NONCOMMUTATIVE RIEMANNIAN GEOMETRY
OF THE ALTERNATING GROUP $A_4$. 

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Abstract: We study the noncommutative Riemannian geometry of the alternating group $A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ using the recent formulation for finite groups in \cite{2}. We find a unique ‘Levi-Civita’ connection for the invariant metric, and find that it has Ricci-flat but non-zero Riemann curvature. We show that it is the unique Ricci-flat connection on $A_4$ with the standard framing (we solve the vacuum Einstein’s equation). We also propose a natural Dirac operator for the associated spin connection and solve the Dirac equation. Some of our results hold for any finite group equipped with a cyclic conjugacy class of 4 elements. In this case the exterior algebra $\Omega(A_4)$ has dimensions $1 : 4 : 8 : 11 : 12 : 12 : 11 : 8 : 4 : 1$ with top-form 9-dimensional. We also find the noncommutative cohomology $H^1(A_4) = \mathbb{C}$.

1 Introduction

A constructive formalism of noncommutative Riemannian geometry has recently been developed in \cite{1} and \cite{2}, using quantum group methods. Here a general and possibly noncommutative algebra or ‘coordinate ring’ is equipped with a ‘quantum Riemannian manifold’ structure consisting of a frame bundle (with quantum group fiber) to which the differential calculus \cite{3} or exterior algebra bundle is associated. The approach builds on the established theory of quantum principal bundles \cite{1} and adds to this notions of ‘framing’, ‘coframing’ (or metric) and Levi-Civita type metric-compatible connections with Riemann and Ricci curvature.

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This approach applies also to finite sets and finite groups, and allows one to endow them with nontrivial Riemannian manifold structures. This would not be possible within conventional differential geometry, but noncommutative differential structures are more general even when the coordinate ring is commutative, and allow a rich structure even for finite sets. Particularly, for a finite group there are natural choices for translation biinvariant differential structure, namely labelled by the nontrivial conjugacy classes (see [3]). There is a natural frame bundle, namely with fiber another copy of the group \([1]\), and there is a natural ‘Killing form’ inducing a metric \([2]\), which may, however, be degenerate. So far, only the case of \(S_3\) in \([4]\) has been worked out in detail it is shown there that this has the natural structure of a noncommutative Einstein manifold with unique Levi-Civita connection for the Killing metric, and with Ricci curvature essentially proportional to the metric. Some other aspects of the noncommutative geometry of \(S_3\) are in \([5]\).

In this paper we extend the repertoire of examples with a detailed study of the alternating group \(A_4\) from this point of view. It turns out that this example is of key interest because, like \(S_3\), it has a natural invariant metric with unique Levi-Civita connection, but this connection is Ricci-flat. Thus, it provides the first concrete example of the solution of the vacuum Einstein equations in this theory. The model is also interesting because the group is nonabelian enough to have nontrivial curvature (the Riemann tensor does not vanish) but simple enough to be fully computable.

Some of the computations are done without reference to the actual group and apply to any group with similar ‘cyclic’ conjugacy class. We include for example \(SL_2(\mathbb{Z}_3)\) in this family. After some preliminaries, we start in Section 3 at this general level. We find the Woronowicz exterior algebra associated to the conjugacy class and show that it is not in fact quadratically generated. It is in fact a good example where we see the absolute necessity of relations in higher degree. We then find the unique form of the invariant metric, namely

\[
\eta^{a,b} = \delta_{a,b} + \mu
\]

in a suitable basis, with \(\mu\) a parameter. We also characterise the torsion free and cotorsion free regular connections. In Section 4 we specialise to \(A_4\) and find the associated Levi-Civita or metric-compatible connection and its Ricci flatness. We also show that it is the unique regular Ricci-flat connection on \(A_4\) independently of metric compatibility.

In Section 5 we look at the Dirac operator appropriate to the metric, as a step towards
comparison with Connes’ approach to noncommutative geometry [6]. As in [2] for $S_3$, it is not naturally hermitian but is fully diagonalisable. Finally, Section 6 contains some further results including the noncommutative de Rahm cohomology and a link to the differential (but not Riemannian) geometry of $S_4$ in [7], as well as concluding remarks.

We note that following [1] there have been some other attempts at a Riemannian geometry on finite groups, but different from the one used here, see for instance [8, 9] and references therein. While the use of a conjugacy class to define an exterior algebra, i.e. the notion of differential structure, is the same in all approaches following [3], see for instance [10], the formulation of [1, 2] is the only one that features some kind of metric compatibility or ‘Levi-Civita’ notion for the spin connection, as well as the only one that features a nontrivial (and nonuniversal) differential structure in the fiber direction, and hence an actual noncommutative geometry of the frame bundle. Other problems solved in [2] were the formulation and computation of the Ricci tensor, see [11] for a discussion.

2 Preliminaries

In this section, we briefly recall the basic definitions of differential structures [3] and of noncommutative Riemannian geometry [2], specialised to the case of finite groups that we need in the paper. Thus, we work with the Hopf algebra $H = \mathbb{C}[G]$ of functions on a finite group $G$. We equip it with its standard basis $(\delta_g)_{g \in G}$ defined by Kronecker delta-functions $\delta_g(h) = \delta_h^g, \forall g, h \in G$. Let $C$ be a nontrivial conjugacy class of $G$, and $\Omega_0$ the vector subspace

$$\Omega_0 = \{\delta_a | a \in C\} = \mathbb{C}C.$$  \hspace{1cm} (1)

Any Ad-stable set not containing the group identity can be used here, but we focus on the irreducible case of a single conjugacy class. From the Woronowicz’s theory in [3], the first order differential calculus associated to $C$ is generated over $\mathbb{C}[G]$ by $\Omega_0$ and given by

$$df = \sum_{a \in C} (\partial^a f) e_a, \quad \partial^a = R_a - \text{id}, \quad e_a f = R_a(f) e_a$$  \hspace{1cm} (2)

$\forall a \in C, f \in H$, where the operator $R_a$ is defined by $R_a(f)(g) = f(ga), \forall g \in G$. The Maurer-Cartan 1-form $e : \Omega_0 \rightarrow \Omega^1(H)$ is given by

$$e_a = e(\delta_a) = \sum_{g \in G} \delta_g d\delta_{ga}$$  \hspace{1cm} (3)
and higher differential forms are obtained from Woronowicz skew-symmetrization procedure \[3\], using the braiding

\[ \Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a \] (4)

The Maurer-Cartan equation takes the form

\[ de_a = \theta \wedge e_a + e_a \wedge \theta, \quad \theta = \sum_{a \in C} e_a. \] (5)

The element \( \theta \) obeys \( \theta \wedge \theta = 0 \) and generates \( d \) in general as graded-commutator with \( \theta \). Lemma 5.3 in \[2\] gives the full set of relations of \( \Omega^2(H) \), namely

\[ \sum_{a, b \in C; \ ab = g} \lambda^g_{\beta} e_{a} \wedge e_{b} = 0, \forall g \in G, \forall \beta. \] (6)

where for \( g \in G \) fixed, \( \{\lambda^g_\beta\} \) is a basis of the invariant subspace of the vector space spanned by \( C \cap gC^{-1} \) under the automorphism \( \sigma(a) = a^{-1}g \). There are also cubic and higher degree relations (which are in fact nontrivial in our case of \( A_4 \)) but we will not need them explicitly (most of Riemannian geometry needs only 1-forms and 2-forms.)

Next, following \[2\], a framing means a basis of \( \Omega^1(H) \) over \( \mathbb{C}[G] \), and an action of the frame group. In our case we chose the framing to be the components \( \{e_a\} \) of the Maurer-Cartan form as above and for frame group we choose \( G \) itself, acting by Ad. This is a canonical choice and its classical meaning is explained in \[1\]. A spin connection is then a collection \( \{A_a\}_{a \in C} \) of component 1-forms. Its associated covariant derivative is defined on an 1-form \( \alpha = \alpha^a e_a \) by

\[ \nabla \alpha = d\alpha^a \otimes H e_a - \alpha^a \sum_{b \in C} A_b \otimes H (e_b^{-1}ab - e_a) \] (7)

with summation on \( a \). The associated torsion tensor \( T : \Omega^1(H) \to \Omega^2(H) \) is defined by \( T \alpha = d \land \alpha - \nabla \alpha \) and the zero-torsion condition is vanishing of

\[ \bar{D}_A e_a \equiv de_a + \sum_{b \in C} A_b \wedge (e_b^{-1}ab - e_a), \quad \forall a \in C \] (8)

The spin connection here has values in the dual space \( \Omega^*_0 \), which is a `braided-Lie algebra' in a precise sense. Associated to this geometrical point of view, there is a regularity condition

\[ \sum_{a, b \in C; \ ab = g} A_a \wedge A_b = 0, \quad \forall g \neq e, g \notin C. \] (9)
The curvature \( \nabla^2 \) associated to a regular connection \( A \) is in frame bundles terms a collection of 2-forms \( \{ F_a \}_{a \in C} \) defined by

\[
F_a = dA_a + \sum_{c,d \in C, cd=a} A_c \wedge A_d - \sum_{c \in C} (A_c \wedge A_a + A_a \wedge A_c) \tag{10}
\]

while the Riemann curvature \( R : \Omega^1(H) \to \Omega^2(H) \otimes_H \Omega^1(H) \) is given by

\[
R \alpha = \alpha_a \sum_{b \in C} F_b \otimes (e_b^{-1} - e_a) \tag{11}
\]

Finally, the Ricci tensor is given by

\[
Ricci = \sum_{a,b,c} i(F_c)^{ab} e_b \otimes_H (e_{a^{-1}} - e_a) \tag{12}
\]

where \( i(F_c) = i(F_c)^{ab} e_a \otimes_H e_b \) and \( i : \Omega^2(H) \to \Omega^1(H) \otimes_H \Omega^1(H) \) is a lifting which splits the projection of \( \wedge \). A canonical choice is \( i \).

\[
i(e_a \wedge e_b) = e_a \otimes_H e_b - \sum_{\beta} \gamma^{\beta,a} \sum_{c,d \in C, cd=ab} \lambda^\beta_c e_c \otimes_H e_d \tag{13}
\]

where \( \{ \gamma^{\beta} \} \) are the dual basis to the \( \{ \lambda^\beta \} \) with respect to the dot product as vectors in \( \mathbb{C} \cap ab \mathbb{C}^{-1} \). Another canonical ‘lift’ is \( i' = \text{id} - \Psi \) but note that in this case \( i' \circ \wedge \) is not a projection operator.

There are two further structures that one may impose in this situation. First of all, given a choice of framing \( \{ e_a \} \), a metric \( g \) is defined as a coframing \( \{ e^*_a \} \), i.e. again a basis of \( \Omega^1 \) but now as a right \( \mathbb{C}[G] \)-module. and transforming under the contragradient action of \( G \). (The corresponding metric is \( g = \sum_a e^{*a} \otimes_H e_a \)). The cotorsion of a spin connection is the torsion with respect to the coframing, and is given by

\[
D_A e^{*a} \equiv de^{*a} + \sum_{b \in C} (e^{*ab^{-1}} - e^{*a}) \wedge A_b. \tag{14}
\]

Vanishing of cotorsion has the classical meaning of a generalisation of metric compatibility of the spin connection, see \[1\]. So we are usually interested in regular torsion-free and cotorsion-free connections.

Finally, a ‘gamma-matrix’ is defined \[2\] as a collection of endomorphisms \( \{ \gamma_a \}_{a \in C} \) of a vector space \( W \) on which \( G \) acts by a representation \( \rho_W \), a ‘spinor field’ is a \( W \)-valued function on \( G \) and the Dirac operator on the spinor fields is

\[
\mathcal{D} = \partial^\alpha \gamma_a - A_b^a \gamma_b \tau^a_W, \tag{15}
\]

where \( A_b = A^b_a e_a \) and \( \tau^a_W = \rho_W(a^{-1} - e) \). There is a canonical choice where \( \gamma \) is built from \( \rho_W \) itself, explained in \[2\].
3 Cyclic Riemannian structures

In this section, we construct Riemannian geometry on groups endowed with conjugacy classes which obey a certain cyclicity condition. For the case when the differential calculus is of degree four, we determine the entire exterior algebra and the moduli space of torsion free connections, and for any degree $n \geq 2$, we give the general form of the invariant metric.

**Definition 3.1** Let $C$ be a conjugacy class with $n$ elements, $n \geq 2$, in a group $G$. We say that $C$ is ‘cyclic’ if there exists at least one $t$ in $C$ such that $\text{Ad}_t$ is a cyclic permutation of $C - \{t\}$ and the map $a \rightarrow \text{Ad}_a(t)$ is a permutation of $C$.

For $n = 4$ we have the following characterisation of $\Omega^2(H)$

**Proposition 3.2** For a cyclic conjugacy class $C = \{t, x, y, z\}$ of order 4 in a finite group $G$, the bimodule $\Omega^2(H)$ of 2-form is 8-dimensional over $\mathbb{C}[G]$ and is defined by the following equations

$$e_a \wedge e_a = 0, \quad \sum_{a,b \in C; \; ab = g} e_a \wedge e_b = 0$$

for all $a \in C, g \in G$, where $(e_a)_{a \in C}$ is the basis of Maurer-Cartan 1-forms.

**Proof:**

We assume the existence of an element $t \in C$ as in Definition 3.1. Without loss of generality, we denote the other elements of $C$ by $x, y, z$ with the following table for $\text{Ad}$:

|   | $t$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $t$ | $t$ | $z$ | $x$ | $y$ |
| $x$ | $y$ | $x$ | $z$ | $t$ |
| $y$ | $z$ | $t$ | $y$ | $x$ |
| $z$ | $x$ | $y$ | $t$ | $z$ |

Table 1

It follows that

$$tx = zt = xz; \quad ty = xt = yx; \quad tz = yt = zy; \quad xy = zx = yz$$

(18)
Using relations (18), we apply the Woronowicz antisymmetrization procedure to obtain the following relations of the form (16) in $\Omega^2(H)$:

\begin{align*}
e_a \wedge e_a &= 0, \quad \forall a \in C \\
e_t \wedge e_x + e_x \wedge e_z + e_z \wedge e_t &= 0 \\
e_x \wedge e_t + e_t \wedge e_y + e_y \wedge e_x &= 0 \\
e_t \wedge e_z + e_z \wedge e_y + e_y \wedge e_t &= 0 \\
e_x \wedge e_y + e_y \wedge e_z + e_z \wedge e_x &= 0
\end{align*}

This form (19) holds for any group since the elements $e_a \otimes e_a$ and $\sum_{ab=g} e_a \otimes e_b$ are in the kernel of $\text{id} - \Psi$. However, using (6) one may see that they are the only relations of $\Omega^2(H)$ which is therefore of dimension 8 as stated. ⬤

From now, we choose a basis of $\Omega^2(H)$ to be

\[\{e_t \wedge e_x, e_t \wedge e_y, e_t \wedge e_z, e_x \wedge e_t, e_y \wedge e_t, e_x \wedge e_y, e_y \wedge e_z, e_x \wedge e_z\},\] (20)

For convenience, we will sometime use indexes 1, 2, 3, 4 to refer to $t, x, y, z$ respectively.

**Proposition 3.3** In the setting of Proposition 3.2, the dimensions of the Woronowicz exterior algebra $\Omega(H)$ are $1 : 4 : 8 : 11 : 12 : 11 : 8 : 4 : 1$ with top-form of degree 9. This algebra is not quadratic, having additional relations in degree $\geq 6$.

**Proof**:

As above, we do not need the group itself but only the matrix for Ad restricted to the conjugacy class (i.e. Table 1). In fact we are computing the invariant part $\Lambda$ of the exterior algebra, with $\Omega(H) = H \otimes \Lambda$ as a vector space. This $\Lambda$ is generated over $\mathbb{C}$ by the $\{e_a\}$ with relations determined by the braiding $\Psi$. Namely we set to zero the kernels of the antisymmetrizers $A_m$ for $m \geq 2$. These $A_m$ are described in [3] as a signed sum over permutations of $\{1, \cdots, m\}$ with transposition replaced by $\Psi$. This is not very convenient for computation and we employ instead a different but equivalent definition of the $A_m$ coming out of the theory of braided groups [12]. As recently discussed in [7], we use the braided-integers

\[ [m, -\Psi] = \text{id} - \Psi_{12} + \Psi_{12}\Psi_{23} - \cdots \pm \Psi_{12} \cdots \Psi_{m-1,m} = \text{id} - \Psi_{12}(\text{id} \otimes [m - 1, -\Psi]), \]
where $\Psi_{12}$ denotes $\Psi$ acting in the first and second places of $\Omega_0^m$, etc. Then

$$A_m = [m, -\Psi]! = (\text{id} \otimes [2, -\Psi])(\text{id} \otimes [3, -\Psi]) \cdots [m, -\Psi].$$

In the braided groups approach to the exterior algebra we set to zero the kernels of all these braided factorials. It is straightforward to program these inductive definitions. We first compute the $16 \times 16$ matrices $\Psi$ acting in the tensor product basis $e_a \otimes e_b$ and then the $A_m$ as above, up to $A_6$. The dimensions $\Omega^m(H)$ over $H$ are then $4^m - \dim \ker A_m$ and found to be as stated. From the general form expected for the exterior algebra we assume the remaining dimensions for $A_7, A_8, A_9$ without explicit computation. Finally, the quadratic exterior algebra is defined by setting to zero only the kernel of $A_2 = \text{id} - \Psi$ without additional relations in higher degree. In that case in degree $m$ we set to zero the union of the null spaces $\text{id} - \Psi_{12}, \cdots, \text{id} - \Psi_{m-1,m}$. Here we find dimensions $1 : 4 : 8 : 11 : 12 : 12 : 12 : \cdots$ i.e., fewer relations in degree $\geq 6$ (it appears that the quadratic one is in fact infinite-dimensional).

In fact for most geometric purposes we need only the exterior algebra up to degree 2, so we limit ourselves to the general result about dimensions. In principle one may go on to compute explicit relations in higher degree and a Hodge * operator as in [5] using the metric below, etc. The result is an important reminder that the degree 2 relations alone may not be enough for a geometrically reasonable exterior algebra.

**Proposition 3.4** In the setting of Proposition 3.2 above and for the framing defined by the Maurer-Cartan 1-form, the moduli space of torsion free connections is $3|G|$-dimensional and is given by the following components 1-forms:

$$
\begin{align*}
A_t &= (1 + \alpha)e_t + \gamma e_x + \lambda e_y + \beta e_z \\
A_x &= \lambda e_t + (1 + \beta)e_x + \alpha e_y + \gamma e_z \\
A_y &= \beta e_t + \lambda e_x + (1 + \gamma)e_y + \alpha e_z \\
A_z &= \gamma e_t + \alpha e_x + \beta e_y + (1 + \lambda) e_z
\end{align*}
$$

where $\alpha, \beta, \gamma, \lambda$ are functions on $G$ such that

$$\alpha + \beta + \gamma + \lambda = -1$$

Thus we have also

$$\sum_{a \in \mathcal{C}} A_a = 0$$
Proof:
We follow the same method as for $S_3$ in [2]. In the framing defined by the Maurer-Cartan 1-form, the torsion free connections obey the following equation (see eq. (8))

$$\sum_{b \in C} A_b \wedge (e_b - 1)_{ab} - e_a) = 0 \quad (24)$$

$\forall a \in C$. Using Table 1, we write (24) as

$$A_x \wedge (e_x - e_t) + A_y \wedge (e_y - e_t) + A_z \wedge (e_z - e_t)$$

$$+ (e_x + e_y + e_z) \wedge e_t + e_t \wedge (e_x + e_y + e_z) = 0 \quad (25)$$

$$A_t \wedge (e_y - e_x) + A_y \wedge (e_z - e_x) + A_z \wedge (e_t - e_x)$$

$$+ (e_t + e_y + e_z) \wedge e_x + e_x \wedge (e_t + e_y + e_z) = 0 \quad (26)$$

$$A_t \wedge (e_x - e_y) + A_x \wedge (e_y - e_t) + A_z \wedge (e_x - e_y)$$

$$+ (e_t + e_x + e_z) \wedge e_y + e_y \wedge (e_t + e_x + e_z) = 0$$

$$A_t \wedge (e_x - e_z) + A_x \wedge (e_y - e_z) + A_y \wedge (e_t - e_z)$$

$$+ (e_t + e_x + e_y) \wedge e_z + e_z \wedge (e_t + e_x + e_y) = 0$$

We just have to solve the first three equations since the fourth one in this system can be obtained from the other by simple summation. We set $A_a = A^b_a e_b$ (sum over $b \in C$) for functions $A^b_a \in H$ with

$$A^t_x = 1 + \alpha, \quad A^x_x = 1 + \beta, \quad A^y_y = 1 + \gamma, \quad A^z_z = 1 + \lambda \quad (26)$$

We put this into the equations to be solved and write them in the basis (20). Using the fact that each coefficient of the basis element has to vanish, we obtain

$$A^t_x = \lambda = A^x_y, \quad A^t_y = \beta = A^y_z = A^z_t, \quad A^t_z = \gamma = A^x_t = A^z_x$$

$$A^x_z = -1 - \lambda - \gamma - \beta = A^y_x = A^z_y, \quad A^y_t = -1 - \alpha - \beta - \gamma$$

$$\alpha + \beta + \gamma + \lambda = -1$$

as stated. Finally using these solutions one checks by simple computation that $A_t + A_x + A_y + A_z = 0$. ∗

We now study the regularity of connections:
Proposition 3.5 Under the hypothesis of Proposition 3.2 the regular connections are either solutions of the system:

\[ A_t \wedge A_t + A_x \wedge A_y + A_y \wedge A_z + A_z \wedge A_x = 0 \]  \hspace{1cm} (28)

\[ A_t \wedge A_x + A_x \wedge A_z + A_z \wedge A_t = 0 \]

\[ A_t \wedge A_y + A_x \wedge A_t + A_y \wedge A_x = 0 \]

\[ A_t \wedge A_z + A_x \wedge A_x + A_y \wedge A_t + A_z \wedge A_y = 0 \]

or solutions of the system:

\[ A_t \wedge A_t = 0, \ A_x \wedge A_x = 0, \ A_y \wedge A_y = 0, \ A_z \wedge A_z = 0 \]  \hspace{1cm} (29)

\[ A_t \wedge A_x + A_x \wedge A_z + A_z \wedge A_t = 0 \]

\[ A_t \wedge A_y + A_x \wedge A_t + A_y \wedge A_x = 0 \]

\[ A_t \wedge A_z + A_y \wedge A_t + A_z \wedge A_y = 0 \]

\[ A_x \wedge A_y + A_y \wedge A_z + A_z \wedge A_x = 0 \]

Proof:

The general form of the regularity’s equation is given by (3). One then needs the multiplication table at least for the elements of the class \( \mathcal{C} \), by enumeration of the cases we find that under the hypothesis of Proposition 3.2, the only possible cases are those shown in Tables 2, 3. These correspond to the two possibilities stated.

\[ \times \begin{array}{cccc} t & x & y & z \\ t & t^2 & zt & xt \\ x & xt & x^2 & yt \\ y & yt & xt & y^2 \\ z & zt & xy & yt \end{array} \]

Table 2: any square is different from the products in (18)

\[ \times \begin{array}{cccc} t & x & y & z \\ t & t^2 & zt & xt \\ x & xt & y^2 & yt \\ y & yt & xt & zt \end{array} \]

Table 3

The case of Table 2 corresponds for instance to the group \( SL(2, \mathbb{Z}/3\mathbb{Z}) \) of the \( 2 \times 2 \) matrices with coefficients in \( \mathbb{Z}/3\mathbb{Z} \), with any of its four-elements conjugacy classes, while the case of Table 3 corresponds for instance to the alternating group \( A_4 \) of order 12, with any of its four-elements conjugacy classes. To explicitly solve these nonlinear systems (28) and (29) one needs more precision on the group \( G \). We solve system (28) in detail.
in Section 4 for $A_4$. There is in fact a fundamental difference between the two case, for instance the connection corresponding in (21) to $\alpha = \beta = \gamma = \lambda$ is a solution of (28) but not a solution of (29).

We also want to find the ‘Levi-Civita connection’, namely a regular torsion free and cotorsion free connection for a natural metric. We need for that end to find a suitable coframing or metric. As shown in [2] a natural choice in the group or quantum group case is to take any Ad-invariant nondegenerate bilinear form $\eta$ defined on $\Omega_0^*$, and indeed [2] provides a general ‘braided-Killing form’ construction that can achieve this. Our ‘cyclic’ conjugacy class $C$ described above is not, however, semi-simple in the sense of Prop.5.4 of [2] (the braided-Killing form is degenerate) and we instead have to determine all possible $\eta$.

**Theorem 3.6** Let $C$ be a cyclic conjugacy class with $n$ elements. Then up to normalisation, all nondegenerate Ad-invariant bilinear forms on $\Omega_0^*$ are given by

$$\eta^{a,b} = \delta_{a,b} + \mu$$

for a constant $\mu \neq -1/n$. The associated metric in the Maurer-Cartan framing is

$$g = \sum_{a \in C} e_a \otimes e_a + \mu \theta \otimes \theta.$$  

**Proof :**

Here $g$ corresponds to an element $\eta \in \Omega_0 \otimes \Omega_0$ with coefficients $\eta^{a,b}$. We require it to be Ad-invariant and for the matrix of coefficients to be invertible (this is said more abstractly in [2] to handle the quantum group case). The first condition is easily seen to be the requirement

$$\eta^{a_{g^{-1}},b_{g^{-1}}} = \eta^{a,gb,g^{-1}}, \quad \forall a, b \in C, \quad g \in G.$$  

This and nondegeneracy is easy to see for the $\eta$ as stated.

Conversely, let us suppose that $\eta$ is Ad-invariant and show that it is of the form (30). Since $\eta$ is Ad-invariant, it obeys (31). We assume the existence of $t \in C$ as in Definition 3.1, then $\text{Ad}_t$ is a cyclic permutation of $C - \{t\}$. From invariance (31) it is obvious that $\eta^{t,a} = \eta^{t,\text{Ad}_t(a)} = \eta^{t,\text{Ad}_t^2(a)} = \ldots = \eta^{t,\text{Ad}_t^{n-2}(a)}$ for any $a \neq t$, and hence by cyclicity

$$\eta^{t,b} = \mu_1, \quad \forall b \neq t,$$
for some constant $\mu_1$. But also from cyclicity we know that for any $a \in C$ there is an element $c \in C$ such that $a = ctc^{-1}$. Hence from (31) we also have

$$\eta^{a,b} = \eta^{ctc^{-1},b} = \eta^{t,ctc^{-1}bc} = \mu_1, \quad \forall a \neq b,$$

so all off-diagonals are $\mu_1$. Similarly, we have $\eta^{a,a} = \eta^{ctc^{-1},ctc^{-1}} = \eta^{t,t} = \mu_2$ for all $a \in C$ by Ad-invariance, for some constant $\mu_2$. Thus $\eta^{a,b} = (\mu_2 - \mu_1)\delta_{a,b} + \mu_1$, which has, up to an overall scaling, the form stated. The remaining condition on the parameter $\mu$ comes from the fact that $\eta$ is invertible. Finally, given $\eta$ we define

$$e^{*a} = \sum_{b \in C} e_b \eta^{ba},$$

as explained in [2] for the associated coframing, which corresponds to the metric $g$ as stated. $\diamond$

One can then observe that this metric is symmetric in the sense

$$\land g = 0 \quad (33)$$

The groups $S_3$, $SL(2, \mathbb{Z}/3\mathbb{Z})$ and $A_4$ are the examples of groups which obey the hypothesis of the previous theorem. The theorem clarifies the observation in [2] for $S_3$ where $\eta^{a,b} = \delta^{a,b}$ is derived as the braided Killing form (up to a normalisation) but it is explained that one may add a multiple $\mu \theta \otimes_H \theta$ to the metric (without changing the connection and Riemmanian curvature). We are now ready to describe torsion free and cotorsion free connections in our cyclic case.

**Proposition 3.7** In the setting of Propositions 3.2 and 3.4 and for the coframing given by $\eta$ as above, the torsion free and cotorsion free connections obey the following relations:

$$R_t^{-1}(\alpha) = R_x^{-1}(\lambda) = R_y^{-1}(\beta) = R_z^{-1}(\gamma) \quad (34)$$

$$R_t^{-1}(\lambda) = R_x^{-1}(\alpha) = R_y^{-1}(\gamma) = R_z^{-1}(\beta)$$

$$R_t^{-1}(\beta) = R_x^{-1}(\gamma) = R_y^{-1}(\alpha) = R_z^{-1}(\lambda)$$

$$R_t^{-1}(\gamma) = R_x^{-1}(\beta) = R_y^{-1}(\lambda) = R_z^{-1}(\alpha)$$

where $\alpha, \beta, \gamma, \lambda$ are as in Proposition 3.4.
Proof:
As in [2], when the coframing is given by the framing and an Ad-invariant $\eta$, one may easily compute the form of the cotorsion. One has,

$$D_A e^a = \eta^b_{ab} de_b + \sum_{b\in C, c\in C} \eta_{b,c}^{ba} c_e \wedge A_c = \sum_{b\in C} \eta^b_{ba} e_b \wedge A_b$$

as a special case of the quantum groups computation in [2]. Since we suppose the connections to be torsion free, equation (23) holds, then (cancelling $\eta^b_{ab}$), vanishing of cotorsion in equation (14) can be written equivalently as

$$de_a + \sum_{b\in C} e_{bab} \wedge A_b = 0 \quad (35)$$

$\forall a \in C$. If we write equation (35) for $a = t, x, y, z$ respectively, using Table 1, equations (19) and the definition of $\eta$, we obtain the following system of equations:

$$e_t \wedge A_t + e_x \wedge A_x + e_y \wedge A_y + e_z \wedge A_z - e_x \wedge e_x - e_y \wedge e_y - e_z \wedge e_z = 0 \quad (36)$$

$$e_t \wedge A_t + e_x \wedge A_x + e_y \wedge A_y + e_z \wedge A_z - e_t \wedge e_t - e_y \wedge e_y - e_z \wedge e_z = 0$$

$$e_t \wedge A_y + e_x \wedge A_y + e_y \wedge A_x + e_z \wedge A_z - e_t \wedge e_t - e_x \wedge e_x - e_y \wedge e_y = 0$$

$$e_t \wedge A_y + e_x \wedge A_y + e_y \wedge A_y + e_z \wedge A_z - e_t \wedge e_t - e_x \wedge e_x - e_y \wedge e_y = 0$$

we can get the fourth equation of system (36) from the three other. We then solve only the first three equations of this system. For that end, we set

$$A_a = e_b A_a^b \quad (37)$$

$\forall a \in C$, with summation understood for $b \in C$, and where we set

$$A_t^t = 1 + \alpha' \quad A_x^x = 1 + \beta' \quad A_y^y = 1 + \gamma' \quad A_z^z = 1 + \lambda' \quad (38)$$

as above. We then proceed in the same manner as we solved sytem (23), using this time the right module structure of $\Omega^1(H)$. We find that the solutions $(A_a)$ take the form

$$A_t = e_t(1 + \alpha') + e_x \lambda' + e_y \beta' + e_z \gamma' \quad (39)$$

$$A_x = e_t \gamma' + e_x (1 + \beta') + e_y \lambda' + e_z \alpha'$$

$$A_y = e_t \lambda' + e_x \alpha' + e_y (1 + \gamma') + e_z \beta'$$

$$A_z = e_t \beta' + e_x \gamma' + e_y \alpha' + e_z (1 + \lambda')$$

with $\alpha' + \beta' + \gamma' + \lambda' = -1$, $\alpha', \beta', \gamma', \lambda' \in H$. 

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we then write these solutions using the left module structure on 1-form via the commutation relation in (2). And we compare the result to that of system (21) to obtain system (34) as stated. ⊗

At this level, we get many torsion free and cotorsion free connections. As one can remark, these equations for the connection do not depend on the coefficient $\mu$ of $\theta \otimes_H \theta$, just as was the case for $S_3$ in [4]. Modulo these modes, we see that there is an essentially unique form of invariant metric on $G$ and we have given some conditions for the associated moduli of torsion free and cotorsion free regular connections.

4 Riemannian geometry of $A_4$

In this section we specialise to the group $A_4$ and present stronger results that depend on its structure and not only on the cyclic form of the conjugacy class. The group is defined by

\[ A_4 = \{e, u, v, w, t, x, y, z, t^2, ut^2, vt^2, wt^2\} \]  

where $e$ is the group identity (this should not be confused with the Maurer-Cartan 1-form) and $t, u, v, w$ are the following permutations of $\{1, 2, 3, 4\}$:

\[ t = (123), \quad u = (14)(23), \quad v = (12)(34), \quad w = (13)(24) \]  

and

\[ x = tv = ut = (134), \quad y = tw = vt = (243), \quad z = tu = wt = (142). \]  

The other products are

\[ v^2 = e, \quad w^2 = e, \quad u^2 = e, \quad t^3 = e, \quad vw = wv = u, \quad vu = uv = w, \quad wu = uw = v \]  

and we choose the conjugacy class

\[ C = \{t, x, y, z\}, \]  

which is ‘cyclic’. Indeed, we have

\[ \text{Ad}_t(x) = ttx^2 = t(ut)t^2 = z, \quad \text{Ad}_t(y) = tgyt^2 = t(vt)t^2 = x \quad \text{Ad}_t(z) = t(wt)t^2 = y \]
and
\[ \text{Ad}_x(t) = xtyt = y, \text{Ad}_y(t) = ytzt = z, \text{Ad}_z(t) = ztxt = x, \text{Ad}_t(t) = t \]
which show that \( C \) obeys conditions of Definition 3.1.

One may also check that the multiplication table of \( C \) corresponds to Table 3 as announced, so that we have at least one regular torsion free and cotorsion free connection on the bundle \( H \otimes H \) where \( H \) denotes from now \( \mathbb{C}[A_4] \).

**Proposition 4.1** For the cyclic conjugacy class on \( A_4 \), framing defined by the Maurer-Cartan form and coframing \( e^* \) by \( \text{Ad} \)-invariant \( \eta \), there exists a unique ‘Levi-Civita’ connection with component 1-forms
\[ A_a = e_a - \frac{1}{4}\theta, \quad \forall a \in C \] (45)

**Proof:**
The connection defined in (45) is easily seen to be a solution of systems (21) and (39). We are going to show that it is the unique torsion free and cotorsion free connection which is solution of system (28). Using the properties of operators \((R_g)_{g \in A_4}\) and equations of system (34) we find that
\[ \alpha = R_u(\lambda), \quad \beta = R_w(\lambda), \quad \gamma = R_v(\lambda), \] (46)
where \( \lambda \) is any function on \( A_4 \) which obeys
\[ (R_u + R_v + R_w + \text{id})(\lambda) = -1. \] (47)

At this level, \( \lambda \) is not necessarily a scalar. To determine it, we set
\[ \lambda = \sum_{g \in A_4} \lambda_g \delta_g \] (48)

hence we get from (47) that
\[ \lambda = (-1 - \lambda_v - \lambda_w - \lambda_u)\delta_e + \lambda_v \delta_v + \lambda_w \delta_w + \lambda_u \delta_u + (-1 - \lambda_x - \lambda_y - \lambda_z)\delta_t + \lambda_x \delta_x \\
+ \lambda_y \delta_y + \lambda_z \delta_z + (-1 - \lambda_{tx} - \lambda_{ty} - \lambda_{tz})\delta_{tx} + \lambda_{tx} \delta_{tx} + \lambda_{ty} \delta_{ty} + \lambda_{tz} \delta_{tz}. \]

We write out the first equation of system (28) in the basis (20) of \( \Omega^2(H) \), passing from the right module structure to the left one, then set to zero each coefficient of the basis
element and obtain the following equations

\[
(1 + \alpha) R_1(\gamma) + \lambda R_1(\lambda) + \beta R_1(\alpha) + \gamma R_1(1 + \beta) = 0
\]

\[
(1 + \alpha) R_1(\lambda) + \lambda R_1(1 + \gamma) + \beta R_1(\beta) + \gamma R_1(\alpha)
\]

\[
- \lambda R_3(\gamma) - \alpha R_3(\lambda) - (1 + \gamma) R_3(\alpha) - \beta R_3(1 + \beta) = 0
\]

\[
(1 + \alpha) R_1(\beta) + \lambda R_1(\alpha) + \beta R_1(1 + \lambda) + \gamma R_1(\gamma)
\]

\[
- \beta R_4(\lambda) - \gamma R_4(1 + \gamma) - \alpha R_4(\beta) - (1 + \lambda) R_4(\alpha) = 0
\]

\[
\gamma R_2(1 + \alpha) + (1 + \beta) R_2(\beta) + \lambda R_2(\gamma) + \alpha R_2(\lambda)
\]

\[
- \lambda R_3(\gamma) - \alpha R_3(\lambda) - (1 + \gamma) R_3(\alpha) - \beta R_3(1 + \beta) = 0
\]

\[
\gamma R_2(\lambda) + (1 + \beta) R_2(1 + \gamma) + \lambda R_2(\beta) + \alpha R_2(\alpha)
\]

\[
- \beta R_4(\gamma) - \gamma R_4(\lambda) - \alpha R_4(\beta) - (1 + \lambda) R_4(1 + \beta) = 0
\]

\[
\gamma R_2(\beta) + (1 + \beta) R_2(\alpha) + \lambda R_2(1 + \lambda) + \alpha R_2(\gamma)
\]

\[
- \beta R_4(1 + \alpha) - \gamma R_4(\beta) - \alpha R_4(\gamma) - (1 + \lambda) R_4(\lambda) = 0
\]

\[
\lambda R_3(1 + \alpha) + \alpha R_3(\beta) + (1 + \gamma) R_3(\gamma) + \beta R_3(\lambda)
\]

\[
- \beta R_4(\lambda) - \gamma R_4(1 + \gamma) - \alpha R_4(\beta) - (1 + \lambda) R_4(\alpha) = 0
\]

\[
\lambda R_3(\beta) + \alpha R_3(\alpha) + (1 + \gamma) R_3(1 + \lambda) + \beta R_3(\gamma)
\]

\[
- \beta R_4(\gamma) - \gamma R_4(\lambda) - \alpha R_4(\alpha) - (1 + \lambda) R_4(1 + \beta) = 0,
\]

where the indexes 1, 2, 3, 4 refer respectively to $t, x, y, z$. We then use (46) to write these equations respectively in terms of $\lambda$, then in terms of its scalar components. A long but straightforward computation of these components leads to $\lambda_g = -\frac{1}{4}, \forall g \in A_4$, hence as an element of $H, \lambda = -\frac{1}{4}$. From (46), we also have $\alpha = \beta = \gamma = -\frac{1}{4}$. The expression of the corresponding connection in (24) is then as stated. To end the proof of the Proposition 4.1, one checks easily that this connection is also solution of the other equations of system (28). \hfill \diamond

We refer to this connection as the ‘Levi-Civita connection’ for the invariant metric on the group $A_4$.

**Proposition 4.2** \( \text{The covariant derivative } \nabla : \Omega^1(H) \rightarrow \Omega^1(H) \otimes_H \Omega^1(H) \text{ for the above ‘Levi-Civita connection’ on } A_4, \text{ and its Riemann curvature } \mathcal{R} : \Omega^1(H) \rightarrow \Omega^2(H) \otimes_H \Omega^1(H) \)
are given by

\[
\nabla(e_t) = -e_t \otimes e_t - e_x \otimes e_z - e_y \otimes e_x - e_z \otimes e_y + \frac{1}{4} \theta \otimes \theta \\
\nabla(e_x) = -e_t \otimes e_y - e_x \otimes e_x - e_y \otimes e_z - e_z \otimes e_t + \frac{1}{4} \theta \otimes \theta \\
\nabla(e_y) = -e_t \otimes e_z - e_x \otimes e_t - e_y \otimes e_y - e_z \otimes e_x + \frac{1}{4} \theta \otimes \theta \\
\nabla(e_z) = -e_t \otimes e_x - e_z \otimes e_y - e_y \otimes e_t - e_z \otimes e_z + \frac{1}{4} \theta \otimes \theta .
\]

\[
\mathcal{R}(e_t) = de_t \otimes e_t + de_x \otimes e_z + de_y \otimes e_x + de_z \otimes e_y \\
\mathcal{R}(e_x) = de_t \otimes e_y + de_x \otimes e_x + de_y \otimes e_z + de_z \otimes e_t \\
\mathcal{R}(e_y) = de_t \otimes e_z + de_x \otimes e_t + de_y \otimes e_y + de_z \otimes e_x \\
\mathcal{R}(e_z) = de_t \otimes e_x + de_x \otimes e_y + de_y \otimes e_t + de_z \otimes e_z
\]

**Proof**: The curvature 2-form \( F \) is defined by equation (10). In the present case, we have \( bc \notin C, \forall b, c \in C \), so that \( \sum_{b,c \in C, bc = a} A_b \wedge A_c = 0, \forall a \in C \). We have also \( \sum_{a \in C} A_a = 0 \) and \( d\theta = 0 \), hence \( F_a = dA_a = de_a \) for the form of the connection in (13). This is exactly the same argument as for \( S_3 \) in [3]. Next, if we replace \( \alpha \) in formula (11) by \( e_t, e_x, e_y, e_z \) respectively, and use Table 1, we obtain relations (51) for the curvature. Finally, we compute the value of the covariant derivative on the basis 1-forms \( \{e_a\} \) by using formula (10). Explicitly, we have

\[
\nabla(e_a) = -\sum_{b \in C} A_b \otimes (e_{b^{-1}ab} - e_a) \\
= -\sum_{b \in C} (e_b - \frac{1}{4} \theta) \otimes (e_{b^{-1}ab} - e_a) \\
= -\sum_{b \in C} e_b \otimes (e_{b^{-1}ab} - e_a) + \frac{1}{4} \theta \otimes \sum_{b \in C} (e_{b^{-1}ab} - e_a) \\
= -\sum_{b \in C} e_b \otimes e_{b^{-1}ab} + \sum_{b \in C} e_b \otimes e_a + \frac{1}{4} \theta \otimes \sum_{b \in C} (e_b - e_a) \\
= -\sum_{b \in C} e_b \otimes e_{b^{-1}ab} + \frac{1}{4} \theta \otimes \theta
\]

According to Table 1, this last equation gives relations (51) as stated.
From the Riemann curvature and the canonical lift \( i \) we can compute the Ricci curvature of the Levi-Civita connection on \( A_4 \) and find that it vanishes. In fact we can prove a slightly stronger result that is it the only Ricci flat connection for this choice of framing.

**Theorem 4.3** For the framing defined by the Maurer-Cartan 1-form, and for the canonical lift \( i \), the above Levi-Civita connection on \( A_4 \) is the unique regular Ricci-flat connection.

**Proof:**

In the present case, the canonical lift takes the form

\[
i(e_a \wedge e_b) = e_a \otimes e_b - \frac{1}{3} \sum_{c,d=ab,c\neq d} e_c \otimes e_d, \quad i(e_a \wedge e_a) = 0.
\]

We have to solve for vanishing of \( \mathcal{L} \)

\[
\text{Ricci} = \sum_{a \in \mathcal{C}} \langle f^a, (i \otimes \text{id}) \mathcal{R}(e_a) \rangle = \sum_{a,b,c \in \mathcal{C}} i(F_c)_{ab} e_b \otimes (e_c - e_a),
\]

where \( i(F_c) = i(F_c)_{ab} e_a \otimes_H e_b \), and the pairing is made between each \( f^a \) and the first factor of the tensor product \((i \otimes_H \text{id}) \mathcal{R}(e_a)\) according to the formula \( \langle f^a, me_b \rangle = m \delta^a_m, \forall m \in H \).

In our case this becomes

\[
\langle f^t, i(F_x) \otimes (e_z - e_t) + i(F_y) \otimes (e_x - e_t) + i(F_z) \otimes (e_y - e_t) \rangle
\]

\[
+ \langle f^x, i(F_t) \otimes (e_y - e_x) + i(F_y) \otimes (e_z - e_x) + i(F_z) \otimes (e_t - e_x) \rangle
\]

\[
+ \langle f^y, i(F_t) \otimes (e_z - e_y) + i(F_z) \otimes (e_t - e_y) + i(F_y) \otimes (e_z - e_y) \rangle
\]

\[
+ \langle f^z, i(F_t) \otimes (e_x - e_z) + i(F_x) \otimes (e_y - e_z) + i(F_y) \otimes (e_t - e_z) \rangle = 0
\]

We first compute \( F_t, F_x, F_y, F_z \) and \( i(F_t), i(F_x), i(F_y), i(F_z) \) for general free torsion connections given in Proposition 3.4, then we rewrite equation (53) in terms of the basic elements \( \{ e_a \otimes_H e_b \}_{a,b \in \mathcal{C}} \) of the left \( H \)-module \( \Omega^1(H) \otimes_H \Omega^1(H) \), the vanishing of each coefficient of the mentioned basic elements leads to 16 equations in terms of \( \alpha, \beta, \gamma, \lambda \) and their ‘first order derivatives’ \( \partial^a \alpha, \partial^a \beta, \partial^a \gamma, \partial^a \lambda \), \( a \in \mathcal{C} \).

We find that it is enough to solve the following 4 equations coming from the coefficients of \( e_t \otimes_H e_t, e_x \otimes_H e_x, e_y \otimes_H e_y, e_z \otimes_H e_z \) respectively:

\[
\beta - 2\gamma + \lambda + \partial^x \alpha + \partial^y \alpha + \partial^z \alpha + \partial^t \beta - 2\partial^x \beta + \partial^y \beta - 2\partial^z \beta + \partial^t \gamma - 2\partial^x \gamma + \partial^y \gamma - 2\partial^z \gamma + \partial^t \lambda - 2\partial^x \lambda - 2\partial^y \lambda - 2\partial^z \lambda = 0
\]

\[
\alpha - 3\beta + \lambda + \gamma + \partial^x \alpha - 2\partial^y \alpha + \partial^y \beta + \partial^x \beta + \partial^z \beta + \partial^y \gamma - 2\partial^x \gamma + \partial^z \gamma - 2\partial^y \gamma - 2\partial^z \gamma - 2\partial^y \lambda - 2\partial^z \lambda = 0
\]

\[
\alpha + \beta - 3\gamma + \lambda + \partial^x \alpha - 2\partial^y \alpha + \partial^y \beta - 2\partial^x \beta + \partial^z \beta + \partial^y \gamma + \partial^z \gamma + \partial^y \lambda - 2\partial^z \lambda = 0
\]

\[
\alpha + \beta + 3\gamma - 3\lambda + \partial^x \alpha - 2\partial^y \alpha + \partial^y \beta - 2\partial^x \beta + \partial^z \beta + \partial^y \gamma - 2\partial^x \gamma + \partial^z \gamma + \partial^y \lambda + \partial^x \lambda + \partial^z \lambda = 0
\]
Indeed, we transform these 4 equations to a system of 48 linear equations where the variables are the components of $\alpha, \beta, \gamma$ and $\lambda$ in the basis $(\delta_g)_{g \in A_4}$. The unique solution of the mentioned system which obeys the condition $\alpha + \beta + \gamma + \lambda = -1$ as in Proposition 3.4 is $\alpha = \beta = \gamma = \lambda = -1/4$. To end the proof one just checks easily that this solution is also a solution of the 12 remaining equations (of the 16 ones mentioned above), coming from the coefficients of $e_a \otimes_H e_b$, $a \neq b$ in equation (53).

One can also check that the Ricci tensor for the Levi-Civita connection with respect to the alternative ‘lift’

$$i'(e_a \wedge e_b) = e_a \otimes_H e_b - e_{aba^{-1}} \otimes_H e_a,$$

also vanishes, i.e. the result does not depend strongly on the choice of lift. This is the same as found for $S_3$, where the two Ricci tensors with respect to $i$ and $i'$ respectively are the same up to a scale [2].

5 The Dirac operator for $A_4$

Following the formalism of reference [2], we write down in this section the ‘gamma matrices’ and the Dirac operator associated to the Maurer-Cartan framing $e$ and the coframing $e^*$ for the invariant metric. We use the associated Levi-Civita connection constructed above.

For the ‘spinor’ representation, we consider the standard 3-dimensional representation of $A_4$ defined on a vector space $W$ by

$$\rho_W(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_W(u) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_W(v) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho_W(w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $e$ is the unit matrix $I$. The Casimir element $C$ associated to the operator $\eta$ is given in [2] by

$$C = \eta^{-1}_{ab} f^a f^b = \eta^{-1}_{ab} (a - e)(b - e)$$
with summation understood, \( b, a \in C \). One checks that it corresponds in the general case of the class \( C = \{ t, x, y, z \} \) as in Prop. 3.2, to the explicit form

\[
C = \frac{1 + 3\mu}{1 + 4\mu} [t^2 + x^2 + y^2 + z^2 - 2(t + x + y + z) + 4e] + \frac{-3\mu}{1 + 4\mu} (tx + ty + tz + xy - 2(t + x + y + z) + 4e)
\]  

(58)

In the case of \( A_4 \), equation (58) reads

\[
C = \frac{1}{1 + 4\mu} [(t - e)^2 + (x - e)^2 + (y - e)^2 + (z - e)^2]
\]

then

\[ \rho_W(C) = \frac{4}{1 + 4\mu} I. \]

Next, we choose our gamma-matrix \( \gamma \) to be the ‘tautological gamma-matrix’ \([2]\) associated to \( \rho_W \) and \( \eta \) defined by

\[
\gamma_a = \eta_{ab}^{-1} \rho_W(f^b) = \sum_{b \in C} \eta_{ab}^{-1} \rho_W(b - e), \quad \forall a \in C.
\]

(59)

In our case we find

\[
\gamma_a = \rho_W(a - e) + \frac{4\mu}{1 + 4\mu}
\]

(60)

and that these matrices obey the relations

\[
\sum_{a \in C} \gamma_a = -\frac{4}{1 + 4\mu},
\]

(61)

\[
\gamma_a \gamma_b + \gamma_b \gamma_a + \frac{2}{1 + 4\mu} (\gamma_a + \gamma_b) + \frac{2}{(1 + 4\mu)^2} = \rho_W(ab + ba)
\]

(62)

following directly from (60).

Equations (60), (61) and (62) hold in the general case considered in Proposition 3.2, providing that the multiplication’s table is that of Table 3. The explicit matrix representation of these gamma-matrices above for \( A_4 \) are:

\[
\gamma_t = \begin{pmatrix}
\frac{-1}{1+4\mu} & 0 & 1 \\
1 & \frac{-1}{1+4\mu} & 0 \\
0 & 1 & \frac{-1}{1+4\mu}
\end{pmatrix}, \quad \gamma_x = \begin{pmatrix}
\frac{-1}{1+4\mu} & 0 & -1 \\
-1 & \frac{-1}{1+4\mu} & 0 \\
0 & 1 & \frac{-1}{1+4\mu}
\end{pmatrix},
\]

(63)

\[
\gamma_y = \begin{pmatrix}
\frac{-1}{1+4\mu} & 0 & -1 \\
1 & \frac{-1}{1+4\mu} & 0 \\
0 & 1 & \frac{-1}{1+4\mu}
\end{pmatrix}, \quad \gamma_z = \begin{pmatrix}
\frac{-1}{1+4\mu} & 0 & 1 \\
-1 & \frac{-1}{1+4\mu} & 0 \\
0 & 1 & \frac{-1}{1+4\mu}
\end{pmatrix}
\]
Proposition 5.1 The Dirac operator (14) on $A_4$ for the gamma-matrices and the Levi-Civita connection on $A_4$ constructed above is given by

$$\slashed{D} = \partial^a \gamma_a - 4$$

(sum over $a \in C$). For $\mu = 0$ we have explicitly

$$\slashed{D} = \begin{pmatrix} -R_t - R_x - R_y - R_z & 0 & R_t - R_x - R_y + R_z \\ R_t - R_x + R_y - R_z & -R_t - R_x - R_y - R_z & 0 \\ 0 & R_t + R_x - R_y - R_z & -R_t - R_x - R_y + R_z \end{pmatrix}.$$  

This has 18 zero modes, 3 modes with eigenvalue $\pm 4$, 3 modes with eigenvalue $\pm 4q$, and 3 modes with eigenvalue $\pm 4\bar{q}$, where $q = e^{2\pi i/3}$.

Proof:

The formula giving the Dirac operator in terms of the gamma-matrices and the representation $\rho_W$ is given by equation (15). We first observe that for the representation $\rho_W$ above, the following two equations hold: $\sum_{a \in C} \rho_W(a) = 0$ and $\sum_{a \in C} \rho_W(a^2) = 0$. Using the $A^b_a$ defined by (45), and the fact that every element of $C$ is of order 3, we obtain

$$\slashed{D} = \partial^a \gamma_a - 4$$

we then replace in equation (64) the representation of the gamma-matrices from (63) to obtain the matrix representation of $\slashed{D}$ as stated.

To compute its eigenvalues we need $R_a$ explicitly as $12 \times 12$ matrices. In the basis spanned by delta-functions at $\{e, u, v, w, t, x, y, z, t^2, ut^2, vt^2, wt^2\}$, the right translation operators take the form

$$R_t = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}, \quad R_x = \begin{pmatrix} 0 & Y & 0 \\ 0 & 0 & Z \\ X & 0 & 0 \end{pmatrix}, \quad R_y = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & Y \\ Z & 0 & 0 \end{pmatrix}, \quad R_z = \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & X \\ Y & 0 & 0 \end{pmatrix}.$$
where $I$ is the $4 \times 4$ identity and

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. $$

We then obtain the eigenvalues as stated. 

The eigenvalues here do in fact depend on $\mu$ and the case $\mu = 0$ seems to be the more natural since it corresponds to the simplest metric $\delta_{a,b}$. The $-4$ in (64) corresponds to the constant curvature of $A_4$ as for $S_3$ in [2]. As for $S_3$, this offset ensures a symmetrical distribution of eigenvalues about zero.

We will now construct the eigenstates of $\hat{D}$. Before doing that we look at the spin 0 or scalar wave equation defined by the corresponding wave operator

$$\Box = -\eta^{-1}_{ab} \partial^a \partial^b = - \sum_a \partial^a \partial^a = \sum_a (2R_a - R_a^2 - \text{id}).$$

(65)

We do not exactly expect a Lichnerowicz formula relating this to the square of $\hat{D}$, but we find that it is the square of a first-order operator with eigenvalues contained in those of $\hat{D}$. It is easy to solve the wave equation directly.

**Proposition 5.2**

$$\Box = -\frac{1}{4} (\sum_a \partial^a)^2 = - \frac{1}{4} (D_0 - 4)^2, \quad D_0 = \sum_a R_a.$$

*There is 1 zero mode, given by the constant function. There is 1 mode of eigenvalue $12q$ and one of eigenvalue $12\bar{q}$ given by the two other 1-dimensional representations of $A_4$. Finally there are 9 modes with eigenvalue $-4$ given by the matrix elements of the remaining 3-dimensional irreducible representation $\rho_W$ above.*

**Proof:**

The square form of $\Box$ follows from the multiplication Table 3. From there one finds that $(\sum_a R_a)^2 = 4 \sum_a R_a^2$, after which the result follows. To solve the wave equation, note that the nontrivial 1-dimensional representations $\rho, \bar{\rho}$ of $A_4$ are given by

$$\rho(t) = q, \quad \rho(u) = \rho(v) = \rho(w) = 1, \quad \bar{\rho}(t) = \bar{q}, \quad \bar{\rho}(u) = \bar{\rho}(v) = \bar{\rho}(w) = 1.$$
Then \( \forall m \in \mathcal{A}_4 \),
\[
\Box \rho(m) = 2 \sum_a \rho(m)\rho(a) - \sum_a \rho(m)\rho(a^2) - 4\rho(m)
\]
\[
= (8q - 4 - 4\bar{q})\rho(m) = 12q\rho(m)
\]
Similarly for \( \bar{\rho} \) with \( q \) replaced by \( \bar{q} \). Finally for the matrix elements \( \{\rho_{kl}\} \) of \( \rho_W \), we have
\[
\Box \rho_{kl}(m) = \sum_a [2R_a \rho_{kl}(m) - R_a^2 \rho_{kl}(m) - \rho_{kl}(m)]
\]
\[
= \sum_a \sum_i [2\rho_{ki}(m)\rho_{il}(a) - \rho_{ki}(m)\rho_{il}(a^2)] - 4\rho_{kl}(m)
\]
since \( \sum_a \rho_W(a^2) = 0 \) and \( \sum_a \rho_W(a) = 0 \). The 9 “waves” \( \rho_{kl} \) are linearly independent because the representation \( \rho_W \) is irreducible. Hence we have completely diagonalised the 12x12 matrix \( \Box \). Equivalently, we have diagonalised \( D_0 \) with corresponding eigenvalues \( 4, 4q, 4\bar{q}, 0 \) as for \( D \) above. ⊗

Moreover, every function \( \phi \) on \( \mathcal{A}_4 \) has a unique decomposition of the form
\[
\phi = p_0 + p_1 \rho + p_2 \bar{\rho} + \sum_{k,l} p_{kl} \rho_{kl}
\]
for some numbers \( p_0, p_1, p_2, p_{kl} \) which are the components of \( \phi \) in the nonabelian Fourier transform. The decomposition above corresponds precisely to the Peter-Weyl decomposition, just as noted for \( S_3 \) in [5].

We now use the preceding results to completely solve the Dirac equation. We set
\[
D_1 = R_t - R_x - R_y + R_z, \quad D_2 = R_t - R_x + R_y - R_z, \quad D_3 = R_t + R_x - R_y - R_z
\]
so that
\[
\mathcal{D} = \begin{pmatrix} -D_0 & 0 & D_1 \\ D_2 & -D_0 & 0 \\ 0 & D_3 & -D_0 \end{pmatrix}.
\]
Let us note first of all that
\[
D_1D_2 = 0, \quad D_2D_3 = 0, \quad D_3D_1 = 0, \quad D_0D_i = 0, \quad D_i^2 = 0, 1 \leq i \leq 3,
\]
from which we see by inspection that the following are 18 linearly-independent zero modes of \( \mathcal{D} \):
\[
\begin{pmatrix} D_2 \rho_{k3} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D_3 \rho_{k1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ D_1 \rho_{k2} \end{pmatrix}, \begin{pmatrix} D_3 \rho_{k1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D_1 \rho_{k2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ D_2 \rho_{k3} \end{pmatrix}
\]
for $1 \leq k \leq 3$. Similarly it is immediate by inspection that

$$
\begin{pmatrix}
\rho^n \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
\rho^n \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
\rho^n
\end{pmatrix}
$$

are 3 modes with eigenvalue $-4q^n$, for $n = 0, 1, 2$. This is because $D_0 \rho = 4q \rho$ (as in Proposition 5.2 above) while $D_i \rho = 0$ for $i > 0$.

It remains only to construct the $+4q^n$ eigenmodes for $n = 0, 1, 2$. Before doing this let us make two observations about the modes already evident. First of all, let $\hat{\rho}$ denote the operator of multiplication by $\rho$. Then $R_a \hat{\rho} = q \hat{\rho} R_a$ since $\rho(a) = q$ for $a = t, x, y, z$. Hence

$$
\hat{\rho} R_t = q \hat{\rho} R_t.
$$

Thus, multiplication of a spinor mode by the function $\rho$ multiplies its eigenvalue by $q$. This generates the $-4q^n$ modes above from the $n = 0$ case.

Secondly, from the multiplication Table 3 we see that

$$
R_t D_1 = D_2 R_t
$$

and its cyclic rotations $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. From this one finds

$$
[\chi, \hat{\rho}] = 0, \quad \chi^3 = \text{id}; \quad \chi = \begin{pmatrix} 0 & 0 & R_t \\ R_t & 0 & 0 \\ 0 & R_t & 0 \end{pmatrix}.
$$

This $\chi$ generates the other two modes from the first in each group of three in (68) and (69). In the case of the zero modes note that

$$
R_t \rho_{k1} = \rho_{k2}
$$

and its cyclic rotations $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$, from the explicit form of $\rho_W(t)$.

We now observe that if we make an ansatz of the form

$$
\psi = \begin{pmatrix} \phi \\ R_t \phi \\ R_{t^2} \phi \end{pmatrix} = \begin{pmatrix} \text{id} \\ R_t \\ R_{t^2} \end{pmatrix} \phi
$$

for function $\phi$ then

$$
\hat{\rho} \psi = \begin{pmatrix} \text{id} \\ R_t \\ R_{t^2} \end{pmatrix} (-D_0 + R_{t^2} D_2) \phi
$$
so eigenspinors are induced by eigenfunctions of the operator

\[-D_0 + R_{t2}D_2 = -D_0 + R_e - R_u - R_v + R_w.\]

All of the \(\rho_{kl}\) are zero modes of \(D_0\) (as in Proposition 5.2), while among them precisely \(\rho_{k1}\) is an eigenmode of \(R_e - R_u - R_v + R_w\), with eigenvalue 4 (this follows from \(\frac{1}{4}(\rho_W(e) - \rho_W(u) - \rho_W(v) + \rho_W(w))\) being a projection matrix of rank 1). Hence \(\phi = \rho_{k1}\) in the ansatz yields three spinor modes

\[
\begin{pmatrix}
\rho_{k1} \\
\rho_{k2} \\
\rho_{k3}
\end{pmatrix}, \quad 1 \leq k \leq 3
\]

with eigenvalue +4 of \(\mathcal{D}\). Applying \(\hat{\rho}\) generates the three with eigenvalue \(4q\) and then the three with eigenvalue \(4\bar{q}\).

This completes our diagonalisation of \(\mathcal{D}\). Finally, we note that there necessarily exists an operator \(\gamma\) with \(\gamma^2 = \text{id}\) and \(\{\gamma, \mathcal{D}\} = 0\), but it is not unique. Thus, diagonalising \(\mathcal{D}\), we can group the eigenbasis into pairs of 3-blocks of zero modes according to the two groups in (68), interchanged by \(\gamma\), and similarly we define \(\gamma\) to interchange the two 3-blocks with eigenvalues \(\pm 4q^n\). This defines at least one choice of \(\gamma\), suggested by our explicit diagonalisation.

6 Cohomology and concluding remarks

In this paper we have concentrated on the Riemannian geometry of \(A_4\). There are also some more elementary geometrical questions that one could look at, related to the differential structure alone. We will discuss some of them here.

Firstly, given an exterior algebra \(\Omega(H)\) one has a noncommutative de Rahm cohomology defined as usual by closed forms modulo exact ones. We find, just as for \(S_3\) in \([F]\), that \(\theta\) generates \(H^1\).

**Proposition 6.1** For \(A_4\) with cyclic conjugacy class \(\{t, x, y, z\}\) the first noncommutative de Rahm cohomology is

\[H^1(A_4) = \mathbb{C}.\theta\]

**Proof:**

We compute \(\partial^a = R_a - \text{id}\) explicitly as four \(12 \times 12\) matrices for their action on \(\mathbb{C}[A_4]\). The
concatenation of these define $d_0 : \mathbb{C}[A_4] \to \mathbb{C}[A_4] \otimes \Omega_0 = \Omega^1(H)$ as a $48 \times 12$-matrix. We also define the $8 \times 16$-matrix $\pi$ which sends $e_a \otimes H e_b \to e_a \wedge e_b$ using the tensor product 16-dimensional basis of $\Omega_0 \otimes \Omega_0$ and the 8-dimensional vector space over $\mathbb{C}$ with basis $(20)$. Similarly, we define an $8 \times 4$-matrix for $d$ acting on $\Omega_0$ again using the basis $(20)$ over $\mathbb{C}$. From these ingredients, we build $d_1 : \Omega^1(H) \to \Omega^2(H)$ as a $96 \times 48$-matrix defined by $d_1(f e_a) = d_0(f) \wedge e_a + f d e_a$. We then compute the kernel of $d_1$ and find it to be 12-dimensional. The image of $d_0$ is necessarily 11-dimensional (its kernel is the constant functions) and hence the cohomology is 1-dimensional. $\theta$ is closed but never exact (for any finite group) and hence represents this class.

Next, whereas the cohomology is a linear problem, one can also consider its non-linear variant called $U(1)$-gauge theory. Here we define the curvature of a 1-form $\alpha \in \Omega^1(H)$ to be the 2-form $F(\alpha) = d\alpha + \alpha \wedge \alpha$. This transforms by conjugation under the gauge transform $\alpha \mapsto u\alpha u^{-1} + u du^{-1}$ for any no-where zero function $u \in H$. One can also impose here unitarity conditions as in [5]. In this context it would be interesting to find the moduli space of (unitary) flat connections. This was done for $S_3$ in [3] and found to have a richer structure than the cohomology alone, i.e. with other solutions beyond multiples of $\theta$ and we would expect something similarly rich for $A_4$. For example, if we focus on flat connections with constant coefficients in the $\{e_a\}$ basis as in [4], a short computation shows that these are given by the five lines

$$\lambda e_t - \theta, \quad \lambda e_x - \theta, \quad \lambda e_y - \theta, \quad \lambda e_z - \theta, \quad (\lambda - 1)\theta$$

(75)

for $\lambda$ a parameter. There is a similar behaviour for any cyclic conjugacy class.

A further question relates to the fact that $A_4 \subset S_4$ as a normal subgroup. Therefore its exterior algebra should be related to that of $S_4$ for a suitable conjugacy class on that. The different differential structures on $S_4, S_5$ for different conjugacy classes are studied in [7] and looking there, one finds that the order 8 conjugacy class containing (123) in $S_4$ has the required exterior algebra. Its eight generators split into two sets, namely $\{e_{123}, e_{134}, e_{243}, e_{142}\}$ generating a subalgebra with relations as in Proposition 3.2 and a complementary set $\{e_{132}, e_{143}, e_{234}, e_{124}\}$ generating the opposite subalgebra. We denote the latter generators by $\{\bar{e}_t, \bar{e}_x, \bar{e}_y, \bar{e}_z\}$. There are nontrivial cross relations between the two sets:

$$e_a \wedge \bar{e}_a + \bar{e}_a \wedge e_a = 0,$$

(76)
\[ e_t \wedge e_z + e_x \wedge e_x + e_y + e_y \wedge e_t = 0 \]
\[ e_t \wedge e_x + e_x \wedge e_y + e_y \wedge e_z + e_z \wedge e_t = 0 \]
\[ e_t \wedge e_y + e_y \wedge e_z + e_z \wedge e_x + e_x \wedge e_t = 0 \]

and their three conjugates (given by applying $\bar{\cdot}$ and reversing products). In other words, the differential geometry of $S_4$ appears to be some form of ‘complexification’ of that of $A_4$.

Indeed, the conjugacy class $\{ i, \bar{x}, \bar{y}, \bar{z} \}$ in $A_4$ is also cyclic and the equations of its associated exterior algebra are of the same form as in Proposition 3.2. It defines a differential geometry on $A_4$ conjugate to the one we have studied above. Indeed, one has

\[ \partial^\dagger_a = \partial_{a-1} = \partial_{a2} = \partial_a \]  

where $\dagger$ denotes transpose with respect to the $l^2$ inner product on $A_4$. Here the first equality is a general feature for any finite group and follows from the braided-Leibniz rule. The second equality is due to all elements of $\mathcal{C}$ in our case having order 3 and the third is a special feature $a^2 = \bar{a}$ of the multiplication Table 3. To complete the analysis let us note that $A_4$ has just one other nontrivial conjugacy class, $\{ u, v, w \}$. This does not generate $A_4$ i.e., the quantum manifold structure that it defines is not connected (not every point can be reached from any other by steps taken from the conjugacy class). The component of this connected to the group identity is $\mathbb{Z}_2 \times \mathbb{Z}_2$ with its universal differential calculus.

Finally, returning to the Riemannian geometry, one can and should consider more general metrics and vierbeins. Since we have found above a canonical invariant $\eta$, one could use this to fix the relationship between a general vierbein and covierbein, i.e. look for metrics of the form $g = \eta^{ab} e_a \otimes_H e_b$ with $\{ e_a \}$ as the free variable (rather than $e, e^*$ independent). Among the moduli space of pairs $(e, A)$ of vierbein and spin connection (or more generally of triples $(e, e^*, A)$), our results above show that there is at least one canonical point where the Ricci tensor vanishes. This motivates the problem of solving the Ricci flat equations in general, i.e. classical ‘gravity’ on $A_4$. Similarly, one can consider functional integrals, now a finite number of usual integrals, with Einstein-Hilbert action, i.e. ‘quantum-gravity’. These are difficult nonlinear questions unsolved even for $S_3$ and beyond our present scope.
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