Abstract

We say that a subset $M$ of $\mathbb{R}^n$ is exponentially Ramsey if there exists $\varepsilon > 0$ and $n_0$ such that $\chi(\mathbb{R}^n, M) \geq (1 + \varepsilon)^n$ for any $n > n_0$, where $\chi(\mathbb{R}^n, M)$ stands for the minimum number of colors in a coloring of $\mathbb{R}^n$ such that no copy of $M$ is monochromatic. One important result in Euclidean Ramsey theory is due to Frankl and Rödl, and states the following (under some mild extra conditions): if both $N_1$ and $N_2$ are exponentially Ramsey then so is their Cartesian product. Applied several times to simple two-point sets $N_i$, this result implies that any subset $M$ of a ‘hyperrectangle’ $N_1 \times \cdots \times N_k$ is exponentially Ramsey.

However, generally, such ‘embeddings’ of $M$ result in very inefficient bounds on the aforementioned $\varepsilon$. In this paper, we present another way of combining exponentially Ramsey sets, which gives much better estimates in some important cases. In particular, we show that the chromatic number of $\mathbb{R}^n$ with a forbidden equilateral triangle satisfies $\chi(\mathbb{R}^n, \triangle) \geq (1.0742... + o(1))^n$, greatly improving upon the previous constant $1.0144$. We also obtain similar strong results for regular simplices of larger dimensions, as well as for related geometric Ramsey-type questions in Manhattan norm.

We then show that the same technique implies several interesting corollaries in other combinatorial problems. In particular, we give an explicit upper bound on the size of a family $F \subset 2^{[n]}$ that contains no weak $k$-sunflowers, i.e. no collection of $k$ sets with pairwise intersections of the same size. This bound improves upon previously known results for all $k \geq 4$. Finally, we also present a simple deduction of the (other) celebrated Frankl–Rödl theorem from an earlier result of Frankl and Wilson. It gives probably the shortest known proof of Frankl and Rödl result with the most efficient bounds.

1 Introduction

1.1 Euclidean Ramsey Theory

Probably the most famous geometric coloring problem is to determine the chromatic number of the plane. For an $n$-dimensional Euclidean space $\mathbb{R}^n$, its chromatic number $\chi(\mathbb{R}^n)$ is the smallest $r$ such that there exists a coloring of the points of $\mathbb{R}^n$ with $r$ colors, i.e. an $r$-coloring, and with no two points of the same color at unit distance apart. Currently, the best bounds on the plane are $5 \leq \chi(\mathbb{R}^2) \leq 7$, where the lower bound was obtained relatively recently, see [7, 13, 39].
In an influential paper [9], Erdős, Graham, Montgomery, Rothschild, Spencer and Straus laid the foundation of *Euclidean Ramsey theory*, a domain that studies different questions about monochromatic patterns in colorings of Euclidean spaces. See a recent survey by Graham [18] on this topic. One of the central notions in this field is as follows: for a subset $\mathcal{M} \subset \mathbb{R}^d$, the *chromatic number* $\chi(\mathbb{R}^n, \mathcal{M})$ is defined as the smallest $r$ such that there exists an $r$-coloring of $\mathbb{R}^n$ with no monochromatic isometric copy of $\mathcal{M}$. We say that a set $\mathcal{M}$ is *Ramsey* if $\chi(\mathbb{R}^n, \mathcal{M}) \to \infty$ as $n \to \infty$, and *exponentially Ramsey* if there exists some $\varepsilon > 0$ such that \[ \chi(\mathbb{R}^n, \mathcal{M}) \geq (1 + \varepsilon + o(1))^n. \]

Frankl and Wilson [17] showed that the simplest configuration $\mathcal{M}$, consisting of 2 points at unit distance apart, is exponentially Ramsey using polynomial method. Actually, they showed a bit more, namely, that this configuration is *super-Ramsey*. We call $\mathcal{M} \subset \mathbb{R}^d$ *super-Ramsey*, if for a sufficiently small $\varepsilon > 0$ and for any dimension $n$, there is a set $V \subset \mathbb{R}^n$ of size that is at most exponential in $n$ such that $|V|/\alpha(V, \mathcal{M}) \geq (1 + \varepsilon + o(1))^n$, where $\alpha(V, \mathcal{M})$ stands for the maximum cardinality $|W|$ of a subset $W \subset V$ with no isometric copy of $\mathcal{M}$. This property is stronger than exponentially Ramsey. Just note that

\[ \chi(\mathbb{R}^n, \mathcal{M}) \geq \frac{|V|}{\alpha(V, \mathcal{M})} \]

by the pigeonhole principle. If $\mathcal{M}$ consists of 2 points at unit distance apart, then we denote $\alpha(V, \mathcal{M})$ simply by $\alpha(V)$ for a shorthand. The current record on the aforementioned value of $\varepsilon$ in this case belongs to Raigorodskii [31] who managed to prove the following.

**Theorem 1.** There is a sequence of subsets $V(n) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, such that $\frac{|V(n)|}{\alpha(V(n))} \geq (\psi_2 + o(1))^n$, where $\psi_2 = \sup_{0 \leq x \leq 1} \frac{1+x+x^3}{1+x^2+x^4} = 1.239...$

More super-Ramsey configurations were found in [15] by Frankl and Rödl, who showed that the set of all $k+1$ vertices of the regular unit $k$-dimensional simplex $\triangle^k$ have this property for all $k \in \mathbb{N}$. In cases $k = 2$ and 3, we denote these sets by $\triangle$ and $\triangledown$ for a shorthand. Though Frankl and Rödl did not provide explicit lower bounds on the corresponding chromatic numbers, Raigorodskii and Sagdeev [37, 38] followed their argument (with some additional optimizations) and extracted the following quantitative result: $\chi(\mathbb{R}^n, \triangle) \geq (1.0140... + o(1))^n$. However, as $k$ grows, the best lower bound on $\chi(\mathbb{R}^n, \triangle^k)$ that one can obtain via this technique decreases extremely rapidly.

Recently, Naslund [25] used the slice-rank method (see [6, 8]) to give a very short argument leading to the following improvement of the lower bound on $\chi(\mathbb{R}^n, \triangle)$:

\[ \chi(\mathbb{R}^n, \triangle) \geq (1.0144... + o(1))^n. \]

It is not clear if one can get lower bounds on $\chi(\mathbb{R}^n, \triangle^k)$ for $k > 2$ via similar technique.

Back to the classicar results, in the paper [16] Frankl and Rödl found much more super-Ramsey configurations by proving the following ‘composition’ statement.

**Theorem 2.** If $\mathcal{N}_1 \subset \mathbb{R}^{d_1}$, $\mathcal{N}_2 \subset \mathbb{R}^{d_2}$ are super-Ramsey, then so is $\mathcal{N}_1 \times \mathcal{N}_2$.

Frankl and Rödl applied this result to show that the vertex set of any non-degenerate simplex is super-Ramsey. These two theorems, together with a simple observation that any subset $\mathcal{M} \subset \mathcal{N}$
of a super-Ramsey set $\mathcal{N}$ inherits this property\footnote{Indeed, any copy of $\mathcal{N}$ also contains a copy of $\mathcal{M}$. Thus $\alpha(V,\mathcal{M}) \leq \alpha(V,\mathcal{N})$ for all $V \subset \mathbb{R}^n$. Similarly, we have $\chi(\mathbb{R}^n,\mathcal{M}) \geq \chi(\mathbb{R}^n,\mathcal{N})$ for all $n \in \mathbb{N}$.}, gives all known examples of super-Ramsey (and even exponentially Ramsey) sets.

In particular, note that $\nabla \subset \square$ and $\triangle^k \subset \square^k + 1$ for all $k \in \mathbb{N}$, where $\square^k$ is the set of all $2^k$ vertices of the $k$-dimensional hypercube with side length of $1/\sqrt{2}$ and $\square$ is simply $\square^1$. Hence, the product theorem of Frankl and Rödl, applied several times to a super-Ramsey two-point configuration, provides an alternative way to prove that $\triangle^k$ is super-Ramsey for all $k \in \mathbb{N}$. Sagdeev \cite{35} showed that a careful choice of the auxiliary parameters in this proof yields that

\[
\chi(\mathbb{R}^n, \nabla) \geq (1.0136\ldots + o(1))^n, \tag{3}
\]

\[
\chi(\mathbb{R}^n, \triangle^k) \geq \left(1 + \frac{1}{(k + 1)^2 \cdot 2^{k+1} + o(1)}\right)^n \text{ for all } k \geq 4. \tag{4}
\]

However, Theorem 2 gives quantitatively inefficient bounds in the case when a set $\mathcal{M}$ can only be embedded into a very large Cartesian product $\mathcal{N}$, in particular, when $\mathcal{M}$ is a simplex. In this paper, we give a much more efficient way to concatenate constructions, using tree-like structures instead of product structures. This allows us to improve the lower bounds on the chromatic number of Euclidean space with any forbidden non-degenerate simplex. In particular, we get the following substantial improvements upon all the inequalities \eqref{inequality2}, \eqref{inequality3}, and \eqref{inequality4}.

**Theorem 3.** For all $k \in \mathbb{N}$, we have

\[
\chi(\mathbb{R}^n, \triangle^k) \geq \left(\psi_2^{1/(k+1)} + o(1)\right)^n
\]
as $n \to \infty$, where $\psi_2$ is from the statement of Theorem 1. In particular, we have

\[
\chi(\mathbb{R}^n, \triangle) \geq (1.0742\ldots + o(1))^n, \quad \text{and} \quad \chi(\mathbb{R}^n, \nabla) \geq (1.0551\ldots + o(1))^n.
\]

Observe that, as $k \to \infty$, the base of the exponent $\psi_2^{1/(k+1)} = 1 + \frac{0.214}{3 + o(1)} + O\left(\frac{1}{k^2}\right)$ tends to 1 with roughly ‘linear’ speed. So, the statement of Theorem 3 is not only significantly stronger than the inequality \eqref{inequality4}, but also than the bound $\chi_m(\mathbb{R}^n, \triangle^k) \geq \left(1 + (3k - 3)^{-2} + o(1)\right)^n$ on the measurable chromatic number recently obtained for all $k \geq 3$ in \cite{5}.

We also provide explicit exponential lower bounds on the chromatic numbers of Euclidean space with forbidden right and acute triangles. Note that the case of an obtuse triangle requires some additional technical propositions from \cite{15}, and so we omit it in the present paper for simplicity.

**Theorem 4.** Let $\mathcal{R}$ and $\mathcal{A}$ be the sets of vertices of a right and an acute triangles, respectively. Then we have

\[
\chi(\mathbb{R}^n, \mathcal{R}) \geq (1.1133\ldots + o(1))^n, \quad \text{and} \quad \chi(\mathbb{R}^n, \mathcal{A}) \geq (1.0742\ldots + o(1))^n.
\]

Let us conclude this sections with a few remarks on the corresponding upper bounds. In case of the ‘classical’ chromatic number $\chi(\mathbb{R}^n)$, the best result belongs to Larman and Rogers \cite{22,30} who showed that $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$. For all finite $\mathcal{M} \subset \mathbb{R}^d$, Prosanov \cite{29} found an explicit $c(\mathcal{M}) \leq 3$ such that $\chi(\mathbb{R}^n, \mathcal{M}) \leq (c(\mathcal{M}) + o(1))^n$. However, these upper bounds usually asymptotically much larger than the best current lower bounds. For instance, in case of an equilateral triangle it is only known that $\chi(\mathbb{R}^n, \triangle) \leq (2.733 + o(1))^n$. Erdős et al. showed that $\mathcal{M}$ is not Ramsey, i.e. that $\chi(\mathbb{R}^n, \mathcal{M})$ is bounded form above by some constant independent of $n$, whenever $\mathcal{M}$ is non-spherical \cite{9} or infinite \cite{10}. Nevertheless, the classification of Ramsey configurations is a wide open problem, and there are even two different conjectures upon the answer, see \cite{23}. Finally, note that it is unknown if there is a Ramsey set that is not exponentially (or super-) Ramsey.
1.2 Weak sunflowers

A collection of \( k \geq 3 \) sets is called a \( k \)-sunflower if all their pairwise intersections coincide. Erdős and Rado [11] introduced this notion, proved that any family \( \mathcal{F} \) of \( r \)-element sets with no \( k \)-sunflower satisfies \( |\mathcal{F}| \leq r!(k-1)^r \), and conjectured that this bound can be further improved to \( |\mathcal{F}| \leq C^r \) for some \( C = C(k) \) depending only on \( k \). Their conjecture remains one of the most famous problems in modern Combinatorics and is wide open even in case \( k = 3 \). Recently, there was a breakthrough result due to Alweiss, Lovett, Wu and Zhang [2] that significantly improved the upper bound.

Various analogues, generalizations, and related notions have been extensively studied over the past decades including the following one. A collection of \( k \geq 3 \) sets is called a weak \( k \)-sunflower if all their pairwise intersections are of the same cardinality. Let \( G_k(n) \) stands for the maximum size of a family \( \mathcal{F} \subset 2^{[n]} \) that contains no weak \( k \)-sunflowers. Kostochka and Rödl [20] proved that

\[
G_k(n) \geq k^{c(n \log n)^{1/3}}
\]

for some absolute constant \( c > 0 \), and their result remains the best known.

Concerning the upper bounds, there was a conjecture of Erdős and Szemerédi [12] stating that \( G_3(n) \leq (2 - \varepsilon)^n \) for a sufficiently small positive \( \varepsilon \). It was also proved by Frankl and Rödl [15]. In fact, they proved that for any \( k \geq 3 \) there exists a sufficiently small \( \varepsilon_k > 0 \) such that \( G_k(n) \leq (2 - \varepsilon_k + o(1))^n \) asymptotically. Frankl and Rödl did not provide explicit estimates for these \( \varepsilon_k \) but, as \( k \) grows, one cannot hope to achieve values larger than \( 2^{-2^{k+o(k)}} \) using their technique, see [34], and even in case \( k = 3 \) the optimal value of \( \varepsilon_3 \) would be smaller than 0.01.

Naslund [26] substantially improved upon this bound for \( k = 3 \) by showing that

\[
G_3(n) \leq (1.837 + o(1))^n.
\]

His proof was based on a relatively recent breakthrough of Ellenberg and Gijswijt [8] in the capset problem via the slice-rank method. See also the prior paper [27] of Naslund and Sawin on the sunflower problem and the original celebrated result of Croot, Lev, and Pach [6] for more examples of using the slice-rank method in Combinatorics. Nevertheless, it is unknown if one can apply this argument to get a non-trivial upper bound on \( G_k(n) \) for \( k > 3 \).

In the present paper, we show that our simple ideas from Section 2.1 combined with the classic theorem by Frankl and Wilson [17] lead to the following exponential upper bound on \( G_k(n) \).

**Theorem 5.** For all \( k \geq 3 \), if \( \mathcal{F} \subset 2^{[n]} \) contains no weak \( k \)-sunflowers, then

\[
|\mathcal{F}| \leq (2\psi^{-1/k} + o(1))^n
\]

as \( n \to \infty \), where \( \psi = \frac{1+\sqrt{2}}{2} = 1.207... \)

Observe that in case \( k = 3 \), this theorem only implies that \( G_3(n) \leq (1.879 + o(1))^n \), while (6) provides a better estimate. Nevertheless, as \( k \to \infty \), the value \( 2\psi^{-1/k} = 2 - \frac{0.376}{k} + o\left(\frac{1}{k^2}\right) \) tends to 2 with roughly ‘linear’ speed, and greatly improves upon the bounds following form the prior techniques for all \( k \geq 4 \).
1.3 Frankl–Rödl from Frankl–Wilson

Given \( r < n \), let \( F \subset \binom{[n]}{r} \) be a family of \( r \)-element subsets of \([n]\). This family is said to be s-avoiding if \( |F_1 \cap F_2| \neq s \) for all \( F_1, F_2 \in F \). It is easy to check that \( F \subset \binom{[n]}{r} \) is s-avoiding if and only if \( \overline{F} := \{ \overline{F} : F \in F \} \subset \binom{[n]}{r-1} \) is \((n-2r+s)\)-avoiding. Hence, we can assume without loss of generality that \( r \leq n/2 \). In the present section, we assume that both \( r \) and \( s \) grow linearly with \( n \), i.e. that \( r \sim \rho n \) and \( s \sim \sigma n \) for some \( 0 < \sigma < \rho \leq 1/2 \) as \( n \to \infty \). Under these conditions, Frankl and Wilson [17] applied polynomial method to show that if \( r-s \) is prime or a prime power, then the size of any s-avoiding family is exponentially smaller than \( \binom{n}{r} \). Their upper bound is known to be asymptotically tight if \( \sigma \leq \rho/2 \), see [4]. However, in case \( \sigma > \rho/2 \), currently the best upper bound is due to Ponomarenko and Raigorodskii [28], but its tightness is indefinite. We summarize these bounds in the following statement. (The asymptotic equivalence between the original technical result of Ponomarenko and Raigorodskii and its simpler form that we use here was shown in [33].)

**Theorem 6.** Given positive \( \sigma < \rho \leq 1/2 \), assume that \( r = r(n) \sim \rho n \), \( s = s(n) \sim \sigma n \) as \( n \to \infty \), and that \( r-s \) is prime or a prime power for all \( n \). Then any s-avoiding family \( F \subset \binom{[n]}{r} \) satisfies

\[
|F| \leq \left( \frac{n}{r} \right) (\delta(\rho, \sigma) + o(1))^n,
\]

where

\[
\delta(\rho, \sigma) = \begin{cases} 
\exp \{H(\rho - \sigma) - H(\rho)\} & \text{if } \sigma < \rho/2; \\
\exp \{H(\rho - \sigma) - H(2\rho - 2\sigma)\} & \text{if } \sigma \geq \rho/2,
\end{cases}
\]

and \( H(x) = -x \ln(x) - (1-x) \ln(1-x) \) is the entropy function.

Frankl and Rödl proved in their seminal paper [15] that the size of any s-avoiding family is still exponentially smaller than \( \binom{n}{r} \) even if \( r-s \) is neither prime nor a prime power. However, it is rather a laborious task to extract some quantitative bounds from their proof (which they prudently omitted). Keevash and Long [19] have recently showed that this general statement in fact follows from the case of prime \( r-s \) settled by Frankl and Wilson [17] combined with the idea that any natural number can be represented as a sum of at most 4 primes. We show that our simple ideas from Section 2.1 combined with the ideas of Keevash and Long give a shorter and more efficient reduction.

**Theorem 7.** Given positive \( \sigma < \rho \leq 1/2 \), assume that \( r = r(n) \sim \rho n \), \( s = s(n) \sim \sigma n \) as \( n \to \infty \). Put \( c = 3 \) if \( r-s \) is odd, and \( c = 4 \) otherwise. Then any s-avoiding family \( F \subset \binom{[n]}{r} \) satisfies

\[
|F| \leq \left( \frac{n}{r} \right) (\delta(\rho, \sigma) + o(1))^{n/c},
\]

where \( \delta(\rho, \sigma) \) is from the statements of Theorem 6.

It might be possible to replace \( c = 4 \) in the last exponent with a better \( c = 2 \) in case \( r-s \) is even, see the discussion in Section 3.3.

Let us briefly numerically compare this bound with prior results in the special ‘centered’ case \( r \sim n/2, s \sim n/4 \). If \( r-s \) is prime or a prime power, then Theorem 6 implies that any s-avoiding family \( F \subset \binom{[n]}{r} \) satisfies \( |F| \leq (1.755 + o(1))^n \) and this is tight. If \( r-s \) is only assumed to be odd, then following the proof from [19], one can only deduce that \( |F| \leq (1.998 + o(1))^n \), while Frankl
and Rödl themselves [15] showed that $|F| \leq (1.99 + o(1))^n$ in this case, and our Theorem 7 yields that $|F| \leq (1.915 + o(1))^n$. Similar quantitative improvements hold for other values of $\rho, \sigma$ as well.

One may consider Theorem 7 as a tool to guarantee an ‘edge’, i.e. a pair of sets with a prescribed cardinality of their intersection, in every sufficiently large uniform family. Under some additional constraints on the parameters, our technique can even guarantee a ‘clique’.

**Theorem 8.** Given positive $\sigma < \rho \leq 1/2$, assume that $r = r(n) \sim \rho n$, $s = s(n) \sim \sigma n$ as $n \to \infty$. Put $c = 1$ if $r - s$ is prime or a prime power, $c = 3$ if $r - s$ is another odd number, and $c = 4$ if $r - s$ is even. Then for all $k \geq 3$, any family $F \subset \binom{[kn]}{kr}$ that contains no collection of $k$ sets with pairwise intersections of cardinality $2s + (k - 2)r$ satisfies

$$|F| \leq \left(\binom{kn}{kr} \left(\delta(\rho, \sigma) + o(1)\right)\right)^{n/c},$$

where $\delta(\rho, \sigma)$ is from the statements of Theorem 6.

### 1.4 Manhattan Ramsey Theory

Let $\mathbb{R}_1^n$ be an $n$-dimensional space equipped with the Manhattan norm$^4$. One can easily translate the range of Euclidean problems considered in Section 1.1 to Manhattan space. Namely, for a subset $M \subset \mathbb{R}^d$, the chromatic number $\chi(\mathbb{R}_1^n, M)$ is defined as the smallest $r$ such that there exists an $r$-coloring of $\mathbb{R}^n$ with no monochromatic $\ell_1$-isometric copy$^5$ of $M$. For a subset $V \subset \mathbb{R}^n$, the independence number $\alpha(V, M)$ stands for the maximum cardinality $|W|$ of a subset $W \subset V$ with no $\ell_1$-isometric copy of $M$. Finally, we call $M \subset \mathbb{R}^d$ super-Ramsey, if for a sufficiently small $\varepsilon > 0$ and for any dimension $n$, there is a set $V \subset \mathbb{R}^n$ of size that is at most exponential in $n$ such that $|V|/\alpha(V, M) \geq (1 + \varepsilon + o(1))^n$.

In the present paper, we show that the following general statement almost trivially follows from Theorem 2 and the classic result of Frankl and Wilson [17], although it has never been stated explicitly to our knowledge.

**Theorem 9.** For all $d \in \mathbb{N}$, any finite $M \subset \mathbb{R}^d$ is super-Ramsey in case of the Manhattan norm. In particular, there exists $\varepsilon = \varepsilon(M) > 0$ such that

$$\chi(\mathbb{R}_1^n, M) \geq (1 + \varepsilon + o(1))^n.$$  

We also use our tree-like structures to provide relatively strong explicit exponential lower bounds on these chromatic numbers in some important special cases, see the exact statements and their proofs in Section 3.2.

### 2 Tools

#### 2.1 Tree-like concatenation

Our main new trick to combine the constructions consists of two elements. First is a simple graph-theoretic lemma. Recall that, for a graph $G = (V, E)$, its independence number $\alpha(G)$ is the size of the largest set of vertices that induces no edges.

$^4$Recall that the Manhattan $\ell_1$-norm is defined by $\|x\|_1 := \sum_{i=1}^n |x_i|$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

$^5$A subset $M' \subset \mathbb{R}^n$ is called an $\ell_1$-isometric copy of $M$ if there exists a bijection $f : M \to M'$ such that $\|x - y\|_1 = \|f(x) - f(y)\|_1$ for all $x, y \in M$. 

6
Lemma 1. Assume that $G_i = (V_i, E_i)$, $i \in [k]$ are graphs on the same vertex set. Fix a tree $T$ on $k$ arbitrarily ordered edges. Then in any subset $W \subset V$ of size strictly bigger than $\alpha(G_1) + \ldots + \alpha(G_k)$ there exists a homomorphic copy $h(T)$ of $T$ (that is, not necessarily with all vertices distinct) in $G = (V, E_1 \cup \ldots \cup E_k)$ such that the image of the $i$-th edge of $T$ in $h(T)$ belongs to $E_i$.

Proof. The proof of this statement is a simple induction on $k$. For $k = 1$ the statement is obvious because the only tree on 2 vertices is an edge, and the collection of vertices that contains no homomorphic copies of an edge must form an independent set.

To do the induction step from $k - 1$ to $k$, take any tree $T$ with $k$ edges and a non-leaf vertex $v \in T$. Then, split the tree $T$ into two non-empty subtrees $T', T''$ arbitrarily such that $v$ is their only common vertex. Clearly, both $T'$ and $T''$ have at most $k - 1$ edges. Let $U \subset [k]$ be the set of indices of edges in $T'$. (Then $[k] \setminus U$ is the set of indices of edges in $T''$.)

By induction, there are at most $\sum_{i \in U} \alpha(G_i)$ vertices $w \in W$ such that no homomorphic copy of $T'$ in $W$ satisfies $h(v) = w$. Indeed, the set of all such $w \in W$ contains no copies of $T'$ since no vertex can play the role of $v$ in the image of $T'$. Similarly, there are at most $\sum_{i \in [k] \setminus U} \alpha(G_i)$ vertices $w \in W$ such that no homomorphic copy of $T''$ in $W$ satisfies $h(v) = w$.

If $|W| > \sum_{i \in [k]} \alpha(G_i)$, then there is a vertex $w \in W$ not falling in either of these two categories, and we get the desired copy of $T$ with $h(v) = w$.

The other important ingredient for us is orthogonality. It allows to turn seemingly ‘non-rigid’ structures like paths or stars into ‘rigid’ cliques. In graph-theoretic terms, we need to work with products of graphs. Below, we introduce it in the form that would be convenient for applications.

Let $G_i = (V_i, E_i)$, $i \in [k]$, be a family of $k$ graphs. For any choice of their edges $(u_i, w_i) \in E_i$, $i \in [k]$, we call the sequence

$$w_0 := (w_1, \ldots, w_k), \ w_1 := (u_1, w_2, \ldots, w_k), \ldots, \ w_k := (w_1, \ldots, w_{k-1}, u_k).$$

an **orthogonal star** and the sequence

$$u_0 := (w_1, \ldots, w_k), \ u_1 := (u_1, w_2, \ldots, w_k), \ u_2 := (u_1, u_2, w_3, \ldots, w_k), \ldots, \ u_k := (u_1, \ldots, u_k)$$

an **orthogonal path** in the Cartesian product $V_1 \times \cdots \times V_k$.

Lemma 2. In the above notation, if $W \subset V_1 \times \cdots \times V_k$ contains no orthogonal stars (or no orthogonal paths), then

$$\frac{|W|}{|V_1 \times \cdots \times V_k|} \leq \sum_{i=1}^k \frac{\alpha(G_i)}{|V_i|}.$$

Proof. For all $i \in [k]$, consider a graph $G'_i := (V_i \times \cdots \times V_k, E'_i)$, where an edge connects two $k$-tuples $(u_1, \ldots, u_k)$ and $(w_1, \ldots, w_k)$ if and only if $(u_i, w_i) \in E_i$ and $u_j = w_j$ for all $j \neq i$.

The first thing to note is that $\frac{\alpha(G'_i)}{|V'_i|} = \frac{\alpha(G'_i)}{|V_i \times \cdots \times V_k|}$. Then $G'_i$ all live on the same set of vertices, and we only need to apply Lemma 1 with $T$ being a star (or a path). It is straightforward to see that any homomorphic copy of such $T$ must be an orthogonal star (or an orthogonal path).

Finally, we conclude with a straightforward technical proposition that we will need.
Proposition 1. Given \( r < n \), let \( F \subseteq \binom{[n]}{r} \) be a family of \( r \)-element subsets of \([n]\). Then for all partitions of \( n \) and \( r \) into \( k \) non-negative summands \( n = n_1 + \cdots + n_k \) and \( r = r_1 + \cdots + r_k \), there is a partition of \([n]\) into \( k \) disjoint subsets \([n] = N_1 \cup \cdots \cup N_k\) satisfying the following two properties. First, \( |N_i| = n_i \) for all \( i \in [k] \). Second, the size of the subfamily
\[
F' := \{ F \in F : |F \cap N_i| = r_i \text{ for all } i \in [k] \}
\]
satisfies
\[
|F'| \cdot \binom{n}{r} \geq |F| \cdot \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}.
\]

Proof. Let us consider all \( \frac{n!}{n_1! \cdots n_k!} \) possible partitions of \([n]\) into \( k \) disjoint subsets having the first desired property. It is easy to see that each \( F \in F \) contributes to the sizes of exactly
\[
\frac{r!}{r_1! \cdots r_k!} \cdot \binom{n-r}{r_1-1} \cdots \binom{n-r}{r_k-1}
\]
corresponding subfamilies. Now the pigeonhole principle finishes the proof. \( \square \)

2.2 Frankl–Rödl product theorem

In the present section we give a proof of Theorem 2 from [16] with minor exposition modifications, allowing the reader to compare our tree-like construction from Lemma 2 with its precursor.

So, assume that both \( \mathcal{N}_1 \subseteq \mathbb{R}^{d_1} \) and \( \mathcal{N}_2 \subseteq \mathbb{R}^{d_2} \) are super-Ramsey. Namely, that there exist constants \( c_i, \varepsilon_i > 0 \) and sets \( V_i(n) \subseteq \mathbb{R}^n \) such that \( |V_i(n)| \leq c_i^n \) and \( |V_i(n)|/\alpha(V_i(n), \mathcal{N}_i) \geq (1 + \varepsilon_i)^n \) for all \( n \in \mathbb{N}, i = 1, 2 \).

For large \( n \), we consider a partition \( n = n_1 + n_2 \), where \( n_1 \sim (1 - \eta)n, n_2 \sim \eta n \), and the value of \( \eta \) will be specified later. Put \( V_1 = V_1(n_1), V_2 = V_2(n_2), \) and \( V = V_1 \times V_2 \subseteq \mathbb{R}^n \). We will show that the Cartesian product \( \mathcal{N}_1 \times \mathcal{N}_2 \) is also super-Ramsey using this set. First, it is clear that \( |V| \leq c_1^{n_1} c_2^{n_2} \), i.e. that the size of \( V \) is at most exponential with \( n \). In what follows, we show that the size of any subset \( W \subseteq V \) with no copies of \( \mathcal{N}_1 \times \mathcal{N}_2 \) is exponentially smaller than \( |V| \).

For all \( x \in V_1 \), let \( w_x \) stand for the number of copies of \( \mathcal{N}_2 \) in the ‘layer’ \( W_x := W \cap \{x\} \times V_2 \). Similarly, for a copy \( Y \subseteq V_2 \) of \( \mathcal{N}_2 \), let \( w_Y \) stand for the number of layers, indexed by \( x \), such that \( \{x\} \times Y \subseteq W_x \). It is clear that \( \sum_x w_x = \sum_Y w_Y \). To give an upper bound on \( |W| \), we now estimate these sums.

On the one hand, each \( w_Y \) does not exceed \( \alpha(V_1, \mathcal{N}_1) \) since otherwise we will find a copy of \( \mathcal{N}_1 \times \mathcal{N}_2 \) in \( W \). Moreover, note that there are at most \( |V_2|^m \) ways to choose \( Y \), where \( m = |\mathcal{N}_2| \). Hence,
\[
\sum_{Y \subseteq V_2} w_Y \leq \alpha(V_1, \mathcal{N}_1) \cdot |V_2|^m \leq \frac{|V_1| \cdot |V_2|^m}{(1 + \varepsilon_1)^n_1} \leq |V| \cdot \frac{c_2^{(m-1)n_2}}{(1 + \varepsilon_1)^{n_1}}.
\]

On the other hand, observe that each layer \( W_x \) contains at least \( |W_x| - \alpha(V_2, \mathcal{N}_2) \) copies of \( \mathcal{N}_2 \). To see this, while \( |W_x| > \alpha(V_2, \mathcal{N}_2) \), discard elements of the layer one by one, reducing the number of copies of \( \mathcal{N}_2 \) by at least 1 in the remainder on each step\(^6\). Therefore,
\[
\sum_{x \in V_1} w_x \geq \sum_{x \in V_1} |W_x| - |V_1| \cdot \alpha(V_2, \mathcal{N}_2) \geq |W| - \frac{|V|}{(1 + \varepsilon_2)^{n_2}}.
\]

\(^6\)Here Frankl and Rödl used a ‘weaker’ bound \( w_x \geq |W_x|/\alpha(V_2, \mathcal{N}_2) \), while one can get a ‘stronger’ estimate than ours by applying a probabilistic argument similar to the one used in the proof of the crossing lemma, see [1]. However, these modifications would not change the resulting exponent asymptotically.
Finally, by combining the inequalities (7) and (8), we conclude that

\[
\frac{|W|}{|V|} \leq \frac{1}{(1 + \varepsilon_2)^{n_2}} + \frac{c_2^{(m-1)n_2}}{(1 + \varepsilon_1)^{n_1}}.
\]

Now it remains only to note that the right-hand side decreases exponentially with \( n \) if the value of \( \eta \) is sufficiently small. More specifically, one can easily check that, by taking 

\[
\eta = \frac{\log(1 + \varepsilon_1)}{\log ((1 + \varepsilon_2)(1 + \varepsilon_2)c_2^{m-1})},
\]

we get that \(|W| \leq |V|/(1 + \varepsilon_2 + o(1))^m\), which is asymptotically an optimal choice here.

3 Applications

3.1 Euclidean Ramsey theory

For all \( k \in \mathbb{N} \), we call the subset \( \{0, e_1, \ldots, e_k\} \subset \mathbb{R}^k \), where \( e_i \) stands for the \( i \)th standard basis vector in \( \mathbb{R}^k \), a \( k \)-semicross and denote it by \( SC^k \). Given \( V \subset \mathbb{R}^n \), consider a unit distance graph \( G(V) = (V, E(V)) \), where two points of \( V \) are connected if and only if they are at unit distance apart. It is easy to see that any orthogonal star in the Cartesian power \( V^k \) is isometric to \( SC^k \). Hence, Lemma 2 implies the following upper bound\(^8\) on the maximum cardinality of a subset of \( V^k \subset \mathbb{R}^{nk} \) with no isometric copies of \( SC^k \):

\[
\alpha(V^k, SC^k) \leq k|V|^{k-1} \alpha(V). \tag{10}
\]

Now we apply the last inequality to the set \( V(n) \subset \mathbb{R}^n \) from the statements of Theorem 1 and combine the result with (1). This shows that

\[
\chi(\mathbb{R}^{kn}, SC^k) \geq \frac{|V(n)|^k}{k^{k-1} \alpha(V(n))} \geq \frac{|V(n)|^k}{k^{k-1} \alpha(V(n))} = \frac{|V(n)|^k}{k^{k-1} \alpha(V(n))} \geq \frac{(\psi_2 + o(1))^n}{k}. \tag{11}
\]

If \( k \in \mathbb{N} \) is fixed and \( n \to \infty \), (11) implies that

\[
\chi(\mathbb{R}^n, SC^k) \geq (\psi_2^{1/k} + o(1))^n.
\]

Finally, a simple observation that \((k + 1)\)-semicross contains \( k + 1 \) vertices of the regular \( k \)-dimensional simplex with side length of \( \sqrt{2} \) completes the proof of Theorem 3.

Let us consider the following ‘asymmetric’ generalization of the above argument. Given \( k \) positive reals \( \lambda_1, \ldots, \lambda_k \), we call the subset \( \{0, \lambda_1 e_1, \ldots, \lambda_k e_k\} \subset \mathbb{R}^k \) a \textit{scaled} \( k \)-semicross and denote it by \( SC^k(\lambda_1, \ldots, \lambda_k) \). One can easily check that any orthogonal star in the scaled Cartesian product \( (\lambda_1 V) \times \cdots \times (\lambda_k V) \) is an isometric copy of \( SC^k(\lambda_1, \ldots, \lambda_k) \). Thus, Lemma 2 yields the following generalization of (10):

\[
\alpha((\lambda_1 V) \times \cdots \times (\lambda_k V), SC^k(\lambda_1, \ldots, \lambda_k)) \leq k|V|^{k-1} \alpha(V).
\]

As earlier, for fixed values of \( k \), the last inequality implies that

\[
\chi(\mathbb{R}^n, SC^k(\lambda_1, \ldots, \lambda_k)) \geq (\psi_2^{1/k} + o(1))^n.
\]

\(^7\)Note that here and in what follows we do not distinguish points and their position vectors.

\(^8\)Observe that two independence numbers, \( \alpha(G(V)) \) and \( \alpha(V) \) defined in the introduction, are identical.
Observe that a right triangle with catheti lengths $a$ and $b$ is isometric to the scaled 2-semicross $\text{SC}^2(a, b)$, while an acute triangle with side lengths of $a, b,$ and $c$ can be embedded into the scaled 3-semicross
\[
\text{SC}^3\left(\sqrt{(a^2 + b^2 - c^2)/2}, \sqrt{(b^2 + c^2 - a^2)/2}, \sqrt{(c^2 + a^2 - b^2)/2}\right).
\]
This completes the proof of Theorem 4.

### 3.2 Manhattan Ramsey theory

Though Frankl and Wilson [17] considered only Euclidean norm, their technique similarly works for all $\ell_p$-spaces. In particular, it can be argued that they proved a pair of points at unit distance apart to be a super-Ramsey configuration in case of Manhattan norm as well. The current quantitative record here is via the following analogue of Theorem 1 (also by Raigorodskii [32]).

**Theorem 10.** There is a sequence of subsets $V(n) \subset \mathbb{R}^n_1$, $n \in \mathbb{N}$, such that $\frac{|V(n)|}{\alpha(V(n))} \geq (\psi_1 + o(1))^n$, where $\psi_1 = \frac{1+\sqrt{3}}{2} = 1.366...$

Similarly, one can easily observe that the proof of Theorem 2 presented in Section 2.2 holds not only for Euclidean but also for Manhattan space and, more generally, for all $\ell_p$-spaces. Hence, when applied several times to a super-Ramsey two-point configuration, Theorem 2 implies that the vertex set of any hyperrectangle, or a box, is super-Ramsey.

For all positive reals $\lambda_1, \ldots, \lambda_k$, we call one-dimensional set $\{0, \lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_{i=1}^k \lambda_i\}$ a *baton* and denote by $\mathcal{B}(\lambda_1, \ldots, \lambda_k)$ for a shorthand. Observe that $\mathcal{B}(\lambda_1, \ldots, \lambda_k)$ can be $\ell_1$-isometrically embedded into a $k$-dimensional box with side lengths of $\lambda_1, \ldots, \lambda_k$. Therefore, all batons are also super-Ramsey. This statement is crucial for Manhattan Ramsey theory due to the following simple observation: each finite set $\mathcal{M} = \{x^0, \ldots, x^k\} \subset \mathbb{R}^d$ is a subset of the $d$-dimensional grid $\prod_{i=1}^d \{x^0_i, \ldots, x^k_i\}$, i.e. a Cartesian product of some batons. Now we apply Theorem 2 again to this product and finish the proof of Theorem 9.

As we discussed in Section 1.1, direct embedding of a simplex or a baton into a box leads to an exponentially small (in terms of the box dimension) value of the corresponding $\varepsilon$ from the statement of Theorem 9, which is rather a pour bound. Using our tree-like structures instead, we provide the following strong quantitative improvements.

**Theorem 11.** For all $k \in \mathbb{N}$ and all positive reals $\lambda_1, \ldots, \lambda_k$, we have
\[
\chi(\mathbb{R}^n_1, \mathcal{B}(\lambda_1, \ldots, \lambda_k)) \geq (\psi_1^{1/k} + o(1))^n, \quad \text{and} \quad \chi(\mathbb{R}^n_1, \triangle^k) \geq (\psi_1^{1/(k+1)} + o(1))^n
\]
as $n \to \infty$, where $\psi_1$ is from the statement of Theorem 10. In particular, we have
\[
\chi(\mathbb{R}^n_1, \triangle) \geq (1.1095... + o(1))^n, \quad \text{and} \quad \chi(\mathbb{R}^n_1, \mathcal{Y}) \geq (1.0810... + o(1))^n.
\]
In fact, the penultimate bound holds not only for regular but also for all triangles.

The proof of this statement basically repeats the ideas from Section 3.1, so we stress only the minor distinctions below. Given $V \subset \mathbb{R}^n$, consider a unit distance graph $G(V) = (V, E(V))$, where two points of $V$ are connected if and only if they are at unit Manhattan distance apart. It is easy to see that any orthogonal star in the scaled Cartesian product $(\lambda_1 V) \times \cdots \times (\lambda_k V)$ is $\ell_1$-isometric to the scaled $k$-semicross $\text{SC}^k(\lambda_1, \ldots, \lambda_k)$, while any orthogonal path there is $\ell_1$-isometric to the
Lemma 2. Hence, as in the previous section, we can apply Lemma 2 to the sets $V(n)$ from the statement of Theorem 10 to conclude that if $\mathcal{M}$ is either a baton $\mathcal{B}(\lambda_1, \ldots, \lambda_k)$ or a scaled $k$-semicross $\mathcal{SC}^k(\lambda_1, \ldots, \lambda_k)$, then

$$\chi(\mathbb{R}^n, \mathcal{M}) \geq (\psi_1^{\frac{1}{k}} + o(1))^n$$

as $n \to \infty$, where $\psi_1$ is from the statement of Theorem 10. To finish the proof of Theorem 11 it remains only to note that regular $k$-dimensional simplex can be $\ell_1$-isometrically embedded into a $(k+1)$-semicross, while any triangle with side lengths of $a$, $b$, and $c$ can be $\ell_1$-isometrically embedded into the scaled $3$-semicross $\mathcal{SC}^3((a + b - c)/2, (b + c - a)/2, (c + a - b)/2)$.

At the end of this section, we mention that, for all $\ell_p$-spaces, one could get similar lower bounds $\chi(\mathbb{R}_p^n, \Delta^k) \geq (\psi_1^{1/(k+1)} + o(1))^n$ with $\psi = \frac{1}{\sqrt{2}} = 1.207\ldots$ using the original result of Frankl and Wilson [17]. In case $p = \infty$ much better bounds are known, see [14].

### 3.3 Frankl–Rödl from Frankl–Wilson

We begin the proof of Theorem 7 with considering the case when $r - s$ is odd. Put $n_1 = n_2 = \lfloor n / 3 \rfloor$, $n_3 = n - n_1 - n_2$ and $n_1 = n_2 = \lfloor r / 3 \rfloor$, $n_3 = n - r_1 - r_2$. Let us consider a partition $[n] = N_1 \cup N_2 \cup N_3$ of $[n]$ into 3 disjoint parts of sizes $n_1$, $n_2$, and $n_3$, respectively, guaranteed by Proposition 1, i.e. such that the subfamily

$$\mathcal{F}' := \{ F \in \mathcal{F} : |F \cap N_i| = r_i \text{ for all } i = 1, 2, 3 \}$$

satisfies

$$|\mathcal{F}'| \cdot \binom{n}{r} \geq |\mathcal{F}| \cdot \binom{n_1}{r_1} \binom{n_2}{r_2} \binom{n_3}{r_3}. \quad (12)$$

Several strengthenings of the ternary Goldbach problem are known, ensuring that every odd number can be expressed as the sum of three ‘almost equal’ primes. One of the best bounds on the error term is due to Matomäki, Maynard, and Shao [24], who showed that every odd $x > 5$ can be expressed as a sum of three primes, each having the form $x/3 + o(x^{0.551})$. In particular, we can write $r - s$ as the sum of three primes $r - s = p_1 + p_2 + p_3$ with $p_i \sim (r - s)/3$ for all $i = 1, 2, 3$. Put $s_i = r_i - p_i$. It is easy to see that $s_1 + s_2 + s_3 = s$ and that $s_i \sim s/3$ asymptotically for all $i = 1, 2, 3$.

Given $i \in \{1, 2, 3\}$, let us consider the following graph $G_i = (V_i, E_i)$. Its set of vertices $V_i = \binom{N_i}{r_i}$ consists of all $r_i$-element subsets of $N_i$. An edge connects two vertices whenever they share exactly $s_i$ common elements. It is clear that the independent sets are exactly the $s_i$-avoiding families by the definition. Recall that $|N_i| = n_i$, $r_i \sim p n_i$, $s_i \sim \sigma n_i$, and that $r_i - s_i = p_i$ is prime. Hence, we can apply Theorem 6 to show that

$$\alpha(G_i) \leq \binom{n_i}{r_i} (\delta(\rho, \sigma) + o(1))^{n_i}. \quad (13)$$

Observe that for all $F \in \mathcal{F}'$, we have $F \cap N_i \in \binom{N_i}{r_i}$ by the definition. Therefore, we can think of $\mathcal{F}'$ as of a subset of the Cartesian product $V_1 \times V_2 \times V_3$, so that $F \in \mathcal{F}'$ corresponds to the triple $(F \cap N_1, F \cap N_2, F \cap N_3)$. Moreover, observe that any two sets $F_0, F_3 \in \mathcal{F}'$, corresponding to the endpoints $u_0 = (w_1, w_2, w_3)$ and $u_3 = (u_1, u_2, u_3)$ of an orthogonal path in $V_1 \times V_2 \times V_3$, respectively, share exactly

$$|w_1 \cap u_1| + |w_2 \cap u_2| + |w_3 \cap u_3| = s_1 + s_2 + s_3 = s$$

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common elements. Being a subfamily of $\mathcal{F}$, the family $\mathcal{F}'$ is also $s$-avoiding, and we conclude that there are no orthogonal paths in $\mathcal{F}'$. Thus, Lemma 2 combined with (13) yields that

$$\frac{|\mathcal{F}'|}{\binom{n_1}{r_1} \binom{n_2}{r_2} \binom{n_3}{r_3}} \leq \sum_{i=1}^{3} (\delta(\rho, \sigma) + o(1))^n_i = (\delta(\rho, \sigma) + o(1))^{n/3}.$$  

Now we use (12) to get the desired upper bound on $|\mathcal{F}|$. This completes the proof of Theorem 7 in case when $r - s$ is odd.

We can argue similarly for even $r - s$ as well. Observe that if it were known that every sufficiently large even number can be expressed as the sum of two ‘almost equal’ primes, then we could prove the statement of Theorem 7 with a better $c = 2$ instead. But as long as this strengthening of the binary Goldbach problem remains only a conjecture, we can only argue as follows. Baker, Harman, and Pintz [3] proved that for all sufficiently large $x$, there is a prime number between $x$ and $x + x^{0.525}$. In particular, their result implies that there is a prime $p_1$ close to $(r - s)/4$. Then we can write an odd number $(r - s) - p_1$ as the sum of three almost equal primes $(r - s) - p_1 = p_2 + p_3 + p_4$ as earlier. Now we split both $n$ and $r$ into 4 almost equal summands, and the rest of our argument goes as before completing the proof of Theorem 7.

Note that every odd number can be expressed not only as the sum of three ‘almost equal’ primes, but also as the sum of three primes in any given asymptotic proportion, see [36] for the details. So, one can consider asymmetric variations of the present technique, hoping to improve the base of the exponent in Theorem 7. Our numerical experiments suggest that this improvement is possible if and only if $\sigma < \rho/2$. For instance, if $r \sim 0.5n$, $s \sim 0.15n$, and $r - s$ is odd, then any $s$-avoiding family $\mathcal{F} \subset \binom{[n]}{r}$ satisfies $|\mathcal{F}| \leq (1.970 + o(1))^n$ by Theorem 7, while the asymmetric partitions $n \sim 0.774n + 0.113n + 0.113n$, $r \sim 0.384n + 0.062n + 0.052n$, $s \sim 0.084n + 0.038n + 0.028n$ give a better bound $|\mathcal{F}| \leq (1.964 + o(1))^n$. However, both of these bounds are still far from the tight upper bound $|\mathcal{F}| \leq (1.911 + o(1))^n$ via Theorem 6, valid in the case when $r - s$ is prime or a prime power.

Now let us implement basically the same ideas to prove Theorem 8. First, we apply Proposition 1 to find a partition $[kn] = N_1 \cup \cdots \cup N_k$ of $[kn]$ into $k$ disjoint parts of sizes $n$ such that the subfamily

$$\mathcal{F}' := \{ F \in \mathcal{F} : |F \cap N_i| = r \text{ for all } i = [k] \}$$

satisfies

$$|\mathcal{F}'| \cdot \binom{kn}{kr} \geq |\mathcal{F}| \cdot \binom{n}{r}^k. \quad (14)$$

Given $i \in [k]$, let us consider the following graph $G_i = (V_i, E_i)$. Its set of vertices $V_i = \binom{N_i}{r}$ consists of all $r$-element subsets of $N_i$. An edge connects two vertices whenever they share exactly $s$ common elements. As earlier, it is clear that the independent sets are exactly the $s$-avoiding families by the definition. Hence, Theorems 6 and 7 imply that

$$\alpha(G_i) \leq \binom{n}{r} (\delta(\rho, \sigma) + o(1))^{n/c}. \quad (15)$$

Observe that for all $F \in \mathcal{F}'$, we have $F \cap N_i \in \binom{N_i}{r}$ by the definition. Therefore, we can consider $\mathcal{F}'$ as a subset of the Cartesian product $V_1 \times \cdots \times V_k$ by corresponding each $F \in \mathcal{F}'$ to the $k$-tuple $(F \cap N_1, \ldots, F \cap N_k)$.  

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Let us consider a collection of sets $F_0, F_1, \ldots, F_k$ that correspond to some orthogonal star

$$w_0 = (w_1, \ldots, w_k), \ w_1 = (u_1, w_2, \ldots, w_k), \ \ldots, \ w_k = (w_1, \ldots, w_{k-1}, u_k).$$

in $V_1 \times \cdots \times V_k$, respectively. Note that for all $1 \leq j_1 < j_2 \leq k$, we have

$$|F_{j_1} \cap F_{j_2}| = |w_{j_1} \cap u_{j_1}| + |w_{j_2} \cap u_{j_2}| + \sum_{i=1 \atop i \neq j_1, j_2}^k |w_i| = 2s + (k - 2)r.$$ 

Since $F$ contains no such collection by the assumption, then so does $F'$. Hence $F'$ also contains no orthogonal stars, and Lemma 2 combined with (15) yields that

$$\frac{|F'|}{\binom{n}{r}} \leq k(\delta(\rho, \sigma) + o(1))^{n/c} = (\delta(\rho, \sigma) + o(1))^{n/c}.$$ 

Now (14) completes the proof of Theorem 8.

### 3.4 Weak sunflowers

In the present section, we prove Theorem 5. Given $k \geq 3$, let $F \subset 2^{[n]}$ be a family of subsets of $[n]$ that does not contain weak $k$-sunflowers.

First, observe that the subfamily $\{F \in F : |F| = r\}$ has size at least $\frac{|F|}{n+1}$ for some $r \leq n$ by the pigeonhole principle. Since this polynomial factor can be easily hidden into the $o(1)$-term of any exponential upper bound of the form $(c + o(1))^n$, we can assume without loss of generality that $F \subset \binom{[n]}{r}$. We can also assume that $2k$ divides $n$ by taking the smallest $n' > n$ that is a multiple of $2k$ and considering $F$ as a family of subsets of $[n'] \supset [n]$.

It is clear that for all $F \in F$ there are exactly $\binom{n}{n/2}$ subsets $G \subset [n]$ such that the cardinality of the symmetric differences $F \triangle G$ equals $n/2$. Hence, the pigeonhole principle implies the for some $G \subset [n]$, the family

$$F' := \{F \triangle G : F \in F\} \cap \binom{[n]}{n/2}$$

satisfies

$$|F'| \cdot 2^n \geq |F| \cdot \binom{n}{n/2}. \tag{16}$$

Observe that $F'$ does not contain weak $k$-sunflowers. Indeed, assume the contrary, namely that for some $F_1, \ldots, F_k \in F$, all the pairwise intersections between $F_1 \triangle G, \ldots, F_k \triangle G \in F'$ are of the same cardinality. Then it is not hard to see that

$$2|F_i \cap F_j| - 2(|F_i \triangle G| \cap (F_j \triangle G)| = |F_i| + |F_j| - |F_i \triangle G| - |F_j \triangle G| = 2r - n$$

for all $i \neq j$. In particular, this implies that the cardinality of $|F_i \cap F_j|$ is independent of $i$ and $j$. Thus the collection $F_1, \ldots, F_k \in F$ is a weak $k$-sunflower, a contradiction.

Let $p$ be the largest prime such that $p < \frac{2 - \sqrt{2} n}{4 \ k}$. Baker, Harman, and Pintz [3] proved that for all sufficiently large $x$, there is a prime number between $x$ and $x + x^{0.525}$. In particular, their result implies that $p \sim \frac{2 - \sqrt{2} n}{4 \ k}$ asymptotically for large $n$. Put $s = \frac{n}{2k} - p \sim \frac{\sqrt{2} n}{4 \ k}$. 

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Since $\mathcal{F}'$ contains no weak $k$-sunflowers, it also does not contain a collection of $k$ sets with pairwise intersections of cardinality $\frac{n}{2} - 2p = 2s + (k-2)\frac{n}{2k}$. Hence, we can apply Theorem 8 with $\frac{n}{k}$ playing the role of $n$, $\frac{n}{2k}$ playing the role of $r$, and with $c = 1$ to show that

$$|\mathcal{F}'| \leq \left(\frac{n}{n/2}\right)(\psi + o(1))^{-n/k},$$

(17)

where

$$\psi = \delta(1/2, \sqrt{2}/4)^{-1} = \exp\{H((2 - \sqrt{2})/2) - H((2 - \sqrt{2})/4)\} = \frac{1 + \sqrt{2}}{2}.$$ \(\text{(16)}\)

Now we combine (16) with (17) to complete the proof of Theorem 5.

Observe that this argument does not fully use the fact that $\mathcal{F}$ contains no weak $k$-sunflowers, but rather relies only on the corollary that $\mathcal{F}$ contains no collection of $k$ sets with a prescribed cardinality of pairwise intersections. Though this cardinality is optimal within the technique, there may be a way to utilize other forbidden intersections to improve the result.

4 Open problems

1. Suppose that the dimension of the simplex $k = k(n)$ depends on $n$. It is not hard to deduce from (11) that $\chi(\mathbb{R}^n, \Delta^k(n))$ tends to infinity with $n$ for $k(n) \leq (\ln n - \varepsilon + o(1))^{\frac{n}{\ln n}}$, where $\varepsilon > 0$. What happens for the larger values of $k$? For instance, does the value $\chi(\mathbb{R}^n, \Delta^k)$ tend to infinity with $n$?

2. For a fixed $k \in \mathbb{N}$, the best upper bound on the chromatic number $\chi(\mathbb{R}^n, \Delta^k)$ for large $n$ is due to Prosanov [29]: $\chi(\mathbb{R}^n, \Delta^k) \leq (1 + 2(k + 1)/k + o(1))^n$. Observe that the base of this exponent does not tend to 1 as $k$ grows, unlike the base of the exponent in the lower bound from Theorem 3. This raises the following question. For an arbitrary small positive $\varepsilon$, is there a sufficiently large $k(k(\varepsilon))$ such that $\chi(\mathbb{R}^n, \Delta^k) \leq (1 + \varepsilon + o(1))^n$ as $n \to \infty$?

3. Note that both problems above can be asked for non-Euclidean $\ell_p$-spaces as well. Let us give another question of this sort.

If $\lambda_1 = \cdots = \lambda_k = 1$ then let us denote the baton $\mathcal{B}(\lambda_1, \ldots, \lambda_k)$ by $\mathcal{B}_k$ for a shorthand. Recall that $\chi(\mathbb{R}^n, \mathcal{B}_k) \geq (\psi^{1/k} + o(1))^n$ by Theorem 11, and the base of this exponent $\psi^{1/k} = 1 + \frac{0.3119}{k} + O(\frac{1}{k^2})$ tends to 1 with roughly ‘linear’ speed. At the same time, the best upper bound that we can get by combining the ideas from [21] and [29] is only $\chi(\mathbb{R}^n, \mathcal{B}_k) \leq (2 + \frac{2}{k} + o(1))^n$.

It is then natural to ask if, for any $\varepsilon > 0$, there exists a sufficiently large $k(k(\varepsilon))$ such that $\chi(\mathbb{R}^n, \mathcal{B}_k) \leq (1 + \varepsilon + o(1))^n$.

4. Recall that for all $k \geq 3$, the best known lower bound on $G_k(n)$ given by (5) is subexponential with $n$. Erdős and Szemerédi wrote in [12] that ‘there is a good chance’ that the correct growth rate of $G_k(n)$ is indeed subexponential. As of now, the following much more modest question is still open. Is it true that, for any $k \geq 3$, a family $\mathcal{F} \subset 2^{[n]}$ of size at least $1.99^n$ contains a weak $k$-sunflower, provided $n = n(k)$ is large enough?

References

[1] N. Alon, J.H. Spencer, The probabilistic method, John Wiley & Sons, 2016.
[2] R. Alweiss, S. Lovett, K. Wu, J. Zhang, Improved bounds for the sunflower lemma, Ann. of Math. (2), 194:3, 795–815, 2021.
[3] R.C. Baker, G. Harman, J. Pintz, *The difference between consecutive primes. II*, Proc. Lond. Math. Soc. (3), 83:3, 552–562, 2001.

[4] A.V. Bobu, A.E. Kupriyanov, A.M. Raigorodskii, *Asymptotic study of the maximum number of edges in a uniform hypergraph with one forbidden intersection*, Sb. Math., 207:5, 652–677, 2016.

[5] D. Castro-Silva, F. de Oliveira Filho, L. Slot, F. Vallentin, *A recursive Lovász theta number for simplex-avoiding sets*, Proc. Amer. Math. Soc., 2022.

[6] E. Croot, V. Lev, P. Pach, *Progression-free sets in \(\mathbb{Z}_n^4\) are exponentially small*, Ann. of Math. (2), 185:1, 331–337, 2017.

[7] A.D.N.J. de Grey, *The chromatic number of the plane is at least 5*, Geombinatorics, 28, 18–31, 2018.

[8] J. Ellenberg, D. Gijswijt, *On large subsets of \(\mathbb{F}_q^n\) with no three-term arithmetic progression*, Ann. of Math. (2), 185:1, 339–343, 2017.

[9] P. Erdős, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, E.G. Straus, *Euclidean Ramsey theorems I*, J. Combin. Theory Ser. A, 14:3, 341–363, 1973.

[10] P. Erdős, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, E.G. Straus, *Euclidean Ramsey theorems II*, Colloq. Math. Soc. J. Bolyai, 10, Infinite and Finite Sets, Keszthely, Hungary and North-Holland, Amsterdam, 520–557, 1975.

[11] P. Erdős, R. Rado, *Intersection theorems for systems of sets*, J. Lond. Math. Soc., 35, 85–90, 1960.

[12] P. Erdős, E. Szemerédi, *Combinatorial properties of systems of sets*, J. Combin. Theory Ser. A, 24:3, 308–313, 1978.

[13] G. Exoo, D. Ismailescu, *The chromatic number of the plane is at least 5: A new proof*, Discrete Comput. Geom., 64:1, 216–226, 2020.

[14] N. Frankl, A. Kupavskii, A. Sagdeev, *Max-norm Ramsey Theory*, preprint arXiv:2111.08949.

[15] P. Frankl, V. Rödl, *Forbidden intersections*, Trans. Amer. Math. Soc., 300:1, 259–286, 1987.

[16] P. Frankl, V. Rödl, *A partition property of simplices in Euclidean space*, J. Amer. Math. Soc., 3:1, 1–7, 1990.

[17] P. Frankl, R.M. Wilson, *Intersection theorems with geometric consequences*, Combinatorica, 1:4, 357–368, 1981.

[18] R.L. Graham, *Euclidean Ramsey theory*, Handbook of Discrete and Computational Geometry, Chapman and Hall/CRC, 281–297, 2017.

[19] P. Keevash, E. Long, *Frankl-Rödl-type theorems for codes and permutations*, Trans. Amer. Math. Soc., 369, 1147–1162, 2017.

[20] A.V. Kostochka and V. Rödl, *On large systems of sets with no large weak \(\Delta\)-subsystems*, Combinatorica, 18:2, 235–240, 1998.

[21] A. Kupavskiy, *On the chromatic number of \(\mathbb{R}^n\) with an arbitrary norm*, Discrete Math., 311:6, 437–440, 2011.

[22] D.G. Larman, C.A. Rogers, *The realization of distances within sets in Euclidean space*, Mathematika, 19, 1–24, 1972.

[23] I. Leader, P.A. Russell, M. Walters, *Transitive sets in Euclidean Ramsey theory*, J. Combin. Theory Ser. A, 119, 382–396, 2012.

[24] K. Matomäki, J. Maynard, X. Shao, *Vinogradov’s theorem with almost equal summands*, Proc. Lond. Math. Soc., 115:2, 323–347, 2017.

[25] E. Nashlund, *Monochromatic Equilateral Triangles in the Unit Distance Graph*, Bull. Lond. Math. Soc., 52:4, 687–692, 2020.

[26] E. Nashlund, *Upper Bounds For Families Without Weak Delta-Systems*, preprint arXiv:2203.13370.

[27] E. Nashlund, W. Sawin, *Upper bounds for sunflower-free sets*, Forum Math. Sigma, 5, e15, 2017.

[28] E.I. Ponomarenko, A.M. Raigorodskii, *New estimates in the problem of the number of edges in a hypergraph with forbidden intersections*, Probl. Inf. Transm., 49:4, 384—390, 2013.

[29] R.I. Prosanov, *Upper Bounds for the Chromatic Numbers of Euclidean Spaces with Forbidden Ramsey Sets*, Math. Notes, 103:2, 243–250, 2018.

[30] R. Prosanov, *A new proof of the Larman–Rogers upper bound for the chromatic number of the Euclidean space*, Discrete Appl. Math., 276, 115–120, 2020.

[31] A.M. Raigorodskii, *On the Chromatic Number of a Space*, Russian Math. Surveys, 55, 351–352, 2000.
[32] A.M. Raigorodskii, On the Chromatic Number of a Space with the Metric $\ell_p$, Russian Math. Surveys, 59, 973–975, 2004.

[33] A.M. Raigorodskii, A.A. Sagdeev, On a Bound in Extremal Combinatorics, Dokl. Math., 97:1, 47–48, 2018.

[34] A.A. Sagdeev, On the Frankl–Rödl Theorem, Izv. Math., 82:6, 1196–1224, 2018.

[35] A.A. Sagdeev, Exponentially Ramsey Sets, Probl. Inf. Transm., 54:4, 372–396, 2018.

[36] A. Sagdeev, On the Partition of an Odd Number into Three Primes in a Prescribed Proportion, Math. Notes, 106:1, 98–107, 2019.

[37] A.A. Sagdeev, A.M. Raigorodskii, On a Frankl–Wilson Theorem and Its Geometric Corollaries, Acta Math. Univ. Comenian., 88:3, 1029–1033, 2019.

[38] A.A. Sagdeev, On a Frankl–Wilson Theorem, Probl. Inf. Transm., 55:4, 376–395, 2019.

[39] A. Soifer, The mathematical coloring book, Springer-Verlag New York, 2009.