The Inverse Transformation of L-Hermite Model and Its Application in Structural Reliability Analysis

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Abstract: In probabilistic analysis, random variables with unknown distributions are often appeared when dealing with practical engineering problem. A Hermite normal transformation model has been proposed to conduct structural reliability assessment without the exclusion of random variables with unknown probability distributions. Recently, linear moments (L-moments) are widely used due to the advantages of stability and insensitivity. In this paper, the complete expressions of the inverse transformation of L-moments Hermite (L-Hermite) model have been proposed. The criteria are proposed to derive the complete inverse transformation of performance function and the complete expressions of the inverse transformation of L-Hermite model are formulated. Moreover, a first-order reliability method for structural reliability analysis based on the proposed inverse transformation of L-Hermite model is then developed using the first four L-moments of random variables. Through the numerical examples, the proposed method is found to be efficient for normal transformations since the results of the proposed L-Hermite are in close agreement with the results of Rosenblatt transformation. Additionally, the reliability index obtained by the proposed method using the first four L-moments of random variables provides a close result to the reliability index obtained by first-order reliability method with known probability density functions in structural reliability assessment.

Keywords: normal transformation; L-moments; complete expression; Hermite model; structural reliability; inverse transformation; complete expressions; first-order reliability method; random variables; practical engineering

MSC: 60K10

1. Introduction

One basic problem of structural reliability analysis is calculating the failure probability. In practical engineering, it is usually necessary to consider the uncertainties both from the structure itself and the external load [1,2]; therefore, the calculation of failure probability is usually a high-dimensional integral, which is difficult to solve directly. Under the background, various reliability approximation methods have been developed during the past five decades, such as Monte Carlo simulation (MCS) [3,4], first-order reliability method (FORM) [5,6], second-order reliability method (SORM) [7–9], the moment methods and simulation methods for estimating the failure probability [10,11]. However, almost all these methods are assumed to have normally distributed random variables. Normal transformation techniques can be applied using the Rosenblatt [12] or Nataf transformation [13] with known probability density functions (PDFs). However, as we all know, the distributions of the random variables are often unknown in many practical engineering problems, so it is a fatal task to realize the x→u and u→x transformation by conventional methods. To solve the
problem, Fleishman suggested the third-order polynomial normal transformation, in which the $u - x$ transformation can be realized by using the first four central moments (C-moments). Due to the complex solution of the polynomial coefficients, some explicit expressions of coefficients have been developed, such as the second-order Fisher-Cornish expansion [14], Winterstein’s formula [15], and the four C-moments standardization function of the Hermite model [16,17]. Among them, the C-moments standardization function of the Hermite model is explicit and efficient for $x - u$ transformation. However, studies have shown that the numerical values of sample C-moments, particularly when the sample size is small, may be very different from those of the probability distribution from which the sample was drawn [18].

The linear moments (L-moments) are an alternative moment system used to characterize the shape of a probability distribution. The concept of L-moments originates from various disconnected results on linear combinations of order statistics. Hosking [18] unified the theory of L-moments and provided guidelines for the practical use of L-moments. L-moments have the advantage of stability and are relatively robust to the effects of outliers [18]. Due to these advantages, L-moments have been used for hydrological and structural reliability [19–28]. For example, Ahmad et al. [18] performed the flood frequency analysis of annual maximum stream flows over the station in Negeri Sembilan by using the L-moments; Pandey et al. [20] evaluated the effectiveness of the method of L-moments for estimating parameters of the Pareto distribution model of peaks over a sufficiently high threshold; MacKenzie and Winterstein [23] compared L-moments and conventional C-moments to model current speeds in the North sea; Winterstein and MacKenzie [24] proposed the Hermite model based on L-moments and estimated extreme response statistics using this Hermite model, in which the polynomial coefficients of $u - x$ transformation have been derived; Tung [25] proposed the simple and explicit expressions for the polynomial coefficients in normal transformation based on L-moments; Tong et al. [28] proposed a method for simulating strongly non-Gaussian and non-stationary processes by combining Karhunen–Loève expansion with the L-moments-based Hermite polynomial model. From past research, it can be concluded that the L-Hermite transformation mainly focuses on the $u - x$ transformation. However, the expression of the inverse transformation (the $x - u$ transformation) for the L-Hermite model is still lacking. Without a clear definition of the complete expression of the polynomial normal transformation by the L-Hermite model and the corresponding monotonic regions of $x$ or $u$, the L-Hermite model will be unsuitable for reliability engineering.

The project of this paper is to find the complete expression of inverse transformation based on the L-Hermite model. Moreover, a combination of L-moments and FORM for calculating the reliability index is proposed. This paper is organized as follows. The definition of L-moments is provided in Section 1. The complete expressions of the transformation, including the $x - u$ transformation and the $u - x$ transformation, are derived in Section 2. Section 3 is to show the method of combination of L-moments and FORM. The accuracy and efficiency of the proposed complete expressions of the inverse transformation of the L-Hermite model and correlated transformation for structural reliability assessment are demonstrated in Section 4. The findings of this paper are finally summarized in Section 5.

2. The L-Hermite Models

2.1. General Results for L-Moments

Let $X$ be a random variable and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics, a random variable distributed as the $j$-th smallest element of a random sample of size $n$ drawn from the distribution of $X$. Hosking [18] defined the L-moments of $X$ to be the quantities

$$
\lambda_r = r^{-1} \sum_{j=0}^{r-1} (-1)^k \binom{r-1}{k} EX_{r-k|r}.
$$

(1)
The L in 'L-moments' emphasizes that \( \lambda_r \) is a linear function of the expected order statistics. The expectation of an order statistic may be written as

\[
E(X_{r-k}) = \frac{r!}{(r-k-1)!k!} \int x[F(x)]^{r-k-1}[1-F(x)]^k dF(x). \tag{2}
\]

The first four L-moments can then be expressed as

\[
\lambda_1 = EX_{1:1} = \int_0^{\infty} x dF(x),
\]

\[
\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^{\infty} x^2 dF(x) - \int_0^{\infty} x dF(x),
\]

\[
\lambda_3 = \frac{1}{2} E(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^{\infty} x^3 dF(x) - 3(\int_0^{\infty} x^2 dF(x) - \frac{1}{2} \int_0^{\infty} x dF(x)),
\]

\[
\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 2X_{2:4} - X_{1:4}) = \int_0^{\infty} x^4 dF(x) - 6(\int_0^{\infty} x^3 dF(x) - 3(\int_0^{\infty} x^2 dF(x) - \frac{1}{2} \int_0^{\infty} x dF(x)),
\]

in which \( \lambda_1 \) and \( \lambda_2 \) are measures of the mean and scale. Higher L-moments reflect different aspects of distribution shape. \( \lambda_3 \) and \( \lambda_4 \) reflect asymmetric and symmetric deviations. The L-moment ratios \( r_3 = \lambda_3/\lambda_2 \) and \( r_4 = \lambda_4/\lambda_2 \) are defined as dimensionless analogs of \( \lambda_3 \) and \( \lambda_4 \), therefore, they are plausible measures of skewness and kurtosis.

In practice, L-moments must usually be estimated from random samples drawn from an unknown distribution. Assuming that \( x_1, x_2, \ldots, x_n \) are the samples and \( x_{(1:n)} \leq x_{(2:n)} \leq \cdots \leq x_{(n:n)} \), the ordered samples, and the first four sample L-moment, i.e., \( l_1, l_2, l_3, \) and \( l_4 \), can be given by \[22\]

\[
l_1 = \frac{1}{n} \sum_{j=1}^{n} x_{(j:n)},
\]

\[
l_2 = \frac{2}{n^2} \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} x_{(j:n)} - \frac{1}{n} \sum_{j=1}^{n} x_{(j:n)},
\]

\[
l_3 = \frac{6}{n^3} \sum_{j=3}^{n} \frac{(j-1)(j-2)}{(n-1)(n-2)} x_{(j:n)} - \frac{6}{n^2} \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} x_{(j:n)} + \frac{1}{n} \sum_{j=1}^{n} x_{(j:n)},
\]

\[
l_4 = \frac{20}{n^4} \sum_{j=4}^{n} \frac{(j-1)(j-2)(j-3)}{(n-1)(n-2)(n-3)} x_{(j:n)} - \frac{30}{n^3} \sum_{j=3}^{n} \frac{(j-1)(j-2)}{(n-1)(n-2)} x_{(j:n)} + \frac{12}{n^2} \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} x_{(j:n)} - \frac{1}{n} \sum_{j=1}^{n} x_{(j:n)}. \tag{10}
\]

2.2. L-Hermite Transformation Models

The conventional Hermite models can be expressed as \[15\]

\[
x_0 = \frac{x - \mu}{\sigma} = \kappa[u + \tilde{h}_3(u^2 - 1) + \tilde{h}_4(u^3 - 3u)],
\]

where \( \kappa, \tilde{h}_3, \tilde{h}_4 \) are Hermite coefficients, and \( \kappa = \frac{1}{\sqrt{1 + 2\tilde{h}_3^2 + 6\tilde{h}_4^2}} \). \( x_0 \) is the standardized random variable, \( \mu \) and \( \sigma \) are the mean and the standard deviation of \( x \), \( u \) is the standard normal variable.

Then regrouping \( u + \tilde{h}_4(u^3 - 3u) \) as \( (1 - 3\tilde{h}_4)u + \tilde{h}_4u^3 \), and dividing by \( (1 - 3\tilde{h}_4) \), Equation (11) can be expressed as

\[
x_0 = \frac{x - \mu}{\sigma} = \kappa[u + b(u^2 - 1) + cu^3], \tag{12}
\]
and the scaling factor $K$ becomes

$$K = \frac{1}{\sqrt{1 + 2b^2 + 6c + 15c^2}}$$  \hspace{1cm} (13)$$

in which $b = \frac{9.21\tau_3}{11.68 - 2.57\gamma}$, $c = \frac{\gamma - 1}{11.68 - 2.57\gamma}$, $\gamma = \tau_4/\tau_{4,Gauss}$, and $\tau_{4,Gauss} = 0.1226$.

Equations (11)–(13) comprise the L-Hermite models. In the conventional Hermite conversion models, both skewness and kurtosis of central moments vary with both $h_3\sim$ and $h_4\sim$, whose expressions are complicated [17]. This leads to approximate results for the coefficients, and then the Hermite model may only approximately match the desired skewness and kurtosis. However, for the L-Hermite models, $\tau_4\sim$ depends only on $c$, and $\tau_3\sim$ depends on $b$ and $c$, therefore, which are explicit expressions. Therefore, the L-moment ratios $\tau_3\sim$ and $\tau_4\sim$ can be preserved without approximation in the results.

3. Complete Expressions of the Inverse Transformation of the L-Hermite Models

3.1. Basic Ideas

To develop the complete expressions of the inverse transformation of the L-Hermite models, i.e., $x-u$ transformation, Equation (12) can be rewritten as follows

$$x' = S^*_u(u) = u^3 + a'_2u^2 + a'_1u - a'_2,$$  \hspace{1cm} (14)$$
in which $S^*_u(u)$ is a cubic function of $u$, $a'_1$, $a'_2$ are polynomial coefficients, $x'$ is the transformed random variable of $x$, and $a'_1$, $a'_2$ and $x'$ are given by

$$a'_1 = \frac{1}{c}, \quad a'_2 = \frac{b}{c}, \quad x' = \frac{x_0}{Kc}.$$  \hspace{1cm} (15)$$

To identify the proper expression of the $x-u$ transformation, the key problem is to find the solution to Equation (14).

3.2. Selection of Suitable Solution for the Inverse Transformation

The $x-u$ transformation can be determined by solving the root of $u$ in Equation (14), and the process of obtaining the root of $u$ is essentially solving the cubic equation. According to the Cardano formula [28], the solution of a cubic equation can be expressed as:

$$u_1 = \sqrt[3]{-\frac{h}{2} + \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}} + \sqrt[3]{-\frac{h}{2} - \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}},$$  \hspace{1cm} (16)$$

$$u_2 = \omega\sqrt[3]{-\frac{h}{2} + \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}} + \omega^2\sqrt[3]{-\frac{h}{2} - \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}},$$  \hspace{1cm} (17)$$

$$u_3 = \omega^2\sqrt[3]{-\frac{h}{2} + \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}} + \omega\sqrt[3]{-\frac{h}{2} - \sqrt{\frac{h^2}{4} - \frac{a'_2}{3}}},$$  \hspace{1cm} (18)$$

where

$$\omega = -\frac{1 + \sqrt{3i}}{2}, \quad h = \frac{2a'_2}{27} - \frac{a'_1a'_2}{3} - a'_2 - x', \quad I = \left(\frac{3a'_1^2 - a'_2^2}{9}\right) + \frac{h^2}{4}. \hspace{1cm} (19)$$

in which $I$ is the discriminant of the roots. When $I > 0$, there exists one real root; when $I = 0$, there are two real roots; and when $I < 0$, there are three real roots.

However, when a cubic equation has three real roots, the roots expressed by Equations (17) and (18) involve complex numbers. In order to solve the problem, the trigonometric function can be used to express the three real roots [29], i.e.,

$$u_1 = 2\sqrt[3]{\frac{a'_2^2 - 3a'_1}{9}} \cos \left(\frac{\alpha}{3}\right) - \frac{a'_2}{3},$$  \hspace{1cm} (20)$$
\[ u_2 = -2 \sqrt[3]{\frac{a_2^2 - 3a_1'}{9}} \cos \left( \frac{\alpha + \pi}{3} \right) - \frac{a_2'}{3}, \]  

(21)

\[ u_3 = -2 \sqrt[3]{\frac{a_2^2 - 3a_1'}{9}} \cos \left( \frac{\alpha - \pi}{3} \right) - \frac{a_2'}{3}, \]  

(22)

in which \( \alpha = \arccos \left( -\frac{h}{2 \sqrt{3 \left( \frac{a_2^2 - 3a_1'}{9} \right)}} \right). \]  

(23)

Using the inequalities \( 0 \leq \arccos(m) \leq \pi \) for a real number \( m \) such that \(-1 \leq m \leq 1\), one can obtain \( u_1 > u_2 > u_3 \) after derivation.

According to the Cardano formula, the number of roots can be determined by the discriminant \( I \). Since the calculation of \( I \) is rather complicated, another concise discriminant method should be found.

In order to find a concise discriminant method, we need to analyze the property of \( S_\nu(u) \). The monotonicity of \( S_\nu(u) \) is relevant to its derivative. According to Equation (14), the first derivative of \( S_\nu(u) \) can be derived as

\[ \frac{dS_\nu(u)}{du} = 3u^2 + 2a_2' u + a_1'. \]  

(24)

One critical point can be defined as the value of \( u \) when \( dS_\nu(u)/du=0 \). The number of solutions is determined by the parameter \( p \), expressed as

\[ p = \frac{3a_1' - a_2'^2}{9}. \]  

(25)

**Selection criterion 1:** the sign of \( p \)

When \( p \geq 0 \), one obtains \( S_\nu(u) \) strictly monotonic, and there will be only one solution of Equation (14), the root can be determined as Equation (16).

When \( p < 0 \), one obtains \( S_\nu(u) \) non-monotonic, there will be two or three solutions of Equation (14), there are two critical points of \( S_\nu(u) \), and the derivative value at the critical point equals zero.

**Selection criterion 2:** the range of \( x \)

Assume that the two critical points are \( \eta_1 \) and \( \eta_2 \) (\( \eta_1 < \eta_2 \)), respectively. The range of \( u \) is divided into three monotonic regions: \((-\infty, \eta_1], [\eta_1, \eta_2], \text{ and } [\eta_2, +\infty)\). The solutions in these sections can be named \( u_{\text{max}}, u_{\text{med}}, u_{\text{min}}(u_{\text{max}} > u_{\text{med}} > u_{\text{min}}) \), respectively.

At critical points, the value of \( x' \) is computed as \( \eta_1^* \) and \( \eta_2^* \).

\[ \eta_1^* = \mu + \sigma Kc \left( \frac{2}{27}a_2'^3 - \frac{a_1' a_2'^2}{3} - a_1' - 2|p|^\frac{2}{3} \right), \]  

(26)

\[ \eta_2^* = \mu + \sigma Kc \left( \frac{2}{27}a_2'^3 - \frac{a_1' a_2'^2}{3} - a_1' + 2|p|^\frac{2}{3} \right), \]  

(27)

The parameter \( I = q^2 + 4p^3 \) can be determined by the value of \( x \), which is shown as

\[
\begin{align*}
I &\geq 0, \ x \geq \max(\eta_1^*, \eta_2^*) \text{ or } x \leq \min(\eta_1^*, \eta_2^*) \\
I &< 0, \min(\eta_1^*, \eta_2^*) < x < \max(\eta_1^*, \eta_2^*)
\end{align*}
\]

(28)

where \( q = \frac{2Kb^3 - 9Kbc - 27Kc^2 - 27x_0c^2}{27Kc^3} \).

**Selection criterion 3:** the sign of \( c \).

From the basic principles of transformation [14]:

\[ F(x) = \Phi(u). \]
where \( F(\cdot) \) and \( \Phi(\cdot) \) are the cumulative density functions (CDF) of \( x \) and \( u \), respectively. Taking the derivative of both sides with respect to \( u \) of Equation (28) leads to

\[
\frac{f(x)dx}{du} = \phi(u).
\] (29)

in which \( f(\cdot) \) and \( \phi(\cdot) \) are the PDF of \( x \) and \( u \), respectively.

According to Equations (14) and (29), the first derivative of \( S_u^*(u) \) can be expressed as

\[
\frac{dS_u^*(u)}{du} = \frac{1}{K\sigma} \phi(u).
\] (30)

Using Equation (30), one can obtain that the sign of \( dS_u^*(u)/du \) can be determined by \( c \) since that \( K, \sigma \), and \( \phi(u)/f(x) \) are greater than zero.

Selection criterion 4: the sign of \( \lambda_3 \).

According to Equation (29), the second-order derivative of \( S_u^*(u) \) can be expressed as

\[
\frac{d^2S_u^*(u)}{du^2} = \frac{1}{K\sigma^2} \left( \phi'(u)f(x) - \phi^2(u)f'(x)/f(x) \right).
\] (31)

where \( f'(x) \) and \( \phi'(u) \) denote the derivatives of \( f(x) \) with respect to \( x \) and \( \phi(u) \) with respect to \( u \), respectively.

When \( u = 0 \), the value of \( dS_u^{*2}(u)/du^2 \) in Equation (31) can be simplified as

\[
\frac{d^2S_u^*(u)}{du^2} = -\frac{\phi^2(0)}{K\sigma f^3(\mu)} f'(\mu).
\] (32)

According to Equation (25), the sign of \( dS_u^{*2}(u)/du^2 \) is determined by \( f'(\mu) \). The sign of \( f'(\mu) \) is determined by the L-skewness \( \lambda_3 \), which measures the symmetry of the distribution. Figure 1a,b give the representative PDFs with positive and negative \( \lambda_3 \). It can be seen from Figure 1a,b that when \( \lambda_3 < 0 \), the random variable has a longer left tail and the maximum of PDF appears on the right side of the mean value, therefore, \( f'(\mu) \) is positive, and when \( \lambda_3 > 0 \), the PDF of a random variable has a longer right tail, so \( f'(\mu) \) is negative.

![PDF plots](image)

(a) L-skewness \( \lambda_3 \leq 0 \)  
(b) L-skewness \( \lambda_3 \geq 0 \)

**Figure 1.** The relationship between the sign of \( f'(\mu) \) and \( \lambda_3 \).

### 3.3. The Complete Expressions of the Inverse Transformation of L-Hermite Models

As detailed above, the solution of the inverse transformation of L-Hermite models is determined by the value of \( p, x, c \), and \( \lambda_3 \).
(i) \( p \geq 0 \)

When \( p \geq 0 \), there is only one root. According to the Cardano formula, the root can be determined as Equation (16).

(ii) \( p < 0 \) and \( I \geq 0 \)

When \( p < 0 \), and based on selection criterion 2, if \( I \geq 0 \), then \( x \leq \min(\eta_1^*, \eta_2^*) \) or \( x \geq \max(\eta_1^*, \eta_2^*) \), and Equation (16) is the appropriate solution.

(iii) \( p < 0 \), \( I < 0 \), and \( c < 0 \)

If \( \Delta < 0 \), then \( \min(\eta_1^*, \eta_2^*) < x < \max(\eta_1^*, \eta_2^*) \), the real roots of Equation (29) may be Equation (21) or Equation (22). Based on Selection criterion 3, if \( c < 0 \), the sign of \( dS_u^*(u)/du \) is negative, and the root is formulated as Equation (21).

(iv) \( p < 0 \), \( I < 0 \), \( c > 0 \) and \( \lambda_3 > 0 \)

If \( c > 0 \), the sign of \( dS_u^*(u)/du \) is positive, both Equation(20) and (22) are possible, and Selection criterion 4 should be considered.

According to Selection criterion 4, when \( \lambda_3 > 0 \), \( dS_u^*(u)/du^2 \) is positive and Equation (20) is the suitable solution;

(v) \( p < 0 \), \( I < 0 \), \( c > 0 \) and \( \lambda_3 < 0 \)

When \( \lambda_3 < 0 \), \( dS_u^*(u)/du^2 \) is negative, and Equation (22) is the suitable solution.

(vi) \( c = 0 \) and \( \lambda_3 \neq 0 \)

When \( c = 0 \), Equation (14) is a quadratic function, and the corresponding \( u \) is expressed as

\[
u = \frac{-1 + \sqrt{1 + 4b^2 + 4b x_0 \sqrt{1 + 2b^2}}}{2b}.\]  

(vii) \( c = 0 \) and \( \lambda_3 = 0 \)

When \( c = 0 \) and \( \lambda_3 = 0 \), the Equation (14) is a linear function, then there is

\[
u = x_0.\]

For convenience, the complete expressions of the inverse transformation of the L-Hermite model can be summarized in Table 1.

| Parameters | Expression of \( u \) | Range of \( x \) |
|------------|-----------------------|-----------------|
| \( p \geq 0 \) \( c \neq 0 \) \( \lambda_3 \geq 0 \) | \( \sqrt{-h/2 + \sqrt{I + \sqrt{-h/2 - \sqrt{I - a_2^*/3}}} \} \) \( \eta_1^* < x < \eta_2^* \) | \( (-\infty, +\infty) \) |
| \( p < 0 \) \( c > 0 \) \( \lambda_3 \geq 0 \) | \( 2\sqrt{|p|} \cos(\pi/3) - a_2^*/3 \) \( \eta_1^* < x < \eta_2^* \) | \( (-\infty, +\infty) \) |
| \( p < 0 \) \( c > 0 \) \( \lambda_3 < 0 \) | \( \sqrt{-h/2 + \sqrt{I + \sqrt{-h/2 - \sqrt{I - a_2^*/3}}} \} \) \( x \geq \eta_2^* \) | \( (-\infty, +\infty) \) |
| \( p < 0 \) \( c < 0 \) \( \lambda_3 \neq 0 \) | \( -2\sqrt{|p|} \cos((\pi/3) - a_2^*/3 \) \( \eta_2^* \leq x \leq \eta_1^* \) | \( (-\infty, +\infty) \) |
| \( c = 0 \) \( \lambda_3 \neq 0 \) | \( -1 + \sqrt{1 + 4b^2 + 4b x_0 \sqrt{1 + 2b^2}} \} / 2b \) \( \eta_1^* < x < \eta_2^* \) | \( (-\infty, +\infty) \) |
| \( c = 0 \) \( \lambda_3 = 0 \) | \( x_0 \) \( \eta_2^* \leq x \leq \eta_1^* \) | \( (-\infty, +\infty) \) |

Table 1. Expressions of \( u \).
4. FORM for Structural Reliability Analysis Based on the L-Hermite Model

Based on the proposed inverse transformation of the L-Hermite model, FORM for reliability analysis, including both independent variables and correlated variables, can be readily realized. The computation procedure for FORM based on the inverse transformation of the L-Hermite model is described as follows, with the flowchart illustrated in Figure 2:

1. Obtain the first four L-moments of each random variable for all variables and the original correlation matrix for correlated variables by the probability information.
2. Determine the polynomial coefficients by Table 1, the correlation matrix $C_Z$, the lower triangular matrix $L_0$, and its inverse matrix $L_0^{-1}$.
3. Assume an initial checking point $x_0$ (generally take the mean value).
4. Obtain the corresponding checking point in the independent standard normal space, $u_0$.
5. Determine the initial reliability index $\beta_0$.
6. Determine the Jacobian matrix $J = \frac{\partial X}{\partial U}$ evaluated at $u_0$, where the Jacobian matrix is given by
   \[
   \frac{\partial x_i}{\partial u_j} = \sigma_{xi} l_{ij} \left[ b_i + 2c_i \sum_{k=1}^{i} l_{ik} u_k + 3d_i \left( \sum_{k=1}^{i} l_{ik} u_k \right)^2 \right] (i, j = 1, 2, \ldots, n). \tag{35}
   \]
7. Determine the value of the gradient vector at $u_0$ and the value of the function in normal space:
   \[
   G_u(u_0) = G(x_0), \tag{36}
   \]
   \[
   \nabla G(u_0) = J^T \nabla G(x_0). \tag{37}
   \]
   in which $G(x_0)$ can be computed directly by taking the derivative for explicit performance functions and can be computed by numerical differentiation, such as the central difference method for implicit functions. The central difference method is presented as follows:
   \[
   \nabla G(x_0) = \begin{pmatrix}
   \frac{G(x_0 + h, x_{02}, \ldots, x_{0n}) - G(x_0 - h, x_{02}, \ldots, x_{0n})}{2h_1} \\
   \vdots \\
   \frac{G(x_{01}, \ldots, x_{0i} + h, \ldots, x_{0n}) - G(x_{01}, \ldots, x_{0i} - h, \ldots, x_{0n})}{2h_i} \\
   \vdots \\
   \frac{G(x_{01}, x_{02}, \ldots, x_{0n} + h) - G(x_{01}, x_{02}, \ldots, x_{0n} - h)}{2h_n}
   \end{pmatrix}. \tag{38}
   \]
8. Calculate the new check point:
   \[
   u^{(k+1)} = u^{(k)} + \frac{\nabla G(u^{(k)})}{\nabla G^T(u^{(k)}) \nabla G(u^{(k)})} \left( \nabla G^T(u^{(k)}) u^{(k)} - \nabla G(u^{(k)}) \right), \tag{39}
   \]
   \[
   u^{(k+1)} = x^{(k)} + J(u^{(k+1)} - u^{(k)}). \tag{40}
   \]
   The corresponding reliability index can be calculated as $\beta = (u_1^T \cdot u_1)^{1/2}$.
9. Calculate the absolute difference between $\beta$ and $\beta_0$ until $|\beta - \beta_0| \leq \epsilon$, where $\epsilon$ is the permissible error (generally $\epsilon = 10^{-6}$).

Otherwise, repeat step 4 through step 9 until convergence is achieved.
(9) Calculate the absolute difference between $\beta$ and $\beta_0$ until $0||\beta - \beta_0\epsilon\leq$, where $\epsilon$ is the permissible error (generally $\epsilon = 10^{-6}$).

Otherwise, repeat step 4 through step 9 until convergence is achieved.

5. Applications in Structural Reliability Analysis
5.1. Comparison of the $x$-$u$ Transformation among Commonly Used Distributions

For a random variable $x$ to have a known PDF, $f(x)$, the relation between $x$ and $u$ can be derived by using normal transformation techniques. Using the proposed complete expressions of the inverse transformation of the L-Hermite model, the $x$–$u$ transformation for random variables with all practical combinations of the first four L-moments can be realized. To demonstrate its accuracy and efficiency, the first example considers four random variables following Weibull, gamma, Gumbel, and exponential distributions. With the first four moments of the random variables, the polynomial coefficients in Equation (11)
can be readily obtained. The distribution parameters and the first four moments of these random variables are listed in Table 2.

### Table 2. Probability distributions and their statistical parameters for Example 1.

| Distribution | The First Four L-Moments | Parameters in Equation (12) |
|--------------|--------------------------|-----------------------------|
|              | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $k$ | $b$ | $c$ |
| Weibull      | 1             | 0.0554        | -0.0065       | 0.0073       | 0.9610 | -0.1211 | 0.0087 |
| Gamma        | 1             | 0.1674        | 0.0164        | 0.0210       | 0.9827 | 0.0991   | 0.0026 |
| Gumbel       | 1             | 0.2702        | 0.0459        | 0.0406       | 0.9001 | 0.1817   | 0.0263 |
| Exponential  | 1             | 0.5           | 0.1667        | 0.0833       | 0.7997 | 0.3707   | 0.0434 |

The changes of the $x$-$u$ transformation function for the four non-normal random variables are shown in Figure 3a–d, obtained from different methods, i.e., the Rosenblatt transformation (The exact), the C-Hermite method (Zhang), the proposed method and the Winterstein formula. Figure 3 reveals the following:

1. The Winterstein formula provides suitable results when the value of $x$ is around the mean value. However, when the absolute value of $x$ is far from the mean, the results obtained by the Winterstein formula exhibit significant differences compared with the exact ones, except for the case of exponential distribution with COV = 1 (Figure 3d).

2. In general, the C-Hermite method can also provide suitable results with the Rosenblatt transformation. For the gamma variable with COV = 0.3 (Figure 3b), the Gumbel variable with COV = 0.5 (Figure 3c), and the exponential variable with COV = 1 (Figure 3d), the results of $x$-$u$ transformation obtained by the C-Hermite method are similar with the exact ones. However, for the Weibull variables, there are large differences between the results obtained by the C-Hermite method and the exact ones when the value of $x$ is far from the mean value (see Figure 3a).

3. The proposed L-Hermite method performs better than the Winterstein formula and the C-Hermite method, especially for the Weibull variable with COV = 0.1 (Figure 3a), and the results of the proposed L-Hermite are in close agreement with the exact ones in the whole investigation range for all the four cases considered.

![Figure 3. Cont.](a) Weibull distribution (COV = 0.1)  
(b) Gamma distribution (COV = 0.3)
5.2. First-Order Reliability Analysis, including Random Independent Variables with Unknown CDFs/PDFs

The second example considers a soil settlement, which has been investigated by B.N. Rao [30]. The structures are shown in Figure 4. Assuming that the contribution of secondary consolidation to settlement is negligible, the settlement at point A is required to be less than 2.5 in. In this example, the performance function is given by

\[ Z = g(C_c, \varepsilon_0, H, p_0, \Delta p) = \frac{2.5}{H} \log_{10}(1 + \frac{\Delta p}{p_0}) - \frac{C_c}{(1 + \varepsilon_0)} \]  

in which \( \varepsilon_0 \) is the void ratio of the clay layer before loading, \( H \) is the thickness of the clay layer, \( C_c \) is the compression index of the clay, \( p_0 \) and \( \Delta p \) are, respectively, the original effective pressure before loading and the pressure increased at point B due to structural construction (mid height of the clay layer). The probability distribution of the random variables is listed in Table 3.

(c) Gumbel distribution (COV = 0.5)  
(d) Exponential distribution (COV = 1)

Figure 3. Comparison of \( x-u \) transformation with different normal transformation methods.

Figure 4. The model of soil settlement.
Table 3. Random variables in Example 2.

| Variable | Distribution | The First Four L-Moments | Parameters in Equation (12) |
|----------|--------------|--------------------------|-----------------------------|
| H        | Normal       | 168 4.7392 0 0.5810      | 1 0 0                       |
| CC       | Normal       | 0.396 0.0558 0 0.0068    | 1 0 0                       |
| e0       | Normal       | 1.19 0.1007 0 0.0123      | 1 0 0                       |
| p0       | Weibull      | 0.5 0.0563 –0.0036 0.0066 | 1.0086 –0.0633 –0.0042      |
| Δp       | Weibull      | 3.72 0.1019 –0.0147 0.0143 | 0.9319 –0.1510 0.0169       |

In this example, the PDFs of random variables are known; therefore, using the general FORM procedure, the reliability index can be obtained as 1.3627.

To investigate the efficiency and accuracy of the proposed method, only the first four L-moments of random variables H, CC, e0, p, Δp are used to calculate the reliability index. With the aid of Table 1 and Equation (12), the x-u and u-X transformations of the random variables can be easily performed by using the first four L-moments. Table 3 lists the parameters for the transformation of the random variables. Then, the reliability index \( \beta_{L-FORM} \) with the known first four L-moments can be readily obtained as 1.3638. Similarly, the reliability index \( \beta_{C-FORM} \) can also be readily obtained as 1.3645 when only the first four C-moments are assumed to be known.

The result shows that the reliability index obtained using the first four L-moments of random variables H, CC, e0, p, Δp are generally close to the result obtained by using the PDFs, and the result is more accurate than that obtained by using the first four C-moments for this example.

5.3. Reliability Analysis of Implicit Performance Function

The third example considers a three-bay six-story frame structure under lateral loads, which is shown in Figure 5.

The performance function of this example is [31]

\[
G(X) = u_{\text{lim}} - u(X).
\]

where \( u(X) \) is the top-floor displacement determined by finite element analysis, and \( u_{\text{lim}} \) is the specified limit, taken as 0.02 m.

Figure 5. Three-bay five-story frame structure considered in Example 3.
Here seven random variables are considered, which includes one Young’s modulus and six applied loads, and their corresponding probabilistic information is listed in Table 4.

Based on the computational procedure shown in Figure 5, the reliability index can be calculated by the proposed method for random variables with unknown PDFs. The parameters of the proposed complete expressions of L-Hermite are listed in Table 4.

In this example, the performance function is implicit. According to Equation (38), the gradient vector of the performance function can be calculated by the central differentiation method. With known PDFs of random variables, the first-order reliability assessment can be calculated as \( \beta_{\text{FORM}} = 2.9009 \). The proposed method is then used to illustrate the accuracy, and the four moments of random variables are listed in Table 4, in which the gradient vector of the performance function is also determined by the central differentiation method according to Equation (38), and the \( \beta_{\text{FORM}} \) is readily obtained as 3.0975. It can be observed that the reliability index obtained by the proposed method using the first four L-moments provides a close result to the one compared with those obtained by FORM using PDFs of random variables.

### 6. Conclusions

This study develops a probability distribution based on the polynomial normal transformation of both independent and correlated variables, and the proposed method is applied to structural reliability analysis. The transformation from non-normal random variables to normal ones, including independent and correlated random variables, is investigated, and the results obtained by the proposed complete expressions of the inverse L-Hermite model are quite accurate compared to the exact values. Numerical examples are presented to demonstrate the application of the proposed complete expressions of the inverse L-Hermite transformation in FORM, including independent and correlated random variables with unknown CDFs/PDFs. It is found that:
1. The polynomial coefficients of the L-Hermite model can be expressed by the first four L-moments explicitly. They are much easier and simpler than those based on C-moments, where the coefficients have to be obtained by nonlinear equations.

2. Numerical examples show that the inverse L-Hermite transformation is close to the exact values in the whole investigation range, which are more accurate than the C-Hermite transformation.

3. The reliability obtained by the proposed FORM based on the L-Hermite model is close to those by using known CDFs/PDFs. It can be concluded that the presented method of inverse L-Hermite transformation is quite efficient for normal transformation and sufficiently accurate to include random variables with unknown probability distributions in structural reliability analysis.

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References
1. Choi, Y.; Ahn, J.; Chang, D. Time–dependent reliability analysis of Plate–stiffened prismatic pressure vessel with corrosion. Mathematics 2021, 9, 1544. [CrossRef]
2. Barbu, V.S.; D’Amico, G.; Gkelsinis, T. Sequential interval reliability for discrete–time homogeneous semi–markov repairable systems. Mathematics 2021, 9, 1997. [CrossRef]
3. Binder, K.; Heermann, D. Monte Carlo Simulation in Statistical Physics; Chapman and Hall: London, UK, 1994.
4. Barker, A.A. Monte carlo calculations of the radial distribution functions for a proton electron plasma. Aust. J. Phys. 1965, 18, 119–134. [CrossRef]
5. Shinozuka, M. Basic analysis of structural safety. J. Struct. Eng. 1983, 109, 721–740. [CrossRef]
6. Cornell, C. A Probability–based structural code. J. Am. Concrr. Inst. 1968, 66, 974–985.
7. Kiureghian, A.; Stefano, M. Efficient Algorithm for Second–Order Reliability Analysis. J. Eng. Mech. ASCE 1988, 117, 2904–2923. [CrossRef]
8. Cai, G.; Elishakoff, I. Refined second–order reliability analysis. Struct. Saf. 1994, 14, 267–276. [CrossRef]
9. Riesch–Oppermann, H.; Bruckner–foit, A. First–and second–order approximations of failure probabilities in probabilistic fracture mechanics. Reliab. Eng. Syst. Saf. 1983, 19, 183–194. [CrossRef]
10. Fleishman, A. A method for simulating non–normal distributions. Psychometrika 1978, 43, 521–532. [CrossRef]
11. Zhao, Y. Third–Moment Standardization for Structural Reliability Analysis. J. Struct. Eng. 2000, 126, 724–732. [CrossRef]
12. Rebb, R.; Mahadevan, S. Validation of models with multivariate output. Reliab. Eng. Syst. Saf. 2006, 91, 861–871. [CrossRef]
13. Liu, P.; Kiureghian, A. Multivariate distribution models with prescribed marginals and covariances. Probabilistic Eng. Mech. 1986, 1, 105–112. [CrossRef]
14. Fisher, R.; Cornish, E. The percentile points of distributions having known cumulants. Technometrics 1960, 2, 209–225. [CrossRef]
15. Winterstein, S. Nonlinear vibration models for extremes and fatigue. J. Eng. Mech. 1988, 114, 1772–1790.
16. Zhao, Y.; Lu, Z. Fourth–moment standardization for structural reliability assessment. J. Struct. Eng. 2007, 133, 916–924. [CrossRef]
17. Zhang, X.; Zhao, Y.; Lu, Z. The inverse transformation of the explicit fourth–moment standardization for structural reliability. Adv. Struct. Eng. 2018, 21, 769–782. [CrossRef]
18. Hosking, J. L-moments: Analysis and estimation of distributions using linear combinations of order statistics. J. R. Stat. Soc. Ser. B Methodol. 1990, 52, 105–124. [CrossRef]
19. Chen, Y.; Huang, G.; Shao, Q. Regional analysis of low flow using L–moments for Dongjiang basin. HydroI. Sci. J. 2006, 51, 1051–1064. [CrossRef]
20. Shanzhad, M.; Asghar, Z. Comparing TL–Moments, L–Moments and Conventional Moments of Dagum Distribution by Simulated data. Rev. Colomb. Estadistica 2013, 36, 79–93.
21. Pandey, M.D.; van Gelder, P.H.A.J.M.; Vrijling, J.K. The estimation of extreme quantiles of wind velocity using L–moments in the peaks–over–threshold approach. Struct. Saf. 2001, 23, 179–192. [CrossRef]
22. Greenwood, J.; Landwehr, J.; Matalas, N. Probability weighted moments: Definition and relation to parameters of several distributions expressable in inverse form. *Water Resour. Res.* 1979, 15, 1049–1054. [CrossRef]

23. MacKenzie, C.; Winterstein, S. Comparing L–Moments and Conventional Moments to Model Current Speeds in the North Sea. In Proceedings of the 2011 Industrial Engineering Research Conference (IERC 2011), Reno, NV, USA, 21–25 May 2011.

24. Winterstein, S.; Machenzie, C. Extremes of nonlinear vibration: Comparing models based on moments, l–moments, and maximum entropy. *J. Offshore Mech. Arct. Eng.* 2013, 135, 21601–21602. [CrossRef]

25. Chen, X.; Tung, Y. Investigation of polynomial normal transform. *Struct. Saf.* 2003, 25, 423–445. [CrossRef]

26. Zhao, Y.; Tong, M.; Lu, Z.; Xu, J. Monotonic Expression of Polynomial Normal Transformation Based on the First Four L–Moments. *J. Eng. Mech. ASCE* 2020, 146, 06020003. [CrossRef]

27. Tong, M.; Zhao, Y.; Lu, Z. Normal Transformation for Correlated Random Variables based on L–moments and its Application in Reliability Engineering. *Reliab. Eng. Syst. Saf.* 2020, 207, 107334. [CrossRef]

28. Tong, M.; Zhao, Y.; Zhao, Z. Simulating strongly non–Gaussian and non–stationary processes using Karhunen–Loève expansion and L–moments–based Hermite polynomial model. *Mech. Syst. Signal Process.* 2021, 160, 107953. [CrossRef]

29. Shelbey, S. *CRC Standard Mathematical Tables*; CRC Press: Boca Raton, FL, USA, 1975.

30. Rao, B.; Chowdhury, R. Factorized high dimensional model representation for structural reliability analysis. *Eng. Comput.* 2008, 25, 708–738. [CrossRef]

31. He, J.; Gao, S.; Gong, J. A sparse grid stochastic collocation method for structural reliability analysis. *Struct. Saf.* 2014, 51, 29–34. [CrossRef]