ON THE ALEXANDER POLYNOMIALS OF HURWITZ CURVES

VIK.S. KULIKOV

Abstract. Properties of the Alexander polynomials of Hurwitz curves are investigated. A complete description of the set of the Alexander polynomials of irreducible Hurwitz curves in the terms of their roots is given.

0. Introduction

In [1], investigation of properties of the Alexander polynomials of Hurwitz curves in \( \mathbb{CP}^2 \) were started. Recall briefly definitions of a Hurwitz curve (with respect to a linear projection \( \text{pr} : \mathbb{CP}^2 \to \mathbb{CP}^1 \)) and its Alexander polynomial. Let \( \mathbb{C}^2_i \) be two copies of the affine plane \( \mathbb{C}^2 \), \( i = 1, 2 \), with coordinates \((u_i, v_i)\), \( u_2 = 1/u_1 \) and \( v_2 = v_1 / u_1 \), which cover \( \mathbb{CP}^2 \setminus p_{\infty} \) (where \( p_{\infty} \) is the center of the projection) such that \( \text{pr} \) is given by \((u_i, v_i) \mapsto u_i \) in the charts \( \mathbb{C}^2_i \). A set \( \bar{H} \subset \mathbb{CP}^2 \setminus \{p_{\infty}\} \), closed in \( \mathbb{CP}^2 \), is called a Hurwitz curve of degree \( m \) if, for \( i = 1, 2 \), \( \bar{H} \cap \mathbb{C}^2_i \) coincides with the set of zeros of an equation

\[
 F_i(u_i, v_i) := v_i^m + \sum_{j=0}^{m-1} c_{j,i}(u_i)v_i^j = 0
\]

such that

(i) \( F_i(u_i, v_i) \) is a \( C^\infty \)-smooth complex valued function in \( \mathbb{C}^2 \);
(ii) the function \( F_i(u_i, v_i) \) has only a finite number of critical values, that is, there are finitely many values of \( u_i \), say \( u_{i,1}, \ldots, u_{i,n_i} \), such that the polynomial equation

\[
 v_i^m + \sum_{j=0}^{m-1} c_{j,i}(u_{i,0})v_i^j = 0 \tag{1}
\]

has no multiple roots for \( u_{i,0} \not\in \{u_{i,1}, \ldots, u_{i,n_i}\} \);
(iii) if \( v_{i,j} \) is a multiple root of equation (1) for \( u_{i,j} \in \{u_{i,1}, \ldots, u_{i,n_i}\} \), then, in a neighbourhood of the point \((u_{i,j}, v_{i,j})\), the set \( \bar{H} \) coincides with the solution a complex analytic equation.

The author was partially supported by the RFBR (02-01-00786).
A Hurwitz curve $\mathcal{H}$ is called \textit{irreducible} if $\mathcal{H} \setminus M$ is connected for any finite set $M \subset \mathcal{H}$, and we say that a Hurwitz curve $\mathcal{H}$ \textit{consists of} $k$ \textit{irreducible components} if

$$k = \max \#\{\text{connected components of } \mathcal{H} \setminus M\},$$

where the maximum is taken over all finite sets $M \subset \mathcal{H}$.

Let $H$ be an \textit{affine Hurwitz curve}, that is, $H = \mathcal{H} \cap (\mathbb{C}P^2 \setminus L_\infty)$, where $L_\infty$ is a line which is a fibre of $pr$ being in general position with respect to $H$. Then the fundamental group $\pi_1 = \pi_1(\mathbb{C}P^2 \setminus (H \cup L_\infty))$ does not depend on the choice of $L_\infty$ and belongs to the class $\mathcal{C}$ of so-called $\mathcal{C}$-groups.

By definition, a $\mathcal{C}$-\textit{group} is a group together with a finite presentation

$$G_W = \langle x_1, \ldots, x_m \mid x_i = w_{i,j,k}^{-1}x_jw_{i,j,k}, w_{i,j,k} \in W \rangle,$$

where $W = \{w_{i,j,k} \in \mathbb{F}_m \mid 1 \leq i, j \leq m, 1 \leq k \leq h(i, j)\}$ is a subset of elements of the free group $\mathbb{F}_m$ (it is possible that $w_{i_1,j_1,k_1} = w_{i_2,j_2,k_2}$ for $(i_1,j_1,k_1) \neq (i_2,j_2,k_2)$), generated by free generators $x_1, \ldots, x_m$ and $h : \{1, \ldots, m\}^2 \rightarrow \mathbb{Z}$ is some function. Such a presentation is called a $\mathcal{C}$-\textit{presentation} ($\mathcal{C}$, since all relations are conjugations). Let $\varphi_W : \mathbb{F}_m \rightarrow G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \leq i \leq m$, and the elements conjugated to them are called the $\mathcal{C}$-generators of the $\mathcal{C}$-group $G$. Let $f : G_1 \rightarrow G_2$ be a homomorphism of $\mathcal{C}$-groups. It is called a $\mathcal{C}$-\textit{homomorphism} if the images of the $\mathcal{C}$-generators of $G_1$ under $f$ are $\mathcal{C}$-generators of the $\mathcal{C}$-group $G_2$. $\mathcal{C}$-groups will be considered up to $\mathcal{C}$-isomorphisms.

Any $\mathcal{C}$-group $G$ can be realized as the fundamental group $\pi_1(S^4 \setminus S)$ of the complement of a closed oriented surface $S$ in the 4-dimensional sphere $S^4$ (see, for example, [3]\textsuperscript{1}).

A $\mathcal{C}$-presentation [2] is called a \textit{Hurwitz $\mathcal{C}$-presentation of degree} $m$ if for each $i = 1, \ldots, m$ the word $w_{i,i,1}$ coincides with the product $x_1 \ldots x_m$, and a $\mathcal{C}$-group $G$ is called a \textit{Hurwitz $\mathcal{C}$-group of degree} $m$ if it possesses a Hurwitz $\mathcal{C}$-presentation of degree $m$. In other words, a $\mathcal{C}$-group $G$ is a Hurwitz $\mathcal{C}$-group of degree $m$ if there are $\mathcal{C}$-generators $x_1, \ldots, x_m$ generating $G$ such that the product $x_1 \ldots x_m$ belongs to the center of $G$. Note that the degree of a Hurwitz $\mathcal{C}$-group $G$ is not defined canonically and depends on the Hurwitz $\mathcal{C}$-presentation of $G$. Denote by $\mathcal{H}$ the class of all Hurwitz $\mathcal{C}$-groups.

Let $\mathcal{H}$ be a Hurwitz curve of degree $m$. A Zariski–van Kampen presentation of $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ (where $\mathbb{C}^2 = \mathbb{C}P^2 \setminus L_\infty$ and $L_\infty$, a fibre of $pr$, is in general position with respect to $\mathcal{H}$) defines on $\pi_1$ a

\textsuperscript{1}Some other properties of $\mathcal{C}$-groups can be found in [2], [4], [6].
structure of a Hurwitz $C$-group of degree $m$ (see [4]), and in [4], it
was proved that any Hurwitz $C$-group $G$ of degree $m$ can be realized
as the fundamental group $\pi_1(C^2 \setminus H)$ for some Hurwitz curve $H$ with
singularities of the form $w^m - z^m = 0$, $\deg H = 2^nm$, where $n$ depends
on the Hurwitz $C$-presentation of $G$. Since we consider $C$-groups up
to $C$-isomorphisms, the class $\mathcal{H}$ coincides with the class $\{\pi_1(C^2 \setminus H)\}$
of fundamental groups of the complements of affine Hurwitz curves.

It is easy to show that $G/G'$ is a finitely generated free abelian group
for any $C$-group $G$. A $C$-group $G$ is called irreducible if $G/G' \simeq \mathbb{Z}$ and
we say that $G$ consists of $n$ irreducible components if $G/G' \simeq \mathbb{Z}^n$. If
a Hurwitz $C$-group $G$ is realized as the fundamental group $\pi_1(C^2 \setminus H)$
of the complement of some affine Hurwitz curve $H$, then the number
of irreducible components of $G$ is equal to the number of irreducible
components of $H$. Similarly, if a $C$-group $G$ consisting of $n$ irreducible
components is realized as $G = \pi_1(S^4 \setminus S)$, then the number of connected
components of the surface $S$ is equal to $n$.

A free group $F_n$ with fixed free generators is a $C$-group and for any
$C$-group $G$ the canonical $C$-epimorphism $\nu : G \to F_1$, sending the $C$-
generators of $G$ to the $C$-generator of $F_1$, is well defined. Denote by $N$
its kernel. Note that if $G$ is an irreducible $C$-group, then $N$ coincides
with $G'$.

Let $G$ be a $C$-group. The $C$-epimorphism $\nu$ induces the following
exact sequence of groups

$$1 \to N/N' \to G/N' \xrightarrow{\nu_*} F_1 \to 1.$$

The $C$-generator of $F_1$ acts on $N/N'$ by conjugation $x^{-1}gx$, where
$g \in N$ and $x$ is one of the $C$-generators of $G$. Denote by $h$ this action
and by $h_C$ the induced action on $(N/N')_C = (N/N') \otimes \mathbb{C}$. The
characteristic polynomial $\Delta(t) = \det(h_C - t\text{Id})$ is called the Alexander polynomial of the $C$-group $G$ (if the vector space $(N/N')_C$
over $\mathbb{C}$ is infinite dimensional, then, by definition, the Alexander polynomial
$\Delta(t) \equiv 0$). For a Hurwitz curve $H$ the Alexander polynomial $\Delta(t)$ of
the group $\pi_1 = \pi_1(C^2 \setminus H)$ is called the Alexander polynomial of $H$.
Note that the Alexander polynomial $\Delta(t)$ of a Hurwitz curve $H$ does not
depend on the choice of the generic line $L_\infty$.

The homomorphism $\nu : \pi_1 \to F_1$ defines an infinite unramified cyclic
covering $f_\infty : X_\infty \to C^2 \setminus H$. We have $H_1(X_\infty, \mathbb{Z}) = N/N'$ and the
action of $h$ on $H_1(X_\infty, \mathbb{Z})$ coincides with the action of a generator of

\[\text{For a group } G \text{ we use the standard notation } G' \text{ for its commutator subgroup and } [g_1, g_2] \text{ for the commutator of elements } g_1, g_2 \in G.\]
the covering transformation group of the covering \( f_\infty \). As it follows from [5], the group \( H_1(X_\infty, \mathbb{Z}) \) is finitely generated.

For any \( n \in \mathbb{N} \) denote by \( \text{mod}_n : F_1 \to \mu_n = F_1/\{h^n\} \) the natural epimorphism to the cyclic group \( \mu_n \) of degree \( n \). The covering \( f_\infty \) can be factorized through the cyclic covering \( f_n : X_n \to \mathbb{C}^2 \setminus H \) associated with the epimorphism \( \text{mod}_n \circ \nu, f_\infty = g_n \circ f_n \). Since a Hurwitz curve \( \bar{H} \) has only analytic singularities, the covering \( f_n \) can be extended to a smooth map \( \bar{f}_n : \bar{X}_n \to \mathbb{C}\mathbb{P}^2 \) branched along \( \bar{H} \) and, maybe, along \( L_\infty \) (if \( n \) is not a divisor of \( \deg \bar{H} \), then \( \bar{f}_n \) is branched along \( L_\infty \)), where \( \bar{X}_n \) is a smooth 4-fold. The action \( h \) induces an action \( h_n \) on \( \bar{X}_n \) and an action \( h_n^* \) on \( H_1(X_n, \mathbb{C}) \).

In [1], it was shown that any such \( \bar{X}_n \) can be embedded as a symplectic submanifold to a projective rational 3-fold on which the symplectic structure is given by an integer Kähler form, and it was proved that the first Betti number \( b_1(\bar{X}_n) = \dim \mathbb{C} H_1(\bar{X}_n, \mathbb{C}) \) of \( \bar{X}_n \) is equal to \( r_n, \neq 1 \), where \( r_n, \neq 1 \) is the number of roots of the Alexander polynomial \( \Delta(t) \) of the curve \( \bar{H} \) which are \( n \)-th roots of unity not equal to 1.

In [1], properties of the Alexander polynomials of Hurwitz curves were investigated. In particular, it was proved that if \( \bar{H} \) is a Hurwitz curve of degree \( d \) consisting of \( n \) irreducible components and \( \Delta(t) \) is its Alexander polynomial, then

(i) \( \Delta(t) \in \mathbb{Z}[t] \);
(ii) \( \Delta(0) = \pm 1 \);
(iii) the roots of \( \Delta(t) \) are \( d \)-th roots of unity;
(iv) the action of \( h_\mathbb{C} \) on \( (N/N') \otimes \mathbb{C} \) is semisimple;
(v) \( \Delta(t) \) is a divisor of the polynomial \( (t-1)(t^d-1)^{d-2} \);
(vi) the multiplicity of the root \( t = 1 \) of \( \Delta(t) \) is equal to \( n-1 \);
(vii) if \( n = 1 \), then \( \Delta(1) = 1 \) and \( \deg \Delta(t) \) is an even number.

The main results of the article are the following theorems.

**Theorem 0.1.** A polynomial \( P(t) \in \mathbb{Z}[t] \) is the Alexander polynomial of an irreducible Hurwitz \( C \)-group \( G \) iff the roots of \( P(t) \) are roots of unity and \( P(1) = 1 \).

\(^3\text{Recall that the class of Hurwitz } C\text{-groups coincides with the class of fundamental groups of the complements of affine Hurwitz curves. Therefore to speak about the Alexander polynomials of Hurwitz curves is the same as to speak about the Alexander polynomials of Hurwitz } C\text{-groups. Hence the results of the article are formulated in terms of Hurwitz } C\text{-groups, since their proves are purely algebraic.} \)
Theorem 0.2. Let a polynomial \( P(t) = (-1)^{m}t^{m} + \sum_{i=0}^{m-1} a_{i}t^{i} \in \mathbb{Z}[t] \) has the following properties:

(i) the roots of \( P(t) \) are roots of unity;

(ii) if \( \zeta \) is a primitive \( p^{k} \)-th root of unity, where \( p \) is a prime number, then the multiplicity of the root \( t = \zeta \) of the polynomial \( P(t) \) is not greater than the multiplicity of its root \( t = 1 \).

Then \( P(t) \) is the Alexander polynomial of some Hurwitz \( C \)-group \( G \).

Theorem 0.3. The polynomial \( P(t) = (-1)^{n+k}(t-1)^{n}(t+1)^{k} \) is the Alexander polynomial of some Hurwitz \( C \)-group \( G \) iff \( n \geq k \).

The proof of Theorem 0.1 is given in section 1. Section 2 is devoted to the proof of Theorem 0.2. In spite of the proves of these theorems use almost the same ideas, the proves are given independently in order to have more clear exposition. The proof of Theorem 0.3 is given in section 3. These theorems allow to formulate the following

Conjecture 0.4. A polynomial \( P(t) = (-1)^{m}t^{m} + \sum_{i=0}^{m-1} a_{i}t^{i} \in \mathbb{Z}[t] \) is the Alexander polynomial of some Hurwitz \( C \)-group \( G \) iff it satisfies the following conditions:

(i) the roots of \( P(t) \) are roots of unity;

(ii) if \( \zeta \) is a primitive \( p^{k} \)-th root of unity, where \( p \) is a prime number, then the multiplicity of the root \( t = \zeta \) of the polynomial \( P(t) \) is not greater than the multiplicity of its root \( t = 1 \).

In section 4, to compare the case of Hurwitz \( C \)-groups with the general case of \( C \)-groups, we give an example (see Example 4.2) of a \( C \)-group consisting of two irreducible components whose Alexander polynomial \( \Delta(t) = (1-t)(t+1)^{2} \) (compare it with Theorem 0.3), and an example (see Example 4.1) of a \( C \)-group consisting also of two irreducible components whose Alexander polynomial \( \Delta(t) = (t-1)^{2} \) (compare it with property (vi) mentioned above, see also Theorem 5.9 in [1]).

1. Alexander polynomials of irreducible Hurwitz \( C \)-groups

In this section, we prove Theorem 0.1.

In [1], it was proved that if a polynomial \( P(t) \in \mathbb{Z}[t] \) is the Alexander polynomial of an irreducible Hurwitz \( C \)-group \( G \), then the roots of \( P(t) \) are roots of unity and \( P(1) = 1 \). Therefore to prove Theorem 0.1, it suffices to prove the inverse statement.

Consider a polynomial \( P(t) \in \mathbb{Z}[t] \) such that the roots of \( P(t) \) are roots of unity and \( P(1) = 1 \).
In the beginning, let us show that it suffices to prove that for any polynomial \( \Psi(t) \) such that

1. The roots of \( \Psi(t) \) are roots of unity,
2. \( \Psi(t) \) has not multiple roots,
3. \( \Psi(1) = 1 \),

there exists an irreducible Hurwitz \( C \)-group with the Alexander polynomial \( \Delta(t) = \Psi(t) \). Indeed, each polynomial \( P(t) \in \mathbb{Z}[t] \) can be factorized into the product \( P(t) = \prod_i \Psi_i(t) \), where each factor \( \Psi_i(t) \in \mathbb{Z}[t] \) has not multiple roots. Since the roots of \( P(t) \) are roots of unity, \( P(1) = 1 \), and \( \Psi_i(1) \in \mathbb{Z} \), we have \( \Psi_i(1) = 1 \) for each \( i \). Next, if \( \Delta_1(t) \) and \( \Delta_2(t) \) are the Alexander polynomials of two irreducible Hurwitz \( C \)-groups \( G_1 \) and \( G_2 \) given by Hurwitz \( C \)-presentations \( G_i = \langle x_1,i, \ldots, x_{m_i,i} \mid R_i \rangle \) of degrees \( m_i, i = 1, 2 \), then, by Proposition 5.12 of [1], the Alexander polynomial of a Hurwitz product \( G = G_1 \circ G_2 \) is equal to \( \Delta_1(t)\Delta_2(t) \), where the Hurwitz product \( G \) of \( G_1 \) and \( G_2 \) is an irreducible Hurwitz \( C \)-group given by \( C \)-presentation

\[
G = \langle x_1,i, \ldots, x_{m_i,i} \mid R_1 \cup R_2, x_{m_1,1} = x_{m_2,2}, \\
[x_{j,i} (x_1,\overline{i} \ldots x_{m\overline{i},i})^{m_i}] = 1 \text{ for } j = 1, \ldots, m_i - 1, i = 1, 2 \rangle,
\]

where \( \overline{i} = \{1, 2\} \setminus \{i\} \) (recall that it follows from the set of relations \( R_\overline{i} \) that the elements \( x_1,\overline{i}, \ldots, x_{m,\overline{i}} \) commute with the product \( x_1,\overline{i} \ldots x_{m,\overline{i}} \).

Fix one of the polynomials \( \Psi_i(t) = \Psi(t) \). Denote by \( k \) the smallest positive integer such that all roots of \( \Psi(t) \) are \( k \)-th roots of unity. Since \( \Psi(t) \) has not multiple roots, \( \Psi(t) \) is a divisor of the polynomial \( t^k - 1 \). To prove the existence of an irreducible Hurwitz \( C \)-group whose Alexander polynomial coincides with \( \Psi(t) \), consider the quotient ring \( M = M_\Psi = \mathbb{Z}[t]/(\Psi(t)) \) of the ring \( \mathbb{Z}[t] \).

The ring \( M \) (considered as an abelian group with respect to additive operation) is a free abelian group freely generated by the elements \( t_0 = t^0, \ldots, t_{d-1} = t^{d-1} \), where \( d = \deg \Psi(t) \). Let \( h_0 \in \text{End}_\mathbb{Z} M \) given by \( h_0(Q(t)) = QT(t) \) and \( i : \mathbb{F}_1 \to \text{End}_\mathbb{Z} M \) given by \( i(x_0) = h_0 \), where \( x_0 \) is a generator of the free group \( \mathbb{F}_1 \). Obviously, \( h_0 \in \text{Aut} M \). It is not hard to show (see, for example, [4], Chapter XV) that \( \det(h_0, C - t\text{Id}) = \Psi(t) \), where \( h_0, C \in \text{Aut}(M \otimes \mathbb{C}) \) is induced by \( h_0 \).
Since $h_0 \in \text{Aut } M$, we can consider (below, we use the multiplicative notation for the group operation in $M$) the semi-direct product

$$G_\Psi = M \rtimes \mathbb{F}_1 \simeq \langle t_0, \ldots, t_{d-1}, x_0 \mid [t_i, t_j] = 1 \quad \text{for } 0 \leq i, j \leq d-1, \]
$$

$$x_0^{-1}t_i x_0 = t_{i+1}, \quad \text{for } i = 0, \ldots, d-2,$$

$$x_0^{-1}t_{d-1} x_0 = \prod_{i=0}^{d-1} t_i^{-a_i},$$

where $a_i$ are taken from the equality $\Psi(t) = t^d + \sum_{i=0}^{d-1} a_i t^i$.

Let $\nu : G_\Psi \to \mathbb{F}_1$ be the canonical epimorphism, $\ker \nu = M \subset G_\Psi$. Obviously, $h_0(g) = x_0^{-1}gx_0$ for $g \in M$.

For $i \geq d$ denote by $t_i = x_0^{-i}t_0 x^i$. Evidently, $x_0^k$ belongs to the center of $G_\Psi$, since $\Psi(t)$ is a divisor of the polynomial $t^k - 1$ and therefore

$$t^{k+i} \equiv t^i \mod \Psi(t)$$

for all $i \geq 0$. Thus, we have $t_i = t_{k+i}$ for all $i \geq 0$.

For $P(t) = \sum_{i=0}^n c_i t^i \in \mathbb{Z}[t]$ denote by $x_{P(t)} = x_0 \prod_{i=0}^n t_i^{c_i}$. Obviously,

$$x_{P(t)} = x_{Q(t)} \quad \text{if } P(t) \equiv Q(t) \mod \Psi(t) \quad (3)$$

and it is easy to check that

$$x_0^{-1} x_{P(t)} x_0 = x_{tP(t)}, \quad (4)$$

and

$$\left(\prod_{i=0}^n t_i^{c_i}\right) x_{Q(t)} \left(\prod_{i=0}^n t_i^{c_i}\right)^{-1} = x_{(t-1)P(t)+Q(t)} \quad (5)$$

for $P(t) = \sum_{i=0}^n c_i t^i$ and for any $Q(t)$. In particular,

$$x_{t^{i+1}} = x_0^{-1} x_{t^i} x_0 \quad (6)$$

for all $i \geq 0$.

Since $\Psi(1) = 1$, one can find a polynomial $P(t) \in \mathbb{Z}[t]$ such that $\Psi(t) = (t-1)P(t) + 1$. Therefore it follows from (3) and (5) that there is $g_{1,0} \in G_\Psi$ such that $x_0 = g_{1,0}^{-1} x_0 g_{1,0}$. Consequently, by (4), the elements $x_0, x_{t^{0}}, \ldots, x_{t^{d-1}}$ are conjugated to each other and therefore $x_i^k$ belongs to the center of $G_\Psi$ for each $i$.

Since $t_i = x_0^{-1} x_{t^i}$ and $x_0, t_0, \ldots, t_{d-1}$ generate $G_\Psi$, the elements $x_0, x_{t^{0}}, \ldots, x_{t^{d-1}}$ also generate $G_\Psi$. Let $w_{1,0}(x_0, x_{t^{0}}, \ldots, x_{t^{d-1}})$ be a word
in letters \( x_0, x_0', \ldots, x_d, \ldots, x_{d-1} \) and their inverses representing the element \( g_{1,0} \). Consider a \( C \)-group \( \tilde{G} \) given by \( C \)-presentation
\[
\tilde{G} = \langle x_1, \ldots, x_{d+1} \mid x_{i+1} = x_i^{-1} x_1 x_i \quad \text{for } 2 \leq i \leq d, \\
x_1^{-k} x_1 x_1^k = x_i \quad \text{for } 2 \leq i \leq d + 1, \\
x_2 = w_{1,0}^{-1}(x_1, \ldots, x_{d+1})x_1 w_{1,0}(x_1, \ldots, x_{d+1}) \rangle. 
\]
(7)

First of all, note that \( \tilde{G} \) is an irreducible \( C \)-group, since all \( C \)-generators of \( \tilde{G} \) are conjugated to each other. Next, the element \( x_1^k \) belongs to the center of \( \tilde{G} \). Therefore \( x_i^k = x_1^k \) for \( 2 \leq i \leq d + 1 \), since the elements \( x_i^k \) are also conjugated to \( x_1^k \) in \( \tilde{G} \). Hence the product \( x_1^k \ldots x_{d+1}^k \) belongs to the center of \( \tilde{G} \). Therefore, by Lemma 5.11 of \( \mathbb{U} \), \( \tilde{G} \) is a Hurwitz \( C \)-group of degree \( k(d+1) \). (Indeed, one can consider another \( C \)-presentation of \( \tilde{G} \) with \( C \)-generators \( x_{i,j} \), \( 1 \leq i \leq d+1, \quad 1 \leq j \leq k \), satisfying the following defining relations:
\[
\begin{align*}
x_{i,j}^{-1} x_{i,j} x_{i,j} & = x_{i,j} \quad \text{for all } i, j, \in \mathbb{Z} \\
x_{1,1}^{-1} x_{1,1} x_{1,1} & = x_{1,1} \quad \text{for } 2 \leq i \leq d; \\
x_{1,1}^{-k} x_{1,1} x_{1,1}^k & = x_{1,1} \quad \text{for } 2 \leq i \leq d + 1; \\
\prod_{i=1}^{d+1} \prod_{j=1}^{k} x_{i,j} & = 1 \quad \text{for all } i, j;
\end{align*}
\]
\[
x_{2,1} = w_{1,0}^{-1}(x_{1,1}, \ldots, x_{d+1,1})x_{1,1} w_{1,0}(x_{1,1}, \ldots, x_{d+1,1}).
\]

Obviously, this presentation becomes a Hurwitz \( C \)-presentation after renumbering the generators.

Denote by \( \tilde{\nu} : \tilde{G} \to \mathbb{F}_1 \) the canonical \( C \)-epimorphism and by \( \tilde{h}_1 \) the action of the \( C \)-generator of \( \mathbb{F}_1 \) on \( \tilde{N} = \ker \tilde{\nu} \) given by \( \tilde{h}_1(g) = x_1^{-1}gx_1 \) for \( g \in \tilde{N} \).

It is easy to see that a map \( \tilde{f} \), sending the set \( \{ x_i \mid i = 1, \ldots, d + 1 \} \) of generators of \( \tilde{G} \) to the set \( \{ x_0, x_{i'} \mid i = 0, \ldots, d - 1 \} \) of generators of \( G_\Phi \) as follows: \( \tilde{f}(x_i) = x_0 \) and \( \tilde{f}(x_i) = x_{i'-2} \) for \( 2 \leq i \leq d + 1 \), can be extended to an epimorphism \( \tilde{f} : \tilde{G} \to G_\Phi \). Obviously, we have \( \tilde{f}(\tilde{N}) = M \) and \( h_0(f(g)) = \tilde{f}(\tilde{h}_1(g)) \) for all \( g \in \tilde{N} \). Consider the restriction \( \tilde{f}_{\tilde{N}} : \tilde{N} \to M \) of \( \tilde{f} \) to \( \tilde{N} \). Since \( M \) is an abelian group, the epimorphism \( \tilde{f}_{\tilde{N}} \) can be decomposed into the composition \( \tilde{f}_{\tilde{N}} = \tilde{f}' \circ \tilde{j} \), where \( \tilde{j} : \tilde{N} \to \tilde{N}/\tilde{N}' \) is the canonical epimorphism and \( \tilde{f}' : \tilde{N}/\tilde{N}' \to M \) is the epimorphism induced by \( \tilde{f}_{\tilde{N}} \). Denote by \( \tilde{K} = \ker \tilde{f}' \).

It follows from [5] that the abelian group \( \tilde{N}/\tilde{N}' \) is finitely generated. Therefore the group \( \tilde{K} \) is also a finitely generated abelian group. Let
Let \( g_1, \ldots, g_n \) be a set of elements of \( \tilde{N} \) such that their images \( \tilde{j}(g_1), \ldots, \tilde{j}(g_n) \) in \( \tilde{N}/\tilde{N}' \) generate the group \( \tilde{K} \) and let a word \( w_i(x_1, \ldots, x_{d+1}) \), 1 \( \leq i \leq n \), in the letters \( x_1^{\pm 1}, \ldots, x_{d+1}^{\pm 1} \) represent the element \( g_i \). Consider a \( C \)-group \( G \) given by \( C \)-presentation

\[
G = \langle x_1, \ldots, x_{d+1} \mid \mathcal{R}, \quad x_j = w_i^{-1}(x_1, \ldots, x_{d+1})x_jw_i(x_1, \ldots, x_{d+1}), \quad 1 \leq j \leq d + 1, \quad 1 \leq i \leq n \rangle,
\]

(8)

where \( \mathcal{R} \) is the set of defining relations in presentation (7). The group \( G \) is also a Hurwitz \( C \)-group. Moreover, it follows from the defining relations for \( G \) that the epimorphism \( \tilde{f} \) can be decomposed into the composition \( \tilde{f} = f \circ r \) where the \( C \)-epimorphism \( r \) sends the generators \( x_i, 1 \leq i \leq d + 1 \) of the group \( \tilde{G} \) to the generators \( x_i, 1 \leq i \leq d + 1 \) of the group \( G \) and \( f : G \to G \) is an epimorphism such that \( f(x_1) = x_0 \) and \( f(x_i) = x_{i-2} \) for \( 2 \leq i \leq d + 1 \). It follows from (8) that the elements \( r(g_i), 1 \leq i \leq n \), belong to the center of the group \( G \).

Denote by \( \nu : G \to F_1 \) the canonical \( C \)-epimorphism and by \( \tilde{h} \) the action of the \( C \)-generator of \( F_1 \) on \( N = \ker \nu \) given by \( \tilde{h}(g) = x_1^{-1}gx_1 \) for \( g \in N \). Obviously, we have \( f(N) = M \) and \( h_0(f(g)) = f(\tilde{h}(g)) \) for all \( g \in N \).

The restriction \( f|_N : N \to M \) of \( f \) to \( N \) can be also decomposed into the composition \( f|_N = f' \circ j \), where \( j : N \to N/N' \) is the canonical epimorphism and \( f' : N/N' \to M \) is the epimorphism induced by \( f|_N \). Denote by \( K = \ker f' \). Obviously, we have \( r'(K) = K \), where \( r' : \tilde{N}/\tilde{N}' \to N/N' \) is the epimorphism induced by \( r \). Let \( h \in \text{Aut} N/N' \) be the automorphism induced by \( \tilde{h} \).

We have the following commutative diagram.

\[
\begin{array}{ccc}
\tilde{N} & \xrightarrow{\tilde{j}} & \tilde{N}/\tilde{N}' \\
\downarrow{r} & & \downarrow{r'} \\
N & \xrightarrow{j} & N/N' \\
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{N}/\tilde{N}' & \xrightarrow{\tilde{f}'} & M \\
\downarrow{r'} & & \downarrow{f'} \\
N/N' & \xrightarrow{j} & M \\
\end{array}
\]

(\( \ast \))

The elements \( \tilde{j}(g_1), \ldots, \tilde{j}(g_n) \) generate the group \( \tilde{K} \). Since the homomorphisms \( \tilde{j}, j, r, \) and \( r' \) in commutative diagram (\( \ast \)) are epimorphisms, the group \( K \) is generated by the elements \( j(r(g_1)), \ldots, j(r(g_n)) \).
By construction of the group $G$, the elements $r(g_1), \ldots, r(g_n)$ belong to the center of $G$. Therefore $h(j(r(g_i))) = j(r(g_i))$, that is, $h|_K = \text{Id}$. But $t = 1$ is not a root of the Alexander polynomial $\Delta(t) = \det(h_C - t\text{Id})$ of $G$, since $G$ is an irreducible $C$-group. Therefore $K$ belongs to the kernel of the natural homomorphism $c : N/N' \to (N/N')_C = (N/N') \otimes \mathbb{C}$ which coincides with the subgroup $\text{Tors}_{N/N'}$ of $N/N'$ consisting of the elements of finite orders. On the other hand, $\text{Tors}_{N/N'} \subset \ker f'$, since $M$ is a free abelian group. Therefore $K = \text{Tors}_{N/N'}$ and, since $f'$ is an epimorphism, $f'_C : (N/N')_C \to M_\mathbb{C}$ is an isomorphism. Thus, $\Delta(t) = \Psi(t)$, since $h_0(f'(g)) = f'(h(g))$ for all $g \in N/N'$.

2. The Alexander polynomials of Hurwitz $C$-groups consisting of several irreducible components

In this section, Theorem 0.2 will be proved.

Consider a polynomial $P(t) = (-1)^{n+\deg P_0}(t - 1)^n P_0(t) \in \mathbb{Z}[t]$ such that $P_0(1) \neq 0$, all roots of $P_0(t)$ are roots of unity, and if $\zeta$ is a primitive $p^k$-th root of unity, where $p$ is a prime number, then the multiplicity of the root $t = \zeta$ of the polynomial $P_0(t)$ is not greater than $n$. The polynomial $P_0(t)$ can be factorized into the product of cyclotomic polynomials. It is well known (see, for example, Lemma 5.3 of [1]) that for a cyclotomic polynomial $\Phi_k(t)$ the value $\Phi_k(1) \neq 1$ iff $k = p^n$ for some prime number $p$. Therefore one can find a factorization $P(t) = \prod_{i=1}^{n+1} \Psi_i(t)$, where for $i = 1, \ldots, n$ each factor $\Psi_i(t) = (-1)^{\deg \Psi_{i,0}}(1 - t)^{\Psi_{i,0}}(t) \in \mathbb{Z}[t]$ has no multiple roots and $\Psi_{n+1}(t) \in \mathbb{Z}[t]$ is a polynomial such that $\Psi_{n+1}(1) = 1$.

By Theorem 0.1 there is a Hurwitz $C$-group whose Alexander polynomial coincides with $\Psi_{n+1}(t)$. Besides, by Proposition 5.12 of [1], the Alexander polynomial of a Hurwitz product $G = G_1 \circ G_2$ of two Hurwitz $C$-groups is equal to the product of their Alexander polynomials. Therefore to prove Theorem 0.2 it suffices to prove the existence of a Hurwitz $C$-group $G$ whose Alexander polynomial $\Delta(t) = \Psi(t)$, where $\Psi(t) = (-1)^{\deg \Psi}(1 - t)^{\Psi}(t) \in \mathbb{Z}[t]$ has no multiple roots and all its roots are roots of unity.

Fix such a polynomial $\Psi(t)$. Let $d = \deg \Psi(t)$. Denote by $k$ the smallest positive integer such that all roots of $\Psi(t)$ are $k$-th roots of unity. Since $\Psi(t)$ has no multiple roots, $\Psi(t)$ is a divisor of the polynomial $t^k - 1$. Therefore, $\Psi(0) = \pm 1$ and $\Psi(t) = \pm t^d + \ldots$, that is, its leading coefficient is equal to $\pm 1$.

As in the proof of Theorem 0.1, to prove the existence of a Hurwitz $C$-group whose Alexander polynomial coincides with $\Psi(t)$, consider the quotient ring $M = M_\Psi = \mathbb{Z}[t]/(\Psi(t))$. Since the leading coefficient of
Ψ(t) is equal to ±1, the ring $M$ (considered as an abelian group with respect to additive operation) is a free abelian group freely generated by the elements $t_0 = t^0, \ldots, t_{d-1} = t^{d-1}$. As above, let $h_0 \in \text{End}_\mathbb{Z} M$ given by $h_0(Q(t)) = tQ(t)$ and $i : F_1 \to \text{End}_\mathbb{Z} M$ given by $i(x_0) = h_0$, where $x_0$ is a generator of the free group $\mathbb{F}_1$. We have $h_0 \in \text{Aut} M$, since $\Psi(0) = \pm 1$ and $\det(h_0, c - t\text{Id}) = \Psi(t)$.

Since $h_0 \in \text{Aut} M$, we can consider the semi-direct product

$$G_{\Psi} = M \rtimes F_1 \simeq \langle t_0, \ldots, t_{d-1}, x_0 \mid [t_i, t_j] = 1 \quad \text{for } 0 \leq i, j \leq d - 1, \quad x_0^{-1}t_xt_0 = t_{i+1}, \quad \text{for } i = 0, \ldots, d - 2, \quad x_0^{-1}t_{d-1}x_0 = \prod_{i=0}^{d-1} t_i^{a_i} \rangle,$$

where $a_i$ are coefficients of the polynomial $\Psi(t) = t^d + \sum_{i=0}^{d-1} a_i t^i$.

Let $\nu_0 : G_{\Psi} \to F_1$ be the canonical epimorphism, $\ker \nu_0 = M \subset G_{\Psi}$. We have $h(g) = x_0^{-1}g x_0$ for all $g \in M$.

For $i \geq d$ denote by $t_i = x_0^{-i}t_0x^i$. Obviously, $x_0^k$ belongs to the center of $G_{\Psi}$, since $\Psi(t)$ is a divisor of the polynomial $t^k - 1$ and therefore

$$t^{k+i} \equiv t^i \mod \Psi(t)$$

for all $i \geq 0$. Thus, we have $t_i = t_{k+i}$ for all $i \geq 0$.

Denote by $x_{ti} = x_0 t_i$. We have

$$x_0^{-1}x_{ti}x_0 = x_{ti+1}. \quad (9)$$

Since each $t_i$ belongs to the normal abelian subgroup $M$ of $G_{\Psi}$, we have the equality $x_0^{-1}g x_0 = x_0^{-1}g x_{ti}$ for all $g \in M$. Therefore all the elements $x_{ti}^k$ also belong to the center of $G_{\Psi}$, since $x_0^k$ belongs to its center.

We have $t_i = x_0^{-1}x_{ti}$ and the elements $x_0, t_0, \ldots, t_{d-1}$ generate $G_{\Psi}$. Therefore the elements $x_0, x_{t_0}, \ldots, x_{t_{d-1}}$ also generate $G_{\Psi}$.

Consider a $C$-group $\tilde{G}$ given by $C$-presentation

$$\tilde{G} = \langle x_1, \ldots, x_{k+1} \mid x_{i+1} = x_1^{-1}x_ix_1 \text{ for } 2 \leq i \leq k, \quad x_1^{-1}x_{k+1}x_1 = x_2, \quad [x_i^k, x_j] = 1 \quad \text{for } 2 \leq i \leq k, \quad 1 \leq j \leq k + 1 \rangle. \quad (10)$$

First of all, notice that $\tilde{G}$ is a $C$-group consisting of two irreducible components. Next, all the elements $x_i^k, i = 1, \ldots, k + 1$, belong to the center of $\tilde{G}$. Thus, the product $x_1^k \ldots x_{k+1}^k$ belongs to the center of $\tilde{G}$. Therefore, as in the proof of Theorem 0.1, one can show that $\tilde{G}$ is a Hurwitz $C$-group of degree $k(k+1)$. 

Denote, as above, by $\tilde{\nu} : \tilde{G} \to \mathbb{F}_1$ the canonical $C$-epimorphism and by $\tilde{h}_1$ the action of the $C$-generator of $\mathbb{F}_1$ on $\tilde{N} = \ker \tilde{\nu}$ given by $\tilde{h}_1(g) = x_1^{-1}g x_1$ for all $g \in \tilde{N}$.

It is easy to see that a map $\tilde{f}$, sending the set $\{x_i \mid i = 1, \ldots, k + 1\}$ of generators of $\tilde{G}$ to the set $\{x_{i-1}x_{i+1} \mid i = 0, \ldots, k + 1\}$ of generators of $G_\Psi$ as follows: $\tilde{f}(x_1) = x_0$ and $\tilde{f}(x_{i-1}) = x_{i-2}$ for $2 \leq i \leq k + 1$, can be extended to an epimorphism $f : \tilde{G} \to G_\Psi$. Obviously, we have $f(\tilde{N}) = M$ and $h_0(f(g)) = f(h_1(g))$ for all $g \in \tilde{N}$.

As in the proof of Theorem 0.1 the restriction $\tilde{f}_{j|\tilde{N}} : \tilde{N} \to M$ of $\tilde{f}$ to $\tilde{N}$ can be decomposed into the composition $\tilde{f}_{j|\tilde{N}} = \tilde{f}' \circ \tilde{j}$, where $\tilde{j} : \tilde{N} \to \tilde{N}/\tilde{N}'$ is the canonical epimorphism and $\tilde{f}' : \tilde{N}/\tilde{N}' \to M$ is the epimorphism induced by $\tilde{f}_{j|\tilde{N}}$. Denote by $\tilde{K} = \ker \tilde{f}'$.

The groups $\tilde{N}/\tilde{N}'$ and $\tilde{K}$ are finitely generated abelian groups. Let $g_1, \ldots, g_n \in \tilde{N}$ be elements such that their images $\tilde{j}(g_1), \ldots, \tilde{j}(g_n)$ in $\tilde{N}/\tilde{N}'$ generate the group $\tilde{K}$ and let a word $w_i(x_1, \ldots, x_{k+1})$, $1 \leq i \leq n$, in letters $x_1, \ldots, x_{k+1}$ and their inverses represents the element $g_i$.

Consider a $C$-group $G$ given by $C$-presentation

$$G = \langle x_1, \ldots, x_{k+1} \mid R, \quad x_j = w_i^{-1}(x_1, \ldots, x_{k+1})x_jw_i(x_1, \ldots, x_{k+1}) \quad \text{for } 1 \leq j \leq k + 1, 1 \leq i \leq n \rangle,$$

(11)

where $R$ is the set of defining relations in presentation (10). The group $G$ is also a Hurwitz $C$-group consisting of two irreducible components. Moreover, by construction of $G$, the epimorphism $\tilde{f}$ can be decomposed into the composition $\tilde{f} = f \circ r$ where the $C$-epimorphism $r$ sends the generators $x_i$, $1 \leq i \leq d + 1$, of the group $\tilde{G}$ to the generators $x_i$, $1 \leq i \leq k + 1$, of the group $G$ and $f : G \to G_\Psi$ is an epimorphism such that $f(x_1) = x_0$ and $f(x_i) = x_{i-2}$ for $2 \leq i \leq k + 1$. It follows from (10) that the elements $r(g_i), 1 \leq i \leq n$, belong to the center of the group $G$.

Let $\nu : G \to \mathbb{F}_1$ be the canonical $C$-epimorphism and let $\tilde{h}$ be an action of the $C$-generator of $\mathbb{F}_1$ on $N = \ker \nu$ given by $\tilde{h}(g) = x_1^{-1}g x_1$ for all $g \in N$. Obviously, we have $f(N) = M$ and $h_0(f(g)) = f(h(g))$ for all $g \in N$.

The restriction $f_{j|N} : N \to M$ of $f$ to $N$ also can be decomposed into the composition $f_{j|N} = f' \circ j$, where $j : N \to N/N'$ is the canonical epimorphism and $f' : N/N' \to M$ is the epimorphism induced by $f_{j|N}$. Denote by $K = \ker f'$. Obviously, we have $r'(K) = K$, where
r' : \tilde{N} / \tilde{N}' \to N / N' is the epimorphism induced by r. Let \( h \in \text{Aut } N / N' \) be the automorphism induced by \( \tilde{h} \).

We have the following commutative diagram.

\[
\begin{array}{ccc}
\tilde{N} & \xrightarrow{\tilde{j}} & \tilde{N} / \tilde{N}' \\
\downarrow{r} & & \downarrow{r'} \\
N & \xrightarrow{j} & N / N' \\
\end{array}
\quad \begin{array}{ccc}
\downarrow{\tilde{K}} & & \downarrow{K} \\
\tilde{N} / \tilde{N}' & \xrightarrow{\tilde{c}} & (\tilde{N} / \tilde{N}')_C \\
\downarrow{\tilde{f}'} & & \downarrow{f'_C} \\
M & \xrightarrow{c} & M_C \\
\end{array}
\quad \begin{array}{c}
(\ast \ast) \\
\end{array}
\end{array}
\]

By Theorem 5.9 of [1], the value \( t = 1 \) is a root of multiplicity one of the Alexander polynomial \( \Delta(t) = \det(h_C - t \text{Id}) \) of the group \( G \), since the Hurwitz \( C \)-group \( G \) consists of two irreducible component. Besides, \( t = 1 \) is also a root of multiplicity one of the characteristic polynomial \( \det((h_0,C - t \text{Id}) = (-1)^d \Psi(t) \). Therefore the eigen-spaces \( ((N/N')_C)_1 \subset (N/N')_C \) and \( (M_C)_1 \subset M_C \) corresponding to the eigenvalue 1 are one-dimensional. Next, \( f_C \) is an epimorphism such that \( f'_C(h_C(v)) = h_0,C(f'_C(v)) \) for any \( v \in (N/N')_C \). Therefore \( ((N/N')_C)_1 \not\subset \ker f'_C \).

The elements \( \tilde{j}(g_1), \ldots, \tilde{j}(g_n) \) generate the group \( \tilde{K} \). Since the homomorphisms \( \tilde{f}, f, r, \) and \( r' \) in commutative diagram \( (\ast \ast) \) are epimorphisms, the group \( K \) is generated by the elements \( j(r(g_1)), \ldots, j(r(g_n)) \).

By construction of the group \( G \), the elements \( r(g_1), \ldots, r(g_n) \) belong to the center of \( G \). Therefore, \( h(j(r(g_i))) = j(r(g_i)) \), that is, \( h_{|K} = \text{Id} \), and consequently, \( c(K) = \ker f'_C \cap ((N/N')_C)_1 = 0 \). We have \( K \subset \ker c = \text{Tors } N / N' \). On the other hand, \( \text{Tors } N / N' \subset K \), since \( M \) is a free abelian group. Therefore, \( K = \text{Tors } N / N' \) and \( f'_C \) is an equivariant isomorphism. Thus, \( \Delta(t) = \Psi(t) \). \( \square \)

3. Hurwitz C–Groups with the Alexander Polynomials

\[
\Delta(t) = (-1)^n t^{n+k} (t - 1)^n (t + 1)^k
\]

This section is devoted to the proof of Theorem 0.3.

Consider a pair \( A = (M, h) \), where \( M \) is a finitely generated free abelian group and \( h \in \text{Aut } M \). In particular, set \( A_\pm = (\mathbb{Z}, \pm \text{Id}) \) and \( A_{\pm \pm} = (\mathbb{Z} \oplus \mathbb{Z}, h_\pm) \), where, in a base \( e_1, e_2 \) of \( \mathbb{Z} \oplus \mathbb{Z} \), the automorphism \( h_\pm \) is given by \( h_\pm(e_1) = e_2 \) and \( h_\pm(e_2) = e_1 \). For two pairs \( A = (M, h) \) and \( A' = (M', h') \) the equality \( A = A' \) will mean that \( A \) and \( A' \) are isomorphic, that is, there is an isomorphism \( g : M \to M' \) such that
Proposition 3.1. Let \( A = (M, h) \) be a pair such that \( h^2 = \text{Id} \). Then there are integers \( n_1, n_2, \) and \( n_3 \) such that \( A = n_1A_+ \oplus n_2A_- \oplus n_3A_+ \).

Proof. Denote by \( M_+ = \{a \in M \mid h(a) = a\} \) and by \( M_- = \{a \in M \mid h(a) = -a\} \). Consider the subgroup \( M' \) of \( M \) generated by the elements \( a \in M_+ \cup M_- \). Obviously, \( M' = M_+ \oplus M_- \). Denote by \( p_+ : M' \to M_+ \) and \( p_- : M' \to M_- \) the projections to the factors. Since \( M \) is a free \( \mathbb{Z} \)-module, its submodules \( M_+ \) and \( M_- \) are also free \( \mathbb{Z} \)-modules. Let \( M_+ \) and \( M_- \) are freely generated over \( \mathbb{Z} \), respectively, by \( e_1, \ldots, e_{k_1} \) and \( e_{k_1+1}, \ldots, e_{k_1+k_2} \).

Let us show that for any \( a \in M \) the element \( 2a \) belongs to \( M' \). Indeed, for any \( a \in M \) we have \( a + h(a) \in M_+ \), \( a - h(a) \in M_- \) and \( 2a = (a + h(a)) + (a - h(a)) \). Therefore all elements of the abelian group \( M/M' \) are of the second order. Note also that, by definition of the subgroups \( M_\pm \), if an element \( a \in M \) is such that \( 2a \in M_+ \) (resp. \( 2a \in M_- \)), then \( a \in M_+ \) (resp. \( a \in M_- \)).

We have \( M/M' \cong (\mathbb{Z}/2\mathbb{Z})^{n_3} \) for some integer \( n_3 \geq 0 \). Let us choose elements \( a_1, \ldots, a_{n_3} \in M \) whose images generate the group \( M/M' \) and let \( 2a_i = (\alpha_{i,1}, \ldots, \alpha_{i,k_1+k_2}) \), \( i = 1, \ldots, n_3 \), be the coordinates of \( 2a_i \) in the base \( e_1, \ldots, e_{k_1+k_2} \). Without loss of generality, adding or subtracting elements from \( M' \), we can assume that for each \( i = 1, \ldots, n_3 \) each coordinate \( \alpha_{i,j} \) is equal to 0 or 1 and there is \( j \) for which \( \alpha_{i,j} = 1 \).

Let us show that the elements \( 2a_1, \ldots, 2a_{n_3} \) are linear independent over \( \mathbb{Z} \). Indeed, assume that for some \( m_{1}, \ldots, m_{n_3} \in \mathbb{Z} \) we have \( \sum m_i(2a_i) = 0 \) in \( M' \). Since \( M' \) is a free abelian group, we can assume that at least one of \( m_i \), say \( m_{n_3} \), is an odd number. In this case the image of \( a_{n_3} \) in \( M/M' \) is a linear combination of the images of the elements \( a_1, \ldots, a_{n_3-1} \), that is, we obtain a contradiction.

Let us show that the elements \( p_+(2a_1), \ldots, p_+(2a_{n_3}) \) (resp. the elements \( p_-(2a_1), \ldots, p_-(2a_{n_3}) \)) are linear independent over \( \mathbb{Z} \). Indeed, assume that for some \( m_1, \ldots, m_{n_3} \in \mathbb{Z} \) we have \( \sum m_i p_+(2a_i) = 0 \) in \( M_+ \). Since \( M_+ \) is a free abelian group, we can assume that at least one of \( m_i \), say \( m_{n_3} \), is an odd number. Then the element \( 2a = 2 \sum m_i a_i \in M_- \) and therefore, \( a = \sum m_i a_i \in M_- \). But it is impossible, since the images of the elements \( a_1, \ldots, a_{n_3-1} \) cannot generate \( M/M' \cong (\mathbb{Z}/2\mathbb{Z})^{n_3} \). As a consequence, we obtain that \( n_3 \leq \min(k_1, k_2) \).

Let us show that we can choose elements \( a_1, \ldots, a_{n_3} \), whose images generate \( M/M' \), so that the system of elements \( p_-(2a_1), \ldots, p_-(2a_{n_3}) \) can be extended to a free base of the free \( \mathbb{Z} \)-module \( M_- \). Indeed, without loss of generality (if it is needed we renumber the elements

\( h = g^{-1} \circ h' \circ g \). For \( A' \) and \( A' \), put \( A \oplus A' = (M \oplus M', h \oplus h') \) and denote by \( nA \) the direct sum of \( n \) copies of \( A \).
that the images of
and by
\(a_{i, k_1 + k_2}\) one can assume that \(\alpha_{1, k_1 + k_2} = 1\). For \(i = 2, \ldots, n_3\) replacing \(a_i\) by
\[
a'_i = a_i - \alpha_{i, k_1 + k_2} (a_1 - \sum_{j=1}^{k_1 + k_2 - 1} \alpha_{i,j} (1 - \alpha_{i,j}) e_j),
\]
we obtain a new system \(a_1, a'_2, \ldots, a'_{n_3}\) such that

(i) the images of \(a_1, a'_2, \ldots, a'_{n_3}\) generate \(M'/M'\);

(ii) the coordinates of the elements \(2a_1, 2a'_2, \ldots, 2a'_{n_3}\) in the base \(e_1, \ldots, e_{k_1 + k_2}\) are equal to 0 or 1;

(iii) for each \(i = 2, \ldots, n_3\) the last coordinate of the element \(2a'_i\) in the base \(e_1, \ldots, e_{k_1 + k_2}\) is equal to 0.

Now, it is clear that repeating similar replacements \(n_3\) times, we can find new elements \(a_1, a_2, \ldots, a_{n_3}\) and a new base \(e_{k_1 + 1}, \ldots, e_{k_1 + k_2}\) of \(M_+\) such that

(i) the images of \(a_1, a_2, \ldots, a_{n_3}\) generate \(M'/M'\);

(ii) for each \(i = 1, \ldots, n_3\) the coordinates \(\alpha_{i,j}\) of the element \(2a_i\) in the new base \(e_1, \ldots, e_{k_1 + k_2}\) are equal to 0 or 1;

(iii) for each \(i = 1, \ldots, n_3\) the coordinate \(\alpha_{i,j}\) of the element \(2a_i\) in the base \(e_1, \ldots, e_{k_1 + k_2}\) is equal to 0 for \(j > k_1 + k_2 - i + 1\) and \(\alpha_{i,k_1 + k_2 - i + 1} = 1\).

As a result, we can decompose \(M_+\) into the direct sum \(M_+ = \tilde{M}_+ \oplus \overline{M}_+\), where \(\overline{M}_+\) is generated by \(e_{k_1 + 1, \ldots, e_{k_1 + k_2 - n_3}}\) and \(\tilde{M}_+\) is generated by \(p_-(2a_1), \ldots, p_-(2a_{n_3})\). Denote by \(\tilde{M}\) a subgroup of \(M\) generated by the elements \(a_1, \ldots, a_{n_3}\) together with the elements \(e_1, \ldots, e_{k_1} \in M_+\). It is easy to see that \(M = \tilde{M} \oplus \overline{M}_+\), \(\tilde{M}\) is invariant under the action of \(h\), \(M_+ = \{a \in \tilde{M} \mid h(a) = a\}\), and \(\overline{M}_+ = \{a \in \overline{M} \mid h(a) = -a\}\). Denote by \(\tilde{M}' = M_+ \oplus \overline{M}_+\) and by \(\tilde{p}_+ : \tilde{M}' \to M_+\) and \(\tilde{p}_- : \tilde{M}' \to \overline{M}_+\) the projections to the factors.

Let us show that, by a linear triangular transformation, we can replace the system of elements \(a_1, \ldots, a_{n_3}\) chosen above by \(\overline{a}_1, \ldots, \overline{a}_{n_3}\) so that the images of \(\overline{a}_1, \ldots, \overline{a}_{n_3}\) generate \(\tilde{M}/\tilde{M}'\), the elements \(\tilde{p}_-(2\overline{a}_1), \ldots, \tilde{p}_-(2\overline{a}_{n_3})\) generate \(\bar{M}_-\), and the system of elements \(\tilde{p}_+(2\overline{a}_1), \ldots, \tilde{p}_+(2\overline{a}_{n_3})\) can be extended to a free base of the free \(\mathbb{Z}\)-module \(M_+\). Indeed, as above, without loss of generality (if it is needed we can renumber the elements \(e_1, \ldots, e_{k_1}\), one can assume that \(\alpha_{1,k_1} = 1\). Then for
It is evident that repeating similar replacements \( n_3 \) times, we can find new elements \( \overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n_3} \) and a new free base \( e_1, \ldots, e_{k_1} \) of the free \( \mathbb{Z} \)-module \( M_+ \) such that

\[
(\text{i}) \text{ the images of } \overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n_3} \text{ generate } \overline{M}/\overline{M}'; \\
(\text{ii}) \text{ the elements } \overline{p}_-(2a_1), \overline{p}_-(2a_2'), \ldots, \overline{p}_-(2a_3') \text{ generate } \overline{M}_-; \\
(\text{iii}) \text{ the coordinates of the elements } 2a_1, 2a_2', \ldots, 2a_3' \text{ in the base } \\
e_1, e_{k_1}, \overline{p}_-(2a_1), \overline{p}_-(2a_2'), \ldots, \overline{p}_-(2a_3') \text{ are equal to 0 or 1;}
\]

\[
(\text{iv}) \text{ for } i = 2, \ldots, n_3 \text{ the coordinate } \alpha_{i,k_1} \text{ of the element } 2a_1' \text{ in the base } \\
e_1, e_{k_1}, \overline{p}_-(2a_1), \overline{p}_-(2a_2'), \ldots, \overline{p}_-(2a_3') \text{ is equal to 0.}
\]

As a result, we can decompose \( M_+ \) into the direct sum \( M_+ = \overline{M}_+ + \overline{M}_- \), where \( \overline{M}_+ \) is generated by \( e_1, \ldots, e_{k_1+n_3} \) and \( \overline{M}_- \) is generated by \( \overline{p}_+(2\overline{a}_1), \ldots, \overline{p}_+(2\overline{a}_{n_3}) \). Denote by \( \overline{M}_{++} \) a subgroup of \( M \) generated by the elements \( \overline{a}_1, \overline{h}(\overline{a}_1), \ldots, \overline{a}_{n_3}, \overline{h}(\overline{a}_{n_3}) \). It is easy to see that \( \overline{M}_{++} \) is invariant under the action of \( h \) and \( M_+ \oplus \overline{M}_- \subset M_{++} \). Therefore \( \text{rk } \overline{M}_{++} = 2n_3 \) and, by construction, \( \overline{M}_{++}, h) \simeq n_3A_{++} \). Moreover, \( (M, h) = (M_+ \oplus \overline{M}_- \oplus M_{++}, h) \simeq n_1A_+ \oplus n_2A_- \oplus n_3A_{++} \), where \( n_1 = k_1 - n_3 \) and \( n_2 = k_2 - n_3 \).

\[\square\]

**Lemma 3.2.** Let \( (M, h) = n_1A_+ \oplus n_2A_- \oplus n_3A_{++} \). Consider the semidirect product \( G = M \rtimes \langle h \rangle \). Then

\[
(i) \ G/G' \simeq \mathbb{Z}^{n_1+n_3+1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n_2}, \\
(ii) \ \det(tIh - h_G) = (t - 1)^{n_1+n_3}(t + 1)^{n_2+n_3}.
\]
Proof. The group \( G \) can be given by presentation
\[
G \simeq \langle e_1, \ldots, e_{n_1+n_2+2n_3}, h \mid [e_i, e_j] = 1, 1 \leq i, j \leq n_1 + n_2 + 2n_3, \\
h^{-\varepsilon}e_ih^{\varepsilon} = e_{n_3+i}, \quad 1 \leq i \leq n_3, \varepsilon = \pm 1, \\
h^{-1}e_{2n_3+i}h = e_{2n_3+i}, \quad 1 \leq i \leq n_1, \\
h^{-1}e_{2n_3+n_1+i}h = e_{2n_3+n_1+i}, \quad 1 \leq i \leq n_2 \rangle.
\]
Therefore the group \( G/G' \) can be given by presentation
\[
G/G' \simeq \langle e_1, \ldots, e_{n_1+n_2+2n_3+1} \mid [e_i, e_j] = 1, 1 \leq i, j \leq n_1 + n_2 + 2n_3 + 1, \\
e_i = e_{n_3+i}, \quad 1 \leq i \leq n_3, \\
e_{2n_3+n_1+i} = e_{2n_3+n_1+i}, \quad 1 \leq i \leq n_2 \rangle
\]
that is, \( G/G' \simeq \mathbb{Z}^{n_1+n_3+1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n_2} \).

Part (ii) of Lemma 3.2 is trivial. \( \square \)

Let us return to the proof of Theorem 0.3. Let a polynomial \( \Delta(t) = (-1)^{n+k}(t-1)^n(t+1)^k \) be the Alexander polynomial of a Hurwitz \( C \)-group \( G \). Then by Theorem 5.9 of [1], \( G \) consists of \( n+1 \) irreducible components, that is, \( G/G' \simeq \mathbb{Z}^{n+1} \).

Consider the canonical \( C \)-epimorphism \( \nu : G \to \mathbb{F}_1, N = \ker \nu \). By [5], the group \( N \) is finitely presented. Therefore \( N/N' \) is a finitely generated abelian group. Let \( T = \text{Tors}(N/N') \) be the subgroup of \( N/N' \) consisting of the elements of finite orders. Then \( T \) is invariant under the action of \( h \). Therefore, \( h \) induces an action (we denote it by the same letter) \( h \) on the free abelian group \( M = (N/N')/T \). Since the action of \( h \) on \( (N/N') \otimes \mathbb{C} \simeq M \otimes \mathbb{C} \) is semi-simple and all roots of the characteristic polynomial \( \det(h - t \text{Id}) = \Delta(t) \) are equal to \( \pm 1 \), we have \( h^2 = \text{Id} \). Consequently, \( h^2 = \text{Id} \) on \( M \). It follows from Proposition 3.1 that \( (M, h) \simeq n_1A_+ \oplus n_2A_- \oplus n_3A_{+-} \) for some non-negative integers \( n_1, n_2, n_3 \).

It is easy to see that \( T \) is a normal subgroup of \( G/N' \). Therefore the exact sequence
\[
1 \to N/N' \to G/N' \xrightarrow{\nu} \mathbb{F}_1 \to 1
\]
implies the following exact sequence
\[
1 \to M \xrightarrow{\overline{\nu}} \mathbb{F}_1 \to 1,
\]
where \( \overline{G} = (G/N')/T \). Denote by \( f : G \to \overline{G} \) and \( g : N \to M \) the canonical epimorphisms, and by \( \nu_1 : \overline{G} \to \mathbb{F}_1 \) the epimorphism induced by \( \nu \). Exact sequence (12) can be included to the following commutative diagram of exact sequences.

\[
\begin{array}{ccc}
1 & \to & N \\
\downarrow g & & \downarrow f \\
1 & \to & M \to \overline{G} \xrightarrow{\nu_1} \mathbb{F}_1 \to 1
\end{array}
\]
It is easy to see that $\overline{G} \simeq M \times \langle h \rangle$. Therefore, by Lemma 3.2, we have
\[ n = n_1 + n_3 \quad \text{and} \quad k = n_2 + n_3. \] (13)

The epimorphism $f$ induces an epimorphism $f^*: G/G' \simeq \mathbb{Z}^{n+1} \to \overline{G}/\overline{G'}$. Applying Lemma 3.2 one more, we have $\overline{G}/\overline{G'} \simeq \mathbb{Z}^{n_1+n_3+1} \oplus (\mathbb{Z}/2\mathbb{Z})^{n_2}$. Therefore,
\[ n_1 + n_2 + n_3 + 1 \leq n + 1. \] (14)

It follows from (13) and (14) that $n_2 = 0$ and $k \leq n$.

In [1], it was shown that the Hurwitz $C$-group $G(2) =< x_1, \ldots, x_4 \mid x_2 x_1 x_2^{-2} = x_4, x_3 = x_2, x_4^2 x_2 x_4^{-2} = x_2$
\[ [x_i, x_1 \ldots x_4, x_i = 1 \text{ for } i = 1, \ldots, 4 > \]
has the Alexander polynomial $\Delta(t) = (t^2 - 1)$. Next, the Alexander polynomial of abelian $C$-group $\mathbb{Z}^n$ is equal to $(-1)^{n-1}(t-1)^{n-1}$. Therefore, for $k \leq n$, to prove the existence of a Hurwitz $C$-group $G$ whose Alexander polynomial $\Delta(t) = (-1)^{n+k}(t-1)^n(t+1)^k$, it suffices to consider a Hurwitz product $G = G(2)^{\otimes k} \circ \mathbb{Z}^{n+1-k}$.

4. Two examples of $C$-groups

In this section, it will be shown that in the general case of $C$-groups, statements similar to Theorem 5.9 of [1] and Theorem 0.3 are not true.

Example 4.1. There is a $C$-group consisting of two irreducible components whose Alexander polynomial $\Delta(t) = (t-1)^2$.

Consider a $C$-group $G$ given by $C$-presentation
\[ G = \langle x_1, x_2, x_3 \mid x_3 = x_1^{-1} x_2 x_1, x_3 = x_1^{-1} x_3 x_2 x_3^{-1} x_1 \rangle. \]

It is easy to see that $G$ is a $C$-group consisting of two irreducible components. Denote by $N$ the kernel of the canonical $C$-epimorphism $\nu: G \to \mathbb{F}_1$. Without loss of generality, we can assume that $\tilde{h}(g) = x_1 g x_1^{-1}$ for $g \in N$.

To find a finite presentation of $N$, let us use the Reidemeister–Schreier method. Let us recall briefly it (see particulars, for example, in [3], section 2.3). In the beginning, so called a Schreier system of representatives should be chosen, that is, a representative $s_i$ of each right coset of the subgroup $N$ in the group $G =< x \in X \mid r = 1, r \in R >$ is chosen so that if a word $s_i$ is a representative, then all its unital
subwords are also representatives of some cosets (in our case we choose the elements $x^k_1$, $k \in \mathbb{Z}$, as Schreier representatives of the cosets of $N$ in $G$). Then the group $N$ is generated by $a_{i,j} = s_i \cdot x_j \cdot s_i^{-1} x_j$, where $s_i x_j$ is the Schreier representative of the coset containing the element $s_i x_j$, and the system of defining relations for $N$ is 
\[ \{ s_r s_i^{-1} = 1 \mid r \in \mathcal{R} \}, \]
where the relations $s_r s_i^{-1}$ are written as words in the letters $a_{i,j}$ and $a_{i,j}^{-1}$.

In our case $N$ is generated by the elements
\[ a_{k,j} = x^k_1 x_j x_1^{-1(k+1)}, \quad (15) \]
where $j = 2, 3$ and $k \in \mathbb{Z}$. It is easy to see that the action $\tilde{h}$ is given by $\tilde{h}(a_{k,j}) = a_{k+1,j}$.

The relation $x_3 = x^{-1}_1 x_2 x_1$ gives rise to relations
\[ a_{k,3} = a_{k-1,2} \quad (16) \]
for $k \in \mathbb{Z}$, and the relation $x_3 = x^{-1}_1 x_3 x_2 x_3^{-1} x_1$ gives rise to relations
\[ a_{k,3} = a_{k-1,3} a_{k,2} a_{k,3}^{-1} \quad (17) \]
for $k \in \mathbb{Z}$. By the Reidemeister–Schreier method, the set consisting of relations (16) and (17) is a set of defining relations for $N$.

Denote by $\overline{a}_{k,j}$ the image of $a_{k,j}$ in $N/N'$. It follows from (16) and (17) that the abelian group $N/N'$ is generated by the elements $\overline{a}_{k,j}$, $j = 2, 3$ and $k \in \mathbb{Z}$, being subject to the relations
\[ \overline{a}_{k,3} = \overline{a}_{k-1,2}, \]
\[ 2 \overline{a}_{k,3} = \overline{a}_{k-1,3} + \overline{a}_{k,2} \quad (18) \]
for $k \in \mathbb{Z}$. The action of $h$ on $N/N'$, induced by $\tilde{h}$, is given by $h(\overline{a}_{k,j}) = \overline{a}_{k+1,j}$ for $j = 2, 3$ and $k \in \mathbb{Z}$.

It follows from (18) that $N/N'$ is generated by $\overline{a}_{0,3}$ and $\overline{a}_{1,3}$ and we have $\overline{a}_{2,3} = -\overline{a}_{0,3} + 2\overline{a}_{1,3}$. Therefore, in the base $\overline{a}_{0,3}$, $\overline{a}_{1,3}$, the automorphism $h$ is given by the matrix
\[ h = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}. \]

One can easily check that $\det(h - t \text{Id}) = (t - 1)^2$.

**Example 4.2.** There is a $C$-group consisting of two irreducible components whose Alexander polynomial $\Delta(t) = (1 - t)(t + 1)^2$. 
Consider a $C$-group $G$ given by $C$-presentation
\[ G = \langle x_1, x_2, x_3 \mid x_3 = x_1^{-1} x_2 x_1, \ [x_1, (x_2^2 x_1^{-1} x_2 x_1^{-1} x_3)] = 1 \rangle. \]

Obviously, $G$ is a $C$-group consisting of two irreducible components. Denote by $N$ the kernel of the canonical $C$-epimorphism $\nu : G \to \mathbb{F}_1$. Without loss of generality, we can assume that $\tilde{h}(g) = x_1 g x_1^{-1}$ for $g \in N$. Applying the Reidemeister – Schreier method, one can show that $N$ is generated by the elements
\[ a_{k,j} = x_1^k x_j x_1^{-(k+1)}, \quad (19) \]
where $j = 2, 3$ and $k \in \mathbb{Z}$. The action $\tilde{h}$ on $N$ is given by $\tilde{h}(a_{k,j}) = a_{k+1,j}$.

As in Example 4.1 the relation $x_3 = x_1^{-1} x_2 x_1$ gives rise to the relations
\[ a_{k,3} = a_{k-1,2} \quad (20) \]
for $k \in \mathbb{Z}$, and the relation $[x_1, (x_2^2 x_1^{-1} x_2 x_1^{-1} x_3)] = 1$ gives rise to the relations
\[ a_{k,3} a_{k+1,3} a_{k+1,2} a_{k+1,3}^{-1} a_{k+2,3}^{-1} a_{k+2,2}^{-1} a_{k+2,3}^{-1} a_{k+1,3}^{-1} = 1 \quad (21) \]
for $k \in \mathbb{Z}$. By the Reidemeister – Schreier method, the set of relations (20) and (21) is a set of defining relations for $N$.

Denote by $\overline{a}_{k,j}$ the image of $a_{k,j}$ in $N/N'$. It follows from (20) and (21) that the abelian group $N/N'$ is generated by the elements $\overline{a}_{k,j}$, $j = 2, 3$ and $k \in \mathbb{Z}$, being subject to the relations
\[ \overline{a}_{k,3} + \overline{a}_{k+1,3} + \overline{a}_{k+1,2} - 2 \overline{a}_{k+2,3} - \overline{a}_{k+2,3} = 0 \quad (22) \]
for $k \in \mathbb{Z}$. The action of $h$ on $N/N'$, induced by $\tilde{h}$, is given by $h(\overline{a}_{k,j}) = \overline{a}_{k+1,j}$ for $j = 2, 3$ and $k \in \mathbb{Z}$.

It follows from (22) that $N/N'$ is generated by $\overline{a}_{0,3}$, $\overline{a}_{1,3}$ and $\overline{a}_{2,3}$ and we have $\overline{a}_{3,3} = \overline{a}_{0,3} + \overline{a}_{1,3} - \overline{a}_{2,3}$. Therefore, in the base $\overline{a}_{0,3}$, $\overline{a}_{1,3}$, $\overline{a}_{2,3}$, the automorphism $h$ is given by the matrix
\[ h = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \]

One can check easily that
\[ \det(h - t \text{Id}) = (1 - t)(t + 1)^2. \]
REFERENCES

[1] Greuel G.-M., Kulikov Vik.S.: On symplectic coverings of the projective plane. ArXiv math. SG/0409027, submitted to Izv. Math.

[2] Kulikov Vik. S.: Alexander polynomials of plane algebraic curves. Izv. Math. 42:1 (1994), 67–90.

[3] Kulikov Vik. S.: Geometric realization of $C$-groups. Izv. Math. 45:1 (1995), 197-206.

[4] Kulikov Vik.S.: A factorization formula of the full twist for double the number of strings. Izv. Math., 68:1 (2004), 125-158.

[5] Kulikova O.V.: On the fundamental groups of the complements of Hurwitz curves. ArXiv math. SG/0409027, will be published in Izv. Math., 69:1 (2005).

[6] Kuzmin Yu.V.: On a method of constructing $C$-groups. Izv. Math., 59:4 (1995), 765-783.

[7] Lange S.: Algebra. Addison-Wesley Publishing Company, 1965.

[8] Magnus W., Karras A., Solitar D.: Combinatorial group Theory: Presentations of Groups in Terms of Generators and Relations, Interscience publishers, New York - London - Sydney, 1966.

Steklov Mathematical Institute
E-mail address: kulikov@mi.ras.ru