A SHORT REMARK ON THE POLARON IN THE SEMI-RELATIVISTIC PAULI-FIERZ MODEL

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Abstract. We consider the polaron of the spinless semi-relativistic Pauli-Fierz model. The Hamiltonian of the model is defined by

\[ H(P) = \sqrt{(P - d\Gamma(k) + eA)^2 + M^2 + d\Gamma(\omega_m)} \]

where \( P \in \mathbb{R}^3 \) is the momentum of the polaron, \( d\Gamma(\cdot) \) denotes the second quantization operator and \( \omega_m = |k| + m \) denotes the dispersion relation of the photon with virtual mass \( m \geq 0 \). Let \( E(P) \) be the lowest energy of \( H(P) \). In this paper, we prove the inequality

\[ E(P - k) - E(P) + \omega_m(k) \geq m, \]

for all \( P, k \in \mathbb{R}^3 \).

1. Definition and Main Result

We consider a system of a charged spinless relativistic particle interacting with the quantized radiation field with fixed total momentum \( P \in \mathbb{R}^3 \). The Hilbert space of the model is the Fock space

\[ \mathcal{F} := \bigoplus_{n=0}^{\infty} \bigotimes_{\text{sym}}^{n} L^2(\mathbb{R}^3 \times \{1, 2\}) \]  

(1)

with \( \otimes_{\text{sym}}^{0} L^2(\mathbb{R}^3 \times \{1, 2\}) =: \mathbb{C} \). Let \( d\Gamma(\cdot) \) be the second quantization operator. The Hamiltonian of the model is defined by

\[ H(P) := \sqrt{(P - d\Gamma(k) + eA)^2 + M^2 + d\Gamma(\omega_m)}, \]

(2)

where \( e \) is the coupling constant, \( M > 0 \) is the mass of the particle, \( \omega_m(k) = |k| + m \) is the photon dispersion relation and \( A \) is the quantized magnetic vector potential at the origin \( x = 0 \). See [2] for more detailed definition. The self-adjointness of \( H(P) \) was studied in [2] and [1, Corollary 7.62]. We assume the following

[H.1] There exist a dense domain \( D \subset \mathcal{F} \) such that, for all \( P \in \mathbb{R}^3 \),

\[ D \subset \text{Dom}((P - d\Gamma(k) + eA)^2) \cap \text{Dom}(d\Gamma(\omega_m)) \]  

(3)

and \( H(P) \) is essentially self-adjoint on \( D \).

Clearly, \( H(P) \) is bounded from below, and we define the ground state energy:

\[ E(P) := \inf \text{spec}(H(P)) \]  

(4)

The main theorem of this paper is the following:
Theorem 1.1. For all $m \geq 0$, the inequality
\[ E(P - k) - E(P) + \omega_m(k) \geq m, \quad k, P \in \mathbb{R}^3 \] (5)
holds.

Remark 1.2. We define
\[ \Delta(P) := \inf_{k \in \mathbb{R}^3} \{E(P - k) - E(P) + \omega_m(k)\} \] (6)
As discussed in [2], $\Delta(P)$ is the spectral gap at the lowest energy of $H(P)$. The inequality [5] implies that the spectral gap is open uniformly in $P$. To establish [5], our photon dispersion relation $|k| + m$ was important. Similar result may not hold for the standard massive dispersion relation $\omega_m(k) = \sqrt{k^2 + m^2}$.

Remark 1.3. This is the remarkable that [5] holds for all $P$ and there is no restriction on the other parameters. This fact is different from the non-relativistic QED where the spectral gap is open only for $P^2/2M < 1$. The uniform spectral gap is a nature of the relativistic dynamics.

Remark 1.4. To prove [5], we only use the operator monotonicity. Unfortunately, our method works only for the spinless case.

Remark 1.5. It is strongly expected that $H(P)$ with $m = 0$ has ground state for all $P \in \mathbb{R}^3$ and $e \in \mathbb{R}$ under suitable conditions including the infrared regularization. By [5], the massive Hamiltonian $H(P), (m > 0)$ has a ground state for all $m > 0$. But, in our opinion, it is difficult to construct a ground state of massless model as the massless limit $m \downarrow 0$.

2. Proof of Theorem 1.1

By the variational principle, [5] follows from the operator inequality
\[ H(P - k) + \omega_m(k) \geq m + H(P), \] (7)
on $D$. We set
\[ K(P) := \sqrt{(P - d\Gamma(k) + eA)^2 + M^2}. \] (8)
Then (7) is equivalent to
\[ K(P - k) \geq K(P) - |k|, \] (9)
on $D$. By Löwner-Heinz inequality, (9) follows form
\[ K(P - k)^2 \geq (K(P) - |k|)^2, \] (10)
in the sense of the quadratic forms on $D$. By expanding the square of the operator, we know that (10) is equivalent to
\[ |k| \cdot K(P) \geq k \cdot (P - d\Gamma(k) + eA). \] (11)
By the Löwner Heinz inequality, (11) follows from
\[ k^2 K(P)^2 \geq |k \cdot (P - d\Gamma(k) + eA)|^2 \] (12)
We set $B = (B_1, B_2, B_3) := P - d\Gamma(k) + eA$. Note that, for all self-adjoint operators $a, b$, it holds that $ab + ba \leq a^2 + b^2$ in the sense of the quadratic forms on $\text{Dom}(a) \cap \text{Dom}(b)$. Then we have

$$\left( \sum_{j=1}^{3} k_j B_j \right)^2 = \sum_{j,l=1}^{3} k_j B_j k_l B_l$$

$$\leq \frac{1}{2} \sum_{j,l} (k_j^2 B_j^2 + k_l^2 B_l^2)$$

$$= \sum_{j} k_j^2 \sum_{l} B_l^2 = |k|^2 \cdot B^2$$

$$\leq |k|^2 \cdot (B^2 + M^2)$$

on $D$. Therefore (12) holds.

References

[1] J. Lörinczi, F. Hiroshima, and V. Betz, *Feynman-Kac-type theorems and Gibbs measures on path space*, vol. 34, Walter De Gruyter, 2011, Seminar on Probability, Studies in Mathematics.

[2] T. Miyao and H. Spohn, *Spectral analysis of the semi-relativistic Pauli-Fierz Hamiltonian*, J. Funct. Anal. 256 (2009), no. 7, 2123–2156.

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