On Hodge decomposition and conformal variational problems✩

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Abstract

The main result is the identification of the orthogonal complement of the subalgebra of conformal vector field inside the algebra of all vector fields of a compact flat 2–manifold. As a fundamental tool, the complete Hodge decomposition for manifold with boundary is used. The identification allows the derivation of governing differential equations for variational problems on the space of conformal vector fields. Several examples are given. In addition, the paper also gives a review, in full detail, of already known vector field decompositions involving subalgebras of volume preserving and symplectic vector fields.

Keywords: Conformal vector fields, vector field subalgebras, Hodge decomposition, Friedrichs decomposition, variational problems, Euler equations

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1. Introduction

It is well known that the Euler equations of fluid dynamics correspond to a Lie–Poisson equation on the infinite dimensional Lie algebra of volume preserving vector fields [1,2]. On domains of \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) the derivation of the

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strong form of the Euler equations starts from a variational principle, and relies on the Helmholtz decomposition in order to identify the $L^2$ orthogonal complement of the space of volume preserving vector fields inside the space of all vector fields. A generalisation to arbitrary compact Riemannian manifolds, with or without boundary, is obtained by identifying vector fields with 1–forms (by contraction with the metric), and then use the Hodge decomposition for manifolds with boundary in order to identifying the $L^2$ orthogonal complement \[3\].

In this paper we give a framework for how various Lie subalgebras of vector fields can be identified with one or several components in the Hodge decomposition by suitable isometric isomorphisms between vector fields and 1–forms, thus identifying the $L^2$ orthogonal complement of the subalgebra. The first example concerns Lie subalgebras of volume preserving vector fields. This example is well known in the literature, although it is hard to find a detailed exposition for the cases that occur for a manifold with boundary. Thus, the first aim of the paper is to give such an exposition. The second example concerns Lie subalgebras of symplectic and Hamiltonian vector fields. It is known that the space of symplectic vector fields can be identified with closed 1–forms (by contraction with the symplectic form). The second aim of the paper is to give a detailed exposition of various $L^2$ orthogonal decompositions involving symplectic and Hamiltonian vector fields, on a compact almost Kähler manifold with boundary. The third example is new and constitutes the main contribution of the paper. We show how the subalgebra of conformal vector fields on a flat 2–manifold can be identified with the space of harmonic fields. As an application, we then give examples of how to derive partial differential equations from variational problems on the space of conformal vector fields. In the main example, which is also the original motivation for this paper, we derive a geodesic equation on the infinite dimensional manifold of planar conformal embeddings of the unit disk.

The paper is organised as follows. In Section 2 we give a detailed review of the three types of vector field subalgebras. We work in the category of Fréchet-Lie algebras, and we give proofs that the subalgebras considered are proper Fréchet-Lie subalgebras. The results in this section does not require the underlying manifold to be compact.

In Section 3 we first review the Hodge decomposition for manifolds with boundary. Thereafter, we use the standard Hodge decomposition for manifolds with boundary in combination with the so called Friedrich decomposition to obtain a complete Hodge decomposition, involving six spaces, and we show that these spaces can be characterised in terms of kernels and im-
ages of the differential and co-differential. The special case of 2–manifolds is
studied in further detail, and the unit disk, the standard annulus, the torus,
and the sphere are given as examples.

In Section 4 we use the Hodge decomposition to obtain various \( L^2 \)
or-orthogonal decompositions of vector fields. Altogether, we derive nine different decompositions, which are summarised in Table 1. The main result is Theorem 4.1 which gives a decomposition of vector fields on a flat 2–manifold, involving conformal vector fields as one of the components in the Hodge decomposition. The section ends with various examples and one counter-example.

In Section 5 we work out the governing differential equations for three variational problems on the space of conformal vector fields on a simply connected bounded domain of \( \mathbb{R}^2 \).

2. Lie algebras of vector fields

In this section, let \( M \) be an \( n \)-manifold, possibly with boundary, such that \( M \) is either compact, or can be equipped with a countable sequence of compact sets \( K_i \subset M \) such that each compact subset \( U \subset M \) is contained in one of \( K_i \). In this case, the linear space \( \mathcal{F}(M) \) of smooth real valued function on \( M \) can be equipped with a sequence of semi-norms, making it a Fréchet space (see e.g. [4] for details). In turn, this induces a Fréchet topology on the space \( \mathcal{T}^r(M) \) of smooth tensor fields on \( M \) of finite order. In particular, the space \( \mathfrak{X}(M) \) of vector fields, and the space \( \Omega^k(M) \) of \( k \)-forms are Fréchet spaces. Furthermore, the topologies are “compatible” with each other, in the sense that any partial differential operator with smooth coefficients between any two spaces of tensor fields is a smooth map [4, Sect. II.2.2]. In particular, the Lie derivative map

\[
\mathfrak{X}(M) \times \mathcal{T}^r(M) \ni (\xi, t) \mapsto \mathcal{L}_\xi t \in \mathcal{T}^r(M)
\]

is smooth, which in turn implies that \( \mathfrak{X}(M) \) is a Fréchet-Lie algebra with Lie bracket given by \([\xi, \eta] = -\mathcal{L}_\xi \eta \) (this bracket fulfils the Jacobi identity).

Recall that a subspace of a Fréchet-Lie algebra is called a Fréchet-Lie subalgebra if it is topologically closed, and also closed under the Lie bracket. In the case when \( M \) has a boundary, it holds that the subspace \( \mathfrak{X}_1(M) \) of vector fields that are tangential to the boundary is a Fréchet-Lie subalgebra. Basically, this is the only subalgebra which can be obtained intrinsically, without introducing any further structures on \( M \). In the remainder of this section we review some other well known subalgebras of vector fields which require extra structure on the manifold.
2.1. Volume preserving vector fields

Assume that $M$ is orientable, and let $M$ be equipped with a volume form, denoted $\text{vol}$. The set of volume preserving vector fields is then given by $\mathfrak{X}_{\text{vol}}(M) = \{ \xi \in \mathfrak{X}(M); \mathcal{L}_\xi \text{vol} = 0 \}$. It is clear that this is a linear subspace of $\mathfrak{X}(M)$. Recall that the divergence with respect to $\text{vol}$ is the partial differential operator $\text{div} : \mathfrak{X}(M) \to \mathcal{F}(M)$ defined by $\mathcal{L}_\xi \text{vol} = \text{div}(\xi)\text{vol}$ for all $\xi \in \mathfrak{X}(M)$. Thus, since the volume form is strictly non-zero, it holds that $\xi \in \mathfrak{X}_{\text{vol}}(M)$ if and only if $\text{div}(\xi) = 0$.

Proposition 2.1. $\mathfrak{X}_{\text{vol}}(M)$ is a Fréchet-Lie subalgebra of $\mathfrak{X}(M)$.

Proof. The differential operator $\text{div} : \mathfrak{X}(M) \to \mathcal{F}(M)$ is smooth in the Fréchet topology. In particular, it is continuous, so the preimage of the closed set $\{0\} \in \mathcal{F}(M)$, which is equal to $\mathfrak{X}_{\text{vol}}(M)$, is also closed. Thus, $\mathfrak{X}_{\text{vol}}(M)$ is a topologically closed subspace of $\mathfrak{X}(M)$.

Next, let $\xi, \eta \in \mathfrak{X}_{\text{vol}}(M)$. Then

\[ \mathcal{L}_{[\xi,\eta]} \text{vol} = \mathcal{L}_\eta \mathcal{L}_\xi \text{vol} = \mathcal{L}_\eta \mathcal{L}_\xi \text{vol} - \mathcal{L}_\xi \mathcal{L}_\eta \text{vol} = 0. \]

Thus, $\mathfrak{X}_{\text{vol}}(M)$ is closed under the Lie bracket, which finishes the proof. \qed

In the case when $M$ has a boundary, it also holds that the subspace $\mathfrak{X}_{\text{vol},t}(M) := \mathfrak{X}_{\text{vol}}(M) \cap \mathfrak{X}_t(M)$ is a Fréchet-Lie subalgebra. This follows immediately since both $\mathfrak{X}_{\text{vol}}(M)$ and $\mathfrak{X}_t(M)$ are Fréchet-Lie subalgebras.

Next, consider the subspace of exact divergence free vector fields given by $\mathfrak{X}^{\text{ex}}_{\text{vol}}(M) = \{ \xi \in \mathfrak{X}(M); i_\xi \text{vol} \in d\Omega^{n-2}(M) \}$.

The following result is well known (see e.g. [2]).

Proposition 2.2. $\mathfrak{X}^{\text{ex}}_{\text{vol}}(M)$ is a Fréchet-Lie subalgebra of $\mathfrak{X}(M)$ and an ideal in $\mathfrak{X}_{\text{vol}}(M)$.

Proof. Topological closeness follows since the maps $\Omega^{n-1}(M) \ni i_\xi \text{vol} \mapsto \xi \in \mathfrak{X}(M)$ and $d : \Omega^{n-2}(M) \to \Omega^{n-1}(M)$ are smooth in the Fréchet topology. Next, if $\xi \in \mathfrak{X}^{\text{ex}}_{\text{vol}}(M)$ then $\xi$ is divergence free since $\mathcal{L}_\xi \text{vol} = di_\xi \text{vol} = 0$. Finally, if $\eta \in \mathfrak{X}_{\text{vol}}(M)$ then $i_{\mathcal{L}_\eta \xi} \text{vol} = \mathcal{L}_\eta i_\xi \text{vol} = \mathcal{L}_\eta d\alpha = d\mathcal{L}_\eta \alpha \in d\Omega^{n-2}(M)$, so $\mathfrak{X}^{\text{ex}}_{\text{vol}}(M)$ is an ideal in $\mathfrak{X}_{\text{vol}}(M)$. \qed

Continuing as before, we also obtain the smaller Fréchet-Lie subalgebra of tangential exact divergence free vector fields, by $\mathfrak{X}^{\text{ex}}_{\text{vol},t}(M) = \mathfrak{X}^{\text{ex}}_{\text{vol}}(M) \cap \mathfrak{X}_t(M)$. 

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The space of volume preserving vector fields is of importance in fluid mechanics. In particular, the motion of an incompressible ideal fluid is described by a differential equation evolving on the phase space $\mathfrak{X}_{\text{vol},t}(M)$, which is the Lie algebra of the infinite dimensional Lie group of volume preserving diffeomorphisms of $M$.

2.2. Symplectic vector fields

Let $M$ be equipped with a symplectic structure, i.e., a closed non-degenerate 2–form $\omega$. Then the subspace of symplectic vector fields on $M$ is given by $\mathfrak{X}_\omega(M) = \{ \xi \in \mathfrak{X}(M); \mathcal{L}_\xi \omega = 0 \}$.

**Proposition 2.3.** $\mathfrak{X}_\omega(M)$ is a Fréchet-Lie subalgebra of $\mathfrak{X}(M)$.

**Proof.** The map $\mathfrak{X}(M) \ni \xi \mapsto \mathcal{L}_\xi \omega \in \Omega^2(M)$ is smooth, so its preimage of $\{0\} \in \Omega^2(M)$ is topologically closed. Thus, $\mathfrak{X}_\omega(M)$ is topologically closed in $\mathfrak{X}(M)$. Further, if $\xi, \eta \in \mathfrak{X}_\omega(M)$, then

$$\mathcal{L}_{[\xi,\eta]} \omega = \mathcal{L}_\eta \mathcal{L}_\xi \omega - \mathcal{L}_\xi \mathcal{L}_\eta \omega = 0.$$

Thus, $\mathfrak{X}_\omega(M)$ is closed under bracket, which concludes the proof. \qed

The space of Hamiltonian vector fields are those who have a globally defined Hamiltonian. That is,

$$\mathfrak{X}_{\text{Ham}}(M) = \{ \xi \in \mathfrak{X}(M); i_\xi \omega \in d\Omega^0(M) \}.$$ 

With the same proof as for Proposition 2.2 but replacing vol with $\omega$, we get the following result.

**Proposition 2.4.** $\mathfrak{X}_{\text{Ham}}(M)$ is a Fréchet-Lie subalgebra of $\mathfrak{X}(M)$ and an ideal in $\mathfrak{X}_\omega(M)$.

Just as in the volume preserving case, we also have the smaller Fréchet-Lie subalgebras of symplectic and Hamiltonian tangential vector fields,

$$\mathfrak{X}_{\omega,t}(M) = \mathfrak{X}_\omega(M) \cap \mathfrak{X}_t(M) \quad \text{and} \quad \mathfrak{X}_{\text{Ham},t}(M) = \mathfrak{X}_{\text{Ham}}(M) \cap \mathfrak{X}_t(M).$$
2.3. Conformal vector fields

Let $M$ be equipped with a Riemannian metric $g$. Then the subspace of conformal vector fields is given by

$$X_{\text{con}}(M) = \{ \xi \in \mathfrak{X}(M); \mathcal{L}_\xi g = Fg, F \in \mathcal{F}(M) \}.$$

Thus, if $\xi \in X_{\text{con}}(M)$ then $\xi$ preserves the metric up to scaling by a function. In turn, this implies that the infinitesimal transformation generated by $\xi$ preserves angles. Indeed, if $\eta, \psi \in \mathfrak{X}(M)$ are everywhere orthogonal, i.e., $i_\eta i_\psi g = g(\eta, \psi) = 0$, then $g(\mathcal{L}_\xi \eta, \psi) + g(\eta, \mathcal{L}_\xi \psi) = 0$, which follows since

$$0 = \mathcal{L}_\xi (g(\eta, \psi)) = Fg(\eta, \psi) + g(\mathcal{L}_\xi \eta, \psi) + g(\eta, \mathcal{L}_\xi \psi) = g(\mathcal{L}_\xi \eta, \psi) + g(\eta, \mathcal{L}_\xi \psi).$$

**Proposition 2.5.** $X_{\text{con}}(M)$ is a Fréchet-Lie subalgebra of $\mathfrak{X}(M)$.

**Proof.** We need to show that $X_{\text{con}}(M)$ is closed under the Lie bracket and that $X_{\text{con}}(M)$ is topologically closed in $\mathfrak{X}(M)$. Let $\xi, \eta \in X_{\text{con}}(M)$. Then

$$\mathcal{L}_{\mathcal{L}_\xi \eta} g = \mathcal{L}_\xi (\mathcal{L}_\eta g) - \mathcal{L}_\eta (\mathcal{L}_\xi g) = \mathcal{L}_\xi (Gg) - \mathcal{L}_\eta (Fg) = (\mathcal{L}_\xi G - \mathcal{L}_\eta F)g,$$

which proves that $[\xi, \eta] = -\mathcal{L}_\eta \eta \in X_{\text{con}}(M)$.

To prove that $X_{\text{con}}(M)$ is topologically closed in $\mathfrak{X}(M)$, we define a map $\Phi : \mathfrak{X}(M) \to \mathcal{F}_2^0(M)$ by $\xi \mapsto \mathcal{L}_\xi g$. This is a smooth map in the Fréchet topology. We notice that $X_{\text{con}}(M) = \Phi^{-1}(\mathcal{F}(M)g)$. Since $\mathcal{F}(M)g$ is topologically closed in $\mathcal{F}_2^0(M)$ it follows from continuity of $\Phi$ that its preimage, i.e., $X_{\text{con}}(M)$, is topologically closed in $\mathfrak{X}(M)$. \qed

Notice that the condition $\mathcal{L}_\xi g = Fg$ for a vector field $\xi$ to be conformal is not as “straightforward” as the conditions for being volume preserving or symplectic, since the function $F$ depends implicitly on $\xi$. We now work out an explicit coordinate version of this condition in the case when the manifold $M$ is conformally flat.

First, recall that a Riemannian manifold is locally conformally flat if for every element $z \in M$ there exists a neighborhood $U$ of $z$ and a function $f \in \mathcal{F}(U)$ such that $e^{2f}g$ is a flat metric on $U$. Thus, we may chose local coordinates mapping $U$ conformally to flat Euclidean space, i.e., such that $g = c \sum dx^i \otimes dx^i$, with $c = e^{2f}$. Next, consider a vector field expressed in
these coordinates $\xi = \sum_i u^i \frac{\partial}{\partial x^i}$. Then

$$\mathcal{L}_\xi g = (\mathcal{L}_\xi c) \sum_i dx^i \otimes dx^i + c \sum_i (du^i \otimes dx^i + dx^i \otimes du^i)$$

$$= \sum_i (2\frac{\partial u^i}{\partial x^i} + i \xi dc) dx^i \otimes dx^i + \sum_{i<j} c\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) (dx^i \otimes dx^j + dx^j \otimes dx^i).$$

From this we get that $\mathcal{L}_\xi g$ is pointwise parallel with $g$ if the components of $\xi$ fulfill the following $n(n+2)/2-1$ relations

$$\begin{cases}
\frac{\partial u^i}{\partial x^i} - \frac{\partial u^{i+1}}{\partial x^{i+1}} = 0 & \forall \ i < n \\
\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} = 0 & \forall \ i < j.
\end{cases}$$

(1)

Notice that these equations are independent of the function $c$. Also, notice that if $M$ is a 2–manifold, these are the Cauchy-Riemann equations.

The coordinate formula (in conformally flat coordinates) for the divergence with respect to the volume form induced by $g$ is given by

$$\text{div}(\xi) = i \xi dc + c \sum_i \frac{\partial u^i}{\partial x^i}.$$ 

Thus, we see that if $M$ is a 2–manifold, and $\xi \in \mathfrak{X}_{\text{con}}(M)$, then locally we have $\mathcal{L}_\xi g = \text{div}(\xi)/c g$. In particular, if $(M, g)$ is a flat 2–manifold, then it holds that $\mathcal{L}_\xi g = \text{div}(\xi) g$.

3. Hodge decomposition

In this section, let $(M, g)$ be a compact oriented $n$–dimensional Riemannian manifold possibly with boundary, and let $\Omega^k(M)$ denote the space of smooth $k$–forms on $M$. We sometimes use the notation $\mathcal{F}(M)$ for $\Omega^0(M)$. Recall the Hodge star operator $\star: \Omega^k(M) \to \Omega^{n-k}(M)$, which is defined in terms of the metric (see [5, Chap. 6]). Using the Hodge star, the space $\Omega^k(M)$ is equipped with the $L^2$ inner product:

$$\langle \alpha, \beta \rangle_M := \int_M \alpha \wedge \star \beta.$$
Up to a boundary integral term, the co–differential $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is the formal adjoint of the differential with respect to the $L^2$ inner product. Indeed, it holds that
\[
\langle d\gamma, \beta \rangle_M = \langle \gamma, \delta \beta \rangle_M + \int_{\partial M} \gamma \wedge \star \beta.
\]
The explicit formula is $\delta = (-1)^{(n-k+1)k} \star d \star$. The subspaces $\Omega^k_t(M), \Omega^k_n(M) \subset \Omega^k(M)$ of tangential and normal $k$–forms are defined by
\[
\Omega^k_t(M) = \{ \alpha \in \Omega^k(M); i^*(\star \alpha) = 0 \}
\]
\[
\Omega^k_n(M) = \{ \alpha \in \Omega^k(M); i^*(\alpha) = 0 \}
\]
where $i : \partial M \rightarrow M$ is the inclusion. We also have the subspace of $k$–forms that vanish on the boundary: $\Omega^k_0(M) = \Omega^k_t(M) \cap \Omega^k_n(M)$. Notice that $\Omega^k_n(M) = \Omega^0_n(M), \Omega^k_t(M) = \Omega^0_t(M), \Omega^k_n(M) = \Omega^0_n(M)$, and $\Omega^k_0(M) = \Omega^0(M)$. It also holds that $\star \Omega^k_t(M) = \Omega^{n-k}_n(M)$.

The following fundamental result is known as the Hodge decomposition theorem for manifolds with boundary (see e.g. [6]):

**Theorem 3.1.** $\Omega^k(M)$ admits the $L^2$ orthogonal decomposition
\[
\Omega^k(M) = d\Omega^{k-1}(M) \oplus \delta \Omega^{k+1}(M) \oplus \mathcal{H}^k(M),
\]
where $\mathcal{H}^k(M) = \{ \alpha \in \Omega^k(M); d\alpha = 0, \delta \alpha = 0 \}$ are the harmonic $k$–fields.

The harmonic fields $\mathcal{H}^k(M)$ can be further decomposed in two different ways, called Friedrich decompositions (see e.g. [6]).

**Theorem 3.2.** $\mathcal{H}^k(M)$ admits the $L^2$ orthogonal decompositions
\[
\mathcal{H}^k(M) = \mathcal{H}^k_t(M) \oplus \{ \alpha \in \mathcal{H}^k(M); \alpha = d\epsilon \}
\]
\[
\mathcal{H}^k(M) = \mathcal{H}^k_n(M) \oplus \{ \alpha \in \mathcal{H}^k(M); \alpha = \delta \gamma \}
\]
where $\mathcal{H}^k_t(M)$ and $\mathcal{H}^k_n(M)$ are the harmonic $k$–fields that are respectively tangential and normal.

Thus, every harmonic field is decomposed into either: (i) a tangential harmonic field plus an exact harmonic field or (ii) a normal harmonic field plus a co-exact harmonic field.
Notice that by combining the first Friedrich decomposition in Theorem 3.2 with the Hodge decomposition in Theorem 3.1 we obtain an L² orthonormal decomposition of exact k–forms as
\[ d\Omega^{k-1}(M) = d\Omega^{k-1}_n(M) \oplus \{ \alpha \in \mathcal{H}^k(M); \alpha = d\epsilon \}. \]
Indeed, for any \( \epsilon \in \Omega^{k-1}(M) \) it cannot hold that \( d\epsilon \) belongs to \( \mathcal{H}^k(M) \) or \( \delta\Omega^{k+1}(M) \) since that would imply that \( d\epsilon \) also belongs to \( \{ \alpha \in \mathcal{H}^k(M); \alpha = d\epsilon \} \), which is disjoint to both. Similarly, we get
\[ \delta\Omega^{k+1}(M) = \delta\Omega^{k+1}_t(M) \oplus \{ \alpha \in \mathcal{H}^k(M); \alpha = \delta\gamma \}. \]

Closely related to the harmonic k–fields \( \mathcal{H}^k(M) \) are the harmonic k–forms, given by \( \mathcal{H}^k(M) = \{ \alpha \in \Omega^k(M); \Delta\alpha = 0 \} \), where \( \Delta := \delta \circ d + d \circ \delta \) is the Laplace–deRham operator. For a closed manifold it holds that \( \mathcal{H}^k(M) = H^k(M) \). However, in the presence of a boundary, \( \mathcal{H}^k(M) \) is strictly smaller than \( H^k(M) \).

By combining the two versions of the Friedrich decomposition in Theorem 3.2 we obtain four mutually orthogonal subspaces of \( \mathcal{H}^k(M) \). Altogether, we thus have six mutually orthogonal subspaces of \( \Omega^k(M) \), which are given in the following table.

| Short name | Definition |
|------------|------------|
| \( A^k_1(M) \) | \( d\Omega^{k-1}_n(M) \) |
| \( A^k_2(M) \) | \( \delta\Omega^{k+1}_t(M) \) |
| \( A^k_3(M) \) | \( \mathcal{H}^k_v(M) \cap \mathcal{H}^k(M) \) |
| \( A^k_4(M) \) | \( \mathcal{H}^k(M) \cap d\Omega^{k-1}(M) \) |
| \( A^k_5(M) \) | \( \mathcal{H}^k_v(M) \cap \delta\Omega^{k+1}(M) \) |
| \( A^k_6(M) \) | \( d\Omega^{k-1}(M) \cap \delta\Omega^{k+1}(M) \) |

Theorem 3.1 together with Theorem 3.2 now yields the following result, which we naturally call the complete Hodge decomposition for manifolds with boundary.

**Corollary 3.1.** \( \Omega^k(M) \) admits the L² orthogonal decomposition
\[ \Omega^k(M) = \bigoplus_{l=1}^{6} A^k_l(M). \]

**Remark 3.1.** If \( M \) does not have a boundary, then \( A^k_4(M), A^k_5(M), A^k_6(M) \) are trivial, and Corollary 3.1 reduced to the ordinary Hodge decomposition.
for closed manifolds. In contrast, if $M$ is the closure of a bounded open subset of $\mathbb{R}^n$ with smooth boundary, then $A^k_3(M)$ is trivial, since a harmonic field on such a manifold which is zero on the boundary, must be zero also in the interior. (The Laplace equation with Dirichlet boundary conditions is well-posed.)

### 3.1. Characterisation in terms of four fundamental subspaces

The differential and co-differential induces four fundamental subspaces of $\Omega^k(M)$ given by

\[
\begin{align*}
\text{im } d &= d\Omega^{k-1}(M), \\
\ker d &= \{ \alpha \in \Omega^k(M); d\alpha = 0 \}, \\
\text{im } \delta &= \delta\Omega^{k+1}(M), \\
\ker \delta &= \{ \alpha \in \Omega^k(M); \delta\alpha = 0 \}.
\end{align*}
\]

Notice that im $d \subset$ ker $d$ and im $\delta \subset$ ker $\delta$. Also notice that the intersection between any two of these subspaces in general is non-empty.

Each of the mutually orthogonal spaces $A^k_l(M)$ can be characterised by orthogonal complements and intersections of the four fundamental subspaces.

**Proposition 3.1.** It holds that

\[
\begin{align*}
A^k_1(M) &= (\ker \delta)^\perp \\
A^k_2(M) &= (\ker d)^\perp \\
A^k_3(M) &= (\text{im } d)^\perp \cap (\text{im } \delta)^\perp \\
A^k_4(M) &= \ker \delta \cap \text{im } d \cap (\text{im } \delta)^\perp \\
A^k_5(M) &= \ker d \cap \text{im } \delta \cap (\text{im } d)^\perp \\
A^k_6(M) &= \text{im } d \cap \text{im } \delta.
\end{align*}
\]

**Proof.** From mutual orthogonality between $A^k_l(M)$ we obtain

\[
\begin{align*}
\text{im } d &= A^k_1(M) \oplus A^k_4(M) \oplus A^k_6(M) \\
\text{im } \delta &= A^k_2(M) \oplus A^k_5(M) \oplus A^k_6(M) \\
\ker d &= A^k_1(M) \oplus A^k_3(M) \oplus A^k_4(M) \oplus A^k_5(M) \oplus A^k_6(M) \\
\ker \delta &= A^k_2(M) \oplus A^k_3(M) \oplus A^k_4(M) \oplus A^k_5(M) \oplus A^k_6(M).
\end{align*}
\]

Using Corollary 3.1 the result now follows from basic set operations. \qed
Remark 3.2. A special case of Proposition 3.1 is given in [7]. Indeed, that paper gives a characterisation of the Helmholtz decomposition of vector fields on bounded domains of $\mathbb{R}^3$ in terms of the kernel and image of the grad and curl operators.

3.2. Special case of 2–manifolds

In this section we analyse in detail the complete Hodge decomposition in the case of 2–manifolds. The de Rham complex and co-complex for a Riemannian 2–manifold $(M, g)$ is

$$
\begin{array}{cccc}
\Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2(M) & \xrightarrow{\delta} & \Omega^1(M) & \xrightarrow{\delta} & \Omega^0(M)
\end{array}
$$

so the Hodge star maps $\Omega^1(M)$ isomorphically to itself. If $\alpha \in \Omega^1(M)$, then $\star \star \alpha = -\alpha$, so the Hodge star induces an almost complex structure on $M$.

Also, since the Hodge star maps normal forms to tangential forms, closed forms to co-closed forms, and exact forms to co-exact forms (and vice-versa), it holds that $\star A^1(M) = A^1(M)$ and $\star A^4(M) = A^4(M)$. It also holds that $\star A^3(M) = A^3(M)$ and $\star A^6(M) = A^6(M)$.

We now work out the complete Hodge decomposition of $\Omega^1(M)$ in some standard examples. These relevant later in the paper, when we discuss the Lie algebra of conformal vector fields.

Example 3.1 (Disk). Let $D$ be the unit disk in $\mathbb{R}^2$, equipped with the Euclidean metric. Since the first co-homology group of $D$ is trivial, it holds that every closed 1–form is exact, and every co-closed 1–form is co-exact. Thus, $A^3(D)$, $A^4(D)$ and $A^6(D)$ are trivial, so all harmonic 1–fields are in $A^2(D)$. Thus, the complete Hodge decomposition for $\Omega^1(D)$ is

$$
\Omega^1(D) = A^1(D) \oplus A^2(D) \oplus A^6(D),
$$

where all the components are infinite dimensional.

Example 3.2 (Annulus). Let $A$ be a standard annulus in $\mathbb{R}^2$, equipped with the Euclidean metric. As for the disk, it holds that $A^3(A)$ is trivial (see Remark 3.1). However, $A^1(A)$ and $A^6(A)$ are not trivial. Indeed, if $\alpha \in A^1(A)$, then $\alpha = df$ where the function $f$ fulfils $\delta f = \Delta f = 0$. Also, since $\alpha$ is normal, it holds that $0 = i^*(df) = d\ast(f)$, so $f$ is constant on each of its two connected components of the boundary. By uniqueness of
the Laplace equation on \( A \) it then holds that \( f \) is determined uniquely by the two boundary constants. Since the differential of a constant is zero, it is no restriction to assume that the constant at the outer boundary is zero, so we have that \( A^1_1(A) = \{df; \Delta f = 0, f|_{\partial A} = 0, f|_{\partial_2 A} = \text{const}\} \). This space has dimension 1, and since \( A^3_0(A) = \ast A^1_1(A) \) it also holds that \( A^3_0(A) \) has dimension 1. Explicitly, in Cartesian coordinates, we have \( A^1_1(A) = \text{span}\{\alpha\} \), where \( \alpha = d \ln(x^2 + y^2) \), and \( A^3_0(A) = \text{span}\{\ast\alpha\} = \text{span}\{(x\,dy - y\,dx)/(x^2 + y^2)\} \). Thus, the complete Hodge decomposition is

\[
\Omega^1(A) = A^1_1(A) \oplus A^2_2(A) \oplus A^3_3(A) \oplus A^4_4(A) \oplus A^5_5(A),
\]

where \( A^1_1(A) \) and \( A^3_3(A) \) are 1–dimensional, and all the other components are infinite dimensional.

**Example 3.3** (Torus). Let \( T^2 \) be the torus, equipped with the metric inherited from \( \mathbb{R}^3 \). Since \( T^2 \) is a closed manifold, \( A^1_1(T^2) \), \( A^3_3(T^2) \) and \( A^5_5(T^2) \) are trivial. However, the first co-homology group is two dimensional, so \( A^3_3(T^2) \) is also two dimensional. Thus, the complete Hodge decomposition for \( \Omega^1(T^2) \) is

\[
\Omega^1(T^2) = A^1_1(T^2) \oplus A^2_2(T^2) \oplus A^3_3(T^2).
\]

**Example 3.4** (Sphere). Let \( S^2 \) be the sphere, equipped with the metric inherited from \( \mathbb{R}^3 \). Since \( S^2 \) is a closed manifold, \( A^1_1(S^2) \), \( A^3_3(S^2) \) and \( A^5_5(S^2) \) are trivial (see Remark 3.1). Since the first co-homology group of \( S^2 \) is trivial, it holds that \( A^3_3(S^2) \) is trivial. Thus, the complete Hodge decomposition for \( \Omega^1(S^2) \) is

\[
\Omega^1(S^2) = A^1_1(S^2) \oplus A^2_2(S^2).
\]

4. **Orthogonal decomposition of vector fields**

The \( L^2 \) inner product on \( \mathfrak{X}(M) \) is given by

\[
\langle \xi, \eta \rangle_M := \int_M g(\xi, \eta)\,\text{vol}
\]

where \( \text{vol} \) is the volume form induced by \( g \). (Notice that we use the same notation as for the \( L^2 \) inner product on forms.)

In this section we show how the Hodge decomposition can be used to obtain the \( L^2 \) orthogonal complement of the Lie subalgebras of vector field discussed in Section 2 above. The approach is to find an isometry \( \mathfrak{X}(M) \to \Omega^1(M) \) which maps the vector field subalgebra under study onto one of the components in the Hodge decomposition.
For the first case of volume preserving vector fields is well known that \( \mathfrak{X}(M) \) can be orthogonally decomposed into divergence free plus gradient vector fields. Using contraction with the metric as an isometric isomorphism \( \mathfrak{X}(M) \to \Omega^1(M) \), this decomposition is expressed by the Hodge decomposition of 1–forms. (Equivalently, one may use contraction with the volume form instead, which corresponds to the Hodge decomposition of \((n-1)\)-forms.)

The decomposition is essential in the derivation of the Euler equations for the motion of an ideal incompressible fluid. In this case, the Lagrange multipliers in the projection has a physical interpretation as the pressure in the fluid.

For the second case, it is well known that symplectic vector fields can be identified with closed 1–forms, by contraction with the symplectic form. Usually, flow equations for symplectic vector fields are written in terms of the Hamiltonian function, i.e., \( \mathfrak{X}_\omega(M) \) is identified with \( \mathcal{F}(M) \), and the equations are expressed on \( \mathcal{F}(M) \) (an example is the equation for quasigeostrophic motion). This approach, corresponding to vorticity formulation in the case of the Euler fluid, is viable in the setting of Hamiltonian vector fields, i.e., those which have a globally defined Hamiltonian. However, on manifolds which are not simply connected (so that not every closed 1–form is exact), this approach may not be feasible. However, representation on the full space of vector fields, using Lagrangian multipliers for orthogonal projection, can always be used.

The third case of conformal vector fields is a new example. For a manifold of dimension larger than 2, it follows from a theorem by Liouville that the space of conformal vector fields is finite dimensional. However, in the case of 2–manifolds, the space of conformal vector fields can be (but does not have to be) infinite dimensional. The approach we follow in this paper works in the case of flat 2–manifolds. For closed manifolds, this includes essentially only the 2–torus, for which the set of conformal vector fields is finite dimensional. However, in the case of a bounded domain of \( \mathbb{R}^2 \), the conformal vector fields correspond to all holomorphic functions on this domain, and is thus infinite dimensional. The application we have in mind is the derivation of Euler-Lagrange equations for variational problems on the space of conformal vector fields. An example is geodesic motion on the infinite dimensional manifold of conformal embeddings.

Each of the vector field subalgebras discussed in Section 2 can be identified with one or several components in the complete Hodge decomposition. However, the choice of isomorphism \( \mathfrak{X}(M) \to \Omega^1(M) \) is different between the three basic cases of divergence free, symplectic, and conformal vector fields. Once the isomorphism has been specified, we use the short-hand notation.
\(\xi \mapsto \xi^\flat\) for the map \(X(M) \to \Omega^1(M)\) and \(\alpha \mapsto \alpha^\sharp\) for its inverse. Table I contains an overview of the decompositions. Detailed expositions for each case are given in the remaining part of this section.

**Remark 4.1.** The requirement that the isomorphism \(X(M) \to \Omega^1(M)\) is an isometry can be weakened. Indeed, our basic requirement is that orthogonality is preserved, so it is enough that the map is conformal. However, in the examples in this paper the isomorphism will be an isometry.

### 4.1. Volume preserving vector fields on a Riemannian manifold

Let \((M, g)\) be a compact Riemannian manifold. As isomorphism we use contraction with the metric, i.e., \(\xi \mapsto \iota_\xi g =: \xi^\flat\).

As before, let \(\text{vol}\) denote the volume form induced by the Riemannian metric \(g\). Since \(\xi\) is divergence free if and only if \(\iota_\xi \text{vol}\) is closed, it follows from the formula \(\iota_\xi \text{vol} = *\xi^\flat\) that \(\xi\) is divergence free if and only if \(\xi^\flat\) is co-closed, i.e., \(\delta \xi^\flat = 0\). Furthermore, \(\xi \in X_{\text{vol}}^0(M)\) if and only if \(\xi^\flat\) is co-exact, i.e., \(\xi^\flat = \delta \alpha\) for some \(\alpha \in \Omega^2(M)\). Thus, \(X_{\text{vol}}(M)^\flat = \ker \delta\) and \(X_{\text{ex}}^0(M)^\flat = \text{im} \delta\), which gives the first and the third decompositions in Table I.

For the case of tangential vector fields, we notice that \(\xi\) is a tangential vector field if and only if \(\xi^\flat\) is a tangential 1–form. Since elements in \(A_{10}^1(M)\) and \(A_{16}^1(M)\) are necessarily non-tangential, we obtain the second and the fourth decompositions in Table I.

### 4.2. Symplectic vector fields on an almost Kähler manifold

Let \((M, g, \omega)\) be an almost Kähler manifold. As isomorphism we use contraction with the symplectic form, i.e., \(\xi \mapsto \iota_\xi \omega =: \xi^\flat\). Due to the almost Kähler structure, this isomorphism is isometric.

Since \(L_\xi \omega = d\iota_\xi \omega = d\xi^\flat\) it follows that \(\xi\) is symplectic if and only if \(\xi^\flat\) is closed. Likewise, \(\xi\) is Hamiltonian if and only if \(\xi^\flat\) is exact. Thus, \(X_{\omega}(M)^\flat = \ker d\) and \(X_{\text{Ham}}(M)^\flat = \text{im} d\). This yields the first and the third symplectic decompositions in Table I.

For the case of tangential symplectic and Hamiltonian vector fields, we notice that \(\xi\) is a tangential vector field if and only if \(\xi^\flat\) is a normal 1–form. Since elements in \(A_{10}^1(M)\) and \(A_{16}^1(M)\) are necessarily non-tangential, we obtain the second and the fourth symplectic decompositions in Table I.

### 4.3. Conformal vector fields on a 2–manifold

Let \((M, \mathfrak{g})\) be a compact Riemannian 2–manifold, possibly with boundary. If \((M, \mathfrak{g})\) is flat, then there exists an orthogonal reflection map \(R \in \mathfrak{T}_1^1(M)\), i.e., an orientation reversing isometry such that \(R^2 = \text{Id}\). In that
| Setting     | Isomorphism       | Decompositions                                                                 |
|------------|-------------------|-------------------------------------------------------------------------------|
| $M^n$, $g$ | $\xi \mapsto i_\xi g$ | $\mathfrak{X}(M) = \mathfrak{X}_{\text{vol}}(M) \oplus \text{grad}(\mathcal{F}_0(M))$ $\bigoplus_{i \neq 1} A_i^1(M)^5$ $A_1^1(M)^5$ |
|            |                   | $\mathfrak{X}(M) = \mathfrak{X}_{\text{vol}}(M) \oplus \text{grad}(\mathcal{F}(M))$ $\bigoplus_{i \in \{2,3,5\}} A_i^1(M)^5$ $\bigoplus_{i \in \{1,4\}} A_i^1(M)^5$ |
|            |                   | $\mathfrak{X}(M) = \mathfrak{X}_{\text{vol}}(M) \oplus \text{grad}(\mathcal{F}_0(M)) \oplus \mathfrak{X}^a_0(M)^5$ $\bigoplus_{i \in \{2,5,6\}} A_i^1(M)^5$ $\bigoplus_{i \in \{3\}} A_i^1(M)^5$ |
| $M^{2n}$, $g$, $\omega$ | $\xi \mapsto i_\xi \omega$ | $\mathfrak{X}(M) = \mathfrak{X}_{\omega}(M) \oplus \delta \Omega^a_0(M)^2$ $\bigoplus_{i \neq 2} A_i^1(M)^5$ $A_2^1(M)^5$ |
| (almost Kähler) |                   | $\mathfrak{X}(M) = \mathfrak{X}_{\omega}(M) \oplus \delta \Omega^2(M)^2$ $\bigoplus_{i \in \{1,3,4\}} A_i^1(M)^5$ $\bigoplus_{i \in \{2,5,6\}} A_i^1(M)^5$ |
|            |                   | $\mathfrak{X}(M) = \mathfrak{X}_{\text{vol}}(M) \oplus \delta \Omega^2_0(M)^2 \oplus \mathfrak{X}^a_0(M)^5$ $\bigoplus_{i \in \{1,4\}} A_i^1(M)^5$ $\bigoplus_{i \in \{3,5\}} A_i^1(M)^5$ |
|            |                   | $\mathfrak{X}(M) = \mathfrak{X}_{\text{vol}}(M) \oplus \delta \Omega^2_0(M)^2 \oplus \mathfrak{X}^a_0(M)^5$ $\bigoplus_{i \in \{1,4\}} A_i^1(M)^5$ $\bigoplus_{i \in \{3,5\}} A_i^1(M)^5$ |
| $M^2$, $g$ | $\xi \mapsto i_\xi g$ | $\mathfrak{X}(M) = \mathfrak{X}_{\text{con}}(M) \oplus \text{grad}(\mathcal{F}_0(M)) \oplus \text{sgrad}(\mathcal{F}_0(M))$ $\bigoplus_{i \neq \{1,2\}} A_i^1(M)^5$ $A_1^1(M)^5$ $A_2^1(M)^5$ |

Table 1: Various $L^2$ orthogonal decompositions of vector fields based on the Hodge decomposition. The first column specifies the required setting. $M$ is a manifold, possibly with boundary, and the upper index denotes its dimension. The second column specifies the isomorphism used to identify vector fields with 1–forms. The third column gives various $L^2$ orthogonal decompositions of $\mathfrak{X}(M)$, and corresponding components in the Hodge decomposition (via the contraction map). The first component in each decomposition is a Lie subalgebra of vector fields.
case, we may chose coordinate charts such that \( g = dx \otimes dx + dy \otimes dy \) and 
\( R = dx \otimes \partial_x - dy \otimes \partial_y \). If \( \xi \in \mathfrak{X}(M) \) then we write \( \bar{\xi} := R\xi \). Since \( R \) is an isometry, it holds that the isomorphism 
\[ \mathfrak{X}(M) \ni \xi \mapsto \bar{\xi} g =: \xi^\flat \in \Omega^1(M) \]
is isometric with respect to the \( L^2 \) inner products on \( \mathfrak{X}(M) \) and \( \Omega^1(M) \).

The following lemma is the key to obtaining the \( L^2 \) orthogonal complement of \( \mathfrak{X}_{\text{con}}(M) \) in \( \mathfrak{X}(M) \).

**Lemma 4.1.** If \((M, g)\) is a flat 2–manifold then \( \mathfrak{X}_{\text{con}}(M)^\flat = \mathfrak{H}^1(M) \).

**Proof.** It is enough to prove the assertion in local coordinates as above. Let 
\( \xi = u \partial_x + v \partial_y \in \mathfrak{X}_{\text{con}}(M) \). Then \((u, v)\) must fulfill the Cauchy–Riemann equations.

It holds that 
\[ d\xi^\flat = u dy \wedge dx - v dx \wedge dy = -(v_x + u_y) dx \wedge dy \]
and
\[ \delta\xi^\flat = \star d \star \xi^\flat = \star d(v dx + u dy) = \star (u_x - v_y) dx \wedge dy = u_x - v_y. \]

Hence, we see that \( d\xi^\flat = 0 \) and \( \delta\xi^\flat = 0 \) if and only if \((u, v)\) fulfills the Cauchy-Riemann equations. This proves the assertion.

We now introduce “reflected” versions of the gradient and the skew gradient. Indeed, for a function \( F \in \mathcal{F}(M) \) we define the reflection gradient as 
\[ \overline{\text{grad}}(F) = (dF)^\sharp \]
and the reflection skew gradient as 
\[ s\text{grad}(F) = (\star dF)^\sharp. \]

In local (flat) coordinates we have 
\[ \overline{\text{grad}}(F) = F_x \partial_x - F_y \partial_y, \quad s\text{grad}(F) = F_y \partial_x + F_x \partial_y. \]

Let \( \mathcal{F}_0(M) = \{ F \in \mathcal{F}(M); F|_{\partial M} = 0 \} \), i.e., the functions that vanish on the boundary. Using Lemma 4.1 together with the Hodge decomposition Theorem 3.1 we obtain the following result.

**Theorem 4.1.** If \((M, g)\) is a flat 2–manifold, then the space of vector fields on \( M \) admits the \( L^2 \) orthogonal decomposition 
\[ \mathfrak{X}(M) = \mathfrak{X}_{\text{con}}(M) \oplus \overline{\text{grad}}(\mathcal{F}_0(M)) \oplus s\text{grad}(\mathcal{F}_0(M)) \]
where each component is closed in \( \mathfrak{X}(M) \) with respect to the Fréchet topology.
Proof. First, we need to verify the following diagram:

\[
\begin{array}{c}
\text{grad}(\mathcal{F}_0(M)) \oplus \text{sgrad}(\mathcal{F}_0(M)) \oplus X_{\text{con}}(M) \\
\downarrow \\
\text{d}\Omega^0_n(M) \oplus \delta\Omega^2_1(M) \oplus \mathcal{H}^1(M)
\end{array}
\]

It follows from the definition of normal forms that $\Omega^0_n(M) = \mathcal{F}_0(M)$. Thus,

\[
\text{grad}(\mathcal{F}_0(M)) = d\mathcal{F}_0(M) = d\Omega^0_n(M).
\]

Also, it holds that $\Omega^0_n(M) = \ast\Omega^2_1(M)$, which gives

\[
\text{sgrad}(\mathcal{F}_0(M)) = \ast d\Omega^0_n(M) = -\delta\Omega^2_1(M) = \delta\Omega^2_1(M).
\]

Next, it follows from Lemma 4.1 that $X_{\text{con}}(M) = H^1(M)$. Now, from the Hodge decomposition of forms (Theorem 3.1) it follows that the subspaces of forms are $L^2$ orthogonal to each other. This implies that the subspaces of vector fields are also $L^2$ orthogonal to each other, since the isomorphism $\xi \rightarrow \xi$ is an isometry. Finally, since the isomorphism $\flat : \mathcal{X}(M) \to \Omega^1(M)$ is a continuous (even smooth) vector space isomorphism with respect to the Fréchet topologies on $\mathcal{X}(M)$ and $\Omega^1(M)$, and since the subspaces of forms are topologically closed in $\Omega^1(M)$, it follows that the vector field subspaces are topologically closed in $\mathcal{X}(M)$. \qed

Example 4.1 (Disk). Let $M = \mathbb{D}$ and let $(x, y)$ be Cartesian coordinates. If $\xi = u \partial_x + v \partial_y$, we take as reflection map $R(\xi) = u \partial_x - v \partial_y$ (corresponding to complex conjugation of $z = x + iy$). It follows from equation (2) that $\mathcal{H}^1(\mathbb{D}) = A^1_0(\mathbb{D})$. Thus, by Theorem 4.1 we get that the set of conformal vector fields on $\mathbb{D}$ (corresponding to holomorphic functions on $\mathbb{D}$) is isometrically isomorphic to the set of simultaneously exact and co-exact 1–forms.

Example 4.2 (Annulus). Let $M = \mathbb{A}$ and let $(x, y)$ be Cartesian coordinates. We use the same reflection map as for the disk. It follows from equation (3) that $\mathcal{H}^1(\mathbb{A}) = A^1_0(\mathbb{A}) \oplus A^1_1(\mathbb{A}) \oplus A^1_2(\mathbb{A})$. The special harmonic fields $\alpha$ and $\ast\alpha$ (see Example 3.2) corresponds to the holomorphic functions $i/z$ and $1/z$. In complex analysis it is well known that these functions have a special role (e.g., in the calculus of residues).

Example 4.3 (Torus). Let $M = \mathbb{T}$ and let $(\theta, \phi)$ be the standard angle coordinates. (Notice that $\theta, \phi$ are not smooth functions on $\mathbb{T}$, but $d\theta$ and $d\phi$ are well defined smooth 1–forms.) We use the reflection map $\partial_\theta, \partial_\phi \mapsto (\partial_\theta, -\partial_\phi)$. It follows from equation (4) and Theorem 4.1 that the set of conformal vector fields is only two dimensional, generated by the pure translations $\partial_\theta$ and $\partial_\phi$.\[17\]
Counter-example 4.4 (Sphere). It follows from equation (5) that the space of harmonic fields on the sphere is trivial. However, the space of conformal vector fields of the sphere corresponds to the Möbius transformations (by identifying the plane with the Riemann sphere) and is thus six dimensional, see e.g. [8]. Thus, it is not possible to identify the space of conformal vector fields with the harmonic $1$–fields in this case. The reason is that $S^2$ is not flat, so Theorem 4.1 does not apply.

5. Application to conformal variational problems

In this section we show how the orthogonal decomposition of conformal vector fields can be used to derive differential equations for variational problems involving conformal vector fields on a bounded domain $U \subset \mathbb{R}^2$ (or equivalently, variational problems involving holomorphic functions on a bounded complex domain $U$). Throughout this section we identify complex valued functions on $U$ with vector fields on $U$.

5.1. Computing the conformal projection

Given a vector field $\xi \in \mathfrak{X}(U)$, there is a direct way to compute the orthogonal projection $\text{Pr}_{\text{con}} : \mathfrak{X}(U) \to \mathfrak{X}_{\text{con}}(U)$, which does not involve solving a set of partial differential equations. The approach is to use the Bergman kernel [9], i.e., the reproducing kernel $K_U(z, \cdot)$ of the Bergman space $A^2(U)$, which is the Hilbert space obtained by completion of $\mathfrak{X}_{\text{con}}(U)$ with respect to the $L^2$ inner product. For any $f \in A^2(U)$ it then holds that $f(z) = \langle \langle f, K_U(z, \cdot) \rangle \rangle_U$, where $\langle \langle f, g \rangle \rangle_U := \int_U f\bar{g} \, dA$ is the complex $L^2$ inner product. The existence of $K_U(z, \cdot)$ follows from Riesz representation theorem since point-wise evaluation of functions in $A^2(U)$ is continuous in the $L^2$ topology. Now, for a general complex valued function $f \in L^2(U)$, its orthogonal projection to $A^2(U)$ is given by $f_{\text{con}}(z) = \langle \langle f, K_U(z, \cdot) \rangle \rangle_U$. Since $\xi \in \mathfrak{X}(U)$ implies that $\text{Pr}_{\text{con}}(\xi) \in \mathfrak{X}_{\text{con}}(U)$ and since $\mathfrak{X}_{\text{con}}(U) \subset A^2(U)$ it must hold that $\text{Pr}_{\text{con}}(\xi) = \langle \langle \xi, K_U(z, \cdot) \rangle \rangle_U$.

For the case of the unit disk, it holds that $K_\mathbb{D}(z, \zeta) = \frac{1}{\pi (1 - z\bar{\zeta})^2}$. For other domains, the kernel function is given by $K_U(z, \zeta) = K_\mathbb{D}(\varphi(z), \varphi(\zeta)) \varphi'(\zeta)\varphi'(z)$, where $\varphi$ is a conformal mapping $U \to \mathbb{D}$.

A basis for $\mathfrak{X}(\mathbb{D})$ is given by $\{z^m \bar{z}^n\}_{m,n \geq 0}$. Thus, if $\xi \in \mathfrak{X}(\mathbb{D})$ is expanded in this basis, we may compute $\text{Pr}_{\text{con}}(\xi)$ by applying $\text{Pr}_{\text{con}}$ to each of the basis
elements. Indeed, if $e_{mn}(z) = z^m z^n$ then
\[
\Pr_{\text{con}}(e_{mn})(z) = \int_{\mathbb{D}} \frac{1}{\pi (1-z\zeta)^2} \zeta^m \bar{\zeta}^n dA(\zeta)
\]
\[
= \int_{\mathbb{D}} \frac{1}{\pi} \sum_{p=1}^{\infty} z^p \bar{\zeta}^p \zeta^m \bar{\zeta}^n dA(\zeta) = \begin{cases} 
\frac{m-n+1}{m+1} z^{m-n} & m \geq n \\
0 & m < n 
\end{cases}
\]

Another result that may be useful is the following.

**Proposition 5.1.** Let $\xi \in \mathfrak{X}(U)$ and $\psi \in \mathfrak{X}_{\text{con}}(U)$. If $\psi(z) \neq 0$ for all $z \in U$ then $\Pr_{\text{con}}(\xi) = 0$ if and only if $\Pr_{\text{con}}(\bar{\psi}\xi) = 0$.

**Proof.** We have
\[
\Pr_{\text{con}}(\xi) = 0 \iff \langle \xi, \eta \rangle = 0 \quad \forall \eta \in \mathfrak{X}_{\text{con}}(U)
\]
\[
\iff \langle \bar{\psi}\xi, \eta/\psi \rangle = 0 \quad \forall \eta \in \mathfrak{X}_{\text{con}}(U)
\]
\[
\iff \langle \bar{\psi}\xi, \rho \rangle = 0 \quad \forall \rho \in \mathfrak{X}_{\text{con}}(U)
\]
\[
\iff \Pr_{\text{con}}(\bar{\psi}\xi) = 0.
\]

\[\square\]

5.2. Integration by parts

Standard variational calculus makes frequent use of integration by parts in order to “isolate” a virtual variation from derivatives. Usually, the boundary term appearing either vanishes (in the case of a space of tangential vector fields), or it can be treated separately giving rise to natural boundary conditions (in the case of a space where vector fields can have arbitrary small compact support). However, in the case of conformal vector fields, there is always a global dependence between interior points, and points on the boundary (due to the Cauchy–Riemann equations). Hence, in the conformal case, we need an appropriate analogue of integration by parts which avoids boundary integrals. Formally, we may proceed as follows. Let $\partial_z : \mathfrak{X}_{\text{con}}(U) \to \mathfrak{X}_{\text{con}}(U)$ be the complex derivative. Then we are looking for the adjoint of this operator with respect to the $L^2$ inner product. That is, an operator $\partial^\dagger_z : \mathfrak{X}_{\text{con}}(U) \to \mathfrak{X}_{\text{con}}(U)$ such that
\[
\langle \xi, \partial_z \eta \rangle_U = \langle \partial^\dagger_z \xi, \eta \rangle_U, \quad \forall \xi, \eta \in \mathfrak{X}_{\text{con}}(U).
\]

Explicitly, it is most easily done on the disk. Indeed, for $\xi, \eta \in \mathfrak{X}_{\text{con}}(\mathbb{D})$, denoting $\partial_z \xi = \xi_z$, we have
\[
\langle \xi, \eta \rangle_\mathbb{D} = \langle (z^2 \xi)_z, \eta \rangle_\mathbb{D},
\]

19
as can be seen from the following. Every element of $X_{\text{con}}(\mathbb{D})$ has a convergent Taylor series, and the monomials $z^n$ form a basis for $X_{\text{con}}(\mathbb{D})$ that is orthogonal with respect to both the real and the complex $L^2$ inner product, for

$$\langle \langle z^m, z^n \rangle \rangle_{\mathbb{D}} = \int_{\mathbb{D}} \xi \bar{\eta} \, dA$$

$$= \int_0^1 \left( r \, dr \int_0^{2\pi} \, d\theta \, r^m e^{im\theta} r^n e^{-in\theta} \right)$$

$$= \frac{2\pi}{m + n + 2} \delta_{m,n}$$

and $\langle \xi, \eta \rangle_{\mathbb{D}} = \text{Re} \langle \langle \xi, \eta \rangle \rangle_{\mathbb{D}}$. Therefore, expanding $\xi = \sum_{n=0}^{\infty} \xi_n z^n$ and $\eta = \sum_{m=0}^{\infty} \eta_m z^m$,

$$\langle \langle \xi, \eta z \rangle \rangle_{\mathbb{D}} = \langle \langle \sum_{n=0}^{\infty} \xi_n z^n, \left( \sum_{m=0}^{\infty} \eta_m z^m \right) z \rangle \rangle_{\mathbb{D}}$$

$$= \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m \langle \langle z^n, m z^{m-1} \rangle \rangle_{\mathbb{D}}$$

$$= \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m \frac{2\pi m}{n + m + 1} \delta_{n,m-1}$$

$$= \sum_{n=0}^{\infty} \pi \xi_n \bar{\eta}_{n+1}$$

while

$$\langle \langle (z^2)z, \eta \rangle \rangle_{\mathbb{D}} = \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m \langle \langle (n + 2) z^{n+1}, z^m \rangle \rangle_{\mathbb{D}}$$

$$= \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m (n + 2) \frac{2\pi}{n + m + 3} \delta_{n+1,m}$$

$$= \sum_{n=0}^{\infty} \pi \xi_n \bar{\eta}_{n+1}.$$

Since the two complex inner products are equal, their real parts are equal, establishing the proposition. On domains other than the unit disk, the formula for $\partial_z^T$ is not as simple. However, it can be computed if the domain is the image of the unit disk under a known conformal embedding $\varphi : \mathbb{D} \to U$. (Due to the Riemann mapping theorem, such an embedding always exists and is unique up composition from the left with the three dimensional submanifold of disk preserving Möbius transformations.) For $\xi, \eta \in X_{\text{con}}(\varphi(\mathbb{D}))$,
there are holomorphic functions
\[ \chi_1 := (z^2 \varphi)_z \circ \varphi^{-1}, \quad \chi_2 := (z^2 \varphi_z^2) \circ \varphi^{-1} \]
on \varphi(\mathbb{D}) such that
\[ \langle \xi, \eta_z \rangle_{\varphi(\mathbb{D})} = \langle |\varphi_z|^{-2}(\chi_1 \xi + \chi_2 \xi z), \eta \rangle. \]
Indeed, we have
\[ \langle \xi, \eta_z \rangle_{\varphi(\mathbb{D})} = \langle \varphi_z \xi \circ \varphi, \varphi_z \eta_z \circ \varphi \rangle_{\mathbb{D}} \]
\[ = \langle (z^2 \varphi_z \xi) \circ \varphi_z, \eta \circ \varphi \rangle_{\mathbb{D}} \]
\[ = \langle (\varphi_z \circ \varphi^{-1})^{-1}(z^2 \varphi_z \xi \circ \varphi_z) \circ \varphi^{-1}, (\varphi_z \circ \varphi^{-1})^{-1} \eta \rangle_{\varphi(\mathbb{D})} \]
\[ = \langle (|\varphi_z|^{-2}(z^2 \varphi_z \xi \circ \varphi_z) \circ \varphi^{-1}, \eta \rangle_{\varphi(\mathbb{D})} \]
\[ = \langle |\varphi_z|^{-2}(\chi_1 \xi + \chi_2 \xi z), \eta \rangle_{\varphi(\mathbb{D})}. \]
Thus, we have \( \partial_z^T \xi = \text{Pr}_{\text{con}}(|\varphi_z \circ \varphi^{-1}|^{-2}(\chi_1 \xi + \chi_2 \xi z)), \) where \( \text{Pr}_{\text{con}} : \mathcal{X}(\varphi(\mathbb{D})) \to \mathcal{X}_{\text{con}}(\varphi(\mathbb{D})) \) is the \( L^2 \) orthogonal projection onto conformal vector fields. Notice that \( \partial_z^T \) depends on the domain \( U \), and is thus non-local. The Hodge decomposition for conformal vector fields, developed in the previous section, together with this formula now allow the calculation of the equations of motion for any Lagrangian density on \( \mathcal{X}_{\text{con}}(\varphi(\mathbb{D})). \)

5.3. Example (Conformal stationary problem)

Let \( V \in \mathfrak{F}(\mathbb{R}^2) \) and consider the Lagrangian density \( \mathcal{L}(\xi, \xi_z) = \frac{1}{2} |\xi_z|^2 + V(\xi) \). Let \( S : \mathcal{X}_{\text{con}}(U) \to \mathbb{R} \) be the corresponding action \( S(\xi) = \int_U \mathcal{L}(\xi(z), \xi_z(z)) \, dA(z) \) and consider the variational problem:

Find \( \xi \in \mathcal{X}_{\text{con}}(U) \) such that \( \frac{\delta S}{\delta \xi}(\xi) \cdot \eta = 0 \) for all variations \( \eta \in \mathcal{X}_{\text{con}}(U). \)

Direct calculations yield
\[ \frac{\delta S}{\delta \xi}(\xi) \cdot \eta = \langle \xi_z, \eta_z \rangle_U + \langle \text{grad}(V) \circ \xi, \eta \rangle_U \]
\[ = \langle \partial_z^T \xi_z, \eta \rangle_U + \langle \text{grad}(V) \circ \xi, \eta \rangle_U \]
\[ = \langle \partial_z^T \xi_z + \text{grad}(V) \circ \xi, \eta \rangle_U. \]
We require this to vanish for all \( \eta \in \mathcal{X}_{\text{con}}(\mathbb{D}). \) That is, the first term in the inner product must be orthogonal to all conformal vector fields, i.e.,
\[ \text{Pr}_{\text{con}} \left( \partial_z^T \xi_z + \text{grad}(V) \circ \xi \right) = 0. \]
Since $\partial_z^\top \xi_z$ is already holomorphic we get
\[
\partial_z^\top \xi_z + \text{Pr}_\text{con} (\text{grad}(V) \circ \xi) = 0.
\]
Now, using the orthogonal decomposition of conformal vector fields, derived in Section 2.3 above, we introduce Lagrange multipliers $F, G \in \mathcal{F}_0(U)$ for the constraints, giving the differential equation
\[
\partial_z^\top \xi_z + \text{grad}(V) \circ \varphi = \overline{\text{grad}(F)} + s\overline{\text{grad}(G)}
\]
\[
\frac{\partial \xi}{\partial \bar{z}} = 0
\]
\[
F|\partial U = G|\partial U = 0
\]
where $\frac{\partial \xi}{\partial t} = 0$ is short way to write the Cauchy-Riemann equations. Notice that it is not certain that the original variational problem is well-posed. That requires additional assumptions on the function $V$.

5.4. Example (Conformal wave equation)

Adding time to the previous example, and denoting $\frac{\partial \xi}{\partial t} = \xi_t$, we consider the Lagrangian density corresponding to a nonlinear conformal wave equation
\[
\mathcal{L}(\xi, \xi_z, \xi_t) = \frac{1}{2} |\xi_t|^2 - \frac{1}{2} |\xi_z|^2 - V(\xi).
\]
Requiring that the action $S(\xi) = \int_0^1 \int_U \mathcal{L}(\xi(z, t), \xi_z(z, t)) \, dA(z) \, dt$ be stationary on paths in $\mathcal{X}_\text{con}(U)$ fixed at the initial and final times yields
\[
\langle -\xi_{tt} - \partial_z^\top \xi_z - \text{grad}(V) \circ \xi, \eta \rangle_U = 0
\]
and so the equations of motion are
\[
\xi_{tt} + \partial_z^\top \xi_z + \text{Pr}_\text{con} (\text{grad}(V) \circ \xi) = 0,
\]
or, spelled out explicitly using Lagrange multipliers, we get the differential equation
\[
\xi_{tt} + \partial_z^\top \xi_z + \text{grad}(V) \circ \xi = \overline{\text{grad}(F)} + s\overline{\text{grad}(G)},
\]
\[
\frac{\partial \xi}{\partial \bar{z}} = 0
\]
\[
F|\partial U = G|\partial U = 0.
\]

Consider now the case $U = \mathbb{D}$ and $V(z) = c|z|^2/2$ for a constant $c \in \mathbb{R}$. This gives the partial differential equation

\[
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\]
\[ \xi_{tt} + (z^2 \xi_z)_z + c \xi = 0. \]

Expanding \( \xi \) in the monomial basis \( \xi = \sum_{m=0}^{\infty} \xi_m z^m \), we get

\[ (\xi_m)_{tt} + (m^2 + m + c)\xi_m = 0, \]

i.e., a set of uncoupled harmonic oscillators. In particular, there are an infinite number of first integrals, given by

\[ I_m(\xi, \dot{\xi}) = \frac{1}{2}|\dot{\xi}_m|^2 + \frac{1}{2}(m^2 + m + c)|\xi_m|^2. \]

### 5.5. Example (Geodesic conformal flow equation)

Let \( \text{Emb}(\mathbb{D}, \mathbb{R}^2) \) denote the set of embeddings \( \mathbb{D} \to \mathbb{R}^2 \). This set has the structure of a Fréchet-Lie manifold. Although this manifold is not a group, it has many similarities with the diffeomorphism group \( \text{Diff}(\mathbb{D}) \). First of all, we notice that \( \text{Diff}(\mathbb{D}) \) is a submanifold of \( \text{Emb}(\mathbb{D}, \mathbb{R}^2) \). Secondly, it holds that the tangent space at the identity is equal to the set of all vector fields on \( \mathbb{D} \), i.e., \( T_{\text{Id}}\text{Emb}(\mathbb{D}, \mathbb{R}^2) = \mathcal{X}(\mathbb{D}) \), which, as reviewed earlier, carries the structure of a Fréchet-Lie algebra with the vector field commutator. (Recall that the subalgebra \( \mathcal{X}_t(\mathbb{D}) \) of tangential vector fields is the tangent space at the identity of \( \text{Diff}(\mathbb{D}) \)).

A Riemannian metric on \( \text{Emb}(\mathbb{D}, \mathbb{R}^2) \) is given by

\[ T_\varphi\text{Emb}(\mathbb{D}, \mathbb{R}^2) \times T_\varphi\text{Emb}(\mathbb{D}, \mathbb{R}^2) \ni (u, v) \mapsto \langle u \circ \varphi^{-1}, v \circ \varphi^{-1} \rangle_{\varphi(\mathbb{D})} \in \mathbb{R}. \quad (7) \]

Notice that this metric is invariant under the group \( \text{Diff}(\mathbb{D}) \) acting on \( T\text{Emb}(\mathbb{D}, \mathbb{R}^2) \) by composition from the right.

Our aim is to derive the geodesic equation with respect to the metric \( (7) \) restricted to the submanifold of conformal embeddings

\[ \text{Con}(\mathbb{D}, \mathbb{R}^2) = \{ \varphi \in \text{Emb}(\mathbb{D}, \mathbb{R}^2); \varphi^*g = Fg \}, \]

where \( g \) is the Euclidean metric on \( \mathbb{R}^2 \). By right translation, the tangent space at \( \varphi \in \text{Con}(\mathbb{D}, \mathbb{R}^2) \) can be identified with a conformal vector field over the domain \( \varphi(\mathbb{D}) \). Indeed, we have the isomorphism

\[ T\text{Con}(\mathbb{D}, \mathbb{R}^2) \ni (\varphi, \dot{\varphi}) \mapsto (\varphi, \dot{\varphi} \circ \varphi^{-1}) \in \text{Con}(\mathbb{D}, \mathbb{R}^2) \times \mathcal{X}_{\text{con}}(\varphi(\mathbb{D})), \]

where \( \text{Con}(\mathbb{D}, \mathbb{R}^2) \times \mathcal{X}_{\text{con}}(\varphi(\mathbb{D})) \) should be thought of as a vector bundle over \( \text{Con}(\mathbb{D}, \mathbb{R}^2) \).
In the language of Lagrangian mechanics, we have the Lagrangian
\[ L(\varphi, \dot{\varphi}) = \frac{1}{2} (\dot{\varphi} \circ \varphi^{-1}, \dot{\varphi} \circ \varphi^{-1})_{\varphi(D)}. \]

We would like to derive the Euler-Lagrange equations, but using the variables \((\varphi, \xi)\) instead of \((\varphi, \dot{\varphi})\). In doing so, we first notice that if \(\varphi_\varepsilon\) is a variation of a curve \(\varphi(t)\) and \(\xi_\varepsilon = \dot{\varphi}_\varepsilon \circ \varphi^{-1}_\varepsilon\), then
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \xi_\varepsilon = \dot{\eta} + \mathcal{L}_\eta \xi,
\]
where \(\varphi_\varepsilon = \exp(\varepsilon \eta) \circ \varphi\) with \(\eta \in \mathcal{X}_{\text{con}}(\varphi(D))\), see [10, 11]. In addition, it holds that
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \frac{1}{2} \langle \xi, \xi \rangle_{\varphi_\varepsilon(D)} = \langle \mathcal{L}_\eta \xi + \text{div}(\eta)\xi, \xi \rangle_{\varphi(D)}.
\]

This equality follows by straightforward calculations and the fact that \(\mathcal{L}_\xi g = \text{div}(\xi)g\) for any \(\xi \in \mathcal{X}_{\text{con}}(\varphi(D))\), as derived in Section 2.3 above.

Using these relations, the variational principle now yields
\[
0 = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_0^1 L(\varphi_\varepsilon, \dot{\varphi}_\varepsilon) \, dt = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_0^1 \frac{1}{2} \langle \xi_\varepsilon, \xi_\varepsilon \rangle_{\varphi_\varepsilon(D)} \, dt
\]
\[
= \int_\varphi(D) \left( \mathcal{L}(\nabla(\|\xi\|^2), \eta)_{\varphi(D)} + \langle \text{grad}(\|\xi\|^2), \eta \rangle_{\varphi(D)} \right).
\]

Next, since
\[
\frac{d}{dt} \langle \eta, \xi \rangle_{\varphi(D)} = \langle \dot{\xi}, \eta \rangle_{\varphi(D)} + \langle \xi, \dot{\eta} \rangle_{\varphi(D)} + \int_{\varphi(D)} \mathcal{L}_\xi (\xi, \eta) \, \text{vol}
\]
\[
= \langle \dot{\xi}, \eta \rangle_{\varphi(D)} + \langle \xi, \dot{\eta} \rangle_{\varphi(D)} + \langle \xi, \mathcal{L}_\xi \eta + 2 \text{div}(\xi)\eta \rangle_{\varphi(D)}
\]
and since the variation \(\eta\) vanish at the endpoints, we get
\[
0 = \langle \dot{\xi} + 2 \text{div}(\xi)\xi - \text{grad}(\|\xi\|^2), \eta \rangle_{\varphi(D)} + \langle \xi, \mathcal{L}_\xi \eta \rangle_{\varphi(D)}.
\]
Using now that \( \mathcal{L}_\xi \eta = \xi' \eta - \eta' \xi = 2\xi' \eta - (\eta \xi)' \) we get

\[
0 = \langle \dot{\xi} + 2 \text{div}(\xi) \xi - \nabla(|\xi|^2) + 2\xi^T \xi - \xi \partial_z^\top \xi, \eta \rangle_{\varphi(D)}
\]

Finally, from the relations \( \text{div}(\xi) = 2 \text{Re}(\xi') \) and \( \nabla(|\xi|^2) = 2\xi^T \xi \), and the decomposition in Theorem 4.1 we obtain the strong geodesic equation

\[
\dot{\xi} + 2 \text{div}(\xi) \xi - \xi \partial_z^\top \xi = \nabla(F) + \text{sggrad}(G) \quad \dot{\varphi} = \xi \circ \varphi \quad \frac{\partial \xi}{\partial z} = 0 \quad F|_{\partial \varphi(D)} = G|_{\partial \varphi(D)} = 0.
\]

Notice that the first equation contains the operator \( \partial_z^\top \), which depends on the domain \( \varphi(D) \). Thus, the first equation for \( \dot{\xi} \) depends on the second equation for \( \dot{\varphi} \). This is different from “usual” Euler equations, where the equation for the reduced variable \( \xi \) is independent of \( \varphi \). From a geometric mechanics point of view (cf. [12]), the reason for this coupling is that the symmetry group of the Lagrangian is smaller than the configuration space.

For further information of this geodesic equation, its application in image registration, and a derivation using the more general class of \( H_1^\alpha \) metrics, see [13, 11].

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