On Teichmüller spaces of Koebe groups

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Abstract

In this paper we parametrize the Teichmüller spaces of constructible Koebe groups, that is Kleinian group that arise as covering of 2-orbifolds determined by certain normal subgroups of their fundamental groups. We also study the covering spaces of the Teichmüller spaces of those Koebe groups. Finally we prove an isomorphisms theorem similar to the Bers-Greenberg theorem for Fuchsian groups. Our method yields a technique to compute explicitly generators of Koebe groups, possibly by programming a computer.

1 Introduction

In this paper we will extend the results of I. Kra [11] and the author [7] to the class of constructible Koebe groups. Our main goal is to produce a set of coordinates for the Teichmüller spaces of Kleinian groups that allow explicit computations. We also prove some results concerning the deformation space of Koebe groups.

One of the most interesting object associated to a Riemann surface $S$ is its moduli or Riemann space, $R(S)$, the space that parametrizes the different complex structures on $S$ modulo biholomorphisms. One possible way of studying $R(S)$ is by passing to its universal covering space, known as Teichmüller space, $T(S)$, which is the set of complex structures on $S$ modulo isometries (we will assume that $S$ has a metric of constant curvature $-1$) isotopic to the identity. In order to get explicit coordinates on $T(S)$, we can study the Riemann surface $S$ via its universal covering space, the upper half plane plane, obtaining in this way that the group of deck transformations becomes a group of Möbius transformations. The points in $T(S)$ are then given by different subgroups of $PSL(2,\mathbb{R})$ (with a preferred set of generators). But groups acting on the upper half plane are quite difficult to handle for
computations, so we need to represent \( S \) in a slightly different way. B. Maskit proved that it is possible to find a finitely generated Kleinian group \( \Gamma \), with a simply connected invariant component \( \Delta \) such that \( \Delta/\Gamma \cong S \). The set of groups quasiconformally conjugated to \( \Gamma \), modulo conjugation by Möbius transformations, is the Teichmüller space of \( \Gamma \), \( T(\Gamma) \). This space is the cartesian product of the Teichmüller spaces of all surfaces represented by \( \Gamma \). But by the same result of B. Maskit, we have that the surfaces uniformized by \( \Gamma \) other than \( S \), are rigid, i.e. their Teichmüller spaces are points. Therefore \( T(\Gamma) \) becomes a model for \( T(S) \) with the advantage that these groups are good for explicit computations. In this line lies the work of I. Kra, where he studied the case of compact surfaces with finitely many punctures. Later, the author extended the results to the case of 2-orbifolds (topological surfaces, with a complex structure, where each point has a neighborhood modeled over a Euclidean disc quotiented a finite group of rotations) in [7]. Observe that orbifolds in particular include the Riemann surfaces, if we take the rotation group to be trivial. In this paper we will study another type of groups related to planar covering of 2-orbifolds. We start by choosing a maximal partition \( P \) on the orbifold \( S \); that is, a set of curves that splits \( S \) into spheres with three marked points or holes. To each curve of the partition \( a_j \), we assign an integer number (bigger than 2) or \( \infty \), \( \mu_j \), called a weight. We then consider the normal subgroup \( H \) of \( \pi_1(S) \) generated by the curves \( a_j^{\mu_j} \) with finite \( \mu_j \). By the same result of B. Maskit we have that the covering determined by \( H \) produces a Kleinian group \( \Gamma \), known as Koebe group, with an invariant component \( \Delta \), which is simply connected if and only if all \( \mu_j = \infty \). The group \( \Gamma \) uniformizes \( S \) and rigid orbifolds, so the study of \( T(\Gamma) \) is somehow equivalent to the study of \( T(S) \). The main result in this line is the following:

**Theorem 1** Let \( S \) be an orbifold with signature \( (p,n;\nu_1,\ldots,\nu_n) \) satisfying \( 2p - 2 + n > 0 \) and \( 3p - 3 + n - \sum 1/\nu_j > 0 \). Let \( P = \{a_1,\ldots,a_{3p-3+n}\} \) be a maximal partition on \( S \), and let \( N = \{\mu_1,\ldots,\mu_{3p-3+n}\} \) be a set of weights. Assume that \( \Gamma \) is a Koebe group uniformizing \( (S,P,N) \). Then there exists a set of coordinates, \((\alpha_1,\ldots,\alpha_{3p-3+n})\), on the Teichmüller space \( T(\Gamma) \), and a set of positive numbers, \((r_1,\ldots,r_{3p-3+n})\), such that

\[
\prod_{j=1}^{3p-3+n} U_j \subset T(\Gamma),
\]

where \( U_j = \{\alpha_j; \text{Im}(\alpha_j) > r_j\} \) if \( \mu_j = \infty \) or \( U_j = \{\alpha_j; 0 < |\alpha_j| < r_j\} \) if \( \mu_j < \infty \). Moreover, we also have the inclusions

\[
T(\Gamma) \subset \prod_{j=1}^{3p-3+n} V_j,
\]
where $V_j$ is the upper half plane if $\mu_j = \infty$, or the unit disc if $\mu_j < \infty$. The entries of a set of generators corresponding to a point in $T(\Gamma)$ can be computed explicitly in terms of the coordinates, and vice versa. The numbers $r_j$ depend only on the topology of $S, \mathcal{P}, \mathcal{N}$.

A Riemann surface $S$, or more generally an orbifold, can be constructed from more basic orbifolds, $S_1$ and $S_2$, by removing discs and identifying their boundaries. In the case that such identification is given by a formula of the type $zw = t$, where $z$ and $w$ are local coordinates on $S_1$ and $S_2$ respectively, we say that $S$ has been constructed from $S_1$ and $S_2$ by plumbing techniques, with parameter $t$. The above study of Koebe groups allows us to understand our computations in terms of plumbing constructions.

**Theorem 2** The orbifold corresponding to the point $(\alpha_1, \ldots, \alpha_{3p-3+n}) \in T(\Gamma)$ can be constructed by plumbing techniques with parameters $(\tau_1, \ldots, \tau_{3p-3+n})$, where $\tau_j = \exp(\pi i c_j \alpha_j)$ if $\mu_j = \infty$, or $\tau_j = c_j \alpha_j$, if $\mu_j < \infty$. The constants $c_j$ depend only on the topology of the orbifold and the partition.

If one of the $\mu'_j$s is finite, then $T(\Gamma)$ will not be $T(S)$, although it can be proven that the latter space is the universal covering of the former. Our result identifies the covering group of $T(S) \to T(\Gamma)$.

**Theorem 3** The covering group of the mapping $T(S) \to T(\Gamma)$ is the normal subgroup of the mapping class group of $S$ generated by the Dehn twists $\tau^\mu_j$ around the curves $a_j$ of the partition with finite $\mu_j$.

The Bers-Greenberg theorem tells us that the complex structure of $T(\Gamma)$ depends only on the pair $(p, n)$ of $S$. This result was first proven for the case of Fuchsian groups in [3]. The author in [8] extended to the case of Koebe groups where all the $\mu_j = \infty$, following [5]. In this paper we generalized it to the case of constructible Koebe groups with finite weights.

**Theorem 4** Let $S, \mathcal{P}$ and $\mathcal{N}$ be as in theorem 1. Let $S_0$ be the surface obtained by removing from $S$ all the points with finite ramification number. Assume that $\Gamma$ and $\Gamma_0$ are two Koebe groups uniformizing $(S, \mathcal{P}, \mathcal{N})$ and $(S_0, \mathcal{P}, \mathcal{N})$ respectively. Then the deformation spaces $T(\Gamma)$ and $T(\Gamma_0)$ are conformally equivalent.

This paper is organized as follows: in §2 we give the necessary background on Teichmüller spaces and Kleinian groups. The Koebe groups we study are constructed from more basic groups, known as triangle groups, which are studied in detail in §3. In §4 we compute the
coordinates of theorem 1; by a theorem of B. Maskit [14], it suffices to consider the cases of dimension 1, which we do in that section, and we also indicate how to proceed in the general situation. We will also give a relation between the coordinates on deformation spaces and plumbing constructions on 2-orbifolds, proving theorem 2. In §5 we prove theorems 3 and 4.

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2 Background on Kleinian groups and Teichmüller spaces

2.1. A Kleinian group \( \Gamma \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \) such that the set of points of \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) where \( \Gamma \) acts discontinuously, the regular set \( \Omega = \Omega(\Gamma) \), is not empty. If \( \hat{\mathbb{C}} - \Omega \) consists of at most 2 points, we say that the group is elementary.

Let \( \Delta \) be a invariant component of \( \Gamma \); that is, a component of \( \Omega \) such that \( \gamma(\Delta) = \Delta \) for all \( \gamma \in \Gamma \). If \( \Gamma \) is finitely generated, the natural mapping \( \pi : \Delta \to \Delta/\Gamma \) is a covering of a compact surface of genus \( p \) with finitely many punctures, ramified over finitely many points. We say that \( \Delta/\Gamma \) is an orbifold of signature \((p, n; \nu_1, \ldots, \nu_n)\), where the \( \nu_j \in \mathbb{Z}^+ \cup \{ \infty \} \), \( \nu_j \geq 2 \). The \( \nu_j \)'s are called the ramification values. We have that \( \pi \) is \( \nu_j \)-to-1 in a neighborhood of a point \( x_j \) whose ramification value is \( \nu_j \). We will assign ramification value equal to \( \infty \) to the punctures of \( S \). The pair \((p, n)\) is called the type of the orbifold. All orbifolds in this paper, except for those of type \((0, 3)\), will satisfy the following two conditions:

\[
\begin{align*}
2p - 2 + n & > 0 \\
3p - 3 + n - \sum \frac{1}{\nu_j} & > 0
\end{align*}
\]

(1)

If \( \Delta/\Gamma \) is an orbifold of signature \((0, 3; \nu_1, \nu_2, \nu_3)\), then \( \Gamma \) is called a triangle group. Triangle groups are divided into hyperbolic, parabolic or elliptic, depending on whether \( 1 - (1/\nu_1 + 1/\nu_2 + 1/\nu_3) \) is positive, zero or negative, respectively. If the group is elliptic or parabolic, then it is elementary, and \( \Delta = \Omega \). If \( \Gamma \) is hyperbolic, \( \Omega \) consists of two discs or half planes, and \( \Delta \) is any of them.

A constructible Koebe group is a Kleinian group \( \Gamma \), with an invariant component \( \Delta \) such that \( \Gamma \) can be built up from elementary and hyperbolic triangle groups by finitely many applications of the Klein-Maskit Combination Theorems (see [17] for the latest version of these theorems). In particular we get that \( \Gamma \) is finitely generated. by the signature of \( \Gamma \) we
will understand the signature of $\Delta/\Gamma$. For the rest of this paper, $\Gamma$ will be a constructible Koebe group, unless otherwise stated.

2.2. A maximal partition with weights on an orbifold $S$, is a pair $(\mathcal{P}, \mathcal{N})$ where:

(i) $\mathcal{P} = \{a_1, \ldots, a_{3p-3+n}\}$ is a set of $3p-3+n$ simple closed disjoint unoriented curves on $S_0 = S - \{x_j; \nu_j < \infty\}$, such that no curve of $\mathcal{P}$ bounds a disc or a punctured disc on $S_0$, and no two curves of $\mathcal{P}$ bound a cylinder on $S_0$;

(ii) $\mathcal{N} = (\mu_1, \ldots, \mu_n) \in (\mathbb{Z} \cup \{\infty\})^{3p-3+n}$, with $\mu_j \geq 3$;

(iii) the weight $\mu_j$ is assigned to the curve $a_j$.

Theorem 5 (Maskit Existence Theorem) Given an orbifold $S$ with signature satisfying (1), and maximal partition $(\mathcal{P}, \mathcal{N})$, there exists a unique (up to conjugation in $\text{PSL}(2, \mathbb{C})$) constructible Koebe group $\Gamma$, with invariant component $\Delta$, such that:

(i) $S \cong \Delta/\Gamma$;

(ii) to each curve $a_j$ of $\mathcal{P}$ corresponds a unique conjugacy class of elements of $\Gamma$ generated by a transformation of order $\mu_j$;

(iii) $(\Omega - \Delta)/\Gamma$ is the union of orbifolds of type $(0, 3)$ obtained by squeezing each curve $a_j$ of $\mathcal{P}$ to a point of ramification $\nu_j$, and discarding all orbifolds of parabolic or elliptic signature that appear.

Proof. It is easy to see that the condition of theorem X.D.15 in [10], pg. 281 are satisfied. Existence is then given in theorem X.F.1 of the same reference, while uniqueness is theorem 1 of [13].

We will say that $\Gamma$ uniformizes the triple $(S, \mathcal{P}, \mathcal{N})$.

2.3. Let $G$ be a finitely generated non-elementary Kleinian group. The Teichmüller or deformation space of $G$ is the set

$$T(G) = \{ w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}; \ w \text{ is quasiconformal, } wGw^{-1} \leq \text{PSL}(2, \mathbb{C}) \}/\sim,$$

where $w_1 \sim w_2$ if there is a Möbius transformation $A$ such that $w_1gw_1^{-1} = Aw_2gw_2^{-1}A$, for all $g \in G$.

If $\Gamma$ is a constructible Koebe group, then $T(\Gamma)$ is a complex manifold of dimension $3p-3+n$.

In the case that all the weights of $\Gamma$ are equal to $\infty$, we have that $T(\Gamma)$ is equivalent to $T(S)$, the deformation space of $S = \Delta/\Gamma$, which is the set of quasiconformal homeomorphisms of $S$ modulo those isotopic to conformal mappings (see [11] and [18]).

2.4. There is a way of decomposing $\Gamma$ into simpler groups as follows. Let $T_j$ be the connected component of $S - \{a_k; a_k \in \mathcal{P}, k \neq j\}$ containing $a_j$. Let $D_j$ be a connected component of
\( \pi^{-1}(T_j) \), where \( \pi : \Delta \to S \) is the natural projection. Denote by \( \Gamma_j \) the stabilizer of \( D_j \) in \( \Gamma \); that is, \( \Gamma_j = \{ \gamma \in \Gamma; \gamma(D_j) = D_j \} \). These groups are Koebe group of type \((0, 4)\) or \((1, 1)\). Therefore \( \dim T(\Gamma_j) = 1 \), and it is clear that there is a mapping from \( T(\Gamma) \) into \( T(\Gamma_j) \) given by restriction. We can choose these subgroups so that \( \Gamma_j \cap \Gamma_{j+1} = F_j \) is a triangle group for \( 1 \leq j \leq 3p - 4 + n \).

**Theorem 6** [Maskit Embedding, [14], [10]] The mapping given by restriction \( T(\Gamma) \to \prod_{j=1}^{3p-3+n} T(\Gamma_j) \) is holomorphic, injective and with open image as long as the groups \( F_j \) do not have signature \((0, 3; 2, 2, \nu)\) for finite \( \nu \).

**Proof.** See [13] and observe that the only triangle groups with non trivial centralizer in \( PSL(2, \mathbb{C}) \) have the above mentioned signatures. \( \square \)

In our case, since we are assuming that the weights are always strictly bigger than 2, we always have such an embedding.

2.5. An element \( A \) of a Kleinian group \( G \) is said to be **primitive** if it has no roots in \( G \); that is, if \( B \in G \) and \( B^n = A \), then \( n = \pm 1 \).

An elliptic element \( C \) of finite order \( n \) is conjugated in \( PSL(2, \mathbb{C}) \) to a rotation of the form \( z \mapsto e^{2k\pi i/n}z \), with \( k \) and \( n \) relatively primes. If \( k = \pm 1 \) we say that \( C \) is **geometric** ([16, pg. 96]).

### 3 Parametrization of triangle groups

3.1. It is a classical fact that two triangle groups with the same signature are conjugate in \( PSL(2, \mathbb{C}) \) [10, pg. 217]. Therefore, to determine a triangle group all we need is its signature and three distinct points of \( \hat{C} \), which we will call **parameters**. We are interested in a having a technique to compute **explicitly** generators for these groups. This requires some canonical choices. The main goal of this section is to develop such techniques. Later, in §4, we will use these generators to compute coordinates on Teichmüller spaces of Koebe groups.

\( \Gamma(\nu_1, \nu_2, \nu_3; a, b, c) \) will denote a triangle group of signature \((0, 3; \nu_1, \nu_2, \nu_3)\) with a pair of canonical generators, \( A \) and \( B \), for the parameters \( a, b, c \). We will always assume that \( A \) and \( B \) are primitive and geometric (if elliptic), and \( |A| = \nu_1, |B| = \nu_2 \) and \( |AB| = \nu_3 \). Here \( |T| \) denotes the order of a Möbius transformation, with parabolic elements considered as elements of order equal to \( \infty \). For technical reasons (see §3.2.) we will always assume that \( \nu_1 > 2 \). It will be clear from our definitions that if \( T \) is a Möbius transformation,
then \(TAT^{-1}\) and \(TBT^{-1}\) are canonical generators for \(\Gamma(\nu_1, \nu_2, \nu_3; T(a), T(b), T(c))\). This will simplify the proofs of this section by taking \(a = \infty\), \(b = 0\) and \(c = 1\).

3.2. If \(A\) is a generator of a triangle group, then it is either parabolic or elliptic. In both cases, there are circles (or more generally, closed Jordan curves), invariant under \(A\). Let \(\tilde{a}\) be one such curve. We orient it by picking a point \(z \in \tilde{a}\), not fixed by \(A\), and then requiring that \(z, A(z)\) and \(A^2(z)\) follow each other in the positive orientation. This is possible since we are assuming that the curves of the partition are always uniformized by elements of order strictly bigger than 2. The cases of groups with partitions curves uniformized by involutions will be treated in a forthcoming publication.

Hyperbolic Groups.

3.3. The results of this section are taken from \([7]\), which are generalization of those in \([11]\).

Given three distinct points \(a, b, c\) in \(\hat{C}\), let \(\Lambda\) be the circle determined by them and oriented so that \(a, b, c\), follow each other in the positive orientation. Let \(\Delta\) be the disc to the left of \(\Lambda\); that is, the set of points \(z\) with \(cr(z, a, b, c) > 0\), where \(cr\) denotes the cross ratio of four distinct point on the Riemann sphere, chosen (there are 6 possible definitions of cross ratio) such that \(cr(\infty, 0, 1, z) = z\). Let \(L\) and \(L'\) be the circles passing through \(\{a, b\}\) and \(\{a, c\}\) respectively, and orthogonal to \(\Lambda\).

Definition 1

Given \(z_1\) an \(z_2\) in \(\Delta \cap L\), we will say that they are well ordered with respect to \((a, b, c)\) if one of the following conditions is satisfied:

(i) \(z_1 = a\),
(ii) \(z_2 = b\),
(iii) \(z_1 \neq a, z_2 \neq b\) and \(cr(a, z_1, z_2, b) > 1\).

For example, if \(a = \infty\), \(b = 0\) and \(c = 1\), then \(\Delta\) is the upper half plane. Two points \(z_1 = \lambda i\) and \(z_2 = \mu i\) are well ordered with respect to these parameters if \(\lambda > \mu\). Naively speaking, this means that \(z_1\) is closer to \(\infty\) than \(z_2\).

Let \(\Gamma\) be a triangle group with signature \((0, 3; \nu_1, \nu_2, \nu_3)\) whose limit set \((\hat{C} - \Omega)\) is \(\Lambda\), and let \(A\) and \(B\) be two generators of \(\Gamma\).

Definition 2

\(A\) and \(B\) are the canonical generators of \(\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)\) if the following conditions are satisfied:

(i) \(A\) and \(B\) have their fixed points in \(L\) and \(AB\) in \(L'\);
(ii) if \(z_1\) and \(z_2\) are the fixed points of \(A\) and \(B\) in \(\Delta \cap L\), then they are well ordered with respect to \((a, b, c)\).
Proposition 1 There exists a unique \( \Gamma(\nu_1, \nu_2, \nu_3; a, b, c) \). In the case of \( a = \infty \), \( b = 0 \) and \( c = 1 \), the canonical generators \( A \) and \( B \) are given by:

1. \( \nu_1 = \infty \),
   \[
   A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_2 & b \\ q_2 + q_3 & -q_2 \end{bmatrix},
   \]
   where \( q_i = \cos(\pi/\nu_i) \), \( b = \frac{q_2^2 - 1}{q_2 + q_3} \);

2. \( \nu_1 \neq \infty \),
   \[
   A = \begin{bmatrix} -q_1 & -kp_1 \\ k^{-1}p_1 & -q_1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_2 & -hp_2 \\ h^{-1}p_2 & -q_2 \end{bmatrix},
   \]
   \( q_i = \cos(\pi/\nu_i) \), \( p_i = \sin(\pi/\nu_i) \),
   \[
   k = \frac{q_2 + q_1q_3 + q_1l}{p_1l}, \quad h = \frac{kp_1p_2}{q_1q_2 + q_3 + l}, \quad l = \sqrt{q_1^2 + q_2^2 + q_3^2 + 2q_1q_2q_3 - 1} > 0.
   \]

Remarks. 1. Observe that the generators \( A \) and \( B \) in the second case converge to those of the first case as \( \nu_1 \to \infty \).

2. The proof of this result (see [6]) is based on a detailed study of hyperbolic triangles in the upper half plane.

3. From the above expression we can see that \( A \) (or \( B \)) fixes the points \( \pm ki \) (or \( \pm hi \)). There is a relation between these formulae and the geometry of the orbifold \( S = H/\Gamma \) as follows.

Put on \( S \) the natural metric of constant curvature \(-1\) coming the Poincaré metric on the upper half place. Then the distance from \( P_1 \) to \( P_2 \) is \( \log(\frac{k}{h}) \), where \( P_j \) is the point with ramification \( \nu_j \), \( j = 1, 2 \), and we are assuming that \( \nu_j < \infty \).

Parabolic Groups.

3.4. Let us consider first the case of signature \((0, 3; \infty, 2, 2)\).

Definition 3 \( A \) and \( B \) are canonical generators of \( \Gamma(\infty, 2, 2; a, b, c) \), if the following conditions are satisfied:

\( A(z) = a \), \( B(b) = b \) and \( AB(c) = c \);

Proposition 2 There exists a unique \( \Gamma(\infty, 2, 2; a, b, c) \)

Proof. Conjugate to get \( a = \infty \), \( b = 0 \) and \( c = 1 \). Then \( A(z) = z + \alpha \), for some complex number \( \alpha \). Since all elements of the group must fixed \( \infty \) ([10, pg. 91]), we get that \( B(z) = -z \), forcing \( \alpha \) to be equal to 2. \( \square \)
3.4. For the other parabolic signatures, as well as for the elliptic cases, we need a new concept due to J. Parkkonen [13].

**Definition 4** Let $M$ be a Möbius transformation of finite order strictly bigger than 2, and let $x$ be a fixed point of $M$. Let $z$ be any point not fixed by $M$. We will say that $x$ is the right (left) fixed point of $M$ if the cross ratio $cr(x, z, M(z), M^2(z))$ has positive (negative) imaginary part.

**Lemma 1** The above definition is invariant under conjugation by elements of $PSL(2, \mathbb{C})$.

This lemma simply means that if $x$ is the right fixed point of $M$, then $T(x)$ is the right fixed point of $TMT^{-1}$ for any Möbius transformations $T$.

**Lemma 2** The above definition does not depend on the point $z$.

*Proof.* By the previous result we can assume that $M(z) = kz$, with $|k| = 1$. Then an easy computation shows that $cr(\infty, z, kz, k^2z) = k + 1$ while $cr(0, z, kz, k^2z) = 1 + \frac{1}{k}$, for all $z \neq \infty, 0$.

3.6. We are now in a position to define the canonical generators for the rest of the parabolic signatures.

**Definition 5** The canonical generators $A$ and $B$ of $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$, where all the ramification values are finite, satisfy the following conditions:

(i) $a$ is the right fixed point of $A$ and $b$ is its left fixed point;
(ii) $B$ fixes $a$ and $c$;

**Proposition 3** There exists a unique $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$, with the signature in the conditions of the above definition.

*Proof.* $A(z) = \lambda z$, with $\lambda = \exp(2\pi i/\nu_1)$, assuming that the parameters are $\infty, 0, 1$ as usual. $B$ must be of the form $B(z) = \mu(z - 1) + z$, where $\mu = \exp(\pm 2\pi i/\nu_2)$. Using the fact that $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} = 1$, one can easily that $\mu = \exp(2\pi i/\nu_2)$, since otherwise $AB$ dose not have the correct order.

**Elliptic Groups**
3.7. By a result of I. Kra and B. Maskit [12], the elliptic triangle groups of signature 
\((0,3;2,2,\nu)\), with finite \(\nu\), cannot be parametrized. Nevertheless, we can have a definition 
of canonical generators, although not a uniqueness statement.

**Definition 6** The canonical generators \(A\) and \(B\) of \(\Gamma(\nu,2,2; a,b,c)\) satisfy:

(i) \(A\) fixes \(a\) and \(b\), and \(a\) is the right fixed point;

(ii) \(B(c) = c\);

Since \(B\) has order 2, there is no way to differentiate between its two fixed points. This is why 
we do not have a uniqueness statement. For computational purposes, the following result is 
enough, although it does not guarantee coordinates on Teichmüller space (see remark 3 in 
§4.4).

**Proposition 4** Given three distinct points \(a, b, c\) on the Riemann sphere, there is a unique 
point \(d \in \hat{\mathbb{C}} - \{a,b,c\}\) such that \(\Gamma(\nu,2,2; a,b,c) = \Gamma(\nu,2,2; a,b,d)\).

**Proof.** We first compute to get \(A(z) = e^{2\pi i/\nu}z\) and \(B(z) = 1/z\), if \(a = \infty\), \(b = 0\), and \(c = 1\). 
Since \(B\) also fixes \(-1\), we get that \(d = -1\). \(\square\)

3.8. The last cases are those elliptic groups with exactly one point of ramification values 2. Without 
of generality, we can ordered the ramification values so that the signature is of the form \((0,3;\nu_1,\nu_2,2)\). This avoids some long computations and does not loose any 
mathematical insight.

**Definition 7** The canonical generators of \(\Gamma(\nu_1,\nu_2,2; a,b,c)\) satisfy:

(i) \(A\) fixes \(a\) and \(b\), and \(a\) is the right fixed point;

(ii) the right fixed point of \(B\) is \(c\);

**Proposition 5** There is a unique \(\Gamma(\nu_1,\nu_2,2; a,b,c)\) in the above conditions.

**Proof.** As usual, we make \(a = \infty\), \(b = 0\) and \(c = 1\) by conjugation. Let \(q_j\) and \(p_j\) denotes 
\(\cos(\pi/\nu_j)\) and \(\sin(\pi/\nu_j)\), \(j = 1,2\), respectively. Then \(A(z) = \lambda^2z\), where \(\lambda = \exp(\pi i/\nu_1)\). 
Since \(AB\) has zero trace, we can assume that \(B\) has negative trace. If \(B\) has a matrix 
expression given by \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\), then we have:

\[
\begin{align*}
  a + b &= c + d \\
  ad - bc &= 1 \\
  a + d &= -2q_2 \\
  -\lambda a - \lambda^{-1}d &= 0
\end{align*}
\]
The last two equations give $a = -\frac{q_2 \lambda i}{p_1}$. We get $b = -q_2 - a \pm ip_2$ and $c = q_2 + a \pm ip_2$. A simple computation shows that

$$cr(1, \infty, \frac{a}{c}, \frac{a^2 + bc}{ac + dc}) = 1 + \frac{1}{-1 + 2q_2^2 \mp 2ip_2q_2}.$$ 

Since 1 has to be the right fixed point of $B$, we get that $b = -q - a - ip$ and $c = q + a - ip$. $\blacksquare$

**Hyperbolic Groups Revisited**

3.9. One can re-write the results about canonical generators of hyperbolic groups using the concept of right and left fixed points. The existence and uniqueness proposition is then as follows:

**Proposition 6** Let $(0, 3; \nu_1, \nu_2, \nu_3)$ be a hyperbolic signature where all the ramification values are finite. Then $A$ and $B$ are the canonical generators of $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$ if and only if:

(i) $A$ and $B$ have their fixed points in $L$, and $AB$ has them in $L'$;

(ii) the left fixed point of $A$ lies in $\Delta$;

The proof is a long but easy computation left to the reader.

**Local Coordinates**

3.10. In order to relate the parametrization of triangle groups and the geometry of orbifolds (§4.9), we need to introduce some local coordinates on the basic orbifolds of type $(0, 3)$. These coordinates were first found by I. Kra in [11]. There he considered an orbifold $S$ of signature $(0, 3; \infty, \infty, \infty)$, with punctures $P_1$, $P_2$, and $P_3$. $S$ has a metric of constant curvature $-1$ coming from its universal covering space, the upper half plane $\mathbb{H}$. As the uniformizing group we can take $\Gamma(\infty, \infty, \infty; \infty, 0, 1)$, generated by $A(z) = z + 2$ and $B(z) = -z/(2z - 1)$. Then, if $\xi \in \mathbb{H}$, the function $z(\xi) = exp(\pi i \xi)$ is invariant under $A$ and induces a local coordinate on a neighborhood of $P_1$ in $S$. We can consider for example those $\xi$ with imaginary part bigger than 1. In $S$ there is a unique geodesic $c$ such that, when it is parametrized by the arc length, $c$ satisfies $lim_{s \to +\infty} c(s) = P_1$ and $lim_{s \to -\infty} c(s) = P_2$. $c$ is characterized by mapping $c$ isometrically into the unit interval $(0, 1)$ in the punctured disc (with its natural Poincaré metric). We say that $z$ (or more precisely, the germ of holomorphic functions determined by it) is a preferred coordinate at $P_1$ relative to $P_2$.

In the case of hyperbolic groups with torsion, we still have uniqueness of geodesic and coordinates. At a point $P_1$ of finite ramification value $\nu_1$, the preferred coordinate looks like
\( z(\xi) = \xi^\mu \), where \( \xi \) lies in a neighborhood of 0 in \( \mathbb{C} \), and the group we are considering is 
\( \Gamma(\nu_1, \nu_2, \nu_3; -1, 1, \frac{1+ki}{1+ki}) \).

In the parabolic and elliptic cases, we do not have uniqueness statements for preferred coordinates, but nevertheless, we can define coordinates by giving a function that generates the corresponding germ of holomorphic mappings. For the parabolic group \( \Gamma(\infty, 2, 2; \infty, 0, 1) \), we define the coordinate at the puncture \( P_1 \) relative to \( P_2 \) (one of whose lifts to \( \mathbb{C} \) is 0) by 
\( z(\xi) = e^{\pi i \xi} \). For parabolic and elliptic groups \( \Gamma(\nu_1, \nu_2, \nu_3) \) with \( \nu_1 < \infty \), we have to take a more general concept of coordinates by allowing our local patch to be modelled on the Riemann sphere. By this we mean that a preferred local coordinate on an orbifold will mapped a neighborhood of the special point into a disc centered at 0, with the point being sent to the origin, or into the exterior of a disc (also centered at 0), with the image of the point under consideration being the point \( \infty \). We then have that the preferred coordinates at \( P_1 \) and relative to \( P_2 \) are given by either 
\( z(\xi) = \xi^\mu \), or \( z(\xi) = (1/\xi)^\mu \).

As the name suggests, for each of the groups of this section we can construct a fundamental domain for their action on the regular set (if the group is elementary) or in one of the two components of \( \Omega \) (in the case of hyperbolic groups) by taking a triangle with angles \( \pi/\nu_j \), and then reflecting it in one of its sides. For example, in the case of the group \( \Gamma(4, 4, 2; \infty, 0, 1) \) we get a rectangle with vertices at the points 0, \((1+i)/2\), 1 and \((1-i)/2\). Consider now the point 0, which project to a point with ramification number 4, say \( P_1 \). Then we can see that the distance from 0 to the line joining 1 and \((1+i)/2\) is 1. This means that the disc 
\{ \( z : |z| < 1 \) \} maps into a neighborhood of \( P_1 \) on the quotient orbifold. Similarly, from we have discs of radius 1 around the other two ramification points. In general, we have the following result:

**Lemma 3** Let \( \Gamma(\nu_1, \nu_2, \nu_3; a, b, c) \) be a triangle group. Let \( P_j \) be a ramification point in the in the quotient orbifold, \( j = 1, 2, 3 \). Let \( p_j \) be a lifting of \( P_j \). Then we can find a positive number \( r \) such that the disc of radius \( r \) around \( p_j \) projects onto a neighborhood of \( P_j \). If \( P_j \) is a puncture, then we can find a horodisc around \( p_j \) (i.e., a disc such that \( p_j \) lies in its boundary) that projects onto a punctured disc on \( S \) containing \( P_j \).

### 4 Coordinates on Teichmüller spaces of Koebe groups

**4.1** In this section we will explain how to construct the Koebe groups given by the Maskit Existence Theorem. We will also give global coordinates for the deformation spaces of these groups, and explain the relation between our coordinates and plumbing parameters. The
computations of §3 allows us to construct an algorithm from which one can get explicitly generators for Koebe groups. This technique was used for I. Kra [11] and the author [7] to compute formulæ for isomorphisms between Teichmüller space. See also [19].

By the Maskit Embedding Theorem, the Teichmüller space of a Koebe group can be embedded into the product of one dimensional space. These latter sets correspond to the groups of type (04, ) and (1, 1), which we will work out in detail, and then indicate how to treat the general case.

**The (0, 4) case**

4.2 Let $S$ be an orbifold of signature $(0, 4; \nu_1, \ldots, \nu_4)$. A maximal partition on $S$ consists on a curve $a_1$, with weight $\mu$. Let $S_1$ and $S_2$ be the two parts of $S - \{a_1\}$. Orient $a_1$ such that $S_1$ lies to its right. If we cut $S$ along the partition curve, and glue to each resulting boundary a disc whose center is a point with ramification value $\mu$, then we obtain that $S_1$ and $S_2$ have been completed to orbifolds of signatures $(0, 3; \mu, \nu_1, \nu_2)$ and $(0, 3; \mu, \nu_3, \nu_4)$ respectively. Let $\Gamma_i$ be triangle groups uniformizing $S_i$, $i = 1, 2$. To recover $S$ be have to do the opposite construction: first we must remove discs from $S_1$ and $S_2$ and then glue along the boundaries. This implies that the elements uniformizing $a_1$ in $S_1$ and $S_2$ must be the same Möbius transformation. In other words, $\Gamma_1 \cap \Gamma_2 = \langle A \rangle$, where $A$ is primitive in both groups. The First Combination Theorem [10, VII.C.2, pg 149] tells us that if we choose the triangle groups properly, then the group $\Gamma = \Gamma_1 *_{<A>} \Gamma_2 := \langle \Gamma_1, \Gamma_2 \rangle$ is a Koebe group of the desired signature. By the classical theory of quasiconformal mapping we have that any orbifold of type $(0, 4)$ can be uniformized by this method.

4.3. We will work out two examples of the above construction, one with weight equal to $\infty$ and the other with finite weight. Let us start by the former case. Assume that $\Gamma_1$ has hyperbolic signature $(0, 3; \infty, \nu_1, \nu_2)$ and $\Gamma$ is a infinite dihedral group, with signature $(0, 3; \infty, 2, 2)$. We can start with $\Gamma_1 = \Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$. Since $S_2$ lies to the right of the partition curve we have that $\Gamma_2$ must have $A^{-1}$ as one of its generators. This implies that $\Gamma_2$ has to be conjugate to $\Gamma(\infty, 2, 2; 0, \infty, 1)$ by a transformation $T$ such that $TAT^{-1} = A^{-1}$. Therefore we get $T(z) = z + \alpha$. The fact that $S_1$ lies to the right of $S_1$ implies that $\text{Im}(\alpha) > 0$.

We get that the Koebe groups uniformizing orbifolds of the above signature with $\infty$ weight are given by the AFP construction

$$\Gamma = \Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1) *_{<A>} \Gamma(\infty, 2, 2; 0, \infty, \alpha),$$

for a proper choice of $\alpha$. Actually, if $\text{Im}(\alpha) > 1$, we can take the line $\{z; \text{Im}(z) = \frac{1}{2}\text{Im}(\alpha)\}$ as the invariant curve needed to apply the First Combination Theorem, and the resulting
group is a Koebe group of the desired type.

Choose an \( \alpha_0 \) such that \( \Gamma_0 = \Gamma_{\alpha_0} \) is a Koebe group, e.g. \( \alpha_0 = i \). Then it is clear that \( \alpha \) is a global coordinate on the deformation space \( T(\Gamma_0) \). This result is also given in \([12]\) and \([10]\), although there the computations are not explicit. \( \alpha \) has a \( \text{PSL}(2, \mathbb{C}) \) invariant expression given by \( \alpha = cr(\infty, 0, 1, \alpha) \). This means that if \( \Gamma \) is a Koebe group of the right signature and infinity weight, given by the AFP \( \Gamma = \Gamma(\infty, \nu_1, \nu_2; a, b, c) \ast \Delta_{A}\Gamma(\infty, 2, 2; d, e, f) \), then \( \Gamma \) is Teichmüller equivalent to \( \Gamma_{\alpha} \), where \( \alpha = cr(a, b, c, e) \).

4.4. As an example of the AFP construction with finite weight, we take the case of a hyperbolic group with signature \((0, 3; 4, \nu_1, \nu_2)\) and a parabolic group of signature \((0, 3; 4, 4, 2)\). We have that \( \Gamma_1 \) is conjugate to \( \Gamma(4, \nu_1, \nu_2; \infty, 0, 1) \). For practical purposes, we choose the transformation \( M(z) = \frac{-z + ki}{z + ki} \), where \( k \) is given in proposition 1, to do such conjugation. In this way we get that \( A(z) = iz \) is a canonical generator of \( \Gamma_1 = \Gamma(4, \nu_1, \nu_2; -1, 1, \frac{-1 + ki}{1 + ki}) \). \( \Gamma_2 \) will be conjugate to \( \Gamma(4, 4, 2; 0, \infty, 1) \) by a mapping of the form \( T(z) = \alpha z \). The orientation requirements imply that \( |\alpha| < 1 \). We then have that \( \alpha \) is a coordinate on the corresponding deformation space, and its invariant expression is given by \( \alpha = \frac{\beta - ki}{-\beta - ki} \), where \( \beta = cr(-1, 1, -\frac{1 + ki}{1 + ki}, \alpha) \).

We can see in this example that \( T(\Gamma_0) \) is not \( T(S_0) \) (\( S_0 = \Delta_0/\Gamma_0 \)). While \( T(S_0) \) is simply connected, it is not hard to see that \( 0 < |\alpha| < 1 \) and the circle \( |\alpha| = r \) is contained in \( T(\Gamma_0) \) for small values of \( r \). See below for an explicit estimate of these values.

4.5. The other cases of type \((0, 4)\) are handled in a similar way. Here we include a table with the results. See the remarks after it for more information.

| \( \Gamma_1 \) | \( \Gamma_2 \) | weight | \( \text{param.} \, \Gamma_1 \) | \( \text{param.} \, \Gamma_2 \) | \( \text{inv. expression} \) |
|----------------|-----------------|-------|----------------|----------------|----------------|
| hyp            | hyp             | \( \infty \) | \((\infty, 0, 1)\) | \((\infty, \alpha, \alpha - 1)\) | \( \beta + 1 \) |
| hyp            | hyp             | \( 2 < \mu < \infty \) | \((-1, 1, \frac{-1 + ki}{1 + ki})\) | \((-\alpha, \alpha, \frac{1 + ki}{1 + ki})\) | \( \frac{(ki - 1)(\beta - ki)}{-\beta(1 + ki) + k^2 - ki} \) |
| hyp            | par             | \( \infty \) | \((\infty, 0, 1)\) | \((\infty, \alpha, \alpha - 1)\) | \( \beta + 1 \) |
| hyp            | par             | \( 3, 4, 6 \) | \((-1, 1, \frac{-1 + ki}{1 + ki})\) | \((0, \infty, \alpha)\) | \( \frac{\beta - ki}{-\beta - ki} \) |
| hyp            | ell             | \( 3, 4, 5 \) | \((-1, 1, \frac{-1 + ki}{1 + ki})\) | \((0, \infty, \frac{\alpha}{x})\) | \( \frac{4kx}{\beta x + 2kxi} \) |
| par            | par             | \( \infty \) | \((\infty, 0, 1)\) | \((\infty, \alpha, \alpha - 1)\) | \( \beta + 1 \) |
| par            | par             | \( 3, 4, 6 \) | \((\infty, 0, 1)\) | \((0, \infty, \alpha)\) | \( \beta \) |
| par            | ell             | \( 3, 4 \) | \((\infty, 0, 1)\) | \((0, \infty, \frac{\alpha}{x})\) | \( \beta x \) |
| ell            | ell             | \( 3, 4, 5 \) | \((\infty, 0, 1)\) | \((0, \infty, \frac{\alpha}{x})\) | \( \beta x \) |

Remarks. 1. In the above table \( x = \frac{(q_1 q_2 - p_1 p_2)}{(q_1 q_2 + p_1 p_2)} \) (\( q_j = \cos \pi/n u_j \), \( p_j = \)}
\[ \sin\left(\frac{\pi}{\mu j}\right). \]

2. If \( \Gamma = \Gamma(\mu, \nu_1, \nu_2; a, b, c) \ast_{<D>} \Gamma(\mu, \nu_3, \nu_4; d, e, f) \), then \( \beta = cr(a, b, c, f) \) and the coordinate on the Teichmüller space is given by the above invariant expression (last column).

3. If one of the triangle groups involved in the construction of \( S \) has elliptic signature \((0, 3; \nu, 2, 2)\), we can still compute the Koebe groups by the above techniques. But since we do not have uniqueness of parameters for these triangle groups, we do not obtain a coordinate on the deformation spaces. We will not work any further these cases.

If \( \mu = \infty \), then the set \( \{ \alpha; \text{Im}(\alpha) > 1 \} \) is contained in the deformation space of the corresponding Koebe group. For the case of finite weights, let us consider a fundamental domain for \( \Gamma_1 \) containing the origin. Let \( d(\mu, \nu_1, \nu_2) \) denote the radius given in lemma 3 of §3.10. Similarly, let \( D(\mu, \nu_3, \nu_4) \) denote the radius of a disc centered at 0 such that the fundamental domain of \( \Gamma_2 \) is contained in that disc. For example, if the signature of \( \Gamma_1 \) is hyperbolic, one can take \( D(\mu, \nu_3, \nu_4) = 1 \). Then we have that the set \( \{ \alpha; |\alpha| < d(\mu, \nu_1, \nu_2)/D(\mu, \nu_3, \nu_4) \} \) is contained in the Teichmüller space of the group under consideration.

**The (1, 1) case**

4.6. To construct a Koebe group \( \Gamma \) of signature \((1, 1; \nu)\) and weight \( \mu \), we start with a triangle group \( \Gamma_1 = \Gamma(\mu, \mu, \nu; a, b, c) \). We then remove two discs around the two points of ramification value \( \mu \), and glue their boundaries. At the group level, this is reflected on finding a transformation \( C \) that conjugates \( B^{\pm 1} \) to \( A \). Choose an orientation of \( a_1 \), the partition curve, so that the ramification point corresponding to \( A (B) \) lies to the left (right) of the invariant lift \( \tilde{a}_1 \) of \( a_1 \). This implies that the conjugation is of the form \( CB^{-1}C^{-1} = A \). Maskit Second Combination Theorem [16, VII.E.5, pg 161] tells us that a proper choice of \( C \) will produce a group \( \Gamma = \Gamma_1 \ast C := < \Gamma_1, C > \) of the desired signature.

4.7. Let us work the most complicated case of the above construction, namely that of \( \Gamma_1 \) having hyperbolic signature \((0, 3; \mu, \mu, \nu)\), with finite weight \( \mu \). We start with

\[ \Gamma_1 = \Gamma(\mu, \nu, \nu; -1, 2, (-1 + ki)/(1 + ki)) \]

The canonical generators of this group are given by

\[ A = \begin{bmatrix} -q - ip & 0 \\ 0 & -q + pi \end{bmatrix}, B = \begin{bmatrix} -q - pmi & pmi \\ -pmi & -q + pmi \end{bmatrix}, \]

where \( q = \cos(\pi/\mu) \) and \( p = \sin(\pi/\mu) \), \( m = k/(2h) + h/(2k) \), \( n = k/(2h) - h/(2k) \), and \( k \) and \( h = p/c \) are given in proposition 1 of §3.3. Observe that \( m^2 - n^2 = 1 \). Actually, these two numbers can be understood in terms of hyperbolic cosines and sines of some geometric
objects on the quotient orbifold, but we are not interested on this line of thought. See [19] for more information. It is not hard to check that the transformation $T$ given by

$$T = i \left[ \frac{R}{n} \begin{array}{c} \frac{(1-m)R}{n} \\ \frac{n}{2R} \\ \frac{-1-m}{2R} \end{array} \right],$$

where $R = \sqrt{m+1/2}$, conjugates $B$ into $A$ (and vice versa). Then, the transformation $C$ needed for the HNN extension will be of the form $C_{\tau} = D_{\tau} T$, where $D_{\tau}(z) = \tau^2/z$. Or in a single expression we have

$$C = i \left[ \frac{\tau n}{2R} \frac{-\tau(m+1)}{2R} \frac{2R}{\tau n} \frac{(1-m)R}{\tau n} \right].$$

An application of the Second Combination Theorem shows that $\Gamma = \Gamma_1 \ast C$ is a Koebe group of the desired type, for a proper choice of $\tau$. It is clear that $\tau^2$ is a global coordinate on the corresponding deformation space. Its invariant expression is given by $(\frac{-n}{2})^{-\beta + ki \tau}$, with $\beta = cr(-1, 1, -\frac{1+ki}{1+ki}, C(-1))$.

To give a bound on the value of $\tau^2$, re-write $C_{\tau}$ as $C_{\tau}(z) = \tau^2 D(z)$, with

$$D = \left[ \begin{array}{c} 1 \\ -n/1 \end{array} \right] \begin{array}{c} (1-m)/n \\ (m-1)/2 \end{array}. $$

The circle of radius $s$ centered at the fixed point of $B$, $x = (m+1)/n$, is mapped onto a circle centered at the origin with radius $|\tau|^2|D(x-s)|$. If these two circles are disjoint, the Second Combination Theorem can be applied. Using basic calculus, we get that the function $|x - s|/|D(x-s)|$ is decreasing as $s$ goes to 0, and its minimum values is $(m-1)/2$. Therefore, the set $\{\tau; |\tau|^2 < (m-1)/2\}$ is contained in the Teichmüller space that we are studying.

4.8. As in the previous case, we include the results of the case $(1, 1)$ in the following table, where $\beta, k, m$ and $n$ are as in §4.7, and $q = \cos(\pi/\mu)$. 

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Assume that and $x$ can be obtained by the above invariant expression, where $\mu, \nu$ given by $\Gamma = \Gamma(\mu, \nu; a, b, c, C(a))$, for a group given by $\Gamma = \Gamma(\mu, \nu; a, b, c)*C$. 

**Plumbing constructions**

4.9. The above group theoretical constructions have a nice geometrical interpretation as follows. Let $S_1$ and $S_2$ be two orbifolds, not necessarily distinct, of type $(0,3)$, and let $x_1$ and $x_2$ be two points or punctures on $S_1$ and $S_2$ respectively, with equal ramification values. Assume that $z_1$ and $z_2$ are local coordinates on $S_1$ and $S_2$ as given in §3.10. Choose a complex number $t$ such that $U_i$ is contained in $S_i$, for $i = 1, 2$, where $U_i = \{p \in S_i; |z_i(p)| < \sqrt{|t|}\}$. If $S_1 = S_2$, then we must also require that $U_1 \cap U_2 = \emptyset$. Let $S = S_1 \sqcup S_2/\sim$, where $p_1 \in S_2$ and $p_2 \in S_2$ are equivalent, $p_1 \sim p_2$, if $|z_i(p_i)| = \sqrt{|t|}$ and $z_1(p_1)z_2(p_2) = t$. Then $S$ is an orbifold of type $(0,4)$ or $(1,1)$, depending on whether $S_1$ and $S_2$ are distinct or not. Naively speaking, we are removing discs from the orbifolds $S_1$ and $S_2$ and gluing them with a twist. We say that $S$ has been constructed from $S_1$ and $S_2$ with plumbing parameter $t$. See [11] for a more detailed explanation of plumbing techniques in the context of Riemann surfaces.

If the plumbing construction requires two different orbifolds, then we take as $z$ the preferred coordinate on $S_1$ centered at the first ramification point and relative to the second ramification point; and similarly for $w$ on $S_2$. If we have $S_1 = S_2$, then we take as $z$ the preferred coordinate centered at the first ramification point and relative to the second, while we take as $w$ the preferred coordinate at the second ramification point and relative to the first one. To compute $w$ all we have to do is find a transformation $T$ that conjugates $B$ to $A$ and such that $T^2 = id$. This guarantees that $T$ conjugates the triangle group $\Gamma_1$ to itself. This

| $\Gamma_1$ | weight | $C$ | inv. expression |
|-----------|--------|-----|-----------------|
| $\Gamma(\infty, \infty, \nu; \infty, 0, 1)$ | $\infty$ | $i\begin{bmatrix} \tau & \sqrt{2/(1+q)} \\ \sqrt{(1+q)/2} & 0 \end{bmatrix}$ | $\sqrt{1+q}/\beta$ |
| $\Gamma(\mu, \mu, \nu; -1, 1, -1+ki, 1+ki)$ | $2 < \mu < \infty$ | $i\begin{bmatrix} \tau & -(m+1)\tau/n \\ -n/(2\tau) & (m-1)/(2\tau) \end{bmatrix}$ | $(-n)/2 - \beta + ki/\beta + ki$ |
| $\Gamma(4, 4, 2; \infty, 0, 1)$ | 4 | $i\begin{bmatrix} 0 & \tau \\ -1/\tau & 1/\tau \end{bmatrix}$ | $\beta$ |
| $\Gamma(3, 3, 3; \infty, 0, 1)$ | 3 | $i\begin{bmatrix} 0 & \tau \\ -1/\tau & 1/\tau \end{bmatrix}$ | $\beta$ |
| $\Gamma(3, 3, 2; \infty, 0, 1)$ | 3 | $i\begin{bmatrix} \tau & -\tau \\ -2/(3\tau) & -1/(3\tau) \end{bmatrix}$ | $-2/(3\beta)$ |
choice of coordinates is different to that of [11] and [7], but it simplifies the computations and agrees with [19]. The construction with $w$ being the coordinate centered at the second ramification point and relative to the third point gives also a plumbing construction.

Let us compute the plumbing parameters of the examples §§4.3, 4.4 and 4.6. In the first case, we have that the coordinate $z$ is given by $z_1 = \exp(\pi i \xi)$, where $\xi$ lies in the upper half plane. Similarly, we have $z_2 \exp(\pi i (\alpha - \xi))$, and therefore we get the plumbing parameter is given by $t = z_1 z_2 = \exp(\pi i \alpha)$. In the situation of §4.4, the coordinates $z_1$ and $z_2$ are given by $z_1(\xi_1) = \xi_1^4$ and $z_2(\xi_2) = (\alpha/\xi_2)^4$, where $\xi_i$ is a point in a fundamental domain of $\Gamma_i, i = 1, 2$. The boundary identification gives $t = \alpha^4$.

For the case of tori, §4.6, we have $z_1(\xi) = \xi^\mu$. The transformation $T$ that conjugates $B$ to $A$ was found in §4.7. The coordinate $w$ is then given by $w(\xi) = z(T(\xi))$. Since $C$ identifies a curve invariant under $B$ with a curve invariant under $A$, we have that the plumbing parameter is computed by means of the expression $t = z(C(\xi)) w(\xi)$. Recalling the transformation $C$ from §4.7 we get the value $t = (\tau^2)^\mu$.

The values of the plumbing parameters for the $(0, 4)$ cases are given by $\exp(\pi i \alpha)$ in the case of infinite weight, or $\alpha^\mu$ for finite weight $\mu$. In this latter case, if one of the triangle groups is elementary and corresponds to $S_1$, then we take a preferred coordinate that maps the ramification point to the origin in $C$, while if the elementary group corresponds to $S_2$, our local coordinate around the ramification point will map such point to the point $\infty$ in $\hat{C}$.

The cases of tori are included in the following table.

| $\Gamma_1$ | $T(\xi)$ | plumber. par. |
|------------|--------|--------------|
| $\Gamma(\infty, \infty, \nu)$ | $-2/(1+q)\xi$ | $\exp(\pi i \tau \sqrt{\frac{2}{1+q}})$ |
| $\Gamma(\mu, \mu, \nu)$ | $(1+m/n)\xi + 1 - m/n\xi - 1 - m$ | $(\tau^2)^\mu$ |
| $\Gamma(4, 4, 2)$ | $1 - \xi$ | $\tau^8$ |
| $\Gamma(3, 3, 3)$ | $1 - \xi$ | $\tau^6$ |
| $\Gamma(3, 3, 2)$ | $2\xi + 1/2\xi - 2$ | $-\frac{27}{8} \tau^6$ |

**The general case**

4.10. Let us finally explain how to compute coordinates for Koebe groups in general and vice versa; that is, given a point in the deformation space, how to find the generators of the corresponding group. Let $\Gamma$ be a Koebe group uniformizing an orbifold $S$ with maximal partition with weights $(P, N)$. In the classical work on deformation spaces of Fuchsian

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groups, one starts with a group, say $F_0$, and then to compute the point corresponding to any other group, $F$, one has to measure how far $F$ is from $F_0$, via the maximal dilatation of quasiconformal mappings. This implies that the starting group $F_0$ plays a special point with respect to coordinatization of deformation spaces. In our case, the coordinates measure how the triangle groups are put together to form the Koebe group $\Gamma$, so it makes sense to compute the position of $\Gamma$ in $T(\Gamma)$, without the need of a reference fixed group. This is not difficult task, since we have the Maskit Embedding Theorem, that reduces everything to the one-dimensional case, and these latter groups are already known by the previous work of this section. At this point the reader should look at §2.4., where it is explained how to find subgroups $\Gamma_j$, $1 \leq j \leq 3p - 3 + n$ of $\Gamma$ that give the Maskit Embedding. Given the group $\Gamma$, we decompose it in simpler subgroups with one-dimensional deformation spaces, say $\Gamma_j$, for $1 \leq j \leq 3p - 3 + n$, and for each of these subgroups we compute the coordinates $\alpha_j$ as explained in this section. The the coordinate of $\Gamma$ will be $(\alpha_1, \ldots, \alpha_{3p-3+n})$. The only point one may wonder is what happens if we start with a different component of $\pi^{-1}(T_j)$, say $D'_j$, giving a different decomposition of $\Gamma$. Let $\Gamma'_j$ be the corresponding stabilizer. We have that there is an element $\gamma \in \Gamma$ such that $\gamma \Gamma_j \gamma^{-1} = \Gamma'_j$. This means that the two stabilizers are conjugated in $PSL(2, \mathbb{C})$, and therefore the coordinate $\alpha_j$ will be the same for both groups, since the expression of this coordinate is invariant under conjugation by Möbius transformations (and the relation that defines Teichmüller spaces kills conjugation by elements of $PSL(2, \mathbb{C})$).

Consider now the inverse problem: let $(\alpha_1, \ldots, \alpha_{3p-3+n})$ be a point in $T(\Gamma)$. We want to find a Koebe group $\Gamma$ corresponding to this point. To do so, we have to consider three different types of partition curves. Suppose we have constructed a Koebe group $\Gamma_{j-1}$, corresponding to the first $j-1$ coordinates, $\alpha_1, \ldots, \alpha_{j-1}$.

Case 1: The curve $a_j$ disconnects $S$. This means that the construction corresponding to this curve is an AFP. Let $S_1$ and $S_2$ be the two parts of $T_j - a_j$. We have that one of the two parts, say $S_1$, has been already uniformized in the previous steps (i.e., in the construction up to $a_{j-1}$). Let $G_1 = \Gamma(\mu_j, \nu(1), \nu(2); a, b, c)$ be the triangle subgroup of $\Gamma_{j-1}$ corresponding to $S_1$. Here $\nu(k)$ are just some ramification numbers of the signature of $S$. Let $G_2 = \Gamma(\mu_j, \nu(3), \nu(4); d, e, f)$ be a triangle group that uniformizes orbifolds with the signature than $S_2$. All we need to do is to find the parameters $(d, e, f)$ such that (i) $G_1$ and $G_2$ share the element $A_j$ uniformizing $a_j$ and (ii) the coordinate of $G_1 \ast_{<A_j>} G_2$, as computed earlier, is precisely $\alpha_j$. The group $\Gamma_j$ will be the group generated by $\Gamma_{j-1}$ and $G_2$.

Case 2: $a_j$ does not disconnect $S$, and $T_j$ is of type $(0, 4)$. The parts of $T_j$ will be uniformized
by two triangle subgroups $G_1 = \Gamma(\mu_j, \nu(1), \nu(2); a, b, c)$ and $G_2 = \Gamma(\mu_j, \nu(3), \nu(4); d, e, f)$ of $\Gamma_{j-1}$. Let $A_1$ and $A_2$ be the elements of $G_1$ and $G_2$, respectively, uniformizing $a_j$ with the correct orientations. Choose a transformation $C$ such that $CA_2C^{-1} = A_1$, and such that the coordinate corresponding to $G_1 *_{A_1} CG_2C^{-1}$ is $\alpha_j$. Then the group $\Gamma_j$ is the group generated by $\Gamma_{j-1}$ and $C$.

**Case 3:** $a_j$ does not disconnect $S$ and the type of $T_j$ is $(1,1)$. Let $G_1 = \Gamma(\mu_j, \mu_j, \nu(1); a, b, c)$ be a triangle subgroup of $\Gamma_{j-1}$ uniformizing $T_j - a_j$. Let $A$ and $B$ be the two canonical generators of $G_1$. Find a transformation $C$ such that $CB^{-1}C^{-1} = A$, and the coordinate of $G_1 * C$ is $\alpha_j$. Then the group $\Gamma_j$ is generated by $\Gamma_{j-1}$ and $C$.

This algorithm completes the proof of theorems 1 and 2 of §1. These constructions give us a way to find explicitly generators of constructible Koebe groups. It is not hard to program a computer to get the computations done fast and easily.

## 5 Some properties of Teichmüller spaces of Koebe groups

As we have remarked in §4.4, the Teichmüller space of a Koebe group is not equivalent to the Teichmüller space of the quotient orbifold, unless all the weights are equal to $\infty$. So let us assume that $\Gamma$ is a constructible Koebe group uniformizing some orbifold with a maximal partition with weights, $(S, \mathcal{P}, \mathcal{N})$, where at least one of the elements of $\mathcal{N}$ is finite. By results of L. Bers [1], B. Maskit [13] and I. Kra [9], we have that the universal covering space of $T(\Gamma)$ is the space $T(S)$. Here we will use Maskit’s version, since his geometric description fits better in our work. See also [4] for an explanation of these results in the context of Schottky groups (another type of Koebe groups not covered in this paper; namely those with weights equal to 1).

Before proceeding any further with the proof of our next result, we need to introduce some background on surface topology. Let $c$ be a simple closed curve on a surface $S$, and let $N_c$ be a tubular neighborhood of $c$, homeomorphic via $h_c$, to the annulus $A_c = \{(r, \theta); 1 < r < 3, 0 \leq \theta \leq 2\pi\}$, with the usual identification modulo $2\pi$. Suppose that $c$ corresponds to the circle $r = 2$. Consider the self-homeomorphisms of $A_c$ given by

$$f_c(r, \theta) = \begin{cases} (r, \theta), & 1 < r \leq 2 \\ (r, \theta + 2\pi(r - 2)), & 2 < r < 3. \end{cases}$$

Since $f_c$ extends to the boundary of $A_c$ as the identity mapping, we can consider the home-
omorphisms of $S$ given by

$$\tau_c(r, \theta) = \begin{cases} 
  id, & \text{on } S - A_c \\
  h_c^{-1}f_ch_c, & \text{on } N_c.
\end{cases}$$

The Dehn twist around $c$ is just the mapping class of $\tau_c$, which we will also denote by $\tau_c$.

Consider the minimal normal subgroup $G$ of the mapping class group of $S$ containing $\tau^\mu_j$ for finite $\mu_j$.

**Theorem 3** $T(\Gamma) \cong T(S)/G$

**Proof.** Let $T(S) \to T(\Gamma)$ be the universal covering of $T(\Gamma)$. Let $H$ denote the covering group of this mapping. In [13] it is proven that $H$ consists of the elements $f$ of the mapping class group of $S$ such that $f$ lifts to $\tilde{f} : \Delta \to \Delta$ and $\tilde{f} \circ \gamma = \gamma \circ \tilde{f}$, for all $\gamma \in \Gamma$. It is well known that for each mapping class we can take $f$ to be quasiconformal. The mapping $f$ induces a quasiconformal self-homeomorphism on the punctured surface $S_0 = S - \{x_j; \nu_j < \infty\}$, which we will also denote by $f$. This mapping satisfies that for each curve $a_j$ of $\mathcal{P}$, $f(a_j)$ is freely homotopic to some $a_{s(j)}$, where $s$ is a permutation of $3p - 3 + n$ elements (this is due to the fact that the parabolic and elliptic elements uniformizing the curves of the partition play a special role in $\Gamma$). Then by a result of L. Bers [2] and I. Kra ([13] we can assume without loss of generality that $f(a_j) = a_{s(j)}$

For each $1 \leq j \leq 3p - 3 + n$, let $T_j$ be the connected component of $S - \{a_k; a_k \in \mathcal{P}, k \neq j\}$ containing $a_j$; these are called the modular parts of $S$. It is clear that $f$ induces a permutation amongst the $T_j$'s. For each $1 \leq j \leq 3p - 3 + n$, it is not hard to find a simple closed curve $b_j$ such that $b_j$ intersects $a_j$ and it is disjoint from $a_k$ for any other curve of the partition $\mathcal{P}$, $k \neq j$ (called “dual” curves in [10, pg. 227]). If $f$ belongs to $H$, then $f$ must commute with the elements $B_j$ uniformizing $b_j$, and therefore we have that $f$ must preserves the sets $T_j$.

So we have reduced the problem to the one dimensional case. Assume then that one of the modular parts, say $T_j$, is of type $(0, 4)$. The subgroup of the mapping class group of $T_j$ that preserves the curve $a_j$ is generated by the ‘half Dehn twist’ around $a_j$. Since we are assuming that the elements of $H$ preserves all the transformations of $\Gamma$, we have that (the restriction of $f$ to $T_j$) must be a power of $\tau_{a_j}$. The action of this mapping on the curve $b_j$ is given by $b_j \mapsto a_jb_ja_j^{-1}$. A similar relation holds on $\Gamma$. Therefore, if $f \in H$, then the restriction of $f$ to $T_j$ is of the form $\tau_{a_j}^{k\mu_j}$, with $k \in \mathbb{Z}$.

Similarly, for $T_j$ of type $(1, 1)$, we have that $f$ reduces to $\tau_j^k$, with $k$ integer; but in this case the action of $f$ will be given by $b_j \mapsto a_jb_j$. So we get $k$ is a multiple of $\mu_j$. 

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Since we have to conjugate in order to get the first step of $f$ fixing the curves $a_j$, we have the normality condition satisfied as well, so we get $H = G$ and the theorem is proven. \qed}

In order to prove the last result, we need to give a slightly different interpretation of the Teichmüller space of a Fuchsian group $F$. See [3] and [4] for more details. A Beltrami coefficient for $F$, $\mu$, is a measurable function of norm less than 1 such that $(\mu \circ \gamma)\overline{\gamma} / \gamma' = \mu$ for all $\gamma \in F$. Given a Beltrami coefficient $\mu$, there exists a unique quasiconformal mapping $w_\mu$ such that $(w_\mu)_\overline{\gamma} = \mu(w_\mu)_\gamma$, and $w$ fixes $\infty$, 0 and 1. It is not hard to see that $w_\mu F w_\mu^{-1}$ is again a group of Möbius transformations. Two Beltrami coefficients $\mu$ and $\nu$, are equivalent if and only if $w_\mu = w_\nu$ on the real axis. The set of equivalence classes of Beltrami coefficients for $F$ is the Teichmüller space of $F$, $T(F)$. On can prove that $\mu$ and $\nu$ are equivalent if and only if there is a Möbius transformation $A$ such that $w_\mu \circ \gamma \circ w_\nu^{-1} = A \circ w_\nu \circ \gamma \circ w_\nu^{-1} \circ A^{-1}$, for all $\gamma \in F$, which fits with our first definition of Teichmüller space of a Kleinian group.

Let $S$ be an orbifold of type $(p, n)$, and let $f$ be a quasiconformal homeomorphism of $S$ onto another orbifold $S'$, such that $f$ respect the ramification values of the point of $S$. This simply means that the ramification value of $x$ and $f(x)$ are equal for all $x$ in $S$. We will call such a mapping a (quasiconformal) deformation of $S$. Two such mappings, $f : S \to S_1$ and $g : S \to S_2$ are equivalent if there exists a conformal mapping $\phi : S_1 \to S_2$, preserving the ramification values, and such that $g^{-1} \circ \phi \circ f$ is homotopic to the identity mapping on $S$. The set of equivalence classes of quasiconformal deformations of $S$ is the Teichmüller space of $S$, $T(S)$.

Let $F$ be a Fuchsian group such that $\mathbf{H}/F \cong S$. Then we have a natural isomorphism $\psi : T(S) \to T(F)$ given by $f \mapsto \mu(f)$, where $f$ is a (quasiconformal) deformation of $S$ and $\mu(f) = f_\overline{\gamma}/f_\gamma$ is the Beltrami coefficient of $f$ computed in local coordinates.

**Proof of theorem 4.** Let $S$ be an orbifold of type $(p, n)$ with at least one point with finite ramification value. Let $F$ be a Fuchsian group uniformizing $S$. Let $S_0$ be the surface of genus $p$ with $n$ punctures obtained by removing from $S$ the points with finite ramification value (by our assumptions we have that $S_0 \neq S$). Let $F_0$ be a Fuchsian group uniformizing $S_0$. Consider the set $\mathbf{H}_F = \mathbf{H} - \{\text{fixed point of elliptic elements of } F\}$. We have that $\mathbf{H}_F/F \cong S_0$. Therefore, there exists a universal covering map $h : \mathbf{H} \to \mathbf{H}_F$, such that $\pi_0 = \pi \circ h$, where $\pi_0 : \mathbf{H} \to S_0$ and $\pi : \mathbf{H}_F \to S_0$ are the natural projection mappings. Actually, here $\pi$ is only the restriction of the canonical mapping from $\mathbf{H}$ onto $S$, but by an abuse of notation we will denote in the same way. In [3] and [4] it is proven that the mapping $h$ induces an isomorphisms between the Teichmüller spaces $T(F)$ and $T(F_0)$, which is defined by the
formula $h^* \mu \circ h = \mu h'/h'$, for all $\mu \in T(F_0)$. It is not hard to understand the isomorphism $h^*$ in terms of the Teichmüller spaces of the orbifolds $S$ and $S_0$. Let $f : S \to \tilde{S}$ be a deformation of $S$. Restrict $f$ to $S_0$, to obtain a deformation $f|_{S_0} := g : S_0 \to \tilde{S}_0$ of $S_0$. This induces a mapping from $r^* : T(S) \to T(S_0)$ by $f \mapsto g$. Then the mapping $h^*$ is the inverse of $r^*$ at the group level. This means that the following diagram is commutative:

$$
\begin{array}{c}
T(S) \xrightarrow{r^*} T(S) \\
\downarrow \psi_0 \quad \quad \quad \quad \quad \quad \downarrow \psi \\
T(F_0) \xleftarrow{h^*} T(F).
\end{array}
$$

Now the theorem follows easily from the definition of the isomorphism $r^*$ as follows. We are given a maximal partition $\mathcal{P}$ and a set of weights $\mathcal{N}$ on $S$. By the Maskit Existence theorem, we have Koebe groups, $\Gamma$ and $\Gamma_0$, uniformizing $(S, \mathcal{P}, \mathcal{N})$ and $(S_0, \mathcal{P}, \mathcal{N})$ in the invariant components $\Delta$ and $\Delta_0$, respectively. Let $\varphi : T(S) \to T(\Gamma)$ and $\varphi_0 : T(S_0) \to T(\Gamma_0)$ be the covering mapping given by theorem 3. The covering groups of these mappings, $H$ and $H_0$ respectively, are the normal subgroup of the mapping class groups of $S$ and $S_0$ generated by the Dehn twists around the curves of $\mathcal{P}$ with finite weight. We have that $r^*$ maps $H$ onto $H_0$, by the own definition of $r^*$. So we can project this mapping to the level of the deformation spaces of Koebe groups, obtaining a function $\tilde{r}$, that makes the following diagram commutative:

$$
\begin{array}{c}
T(S) \xrightarrow{r^*} T(S) \\
\downarrow \varphi_0 \quad \quad \quad \quad \quad \quad \downarrow \varphi \\
T(\Gamma_0) \xrightarrow{\tilde{r}} T(\Gamma).
\end{array}
$$

It is clear that $\tilde{r}$ is a bijection, given the desired result.

$\square$

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