \delta(\alpha) \), there exist positive constants \( C_1 = C_1(\sigma_1, \sigma_2; \lambda, \alpha) \) and \( C_2 = C_2(\sigma_1, \sigma_2; \lambda, \alpha) \) such that

\[
C_1 T < N(T, \sigma_1, \sigma_2; \lambda, \alpha) < C_2 T
\]

holds for sufficiently large \( T \).

In this paper, we study the value-distribution of \( L(\lambda, \alpha, s) \) and the asymptotic behavior of \( N(T, \sigma_1, \sigma_2; \lambda, \alpha) \) according to analogues of Theorems 1.1 and 1.2 for \( L(\lambda, \alpha, s) \). Let \( P_{\sigma,T}(\cdot; \lambda, \alpha) \) be a probability measure on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) defined as

\[
P_{\sigma,T}(A; \lambda, \alpha) = \frac{1}{T} \mu_1 \left( \{ t \in [0, T] \mid L(\lambda, \alpha, \sigma + it) \in A \} \right).
\]

Then Garunkštis and Laurinčikas [9] proved the following result.

**Theorem 1.4** (Garunkštis–Laurinčikas). Let \( \sigma > 1/2, \lambda \in \mathbb{R} \), and \( 0 < \alpha \leq 1 \). Then there exists a probability measure \( P_\sigma(\cdot; \lambda, \alpha) \) on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) such that \( P_{\sigma,T}(\cdot; \lambda, \alpha) \) converges weakly to \( P_\sigma(\cdot; \lambda, \alpha) \) as \( T \to \infty \).

Note that Theorem 1.4 holds with an arbitrary \( 0 < \alpha \leq 1 \), but it does not ensure the absolutely continuity of the limit measure \( P_\sigma(\cdot; \lambda, \alpha) \). If \( P_\sigma(\cdot; \lambda, \alpha) \) is absolutely continuous, we obtain

\[
(1.7) \quad \lim_{T \to \infty} P_{\sigma,T}(A; \lambda, \alpha) = P_\sigma(A; \lambda, \alpha)
\]

for any \( A \in \mathcal{B}(\mathbb{C}) \) with \( \mu_2(\partial A) = 0 \) as we have seen in (1.3). However, Theorem 1.3 tells us only that (1.7) is valid for \( A \in \mathcal{B}(\mathbb{C}) \) with \( P_\sigma(\partial A; \lambda, \alpha) = 0 \) without the absolutely continuity.

We have assumed that the parameter \( \alpha \) is transcendental in Theorem 1.3. In the study of the value-distributions of \( \zeta(s, \alpha) \) and \( L(\lambda, \alpha, s) \), this assumption sometimes can be replaced with the assumption that the system \( \{ \log(n + \alpha) \}_{n \geq 0} \) is linearly independent over \( \mathbb{Q} \). Then, we obtain the following result as an analogue of Theorem 1.2 under the latter assumption.

**Theorem 1.5.** Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Assume that the system \( \{ \log(n + \alpha) \}_{n \geq 0} \) is linearly independent over \( \mathbb{Q} \). Then the limit value

\[
(1.8) \quad C(\sigma_1, \sigma_2; \alpha) = \lim_{T \to \infty} \frac{1}{T} N(T, \sigma_1, \sigma_2; \lambda, \alpha)
\]

exists for any \( 1/2 < \sigma_1 < \sigma_2 < \infty \).
It is notable that the limit value is determined only from $\sigma_1$, $\sigma_2$, $\alpha$ and does not depend on $\lambda$ although the constants $C_1$ and $C_2$ of Theorem 1.3 may depend on $\lambda$. One can prove Theorem 1.4 by just adapting the method of Borchsenius–Jessen [6]. However, we give a full proof in Appendix of this paper since the author could not find a suitable reference for the proof.

1.3. Statements of results. Comparing the results on $\log \zeta(s)$ in Section 1.1 and on $L(\lambda, \alpha, s)$ in Section 1.2, we hope that there is room for improvement of limit formulas (1.7) and (1.8) as well as (1.5) and (1.6). The main purpose of this paper is to refine Theorems 1.4 and 1.5 in this sense. For this, we need a further restriction of the parameter $\alpha$.

Definition 1.6. We define the admissible subset $\mathcal{S}$ as the collection of all $\alpha \in (0, 1]$ which satisfy the following conditions (1) and (2).

(1) Independence. The system $\{\log(n+\alpha)\}_{n \geq 0}$ is linearly independent over $\mathbb{Q}$.

(2) Spacing. There exists a constant $\Omega(\alpha) > 0$ such that for large $X > 0$ and for any positive integer $N$, we have

$$\left| \sum_{j=1}^{N} \epsilon_j \log(n_j + \alpha) \right| \geq X^{-\Omega(\alpha)N^2}$$

provided $\sum_{j=1}^{N} \epsilon_j \log(n_j + \alpha) \neq 0$ with $0 \leq n_1, \ldots, n_N \leq X$ arbitrary integers and $\epsilon_1, \ldots, \epsilon_N \in \{-1, 1\}$.

We see that almost all $\alpha \in (0, 1]$ belong to $\mathcal{S}$ with respect to $\mu_1$ by applying the results of transcendental number theory. For the proof of this fact and more information about the admissible subset $\mathcal{S}$; see Section 2.1. Then, we prove the following result as an improvement of Theorem 1.4.

Theorem 1.7. Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. Let $\sigma_1 > 0$ be a large fixed real number. Let $\epsilon_1 > 0$ be a small fixed real number. Assume that $\alpha$ is a member of $\mathcal{S}$. Then for each $\epsilon > 0$, there exists a positive constant $T_0 = T_0(\alpha, \sigma_1, \epsilon_1, \epsilon)$ such that for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$ and for all $T \geq T_0$, we have

$$P_{\sigma, T}(R; \lambda, \alpha) = P_{\sigma}(R; \lambda, \alpha) + O\left((\mu_2(R) + 1)(\log T)^{-1/4+\epsilon}\right),$$

where $R$ is any rectangle in $\mathbb{C}$ whose edges are parallel to the axes and have length greater than $(\log T)^{-1/4+\epsilon}$. The implied constant depends only on $\alpha$, $\lambda$, $\sigma_1$, and $\epsilon_1$. Moreover, there exists a non-negative smooth function $M_{\sigma}(z; \alpha)$ on $\mathbb{C}$ such that we have

$$P_{\sigma}(A; \lambda, \alpha) = \int_A M_{\sigma}(z; \alpha) |dz|$$

for every $\sigma > 1/2$ and $A \in \mathcal{B}(\mathbb{C})$.

The second statement of Theorem 1.7 assert that the limit measure $P_{\sigma}(\cdot; \lambda, \alpha)$ is absolute continuous with the density $M_{\sigma}(z; \alpha)$ if $\alpha \in \mathcal{S}$. The function $M_{\sigma}(z; \alpha)$ is an analogue of $F_{\sigma}(z)$ of Theorem 1.1. We again remark that it does not depend on $\lambda$. The final result of this paper is as follows, which is an improvement of Theorems 1.3 and 1.5 in the case that $\alpha$ belongs to the admissible subset $\mathcal{S}$. 

Theorem 1.8. Let $1/2 < \sigma_1 < \sigma_2 < \infty$, $\lambda \in \mathbb{R}$, and $\alpha \in \mathbb{S}$. Then there exists an absolute constant $A > 0$ such that

$$N(T, \sigma_1, \sigma_2; \lambda, \alpha) = C(\sigma_1, \sigma_2; \alpha) T + O\left(T(\log T)^{-A}\right)$$

as $T \to \infty$, where the implied constant depends only on $\sigma_1$, $\sigma_2$, $\lambda$ and $\alpha$. Furthermore, the constant $C(\sigma_1, \sigma_2; \alpha)$ is given by

$$C(\sigma_1, \sigma_2; \alpha) = \frac{1}{2\pi} \int_{\sigma_1}^{\sigma_2} \left( \int_{\mathcal{C}} \log |z| \frac{\partial^2}{\partial \sigma^2} M_\sigma(z; \alpha) |dz| \right) d\sigma.$$ 

This result is also an analogue of the result of Guo [13, Theorem 1.1.2] on the distribution of zeros of $\zeta'(s)$. We adapt the method of Guo for a part of the proof of Theorem 1.8.

This paper consists of seven sections. Section 2 contains preliminaries for the proofs of the main results. In Section 2.1 we study the admissible subset $\mathcal{S}$ more precisely. We recall the notion of $S$-numbers introduced by Mahler [24] and show that every $S$-number in $(0, 1]$ belongs to $\mathcal{S}$. In Section 2.2 we construct the density function $M_\sigma(z; \alpha)$. For this, we review the theory of equidistributions on circles studied by Jessen and Wintner [20]. The function $M_\sigma(z; \alpha)$ is defined as an infinite convolution of such distributions. In Section 3 we consider several mean values of the Lerch zeta-function. We prove Theorem 3.2 in this section which assert that the mean values are expressed as the integrals involving the density function $M_\sigma(z; \alpha)$. Both of the proofs of Theorems 1.7 and 1.8 are based on Theorem 3.2. Theorem 1.7 is proved in Section 4, where we use the Beurding–Selberg functions which approximate well the signum function $\text{sgn}(x)$ on $\mathbb{R}$. In order to prove Theorem 1.8 we need to examine the analytic properties of the function $M_\sigma(z; \alpha)$ more precisely. Further results on $M_\sigma(z; \alpha)$ are given in Section 5. Finally, we prove Theorem 1.8 in Section 6. A mean value theorem for $\log |L(\lambda, \alpha, s)|$ is a key for the proof. The last section is an appendix. We give a proof of Theorem 1.5 in this section by following the method of Borchsenius–Jessen [12].

Throughout this paper, we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ by the bijection $x + iy \mapsto (x, y)$. Hence we regard functions on $\mathbb{C}$ as functions on $\mathbb{R}^2$, and for example, we write the two-dimensional $L^p$-space by $L^p(\mathbb{R}) = L^p(\mathbb{R}^2)$. We also use the symbols $\ll$ and $\gg$ as the usual Vinogradov symbols.

2. Preliminaries

2.1. Admissible subset $\mathcal{S}$. Definition [13] is motivated by transcendental number theory. In fact, we find that any transcendental number satisfies condition [1]. For condition (2), we recall the notion of $S$-numbers introduced by Mahler [24]. According to the notation of the book of Baker [2], for a complex number $\xi$, and for positive integers $n$ and $h$, let $P(x) \in \mathbb{Z}[x]$ be a polynomial with degree at most $n$ and height at most $h$ for which $|P(\xi)|$ takes the smallest non-zero value. Here the height of a polynomial is the maximal value of the absolute values of the coefficients. Then, we determine a real number $\omega_{n,h}(\xi) \in (0, \infty)$ by the equation $|P(\xi)| = h^{-n\omega_{n,h}(\xi)}$. We further define $\omega(\xi) \in [0, \infty]$ as

$$\omega(\xi) = \lim_{n \to \infty} \sup_{h \to \infty} \omega_{n,h}(\xi).$$
Definition 2.1. A complex number $\xi$ is called an $S$-number if $0 < \omega(\xi) < \infty$.

Hence, if $\xi$ is an $S$-number, there exists a positive real number $\omega'(\xi)$ such that we have

$$
|P(\xi)| \geq h^{-\omega'(\xi)}
$$

for every polynomial $P(x) \in \mathbb{Z}[x]$ with degree at most $n$ and height at most $h$ if $P(\xi) \neq 0$. Then, the following result holds.

Lemma 2.2. Every $S$-number in $(0, 1]$ belongs to the admissible subset $\mathcal{S}$.

Proof. It is known that all $S$-numbers are transcendental [2, Section 8.2], and therefore every $S$-number $\alpha \in (0, 1]$ satisfies condition (1). Then we prove that it also satisfies condition (2). Let $\mu$ and $\nu$ be non-negative integers with $\mu + \nu = N$, where $N \geq 1$. Then we take $N$ integers $0 \leq m_1, \ldots, m_\mu, n_1, \ldots, n_\nu \leq X$ which satisfy $(m_1 + \alpha) \cdots (m_\mu + \alpha) \neq (n_1 + \alpha) \cdots (n_\nu + \alpha)$. Here, if one of $\mu$ and $\nu$ is equal to 0, the product $(m_1 + \alpha) \cdots (m_\mu + \alpha)$ or $(n_1 + \alpha) \cdots (n_\nu + \alpha)$ is interpreted as 1. We have

$$
\sum_{j=1}^\mu \log(m_j + \alpha) - \sum_{k=1}^\nu \log(n_k + \alpha) = \log \left( \frac{(m_1 + \alpha) \cdots (m_\mu + \alpha)}{(n_1 + \alpha) \cdots (n_\nu + \alpha)} \right)
$$

with the polynomial

$$
P(x) = (m_1 + x) \cdots (m_\mu + x) - (n_1 + x) \cdots (n_\nu + x).
$$

The degree of $P(x)$ is at most $\mu + \nu = N$, and the height is at most

$$
\left( \begin{array}{c} N \\ \lfloor N/2 \rfloor \end{array} \right) X^N \leq (2X)^N,
$$

where $\lfloor x \rfloor$ is the maximum integer not greater than $x$. Hence, by (2.1), we have

$$
|P(\alpha)| \geq (2X)^{-\omega'(\alpha)N^2}.
$$

Therefore we obtain the desired inequality by (2.2) when $X$ is large. \hfill \Box

Almost all real numbers are $S$-numbers with respect to $\mu_1$; see [2, Theorem 8.2]. Hence we deduce the following corollary.

Corollary 2.3. Almost all $\alpha$ in $(0, 1]$ belong to $\mathcal{S}$ with respect to $\mu_1$.

A typical example of $S$-numbers is Napier’s constant $e$, and we see that its fractional part $\{e\}$ is a member of $\mathcal{S}$. On the other hand, Liouville numbers such as $\sum_k 10^{-k!}$ are transcendental but not $S$-numbers. However, it seems difficult to see Liouville numbers in $(0, 1]$ are belongs to $\mathcal{S}$ or not.
2.2. Equidistributions on circles. According to [20, Section 5], let $S$ be the circle $|z| = r$ in the complex plane with $r > 0$, and define the equidistribution on $S$ as the probability measure $\phi$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ given by

$$\phi(A) = \tilde{\mu}_S(A \cap S),$$

where $\tilde{\mu}_S$ is the normalized Haar measure of $S$. For $0 < \alpha \leq 1$, let $S_n$ be the circles $|z| = (n + \alpha)^{-\sigma}$ for $n \geq 0$. Denote the equidistributions on $S_n$ by $\phi_n$. Then we consider the infinite convolution $\phi = \phi_0 * \phi_1 * \cdots$.

**Proposition 2.4.** For any $\sigma > 1/2$, the convolution measure $\phi_0 * \phi_1 * \cdots * \phi_n$ converges weakly to a probability measure $\phi$ as $n \to \infty$. Moreover, the limit measure $\phi$ is absolutely continuous with a continuous density function $M_\sigma(z; \alpha)$, i.e. there exists a non-negative real valued continuous function $M_\sigma(z; \alpha)$ such that

$$\phi(A) = \int_A M_\sigma(z; \alpha) \, |dz|, \quad A \in \mathcal{B}(\mathbb{C}).$$

**Proof.** This is a simple application of [20, Theorem 10].

In general, we define the Fourier transform

$$\tilde{f}(z) = \int_{\mathbb{C}} f(w) \psi_z(w) \, |dw|$$

for $f \in L^1(\mathbb{C})$ with $\psi_z(w) = \exp(i \text{Re}(z \overline{w}))$. Then we have

$$\tilde{M}_\sigma(z; \alpha) = \int_{\mathbb{C}} M_\sigma(w; \alpha) \psi_z(w) \, |dw| = \int_{\mathbb{C}} \psi_z(w) \, d\phi(w).$$

Since $\phi = \phi_0 * \phi_1 * \cdots$, we find that

$$\int_{\mathbb{C}} \psi_z(w) \, d\phi(w) = \prod_{n=0}^{\infty} \int_{\mathbb{C}} \psi_z(w) \, d\phi_n(w)$$

by the argument in [20, Section 4]. Furthermore, by [20, Section 5], we have

$$\int_{\mathbb{C}} \psi_z(w) \, d\phi_n(w) = J_0(|z|(n + \alpha)^{-\sigma}),$$

where $J_0(x)$ is the Bessel function of the first kind of order 0. Therefore, the Fourier transform $\tilde{M}_\sigma(z; \alpha)$ is expressed as the infinite product

$$\tilde{M}_\sigma(z; \alpha) = \prod_{n=0}^{\infty} J_0(|z|(n + \alpha)^{-\sigma}).$$

More detailed properties on the function $M_\sigma(z; \alpha)$ are as follows.

**Proposition 2.5.** We have the following properties on $M_\sigma(z; \alpha)$.

(i) For any $\sigma > 1/2$ and $0 < \alpha \leq 1$, the function $M_\sigma(z; \alpha)$ possesses continuous partial derivative of arbitrarily high order as a function of $x$ and $y$ with $z = x + iy$. Their growths are estimated as for any $c > 0$ and for $\sigma \geq 1/2 + \epsilon_1$ with small $\epsilon_1 > 0$,

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} M_\sigma(z; \alpha) \ll e^{-c|z|^2}, \quad |z| \to \infty$$

for all non-negative integers $m$ and $n$, where the implied constant depends only on $\alpha$, $m$, $n$, and $\epsilon_1$. 

(ii) For any \( \sigma > 1/2 \) and \( 0 < \alpha \leq 1 \), we have
\[
\int_{C} M_\sigma(z; \alpha) \, |dz| = 1.
\]

(iii) If \( 1/2 < \sigma \leq 1 \), then \( M_\sigma(z; \alpha) > 0 \) for all \( z \in \mathbb{C} \).

(iv) \( \widetilde{M}_\sigma(z; \alpha) \) is a real valued function with \( |\widetilde{M}_\sigma(z; \alpha)| \leq 1 \). The values \( \widetilde{M}_\sigma(z; \alpha) \)
are determined only from \( \sigma, \alpha \), and \( |z| \), i.e. \( M_\sigma(z_1; \alpha) = \widetilde{M}_\sigma(z_2; \alpha) \) holds if
\( |z_1| = |z_2| \).

Proof. (i) The former statement is deduced from [20, Theorem 10], and the latter \((2.5)\) is due to [20, Theorem 16].

(ii) By \((2.3)\) and \((2.4)\), we have
\[
\int_{C} M_\sigma(z; \alpha) \, |dz| = \widetilde{M}_\sigma(0; \alpha) = \prod_{n=0}^{\infty} J_0(0).
\]

Thus we obtain \((2.6)\) since \( J_0(0) = 1 \).

(iii) This is again deduced from [20, Theorem 10].

(iv) We know that \( J_0(x) \) is real valued for \( x \in \mathbb{R} \) and satisfies \( |J_0(x)| \leq 1 \). Hence
formula \((2.4)\) yields the first statement. The remaining statement is also
deduced from \((2.3)\).

\( \square \)

3. Mean values of the Lerch zeta-function

In this section, we study the mean values
\[
\frac{1}{T} \int_{0}^{T} \Phi(L(\lambda, \alpha, \sigma + it)) \, dt
\]
for test functions \( \Phi(z) \) via the density function \( M_\sigma(z; \alpha) \) defined in Section 2.2.

First, we define two classes of test functions. Let \( \mathcal{S} = \mathcal{S}(\mathbb{C}) = \mathcal{S}(\mathbb{R}^2) \) denote the
Schwartz space on \( \mathbb{C} \), which consists of all rapidly decreasing \( C^\infty \)-functions whose
partial derivatives of arbitrarily high order also rapidly decrease. Next, according to
[17, Section 9], we also define the class \( \Lambda \) as the set of all functions \( f \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C}) \)
with \( \widetilde{f} \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C}) \) that satisfy the inverse formula
\[
f(z) = \int_{C} \widetilde{f}(w) \psi_{-z}(w) \, |dw|.
\]

Note that the Schwartz space \( \mathcal{S} \) is included in the class \( \Lambda \), and especially, any
compactly supported \( C^\infty \)-function belongs to the class \( \Lambda \). By Proposition 2.5. the
function \( M_\sigma(z; \alpha) \) belongs to \( \mathcal{S} \), and hence its Fourier transform \( \widetilde{M}_\sigma(z; \alpha) \)
does.

Then, we state the following two results on the mean values [33.1].

**Theorem 3.1.** Let \( \lambda \in \mathbb{R} \) and \( \alpha \in \mathcal{S} \). Let \( \sigma_1 \) be a large fixed positive constant, and
let \( \theta, \delta > 0 \) with \( \theta + \delta < 1/4 \). Then there exists \( T_0 = T_0(\alpha, \sigma_1, \theta, \delta) > 0 \) such that for
all \( T \geq T_0 \) and for all \( \sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1] \), we have
\[
\frac{1}{T} \int_{0}^{T} \psi_{z}(L(\lambda, \alpha, \sigma + it)) \, dt = \widetilde{M}_\sigma(z; \alpha) + O\left(\exp\left(-\frac{1}{2}(\log T)^{\theta/2}\right)\right)
\]
for any \( z \in \Omega \), where
\[
\Omega = \{ x + iy \in \mathbb{C} \mid |x| \leq (\log T)^{\delta}, |y| \leq (\log T)^{\delta} \}.\]
The implied constant depends only on $\lambda$ and $\sigma_1$.

**Theorem 3.2.** Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathcal{S}$. Let $\sigma_1$ be a large fixed positive constant, and let $\theta, \delta > 0$ with $\theta + \delta < 1/4$. Then there exists $T_0 = T_0(\alpha, \sigma_1, \theta, \delta) > 0$ such that for all $T \geq T_0$ and for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have

$$\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt = \int_{\mathcal{C}} \Phi(z) M_\sigma(z; \alpha) \, |dz| + E$$

for any function $\Phi$ in the class $\Lambda$, where $E$ is estimated as

$$E \ll \exp \left( -\frac{1}{2} \left( \frac{1}{2} \log T \right)^2 \right) \int_{\Omega} \left| \tilde{\Phi}(z) \right| |dz| + \int_{\mathcal{C} \setminus \Omega} \left| \tilde{\Phi}(z) \right| |dz|.$$

The implied constant depends only on $\lambda$ and $\sigma_1$.

Theorem 3.2 is an analogue of [12, Theorem 1.1.1] for Lerch zeta-functions. We first prove that Theorem 3.1 implies Theorem 3.2.

**Proof of Theorem 3.2 assuming Theorem 3.1.** By definition of the class $\Lambda$, we have

$$\Phi(L(\lambda, \alpha, s)) = \int_{\mathcal{C}} \tilde{\Phi}(z) \psi_{-z}(L(\lambda, \alpha, s)) \, |dz|$$

for any $\Phi \in \Lambda$. Then we use Theorem 3.1. For all $T \geq T_0$ and $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, we have

$$\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt = \int_{\mathcal{C}} \tilde{\Phi}(z) \frac{1}{T} \int_0^T \psi_{-z}(L(\lambda, \alpha, \sigma + it)) \, dt |dz|$$

$$= \int_{\Omega} \tilde{\Phi}(z) \frac{1}{T} \int_0^T \psi_{-z}(L(\lambda, \alpha, \sigma + it)) \, dt |dz| + E_1$$

$$= \int_{\Omega} \tilde{\Phi}(z) \tilde{M}_\sigma(-z; \alpha) |dz| + E_1 + E_2$$

$$= \int_{\mathcal{C}} \tilde{\Phi}(z) \tilde{M}_\sigma(-z; \alpha) |dz| + E_1 + E_2 + E_3.$$}

The above error terms are estimated as

$$E_1, E_3 \ll \int_{\mathcal{C} \setminus \Omega} \left| \tilde{\Phi}(z) \right| |dz|$$

$$E_2 \ll_{\lambda, \sigma_1} \exp \left( -\frac{1}{2} \left( \frac{1}{2} \log T \right)^2 \right) \int_{\Omega} \left| \tilde{\Phi}(z) \right| |dz|$$

due to the inequalities $|\psi_z(w)| \leq 1$ and $|\tilde{M}_\sigma(z; \alpha)| \leq 1$. Finally, we have

$$\int_{\mathcal{C}} \tilde{\Phi}(z) \tilde{M}_\sigma(-z; \alpha) |dz| = \int_{\mathcal{C}} \tilde{\Phi}(z) \tilde{M}_\sigma(z; \alpha) |dz| = \int_{\mathcal{C}} \Phi(z) M_\sigma(z; \alpha) |dz|$$

by (iv) of Proposition 2.5 and Parseval’s identity. Hence the result follows. 

In the remaining part of this section, we prove Theorem 3.1. The proof consists of the proofs of the following four Propositions 3.3, 3.4, 3.6, and 3.7.
Proposition 3.3. Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Let \( \sigma_1 \) be a large fixed positive constant, and let \( \theta, \delta > 0 \) with \( \theta < 2/3 \). Then there exists \( T_0 = T_0(\theta) > 0 \) such that for all \( T \geq T_0 \) and for all \( \sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1] \), we have

\[
\frac{1}{T} \int_0^T \psi_z(L(\lambda, \alpha, \sigma + it)) \, dt = \frac{1}{T} \int_0^T \psi_z \left( \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} \right) \, dt + E_1
\]

for any \( z \in \Omega \), where \( X = \exp((\log T)^{3\theta/2}) \), and \( E_1 \) is estimated as

\[
E_1 \ll_{\lambda, \sigma_1} \exp \left( -\frac{1}{2}(\log T)^{\theta/2} \right).
\]

To prove Proposition 3.3, we use the following lemma.

Lemma 3.4. Let \( 0 < \lambda \leq 1 \) and \( 0 < \alpha \leq 1 \). Then, for any \( \sigma > 1/2 \) and \( 2\pi \leq |t| \leq \pi \lambda Y \), we have

\[
L(\lambda, \alpha, s) = \sum_{0 \leq n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^s} + \delta_{\lambda} \frac{Y^{1-s}}{s-1} + O_{\lambda}(Y^{-\sigma}),
\]

where \( \delta_{\lambda} = 1 \) if \( \lambda = 1 \), \( \delta_{\lambda} = 0 \) otherwise.

Proof. This is \([22] \) Theorem 3.1.2 if \( 0 < \lambda < 1 \). When \( \lambda = 1 \), the Lerch zeta-function is the Hurwitz zeta-function, and hence the result is deduced from \([22] \) Theorem 3.1.3.

Proof of Proposition 3.3. Note that we may assume that \( 0 < \lambda \leq 1 \) without loss of generality. By the definition of \( \psi_z(w) \), we have \( |\psi_z(w)| = 1 \) and \( |\psi_z(w) - \psi_z(w')| \leq |z||w - w'| \) for any \( z, w, w' \in \mathbb{C} \). Hence we have

\[
|E_1| \leq \frac{2\pi}{T} + \left| \frac{z}{T} \right| \int_0^T \left| L(\lambda, \alpha, \sigma + it) - \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} \right| \, dt.
\]

Let \( Y = T/\lambda \). Then we have \( X < Y \) due to \( \theta < 2/3 \). We deduce from Lemma 3.4 that

\[
\left| L(\lambda, \alpha, \sigma + it) - \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} \right| \ll_{\lambda} \sum_{X < n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} + \frac{T^{1-\sigma}}{t} + T^{-\sigma}
\]

for \( \sigma > 1/2 \) and \( 2\pi \leq t \leq T \). Applying Cauchy’s inequality, we obtain

\[
E_1 \ll_{\lambda} \frac{1}{T} + \frac{|z|}{T^{1/2}} \left( \int_0^T \left| \sum_{X < n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} \right|^2 \, dt \right)^{1/2} + |z|T^{-\sigma} \log T
\]

By the inequality \([30] \) Theorem 2, we see that

\[
\int_0^T \left| \sum_{X < n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma+it}} \right|^2 \, dt = \sum_{X < n \leq Y} \frac{T + O(n)}{(n + \alpha)^{2\sigma}}.
\]
Hence we have
\[
\int_0^T \left| \sum_{X < n \leq Y} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma + it}} \right|^2 dt \ll_{\lambda} T \sum_{n > X} (n + \alpha)^{-2\sigma} \ll_{\sigma_1} \frac{T}{2\sigma - 1} X^{1 - 2\sigma}.
\]

Therefore, by (3.2), we obtain
\[
E_1 \ll_{\lambda, \sigma_1} \frac{1}{T} + \frac{|z|}{\sqrt{2\sigma - 1}} X^{1/2 - \sigma} + |z| \sigma \log T.
\]

Since \( X = \exp((\log T)^{\frac{3\theta}{2}}) \), \(|z| \leq 2(\log T)^\delta\), and \( \sigma \geq 1/2 + (\log T)^{-\theta}\), we conclude
\[
E_1 \ll_{\lambda, \sigma_1} \exp \left( -\frac{1}{2} (\log T)^{\theta/2} \right)
\]
for any \( T \geq T_0 \) with a constant \( T_0 = T_0(\theta) > 0 \).

\section*{Proposition 3.5}

\textit{Proposition 3.5.} Let \( \lambda \in \mathbb{R} \) and \( \alpha \in \mathbb{G} \). Let \( \sigma_1 \) be a large fixed positive constant, and let \( \theta, \delta > 0 \) with \( \theta + \delta < 1/4 \). Then there exists \( T_0 = T_0(\alpha, \sigma_1, \theta, \delta) > 0 \) such that for all \( U \geq T \geq T_0 \) and for all \( \sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1] \), we have
\[
\frac{1}{T} \int_0^T \psi_z \left( \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma + it}} \right) dt = \frac{1}{U} \int_0^U \psi_z \left( \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma + it}} \right) du + E_2
\]
for any \( z \in \Omega \), where \( X = \exp((\log T)^{3\theta/2}) \), and \( E_2 \) is estimated as
\[
E_2 \ll \exp \left( -\frac{1}{2} (\log T)^{\theta/2} \right).
\]

\textit{Proof.} For simplicity, we write
\[
F(t) = F(t, \sigma, X; \lambda, \alpha) = \sum_{0 \leq n \leq X} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^{\sigma + it}}.
\]

Then we have
\[
\psi_z(F(t)) = \exp \left( \frac{iz}{2} F(t) + \frac{iz}{2} F(t) \right)
= \sum_{0 \leq \mu + \nu < N} \frac{1}{\mu! \nu!} \left( \frac{iz}{2} \right)^\mu \left( \frac{i\bar{z}}{2} \right)^\nu F(t)^\mu \overline{F(t)}^\nu + O \left( \frac{|z|^N}{N!} |F(t)|^N \right)
\]
for a sufficiently large positive even integer \( N \). Hence \( E_2 \) is estimated as
\[
E_2 = \sum_{0 \leq \mu + \nu < N} \frac{1}{\mu! \nu!} \left( \frac{iz}{2} \right)^\mu \left( \frac{i\bar{z}}{2} \right)^\nu H(\mu, \nu)
+ O \left( \frac{|z|^N}{N!} \left( \frac{1}{T} \int_0^T |F(t)|^N dt + \frac{1}{U} \int_0^U |F(u)|^N du \right) \right),
\]
where
\[
H(\mu, \nu) = \frac{1}{T} \int_0^T F(t)^\mu \overline{F(t)}^\nu dt - \frac{1}{U} \int_0^U F(u)^\mu \overline{F(u)}^\nu du.
\]
First, we estimate the first term of (3.3). We have

\[
H(\mu, \nu) \ll \frac{1}{T} \sum_{0 \leq n \leq X} \sum_{0 \leq m_{1} \leq X} \cdots \sum_{0 \leq m_{\mu} \leq X} \sum_{0 \leq n_{1} \leq X} \cdots \sum_{0 \leq n_{\nu} \leq X} \frac{1}{(m_{1} + \alpha) \cdots (m_{\mu} + \alpha)} \times \frac{1}{(n_{1} + \alpha) \cdots (n_{\nu} + \alpha)} \log \frac{(m_{1} + \alpha) \cdots (m_{\mu} + \alpha)}{(n_{1} + \alpha) \cdots (n_{\nu} + \alpha)}^{-1}
\]

for \((\mu, \nu) \neq (0, 0)\). By condition (2) of Definition 1.6, this is

\[
\ll \frac{1}{T} \left\{ \sum_{0 \leq n \leq X} \frac{1}{(n + \alpha)^{\sigma}} \right\}^{N} X^{\Omega(\alpha)N^{2}} \leq \frac{1}{T} X^{(\Omega(\alpha) + 1)N^{2}}
\]

for \(X \geq X(\alpha, \sigma_{1})\) with a positive constant \(X(\alpha, \sigma_{1})\). By this and \(H(0, 0) = 0\), the first term of (3.3) is estimated as

\[
(3.4) \quad \ll \frac{(1 + |z|^{2})^{N/2}}{T} X^{(\Omega(\alpha) + 1)N^{2}}.
\]

We consider the second term of (3.3). Let \(N = 2M\). Then we have

\[
(3.5) \quad \frac{1}{U} \int_{0}^{U} |F(u)|^{2M} du
\]

\[
= \sum_{0 \leq m_{1} \leq X} \cdots \sum_{0 \leq m_{M} \leq X} \sum_{0 \leq n_{1} \leq X} \cdots \sum_{0 \leq n_{M} \leq X} e^{2\pi i \lambda m_{1}} \cdots e^{2\pi i \lambda m_{M}} e^{-2\pi i \lambda n_{1}} \cdots e^{-2\pi i \lambda n_{M}} \frac{1}{\{(n_{1} + \alpha) \cdots (n_{M} + \alpha)\}^{2\sigma}}
\]

\[
+ O \left( \frac{1}{U} \sum_{0 \leq n_{1} \leq X} \cdots \sum_{0 \leq m_{1} \leq X} \cdots \sum_{0 \leq n_{M} \leq X} \frac{1}{(m_{1} + \alpha) \cdots (m_{M} + \alpha)} \cdot \frac{1}{(n_{1} + \alpha) \cdots (n_{M} + \alpha)} \log \frac{(m_{1} + \alpha) \cdots (m_{M} + \alpha)}{(n_{1} + \alpha) \cdots (n_{M} + \alpha)}^{-1} \right).
\]

Due to condition (11) of Definition 1.6, the equation \((m_{1} + \alpha) \cdots (m_{M} + \alpha) = (n_{1} + \alpha) \cdots (n_{M} + \alpha)\) is equivalent to \(\{m_{1}, \ldots, m_{M}\} = \{n_{1}, \ldots, n_{M}\}\). Thus the diagonal term of (3.5) is

\[
(3.6) \quad \ll M! \left( \sum_{0 \leq n \leq X} \frac{1}{(n + \alpha)^{2\sigma}} \right)^{M} \leq M! \left( \frac{c(\alpha, \sigma_{1})}{2\sigma - 1} \right)^{M}
\]

with a positive constant \(c(\alpha, \sigma_{1})\). The off-diagonal term is estimated as

\[
(3.7) \quad \ll \frac{1}{U} \left( \sum_{0 \leq n \leq X} \frac{1}{(n + \alpha)^{\sigma}} \right)^{N} X^{\Omega(\alpha)N^{2}} \ll \frac{1}{T} X^{(\Omega(\alpha) + 1)N^{2}}
\]
for \( X \geq X(\alpha, \sigma_1) \) by applying condition (2) again. Therefore we have
\[
E_2 \ll \frac{(1 + |z|^2)^{N/2}}{T} X^{(\Omega(\alpha)+1)N^2} + |z|^N \left( \frac{N}{2} \right)! \left( \frac{c(\alpha, \sigma_1)}{2 \sigma - 1} \right)^{N/2}
\]
by (3.3), (3.4), and (3.7). By the assumption \( \theta + \delta < 1/4 \), we have \( \theta + 2\delta < 1/2 - (3/4)\theta \). We take \( \eta = \{(\theta + 2\delta) + (1/2 - (3/4)\theta)\}/2 \) and \( N = 2[(\log T)^\eta] \).

Recalling \( X = \exp((\log T)^{3\theta/2}) \), \( |z| \leq 2(\log T)^\delta \), and \( \sigma \geq 1/2 + (\log T)^{-\theta} \), we obtain
\[
E_2 \ll \exp \left( -\frac{1}{2}(\log T)^{\theta/2} \right)
\]
for \( T \geq T_0 \) with a positive constant \( T_0 = T_0(\alpha, \sigma_1, \theta, \delta) \). \( \square \)

**Proposition 3.6.** Let \( \lambda \in \mathbb{R} \) and \( \alpha \in \mathcal{G} \). Then, for any \( \sigma > 1/2 \) and \( X \geq 1 \), we have
\[
\lim_{U \to \infty} \frac{1}{U} \int_0^U \psi \left( \sum_{0 \leq \eta \leq X} \frac{e^{2\pi i \eta \lambda}}{(n + \alpha)^{\sigma + i \eta}} \right) \, du = \prod_{0 \leq n \leq X} J_0(|z|(n + \alpha)^{-\sigma}).
\]

**Proof.** Let \( \gamma_n = -(2\pi)^{-1}\log(n + \alpha) \) for \( n \geq 0 \). Then \( \{\gamma_n\} \) is linearly independent over \( \mathbb{Q} \) by condition (1) of Definition 1.6. Then, by applying [15, Lemma 2], we have
\[
\frac{1}{U} \int_0^U \psi \left( \sum_{0 \leq \eta \leq X} \frac{e^{2\pi i \eta \lambda}}{(n + \alpha)^{\sigma + i \eta}} \right) \, du = \frac{1}{U} \int_0^U \prod_{0 \leq n \leq X} \psi \left( \frac{e^{2\pi i \eta \lambda}}{(n + \alpha)^{\sigma} e^{2\pi i \gamma_n \eta}} \right) \, du
\]
\[
\to \prod_{0 \leq n \leq X} \int_0^1 \psi \left( \frac{e^{2\pi i \lambda \eta}}{(n + \alpha)^{\sigma} e^{2\pi i \theta}} \right) \, d\theta
\]
as \( U \to \infty \). Moreover, we see that
\[
\int_0^1 \psi \left( \frac{e^{2\pi i \lambda \eta}}{(n + \alpha)^{\sigma} e^{2\pi i \theta}} \right) \, d\theta = \int_0^1 \exp(i|z|(n + \alpha)^{-\sigma} \cos(2\pi \varphi)) \, d\varphi
\]
by the changes of integral variables. The right-hand side is equal to \( J_0(|z|(n + \alpha)^{-\sigma}) \), and hence the result follows. \( \square \)

**Proposition 3.7.** Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Let \( \sigma_1 \) be a large fixed positive constant, and let \( \theta, \delta > 0 \). Then there exists \( T_0 = T_0(\theta) > 0 \) such that for all \( T \geq T_0 \) and for all \( \sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1] \), we have
\[
\prod_{0 \leq n \leq X} J_0(|z|(n + \alpha)^{-\sigma}) = \tilde{M}_\sigma(z; \alpha) + E_3
\]
for any \( z \in \Omega \), where \( X = \exp((\log T)^{3\theta/2}) \), and \( E_3 \) is estimated as
\[
E_3 \ll_{\sigma_1} \exp \left( -\frac{1}{2}(\log T)^{\theta/2} \right).
\]

**Proof of Theorem 3.7.** It is easily directly deduced by combining Propositions 3.3, 3.5, 3.6 and 3.7. \( \square \)
4. Proof of Theorem 1.7

We deduce Theorem 1.7 from Theorem 3.2 by approximating the characteristic function \(1_R(z)\) by functions in the class \(\Lambda\). For this, we use the following function \(F_{a,b}(x)\).

**Lemma 4.1** (Lemma 4.1 of [23]). Let

\[ K(x) = \left( \frac{\sin \pi x}{\pi x} \right)^2. \]

Then, for any \(a, b, \omega \in \mathbb{R}\) with \(a < b\) and \(\omega > 0\), there exists a continuous function \(F_{a,b,\omega} : \mathbb{R} \to \mathbb{R}\) such that the following conditions hold:

1. \(F_{a,b,\omega}(x) - 1_{[a,b]}(x) \ll K(\omega(x-a)) + K(\omega(x-b))\) for any \(x \in \mathbb{R}\),
2. \(\int_{\mathbb{R}} (F_{a,b,\omega}(x) - 1_{[a,b]}(x)) \, dx \ll \omega^{-1}\),
3. if \(|x| \geq \omega\), then \(\tilde{F}_{a,b,\omega}(x) = 0\),
4. \(\tilde{F}_{a,b,\omega}(x) \ll (b-a) + \omega^{-1}\).

Here

\[ \tilde{F}_{a,b,\omega}(x) = \int_{\mathbb{R}} F_{a,b,\omega}(u)e^{ixu} \, |du| \]

is the Fourier transformation of \(F_{a,b,\omega}(x)\) with \(|du| = (2\pi)^{-1/2} du\).

The function \(F_{a,b,\omega}(x)\) is constructed by using the (refined) Beurding–Selberg functions

\[ H(x) = \left( \frac{\sin \pi x}{\pi} \right)^2 \left\{ \sum_{m=-\infty}^{\infty} \text{sgn}(m)(x-m)^{-2} + 2x^{-1} \right\} \]

which approximate well the signum function \(\text{sgn}(x)\); see [32].

**Proof of Theorem 1.7.** We take the constants \(\lambda, \alpha, \sigma_1, \theta, \delta > 0\) as in the assumption of Theorem 3.2. For \(T \geq T_0(\alpha, \sigma_1, \theta, \delta)\), let \(R\) be any rectangle

\[ R = \{ z = x + iy \mid a \leq x \leq b, \ c \leq y \leq d \} \]

with \(b-a, \ d-c \gg (\log T)^{-\delta}\). We also take \(\omega = (2\pi)^{-1}(\log T)^{\delta}\). Note that condition (4) of Lemma 4.1 deduces \(\tilde{F}_{a,b,\omega}(x) \ll (b-a)\) in this case. Then, we define

\[ \Phi(z) = F_{a,b,\omega}(x)F_{c,d,\omega}(y). \]

Since we have

\[ \Lambda = \{ f \in L^1(\mathbb{C}) \mid f \text{ is continuous and } \tilde{f} \in L^1(\mathbb{C}) \} \]

by [17, Section 9], Lemma 4.1 deduces that \(\Phi\) is a member of the class \(\Lambda\). Moreover, we have

\[ (4.1) \quad \Phi(z) - 1_R(z) \ll K(\omega(x-a)) + K(\omega(x-b)) + K(\omega(y-c)) + K(\omega(y-d)) \]

and \(\tilde{\Phi}(z) \ll (b-a)(d-c) = \mu_2(R)\) also by Lemma 4.1.
In the above setting, we apply Theorem \[3.2\]. Let \( \epsilon_1 > 0 \) arbitrarily, and we take
\[ T_1 = T_1(\alpha, \sigma_1, \theta, \delta, \epsilon_1) \geq T_0(\alpha, \sigma_1, \theta, \delta) \]
so that \( (\log T_1)^{-\theta} < \epsilon_1 \) holds. Then, for all \( T \geq T_1 \) and for all \( 1/2 + \epsilon_1 \leq \sigma \leq \sigma_1 \), we have

\[
\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt = \int_{\mathbb{C}} \Phi(z) M_\sigma(z; \alpha) \, |dz| + E,
\]
where
\[
E \ll_{\lambda, \sigma_1} \exp\left( -\frac{1}{2} (\log T)^{\theta/2} \right) \int_{\Omega} \left| \tilde{M}_\sigma \right| \, |z| + \int_{C \setminus \Omega} \left| \tilde{M}_\sigma \right| \, |z|.
\]
By Lemma \[4.1\] we see that
\[
E \ll_{\lambda, \sigma_1} \exp\left( -\frac{1}{2} (\log T)^{\theta/2} \right) (\log T)^{2 \delta} \mu_2(R) + 0 \ll \mu_2(R) (\log T)^{-\delta}
\]
for \( T \geq T_2 \) with a constant \( T_2 = T_2(\alpha, \sigma_1, \theta, \delta, \epsilon_1) \geq T_1 \). We estimate the error between the left-hand side of (4.2) and
\[
P_{\sigma, T}(R; \lambda, \alpha) = \frac{1}{T} \int_0^T 1_R(L(\lambda, \alpha, \sigma + it)) \, dt.
\]
Applying inequality (4.1), it is
\[
\ll \frac{1}{T} \int_0^T K(\omega(\text{Re } L(\lambda, \alpha, \sigma + it) - a)) \, dt
\]
\[
+ \frac{1}{T} \int_0^T K(\omega(\text{Re } L(\lambda, \alpha, \sigma + it) - b)) \, dt
\]
\[
+ \frac{1}{T} \int_0^T K(\omega(\text{Im } L(\lambda, \alpha, \sigma + it) - c)) \, dt
\]
\[
+ \frac{1}{T} \int_0^T K(\omega(\text{Im } L(\lambda, \alpha, \sigma + it) - d)) \, dt.
\]
Here, we consider only the term
\[
\frac{1}{T} \int_0^T K(\omega(\text{Re } L(\lambda, \alpha, \sigma + it) - a)) \, dt,
\]
since the other terms are estimated in a similar way. We have
\[
K(\omega x) = \frac{2}{\omega^2} \int_0^\omega (\omega - u) \cos(2\pi xu) \, du = \frac{2}{\omega^2} \text{Re} \int_0^\omega (\omega - u) e^{2\pi i xu} \, du,
\]
and hence (4.4) is estimated as
\[
\ll \frac{1}{\omega^2} \int_0^\omega (\omega - u) \left| \frac{1}{T} \int_0^T \exp(2\pi i u (\text{Re } L(\lambda, \alpha, \sigma + it) - a)) \, dt \right| \, du
\]
\[
= \frac{1}{\omega^2} \int_0^\omega (\omega - u) \left| \frac{1}{T} \int_0^T \exp(2\pi i u \text{Re } L(\lambda, \alpha, \sigma + it)) \, dt \right| \, du
\]
\[
\ll_{\lambda, \sigma_1} \frac{1}{\omega^2} \int_0^\omega (\omega - u) \left\{ \left| \tilde{M}_\sigma(2\pi u; \alpha) \right| + \exp \left( -\frac{1}{2} (\log T)^{\theta/2} \right) \right\} \, du
\]
\[
\ll_{\alpha, \epsilon_1} \frac{1}{\omega} + \exp \left( -\frac{1}{2} (\log T)^{\theta/2} \right) \ll (\log T)^{-\delta}.
\]
Here we used Theorem \ref{thm3.1} for the second inequality and used the fact that $\tilde{M}_\sigma(z; \alpha)$ rapidly decreases for the third inequality. Therefore we have

\begin{equation}
\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt - P_{\sigma, T}(R; \lambda, \alpha) \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} (\log T)^{-\delta}.
\end{equation}

(4.5)

At the end of the proof, we estimate the difference

$$\int_C \Phi(z) M_\sigma(z; \alpha) \, |dz| - \int_R M_\sigma(z; \alpha) \, |dz| = \int_C (\Phi(z) - 1_{R(z)}) M_\sigma(z; \alpha) \, |dz|$$

coming from the right-hand side by applying (4.1) again. We have

\begin{equation}
\int_C K(\omega(x - a)) M_\sigma(z; \alpha) \, |dz| = \int_R K(\omega(x - a)) m_\sigma(x; \alpha) \, |dx|,
\end{equation}

(4.6)

where

$$m_\sigma(x; \alpha) = \int_R M_\sigma(x + iy; \alpha) \, |dy|.$$ 

By estimate \eqref{2.5}, the function $m_\sigma(x; \alpha)$ is estimated as

$$m_\sigma(x; \alpha) \ll_{\alpha, \epsilon_1} 1.$$ 

Hence we obtain that (4.6) is

$$\ll_{\alpha, \epsilon_1} \int_R K(\omega(x - a)) \, |dx| \ll \omega^{-1} = (\log T)^{-\delta}.$$ 

The remaining terms are estimated similarly. We have

\begin{equation}
\int_C \Phi(z) M_\sigma(z; \alpha) \, |dz| - \int_R M_\sigma(z; \alpha) \, |dz| \ll_{\alpha, \epsilon_1} (\log T)^{-\delta}.
\end{equation}

(4.7)

Therefore, we conclude

$$P_{\sigma, T}(R; \lambda, \alpha) = \int_R M_\sigma(z; \alpha) \, |dz| + O_{\lambda, \alpha, \sigma_1, \epsilon_1} \left( (\mu_2(R) + 1)(\log T)^{-\delta} \right)$$

by estimates (4.3), (4.5), and (4.7). We take $\theta = \epsilon/2$ and $\delta = 1/4 - \epsilon$ for each $0 < \epsilon < 1/4$. Then we have $\theta, \delta > 0$ and $\theta + \delta < 1/4$, and the desired result follows.

\section{5. Further results on the density function}

We have obtained several results on the function $M_\sigma(z; \alpha)$ in Section \ref{2}. In this section, we study it as a function of $\sigma$. The ultimate goal of this section is the following estimate.

\textbf{Proposition 5.1.} Let $0 < \alpha \leq 1$ and $\sigma > 1/2$. Then there exists a positive constant $c(\sigma)$ such that for any integers $k, l, m \geq 0$, we have

$$\frac{\partial^{k+l+m}}{\partial x^k \partial y^l \partial \sigma^m} M_\sigma(z; \alpha) \ll_{k, l, m} \exp \left( -c(\sigma)|z|^{1/\sigma} \right), \quad |z| \to \infty$$

with $z = x + iy$. 

To begin with, we recall that

\[ J_0(|z|(n + \alpha)\sigma) = \int_0^1 \psi_z((n + \alpha)^{-\sigma}e^{2\pi\theta}) d\theta \]

\[ = \int_0^1 \exp\{ix(n + \alpha)^{-\sigma}\cos(2\pi\theta) + iy(n + \alpha)^{-\sigma}\sin(2\pi\theta)\} d\theta. \]

We then define

(5.1) \[ \tilde{M}_n(s, z_1, z_2; \alpha) \]

\[ = \int_0^1 \exp\{iz_1(n + \alpha)^{-s}\cos(2\pi\theta) + iz_2(n + \alpha)^{-s}\sin(2\pi\theta)\} d\theta \]

\[ = 1 - \frac{z_1^2 + z_2^2}{4}(n + \alpha)^{-2s} \]

\[ + \int_0^1 \sum_{k=3}^{\infty} i^k (n + \alpha)^{-ks}(z_1\cos(2\pi\theta) + z_2\sin(2\pi\theta))^k d\theta \]

for \( s, z_1, z_2 \in \mathbb{C} \) with \( \text{Re}\ s > 0 \). Note that \( \tilde{M}_n(\sigma, x, y; \alpha) = J_0(|x + iy|(n + \alpha)^\sigma) \) if \( \sigma, x, y \in \mathbb{R} \). We investigate the function

(5.2) \[ \tilde{M}(s, z_1, z_2; \alpha) = \prod_{n=0}^{\infty} \tilde{M}_n(s, z_1, z_2; \alpha). \]

**Lemma 5.2.** If we fix two of the variables, the local parts \( \tilde{M}_n(s, z_1, z_2; \alpha) \) are holomorphic with respect to the remaining variable for any \( s, z_1, z_2 \in \mathbb{C} \) with \( \text{Re}\ s > 0 \). Let \( K \) be any compact subset on the half plane \( \{\text{Re}\ s > 1/2\} \), and let \( K_1, K_2 \) be any compact subsets on \( \mathbb{C} \). Then the infinite product \( (5.2) \) uniformly converges on \( K \times K_1 \times K_2 \). Therefore, if we again fix two of the variables, the function \( \tilde{M}(s, z_1, z_2; \alpha) \) is holomorphic with respect to the remaining variable for any \( s, z_1, z_2 \in \mathbb{C} \) with \( \text{Re}\ s > 1/2 \).

**Proof.** The first statement is clear from the definition \( (5.1) \). Assume that \( (s, w_1, w_2) \) varies in \( K \times K_1 \times K_2 \), and let \( \sigma_0 \) be the smallest real parts of \( s \in K \). Then there exists a sufficiently large constant \( N = N(K, K_1, K_2; \alpha) \) such that

\[ \tilde{M}_n(s, z_1, z_2; \alpha) = 1 + O_{K_1, K_2} ( (n + \alpha)^{-2\sigma_0} ) \]

for \( n \geq N \) by \( (5.1) \). Since the series \( \sum_{n}(n + \alpha)^{-2\sigma_0} \) converges due to \( \sigma_0 > 1/2 \), the second statement follows. The last statement directly follows from the preceding two statement. \( \square \)

In the remaining part of this section, we prove the following estimate.

**Lemma 5.3.** Let \( 0 < \alpha \leq 1 \) and \( \sigma > 1/2 \). Then there exist positive constants \( K(\sigma; \alpha) \) and \( c(\sigma) \) such that for any \( x, y \in \mathbb{R} \) with \( |x| + |y| \geq K(\sigma; \alpha) \), we have

\[ \left| \tilde{M}(s, z_1, z_2; \alpha) \right| \leq \exp \left( -c(\sigma)(|x| + |y|)^{1/\text{Re}(s)} \right) \]

for any \( s, z_1, z_2 \in \mathbb{C} \) with \( |s - \sigma| < 1/(x^2 + y^2) \), \( |z_1 - x| < 1/2 \), \( |z_2 - y| < 1/2 \).
Proof. Let \( n_0 \) be a real number satisfying
\[
n_0 + \alpha = \left( \frac{|x| + |y|}{c_0} \right)^{1/\text{Re}(s)},\]
where \( c_0 \) is a positive absolute constant suitably chosen later. At first, we take \( c_0 < 1/2 \) and \( K(\sigma; \alpha) \geq 1 \). Then, for \( n \geq n_0 \), we have
\[
(|x| + |y|)(n + \alpha)^{-\text{Re}(s)} \leq (|x| + |y|)n_0^{-\text{Re}(s)} = c_0.
\]
Hence we see that
\[
\left| \frac{z_1^2 + z_2^2}{4} (n + \alpha)^{-2s} \right| \leq c_0^2,
\]
\[
\left| \sum_{k=3}^{\infty} \frac{\iota^k}{k!} (n + \alpha)^{-ks} (z_1 \cos(2\pi\theta) + z_2 \sin(2\pi\theta))^k \right| \leq 4c_0^3
\]
for \( n \geq n_0 \) and \( |z_1 - x| < 1/2, |z_2 - y| < 1/2 \). By formula (5.1), we can define
\[
\text{Log} \, \overline{M}_n(s, z_1, z_2; \alpha)
\]
and obtain
\[
(5.3) \quad \text{Log} \, \overline{M}_n(s, z_1, z_2; \alpha) = -\frac{z_1^2 + z_2^2}{4} (n + \alpha)^{-2s} + O \left( (|x| + |y|)^3 (n + \alpha)^{-3\text{Re}(s)} \right).
\]
Since we see that
\[
\left| \frac{z_1^2 + z_2^2}{4} (n + \alpha)^{-2s} - \frac{x^2 + y^2}{4} (n + \alpha)^{-2\text{Re}(s)} \right| \leq \left| \frac{z_1^2 + z_2^2}{4} - \frac{x^2 + y^2}{4} \right| (n + \alpha)^{-2\text{Re}(s)} + \frac{x^2 + y^2}{4} \left| (n + \alpha)^{-2s} - (n + \alpha)^{-2\text{Re}(s)} \right|
\]
\[
\ll (|x| + |y|)(n + \alpha)^{-2\text{Re}(s)} + \log(n + \alpha)(n + \alpha)^{-2\text{Re}(s)}
\]
for \( |s - \sigma| < 1/(x^2 + y^2) \), there exists an absolute constant \( A > 0 \) such that
\[
\left| \text{Log} \, \overline{M}_n(s, z_1, z_2; \alpha) + \frac{x^2 + y^2}{4} (n + \alpha)^{-2\text{Re}(s)} \right| \leq A \left( (|x| + |y|)(n + \alpha)^{-\text{Re}(s)} + (|x| + |y|)^{-1} \right) (|x| + |y|)^2 (n + \alpha)^{-2\text{Re}(s)}
\]
\[
+ A \log(n + \alpha)(n + \alpha)^{-2\text{Re}(s)}.
\]
Then, assuming further \( c_0 < (32A)^{-1} \) and \( K(\sigma, \alpha) \geq 32A \), we have
\[
A \left( (|x| + |y|)(n + \alpha)^{-\text{Re}(s)} + (|x| + |y|)^{-1} \right) \leq A(c_0 + K^{-1}) \leq \frac{1}{16}
\]
for \( |x| + |y| \geq K(\sigma; \alpha) \). Hence, by (5.3), we obtain
\[
\text{Re} \, \text{Log} \, \overline{M}_n(s, z_1, z_2; \alpha) \leq -\frac{(|x| + |y|)^2}{16} (n + \alpha)^{-2\text{Re}(s)} + A \log(n + \alpha)(n + \alpha)^{-2\text{Re}(s)}
\]
for \( n \geq n_0 \), and moreover,
\[
(5.4) \quad \prod_{n \geq n_0} \overline{M}_n(s, z_1, z_2; \alpha) \leq \exp \left( -\frac{(|x| + |y|)^2}{16} \sum_{n \geq n_0} (n + \alpha)^{-2\text{Re}(s)} \right.
\]
\[
\left. + A \sum_{n \geq n_0} \log(n + \alpha)(n + \alpha)^{-2\text{Re}(s)} \right).
\]
Since we have
\[ |\text{Re}(s) - \sigma| \leq |s - \sigma| \leq \frac{1}{x^2 + y^2} \leq \frac{4}{K(\sigma; \alpha)^2}, \]
we deduce
\[ 2 \text{Re}(s) - 1 > \sigma - 1/2 > 0 \]
if we take \( K(\sigma; \alpha) > 4/\sqrt{2\sigma - 1}. \) Then, for the first term of (5.1), we have
\[ (|x| + |y|)^2 \sum_{n \geq n_0} (n + \alpha)^{-2 \text{Re}(s)} \geq \frac{1}{2 \text{Re}(s) - 1} (n_0 + \alpha)^{1 - 2 \text{Re}(s)} \]
\[ \geq \frac{2 \text{Re}(s) - 1}{2 \text{Re}(s) - 1} (|x| + |y|)^{1/\text{Re}(s)}. \]
On the other hand, we have
\[ \sum_{n \geq n_0} \log(n + \alpha)(n + \alpha)^{-2 \text{Re}(s)} \]
\[ \leq \frac{1}{2 \text{Re}(s) - 1} \log(n_0 + \alpha)(n_0 + \alpha)^{1 - 2 \text{Re}(s)} \]
\[ \times \left( 1 + \frac{2 \text{Re}(s) - 1}{n_0 + \alpha} + \frac{1}{2 \text{Re}(s) - 1} \log(n_0 + \alpha) \right). \]
Therefore, we see that there exists \( K(\sigma; \alpha) > 0 \) such that for any \( |x| + |y| \geq K(\sigma; \alpha), \)
\[ \sum_{n \geq n_0} \log(n + \alpha)(n + \alpha)^{-2 \text{Re}(s)} \leq \Theta \]
with a positive constant \( \Theta < (16\lambda)^{-1}. \) Thus (5.1) deduces
\[ (5.5) \quad \prod_{n \geq n_0} \tilde{M}_n(s, z_1, z_2; \alpha) \leq \exp \left( -c(\sigma)(|x| + |y|)^{1/\text{Re}(s)} \right), \]
where \( c(\sigma) \) is a positive constant depending only on \( \sigma. \)

Then, we estimate \( \tilde{M}_n(s, z_1, z_2; \alpha) \) for \( n < n_0. \) By (5.1), we have
\[ \left| \tilde{M}_n(s, z_1, z_2; \alpha) \right| \leq \exp \left\{ (\text{Im}(z_1) + \text{Im}(z_2))(n + \alpha)^{-\text{Re}(s)} \right\} \]
\[ \leq \exp \left( (n + \alpha)^{-\text{Re}(s)} \right) \]
since \( |z_1 - x| < 1/2 \) and \( |z_2 - y| < 1/2 \) with \( x, y \in \mathbb{R}. \) Therefore the contribution from the terms for \( n < n_0 \) is estimated as
\[ \prod_{n < n_0} \tilde{M}_n(s, z_1, z_2; \alpha) \leq \exp \left( \sum_{n < n_0} (n + \alpha)^{-\text{Re}(s)} \right) \]
\[ \leq \exp \left( c'(\sigma; \alpha)(|x| + |y|)^{1/2\text{Re}(s)} \right), \]
where \( c'(\sigma; \alpha) \) is a suitable positive constant. Hence we have the desired result from estimates (5.5) and (5.6). \( \Box \)
Proof of Proposition 5.1. By Lemma 5.2, we can apply Cauchy’s integral formula for the function $\tilde{M}(s, z_1, z_2; \alpha)$. Proposition 5.1 is easily deduced from this and the estimate of Lemma 5.3.

\[\square\]

Corollary 5.4. Let $0 < \alpha \leq 1$ and $\sigma > 1/2$. Then, for any integer $m \geq 0$, the function

\[(5.7) \quad \frac{\partial^m}{\partial \sigma^m} M_\sigma(z; \alpha)\]

belongs to $\mathcal{S}$ as a function in $x$ and $y$ with $z = x + iy$.

Proof. By Proposition 5.1 the function

\[(5.8) \quad \frac{\partial^m}{\partial \sigma^m} \tilde{M}_\sigma(z; \alpha)\]

belongs to $\mathcal{S}$. Thus we have

\[
\frac{\partial^m}{\partial \sigma^m} M_\sigma(z; \alpha) = \frac{\partial^m}{\partial \sigma^m} \int_C \tilde{M}_\sigma(w; \alpha) \psi_{-z}(w) |dw| = \int_C \frac{\partial^m}{\partial \sigma^m} \tilde{M}_\sigma(w; \alpha) \psi_{-z}(w) |dw|.
\]

In other words, the Fourier inverse of function (5.8) is equal to function (5.7). Hence function (5.7) belongs to $\mathcal{S}$. \[\square\]

6. Proof of Theorem 1.8

Finally we prove Theorem 1.8. The following proposition is an analogue of [13, Theorem 1.1.3] for Lerch zeta-functions and is a key for the proof of Theorem 1.8.

Proposition 6.1. Let $\lambda \in \mathbb{R}$ and $\alpha \in \mathcal{S}$. Let $\sigma_1$ be a large fixed positive constant. Let $\epsilon_1 > 0$ be a small fixed real number. Then there exists $T_0 = T_0(\lambda, \alpha, \sigma_1, \epsilon_1) > 0$ such that for all $T \geq T_0$ and for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, we have

\[(6.1) \quad \frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)| dt = \int_C \log |z|M_\sigma(z; \alpha)|dz| + O_{\lambda, \alpha, \sigma_1, \epsilon_1}((\log T)^{-A})\]

with an absolute constant $A > 0$.\]

Before the proof of Proposition 6.1 we estimate the upper bound of the integral

\[
\int_0^T \log^2 |L(\lambda, \alpha, \sigma + it)| dt
\]

for $\sigma > 1/2$ in Section 6.1. It is used to bound the error term coming from the left-hand side of (6.1). We then prove Proposition 6.1 in Section 6.2. Proposition 6.1 is connected to Theorem 1.8 by Lemma 6.8 which is a consequence of well-known Littlewood’s lemma [22, Lemma 8.4.9]; see also [31, p. 221]. The proof of Theorem 1.8 is completed in Section 6.3.
6.1. Mean square of the logarithm of the Lerch zeta-function.

**Proposition 6.2.** Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. Let $\sigma_1 > 0$ be a large fixed real number. Let $\epsilon_1 > 0$ be a small fixed real number. Then there exists $T_0 = T_0(\lambda, \alpha, \sigma_1, \epsilon_1) > 0$ such that for all $T \geq T_0$ and for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, we have

$$\int_0^T \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} T.$$

The proof is based on the method in [21, Section 5]. Let

$$f(z) = \alpha^2 L(\lambda, \alpha, z).$$

By [22, Lemma 8.5.2], the function $f(z)$ satisfies

$$(6.2) \quad 1 - \frac{\alpha}{x - 1} < |f(z)| < 1 + \frac{\alpha}{x - 1}, \quad z = x + iy$$

for $x > 1 + \alpha$. Moreover, let $\sigma_0 = \max(\sigma_1, 3)$, and let

$$r = \sigma_0 - \frac{1}{2} \left( \sigma + \frac{1}{2} \right), \quad \delta = \frac{1}{\sigma_1 + 4} \left( \sigma - \frac{1}{2} \right), \quad r_1 = r - \delta, \quad r_2 = r - 2\delta.$$

Note that we have $r > r_1 > r_2 > 0$, $0 < \delta < 1$, and $r + 2\delta < \sigma_0$. We define $s_0 = s_0(t) = \sigma_0 + it$. If $t \geq \sigma_0$, the function $f(z)$ is analytic in $|z - s_0| \leq r + 2\delta$ and $1/2 \leq |f(s_0)| \leq 3/2$ by inequality (6.2). Then, for $t \geq \sigma_0$ and $0 < u \leq r + 2\delta$, we define

$$M_u(t) = \max_{|z - s_0(t)| \leq u} \left| \frac{f(z)}{f(s_0(t))} \right| + 3$$

and

$$N_u(t) = \sum_{|\rho - s_0(t)| \leq u} 1,$$

where $\rho$ runs through the zeros of $L(\lambda, \alpha, s)$. We begin with the following formula.

**Lemma 6.3.** Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. Let $\sigma_1 > 0$ be a large fixed real number. Let $\epsilon_1 > 0$ be a small fixed real number. Then for all $t \geq \sigma_0$ and for all $1/2 + \epsilon_1 \leq \sigma \leq \sigma_1$, we have

$$\log |L(\lambda, \alpha, \sigma + it)| = \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| + O_{\alpha, \sigma_1, \epsilon_1} (\log M_r(t)),$$

where $\rho$ runs through the zeros of $L(\lambda, \alpha, s)$.

**Proof.** We apply [13, Lemma 2.2.1]. Then we have for $|z - s_0| \leq r_2$,

$$(6.3) \quad \left| \frac{f'(z)}{f(z)} - \sum_{|\rho - s_0| \leq r_1} \frac{1}{z - \rho} \right| \leq \frac{36r_1}{(r_1 - r_2)^2} \left\{ \log M_r(t) + N_{r_1}(t) \log \left( \frac{r_1}{r - r_1} + 1 \right) \right\} \leq \frac{36\sigma_0}{\delta^2} \left( \log M_r(t) + N_{r_1}(t) \frac{\sigma_0}{\delta} \right).$$
Jensen’s formula yields that the equation
\[ \int_0^r \frac{N_x(t)}{x} \, dx + \log |f(s_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(s_0 + re^{i\theta})| \, d\theta \]
holds, and its left-hand side is estimated as
\[ \geq \int_{r_1}^{r} \frac{N_x(t)}{x} \, dx + \log |f(s_0)| \geq N_{r_1}(t) \frac{r - r_1}{r} + \log |f(s_0)|. \]
Also, the right-hand side is
\[ \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(s_0 + re^{i\theta})}{f(s_0)} \right| \, d\theta + \log |f(s_0)| \leq \log M_r(t) + \log |f(s_0)|. \]
Hence we have
\[ (6.4) \quad N_{r_1}(t) \leq \frac{r}{r - r_1} \log M_r(t) \leq \frac{\sigma_0}{\delta} \log M_r(t). \]
By (6.3) and (6.4), we obtain
\[ \frac{f'(z)}{f(z)} - \sum_{|\rho - s_0| \leq r_1} \frac{1}{z - \rho} \ll_{\sigma_1, \epsilon_1} \log M_r(t) \]
for all \(|z - s_0| \leq r_2\), since \(\sigma_0 \ll \sigma_1\) and \(\delta \gg \sigma_1, \epsilon_1\). Note that \(|\sigma + it - s_0| = \sigma_0 - \sigma \leq r_2\). Therefore, integrating from \(s_0\) to \(\sigma + it\), we have
\[ \log |f(\sigma + it)| - \sum_{|\rho - s_0| \leq r_1} \log |\sigma + it - \rho| \]
\[ = \log |f(s_0)| - \sum_{|\rho - s_0| \leq r_1} \log |s_0 - \rho| + \int_{s_0}^{\sigma + it} \left( \frac{f'(z)}{f(z)} - \sum_{|\rho - s_0| \leq r_1} \frac{1}{z - \rho} \right) \, dz \]
\[ = \log |f(s_0)| - \sum_{|\rho - s_0| \leq r_1} \log |s_0 - \rho| + O_{\sigma_1, \epsilon_1}(\log M_r(t)). \]
Since \(1/2 \leq |f(s_0)| \leq 3/2\), we have \(\log |f(s_0)| \ll 1\). Also, since \(1 \leq |s_0 - \rho| \leq \sigma_0\) for all zeros \(\rho\) with \(|s_0 - \rho| < r_1\), we have
\[ \sum_{|\rho - s_0| \leq r_1} \log |s_0 - \rho| \ll_{\sigma_1} N_{r_1}(t) \ll_{\sigma_1, \epsilon_1} \log M_r(t) \]
by (6.4). By the definition of \(f(z)\), the desired result follows. \( \square \)

By Lemma (6.3) we have
\[ (6.5) \quad \int_T^{2T} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \]
\[ \ll \int_T^{2T} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| \right)^2 \, dt + \int_T^{2T} \log^2 M_r(t) \, dt \]
for \(T \geq \sigma_0\). Next we estimate two integrals of the right-hand side of (6.5).
Lemma 6.4. Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Let \( \sigma_1 \) be a large fixed positive constant. Let \( \epsilon_1 > 0 \) be a small fixed real number. Then there exists \( T_0 = T_0(\lambda, \alpha, \sigma_1, \epsilon_1) > 0 \) such that for all \( T \geq T_0 \) and for all \( 1/2 + \epsilon_1 \leq \sigma \leq \sigma_1 \), we have

\[
\int_{T}^{2T} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| \right)^2 dt \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} T,
\]

where \( \rho \) runs through the zeros of \( L(\lambda, \alpha, s) \).

Proof. We have

\[
\int_{T}^{2T} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| \right)^2 dt 
\leq \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \int_{n}^{n+1} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| \right)^2 dt.
\]

According to the method in [21], let

\[
D_n = \bigcup_{l=0}^{\lfloor 1/\sqrt{3} \rfloor + 1} D_r \left( s_0(t) + l\sqrt{3} \right),
\]

where \( D_r(c) = \{ z \in \mathbb{C} \mid |z - c| \leq r \} \). Then we see that

\[
D_n \supset \bigcup_{n \leq t \leq n + 1} D_{r_1}(s_0(t)),
\]

and therefore,

\[
\int_{n}^{n+1} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\sigma + it - \rho| \right)^2 dt 
\leq \int_{n}^{n+1} \left( \sum_{\rho \in D_n} \log |\sigma + it - \rho| \right)^2 dt 
\leq \left( \sum_{\rho \in D_n} \left( \int_{n}^{n+1} \log^2 |\sigma + it - \rho| dt \right)^{1/2} \right)^2
\]

due to Minkowski’s inequality. If \( n \leq t \leq n + 1 \) and \( \rho = \beta + i\gamma \in D_n \), we have

\[
|t - \gamma| \leq |\sigma + it - \rho| \leq c
\]

for a constant \( c = c(\sigma_1) > 1 \). Thus

\[
\log^2 |\sigma + it - \rho| \leq \log^2 |t - \gamma| + \log^2 c.
\]

Moreover, we see that

\[
\int_{n}^{n+1} \log^2 |t - \gamma| dt \leq \int_{n-r-\gamma}^{n+r+2-\gamma} \log^2 x \, dx \leq \int_{-2r-2}^{-2r-2} \log^2 x \, dx \ll_{\sigma_1} 1,
\]
since \( n - 1 \leq \gamma \leq n + 2 + r \) for \( \rho = \beta + i\gamma \in \mathcal{D}_n \). From the above, we deduce that the left-hand side of (6.6) is

\[
(6.7) \quad \int_{T}^{2T} \left( \sum_{|\rho - s_0(t)| \leq \tau_1} \log |\sigma + it - \rho| \right)^2 dt \ll_{\sigma_1} \sum_{n=T}^{[2T]+1} \left( \sum_{\rho \in \mathcal{D}_n} 1 \right)^2 \\
\leq \sum_{n=T}^{[2T]+1} \left( \sum_{l=0}^{[1/\sqrt{3}]+1} N_r \left( n + l\sqrt{3} \right) \right)^2.
\]

By an argument similar to (6.4), we see that

\[
N_r \left( n + l\sqrt{3} \right) \leq \frac{\sigma_0}{\delta} \log M_{r+\delta} \left( n + l\sqrt{3} \right).
\]

Applying this inequality, we obtain

\[
(6.8) \quad \sum_{n=T}^{[2T]+1} \left( \sum_{l=0}^{[1/\sqrt{3}]+1} N_r \left( n + l\sqrt{3} \right) \right)^2 \ll_{\sigma_1, \epsilon_1} \sum_{n=T}^{[2T]+1} \log^2 \left( 2 \max_{z \in E_n} |f(z)| + 3 \right),
\]

where

\[ E_n = \bigcup_{l=0}^{[1/\sqrt{3}]+1} D_{r+\delta} \left( \sigma_0 + i \left( n + l\sqrt{3} \right) \right). \]

We take \( s_n = \sigma_n + it_n \in E_n \) so that \( |f(z)| \) takes the maximum value at \( s_n \). The function \( F(x) = \log^2(x + 3) \) is convex for \( x \geq 0 \). Hence we have

\[
(6.9) \quad \sum_{n=T}^{[2T]+1} \log^2 \left( 2 \max_{z \in E_n} |f(z)| + 3 \right) = \sum_{n=T}^{[2T]+1} F(2|f(s_n)|) \\
\ll TF \left( \frac{1}{T} \sum_{n=T}^{[2T]+1} 2|f(s_n)| \right) \\
\leq TF \left( \frac{2}{T^2} \sum_{n=T}^{[2T]+1} |f(s_n)|^2 \right)^{1/2}.
\]

The remaining work is to estimate \( \sum_n |f(s_n)|^2 \). For this, we define

\[ S_j = \{ s_n \mid n \equiv j \mod (4|r + 2\delta| + 6) \}. \]

Then we have

\[
(6.10) \quad \sum_{n=T}^{[2T]+1} |f(s_n)|^2 \ll_{\sigma_1} \sum_{s_n \in S_j \mid T \leq n \leq [2T]+1} |f(s_n)|^2.
\]

Note that \( f(z) \) is regular for \( |z - (\sigma_0 + it_n)| \leq r + 2\delta \) and \( |s_n - (\sigma_0 + it_n)| \leq r + \delta \). Hence, the inequality of [31] Lemma in p. 256] deduces

\[
|f(s_n)|^2 \leq \frac{1}{\pi \delta^2} \int_{D_{r+2\delta}(\sigma_0+it_n)} |f(z)|^2 \, dx \, dy.
\]
Moreover, if \( s_m, s_n \in S_j \) with \( m > n \), then we have
\[
|t_m - t_n| \geq \{m - (r + \delta)\} - \{n + (\lfloor \sqrt{j} \rfloor + 1)\sqrt{j} + (r + \delta)\} > 2r + 4\delta.
\]
Thus \( D_{r + 2\delta}(\sigma_0 + it_m) \) and \( D_{r + 2\delta}(\sigma_0 + it_m) \) are disjoint, and therefore,
\[
(6.11) \quad \sum_{s_n \in S_j, [T] \leq n \leq \lfloor 2T \rfloor + 1} |f(s_n)|^2 \leq \frac{1}{\pi \delta^2} \int_{\sigma_0 - (r + 2\delta)}^{\sigma_0 + (r + 2\delta)} \int_{T-1-(2r+3\delta)}^{2T+3+(2r+3\delta)} |f(x + iy)|^2 \, dy \, dx
\]
for a positive constant \( c_1 = c_1(\sigma_1) \). With regard to the inner integral, we recall that for all \( \sigma \geq 1/2 + \epsilon_1 \),
\[
\int_0^T |L(\lambda, \alpha, \sigma + it)|^2 \, dt \ll_{\lambda, \alpha, \epsilon_1} T
\]
for \( T \geq T_0 \) with a positive constant \( T_0 = T_0(\lambda, \alpha, \epsilon_1) \) by [22] Theorem 3.3.1. Hence we see that
\[
(6.12) \quad \int_{\epsilon_1}^{2\epsilon_1 + c_1 \epsilon_1} \int_{T-1-(2r+3\delta)}^{2T+3+(2r+3\delta)} |L(\lambda, \alpha, x + iy)|^2 \, dy \, dx \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} T.
\]
From the above \((6.7) - (6.12)\), we conclude
\[
\int_T^{2T} \left( \sum_{|\rho - s_0(t)| \leq r_1} \log |\rho + it - \rho| \right)^2 \, dt \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} T
\]
as desired.

For the second integral of the right-hand side of \((6.5)\), we obtain the following estimate.

**Lemma 6.5.** Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha < 1 \). Let \( \sigma_1 \) be a large fixed positive constant. Let \( \epsilon_1 > 0 \) be a small fixed real number. Then there exists \( T_0 = T_0(\lambda, \alpha, \sigma_1, \epsilon_1) > 0 \) such that for all \( T \geq T_0 \) and for all \( 1/2 + \epsilon_1 \leq \sigma \leq \sigma_1 \), we have
\[
\int_T^{2T} \log^2 M_r(t) \, dt \ll_{\lambda, \alpha, \sigma_1, \epsilon_1} T.
\]

**Proof.** We find that the function \( G(x) = \log^2 x \) is convex for \( x > e \) and \( M_r(t)^2 > e \). Then, applying Jensen’s inequality, we have
\[
\int_T^{2T} \log^2 M_r(t) \, dt \ll \int_T^{2T} \log^2 M_r(t)^2 \, dt \leq T \log^2 \left( \frac{1}{T} \int_T^{2T} M_r(t)^2 \, dt \right).
\]
Also we have
\[
\int_T^{2T} M_r(t)^2 \, dt \leq \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} \int_n^{n+1} M_r(t)^2 \, dt.
\]
Let
\[
F_n = \bigcup_{n \leq t \leq n+1} D_r(s_0(t)),
\]
and let \( s'_n \in F_n \) be a point at which \(|f(z)|\) takes the maximum value. Then we have

\[
\sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} M_r(t)^2 dt \ll \sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} |f(s'_n)|^2 + T,
\]

and by a similar argument in the proof of Lemma 6.3 we see that

\[
\sum_{n=\lfloor T \rfloor}^{\lfloor 2T \rfloor + 1} |f(s'_n)|^2 \ll_{\lambda, \alpha, \sigma_1, \sigma_2} T
\]

for \( T \geq T_0 \) with a positive constant \( T_0 = T_0(\lambda, \alpha, \sigma_1) \). Hence the result follows.

By Lemmas 6.4 and 6.5, we have

\[
\int_{2T}^{T} \log^2 |L(\lambda, \alpha, \sigma + it)| dt \ll T
\]

for large \( T \). To finish the proof of Proposition 6.2, we need an estimate for small \( T \).

**Lemma 6.6.** Let \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Let \( \sigma_1 \) be a large fixed positive constant. Let \( \epsilon_1 > 0 \) be a small fixed real number. Then for any fixed constant \( T_0 > 0 \), and for all \( 1/2 + \epsilon_1 \leq \sigma \leq \sigma_1 \), we have

\[
\int_0^{T_0} \log^2 |L(\lambda, \alpha, \sigma + it)| dt \ll 1.
\]

The implied constant depends only on \( \lambda, \alpha, \sigma_1, \epsilon_1, \) and \( T_0 \).

**Proof.** Let \( \rho_1, \ldots, \rho_n \) be all zeros of \( L(\lambda, \alpha, s) \) in the rectangle \( 1/2 + \epsilon_1 \leq \sigma \leq \sigma_1, 0 \leq t \leq T_0 \). Here \( n \geq 0 \) is a finite integer determined from \( \lambda, \alpha, \sigma_1, \epsilon_1, \) and \( T_0 \). We take a positive real number \( r_0 = r_0(\lambda, \alpha, \sigma_1, \epsilon_1, T_0) \) so that the disks \( D_{r_0}(\rho_1), \ldots, D_{r_0}(\rho_n) \) are distinct. Let \( \Omega \) be the closure of the set

\[
\{1/2 + \epsilon_1 \leq \sigma \leq \sigma_1, 0 \leq t \leq T_0 \} \setminus \bigcup (D_{r_0}(\rho_1) \cup \cdots \cup D_{r_0}(\rho_n))
\]

and define

\[
M = M(\lambda, \alpha, \sigma_1, \epsilon_1, T_0) = \max_{s \in \Omega} \log^2 |L(\lambda, \alpha, s)|.
\]

We divide the integral in (6.13) into

\[
\int_{I_1} \log^2 |L(\lambda, \alpha, \sigma + it)| dt \quad \text{and} \quad \int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| dt,
\]

where

\[
I_1 = \{\sigma + it \mid 0 \leq t \leq T_0\} \cap \Omega,
\]

\[
I_2 = \{\sigma + it \mid 0 \leq t \leq T_0\} \cap \Omega^c.
\]

The integral over \( I_1 \) is estimated as

\[
\int_{I_1} \log^2 |L(\lambda, \alpha, \sigma + it)| dt \leq MT_0 \ll 1,
\]

where the implied constant depends only on \( \lambda, \alpha, \sigma_1, \epsilon_1, \) and \( T_0 \). Let

\[
L(\lambda, \alpha, s) = (s - \rho_k)^{\nu_k} L_1(\lambda, \alpha, s),
\]
where \( m_k \) is the order of the zero at \( s = \rho_k \). Then we have
\[
\log^2 |L(\lambda, \alpha, s)| \ll m_k^2 \log^2 |s - \rho_k| + \log^2 |L_1(\lambda, \alpha, s)|.
\]
Therefore, the integral over \( I_2 \) is
\[
\int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \leq \sum_{k=1}^n m_k^2 \int_{\gamma_k - r_0}^{\gamma_k + r_0} \log^2 |\sigma + it - \rho_k| \, dt + \sum_{k=1}^n \int_{\gamma_k - r_0}^{\gamma_k + r_0} \log^2 |L_1(\lambda, \alpha, \sigma + it)| \, dt,
\]
where \( \rho_k = \beta_k + i \gamma_k \). We see that
\[
\int_{\gamma_k - r_0}^{\gamma_k + r_0} \log^2 |\sigma + it - \rho_k| \, dt \ll 1
\]
and
\[
\int_{\gamma_k - r_0}^{\gamma_k + r_0} \log^2 |L_1(\lambda, \alpha, \sigma + it)| \, dt \ll 1
\]
with the implied constant depending on \( \lambda, \alpha, \sigma_1, \epsilon_1, \) and \( T_0 \). The maximum value of \( m_k \) is determined only from \( \lambda, \alpha, \sigma_1, \epsilon_1, T_0 \), and hence we have
\[
\int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \ll 1
\]
due to (6.15). Lemma 6.6 is deduced from estimates (6.14) and (6.16).

**Proof of Proposition 6.2.** By (6.5) and Lemmas 6.4 and 6.5, we have
\[
\int_{I_2} \log^2 |L(\lambda, \alpha, \sigma + it)| \, dt \ll 2^{k - 1} T_0
\]
for \( k \geq 1 \). Together with Lemma 6.6 by summing up over \( k \), we have the result.

**Proof of Proposition 6.1.** We begin with constructing certain functions that approximate \( \log |z| \) by following the way in [13]. Let
\[
f(u) = \begin{cases} 
\exp \left( - (b - a) \left( \frac{1}{u - a} + \frac{1}{b - u} \right) \right) & \text{if } a < u < b, \\
0 & \text{otherwise}
\end{cases}
\]
with \( a, b \in \mathbb{R} \) and \( 0 < b - a < 1 \). Then we define
\[
F(x) = \frac{\int_{-\infty}^{x + b - a_1} f(u) \, du}{\int_{-\infty}^{x + a - b_1} f(u) \, du} \cdot \frac{\int_{x + a - b_1}^{\infty} f(u) \, du}{\int_{x + b - a_1}^{\infty} f(u) \, du}
\]
for \( a_1, b_1 \in \mathbb{R} \) with \( a_1 < b_1 \). With the above setting, we define
\[
\Phi(z) = F(|z|) \log |z|,
\]
which is infinitely differentiable and supported on \(a_2 \leq |z| \leq b_2\) with \(a_2 = a_1 - (b-a)\) and \(b_2 = b_1 + (b-a)\). Moreover, we take the above \(a, b, a_1, b_1, a_2, b_2\) as the functions \(a = 1 - (\log T)^{-\gamma}, b = 1, a_1 = 2(\log T)^{-\gamma}, b_1 = (\log T)^{\gamma}, a_2 = (\log T)^{-\gamma}, b_2 = (\log T)^{\gamma} + (\log T)^{-\gamma}\) with a constant \(0 < \gamma < 1\). Then the following lemma holds.

**Lemma 6.7** (Lemma 3.1.1 of [13]). The function \(\Phi(z)\) satisfies the following conditions.

1. \(\Phi(z) = \log |z|\) for \(a_1 \leq |z| \leq b_1\).
2. \(|\Phi(z)| \leq |\log |z||\) for \(a_2 \leq |z| \leq a_1, b_1 \leq |z| \leq b_2\).
3. The Fourier transform of \(\Phi(z)\) is estimated as

\[
\tilde{\Phi}(z) \ll (|\log a_2| + |\log b_2|) \min \left( \frac{b_2^2}{(b-a)2x^2}, \frac{b_2^2 + a_2^{-2}}{(b-a)^2y^2}, \frac{b_2^2 + a_2^{-2}}{(b-a)^4x^2y^2} \right)
\]

for large \(T\) with \(z = x + iy\).

**Proof of Proposition 6.7** Since the function \(\Phi(z)\) belongs to the class \(\Lambda\), we have

\[
\frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt = \int_{\mathbb{C}} \Phi(z) M_{\sigma}(z; \alpha) \, |dz| + E,
\]

by Theorem 3.2, where

\[
E \ll_{\lambda, \sigma} \exp \left( -\frac{1}{2} (\log T)^{\delta/2} \right) \int_{\Omega} \left| \tilde{\Phi}(z) \right| \, |dz| + \int_{\mathbb{C}\setminus\Omega} \left| \tilde{\Phi}(z) \right| \, |dz|.
\]

By the first inequality of condition (3) in Lemma 6.7, the first integral is estimated as

\[
\int_{\Omega} \left| \tilde{\Phi}(z) \right| \, |dz| \ll \int_{-(\log T)^{\delta/2}}^{(\log T)^{\delta/2}} (\log T)^{2\gamma + \epsilon} \, dx \, dy \ll (\log T)^{2\gamma + 2\delta + \epsilon}
\]

with sufficiently small \(\epsilon > 0\). For the estimate of the second integral, we consider a covering of the region \(\mathbb{C}\setminus\Omega\) as follows:

\[
\mathbb{C}\setminus\Omega \subset U_1 \cup U_2 \cup U_3,
\]

where

\[
U_1 = \left\{ z = x + iy \mid |x| \geq (\log T)^{\delta}, |y| \leq (\log T)^{\delta/2} \right\},
U_2 = \left\{ z = x + iy \mid |x| \leq (\log T)^{\delta/2}, |y| \geq (\log T)^{\delta} \right\},
U_3 = \left\{ z = x + iy \mid |x| \geq (\log T)^{\delta/2}, |y| \geq (\log T)^{\delta/2} \right\}.
\]

Then we have

\[
\int_{U_1} \left| \tilde{\Phi}(z) \right| \, |dz| \ll \int_{-(\log T)^{\delta/2}}^{(\log T)^{\delta/2}} \int_{-(\log T)^{\delta/2}}^{(\log T)^{\delta/2}} \frac{(\log T)^{4\gamma + \epsilon}}{x^2} \, dx \, dy \ll (\log T)^{4\gamma - \delta/2 + \epsilon}
\]

by the second inequality of condition (3). Similarly we have

\[
\int_{U_2} \left| \tilde{\Phi}(z) \right| \, |dz| \ll (\log T)^{4\gamma - \delta/2 + \epsilon}
\]
by using the third inequality of condition (3), and furthermore, the fourth inequality
deduces
\[
\int_{U_3} |\Phi(z)| \, |dz| \ll \int_{(\log T)^{\delta/2}}^{\infty} \int_{(\log T)^{\delta/2}}^{\infty} \frac{(\log T)^{6\gamma+\epsilon}}{x^2y^2} \, dx \, dy \ll (\log T)^{6\gamma-\delta+\epsilon}.
\]
Assuming \( \gamma < \delta/4 \), we have \( 4\gamma - \delta/2 > 6\gamma - \delta \). Thus we obtain
\[
(6.18) \quad E \ll_{\lambda,\sigma_1} \exp \left( \frac{1}{2} (\log T)^{\delta/2} \right) (\log T)^{2\gamma+2\delta+\epsilon} + (\log T)^{4\gamma-\delta/2+\epsilon}
\]
for any \( T \geq T_0 \) with a positive constant \( T_0 = T_0(\alpha, \theta, \delta, \sigma_1, \gamma) \).
Next, we consider the difference
\[
E_L = \frac{1}{T} \int_0^T \Phi(L(\lambda, \alpha, \sigma + it)) \, dt - \frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)| \, dt
\]
coming from the left-hand side of (6.17). Let
\[
I = \{ t \in [0, T] \mid |L(\lambda, \alpha, \sigma + it)| \leq a_1 \},
\]
\[
J = \{ t \in [0, T] \mid |L(\lambda, \alpha, \sigma + it)| \geq b_1 \}.
\]
Then Lemma 6.7 gives
\[
|E_L| \leq \frac{1}{T} \int_{I \cup J} |\log |L(\lambda, \alpha, \sigma + it)|| \, dt.
\]
Applying Cauchy’s inequality, we see that this is
\[
(6.19) \quad \leq \left( \frac{\mu_1(I) + \mu_1(J)}{T} \right)^{1/2} \left( \frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)|^2 \, dt \right)^{1/2}.
\]
Let
\[
R = \{ z = x + iy \mid |x| \leq (\log T)^{-\gamma}, \, |y| \leq (\log T)^{-\gamma} \},
\]
\[
R' = \{ z = x + iy \mid |x| \leq \frac{1}{\sqrt{2}} (\log T)^{\gamma}, \, |y| \leq \frac{1}{\sqrt{2}} (\log T)^{\gamma} \}.
\]
Since we have \((\log T)^{-\gamma} \gg (\log T)^{-1/16}\), it is deduced from Theorem 1.7 that
\[
(6.20) \quad \frac{\mu_1(I)}{T} \ll_{\lambda,\alpha,\sigma_1,\epsilon_1} \int_R M_\sigma(z; \alpha) \, |dz| + (\mu_2(R) + 1)(\log T)^{-1/8}
\]
\[
\ll \int_R M_\sigma(z; \alpha) \, |dz| + (\log T)^{-1/8}
\]
and
\[
(6.21) \quad \frac{\mu_1(J)}{T} = 1 - \frac{\mu_1([0, T] \setminus J)}{T}
\]
\[
\ll_{\lambda,\alpha,\sigma_1,\epsilon_1} 1 - \int_{R'} M_\sigma(z; \alpha) \, |dz| + (\mu_2(R') + 1)(\log T)^{-1/8}
\]
\[
\ll \int_{C \setminus R'} M_\sigma(z; \alpha) \, |dz| + (\log T)^{2\gamma-1/8}.
\]
for any $T \geq T_0^{(1)}$ with a positive constant $T_0^{(1)} = T_0^{(1)}(\alpha, \sigma_1, \epsilon_1)$. The integrals in (6.20) and (6.21) are estimated as

\begin{align*}
\int_R M_\sigma(z; \alpha) \, |dz| &\ll_{\alpha, \epsilon_1} \int_R 1 \, dx \, dy \ll (\log T)^{-2\gamma} \\
\int_{C\setminus R'} M_\sigma(z; \alpha) \, |dz| &\ll_{\alpha, \epsilon_1} \int_{\frac{1}{\sqrt{2}}(\log T)^{\gamma}} e^{-r^2} \frac{r \, dr}{\sqrt{\gamma}} < \exp \left( -\frac{1}{4} (\log T)^{2\gamma} \right)
\end{align*}

and due to Proposition 2.5. We then use Proposition 6.2. It follows from Proposition 6.2 and estimates (6.19)–(6.23) that $E_L$ is estimated as

\begin{align*}
E_L &\ll_{\lambda, \alpha, \sigma_1, \epsilon_1} (\log T)^{2\gamma - 1/8}
\end{align*}

for any $T \geq T_0^{(2)}$ with a constant $T_0^{(2)} = T_0^{(2)}(\alpha, \sigma_1, \epsilon_1, \gamma) \geq T_0^{(1)}$.

Finally, we estimate

\begin{align*}
E_R &= \int_C \Phi(z) M_\sigma(z; \alpha) \, |dz| - \int_C \log |z| M_\sigma(z; \alpha) \, |dz|.
\end{align*}

It is estimated as

\begin{align*}
|E_R| &\leq \int_{|z| \leq a_1} |\log |z|| M_\sigma(z; \alpha) \, |dz| + \int_{|z| \geq b_1} |\log |z|| M_\sigma(z; \alpha) \, |dz|
\end{align*}

by Lemma 6.7. Again applying Proposition 2.5, we see that the first integral is

\begin{align*}
\ll_{\alpha, \epsilon_1} |\log a_1| a_1^2 \ll (\log T)^{-2\gamma + \epsilon},
\end{align*}

and the second integral is

\begin{align*}
\ll_{\alpha, \epsilon_1} \int_{b_1}^{\infty} \log r e^{-r^2} \, r \, dr &\ll \int_{b_1}^{\infty} r^{-2} \, dr \ll (\log T)^{-\gamma}.
\end{align*}

If we take $\epsilon < \gamma$, we have

\begin{align*}
E_R &\ll_{\alpha, \epsilon_1} (\log T)^{-\gamma}.
\end{align*}

We take $\gamma = \delta/12$ and fix $\theta, \delta > 0$ with $\theta + \delta < 1/4$. Combining estimates (6.18), (6.24), and (6.25), we conclude that

\begin{align*}
\frac{1}{T} \int_0^T \log |L(\lambda, \alpha, \sigma + it)| \, dt - \int_C \log |z| M_\sigma(z; \alpha) \, |dz| &\ll_{\lambda, \alpha, \sigma_1, \epsilon_1} (\log T)^{-\frac{\delta}{12}}.
\end{align*}

Hence the result follows. □

6.3. Completion of the proof. We use the following lemma to prove Theorem 1.8.

**Lemma 6.8** (Lemma 8.4.11 of [22]). Let $1/2 \leq \sigma \leq 1 + \alpha$. Then we have

\begin{align*}
2\pi \int_{\sigma}^{1+\alpha} N(T, u; \lambda, \alpha) \, du &= \sigma T \log \alpha + \int_0^T \log |L(\lambda, \alpha, \sigma + it)| \, dt + O(\log T)
\end{align*}

for sufficiently large $T$, where $N(T, u; \lambda, \alpha)$ is the number of zeros of $L(\lambda, \alpha, \sigma + it)$ in the region $\sigma > u$, $0 < t < T$. 
Proof of Theorem 1.8. Let $\sigma > 1/2$ be a fixed real number. By Proposition 6.1 and Lemma 6.8 we have

$$2\pi \int_{\sigma}^{1+\alpha} N(T, u; \lambda, \alpha) \, du$$

$$= \sigma T \log \alpha + T \int_{C} \log |z| |M_{\sigma}(z; \alpha)| \, |dz| + O_{\lambda, \alpha, \sigma} \left(T(\log T)^{-A}\right)$$

for $T \geq T_0$. If a small positive real number $h$ satisfies

$$\frac{1}{2} < \frac{1}{2} \left(\sigma + \frac{1}{2}\right) \leq \sigma - h < \sigma < \sigma + h \leq \sigma + 1,$$

we have also

$$2\pi \int_{\sigma \pm h}^{1+\alpha} N(T, u; \lambda, \alpha) \, du$$

$$= \sigma T \log \alpha + T \int_{C} \log |z| |M_{\sigma \pm h}(z; \alpha)| \, |dz| + O_{\lambda, \alpha, \sigma} \left(T(\log T)^{-A}\right).$$

Hence the formula

$$(6.26) \quad 2\pi \int_{\sigma}^{\sigma + h} N(T, u; \lambda, \alpha) \, du$$

$$= h T \log \alpha + T (\phi_{\alpha}(\sigma + h) - \phi_{\alpha}(\sigma)) + O_{\lambda, \alpha, \sigma} \left(T(\log T)^{-A}\right)$$

holds, where

$$\phi_{\alpha}(\sigma) = \int_{C} \log |z| |M_{\sigma}(z; \alpha)| \, |dz|.$$

By Corollary 5.4 we see that $\phi_{\alpha}(\sigma)$ is infinitely differentiable, and therefore

$$\frac{\partial^n}{\partial \sigma^n} \phi_{\alpha}(\sigma) = \int_{C} \log |z| \frac{\partial^n}{\partial \sigma^n} |M_{\sigma}(z; \alpha)| \, |dz|.$$

Hence we have

$$\phi_{\alpha}(\sigma + h) - \phi_{\alpha}(\sigma) = h \int_{C} \log |z| \frac{\partial}{\partial \sigma} |M_{\sigma}(z; \alpha)| \, |dz| + O_{\alpha, \sigma}(h^2).$$

Since the function $N(T, u; \lambda, \alpha)$ is decreasing in $u$, we have, by (6.26),

$$N(T, \sigma; \lambda, \alpha)$$

$$\geq T \frac{\log \alpha}{2\pi} + T \int_{C} \log |z| \frac{\partial}{\partial \sigma} |M_{\sigma}(z; \alpha)| \, |dz| + O_{\lambda, \alpha, \sigma} \left(h^{-1}T(\log T)^{-A} + hT\right).$$

Similarly, we have

$$N(T, \sigma; \lambda, \alpha)$$

$$\leq T \frac{\log \alpha}{2\pi} + T \int_{C} \log |z| \frac{\partial}{\partial \sigma} |M_{\sigma}(z; \alpha)| \, |dz| + O_{\lambda, \alpha, \sigma} \left(h^{-1}T(\log T)^{-A} + hT\right)$$

by considering $\sigma - h$ instead of $\sigma + h$. We take $h = (\log T)^{-A/2}$. The above error terms are $\ll T(\log T)^{-A/2}$, and hence we have

$$N(T, \sigma; \lambda, \alpha) = T \frac{\log \alpha}{2\pi} + T \int_{C} \log |z| \frac{\partial}{\partial \sigma} |M_{\sigma}(z; \alpha)| \, |dz| + O_{\lambda, \alpha, \sigma} \left(T(\log T)^{-A/2}\right).$$
Thus we obtain for any fixed $1/2 < \sigma_1 < \sigma_2$, 

\[ N(T, \sigma_1, \sigma_2; \lambda, \alpha) = N(T, \sigma_2; \lambda, \alpha) - N(T, \sigma_1; \lambda, \alpha) \]

\[
= \frac{T}{2\pi} \int_{\mathbb{C}} \log |z| \left( \frac{\partial}{\partial \sigma} M_{\sigma_2}(z; \alpha) - \frac{\partial}{\partial \sigma} M_{\sigma_1}(z; \alpha) \right) |dz| + O \left( T (\log T)^{-A/2} \right)
\]

\[
= \frac{T}{2\pi} \int_{\mathbb{C}} \log |z| \frac{\partial^2}{\partial \sigma^2} M_{\sigma}(z; \alpha) |dz| d\sigma + O \left( T (\log T)^{-A/2} \right)
\]

as desired, where the implied constant depends only on $\lambda, \alpha, \sigma_1$ and $\sigma_2$. □

**Appendix A. The study of Borchsenius and Jessen**

In this paper, we have proved an asymptotic formula for zeros of $L(\lambda, \alpha, s)$ of the form

\[
N(T, \sigma_1, \sigma_2; \lambda, \alpha) = CT + O \left( T (\log T)^{-A} \right), \quad T \to \infty
\]
in the case of $\alpha \in \mathbb{S}$. As we have remarked in Section 1.2, we may also obtain the formula

\[
C(\sigma_1, \sigma_2; \lambda) = \lim_{T \to \infty} \frac{1}{T} N(T, \sigma_1, \sigma_2; \lambda, \alpha)
\]

if $\alpha$ satisfies condition (1) of Definition 1.6. Since the proof is a simple analogue of the method of Borchsenius and Jessen [6], we follow it in this section as an appendix of this paper. For the proof, we refer to three results from [6] as follows.

**Lemma A.1.** Let $-\infty \leq \alpha < \alpha_0 < \beta_0 < \beta \leq +\infty$ and $-\infty < \gamma_0 < +\infty$, and let $f_1(s), f_2(s), \ldots$ be a sequence of functions almost periodic in $[\alpha, \beta]$ converging uniformly in $[\alpha_0, \beta]$ towards a function $f(s)$. Suppose that none of the functions is identically zero. Suppose further, that $f(s)$ is continued as a regular function in the half strip $\alpha < \sigma < \beta$, $t > \gamma_0$, and that for any fixed $\alpha < \alpha_1 < \beta_1 < \beta$ and $\gamma > \gamma_0$,

\[
\limsup_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \left\{ \int_{\alpha_1}^{\beta_1} |f(\sigma + it) - f_N(\sigma + it)|^p d\sigma \right\}^{1/p} dt \to 0
\]

as $N \to \infty$ with an index $p > 0$.

Then the function

\[
\phi_f(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log |f(\sigma + it)| dt
\]

converges as $\delta \to \infty$ for any fixed $\gamma > \gamma_0$ uniformly in $[\alpha, \beta]$ towards a function $\phi_f(\sigma)$, which is called the Jensen function of $f(s)$. The Jensen function $\phi_{f_N}(\sigma)$ of $f_N(s)$ converges as $N \to \infty$ uniformly in $[\alpha, \beta]$ towards $\phi_f(\sigma)$.

Moreover, for every strip $(\sigma_1, \sigma_2)$ with $\alpha < \sigma_1 < \sigma_2 < \beta$, if $\phi_f(\sigma)$ is differentiable at $\sigma_1$ and $\sigma_2$, then the limit value

\[ C_f(\sigma_1, \sigma_2) = \lim_{\delta \to \infty} \frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma} \]

exists for any fixed $\gamma > \gamma_0$ and determined by the formula

\[ C_f(\sigma_1, \sigma_2) = \frac{1}{2\pi} (\phi_f'(\sigma_2) - \phi_f'(\sigma_1)) \]
where \( N_f(\sigma_1, \sigma_2; \gamma, \delta) \) denotes the number of zeros of \( f(s) \) in the rectangle \( \sigma_1 < \sigma < \sigma_2, \gamma < t < \delta \).

Proof. The first part of the conclusion is from [6, Theorem 1], and the second part is seen in the argument following the statement of [6, Theorem 1]. \( \square \)

Lemma A.2 (Theorem 5 of [9]). Let \( \ell(z) = \ell_1 z + \ell_2 z^2 + \cdots \) and \( m(z) = m_1 z + m_2 z^2 + \cdots \) be power series converging in a circle \(|z| < \rho\), and such that \( \ell_1 \neq 0 \) and \( m_1 \neq 0 \). Let \( r_0, r_1, \ldots \) be a sequence of real numbers such that \( 0 < r_n < \rho \) for all \( n \), and let \( \lambda_0, \lambda_1, \ldots \) be a sequence of real numbers differing from each other and from zero.

For every \( N \), we define

\[
f_N(t_0, \ldots, t_N) = \sum_{n=0}^{N} \ell(r_n t_n) \quad \text{and} \quad g_N(t_0, \ldots, t_N) = \sum_{n=0}^{N} \lambda_n m(r_n t_n),
\]

where \((t_0, \ldots, t_N)\) describes the torus \( T^{N+1} = \prod_{n=0}^{N} \mathbb{C}^1 \), where \( \mathbb{C}^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \).

Let \( \mu_N \) be the normalized Haar measure of \( T^{N+1} \). We define a probability measure \( \mu_N \) on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) as

\[
\mu_N(A) = \int_{T^{N+1}} 1_A(f_N(t_0, \ldots, t_N)) \, dm_N(t_0, \ldots, t_N).
\]

We also define \( \nu_N \) as a measure on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) such that

\[
\nu_N(A) = \int_{T^{N+1}} 1_A(f_N(t_0, \ldots, t_N))|g_N(t_0, \ldots, t_N)|^2 \, dm_N(t_0, \ldots, t_N).
\]

Then, if \( r_n \to 0 \), the measures \( \mu_N \) and \( \nu_N \) are absolutely continuous with continuous density functions \( F_N(z) \) and \( G_N(z) \), respectively, for \( N \geq N_0 \) with a positive integer \( N_0 \), and \( F_N(z) \) and \( G_N(z) \) possess continuous partial derivatives of order \( \leq p \) for \( N \geq N_p \) with a positive integer \( N_p \).

Moreover, if the series

\[
S_0 = \sum_{n=0}^{\infty} r_n^2, \quad S_1 = \sum_{n=0}^{\infty} |\lambda_n| r_n^2, \quad S_2 = \sum_{n=0}^{\infty} \lambda_n^2 r_n^2
\]

converge, then \( \mu_N \) and \( \nu_N \) converge weakly to \( \mu \) and \( \nu \) as \( N \to \infty \), which are absolutely continuous measures with continuous density functions \( F(z) \) and \( G(z) \), respectively. The functions \( F(z) \) and \( G(z) \) possess continuous partial derivatives of arbitrarily high order, and the functions \( F_N(z) \) and \( G_N(z) \) and their partial derivatives converge uniformly towards \( F(z) \) and \( G(z) \) and their partial derivatives as \( N \to \infty \).

Lemma A.3. Let \( f_N(s) \) be a function as in Lemma A.1. We define measures \( \mu_{\sigma, N; \gamma, \delta} \) and \( \nu_{\sigma, N; \gamma, \delta} \) as

\[
\mu_{\sigma, N; \gamma, \delta}(A) = \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} 1_A(f_N(\sigma + it)) \, dt,
\]

\[
\nu_{\sigma, N; \gamma, \delta}(A) = \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} 1_A(f_N(\sigma + it))|f_N'(\sigma + it)|^2 \, dt.
\]
for $A \in \mathcal{B}(\mathbb{C})$. Suppose that there exist measures $\mu_{\sigma,N}$ and $\nu_{\sigma,N}$ to which $\mu_{\sigma,N;\gamma,\delta}$ and $\mu_{\sigma,N;\gamma,\delta}$ converges weakly as $\delta \to \infty$. Suppose further, that $\mu_{\sigma,N}$ and $\nu_{\sigma,N}$ are absolutely continuous with continuous density functions $F_{\sigma,N}(z)$ and $G_{\sigma,N}(z)$, respectively.

Then the Jensen function $\phi_{f_N}(\sigma)$ is expressed as

$$\phi_{f_N}(\sigma) = \int_{\mathbb{C}} \log |z| F_{\sigma,N}(z) \, |dz|,$$

and $\phi_{f_N}(\sigma)$ is twice differentiable with the second derivative

$$\phi''_{f_N}(\sigma) = G_{\sigma,N}(0).$$

Proof. They are direct consequences of the arguments in [6, Sections 8 and 9]. □

Proof of Theorem 1.5. We take the function $f_N(s)$ in Lemma A.1 as

$$f_N(s) = \sum_{n=0}^{N} \frac{e^{2\pi i \lambda_n}}{(n + \alpha)^s}.$$
converges weakly to $\mu_{\sigma,N}$ as $T \to \infty$. Similarly, the measure
\[ \nu_{\sigma,N,T}(A) = \frac{1}{T} \int_0^T 1_A(f_N(\sigma+it))|f'_N(\sigma+it)|^2 \, dt \]
converges weakly to
\[ \nu_{\sigma,N}(A) = \int_{T^{N+1}} 1_A(f_N(t_0,\ldots,t_N))|g_N(t_0,\ldots,t_N)|^2 \, dm_N(t_0,\ldots,t_N). \]

Then, we use Lemma A.2. Since $r_n \to 0$ as $n \to \infty$, we find that $\mu_{\sigma,N}$ and $\nu_{\sigma,N}$ are absolutely continuous with continuous density functions $F_{\sigma,N}(z)$ and $G_{\sigma,N}(z)$ for large $N$. Therefore, by Lemma A.3, the Jensen function $\phi_{f_N}(\sigma)$ is expressed as
\[ \phi_{f_N}(\sigma) = \int_{C} \log |z| F_{\sigma,N}(z) \, dz, \]
and $\phi_{f_N}(\sigma)$ is twice differentiable with the second derivative
\begin{equation}
\phi''_{f_N}(\sigma) = G_{\sigma,N}(0). 
\end{equation}
Moreover, we see that
\begin{align*}
S_0 &= \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^2}, \\
S_1 &= \sum_{n=0}^{\infty} \frac{\log(n+\alpha)}{(n+\alpha)^2}, \\
S_2 &= \sum_{n=0}^{\infty} \frac{\log^2(n+\alpha)}{(n+\alpha)^2}
\end{align*}
are convergent if $\sigma > 1/2$. Hence the last part of Lemma A.2 can be applied, and thus $G_{\sigma,N}(z)$ converges uniformly towards a non-negative continuous function $G_{\sigma}(z)$ as $N \to \infty$. Since the Jensen function $\phi_{f_N}(\sigma)$ converges uniformly to $\phi_f(\sigma)$ by Lemma A.1, we conclude from A.3 that $\phi_f(\sigma)$ is twice differentiable with the second derivative
\[ \phi''_f(\sigma) = G_{\sigma}(0). \]
Therefore Theorem 1.5 follows from the second part of Lemma A.1.

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