THE LOWEST-ORDER STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD

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Abstract. Recently, a new stabilizer free weak Galerkin method (SFWG) is proposed, which is easier to implement and more efficient. The main idea is that by letting \( j \geq j_0 \) for some \( j_0 \), where \( j \) is the degree of the polynomials used to compute the weak gradients, then the stabilizer term in the regular weak Galerkin method is no longer needed. Later on in [1], the optimal of such \( j_0 \) for certain type of finite element spaces was given. In this paper, we propose a new efficient SFWG scheme using the lowest possible orders of piecewise polynomials for triangular meshes in 2D with the optimal order of convergence.

Key words. stabilizer free, weak Galerkin finite element methods, lowest-order finite element methods, weak gradient, error estimates.

AMS subject classifications. Primary: 65N15, 65N30; Secondary: 35J50

1. Introduction. In this paper, we are concerned with the development of an SFWG finite element method using the following Poisson equation

\[
\begin{align*}
-\Delta u &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

as the model problem, where \( \Omega \) is a polygonal domain in \( \mathbb{R}^2 \). The variational formulation of the Poisson problem (1.1)-(1.2) is to seek \( u \in H^1_0(\Omega) \) such that

\[
(\nabla w, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega).
\]

The standard weak Galerkin (WG) method for the problem (1.1)-(1.2) seeks weak Galerkin finite element solution \( u_h = \{u_0, u_b\} \) such that

\[
(\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v),
\]

for all \( v = \{v_0, v_b\} \) with \( v_b = 0 \) on \( \partial \Omega \), where \( \nabla_w \) is the weak gradient operator and \( s(u_h, v) \) in (1.4) is a stabilizer term that ensures a sufficient weak continuity for the numerical approximation. The WG method has been developed and applied to different types of problems, including convection-diffusion equations [7, 6], Helmholtz equations [9, 12, 5], Stokes flow [11, 10], and biharmonic problems [8]. Recently, Al-Taweel and Wang in [2], proposed the lowest-order weak Galerkin finite element method for solving reaction-diffusion equations with singular perturbations in 2D. One of major sources of the complexities of the WG methods and other discontinuous finite element methods is the stabilization term.

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A stabilizer free weak Galerkin finite element method is proposed by Ye and Zhang in [3] as a new method for the solution of the Poisson equation on polytopal meshes in 2D or 3D, where \((P_k(T), P_k(e), [P_j(T)]^2)\) elements are used. It is shown that there is a \(j_0 > 0\) so that the SFWG method converges with optimal order of convergence for any \(j \geq j_0\). However, when \(j\) is too large, the magnitude of the weak gradient can be extremely large, causing numerical instability. In [4], the optimal \(j_0\) is given to improve the efficiency and avoid unnecessary numerical difficulties. In this setting, if \((P_k(T), P_k(e), [P_j(T)]^2)\) elements are used for a triangular mesh, \(j_0 = k + 1\), where \(k \geq 1\). In this paper, we propose a scheme using \((P_0(T), P_1(e), [P_1(T)]^2)\) elements for triangular meshes with the optimal order of convergence, which is more efficient than using the regular WG method \((P_0(T), P_1(e), [P_1(T)]^2)\) elements.

The goal of this paper is to develop the theoretical foundation for using the lowest-order SFWG scheme to solve the Poisson equation (1.1)-(1.2) on a triangle mesh in 2D. The rest of this paper is organized as follows. In Section 2 the notations and finite element space are introduced. Section 3 is devoted to investigating the error equations and several other required inequalities. The main error estimate is studied in Section 4. In Section 5 we will derive the optimal order \(L^2\) error estimates for the SFWG finite element method for solving the equations (1.1)-(1.2). Several numerical tests are presented in Section 6. Conclusions and some future research plans are summarized in Section 7.

2. Notations. In this section, we shall introduce some notations, and definitions.

Suppose \(\mathcal{T}_h\) is a quasi uniform triangular partition of \(\Omega\). For every element \(T \in \mathcal{T}_h\), denote \(h_T\) as its diameter and \(h = \max_{T \in \mathcal{T}_h} h_T\). Let \(\mathcal{E}_h\) be the set of all the edges in \(\mathcal{T}_h\). The weak Galerkin finite element space is defined as follows:

\[
V_h = \{ (v_0, v_b) : v_0 \in P_0(T), \forall T \in \mathcal{T}_h, \text{ and } v_b \in P_1(e), \forall e \in \mathcal{E}_h \}.
\]

In this instance, the component \(v_0\) symbolizes the interior value of \(v\), and the component \(v_b\) symbolizes the edge value of \(v\) on each \(T\) and \(e\), respectively. Let \(V_h^0\) be the subspace of \(V_h\) defined as:

\[
V_h^0 = \{ v : v \in V_h, v_b = 0 \text{ on } \partial \Omega \}.
\]

For each element \(T \in \mathcal{T}_h\), let \(Q_0\) be the \(L^2\)-projection onto \(P_0(T)\) and let \(Q_1(h)\) be the \(L^2\)-projection onto \([P_1(T)]^2\). On each edge \(e\), denote by \(Q_0\) the \(L^2\)-projection operator onto \(P_1(e)\). Combining \(Q_0\) and \(Q_0\), denote by \(Q_h = \{Q_0, Q_1\}\) the \(L^2\)-projection operator onto \(V_h\).

For any \(v = \{v_0, v_b\} \in V_h\), the weak gradient \(\nabla_w v \in [P_1(T)]^2\) is defined on \(T\) as the unique polynomial satisfying

\[
(\nabla_w v, \bar{q})_T = -(v_0, \nabla \cdot \bar{q})_T + \langle v_b, \bar{q} \cdot \vec{n} \rangle_{\partial T}, \quad \forall \bar{q} \in [P_1(T)]^2,
\]

where \(\vec{n}\) is the unit outward normal vector of \(\partial T\).

For simplicity, we adopt the following notations,

\[
(v, w)_T = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T v w dx,
\]
\[ \langle v, w \rangle_{\partial T_h} = \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} w \, dx. \]

**Stabilizer free Weak Galerkin Algorithm 1.** A numerical solution for (1.1)-(1.2) can be obtained by finding \( u_h = \{v_0, v_b\} \in V_h^0 \), such that the following equation holds

(2.4) \[ (\nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_h^0. \]

We define an energy norm \( \| \cdot \| \) on \( V_h \) as:

(2.5) \[ \| v \|^2 = \sum_{T \in \mathcal{T}_h} (\nabla_w v, \nabla_w v)_T. \]

An \( H^1 \) semi norm on \( V_h \) is defined as:

\[ \| v \|_{1, h}^2 = \sum_{T \in \mathcal{T}_h} \left( \left\| \nabla v_0 \right\|_T^2 + h_T^{-1} \| v_0 - v_b \|^2_{\partial T} \right). \]

**Remark 1.** \( \nabla v_0 \) in the above definition is simply a placeholder, since \( \nabla v_0 |_{\partial T} = 0 \) for all \( T \in \mathcal{T}_h \).

**Lemma 2.1.** There exists \( C > 0 \) so that

\[ \| v \| \leq C \| v \|_{1, h}, \quad \forall v \in V_h. \]

**Proof.** For any \( T \in \mathcal{T}_h \) and \( \tilde{q} \in [P_1(T)]^2 \), it follows from integration by part, trace inequality, and inverse inequality that

\[ (\nabla_w v, \tilde{q})_T = (\nabla v, \tilde{q})_T + (v_0 - v_b, \tilde{q} \cdot n)_{\partial T} \]

\[ \leq \| \nabla v \|_T \| \tilde{q} \|_T + C h_T^2 \| v_0 - v_b \|_{\partial T} \| \tilde{q} \|_T. \]

Letting \( \tilde{q} = \nabla_w v \) yields the result. \( \square \)

The following lemma is one of the major results of this paper.

**Lemma 2.2.** For any \( v \in V_h \), if \( \nabla_w v|_{T_i} \in [P_{k+1}(T_i)]^2 \), \( \forall i = 1, 2, T_1 \cup T_2 = e_1 \), then

(2.6) \[ \| v^{(i)}_0 - v^{(i)}_0 \|_{e_1}^2 \leq C h_{T_i} \| \nabla_w v \|_{T_1 \cup T_2}, \]

where \( v^{(i)}_0 = v_0|_{T_i}, i = 1, 2 \).

**Proof.** Without loss, we may assume that the vertices of \( T_2 \) are \((0, 0), (1, 0)\), and \((0, 1)\),

\[ e_1 = \{(x, 0) | 0 \leq x \leq 1\}, \]

and the other edge of \( T_1 \) is \((a_1, b_1)\), where \( b_1 < 0 \). Denote \( v^{(i)}_0 = v_0|_{T_i}, i = 1, 2 \). Let
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Fig. 2.1: The two triangular mesh

\( \vec{t}_2 \) and \( \vec{t}_4 \) be unit tangents to \( e_2 \) and \( e_4 \), respectively; \( L_3 \) and \( L_5 \) be linear functions such that \( L_3|_{e_3} = 0 \) and \( L_5|_{e_5} = 0 \). Let

\[
\bar{q}_1 = q|_{T_1} = L_3(x,y)(v^{(2)}_0 - v^{(1)}_0)\vec{t}_2
\]

and

\[
\bar{q}_2 = q|_{T_2} = L_5(x,y)(v^{(2)}_0 - v^{(1)}_0)\vec{t}_4.
\]

Then

\[
\bar{q}_i \cdot \vec{n}_{2i} = 0, \quad \bar{q}_i|_{e_{2i+1}} = 0, \quad i = 1, 2.
\]

Scale \( L_1 \) and \( L_3 \) if necessary so that

\[-L_3(0,0)\vec{t}_2 \cdot \vec{n}_1^{(1)} = 1 = L_5(0,0)\vec{t}_4 \cdot \vec{n}_1^{(2)},\]

where \( \vec{n}_1^{(i)} \) is the unit outwards normal vector of \( e_1 \in \partial T_i, i = 1, 2 \).

Since \( L_5(1,0)\vec{t}_2 \cdot \vec{n}_1^{(2)} = 0 \),

\[
L_5(x,y)\vec{t}_4 \cdot \vec{n}_1^{(2)} = \hat{L}_5(x,y) = 1 - x - y.
\]

Similarly

\[
L_3(x,y)\vec{t}_2 \cdot \vec{n}_1^{(2)} = \hat{L}_3(x,y) = 1 - x + \alpha y, \quad \text{for some } \alpha.
\]

It follows from the shape regularity assumptions that the slope of \( e_3, \frac{1}{\alpha} \), satisfies \( |\frac{1}{\alpha}| \geq \alpha_0 > 0 \) for some \( \alpha_0 \). Since \( \hat{L}_3|_{e_1} = \hat{L}_5|_{e_1} \),

\[
(\nabla_w q)|_{T_1 \cup T_2} = \left< v_b - v_0, \bar{q} \cdot \vec{n} \right>_{\partial T_1} + \left< v_b - v_0, \bar{q} \cdot \vec{n} \right>_{\partial T_2} = \left< v^{(2)}_0 - v^{(1)}_0, (v^{(2)}_0 - v^{(1)}_0)\hat{L}_3 \right>_{e_1}.
\]
Note that $0 \leq \tilde{L}_5(x, y) \leq 1$ on $T_1$ and $0 \leq \tilde{L}_3(x, y) \leq 1$ on $T_2$. Then
\[
(\nabla w, \tilde{q})_{T_1 \cup T_2} = \left( v_0^{(2)} - v_0^{(1)}, (v_0^{(2)} - v_0^{(1)}) \tilde{L}_3(x, y) \right)_{e_1}
\]
\[
= \int_0^1 (v_0^{(2)} - v_0^{(1)})^2 (1 - x) \, dx
\]
\[
= \frac{1}{2} \|v_0^{(2)} - v_0^{(1)}\|_{e_1}^2.
\]
Thus
\[
\|\tilde{q}\|_{T_2}^2 = \iint_{T_2} (1 - x - y)^2 (v_0^{(2)} - v_0^{(1)})^2 \, dA = \frac{1}{12} \|v_0^{(2)} - v_0^{(1)}\|_{e_1}^2.
\]
Similarly,
\[
\|\tilde{q}\|_{T_1}^2 = \frac{1}{4} \left(1 + \alpha + \frac{\alpha^2}{3}\right)(v_0^{(2)} - v_0^{(1)})^2 \leq \frac{1}{4} \left(1 + \frac{1}{\alpha_0} + \frac{1}{3\alpha_0}\right) \|v_0^{(2)} - v_0^{(1)}\|_{e_1}^2.
\]
Thus
\[
| (\nabla w, \tilde{q})_{T_1 \cup T_2} | \leq \|\nabla w\|_{T_1} \|\tilde{q}\|_{T_1} + \|\nabla w\|_{T_2} \|\tilde{q}\|_{T_2}
\]
\[
\leq C \|\nabla w\|_{T_1 \cup T_2} \cdot \|v_0^{(2)} - v_0^{(1)}\|_{e_1}.
\]
Thus after a scaling we have
\[
\|v_0^{(2)} - v_0^{(1)}\|_{e_1} \leq Ch_{T_2}^\frac{1}{2} \|\nabla w\|_{T_1 \cup T_2}.
\]

**Lemma 2.3.** Let $T_1$ and $T_2$ be such as in Lemma 2.2. Then
\[
(2.7) \quad \|v - v_0\|^2_{\partial T_1 \cup \partial T_2} \leq Ch_{T_1} \|\nabla w\|^2_{T_1 \cup T_2}.
\]

**Proof.** Without loss, we may assume that $T_1$ and $T_2$ are as shown in the Figure 2.2 where $e_1 \cup e_2 \cup e_3 = \partial T_1, e_1 \cup e_4 \cup e_5 = \partial T_2$ and $e_1 = \partial T_1 \cap \partial T_2$. If follows from Lemma 2.2 that
\[
\|v_0^{(1)} - v_0^{(2)}\|_{e_1}^2 \leq Ch_{T_1} \|\nabla w\|^2_{T_1 \cup T_2}.
\]
We want to show
\[
(2.8) \quad \|v - v_0\|^2_{e_4} \leq Ch_{T_1} \|\nabla w\|^2_{T_1 \cup T_2}
\]
fist. Let $L_2(x, y) = 1 - y$, then $L_2 = 0$ on $e_2$. Denote by $\tilde{n}_j^{(i)}$ the unite outer ward normal vector to $\partial T_i \cap e_j, i = 1, 2, j = 1, \ldots, 5$. Let $\tilde{t}_3$ and $\tilde{t}_5$ be unit tangent vectors to $e_3$ and $e_5$, respectively. Let
\[
\tilde{q} |_{T_1} = \tilde{q}_1 = Q^{(1)} \tilde{t}_3,
\]
where $Q^{(1)} = a_2 L_2$. Let $a$ be such that $a \tilde{t}_5 \cdot \tilde{n}_1^{(2)} = -\tilde{t}_3 \cdot \tilde{n}_1^{(1)}$ and
\[
\tilde{q} |_{T_2} = \tilde{q}_2 = Q^{(2)} \tilde{t}_5,
\]
where \( Q^{(2)}(x, y) = a(a_2(1 - x) + (x - y)b_2) \). Then \( \vec{q}_1 \cdot \vec{n}_1^{(1)} = -\vec{q}_2 \cdot \vec{n}_1^{(2)} \) on \( e_1 \). Thus

\[
\left| \left< v_b - v_0^{(1)}, \vec{q}_1 \cdot \vec{n}_1^{(1)} \right>_{e_1} + \left< v_b - v_0^{(2)}, \vec{q}_2 \cdot \vec{n}_1^{(2)} \right>_{e_1} \right|
\]

\[
= \left| \left< v_0^{(2)} - v_0^{(1)}, \vec{q}_1 \cdot \vec{n}_1^{(2)} \right>_{e_1} \right|
\]

\[
\leq ||v_0^{(2)} - v_0^{(1)}||_{e_1} ||\vec{q}_1||_{e_1}
\]

\[
\leq Ch_{T_1} ||\nabla w v||_{T_1 \cup T_2} ||\vec{q}_1||_{e_1}
\]

\[
\leq Ch_{T_1} ||\nabla w v||_{T_1 \cup T_2} ||\vec{q}_2||_{e_1}
\]

for any choice of \( a_2 \) and \( b_2 \). Write

\[
(v_b - v_0)|_{e_4} = c_0 + c_1 x.
\]

Without loss, we assume \( \vec{t}_4 \cdot \vec{n}_4 = 1 \). Let \( aa_2 = c_0, a(-a_2 + b_2) = c_1 \). Then

\[
\vec{q}_2 \cdot \vec{n}_1^{(2)} = (v_b - v_0)|_{e_4}.
\]

Since

\[
(\nabla_w v, \vec{q})_{T_1 \cup T_2} = \left< v_0^{(2)} - v_0^{(1)}, \vec{q}_1 \cdot \vec{n}_1^{(2)} \right>_{e_1} + \left< v_b - v_0, \vec{q}_2 \cdot \vec{n}_4 \right>_{e_4}
\]

\[
= \left< v_0^{(2)} - v_0^{(1)}, \vec{q}_1 \cdot \vec{n}_1^{(2)} \right>_{e_1} + ||v_b - v_0||^2_{e_4},
\]

\[
||v_b - v_0||^2_{e_4} \leq ||\nabla_w v||_{T_1 \cup T_2} ||\vec{q}||_{T_1 \cup T_2} + C||\nabla_w v||_{T_1 \cup T_2} ||v_b - v_0||_{e_4}.
\]

Note that

\[
||v_b - v_0||^2_{e_4} = \int_1^a (c_0 + c_1 x)^2 dx
\]

\[
= \frac{\alpha}{3} \left( c_0^2 \alpha^2 + \alpha c_0 c_1 + c_1^2 \right)
\]

\[
\geq \frac{\alpha}{6} \left( 1 + \alpha^2 - \sqrt{\alpha^4 - \alpha^2 + 1} \right) (c_0^2 + c_1^2)
\]

\[
\geq \frac{\alpha^3}{6(1 + \alpha^2)} (c_0^2 + c_1^2).
\]
It is easy to see that
\[ \|q\|_{T_1}^2 = \frac{1}{a^2} \int_{T_1} c_0^2 (1 - y)^2 dA = \frac{c_0^2}{12a^2}, \]
where \( a = -t_5 \cdot \hat{n}_1^{(1)} / (t_5 \cdot \hat{n}_1^{(2)}) \). It can be shown that
\[ \|q\|_{T_2}^2 = \int_{T_2} (c_0(1 - x) + (x - y)(c_0 + c_1))^2 dA \]
\[ = \frac{1}{12(a - b)^2} \left( c_0^2(3a^2 - 3ab + b^2) + (3a - b)c_1c_0 + c_1^2 \right) \]
\[ \leq \frac{1}{12(a - b)^2} (4(a^2 + b^2) + 1) \left( c_0^2 + c_1^2 \right). \]

It follows from the shape regularity conditions that
\[ a \geq \alpha_0, \]
\[ (a - b) \geq \alpha_0, \]
\[ a^2 + b^2 \leq \beta_0, \]
for some \( \alpha_0 > 0 \) and \( \beta_0 > 0 \). Thus
\[ \|q\|_{T_1 \cup T_2} \leq C \|v_b - v_0\|_4^2 \]
for some \( C \). Thus
\[ \|v_b - v_0\|_{e_4} \leq C \|\nabla_w v\|_{T_1 \cup T_2}. \]

Using a scaling argument, we have
\[ \|v_b - v_0\|_{e_4} \leq Ch_{T_2} \|\nabla_w v\|_{T_1 \cup T_2}. \]

Similarly, we can show that
\[ \|v_b - v_0\|_{e_i} \leq Ch_{T_1} \|\nabla_w v\|_{T_1 \cup T_2}, \quad i = 2, 3, 5. \]

Now let’s look at \( \|v_b - v_0\|_{e_i \in \partial T_2} \). Let \( \check{q}_2 = (a + bx)t_5 \), where \( t_5 \cdot \hat{n}_1^{(2)} = \frac{1}{\sqrt{2}} \). Then
\[ (\nabla_w v, \check{q}_2)_{T_2} = \left\langle v_b - v_0, \frac{1}{\sqrt{2}} (a + bx) \right\rangle_{e_1} + \left\langle v_b - v_0, a + bx \right\rangle_{e_4}, \]
choose \( a \) and \( b \) so that
\[ \left\langle v_b - v_0, \frac{1}{\sqrt{2}} (a + bx) \right\rangle_{e_1} = \|v_b - v_0\|_{e_1}^2. \]

Using a similar argument as when we were deriving estimate for \( \|v_b - v_0\|_{e_4} \), we can get
\[ \|v_b - v_0\|_{e_i \in \partial T_2} \leq Ch_{T_2} \|\nabla_w v\|_{T_1 \cup T_2}. \]

Thus
\[ \|v_b - v_0\|_{\partial T_1 \cup \partial T_2} \leq Ch_{T_1} \|\nabla_w v\|_{T_1 \cup T_2}. \]
Corollary 2.4.

\[ \sum_{T \in \mathcal{T}_h} h_T^{-1} \| v_h - v_0 \|_{\mathcal{B}T}^2 \leq C \| v \|_T^2. \]  

Combining Lemmas 2.1 and Corollary 2.4, we have the following theorem.

Theorem 2.5. There exists \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[ C_1 \| v \|_{1,h} \leq \| v \| \leq C_2 \| v \|_{1,h}, \quad \forall v \in V_h^0. \]

3. Error equation. In this section, we derive the error estimate of Algorithm 1.

Lemma 3.1. For any function \( \psi \in H^1(T) \), the following trace inequality holds true:

\[ \| \psi \|_T^2 \leq C \left( h_T^{-1} \| \psi \|_T^2 + h_T \| \nabla \psi \|_T^2 \right). \]  

Lemma 3.2. (Inverse Inequality) There exists a constant \( C \) such that for any piecewise polynomial \( \psi|_T \in P_k(T) \),

\[ \| \nabla \psi \|_T \leq C h_T^{-1} \| \psi \|_T, \quad \forall T \in \mathcal{T}_h. \]  

Lemma 3.3. Let \( u \in H^{2+i}_0(\Omega), i = 0, 1 \), be the solution of the problem and \( \mathcal{T}_h \) be a finite element partition of \( \Omega \) satisfying the shape regularity assumptions. Then, the \( L^2 \) projections \( Q_0 \) and \( Q_h \) satisfy

\[ \sum_{T \in \mathcal{T}_h} \left( \| u - Q_0 u \|_T^2 + h_T^2 \| \nabla (u - Q_0 u) \|_T^2 \right) \leq C h^2 \| u \|_{1,i}^2, \]

\[ \sum_{T \in \mathcal{T}_h} \left( \| \nabla u - Q_h \nabla u \|_T^2 + h_T^2 \| \nabla (u - Q_h \nabla u) \|_{1,T}^2 \right) \leq C h^{2(1+i)} \| u \|_{2+i}^2, \quad i = 0, 1. \]  

Lemma 3.4. Let \( \phi \in H^1(\Omega) \). Then for each element \( T \in \mathcal{T}_h \), we have

\[ Q_h(\nabla \phi) = \nabla_w Q_h \phi. \]  

Proof. By definition (2.3) and integration by parts, for each \( \tilde{q} \in [P_1(T)]^2 \) we have

\[ (Q_h(\nabla \phi), \tilde{q})_T = - (\phi, \nabla \cdot \tilde{q})_T + (\phi, \tilde{q} \cdot \vec{n})_{\partial T} \]

\[ = - (Q_0 \phi, \nabla \cdot \tilde{q})_T + (Q_0 \phi, \tilde{q} \cdot \vec{n})_{\partial T} \]

\[ = (\nabla_w Q_h \phi, \tilde{q})_T. \]
which implies (3.5). □

**Lemma 3.5.** Let \( \phi \in H^1(\Omega) \). Then for all \( v \in V_h^0 \), we have

\[
(\nabla \phi, \nabla v_0)_T = (\nabla w(Q_h \phi), \nabla w v)_T + \langle (Q_h (\nabla \phi) \cdot \vec{n}, v_0 - v_b)_{\partial T} \rangle.
\]

**Proof.** Let \( Q_h (\nabla \phi) = \vec{q} \) and \( Q_h \phi = P \). By Lemma 3.4 \( \vec{q} = \nabla w P \).

\[
(\vec{q}, \nabla v_0)_T = -(\vec{q} \cdot \vec{n}, v_0 - v_b)_{\partial T} = (\nabla w P, \nabla w v)_T + \langle \vec{q} \cdot \vec{n}, v_0 - v_b \rangle_{\partial T},
\]

implies

\[
(\vec{q}, \nabla v_0)_T = (\vec{q}, \nabla w v)_T + \langle \vec{q} \cdot \vec{n}, v_0 - v_b \rangle_{\partial T},
\]

which completes the proof. □

**Lemma 3.6.** Let \( e_h = Q_h u - u_h \in V_h \). Then for any \( v \in V_h^0 \), we have

\[
(\nabla w e_h, \nabla w v)_T = \ell(u, v),
\]

where \( \ell(u, v) \) is defined as follows,

\[
\ell(u, v) = \sum_{T \in T_h} \langle (\nabla u - Q_h \nabla u) \cdot \vec{n}, v_0 - v_b \rangle_{\partial T}.
\]

**Proof.** Testing the equation (1.1) by \( v = \{v_0, v_b\} \in V_h \) and using the fact that \( \sum_{T \in T_h} (\nabla u \cdot \vec{n}, v_0 - v_b)_{\partial T} = 0 \), we arrive at

\[
(\nabla u, \nabla v_0)_{T_h} - (\nabla u \cdot \vec{n}, v_0 - v_b)_{\partial T} = (f, v_0).
\]

It follows from Lemma 3.5 that

\[
(\nabla u, \nabla v_0)_T = (\nabla w(Q_h u), \nabla w v)_T + \langle (Q_h (\nabla u) \cdot \vec{n}, v_0 - v_b)_{\partial T} \rangle.
\]

Combining (3.10) and (3.11) gives

\[
(\nabla w(Q_h u), \nabla w v)_T = (f, v_0) + \ell(u, v).
\]

Subtracting (2.4) from the above equation yields the error equation (3.9), and this completes the proof. □

4. **Error Estimates.** We will derive error estimates in this section.

**Lemma 4.1.** Let \( u \in H_0^{2+i}(\Omega), i = 0, 1 \), be the solution of the problem (1.1), (1.2). Then for \( v \in V_h^0 \),

\[
\ell(u, v) \leq C h^{1+i} \|u\|_{2+i} \|v\|, \quad i = 0, 1,
\]

respectively.
Proof. It follows from the definition of $Q_h, Q_0, Q_b$, the Cauchy-Schwarz inequality, trace inequality (3.1), and Theron (2.5)

$$|\ell(u, v)| \leq \sum_{T \in \mathcal{T}_h} |\langle (\nabla u - Q_h \nabla u) \cdot \vec{n}, v_0 - v_b \rangle_{\partial T}|$$

$$= C \sum_{T \in \mathcal{T}_h} \|\nabla u - Q_h \nabla u\|_{\partial T} v_0 - v_b \|_{\partial T}$$

$$\leq C \left( \sum_{T \in \mathcal{T}_h} h_T \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} v_0 - v_b \|_{\partial T}^2 \right)^{1/2}$$

$$\leq C h^{1+i} \|u\|_{2+i} \|v\|, i = 0, 1.$$  

This completes the proof. □

**Theorem 4.2.** Let $u$ and $u_h \in V_h$ be the exact solution and SFWG finite element solution of the problem (1.1)-(1.2) and (2.4). In addition, assume the regularity of exact solution $u \in H^{2+i}_0(\Omega), i = 0, 1$, then there exists a constant $C$ such that

$$\|Q_h u - u_h\| \leq C h^{1+i} \|u\|_{2+i}, i = 0, 1,$$

respectively.

Proof. It follows from error equation (3.9) that

$$\|Q_h u - u_h\|^2 = \langle \nabla_w e_h, \nabla_w e_h \rangle_{\mathcal{T}_h} = \ell(u, e_h).$$

Letting $v = e_h$ in (4.1), yields

$$\|Q_h u - u_h\|^2 \leq C h^{1+i} \|u\|_{2+i} \|e_h\|, i = 0, 1,$$

and this implies the conclusion. □

### 5. Error Estimates in $L^2$ norm

The duality argument is utilized to get $L^2$ error estimate. Let $e_h = \{e_0, e_b\} = Q_h u - u_h$. The dual problem seeks $\Phi \in H^1_0(\Omega)$ satisfying

$$-\Delta \Phi = e_0, \quad \text{in} \ \Omega$$

$$\Phi = 0, \quad \text{on} \ \partial \Omega.$$  

Suppose that the following $H^2$-regularity holds true

$$\|\Phi\|_2 \leq C \|e_0\|.$$

**Theorem 5.1.** Let $u_h = \{u_0, u_b\}$ be the SFWG finite element solution of (2.4). Assume that the exact solution $u \in H^{2+i}_0(\Omega), i = 0, 1$ and (5.2) holds true. Then, there exists a constant $C$ such that

$$\|Q_0 u - u_0\| \leq C h^{1+i} \|u\|_{2+i}, i = 0, 1,$$

and

$$\|u - u_0\| \leq C \|u\|_2.$$
Proof. Testing (5.1) by $e_0$ and using the fact that $\sum_{K \in T_h} \langle \nabla \Phi \cdot \vec{n}, e_b \rangle_{\partial T_h} = 0$, we obtain

$$\|e_0\|^2 = (-\Delta \Phi, e_0) = (\nabla \Phi, \nabla e_0)_{T_h} - (\nabla \Phi \cdot \vec{n}, e_0 - e_b)_{\partial T_h}.$$  \hfill (5.5)

Setting $\phi = \Phi$ and $v = e_h$ in (3.6) yields

$$(\nabla \Phi, \nabla e_0)_{T_h} = (\nabla_w (Q_h \Phi), \nabla_w e_h)_{T_h} + \langle Q_h (\nabla \Phi) \cdot \vec{n}, e_0 - e_b \rangle_{\partial T_h}.$$  \hfill (5.6)

Substituting (5.6) into (5.5) gives

$$\|e_0\|^2 = \langle \nabla_w (Q_h \Phi), \nabla_w e_h \rangle_{T_h} + \ell(\Phi, e_h).$$  \hfill (5.7)

Using equation 2.4 and the error equation (3.9), we have

$$(\nabla_w (Q_h \Phi), \nabla_w e_h)_{T_h} = \ell(u, Q_h \Phi).$$  \hfill (5.8)

By combining (5.7) with (5.8), we obtain

$$\|e_0\|^2 = \ell(u, Q_h \Phi) + \ell(\Phi, e_h).$$  \hfill (5.9)

To bound the terms on the right-hand side of equation (5.9). We use the Cauchy-Schwarz inequality, the trace inequality (3.1) and the definition of $Q_h$ and $Q_b$ to get

$$|\ell(u, Q_h \Phi)| = \sum_{T \in T_h} \langle \nabla u - Q_h (\nabla u) \cdot \vec{n}, Q_0 \Phi - Q_b \Phi \rangle_{\partial T}$$

$$\leq \left( \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \|Q_0 \Phi - Q_b \Phi\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq C \left( \sum_{T \in T_h} h_T \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \|Q_0 \Phi - \Phi\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq Ch^{1+i} \|u\|_{2+i} \|\Phi\|_1,$$

which implies

$$|\ell(u, Q_h \Phi)| \leq Ch^{1+i} \|u\|_{2+i} \|\Phi\|_2, i = 0, 1.$$  \hfill (5.10)

The estimates (4.1), (4.2), and Lemma 3.3 give

$$|\ell(\Phi, e_h)| = \|\nabla \Phi - Q_h (\nabla \Phi) \cdot \vec{n}, e_0 - e_b\|_{\partial T_h}$$

$$\leq Ch^{1+i} \|u\|_{2+i} \|\Phi\|_2, i = 0, 1.$$  \hfill (5.11)

Now combining (5.9) with the estimates (5.10)-(5.11), we obtain

$$\|e_0\|^2 \leq Ch^{1+i} \|u\|_{2+i} \|\Phi\|_2, i = 0, 1,$$

which combined with (5.2) and the triangle inequality, provides the required error estimate (5.3). (5.4) follows from

$$\|u - u_0\| \leq \|u - Q_0 u\| + \|Q_0 u - u_0\| \leq Ch \|u\|_2."
6. Numerical Experiments. In this section, two numerical examples in two dimensional uniform triangular meshes are presented to validate the theoretical results derived in previous sections. Since the regular SFWG method does not work for \((P_0(T), P_1(e), [P_1(T)]^2)\) elements, we'll compare our new SFWG method with the standard WG method.

Example 6.1. In this example, we use the SFWG scheme \([2.4]\) to solve the Poisson problem \(1.1-1.2\) posed on the unit square \(\Omega = (0,1) \times (0,1)\) with the analytic solution

\[
u(x, y) = \sin(\pi x) \sin(\pi y).
\]

The boundary conditions and the source term \(f(x, y)\) are computed accordingly. The first two levels of meshes are plotted in Fig. 6.1. Table 6.1 shows errors and convergence rates in \(H^1\)-norm and \(L^2\)-norm comparison between the SFWG finite element method and the WG finite element method proposed in [2]. As we can see in Fig. 6.1, the error between \(u_0\), the numerical solution obtain from SFWG method, and \(Q_0u\), the \(L^2\)-projection of \(u\) is \(\|Q_0u - u_0\| = O(h^2)\). On the other hand, if the regular WG method is used, \(\|Q_0u - u_0\| = O(h)\), much lower than \(O(h^2)\). Thus our new SFWG method is much more accurate.

| 1/\(h\) | \(\|Q_0u - u_0\|\) | Rate | \(\|Q_0u - u_0\|\) | Rate | \(\|Q_0u - u_0\|\) | Rate |
|-------|----------------|-------|----------------|-------|----------------|-------|
| 2     | 6.2075E-01     | -     | 8.8329E-02     | -     | 1.1717E-00     | -     |
| 4     | 1.8108E-01     | 1.78  | 3.0651E-02     | 1.53  | 5.7311E-01     | 1.03  |
| 8     | 4.7252E-02     | 1.94  | 8.3544E-03     | 1.88  | 2.7088E-01     | 1.08  |
| 16    | 1.1952E-02     | 1.98  | 2.1351E-03     | 1.97  | 1.2974E-01     | 1.06  |
| 32    | 2.9971E-03     | 2.00  | 5.3676E-04     | 1.99  | 6.3299E-02     | 1.04  |
| 64    | 7.5022E-04     | 2.00  | 1.3438E-04     | 2.00  | 3.1159E-02     | 1.02  |

Figure 6.2 shows the computational time (in seconds) comparison between SFWG finite element method and weak Galerkin finite element method. As we can see in Figure 6.2, that the SFWG algorithm is running faster than the standard weak Galerkin algorithm. We can also see in Figure 6.2 that the computation time with 8192 elements
by using the SFWG is $15.0469$, which is much less than $16.5156$, needed by using the standard weak Galerkin algorithm. Therefore, when a large number of elements are used the computation time becomes a significant factor. Thus the SFWG method is more efficient in both accuracy and computation time. Numerical example is carried out on a Laptop computer with 12.0 GB memory and Intel(R) Core (TM) i7-8550U CPU @ 1.80 GHz.

![Comparison of computation times](image)

Fig. 6.2: Comparison of computation times for $(P_0(T), P_1(e), [P_1(T)]^2)$ elements.

**Example 6.2.** This example is adopted from [4]. Let $\Omega = (0,1)^2$ and the boundary value condition (1.1) is chosen such that the exact solution is

$$u(x,y) = r^{2/3} \sin \frac{2\theta}{3}, \quad (x,y) \in \Omega,$$

where polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ are used.

| $\frac{1}{h}$ | $\|Q_h u - u_h\|$ | Rate | $\|Q_0 u - u_0\|$ | Rate | $\|Q_h u - u_h\|$ | Rate | $\|Q_0 u - u_0\|$ | Rate |
|---------------|----------------|------|----------------|------|----------------|------|----------------|------|
| 2             | 1.6754E-02     | -    | 1.1548E-03     | -    | 9.3655E-02     | -    | 3.5650E-03     | -    |
| 4             | 1.0645E-02     | 0.65 | 3.7097E-04     | 1.64 | 5.6604E-02     | 1.03 | 1.6153E-03     | 1.14 |
| 8             | 6.7121E-03     | 0.67 | 1.1709E-04     | 1.66 | 3.1030E-02     | 1.08 | 7.5599E-04     | 1.10 |
| 16            | 4.2294E-03     | 0.67 | 3.6893E-05     | 1.67 | 1.6352E-02     | 1.06 | 3.7573E-04     | 1.00 |
| 32            | 2.6644E-03     | 0.67 | 1.1621E-05     | 1.67 | 8.4922E-03     | 1.04 | 1.8985E-04     | 0.98 |
| 64            | 1.6784E-03     | 0.67 | 3.6605E-06     | 1.67 | 4.4059E-03     | 1.02 | 9.5830E-05     | 0.99 |
The exact solution \( u \in H^{1+2/3}(\Omega) \). Table 6.2 shows the convergence rates in the \( H^1 \)-norm is \( O(h^{2/3}) \) and \( L^2 \)-norm is \( O(h^{5/3}) \) by using the SFWG algorithm.

7. Conclusion and Remark. In this paper, we have developed a new SFWG finite element methods for the Poisson equation (1.1)–(1.2) on triangle mesh. The stabilizer free setting has been used in the numerical scheme. The error estimate in energy norm has been provided and validated in the numerical tests.

Numerical experiments have shown that the new SFWG finite element method works for \( (P_k(T), P_{k+1}(e), [P_{k+1}(T)^2]) \) elements in general. One of our future projects is to extend the theoretical results to the general cases.

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