Parameter estimation of Nash IUH for multiple storm events using particle swarm optimization method

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Abstract. In present study, we proposed to optimize the parameters of Nash Instantaneous Unit Hydrograph using particle swarm optimization method. The application of particle swarm optimization method made the optimal parameter estimation for multiple storm events possible. The effectiveness of the proposed method was tested by published data sets. Compared with the reported results, the simulation accuracy in terms of peak magnitude and peak time is all improved, and the convergence rate is significantly improved, which demonstrates that the superiority of particle swarm method in parameter optimization of Nash IUH model.

1. Introduction

The unit hydrograph (UH) concept (Sherman et al., 1932) is one of the most practical ways in estimating the direct runoff responses to the excess rainfall (i.e., total rainfall minus rainfall loss). Instantaneous Unit Hydrograph (IUH) overcomes the shortcomings of UH on changing hydrograph duration, and provides an objective and rigorous framework for estimating unit hydrographs of any desired duration. The IUH is the direct runoff hydrograph resulting from unit excess rainfall applied uniformly over a watershed for short duration. IUH summarizes both runoff generation and flow routing processes based on the complex characteristics of drainage basin, which is favorable for the theoretic research and regional synthetic analysis of rainfall-runoff relationship, and of the importance in the computation of water resources engineering on flood forecast and prevention. The Nash IUH model offers a theoretically sound and practical approach to estimate unit hydrographs for a wide variety of watersheds (Boufadel, 1998).

The Nash IUH has been considered as the better one to any other IUH method (Mohan and Vijayalakshmi, 2008). One important step to apply Nash IUH is to determine its parameters. There are many methods available to estimate these parameters. Nash (1957) proposed a cascade of several linear reservoirs as a model to derive IUH for natural watersheds, with two parameters of gamma distribution function ‘n’ and ‘k’, where ‘n’ is the shape parameter and ‘k’ is the scale parameter. The assumption here is that the catchment is made up of a series of ‘n’ identical linear reservoirs each having the same storage constant ‘k’. There are a number of methods have been applied to estimate the Nash IUH parameters, including conventional parameter methods (Singh, 1988), such as moment method, least square error, maximum likelihood, gradient search method (Ma et al., 2014), and artificial intelligent
methods, such as genetic algorithm (Jin et al., 2003). Nourani (2008) reported that genetic algorithm was more accurate in estimating Nash IUH parameters for the Ammameh Watershed in Central Iran. All the targets of these methods are to search for the IUH parameters in better reflecting the basin confluence characteristics. Boufadel (1998) stated that nonlinear constrained optimization provides better estimations of the IUH parameters than the method of moments. The evaluation of an optimum method could be an instructive study for real hydrological applications.

Here we proposed to optimize the parameters of Nash IUH model using Particle Swarm Optimization (PSO) method. Using runoff data from several published data sets, this study compares the performance of PSO on parameter estimations of Nash IUH against those obtained by other means.

2. Method

2.1 Nash Instantaneously Unit Hydrograph (IUH)

The Nash Instantaneously Unit Hydrograph (IUH) model is defined as:

\[ u(t) = \frac{1}{kt(n)} \Gamma \left( \frac{t}{k} \right) \exp \left( -\frac{t}{k} \right) \]

Where \( t \) is time, \( u(t) \) is IUH value corresponds to \( t \); \( n \) and \( k \) are two model parameters which stand for the number of reservoirs and the spread time of linear reservoirs respectively; \( \Gamma(n) \) is the gamma function of \( n \); \( e \) is the base-number of Napierian logarithm. In practical engineering application, the IUH \( u(t) \) is firstly transformed into the \( S(t) \) curve,

\[ S(t) = \int_0^t u(t) dt = \int_0^t \frac{1}{kt(n)} \Gamma \left( \frac{t}{k} \right) \exp \left( -\frac{t}{k} \right) dt = \frac{1}{\Gamma(n)} \int_0^t \Gamma \left( \frac{t}{k} \right) \exp \left( -\frac{t}{k} \right) d\left( \frac{t}{k} \right) \]

To obtain the \( S(t) \) curve, the look-up table method or approximate formula method are frequently employed. However, both of two methods will introduce errors (Geng and Qi, 2003). The GAMMADIST\((x, \alpha, \beta, \gamma)\) function in Excel is used to resolve \( S(t) \) curve, in which the \( x \) is the calculated Gamma distribution value, \( \alpha \) and \( \beta \) are the distribution parameters, and \( \gamma \) is a logical value. If false is obtained for \( \gamma \), the GAMMADIST returns the probability density function:

\[ f(x, \alpha, \beta) = \frac{1}{\beta^n \Gamma(n)} x^{n-1} \exp \left( -\frac{x}{\beta} \right) \]

If true, GAMMADIST returns cumulative distribution function:

\[ F(x, \alpha, \beta) = \int_0^x f(x, \alpha, \beta) dx = \int_0^x \frac{1}{\beta^n \Gamma(n)} x^{n-1} \exp \left( -\frac{x}{\beta} \right) dx \]

Combing the equations (2) and (4),

\[ S(t) = \text{GAMMADIST}(t / k, n, 1, \text{true}) \]

Where \( n \) and \( k \) are Nash IUH model parameters. Then, the \( S(t) \) is further converted to nondimensional time step unit hydrograph \( u(\Delta t, t) \), and the dimensional time step unit hydrograph \( q(\Delta t, t) \) is finally obtained from \( u(\Delta t, t) \).

2.2 Particle Swarm Optimization

If we assume the searching space is in \( K \) dimensions, the population size of the swarm is \( L \), then the position of the \( i \)th swarm particle is expressed as a \( K \)-dimensional vector

\[ X_i = (X_{i1}, X_{i2}, \ldots, X_{ik}) \]

The best position of the \( i \)th swarm particle is

\[ P_{best_i} = (P_{best_{i1}}, P_{best_{i2}}, \ldots, P_{best_{ik}}) \]

The best position obtained so far is given as

\[ G_{best} = (G_{best_{1}}, G_{best_{2}}, \ldots, G_{best_{K}}) \]

The “fly” velocity of the \( i \)th swarm particle is written as
\[ V_i = (V_{i1}, V_{i2}, \cdots, V_{ik}) \]  

The position of the swarm particle is updated using the following iteration equation:

\[ V_{ik}^{t+1} = \omega V_{ik}^t + c_1 r_1 (P_{besh}^t - V_{ik}^t) + c_2 r_2 (G_{besh}^t - V_{ik}^t) \]  

\[ X_{ik}^{t+1} = X_{ik}^t + V_{ik}^{t+1} \]

Where \( i=1, \ldots, L \), \( k=1, \ldots, K \), and \( t \) is the iteration number. \( \omega \) is the inertial weight set to in the range of \([0.9, 1.2]\) (Shi and Eberhart, 1998a,b), and decrease with \( t \). \( r_1 \) and \( r_2 \) is the random numbers in the range of \([0, 1]\). \( c_1 \) and \( c_2 \) are the constants as the cognitive and social parameters respectively, and the sum of them is less than or equal to 4 is proposed by Kennedy (1998) and Carlisle and Dozier (2001) to have reasonable compromise between local and global search regions.

2.3 Optimization formulation

In practical flood forecasting, it requires very high accuracy on the magnitude and occurrence time of peak flood. Therefore, the objective of our study is to minimize the differences between observed flood evolution and the stream flow simulated by Nash IUH. Meanwhile, we will perform the optimization processes for multi-storm events simultaneously. The optimization can be given as:

\[ F(n,k) = \begin{cases} 
\min f(n,k) = \sum_{j=1}^{M} \sum_{i=1}^{Nj} w(i,j) |Q(i,j) - Q_o(i,j)| & i = 1,2, \ldots, N \\
\theta(i,j) = Q(i,j)/\sum_{i=1}^{Nj} Q_o(i,j) 
\end{cases} \]

Where \( M \) is the number of storm events involved, \( Nj \) is the number of time steps in the jth storm event, \( Q(i,j) \) is the calculated stream flow using IUH, \( Q_o(i,j) \) is the observations, and \( w(i,j) \) is the weight function. The optimization is difficult to be reached by traditional optimization method. In this study, we proposed to optimize it by employing the PSO method.

The algorithmic steps of PSO for Nash IUH model parameters are outlined as follows:

1. Initialize the population of particles \( X_i \) by assigning random positions in the parameter ranges of the Nash IUH model parameters, and set the initial velocities to the particles.
2. Obtain the fitted value of each particle for the functions by running the Nash IUH model and routing scheme with the parameter values in step (1).
3. Initialize the best position \( P_{besh} \) of the ith particle swarm, and their global best position \( G_{besh} \).
4. Create a storage archive, and select the nondominated particles in \( P \) and create a copy using their location information.
5. Compute the crowding distance of each nondominated particle, and rank it with the decrease sequence.
6. Select ten percent of nondominated solutions as \( G_{besh} \), and update the particle position and velocity using equation (9) and (10) respectively. In case the particle touches the space edge, the opposite direction of velocity is exerted.
7. Run the Nash IUH model to get the fitted values for objections till the max loop number is reached.

3. Application

3.1 Data

The proposed method was applied on several sets of multi-storm data consisted of sets of effective rainfall and direct runoff. Set 1 includes 8 storms, which was taken from Diskin and Boneh (1975). Set 2 includes 4 storms, which was taken from Singh (1976). Set 3 includes 22 storms, which was taken from Bree (1978).

3.2 Parameter optimization using ideal data

To eliminate the uncertainty produced by model structure, we examine the issue that the similar results may be generated by different parameters, by using ideal data. The ideal data is the model generated data.
with a set of forcing data with the fixed parameters. For the Nash IUH model studied, we set the values of \( N \) and \( K \), and drive the model with effective rainfall processes, and the model output is so-called ideal runoff processes.

Then, we use the ideal runoff data as target to calibrate the model parameters, to see whether the calibrated value is the same with the previously fixed \( N \) and \( K \) value. The purpose of this step is to verify the proposed method can avoid equifinality and obtain the unique set of optimal parameter values.

Fig.1 The object function (upper panel) and model parameters (lower panel) during the iteration procedures for (a) 8 storm events in Diskin and Boneh (1975), (b) 4 storm events in Singh (1976), and (c) 21 storm events in Bree (1978).

For Set 1 data, the assumed parameter value for \( N \) and \( K \) is 3.56 and 20.78 respectively. The 8 ideal runoff hydrographs are obtained with the 8 storms in Set 1 and the assumed values of \( N \) and \( K \). The detailed optimization processes can be observed in Fig.1(a), which shows that the \( N \) and \( K \) converges the assumed value after 55 model loops. The similar procedures were performed for Set 2. The assumed parameter values of \( N=2.56 \), and \( K=8.23 \) is achieved after 48 loops with 4 storms and the related runoff hydrographs, as shown in Fig.1(b). Fig.1(c) demonstrates the calibration processes for Set 3 by obtaining the steady parameter values of \( N=3.76 \) and \( K=5.82 \) with 56 loops.

### 3.3 Parameter optimization using real data

**Case 1**: For Set 1 data, the drainage area is 349km\(^2\), and the time step is daily. We optimize and evaluate the parameters of IUH based on 8 storm events, and compare the optimization results of the proposed method with PSO, the method in literature (Diskin and Boneh, 1975), and the method in literature (Singh, 2006). The optimized parameters obtained \( N \) and \( K \) are 3.36 and 22.41 respectively.

The calculated peak magnitude and peak time are listed in Table 1, comparing with the results reported in the literatures. For the peak time estimated, the results are comparable for the three methods performed. It is observed that, the simulated errors for 6 storms (i.e., Storm 3-8) are significantly minimized by the proposed method. And the relative errors for the 8 storm events are all less than 20%.
TABLE 1 Error in Peak and Time to Peak of Case 1

| Event | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| **Observed** |       |       |       |       |       |       |       |
| Peak (mm/day) | 40.64 | 29.97 | 21.08 | 13.97 | 12.7  | 11.68 | 18.54 | 12.7  |
| Time to peak (day) | 4     | 5     | 6     | 4     | 4     | 5     | 3     | 5     |
| **Simulated (Diskin and Boneh, 1975)** |       |       |       |       |       |       |       |
| Peak (mm/day) | 34.63 | 25.9  | 22.1  | 15.21 | 14.68 | 14.97 | 21.69 | 13.8  |
| Time to peak (day) | 4     | 6     | 6     | 4     | 3     | 5     | 3     | 5     |
| Relative errors on Peak(%) | -14.8 | -13.6 | 4.8   | 8.9   | 15.6  | 28.2  | 17.0  | 8.7   |
| **Simulated (Singh, 2006)** |       |       |       |       |       |       |       |
| Peak (mm/day) | 33.65 | 25.24 | 21.65 | 14.62 | 14.14 | 14.4  | 20.68 | 13.16 |
| Time to peak (day) | 4     | 6     | 6     | 4     | 3     | 5     | 3     | 5     |
| Relative errors on Peak(%) | -17.2 | -15.8 | 2.7   | 4.7   | 11.3  | 23.3  | 11.5  | 3.6   |
| **Simulated (present study)** |       |       |       |       |       |       |       |
| Peak (mm/day) | 32.7  | 24.57 | 21.06 | 14.18 | 13.71 | 13.99 | 20.04 | 12.77 |
| Time to peak (day) | 4     | 6     | 6     | 4     | 3     | 5     | 3     | 5     |
| Relative errors on Peak(%) | -19.5 | -18.0 | -0.1  | 1.5   | 8.0   | 19.8  | 8.1   | 0.6   |

**Case 2:** For Set 2 data, the N and K values are optimized from 4 storm events, and are used to calculate the unit hydrograph and runoff processes. The N and K, calculated by the proposed method are 3.11 and 6.22 respectively.

Fig.2 shows that the UHs computed by Linear Programming and Least squares in the literature (Singh, 1976) and the method in literature (Singh, 2006) for each storm event. For the first and fourth storm events, the negative value appeared for UH calculated by Least Squares, and the UHs were not smooth using both Linear Programming and Least Squares. The UHs, calculated in the study of Singh (2006) and by the proposed method, have non-negative value and smooth processes.

Table 2 shows the predicted runoff processes using the Linear Programming, Least Squares, the method in the literature(Singh,2006), and the proposed method. Regarding to peak time, the proposed method improves the simulation results for the second storm event. The relative error for the first storm event is within 20%, and within 5% for other 3 storm events.

**Case 3:** For Set 3 data, the parameters of UH are optimized by 21 storm events reported in Bree (1978), and the N and K are 1.94 and 8.83 respectively. The statistical results for the twenty-second storm in Table 3 show that the accuracy of the proposed method is higher than the results in both Singh (2006) and Bree (1978).
Fig. 2 Unit Hydrograph derived for the 4 storms in Singh (1976).

**TABLE 2** Error in Peak and Time to Peak of Case 2

| Event | 1       | 2       | 3       | 4       |
|-------|---------|---------|---------|---------|
|       | Observed Peak (mm/h) | 3.56 | 2.24 | 0.65 | 2.26 |
|       | Time to peak (h) | 20 | 24 | 20 | 36 |
|       | Simulated (Linear Programming in Singh, 1976) Peak (mm/h) | 3.55 | 2.23 | 0.65 | 2.49 |
|       | Time to peak (h) | 24 | 24 | 20 | 48 |
|       | Relative errors on Peak (%) | -0.3 | -0.4 | 0 | 10.2 |
|       | Simulated (Least Squares in Singh, 1976) Peak (mm/h) | 3.61 | 2.24 | 0.65 | 2.28 |
|       | Time to peak (h) | 24 | 24 | 20 | 44 |
|       | Relative errors on Peak (%) | 1.4 | 0 | 0 | 0.9 |
|       | Simulated (Singh, 2006) Peak (mm/h) | 3.09 | 2.32 | 0.66 | 2.42 |
|       | Time to peak (h) | 24 | 20 | 20 | 44 |
|       | Relative errors on Peak (%) | -13.2 | 3.6 | 1.5 | 7.1 |
|       | Simulated (present study) Peak (mm/h) | 2.93 | 2.21 | 0.63 | 2.37 |
|       | Time to peak (h) | 24 | 24 | 20 | 44 |
|       | Relative errors on Peak (%) | -17.7 | -1.3 | -3.1 | 4.9 |

**TABLE 3** Error in Peak and Time to Peak of Case 3

| P     | T     | P     | T     | RP    | P     | T     | RP    | P     | T     | RP    |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 22.1  | 18    | 22.9  | 21    | 3.62  | 21.3  | 21    | -3.62 | 22    | 18    | -0.45 |

**Remarks:** P-Peak (m$^3$/s), T-Time to peak (h), RP-Relative errors on Peak (%)
4. Conclusions
This paper proposed a PSO-based method to optimize the parameters of Nash IUH model for multi-storm events. The effectiveness and accuracy are illustrated and demonstrated by several real data sets reported in the literature. The calculating results, for both unit hydrograph and runoff processes, are compared with the reported results from other methods, such as moment method, approximate method, and genetic algorithm toolbox. The proposed method shows its good capability to calculate instantaneous unit hydrograph for surface runoff process. Through the validation by the ideal data, the proposed method can obtain the unique parameter set and prevent to have same results from different parameter sets. Moreover, the relationship of S(t) curve of instantaneous unit hydrograph and the statistical function GAMMADIST(x, α, β, γ) was analyzed through the comparisons. The function GAMMADIST(x, α, β, γ) is helpful to avoid errors induced by integral formula method.

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