Explicit formulas for isoperimetric deformations of smooth and discrete elasticae

Shota Shigetomi\textsuperscript{1*} and Kenji Kajiwara\textsuperscript{2}

\textsuperscript{1} Graduate School of Mathematics, Kyushu University, 744 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan
\textsuperscript{2} Institute of Mathematics for Industry, Kyushu University, 744 Motoooka, Nishi-ku, Fukuoka 819-0395, Japan

*Corresponding author: 3MA20001N@s.kyushu-u.ac.jp

Received September 01, 2021, Accepted October 11, 2021

Abstract

We construct the explicit formula for the isoperimetric deformation of elastica described by the modified KdV equation. We also construct the explicit formulas for the continuous and discrete deformations of the discrete analogue of elastica described by the semi-discrete potential modified KdV equation and the discrete potential modified KdV equation, respectively. The formulas are given in terms of the elliptic theta functions.

Keywords

Euler’s elastica, discrete curve, discrete differential geometry

Research Activity Group

Applied Integrable Systems

1. Introduction

Elastica is a well-known class of planar curves that describes the shape of thin elastic rods [1–3]. Mathematically, it is characterized by the differential equation for its curvature

\[
\kappa_{xx} + \frac{1}{2} \kappa^3 + \lambda \kappa = 0, \quad \lambda \in \mathbb{R},
\]

where \( \kappa = \kappa(x) \) and \( x \) is the arc length. On the other hand, it is well known that the modified KdV (mKdV) equation

\[
\kappa_{xxx} + \frac{3}{2} \kappa^2 \kappa_x + \kappa_t = 0.
\]

describes an isoperimetric deformation of planar curves [4, 5]. The explicit formulas for the isoperimetric deformations of curves have been constructed in terms of the \( \tau \) functions [5]. Since the traveling wave solutions to the modified KdV equation satisfy the equation for elastica (1), it is possible to construct explicit formula for the isoperimetric deformation of elastica.

In this paper, we construct an explicit formula for the isoperimetric deformation of elastica described by the modified KdV equation. Thanks to its integrability, it is also possible to construct a discrete version of elastica characterized by the integrable discrete analogue of the modified KdV equation [6–8]. We also construct the explicit formulas to both continuous and discrete deformations of the discrete elastica described by the semi-discrete potential mKdV equation and the discrete potential mKdV equation, respectively. The formulas can be regarded as the extended versions of the results by Mumford [9] and Matsuura [10].

2. Isoperimetric deformations of planar curves

We first summarize the formulations of three kinds of planar curve deformations [5, 11, 12].

2.1 Continuous deformation of smooth curves

Let \( \gamma(x) \in \mathbb{R}^2 \) be an arc length parameterized curve in Euclidean plane \( \mathbb{R}^2 \), and \( x \) be the arc length. Then the Frenet equation of \( \gamma \) is

\[
\gamma_{xx} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \gamma_x
\]

and the function \( \kappa \) is the curvature of \( \gamma \). We consider the following deformation, where \( t \) is the deformation parameter [3, 4],

\[
\frac{\partial}{\partial t} \gamma_x = \begin{pmatrix} 0 & \kappa_x \\ -\kappa_x & 0 \end{pmatrix} \kappa_x + \frac{1}{2} \kappa^3 \right) \gamma_x. \]

Then the potential function \( \theta(x,t) \) defined by \( \kappa = \theta_x \) satisfies the potential mKdV equation

\[
\theta_{xxx} + \frac{1}{2} (\theta_x)^3 + \theta_t = 0,
\]

so that \( \kappa \) satisfies the mKdV equation

\[
\kappa_{xxx} + \frac{3}{2} \kappa^2 \kappa_x + \kappa_t = 0.
\]

The function \( \theta \) is also called the angle function, as \( \gamma_x \) can be expressed as

\[
\gamma_x = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.
\]
2.2 Continuous deformation of discrete curves

Let \( \gamma_n \in \mathbb{R}^2 \) \((n \in \mathbb{Z})\) be a discrete plane curve with constant segment length

\[
|\gamma_{n+1} - \gamma_n| = a,
\]

where \(a \in \mathbb{R}\) is a constant. We introduce the angle function \(\Theta_n\) of a discrete curve \(\gamma_n\) by

\[
\frac{\gamma_{n+1} - \gamma_n}{a} = \left(\frac{\cos \Theta_n}{\sin \Theta_n}\right).
\]

A discrete curve \(\gamma_n\) satisfies

\[
\frac{\gamma_{n+1} - \gamma_n}{a} = R(K_n) \frac{\gamma_n - \gamma_{n-1}}{a} \tag{8}
\]

for \(K_n = \Theta_n - \Theta_{n-1}\), where \(R(K_n)\) denotes the rotation matrix given by

\[
R(K_n) = \left(\frac{\cos K_n}{\sin K_n}, \frac{\sin K_n}{\cos K_n}\right).
\]

Now consider the following continuous deformation of discrete curve [11]

\[
\frac{d\gamma_n}{dt} = R(W_n) \frac{\gamma_{n+1} - \gamma_n}{a}. \tag{9}
\]

Compatibility of the system (6), (8) and (9) implies \(W_n = -W_{n-1} - K_n\), so that the potential function \(\theta_n\) is introduced by

\[
K_n = \frac{\theta_{n+1} - \theta_{n-1}}{2}, \quad W_n = \frac{\theta_n - \theta_{n+1}}{2}, \tag{10}
\]

and it follows that \(\theta_n\) satisfies the semi-discrete potential mKdV equation [11]

\[
\frac{d}{dt}(\theta_{n+1} + \theta_{n}) = \frac{4}{a} \sin \left(\frac{\theta_{n+1} - \theta_{n}}{2}\right). \tag{11}
\]

Note that the angle function \(\Theta_n\) can be expressed as

\[
\Theta_n = \frac{\theta_{n+1} + \theta_{n}}{2}.
\]

2.3 Discrete deformation of discrete curves

Next we consider the following discrete deformation of discrete curve [5, 12]

\[
|\gamma_{n+1}^m - \gamma_n^m| = a, \tag{12}
\]

\[
\frac{\gamma_{n+1}^m - \gamma_n^m}{a} = R(K_n^m) \frac{\gamma_n^m - \gamma_{n-1}^m}{a}, \tag{13}
\]

\[
\frac{\gamma_{n+1}^m - \gamma_n^m}{b} = R(W_n^m) \frac{\gamma_n^m - \gamma_{n-1}^m}{a}, \tag{14}
\]

where \(m \in \mathbb{Z}\) is a discrete deformation parameter. Compatibility of the system (12)–(14) implies the existence of the potential function \(\theta_n^m\) defined by

\[
K_n^m = \frac{\theta_{n+1}^m - \theta_{n-1}^m}{2}, \quad W_n^m = \frac{\theta_n^m - \theta_{n+1}^m}{2},
\]

and it follows that \(\theta_n^m\) satisfies the discrete potential mKdV equation

\[
\tan \left(\frac{\theta_n^{m+1} - \theta_n^m}{4}\right) = \frac{b + a}{b - a} \tan \left(\frac{\theta_n^{m+1} - \theta_n^{m+1}}{4}\right). \tag{15}
\]

Note that the angle function \(\Theta_n^m\) can be expressed as

\[
\Theta_n^m = \frac{\theta_{n+1}^m + \theta_n^m}{2}.
\]

3. Explicit formulas in terms of \(\tau\) function

We summarize the explicit formulas of isoperimetric deformations of planar curves in terms of the \(\tau\) functions [5].

3.1 Continuous deformation of smooth curves

Let \(\tau = \tau(x, t; y)\) be a complex function depending on three continuous variables \(x, t\) and \(y\), which satisfy the following system of bilinear equations of Hirota type,

\[
\frac{1}{2} D_x D_y \tau \cdot \tau = -\tau^* \tau^*, \tag{16}
\]

\[
D_x^2 \tau \cdot \tau^* = 0, \tag{17}
\]

\[
(D_x^2 + D_t) \tau \cdot \tau^* = 0, \tag{18}
\]

where \(*\) denotes the complex conjugate. Here, \(D_x, D_y\) and \(D_t\) are Hirota’s bilinear differential operators [13]. Let \(\tau\) be a solution to (16)–(18). Define a real function \(\theta(x, t; y)\) and an \(\mathbb{R}^2\) valued function \(\gamma(x, t; y)\) by

\[
\theta(x, t; y) = \frac{2}{\sqrt{-1} \log \tau^*}, \tag{19}
\]

\[
\gamma(x, t; y) = \left(\frac{1}{\sqrt{-1} \log \tau^*}, \log \frac{\tau^*}{\tau}\right). \tag{20}
\]

Then for any \(x, t, y \in \mathbb{R}\), the functions \(\theta\) and \(\gamma\) satisfy (2)–(4) and (5) [5].

3.2 Continuous deformation of discrete curves

Let \(\tau_n = \tau_n(t; y)\) be a complex function depending on the discrete variable \(n\) and two continuous variables \(t\) and \(y\), which satisfies the following system of bilinear equations

\[
\frac{1}{2} D_t D_y \tau_n \cdot \tau_n = -\tau_n^\ast \tau_n^\ast, \tag{19}
\]

\[
D_y \tau_{n+1} \cdot \tau_n = -a \tau_{n+1}^\ast \tau_n^\ast, \tag{20}
\]

\[
D_t \tau_{n+1} \cdot \tau_n^\ast = -\frac{1}{a} \tau_{n+1} \tau_n^\ast. \tag{21}
\]

**Theorem 1** Let \(\gamma_n\) be a solution to eqs. (19)–(21). Define a real function \(\theta_n(t; y)\) and an \(\mathbb{R}^2\) valued function \(\gamma_n(t; y)\) by

\[
\theta_n(t; y) = \frac{2}{\sqrt{-1} \log \tau_n}, \tag{22}
\]

\[
\gamma_n(t; y) = \left(\frac{1}{\sqrt{-1} \log \tau_n}, \log \frac{\tau_n^\ast}{\tau_n}\right). \tag{23}
\]

Then for any \(t, y \in \mathbb{R}\) and \(n \in \mathbb{Z}\), the functions \(\theta_n\) and \(\gamma_n\) satisfy (7)–(9) and (11).

**Proof** Express \(\gamma_n = \ell(X_n, Y_n)\). From (20) we have

\[
\left(\log \frac{\tau_{n+1}}{\tau_n}\right)_y = -\frac{a \tau_{n+1} \tau_n^\ast}{\tau_{n+1} \tau_n^\ast}. \tag{24}
\]
Adding (24) and its complex conjugate, we obtain by using (22) and (23)
\[
\frac{X_{n+1} - X_n}{a} = \cos \Theta_n, \quad \Theta_n = \frac{\theta_{n+1} + \theta_n}{2}.
\]
Subtracting the complex conjugate of (24) from (24), we have
\[
\frac{Y_{n+1} - Y_n}{a} = \sin \Theta_n.
\]
Then we obtain
\[
\frac{\gamma_{n+1} - \gamma_n}{a} = \left( \frac{\cos \Theta_n}{\sin \Theta_n} \right),
\]
which gives eq. (7). It is easy to see that (8) follows from (25). In order to show (9), it is convenient to identify \(\mathbb{R}^2\) as \(\mathbb{C}\). Then by using (10) and (25), we see that (9) is rewritten as
\[
\frac{d\gamma_n}{dt} = e^{\sqrt{-1}w_n} \frac{\gamma_{n+1} - \gamma_n}{a} = e^{\sqrt{-1}\theta_n}.
\]
Noticing that
\[
\gamma_n = X_n + \sqrt{-1}Y_n = (\log \tau_n^* y),
\]
the left hand side of (26) can be rewritten by using (19) as
\[
\frac{d\gamma_n}{dt} = - (\log \tau_n^*) y = - \frac{1}{2} D_1 D_2 \tau_n^* \cdot \tau_n^* = e^{\sqrt{-1}\theta_n},
\]
which implies (26) and thus (9). Finally, by using (21) and (22), we see that
\[
\frac{d}{dt} \left( \theta_{n+1} + \theta_n \right) = \frac{2}{\sqrt{-1}} \frac{d}{dt} \left( \log \frac{\tau_{n+1}}{\tau_n} \right) = \frac{2}{\sqrt{-1}} \left( \frac{D_1 \tau_{n+1} \cdot \tau_n}{\tau_{n+1} \tau_n} - \frac{D_2 \tau_n}{\tau_{n+1} \tau_n} \right)
\]
\[
= \frac{2}{\sqrt{-1}} \left( \frac{\tau_{n+1} \tau_n}{\tau_{n+1} \tau_n} - \frac{\tau_n}{\tau_{n+1} \tau_n} \right)
\]
\[
= \frac{4}{a} \sin \left( \theta_{n+1} - \theta_n \right),
\]
which is the semi-discrete potential mKdV equation. □ (QED)

3.3 Discrete deformation of discrete curves

Let \(\tau_n^m = \tau_n^m(y)\) be a complex function depending on two discrete variables \(n, m\) and one continuous variable \(y\), which satisfies the following bilinear equations
\[
D_y \tau_{n+1}^m = -ar_{n+1}^m \tau_n^m, \quad (27)
\]
\[
D_y \tau_{n+1}^m = -br_{n+1}^m \tau_n^m, \quad (28)
\]
\[
b \tau_{n+1}^m \tau_n^m = -ar_{n+1}^m \tau_n^m, \quad (29)
\]
Let \(\tau_n^m\) be a solution to eqs. (27)–(29). Define a real function \(\theta_n^m(y)\) and an \(\mathbb{R}^2\) valued function \(\gamma_n^m(y)\) by
\[
\theta_n^m(y) = \frac{1}{2} \log \frac{\tau_n^m}{\tau_n^{*m}},
\]
\[
\gamma_n^m(y) = \left( \frac{\tau_n^m \tau_n^{*m}}{\tau_n^{*m} \tau_n^m} \right).
\]
Then for any \(y \in \mathbb{R}\) and \(m \in \mathbb{Z}\), the functions \(\theta_n^m\) and \(\gamma_n^m\) satisfy (12)–(14) and (15) [5].

4. Explicit Solutions

In this section, we construct the solutions to the bilinear equations in terms of the elliptic theta functions. Let \(H_i\) be the upper half plane and \(\vartheta_i(z) = \vartheta_i(z|\omega) (z \in \mathbb{C}, \omega \in \mathbb{H}, i = 1, 2, 3, 4)\) are the elliptic theta functions (see, for example, [14,15]). We define constants \(\omega_1, \omega_2 \in \mathbb{C}\) as
\[
\omega_1 = \sqrt{-1}r, \quad \omega_2 = \frac{1}{2} + \sqrt{-1}r, \quad r \in \mathbb{R}_{>0}.
\]

4.1 Continuous deformation of elastics

(1) Let \(\omega = \omega_1\). Define
\[
h_1 = \frac{\vartheta_3(0)}{\vartheta_3(0)} + \frac{\vartheta_4(0)}{\vartheta_4(0)} + \frac{\vartheta_3(0)}{\vartheta_3(0)}, \quad c_1 = \frac{\vartheta_4(0)}{\vartheta_4(0)},
\]
\[
Q_1 = \frac{1}{4} \left( \frac{\vartheta_1(0)}{\vartheta_1(0)} + \frac{\vartheta_3(0)}{\vartheta_3(0)} \right),
\]
\[
q_1 = 3 \frac{\vartheta_3(0)}{\vartheta_3(0)} + 3 \frac{\vartheta_4(0)}{\vartheta_4(0)} - \frac{\vartheta_3(0)}{\vartheta_3(0)} - 3 \frac{\vartheta_4(0)}{\vartheta_4(0)}.
\]

(2) Let \(\omega = \omega_2\). Define
\[
h_2 = \frac{\vartheta_3(0)}{\vartheta_3(0)} - \frac{\vartheta_4(0)}{\vartheta_4(0)} + \frac{\vartheta_3(0)}{\vartheta_3(0)}, \quad c_2 = \frac{\vartheta_4(0)}{\vartheta_4(0)},
\]
\[
Q_2 = \frac{1}{4} \left( \frac{\vartheta_1(0)}{\vartheta_1(0)} + \frac{\vartheta_3(0)}{\vartheta_3(0)} \right),
\]
\[
q_2 = 3 \frac{\vartheta_3(0)}{\vartheta_3(0)} + 3 \frac{\vartheta_4(0)}{\vartheta_4(0)} - \frac{\vartheta_3(0)}{\vartheta_3(0)} - 3 \frac{\vartheta_4(0)}{\vartheta_4(0)}.
\]

Theorem 2 Let \(\omega = \omega_1\). Consider the \(\tau\) function
\[
\tau = \exp \left[ \frac{\pi i}{2} \left( z_{11} - \frac{\omega}{4} \right) - h_1 c_1 xy + Q_1 p^2 x^2 \right] \times \vartheta_3(z_{11}),
\]
\[
z_{11} = px + qy + \frac{c_1}{p} y - \frac{1}{2} + \frac{\omega}{4},
\]
where \(p \neq 0\) is a parameter. Then, (30) and its complex conjugate satisfy the bilinear equations (16)–(18).

Theorem 3 Let \(\omega = \omega_2\). Consider the \(\tau\) function
\[
\tau = \exp \left[ \frac{\pi i}{4} - h_2 c_2 xy + Q_2 p^2 x^2 \right] \vartheta_3(z_{12}),
\]
\[
z_{12} = px + qy + \frac{c_2}{p} y + \frac{1}{4}.
\]
Then, (31) and its complex conjugate satisfy the bilinear equations (16)–(18).

4.2 Continuous deformation of discrete elastics

(1) Let \(\omega = \omega_1\). Define
\[
\epsilon_1(\alpha) = \frac{\vartheta_1(0) \vartheta_3(0)}{\vartheta_1(0) \vartheta_3(0)}, \quad \delta_1(\alpha) = \frac{\vartheta_4(0) \vartheta_4(0)}{\vartheta_4(0) \vartheta_4(0)},
\]
\[
a_1(\alpha) = \frac{1}{p} \vartheta_3(0),
\]
Let \( \alpha \in \mathbb{R}\setminus(1/2)\mathbb{Z} \) is a parameter.

(2) Let \( \omega = \omega_2 \). Define
\[
\varepsilon_2(\alpha) = \frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)} - \vartheta_1(\alpha) = \vartheta_2(0) \vartheta_1'(\alpha) - \vartheta_1(\alpha) \vartheta_2(0) \vartheta_1(\alpha),
\]
\[
a_2(\alpha) = \frac{1}{p} \frac{\vartheta_1'(\alpha)}{\vartheta_1(\alpha)}.
\]

**Theorem 4** Let \( \omega = \omega_1 \). Consider the \( \tau \) function
\[
\tau_n = \exp \left[ \frac{\pi i}{2} \left( z_{21} - \frac{\omega}{4} \right) - h_1 c_1 t y \right] \times \exp \left[ -n \left( \varepsilon_2(\alpha) p + \delta_2(\alpha) \frac{y}{p} \right) \right] \vartheta_3(3z_{21}),
\]
with \( \omega = a_1(\alpha) \).

Then, (32) and its complex conjugate satisfy the bilinear equations (19)–(21) with \( a = a_1(\alpha) \).

**Theorem 5** Let \( \omega = \omega_2 \). Consider the \( \tau \) function
\[
\tau_n = \exp \left[ \frac{\pi i}{4} - b_2 c_2 t y - n \left( \varepsilon_2(\alpha) p + \delta_2(\alpha) \frac{y}{p} \right) \right] \times \vartheta_3(z_{22}),
\]
with \( \omega = a_2(\alpha) \).

Then, (33) and its complex conjugate satisfy the bilinear equations (19)–(21) with \( a = a_2(\alpha) \).

### 4.3 Discrete deformation of discrete elasticA

(1) Let \( \omega = \omega_1 \). Define
\[
c_1(\alpha, \beta) = \frac{1}{p} \vartheta_4(0) \vartheta_1(\alpha - \beta),
\]
where \( \beta \in \mathbb{R}\setminus(1/2)\mathbb{Z} \) is a parameter.

(2) Let \( \omega = \omega_2 \). Define
\[
c_2(\alpha, \beta) = \frac{1}{p} \vartheta_4(0) \vartheta_1(\alpha - \beta). \]

**Theorem 6** Let \( \omega = \omega_1 \). Consider the \( \tau \) function
\[
\tau_n^m = \exp \left[ \frac{\pi i}{4} \left( z_{31} - \frac{\omega}{4} \right) - (\delta_1(\alpha)n + \delta_1(\beta)m) \frac{y}{p} \right] \vartheta_3(z_{31}),
\]
with \( \omega = a_1(\beta) \), \( b = a_1(\alpha) \) and \( c = c_1(\alpha, \beta) \).

Then, (34) and its complex conjugate satisfy the bilinear equations (27)–(29) with \( a = a_1(\beta) \), \( b = a_1(\alpha) \) and \( c = c_1(\alpha, \beta) \).

**Theorem 7** Let \( \omega = \omega_2 \). Consider the \( \tau \) function
\[
\tau_n^m = \exp \left[ \frac{\pi i}{4} - (\delta_2(\alpha)n + \delta_2(\beta)m) \frac{y}{p} \right] \vartheta_3(z_{32}),
\]
with \( \omega = a_2(\beta) \), \( b = a_2(\alpha) \) and \( c = c_2(\alpha, \beta) \).

For the proof of the above results, we use identities of elliptic theta functions. Here, we show an example of such identities \([14, 15]\)
\[
2\vartheta_3(x_1) \vartheta_3(y_1) \vartheta_2(u_1) \vartheta_2(v_1) = \vartheta_3(x) \vartheta_3(y) \vartheta_2(u) \vartheta_2(v) + \vartheta_4(x) \vartheta_4(y) \vartheta_1(u) \vartheta_1(v)
+ \vartheta_2(x) \vartheta_2(y) \vartheta_3(u) \vartheta_3(v) + \vartheta_1(x) \vartheta_1(y) \vartheta_4(u) \vartheta_4(v),
\]
where \( x, y, u, v \in \mathbb{C} \) and
\[
x_1 = \frac{1}{2} (x + y + u + v), \quad y_1 = \frac{1}{2} (x + y - u - v),
\]
\[
u_1 = \frac{1}{2} (x - y + u - v), \quad v_1 = \frac{1}{2} (x - y - u + v).
\]

The detailed derivation will be given in the forthcoming paper.

**Acknowledgments**

The author thanks Professor Nozomu Matsuura for his helpful comments and discussions. This work was supported by JST CREST No. JPMJCR1911 and JSPS KAKENHI No.21K03329.

**References**

[1] D. A. Singer, Lectures on elastic curves and rods, Curvature and variational modeling in physics and biophysics, AIP Conf. Proc., 1002 (2008), 3–32.
[2] S. Matsuani, Euler’s elastica and beyond, J. Geom. Symmetry Phys., 17 (2010), 45–86.
[3] J. Inoguchi, Plane Curves and Solitons (in Japanese), Asakura Publishing, Tokyo, 2010.
[4] R. E. Goldstein and D. M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett., 67 (1991), 3203–3206.
[5] J. Inoguchi, K. Kajiwara, N. Matsuura and Y. Ohta, Motion and Bäcklund transformations of discrete plane curves, Kyushu J. Math., 66 (2012), 303–324.
[6] A. I. Bobenko and Yu. B. Suris, Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top, Comm. Math. Phys., 204 (1999), 147–188.
[7] K. Sogo, Variational discretization of Euler’s elastica problem, J. Phys. Soc. Jpn., 75 (2006), 064007.
[8] S. E. Graiff Zurita and K. Kajiwara, Fairing of discrete planar curves by discrete Euler’s elasticae, JSIAM Lett., 11 (2019), 73–76.
[9] D. Mumford, Elastica and computer vision, in: Algebraic Geometry and Its Applications, C. L. Bajaj eds., pp. 491–506, Springer-Verlag, New York, 1994.
[10] N. Matsuura, Explicit formula for planar discrete elasticae (in Japanese), Proc. MSJ Spring Meeting 2020, 2020.
[11] S. Kaji, K. Kajiwara and H. Park, Linkage mechanisms governed by integrable deformations of discrete space curves, in: Nonlinear Systems and Their Remarkable Mathematical Structures Vol. 2, N. Euler and M. C. Nucci eds., pp. 356–381, CRC Press, Boca Raton, 2019.
[12] N. Matsuura, Discrete KdV and discrete modified KdV equations arising from motions of discrete planar curves, Int. Math. Res. Not., 2012 (2012), 1681–1698.
[13] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[14] D. Mumford, Tata Lectures on Theta I, Birhäuser, Basel, 2007.
[15] S. Kharchev and A. Zabrodin, Theta vocabulary I, J. Geom. Phys., 94 (2015), 19–31.