The Piecewise Polynomial Collocation Method for The Solution of Fredholm Equation of Second Kind By Using AGE Iteration

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Abstract. In this study, we study a piecewise approximate solution of Fredholm integral equation of the second kind by using the collocation discretization scheme. The discretization scheme has been used to build up a piecewise approximation equation which is the approximation equation form a system of linear equation. In the effort to get the numerical solution, the Gauss-Seidel iterative method and Alternating Group Explicit (AGE) have been treated as a linear solver. In addition to that, the formulation and the implementation of AGE method are included. Based on the numerical finding, the result showed that Alternating Group Explicit (AGE) iterative method is better than the Gauss-Seidel method.

1. Introduction

According to Starke and Niethammer, integral equations were found in early 1960’s [1]. The integral equations basically are the most important functions in various fields such as in mathematics, physics, and engineering [2]. The studies of integral equation have been used widely at both theoretical and computation levels since there are many fields were related to the understandings and concepts of integral functions. In this study, the second kind of Fredholm linear equation has been solved by collocation method of discretization scheme and piecewise constant polynomial. There are a few methods that were introduced to solve the linear Fredholm equation of second kind. Based on Dagan [3], the spurious solution has been aware in solving linear Fredholm equation of second kind. Besides, Zhong had applied integral mean value method to solve the linear Fredholm equation of second kind [4]. Not only that, Legendre and Block-Pulse functions also was introduced by Maleknejad and Tavasoli to solve the linear integral equation of second kind [5].

We introduced the linear Fredholm equation of second kind:

$$U(x) + \lambda \int_a^b K(x,t)U(t)dt = g(x), x \in [a,b]$$  \hspace{1cm} (1)
In the linear Fredholm equation of second kind, \( \lambda \) is provided with any real number based on the example that are given [6], where \( a \) and \( b \) were fixed values and kernel is a function for \( K(x, t) \)[7]. The collocation approximation equation of problem (1) came from the ideas of substituting the piecewise constants polynomial in the integral functions. Basically the coefficient matrix of linear system it dense and large matrix [8].

2. Derivative of Piecewise constant approximation equation

Principal of piecewise constant approximation is the basic understanding in this study to extract the information at various findings from the functions. The size of interval can be bigger or smaller according to the problems that will be operated. In effort to get the numerical solution of linear Fredholm equation of second kind, the range of integral is fitted at \( a \leq x \leq b \) as shown in Fig. 1.

![Figure 1. Distribution of uniformly nodes points on the integral \([a, b]\) at \( m = 10\)](image)

Let \( I_{[0,1]} = [a, b] \) be a class of piecewise constant function on the axis that defined on interval.

\[ I : a = x_1 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \]

Since this study considers the application of the collocation method to construct an approximation equations, the piecewise constant function can be declared as below. This will be previewed in every collocation points of the interval as follows

\[ T_i(X_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, j = 1, 2, \ldots, n \] (2)

Based on the piecewise constant function in equation (2), the function \( U(x) \) is a polynomial basis functions \( T_i(X_j), i = 1, 2, \ldots, n \). Then the approximation function of \( U(t) \) based on equation (2) can be stated as

\[ U(t) = \sum_{j=1}^{n} U_j T_j(t) \text{ Where } U_j = U(t_j) \] (3)

Eq. (3) is called as the piecewise constant polynomial function.

By substituting the polynomial function (3) into the function of \( U(x) \) in problem (1), we get

\[ \sum_{j=1}^{n} U_j T_j(t) + \lambda \int K(x, t) \sum_{j=1}^{n} U_j T_j(t) dt = g(x) \] (4)

And it follows by
\[
\sum_{j=1}^{n} U_j T_j(x) + \lambda \sum_{j=1}^{n} U_j \int_{a}^{b} K(x,t) T_j(t) dt = g(x)
\] (5)

Consider the collocation points \(x=x_i, i = 1,2,...,n\) and substituted into equation (5). It can be simplified as follows

\[
U_i + \lambda \sum_{j=1}^{n} U_j \left( \int_{a}^{b} K(x_i,t) T_j(t) dt \right) = g_i
\] (6)

For \(i=1, 2, 3, ..., n\) where \(g_i = g(x_i)\) and \(U_i = U(x_i)\). Then the equation (6) can be rewrite again as the following equation

\[
U_i + \lambda \sum_{j=1}^{n} K_{i,j} U_j = g_i
\] (7)

Where,

\[
K_{i,j} = \int_{a}^{b} K(x_i,t) T_j(t) dt
\]

By considering the approximation equation (7) this approximation equation can lead a generated linear system as

\[
KU = G
\] (8)

Where

\[
K = \begin{bmatrix}
1 - \lambda.K11 & -\lambda.K21 & -\lambda.K31 & -\lambda.K41 & -\lambda.K51 & -\lambda.K61 & -\lambda.K71 & -\lambda.K81 & \cdots & -\lambda.Kn1 \\
-\lambda.K12 & 1 - \lambda.K22 & -\lambda.K32 & -\lambda.K42 & -\lambda.K52 & -\lambda.K62 & -\lambda.K72 & -\lambda.K82 & \cdots & -\lambda.Kn2 \\
-\lambda.K13 & -\lambda.K23 & 1 - \lambda.K33 & -\lambda.K43 & -\lambda.K53 & -\lambda.K63 & -\lambda.K73 & -\lambda.K83 & \cdots & -\lambda.Kn3 \\
-\lambda.K14 & -\lambda.K24 & -\lambda.K34 & 1 - \lambda.K44 & -\lambda.K54 & -\lambda.K64 & -\lambda.K74 & -\lambda.K84 & \cdots & -\lambda.Kn4 \\
-\lambda.K15 & -\lambda.K25 & -\lambda.K35 & -\lambda.K45 & 1 - \lambda.K55 & -\lambda.K65 & -\lambda.K75 & -\lambda.K85 & \cdots & -\lambda.Kn5 \\
-\lambda.K16 & -\lambda.K26 & -\lambda.K36 & -\lambda.K46 & -\lambda.K56 & 1 - \lambda.K66 & -\lambda.K76 & -\lambda.K86 & \cdots & -\lambda.Kn6 \\
-\lambda.K17 & -\lambda.K27 & -\lambda.K37 & -\lambda.K47 & -\lambda.K57 & -\lambda.K67 & 1 - \lambda.K77 & -\lambda.K87 & \cdots & -\lambda.Kn7 \\
-\lambda.K18 & -\lambda.K28 & -\lambda.K38 & -\lambda.K48 & -\lambda.K58 & -\lambda.K68 & -\lambda.K78 & 1 - \lambda.K88 & \cdots & -\lambda.Kn8 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & -\lambda.Kn & 1 - \lambda.Knn
\end{bmatrix}
\]
Before deriving the formulation of AGE method, the full matrix of linear systems (8) needs to reduce into tridiagonal linear systems.

\[
KU = G
\]  

(9)

Where

\[
K = \begin{bmatrix}
1 - \lambda K11 & -\lambda K21 \\
-\lambda K12 & 1 - \lambda K22 & -\lambda K32 \\
-\lambda K13 & 1 - \lambda K23 & -\lambda K33 & -\lambda K43 \\
-\lambda K14 & 1 - \lambda K24 & -\lambda K34 & 1 - \lambda K44 & -\lambda K54 \\
-\lambda K15 & 1 - \lambda K25 & -\lambda K35 & 1 - \lambda K45 & -\lambda K55 & -\lambda K65 \\
-\lambda K16 & 1 - \lambda K26 & -\lambda K36 & 1 - \lambda K46 & -\lambda K56 & 1 - \lambda K66 & -\lambda K76 \\
-\lambda K17 & 1 - \lambda K27 & -\lambda K37 & 1 - \lambda K47 & -\lambda K57 & 1 - \lambda K67 & -\lambda K77 & -\lambda K87 \\
-\lambda K18 & 1 - \lambda K28 & -\lambda K38 & 1 - \lambda K48 & -\lambda K58 & 1 - \lambda K68 & 1 - \lambda K78 & 1 - \lambda K88 \\
1 - \lambda K99 & 1 - \lambda K109 \\
1 - \lambda K100 & 1 - \lambda K1010
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
U1 & U2 & U3 & U4 & U5 & U6 & U7 & U8 & \cdots & Un
\end{bmatrix}^T
\]

\[
G = \begin{bmatrix}
g1 & g2 & g3 & g4 & g5 & g6 & g7 & g8 & \cdots & gn
\end{bmatrix}^T
\]

3. Formulation of AGE Method

In this part, Alternating group Explicit and Gauss-seidel methods were applied in this study to solve the linear Eq. (8). GS iterative method was appointed to be control method in this research. In this section, the maximum value of \( r \) is fixed in between the range of \( 0 \leq r \leq 1 \). The values can be chosen freely between the ranges and will be executed by the computer programmer. The smallest number of iterations will be choosing as the best approximation.

We split the matrix \( K \) into the sum of two matrices.
\[ K = G_1 + G_2 \]  \hspace{1cm} (10)

 Basically, the AGE method was introduced by Evans [9] to solve any tridiagonal linear system as in equation (9). This method is undergone two steps to get the numerical results. Therefore, the basic formulation of this method is given by

\[ (G_1 + rI)u(k + \frac{1}{2}) = f - (G_2 - rI)u^{(k)} \]

\[ (G_2 + rI)u(k + \frac{1}{2}) = f - (G_1 - rI)u^{(k+1/2)} \]  \hspace{1cm} (11)

From the Eq. (12), the coefficient matrix K in linear system Eq. (9) will be decomposed into the sum of two matrices:

\[ A = G_1 + G_2 \]  \hspace{1cm} (12)

Where matrices \( G_1 \) and \( G_2 \) need to follow the following conditions:

i. \( (G_1 + rI) \) and \( (G_2 + rI) \) are non-singular for any \( r > 0 \).

ii. for any vectors \( c \) and \( d \) and for any constants \( r > 0 \) it is convenient to solve the systems explicitly, i.e. :

\[ u^{(k+1/2)} = G_1^{-1}c \]

\[ u^{(k+1)} = G_2^{-1}d \]

for \( u^{(k+1)} \) and \( u^{(k+1/2)} \), respectively.

\( G_1 \) and \( G_2 \) are small (2 x 2) block systems or made up according to the changes of their rows and corresponding columns[10]. In general, by considering the splitting of the coefficient matrix, \( K \) in eq. (9), let us define two sub-matrices \( G_1 \) and \( G_2 \) being given as

\[
G_1 = \begin{bmatrix}
q_0 & q_1 & r_1 \\
p_1 & q_1 & q_2 \\
p_2 & r_2 & q_2 \\
q_3 & r_3 & q_3 \\
p_3 & q_3 & q_4 \\
p_4 & q_4 & q_4 \\
q_5 & & & \\
\end{bmatrix}
\]

\[
G_2 = \begin{bmatrix}
q_0 & r_0 & p_0 \\
q_0 & p_0 & q_1 \\
q_1 & p_1 & q_1 \\
q_2 & r_2 & q_2 \\
q_2 & p_2 & q_2 \\
q_3 & r_3 & q_3 \\
q_3 & p_3 & q_3 \\
q_4 & r_4 & q_4 \\
q_4 & p_4 & q_4 \\
\end{bmatrix}
\]

If \( n \) is even, with \( q_n = \frac{q_n}{2} \), where \( G_1 \) and \( G_2 \) satisfy the condition that \( G_1 + rI \) and \( G_2 + rI \) non-singular for any \( r > 0 \). By substituting equation (13) into tridiagonal linear system (9), now the linear system can be rewritten in the form:

\[ (G_1 + G_2)u = G \]
And by following a strategy similar to the ADI method [11], $u^{(k+1/2)}$ and $x^{(k+1)}$ can be determined implicitly by:

\[(G_1 + rl) u(k + \frac{1}{2}) = G - (G_2 - rl) u^{(k)}\]

\[(G_2 + rl) u(k + \frac{1}{2}) = G - (G_1 - rl) u^{(k+1/2)}\]  

(13)

Explicitly by:

\[u^{(k+1/2)} = (G_1 + rl)^{-1} (G - (G_2 - rl) u^{(k)})\]

\[u^{(k+1)} = (G_2 + rl)^{-1} (G - rl) u^{(k+1/2)}\]  

(14)

Where $r$ is the iteration parameter, given by [12]:

\[r = (uv)^{1/2}\]

In this case, $u$ and $v$ are the minimum and maximum eigenvalue of the sub matrices of $G_1$ and $G_2$. For the other case, the second smallest eigenvalue is considered if the sub matrices are singular and the smallest eigenvalue is zero [13].

Based on the formulation (15), the implementation of AGE method can be represented by Algorithm 1 as follow.

Algorithm 1. AGE method

i. Initialize $u^{(0)} = 0, \varepsilon = 10^{-10}$.

ii. Calculate matrix K and G.

iii. For first stage calculate,

\[u^{(k+1/2)} = (G_1 + rl)^{-1} (G - (G_2 - rl) u^{(k)})\]

iv. For second stage calculate,

\[u^{(k+1)} = (G_2 + rl)^{-1} (G - rl) u^{(k+1/2)}\]

v. Convergence test $\left| u^{(k+1)} - u^{(k)} \right| < \varepsilon = 10^{-10}$. If the convergence criterion is satisfied, go to step vi. Otherwise go back to step iii.

vi. Display approximation solution.

4. Computational Experiments

In order to test the applicability of the presented methods, we are considering three examples and three parameters in this section which are number of iterations (K), time taken (Time) and the maximum absolute errors (Abs error) to distinguish with the three examples. We are performing the GS method and AGE method in this study. The value of tolerance error set up as constant $\varepsilon = 10^{-10}$ at different sizes grids, $m = 512, 1024, 2048, 4096$ and $8192$.

Example 1

Consider the following equation is Fredholm integral equation.
\[ U(x) = e^{-x} - \frac{1}{2} + \frac{1}{2} e^{(x+1)} + \frac{1}{2} \int_0^1 (x+1)e^{-y}U(y)dy \]  
\[ \text{(15)} \]

The exact value of problem (11) is given by:

\[ U(x) = e^{-x} \]
\[ \text{(16)} \]

**Example 2**[14]

Consider the following equation is Fredholm integral equation.

\[ U(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xtU(t)dt, \; 0 < x < 1 \]
\[ \text{(17)} \]

The exact value of problem (13) is given by:

\[ U(x) = e^{3x} \]
\[ \text{(18)} \]

**Example 3**[15]

Consider the following equation is Fredholm integral equation.

\[ U(x) = e^x - 1 + \int_0^1 tU(t)dt \]
\[ \text{(19)} \]

The exact value of problem (15) is given by

\[ U(x) = e^x \]
\[ \text{(20)} \]

**Table 1**: Comparison of a number of iterations, execution time (seconds) and maximum absolute error

| EXAMPLE | M   | NUMBER OF ITERATIONS (K) | TIME(second)(T) | ERROR (Abs error) |
|---------|-----|--------------------------|-----------------|-------------------|
|         |     | GS AGE GS AGE GS AGE GS AGE |                 |                   |
| 1       | 512 | 23 7 9.05 6.86 3.76216E-07 4.723211e-07 |               |                   |
|         | 1024| 24 7 37.72 27.18 9.420403E-08 1.185815e-07 |               |                   |
|         | 2048| 24 7 150.63 108.54 2.356128E-08 2.970804e-08 |               |                   |
|         | 4096| 24 7 602.66 434.2 5.898283E-09 7.434852e-09 |               |                   |
|         | 8192| 24 7 2411.21 1736.7 1.482500E-09 1.859704e-09 |               |                   |
| 2       | 512 | 15 8 1.02 0.83 1.909286e-05 8.899050e-05 |               |                   |
|         | 1024| 15 6 3.24 3.13 4.751341e-06 5.593890e-06 |               |                   |
|         | 2048| 15 6 12.79 12.46 1.185067e-06 5.938900e-06 |               |                   |
for the iterative methods

| Example | Percentage | Number of Iterations (%) | Time (second)(%) |
|---------|------------|--------------------------|-----------------|
| 1       | 69.57-70.83| 24.20-27.97              |
| 2       | 46.67-60.00| 2.58-18.63               |

the number of iterations and execution time for SOR iteration methods compared with GS method.
From the three examples above, the results of numerical computational were calculated from the application of GS and AGE iterative methods. The results have been setting up in the Table 1 with different grid sizes, \( m = 512, 1024, 2048, 4096 \) and 8192. Meanwhile, Table 2 is showed the reduction percentage of AGE iteration as compared with GS iteration. Based on the numerical results from Tables 1 and 2, the results obviously explained that the number of iterations and execution time of AGE iteration are smaller than GS method. The number of iterations of AGE iterative method compared to GS iterative method has declined by 69.57%-70.83%, 46.67%-60.00%, 66.67-66.67% respectively. We also can see that the execution time of AGE iterative method also has declined by 24.20%-27.97%, 2.58%-18.63%, 13.93%-15.18% respectively as compared with GS iterative methods.

5. Conclusion
In this paper, the approximate solution of linear Fredholm integral equations of the second kind is derived by combining collocation method and piecewise approximation. The reliability of the method is shown by operating the three examples in the computational experiments. From the result of numerical simulations, the AGE method is previewed an efficient solution compared to GS method which is showed less accuracy. The comparison between AGE method and GS method obviously showed that AGE method is the best method to carry out the best solutions.

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