Optimal Model Averaging of Support Vector Machines in Diverging Model Spaces

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Abstract

Support vector machine (SVM) is a powerful classification method that has achieved great success in many fields. Since its performance can be seriously impaired by redundant covariates, model selection techniques are widely used for SVM with high dimensional covariates. As an alternative to model selection, significant progress has been made in the area of model averaging in the past decades. Yet no frequentist model averaging method was considered for SVM. This work aims to fill the gap and to propose a frequentist model averaging procedure for SVM which selects the optimal weight by cross validation. Even when the number of covariates diverges at an exponential rate of the sample size, we show asymptotic optimality of the proposed method in the sense that the ratio of its hinge loss to the lowest possible loss converges to one. We also derive the convergence rate which provides more insights to model averaging. Compared to model selection methods of SVM which require a tedious but critical task of tuning parameter selection, the model averaging method avoids the task and shows promising performances in the empirical studies.

Keywords: Asymptotic Optimality, Cross Validation, Model Averaging, Support Vector Machines

1 Introduction

Classification methods have been widely used in many fields such as medical diagnosis, facial recognition and fraud detection. In general, these methods can be classified into two

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types: model-based classification and algorithmic methods. The former is represented by logistic regression, linear discriminant analysis and other density-based methods, which derives a determination rule according to the assumed probability model estimate. In contrast, the algorithmic methods directly solve the classification rule by minimizing the margin-based loss plus a penalty. SVM (Cortes and Vapnik, 1995), boosting (Freund, 1995), random forests (Breiman, 2001) and other margin-based classifiers exemplify the approaches. As a typical algorithmic method, the standard SVM uses hinge loss and $L_2$ penalty. SVM has achieved great success in many fields and significant theoretical advances have been made in the literature. To just name a few, Wang and Shen (2006) and Wang et al. (2008) denote that it has a high generalization ability. Bayes risk consistency and rate of convergence are shown in Lin (2004), Zhang (2004) and Blanchard et al. (2008). Koo et al. (2008) gives a bahadur representation of linear SVM and derives the asymptotic behavior of the coefficient estimates. Wang et al. (2019) studies a distributed inference for linear SVM in large-scale data. Since the performance of SVM can be seriously impaired by redundant covariates, many model selection methods have been proposed for SVM. For example, Zhu et al. (2004), Zou (2007) and Wegkamp and Yuan (2011) consider the $L_1$-penalized SVM. Zhang et al. (2016a) and Zhang et al. (2016b) propose to use a modified BIC criterion and a general class of nonconvex penalized SVM (specifically, SCAD and MCP) respectively to select the covariates and show the model selection consistency and oracle property.

Model selection aims to choose a single optimal model but it ignores the information in other models and often produces a rather unstable estimator. Alternatively, “without putting all inferential eggs in one unevenly woven basket” (Longford, 2005), model averaging takes all the candidate models into account and makes prediction by a weighted average. A lot of research work has been done in model averaging which can be classified into Bayesian and frequentist model averaging. Bayesian model averaging determines the model weight by the posterior model probability and Hoeting et al. (1999) provides a comprehensive literature review. Frequentist model averaging selects model weight from a frequentist perspective and many methods have been proposed in the past decades such as SAIC and SBIC (Buckland et al., 1997), mallows model averaging (Hansen, 2007) and jackknife model averaging (Hansen and Racine, 2012). Ando and Li (2014) and Zhang et al. (2020) consider least squares model averaging with high dimensional data. For
more complicated nonlinear models and semiparametric models, [Zhang et al. (2016)](2016c) and [Ando and Li (2017)](2017) handle model averaging with generalized linear models. [Lu and Su (2015)](2015) and [Wang et al. (2021)](2021) propose jackknife model averaging methods for quantile regression. [Li et al. (2018)](2018a), [Li et al. (2018b)](2018b) and [Zhu et al. (2019)](2019) propose model averaging procedures for varying coefficient models. Following [Li et al. (2015)](2015) and [Chen et al. (2018)](2018), [Fang et al. (2020)](2020) proposes a semiparametric model averaging method for dichotomous responses.

The frequentist model averaging procedures mentioned above all focus on model-based methods. On the other hand, model selection in algorithmic methods also ignores the information in discarded covariates and is suffer from model selection uncertainty. To overcome the disadvantages and specifically for classification problems, we can construct a weighted or combined classifier to predict the label. Some work has been done such as aggregation [Lecué and Mendelson (2013)](2013), [Lecué and Mitchell (2012)](2012), [Tsybakov (2004)](2004), ensemble learning [Friedman and Popescu (2008)](2008), boosting [Schapire et al. (1998)](1998) and combing [Kittler et al. (1998)](1998), [Wang and Shen (2006)](2006). They consider different weight choices and candidate classifiers. For example, [Friedman and Popescu (2008)](2008) constructs a linear combination of the predictions with the parameters obtained by a regularized linear regression on the training data without restrictions. [Lecué and Mendelson (2013)](2013) focuses on the aggregation procedure with exponential weights (AEW) and proves that it satisfies the oracle inequality of the excess risk. AEW provides a simple weight choice method, but the asymptotic optimality is not proved in terms of minimizing the risk among all the weighted classifiers. [Wang and Shen (2006)](2006) estimates the generalization error directly and uses data perturbation techniques to attain high accuracy. Nevertheless, the estimated generalization error is very complicated and non-smooth. It is almost non-feasible to combine more than two classifiers.

In this paper, we propose a frequentist model averaging method for SVM which consists of two steps. In the first step, a linear classifier is developed by applying SVM without the $L_2$ penalty to each candidate set of covariates. In the second step, a weighted combination of the classifiers obtained in the first step is built. The optimal weight is selected by leave-one-out cross-validation. The proposed method is asymptotically optimal in the sense of achieving the lowest possible hinge loss in prediction.

There are several significant theoretical and practical contributions of our work. From
the perspective of model averaging, first, we expand the territory of frequentist model averaging to algorithmic methods. Second, the asymptotic optimality is established when the number of covariates diverges at an exponential rate of the sample size, which is not quite common in the existing literature. To show the asymptotic optimality, we need to establish a uniform convergence of SVM estimates from all the candidate models, which is not trivial with a diverging model space. Third, other than showing the asymptotic optimality of the proposed method, i.e., the ratio of its hinge loss to the lowest possible loss converges to one, we also derive the convergence rate which provides more insights to model averaging. Fourth, the asymptotic optimality can be achieved even when the “true” model is included in the candidate models as long as a non-separable case is considered. Note that regular model averaging methods usually require that all candidate models are misspecified (Hansen, 2007; Zhang et al., 2016c) to achieve asymptotic optimality.

From the perspective of SVM, first, we provide a useful alternative to penalized SVM when the number of covariates is large. Our proposal does not need to select tuning parameters which is a tedious but critical task in penalized SVM. Second, the flexible way of preparing candidate models makes the result of penalized SVM belong to the model space considered by our method, which brings potentials for the prediction improvement. Third, the penalized SVM assumes sparsity of the “truth” while our method works well for both sparse and dense situations. Fourth, we also provide a modification of the proposed model averaging procedure so that the optimal weight will be put on a single candidate model, which gives us a sparse result. Actually, even for our original proposal, we find in the simulations that the optimal weights of many candidate models are zeros which is probably due to the special form of hinge loss. It means that our method can screen out some poor candidate classifiers automatically and save the computational cost in prediction.

Section 2 introduces the proposed model averaging method. The asymptotic optimality is established in Section 3. Section 4 discusses some implementation details. Results of a simulation study and some real data analyses are presented in Section 5. Section 6 concludes the paper with some remarks. All the technical details are provided in the Appendix.
2 Model Averaging of SVM

We consider a binary classification problem with a training set of observations $D_n = \{(X_i, Y_i), i = 1, \cdots, n\}$ independently drawn from the distribution of a random pair $(X, Y)$, where $X^T = (1, X_1, \cdots, X_p) = (1, (X^+)^T) \in \mathbb{R}^{p+1}$ and $Y \in \{1, -1\}$ is the class label. The target is to find a classifier $\phi$ which maps the input space $X$ to $\{-1, 1\}$. For the standard linear SVM, the classifier is defined as $\phi(X) = \text{sign}(X^T\beta)$ with $\beta^T = (\beta_0, \beta_1, \cdots, \beta_p) = (\beta_0, (\beta^+)^T)$. Let $\|\cdot\|$ be the Euclidean norm operator of a vector. SVM estimates $\beta$ by solving the optimization problem
\[
\min_{\beta} \left\{ \frac{1}{n} \sum_{i=1}^{n} (1 - Y_i X_i^T \beta)_{+} + \frac{\lambda_n}{2} \|\beta^+\|^2 \right\},
\]
where $(1 - u)_+ = \max\{1 - u, 0\}$ denotes the hinge loss and $\lambda_n > 0$ is a tuning parameter that needs to be carefully tuned for a good performance.

Following Koo et al. (2008), we define the “true” parameter value as
\[
\beta^* = \arg\min_{\beta} E(1 - Y X^T \beta)_{+},
\]
where $(\beta^*)^T = (\beta_0^*, \beta_1^*, \cdots, \beta_p^*) = (\beta_0^*, (\beta^+)^T)$ and $E(1 - Y X^T \beta)_{+}$ is the population hinge loss. We only consider the non-separable case in the limit to ensure the uniqueness of the truth $\beta^*$. Here by non-separable we mean that $E(1 - Y X^T \beta^*)_{+} > 0$.

The model selection methods of SVM assume $\beta^*$ is sparse in the sense that most of its components are exactly zero. To recover the non-zero part of $\beta^*$, either information criteria such as BIC (Zhang et al., 2016a) are used or the $L_2$ penalty in (1) is replaced by the $L_1$ penalty (Zhu et al., 2004) or a nonconvex penalty such as SCAD (Zhang et al., 2016b). If the sparsity assumption is true, model selection consistency and oracle property can be established well. However, if the sparsity assumption does not hold which is most likely true in practice, there is no theoretical guarantee for the model selection methods.

Alternatively, we propose a model averaging method for SVM. Specifically, consider a sequence of $K$ candidate models/classifiers $\{\phi_k(X) = \text{sign}(X^T_{(k)} \beta_{(k)}), k = 1, \cdots, K\}$, where $X^T_{(k)} = (1, (X^+_{(k)})^T)$ is a $(p_k + 1)$-dimensional sub-vector of $X^T$ that always includes the constant one, $\beta^T_{(k)} = (\beta_0(k), \beta_1(k), \cdots, \beta_{p_k}(k)) = (\beta_0(k), (\beta^+_{(k)})^T)$ and $X_{i,(k)} = (1, X_{i,1(k)}, \cdots, X_{i,p_k(k)})^T$ is the $i$th sample realization of $X_{(k)}$. Note that $K$ can be diverging as $n$ goes to infinity. So far we assume the covariate set in each candidate model is already prepared. We will discuss the preparation of the candidate models in Section 4.
For the $k$th candidate model, we estimate $\beta(k)$ by

$$
\hat{\beta}(k) = \arg \min_{\beta(k)} \left\{ \frac{1}{n} \sum_{i=1}^{n} (1 - Y_i X^T_{i,(k)} \beta(k))_+ \right\},
$$

which is the analogue of (1) restricted to the $k$th model but no penalty term is used. So we do not need to worry about the choice of tuning parameter $\lambda_n$ since it is set to be 0. This setting seems to be weird at the first glance. But actually it is quite common in model averaging literature noting that least squares model averaging \cite{Hansen2007} and generalized linear model averaging \cite{Zhang2016}, along with many other model averaging methods, do not use any penalty term for candidate model fitting neither.

Let $w^T = (w_1, \cdots, w_K)$ be a vector belonging to $W = \{w \in [0, 1]^K : \sum_{k=1}^{K} w_k = 1 \}$. The weighted classifier is defined as $\hat{\phi}_w(X) = \text{sign} \left( \sum_{k=1}^{K} w_k X^T_{(k)} \hat{\beta}(k) \right)$. Ideally, the optimal weight vector $w$ should minimize the conditional expected hinge risk

$$
EHR_n(w) = E \left[ \left( 1 - Y \sum_{k=1}^{K} w_k X^T_{(k)} \hat{\beta}(k) \right)_+ D_n \right].
$$

However, a direct minimization over $EHR_n(w)$ is infeasible since the distribution of $(X, Y)$ is unknown. We propose to use a leave-one-out cross validation criterion to approximate $EHR_n(w)$.

Specifically, for each candidate model $k$, let $\hat{\beta}_{i(k)}$ be the estimator of $\beta(k)$ with the $i$th observation deleted, i.e.,

$$
\hat{\beta}_{i(k)} = \arg \min_{\beta(k)} \left\{ \frac{1}{n-1} \sum_{j \neq i} (1 - Y_j X^T_{j,(k)} \beta(k))_+ \right\}.
$$

Define the leave-one-out cross validation criterion as

$$
CV_n(w) = \frac{1}{n} \sum_{i=1}^{n} \left( 1 - Y_i \sum_{k=1}^{K} w_k X^T_{i,(k)} \hat{\beta}_{i(k)} \right)_+.
$$

Note that the hinge loss is convex, so $CV_n(w)$ is convex in $w$. The optimal weight $\hat{w}$ is obtained by

$$
\hat{w} = \arg \min_{w \in W} CV_n(w).
$$

The final weighted classifier is given by

$$
\hat{\phi}_w(X) = \text{sign} \left( \sum_{k=1}^{K} \hat{w}_k X^T_{(k)} \hat{\beta}(k) \right).
$$
3 Theoretical Results

In this section we study the asymptotic properties of $\hat{\beta}(k)$ and $\hat{\beta}_i(k)$ and then prove the asymptotic optimality of $\hat{w}$ in the sense of achieving the lowest possible conditional expected hinge loss $EHR_n(w)$.

For the asymptotic properties of the estimated SVM parameters, Koo et al. (2008) obtains the Bahadur representation and establishes the asymptotic normality with fixed parameter dimensions. Zhang et al. (2016a) gives uniform convergence rate of $\hat{\beta}(k)$ for “correct” models when the dimension is diverging and the true parameter $\beta^*$ is sparse. However, for model averaging of SVM, we do not need to assume the sparsity of $\beta^*$. Also, we do not assume that the candidate models contain a “correct” model or not. So the previous theoretical results cannot be applied directly. Further, we need to consider the asymptotic properties of cross validation estimator $\hat{\beta}_i(k)$, which is more challenging.

Define the population version of (2) as $L_k(\beta(k)) = E(1 - YX^T(k)\beta(k)) +$ and denote its minimizer as $\beta^*_k = \arg\min_{\beta(k)} L_k(\beta(k)),$ which is assumed to be unique. Define $S_k(\beta(k)) = -E\{I(1 - YX^T(k)\beta(k) \geq 0)YX(k)\}$ and $H_k(\beta(k)) = E\{\delta(1 - YX^T(k)\beta(k))X(k)X^T(k)\}$, where $\delta(\cdot)$ denotes the Dirac delta function. Denote $f_{k+}$ and $f_{k-}$ as the densities of $X^+_i(k)$ conditioning on $Y = 1$ and $Y = -1$, respectively. Let

$$\bar{k} = \max_{1 \leq k \leq K} p_k + 1 \text{ and } \xi_n = \inf_{w \in W} EHR_n(w).$$

We now present the technical conditions.

(A1) $f_{k+}$ and $f_{k-}$ are continuous and have some common support in $\mathbb{R}^{p_k}$.

(A2) $|X_j| \leq M \leq \infty$ for a positive constant $M$ and $1 \leq j \leq p$.

(A3) The densities of $X^T_{i,(k)}\beta^*_k$ conditioning on $Y = 1$ and $Y = -1$ are uniformly bounded away from zero and infinitely at the neighborhood of $X^T_{i,(k)}\beta^*_k = 1$ and $X^T_{i,(k)}\beta^*_k = -1$, respectively.

(A4) $K = O(n)$, $\bar{k} = O(n^\kappa)$ for a constant $0 < \kappa < 1/4$, and $p = O(\exp(n^\gamma))$ for a constant $0 < \gamma < (1 - 4\kappa)/3$.

(A5) There exists a positive constant $c_1$ such that $\lambda_{\min}(H_k(\beta^*_k)) \geq c_1$, where $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a matrix.
There exists a positive constant $c_2$ such that $\lambda_{\text{max}}(E(X^{(k)}X^{(k)T})) \leq c_2$, where $\lambda_{\text{max}}(\cdot)$ is the largest eigenvalue of a matrix.

(A7) $\xi_n^{-1}\sqrt{n^{-1}k^2\log p} = o_p(1)$.

Most of the conditions are similar to conditions in Zhang et al. (2016a). Condition (A1) is required so that $S_k(\beta^{(k)})$ and $H_k(\beta^{(k)})$ are well-defined and Condition (A5) requires that the Hessian matrix is well-behaved at $\beta^*_k$, see Koo et al. (2008) for more details. Condition (A2) is also assumed in the literature of high dimensional model selection consistency. Condition (A3) assumes that as the sample size increases, there is enough information around the non-differentiable point of the hinge loss function. Condition (A4) specifies the divergence rates of $K$, $\bar{k}$ and $p$. The number of covariates, number of candidate models and number of covariates at each candidate model are all allowed to be diverging. Specifically, the number of covariates $p$ can diverge at an exponential rate of the sample size. Condition (A6) on the largest eigenvalue is similar to the sparse Riesz condition. Condition (A7) implies that the order of $\xi_n$ is larger than $\sqrt{n^{-1}k^2\log p}$. If $p$ and $\bar{k}$ are bounded, this order is $n^{-1/2}$ as usual.

**Theorem 3.1** Suppose Conditions (A1) $\sim$ (A5) hold. Then

$$\max_{1 \leq k \leq K} \|\hat{\beta}^{(k)} - \beta^*_k\| = O_p\left(\sqrt{n^{-1}\bar{k}\log p}\right)$$

and

$$\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \|\hat{\beta}_{i(k)} - \beta^*_k\| = O_p\left(\sqrt{(n-1)^{-1}\bar{k}\log p}\right).$$

Theorem 3.1 establishes the uniform convergence of $\hat{\beta}^{(k)}$ and $\hat{\beta}_{i(k)}$. The uniform convergence rate is related to sample size $n$, the number of covariates $p$, and the largest number of covariates in all $K$ candidate models under consideration. If $p$ and $\bar{k}$ are bounded, the convergence rate will be the common $n^{-1/2}$.

Based on the uniform convergence, we are able to prove the asymptotic optimality of our model averaging estimator. First we present a lemmas as follow.

**Lemma 3.2** Suppose Conditions (A1) $\sim$ (A6) hold. Then

$$\sup_{w \in W} \left| EHR_n(w) - E \left( 1 - Y \sum_{k=1}^{K} w_k X^{(k)T}_j \beta^*_k \right) \right| = O_p\left(\sqrt{n^{-1}\bar{k}\log p}\right)$$

Based on the uniform convergence, we are able to prove the asymptotic optimality of our model averaging estimator. First we present a lemmas as follow.
\[
\sup_{w \in W} |CV_n(w) - EHR_n(w)| = O_p\left(\sqrt{n^{-1}k^2 \log p}\right).
\] (8)

This lemma shows that \(CV_n(w)\) is a good approximation of \(EHR_n(w)\) as we need. Finally we present the asymptotic optimality theorem.

**Theorem 3.3** Suppose Conditions (A1) \(\sim\) (A7) hold. Then

\[
\frac{EHR_n(\hat{w})}{\inf_{w \in \mathcal{W}} EHR_n(w)} = 1 + O_p\left(\xi_n^{-1} \sqrt{n^{-1}k^2 \log p}\right) \rightarrow_p 1.
\]

Theorem 3.3 shows that the proposed model averaging classifier is asymptotic optimal in the sense of achieving the lowest possible hinge loss in prediction, i.e., the ratio of its hinge loss to the lowest possible loss converges to one. Further, we provide the convergence rate \(O_p\left(\xi_n^{-1} \sqrt{n^{-1}k^2 \log p}\right)\). This kind of result is not quite common in the literature since most of the asymptotic optimality theorems only show the convergence to 1 but do not provide the convergence rate. Based on the convergence rate, we can see that the convergence speed will be slower if the number of all considered covariates and the number of covariates in each candidate model diverge faster. If the lowest possible hinge loss \(\xi_n\) is larger, the asymptotic optimality is easier to achieve. This is consistent to a general consensus that model averaging usually shows more advantages when all the candidate models are not that good. The convergence rate does not depend on the number of candidate models directly. However, if more candidate models are considered, \(\xi_n\) is possibly smaller and the asymptotic optimality is harder to achieve.

### 4 Implementation Details

In this section we discuss some implementation details of the proposed model averaging method which is denoted as MA-SVM from now on.

First we discuss the preparation of candidate models. We suggest to use nested candidate models in practice for convenience, although the theoretical results are also applicable to non-nested candidate models. There has been a long history in the model averaging area to use the nested candidate models and several strategies have been proposed (Hansen 2014, Zhang et al. 2016c, 2020). One common way is to order the covariates by a screening criterion and then take the first \(K\) covariates that are considered to be important for the classification. For a binary response variable, we may use
the Kolmogorov filter (Mai and Zou, 2013), mean-variance index (Cui et al., 2015) or information gain (Ni and Fang, 2016) as the screening criterion. A more informative way is to take advantage of model selection methods and use the solution path of penalized SVMs. In the simulation and real data examples, we apply the $L_1$ penalized SVM to obtain a solution path and take the first $K$ covariates in the path. By doing so, the selected model by the penalized SVM belongs to the model space considered by MA-SVM, which brings potentials for the prediction improvement. As for the choice of $K$, we take $K = p$ for a relatively small $p$ and $K = \sqrt{n}$ for a large $p$. It is a little bit fuzzy for the concept of “small” and “large” and the users of MA-SVM may handle it flexibly.

The second issue is the optimization problem of (4). Note that $CV_n(w)$ has a similar form to the hinge loss of a regular SVM with no penalization. So it can be solved by the constrained linear program as follows:

\[
\begin{align*}
\min_{\zeta_i, w_k} & \sum_{i=1}^{n} \zeta_i, \\
\text{s.t.} & \quad \zeta_i \geq 0, \ i = 1, \cdots, n, \\
& \quad Y_i \left( \sum_{k=1}^{K} w_k X_{i(k)}^T \hat{\beta}_{i(k)} \right) \geq 1 - \zeta_i, \ i = 1, \cdots, n, \\
& \quad \sum_{k=1}^{K} w_k = 1 \quad \text{and} \quad 0 \leq w_k \leq 1, \ k = 1, \cdots, K.
\end{align*}
\]

The third issue is about how to achieve a sparse solution. Model averaging methods usually can achieve better prediction performance. But in most cases it can not produce a sparse solution since usually all the regression coefficients of the covariates included in the candidate models are not zero. To overcome the shortcoming, we modify the MA-SVM a little bit to achieve a sparse solution. Specifically, similar to Lecue (2007), we define $\Psi(x) = \max(-1, \min(x, 1))$ and replace $X_{i(k)}^T \hat{\beta}_{i(k)}$ by $\Psi(X_{i(k)}^T \hat{\beta}_{i(k)})$ in (3) and (9).

Note that a linear function achieves its minimum over a convex polygon at one of the vertices of the polygon, the hinge loss is linear on $[-1, 1]$ and $W$ is a convex set. Thus the solution to the optimization problem (4) after the replacement will put all the weight one to a single candidate model. This modified method can be considered as a combination of model averaging and model selection and it is denoted as MS-SVM.
5 Simulation and Real Data Examples

5.1 A Simulation Study

In this section, we conduct a simulation study to compare the finite sample performance of our proposed methods MA-SVM and MS-SVM with the following four methods.

- SVM: the standard linear support vector machine with $L_2$ penalty.
- AEW: the aggregated exponential weight method in Lecue and Mendelson (2013).
- SCAD-SVM: the penalized SVM with a SCAD penalty in Zhang et al. (2016b).
- $L_1$-SVM: the penalized SVM with a $L_1$ penalty in Wegkamp and Yuan (2011).

We consider two training data generation models similar to Zhang et al. (2016b). The first one is essentially the setting for a standard linear discriminant analysis. The second one is related to the probit regression model.

**Model 1:**

$P(Y = 1) = P(Y = -1) = 0.5$, $X^+ | (Y = 1) \sim MN(\mu, \Sigma)$, $X^+ | (Y = -1) \sim MN(r\mu, \Sigma)$, $\mu = (1, 1, \ldots, 1, 0, \ldots, 0)^T \in \mathbb{R}^p$ with first $q$ elements being ones, $\Sigma = (\sigma_{ij})$ with nonzero elements $\sigma_{ii} = 1$, $i = 1, 2, \ldots, p$ and $\sigma_{ij} = 0.5^{|i-j|}$ for $1 \leq i \neq j \leq q$, and $r = \Delta / \sqrt{\mu^T \Sigma^{-1} \mu}$ so that the Bayes error is $\Phi(-\Delta/2)$, where $\Phi(\cdot)$ denotes the cumulative distribution of the standard normal distribution. We take $\Delta = 0.5, 1, 1.5, 2, 2.5, 3$ and the corresponding Bayes errors are 40.13%, 30.85%, 22.66%, 15.87%, 10.56%, 6.68%, respectively. We consider sample size $n = 100, 200$ and three different cases for the values of $p$ and $q$.

- **Case 1:** $p = 7, q = 7$, but we assume only the first 5 covariates are considered.
- **Case 2:** $p = 10, q = 5$.
- **Case 3:** $p = 400, q = 5$.

**Model 2:**

$X^+ \sim MN(0_p, \Sigma)$, $\Sigma = (\sigma_{ij})$ with nonzero elements $\sigma_{ii} = 1$ for $i = 1, 2, \ldots, p$ and $\sigma_{ij} = 0.4^{|i-j|}$ for $1 \leq i \neq j \leq p$, $P(Y = 1|X^+) = \Phi((X^+)^T \beta^+)$ and $\beta^+ = (1.1, 1.1, 1.1, 0, \ldots, 0)^T \in \mathbb{R}^p$. The Bayes error is 10.4%. We consider $(n, p) = (100, 400), (100, 800), (200, 800)$ and $(200, 1600)$.

For each case, we generate a test data with sample size 10000. All the considered methods are fitted in the training data and applied to the test data. We evaluate the methods by the test error (missclassification rate in the test data) and the hinge loss in the test data. For model 1, we evaluate the performance of the methods by the relative
Figure 1: Averaged relative test error for different $n$, $p$, $q$ and $\Delta$ in Model 1 over 100 simulation replications.
Figure 2: Averaged hinge loss in test data for different $n$, $p$, $q$ and $\Delta$ in Model 1 over 100 simulation replications.
Table 1: Averaged test error and hinge loss for different $n$ and $p$ in Model 2 over 100 simulation replications.

| Method    | $n$  | $p$  | Test error | s.d.  | Hinge loss | s.d.  |
|-----------|------|------|------------|-------|------------|-------|
| SVM       | 100  | 400  | 33.920%    | 0.1365% | 0.8031     | 0.0018 |
| AEW       |      |      | 15.173%    | 0.1810% | 0.3611     | 0.0044 |
| MA-SVM    |      | 100  | 14.866%    | 0.2303% | 0.3360     | 0.0015 |
| MS-SVM    | 100  | 400  | 16.581%    | 0.3122% | 0.3355     | 0.0062 |
| SCAD-SVM  |      |      | 18.895%    | 0.4818% | 0.4760     | 0.0170 |
| $L_1$-SVM |      |      | 21.120%    | 0.5273% | 0.5587     | 0.1298 |
| SVM       | 100  | 800  | 33.760%    | 0.1067% | 0.8021     | 0.0012 |
| AEW       |      |      | 15.603%    | 0.1564% | 0.3699     | 0.0035 |
| MA-SVM    |      | 100  | 15.245%    | 0.2394% | 0.3734     | 0.0021 |
| MS-SVM    | 100  | 800  | 17.053%    | 0.3040% | 0.3463     | 0.0059 |
| SCAD-SVM  |      |      | 19.590%    | 0.4116% | 0.5416     | 0.0182 |
| $L_1$-SVM |      |      | 23.470%    | 0.4911% | 0.5824     | 0.0599 |
| SVM       | 200  | 800  | 37.895%    | 0.0771% | 0.8885     | 0.0007 |
| AEW       |      |      | 12.232%    | 0.2001% | 0.2924     | 0.0020 |
| MA-SVM    |      | 200  | 11.325%    | 0.0702% | 0.2731     | 0.0018 |
| MS-SVM    | 200  | 800  | 13.922%    | 0.1396% | 0.2903     | 0.0026 |
| SCAD-SVM  |      |      | 11.490%    | 0.2725% | 0.2799     | 0.0175 |
| $L_1$-SVM |      |      | 18.935%    | 0.3374% | 0.4766     | 0.0485 |
| SVM       | 200  | 1600 | 37.895%    | 0.0771% | 0.8885     | 0.0007 |
| AEW       |      |      | 12.232%    | 0.2001% | 0.2924     | 0.0020 |
| MA-SVM    |      | 200  | 11.325%    | 0.0702% | 0.2731     | 0.0018 |
| MS-SVM    | 200  | 1600 | 13.516%    | 0.1414% | 0.2816     | 0.0027 |
| SCAD-SVM  |      |      | 12.505%    | 0.2911% | 0.3148     | 0.0172 |
| $L_1$-SVM |      |      | 19.045%    | 0.2611% | 0.4780     | 0.0090 |

test error which is computed by (test error - Bayes error)/Bayes error. For model 2, we directly use test error for evaluation. The number of simulation runs is 100. We generate a new training data at each simulation run but the test data is only generated once.
The simulation results are presented in Figure 1 and Figure 2 for model 1 and Table 1 for model 2. The proposed MA-SVM achieved the smallest test errors in all cases, regardless sparse or dense signals, low or high dimensions. MS-SVM usually achieves the best hinge loss followed by MA-SVM and AEW. The regular SVM performs really bad indicating that model selection or model averaging is absolutely necessary. The improvement of MA-SVM compared to $L_1$-SVM shows the advantage of combing different models over choosing only one model. As a method that also chooses one single model, MS-SVM performs better or similar to SCAD-SVM and $L_1$-SVM in most cases.

A quite interesting phenomenon we figure out for MA-SVM in the simulations is that the optimal weights of many candidate models are zeros. We conjecture that is probably due to the special form of hinge loss but more theoretical investigations are needed in our future research. It means that our method can screen out some poor candidate classifiers automatically and save the computational cost in prediction.

5.2 Real Data Examples

To check the practical performances of our proposed methods, we consider four real data sets. Three of them are from kaggle [https://www.kaggle.com/] Red Wine Quality, Diabetes and Ionosphere. The other one is Gisette from [https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html#gisette] for a high dimensional case. For the Wine data, the input variables are about physicochemical traits and the output variable is the quality of wine which is classified into “good” or “bad”. For the Diabetes data, the inputs are patients’ age, BMI and so on, and the output variable indicates whether a patient has diabetes or not. The Ionosphere is a radar data which has 32 continuous inputs and the output is either “good” or ”bad” for the radar. The data Gisette is also for binary classification with 5000 covariates. The target is to separate highly confusable digits “4” and “9”.

For each data set, we randomly select 50% of the samples as the training data and the rest samples are used for testing. The six methods considered in the simulation study are fitted in the training data and applied to the test data. The test error and hinge loss are calculated. We repeat this procedure for 100 times and report the results in Table 2. The proposed MA-SVM performs very well especially for the test error.
Table 2: Averaged testing error and hinge loss for four real data examples.

| Data     | Method   | $n$  | $p$  | Test error | s.d.  | Hinge loss | s.d.  |
|----------|----------|------|------|------------|-------|------------|-------|
| Wine     | SVM      | 1600 | 11   | 0.38688    | 0.00453| 1.86933    | 0.00636|
|          | AEW      |      |      | 0.26688    | 0.00121| 0.60680    | 0.00222|
|          | MA-SVM   |      |      | 0.25313    | 0.00105| 0.60731    | 0.00236|
|          | MS-SVM   |      |      | 0.27625    | 0.00130| 0.61357    | 0.00178|
|          | SCAD-SVM |      |      | 0.26938    | 0.00103| 0.60936    | 0.00214|
|          | L1-SVM   |      |      | 0.26188    | 0.00972| 0.62473    | 0.00397|
|          |          |      |      |            |       |            |       |
| Diabetes | SVM      | 768  | 8    | 0.24142    | 0.00216| 0.85242    | 0.00254|
|          | AEW      |      |      | 0.22961    | 0.00205| 0.53288    | 0.00277|
|          | MA-SVM   |      |      | 0.21785    | 0.00211| 0.53326    | 0.00304|
|          | MS-SVM   |      |      | 0.23225    | 0.00207| 0.49510    | 0.00220|
|          | SCAD-SVM |      |      | 0.23117    | 0.00210| 0.54123    | 0.00250|
|          | L1-SVM   |      |      | 0.22559    | 0.00187| 0.52810    | 0.00230|
|          |          |      |      |            |       |            |       |
| Ionosphere | SVM    | 350  | 32   | 0.21429    | 0.00309| 0.54750    | 0.00562|
|          | AEW      |      |      | 0.15714    | 0.00157| 0.39549    | 0.00456|
|          | MA-SVM   |      |      | 0.15000    | 0.00239| 0.39986    | 0.00534|
|          | MS-SVM   |      |      | 0.16071    | 0.00195| 0.34463    | 0.00330|
|          | SCAD-SVM |      |      | 0.17857    | 0.00125| 0.63355    | 0.00842|
|          | L1-SVM   |      |      | 0.15317    | 0.00193| 0.41220    | 0.00508|
|          |          |      |      |            |       |            |       |
| Gisette  | SVM      | 1000 | 5000 | 0.23280    | 0.006251| 0.53885    | 0.00143|
|          | AEW      |      |      | 0.12861    | 0.001670| 0.31857    | 0.00335|
|          | MA-SVM   |      |      | 0.11325    | 0.001679| 0.26113    | 0.00418|
|          | MS-SVM   |      |      | 0.12750    | 0.001579| 0.23737    | 0.00333|
|          | SCAD-SVM |      |      | 0.10125    | 0.001765| 0.25191    | 0.00601|
|          | L1-SVM   |      |      | 0.12500    | 0.001802| 0.26652    | 0.00579|

6 Concluding Remarks

In this paper we propose a new weighted binary classifier based on model averaging for support vector machines, and the weights are obtained by leave-one-out cross validation.
A modified method is also proposed to achieve sparse results. In the theoretical perspective, the asymptotic optimality is established when the number of covariates diverges at an exponential rate of the sample size and its convergence rate is also provided. The asymptotic optimality can be achieved even when the true model is included in the candidate models as long as a non-separable case is considered. In the practical perspective, the proposed model averaging method performs quite well in either sparse or dense cases, low or high dimensions.

There are a couple of issues to be investigated in our future study. In this paper we only consider the non-separable case as Koo et al. (2008), Zhang et al. (2016a) and Zhang et al. (2016b) considered. What will be the case if the lowest possible hinge loss can achieve zero as the model space diverges needs more study. Another interesting extension is to consider model averaging for non-linear support vector machines.

Appendix: Proofs

Proof of Theorem 3.1

We just present the proof of (6). The proof of (5) is very similar to (6) and we omit it here. If we can prove that for any $0 < \eta < 1$, there exists a large constant $\Delta$ such that for sufficient large $n$:

$$P\left( \left| \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \left\| \hat{\beta}_i(k) - \beta^*_k \right\| \sqrt{\frac{(n-1)^{-1}k \log p}{(n-1)^{-1}k \log p}} \right| < \Delta \right) \geq 1 - \eta, \quad (10)$$

then (6) holds.

Denote $l_{ik}(\beta_k) = \frac{1}{n-1} \sum_{j \neq i} (1 - Y_j X_j^T \beta_k)^+$ Then $\hat{\beta}_i(k) = \arg \min_{\beta_k} l_{ik}(\beta_k)$. To show (10), we just need to prove

$$P\left( \left| \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \left\| \hat{\beta}_i(k) - \beta^*_k \right\| \sqrt{\frac{(n-1)^{-1}k \log p}{(n-1)^{-1}k \log p}} \right| < \Delta \right) \geq P\left( \min_{1 \leq i \leq n} \min_{1 \leq k \leq K} \inf_{\|u\| = \Delta} \left[ l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) \right] > 0 \right) \quad (11)$$

and

$$P\left( \min_{1 \leq i \leq n} \min_{1 \leq k \leq K} \inf_{\|u\| = \Delta} \left[ l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) \right] > 0 \right) \geq 1 - \eta. \quad (12)$$
First, we prove (11). Denote
\[ A = \left\{ (x, y) : \min_{1 \leq i \leq n} \min_{1 \leq k \leq K} \inf_{\|u\| = \Delta} \left[ l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) \right] > 0 \right\} \]
and
\[ B = \left\{ (x, y) : \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \|\beta_{i(k)} - \beta^*_k\| < \Delta \sqrt{k \log p/(n-1)} \right\}. \]
We just need to show \( A \subseteq B \). By the definition of \( A \), for any element \((x, y)\) in \( A \), \(1 \leq i \leq n, 1 \leq k \leq K\) and \(\|u\| = \Delta\), we have
\[ l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) > 0. \]
By the convexity of \( l_{ik}(\cdot) \) and the fact that \( \hat{\beta}_{i(k)} = \arg\min_{\beta^*_k} l_{ik}(\beta^*_k) \), for every \( 1 \leq i \leq n, 1 \leq k \leq K \), we have
\[ \|\hat{\beta}_{i(k)} - \beta^*_k\| < \Delta \sqrt{k \log p/(n-1)}. \]
As a result,
\[ \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \|\hat{\beta}_{i(k)} - \beta^*_k\| < \Delta \sqrt{k \log p/(n-1)} \]
and \( A \subseteq B \) holds.

Next, we prove (12). Note that we can decompose \( l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) \) as
\[
\begin{align*}
    & l_{ik}(\beta^*_k) + \sqrt{k \log p/(n-1)u} - l_{ik}(\beta^*_k) \\
    &= \frac{1}{n-1} \sum_{j \neq i} \left\{ (1 - Y_j X^T_{j,(k)}(\beta^*_k) + \sqrt{k \log p/(n-1)u})_+ - (1 - Y_j X^T_{j,(k)}(\beta^*_k))_+ \right\} \\
    &\quad + \frac{1}{n-1} \sum_{j \neq i} \left\{ (1 - Y_j X^T_{j,(k)}(\beta^*_k) + \sqrt{k \log p/(n-1)u})_+ - (1 - Y_j X^T_{j,(k)}(\beta^*_k))_+ \right\} \\
    &\quad + Y_j X^T_{j,(k)} \sqrt{k \log p/(n-1)u} I\{1 - Y_j X^T_{j,(k)}(\beta^*_k) \geq 0\} \\
    &\quad - E \left[ (1 - Y_j X^T_{j,(k)}(\beta^*_k) + \sqrt{k \log p/(n-1)u})_+ - (1 - Y_j X^T_{j,(k)}(\beta^*_k))_+ \right] \\
    &\quad + E \left[ (1 - Y_j X^T_{j,(k)}(\beta^*_k) + \sqrt{k \log p/(n-1)u})_+ - (1 - Y_j X^T_{j,(k)}(\beta^*_k))_+ \right] \\
    &\quad - Y_j X^T_{j,(k)} \sqrt{k \log p/(n-1)u} I\{1 - Y_j X^T_{j,(k)}(\beta^*_k) \geq 0\},
\end{align*}
\]
where $I\{\cdot\}$ is the indicator function. Denote
\[
g_{j,k}(u) = \left[ (1 - Y_jX_{j,k}^T(\beta_{(k)}^*) + \sqrt{k\log p/(n - 1)u})_+ - (1 - Y_jX_{j,k}^T(\beta_{(k)}^*))_+ \right]
+ Y_jX_{j,k}^T\sqrt{k\log p/(n - 1)u}I\{1 - Y_jX_{j,k}^T(\beta_{(k)}^*) \geq 0\}
- E\left[ (1 - Y_jX_{j,k}^T(\beta_{(k)}^*) + \sqrt{k\log p/(n - 1)u})_+ - (1 - Y_jX_{j,k}^T(\beta_{(k)}^*))_+ \right],
\]
\[A_{n,i,k} = \sum_{j \neq i} g_{j,k}(u)\]
and
\[B_{n,i,k} = \sum_{j \neq i} \left\{ E\left[ (1 - Y_jX_{j,k}^T(\beta_{(k)}^*) + \sqrt{k\log p/(n - 1)u})_+ - (1 - Y_jX_{j,k}^T(\beta_{(k)}^*))_+ \right]
- Y_jX_{j,k}^T\sqrt{k\log p/(n - 1)u}I\{1 - Y_jX_{j,k}^T(\beta_{(k)}^*) \geq 0\} \right\}.\]
Thus
\[l_{ik}(\beta_{(k)}^*) + \sqrt{k\log p/(n - 1)u} - l_{ik}(\beta_{(k)}^*) = \frac{1}{n - 1}(A_{n,i,k} + B_{n,i,k}).\]
The rest of proof consists of three steps.

**Step 1:** We show that
\[\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta} |A_{n,i,k}| = \bar{k}o_p(1).\] (13)

For (13), we need to prove that for any $\epsilon > 0$,
\[P(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta} \bar{k}^{-1}|A_{n,i,k}| > \epsilon) \to 0 \text{ as } n \to \infty.\] (14)

By lemma 2.5 of [van de Geer (2000)](http://example.com), the ball $S = \{u : \|u\| \leq \Delta\}$ in $\mathbb{R}^{p_k+1}$ can be covered by $N$ balls with radius $\delta$, where $N \leq ((4\Delta + \delta)/\delta)^{p_k+1}$. Denote $u^1, \ldots, u^N$ the centers of the $N$ balls. For any $u \in S$, we can find a $u^* \in \{u^1, \ldots, u^N\}$ such that $\|u - u^*\| \leq \delta$. So
\[
P\left(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta} \bar{k}^{-1}|A_{n,i,k}| > \epsilon\right)
= P\left(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \bar{k}^{-1} \left| \sum_{j \neq i} (g_{j,k}(u) - g_{j,k}(u^*) + g_{j,k}(u^*)) \right| > \epsilon\right)
\leq P\left(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \bar{k}^{-1} \left[ \sum_{j \neq i} |g_{j,k}(u) - g_{j,k}(u^*)| + \left| \sum_{j \neq i} g_{j,k}(u^*) \right| \right] > \epsilon\right)
\leq P\left(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \bar{k}^{-1} \left| \sum_{j \neq i} g_{j,k}(u) - g_{j,k}(u^*) \right| > \epsilon/2\right)
+ P\left(\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \bar{k}^{-1} \left| \sum_{j \neq i} g_{j,k}(u^*) \right| > \epsilon/2\right).\]
In order to prove (14), we just need to prove the following two conclusions:

\[
P\left( \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \tilde{k}^{-1} \sum_{j \neq i} |g_{j,k}(u) - g_{j,k}(u^*)| > \epsilon/2 \right) \to 0 \quad (15)
\]

and

\[
P\left( \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \max_{1 \leq s \leq N} \tilde{k}^{-1} \sum_{j \neq i} g_{j,k}(u^*) > \epsilon/2 \right) \to 0. \quad (16)
\]

Consider (15) first. When \( \|u - u^*\| \leq \delta \), we have

\[
|g_{j,k}(u) - g_{j,k}(u^*)| \\
\leq \left| \left( 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* + \sqrt{k \log p/(n - 1)u}) \right) + \left( 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* + \sqrt{k \log p/(n - 1)u}) \right)^+ \right| \\
+ \sqrt{k \log p/(n - 1)} \left| Y_jX_{j,(k)}^T(u - u^*)I \{ 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* \geq 0) \} \right| \\
+ \left| E \left( 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* + \sqrt{k \log p/(n - 1)u}) \right) \right| \\
- \left| E \left( 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* + \sqrt{k \log p/(n - 1)u}) \right)^+ \right| \\
\leq 2 \|X_{j,(k)}\| \sqrt{k \log p/(n - 1)} \|u - u^*\| + E \|X_{j,(k)}\| \sqrt{k \log p/(n - 1)} \|u - u^*\| \\
\leq 3\tilde{k}M\sqrt{k \log p/(n - 1)} \|u - u^*\| \\
\leq 3\tilde{k}M\sqrt{\log p/(n - 1)} \delta,
\]

where the second inequality holds because the hinge loss satisfies the lipschitz condition and the third inequality holds because of the Condition (A2). Then

\[
\max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \sup_{\|u\| = \Delta, \|u - u^*\| \leq \delta} \tilde{k}^{-1} \sum_{j \neq i} |g_{j,k}(u) - g_{j,k}(u^*)| \leq (n - 1)\tilde{k}^{-1} \cdot 3\tilde{k}M\sqrt{k \log p/(n - 1)} \delta.
\]

Let the radius \( \delta = \frac{\epsilon}{6\tilde{k}M\sqrt{\log p}} \). Then (15) holds.

Now consider (16). Note that \( E[g_{j,k}(u^*)] = 0 \) and

\[
E[g_{j,k}(u^*)]^2 \leq E \left[ (1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* + \sqrt{k \log p/(n - 1)u^*})) + (1 - Y_jX_{j,(k)}^T(\beta_{(k)}^*))^+ \right] \\
+ Y_jX_{j,(k)}^T\sqrt{k \log p/(n - 1)u^*}I \{ 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^* \geq 0) \}^2 \\
= E \left[ I \left\{ 1 - Y_jX_{j,(k)}^T(\beta_{(k)}^*) \leq \sqrt{k \log p/(n - 1)} \max_j \|X_{j,(k)}\| \Delta \right\} \right].
\]
\[
\left\{ (1 - Y_j X_{j,(k)}^T \beta^*_k) + \sqrt{k \log p/(n-1) u^*})_+ - (1 - Y_j X_{j,(k)}^T \beta^*_k) + \\
+ Y_j X_{j,(k)}^T \sqrt{k \log p/(n-1) u^*} I\{1 - Y_j X_{j,(k)}^T \beta^*_k \geq 0\} \right\}^2
\]

\[
\leq E \left[ I \left\{ |1 - Y_j X_{j,(k)}^T \beta^*_k| < \sqrt{k \log p/(n-1) \max_j \| X_{j,(k)} \| \Delta} \right. \right. \\
\left. \left. \left( 2 \sqrt{k \log p/(n-1) Y_j X_{j,(k)}^T u^*} \right)^2 \right\} \right].
\]

The first equality holds because when

\[
|1 - Y_j X_{j,(k)}^T \beta^*_k| \geq \sqrt{k \log p/(n-1) \max_j \| X_{j,(k)} \| \Delta},
\]

we have

\[
(1 - Y_j X_{j,(k)}^T \beta^*_k) + \sqrt{k \log p/(n-1) u^*})_+ - (1 - Y_j X_{j,(k)}^T \beta^*_k) + \\
+ Y_j X_{j,(k)}^T \sqrt{k \log p/(n-1) u^*} I\{1 - Y_j X_{j,(k)}^T \beta^*_k \geq 0\} = 0.
\]

The second inequality holds because the hinge loss satisfies the lipschitz condition. As a result,

\[
\sum_{j \neq i} E[g_{j,k}(u^*)]^2 \\
\leq \sum_{j \neq i} E \left[ \left( 2 \sqrt{k \log p/(n-1) Y_j X_{j,(k)}^T u^*} \right)^2 \cdot \\
I \left\{ |1 - Y_j X_{j,(k)}^T \beta^*_k| < \sqrt{k \log p/(n-1) \max_j \| X_{j,(k)} \| \Delta} \right. \right. \\
\left. \left. \left| \right\} \right].
\]

\[
\leq \sum_{j \neq i} 4 \tilde{k} \log p/(n-1) \cdot \tilde{k} M^2 \Delta^2 \\
\cdot E \left[ I \left\{ |1 - Y_j X_{j,(k)}^T \beta^*_k| < \sqrt{k \log p/(n-1) \max_j \| X_{j,(k)} \| \Delta} \right. \right. \\
\left. \left. \left| \right\} \right]
= 4 \tilde{k}^2 \log p M^2 \Delta^2 \cdot \frac{p_k}{(p_k + 1)} \frac{M^2}{\Delta^2} \leq \tilde{k} M^2 \Delta^2.
\]

By the bounded conditional density condition (A3), we have

\[
P \left( |1 - Y_j X_{j,(k)}^T \beta^*_k| < \sqrt{k \log p/(n-1) \max_j \| X_{j,(k)} \| \Delta} \right) \leq C \tilde{k} \log n \sqrt{\log p/(n-1)},
\]

where \( C \) is a constant. Then following (17) we have

\[
\sum_{j \neq i} E[g_{j,k}(u^*)]^2 \leq 4 \tilde{k}^2 \log p M^2 \Delta^2 \cdot C \tilde{k} \log n \sqrt{\log p/(n-1)} = 4 C M^2 \Delta^2 \cdot \tilde{k}^3 \log n (\log p)^{\frac{3}{2}} (n-1)^{-\frac{1}{2}}.
\]

21
Note \(|g_{j,k}(u^*)| \leq 3M\Delta k\sqrt{\log p/(n-1)}\). Then by Bernstein inequality and (A4), for large enough \(n\), we have

\[
P \left( \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \max_{1 \leq s \leq N} \sum_{j \neq i} g_{j,k}(u^*) \right) > \epsilon/2
\]

\[
\leq \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s=1}^{N} P \left( \left| \frac{1}{n-1} \sum_{j \neq i} g_{j,k}(u^*) \right| > \frac{\epsilon k}{2(n-1)} \right)
\]

\[
\leq 2 \exp \left\{ \log n + \log K + \log N - C_1 k^{-1} (\log n)^{-1} (\log p)^{-3/2} (n-1)^{1/2} \right\} \to 0,
\]

where \(C_1\) is a constant. This finishes the proof of (16) and (13).

**Step 2:** We show that \(\min_{1 \leq i \leq n} \min_{1 \leq k \leq K} \inf_{\|u\| = \Delta} B_{n,i,k}\) dominates the term of \(\tilde{k}o_p(1)\). By Cauchy inequality, we have

\[
\left| \sum_{j \neq i} Y_j X_{j,(k)}^T u I \{ 1 - Y_j X_{j,(k)}^T \beta_{(k)}^* \geq 0 \} \right|
\]

\[
\leq \|u\| \left\| \sum_{j \neq i} Y_j X_{j,(k)} I \{ 1 - Y_j X_{j,(k)}^T \beta_{(k)}^* \geq 0 \} \right\|
\]

\[
\leq \Delta \sqrt{p_k + 1} \max_{0 \leq l \leq p_k} \left| \sum_{j \neq i} Y_j X_{j,(k)} I \{ 1 - Y_j X_{j,(k)}^T \beta_{(k)}^* \geq 0 \} \right|.
\]

By \(E(Y_j X_{j,(k)} I \{ 1 - Y_j X_{j,(k)}^T \beta_{(k)}^* > 0 \}) = 0\) and Lemma 14.24 of Buhlmann and van de Geer (2011), we have

\[
\max_{0 \leq l \leq p_k} \left| \sum_{j \neq i} Y_j X_{j,(k)} I \{ 1 - Y_j X_{j,(k)}^T \beta_{(k)}^* \geq 0 \} \right| = O_p(\sqrt{(n-1) \log (p_k + 1)}) = O_p(\sqrt{(n-1) \log k}).
\]

So the second term of \(B_{n,i,k}\) has order \(\sqrt{k \log p/(n-1)} \cdot \Delta \sqrt{p_k + 1} \cdot O_p(\sqrt{(n-1) \log k}) = O_p(\tilde{k} \sqrt{\log p \log k}) \Delta = O_p(\tilde{k} \log p) \Delta\). By Taylor expansion of hinge loss function at \(\beta_{(k)}^*\), we have

\[
\sum_{j \neq i} E \left[ (1 - Y_j X_{j,(k)}^T \beta_{(k)}^*) + \sqrt{\tilde{k} \log p/(n-1)u}) + (1 - Y_j X_{j,(k)}^T \beta_{(k)}^*)^+ \right]
\]

\[
= 0.5 \tilde{k} \log p \cdot u^T H_k \left( \beta_{(k)}^* + t \sqrt{\tilde{k} \log p/(n-1)u} \right) u
\]

\[
= 0.5 \tilde{k} \log p \cdot u^T \left[ H_k(\beta_{(k)}^*) + o_p(1) \right] u
\]

\[
\geq 0.5 c_1 \tilde{k} \log p \Delta^2 (1 + o_p(1))
\]

for some \(0 \leq t \leq 1\) and \(c_1\) is from Condition (A5). As a result, \(0.5 c_1 \tilde{k} \log p \Delta^2\) dominates the other terms in \(B_{n,i,k}\) for sufficiently large \(\Delta\). This finishes the proof of Step 2.

22
Step 3: From the results above, we know that $A_{n,i,k} + B_{n,i,k}$ is dominated by $0.5c_1\bar{k}\log p\Delta^2$ for sufficiently large $n$ and $\Delta$, which is positive. Then we prove (12) and finish the proof of Theorem 3.1.

Proof of Lemma 3.2

\[
\left| EHR_n(w) - E \left( 1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T \right) \right| 
\leq E \left[ (1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T )^+ | D_n \right] - E \left( 1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T \right) + \\
= E \left[ (1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T )^+ | D_n \right] - E \left( 1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T \right) + \\
= E \left[ (1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T (\beta^*_T - \hat{\beta}(k)) )^+ \right] - \int_0^{-\sum_{k=1}^{K} w_k X_{(k)}^T (\beta^*_T - \hat{\beta}(k))} \left( I \{ 1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T \geq 0 \} - I \{ 1 - Y \sum_{k=1}^{K} w_k X_{(k)}^T \beta^*_T \geq s \} \right) ds | D_n 
\leq 3E \left[ \sum_{k=1}^{K} w_k \left| X_{(k)}^T (\beta^*_T - \hat{\beta}(k)) \right| | D_n \right] 
\leq 3 \sum_{k=1}^{K} w_k \left( E \left[ X_{(k)}^T (\beta^*_T - \hat{\beta}(k))^2 | D_n \right] \right)^{1/2} 
= 3 \sum_{k=1}^{K} w_k \left[ (\beta^*_T - \hat{\beta}(k))^T E \left( X_{(k)} X_{(k)}^T \right) (\beta^*_T - \hat{\beta}(k)) \right]^{1/2} 
\leq 3 \sum_{k=1}^{K} w_k \sqrt{2} O_p \left( \sqrt{n^{-1}\bar{k}\log p} \right) = O_p \left( \sqrt{n^{-1}\bar{k}\log p} \right),
\]
where the third equality holds because of the Knight’s (1998) identity
\[(u + v)I\{u + v \geq 0\} - uI\{u \geq 0\} = vI\{u \geq 0\} - \int_0^{-v} [I\{u \geq s\} - I\{u \geq 0\}]ds\]
and the last inequality is from condition (A6) and Theorem 3.1. This shows (7).

Now we show (8). Rewrite
\[CV_n(w) - EH_R_n(w) = CV_{1n}(w) + CV_{2n}(w) + CV_{3n}(w),\]
where
\[CV_{1n}(w) = \frac{1}{n} \sum_{i=1}^{n} \left\{ (1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^*) - (1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^*) \right\},\]
\[CV_{2n}(w) = - \left\{ EH_R_n(w) - E(1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^*) \right\},\]
and
\[CV_{3n}(w) = \frac{1}{n} \sum_{i=1}^{n} \left\{ (1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^*) - E(1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^*) \right\}.

(i) We show sup_{w \in \mathcal{W}} |CV_{1n}(w)| = O_p \left( \sqrt{n^{-1}k^2 \log p} \right).

According to Knight’s (1998) identity, we have
\[CV_{1n}(w) = \frac{1}{n} \sum_{i=1}^{n} \left\{ Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T (\beta_{i(k)}^* - \hat{\beta}_{i(k)}) I\{1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \geq 0\} \right\}
- \int_{0}^{-Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T (\beta_{i(k)}^* - \hat{\beta}_{i(k)})} \left( I\{1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \geq s\} \right) ds \}
- \left\{ 1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \right\} ds \}
= CV_{1n,1}(w) + CV_{1n,2}(w),

where
\[CV_{1n,1}(w) = \frac{1}{n} \sum_{i=1}^{n} \left\{ Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T (\beta_{i(k)}^* - \hat{\beta}_{i(k)}) I\{1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \geq 0\} \right\}

and
\[CV_{1n,2}(w) = - \frac{1}{n} \sum_{i=1}^{n} \left\{ \int_{0}^{-Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T (\beta_{i(k)}^* - \hat{\beta}_{i(k)})} \left( I\{1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \geq s\} \right) ds \}
- \left\{ 1 - Y_i \sum_{k=1}^{K} w_k X_{i,(k)}^T \beta_{i(k)}^* \right\} ds \}.\]
For $CV_{1n,1}(w)$, we have
\[
\sup_{w \in W} |CV_{1n,1}(w)| \leq \sup_{w \in W} \frac{1}{n} \sum_{k=1}^{K} w_k \sum_{i=1}^{n} |Y_i \mathbf{X}_{i,(k)}^T (\beta^*_k - \hat{\beta}_i(k))|
\]
\[
\leq \max_{1 \leq i \leq n} \max_{1 \leq k \leq K} \|\beta^*_k - \hat{\beta}_i(k)\| \max_{1 \leq k \leq K} \frac{1}{n} \sum_{i=1}^{n} \|\mathbf{X}_{i,(k)}\|
\]
\[
= O_p \left( \sqrt{(n-1)^{-1}k \log p} \right) O(\sqrt{k}) = O_p \left( \sqrt{n^{-1}k^2 \log p} \right). \tag{18}
\]

For $CV_{1n,2}(w)$, we have
\[
\sup_{w \in W} |CV_{1n,2}(w)| \leq \sup_{w \in W} \frac{2}{n} \sum_{k=1}^{K} w_k \sum_{i=1}^{n} |Y_i \mathbf{X}_{i,(k)}^T (\beta^*_k - \hat{\beta}_i(k))| = O_p \left( \sqrt{n^{-1}k^2 \log p} \right),
\]
where the equality holds because of (18).

(ii) By (i) we have $\sup_{w \in W} |CV_{2n}(w)| = O_p \left( \sqrt{n^{-1}k \log p} \right)$.

(iii) By the weak law of large numbers, $CV_{3n}(w) = O_p(\sqrt{1/n})$.

By (i), (ii) and (iii), we finish the proof of (8).

**Proof of Theorem 3.3** By (8), we have
\[
\sup_{w \in W} \left| \frac{CV_n(w)}{EH_{R_n}(w)} - 1 \right| \leq \sup_{w \in W} \frac{|CV_n(w) - EH_{R_n}(w)|}{\inf_{w \in W} EH_{R_n}(w)} = O_p \left( \xi_n^{-1} \sqrt{n^{-1}k^2 \log p} \right). \tag{19}
\]

By the fact that $\inf_{w \in W} EH_{R_n}(w) \geq E(1 - Y^T \mathbf{x}^*)_+ > 0$ with probability 1, and the Lemma 3 of [Feng et al. (2021)], we have
\[
\frac{\inf_{w \in W} CV_n(w)}{\inf_{w \in W} EH_{R_n}(w)} = 1 + O_p \left( \xi_n^{-1} \sqrt{n^{-1}k^2 \log p} \right). \tag{19}
\]
Then
\[
\left| \frac{EH_{R_n}(\hat{w})}{\inf_{w \in W} EH_{R_n}(w)} - 1 \right| = \left| \frac{CV_n(\hat{w})}{\inf_{w \in W} EH_{R_n}(w)} - 1 + \frac{EH_{R_n}(\hat{w}) - CV_n(\hat{w})}{\inf_{w \in W} EH_{R_n}(w)} \right|
\]
\[
\leq \left| \frac{\inf_{w \in W} CV_n(w)}{\inf_{w \in W} EH_{R_n}(w)} - 1 \right| + \sup_{w \in W} |CV_n(w) - EH_{R_n}(w)| \inf_{w \in W} EH_{R_n}(w)
\]
\[
= O_p \left( \xi_n^{-1} \sqrt{n^{-1}k^2 \log p} \right),
\]
where the last line holds because of (19) and (8). This finishes the proof of Theorem 3.3.

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