Training Teachers To Teach Probability

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**Key Words:** Professional knowledge.

**Abstract**

In this paper we analyze the reasons why the teaching of probability is difficult for mathematics teachers, describe the contents needed in the didactical preparation of teachers to teach probability and analyze some examples of activities to carry out this training. These activities take into account the experience at the University of Granada, in courses directed to primary and secondary school teachers as well as in an optional course on Didactics of Statistics, which is included in the Major in Statistical Sciences and Techniques course since 1996. The aim is encouraging other colleagues to organize similar courses at their universities, either as part of their official programs or in their postgraduate training.

**1. Introduction**

Nowadays probability and statistics are part of mathematics curricula for primary and secondary school classes in many countries. The reasons to include probability and statistics teaching have been repeatedly highlighted over the past 20 years (by Holmes 1980; Hawkins, et al., 1991; Vere-Jones 1995; Wild and Pfankuch 1999, Gal 2002), and include the usefulness of statistics and probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions and its role in developing a critical reasoning.

In primary and secondary school levels, probability and statistics is part of the mathematics curriculum and mathematics teachers frequently lack specific preparation in statistics education. For example, in Spain, prospective secondary teachers with a major in Mathematics do not receive specific training in statistics education. The situation is even worse for primary teachers, most of whom have not had basic training in statistics and this problem is common to many countries. Textbooks and curriculum documents prepared for primary and secondary teachers do not offer enough support, as shown in Ortiz (1999) and Ortiz, et al. (2002). The textbooks sometimes present a too narrow view of probability (only the classical approach), and applications are at other times restricted to games of chance and in some of them the definitions of concepts are incorrect.
Consequently, it is urgent to offer these teachers a better prior training as well as continuous support from University departments and research groups. In this paper we discuss what type of didactical knowledge these teachers need, beyond the knowledge of statistics and probability itself, and analyze some activities that we found useful in training primary and secondary teachers at the University of Granada. We will concentrate on probability, although the main ideas are also useful for statistics. As a previous step, we describe the main characteristics of stochastic knowledge and reasoning.

2. The Nature of Stochastic Reasoning and Knowledge

A main point in preparing teachers is the epistemological reflection, which can help them to understand the role of concepts within statistics and other areas, its importance in students' learning and students' conceptual difficulties in problem solving. Probability is a young area and its formal development was linked to a large number of paradoxes, which show the disparity between intuition and conceptual development in this field (Borovcnik, et al. 1991). This comparative difficulty is also shown in the fact that, even when Kolmogorov axiomatic was generally accepted in 1933, professional statisticians still debate about different views of probability and different methodologies of inference (Fine 1971).

Borovcnik and Peard (1996) remark that counterintuitive results in probability are found even at very elementary levels, whereas in other branches of mathematics counterintuitive results are encountered only when working at a high degree of abstraction. For example, the fact that having obtained a run of four consecutive heads when tossing a coin does not affect the probability that the following coin will result in heads is counterintuitive. These authors also suggest that probabilistic reasoning is different from logical reasoning because in a logical reasoning a proposition is always true or false and we have no complete certitude about a proposition concerning a random event.

In arithmetic or geometry an elementary operation can be reversed and this reversibility can be represented with concrete materials. This is very important for young children, who still are very linked to concrete situations in their mathematical thinking. For example, when joining a group of two apples with another group of three apples, a child always obtain the same result (5 apples); if separating the second set from the total he/she always returns to the original set; no matter how many times this operation is repeated. These experiences are very important to help children progressively abstract the mathematical structure behind them.

In the case of random experiment we obtain different results each time the experiment is carried out and the experiment cannot be reversed (we can not get the first result again when repeating the experiment). Even though simulation or experimentation with random generators, such as dice and coins have a very important function in stabilizing children’s intuition and in materializing probabilistic problems, these experiences do not provide the key to how and why the problems are solved. It is only with the help of combinatorial schemes or tools like tree diagrams that children start to understand the solution of probabilistic problems. This indicates the complementary nature of classical and frequentist approaches to probability.

Another reason for this difficulty is that stochastics is quickly moving away from pure mathematics, and being more related to applications. For example, although independence is mathematically reduced to the multiplicative rule, this definition does not include all the causality problems that subjects often relate to independence nor always serve to decide if there is independence in a particular experiment.

In summary, stochastics is difficult to teach, because we should not only present different models and show their applications, but we have to go deeper into wider questions, consisting of how to obtain knowledge from data, why a model is suitable, how to help students develop correct intuitions in this field and deal with controversial ideas, such as randomness or causality.
3. What is Didactical Knowledge and how Can it be Taught? The Experience in Training Statistics Teachers at the University of Granada

The teaching of statistics and probability takes place in mathematics classrooms, and teachers tend to adapt their vision of stochastics and its teaching, to problem-solving methods and reasoning standards used in mathematics. A wide statistical knowledge, even when essential, is not enough for teachers to be able to teach probability.

According to Cooney (1999), the current vision of teacher’s knowledge is complex, as compared with the vision held in past decades. Research focused on teacher's training is producing a great deal of information about 'didactical knowledge', which includes the following complementary aspects (NCTM 1991; Aichele and Coxford 1994):

- Epistemological reflection on the meaning of concepts to be taught (e.g. reflection on the different meaning of probability). For the particular case of statistics, Biehler (1990) also suggests that teachers require meta-knowledge about statistics, including a historical, philosophical, cultural and epistemological perspective on statistics and its relations to other domains of science.
- Experience in adapting this knowledge to different teaching levels, and students’ various levels of understanding. This include, according Steinbring (1990), organizing and implementing statistics projects, experiencing students’ multiple forms of work and understanding experiments, simulations and graphical representations not just as methodological teaching aids, but rather as essential means of knowing and understanding statistics.
- Critical capacity to analyze textbooks and curricular documents.
- Prediction of students' learning difficulties, errors, obstacles and strategies in problem solving (e.g., students strategies in comparing two probabilities; students' confusion between the two terms in a conditional probability).
- Capacity to develop and analyze assessment tests and instruments and interpret students’ responses to the same.
- Experience with good examples of teaching situations, didactic tools and materials (e.g., challenging and interesting problems; Galton board, simulation, calculators, etc.).

It is important to find suitable and effective ways to teach this "didactical knowledge" to teachers. Since students build their knowledge in an active way, by solving problems and interacting with their classmates we should use this same approach in training the teachers especially if we want them later use a constructivist and social approach in their teaching (Even and Lappan 1994; Jaworski 2001). An important view is that we should give teachers more responsibility in their own training and help them to develop creative and critical thinking (Shulman 1986). That is why we should create suitable conditions for teachers to reflect on their previous beliefs about teaching and discuss these ideas with other colleagues (Thompson 1992).

Below we describe two examples of didactical activities to train teachers in probability. These activities are complementary from various viewpoints and can be used to provoke teachers' reflection about the meaning of elementary stochastic notions, students' difficulties and obstacles, didactical methodology and materials. These activities have been experimented along the past 10 years at different courses in Statistics Education directed at primary or secondary school teachers at the University of Granada, Spain.

One of these courses has been included since 1996 as an optional topic within the Major in Statistics Sciences. The course has a length of 60 teaching hours and the majority of students are in the 4º or 5º year of studies, and, therefore, have a good training in statistics. Consequently this course is focused only
in the didactical content, which has been developed by Batanero (2001) and is divided into 5 chapters:

1. **Introduction**: Statistics Education, historical perspective, associations, journals, conferences.
2. **Epistemological foundations**: Statistics. Current tendencies. Different conceptions of randomness and probability. Fundamental stochastic ideas. Exploratory data analysis. Association and causality. Inference and induction.
3. **Research on statistical reasoning and learning difficulties**: Cognitive development: Piaget and Fischbein. Heuristics and biases in stochastic reasoning. Didactical research: errors, difficulties, misconceptions in probability, graphing, averages, association, distributions and inference.
4. **Curriculum and instruction**: Goals in the teaching of statistics. Stochastics Phenomenology. Educational theories and teaching approaches. Assessment. Teaching resources. Computers and calculators. Internet.
5. **Teaching statistics through project work**: Examples for secondary education.

The course is organized around practical activities that are described in the aforementioned text (Batanero 2001). Below we analyze two of these activities.

### 4. Situation A: Students' Perception of Randomness

In this situation we use answers given by secondary school students to a classical item in research on subjective perception of randomness (for a review of these investigations, see Falk and Konold 1997). The aim is to reflect on the complex meaning of stochastic notions, particularly that of randomness, show the utility of this situation in teaching and assessment and predict some learning difficulties.

#### 4.1. How the Idea of Randomness is Useful to Different People

To start the activity we give the teachers the following item taken from Green (1991):

**Item 1**: Some children were each told to toss a coin 40 times. Some did it properly. Others just made it up. They put H for Heads and T for tails. These are Daniel and Diana's results:

- Daniel: H T H T H H T H T T H T H H T T H T H T H T H T T H T H T
- Diana: H T T T H T T H T H T T T H T T T T H T T T H T T T T H T T T H T

Did either or both Daniel or Diana make it up? How can you tell?

We explain the teachers how this item has been extensively used in educational research to assess secondary school students' conceptions about random results. We then discuss with them the following question:

**Question 1.** What type of people do you think are interested in problems similar to item 1?

The aim of this question is to make teachers reflect on the diversity of people and institutions interested in randomness, with various purposes: Educational institutions recommend a frequentist approach to the teaching of probability, where students are encouraged to experiment with "random devices", and use "random number tables". In games of chance (lotteries, etc.) ordinary people are interested in the properties of randomness because they try to win money, or they like to be implied in exciting risk situations. Since it is quite difficult and painstaking to obtain long sequences of random results with mechanical devices, statisticians use random number tables, or computer programs to produce pseudo
random numbers generators, and they need to assess their "quality". Scientists and professionals also use random number tables, to solve complex probabilistic problems by simulation.

4.2. Task Variables in Assessment

To continue the activity we show the teachers the information in Table 1. This table shows the responses to item 1 obtained by Serrano (1996) from 277 secondary school students. The results show that more students considered Daniel’s sequence to be random, as well as different percentages in the two groups of students.

|                      | 14 year-old (n=147) | 18 year-old (n=130) |
|----------------------|----------------------|----------------------|
| He/she made it up    | Daniel 54 (37)       | Daniel 30 (23)       |
|                      | Diana 83 (56)        | Diana 63 (49)        |
| He/she did it correctly | 86 (58)            | 82 (63)             |
|                      | 53 (36)             | 48 (37)             |
| I don't know         | 7 (5)               | 18 (14)             |
|                      | 11 (8)              | 19 (14)             |

We discuss with the teachers the questions below.

**Question 2.** How would you explain the changes in the percentage of answers to whether Diana or Daniel made it up in item 1?

**Question 3.** Do you think we can do other changes in the item and then obtain different responses from the students?

**Question 4.** What might explain why the two groups of students answered differently?

In spite of the similarity of the two sequences in item 1, more students in Serrano's research considered that Diana was cheating than in the case of Daniel. Some students might expect the number of heads and tails in a sequence of random results to be very close and then consider Daniel's sequence to be random (19 heads, which is very close to the expected value of 20 heads), while Diana’s sequence with just 12 heads might seems suspicious to them.

We can show the teachers how slight changes in the item statement produce a change in students' answers. For example, research by Gigerenzer (Gigerenzer 1994; Gigerenzer and Hoffrage 1995) has shown how the difficulty of Bayes problems disappear when data are given in frequency format, instead of using probabilities. Apart from changing the sequence itself in Item 1, we might reword the item, include more than two events in the sequence or provide students with a simulation tool to observe different repetitions of random sequences, before reply the item.

All these task variables should be taken into account in assessment, as well as students’ ages, capacities,
knowledge and interest. In this example, differences between the two groups of students might be explained by age, but also by the fact that 18-year old students had been taught probability during their secondary education.

4.3. Students' strategies and conceptions

The percentages of positive or negative answers to item 1 in each group of students provide us with some information. However, a deeper understanding of the underlying reasoning is achieved when we analyze the arguments given by the students to support their decision.

Some reasons given by students in Serrano’s research to justify that Daniel or Diana were cheating were the following:

- The sequence pattern is too regular to be random, results almost alternate;
- The frequencies of heads and tails are too different;
- There are too long runs; heads and tails should alternate more frequently.

After giving the teachers samples of student’s responses in each of the above categories, we can propose them the following questions:

**Question 5.** Which of the above arguments are right? How would you explain the wrong answers?

**Question 6.** Which of these arguments do you think were mainly employed in each item? What other correct and wrong arguments would you expect in this item?

**Question 7.** Are these arguments similar or different to those used by professional statistician in testing randomness?

In the students' arguments, the complexity of randomness is observed, since different students focus their attention on different properties of the sequences, and we cannot strictly classify the argument as correct or wrong, since they not just depend on whether the response is positive or negative, but on which sequence it refers to.

Moreover, students used very sophisticated strategies. Since in Daniel’s sequence heads and tails are balanced, most students considered that Daniel tossed his coin correctly. Since the number of tails is more imbalanced in the second sequence, most students argued that Diana is cheating. These students computed the frequencies of heads or tails in the given sequence and compared these with the expected frequencies in a theoretical model of equiprobability. When the observed frequencies were too different from their expectation, they rejected the hypothesis that the sequence was random.

Other students analyzed the runs length. When finding a run of four or five consecutive similar results, some of them considered the sequence was not random (30% in Diane’s sequence). An underlying mechanism here is the **negative recency** explained by representativeness heuristic (Kahneman et al. 1982), where people expect a sample of random results to be very similar to the generating process, even in a short series of trials. Other possible strategies to solve this problem are:

- Analyzing the order of heads and tails and deciding that the sequence is too regular to be random (35% in Daniel’s sequence and 6% in Diana’s). Again representativeness heuristics is shown in this response;
- Suggesting we cannot decide if the child is cheating, since random results are unpredictable; here students reason according to the outcome approach described by Konold (1989; 1995) (31% in
These strategies are quite different from those of professional statisticians when solving a similar problem. Though in students' responses we see some intuitive elements of hypothesis testing (theoretical probability, expected frequencies in the case of a true hypothesis, observed frequencies, ...), professional statisticians employ these elements in an explicit and formal way. For example, to compare observed and expected frequencies, they would apply a goodness of fit test, choosing a significance level, and, by using the Chi squared critical values, they would take an objective decision, as regards rejecting or not the sequence's randomness.

The number and types of strategies of professional statisticians are more complex and complete than those of students. Whilst a student can think he/she has "proven" that a sequence is random, the statistician would just say that there is no reason for rejecting the sequence randomness. Moreover, some criteria subjectively used by students to reject randomness are different from what statistics suggest. For example, some students assumed Diana’s sequence is not random because it contains too long runs, while the runs are not long enough to rule out randomness with a statistical test.

4.4. Complexity of randomness

When the previous analysis is finished, we can discuss with the teachers the epistemological nature of randomness. We continue the activity with the following questions:

**Question 8.** How can we define randomness? Do you think it is possible to find "absolute" randomness? Is randomness a property of some phenomena or is it a model to analyze them?

**Question 9.** How can we know with certainty that a dice or a coin, is producing random results?

There is not a general accepted definition of randomness and mathematical definitions do not allow us to determine with certainty whether a given sequence is random or not. Fine (1971) describes different mathematical definitions of randomness e.g., using computational complexity and selection algorithms.

Furthermore, 'randomness' have different meaning for various people and in different contexts. In secondary school teaching, a sequence is random only if it is produced by a random process. A statistician interested in building an algorithm to generate random numbers, can separate the random process from the random sequence, since he/she uses a determinist algorithm to generate randomness.

According to Kyburg (1974) randomness is composed of the four following terms:

- the object that is supposed to be a random member of a class;
- the set of which the object is a random member (population or collective);
- the property with respect to which the object is a random member of the given class;
- the knowledge of the person giving the judgment of randomness.

For Kyburg, what is considered to be random depends on our knowledge. For example, if I have tossed a die and I observe the result obtained, this result is no longer random for me, though it may be random for another person that does not know the result. A child would think that the decimal figures of number Pi are random, due to their disorganized appearance. Furthermore, psychological research shows that, as a rule, subjects assign more alternations to different results than what is theoretically expected in random sequences ('gambler fallacy').

In fact, it is preferable to consider randomness as a mathematical model that we apply to understand some
situations, and not as a property of these situations. As pointed out by McLean (2001), in relation to statistical modeling, a model is only “true” internally, and the purpose in creating any model is to achieve some understanding of a particular phenomena and hence to gain some control over it.

5. Situation B: Winning Strategy in a Probabilistic Game

In the second activity we invite teachers to play a game and find the best possible strategy to win. The game is based on Bertrand's paradox and serves to compare the frequentist and Laplace’s conceptions of probability, and to reflect on the concepts of dependent experiments and conditional probability, fundamental stochastic ideas (Heitele 1975) and the role of problem-solving in the construction of mathematical knowledge.

5.1. Introducing the game

We propose the teachers to experiment themselves a didactic situation that has been designed to teach probability at secondary level. The teachers would play the role of students and the teachers trainer would take the teacher's place. We start the activity with the following game:

**Game:** We take three counters of the same shape and size. One is blue on both sides, the second is red on both sides and the third is blue on one side and red on the other.

We put the three counters into a box, and shake the box, before selecting a counter at random.

After selecting the counter we show one of the sides. The aim of the game is to guess the color of the hidden side. We repeat the process, putting the counter again in the box after each new extraction. We make predictions about the hidden side color and win a point each time our prediction is right.

Once the teachers understand these rules, and after doing some trials, we ask them to find the strategy that produces the better chances to win over a long series of trials. Each teacher will think of a strategy individually, since this is a competition. The work is organized in the stages described below.

5.2. Starting the game and looking for the best strategy

We make 10 trials. In each trial the teachers educator pick a counter at random and show the teachers one side. The teachers write down the visible side color and then the color predicted for the hidden side (See the recording sheet in Figure 1). After the teachers make their predictions, we show them the hidden side and they write down the color. When we finish the first series of ten drawings, each teacher compares his/her score with that of his/her colleagues and we find out who the winners are.

| Trial | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| Seen color |   |   |   |   |   |   |   |   |   |    |
| Predicted color |   |   |   |   |   |   |   |   |   |    |
| Hidden color |   |   |   |   |   |   |   |   |   |    |

Figure 1. Recording the game results
5.3. Analysis of strategies and new trials

We give the teachers some time to think and discuss their strategies (if any) and to describe them on a sheet of paper. The teachers work in pairs. Common strategies in this game are:

A: Predicting red and blue alternatively;

B: Choosing red (blue) in all the trials

C: Predicting the color at random;

D: Two blues, one red (or something similar)

E: Predicting the color shown on the visible side

F: Predicting the color contrary to that shown on the visible side.

Once the teachers have described their strategies, we play 10 more trials, in which each teacher can choose one of the strategies above. The aim is to give the teachers the opportunity of expressing their ideas and check their conjectures. Results are compared and, when necessary, this phase is repeated to increase the total number of experiments. Generally, when increasing the number of trials, some of the strategies are discarded, because results contradict the teachers' initial expectations. Finally, the teachers have a clear preference for one or several favorite strategies, though some of them might keep to a wrong strategy, by thinking that failure is due to the random nature of the experiment.

5.4. Proving which is the best strategy

The final strategies are written on the blackboard. Different teams support each strategy and discuss their arguments in favor or against them. We ask the teachers to describe their reasoning in deciding their strategy is the best. In this discussion, both correct reasoning and possible misconceptions are revealed.

Another discussion is where it is possible to prove that one strategy is optimal using only empirical results. Even when a given strategy has won in a particular series of trials, this does not prove that such a strategy will be optimal in future trials. On the other hand, and even when a simple analysis with tree diagrams or listing the sample space shows that E is the best strategy, this does not mean that in a series of 10, 20, ... 50 games, strategy E will certainly win, due to the game random nature. However, a rational person would prefer the systematic use of strategy E.

5.5. Didactic reflection

The activity carried out with the teachers until this point can be used at secondary school level to teach some ideas of conditional probability and compound experiments. We have thus shown the teachers an example of a didactic situation, and a teaching pattern in the field of probability. At the same time this activity can reinforce the teachers' probabilistic knowledge.

In the problem posed -looking for the best strategy in a game- we can work with different types of knowledge. It is important that teachers identify the type of reasoning that serves to validate the best strategy and compare empirical confirmation as a support for decision with logical or combinatorial deductive arguments, which can "prove" our solution.
It is possible to apply here the equiprobability principle. The classical Laplace's conception of probability of an event as the 'quotient between the number of cases favorable to that event and the number of possible cases' is useful. However, the teachers should conventionally assume the elementary events symmetry, and independence of successive experiments, which often requires a subjective judgment.

If, instead of using counters, we had used a biased die or a thumbtack, the frequentist conception, where probability is the value to which the event relative frequency approach over a long series of trials, would be preferable. However, it would be necessary to be aware of the oscillation of relative frequency, especially, for a limited number of trials (see Batanero, Henry, and Parzyzc in press for a detailed analysis of different meanings of probability).

Furthermore, once the probabilities of elementary events were known, we could apply probabilistic and combinatorial rules again to calculate more complex probabilities. However, in situations where probability calculus is too complex, simulation allows us to obtain an estimate for the events probabilities, when the number of trials is high enough. Consequently, this game contextualizes the debate between different conceptions of probability, and shows the complementarity of these conceptions.

In this situation - and, moreover, in any simple random situation, fundamental stochastic ideas described by Heitele (1975) appear. Fundamental ideas are implicit through the curriculum from school until university level, at various degrees of formalization. Besides the ideas of event, probability and convergence, Heitele mentions combinatorial operations, addition and multiplication rules, independence conditional probability, random variable, equidistribution and symmetry, expectation and sampling. All of these ideas appear in this game and the analysis of the game will serve to reflect with teachers about which the fundamental stochastic ideas are, and what degree of formalization of the same is appropriate to present these ideas to secondary students.

6. Evaluation

The assessment criteria, including assessment aims, contents and procedures are essential components in teaching. Assessment is a continuous process in our Statistics Education course, and different sources of data, including the teacher’s written answers to some activities are used for this purpose. The goal is providing information about teachers’ learning and the change in their conceptions and adequacy of the activities.

In the following we provide an example, in analyzing the data taken from written solutions to Activity 2 of 49 future teachers taking part in the Statistics Education course in the academic year (2000-2001). The frequencies of strategies considered correct at the beginning (see 5.2) and at the end (see 5.4) of the activity by these teachers are presented in Table 2.

| Strategies considered correct by the future teachers | Number of future teachers choosing the strategy (n=47) |
|-----------------------------------------------------|-----------------------------------------------------|
| Predicting the color shown on the visible side (correct) | Initial stage | Last stage |
|                                                     | 18          | 32         |
It is surprising that less than half the future teachers chose the correct strategy at the beginning of the game, in spite of the apparent simplicity of the problem and their high statistical training (they were majoring in statistics, and in their 4º or 5º year of study). A variety of starting strategies appear and this suggest we achieved our goal of setting a problematic situation that serve these teachers to confront their different solutions and help the debate and reflection.

In the final step the majority of teachers stick to the correct strategy. It is important to notice that they were asked to write and justify their solutions as a part of their homework and therefore had some time to reflect in the problem. We can observe a general positive change in these teachers’ conceptions about the activity and the concepts involved. However about a third of them were still convinced that their own (incorrect) strategies were better than that of their classmates.

This serve to start the phase of debating different solutions by the teachers. The aim was not just learning probability (although this was also achieved), but to make teachers aware of their own probability misconceptions that possibly they will later find in their own students. The analysis of arguments in favor of each strategy serves to make these conceptions emerge, as well as to analyze the statistical concepts involved in the task. Teachers were asked to discover by themselves the errors underlying their classmates arguments. Some examples are given below.

"Taking the color contrary to that shown in the visible side, since the probability that the hidden side is same color is smaller. That is, the probability of same color is 2/5 and the probability of different color is 3/5".

"I think there is no valid strategy, since the probability is the same for the two colors, since each trial is independent of the other trials and the experiment is random".

"Because we do not change the probability for the different colors when showing a side".

"I think alternation is easier, since there is the same number of sides for each color: 3 possibilities for red and 3 possibilities for blue".

Even when most of the teachers finally got the correct solution, not all of them were able to provide a sound mathematical proof of the same. The following responses and other similar reasoning served to make the teachers reflect about the nature and types of mathematical proof, as well as about the evidence provided by theoretical and empirical probability:

"I bet to the color on the visible side. I recorded my results and experimentally got more successes than failures, so I should be right, but I cannot find a mathematical explanation".
“At the beginning I bet to the contrary color, but I was wrong, since the probability to get a card with two equal sides is 2/3”.

“The sample space in this experiment is \{(BB), (BR), (RB), (RR)\} (note that these are the combinations of 2 elements with repetition). The probabilities of these events are: P(BB) = 1/3; P(BR)=1/6; P(RB)=1/6; P(RR)=1/3; P(S)= P(BB)+ P(RR)=2/3”.

“P(BB) = P(B). P(B/B)=1/2 . 2/3; P(RR)= P(R). P(R/R)=1/2 . 2/3
P(BR)= P(B). P(R/B)=1/2 . 1/3; P(RB)= P(R). P(B/R)=1/2 . 1/3 and then it is more likely to have the same color in both sides. I also did an empirical checking with 204 trials and these are my results:

| Same color | Different Color |
|------------|-----------------|
| Observed   | 138             | 66             |
| Expected   | 132             | 68             |

\[ \chi^2 = \frac{(138 - 132)^2}{132} + \frac{(66 - 68)^2}{68} = 0.3325 \]

and this is a confirmation”.

In addition to this continuous evaluation, the future teachers were given a final exam. They should also develop a didactical unit for an optional statistical topic and educational grade.

The exam provided us information about the theoretical knowledge acquired by the future teachers as regards epistemology, psychology and didactics of statistics. Although performance was very varied, our general impression is that they acquire some knowledge that will be useful in their future professional work, either as statistics teachers or as consultants.

The didactic units prepared by the future teachers were also very variable as regards content and quality. While some of them preferred to prepare a teaching unit for primary or secondary school level, the majority developed a theme for the basic statistics course at University level. Some of them used or even elaborated some didactic software; in some cases they presented their work in a local education congress. Finally we recognize that the course length is too short and should be expanded in order that these future teachers achieve a real competence in planning and writing these didactic units.

7. Final Remarks

The examples described are complementary from different perspectives. As regards randomness, the first situation starts out from an experiment that has already been carried out, and randomness must be judged after data has been obtained (a posteriori statistical study of the experiment). In the second situation, the subject is asked to predict the experiment results, after analyzing the problem-situation structure (a priori probabilistic study of the experiment). The two situations also show examples of different visions of stochastics:

- The formal vision of stochastical knowledge, which serves to validate the best strategy in the game using an existing mathematical theory, in this case, combinatorics.
- The empirical vision, which emphasizes the role of experimenting in probability, and the type of validation that it provides: a mathematical solution (a strategy) is validated through statistical
knowledge, when its provides better results in the long run.

- The *structural - analytical* vision: the trials sequence is judged to be random through the analysis of its structure and comparison with properties of the probability model expected.

The student is less involved in the first activity (to decide if another child was cheating) than in the second one (to play and choose a strategy for winning a game). This second activity also shows the difficulty of matching students' attitudes and intuitions with their formal thinking, in the field of probability.

An important conclusion is that the didactic training of teachers must show them how to carry out didactic analyses similar to those presented here. This type of analysis should be a main component of teacher training courses, from the stochastical and didactic point of view.

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