INFLUENCES OF MONOTONE BOOLEAN FUNCTIONS

DEMETRES CHRISTOFIDES

Abstract. Recently, Keller and Pilpel conjectured that the influence of a monotone Boolean function does not decrease if we apply to it an invertible linear transformation. Our aim in this short note is to prove this conjecture.

1. Introduction

Given a positive integer \( n \), a Boolean function on \( n \) variables is a function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \). The function is called monotone if for all \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \{0, 1\}^n \) satisfying \( x_i \leq y_i \) for each \( 1 \leq i \leq n \), we have \( f(x) \leq f(y) \).

For an \( n \)-variable Boolean function \( f \), the influence of the \( i \)-th variable on \( f \) is defined to be

\[
I_i(f) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |f(x + e_i) - f(x)|,
\]

where \( e_i \) denotes the element of \( \{0,1\}^n \) whose only non-zero coordinate is in the \( i \)-th position, and addition is done coordinate-wise modulo two. The total influence of \( f \) is defined to be

\[
I(f) = \sum_{i=1}^n I_i(f).
\]

For the proof of our result it will be convenient to introduce the following definition: Given \( y \in \{0,1\}^n \) we define the influence of \( y \) on \( f \) to be

\[
I_y(f) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |f(x + y) - f(x)|.
\]

[We remark that if we consider the correspondence between the elements of \( \{0,1\}^n \) and the subsets of \( \{1, \ldots, n\} \) then the influence of \( y \) on \( f \) is not the same as the usual definition of the influence of the set \( Y \) (corresponding to \( y \)) over \( f \). Since we will not be using the latter definition, we hope that no confusion arises.]

The notion of influence of a variable on a Boolean function was introduced by Ben-Or and Linial [1]. It has since found many application in discrete mathematics, theoretical computer science and social choice theory. We refer the reader to [2] for a survey of some of these applications. In this note we study the effect on the influence after applying an invertible linear transformation on a monotone Boolean function.

Given an \( n \)-variable Boolean function \( f \) and an invertible linear transformation \( L \in GL_n(\mathbb{F}_2) \), the function \( Lf \) is defined by \( Lf(x) = f(Lx) \). In [3] Keller and Pilpel raised the following conjecture.

Conjecture 1 (Keller and Pilpel [3]). If \( f \) is an \( n \)-variable monotone Boolean function and \( L \in GL_n(\mathbb{F}_2) \) then \( I(f) \leq I(Lf) \).

We prove this conjecture in the next section.

Date: September 30, 2009.

2000 Mathematics Subject Classification. 05D05; 06E30.

Key words and phrases. Discrete cube; Boolean functions; Influence.

Supported by the EPSRC, grant no. EP/E02162X/1.
2. Proof of the conjecture

To prove the conjecture we will use the following simple combinatorial lemma. We will prove the lemma using the well-known Hall’s marriage theorem. One can obtain short proofs of the lemma using other equivalent statements.

**Lemma 2.** Let $L \in GL_n(\mathbb{F}_2)$. Then we can permute the columns of $L$ to obtain a new matrix $L'$ whose diagonal entries are non-zero.

**Proof.** Let us define a bipartite graph $B$ on $\{r_1, \ldots, r_n\} \times \{c_1, \ldots, c_n\}$ by joining $r_i$ to $c_j$ if and only if $L_{ij} = 1$. It is enough to prove that $B$ contains a perfect matching. Indeed, if $r_i$ is matched to $c_{\pi(i)}$, then $\pi^{-1}$ provides the required permutation of the columns. The existence of this perfect matching is an immediate consequence of Hall’s marriage theorem. Indeed, if this is not the case, then there is a set $R$ of $k$ rows and a set $C$ of $\ell < k$ columns such that every row of $R$ has a non-zero entry only in a column of $C$. But since $\ell < k$, the rows in $R$ are linearly dependent contradicting the fact that $L$ is invertible. □

It is immediate that if $L'$ is obtained from $L$ by permuting its columns then $I(Lf) = I(L'f)$. Indeed, if $L' = LP$ where $P$ is the permutation matrix which maps $e_i$ to $e_{\pi(i)}$, then $I_i(L'f) = I_{\pi(i)}(Lf)$ and so the total influences are equal. Thus to prove the conjecture we may assume by the previous lemma that each diagonal entry of $L$ is non-zero. In this case, we will prove the stronger assertion that $I_i(f) \leq I_i(Lf)$ for each $1 \leq i \leq n$. We claim that $I_i(Lf) = I_{Le_i}(f)$. Indeed,

\[
I_i(Lf) = \frac{1}{2^n} \sum_x |Lf(x + e_i) - Lf(x)| = \frac{1}{2^n} \sum_x |f(Lx + Le_i) - f(Lx)| = \frac{1}{2^n} \sum_y |f(y + Le_i) - f(y)| = I_{Le_i}(f).
\]

Splitting the sum in the definition of $I_{Le_i}(f)$ into two parts depending on whether the $i$-th coordinate is equal to zero or not we obtain that

\[
I_{Le_i}(f) = \frac{1}{2^n} \sum_y |f(y + Le_i) - f(y)| = \frac{1}{2^n} \left( \sum_{\{y : y_i = 0\}} |f(y + Le_i) - f(y)| + \sum_{\{y : y_i = 1\}} |f(y) - f(y + Le_i)| \right) = \frac{1}{2^n} \sum_{\{z : z_i = 0\}} (|f(z + Le_i) - f(z)| + |f(z + e_i) - f(z + e_i + Le_i)|)
\]

\[
\geq \frac{1}{2^n} \sum_{\{z : z_i = 0\}} |f(z + e_i) + f(z + Le_i) - f(z) - f(z + e_i + Le_i)|.
\]

Observe that since each diagonal entry of $L$ is non-zero, the $i$-th coordinate of $Le_i$ is equal to one and so if the $i$-th coordinate of $z$ is zero, then the $i$-th coordinate of $z + e_i + Le_i$ is also zero and so by the monotonicity of $f$ we have $f(z) \leq f(z + e_i)$ and $f(z + e_i + Le_i) \leq f(z + Le_i + e_i) + f(z + e_i) - f(z + e_i + Le_i)$. Therefore, $I_i(Lf) \leq I_{Le_i}(f)$.
It follows that
\[ I_{Le_i}(f) \geq \frac{1}{2^n} \sum_{\{z: z_i = 0\}} |f(z + e_i) + f(z + Le_i) - f(z) - f(z + e_i + Le_i)| \]
\[ = \frac{1}{2^n} \sum_{\{z: z_i = 0\}} |f(z + e_i) - f(z)| + \frac{1}{2^n} \sum_{\{z: z_i = 0\}} |f(z + Le_i) - f(z + e_i + Le_i)| \]
\[ = \frac{1}{2^n} \sum_{\{z: z_i = 0\}} |f(z + e_i) - f(z)| + \frac{1}{2^n} \sum_{\{w: w_i = 1\}} |f(w) - f(w + Le_i)| \]
\[ = I_i(f), \]
as required. This completes the proof of Conjecture 1.

References

[1] Ben-Or and N. Linial, Collective coin flipping, in Randomness and computation, Academic Press 1990, 91–115.
[2] G. Kalai and S. Safra, Threshold phenomena and influence: perspectives from mathematics, computer science, and economics, in Computational complexity and statistical physics, Oxford Univ. Press 2006, 25–60.
[3] N. Keller and H. Pilpel, Linear transformations of monotone functions on the discrete cube, Discrete Math. 309 (2009), 4210–4214.

E-mail address: christod@maths.bham.ac.uk

School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK