CONSTRUCTIONS OF ASYMPTOTICALLY OPTIMAL CODEBOOKS WITH RESPECT TO WELCH BOUND AND LEVENShtein BOUND

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(Communicated by Sihem Mesnager)

Abstract. Codebooks with small maximum cross-correlation amplitudes are used to distinguish the signals from different users in code division multiple access communication systems. In this paper, several classes of codebooks are introduced, whose maximum cross-correlation amplitudes asymptotically achieve the corresponding Welch bound and Levenshtein bound. Specially, a class of optimal codebooks with respect to the Levenshtein bound is obtained. These classes of codebooks are constructed by selecting certain rows deterministically from circulant matrices, Fourier matrices and Hadamard matrices, respectively. The construction methods and parameters of some codebooks provided in this paper are new.

1. Introduction

Codebooks with maximal cross-correlation amplitude achieving the Welch bound and Levenshtein bounds are desirable in code division multiple access (CDMA) communication systems ([25], [20]). Precisely, an \((N,K)\) codebook \(\mathbb{C}\) is a set \(\{c_0, c_1, \cdots, c_{N-1}\}\) of \(N\) unit norm \(1 \times K\) complex-valued vectors over an alphabet \(A\). The size of alphabet \(A\) is called the alphabet size of the codebook \(\mathbb{C}\).

As a performance measure of a codebook in practical applications, the maximal cross-correlation amplitude of an \((N,K)\) codebook \(\mathbb{C}\), denoted by \(I_{\text{max}}(\mathbb{C})\), is defined by

\[
I_{\text{max}}(\mathbb{C}) = \max_{0 \leq i \not= j \leq N-1} |c_i c_j^H|,
\]

2020 Mathematics Subject Classification: Primary: 94B15; Secondary: 11T71.

Key words and phrases: Codebooks, maximal cross-correlation amplitude, Welch bound, Levenshtein bound, circulant matrices, Fourier matrices, Hadamard matrices, character sums.

G. Wang is supported by the Doctoral Foundation of Tianjin Normal University (Grant No. 52XB2014). F.-W Fu is supported by the National Key Research and Development Program of China (Grant No. 2018YFA0704703), the National Natural Science Foundation of China (Grant No. 61971243), the Natural Science Foundation of Tianjin (20JCZDJC00610), the Fundamental Research Funds for the Central Universities of China (Nankai University), and the Nankai Zhiding Foundation.

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where $c_j^H$ is the conjugate transpose of the complex vector $c_j$. In practical applications, such $(N, K)$ codebooks $C$ that $N$ is large and $I_{\text{max}}(C)$ is small are desirable for a fixed $K$. However, the well-known Welch bound gives the lower bound for $I_{\text{max}}(C)$.

**Lemma 1.1.** [34] For any $(N, K)$ codebook $C$ with $N > K$,

$$I_{\text{max}}(C) \geq I_W = \sqrt{\frac{N - K}{(N - 1)K}},$$

where the equality holds if and only if

$$|c_i c_j^H| = \sqrt{\frac{N - K}{(N - 1)K}}$$

for any $0 \leq i, j \leq N - 1$, $i \neq j$.

A codebook whose maximal cross-correlation amplitude achieves the equality of the Welch bound is called a maximum-Welch-bound-equality (MWBE) codebook. The MWBE codebooks have applications in many fields, such as CDMA communications ([25]), space-time codes ([30]), compressed sensing ([19]) and so on ([7], [4], [5]). Constructing MWBE codebooks is very hard in general, as pointed out by Sarwate ([27]). To our knowledge, only the following classes of MWBE codebooks are provided in the previous literatures.

- The $(N, N)$ orthogonal codebooks and $(N, N - 1)$ MWBE codebooks with $N > 1$ from the discrete Fourier transform matrices or $m$-sequences ([36]);
- The $(N, K)$ MWBE codebooks from conference matrices, where $N = 2K = 2^{d+1}$ with a positive integer $d$ or $N = 2K = p^d + 1$ with a prime $p$ and a positive integer $d$ ([28]);
- The $(N, K)$ MWBE codebooks from $(N, K, \lambda)$ difference sets in cyclic groups and abelian groups ([6]);
- The $(N, K)$ MWBE codebooks from $(2, k, \nu)$-Steiner systems ([10]);
- The $(N, K)$ MWBE codebooks from graph theory and finite geometries ([9]).

Hence, there have been a number of attempts to construct asymptotically optimal codebooks, i.e., $I_{\text{max}}(C)$ of the codebooks asymptotically achieves the Welch bound for sufficiently large $K$. The asymptotically optimal codebooks are constructed mainly based on combinatorial design ([42], [18]), character sums ([12], [22], [26]), and sequences ([39]). For the sake of comparisons, Table 1 lists the parameters of the known asymptotically optimal codebooks with respect to Welch bound in the existed literatures of recent years.

The Welch bound on the maximum cross-correlation amplitude of codebooks cannot be achieved when $N > \frac{K(K+1)}{2}$ for real codebooks and $N > K^2$ for all codebooks ([28]). The following Levenshtein bounds turn out to be tighter than Welch bound in the case that $N$ is large.

**Lemma 1.2.** [17] For any real-valued $(N, K)$ codebook $C$ with $N > \frac{K(K+1)}{2}$, we have

$$I_{\text{max}}(C) \geq I_L = \sqrt{\frac{3N - K^2 - 2K}{(K + 2)(N - K)}}.$$
| Parameters $(N,K)$ | $I_{\text{max}}$ | Constraints | Ref. |
|-------------------|-----------------|-------------|-----|
| $(p^n, K = \frac{p^n - 1}{p-1}(p^n + p^{\frac{n}{2}}) + 1)$ | $\frac{(p+1)p^\frac{n}{2}}{2pn}$ | $p$ is an odd prime | 14 |
| $(q^2, \frac{(q-1)^2}{2})$ | $\frac{q+1}{q-1}$ | $q$ is an odd prime power | 40 |
| $(q(q+4), \frac{(q+3)(q+1)}{2})$ | $\frac{1}{q+1}$ | $q$ and $q+4$ are two prime powers | 18 |
| $(q, \frac{q^n-1}{2})$ | $\sqrt{q+1}$ | $q$ is a prime power | 18 |
| $(q^l + q^{l-1} - 1, q^{l-1})$ | $\frac{1}{q-1}$ | $q$ is a prime power and $l > 2$ | 42 |
| $((q-1)^k + q^{k-1}, q^{k-1})$ | $\frac{\sqrt{q^{k+1}}}{q}$ | $q \geq 4$ is a prime power and $k > 2$ | 12 |
| $((q-1)^k + K, K)$ | $\frac{\sqrt{q^{k+1}}}{K}$ | $q$ is a prime power and $k > 2$ | 12 |
| $((q^s-1)^n + K, K)$ | $\frac{\sqrt{q^{sn+n-1}}}{q}$ | $q$ is a prime power, $s > 1, n > 1$ | 22 |
| $(q^n + q^s - q, q^s - q)$ | $\frac{1}{q-1}$ | $q$ is a prime power | 21 |
| $(q^n - q, q^s - q)$ | $\frac{1}{q-1}$ | $q$ is a prime power | 21 |
| $(q^n - 2q + 1, (q-1)^s)$ | $\frac{1}{(q-1)^2}$ | $q$ is a prime power | 21 |
| $((p_{\text{min}} + 1)Q^2, Q^2)$ | $\frac{1}{Q}$ | $Q > 1$ is an integer and $p_{\text{min}}$ is the smallest prime factor of $Q$ | 32 |
| $((p_{\text{min}} + 1)Q^2 - Q, Q(Q - 1))$ | $\frac{1}{Q-1}$ | $Q > 2$ is an integer and $p_{\text{min}}$ is the smallest prime factor of $Q$ | 32 |
| $(N_1, N_2, \frac{N_1N_2-1}{2})$ | $\sqrt{(N_1+1)(N_2+1)}$ | $N_1 \equiv 3 \text{ mod } 4$ and $N_2 \equiv 3 \text{ mod } 4$ | 15 |
| $(N_1 \cdots N_l, \frac{N_1 \cdots N_l-1}{2})$ | $\sqrt{(N_1+1) \cdots (N_l+1)}$ | $N_i \equiv 3 \text{ mod } 4$ for any $l \geq 1$ | 15 |
| $(2K + 1, K)$ | $\frac{2^{s_1s_2n} \cdots s_{l}n}{2^{s_1s_2n} \cdots s_{l}n - 1}$ | $n \geq 1, s_1, s_2 > 1$ | 23 |
| $(2K + (-1)^{m_1}, K)$ | $\frac{2^{s_1s_2n} \cdots s_{m}n}{2^{s_1s_2n} \cdots s_{m}n - 1}$ | $n \geq 1, l > 1$ and $s_i > 1$ for any $1 \leq i \leq m$ | 23 |
| $(kp^n + p^s, p^s)$ | $\frac{1}{p}$ | $p$ is a prime and $k | (p+1)$ | 24 |
| $(p^n - 1, \frac{p^n-1}{2})$ | $\frac{p^n-1}{p-1}$ | $p$ is an odd prime | 39 |
Table 2. The parameters of codebooks asymptotically achieving Levenshtein bound.

| Parameters $(N,K)$ | $I_{\text{max}}$ | Constraints | Ref. |
|-------------------|------------------|-------------|------|
| $(2^{2m} + 2^m, 2^m)$ | $\sqrt{2^{2m}}$ | $m$ is a positive integer | [37] |
| $(q^2 - 1, q - 1)$ | $\sqrt{q}$ | $q$ is a prime power | [29] |
| $(q^2 - q - 1, q - 2)$ | $\sqrt{q}$ | $q$ is a prime power | [12] |
| $(q^2 + q - 1, q)$ | $\sqrt{q}$ | $q$ is a prime power | [42] |

For any complex-valued $(N,K)$ codebook $C$ with $N > K^2$, we have

$$I_{\text{max}}(C) \geq I_L = \sqrt{\frac{2N - K^2 - K}{(K+1)(N-K)}}.$$ 

In general, it is very difficult to construct a codebook achieving the Levenshtein bounds. To the best of our knowledge, the constructions of codebooks that can achieve the Levenshtein bound are presented from binary Kerdock codes, perfect nonlinear functions, sets of bent functions, $\mathbb{Z}_4$-Kerdock codes and $\mathbb{Z}_4$-valued quadratic forms. All the known codebooks achieving the Levenshtein bound are presented in the following.

- Optimal $(2^{2m-1} + 2^m, 2^m)$ codebooks with alphabet size 4 generated from the binary Kerdock codes for even $m$ ([35]).
- Optimal $(p^{2m} + p^m, p^m)$ codebooks with alphabet size $p + 2$ given from perfect nonlinear functions for odd prime $p$ ([8]).
- Optimal $(2^{2m-1} + 2^m, 2^m)$ codebooks with alphabet size 4 for even $m$ and optimal $(p^{2m} + p^m, p^m)$ codebooks with alphabet size $p + 2$ for odd prime $p$ constructed from sets of bent functions ([41]).
- Optimal $(2^{2m-1} + 2^m, 2^m)$ codebooks with alphabet size 4 given from subcodes of binary Kerdock codes for even $m$ ([37]).
- Optimal $(2^{2m} + 2^m, 2^m)$ complex codebooks with alphabet size 6 derived from $\mathbb{Z}_4$-Kerdock codes ([2]).
- Optimal $(2^{2m} + 2^m, 2^m)$ codebooks with alphabet size 6 obtained from $\mathbb{Z}_4$-valued quadratic forms ([13]).

In addition, constructions of codebooks asymptotically achieving the Levenshtein bounds are also provided, which are mainly based on characters of finite fields and binary codes. Table 2 lists the parameters of all the known codebooks asymptotically achieving the Levenshtein bounds in the existed papers.

In [39], the authors described a framework for constructing a codebook by selecting certain rows from a matrix associated with a binary sequence. Through the framework, the authors could provide a freedom of choosing a variety of binary sequences to construct new partial Fourier and Hadamard codebooks $C$ with nontrivially bounded $I_{\text{max}}(C)$. Motivated by this paper, several classes of asymptotically optimal codebooks with respect to Welch bound and Levenshtein bound are constructed by giving a new choosing method of certain rows from deterministic circulant matrices, Fourier matrices and Hadamard matrices, respectively. Specially, a class of optimal codebooks with respect to Levenshtein bound is obtained. The construction methods and parameters of some codebooks provided in this paper are new.
The rest of the paper is organized as follows. In Section 2, some preliminaries are given which will be needed in subsequent sections. In Section 3, several classes of asymptotically optimal codebooks with respect to Welch bound and Levenshtein bound are constructed. In Section 4, we conclude this paper.

2. Preliminaries

Suppose that $F_q$ is a finite field with $q$ elements, where $q = p^n$ is a prime power and $n$ is a positive integer. Define the trace function $Tr^n_1(\cdot)$ from the finite field $F_q$ to the finite field $F_p$ as

$$Tr^n_1(x) = x + x^p + \cdots + x^{p^{n-1}}, x \in F_q.$$ 

For $a \in F_q$, an additive character of $F_q$ can be described by

$$\chi_a(x) = \zeta_p^{Tr^n_1(ax)}, x \in F_q,$$

where $\zeta_p = e^{2\pi i/p}$. When $a = 0$, $\chi_0(x) = 1$ for all $x \in F_q$ and is called the trivial additive character of $F_q$. when $a = 1$, $\chi_1$ is called the canonical additive character of $F_q$. The orthogonal relation of additive characters of $F_q$ is

$$\sum_{x \in F_q} \chi_a(x) = \begin{cases} q, & \text{if } a = 0; \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $g$ is a generator of the multiplicative cyclic group $F_q^* = F_q \setminus \{0\}$, i.e., $F_q^* = \langle g \rangle$. For $j = 0, 1, \ldots, q - 2$, a multiplicative character of $F_q$ can be described by

$$\psi_j(g^k) = \zeta_q^{jk},$$

where $\zeta_q = e^{2\pi i/q}$ and $k = 0, 1, \ldots, q - 2$. When $j \neq 0$, $\psi_j$ is called the nontrivial multiplicative character of $F_q$. The orthogonal relation of multiplicative characters of $F_q$ is

$$\sum_{x \in F_q^*} \psi_j(x) = \begin{cases} q - 1, & \text{if } j = 0; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $F_q$ can be regarded as an additive subgroup of $F_{q^n}$. For a nontrivial multiplicative character of $F_{q^n}$, Katz [16] presented the magnitude of summation of the character values over a special coset of $F_q$.

Lemma 2.1. [16] For a nontrivial multiplicative character $\psi_j$ of $F_{q^n}$ and an element $\alpha$ of $F_{q^n}$ with $F_{q^n} = F_q(\alpha)$, there is an estimation

$$\left| \sum_{t \in F_q} \psi_j(t - \alpha) \right| \leq (n - 1)\sqrt{q}.$$ 

In the case of quadratic extension of the fields, a more precise estimation of Katz sum is given by Xu and Xu [38].

Lemma 2.2. [38] Suppose that $\psi_j$ is a nontrivial multiplicative character of $F_{q^2}$. For an element $\alpha$ of $F_{q^2}$ with $F_{q^2} = F_q(\alpha)$, we have

$$\left| \sum_{t \in F_q} \psi_j(t - \alpha) \right| = \sqrt{q}, \quad \text{if } (q - 1) \nmid j,$$
\[ \sum_{t \in \mathbb{F}_q} \psi_j(t - \alpha) = -1, \quad \text{if } (q - 1)|j. \]

Bombieri [1] gave the following estimation of exponential sum in finite fields.

**Lemma 2.3.** [1] Let \( g(x) \) be a rational polynomial over \( \mathbb{F}_p \) which is non-constant on \( \mathbb{F}_p \). Let \( \chi_p \) be a nontrivial additive character of the prime field \( \mathbb{F}_p \). Then we have the following estimation

\[ \left| \sum_{x \in \mathbb{F}_p} \chi_p(g(x)) \right| \leq (t + \deg (g)_{\infty} - 2) \sqrt{p}, \]

where \( \deg (g)_{\infty} \) is the degree of the pole divisor of \( g(x) \) over \( \mathbb{F}_p \) and \( t \) is the number of different poles of \( g(x) \) over \( \mathbb{F}_p \).

By taking \( g(x) = Tr_{1}^{n}(\frac{\gamma x}{x - \alpha}) \) for some field extension \( \mathbb{F}_q = \mathbb{F}_p(\alpha) \) and \( \alpha, \gamma \in \mathbb{F}_q^* \), the following lemma is given by Wang et al.

**Lemma 2.4.** [33] Let \( p \) be an odd prime integer and \( q = p^n \) where \( 1 < n \leq \sqrt{p} \). Let \( \alpha \in \mathbb{F}_q \) such that \( \mathbb{F}_q = \mathbb{F}_p(\alpha) \). Let \( \gamma \in \mathbb{F}_q^* \) and \( \chi_p \) be a nontrivial additive character of \( \mathbb{F}_p \), then

\[ \left| \sum_{x \in \mathbb{F}_p} \chi_p(Tr_{1}^{n}(\frac{\gamma}{x - \alpha})) \right| \leq (n - 1) \sqrt{p}. \]

A Zadoff-Chu sequence is a complex-valued sequence which generates an electromagnetic signal with constant amplitude. When cyclically shifted versions of the Zadoff-Chu sequence are imposed on a signal, they result in zero correlation with one another at the receiver ([3]). Precisely, let \( s \) be an integer, \( N \) be the sequence length and \( M \) be an integer prime to \( N \). The \( M \)th sequence within the Zadoff-Chu family is given by

\[ Z_M[n] = \begin{cases} 
\exp \left( -i \frac{\pi M n (n+2s)}{N} \right), & n = 0, 1, \ldots, N - 1, \text{for even } N; \\
\exp \left( -i \frac{\pi M n (n+1+2s)}{N} \right), & n = 0, 1, \ldots, N - 1, \text{for odd } N.
\end{cases} \]

A \( p \)-ary \( m \)-sequence of period \( p^n - 1 \) has been popular in many communication systems as it has the ideal two-level autocorrelation ([11]). A \( p \)-ary \( m \)-sequence can be represented by a single-term trace function \( Tr_{1}^{n}(x) \) from \( \mathbb{F}_{p^n} \) to \( \mathbb{F}_p \), i.e.,

\[ s_m = Tr_{1}^{n}(\beta g^m), \quad m = 0, 1, \ldots, p^n - 2, \]

where \( \beta \in \mathbb{F}_p^* \), and \( g \) is a primitive element of \( \mathbb{F}_{p^n} \).

3. The constructions of asymptotically optimal codebooks

3.1. The first class of asymptotically optimal codebooks. A circulant matrix \( A \) can be described as

\[ A = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{N-1} \\
a_{N-1} & a_0 & \cdots & a_{N-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_0
\end{pmatrix} = \begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
A_{N-1}
\end{pmatrix}, \]

where the whole matrix \( A \) can be determined completely by the first row of the matrix \( A \). Consider the case of the Zadoff-Chu sequence where \( M = 1, s = 0 \) and
N is even. Let $a_n = \exp\left[\frac{-i\pi n^2}{N}\right]$, $n = 0, 1, \cdots, N - 1$ in which $N$ is even. Hence, we obtain a circulant matrix $A$ with its $k$th row and $j$th column element

$$a_{k,j} = \exp\left[\frac{-i\pi(k-j)^2}{N}\right]$$

for any $k, j \in \{0, 1, \cdots, N - 1\}$.

Let $q$ be a prime power and let $\alpha \in \mathbb{F}_{q^2}$ such that $\mathbb{F}_{q^2} = \mathbb{F}_q(\alpha)$. Assume that $g$ is a generator of the cyclic group $\mathbb{F}_{q^2}^*$, i.e., $\mathbb{F}_{q^2}^* = \langle g \rangle$. Let $N = q^2 - 1$ be even and

$$M = \{m = \log_q(t - \alpha) : t \in \mathbb{F}_q\} = \{m_0, m_1, \ldots, m_{q-1}\}.$$ 

By this means, we select $q$ rows from the matrix $A$ forming a partial circulant matrix associated with $M$. With the scaling factor of $\frac{1}{\sqrt{q}}$, the $l$th column vector of the partial circulant matrix is given as

$$(1) \quad c_l = \frac{1}{\sqrt{q}}(a_{m_0,l}, a_{m_1,l}, \cdots, a_{m_{q-1},l})^T, \quad 0 \leq l \leq N - 1.$$ 

**Theorem 3.1.** Let $C_1 = \{c_0, c_1, \cdots, c_{N-1}\}$, where $c_l$ is defined by (1) and $0 \leq l \leq N - 1$. Then $C_1$ is an $(N_1 = q^2 - 1, K_1 = q)$ codebook with $I_{\text{max}}(C_1) = \frac{\sqrt{q}}{q}$ and alphabet size $q^2 - 1$.

**Proof.** By the definition of the set $C_1$, $C_1$ contains $(q^2 - 1)$ codewords of length $q$, thus $N_1 = q^2 - 1$, $K_1 = q$ and alphabet size $(q^2 - 1)$. For any $c_k, c_j \in C_1, 0 \leq k \neq j \leq N - 1$, we have

$$|c_k^H c_j| = \frac{1}{q} \sum_{r=0}^{q-1} e^{i\pi(m_{-r} - k)^2/q^2} - e^{i\pi(m_{-r} - j)^2/q^2} \leq \frac{1}{\sqrt{q}},$$

where $\psi_{(j-k)}$ is a nontrivial multiplicative character of $\mathbb{F}_{q^2}$ and the last inequality holds from Lemma 2.1. The equality holds if and only if $(q - 1) \nmid (j - k)$.

Hence, $I_{\text{max}}(C_1) = \frac{\sqrt{q}}{q}$.\hfill \Box

**Theorem 3.2.** The $(N_1 = q^2 - 1, K_1 = q)$ codebook $C_1$ with $I_{\text{max}}(C_1) = \frac{\sqrt{q}}{q}$ in Theorem 3.1 is asymptotically optimal with respect to the Welch bound.

**Proof.** Obviously, $K_1 < N_1 < K_1^2$. The corresponding Welch bound of the codebook $C_1$ is

$$I_W = \sqrt{\frac{N_1 - K_1}{(N_1 - 1)K_1}} = \sqrt{\frac{q^2 - q - 1}{q^3 - 2q}}.$$

Then

$$\lim_{q \to \infty} \frac{I_W}{I_{\text{max}}(C_1)} = \lim_{q \to \infty} \sqrt{\frac{q^3 - q^2 - q}{q^3 - 2q}} = 1,$$

which implies that the codebook $C_1$ is asymptotically optimal with respect to the Welch bound.\hfill \Box

Note that $0 \notin M$ in the codebook $C_1$ in Theorem 3.1. We can select $q + 1$ rows from the matrix $A$ forming a partial circulant matrix $A_{M \cup \{0\}}$ associated with
Proof. We can obtain an \((N_1' = q^2 - 1, K_1' = q + 1)\) codebook \(C_1'\) with \(I_{\max}(C_1') = \frac{1+\sqrt{q}}{q+1}\), which is asymptotically optimal with respect to the Welch bound.

The set \(B = \{B_1, B_2, \cdots, B_m\}\), where for \(1 \leq i \leq m\), \(B_i\) is an orthonormal basis of \(C^{K_i}\), is called an approximately mutually unbiased bases (AMUBs) if

\[
|v_i v_j^H| \leq \frac{1}{K}(1 + o(1))
\]

holds for all \(v_i \in B_i\) and \(v_j \in B_j\).

Proposition 3.3. The columns of the partial circulant matrix \(\Phi = \frac{1}{\sqrt{q+1}} A_{M \cup \{0\}}\) associated with \(M \cup \{0\}\) form an AMUBs of \(C^{q+1}\).

Proof. Firstly, we divide the \(q^2 - 1\) columns of \(\Phi = \frac{1}{\sqrt{q+1}} A_{M \cup \{0\}}\) into \(q-1\) parts, each of which consists of \(q + 1\) columns in the following way. For each \(j = 0, 1, \cdots, q - 2\),

\[
\varepsilon_j = \{j + i \cdot (q - 1) : 0 \leq i \leq q\}.
\]

For any \(i_1 \neq i_2 \in \varepsilon_j\), \(j = 0, 1, \cdots, q - 2\), we have \((q - 1)\{i_2 - i_1\}\). Furthermore, \(|\Phi_{i_1}\|_2 = 1\), where \(\Phi_{i_1}\) denotes the \(i_1\)th column vector of \(\Phi\), and

\[
\Phi_{i_1}^H \cdot \Phi_{i_2} = \frac{1}{q + 1} \left( e^{\frac{i \pi r^2}{q^2 - 1}} + \sum_{r=0}^{q-1} e^{\frac{i \pi r (m_1 - m_2)}{q^2 - 1}} \cdot e^{\frac{i \pi r (m_1^2 - m_2^2)}{q^2 - 1}} \right)
\]

\[
= \frac{1}{q + 1} \left( e^{\frac{i \pi r^2}{q^2 - 1}} + \sum_{r=0}^{q-1} e^{\frac{i \pi r (m_1 - m_2)}{q^2 - 1}} \cdot e^{\frac{i \pi r (m_1^2 - m_2^2)}{q^2 - 1}} \right)
\]

\[
= \frac{1}{q + 1} e^{\frac{i \pi r (m_1^2 - m_2^2)}{q^2 - 1}} \cdot (1 + \sum_{t \in F_q} \psi(t_2 - t_1) (t - \alpha))
\]

\[
= 0.
\]

Therefore, \(\{\Phi_{i_1} : i_1 \in \varepsilon_j\}\) is an orthonormal basis of \(C^{q+1}\) for \(j = 0, 1, \cdots, q - 2\).

For \(i_1 \in \varepsilon_j\) and \(i_2 \in \varepsilon_{j'}\) with any \(j_1 \neq j_2 \in \{0, 1, \cdots, q - 2\}\), we have \((q - 1) \nmid (i_2 - i_1)\).

\[
|\Phi_{i_1}^H \cdot \Phi_{i_2}| = \left| \frac{1}{q + 1} \left( 1 + \sum_{r=0}^{q-1} e^{\frac{i \pi r (m_1 - m_2)}{q^2 - 1}} \right) \right|
\]

\[
= \left| \frac{1}{q + 1} \left( 1 + \sum_{t \in F_q} \psi(t_2 - t_1) (t - \alpha) \right) \right|
\]

\[
\leq \frac{1 + \sqrt{q}}{q + 1}.
\]

Then \(|\Phi_{i_1}^H \cdot \Phi_{i_2}|^2 \leq \frac{1 + \sqrt{q}}{(q+1)^2} = \frac{1}{q+1} + \frac{2\sqrt{q}}{(q+1)^2} = \frac{1}{q+1}(1 + o(1))\).

Therefore, the columns of the partial circulant matrix \(\Phi = \frac{1}{\sqrt{q+1}} A_{M \cup \{0\}}\) form an AMUBs of \(C^{q+1}\). \(\Box\)

Consider the case of the \(p\)-ary \(m\)-sequence with \(\beta = 1\) and \(n = 2\), i.e., \(s_m = Tr_1^\beta(g^m)\), where \(m = 0, 1, \cdots, p^2 - 2\) and \(g\) is a primitive element of \(F_{p^2}\). Let
We can obtain an asymptotically optimal codebook with respect to the Welch bound. Hence, let \( N = p^2 - 1 \) and we obtain a circulant matrix

\[
B = \begin{pmatrix}
  b_0 & b_1 & \cdots & b_{N-1} \\
  b_{N-1} & b_0 & \cdots & b_{N-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_1 & b_2 & \cdots & b_0
\end{pmatrix}
\]

with its \( k \)th row and \( j \)th column element

\[
b_{k,j} = \chi_1(g^{j-k})
\]

for any \( k, j \in \{0, 1, \cdots, N-1\} \).

Let \( p \) be an odd prime and \( \alpha \in \mathbb{F}_{p^2} \) such that \( \mathbb{F}_{p^2} = \mathbb{F}_p(\alpha) \). Assume that \( g \) is a generator of \( \mathbb{F}_{p^2}^* \), i.e., \( \mathbb{F}_{p^2}^* = \langle g \rangle \). Let

\[
M = \{ m = \log_p(t - \alpha) : t \in \mathbb{F}_p \} = \{ m_0, m_1, \ldots, m_{p-1} \}.
\]

We obtain a partial circulant matrix associated with \( M \). With the scaling factor of \( \frac{1}{\sqrt{p}} \), the \( l \)th column vector of the partial circulant matrix is given as

\[
c_l = \frac{1}{\sqrt{p}} (b_{m_0,l}, b_{m_1,l}, \cdots, b_{m_{p-1},l})^T, 0 \leq l \leq N - 1.
\]

**Theorem 3.4.** Let \( C_2 = \{ c_0, c_1, \cdots, c_{N-1} \} \), where \( c_l \) is defined by (2) and \( 0 \leq l \leq N - 1 \). Then \( C_2 \) is an \( (N_2 = p^2 - 1, K_2 = p) \) codebook with \( I_{\text{max}}(C_2) = \frac{1}{\sqrt{p}} \) and alphabet size \( p \).

**Proof.** By the definition of the set \( C_2 \), \( C_2 \) contains \( p^2 - 1 \) codewords of length \( p \), thus \( N_2 = p^2 - 1, K_2 = p \) and alphabet size \( p \). For any \( c_k, c_j \in C_2 \), \( 0 \leq k \neq j \leq N - 1 \),

\[
|c_k^H \cdot c_j| = \left| \frac{1}{p} \sum_{r=0}^{p-1} \chi_1(-g^{k-m_r}) \cdot \chi_1(g^{j-m_r}) \right| = \left| \frac{1}{p} \sum_{r=0}^{p-1} \chi_1(g^{m_r} \cdot (g^j - g^k)) \right| = \left| \frac{1}{p} \sum_{r=0}^{p-1} \chi_1(g^{j-m_r} g^{k-m_r}) \right| = \left| \frac{1}{p} \sum_{t \in \mathbb{F}_p} \chi_1 \left( \frac{g^j - g^k}{t - \alpha} \right) \right| \leq \frac{\sqrt{p}}{p},
\]

where \( \chi_p \) is the canonical additive character of \( \mathbb{F}_p \) and the last inequality holds from Lemma 2.4.

Hence, \( I_{\text{max}}(C_2) \leq \frac{\sqrt{p}}{p} \). \( \square \)

Similar to the proof of Theorem 3.2, we have the following result.

**Theorem 3.5.** The \( (N_2 = p^2 - 1, K_2 = p) \) codebook \( C_2 \) with \( I_{\text{max}}(C_2) \leq \frac{\sqrt{p}}{p} \) in Theorem 3.4 is asymptotically optimal with respect to the Welch bound.

Note that \( 0 \notin M \) in the discussion of Theorem 3.4. We can select \( p+1 \) rows from the matrix \( B \) forming a partial circulant matrix associated with \( M \cup \{0\} \). Similarly, we can obtain an \( (N_2 = p^2 - 1, K_2 = p + 1) \) codebook \( C'_2 \) with \( I_{\text{max}}(C'_2) \leq \frac{1+\sqrt{p}}{p+1} \), which is asymptotically optimal with respect to the Welch bound.
3.2. The second class of asymptotically optimal codebooks. Let \( \mathcal{F}^{(N)} \) be the \( N \times N \) Fourier matrix whose \((k, j)\)-th entry is given by
\[
(\mathcal{F}^{(N)})_{k,j} = \exp \left[ \frac{2\pi i k j}{N} \right], \quad k, j \in \{0, 1, \cdots, N - 1\}
\]
and denote the Fourier matrix \( \mathcal{F}^{(N)} \) by
\[
\begin{pmatrix}
F_0 \\
F_1 \\
\vdots \\
F_{N-1}
\end{pmatrix}.
\]

Suppose that \( q \) is a prime power and let \( \alpha \in \mathbb{F}_{q^2} \) such that \( \mathbb{F}_{q^2} = \mathbb{F}_q(\alpha) \). Assume that \( g \) is a generator of the cyclic group \( \mathbb{F}_{q^2}^* \), i.e., \( \mathbb{F}_{q^2}^* = \langle g \rangle \). Let \( N = q^2 - 1 \) and
\[
M = \{ m = \log_q (t - \alpha) : t \in \mathbb{F}_q \}.
\]
 Obviously, \( 0 \notin M \). By this means, we select \(|M \cup \{0\}| = q + 1\) rows from \( \mathcal{F}^{(N)} \) forming a partial Fourier matrix \( \mathcal{F}_{M \cup \{0\}}^{(N)} \) which has the size of \((q+1) \times (q^2 - 1)\) and denote \( A_{M \cup \{0\}} = \frac{1}{\sqrt{q^2}} \mathcal{F}_{M \cup \{0\}}^{(N)} = (c_0, c_1, \cdots, c_{N-1}) \).

**Theorem 3.6.** Let \( C_3 = \{c_0, c_1, \cdots, c_{N-1}\} \). Then \( C_3 \) is an \((N_3 = q^2 - 1, K_3 = q + 1)\) codebook with \( I_{\max}(C_3) = \frac{1 + \sqrt{3}}{q+1} \) and alphabet size \( q^2 - 1 \).

**Proof.** By the definition of \( C_3 \), \( C_3 \) contains of \( q^2 - 1 \) codewords of length \( q + 1 \), thus \( N_3 = q^2 - 1, K_3 = q + 1 \) and alphabet size \( q^2 - 1 \). We divide the \( q^2 - 1 \) columns of the matrix \( A_{M \cup \{0\}} \) into \( q - 1 \) parts, each of which consists of \( q + 1 \) columns in the following way. For each \( l = 0, 1, \cdots, q - 2 \),
\[
\varepsilon_l = \{ l+i \cdot (q-1) : 0 \leq i \leq q \}.
\]
For any \( c_k, c_j \in C_3, 0 \leq k \neq j \leq N - 1 \),
1) if \( k, j \in \varepsilon_l \) for \( l = 0, 1, \cdots, q - 2 \), we have \( k = l + i_1(q-1) \) and \( j = l + i_2(q-1) \), \( i_1 \neq i_2 \). Then
\[
|c_k^H \cdot c_j| = \left| \frac{1}{q+1} \left( 1 + \sum_{r=0}^{q-1} e^{-2\pi i m k \cdot \frac{j-r}{q^2-1}} \right) \right|.
\]
Since \( (q-1)|(j-k) \), \( \sum_{t \in \mathbb{F}_q} \psi_{j-k}(t - \alpha) = -1 \) from Lemma 2.2, where \( \psi_{j-k} \) is a nontrivial multiplicative character of \( \mathbb{F}_{q^2} \), hence \( |c_k^H \cdot c_j| = 0 \).
2) if \( k \in \varepsilon_{l_1} \) and \( j \in \varepsilon_{l_2} \), where \( l_1 \neq l_2 \in \{0, 1, \cdots, q - 2 \} \). Then
\[
|c_k^H \cdot c_j| = \left| \frac{1}{q+1} \left( 1 + \sum_{r=0}^{q-1} e^{-2\pi i m k \cdot \frac{j-r}{q^2-1}} \right) \right|.
\]
Since \( (j-k)(t-\alpha) \), \( \sum_{t \in \mathbb{F}_q} \psi_{(j-k)}(t - \alpha) = -1 \) from Lemma 2.2, where \( \psi_{j-k} \) is a nontrivial multiplicative character of \( \mathbb{F}_{q^2} \), hence \( |c_k^H \cdot c_j| = 0 \).
Table 3. The explicit parameter values of codebooks $C_3$ in Theorem 3.6.

| $q$ | $N_3$ | $K_3$ | $I_{\text{max}}(C_3)$ | $I_W$ | $\frac{I_W}{I_{\text{max}}(C_3)}$ |
|-----|------|------|------------------------|------|-----------------------------|
| 23  | 528  | 24   | 0.241492980            | 0.199620133 | 0.826608429 |
| 43  | 1848 | 44   | 0.171759966            | 0.148990467 | 0.867434190 |
| 61  | 3720 | 62   | 0.142100801            | 0.125954276 | 0.886372738 |
| 97  | 9408 | 98   | 0.110702631            | 0.100493097 | 0.907751464 |
| 125 | 15624| 126  | 0.096669364            | 0.088729970 | 0.917870629 |
| 169 | 28560| 170  | 0.082352941            | 0.076469234 | 0.928554986 |
| 343 | 117648| 344 | 0.056744939            | 0.053837732 | 0.948767114 |

Since $(q - 1) \nmid (j - k)$, $\left| \sum_{t \in \mathbb{F}_q^*} \psi_{j-k}(t - \alpha) \right| = \sqrt{q}$ from Lemma 2.2, where $\psi_{j-k}$ is a nontrivial multiplicative character of $\mathbb{F}_q^*$, hence $|c_k^H \cdot c_j| = \frac{1 + \sqrt{q}}{q+1}$.

From the above discussions, we have $I_{\text{max}}(C_3) = \frac{1 + \sqrt{q}}{q+1}$.

Similar to the proof of Theorems 3.2 and 3.5, we have the following result.

**Theorem 3.7.** The $(N_3 = q^2 - 1, K_3 = q + 1)$ codebook $C_3$ with $I_{\text{max}}(C_3) = \frac{1 + \sqrt{q}}{q+1}$ in Theorem 3.6 is asymptotically optimal with respect to the Welch bound.

In Table 3, for some $q$, the explicit parameter values of the codebook $C_3$ in Theorem 3.6 are listed. The numerical results indicate that the codebook $C_3$ asymptotically achieves the Welch bound as $q$ increases.

In the discussions of Theorem 3.6, we can select $q$ rows from the Fourier matrix $\mathcal{F}^{(N)}$ forming a partial Fourier matrix $\mathcal{F}^{(N)}_M$ associated with $M$. We obtain an $(N_3' = q^2 - 1, K_3' = q)$ codebook $C_3'$ with $I_{\text{max}}(C_3') = \frac{\sqrt{q}}{q}$, which is asymptotically optimal with respect to the Welch bound.

**Remark 3.1.** The $(N_1 = q^2 - 1, K_1 = q)$ codebook $C_1$, the $(N_1' = q^2 - 1, K_1' = q+1)$ codebook $C_1'$, where $q$ is a power of an odd prime, and the $(N_2 = p^2 - 1, K_2 = p)$ codebook $C_2$, the $(N_2' = p^2 - 1, K_2' = p+1)$ codebook $C_2'$, where $p$ is an odd prime, in Section 3.1, are the special cases of the $(N_3 = q^2 - 1, K_3 = q)$ codebook $C_3$ and $(N_3 = q^2 - 1, K_3 = q + 1)$ codebook $C_3$, where $q$ is a prime power.

Let $\mathbb{E}_{q+1}$ denote the set formed by the standard basis of the $(q + 1)$-dimensional Hilbert space:

$$(1, 0, 0, \ldots, 0, 0),$$
$$(0, 1, 0, \ldots, 0, 0),$$
$$\vdots$$
$$(0, 0, 0, \ldots, 0, 1).$$

Let $\mathbb{E}_3 = C_3 \cup \mathbb{E}_{q+1}$, where $C_3$ is the $(N_3 = q^2 - 1, K_3 = q + 1)$ codebook constructed in Theorem 3.6, then $\mathbb{E}_3$ is an $(\tilde{N}_3 = q^2 + q, \tilde{K}_3 = q + 1)$ codebook with $I_{\text{max}}(\mathbb{E}_3) = \frac{1 + \sqrt{q}}{q+1}$. Obviously, $\mathbb{E}_3$ is a complex-valued codebook with $\tilde{K}_3 < \tilde{N}_3 < \tilde{K}_3^2$. The corresponding Welch bound of $\mathbb{E}_3$ is $I_W = \sqrt{\frac{q-1}{q^2+q-1}}$. Then $\lim_{q \to \infty} \frac{I_W}{I_{\text{max}}(\mathbb{E}_3)} = \lim_{q \to \infty} \sqrt{\frac{(q-1)(q+1)^2}{(q^2+q-1)(1+\sqrt{q})^2}} = 1.$
Let $E_q$ denote the set formed by the standard basis of the $q$-dimensional Hilbert space:

$$(1,0,\cdots,0,0),$$

$$(0,1,\cdots,0,0),$$

$$\vdots$$

$$(0,0,\cdots,0,1).$$

Let $B'_3 = C'_3 \cup E_q$, where $C'_3$ is the $(N'_3 = q^2 - 1, K'_3 = q)$ codebook in the previous construction, then $B'_3$ is an $(N'_3 = q^2 + q - 1, K'_3 = q)$ codebook with $I_{\text{max}}(B'_3) = \frac{1}{\sqrt{q}}$. Obviously, $E'_q$ is a complex-valued codebook with $N'_3 > K'_3$. The corresponding Levenshtein bound of $E'_3$ is $I_L = \frac{\sqrt{q+2}}{q+1}$. Then $\lim_{q \to \infty} \frac{I_{\text{min}}(E'_3)}{I_{\text{max}}(E'_3)} = \frac{\sqrt{q(q+2)}}{q+1} = 1$.

**Remark 3.2.** We are easy to see that removing any codeword in the optimal $(q^2 + q, q)$ codebooks from perfect nonlinear functions and sets of bent functions over $\mathbb{F}_q$ yields $(q^2 + q - 1, q)$ codebooks asymptotically meeting the Levenshtein bound. However, in this paper, we give a class of $(q^2 + q - 1, q)$ codebooks $E'_3$ asymptotically optimal with respect to the Levenshtein bound by different ways from Fourier matrices.

### 3.3. The Third Class of Asymptotically Optimal Codebooks

Let $N = p^2$ for a prime $p$ and $q$ be a primitive element of $\mathbb{F}_{p^2}$. An $N \times N$ cyclic Hadamard matrix $H$ has the entries of

$$h_{j,l} = \begin{cases} 1, & \text{if } j = 0 \text{ or } l = 0; \\ e^{\frac{2\pi i}{p} \cdot Tr_1^2(g^{j-1} \cdot g^{l-1})}, & \text{otherwise}, \\ \end{cases}$$

for $0 \leq j, l \leq N - 1$.

Let $p$ be an odd prime. Let $\alpha \in \mathbb{F}_{p^2}$ such that $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$ and

$$M = \left\{ m = \log_p \frac{1}{t - \alpha} : t \in \mathbb{F}_p \right\}.$$

Obviously, $0 \not\in M$. Next we choose $|M \cup \{0\}| = p + 1$ rows from the Hadamard matrix $H$ to form a partial Hadamard matrix $H_{M \cup \{0\}}$ of size $(p+1) \times p^2$ and denote $A_{M \cup \{0\}} = \frac{1}{\sqrt{p^2+1}}H_{M \cup \{0\}} = (c_0, c_1, \cdots, c_{N-1})$.

**Theorem 3.8.** Let $C_4 = \{c_0, c_1, \cdots, c_{N-1}\}$. Then $C_4$ is an $(N_4 = p^2, K_4 = p + 1)$ codebook with $I_{\text{max}}(C_4) = \frac{1+\sqrt{p^2+1}}{p+1}$ and alphabet size $p$.

**Proof.** By the definition of $C_4$, $C_4$ contains $p^2$ codewords of length $p + 1$, thus $N_4 = p^2$, $K_4 = p + 1$ and alphabet size $p$. For any $c_j, c_k \in C_4, 0 \leq j \neq k \leq N - 1$, $|c_j^H \cdot c_k| = \frac{1}{p+1} \left( 1 + \sum_{r=0}^{p-1} e^{-\frac{2\pi i}{p} \cdot Tr_1^2(g^{m+r+j-2})} \cdot e^{\frac{2\pi i}{p} \cdot Tr_1^2(g^{m+r+k-2})} \right)$

$$= \frac{1}{p+1} \left( 1 + \sum_{r=0}^{p-1} e^{\frac{2\pi i}{p} \cdot Tr_1^2(g^{m+r-k-2} - g^{j-2} - g^{i-2})} \right)$$

$$= \frac{1}{p+1} \left( 1 + \sum_{t=0}^{p-1} e^{\frac{2\pi i}{p} \cdot Tr_1^2(\frac{g^{k-2} - g^{j-2}}{t})} \right)$$
Table 4. The explicit parameter values of codebooks $C_4$ in Theorem 3.8.

| $p$ | $N_4$ | $K_4$ | $I_{\text{max}}(C_4)$ | $I_W$ | $\frac{I_W}{I_{\text{max}}(C_4)}$ |
|-----|-------|-------|------------------------|-------|-------------------------------|
| 7   | 49    | 8     | 0.455718914           | 0.326758065 | 0.717016685 |
| 17  | 289   | 18    | 0.284616979           | 0.228639967 | 0.803325114 |
| 23  | 529   | 24    | 0.186388488           | 0.160012599 | 0.858505805 |
| 37  | 1369  | 38    | 0.142100801           | 0.125954559 | 0.884263552 |
| 61  | 3721  | 62    | 0.110702631           | 0.100493153 | 0.907775652 |
| 97  | 9409  | 98    | 0.085632684           | 0.079301951 | 0.926071067 |
| 157 | 24649 | 158   | 0.056744939           | 0.053837733 | 0.948767131 |

\[
= \frac{1}{p+1} \left( 1 + \sum_{t \in \mathbb{F}_p} \chi_p(T r^2(\frac{g^k - 2 - g^{j - 2}}{t - \alpha})) \right)
\]
\[
\leq \frac{1 + \sqrt{p}}{p+1},
\]

where $\chi_p$ is the canonical additive character of $\mathbb{F}_p$.

Hence, $I_{\text{max}}(C_4) \leq \frac{1 + \sqrt{p}}{p+1}$.

Similar to the proof of Theorems 3.2, 3.5 and 3.7, we have the following result.

**Theorem 3.9.** The $(N_4 = p^2, K_4 = p + 1)$ codebook $C_4$ with $I_{\text{max}}(C_4) \leq \frac{1 + \sqrt{p}}{p+1}$ in Theorem 3.8 is asymptotically optimal with respect to the Welch bound.

For some prime numbers $p$, Table 4 presents the parameter values of codebook $C_4$ in Theorem 3.8. The numerical results indicate that the codebook $C_4$ asymptotically achieves the Welch bound as $p$ increases.

We can select $p$ rows from the Hadamard matrix $H$ forming a partial Hadamard matrix $H_M$ associated with $M$. We obtain an $(N'_4 = p^2, K'_4 = p)$ codebook $C'_4$ with $I_{\text{max}}(C'_4) \leq \frac{1 + \sqrt{p}}{p+1}$, which is asymptotically optimal with respect to the Welch bound.

Let $E_{p+1}$ denote the set formed by the standard basis of the $(p+1)$-dimensional Hilbert space:

\[
\begin{align*}
(1, 0, 0, \ldots, 0, 0), \\
(0, 1, 0, \ldots, 0, 0), \\
\vdots \\
(0, 0, 0, \ldots, 0, 1).
\end{align*}
\]

Let $B_4 = C_4 \cup E_{p+1}$, where $C_4$ is the $(N_4 = p^2, K_4 = p + 1)$ codebook in Theorem 3.8, then $B_4$ is an $(\tilde{N}_4 = p^2 + p + 1, \tilde{K}_4 = p + 1)$ codebook with $I_{\text{max}}(B_4) = \frac{1 + \sqrt{p}}{p+1}$. Obviously, $B_4$ is a complex-valued codebook with $\tilde{K}_4 < \tilde{N}_4 < \tilde{K}_4^2$. The corresponding Welch bound of $B_4$ is $I_W = \frac{\sqrt{p}}{1 + \sqrt{p}}$. Then $\lim_{p \to \infty} \frac{I_W}{I_{\text{max}}(B_4)} = \lim_{p \to \infty} \frac{\sqrt{p}}{1 + \sqrt{p}} = 1$. 


Table 5. The parameters of codebooks asymptotically achieving Welch bound.

| $(N, K)$                     | $I_{\text{max}}$ | Constraints | References |
|-----------------------------|------------------|-------------|------------|
| $(q^2 - 1, q + 1)$          | $\frac{1 + \sqrt{q}}{q+1}$ | $q$ is a prime power | Theorem 3.6 |
| $(q^2 - 1, q)$              | $\frac{\sqrt{q}}{q}$ | $q$ is a prime power | codebooks $\mathbb{C}_4'$ |
| $(q^2 + q, q + 1)$          | $\frac{1 + \sqrt{q}}{q+1}$ | $q$ is a prime power | codebooks $\mathbb{B}_3$ |
| $(p^2, p + 1)$              | $\frac{1 + \sqrt{p}}{p+1}$ | $p$ is an odd prime | Theorem 3.8 |
| $(p^2, p)$                  | $\frac{\sqrt{p}}{p}$ | $p$ is an odd prime | codebooks $\mathbb{C}_4'$ |
| $(p^2 + p + 1, p + 1)$      | $\frac{1 + \sqrt{p}}{p+1}$ | $p$ is an odd prime | codebooks $\mathbb{B}_4$ |

Let $\mathbb{E}_p$ denote the set formed by the standard basis of the $p$-dimensional Hilbert space:

$$(1, 0, \ldots, 0, 0),$$
$$(0, 1, \ldots, 0, 0),$$

$$(0, 0, \ldots, 0, 1).$$

Let $\mathbb{B}_4' = \mathbb{C}_4' \cup \mathbb{E}_p$, where $\mathbb{C}_4'$ is the $(N_4' = p^2, K_4' = p)$ codebook, then $\mathbb{B}_4'$ is an $(N_4' = p^2 + p, K_4' = p)$ codebook with $I_{\text{max}}(\mathbb{B}_4') \leq \frac{1}{\sqrt{p}}$. Note that $N_4' = p^2 + p$ and $K_4' = p$, then the corresponding Levenshtein bound of $\mathbb{B}_4'$ is

$$I_L = \frac{1}{\sqrt{p}} = I_{\text{max}}(\mathbb{B}_4').$$

The codebook $\mathbb{B}_4'$ is optimal with respect to the Levenshtein bound.

**Remark 3.3.** The optimal $(p^2 + p, p)$ codebook $\mathbb{B}_4'$ with respect to the Levenshtein bound, where $p$ is an odd prime, is indeed the special case of the optimal $(p^{2m} + p^m, p^m)$ codebooks from perfect nonlinear functions [8] and sets of bent functions [41] for any odd prime $p$ and positive integer $m$. However, in this paper, we provide completely different methods of constructing optimal $(p^2 + p, p)$ codebook $\mathbb{B}_4'$ from cyclic Hadamard matrices.

4. Conclusion

In this paper, constructions of asymptotically optimal codebooks with respect to Welch bound and Levenshtein bound are presented. These asymptotically optimal codebooks are constructed by selecting certain rows deterministically from circulant matrices, Fourier matrices and Hadamard matrices, respectively. The asymptotically optimal codebooks with respect to Welch bound constructed in this paper are summarized in Table 5. In addition, with respect to the Levenshtein bound, the optimal $(p^2 + p, p)$ codebooks $\mathbb{B}_4'$ with $I_{\text{max}}(\mathbb{B}_4') = \frac{1}{\sqrt{p}}$, where $p$ is an odd prime and asymptotically optimal $(q^2 + q - 1, q)$ codebooks $\mathbb{B}_3'$ with $I_{\text{max}}(\mathbb{B}_3') = \frac{1}{\sqrt{q}}$, where $q$ is a prime power, are obtained from cyclic Hadamard matrices and Fourier matrices, respectively.
ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers and the Editor Professor Sihem Mesnager for their valuable suggestions and comments that helped to greatly improve the paper.

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Received June 2021; revised October 2021; early access December 2021.

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