Abstract

We investigate the phenomenon of the decay of a supercurrent through homogeneous nucleation of vortex-antivortex pairs in a 2-D like superconductor or superfluid by means of a quantum electrodynamic formulation for the decay of the 2-D vacuum. The case in which both externally-driven current and Magnus force are present is treated exactly, taking the vortex activation energy and its inertial mass as independent parameters. Quantum dissipation is included through the formulation introduced by Caldeira and Leggett. The most relevant consequence of quantum dissipation is the elimination of the threshold for vortex production due to the Magnus force. In the dissipation-dominated case, corresponding formally to the limit of zero inertial mass, an exact formula for the pair production rate is given. If however the inertial mass is strictly zero we find that vortex production is inhibited by a quantum effect related to the Magnus force. The possibility of including vortex pinning is investigated by means of an effective harmonic potential. While an additional term in the vortex activation energy can account for the effect of a finite barrier in the direction perpendicular to the current, pinning along the current depresses the role of the Magnus force in the dissipation-dominated dynamics, except for the above-mentioned quantum effect. A possible description of vortex nucleation due to the combined effects of temperature and externally-driven currents is also presented along with an evaluation of the resulting voltage drop.
I. INTRODUCTION

The layered structure and high critical temperature of the oxide superconductors has led to much renewed interest in the behaviour of these materials under applied magnetic fields. Of particular interest are the statics and dynamics of the magnetic vortices in the presence of pinning impurity centers, where issues such as the formation of a glassy vortex phase \[\text{[1]}\] and those connected to transport properties such as flux-creep \[\text{[2]}\] and vortex tunneling \[\text{[3-5]}\] have been vivaciously debated in the recent literature. Similar questions of vortex dynamics have attracted the interest of researchers in the field of superfluidity for the past three decades \[\text{[6]}\].

An issue that appears to lend itself to full analytic treatment by means of a path-integral formulation is the quantum tunneling of a vortex trapped by a pinning center in a 2-D superfluid or superconductor structure \[\text{[4,5]}\]. This problem has been investigated both with and without the inclusion of quantum dissipation and discussions that give full weight to the inertial mass of the moving vortex can be found in the literature alongside treatments \[\text{[5]}\] in which the inertial mass is taken to be negligible. The physics of the problem is captured by semi-classical evaluations of the tunneling rate and discussion has centered on the effects of dissipation and pinning in opposing the stabilising influence of the Magnus force on the classical orbits of a vortex.

Mathematically, and in the classical framework for a single vortex moving in a supercurrent at relatively low velocity, the equation of motion in the 2-D plane reads:

\[
m\ddot{q} = -\nabla V(q) - e\dot{q}\times B - \eta\dot{q}
\] (1.1)

Here \(m\) is the inertial mass of the vortex of topological charge \(e = \pm 2\pi\), treated as a single, point-like particle of 2-D coordinate \(q(t)\). Also, \(V(q) = V_p(q) + V_v(q, J)\) is the phenomenological pinning potential plus the electric-like potential due the supercurrent \(J\), namely \(-\nabla V_v = eE = \times eJ\) (the latter being a compact notation for the 2-D dual of a vector: \((\times J)_\mu = \epsilon_{\mu\nu}J_\nu\)). Finally \(B = \hat{z}d\rho_s^{(3)}\) is the magnetic-like field of the Magnus force \[\text{[8-10]}\] acting on the vortex in the plane of the film of thickness \(d\) and directed along the vector \(\hat{z}\) orthogonal to it and associated with a superfluid component having 3-D number density \(\rho_s^{(3)}\); and \(\eta\) is the phenomenological friction coefficient. We neglect in this work the additional magnetic field intrinsic to the vortex since, as discussed by Clem \[\text{[11]}\], the flux quantum is homogeneously spread over a very large distance for a single flux vortex in a 2-D superconductor. The quantum-mechanical version of Eq. (1.1) is attained via the Feynman path-integral transposition in which the dissipation is treated quantistically through the formulation due (for the ohmic case) to Caldeira and Leggett \[\text{[12]}\].

A situation closely related to the quantum tunneling of a vortex in the presence of dissipation is the decay of a supercurrent, whether in a charged superconductor or in a neutral superfluid, due to the spontaneous homogeneous...
nucleation of vortex-antivortex pairs from the “vacuum” of the perfect superfluid wavefunction, in the presence of an external uniform field. In this case the field corresponds – in its electric part – to the dual of the external supercurrent $J$, which may be kept fixed in the system if a voltage drop is measured at the edge of the superconducting sample, and – in its magnetic part – to the Magnus force field term $\mathbf{B}$. This situation is the subject of the present article, in which we wish to show that the lifetime $\Gamma^{-1}$ of the supercurrent can be calculated explicitly in terms of the phenomenological parameters of the material by means of a straightforward application of the field-theoretic relativistic formalism leading to the decay rate of the vacuum in scalar quantum electrodynamics. Indeed, the excitations of a relativistic quantum field encode in a rather natural way the particle-antiparticle quantum fluctuations of the vacuum and therefore the particle-antiparticle creation. This formalism, independently already proposed by Ping Ao [13], albeit for the simplest of the cases treated herewith, allows us to evaluate $\Gamma$ for a number of vortex dynamics cases relevant to experimentally accessible situations. With reference to the work of Ping Ao [13], it would appear that a main difference with the present work is the fact that the effects of the Magnus force, of dissipation and/or of pinning were not included in the resulting formula for the vortex pair production rate. For all the above-mentioned cases, the central technical ingredient of our formulation for the zero-temperature dynamics is an effective screened interaction and the resulting quadratic form of the effective-particle Lagrangian for the Feynman path-integral formulation of vacuum decay.

Indeed, our article is organised as follows. In Section II we sketch some basic facts about the vortex dynamics and introduce our quantum field theory (QFT) formulation. This is expanded in Section III, where we treat in detail the situation in which dissipation and pinning are both absent, with the supercurrent inducing a vortex-antivortex pair production in the presence of the homogeneous fields of the supercurrent and of the Magnus force. Two phenomenological parameters are introduced to describe the isolated vortex: its nucleation energy $E_0$, suitably renormalised (e.g. to include screening effects by the external current or by the vortex plasma above the Kosterlitz-Thouless (KT) transition and pinning by impurities), and the vortex inertial mass $m$. We consider $m$ and $E_0$ to be independent phenomenological parameters so as to account for the issue, debated in the recent vortex-dynamics literature [5,14], where the inertial mass $m$ may be taken as negligible (but see also [2]). It is seen that the Magnus force raises a threshold for the magnitude of the supercurrent (or, the “electric” field) in which the pair production may occur. This was also qualitatively observed in [4,5]). In Section IV we include the effects of quantum dissipation through the formulation of Caldeira and Leggett [12], in which the drag force experienced by the moving vortex is mimicked by a linear coupling to the coordinates of an infinite set of harmonic oscillators. The main qualitative effect is seen to be the suppression of the threshold for pair production, as the Landau-level stability of the vortex in the field of the Magnus force is destroyed by any infinitesimal amount of dissipation. The
formula for the production rate $\Gamma$ is worked out in detail for the dissipation-dominated case, that is in the limit $\gamma = m/\mathcal{E}_0 \to 0$. We deal mainly with the ohmic-case, to then show that sub- and super-ohmic cases can be treated in a similar way to yield a less pronounced pair-production phenomenon. In Section V we discuss the possibility of including the effects of pinning centres. A finite pinning barrier in the direction perpendicular to the current essentially results in a contribution to the activation energy $\mathcal{E}_0$, however pinning in the direction of the current has the consequence of suppressing the effect of the Magnus force in the dissipation-dominated dynamics. We also discuss the extreme case where pinning can be mimicked via an (unbounded) anisotropic harmonic force. For large enough separation, this corresponds to a confining linear potential in our relativistic formulation and this may mimic a situation in which a uniform pinning pressure is exerted. The result is that, as soon as a component of the force in the direction perpendicular to the current is present, a threshold for vortex production appears. Section VI contains a more tentative discussion on how to implement the results obtained by the QFT approach in order to evaluate the vortex-induced voltage drop in the direction of the current across the sample, including also the effect of vortices which may be activated by thermal fluctuations. Finally, in the Appendix we show that some of the qualitative features observed within the QFT approach can also be seen to agree with a study of the classical equations of motion for a single vortex.

II. VORTEX DYNAMICS: A QUANTUM FIELD THEORY APPROACH

We begin this Section by briefly recalling the description of vortex dynamics based on the time-dependent Landau-Ginzburg (TDLG) formulation for the superfluid system in the presence of vortex solutions and of a supercurrent. For more extensive discussions, also of related theoretical issues, the reader is referred for instance to the articles of the Seattle group [8], [4] and that of Lee and Fisher [15] (see also [10]). For a superconductor, the TDLG effective action for the wavefunction $\psi(x, y, t) = \sqrt{\rho_s} e^{i\theta(x, y, t)}$ corresponds to the Lagrangian (in the units in which $\hbar = 1$ and the carriers’ charge $e^* = 1$):

$$\mathcal{L} = -\rho_s \dot{\theta} - \frac{1}{2m_0} \rho_s (\nabla \theta - A)^2 + f(\rho_s, \dot{\rho}_s, \nabla \rho_s)$$  \hspace{1cm} (2.1)

Here $m_0$ is the effective mass of the carriers, $\rho_s$ the 2-D superfluid density and $f$ contains the additional, phase independent, terms of the LG expansion in $|\psi|$ and its derivatives. In the following, we shall make the approximation of constant $\rho_s$, thus dropping $f(\rho_s, \dot{\rho}_s, \nabla \rho_s)$, and treat the vortices as point-like objects. The external supercurrent is related to the external vector potential $A$ via

$$\mathbf{J} = \frac{\delta \mathcal{L}}{\delta A} = \frac{\rho_s}{m_0} (\nabla \theta - A)$$  \hspace{1cm} (2.2)
Our situation consists in assuming that \( J \) is fixed by some external device “driving” a constant current which we take to be uniform throughout the sample. We are then interested in estimating the longitudinal potential drop and thus the ohmic resistance (in general current-dependent) due to the production and motion of the vortices. We implement the requirement of fixed current \( J \) by adding the term \( \mathbf{A} \cdot \mathbf{J} \) to the Lagrangian and treating \( \mathbf{A} \) as a Lagrange multiplier:

\[
\mathcal{L} = -\rho_s \dot{\theta} - \frac{1}{2m_0} \rho_s (\nabla \theta - \mathbf{A})^2 - \mathbf{A} \cdot \mathbf{J}
\]  

Then \( \delta \mathcal{L}/\delta \mathbf{A} = 0 \) fixes the current: \( J = \frac{\rho_s}{m_0} (\nabla \theta - \mathbf{A}) \); substituting back and disregarding a constant term \( \frac{m_0}{2\rho_s} J^2 \), we get

\[
\mathcal{L} = -\rho_s \dot{\theta} - \nabla \theta \cdot \mathbf{J}
\]  

A vortex is introduced as a configuration of the phase \( \theta(\mathbf{r}, t) \) having non-trivial topology and such that \( \nabla \theta \) has formally a non-vanishing rotational (a scalar, in 2D):

\[
\nabla \times \nabla \theta = \rho_v(\mathbf{r})
\]  

In the presence of such singular vortex configurations with topological charges \( e_n = 2\pi n \) (where, in the following, \( n = \pm 1 \)) and treating the vortices as point-like particles, we can rewrite the action in terms of a vortex density \( \rho_v(\mathbf{r}) = \sum_n e_n \delta(\mathbf{r} - \mathbf{r}_n) \) and vortex current density \( \mathbf{J}_v(\mathbf{r}) = \sum_n e_n \mathbf{r}_n \cdot \delta(\mathbf{r} - \mathbf{r}_n) \), with \( \mathbf{r}_n(t) \) the vortex position. The result is as follows

\[
\mathcal{S} = \int dt d^2r \mathcal{L} = \int dt d^2r (-\rho_v \mathbf{V}_v - \mathbf{J}_v \cdot \mathbf{A}_v)
\]

\[
= \int dt \left\{ -\sum_n e_n \mathbf{V}_v(\mathbf{r}_n) - \sum_n e_n \mathbf{r}_n \cdot \mathbf{A}_v(\mathbf{r}_n) \right\}
\]

where we have introduced the vortex fields \( \mathbf{A}_v \) and \( \mathbf{V}_v \) through \( \nabla \times \mathbf{A}_v = B = \rho_s \) and \( \mathbf{E} = -\nabla V_v = \times \mathbf{J} \). One can formally obtain the Magnus force term \(-\mathbf{J}_v \cdot \mathbf{A}_v\) from the \(-\rho_s \dot{\theta}\) term in the TDGL expansion \((2.1)\), in a shorthand way and disregarding possible subtle questions related to boundary conditions (for a more thorough approach see for instance \([8-10,15]\)). If \( \dot{\theta}(\mathbf{r}) = \sum_n e_n \dot{\theta}(\mathbf{r} - \mathbf{r}_n(t)) \), then \( \dot{\theta} = -\sum_n e_n \dot{\mathbf{r}}_n \cdot \nabla \theta(\mathbf{r} - \mathbf{r}_n) \). If we denote by \( \Delta^{-1} \) the inverse of the Laplacian operator, then \( \dot{\theta} = \epsilon_{\mu\nu} \Delta^{-1} \partial_{\nu} J_{\nu\mu} \), where the vortex current is \( J_{\nu\mu} = \sum_n e_n \dot{\mathbf{r}}_{\mu\nu}(\mathbf{r} - \mathbf{r}_n) \) and we have written, from Eq. \((2.3)\), \( \partial_{\mu} \theta(\mathbf{r} - \mathbf{r}_n) = -\epsilon_{\mu\nu} \Delta^{-1} \partial_{\nu} \delta(\mathbf{r} - \mathbf{r}_n) \). Therefore

\[
\int d^2 r \rho_s \dot{\theta} = \int d^2 r \mathbf{A}_v \cdot \mathbf{J}_v
\]  

having formally put \( \mathbf{A}_{\nu\mu} = -\epsilon_{\mu\nu} \Delta^{-1} \partial_{\nu} \rho_s \). Also, it is straightforward to show that
\[
\int d^2r \mathbf{J} \cdot \nabla \theta = - \int d^2r \nabla V_v \cdot (\times \nabla \theta) = \int d^2r V_v \rho_v
\]  

(2.8)

Adding a kinetic energy term \( \sum_n \frac{1}{2} m_n \dot{r}_n^2 \) to the Lagrangian, Eq. (2.6) shows that on the vortex acts a Lorentz-like force linked to a magnetic-like field \( \mathbf{B} \) of intensity \( \rho_s \). We also recover the classical equation of motion (1.1) for the vortex: the electric part of the field acting on the vortex is associated with the dual vector of the externally-applied supercurrent \( \mathbf{J} \). The picture is therefore quite reminiscent of the situation for scalar two-space-dimensional quantum electrodynamics (QED).

In practice, the “electric field” \( \mathbf{E} = \times \mathbf{J} \) will be really uniform only on some scale for which the current \( \mathbf{J} \) is uniform. For scales less than that, the vortices themselves will alter the uniformity of the current. In the language of QED, a vortex at position \( \mathbf{r} = \mathbf{r}_1 \) near an antivortex at position \( \mathbf{r} = \mathbf{r}_2 \) will also feel – beside the background electrostatic potential \( V_v(\mathbf{r}_1) \) corresponding to the uniform electric field – the two-dimensional electrostatic attraction due to the antivortex

\[
\Delta V_v(\mathbf{r}_1) = \frac{e^2 \rho_s}{2\pi m} \ln \left| \frac{\mathbf{r}_1 - \mathbf{r}_2}{a} \right|
\]

(2.9)

where \( a \) is some scale which we can conveniently take to represent the vortex size. The above vortex-antivortex “electrostatic” interaction gives rise to an additional barrier, which can be taken into account \[16,3\] by an additive renormalization term in the activation energy. Following Minnhagen \[16\] (see also \[3\]), we make use of the fact that the barrier maximum occurs for \( |\mathbf{r}_1 - \mathbf{r}_2| \sim \frac{\rho_s}{m_0 J} = \frac{\rho_s m_0}{\rho_s} \) thus contributing a term to be included in the renormalization of the activation energy of a vortex (\( e_1 = 2\pi \))

\[
\mathcal{E}_{0R} = \mathcal{E}_0 - \frac{\pi \rho_s}{\tilde{\epsilon} m_0} \ln \left( \frac{m_0 J}{\rho_s a} \right) + \cdots
\]

(2.10)

Notice the appearance of the dielectric constant \( \tilde{\epsilon} = \tilde{\epsilon}(T) \) which keeps into account the polarization due to the thermal effects below the KT transition. Above the KT transition, the “electrostatic” interaction is screened by the KT screening length \( \lambda_{KT} \) and the renormalized activation energy will be, for \( J \ll \rho_s/m_0 \lambda_{KT} \)

\[
\mathcal{E}_{0R} = \mathcal{E}_0 - \pi \frac{\rho_s}{m_0} \ln \left( \frac{a}{\lambda_{KT}} \right) + \cdots
\]

(2.11)

In the following we will lump these renormalization effects in the definition of the activation energy, which we call simply \( \mathcal{E}_0 \).

The central issue of the present work is that the “electric” field term \( \rho_v V_v \) in Eq. (2.6) generates an instability in the system, which will have a tendency of producing vortex-antivortex pairs. We propose to describe pair creation through the formalism of relativistic QED. In this formalism, pairs are created out of the vacuum in the presence of a constant electromagnetic field, much as in our electromagnetic-analogue situation for the vortex dynamics. Further
terms will have to be added to the action (2.4), which describes only the interaction with the electromagnetic field, in order to account for the vortex motion. There will be, first of all, the energy $\mathcal{E}_0$ of the isolated vortex at rest in the absence of the external current. This we take as an effective renormalised activation energy $\mathcal{E}_{0R}$ to include the electrostatic screening energy $\Delta \mathcal{E}$ as well as the average energy required to overcome possible pinning potential barriers. We consider, to keep the discussion as general as possible, also a kinetic energy term $m^2 \dot{r}^2$, where $m$ is the possible inertial mass of the vortex.

In the relativistic formalism, which we employ, the total “free particle” energy for a vortex having momentum $\mathbf{p}$ would then be

$$\mathcal{E} = \sqrt{\mathcal{E}_0^2 + \frac{1}{\gamma} \mathbf{p}^2} \simeq \mathcal{E}_0 + \frac{1}{2m} \mathbf{p}^2 + \cdots = \mathcal{E}_0 + \frac{m}{2} \dot{\mathbf{r}}^2 + \cdots$$  

(2.12)

Notice that in (2.12) the parameter $\frac{1}{\gamma} = \mathcal{E}_0/m$ plays the role of the square of the speed of light $c$, in that the limit $\gamma \to 0$ reproduces the non-relativistic ($c \to \infty$) expression for the kinetic energy. We imagine that the vortices will have a rather small kinetic energy as compared to $\mathcal{E}_0$. Indeed we will introduce the important effects of dissipation, by means of the Caldeira-Leggett formulation, and consider in particular the interesting case in which dissipation is more important than inertia [5]. This formally corresponds to the limit $m \to 0$ or $\gamma \to 0$ at fixed $\eta$. Note that in this case the drift velocity in the direction parallel to the electric field is, classically

$$\langle \dot{\mathbf{r}} \rangle_\parallel = \frac{eE}{\eta + B^2/\eta}$$  

(2.13)

Thus, we will implicitly look at the non-relativistic limit of a relativistic formulation. Notice that in the above formula and in the remainder of the paper we redefine the fields according to $E \to 2\pi E$ and $B \to 2\pi B$, unless otherwise stated, so that our particles have formally charges $e = \pm 1$.

The free particle relativistic expression of the energy, Eq. (2.12), corresponds to a space-time anisotropic scalar QFT with Lagrangian

$$\mathcal{L}_0(\phi) = \partial_0 \phi^* \partial_0 \phi - \frac{1}{\gamma} \nabla \phi^* \cdot \nabla \phi - \mathcal{E}_0^2 \phi^* \phi$$  

(2.14)

Notice that we have introduced a relativistic complex scalar field $\phi$ describing both vortices and antivortices as its particle-antiparticle content. Thus, for a relativistic QFT in which particle pairs corresponding to vortices and antivortices are nucleated homogeneously from an external field $A_\mu \equiv (V_\nu, A_\nu)$, the Lagrangian is given by

$$\mathcal{L}(\phi) = D_0 \phi^* D_0 \phi - \frac{1}{\gamma} D \phi^* \cdot D \phi - \mathcal{E}_0^2 \phi^* \phi$$  

(2.15)

in which we have introduced the covariant derivative $D_\mu = \partial_\mu - i A_\mu$. Therefore, we see that the relativistic formulation captures quite naturally the quantum version of a plausible situation in which the activation energy and the
inertial mass are in general unrelated in the vortex motion through the
supercurrent. We are therefore in a position to evaluate the production rate $\Gamma$
of the vortex-antivortex pairs from a uniform “electromagnetic” field $A_\mu$ as a
function of the field strengths $E$ and $B$ and of the ratio $\gamma = m/\mathcal{E}_0$.

III. AN EXACT FORMULA FOR THE VORTEX-ANTIVORTEX PRODUCTION
RATE

The calculation that follows is the two-dimensional scalar version of the
well-known Schwinger calculation for the three-dimensional quantum electro-
dynamic problem of vacuum decay by production of electron-positron pairs
that are taken to infinity upon creation \cite{17}. We evaluate the probability
amplitude for the vacuum decay in time $T$

$$Z = \langle 0 | e^{-iHT} | 0 \rangle \equiv e^{-iTW_0} \tag{3.1}$$

where $W_0 = \mathcal{E}(\text{vac}) - i\frac{T}{2}$ is taken to give the energy of the vacuum $\mathcal{E}(\text{vac})$
and its decay rate $\Gamma$. With a suitable normalization factor, the probability
amplitude is given by the functional integral over field configurations

$$Z = N \int D\phi \exp \left\{ -i \int d^2 r dt \phi^* \left( -D_0^2 + \frac{1}{\gamma} D^2 - \mathcal{E}_0^2 \right) \phi \right\} \tag{3.2}$$

which is conveniently evaluated – formally – in the Euclidean metric

$$Z = \exp \left\{ -Tr \ln \left( -\frac{1}{\gamma} D^2 - D_3^2 + \mathcal{E}_0^2 \right) \right\}$$
$$= \exp(-iTW_0) \tag{3.3}$$

by means of the known identity

$$Tr \ln \frac{-D_3^2 + \mathcal{E}_0^2}{\Lambda^2} = \lim_{\epsilon \to 0} \int_{\epsilon}^\infty \frac{d\tau}{\tau} Tr \left\{ e^{-\left( -D_3^2 + \mathcal{E}_0^2 \right)\tau} - e^{-\Lambda^2 \tau} \right\} \tag{3.4}$$
in which we have written the square of the covariant anisotropic Euclidean Laplacian (with $x_3 \equiv it$): $D_3^2 = D_3^2 + \frac{1}{\gamma}(D_1^2 + D_2^2)$.

The evaluation of the trace is straightforward with the method of the
Feynman path-integral; for the $(d+1)$-dimensional free particle the result is,
for a system of size $L^d \times T$:

$$Tr \left\{ e^{-\left( -D_3^2 \right)\tau} \right\} = \int dq_0(q_0) e^{-\left( -D_3^2 \right)\tau q_0}$$
$$= \int_{q(0) = q(\tau) = q_0} Dq(t) \exp \left\{ -\int_0^\tau dt \frac{1}{2} m_\mu \dot{q}_\mu \dot{q}_\mu \right\}$$
$$= iTL^d \left( \frac{1}{4\pi\tau} \right)^{1/2} \left( \frac{\gamma}{4\pi\tau} \right)^{d/2} \tag{3.5}$$

where we have employed a fictitious anisotropic mass term $m_\mu$: $m_1 = m_2 = \frac{\gamma}{2}$
and $m_3 = \frac{1}{2}$. Hence $W_0$ is real and, of course, there is no vacuum decay. In
the above manipulations $q$ is taken to be a $(d + 1)$-dimensional coordinate with anisotropic Euclidean metric; for the particle coupled to a gauge field case, we have the analogous path-integral construction

$$Tr \left\{ e^{-(-D_E^2)\tau} \right\} = \int dq_0 \int_{q(0)=q_0} Dq(t) e^{-\int_0^\tau dt L_E}$$

(3.6)

where, with the anisotropic fictitious mass terms, the Euclidean version of the relativistic Lagrangian for the vortex is

$$L_E = \frac{1}{2} m_\mu \dot{q}_\mu \dot{q}_\mu - i \dot{q}_\mu A_\mu(q)$$

(3.7)

The uniform-field situation corresponds to $A_\mu(q) = \frac{1}{2} F_{\mu\nu}(q)$, with the $(2+1)$-dimensional field tensor given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & B & iE_x \\ -B & 0 & iE_y \\ -iE_x & -iE_y & 0 \end{pmatrix}$$

(3.8)

after the analytic continuation to Euclidean time. The diagonalization of the operator $(-D_E^2)\tau$ is then achieved via Fourier transformation in the Euclidean-time variable:

$$\int_0^\tau dt L_E = \frac{\tau}{2\pi} \sum_n \Omega_{\mu\nu}(\omega_n) q_\mu(\omega_n) q_\nu(-\omega_n)$$

(3.9)

where

$$\Omega_{\mu\nu}(\omega_n) = \begin{pmatrix} \frac{\gamma}{4} \omega^2_n & \omega_n B & \frac{\omega_n}{2} iE_x \\ -\frac{\omega_n}{2} B & \frac{\gamma}{4} \omega^2_n & \frac{\omega_n}{2} iE_y \\ -\frac{\omega_n}{2} iE_x & -\frac{\omega_n}{2} iE_y & \frac{\omega^2_n}{4} \end{pmatrix}$$

(3.10)

The path-integral is now readily evaluated, to give

$$Tr \left\{ e^{-(-D_E^2)\tau} \right\} = iT L^2 N(\tau) \prod_{n=1}^\infty \left\{ \frac{\tau}{2\pi} \det \Omega \right\}^{-1}$$

$$= iT L^2 N(\tau) \prod_{n=1}^\infty \left\{ \frac{\omega^2_n}{4} \right\}^{3/2} \prod_{n=1}^\infty \left\{ 1 + \frac{\tau^2}{\pi^2 n^2} \left( -\frac{E^2}{\gamma} + \frac{B^2}{\gamma^2} \right) \right\}^{-1}$$

(3.11)

where the normalization factor $N(\tau)$ is adjusted so as to give the free-particle result, Eq. (3.5), in the limit of zero field. Using the known infinite-product formula $\prod_{n=1}^\infty (1 + x^2/\pi^2 n^2) = \sinh(x)/x$, we see that we get

$$Tr \left\{ e^{-(-D_E^2)\tau} \right\} = iT L^2 \gamma \left( \frac{1}{4\pi\tau} \right)^{3/2} \frac{\tau \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}}}{\sin \left( \tau \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}} \right)}$$

(3.12)

At this point, the formula for the supercurrent decay rate becomes, from (3.3), (3.4) and (3.12)
\[
\frac{\Gamma}{2L^2} = Re \int_\epsilon^\infty \frac{d\tau}{\tau} \frac{1}{i\gamma} \left( \frac{1}{4\pi\tau} \right)^{3/2} e^{-\epsilon_0^2\tau} \left\{ \frac{\tau \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}}}{\sin \left( \frac{\tau \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}}}{1/4\pi} \right)} - 1 \right\}
\]

(3.13)

where a suitable vacuum subtraction has been inserted (so that no pair production takes place in the absence of an external field). This formula can be evaluated by summing over the residues at the poles of the \(x/\sin(x)\) function, namely \(x \equiv \tau \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}} = n\pi\), except for the pole at \(x = 0\). To give meaning to the poles one attributes a small negative imaginary part to \(x\), so that \(\text{Im} \frac{1}{\sin x} = \text{Im} \left( \frac{(-1)^n}{x - \pi n - i\epsilon} \right) = (-)^n \pi \delta(x - \pi n)\). The end result is, for the pair production rate per unit area

\[
\frac{\Gamma}{L^2} = \frac{\gamma}{4\pi^2} \left( \sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}} \right)^{3/2} \sum_{n=1}^{\infty} (-)^n+1 n^{-3/2} \exp \left( -\frac{\pi \epsilon_0^2}{\sqrt{\frac{E^2}{\gamma} - \frac{B^2}{\gamma^2}}} n \right)
\]

(3.14)

The main result in this exact expression, valid for any ratio \(\gamma\), is the presence of a threshold for the decay of the supercurrent, \(E > E_c \equiv B/\sqrt{\gamma}\). This conclusion could have been obtained from the classical equation of motion, Eq. (1.1). In fact, from (1.1) the classical trajectory, without dissipation i.e. \(\eta = 0\), is a cycloid describing a motion on average orthogonal to \(E\), that is with the vortices dragged along with the current \(J\) (see Appendix). The average displacement in the direction parallel to \(E\) turns out to be \(\langle \Delta x \rangle = Em/B^2\), thus we see that a vortex can nucleate if the energy gained from the electric field can at least equal the nucleation energy: \(E\langle \Delta x \rangle \geq \epsilon_0\), which gives the above threshold condition. However, even though vortices can nucleate above threshold, the fact that there is no net transport in the direction orthogonal to the current in the absence of dissipation implies that no voltage drop will result in this case, hence no resistance.

In the following we will discuss the interesting and realistic case in which there is always friction, induced by the quantum dissipation (for a discussion of a microscopic theory see [14]). In this case particles will drift, classically and after a transient during which a few cycloidal spirals are noted, along a straight line with drift velocity in the direction of \(E\) given by Eq. (2.13). Therefore we anticipate the absence of a threshold for this case, and since a net transport in the direction of \(E\) is activated we conclude that a voltage drop will ensue.

As a final comment on the Eq. (3.14) we note that, in the absence of the Magnus field \(B\) and at the \(n = 1\) order – also remembering that below the KT transition \(\epsilon_0\) depends on \(J\) as in Eq. (2.10) – the argument of the exponential has the same leading dependence on the current \(J\) as reported in the work of Ping Ao [13].
IV. VORTEX NUCLEATION IN THE PRESENCE OF QUANTUM DISSIPATION

Much work in the physics of vortex dynamics assumes the presence of an amount of dissipation. At a quantum level, this can be treated with the formalism introduced by Caldeira and Leggett [12]. This assumes the moving quantum particle to interact with a bath of quantum harmonic oscillators having an arbitrary distribution of masses $m_k$ and frequencies $\omega_k$, the only physical constraints being that the coupling between particle coordinate $q$ and harmonic coordinates $x_k$ is linear and the classical equation of motion reproduces the form (1.1). For the problem at hand, therefore, the Lagrangian describing vortex motion in the presence of a supercurrent and dissipation is, in Euclidean time:

$$\mathcal{L}_E^D = \frac{1}{2} m_\mu \dot{q}_\mu \dot{q}_\mu - \frac{1}{2} i \dot{q}_\mu F_{\mu\nu} q_\nu + \sum_k \left\{ \frac{1}{2} m_k (\dot{x}_k^2 + \omega_k^2 x_k^2) + c_k x_k \cdot \dot{q} + \frac{c_k^2}{2m_k \omega_k^2} q^2 \right\}$$  (4.1)

where, as is well known, the last term in the oscillators Lagrangian is added to avoid the lowering of the minimum in the particle potential by means of the coupling to the external bath, and where the $c_k$ are constrained by

$$\frac{\pi}{2} \sum_k \frac{c_k^2}{m_k \omega_k} \delta(\omega - \omega_k) \equiv J(\omega) = \eta \omega$$  (4.2)

$\eta$ being the phenomenological friction coefficient. The above constraint corresponds to the so-called ohmic case for the dissipation; more generally, the spectral function can have a frequency-dependent form ($\omega_c$ is a cutoff frequency) $J(\omega) = \eta \omega^s \exp(-\omega/\omega_c)$, where $s > 1$ corresponds to the super-ohmic and $s < 1$ to the sub-ohmic cases. Although we deal mainly with the ohmic, $s = 1$ case in what follows we will also extend our results in a rather simple fashion to the non-ohmic cases in order to show how vortex-pair production is optimised in the ohmic case. After Fourier transformation, the $x_k$-modes can be integrated out, for example by means of their equation of motion in terms of the action $S_E^D = \int_0^\tau dt \mathcal{L}_E^D$

$$\frac{\partial S_E^D}{\partial x_k(\omega)} = \frac{\tau}{2\pi} \left( m_k (\omega^2 + \omega_k^2) x_k(-\omega) + c_k q(-\omega) \right) = 0$$  (4.3)

leading to the effective action

$$S_E^D = \frac{\tau}{2\pi} \sum_n \left\{ q_\mu(\omega_n) \Omega_{\mu\nu}(\omega_n) q_\nu(-\omega_n) + \sum_k \frac{c_k^2 \omega_n^2}{2m_k \omega_k^2 (\omega_n^2 + \omega_k^2)} q_a(\omega_n) q_a(-\omega_n) \right\}$$  (4.4)

where the oscillator part contains a summation over the space-like components $a=1, 2$ only of the particle coordinate. The sum over $\omega_k$ can be carried out by resorting to the constraint (1.2); in fact

$$\sum_k \frac{c_k^2}{2m_k \omega_k^2 (\omega_n^2 + \omega_k^2)} = \int_0^{+\infty} \frac{d\omega}{\omega_n^2 + \omega^2} \sum_k \frac{c_k^2 \omega^2}{2m_k \omega_k^2} \delta(\omega - \omega_k) = \frac{\eta}{2|\omega_n|}$$  (4.5)
We therefore attain the form
\[ S_E^D = \frac{\tau}{2\pi} \sum_n q_\mu(\omega_n) \Omega^D_{\mu\nu}(\omega_n)q_\nu(-\omega_n) \] (4.6)

with \( \Omega^D \) modified only in its diagonal space-like elements by the inclusion of dissipation; for \( \omega_n > 0 \)
\[ \Omega^D_{\mu\nu}(\omega_n) = \begin{pmatrix}
\frac{\gamma}{4} \omega_n^2 + \frac{\eta}{2} \omega_n & \frac{\omega_n B}{2} & \frac{\omega_i E_x}{2} \\
-\frac{\omega_n B}{2} & \frac{\gamma}{4} \omega_n^2 + \frac{\eta}{2} \omega_n & \frac{\omega_i E_y}{2} \\
-\frac{\omega_i E_x}{2} & -\frac{\omega_i E_y}{2} & \frac{\omega_n^2}{4}
\end{pmatrix} \] (4.7)

Notice that this implies that in Euclidean time dissipation processes can be represented by means of an effective action. At this point, the formula (3.13) for the decay rate of the supercurrent, in the presence of both the uniform external fields and of dissipation, makes use of the matrix determinant
\[ \det \Omega^D = \frac{\omega^2}{4} \left( \frac{\gamma}{4} \omega^2 + \frac{\eta}{2} \omega \right)^2 - \frac{\omega^2}{4} \left( \frac{\gamma}{4} \omega^2 + \frac{\eta}{2} \omega \right) E^2 + \frac{\omega^4}{16} B^2 \] (4.8)

which, adopting a factorization similar to the one employed without dissipation, leads to
\[ \frac{\Gamma}{2L^2} = \text{Re} \int_0^\infty \frac{d\tau}{\tau} i\mathcal{N}'(\tau)e^{-\epsilon^2/\tau} \left\{ \prod_{n=1}^\infty \left[ 1 - \frac{E^2}{\eta \omega_n/2 + \gamma \omega_n^2/4} + \frac{\omega_n^2 B^2}{4(\eta \omega_n/2 + \gamma \omega_n^2/4)^2} \right]^{-1} - 1 \right\} \] (4.9)

with the normalization factor \( \mathcal{N}'(\tau) \) given by
\[ \mathcal{N}'(\tau) = \mathcal{N}(\tau) \prod_{n=1}^\infty \left\{ \frac{\omega_n^2}{4} \left( \frac{\eta}{2} \omega_n + \frac{\gamma}{4} \omega_n^2 \right)^2 \right\}^{-1} = \gamma \left( \frac{1}{4\pi \tau} \right)^{3/2} \prod_{n=1}^\infty \left( 1 + \frac{2\eta}{\gamma \omega_n} \right)^{-1} \] (4.10)

Notice that we have explicitly rewritten the vacuum subtraction term in such a way that no pair production takes place in the absence of an external field, \( E = B = 0 \). In this case a closed-form expression for \( \Gamma \) does not appear possible; nevertheless the leading contribution to the \( \tau \)-integral can be found from the residue of the main, \( n = 1 \) pole of its argument. No relevant pole arises from \( \mathcal{N}'(\tau) \), hence the main contribution is from \( \omega = \omega_1 = 2\pi/\tau_1 \) which is the real positive zero of
\[ 1 - \frac{E^2}{\eta \omega/2 + \gamma \omega^2/4} + \frac{\omega^2 B^2}{4(\eta \omega/2 + \gamma \omega^2/4)^2} = 0 \] (4.11)

For given \( B, \eta \) and \( \gamma \), this cubic equation has a real zero for any infinitesimal electric field \( E \), as indeed for \( E \to 0 \) we get
\[ \omega_1 = \frac{2\eta}{\eta^2 + B^2} E^2 + \cdots \] (4.12)

The physical meaning of this is that while in the absence of dissipation, with \( \omega_1 = 2\sqrt{B^2 - \eta^2} \), a real root appears only for \( E > B/\sqrt{\gamma} \), now any infinitesimal \( E \), therefore any infinitesimal supercurrent, gives rise to vortex
pair production and decay. The main consequence of (ohmic) dissipation is therefore the elimination of the threshold for a non-vanishing $\Gamma$, as was also noted by, e.g., Ao and Thouless [11]. Noticing that $\omega_1$ is independent of $\gamma$, we work out the final formula in the “dissipaton-dominated” case, that is in the limit $\gamma \to 0$ also considered by Stephen [12], although a less instructive analytic result could be obtained for arbitrary $\gamma$ to leading order in $E^2$. The formula for $\Gamma$ is first of all rewritten, more conveniently, as

$$\frac{\Gamma}{2L^2} = Re \int_\epsilon^\infty \frac{d\tau}{\tau} i \gamma \left( \frac{1}{4\pi \tau} \right)^{3/2} \prod_{n=1}^{\infty} \left( 1 + 4 \frac{B^2 + \eta^2 + \gamma(\eta \omega_n - E^2)}{\gamma^2 \omega_n^2} \right)^{-1} \cdot e^{-\gamma^2 \tau} \left[ \prod_{n=1}^{\infty} \left( 1 - \frac{8\eta E^2}{\gamma^2 \omega_n^3 + 4\omega_n (B^2 + \eta^2 + \gamma(\eta \omega_n - E^2))} \right)^{-1} - 1 \right]$$

(4.13)

which for $\gamma \to 0$ simplifies to

$$\lim_{\gamma \to 0} \frac{\Gamma}{2L^2} = Re \int_\epsilon^\infty \frac{d\tau}{\tau} i \gamma (4\pi \tau)^{-3/2} \prod_{n=1}^{\infty} \left( 1 - \frac{2\eta E^2}{\omega_n (B^2 + \eta^2)} \right)^{-1}$$

(4.14)

with the following asymptotic form of the normalization factor

$$\mathcal{N}(\tau) = \gamma (4\pi \tau)^{-3/2} \prod_{n=1}^{\infty} \left( 1 + 4 \frac{B^2 + \eta^2}{\gamma^2 \omega_n^2} \right)^{-1}$$

$$\to \gamma (4\pi \tau)^{-3/2} \frac{2\tau}{\gamma} e^{-\frac{\tau}{2\gamma} \sqrt{B^2 + \eta^2}}$$

(4.15)

in which the limit form $x/\sinh(x) \to 2xe^{-x}$ has been used. The leading pole has the precise form (4.12), but for arbitrary $E$, and evaluating the residue at this pole the formula for $\Gamma$ reads, with $\mathcal{E}_0^2 = \mathcal{E}_0^2 + \frac{1}{2}\sqrt{B^2 + \eta^2}$

$$\lim_{\gamma \to 0} \frac{\Gamma}{2L^2} = 2\pi \tau \left( 1 + \frac{2\eta E^2}{\sqrt{B^2 + \eta^2}} \right)^{3/2} \sqrt{B^2 + \eta^2} \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n} \right)^{-1}$$

(4.16)

The last infinite product is clearly divergent unless a frequency cutoff is introduced, $\omega_n \leq \omega_c$ where $\omega_c$ is a large frequency above which the effects of dissipation can be neglected [13]. This implies the existence of a cutoff integer $n^* = [\omega_c/\omega_1]$ and the last factor will thus be a cutoff-dependent dimensionless number $C(n^*) = C^*$. For small $E^2$ we see that $n^* = \frac{2\eta}{\sqrt{\eta^2 + B^2} E^2}$ is large and in this case we can estimate the asymptotic behaviour of $C^*$ by

$$C^* = \prod_{n=2}^{n^*} \left( 1 - \frac{1}{n} \right)^{-1} \to \exp -n^* \int_{2/n^*}^{1} dx \ln \left( 1 - \frac{1}{n^* x} \right) \sim \frac{e}{4} n^* = \frac{e\eta}{\eta^2 + B^2} \omega_c$$

(4.17)

If, on the other hand, $E^2$ is very large and formally $n^*$ shrinks to zero, it would mean that our situation is such that the Caldeira-Leggett dissipative regime breaks down. We assume here to be always in a range of $E$ compatible with the Caldeira-Leggett regime, thus $n^*$ can be either large (corresponding
to $C^*$ given by (4.17) or finite (corresponding to $C^*$ of order unity). The final asymptotic form for the pair production rate when $\gamma \to 0$ will be, therefore, substituting $\tau_1 = 2\pi/\omega_1$ from Eq. (4.12) in Eq. (4.16),

$$\lim \frac{\Gamma}{L^2} \sim \frac{1}{2\pi} C^* \sqrt{\eta E} \exp \left\{-\mathcal{E}_{0R}^2 \frac{\pi (B^2 + \eta^2)}{\eta E^2} \right\}$$

(4.18)

where we have defined the “renormalized” activation energy

$$\mathcal{E}_{0R}^2 = \mathcal{E}_0^2 + \frac{1}{\gamma} \sqrt{B^2 + \eta^2}$$

(4.19)

This has a physical interpretation connected with the energy $B/2m$ of the lowest Landau level which becomes infinite in the limit of zero inertial mass and contributes a diverging zero-point energy to the total activation energy. Indeed, for $\eta = 0$, $\mathcal{E}_{0R} \sim \mathcal{E}_0 + B^2/2m$. If an infinite barrier for placing a vortex in the lowest Landau level does really develop, then vortex pair production will cease and there will be no supercurrent decay. We therefore generically assume in the following that $\mathcal{E}_{0R}$ is ultimately just a phenomenological finite-value parameter of the theory.

At this point we can generalise the results obtained above for the ohmic, $s = 1$, case to the non-ohmic, $s \neq 1$, situations (for a recent discussion on the relevance of non-ohmic cases, see [18]). Indeed, Eq. (4.12) holds good if $J(\omega) = \eta \omega^s$ provided we replace $\eta$ with a frequency-dependent friction coefficient $\eta(\omega) = \eta \omega^p$, with $p = s - 1$. Then we have the following equation for the main pole $\omega_1 = 2\pi/\tau_1$

$$\omega_1 \sim \frac{2\eta \omega_1^p}{\eta^2 \omega_1^{2p} + B^2 E^2 + \cdots}$$

(4.20)

which we can qualitatively handle in the limit $E \to 0$. First, we observe from Eq. (4.20) that the role of $B$ is enhanced in the super-ohmic case, while it is suppressed if dissipation is sub-ohmic, in agreement with [1]. Then, both in the super-ohmic $p > 0$ and in the sub-ohmic $p < 0$ cases, we get that $\omega_1 \sim E^{2/(1-|p|)}$ so that in the relevant formula for $\Gamma$, e.g. Eq. (4.18), we must substitute $E$ with $E^{1/(1-|p|)}$. For small $E$ this leads to a suppression of the vortex production rate as soon as $p \neq 0$, compared to the ohmic case which therefore represents the situation with the maximum production rate.

To conclude this Section, we point out that the argument of the exponential appearing in Eq. (4.18) has a direct physical interpretation. Apart from constants, it is the Action ($=\text{Energy} \times \text{Time}$) required for the nucleation of a vortex, since $\text{Energy} = \mathcal{E}_{0R}$ and $\text{Time} = \ell_N/v_N = \frac{\mathcal{E}_{0R}}{E} \frac{\eta(B)}{E}$. Indeed, $\ell_N = \frac{\mathcal{E}_{0R}}{E} \frac{\eta(B)}{E}$ is the distance involved for the nucleation in a field $E$ whereas $v_N = \frac{E}{\eta(B)}$ is the drift velocity (if we indicate $\eta(B) = (\eta^2 + B^2)/\eta$).
V. VORTEX NUCLEATION IN THE PRESENCE OF A PINNING POTENTIAL

Most superconducting materials in which vortex dynamics plays a role contain a finite density of defects of some sort. Many discussions in the literature deal with the tunneling effect of vortices trapped by a potential barrier on the resistance of a superconductor. A finite height and width of the pinning barrier in the direction of the “electric field” $E$, that is perpendicular to the current $J$, ultimately give rise to a finite additional term which renormalises $\mathcal{E}_{0R}$. Pinning barriers in the direction of the current have instead a different physical effect, in particular in the dissipation-dominated case. In our own relativistic formulation we can account for an additional effect due to pinning if we consider the “extreme” case of the unbounded harmonic-well potential

$$U(q) = \frac{1}{2} k_x q_x^2 + \frac{1}{2} k_y q_y^2$$  \hspace{1cm} (5.1)

which we add to our quantum field Lagrangian through the term $U(r)\phi^*\phi$. Strictly-speaking, in our relativistic treatment this amounts to a confinement force that, while linear in $q$ at small distances, at large distances remains constant and thus simulates the presence of a uniform “confinement pressure” on the vortices. Thus, in this “extreme” case a threshold for the pair production may reappear. In what follows, we first treat the case of the unbounded potential in the absence of dissipation, and then discuss the more realistic case of a finite potential barrier. Technically, with the form (5.1) we can formally repeat the calculation of the previous Sections, with a quadratic form in the path integral defined by the matrix

$$\tilde{\Omega}_{\mu\nu}(\omega_n) = \begin{pmatrix} \frac{\gamma}{4} \omega_n^2 + \frac{\eta}{2} \omega_n + \frac{1}{2} k_x & \frac{\omega_n}{2} B & \frac{\omega_n}{2} i E_x \\ -\frac{\omega_n}{2} B & \frac{\gamma}{4} \omega_n^2 + \frac{\eta}{2} \omega_n + \frac{1}{2} k_y & \frac{\omega_n}{2} i E_y \\ -\frac{\omega_n}{2} i E_x & -\frac{\omega_n}{2} i E_y & \frac{\omega_n^2}{4} \end{pmatrix}$$  \hspace{1cm} (5.2)

Having the limit $\gamma \rightarrow 0$ always in mind, we begin the discussion of this extreme case of pinning by considering the absence of dissipation, $\eta = 0$. Then, with a suitable factorization for $\text{det} \, \tilde{\Omega}$ we can write, for the pair production rate and in the limit $\gamma \rightarrow 0$

$$\lim \frac{\Gamma}{2L^2} = Re \int_{\mathcal{C}} d\tau e^{-\gamma^2 \tau^2} \left\{ \prod_{n=1}^{\infty} \left( 1 + \frac{k_x k_y - 2(k_x E_y^2 + k_y E_x^2)}{\omega_n^2 B^2} \right) \right\}^{-1} - \text{const.} \right\}$$  \hspace{1cm} (5.3)

The normalization factor has again a singular limit form

$$\mathcal{N}'(\tau) = \gamma \left( \frac{1}{4\pi \tau} \right)^{3/2} \prod_{n=1}^{\infty} \left\{ 1 + \frac{4}{\gamma^2 \omega_n^2} \left( \frac{k_x + k_y}{2} - \gamma E^2 + B^2 \right) \right\}^{-1}$$

$$\rightarrow \left( \frac{1}{4\pi \tau} \right)^{3/2} 2\tau B e^{-\gamma^2 \tau}$$  \hspace{1cm} (5.4)

in which we see that the very same quantum zero-point energy fluctuations in the Landau levels give rise to a divergent renormalization in the rest mass.
for $\gamma \rightarrow 0$: $\mathcal{E}_0^2 \rightarrow \mathcal{E}_0^{2_R} = \mathcal{E}_0^2 + B/\gamma$. The remainder of the integral in $\tau$ can be evaluated as in Section III in this limit, and we end up with the exact formula

$$
\lim \frac{E}{\mathcal{E}_0^{2+}} = \frac{1}{2} \left( k_x E_y^2 + k_y E_x^2 - k_x k_y/2 \right)^{1/4} 
\cdot \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n - 1/2} \exp \left\{ -\pi B \mathcal{E}_0^{2_R} \sqrt{\frac{k_x E_y^2 + k_y E_x^2 - k_x k_y/2}{k^2/2}} \right\}
$$

in which we see that $\Gamma$ is zero unless the condition $2E_x^2/k_x + 2E_y^2/k_y > 1$ is satisfied, that is no production of vortex-antivortex pairs takes place inside an ellipse in $(E_x, E_y)$ having half-axes $\sqrt{k_x/2}$ and $\sqrt{k_y/2}$. The physical interpretation of this phenomenon is that the threshold in the minimum value of the electric field for a non-vanishing pair production is now restored by confinement. Indeed, while particles get trapped in stable Landau levels for a vanishing inertial mass, under the action of a pinning potential the trapping role of the magnetic field is restricted to a renormalization of the rest mass. Transitions out of these levels can occur for strong enough electric fields since the situation becomes now a quasi-static equilibrium between the confinement and the electrostatic forces, the dynamics being separated away with the magnetic field term. Notice, however, that for $E_y = 0$ and $k_x = 0$ (current and confinement both in the $y$-direction) there is a non-vanishing rate $\Gamma$ for any $E_x \neq 0$, hence there is no threshold. Nevertheless, no net transport takes place in the direction of the electric field and the supercurrent does not decay if dissipation is absent (see Appendix).

We now consider the case in which ohmic dissipation is present in a more realistic setting, by specializing to $E_y = k_x = 0$. In fact, one can take into account a finite-width pinning barrier in the $x$-direction by a renormalization of $\mathcal{E}_0$, but a new qualitative feature appears if pinning is in the direction of the current, $J = -\times \mathbf{E}$. We adopt the following factorization for the matrix determinant:

$$
\det \tilde{\Omega}^D = \frac{\omega^2}{4} \left( \frac{\gamma}{\omega^2} \right)^2 \left\{ 1 + \frac{4\eta^2 + 4B^2 + \gamma(4\eta \omega + 2k_y - 4E^2)}{\gamma^2 \omega^2} \right\} 
\times \left\{ 1 + \frac{(4k_y - 8E^2) \eta \omega^{-1} - 8k_y E^2 \omega^{-2}}{\gamma^2 \omega^2 + 4\eta^2 + 4B^2 + \gamma(4\eta \omega + 2k_y - 4E^2)} \right\}
$$

since for $\gamma \rightarrow 0$ this becomes

$$
\lim \det \tilde{\Omega}^D = \frac{\omega^2}{4} \left( \frac{\gamma}{\omega^2} \right)^2 \left\{ 1 + \frac{B^2 + \eta^2}{\gamma^2 \omega^2} \right\} 
\times \left\{ 1 + \frac{1}{B^2 + \eta^2} \left[ \frac{k_y - 2E^2}{\omega} \eta + \frac{-2k_y E^2}{\omega^2} \right] \right\}
$$

(5.7)

Here we see that the prefactor is of the type already seen for dissipation, hence it gives rise to a nucleation energy renormalization of the type (4.19). Denoting
\[
\tilde{B}^2 = B^2 + \eta^2 \\
\tilde{E}^2 = E^2 - \frac{1}{2} k_y \\
\tilde{K}^2 = \frac{1}{2} k_y E^2
\]

we see that the dominant pole in the integral formula for \( \Gamma \) is the solution of \( \tilde{B}^2 \omega_1^2 - 2\eta \tilde{E}^2 \omega_1 - 4\tilde{K}^2 = 0 \). The only relevant principal pole is then for

\[
\omega_1 = \frac{2\pi}{\tau_1} = \left( \eta \tilde{E}^2 + \sqrt{\eta^2 \tilde{E}^4 + 4\tilde{B}^2 \tilde{K}^2} \right) / \tilde{B}^2
\]

for which the residue is readily calculated and yields the main contribution to the vacuum decay rate

\[
\lim L^2 \Gamma \sim 2\pi e^{-\tilde{E}^2 \tau_1} \left( \frac{1}{4\pi \tau_1} \right)^{3/2} \sqrt{B^2 + \eta^2}
\]

where \( C^* \) is expressed by an infinite product, cutoff at \( n^* = \omega_c / \omega_1 \), similarly to what seen in Section IV. The formula for the pole, Eq. (5.9), reveals its physical contents in the limit in which the \( E^2 \ll k_y \). Expanding the square-root to lowest order, we always obtain a pole for small \( E \)

\[
\omega_1 = \frac{2E^2}{\eta}
\]

which is of the same form as Eq. (4.12) for the purely dissipative case, except that – due to pinning – the renormalization effect induced by the Magnus field on the friction coefficient has disappeared. This result is in agreement with the analysis based on the classical equations presented in the Appendix, which, furthermore, shows that Eq. (5.11) holds (for small \( E \)) also for a finite-height and finite-width pinning barrier. In the general case in which the pinning centres will be distributed with some density, this will correspond to an effective friction coefficient that will present a dependence on \( B \) intermediate between the two cases: \( \eta < \eta_{eff} < (\eta^2 + B^2) / \eta \).

VI. AN APPROACH TO THERMAL EFFECTS AND CONCLUSIONS

To complete the picture one would need to evaluate the vortex nucleation rate under the effect of both the external current \( J \) (resulting in the “electric” field \( E \)) and the temperature \( T \) in order to estimate the voltage drop. This appears to be a rather difficult task, since it would involve a theoretical description of how the thermal distribution of vortices is obtained from a state with no vortices, a process of non-equilibrium thermodynamics.

Here we will take a less ambitious view and try to obtain a phenomenological description of the process of vortex pair production in the presence of thermal fluctuations. By a detailed balance argument, to be developed below, one finds that the density of free vortices \( \rho_f \) is proportional to the
square root of the production rate, $\rho_f \sim \sqrt{\Gamma}$. Considering the relevant case of the dissipation-dominated dynamics described in Section III, we tentatively propose to simulate the effect of temperature on the vortex production by introducing thermal currents $J(T)$ (induced for instance by phonons, or otherwise) and using the result of Eq. (4.18), with $E^2 = (2\pi J(T))^2$, namely $\Gamma \sim \exp \left[ -\mathcal{E}_0 / \sqrt{\eta_{\text{eff}}} J(T) \right]$. The requirement of obtaining a Boltzmann distribution for $\rho_f$ fixes $J^2(T) = \frac{1}{8\eta_{\text{eff}}} \mathcal{E}_0 J(T)$. In the general case, where we have both temperature $T$ and external current $J$, we will use Eq. (4.18) with $E^2 = (2\pi J_{\text{tot}})^2$ by adding incoherently the two contributions: $J_{\text{tot}}^2 = J(T)^2 + J^2$. We can thus rewrite, in the general case

$$\frac{\Gamma}{L_x L_y} \sim C^* \sqrt{\eta_{\text{eff}}} J_{\text{tot}} \exp \left[ -2\mathcal{E}_0 / (T + \Delta T(J)) \right]$$

(6.1)

where $L_{x,y}$ are the linear sizes of the sample; also

$$\Delta T(J) = \frac{8\pi J^2}{\eta_{\text{eff}} \mathcal{E}_0 R} \sim t_n(J)^{-1}$$

(6.2)

and we have indicated that $\Delta T(J)$ is proportional to the inverse of the nucleation time for a vortex (see the discussion at the end of Section IV). Let us recall that below the KT transition $\mathcal{E}_0$ depends on $J$ (see Eq. (2.10)) and in this dependence we include just the external coherent current as $<J_{\text{tot}}>$ = $J$. Therefore, vortices are not produced for zero current and temperatures below the KT transition.

We can then compute the free vortex number density $\rho_f$ by a detailed balance argument: the production rate must equal the annihilation rate. Neglecting the rate at which the vortices disappear from the sample due to the drift described by Eq. (2.13), thus assuming macroscopic dimensions for the sample, the annihilation rate per unit area will be due to two mechanisms:

1) A possible vortex-antivortex annihilation, and this rate will be proportional to the product of an annihilation length $\sigma$ (the 2-D analogue of the familiar 3-D cross section), times the density of the free vortices $\rho_f$, times the incident flux $\rho_f 2 <v>$, where the average drift velocity will be given, according to (2.13), by $<v> = 2\pi J_{\text{tot}} / \eta_{\text{eff}}$. Hence:

$$\frac{\Gamma_{Af}}{L_x L_y} \sim 2\rho_f^2 <v> / \sigma$$

(6.3)

2) A possible annihilation of vortices with pinned antivortices, with a rate

$$\frac{\Gamma_{Ap}}{L_x L_y} \sim \rho_f \rho_p \langle v \rangle / \sigma$$

(6.4)

where $\rho_p$ is the density of pinned vortices. In turn, at equilibrium we will have $\rho_f \Gamma_p = \rho_p \Gamma_u$, where $\Gamma_p$ and $\Gamma_u$ are the rates for pinning and unpinning, respectively. In conclusion, the detailed balance will be
\[
\frac{\Gamma}{L_x L_y} = \rho_f^2 \frac{2\pi J_{tot}}{\eta_{eff}} (2 + \Gamma_p/\Gamma_u) \quad (6.5)
\]

which gives
\[
\rho_f = \left[ \frac{C^* \eta^{\frac{1}{2}}}{2\pi \sigma (2 + \Gamma_p/\Gamma_u)} \right]^{\frac{1}{2}} \exp -\frac{\mathcal{E}_{0R}}{T + \Delta T(J)} \quad (6.6)
\]

Thus, the vortex current \( J_v \) in the \( x \)-direction orthogonal to the supercurrent density \( \mathbf{J} \) will be \( J_v = 2\pi \rho_f J/\eta_{eff} \), yielding the potential drop \( \Delta V = L_x J_v \) in the \( y \)-direction parallel to \( \mathbf{J} \). Thus:
\[
\Delta V = 2\pi L_x J \left[ \frac{C^* \eta^{\frac{1}{2}}}{\eta_{eff} \sigma (2 + \Gamma_p/\Gamma_u)} \right]^{\frac{1}{2}} \exp -\frac{\mathcal{E}_{0R}}{T + \Delta T(J)} \quad (6.7)
\]

For small external currents such that \( \Delta T(J) \ll T \) one gets, above the KT transition, in terms of the current \( I = L_y J \)
\[
\Delta V \sim R(T) I + Q(T) I^3 + \cdots \quad (6.8)
\]

where
\[
R(T) = 2\pi \frac{L_x}{L_y} \left[ \frac{C^* \eta^{\frac{1}{2}}}{\eta_{eff} \sigma (2 + \Gamma_p/\Gamma_u)} \right]^{\frac{1}{2}} e^{-\mathcal{E}_{0R}/T}
\]
\[
Q(T) = \frac{R(T)}{L_y^2} \frac{8\pi}{\eta_{eff} T^2} \quad (6.9)
\]

It is to be stressed at this point that the above considerations have led us to a rather predictable result: there is no resistance in the sample if dissipation is absent. As we have previously pointed out, this qualitative conclusion could have been obtained from the classical equations of motion, Eq.s (1.1) and (2.13) (see Appendix). We recall that, as discussed at the end of Section V and in the Appendix, in the general case of a distribution of pinning barriers one will have \( \eta < \eta_{eff} < (\eta^2 + 4\pi^2 \rho_s^2)/\eta \) (remembering that \( B = 2\pi \rho_s \)).

We finally point out that, for temperatures below the KT transition, \( \mathcal{E}_{0R} \) depends on \( J \), as in Eq. (2.10). Thus, for low \( J \) such that \( \Delta T(J) \ll T \), we recover the standard result that \( \Delta V \sim I^{1+a} \), where \( a \rightarrow 2 \) for \( T \rightarrow T_{KT} \) [19]. Our formula includes dynamical effects due to current-induced nucleation. In particular, by expanding for low \( J \), we can repeat the above calculations to arrive at the following dependence on \( J \)
\[
\Delta V \sim c_1 I^{1+a} + c_2 I^{3+a} \left( \ln \frac{m_0 J}{\rho_s a} \right)^2 + \cdots \quad (6.10)
\]

where \( c_1 \) and \( c_2 \) have expressions similar to those reported in (6.3).

To conclude, we have presented a plausible physical scenario for the applicability of our exact formalism giving the decay rate of a superconducting
current and thus the resistance, or voltage drop, observable in a real sample. Thermal effects have been accounted for phenomenologically through thermally-induced currents, and pinning forces by means of a harmonic potential. We find that in some extremal cases there is a threshold for the production of vortex-antivortex pairs, in that these are trapped in Landau or harmonic-oscillator levels, but that the threshold is eliminated, if extremal pinning is absent, by the addition of any infinitesimal amount of dissipation. In the case of finite pinning, or even infinite-pinning barrier in the direction of the current, the threshold disappears. Pinning barriers in the direction of the current suppress the effect of the Magnus force (confined to a renormalization of the vortex nucleation energy). The limit of a vanishing inertial mass has been shown to lead to considerable simplification in the algebra, as well as to new physical effects like the divergent renormalization of the vortex nucleation energy, resulting in the inhibition of vortex production for all the cases considered when the inertial mass is strictly zero. Further work is needed to account for the thermal effects in the nucleation of vortices within a fully microscopic theoretical approach.
APPENDIX

Here we summarize some relevant results from the solution of the classical equations of motion for the vortex dynamics, Eq. (1.1) (for charges with $e^2 = 1$). We begin by recalling that in the presence of the fields $E = E \hat{x}$ and $B = B \hat{z}$ the solution corresponding to the initial conditions $x(0) = y(0) = 0$ and $\dot{x}(0) = \dot{y}(0) = 0$ is of the form

$$x(t) = \frac{mE}{eB^2}(1 - \cos \omega_c t)$$
$$y(t) = -\frac{mE}{eB^2} \sin \omega_c t + \frac{E}{B} t$$

which represents a cycloid with frequency $\omega_c = eB/m$. This shows that the average displacement along the electric field’s $x$-direction is fixed at $\langle x(t) \rangle = x_0 = mE/eB^2$ so that there is no transport in the direction orthogonal to the current, hence no voltage drop. Also, we require the field to be such that $Ex_0 \geq E_0$ to extract the particle from its classical trajectory, in agreement with our conclusion on the existence of a threshold in Section III. If on the contrary friction is switched on, the particle will drift after a transient along a straight line with a drift velocity in the $x$-direction given precisely by Eq. (2.13); in this case transport is activated and a voltage drop ensues. Indeed, with the same initial conditions as above, the motion in the presence of friction is represented by

$$x(t) = \frac{meE}{\eta^2 + B^2} \left\{ e^{-\left(\frac{\eta}{m}\right) t} \cos(\omega_c t - 2\alpha) - \cos 2\alpha \right\} + \frac{\eta eE}{\eta^2 + B^2} t$$
$$y(t) = \frac{meE}{\eta^2 + B^2} \left\{ e^{-\left(\frac{\eta}{m}\right) t} \sin(\omega_c t - 2\alpha) + \sin 2\alpha \right\} + \frac{BE}{\eta^2 + B^2} t$$

where $\tan \alpha = \eta/B$, showing that (after a transient damped-cycloidal motion that can be neglected for $t \geq m/\eta$) we have an average $x$-component of the velocity $\langle \dot{x}(t) \rangle = eE/\eta(B)$ with $\eta(B) = (\eta^2 + B^2)/\eta$ as in Eq. (2.13). Also, there is no threshold value of $E$ for the vortex-antivortex nucleation.

Next we consider what might be, classically, the main effect of pinning on the vortex motion. We introduce a pinning potential approximately described by $U(x, y) = \frac{1}{2}k_x(x - x_p)^2 + \frac{1}{2}k_y(y - y_p)^2$ for $|x - x_p| < \ell_x$ and $|y - y_p| < \ell_y$ and zero otherwise, where $(x_p, y_p)$ are the coordinates of the pinning centre. We imagine that there will be many pinning centres in the sample, but discuss the local dynamics of a vortex around one of the impurity sites. Our particles, i.e. the vortices, will feel a potential barrier around $(x_p, y_p)$. Like in Section V, we assume that the potential barrier in the direction of the electric field $E = E \hat{x}$ will have the main effect of increasing the activation energy $E_0R$, thus renormalising it by way of the addition of a further term. We can thus treat this part of the problem phenomenologically. The barrier in the $y$-direction has a different implication for the dynamics. Consider first the frictionless case, $\eta = 0$. The general solution of the classical equation, including the force due to $U(y)$, is
\[ x(t) - x_p = -\frac{\omega_c}{\omega} A \cos \omega \tau + \frac{1}{2 m k_y} + B^2 \tau^2 + \frac{k_y e E}{m k_y + B^2} y_0 \tau + x_0 \]
\[ y(t) - y_p = -A \sin \omega \tau + \frac{E B}{m k_y + B^2} \tau + y_0 \]

(A.3)

where \( \tau = t - t_0 \) and \( \omega = \sqrt{\omega_c^2 + k_y/m} \). We see that, after averaging over the oscillations and with general initial conditions, \( \langle y(t) - y_p \rangle = \frac{E B}{m k_y + B^2} \tau + y_0 \) and \( \langle x(t) - x_p \rangle = x_0 + \frac{k_y}{e B} y_0 \tau + \frac{1}{2} \frac{k_y e E}{m k_y + B^2} \tau^2 \), so that under the effect of pinning in the \( y \)-direction \( \langle x(t) - x_p \rangle \) can reach, after a suitable time, a large enough value and allow the vortex to gain nucleation energy from the electric field. Depending on the values of the parameters this can occur within the pinning potential width \( \ell_y \), in which case the pinning in the \( y \)-direction removes the threshold for vortex nucleation which occurred without friction and pinning. However, the classical equations indicate that even in the present case there is no net transport in the \( x \)-direction if there is no friction. In fact, the previous solution holds up to \( |\langle y(t) - y_p \rangle| \approx \ell_y \), beyond which either there is no longer any pinning, and thus the motion will fall back on the familiar cycloid (resulting in no transport), or else the vortex will feel the effect of another pinning centre at \((x'_p, y'_p)\). In the latter case we will have, at the initial time \( t_0 \) of the new piece of the trajectory, \( \langle y(t_0) - y'_p \rangle = -\ell'_y \). Therefore, at the later time \( t_1 \) for which \( \langle y(t_1) - y'_p \rangle = \ell'_y \) (that is as the vortex moves through the \( y \)-range of the new pinning centre from \( y = y'_p - \ell'_y \) to \( y = y'_p + \ell'_y \)), we find that \( \langle x(t_1) - x'_p \rangle = \langle x(t_0) - x'_p \rangle \). The conclusion is that there has been no net transport in the \( x \)-direction. The situation is completely different if there is friction, \( \eta \neq 0 \), as can be easily seen by solving the classical equations by formally taking \( m \ddot{x} = 0 \) and \( m \ddot{y} = 0 \) for long times and adding the pinning force due to \( U(y) \). The solution is that, for long times \( (t > m/\eta) \), \( \langle \dot{x}(t) \rangle = eE/\eta \) whilst \( \langle y(t) - y'_p \rangle = BE/\eta \). Thus, there is always the possibility of nucleation and transport. Comparing with the case without pinning, we see that the drift velocity in the \( x \)-direction is now determined by the actual friction coefficient \( \eta < \eta(B) = (\eta^2 + B^2)/\eta \), whilst in \( y \) one has simply a displacement (thus, for low \( E \) at least, \( y(t) \) will remain in the range where \( U(y) \neq 0 \)). In the general case, where there can be pinning centers distributed with some density, the effective friction coefficient \( \eta_{\text{eff}} \) will be somehow in between the extreme cases, \( \eta < \eta_{\text{eff}} < (\eta^2 + B^2)/\eta \).

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