On divisibility by primes in columns of character tables of symmetric groups

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Abstract. For an arbitrary prime $p$, we prove that the proportion of entries divisible by $p$ in certain columns of the character table of the symmetric group $S_n$ tends to 1 as $n \to \infty$. This is done by finding lower bounds on the number of $k$-cores for $k$ large enough with respect to $n$.

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1. Introduction. In [3], Miller formulated the following conjecture about the character table of symmetric groups:

Conjecture 1. Let $p$ be a prime and $E_p(n)$ be the number of entries divisible by $p$ in the character table of $S_n$. Then $E_p(n)/(p(n))^2 \to 1$ as $n \to \infty$.

Here, as in the rest of the paper, $p(n)$ is the number of partitions of $n$.

In [1], Gluck proved that in certain columns of the character table of $S_n$, the proportion of even entries tends to 1. The main results of this paper extend this to a larger set of columns of the character table of $S_n$ and hold for any prime $p$. These results however are not sufficient to prove Conjecture 1. In order to state our main results, we need the following notation. Let $P(n)$ be the set of all partitions of $n$ and

$$\Omega_p(n) := \{ \text{partitions of } n \text{ into parts not divisible by } p \}.$$

Further, for any partition $\mu$ of $n$, let $\mu^* \in \Omega_p(n)$ be obtained from $\mu = (\mu_1, \mu_2, \ldots)$ by replacing each part $\mu_i = p^{k_i} a_i$ with $p \nmid a_i$ by $p^{k_i}$ parts $a_i$. Moreover, for $\lambda \in \Omega_p(n)$, let

$$K_p(\lambda) := \{ \mu \in P(n) : \mu^* = \lambda \}.$$

For $\alpha, \beta \in P(n)$, let $\chi^\alpha$ be the irreducible character of $S_n$ indexed by $\alpha$ and $\chi^\alpha_\beta$ be the value that $\chi^\alpha$ takes on the conjugacy class with cycle partition $\beta$. 
Lemma 4. Let $p$ be a prime, $n \geq 2$, $c > \sqrt{3/2} \pi$, and $\lambda = (a_1^1, \ldots, a_h^h) \in \Omega_p(n)$. Assume that for some $1 \leq i \leq h$, there exists $s$ with $p^s - b_i$ and $a_i p^s \geq c \sqrt{n} \log(n)$. Then for any $\mu \in K_p(\lambda)$, we have that
\[
\frac{|\{\alpha \in P(n) : \chi_\mu^\alpha \equiv 0 \mod p\}|}{p(n)} \geq 1 - \frac{c^4 d \log(n)}{n^{c\pi/\sqrt{6} - 1/2}}
\]
for some constant $d$.

Note that $\frac{1}{n^{c\pi/\sqrt{6} - 1/2}} \to 0$ since $c > \sqrt{3/2} \pi$.

Corollary 3. Let $p$ be a prime, $n \geq 2$, $c > \sqrt{3/2} \pi$, and $\lambda = (a_1^1, \ldots, a_h^h) \in \Omega_p(n)$. If $h \leq \frac{\sqrt{n}}{cp \log(n)}$, then, for any $\mu \in K_p(\lambda)$, we have that
\[
\frac{|\{\alpha \in P(n) : \chi_\mu^\alpha \equiv 0 \mod p\}|}{p(n)} \geq 1 - \frac{c^4 d \log(n)}{n^{c\pi/\sqrt{6} - 1/2}}
\]
for some constant $d$.

Corollary 3 easily follows from Theorem 2 since under the assumptions of Corollary 3 there exists $i$ with $a_i b_i \geq cp \sqrt{n} \log(n)$ and then the assumptions of Theorem 2 are satisfied.

If $\mu, \nu \in K_p(\lambda)$, then the two columns of the character table of $S_n$, corresponding to conjugacy classes with cycle partitions $\mu$ and $\nu$ are congruent modulo $p$ (see [3, Proposition 1]). In particular, the numbers of character values divisible by $p$ in the two columns are equal. This explains why Theorem 2 and Corollary 3 only have assumptions on $\lambda = \mu^*$ and not on $\mu$.

2. Proof of Theorem 2. Given a positive integer $k$ and a partition $\gamma$, we say that $\gamma$ is a $k$-core if $\gamma$ has no hook of length divisible by $k$. For any partition $\beta$ of $n$ and a positive integer $k$, one can define its $k$-core partition $\gamma$ to be the partition obtained from $\beta$ by recursively removing as many $k$-hooks as possible ($\gamma$ does not depend on which maximal sequence of $k$-hooks is removed from $\beta$), thus $|\beta| = |\gamma| + mk$ for a certain non-negative integer $m$ (see for example [4, Section 3]).

For any integer $m \geq 0$, let $p_k(m)$ be the number of multipartitions of $m$ into $k$ parts. For any non-negative integer $m$ and any $k$-core partition $\gamma$ of $n - km$, the number of partitions of $n$ with $k$-core $\gamma$ is always equal to $p_k(m)$ (see for example [4, Proposition 3.7]).

We start by finding bounds on the number of $k$-core partitions of $n$ when $k$ is large enough. To obtain these bounds, we will need bounds on the growth of the number of multipartitions, which will allow us to find lower bounds on $c_k(n)$, the number of $k$-core partitions of $n$. These results will then allow us to prove Theorem 2 at the end of this section.

Lemma 4. Let $k \geq 1$ and $m \geq 1$. Then $p_k(m) \leq (k + 1)p_k(m - 1)$.

Proof. For $\lambda = (\lambda^1, \ldots, \lambda^h)$ a multipartition of $m - 1$, let $h$ be maximal such that $|\lambda^h| > 0$ (set $h = 0$ if $m = 1$) and let $A(\lambda)$ be the set of multipartitions of $m$ which can be obtain by adding a node either to $\lambda^h$ on the last row or
the first column or by adding one node to some \( \lambda^i \) with \( i > h \). Note that \( |A(\lambda)| \leq k + 1 \) for each \( \lambda \) and any multipartition of \( m \) is contained in \( A(\lambda) \) for some multipartition \( \lambda \) of \( m - 1 \). The result follows. \( \square \)

**Lemma 5.** For any \( 1 \leq k \leq n \), we have \( p(n) - c_k(n) \leq (k + 1)p(n - k) \).

**Proof.** It follows from Lemma 4 and the classification of partitions with the same \( k \)-core (see for example [4, Proposition 3.7]) since

\[
p(n) - c_k(n) = \sum_{m=1}^{\lfloor n/k \rfloor} c_k(n - mk)p_k(m) \leq (k + 1) \sum_{m=1}^{\lfloor n/k \rfloor} c_k(n - k - (m - 1)k)p_k(m - 1) = (k + 1)p(n - k).
\]

\( \square \)

**Lemma 6.** Let \( n \geq 2 \) and \( c > \sqrt{3/\pi} \). If \( k \geq c \sqrt{n \log(n)} \), then

\[
\frac{c_k(n)}{p(n)} \geq 1 - \frac{c^4d \log(n)}{n^{c \pi / \sqrt{6} - 1/2}}
\]

for some constant \( d \).

**Proof.** From Lemma 5, we have that

\[
\frac{p(n) - c_k(n)}{p(n)} \leq \frac{(k + 1)p(n - k)}{p(n)}.
\]

Note that there exist constants \( d_1, d_2 > 0 \) such that for any \( m \geq 1 \),

\[
\frac{d_1}{m} e^{\pi \sqrt{2m/3}} \leq p(m) \leq \frac{d_2}{m} e^{\pi \sqrt{2m/3}}
\]

(see [2, (1.41)]). Using the inequalities displayed above, we see that the statement holds for \( k = n \), so we may assume that \( n - k \geq 1 \). Then

\[
\frac{(k + 1)p(n - k)}{p(n)} \leq \frac{d_2(k + 1)n}{d_1(n - k)} e^{-\pi \sqrt{2n/3}} (1 - \sqrt{1 - \frac{k}{n}}) \leq \frac{2d_2 kn}{d_1(n - k)} e^{-\pi \sqrt{6n}}.
\]

If \( k \geq 4c \sqrt{n \log(n)} \), then

\[
\frac{p(n) - c_k(n)}{p(n)} \leq \frac{2d_2 n^2}{d_1} e^{-\frac{4c \pi \log(n)}{\sqrt{6}}} = \frac{2d_2}{d_1 n^{4(\pi / \sqrt{6} - 1/2)}}.
\]

so in this case, the lemma holds since \( \frac{c \pi}{\sqrt{6}} - \frac{1}{2} > 0 \) by assumption on \( c \). If \( k = c \sqrt{n \log(n)} \) with \( c \leq c < 4c \) and \( k \leq n/2 \), then

\[
\frac{p(n) - c_k(n)}{p(n)} \leq \frac{4d_2 c \sqrt{n \log(n)}}{d_1} e^{-\frac{c \pi \log(n)}{\sqrt{6}}} \leq \frac{16d_2 \log(n)}{d_1 n^{c \pi / \sqrt{6} - 1/2}},
\]

so that also in this case, the lemma holds.
If \( k = \bar{c}\sqrt{n}\log(n) \) with \( c \leq \bar{c} < 4c \) and \( k > n/2 \), then \( \bar{c} > \sqrt{n/(2\log(n))} \). Since \( d_3\sqrt{n} > 8(\log(n))^3 \) for \( d_3 \) large enough, there exists a constant \( d_3 \) such that \( n < d_3^3 \). It then follows that

\[
\frac{p(n) - c_k(n)}{p(n)} \leq \frac{2d_2d_3^4\sqrt{n}\log(n)}{d_1} e^{-\frac{\pi\log(n)}{\sqrt{6}}} \leq \frac{512c^4d_2d_3\log(n)}{d_1n^{c\pi/\sqrt{6}-1/2}}.
\]

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Let \( \lambda = (a_1^{b_1}, \ldots, a_h^{b_h}) \in \Omega_\mu(n) \) and assume that there exist \( i \) and \( s \) with \( p^s \leq b_i \) and \( a_i p^s \geq c\sqrt{n}\log(n) \). For any \( 1 \leq j \leq h \), let \( b_j = f_{j,0}p^0 + \cdots + f_{j,s}p^s \) be the \( p \)-adic decomposition of \( b_j \) and set \( \delta_j := ((p^{s_j})^{f_{j,s_j}}, \ldots, 1^{f_{j,s}}) \) and \( \lambda := a_1\delta_1 \cup \cdots \cup a_h\delta_h \) (if \( \phi = (\phi_1, \ldots, \phi_r) \) and \( \psi = (\psi_1, \ldots, \psi_s) \) are partitions and \( t \) is a non-negative integer, then \( t\phi = (t\phi_1, \ldots, t\phi_r) \) and \( \phi \cup \psi \) is the partition obtained by rearranging the parts of \( (\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_s) \)). Then \( \lambda \in K_p(\lambda) \) and by assumption \( \lambda_1 \geq c\sqrt{n}\log(n) \).

Note that for any partition \( \mu \in K_p(\lambda) \), we have from [3, Proposition 1] that

\[
\frac{|\{\alpha \in P(n) : \chi_\mu^\alpha \equiv 0 \mod p\}|}{p(n)} = \frac{|\{\alpha \in P(n) : \chi_\lambda^\alpha \equiv 0 \mod p\}|}{p(n)}.
\]

Since \( \lambda_1 \geq c\sqrt{n}\log(n) \), the theorem holds for \( \lambda \) by Lemma 6 and the Murnaghan-Nakayama formula. So the statement of the theorem holds also for \( \mu \).

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