BRESSOUD-SUBBARAO TYPE WEIGHTED PARTITION IDENTITIES
FOR A GENERALIZED DIVISOR FUNCTION

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Abstract. In 1984, Bressoud and Subbarao obtained an interesting weighted partition identity for a generalized divisor function, by means of combinatorial arguments. Recently, the last three named authors found an analytic proof of the aforementioned identity of Bressoud and Subbarao starting from a q-series identity of Ramanujan. In the present paper, we revisit the combinatorial arguments of Bressoud and Subbarao, and derive a more general weighted partition identity. Furthermore, with the help of a fractional differential operator, we establish a few more Bressoud-Subbarao type weighted partition identities beginning from an identity of Andrews, Garvan and Liang. We also found a one-variable generalization of an identity of Uchimura related to Bell polynomials.

1. Introduction

Ramanujan [12, pp. 354–355], [13, pp. 302–303], noted down five interesting q-series identities at the end of his second notebook. These identities have been recently systematically studied in [3], [6]. Here we mention one of these five q-series identities, namely, for \(|q| < 1, c \in \mathbb{C}, 1 - cq^n \neq 0, n \geq 1,\)

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}c^n q^{\frac{n(n+1)}{2}}}{(1 - q^n)(q)_n} = \sum_{n=1}^{\infty} \frac{c^n q^n}{1 - q^n},
\]

(1.1)

This identity was rediscovered by Uchimura [14, Equation (3)] and Garvan [8], whereas the special case \(c = 1\) of (1.1) goes back to Kluyver [11],

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1 - q^n)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.
\]

(1.2)

Uchimura [14, 15] established a new expression for the above identity with the help of a sequence of polynomials \(U_1(x) = x, U_n(x) = nx^n + (1 - x^n)U_{n-1}(x)\). These polynomials have connections to the analysis of the data structure called “heap”. Mainly, he [13, Theorem 2] found that

\[
\sum_{n=1}^{\infty} nq^n(q^{n+1})_\infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1 - q^n)(q)_n} = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.
\]

(1.3)
In 1984, Bressoud and Subbarao [4] extracted a beautiful partition theoretic interpretation of the above identity. Before presenting their identity, let us first write down a few notations that will be essential throughout the article.

- \( \pi \): an integer partition,
- \( p(n) \): the number of integer partitions of \( n \),
- \( p(t)(n) \): the number of partitions of \( n \) into exactly \( t \) distinct part sizes,
- \( s(\pi) \): the smallest part of \( \pi \),
- \( \ell(\pi) \): the largest part of \( \pi \),
- \( \#(\pi) \): the number of parts of \( \pi \),
- \( \nu_d(\pi) \): the number of parts of \( \pi \) without multiplicity,
- \( \mathcal{P}(n) \): collection of all integer partitions of \( n \),
- \( \mathcal{D}(n) \): collection of all partitions of \( n \) into distinct parts.

Let \( d(n) \) be the number of positive divisors of \( n \). Then comparing the coefficient of \( q^n \) in (1.3), one has

\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} s(\pi) = d(n). \tag{1.4}
\]

This identity was rediscovered by Fokking, Fokking and Wang [7] in 1995. Over the years, the above identity motivated mathematicians to find different weighted partition identities for divisor functions. Interested readers can see [1, 4, 5, 6, 9, 16]. Using purely combinatorial explanations, Bressoud and Subbarao [4] found an elegant weighted partition identity for a generalized divisor function \( \sigma_z(n) = \sum_{d|n} d^z \). Mainly, they derived that, for any non-negative integer \( z \),

\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z = \sigma_z(n), \tag{1.5}
\]

whereas they did not point out a generating function identity for (1.5). Recently, Bhoria, Eyyunni and Maji [3] Remark 4] found a generating function identity for (1.5) inspired from the identity (1.1) of Ramanujan and also derived a one-variable generalization of (1.5). They [3] Theorem 2.4] proved that, for any integer \( z \), complex number \( c, n \in \mathbb{N} \),

\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z c^{\ell(\pi)} = \sigma_{z,c}(n), \tag{1.6}
\]

where

\[
\sigma_{z,c}(n) = \sum_{d|n} d^z c^d. \tag{1.7}
\]

To derive (1.6), they utilized a differential and an integral operator successively on the weighted partition implication of (1.1). One of the major goals of the current paper is to show that the above identity (1.6) is in fact true for any complex number \( z \). In the present paper, we shall give detailed explanations of the combinatorial argument developed by Bressoud and Subbarao [4] to show that (1.6) is indeed true for any complex number \( z \), see Theorem 2.1.
Apart from obtaining the leftmost expression in (1.3), Uchimura also generalized this identity as a whole. He defined, for each non-negative integer \( m \),

\[
M_m := M_m(q) = \sum_{n=1}^{\infty} n^m q^n (q^{n+1})_\infty, \quad \text{and} \quad K_{m+1} := K_{m+1}(q) = \sum_{n=1}^{\infty} \sigma_m(n) q^n. \tag{1.8}
\]

He proved the following theorem.

**Theorem 1.1.** We have the following properties:

1. \[
\exp \left( \sum_{m=1}^{\infty} K_m t^m / m! \right) = 1 + \sum_{m=1}^{\infty} M_m t^m / m!.
\]
2. Let \( Y_m \) be the Bell polynomial defined by
   \[
   Y_m(u_1, u_2, \ldots, u_m) = \sum_{\Pi(m)} \frac{n!}{k_1! \cdots k_m!} \left( \frac{u_1}{1!} \right)^{k_1} \cdots \left( \frac{u_m}{m!} \right)^{k_m},
   \]
   where \( \Pi(m) \) denotes a partition of \( m \) with
   \[
k_1 + 2k_2 + \cdots + mk_m = m.
   \]

Then for any \( m \geq 1 \), \( M_m = Y_m(K_1, \ldots, K_m) \).

Uchimura did not study the partition-theoretic implications of the above theorem, that is, he did not state the corresponding generalization of (1.4). Dilcher [5, Corollary 1] recorded a few combinatorial identities arising out of Theorem 1.1. For \( m \leq 4 \), he connected the coefficients \( C_m(n) \) of \( q^n \) in \( M_m \) with the divisor functions as follows:

\[
\begin{align*}
C_1(n) &= d(n), \quad C_2(n) = \sigma_1(n) + \sum_{j=1}^{n-1} d(j)d(n-j), \tag{1.9} \\
C_3(n) &= \sigma_2(n) + 3 \sum_{j=1}^{n-1} d(j)\sigma_1(n-j) + \sum_{j+k+\ell=n, j,k,\ell\geq 1} d(j)d(k)d(\ell), \\
C_4(n) &= \sigma_3(n) + 3 \sum_{j=1}^{n-1} \sigma_1(j)\sigma_1(n-j) + 4 \sum_{j=1}^{n-1} d(j)\sigma_2(n-j) + 6 \sum_{j+k+\ell=n, j,k,\ell\geq 1} d(j)d(k)\sigma_1(\ell) \\
&\quad + \sum_{j_1+\cdots+j_4=n, j_1,\ldots,j_4\geq 1} d(j_1) \cdots d(j_4).
\end{align*}
\]

More importantly, the coefficient \( C_m(n) \) has an interesting partition theoretic interpretation, namely,

\[
C_m(n) = \sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^m e^{s(\pi)}. \tag{1.10}
\]

Dilcher [5, Corollary 2] derived two different generalization of (1.3). The first one is the following identity involving the binomial coefficient \( \binom{n}{k} \).
Theorem 1.2. Let $k$ be a positive integer. Then
\[
\sum_{n=k}^{\infty} \binom{n}{k} q^n(q^{n+1})_\infty = q^{-\binom{k}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{(n+1)k}}{(1-q^n)^k(q)_n} = \sum_{j_1=1}^{\infty} q^{j_1} \sum_{j_2=1}^{j_1} q^{j_2} \cdots \sum_{j_k=1}^{j_{k-1}} q^{j_k}.
\]

Employing the above identity and Theorem 1.1 of Uchimura, Dilcher obtained the below identity.

Theorem 1.3. Let $m \in \mathbb{N}$ and $K_i$’s are defined as in (1.8). Then
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{(n+1)/2}}{(1-q^n)^m(q)_n} = N_m(K_1, K_2, \cdots, K_m),
\]
(1.11)
where $N_m(x_1, x_2, \cdots, x_m)$ is a polynomial in $m$ variables with rational coefficients.

Recently, Gupta and Kumar [10] found an interesting generalization of Theorem 1.3 and encountered the same generalized divisor function $\sigma_{z,c}(n)$ (1.7). In the same paper, they further studied many analytic properties of $\sigma_{z,c}(n)$. In the present paper, we mainly focus on the weighted partition representation for $\sigma_{z,c}(n)$ and one of the other main results is a one-variable generalization of Uchimura’s identity i.e., Theorem 1.1. Surprisingly, we see the presence of the same generalized divisor function in the generalization of Theorem 1.1.

In the next section, we mention all the main results of this paper.

2. MAIN RESULTS

Theorem 2.1. Let $c$ and $z$ be two complex numbers. For any $n \in \mathbb{N}$, we have
\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{1-s(\pi)} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^z c^{\ell(\pi) - s(\pi) + j} = \sigma_{z,c}(n).
\]
(2.1)

The next theorem is a one-variable generalization of Uchimura’s result in Theorem 1.1.

Theorem 2.2. For a non-negative integer $m$ and a complex number $c$ with $|cq| < 1$, we define
\[
M_{m,c} := M_{m,c}(q) = \sum_{n=1}^{\infty} n^m c^n q^n(q^{n+1})_\infty, \quad K_{m+1,c} := K_{m+1,c}(q) = \sum_{n=1}^{\infty} \sigma_{m,c}(n) q^n.
\]
(2.2)

Also, let $Y_m$ be the Bell polynomial defined by
\[
Y_m(u_1, u_2, \cdots, u_m) := \sum_{\Pi(m)} \frac{m!}{k_1! \cdots k_m!} \left( \frac{u_1}{1!} \right)^{k_1} \cdots \left( \frac{u_m}{m!} \right)^{k_m},
\]
where $\Pi(m)$ denotes a partition of $m$ with
\[
k_1 + 2k_2 + \cdots + mk_m = m.
\]
Then the exponential generating functions of $M_{m,c}$ and $K_{m,c}$ are related by:

$$
\left(\frac{q}{cq}\right)^{\infty} \exp \left( \sum_{m=1}^{\infty} K_{m,c} \frac{t^m}{m!} \right) = \left(\frac{q}{cq}\right)^{\infty} + \sum_{m=1}^{\infty} M_{m,c} \frac{t^m}{m!}, \quad \text{and}
$$

(2.3)

for any $m \geq 1$, we have

$$
M_{m,c} = \left(\frac{q}{cq}\right)^{\infty} Y_m(K_{1,c}, \cdots, K_{m,c}).
$$

(2.4)

Next, we mention a few Bressoud-Subbarao type weighted partition identities.

**Theorem 2.3.** Let $k$ and $c$ be two complex numbers. We have the following identity:

$$
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^k c^s(\pi) = \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^{\ell(\pi) - j}.
$$

(2.5)

Note that the left hand side above arises precisely from the partition-theoretic interpretation of $M_{m,c}$ in Theorem 2.2. Setting $c = 1$ in (2.5) naturally leads to the partition-theoretic explanation of the coefficient of $q^n$ in the series $M_m$ in (1.8).

**Corollary 2.4.** Suppose that $k$ is a complex number. Then

$$
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^k = \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k.
$$

(2.6)

This corollary is also a generalization of (1.4). We explain why this is so. The left hand side readily reduces to the left side of (1.4) for $k = 1$. The right hand side takes the form $\sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)$. This can be written as

$$
\sum_{\pi \in P(n)} \ell(\pi) \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} - \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j
$$

$$
= - \sum_{\pi \in P(n)} \ell(\pi)(1 - \nu_d(\pi)) - \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j
$$

(2.7)

$$
= - \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j.
$$

Now, for a complex number $\alpha$ and a partition $\pi$ of a positive integer, consider the identity

$$(1 - \alpha)^{\nu_d(\pi)} = \sum_{j=0}^{\nu_d(\pi)} (-\alpha)^j \binom{\nu_d(\pi)}{j}.
$$

Differentiating with respect to $\alpha$, we get

$$
- \nu_d(\pi)(1 - \alpha)^{\nu_d(\pi) - 1} = \sum_{j=0}^{\nu_d(\pi)} (-1)^j j\alpha^{\nu_d(\pi) - 1} \binom{\nu_d(\pi)}{j}.
$$

(2.8)
We let $\alpha$ approach 1, and see that the right side above reduces to the inner sum in (2.7). The behaviour of the left side above depends on the value of $\nu_d(\pi)$. Note that, as $\alpha \to 1$, $-\nu_d(\pi)(1-\alpha)^{\nu_d(\pi)-1}$ tends to 0 if $\nu_d(\pi) > 1$ and to $-\nu_d(\pi) = -1$ when $\nu_d(\pi) = 1$. Hence, putting this information in (2.7), we can write

$$\sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j) = - \sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j$$

$$= - \sum_{\pi \in \mathcal{P}(n)} \sum_{\nu_d(\pi)=1}^{\nu_u(\pi)=1} (-1)^j = \sum_{\pi \in \mathcal{P}(n)}^{\nu_d(\pi)=1} 1,$$  \hspace{0.5cm} (2.9)

the number of partitions $\pi$ of $n$ with $\nu_d(\pi) = 1$. But the partitions of $n$ with only one distinct part are of the form $m + m + \cdots + m$, where $m$ is a positive divisor of $n$, and conversely, for each positive divisor $m$ of $n$, we do have the partition $m + m + \cdots + m$ with $n/m$ summands, which is a partition with one distinct part. Thus, the last sum in (2.9) equals $d(n)$, and we have shown that (2.6) is indeed a generalization of (1.4).

Recall that $p^{(2)}(n)$ denotes the number of partitions of $n$ with exactly two distinct part sizes. From the identity (1.9) of Dilcher, for $k = 2$, namely,

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} - 1 s(\pi)^2 = \sigma(n) + \sum_{j=1}^{n-1} d(j)d(n-j),$$  \hspace{0.5cm} (2.10)

and the special case $k = 2$ of (2.6), we obtain a representation for $p^{(2)}(n)$.

**Corollary 2.5.** For each positive integer $n$, we have

$$p^{(2)}(n) = \frac{1}{2} \left\{ \sum_{j=1}^{n-1} d(j)d(n-j) + d(n) - \sigma(n) \right\}.$$  \hspace{0.5cm} (2.11)

Again, we mention a Bressoud-Subbarao type weighted partition identity for the generalized divisor function $\sigma_{k,c}(n)$.

**Theorem 2.6.** Let $n \in \mathbb{N}$ and $k, c \in \mathbb{C}$. Then we get

$$\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} - 1 s(\pi)^j = \sum_{\pi \in \mathcal{P}(n)} \sum_{\nu_d(\pi) \geq 2} (-1)^j \binom{\nu_d(\pi)-1}{j} (\ell(\pi) - j) c^{\ell(\pi)-j} + \sigma_{k,c}(n).$$  \hspace{0.5cm} (2.12)

**Remark 1.** Corresponding to $k = 0$ and $c = 1$, the above identity reduces to (1.4). Note that $d(n) = p^{(1)}(n)$, where $p^{(1)}(n)$ denotes the number of partitions of $n$ into exactly 1 part size. Hence, identity (1.4) can be rewritten as

$$p^{(1)}(n) = \sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} - 1 s(\pi).$$  \hspace{0.5cm} (2.13)

Letting $k = 1$ and $c = 1$ in Theorem 2.6, we obtain an interesting analog of (2.13).
Corollary 2.7. Let \( n \) be a positive integer. Then we have

\[
p^{(2)}(n) = \sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)} s(\pi) (\ell(\pi) - s(\pi)).
\] (2.14)

In the next section, we demonstrate proofs of all main results.

3. Proof of the main results

Proof of Theorem 2.1. The main idea of the proof of this theorem is due to Bressoud and Subbarao. Here we explain the details while simultaneously extending their result to complex number \( z \). We first define, for each positive integer \( N \), the set \( \mathcal{C}(N) \) of partitions \( \pi \) into distinct parts satisfying the following inequalities:

\[
\ell(\pi) \geq N > \ell(\pi) - s(\pi).
\]

Let \( \pi \in \mathcal{C}(N) \) be any partition into distinct parts. The definition of \( \mathcal{C}(N) \) implies that the only possibilities for \( N \) are \( \ell(\pi) - s(\pi) + j \), with \( 1 \leq j \leq s(\pi) \). Therefore for any partition \( \pi \) into distinct parts, there are exactly \( s(\pi) \) many integers \( N \) such that \( \pi \in \mathcal{C}(N) \). With the help of this fact the left hand side of (2.1) can be written as

\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi)}\sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j)^{\ell(\pi) - s(\pi) + j} c^{\ell(\pi) - s(\pi) + j} = \sum_{\pi \in \mathcal{D}(n)} (-1)^{\#(\pi) - 1} \sum_{\pi \in \mathcal{C}(N)} N^n c^N.
\] (3.1)

Now our goal is to show that

\[
\sum_{\pi \in \mathcal{D}(n) \cap \mathcal{C}(N)} (-1)^{\#(\pi) - 1} = \begin{cases} 
1, & \text{if } N | n, \\
0, & \text{if } N \nmid n.
\end{cases} \quad (3.2)
\]

To prove (3.2), we shall try to pair the distinct parts partitions of \( n \) in \( \mathcal{C}(N) \) that have opposite parity in their respective number of parts. The contribution of such pairs will be a +1 and a -1 to the sum in the left side of (3.2), since the weight is \((-1)^{\#(\pi) - 1}\). Thus, the sum of the contributions of each such pair will vanish. When \( N | n \), the only distinct parts partition that will remain is \( n \) itself. And when \( N \nmid n \), all the distinct parts partitions of \( n \) will be paired.

Now we shall explain the algorithm to create such pairs. We consider two different cases.

Case 1: Let \( \pi \in \mathcal{D}(n) \cap \mathcal{C}(N) \). If \( \pi \) contains a part which is divisible by \( N \) and if \( \pi \) has at least one other part, (note that the other part cannot be a multiple of \( N \) since \( \ell(\pi) - s(\pi) < N \)) then remove the part which is divisible by \( N \) and add \( N \) to the smallest remaining part. Continue this way to create new partitions by adding \( N \) to the smallest part in the previous partition until we get a distinct parts partition of \( n \).
A natural question arises here as to what the guarantee is that this algorithm gives a new distinct parts partition of \( n \) that also lies in \( \mathcal{C}(N) \). In the next paragraph, we shall explain by taking a general distinct parts partition of \( n \).

Let \( \pi \in \mathcal{D}(n) \cap \mathcal{C}(N) \) with parts \( a_1 < a_2 < \cdots < a_k \) and suppose \( N \) divides \( a_i \) for some \( 1 \leq i \leq k \), say \( a_i = j \cdot N \) for some \( j \geq 1 \). First, we remove \( a_i \) and add \( N \) to the smallest remaining part. We continue adding \( N \) to the smallest part in the previous partition to create a new partition until we again have a partition of \( n \). Since \( a_i = j \cdot N \), we have to do this process \( j \) times. Let \( \pi_j' \) denote the new partition of \( n \).

Here we look at various cases depending on the value of \( j \). If \( j < k - 1 \), then \( \pi_j' \) equals

\[
\begin{cases}
    a_{j+1} + \cdots + a_{i-1} + a_{i+1} + \cdots + a_k + (a_1 + N) + \cdots + (a_j + N), & \text{if } 1 \leq j < i - 1, \\
    a_{j+2} + \cdots + a_k + (a_1 + N) + \cdots + (a_{i-1} + N) + (a_{i+1} + N) + \cdots + (a_{j+1} + N), & \text{if } i - 1 \leq j < k - 1.
\end{cases}
\]

One can easily check that the new partition \( \pi_j' \) is a distinct parts partition of \( n \) and lies in \( \mathcal{C}(N) \). Now, if \( j = k - 1 \), then we simply add \( N \) to each part and get a new distinct parts partition of \( n \), namely, \( \pi_j' = (a_1 + N) + \cdots + (a_{i-1} + N) + (a_{i+1} + N) + \cdots + (a_k + N) \), which also belongs to \( \mathcal{C}(N) \). Note that \( \pi \) and \( \pi_j' \) have opposite parity in their number of parts.

Again, if \( j > k - 1 \), then with the help of division algorithm and utilizing previous cases, one can construct \( \pi_j' \in \mathcal{D}(n) \cap \mathcal{C}(N) \) such that \( \pi \) and \( \pi_j' \) will have opposite parity in their number of parts.

**Case 2:** Let \( \pi \in \mathcal{D}(n) \cap \mathcal{C}(N) \) with parts \( a_1 < a_2 < \cdots < a_k \) such that \( N \nmid a_i \) for all \( 1 \leq i \leq k \). In this case, we must reverse the procedure, that is, we subtract \( N \) from the largest part in the previous partition until we reach a unique partition \( \pi' \) for which

\[
\ell(\pi') - N < \text{the total amount subtracted} < s(\pi') + N. \tag{3.3}
\]

Finally, this total amount subtracted is then inserted as a new part. The above condition [3.3] may look artificial, but soon we will explain why this condition comes in naturally.

First, we subtract \( N \) from the largest part \( a_k \). Then we have \( a_k - N < a_1 \) since \( N > \ell(\pi) - s(\pi) = a_k - a_1 \). Let \( \pi_1 := (a_k - N) + a_1 + \cdots + a_{k-1} \). Note that \( \pi_1 \) is not a partition of \( n \), so we need to add \( N \) to \( \pi_1 \) to get a distinct parts partition of \( n \). This brings up three possibilities.

**Sub case 1:** Firstly, suppose that \( N < a_k - N \). Let us define \( \pi_1' := N + (a_k - N) + a_1 + \cdots + a_{k-1} \). See that \( \pi_1' \in \mathcal{D}(n) \). Moreover, \( \pi_1' \in \mathcal{C}(N) \) if and only if \( \ell(\pi_1') = a_k - 1 \geq N > \ell(\pi_1') - s(\pi_1') = \ell(\pi_1') - N \). In this sub case, the first inequality will be satisfied naturally since we assumed that \( N < a_k - N \), but the second inequality may or may not be true. The second inequality will be true if \( N > \ell(\pi_1') - N \text{(the amount subtracted)} \), that is, \( N \text{(the amount subtracted)} > \ell(\pi_1') - N \).

**Sub case 2:** Next, suppose we have \( a_k - N < N < a_{k-1} \). In this sub case, let us define \( \pi_1 := (a_k - N) + \cdots + N + \cdots + a_{k-1} \), which is a distinct parts partition of \( n \). Now \( \pi_1' \in \mathcal{C}(N) \)
if and only if \( \ell(\pi'_1) = a_{k-1} \geq N > \ell(\pi'_1) - s(\pi'_1) = a_{k-1} - (a_k - N) \). In this sub case, one can easily see that both of the inequalities are true.

**Sub case 3:** Finally, suppose \( a_{k-1} < N \) is true. Now we define \( \pi'_2 := (a_k - N) + a_1 + \cdots + a_{k-1} + N \), which is a distinct parts partition of \( n \). Here \( N \) is the largest part. Now \( \pi'_2 \in C(N) \) if and only if \( \ell(\pi'_1) \geq N > \ell(\pi'_1) - s(\pi'_1) = N - s(\pi'_1) \). The first inequality is true since \( \ell(\pi'_1) = N \), but the second inequality will be true if \( s(\pi'_1) + N > N \)(the amount subtracted). Combining all three sub cases, we can clearly see that \( \pi'_1 \in C(N) \) if and only if

\[
\ell(\pi'_1) - N < N \text{ (the amount subtracted)} < s(\pi'_1) + N.
\]  

(3.4)

Thus, the above condition justifies why we need (3.3). If the chain of inequalities in (3.4) does not hold at this stage, then again we subtract \( N \) from the largest part \( a_{k-1} \) in \( \pi_1 \). And we define \( \pi_2 := (a_{k-1} - N) + (a_k - N) + a_1 + \cdots + a_{k-2} \). Now one can observe that we have subtracted \( 2N \) from the original partition \( \pi \) of \( n \), so we have to add \( 2N \) to obtain a distinct parts partition of \( n \). Again, we face three sub cases. Firstly, if \( 2N < (a_{k-1} - N) \), then we write \( \pi'_2 := 2N + (a_{k-1} - N) + \cdots + a_{k-2} \). Secondly, if \( (a_{k-1} - N) < 2N < a_{k-2} \), then put \( \pi'_2 := (a_{k-1} - N) + \cdots + 2N + \cdots + a_{k-2} \). And thirdly, if \( a_{k-2} < 2N \), then set \( \pi'_2 := (a_{k-1} - N) + \cdots + a_{k-2} + 2N \). Along the same lines, we can show that \( \pi'_2 \in C(N) \) if and only if

\[
\ell(\pi'_2) - N < 2N \text{ (the amount subtracted)} < s(\pi'_2) + N.
\]  

(3.5)

If the partition \( \pi'_2 \) does not satisfy the above conditions, then we apply the algorithm once again. More generally, suppose that we have subtracted the integer \( N \) \( j \) many times. We then have to add \( j \cdot N \) as a new part. Note that this part may be the smallest part, the largest part or neither of them, and so we have to consider three subcases. If \( j < k \), then our new distinct parts partition of \( n \), again denoted by \( \pi'_j \), will be

\[
\begin{cases}
  j \cdot N + (a_{k-j+1} - N) + \cdots + a_k + \cdots + a_{k-j}, & \text{if } j \cdot N \text{ is the smallest part}, \\
  (a_{k-j+1} - N) + \cdots + j \cdot N + \cdots + a_{k-j}, & \text{if } j \cdot N \text{ is neither the smallest nor the largest part}, \\
  (a_{k-j+1} - N) + \cdots + a_k + \cdots + a_{k-j} + j \cdot N, & \text{if } j \cdot N \text{ is the largest part}.
\end{cases}
\]

In a similar vein, one can show that \( \pi'_j \in C(N) \) if and only if

\[
\ell(\pi'_j) - N < j \cdot N \text{ (the total amount subtracted)} < s(\pi'_j) + N.
\]  

(3.6)

This justifies why condition (3.3) is necessary. If \( j \geq k \), then we can use division algorithm and give similar arguments. Hence, summarizing Case 2, we started with a distinct parts partition \( \pi \) with no part divisible by \( N \) and then constructed another distinct parts partition \( \pi'_j \) which has only one part divisible by \( N \), namely, \( j \cdot N \) for some suitable positive integer \( j \).

The above algorithm explains that if \( N \nmid n \), then any partition \( \pi \in D(n) \cap C(N) \) can be paired with another partition \( \pi'_j \in D(n) \cap C(N) \) such that they have opposite parity in their number of parts. And if \( N|n \), the only partition of \( n \) which will remain unpaired is
the partition $n$ itself since the partition $n$ neither belongs to Case 1 nor to Case 2. This completes the proof of (3.2). Finally, combining (3.1) and (3.2), one can obtain (2.1). □

Remark 2. In [3, Section 5], the authors showed that Theorem 2.1 is valid for any integer $z$, by applying the differential operator $D[f(c)] := c \frac{d}{dc} \{f(c)\}$, and the integral operator $I[f(c)] := \int_{0}^{c} \frac{f(t)}{t} \, dt$, on the partition theoretic interpretation of (1.1). Here we have stated that Theorem 2.1 is in fact true for any complex number $z$. Moreover, for positive real numbers $z$ and $j$, one can define the fractional derivative

$$\frac{d^z}{dc^z}(c^j) := \frac{\Gamma(j + 1)}{\Gamma(j - z + 1)} c^{j-z}. \tag{3.7}$$

This definition matches with the usual definition of the derivative when $z$ is any positive integer. Therefore, using this fractional derivative, we have $D^z(c^j) = j^z c^j$ for any $z > 0$.

Hence applying this fractional operator on the partition theoretic interpretation of (1.1), one can first show that, Theorem 2.1 is true for any positive real number $z$ and then using analytic continuation, we can prove that the identity (2.1) is in fact valid for any complex $z$.

Proof of Theorem 2.2. Let us define a function

$$A(x, q) := \frac{(q)_{\infty}}{(xq)_{\infty}}. \tag{3.7}$$

We next invoke Euler’s identity, which is a special case of the $q$-binomial theorem. For $|q|, |t| < 1$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}. \tag{3.8}$$

Replacing $t$ by $xq$ in (3.8) and utilizing in (3.7), we get, for $|xq| < 1$,

$$A(x, q) = (q)_{\infty} \sum_{n=0}^{\infty} \frac{x^n q^n}{(q)_n} = \sum_{n=0}^{\infty} x^n q^n (q^{n+1})_{\infty}. \tag{3.9}$$

Putting $x = ce^t$, for $m \geq 1$, one can see that

$$\frac{\partial^m}{\partial t^m} A(ce^t, q) \bigg|_{t=0} := \sum_{n=1}^{\infty} n^m c^n q^n (q^{n+1})_{\infty} = M_{m,c}. \tag{3.10}$$

Thus, the power series for $A(ce^t, q)$ in the variable $t$ takes the shape as

$$A(ce^t, q) = A(c, q) + \sum_{m=1}^{\infty} M_{m,c} \frac{t^m}{m!}. \tag{3.9}$$

Suppose we write

$$\log(A(x, q)) = \sum_{n=0}^{\infty} h_n(c) \frac{t^n}{n!}. \tag{3.10}$$
From (3.7), we get

\[
\log(A(x, q)) = \log((q)_\infty) - \log((xq)_\infty) = \log((q)_\infty) - \sum_{n=1}^{\infty} \log(1 - xq^n)
\]

\[
= \log((q)_\infty) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(xq^n)^m}{m} \frac{1}{1 - q^n}
\]

\[
= \log((q)_\infty) + \sum_{m=1}^{\infty} x^m \frac{q^m}{m} \frac{1}{1 - q^m}.
\]

Putting \(x = ce^t\), we find that

\[
\log(A(ce^t, q)) = \log((q)_\infty) + \sum_{m=1}^{\infty} x^m \frac{q^m}{m} \frac{e^{tm}}{1 - q^m}
\]

\[
= \log((q)_\infty) + \sum_{m=1}^{\infty} \frac{q^m}{m} \frac{e^{tm}}{1 - q^m} \sum_{n=0}^{\infty} \frac{(tm)^n}{n!}
\]

\[
= \log((q)_\infty) + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{c^m m^{n-1} q^m}{1 - q^m} \frac{t^n}{n!}
\]

\[
= \log((q)_\infty) - \log((cq)_\infty) + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{c^m m^{n-1} q^m}{1 - q^m} \right) \frac{t^n}{n!}
\]

\[
= \log \left( \frac{(q)_\infty}{(cq)_\infty} \right) + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{c^m m^{n-1} q^m}{1 - q^m} \right) \frac{t^n}{n!}.
\]

By comparing coefficients in (3.10), we get

\[ h_0(c) = \log \left( \frac{(q)_\infty}{(cq)_\infty} \right), \tag{3.11} \]

and for any \(n \geq 0\), we have

\[ h_{n+1}(c) = \sum_{m=1}^{\infty} \frac{c^m m^n q^m}{1 - q^m} = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} c^m m^n q^k = \sum_{\ell=1}^{\infty} \sum_{d|\ell} c^d q^\ell q^d = K_{n+1,c}. \tag{3.12} \]

Now in view of (3.10), (3.11), and (3.12), one can see that

\[ A(ce^t, q) = \frac{(q)_\infty}{(cq)_\infty} \exp \left( \sum_{n=1}^{\infty} K_{n,c} \frac{t^n}{n!} \right), \]

and finally using (3.9), we arrive at

\[ \frac{(q)_\infty}{(cq)_\infty} \exp \left( \sum_{n=1}^{\infty} K_{n,c} \frac{t^n}{n!} \right) = \frac{(q)_\infty}{(cq)_\infty} + \sum_{n=1}^{\infty} M_{n,c} \frac{t^n}{n!}. \]

Finally comparing the coefficients of \(t^n/n!\), for \(n \geq 1\), and using the definition of the Bell polynomial, one can obtain (2.4). \(\square\)
Proof of Theorem 2.3. We start with the partition-theoretic interpretation of an identity due to Andrews, Garvan and Liang [2], first noted down by Dixit and Maji [6, Equation (2.6), Corollary 2.4]

\[
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} \left(1 + c + \cdots + c^{s(\pi)-1}\right) = \sum_{\pi \in P(n)} c^{\ell(\pi)-\nu_d(\pi)}(c-1)^{\nu_d(\pi)-1}.
\] (3.13)

Multiplying throughout by \((c-1)\), we get an identity with which we are going to work.

\[
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} \left(c^{s(\pi)} - 1\right) = \sum_{\pi \in P(n)} c^{\ell(\pi)-\nu_d(\pi)}(c-1)^{\nu_d(\pi)}.
\] (3.14)

The left side is clearly a polynomial in \(c\). The same is true of the right side as well, as \(\ell(\pi) \geq \nu_d(\pi)\) for any partition \(\pi\). To see why, observe that if \(\ell(\pi)\) is the largest part in a partition, then the possible parts in the partition come from the set \([1, 2, \ldots, \ell(\pi)]\). This means that \(\nu_d(\pi)\), the number of distinct parts appearing in the partition \(\pi\) is at most \(\ell(\pi)\), with equality occurring if and only if each of the integers from 1 to \(\ell(\pi)\) appears in the partition. We now apply the fractional differential operator \(D^k := \left(c \frac{d}{dc}\right)^k\), with \(k > 0\), to both sides of (3.14). The left side clearly transforms into

\[
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^k \ c^{\ell(\pi)}.
\] (3.15)

Before applying \(D^k\) to the right side of (3.14), we expand it using the binomial theorem to get

\[
\sum_{\pi \in P(n)} c^{\ell(\pi)-\nu_d(\pi)} \sum_{j=0}^{\nu_d(\pi)} \binom{\nu_d(\pi)}{j} (-1)^j c^{\nu_d(\pi)-j} = \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} c^{\ell(\pi)-j} (-1)^j \binom{\nu_d(\pi)}{j}.
\] (3.16)

Operating by \(D^k\), we obtain

\[
\sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (\ell(\pi) - j)^k c^{\ell(\pi)-j} (-1)^j \binom{\nu_d(\pi)}{j}.
\] (3.17)

Equating the differentiated expressions in (3.15) and (3.17), we get (2.5) for any \(k > 0\). Finally, making use of analytic continuation on the variable \(k\), the proof of Theorem 2.3 is over.

\(\Box\)

Proof of Corollary 2.5. The idea of this proof is to consider the case \(k = 2\) of (2.6) in Corollary 2.4 and compare it with (2.10) due to Dilcher. Begin by setting \(k = 2\) in (2.6) to
Now differentiate this with respect to \( c \)

\[
\sum_{\pi \in D(n)} (-1)^{\#(\pi)} s(\pi)^2 = \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^2 = \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^2 + \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2 
\]

\[
= \sum_{\pi \in P(n)} \ell(\pi)^2 (1 - \nu_d(\pi) - 2 \sum_{\pi \in P(n)} \ell(\pi) \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j}) + \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2.
\]

\[
\sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2 = \sum_{d \mid n} d. 
\]

Putting this in (3.18) brings us to

\[
\sum_{\pi \in D(n)} (-1)^{\#(\pi)} s(\pi)^2 = 2 \sum_{d \mid n} d + \sum_{\pi \in P(n)} \sum_{j=0}^{\nu_d(\pi)} (-1)^j \binom{\nu_d(\pi)}{j} j^2.
\]

Recall the identity (2.8) and multiply it with \( c \) to get

\[
-\nu_d(\pi) \cdot c \cdot (1 - c)^{\nu_d(\pi) - 1} = \sum_{j=0}^{\nu_d(\pi)} (-1)^j j c^j \binom{\nu_d(\pi)}{j}.
\]

Now differentiate this with respect to \( c \) to derive the identity

\[
-\nu_d(\pi) \left[ (1 - c)^{\nu_d(\pi) - 1} - c(\nu_d(\pi) - 1)(1 - c)^{\nu_d(\pi) - 2} \right] = \sum_{j=0}^{\nu_d(\pi)} (-1)^j j^2 c^{j-1} \binom{\nu_d(\pi)}{j}.
\]
Letting $c \to 1$ in (3.21), we see that

$$
\sum_{j=0}^{\nu_d(\pi)} (-1)^j \cdot j^2 \cdot \binom{\nu_d(\pi)}{j} = F(\nu_d(\pi)),
$$

(3.22)

where

$$
F(\nu_d(\pi)) = \lim_{c \to 1} -\nu_d(\pi) \left[ (1-c)^{\nu_d(\pi)-1} - c(\nu_d(\pi) - 1) - (1-c)^{\nu_d(\pi)-2} \right].
$$

If $\nu_d(\pi) > 2$, $F(\nu_d(\pi))$ clearly approaches 0. When $\nu_d(\pi) = 2$, the first term in the parenthetical sum in $F(\nu_d(\pi))$ vanishes and the second term nears the value $-1$, so that in effect, $F(2) = 2$. Finally, when $\nu_d(\pi) = 1$, the second term in $F(\nu_d(\pi))$ goes to 0 and the first term to 1, thereby giving us the value of $F(1)$ as $-1$. In summary,

$$
F(\nu_d(\pi)) = \begin{cases} 
-1, & \text{if } \nu_d(\pi) = 1, \\
2, & \text{if } \nu_d(\pi) = 2, \\
0, & \text{if } \nu_d(\pi) > 2.
\end{cases}
$$

Using this, via (3.22), in (3.20), we arrive at

$$
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^2 = 2 \sum_{d \mid n} d - \sum_{\pi \in P(n)} 1 + \sum_{\pi \in P(n)} 2 = 2\sigma(n) - d(n) + 2p_d^{(2)}(n).
$$

(3.23)

Comparing (3.23) with (2.10), we see that

$$
\sigma(n) + \sum_{j=1}^{n-1} d(j) d(n-j) = 2\sigma(n) - d(n) + 2p_d^{(2)}(n),
$$

(3.24)

which on rearrangement yields (2.11) and the proof is complete.

\textit{Proof of Theorem 2.6} We start with (3.13) and multiplying throughout by $c$, we get

$$
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} s(\pi)^2 = \sum_{\pi \in D(n)} c = \sum_{\pi \in P(n)} c^{\ell(\pi)-\nu_d(\pi)+1} (c-1)^{\nu_d(\pi)-1}.
$$

(3.25)

Now apply the fractional differential operator $D^k := (c \frac{d}{dc})^k$, with $k > 0$, to both sides of (3.25). The left side clearly transforms into

$$
\sum_{\pi \in D(n)} (-1)^{\#(\pi)-1} \sum_{j=1}^{s(\pi)} j^k c^j.
$$

(3.26)

Before applying the fractional differential operator to the right hand side of (3.25), we use binomial theorem to expand

$$
\sum_{\pi \in D(n)} c^{\ell(\pi)-\nu_d(\pi)+1} (c-1)^{\nu_d(\pi)-1} = \sum_{\pi \in D(n)} \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)-1}{j} c^{\ell(\pi)-j}.
$$

(3.27)
Now applying the fractional differential operator $D^k$, we obtain
\[
\sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^j = \sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^j + \sum_{d \mid n} d^k c^d.
\] (3.28)

We have previously observed that partitions of a positive integer $n$ with one distinct part correspond to divisors of $n$. This means that if $m + \cdots + m$ is a partition of $n$, corresponding to a divisor $m$ of $n$, then the largest part is also $m$. Thus,
\[
\sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^j = \sum_{d \mid n} d^k.
\] (3.29)

Putting this in (3.28), we get
\[
\sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j)^k c^j + \sum_{d \mid n} d^k c^d.
\] (3.30)

Equating the differentiated expressions in (3.26) and (3.30), we can obtain (2.12) for $k > 0$. Finally, using analytic continuation on $k$, one can see that Theorem 2.6 is valid for any complex $k$. $\square$

**Proof of Corollary 2.7.** Putting $k = 1$ and $c = 1$ in Theorem 2.6, the identity becomes
\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{#(\pi)-1} \sum_{j=1}^{s(\pi)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j) + \sum_{d \mid n} d.
\] (3.31)

Letting $z = c = 1$ in Theorem 2.3, we have
\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{#(\pi)-1} \sum_{j=1}^{s(\pi)} (\ell(\pi) - s(\pi) + j) = \sum_{d \mid n} d.
\] (3.32)

From (3.31) and (3.32), one can see that
\[
\sum_{\pi \in \mathcal{D}(n)} (-1)^{#(\pi)} s(\pi) (\ell(\pi) - s(\pi)) = \sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j).
\] (3.33)

Now we will divide the right hand sum into two parts corresponding to $\nu_d(\pi) = 2$ and $\nu_d(\pi) \geq 3$. Thus, we have
\[
\sum_{\pi \in \mathcal{P}(n)} \sum_{j=0}^{1} (-1)^j \binom{1}{j} (\ell(\pi) - j) + \sum_{\pi \in \mathcal{P}(n)} \nu_d(\pi) \sum_{j=0}^{\nu_d(\pi)-1} (-1)^j \binom{\nu_d(\pi)}{j} (\ell(\pi) - j).
\] (3.34)
One can easily see that the first sum reduces to $p^{(2)}(n)$. To show that the second sum vanishes, we mention the following expressions:

\[(1 - x)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x^j,\]  
\[-m(1 - x)^{m-1} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} j x^{j-1}.\]  

(3.35)

Now letting $x = 1$ and $m = \nu_d(\pi) - 1$ in (3.35) and using them, one can see that the second sum in (3.34) vanishes. □

4. Concluding Remarks

In this paper, we established an interesting weighted partition identity (2.1) for a generalized divisor function $\sigma_{z,c}(n) := \sum_{d|n} d^z c^{\ell}$. Mainly, we adopt the combinatorial proof of Bressoud and Subbarao. We also point out that Theorem 2.1 can also be derived from the partition theoretic interpretation of (1.1) by using a fractional differentiation operator. Motivated from this observation, we applied the aforementioned operator on the partition theoretic interpretation of an identity of Andrews, Garvan and Liang, and obtained a few Bressoud-Subbarao type weighted partition identities.

We also generalize Uchimura’s identity i.e., Theorem 1.1 by introducing a complex parameter $c$ and surprisingly, the generating functions $K_n$ for the divisor functions $\sigma_{z,1}(n)$ in Theorem 1.1 are replaced by the generating functions for the generalized divisor functions $\sigma_{z,c}(n)$, for $z \in \mathbb{N}$. This raises a natural question. Since the generalization of the identity of Bressoud and Subbarao in (1.6) holds true for complex numbers $z$ as well, it would be interesting to see if such a generalization exists for Theorem 1.1 (i.e., an analogue of Theorem 2.2 for complex numbers $m$).

In (2.13), we mentioned an alternate form of Bressoud-Subbarao’s identity (1.4). We found an interesting analogue of (2.13) i.e., Corollary 2.7 which is a weighted partition identity for $p^{(2)}(n)$. It would be highly desirable to find a Bressoud-Subbarao type combinatorial proof for Corollary 2.7.

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6. Data availability statement

The authors declare that the manuscript has no associated data.

7. Statement of Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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