Abstract. In the search for hypercomplex analytic functions on the half-plane, we review the construction of induced representations of the group $G = \text{SL}_2(\mathbb{R})$. Firstly we note that $G$-action on the homogeneous space $G/H$, where $H$ is any one-dimensional subgroup of $\text{SL}_2(\mathbb{R})$, is a linear-fractional transformation on hypercomplex numbers. Thus, we investigate various hypercomplex characters of subgroups $H$. The correspondence between the structure of the group $\text{SL}_2(\mathbb{R})$ and hypercomplex numbers can be illustrated in many other situations as well. We give examples of induced representations of $\text{SL}_2(\mathbb{R})$ on spaces of hypercomplex valued functions, which are unitary in some sense. Raising/lowering operators for various subgroup prompt hypercomplex coefficients as well.

Keywords. Induced representation, unitary representations, $\text{SL}_2(\mathbb{R})$, semisimple Lie group, complex numbers, dual numbers, double numbers, Möbius transformations, split-complex numbers, parabolic numbers, hyperbolic numbers, raising/lowering operators, creation/annihilation operators.

1. Introduction

Analytic functions of a complex variable form a beautiful theory with rich applications in many fields ranging from number theory to electrical engineering. Thus, it is natural to look for its analogs and generalisations in different directions. The most basic (or fundamental?) situation appears if we replace the complex imaginary unit $i^2 = -1$ with either the hyperbolic one $j^2 = +1$ or the nilpotent $\varepsilon^2 = 0$.

Two-dimensional commutative associative algebra over reals generated by 1 and $j$ consists of numbers $x + jy$, where $x, y \in \mathbb{R}$. They are known as split-complex, duplex, hyperbolic or double numbers [4, 14, 43, 50]. The algebra has zero divisors $j_{\pm} = \frac{1}{2}(1 \pm j)$ with the properties $j_{\pm}^2 = j_{\pm}$ and $j_+j_- = 0$. Thus, double numbers are isomorphic to $\mathbb{R} \oplus \mathbb{R}$—the direct sum
of two copies of the real line spanned by $j_+$ and $j_-$. This explains the names “split-complex” and “double”.

The analogous algebra associated to the nilpotent unit $\varepsilon$ consists of elements $x + \varepsilon y$, which are called dual numbers [6, 12, 54]. All zero divisors in the algebra are $\varepsilon y, y \in \mathbb{R}$. Physical applications of hypercomplex numbers are scattered through classical mechanics [31, 54], non-linear dynamics [43, 44], relativity [4, 50], cosmology [9, 12] and quantum mechanics [14, 34, 51].

**Remark 1.1.** Unfortunately, there are no common notations for hypercomplex units. Moreover, it would be difficult simply to list the whole variety of symbols employed for this. Even the complex imaginary unit $i$ is oftenly written as $j$ in engineering. The hyperbolic unit is denoted by $j$ in many papers starting at least from the foundational article [53]; although a different letter $e$ is used in the remarkable book [54]. The symbol $\iota$ is used for the nilpotent unit in [10, 45], however we chose $\varepsilon$ following Yaglom [54]. The latter notation is also suggestive in light of the following remark.

**Remark 1.2.** The parabolic unit $\varepsilon$ is a close relative to the infinitesimal number $\varepsilon$ from non-standard analysis [8, 52]. The former has the property that its square is exactly zero, meanwhile the square of the latter is almost zero at its own scale. In fact, there is a version of non-standard analysis [3] employing the nilpotent unit $\varepsilon$ as an infinitesimal$^1$. Also, some non-standard proofs of the main calculus theorems are given in [6]. A similar property allows to obtain classical mechanics from the representations of the Heisenberg group [31, 34].

What kind of “analytic” functions can be associated with dual and double numbers? Since this question is very natural it was addressed over a prolonged period of time by various researchers. Many of them were unaware of works of their predecessors, neither I can claim to possess the complete knowledge. Below is a brief summary of several works known to me.

For double numbers, a systematic study was already accomplished in [53], there are also numerous later investigations and surveys, see [4, 14], [37, Part IV], [39, 42, 43, 48] and further references therein. The existing consensus is based on the factorisation of the wave equation $\partial_x^2 - \partial_y^2 = (\partial_x - j\partial_y)(\partial_x + j\partial_y)$ into a product of two linear differential operators. This is an analog of the factorisation of the Laplacian $\partial_x^2 + \partial_y^2 = (\partial_x - i\partial_y)(\partial_x + i\partial_y)$ into the product of the Cauchy–Riemann operator and its adjoint. Thus, hyperbolic analytic functions are defined to be null solutions of the operator $\partial_x + j\partial_y$. However, the split of dual number in the basis $j_{\pm}$ reduces an “analytic” function $f(x, y) = j_+ f_+(x + y) - j_- f_-(x - y)$ to the sum, where $f_{\pm}$ are two generic differentiable real-valued functions of a real variable. This is related to the representation of the generic solution of the wave equation on the infinite string as a sum of a wave traveling to the left and another traveling to the right.

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$^1$I am grateful to the anonymous referee for pointing my attention to the book [3] by J.L. Bell.
For the nilpotent unit \( \varepsilon \) the situation is even more trivial. The above factorisation approach does not lead anywhere useful\(^2\), since \((\partial_x - \varepsilon \partial_y)(\partial_x + \varepsilon \partial_y) = \partial_x^2\). An attempt to define analytic functions in terms of power series of \( x + \varepsilon y \) produces only functions of the form \( f(x) + \varepsilon y f'(x) \) for a real-analytic function \( f(x) \) of a single variable \([6]\).

In the series of previous works \([18, 19, 20, 21]\) we proposed to develop an analytic function theory in the spirit of the Erlangen Programme of F. Klein: from an appropriate group action. The present paper is a further step towards that goal. We observe that classical spaces of complex analytical functions—the Hardy and Bergman spaces—are irreducible moduli under certain representations of the group \( \text{SL}_2(\mathbb{R}) \). Moreover, those representations are induced by complex-valued characters of the compact subgroup \( K \subset \text{SL}_2(\mathbb{R}) \). Thus, we hope to find other types of analytic function spaces among all irreducible (or primary) \( \text{SL}_2(\mathbb{R}) \)-moduli under induced representations.

The paper outline is as follows. We recall the structure of \( \text{SL}_2(\mathbb{R}) \) and list its tree non-isomorphic one-dimensional continuous subgroups in Section 2. This information is employed to classify tree non-isomorphic two-dimensional \( \text{SL}_2(\mathbb{R}) \)-homogeneous spaces in Section 3. We discover that three kinds of hypercomplex numbers are perfectly suited to describe \( \text{SL}_2(\mathbb{R}) \)-action on those homogeneous spaces. To induce representations from one-dimensional subgroups we describe their characters (from the algebraic point of view) in Section 4. The next Section 5 provides additional objects, which can be also viewed as characters with geometrical spirit. An induced representation can be linear if the inducing characters is also linear. For geometric characters this is not straightforward and we discuss various possibilities in Sections 6 and 7. Finally, we are able to write down various induced representations in Section 8. A convenient description of an irreducible \( \text{SL}_2(\mathbb{R}) \)-module can be produced in terms of a linear basis, consisting of eigenvectors of the subgroup inducing the representation. The Lie algebra \( \mathfrak{sl}_2 \) of \( \text{SL}_2(\mathbb{R}) \) acts transitively on this basis by means of ladder operators, which are important in quantum mechanics. We describe the respective structures in Section 9. The paper is concluded by an (incomplete) list of interesting directions for further research.

2. The Group \( \text{SL}_2(\mathbb{R}) \) and Its Subgroups

Let \( \text{SL}_2(\mathbb{R}) \) be the group of \( 2 \times 2 \) matrices with real entries and of determinant one \([38]\). This is the smallest semisimple Lie group. Any matrix in \( \text{SL}_2(\mathbb{R}) \) admits a (unique) decomposition of the form \([13, \text{Exer. I.14}]\):

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},
\]

for the following real values of the parameters:

\[\alpha = \sqrt{c^2 + d^2} \in (0, \infty), \ \nu = ac + bd \in (-\infty, \infty), \ \phi = -\arctan \frac{c}{d} \in (-\pi, \pi).\]

\(^2\)However, there is an interesting factorisation of a parabolic operator \( \partial_x^2 + \partial_y \) which requires Clifford algebras of higher dimensions \([7]\).
Formula (1) rewritten in a way $\text{SL}_2(\mathbb{R}) = \text{ANK}$ is known as Iwasawa decomposition [38, § III.1] and can be generalised to any semisimple Lie group.

Each out of the three types of matrices in the right-hand side of (1) forms a one-parameter subgroup $A$, $N$ and $K$. They are obtained by the exponentiation of the respective zero-trace matrices:

\begin{align*}
A &= \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, t \in \mathbb{R} \right\}, \\
N &= \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, t \in \mathbb{R} \right\}, \\
K &= \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, t \in (-\pi, \pi] \right\}.
\end{align*}

The following simple result has an instructive proof.

**Proposition 2.1.** Any continuous one-parameter subgroup of $\text{SL}_2(\mathbb{R})$ is conjugate to one of subgroups $A$, $N$ or $K$.

**Proof.** Any one-parameter subgroup is obtained through the exponentiation

\[ e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \]

of an element $X$ of the Lie algebra $\mathfrak{sl}_2$ of $\text{SL}_2(\mathbb{R})$. Such $X$ is a $2 \times 2$ matrix with the zero trace. The behaviour of the Taylor expansion (5) depends from properties of powers $X^n$. This can be classified by a straightforward calculation:

**Lemma 2.2.** The square $X^2$ of a traceless matrix $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is the identity matrix times $a^2 + bc = -\det X$. The factor can be negative, zero or positive, which corresponds to the three different types of the Taylor expansion (5) of $e^{tX}$.

It is a simple exercise on characteristic polynomials to see that through the matrix similarity we can obtain from $X$ a generator

- of the subgroup $K$ if $(-\det X) < 0$;
- of the subgroup $N$ if $(-\det X) = 0$;
- of the subgroup $A$ if $(-\det X) > 0$.

The determinant is invariant under the similarity, thus these cases are distinct. \qed

**Example 2.3.** The following two subgroups are conjugated to $A$ and $N$ respectively:

\begin{align*}
A' &= \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\}, \\
N' &= \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\}.
\end{align*}
3. Action of $\text{SL}_2(\mathbb{R})$ as a Source of Hypercomplex Numbers

Let $H$ be a subgroup of a group $G$. Let $\Omega = G/H$ be the corresponding homogeneous space and $s : \Omega \to G$ be a smooth section [15, §13.2], which is a left inverse to the natural projection $p : G \to \Omega$. The choice of $s$ is inessential in the sense that by a smooth map $\Omega \to \Omega$ we can always reduce one to another.

Any $g \in G$ has a unique decomposition of the form $g = s(\omega)h$, where $\omega = p(g) \in \Omega$ and $h \in H$. Note that $\Omega$ is a left homogeneous space with the $G$-action defined in terms of $p$ and $s$ as follows:

$$g : \omega \mapsto g \cdot \omega = p(g \ast s(\omega)),$$

where $\ast$ is the multiplication on $G$. This is also illustrated by the following commutative diagram:

For $G = \text{SL}_2(\mathbb{R})$, as well as for other semisimple groups, it is common to consider only the case of $H$ being the maximal compact subgroup $K$. However in this paper we admit $H$ to be any one-dimensional subgroup. Due to the previous Proposition it is sufficient to take $H = K$, $N'$ or $A'$. Then $\Omega$ is a two-dimensional manifold and for any choice of $H$ we define [19, Ex. 3.7(a)]:

$$s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2, \ v > 0. \tag{9}$$

A direct (or computer algebra [26]) calculation shows that:

**Proposition 3.1.** The $\text{SL}_2(\mathbb{R})$ action (8) associated to the map $s$ (9) is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \left( \frac{au + b}{(cu + d)^2 - \sigma(cv)^2}, \frac{v}{(cu + d)^2 - \sigma(cv)^2} \right), \tag{10}$$

where $\sigma = -1$, 0 and 1 for the subgroups $K$, $N'$ and $A'$ respectively.

The expression in (10) does not look very appealing, however an introduction of hypercomplex numbers makes it more attractive:

**Proposition 3.2.** Let a hypercomplex unit $\iota$ be such that $\iota^2 = \sigma$, then the $\text{SL}_2(\mathbb{R})$ action (10) becomes:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \quad \text{where } w = u + \iota v, \tag{11}$$

for all three cases parametrised by $\sigma$ as in Prop. 3.1.

**Remark 3.3.** We wish to stress that the hypercomplex numbers were not introduced here by our intention, arbitrariness or “generalising attitude” [46, p. 4]. They were naturally created by the $\text{SL}_2(\mathbb{R})$ action.
Notably the action (11) is a group homomorphism of the group $\text{SL}_2(\mathbb{R})$ into transformations of the “upper half-plane” on hypercomplex numbers. Although dual and double numbers are algebraically trivial, the respective geometries in the spirit of Erlangen programme are refreshingly inspiring [16, 27, 33] and provide useful insights even in the elliptic case [23]. In order to treat divisors of zero, we need to consider M"obius transformations (11) of conformally completed plane [12, 25].

Now we wish to linearise the action (8) through the induced representations [15, §13.2], [19, §3.1]. We define a map $r: G \to H$ associated to $p$ and $s$ from the identities:

$$r(g) = (s(\omega))^{-1} g, \quad \text{where } \omega = p(g) \in \Omega.$$  \hfill (12)

Let $\chi$ be an irreducible representation of $H$ in a vector space $V$, then it induces a representation of $G$ in the sense of Mackey [15, §13.2]. This representation has the realisation $\rho_\chi$ in the space of $V$-valued functions by the formula [15, §13.2.(7)–(9)]:

$$[\rho_\chi(g)f](\omega) = \chi(r(g^{-1} * s(\omega))) f(g^{-1} \cdot \omega),$$  \hfill (13)

where $g \in G$, $\omega \in \Omega$, $h \in H$ and $r: G \to H$, $s: \Omega \to G$ are maps defined above; $*$ denotes multiplication on $G$ and $\cdot$ denotes the action (8) of $G$ on $\Omega$.

In our consideration $H$ is always one-dimensional. Traditionally, an irreducible representation of such a subgroup is supposed to be a complex valued character. However, hypercomplex numbers naturally appeared in the $\text{SL}_2(\mathbb{R})$ action (11), why shall we admit only $i^2 = -1$ to deliver a character then?

4. Hypercomplex Characters—an Algebraic Approach

As we already mentioned the typical discussion of induced representations of $\text{SL}_2(\mathbb{R})$ is centred around the case $H = K$ and a complex valued character of $K$. A linear transformation defined by a matrix (4) in $K$ is a rotation of $\mathbb{R}^2$ by the angle $t$. After identification $\mathbb{R}^2 = \mathbb{C}$ this action is given by the multiplication $e^{it}$, with $i^2 = -1$. The rotation preserves the (elliptic) metric given by:

$$x^2 + y^2 = (x + iy)(x - iy).$$  \hfill (14)

Therefore the orbits of rotations are circles, any line passing the origin (a “spoke”) is rotated by the angle $t$, see Fig. 1(E).

Introduction of hypercomplex numbers produces the most straightforward adaptation of this result.

**Proposition 4.1.** The following table shows correspondences between three types of algebraic characters:
Induced Representations and Hypercomplex Numbers

Figure 1. Rotations of algebraic wheels, i.e. the multiplication by $e^{it}$: elliptic (E), trivial parabolic ($P_0$) and hyperbolic (H). All blue orbits are defined by the identity $x^2 - \iota^2 y^2 = r^2$. Green “spokes” (straight lines from the origin to a point on the orbit) are “rotated” from the real axis.

|                | Elliptic | Parabolic | Hyperbolic |
|----------------|----------|-----------|------------|
| $j^2 = -1$     | $\varepsilon^2 = 0$ | $j^2 = 1$ |
| $w = x + iy$   | $w = x + \varepsilon y$ | $w = x + jy$ |
| $\bar{w} = x - iy$ | $\bar{w} = x - \varepsilon y$ | $\bar{w} = x - jy$ |
| $e^{it} = \cos t + i \sin t$ | $e^{\varepsilon t} = 1 + \varepsilon t$ | $e^{jt} = \cosh t + j \sinh t$ |
| $|w|^2_e = w\bar{w} = x^2 + y^2$ | $|w|^2_p = w\bar{w} = x^2$ | $|w|^2_h = w\bar{w} = x^2 - y^2$ |
| $\arg w = \tan^{-1} \frac{y}{x}$ | $\arg w = \frac{y}{x}$ | $\arg w = \tanh^{-1} \frac{y}{x}$ |
| unit circle    | “unit” strip $x = \pm 1$ | unit hyperbola $|w|^2_h = 1$ |

Geometrical action of multiplication by $e^{it}$ is drawn on Fig. 1 for all three cases.

Explicitly parabolic rotations associated with $e^{\varepsilon t}$ acts on dual numbers as follows:

$$e^{\varepsilon x} : a + \varepsilon b \mapsto a + \varepsilon(ax + b).$$

This links the parabolic case with the Galilean group [54] of symmetries of the classic mechanics, with the absolute time disconnected from space.

The obvious algebraic similarity and the connection to classical kinematic is a wide spread justification for the following viewpoint on the parabolic case, cf. [11, 54]:

- the parabolic trigonometric functions are trivial:
  $$\cosp t = \pm 1, \quad \sint t = t;$$

- the parabolic distance is independent from $y$ if $x \neq 0$:
  $$x^2 = (x + \varepsilon y)(x - \varepsilon y);$$
• the polar decomposition of a dual number is defined by [54, App. C(30')]:
\[ u + \varepsilon v = u(1 + \varepsilon \frac{v}{u}), \quad \text{thus} \quad |u + \varepsilon v| = u, \quad \arg(u + \varepsilon v) = \frac{v}{u}; \quad (18) \]

• the parabolic wheel looks rectangular, see Fig. 1(P0).

Those algebraic analogies are quite explicit and widely accepted as an ultimate source for parabolic trigonometry [11, 39, 54]. Moreover, those three rotations are all non-isomorphic symplectic linear transformations of the phase space, which makes them useful in the context of classical and quantum mechanics [30, 34].

However we will see shortly that there exists an alternative with geometric motivation and connection to equations of mathematical physics.

5. A Parabolic Wheel—A Geometrical Viewpoint

We make another attempt to describe parabolic rotations. If multiplication (a linear transformation) is not sophisticated enough for this we can advance to the next level of complexity: linear-fractional.

Hypercomplex units do not need to be seen as abstract quantities. As follows from Lem. 2.2 the generators of subgroup $K$, $N$ and $A$ represent units of complex, dual and double numbers respectively. Their exponentiation to one-parameter subgroups $K$, $N'$ and $A'$ of SL$_2(\mathbb{R})$ produce matrix forms of the Euler identities from the fifth row of the table in Prop. 4.1.

Thus we attempt to define characters of subgroups $K$, $N'$ and $A'$ in term of geometric action of SL$_2(\mathbb{R})$ by Möbius transformations. The action (11) is defined on the upper half-plane and to relate it to unitary characters we wish to transfer it to the unit disk. In the elliptic case this is done by the Cayley transform, its action on the subgroup $K$ is:

\[
\frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}. \quad (19)
\]

The diagonal matrix in the right hand side defines the Möbius transformation which reduces to multiplication by $e^{2it}$, i.e. the elliptic rotation.

A hyperbolic cousin of the Cayley transform is:

\[
\frac{1}{2} \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad (20)
\]

similarly produces a Moebius transformation which is the multiplication by $e^{2jt}$, which a unitary (Lorentz) transformation of two-dimensional Minkowski space-time.

In the parabolic case we use the similar pattern and define the Cayley transform from the matrix:

\[
C_\varepsilon = \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}
\]

The Cayley transform of matrices (3) from the subgroup $N$ is:

$$
\begin{pmatrix}
1 & -\varepsilon \\
-\varepsilon & 1
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \varepsilon \\
\varepsilon & 1
\end{pmatrix}
= \begin{pmatrix}
1 + \varepsilon t & t \\
0 & 1 - \varepsilon t
\end{pmatrix}
= \begin{pmatrix}
e^{\varepsilon t} & t \\
0 & e^{-\varepsilon t}
\end{pmatrix}.
(21)
$$

This is not far from the previous identities (19) and (20), however, the off-diagonal $(1,2)$-term destroys harmony. Nevertheless we will continue a unitary parabolic rotation to be the Möbius transformation with the matrix (21), which will not be a multiplication by a scalar anymore.

![Figure 2. Rotation of geometric wheels: elliptic ($E$), two parabolic ($P$ and $P'$) and hyperbolic ($H$). Blue orbits are level lines for the respective moduli. Green straight lines join points with the same value of argument and are drawn with the constant “angular step” in each case.](image)

**Example 5.1.** The parabolic rotations with the upper-triangular matrices from the subgroup $N$ becomes:

$$
\begin{pmatrix}
e^{\varepsilon t} & t \\
0 & e^{-\varepsilon t}
\end{pmatrix} : -\varepsilon \mapsto t + \varepsilon(t^2 - 1).
(22)
$$

This coincides with the *cyclic rotations* defined in [54, §8]. A comparison with the Euler formula seemingly confirms that $\sin \varepsilon t = t$, but suggests a new expression for $\cos \varepsilon t$:

$$
\cos \varepsilon t = 1 - t^2,
\sin \varepsilon t = t.
$$

Therefore the parabolic Pythagoras’ identity would be:

$$
\sin^2 \varepsilon t + \cos \varepsilon t = 1,
(23)
$$
which nicely fits in between the elliptic and hyperbolic versions:
\[ \sin^2 t + \cos^2 t = 1, \quad \sinh^2 t - \cosh^2 t = -1. \]
The identity (23) is also less trivial than the version \( \cosh^2 t = 1 \) from [11] (see also (16), (17)).

**Example 5.2.** There is the second option to define parabolic rotations for the lower-triangular matrices from the subgroup \( N' \). The important difference now is: the reference point cannot be \(-\varepsilon\) since it is a fixed point (as well as any point on the vertical axis). Instead we take \( \varepsilon^{-1} \), which is an ideal element (a point at infinity [54, App. C]) since \( \varepsilon \) is a divisor of zero. The proper compactifications by ideal elements for all three cases were discussed in [25].

We get for the subgroup \( N' \):
\[
\begin{pmatrix}
e^{-et} & 0 \\
t & e^{et}
\end{pmatrix} : \frac{1}{\varepsilon} \mapsto \frac{1}{t} + \varepsilon \left(1 - \frac{1}{t^2}\right).
\]

A comparison with (22) shows that this form is obtained by the change \( t \mapsto t^{-1} \). The same transformation gives new expressions for parabolic trigonometric functions. The parabolic “unit circle” (or cycle [27, 33, 54]) is defined by the equation \( x^2 - y^2 = 1 \) in both cases, see Fig. 2(\( P \)) and (\( P' \)). However other orbits are different and we will give their description in the next Section.

Fig. 2 illustrates Möbius actions of matrices (19), (20) and (21) on the respective “unit disk”, which are images of the upper half-planes under respective Cayley transforms [27, § 8], [33, Ch. 10].

### 6. Rebuilding Algebraic Structures from Geometry

We want induced representations to be linear, to this end the inducing character shall be linear as well. Rotations in elliptic and hyperbolic cases are given by products of complex or double numbers respectively and thus are linear. However non-trivial parabolic rotations (22) and (24) (Fig. 2(\( P \)) and (\( P' \))) are not linear. Can we find algebraic operations for dual numbers, which will linearise those Möbius transformations?

It is common in mathematics to “revert a theorem into a definition” and we will use this systematically to recover a compatible algebraic structure.

**6.1. Modulus and Argument**

In the elliptic and hyperbolic cases orbits of rotations are points with the constant norm (modulus): either \( x^2 + y^2 \) or \( x^2 - y^2 \). In the parabolic case we employ this point of view as well:

**Definition 6.1.** Orbits of actions (22) and (24) are contour lines for the following functions which we call respective moduli (norms):

for \( N \) : \( |u + \varepsilon v| = u^2 - v \),
for \( N' \) : \( |u + \varepsilon v|' = \frac{u^2}{v + 1} \).
Remark 6.2. (1) The expression $|(u,v)| = u^2 - v$ represents a parabolic distance from $(0, \frac{1}{2})$ to $(u,v)$, see [27, Lem. 8.3], [33, Ex. 10.6], which is in line with the “parabolic Pythagoras’ identity” (23).

(2) Modulus for $N'$ expresses the parabolic focal length from $(0, -1)$ to $(u,v)$ as described in [27, Lem. 8.4], [33, Ex. 10.7].

The only straight lines preserved by both the parabolic rotations $N$ and $N'$ are vertical lines, thus we will treat them as “spokes” for parabolic “wheels”. Elliptic spokes in mathematical terms are “points on the complex plane with the same argument”, thus we again use this for the parabolic definition:

**Definition 6.3.** Parabolic arguments are defined as follows:

\[
\text{for } N : \arg(u + \varepsilon v) = u, \quad \text{for } N' : \arg'(u + \varepsilon v) = \frac{1}{u}.
\]

Both Definitions 6.1 and 6.3 possess natural properties with respect to parabolic rotations:

**Proposition 6.4.** Let $w_t$ be a parabolic rotation of $w$ by an angle $t$ in (22) or in (24). Then:

\[
|w_t|^{(t)} = |w|^{(t)}, \quad \arg^{(t)} w_t = \arg^{(t)} w + t,
\]

where primed versions are used for subgroup $N'$.

All proofs in this and the following Sections were performed through symbolic calculations on a computer. Details can be found in the earlier (more heuristic) paper on this topic [26].

**Remark 6.5.** Note that in the commonly accepted approach [54, App. C(30')] parabolic modulus and argument are given by expressions (18), which are, in a sense, opposite to our agreements.

6.2. Rotation as Multiplication

We revert again theorems into definitions to assign multiplication. In fact, we consider parabolic rotations as multiplications by unimodular numbers thus we define multiplication through an extension of properties from Proposition 6.4:

**Definition 6.6.** The product of vectors $w_1$ and $w_2$ is defined by the following two conditions:

\[
(1) \quad \arg^{(t)}(w_1w_2) = \arg^{(t)} w_1 + \arg^{(t)} w_2;
\]

\[
(2) \quad |w_1w_2|^{(t)} = |w_1|^{(t)} \cdot |w_2|^{(t)}.
\]

We also need a special form of parabolic conjugation, which coincides with sign reversion of the argument.

**Definition 6.7.** Parabolic conjugation is given by

\[
\overline{u + \varepsilon v} = -u + \varepsilon v.
\]
Obviously we have the properties: \(|w|^{(t)} = |\overline{w}|^{(t)}\) and \(\arg^{(t)} w = -\arg^{(t)} \overline{w}\). A combination of Definitions 6.1, 6.3 and 6.6 uniquely determine expressions for products.

**Proposition 6.8.** The parabolic product of vectors is defined by formulae:

for \(N\): \[(u, v) \ast (u', v') = (u + u', (u + u')^2 - (u^2 - v)(u'^2 - v')); \quad (28)\]

for \(N'\): \[(u, v) \ast (u', v') = \left(\frac{uu'}{u + u'}, \frac{(v + 1)(v' + 1)}{(u + u')^2} - 1\right). \quad (29)\]

Although both expressions look unusual they have many familiar properties:

**Proposition 6.9.** Both products (28) and (29) satisfy the following conditions:

1. They are commutative and associative;
2. The respective rotations (22) and (24) are given by multiplications with a dual number with the unit norm.
3. The product \(w_1 \overline{w}_2\) is invariant under the respective rotations (22) and (24).
4. For any dual number \(w\) the following identity holds:

\[|w\overline{w}| = |w|^2.\]

In particular, the property (3) will be crucial below for an inner product (38), which makes induced representations unitary.

### 7. Invariant Linear Algebra

Now we wish to define a linear structure on \(\mathbb{R}^2\) which would be invariant under point multiplication from the previous Subsection (and thus under the parabolic rotations, cf. Prop. 6.9 (2)). Multiplication by a real scalar is straightforward (at least for a positive scalar): it should preserve the argument and scale the norm of a vector. Thus we have formulae for \(a > 0\):

\[a \cdot (u, v) = (u, av + u^2(1 - a))\quad \text{for } N; \quad (30)\]

\[a \cdot (u, v) = \left(u, \frac{v + 1}{a} - 1\right)\quad \text{for } N'. \quad (31)\]

On the other hand, the addition of vectors can be done in several different ways. One of them, related to tropical mathematics [5, 40], is outlined in Example 10.1. Here, we present another alternative with all due details.

Addition of vectors for both subgroups \(N\) and \(N'\) can be defined by the common rules, where subtle differences are hidden within corresponding Definitions 6.1 (norms) and 6.3 (arguments).
**Definition 7.1.** Parabolic addition of vectors is defined by the following formulæ:

\[
\arg^{(i)}(w_1 + w_2) = \frac{\arg^{(i)} w_1 \cdot |w_1|^{(i)} + \arg^{(i)} w_2 \cdot |w_2|^{(i)}}{|w_1 + w_2|^{(i)}},
\]

\[
|w_1 + w_2|^{(i)} = |w_1|^{(i)} \pm |w_2|^{(i)},
\]

where primed versions are used for the subgroup \(N'\).

The rule for the norm of sum (33) may look too trivial at the first glance. We should say in its defence that it nicely sits in between the elliptic \(|w + w'| \leq |w| + |w'|\) and hyperbolic \(|w + w'| \geq |w| + |w'|\) triangle inequalities for norms.

The rule (32) for argument of the sum is not arbitrary as well. From the Sine Theorem in the Euclidean geometry we can deduce that:

\[
\sin(\phi - \psi') = \frac{|w| \cdot \sin(\psi - \psi')}{|w + w'|}, \quad \sin(\psi' - \phi) = \frac{|w'| \cdot \sin(\psi - \psi')}{|w + w'|},
\]

where \(\psi^{(i)} = \arg^{(i)} w\) and \(\phi = \arg(w + w^{(i)})\). Using parabolic expression (16) for the sine \(\sin \theta = \theta\) we obtain the arguments addition formula (32).

A proper treatment of zeros in denominator of (32) can be achieved through a representation of a dual number \(w = u + \varepsilon v\) as a pair of homogeneous polar coordinates \([a, r] = [|w|^{(i)} \cdot \arg^{(i)} w, |w|^{(i)}]\) (dashed version for the subgroup \(N'\)). Then the above addition is defined component-wise in the homogeneous coordinates:

\[
w_1 + w_2 = [a_1 + a_2, r_1 + r_2], \quad \text{where } w_i = [a_i, r_i].
\]

The multiplication from Defn. 6.6 is given in the homogeneous polar coordinates by:

\[
w_1 \cdot w_2 = [a_1 r_2 + a_2 r_1, r_1 r_2], \quad \text{where } w_i = [a_i, r_i].
\]

Thus homogeneous coordinates linearise the addition (32)–(33) and multiplication by a scalar (30). A transition to other more transparent coordinates shall be treated withing birational geometry framework [35].

Both formulæ (32)–(33) together uniquely define explicit expressions for addition of vectors. However those expressions are rather cumbersome and not really much needed. Instead we list properties of these operations:

**Proposition 7.2.** Vector additions for subgroups \(N\) and \(N'\) defined by (32)–(33) satisfy the following conditions:

1. They are commutative and associative.
2. They are distributive for multiplications (28) and (29); consequently:
3. They are parabolic rotationally invariant;
4. They are distributive in both ways for the scalar multiplications (30) and (31) respectively:

\[
a \cdot (w_1 + w_2) = a \cdot w_1 + a \cdot w_2, \quad (a + b) \cdot w = a \cdot w + b \cdot w.
\]
To complete the construction we need to define the zero vector and the inverse. The inverse of \( w \) has the same argument as \( w \) and the opposite norm.

**Proposition 7.3.** \((N)\) The zero vector is \((0, 0)\) and consequently the inverse of \((u, v)\) is \((u, 2u^2 - v)\).
\[(N')\] The zero vector is \((\infty, -1)\) and consequently the inverse of \((u, v)\) is \((u, -v - 2)\).

Thereafter we can check that scalar multiplications by negative reals are given by the same identities (30) and (31) as for positive ones.

**Remark 7.4.** The irrelevance of the standard linear structure for parabolic rotations manifests itself in many different ways, e.g. in an apparent “non-conformality” of lengths from parabolic foci, that is with the parameter \( \sigma = 0 \) in [27, Prop. 5.1.3], [33, Ex. 7.14.iii] . An adjustment of notions to the proper framework restores the clear picture.

The initial definition of conformality [27, Defn. 5.4], [33, Defn. 7.13] considered the usual limit \( y' \to y \) along a straight line, i.e. “spoke” in terms of Fig. 1. This is justified in the elliptic and hyperbolic cases. However in the parabolic setting the proper “spokes” are vertical lines, see Fig. 2 \((P)\) and \((P')\), so the limit should be taken along them [27, Prop. 5.2], [33, § 11.6.1].

8. Induced Representations

We discussed above various implementations of hypercomplex unitary characters. Now we can return to consideration of induced representations. We can notice that only the subgroup \( K \) requires a complex valued character due to the fact of its compactness. For subgroups \( N' \) and \( A' \) we can consider characters of all three types—elliptic, parabolic and hyperbolic. Moreover a parabolic character can be taken either as algebraic (15) or any of two geometric (22) and (24). Therefore we have seven essentially different induced representations, which multiply types to eleven (counting flavours of parabolic characters).

**Example 8.1.** Consider the subgroup \( H = K \), then we are limited to complex valued characters of \( K \) only. All of them are of the form \( \chi_k \):

\[
\chi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-ikt}, \quad \text{where } k \in \mathbb{Z}. \tag{34}
\]

Using the explicit form (9) of the map \( s \) we find the map \( r \) given in (12) as follows:

\[
r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{c^2 + d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \in K.
\]

Therefore:

\[
r(g^{-1} * s(u, v)) = \frac{1}{\sqrt{(cu + d)^2 + (cv)^2}} \begin{pmatrix} cu + d & -cv \\ cv & cu + d \end{pmatrix}, \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Substituting this into (34) and combining with the Möbius transformation of the domain (11) we get the explicit realisation $\rho_k$ of the induced representation (13):

$$\rho_k(g)f(w) = \left|\frac{cw + d}{cw + d}\right|^k \left(\frac{aw + b}{cw + d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = u + iv. \quad (35)$$

This representation acts on complex valued functions in the upper half-plane $\mathbb{R}_+^2 = \text{SL}_2(\mathbb{R})/K$ and belongs to the discrete series [38, § IX.2].

**Proposition 8.2.** Let $f_k(w) = \left|\frac{w-i}{w-i}\right|^k$ for $k = 2, 3, \ldots$, then

1. $f_k$ is an eigenvector for any operator $\rho_k(h)$, where $h \in K$, with the eigenvalue $\chi_k(h)$ [38, § IX.2].
2. The function $K(z,w) = \rho_k(s(z))f_k(w)$, where $s(z)$ is defined in (9), is the Bergman reproducing kernel up to the factor $\left|\frac{z-i}{w-i}\right|^k$ in the upper half-plane [19, § 3.2].

Similarly we can get the Cauchy kernel for the limiting case $k = 1$ of the mock discrete series [38, Ch. IX]. There are many other important connections of representation (35) with complex analysis and operator theory. For example, Möbius transformations of operators lead to Riesz-Dunford functional calculus and associated spectrum [22].

**Example 8.3.** In the case of the subgroup $N$ there is a wider choice of possible characters.

1. Traditionally only complex valued characters of the subgroup $N$ are considered, they are:

$$\chi_C^\tau \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{i\tau t}, \quad \text{where } \tau \in \mathbb{R}. \quad (36)$$

A direct calculation shows that:

$$r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in N'. \quad (37)$$

Thus:

$$r(g^{-1} * s(u,v)) = \begin{pmatrix} 1 & 0 \\ \frac{cu}{d+cu} & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (37)$$

A substitution of this value into the character (36) together with the Möbius transformation (11) we obtain the next realisation of (13):

$$\rho_C^\tau(g)f(w) = \exp \left(i \frac{\tau cw}{cu + d}\right) f \left(\frac{aw + b}{cw + d}\right), \quad \text{where } w = u + \varepsilon v, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (37)$$

The representation acts on the space of complex valued functions on the upper half-plane $\mathbb{R}_+^2$, which is subset of dual numbers as a homogeneous space $\text{SL}_2(\mathbb{R})/N$. The mixture of complex and dual numbers in the same expression is confusing.
(2) The parabolic character $\chi_\tau$ with the algebraic flavour is provided by multiplication (15) with the dual number:

$$\chi_\tau \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{\varepsilon \tau t} = 1 + \varepsilon \tau t,$$

where $\tau \in \mathbb{R}$.

As before substitute the value (37) into this character we receive the representation:

$$\rho_\tau(g) f(w) = \left(1 + \varepsilon \frac{\tau cv}{cu + d}\right) f \left(\frac{aw + b}{cw + d}\right),$$

where $w$, $\tau$ and $g$ are as above. The representation is defined on the space of dual numbers valued functions on the upper half-plane of dual numbers. Thus the expression contains only dual numbers with their usual algebraic operations. Thus it is linear with respect to them.

(3) The geometric character $\chi^g_\tau$ is given by the action (22). Then the corresponding representation acts again on the space of dual numbers valued functions on the upper half-plane of dual numbers as follows:

$$\rho^g_\tau(g) f(w) = \left(1 + \varepsilon \frac{2\tau cv}{cu + d}\right) f \left(\frac{aw + b}{cw + d}\right) + \frac{\tau cv}{cu + d} + \varepsilon \left(\frac{(\tau cv)^2}{(cu + d)^2}\right),$$

where $w$, $\tau$ and $g$ are as above. This representation is linear with respect to operations (30), (32) and (33).

All characters in the previous Example are unitary, the first two in a conventional sense and the last one in the sense of Prop. 6.9. Then the general scheme of induced representations [15, § 13.2] implies their unitarity in proper senses.

**Theorem 8.4.** All three representations of $\text{SL}_2(\mathbb{R})$ from Example 8.3 are unitary on the space of function on the upper half-plane $\mathbb{R}^2_+$ of dual numbers with the inner product:

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^2_+} f_1(w) \overline{f_2}(w) \frac{du \, dv}{v^2}, \quad \text{where } w = u + \varepsilon v,$$

and we use

(1) the conjugation and multiplication of functions’ values in algebras of complex and dual numbers for representations $\rho^\mathbb{C}_\tau$ and $\rho_\tau$ respectively;

(2) conjugation (27) and multiplication (28) of functions’ values for the representation $\rho^g_\tau$.

The inner product (38) is positive defined for the representation $\rho^\mathbb{C}_\tau$ but is not for two others. The respective spaces are parabolic cousins of the Krein spaces [1], which are hyperbolic in our sense.
9. Similarity and Correspondence: Ladder Operators

From the above observation we can deduce the following empirical principle, which has a heuristic value.

Principle 9.1 (Similarity and correspondence). (1) Subgroups $K$, $N$ and $A$ play the similar rôle in a structure of the group $\text{SL}_2(\mathbb{R})$ and its representations.

(2) The subgroups shall be swapped simultaneously with the respective replacement of hypercomplex unit $\iota$.

The first part of the Principle (similarity) does not look sound alone. It is enough to mention that the subgroup $K$ is compact (and thus its spectrum is discrete) while two other subgroups are not. However in a conjunction with the second part (correspondence) the Principle have received the following confirmations so far:

- The action of $\text{SL}_2(\mathbb{R})$ on the homogeneous space $\text{SL}_2(\mathbb{R})/H$ for $H = K$, $N'$ or $A'$ is given by linear-fractional transformations of complex, dual or double numbers respectively (Prop. 3.2).
- Subgroups $K$, $N'$ or $A'$ are isomorphic to the groups of unitary rotations of respective unit cycles in complex, dual or double numbers (Prop. 4.1).
- Representations induced from subgroups $K$, $N'$ or $A'$ are unitary if the inner product spaces of functions with values in complex, dual or double numbers (Thm. 8.4).

Remark 9.2. The principle of similarity and correspondence resembles supersymmetry between bosons and fermions in particle physics, but we have similarity between three different types of entities in our case.

Let us give another illustration to the Principle. Consider the Lie algebra $\mathfrak{sl}_2$ of the group $\text{SL}_2(\mathbb{R})$. Pick up the following basis in $\mathfrak{sl}_2$ [49, § 8.1]:

\[ A = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

The commutation relation between those elements are:

\[ [Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2}Z. \quad (39) \]

Let $\rho$ be a representation of the group $\text{SL}_2(\mathbb{R})$ in a space $V$. Consider the derived representation $d\rho$ of the Lie algebra $\mathfrak{sl}_2$ [38, § VI.1] and denote $\tilde{X} = d\rho(X)$ for $X \in \mathfrak{sl}_2$. To see the structure of the representation $\rho$ we can decompose the space $V$ into eigenspaces of the operator $\tilde{X}$ for some $X \in \mathfrak{sl}_2$, cf. Prop. 8.2 or the Taylor series in complex analysis.

Example 9.3. It would not be surprising that we are going to consider three cases:

(1) Let $X = Z$ be a generator of the subgroup $K$ (4). Since this is a compact subgroup the corresponding eigenspaces $\tilde{Z}v_k = kv_k$ are parametrised
by an integer $k \in \mathbb{Z}$. The raising/lowering (ladder) operators $L^\pm$ [38, § VI.2], [49, § 8.2] are defined by the following commutation relations:

$$[\tilde{Z}, L^\pm] = \lambda \pm L^\pm. \quad (40)$$

In other words $L^\pm$ are eigenvectors for operators $\text{ad} Z$ of adjoint representation of $\mathfrak{sl}_2$ [38, § VI.2]. From the commutators (40) we deduce that $L^\pm v_k$ are eigenvectors of $\tilde{Z}$ as well:

$$\tilde{Z}(L^\pm v_k) = (L^\pm \tilde{Z} + \lambda L^\pm)v_k = L^\pm(\tilde{Z}v_k) + \lambda L^\pm v_k = ikL^\pm v_k + \lambda L^\pm v_k = (ik + \lambda)L^\pm v_k.$$ 

Thus the action of ladder operators on respective eigenspaces can be visualised by the diagram:

![Diagram](...)

Assuming $L^\pm = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ from the relations (39) and defining condition (40) we obtain linear equations with unknown $a$, $b$, and $c$:

$$c = 0, \quad 2a = \lambda b, \quad -2b = \lambda a.$$ 

The equations have a solution if and only if $\lambda^2 + 4 = 0$, and the raising/lowering operators are $L^\pm = \pm i\tilde{A} + \tilde{B}$.

(2) Consider the case $X = 2B$ of a generator of the subgroup $A'$ (6). The subgroup is not compact and eigenvalues of the operator $\tilde{B}$ can be arbitrary, however raising/lowering operators are still important [13, § II.1], [41, § 1.1]. We again seek a solution in the form $L^+_h = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ for the commutator $[2\tilde{B}, L^+_h] = \lambda L^+_h$. We will get the system:

$$4c = \lambda a, \quad b = 0, \quad a = \lambda c.$$ 

A solution exists if and only if $\lambda^2 = 4$. There are obvious values $\lambda = \pm 2$ with the ladder operators $L^\pm_h = \pm 2\tilde{A} + \tilde{Z}$, see [13, § II.1], [41, § 1.1]. Each indecomposable $\mathfrak{sl}_2$-module is formed by a one-dimensional chain of eigenvalues with a transitive action of ladder operators.

Admitting double numbers we have an extra possibility to satisfy $\lambda^2 = 4$ with values $\lambda = \pm 2j$. Then there is an additional pair of hyperbolic ladder operators $L^\pm_j = \pm 2j\tilde{A} + \tilde{Z}$, which shift eigenvectors in the “orthogonal” direction to the standard operators $L^\pm_h$. Therefore an indecomposable $\mathfrak{sl}_2$-module can be parametrised by a two-dimensional
lattice of eigenvalues on the double number plane:

\[
\begin{array}{cccc}
L^+_h & L^+_j & L^+_h & L^+_j \\
L^- & L^- & L^- & L^- \\
V_{(n-2)+j(k-2)} & V_{n+j(k-2)} & V_{(n+2)+j(k-2)} & V_{n+j(k+2)} & V_{(n+2)+j(k+2)} & \ldots
\end{array}
\]

(3) Finally consider the case of a generator \( X = \mathcal{B} + \mathcal{Z}/2 \) of the subgroup \( N' \) (7). According to the above procedure we get the equations:

\[
b + 2c = \lambda a,
-a = \lambda b,
\frac{a}{2} = \lambda c,
\]

which can be resolved if and only if \( \lambda^2 = 0 \). If we restrict ourselves with the only real (complex) root \( \lambda = 0 \), then the corresponding operators \( L^\pm = -\mathcal{B} + \mathcal{Z}/2 \) will not affect eigenvalues and thus are useless in the above context. However the dual number roots \( \lambda = \pm \varepsilon \) lead to the operators \( L^\pm_\varepsilon = \pm \varepsilon \mathcal{A} - \mathcal{B} + \mathcal{Z}/2 \). These operators are suitable to build an \( \mathfrak{sl}_2 \)-modules with a one-dimensional chain of eigenvalues.

Remark 9.4. It is noteworthy that:

- the introduction of complex numbers is a necessity for the existence of ladder operators in the elliptic case;
- in the parabolic case we need dual numbers to make ladder operators useful;
- in the hyperbolic case double numbers are required for neither existence nor usability of ladder operators, but do provide an enhancement.

We summarise the above consideration with a focus on the Principle of similarity and correspondence:

**Proposition 9.5.** Let a vector \( X \in \mathfrak{sl}_2 \) generate the subgroup \( K, N' \) or \( A' \), that is \( X = \mathcal{Z}, \mathcal{B} - \mathcal{Z}/2, \) or \( \mathcal{B} \) respectively. Let \( \iota \) be the respective hypercomplex unit.

Then raising/lowering operators \( L^\pm \) satisfying to the commutation relation:

\[
[X, L^\pm] = \pm \iota L^\pm,
[L^-, L^+] = 2\iota X.
\]

are:

\[
L^\pm = \pm \iota \mathcal{A} + \mathcal{Y}.
\]
Here $Y \in \mathfrak{sl}_2$ is a linear combination of $B$ and $Z$ with the properties:

- $Y = [A, X]$.
- $X = [A, Y]$.
- Killings form $K(X, Y)$ \cite{15, § 6.2} vanishes.

Any of the above properties defines the vector $Y \in \text{span}\{B, Z\}$ up to a real constant factor.

It is natural to expect that the usability of the Principle of similarity and correspondence will not be limited to the considered examples only. For example, similar types of ladder operators appeared in relation to the Heisenberg group as well \cite{30}.

10. Open Problems

We start from an illustration, that the invariant linear algebra presented in Section 7 is not unique.

Example 10.1 (Tropical form). Let us introduce the lexicographic order on $\mathbb{R}^2$:

$$(u, v) \prec (u', v')$$ if and only if \begin{align*}
\text{either} & \quad u < u' ; \\
\text{or} & \quad u = u', \ v < v'.
\end{align*}

One can define functions min and max of a pair of points on $\mathbb{R}^2$ respectively. Then an addition of two vectors can be defined either as their minimum or maximum. A similar definition is used in tropical mathematics, also known as Maslov dequantisation or $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$ algebras, see \cite{40} for an energetic survey and \cite{5} for a comprehensive coverage. It is easy to check that such an addition is distributive with respect to scalar multiplications (30)—(31) and consequently is invariant under parabolic rotations. This approach looks promising and definitely deserves a further careful investigation.

There are many other interesting questions to be investigated for the induced representations built in this paper. Here is a sketchy list of some of them:

- All complex-valued irreducible unitary representations of $\text{SL}_2(\mathbb{R})$ split into the three series: discrete, principal and complementary \cite{13, 38}. It is useful to find relations of new hypercomplex representations to these series.
- A generic irreducible $\mathfrak{sl}_2$-module \cite{13, 41} may not be unitarisable for any complex-valued inner product. How many of those non-unitarisable moduli become unitary for some hypercomplex-valued product?
- Most of the above irreducible hypercomplex-unitary $\text{SL}_2(\mathbb{R})$-moduli can be realised as spaces of functions either on the real line or the upper half-plane \cite{19}, \cite[§ 5]{32}. The concept of covariant transform \cite[§ 4]{32} provides the universal tool for a construction of associated integral formulae and reproducing kernels.
The described spaces of functions are null-solutions to certain differential operators, which possess $\text{SL}_2(\mathbb{R})$-symmetry. In particular, the covariant transform with a mother wavelet, which is annihilated by a ladder operator (or any of its power), creates the space of null-solutions to an associated differential equation, see [32, § 5.3]. This shall be useful in analysis of solutions of partial differential equations [36, 43].

Analytical spaces of complex-valued functions work as models for functional calculus of normal operators, e.g Dunford–Riesz calculus. The newly build $\text{SL}_2(\mathbb{R})$-moduli support covariant functional calculi and associated spectra [17, 22], [32, § 6] for non-selfadjoint operators.

These directions can be viewed as parts of the Erlangen programme at large [23, 32]. The Principle of similarity and correspondence may help to find most harmonious constructions.

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