CHEEGER-CHERN-SIMONS THEORY AND DIFFERENTIAL STRING CLASSES

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ABSTRACT. We introduce certain relative differential characters which we call Cheeger-Chern-Simons characters. These combine the well-known Cheeger-Simons characters with Chern-Simons forms. In the same way as the Cheeger-Simons characters generalize Chern-Simons invariants of oriented closed manifolds, the Cheeger-Chern-Simons characters generalize Chern-Simons invariants of oriented manifolds with boundary.

Using Cheeger-Chern-Simons characters, we introduce the notion of differential trivializations of universal characteristic classes. Specializing to the class $\frac{1}{2}p_1 \in H^4(BSpin_n;\mathbb{Z})$ this yields a notion of differential String classes. Differential String classes turn out to be stable isomorphism classes of geometric String structures.

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1. INTRODUCTION

The present article contributes to the program of String geometry. By String geometry we understand the study of geometric structures on a smooth manifold \( X \) that correspond to so-called Spin structures on the free loop space \( \mathcal{L}(X) \). Given a principal Spin\(_ n \)-bundle \( \pi: P \to X \), the loop space functor yields a principal \( \mathcal{L}(\text{Spin}_n) \)-bundle \( \pi: \mathcal{L}(P) \to \mathcal{L}(X) \).

To study the geometry of the free loop space, one would like to construct associated vector bundles from loop group bundles. However, all positive energy representations of the loop group \( L(\pi) \) are projective [25]. Therefor one needs to lift the structure group of the loop bundle from \( \mathcal{L}(\text{Spin}_n) \) to its universal central extension \( \mathcal{L}(\text{Spin}_n) \). The obstruction to such lifts is a certain cohomology class in \( H^3(\mathcal{L}(X); \mathbb{Z}) \), most easily described as the transgression to loop space of the class \( \frac{1}{2}p_1(P) \in H^4(X; \mathbb{Z}) \). Actual bundle lifts are sometimes referred to as Spin structures on the loop bundle.

Instead of those lifts one may also consider trivializations of the class \( \frac{1}{2}p_1(P) \) on the manifold \( X \) itself. This is the program of String geometry. On the one hand, \( \frac{1}{2}p_1(P) \) is the obstruction to lift the structure group of \( \pi: P \to X \) from Spin\(_ n \) to its 3-connected cover String\(_ n \). As is well known, String\(_ n \) cannot be realized as a finite dimensional Lie group, but as an infinite dimensional Fréchet Lie group [24]. From this perspective, String geometry is the study of principal String\(_ n \)-bundles (with connection) lifting the given principal Spin\(_ n \)-bundle (with connection). Actual such lifts may be called (geometric) String structures in the Lie theoretic sense. On the other hand, \( \frac{1}{2}p_1(P) \) is the characteristic class of a higher categorical geometric structure, the so-called Chern-Simons bundle 2-gerbe [8,34].

From this perspective, String geometry is the study of trivializations of the Chern-Simons bundle 2-gerbe together with a compatible connection. Actual such trivializations (with connection) are called (geometric) String structures in the gerbe theoretic sense.

Isomorphism classes of Spin\(_ n \)-bundles \( \pi: P \to X \) are in 1-1 correspondence with homotopy classes of maps \( f: X \to B\text{Spin}_n \) to the classifying space. Likewise, isomorphism classes of String\(_ n \)-lifts \( \pi: P \to X \) are in 1-1 correspondence with homotopy classes of lifts

\[
\begin{array}{ccc}
X & \xrightarrow{f} & B\text{Spin}_n \\
\hat{\pi} & \Downarrow & \\
\tilde{f} & \Downarrow & B\text{String}_n \\
\end{array}
\]

By work of Redden [27], the latter are also in 1-1 correspondence with certain cohomology classes in \( H^3(P; \mathbb{Z}) \), called String classes. The set of all String classes on \( \pi: P \to X \) is a torsor for the cohomology \( H^3(X; \mathbb{Z}) \) of the base. String classes are recovered by String structures, in both senses mentioned above. Associated with a String class on a compact riemannian manifold is a canonical 3-form \( \rho \in \Omega^3(X) \) which trivializes the class \( \frac{1}{2}p_1 \), see [27]. Transgression to loop space maps a String class to the Chern class of a line bundle over \( \mathcal{L}(P) \). The total space of this line bundle is the total space of the \( \mathcal{L}(\text{String}_n) \)-lift of the loop bundle \( \pi: \mathcal{L}(P) \to \mathcal{L}(X) \).

String classes are a special case of the more general concept of trivialization classes for universal characteristic classes for principal \( G \)-bundles [27]. The present article introduces a notion of differential trivialization classes for universal characteristic classes of principal \( G \)-bundles with connection \( \pi: (P, \theta) \to X \). These are certain differential cohomology classes \( \hat{\rho} \in \tilde{H}^3(P; \mathbb{Z}) \) whose characteristic classes are trivialization classes as in [27]. Associated with a differential trivialization class \( \hat{\rho} \) is a form \( \rho \in \Omega^3(X) \) on the base which trivializes the given characteristic class.

Specializing to the class \( \frac{1}{2}p_1 \in H^3(B\text{Spin}_n; \mathbb{Z}) \) we obtain our notion of differential String classes. These are differential cohomology classes \( \hat{\rho} \in \tilde{H}^3(P; \mathbb{Z}) \) on the total space of the Spin\(_ n \)-bundle that restrict on any fiber to the so-called basic class in \( \tilde{H}^3(\text{Spin}_n; \mathbb{Z}) \): the
stable isomorphism class of the basic gerbe \[20\]. The characteristic class of a differential String class is a String class in the sense above. Associated with a differential String class \( \hat{q} \) is a uniquely determined 3-form \( \rho \in \Omega^3(X) \) that trivializes the class \( \frac{1}{2} p_1(P) \). The set of all differential String classes on \( (P, \theta) \) is a torsor for the differential cohomology \( \hat{H}^3(X; \mathbb{Z}) \). String classes in our sense recover canonical differential refinements of String classes on a compact Riemannian manifold, that are implicit in \[27\]. They coincide with stable isomorphism classes of geometric String structures in the sense of \[34\] and thus coincide with the differential String classes from \[35\].

The key step in the notion of differential trivialization classes is the notion of Cheeger-Chern-Simons characters \( CCS_\theta \in \hat{H}^3(\pi; \mathbb{Z}) \) that we introduce in the present work. Cheeger-Chern-Simons characters are certain relative differential characters in the sense of \[6\]. Differential characters were introduced by Cheeger and Simons in \[9\] as certain \( U(1) \)-valued characters on the group of smooth singular cycles in \( X \). The ring of differential characters on a manifold \( X \) is nowadays called the differential cohomology of \( X \). We denote it by \( \hat{H}^\ast(X; \mathbb{Z}) \). Out of a differential character \( h \in \hat{H}^3(X; \mathbb{Z}) \) one obtains a smooth singular cohomology class \( \hat{c}(h) \in H^3(X; \mathbb{Z}) \) – its characteristic class – as well as a differential 3-form \( \text{curv}(h) \in \Omega^3(X) \) with integral periods – its curvature. Both the characteristic class and the curvature map are well-known to be surjective. In this sense, differential characters are refinement of smooth singular cohomology classes by differential forms.

As a particular example, Cheeger and Simons construct certain even degree differential characters on the base \( X \) of a principal \( G \)-bundle with connection \( (P, \theta) \rightarrow X \) with curvature given by the Chern-Weil forms of \( (P, \theta) \rightarrow X \). This construction thus lifts the Chern-Weil map to a differential character valued map. We refer to these particular characters as Cheeger-Simons characters and denote them by \( \tilde{C}W_\theta \) to emphasize their relation to the Chern-Weil map. They are also called differential characteristic classes by some authors \[7\] \[17\].

Relative differential characters were introduced by Brightwell and Turner in \[6\] as differential characters on the mapping cone cycles of a smooth map \( \varphi : A \rightarrow X \). Thus it would also be appropriate to call them mapping cone characters. A relative character \( h \in \hat{H}^3(\varphi; \mathbb{Z}) \) determines an absolute character \( \hat{p}(h) \in \hat{H}^3(X; \mathbb{Z}) \). Out of a relative character \( h \in \hat{H}^3(\varphi; \mathbb{Z}) \) one obtains an additional differential form \( \text{cov}(h) \in \Omega^{d-1}(A) \) – its covariant derivative. Relative differential characters in \( \hat{H}^3(\varphi; \mathbb{Z}) \) may be regarded as sections of the absolute characters \( \hat{p}(h) \in \hat{H}^3(X; \mathbb{Z}) \) along the map \( \varphi : A \rightarrow X \). An absolute character in \( \hat{H}^3(X; \mathbb{Z}) \) has sections along a smooth map \( \varphi \) if and only if its characteristic class vanishes upon pull-back by \( \varphi \). See \[1\] \[2\] for further details. Bundle gerbes on \( X \) provide a particular class of examples of relative differential characters: any bundle gerbe \( \mathcal{G} \) with connection, defined by a submersion \( \pi : Y \rightarrow X \), determines a relative differential character \( h_\mathcal{G} \in \hat{H}^3(\pi; \mathbb{Z}) \) with covariant derivative the curving \( H \in \Omega^2(Y) \) of the bundle gerbe. The absolute character \( \hat{p}(h_\mathcal{G}) \in \hat{H}^3(X; \mathbb{Z}) \) corresponds to the stable isomorphism class of the bundle gerbe.

In the present paper we refine the above mentioned construction of Cheeger-Chern-Simons characters to relative characters for the bundle projection \( \pi : P \rightarrow X \). The resulting relative characters will be called Cheeger-Chern-Simons characters, and we denote them by \( CCS_\theta \).

Two observations lead to the construction of Cheeger-Chern-Simons characters: First of all, universal characteristic classes of principal \( G \)-bundles vanish upon pull-back to the total space. Thus Cheeger-Simons characters admit sections along the bundle projection. Secondly, the pull-back of a Chern-Weil form \( \tilde{C}W_\theta \) to the total space has a canonical trivialization: the associated Chern-Simons form \( CS_\theta \). Putting these observations together, we obtain the notion of Cheeger-Chern-Simons characters:

Let \( G \) be a Lie group with finitely many components and \( (P, \theta) \rightarrow X \) a principal \( G \)-bundle with connection. Associated with an invariant polynomial \( \lambda \) on the Lie algebra \( \mathfrak{g} \)
and corresponding universal characteristic class \( u \in H^{2k}(BG; \mathbb{Z}) \) is a unique natural relative differential character \( CCS_{\theta}(\lambda, u) \in \tilde{H}^{2k}(\pi; \mathbb{Z}) \) with covariant derivative the Chern-Simons form \( CS_{\theta}(\lambda) \) that maps to the Cheeger-Simons character \( \tilde{CW}_{\theta}(\lambda, u) \) under the map \( \tilde{H}^4(\pi; \mathbb{Z}) \to H^*(X; \mathbb{Z}) \). The construction of the Cheeger-Chern-Simons character \( CCS_{\theta} \) relies on the same arguments as the construction of the Cheeger-Simons character \( CW_{\theta} \) in \([9]\). We also discuss multiplicativity and dependendence upon the connection.

In the same way as the Cheeger-Simons character \( CW_{\theta} \) generalizes the Chern-Simons invariants of oriented closed manifolds, the Cheeger-Chern-Simons character \( CCS_{\theta} \) generalizes the Chern-Simons invariants of oriented manifolds with boundary. Specializing to the universal characteristic class \( u = \frac{1}{2} p_1 \in H^4(B\text{Spin}_n, \mathbb{Z}) \), the Cheeger-Chern-Simons character \( CCS_{\theta}(\frac{1}{2} p_1) \) coincides with the relative differential cohomology class \( h_{\theta, 2} \in \tilde{H}^4(\pi; \mathbb{Z}) \) represented by the Chern-Simons bundle 2-gerbe \( \mathcal{F}, \mathcal{G} \) on \( X \).

For the application of the Cheeger-Chern-Simons characters to differential trivializations of universal characteristic classes and to differential String classes, we need two kinds of transgression: Transgression to loop space of (absolute and relative) characters was discussed in \([1, 2]\). We also construct transgression in the universal bundle as a map from Cheeger-Simons characters on \( BG \) to certain characters on \( G \). This is done by modifying the usual construction of the transgression \( T : H^*(BG; \mathbb{Z}) \to H^{*-1}(G; \mathbb{Z}) \) using the mapping cone cohomology of the bundle projection.

The paper is organized as follows: Section 2 reviews the well-known Chern-Weil, Chern-Simons and Cheeger-Simons constructions and constructs Cheeger-Chern-Simons characters. Section 3 discusses several notions of transgression for Cheeger-Simons characters: transgression to loop space and transgression in the universal bundle. Section 4 introduces differential trivializations of universal characteristic classes of principal \( G \)-bundles. Section 5 specializes to the case of differential String classes. The appendices provide background information on differential characters, bundle 2-gerbes and transgression.

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2. CHEEGER-CHERN-SIMONS THEORY

Let \( G \) be a Lie group with finitely many components. Let \( g \) be its Lie algebra. Let \( (P, \theta) \to X \) be a principal \( G \) bundle with connection. Let \( \lambda \) be an invariant polynomial on \( g \), homogeneous of degree \( k \). By the classical Chern-Weil construction, the polynomial \( \lambda \) associates with \((P, \theta)\) the Chern-Weil form \( CW_{\theta}(\lambda) \in \Omega^{2k}(X) \). In fact, the Chern-Weil form \( CW_{\theta}(\lambda) \) is closed and its de Rham cohomology class \( [CW_{\theta}(\lambda)] \in H^{2k}_{\text{dR}}(X) \) does not depend upon the connection \( \theta \).

The Chern-Weil construction has two well-known refinements, the Chern-Simons and Cheeger-Simons construction: The pull-back of the Chern-Weil form \( CW_{\theta}(\lambda) \) along the bundle projection \( \pi : P \to X \) is an exact form. The Chern-Simons form \( CS_{\theta}(\lambda) \in \Omega^{2k-1}(P) \), constructed in \([10]\), satisfies \( dCS_{\theta}(\lambda) = \pi^*CW_{\theta}(\lambda) \). Moreover, the Chern-Weil construction has a unique lift to the differential cohomology \( \tilde{H}^{2k}(X; \mathbb{Z}) \), constructed in \([9]\). In other words, the Chern-Weil form \( CW_{\theta}(\lambda) \) is the curvature of a differential character \( \tilde{CW}_{\theta}(\lambda) \in \tilde{H}^{2k}(X; \mathbb{Z}) \).

In this section, we further refine these well-known constructions: we show that there is a canonical natural relative differential character for the bundle projection \( \pi : P \to X \).
with covariant derivative $CS_\theta(\lambda)$, which maps to $CW_\theta(\lambda)$ under the map $\tilde{p} : \tilde{H}^{2k}(\pi; \mathbb{Z}) \to \tilde{H}^{2k}(X; \mathbb{Z})$. Since this construction combines the Chern-Simons form $CS_\theta(\lambda)$ with the Cheeger-Simons differential character $CW_\theta(\lambda)$, we call it the Cheeger-Chern-Simons construction.

For convenience of the reader, we review the classical Chern-Weil and Chern-Simons constructions. The basic notions of relative and absolute differential characters are reviewed in Appendix A.

2.1. The Chern-Weil, Chern-Simons and Cheeger-Simons constructions. In this section, we briefly review the results of the classical Chern-Weil and Chern-Simons constructions. We review in more detail the refinement of the classical Chern-Weil map to a differential character valued map, constructed by Cheeger and Simons in [9]. We term this the Cheeger-Simons construction.

2.1.1. Universal bundles and connections. Let $G$ be a Lie group with finitely many components. Let $\mathfrak{g}$ be its Lie algebra. Let $x \in BG$. We denote by $\pi_{EG} : EG \to BG$ the universal principal $G$-bundle over the classifying space of $G$. We denote by $EG_x := \pi_{EG}^{-1}(x)$ the fiber of $EG$ over $x$.

Any principal $G$-bundle $\pi : P \to X$ can be written as pull-back of the universal bundle via a pull-back diagram

$$
P = f^*EG \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
A completely different point of view on classifying spaces, universal bundles and universal connections is taken in [12]. By means homotopical algebra, Freed and Teleman construct a universal principal $G$-bundle, denoted $EV_G \to B_G$, where both the total space and base are simplicial sheaves (instead of infinite dimensional manifolds). In this setting, there is a canonical universal connection for principal $G$-bundles, which is a $g$-valued 1-form on $EV_G$. Moreover, this universal connection induces unique classifying maps for principal $G$-bundles with connection.

Although it would be interesting to study the Cheeger-Simons and Cheeger-Chern-Simons characters in terms of this new notion of universal connection, we will not pursue this approach in the present paper. For a fairly general exposition of generalized differential cohomology in terms of homotopical algebra, we refer to [7].

2.1.2. The Chern-Weil construction. Let $G$ be a Lie group with finitely many components. Let $g$ be its Lie algebra. Following the notation of [9], we set

$$I^k(G) := \left\{ \lambda : g \otimes \ldots \otimes g \to \mathbb{R} \mid \lambda \text{ symmetric, multilinear, Ad}_G\text{-invariant} \right\}$$

for the space of $Ad_G$-invariant symmetric multilinear real valued functions from the $k$-fold tensor product of $g$. Such functions are called invariant homogeneous polynomials of degree $k$ on $g$.

Let $(\pi, \theta) : P \to X$ be a principal $G$-bundle with connection. We denote by $F_{\theta} \in \Omega^2(X, \text{Ad}(P))$ the curvature 2-form of the connection $\theta$. The Chern-Weil map $CW_{\theta} : I^k(G) \to \Omega^2_k(X)$ associates to an invariant polynomial $\lambda \in I^k(G)$ the Chern-Weil form

$$CW_{\theta}(\lambda) := \lambda(F_{\theta}^k) = \lambda(F_{\theta} \wedge \ldots \wedge F_{\theta}) \in \Omega^2_k(X).$$

The Chern-Weil form $CW_{\theta}(\lambda)$ is a closed differential form whose de Rham cohomology class does not depend upon the choice of connection $\theta$.

More precisely, the Chern-Weil forms for two connections $\theta_0$, $\theta_1$ differ by the differential of the Chern-Simons form $CS(\theta_0, \theta_1; \lambda)$ for these connections. The construction of the Chern-Simons form is reviewed in Section 2.1.3 below.

2.1.3. The Cheeger-Simons construction. As above, let $G$ be a compact Lie group with Lie algebra $g$. Let $\Theta$ be a fixed universal connection on the universal principal $G$-bundle $\pi_{EG} : EG \to BG$. We denote the universal Chern-Weil map to the real cohomology of $BG$

$$CW : I^k(G) \to H^{2k}(BG; \mathbb{R}), \quad \lambda \mapsto [\lambda(F_{\Theta}^k)]_{dR} \in H^{2k}_{dR}(BG) \cong H^{2k}(BG; \mathbb{R}).$$

For any principal $G$-bundle with connection $\pi : (P, \theta) \to X$, we have the commutative diagram:

$$
\begin{array}{c}
I^k(G) \xrightarrow{CW} H^{2k}(BG; \mathbb{R}) \xrightarrow{c_{dR}} H^{2k}(BG; \mathbb{Z}) \\
\Omega^2_k(X) \xrightarrow{dR} H^{2k}(X; \mathbb{R}) \xrightarrow{c_{dR}} H^{2k}(X; \mathbb{Z}).
\end{array}
$$

The maps $c_{dR}$ and $c_{dR}$ are induced by a classifying map $f : X \to BG$ for the bundle with connection $(P, \theta)$. They do not depend upon the choice of classifying map. The map $dR : \Omega^2_k(X) \to H^{2k}(X; \mathbb{R})$ is the projection $\Omega^2_k(X) \to H^{2k}_{dR}(X)$, followed by the de Rham isomorphism. The horizontal maps in the right square are the change of coefficients maps induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. For an integral cohomology class $u \in H^{2k}(X; \mathbb{Z})$, we denote by $u_{dR}$ its image in $H^{2k}(X; \mathbb{R})$.

\footnote{In [10] this form is denoted $P(\theta)$ where $P$ denotes an invariant polynomial on $g$.}
The question arises whether the Chern-Weil form $CW_\theta(\lambda)$ may be represented as curvature of an appropriate differential character (in case it has integral periods). The question is answered affirmatively in [9] by what we term the Cheeger-Simons construction. Following the notation established there, we put:

$$K^{2k}(G; \mathbb{Z}) := \{ (\lambda, u) \in \hat{H}^3(G) \times H^{2k}(BG; \mathbb{Z}) \mid CW(\lambda) = u \}$$

for the set of pairs of invariant polynomials and integral universal characteristic classes that match in real cohomology. Moreover, denote by

$$R^n(X; \mathbb{Z}) := \{ (\omega, w) \in \Omega^n_0(X) \times H^n(X; \mathbb{Z}) \mid [\omega]_{dR} = w \}$$

the set of pairs of closed forms with integral periods $\omega$ and smooth singular cohomology classes $w$ that match in real cohomology.

It is shown in [9] that the Chern-Weil map $CW_\theta$ has a unique natural lift to a differential character valued map in the following sense: For any $k \geq 1$ and any principal $G$-bundle with connection $\pi : (P, \theta) \to X$, there exists a unique natural map $\hat{CW}_\theta$ such that the diagram

$$\begin{array}{ccc}
K^{2k}(G; \mathbb{Z}) & \xrightarrow{\hat{CW}_\theta} & \hat{H}^{2k}(X; \mathbb{Z}) \\
\downarrow \scriptstyle{\text{curv}} & & \downarrow \scriptstyle{(\text{curv})} \\
R^{2k}(X; \mathbb{Z}) & \xrightarrow{\text{CW}_\theta \times \text{c}_\mathbb{Z}} & R^{2k}(X; \mathbb{Z})
\end{array}$$

commutes. We call the differential character $\hat{CW}_\theta(\lambda, u) \in \hat{H}^{2k}(X; \mathbb{Z})$ the **Cheeger-Simons character** associated with $(\lambda, u) \in K^{2k}(G; \mathbb{Z})$. By diagram (2), the curvature of the Cheeger-Simons character is the Chern-Weil form

$$\text{curv}(\hat{CW}_\theta(\lambda, u)) = CW_\theta(\lambda)$$

and its characteristic class is given by

$$c(\hat{CW}_\theta(\lambda, u)) = c_Z(u) = f^*u = u(P).$$

Naturality of the map $\hat{CW}_\theta$ means that for any smooth map $g : X' \to X$ and the pull-back bundle $g^*(P, \theta) \to X'$, we have

$$g^*(\hat{CW}_\theta(\lambda, u)) = \hat{CW}_{g^*\theta}(\lambda, u) \in \hat{H}^{2k}(X' ; \mathbb{Z}).$$

We call the map $\hat{CW}_\theta : K^*(G; \mathbb{Z}) \to \hat{H}^*(X ; \mathbb{Z})$ the **Cheeger-Simons construction**. By [9, Cor. 2.3], it is a ring homomorphism with respect to the ring structures of $K^*(G; \mathbb{Z})$ and $\hat{H}^*(X; \mathbb{Z})$.

### 2.1.4. The Chern-Simons construction

As before, let $\pi : (P, \theta) \to X$ be a principal $G$-bundle with connection and $\lambda \in \hat{H}^3(G)$ an invariant polynomial. The pull-back bundle $\pi^*P \to P$ has a tautological section $\sigma_{\text{uant}}$, which maps any point $p \in P$ to itself, now considered as a point in the fiber $(\pi^*P)_p = P_{\pi(p)}$. The tautological section $\sigma_{\text{uant}}$ yields a trivialization $P \times G \xrightarrow{\cong} \pi^*P$, $(p, g) \mapsto \sigma_{\text{uant}}(p) \cdot g$. In particular, $\pi^*P$ carries a canonical flat connection $\theta_{\text{uant}}$, obtained from the trivial connection on $P \times G$ by pull-back via the inverse of the trivialization.

Since the de Rham cohomology classes of Chern-Weil forms do not depend upon the choice of connection, all Chern-Weil classes of $\pi^*P$ vanish. In particular, the pull-backs $\pi^*CW_\theta(\lambda) = CW_{\pi^*\theta}(\lambda) \in \Omega^{2k}(P)$ of Chern-Weil forms are exact forms on $P$. They are the differentials of the Chern-Simons forms to be reviewed next.

The **Chern-Simons form** of a connection $\theta$ was first constructed in [10] as an invariant $(2k - 1)$-form on $P$ whose differential is the pull-back $\pi^*CW_\theta(\lambda)$ of the Chern-Weil form.

---

3In [9] this character is denoted $S_{\mu}(\alpha)$ where $P$ denotes an invariant polynomial, $\mu$ a universal characteristic class and $\alpha$ a principal $G$-bundle with connection.
In fact, there are two different notions of Chern-Simons forms, closely related to one another: the Chern-Simons form of two connections $\theta_0, \theta_1$ is a $(2k-1)$-form on the base $CS(\theta_0, \theta_1; \lambda) \in \Omega^{2k-1}(X)$ with differential the difference of the corresponding Chern-Weil forms, while the Chern-Simons form for one connection $\theta$ is a $(2k-1)$-form on the total space $CS_\theta(\lambda) \in \Omega^{2k-1}(P)$ with differential the pull-back of the Chern-Weil form.

Denote by $\mathcal{A}(P)$ the space of connections on the principal $G$-bundle $P \to X$. It is an affine space for the vector space $\Omega^1(X; \text{Ad}(P))$ of 1-forms on the base $X$ with values in the associated bundle $\text{Ad}(P) := P \times_{\text{Ad}} \mathfrak{g}$. In particular, $\mathcal{A}(P)$ is path connected.

Let $\theta_0, \theta_1$ be connections on $\pi : P \to X$. Let $\lambda : [0, 1] \to \mathcal{A}(P)$ be a smooth path joining them. It defines a connection $\theta \in \mathcal{A}(P \times [0, 1])$ on the principal $G$-bundle $P \times [0, 1] \to X \times [0, 1]$. Integrating the Chern-Weil form of this connection over the fiber of the trivial bundle $X \times [0, 1] \to X$, we obtain a $(2k-1)$-form $\int_{[0,1]}\text{CW}_{\theta(t)}(\lambda) \in \Omega^{2k-1}(X)$. By the fiberwise Stokes theorem, we have:

\[
\text{CW}_{\theta_0}(\lambda) - \text{CW}_{\theta_1}(\lambda) = -d \int_{[0,1]} \text{CW}_\theta(\lambda) + \int_{[0,1]} d \text{CW}_\theta(\lambda).
\]

Since $\mathcal{A}(P)$ is an affine space, there is a canonical path joining two connections $\theta_0$ and $\theta_1$, namely the straight line $\theta(t) := (1-t)\theta_0 + t\theta_1$ from $\theta_0$ to $\theta_1$. The Chern-Simons form for two connections is the $(2k-1)$-form on the base, obtained as above, for the straight line:

\[
CS(\theta_0, \theta_1; \lambda) := -\int_{[0,1]} \text{CW}_{(1-t)\theta_0 + t\theta_1}(\lambda) \in \Omega^{2k-1}(X).
\]

As above, it satisfies $dCS(\theta_0, \theta_1; \lambda) = \text{CW}_{\theta_0}(\lambda) - \text{CW}_{\theta_1}(\lambda)$.

The Chern-Simons form $CS_\theta(\lambda) \in \Omega^{2k-1}(P)$ for one connection $\theta$ on the bundle $P \to X$ is defined as the Chern-Simons form for the two connections $\theta_{\text{flat}}$ and $\pi^*\theta$ on the pull-back bundle $\pi^*P \to P$:

\[
CS_\theta(\lambda) := CS(\theta_{\text{flat}}; \pi^*\theta; \lambda).
\]

Since the connection $\theta_{\text{flat}}$ is flat, we have $dCS_\theta(\lambda) = \text{CW}_{\pi^*\theta}(\lambda) = \pi^*\text{CW}_\theta(\lambda)$.

We call the map $CS_\theta : P(G) \to \Omega^{2k-1}(P)$, $\lambda \mapsto CS_\theta(\lambda)$, the Chern-Simons construction. Up to an exact remainder, the Chern-Simons form $CS_\theta(\lambda)$ is the unique natural $(2k-1)$-form on $P$ with differential $\pi^*\text{CW}_\theta(\lambda)$. In fact, any two $(2k-1)$-forms on $EG$ with differential $\pi^*_{E\mathfrak{g}}\text{CW}_\theta(\lambda)$ differ by an exact form, since $EG$ is contractible.

In general, the Chern-Simons form $CS_\theta(\lambda)$ is an invariant form on $P$, but it is neither horizontal nor closed. It depends in a well-known manner upon the connection $\theta$ (see [10] Prop. 3.8 and Section 2.3 below). The pull-back of the Chern-Simons form $CS_\theta(\lambda)$ to the fiber $P_x \cong G$ over any point $x \in X$ does not depend upon the choice of connection $\theta$. It can be expressed solely (and explicitly) in terms of the Maurer-Cartan form of $G$. Moreover, for low dimensional bases $X$, the Chern-Simons form $CS_\theta(\lambda)$ is independent of the connection $\theta$ and is itself a closed form, see [10] Thm. 3.9).

Let $\Theta$ be a universal connection on the universal principal $G$-bundle $\pi_{EG} : EG \to BG$. Then we have $dCS_\Theta(\lambda) = \pi_{EG}^*\text{CW}_\Theta(\lambda) = \text{curv}(\pi_{EG}^*\text{CW}_\Theta(\lambda, u))$. From the exact sequence $\text{Che} \mathfrak{g}$ and contractibility of $EG$ we conclude $t(CS_\Theta(\lambda)) = \pi_{EG}^*\text{CW}_\Theta(\lambda, u)$. Thus the Chern-Simons form provides a topological trivialization of the pull-back of the Cheeger-Simons character $\text{CW}_\Theta(\lambda, u)$ to $EG$.

Now let $(P, \theta) \to X$ be a principal $G$-bundle with connection. Let $f : X \to BG$ be a classifying map for the bundle with connection and denote by $F : P = f^*EG \to EG$ the

\[\text{In [10] the invariant polynomials are denoted by } P. \text{ The corresponding Chern-Simons form is denoted TP}(\theta) \text{ to emphasize its relation to the transgression of the Chern-Weil form.} \]
induced bundle map. Then we have:

\[ (9) \quad t(\lambda) = F^* t(\lambda) = \pi^* \pi_{EG}^* \hat{CW}_\phi(\lambda, u) = \pi^* f^* \hat{CW}_\phi(\lambda, u) = \pi^* \hat{CW}_\phi(\lambda, u). \]

Here \( t : \Omega^{2k-1}(EG) \to \tilde{H}^{2k}(EG; \mathbb{Z}) \) denotes topological trivialization of differential characters, as explained in the Appendix A.

2.1.5. The Chern-Simons action. Let \( (P, \theta) \to X \) be a principal G-bundle with connection. Let \( f : M \to X \) be a smooth map. Suppose that the pull-back bundle \( \pi : f^* P \to M \) admits a section \( \sigma : M \to f^* P \), hence \( \pi \circ \sigma = \text{id}_M \). Then we obtain a trivialization \( M \times G \to f^* P \) by \( (x, g) \mapsto \sigma(x) \cdot g \). Thus the bundle \( \pi : f^* P \to M \) can be represented by a constant map \( f : M \to BG \) and hence all its characteristic classes vanish.

In particular, any Cheeger-Simons character \( \hat{CW}_{f^*}\theta(\lambda, u) \) is topologically trivial. In fact, topological trivializations are given by pull-back via \( \sigma \) of the corresponding Chern-Simons form \( CS_{f^*}\theta(\lambda) \). Namely, from (9) we obtain:

\[ (10) \quad \hat{CW}_{f^*}\theta(\lambda, u) = \sigma^* (\pi^* \hat{CW}_{f^*}\theta(\lambda, u)) = t(\sigma^* CS_{f^*}\theta(\lambda)). \]

Since the left hand side is independent of the choice of section \( \sigma \), the same holds for the right hand side.

In particular, if \( M \) is a closed oriented \((2k - 1)\) manifold and the pull-back bundle \( f^* P \to M \) admits sections, then we obtain the Chern-Simons invariant of \( M \) by evaluating the Cheeger-Simons character on the fundamental class:

\[ \hat{CW}_{f^*}\theta(\lambda, u)[M] = \exp \left( 2 \pi i \int_M \sigma^* CS_{f^*}\theta(\lambda) \right). \]

This happens e.g. if \( G \) is simply connected and \( M \) is a closed oriented 3-manifold, for in this case, any principal \( G \)-bundle \( \pi : f^* P \to M \) admits sections. In this sense, the Cheeger-Simons character \( \hat{CW}_\theta(\lambda, u) \) generalizes the classical Chern-Simons invariants of closed oriented 3-manifolds.

In Section 2.3, we generalize this observation to the Chern-Simons action of oriented manifolds with boundary.

2.2. The Cheeger-Chern-Simons construction. In this section, we combine the Cheeger-Simons and Chern-Simons constructions to a relative differential character valued map. This map will be called the Cheeger-Chern-Simons construction.

As above, let \( G \) be a Lie group with finitely many components. Fix a classifying connection \( \Theta \) on the universal principal \( G \)-bundle \( \pi_{EG} : EG \to BG \). Let \( \pi : (P, \theta) \to G \) be principal \( G \)-bundle with connection and let \( f : X \to BG \) be a classifying map for the bundle with connection. Since \( EG \) is contractible, universal characteristic classes for principal \( G \)-bundles vanish upon pull-back to the total space. Thus any differential character on \( X \) with characteristic class a universal characteristic class for principal \( G \)-bundles is topologically trivial along the bundle projection.

This holds in particular for the Cheeger-Simons character \( \hat{CW}_\theta(\lambda, u) \in \tilde{H}^{2k}(X; \mathbb{Z}) \): since \( u \in H^{2k}(BG; \mathbb{Z}) \), we have

\[ \pi^* c(\hat{CW}_\theta(\lambda, u)) \equiv \pi^* c_{BG} u = f^* \pi^* u = 0. \]

From the exact sequence (40) we conclude that \( \hat{CW}_\theta(\lambda, u) \) admits sections along \( \pi \). A canonical such section will be obtained by the Cheeger-Chern-Simons construction below.

2.2.1. Prescribing the covariant derivative. To begin with, we lift the map \( CW_\theta \times c_{BG} \) to \( R^{2k}(\pi; \mathbb{Z}) \):
Proposition 1. Let \((P, \theta) \to X\) be a principal \(G\)-bundle with connection. Then the Chern-Weil map \(CW_\theta\) has a canonical natural lift \(CCS_\theta\) such that the diagram

\[
\begin{array}{ccc}
K^{2k}(BG; \mathbb{Z}) & \xrightarrow{cw \times cZ} & R^{2k}(X; \mathbb{Z}) \\
\downarrow{CCS_\theta} & & \downarrow{R^{2k}(\pi; \mathbb{Z})} \\
R^{2k}(\pi; \mathbb{Z}) & & \\
\end{array}
\]

commutes.

Proof. Since \(EG\) is contractible, the long exact sequence for the mapping cone complex of the bundle projection \(\pi_{EG} : EG \to BG\) reads:

\[
\cdots \to H^{2k-1}(EG; \mathbb{Z}) \to H^{2k}(\pi; \mathbb{Z}) \overset{\delta}{\to} H^{2k}(BG; \mathbb{Z}) \to H^{2k}(EG; \mathbb{Z}) \to \cdots
\]

In particular, we obtain isomorphisms \(p : H^{2k}(\pi_{EG}; \mathbb{Z}) \xrightarrow{\simeq} H^{2k}(BG; \mathbb{Z})\). For a universal characteristic class \(u \in H^{2k}(BG; \mathbb{Z})\), we denote by \(\bar{u} := p^{-1}(u) \in H^{2k}(\pi_{EG}; \mathbb{Z})\) its pre-image under this isomorphism.

Now let \((P, \theta) \to X\) be a principal \(G\)-bundle with connection. We define the lift \(CCS_\theta : K^{2k}(BG; \mathbb{Z}) \to R^{2k}(\pi; \mathbb{Z})\) by

\[
CCS_\theta(\lambda, u) := (CW_\theta(\lambda), CS_\theta(\lambda), cl_\mathbb{Z}(\bar{u})).
\]

As above, \(cl_\mathbb{Z} : H^{2k}(\pi_{EG}; \mathbb{Z}) \to H^{2k}(\pi; \mathbb{Z})\) denotes the pull-back with the classifying map \(f : X \to BG\). In terms of a universal connection \(\Theta\) on \(\pi_{EG} : EG \to BG\) we have:

\[
CCS_\theta(\lambda, u) = f^*CCS_\Theta(\lambda, u)
= f^*(CW_\Theta(\lambda), CS_\Theta(\lambda), \bar{u}).
\]

Clearly, the composition of the map \(CCS_\theta\) with the forgetful map \(R^{2k}(\pi; \mathbb{Z}) \to R^{2k}(X; \mathbb{Z})\) yields the map \(CW_\theta \times cl_\mathbb{Z}\). By definition, the map \(CCS_\theta\) is natural with respect to pull-back of bundles with connection by smooth maps \(g : X' \to X\).

It remains to check that \(CCS_\theta\) takes values in \(R^{2k}(\pi; \mathbb{Z})\). Let \((\lambda, u) \in K^{2k}(G; \mathbb{Z})\). We show that \((CW_\Theta(\lambda), CS_\Theta(\lambda)) \in \Omega^{2k}(\pi_{EG})\) is \(d_{\pi_{EG}}\)-closed with integral periods.\(^5\) By definition of the Chern-Weil and Chern-Simons forms, we have:

\[
d_{\pi_{EG}}(CW_\Theta(\lambda), CS_\Theta(\lambda)) = (dCW_\Theta(\lambda), \pi_{EG}^*CW_\Theta(\lambda) - dCS_\Theta(\lambda)) = 0.
\]

Moreover, since \((\lambda, u) \in K^{2k}(G; \mathbb{Z})\), the Chern-Weil form \(CW_\Theta(\lambda)\) has integral periods. The \(U(1)\)-valued cocycle \(\exp(2\pi i CW_\Theta(\lambda)) \in Z^{2k}(BG; U(1))\) thus vanishes on integral cycles and represents the trivial class in \(H^{2k}(BG; U(1)) \cong \text{Hom}(H_{2k}(BG; \mathbb{Z}), U(1))\). Hence there exists a \(U(1)\)-valued cochain \(w \in C^{2k-1}(BG; U(1))\) satisfying \(\exp(2\pi i CW_\Theta(\lambda)) = \delta w\). We then have:

\[
\delta(\pi^*w) = \pi^*(\delta w)
= \pi^*(\exp(2\pi i CW_\Theta(\lambda)))
= \exp(2\pi i \pi^* CW_\Theta(\lambda))
= \exp(2\pi i dCS_\Theta(\lambda))
= \delta \exp(2\pi i CS_\Theta(\lambda)).
\]

\(^5\)This essentially follows from the proof of [9, Prop. 3.15], but for convenience of the reader, we give the full argument.
Thus \((\exp(2\pi i CS(\lambda)) - \pi^* w)\) is a cocycle in \(Z^{2k-1}(EG; U(1))\). Since \(EG\) is contractible, we find a cochain \(v \in C^{2k-2}(EG; U(1))\) such that \((\exp(2\pi i CS(\lambda)) - \pi^* w) = \delta v\).

Now let \((s, t) \in C_{2k}(\pi_{EG}; \mathbb{Z})\) be a relative cycle. Then we have:

\[
\exp(2\pi i \int_{(s, t)} (\text{CW}_\Theta(\lambda), CS_\Theta(\lambda))) = (\delta w, \pi^* w + \delta v)(s, t)
\]

\[
= (w, v)(\partial_\pi(s, t))
\]

This yields \(\int_{(s, t)} (\text{CW}_\Theta(\lambda), CS_\Theta(\lambda)) \in \mathbb{Z}\). Hence the pair \((\text{CW}_\Theta(\lambda), CS_\Theta(\lambda)) \in \Omega^{2k}(\pi_{EG})\) has integral periods. In particular, the image of the relative de Rham class \([\text{CW}(\lambda), CS(\lambda)]_{\text{dR}}\) under the de Rham isomorphism lies in the image of the reduction of coefficients map \(H^{2k}(\pi_{EG}; \mathbb{Z}) \to H^{2k}(\pi_{EG}; \mathbb{R})\), \(\bar{u} \mapsto u\).

It remains to show that \((\text{CW}_\Theta(\lambda), CS_\Theta(\lambda))\) and \(\bar{u}\) match in real cohomology, i.e. that \(dR(\text{CW}_\Theta(\lambda), CS_\Theta(\lambda)) = u\) from \(2.1.3\) and hence \(dR(\text{CW}_\Theta(\lambda), CS_\Theta(\lambda)) = u\).

2.2.2. The Cheeger-Chern-Simons character. In the same way as the Cheeger-Simons construction \(\text{CW}_\Theta\) lifts the map Chern-Weil construction \(\text{CW}_\Theta \times cl\mathbb{Z}\) to a differential character valued map, there is a canonical natural lift of \(\text{CCS}_\Theta\) to a relative differential character valued map \(\text{CCS}_\Theta\). The relative differential character \(\text{CCS}_\Theta(\lambda, u)\) trivializes the differential character \(\text{CW}(\lambda, u)\) along the bundle projection \(\pi\) like the Chern-Simons form \(CS_\Theta(\lambda)\) trivializes the Chern-Weil form \(\pi^* \text{CW}(\lambda)\) in relative de Rham cohomology. It is the unique natural section of the Cheeger-Simons character with prescribed covariant derivative equal to the Chern-Simons form. It is thus uniquely determined by the Chern-Simons construction, which itself is not unique, but canonical.

Theorem 2 (Cheeger-Chern-Simons construction). Let \(G\) be a Lie group with finitely many components. Let \((\lambda, u) \in K^{2k}(G; \mathbb{Z})\). For any principal \(G\)-bundle with connection \(\pi : (P, \Theta) \to X\), there exists a unique relative differential character \(\text{CCS}_\Theta(\lambda, u) \in H^{2k}(\pi; \mathbb{Z})\) such that the following holds:

The curvature and covariant derivative of \(\text{CCS}_\Theta(\lambda, u)\) are given by the Chern-Weil and Chern-Simons form:

\[
(\text{curv}, \text{cov})(\text{CCS}_\Theta(\lambda, u)) = (\text{CW}_\Theta(\lambda), CS_\Theta(\lambda)).
\]

The Cheeger-Chern-Simons character \(\text{CCS}_\Theta(\lambda, u)\) trivializes the Chern-Simons character \(\text{CW}_\Theta(\lambda, u)\) along the bundle projection \(\pi\):

\[
\bar{\rho}_\pi(\text{CCS}_\Theta(\lambda, u)) = \text{CW}_\Theta(\lambda, u).
\]

The Cheeger-Chern-Simons construction \(\text{CCS}_\Theta\) is natural with respect to pull-back by smooth maps, i.e., for any smooth map \(f : X' \to X\) and the pull-back bundle \(f^*P\), we have

\[
f^* \text{CCS}_\Theta(\lambda, u) = \text{CCS}_{f^*\Theta}(\lambda, u).
\]

\(^6\)Here we denote the pull-back along the pull-back diagram of \(f : X \to X'\) simply by \(f^*\). Strictly speaking, we would have \((f, f^*)\), where \(F : f^*P \to P\) is the induced bundle map on the pull-back bundle. Likewise, the pull-back connection \(f^*\Theta\) is given by the connection 1-form \(F^*\theta \in \Omega^1(f^*P)\).
From (12) and (13), we obtain the commutative diagram:

\[
\begin{array}{ccc}
K^{2k}(G; \mathbb{Z}) & \xrightarrow{CCS} & \tilde{H}^{2k}(\pi; \mathbb{Z}) \\
\xrightarrow{\text{curv.cov.c}} & & \\
\xrightarrow{\text{curv.cov.c}} & & \\
\xrightarrow{\text{curv.cov.c}} & & R^{2k}(\pi; \mathbb{Z})
\end{array}
\]

In particular, we have

\[
(\text{curv.cov.c})(CCS_\Theta(\lambda, u)) = CCS_\Theta(\lambda, u).
\]

**Proof.** We first prove uniqueness. By the requirement (14) that the Cheeger-Chern-Simons construction be natural with respect to pull-back of principal G-bundles with connections, it is uniquely determined by the map $\tilde{CCS_\Theta}: K^{2k}(G; \mathbb{Z}) \to \tilde{H}^{2k}(\pi_{EG}; \mathbb{Z})$ on the universal principal G-bundle $\pi_{EG}: EG \to BG$ with a fixed universal connection $\Theta$. We show that this map is uniquely determined by (12) and (13).

It is well-known, that $H^{2k-1}(BG; \mathbb{R}) = \{0\}$ for any $k \geq 1$. Consider the long exact sequence of the mapping cone complex of the bundle projection $\pi_{EG}: EG \to BG$:

\[
\cdots \to H^{2k-2}(EG; \mathbb{R}) \to H^{2k-1}(\pi_{EG}; \mathbb{R}) \to H^{2k-1}(BG; \mathbb{R}) \to H^{2k-1}(EG; \mathbb{R}) \to \cdots
\]

Thus $H^{2k-1}(\pi_{EG}; \mathbb{R}) = \{0\}$ and the exact sequence (19) reads:

\[
0 \to \frac{H^{2k-1}(\pi_{EG}; \mathbb{R})}{H^{2k-1}(\pi_{EG}; Z)} \xrightarrow{\text{curv.cov.c}} \frac{\tilde{H}^{2k}(\pi_{EG}; \mathbb{Z})}{\tilde{H}^{2k}(\pi_{EG}; Z)} \xrightarrow{\text{curv.cov.c}} R^{2k}(\pi_{EG}; \mathbb{Z}) \to 0.
\]

Hence the map $(\text{curv.cov.c}): \tilde{H}^{2k}(\pi_{EG}; \mathbb{Z}) \to R^{2k}(\pi_{EG}; \mathbb{Z})$ is an isomorphism.

Now let $CCS_\Theta: K^{2k}(G; \mathbb{Z}) \to \tilde{H}^{2k}(\pi_{EG}; \mathbb{Z})$ be any map satisfying (12) and (13). Let $(\lambda, u) \in K^{2k}(BG; \mathbb{Z})$ and $\tilde{u} \in H^{2k}(\pi_{EG}; \mathbb{Z})$ as in the proof of Lemma 1. By (13) we have $c(\tilde{p}(CCS_\Theta(\lambda, u))) = c(CW(\lambda, u)) = u$. The isomorphism $p: H^{2k}(\pi_{EG}; \mathbb{Z}) \to H^{2k}(BG; \mathbb{Z})$, $\tilde{u} \mapsto u$, from the mapping cone exact sequence yields the identification $c(CCS_\Theta(\lambda, u)) = \tilde{u}$. Together with (12), we obtain $CCS_\Theta(\lambda, u) = (\text{curv.cov.c})^{-1}(CW_\Theta(\lambda), CCS_\Theta(\lambda, u))$. Thus the Cheeger-Chern-Simons map $CCS_\Theta: K^{2k}(G; \mathbb{Z}) \to \tilde{H}^{2k}(\pi_{EG}; \mathbb{Z})$ for a universal bundle with universal connection $\pi_{EG}: (EG, \Theta) \to BG$ is the unique lift in the diagram

\[
\begin{array}{ccc}
K^{2k}(G; \mathbb{Z}) & \xrightarrow{CCS} & \tilde{H}^{2k}(\pi_{EG}; \mathbb{Z}) \\
\xrightarrow{\text{curv.cov.c}} & & \\
\xrightarrow{\text{curv.cov.c}} & & \\
\xrightarrow{\text{curv.cov.c}} & & R^{2k}(\pi_{EG}; \mathbb{Z})
\end{array}
\]

In other words, $CCS = (\text{curv.cov.c})^{-1} \circ CCS$.

To prove existence, we define the Cheeger-Chern-Simons map by the above formula and show that this construction satisfies the statements: Let $\pi: (P, \Theta) \to X$ be a principal G-bundle with connection. Fix a classifying map $f: X \to BG$ such that $(P, \Theta) = f^\ast(EG, \Theta)$.
as principal $G$-bundles with connection. Then put:

$$(18) \quad CCS_{\Theta}(\lambda, u) := f^*(CCS_{\Theta}(\lambda, u)) = f^*((\text{curv}, \text{cov}, c)^{-1}(CW_{\Theta}(\lambda), CS_{\Theta}(\lambda), \bar{u})) .$$

By the very definition, the construction is natural with respect to pull-back of $G$-bundles with connection, hence it satisfies $(14)$. Moreover,

$$(\text{curv}, \text{cov})(CCS_{\Theta}(\lambda, u)) = f^*(\text{curv}, \text{cov})(CCS_{\Theta}(\lambda, u))
= f^*(CW_{\Theta}(\lambda), CS_{\Theta}(\lambda))
= (CW_{\Theta}(\lambda), CS_{\Theta}(\lambda)) .$$

This shows $(12)$.

Exactness of the sequence

$$0 \to H^{2k-1}(BG; \mathbb{R}) \to H^{2k}(BG; \mathbb{Z}) \to R^{2k}(BG; \mathbb{Z}) \to 0$$

implies that the Cheeger-Simons character $CW_{\Theta}(\lambda, u)$ is uniquely determined by its curvature $\text{curv}(CW_{\Theta}(\lambda, u)) = CW(\lambda)$ and characteristic class $c(CW_{\Theta}(\lambda, u)) = u$. Since $c(\tilde{\rho}(CCS_{\Theta}(\lambda, u))) = u$ and $\text{curv}(\tilde{\rho}(CCS_{\Theta}(\lambda, u))) = CW(\lambda)$ we conclude $\tilde{\rho}(CCS_{\Theta}(\lambda, u)) = CW_{\Theta}$. By naturality of the Cheeger-Simons construction, we thus obtain:

$$\tilde{\rho}(CCS_{\Theta}) = \tilde{\rho}(f^*CCS_{\Theta}) = f^*\tilde{\rho}(CCS_{\Theta}) = f^*(CW_{\Theta}) = CW_{\Theta} .$$

This proves $(13)$.

Finally, diagram $(15)$ and formula $(16)$ are immediate from $(18)$.

\[\square\]

**Remark 3.** Existence of sections of the Cheeger-Simons character $CW_{\Theta}(\lambda, u)$ with prescribed covariant derivative $CS_{\Theta}(\lambda)$ also follows from $(1)$ Prop. 75), since the pair $(CW_{\Theta}(\lambda), CS_{\Theta}(\lambda)) \in \Omega^{2k-2}(\pi_{EG})$ is closed with integral periods.

2.2.3. Multiplicativity. It is well-known that the Cheeger-Simons construction is multiplicative: it defines a ring homomorphism $CW : K^*(G; \mathbb{Z}) \to \tilde{H}^*(X; \mathbb{Z})$. In $(2)$ we show that for any smooth map $\varphi : A \to X$, the graded group $\tilde{H}^*(\varphi; \mathbb{Z})$ is a right module over the ring $\tilde{H}^*(X; \mathbb{Z})$. It is easy to see that the Cheeger-Chern-Simons construction is (almost) multiplicative with respect to this module structure:

**Proposition 4** (Multiplicativity). Let $\pi : (P, \Theta) \to X$ be a principal $G$-bundle with connection. Let $(\lambda_1, u_1) \in K^{2k_1}(G; \mathbb{Z})$ and $(\lambda_2, u_2) \in K^{2k_2}(G; \mathbb{Z})$. Then there exists a differential form $\rho \in \Omega^{2k-2}(EG)$ such that we have:

$$(19) \quad CCS_{\Theta}(\lambda_1, u_1) \ast CCS_{\Theta}(\lambda_2, u_2) = CCS_{\Theta}(\lambda_1 \cdot \lambda_2, u_1 \cup u_2) + t_{\pi}(0, \rho) .$$

\[\text{Proof.} \quad \text{It suffices to prove this for the universal $G$-bundle $\pi_{EG} : EG \to BG$ with universal connection $\Theta$. Relative differential characters in $\tilde{H}^*(\pi_{EG}; \mathbb{Z})$ are uniquely determined by their curvature, covariant derivative and characteristic class. Hence it suffices to compare those data for the two sides of $(19)$.

The Chern-Weil map $CW_{\Theta}$ is multiplicative while the Chern-Simons map $CS_{\Theta}$ is multiplicative only up to an exact form. Thus there exists a differential form $\rho \in \Omega^{2k-2}(EG)$ such that $CS_{\Theta}(\lambda_1) \wedge \pi^*CW_{\Theta}(\lambda_2) = CS_{\Theta}(\lambda_1 \cdot \lambda_2) - d\rho$. This yields:

$$(\text{curv, cov})(CCS_{\Theta}(\lambda_1, u_1) \ast CCS_{\Theta}(\lambda_2, u_2))
= (\text{curv, cov})(CCS_{\Theta}(\lambda_1, u_1)) \ast \pi^*\text{curv}(\tilde{CW}_{\Theta}(\lambda_2, u_2))
= (CW_{\Theta}(\lambda_1) \wedge CW_{\Theta}(\lambda_2), CS_{\Theta}(\lambda_1) \wedge \pi^*CW_{\Theta}(\lambda_2))
= (CW_{\Theta}(\lambda_1 \cdot \lambda_2), CS_{\Theta}(\lambda_1 \cdot \lambda_2)) + d\rho(0, \rho)
= (\text{curv, cov})(CCS_{\Theta}(\lambda_1 \cdot \lambda_2) + t_{\pi}(0, \rho))$$

as principal $G$-bundles with connection. Then put:
and
\[
c(\text{CCS}_\theta(\lambda_1, u_1) \ast \text{CW}_\theta(\lambda_2, u_2)) = c(\text{CCS}_\theta(\lambda_1, u_1)) \cup \pi^* c(\text{CW}_\theta(\lambda_2, u_2)) \\
= \delta_{\lambda_1} \cup \pi^* u_2 \\
= \delta_{\lambda} \cup u_2 \\
= c(\text{CCS}_\theta(\lambda_1 \cdot \lambda_2, u_1 \cup u_2) + \iota_\theta(0, \rho)). \quad \square
\]

2.3. The Chern-Simons action. In the same way as the Cheeger-Simons character \(\text{CW}_\theta(\lambda, u)\) generalizes the classical Chern-Simons action along oriented closed \((2k - 1)\)-manifolds, the Cheeger-Chern-Simons character generalizes the classical Chern-Simons action along oriented manifolds with boundary:

Thus let \(\pi : (P, \theta) \to X\) be a principal \(G\)-bundle with connection, and \((\lambda, u) \in K^{2k}(G; \mathbb{Z})\). Let \(M\) be a compact oriented \((2k - 1)\)-manifold with boundary, and denote by \(i_\partial M : \partial M \to M\) the inclusion of the boundary. Let \(f : M \to X\) be a smooth map and \(F : f^* P \to P\) the induced bundle map. Let \(\sigma : M \to f^* P\) be a smooth section of the pull-back bundle. Put \(g := F \circ \sigma|_{\partial M} : \partial M \to P\). This way we obtain a map of pairs \((M, \partial M) \xrightarrow{(f, g)} (X, P)\). The character \((f, g)^* \text{CCS}_\theta(\lambda, u) \in \hat{H}^{2k}(i_\partial M; \mathbb{Z})\) is topologically trivial, since \(H^{2k}(i_\partial M; \mathbb{Z}) = 0\) for dimensional reasons. We factorize the map \((f, g)\) as

\[
\begin{array}{ccc}
\(M, f^* P\) & \xrightarrow{(\sigma, \text{id})} & \(M, f^* P\) \\
\(\text{id}, \sigma|_{\partial M}\) & \xrightarrow{(\text{id}, \text{id})} & \(M, \partial M\) \\
\(M, \partial M\) & \xrightarrow{(f, g)} & (X, P)
\end{array}
\]

Since characters in \(\hat{H}^{2k}(i_\partial M; \mathbb{Z})\) are uniquely determined by their covariant derivative, we have \((\pi, \text{id}_M)^*(f, F)^* \text{CCS}_\theta(\lambda, u) = \iota_\theta(F^* \text{CS}_\theta(\lambda), 0)\) and hence \((f, g)^* \text{CCS}_\theta(\lambda, u) = \iota_{\partial M}(\sigma^* \text{CS}^\theta(\lambda), 0)\). This yields

\[
\langle \text{CCS}_\theta(\lambda, u) \rangle ((f, g)^* [M, \partial M]) = ((f, g)^* \text{CCS}_\theta(\lambda, u))[M, \partial M] \\
= \exp\left(2\pi i \int_M \sigma^* \text{CS}^\theta\right).
\]

Since the left hand side only depends upon \(\sigma|_{\partial M}\), so does the right hand side.

2.4. Dependence upon the connection. As above let \((P, \theta) \to X\) be a principal \(G\)-bundle with connection and \((\lambda, u) \in K^{2k}(G; \mathbb{Z})\). In this section we discuss the dependence of the Cheeger-Chern-Simons character \(\text{CCS}_\theta(\lambda, u)\) upon the connection \(\theta\). We first review the well-known dependencies of the Chern-Weil form \(\text{CW}_\theta(\lambda)\), the Chern-Simons form \(\text{CS}_\lambda(\lambda)\) and the Cheeger-Simons character \(\text{CW}_\theta(\lambda, u)\) upon the connection \(\theta\).

Let \(\theta_0, \theta_1 \in \mathfrak{g}(P)\) be connections on \(\pi : P \to X\). As explained in Section 2.1.4, the Chern-Weil forms for the two connections differ by the differential of the Chern-Simons form:

\[
\text{CW}_{\theta_1}(\lambda) - \text{CW}_{\theta_0}(\lambda) = d\text{CS}(\theta_0, \theta_1; \lambda).
\]
Analogously, we find for the Chern-Simons forms of the two connections:

\[ CS_{\theta_1}(\lambda) - CS_{\theta_2}(\lambda) = d\left( \int_{[0,1]} CS_{1-t}(\theta_0 + t\theta_1) \right) - d\int_{[0,1]} dCS_{1-t}(\theta_0 + t\theta_1) \]

\[ = -d\alpha(\theta_0, \theta_1; \lambda) - \int_{[0,1]} \pi^*CW_{1-t}(\theta_0 + t\theta_1) \]

(20)

Combining the formulae for the Chern-Weil and Chern-Simons forms, we thus have

\[ (CW_{\theta_1}(\lambda), CS_{\theta_1}(\lambda)) - (CW_{\theta_2}(\lambda), CS_{\theta_2}(\lambda)) = d\pi(CS(\theta_0, \theta_1, \lambda), \alpha(\theta_0, \theta_1, \lambda)). \]

Now consider the Cheeger-Simons characters for the two connections \( \theta_0, \theta_1 \in \mathcal{A}(P) \). Choose smooth classifying maps \( f_i : X \to BG, i = 0, 1 \), for the bundle with connection. Let \( \theta(t) := (1-t)\theta_0 + t\theta_1 \) be the straight line joining the two connections, and \( f_i : X \to BG \) a smooth family of smooth classifying maps for the connections \( \theta, t \in [0,1] \). Then the map \( F : X \times [0,1] \to BG, F(t, \cdot) := f_t \), is a smooth homotopy from \( f_0 \) to \( f_1 \). The homotopy formula (42) yields

\[ \text{CW}_{\theta_t}(\lambda, u) - \text{CW}_{\theta_0}(\lambda, u) = f_t^*\text{CW}_{\theta_0}(\lambda, u) - f_0^*\text{CW}_{\theta_0}(\lambda, u) = t_\pi(F^*\text{CW}_{\theta_0}(\lambda)) \]

(21)

We have the analogous result for the Cheeger-Chern-Simons characters of two connections:

**Proposition 5.** Let \( \pi : P \to X \) be a principal G-bundle with connections \( \theta_0, \theta_1 \in \mathcal{A}(P) \). Let \( (\lambda, u) \in K^{2k}(G, \mathbb{Z}) \). Then we have:

\[ \text{CCS}_{\theta_t}(\lambda, u) - \text{CCS}_{\theta_0}(\lambda, u) = t_\pi(CS(\theta_0, \theta_1; \lambda), \alpha(\theta_0, \theta_1; \lambda)). \]

**Proof.** As above choose classifying maps \( f_i \) for the connections \( \theta_t := (1-t)\theta_0 + t\theta_1 \) for \( t \in [0,1] \). Denote by \( F : (X, P) \times [0,1] \to (BG, EG) \) the induced homotopy from \( f_0 \) to \( f_1 \). Using the homotopy formula (42) for relative characters, we find:

\[ \text{CCS}_{\theta_t}(\lambda, u) - \text{CCS}_{\theta_0}(\lambda, u) = f_t^*\text{CCS}_{\theta_0}(\lambda, u) - f_0^*\text{CCS}_{\theta_0}(\lambda, u) = t_\pi(F^*\text{CS}_{\theta_0}(\lambda)) - t_\pi(F^*\text{CS}_{\theta_0}(\lambda)) \]

(21)

3. **Transgression**

In this section we discuss transgression of Cheeger-Simons characters. On the one hand we have the usual transgression of absolute and relative characters to (free and based) loop spaces. On the other hand, we derive a generalization of the transgression map in the universal principal G-bundle from integral cohomology to Cheeger-Simons characters. The two transgressions coincide only topologically under the homotopy equivalence between

\[ \text{Note that by the orientation conventions, we have } f_{[0,1]} \omega = (-1)^{k-1} \int_0^1 \omega ds \text{ for any k-form } \omega. \]
G and $L_0(BG)$. A further notion of transgression may be obtained from work of Murray and Vozzo [21] in combination with fiber integration of differential characters.

3.1. Transgression to loop space. In [1] Ch. 9] we construct transgression of (absolute) differential characters on $X$ to mapping spaces and in particular to the free loop space $L(X)$:

$$\tau : \hat{H}^*(X;\mathbb{Z}) \rightarrow \hat{H}^{*-1}(L(X);\mathbb{Z}), \quad h \mapsto \hat{\pi}_*(ev^*h).$$

Here $ev : L(X) \times S^1 \rightarrow X$, $(\gamma, t) \mapsto \gamma(t)$, denotes the evaluation map and $\hat{\pi}_*$ the fiber integration for the trivial bundle $\pi : L(X) \times S^1 \rightarrow L(X)$. In [2] Ch. 5 we generalized the concept of fiber integration to relative differential cohomology. For a smooth map $\varphi : A \rightarrow X$, we thus have the transgression map

$$\tau : \hat{H}^*(\varphi;\mathbb{Z}) \rightarrow \hat{H}^{*-1}(L(X,A);\mathbb{Z}), \quad h \mapsto \hat{\pi}_*(ev^*h).$$

Here $L(X,A)$ denotes Fréchet manifold of (pairs of) smooth maps $\gamma : S^1 \rightarrow (X,A)$ and $ev : L(X,A) \times S^1 \rightarrow (X,A)$, $(\gamma, t) \mapsto \gamma(t)$, denotes the evaluation map.

Note that both transgression maps have natural restrictions to transgression maps to based loop spaces. We will apply these transgression maps in Section 5 to the Cheeger-Chern-Simons characters.

3.2. Transgression in the universal bundle. Let $G$ be a Lie group with finitely many components. Let $\pi_{EG} : EG \rightarrow BG$ be the universal principal $G$-bundle over the classifying space $BG$. In this section we consider various ways to define the cohomology transgression in the universal principal $G$-bundle. We add a new description of the transgression by using the mapping cone cohomology of the bundle projection. This enables us to define in the following section the universal transgression of Cheeger-Simons characters.

In the literature there appear two different conventions for the transgression, which are (almost) inverse to each other. We follow the notion of transgression used in [10, 15, 9], which is a the left inverse to the one considered in [3]. The former is called suspension by some authors, since it is closely related to the suspension isomorphism. Moreover, it is related to the transgression to loop space as considered in the previous section. We review these relations in the Appendix below.

The transgression $T : H^*(BG;\mathbb{Z}) \rightarrow H^{*-1}(G;\mathbb{Z})$ is usually defined as the composition of the following maps:

$$H^*(BG;\mathbb{Z}) \xrightarrow{\pi_{EG}^*} H^*(EG;\mathbb{Z}) \xrightarrow{T} H^{*-1}(EG;\mathbb{Z}) \xrightarrow{\pi_{EG}^*} H^{*-1}(G;\mathbb{Z}).$$

Here $x \in BG$ is an arbitrary point. The third map is the inverse of the connecting homomorphism $H^{*-1}(EG;\mathbb{Z}) \rightarrow H^*(EG;\mathbb{Z})$ in the long exact sequence of the pair $(EG,EG)_x$. By contractibility of $EG$, the connecting homomorphism is an isomorphism. The isomorphism $H^{*-1}(EG;\mathbb{Z}) \xrightarrow{\sim} H^{*-1}(G;\mathbb{Z})$ is induced by the diffeomorphisms $G \rightarrow EG_x$ obtained from the group action.

There are several other constructions of the transgression map. We use the following: Let $u \in H^*(BG;\mathbb{Z})$ and $x \in BG$. Choose a cocycle $\mu \in C^*(BG;\mathbb{Z})$ that represents the class $u$. Since $EG$ is contractible, we find a chain $\nu \in C^{-1}(EG;\mathbb{Z})$ such that $\pi_{EG}^*\mu = \delta \nu$. Thus the pair $(\mu, \nu) \in C^*(\pi_{EG};\mathbb{Z})$ is a cocycle of the mapping cone complex.

Let $i_{EG} : EG_x \rightarrow EG$ denote the inclusion of the fiber over $x$. Then we have $\delta(i_{EG};\nu) = i_{EG}^*\pi_{EG}^*\mu = \pi_{EG}^*(i_{EG}^*\mu)$. For dimensional reasons, we have $i_{EG}^*\mu = \delta \alpha$ for some $\alpha \in C^{-1}(\{x\};\mathbb{Z})$. Thus $\pi_{EG}^*\alpha - i_{EG}^*\nu$ is a cocycle and we set

$$T(u) := [\pi_{EG}^*\alpha - i_{EG}^*\nu] \in H^{*-1}(EG;\mathbb{Z}) \xrightarrow{\sim} H^{*-1}(G;\mathbb{Z}).$$
It is well-known that the two definitions of the transgression given above are independent of the various choices and that they yield the same map.\footnote{A classical reference is\cite{[3]} where it is proved not for the transgression considered here but for a right inverse, which is a map from a submodule of $H^{1-1}(G;\mathbb{Z})$ to a quotient of $H^1(BG;\mathbb{Z})$. Note also that the mapping cone complex comes with a different sign than the usual relative cohomology: for the inclusion of a subset $\varphi = i_k : A \hookrightarrow X$, the homomorphism $H^{1-1}(A;\mathbb{Z}) \rightarrow H^1(X,A;\mathbb{Z})$ in the usual long exact sequence coincides with the negative of the homomorphism $H^{1-1}(A;\mathbb{Z}) \rightarrow H^1(i_k;\mathbb{Z})$ in the mapping cone sequence.}

We now describe a modification of the second construction by using the mapping cone cohomology of the bundle projection $\pi_{EG}$. As in Section 2.2 we note that the mapping cone long exact sequence

$$
\ldots \rightarrow H^{1-1}(EG;\mathbb{Z}) \rightarrow H^*(\pi_{EG};\mathbb{Z}) \rightarrow H^*(BG;\mathbb{Z}) \rightarrow H^*(EG;\mathbb{Z}) \rightarrow \ldots
$$

yields an isomorphism $H^*(BG;\mathbb{Z}) \rightarrow H^*(\pi_{EG};\mathbb{Z})$. Let $x \in BG$ be an arbitrary point and $i_\ast : \{x\} \rightarrow BG$ the inclusion. The induced bundle map is the inclusion $i_{EG} : EG_\ast \rightarrow EG$ of the fiber over $x$. Thus we have the pull-back diagram:

$$
\begin{array}{ccc}
EG_\ast & \xrightarrow{i_{EG}} & EG \\
\pi_{EG}|_{EG_\ast} & \downarrow & \pi_{EG} \\
\{x\} & \xrightarrow{i_\ast} & BG
\end{array}
$$

Pull-back along $(i_\ast,i_{EG})$ yields homomorphisms $H^*(\pi_{EG};\mathbb{Z}) \xrightarrow{(i_\ast,i_{EG})^*} H^*(\pi_{EG}|_{EG_\ast};\mathbb{Z})$. The mapping cone sequence for the left vertical map $\pi_{EG}|_{EG_\ast}$ yields isomorphisms $H^{1-1}(EG_\ast;\mathbb{Z}) \rightarrow H^*(\pi_{EG}|_{EG_\ast};\mathbb{Z})$. Thus we may regard the transgression map as the composition:

$$
(24) \quad H^*(BG;\mathbb{Z}) \xrightarrow{\approx} H^*(\pi_{EG};\mathbb{Z}) \xrightarrow{\approx} H^*(\pi_{EG}|_{EG_\ast};\mathbb{Z}) \approx H^{1-1}(EG_\ast;\mathbb{Z}) \approx H^{1-1}(G;\mathbb{Z}).
$$

We show that this representation of the transgression coincides with the previous one: Let $\mu \in Z^*(BG;\mathbb{Z})$ be a cocycle representing $u \in H^*(BG;\mathbb{Z})$. As above we find a cochain $v \in C^*(EG;\mathbb{Z})$ such that the pair $(\mu, v) \in C^*(\pi_{EG};\mathbb{Z})$ is a mapping cone cocycle. It represents the class $\tilde{u} \in H^*(\pi_{EG};\mathbb{Z})$, since $\mu$ represents the class $u \in H^*(BG;\mathbb{Z})$. Pull-back along $-(i_\ast,i_{EG})$ yields

$$
-(i_\ast,i_{EG})^*(\mu, v) = -(-\delta \alpha, \delta i_{EG}, v) = (0, \pi_{EG}^* \alpha - i_{EG}^* v) = \delta_{EG}(\alpha,0)
$$

for any $\alpha \in C^{-1}(\{x\};\mathbb{Z})$ with $\delta \alpha = i_\ast \mu$. Now the inverse of the isomorphism $H^{1-1}(EG_\ast;\mathbb{Z}) \rightarrow H^*(\pi_{EG}|_{EG_\ast};\mathbb{Z})$ maps the class represented by $-(i_\ast,i_{EG})^*(\mu, v)$ to the class represented by $\pi_{EG}^* \alpha - i_{EG}^* v$.

3.3. Universal transgression of Cheeger-Simons characters. In this section we define transgression of Cheeger-Simons characters $\text{CW}_\Theta(\lambda, u) \in H^2k(BG;\mathbb{Z})$ for a universal principal $G$-bundle $\pi_{EG} : EG \rightarrow BG$ with universal connection $\Theta$.

A first attempt to define the universal transgression in differential cohomology is to replace singular cohomology groups in (23) by differential cohomology groups. As above let $i_\ast : \{x\} \rightarrow BG$ be the inclusion of an arbitrary point in the base and $i_{EG} : EG_\ast \rightarrow EG$ the inclusion of the fiber over $x$. Now the long exact sequences (40) for the inclusion $i_{EG}$ reads:

$$
\ldots \rightarrow H^{-2}(EG;U(1)) \rightarrow H^{1-1}(EG_\ast;\mathbb{Z}) \rightarrow H^*(\pi_{EG}|_{EG_\ast};\mathbb{Z}) \rightarrow H^*(EG;\mathbb{Z}) \rightarrow \ldots
$$

Thus the map $\tilde{\gamma}_{EG} : H^{1-1}(EG_\ast;\mathbb{Z}) \rightarrow H^*(i_{EG}|_{EG_\ast};\mathbb{Z})$ is not an isomorphism. But it is injective and thus has a canonical left inverse. This way we obtain a transgression map defined on all
those characters $h \in \hat{H}^*(BG; \mathbb{Z}) \cong \hat{H}^*(i_\ast; \mathbb{Z})$ for which the pull-back $(\pi_{EG}, \pi_{EG}|_{EG})^\ast h \in \hat{H}^*(i_{EG}; \mathbb{Z})$ lies in the image of the map $i_{EG} : \hat{H}^{n-1}(EG; \mathbb{Z}) \to \hat{H}^*(i_{EG}; \mathbb{Z})$. It maps the character $h$ to the unique preimage of $(\pi_{EG}, \pi_{EG}|_{EG})^\ast h$.

If $H^{2k-1}(G; \mathbb{R}) = \{0\}$, then the Cheeger-Simons character $\hat{C}W_\Theta(\lambda, u) \in \hat{H}^{2k}(BG; \mathbb{Z})$ associated with $(\lambda, u) \in K^{2k}(G; \mathbb{Z})$ belongs to the domain of definition of this transgression. Its transgression is the unique differential character in $\hat{H}^{2k-1}(G; \mathbb{Z})$ with characteristic class $T(u)$ and curvature $i_{EG}^\ast CS_\Theta(\lambda)$: From the mapping cone exact sequence

$$\ldots \to \hat{H}^{2k-2}(EG; \mathbb{R}) \to \hat{H}^{2k-1}(i_{EG}; \mathbb{R}) \to \hat{H}^{2k-1}(EG; \mathbb{R}) \to \ldots$$

we obtain $\hat{H}^{2k-1}(i_{EG}; \mathbb{R}) = \{0\}$. From the exact sequence (39) we conclude that relative characters in $\hat{H}^{2k}(i_{EG}; \mathbb{Z})$ are uniquely determined by their curvature, covariant derivative and characteristic class. The character $(\pi_{EG}, \pi_{EG}|_{EG})^\ast \hat{C}W_\Theta(\lambda, u) \in \hat{H}^{2k}(i_{EG}; \mathbb{Z})$ has curvature $\pi_{EG}^\ast CW_\Theta(\lambda)$, covariant derivative $i_{EG}^\ast CS_\Theta(\lambda)$ and characteristic class $(\pi_{EG}, \pi_{EG}|_{EG})^\ast u$. The same holds for the image of the absolute character with curvature $i_{EG}^\ast CS_\Theta(\lambda)$ and characteristic class $T(u)$ under the map $i_{EG}^\ast : \hat{H}^{2k-1}(EG; \mathbb{Z}) \to \hat{H}^{2k}(i_{EG}; \mathbb{Z})$.

Instead of using (23), we construct the transgression map analogously to the one for singular cohomology in (24). Replacing singular cohomology groups by differential cohomology groups, we obtain the following sequence of maps:

$\hat{H}^n(BG; \mathbb{Z}) \xrightarrow{\hat{\pi}_{EG}} \hat{H}^n(\pi_{EG}; \mathbb{Z}) \xrightarrow{-(i_{EG})^\ast} \hat{H}^n(\pi_{EG}|_{EG}; \mathbb{Z}) \xrightarrow{i_{EG}^\ast} \hat{H}^{n-1}(EG; \mathbb{Z}) \xrightarrow{\hat{C}W_\Theta} \hat{H}^{n-1}(BG; \mathbb{Z})$.

Note that for $n \geq 3$, the third map is an isomorphism. For in this case, the long exact sequence (40) for the projection $\pi_{EG}|_{x}$ reads:

$$\hat{H}^{n-2}((x); U(1)) \xrightarrow{i_{EG}^\ast} \hat{H}^{n-1}(EG; \mathbb{Z}) \xrightarrow{(i_{EG})^\ast} \hat{H}^n(\pi_{EG}|_{EG|x}; \mathbb{Z}) \xrightarrow{\hat{\pi}_{EG|x}} \hat{H}^n((x); \mathbb{Z}) \xrightarrow{=0} \ldots$$

On the other hand, the homomorphism $\hat{\pi}_{EG} : \hat{H}^n(\pi_{EG}; \mathbb{Z}) \to \hat{H}^n(BG; \mathbb{Z})$ is not injective and hence does not have a canonical left inverse: The long exact sequence (40) for the bundle projection $\pi_{EG}$ reads:

$$\ldots \to \hat{H}^{n-1}(EG; \mathbb{Z}) \xrightarrow{i_{EG}^\ast} \hat{H}^n(\pi_{EG}; \mathbb{Z}) \xrightarrow{\hat{\pi}_{EG}} \hat{H}^n(BG; \mathbb{Z}) \xrightarrow{c_{\pi_{EG}}} \hat{H}^n(EG; \mathbb{Z}) \ldots$$

Thus the homomorphism $\hat{\pi} : \hat{H}^n(\pi_{EG}; \mathbb{Z}) \to \hat{H}^n(BG; \mathbb{Z})$ is surjective with kernel $d\Omega^{n-2}(EG)$. Although an arbitrary character $\hat{H}^n(BG; \mathbb{Z})$ does not have a canonical preimage under the map $\hat{\pi}$, the Cheeger-Simons characters have:

To the Cheeger-Simons character $\hat{C}W_\Theta(\lambda, u) \in \hat{H}^{2k}(BG; \mathbb{Z})$ is canonically associated the Cheeger-Chern-Simons character $\hat{CS}_\Theta(\lambda, u) \in \hat{H}^{2k}(\pi_{EG}; \mathbb{Z})$. This yields a canonical notion of universal transgression for Cheeger-Simons characters:

**Definition 6.** Let $G$ be a Lie group with finitely many components. Let $\pi_{EG} : EG \to BG$ be a universal principal $G$-bundle with universal connection $\Theta$. Let $(\lambda, u) \in K^{2k}(G; \mathbb{Z})$ with $k \geq 2$. The **universal transgression** of the Cheeger-Simons character $\hat{C}W_\Theta(\lambda, u) \in \hat{H}^{2k}(BG; \mathbb{Z})$ is the differential character on $G$ defined by

$$T(\hat{C}W_\Theta(\lambda, u)) := -1^{-1}((i_{EG})^\ast \hat{CS}_\Theta(\lambda, u)) \in \hat{H}^{2k-1}(EG; \mathbb{Z}) \cong \hat{H}^{2k-1}(G; \mathbb{Z}).$$

The Chern-Simons form $CS_\Theta(\lambda) \in \Omega^{2k-1}(EG)$ was initially constructed in (10) as a de Rham representative of the transgression $T : \hat{H}^n(BG; \mathbb{R}) \to \hat{H}^{n-1}(G; \mathbb{R})$. The universal transgression of Cheeger-Simons characters as in Definition 6 is a refinement of the
transgression $T : H^* (BG; \mathbb{Z}) \to H^{*-1} (G; \mathbb{Z})$ and of the Chern-Simons construction: for any $(\lambda, u) \in K^{2k} (G; \mathbb{Z})$ we have

$$
(26) \quad \lambda \in \mathbb{Z} \quad \implies \quad T (c (CW_{\Theta} (\lambda, u))) = \lambda \quad \text{and} \quad \text{curv} (T (cw_{\Theta} (\lambda, u))) = \text{curv} (\lambda).
$$

In case $H^{2k-2} (G; \mathbb{R}) = \{0\}$, the transgressed character $T (cw_{\Theta} (\lambda, u))$ is the unique character in $\hat{H}^{2k-1} (G; \mathbb{Z})$ with characteristic class $T (u)$ and curvature $i_{EG}^{\prime} CS_{\Theta} (\lambda, u)$. It does not depend upon the connection $\Theta$, since the form $i_{EG}^{\prime} CS_{\Theta} (\lambda, u)$ can be expressed purely in terms of the Maurer-Cartan form of $G$, see [10].

3.4. Transgression via the caloron correspondence. Let $G$ be a compact connected Lie group and $\mathcal{L} (G)$ and $\mathcal{L}_0 (G)$ the free and the based loop group. Calorons are certain periodic $G$-instantons on a manifold of the form $X \times S^1$. They were introduced in theoretical physics [14, 22]. Later it was observed [13] that they are in 1-1 correspondence with $\mathcal{L} (G)$-instantons on $X$. In mathematical terms, the so-called caloron correspondence [21, 16] is a 1-1 correspondence between principal $\mathcal{L} (G)$- or $\mathcal{L}_0 (G)$-bundles over a manifold $X$ and principal $G$-bundles over $X \times S^1$. Both sides of the correspondence may be equipped with connections: in this case, the caloron correspondence is a 1-1 correspondence between $G$-bundles over $X \times S^1$ with connection and $\mathcal{L} (G)$- or $\mathcal{L}_0 (G)$-bundles over $X$ with connection and a so-called Higgs field.

A particular $\mathcal{L} (G)$-bundle with connection arises from the path fibration $\mathcal{P} (G) \to G$. This bundle also carries a canonical Higgs field. Thus the caloron correspondence gives rise to a canonical principal $G$-bundle with connection $(\hat{P}, \hat{\theta}) \to G \times S^1$, see [21]. Since the base of this $G$-bundle is a (trivial) fiber bundle with compact oriented fibers, we may apply fiber integration for differential characters as constructed in [1] to the Cheeger-Simons characters $\hat{CW}_{\Theta} (\lambda, u) \in \hat{H}^{2k} (G \times S^1)$. This yields another notion of transgression for Cheeger-Simons characters: Let $(\lambda, u) \in K^{2k} (G; \mathbb{Z})$. Then we put:

$$
(27) \quad T' (\hat{CW}_{\Theta} (\lambda, u)) := \int_{S^1} \hat{CW}_{\Theta} (\lambda, u).
$$

It would be interesting to know whether the two transgression maps $T$ (defined via the Cheeger-Chern-Simons characters) and $T'$ (defined via the caloron correspondence) coincide. So far, we have no answer to this question in general. However, one easily sees that both constructions yield characters with the same curvature and characteristic class: It is shown in [8, Prop. 3.4] that $c (T' (\hat{CW}_{\Theta} (\lambda, u))) = \int_{S^1} u (P) = T (u)$ and in [21, Prop. 4.11] that also $\text{curv} (T' (\hat{CW}_{\Theta} (\lambda, u))) = \text{curv} (\lambda)$. Thus from (26) and (27) we conclude that $c (T (\hat{CW}_{\Theta} (\lambda, u))) = c (T' (\hat{CW}_{\Theta} (\lambda, u)))$ and $\text{curv} (T (\hat{CW}_{\Theta} (\lambda, u))) = \text{curv} (T' (\hat{CW}_{\Theta} (\lambda, u)))$. Hence the two transgression maps $T, T'$ from Cheeger-Simons characters to differential characters on $G$ agree up to topologically trivial flat characters. In case $H^{2k-2} (G; \mathbb{R}) = \{0\}$, they coincide.

Note that we do not have a general notion of transgression of differential characters $\hat{H}^{2k} (BG; \mathbb{Z}) \to \hat{H}^{2k-1} (G; \mathbb{Z})$. Both transgressions discussed here do not immediately extend to all of $\hat{H}^{2k} (BG; \mathbb{Z})$: The transgression $T$ from Section 3.3 uses the fact that Cheeger-Simons characters have canonical sections along the projection map $\pi_{EG} : EG \to BG$. An arbitrary character in $\hat{H}^{2k} (BG; \mathbb{Z})$ does not have a canonical section along $\pi_{BG}$. The transgression $T'$ in the current section uses the fact that the construction of Cheeger-Simons characters is natural with respect to connection preserving bundle maps: any classifying map of the bundle with connection $(\hat{P}, \hat{\theta}) \to G \times S^1$ maps the character $\hat{CW}_{\Theta} (\lambda, u) \in
\[ \hat{H}^{2k}(BG; \mathbb{Z}) \text{ to the character } \hat{\mathcal{C}}W g(\lambda, u) \in \hat{H}^{2k}(G \times S^1; \mathbb{Z}). \]

For an arbitrary character in \( \hat{H}^{2k}(BG; \mathbb{Z}) \), the pull-backs by different classifying maps might be different characters in \( \hat{H}^{2k}(G \times S^1; \mathbb{Z}) \). However, they have the same characteristic class, as any two classifying maps are homotopic.

4. DIFFERENTIAL TRIVIALIZATIONS OF UNIVERSAL CHARACTERISTIC CLASSES

In this section we use Cheeger-Chern-Simons characters to discuss differential refinements of universal characteristic classes for principal \( G \)-bundles. Specializing to the class \( \frac{1}{2}p_1 \in H^4(BSpin_n; \mathbb{Z}) \) this yields our notion of differential String classes, see Section 3 below.

4.1. Trivializations of universal characteristic classes. Throughout this section let \( G \) be a Lie group with finitely many components and \( \pi : P \to X \) a principal \( G \)-bundle. As above let \( \pi_{EG} : EG \to BG \) be a universal principal \( G \)-bundle over the classifying space of \( G \), i.e. a principal \( G \)-bundle with contractible total space. Let \( u \in H^n(BG; \mathbb{Z}) \) be a universal characteristic class for principal \( G \)-bundles. Equivalently, we may consider \( u \) as a homotopy class of maps \( BG \overset{u}{\to} K(\mathbb{Z}, n) \). In the following we briefly review the notion and basic properties of trivialization of universal characteristic classes from \[27\].

Let \( BGu \) be the homotopy fiber of \( BG \overset{u}{\to} K(\mathbb{Z}, n) \). Let \( f : X \to BG \) be a classifying map of the bundle \( \pi : P \to X \), i.e. \( f^*EG \cong P \) as principal \( G \)-bundles over \( X \). A trivialization of the class \( u(P) := f^*u \in H^n(X; \mathbb{Z}) \) is by definition a homotopy class of lifts

\[
\begin{array}{ccc}
X & \xrightarrow{f} & BG \\
\downarrow & & \downarrow \\
BGu & \xrightarrow{\tilde{\gamma}} & X
\end{array}
\]

The class \( u(P) \) admits trivializations if and only if it is trivial, i.e. \( u(P) = 0 \). By \[27\], Prop. 2.3, a trivialization of \( u(P) \) gives rise to a cohomology class \( q \in H^{n-1}(P; \mathbb{Z}) \) such that for any \( x \in X \) we have:

\[
(29) \quad H^{n-1}(P; \mathbb{Z}) \ni i^*_p q = T(u) \in H^{n-1}(G; \mathbb{Z}).
\]

Here \( i_p : P_x \to P \) denotes the inclusion of the fiber \( P_x := \pi^{-1}(x) \subset P \) over \( x \in X \). A cohomology class \( q \in H^{n-1}(P; \mathbb{Z}) \) satisfying (29) is called a \( u \)-trivialization class.

The cohomology of the base acts on \( u \)-trivialization classes by \( q \mapsto q + \pi^*w \), where \( w \in H^{n-1}(X; \mathbb{Z}) \). If \( H^j(G; \mathbb{Z}) = \{0\} \) for \( j < n - 1 \), then trivializations of \( u(P) \) are classified up to homotopy by \( u \)-trivialization classes \( q \in H^{n-1}(P; \mathbb{Z}) \). In this case, the transgression \( T : H^n(BG; \mathbb{Z}) \to H^{n-1}(G; \mathbb{Z}) \) is an isomorphism and we have the Serre exact sequence

\[
\begin{array}{ccccccc}
\{0\} & \to & H^{n-1}(X; \mathbb{Z}) & \xrightarrow{\pi^*} & H^{n-1}(P; \mathbb{Z}) & \xrightarrow{i_p} & H^{n-1}(G; \mathbb{Z}) & \to & H^n(X; \mathbb{Z}).
\end{array}
\]

In particular, the set of \( u \)-trivialization classes is a torsor for \( H^{n-1}(X; \mathbb{Z}) \).

4.2. Differential trivializations. We are looking for an appropriate notion of differential refinements of \( u \)-trivialization classes. Naively, one could define a differential \( u \)-trivialization to be any differential character \( \tilde{q} \in \hat{H}^{n-1}(P; \mathbb{Z}) \) whose characteristic class \( c(\tilde{q}) \) is a \( u \)-trivialization class. However, by the exact sequences \[33\] this would determine those differential characters only up to an infinite dimensional space of differential forms on \( P \). Instead, we expect that for an appropriate notion of differential \( u \)-trivializations, the space of all those is a torsor for the differential cohomology \( \hat{H}^{n-1}(X; \mathbb{Z}) \) (respectively \( \pi^*\hat{H}^{n-1}(X; \mathbb{Z}) \), in case pull-back by \( \pi \) is not injective).

Let \( u \in H^n(BG; \mathbb{Z}) \) be a universal characteristic class in the image of the Chern-Weil map, i.e. \( n = 2k \) and there exists an invariant polynomial \( \lambda \in \mathfrak{f}(g) \) such that
For any fiber, the Chern-Simons form is restricted to any fiber: $H^2_{\text{trivial}}(P) \ni [\tilde{\imath}_* \theta] = T(u)_{\text{trivial}} \in H^{2k-1}(G;\mathbb{R})$, where $x \in X$ is an arbitrary point. Thus we expect the curvature of a differential $u$-trivialization $\hat{\theta}$ to be related to the Chern-Simons form $CS_\theta(\lambda)$. However, this form is not closed, since $dCS_\theta(\lambda) = \pi^* C\theta(\lambda)$.

By assumption, we have $u(P) = f^* u = 0$, and hence $[C\theta(\lambda)]_{\text{trivial}} = u_{\mathbb{R}} = 0$. Thus there exist differential forms $\rho \in \Omega^{2k-1}(X)$ such that $d\rho = C\theta(\lambda)$. Then the form $CS_\theta(\lambda) - \pi^* \rho$ is closed. Moreover, we may choose $\rho$ such that $CS_\theta(\lambda) - \pi^* \rho$ has integral periods. This follows from the long exact sequence for mapping cone de Rham cohomology with integral periods:

$$\ldots \to H^{2k-1}_{\text{trivial}}(P;\mathbb{Z}) \to H^{2k}_{\text{trivial}}(\pi;\mathbb{Z}) \to H^{2k}_{\text{trivial}}(X;\mathbb{Z}) \to \ldots$$

Since $[C\theta(\lambda)]_{\text{trivial}} = 0$, the mapping cone class $[C\theta(\lambda), CS_\theta(\lambda)]_{\text{trivial}}$ lies in the image of the homomorphism $H^{2k-1}_{\text{trivial}}(P;\mathbb{Z}) \to H^{2k}_{\text{trivial}}(\pi;\mathbb{Z})$. We thus find a pair of forms $(\rho, \eta) \in \Omega^{2k-1}(\pi)$ such that $(C\theta(\lambda), CS_\theta(\lambda)) = d\rho(\rho, \eta)$ lies in the image of $H^{2k-1}_{\text{trivial}}(P) \to \Omega^{2k}_{\text{trivial}}(\pi)$. Thus $C\theta(\lambda) = d\rho$ and $CS_\theta(\lambda) - \pi^* \rho + d\eta$ has integral periods. But then also $CS_\theta(\lambda) - \pi^* \rho$ has integral periods.

The space of forms $\rho \in \Omega^{2k-1}(X)$ with $d\rho = C\theta(\lambda)$ and $CS_\theta(\lambda) - \pi^* \rho \in \Omega^{2k-1}_0(P)$ is a torsor for the infinite dimensional group $\Omega^{2k-1}_0(X)$. For a fixed such form and a fixed $u$-trivialization class $\hat{\theta}$, the set of differential characters $q = \hat{CS}_\theta(\lambda) - \pi^* \rho$ and characteristic class $c(q) = q$ is a torsor for the torus $H^{2k-1}_0(P;\mathbb{R}/\mathbb{Z})$.

The condition that $CS_\theta(\lambda) - \pi^* \rho$ has integral periods has a nice interpretation in terms of global sections of the Cheeger-Simons character $CW_\theta(\lambda, u) \in \hat{H}^{2k}(X;\mathbb{Z})$: As above assume that $u(P) = 0$. Thus $CW_\theta(\lambda, u)$ is topologically trivial. By the long exact sequence for the map $\text{id}_X$ it has a global section. By (41) global sections are uniquely determined by their covariant derivative. Thus for any form $\rho \in \Omega^{2k-1}(X)$ with $d\rho = C\theta(\lambda)$, we have $\hat{\rho}_u(\text{id}_X(\rho, 0)) = CW_\theta(\lambda, u)$.

Now consider the long exact sequence (40) once for the map $\text{id}_X$ and once for the bundle projection $\pi: P \to X$. Pull-back along the map $(\text{id}_X, \pi): (X, \pi) \to (X, X)$ yields the commutative diagram:

\[
\begin{array}{ccc}
H^{2k-2}(X; U(1)) & \xrightarrow{\pi^*} & \hat{H}^{2k-1}(P; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^{2k-2}(X; U(1)) & \xrightarrow{\text{id}_X^*} & \hat{H}^{2k}(X; \mathbb{Z})
\end{array}
\]

Pull-back along $(\text{id}_X, \pi)$ maps the global section $\text{id}_u(\rho, 0) \in \hat{H}^{2k}(\text{id}_X; \mathbb{Z})$ of the Cheeger-Simons character $CW_\theta(\lambda, u)$ maps to the relative character $\tau_\theta(\rho, 0) \in \hat{H}^{2k}(\pi; \mathbb{Z})$. Thus commutativity of the right square yields $\hat{\rho}(\tau_\theta(\rho, 0)) = \hat{\rho}(\text{id}_u(\rho, 0)) = CW_\theta(\lambda, u)$. On the other hand we have $\hat{\rho}(\text{CCS}_\theta(\lambda, u)) = CW_\theta(\lambda, u)$. Thus the relative characters $\hat{\rho}(\text{CCS}_\theta(\lambda, u))$ and $\tau_\theta(\rho, 0)$ differ by a character in the image of the homomorphism $1: \hat{H}^{2k-1}(P; \mathbb{Z}) \to \hat{H}^{2k}(\pi; \mathbb{Z})$. This observation yields our notion of differential $u$-trivializations:

**Definition 7.** Let $G$ be a Lie group with finitely many components and $\pi: (P, \theta) \to X$ a principal $G$-bundle with connection. Let $u \in H^{2k}(BG; \mathbb{Z})$ be a universal characteristic
class for principal $G$-bundles and $(\lambda, u) \in K^2_k(G; \mathbb{Z})$. A differential $u$-trivialization is a differential character $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ such that

\begin{equation}
-\iota_\pi(\tilde{q}) = \tilde{CCS}_\theta(\lambda, u) - \iota_\pi(\rho, 0)
\end{equation}

for some $\rho \in \Omega^{2k-1}(X)$.

To establish our notion of differential $u$-trivializations, we started from a global section of the Cheeger-Simons character $\tilde{CW}_\theta(\lambda, u)$. We show that any differential $u$-trivialization uniquely determines a global section:

**Lemma 8.** Let $G$ be a Lie group with finitely many components and $(\lambda, u) \in K^2_k(G; \mathbb{Z})$. Let $\pi : (P, \theta) \to X$ be a principal $G$-bundle with connection. Let $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ be a differential $u$-trivialization. Then the differential form $\rho \in \Omega^{2k-1}(X)$ in (31) is uniquely determined by $\tilde{q}$ and satisfies $\tilde{\rho}(\iota_\pi(\rho, 0)) = \tilde{CW}_\theta(\lambda, u)$. Thus $\iota_\pi(\rho, 0) \in H^{2k}([\mathrm{id}_X; \mathbb{Z}])$ is the unique global section of the Cheeger-Simons character $\tilde{CW}_\theta(\lambda, u)$ with covariant derivative $\rho$. Conversely, any such global section determines differential $u$-trivializations, uniquely up to characters of the form $j(\pi^* w)$ in $H^{2k-1}(P; \mathbb{Z})$ for some $w \in H^{2k-2}(X; \mathbb{U}(1))$.

**Proof.** Assume that (31) holds for two differential forms $\rho, \rho' \in \Omega^{2k-1}(X)$. Then we have $\iota_\pi(\rho - \rho', 0) = 0$ and hence $\text{cov}(\iota_\pi(\rho - \rho', 0)) = \pi^*(\rho - \rho') = 0$. Since pull-back of differential forms along the bundle projection $\pi : P \to X$ is injective, we conclude $\rho = \rho'$.

From the commutative diagram (30) and (31) we conclude

\[ \tilde{\rho}(\iota_\pi(\rho, 0)) = \tilde{\rho}(\iota_\pi(\rho, 0)) = \tilde{\rho}(\tilde{CCS}_\theta(\lambda, u) + \iota(\tilde{q})) = \tilde{CW}_\theta(\lambda, u). \]

Thus we have a global section of $\tilde{CW}_\theta(\lambda, u)$ with covariant derivative $\rho$.

Conversely, let $u(P) = 0$ and $\rho \in \Omega^{2k-1}(X)$ such that $d\rho = CW_\theta(\lambda, u)$. Then $\iota_\pi(\rho, 0) \in H^{2k}([\mathrm{id}_X; \mathbb{Z}])$ is the unique global section of $\tilde{CW}_\theta(\lambda, u)$ with covariant derivative $\rho$. Hence $\iota_\pi(\rho, 0) \in H^{2k}(\pi; \mathbb{Z})$ is a section along $\pi$. Thus we have $\tilde{\rho}(\tilde{CCS}_\theta(\lambda, u) - \iota_\pi(\rho, 0)) = 0$. By the long exact sequence (40) we find a differential character $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ such that $\iota(\tilde{q}) = \tilde{CCS}_\theta(\lambda, u) - \iota_\pi(\rho, 0)$. By (40) again it is uniquely determined up to a character of the form $j(\pi^* w)$ for some $w \in H^{2k-2}(X; \mathbb{U}(1))$. \qed

We show that differential $u$-trivializations are differential refinements of $u$-trivialization classes, i.e. its characteristic classes are $u$-trivialization classes. Conversely, any $u$-trivialization class is the characteristic class of a differential $u$-trivialization. In particular, the property for a principal $G$-bundle with connection $\pi : (P, \theta) \to X$ to admit differential $u$-trivializations is a purely topological condition, namely vanishing of the characteristic class $u(P) \in \hat{H}^2(X; \mathbb{Z})$.

**Proposition 9.** Let $G$ be a Lie group with finitely many components and $(\lambda, u) \in K^2_k(G; \mathbb{Z})$. Let $\pi : (P, \theta) \to X$ be a principal $G$-bundle with connection. Then the following holds:

If $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ is a differential $u$-trivialization, then $c(\tilde{q})$ is a $u$-trivialization class. Conversely, for any $u$-trivialization class $q \in H^{2k-1}(P; \mathbb{Z})$ there exist differential $u$-trivializations $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ with characteristic class $c(\tilde{q}) = q$. In particular, $\pi : (P, \theta) \to X$ admits differential $u$-trivializations if and only if $u(P) = 0$. Moreover, the curvature of a differential $u$-trivialization $\tilde{q}$ with differential form $\rho$ satisfies

\begin{equation}
\text{curv}(\tilde{q}) = CS_\theta(\lambda) - \pi^* \rho.
\end{equation}

For a differential $u$-trivialization $\tilde{q} \in H^{2k-1}(P; \mathbb{Z})$ and an arbitrary point $x \in X$, we have:

\begin{equation}
\hat{H}^{2k-1}(P_x; \mathbb{Z}) \ni \iota_{\pi^*} \tilde{q} = T(\tilde{CW}_\theta(\lambda, u)) \in \hat{H}^{2k-1}(G; \mathbb{Z}).
\end{equation}

The set of all differential $u$-trivializations is a torsor for the group $\pi^* \hat{H}^{2k-1}(X; \mathbb{Z})$. 
Proof. Assume \( u(P) = 0 \). Thus the Cheeger-Simons character \( \overline{CW}_\theta(\lambda, u) \) is topologically trivial. Choose \( \rho \in \Omega^{2k-1}(X) \) such that \( t(\rho) = \overline{CW}_\theta(\lambda, u) \). By Lemma \([5]\) there exist differential \( u \)-trivializations with differential form \( \rho \).

Next we prove \([33]\). Let \( \tilde{q} \in \tilde{H}^{2k-1}(P, \mathbb{Z}) \) be a differential character satisfying \([29]\). We compute the pull-back to the fiber \( P_x \) over any point \( x \in X \). From the commutative diagram

\[
\begin{array}{ccc}
\{x\}, P_x & \xrightarrow{(i_x, \pi_x)} & (X, P) \\
(id_{\{x\}}, \pi_{P_x}) & \downarrow & (id_X, \pi) \\
\{x\}, \{x\} & \xrightarrow{(i_x, i_x)} & (X, X)
\end{array}
\]

and naturality of the long exact sequence \([40]\) we obtain

\[
i_{\pi(P_x)}(i_x' \tilde{q}) = (i_x, i_P)^* T(\tilde{q})
\]

\[(i_x, i_P)^* (\tilde{q}) = i_{\pi(P_x)}(T(\overline{CW}_\theta(\lambda, u))) + (id_{\{x\}}, \pi_{P_x})^* i_{id}(i_x' \rho, 0).
\]

By the long exact sequence \([40]\) for the bundle projection \( \pi_{P_x} \) over a point \( x \in X \) the map \( i_{\pi(P_x)} : \tilde{H}^{2k-1}(P; \mathbb{Z}) \to \tilde{H}^{2k}(\pi_{P_x}; \mathbb{Z}) \) is an isomorphism. We thus conclude \( i_{\pi(P_x)} \tilde{q} = T(\overline{CW}_\theta(\lambda, u)) \).

From \([33]\) we conclude

\[
i_{\pi_x}(c(\tilde{q})) = c(i_{\pi(P_x)} \tilde{q}) = c(T(\overline{CW}_\theta(\lambda, u))) = T(c(\overline{CW}_\theta(\lambda, u))) = T(u).
\]

Thus the characteristic class \( c(\tilde{q}) \) of any differential \( u \)-trivialization \( \tilde{q} \) is a differential class. In particular, \( u(P) = 0 \).

Conversely, let \( q \in H^{2k-1}(P; \mathbb{Z}) \) be a differential \( u \)-trivialization class. We construct a differential \( u \)-trivialization \( \tilde{q} \) with \( c(\tilde{q}) = q \). Let \( q' \in \tilde{H}^{2k-1}(P; \mathbb{Z}) \) be any differential \( u \)-trivialization with differential form \( \rho' \). Put \( q' := c(\tilde{q}) \). Since \( q \) and \( q' \) are both differential \( u \)-trivialization classes, we have \( q - q' = \pi^* w \) for some \( w \in H^{2k-1}(X; \mathbb{Z}) \). Choose a differential character \( \tilde{w} \in \tilde{H}^{2k-1}(X; \mathbb{Z}) \) with characteristic class \( c(\tilde{w}) = w \). Now put \( \tilde{q} := q' + \pi^* \tilde{w} \). Then we have \( c(\tilde{q}) = q \). Put \( \rho := \rho' - \text{curv}(\tilde{w}) \). Then we have \( i_{\text{id}}(\tilde{w}) = i_{\text{id}}(-\text{curv}(w), 0) \) and hence

\[
i_{\text{id}}(\tilde{q}) = -i_{\text{id}}(\tilde{q}) - i_{\text{id}}(\pi^* \tilde{w})
\]

\[= CCS_B(\lambda, u) - i_{\text{id}}(\rho, 0) - (id_X, \pi)^* i_{\text{id}}(\tilde{w})
\]

\[= CCS_B(\lambda, u) - i_{\text{id}}(\rho' - \text{curv}(\tilde{w}), 0)
\]

\[= CCS_B(\lambda, u) - i_{\text{id}}(\rho, 0).
\]

Thus \( \tilde{q} \) is a differential \( u \)-trivialization.

The equation \( \text{curv}(\tilde{q}) = CCS_B(\lambda) - \pi^* \rho \) follows immediately from \([31]\) and \( \text{cov}(i(\tilde{q})) = -\text{curv}(\tilde{q}) \).

Let \( \tilde{q} \) be a differential \( u \)-trivialization with differential form \( \rho \) and \( h \in \tilde{H}^{2k-1}(X; \mathbb{Z}) \). As above we have \( i_{\text{id}}(h) = i_{\text{id}}(-\text{curv}(h), 0) \) and hence

\[
i_{\text{id}}(\tilde{q} + \pi^* h) = CCS_B(\lambda, u) - i_{\text{id}}(\rho, 0) - i_{\text{id}}(-\text{curv}(h), 0)
\]

\[= CCS_B(\lambda, u) - i_{\text{id}}(\rho - \text{curv}(h)).
\]

Thus \( \tilde{q} + \pi^* h \) is a differential \( u \)-trivialization with differential form \( \rho - \text{curv}(h) \). Hence the differential cohomology group \( \tilde{H}^{2k-1}(X; \mathbb{Z}) \) acts on the set of differential \( u \)-trivializations.

The action of the group \( \tilde{H}^{2k-1}(X; \mathbb{Z}) \) on the set of differential \( u \)-trivializations is in general not free. The kernel of the map \( \pi^* : \tilde{H}^{2k-1}(X; \mathbb{Z}) \to \tilde{H}^{2k-1}(P; \mathbb{Z}) \) is contained in the image of the map \( j : H^{2k-2}(X; U(1)) \to \tilde{H}^{2k-1}(X; \mathbb{Z}) \). By injectivity of the latter, the kernel
consists of characters of the form $j(v)$, where $v$ is in the kernel of $\pi^*: H^{2k-2}(X; U(1)) \to H^{2k-2}(P; U(1))$.

It remains to show that the action is transitive. Let $\hat{q}$ and $\hat{q}'$ be differential $u$-trivializations with differential forms $\rho$ and $\rho'$, respectively. By Lemma \ref{lem:trivializations} we have $\hat{p}_u(t_u(\rho, 0)) = C\hat{W}_\theta(\lambda, u) = \hat{p}_u(t_u(\rho', 0))$. From the long exact sequence \eqref{eq:long_exact_sequence} we thus obtain a differential character $h' \in H^{2k-1}(X; \mathbb{Z})$ such that $t_u(h') = t_u(\rho - \rho', 0)$. This yields

$$t_\pi(\hat{q} - \hat{q}') = t_\pi(\rho - \rho', 0) = i(\pi^*h').$$

From the upper row in \eqref{eq:long_exact_sequence}, we conclude $\hat{q} - \hat{q}' = \pi^*(h' + j(v))$ for some $v \in H^{2k-2}(X; U(1))$. Put $h := h' + j(u) \in H^{2k-1}(X; \mathbb{Z})$. Then we have $\hat{q} - \hat{q}' = \pi^*h$. \hfill $\Box$

**Remark 10.** In general, the condition \eqref{eq:condition} is weaker than \eqref{eq:condition}: Let $\mu \in \Omega^{2k-2}(X)$ and $f \in C^\infty(P)$ not constant along the fibers. Put $\eta := f \cdot \pi^*\mu$. Then $\eta$ vanishes upon pull-back to any fiber $P_\lambda$. Moreover, $d\eta = df \wedge \pi^*\mu + f \cdot \pi^*d\mu$ is not the pull-back of a form on the base $X$. Now let $\hat{q}$ be any differential $u$-trivialization. Then $\hat{q} + t(\eta) \in H^{2k-1}(P;\mathbb{Z})$ still satisfies \eqref{eq:condition}, since $\hat{q}_\rho(\hat{q} + t(\eta)) = \hat{q}_\rho \hat{q} + t(\hat{q}_\rho \eta) = T(C\hat{W}_\theta(\lambda, u))$. But $\hat{q} + t(\eta)$ does not satisfy \eqref{eq:condition} since $\text{curv}(\hat{q} + t(\eta)) - C\hat{S}_\theta(\lambda) = \pi^*\rho + d\eta$ is not the pull-back of a form on $X$. Thus \eqref{eq:condition} is violated.

In general, even the two conditions \eqref{eq:condition} and \eqref{eq:condition} together do not imply \eqref{eq:condition}: Suppose there exists a closed form $v \in \Omega^{2k-1}(X)$ such that $\pi^*v = d\eta$ for some $\eta \in \Omega^{2k-2}(P)$ and $\hat{q}_\rho \eta$ is exact for any $x \in X$. Without loss of generality we assume that $v$ does not have integral periods. Let $\hat{q}$ be a differential $u$-trivialization with differential form $\rho$. Put $h := \hat{q} + t(\eta)$. Then we have $\hat{q}_\rho h = \hat{q}_\rho \hat{q} + t(\hat{q}_\rho \eta) = T(C\hat{W}_\theta(\lambda, u))$, since $\hat{q}_\rho \eta$ is exact and thus $t(\hat{q}_\rho \eta) = 0$. Thus $h$ satisfies \eqref{eq:condition}. Moreover, $h$ satisfies \eqref{eq:condition}, since

$$\text{curv}(h) = \text{curv}(\hat{q}) + d\eta = C\hat{S}_\theta(\lambda) - \pi^*(\rho - v).$$

But we have $-\pi(h) = -\pi(\hat{q}) - i_\rho(0, \eta) = C\hat{S}_\theta(\lambda, u) - i_\rho(\rho, \eta)$. Thus $h$ satisfies \eqref{eq:condition} if and only if $t_\pi(\rho, \eta) = t_\pi(\rho', 0)$ for some $\rho' \in \Omega^{2k-1}(X)$. The latter condition is equivalent to $(\rho - \rho', \eta)$ being closed with integral periods. By assumption, $d\eta = \pi^*v$ and $d\sigma = 0$. Thus we necessarily have $\rho - \rho' = v$. But by assumption $v$ does not have integral periods, and so neither does $(\rho, \pi^*v) = (\rho - \rho', \eta)$.

### 4.3. Dependence upon the connection

Since the Cheeger-Chern-Simons character $C\hat{S}_\theta(\lambda, u)$ depends upon the connection $\theta$, so do differential $u$-trivializations:

**Proposition 11.** Let $\theta_0, \theta_1 \in A^1(P)$ be connections on a principal $G$-bundle $\pi: P \to X$. Let $(\lambda, u) \in K^2(G; \mathbb{Z})$. Let $\alpha(\theta_0, \theta_1; \lambda) \in \Omega^{2k-2}(P)$ as in Section \ref{sec:connections}. Let $\hat{q} \in H^{2k-1}(P;\mathbb{Z})$ be a differential $u$-trivialization on $(P, \theta_0)$ with differential form $\rho$. Then $\hat{q} - t(\alpha(\theta_0, \theta_1; \lambda))$ is a differential $u$-trivialization on $(P, \theta_1)$ with differential form $\rho + CS(\theta_0, \theta_1; \lambda)$.

**Proof.** By definition, we have

$$-\pi(\hat{q}) = C\hat{S}_\theta(\lambda, u) - \pi_\rho(\rho, 0)$$

$$C\hat{S}_\theta(\lambda, u) - \pi_\rho(\rho + CS(\theta_0, \theta_1; \lambda), 0) - \pi_\rho(0, \alpha(\theta_0, \theta_1; \lambda))$$

$$= C\hat{S}_\theta(\lambda, u) - \pi_\rho(\rho + CS(\theta_0, \theta_1; \lambda), 0) - \pi_\rho(0, \alpha(\theta_0, \theta_1; \lambda))$$

Thus the character $\hat{q} - t(\alpha(\theta_0, \theta_1; \lambda)) \in H^{2k-1}(P;\mathbb{Z})$ and the form $\rho + CS(\theta_0, \theta_1; \lambda) \in \Omega^{2k-1}(X)$ together satisfy condition \eqref{eq:condition} on $(P, \theta_1)$. \hfill $\Box$
5. Differential String Classes and Chern-Simons Theory

In this section, we establish our notion of differential String classes on a principal Spin$_n$-bundle with connection $\pi : (P, \theta) \to X$, where $n \geq 3$. We obtain this notion by specializing the notion of differential $u$-trivialization to the case $u = \frac{1}{2} p_1 \in H^3(\text{BSpin}_n; \mathbb{Z})$. Our notion of differential String classes corresponds to the notion of geometric String structures from [34]. A geometric String structure is a trivialization of the Cheeger-Simons bundle $2$-gerbe with compatible connection, i.e. a certain bundle gerbe with connection on $P$. The stable isomorphism classes of geometric String structures are in 1-1 correspondence with differential String structures.

By [27], for any fixed Riemannian metric $g$ on $X$ there is another way to obtain differential refinements of String classes. Thus assume that $\rho_1(P) = 0$. Hodge decomposition on $X$ provides us with a canonical $3$-form $\rho_0 \in \Omega^3(X)$ with differential the Chern-Weil form $d\rho_0 = \text{CW}(\frac{1}{2} p_1)$, namely the unique coexact such form. Using adiabatic limits and Hodge theory on $P$, Redden constructs for any String class $q \in H^3(P; \mathbb{Z})$ a canonical closed form $\omega \in \Omega^3(P)$ such that $[\omega]_{\text{dR}} = q$. In this setting, one may define a differential String class to be a differential character $\tilde{q} \in \widehat{H}^3(P; \mathbb{Z})$ with curvature $\omega$ and characteristic class $q$. It turns out that this notion of differential String classes coincides with our notion. Moreover, these characters are uniquely determined by the condition that the pull-back to any fiber is the basic character in $\widehat{H}^3(\text{Spin}_n; \mathbb{Z})$.

5.1. String structures and String classes. The group String$_n$ is by definition a 3-connected cover of Spin$_n$. It is defined only up to homotopy. As is well known, the homotopy type String$_n$ cannot be represented a finite dimensional Lie group since any such group has non-vanishing $\pi_3$. There exist several models of String$_n$, either as a topological group [32], as a finite dimensional Lie $2$-group [29] or as an infinite dimensional Fréchet Lie group [24]. In the latter case a String structure (in the Lie theoretic sense) is defined as a lift of the structure group of $\pi : P \to X$ from Spin$_n$ to String$_n$.

String$_n$ is defined as the homotopy fiber of a classifying map $\lambda : B\text{Spin}_n \to K(\mathbb{Z}; 4)$ for the generator $\frac{1}{2} p_1 \in H^3(\text{BSpin}_n; \mathbb{Z}) \cong H^3(\text{Spin}_n; \mathbb{Z}) \cong \pi_3(\text{Spin}_n) \cong \mathbb{Z}$. A String structure (in the homotopy theoretic sense) on a principal Spin$_n$-bundle $\pi : P \to X$ is a homotopy class of lifts $\tilde{f}$ of classifying maps $f$ of the bundle $\pi : P \to X$:

$$
\begin{diagram}
\node{X} \arrow{e,东南}{f} \node{B\text{Spin}_n} \arrow{se,近似}{\tilde{f}} \\
\node{f} \node{\lambda} \node{K(\mathbb{Z}; 4)}
\end{diagram}
$$

Isomorphism classes of String structures in the Lie theoretic sense correspond to String structures in the homotopy theoretic sense. Clearly, a principal Spin$_n$-bundle $\pi : P \to X$ admits a String structure (in either sense) if and only if $\frac{1}{2} p_1(P) = 0$. Since $H^3(\text{Spin}(n); \mathbb{Z}) = 0$ for $j = 1, 2$, by [27] there is a 1-1 correspondence between isomorphism classes of String structures on $P$ and $\frac{1}{2} p_1$-trivialization classes $q \in H^3(P; \mathbb{Z})$. These classes are called String classes.

5.2. Differential String classes. We now derive our notion of differential String classes by specializing the concept of differential trivializations of universal characteristic classes of principal $G$-bundles from Section 4. For $G = \text{Spin}_n$, $n \geq 3$, the Chern-Weil construction yields an isomorphism $\frac{1}{2} p_1^G(\text{Spin}_n) \Rightarrow H^3_{\text{dR}}(\text{BSpin}_n; \mathbb{Z}) \cong H^3(\text{BSpin}_n; \mathbb{Z}) \cong \mathbb{Z}$. We thus write the elements of $K^4(\text{Spin}_n; \mathbb{Z})$ simply as $\lambda$ or $\omega$ instead of pairs $(\lambda, \omega)$.

---

9Here $\delta_k^G(G)$ denotes the space of invariant polynomials of degree $k$, the Chern-Weil forms of which have integral periods.
Let \((P, \theta) \to X\) be a principal \(\text{Spin}_n\)-bundle with connection. The invariant polynomial \(\lambda \in \Omega^2(\text{Spin}_n) \cong H^4(\text{BSpin}_n; \mathbb{Z})\) yields the Chern-Weil form \(CW_\theta(\lambda) \in \Omega^3_\theta(X)\) and the Cheeger-Simons character \(CW_\theta(\lambda) \in \hat{H}^4(X; \mathbb{Z})\) with curvature \(\text{curv}(CW_\theta(\lambda)) = CW_\theta(\lambda)\) and characteristic class \(c(CW_\theta(\lambda)) = u\). Moreover, we have the Chern-Simons form \(CS_\theta(\lambda) \in \Omega^3(P)\) and the Cheeger-Chern-Simons character \(CCS_\theta(\lambda) \in \hat{H}^4(\pi; \mathbb{Z})\) with covariant derivative \(\text{cov}(CCS_\theta(\lambda)) = CS_\theta(\lambda)\). Since \(H^4(\text{BSpin}_n; \mathbb{Z}) \cong i_0^*(\text{Spin}_n) \cong K^*(\text{Spin}_n; \mathbb{Z}) \cong \mathbb{Z}\) with generator \(\frac{1}{2}p_1\), we may write \(\lambda = \ell \cdot \frac{1}{2}p_1\) for some \(\ell \in \mathbb{Z}\). We call \(\ell\) the level of \(\lambda\).

**Definition 12.** Let \(\pi: (P, \theta) \to X\) be a principal \(\text{Spin}_n\)-bundle with connection, where \(n \geq 3\). A **differential String class on** \((P, \theta)\) **is a differential** \(\frac{1}{2}p_1\)-**trivialization**, i.e. a differential character \(\tilde{q} \in \hat{H}^3(P; \mathbb{Z})\) such that

\[
-\iota_x(\tilde{q}) = CCS_\theta(\frac{1}{\ell}p_1) - \iota_x(\rho, 0)
\]

for some \(\rho \in \Omega^1(X)\).

Analogously, we define differential String classes at level \(\ell\) by replacing \(\frac{1}{2}p_1\) in (34) by \(\lambda = \ell \cdot \frac{1}{2}p_1\).

From the results of Section 4, we conclude that a bundle \((P, \theta)\) admits differential String classes if and only if it is String, i.e. \(\frac{1}{2}p_1(P) = 0\). The differential form \(\rho\) in the notion of differential String classes is uniquely determined by the character \(\tilde{q}\). It satisfies \(d\rho = CW_\theta(\frac{1}{2}p_1)\). Conversely, for any differential form \(\rho \in \Omega^1(X)\) with \(d\rho = CW_\theta(\frac{1}{2}p_1)\), there exist differential String classes \(\tilde{q} \in \hat{H}^3(P; \mathbb{Z})\) with differential form \(\rho\).

Since \(H^4(\text{Spin}_n; \mathbb{Z}) = \{0\}\) for \(i \leq 2\), the Leray-Serre sequence yields the following exact sequence [24, Prop. 2.5]:

\[
\begin{align*}
0 \to H^3(X; \mathbb{Z}) \xrightarrow{\pi^*} H^3(P; \mathbb{Z}) \xrightarrow{i_*} H^3(\text{Spin}_n; \mathbb{Z}) \xrightarrow{T^{-1}} H^4(X; \mathbb{Z}).
\end{align*}
\]

From the long exact sequence (38) we conclude that the pull-back \(\pi^*: \hat{H}^3(X; \mathbb{Z}) \to \hat{H}^3(P; \mathbb{Z})\) is injective. Thus by Proposition 9 the set of all differential String classes on \((P, \theta)\) is a torsor for the differential cohomology group \(\hat{H}^3(X; \mathbb{Z})\).

Let \(\theta_0, \theta_1 \in \mathcal{A}(P)\) be connections on the principal \(\text{Spin}_n\)-bundle \(\pi: P \to X\). Let \(CS(\theta_0, \theta_1; \frac{1}{2}p_1) \in \Omega^3(X)\) and \(\alpha(\theta_0, \theta_1; \frac{1}{2}p_1) \in \Omega^2(P)\) be as in Section 2.4. As discussed in Section 4.4, differential String classes depend upon the connection: if \(\tilde{q}\) is a differential String class on \((P, \theta_0)\) with differential form \(\rho\), then \(\tilde{q} - \iota_1(\alpha(\theta_0, \theta_1; \frac{1}{2}p_1))\) is a differential String class on \((P, \theta_1)\) with differential form \(\rho + CS(\theta_0, \theta_1; \frac{1}{2}p_1)\).

The transgression map \(T: H^4(\text{BSpin}_n; \mathbb{Z}) \to H^3(\text{Spin}_n; \mathbb{Z})\) is an isomorphism. The restriction of the Chern-Simons form \(CS_\theta(\frac{1}{2}p_1)\) to any fiber \(P_s\) represents the class \(T(\frac{1}{2}p_1) \in H^3(\text{Spin}_n; \mathbb{Z})\) in de Rham cohomology. The form \(i_\rho^*CS_\theta(\frac{1}{2}p_1) \in \Omega^3(\text{Spin}_n)\) can be expressed purely in terms of the Maurer-Cartan form of \(\text{Spin}_n\), see [10]. In particular, it is independent of the connection \(\theta\) on \(P\).

Since \(H^2(\text{Spin}_n; \mathbb{R}) = \{0\}\), we conclude from the long exact sequence (40) that \(T(CW_\theta(\frac{1}{2}p_1))\) is the unique differential character in \(\hat{H}^3(\text{Spin}_n; \mathbb{Z})\) with curvature \(i_\rho^*CS_\theta(\frac{1}{2}p_1)\) and characteristic class \(T(\frac{1}{2}p_1)\). We call \(T(CW_\theta(\frac{1}{2}p_1))\) the **basic** 3-character on \(\text{Spin}_n\), since it coincides with the stable isomorphism class of the so-called basic bundle gerbe on \(\text{Spin}_n\), see [20].

By Lemma 8 any differential String structure \(\tilde{q}\) satisfies \(i_\rho^*\tilde{q} = T(CW_\theta(\frac{1}{2}p_1))\). By Remark 10 this condition is not sufficient to determine differential String classes. By (32) any differential String class \(\tilde{q}\) satisfies \(\text{curv}(\tilde{q}) = CS_\theta(\frac{1}{2}p_1) - \pi^*\rho\) for some \(\rho \in \Omega^1(X)\). With this additional requirement, we obtain differential String classes:

\[\text{for } \lambda = \frac{1}{2}p_1\text{ the Chern-Simons form } CS_\theta(\frac{1}{2}p_1)\text{ is the usual Chern-Simons 3-form for } \text{Spin}_n.\]
Proposition 13. Let \( \pi : (P, \theta) \to X \) be a principal Spin\(_n\)-bundle with connection. Let \( h \in \check{H}^3(P, \mathbb{Z}) \) be a differential character with curvature \( \text{curv}(h) = CS_0(\frac{1}{2}P_1) - \pi^* \rho \) for some \( \rho \in \Omega^3(X) \) and characteristic class \( c(h) \) a String class. Then \( h \) is a differential String class with differential form \( \rho \).

Let \( h \in \check{H}^3(P, \mathbb{Z}) \) be a differential character satisfying \( i_{\pi*} h = T(\check{CW}_0(\frac{1}{2}P_1)) \) for any \( x \in X \) and \( \text{curv}(h) = CS_0(\frac{1}{2}P_1) - \pi^* \rho \) for some \( \rho \in \Omega^3(X) \). Then \( h \) is a differential String class with differential form \( \rho \).

Proof. Let \( h \in \check{H}^3(P, \mathbb{Z}) \) be a differential character with \( \text{curv}(h) = CS_0(\frac{1}{2}P_1) - \pi^* \rho \) for some \( \rho \in \Omega^3(X) \) and characteristic class \( c(h) \) a String class. Then we have \( \text{curv}(i_{\pi*} h) = i_{\pi*} CS_0(\frac{1}{2}P_1) \) and \( c(i_{\pi*} h) = T(\frac{1}{2}P_1) \). Since \( H^2(\text{Spin}_n; \mathbb{R}) = \{0\} \), this implies that \( i_{\pi*} h = T(\check{CW}_0(\lambda, u)) \) for any \( x \in X \). Thus it suffices to show prove the second statement.

Since \( H^i(\text{Spin}_n; \mathbb{Z}) = \{0\} \) for \( i = 1, 2 \), the bundle projection provides an isomorphism \( \pi^* : H^2(X; \mathbb{R}) \to H^2(P; \mathbb{R}) \). By the exact sequences \( (38) \), differential characters in \( \check{H}^3(P, \mathbb{Z}) \) are thus uniquely determined by their curvature and characteristic class up to characters of the form \( \pi^* \iota^2(\nu) \) for some closed form \( \nu \in \Omega^2(X) \).

Since \( i_{\pi*} h = T(\check{CW}_0(\frac{1}{2}P_1)) \), the characteristic class \( c(h) \) is a String class. By Lemma\( [8] \) there exists a differential String class \( \check{q} \in \check{H}^3(P, \mathbb{Z}) \) with characteristic class \( c(h) \). Let \( \rho' \in \Omega^3(X) \) be the differential form of the differential String class \( \check{q'} \). By the condition on \( \text{curv}(h) \), we have \( \text{curv}(h) - \text{curv}(\check{q'}) = \pi^* (\rho - \rho') \). Since the pull-back \( \pi^* : \Omega^*(X) \to \Omega^*(P) \) is injective, the form \( \rho - \rho' \) is closed. On the other hand, \( \text{curv}(h) - \text{curv}(\check{q'}) \) is exact, since both forms represent the cohomology class \( c(h) \mathbb{R} \). By the exact sequence \( (35) \) the pull-back \( \pi^* : H^3(X; \mathbb{Z}) \to H^3(P; \mathbb{Z}) \) is injective. Thus we find a form \( \eta \in \Omega^3(X) \) such that \( \rho - \rho' = d\eta \).

Put \( \hat{q} := \check{q'} + i(\pi^* \eta) \). Then we have \( \text{curv}(\hat{q}) = \text{curv}(\check{q'}) + d\pi^* \eta = \text{curv}(h) \) and \( c(\hat{q}) = c(\check{q'}) = c(h) \). Thus there exists a closed form \( \nu \in \Omega^2(X) \) such that \( h = \hat{q} + i(\pi^* \nu) = \check{q'} + i(\pi^* (\eta + \nu)) \). Since \( \check{q'} \) is a differential String class, we have:

\[
-i_\pi(h) = -i_\pi(q' + i(\pi^* (\eta + \nu)))
= \check{CCS}_0(\frac{1}{2}P_1) - i_\pi(\rho', 0) + i_\pi(0, \pi^*(\eta + \nu))
= \check{CCS}_0(\frac{1}{2}P_1) - i_\pi(\rho' - d\eta, 0) + i_\pi(d\eta(\eta + \nu, 0))
= \check{CCS}_0(\frac{1}{2}P_1) - i_\pi(\rho, 0).
\]

Thus \( h \) is a differential String class with differential form \( \rho \). \( \square \)

5.3. Related concepts. In this section we relate differential String classes as defined above to a similar concept obtained from \( [27] \) and to geometric String structures in the sense of \( [35] \). The latter are certain bundle gerbes with connection on \( P \). We show that their stable isomorphism classes are differential String classes and vice versa, any differential String class is the stable isomorphism class of a geometric String structure. In particular, differential String classes as above coincide with the differential String classes from \( [35] \). In \( [27] \), Redden constructs for any String class \( q \) on a principal Spin\(_n\)-bundle over a compact Riemannian manifold \( (X, g) \) a canonical 3-form on \( P \) which represents \( q \) in de Rham cohomology. Any differential character with curvature this form and characteristic class \( q \) may be considered a canonical differential refinement of \( q \). We show that these are differential String classes.

5.3.1. Canonical differential refinements. Let \( (X, g) \) be a compact Riemannian manifold and \( q \in H^3(P, \mathbb{Z}) \) a fixed String class. Let \( p_0 \) be the unique coexact 3-form satisfying \( dp_0 = CW_0(\lambda) \). Denote by \( \mathcal{H}^3(X) \) the space of harmonic 3-forms on \( X \) with respect to
the metric $g$. By \cite[Thm. 3.7]{27}, there is a unique harmonic form with integral periods $\rho \in H^1(X; \mathbb{Z})$ such that
\begin{equation}
H^4_{\text{dR}}(P; \mathbb{Z}) \ni [CS_0(\hat{\lambda}) - \pi^*(\rho_0 + \rho)]_{\text{dR}} = q_\mathbb{R} \in H^3(P; \mathbb{Z})_{\text{dR}}.
\end{equation}
Thus the form $CS_0(\hat{\lambda}) - \pi^*(\rho_0 + \rho)$ is a canonical representative of the String class $q$ in de Rham cohomology. In particular, it has integral periods. As a consequence one might consider any differential character $h \in H^3(P; \mathbb{Z})$ with characteristic class a String class $q$ and curvature the associated Redden form $CS_0(\hat{\lambda}) - \pi^*(\rho_0 + \rho)$ as a canonical differential refinement of the String class $q$.

By Proposition \ref{prop:5.3.1} any such character $h$ is a differential String class in the sense of Definition \ref{def:5.3.2}. In this sense our notion of differential String classes recovers the notion of (stable isomorphism classes of) geometric String structures from \cite{34}. In particular, it coincides with the notion of differential String classes from \cite{35}. By Definition \ref{def:5.3.2} this geometric String structure is the stable isomorphism class of a geometric String structure. In this sense our notion of differential String classes recovers the notion of differential String classes from \cite{35}. In particular, it coincides with the notion of differential String classes from \cite{35}.

5.3.2. Geometric String structures. A geometric String structure in the sense of \cite{34} is a certain bundle gerbe with connection on $P$. More precisely, it is a trivialization of the so-called Cheeger-Simons bundle 2-gerbe $\mathcal{G}$ with compatible connection. The latter represents the Cheeger-Chern-Simons character $CCS_0(\frac{1}{2}p_1) \in \hat{H}^4(\pi; \mathbb{Z})$: Note that both the Cheeger-Simons bundle 2-gerbe and the Cheeger-Chern-Simons bundle $CCS_0(\frac{1}{2}p_1)$ are natural with respect to bundle maps. Thus they are defined on the universal principal Spin$\mathfrak{n}$-bundle. Moreover, they have the same curvature $CW_0(\frac{1}{2}p_1)$ and characteristic class $\frac{1}{2}p_1$. Finally, the curving of the Cheeger-Simons bundle 2-gerbe $\mathcal{G}$ is the Chern-Simons form $CS_0(\frac{1}{2}p_1)$ and thus coincides with the covariant derivative of the Cheeger-Chern-Simons character $CCS_0(\frac{1}{2}p_1)$. As explained in Appendix \ref{app:A} the Cheeger-Simons bundle 2-gerbe $\mathcal{G}$ defines a differential character $h_{\mathcal{G},\mathcal{G}} \in \hat{H}^4(\pi_{EG}; \mathbb{Z})$. Since $H^4(\pi_{EG}; \mathbb{R}) = 0$, such characters are uniquely determined by their characteristic class, curvature and covariant derivative. Thus we conclude $h_{\mathcal{G},\mathcal{G}} = CCS_0(\frac{1}{2}p_1) \in \hat{H}^4(\pi_{EG}; \mathbb{Z})$.

A geometric String structure on $(P, \theta)$ is by definition a trivialization of the Cheeger-Simons bundle 2-gerbe $\mathcal{G}$ together with a compatible connection. In particular, it is a bundle gerbe with connection $\mathcal{G}$ on $P$. Its Dixmier-Douady class is a String class. Moreover, there is a uniquely determined 3-form $\rho \in \Omega^3(X)$ such that $\text{curv}(\mathcal{G}) = CS_0(\frac{1}{2}p_1) - \pi^*\rho$. Thus the stable isomorphism class of $\mathcal{G}$ is a differential character with characteristic class a String class $\mathcal{G}$ and curvature $CS_0(\frac{1}{2}p_1) - \pi^*\rho$. By Proposition \ref{prop:5.3.1} it is a differential String class in the sense of Definition \ref{def:5.3.2}. Conversely, since both the set of differential String classes on $(P, \theta)$ and the set of stable isomorphism classes of geometric String structures are torsors for the differential cohomology group $\hat{H}^4(X; \mathbb{Z})$, any differential String class is the stable isomorphism class of a geometric String structure. In this sense our notion of differential String classes recovers the notion of (stable isomorphism classes of) geometric String structures from \cite{34}. In particular, it coincides with the notion of differential String classes from \cite{35}.

5.4. Chern-Simons theory. The Cheeger-Simons character $\hat{CW}_0(\lambda) \in \hat{H}^4(X; \mathbb{Z})$ for $\lambda = \ell \cdot \frac{1}{2}p_1 \in L^2_0(\text{Spin}_n) \cong H^4(B\text{Spin}_n; \mathbb{Z}) \cong \mathbb{Z}$ may be considered as defining a Chern-Simons theory at level $\ell$ for the group Spin$_n$ with target space $X$, extended down to points. Its evaluations on lower dimensional closed oriented manifolds are given by the transgressions to mapping spaces \cite{11}.

For a point $*$, the mapping space $\mathcal{C}^\infty(\ast, X)$ is canonically identified with $X$ itself. Thus the evaluation of $\hat{CW}_0(\lambda)$ on a point $*$ is just the character $\hat{CW}_0(\lambda)$ itself. The evaluation of $\hat{CW}_0(\lambda)$ on a closed oriented 1-manifold $S$ is the transgression $\tau_0(\hat{CW}_0(\lambda)) \in \hat{H}^3(C^\infty(S, X); \mathbb{Z})$ to the space of smooth maps $f : S \to X$. Similarly, evaluation of $\hat{CW}_0(\lambda)$

\cite{11}There are several ways to construct transgression of differential cohomology to loop spaces and higher mapping spaces. For a nice geometric method, see \cite{1}.
The obstruction to such lifts is precisely the transgression to loop space of the class 
\[ \tau : \Sigma \to X. \] It thus yields an isomorphism class of hermitean line bundles with connection over the mapping space \( C^\omega(\Sigma, X). \) The evaluation of \( CW_\theta(\lambda) \) on a closed oriented 3-manifold \( M \) is the transgression \( \tau_M(CW_\theta(\lambda)) \in \tilde{H}^1(C^\omega(M, X); \mathbb{Z}) \equiv C^\omega(C^\omega(M, X), U(1)) \) to the space of smooth maps \( f : M \to X. \) It coincides with the holonomy along closed oriented 3-manifolds, considered as a smooth function \( \text{hol}_M(CW_\theta(\lambda)) : C^\omega(M, X) \to U(1). \)

By [12], a reasonable notion of geometric String structure on \( (P, \theta) \) should provide (a notion of) trivializations of the corresponding extended Chern-Simons theory. In the present case, a trivialization of a differential character is a global section (in the same way as principal bundles are trivialized by global sections). See [1] for more details and examples. A character \( h \in \tilde{H}^3(X; \mathbb{Z}) \) admits a trivialization if and only if it is topologically trivial, i.e. \( c(h) = 0. \) There is a 1-1 correspondence between global sections of a given character \( h \) and differential forms \( \omega \in \Omega^{1, -1}(X) \) satisfying \( d\omega = \text{curv}(h). \)

Trivializations of the extended Chern-Simons theory associated with the Cheeger-Simons character \( CW_\theta(\frac{1}{2} p_1) \) are obtained by transgression of differential String classes: Any differential String class \( \tilde{q} \) on \( (P, \theta) \) with differential form \( \rho \in \Omega^2(X) \) provides a real lift of the holonomy of \( CW_\theta(\frac{1}{2} p_1) \). By Lemma 8, the differential form \( \rho \) defines a global section of the character \( CW_\theta(\frac{1}{2} p_1) \). In particular, if \( \rho \) yields a real lift for the evaluation on closed oriented 3-manifolds \( M \): for any smooth map \( f : M \to X \) we have

\[ \text{hol}_M(CW_\theta(\frac{1}{2} p_1))(f) := \left( f^* \text{hol}_M(CW_\theta(\frac{1}{2} p_1)) \right)[M] = \exp \left( 2\pi i \int_M f^* \rho \right). \]

Analogously, transgression of differential String classes to mapping spaces yields trivializations of the corresponding evaluations: Note that the transgression maps constructed in [11] and [12] commute with the long exact sequence (40). Thus for an oriented closed surface \( \Sigma \) the transgression \( \tau_2(\tilde{q}) \in \tilde{H}^2(C^\omega(\Sigma, X); \mathbb{Z}) \) of a differential String class satisfies:

\[ -\text{id}(\tau_2(\tilde{q})) = \tau_2(CCS_\theta(\frac{1}{2} p_1)) - \text{id}(\tau_2(\rho)), 0. \]

In particular, \( \tau_2(\rho) \in \Omega^2(C^\omega(\Sigma, X)) \) defines an isomorphism class of sections of the line bundle over \( C^\omega(\Sigma, X) \) associated with the transgressed Cheeger-Simons character \( \tau_2(CW_\theta(\frac{1}{2} p_1)). \)

Likewise, for a closed oriented 1-manifold \( S \), transgression along \( S \) of a differential String class \( \tilde{q} \) satisfies

\[ -\text{id}(\tau_3(\tilde{q})) = \tau_3(CCS_\theta(\frac{1}{2} p_1)) - \text{id}(\tau_3(\rho)), 0. \]

Again \( \tau_3(\rho) \in \Omega^3(C^\omega(S, X)) \) yields a global section of the transgressed character \( \tau_3(CW_\theta(\frac{1}{2} p_1)). \)

5.5. Transgression to loop space. Let \( \pi : P \to X \) be a principal Spin\(_n\)-bundle. Applying the loop space functor yields a principal \( \mathcal{L}(\text{Spin}_n) \)-bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X). \) In mathematical physics one would like to construct vector bundles associated to the loop group bundle in the same manner as the spinor bundle is constructed from a Spin structure on \( X. \) However, the positive energy representations of the loop group \( \mathcal{L}(\text{Spin}_n) \) are all projective. Therefore, one needs to lift the structure group of the loop bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X) \) from \( \mathcal{L}(\text{Spin}_n) \) to its universal central extension

\[ 1 \to U(1) \to \mathcal{L}(\text{Spin}_n) \to \mathcal{L}(\text{Spin}_n) \to 1. \]

The obstruction to such lifts is precisely the transgression to loop space of the class \( \frac{1}{4} p_1(P) \in H^4(X; \mathbb{Z}). \) Such lifts are sometimes called Spin structures on the loop bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X). \) A manifold \( X \) is called String if it admits a Spin structure
\[ \pi : P \to X \text{ and } \frac{1}{2}p_1(P) = 0. \] More generally, one may call a principal Spin\(_n\)-bundle String if \( \frac{1}{2}p_1(P) = 0. \)

If \( X \) is a String manifold, then it is possible to lift the structure group of the loop bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X) \) from \( \mathcal{L}(\text{Spin}_n) \) to its universal central extension. In a next step one may want to construct associated vector bundles and interesting operators on sections of those. However, there remain serious analytical difficulties when dealing with differential operators on the infinite dimensional loop space \( \mathcal{L}(X) \). A famous conjecture of Witten says that the \( S^1 \)-equivariant index of a hypothetical Dirac operator on \( \mathcal{L}(X) \) should be given by the so-called Witten genus \([36]\). So far, construction of Dirac operators on loop space is far beyond reach, let alone analytical features like the Fredholm property, which are required to talk about the index. A related conjecture due to Höhn and Stolz \([31]\) (which can be formulated without using those hypothetical Dirac operators) expects the Witten genus on a String manifold to be an obstruction against positive Ricci curvature.

Instead of struggling with the analysis on the free loop space \( \mathcal{L}(X) \), one may also study the obstruction \( \frac{1}{2}p_1(P) \) and its trivializations on the manifold \( X \) itself. This is the program of String geometry: The universal characteristic class \( \frac{1}{2}p_1 \) generates \( H^4(\text{BSpin}_n; \mathbb{Z}) \cong H^4(\text{Spin}_n; \mathbb{Z}) \cong \pi_3(\text{Spin}_n) \cong \mathbb{Z} \). Therefor, \( \frac{1}{2}p_1(P) \in H^4(X; \mathbb{Z}) \) is the obstruction to lift the structure group of a principal Spin\(_n\)-bundle \( \pi : P \to X \) to its 3-connected cover String\(_n \to \text{Spin}_n\).

As explained in Section 4.1 homotopy classes of lifts of classifying maps

\[ \begin{array}{ccc}
X & \xrightarrow{f} & BG \xrightarrow{u} K(\mathbb{Z}, n).
\end{array} \]

give rise to so-called \( u \)-trivialization classes \( q \in H^{n-1}(P; \mathbb{Z}) \). In the special case of \( G = \text{Spin}_n \), \( n \geq 3 \), and \( u = \frac{1}{2}p_1 \in H^4(\text{BSpin}_n; \mathbb{Z}) \), such lifts are called String structures (in the homotopy theoretic sense). By \([27]\), there is a 1-1 correspondence between String structures in the homotopy theoretic sense and String classes \( q \in H^3(P; \mathbb{Z}) \). Similarly, one may consider differential String classes on a principal Spin\(_n\)-bundle with connection \((P, \theta)\) as isomorphism classes of String structures with additional geometric structure. As explained in Section 5.3 String classes on \((P, \theta)\) are precisely the stable isomorphism classes of so-called geometric String structures.

In the String geometry program one may regard (geometric) String structures on \((P, \theta) \to X\) as replacements of (geometric) Spin structures on the loop bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X) \). More explicitly, transgression to loop space can be applied not only to the obstruction class \( \frac{1}{2}p_1 \) but also to its trivializations: (geometric) String structures on \( \pi : P \to X \) are transgressed to (geometric) Spin structures on the loop bundle \( \pi : \mathcal{L}(P) \to \mathcal{L}(X) \). For more details, see \([33][35]\).

Here we notice that transgression of relative and absolute differential characters fits into that picture: differential String classes on \((P, \theta) \to X\) are transgressed to differential refinements of the isomorphism classes of the universal central extension \([37]\): given a differential String class \( \tilde{q} \in \tilde{H}^3(P; \mathbb{Z}) \), the transgressed character \( \tau_{\mathcal{L}}(\tilde{q}) \) satisfies

\[ -i_\pi(\tau) = \tau(\overline{C\mathcal{C}S_\theta(\frac{1}{2}p_1)}) - i_\pi(\tau(\rho), 0). \]

In particular, the transgressed differential form \( \tau(\rho) \) yields a global section of the transgressed character \( \tau(\overline{C\mathcal{C}S_\theta(\frac{1}{2}p_1)}) \). Moreover, by naturality of the transgression, the transgressed character \( \tau(\tilde{q}) \in \tilde{H}^2(\mathcal{L}(P); \mathbb{Z}) \) satisfies \( i_{\mathcal{L}^2(\rho)}(\tau) = \tau(i_{\mathcal{L}^2(\rho)}\tilde{q}) \in \)
$H^2(L(\text{Spin}_n;\mathbb{Z})$. This follows from the commutative diagram of evaluation maps

$$
\begin{array}{ccc}
L(\text{Spin}_n) \times S^1 & \xrightarrow{\text{ev}} & L(P) \times S^1 \\
\downarrow \tilde{\pi} & & \downarrow \tilde{\pi} \\
L(\text{Spin}_n) & \longrightarrow & L(P)
\end{array}
$$

and the identification of the fiber $L(P)_\gamma = \pi^{-1}(\gamma)$ over $\gamma \in L(X)$ with the loop group $L(\text{Spin}_n)$.

In particular, since transgression yields an isomorphism $\tau : H^3(\text{Spin}_n;\mathbb{Z}) \to H^2(L(\text{Spin}_n);\mathbb{Z})$, the (isomorphism class of) line bundle with connection associated with the transgressed String class represents on every fiber of the loop bundle the (isomorphism class of) the universal central extension $L(\text{Spin}_n) \to L(\text{Spin}_n)$. Thus $\tau(\tilde{q})$ represents (isomorphism classes of) $L(\text{Spin}_n)$-lifts of the loop bundle. In this sense, String classes transgress to Spin classes on the loop bundle, and differential String classes transgress to differential refinements of those.

**Appendix A. Differential Characters**

In this section we briefly recall the notion of (absolute and relative) differential characters as introduced in [9] and [6]. We recall some facts on the relation between absolute and relative characters from [1] and [2].

Let $\phi : A \to X$ be a smooth map. Let $Z_n(\phi;\mathbb{Z})$ be the group of smooth singular cycles of the mapping cone complex. Denote the differential of the mapping cone complex by $\partial_\phi$. Similarly, denote by $\Omega^k(\phi)$ the mapping cone de Rham complex with differential $d_\phi$.

Let $k \geq 2$. The group of degree-$k$ relative or mapping cone differential characters $H^k(\phi;\mathbb{Z})$ is defined as:

$$H^k(\phi;\mathbb{Z}) := \{ h \in \text{Hom}(Z_{k-1}(\phi;\mathbb{Z}), U(1)) \mid h \circ \partial_\phi \in \Omega^k(\phi) \}.$$ 

The notation $h \circ \partial_\phi \in \Omega^k(\phi)$ means that there exist differential forms $(\omega, \theta) \in \Omega^k(\phi)$ such that for any chain $(v, w) \in C_k(\phi;\mathbb{Z})$ we have $h(\partial_\phi(v, w)) = \exp \left( 2\pi i \int_{(v, w)} (\omega, \theta) \right)$.

It turns out that the pair of forms $(\omega, \theta) \in \Omega^k(X) \times \Omega^{k-1}(A)$ is uniquely determined by the character $h$ and is closed with integral periods. We call $\omega := \text{curv}(h)$ the curvature of the character $h$ and $\theta =: \text{cov}(h)$ its covariant derivative. We also have a homomorphism $c : H^k(\phi;\mathbb{Z}) \to H^k(\phi;\mathbb{Z})$, called characteristic class.

The group $H^k(X;\mathbb{Z})$ of absolute differential characters on $X$ is obtained as above by replacing the mapping cone complexes by the smooth singular and the de Rham complex of $X$. A character $h \in H^k(X;\mathbb{Z})$ then has a characteristic class in $c(h) \in H^k(X;\mathbb{Z})$ in integral cohomology and a curvature $\text{curv}(h) \in \Omega^k(X)$ in the closed forms with integral periods.

In [1] Ch. 8 we establish a 1-1 correspondence between $H^2(\phi;\mathbb{Z})$ and the group of isomorphism classes of hermitean line bundles with connection and section along $\phi$. By a section we mean a nowhere vanishing section or a section of the associated $U(1)$-bundle. The curvature of a character corresponds to the (normalized) curvature form of the line bundle. Its covariant derivative corresponds to the covariant derivative of the section. Hence the name. The characteristic class corresponds to the first Chern class of the line bundle. Similarly, $H^2(X;\mathbb{Z})$ corresponds to the group of isomorphism classes of hermitean line bundles with connection.
The group $\check{H}^k(\varphi; \mathbb{Z})$ fits into the following exact sequences:

$\begin{align*}
0 & \longrightarrow H^{k-1}(\varphi; \mathbb{R}(1)) \xrightarrow{j} \check{H}^k(\varphi; \mathbb{Z}) \xrightarrow{(\text{curv, cov})} \Omega^k(\varphi) \xrightarrow{\iota} 0 \\
0 & \longrightarrow \frac{\Omega^{k-1}(\varphi)}{\Omega^k(\varphi)} \xrightarrow{c} \check{H}^k(\varphi; \mathbb{Z}) \xrightarrow{\psi} H^k(\varphi; \mathbb{Z}) \xrightarrow{e} 0.
\end{align*}$

The mapping cone de Rham cohomology class of the curvature and covariant derivative coincides with the image of the characteristic class in real cohomology. Thus the two sequences above may be joined to the following exact sequence:

$\begin{align*}
0 & \longrightarrow H^{k-1}(\varphi; \mathbb{Z}) \xrightarrow{i} \check{H}^k(\varphi; \mathbb{Z}) \xrightarrow{c} R^k(\varphi; \mathbb{Z}) \longrightarrow 0.
\end{align*}$

Here $R^k(\varphi; \mathbb{Z})$ denotes the set of pairs of differential forms and integral cohomology classes that match in real cohomology. We also obtain the corresponding short exact sequences for the group of absolute characters. For details, see [6] and [1, Ch. 8].

Moreover, we have natural homomorphisms between absolute and relative characters. These fit into the following long exact sequence [1, Ch. 8]:

$\begin{align*}
H^{k-2}(X; \mathbb{R}(1)) & \xrightarrow{\rho \circ j} \check{H}^{k-1}(A; \mathbb{Z}) \xrightarrow{\iota_0} \check{H}^k(\varphi; \mathbb{Z}) \xrightarrow{\rho_\varphi} \check{H}^k(X; \mathbb{Z}) \xrightarrow{\rho^* c} H^k(A; \mathbb{Z}).
\end{align*}$

The sequence proceeds by the long exact sequences for smooth singular cone cohomology with $\mathbb{R}(1)$-coefficients on the left and with integer coefficients on the right. In degree 2, the homomorphism $\rho_\varphi$ corresponds to the forgetful map that ignores sections.

A relative character $h' \in \check{H}^k(\varphi; \mathbb{Z})$ is called a section along $\varphi$ of the absolute character $h = \check{\rho}_\varphi(h')$. The sequence (40) in particular tells us that a character $h \in \check{H}^k(X; \mathbb{Z})$ admits sections along $\varphi$ if and only if it is topologically trivial along $\varphi$, i.e., $\varphi^* c(h) = 0$. Moreover, it is shown in [1, Ch. 8] that global sections are uniquely determined by their covariant derivative, i.e., we have an isomorphism

$\begin{align*}
\text{cov} : \check{H}^k(\text{id}_X; \mathbb{Z}) & \cong \Omega^{2k-1}(X).
\end{align*}$

The inverse is given by $\text{cov}^{-1}(\rho) = \iota_{ad}(\rho, 0)$.

Differential cohomology is not homotopy invariant. Instead, there is the following homotopy formula [1, Ex. 56]: for a homotopy $f : [0, 1] \times X \to Y$ between smooth maps $f_0, f_1 : X \to Y$ and a differential character $h \in \check{H}^k(Y; \mathbb{Z})$, we have:

$\begin{align*}
f_1^* h - f_0^* h = \iota \left( \int_0^1 f_s^* \text{curv}(h) ds \right).
\end{align*}$

Likewise, for a homotopy $(f, g) : [0, 1] \times (X, A) \to (Y, B)$ between smooth maps $(f_0, g_0), (f_1, g_1) : (X, A) \to (Y, B)$, and a relative differential character $h \in \check{H}^k(\psi; \mathbb{Z})$, where $\psi : B \to Y$, we have [2, Cor. 40]:

$\begin{align*}
(f_1, g_1)^* h - (f_0, g_0)^* h = \iota \left( \int_0^1 f_s^* \text{curv}(h) ds, - \int_0^1 g_s^* \text{cov}(h) ds \right).
\end{align*}$

The graded abelian group $\check{H}^*(X; \mathbb{Z})$ of absolute differential characters carries a ring structure compatible with the exact sequences in (38) and with the wedge product of differential forms and the cup product on singular cohomology. We derived a nice characterization of the ring structure in [1, Ch. 6]. Moreover, the graded abelian group $\check{H}^*(\varphi; \mathbb{Z})$ of relative or mapping cone characters carries the structure of a right module over the ring $\check{H}^*(X; \mathbb{Z})$, compatible with the module structures on mapping cone differential forms and mapping cone cohomology, see [2, Ch. 4].
In [1] Ch. 7 and [2] Ch. 5] we construct fiber integration and transgression maps for absolute and relative differential characters. Thus on any fiber bundle \( \pi : E \to X \) with compact oriented fibers \( F \) we obtain fiber integration homomorphisms

\[
\hat{\tau} : \hat{H}^{k}(E; \mathbb{Z}) \to \hat{H}^{k-\dim F}(X; \mathbb{Z})
\]

\[
\hat{\tau} : \hat{H}^{k}(\Phi; \mathbb{Z}) \to \hat{H}^{k-\dim F}(\varphi; \mathbb{Z}).
\]

Here \( \varphi : A \to X \) denotes a smooth map and \( \Phi : \varphi^{\ast} E \to E \) the induced bundle map. The fiber integration maps commute with the usual fiber integrations on differential forms and cohomology and with the homomorphisms in the short exact sequences \((40)\) and the long exact sequence \((41)\).

Transgression to the free loop space \( \mathcal{L}(X) := \{ \gamma : S^{1} \to X \text{ smooth} \} \) is defined via pullback by the evaluation map \( ev : \mathcal{L}(X) \times S^{1} \to X, (\gamma, t) \mapsto \gamma(t) \), and fiber integration in the trivial bundle:

\[
\tau : \hat{H}^{k}(X; \mathbb{Z}) \to \hat{H}^{k-1}(\mathcal{L}(X); \mathbb{Z}), \quad h \mapsto \hat{\tau}(\text{ev}^{\ast}h).
\]

Likewise, transgression to the free loop space is defined for relative or mapping cone characters: For a smooth map \( \varphi : A \to X \) let \( \mathcal{L}(\varphi) : \mathcal{L}(A) \to \mathcal{L}(X), \gamma \mapsto \varphi \circ \gamma \), be the induced map of loop spaces. Then we have the transgression \( \tau : \hat{H}^{k}(\varphi; \mathbb{Z}) \to \hat{H}^{k-1}(\mathcal{L}(\varphi); \mathbb{Z}), h \mapsto \hat{\tau}(\text{ev}^{\ast}h) \).

Here \( ev : \mathcal{L}(X, A) \times S^{1} \to (X, A) \) and \( \mathcal{L}(X, A) \) denotes the set of pairs of smooth maps \( (f, g) : S^{1} \to (X, A) \) such that \( \varphi \circ g = f \). Moreover, the transgression maps for absolute and relative characters commute with the maps in the exact sequence \((40)\). For details, see [2] Ch. 5.

**APPENDIX B. BUNDLE 2-GERBES**

In [2] Sec. 3.2.2] we show that a bundle gerbe with connection \( \mathcal{G} \), represented by a submersion \( \pi : Y \to X \), defines a relative differential character \( h_{\mathcal{G}} \in \hat{H}^{3}(\pi; \mathbb{Z}) \). Moreover, we have \( (\text{curv}, \text{cov})(h_{\mathcal{G}}) = (H, B)(\mathcal{G}) \), i.e. the curvature and covariant derivative of the character \( h_{\mathcal{G}} \) coincide with the curvature and curving of the bundle gerbe \( \mathcal{G} \). The image of the relative character under the map \( \hat{\rho} : \hat{H}^{3}(\pi; \mathbb{Z}) \to \hat{H}^{3}(X; \mathbb{Z}) \) coincides with the stable isomorphism class of the bundle gerbe.

In this section, we describe the analogous statement for bundle 2-gerbes with connection.\(^{12}\) As a particular instance of this fact, we conclude that for any principal \( \operatorname{Spin}_{n} \)-bundle with connection \( \pi : (P, \theta) \to X \) the Cheeger-Simons bundle 2-gerbe with respect to the connection \( \theta \) represents the Cheeger-Chern-Simons character \( \text{CCS}_{\theta}(\frac{1}{2}p_{1}) \in \hat{H}^{4}(\pi; \mathbb{Z}) \). This in turn implies that differential String structures in the sense of [34] represent differential String classes in our sense.

Recall that a bundle 2-gerbe with connection \( \mathcal{G} \), represented by a submersion \( \pi : Y \to X \), consists of a bundle gerbe with connection \( \mathcal{G} \to Y[2] \) and a 3-form \( B \in \Omega^{3}(Y) \), subject to several compatibility conditions for tensor products of pull-backs to the various higher fiber products. The 3-form \( B \) is called the *curving* of the connection on \( \mathcal{G} \). Moreover, the connection of the bundle 2-gerbe has a *curvature* 4-form \( H \in \Omega^{4}(X) \). The curvature and curving are related by \( \pi^{\ast}H = dB \). The *characteristic class* of a bundle 2-gerbe is a cohomology class \( CC(\mathcal{G}) \in \hat{H}^{4}(\pi; \mathbb{Z}) \).

A *trivialization* with connection of a bundle 2-gerbe with connection is a bundle gerbe with connection \( \mathcal{G} \) over \( Y \), subject to several compatibility conditions for tensor products of pull-backs to the various higher fiber products. In particular, any trivialization with connection comes together with a uniquely determined 3-form \( \rho \in \Omega^{3}(X) \) such that \( \pi^{\ast}\rho = \text{curv}(\mathcal{G}) + B \) and \( d\rho = H \). A bundle 2-gerbe admits trivializations if and only if its characteristic class vanishes, and any trivialization admits compatible connections.

\(^{12}\)For more details on bundle 2-gerbes with connection and their trivializations see [20][34].
Now let $\mathcal{G}$ be a bundle 2-gerbe with connection, represented by a submersion $\pi : Y \to X$. We define the differential character $h_\mathcal{G} \in \check{H}^4(\pi; \mathbb{Z})$ as follows\(^{13}\). For a cycle $(s, t) \in Z_3(\pi; \mathbb{Z})$ choose a geometric relative cycle $(\xi, \tau) \in \mathcal{L}_3(\pi)$ that represents the homology class of $(s, t)$. Let $(\xi, \tau)$ be represented by a smooth map $(S, T) \xrightarrow{\langle \xi, \tau \rangle} (X, Y)$. Choose a chain $(a, b) \in C_4(\pi, \mathbb{Z})$ such that $[(s, t) - \partial_T(a, b)]_{\partial_S} = [\xi, \tau]_{\partial_S}$. Since $\dim(S) = 3$, the pull-back bundle 2-gerbe $f^*\mathcal{G}$ is trivial. Choose a trivialization $\mathcal{G}$ with compatible connection and 3-form $\theta \in \Omega^3(T)$. The map $g : T \to Y$ factors through the induced map $f^*Y \xrightarrow{g} Y$ and thus induces a map $g : T \to f^*Y$. Since $\dim(T) = 2$, the pull-back bundle gerbe $g^*\mathcal{G}$ is trivial. Choose a trivialization with compatible connection and 2-form $\theta \in \Omega^2(T)$.

Now put:

\[
(44) \quad h_\mathcal{G}(s, t) := \exp(2\pi i \left( \int_{(s, t)} (\rho, \theta) + \int_{(a, b)} (H, B) \right)).
\]

In the same way as for bundle gerbes with connection [2, Ch. 3], one can show that $h_\mathcal{G}$ is indeed a differential character in $\check{H}^4(\pi; \mathbb{Z})$ with

\[
(\text{curv, cov})(h_\mathcal{G}) = (H, B).
\]

**APPENDIX C. TRANSGRESSION**

In this section, we discuss the relation of transgression to loop space with transgression in universal bundles. We show that the two transgressions agree in singular cohomology. This is of course well-known. For convenience of the reader, we give the argument here. Most of the material can be found e.g. in [11, Ch. 10.3].

Let $X$ be a smooth manifold and $x_0 \in X$ an arbitrary base point. Transgression to the based loop space $\mathcal{L}_0(X) := \{ \gamma \in \mathcal{L}(X) | \gamma(1) = x_0 \}$ is defined as for the free loop space by pull-back via the evaluation map $\ev : \mathcal{L}_0(X) \times S^1 \to X$, $(\gamma, t) \mapsto \gamma(t)$ and fiber integration in the trivial bundle:

\[
\mathcal{L}^0 : H^*(X; \mathbb{Z}) \to H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}), \quad u \mapsto \pi_!(\ev^*u).
\]

On the other hand, for any fibration $P \xrightarrow{\pi} X$ with fiber $F$ and contractible total space, we have the transgression defined by

\[
(45) \quad T_\pi : H^*(X; \mathbb{Z}) \xrightarrow{\pi^*} H^*(X, x_0; \mathbb{Z}) \xrightarrow{\pi_!} H^*(P, P_0; \mathbb{Z}) \xrightarrow{\delta^{-1}} H^{*-1}(P_0; \mathbb{Z}).
\]

The connecting homomorphism $\delta : H^{*-1}(P_0; \mathbb{Z}) \to H^*(P, P_0; \mathbb{Z})$ is an isomorphism since $P$ is assumed to be contractible.

For a topological space $Y$ let $CY := Y \times [0, 1]/Y \times \{0\}$ denote the cone over $Y$ and $SY := CY/Y \times \{1\}$ the suspension. Let $p : CY \to SY$ denote the projection. The suspension isomorphism $S^* : H^*(SY; \mathbb{Z}) \to H^{*+1}(Y; \mathbb{Z})$ is the concatenation of isomorphisms

\[
(46) \quad H^*(SY; \mathbb{Z}) \xrightarrow{\gamma^*} H^*(CY, Y; \mathbb{Z}) \xrightarrow{\delta^{-1}} H^{*-1}(Y; \mathbb{Z}).
\]

The connecting homomorphism $\delta : H^{*-1}(Y; \mathbb{Z}) \to H^*(CY, Y; \mathbb{Z})$ is an isomorphism since the cone $CY$ is contractible.

Now let $Y = \mathcal{L}_0(X)$. The suspension is a quotient of the trivial fiber bundle with fiber $S^1$. In other words, $S\mathcal{L}_0(X) = \mathcal{L}_0(X) \times S^1/\mathcal{L}_0(X) \times \{1\}$. Let $pr : \mathcal{L}_0(X) \times S^1 \to S\mathcal{L}_0(X)$ denote the projection. Define a map $\ell : S\mathcal{L}_0(X) \to X$ by $[s, \gamma] \mapsto \gamma(s)$. Then we have the

\(^{13}\text{We use the notations from [2].}\)
commutative diagram:

\[
\begin{array}{c}
\mathcal{L}_0(X) \times S^1 \\
\downarrow \text{pr} \\
S\mathcal{L}_0(X)
\end{array}
\xrightarrow{e^*} \xrightarrow{\ell^*} \xrightarrow{\text{pr}^*} X
\]

Fiber integration \( \pi \) in smooth singular cohomology is defined by the Leray-Serre spectral sequence, see [2] p. 482f. It is realized by pre-composition of cocycles with the transfer map on smooth singular chains, see [1] Ch. 4. This identification yields the commutative diagram:

\[
H^*(S\mathcal{L}_0(X); \mathbb{Z}) \xrightarrow{\pi^*} H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}) \]

Thus we obtain

\[\tau^0 = \pi \circ \text{ev}^* = \pi \circ \text{pr} \circ \ell^* = S^* \circ \ell^* : H^*(X; \mathbb{Z}) \to H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}).\]

Now let \( \mathcal{P}_0(X) := \{ \gamma : [0, 1] \to X \text{ smooth}, \gamma(0) = x_0 \} \) be the based path space of \( X \). Let \( e : \mathcal{P}_0(X) \to X, \gamma \mapsto \gamma(1) \), be the path fibration with fiber \( \mathcal{L}_0(X) \). Since \( \mathcal{P}_0(X) \) is contractible we obtain from (45) the transgression \( T_{\mathcal{P}_0} : H^*(\mathbb{C}; \mathbb{Z}) \to H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}) \) for the path fibration.

We have a map of pairs \( g : (C\mathcal{L}_0(X), \mathcal{L}_0(X)) \to (\mathcal{P}_0(X), \mathcal{L}_0(X)) \) induced by the map \( g : C\mathcal{L}_0(X) \to \mathcal{P}_0(X), (t, \gamma) \mapsto (s \to \gamma(st)) \). Since \( C\mathcal{L}_0(X) \) and \( \mathcal{P}_0(X) \) are contractible total spaces of fibrations with the same fiber, the induced map on cohomology is an isomorphism due to the five lemma. This yields the commutative diagram of isomorphisms:

\[\begin{array}{ccc}
H^*(\mathcal{P}_0(X), \mathcal{L}_0(X); \mathbb{Z}) & \xrightarrow{g^*} & H^*(C\mathcal{L}_0(X), \mathcal{L}_0(X); \mathbb{Z}) \\
\downarrow \delta & & \downarrow \delta \\
H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}) & & H^{*-1}(\mathcal{L}_0(X); \mathbb{Z})
\end{array}\]

Moreover, we have the commutative diagram

\[\begin{array}{ccc}
C\mathcal{L}_0(X) & \xrightarrow{e \circ g} & X \\
\downarrow \text{pr} & & \downarrow \ell \\
S\mathcal{L}_0(X) & & \mathcal{L}_0(X)
\end{array}\]

This yields:

\[\tau^0 = S^* \circ \ell^* = \delta^{-1} \circ \text{pr}^* \circ \ell^* = \delta^{-1} \circ (e \circ g)^* = \delta^{-1} \circ e^* = T_{\mathcal{P}_0}.
\]

As above, let \( \pi_{\mathcal{E}G} : \mathcal{E}G \to \mathcal{B}G \) be a universal principal \( G \)-bundle, i.e. a principal \( G \)-bundle with contractible total space \( \mathcal{E}G \). A contraction of \( \mathcal{E}G \) to \( y_0 \in \mathcal{E}G \) associates to any point \( y \in \mathcal{E}G \) a path \( \gamma_y : [0, 1] \to \mathcal{E}G \) with \( \gamma_y(0) = y_0 \) and \( \gamma_y(1) = y \). Thus \( \gamma_y \in \mathcal{P}_0(\mathcal{E}G) \).

Let \( x_0 := \pi_{\mathcal{E}G}(y_0) \). Then we obtain a map \( H : \mathcal{E}G \to \mathcal{P}_0(\mathcal{B}G), y \mapsto \pi_{\mathcal{E}G} \circ \gamma_y \). Since \( e(\pi_{\mathcal{E}G} \circ \gamma_y) = \pi_{\mathcal{E}G}(\gamma_y(1)) = \pi_{\mathcal{E}G}(y) \), the map preserves the fibers. It is in fact a homotopy equivalence of fibrations from the universal principal \( G \)-bundle \( \pi_{\mathcal{E}G} : \mathcal{E}G \to \mathcal{B}G \) to the path fibration \( e : \mathcal{P}_0(\mathcal{B}G) \to \mathcal{B}G \) over the classifying space \( \mathcal{B}G \). In particular, it yields a homotopy equivalence of the fibers and thus isomorphisms \( H^* : H^*(\mathcal{L}_0(\mathcal{B}G); \mathbb{Z}) \to H^*(\mathcal{G}; \mathbb{Z}) \). Moreover, as a homotopy equivalence of fibrations with contractible total spaces the map \( H \) identifies the transgressions \( H^* \circ T_{\mathcal{P}_0} = T \).
Summarizing, we obtain the following identifications of transgressions:

\[ \hat{t}^0 = S^* \circ t^r = T_{\rho_0}: H^*(X; \mathbb{Z}) \to H^{*-1}(\mathcal{L}_0(X); \mathbb{Z}) \]

\[ H^* \circ T_{\rho_0} = T : H^*(BG; \mathbb{Z}) \to H^{*-1}(G; \mathbb{Z}). \]

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