Some new results for Horn’s hypergeometric functions $\Gamma_1$ and $\Gamma_2$

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Abstract

The object of the present work is to deduce several important developments in various recursion relations, relevant differential recursion formulas, infinite summation formulas, integral representations, and integral operators for Horn’s hypergeometric functions $\Gamma_1$ and $\Gamma_2$.

Keywords: Horn’s functions, recursion formulas, infinite summation formulas, integral operators.

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1. Introduction, notations, and preliminaries

The problem of recursion formulas of hypergeometric type functions with respect to their parameters was considered in numerous papers, see for example \cite{10, 20, 24}. Recently, Opps et al. \cite{11, 12} obtained recursion formulas for Appell’s function $F_2$. Brychkov \cite{2} and Brychkov et al. \cite{3–6} gave the recursion formulas for Appell’s hypergeometric functions $F_1$, $F_2$, $F_3$ and $F_4$. Brychkov and Savischenko \cite{7} obtained some formulas for Horn functions $H_1(a,b,c;d;w,z)$ and $H_1^{(c)}(a,b;d;w,z)$. Mullen \cite{10}, Sharma \cite{20} and Wang \cite{24} gave various recursion formulas for Appell’s functions. Sahin \cite{17} and Sahai and Verma \cite{13–16} gave some recursion formulas for three variables hypergeometric functions and $k$-Lauricella’s hypergeometric functions. Sahin and Agha \cite{18}, studied the recursion relations of $G_1$ and $G_2$ Horn’s hypergeometric functions. Srivastava et al. \cite{23} introduced incomplete Hurwitz-Lerch zeta functions of two variables.

Recall that the following abbreviated notations, the Pochhammer symbol $(a)_n$ is defined in \cite{21, 22} by

\[ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\cdots(a+n-1), & n \in \mathbb{N}; a \in \mathbb{C} - \{0\}; \\ 1, & n = 0; a \in \mathbb{C} - \{0\}, \end{cases} \]  

(1.1)

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where \( \Gamma(a) \) is a gamma function, the symbols \( \mathbb{N} \) and \( \mathbb{C} \) denote the sets of natural numbers and complex numbers, respectively, and
\[
(a)_n = \frac{(-1)^n}{(1-a)_n}, (a \neq 0, \pm 1, \pm 2, \pm 3, \ldots, \forall a > n; n \in \mathbb{N}).
\]
(1.2)

Its well known form are also used in [8] as
\[
(a)_{n+1} = a(a+1)_{n} = (a+1)(a)_n, \quad (a)_n = \left(1 + \frac{n}{a}\right)(a)_n; a \neq 0.
\]
For \( n, m \in \mathbb{N} \), we have
\[
(a)_{n+m} = (a)(a+n)_m = (a)_m(a+m)_n, \quad (b)_{n-m} = \frac{(-1)^m(b)_n}{(1-b-n)_m}, \quad 0 \leq m \leq n.
\]
(1.3)

The Horn’s hypergeometric functions \( \Gamma_1 \) and \( \Gamma_2 \) were defined by the series [8, 9]
\[
\Gamma_1(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m-n}{m!n!} x^m y^n; |x| < 1, |y| < \infty \quad (1.4)
\]
and
\[
\Gamma_2(b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(b)_n(c)_m-n}{m!n!} x^m y^n; |x| < \infty, |y| < \infty. \quad (1.5)
\]
Throughout this work, supposing that none of the numerator parameters \( a, b, \) and \( c \) are zero or a negative integer, and with the usual restrictions (1.1), (1.2), and (1.3), the Horn’s hypergeometric functions \( \Gamma_1 \) and \( \Gamma_2 \) were defined by the series (1.4) and (1.5) taking into consideration the parameter \( a \) satisfies the condition in (1.1) and the parameters \( b \) and \( c \) satisfy the conditions in (1.2) and (1.3).

To simplify the notations, we write \( \Gamma_1 \) for the series \( \Gamma_1(a, b, c; x, y) \), \( \Gamma_1(a \pm n) \) for the series \( \Gamma_1(a \pm n, b, c; x, y) \), and \( \Gamma_2(b, c \pm n) \) stands for the series \( \Gamma_2(b, c \pm n; x, y) \) etc.

Abul-Ez and Sayyed [1] and Sayyed [19] introduced an integral operator \( \hat{\mathcal{I}} \) as follows
\[
\hat{\mathcal{I}} = \frac{1}{x} \int_0^x dx + \frac{1}{y} \int_0^y dy.
\]
For \( |t| < 1 \) and \( \alpha \) satisfying the condition in (1.1), we have the binomial theorem (see [22])
\[
(1 - t)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} t^r. \quad (1.6)
\]
Motivated essentially by the demonstrated potential for applications of the Horn’s functions \( \Gamma_1 \) and \( \Gamma_2 \) in many diverse areas of physical, mathematical, statistical sciences and engineering, this paper is organized as follows. In Section 2, we establish a number of new recursion formulas, various differential recursion formulas and generating relations for Horn’s functions \( \Gamma_1 \). In Section 3, we present new an investigation of several classes of recursion formulas, differential recursion formulas, infinite summation formulas and integral representations for Horn’s functions \( \Gamma_2 \). In Section 4, we introduce and investigate of integral operators for Horn’s functions \( \Gamma_1 \) and \( \Gamma_2 \). Finally, some concluding remarks and observations of the results are obtained in Section 5.

2. Recursion formulas and differential recursion formulas of \( \Gamma_1 \)

In this section, we give some investigation of several classes of recursion formulas, generating relations and relevant differential formulas for Horn’s function \( \Gamma_1 \).
The proofs are omitted (mathematical induction method).

**Theorem 2.1.** For \( n \in \mathbb{N} \) and \( b \neq 1 \), recursion formulas for Horn's function \( \Gamma_1 \) are as follows

\[
\Gamma_1(a + n) = \Gamma_1 + \frac{cx}{b - 1} \sum_{k=1}^{n} \Gamma_1(a + k, b - 1, c + 1; x, y) \tag{2.1}
\]

and

\[
\Gamma_1(a - n) = \Gamma_1 - \frac{cx}{b - 1} \sum_{k=0}^{n-1} \Gamma_1(a - k, b - 1, c + 1; x, y). \tag{2.2}
\]

**Proof.** From the definition of the \( \Gamma_1 \) (1.4) and transformation

\[(a + 1)_m = \left(1 + \frac{m}{a}\right)(a)_m, \ a \neq 0,\]

we get

\[
\Gamma_1(a + 1) = \Gamma_1 + \frac{cx}{b - 1} \Gamma_1(a + 1, b - 1, c + 1; x, y), \ b \neq 1. \tag{2.3}
\]

If we compute the Horn's function \( \Gamma_1 \) with the parameter \( a + n \) by relation (2.3) for \( n \) times, we obtain the formula (2.1). From the contiguous relation (2.3) and replacing \( a \) by \( a - 1 \), we get

\[
\Gamma_1(a - 1) = \Gamma_1 - \frac{cx}{b - 1} \Gamma_1(a, b - 1, c + 1; x, y), \ b \neq 1.
\]

If we apply this relation to the \( \Gamma_1 \) with the parameter \( a - n \) for \( n \) times, we obtain (2.2). \( \square \)

**Theorem 2.2.** For \( b \neq 1, 2, 3, \ldots \), the Horn's function \( \Gamma_1 \) satisfies the identity:

\[
\Gamma_1(a + n) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k(c)_k}{(1-b)_k} \Gamma_1(a + k, b - k, c + k; x, y)
\]

and

\[
\Gamma_1(a - n) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k(c)_k}{(1-b)_k} \Gamma_1(a, b - k, c + k; x, y).
\]

**Proof.** The proofs are omitted (mathematical induction method). \( \square \)

**Theorem 2.3.** For \( n \in \mathbb{N} \) and \( c \neq 1 \), the recursion relations hold true for Horn's function \( \Gamma_1 \):

\[
\Gamma_1(b + n) = \Gamma_1 + \frac{y}{c - 1} \sum_{k=1}^{n} \Gamma_1(a, b + k, c - 1; x, y) \tag{2.4}
\]

\[-acx \sum_{k=1}^{n} \frac{\Gamma_1(a + 1, b + k - 2, c + 1; x, y)}{(b + k - 1)(b + k - 2)}, \ b \neq 1 - k, \ b \neq 2 - k \forall k \in \mathbb{N}
\]

and

\[
\Gamma_1(b - n) = \Gamma_1 - \frac{y}{c - 1} \sum_{k=1}^{n} \Gamma_1(a + 1, b - k + 1, c - 1; x, y) \tag{2.5}
\]

\[+acx \sum_{k=1}^{n} \frac{\Gamma_1(a + 1, b - k - 1, c + 1; x, y)}{(b - k)(b - k - 1)}, \ b \neq k, \ b \neq 1 + k \forall k \in \mathbb{N}.
\]
Proof. From (1.4) and the relation

\[(b + 1)_{n-m} = \left(1 + \frac{n}{b} - \frac{m}{b}\right)(b)_{n-m}, b \neq 0,\]

we obtain the contiguous function

\[\Gamma_1(b + 1) = \Gamma_1 + \frac{y}{c - 1} \Gamma_1(a, b + 1, c - 1; x, y)\]

\[\quad - \frac{acx}{b(b - 1)} \Gamma_1(a + 1, b - 1, c + 1; x, y); b \neq 0, b \neq 1, c \neq 1.\]  

(2.6)

By iterating this method on the above contiguous relation for \(\Gamma_1\) with \(b + n\) for \(n\) times, we get (2.4). Replacing \(b\) by \(b - 1\) in contiguous relation (2.6), we obtain

\[\Gamma_1(b - 1) = \Gamma_1 - \frac{ay}{c - 1} \Gamma_1(a + 1, b, c - 1; x, y)\]

\[\quad + \frac{acx}{(b - 1)(b - 2)} \Gamma_1(a + 1, b - 2, c + 1; x, y), c, b \neq 1, b \neq 2.\]  

(2.7)

If we compute the Horn’s function \(\Gamma_1\) with the parameter \(b - n\) by the contiguous relation (2.7) for \(n\) times, we obtain the recursion formula (2.5).

\[\Box\]

Theorem 2.4. For \(n \in \mathbb{N}\) and \(b \neq 1\), the Horn’s function \(\Gamma_1\) satisfies the relations:

\[\Gamma_1(c + n) = \Gamma_1 + \frac{ax}{b - 1} \sum_{k=1}^{n} \Gamma_1(a + 1, b - 1, c + k; x, y)\]

\[\quad - by \sum_{k=1}^{n} \frac{\Gamma_1(a, b + 1, c + k - 2; x, y)}{(c + k - 1)(c + k - 2)}, c \neq 1 - k, c \neq 2 - k \forall k \in \mathbb{N}\]  

(2.8)

and

\[\Gamma_1(c - n) = \Gamma_1 - \frac{x}{b - 1} \sum_{k=1}^{n} \Gamma_1(a + 1, b - 1, c + k - 1; x, y)\]

\[\quad - aby \sum_{k=1}^{n} \frac{\Gamma_1(a, b + 1, c - k - 1; x, y)}{(c - k)(c - k - 1)}, c \neq k, c \neq 1 + k \forall k \in \mathbb{N}\]  

(2.9)

Proof. Using the definition of the \(\Gamma_1\) and the equality

\[(c + 1)_{m-n} = \left(1 + \frac{m}{c} - \frac{n}{c}\right)(c)_{m-n}, c \neq 0,\]

we obtain the contiguous function

\[\Gamma_1(c + 1) = \Gamma_1 + \frac{ax}{b - 1} \Gamma_1(a + 1, b - 1, c + 1; x, y)\]

\[\quad - \frac{by}{c(c - 1)} \Gamma_1(a, b + 1, c - 1; x, y), b \neq 1, c \neq 0, c \neq 1.\]  

(2.10)

By iterating this method on the above contiguous relation for \(\Gamma_1\) with the parameter \(c + n\) for \(n\) times, we obtain (2.8).

Replacing \(c\) by \(c - 1\) in the contiguous relation (2.10), we get

\[\Gamma_1(c - 1) = \Gamma_1 - \frac{ax}{b - 1} \Gamma_1(a + 1, b - 1, c; x, y)\]

\[\quad + \frac{by}{(c - 1)(c - 2)} \Gamma_1(a, b + 1, c - 2; x, y), b, c \neq 1, b \neq 2.\]  

(2.11)
If we compute the Horn’s function $\Gamma_1$ with the parameter $c - n$ by contiguous relation (2.11) for $n$ times, we obtain (2.9).

Now, we apply differential operators $\theta_x = x \frac{\partial}{\partial x}$ and $\theta_y = y \frac{\partial}{\partial y}$ and state the following theorem.

**Theorem 2.5.** Differential recursion formula for the function $\Gamma_1$ is as follows

$$\Gamma_1(a + 1) = \left(1 + \frac{\theta_x}{a}\right) \Gamma_1, a \neq 0.$$  \hspace{1cm} (2.12)

**Proof.** Define the differential operators

$$\theta_x x^m = mx^m, \text{ and } \theta_y y^n = ny^n.$$

By using the above differential operators and the transformation

$$(a + 1)_m = \left(1 + \frac{m}{a}\right)(a)_m, a \neq 0,$$

we get

$$\Gamma_1(a + 1) = \sum_{m,n=0}^{\infty} \frac{(a + 1)_m (b)_n (c)_m - n x^m y^n}{m!n!}$$

$$= \sum_{m,n=0}^{\infty} \left(1 + \frac{m}{a}\right) \frac{(a)_m (b)_n (c)_m - n x^m y^n}{m!n!}$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m - n x^m y^n + \frac{1}{a} \sum_{m,n=0}^{\infty} \frac{m (a)_m (b)_n (c)_m - n y^n}{m!n!}}{m!n!}$$

$$= \Gamma_1 + \frac{1}{a} \theta_x \Gamma_1, a \neq 0.$$  

Hence, we obtain the differential recursion formula (2.12). \hfill \square

**Theorem 2.6.** The differential recursion formulas hold true for the Horn’s function $\Gamma_1$:

$$\Gamma_1(b + 1) = \left(1 + \frac{\theta_y}{b} - \frac{\theta_x}{b}\right) \Gamma_1, b \neq 0$$  \hspace{1cm} (2.13)

and

$$\Gamma_1(c + 1) = \left(1 + \frac{\theta_x}{c} - \frac{\theta_y}{c}\right) \Gamma_1, c \neq 0.$$  \hspace{1cm} (2.14)

**Proof.** By defining the transformation

$$(b + 1)_{n-m} = \left(1 + \frac{n-m}{b}\right)(b)_{n-m}, b \neq 0$$

and using the differential operators for the Horn’s function $\Gamma_1$, we get

$$\Gamma_1(b + 1) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m - n x^m y^n}{m!n!}$$

$$= \sum_{m,n=0}^{\infty} \left(1 + \frac{n-m}{b}\right) \frac{(a)_m (b)_n (c)_m - n x^m y^n}{m!n!}$$

$$= \Gamma_1 + \frac{1}{a} \theta_x \Gamma_1, a \neq 0.$$  

Hence, we obtain the differential recursion formula (2.13).
\[
\Gamma_1 + \frac{1}{b} \gamma_1 - \frac{1}{b} \theta_x \Gamma_1, \quad b \neq 0.
\]

Thus, we obtain (2.13). Using the relation
\[
(c + 1)_{m-n} = \left(1 + \frac{m-n}{c}\right)(c)_{m-n}, \quad c \neq 0
\]
and (1.4), after simplification we get (2.14).

**Theorem 2.7.** The derivative formulas hold true for Horn’s function \(\Gamma_1\),
\[
\frac{\partial^r}{\partial x^r} \Gamma_1 = \frac{(-1)^r(a_r,c)_r}{(1-b)_r} \Gamma_1(a+r,b-r,c+r;x,y), \quad b \neq 1,2,3,\ldots \tag{2.15}
\]
and
\[
\frac{\partial^r}{\partial y^r} \Gamma_1 = \frac{(-1)^r(b_r,c)_r}{(1-c)_r} \Gamma_1(a+b+r,c-r;x,y), \quad c \neq 1,2,3,\ldots. \tag{2.16}
\]

**Proof.** Differentiating (1.4) with respect to \(x\), we get
\[
\frac{\partial}{\partial x} \Gamma_1 = \frac{ac}{b-1} \Gamma_1(a+1,b-1,c+1;x,y), \quad b \neq 1.
\]

Repeating the above process, we eventually arrive at
\[
\frac{\partial^r}{\partial x^r} \Gamma_1 = \frac{a(a+1)\ldots(a+r-1)c(c+1)\ldots(c+r-1)}{(b-1)(b-2)\ldots(b-r)} \Gamma_1(a+r,b-r,c+r;x,y)
\]
\[
= \frac{(-1)^r(a_r,c)_r}{(1-b)_r} \Gamma_1(a+r,b-r,c+r;x,y), \quad b \neq 1,2,3,\ldots.
\]
The differential recursion formulas (2.16) can be proved in similar manner (2.15).

**Theorem 2.8.** The infinite summation formulas of Horn’s function \(\Gamma_1\) hold true:
\[
\sum_{k=0}^{\infty} \frac{(a)_k}{k!} \Gamma_1(a+k)t^k = (1-t)^{-a} \Gamma_1(a,b,c; \frac{x}{1-t}, y), \quad |t| < 1, \tag{2.17}
\]
\[
\sum_{k=0}^{\infty} \frac{(b)_k}{k!} \Gamma_1(b+k)t^k = (1-t)^{-b} \Gamma_1(a,b,c; x(1-t), \frac{y}{1-t}), \quad |t| < 1, \tag{2.18}
\]
and
\[
\sum_{k=0}^{\infty} \frac{(c)_k}{k!} \Gamma_1(c+k)t^k = (1-t)^{-c} \Gamma_1(a,b,c; x, y(1-t)), \quad |t| < 1. \tag{2.19}
\]

**Proof.** Using (1.4), (1.3), and (1.6), we obtain (2.17), (2.18), and (2.19).

3. Recursion formulas and differential recursion formulas for Horn’s function \(\Gamma_2\)

In this section, we derive the recursion formulas, differential recursion formulas, generating relations and integral representations for Horn’s function \(\Gamma_2\) with five theorems as follows. First, we present the recursion formulas of Horn’s function \(\Gamma_2\).
Theorem 3.1. For \( n \in \mathbb{N} \), the recursion formulas of \( \Gamma_2 \) with parameters \( b \) and \( c \) are

\[
\Gamma_2(b + n) = \Gamma_2 + \frac{y}{c-1} \sum_{k=1}^{n} \Gamma_2(b + k, c - 1; x, y)
\]

\[
- c x \sum_{k=1}^{n} \frac{\Gamma_2(b + k - 2, c + 1; x, y)}{(b + k - 1)(b + k - 2)}, \quad c \neq 1, b \neq 1 - k, b \neq 2 - k, \forall k \in \mathbb{N},
\]

\[
\Gamma_2(b - n) = \Gamma_2 - \frac{y}{c-1} \sum_{k=1}^{n} \Gamma_2(b - k + 1, c - 1; x, y)
\]

\[
- c x \sum_{k=1}^{n} \frac{\Gamma_2(b - k - 1, c + 1; x, y)}{(b - k)(b - k - 1)}, c \neq 1, b \neq k, b \neq 1 + k, \forall k \in \mathbb{N},
\]

\[
\Gamma_2(c + n) = \Gamma_2 + \frac{x}{b-1} \sum_{k=1}^{n} \Gamma_2(b - 1, c + k; x, y)
\]

\[
- b y \sum_{k=1}^{n} \frac{\Gamma_2(b + 1, c + k - 2; x, y)}{(c + k - 1)(c + k - 2)}, b \neq 1, c \neq 1 - k, c \neq 2 - k, \forall k \in \mathbb{N},
\]

\[
\Gamma_2(c - n) = \Gamma_2 - \frac{x}{b-1} \sum_{k=1}^{n} \Gamma_2(b - 1, c + k - 1; x, y)
\]

\[
- b y \sum_{k=1}^{n} \frac{\Gamma_2(b + 1, c - k - 1; x, y)}{(c - k)(c - k - 1)}, b \neq 1, c \neq k, c \neq 1 + k, \forall k \in \mathbb{N}.
\]

Proof. By the similar method in the Theorems 2.3-2.4, we obtain the recursion formulas for Horn’s’s function \( \Gamma_2 \), (3.1).

Theorem 3.2. The derivative formulas for the Horn’s function \( \Gamma_2 \) hold true:

\[
\Gamma_2(b + 1) = \left(1 + \frac{\theta_y}{b} - \frac{\theta_x}{b}\right) \Gamma_2, \quad b \neq 0
\]

and

\[
\Gamma_2(c + 1) = \left(1 + \frac{\theta_x}{c} - \frac{\theta_y}{c}\right) \Gamma_2, \quad c \neq 0.
\]

Proof. The proof of the current theorem is similar to the proof of Theorem 2.6.

Theorem 3.3. For the Horn’s function \( \Gamma_2 \), the differential recurrence relations hold true

\[
\frac{\partial^r}{\partial x^r} \Gamma_2 = \frac{(-1)^r c_r}{(1 - b)_r} \Gamma_2(b - r, c + r; x, y), \quad b \neq 1, 2, 3, \ldots
\]

and

\[
\frac{\partial^r}{\partial y^r} \Gamma_2 = \frac{(-1)^r b_r}{(1 - c)_r} \Gamma_2(b + r, c - r; x, y), \quad c \neq 1, 2, 3, \ldots
\]

Proof. We obtain results (3.2) and (3.3) as same way as the proof of (2.15) and (2.16).

In a similar manner (3.2) and (3.3) as same way as the proof of (2.15) and (2.16).

Theorem 3.4. The infinite summation formulas for Horn’s function \( \Gamma_2 \) hold true:

\[
\sum_{k=0}^{\infty} \frac{(b)_k}{k!} \Gamma_2(b + k) t^k = (1 - t)^{-b} \Gamma_2 \left( b, c; (1 - t), \frac{y}{1 - t} \right), |t| < 1
\]
and
\[
\sum_{k=0}^{\infty} \frac{(c)^k}{k!} I_2(c + k) t^k = (1 - t)^{-c} I_2\left(b, c; \frac{x}{1-t}, y(1-t)\right), |t| < 1.
\]

4. Integral operators of $\Gamma_1$ and $\Gamma_2$

Here we apply integral operator of $\Gamma_1$ and $\Gamma_2$ and state the following theorem.

**Theorem 4.1.** For $a \neq 1, 2, b \neq 1, 2$ and $c \neq 1, 2$, the integration formula of the function $\Gamma_1$ holds true:

\[
\hat{I}^2 \Gamma_1 = \frac{b(b+1)}{x^2(a-1)(a-2)(c-1)(c-2)} \Gamma_1(a-2, b+2, c-2; x, y) + \frac{2}{xy(a-1)} \Gamma_1(a-1, b, c; x, y)
+ \frac{c(c+1)}{y^2(b-1)(b-2)} \Gamma_1(a, b-2, c+2; x, y), x, y \neq 0.
\]

(4.1)

**Proof.** Let $\hat{I}$ acts on the Horn's function $\Gamma_1$, then we have

\[
\hat{I} \Gamma_1 = \sum_{m,n=0}^{\infty} \left(\frac{1}{m+1} + \frac{1}{n+1}\right) \frac{(a)_{m(b)n-m}(c)_{m-n} x^m y^n}{m!n!}
+ \sum_{m=0, n=1}^{\infty} \frac{(a)_{m(b)n-m}(c)_{m-n} x^m y^n}{m!(n+1)!}
+ \sum_{m=1, n=0}^{\infty} \frac{(a)_{m(b)-1}(c)_{m-n-1} x^m y^n}{m!(n+1)!}
= \frac{b}{x(a-1)(c-1)} \sum_{m,n=0}^{\infty} \frac{(a-1)_{m(b)+1}(c-1)_{m-n} x^m y^n}{m!n!}
+ \frac{c}{y(b-1)} \sum_{m,n=0}^{\infty} \frac{(a)_{m(b)-1}(c+1)_{m-n} x^m y^n}{m!n!}
- \frac{b}{x(a-1)(c-1)} \Gamma_1(a-1, b+1, c-1; x, y) + \frac{c}{y(b-1)} \Gamma_1(a, b-1, c+1; x, y), x, y \neq 0, a, b \neq 1.
\]

We can write $\hat{I} = \hat{I}_x + \hat{I}_y$, where $\hat{I}_x = \int_0^x dx$ and $\hat{I}_y = \int_0^y dy$, then the operator $\hat{I}^2$ is such that

\[
\hat{I}^2 = \hat{I}^2 = (\hat{I}_x)^2 + 2\hat{I}_x \hat{I}_y + (\hat{I}_y)^2 = \frac{1}{x^2} \int_0^x \int_0^x dx dy + \frac{2}{xy} \int_0^y \int_0^x dx dy + \frac{1}{y^2} \int_0^y \int_0^y dy dy.
\]

By using the above integral operator $\hat{I}^2$, we obtain the relation (4.1). \qed

**Theorem 4.2.** The integration formula of the function $\Gamma_1$ holds true:

\[
\hat{I}^r \Gamma_1 = \frac{(-1)^r}{x^r y^r(1-a)^r} \prod_{k=1}^{r} (\theta_x + \theta_y - k + 1) \Gamma_1(a-r), x, y \neq 0, a \neq 1, 2, 3, \ldots
\]

(4.2)

**Proof.** With the help of the integral operator $\hat{I}$ and differential operators, we get the formula

\[
\hat{I} \Gamma_1 = \sum_{m,n=0}^{\infty} \frac{(m+n+2)(a)_{m(b)n-m}(c)_{m-n} x^m y^n}{(m+1)!(n+1)!}
= \sum_{m,n=0}^{\infty} \frac{(m+n)(a)_{m-1}(b)_{n-m}(c)_{m-n} x^{m-1} y^{n-1}}{m!n!}
= \frac{\theta_x + \theta_y}{xy(a-1)} \Gamma_1(a-1), x, y \neq 0, a \neq 1.
\]
Iterating this integral operator \( \hat{I} \) and differential operators on \( \Gamma_1 \) for \( r \)-times, we get the formula (4.2).

**Theorem 4.3.** For Horn’s function \( \Gamma_1 \), we have the integral operators \( \hat{I}_x^r \) and \( \hat{I}_y^r \):

\[
\hat{I}_x^r \Gamma_1 = \frac{(b)_r}{x^r(1-a)_r(1-c)_r} \Gamma_1(a - r, b + r, c - r; x, y), \ x \neq 0, a, c \neq 1, 2, 3, \ldots , \\
\hat{I}_y^r \Gamma_1 = \frac{(-1)^r(c)_r}{y^r(1-b)_r} \Gamma_1(a, b - r, c + r; x, y), \ y \neq 0, b \neq 1, 2, 3, \ldots
\]

and

\[
\left( \hat{I}_x \hat{I}_y \right)^r \Gamma_1 = \frac{(-1)^r}{x^ry^r(1-a)_r} \Gamma_1(a - r), \ x, y \neq 0, a \neq 1, 2, 3, \ldots
\]

**Proof.** Also, we get relations concerning \( \hat{I}_x \) and \( \hat{I}_y \) individually, we get

\[
\hat{I}_x \Gamma_1 = \frac{b}{x(a - 1)(c - 1)} \Gamma_1(a - 1, b + 1, c - 1; x, y), \ x \neq 0, a, c \neq 1.
\]

By applying the operation of the above relation for \( r \)-times, we obtain the desired (4.3). Similarly, for \( \Gamma_1 \), applying operation in the proof of the relation (4.3), we obtain (4.4) and (4.5).

**Theorem 4.4.** For \( b \neq 1, 2 \) and \( c \neq 1, 2 \), the integration formula of the function \( \Gamma_2 \) holds true:

\[
\hat{I}_x^r \Gamma_2 = \frac{b(b+1)}{x^2(c-1)(c-2)} \Gamma_2(b+2, c-2; x, y) + \frac{2}{xy} \Gamma_2 \\
+ \frac{c(c+1)}{y^2(b-1)(b-2)} \Gamma_2(b-2, c+2; x, y), \ x, y \neq 0.
\]

**Theorem 5.** The integration formula of the function \( \Gamma_2 \) holds true:

\[
\hat{I}_x^r \Gamma_2 = \frac{1}{x^r y^r} \prod_{k=1}^{r} (\theta_x + \theta_y - k + 1) \Gamma_2, \ x, y \neq 0.
\]

**Theorem 6.** For Horn’s function \( \Gamma_2 \), we have the integral operators \( \hat{I}_x^r \) and \( \hat{I}_y^r \):

\[
\hat{I}_x^r \Gamma_2 = \frac{(b)_r}{x^r(1-c)_r} \Gamma_2(b + r, c - r; x, y), \ x \neq 0, c \neq 1, 2, 3, \ldots , \\
\hat{I}_y^r \Gamma_2 = \frac{(-1)^r(c)_r}{y^r(1-b)_r} \Gamma_2(b - r, c + r; x, y), \ y \neq 0, b \neq 1, 2, 3, \ldots , \\
\left( \hat{I}_x \hat{I}_y \right)^r \Gamma_2 = \frac{(-1)^r}{x^ry^r} \Gamma_2, \ x, y \neq 0.
\]

5. Concluding remarks and observations

As a direct consequence of the several recursion formulas and differential recursion formulas of Horn’s functions \( \Gamma_1 \) and \( \Gamma_2 \) which we have established here, the infinite summation formulas, integral operators and integral representations for \( \Gamma_1 \) and \( \Gamma_2 \) have been discussed. Our analytic expressions can be used as a benchmark for an accuracy of a different approximation techniques designed especially for an investigation of radiation field problems.

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