New families of singularity-free cosmological models

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In this talk we extend a family of geodesically complete $G_2$ stiff fluid cosmological models to the case in which the velocity of the fluid is not orthogonal to the gradient of the transitivity surface element.

1 Introduction

During the previous decade the interest for singularity-free cosmological models was supported by the publication of the first known geodesically complete perfect fluid cosmology \cite{1}. Until then such possibility had been overlooked due to the restrictive conditions imposed by singularity theorems \cite{2}, \cite{3}. These theorems required physically reasonable restrictions such as energy, causality and generic conditions, but they also imposed the existence of certain trapped sets, such as compact achronal sets without edge or closed trapped surfaces, which were not so obvious. Namely this latter condition is the one which has been used to avoid the formation of singularities. More details about geodesic completeness of the Senovilla spacetime may be found in \cite{4}.

Since 1990, the number of new cosmological models which were singularity-free did not increase very much, most of them within the framework of $G_2$ orthogonally transitive spacetimes \cite{5}. But in 2002 a family of stiff fluid cosmologies depending on two almost arbitrary functions was shown to be geodesically complete \cite{6}. It is so far the largest family of singularity-free perfect fluid cosmologies. It was obtained requiring that the velocity of the fluid should be orthogonal to the gradient of the transitivity surface element at every point of the spacetime. In this talk we show that this restriction may be removed and thereby the family of singularity-free cosmologies is enlarged.

In the next section we show the Einstein equations for $G_2$ orthogonally transitive stiff fluid spacetimes and simplify them so that the analysis of the geodesics may be carried out conveniently. In Section 3 regularity theorems \cite{7} are used to derive conditions on the spacetimes to be geodesically complete.
2 Einstein equations for $G_2$ stiff fluid cosmologies

We shall write the Einstein equations for the metric of a spacetime endowed with an Abelian orthogonally transitive group of isometries $G_2$ acting on spacelike surfaces. Except for some spherically symmetric spacetimes, every other singularity-free cosmology belongs to that group. We shall also require that the metric be diagonal.

We choose a coordinate chart in which the metric is isotropic in the non-ignorable coordinates, $t, r$. The coordinates $z$ and $\phi$ are adapted to the generators of the isometry group and therefore metric functions do not depend on them,

$$g = e^{2K} (-dt^2 + dr^2) + e^{-2U} dz^2 + \rho^2 e^{2U} d\phi^2. \tag{1}$$

The range of the coordinates is the standard,

$$-\infty < t, z < \infty, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi, \tag{2}$$

for cylindric symmetry.

The matter content of the spacetime is a perfect fluid,

$$T = \mu u \otimes u + p (g + u \otimes u), \tag{3}$$

for which pressure, $p$, and energy density, $\mu$, are equal. The velocity of the fluid $u$ is parametrized with the help of a function $\xi$,

$$u = -e^K (\cosh \xi dt + \sinh \xi dr), \tag{4}$$

which allows to write the Einstein equations for a stiff perfect fluid as

$$U_{tt} - U_{rr} + \frac{1}{\rho} (U_t \rho_t - U_r \rho_r) = 0, \tag{5a}$$

$$\rho_{tt} - \rho_{rr} = 0, \tag{5b}$$

$$\frac{K_t \rho_t + K_r \rho_r}{\rho} = \frac{\rho_{tt} + U_t \rho_t + U_r \rho_r}{\rho} + 2U_t U_r + e^{2K} p \sinh 2\xi, \tag{5c}$$

$$\frac{K_t \rho_t + K_r \rho_r}{\rho} = \frac{\rho_{tt} + \rho_{rr}}{2\rho} + \frac{U_t \rho_t + U_r \rho_r}{\rho} + U_t^2 + U_r^2 + e^{2K} p \cosh 2\xi, \tag{5d}$$

$$K_{rr} - K_{tt} + \frac{U_r \rho_r - U_t \rho_t}{\rho} + U_t^2 - U_r^2 = p e^{2K}, \tag{5e}$$

$$K_r - \xi_t + \frac{p_r}{2p} + \frac{p_t \cosh \xi - \rho_t \sinh \xi}{\rho} \sinh \xi = 0, \tag{5f}$$

$$K_t - \xi_r + \frac{p_t}{2p} + \frac{p_r \cosh \xi - \rho_r \sinh \xi}{\rho} \cosh \xi = 0. \tag{5g}$$
This differential system may be simplified by taking $\rho$ as our coordinate $r$. We thereby restrict the solutions to the set for which the gradient of the transitivity surface element, $\rho$, is spacelike. This is the set where singularity-free spacetimes have been found so far.

$$U_{tt} - U_{rr} - \frac{U_r}{r} = 0,$$

(6a)

$$K_t = U_t + 2rU_tU_r + e^{2K} pr \sinh 2\xi,$$

(6b)

$$K_r = U_r + r(U_t^2 + U_r^2) + e^{2K} pr \cosh 2\xi,$$

(6c)

$$K_{rr} - K_{tt} + \frac{U_r}{r} + U_r^2 - U_t^2 = pe^{2K},$$

(6d)

$$K_r - \xi_t + \frac{p_r}{2p} - \frac{\sinh^2 \xi}{r} = 0,$$

(6e)

$$K_t - \xi_r + \frac{p_t}{2p} - \frac{\sinh \xi \cosh \xi}{r} = 0.$$

(6f)

We may further write an exact differential,

$$dH = e^{2K}rp(\sinh 2\xi \, dt + \cosh 2\xi \, dr),$$

(7)

which provides $\xi$ and $p$ once $K$ is known,

$$\tanh 2\xi = \frac{H_t}{H_r}, \quad |p| = \frac{e^{-2K}}{r} \sqrt{H_r^2 - H_t^2}.$$

(8)

This allows another simplification of the system of differential equations,

$$U_{tt} - U_{rr} - \frac{U_r}{r} = 0,$$

(9a)

$$H_{rr} - H_{tt} = \frac{\sqrt{H_r^2 - H_t^2}}{r},$$

(9b)

$$K_t = U_t + 2rU_tU_r + H_t,$$

(9c)

$$K_r = U_r + r(U_t^2 + U_r^2) + H_r,$$

(9d)

which is formed by an inhomogeneous 2-D wave equation for $U$ and a non-linear wave equation for $H$. The metric function $K$ is obtained after integrating a quadrature. It may be checked that this system is consistent.

It can be shown that the axis can be made regular just rescaling the angular coordinate $\phi$.

Although it is not necessary for our purposes, we point out that the Wainright-Ince-Marshman formalism [8] may be used for generating solutions to the non-linear wave equation.
3 Singularity-free stiff fluid models

The simplified system of Einstein equations that we have shown is suitable for the analysis of geodesic completeness. The case of diagonal Abelian orthogonally transitive spacetimes has been thoroughly studied in [7] and we shall make use of those results, which may be written in the form of a theorem,

**Theorem:** A diagonal Abelian orthogonally transitive spacetime with spacelike orbits endowed with a metric in the form (1) with $C^2$ metric functions $K, U, \rho$, where $\rho$ has a spacelike gradient, is future causally geodesically complete provided that along causal geodesics:

1. For large values of $t$ and increasing $r$,
   (a) $(K - U - \ln \rho)_r + (K - U - \ln \rho)_t \geq 0$, and either $(K - U - \ln \rho)_r \geq 0$ or $|K - U - \ln \rho|_r \lesssim (K - U - \ln \rho)_r + (K - U - \ln \rho)_t$.
   (b) $K_r + K_t \geq 0$, and either $K_r \geq 0$ or $|K_r| \lesssim K_r + K_t$.
   (c) $(K + U)_r + (K + U)_t \geq 0$, and either $(K + U)_r \geq 0$ or $|(K + U)_r| \lesssim (K + U)_r + (K + U)_t$.

2. For large values of $t$, a constant $b$ exists such that

$$\begin{align*}
K(t, r) - U(t, r) &\geq -\ln |t| + b, \\
2K(t, r) &\geq -\ln |t| + b.
\end{align*}$$

For past-pointing geodesics the corresponding theorem is quite similar, just exchanging the sign of the time derivatives and imposing conditions for small values of $t$, instead of large values. A generalization to nondiagonal metrics may be found in [9].

The results that are drawn from applying these theorems to the stiff fluid case may be summarized as follows:

**Theorem 1:** A cylindrical spacetime with a stiff perfect fluid as matter content, endowed with a metric in the form (1) with $C^2$ metric functions $K, U, \rho$ is future geodesically complete if the gradient of the surface element is spacelike and

1. For large values of $t$, $U(t, 0) \geq -\frac{1}{2} \ln |t| + b$.

2. Either $r^{1+\varepsilon}|U_r + U_t|$ or $r^{1+\varepsilon}(H_r + H_t)$ does not tend to zero for large values of $t$ and $r$. 
Theorem 2: A cylindrical spacetime with a stiff perfect fluid as matter content, endowed with a metric in the form (1) with $C^2$ metric functions $K, U, \rho$ is past geodesically complete if the gradient of the surface element is spacelike and

1. For small values of $t$, $U(t, 0) \geq -\frac{1}{2} \ln |t| + b$.
2. Either $r^{1-\varepsilon}|U_r - U_t|$ or $r^{1-\varepsilon}(H_r - H_t)$ does not tend to zero for small values of $t$ and large values of $r$.

It is not much difficult to find metric functions that comply with these requirements:

Corollary: A metric with arbitrary $H$ and a function $U$ which grows for large $|t|$ and for large $r$ makes the spacetime geodesically complete.

As it is shown in [6], it is not difficult to provide solutions of the inhomogeneous wave equation in a form amenable to check the premises of the theorems. We may consider the initial value problem,

$$U_{tt} - U_{rr} - \frac{U_r}{r} = 0,$$

$$U(0, r) = f(r), \quad U_t(0, r) = g(r),$$

(10)

which can be solved in closed integral form [10],

$$U = U_f + U_g,$$

$$U_f(x, y, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^t dRR \frac{g(x + R \cos \phi, y + R \sin \phi)}{\sqrt{t^2 - R^2}},$$

$$U_g(x, y, t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int_0^{2\pi} d\phi \int_0^t dRR \frac{f(x + R \cos \phi, y + R \sin \phi)}{\sqrt{t^2 - R^2}},$$

(11)

for initial data $U(x, y, 0) = f(x, y)$, $U_t(x, y, 0) = g(x, y)$ with circular symmetry.

The term depending respectively on the initial data $f$ is even in $t$. On the contrary, $U_g$ is odd. Since the condition of the corollary is symmetric in $t$, this means that either $U_f$ and $U_g$ must comply the condition separately or just $U_f$ complies it, but overcoming the term $U_g$.

In [6] it is shown how this can be done for polynomial initial data. For instance if $f$ and $g$ are polynomials and the degree of $f$ is larger than the one of $g$ and the coefficient of the leading term is positive, we generate a geodesically complete spacetime.
4 Final remarks

In this talk sufficient conditions for an Abelian diagonal orthogonally transitive spacetime with spacelike orbits and with a stiff perfect fluid as matter content to be geodesically complete have been produced. The resulting conditions are simple and can be implemented easily on examples. These results generalize previous work by the authors. More details will be found elsewhere [11].

The main consequence of this talk is that geodesically complete spacetimes are more abundant than it was thought before. Since stiff fluids are a special case of perfect fluids, that may also be interpreted as scalar fields, further work is needed and other sources and symmetries are to be considered in the future.

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