$R^4$ corrections to conifolds and $G_2$-holonomy spaces

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Abstract

Motivated by examples that appeared in the context of string theory – gauge theory duality, we consider corrections to supergravity backgrounds induced by higher derivative ($R^4+...$) terms in superstring effective action. We argue that supersymmetric solutions that solve BPS conditions at the leading (supergravity) order continue to satisfy a 1-st order “RG-type” system of equations with extra source terms encoding string (or M-theory) corrections. We illustrate this explicitly on the examples of $R^4$ corrections to generalized resolved and deformed 6-d conifolds and a class of non-compact 7-d spaces with $G_2$ holonomy. Both types of backgrounds get non-trivial modifications which we study in detail, stressing analogies between the two cases.

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1. Introduction

String theory – gauge theory duality implies certain correspondence between perturbative expansions on the both sides of the duality. For example, in the case of the AdS/CFT duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory, the $\alpha'$ and $g_s$ expansions on the string side translate into $(g_{YM}^2 N)^{-1/2}$ and $1/N$ expansions on the gauge theory side [1]. In the absence of detailed microscopic understanding of string models in curved Ramond-Ramond backgrounds, one available source of non-trivial string-theoretic information is the low-energy space-time effective action, which, in principle, is universal, i.e. does not depend on a particular background. The leading correction term in the type II string effective action is of 4-th power in curvature ($\alpha'^3 R^4$ plus additional terms depending on other supergravity fields as required by supersymmetry).

Provided the basic supergravity background has regular (and small) curvature, the $\alpha'$-expansion is well-defined and should contain useful information about strong-coupling expansion on the gauge theory side. How the effect of $R^4$ corrections on the string theory side translates to the gauge-theory side was illustrated in [2] on the example of the near-extremal (finite-temperature) version of the AdS/CFT correspondence (the $R^4$ correction is related to strong-coupling correction to the entropy of the $\mathcal{N} = 4$ SYM theory, see also [3] for some related work).

Another example of the important role of the $R^4$ term in the string theory – gauge theory duality was given in [4]. There we considered the example of duality between string theory in the “fractional D3-brane on conifold” background [5] and $\mathcal{N} = 1$ supersymmetric $SU(N+M) \times SU(N)$ gauge theory with bifundamental matter chiral multiplets [6]. It was explained how the $\alpha'$-corrections to the radial dependence of the supergravity fields should translate into higher-order terms in the RG flow equations on the gauge theory side. In particular, the $R^4$ term was related to the leading term in strong-coupling expansion of the anomalous dimension of matter multiplets that enters the NSVZ beta-functions [1].

The present work was motivated by [4]. One general question that was left open in [4] is how the $\alpha'$-corrected supergravity equations may be related to the RG flow equations on the gauge theory side given that the former contain higher derivatives while the latter – only first order ones. As we shall argue in section 2, in the supersymmetric

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1 In contrast to $\mathcal{N} = 2$ examples (see, e.g., [7]) where the 1-loop beta-functions are exact (and thus should be reproduced exactly by the dual supergravity backgrounds), the beta-functions of $\mathcal{N} = 1$ gauge theories with matter receive non-trivial higher-order corrections.
cases where the leading supergravity background is a solution of 1-st order BPS system
\[ \dot{\varphi}^a = G^{ab}(\varphi) \frac{\partial W(\varphi)}{\partial \varphi^b} \] the solutions of the \( \alpha' \) corrected effective action equations should satisfy
\[ \dot{\varphi}^a = \partial^a W(\varphi) + J^a(\varphi, \alpha') , \quad \partial^a \equiv G^{ab}(\varphi) \frac{\partial}{\partial \varphi^b} . \] (1.1)

The “source” term \( J^a \) which encodes information about string \( \alpha' \) corrections should depend only algebraically on the fields \( \varphi^a \). There are two steps involved in arriving at this conclusion. First, one believes that, in a supersymmetric theory, if a leading-order solution is supersymmetry-preserving, i.e. solves a 1-st order system, the same should be true for its exact \( \alpha' \)-deformation. Indeed, one expects that since the \( \alpha' \)-corrections to superstring effective action (\( l_P \) corrections in 11-d) should preserve a deformed version of local supersymmetry, the corrections to a globally-supersymmetric background can be found from the deformed version of the Killing spinor equation (\( \nabla \epsilon + \alpha'^3 RR \nabla \Re + ... = 0 \)). The latter starts with 1-st derivative term and contains higher derivatives only in \( \alpha' \)-dependent terms, \[ \dot{\varphi}^a = \partial^a W(\varphi) + \alpha'^3 B^a(\varphi, \dot{\varphi}, \ddot{\varphi}, ... ) + ... . \] As we shall see, one necessary condition for this to happen is that the \( R^4 \)-correction should vanish when evaluated on the leading BPS background. Second, the fact that the leading-order background solves the first-order equation \( \dot{\varphi}^a = \partial^a W(\varphi) \) allows one to express, order by order in \( \alpha' \), all derivatives of \( \varphi^a \) in \( \alpha' \)-correction terms \( B^a \) as algebraic functions of \( \varphi^a \) only.

We shall explicitly demonstrate how this happens on the two examples: leading \( \alpha'^3 R^4 \) corrections in 10-d (or similar \( l_6^P R^4 \) corrections in 11-d) to (i) generalized 6-d conifold metrics \[ [8,9,10,11,12] \] and (ii) a class of 7-d metrics with \( G_2 \) holonomy \[ [13,14,15] \]. Both cases are Ricci flat metrics preserving part of global supersymmetry (8 and 4 supercharges, respectively), and thus can be obtained as solutions of 1-st order systems. These spaces have regular curvature so that \( \alpha' \)-expansion is well-defined. A priori, one would expect that once \( \alpha' \)-corrections are included, one should go back to the Einstein equations corrected by higher-derivative terms and solve them to find the corrections to the metric (this is, in fact, what happens in bosonic string theory). As we shall find, in agreement with the above remarks, the situation in the superstring case is much simpler: the corrected solutions can be found from the corresponding “corrected” BPS equations \( \dot{\varphi}^a = \partial^a W(\varphi) + \varepsilon J^a(\varphi) \), \( \varepsilon \sim \alpha'^3 \). We shall solve these equations explicitly to leading order in \( \varepsilon \).

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2 The functions \( \varphi^a \) parametrize the supergravity fields which are assumed to depend on one radial coordinate \( t \) only. This will be the case for the examples in this paper where the metrics possess large global symmetry and only the radial direction dependence is a non-trivial one.
While the standard singular conifold metric does not receive $\alpha'^3$ corrections and should be an exact string solution \cite{4}, the scale-dependent (generalized \cite{11,12}, resolved \cite{10} and deformed \cite{3}) conifolds get non-trivial modifications. We shall find that in these cases the “source” $J^\alpha(\phi)$ is expressed in terms of a single function $E \sim RRR$ (6-d Euler density). We shall relate this to the fact that the conifolds are Kähler manifolds, and thus (in an appropriate scheme) their $\alpha'$-deformation may be represented as a change of the Kähler potential \cite{10,11,18}. The $\alpha'$-corrections to conifold metrics we find provide probably the first explicit examples of the expected \cite{10,11,18} deformation of 6-d Calabi-Yau metrics by string $\alpha'$ corrections.

These results should guide the study of $\alpha'$-corrections to more general backgrounds involving p-form fluxes on conifolds (like the one in \cite{3}) which are of interest from the point of view of string theory – gauge theory duality. There the corresponding string sigma model will no longer be Kähler, but the above remarks about the corrected 1-st order form of the effective equations will still apply. The $\alpha'$-deformation of the metric and other fields will then be of physical significance, being related to higher order corrections to RG equations on the gauge theory side \cite{4}.

Below we shall also consider the deformation under $R^4$ corrections of some known examples of non-compact 7-d Ricci flat metrics with $G_2$-holonomy \cite{13,14,15,19,20} (these may also have potential gauge-theory duality applications as discussed, e.g., in \cite{21,22,15}). In the tree-level string theory context, the corresponding NS string sigma model has 2-d $n = 1$ world-sheet supersymmetry, and, like in the conifold (Kähler, i.e. $n = 2$) case it is expected \cite{23,24} to be deformed by $\alpha'$-corrections.\footnote{Our general discussion of $\alpha'$ deformation of radially dependent Ricci flat metrics applies also to simpler examples of hyperKähler 4-d metrics like the Eguchi-Hanson and Taub-NUT (or “KK monopole”) which correspond to 2-d $n = 4$ supersymmetric (finite \cite{25}) sigma model. Here one finds no corrections coming from $\alpha'^3 R^4 + ...$ terms, i.e. in these cases there are no corrections to the 1-st order system. In general, one expects that all 1/2 supersymmetric backgrounds of type II string theory (preserving 16 supercharges) should not receive corrections in an appropriate scheme, while 1/4 and 1/8 supersymmetric backgrounds should get corrections.} We indeed find that these $G_2$ metrics are deformed by $\alpha'$ corrections, implying, in particular, a “deformation” of the classical $G_2$-holonomy structure which should go along with the deformation of the supersymmetry transformation rules and the form of the Killing spinor equation. At the same time, the...
number of Killing spinors, i.e. the number of corresponding global supersymmetries should remain the same, and that should be reflected in the structure of the associated 2-d CFT.\footnote{It would be interesting to compare the sigma model approach with the direct conformal field theory constructions of $G_2$ CFT’s in \cite{20,27,28,29}. The classical “W-type” symmetry of $G_2$ sigma models \cite{23} (associated with covariantly constant 3-form on target space) was promoted to a quantum chiral algebra in \cite{26}, i.e. \cite{26} defined the corresponding general class of 2-d CFT’s, with particular examples given by particular representations of this algebra. One expects that starting with the quantum sigma model, there should be a way to define the corresponding quantum $G_2$ algebra generators (with the definition involving $\alpha'$ corrections and depending on a scheme) so that they should form the same chiral quantum algebra to all orders in $\alpha'$, as required by the conformal bootstrap construction \cite{26} (we are grateful to S. Shatashvili for a clarifying discussion of this point). The situation should be the same for the 6-d CY case, where the corresponding algebra was given in \cite{30} (see also \cite{28}). There the relation between the exact CFT and particular 2-d sigma model may be more transparent since the $\alpha'$ corrections preserve at least the Kähler structure of the target space metric (assuming one uses a renormalization scheme where $n = 2$ world-sheet supersymmetry is manifest).}

The analysis of the $\alpha'$-corrections to the $G_2$ metrics has close analogy with the simpler one in the conifold case, but here we do not have the advantage of existence of a special scheme where corrections are parametrized by a deformation of a single function – the Kähler potential. As in the CY case, the deformed metric should still possess the same number (four) of global supercharges as the leading-order one, i.e. it should solve the corresponding $\alpha'$-deformed version of the Killing spinor equation. Again, as in the conifold case (and, more generally, as for Kähler Ricci flat spaces \cite{31}), we shall find that there exists a scheme where the dilaton is not changed from its constant value; also, the $R^4$ term evaluated on the leading-order solution will vanish, and thus, as expected, there will be no shift of the central charge. Considering $G_2$ spaces as solutions of the 11-d theory, we shall find that there exists a scheme in which the corrected solution preserves the direct-product $R^{1,3} \times M^7$ form, i.e. there is no generation of a warp factor.

The paper is organized as follows. In section 2 we shall describe the general approach to deriving $\alpha'$-corrected form of the 1-st order system of equations for supersymmetric (BPS) backgrounds.

In section 3 we shall illustrate our approach on the example of conifold metrics as solutions of 10-d superstring theory. We shall first review (in section 3.1) the structure of $R^4$ string tree-level corrections to the supergravity action, and prove that for any Ricci-flat
leading-order solution for which $R^4$ invariant vanishes, there is a scheme where there is no correction to the dilaton. This claim is based on certain identity (proved in Appendix A) between second covariant derivatives of $R^3$ invariants (which is, in turn, related to the fact that the 4-point superstring amplitude involving 3 gravitons and 1 dilaton vanishes).

In section 3.2 we shall determine the corrected form of 1-st order equations for the generalized resolved conifold metric and explain that its structure is indeed consistent with the expected \([16,17,18]\) deformation of the Kähler structure. We shall explicitly compute the corresponding correction to the Kähler potential and thus the metric of this non-compact CY space. In section 3.3 we shall briefly discuss analogous computation for the deformed conifold case.

In section 4 we shall carry out similar analysis for a class of $G_2$ holonomy spaces viewed as solutions of 11-d supergravity modified by $R^4$ terms (similar results are found in 10-d tree level string theory framework). In section 4.1 we shall review the structure of the $R^4$ terms in 11-d theory and mention correspondence upon dimensional reduction with the string one-loop $R^4$ corrections in 10 dimensions. In section 4.3 will find explicitly the simple corrected form of the BSGPP solution \([13,14]\). The analysis of the corrections to the BGGG \([15]\) metric will be more involved and less explicit (due to the fact that the general solution for the homogeneous system of 1-st order equations describing small perturbations near the BGGG solution is not known in an analytic form).

2. String corrections to first-order equations

The gravity backgrounds we shall consider in this paper will have nontrivial dependence on only one “radial” coordinate, and may be derived as solutions of equations of motion following from 1-dimensional action obtained by plugging the ansatz for the metric (and the dilaton) into the string effective action

$$S = S^{(0)} + S^{(1)} + ... = \int dt \left[ \frac{1}{2} G_{a\bar{b}}(\varphi) \dot{\varphi}^a \dot{\varphi}^\bar{b} - V(\varphi) + \varepsilon L^{(1)}(\varphi, \dot{\varphi}, \ddot{\varphi}, ...) + ... \right]. \tag{2.1}$$

Here $\varphi^a(t)$ are functions of the radial coordinate that parametrize the metric and the dilaton, $V$ is a scalar potential following from the Einstein term in the action, and $L^{(1)}$ stands for the contribution of the leading higher-derivative ($R^4 + ...$) correction term. The expansion parameter $\varepsilon$ will be proportional to $\alpha'^3$ in $d = 10$ or $l_6^P$ in $d = 11$. We will be interested in finding the corrected form of the solutions to leading order in $\varepsilon$ but the discussion that follows should have a direct generalization to higher-order corrections. The
examples of spaces we shall consider will be non-singular, and thus perturbation theory in dimensionless ratio of $\varepsilon$ and an appropriate power of the curvature scale will be well-defined.

In the cases of interest, $V$ will be expressed in terms of a superpotential (reflecting the fact that a fraction of global supersymmetry is preserved by the corresponding solutions)

$$V = -\frac{1}{2} G^{ab} \frac{\partial W}{\partial \varphi^a} \frac{\partial W}{\partial \varphi^b} \quad (2.2)$$

Then the action $S^{(0)}$ may be rewritten, up to a total derivative, as

$$S^{(0)} = \frac{1}{2} \int dt \ G_{ab}(\dot{\varphi}^a - G^{ac} \frac{\partial W}{\partial \varphi^c})(\dot{\varphi}^b - G^{bd} \frac{\partial W}{\partial \varphi^d}) \quad (2.3)$$

and thus solutions of

$$\varphi^a = G^{ab} \frac{\partial W}{\partial \varphi^b} \quad (2.4)$$

provide its extrema, and satisfy the usual “zero-energy” constraint $T + V = 0$.

The string or M-theory higher-derivative corrections to the 10-d and 11-d supergravity actions should preserve a “deformed” version of the original local supersymmetry, so that the corrected versions of globally-supersymmetric supergravity solutions are expected to solve a “deformed” version of the Killing spinor equations, $((\nabla_m + RR\nabla R\gamma_m + \ldots)\varepsilon = 0$, see, e.g., [32,33,34]), and thus a “deformed” version of the first-order BPS equation (2.4). This suggests a conjecture (which will be justified in the explicit examples considered below) that the corrected effective action (2.1) may be rewritten in the form similar to (2.3)

$$S = \frac{1}{2} \int dt \ G_{ab}(\dot{\varphi}^a - G^{ac} \frac{\partial W}{\partial \varphi^c} - \varepsilon G^{ac} W_c^{(1)})(\dot{\varphi}^b - G^{bd} \frac{\partial W}{\partial \varphi^d} - \varepsilon G^{bd} W_d^{(1)}) + O(\varepsilon^2) \quad (2.5)$$

where $W_a^{(1)}$ are some functions of $\varphi^a$’s and their derivatives. Then (2.4) will be replaced by

$$\varphi^a = G^{ab} \frac{\partial W}{\partial \varphi^b} + \varepsilon J^a \quad (2.6)$$

$J^a$ will play the role of sources if one solves these corrected equations (2.6) in perturbation theory in $\varepsilon$. It is natural to expect that with higher order correction terms added to (2.1) and thus to (2.3), one should be able to find the exact solutions to the resulting effective equations from the system (2.3) with $\varepsilon J^a$ replaced by a power series in $\varepsilon$. 

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Comparing (2.1) to (2.5), we derive the following condition for (2.5) and (2.6) to be true (modulo total-derivative terms which we shall always ignore)

\[ L^{(1)} = -\left( \dot{\varphi}^a - G^{ab} \frac{\partial W}{\partial \varphi^b} \right) W_{a}^{(1)}. \] (2.7)

One consequence of this relation is that the value of the correction \( L^{(1)} \) evaluated on the leading-order BPS solution of (2.4) must vanish. As we shall see, (2.7) will indeed be satisfied in all examples we will consider. We expect that this condition should hold for any supersymmetric solution and any correction preserving supersymmetry. In particular, the correction to the action should vanish on the leading-order BPS solution.

The condition (2.7) suggests the general strategy of computing the corrections \( J^a \) to the first-order equations (2.6): one should introduce the variables

\[ Q^a = \dot{\varphi}^a - G^{ab} \frac{\partial W}{\partial \varphi^b}, \] (2.8)

and express the correction \( L^{(1)} \) as a function of \( \varphi^a, Q^a \) and derivatives of \( Q^a \). Since \( L^{(1)} \) should vanish for \( Q^a = 0 \), expanding it in powers of \( Q^a \) and its derivatives we get

\[ L^{(1)} = -Q^a W_{a}^{(1)}(\varphi) + Q^a Q^b W_{ab}^{(1)}(\varphi) + \cdots + \text{total derivative terms}. \] (2.9)

Since we are interested in solving (2.6) to leading order in \( \varepsilon \), all we need to know is the first \( W_{a}^{(1)}(\varphi) \) term which we may thus identify with \( W_{a}^{(1)} \) in (2.7).

In each particular case discussed below the functions \( W_{a}^{(1)}(\varphi) \) will be found by a straightforward (computer-assisted) computation. It is important to note that while the correction term \( L^{(1)} \) in (2.1) depends, in general, not only on \( \varphi^a \) but also on its derivatives, the leading term \( W_{a}^{(1)}(\varphi) \) in (2.4), and, therefore, \( J^a \), computed in this way will depend only on \( \varphi^a \) — since the leading-order equations are first-order, all higher derivatives of \( \varphi^a \) can be expressed in terms of \( \varphi^a \) using (2.4) order-by-order in \( \varepsilon \).

It is clear that the same should be true also at higher orders in \( \varepsilon \), provided one uses perturbation theory in \( \varepsilon \). This suggests that the exact (all-order in \( \varepsilon \)) form of the BPS equations (2.4) may admit an equivalent representation where all terms in the r.h.s. are simply algebraic functions of \( \varphi^a \), just as in the case at the leading supergravity order in

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\footnote{A heuristic reason why this should be the case is that the value of the correction may be regarded as a shift of energy which should vanish for a supersymmetric solution.}
One is tempted to conjecture that such “RG equations” which do not involve higher-derivative terms (and resulting after a non-trivial rearrangement of the $\alpha'$ or $l_P$ expansion) should follow from a more fundamental definition of the supersymmetry/BPS condition in string theory which is not referring to low-energy effective action expansion.

At the leading order in perturbation theory in $\varepsilon$ the corrected equations (2.6) take the form of first-order equations. It is useful to note that to compute $W_a^{(1)}(\varphi)$ or $J^a$ we will not need to know the explicit form of the solution to the original equations (2.4): after fixing a particular ansatz for the metric involving functions $\varphi^a$ and computing the corresponding curvature invariants entering (2.4) we will be able to express them in terms of $\varphi^a$ using only the general form of the first-order system (2.4).

3. $R^4$ corrections to conifold metrics in type II superstring theory

In this section we will consider type II string theory on manifolds $R^{1,3} \times M^6$, where $M^6$ is either a resolved or deformed conifold. Our aim will be to determine explicitly how the metric on these spaces is changed by the leading $R^4$ correction to the supergravity action. For definiteness, we will discuss the effect of the tree-level $\alpha'^3$ string correction [35,16,36,37], but all is the same for the similar 1-loop correction (the difference between the tree-level and 1-loop $R^4$ invariants in type IIA theory vanishes for the 6-d backgrounds we consider).

Since the conifolds are Ricci flat Kähler manifolds [8], one in general expects that (in an appropriate scheme) the deformation of these metrics will be determined, as for all CY metrics [10,17,18], by a modification of the Kähler potential. We shall start addressing this problem in the 1-st order equation framework of the previous section: this approach is more universal as it applies to less supersymmetric (non-Kähler) spaces like $G_2$ holonomy spaces discussed later in section 4.

3.1. $\alpha'^3R^4$ terms in type II superstring effective action

We begin by recalling the structure of $\alpha'^3$ corrections to the relevant part of type II effective action – the one which depends on the graviton and the dilaton only. Using the Einstein frame, the leading $\alpha'^3$ corrections to the tree-level string effective action implied by the structure of the Green-Schwarz 4-point massless string scattering amplitude can be written as [35,38,39]

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 + \varepsilon \mathcal{L}^{(1)} \right], \quad \mathcal{L}^{(1)} = e^{-2\phi} I_4(\overline{C}) ,$$

$$I_4(\overline{C}) = \overline{C}^{hmnk} \overline{C}_{pqn} \overline{C}^l_{h r lp} \overline{C}_{q rs k} + \frac{1}{2} \overline{C}^{hkmn} \overline{C}_{pqn} \overline{C}^l_{h r lp} \overline{C}_{q rs k} ,$$

(3.1)
where $\kappa = 8\pi^{7/2}g_s\alpha'^2$, $\varepsilon = \frac{1}{8}\alpha'^3\zeta(3)$ and
\[
\overline{C}_{ijkl} = C_{ijkl} - \frac{1}{4}(\nabla^2 \phi)_{ijkl}, \quad (3.2)
\]
\[
(\nabla^2 \phi)_{ij}^{kl} = \delta^k_i \nabla_j \nabla^l \phi - \delta^j_i \nabla_i \nabla^l \phi - \delta^l_i \nabla_j \nabla^k \phi + \delta^j_i \nabla_i \nabla^k \phi. \quad (3.3)
\]
Here $C_{ijkl}$ is the Weyl tensor, and we omit terms proportional to $\nabla_m \phi \nabla^m \phi$ in the definition of $\overline{C}_{ijkl}$ (our leading-order backgrounds will have trivial dilaton field).

Due to the field redefinition ambiguity [40,35], we can assume that all the terms in (3.1) depend only on the Weyl tensor: the Ricci tensor terms can be expressed in terms of the dilaton terms using leading-order equations of motion, i.e. changing the scheme. For the same reason, we shall also assume that in the scheme we use there are no terms proportional to the dilaton equation of motion $\nabla_m \nabla^m \phi$. We will see, however, that to preserve the Kähler structure of $M^6$ one will have to change the scheme, adding a certain Ricci tensor dependent term to the effective action.

Beyond the 4-point level, the above form of the leading $\alpha'^3$ corrections is dictated by the sigma model considerations. As follows from [36,37], there exists a scheme in which the metric and dilaton dependent terms in the action (reproducing the 4-loop correction to the beta-function [16]) are given, in the string frame, by
\[
S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \ e^{-2\phi} \left[ R + 4(\partial \phi)^2 + \varepsilon I_4(R) \right], \quad (3.4)
\]
\[
I_4(R) = R^{hmnk} R_{pmnq} R^r_{sp} R^q_{rsk} + \frac{1}{2} R^{hknm} R_{pmnq} R^r_{hspm} R^q_{rsk}. \quad (3.5)
\]
The action that follows from (3.4) upon the transformation $g_{mn} \to e^{\phi/2}g_{mn}$ gives the corresponding action in the Einstein frame that differs from the one in (3.1) by a change of the scheme (assuming one restores back the presently irrelevant $(\partial \phi)^2$ terms omitted in (3.2)).

Since the leading-order Ricci-flat backgrounds we are interested in have $\phi = 0$, we need to keep only the terms $e^{-\frac{3}{2}\phi}C^4$ and $C^3 \nabla^2 \phi$ in (3.1) as these may give corrections to the dilaton equation. Explicitly, the relevant terms in (3.1) are found to be
\[
\mathcal{L}^{(1)} = e^{-\frac{3}{2}\phi} \left[ I_4(C) + 2E^{ij} \nabla_i \nabla_j \phi + O((\nabla \phi)^2) \right], \quad (3.6)
\]
where $I_4(C)$ is given by (3.5) with $R_{ijkl} \to C_{ijkl}$, and $E_{ij}$ is defined by
\[
E_{ij} = -C^{mkl}_i C_{jpq} C^m_{j} C^{pq}_{l} + \frac{1}{4} C^m_{i} C^{kl}_j C_{mpq} C_{k}^{pq} - \frac{1}{2} C_{ijkl} C^{kmpq} C^{l}_{mpq}. \quad (3.7)
\]
The tensor $E_{ij}$ has the property
\[ g^{ij} E_{ij} = E , \]  
where $E \sim \epsilon_6 \epsilon_6 RRR$ is the 6-d Euler density, which, for a Ricci-flat space and up to a numerical coefficient, is given by
\[ E = C_{jmnk} C^{mpqn} C_{pj}^k q + \frac{1}{2} C_{jkmn} C^{pqmn} C_{pj}^k q . \]  
As we will show in Appendix A, for any Einstein (in particular, any Ricci-flat) manifold the tensor $E_{ij}$ satisfies also the following identity
\[ \nabla_i \nabla_j E_{ij} = \frac{1}{6} \nabla^i \nabla_i E . \]  
Thus (3.6) may be rewritten, up to a total derivative term, as
\[ \mathcal{L}^{(1)} = e^{-\frac{2}{3} \phi} \left[ I_4(C) + \frac{1}{3} E \nabla^2 \phi + O((\nabla \phi)^2) \right] . \]  
As a result, the second term can be removed by the following shift of the dilaton in (3.11) by a local curvature invariant $E$
\[ \phi \to \phi - \frac{1}{3} \epsilon E . \]  
Then, in such a scheme the dilaton can get corrections only from the first term $e^{-\frac{2}{3} \phi} I_4(C)$ in (3.11). In fact, there will be no correction to the dilaton coming from the exponential coupling of $\phi$ to the $I_4(C)$ invariant in (3.6): for all supersymmetric backgrounds we will study in this paper we will find by direct computation that $I_4(C)$ vanishes. This invariant should vanish for all special holonomy, supersymmetry-preserving metrics (see also [4])
\[ I_4(C) \big|_{\text{special holonomy Ricci flat metrics}} = 0 . \]  
Thus, in the scheme where the term $E \nabla^2 \phi$ in (3.11) is absent, the dilaton of these supersymmetric backgrounds is not modified from its constant leading-order value. Therefore, in what follows we will set $\phi = 0$.

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\[ 6 \] In general, the Euler number for a (compact without boundary) manifold of even dimension $d = 2m$ is given by $\chi_d = \frac{1}{2^{d/2}} (d/2)! (4\pi)^{d/2} \int d^d x \sqrt{g} \epsilon_{d} \epsilon_d R \ldots R$. For $d = 6$ we get $\chi_6 = \frac{1}{3 \cdot 2^4 \cdot (4\pi)^3} \int d^6 x \sqrt{g} E_6$, where $E_6 = \epsilon_6 \epsilon_6 RRR = 64 E + O(R_{mn})$, with $E = CCC + \frac{1}{2} CCC$ given by the expression below. Thus $\chi_6 = \frac{4}{3 \cdot 24 \cdot (4\pi)^3} \int d^6 x \sqrt{g} E$.

\[ 7 \] Note that since the 4-point on-shell superstring scattering amplitude involving 3 gravitons and 1 dilaton has the same kinematic factor as the one following from the second term in (3.6) or (3.11), we conclude that this amplitude is always zero.

\[ 8 \] For 6-d conifolds, these conclusions are in agreement with the previous general statements about the dilaton shift (and the non-renormalization of central charge) for the CY spaces [31].
3.2. Corrections to the resolved conifold metric

The metric on resolved conifold that solves the $R_{mn} = 0$ equations is a special case of the following metric:

$$ds_6^2 = e^{10y}du^2 + e^{2y}ds_5^2 ,$$  \hspace{1cm} (3.14)$$

$$ds_5^2 = e^{-8w}e^2 + e^{2w+2v}(e_{\theta_1}^2 + e_{\phi_1}^2) + e^{2w-2v}(e_{\theta_2}^2 + e_{\phi_2}^2) ,$$  \hspace{1cm} (3.15)$$

where

$$e_{\psi} = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 , \quad e_{\theta_i} = d\theta_i , \quad e_{\phi_i} = \sin \theta_i d\phi_i ,$$

and $y, w, v$ are the functions of the radial coordinate $u$. Since the leading order 10-d background is the direct product $R^{1,3} \times M^6$, it is clear from the sigma model considerations that there should exist a scheme where the $\alpha'$-corrected string-frame metric is also a direct product of $R^{1,3}$ and a corrected 6-d metric. A priori, the metric need not remain a direct product in the Einstein frame (as this property is scheme-dependent), so the most general ansatz should be:

$$ds_{10}^2 = e^{2p}(ds_4^2 + ds_6^2) ,$$  \hspace{1cm} (3.16)$$

where $ds_4^2$ is the 4-dimensional Minkowski metric and $p(u)$ is an additional “warp factor” function.

As was already mentioned above, corrections to the metric and the dilaton can be studied separately. Computing the scalar curvature of (3.16),(3.14), we find, as in [10], the corresponding 1-d action

$$S^{(0)} = \frac{1}{2} \int du \, e^{8p} [5y'^2 - 5w'^2 - v'^2 + 18p'^2 + 20y'p' + \frac{1}{4} e^{8y} (4e^{-2w} \cosh 2v - e^{-12w} \cosh 4v)] .$$  \hspace{1cm} (3.17)$$

This action has the form (2.1) with $\varphi^a = (y, v, w, p)$ and $u$ playing the role of the coordinate $t$. It admits (cf. (2.3)) the following superpotential [10]

$$W = -\frac{1}{4} e^{8p+4y}(e^{4w} + e^{-6w} \cosh 2v) .$$  \hspace{1cm} (3.18)$$

The corresponding system (2.4) of 1-st order equations is then

$$y' + \frac{1}{5} e^{4y}(e^{4w} + e^{-6w} \cosh 2v) = 0 , \quad w' - \frac{1}{10} e^{4y}(2e^{4w} - 3e^{-6w} \cosh 2v) = 0 ,$$  \hspace{1cm} (3.19)$$

\text{footnote}{This form of the ansatz is of course equivalent (by a redefinition of $y$ and $w$ in (3.14)) to $ds_{10}^2 = e^{2p}ds_4^2 + ds_6^2$.}
\[ v' - \frac{1}{2}e^{4y-6w} \sinh 2v = 0 , \quad p' = 0 . \]  
(3.20)

The general solution of these equations depends on two non-trivial integration constants (the third one is a shift of the radial coordinate) and can be written as

\[ e^{-4v} = 1 + \frac{6a^2}{\rho^2} , \quad e^{-10w} = \frac{2}{3} \kappa(\rho)e^{2v} , \quad e^{2y} = \frac{1}{9} \kappa(\rho)e^2 e^{8w} , \quad p = 0 . \]  
(3.21)

The coordinate \( \rho \) and the original coordinate \( u \) are related by \( \frac{d\rho}{du} = -\sqrt{\frac{3}{2}}e^{5y-5w-v} \), i.e. the metric (3.14) takes the form \[ ds^2 = \kappa^{-1}(\rho)d\rho^2 + \rho^2 \left[ \frac{1}{9} \kappa(\rho)e_{\psi}^2 + \frac{1}{6} (e_{\psi_1}^2 + e_{\psi_2}^2) + \frac{1}{6} (1 + \frac{6a^2}{\rho^2}) (e_{\phi_2}^2 + e_{\phi_2}^2) \right] , \]  
(3.22)

where \[ \kappa(\rho) = \frac{1 + \frac{9a^2}{\rho^2} - \frac{b^6}{\rho^6}}{1 + \frac{6a^2}{\rho^2}} , \quad \rho_0 \leq \rho < \infty , \quad \rho_0^6 + 9a^2 \rho_0^4 - b^6 = 0 . \]  
(3.23)

The case of \( a = b = 0 \) corresponds to the standard singular conifold [8]; \( b = 0 \) gives the standard resolved conifold metric [11]; \( a = 0, b \neq 0 \) corresponds to the non-singular generalized conifold [11,12] (a 6-d analog of the 4-d Eguchi-Hanson metric); \( a \neq 0, b \neq 0 \) is the generalized resolved conifold metric.

The minimal value of \( \rho \), i.e. \( \rho_0 = \rho_0(a,b) \geq 0 \) is positive and becomes zero when \( b = 0 \). For \( b \neq 0 \) we are to assume that \( \psi \in [0,2\pi) \) to avoid the conical bolt singularity at \( \rho = \rho_0 \). The curvature invariants for this metric are regular (unless both \( a \) and \( b \) are zero when we get back to the singular conifold metric).

Corrected form of 1-st order equations

To find the deformation of this metric under the \( R^4 \) correction to the Einstein action we are to determine the source terms in (2.6) as described in section 2 (see (2.7)–(2.9)). Computing the \( R^4 \) curvature invariant in (3.1) for the metric (3.15) and then using the equations (3.19), (3.20) to express the derivatives of functions in terms of functions themselves we find that it indeed vanishes as required by (2.7). Its first variation (cf. (2.9)) gives the sources \( J^a = (J_y, J_v, J_w, J_p) \) in (2.8)

\[ J_y = 0 , \quad J_v = 0 , \quad J_p = 0 , \quad J_w = \frac{8}{5} e^{-8v-42w-2y} (35e^{2v} + 73e^{6v} + 73e^{10v} + 35e^{14v} - 18e^{10w} - 114e^{4v+10w} - 168e^{8v+10w} - 114e^{12v+10w} - 18e^{16v+10w} + 36e^{2v+20w} + 111e^{6v+20w} + 111e^{10v+20w} + 36e^{14v+20w} - 18e^{4v+30w} - 32e^{8v+30w} - 18e^{12v+30w} ) . \]  
(3.24)
Since the source for the $p$ equation vanishes, we can thus set it to zero $p = 0$, i.e. the 10-d space retains indeed its direct-product structure. This is a scheme-dependent property: if we used another scheme, e.g., the one with the Riemann tensor $R_{ijkl}$ instead of the Weyl tensor $C_{ijkl}$ in (3.7), we would get a nontrivial expression for the warp factor $p$.

The equation for $w$ is thus the only one that gets corrected,

$$w' - \frac{1}{10} e^{4y}(2e^{4w} - 3e^{-6w} \cosh 2v) = \varepsilon J_w. \quad (3.25)$$

This structure of the corrected equations is related to the fact that the resolved conifold is, in fact, a Kähler manifold (see below). Moreover, $J_w$ can be represented as the $u$-derivative of the 6-d Euler density $E$ (3.9) evaluated on the metric (3.22) (i.e. on (3.14) which solves the system (3.13), (3.20))

$$J_w = -\frac{4}{15} \frac{d}{du} E. \quad (3.26)$$

Computing the 6-d Euler density for the metric (3.14) and using (3.19), (3.20) to eliminate the derivatives of $v, w, y$, we get

$$E = 6e^{-6v-36w-6y}(-5e^{2v} - 8e^{6v} - 5e^{10v} + 3e^{10w} + 15e^{4v+10w} + 15e^{8v+10w} + 3e^{12v+10w} - 6e^{2v+20w} - 13e^{6v+20w} - 6e^{10v+20w} + 3e^{4v+30w} + 3e^{8v+30w}). \quad (3.27)$$

Explicitly, for the metric (3.22) we get

$$E = \frac{864}{\rho^{14}(6a^2 + \rho^2)}(80a^4b^{18} + 2592a^8b^{12}\rho^2 + 48a^2b^{18}\rho^2 + 1728a^6b^{12}\rho^4 + 8b^{18}\rho^4 + 336a^4b^{12}\rho^6 + 168a^4b^6\rho^{12} + 1296a^8\rho^{14} + 24a^2b^6\rho^{14} + 216a^6\rho^{16} + b^6\rho^{16} + 10a^4\rho^{18}). \quad (3.28)$$

It follows that $E$ vanishes (and thus there are no corrections) [4] for the standard (singular) conifold which corresponds to $a = b = 0$, and is regular in all other cases ($E(\rho_0) \geq E \geq E(\infty) = 0$).[4]

10 Note that since these metrics are non-compact, the integral of (3.27) that appears in the formal definition of the Euler number (given in the footnote above eq. (3.9)) need not produce an integer (we are grateful to R. Minasian and P. Vanhove for drawing our attention to this fact). Indeed, taking into account that the angular part of the 6-d integral gives $\frac{1}{4\pi}(4\pi)^3$ for $b = 0$ case (where $\psi$ is $4\pi$ periodic as in the standard conifold case) and $\frac{1}{2\cdot3\cdot5\cdot2\pi}(4\pi)^3$ for $b \neq 0$ case (where $\psi$ is $2\pi$ periodic to avoid the bolt singularity) and doing the integral over $\rho$ from $\rho_0$ to $\infty$ one finds: $\chi_6(a, b = 0) = 14/27$, $\chi_6(a, b = 0) = 88/27$. To get an integer value for the Euler number one needs to introduce a boundary at some $\rho = \rho_*$ and take into account the boundary terms in the expression for the Euler number.
Kähler structure

The resulting system of linear equations for $y$ in (3.19), $v$ in (3.20) and (3.25) looks rather complicated, but it can be solved explicitly if one is guided by the information provided by the existence of the Kähler structure \[8\] on the resolved conifold. Indeed, the resolved conifold is a CY manifold with the following Kähler potential \[8,10\]

$$K = 4[K(t) + a^2 \ln(1 + |\Lambda|^2)] \, ,$$

(3.29)

where $\Lambda$ is a function of angles and $t$ is related to the radial coordinate $r$ used in \[10\] as $r^2 = e^t$, i.e. it is related to $\rho$ in (3.22) by

$$e^{2t} = 6^{-3} \left(\rho^6 + 9a^2 \rho^4 - b^6\right) \, , \quad -\infty < t < \infty \, .$$

(3.30)

Written in terms of $K$ the resolved conifold metric is \[10\]

$$ds_6^2 = K''(dt^2 + e^{2\psi}) + K'(e_{\theta_1}^2 + e_{\phi_1}^2) + (K' + a^2)(e_{\theta_2}^2 + e_{\phi_2}^2) \, , \quad K' = \frac{dK}{dt} \, .$$

(3.31)

It is clear from the metric that the radii of the spheres at $t = -\infty$ (i.e. at $\rho = \rho_0(a,b)$) are determined by $K'(\infty)$ which can be expressed in terms of the constants $a$ and $b$. Comparing (3.31) with the general ansatz for the metric (3.14) written in terms of the $t$ coordinate,

$$ds^2 = e^{2y}[e^{-8w}(dt^2 + e^{2\psi}) + e^{2w+2v}(e_{\theta_1}^2 + e_{\phi_1}^2) + e^{2w-2v}(e_{\theta_2}^2 + e_{\phi_2}^2)] \, , \quad \frac{du}{dt} = e^{-4(y+w)} \, ,$$

(3.32)

we get the following relations

$$e^{10y} = K''K'(K' + a^2)^2 \, , \quad e^{20w} = \frac{K'(K' + a^2)}{K''^2} \, , \quad e^{-4v} = 1 + \frac{a^2}{K'} \, .$$

(3.33)

Substituting these relations into (3.19),(3.20), we find that the equations for $v$ and $y + w$ are satisfied identically for any function $K$, while the equation for $w$ reduces to

$$\frac{1}{10}(2t - \ln[K'(K' + a^2)K''])' = 0 \,.$$ 

(3.34)

\[11\] The standard form of the Kähler metric is $g_{\rho\bar{q}} = \partial_{\rho} \partial_{\bar{q}} K$ where in the present case the three complex coordinates are $U, Y, \Lambda$, with $U = e^{\frac{i}{2}} e^{\frac{i}{2}(\psi + \phi_1 + \phi_2)} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}$, $Y = e^{\frac{i}{2}} e^{\frac{i}{2}(\psi - \phi_1 + \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}$, $\Lambda = e^{-i\phi_2} \tan \theta_2$. 

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This is, of course, equivalent to first integral of the only non-trivial component of the Einstein equation for the Kähler metric (3.31), \( R_{m\bar{n}} = -\partial_m \partial_{\bar{n}} \ln \det g = 0 \), since the determinant of (3.31) (defined with respect to the complex coordinates \( Y, U, \Lambda \)) is \( e^{-4t} [K'(K' + a^2)K'']^2 \).

The equations (3.19), (3.20) corrected by the sources (3.24), (3.25) can be written in the form

\[
\frac{d}{dt}(\tilde{y} + w) + \frac{1}{2} e^{-10w} \cosh 2v = 0 , \quad \frac{dv}{dt} - \frac{1}{2} e^{-10w} \sinh 2v = 0 , \quad (3.35)
\]

\[
\frac{d}{dt}(w - \frac{3}{2} \tilde{y}) - \frac{1}{2} = -\frac{2}{3} \varepsilon \frac{d}{dt} E , \quad \tilde{y} \equiv y + \frac{4}{15} \varepsilon E . \quad (3.36)
\]

Note that eq. (3.36) can be easily integrated.

The corrected 6-d metric corresponding to the solution of this system may be interpreted as follows: up to a specific conformal factor (which is related to the redefinition \( y \rightarrow \tilde{y} \), cf. (3.32)), it is again a Kähler metric with the corresponding Kähler potential satisfying the corrected version (3.36) of (3.34):

\[
t - t_0 - \frac{1}{2} \ln [K'(K' + a^2)K''] = -\frac{4}{3} \varepsilon E . \quad (3.37)
\]

Here \( t_0 \) is an integration constant which is convenient to fix so that \( e^{2t_0} = \frac{3}{2} \).

Indeed, the Weyl transformation \( g_{mn} = e^{2h} \tilde{g}_{mn} \) of the metric (3.32), i.e., of (3.14), (3.15) written in terms of the coordinate \( t \), amounts to the redefinition: \( y = \tilde{y} + h \).

Choosing \( h = -\frac{4}{15} \varepsilon E \) we may thus express \( \tilde{y}, w, v \) in terms of the corrected Kähler potential function \( K \) using the same relations as in (3.33); then (3.36) reduces to (3.37).

Note that \( w - \frac{3}{2} \tilde{y} \) is essentially the same as \( -\frac{1}{8} \ln \det \tilde{g} \), where \( \det \tilde{g} \) is the determinant of the rescaled 6-d metric. Eq. (3.37) is thus a special case of the general equation for the deformation of the Kähler potential of a CY space due to the string \( \alpha'^3 R^4 \) term \([17,18]\): the corrected form of the beta-function equations (in the scheme where the Kähler structure of the metric is preserved) is \( R_{m\bar{n}} + \partial_m \partial_{\bar{n}} H = 0 \), where \( H \) is a series in \( \alpha' \) of local curvature invariants, starting with the \( \alpha'^3 E \) term. That means \([18]\) that one can prepare such a Kähler potential \( K(\alpha') \), that after all \( \alpha' \)-corrections taken into account one is left simply with \( R_{mn}(K_0) = -\partial_m \partial_{\bar{n}} \ln \det g(K_0) = 0 \), where \( K_0 \) corresponds to the standard CY metric, \([12]\) i.e., integrating the above equation,

\[
\ln \det g(K_0) = \ln \det g(K) + k_1 \varepsilon E + ... , \quad k_1 = -\frac{16}{3} . \quad (3.38)
\]

\[12\] The two metrics – \( g(K_0) \) and \( g(K) \) are, of course, related in a complicated non-local way.
We shall use the alternative interpretation: one starts with \( K_0 \) and finds \( K = K_0 + \varepsilon K_1 + \ldots \) as a solution of the corrected string (beta-function) equations.

As for the meaning of the Weyl rescaling of the metric, it is related to the particular scheme choice used in (3.1), which is not the same scheme in which the metric retains its Kähler structure (see below).

**Corrected form of the metric**

Since the curvature of the metric (3.22) is regular (unless \( a = b = 0 \) but then \( E = 0 \)) and the scale of the curvature is determined by either \( a \) or \( b \), the \( \alpha' \)-corrections are small for small enough \( \alpha' \) or \( \beta' \) (recall that \( \rho \geq \rho_0 \), see (3.23)), the \( \alpha' \)-expansion is well-defined and thus it makes sense to concentrate on the leading deformation of the metric (3.22) caused by the first non-zero \( R^4 \) correction term.

Integrating (3.37) one more time, we get (we always expand to leading order in \( \varepsilon \))

\[
K' + \frac{3}{2} a^2 K'' = 2 \int_{-\infty}^{t} dt' e^{2t'} [1 + \frac{8}{3} \varepsilon E(t') + c^3], \tag{3.39}
\]

where \( c \) is another integration constant which we choose so that \( c^3 = (K'^3 + \frac{3}{2} a^2 K'^2) (\infty) \).

Since \( K' (\infty) \) determines the radii of the spheres at \( t = \infty \) in the metric (3.32), such choice of \( c \) means that we keep the radii unchanged by the \( R^4 \) correction. For \( \varepsilon = 0 \) eq. (3.39) is the equation for the Kähler potential of the (generalized) resolved conifold [10,11,12]: the relation to the metric (3.22),(3.23) is established by setting \( c = \frac{1}{6} b^2 \), \( \rho^2 = 6K' \) (see also (3.30)). Eq. (3.39) implies

\[
K = K_0 + \varepsilon K_1, \quad K'_1 = \frac{16}{9} ((K'_0 + a^2)K'_0)^{-1} \int_{-\infty}^{t} dt' e^{2t'} E(t'). \tag{3.40}
\]

Changing from \( t \) to \( \rho \) coordinate (3.30) and using (3.28) for \( E \) we get

\[
K'_1 = \frac{64}{\rho^2 (6a^2 + \rho^2)} [E(\rho) - E(\rho_0)], \quad K'_1 \equiv \frac{d}{dt} K_1, \tag{3.41}
\]

where \( \rho_0 = \rho_0(a,b) \) is defined in (3.23) and

\[
E(\rho) = \int_{+\infty}^{t(\rho)} dt' e^{2t'} E(t') = -\frac{2}{3\rho^{10} (6a^2 + \rho^2)^5} \left[ 8b^{18} \rho^2 + 6804a^{10} \rho^{10} + 5670a^8 \rho^{12} + 3b^6 \rho^{14} + 9a^4 \rho^4 (42b^{12} + 42b^6 \rho^6 + 5 \rho^{12}) + a^2 (24b^{18} + 63b^6 \rho^{12}) + 54a^6 (18b^{12} \rho^2 + 17 \rho^{14}) \right]. \tag{3.42}
\]

This determines the form of the metric (3.31),(3.33) (which depends only on \( K' \) and \( K'' \)).
Note that the corrections vanish at both ends of the interval of values of \( \rho \): \( \rho_0 \leq \rho < \infty \). Eq. (3.41) simplifies in the two special cases: the generalized conifold \( a = 0, \ b \neq 0 \), where

\[
K'_1 = \frac{128}{\rho^4} \left( \frac{11}{3} - \frac{b^6}{\rho^6} - \frac{8b^{18}}{3\rho^{18}} \right), \quad b \leq \rho < \infty, \quad (3.43)
\]

and the standard resolved conifold \( a \neq 0, \ b = 0 \), where

\[
K'_1 = \frac{16\rho^2 (12a^2 + \rho^2) (648a^4 + 126a^2\rho^2 + 7\rho^4)}{3(6a^2 + \rho^2)^6}, \quad 0 \leq \rho < \infty. \quad (3.44)
\]

These expressions give the explicit form of the corresponding metrics (3.31).

**Scheme dependence**

Let us now return to the question of the Weyl rescaling of the metric we needed to do to make its Kähler structure manifest. This is related to the issue of scheme dependence. We have shown that the \( R^4 \) corrections (chosen in the specific scheme (3.1)) modify the 6-dimensional metric (3.22) into

\[
ds^2_6 = \left[ 1 - \frac{4}{15} E(t) \right] ds^2_6(K), \quad (3.45)
\]

where \( ds^2_6(K) \) is the Kähler metric defined by the Kähler potential satisfying (3.39). The \( E \)-dependent conformal factor in front of the metric is scheme-dependent, i.e. it can be removed by a redefinition of the 10-d metric. Indeed, let us consider the following redefinition of the 10-d Einstein-frame metric

\[
g_{mn} \rightarrow g_{mn} + s_1 \varepsilon E_{mn}, \quad (3.46)
\]

where \( E_{mn} \sim RRR \) is the tensor defined in (3.4). This redefinition changes the form of the \( R^4 \) correction in (3.1) by terms of the structure \( R^{mn}E_{mn} + \nabla^m \nabla^n E_{mn} + \cdots \). Since our leading-order metric has a direct-product \( R^1,3 \times M^6 \) form, \( E_{mn} \) has non-zero components in \( M_6 \) directions only. Computing \( E_{mn} \) for the resolved conifold metric (3.22), we find that in this case

\[
E_{ij} = \frac{1}{6} g_{ij} E, \quad i, j = 1, \ldots, 6. \quad (3.47)
\]

As a result, the redefinition (3.46) does not change the 4-d components of the 10-d metric, but rescales the 6-d components by \( E \). This implies that there is an explicit choice of scheme in which the \( R^4 \)-corrected metric preserves its Kähler structure.

\[\text{Note that this relation is of course consistent with the general identities (3.8) and (3.10).}\]
### 3.3. Deformed conifold case

The relevant ansatz for the 6-d metric in this case is again parametrized by 3 functions $y, w, q$ of a radial coordinate $u$ [10,11] (cf. (3.14),(3.15))

\[ ds^2_6 = e^{10y}du^2 + e^{2y}ds^2_5 , \]

\[ ds^2_5 = e^{-8w}e^2_\psi + e^{2w+2q}(g_1^2 + g_2^2) + e^{2w-2q}(g_3^2 + g_4^2) , \]

where [4]

\[ g_1 = -\frac{\epsilon_2 + e\phi_1}{\sqrt{2}} , \quad g_2 = -\frac{\epsilon_1 - e\phi_1}{\sqrt{2}} , \quad g_3 = \frac{\epsilon_2 - e\phi_1}{\sqrt{2}} , \quad g_4 = \frac{\epsilon_1 + e\phi_1}{\sqrt{2}} , \quad g_5 = e\psi , \]

\[ \epsilon_1 \equiv \sin \psi \sin \theta_2 d\phi + \cos \psi d\theta , \quad \epsilon_2 \equiv \cos \psi \sin \theta_2 d\phi - \sin \psi d\theta . \]

For $q = 0$ this is equivalent to the standard conifold ansatz or (3.14),(3.15) with $v = 0$. Choosing again the 10-d Einstein-frame metric as (3.16) we find that the analog of the Einstein action (3.17) here is

\[ S(0) = \frac{1}{2} \int du \ e^{8p}[5y'^2 - 5w'^2 - q'^2 + 18p'^2 + 20y'p'] + \frac{1}{4} e^{8y}(4e^{-2w} \cosh 2q - e^{-12w} - e^{8w} \sinh^2 2q)] . \]

This action admits the following superpotential [10]

\[ W = -\frac{1}{4} e^{4p+4y}(e^{4w} \cosh 2q + e^{-6w}) . \]

Note that this $W$ is very similar to the one in the resolved conifold case being related to (3.18) by a formal transformation $v \rightarrow q, e^{4w} \leftrightarrow e^{-6w}$. The corresponding 1-st order system (2.4) is

\[ y' + \frac{1}{5} e^{4y}(e^{4w} \cosh 2q + e^{-6w}) = 0 , \quad w' - \frac{1}{10} e^{4y}(2e^{4w} \cosh 2q - 3e^{-6w}) = 0 , \]

\[ q' - \frac{1}{2} e^{4p+4w} \sinh 2q = 0 , \quad p' = 0 . \]

The solution of this system gives the generalized [10,11] deformed conifold metric depending on the two parameters $(b, \epsilon)$ and having regular curvature. For $b = 0$ it becomes the metric of the standard deformed conifold [3] while for $\epsilon = 0$ it gives the metric of generalized standard conifold (i.e. (3.22),(3.23) with $a = 0$).
The system (3.53),(3.54) is very similar to (3.19),(3.20) in the resolved conifold case and the analysis of the $R^4$ correction to the deformed conifold metric is thus closely parallel to the one in the previous section, so we will omit the details.

We find again that the $R^4$ invariant in (3.3) vanishes when evaluated on the solution of (3.53),(3.54), and that the corrected form of the 1-st order equations is (2.6) with the source terms

$$J_y = 0 , \quad J_q = 0 , \quad J_p = 0 , \quad J_w = -\frac{4}{15} \frac{d}{du} E .$$  (3.55)

$E$ is again the 6-dimensional Euler density (3.9), which, evaluated on the solution of (3.53),(3.54), is given by (cf. (3.27))

$$E = \frac{3}{32} e^{-12q-36w-6y} \left( -1152 e^{12q} + 9 e^{60w} + 1152 e^{10q+10w} + 1152 e^{14q+10w} \\
- 240 e^{8q+20w} - 1120 e^{12q+20w} - 240 e^{16q+20w} + 192 e^{10q+30w} + 192 e^{14q+30w} \\
- 16 e^{8q+40w} + 32 e^{12q+40w} - 16 e^{16q+40w} - 24 e^{2q+50w} + 72 e^{6q+50w} \\
- 48 e^{10q+50w} - 48 e^{14q+50w} + 72 e^{18q+50w} - 24 e^{22q+50w} - 22 e^{4q+60w} \\
+ 7 e^{8q+60w} + 12 e^{12q+60w} + 7 e^{16q+60w} - 22 e^{20q+60w} + 9 e^{24q+60w} \right) .$$  (3.56)

We conclude that as in the resolved conifold case: (i) this form of the corrected equations is consistent with the expectation that there should exists a scheme where the metric preserves its Kähler structure; (ii) in the scheme we are using the warp factor $p$ can be set equal to zero, i.e. the 10-d metric preserves its 4+6 factorized form.

4. $R^4$ corrections to a class of 7-d metrics with $G_2$ holonomy

In this section we shall analyze the corrections induced by the $R^4$ terms in the effective action to another class of supersymmetric leading order Ricci flat solutions – spaces with $G_2$ holonomy found in [13,14] and in [15]. We shall phrase our discussion in the 11-d framework, i.e. look at solutions of the $R + l_6^2 R^4 + \ldots$ low-energy effective action of M-theory, but the analysis of the corresponding 10-d string solutions is essentially the same (we shall comment on this explicitly in section 4.5). We shall see that $G_2$ spaces get non-trivial corrections, implying that $G_2$ structure should be deformed (in line with the deformation of the local supersymmetry transformation rule and thus of the form of the Killing spinor equation).

We shall derive the corresponding corrected (“inhomogeneous”) form of 1-st order equations for the functions parametrizing these metrics and discuss their solutions. The important role will be played by the analysis of solutions of the homogeneous part of the equations, i.e. the equations for small perturbations near the leading-order solution. In section 4.4 we will extend (and correct) the analysis of this homogeneous system given previously in [15].
4.1. $R^4$ terms in 11-dimensional theory

Let us start with recalling the structure of the leading $R^4$ correction terms in the M-theory effective action (see, e.g., [41,84] and references there)

$$S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2 \cdot 4!} F_4^2 + \cdots \right) + S^{(1)}, \quad (4.1)$$

$$S^{(1)} = b_1 T_2 \int d^{11}x \sqrt{-g} \left( J_0 - 2 I_2 \right),$$

$$J_0 = t_8 \cdot t_8 \; CCCC + \frac{1}{4} E_8 + \cdots, \quad E_8 \equiv \frac{1}{3!} \epsilon_11 \cdot \epsilon_11 CCCC,$$  

$$I_2 = \frac{1}{4} E_8 + 2 \epsilon_11 C_3 [CCCC - \frac{1}{4} (CC)^2] + \cdots. \quad (4.2)$$

Here $C = (C_{mnkl})$ is the Weyl tensor and dots in the two (super)invariants $J_0$ and $I_2$ stand for other (not completely known) terms depending on $F_4 = dC_3$. Also, $b_1 = (2\pi)^{-4} \cdot 3^{-2} \cdot 2^{-13}$, and $T_2 = (2\pi)^{2/3} (2\kappa^2)^{-1/3}$ (membrane tension). The backgrounds we will be interested in will have $F_4 = 0$ and the direct-product structure $R^{1,3} \times M^7$, where at the leading order $M^7$ will have $G_2$-holonomy. That means, in particular, that terms depending on $F_4$ and $\epsilon_11$ in (4.1) will not contribute, and the relevant part of the 11-d effective action may be written as (cf. (3.1),(3.4))

$$S = S^{(0)} + S^{(1)} = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left[ R + \epsilon I_4(C) \right], \quad (4.3)$$

$$I_4(C) = C^{hmnk} C_{p mnq} C_h^{\; r sp} C^q_{\; rsk} + \frac{1}{2} C^{hk mn} C_{p q mn} C_h^{\; r sp} C^q_{\; rsk}, \quad (4.4)$$

where $\epsilon = 3^{-1} 2^{-5} (2\pi)^{-10/3} (2\kappa^2)^{2/3}$. The direct reduction of (4.3) to 10 dimensions should give the 1-loop $R^4 + \cdots$ term in type IIA superstring theory. In Appendix A we perform a check of consistency of this reduction in the sector of $CC \nabla \nabla \phi$ terms.

4.2. General ansatz for the metric

We shall consider a general class of 7-d spaces of $G_2$ holonomy studied in [13,14,15,19]. They have the global symmetry $SU(2) \times SU(2) \times U(1) \times Z_2$. The general ansatz for the

\footnote{We choose the scheme where the curvature tensor is expressed in terms of the Weyl tensor and Ricci tensor terms, with the latter replaced by the $F_4$-dependent terms using leading-order field equations.}
11-d metric deformed by the quantum corrections that preserves this global symmetry is (cf. (3.14)–(3.16) [15])

\[ ds_{11}^2 = e^{2p}(ds_4^2 + ds_7^2) , \] (4.5)

where \( ds_4^2 \) is the metric of the 4-dimensional Minkowski space, and

\[
\begin{align*}
\sum_{i=1}^{7} ds_i^2 &= e^{4\alpha+4\beta+2\gamma+2\delta} dt^2 + e^{2\alpha}[(\sigma_1 - \tilde{\sigma}_1)^2 + (\sigma_2 - \tilde{\sigma}_2)^2] + e^{2\delta}(\sigma_3 - \tilde{\sigma}_3)^2 \\
&+ e^{2\beta}[(\sigma_1 + \tilde{\sigma}_1)^2 + (\sigma_2 + \tilde{\sigma}_2)^2] + e^{2\gamma}(\sigma_3 + \tilde{\sigma}_3)^2 .
\end{align*}
\] (4.6)

Here \( \sigma_i \) and \( \tilde{\sigma}_i \) are the basis one-forms invariant under the left action of the groups \( SU(2) \) and \( \tilde{SU}(2) \), respectively, i.e.

\[
\begin{align*}
\sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi , \quad \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi , \quad \sigma_3 = d\psi + \cos \theta d\phi ,
\end{align*}
\] (4.7)

with the analogous formulas for \( \tilde{\sigma}_i \) in terms of \( \tilde{\psi}, \tilde{\theta}, \tilde{\phi} \). The functions \( \alpha, \beta, \gamma, \delta, p \) depend only on the “radial” coordinate \( t \).

The warp-factor field \( p \) is introduced to account for the fact that the \( R^4 \) corrections could, in principle, destroy the direct product structure of the original \( R^{1,3} \times M^7 \) background (this will not happen in a particular scheme we are using but \( p \) may be non-vanishing in other schemes).

Computing the \( R \)-term in the 11-d action (4.3) on this ansatz, one finds the following 1-d action, cf. (3.17) (this is the generalization of the expression derived in [15] to the case of nonvanishing function \( p \))

\[ S^{(0)} = \int dt \ (T - V) , \] (4.8)

\[
\begin{align*}
T &= e^{9p}(2\alpha^2 + 8\alpha\beta + 2\beta^2 + 4\alpha\delta + 4\beta\delta + 4\alpha\gamma + 4\beta\gamma + 2\delta\gamma + 36\alpha\delta + 36\beta\delta + 18\delta\gamma + 18\gamma\delta + 90p^2) , \\
V &= \frac{1}{8} e^{9p} (2e^{6\alpha+\gamma+2\gamma} - 4e^{4\alpha+\gamma+2\gamma} + 2e^{2\alpha+6\beta+2\gamma} - 8e^{4\alpha+2\beta+2\delta+2\gamma} \\
&\quad - 8e^{2\alpha+4\beta+2\delta+2\gamma} + 2e^{2\alpha+2\beta+4\delta+2\gamma} + e^{4\alpha+2\beta+4\gamma} + e^{4\beta+2\delta+4\gamma} ) .
\end{align*}
\] (4.9)

Comparing to (2.11), here \( \varphi^a = (\alpha, \beta, \gamma, \delta, p) \), \( \dot{\varphi}^a = \frac{d\varphi^a}{dt} \).

This action admits a superpotential (in fact, two simple superpotentials related by interchanging \( \alpha \leftrightarrow \beta \)). The one that leads to the system of first-order equations obtained in [13] is

\[ W = \frac{1}{2} e^{9p} \left( 2e^{3\alpha+\beta+\gamma} + 2e^{\alpha+3\beta+\gamma} + 2e^{\alpha+\beta+2\delta+\gamma} - e^{2\alpha+\delta+2\gamma} + e^{2\beta+\delta+2\gamma} \right) . \] (4.10)
The variables and the radial coordinate used in [15] are related to \( \varphi \) and \( t \) as follows

\[
A = e^\alpha , \quad B = e^\beta , \quad C = e^\gamma , \quad D = e^\delta , \quad dr = e^{2\alpha+2\beta+2\gamma+\delta} dt ,
\]

\[
ds^2_I = C^{-2}(r)dr^2 + A^2(r)[(\sigma_1 - \bar{\sigma}_1)^2 + (\sigma_2 - \bar{\sigma}_2)^2] + D^2(r)(\sigma_3 - \bar{\sigma}_3)^2
+ B^2(r)[(\sigma_1 + \bar{\sigma}_1)^2 + (\sigma_2 + \bar{\sigma}_2)^2] + C^2(r)(\sigma_3 + \bar{\sigma}_3)^2.
\]

(4.12)

The superpotential (4.11) expressed in terms of \( A, B, C, D \) takes the form

\[
W = e^{9p}( A^3 BC + AB^3 C + ABCD^2 - \frac{1}{2} A^2 C^2 D + \frac{1}{2} B^2 C^2 D ).
\]

(4.13)

The corresponding system of first-order equations (2.4) is the one studied in [15]

\[
\frac{dA}{dr} = \frac{1}{4} \left( B^2 - A^2 + D^2 \right) + \frac{1}{A} , \quad \frac{dB}{dr} = \frac{1}{4} \left( \frac{A^2 - B^2 + D^2}{ACD} - \frac{1}{B} \right),
\]

\[
\frac{dC}{dr} = \frac{1}{4} \left( \frac{C}{B^2} - \frac{C}{A^2} \right) , \quad \frac{dD}{dr} = \frac{A^2 + B^2 - D^2}{2ABC} , \quad \frac{dp}{dr} = 0.
\]

(4.14)

The superpotential given in [15] is obtained from (4.10) (or (4.13)) by interchanging \( \alpha \leftrightarrow \beta \) (or \( A \leftrightarrow B \)), and setting \( p = 0 \).

There are two simple known solutions of the system (4.14): the one found in [13], and another one found in [15]. Any 7-dimensional manifold \( M^7 \) with the metric (4.12) has a particular \( U(1) \) isometry acting by the same shift on the angular coordinates \( \psi \) and \( \tilde{\psi} \): \( \psi \rightarrow \psi + \nu, \quad \tilde{\psi} \rightarrow \tilde{\psi} + \nu \). The field \( C(r) \) in (4.12) determines the radius of the associated circle (the scale of the \( \sigma_3 + \bar{\sigma}_3 \) direction). The 11-dimensional manifold of the form \( R^{1,3} \times M^7 \) can be reduced along this \( U(1) \) direction to a 10-d background of type IIA superstring theory. The solutions found in [13,14,15] correspond after this reduction to a D6-brane wrapped a three-sphere \( S^3 \) of deformed conifold [21,22]. Since the type IIA dilaton is given by \( e^{\phi} = C^3/2 \), the field \( C \) determines also the value of the string coupling (see [15,22] for details).

\textit{Vanishing of correction to warp factor}

The contribution of the \( R^4 \) correction (4.4) to the 1-d action corresponding to the ansatz (4.3) can be obtained using the method described in section 2. We have checked that the combination \( I_4(C) \) given by (4.4) \textit{vanishes} for \textit{any} solution of the 1-st order system (4.14).

\footnote{Note that the factor 2 in eq. (4.6) and \( \sqrt{2} \) in eq. (4.8) of [15] should be omitted to get the correct expression.}
Since (4.14) can be used to express any derivative of the metric and thus its curvature as an algebraic function of $\alpha, \beta, \gamma, \delta$, to check this one does not need to know the explicit form of the solutions of (4.14).

Since $I_4(C)$ in (4.14) depends only on the Weyl tensor, it is easy to see that then the correction $W^{(1)}_5$ in (2.9) corresponding to the warp factor $p = \varphi^5$ in (4.5) also vanishes. This does not yet imply that the corresponding source component $J^5$ in (2.6) should also vanish since the components $G^{5a}$ of the metric are non-trivial. Nevertheless, computing all the corrections $W^{(1)}_a$ and the corresponding sources $J^a$, we have found that indeed $J^5 = 0$. As a result, we conclude that the $R^4$ correction does not modify the equation for $p$, i.e. we can set $p = 0$ so that the direct product structure of the background $R^{1,3} \times M^7$ is preserved at the $R^4$ level. This is a scheme-dependent statement: if we used the action with the Riemann tensor $R_{ijkl}$ instead of the Weyl tensor $C_{ijkl}$ in $I_4$ in (4.4), we would get a nontrivial correction to the warp factor $p$. The two schemes are, in general, related by a redefinition similar to (3.46),

$$g_{mn} \rightarrow g_{mn} + s_1 \varepsilon(RRR)_{mn} + s_2 \varepsilon RRR g_{mn} + \ldots,$$

which may rescale the 11-d metric by a factor $1 + k_1 \varepsilon RRR$.

### 4.3. Corrections to BSGPP solution

We shall first study the $R^4$ corrections to the $G_2$ holonomy metric found in [13,14]. The manifold has an enhanced $SU(2) \times SU(2) \times SU(2) \times Z_2$ global symmetry which is achieved by setting $\alpha = \delta, \beta = \gamma$ in (4.6), i.e. $D = A, C = B$ in (4.12). Then the system (4.14) reduces to

$$\frac{dA}{dr} = \frac{1}{2A}, \quad \frac{dB}{dr} = \frac{1}{4B}(1 - \frac{B^2}{A^2}), \quad D = A, \quad C = B. \quad (4.15)$$

The general solution of this leading-order system ($A = A_0, B = B_0$) depends on two parameters $r_0$ and $s$

$$A_0(r; s, r_0) = \sqrt{r - s}, \quad B_0(r; s, r_0) = 3^{-1/2}(r - s)^{-1/4}(r - s)^{3/2} - r_0^{3/2}. \quad (4.16)$$

In the special case $r_0 = 0, s = 0$ the metric (4.12) takes the form:

$$ds^2 = d\rho^2 + \rho^2 \left( \frac{1}{12} \left[ (\sigma_1 - \bar{\sigma}_1)^2 + (\sigma_2 - \bar{\sigma}_2)^2 + (\sigma_3 - \bar{\sigma}_3)^2 \right] 
+ \frac{1}{36} \left[ (\sigma_1 + \bar{\sigma}_1)^2 + (\sigma_2 + \bar{\sigma}_2)^2 + (\sigma_3 + \bar{\sigma}_3)^2 \right] \right), \quad r \equiv \frac{1}{12}\rho^2. \quad (4.17)$$

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This space is a cone which is singular at $\rho = 0$. It follows from the analysis given below that, like the standard singular conifold, this singular space is not corrected by the $R^4$ term in the effective action.\footnote{In contrast to the case of the singular conifold the invariant $E$ \eqref{eq:E_conifold} is non-zero for this solution.}

The general non-singular solution \eqref{eq:4.16} can be obtained from the special solution with $s = 0, \ r_0 = 1$ by using the translational invariance in $r$ and the invariance of the system \eqref{eq:4.15} under the following scaling transformation: $r \to \lambda^2 r, \ A \to \lambda A, \ B \to \lambda B$, which corresponds to the rescaling of the $7$-d metric \eqref{eq:4.12} by $\lambda^2$. Thus, without loss of generality, one can choose $s = 0$ and $r_0 = 1$. The resulting metric \eqref{eq:4.12} (with $1 \leq r < \infty$) has regular curvature, and thus computing corrections in expansion in powers of curvature is well-defined.

The corrected analog \eqref{eq:2.6} of the system \eqref{eq:4.15} is found to be

$$
\frac{dA}{dr} = \frac{1}{2A} + \varepsilon J_A, \quad \frac{dB}{dr} = \frac{1}{4B} \left(1 - \frac{B^2}{A^2}\right) + \varepsilon J_B, \quad \varepsilon \ll 1
$$

where $J_A = 2^{-11} \cdot 3 A^{-13} (A^2 - 3B^2) (133A^4 - 414A^2B^2 + 301B^4), \quad J_B = -A^{-1}BJ_A.$ \eqref{eq:4.19}

One is supposed to solve this system to leading order in $\varepsilon$ only (since we ignored higher order in $\varepsilon$ corrections to the effective action). Setting

$$
A = e^a = A_0 e^{s a}, \quad B = e^b = B_0 e^{s b}, \quad (A, B) \to (A_0, B_0)
$$

where $A_0$ and $B_0$ in \eqref{eq:4.16} solve \eqref{eq:4.13}, we find that $a$ and $b$ should satisfy

$$
(A_0 \frac{d}{dr} + \frac{1}{2A_0}) a = J_{A_0}, \quad [B_0 \frac{d}{dr} + \frac{1}{4B_0} \left(1 + \frac{B_0^2}{A_0^2}\right)] b - \frac{B_0}{2A_0^2} a = J_{B_0}, \quad (A_0, B_0)
$$

where $J_{A_0}, J_{B_0}$ are given by \eqref{eq:4.19} with $A, B \to A_0, B_0$. The general solution of the inhomogeneous system of linear equations \eqref{eq:4.21} is given by the sum of its particular solution and a general solution of the homogeneous system obtained by setting the sources to zero. Since we know the general two-parameter solution \eqref{eq:4.16} of the nonlinear equations \eqref{eq:4.13}, we can easily obtain the general solution of the corresponding linearized system by differentiating \eqref{eq:4.16} with respect to the parameters $s$ and $r_0$:

$$
a_0(r) = c_s \frac{\partial \ln A_0}{\partial s} + c_{r_0} \frac{\partial \ln A_0}{\partial r_0} = -\frac{c_s}{2r},
$$

$$
b_0(r) = c_s \frac{\partial \ln B_0}{\partial s} + c_{r_0} \frac{\partial \ln B_0}{\partial r_0} = -\frac{c_s}{2r} \frac{r^{3/2} + \frac{1}{2}}{r^{3/2} - 1} = \frac{3c_{r_0}}{4(r^{3/2} - 1)}.
$$

\footnote{In contrast to the case of the singular conifold the invariant $E$ \eqref{eq:E_conifold} is non-zero for this solution.}
Here $c_s$ and $c_{r_0}$ are arbitrary constants, and we have chosen $s = 0$ and $r_0 = 1$ here and in what follows (so that $1 \leq r < \infty$). Evaluating (4.19) on the solution (4.16), one finds

$$J_{A_0} = \frac{301 + 640r^{3/2} + 256r^3}{6144r^8}, \quad J_{B_0} = -\frac{\sqrt{r^{3/2} - 1}}{\sqrt{3}r^{3/4}}J_{A_0}.$$

Then the general solution of (4.21) is given by

$$a(r) = -\frac{c_s}{2r} - \frac{1}{84r^2} - \frac{1}{48r^6} - \frac{301}{39936r^{14}},$$

$$b(r) = \frac{1}{r^{3/2} - 1} \left[ -\frac{c_s}{2} \left( r^{1/2} + \frac{1}{2r} \right) - \frac{3c_{r_0}}{4} + \frac{1}{63r^3} + \frac{5}{336r^{7/2}} - \frac{823}{79872r^6} - \frac{2107}{299520r^{11/2}} \right].$$

The corrections vanish at large $r$. We can fix the constant $c_s$ and $c_{r_0}$ by imposing the boundary conditions that imply that the short-distance ($r \to 1$) limit of the metric is also not changed by the $R^4$ correction, namely,

$$a(1) = 0, \quad b(1) = 0, \quad i.e. \quad A(1) = 1, \quad B(1) = 0.$$

The second condition $b(1) = 0$ is a non-trivial one: for generic values of $c_s$ and $c_{r_0}$ the function $b(r)$ has a pole at $r = 1$, so that, in general, one could only require regularity of $b(r)$ at this point. A simple computation shows that in the present case the boundary conditions (4.25) can indeed be satisfied, provided we choose:

$$c_s = -\frac{3753}{46592}, \quad c_{r_0} = \frac{63}{640}.$$

4.4. Corrections to BGGG solution

Next, let us analyze corrections to another class of spaces with $G_2$ holonomy found in [15] (see also [19]). The corresponding metric is given by a special solution to the system (4.14) depending only on two (out of possible four) parameters $s, r_0$

$$A_0(r; s, r_0) = \sqrt{\frac{(r - s - r_0)(r - s + 3r_0)}{8r_0}}, \quad B_0(r; s, r_0) = \sqrt{\frac{(r - s + r_0)(r - s - 3r_0)}{8r_0}},$$

$$C_0(r; s, r_0) = \sqrt{\frac{2r_0(r - s - 3r_0)(r - s + 3r_0)}{3(r - s - r_0)(r - s + r_0)}}, \quad D_0(r; s, r_0) = \frac{r - s}{\sqrt{6r_0}}.$$

The parameter $r_0$ we are using differs from the one in [13] by a factor of $\frac{3}{2}$. 
As in the BSGPP case, the presence of the two free parameters $s$ and $r_0$ simply reflects the invariance of (4.14) under a shift and a rescaling of the coordinate $r$. In what follows we shall set $s = 0$ and $r_0 = 1$, so that the range of $r$ will be $3 \leq r < \infty$.

To find the corrections we need to know the general solution of the homogeneous system of linear equations describing small perturbations of $A$, $B$, $C$ and $D$ around the solution (4.27), as well as a particular solution of the inhomogeneous system with sources corresponding to $R^4$ correction expanded near the solution (4.27). Unfortunately, in the present case it does not seem possible to find the exact general solutions of these two systems. That makes the computation of the corrections much more complicated. Instead of finding the solutions numerically, we will determine them in the vicinity of $r = 3$ by expanding in powers of $r - 3$, and also at large $r$ by expanding in powers of $1/r$. We will then sew the two expansions. This will allows us to get all essential information for determining the corrections, though the explicit analytic form of the corrected solution in this case will not be available.

**Linearized perturbations near the leading solution**

Let us start with the system of linear equations describing small perturbations of $A$, $B$, $C$ and $D$ around the solution (4.27). Representing the fields as in (4.20),

$$A = e^\alpha = A_0 e^{\varepsilon\alpha}, \quad B = e^\beta = B_0 e^{\varepsilon\beta}, \quad C = e^\gamma = C_0 e^{\varepsilon\gamma}, \quad D = e^\delta = D_0 e^{\varepsilon\delta},$$

we find the system of equations for the perturbations near (4.27) with $s = 0$, $r_0 = 1$ (cf. (4.21))

$$\frac{da}{dr} = -\frac{5r^2 - 12r + 3}{2r(r - 1)(r - 3)} a + \frac{b}{2r} - \frac{c}{r + 3} + \frac{d}{r - 3},$$

$$\frac{db}{dr} = \frac{a}{2r} - \frac{5r^2 + 12r + 3}{2r(r + 1)(r + 3)} b - \frac{c}{r - 3} + \frac{d}{r + 3},$$

$$\frac{dc}{dr} = \frac{4a}{(r - 1)(r + 3)} - \frac{4b}{(r - 1)(r + 3)}, \quad \frac{dd}{dr} = \frac{2a}{r - 3} + \frac{2b}{r + 3} - \frac{c}{r} - \frac{5r^2 - 9}{r(r^2 - 9)} d.$$

Differentiating the solution (4.27) with respect to the parameters $s$ and $r_0$ (and then setting them equal to $s = 0$, $r_0 = 1$) as in (4.22), we obtain the following special two-parameter solution to (4.27)

$$a_0(r) = c_s \frac{r + 1}{(r - 1)(r + 3)} - c_{r_0} \frac{r^2 + 3}{(r - 1)(r + 3)}, \quad b_0(r) = c_s \frac{r - 1}{(r + 1)(r - 3)} - c_{r_0} \frac{r^2 + 3}{(r + 1)(r - 3)},$$

18 The analysis below extends and corrects the previous discussion in [15].

19 Here $a_0(r) \equiv -c_s \frac{\partial \ln A_0}{\partial s} + 2c_{r_0} \frac{\partial \ln A_0}{\partial r_0}$, etc.
\[ c_0(r) = c_s \frac{8r}{(r^2 - 1)(r^2 - 9)} + c_{r_0} \frac{r^4 - 26r^2 + 9}{(r^2 - 1)(r^2 - 9)} , \quad d_0(r) = \frac{c_s}{r} - c_{r_0} . \quad (4.30) \]

The system of four linear equations (4.29) must have four independent solutions. Since we do not know how to find the remaining two solutions in a closed form, we shall analyze them in the vicinity of \( r = 3 \) and at large \( r \), and then sew the solutions obtained. Let us write the (4.29) in the form

\[ \frac{d\phi}{dr} = M_s(r)\phi , \quad \phi^q = (a, b, c, d) . \quad (4.31) \]

The matrix \( M(r) \) has the following expansion near \( r = 3 \)

\[ M(r) = \frac{\mathcal{M}}{r - 3} + O(1) , \quad \mathcal{M} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} . \quad (4.32) \]

\( \mathcal{M} \) has the following eigenvalues \( \lambda_q \) and the corresponding eigenvectors \( v_q \)

\[ \lambda_1 = -3 , \quad v_1 = (-1, 0, 0, 2) ; \quad \lambda_2 = -1 , \quad v_2 = (0, 1, 1, 0) ; \]
\[ \lambda_3 = 0 , \quad v_3 = (1, 0, 0, 1) ; \quad \lambda_4 = 1 , \quad v_4 = (0, -1, 1, 0) . \quad (4.33) \]

Near \( r = 3 \) the solution corresponding to an eigenvalue \( \lambda \) behaves as \((r - 3)^\lambda[1 + O(r - 3)]\).

The solutions corresponding to \( \lambda_2 = -1 \) and \( \lambda_3 = 0 \) can be obtained from the two-parameter solution (4.30) by choosing \( c_s = 2 \), \( c_{r_0} = 2/3 \) and \( c_s = 6 \), \( c_{r_0} = 1 \), respectively. Thus we know their exact form (away from \( r = 3 \))

\[ a_2 = -\frac{2r(r - 3)}{3(r - 1)(r + 3)} , \quad b_2 = -\frac{2(r^2 - 3r + 6)}{3(r + 1)(r - 3)} , \]
\[ c_2 = \frac{2(r^4 - 26r^2 + 24r + 9)}{3(r^2 - 1)(r^2 - 9)} , \quad d_2 = -\frac{2(r - 3)}{3r} , \quad (4.34) \]

and

\[ a_3 = -\frac{r^2 - 6r - 3}{(r - 1)(r + 3)} , \quad b_3 = -\frac{r - 3}{r + 1} , \quad c_3 = \frac{(r - 3)(r^2 + 6r + 1)}{(r^2 - 1)(r + 3)} , \quad d_3 = \frac{6 - r}{r} . \quad (4.35) \]

Note that, contrary to the claim in [13], the perturbation (4.35) does not describe a new deformation of the two-parameter solution (4.27) but corresponds simply to a change of the parameters \( s \) and \( r_0 \).

\footnote{We will denote the solution corresponding to the eigen-value \( \lambda_k \) as \((a_k, b_k, c_k, d_k)\).}
The perturbation corresponding to $\lambda_1 = -3$ is singular at $\tau = r - 3 = 0$ (i.e. it does not represent a smooth deformation of the solution (4.27)); it is found to be

$$a_1 = -\frac{1}{\tau^3} + \frac{2}{3\tau^2} - \frac{7}{36\tau} - \frac{2\tau}{27} + \frac{661\tau^2}{12960} + O(\tau^3) + \frac{(4 + \tau) \ln \tau}{3(2 + \tau)(6 + \tau)}, \quad \tau \equiv r - 3,$$

$$b_1 = \frac{17\tau}{216} - \frac{17\tau^2}{72} - \frac{37\tau^3}{1296} + O(\tau^3) + \frac{(2 + \tau) \ln \tau}{3\tau(4 + \tau)},$$

$$c_1 = \frac{1}{6\tau^2} - \frac{5}{18\tau} + \frac{5}{108} - \frac{5\tau^2}{288} + O(\tau^3) + \frac{8(3 + \tau) \ln \tau}{3\tau(2 + \tau)(4 + \tau)(6 + \tau)},$$

$$d_1 = \frac{2}{\tau^3} - \frac{4}{3\tau^2} + \frac{7}{9\tau} - \frac{5}{12} + \frac{55\tau}{648} - \frac{13\tau^2}{1215} + O(\tau^3) + \frac{\ln \tau}{3(3 + \tau)}. \quad (4.36)$$

The $\ln \tau$ terms in (4.36) (which are exact) are multiplied by the functions of the explicitly known two-parameter solution (4.30) with $c_s = 1/3$, $c_{r_0} = 0$.

The fourth linearized solution corresponding to $\lambda_4 = 1$ vanishes at $r = 3$ and (as we shall see below) goes to a constant at $r = \infty$. Since the value of the function $C$ in (4.12) may be interpreted as in [15] as determining the radius of the M-theory circle, this perturbation describes a nontrivial deformation of the manifold corresponding to a finite change in this radius at infinity [21] (cf. [15]). Near $\tau = r - 3 = 0$ it is given by a series in powers of $\tau$ with the radius of convergence $|\tau| < 2$. To match the solution to a solution found at large $r$ we computed it up to the order $\tau^{20}$. The first few terms of the series are

$$a_4 = -\frac{\tau^2}{5} + \frac{2\tau^3}{15} - \frac{131\tau^4}{1890} + O(\tau^5), \quad b_4 = -\tau + \frac{17\tau^2}{36} - \frac{29\tau^3}{144} + \frac{1141\tau^4}{14400} + O(\tau^5),$$

$$c_4 = \tau - \frac{13\tau^2}{36} + \frac{227\tau^3}{2160} - \frac{103\tau^4}{4800} + O(\tau^5), \quad d_4 = -\frac{4\tau^2}{15} + \frac{8\tau^3}{45} - \frac{244\tau^4}{2835} + O(\tau^5). \quad (4.37)$$

Next, let us solve the system (4.29) at large $r$, where $M(r)$ in (4.31) has the form

$$M(r) = \frac{M_\infty}{r} + O\left(\frac{1}{r^2}\right), \quad M_\infty = \left(\begin{array}{cccc}
-\frac{5}{2} & \frac{1}{2} & -1 & 1 \\
\frac{1}{2} & -\frac{5}{2} & -1 & 1 \\
0 & 0 & 0 & 0 \\
2 & 2 & -1 & -5 \\
\end{array}\right). \quad (4.38)$$

The eigenvalues and eigenvectors of $M_\infty$ are

$$\lambda_1 = -6, \quad v_1 = (1, 1, 0, -4); \quad \lambda_2 = -3, \quad v_2 = (1, -1, 0, 0);$$

$$\lambda_3 = -1, \quad v_3 = (1, 1, 0, 1); \quad \lambda_4 = 0, \quad v_4 = (-1, -1, 1, -1). \quad (4.39)$$
Here the solution associated to an eigenvalue \( \lambda \) goes as \( r^\lambda[1 + O(\frac{1}{r})] \). The solutions corresponding to \( \lambda_3 = -1 \) and \( \lambda_4 = 0 \) are obtained from the two-parameter solution (4.30) by imposing the relations \( c_s = 1, c_{r_0} = 0 \) and \( c_s = 0, c_{r_0} = 1 \), respectively. The solution corresponding to \( \lambda_1 = -6 \) is given by a power series in \( 1/r \). The series converges very slowly for \( r > 3 \), and to match the solutions with the ones found near \( r = 3 \) we computed the terms up to the order \( 1/r^{30} \). The first few terms are

\[
\begin{align*}
    a_{1\infty} &= \frac{1}{r^6} + \frac{4}{r^7} + \frac{205}{7r^8} + \frac{754}{7r^9} + O(\frac{1}{r^{10}}), \\
    b_{1\infty} &= \frac{1}{r^6} - \frac{4}{r^7} + \frac{205}{7r^8} - \frac{754}{7r^9} + O(\frac{1}{r^{10}}), \\
    c_{1\infty} &= -\frac{2}{r^8} + O(\frac{1}{r^{10}}), \\
    d_{1\infty} &= -\frac{4}{r^6} - \frac{810}{7r^8} + O(\frac{1}{r^{10}}).
\end{align*}
\]

(4.40)

The \( 1/r \) expansion for the solution for \( \lambda_1 = -3 \) breaks down at the order \( 1/r^6 \) where a \( \frac{1}{r} \ln r \) term appears. The coefficients of the \( \ln r \) terms are proportional to the solution (4.40) (cf. (4.36)), i.e.

\[
\begin{align*}
    a_{2\infty} &= \frac{1}{r^3} - \frac{2}{r^4} - \frac{5}{r^5} + O(\frac{1}{r^7}) + \frac{324}{5} a_{1\infty} \ln r, \\
    b_{2\infty} &= -\frac{1}{r^3} - \frac{2}{r^4} + \frac{5}{r^5} + O(\frac{1}{r^7}) + \frac{324}{5} b_{1\infty} \ln r, \\
    c_{2\infty} &= -\frac{2}{r^4} - \frac{8}{r^6} + O(\frac{1}{r^8}) + \frac{324}{5} c_{1\infty} \ln r, \\
    d_{2\infty} &= \frac{6}{r^4} - \frac{136}{5r^6} + O(\frac{1}{r^8}) + \frac{324}{5} d_{1\infty} \ln r.
\end{align*}
\]

(4.41)

Now we are ready to determine the large \( r \) behavior of the nontrivial solution (4.37) vanishing at \( r = 3 \). At large \( r \) it is represented by a superposition of the independent solutions (1.30), (4.40) and (4.41), i.e.

\[
\phi_4^i = \phi_0^i(r) + c_1 \phi_{1\infty}^i(r) + c_2 \phi_{2\infty}^i(r),
\]

(4.42)

where \( \phi_0^i(r), \phi_{1\infty}^i(r) \) and \( \phi_{2\infty}^i(r) \) are given by (4.30), (4.40) and (4.41), respectively. Computing the l.h.s and the r.h.s. of (4.42) at different values of \( r = \tau + 3 \), we find the constants \( c_{r_0}, c_s, c_1 \) and \( c_2 \)

\[
c_{r_0} = -2.00 \pm 0.02, \quad c_s = -9.31 \pm 0.09, \quad c_1 = -7050 \pm 70, \quad c_2 = 54.3 \pm 0.5.
\]

(4.43)

These values were obtained by matching the solutions at \( r = 17/4 \).

In the process of the analysis of small perturbations near the special solution (4.27) we have thus shown that the system (4.14) has a three-parameter family of regular solutions, with the third non-trivial parameter corresponding to the perturbation \( \phi_{4}^i \). The existence of a 3-parameter family of solutions of the 1-st order system (4.14) was also demonstrated in [19] by using numerical analysis of (4.14).
**Solution of the inhomogeneous system**

Now we are ready to study the $R^4$ corrections to the solution (4.27). Computing the source terms $J^s$ in the corrected analog (2.6) of (4.29) from (4.4) expanded (2.9) near the leading-order solution (4.27), we find the following inhomogeneous system of linear equations that replaces (4.31)

\[
\frac{d\phi^s}{dr} = M^s_q(r)\phi^q + J^s(r) ,
\]

where the matrix $M(r)$ is the same as in (4.31), and the sources $J^s$ are given by

\[
\begin{align*}
J^1 &= -\frac{1024\left(-78 - 405r - 286r^2 - 549r^3 - 58r^4 + 57r^5 - 26r^6 + r^7\right)}{27(r^2 - 1)^8}, \\
J^2 &= -\frac{1024\left(78 - 405r + 286r^2 - 549r^3 + 58r^4 + 57r^5 + 26r^6 + r^7\right)}{27(r^2 - 1)^8}, \\
J^3 &= -\frac{2048r\left(577 + 1089r^2 - 101r^4 + 3r^6\right)}{27(r^2 - 1)^8}, \\
J^4 &= \frac{2048r\left(-233 - 9r^2 + 13r^4 + 5r^6\right)}{27(r^2 - 1)^8}.
\end{align*}
\]

To solve the resulting system we follow the same strategy that we used to analyze the small perturbations near the leading solution: find approximate solutions of (4.44) near $r = 3$ and at large $r$, and then sew the two expansions.

To solve (4.44) near $r = 3$ we impose the conditions

\[
\phi^s(\tau)|_{\tau=0} = 0 , \quad \tau \equiv r - 3 .
\]

As follows from the analysis of small perturbations, this condition fixes the solution modulo the solution (4.37) of the system (4.31) which also vanishes at $\tau = 0$. We computed the expansion in $\tau$ up to the order $\tau^{20}$. The series has a radius of convergence $|\tau| < 1$, and its first few terms are given by

\[
\begin{align*}
a(\tau) &= \frac{137\tau}{2304} - \frac{617\tau^2}{3840} + O(\tau^3) + c_4a_4(\tau) , \\
b(\tau) &= -\frac{137\tau}{2304} + \frac{5933\tau^2}{82944} + O(\tau^3) + c_4b_4(\tau) , \\
c(\tau) &= \frac{11213\tau^2}{82944} + O(\tau^3) + c_4c_4(\tau) , \\
d(\tau) &= \frac{137\tau}{2304} - \frac{5189\tau^2}{34560} + O(\tau^3) + c_4d_4(\tau) .
\end{align*}
\]

\footnote{We first found $J^s$ for an arbitrary solution of (4.14) as functions of $\phi^s$ by using the method described in section 2, and then computed them on the special solution (4.27) with $s = 0$ and $r_0 = 1$.}
Here $c_4$ is an arbitrary constant which is not fixed by the boundary conditions, and $a_4, b_4, c_4, d_4$ is the solution (4.37) of the homogeneous system. We will later fix the constant $c_4$ by requiring that the corrected solution also vanishes at $r = \infty$. This is a natural requirement since corrections to the BGGG background vanish at large $r$.

Now let us study the large $r$ region. Since the sources $J^q$ go to zero at large $r$ as $1/r^9$, there is a solution starting with $1/r^8$. The series converges very slowly for $r > 3$, and we computed it up to the order $1/r^{40}$. The leading terms are

$$
a_\infty(r) = \frac{7424}{189r^8} - \frac{12800}{189r^9} + O\left(\frac{1}{r^{10}}\right), \quad b_\infty(r) = \frac{7424}{189r^8} + \frac{12800}{189r^9} + O\left(\frac{1}{r^{10}}\right),$$

$$
c_\infty(r) = \frac{256}{9r^8} + O\left(\frac{1}{r^{10}}\right), \quad d_\infty(r) = -\frac{32000}{189r^8} + O\left(\frac{1}{r^{10}}\right). \quad (4.48)$$

Now it is straightforward to find the large $r$ behavior of the solution (4.47) vanishing at $r = 3$. At large $r$ it is given by a sum of the solution (4.48) and a solution of the homogeneous system (4.31), i.e.

$$\phi^q = \phi^q_\infty(r) + \phi^q_0(r) + c_1\phi^q_1\infty(r) + c_2\phi^q_2\infty(r), \quad (4.49)$$

where $\phi^q_\infty(r), \phi^q_0(r), \phi^q_1\infty(r)$ and $\phi^q_2\infty(r)$ are given by (4.48), (4.30), (4.40) and (4.41), respectively. Computing the l.h.s and the r.h.s. of (4.49) at different values of $r = \tau + 3$, we find that the constants $c_{r_0}, c_s$ in (4.30) and $c_1$ and $c_2$ are expressed in terms of the constant $c_4$ as follows

$$c_{r_0} = 0.10 \pm 0.01 - (2.0 \pm 0.2)c_4, \quad c_s = 0.51 \pm 0.05 - (9 \pm 1)c_4,$$

$$c_1 = 320 \pm 30 - (7000 \pm 700)c_4, \quad c_2 = -2.5 \pm 0.3 + (54 \pm 5)c_4. \quad (4.50)$$

These values were obtained by sewing the solutions at $\tau = 99/100$. Comparing these with (4.43), we see that they match. We also see that if one chooses $c_4 \approx 0.05$ then the coefficient $c_{r_0}$ vanishes, and, therefore, the corrections to the metric vanish not only at $r = 3$ but also at $r = \infty$. The resulting corrected solution then smoothly interpolates between the same short-distance and large-distance asymptotic values.
4.5. Corrections to $G_2$ spaces as solutions of 10-d superstring theory

It is straightforward to consider the leading $R^4$ corrections to the 10-d backgrounds of the form $R^{1,2} \times M^7$ in type II superstring theory, in the same way as this was done for the conifolds in section 3. One can check that with the scheme choice corresponding to (3.1) (i) the direct product structure of the manifold $R^{1,2} \times M^7$ is again preserved, i.e. the warp-factor is zero; (ii) the metric of $M^7$ space gets the same corrections as in 11 dimensions; (iii) as in the conifold case (see section 3.2) the dilaton is again shifted by $\frac{1}{3} \varepsilon E$, where $E$ is the cubic curvature invariant in (3.9) (so that there is a scheme where the dilaton is unchanged). The explicit expressions for $E$ for the two spaces (4.16) and (4.27) are ($s = 0$, $r_0 = 1$

\[ E_{BSGPP} = -\frac{553 + 915 r^3 + 480 r^3 + 320 r^2}{3072 r^{12}}, \quad E_{BGGG} = \frac{512 (99 + 207 r^2 + 9 r^4 + 5 r^6)}{9 (r^2 - 1)^7}. \]

One may ask about the connection between the 10-d and 11-d results. Earlier in this section we have shown that the direct product structure of the space $R^{1,3} \times M^7$ is preserved by the $R^4$ corrections to the effective action in 11 dimensions. If we reduce the 11-d background to 10 dimensions along one of the “free” spatial directions of $R^{1,3}$ we get a 10-d type IIA string background of the form $R^{1,2} \times M^7$ with constant dilaton. Thus, the $R^4$ corrections (4.4) in 11 dimensions reduced to 10 dimensions should give the (one-loop) $R^4$ corrections in 10 dimensions in the scheme where the dilaton is not modified, i.e. where there is no $E \nabla^2 \phi$ term in (3.11).

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Appendix A. Identity for $E_{ij}$ and 11 $\rightarrow$ 10 dimensional reduction of $R^4$ terms

Here we shall first prove the identity (3.10) for the $E_{ij}$ tensor defined in (3.7). We shall then comment on the role of this identity in checking the consistency of the relation by dimensional reduction between the $R^4$ terms in the M-theory action and the superstring effective action, which is implied by local supersymmetry.
Let us first recall the origin of $E_{ij}$: it appears from the $R^4$ invariant (3.3) upon performing a conformal variation of the metric: $\delta g_{ij} = \psi g_{ij}$, $\delta I_4(R) = 4E^{ij}\nabla_i \nabla_j \psi$. Since the structure of $I_4(R)$ is strongly constrained by supersymmetry, the same should be true for $E_{ij}$. We suspect that there may be a way to understand the existence of the identity (3.10) from the fact that the structure of $E_{ij}$ is constrained by the local supersymmetry.

First, it is easy to check that if (3.10) holds for any Einstein manifold, then the same identity should hold also with the Weyl tensors in (3.7) and (3.9) replaced by the Riemann ones. Then omitting terms that vanish for $R_{ij} = kg_{ij}$ we obtain

$$\nabla_i \nabla_j E_{ij} - \frac{1}{6} \nabla_i \nabla_i E = I_1 + I_2 + I_3 + I_4 , \quad (A.1)$$

where

$$I_1 = -\nabla_i \nabla_j R_{imkl}R_{jpkq}R_{lpmq} + \frac{1}{4} \nabla_i \nabla_j R_{imkl}R_{jmpq}R_{klpq}$$

$$+ \frac{1}{2} \nabla_i R_{jmk} R_{mpq} R_{pjkq} + \frac{1}{4} \nabla_i R_{jmk} R_{mpq} R_{pqkl} , \quad (A.2)$$

$$I_2 = -R_{imkl} R_{jpkq} \nabla_i \nabla_j R_{lpmq} + \frac{1}{4} R_{imkl} R_{jmpq} \nabla_i \nabla_j R_{klpq}$$

$$+ \frac{1}{2} R_{kijl} \nabla_i \nabla_j R_{kmpq} R_{lpmq} + \frac{1}{2} R_{kijl} \nabla_i \nabla_j R_{kmpq} R_{lmpq} , \quad (A.3)$$

$$I_3 = \frac{1}{4} \nabla_j R_{imkl} \nabla_i (R_{jmpq} R_{klpq}) - \frac{1}{4} \nabla_i R_{jmk} \nabla_i (R_{jmpq} R_{klpq}) , \quad (A.4)$$

$$I_4 = -\nabla^j R_{imkl} \nabla^i (R_{jpkq} R_{lpmq}) - R_{imkl} \nabla^i R_{jpkq} \nabla^j R_{lpmq}$$

$$+ \frac{1}{4} R_{imkl} \nabla_i R_{jmpq} \nabla_j R_{klpq} + \frac{1}{2} (R_{kijl} + R_{lijk}) \nabla_j R_{kmpq} \nabla_i R_{lpmq} \quad (A.5)$$

$$+ \frac{1}{2} \nabla_i R_{jmk} \nabla_i R_{mpq} R_{pjkq} + \frac{1}{2} \nabla_i R_{jmk} \nabla_i R_{mpq} R_{lmpq} \nabla_i R_{pjkq} .$$

We are going to show that $I_1 = 0$, $I_2 = 0$, $I_3 = -I_4$, thus demonstrating that the r.h.s. of (A.1) is zero.

Using the Bianchi identity we find $\frac{1}{2} \nabla_i \nabla_j R_{jmkl} R_{mpql} R_{pjqk} = \nabla_i \nabla_j R_{jmkl} R_{jpkq} R_{lpmq}$, $\frac{1}{4} \nabla_i \nabla_j R_{jmkl} R_{jmpq} R_{mpkl} = \frac{1}{4} \nabla_i \nabla_j R_{jmkl} R_{jmpq} R_{lmpq}$. Using then cyclic identity, we get

$$I_1 = \nabla_i \nabla_j R_{imkl} R_{jpkq} R_{lpmq} - \frac{1}{4} \nabla_i \nabla_j R_{imkl} R_{jmpq} R_{klpq} .$$

The cyclic identity gives also $\nabla_i \nabla_j R_{imkl} R_{jpkq} R_{lpmq} = \frac{1}{2} \nabla_i \nabla_j R_{ikml} R_{jmpq} R_{klpq} , so that finally $I_1 = 0$. To show that $I_2 = 0$ we need the following identities

$$R_{kijl} \nabla_i \nabla_j R_{kmpq} R_{lpmq} = R_{kiml} \nabla_i \nabla_j R_{kmpq} R_{ljpq} - \frac{1}{2} R_{kijkl} \nabla_i \nabla_j R_{klpq} R_{mpqk} ,$$
\[ R_{ijjk} \nabla_i \nabla_j R_{kmpq} R_{ltmq} = \frac{1}{2} R_{ijlk} \nabla_i \nabla_j R_{kmpq} R_{ltpq} , \]
\[ \nabla_i \nabla_j R_{lpmq} R_{ipkl} R_{jkmp} = 2 \nabla_i \nabla_j R_{lpmq} R_{imkl} R_{jkpq} , \]
and \[ \nabla_i \nabla_j R_{mlpq} R_{imkl} R_{jqpk} = \frac{1}{2} R_{ijkl} [\nabla_i, \nabla_j] R_{lpmq} R_{mqpk} . \] It is easy to see that
\[ I_3 = -\frac{1}{4} \nabla_j R_{imkl} \nabla_i (R_{jmpq} R_{klpq}) . \]
Since \[ \frac{1}{2} (R_{ijkl} + R_{ijlk}) \nabla_i R_{kmpq} \nabla_j R_{ltmq} = \nabla_i R_{jqkm} \nabla_j R_{lmpq} (R_{kipl} + R_{lipk}) , \] we get
\[ I_3 + I_4 = \nabla_i R_{jpkq} (\nabla_j R_{imkl} R_{lpmq} - \frac{1}{4} \nabla_j R_{ipml} R_{kqml} + \nabla_j R_{ipml} R_{imkl} - \nabla_i R_{jmkl} R_{ltqm}) . \]
This is simplified using \[ -\frac{1}{4} \nabla_i R_{jpkq} \nabla_j R_{ipml} R_{kqml} = \frac{1}{2} \nabla_i R_{jpkq} \nabla_j R_{pqml} R_{ikml} \] and
\[ \nabla_i R_{jpkq} \nabla_j R_{jmkl} R_{lqpm} = \nabla_i R_{jpkq} (\nabla_j R_{imkl} R_{lqpm} - \nabla_j R_{pqml} R_{imkl}) . \] Then finally
\[ I_3 + I_4 = \nabla_i R_{jpkq} \nabla_j R_{pmkl} R_{ikml} + \frac{1}{2} \nabla_i R_{jpkq} \nabla_j R_{pqml} R_{ikml} = 0 . \]
This completes the proof of
\[ \nabla^i \nabla^j E_{ij} = \frac{1}{6} \nabla^2 E . \quad (A.6) \]
As is well known, the 11-d and 10-d supergravities are related by dimensional reduction along an isometric direction. Since the \( R^4 \) invariants (3.1) in 10 and (4.2) in 11 dimensions should be consistent with the respective supersymmetries, one expects them to be also related by dimensional reduction. While this is obviously true for the purely metric-dependent terms, let us compare the simplest dilaton-dependent terms \( R^3 \nabla^2 \phi \) as they appear upon dimensional reduction from (4.2) with the similar terms present in the string effective action (3.1). As usual, we shall assume that the 11-d metric can be written as
\[ ds_{11}^2 = e^{-\phi/6} ds_{10E}^2 + e^{4\phi/3}(dx_{11} + C_m dx_m)^2 , \]
where \( \phi \) and \( C_m \) are the dilaton and the RR vector field, respectively, and \( ds_{10E}^2 \) is the 10-d metric in the Einstein frame. Since we want to consider the linear dilaton term \( C^3 \nabla^2 \phi \) that follows from the \( C^4 \) term in 11 dimensions, we can consider only the components of

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\[ \text{Footnote 22: The 11-d } R^4 \text{ term is related to string one-loop } R^4 \text{ term whose form is different from the tree-level invariant (3.1) by the } \epsilon_{10c10} RRRR \text{ term. However, this difference is irrelevant in the present case as the term } \epsilon_{10c10} RRRR \nabla^2 \phi \text{ vanishes due to Bianchi identity, and thus } C^3 \nabla^2 \phi \text{ term following from (3.1)} \]

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the Weyl tensor that have 10-dimensional indices. Modulo field redefinition ambiguity we can replace the Weyl tensor $C_{ijkl}$ by the Riemann one $R_{ijkl}$. Using the general relation between curvatures of the two conformally-equivalent spaces ($\tilde{g}_{ij} = e^{2\phi} g_{ij}$)

$$\tilde{R}^i_{jkl} = R^i_{jkl} - \delta^i_k (\nabla_j \nabla_l \varphi - \nabla_j \varphi \nabla_l \varphi) + \delta^i_l (\nabla_j \nabla_k \varphi - \nabla_j \varphi \nabla_k \varphi) - g_{jl} (\nabla^i \nabla_k \varphi - \nabla^i \varphi \nabla_k \varphi)$$

$$+ g_{jk} (\nabla^i \nabla_l \varphi - \nabla^i \varphi \nabla_l \varphi) - \nabla^m \varphi \nabla_m \varphi (\delta^i_k g_{jl} - \delta^i_l g_{jk}) ,$$

we derive the following relation between 11-d and 10-d Riemann tensors

$$R^{(11)m}_{nrs} = R^{(10)m}_{nrs} + \frac{1}{12} (\delta^m_r \nabla_n \nabla_s \phi - \delta^m_s \nabla_n \nabla_r \phi - \delta^n_r \nabla_m \nabla_s \phi + \delta^n_s \nabla_m \nabla_r \phi) , \quad (A.7)$$

where we have omitted terms quadratic in $\phi$ and terms proportional to the dilaton equation of motion (i.e. $\nabla^m \nabla_m \phi$). Comparing $(A.7)$ with $(3.2)$, we conclude that the coefficients in front of the tensor $(\nabla^2 \phi)^m_{nrs}$ differ by the factor $-\frac{1}{3}$. Therefore, if we start from the 11-dimensional $R^4$ term and dimensionally reduce, we get $-\frac{2}{3} E^{ij} \nabla_i \nabla_j \phi$ term, while we get $2 E^{ij} \nabla_i \nabla_j \phi$ term if we start directly in 10 dimensions (see $(3.6)$). That would be a puzzling contradiction if not for the fact that the term $E^{ij} \nabla_i \nabla_j \phi$ is, in fact, vanishing on-shell, thanks to the non-trivial identity $(3.10),(A.6)$ proved above. We suspect that there should be several similar identities related to supersymmetry of the $R + R^4 + ...$ actions which may explicitly appear in constructing these actions in the component approach as in [33].
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