Perturbation Theory around Non–Nested Fermi Surfaces

I. Keeping the Fermi Surface Fixed

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This paper is dedicated to the memory of Ansgar Schnizer

Abstract

The perturbation expansion for a general class of many–fermion systems with a non–nested, non–spherical Fermi surface is renormalized to all orders. In the limit as the infrared cutoff is removed, the counterterms converge to a finite limit which is differentiable in the band structure. The map from the renormalized to the bare band structure is shown to be locally injective. A new classification of graphs as overlapping or non–overlapping is given, and improved power counting bounds are derived from it. They imply that the only subgraphs that can generate $r$ factorials in the $r^{\text{th}}$ order of the renormalized perturbation series are indeed the ladder graphs and thus give a precise sense to the statement that ‘ladders are the most divergent diagrams’. Our results apply directly to the Hubbard model at any filling except for half–filling. The half–filled Hubbard model is treated in another place.
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1. Introduction and Overview

1.1 The Problem

Consider the following problem in many-body physics. Let \( \Lambda \) be a finite box in \( d \)-dimensional space, i.e. \( \Lambda \subset \mathbb{R}^d \) or \( \Lambda \subset \Gamma \), where \( \Gamma \) is a lattice in \( \mathbb{R}^d \), and let \( c_\sigma(x) \) and \( c_\sigma^+(x) \) be fermionic annihilation and creation operators obeying the canonical anti-commutation relations

\[
\{ c_\sigma(x), c_{\sigma'}^+(x') \} = \delta_{\sigma\sigma'} \delta(x-x')
\]

and let \( F \) be the fermionic Fock space generated by this algebra [BR]. Let \( H_\Lambda = H_0 + \lambda V \) be the operator on \( F \) given by

\[
H_0 = \sum_\sigma \int ds(x)c_\sigma^+(x)(T+U)c_\sigma(x) \tag{1.1}
\]

where \( T \) is an operator describing the one-particle kinetic energy, \( U \) is multiplication by a periodic potential, and \( \int ds(x) \) denotes \( \int_\Lambda dx \) for a continuous system and \( \sum_{x \in \Lambda} \) for a system on a lattice. Let \( n_\sigma(x) = c_\sigma^+(x)c_\sigma(x) \) be the number operator at \( x \) for spin \( \sigma \). The interaction

\[
V = \sum_{\sigma,\sigma'} \int ds(x) \int ds(x')n_\sigma(x)v_{\sigma\sigma'}(x-x')n_{\sigma'}(x') \tag{1.2}
\]

is assumed to be short-ranged (see Assumption A1 below). The Hamiltonian \( H_\Lambda \) describes many electrons in a crystal or on a lattice, that interact with a stationary ionic background through \( U \) and with each other through the pair potential \( V \). If the coupling strength of the electron–electron interaction \( \lambda = 0 \), the electrons move independently according to the one-particle Schrödinger operator \( T + U(x) \). In the continuum system \( T = -\Delta/2m \) is the Laplacean and \( U(x+\gamma) = U(x) \) for all \( \gamma \in \Gamma \), where the lattice \( \Gamma \) is generated by \( d \) linearly independent vectors in \( \mathbb{R}^d \) (e.g. \( \Gamma = \mathbb{Z}^d \)); in the case of a lattice system, \( U = 0 \) and the kinetic energy \( T \) is defined by the hopping matrix between the sites of the lattice. For \( \lambda \neq 0 \), the potential \( V \) takes into account interactions such as screened electromagnetic interactions. A slight generalization of (1.2) allows for inclusion of phonon-mediated interactions.
Let $\beta = 1/kT$ be the inverse temperature and define the grand canonical partition function $Z_\Lambda$ as

$$Z_\Lambda = \text{tr} e^{-\beta(H_\Lambda - \mu N_\Lambda)}$$  \hspace{1cm} (1.3)

where

$$N_\Lambda = \sum_\sigma \int_{\Lambda} ds(x)n_\sigma(x)$$  \hspace{1cm} (1.4)

is the number operator, $\mu$ is the chemical potential and the trace is over Fock space. For an observable $\mathcal{O}$, i.e. a polynomial in the fermion operators, the thermal expectation value is defined as

$$\langle \mathcal{O} \rangle_\Lambda = \frac{1}{Z_\Lambda} \text{tr} \left( e^{-\beta(H_\Lambda - \mu N_\Lambda)} \mathcal{O} \right)$$  \hspace{1cm} (1.5)

The question we are interested in is whether the thermodynamic limit $\mathcal{G} = \lim_{\Lambda \to \infty} \mathcal{G}_\Lambda$ of the connected Green functions $\mathcal{G}_\Lambda = \langle c_{\sigma_1}^+(x_1) \ldots c_{\sigma_m}^+(x_m) \rangle_{\Lambda,\text{conn}}$, which are special cases of $\mathcal{O}$ above, exists and whether in infinite volume a weak-coupling expansion

$$\mathcal{G} = \sum_{r=0}^{\infty} \lambda^r G_r$$  \hspace{1cm} (1.6)

can be used to determine the dependence of $\mathcal{G}$ on $\lambda$.

For this question the most interesting, because most singular, case is that of zero temperature, $T = 0$. For positive temperature or the finite volume lattice case the expansion obtained by expanding the factor $e^{\lambda V}$ in $\lambda$ is convergent, but its radius of convergence shrinks to zero in the thermodynamic and zero-temperature limit: at $T = 0$ and in infinite volume, one can not even pose the question of convergence of the expansion in $\lambda$ because the coefficients $G_r$ already diverge for $r \geq 3$. In the limit $T \to 0$, (1.5) reduces to expectation values in the ground state of the system, so physically the question is about the nature of the many–particle ground state of the system and the validity of perturbation theory to calculate $n$–point-functions. The radius of convergence of the unrenormalized expansion in finite volume shrinks to zero as the volume goes to infinity. Thus, although the expansion converges for the large but finite systems which these models are to describe, this is true only if $\lambda$ is of order $1/volume$, which is obviously unrealistic for any macroscopic system. Consequently, the unrenormalized expansion will not give insight into the properties of the ground state.

In this paper we consider formal perturbation theory. That is, we study the thermodynamic limit of the coefficient functions $G_r$. By an analysis similar to [FT1], the
expansion is renormalized so that these functions converge as the volume goes to infinity. More precisely, we introduce a well–defined infinite volume model obtained by cutting off the singularity at the Fermi surface (i.e. introducing an infrared cutoff) and renormalized by including counterterms $K$ in the action, and then show that all coefficients $G_r$ have limits as the infrared cutoff is removed. These counterterms are bilinear in the fermions and can therefore be viewed as a modification of $H_0$ (although they are treated as extra interaction vertices in the formal expansion). They also have finite limits as the infrared cutoff is removed. The limiting counterterms reflect the modification of the band structure due to the interaction. The precise meaning of this will be discussed in much more detail below. Although we do not go through the finite–volume bounds here, it will be clear from the way our bounds are derived that the same procedure can be applied to obtain an expansion in finite volume with coefficients that converge in the thermodynamic limit.

Except for special cases, the renormalized expansion is, as an expansion in $\lambda$, not convergent but only locally Borel summable because the coefficients behave as $G_r \sim r!$. The occurrence of these factorials indicates that the nonperturbative ground state may exhibit symmetry breaking. For example, if the interaction is attractive in the zero angular momentum sector, this is the case [FT2]. One of the main results we shall prove here is that for a very wide class of models (and regardless of the sign of the interaction), the $r$ factorials in individual graphs come only from ladder diagrams.

Renormalization has been done in [FT1] for the continuum case where $T = -\Delta/2m$ and $U = 0$. We shall refer to this case as the spherical case since the band structure (defined below) has an $O(d)$ rotational symmetry. The procedure for removing the divergences in the present case is similar to the spherical case in that we have to renormalize two–legged insertions. However, the present work is a nontrivial extension of [FT1] because in contrast to the spherical case the counterterms are not constants. In brief, subtracting functions is much more complicated than subtracting constants. In particular, the regularity properties of the counterterms are quite subtle.

In the remainder of this introductory section, we give a non–rigorous, physical discussion of why divergences occur and how they may be removed by renormalization. We hope that this will convince the reader, before going through all the details, that the renormalization subtractions are natural and the divergences of the naive expansion are artificial in these models. We state our main results in Section 1.5 and then discuss their physical interpretation. Finally, we give an overview of the sections containing the proofs. Every
section begins with a brief explanation of what is done and how it fits into the general strategy.

1.2 The Formal Perturbation Expansion

The models have the formal functional integral representation

$$P(\eta, \bar{\eta}) = \int D\psi D\bar{\psi} e^{\mathcal{A}^+(\bar{\eta}, \psi) + (\eta, \bar{\psi})}$$

where \( \mathcal{A} = -\langle \bar{\psi}, C^{-1} \psi \rangle - \lambda V \), \( D\psi D\bar{\psi} \) is the formal measure \( \prod_{x, \alpha} d\psi_\alpha(x) d\bar{\psi}_\alpha(x) \),

\[
(\bar{\psi}, C^{-1} \psi) = \int ds(x) ds(y) \sum_{\alpha, \beta} \bar{\psi}_\alpha(x) (C^{-1})_{\alpha \beta} (x, y) \psi_\beta(y),
\]

and

\[
V = \int ds(x) ds(x') \sum_{\alpha, \beta, \alpha', \beta'} \bar{\psi}_\alpha(x) \psi_\beta(x) \tilde{v}_{\alpha \beta, \alpha' \beta'}(x, x', x') \bar{\psi}_{\alpha'}(x') \psi_{\beta'}(x'),
\]

where now \( \int ds(x) F(x) \) stands for the integral over the spatial variable \( x \) and imaginary time \( \tau \), \( x = (\tau, x) \), with an appropriate measure, e.g.

\[
ds(x) = d\tau d^d x
\]

for a continuous system on \([0, \beta] \times \mathbb{R}^d\) and

\[
\int ds(x) F(x) = \int_0^\beta d\tau \sum_{x \in \Gamma} F(\tau, x)
\]

for a lattice system on \([0, \beta] \times \Gamma\), e.g. \( \Gamma = \mathbb{Z}^d \). Here \( \beta = 1/k_B T \) is the inverse temperature. The imaginary time is introduced to get a functional integral representation for the trace over Fock space in the standard way. The connected Green functions can formally be calculated as derivatives of \( \log P \) with respect to the sources \( \eta \) and \( \bar{\eta} \).

In this paper, we consider the limiting case \( T = 0 \), so \( \beta = \infty \) and the configuration spaces are \( \mathbb{R}^{d+1} \) and \( \mathbb{R} \times \Gamma \) (e.g. \( \mathbb{R} \times \mathbb{Z}^d \)), respectively. The spin index \( \alpha \in \{\uparrow, \downarrow\} \), the interaction is assumed to be translation invariant, so that

\[
\tilde{v}_{\alpha \beta, \alpha' \beta'}(x, x') = v_{\alpha \beta, \alpha' \beta'}(\tau - \tau', x - x'),
\]
and short-range, i.e. \( v \) decreasing so fast that its Fourier transform \( \hat{v} \) is at least twice differentiable (see Assumption A1 below). Note that we do not assume that it is instantaneous. For simplicity, we also assume that it is spin-diagonal, i.e. \( v_{\alpha \beta, \alpha' \beta'} = \delta_{\alpha \alpha'} \delta_{\beta \beta'} v \).

In contrast to the assumption about the decay of \( v \), the latter assumption is merely for notational convenience and can easily be dropped.

One may imagine \( v \) to arise from exchange of (quasi)particles like photons or phonons and formalize this by a Hubbard–Stratonovich transformation, introducing one or more scalar fields with covariance \( v \) so that the interaction vertex is resolved as an exchange of fields and the interaction becomes bilinear in the fermion fields. For the purposes of the perturbation expansion we shall not need this. In particular since we assume smoothness of \( \hat{v} \), we shall not need a cutoff on the interaction lines, and we shall often draw graphs with four-legged vertices instead of ones with interaction lines.

For the lattice models, we take

\[
(C^{-1})(x, x') = \delta_{\alpha \beta} (\delta_{xx'} (\partial_{\tau'} - \mu) - T_{x-x'}) \delta(\tau - \tau'),
\]

(1.13)

where \( \mu \) is the chemical potential and \( T_{x-x'} \) is the amplitude for hopping from site \( x \) to site \( x' \), which we assume to be symmetric and short-ranged (see Assumptions A2 and A3 on \( e \) below).

A model of particular interest that is easy to formulate but difficult to analyze is the Hubbard model, for which

\[
T_x = \sum_{|y|=1} t_y \delta_{x,y}
\]

(1.14)

with \( t_y \) the so-called hopping parameters. In the simplest version of the model, \( t_y = t \) is the same for all \( y \) of length one, so the operator \( T \) is just the discrete Laplacean on \( \mathbb{Z}^d \), with the diagonal term omitted since it can be absorbed in the chemical potential \( \mu \), and the interaction term is on-site and spin-diagonal,

\[
v_{\alpha \beta, \alpha' \beta'}(x - x') = \delta_{xx'} \delta_{\alpha \beta} \delta_{\alpha' \beta'}.
\]

(1.15)

Various extensions of this model, e.g. with more complicated finite range hopping have been studied in connection to high-temperature superconductivity. For suitable values of the filling factor, they all fall into the class of band structures discussed here. For a review of mathematically rigorous results about the Hubbard model, see [L].
Formally equivalent to $P$, but in fact much more convenient is the generating functional for connected amputated Green functions

$$G(\psi_e, \bar{\psi}_e) = \log \frac{1}{Z} \int D\psi D\bar{\psi} e^{-(\bar{\psi}, C^{-1} \psi)} e^{\lambda V(\psi + \psi_e, \bar{\psi} + \bar{\psi}_e)}$$  \hspace{1cm} (1.16)$$

where the constant $Z$ takes out the field–independent term so that $G(0,0) = 0$. $G$, as written above, is not a well–defined object in infinite volume; it can be made well-defined by restricting to a finite volume $\Lambda$, or by introducing a suitable cutoff. If the free covariance $C$ is bounded and any power of it is integrable, $\frac{1}{\Lambda}G_A$ exists and is analytic in $\lambda$, as was first observed by Caianiello. However, for any realistic model, $C$ will not have this properties, unless cutoffs are imposed. The radius of convergence obtained using naive bounds shrinks to zero when the cutoffs are removed, and establishing analyticity uniformly in the cutoffs requires techniques as in [FMRT].

Our analysis is done in momentum space, where from now on momentum is short for Bloch’s quasi–momentum, which can be used to label one–particle states because of the periodicity of the one–particle potential $U$. In infinite volume, momentum space is the first Brillouin zone $B$, i.e. the torus

$$B = \mathbb{R}^d / \Gamma^\#$$  \hspace{1cm} (1.17)$$

where $\Gamma^\#$ is the dual lattice to $\Gamma$, e.g. $\Gamma^\# = 2\pi \mathbb{Z}^d$ for $\Gamma = \mathbb{Z}^d$. In finite volume, the momenta are in a finite subset of $B$, $p = 2\pi n/L$ with $n \in \mathbb{Z}^d \cap B$ if the volume is a box of sidelength $L$. The eigenfunction expansions used to transform into momentum space are discussed briefly in Appendix B for the general case; for the purposes of this introduction, we just give the formulas for the case of a lattice model on $\mathbb{Z}^d$, where we can simply do a Fourier expansion. The only changes in the general case are (of course) that the Brillouin zone will differ with the lattice and that the formulas for switching between position and quasi–momentum space involve the eigenfunctions of the one–particle Hamiltonian $H_o$ with the periodic potential. Under the Fourier transform

$$\psi(x) = (2\pi)^{-(d+1)} \int d^d p d\rho e^{-ip_o \tau + ip x} \hat{\psi}(p)$$  \hspace{1cm} (1.18)$$

$$\bar{\psi}(x) = (2\pi)^{-(d+1)} \int d^d p d\rho e^{ip_o \tau - ip x} \hat{\bar{\psi}}(p)$$

the quadratic part of the action becomes

$$(\bar{\psi}, C^{-1} \psi) = (2\pi)^{-(d+1)} \int d^d p d\rho \bar{\psi}(p)(ip_o - \epsilon(p))\psi(p)$$  \hspace{1cm} (1.19)$$
where we have dropped the hats and introduced the band structure

\[ e(p) = \varepsilon(p) - \mu \]  

(1.20)

where

\[ \varepsilon(p) = \int ds(x)e^{-ipx}T_x, \]  

(1.21)

and the interaction becomes, with \( p_i = (p_i^0, p_i) \), and

\[ d^{d+1}p = dp^0 dp p = \frac{dp^0}{2\pi} \frac{dp}{(2\pi)^d}, \]  

(1.22)

\[ V = \int d^{d+1}p_1 \ldots d^{d+1}p_4 (2\pi)^{d+1} \delta((p_2 + p_4 - p_1 - p_3)_0) \delta^#(p_2 - p_1 + p_4 - p_3) \]  

\[ \hat{\psi}(p_3 - p_1)\psi(p_1)\psi(p_2)\bar{\psi}(p_3)\psi(p_4). \]

(1.23)

here \( \delta^# \) is the delta function on \( B \), more explicitly

\[ \delta^#(p) = \frac{1}{(2\pi)^d} \sum_{x \in \mathbb{Z}^d} e^{ipx} = \sum_{\gamma \in \Gamma^#} \delta(p + \gamma) \]  

(1.24)

where the \( \delta \) on the right side denotes that on \( \mathbb{R}^d \). In general, the solution of the one-particle problem will produce crossing bands. We exclude this case here, and we also introduce an ultraviolet cutoff that removes the high energy bands. For the lattice systems, such a cutoff is already built in as the lattice spacing; for continuous systems it is not a real physical restriction since high energies do not occur in a crystal. If there are finitely many bands that do not cross, the band index is just a bookkeeping device dragged along, so, without loss, we restrict to the one–band case here.

For \( \lambda = 0 \), the fermions do not influence each other and the model is completely characterized by the covariance \( \tilde{C} \),

\[ \tilde{C}(x) = \int d^{d+1}p \frac{e^{-ip_0^+ + ipx}}{ip_0 - e(p)} \]  

(1.25)

in the sense that all \( 2n \)-point functions are determinants of matrices with elements \( \tilde{C}(x_i - x_j) \).

The propagator in momentum space, \( C(p) = e^{ip_0^+ + (ip_0 - e(p))} \), has a singularity at \( p_0 = 0 \) for all \( p \in S \), where \( S = \{ p : e(p) = 0 \} \) is the Fermi surface of the independent electron approximation. Although the function \( 1/(ip_0 - e(p)) \) is in \( L^{1+\delta}_{loc}(\mathbb{R} \times B) \) for all \( \delta \in [0, 1) \), graphs in the perturbation expansion diverge because of the singularity on \( S \) and
because in the expansion, arbitrary powers of \( C \) are integrated. The numerator \( e^{ip_0} \) is included in the standard way since we want to consider the expansion around the situation where all states inside the Fermi surface, i.e. those with \( \varepsilon(p) < 0 \), are already occupied.

Expanding \( G \) in a formal power series in \( \lambda \), we can write

\[
G(\psi_e, \bar{\psi}_e) = \sum_{r \geq 0} \lambda^r G_r(\psi_e, \bar{\psi}_e) \tag{1.26}
\]

with

\[
G_r(\psi, \bar{\psi}) = \sum_{m \geq 1} \sum_{\alpha_1, \ldots, \alpha_m} \int \prod_{i=1}^{2m} d^{d+1} p_i (2\pi)^{d+1} \delta^\# \left( \sum_{i=1}^{m} p_i - \sum_{i=m+1}^{2m} p_i \right) (G_{2m,r})_{\alpha_1, \ldots, \alpha_m} (p_1, \ldots, p_{2m-1}) \prod_{i=1}^{m} \psi_{\alpha_i}(p_i) \bar{\psi}_{\bar{\alpha}_i}(p_{m+i}), \tag{1.27}
\]

where the coefficient function \( G_{2m,r} \) is totally antisymmetric in the simultaneous exchange of momenta and spin indices (see Section 2.3). Again, the \( \delta^\# \) is periodic with respect to \( \Gamma^\# \) in the spatial part of the momentum. The coefficient \( G_{2m,r} \) can be expressed in the usual way as a sum over values of connected Feynman diagrams. The sum over \( m \) runs over a finite index set for each fixed \( r \) because the number of vertices is \( r \) and the graphs are connected with \( 2m \) external legs.

The Feynman graphs are similar to those in quantum electrodynamics: there are two types of lines, namely fermion lines (drawn solid), carrying a direction, and interaction lines (drawn dashed). The vertices have two legs to which fermion lines can be connected (one incoming, one outgoing), and one leg for an interaction line. The action determines the assignment of propagators \( C(p) \) to fermion lines, \( \hat{v}(p) \) to interaction lines, and momentum conservation delta functions to vertices. Equivalently, one can replace two vertices that are joined by an interaction line by a single four–fermion vertex with exactly two incoming fermion legs and exactly two outgoing fermion legs. The graphs then have only four–legged fermion vertices and only fermion lines. There is one notable difference between the cases \( U = 0 \) and \( U \neq 0 \): In the spherical case (\( U = 0 \)), where \( \varepsilon(p) = p^2/2m, \ p \in \mathbb{R}^d \). The corresponding ultraviolet problem (behaviour at large \(|p|\)) was solved in [FT1]. In presence of a crystal potential (\( U \neq 0 \)), the integrals over the spatial part of the momentum are over the first Brillouin zone \( \mathcal{B} \), which is a compact set. Thus there is no case of large \( p \) here. Momentum conservation at every vertex means conservation in \( \mathcal{B} \), as given by \( \delta^\# \) above. If one prefers to think of the momenta in \( \mathbb{R}^d \), fixing momenta with \( \delta^\# \) means that
at every vertex, there remains a sum over $\gamma \in \Gamma^\#$. Although formally infinite, this sum always contains only one nonzero term since there is a unique $\gamma \in \Gamma^\#$ that translates back a vector in $\mathbb{R}^d$ into the fundamental domain of the translational group $\Gamma^\#$. However, it is natural and simpler to consider momentum space as the torus $\mathcal{B}$ since $e$ is $\Gamma^\#$-periodic.

For example, in the Hubbard model,

$$e(p) = 2t \sum_{i=1}^{d} \cos p_i - \mu$$

is the tight-binding band relation and $\hat{v}(p) = 1$.

The much more general class of models and the range of chemical potential $\mu$ that we treat in this paper is given by the following assumptions.

1.3 Assumptions

We assume that the one-particle problem (discussed in Appendix B) is such that we have a Brillouin zone $\mathcal{B}$ which is a $d$-dimensional torus of type (1.17). We assume that $e = \varepsilon - \mu$ (see (1.20)) is a continuous function on $\mathcal{B}$ and that for some value $\mu_0$ of the chemical potential, the Fermi surface

$$S = \{ p \in \mathcal{B} : e(p) = 0 \}$$

has only a finite number of connected components. Furthermore, there is $k \geq 2$ and a neighbourhood $\mathcal{N}$ of $S$ such that:

A1 The interaction $\hat{v} \in C^k(\mathbb{R} \times \mathcal{B}, \mathcal{C})$. The sup norm over $\mathbb{R} \times \mathcal{B}$ of the first $k$ derivatives is finite.

A2 The band structure $e \in C^k(\mathcal{N}, \mathbb{R})$, and $\nabla e(p) \neq 0$ for all $p \in S$.

The third assumption is a geometrical condition on the Fermi surface. It is very simple to understand and is fulfilled for generic surfaces. Let $n: S \to \mathbb{R}^d$, $\omega \mapsto n(\omega) = \frac{\nabla e}{|\nabla e|}(\omega)$, be the unit normal to the surface. By A2, $S$ is a $C^k$ submanifold of $\mathcal{B}$, and $n$ is a $C^{k-1}$ unit
vector field. If $S$ consists of more than one connected component, choose a normal field for any component. For $\omega, \omega' \in S$, define the angle between $n(\omega)$ and $n(\omega')$ by
\[
\theta(\omega, \omega') = \arccos(n(\omega) \cdot n(\omega')).
\] (1.30)

Let
\[
D(\omega) = \{ \omega' \in S : |n(\omega) \cdot n(\omega')| = 1 \} = \{ \omega' \in S : n(\omega) = \pm n(\omega') \},
\] (1.31)
and denote the $(d-1)$-dimensional measure of $A \subset S$ by $\text{vol}_{d-1} A$. Also, for any $A \subset \mathbb{R}^d$ and $\beta > 0$ denote by $U_\beta(A) = \{ p \in \mathbb{R}^d : \text{distance}(p, A) < \beta \}$ the open $\beta$-neighbourhood of $A$. For fixed $\varepsilon$ and $\mu_0$, we assume:

**A3** There is an open interval $\mathcal{M}$ around $\mu_0$ and there are strictly positive numbers $Z_0, Z_1, \rho, \beta_0$ and $\kappa$ such that for all $\mu \in \mathcal{M}$, the Fermi surface $S = S(\mu) = \{ p \in \mathcal{B} : e(p) = 0 \}$ has the following properties: $S(\mu) \subset \mathcal{N}$, and for all $\beta \leq \beta_0$ and all $\omega \in S$,

(i) $\text{vol}_{d-1} (U_\beta(D(\omega)) \cap S) \leq Z_0 \beta^\kappa$

(ii) if $\omega' \notin U_\beta(D(\omega)) \cap S$, then $|\sin \theta(\omega, \omega')| = \sqrt{1 - (n(\omega) \cdot n(\omega'))^2} \geq Z_1 \beta^\rho$.

Throughout this paper, A1–A3 will be assumed to hold, and $\mu$ will be assumed to lie in the interval $\mathcal{M}$ specified in A3. We now explain what these assumptions mean.

Assumption A1 on $\hat{v}$ is a decay assumption in position space, e.g. for an instantaneous interaction $V$ on a lattice system on $\mathbb{Z}^d$ and $k = 2$, A1 holds if
\[
\sum_{x \in \mathbb{Z}^d} |x|^2 |V(x)| < \infty.
\] (1.32)

For continuous systems, A1 is implied by a similar integral condition.

Assumption A2 excludes singular points. For example, a point $p$ on $S$ where $\nabla e(p) = 0$ is called a van Hove singularity.

The condition that $e$ is continuously differentiable is fulfilled for the case where $e$ comes from a Schrödinger equation for the one-body problem with a regular periodic potential, if there is no level-crossing. Indeed, it is real analytic. In lattice models with finite-range
hopping, $e$ is analytic. However, infinite-range hopping is also allowed: $e \in C^k$ if the $k^{th}$ moment of the hopping amplitude exists, i.e. $\sum_{|x|} |x|^k |T_x| < \infty$.

Assumption $A3$ is, more informally, that for every $\omega \in S$

(i) the set of points $\omega'$ where the normal $n(\omega')$ is parallel or antiparallel to $n(\omega)$, has positive codimension $\kappa > 0$ in $S$ and

(ii) if $\omega'$ is not in the set $D(\omega)$, where the normal is (anti)parallel to $n(\omega)$, the angle between $n(\omega)$ and $n(\omega')$ increases with some power of the distance between $\omega'$ and $D(\omega)$.

Thus in order to violate these assumptions, the surface $S$ must have flat regions or subsets where $\theta(\omega, \omega')$ vanishes exponentially fast as $|\omega - \omega'| \to 0$. To illustrate $A3$, we draw an example of a Fermi surface that satisfies $A3$ in $d = 2$ (i.e. a Fermi curve) on $B = \mathbb{R}^2/2\pi \mathbb{Z}^2$ (the square bounds the fundamental region $[-\pi, \pi)^2$ for the torus $B$, and the shaded areas indicate $e(p) < 0$):