Long-time behavior and relaxation of power-law correlation in one-dimensional self-gravitating system

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Abstract

Long-time behavior of spatial power-law correlation in one-dimensional self-gravitating system is numerically investigated. The power-law structure persists even after the system is virialized. The structure gradually disappears with energy exchange among particles. Lifetime of the power-law structure is estimated to be proportional to the system size.

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1 Introduction

Spatial structure in many-body systems is an important subject in many branches of physics, such as molecular systems, gravitational systems, and so on. For various types of such structures, their origins are interesting subject of study. Some of them may be realized as equilibrium distribution, and others may be already set at initial conditions.

Recently we have discovered that spatial structure with power-law correlation spontaneously emerges from uniformly random initial conditions in

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one-dimensional self-gravitating system [1]. What is important in this novel phenomenon is that the spatial structure is not given at the initial condition, but dynamically created from a state without spatial correlation. Succeeding research clarified that the structure is created first in small spatial scale then grows up to large scale through hierarchical clustering [2].

The question to be discussed now is the properties of the state which has power-law correlation. In particular we have to clarify whether the state persists for long time, or just a short transient and disappears quickly. If the state lasts for long time, we have good chance to observe it and it has much importance. On the other hand, if the state disappears quickly, it may not be observable easily and it is less important to the whole dynamics of the system.

Another important question is on the relaxation process (See [3–5] and references therein.). Since the system is an isolated many-body system without dissipation, one would expect that the system will eventually relax to thermal equilibrium for sufficiently long time scale. Hence relation between decay (if it does) of power-law structure and relaxation processes of thermalization is also important.

The purpose of this paper is to investigate the long-time behavior of the power-law structure; its lifetime and mechanism of relaxation. The model is introduced in the next section. In section 3 we introduce several quantities which represent relaxation and investigate their dynamics to clarify the mechanism by which the power-law structure disappears. The final section is devoted to summary and discussions.
2 Model

The model we use in this paper is the self-gravitating one-dimensional system [3–9]. It represents a system of particles on one-dimension interacting through Newtonian gravity. It can also be interpreted as a system consists of equivalent mass-sheets placed parallel in 3-dimensional space, hence it is also called as the sheet model.

The Hamiltonian of the model is

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + 2\pi G m^2 \sum_{i>j} |x_i - x_j|, \quad -\infty < x_i < \infty , \]

where \( x_i \) and \( p_i \) represent coordinate and momentum of \( i \)'th particle, \( N \) is the number of particles, \( m \) stands for mass of each particle, and \( G \) is a positive constant. Force between two particles is attractive and long-ranged.

In this paper we set \( m \equiv 1/N \) and \( 4\pi G \equiv 1 \). Time is measured in unit \( t_c \equiv 1/\sqrt{4\pi GM/L} \), where \( M \equiv mN \) is total mass, \( L \) is the spatial length on which particles are distributed initially.

3 Numerical results

Now we show temporal evolution of the power-law correlation. Since the formation process is already studied in our previous work [2], here we focus on the long-time behavior.

In what follows we use a typical example whose initial condition and parameters are defined as follows:

\begin{align*}
\text{system size} & : N = 2048 , \\
\text{initial condition} & : \forall i \ u_i = 0 , \tag{2}
\end{align*}
\[ x_i = \text{uniformly random in } [0, 1). \]

### 3.1 Power-law correlation

In our previous works we showed that spatial structure where two-point correlation function obey power-law

\[ \xi(r) \propto r^{-\alpha} \]  \hspace{1cm} (3)

is formed in the model (1) from uniformly-random initial conditions [1, 2]. Here the two-point correlation function \( \xi(r) \) is defined as

\[ dP = ndV(1 + \xi(r)) \]  \hspace{1cm} (4)

where \( dP \) stands for probability to find another particle in volume \( dV \) at distance \( r \) from a particle. \( n \) is the average number density.

Using this correlation function we found that the model (1) with initial conditions with zero velocity dispersion evolves into a state where \( \xi(r) \) obeys power-law (3). In Figure 1 we show a typical example of such power-law correlation.

\[ \xi(r) \]

\[ r \]

Figure 1: An example of power-law behavior of correlation function (4) for parameters (2). \( t = 250. \) \( \xi(r) \propto r^{-0.11}. \)
3.2 $V_r$ : virial ratio

The main question we would like to discuss in this paper is whether the power-law structure lasts for long time or not. For that purpose we need to define a time scale for reference.

As for the one-dimensional self-gravitating system (1), it is known that relaxation proceeds through three stages; first virialization process occurs, then “microscopic relaxation”, then “macroscopic relaxation” [4, 5]. (Each process is explained later.) We will compare the decay of power-law structure and each relaxation process.

First we adopt a time scale called virial time $t_{vr}$ defined from virial ratio $V_r(t)$. Virial ratio is defined as

$$V_r(t) \equiv \frac{2E_{\text{kin}}(t)}{E_{\text{pot}}(t)} ,$$

(5)

where $E_{\text{kin}}(t)$ and $E_{\text{pot}}(t)$ are kinetic and potential energy of the system at time $t$, respectively.

Since the total energy is conserved and finite, it follows from virial theorem that

$$2\overline{E_{\text{kin}}}/\overline{E_{\text{pot}}} = 1 ,$$

(6)

where $\overline{E_{\text{kin}}}$ and $\overline{E_{\text{pot}}}$ are long time average of $E_{\text{kin}}(t)$ and $E_{\text{pot}}(t)$, respectively. Thus convergence of $V_r(t)$ to 1 is one good measure of relaxation of the system, and the time scale $t_{vr}$ at which $V_r(t)$ converges to 1 is a good reference timescale. We call $t_{vr}$ virial time.

Figure 2 shows temporal evolution of the virial ratio (5). It is seen that the virial ratio $V_r(t)$ converges to 1 at about $t \sim 150$. At this time it can be said that the system has relaxed to a state in the sense of energy partition.
The important point to note is that the power-law correlation shown in Fig. 1 exists after the virial ratio is converged to 1. (Remember that the figure is taken at $t = 250$. Also it is confirmed in other examples.) That is, power-law correlation lasts longer than virial time. In this sense the power-law correlation persists long.

3.3 $\alpha(t)$: exponent of power-law

The next problem is the length of time it lasts. For that purpose we investigate the long-time behavior of the exponent $\alpha$ of power-law (3).

Fig. 3 represents temporal evolution of the exponent $\alpha$ for time scale much longer than the one shown in Fig. 2 for 10 independent orbits. Initial conditions are the same as (2) with different realization of random numbers. From this figure we see that the power-law structure is not stationary but transient and gradually fading in this time scale. Decrease of the exponent $\alpha(t)$ can be considered as relaxation process.
Figure 3: Long-time behavior of the exponent $\alpha(t)$ of correlation \([3]\) for $0 \leq t \leq 10000$ for 10 initial conditions.

Figure 4: Temporal evolution of $\Delta(t)$ for $0 \leq t \leq 10000$.

### 3.4 $\Delta(t)$ : energy exchange

Next we investigate the process of relaxation of power-law correlation. For that purpose we measure $\Delta(t)$ defined below, by which we can see the degree of energy exchange between particles \([4, 5]\).

$\Delta(t)$ is defined as follows. First we define $\varepsilon_i(t)$, 1-particle energy per unit mass as

$$\varepsilon_i(t) \equiv \frac{1}{2} v_i^2(t) + 2\pi Gm \sum_{j=1}^{N} |x_j(t) - x_i(t)| \quad .$$  \hspace{1cm} (7)

If the system is ergodic the long time average of $\varepsilon_i(t)$ converges to a unique value

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \varepsilon_i(t) dt = \frac{5E_{\text{total}}}{3} \equiv \varepsilon_0 \quad \hspace{1cm} (8)$$
for all $i$.

Using this $\varepsilon_0$, $\Delta(t)$ is defined as

$$
\Delta(t) \equiv \frac{1}{\varepsilon_0} \left( \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{t} \int_{0}^{t} \varepsilon_i(t') \, dt' - \varepsilon_0 \right] \right)^2.
$$

(9)

$\Delta(t)$ represents deviation from equipartition of energy among particles. At each time step $\varepsilon_i(t)$ varies from particle to particle. A particle which initially has high energy may give the energy to particles which initially have low energy, and vice versa. Hence, in the course of time evolution, if the value 

$$
\left( \frac{1}{t} \int_{0}^{t} \varepsilon_i(t') \, dt' - \varepsilon_0 \right)
$$

decays to zero for most of the particles, we can say that energy is well exchanged among particles. On the other hand, $\Delta(t)$ remains constant if energy is not exchanged between particles and every $\varepsilon_i(t)$ is kept constant. Thus decrease of $\Delta(t)$ indicates energy exchange among particles.

Fig. 4 shows temporal evolution of $\Delta(t)$ for the parameters (2). Initial increase of $\Delta(t)$ represents that, instead of becoming equipartitioned, differences of one-particle energy among particles are temporarily enhanced. Let us turn our attention to behavior of $\Delta(t)$ in later time. We see that $\Delta(t)$ begins to decrease from $t \sim 1500$, which is of the same order of the time when the power-law fades away as seen in Fig. 3. Hence we can say that the power-law structure relaxes as energy is exchanged among particles.

3.5 $\nu(\varepsilon)$: cumulative energy distribution

Now we investigate further the process of relaxation of power-law structure. We have seen that the power-law structure begins to disappear as energy exchange among particle begins, as indicated by the decrease of $\Delta(t)$. If, through this relaxation, energy distribution is kept constant, this relaxation
phase corresponds to "microscopic relaxation" discussed by Tsuchiya et al. \[4, 5\]. If, on the other hand, energy distribution is also relaxed to that of thermal equilibrium, then the phase corresponds to "macroscopic relaxation".

We measure energy distribution by $\nu(\varepsilon)$ defined as follows:

$$\nu(\varepsilon) \equiv \frac{1}{N} \text{ (number of particles whose energy } \varepsilon_i \text{ is less than } \varepsilon), \quad (10)$$

where $N$ is the total number of particles. Since $\nu(\varepsilon)$ is a cumulative distribution, $\nu(\varepsilon)$ increases monotonically with respect to $\varepsilon$. If the graph of $\nu(\varepsilon)$ does not change with respect to time through relaxation, then the relaxation is microscopic relaxation.

![Figure 5: $\nu(\varepsilon)$ for $t = 250, 300, \cdots, 2000$](image)

![Figure 6: $\nu(\varepsilon)$ for $t = 250, 300, \cdots, 2000, 8000, 8050, \cdots, 9750$](image)

Figs. 5 and 6 show $\nu(\varepsilon)$ for two temporal intervals. Two well-known quasi-stationary state, water-bag distribution and isothermal distribution \[4, 5\] are
also shown as references.

Fig. 5 represents $\nu(\varepsilon)$ for $250 < t < 2000$, when the system has power-law correlation and the system is virialized (see Fig. 2). Please note that $\nu(\varepsilon)$ does not change for this time.

Fig. 6 represents $\nu(\varepsilon)$ for $8000 < t < 9750$ overlaid on the previous figure. In this temporal interval power-law correlation is lost (see Fig. 3). It is clearly seen that the energy distribution remains the same regardless of the existence of power-law correlation. Thus we can say that the relaxation process of power-law structure can be explained by the microscopic relaxation caused by energy exchange among particles while keeping the global energy distribution fixed.

4 Summary and discussions

In this paper we numerically investigated the long-time behavior of power-law correlation structure [1, 2] in one-dimensional self-gravitating system. The power-law correlation exists even after the system is virialized. In this sense the structure persists for long time. The lifetime of the structure is, however, not infinite and it gradually fades away. By comparing the timescale of the lifetime with other time scales of relaxation we see that the decay of the power-law structure is not caused by virialization process but by energy exchange among particles. The one-particle energy distribution $\nu(\varepsilon)$ is invariant through the relaxation of the structure, hence it is the microscopic relaxation [4, 5]. The system is not stationary if we see it by the exponent $\alpha$ of correlation, but it is at the same time quasi-stationary if we see it by the energy distribution $\nu(\varepsilon)$ after the system is virialized.
The fact that power-law correlation structure lasts long implies that the structure is important in the whole dynamical evolution of the system and we may be able to observe such structure in other systems even if the system is virialized.

Relation between the decay of power-law correlation and microscopic relaxation has an important consequence. It is known that the microscopic relaxation time depends linearly on the system size \( t_{\text{micro}} \propto N \). Hence it is likely that the average lifetime of the power-law correlation has similar dependence on \( N \), which is quite long for large \( N \).

Let us now summarize schematically the entire life of the spatial power-law correlation structure in the model (1) clarified through a series of our works.

1. From initial condition with zero velocity dispersion (1), spatial structure with power-law correlation emerges in small spatial scale (2). Typical example of such initial condition is \( \forall i \ v_i = 0 \), \( x_i = \text{random} \).

2. The power-law correlation structure grows up to large scale through hierarchical clustering (2).

3. The system is virialized. Power-law correlation structure persists.

4. Microscopic relaxation takes place and power-law correlation structure disappears. Energy distribution remains invariant.

Related models with a uniform background or friction (12, 13) are investigated from cosmological interest. In those models similar fractal structure has been observed and is more persistent. Comparison with our results may
be helpful to understand mechanism by which the structure is retained in their models.

It should also be noted that the structure formation we have studied have origin in dynamics. It is interesting that, instead of monotonously relaxing to thermal equilibrium the system, initially does not have spatial correlation, creates spatial correlation of power-law type through dynamics. It seems difficult to explain it in terms of thermodynamics or equilibrium statistical physics, since the structure appears in states which are not thermally relaxed. Recently dynamical formation of spatial structure in long-range interacting systems is actively investigated. Relation between our results and other examples of structure formation with dynamical origin [10, 11, 14, 15] will be interesting.

In this paper we have focused on typical behavior of the system. Several problems related to details of the behavior remain unsolved. Among them there are, for example, initial condition dependence, system size dependence, and so on. These are beyond the scope of this short paper and of important subjects of future study.

In this paper we have studied the behavior of the system until power-law structure disappears through energy exchange and energy distribution. It will be interesting to investigate whether there are other relaxation processes effectively working [16, 17]; and the behavior of the system in the course of approach to equilibrium [4, 5, 18, 19].

The spontaneous emergence of power-law correlation we have studied so far is a novel phenomenon with rich implication. We hope our research serves as a foundation for future study, both fundamental and applied.
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