Robust Model Predictive Control with Adjustable Uncertainty Sets

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Abstract—In this paper, we present Robust Model Predictive Control (MPC) problems with adjustable uncertainty sets. In contrast to standard Robust MPC problems with known uncertainty sets, we treat the uncertainty sets in our problems as additional decision variables. In particular, given a metric for adjusting the uncertainty sets, we address the question of determining the optimal size and shape of those uncertainty sets, while ensuring robust constraint satisfaction. The focus of this paper is on ensuring constraint satisfaction over an infinite horizon, also known as persistent feasibility. We show that, similar as in standard Robust MPC, persistent feasibility can be guaranteed if the terminal set is an invariant set with respect to both the state of the system and the adjustable uncertainty set. We also present an algorithm for computing such invariant sets, and illustrate the effectiveness of our approach in a cooperative adaptive cruise control application.

I. INTRODUCTION

Robust control concerns itself with designing controllers for uncertain systems [1], [2]. In the field of Model Predictive Control (MPC), Robust MPC has been first introduced in [3]–[5]. Commonly, Robust MPC deals with problems where the uncertainty sets are known a priori. The objective in this case is to design a controller that is robust against all disturbance realizations from this uncertainty set.

Recently, a new paradigm in robust control has been introduced where uncertainty sets are not assumed to be known, but can be adjusted instead, i.e., their size is not fixed but to be determined. These kinds of problems are referred to as robust control problems with adjustable uncertainty sets [6], [7].

Robust control problems with adjustable uncertainty sets arise in several applications: for example, in reserve provision problems, the adjustable uncertainty set is interpreted as a reserve capacity that a system can offer to third parties and for which it receives (monetary) reward [6]–[10]. In this case, the maximal reserve capacity can be computed by maximizing the size of the uncertainty set. The challenge when determining a reserve capacity is to ensure that, for every admissible reserve demand, the system is indeed able to meet this demand without violating its constraints. Another problem that can be formulated as a robust control problem with adjustable uncertainty sets arises in a robustness analysis for determining the limits of robustness of a given system (a system’s resilience against disturbances).

The existing literature on robust control with adjustable uncertainty sets has mainly focused on designing controllers over a finite horizon and deriving computationally tractable convex approximations [6]–[9]. In this paper, we consider the infinite-horizon case, where our objective is to design a controller that satisfies the constraints for all times.

Specifically, we extend the work of [7] to the infinite horizon case by employing ideas from standard Robust MPC. The resulting algorithm, called Robust MPC with Adjustable Uncertainty Sets (RMPC-AU), ensures satisfaction of state and input constraints for all future time steps, a notion known as persistent feasibility in the literature [11]. The contributions of this paper can be summarized as follows:

• We introduce the notion of adjustable control invariant set and adjustable positive invariant sets, and show that these sets can be used to ensure persistent feasibility. We also provide algorithms for computing these sets.

• Using ideas from standard Robust MPC, we show that, by appropriately parametrizing the control policies and uncertainty sets, computationally tractable reformulations can be derived.

We illustrate our approach on a cooperative adaptive cruise control (CACC) problem where the objective is to minimize the distance gap between vehicles while maximizing the uncertainty set for the future states of the other vehicles. Here, the challenge is that there is a trade-off between these two objectives and it is not straightforward to address it with standard methodologies [12]–[14]. Using our approach, we can easily handle this challenge by simultaneously computing the input sequence and the allowable uncertainty set in a single optimization problem. Moreover, we can gain insights on the relationship between the distance tracking performance and the uncertainty set for the predicted states of the other vehicles.

The remainder of this paper is organized as follows. Section III defines the RMPC-AU formulation and introduces the conditions which guarantee recursive feasibility for our problem. Section IV introduces the policy and the uncertainty set approximations and the terminal constraint formulation, and shows the final tractable formulation of the optimization problem for RMPC-AU. Section V illustrates our method on a CACC example. Section VI concludes the work. The Appendix provides the derivations for tractable reformulation of our optimal control problem and the method to compute the adjustable control invariant set.

A. Notation

We denote by \( x_{kl} \in \mathbb{R}^{n_x} \) the state at time \( k + t \), predicted at time \( t \). Furthermore, we define
\[
[x_0], x_1, \ldots, x_{k-1}^T \in \mathbb{R}^{kn_x}.
\]
Given a set \( \mathbb{X} \subseteq \mathbb{R}^{n_x} \), we...
denote by $\mathbb{X}^k$ its $k$-fold Cartesian product, i.e., $\mathbb{X}^k := \bigtimes_{i=0}^{k-1} \mathbb{X}$.

Finally, we denote by $\mathcal{P}(\mathbb{R}^{n_u})$ the power set of $\mathbb{R}^{n_u}$, and $\mathbb{N}_+$ is the set of non-negative integers. Also, $A \Rightarrow B$ means that $A$ implies $B$, i.e., if $A$ is true then $B$ must be true.

II. Problem Formulation

In this section we first review the finite-horizon robust optimal control problem with adjustable uncertainty sets from [7], before we introduce the concept of Robust Model Predictive Control with adjustable uncertainty sets. Also, $A \Rightarrow B$ means that $A$ implies $B$, i.e., if $A$ is true then $B$ must be true.

A. Robust Control with Adjustable Uncertainty Sets

Following [7], we consider discrete-time uncertain systems of the form

$$x_{k+1} = Ax_k + Bu_k + Ew_k,$$  \hspace{1cm} (1)

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the input, and $w_k \in \mathbb{R}^{n_w}$ is the uncertain disturbance at time step $k$. We consider polytopic state and input constraints of the form

$$x_k \in \mathbb{X} := \{ x \in \mathbb{R}^{n_x} : F_x x \leq f_x \},$$  \hspace{1cm} (2a)

$$u_k \in \mathbb{U} := \{ u \in \mathbb{R}^{n_u} : F_u u \leq f_u \},$$  \hspace{1cm} (2b)

where $F_x \in \mathbb{R}^{n_x \times n_x}, f_x \in \mathbb{R}^{n_x}, F_u \in \mathbb{R}^{n_u \times n_u}, f_u \in \mathbb{R}^{n_u}$ are known matrices. The uncertainty $w_k$ is assumed to belong to the adjustable uncertainty set $\mathbb{W}_k$, i.e.,

$$w_k \in \mathbb{W}_k \subseteq \mathbb{R}^{n_w}.$$  \hspace{1cm} (3)

Here, $\mathbb{W}_k$ is called “adjustable” because its size is not defined a priori, but to be determined later. Given the planning horizon $N$, we denote by $\varphi_{k|t}(u_t, w_t)$ the state at time $t + k$ predicted at time $t$ using the model (1) subject to the inputs $u_t = [u_0, u_1, ..., u_{N-1}]$ and the uncertainties $w_t = [w_0, w_1, ..., w_{N-1}]$.

Our goal is to determine the optimal input policy and the uncertainty sets, over the horizon $N$, such that constraints (2) are satisfied. We consider state feedback policies $\pi_t(\cdot) := [\pi_0(\cdot), \pi_1(\cdot), ..., \pi_{N-1}(\cdot)],$ where $\pi_k(\cdot) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$, such that the input at time step $k$ is given by $u_k = \pi_k(x_k)$.

Given a metric $\rho(\cdot) : \mathcal{P}(\mathbb{R}^{n_w}) \rightarrow \mathbb{R}$ for adjusting the uncertainty set, the control objective is to minimize a “nominal” cost $J(\cdot, \cdot)$ while maximizing the uncertainty set with respect to $\rho(\cdot)$. For instance, $\rho(\cdot)$ can represent the volume of the uncertainty set, but is in general problem-dependent [7]. Therefore, the finite time constrained optimal control problem with adjustable uncertainty sets at time $t$ is given by

$$\min_{\pi_t(\cdot), \mathbb{W}_t} \left\{ \max_{w_t \in \mathbb{W}_t} J(\pi_t(x_t), w_t) \right\} - \sum_{k=0}^{N-1} \lambda_k \rho(\mathbb{W}_k(t))$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + Ew_k,$$

$$x_k \in \mathbb{X}, \pi_k(x_k) \in \mathbb{U}, \forall w_k \in \mathbb{W}_k,$$

$$\forall k = 0, ..., N - 1,$$

$$x_0 = x(t).$$

Finally, we denote by $\mathcal{N}$ uncertainty sets, over the horizon $N$, such that $A$ implies $B$, i.e., if $A$ is true then $B$ must be true.

B. Robust MPC with Adjustable Uncertainty Set

The Robust MPC with Adjustable Uncertainty Sets (RMPC-AU) solves, at time $t$, the finite time constrained optimal control problem (4). Let

$$\pi^*_t(\cdot) = [\pi^*_0(\cdot), \pi^*_1(\cdot), ..., \pi^*_{N-1}(\cdot)],$$  \hspace{1cm} (6a)

$$\mathbb{W}^*_t = [\mathbb{W}^*_0, \mathbb{W}^*_1, ..., \mathbb{W}^*_t],$$  \hspace{1cm} (6b)

be the solution of the optimal control problem (4). Then, the first input $\pi^*_0(x_t)$ is applied to the system. At the next time step $t + 1$, a new optimal control problem in the form of (4) is solved based on new measurements of the state is solved over a shifted horizon, yielding a moving or receding horizon control strategy. The control law is given by

$$u_t = \pi^*_0(x_t),$$  \hspace{1cm} (7)

and the closed loop system can be written as

$$x_{t+1} = Ax_t + Bu^*_0(x_t) + Ew_t,$$  \hspace{1cm} (8)

where $w_t \in \mathbb{W}^*_0$.

C. Recursive Feasibility of RMPC-AU

Our objective is to design a controller which ensures state and input constraint satisfaction at all times. In this section we formally define the notion of recursive feasibility for RMPC-AU (4)-(8), before we provide sufficient conditions which guarantee it.

Definition II.1 (Recursive feasibility for RMPC-AU). RMPC-AU (4)-(6) is said to be recursively feasible, if the existence of a solution $(\pi^*_1, \mathbb{W}^*_1)$ for (4) at time $t = 0$ implies feasibility of RMPC-AU (4)-(8) for all time $t > 0$.

In general, recursive feasibility is not guaranteed without imposing additional conditions on the terminal constraint $\mathbb{F}$. Similar as in standard Robust MPC [11], the recursive
feasibility of RMPC-AU can be achieved by choosing the terminal set $\mathcal{F}$ to be an invariant set.

**Definition II.2 (Adjustable Control Invariant Set).** A set $\mathcal{C}_{\text{adj}} \subseteq X \times P(\mathbb{R}^{n_u})$ is said to be an adjustable control invariant set for the system (11) subject to the constraints (2) and the uncertainty (3), if

$$
\forall (x_t, w_{t-1}) \in \mathcal{C}_{\text{adj}} \Rightarrow \exists u_t \in U, \mathcal{W}_t \subseteq \mathbb{R}^{n_w} \quad (x_{t+1}, \mathcal{W}_t) \in \mathcal{C}_{\text{adj}}, \forall v_t \in \mathcal{W}_t, t \in \mathbb{N}_+.
$$

Notice that Definition II.2 is a generalization of the standard robust control invariant set, which can be recovered by fixing $\mathcal{W}_{t-1} = \mathcal{W}_t$ in (9). Furthermore, when $\mathcal{W}_{t-1} = \mathcal{W}_t = \{0\}$, (9) recovers the standard control invariant set [15]. We now have the following result:

**Theorem 1.** Consider RMPC-AU (4)–(8) with $N \geq 1$. If the terminal set $\mathcal{F}$ in (4) is an adjustable control invariant set for the system (11), then the RMPC-AU is recursively feasible.

**Proof.** We prove Theorem 1 by induction, showing that taking the terminal set $\mathcal{F}$ as an adjustable control invariant set is a sufficient condition for persistent feasibility of RMPC-AU (4)—(8).

First, consider the optimal control problem (4) at time $t$ with its optimal solution $(\pi^*_t(\cdot), \mathcal{W}^*_t)$ as described in (7). At the next time step $t + 1$, for all $w_t \in \mathcal{W}_t$, the optimal control problem (4) is guaranteed to have at least one feasible solution with the shifted input policy and uncertainty set and the added solution, $(\hat{u}, \hat{\mathcal{W}})$, at the last step. Here, $\hat{u}$ and $\hat{\mathcal{W}}$ are the feasible input and the uncertainty set, respectively, which are guaranteed to exist by Definition II.2. Explicitly, this feasible solution at time $t + 1$ can be written as:

$$
\pi_{t+1}(\cdot) = [\pi^*_t(\cdot), \pi^*_t(\cdot), \ldots, \hat{u}], \quad (10a)
$$

$$
\mathcal{W}_{t+1} = [\mathcal{W}^*_t, \mathcal{W}^*_t, \ldots, \hat{\mathcal{W}}], \quad (10b)
$$

Therefore, by induction, RMPC-AU is recursively feasible.

While Theorem 1 establishes recursive feasibility, solving the optimization problem (4) is computationally intractable since (i) the optimization is performed over general feedback policies, (ii) the optimization of the uncertainty set is performed over arbitrary subsets of $\mathbb{R}^{n_u}$, and (iii) the constraints must be satisfied robustly for every uncertainty realization [7]. In the following, we present approximations of RMPC-AU (4)–(8) that are computationally tractable while ensuring recursive feasibility.

### III. Tractable Approximation of RMPC-AU

In this section, we provide a tractable convex formulation for the optimal control problem in (4).

#### A. Uncertainty Set and Policy Approximation

Following [7], we consider uncertainty sets that are obtained as the affine transformation of a so-called primitive set $\mathcal{S}$, i.e.,

$$
\mathcal{W}_t = Y_t \mathcal{S} + y_t, \text{ where } (Y_t, y_t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_u \times n_w} \times \mathbb{R}^{n_w},
$$

and

$$
\mathcal{S} := \{ s \in \mathbb{R}^{n_u} : G s \leq g \}
$$

is a fixed polytope with given matrices $G \in \mathbb{R}^{l \times n_u}$ and $g \in \mathbb{R}^l$. The set $\mathcal{Y}$ in (11) allows us to control the shape of the uncertainty set, see [7, Section 3.2] for details. Given the planning horizon $N$, we define by $Y_t$ and $y_t$ the compact forms of the $N$ time steps forward consecutive $Y_{k|t}$ and $y_{k|t}$ given the time $t$, respectively; i.e., $Y_t := \text{diag}(Y_{0|t}, Y_{1|t}, \ldots, Y_{N-1|t})$ and $y_t := [y_{0|t}, y_{1|t}, \ldots, y_{N-1|t}]^T$.

It is well-known that optimizing over general feedback policies is computationally intractable. To overcome this issue, we adopt the affine disturbance feedback policy of [16], which is given by

$$
u_{k|t} = \tilde{\pi}_{k|t}(s_{k|t}) := p_{k|t} + \sum_{j=0}^{k-1} P_{k,j}(s_{j|t}),
$$

where $p_{k|t} \in \mathbb{R}^{n_u}$ and $P_{k,j} \in \mathbb{R}^{n_u \times n_u}$ are parameters. Note that the control policy (13) is defined with respect to $s_k$, and not with respect to $w_k$. We denote by $\tilde{\pi}_t(s_{N-1|t})$ the concatenated form of the $N$ time steps forward consecutive policies at time $t$; i.e., $\tilde{\pi}_t(s_{N-1|t}) := [\tilde{\pi}_0(t)(s_{N-1|t}), \ldots, \tilde{\pi}_{N-1|t}(s_{N-1|t})]^T = p_t + P_t s_{N-1|t}$, where $P_t \in \mathbb{R}^{n_u \times n_u}$ is a lower triangular block matrix whose elements are defined by $P_{k,j}$ and $p_t := [p_0(t), \ldots, p_{N-1|t}]^T \in \mathbb{R}^{n_u}$.

In the remainder of this paper, we assume that the uncertainty set can be represented as in (12). Furthermore, by considering the policies (13), problem (4) can be rewritten as

$$
\min \left\{ \max_{w_t \in \mathcal{W}_t} c^T \tilde{\pi}_t(s_{N|t}) \right\} = \sum_{k=0}^{N-1} \lambda_k r(Y_{k|t} \mathcal{S} + y_{k|t})
$$

s.t.

$$
x_{k+1|t} = Ax_{k|t} + B \tilde{\pi}_{k|t}(s_{k|t}) + E w_{k|t},
\tilde{\pi}_{k|t}(s_{k|t}) = p_{k|t} + \sum_{j=0}^{k-1} P_{k,j}(s_{j|t}),
\nu_{k|t} = Y_{k|t} s_{k|t} + y_{k|t}, (Y_{k|t}, y_{k|t}) \in \mathcal{Y},
\forall k = 0, 1, \ldots, N - 1,
\mathcal{X} \subseteq \mathcal{F},
$$

with $(P_t, p_t, Y_t, y_t)$ as the decision variables. $c \in \mathbb{R}^{n_n}$ is a matrix constructed to represent the linear stage cost from the problem data, see e.g. [16] for details on the construction.

#### B. Terminal Set Approximation

1) **Adjustable Positive Invariant Set:** Recall that we restricted our input to be in the form (13) to attain computational tractability. Unfortunately, this restriction also implies that Theorem 1 cannot be directly applied because it guarantees the existence of an input that may not depend on the uncertainty in a linear fashion.
Remark 1. If the uncertainty set (11) is assumed to always include {0}, then Theorem 1 can guarantee recursive feasibility of the reformulated RMPC-AU (14) (6) (8), at least with \( \mathbb{W}_k = \{0\} \) for \( k = 0, \ldots, N - 1 \). This is because the control policy (13) becomes equivalent to the deterministic input actions when the uncertainty set is \{0\}.

This motivates the following definition:

Definition III.1 (Adjustable Positive Invariant Set). A set \( \mathcal{O}_{\text{adj}} \subseteq \mathbb{X} \times \mathbb{P}(\mathbb{R}^{n_*}) \) is said to be an adjustable positive invariant set for the uncertain autonomous system \( x_{k+1} = g(x_k, u_k) \) subject to the constraints (2a) and the uncertainty (3), if

\[
\forall (x_t, W_{t-1}) \in \mathcal{O}_{\text{adj}} \Rightarrow \exists W_t \subseteq \mathbb{R}^{n_*} \quad (x_{t+1}, W_t) \in \mathcal{O}_{\text{adj}}, \forall u_t \in W_t, t \in \mathbb{N}_+.
\]

Note that an autonomous system \( x_{t+1} = g(x_t, u_t) \) can be achieved from the control system (1) by fixing a feedback control policy.

Now, we are in place to state the following result.

Corollary 1. Consider the RMPC-AU (14), (6) (8) for \( N \geq 1 \). If the terminal set \( \mathcal{F} \) in (3) is an adjustable positive invariant set for the system (1) (3) in a closed loop with an affine state feedback law \( u_t = Kx_t + b \), where \( K \in \mathbb{R}^{n_\ast \times n_*} \) and \( b \in \mathbb{R}^{n_*} \) are given, then the RMPC-AU (14), (6) (8) is recursively feasible.

Proof. By Definition II.2 and III.1, an adjustable positive invariant set for a control system with a fixed policy is an adjustable control invariant set for that control system. Therefore, it follows from Theorem 1 that the RMPC-AU (4) (8) is recursively feasible when its terminal set is an adjustable positive invariant set.

It is known that the class of admissible affine state feedback policies is equivalent to the class of admissible affine disturbance feedback policies [16]. Moreover, any affine disturbance feedback policy can be represented as the feedback policy (17) when the uncertainty set is approximated by (11) and (12), as proven in [7]. Therefore, the existence of the affine state feedback policy guarantees the existence of a feedback policy (13) with the uncertainty approximation (11) and (12). Hence, the RMPC-AU (14), (6) (8), which is subject to the feedback policy (13), is recursively feasible. \( \square \)

2) Computation of Adjustable Positive Invariant Set: In this section, we present an algorithm for computing an adjustable positive invariant set \( \mathcal{O}_{\text{adj}} \). In particular, we propose the method to compute the size of \( \mathcal{O}_{\text{adj}} \) through which we can ensure recursive feasibility (Corollary 1).

Consider the affine state feedback policy \( u_k = Kx_k + b \) for the system dynamics (14). Using the uncertainty approximation (11), (12), the closed loop system dynamics at time step \( k \) can be written as:

\[
x_{k+1} = (A + BK)x_k + E(Y_k s_k + y_k) + Bb
= (A + BK)x_k + E\vec{v}(Y_k) + Ey_k + Bb,
\]

where \( \mathcal{S} := \text{diag}(s_1^T, \ldots, s_k^T) \in \mathbb{R}^{n_* \times n_*} \) and \( \vec{v}(\cdot) \) is a linear transformation which converts the matrix into a column vector in a column-wise manner. This system dynamics can be reformulated as:

\[
z_{k+1} := \frac{x_{k+1} - \vec{v}(Y_{k+1})}{y_{k+1}} = g(z_k, s_k) := (17)
= \begin{bmatrix}
A + BK & ES & E \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
x_k \\
\vec{v}(Y_k) \\
y_k
\end{bmatrix}
+ \begin{bmatrix}
Bb
\end{bmatrix}.
\]

The robust positive invariant set of the model (17) represents the adjustable positive invariant set of our system (1) (3) with the uncertainty set approximation (11). We also denote by \( \mathcal{Z} := \mathbb{X} \times \mathbb{M} \) the feasible set of (17) where \( \mathbb{M} := \{y_k \in \mathbb{Y} \subseteq \mathbb{R}^{n_*} \} \). Therefore, \( z_k \) has dimension \( n := n_x + n_\ast + n_* \).

Finally, the robust control invariant set for the system (17) constrained to the feasible set \( \mathcal{Z} \) represents the adjustable positive invariant set for (1) (3) with the uncertainty set approximation (11).

In order to compute the robust positive invariant set for (17), we first need the following definition from [15].

Definition III.2 (Robust Precursor Set). For the autonomous system (17), we define the robust precursor set of the set \( \mathcal{Z} \)

\[
\text{Pre}(\mathcal{Z}, \mathcal{S}) = \{z \in \mathbb{R}^n : g(z, s) \in \mathcal{Z}, \forall s \in \mathcal{S}\}
\]

(18)

Pre(\( \mathcal{Z}, \mathcal{S} \)) defines the set of states \( z \) which can be driven into the target set \( \mathcal{Z} \) in one step for all uncertainty \( s \in \mathcal{S} \). Note that the robust precursor set of the system (17) can be computed using [15, Lemma 10.1]. Algorithm 1 outlines the procedure for computing the adjustable robust control invariant set. If Algorithm 1 converges, then \( \mathcal{O}_{\text{adj}} \) is an adjustable positive invariant set.

Algorithm 1 Computation of \( \mathcal{O}_{\text{adj}} \)

1: InputSystem model (17), \( \mathcal{Z}, \mathcal{S} \)
2: Output \( \mathcal{O}_{\text{adj}} \)
3: \( \Omega_0 \leftarrow \mathcal{Z}, k \leftarrow -1 \)
4: Repeat
5: \( k \leftarrow k + 1 \)
6: \( \Omega_{k+1} \leftarrow \text{Pre}(\Omega_k, \mathcal{S}) \cap \Omega_k \)
7: Until \( \Omega_{k+1} = \Omega_k \)
8: \( \mathcal{O}_{\text{adj}} \leftarrow \Omega_k \)

Remark 2. The adjustable positive invariant set computed by Algorithm 1 can be used to find the largest constant uncertainty set which the system can indefinitely tolerate without violating its constraints. This is because the uncertainty set \( \mathbb{W}_k = Y_k \mathcal{S} + y_k \) is time-invariant in the dynamics (17).

Algorithm 1 returns us, if it converges, a polytopic description of \( \mathcal{O}_{\text{adj}} \) with \( n_t \) linear constraint, which we can directly...
integrate into the optimization problem as
\[ F(x_k; \text{vec}(Y_k); y_k) \leq f \]  
(19)
where \( F \in \mathbb{R}^{n_x \times (n_x + n_u)} \) and \( f \in \mathbb{R}^{n_x} \).

C. Tractable Reformulation

By combining the uncertainty set approximation \([11]\), the feedback policy \([13]\), and the polytopic terminal constraints \([19]\), problem \([14]\) can be rewritten as

\[
\min \tau_t - \sum_{k=0}^{N-1} \lambda_k \rho(Y_{k|t} S + y_{k|t}) \\
\text{s.t.} \\
\tau_t \in \mathbb{R}, \\
P_t \in \mathbb{R}^{n_y \times n_u}, \ p_t \in \mathbb{R}^{n_y}, \ (Y_t, y_t) \in \mathcal{Y}, \\
\mu_t \in \mathbb{R}^{n_y}, \ \Lambda_t \in \mathbb{R}^{(n_y+n_u) \times n_y}, \ \Gamma_t \in \mathbb{R}^{n_x \times n_y}, \\
\mu_t \geq 0, \ \Lambda_t \geq 0, \ \Gamma_t \geq 0, \\
\eta_t^T p_t + \mu_t^T y_t \leq \tau_t, \ \mu_t G = c_t^T P_t, \\
\sum_{t=0}^{T} y_t \leq d_t, \ \Lambda_t G = C_t P_t + D_t Y_t, \\
\Gamma_t G = F[M_t P_t + N_t Y_t; \text{vec}(Y_{N-1|t})]; \ y_{N-1|t} \leq f, \\
\text{where} \ (\tau_t, P_t, y_t, \mu_t, \Lambda_t, \Gamma_t) \text{ are the decision variables}; \\
\mathcal{Y} := \{ (Y_t, y_t) : \{Y_{k|t}, y_{k|t} \in \mathcal{Y} \}_{k=0}^{N-1}, \ Y_t = \text{diag}(Y_0, ..., Y_{N-1}) \in \mathbb{R}^{n_y \times n_y}, \ y_t = \text{vec}(Y_{N-1|t}); \ y_{N-1|t} \leq f, \\
G := \text{diag}(G, ..., G) \in \mathbb{R}^{n_y \times n_y}; \ g := \text{vec}(g) \in \mathbb{R}^{n_y}; \ (C_t, D_t, M_t, N_t) \text{ are given in Appendix A. Problem } (20) \text{ is a convex optimization problem with a finite number of constraints and decision variables; therefore, it can be solved using off-the-shelf solvers such as GUROBI [17] or MOSEK [18].}

IV. EXAMPLE: VEHICLE PLATOONING

In this section we illustrate the proposed RMPC-AU on a cooperative adaptive cruise control (CACC) application. We consider a group (or platoon) of two vehicles featuring bi-directional Vehicle-to-Vehicle (V2V) communication. The ego-vehicle is positioned in the back, and receives the front vehicle’s velocity, \( v_f \). In return, the ego vehicle sends the maximum acceleration or deceleration which the front vehicle must not exceed in order to avoid a rear collision and, on the other hand, loss of contact because the spacing has become excessive. The front vehicle can then enforce this constraint on the acceleration/deceleration as done in \([12], [19]\), thus ensuring safety in the platoon operation. Our platoon setup is depicted in Fig. 1.

The control objective of our problem is to minimize the tracking distance while maximizing the magnitude of the acceleration/deceleration that the front vehicle can undertake. However, there is a trade-off between these two factors: the framework introduced in the previous sections allows us to systematically address this trade-off. For instance, if the objective is energy saving, then tracking distance should be minimized, thereby reducing the aerodynamic drag on the rear vehicle. This will result in smaller bounds on the acceleration/deceleration of the front vehicle. If the objective is to allow more freedom to the movements of the front vehicle, then its maximum acceleration/deceleration should be maximized, which will result in a larger distance gap.

In our optimization problem, the adjustable uncertainty set is interpreted as a constraint set for the input of the front vehicle.

We consider the longitudinal dynamics of the ego vehicle when following the front vehicle. We assume a point mass model with two states: \( d \), the distance between the ego and the front vehicle, and \( \Delta v \), the difference between the ego vehicle’s velocity, \( v_{ego} \) and the front vehicle’s velocity, \( v_f \); i.e., \( \Delta v = v_f - v_{ego} \). There is one input, \( u \), which represents the acceleration of the ego vehicle, and one additive disturbance, \( w \), which represents the acceleration or deceleration of the front vehicle. The model can be written in the form of (1) with

\[
A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ -\Delta t \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ \Delta t \end{bmatrix}.
\] 
(21)

where \( \Delta t = 0.2s \) represents the discretization time. The uncertainty \( u_k \) is bounded by some symmetric uncertainty set around zero, i.e., \( u_k \in \mathbb{W}_k \subseteq \mathbb{R} \) where each \( \mathbb{W}_k := \{ w : |w| \leq y_k \in \mathbb{R}_{\geq 0} \} \); \( y_k \) represents the maximum acceleration and deceleration of the front vehicle. Given this shape of the uncertainty sets, \( y_k \) quantifies the size of \( \mathbb{W}_k \). Moreover, we enforce that the uncertainty set has a constant size over the horizon, i.e., \( y_k = y_0 = \cdots = y_N \).

The optimal control problem at time \( t \) can be formulated as

\[
\min_{\pi_t(\cdot), y_t} \left\{ \max_{w_t \in \mathbb{W}^N} \sum_{k=0}^{N} [1,0] x_{k|t} - \lambda y_t \right\} \\
\text{s.t.} \\
x_{k+1|t} = A x_{k|t} + B \pi_k(t)(x_{k|t}) + E w_{k|t}, \\
\begin{bmatrix} 10 \\ -5 \end{bmatrix} \leq x_{k|t} \leq \begin{bmatrix} 20 \\ 5 \end{bmatrix}, -10 \leq \pi_k(t)(x_{k|t}) \leq 10, \\
\forall \ w_{k|t} \in [-y_k, y_k], \\
\forall \ k = 0,1, ..., N-1, \\
x_{0|t} = x(t), \ (x_{N|t}, y_t) \in F,
\] 
(22)
with \( \pi_t(\cdot), y_t \) as the optimization variables. Clearly, \( \pi_t(\cdot), y_t \) is an instance of \([4]\).
A. Effects of Adjustable Invariant Sets

The RMPC-AU (8)-(8), and (22) is solved with adjustable positive/control invariant sets and without any terminal constraints. Figure 2 illustrates the adjustable positive invariant set and the adjustable control invariant set computed using Algorithm 1 and Algorithm 2, respectively. To construct a closed loop system for the adjustable positive invariant set, we used some stabilizing affine state feedback policy, \( \pi_{k|t}(x_{k|t}) = Kx_{k|t} \), obtained by pole-placement method. As expected, the sets in the state space shrink as the size of the uncertainty set, \( \lambda \), increases. In fact, when the uncertainty set is only \( \{0\} \), i.e. \( \lambda = 0 \), it can be verified that the adjustable positive and control invariant sets are equivalent to the positive and control invariant sets for a deterministic system, respectively. Also, the adjustable positive invariant set is much smaller than the adjustable control invariant set because we fixed the control law to be affine for computing the positive adjustable invariant set, whereas the control adjustable invariant set allows any (also nonlinear) control policy.

In the closed loop simulation, RMPC-AU (8)-(8), and (22) was persistently feasible when the optimization problem (22) is solved with the adjustable control or positive invariant set. Note that even with the adjustable control invariant set the RMPC-AU was persistently feasible because our setting allows that \( \{0\} \) is a subset of \( \mathbb{W} \) at all times; see Remark 1. It’s also noted that with the adjustable positive invariant set, RMPC-AU was persistently feasible with the constant-size uncertainty set by virtue of Algorithm 1.

B. Effects of \( \lambda \)

Recall that \( \lambda \) is the trade-off parameter between minimization of \( d \) and the maximization of \( y \). Fig. 3 shows the effect of \( \lambda \) on the solution of the optimal control problem (22) with the terminal set as the positive adjustable invariant set in Fig. 2. The system dynamics is initialized at \( d = 15m \) and \( \Delta v = 0m/s \). In the upper plot in Fig. 3, \( y \) is plotted at different values of \( \lambda \); higher values of \( \lambda \) bring about a larger \( y \), which is consistent with the formulation in (22) and to be expected. The lower plot in Fig. 3 shows, for different values of \( \lambda \), the average distance when the front vehicle does not decelerate or accelerate; i.e., \( y = 0 \). Also, \( d \) increases as the value of \( \lambda \) increases; since the front vehicle is allowed greater freedom in maneuvering, the tracking distance has to be correspondingly increased in order to preserve safety.

V. CONCLUSIONS

In this work, we studied the Robust Model Predictive Control problem with adjustable uncertainty sets. Our method optimizes over uncertainty sets as well as control policies, and guarantees robust recursive feasibility by enforcing an adjustable invariant set as a terminal constraint. We applied our framework to a cooperative adaptive cruise control and showed the validity of our approach in simulation. Future work includes new methods to construct adjustable invariant set for time-varying uncertainty sets and the actual experiment of our approach on CACC and examining our approach’s effects on energy saving by adjusting the inter-vehicular distance.

APPENDIX A: TRACTABLE REFORMULATION OF RMPC-AU

The optimization problem (4) can be formulated into a computationally tractable convex problem through the following steps from the work of [7]. First, we can reformulate our optimization problem (14) using the polytopic terminal constraints (19) as

\[
\min \left\{ \max_{s_k \in \mathbb{S}^N} \mathbf{c}^T(P_ts_t + p_t) \right\} - \sum_{k=0}^{N-1} \lambda_k \rho(y_{k|t} \mathbb{S} + y_{k|t}) \\
\text{s.t.} \quad P_t \in \mathbb{R}^{N_{n_x} \times N_{n_x}}, \quad p_t \in \mathbb{R}^{N_{n_u}}, \quad (Y_t, y_t) \in \mathcal{Y}, \quad C_t(P_ts_t + p_t) + D_t(Y_t, y_t) \leq d_t, \quad M_t(P_ts_t + p_t) + N_t(Y_t, y_t) \leq f, \quad f \leq 0, \\
\forall s_t \in \mathbb{S}^N.
\]

(23)
The robust control invariant set of the model (25) represents the adjustable control invariant set of our system (11) with the uncertainty set approximation (11). This set can be computed by Algorithm 2 which uses the precursor set of the control system (25) instead of the autonomous system (11). In this case, a precursor set is defined by the following.

**Definition V.1 (Robust Precursor Set).** For the system (25), we denote the robust precursor set to the set \( Z \) as

\[
\text{Pre}(Z, S) = \{ z \in \mathbb{R}^n : \exists u \in U \text{ s.t. } h(z, u, s) \in Z, \forall s \in S \}.
\]

(26)

\( \text{Pre}(Z, S) \) defines the set of states \( z \) which can be driven into the target set \( Z \) in one step while satisfying the system input constraints. If Algorithm 2 converges, then \( C_{\text{adj}} \) is an adjustable control invariant set.

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