SINGULAR HOLOMORPHIC FOLIATIONS BY CURVES. III: ZERO LELONG NUMBERS

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ABSTRACT. Let \( F \) be a holomorphic foliation by curves defined in a neighborhood of 0 in \( \mathbb{C}^n \) \((n \geq 2)\) having 0 as a weakly hyperbolic singularity. Let \( T \) be a positive harmonic current directed by \( F \) which does not give mass to any of the \( n \) coordinate invariant hyperplanes \( \{z_j = 0\} \) for \( 1 \leq j \leq n \). Then we show that the Lelong number of \( T \) at 0 vanishes. Moreover, an application of this local result in the global context is given. We discuss also the relation between several basic notions such as directed positive harmonic currents, directed positive \( d\bar{d} \)-closed currents, Lelong numbers etc. in the framework of singular holomorphic foliations.

MSC 2020: Primary: 37F75, 37A30; Secondary: 57R30.

Keywords: singular holomorphic foliation, (weakly) hyperbolic singularity, directed positive harmonic current, directed positive \( d\bar{d} \)-closed current, Lelong number.

1. INTRODUCTION

The aim of this article is twofold. Its first (but not main) purpose is to revisit the basis of several fundamental notions in the theory of singular holomorphic foliations such as: directed positive harmonic currents, directed positive \( d\bar{d} \)-closed currents, Lelong numbers etc. The second (but main) purpose of the article is to prove the following local result and apply it to several global contexts.

**Theorem 1.1.** (Main Theorem) Let \( F := (\mathbb{D}^n, \mathbb{Z}, \{0\}) \) with \( n \geq 2 \), be a holomorphic foliation, which is defined on the unit polydisc \( \mathbb{D}^n \) of \( \mathbb{C}^n \) and which is associated to the linear vector field

\[
\Phi(z) = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j}, \quad z = (z_1, \ldots, z_n),
\]

where \( \lambda_j \) are all nonzero complex numbers and there are some \( 1 \leq l \neq k \leq n \) with \( \lambda_k/\lambda_l \notin \mathbb{R} \). Let \( T \) be a positive harmonic current directed by \( F \) which does not give mass to any of the \( n \) coordinate invariant hyperplanes \( \{z_j = 0\} \). Then the Lelong number of \( T \) at the origin \( 0 := (0, \ldots, 0) \) vanishes.

Note that the hypothesis on the linear vector field means that 0 is an isolated weakly hyperbolic singularity of \( F \) and \( F \) has no other singularity.

It is natural to ask how and to what extent the value of the current \( T \) near the union

\[
\mathcal{Z} := \mathbb{D}^n \cap \bigcup_{j=1}^{n} \{z_j = 0\}
\]

**Date:** March 25, 2022.
of the coordinate invariant hyperplanes on $\mathbb{D}^n$ affects the conclusion of the Main Theorem. The next result, which gives also a stronger version of the Main Theorem, answers this question.

**Theorem 1.2.** Let $\mathcal{F}$ be the foliation as in Theorem 1.1 and $\mathcal{F}$ the restriction of $\mathcal{F}$ to $\mathbb{D}^n \setminus \mathcal{Z}$. Let $T$ be a positive harmonic current on $\mathbb{D}^n \setminus \mathcal{Z}$ directed by $\mathcal{F}$ such that the mass of $T$ on $\mathbb{D}^n \setminus (r\mathbb{D})^n$ is finite for some $r \in (0, 1)$. Here $(r\mathbb{D})^n$ denotes the polydisc of polyradius $r$ in $\mathbb{C}^n$. Then the following assertions hold:

1. The mass of $T$ on $\mathbb{D}^n$ is finite.
2. The Lelong number of $T$ at every point of $\mathcal{Z}$ vanishes.

Combining Theorem 1.1 and some results of Fornæss–Sibony [18, 20], the following global picture is obtained for directed positive harmonic currents living on compact complex manifolds.

**Theorem 1.3.** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation with the set of singularities $E$ in a compact complex manifold $X$. Assume that

1. there is no invariant analytic curve;
2. all the singularities are hyperbolic linearizable;
3. there is no non-constant holomorphic map $\mathbb{C} \to X$ such that out of $E$ the image of $\mathbb{C}$ is locally contained in a leaf.

Then, for every positive harmonic current $T$ directed by $\mathcal{F}$, $T$ is diffuse and the Lelong number of $T$ vanishes everywhere in $X$.

The above theorem and results by Brunella [5], Jouanolou [25] and Lins Neto-Soares [29], give us the following corollary. It can be applied to every generic foliation in $\mathbb{P}^n$ with a given degree $d > 1$.

**Corollary 1.4.** Let $\mathcal{F} = (\mathbb{P}^n, \mathcal{L}, E)$ be a singular foliation by Riemann surfaces on the complex projective space $\mathbb{P}^n$ with $n \geq 2$. Assume that all the singularities are hyperbolic and that $\mathcal{F}$ has no invariant algebraic curve. Then for every positive harmonic current $T$ directed by $\mathcal{F}$, $T$ is diffuse and the Lelong number of $T$ vanishes everywhere in $\mathbb{P}^n$.

It is worthy noting that the above results generalize our previous work [31] to all dimensions. The last two decades witness many important advances in the theory of holomorphic foliations by curves on ambient complex surfaces emphasizing on singular holomorphic foliations. The reader is invited to consult the surveys [14, 19, 33, 35] for systematic expositions. The present work is motivated by these exciting developments. Our distant goal is trying to understand the theory in the general case of higher dimensions $n > 2$. Therefore, one of the first steps should be to investigate local situations near the singularities of the foliation in question.

This point of view seems to be fruitful in dimension $n = 2$. Indeed, the work of Fornæss–Sibony [18, 19, 20] initiates the local study of positive harmonic measure near a hyperbolic singularity for this dimension. This study is an important tool for further developments of the theory, see e.g. [12, 13, 20]. A typical feature in dimension $n = 2$ is that the phase spaces are not only simple, but also essentially unique modulo a translation and a rotation, see Figure 1. This makes the analysis in dimension $n = 2$ feasible. Roughly speaking, a phase space $\Pi_x$ is a domain in $\mathbb{C}$ which parametrizes the part of
FIGURE 1. On the left: the phase space of a foliation with a hyperbolic singularity in dimension $n = 2$: a sector with central angle $\theta$. On the right: the phase space is unique (i.e. with vertex at the origin $O$ and with $Ox$ the real axis) modulo a translation and a rotation.

the leaf $L_x$ inside the unit polydisc $\mathbb{D}^n$, where $x$ is a point in $\mathbb{D}^n \setminus \{0\}$ (see Section 3 for more details). In fact, in dimension $n = 2$, the (unique) phase space is a sector and the behavior of its Poisson kernel determines the mass-repartition of the positive harmonic currents near the hyperbolic singularity. In [31] the author revisits this question and obtains a complete behavior of the Poisson kernel of the phase space. This result plays a vital role in the author’s several subsequent developments [32, 34] when he studies the Lyapunov exponent of singular holomorphic foliations living on complex surfaces.

Unfortunately, in higher dimension $n > 2$, the geometry of phase spaces $\Pi_x$ is not simple as well as not unique any more. Figures 2 and 3 describe all possible phase spaces in dimension $n = 3$. As we will see in Section 3, $\Pi_x$ is a convex $m(x)$-gon, where the integer $m(x)$ varies between 2 and $n$. Moreover, $\Pi_x$ may be bounded or unbounded. The Poisson kernels of these phase spaces are difficult to study. Although they are all conformally equivalent to the unit disc $\mathbb{D}$ by a Schwarz-Christoffel mapping (see e.g. [16]), this tentative attempt turns out to be not realistic. Indeed, we have, in principle, Schwarz-Christoffel formula in order to compute this mapping and hence the Poisson kernel of $\Pi_x$. But this formula is useful only when we understand very well the shape of the phase space in question, and even if it is the case, when $n$ is large, the formula only gives us a small information on the Poisson kernel of the domain $\Pi_x$ near its boundary. However, the shape of $\Pi_x$ changes drastically in terms of $x \in \mathbb{D}^n \setminus \{0\}$, in particular, when $x$ approaches the coordinate hyperplanes. Therefore, this formula alone does not work.

Our main idea is to use the comparison principle of Poisson kernels and to combine it with the complete behavior of Poisson kernel in dimension 2 provided by [31] and Schwarz-Christoffel formula. The comparison principle of Poisson kernels is a well-known technique in Harmonic Analysis where it often applies to bounded smooth domains. In the present work, the principle applies to phase spaces which are, in general, neither smooth nor bounded. We do not obtain a complete behavior of the Poisson kernel of the phase spaces as in dimension 2, but instead we get their asymptotic behavior which suffices for our purpose. We hope that the techniques developed in this work will be useful in many other problems.
Figure 2. The phase spaces of a foliation with a hyperbolic singularity in dimension $n = 3$: the first (the triangle $ABC$) is unique modulo the composition of a translation and a dilation, whereas the remaining three sectors are unique modulo a translation.

The article is organized as follows. In Section 2 we strengthen the basis of the theory of singular holomorphic foliations and set up the background of the article. So this section fulfills the first purpose of this article. The rest of the article is devoted to the second (and main) purpose. More specifically, Section 3 studies the geometry of a singular flow box. Here, we will see that the phase spaces as well as other related objects encountered in dimension $n > 2$ are much more complicated to understand than those in dimension $n = 2$. Our main estimates are developed in Section 4 which are the core of the work. The proofs of Theorem 1.1 and Theorem 1.2 will be provided in Section 5. The proofs of Theorem 1.3 and Corollary 1.4 will be given in Section 6. The article is concluded with some remarks and open questions.

Notation. Throughout the paper,

- $\mathbb{D}$ denote the unit disc in $\mathbb{C}$, and for $r > 0$, $r\mathbb{D}$ denotes the disc of center 0 and of radius $r$.
- For an open set $\Omega \subset \mathbb{C}$, $\partial \Omega$ denotes the topological boundary of $\Omega$ and $P_{\Omega}$ denotes its Poisson kernel.
- $\text{Leb}_1$ (resp. $\text{Leb}_2$) denotes the one-dimensional (resp. two-dimensional) Lebesgue measure.
- The letters $c$, $c'$, $c''$, $c_0$, $c_1$, $c_2$ etc. denote positive constants, not necessarily the same at each occurrence.
- The notation $\gtrsim$ and $\lesssim$ means inequalities up to a multiplicative constant, whereas we write $\approx$ when both inequalities are satisfied.

Acknowledgments. The author acknowledges support by the Labex CEMPI (ANR-11-LABX-0007-01) and by the project QuaSiDy (ANR-21-CE40-0016). The paper was partially prepared during the visit of the author at the Vietnam Institute for Advanced Study.
in Mathematics (VIASM). He would like to express his gratitude to this organization for hospitality and for financial support.

2. Background

In this section we undertake the first (not main) task of this work. Namely, we revisit the background of the theory of singular holomorphic foliations emphasizing the relations between some basic notions such as directed positive harmonic currents, directed positive $dd^c$-closed currents etc. The survey [35] gives a unified treatment in a more general context of a laminations which is holomorphically immersed in a complex manifold. See also the survey [19] for the original discussion of these notions.

2.1. Positive $dd^c$-closed currents and Lelong number. Let $X$ be a complex manifold of dimension $n$. We fix an atlas of $X$ which is locally finite. Up to reducing slightly the charts, we can assume that the local coordinate system associated to each chart is defined on a neighbourhood of the closure of this chart. For $0 \leq p, q \leq k$ and $l \in \mathbb{N}$, denote by $\mathcal{D}^{p,q}(X)$ the space of $(p, q)$-forms of class $\mathcal{C}^l$ with compact support in $X$, and $\mathcal{D}^{p,q}(X)$ their intersection for $l \in \mathbb{N}$. If $\alpha$ is a $(p, q)$-form on $X$, denote by $\|\alpha\|_{q^l}$ the sum of the $\mathcal{C}^l$-norms of the coefficients of $\alpha$ in the local coordinates. These norms induce a topology on $\mathcal{D}^{p,q}(X)$ and $\mathcal{D}^{p,q}(X)$. In particular, a sequence $\alpha_j$ converges to $\alpha$ in $\mathcal{D}^{p,q}(X)$ if these forms are supported in a fixed compact set and if $\|\alpha_j - \alpha\|_{q^l} \to 0$ for every $l$.

A $(p, q)$-current on $X$ (or equivalently, a current of bidegree $(p, q)$, or equivalently, a current of bidimension $(n-p, n-q)$) is a continuous linear form $T$ on $\mathcal{D}^{n-p,n-q}(X)$ with values in $\mathbb{C}$.

A $(p, p)$-form on $X$ is positive if it can be written at every point as a combination with positive coefficients of forms of type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

where the $\alpha_j$ are $(1, 0)$-forms. A $(p, p)$-current or a $(p, p)$-form $T$ on $X$ is weakly positive if $T \wedge \varphi$ is a positive measure for any smooth positive $(n-p, n-p)$-form $\varphi$. A $(p, p)$-current $T$ is positive if $T \wedge \varphi$ is a positive measure for any smooth weakly positive $(n-p, n-p)$-form $\varphi$. If $X$ is given with a Hermitian metric $\beta$ and $T$ is a positive $(p, p)$-current on $X$, $T \wedge \beta^{n-p}$ is a positive measure on $X$. The mass of $T \wedge \beta^{n-p}$ is a measurable set $A$ is denoted by $\|T\|_A$ and is called the mass of $T$ on $A$. The mass $\|T\|$ of $T$ is the total mass of $T \wedge \beta^{n-p}$ on $X$.

A $(p, p)$-current on $X$ is closed if $dT = 0$ in the weak sense (namely, $T(d\alpha) = 0$ for every test form $\alpha \in \mathcal{D}^{n-p,n-p-1}(X) \oplus \mathcal{D}^{n-p-1,n-p}(X)$). A $(p, p)$-current on $X$ is $dd^c$-closed if $dd^c T = 0$ in the weak sense (namely, $T(dd^c \alpha) = 0$ for every form $\alpha \in \mathcal{D}^{n-p-1,n-p}(X)$).

Let $T$ be a positive $dd^c$-closed current on $X$. A fundamental theorem of Skoda [38] says that the Lelong number of $T$ at a point $x \in X$, defined by

$$\nu(T, x) := \lim_{r \to 0^+} \frac{1}{\pi^{n-p} r^{2(n-p)}} \int_{B(x,r)} T \wedge (dd^c z)^n.$$  

always exists and is finite non-negative. Here, we identify, via a local coordinate $z$, a neighborhood of $x$ in $X$ to an open neighborhood of $0$ in $\mathbb{C}^n$, and $B(x, r)$ is thus identified with the Euclidean ball in $\mathbb{C}^n$ with center 0 and radius $r$. In fact, Siu [37] (see also [7]) shows that when $T$ is a positive closed current, the Lelong number $\nu(T, x)$ is independent
of the choice of local coordinates near \( x \). The same result for positive \( dd^c \)-closed currents is proved by Alessandrini–Bassanelli [2].

The next simple result allows for extending positive \( dd^c \)-closed currents of bidimension \((1,1)\) through isolated points.

**Theorem 2.1.** (Dinh-Nguyen-Sibony [10] Lemma 2.5, Fornæss-Sibony-Wold [21] Lemma 17) Let \( T \) be a positive current of bidimension \((1,1)\) with compact support on a complex manifold \( X \). Assume that \( dd^c T \) is a negative measure on \( X \setminus E \) where \( E \) is a finite set. Then \( T \) is a positive \( dd^c \)-closed current on \( X \).

When the support of \( T \) is not compact, we only have the following local mass finiteness.

**Theorem 2.2.** (Alessandrini–Bassanelli [11] Main Theorem 5.6) Let \( T \) be a positive \( dd^c \)-closed current of bidimension \((1,1)\) outside a single point \( x \) on a complex manifold \( X \). Then the mass of \( T \) is finite near \( x \).

### 2.2. Singular holomorphic foliations and hyperbolic singularities

Let \( X \) be a complex manifold of dimension \( n \). A holomorphic foliation (by Riemann surfaces, or equivalently, by curves) \( \mathcal{F} = (X, \mathcal{L}) \) on \( X \) is the data of a foliation atlas with foliated charts

\[
\Phi_p : U_p \to \mathbb{B}_p \times \mathbb{T}_p.
\]

Here, \( \mathbb{T}_p \) and \( \mathbb{B}_p \) are domains in \( \mathbb{C}^{n-1} \) and in \( \mathbb{C} \) respectively, \( U_p \) is a domain in \( X \), and \( \Phi_p \) is biholomorphic, and all the changes of coordinates \( \Phi_p \circ \Phi_q^{-1} \) are of the form

\[
x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t).
\]

The open set \( U_p \) is called a flow box and the Riemann surface \( \Phi_p^{-1}\{t = c\} \) in \( U_p \) with \( c \in \mathbb{T}_p \) is a plaque. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. Two plaques are adjacent if they have non-empty intersection.

A leaf \( L \) is a minimal connected subset of \( X \) such that if \( L \) intersects a plaque, it contains that plaque. So a leaf \( L \) is a Riemann surface immersed in \( X \) which is a union of plaques. A leaf through a point \( x \) of this foliation is often denoted by \( L_x \). A transversal is a complex submanifold of codimension 1 in \( X \) which is transverse to the leaves of \( \mathcal{F} \).

A holomorphic foliation with singularities is the data \( \mathcal{F} = (X, \mathcal{L}, E) \), where \( X \) is a complex manifold, \( E \) a closed subset of \( X \) and \((X \setminus E, \mathcal{L})\) is a holomorphic foliation. Each point in \( E \) is said to be a singular point, and \( E \) is said to be the set of singularities of the foliation. We always assume that \( X \setminus E = X \), see e.g. [10, 19, 33, 35] for more details.

Consider a holomorphic foliation \( \mathcal{F} = (X, \mathcal{L}, E) \) and an isolated point \( x \) of \( E \). We say that a \( x \) is linearizable if there is a (local) holomorphic coordinates system of \( X \) on an open neighborhood \( U \) of \( x \) on which \( \mathcal{F} |_{U_x} = (U_x, \mathcal{L} |_{U_x}, \{x\}) \) is identified with \( \mathcal{F} : = (\mathbb{D}^n, \mathcal{L}, \{0\}) \) and the leaves of \( \mathcal{F} \) are, under this identification, integral curves of a linear vector field

\[
\Phi(z) = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j}, \quad z = (z_1, \ldots, z_n),
\]

where \( \lambda_j \) are some nonzero complex numbers. Such a neighborhood \( U_x \) is called a singular flow box of \( x \). \( \mathcal{F} = (\mathbb{D}^n, \mathcal{L}, \{0\}) \) is called a local model of the linearizable singularity \( x \).
Definition 2.3. Let \( x \in E \) a linearizable singular point of \( \mathcal{F} \) as above.

- We say that is \( x \) is weakly hyperbolic if there are some \( 1 \leq j + k \leq n \) with \( \lambda_j/\lambda_k \notin \mathbb{R} \).
- We say that \( x \) is hyperbolic if \( \lambda_j/\lambda_k \notin \mathbb{R} \) for all \( 1 \leq j + k \leq n \).

Remark 2.4. When \( n = 2 \), weakly hyperbolic singular point = hyperbolic singular point. But for \( n > 2 \), the weakly hyperbolic singularity is strictly weaker than the hyperbolic singularity.

2.3. Positive harmonic currents and directed positive \( dd^c \)-closed currents. Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation on a complex manifold \( X \) of dimension \( n \).

A (directed) \((p, q)\)-form on \( \mathcal{F} \) can be seen on the flow box \( U \simeq B \times \mathbb{T} \) as a \((p, q)\)-form on \( B \) depending on the parameter \( t \in \mathbb{T} \). For \( 0 \leq p, q \leq 1 \), denote by \( P^{p,q}(\mathcal{F}) \) the space of \((p, q)\)-form \( f \) with compact support in \( X \setminus E \) satisfying the following property: \( f \) restricted to each flow box \( U \simeq B \times \mathbb{T} \) is a \((p, q)\)-form of class \( \mathcal{C}^l \) on the plaques whose coefficients and all their derivatives up to order \( l \) depend continuously on the plaque. The norm \( \| \cdot \|_{C^l} \) on this space is defined as in the case of manifold using a locally finite atlas of \( \mathcal{F} \). We also define \( P^{p,q}(\mathcal{F}) \) as the intersection of \( P_l^{p,q}(\mathcal{F}) \) for \( l \geq 0 \). In particular, a sequence \( f_j \) converges to \( f \) in \( P^{p,q}(\mathcal{F}) \) if these forms are supported in a fixed compact set of \( X \setminus E \) and if \( \| f_j - f \|_{C^l} \to 0 \) for every \( l \). A (directed) current of bidegree \((p, q)\) (or equivalently, of bidimension \((1 - p, 1 - q)\)) on \( \mathcal{F} \) is a continuous linear form on the space \( P^{1-p,1-q}(\mathcal{F}) \) with values in \( \mathbb{C} \). We often write for short \( \mathcal{D}(\mathcal{F}) \) instead of \( P^{0,0}(\mathcal{F}) \).

A form \( \alpha \in P^{1,1}(\mathcal{F}) \) is said to be positive if its restriction to every plaque is a positive \((1, 1)\)-form in the usual sense.

Definition 2.5. Let \( T \) be a directed current of bidimension \((1, 1)\) on \( \mathcal{F} \).
- \( T \) is said to be positive if \( T(f) \geq 0 \) for all positive forms \( f \in P^{1,1}(\mathcal{F}) \).
- \( T \) is said to be closed if \( d^c T = 0 \) in the weak sense (namely, \( T(df) = 0 \) for all directed forms \( f \in P^1(\mathcal{F}) \)).
- \( T \) is said to be harmonic if \( dd^c T = 0 \) in the weak sense (namely, \( T(dd^c f) = 0 \) for all functions \( f \in \mathcal{D}(\mathcal{F}) \)).

We have the following decomposition.

Proposition 2.6. (see e.g. [10] Propositions 2.1, 2.2 and 2.3) Let \( T \) be a directed harmonic current on \( \mathcal{F} \). Let \( U \simeq B \times \mathbb{T} \) be a flow box which is relatively compact in \( X \).

1. (Existence) Then, there is a positive Radon measure \( \mu \) on \( \mathbb{T} \) and for \( \mu \)-almost every \( t \in \mathbb{T} \), there is a harmonic function \( h_t \) on \( B \) such that if \( K \) is compact in \( B \), the integral \( \int_K \| h_t \|_{L^1(K)} d\mu(t) \) is finite and

\[
T(\alpha) = \mathbb{I}_{\mathbb{T}} \left( \int_B h_t(y) \alpha(y, t) \right) d\mu(t)
\]

for every form \( \alpha \in P^{1,1}(\mathcal{F}) \) compactly supported on \( U \).

2. (Uniqueness) If \( \mu' \) and \( h'_t \) are associated with another decomposition of \( T \) in \( U \), then there is a measurable function \( \theta > 0 \) on a measurable subset \( \mathbb{T}' \subset \mathbb{T} \) such that \( h_t = 0 \) for \( \mu \)-almost every \( t \notin \mathbb{T}' \), \( h'_t = 0 \) for \( \mu' \)-almost every \( t \notin \mathbb{T}' \), and \( h_t = \theta(t) h'_t \) for \( \nu \) and \( \nu' \)-almost every \( t \in \mathbb{T}' \).

3. If moreover, \( T \) is positive, then for \( \mu \)-almost every \( t \in \mathbb{T} \), the harmonic function \( h_t \) is positive on \( B \).
Moreover, we say that following properties (i)-(ii) are satisfied:

(i) A directed positive harmonic current $T$ on $\mathcal{F} = (X, \mathcal{L}, E)$ is said to be diffuse if for any decomposition of $T$ in any flow box $U \simeq \mathbb{B} \times T$ as in Proposition 2.6 the measure $\nu$ has no mass on each single point of the transversal $T$.

(ii) Property (ii) of Definition 2.8 means the following two properties (ii-a)-(ii-b):

(ii-a) If $T$ is diffuse, then for $\mu$-almost every $t \in T$, the harmonic function $h_t$ is constant on $\mathbb{B}$.

Definition 2.7. A directed positive harmonic current $T$ on $\mathcal{F} = (X, \mathcal{L}, E)$ is said to be diffuse if for any decomposition of $T$ in any flow box $U \simeq \mathbb{B} \times T$ as in Proposition 2.6 the measure $\nu$ has no mass on each single point of the transversal $T$.

For a complex manifold $M$, let $\mathcal{D}^{1,1}(M)$ denote the space of smooth $(1, 1)$-forms $\alpha$ compactly supported in $M$ endowed with the semi-norms $\| \cdot \|_{\mathcal{F}(M)}$, where $l \in \mathbb{N}$ and $(M_k)_{k \in \mathbb{N}}$ is an increasing sequence of relatively compact open subsets of $M$ such that $M = \bigcup_{k \in \mathbb{N}} M_k$. For $x \in X \setminus E$, let $j_x : L_x \hookrightarrow X$ be the canonical injective immersion from $L_x$ into $X$. The aggregate of the pull-back via $j_x$ of each test form $\alpha \in \mathcal{D}^{1,1}(X \setminus E)$ defines a form in $\mathcal{D}^{1,1}(\mathcal{F})$, denoted by $j^* \alpha$. So we obtain a canonical restriction map

$$j^* : \mathcal{D}^{1,1}(X \setminus E) \to \mathcal{D}^{1,1}(\mathcal{F})$$

given by $\alpha \mapsto j^* \alpha$.

We see easily that the image $\mathcal{I}$ of $j^*$ is dense in $\mathcal{D}^{1,1}(\mathcal{F})$.

The original notions of directed positive $dd^c$-closed currents for singular holomorphic foliations (resp. for singular laminations which are holomorphically immersed in a complex manifold) with a small set of singularities were introduced by Berndtsson-Sibony [41] (resp. by Fornæss-Sibony [18, 19]). In [35] we give another notion of directed positive $dd^c$-closed currents for singular Riemann surface laminations which are holomorphically immersed in a complex manifold. Our notion coincides with the previous ones when the lamination is $C^2$-transversally smooth, this condition is automatically fulfilled when $\mathcal{F}$ is a singular holomorphic foliation. The advantage of our (slightly improved) notion is that it is relevant even when the set of singularities is not small. We recall our definition in the present context of singular holomorphic foliations.

Definition 2.8. Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation. A directed positive $dd^c$-closed current (resp. a directed positive closed current) on $\mathcal{F}$ is a positive $dd^c$-closed current $T$ (resp. a positive closed current $T$) of bidimension $(1, 1)$ on $X$ such that the following properties (i)-(ii) are satisfied:

(i) $T$ does not give mass to $E$, i.e. the mass $\|T\|_E$ of $T$ on $E$ is zero;

(ii) $T$ is a directed positive harmonic current (resp. a directed positive closed current) on $\mathcal{F}$ in the sense of Definition 2.5.

Moreover, we say that $T$ is diffuse if it is diffuse in the sense of Definition 2.7 as a directed positive harmonic current (resp. a directed positive closed current) on $\mathcal{F}$.

Remark 2.9. When $E = \emptyset$, property (i) of Definition 2.8 is trivially satisfied and property (ii) says that a directed positive $dd^c$-closed current $T$ may be regarded as a positive harmonic current. The converse statement (when $E = \emptyset$) is also true, see Subsection 2.4 below.

Remark 2.10. Property (ii) of Definition 2.8 means the following two properties (ii-a)-(ii-b):

(ii-a) $\langle T, \alpha \rangle = \langle T, \beta \rangle$ for $\alpha, \beta \in \mathcal{D}^{1,1}(X \setminus E)$ such that $j^* \alpha = j^* \beta$; so the current

$$\tilde{T} : \mathcal{I} \to \mathbb{C}$$

given by $\langle \tilde{T}, j^* \alpha \rangle := \langle T, \alpha \rangle$ for $\alpha \in \mathcal{D}^{1,1}(X \setminus E)$,

is well-defined;
(ii-b) the current \( \hat{T} \) defined in (ii-a) can be uniquely extended from \( \mathcal{I} \) to \( \mathcal{D}^{1,1}(\mathcal{F}) \) by continuity (as \( \mathcal{I} \) is dense in \( \mathcal{D}^{1,1}(\mathcal{F}) \)) to a current \( \hat{T} \) of order zero, and \( \hat{T} \) is a directed positive harmonic current (resp. directed positive closed current) on \( \mathcal{F} \) in the sense of Definition 2.5.

Property (ii-b) holds automatically since \( T \) is a positive \( dd^c \)-closed current (resp. positive closed current) on \( X \). So property (ii) is equivalent to the single property (ii-a). If there is no confusion, we often denote \( \hat{T} \) and \( \hat{T} \) simply by \( T \).

As an immediate consequence of Proposition 2.6 and Definition 2.8, we obtain the following characterization of directed positive \( dd^c \)-closed currents directed by a singular holomorphic foliation.

**Proposition 2.11.** Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation and \( T \) a positive \( dd^c \)-closed current (resp. a positive closed current) of bidimension \( (1, 1) \) on \( X \). Then \( T \) is a directed positive \( dd^c \)-closed current (resp. a directed positive closed current) on \( \mathcal{F} \) if and only if the following properties (i)-(ii)' are satisfied:

(i) \( T \) does not give mass to \( E \), i.e. the mass \( \|T\|_E \) of \( T \) on \( E \) is zero;

(ii)' For any flow box \( U \simeq B \times T \) which is relatively compact in \( X \), there is a positive Radon measure \( \mu \) on \( T \) and for \( \mu \)-almost every \( t \in T \), there is a harmonic function \( h_t \) on \( B \) such that if \( K \) is compact in \( B \), the integral \( \int_K h_t \|\cdot\|_{L^1(K)} d\mu(t) \) is finite and

\[
T(\alpha) = \int_T \left( \int_B h_t(y) \alpha(y, t) \right) d\mu(t)
\]

for every form \( \alpha \in \mathcal{D}^{1,1}(X) \) compactly supported on \( U \).

Moreover, if \( T \) is positive \( dd^c \)-closed current (resp. positive closed current), then for \( \mu \)-almost every \( t \in T \), the harmonic function \( h_t \) is positive on \( B \) (resp. the harmonic function \( h_t \) is constant on \( B \)).

### 2.4. Lelong number of positive harmonic currents

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation and \( T \) a positive harmonic current. By Proposition 2.6 (1), we see that

\[
T'(\alpha) := T(j^*\alpha) \quad \text{for} \quad \alpha \in \mathcal{D}^{1,1}(X \setminus E)
\]

is a positive \( dd^c \)-closed current of bidimension \( (1, 1) \) on \( X \setminus E \). In fact, Remark 2.9 says that outside the singularities, directed positive \( dd^c \)-closed currents coincide with positive harmonic currents. In other words, \( T' \) may be canonically identified with \( T \) outside \( E \).

The mass of \( T \) on a (measurable) set \( A \), denoted by \( \|T\|_A \), is the mass of \( T \) (as a positive \( dd^c \)-closed current) on \( A \setminus E \). The mass \( \|T\| \) of \( T \) is the total mass of \( T \) on \( X \).

For every \( x \in X \setminus E \), we can define

\[
\nu(T, x) := \nu(T', x),
\]

where the right-hand side is given by (2.1) with \( p = n - 1 \).

For \( x \in E \), we cannot use (2.1) immediately because \( T \) may not be extended through an open neighborhood of \( x \) as a positive \( dd^c \)-closed current. However, following the model formula (2.1) we still define

\[
\nu(T, x) := \limsup_{r \to 0^+} \frac{1}{\pi r^2} \int_{B(x, r) \setminus E} T \wedge (dd^c \|z\|^2) \in \mathbb{R}^+ \cup \{\infty\}.
\]
as the Lelong number of $T$ at $x$. Here, we identify, through a local coordinate $z$, a neighborhood of $x$ in $X$ to an open neighborhood of 0 in $\mathbb{C}^n$, and $\mathbb{B}(x, r)$ is thus identified with the Euclidean ball in $\mathbb{C}^n$ with center 0 and radius $r$. A priori, the Lelong number $\nu(T, x)$ may depend on the choice of local coordinates near $x$. However, it is easy to check that if $\nu(T, x)$ is equal to either 0 or $\infty$, then it is independent of local coordinates near $x$.

**Example 2.12.** Let $\mathcal{F} := (\mathbb{D}^2, \mathcal{L}, \{0\})$ be a singular holomorphic foliation associated to a linear vector field in $\mathbb{D}^2$. Consider the current $T(z) := (-\log |z_2|) \cdot [z_1 = 0]$ for $z = (z_1, z_2) \in \mathbb{D}^2 \setminus \{0\}$. Here $[z_1 = 0]$ is the current of integration on the complex line $\{z_1 = 0\}$. We can check that $T$ is a positive harmonic current for $\mathcal{F}$, but $\nu(T, 0) = \infty$. In fact, the simple extension $\tilde{T}$ of $T$ through $\{0\}$ satisfies $dd^c\tilde{T} = -\delta_0$, where $\delta_0$ is the Dirac mass at the origin. So $\tilde{T}$ is positive, but not $dd^c$-closed.

### 2.5. Green function and Poisson kernel.

**Definition 2.13.** Let $\Omega \subset \mathbb{C}$ be either a (not necessarily bounded) convex polygon or a $C^2$-smooth bounded domain. The diagonal $\Delta$ of $\Omega$ is the set $\{(\zeta, \zeta) : \zeta \in \Omega\} \subset \Omega \times \Omega$. A function $U = (\Omega \times \overline{\Omega}) \setminus \Delta \to \mathbb{R}$ is the Green’s function on $\Omega$ if:

1. for each fixed $\zeta \in \Omega$ the function $U(\zeta, \zeta) + \frac{1}{2\pi} \log |\zeta - \zeta|$ is harmonic as a function of $\zeta \in \Omega$ (even at the point $\zeta$);
2. $U(\zeta, \zeta)|_{\zeta \in \partial \Omega} = 0$ for each fixed $\zeta \in \Omega$.

Let $\text{Nor} = \text{Nor}_\partial\Omega$ represent the unit outward normal vector field on $\partial\Omega$. The following classical result gives the Poisson kernel of bounded smooth domains in $\mathbb{C}$ (see e.g. [26]).

**Proposition 2.14.** Let $\Omega \subset \mathbb{C}$ be a $C^2$-piecewise smooth bounded domain. Then

1. There is a unique Green function $G$ on $\Omega$.
2. Let the Poisson kernel on $\Omega$ be the function

   $$P(\zeta, \xi) = P_\Omega(\zeta, \xi) := -\text{Nor}_\xi U(\zeta, \xi) \quad \text{for} \quad \zeta \in \Omega, \, \xi \in \partial\Omega.$$  

   Then the following two assertions hold:
   (2-i) If $u \in C(\overline{\Omega})$ is harmonic on $\Omega$, then
   $$u(\zeta) = \int_{\partial\Omega} P(\zeta, \xi)u(\xi)d\text{Leb}_1(\xi) \quad \text{for} \quad \zeta \in \Omega.$$  

   Here, $d\text{Leb}_1$ is the 1-dimensional Lebesgue measure on $\partial\Omega$.
   (2-ii) $P(\cdot, \xi)$ is a positive harmonic function on $\Omega$ when $\xi \in \partial\Omega$ is fixed.

### 3. Geometry of singular flow boxes

Let $\mathcal{U}$ be an open neighborhood of the closed $n$-unit polydisc $\mathbb{D}^n$. Consider the foliation $\mathcal{F} = (\mathcal{U}, \mathcal{L}, \{0\})$ which is the restriction to $\mathcal{U}$ of the foliation associated to the vector field

\begin{equation}
\Phi(z) = \sum_{j=1}^n \lambda_j z_j \frac{\partial}{\partial z_j}, \quad z \in \mathbb{C}^n,
\end{equation}

with $\lambda_j \in \mathbb{C}^*$. The foliation is singular at the origin. We often call $\mathcal{F}$ a local model of a linearizable singularity.
A
B
C
x
y

Figure 3. The phase spaces of a foliation with a hyperbolic singularity in dimension $n=3$: the first (the unbounded polygon $yCBx$) is unique modulo the composition of a translation and a dilation, whereas the second (the sector $xAy$) is unique modulo a translation.

Write $\lambda_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$. For $x = (x_1, \ldots, x_n) \in \mathbb{U}\setminus\{0\}$, define the holomorphic map $\varphi_x : \mathbb{C} \to \mathbb{C}^n \setminus \{0\}$ by

$$
\varphi_x(\zeta) := \left( x_1 e^{\lambda_1 \zeta}, \ldots, x_n e^{\lambda_n \zeta} \right) \quad \text{for} \quad \zeta \in \mathbb{C}.
$$

It is easy to see that $\varphi_x(\mathbb{C})$ is the integral curve of $\Phi$ which contains $\varphi_x(0) = x$. The leaf of $\mathcal{F}$ through $x$ is given by $L_x := \varphi_x(\mathbb{C}) \cap \mathbb{U}$.

Write $\zeta = u + iv$ with $u, v \in \mathbb{R}$. Fix $x = (x_1, \ldots, x_n) \in \mathbb{U}\setminus\{0\}$. Let $j = 1, \ldots, k$. If $x_j \neq 0$, let $H_{x,j}$ be the open half-plane defined by

$$
H_{x,j} := \{ \zeta = u + iv \in \mathbb{C} : -a_j u + b_j v - \log |x_j| > 0 \} = \{ \zeta \in \mathbb{C} : |\varphi_x(\zeta)_j| < 1 \},
$$

where $\varphi_x(\zeta)_j$ is the $j$-th coordinate of $\varphi_x(\zeta) \in \mathbb{C}^n$. If $x_j = 0$, then set simply $H_{x,j} := \mathbb{C}$.

The phase space of a point $x \in \mathbb{D}^n \setminus \{0\}$ is the domain $\Pi_x := \varphi_x^{-1}(\mathbb{D}^n)$ in $\mathbb{C}$. Observe that

$$
\Pi_x = \bigcap_{j=1}^n H_{x,j}.
$$

Remark 3.1. So, $\Pi_x$ is a convex $m(x)$-gon which is not necessarily bounded, where $0 \leq m(x) \leq n$ is an integer depending only on $x$. Moreover, for $x, x' \in \mathbb{D}^n \setminus \{0\}$ and $1 \leq j \leq n$ such that $x_j \neq 0, x'_j \neq 0$, $\partial H_{x,j}$ is either equal to or parallel to $\partial H_{x',j}$.

When $\{0\}$ is a weakly hyperbolic singularity, there are at least two edges of $\Pi_x$ which are not parallel. In particular when $n = 2$ and $\{0\}$ is a weakly hyperbolic singularity, the geometry of phase spaces are very simple: $\Pi_x$ is a sector and $m(x) = 1$ and for every $y \in \mathbb{D}^n \setminus \{0\}$, there is a unique translation $T_{x,y}$ on the plane such that $\Pi_y = T_{x,y}(\Pi_x)$.

All possible phase spaces in dimension $n = 3$ are illustrated in Figures 2 and 3.

For $x \in \mathbb{D}^n \setminus \{0\}$ consider

$$
\mathcal{L}_x := \varphi_x(\Pi_x) = L_x \cap \mathbb{D}^n.
$$

In particular, it is a leaf of the restriction foliation $\mathcal{F}|_{\mathbb{D}^n}$, where $\mathcal{F}|_{\mathbb{D}^n} := (\mathbb{D}^n, \mathcal{L}|_{\mathbb{D}^n}, \{0\})$. 

We use the standard Euclidean metric dist on \( C \), that is, for \( \zeta, \xi \in C \) and for \( A, B \subset C \),
\[
\text{dist}(\zeta, \xi) := |\zeta - \xi| \quad \text{and} \quad \text{dist}(A, B) := \inf_{\zeta \in A, \xi \in B} \text{dist}(\zeta, \xi).
\]
Let \( \zeta \in \Pi_x \) and \( 1 \leq j \leq n \). If \( x_j \neq 0 \), let \( \text{dist}_{x,j}(\zeta) \) be the distance from \( \zeta \) to the real line \( \partial \mathbb{H}_{x,j} \), that is,
\[
\text{(3.6)} \quad \text{dist}_{x,j}(\zeta) := \text{dist}(\zeta, \partial \mathbb{H}_{x,j}) = |\lambda_j|^{-1} - a_j u + b_j v - \log |x_j|.
\]
If \( x_j = 0 \), set simply \( \text{dist}_{x,j}(\zeta) := \infty \).

Note that if \( y = (y_1, \ldots, y_n) := \varphi_x(\zeta) \), then
\[
(3.7) \quad \log |y_j| := |\lambda_j|\text{dist}_{x,j}(\zeta).
\]
Note that \( \Pi_x \) contains 0 since \( \varphi_x(0) = x \). Moreover, for \( x \in \mathbb{D}^n \),
\[
(3.8) \quad \text{dist}(0, \partial \Pi_x) = \min \left\{ -\frac{\log |x_1|}{|\lambda_1|}, \ldots, -\frac{\log |x_n|}{|\lambda_n|} \right\}.
\]
For each \( x \in \mathbb{D}^n \) and \( \zeta \in \Pi_x \), there is a permutation \( \{k_1, \ldots, k_n\} \) of \( \{1, \ldots, n\} \) such that
\[
|\lambda_{k_1}|\text{dist}_{x,k_1}(\zeta) < |\lambda_{k_2}|\text{dist}_{x,k_2}(\zeta) < \ldots < |\lambda_{k_n}|\text{dist}_{x,k_n}(\zeta).
\]
Define
\[
(3.9) \quad \text{dist}_x(\zeta) := 1 + |\lambda_{k_1}|\text{dist}_{x,k_1}(\zeta) \quad \text{and} \quad \text{dist}^*_x(\zeta) := 1 + |\lambda_{k_n}|\text{dist}_{x,k_n}(\zeta).
\]
So we obtain the following useful estimates:
\[
(3.10) \quad \min_{1 \leq k \leq n} |\lambda_k|\text{dist}(x, \partial \Pi_x) = \text{dist}_x(\zeta) - 1 \leq \max_{1 \leq k \leq n} |\lambda_k|\text{dist}(x, \partial \Pi_x) \quad \text{and} \quad \text{dist}_x(\zeta) \leq \text{dist}^*_x(\zeta).
\]
The first inequality of implies that
\[
(3.11) \quad \left( \min_{1 \leq k \leq n} |\lambda_k| \right) \cdot \text{dist}(x, \partial \Pi_x) \leq \text{dist}_x(\zeta) - 1 \leq \left( \max_{1 \leq k \leq n} |\lambda_k| \right) \cdot \text{dist}(x, \partial \Pi_x).
\]
Observe that when the ratio \( \lambda_i/\lambda_j \) are not all rational and all the coordinates of \( x \) do not vanish, \( \varphi_x : \Pi_x \to \mathcal{L}_x \) is bijective and hence \( \mathcal{L}_x \) is simply connected. Otherwise, when the ratios \( \lambda_i/\lambda_j \) are all rational, all the leaves of \( \mathcal{F}_{|\mathbb{D}} \) are closed submanifolds of \( \mathbb{D}^n \setminus \{0\} \) and are biholomorphic to annuli.

Let \( \mathbb{I} \) denote the quotient of \( (\overline{\mathbb{D}} \setminus \{0\})^n \) by the equivalence relation \( x \sim y \) if \( \mathcal{L}_x = \mathcal{L}_y \) for \( x, y \in (\overline{\mathbb{D}} \setminus \{0\})^n \). Let \( [x] \) be the class of \( x \) in this equivalence relation. Let \( \pi : (\overline{\mathbb{D}} \setminus \{0\})^n \to \mathbb{I} \) by the canonical projection given by \( \pi(x) := [x] \), \( x \in (\overline{\mathbb{D}} \setminus \{0\})^n \). We endow \( \mathbb{I} \) with the complex structure induced from \( \mathbb{C}^n \).

**Lemma 3.2.** Suppose that \( \{0\} \) is weakly hyperbolic. Then:

1. \( \mathbb{I} \) is a complex manifold of dimension \( n - 1 \). In particular, when \( n = 2 \), \( \mathbb{I} \) is a complex torus of dimension 1.
2. For every point \( x \in (\overline{\mathbb{D}} \setminus \{0\})^n \), there is a neighborhood \( U_x \) of \( x \) such that \( U_x \) is a flow box \( U_x \simeq B_x \times T_x \) and that the restriction \( \pi|_{T_x} \) of \( \pi \) on the transversal \( T_x \) is biholomorphic onto its image which is open subset of \( \mathbb{I} \). In other words, \( \pi \) is locally biholomorphic in the transversal direction.
Proof. We only need to prove the following

Fact. For every \( x \in (\mathbb{D}\setminus\{0\})^n \) and every \( \epsilon > 0 \), there is a neighborhood \( U_{x,\epsilon} \) of \( x \) in \((\mathbb{D}\setminus\{0\})^n\) such that if \( y, z \in U_{x,\epsilon} \) and \( y \approx z \), then there is \( \zeta \in \mathbb{C} \) such that \( |\zeta| < \epsilon \) and \( z = \varphi_y(\zeta) \), where \( \varphi_y \) is defined in \((3.2)\).

Indeed, taking the fact for granted and shrinking \( U_{x,\epsilon} \) if necessary, we may assume that \( U_{x,\epsilon} \) is a flow box \( \mathbb{R} \times \mathbb{T} \). Consider the restriction \( \pi|_{U_{x,\epsilon}} : U_{x,\epsilon} \to \mathbb{T} \). We deduce from from the fact that the restriction \( \pi|_{\mathbb{T}} \) of \( \pi|_{U_{x,\epsilon}} \) to \( \mathbb{T} \) is homeomorphic onto its image which is open subset of \( \mathbb{T} \). The assertions of the lemma follow modulo the above fact.

To prove the fact, we infer from \( y \approx z \) that there is a \( \zeta \in \mathbb{C} \) such that \( z = \varphi_y(\zeta) \). Since \( \{0\} \) is a weakly hyperbolic singularity, there are \( l, l' \) with \( 1 \leq l < l' \leq n \) such that \( t := \lambda_l/\lambda_l \notin \mathbb{R} \). Write \( t = a + ib \), with \( a, b \in \mathbb{R} \) and \( b \neq 0 \). We choose a neighborhood \( U_{x,\epsilon} \in (\mathbb{D}\setminus\{0\})^n \) of \( x \) so that

\[
\frac{|y_j - x_j|}{|x_j|} < \epsilon \quad \text{for} \quad y = (y_1, \ldots, y_n) \in U_{x,\epsilon}, 1 \leq j \leq n.
\]

We infer from the above inequalities and \( y, z \in U_{x,\epsilon} \) that for \( \epsilon > 0 \) small enough (depending only on \( x \)),

\[
|y_j - x_j| < \epsilon |x_j| \quad \text{and} \quad |y_j - x_j| < \epsilon |x_j|.
\]

Hence,

\[
\frac{|z_j - y_j|}{|y_j|} < \epsilon \quad \text{for} \quad 1 \leq j \leq n.
\]

Using this, we infer from the equalities \( z_j = y_j e^{\lambda_j \zeta} \) for \( j \in \{l, l'\} \) that there are \( k, k' \in \mathbb{Z} \) such that

\[
|\lambda_l \zeta - 2\pi k'| < \epsilon \quad \text{and} \quad |\lambda_l \zeta - 2\pi k| < \epsilon.
\]

So \( |tk - k'| < \epsilon \), which is equivalent to \( |k(a + ib) - k'| < \epsilon \). Hence, \( |k'| < |b|^{-1} \epsilon \).

When \( \epsilon < \min(|b|^{-1}, 1) \), we deduce from \( k' \in \mathbb{Z} \) that \( k' = 0 \). This, combined with the previous inequality \( |\lambda_l \zeta - 2\pi k'| < \epsilon \), yields that \( |\zeta| < \epsilon \). Hence, the fact follows.

\[\square\]

Lemma 3.3. By shrinking \( U \) if necessary, there is a Borel subset \( \mathcal{X} \subset (\mathbb{D}\setminus\{0\})^n \) with the following properties:

1. For every \( x = (x_1, \ldots, x_n) \in \mathcal{X} \), there are two indexes \( 1 \leq j < l \leq n \) such that \( |x_j| = 1 \) and \( |x_l| = 1 \).

2. The restriction of the canonical projection \( \pi \) to \( \mathcal{X} \) (still denoted by \( \pi \)) maps \( \mathcal{X} \) onto \( \mathbb{I} \) bijectively. Moreover, \( \mathbb{I} \) is a Borel set and the map \( \pi : \mathcal{X} \to \mathbb{I} \) and its inverse are Borel.

3. For \( x, y \in \mathcal{X} \) with \( x \neq y \), \( L_x \) and \( L_y \) are disjoint.

4. The union of \( L_x, x \in \mathcal{X} \) is equal to \((\mathbb{D}\setminus\{0\})^n\).

Proof. To each \( x \in (\mathbb{D}\setminus\{0\})^n \), we associate a finite subset \( \mathcal{S}(x) \subset (\mathbb{D}\setminus\{0\})^n \) as follows:

\( y \in \mathcal{S}(x) \) if and only if \( y \approx x \) and there are at least two indexes \( 1 \leq j < l \leq n \) such that \( |y_j| = 1 \) and \( |y_l| = 1 \).

Observe that if \( x \approx x' \) then \( \mathcal{S}(x) = \mathcal{S}(x') \) and if \( \mathcal{S}(x) \cap \mathcal{S}(x') \neq \emptyset \) then \( x \approx x' \). Moreover, we have the following geometric interpretation: \( y \in \mathcal{S}(x) \) if and only if \( y = \varphi_x(\zeta) \), where \( \zeta \) is a vertex of the convex polygon \( \Pi_x \). So the cardinality of \( \mathcal{S}(x) \) is equal to \( m(x) \). In particular, \( 0 \leq \# \mathcal{S}(x) \leq n \).
On the other hand, the weak hyperbolicity of the single singularity \( \{0\} \) implies that for every \( x \in (\overline{D\setminus\{0\}})^n \), there are at least two lines among \( n \) lines \( \mathbb{H}_{x,j} \) which are not parallel. Suppose that they are \( \partial \mathbb{H}_{x,l} \) and \( \partial \mathbb{H}_{x,l'} \). So the convex polygon \( \Pi_x \) admits at least one (finite) vertex. Moreover, this vertex is either the intersection of \( \partial \mathbb{H}_{x,l} \) and \( \partial \mathbb{H}_{x,l'} \), or the intersection of one of them with another line \( \partial \mathbb{H}_{x,j} \). Therefore, \( \mathcal{S}(x) \neq \emptyset \), and hence \( 1 \leq \# \mathcal{S}(x) \leq n \).

The weak hyperbolicity of the single singularity \( \{0\} \) also implies that for \( y, z \in (\overline{D\setminus\{0\}})^n \) with \( y \sim x, z \sim x \), if \( |y_j| = |z_j| \) for all \( 1 \leq j \leq n \), then \( y = z \). Indeed, write \( y = \varphi_x(\zeta) \) and \( z = \varphi_x(\xi) \) for some \( \zeta, \xi \in \Pi_x \). We infer from \( |y_j| = |z_j| \) and equality (3.7) that \( \text{dist}(\zeta, \partial \mathbb{H}_{x,j}) = \text{dist}(\xi, \partial \mathbb{H}_{x,j}) \). This equality for \( j = l \) and \( j = l' \), coupled with the fact that two real lines \( \partial \mathbb{H}_{x,l} \) and \( \partial \mathbb{H}_{x,l'} \) are not parallel and \( \zeta, \xi \in \Pi_x \), implies that \( \zeta = \xi \). So \( y = z \) as asserted.

Therefore, we order the elements of \( \mathcal{S}(x) \) using a lexicographical order. More specifically, for \( y, z \in \mathcal{S}(x) \) we say that \( y \succ z \) if and only if there is an index \( 1 \leq j \leq n \) such that \( |y_k| = |z_k| \) for all \( k < j \) and \( |y_j| > |z_j| \). Let \( x^* \) be the greatest element of \( \mathcal{S}(x) \) with respect to this total order. Let

\[
\mathbb{X} := \{ x^* : x \in (\overline{D\setminus\{0\}})^n \}.
\]

We can check all properties of the lemma.

The following result is one of the main ingredients in the proofs of Theorem 1.1 and Theorem 1.2.

**Lemma 3.4.** Let \( T \) be a positive harmonic current \( T \) directed by \( \mathcal{F} \) on \( (\overline{D\setminus\{0\}})^n \) such that the mass of \( T \) on \( \overline{D^n \setminus (r_0 \overline{D})^n} \) is finite for some \( r_0 \in (0, 1) \). Then there is a positive measure \( \mu \) on \( \mathbb{X} \) and positive harmonic functions \( \tilde{h}_x \) on \( \mathcal{L}_x \) for \( \mu \)-almost every \( x \in \mathbb{X} \) such that in \( (\overline{D\setminus\{0\}})^n \)

\[
T = \int_\mathbb{X} T_x d\mu(x), \quad \text{where} \quad T_x := \tilde{h}_x[\mathcal{L}_x].
\]

Moreover, the mass of \( T_x \) on \( \overline{D^n \setminus (r_0 \overline{D})^n} \) is 1 for \( \mu \)-almost every \( x \in \mathbb{X} \).

**Proof.** Suppose without loss of generality that \( T \) is defined on a neighborhood \( U \) of \( (\overline{D\setminus\{0\}})^n \) as in Lemma 3.3. By Lemma 3.2, the map \( \pi \) which associates to a point in \( L_x \) the image \( [x] \) in \( \Pi \) is a holomorphic map and \( T \) is directed by the fibers of \( \pi \). We regard \( \pi : (\overline{D\setminus\{0\}})^n \to \Pi \) as a simple foliation \( \mathcal{F} \) whose leaves are \( \mathcal{L}_x \times \{x\}, [x] \in \Pi \). Applying Proposition 2.6 to \( T \) in this context, we obtain the following decomposition of \( T \). There is a positive measure \( \mu' \) on \( \Pi \) and positive harmonic functions \( \tilde{h}_x \) on \( L_x \) for \( \mu' \)-almost every \( x \in \mathbb{X} \) such that in a neighborhood of \( \overline{D^n} \)

\[
T = \int_\Pi T_x d\mu'([x]), \quad \text{where} \quad T_x := \tilde{h}_x[\mathcal{L}_x]. \tag{3.12}
\]

Since we know by Lemma 3.3 that the restriction of \( \pi \) to the Borel set \( \mathbb{X} \) is Borel measurable bijective map, we define a measure \( \nu \) on \( \mathbb{X} \) as follows:

\[
\mu(A) = \mu'((\pi(A)) \quad \text{for any Borel set} \quad A \subset \mathbb{X}.
\]

Consequently, the above decomposition of \( T \) can be rewritten as

\[
T = \int_\mathbb{X} T_x d\mu(x), \quad \text{where} \quad T_x := \tilde{h}_x[\mathcal{L}_x].
\]
Finally, since \( T \) may be regarded as a positive \( dd^c \)-closed current on an open neighborhood of \( (\mathbb{D}^n \setminus (r_0 \mathbb{D}))^n \), the mass of \( T \) on the last set is finite. Therefore, we multiply \( \nu \) by a suitable positive function \( \theta(x) \) and divide \( h_x \) by \( \theta(x) \) in order to assume that the mass of \( T_x \) on \( \mathbb{D}^n \setminus (r_0 \mathbb{D})^n \) is 1 for \( \mu \)-almost every \( x \in \mathbb{X} \). This completes the proof of the lemma.

For the sake of completeness, we give here an alternative argument which permit us to get the decomposition (3.12) directly without using Proposition [2.6]. Since \( T \) is a directed current of bidegree \((1,1)\), we see that if \( \alpha \) is any smooth form of degree 1 or 2 on \( \mathbb{I} \), then \( T \wedge \pi^*(\alpha) = 0 \).

Consider the family \( \mathcal{F} \) of all positive \( dd^c \)-closed \((1,1)\)-currents \( R \) on \( U \) which are vertical in the sense that \( R \wedge \pi^*(\alpha) = 0 \) for any smooth form \( \alpha \) of degree 1 or 2 on \( \mathbb{I} \).

**Claim.** If \( S \) is any current in \( \mathcal{F} \) and \( u \) is a smooth positive function on \( \mathbb{I} \), then \( (u \circ \pi)S \) also belongs to \( \mathcal{F} \).

Indeed, it is clear that \((u \circ \pi)S\) is positive and vertical. The only point to check is that \((u \circ \pi)S\) is \( dd^c \)-closed.

Define \( \tilde{u} := u \circ \pi \). Since \( S \) is \( dd^c \)-closed, we have \( dd^c S = 0 \). Moreover, as \( S \) is vertical, we also get that \( d \tilde{u} \wedge S = 0 \) and \( d^c \tilde{u} \wedge S = 0 \) and \( dd^c \tilde{u} \wedge S = 0 \). Therefore, a straightforward calculation gives

\[
d d^c(\tilde{u}S) = d(d^c \tilde{u} \wedge S) - d^c (d \tilde{u} \wedge S) - dd^c \tilde{u} \wedge S + \tilde{u} dd^c S = 0,
\]

which proves the claim.

It follows from the claim that every extremal element in \( \mathcal{F} \) is supported by a fiber \( \pi^{-1}(\{x\}) \) of \( \pi \), which is biholomorphic to the convex polygon \( \Pi_x \subset \mathbb{C} \). A positive \( dd^c \)-closed current on a Riemann surface is defined by a positive harmonic function. We conclude that every extremal element in \( \mathcal{F} \) is of the form \( \tilde{h}_x[\mathcal{L}_x] \), where \( \tilde{h}_x \) is a positive harmonic function on \( \mathcal{L}_x \). The set of all positive \( dd^c \)-closed vertical currents \( S \in \mathcal{F} \) such that \( \|S\|_{\mathbb{D}^n \setminus (r_0 \mathbb{D})^n} = 1 \) is a convex compacts set. Therefore, by Choquet’s representation theorem, \( S \) is an average of those extremal currents. The decomposition (3.12) follows.

\[\square\]

**Remark 3.5.** We will see later on that the mass of \( T \) on \( \mathbb{D}^n \) is finite. Hence, we can even assume that the mass of \( T_x \) in \( \mathbb{D}^n \) is 1 for \( \mu \)-almost every \( x \in \mathbb{X} \).

For \( \nu \)-almost every \( x \in \mathbb{X} \), consider the function \( h_x : \Pi_x \to \mathbb{R}^+ \) given by

\[
h_x(\zeta) := \tilde{h}_x(\varphi_x(\zeta)) \quad \text{for} \quad \zeta \in \Pi_x.
\]

So \( h_x \) is a positive harmonic function on \( \Pi_x \). For \( x \in \overline{\mathbb{D}^n} \), let \( P_x(\cdot, \cdot) \) be the Poisson kernel of \( \Pi_x \), that is, \( P_x := P_{\Pi_x} \). For \( x \in \mathbb{X} \) and \( r > 0 \) consider the following (eventually empty) sub-domain of \( \Pi_x \):

\[
\Pi_x^r := \{ \zeta \in \Pi_x : \text{dist}_x(\zeta) > r \}.
\]

Note that \( \Pi_x^r \) is an (eventually empty) polygon whose edges are parallel to some edges of \( \Pi_x \).

Since \( \mathbb{D}^n \subset U \), we can find \( r_0' > 0 \) such that \( U' := ((1 + r_0') \mathbb{D})^n \subset U \). For \( x \in \mathbb{X} \) let \( \Pi_x^r := \varphi_x^{-1}(U') \).

**Lemma 3.6.** There exists \( r_1 > 0 \) such that the following assertions hold for \( x \in \mathbb{X} \).

1. If \( \Pi_x \) contains a disc of radius \( 2r_1 \), then \( \Pi_x^r \neq \emptyset \).
2. \( \Pi_x^r \) contains a disc of radius \( 2r_1 \).
Proof. The first assertion holds for all $r_1 > 0$ using (3.14).

Fix $x = (x_1, \ldots, x_n) \in \mathbb{X}$. Let $j = 1, \ldots, k$. If $x_j \neq 0$, following (3.3) let $\mathbb{H}'_{x,j}$ be the open half-plane defined by (3.15)

$$\mathbb{H}'_{x,j} := \{z = u + iv \in \mathbb{C} : -a_j u + b_j v - \log |x_j| > \log (1 + r'_0)\} = \{z \in \mathbb{C} : |\varphi(z)| < 1 + r'_0\}.$$  

Consequently, $\partial \mathbb{H}_{x,j}$ and $\partial \mathbb{H}'_{x,j}$ are parallel lines and $\text{dist}(\partial \mathbb{H}_{x,j}, \partial \mathbb{H}'_{x,j}) = |\lambda_j|^{-1} \log (1 + r'_0)$. Therefore, we get for $z \in \Pi_x$,

$$\text{dist}(z, \partial \mathbb{H}_{x,j}) = \text{dist}(z, \partial \mathbb{H}'_{x,j}) + \log (1 + r'_0).$$

If $x_j = 0$, then set simply $\mathbb{H}'_{x,j} := \mathbb{C}$. As in (3.4) we have that $\Pi'_x = \bigcap_{j=1}^n \mathbb{H}'_{x,j}$. We infer from the above consideration that for $z \in \Pi_x$,

$$\text{dist}(z, \partial \Pi'_x) = \text{dist}(z, \partial \Pi_x) + \log (1 + r'_0).$$

This inequality implies the second assertion for $0 < r_1 \ll (\max_{1 \leq j \leq n} |\lambda_j|^{-1}) \log (1 + r'_0)$. \hfill $\square$

Definition 3.7. Let $\mathbb{X}'$ be the set of all $x \in \mathbb{X}$ such that the convex polygon $\Pi_x$ contains a disc of radius $2r_1$.

Remark 3.8. Roughly speaking, the fact that the convex polygon $\Pi_x$ contains a disc of radius $2r_1$ means that $\Pi_x$ is non-degenerate (i.e., not so thin). In Section 4, we can obtain good estimates on Poisson kernel only for such polygons.

If $x \in \mathbb{X} \setminus \mathbb{X}'$, then $\Pi_x$ does not contain a disc of radius $2r_1$. Consequently, the definition of $r_1$ in Lemma 3.6 implies that $\mathbb{P}_x$ contains a disc of radius $2r_1$. This means that by passing from $\Pi_x$ to $\Pi'_x$, if necessary, we may assume that for “every” $x \in \mathbb{X}$, $\Pi_x$ contains a disc of radius $2r_1$.

Lemma 3.9. For $\mu$-almost every $x \in \mathbb{X}$, the following properties hold.

1. The function $h_x$ is the Poisson integral of its boundary values, that is,

$$h_x(z) = \int_{\partial \Pi_x} P_x(z, \xi) h_x(\xi) d\text{Leb}_1(\xi).$$

2. There is a constant $c > 0$ independent of $x \in \mathbb{X}$ such that

$$\int_{\xi \in \partial \Pi_x} h_x(\xi) d\text{Leb}_1(\xi) < c.$$

Proof. Proof of assertion (1). We only consider $x \in \mathbb{X}$ such that the mass of $T_x$ on $\mathbb{D}^n \setminus (r_0\mathbb{D})^n$ is 1. By Lemma 3.4, $\mu$-almost every $x \in \mathbb{X}$, satisfies this condition. By definition, the mass of $T_x$ in $\mathbb{D}^n \setminus (r_0\mathbb{D})^n$ is the mass of the following positive measure in $\mathbb{D}^n \setminus (r_0\mathbb{D})^n$:

$$T_x \wedge (idz_1 \wedge d\bar{z}_1 + \ldots + idz_n \wedge d\bar{z}_n) = h(x)(idz_1 \wedge d\bar{z}_1 + \ldots + idz_n \wedge d\bar{z}_n) \wedge [L_x].$$

Using the parametrization (3.2) of $L_x$ by $\Pi_x$, we get that the mass of this measure is equal to the one of its pull-back to $\Pi_x$. Using (3.13) and writing $z = u + iv$, the last measure on $\Pi_x$ is

$$h_x(z) \left( e^{-2|\lambda_1| \text{dist}_{a,1}(z)} + \ldots + e^{-2|\lambda_n| \text{dist}_{a,n}(z)} \right) d\text{Leb}_2(z).$$

Since the mass of $\mathbb{D}^n \setminus (r_0\mathbb{D})^n$ is the integral of the above expression on the domain $\Pi_x \setminus \Pi_x^0$ and $r_1 < r_0$, this mass is larger than the integral on the sub-domain $\Pi_x \setminus \Pi_x^{r_1}$. Moreover,
\[ e^{-2|\lambda_1 \text{dist}_{\omega_1}(\zeta)|} + \ldots + e^{-2|\lambda_n \text{dist}_{\omega_n}(\zeta)|} \approx 1 \text{ on } \Pi_x \setminus \Pi_x^1. \] Hence, there is a constant \( c > 0 \) independent of \( x \) such that

\[ (3.16) \quad \int_{\Pi_x \setminus \Pi_x^1} T_x \wedge (idz_1 \wedge d\bar{z}_1 + \ldots + idz_n \wedge d\bar{z}_n) \geq c \int_{\Pi_x \setminus \Pi_x^1} h_x(\zeta)d\text{Leb}_2(\zeta). \]

So there is a constant \( c' > 0 \) independent of \( x \in X \) such that

\[ (3.17) \quad \int_{\Pi_x \setminus \Pi_x^1} h_x(\zeta)d\text{Leb}_2(\zeta) < c'. \]

Since \( h_x \) is a positive harmonic function on \( \Pi_x \) and continuous up to the boundary \( \partial \Pi_x \), it follows from Proposition 4.1 that one of the following two cases happens.

1. Case 1: \( \Pi_x \) is bounded. In this case we have
   \[ h_x(\zeta) = \int_{\partial \Pi_x} P_x(\zeta, \xi)h_x(\xi)d\text{Leb}_1(\xi) \quad \text{for} \quad \zeta \in \Pi_x. \]

2. Case 2: \( \Pi_x \) is unbounded. Let \( \phi_x \) be a biholomorphic map from \( \Pi_x \) onto \( \mathbb{D} \) which sends \( \infty \) to \( 1 \in \partial \mathbb{D} \). Then, there is a constant \( c_x > 0 \) such that
   \[ h_x(\zeta) = \int_{\partial \Pi_x} P_x(\zeta, \xi)h_x(\xi)d\text{Leb}_1(\xi) + c_x \Gamma_x(\zeta) \quad \text{for} \quad \zeta \in \Pi_x. \]

Here, \( \Gamma_x \) is the positive harmonic function on \( \Pi_x \) defined in (4.4) by

\[ \Gamma_x(\zeta) := \frac{1 - |\phi_x(\zeta)|^2}{|\phi_x(\zeta) - 1|^2} \quad \text{for} \quad \zeta \in \Pi_x. \]

To complete the proof of assertion (1), we only need to check that if Case (2) happens, then \( c_x = 0 \). There are two subcases to consider.

Subcase \( x \in X' \): We infer from the above equality of Case (2) and the fact that \( h_x \geq 0 \), \( P_x(\zeta, \xi) \geq 0 \) that

\[ h_x(\zeta) = \int_{\partial \Pi_x} P_x(\zeta, \xi)h_x(\xi)d\text{Leb}_1(\xi) + c_x \Gamma_x(\zeta) \geq c_x \Gamma_x(\zeta) \quad \text{for} \quad \zeta \in \Pi_x. \]

By Proposition 4.2, \( \Gamma_x(\zeta) \geq c_x^* \) for \( \zeta \in \partial \Pi_x^1 \). Hence, we get that

\[ \int_{\Pi_x \setminus \Pi_x^1} h_x(\zeta)d\text{Leb}_2(\zeta) \geq c_x^* c_x \int_{\partial \Pi_x^1} \Gamma_x(\zeta)d\text{Leb}_1(\zeta) \geq c_x^* c_x \int_{\partial \Pi_x^1} d\text{Leb}_1(\zeta). \]

Since the first integral is finite and the last one is infinite (as \( \Pi_x \) is unbounded and \( \Pi_x^1 + \emptyset \) and \( c_x^* > 0 \), we infer that \( c_x = 0 \). This proves assertion (1).

Since \( h_x \) is positive harmonic on an open neighborhood of \( \Pi_x \), by Harnack’s inequality \( h_x(\zeta)/h_x(\zeta - \xi) \) is bounded from below by a strictly positive constant independent of \( x \) for \( |\xi| \lesssim 1 \). We infer from (3.17) that its integral on \( \partial \Pi_x \) is also bounded by a constant.

Assertion (2) follows in this subcase.

Subcase \( x \notin X' \): By Remark 3.8, \( \Pi_x \) is a unbounded polygon which contains a disc of radius \( 2r_1 \). Therefore, we are still able to apply Proposition 4.2 as in the previous subcase in order to prove assertion (1).

Assertion (2) can be proved in the same way as in the previous subcase.

The proof of the lemma is complete modulo Proposition 4.2. \( \square \)
\textbf{Remark 3.10.} Lemma 3.9 in dimension \( n = 2 \) has previously been obtained by Fornæss–Sibony in [20, Proposition 1] (see also [13, Lemma 4.2] for another proof). In their analysis, these authors make a full use of the fact that \( \Pi_\pi \) is independent of \( x \in X \) modulo a translation. However, this peculiar fact in dimension 2 does not hold in higher dimensions.

For \( 0 < r < 1 \), let
\begin{equation}
F(r) := \int_{\mathbb{D}(0,r)} T \wedge dd^c \|x\|^2.
\end{equation}
Consider also the function
\begin{equation}
f(r) := \frac{1}{\pi r^2} F(r).
\end{equation}
By (2.3) the Lelong number \( \nu(T,0) \) of \( T \) at \( 0 \) is \( \limsup_{r \to 0} f(r) \). Let
\[ K := \{ (x, \xi) : \ x \in X \quad \text{and} \quad \xi \in \partial \Pi_x \}. \]
For each \( s > 0 \), consider the function \( K_s : K \to \mathbb{R}^+ \) given by
\begin{equation}
K_s(x, \xi) := \begin{cases} 
\int_{\xi \in \Pi_x} e^{2s - 2d_{\Pi_x}(\xi)} P_\pi(\xi, \xi) d\text{Leb}_2(\xi), & \text{if } \Pi_x \neq \emptyset; \\
0, & \text{if } \Pi_x = \emptyset.
\end{cases}
\end{equation}

\textbf{Lemma 3.11.} For every \( 0 < r < 1 \),
\[ f(r) \leq n \int_{x \in X} \left( \int_{\xi \in \Pi_x : |\varphi_x(\xi)| \leq r} K_{-\log r}(x, \xi) h_x(\xi) d\text{Leb}_1(\xi) \right) d\mu(x). \]

\textbf{Proof.} Applying Lemma 3.9 to (3.18) and using (3.13), we see that for \( 0 < r \ll 1 \),
\[ F(r) = \int_{x \in X} \int_{\xi \in \Pi_x} h_x(\xi) |\varphi_x'(\xi)|^2 d\text{Leb}_2(\xi) d\mu(\alpha). \]
On the other hand, we infer from (3.2) and (3.9)–(3.10) that for \( y = \varphi_x(\xi), |y| \leq r \) implies \( d_{\Pi_x}(\xi) \geq -\log r \). Moreover, using (3.13) and (3.9)–(3.10) again, we get that
\[ |\varphi_x'(\xi)|^2 = e^{-2|\lambda_1|d_{\Pi_x}(\xi)} + \ldots + e^{-2|\lambda_n|d_{\Pi_x}(\xi)} \leq ne^{-2d_{\Pi_x}(\xi)}. \]

Consequently,
\[ F(r) \leq n \int_{x \in X} \int_{\xi \in \Pi_x} h_x(\xi) e^{-2d_{\Pi_x}(\xi)} d\text{Leb}_2(\xi) d\mu(\alpha). \]
Applying Lemma 3.9 to the inner integral of the last line and using (3.19), the lemma follows. \( \Box \)

\textbf{Remark 3.12.} It is worthy noting that the above proof also shows the following estimate. For every \( 0 < r < 1 \),
\[ \hat{f}(r) \leq n \int_{x \in X} \left( \int_{\xi \in \Pi_x} K_{-\log r}(x, \xi) h_x(\xi) d\text{Leb}_1(\xi) \right) d\mu(x), \]
where \( \hat{f}(r) := \frac{1}{\pi r^2} \hat{F}(r) \) and \( \hat{F}(r) := \int_{(rD)^n} T \wedge dd^c \|x\|^2. \) This estimate is stronger than Lemma 3.11 because \( \mathbb{D}(0,r) \subset (rD)^n. \)
4. POISSON KERNELS OF CONVEX POLYGONS: MAIN ESTIMATES

Let $\Gamma$ be the positive harmonic function on $\mathbb{D}$ given by

$$
\Gamma(\zeta) := \frac{1 - |\zeta|^2}{|\zeta - 1|^2} \quad \text{for} \quad \zeta \in \mathbb{D}.
$$

**Proposition 4.1.** Let $\Omega \subset \mathbb{C}$ be a (not necessarily bounded) convex polygon with the Green function $G(\zeta, \xi)$. Following the model of $C^2$-smooth bounded domains of Proposition 2.14, let the Poisson kernel on $\Omega$ be the function

$$
P(\zeta, \xi) := -\operatorname{Nor}_\xi G(\zeta, \xi) \quad \text{for} \quad \zeta \in \Omega, \ \xi \in \partial \Omega.
$$

There are two cases.

1. **Case $\Omega$ is bounded:** Then, for every positive function $u \in \mathscr{C}(\overline{\Omega})$ which is harmonic on $\Omega$, we have

$$
u(\zeta) = \int_{\Omega} P(\zeta, \xi)u(\xi)\mathrm{d}\text{Leb}_1(\xi) \quad \text{for} \quad \zeta \in \Omega.
$$

2. **Case $\Omega$ is unbounded:** Let $\phi$ be a biholomorphic map from $\Omega$ onto $\mathbb{D}$ which sends $\infty$ to $1 \in \partial \mathbb{D}$. Then, for every positive function $u \in \mathscr{C}(\overline{\Omega})$ which is harmonic on $\Omega$, there is a constant $c = c_u > 0$ such that

$$
u(\zeta) = \int_{\Omega} P(\zeta, \xi)u(\xi)\mathrm{d}\text{Leb}_1(\xi) + c(\Gamma \circ \phi)(\zeta) \quad \text{for} \quad \zeta \in \Omega.
$$

Moreover, in both cases, $P(\cdot, \xi)$ is a positive harmonic function on $\Omega$ when $\xi \in \partial \Omega$ is fixed.

**Proof.** Let $\phi$ be a biholomorphic map from $\Omega$ onto $\mathbb{D}$. Since $\Omega$ is a convex polygon, $\phi$ extends continuously to $\overline{\Omega}$ and $\phi|_{\partial \Omega}$ is one-to-one onto its image in $\partial \mathbb{D}$. Let $G_{\mathbb{D}}$ be the Green function of the unit-disc $\mathbb{D}$. It follows from the definition of Green function that

$$
G(\zeta, \xi) = G_{\mathbb{D}}(\phi(\zeta), \phi(\xi)) \quad \text{for} \quad (\zeta, \xi) \in \Omega \times \overline{\mathbb{D}} \setminus \Delta.
$$

To prove assertion (1), Let $u$ be a function in $\mathscr{C}(\overline{\Omega})$ which is harmonic on $\Omega$, and let $\zeta \in \Omega$. Observe that $\phi$ extends to a diffeomorphism from $\overline{\Omega}$ onto $\partial \mathbb{D}$. Consider the function $v : \overline{\mathbb{D}} \rightarrow \mathbb{R}$ defined by

$$
v(\hat{\zeta}) := u(\phi^{-1}(\hat{\zeta})) \quad \text{for} \quad \hat{\zeta} \in \overline{\mathbb{D}}.
$$

By Proposition 2.14 applied to $v$, we have that

$$
v(\phi(\zeta)) = \int_{\partial \mathbb{D}} P_{\mathbb{D}}(\phi(\zeta), \hat{\xi})v(\hat{\xi})\mathrm{d}\text{Leb}_1(\hat{\xi}) = \int_{\partial \Omega} -\operatorname{Nor}_{\hat{\xi}} G_{\mathbb{D}}(\phi(\zeta), \hat{\xi})v(\hat{\xi})\mathrm{d}\text{Leb}_1(\hat{\xi}).
$$

Using the change of variable $\hat{\xi} := \phi(\xi)$ for $\xi \in \partial \Omega$, we see that the RHS of the last line is equal to

$$
\int_{\partial \Omega} \left( -\operatorname{Nor}_{\hat{\xi}} G_{\mathbb{D}}(\phi(\zeta), \hat{\xi}) \right)_{\hat{\xi} = \phi(\xi)}v(\phi(\xi))\mathrm{d}\text{Leb}_1(\phi(\xi)).
$$

Since $\phi$ is conformal on $\overline{\Omega}$, we see that $\mathrm{d}\text{Leb}_1(\phi(\xi))|_{\partial \mathbb{D}} = \mathrm{d}\text{Leb}_1(\hat{\xi})|_{\partial \Omega}|\phi'(\xi)|$. Moreover, using identity (4.3) and Proposition 2.14 we also get that

$$
\left( -\operatorname{Nor}_{\hat{\xi}} G_{\mathbb{D}}(\phi(\zeta), \hat{\xi}) \right)_{\hat{\xi} = \phi(\xi)} = \left( -\operatorname{Nor}_{\xi} G_{\mathbb{D}}(\zeta, \xi) \right)|\phi'(\xi)|^{-1} = P_{\mathbb{D}}(\zeta, \xi)|\phi'(\xi)|^{-1}.
$$

So the last integral is equal to

$$
\int_{\partial \Omega} P_{\mathbb{D}}(\zeta, \xi)u(\xi)\mathrm{d}\text{Leb}_1(\xi).
$$
Consequently, assertion (1) follows.

Now we turn to the proof of assertion (2). Observe that \( \phi \) maps \( \partial \Omega \) bijectively onto \( \partial \mathbb{D}\setminus\{0\} \). On the other hand, for every positive function \( u \in C(\overline{\mathbb{D}}\setminus\{1\}) \) which is harmonic on \( \mathbb{D} \), there is a constant \( c = c_u \geq 0 \) such that \( u \) is the Poisson integral of the measure \( u(y)\sigma_{\mathbb{D}}(y) + \delta_1 \), where \( \sigma_{\mathbb{D}} \) is the Lebesgue measure on \( \partial \mathbb{D} \), and \( \delta_1 \) is the Dirac mass at 1. 

So using (4.1) and the explicit formula of \( P_\mathbb{D} \), we get

\[
    u(\zeta) = \int_{\partial \Omega} P_\mathbb{D}(\zeta, \xi)u(\xi)d\text{Leb}_1(\xi) + c\Gamma(\zeta) \quad \text{for} \quad \zeta \in \mathbb{D}.
\]

Using this and identity (4.3), we infer from (4.1) that there is a biholomorphic map \( \phi_x \) from \( \Pi_x \) onto \( \mathbb{D} \) which sends \( \infty \) to 1 in \( \partial \mathbb{D} \). Let \( \Gamma_x \) be positive harmonic function on \( \Pi_x \) defined by

\[
    \Gamma_x(\zeta) = c_x^* \quad \text{for} \quad \zeta \in \Pi_x^\Gamma.
\]

where \( \Gamma \) is given in (4.1). Then, there is a constant \( c_x^* > 0 \) dependent on \( x \) such that

\[
    \Gamma_x(\zeta) \geq c_x^* \quad \text{for} \quad \zeta \in \Pi_x^\Gamma.
\]

**Proof.** We rephrase the problem differently but equivalently. So we only need to prove that for every \( x \) as in the assumption, there is a constant \( c_x^* > 0 \) dependent on \( x \in \mathbb{X} \) such that there is a biholomorphic map \( \tilde{\phi}_x \) from the upper-half plane \( \mathbb{H} := \{ \zeta \in \mathbb{C} : \text{Im} \zeta > 0 \} \) onto \( \Pi_x \) sending \( \infty \) to \( 1 \) in \( \partial \mathbb{D} \) such that

\[
    \tilde{\phi}_x(\mathbb{H}_c) \cap \Pi_x^\Gamma = \emptyset, \quad \text{where} \quad \mathbb{H}_c := \{ \zeta \in \mathbb{C} : \text{Im} \zeta \in (0, c) \}.
\]

Indeed, since \( \tilde{\phi}_x := \tilde{\phi}_x \circ \phi_x \) is a biholomorphic map from \( \mathbb{H} \) onto \( \mathbb{D} \) sending \( \infty \) to 1 in \( \partial \mathbb{D} \), we infer from (4.1) that

\[
    \Gamma(\tilde{\phi}_x(\zeta)) = \text{Im} \zeta \quad \text{for} \quad \zeta \in \mathbb{H}.
\]

Therefore, it follows that

\[
    \Gamma_x(\zeta) = \text{Im} \tilde{\phi}_x^{-1}(\zeta) \quad \text{for} \quad \zeta \in \Pi_x.
\]

Thus (4.5) implies the proposition.

Next, we will prove that there is constant \( c_x^* > 0 \) dependent on \( x \in \mathbb{X} \) such that for \( \theta \in \mathbb{C} \) with \( |\theta| < c_x^* \),

\[
    |\tilde{\phi}_x(\zeta) - \tilde{\phi}_x(\zeta + \theta)| \leq 1 \quad \text{for} \quad \zeta \in \partial \mathbb{H}.
\]

Taking (4.6) for granted, (4.5) will follow because \( \tilde{\phi}_x(\zeta) \in \partial \Pi_x \) for \( \zeta \in \partial \mathbb{H} \), which completes the proof of the proposition.

To prove the reduction (4.6). Let \( w_1, \ldots, w_k \) be all finite vertices of the convex polygon \( \Pi_x \) in counterclockwise order and set \( w_0 = w_{k+1} := \infty \), see Figure 4. Let \( \alpha_j := \angle(w_{j-1}w_jw_{j+1}, w_{j+1}w_{j-1}) \), for \( 1 \leq j \leq k + 1 \), be their corresponding interior angles in counterclockwise order, with the convention that \( w_{k+2} := w_1 \). Observe that \( \alpha_j \in (0, \pi) \) for \( 1 \leq j \leq k \) and \( \alpha_{k+1} \in (-\pi, 0] \). Write

\[
    \alpha_j := \frac{\pi}{\gamma_j} \quad \text{for} \quad 1 \leq j \leq k.
\]
Figure 4. On the left: the upper-half plane \( \mathbb{H} \) and the points \( z_1, \ldots, z_k \in \partial \mathbb{H} \), where the dotted points correspond to intermediate points \( z_3, \ldots, z_{k-2} \). On the right: the image of \( \mathbb{H} \) by \( \tilde{\phi}_x \) : the unbounded convex polygon \( \Pi_x \) with vertices \( w_1 = \tilde{\phi}_x(z_1), \ldots, w_k = \tilde{\phi}_x(z_k) \), where the dashed line corresponds to the (not necessarily aligned) intermediate points \( w_3, \ldots, w_{k-2} \).

So \( \gamma_1, \ldots, \gamma_k > 1 \). By the classical Schwarz-Christoffel formula (see e.g. [16, formula (22) p.10]), we can write

\[
\tilde{\phi}_x(\zeta) = c'_x + c''_x \int_\zeta^\zeta \prod_{j=1}^k (\eta - z_j)^{\frac{1}{\gamma_j}-1} d\eta,
\]

for some complex constants \( c'_x \) and \( c''_x \), where \( z_1, \ldots, z_k \in \partial \mathbb{H} \) and \( \tilde{\phi}_x(z_j) = w_j \) for \( 1 \leq j \leq k \).

Clearly, \( c''_x \neq 0 \). Consequently, we infer from (4.8) that for every \( \zeta \in \partial \mathbb{H} \) and \( \theta \in \mathbb{C} \),

\[
|\tilde{\phi}_x(\zeta) - \tilde{\phi}_x(\zeta + \theta)| = |c''_x| \int_\zeta^{\zeta+\theta} \prod_{j=1}^k (\eta - z_j)^{\frac{1}{\gamma_j}-1} d\eta.
\]

In order to prove (4.6), we need the following auxiliary result.

**Lemma 4.3.** Let \( p \geq 1 \) and let \( s_1, \ldots, s_p \in (-1, 0) \). Let \( t_1 < \ldots < t_p \) be real numbers. Then, for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that for \( a \in [t_1 - 1, t_p + 1] \), we have

\[
\int_{a}^{a+\delta} \prod_{j=1}^p |t - t_j|^s dt < \epsilon.
\]

**Proof.** The proof is elementary and we leave it to the interested reader. \( \square \)

Resuming the proof of (4.6), we consider two cases according to the position of \( \zeta \in \partial \mathbb{H} \) with respect to the set \( Z := \{z_1, \ldots, z_k\} \subset \partial \mathbb{H} \). Observe that \( z_1 < \cdots < z_k \) since \( \tilde{\phi}_x(z_j) = w_j \) and the \( w_j \) are in counterclockwise order. By a change of variables, we rewrite (4.9) as

\[
|\tilde{\phi}_x(\zeta) - \tilde{\phi}_x(\zeta + \theta)| = \int_0^{|\theta|} \prod_{j=1}^k |(\zeta + \frac{\theta}{|\theta|} t) - z_j|^{\frac{1}{\gamma_j}-1} dt.
\]

**Case** \( \text{dist}(\zeta, Z) \ll 1 \).
Let \( 1 \leq l < m \leq k \) be such that \( \text{dist}(\zeta, z_j) \ll 1 \) for \( l \leq j \leq m \) and \( \text{dist}(\zeta, z_j) \gg 1 \) otherwise. So \( \left| (\zeta + \frac{\theta}{|\theta|} t) - z_j \right| \gtrsim 1 \) for \( j \notin [l, m] \), we deduce from (4.10) that

\[
|\tilde{\phi}_x(\zeta) - \tilde{\phi}_x(\zeta + \theta)| \lesssim |c|^{\prime} \int_{t_0}^{t_1} \int_{|\theta|}^{|\theta|} |(\zeta + \frac{\theta}{|\theta|} t) - z_j|^{\frac{1}{2j}} dt.
\]

Moreover, the RHS is dominated by a constant times

\[
|c|^{\prime} |s - \zeta|^\sum_{j=0}^{k} \frac{1}{j} ds.
\]

Using (4.7) we may apply Lemma 4.3. Consequently, the last integral is small provided that \( c := c_0 \) is small enough for \( \theta \in \mathbb{C} \) with \( |\theta| \in (0, c) \). This proves (4.6) in this case.

**Case** \( \text{dist}(\zeta, Z) \gg 1 \). So for a constant \( 0 < c \ll 1 \), we get that \( |(\zeta + \frac{\theta}{|\theta|} t) - z_j| \gtrsim |\zeta - z_j| - |t| 
\approx 1 - c > 0 \) for \( t \in [0, |\theta|] \) and \( |\theta| < c \). Therefore, we deduce from (4.10) that

\[
|\tilde{\phi}_x(\zeta) - \tilde{\phi}_x(\zeta + \theta)| \lesssim c'(1 - c)^\sum_{j=1}^{k} \frac{1}{j},
\]

where \( c' \) is a constant depending only on \( n \). Choosing \( 0 < c \ll 1 \) and \( c_0 := c \) small enough, (4.6) holds in this last case. \( \square \)

The following result due to Widder [40] gives the Poisson kernel for strips.

**Proposition 4.4.** For \( (a, b, c, d) \in \mathbb{R}^4 \) with \( a^2 + b^2 > 0 \) and \( c < d \), consider the strip

\[
S = S_{a,b,c,d} := \{ \zeta = u + iv \in \mathbb{C} : c < au + bv < d \},
\]

which is limited by two parallel lines \( L_1 = \{ au + bv = c \} \) and \( L_2 = \{ au + bv = d \} \). Let \( R := \text{dist}(L_1, L_2) \) be the distance between \( L_1 \) and \( L_2 \). See Figure 5. Then the following assertions hold:

1. The Poisson kernel of \( S \) is given by

\[
P_S(\zeta, \xi) = \frac{\pi}{R} \cdot \frac{\sin \left( \frac{\pi \text{dist}(\zeta, \xi)}{R} \right)}{\cosh \left( \frac{\pi \text{dist}(\zeta, \xi)}{R} \right) - \cos \left( \frac{\pi \text{dist}(\zeta, \xi)}{R} \right)} \quad \text{for} \quad \zeta \in S, \quad \xi \in \partial S.
\]

Here, if \( \xi \in L_j \) then \( \xi_j \) is the orthogonal projection of \( \zeta \) onto \( L_j \).

2. For \( \zeta \in S \) and \( \xi \in \partial S \), we have

\[
P_S(\zeta, \xi) \lesssim 2 \cdot \frac{\text{dist}(\zeta, \partial S)}{(\text{dist}(\zeta, \xi))^2}.
\]

**Proof.** Using a rotation and a translation, we may suppose that the strip \( S \) is given by \( S_R := \{ \zeta = u + iv \in \mathbb{C} : 0 < v < R \} \). The change of variable \( \zeta \mapsto \frac{\pi \zeta}{R} \) maps \( S_R \) biholomorphically onto \( S_R \). Using this and the explicit formula of the Poisson kernel of \( S_R \) established in [40] formula (1)], we get that

\[
P_{S_R}(\zeta, \xi) = \frac{\pi}{R} \cdot \frac{\sin \left( \frac{\pi |\text{Im}(\zeta - \xi)|}{R} \right)}{\cosh \left( \frac{\pi |\text{Re}(\zeta - \xi)|}{R} \right) - \cos \left( \frac{\pi |\text{Im}(\zeta - \xi)|}{R} \right)} \quad \text{for} \quad \zeta \in S_R, \quad \xi \in \partial S_R.
\]

Since \( |\text{Im}(\zeta - \xi)| = \text{dist}(\zeta, \xi) \) and \( |\text{Re}(\zeta - \xi)| = \text{dist}(\xi, \xi_j) \), assertion (1) follows from this formula.
Proposition 4.5. Let $\Omega$ contains in Poisson kernels (see e.g. Krantz [26] for the principle in the case of smooth bounded domains). By Definition 2.13 and by Proposition 4.1 (see also identity (4.3)), we see that

$$\Omega = \{ au + bv = c \} \quad \text{and} \quad \Omega = \{ au + bv = d \},$$

a point $\zeta \in \Omega$ and a point $\xi \in \partial \Omega$. In this figure $\zeta \in \Omega$, and hence $\zeta$ is the orthogonal projection of $\zeta$ onto $\Omega$. Moreover, in this figure we see that $\text{dist}(\zeta, \partial \Omega) = \text{dist}(\zeta, \Omega)$. Hence, since $\frac{2}{\pi} \leq \text{min}(t, \pi - t) \sin t \leq \text{min}(t, \pi - t)$ for $t \in [0, \pi]$, we infer that

$$\frac{2}{\pi} \leq \sin \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right) \leq 1.$$

Next, observe that $\cosh t \geq 1 \geq \cosh t$ for $t \in \mathbb{R}$. Moreover, Taylor expansion of the function $\cosh t$ gives that $\cosh t \geq t^2$ for $t \in \mathbb{R}$. Writing $1 - \cos t = 2 \sin^2 \frac{t}{2}$, we see that $\frac{4}{\pi} \leq 2(1 - \cos t) \leq 1$ for $t \in [0, \pi]$. Using the above estimates, we obtain, for $\zeta \in \Omega$ and $\xi \in \partial \Omega$

$$\cosh \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right) - \cos \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right) = \left( \cosh \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right) - 1 \right) + \left( 1 - \cos \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right) \right) \geq \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right)^2 + \frac{1}{2} \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right)^2 \geq \frac{1}{2} \left( \frac{\pi(\text{dist}(\zeta, \Omega))}{R} \right)^2,$$

where the equality in the last line holds by Pythagorean Theorem. Inserting these estimates in the expression of $P_{\Omega}(\zeta, \xi)$ given in assertion (1), we obtain assertion (2). \qed

The following results presents the basic technique to compare Green functions and Poisson kernels (see e.g. Krantz [26] for the principle in the case of smooth bounded domains in $\mathbb{R}^N$).

**Proposition 4.5.** Let $\Omega_1$ and $\Omega_2$ be (not necessarily bounded) convex polygons in $\mathbb{C}$. Suppose that $\Omega_1 \subset \Omega_2$ and there is a point $\xi \in \partial \Omega_1 \cap \partial \Omega_2$ which is not a vertex of $\Omega_1$ and $\Omega_2$. Then, for every $\zeta \in \Omega_1$,

$$P_{\Omega_1}(\zeta, \xi) \leq P_{\Omega_2}(\zeta, \xi).$$

**Proof.** By Definition 2.13 and by Proposition 4.1 (see also identity (4.3)), we see that $G_{\Omega_1}(\zeta, \cdot)$ equals 0 on $\partial \Omega_1$. Moreover, by the Maximum Principle for harmonic functions, $G_{\Omega_2}(\zeta, \cdot) \geq 0$ on $\Omega_2$. Since $\Omega_1 \subset \Omega_2$, it follows that $G_{\Omega_2}(\zeta, \cdot) \geq 0$ on $\partial \Omega_1$. Since the function $G_{\Omega_1}(\zeta, \cdot) - G_{\Omega_2}(\zeta, \cdot)$ is harmonic on $\Omega_1$, it follows from the Maximum Principle again that $G_{\Omega_2}(\zeta, \cdot) - G_{\Omega_1}(\zeta, \cdot) \geq 0$ on $\Omega_1$. On the other hand, since $\xi \in \partial \Omega_1 \cap \partial \Omega_2$, we infer from Definition 2.13 again that $G_{\Omega_2}(\zeta, \xi) - G_{\Omega_1}(\zeta, \xi) = 0 - 0 = 0$. Therefore, by the Hopf lemma
Proposition 4.6. Suppose that in dimension some $\gamma > 1$. Then, combined with (4.2), completes the proof. □

The next result describes the complete behavior of the Poisson kernel of a phase space in dimension $n = 2$.

**Proposition 4.6.** Suppose that $n = 2$. So $\Pi_x$ is a sector in $\mathbb{C}$ with aperture angle $\pi/\gamma$ for some $\gamma > 1$. Then there is a constant $c > 1$ which depends only on $\lambda_1$ and $\lambda_2$ such that for every $x \in X$, $\zeta \in \Pi_x$ and $\xi \in \partial \Pi_x$,

$$c^{-1} \leq \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2} \cdot \left( \frac{\min (\text{dist}_x^*(\zeta), \text{dist}_x^*(\xi))}{\max (\text{dist}_x^*(\zeta), \text{dist}_x^*(\xi))} \right)^{\gamma - 1} \leq c.$$ 

**Proof.** We use the proof and the notation of Lemma 3.3 in [31]. By using a translation and a rotation in $\mathbb{C}$ if necessary, we may assume without loss of generality that $\lambda_1 = 1$ and $\lambda_2 = a - ib$ with $a, b \in \mathbb{R}$ and $b > 0$. So the aperture angle of $\Pi_x$ is $\alpha := \pi/\gamma = \arctan(-b/a)$ and $\Pi_x$ is given by

$$\Pi_x := \{ \zeta \in \mathbb{C} : \arg \zeta \in (0, \pi/\gamma) \}.$$ 

Write $\zeta := u + iv$ with $u, v \in \mathbb{C}$. So $\Pi_x = \{ \zeta \in \mathbb{C} : v > 0 \text{ and } bu + av > 0 \}$ and $\partial \Pi_{x,1} = \{ v = 0 \}$, $\partial \Pi_{x,2} = \{ bu + av = 0 \}$. The map

$$\tau : z \mapsto z^\gamma$$

maps the sector $\Pi_x$ to the upper half plane $\mathbb{H} := \{ U + iV \in \mathbb{C} : V > 0 \}$ with coordinates $(U, V)$. Write

$$y := \tau(\xi) \quad \text{and} \quad U + iV := \tau(\zeta).$$

So $y$ lies on the real line $\partial \mathbb{H} = \{ U + iV \in \mathbb{C} : V = 0 \}$. Note that

$$(4.12) \quad \text{dist}_x(\zeta) \approx \min \{ v, bu + av \} \quad \text{and} \quad \text{dist}_x^*(\zeta) \approx \max \{ v, bu + av \}.$$ 

We deduce from (4.11) and $y := \tau(\xi)$ that

$$P_{\mathbb{H}}(U + iV, y)dy = P_x(\zeta, \xi)d\text{Leb}_1(\xi)$$

and $dy = |\xi|^\gamma - 1 d\xi$. So it follows from the explicit formula $P_{\mathbb{H}}(U + iV, y) = \frac{V}{\sqrt{U^2 + (y - U)^2}}$ that

$$(4.13) \quad P_x(\zeta, \xi) = \frac{V}{\sqrt{U^2 + (y - U)^2}}|\xi|^{\gamma - 1}.$$ 

Let $c_1, c_2, c_3 > 1$ be constants large enough with $c_3 > c_2$ given by Lemma 3.3 in [31]. By assertion (1) of that lemma and by (4.12), we get that

$$(4.14) \quad \frac{1}{c_1} \leq \frac{(\text{dist}_x^*(\zeta))^\gamma}{\sqrt{V^2 + U^2}} \leq c_1 \quad \text{and} \quad \frac{1}{c_1} \leq \frac{(\text{dist}_x(\zeta))^\gamma - 1 \text{dist}_x(\zeta)}{V} \leq c_1.$$ 

Note that by equality $y = \tau(\xi) = \xi^\gamma$ and by (4.12),

$$(4.15) \quad (1 + |y|)^{1/\gamma} \approx |\xi| \quad \text{and} \quad \text{dist}_x^*(\xi) \approx |\xi|.$$
According to Lemma 3.3 in [31], we consider four cases.

Case $\text{dist}_x^*(\zeta) \geq c_2 |\zeta|$: In this case $\text{dist}_x^*(\zeta) \approx |\zeta|$ and $|\zeta - \xi| \approx \text{dist}_x^*(\zeta)$. This, combined with assertion (2) of Lemma 3.3 in [31] and (4.13) and (4.12), (4.15), implies that

$$P_x(\zeta, \xi) \approx \frac{\text{dist}_x(\zeta)}{(\text{dist}_x^*(\zeta))^{\gamma+1}} |\xi|^{\gamma-1} \approx \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2} \cdot \left(\frac{|\xi|}{\text{dist}_x^*(\zeta)}\right)^{\gamma-1}.$$  

So the conclusion of the lemma is true in this case.

Case $\text{dist}_x^*(\zeta) \leq c_2^{-1} |\xi|$: In this case $|\zeta - \xi| \approx |\zeta| \approx \text{dist}_x^*(\zeta)$. This, combined with assertion (3) of Lemma 3.3 in [31] and (4.13) and (4.12), (4.15), implies that

$$P_x(\zeta, \xi) \approx \frac{\text{dist}_x(\zeta)(\text{dist}_x^*(\zeta))^{\gamma-1}}{|\xi|^{2\gamma}} \frac{|\xi|^{\gamma-1}}{\text{dist}_x(\zeta)} \approx \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2} \cdot \left(\frac{\text{dist}_x^*(\zeta)}{|\xi|}\right)^{\gamma-1}.$$  

So the conclusion of the lemma is also true in this second case.

Case $c_2^{-1} |\xi| \leq \text{dist}_x(\zeta), \text{dist}_x^*(\zeta) \leq c_2 |\xi|$: In this case $\text{dist}_x(\zeta) \approx \text{dist}_x^*(\zeta) \approx |\zeta|$ and $|\zeta - \xi| \approx |\xi|$. This, combined with assertion (4) of Lemma 3.3 in [31] and (4.13) and (4.12), (4.15), implies that

$$P_x(\zeta, \xi) \approx \frac{|\xi|^{\gamma-1}}{|\xi|^\gamma} \approx \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2}.$$  

So the conclusion of the lemma is also true in this third case.

Case $\text{dist}_x(\zeta) \leq c_2^{-1} |\xi|$ and $c_2^{-1} |\xi| \leq \text{dist}_x^*(\zeta) \leq c_2 |\xi|$: Following the proof of assertion (5) in Lemma 3.3 in [31], we may suppose without loss of generality that $v \leq bu + av$. For every $1 \leq v \leq c_3 (1 + |y|)^{1/\gamma}$, there exists a solution $\hat{u} := \hat{u}(y, v)$ of the following equation

$$\hat{U} = y, \quad \text{where} \quad \hat{U} + i \hat{V} = (\hat{u} + iv)^\gamma = \tau(\hat{u} + iv)$$

which satisfies

$$c_2^{-1} (1 + |y|)^{1/\gamma} \leq \hat{u}(y, v), \quad \rho(y, v) \leq c_2 (1 + |y|)^{1/\gamma},$$

where $\rho(y, v) := b\hat{u}(y, v) + av$. Let $\rho := \rho(y, v)$.

There are two subcases.

Subcase: $y \geq 0$.

As $y = \tau(\xi)$, we see that $\xi$ lies on the ray $\{v = 0, \ u > 0\}$. So by (4.12), $|\xi| \approx |y|^{1/\gamma} \approx \text{dist}_x(\zeta)$ and $\text{dist}_x(\zeta) \approx v$. By Lemma 3.4 in [31] applied to $y + i \hat{V} = \hat{U} + i \hat{V} = (\hat{u} + iv)^\gamma$ and $y = \xi^\gamma$, we have that

$$\hat{V} \approx |(\hat{u}(y, v) + iv) - \xi| \xi^{\gamma-1}.$$  

On the other hand, inequality (13) in Lemma 3.4 and assertion (1) of Lemma 3.3 in [31] together imply that

$$\hat{V} \approx v \approx v \xi^{\gamma-1}.$$  

Hence, we infer that

$$(4.16) \quad |(\hat{u}(y, v) + iv) - \xi| \approx |v|$$

when $c_3$ is large enough. Consequently, we get that

$$|\zeta - \xi| = |u + iv - \xi| \leq |(\hat{u}(y, v) + iv) - \xi| + |u - \hat{u}(y, v)| \lesssim |v| + |(bu + av) - \rho|.$$  

On the other hand, since $|\zeta - \xi| \geq \text{dist}(\zeta, \partial \Pi_x) \gtrsim \text{dist}_x(\zeta) \approx v$, we deduce from (4.16) that

$$|\zeta - \xi| \approx |\zeta - \xi| + v \approx |\zeta - \xi| + |(\hat{u}(y, v) + iv) - \xi| \approx |u - \hat{u}(y, v)| \gtrsim |v| + |(bu + av) - \rho|.$$
Figure 6. We apply Proposition 4.5 to $\Omega_1 := \Pi_x$ and $\Omega_2 := \mathbb{H}_{x,k} \cap \mathbb{H}_{x,l}$, and to $\zeta \in \Pi_x$ and $\xi \in \mathring{\Pi}_x \cap \mathring{\mathbb{H}}_{x,l}$.

So $|v| + |(bu + av) - \rho| \approx |\zeta - \xi|$. This, combined with assertion (5) of Lemma 3.3 in [31] and (4.13) and (4.12), (4.15), implies that

$$P_x(\zeta, \xi) \approx \frac{v}{\sqrt{v^2 + |(bu + av) - \rho|^2}} \approx \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2}.$$ 

So the conclusion of the lemma is also true in this first subcase of the fourth case.

**Subcase:** $y \leq 0$.

As $y = \tau(\xi)$, we see that $\xi$ lies on the ray $\{bu + av = 0\} \cap \mathring{\Pi}_x$ and $|(\hat{u}(y, v) + iv) - \xi| \leq c_2|v|$ when $c_3$ is large enough. So $|\xi| \approx |y|^{1/\gamma} \approx \text{dist}^*_x(\zeta)$ and $\text{dist}_x(\zeta) \approx v$. Moreover,

$$v + |(bu + av) - \rho| \approx v + |bu + av| \approx |\xi| \approx |\zeta - \xi|.$$ 

This, combined with assertion (5) of Lemma 3.3 in [31] and (4.13) and (4.12), (4.15), implies that

$$P_x(\zeta, \xi) \approx \frac{v}{\sqrt{v^2 + |(bu + av) - \rho|^2}} \approx \frac{\text{dist}_x(\zeta)}{|\zeta - \xi|^2}.$$ 

So the conclusion of the lemma is also true in this last subcase of the fourth case. □

**Proposition 4.7.** Suppose that $n \geq 2$. Then there are constants $c, \gamma > 1$ which depend only on $\lambda_1, \ldots, \lambda_n$ with the following property. For every $x \in \mathbb{X}$ and for $\zeta \in \Pi_x$ and $\xi \in \mathring{\Pi}_x \cap \mathring{\mathbb{H}}_{x,l}$, and for every $1 \leq k \leq n$ such that $\lambda_i/\lambda_k \notin \mathbb{R}$, we have

$$P_x(\zeta, \xi) \leq c \left( \frac{\min \{ \text{dist}_{x,l}(\zeta), \text{dist}_{x,k}(\zeta) \} \text{dist}_{x,k}(\xi)}{|\zeta - \xi|^2} \right) \gamma^{-1}.$$ 

**Proof.** Consider the sector $\Omega_2 := \mathbb{H}_{x,k} \cap \mathbb{H}_{x,l}$. Applying Proposition 4.5 to $\Omega_1 := \Pi_x$ and $\Omega_2$ yields that

$$P_x(\zeta, \xi) \leq P_{\Omega_2}(\zeta, \xi).$$ 

Applying Proposition 4.6 to the RHS and using (3.9), the result follows. See Figure 6 for an illustration of this proof. □
5. Proof of the Main Theorem

Suppose without loss of generality that \( \mathcal{F} = (\mathbb{D}^n, \mathcal{F}, \{0\}) \) is a singular holomorphic foliation associated to a linear vector field which is defined on an open neighborhood of \( \mathbb{D}^n \). Suppose also that \( \mathcal{F} \) is the restriction of \( \mathcal{F} \) on a possibly smaller open neighborhood of \( \mathbb{D}^n \). Prior to the proof of the Main Theorem, some auxiliary estimates are needed.

**Lemma 5.1.** There are constants \( \gamma, c > 1 \) independent of \( x \in X' \) and \( \zeta \in \Pi_x \) and \( \xi \in \partial \Pi_x \) such that

\[
P_x(\zeta, \xi) \leq c \min \left( 1, \left( \frac{\text{dist}_x^*(\xi)}{\text{dist}_x(\zeta)} \right)^{\gamma - 1} \right) \text{dist}_x(\zeta) \frac{1}{|\zeta - \xi|^2}.
\]

**Proof.** Let \( 1 \leq l, k \leq n \) be determined by \( \xi \in \partial \mathbb{H}_{x,l} \) and \( \text{dist}_x(\zeta) = \text{dist}_{x,k}(\zeta) \). There are three cases to consider.

**Case** \( \lambda_l / \lambda_k \notin \mathbb{R} \:
Applying Proposition 4.7 yields that

\[
P_x(\zeta, \xi) \leq c \frac{\text{dist}_{x,k}(\zeta)}{|\zeta - \xi|^2} \left( \min \left( \frac{\max \left( \text{dist}_{x,l}(\zeta), \text{dist}_{x,k}(\zeta) \right)}{\text{dist}_{x,l}(\zeta)}, \frac{\text{dist}_{x,k}(\xi)}{\text{dist}_{x,k}(\zeta)} \right) \right)^{\gamma - 1}.
\]

The expression in big parenthesis in the RHS is smaller than or equal to \( \frac{\text{dist}_{x,k}(\xi)}{\text{dist}_{x,k}(\zeta)} \), which is, in turn, bounded from above by \( \frac{\text{dist}_x^*(\xi)}{\text{dist}_x(\zeta)} \). Hence, the desired inequality follows.

**Case** \( l = k \:
Since the singularity \( \{0\} \) is weakly hyperbolic, there is \( 1 \leq j \leq n \) such that \( \lambda_l / \lambda_j \notin \mathbb{R} \).
We argue as in the first case for \( j \) instead of \( k \), and the proof in this case is thereby completed.

**Case** \( l \neq k \) and \( \lambda_l / \lambda_k \in \mathbb{R} : \nLet \( \mathcal{S} \) be the strip limited by two parallel lines \( \partial \mathbb{H}_{x,l} \) and \( \partial \Pi_{x,k} \). By Proposition 4.5 applied to \( \Omega_1 := \Pi_x \) and \( \Omega_2 := \mathcal{S} \), we get

\[
P_x(\zeta, \xi) \leq P_\mathcal{S}(\zeta, \xi).
\]

Since \( \xi \in \partial \mathbb{H}_l \), and \( \partial \Pi_{x,l}, \partial \Pi_{x,k} \) are parallel, we see that \( \text{dist}_x^*(\xi) \geq \text{dist}(\partial \Pi_{x,l}, \partial \Pi_{x,k}) \). On the other hand, since \( \text{dist}_x(\zeta) = \text{dist}_{x,k}(\zeta) \) and the strip limited by two parallel lines \( \partial \Pi_{x,l}, \partial \Pi_{x,k} \) contains \( \Pi_x \), it follows that

\[
\text{dist}_x(\zeta) = \text{dist}_{x,k}(\zeta) \leq \text{dist}_{x,l}(\zeta) = \text{dist}(\partial \Pi_{x,l}, \partial \Pi_{x,k}) - \text{dist}_{x,k}(\zeta),
\]

and hence

\[
\text{dist}_x(\zeta) \leq \frac{1}{2} \text{dist}(\partial \Pi_{x,l}, \partial \Pi_{x,k}).
\]

Combining these estimates, we get \( \frac{\text{dist}_x^*(\xi)}{\text{dist}_x(\zeta)} \geq 2. \) The desired conclusion follows from Proposition 4.4 (2). \( \square \)

The following result gives a precise behavior of \( K_s(x, \xi) \) (introduced in (3.20)) when the leaves get close to the hyperplanes \( \{z_j = 0\} \).

**Lemma 5.2.** There is a constant \( c, \gamma > 1 \) such that for all \( x \in X', s > 1 \) and \( \xi \in \partial \Pi_x \),

\[
K_s(x, \xi) \leq c \min \left( 1, \left( \frac{\text{dist}_x^*(\xi)}{s} \right)^{\gamma - 1} \right).
\]
Applying Lemma 5.1, we deduce from the above estimate and from the inequality
\[ \zeta \]
that
\[ P_x(\zeta, \xi) \leq c' P_x(\zeta', \xi) \quad \text{for} \quad \zeta, \zeta' \in \Pi^*_x, \quad |\zeta - \zeta'| \leq 1. \]
Combining all these inequalities, we infer from (3.20) that
\[ (\forall \epsilon > 0) \quad K_s(x, \xi) \leq e^{-\epsilon^2} P_x(\zeta, \xi) \text{Leb}_1(\zeta). \]
On the other hand, for every \( \zeta \in \Pi^*_x \) there is exactly one \( j \in \mathbb{N} \) such that \( \zeta \in \Pi^*_x \setminus \Pi^*_x+j \), and for \( \zeta \in \partial \Pi^*_x+j \), we have \( \text{dist}_x(\zeta) = s+j \) and hence \( e^{2s-2\text{dist}_x(\zeta)} = e^{-2j} \). Moreover, there is a constant \( c > 0 \) independent of \( x \) and \( s \) such that the following inequality holds
\[ \int_{\Pi^*_x} f(\zeta) \text{Leb}_2(\zeta) \leq c \sum_{j=0}^{\infty} \int_{\zeta \in \Pi^*_x+j} e^{-2j} P_x(\zeta, \xi) \text{Leb}_1(\zeta), \]
for every positive function \( f \) defined on \( \Pi^*_x \) satisfying
\[ f(\zeta) \leq c' f(\zeta') \quad \text{for} \quad \zeta, \zeta' \in \Pi^*_x, \quad |\zeta - \zeta'| \leq 1. \]
Combining all these inequalities, we infer from (3.20) that
\[ K_s(x, \xi) \leq \sum_{j=0}^{\infty} \int_{\zeta \in \Pi^*_x+j} e^{-2j} P_x(\zeta, \xi) \text{Leb}_1(\zeta). \]
Applying Lemma 5.1, we deduce from the above estimate and from the inequality \( \text{dist}_x(\zeta) \leq s+j \) for \( \zeta \in \partial \Pi^*_x+j \) that
\[ K_s(x, \xi) \leq c \min \{ 1, \left( \frac{\text{dist}_x(\zeta)}{s} \right)^{-1} \} \cdot \sum_{j=0}^{\infty} \int_{\zeta \in \Pi^*_x+j} \frac{e^{-2j(s+j)}}{|\zeta - \xi|^2} \text{Leb}_1(\zeta). \]
To conclude the proof of the lemma, we only need to show that the last sum is uniformly bounded independently of \( x \in \mathbb{R} \). To this end we will show that there is a constant \( c > 0 \) independent of \( x \in \mathbb{R} \) such that
\[ \int_{\zeta \in \partial \Pi^*_x} \frac{1}{|\zeta - \xi|^2} \text{Leb}_1(\zeta) \leq ct^{-1}. \]
Indeed, taking for granted this inequality, we apply it to \( t = s+j \) for \( j \in \mathbb{N} \) and sum up the results. This will imply that the above sum is uniformly bounded. To prove (5.1), we observe that the edges of \( \partial \Pi^*_x \) are parallel to those of \( \Pi^*_x \), and \( \Pi^*_x \) possesses at most \( n \) edges. Moreover, \( |\zeta - \xi| \geq t \) for \( \zeta \in \partial \Pi^*_x \). So the LHS of (5.1) is bounded by
\[ n \cdot \sup_{\ell} \int_{|\zeta - \xi| \geq t} \frac{1}{|\zeta - \xi|^2} d\text{Leb}_1(\zeta), \]
the supremum being taken over all real lines \( \ell \subset \mathbb{C} \). A straightforward computation shows that the above supremum is bounded by \( O(t^{-1}) \). Hence, (5.1) follows, and the proof is complete.

End of the proof of Theorem 1.1. By Lemma 5.2, the family of functions \( (K_s)_{s>0} : \mathbb{R} \rightarrow \mathbb{R}^+ \), is uniformly bounded. Moreover, \( \lim_{s \rightarrow \infty} K_s(x, \xi) = 0 \) for \( (x, \xi) \in \mathbb{R} \).
On the other hand, consider the measure \( \chi \) on \( \mathbb{R} \), given by
\[ \int_{\mathbb{R}} \alpha d\chi = \int_{\mathbb{R}} \left( \int_{\zeta \in \partial \Pi_x} \alpha(x, \xi) h_x(\xi) \text{Leb}_1(\xi) d\mu(x) \right) d\mu(x). \]
for every continuous bounded test function $\alpha$ on $\mathbb{K}$. By inequality (3.16) in the proof of Lemma 3.9, there is a constant $c > 0$ (independent of $x$) such that

$$\|T\|_{\mathbb{D}^{n}(r_{0}\mathbb{D})^{n}} = \int_{\mathbb{D}^{n}(r_{0}\mathbb{D})^{n}} T \wedge (idz_{1} \wedge d\bar{z}_{1} + \ldots + idz_{n} \wedge d\bar{z}_{n})$$

$$\geq \int_{x \in \mathbb{K}} \left( \int_{\Pi_{x}} T_{x} \wedge (idz_{1} \wedge d\bar{z}_{1} + \ldots + idz_{n} \wedge d\bar{z}_{n}) \right) d\mu(x)$$

$$\geq c \int_{x \in \mathbb{K}} \left( \int_{\Pi_{x}} h_{x}(\zeta) d\text{Leb}_{2}(\zeta) \right) d\mu(x)$$

$$\geq c' c \int_{x \in \mathbb{K}} \left( \int_{\xi \in \partial\Pi_{x}} h_{x}(\zeta) d\text{Leb}_{1}(\zeta) \right) d\mu(x),$$

where the last inequality holds for a constant $c' > 0$ (independent of $x$) by an application of Harnack’s inequality for positive harmonic functions. So $\chi$ is a finite positive measure.

Consequently, we get, by dominated convergence, that $\lim_{s \to \infty} \int_{\mathbb{K}} K_{s} d\chi = 0$. This, combined with Lemma 3.11, implies that

$$0 \leq \lim_{r \to 0^{+}} f(r) \leq n \lim_{s \to \infty} \int_{\mathbb{K}} K_{s} d\chi = 0,$$

which, coupled with (3.18)-(3.19) and (2.3), gives that $\nu(T, 0) = 0$, as desired. \hfill $\Box$

**Proof of Theorem 1.2** Using Remark 3.12 and arguing as in the proof of Theorem 1.1 we see that $\tilde{F}(r) < \infty$ for $0 < r < 1$. Hence, the mass of $T$ on $(r\mathbb{D})^{n}$ is finite for all $r \in [0, 1]$. This proves assertion (1).

To prove assertion (2) pick a point $x \in \mathbb{Z}$. There are two cases.

**Case 1:** $x = 0$. The proof of Theorem 1.1 also work in this context and we get that $\nu(T, 0) = 0$.

**Case 2:** $x \neq 0$.

Let $U_{x}$ be a regular flow box of $\mathcal{F}$ which contains $x$ and which is away from $\{0\}$. Let $\mathcal{T}$ be a transversal $U_{x}$ which contains $x$. By shrinking $U_{x}$ if necessary, we may assume without loss of generality that $\mathcal{T}$ is a complex manifold of dimension $n - 1$. Let $\mathcal{T} \subset \mathcal{F}$ be the transversal of $\mathcal{F}$ on $U_{x}$. Since $x \in \mathbb{Z}$, it follows that $x \notin \mathcal{T}$. By Proposition 2.6 we can write in $U_{x}$

$$T = \int_{t \in \mathcal{T}} h_{t}[\mathbb{B}_{l}] d\nu(t),$$

where, $\nu$ is a positive Borel measure on $\mathcal{T}$, and for $\nu$-almost every $t \in \mathcal{T}$, $h_{t}$ denote the positive harmonic function associated to the current $T$ on the plaque $\mathbb{B}_{l}$. We may assume without loss of generality that $h_{t}(t) = 1$. By Harnack’s inequality, there is a constant $c > 0$ independent of $t$ such that

$$c^{-1} h_{t}(z) \leq h_{t}(w) \leq c h_{t}(z), \quad z, w \in \mathbb{B}_{l}.$$ 

In particular, we get that $h_{t}(z) \approx 1$ for $z \in \mathbb{B}_{l}$ as $h_{t}(t) = 1$. Using this and the above local description of $T$ on $U_{x}$ and formulas (2.1)-(2.2)-(2.3), we infer easily that $\nu(T, x) \leq c \nu(\{x\})$, where

$$\nu(\{x\}) := \lim_{\epsilon \to 0} \nu(\{y \in \mathcal{T} : \text{dist}(y, x) < \epsilon\}),$$

$$\nu(\{x\}) := \lim_{\epsilon \to 0} \nu(\{y \in \mathcal{T} : \text{dist}(y, x) < \epsilon\}).$$
and \( \text{dist}(\cdot, \cdot) \) is a distance induced by a fixed Hermitian metric on the complex manifold \( T \). Therefore, if one can show that \( \nu(\{x\}) = 0 \), then it follows from the last inequality that \( \nu(T, x) = 0 \), and we are done. So it remains to prove that \( \nu(\{x\}) = 0 \).

By assertion (1), the mass of \( T \) on \( U_x \) is finite. This, combined with the above description of \( T \) on \( U_x \) and the above estimate \( h_t(z) \approx 1 \) for \( z \in \mathbb{B}_x \), implies that \( \nu \) is a finite measure. Therefore, we deduce from the equality \( \bigcap_{j=1}^{\infty} \{ y \in \mathbb{T} : \text{dist}(y, x) < j^{-1} \} = \emptyset \) and from (5.2) that \( \nu(\{x\}) = 0 \) as desired. \( \square \)

6. Proof of the Global Result and Concluding Remarks

First we recall two results of Fornæss–Sibony [18, 20]. The proof given here makes an emphasis on the generalization of these results in the higher dimension \( n \geq 2 \). These results will be needed in the proof of Theorem 1.3.

In the following proposition we are concerned with directed positive harmonic currents \( T \) of the form

\[
T = h[L_a]
\]

where \( a \) is a point in \( X \setminus E \) and \( h \) is a positive harmonic function on the leaf \( L_a \). Let \( f \) be the lifting of the harmonic function to the unit disc \( \mathbb{D} \), that is,

\[
f = h \circ \varphi \quad \text{on} \quad \mathbb{D},
\]

where \( \varphi : \mathbb{D} \to L \) is a universal covering map.

**Proposition 6.1.** (Fornæss–Sibony [18, 20]) Let \( \mathcal{F} = (X, \mathcal{F}, E) \) be a singular holomorphic foliation with the set of singularities \( E \) in a compact complex manifold \( X \). Assume that

1. \( E \) is a finite set;
2. there is no invariant analytic curve;
3. there is no non-constant holomorphic map \( \mathbb{C} \to X \) such that out of \( E \) the image of \( \mathbb{C} \) is locally contained in a leaf.

Let \( T \) be a positive harmonic current directed by \( \mathcal{F} \) which has the form (6.1). Then:

(i) For every neighborhood \( U \) of \( E \), there is a constant \( c_U > 0 \) such that \( h(x) \leq c_U \) for \( x \in L_a \setminus U \);

(ii) The function \( f \) given in (6.2) has nontangential limits \( 0 \) \( \text{Leb}_1 \)-almost everywhere on \( \mathbb{C} \mathbb{D} \).

**Remark 6.2.** The condition (3) in Theorem 1.3 is equivalent to the Brody hyperbolicity of \( \mathcal{F} \) in the sense of [11], that is, there is a constant \( c > 0 \) such that for every holomorphic map \( \varphi \) from \( \mathbb{D} \) to a leaf,

\[
|\varphi'(\zeta)| < \frac{c}{1 - |\zeta|} \quad \text{for} \quad \zeta \in \mathbb{D}.
\]

It is my mistake to omit this condition in my previous work [31, Theorems 1.1 and 1.3].

**Proof.** Let \( U \) be a flow box. By Harnack’s inequality, there is a constant \( c > 0 \) which depends only on \( U \) such that

\[
c^{-1} h(z) \leq h(w) \leq ch(z), \quad z, w \in P,
\]
where \( P \) is a plaque of \( \mathbb{U} \) which is contained in \( L_a \). Since the mass of \( T \) on \( \mathbb{U} \) is finite, we infer from the above inequality and (6.1) that if the leaf \( L_a \) intersect \( \mathbb{U} \) infinitely many times in pairwise different plaques \( P_j \) with \( j \in \mathbb{N} \), then the harmonic functions \( h_j := h|_{P_j} \) must go uniformly to zero as \( j \to \infty \). Hence, assertion (i) follows.

To prove assertion (ii), consider the set \( S \) consisting of all \( \zeta \in \partial \mathbb{D} \) of such that \( f \) has nontangential limits \( f(\zeta) \) at \( \zeta \). Since \( h > 0 \) on \( L_a \), it follows that the harmonic function \( f \) is positive on \( \mathbb{D} \). Hence \( S \) is of full \( \text{Leb}_1 \)-measure on \( \partial \mathbb{D} \). Consider

\[
S_0 := \{ \zeta \in S : f(\zeta) > 0 \}.
\]

Suppose in order to reach a contradiction that \( \text{Leb}_1(S_0) > 0 \). We consider the curve \( \{ \varphi(re^{i\theta}) : r \in [0,1] \} \). By the above argument, it follows that this curve can only intersect finitely many plaques in any flow box.

Suppose that some plaque \( P \) is visited by this curve infinitely many times as \( r \to 1 \). Note that \( h \) must be constant on \( P \), hence constant on the leaf \( L_a \), hence \( T \) is a positive closed current whose Lelong numbers at all points of \( L_a \) are \( \geq 1 \). By Siu’s theorem [37], \( L_a \) is contained in a proper analytic set of \( X \). Since \( X \) is compact, \( L_a \) is an analytic subset of \( X \setminus E \). As \( E \) is a finite set and \( \dim_{\mathbb{C}}(X) \geq 2 \), we deduce from Remmert–Stein theorem that \( L_a \) is an invariant analytic curve in \( X \). This contradicts assumption (2).

Consequently, we infer from the two previous paragraphs that the curve \( \varphi(re^{i\theta}) \) converges as \( r \nearrow 1 \) to a singular point. Since \( E \) is a finite set, there are a set \( S_1 \subset S_0 \) with \( \text{Leb}_1(S_1) > 0 \) and a point \( a_0 \in E \) such that \( \varphi(re^{i\theta}) \to a_0 \) as \( r \nearrow 1 \) for all \( \theta \in S_1 \).

By assumption (3), \( \mathcal{F} \) is Brody hyperbolic. So by (6.3), there is a constant \( c > 0 \) such that

\[
|\varphi'(\zeta)| < \frac{c}{1 - |\zeta|} \quad \text{for} \quad \zeta \in \mathbb{D}.
\]

Since \( X \) is compact, we infer from a theorem of Lehto–Virtanen [27] that \( \varphi \) is bounded in any angle with vertex \( \zeta \) for every \( \zeta \in S_1 \). Using the argument in Privalov [9], we can construct a subset \( S_2 \subset S_1 \) with \( \text{Leb}_1(S_2) > 0 \) and a Jordan subdomain \( G \subset \mathbb{D} \) with rectifiable boundary such that \( S_2 \subset \partial G \), and that \( G \) contains an angle with vertex at \( \zeta \) for all \( \zeta \in S_2 \), and that \( \varphi(G) \) is contained in a local chart around \( a_0 \). By Lindelöf’s theorem, \( \varphi \) has nontangential limits \( a_0 \) on \( S_2 \). By Privalov’s theorem applied to \( \varphi|_{G} \), we see that \( \varphi \equiv a_0 \). Hence, \( \varphi \equiv a_0 \) on \( \mathbb{D} \), which contradicts \( \varphi(\mathbb{D}) = L_a \). So \( \text{Leb}_1(S_1) = 0 \), and hence \( \text{Leb}_1(S_0) = 0 \). This proves assertion (ii).

\[\square\]

**Theorem 6.3. (Fornæss–Sibony [20, Corollary 2])** Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation as in Proposition 6.1. Assume in addition that \( \dim X = 2 \). Then every positive harmonic current directed by \( \mathcal{F} \) is diffuse.

**Proof.** Assume in order to get a contradiction that there is a positive harmonic current \( T \) directed by \( \mathcal{F} \) which is not diffuse. So \( T \) has an atomic part, i.e. a Dirac mass at a point \( a \in X \setminus E \). The restriction \( T \) to the leaf \( L_a \) is a non-zero positive harmonic current. We can normalize so that the transverse measure is the Dirac mass at \( a \). Then we have a positive harmonic function \( h \) defined on \( L_a \).
By Proposition 6.1(ii) (see e.g. [3, Corollaries 6.15 and 6.44]), there is positive Borel measure \( \nu \) on \( \partial \mathbb{D} \) with support \( S \) such that \( \text{Leb}_1(S) = 0 \) and
\[
f(\zeta) = \int_{\partial \mathbb{D}} P_\partial(\zeta, \xi)d\nu(\xi),
\]
where \( P_\partial \) is the Poisson kernel of \( \mathbb{D} \). The function \( f \) should be unbounded, since otherwise \( \nu \) would have a bounded density with respect to \( d\text{Leb}_1 \), which would contradict that \( \text{supp}(\nu) = S \) and \( \text{Leb}_1(S) = 0 \).

On the other hand, let \( a_0 \in E \) be a singular point. Fix a local holomorphic coordinates system of an open neighborhood \( U \) of \( a_0 \) on which \( \mathcal{F} \) is identified with a local model \((\mathbb{D}^n, \mathcal{L}, 0)\) of the form (3.1). Let \( X \) be given by Lemma 3.3 By Proposition 2.6(2), for every \( x \in X \), there is a constant \( c_x > 0 \) such that \( h = c_x h_x \) on \( L_x \). Consequently, by Remark 3.10 (that is, by Lemma 3.9 for \( n = 2 \)), we have
\[
h(\zeta) = \int_{\partial \Pi_x} P_x(\zeta, \xi)h(\xi)d\text{Leb}_1(\xi).
\]
On the other hand, by Proposition 6.1(i) and by the inclusion \( X \subset \partial \mathbb{D}^n \), \( h \) is uniformly bounded on \( \partial \Pi_x \) independently of \( x \in X \). So by the above integral representation, \( h \) is also uniformly bounded on \( \Pi_x \). Hence, \( h \) must be uniformly bounded on a neighborhood of each singular point \( a_0 \in E \). This, combined with Proposition 6.1(i), implies that \( h \) must be uniformly bounded on \( L_a \). This contradicts the unboundedness of \( f \).

Hence, \( T \) is diffuse.

End of the proof of Theorem 1.3 To prove that \( T \) is diffuse, we argue as in the proof of Theorem 6.3 using Lemma 3.9 for all \( n \geq 2 \).

Now we prove that \( \nu(T, x) = 0 \) for all \( x \in X \). Let \( x \in X \). Consider two cases.

Case 1: \( x \notin E \).

Let \( U \simeq \mathbb{B} \times \mathbb{T} \) be a regular flow box with transversal \( \mathbb{T} \) which contains \( x \). By Proposition 2.6(1), we can write in \( U \)
\[
T = \int h_t[\mathbb{B}_t]d\mu(t),
\]
where \( \mu \) is a positive Radon measure on \( \mathbb{T} \), and for \( \mu \)-almost every \( t \in \mathbb{T} \), \( h_t \) is a positive harmonic function on the plaque \( \mathbb{B}_t \simeq \mathbb{B} \times \{t\} \). By Harnack’s inequality, there is a constant \( c > 0 \) independent of \( t \) such that
\[
c^{-1}h_t(z) \leq h_t(w) \leq ch_t(z), \quad z, w \in \mathbb{B}_t.
\]
Using this and the above local description of \( T \) on \( U \) and formula (2.3), we infer easily a constant \( c > 0 \) depending only on \( U \) such that \( \nu(T, x) \leq c\mu(\{x\}) \). On the other hand, since we have shown that \( T \) is diffuse, \( \mu(\{x\}) = 0 \). Hence, \( \nu(T, x) = 0 \).

Case 2: \( x \in E \).

Fix a (local) holomorphic coordinates system of \( X \) on a singular flow box \( U_x \) of \( x \) such that \( (U_x, x) \) is identified with \((\mathbb{D}^n, 0)\) and the leaves of \( \mathcal{F} \) on this box are integral curves of the linear vector field \( \Phi \) given by (3.1). Consider
\[
J := \{j : 1 \leq j \leq n \quad \text{and} \quad T \quad \text{gives mass to the invariant hyperplane} \quad \{z_j = 0\}\}.
\]
If \( J = \emptyset \), then we are able to apply Theorem 1.1 which gives \( \nu(T, x) = 0 \) as desired.
Consider the case $J \neq \emptyset$. Let $T_j$ be the restriction of $T$ on the invariant hyperplane \( \{ z_j = 0 \} \). This is a directed positive harmonic current. Consider $T' := T - \sum_{j \in J} T_j$. So $T'$ is also a directed positive harmonic current giving no mass to any coordinate invariant hyperplane \( \{ z_j = 0 \} \), and we have by (2.2)–(2.3),

$$\nu(T, x) \leq \nu(T', x) + \sum_{j \in J} \nu(T_j, x).$$

By Theorem [1.1], $\nu(T', 0) = 0$. Therefore, in order to prove that $\nu(T, x) = 0$, we only need to show that $\nu(T_j, 0) = 0$ for $j \in J$. Observe that since $0 \in \mathbb{C}^n$ is a hyperbolic singularity, the restriction of $\mathcal{F}$ on \( \{ z_j = 0 \} \) admits $0 \in \mathbb{C}^{n-1}$ as a hyperbolic singularity. We can argue as above by going down in one dimension by restricting $\mathcal{F}$ and $T$ to the invariant hyperplane \( \{ z_j = 0 \} \). We repeat this procedure. It should stop after a finite steps. Otherwise, we would go to a plane $H$ of dimension $n = 2$. Then, the two invariant hyperplanes (i.e. two separatrices in this context) of $\mathcal{F}|_H$ are two leaves of $\mathcal{F}$, each one of these leaves is of the form

$$\{ z = (z_1, \ldots, z_n) \in \mathbb{D}^n : z_l = 0 \quad \text{for all} \quad l \neq j \} \quad \text{for some} \quad 1 \leq j \leq n.$$

$T|_H$ cannot give mass to none of them, otherwise $T$ would give mass to a leaf, which in turn implies that this leaf is an invariant analytic curve, which is impossible by assumption (1). Consequently, we are able to apply Theorem [1.1] to get that $\nu(T|_H, 0) = 0$. This completes the proof. \qed

**Proof of Corollary 1.4.** By Brunella [5], if all the singularities of a foliation $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k)$ are hyperbolic and $\mathcal{F}$ does not possess any invariant algebraic curve, then $\mathcal{F}$ admits no nontrivial directed positive closed current. In particular, assumption (3) of Theorem 1.3 is fulfilled. Clearly, all two other assumptions of this theorem are also fulfilled. This theorem implies the corollary.

Let $\mathcal{F}_d(\mathbb{P}^n)$ be the space of all singular holomorphic foliations of degree $d$ in $\mathbb{P}^n$. By Jouanolou [25] and Lins Neto-Soares [29], there is a real Zariski dense open set $\mathcal{H}(d) \subset \mathcal{F}_d(\mathbb{P}^k)$ such that for every $\mathcal{F} \in \mathcal{H}(d)$, all the singularities of $\mathcal{F}$ are hyperbolic and $\mathcal{F}$ does not possess any invariant algebraic curve. So a generic foliation in $\mathcal{F}_d(\mathbb{P}^n)$ satisfies the assumptions of Corollary 1.4. \qed

We conclude the article with a remark and an open question.

**Remark 6.4.** By Dinh–Wu [15, Theorem 1.1], our main result (Theorem 1.1) is essentially sharp.

When the singularities are linearizable but not weakly hyperbolic, the study of Lelong numbers seems difficult. Chen’s recent article [8] gives a partial result in this direction for dimension $n = 2$. It seems to be interesting to find sufficient conditions on the nature of the singularity $\{ 0 \}$ to ensure that the Lelong number of $T$ at the origin is zero.

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