Abstract. We generalize our method for $GL_2 \times GL_1$ to the subconvexity for $L$-functions appearing in Waldspurger’s formulae, a special case for $GL_2 \times GL_2$. In this sense, the case for $GL_2 \times GL_1$ is regarded as the subconvexity for split toric integral. Both were sketched in Venkatesh’s paper. Surprisingly enough, this bound survives from the best known bounds for $GL_2 \times GL_1$ and for $GL_2 \times GL_2$ with a large “probability”. This is in some sense equivalent to saying that Linnik’s conditional ergodic approach for his equidistribution theorem has so far not yet been completely covered by other methods.

1. Introduction

1.1. Statement of the Main Result. Let $\pi_1, \pi_2$ be two generic automorphic representation of $GL_2$ over a number field $F$. The subconvexity problem for the Rankin-Selberg $L$-function associated with $\pi_1 \times \pi_2$ concerns the estimation of $L(1/2, \pi_1 \times \pi_2)$. If we fix $\pi_1$ and let $\pi_2$ vary, this is an important sub-problem. The convex bounds states

\[ L(\frac{1}{2}, \pi_1 \times \pi_2) \ll F, \pi_1, \epsilon \ C(\pi_2)^{\frac{1}{2} + \epsilon}, \]

where $C(\pi_2)$ is the analytic conductor of $\pi_2$. Any type of estimation as follows, for certain constant $\delta > 0$,

\[ L(\frac{1}{2}, \pi_1 \times \pi_2) \ll F, \pi_1, \epsilon \ C(\pi_2)^{\frac{1}{2} - \delta + \epsilon}, \]

is called a subconvex bound.

The subconvexity problem for Rankin-Selberg $L$-functions has been an active and important research problem since decades ago. It’s important not only because it’s a natural generalization of subconvexity to higher degree $L$-functions, but also because of its applicability to other important problems. For example, Sarnak [33, Theorem 0.2] established (1.1) in the special case over $F = \mathbb{Q}$, $\pi_1, \pi_2$ being cusp forms, $\pi_1$ holomorphic of weight $k$ and the bound with respect to $k$. He also pointed out its relation with the quantum unique ergodicity problem. This essentially marked the starting point of the investigation of subconvex bounds like (1.1). Later, in the series of papers [24, 27, 15], the authors established effective versions of (1.1) in the $q$-aspect, which implies effective versions of the equidistribution of Heegner points. Harmless to say, there are a lot of other sophisticated versions of (1.1).

The first remarkable general version of (1.1) appears in [28], with an unspecified exponent saving $\delta$ (but can be deduced explicitly as we shall see later in this paper) but completely general with respect to all aspects of $\pi_2$.

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So far, all the above mentioned bounds were established using the integral representation of the concerned \(L\)-functions on \(\text{GL}_2\), which may have other integral representations on other groups if we restrict to some sub-class of \(\pi_1, \pi_2\). Although Langlands’ functoriality principle indicates that the largest family of \(L\)-functions should be on the matrix groups \(\text{GL}_n\), would the integral representations on other groups be completely covered by those on \(\text{GL}_n\) in establishing subconvex bounds like \([1.1]\)?

In this paper, we are interested in a subconvexity problem, whose generality lies in between the one considered in \([44]\) (for \(\text{GL}_2 \times \text{GL}_1\)) and the one in \([28]\) (for \(\text{GL}_2 \times \text{GL}_2\)). We shall be particularly interested in its integral representation not on the matrix groups.

Let \(\mathbb{A}\) be the adele ring of a number field \(F\) and \(\pi\) be a cuspidal representation of \(\text{GL}_2(\mathbb{A})\) with central character \(\omega\). Let \(E\) be a quadratic field extension of \(F\) and \(\Omega\) a Hecke character of the idele group of \(E\) such that the restriction of \(\Omega\) on the idele group of \(F\) coincides with \(\omega^{-1}\). Denote by \(\pi_E\) the base change of \(\pi\) to \(\text{GL}_2(\mathbb{A}_E)\), where \(\mathbb{A}_E\) is the adele ring of \(E\). By a theorem of Tunnell \([36]\) and Saito \([32]\), if the epsilon factor

\[
\epsilon(1/2, \pi_E \otimes \Omega) = 1,
\]

then there exists a unique quaternion algebra \(B\) defined over \(F\) containing \(E\) such that:

1. The Jacquet-Langlands lifting \(\text{JL}(\pi) = \text{JL}(\pi; B)\) of \(\pi\) to \(G(\mathbb{A})\) exists, where \(G = G_B\) is the \(F\)-group of the invertible elements in \(B\);
2. The central value \(L(1/2, \pi_E \otimes \Omega)\) (or the corresponding complete \(L\)-function \(\Lambda(1/2, \pi_E \otimes \Omega)\)) admits a period representation on the space of \(\text{JL}(\pi)\), called Waldspurger’s formula \([39]\).

Our main result is a conditional hybrid subconvex bound of \(L(1/2, \pi_E \otimes \Omega)\):

**Theorem 1.1.** Let \((E, \Omega)\) vary and satisfy \([1.2]\). Assume that for some fixed constants \(0 \leq \delta_0 < 1, d > 0\) we have (writing \(q_v\) for the cardinality of the residue field of \(F_v\))

\[
\left| \{ v \in V_F : v < \infty, q_v \leq E, v \text{ splits in } E \} \right| \gg \frac{E^{1-\delta_0}}{\log E}
\]

as long as \(E \geq D(E)^{d}\). For any \(\epsilon > 0\) and sufficiently large \(A > 0\), with \(B, \delta\) as in Theorem \([4.5]\) and \(\delta'\) as in Theorem \([4.2]\) we have

\[
L\left(\frac{1}{2}, \pi_E \otimes \Omega \right) \ll_{\epsilon, \delta, \delta'} \left( D(E)C(\Omega) \right)^{\frac{1}{2}+\epsilon} \max( D(E)^{-\frac{\delta(1-\delta_0)}{2(2+\beta-\delta_0)}}, C(\Omega)^{-\frac{1}{2}(1-\delta_0)}D(E)^{-\frac{d}{4(1-\delta_0)}} )
\]

under the condition

\[
C(\Omega) \geq D(E)^{-\frac{d(2+\beta-\delta_0)-2\delta}{1-2\beta}}
\]

where \(D(E)\) is the discriminant of \(E\), and \(C(\Omega)\) is the analytic conductor of \(\Omega\).

**Remark 1.2.** It should be noted that this is just an effective version of the discussion in \([37]\) Section 7. In some sense, this subconvex bound is related to Linnik’s ergodic approach of his equidistribution theorem. Similarly, \([44]\) is an effective version of the discussion in \([37]\) Section 6.

**Remark 1.3.** The relation between Theorem \([4.4]\) and Theorem \([4.5]\) seems to have a qualitative version in terms of ergodic theory \([28]\) Section 1.2 & 1.3. In this sense, Theorem \([4.4]\) can also be considered as a quantitative version of the implication (B) \(\Rightarrow\) (C) in \([28]\) Section 1.2.4.

**Remark 1.4.** Note that the possible \(B\) to which we apply Theorem \([4.5]\) is finitely many once \(\pi\) is fixed, hence the implicit constant depending on \(B\) actually depends on \(\pi\).

**Remark 1.5.** Still by Tunnell \([36]\) and Saito \([32]\), if \(1.2\) is not satisfied, then

\[
\epsilon(1/2, \pi_E \otimes \Omega) = -1.
\]

In this case, the derivative \(\frac{d}{ds}L(s, \pi_E \otimes \Omega)\) \(|_{s=1/2}\) admits a period representation, known as a Gross-Zagier formula (c.f. \([46]\)). It would be interesting to see if our method can give some non-trivial estimation of
the derivative in this case. Note that if in addition \( \Omega^{-2} = \omega \circ \text{Nf}_E \), necessarily we have by the functional equation

\[
L\left( \frac{1}{2}, \pi_E \otimes \Omega \right) = 0,
\]

which is a trivial case of the subconvexity problem.

**Remark 1.6.** The assumption (1.3) for \( \delta_0 = 0 \) is a strong effective version of the Chebotarev density theorem. It may require a (double)-logarithmic power saving in the estimation of the error term in [17] Theorem 5.13 (5.51) for example to

\[
O\left( x \exp\left( -\frac{cd^{-4}(\log x)(\log \log x)}{\sqrt{\log x} + 3 \log q(f)} \right) \right)
\]

for the Dedekind zeta-function \( \zeta_E(s) \). Without this saving, i.e. with the current zero-free region, one would need to assume

\[
C(\Omega) \geq D(E)^{c \log \log D(E)}
\]

for some constant \( C > 0 \), which makes the contribution of \( D(E) \) on the right side completely absorbed by the one of \( C(\Omega) \). By contrast, a saving with \( \log \log x \) replaced by \( (\log \log x)^{1+\epsilon_0} \) for any \( \epsilon_0 > 0 \) would make the restriction on the relation between \( D(E) \) and \( C(\Omega) \) disappear. We will see that with a large “probability” \( \delta_0 = 0, d = 1/44 \) is permitted in our future paper [25].

For historical remarks and the intuition of the method, we suggest [44, Section 1.1 & Section 3]. We closely follow our method in [44], except two differences:

1. We no longer constrain ourselves to test functions made from new vectors, but use some subspaces. This is due to the fact that the corresponding local estimates are much harder with the new vectors than in [44]. The method is inspired from [12].

2. In order to emphasize on the intuition of equidistribution properties, we no longer translate the test vector/function but conjugate the embedding of \( E \) into \( B \), the two viewpoints being equivalent.

**Remark 1.7.** The reason for which the use of subspaces simplifies the estimation of local factors is roughly as follows: in the induced model of a local representation \( \pi_v = \text{Ind}_{v_0}^{G_v} \rho_v \), a subspace interesting to us is the space of functions supported in \( J_{v_0} K_v \) for some \( v_0 \in G_v \), while an interesting vector, the new vector for example, is such a function with specific values on \( K_v \). The local factor is roughly an orthogonal projector onto the subspace resp. the interesting vector. It is reasonable that projection onto a subspace is easier to calculate than the one onto a specific vector in that subspace.

1.2. Comparison with Known Results. It is necessary to compare the result given here with the one in [28], in which the following more general type of subconvex bound is obtained. Let \( \pi \) be a fixed cuspidal representation of \( \text{GL}_2(A) \) as above. Let \( \pi' \) vary over generic automorphic representation of \( \text{GL}_2(A) \). The main result of [28] is that for any \( \delta > 0 \), we have the following subconvex bound for the Rankin-Selberg

\[
L\left( \frac{1}{2}, \pi \times \pi' \right) \ll_{F, \epsilon} C(\pi')^{\frac{1}{2} - \delta + \epsilon}
\]

for some constant \( \delta > 0 \). In particular, if \( \pi' = \pi(\Omega) \) is the one associated with \( \Omega \) via theta correspondence, we find

\[
L\left( \frac{1}{2}, \pi_E \otimes \Omega \right) = L\left( \frac{1}{2}, \pi \times \pi(\Omega) \right) \ll_{F, \epsilon} C(\pi(\Omega))^{\frac{1}{2} - \delta + \epsilon}.
\]

First, we relate \( C(\pi(\Omega)) \) with \( D(E)C(\Omega) \).

**Proposition 1.8.** Let \( \pi = \otimes_v \pi_v \) be a generic automorphic representation of \( \text{GL}_d(A) \) for \( d \in \mathbb{N} \). Let \( \psi = \otimes_v \psi_v \) be an additive character of \( F \setminus A \).

1. At \( v < \infty \), the epsilon factor takes the form

\[
\epsilon_v(s, \pi_v; \psi_v) = \epsilon_v(\frac{1}{2}, \pi_v; \psi_v)(C(\psi_v)^dC(\pi_v))^{\frac{1}{2} - s},
\]

where \( \epsilon_v(1/2, \pi_v; \psi_v) \) is a complex number of absolute value 1, the so-called root number.
Remark 1.9. This basically gives a relation between the (analytic) cond uctor \( C(\pi_v) \) of an automorphic representation \( \pi \) and the one \( C_L(s, \pi) \) of its \( L \)-function \( L(s, \pi) \), defined for example in [17 (5.7)]:

\[
C_L(0, \pi) = C(\pi)D(F)^d.
\]

Proof. (1) For the case \( d = 1 \) resp. \( d = 2 \), see the calculation in [31 Section 7.1] resp. the remark just before [6 Section 2]. For general case \( d > 2 \), see [21].

(2) The assertion concerning the epsilon factor is in the discussion of [14 Theorem 8.7] appeared on the page where [14 Theorem 8.8] appears. It then follows from the proof of [14 Theorem 8.7] that the local \( L \)-function \( L_v(s, \pi_v) \) takes the form

\[
L_v(s, \pi_v) = \prod_{i=1}^d \Gamma_{F_v}(s + s_i),
\]

where \( \Gamma_{F_v} = \zeta_v \) is the local zeta-function defined below in Section 2.1 and the constants \( s_i \in \mathbb{C} \) encode the spectral parameters of \( \pi_v \). By definition, we have

\[
C(\pi_v) = \prod_{i=1}^d (|s_i| + 3),
\]

which is visibly determined by \( L_v(s, \pi_v) \).

\( \square \)

Corollary 1.10. Let \( v \) denote a place of \( F \) and \( \psi_v \) be à la Tate. Define \( C(\Omega_v) = \Pi_w|_vC(\Omega_w), E_v = \Pi_w|_vE_w, \) etc.

(1) If \( v < \infty \), then we have

\[
C(\pi(\Omega_v)) = C(\Omega_v)D(E_v)D(F_v)^{-2}.
\]

(2) If \( v | \infty \), then we have

\[
C(\pi(\Omega_v)) = C(\Omega_v).
\]

Consequently, we have \( C(\pi(\Omega)) = C(\Omega)N_{E/F}D(E/F) \), where \( d(E/F) \) is the relative discriminant.

Proof. (1) By [19 Theorem 4.7 (iii)] we have a relation of epsilon factors, writing \( \pi_v = \pi(\Omega_v) \)

\[
\epsilon(s, \pi_v; \psi_v) = \epsilon(s, \Omega_v; \psi_{E_v})\lambda(E_v/F_v; \psi_v),
\]

where \( \psi_{E_v}(x) = \psi_v(\text{Tr}_{F_v}^{E_v}(x)) \) and \( \lambda(E_v/F_v; \psi_v) \) is the local Weil index for quadratic spaces (whose product over \( v \) equals 1). By the Proposition we get

\[
\epsilon(\frac{1}{2}, \pi_v; \psi_v) = \epsilon(\frac{1}{2}, \Omega_v; \psi_{E_v})\lambda(E_v/F_v; \psi_v)C(\psi_{E_v})^2C(\pi_v) = C(\psi_{E_v})C(\pi_v).
\]

The assertion follows from \( C(\psi_v) = D(F_v), C(\psi_{E_v}) = D(E_v), \) almost by definition.

(2) This is obvious by the Proposition.

The last assertion follows by [34 Proposition III.8].

\( \square \)

Next, we compare the quality of subconvex bounds in the two results. Recall that the method of [28 is based on the integral representation [28 (4.21)] of the triple product \( L \)-function

\[
L(\frac{1}{2}, \pi' \times \pi \times \pi_3) = L(\frac{1}{2}, \pi' \times \pi)^2;
\]
where \( \pi_3 = \pi(1, (\omega^J)^{-1}) \) with \( \omega^J \) resp. \( \omega \) the central character of \( \pi' \) resp. \( \pi \). The bound for the local terms compensates the convex bound as usual and they were reduced to bounding the integral at the right hand side of

\[
L\left(\frac{1}{2}, \pi' \times \pi \right) \ll_{F, \epsilon, \pi} C(\pi') \frac{1}{2} \int_{GL_2(F)Z \backslash GL_2(\mathbb{A})} \varphi'(g) \varphi(g) E(g) dg,
\]

where \( \varphi' \in \pi' \), \( \varphi \in \pi \), \( E \in \pi_3 \) are properly chosen test functions. To this end, they applied a method of amplification due to Duke and Iwaniec, which is explained in detail in [37, Section 4.1]. As we do not intend to be too precise, we may assume the Ramanujan-Petersson conjecture for simplicity. Then the bound \([37, (4.2)]\) for the above period roughly gives

\[
L\left(\frac{1}{2}, \pi' \times \pi \right) \ll_{F, \epsilon, \pi} C(\pi') \frac{1}{2} + \epsilon.
\]

Specializing to the case \( \pi' = \pi(\Omega) \) it gives

\[
(1.4) \hspace{2cm} L\left(\frac{1}{2}, \pi \otimes \Omega \right) \ll_{F, \epsilon} (D(E)C(\Omega)) \frac{1}{2} + \epsilon.
\]

If we can obtain (for example, by making \([44] \) effective on the dependence on \( \pi \) and \( F \)) for the character-twisted \( L \)-function (still assuming the Ramanujan-Petersson conjecture)

\[
L\left(\frac{1}{2}, \pi \otimes \chi \right) \ll_{\epsilon} D(F)B^{\epsilon + \epsilon}(C(\pi)C(\chi)^2 \frac{1}{2} + \epsilon (C(\pi))B^{-\frac{1}{2}} C(\chi)^{-\frac{1}{2}})
\]

then we can deduce from it the following bound

\[
(1.5) \hspace{2cm} L\left(\frac{1}{2}, \pi \otimes \Omega \right) \ll_{F, \epsilon} (D(E)B^{\epsilon + 2(B-\frac{1}{2})+\epsilon} C(\Omega) \frac{1}{2} + \epsilon,
\]

and Theorem \([4, 3]\) for \( \delta = 1/8 \). We assume Theorem \([4, 3]\) holds for \( \delta' \leq \delta/2 \), then Theorem \([11]\) gives (for \( \delta_0 = 0, d = 1/16(2 + B) \))

\[
L\left(\frac{1}{2}, \pi \otimes \Omega \right) \ll_{F, \epsilon} (D(E)C(\Omega)) \frac{1}{2} + \epsilon \cdot D(E) C(\pi) C(\chi)^{-\frac{1}{2} + \epsilon}
\]

which is better than both \([1.4] \) and \([1.5] \) for a large “probability” as long as \( B < 1 \) and

\[
D(E)^{\frac{1}{2} + \epsilon} < C(\Omega) < D(E)^{8(2 + B)(2 + \frac{4H}{2H}) + \frac{B}{2H}}
\]

The expected \( B \), best using the current technology would be \( B = 3/4 \) (c.f. discussion in Section 5.1), in which case the interval becomes

\[
D(E)^{8} < C(\Omega) < D(E)^{\frac{17}{2} + \frac{4B}{2H}}
\]

which is non trivial even for \( B' = 0 \).

1.3. Plan of the Paper. The paper is organized as follows.

Section 2 consists of some recall and adaptation of known results from the literature. After fixing notations in Section 2.1, we recall our departure point, the Waldspurger’s formula, in Section 2.2 with its version in terms of subspaces. We then develop its counterpart of Eisenstein series in Section 2.3 and 2.4, historically first considered by Wielonsky (although the formula is different from Wielonsky’s, we still call it Wielonsky Formula). In Section 2.5 we give a specific parametrization of local embeddings of a quadratic separable extension into the matrix algebra, with respect to which we optimize the local factors (c.f. point (2) before Remark \([147] \)). We point out the geometric meaning of the local factors with subspaces in Section 2.6. We finally recall the classification of the supercuspidal representations of \( GL_2 \) in Section 2.7, presented in a way convenient for our purpose.

We make the necessary local estimations case by case in Section 3. In Section 3.1 a generalization of \([44, Section 4.1] \) to \( K_\infty \)-finite vectors is given. The main tool is a variant of the classical stationary phase method. Section 3.3 and 3.4 are easy cases of estimations at non-archimedean places. Section 3.5 and 3.6 treat the essential part of the local estimations, one for ramified non-supercuspidals and the other for supercuspidals. They are similar in structure: we first identify our subspaces in the induced model, then transform the problem into a measure calculation of the intersection of two double cosets inside \( G_v \).
The global estimation in Section 4 is only slightly different from \cite{12} Section 6 in principle. We insert the results from two problems of subconvexity on average Theorem \cite{13} and \cite{19} together with one arithmetic problem \cite{13} of split places in a quadratic field extension into the estimation in Section 4.4, 4.5 and 4.6, leaving their proofs as research topics in the near future.

Finally we discuss known special cases and possible ways to resolve the two above mentioned problems of subconvexity on average in Section 5.

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2. SOME PRELIMINARIES

2.1. Notations and Conventions.

2.1.1. **Algebras, Groups and Spaces.** We fix a number field \( \mathbb{F} \). We reserve \( v \) to denote a place of \( \mathbb{F} \), and \( \mathbb{F}_v \) is the completion of \( \mathbb{F} \) at \( v \) with respect to the associated absolute valuation \( | \cdot |_v \). \( \mathfrak{o}_v \) is the ring of integers of \( \mathbb{F}_v \). For \( v \leq \infty \), \( \mathfrak{o}_v \) is its completion at \( v \) with a uniformizer \( \varpi_v \) and maximal ideal \( \mathfrak{p}_v = \varpi_v \mathfrak{o}_v \), \( q_v \) is the cardinality of the residue class field \( \mathfrak{o}_v / \mathfrak{p}_v \). \( \mathbb{A}_v \) is the adele ring of \( \mathbb{F}_v \). \( \mathbb{A}_v^\times \) is the idele group. Note that all the notations here make sense for any number field, especially for a quadratic extension \( \mathbb{E}/\mathbb{F} \).

For any \( \mathbb{F} \)-variety \( V \), we denote by \( V(\mathbb{F}_v) \) resp. \( V(\mathbb{A}_v) \) the \( \mathbb{F}_v \)- resp. \( \mathbb{A}_v \)-points of \( V \). This notation applies in particular to \( V = G/H \) or \( V = H \backslash G \), where \( G \) is an \( \mathbb{F} \)-group and \( H \) is a closed \( \mathbb{F} \)-subgroup of \( G \).

For a quaternion algebra \( \mathbb{B} \) over \( \mathbb{F} \), Ram(\( \mathbb{B} \)) is the set of places of \( \mathbb{F} \) at which \( \mathbb{B} \) is ramified, i.e. at which \( \mathbb{B}_v \) is a division algebra over \( \mathbb{F}_v \). Note that Ram(\( \mathbb{B} \)) is a finite set of even cardinality and determines \( \mathbb{B} \) up to \( \mathbb{F} \)-isomorphisms. We choose a maximal \( \mathfrak{o} \)-order \( \mathfrak{O} = \mathfrak{O}_\mathbb{B} \) of \( \mathbb{B} \) together with \( \mathbb{B} \), and at each \( v \notin \text{Ram}(\mathbb{B}) \), we fix an \( \mathbb{F}_v \)-isomorphism

\[
\delta_v : \mathbb{B}_v \rightarrow M_2(\mathbb{F}_v)
\]

such that if \( v < \infty \) then \( \delta_v(\mathfrak{O}_v) = M_2(\mathfrak{o}_v) \). Hence each time when we mention \( \mathbb{B} \) we talk about the triple (\( \mathbb{B}, \mathfrak{O}, \delta \)). We will specify the choice of \( \mathfrak{O} \) when necessary. We don’t distinguish \( \mathbb{B}(\mathbb{F}_v) \) and \( M_2(\mathbb{F}_v) \) at \( v \notin \text{Ram}(\mathbb{B}) \) in the sequel. The group \( G = G_\mathbb{B} = \mathbb{B}^\times \) of invertible elements of \( \mathbb{B} \) is an algebraic group defined over \( \mathbb{F} \), with center \( \mathbb{Z} \) isomorphic to the multiplicative group scheme \( \mathbb{G}_m \). At each \( v \notin \text{Ram}(\mathbb{B}) \), let \( K_v \) be the (maximal) compact subgroup of \( G(\mathbb{F}_v) \) corresponding to the standard (maximal) compact subgroup of \( \text{GL}_2(\mathbb{F}_v) \) under \( \delta_v \), i.e. to \( \text{SO}_2(\mathbb{R}) \) if \( \mathbb{F}_v = \mathbb{R} \), to \( \text{SU}_2(\mathbb{C}) \) if \( \mathbb{F}_v = \mathbb{C} \) or to \( M_2(\mathfrak{o}_v) \) if \( v < \infty \); at each \( v < \infty \), let \( K_v = \mathfrak{O}_v^\times \). Note that at \( v | \infty, v \in \text{Ram}(\mathbb{B}) \), \( \mathbb{Z}(\mathbb{F}_v) \) is compact. We equip the quotient space \( X(\mathbb{B}) = G(\mathbb{F})Z(\mathbb{A}) \backslash G(\mathbb{A}) \) with the Tamagawa measure \( dg \) on \( Z(\mathbb{F}) \) quotient by the discrete measure of \( Z \backslash G(\mathbb{F}) \), still denoted by \( dg \). The space \( L^2(X(\mathbb{B}), dg) \), or more generally \( L^2(G_\mathbb{B}, \omega) = L^2(G_\mathbb{B}, \omega; dg) \) of functions on \( G(\mathbb{A}) \) covariant as \( \omega \) under \( Z(\mathbb{A}) \), is then a Hilbert space equipped with a \( G(\mathbb{A}) \)-invariant inner product \( (\cdot, \cdot)_X(\mathbb{B}) \). The total mass, i.e. the Tamagawa number \( \text{Vol}(X(\mathbb{B}), dg) = 2 \). If \( \mathbb{B} \) is split over \( \mathbb{F} \), then we have the usual notion of \( L^2_0(G_\mathbb{B}, \omega) \); if \( \mathbb{B} \) is non-split over \( \mathbb{F} \) we recall that \( L^2_0(G_\mathbb{B}, \omega) \) is the ortho-complement of the one-dimensional \( G(\mathbb{A}) \)-characters in \( L^2(G_\mathbb{B}, \omega) \). Any unitary irreducible representation \( \sigma \) occurring in \( L^2_0(G_\mathbb{B}, \omega) \) is called a cuspidal (automorphic) representation of \( G(\mathbb{A}) \).

Consider the varying pairs \((E, \iota)\), where \( E \) is a quadratic field extension of \( \mathbb{F} \) and \( \iota : E \rightarrow B \) is an \( \mathbb{F} \)-embedding. Such a pair is called a(n) \((B)\)-admissible pair. We reserve \( w \) to denote a place of \( E \) and write \( E_v = \prod_{w \mid v} E_w \). At each \( v \in \text{Ram}(\mathbb{B}) \), \( E_v \) must be non-split. Conversely, for every \( E \) such that \( E_v \) is
non-split at each \( v \in \text{Ram}(B) \) one can find an \( F \)-embedding \( \iota : E \to B \). \( \iota(B) \) is a non-split \( F \)-subtorus \( T \) in \( G \). Via \( \iota \) we have

\[ E^x K^x \backslash A_E \simeq T(F)Z \backslash T(A), \]

which is a compact group. We denote by \( \text{Ram}_\infty(E/F) \) the set of \( v \mid \infty \) such that \( F_v = \mathbb{R}, E_v = \mathbb{C} \); by \( \text{Ram}(E/F) \) the set of \( v < \infty \) such that \( E_v/F_v \) is a ramified field extension. Let \( \text{Ram}(E/F) = \text{Ram}_\infty(E/F) \cup \text{Ram}(E/F) \). The quadratic Hecke character associated to \( E/F \) is denoted by \( \eta = \eta_{E/F} \).

The ring of integers of \( E \) is defined as follows. At \( \infty \) which is a compact group. We denote by \( \text{Ram}(\psi \psi^{-1}) \) the set of \( v \mid \infty \) such that \( F_v = \mathbb{R} \). The ring of integers of \( E \) will be denoted by \( O \). The local maximal ideals will be denoted by \( \mathcal{P}_w \).

Once \( (E, \iota) \) is chosen, one can find a unique coset \( \varepsilon \eta \text{Norm}_E(E^x) \in \text{F}^x/\text{F}_E(E^x) \) such that \( B = (E, \varepsilon) \) in the sense of the first DEFINITION on \( [38, \text{p.1}] \). \( B \) can be thus identified with the matrix group

\[ B_\varepsilon = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b \in E \right\} \]

where \( x \mapsto x, x \in E \) is the action of the unique non trivial element in the Galois group of \( E/F \). Under this identification, \( G \) is identified with \( G_\varepsilon = B_\varepsilon^\times \). We write

\[ u = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}. \]

This setting is particularly practical for the application of the relative trace formulae.

In \( \text{GL}_2 \), for local or global variables \( x \in F_v \) or \( \mathbb{A} \), \( y \in F_v^x \) or \( \mathbb{A}_F^x \), we define

\[ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, n_-(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}. \]

\( B \) is the subgroup of upper triangular matrices, \( A \) is the subgroup of diagonal matrices, \( A_1 \) is the diagonal matrices with lower element 1, \( N \) resp. \( N_- \) is the subgroup of unipotent upper resp. lower triangular matrices, \( B_1 = N A_1 \). They are \( F_v \)-(or \( F \))-groups. Locally, \( K_v \) is the standard maximal compact subgroup of \( \text{GL}_2(F_v) \). If \( v < \infty \), we define subgroups of \( K_v \) for an integer \( m \geq 0 \)

\[ K_{0,v}(p_v^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v : c \in p_v^m \right\}, K_v(p_v^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v : b, c \in p_v^m, a, d \in 1 + p_v^m \right\}; \]

\[ \mathcal{K}_{0,v}(p_v^m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v : b \in p_v^m \right\}. \]

For any \( F \)-algebraic group \( H \), we may sometimes write \( H_v = H(F_v) \) for simplicity. For any Hilbert space \( \pi \), we write by \( B(\pi) \) an orthogonal basis of \( \pi \).

**2.1.2. Measures.** We follow the normalization of measures in \([16, \text{Section 1.6.1-1.6.3}] \), hence omit most of the details. We recall their definitions for \( F \) and \( E \) as follows, since another normalization based on them will also be used.

For \( F = Q \), we take the standard additive character \( \psi_Q \) of \( Q \backslash A_Q \) which is \( x \mapsto e^{2\pi i x} \) on \( \mathbb{R} \). Take the additive character \( \psi = \psi_F = \psi \circ \text{Tr}_Q^F \) of \( A_F \) which is trivial on \( F \). We have a decomposition \( \psi = \prod_v \psi_v \).

The additive Haar measure \( dx_v \) on \( F_v \) is the one self-dual w.r.t. \( \psi_v \). More concretely, it is the Lebesgue measure if \( F_v = \mathbb{R} \); it is twice the Lebesgue measure if \( F_v = \mathbb{C} \); it gives \( \varepsilon_v \) the total mass \( q_v^{-d_v/2} \) if \( v < \infty \) where \( \prod_v q_v^{d_v} = D(F) \) is the discriminant of \( F \). We take \( dx = \prod_v dx_v \) as the additive Haar measure on \( A_F \), then \( A_F \backslash A \) has total mass 1 (c.f. Chapter XIV Proposition 7 of \([20]\) ). We take the Haar measure \( d^x x_v \) on \( F_v^x \) by \( d^x x_v = \zeta_v(1)|x_v|_v^{-1} dx_v \), where \( \zeta_v(s) = (1 - q_v^{-s})^{-1} = \frac{1}{(1 - q_v^{-s})} \) if \( v < \infty \); \( \zeta_v(s) = \pi^{-s/2} \Gamma(s/2) \) if \( F_v = \mathbb{R} \); \( \zeta_v(s) = 2(2\pi)^{-1} \Gamma(s) \) if \( F_v = \mathbb{C} \). This is the standard Tamagawa measure on \( \mathcal{Z}_m \). Another normalization is \( d^x x_v \), defined as follows. At \( v \mid \infty \), take \( d^x x_v = d^x x_v \); at \( v < \infty \), take \( d^x x_v = q_v^{d_v/2} d^x x_v \), hence it gives \( \varepsilon_v^{-1} \) total mass 1.

The above normalization also makes sense if \( F \) is replaced by \( E \). At each place \( v \), we endow \( F_v^x \backslash E_v^x \) with the quotient measure, which is transported to a measure \( dt_v \) on \( Z \backslash T(F_v) \) via \( \iota \). We then have a
product measure \( dt = \prod_v dt_v \) on \( Z \setminus T(\mathbb{A}) \). It is the Tamagawa measure on \( Z \setminus T(\mathbb{A}) \) hence

\[
(2.4) \quad \text{Vol}(T(F)Z(\mathbb{A}) \setminus T(\mathbb{A}), dt) = 2\Lambda(1, \eta).
\]

There is another natural measure \( \widetilde{dt}_v \) on \( F_v^\times \setminus E_v^\times = Z(T(F_v)) \), which is normalized as follows. At \( v \) such that \( E_v/F_v \) is split, we identify \( F_v^\times \setminus E_v^\times \) with \( F_v^\times \) and \( \widetilde{dt}_v \) is \( d^x x_v \). At \( v \) such that \( E_v/F_v \) is non-split, \( \widetilde{dt}_v \) gives \( F_v^\times \setminus E_v^\times \) the toal mass 1. Their product \( \widetilde{dt} = \prod_v \widetilde{dt}_v \) gives another Haar measure on \( Z \setminus T(\mathbb{A}) \), related to \( dt \) by

\[
(2.5) \quad dt = D(F)^{1/2}D(E)^{-1/2} \text{Vol}(\text{Ram}(E/F))^{1/2} \widetilde{dt}.
\]

2.1.3. Characters, Representations and Conductors. At \( v \notin \text{Ram}(B) \), the conductor \( C(\pi_v) \) of a unitary irreducible representation \( \pi_v \) of \( G_v \) is in the usual sense for \( GL_2 \). At \( v \in \text{Ram}(B) \), we define \( C(\pi_v) = C(JL(\pi_v)) \) where \( JL(\pi_v) = JL(\pi_v; GL(F_v)) \) is the local Jacquet-Langlands lifting of \( \pi_v \) to \( GL_2(F_v) \). In both cases if \( v < \infty \), we define the logarithmic conductor \( c(\pi_v) \) such that \( C(\pi_v) = q_v^{c(\pi_v)} \).

Recall that we fix a cuspidal automorphic representation \( \pi = \otimes_v \pi_v \) of \( GL_2(\mathbb{A}) \) with central character \( \omega = \omega_\pi \). Let \( \Omega \) be a Hecke character of \( E \) such that \( \omega \cdot \Omega \mid_{Z(\mathbb{A})} = 1 \) and \( \epsilon(1/2, \pi_E \otimes \Omega) = (1/2) \). Then there is a unique quaternion \( F \)-algebra \( B \) containing \( E \) via some \( F \)-embedding \( \iota: E \to B \) with the image \( \iota(E^\times) \) denoted by \( T \) and \( \Omega \) viewed as a character of \( T(\mathbb{A}) \) via \( \iota \), such that \( JL(\pi) = JL(\pi; B) \) exists and \( \text{Hom}_T(B)(JL(\pi_v), \Omega_v^{-1}) \neq \{0\} \) for all \( v \). We say that \( B \) belongs to \( (\pi, E, \Omega) \) in this case, a terminology which also applies to the local case.

At \( v \) for \( \chi_v \) a character of \( F_v^\times \), we define its conductor \( C(\chi_v) \) as follows:

- If \( F_v = \mathbb{R} \), then there is \( \sigma \in \mathbb{R}, m \in \{0, 1\} \) such that \( \chi_v(t) = |t|^\sigma \text{sgn}(t)^m, t \in \mathbb{R}^\times \). We define
  
  \( C(\chi_v) = 2 + |i\sigma + m|/2 \).

- If \( F_v = \mathbb{C} \), then there is \( \sigma \in \mathbb{R}, m \in \mathbb{Z} \) such that \( \chi_v(\rho e^{i\theta}) = \rho^{i\sigma} e^{im\theta}, \rho > 0, \theta \in [0, 2\pi) \). We define
  
  \( C(\chi_v) = (2 + |i\sigma + m|)/2 \).

- If \( v < \infty \), we define the logarithmic conductor
  
  \( c(\chi_v) = \min\{n \in \mathbb{N} : \chi_v \mid_{(1 + p_v^m) \otimes E_v} = 1\} \),

  and put \( C(\chi_v) = q_v^{c(\chi_v)} \).

Let \( E_v \) be an etale quadratic extension of \( F_v \), and \( \Omega_v \) be a character of \( E_v^\times \). We define its \( E_v \)-conductor \( C(\Omega_v) \) as follows:

- If \( E_v \simeq F_v \times F_v \) is split, then there is a unique pair \( \{\chi_1, \chi_2\} \) of characters of \( F_v^\times \) such that
  \( \Omega_v : F_v \times F_v \simeq E_v \to \mathbb{C}^\times, (x_1, x_2) \mapsto \chi_1(x_1)\chi_2(x_2) \).

  We define \( C(\Omega_v) = \min(C(\chi_1), C(\chi_2))^2 \); \( c(\Omega_v) = \min(c(\chi_1), c(\chi_2)) \) if \( v < \infty \).

- If \( F_v = \mathbb{R}, E_v = \mathbb{C} \), then there is \( \sigma \in \mathbb{R}, m \in \mathbb{Z} \) such that \( \Omega_v(\rho e^{i\theta}) = \rho^{i\sigma} e^{im\theta}, \rho > 0, \theta \in [0, 2\pi) \). We define
  
  \( C(\Omega_v) = (2 + |i\sigma + m|)/2 \).

- If \( v < \infty \) and \( E_v \) is non split with ring of integers \( \mathcal{O}_v \), we define the logarithmic conductor
  
  \( c(\Omega_v) = \min\{n \in \mathbb{N} : \Omega_v \mid_{(1 + p_v^m) \otimes \mathcal{O}_v} = 1\} \),

  and put \( C(\Omega_v) = q_v^{2c(\Omega_v)} \).

2.1.4. L-Functions. L-functions with symbol \( L(\cdot) \) are the usual ones without factors at infinity. \( \Lambda(\cdot) \) are the complete ones with factors at infinity. For simplicity of notations we may write functions like \( L(s, \varrho, Ad) \) of an automorphic representation \( \varrho \) of \( G_B(\mathbb{A}) \) to denote the corresponding one \( L(s, JL(\varrho), Ad) \) of \( JL(\varrho; M_2) \) of \( GL_2(\mathbb{A}) \).
2.2. Waldspurger Formula. Let $B$ belong to $(\pi, E, \Omega)$ with $\omega \cdot \Omega\big|_{Z(\mathbb{A})} = 1$ and $\epsilon(1/2, \pi_E \otimes \Omega) = \Omega(-1)$, Waldspurger [39] Proposition 7] proved the following formula:

**Theorem 2.1.** For any $\varphi = \otimes_v \varphi_v \in \text{JL}(\pi) = \text{JL}(\pi; B)$ decomposable, we have

$$|\ell(\varphi; \Omega, t)|^2 = \frac{\Lambda(1/2, \pi_E \otimes \Omega)}{2\Lambda(1, \pi, \text{Ad})} \cdot \prod_{v \in V_p} \frac{L(1, \eta_v)\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)}{\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)} \alpha_v(\varphi_v; \Omega_v, \iota_v),$$

where

- $\ell(\varphi; \Omega, t) = \int_{T(F)Z(\mathbb{A})} \varphi(t)\Omega(t)dt$.
- $\alpha_v(\varphi_v; \Omega_v, \iota_v) = \int_{Z(T(F_v))} \frac{\langle \pi_v(t)\varphi_v, \varphi_v \rangle_{\Omega_v}}{\langle \varphi_v, \varphi_v \rangle_{\Omega_v}} \Omega_v(t)dt$ and $\langle \cdot, \cdot \rangle_v$ is any $G_v$-invariant inner product on $\text{JL}(\pi_v)$.

**Remark 2.2.** Note that the analytic $F$-conductor of $\Omega$ is equivalent to its analytic conductor, with implied constants depending only on $\omega = \omega_\pi$ since $\Omega|_{\mathbb{A}^*} = \omega^{-1}$. We do not distinguish them in the sequel.

**Remark 2.3.** Waldspurger proved the self-dual case. For an explication of the general case, see [66] Chapter 2. The difference of constants is due to our different normalization of measures.

**Remark 2.4.** It is useful to consider the formula under another measure normalization. Namely, we define

- $\hat{\ell}(\varphi; \Omega, t) = \int_{T(F)Z(\mathbb{A})} \varphi(t)\Omega(t)[dt]$ where $[dt]$ is the probability measure on $T(F)Z(\mathbb{A})$.
- $\hat{\alpha}_v(\varphi_v; \Omega_v, \iota_v) = \int_{Z(T(F_v))} \frac{\langle \pi_v(t)\varphi_v, \varphi_v \rangle_{\Omega_v}}{\langle \varphi_v, \varphi_v \rangle_{\Omega_v}} \Omega_v(t)[dt]$.

By [2.3] and [2.20], the above formula reads

$$\left(\frac{\hat{\ell}(\varphi; \Omega, t)}{\langle \varphi, \varphi \rangle_{X(B)}}\right)^2 = \frac{D(F)^{1/2}\text{Ram}(E/F)||A(1/2, \pi_E \otimes \Omega)}}{8D(E)^{1/2}\text{Ram}(E/F)\Lambda(1/2, \pi_E \otimes \Omega)} \cdot \prod_{v \in V_p} \frac{L(1, \eta_v)\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)}{\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)} \hat{\alpha}_v(\varphi_v; \Omega_v, \iota_v).$$

We will use Theorem 2.1 with some subspace instead of some individual vector of $\text{JL}(\pi)$. To this end, we introduce the following notations. If $\sigma_v$ is a finite dimensional subspace of $\pi_v$ and $B_v$ is an orthogonal basis of $\sigma_v$, we define

$$\alpha_v(\sigma_v; \Omega_v, \iota_v) = \sum_{c \in B_v} \alpha_v(c; \Omega_v, \iota_v).$$

Similarly we define $\hat{\alpha}_v(\sigma_v; \Omega_v, \iota_v)$ resp. $\hat{\alpha}(\sigma; \Omega, \iota)$ resp. $\hat{\alpha}(\sigma; \Omega, t)$ in terms of $\hat{\alpha}_v(c; \Omega_v, \iota_v)$ resp. $|\hat{\ell}(\varphi_v; \Omega_v, t)|^2$ resp. $|\hat{\ell}(\varphi_v; \Omega_v, t)|^2$ for $\sigma = \otimes_v \sigma_v$. Note that $\alpha_v(\sigma_v; \Omega_v, \iota_v)$ (resp. $\hat{\alpha}(\sigma_v; \Omega_v, \iota_v)$) is independent of the choice of $B_v$, as the notation suggests. The formula in Theorem 2.1 resp. (2.6) remains valid if we replace the left hand side by $\alpha(\sigma; \Omega, \iota)$ resp. $\hat{\alpha}(\sigma; \Omega, \iota)$ and the $\alpha(\cdot)$ terms at the right hand side by $\alpha_v(\sigma_v; \Omega_v, \iota_v)$ resp. $\hat{\alpha}_v(\sigma_v; \Omega_v, \iota_v)$.

We are particularly interested in two types of sub $K$-representations of $\pi_v$:

1. $\sigma_v = \sigma_0 = \sigma_0(\pi_v)$ is the subspace generated by the new vector $\nu_0$ and the action of $K$. We denote the orthogonal projector from $\pi_v$ to $\sigma_0$ by $\text{Pr}_{\sigma_0}$.\[
\]
2. $\sigma_v = \sigma_v^K(\nu_0) = [\nu_0; c]$ is the subspace of vectors invariant by $K(\nu_0)$. We denote the orthogonal projector from $\pi_v$ to $[\nu_0; c]$ by $\text{Pr}_{\nu_0}$. If $c = c(\nu_0)$, we write $[\nu_0] = [\nu_0; c]$.

The formula in its practical version is written as

$$\hat{\alpha}(\sigma; \Omega, t) = \frac{D(F)^{1/2}\text{Ram}(E/F)||A(1/2, \pi_E \otimes \Omega)}}{8D(E)^{1/2}\text{Ram}(E/F)\Lambda(1/2, \pi_E \otimes \Omega)} \cdot \prod_{v \in V_p} \frac{L(1, \eta_v)\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)}{\zeta_v(2)L(1/2, \pi_{E_v} \otimes \Omega_v)} \hat{\alpha}_v(\sigma_v; \Omega_v, \iota_v).$$
A Measure Comparison. Two standard measures on \( GL_r, r \geq 1 \) are often used in the literature. They both appear naturally: one, the Tamagawa measure, is of algebraic geometric nature; the other, that we call hyperbolic measure, is of analytic nature, i.e., in Plancherel formulas. Since we do not find an explicit comparison of the two measures in the literature and since we need the case \( r = 2 \) in the next subsection, we propose to give an explicit comparison in this subsection.

We take the standard Tamagawa measure \( dg = \prod_v dg_v \), i.e., we fix the standard additive character \( \psi \) as in Section 2.1.2 and at each place \( v \) we take the differential form \( \zeta_v(1) \frac{dx_v}{|\det x_v|_v^r} \) on \( M_r(F_v) \), where \( dx_v \) is the \( r^2 \)-dimensional Haar measure on the additive group of \( F_v \), with respect to the standard additive character \( \psi_v \). Then we define \( dg_v = \frac{dx_v}{|\det x_v|_v^r} \) restricted to \( GL_r(F_v) \).

The hyperbolic measure is defined as follows.

1. On \( N_r \), the upper triangular unipotent radical we take the \( r(r - 1)/2 \)-dimensional additive Haar measure of \( F_v \), as \( N_r(F_v) \cong F_v^{r(r - 1)/2} \) topologically.

2. On \( K_r \), which is the standard maximal compact subgroup of \( GL_r(F_v) \) at each place \( v \), we take the Haar measure \( dk_v \), which gives \( K_{r,v} \) the total mass 1 at each place \( v \).

3. On \( Z_r \cong G_m^{r-1} \) the center of \( GL_r \), we take the Tamagawa measure \( d^\times(z) \) which is \( \zeta_v(1) \frac{dz_v}{|z_v|_v^r} \) on \( F_v^\times \) at each place \( v \).

4. From the surjective map (Iwasawa decomposition)

\[
Z_r \times N_r \times A_{r-1} \times K_r \rightarrow GL_r, (z, n, a, k) \mapsto zn \begin{pmatrix} a & \vdots \\ \vdots & 1 \end{pmatrix} k,
\]

where \( A_{r-1} \cong G_m^{r-1} \) is the \( r - 1 \)-dimensional maximal split torus contained in \( GL_{r-1} \), we know that \( d^\times(z) \cdot dn \cdot \delta(a)^{-1} da \cdot dk \) is also a Haar measure on \( GL_r \). Here the modulus \( \delta(a) = \prod_v \delta(a_v) \) is given by

\[
\delta(\text{diag}(a_1, \cdots, a_{r-1}, 1)) = \prod_{k=1}^{r-1} |a_k|_v^{r-2k+1}, a_i \in F_v^\times
\]

at each place \( v \), and \( da \) is the Tamagawa measure on \( A_{r-1} \).

**Proposition 2.5.** We have the relation

\[
dg = c_r d^\times z \cdot dn \cdot \delta(a)^{-1} da \cdot dk,
\]

\[
c_r = \text{D}(F) \cdot \prod_{j=2}^{r} \Lambda_F(j)^{-1}.
\]

**Proof.** Let \( c_{r,v} \) be the constant appearing locally as

\[
dg_v = c_{r,v} d^\times z_v \cdot dn_v \cdot \delta(a_v)^{-1} da_v \cdot dk_v.
\]

We calculate \( c_{r,v} \) place by place.

- At \( v < \infty \), we have

\[
\text{Vol}(K_{r,v}, dg) = \text{Vol}(M_r(o_v), dx) \zeta_v(1) \left| \frac{G_r(F_v)}{M_r(F_v)} \right| = C(\psi_v)^{-\frac{r}{2}} \prod_{k=2}^{r} \zeta_v(k)^{-1}.
\]

On the other hand, under the Iwasawa decomposition, the preimage of \( K_{r,v} \) is \( Z_r(o_v) \times N_r(o_v) \times A_{r-1}(o_v) \times K_{r,v} \), hence of total mass \( C(\psi_v)^{-1/2} C(\psi_v)^{-r(r-1)/4} C(\psi_v)^{-(r-1)/2} \). Whence we get

\[
c_{r,v} = C(\psi_v)^{-\frac{r(r-1)}{4}} \prod_{k=2}^{r} \zeta_v(k)^{-1}.
\]
- At $\mathbb{F}_v = \mathbb{R}$, we take a function $f$ defined by
\[
 f : \text{GL}_r(\mathbb{R}) \to \mathbb{C}, \ X \mapsto \exp(-\text{Tr}(XX^t))|\det X|^r.
\]

On the one hand, we have
\[
 \int_{\text{GL}_r(\mathbb{R})} f(X)\zeta(1) \frac{dX}{|\det X|^r} = \int_{M_r(\mathbb{R})} \exp(-\text{Tr}(XX^t))dX = \pi^{2r}.
\]
On the other hand, on $X = \text{znak}$ with
\[
 n = (x_{i,j}), x_{i,j} = 0 \text{ if } i > j, x_{i,i} = 1; \\
 a = \text{diag}(a_1, \ldots, a_{r-1}, 1);
\]
we obtain
\[
 \text{Tr}(XX^t) = z^2 \left( 1 + \sum_{j=1}^{r-1} a_j^{-2} \right) + z^2 \sum_{i=1}^{r-1} x_{i,i}^2 + z^2 \sum_{1 \leq i < j < r} a_j^2 x_{i,j}^2;
\]
| det $X$ | = $|z|^r \prod_{j=1}^{r-1} |a_j|$

Hence, we can calculate the integral in another way as
\[
c_{r,v} \int_{\mathbb{R}^r} \int_{(\mathbb{R}^*)^r} \exp(-\text{Tr}(XX^t))|z|^r \prod_{j=1}^{r-1} |a_j|^{2r} \, dx \, da \, dz
\]
\[
= c_{r,v} \prod_{j=1}^{r-1} \Gamma(j) \frac{\zeta(j)}{2},
\]
Thus we get
\[
c_{r,v} = \prod_{j=2}^{r} \Gamma(j)^{-1} = \prod_{j=2}^{r} \zeta(j)^{-1}.
\]
- At $\mathbb{F}_v = \mathbb{C}$, we calculate similarly by taking
\[
 f : \text{GL}_r(\mathbb{C}) \to \mathbb{C}, \ X \mapsto \exp(-\text{Tr}(XX^t))|\det X|^{2r},
\]
and get
\[
c_{r,v} = \prod_{j=2}^{r} \Gamma(j)^{-1} = \prod_{j=2}^{r} \zeta(j)^{-1}.
\]

We conclude by noting $C(\psi) = D(F)$. \(\square\)

2.4. Wielonsky Formula. We need an analogue of Waldspurger Formula for Eisenstein series, which is a generalization of [44, Lemma 2.8]. Wielonsky [40] obtained such a formula. We call our formula “Wielonsky formula” although it is actually different from that one.

Let $\pi = \pi_{s, \xi} = \text{Ind}_{\text{GL}_2(\mathbb{A})}^{\text{GL}_2(\mathbb{A})}(\xi \cdot |^{s}, \xi \cdot |^{-s})$ with $\xi$ a Hecke character of $\mathbb{F}_v^\times \backslash \mathbb{A}_v^\times$ and $s \in \mathbb{C}$. Recall the Eisenstein series defined for a flat section $f_s \in \pi_{s, \xi}$ with $f = f_0 \in \pi_{0, \xi}$ by
\[
 E(s, f)(g) = \sum_{\gamma \in B(\mathbb{F}) \backslash \text{GL}_2(\mathbb{F})} f_s(\gamma g) = \sum_{\gamma \in T(\mathbb{F})} f_s(\gamma g), \forall g \in \text{GL}_2(\mathbb{A}).
\]

Hence we get
\[
 \ell(E(s, f); t, \Omega) = \int_{Z(\mathbb{A}) T(\mathbb{F}) \backslash T(\mathbb{A})} \sum_{\gamma \in T(\mathbb{F})} f_s(\gamma t)\Omega(t)d^\times t = \int_{Z(\mathbb{A}) T(\mathbb{F})} f_s(t)d^\times t.
\]
Thus we have
\[
|\ell(E(s, f); t, \Omega)|^2 = \prod_v \int_{(Z \setminus T(F_v))^2} f_{v, s}(t_1) \overline{f_{v, s}(t_2)} \Omega_v(t_1 t_2^{-1}) d^x t_1 d^x t_2
\]
\[
= \prod_v \int_{Z \setminus T(F_v)} (\pi_v(t)f_{v, s}, f_{v, s})_{v, T} \Omega_v(t) d^x t,
\]
where \((f_1, f_2)_{v, T}\) is the hermitian form defined on \(\pi_v\) by
\[
(f_1, f_2)_{v, T} = \int_{Z \setminus T(F_v)} f_1(t) \overline{f_2(t)} d^x t.
\]

We remark that the decomposition
\[
B_1 \times T \to \text{GL}_2, (b, t) \mapsto bt,
\]
where we recall \(B_1 = NA_1\), tells us that \(db d^x t\) is also a Haar measure on \(\text{GL}_2\) by the same argument as for the Iwasawa decomposition. The exact relation is given by the following lemma.

**Lemma 2.6.** On \(\text{GL}_2(\mathbb{A})\), the above measure is related to the standard Tamagawa measure by
\[
dg = \prod_v dg_v = \prod_v \zeta_v(1)^{-1} L(1, \eta_v)^{-1} \delta(b_v)^{-1} db_v d^x t_v.
\]

**Proof.** Assume \(E = \mathbb{F}[\sqrt{d}]\) with \(d \in \mathbb{F}^\times - (\mathbb{F}^\times)^2\). We claim that the result is independent of the embedding and prove it for the specific embedding
\[
\ell(\sqrt{d}) = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}.
\]
Then we get the coordinates \((x, y, s, t)\) on \(\text{GL}_2\) defined by
\[
\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & t \\ dt & s \end{pmatrix} = \begin{pmatrix} ys + dt & yt + xs \\ dt & s \end{pmatrix}.
\]
Hence locally we have
\[
dg_v = \zeta_v(1) \frac{|d_v| dxdydsdt}{|y_v|^2 |s^2 - dt^2|_v} = |d_v| \zeta_{E, v}(1)^{-1} \delta(b_v)^{-1} db_v d^x t_v.
\]
Note that \(\prod_v |d_v| = 1\) and \(\zeta_{E, v}(s) = \zeta_v(s) L(s, \eta_v)\), we are done. \(\square\)

This lemma together with Proposition 2.5 implies that on \(Z \setminus T \simeq B \setminus \text{GL}_2 \simeq B \cap K \setminus K\), we have
\[
\prod_v d^x t_v = \prod_v \zeta_v(1) L(1, \eta_v) \frac{dg_v}{\delta(b_v)^{-1} db_v} = \prod_v c_{2, v} \zeta_v(1) L(1, \eta_v) dk_v.
\]
Consequently, we get
\[
\prod_v (f_{1, v}, f_{2, v})_{v, T} = \prod_v c_{2, v} \zeta_v(1) L(1, \eta_v) \int_{K_v} f_{1, v}(k_v) \overline{f_{2, v}(k_v)} dk_v.
\]
Recall the Eisenstein norm [44] Lemma 2.8] defined for \(s \in \mathbb{R}\)
\[
\|E(s, f)\|_{\text{Ein}}^2 = \int_K |f_s(k)|^2 dk = \prod_v \int_{K_v} |f_{v, s}(k_v)|^2 dk_v.
\]
We have proved
Proposition 2.7. Let $f = \otimes E f_s \in \pi = \pi_{\mathcal{O}}$ in the induced model such that $f_s \in \pi_{s,\xi}$ defines a flat section for $s \in \mathbb{C}$. Then for any $s \in i\mathbb{R}$, we have

$$\frac{\|\ell(E(s, f); \Omega, \iota)\|^2}{\|E(s, f)\|^2_{\text{fin}}} = \frac{\Lambda(1/2 + s, \xi_E \otimes \Omega)\Lambda(1/2 - s, \xi_E^{-1} \otimes \Omega)}{D(F)^2\Lambda(1 + 2s, \xi^2)\Lambda(1 - 2s, \xi^{-2})} \cdot \prod_{v \in \mathcal{V}_E} \frac{\zeta_v(1)L(1, \eta_v)L(1 + 2s, \xi_v^2)L(1 - 2s, \xi_v^{-2})}{\zeta_v(2)L(1 + s, \xi_E \otimes \Omega_v)L(1/2 - s, \xi_{E, v} \otimes \Omega_v)} \alpha_v(f_s; \Omega_v, t_v),$$

where

- $\xi_E$ resp. $\xi_{E, v}$ is the base change of $\xi$ resp. $\xi_v$ to $\mathbb{A}^\mathbb{F}_v$ resp. $\mathbb{A}_v^E$. $\xi_{E, v} = \xi_v \circ \mathfrak{N}_{E, v}^{E}$. 
- $\alpha_v(f_s; \Omega_v, t_v) = \frac{\langle \pi_{s, \xi_v}(t), f_s, f_s, v \rangle_v}{\langle f_s, f_s, v \rangle_v} \Omega_v(t) dt$ and $\langle \cdot, \cdot \rangle_v$ is any $\text{GL}_2(F_v)$-invariant inner product on $\pi_{s, \xi_v}$. 

Remark 2.8. The formula extends to $s \in \mathbb{C}$ if we replace on the left hand side $|\ell(E(s, f); \Omega, \iota)|^2$ by $\ell(E(s, f); \Omega, \iota)\ell(E(-s, f); \Omega^{-1}, \iota)$, and replace/extend $\langle \cdot, \cdot \rangle_v$ on the right hand side by a $\text{GL}_2(F_v)$-invariant pairing on $\pi_{s, \xi_v} \times \pi_{s, \xi_v} = \pi_{s, \xi_v} \times \pi_{-s, \xi^{-1}}$. 

Remark 2.9. There are versions for $\hat{\ell}, \hat{\alpha}$ of the formula in Proposition 2.7, also for subspaces instead of a single vector. We omit the details.

2.5. Embeddings of a Quadratic Separable Algebra into Matrix Algebra. We omit the subscript $v$ since we work locally in this subsection. We define a set of $F$-embeddings $\iota_r : E \to B$ in the case that $B$ is split, i.e. $B = M_2(F)$.

2.5.1. Archimedean Place. Consider first the case that $E$ is split. Fix a splitting $s : E \cong F \times F$, and define

$$\iota_0 : E \to B, \iota_0(s^{-1}(t_1, t_2)) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix},$$

Then for $r \in F$, we define $\iota_r : E \to B$ such that

$$\iota_r(x) = n(-r)t_0(x)n(r), \forall x \in E. \tag{2.8}$$

Next consider the case that $E$ is non split. Hence $F = \mathbb{R}, E = \mathbb{C} = \mathbb{R}[i]$. We define $\iota_0$ by

$$\iota_0(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.9}$$

2.5.2. Finite Place. First consider the case that $E$ does not split. Hence $\nu \nu$ extends to a unique valuation $\mathfrak{v}_E$ on $E$ with uniformiser $\varpi_E$. By Definition on [83, p.44], there is $\beta \in \mathcal{O} - \mathfrak{v}$ such that

$$\mathcal{O} = \mathfrak{v} + \beta \mathfrak{v}, \tag{2.10}$$

and any other order of $E$ can be written as $\mathcal{O}_r = \mathfrak{v} + \varpi^r \beta \mathfrak{v}$ for a unique $r \in \mathbb{N}$ called the conductor of the order. $\beta$ is a root of an irreducible polynomial $X^2 - bX + a$ in $\mathfrak{v}[X]$. For any $x \in E$, let $\iota(x) \in M_2(F)$ be the matrix such that

$$x(\beta, 1) = (\beta, 1)\iota(x).$$

Note that the choice of $\beta$ is not unique. All the other possible choices are $\beta' = u + v\beta$ for $u \in \mathfrak{o}$ and $v \in \mathfrak{o}^\times$, hence

$$\iota'(\beta') = \begin{pmatrix} b' & 1 \\ -a' & 1 \end{pmatrix}, b' = 2u + vb, a' = u^2 + uvb + v^2a.$$
embedding. The local principality of lattices in \( \mathbf{E} \), which states that any \( \mathfrak{o} \)-lattice in \( \mathbf{E} \) is principal for some order, gives the decomposition

\[
\text{GL}_2(\mathbf{F}) = \bigsqcup_{r=0}^{\infty} \iota_0(\mathbf{E}^\times) \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \text{GL}_2(\mathfrak{o}).
\]  

(2.11)

For any \( r > 0 \), we see

\[
\begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(1 + \varpi^{-r} \beta) \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} = \begin{pmatrix} \varpi^{2r} + \varpi^r \mathfrak{b} & -\mathfrak{a} \\ -\mathfrak{a} & 1 \end{pmatrix} \in \text{GL}_2(\mathfrak{o}), \ i.e.
\[
\begin{pmatrix} \varpi^{-r} & \ast \\ \ast & 1 \end{pmatrix} \in \text{GL}_2(\mathfrak{o}) \left( \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(\mathbf{E}^\times) \right).
\]

Applying the inverse map to (2.11) and taking into account of the above observation, we get

\[
\text{GL}_2(\mathbf{F}) = \bigsqcup_{r=0}^{\infty} \text{GL}_2(\mathfrak{o}) \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(\mathbf{E}^\times).
\]  

(2.12)

[12] Proposition 4.3] gives, with

\[
\mathfrak{o} - \mathfrak{p} \ni u_0 \equiv \beta \pmod{\mathfrak{p}}, \mathfrak{a} - \mathfrak{b}u_0 + u_0^2 \in \mathfrak{p} - \mathfrak{p}^2
\]

if \( \mathbf{E}/\mathbf{F} \) is ramified,

\[
K \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(\mathbf{E}^\times) = K_0(\mathfrak{p}) \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(\mathbf{E}^\times) \bigsqcup K_0(\mathfrak{p}) \mathfrak{w} \begin{pmatrix} \varpi^r & \ast \\ \ast & 1 \end{pmatrix} \iota_0(\mathbf{E}^\times), r > 0;
\]  

(2.14)

\[
K \iota_0(\mathbf{E}^\times) = \begin{cases} 
K_0(\mathfrak{p}) \iota_0(\mathbf{E}^\times) & \text{if } \mathbf{E}/\mathbf{F} \text{ unramified}; \\
K_0(\mathfrak{p}) \iota_0(\mathbf{E}^\times) \bigsqcup K_0(\mathfrak{p}) \begin{pmatrix} 1 \\ -u_0^{-1} \end{pmatrix} \mathfrak{w} \iota_0(\mathbf{E}^\times) & \text{if } \mathbf{E}/\mathbf{F} \text{ ramified}.
\end{cases}
\]  

(2.15)

For any \( r \in \mathbb{N} \), we define an embedding \( \iota_r \) by

\[
\iota_r(t) = a(\varpi^r) \iota_0(t) a(\varpi^{-r}), \forall t \in \mathbf{E}.
\]  

(2.16)

In particular, we have

\[
\iota_r(\beta) = \begin{pmatrix} \mathfrak{b} \\ -\mathfrak{a} \varpi^{-r} \\ 0 \end{pmatrix},
\]

and

\[
\iota_r(\mathbf{E}) \cap M_2(\mathfrak{o}) = \iota_r(\mathcal{O}_r).
\]

In other words, \( \mathcal{O}_r \) is optimally embedded in \( M_2(\mathfrak{o}) \) via \( \iota_r \) in the sense of Section II.3 [38]. For any embedding \( \iota : \mathbf{E} \to \mathbf{B} \), \( \iota^{-1}(M_2(\mathfrak{o})) \) must be an \( \mathfrak{o} \)-order of \( \mathbf{E} \) since \( M_2(\mathfrak{o}) \) is. Hence there is a unique \( r \in \mathbb{N} \), called the conductor of \( \iota \), such that

\[
\iota(\mathbf{E}) \cap M_2(\mathfrak{o}) = \iota(\mathcal{O}_r).
\]  

[38] Theorem II.3.2] says that there is \( k \in K = \text{GL}_2(\mathfrak{o}) \) such that

\[
\iota(t) = k \iota_r(t) k^{-1}, \forall t \in \mathbf{E}.
\]

(2.17)

Next, consider the case that \( \mathbf{E} \simeq \mathbf{F} \times \mathbf{F} \) is split. Any \( \mathfrak{o} \)-order of \( \mathbf{E} \) can be written as \( \mathfrak{o}(1,1) + \varpi r \mathfrak{o}(1,0) \) for some \( r \) satisfying a split separable monic polynomial in \( \mathfrak{o}[X] \). After replacing \( \tau \) by \( u + \varpi v \) for suitable \( u \in \mathfrak{o}, v \in \mathfrak{o}^\times \), we can assume \( \tau \) satisfies \( \tau(\varpi^r) = 0 \) for some \( r \in \mathbb{N} \), thus \( \tau = (\varpi^r, 0) = \varpi^r \beta \) with

\[
\beta = (1, 0).
\]

(2.18)

Hence for any \( \mathfrak{o} \)-order of \( \mathbf{E} \) there is a unique \( r \in \mathbb{N} \), called the conductor, such that it can be written as

\[
\mathcal{O}_r = \mathfrak{o}(1, 1) + \varpi^r \mathfrak{o}(1, 0).
\]

Define \( \iota_0 \) to be the embedding

\[
\iota_0(t_1, t_2) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \forall (t_1, t_2) \in \mathbf{F} \times \mathbf{F} = \mathbf{E}.
\]
The usual Iwasawa decomposition gives
\[
GL_2(F) = \bigcup_{r=0}^{\infty} \iota_0(E^\times) \begin{pmatrix} 1 & \omega^{-r} \\ 0 & 1 \end{pmatrix} GL_2(\mathfrak{o}).
\]

For any integer \( r \geq 1 \), we define an embedding \( \iota_r \) by
\[
(2.18) \quad \iota_r(t) = n(\omega^{-r})\iota_0(t)n(-\omega^{-r}), \forall t \in F \times F = E.
\]

Then we have
\[
\iota_r(E) \cap M_2(\mathfrak{o}) = \iota_r(\mathcal{O}_r).
\]
The same argument with the same reference in [38] then leads to:

**Lemma 2.10.** For any embedding \( \iota : E \to B = M_2(F) \), there is a unique integer \( r =: c(\iota) \in \mathbb{N} \), called the **conductor** of \( \iota \), and \( k \in K = GL_2(\mathfrak{o}) \) such that
\[
\iota(t) = kt_r(t)k^{-1}, \forall t \in E,
\]
where \( \iota_r \) is the canonical embedding defined by \( (2.10) \) if \( E \) is non-split or \( (2.18) \) if \( E \) is split. Consequently,
\[
\alpha(\sigma; \Omega, \iota) = \alpha(\sigma; \Omega, \iota_r)
\]
depends only on the conductor of \( \iota \) if \( \sigma \) is stable under \( K \).

For our purpose, the following sets are interesting:
\[
(2.19) \quad \mathcal{O}_r^{(n)} = \left\{ t \in \mathcal{O}_r - \omega \mathcal{O}_r : N_{F/\mathbb{F}}(t) \in \omega^n \mathfrak{o}^\times \right\}, \mathcal{O}_r^{(\leq n)} = \bigcup_{m=0}^{n} \mathcal{O}_r^{(m)}, n \in \mathbb{N}.
\]

**Lemma 2.11.** The sets \( \mathcal{O}_r^{(n)} \) can be characterized as follows:

1. **If** \( E/F \) **is split**, then for \( n \in \mathbb{N} \)
   \[
   \mathcal{O}_r^{(n)} = \begin{cases} 
   \mathfrak{o}^\times (1 + p^r, 1) & \text{if } n = 0 \\
   \emptyset & \text{if } 2 \mid n, n < 2r \\
   \omega^{n/2} \mathfrak{o}^\times (1 + \omega^{-n/2} \mathfrak{o}^\times, 1) & \text{if } 2 \mid n, 0 < n \leq 2r \\
   (\omega^r \mathfrak{o}^\times, \omega^{n-r} \mathfrak{o}^\times) \cup (\omega^{-r} \mathfrak{o}^\times, \omega^r \mathfrak{o}^\times) & \text{if } n > 2r.
   \end{cases}
   \]

2. **If** \( E/F \) **is an unramified field extension**, then for \( n \in \mathbb{N} \)
   \[
   \mathcal{O}_r^{(n)} = \begin{cases} 
   \emptyset & \text{if } 2 \mid n \text{ or } n > 2r \\
   \mathcal{O}_r^\times & \text{if } n = 0 \\
   \omega^{n/2}(\mathcal{O}_r^\times - \mathcal{O}_r^\times_{r-n/2+1}) & \text{if } 2 \mid n, 0 < n \leq 2r.
   \end{cases}
   \]

3. **If** \( E/F \) **is a ramified field extension**, then for \( n \in \mathbb{N} \)
   \[
   \mathcal{O}_r^{(n)} = \begin{cases} 
   \emptyset & \text{if } n > 2r + 1 \text{ or } 0 \leq n \leq 2r, 2 \nmid n \\
   \mathcal{O}_r^\times & \text{if } n = 0 \\
   \omega^{n/2}(\mathcal{O}_r^\times_{r-n/2} - \mathcal{O}_r^\times_{r-n/2+1}) & \text{if } 2 \mid n, 0 < n \leq 2r \\
   \omega^{k+1} \mathcal{O}_\mathbb{E}^\times & \text{if } n = 2r + 1.
   \end{cases}
   \]

This is a simple exercise in field theory. We omit the proof.

Note that \( \mathcal{O}_r^\times \) acts on \( \mathcal{O}_r^{(n)} \) by multiplication, with finite quotient.

**Corollary 2.12.** The cardinality of \( \mathcal{O}_r^\times \setminus \mathcal{O}_r^{(n)} \) is:

1. **If** \( E/F \) **is an unramified field extension**, then for \( n \in \mathbb{N} \)
   \[
   |\mathcal{O}_r^\times \setminus \mathcal{O}_r^{(n)}| = \begin{cases} 
   0 & \text{if } 2 \nmid n \text{ or } n > 2r \\
   1 & \text{if } n = 0 \\
   q^r & \text{if } n = 2r \\
   q^{n/2}(1 - q^{-1}) & \text{if } 2 \mid n, 0 < n < 2r.
   \end{cases}
   \]
(2) If $E/F$ is a ramified field extension, then for $n \in \mathbb{N}$

$$|O^n_r \setminus O^\infty_r| = \begin{cases} 0 & \text{if } n > 2r + 1 \text{ or } 0 \leq n \leq 2r, 2 \nmid n \\ 1 & \text{if } n = 0 \\ q^{n/2}(1 - q^{-1}) & \text{if } 2 \mid n, 0 < n \leq 2r \\ q^r & \text{if } n = 2r + 1. \end{cases}$$

Proof. We only need to calculate $|O^n_r \setminus O^\infty_r|$ for $r \in \mathbb{N}, n \geq 1$ and insert it to the lemma. Let

$$U^{(r)}_F = 1 + \varpi^r \mathfrak{o}, r > 0; U^{(0)}_F = \mathfrak{o}^\times.$$

$$U^{(r)}_E = 1 + \varpi^r \mathfrak{O}, r > 0; U^{(0)}_E = \mathcal{O}^\times.$$

Since $O^\infty_r = \mathfrak{o}^\times U^{(r)}_E, \mathfrak{o}^\times \cap U^{(r)}_E = U^{(r)}_F$, we get

$$|U^{(r)}_E \setminus O^\infty_r| = |U^{(r)}_F \setminus \mathfrak{o}^\times| = \begin{cases} q^r(1 - q^{-1}) & \text{if } r > 0; \\ 1 & \text{if } r = 0. \end{cases}$$

It is easy to see

$$|U^{(r+n)}_E \setminus U^{(r)}_E| = \begin{cases} q^{2n} & \text{if } r > 0; \\ q^{2n}(1 - q^{-2}) & \text{if } r = 0, E/F \text{ ramified}; \\ q^{2n}(1 - q^{-1}) & \text{if } r = 0, E/F \text{ unramified}. \end{cases}$$

We deduce that

$$|O^n_{r+n} \setminus O^\infty_r| = \frac{|U^{(r)}_E \setminus O^\infty_r| \cdot |U^{(r+n)}_E \setminus U^{(r)}_E|}{|U^{(r+n)}_E \setminus O^\infty_{r+n}|} = \begin{cases} q^n & \text{if } r > 0 \text{ or } E/F \text{ ramified}; \\ q^n(1 + q^{-1}) & \text{if } r = 0, E/F \text{ unramified}. \end{cases}$$

\[\square\]

2.6. Relation with Waldspurger Functional. We omit the subscript $v$ since we work locally in this subsection.

Assume $\text{Hom}_T(\pi, \Omega^{-1}) \neq 0$, hence of dimension 1 by [39].

Let’s first consider the case $E$ is non-split, hence a field. In this case, $Z \setminus T \simeq F^\infty \setminus E^\infty$ is compact. Hence $\text{Res}^T_{T} \pi$ is a direct sum of characters of $T$. Any $\ell \in \text{Hom}_T(\text{Res}^T_{T} \pi, \Omega^{-1})$ can be written as

$$\ell(v) = \frac{(v, \hat{\upsilon})}{(\hat{\upsilon}, \hat{\upsilon})}\ell(\hat{\upsilon}), \forall v \in \pi^\infty$$

where $0 \neq \hat{\upsilon} = \hat{\upsilon}_t, \Omega \in \pi$ is the unique up to scalar vector such that

$$\pi(t) \hat{\upsilon} = \Omega^{-1}(t) \hat{\upsilon}, \forall t \in T.$$

We fix an $\ell \neq 0$, i.e. $\ell(\hat{\upsilon}) \neq 0$. The operator

$$\Pr : \pi \to \pi, \Pr(v) = \int_{Z \setminus T} \Omega(t) \pi(t) . v \hat{\upsilon} dt$$

satisfies $\Pr^2 = \Pr, \Pr^* = \Pr$ with image equal to the $\Omega^{-1}$-isotypical subspace of $\text{Res}^T_{T} \pi$. Hence it is the orthogonal projector onto the $\Omega^{-1}$-isotypical subspace $\mathbb{C}\hat{\upsilon}$. We deduce

$$\Pr(v) = \frac{(v, \hat{\upsilon})}{(\hat{\upsilon}, \hat{\upsilon})}\hat{\upsilon}, \forall v \in \pi.$$

Hence we have

$$\hat{\alpha}(v; \Omega, t) = \int_{Z \setminus T} \frac{\pi'(t) . v . v}{(v, v)} \Omega(t) \hat{\upsilon} dt = \frac{(\Pr(v), v)}{(v, v)} = \frac{|(v, \hat{\upsilon})|^2}{\|v\|^2}, \frac{|\ell(v)|^2}{\|\hat{\upsilon}\|^2}.$$
In particular we see the positivity of \( \hat{\alpha}(\cdot; \Omega, \iota) \) in this case. If \( \sigma \) is any finite dimensional subspace of \( \pi \) with orthogonal projector \( \text{Pr}_\sigma \), we deduce that
\[
(2.22) \quad \hat{\alpha}(\sigma; \Omega, \iota) = \frac{\|\text{Pr}_\sigma(\hat{v})\|^2}{\|\hat{v}\|^2}.
\]

**Definition 2.13.** If \( \iota = \iota_r \), we write the corresponding \( T = T_r, \hat{v}_r = \hat{v}_{r,\Omega} = \hat{v}_{r,\Omega} \). If we are given a model of \( \pi \) as a space of functions, the function corresponding to \( \hat{v}_r \) is denoted by \( \hat{f}_r \).

Next, let’s turn to the case where \( E \) splits. In this case \( E \cong F \times F, B \) must split. There exist \( g_0 \in \text{GL}_2(F) \) and characters \( \chi_1, \chi_2 \) of \( F^\times \) with \( C(\chi_1) \leq C(\chi_2) \) such that
\[
(2.23) \quad \iota(t_1, t_2) = g_0^{-1} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \forall t_1, t_2 \in F,
\]
and
\[
\Omega(t_1, t_2) = \chi_1(t_1)\chi_2(t_2), \forall t_1, t_2 \in F^\times.
\]
A Waldspurger functional \( \ell \in \text{Hom}_T(\pi, \Omega^{-1}) \) can be defined via a(any) Whittaker model \( W(\pi, \bar{\psi}) \) of \( \pi \) as
\[
(2.24) \quad \ell(W) = \int_{F^\times} W(a(y)g_0)\chi_1(y)d^\times y,
\]
where \( \psi \) is an additive character of \( F \) usually chosen to have conductor \( \mathfrak{o} \). For any \( v \in \pi \), if \( W_v \) is the corresponding function in the Whittaker model, then it is easy to show
\[
(2.25) \quad \hat{\alpha}(v; \Omega, \iota) = \int_{Z \backslash T} \frac{\langle \pi(t)v, v \rangle \Omega(t)dt}{\langle v, v \rangle} = \frac{\|\ell(W_v)\|^2}{\|W_v\|^2}
\]
where \( \|W_v\|^2 = \int_{F^\times} |W_v(a(y))|^2d^\times y. \) We see the positivity of \( \alpha(\cdot; \Omega, \iota) \) in this case.

2.7. **Classification of Supercuspidal Representations.** We omit the subscript \( v \) since we work locally in this subsection. We briefly recall the classification of supercuspidal representations \( \pi \) of \( \text{GL}_2 \). For more details, one may consult [12 Section 5.1-5.4] or [5 Chapter 4]. We fix an additive character \( \psi \) of \( F \) such that \( \psi \) is trivial on \( p \) but not trivial on \( \mathfrak{o} \). It is equivalent to considering a place \( v \notin \text{Ram}(B), \) i.e. \( B = \text{M}_2(F), G = \text{GL}_2 \) in the following discussion.

2.7.1. **Type 0 minimal supercuspidal.** There is a representation \( \rho \) of \( J = ZK \) inflated from a cuspidal representation \( \tilde{\rho} \) of \( \text{GL}_2(\mathfrak{o}/p) \), i.e. \( \rho \) is trivial on \( K(p) \) and factors through \( \tilde{\rho} \). We have
\[
\pi \simeq c - \text{Ind}^G_J \rho.
\]
Consequently, we have
\[
\rho |_Z = \omega_\pi, c(\omega_\pi) \leq 1.
\]

The character table of \( \tilde{\rho} \) is given in [5 Section 6.4.1] or [12 Proposition 5.1]. We do not need the full information of the table but its restriction to \( n_-(\mathfrak{o}) \). We have
\[
\text{Tr} \tilde{\rho}(n_-(x)) = \begin{cases} 
-1 & \text{if } x \notin p; \\
q - 1 & \text{if } x \in p,
\end{cases}
\]
from which we deduce
\[
(2.26) \quad \rho |_{n_-(\mathfrak{o})} \cong \bigoplus_{u \in \mathfrak{o}^\times/(1+p)} \psi_u \text{ with } \psi_u(x) = \psi(ux),
\]
where we have identified \( n_-(\mathfrak{o}) \) with \( \mathfrak{o} \). In other words, there is an orthonormal basis \( \{e_u\}_{u \in \mathfrak{o}^\times/(1+p)} \) of \( \rho \) such that
\[
\rho(n_-(x))e_u = \psi(ux)e_u.
\]
2.7.2. Type 1 minimal supercuspidal. There exist \(a_0, a_1 \in \mathfrak{a}\) such that

\[
\alpha = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}
\]

generates an unramified field extension \(L/F\) in \(B\) with \(L = \mathbb{F}[\alpha]\), and the ring of integers \(\mathcal{O}_L = \mathfrak{o}[\alpha]\).

There is an integer \(m \geq 0\) and a character \(\lambda\) of the group \(J = L^\times K(p^{m+1})\) with

\[
\lambda \big|_{K(p^{m+1})} : K(p^{m+1}) \to \mathbb{C}^\times, x \mapsto \psi(\varpi^{-2m+1} \text{Tr}(\alpha(x-1))).
\]

Note that \(\lambda\) is trivial on \(K(p^{2m+2})\) but not trivial on \(K(p^{2m+1})\). We have

\[
\pi \simeq c - \text{Ind}_L^J \lambda.
\]

Consequently, we must have

\[
\lambda \big|_x = \omega_\pi, c(\omega_\pi) \leq 2m + 2.
\]

The integer \(2m + 1\) is called the level/depth of \(\pi\). The conductor \(c(\pi) = 4m + 4\).

Since \(\mathcal{O}_L\) is optimally embeded in \(M_2(\mathfrak{a})\) and \(L/F\) is unramified, we have

\[
L^\times = Z\mathcal{O}_L^\times, J^0 = J \cap K = \mathcal{O}_L^\times K(p^{m+1}), J = ZJ^0.
\]

2.7.3. Type 2 minimal supercuspidal. We have \(a_0, a_1, \alpha, L, \mathcal{O}_L\) the same as in the Type 1 case. There is a character \(\chi\) of \(L\) and an integer \(m > 0\) such that

\[
\chi \big|_{1+\varpi^{-m+1}\mathcal{O}_L} : 1 + \varpi^{-m+1}\mathcal{O}_L \to \mathbb{C}^\times, x \mapsto \psi(\varpi^{-2m+1} \text{Tr}(\alpha(x-1))).
\]

It determines a character \(H^1 = (1 + \varpi\mathcal{O}_L)K(p^{m+1})\) by

\[
\lambda : H^1 \to \mathbb{C}^\times, ux \mapsto \chi(u)\psi(\varpi^{-2m+1} \text{Tr}(\alpha(x-1))), \forall u \in 1 + \varpi\mathcal{O}_L, x \in K(p^{m+1}).
\]

Set \(A^n = a(1 + p^n), n \in \mathbb{N}\). \(\lambda\) can be extended to a character

\[
\tilde{\lambda} : A^m H^1 \to \mathbb{C}^\times, yx \mapsto \lambda(x), \forall y \in A^m, x \in H^1.
\]

Let \(J^1 = (1 + \varpi\mathcal{O}_L)K(p^m), J = L^\times K(p^m)\). Define

\[
\eta = \text{Ind}_{A^m H^1}^{J^1} \tilde{\lambda},
\]

then \(\eta\) is an irreducible representation of \(J^1\). It has the property that (c.f. [5 Lemma 15.6])

\[
(2.27) \quad \eta \big|_{H^1} \simeq \lambda^\oplus q.
\]

There is an irreducible representation \(\rho\) of \(J\) such that

\[
\pi \simeq c - \text{Ind}_J^G \rho, \rho \big|_{J^1} \simeq \eta.
\]

Consequently, we must have

\[
\rho \big|_x = \omega_\pi, c(\omega_\pi) \leq 2m + 1.
\]

The integer \(2m\) is called the level/depth of \(\pi\). The conductor \(c(\pi) = 4m + 2\).

We still write \(J^0 = J \cap K\).

We need some more information about \(\eta\). Since we have

\[
J^1 = A^m H^1 n_-(p^m), A^m H^1 \cap n_-(p^m) = n_-(p^{m+1}),
\]

we see that \(\eta \big|_{n_-(p^m)}\) can be identified as

\[
\eta \big|_{n_-(p^m)} \simeq \text{Ind}_{n_-(p^{m+1})}^{n_-(p^m)} \lambda.
\]

For \(x \in p^{m+1}\), we have

\[
\lambda(n_-(x)) = \psi(\varpi^{-2m} x).
\]

Hence, identifying \(n_-(p^m)\) with \(p^m\), we see that \(\eta\) is \(\text{Ind}_{p^{m+1}}^{p^m} \psi(\varpi^{-2m})\). We deduce

\[
(2.28) \quad \eta \big|_{n_-(p^m)} \simeq \bigoplus_{u \in 1 + p^m/1 + p^{m+1}} \psi(\varpi^{-2m} u).
\]
In other words, there is an orthonormal basis \( \{e_u\}_{u \in 1+p^m/1+p^{m+1}} \) of \( \eta \) such that
\[
\eta(n-x).e_u = \psi(\omega^{-2m}ux)e_u, \forall x \in p^m.
\]

2.7.4. Type 3 minimal supercuspidal. There is \( a_0 \in \omega \alpha^\times, a_1 \in p \) and
\[
\alpha = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}
\]
generates a (totally) ramified field extension \( L/F \) in \( B \) with \( L = F[\alpha] \), and the ring of integers \( \mathcal{O}_L = \mathfrak{o}[\alpha] \).
Write the following Eichler order and its prime element as
\[
\mathfrak{J} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}, \Pi = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}.
\]
Let \( U^m_3 = U_3 = \mathfrak{J}^\times = K_0(p) \) and \( U^n_3 = 1 + \Pi^n \mathfrak{J}, n \geq 1 \). There is an integer \( m \geq 0 \) and a character \( \lambda \) of the group \( J = L^\times U^{m+1}_3 \) with
\[
\lambda |_{U^{m+1}_3}: U^{m+1}_3 \to \mathbb{C}^\times, x \mapsto \psi(\omega^{-m-1}\text{Tr}(\alpha(x-1))).
\]
Note that \( \lambda \) is trivial on \( K(p^{m+2}) \) but not trivial on \( K(p^{m+1}) \). We have
\[
\pi \simeq c - \text{Ind}^G_J \lambda.
\]
Consequently, we must have
\[
\lambda |_Z = \omega_\pi, c(\omega_\pi) \leq m + 2.
\]
The half integer \( (2m+1)/2 \) is called the level/depth of \( \pi \). The conductor \( c(\pi) = 2m + 3 \).
\( \mathcal{O}_L \) is optimally embedded in \( \mathfrak{J} \) hence in \( M_2(\mathfrak{o}) \) (since \( \alpha \in \mathfrak{J} \)). \( \alpha \) is a prime element of \( \mathcal{O}_L \). The normalizer of \( \mathfrak{J} \) in \( G \) is
\[
K_\mathfrak{J} = \langle \Pi \rangle \ltimes K_0(p).
\]
We have
\[
L^\times = \alpha^Z \mathcal{O}_L^\times, J^0 = J \cap K_0(p) = J \cap \mathfrak{J}^\times = \mathcal{O}_L^\times U^{m+1}_3, J = \alpha^Z J^0.
\]

2.7.5. Non minimal supercuspidal. There is a minimal supercuspidal \( \vartheta \) and a character \( \chi \) of \( F^\times \), such that
\[
\pi \simeq \vartheta \otimes (\chi \circ \det), c(\pi) = 2c(\chi) > c(\vartheta).
\]

3. Local Choice and Estimation
We omit the subscript \( v \) since we work locally in this section. We assume \( B \) belongs to \( (\pi, E, \Omega) \) in all the following statements and write \( \pi \) instead of \( JL(\pi; B) \) for abbreviation.

3.1. B-nonsplit Place. At any place \( v \in \text{Ram}(B) \), i.e. \( B_v \) is a division quaternion algebra, we have by Waldspurger \[39\]

**Lemma 3.1.** For any embedding \( \iota \),
\[
\tilde{a}(\pi; \Omega, \iota) = 1.
\]
3.2. B-Split, Archimedean Place.

**Lemma 3.2.** If \( E \) is split, then for any \( v_0 \in \pi \) which is \( K \)-finite there is \( r \in F \) with \( C(\Omega) \leq |r|_v \leq 2C(\Omega) \), such that

\[ \hat{\alpha}(v_0; \Omega, \iota_r) \gg v_0, C(\Omega)^{-1/2}. \]

If \( F = \mathbb{R}, E = \mathbb{C} \), then we have, with \( \iota_0 \) defined in (2.9)

\[ \hat{\alpha}(v_0; \Omega, \iota_0) = 1. \]

It is easy to see that the second part follows (2.21). The only non-trivial part is the first one, on which we will focus. In fact it concerns an extension of the stationary phase method to the case of test functions with non compact support.

**Definition 3.3.** Let \( S \in C^\infty(\mathbb{R}^n) \) be a smooth real valued (phase) function. Associated to it there are \( n \) weight functions \( \omega_i \) and \( n \) differential operators \( L^*_i \) defined by

\[ \omega_i(x) = \frac{\partial S}{\partial x_i}, L^*_i = \frac{\partial}{\partial x_i} \circ \omega_i, 1 \leq i \leq n. \]

We define \( C^\infty_0(\mathbb{R}^n, S) \) to be the space of smooth (complex valued) functions \( \phi \) on \( \mathbb{R}^n \) such that for any polynomial/word in \( n \) variables \( P \) we have

\[ \lim_{x_k \to \pm \infty} \omega_i(x) P(L^*_1, \ldots, L^*_n) \phi(x) = 0, 1 \leq i, k \leq n. \]

**Lemma 3.4.** (stationary phase) Let \( S \in C^\infty(\mathbb{R}^n) \) be a smooth real valued (phase) function and \( \phi \in C^\infty(\mathbb{R}) \). We consider the oscillatory integral for \( \mu \in \mathbb{R} \)

\[ I(\mu, \phi, S) = \int_{\mathbb{R}^n} \phi(x)e^{i\mu S(x)} \, dx. \]

(1) If \( \phi \in C^\infty_0(\mathbb{R}^n, S) \) and \( \nabla S \) does not vanish in the support of \( \phi \), then we have for any \( N \in \mathbb{N} \)

\[ |I(\mu, \phi, S)| \ll_{N, \phi, S} \mu^{-N}, \]

where the implicit constant can be taken as a product of some Sobolev norm of \( \phi \) and of \( S \) of order \( N \).

(2) If \( \phi \in C^\infty_0(\mathbb{R}^n, S) \) and \( x_0 \) lies in the support of \( \phi \) such that

\[ \nabla S(x_0) = 0, \det \nabla^2 S(x_0) \neq 0. \]

 Assume further that \( \nabla S \neq 0 \) on the support of \( \phi \) outside \( \{x_0\} \). Then as \( |\mu| \to \infty \) we have

\[ I(\mu, \phi, S) = (2\pi)^{\frac{n}{2}} \det \nabla^2 S(x_0)^{-\frac{1}{2}} e^{\frac{i}{2} \text{sgn}(\nabla^2 S(x_0))} e^{i\mu S(x_0)} \phi(x_0) \mu^{-\frac{n}{2}} + O(\mu^{-\frac{n}{2} - 1}), \]

where the implicit constant can be taken as a product of some Sobolev norm of \( \phi \) and of \( S \).

**Remark 3.5.** A point \( x_0 \) satisfying the condition (3.1) is called a **nondegenerate critical point** of \( S \).

**Proof.** (1) follows the proof of [10, Lemma 3.12]. Our definition is made to ensure that the integration by parts is still valid for that argument. (2) follows from (1) and [10, Theorem 3.14 (ii)] by a standard argument with partition of unity subordinate to \( B(x_0, 2\epsilon) \cap \overline{B(x_0, \epsilon)} \).

In the case \( F = \mathbb{R} \), we suppose \( \chi(y) = |y|^m \text{sgn}(y)^m \) for some \( \mu \in \mathbb{R}, m \in \{0, 1\} \). For \( t \in [1, 2] \), the zeta-integral we consider is

\[ \int_{\mathbb{R}^n} \pi(n(\mu)). \phi(y) \chi(y) \frac{dy}{|y|} = I(\mu, \phi_+, S_+) + (-1)^m I(\mu, \phi_-, S_-) \]

where we have denoted

\[ \phi_+(x) = \phi(e^{\pi t}), \phi_-(x) = \phi(-e^{\pi t}), S_+(x; t) = x + 2\pi tet^2, S_-(x; t) = x - 2\pi tet^2. \]
Lemma 3.6. Any $\phi : \mathbb{R}^x \to \mathbb{C}$ lying in the Kirillov model of $\pi$ corresponding to a $K$-finite vector is a smooth function admitting the asymptotic behaviors

1. at $0$: there are $\lambda_1, \lambda_2 > 0$, $\sigma_1, \sigma_2$ analytic in a small ball around $0$ such that
   $\phi(y) = |y|^{\lambda_1} \sigma_1(y), y \to 0^+$; $\phi(y) = |y|^{\lambda_2} \sigma_2(y), y \to 0^-$,
   or a finite linear combination of the above type.

2. at $\infty$: there are $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2$ analytic in a small ball around $0$ such that
   $\phi(y) = e^{-2\pi |y|} |y|^{\lambda_1} \sigma_1(\frac{1}{y}), y \to +\infty$; $\phi(y) = e^{-2\pi |y|} |y|^{\lambda_2} \sigma_2(\frac{1}{y}), y \to -\infty$,
   or a finite linear combination of the above type.

Proof. Let $\phi_k, k \in \mathbb{Z}$ be a non-zero function in the Whittaker model of $\pi$ such that
   $\phi_k(g) \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = e^{ik\alpha} \phi_k(g)$.

Its restriction to the Kirillov model is still denoted by $\phi_k$. Since the $(g, K)$-module is simply the algebraic span of $\phi_k$’s over $\mathbb{C}$, it suffices to verify the asymptotic behaviors of $\phi_k$. This is classical and can be verified by the proof of [4, Proposition 2.8.1] (or by using [4, (8.2)] and [8, Theorem 4.4.5 & 4.6.4]).

In the case $\mathbb{F} = \mathbb{C}$, we need to take care of the two possible directions in which $C(\chi) \to \infty$.

Definition 3.7. We call a function $\sigma : \mathbb{C}^x \to \mathbb{C}$ finite analytic if under the polar decomposition $\mathbb{C}^x \simeq \mathbb{R}_+ \times \mathbb{C}^1$, $z \mapsto (\rho, u)$ it is a finite sum
   $\sigma(z) = \sum_{k \in \mathbb{Z}} u^k \sigma_k(\rho)$

with $\sigma_k : [0, \infty) \to \mathbb{C}$ analytic. We similarly define finite smooth functions by weakening the condition on $\sigma_k$ as $\sigma_k : (0, \infty) \to \mathbb{C}$ smooth.

We choose a constant $0 < C < 1$. We write $\chi(z) = y^{i\mu} e^{i\alpha}$ if $z = y e^{i\alpha}$ is the polar coordinates and propose to study for $t \in [1, 2]$ and $\phi$ finite smooth, if $|\mu| \geq C|k+m|$

\[
\int_{\mathbb{C}^x} \pi(n(t\mu)).\phi_k(z) \chi(z) \frac{dz}{|z|} = I(t, \phi_k, S_1)
\]

for $\epsilon_0 = (k+m)/\mu \in [-C^{-1}, C^{-1}]$ and $\hat{\phi}_k(x, \alpha) = \phi_k(e^x), S_1(x, \alpha; \epsilon_0, t) = x + \epsilon_0 \alpha + 2te^x \cos \alpha$.

if $|k+m| \geq C|\mu|$

\[
\int_{\mathbb{C}^x} \pi(n(t(k+m))).\phi_k(z) \chi(z) \frac{dz}{|z|} = I(k+m, \phi_k, S_2)
\]

for $\epsilon_0 = \mu/(k+m) \in [-C^{-1}, C^{-1}]$ and $S_2(x, \alpha; \epsilon_0, t) = \epsilon_0 x + \alpha + 2te^x \cos \alpha$.

It is clear that as $C(\chi) \to \infty$ we have eventually either $|\mu| \geq C|k+m|$ or $|k+m| \geq C|\mu|$ for all $k \in \mathbb{Z}$ such that $\phi_k \neq 0$.

Lemma 3.8. Any $\phi : \mathbb{C}^x \to \mathbb{C}$ lying in the Kirillov model of $\pi$ corresponding to a $K$-finite vector is a smooth function admitting the asymptotic behaviors as in Lemma 3.6 with $\sigma_1, \sigma_2$ “analytic” replaced by “finite analytic”.
Proof. Again, it suffices to consider the $K$-isotypic vectors. If $\pi = \pi(\mu_1, \mu_2)$ with $\mu = \mu_1 \mu_2^{-1}$ satisfying
$$\mu(z) = z^{p+q}, \quad \frac{p+q}{2} \in i\mathbb{R} \cup (0, \frac{1}{2}), p - q \in \mathbb{N},$$
and if $\phi$ correspond to a vector in $\rho_N$, the unique $N + 1$ dimensional irreducible representation of $K = SU_2(\mathbb{C})$ on the space of homogeneous polynomials in two variables of degree $N$, with
$$\phi(g \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}) = u^{2k} \phi(g), \forall u \in \mathbb{C}^1,$$
then $\phi$ in the Kirillov model is determined by (c.f. [30, Section 5])
$$\phi(t) = \omega(t^2) t^{1+\frac{p}{2}} K_p(4\pi t), t \in \mathbb{R}_+ \text{ if } k = -\frac{N}{2};$$
$$\phi(t) = \omega(t^2) t^{1+\frac{q}{2}} K_q(4\pi t), t \in \mathbb{R}_+ \text{ if } k = \frac{N}{2};$$
$$\phi(tu) = \omega(u^2) u^k \phi(t), u \in \mathbb{C}^1, t \in \mathbb{R}_+;$$
where $K$ is the usual K-bessel functions, $\omega$ is the central character of $\pi$ and $-N/2 \leq k \leq N/2, k-N/2 \in \mathbb{Z}$, $N \equiv p - q \pmod{2}, N \geq p - q$. And for different $k$'s the functions $t \in \mathbb{R}_+ \mapsto \phi(t)\omega(t^{1/2})$ are related by differential-difference equations [30] (15 & (16)). We conclude by the asymptotic behaviors of K-bessel functions. \hfill \Box

Proof of (the first part of) Lemma 3.2. We need to apply Lemma 3.4 using Lemma 3.6 ($\mathbb{F} = \mathbb{R}$) or Lemma 3.8 ($\mathbb{F} = \mathbb{C}$). Let $\phi$ be the function in the Kirillov model of $\pi$ corresponding to $\nu_0$.

1. $\mathbb{F} = \mathbb{R}$: By induction on $n \in \mathbb{N}$ it is easy to see that for the differential operators $L_\pm^k = \frac{d}{dx} (1 \pm 2\pi t e^x)^{-1}$ associated with $S_\pm$ as in Definition 3.3 there are polynomials $P_k^\pm, 0 \leq k \leq n$ such that
$$(L_\pm^k)^n = (1 + 2\pi t e^x)^{-2n} \sum_{k=0}^{n} P_k^\pm (e^x) \frac{d^k}{dx^k}.$$ 
By Lemma 3.6 $\phi_\pm$ has asymptotic expansions
$$\phi_\pm(x) = e^{\lambda_1^\pm x} \sigma_1^\pm(e^x), x \to -\infty; \phi_\pm(x) = e^{-\lambda_2^\pm x} \sigma_2^\pm(e^{-x}), x \to +\infty$$
for $\Re \lambda_1^\pm > 0$ and $\sigma_1^\pm, \sigma_2^\pm$ analytic about $0$. Hence $\frac{d^k}{dx^k} \phi_\pm(x)$ has the same type of asymptotic expansions as $\phi_\pm(x)$. Therefore $\phi_\pm \in C^\infty(\mathbb{R}, S_\pm)$. Since the only (nondegenerate) critial point of $S_\pm$ is $x_0 = -\log 2\pi |t|$, we only need to take $r = t\mu$ for $t \in [1, 2]$ such that $\phi(1/(2\pi t)) \neq 0$ by Lemma 3.4.

2. $\mathbb{F} = \mathbb{C}$: We take $S_1$ for example. By induction on the length $n$ of the word $P$, there are polynomials $P_{l_1,l_2}$ such that
$$P(L_1^+, L_2^+) = \|\nabla S_1\|^{-4n} \sum_{l_1+l_2 \leq n} P_{l_1,l_2}(e^x, \cos \alpha, \sin \alpha; t, \epsilon_0) \frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial \alpha^{l_2}}$$
which applied to $\phi_k$ gives
$$\|\nabla S_1\|^{-4n} \sum_{l=0}^{n} P_{l,0}(e^x, \cos \alpha, \sin \alpha; t, \epsilon_0) \frac{d^l}{dx^l} \phi_k(e^x).$$
It is easy to see that $\frac{d^l}{dx^l} \phi_k(e^x)$ has the same type of asymptotic expansions as $\phi_k(e^x)$ at $\pm \infty$ by Lemma 3.8 hence $\phi_k \in C^\infty(\mathbb{R}, S_1)$ is verified. Since the only (nondegenerate) critical point of $S_1$ is $(x, \alpha)$ such that $2te^x = \sqrt{1 + \epsilon_0^2} e^{-i\alpha} = \frac{-1 + i\epsilon_0}{\sqrt{1 + \epsilon_0^2}}$, we only need to take $r = t\mu$ for $t \in [1, 2]$
L

Lemma 3.12. Let \( \sigma \) be a new vector of \( \pi \), then we have for \( r = c(\Omega) \) with absolute implicit constant \( (\sigma_0 \text{ being defined before (2.7)}) \)

\[
\hat{a}(\sigma_0(\pi); \Omega, _\tau) \geq \hat{a}(v_0; \Omega, _\tau) \gg C(\Omega)^{-1/2}.
\]

If \( \pi \) is spherical and \( \Omega \) is unramified i.e. \( r = 0 \), then we have

\[
\hat{a}(\sigma_0(\pi); \Omega, _\tau) = \hat{a}(v_0; \Omega, _\tau) = \frac{\zeta(2)L(1/2, \pi_E \otimes \Omega)}{L(1, \eta)L(1, \pi, Ad)}.
\]

Proof. By (2.26), the first part is a re-formulation of [12] Proposition 3.1 or [37] Lemma 11.7 or [44] Corollary 4.8. For the second part, by the definition of the new vector, we have

\[
\hat{a}(v_0; \Omega, _\tau) = |L(1/2, \pi \otimes \chi)|^2\\|v_0|^2
\]

where \( \chi \) is such that \( \Omega \) corresponds to \( (\chi, \omega^{-1} \chi^{-1}) \) via the identification \( E \simeq F \times F \) and \( |v_0| \) is calculated in the Whittaker model by

\[
|v_0|^2 = \zeta(2)^{-1}L(1, \pi \times \hat{\pi}) = \zeta(2)^{-1}\zeta(1)L(1, \pi, Ad).
\]

We conclude by noting that \( L(1, \eta) = \zeta(1) \) in the \( E \) split case, and \( L(1/2, \pi \otimes \omega^{-1} \chi^{-1}) = \overline{L(1/2, \pi \otimes \chi)} \). \( \square \)

3.3. B-Split, E-Split, Finite Place.

Lemma 3.12. Let \( v_0 \) be a new vector of \( \pi \), then we have for \( r = c(\Omega) \) with absolute implicit constant \( (\sigma_0 \text{ being defined before (2.7)}) \)

\[
\hat{a}(\sigma_0(\pi); \Omega, _\tau) \geq \hat{a}(v_0; \Omega, _\tau) \gg C(\Omega)^{-1/2}.
\]

If \( \pi \) is spherical and \( \Omega \) is unramified i.e. \( r = 0 \), then we have

\[
\hat{a}(\sigma_0(\pi); \Omega, _\tau) = \hat{a}(v_0; \Omega, _\tau) = \frac{\zeta(2)L(1/2, \pi_E \otimes \Omega)}{L(1, \eta)L(1, \pi, Ad)}.
\]

Proof. By (2.26), the first part is a re-formulation of [12] Proposition 3.1 or [37] Lemma 11.7 or [44] Corollary 4.8. For the second part, by the definition of the new vector, we have

\[
\hat{a}(v_0; \Omega, _\tau) = |L(1/2, \pi \otimes \chi)|^2\\|v_0|^2
\]

where \( \chi \) is such that \( \Omega \) corresponds to \( (\chi, \omega^{-1} \chi^{-1}) \) via the identification \( E \simeq F \times F \) and \( |v_0| \) is calculated in the Whittaker model by

\[
|v_0|^2 = \zeta(2)^{-1}L(1, \pi \times \hat{\pi}) = \zeta(2)^{-1}\zeta(1)L(1, \pi, Ad).
\]

We conclude by noting that \( L(1, \eta) = \zeta(1) \) in the \( E \) split case, and \( L(1/2, \pi \otimes \omega^{-1} \chi^{-1}) = \overline{L(1/2, \pi \otimes \chi)} \). \( \square \)

3.4. B-Split, E-nonsplit, Finite Place, \( \pi \) spherical. There exist unramified quasi-characters \( \mu_1, \mu_2 \) of \( F^\times \) such that \( \pi = \pi(\mu_1, \mu_2) = \text{Ind}_{\mu_2}^{\mu_1}(\mu_1, \mu_2) \). Write

\[
\alpha_1 = \mu_1(\varpi), \alpha_2 = \mu_2(\varpi).
\]

Let \( v_0 \in \pi \) be a spherical vector. The function

\[
g \mapsto \frac{\langle \pi(g) v_0, v_0 \rangle}{\langle v_0, v_0 \rangle}
\]

is \( K \)-bi-invariant. Its value on \( Ka(\varpi^m)K, m \in \mathbb{N} \) is given by Macdonald formula ([4] Theorem 4.6.1])

\[
\sigma(\pi, m) = q^{-m/2} \frac{1 - q^{-1} \alpha_2 \alpha_1^{-1}}{1 - \alpha_2 \alpha_1} + q^{-m} \frac{1 - q^{-1} \alpha_1 \alpha_2^{-1}}{1 - \alpha_1 \alpha_2^{-1}}.
\]

Since we have

\[
F^\times \setminus B^\times = \sigma^\times \bigsqcup_{m \geq 0} Ka(\varpi^m)K,
\]

for any embedding \( \nu : E \to B \) we get

\[
\hat{a}(v_0; \Omega, _\tau) = \int_{F^\times \setminus \sigma^\times} \frac{\langle \pi(\nu(t)) v_0, v_0 \rangle}{\langle v_0, v_0 \rangle} \Omega(t) \tilde{dt} = \sum_{m \geq 0} \sigma(\pi, m) \int_{0 \setminus \tau^{-1}(Ka(\varpi^m)K)} \Omega(t) \tilde{dt}.
\]
If the conductor of \( \iota \) is \( r \), then it is easy to see
\[
\iota^{-1}(K\alpha(\varpi^m)K) = \mathcal{O}^{(m)}
\]
where the right hand side is defined in (2.19). We get
\[
(3.2) \quad \hat{\alpha}(v_0; \Omega, \iota) = \sum_{m \geq 0} \sigma(\pi, m) \int_{\alpha^{m} \setminus \mathcal{O}^{(m)}} \Omega(t) \, dt.
\]
By Lemma 2.11 and Corollary 2.12, for \( r \geq 1 \), (3.2) becomes, if \( \mathcal{E}/\mathcal{F} \) is unramified
\[
\hat{\alpha}(v_0; \Omega, \iota) = \sigma(\pi, 0) \cdot \frac{1 - c(\Omega)}{q^r(1 + q^{-1})} + \frac{\sigma(\pi, 2r)}{(\alpha_1 \alpha_2)^r} \left( \frac{1 - 1_0 > c(\Omega)}{q(1 + q^{-1})} \right)
\]
\[+ \sum_{m=1}^{r-1} \frac{\sigma(\pi, 2m)}{(\alpha_1 \alpha_2)^m} \left( \frac{1 - m > c(\Omega)}{q^{r-m}(1 + q^{-1})} - \frac{1 - m + 1 > c(\Omega)}{q^{r-m+1}(1 + q^{-1})} \right).
\]
while if \( \mathcal{E}/\mathcal{F} \) is ramified
\[
\hat{\alpha}(v_0; \Omega, \iota) = \frac{1 - \sigma(\pi, 2)}{q^c(\Omega)} \left( \frac{1 - 1_0 > c(\Omega)}{2q^r} \right) + \frac{\sigma(\pi, 2r + 1)}{\Omega(\varpi_E)(\alpha_1 \alpha_2)^r} \frac{1 - c(\Omega)}{2}
\]
\[+ \sum_{m=1}^{r} \frac{\sigma(\pi, 2m)}{(\alpha_1 \alpha_2)^m} \left( \frac{1 - m > c(\Omega)}{2q^{r-m}} - \frac{1 - m + 1 > c(\Omega)}{2q^{r-m+1}} \right).
\]
Thus if \( r = c(\Omega) > 0 \), we get
\[
\hat{\alpha}(v_0; \Omega, \iota) = \frac{1 - \sigma(\pi, 2)(\alpha_1 \alpha_2)^{-1}}{q^c(\Omega)} \left( 1 + \frac{1}{1 - q^{-1}} \sum_{m=1}^{r} \frac{q^m \sigma(\pi, 2m)}{(\alpha_1 \alpha_2)^m} \right) \text{ if } \mathcal{E}/\mathcal{F} \text{ unramified;}
\]
\[1/2 \quad \text{ if } \mathcal{E}/\mathcal{F} \text{ ramified.}
\]
Lemma 3.13. If \( c(\Omega) > 0 \) and the conductor of \( \iota \) is \( c(\Omega) \), then we have
\[
\hat{\alpha}(v_0; \Omega, \iota) = q^{-c(\Omega)} \frac{\zeta(2)L(1, \eta)}{c(\mathcal{E}/\mathcal{F})L(1, \pi, \text{Ad})},
\]
where \( c(\mathcal{E}/\mathcal{F}) \) is the ramification index of \( \mathcal{E}/\mathcal{F} \) and \( \eta \) is the quadratic character associated with \( \mathcal{E}/\mathcal{F} \).

If \( c(\Omega) = 0 \) and \( \mathcal{E}/\mathcal{F} \) is unramified, we get for \( r \geq 1 \)
\[
\hat{\alpha}(v_0; \Omega, \iota) = \frac{1}{q^r(1 + q^{-1})} \left( 1 + (1 - q^{-1}) \sum_{m=1}^{r} q^m \sigma(\pi, 2m) \right)
\]
while if \( \mathcal{E}/\mathcal{F} \) is ramified, we get for \( r \geq 1 \)
\[
\hat{\alpha}(v_0; \Omega, \iota) = \frac{1}{2q^r} \left( 1 + q^r \sigma(\pi, 2r + 1) \frac{\Omega(\varpi_E)^2}{\Omega(\varpi_E)^{2r+1}} + (1 - q^{-1}) \sum_{m=1}^{r} q^m \sigma(\pi, 2m) \right).
\]
If \( c(\Omega) = r = 0 \) and \( \mathcal{E}/\mathcal{F} \) is unramified, we get
\[
\hat{\alpha}(v_0; \Omega, \iota) = 1 = \frac{\zeta(2)L(1/2, \pi_E \otimes \Omega)}{L(1, \eta)L(1, \pi, \text{Ad})}
\]
since in this case \( L(1, \eta) = \zeta(2)\zeta(1)^{-1} \) and
\[
L(1/2, \pi_E \otimes \Omega) = (1 - \alpha_1 \alpha_2^{-1} q^{-1/2})(1 - \alpha_2 \alpha_1^{-1} q^{-1/2})^{-1} = \zeta(1)^{-1} L(1, \pi, \text{Ad});
\]
while if \( \mathcal{E}/\mathcal{F} \) is ramified, we get
\[
\hat{\alpha}(v_0; \Omega, \iota) = 1 + \sigma(\pi, 1) \Omega(\varpi_E) = \frac{(1 + \alpha_1 \Omega(\varpi_E)q^{-1/2})(1 + \alpha_2 \Omega(\varpi_E)q^{-1/2})}{1 + q^{-1}}
\]
\[= \frac{(1 - \alpha_1 \alpha_2^{-1} q^{-1})(1 - \alpha_2 \alpha_1^{-1} q^{-1})(1 - q^{-1})}{(1 - q^{-1})(1 - \alpha_1 \Omega(\varpi_E)q^{-1/2})(1 - \alpha_2 \Omega(\varpi_E)q^{-1/2})} = \frac{\zeta(2)L(1/2, \pi_E \otimes \Omega)}{L(1, \eta)L(1, \pi, \text{Ad})}
\]
since \( \Omega(\varpi_E)^2 = (\alpha_1 \alpha_2)^{-1} \) and \( L(1, \eta) = 1 \) in this case.
Lemma 3.14. If \( c(\Omega) = 0 \) and the conductor of \( \iota \) is \( r \geq 1 \), then we have the estimation with absolute implicit constant
\[
\hat{\alpha}(v_0; \Omega, \iota) \ll q^{-(1-2\theta)r},
\]
where \( \theta \) is a constant towards the Ramanujan conjecture. If \( c(\Omega) \) and the conductor of \( \iota \) are both 0, then we have
\[
\hat{\alpha}(v_0; \Omega, \iota) = \frac{(2) L(1/2, \pi_\Omega \otimes \Omega)}{L(1, \eta)L(1, \pi, \text{Ad})}.
\]

Remark 3.15. The second part of the previous lemma is a slight generalization of [7] Proposition 5.9 (a) & Proposition 5.10.

3.5. B-Split, E-nonsplit, Finite Place, \( \pi \) ramified non supercuspidal. There exist quasi-characters \( \mu_1, \mu_2 \) of \( F^\times \) such that \( \pi = \pi(\mu_1, \mu_2) \) is (maybe a subquotient of) \( \text{Ind}_{G}^{\G}(\mu_1, \mu_2) \). The function \( \hat{f}_\tau \) (Definition 2.13) is defined by (c.f. [12, (4.4)])
\[
\hat{f}_\tau \left( \left( \begin{array}{cc} a \\ 0 \\ -d \end{array} \right) \right)(t) = \frac{1}{d} \mu_1(a) \mu_2(d) \Omega^{-1}(t), \forall a, d \in F^\times, t \in E^\times.
\]

Lemma 3.16. Let \( c \geq \max(c(\mu_1), c(\mu_2), 1) \).

1. Assume
   - \( c(\Omega) \leq c \) if \( E/F \) is unramified;
   - \( \Omega |_{1+w_E^{-1}O^\times} = 1 \) if \( E/F \) is ramified.

   Then we have \( \hat{v}_0 = \hat{v}_{0, \Omega} \in [\pi; c] \). Hence \( Pr_c(\hat{v}_0) = \hat{v}_0 \). Consequently, by (2.22)
   \[
   \hat{\alpha}(\pi; c; \Omega, \iota_0) = 1.
   \]

2. Assume
   - \( c(\Omega) > c \) if \( E/F \) is unramified;
   - \( \Omega |_{1+w_E^{-1}O^\times} \neq 1 \) if \( E/F \) is ramified.

   Take \( r = c(\Omega) - c \). Then we have
   \[
   \| Pr_c(\hat{v}_r) \| = (1 + q^{-1})^{-1} \| \hat{v}_r \|^2.
   \]

   Consequently, by (2.22)
   \[
   \hat{\alpha}(\pi; c; \Omega, \iota_r) = (1 + q^{-1})^{-1}.
   \]

Proof of the first part of Lemma 3.16. For any \( k \in K(p^\epsilon), b \in B, t \in E^\times \), we have
\[
f(k).f_b(\iota_0(t)) = f_b(\iota_0(t)k_{10}(t)^{-1}) = f_b(\iota_0(t)k_{10}(t)^{-1}).
\]
Hence it suffices to prove
\[
f_b(t_0(10)(t)^{-1}) = 1, \forall k \in K(p^\epsilon), t \in E^\times.
\]

Note that
\[
\iota_0(\mathcal{O}^\times) = \left\{ \begin{array}{ll} Z_{t_0}(\mathcal{O}^\times) & \text{if } E/F \text{ is unramified}, \\
Z_{t_0}(\mathcal{O}^\times) \bigcap [\iota_0(\mathcal{O}^\times)Z_{t_0}(\mathcal{O}^\times)] & \text{if } E/F \text{ is ramified},
\end{array} \right.
\]
and \( \iota_0(\mathcal{O}^\times) \subseteq K \) which stabilizes \( K(p^\epsilon) \). We need only to show that
\[
f_b(k) = 1, \forall k \in K(p^\epsilon),
\]
and if \( E/F \) is ramified
\[
f_b(t_0(\iota_0(\mathcal{O}^\times))^{-1}) = 1, \forall k \in K(p^\epsilon).
\]

Note that we have the decomposition
\[
k = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \left( \begin{array}{ccc} \det k \cdot \text{Nr}(x_4 - a^{-1}x_3\beta)^{-1} & \ast \\ 0 & 1 \end{array} \right) \iota_0(x_4 - a^{-1}x_3\beta).
\]
If $k \in K(p^c)$, then det $k \in 1 + p^c$, $x_3 \in p^c$, $x_4 \in 1 + p^c$, hence $x_4 - a^{-1}x_3 \beta \in 1 + \varpi^c O$, $Nr(x_4 - a^{-1}x_3 \beta) \in 1 + p^c$. We get (3.3) by noting $c \geq c(\mu_1), c(\Omega)$ hence

$$f_0(k) \in \mu_1(1 + p^c)\Omega^{-1}(1 + \varpi^c O) = \{1\}.$$ 

If $E/F$ is ramified, we can take $\varpi_E = \beta - u_0$ with $u_0$ given by (2.17). Hence

$$\iota_0(\varpi_E)K(p^c)\iota_0(\varpi_E)^{-1} \subseteq \left( \begin{pmatrix} b - u_0 - a u_0^{-1} & 1 & 0 \\ 0 & -u_0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a u_0^{-1} & 1 \end{pmatrix}, K(p^c) \right).$$

and

$$\left( \begin{pmatrix} 1 & 0 \\ a u_0^{-1} + p^c & 1 \end{pmatrix}, \begin{pmatrix} -u_0 & 1 \\ a u_0^{-1} + p^c & 1 \end{pmatrix} \right) \subseteq \iota_0(\varpi_E)^{-1}.$$ 

But $1 - u_0^{-1} \beta + p^c \beta \subseteq (1 - u_0^{-1} \beta)(1 + \varpi^c E - 1 O)$, and $\Omega$ is trivial on $1 + \varpi^c E - 1 O$ by assumption. We thus get

$$f_0(\iota_0(\varpi_E)K(p^c)\iota_0(\varpi_E)^{-1}) = \left| \begin{pmatrix} b - u_0 - a u_0^{-1} & 1 \\ 0 & -u_0 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix} \right|^{1/2} \mu_1 \left( \begin{pmatrix} b - u_0 - a u_0^{-1} \\ N r(1 - u_0^{-1} \beta) \end{pmatrix} \right),$$

which is 1 since $\Omega(-u_0) = \omega(-u_0)^{-1} = \mu_1 \mu_2^{-1}(-u_0)$, proving (3.4). \(\square\)

For the second part of Lemma 3.16 we need to establish the branching law for $[\pi; c]$ restricted to $\iota_r(\mathcal{O}_c^x)$. We define a character

$$\Omega_c : \mathcal{O}_c^x \rightarrow \mathbb{C}, xy \mapsto \omega(x), \forall x \in \mathfrak{o}^x, y \in 1 + \varpi^c O.$$ 

It is well-defined because $\omega = \mu_1 \mu_2$ is trivial on $1 + p^c = \mathfrak{o}^x \cap 1 + \varpi^c O$. Similarly, if $E/F$ is ramified, we can define

$$\hat{\Omega}_c : \mathcal{O}_c^{-1} \rightarrow \mathbb{C}, xy \mapsto \omega(x), \forall x \in \mathfrak{o}^x, y \in 1 + \varpi^c E - 1 O.$$ 

A double coset decomposition with $k_1 = 1$

$$K = \bigsqcup_i K(p^c)k_{i \iota_r}(\mathcal{O}_c^x)$$

gives an orthogonal decomposition

$$\text{Ind}_{K_0(p^c)}^{K}(\mu_1, \mu_2) = \bigoplus \mathcal{S}_r(c),$$

where $\mathcal{S}_r(c) = \mathcal{S}_r(c)$ is the subspace of functions supported in $K_0(p^c)k_{i \iota_r}(\mathcal{O}_c^x)$. Recall the coset decomposition

$$K = \bigsqcup_{u \in \mathfrak{o}^x / p^c} K_0(p^c) \begin{pmatrix} 1 \\ u \end{pmatrix} \bigsqcup_{v \in p^c} K_0(p^c) \begin{pmatrix} 1 \\ v \end{pmatrix} w.$$ 

Note that

$$\mathcal{O}_c^{x} = \begin{cases} \mathfrak{o}^x(1 + \varpi^c O) = \mathfrak{o}^x + \varpi^c \mathfrak{o}^x & \text{if } r > 0 \\ \mathfrak{o}^x & \text{if } r = 0. \end{cases}$$

**Case 1:** $r > 0$. For $x \in \mathfrak{o}^x, y \in \mathfrak{o}$ we have

$$\iota_r(x + \varpi^c \beta y) \in K_0(p^c) \begin{pmatrix} 1 \\ a y / x \end{pmatrix}.$$ 

Hence independently of $c > 0$, we get

$$K_0(p^c)\iota_r(\mathcal{O}_c^x) = \bigsqcup_{u \in \mathfrak{o}^x / p^c} K_0(p^c) \begin{pmatrix} 1 \\ u \end{pmatrix} = \left\{ \begin{pmatrix} c_1 \\ c_3 \\ c_2 \\ c_4 \end{pmatrix} \in K \mid c_4 \in \mathfrak{o}^x \right\}.$$
Lemma 3.17. We have proved: and is of dimension \( (3.8) \) we get independently of \( r \) and \( \beta \) is isomorphic to \( \text{Ind}_{\tilde{\Omega}_{r+c}}^{\Omega_{r+c}} \tilde{\Omega}_{r+c} \), hence contains each character \( \Omega' \) of \( \Omega_{r+c} \) restricting to \( \omega \) on \( \sigma^\times \leq r + c \) once.

- If \( k = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in K - K_0(p')I_0(\Omega_{r+c}^\times), \) then \( c_4 \in p. \) For any \( x \in \sigma, y \in \sigma^\times, x, y \in \sigma^\times \) if \( c(\Omega) \leq r + c \) once.

Case 2: \( r = 0. \) We have

\[
\Omega_{r+c}^\times = \{ x + y \beta | \min(v(x), v(y)) = 0 \}, \text{if } E/F \text{ is unramified; } \{ x + y \beta | \min(v(x), v(y)) = 0, -x/y \neq \beta \pmod{P} \}, \text{if } E/F \text{ is ramified.}
\]

Since \( \beta \neq 0 \pmod{P} \) by our choice and

\[
K_0(p') \begin{pmatrix} 1 & 1 \\ v & 1 \end{pmatrix} w = K_0(p')I_0(x + y \beta), \text{if } x, y \in \sigma, x/(ay) = v \pmod{p'},
\]

\[
K_0(p') \begin{pmatrix} 1 & 1 \\ u & 1 \end{pmatrix} w = K_0(p')I_0(x + y \beta), \text{if } x, y \in \sigma, -ax/y = u \pmod{p'},
\]

we get independently of \( c > 0 \)

\[
K = \begin{cases} 
K_0(p')I_0(\Omega_{r+c}^\times) & \text{if } E/F \text{ is unramified; } \\
\bigcup K_0(p') \begin{pmatrix} 1 & 1 \\ u_0 & 1 \end{pmatrix} I_0(\Omega_{r+c}^\times) & \text{if } E/F \text{ is ramified. }
\end{cases}
\]

- \( K_0(p') \cap I_0(\Omega_{r+c}^\times) = I_0(\Omega_{r+c}^\times) , \) and the restriction of the character \( (\mu_1, \mu_2) \) of \( K_0(p') \) to \( I_0(\Omega_{r+c}^\times) \) corresponds to the character \( \tilde{\Omega}_{r+c} \) of \( \Omega_{r+c}^\times. \) We deduce that \( \tilde{\Omega}_{r+c} \) is isomorphic to \( \text{Ind}_{\Omega_{r+c}}^{\Omega_{r+c}} \Omega_{r+c} \), hence contains each character \( \Omega' \) of \( \Omega_{r+c}^\times \) restricting to \( \omega \) on \( \sigma^\times \) with \( c(\Omega') \leq r + c \) once.

- Assume \( E/F \) ramified. For any \( x + y \beta \in \Omega_{r+c}^\times \) we have

\[
n_-(u_0)I_0(x + y \beta)n_-(u_0) = \begin{pmatrix} x + y(b - u_0) \\ -(u_0^2 - bu_0 + a)y \end{pmatrix} \in K_0(p'),
\]

on which the character \( (\mu_1, \mu_2) \) acts as \( \tilde{\Omega}_c'. \) For \( x, y \in \sigma \) we have

\[
1 + \sigma^{c-1}(x + y \beta) = 1 + \sigma^{c-1}(x + yu_0) + \sigma^{c-1}(\beta - u_0)y \in 1 + p^{c-1} + \sigma^{c-1} \sigma E \sigma,
\]

thus

\[
1 + \sigma^{c-1} \sigma = (1 + \sigma^{c-1} \sigma E \sigma)(1 + \sigma^{c-1} \sigma).
\]

Consequently, \( \tilde{\Omega}_c' \) is trivial on \( 1 + \sigma^{c-1} \sigma \) unless \( c = c(\Omega) \). Hence \( S_{n_-(u_0)} \) is isomorphic to \( \text{Ind}_{\Omega_{r+c}^\times}^{\Omega_{r+c}} \Omega_{r+c} \), and contains only characters \( \Omega' \) of \( \Omega_{r+c}^\times \) restricting to \( \omega \) on \( \sigma^\times \) and trivial on \( 1 + \sigma^{2c-1} \sigma E \sigma \).

Any such \( \Omega' \) is indeed trivial on \( 1 + \sigma^{c-1} \sigma \) if \( c > c(\Omega) \).

We have proved:

**Lemma 3.17.** The restriction of \( \text{Ind}_{K_0(p')}^{K_0(p')}(\mu_1, \mu_2) \) to \( \sigma(\Omega_{r+c}^\times) \) only contains characters \( \Omega' \) of \( \Omega_{r+c}^\times \) with \( c(\Omega') \leq r + c \). If \( c(\Omega') = r + c \), \( \Omega' \)-isotypic space lies in \( S_{n_-(u_0)}(\Omega' \text{ with support } (3.7)) \) (including the case \( r = 0 \)) and is of dimension 1 except in the case \( E/F \) ramified, \( r = 0, c = c(\Omega) \) and \( \tilde{\Omega}_c' \) is trivial on \( 1 + \sigma^{2c-1} \sigma E \sigma \).

In the exceptional case, an extra \( \Omega' \)-isotypic space lies in \( S_{n_-(u_0)} \).
Proof of the second part of Lemma 3.16. For any $t_0 \in \Omega_\nu^c$, we have $\iota_r(t_0) \in K$. Hence $\Pr_c(\hat{v}_r)$, calculated by

$$\Pr_c(\hat{v}_r) = \int_{K(p^c)} \pi(k).\hat{v}_r dk$$

with normalized Haar measure $\text{Vol}(K(p^c), \hat{d}k) = 1$, satisfies

$$\pi(\iota_r(t_0)).\Pr_c(\hat{v}_r) = \int_{K(p^c)} \pi(\iota_r(t_0)k_{\iota_r}(t_0)^{-1}).\pi(\iota_r(t_0)).\hat{v}_r dk = \Omega^{-1}(t_0)\int_{K(p^c)} \pi(k).\hat{v}_r dk = \Omega^{-1}(t_0)\Pr_c(\hat{v}_r).$$

Hence $\Pr_c(\hat{v}_r)$ is a vector in $[\pi;c]$ which lies in the $\Omega^{-1}$-isotypic subspace under the action of $\iota_r(\Omega_\nu^c)$. But in our case we have isomorphism of $K$-representations

$$[\pi] \simeq \text{Ind}_{K_0(p^c)}^K(\mu_1, \mu_2).$$

Applying Lemma 3.17 we see $\Pr_c(\hat{f}_r)$ lies in $S_1(c)$ with

$$\Pr_c(\hat{f}_r)(k_{\iota_r}(t_0)) = (\mu_1, \mu_2)(k)\Omega^{-1}(t_0)\Pr_c(\hat{f}_r)(1), \forall k \in K_0(p^c), t_0 \in \Omega_\nu^c.$$

Note for any $k = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in K(p^c)$, we have

$$k = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \left( \begin{array}{cc} \det k \cdot \text{Nr}(x_4 - a^{-1}\omega^r x_3 \beta)^{-1} & \phi \\ 0 & 1 \end{array} \right) \iota_0(x_4 - a^{-1}\omega^r x_3 \beta),$$

and $\det k \in 1 + p^c$, $x_4 - a^{-1}\omega^r x_3 \beta \in 1 + p^c + p^{r+c} \subseteq (1 + p^c)(1 + \omega^c + \mathcal{O})$ hence $\text{Nr}(x_4 - a^{-1}\omega^r x_3 \beta) \in 1 + p^c$.

We deduce that $\hat{f}_r(k) = 1, \forall k \in K(p^c)$. Therefore

$$\Pr_c(\hat{f}_r)(1) = 1.$$

If $\pi$ is in the principal unitary series, then the norm of a function $f \in \pi$ is calculated by

$$\int_{B \backslash G} |f(g)|^2 dg.$$

Otherwise, the norm is calculated via the intertwining operator $M$

$$Mf(g) = \int_{F} f(wn(x)g) dx,$$

as

$$\int_{B \backslash G} f(g)Mf(g) dg,$$

or some limit process using the above formula (c.f. [13 §1.17-1.20]). In the non unitary series case, if $\hat{f}_r \in \hat{\pi} \simeq \pi(\mu_2, \mu_1)$ is the function defined by

$$\hat{f}_r \left( \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \iota_r(t) \right) = \left| \frac{a}{d} \right|^{1/2} \mu_2(a)\mu_1(d)\Omega^{-1}(t), \forall a, d \in F^\times, t \in E^\times,$$

then by the uniqueness of $\hat{v}$, we have

$$M(\hat{f}_r) = M(\hat{f}_r)(1)\hat{f}_r.$$

Since $\Pr_c$ commutes with $M$, we get

$$M(\Pr_c(\hat{f}_r)) = M(\hat{f}_r)(1)\Pr_c(\hat{f}_r),$$

where $\Pr_c(\hat{f}_r)$ is supported in $K_0(p^c)\iota_r(\Omega_\nu^c)$ and

$$\Pr_c(\hat{f}_r)(k_{\iota_r}(t_0)) = (\mu_2, \mu_1)(k)\Omega^{-1}(t_0), \forall k \in K_0(p^c), t_0 \in \Omega_\nu^c.$$
Thus, Lemma 3.18. It is easy to see
\[ [K : K_0(p^c)L_r(O_r^\times)] = (1 + q^{-1})^{-1}. \]
\[ \square \]

3.6. B-Split, E-nonsplit, Finite Place, \( \pi \) supercuspidal.

**Lemma 3.18.** Let \( c > c(\pi)/2 \).

(1) **Assume**
- \( c(\Omega) < c \) if \( E/F \) is ramified;
- \( c(\Omega) \leq c \) if \( E/F \) is unramified.

Then we have \( \hat{\nu}_0 = \hat{\nu}_{0,\Omega} \in [\pi;c] \). Hence \( \Pr_r(\hat{\nu}_0) = \hat{\nu}_0 \). Consequently, by (2.22)
\[ \hat{\alpha}(\pi; c; \Omega, \nu_0) = 1. \]

(2) **Assume**
- \( c(\Omega) > c \) if \( E/F \) is ramified;
- \( c(\Omega) > c \) if \( E/F \) is unramified.

We have
\[ \| \Pr_r(\hat{\nu}_0) \| \leq \| \hat{\nu}_0 \|^2 q^{c - c(\Omega)} \cdot \begin{cases} (1 + q^{-1})^{-1} & \text{if } E/F \text{ unramified;} \\ 1/2 & \text{if } E/F \text{ ramified.} \end{cases} \]

Equality holds if and only if \( r = c(\Omega) - c \). Consequently, by (2.22),
\[ \hat{\alpha}(\pi; c; \Omega, \nu_r) \leq q^{c - c(\Omega)} \cdot \begin{cases} (1 + q^{-1})^{-1} & \text{if } E/F \text{ unramified;} \\ 1/2 & \text{if } E/F \text{ ramified.} \end{cases} \]

Equality holds if and only if \( r = c(\Omega) - c \).

First note that if \( \pi \) is not minimal supercuspidal, then there is a minimal supercuspidal \( \vartheta \) and a character \( \chi \) of \( F^\times \), such that
\[ \pi \simeq \vartheta \otimes (\chi \circ \det), c(\pi) = 2c(\chi) > c(\vartheta). \]

For \( c > c(\pi)/2 = c(\chi) > c(\vartheta)/2 \), \( \chi \) is trivial on \( K(p^c) \). We must have
\[ [\pi; c] = [\vartheta; c] \otimes (\chi \circ \det). \]

It is also easy to see
\[ \hat{\nu}_{r,\Omega} = \hat{\nu}_{r,\Omega\chi} \otimes 1, \]
where \( \hat{\nu}_{r,\Omega\chi} \) lies in \( \vartheta \) and we have identified \( \chi \) with \( \chi \circ N_{1_F} \). Since \( c > c(\chi) \geq c(\chi \circ N_{1_F}) \), we have
\[ c(\Omega) \geq c \Leftrightarrow c(\Omega\chi) \geq c; c(\Omega) > c \Leftrightarrow c(\Omega\chi) > c. \]

Thus, Lemma 3.18 for \( \pi \) is a consequence of it for \( \vartheta \).

For different types of minimal supercuspidals, the proofs of Lemma 3.18 are similar with each other. We treat the case of Type 1 minimal supercuspidals in detail, and leave the other cases to the reader. For \( n \in \mathbb{N} \), let
\[ B_0^{(n)} = Ka(z^n)K, B_0^{(\leq n)} = \bigcup_{l \leq n} B_0^{(l)}; \]
\[ B^{(n)} = ZKa(z^n)K, B^{(\leq n)} = \bigcup_{l \leq n} B^{(l)}. \]

It is easy to see
\[ B^{(n)} = ZB_0^{(n)}, B^{(\leq n)} = ZB_0^{(\leq n)}, \bigcup_{n \geq 0} B_0^{(n)} \subset M_2(\mathcal{O}) - \pi M_2(\mathcal{O}). \]
Consider the following mirabolic subgroups
\[ B_1(o) = \left\{ \begin{pmatrix} z & z' \\ 0 & 1 \end{pmatrix} : z \in o^\times, z' \in o \right\}; \]
\[ B_2(o) = \left\{ \begin{pmatrix} 1 & z' \\ z & 0 \end{pmatrix} : z \in o^\times, z' \in o \right\}; \]
\[ B_3(o) = \left\{ \begin{pmatrix} z & 0 \\ z' & 1 \end{pmatrix} : z \in o^\times, z' \in o \right\}; \]
\[ B_4(o) = \left\{ \begin{pmatrix} 1 & 0 \\ z' & z \end{pmatrix} : z \in o^\times, z' \in o \right\}. \]

Lemma 3.19. For each \( i = 1, 2, 3, 4 \), we have
\[ K = O_L^\times B_i(o), O_L^\times \cap B_i(o) = \{1\}. \]

Proof. For any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \), we have \( \min(v(a), v(c)) = 0 \), hence \( a_0a + c \alpha \in O_L^\times \). Thus
\[ (a_0a + c \alpha)^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0a & c \\ a_0c & a_1c \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ -a_0c & a_0a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O_L^\times \begin{pmatrix} a_0a + a_1c & -c \\ -a_0c & a_0a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
\[ \subseteq o^\times \begin{pmatrix} Nr(a + a_0^{-1}cx) & 0 \\ ab + a_0^{-1}ab - a_0^{-1}cd \\ ad - bc \\ 0 \end{pmatrix} \]
\[ \begin{pmatrix} Nr(a + a_0^{-1}cx) & ab + a_0^{-1}ab - a_0^{-1}cd \\ ad - bc \\ 0 \end{pmatrix} \]
where \( Nr \) is the norm map for \( L/F \). The other cases follow similar argument by noting \( \min(v(b), v(d)) = 0 \). □

Corollary 3.20. For any integer \( l \geq 0 \), we have
\[ K \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K = O_L^\times \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K = J^0 \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K, \]
\[ G = \bigsqcup_{l \geq 0} J \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K. \]

Proof. We only need to note that
\[ K \cap \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right)^{-1} \supseteq B_3(o). \]
□

Corollary 3.21. If there is some non zero function \( f \in [\pi; c] \), then we must have \( c \geq 2m + 2 \) (i.e. \( c \geq c(\pi)/2 \)) and the support of \( f \) is contained in
\[ B^{(c-2m-2)} = \bigsqcup_{l=0}^{c-2m-2} J \left( \begin{pmatrix} \omega^l \\ 1 \end{pmatrix} \right) K. \]

Reciprocally, any function supported in the above set is \( K(\sigma^c) \)-invariant. Consequently, the projector \( P_{\sigma^c} \) is given by
\[ f \mapsto f 1_{B^{(c-2m-2)}}, \]
The projection from \( \pi \) to \( \sigma_0 = \sigma_0(\pi) \) is given by
\[ f \mapsto f 1_{B^{(2m+2)}}, \]
Proof. If we have some $K(p^c)$-invariant $f \neq 0$ supported in $Ja(\omega^l)K$ for some $l \geq 0$, we may assume $f(\omega^l)k_0 \neq 0$ for some $k_0 \in K$. If $c - l \leq 2m + 1$, then there would be some $y \in p^{2m+1} \subset p^{-l}$ such that $\psi(\omega^{-2m-1}y) \neq 1$. But
\[
f(\omega^l)k_0 = f(\omega^l)k_0(1 + n_-(\omega^l)y)k_0 = f(1 + n_-(y))a(\omega^l)k_0 = \lambda(1 + n_-(y))f(a(\omega^l))k_0 \quad \text{since } 1 + n_-(y) \in K(p^{2m+1}) \subset J
\]

which is a contradiction. Hence $c - l \geq 2m + 2$. We conclude the first part by noting that functions supported in $Ja(\omega^l)K$ forms a direct sum decomposition of $\pi$ as $K$-sub-representations by the last equality of the previous corollary.

The reciprocal follows by noting that $B(\leq c-2m-2)$ as well as its complement are stable by multiplication by $K(p^c)$ at right, and for any $k \in K, l \leq c - 2m - 2$
\[
a(\omega^l)kK(p^c)k^{-1}a(\omega^l) \subset K(p^{-l}) \subset K(p^{2m+2})
\]
on which $\lambda$ is trivial.

The assertion concerning $\sigma_0$ follows from $\pi^K(p^{4m+3}) \oplus \sigma_0 = \pi^K(p^{4m+4})$, c.f. [44, Theorem 5.2]. □

Lemma 3.22. Let $s > 0$. If $z \in \sigma^\times / (1 + p^{m+1})$ runs over a system of representatives, then we have
\[
K = \bigsqcup_z J^0a(z)\iota_s(\mathcal{O}_s^\times).
\]

Proof. By Lemma 3.19 we have a projection map
\[
\tau : K \to B_1(\sigma).
\]
Choosing a system of representatives $[B_1(\sigma)]$ of $B_1(\sigma)$ (mod $p^{m+1}$), we get a decomposition
(3.9)
\[
K = \bigsqcup_{b \in [B_1(\sigma)]} Jb
\]
as well as a map
\[
\tilde{\tau} : K \to [B_1(\sigma)]
\]
determined by $\tilde{\tau}(k) = \tau(k) \equiv 0 \mod p^{m+1}$.

If $z_1, z_2 \in \sigma^\times, t_1, t_2 \in \mathcal{O}_s^\times$ are such that $\tilde{\tau}(a(z_1)\iota_s(t_1)) = \tilde{\tau}(a(z_2)\iota_s(t_2))$, then $a(z_1)\iota_s(t_1) a(z_2)^{-1} \in J^0$, i.e. $\tilde{\tau}(a(z_1)\iota_s(a(t_1) a(z_2)^{-1}) = 1$, or
\[
\tilde{\tau}(a(z_1)\iota_s(t_1) a(z_2)^{-1}) = 1 \mod p^{m+1}.
\]

Writing $t = x + y\beta$ with $x \in \sigma^\times, y \in \sigma^+$ and using the formula in the proof of Lemma 3.19, we see that the (1,2) term of $\tau(a(z_1)\iota_s(t) a(z_2)^{-1})$ is the product of $\sigma^{-\ast}y, z_1^{-1}z_2^{-1}(x + by)\sigma^{2s} - z_1^{-1}a_0^{-1}a_1 a_y \sigma^s + z_2^{-1}a_0^{-1}ax$ and $z_1^{-1}z_2\text{Nr}(x + y\beta)^{-1}$. The two later being in $\sigma^\times$ since $s > 0$, we must have $\sigma^{-\ast}y \in \sigma^{m+1}$ hence $y \in p^{m+1}$. The (1,1) term is $z_1^{-1}z_2^{-1}$ times $\text{Nr}(x + by - z_1^{-1}a_0^{-1}a_y \sigma^s)\text{Nr}(x + y\beta)^{-1}$. The later is in $1 + p^{m+1}$ since $y \in p^{m+1}$, hence $z_1^{-1}z_2^{-1} \in 1 + p^{m+1}$. Thus $\tilde{\tau}$ induces an injection
\[
\sigma^\times / (1 + p^{m+1}) \times \mathcal{O}_s^\times / \mathcal{O}_{s+m+1} \to [B_1(\sigma)], (z, t) \mapsto \tilde{\tau}(a(z)\iota_s(t)).
\]
Both sides being finite and having the same cardinality, it must be surjective as well. Since $a(1 + p^{m+1}), \iota_s(\mathcal{O}_{s+m+1}) \subset K(p^{m+1})$, we conclude by (3.9). □

Lemma 3.23. Let $s = \max(c(\Omega) - 2m - 2, 0)$ and $\iota_0$ correspond to a function $\hat{f}_0$. Up to a constant multiple, we have:

- If $s > 0$, then there is a unique $z \in \sigma^\times / (1 + p^{m+1})$ such that $\hat{f}_0$ is supported in $Ja(z\omega^s)T_0$ and
\[
\hat{f}_0(xa(z\omega^s)\iota_0(t)) = \lambda(x)\Omega^{-1}(t), \forall x \in J, t \in \mathcal{E}^\times.
\]
- If $s = 0$, then there is a unique double coset $J^0k_0\iota_0(\mathcal{O}^\times) \in J^0/K_0(\mathcal{O}^\times)$ such that $\hat{f}_0$ is supported in $Jk_0T_0$ and
\[
\hat{f}_0(xk_0\iota_0(t)) = \lambda(x)\Omega^{-1}(t), \forall x \in J, t \in \mathcal{E}^\times.
\]
Proof. Note that we have the following identity of double coset decompositions for any \( s \in \mathbb{N} \)
\[
J\backslash K a(\varpi^s)T_0/T_0 \simeq J\backslash ZK/ZK \cap T_s \simeq J^0\backslash K/\iota_0(O_s^\times).
\]
Together with (2.12) and the previous lemma, we get the following decomposition
\[
G = \bigsqcup_{k_0} Jk_0 T_0 \bigsqcup \bigsqcup_{s \geq 1} z J a(\varpi^s)T_0,
\]
where \( k_0 \) runs over a set of representatives of \( J^0\backslash K/\iota_0(O^\times) \) and \( z \) runs over a set of representatives of \( \mathfrak{o}^\times/(1 + p^m + 1) \). Let \( S(s, z) \) be the space of functions supported in \( J a(\varpi^s)T_0 \) if \( s > 0 \); \( S(k_0) \) be the space of functions supported in \( Jk_0 T_0 \). Then we have a decomposition as \( T_0 \)-representations
\[
\text{Res}_{T_0}^G \pi = \bigoplus_{k_0} S(k_0) \bigoplus \bigoplus_{s \geq 1, z} S(s, z).
\]
Consider \( s > 0 \). One can check, in fact the proof of the previous lemma already shows (with \( z_1 = z_2 = z \)), that
\[
a(z\varpi^s)T_0 a(z\varpi^s)^{-1} \cap J = Z(a(z)T_s a(z)^{-1} \cap K \cap J^0)
\]
\[
= Z(a(z)\iota_s(O^\times) a(z)^{-1} \cap J^0) = Z(a(z)\iota_s(O_{s+m+1}^\times) a(z)^{-1}),
\]
only where \( \lambda \) acts as a character \( \Omega_\lambda \) of \( F^\times O_{s+m+1}^\times = F^\times (1 + p^{s+m+1} \beta) \) defined by
\[
\Omega_\lambda(x(1 + y\beta)) = \omega(x)(\varpi^{-s-2m-1}y(-az^{-1} + a_0\varpi^2z)), \ x \in F^\times, \ y \in p^{s+m+1}.
\]
The function
\[
\varpi^s/(1 + p^{m+1}) \to \varpi^s/(1 + p^{m+1}), \ z \mapsto -az^{-1} + a_0\varpi^2z
\]
is injective, since
\[
-az_1^{-1} + a_0\varpi^2z_1 = -az_2^{-1} + a_0\varpi^2z_2 \pmod{p^{m+1}} \text{ if and only if } (z_1 - z_2)(a_1^{-1} - a_2^{-1} + a_0\varpi^2z) \in p^{m+1} \text{ if and only if } z_1 = z_2 \pmod{p^{m+1}}.
\]
It is thus bijective since \( \mathfrak{o}^\times/(1 + p^{m+1}) \) is finite. As \( z \) runs over \( \mathfrak{o}^\times/(1 + p^{m+1}) \), the function
\[
p^{s+m+1}/p^{s+2m+2} \to \mathbb{C}^\times, \ y \mapsto \psi(\varpi^{-s-2m-1}y(-az^{-1} + a_0\varpi^2z))
\]
runs over all the characters of \( p^{s+m+1}/p^{s+2m+2} \) not trivial on \( p^{s+2m+1}/p^{s+2m+2} \). Hence \( \Omega_\lambda \) runs over all the characters of \( F^\times O_{s+m+1}^\times \) restricting to \( \omega \) on \( F^\times \) with \( c(\Omega_\lambda) = s + 2m + 2 \). As \( S(s, z) \simeq \text{Ind}_{F^\times O_{s+m+1}^\times}^{F^\times \times} \Omega_\lambda \) which is the direct sum of all characters \( \Omega' \) of \( E^\times \) restricting to \( \Omega_\lambda \) on \( F^\times O_{s+m+1}^\times \), we see that
\[
\bigoplus_z S(s, z)
\]
contains all characters \( \Omega' \) of \( E^\times \) with \( c(\Omega') = s + 2m + 2 \) with multiplicity one. We conclude by the multiplicity one result of Waldspurger [39].

Proof of Lemma 3.18. For the first part, by Corollary 3.20 and 3.21 it suffices to show that the support of \( \tilde{f}_0 \) is contained in
\[
\bigsqcup_{l \leq c-2m-2} ZK a(\varpi^l)K.
\]
Note that for any \( k_0 \in K \), we have
\[
Jk_0 a(\varpi^l)T_0 \subseteq ZK a(\varpi^l)\iota_0(O - \varpi O).
\]
If \( E/F \) is unramified, then \( \iota_0(O - \varpi O) = \iota_0(O^\times) \subseteq K \). If \( E/F \) is ramified, then \( \iota_0(O - \varpi O) = \iota_0(O^\times) \cup \iota_0(\varpi E) \iota_0(O^\times) \subseteq K \cup \iota_0(\varpi E)K \), and we can take \( \varpi E = \beta - u_0 \) such that
\[
a(\varpi^l)\iota_0(\varpi E) = \left(\varpi^l(b - u_0) - a \over a \right) \in K a(\varpi^{l+1})K,
\]
since \( a(\varpi^l)\iota_0(\varpi E) \in M_2(\mathfrak{o}) - \varpi M_2(\mathfrak{o}) \) with determinant in \( \varpi^0 \mathfrak{o}^\times \). Take \( s = \max(0, c(\Omega) - 2m - 2) \), then the support of \( \tilde{f}_0 \) is contained in \( ZK a(\varpi^l)K \) if \( E/F \) is unramified; or in \( ZK a(\varpi^l)K \cup ZK a(\varpi^{l+1})K \) if \( E/F \) is ramified. Under the assumption of the lemma, they both are subsets of \( \sqcup_{l \leq c-2m-2} ZK a(\varpi^l)K. \)
For the second part, let \( s = c(\Omega) - 2m - 2 > 0 \). Note that \( \tilde{f}_0 \) is of modulus 1 on its support, hence the same is true for \( \tilde{f}_r = \pi(a(z\omega)) \tilde{f}_0 \) for any \( r \in \mathbb{N} \). Since, by Lemma 3.21, the support of \( \tilde{f}_0 \) is \( JA(z\omega^*)T_0 \) for some \( z \in \mathfrak{o}^\times \), we get, by Corollary 3.23 that
\[
\|f_r\|^2 = \text{Vol}(Z \backslash JA(z\omega^*)T_0a(z\omega^-)) = \text{Vol}(Z \backslash JA(z\omega^*)T_0);
\]
\[
\|\text{Pr}_c(f_r)\|^2 = \text{Vol}(Z \backslash JA(z\omega^*)T_0a(z\omega^-) \cap B^{(\leq c-2m-2)}).
\]
We have seen in the proof of Lemma 3.23 that
\[
J \backslash JA(z\omega^*)T_0 \simeq F^\times \mathcal{O}_{s+m+1} \setminus E^\times \simeq \mathcal{O}_{s+m+1} \setminus (\mathcal{O} - \omega \mathcal{O}),
\]
hence we get
\[
\text{Vol}(Z \backslash JA(z\omega^*)T_0) = \text{Vol}(Z \backslash J : |\mathcal{O}_{s+m+1} \setminus (\mathcal{O} - \omega \mathcal{O})|).
\]
For the projection, note that
\[
J \backslash JA(z\omega^*)T_0a(z\omega^-) \cap B^{(\leq c-2m-2)} \simeq Z(z\omega^*)t_0(\mathcal{O}_{s+m+1} \setminus (\mathcal{O} - \omega \mathcal{O})).
\]
But it is easy to see
\[
a(z\omega^*)t_0(E)a(z\omega^-) \cap M_2(\mathfrak{o}) = \begin{cases} a(z\omega^*)t_0(\mathcal{O}^{s-2m-2} a(z\omega^-) & \text{if } r \geq s; \\ a(z\omega^*)t_0(\mathcal{O}_s a(z\omega^-) & \text{if } r < s. 
\end{cases}
\]
Thus we get
\[
a(z\omega^*)T_0a(z\omega^-) \cap B^{(\leq c-2m-2)} = \begin{cases} \mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O}^{s-2m-2 - (r-s)}) a(z\omega^-) & \text{if } r \geq s; \\ \mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O}_s^{s-2m-2 + r-s}) a(z\omega^-) & \text{if } r < s.
\end{cases}
\]
Consequently, we obtain
\[
J \backslash JA(z\omega^*)T_0a(z\omega^-) \cap B^{(\leq c-2m-2)} \simeq \mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O}_{\min(r,s)}^{s-2m-2 - [r-s]}).
\]
\[
\text{Vol}(Z \backslash JA(z\omega^*)T_0a(z\omega^-) \cap B^{(\leq c-2m-2)}) = \text{Vol}(Z \backslash J : |\mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O}^{s-2m-2 - [r-s]})|).
\]
We conclude that
\[
\frac{\|\text{Pr}_c(f_r)\|^2}{\|f_r\|^2} = \frac{|\mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O}_{\min(r,s)}^{s-2m-2 - [r-s]})|}{|\mathcal{O}_{s+m+1}^{\mathfrak{o}} \setminus (\mathcal{O} - \omega \mathcal{O})|} = \frac{|\mathcal{O}_{\min(r,s)}^{\mathfrak{o}} \setminus (\mathcal{O}^{s-2m-2 - [r-s]})|}{|\mathcal{O}_{\min(r,s)}^{\mathfrak{o}} \setminus (\mathcal{O} - \omega \mathcal{O})|},
\]
and the formula given in Corollary 2.12 gives the result.

\[ \square \]

4. Proof of the Main Results

4.1. Local Estimation. Recall that we fix a number field \( F \), an automorphic cuspidal representation \( \pi \) of \( \text{GL}_2(\mathbb{A}) \). We let vary the pairs of \((E, \Omega)\) where \( E \) is a quadratic field extension of \( F \) and \( \Omega \) is a Hecke character of \( \mathbb{A}_F^\times \) which coincides with the central character \( \omega = \omega_\pi \) of \( \pi \) under the diagonal embedding \( \mathbb{A}^\times \to \mathbb{A}_F^\times \). Furthermore, we assume that (1.1) holds.

There is a unique quaternion algebra \( B \) defined over \( F \) containing \( E \) via some embedding \( \iota : E \to B \), for which the Jacquet-Langlands lifting \( \pi' = JL(\pi ; B) \) exists as an automorphic cuspidal representation of \( G = B^\times \) such that
\[
\text{Hom}_{T(F)}(\pi'_v, \Omega_v) \neq 0
\]
at every \( v \), where \( T \) is the \( F \)-subtorus of \( G \) defined by the image of \( E^\times \) under \( \iota \). We also fix for any such \( B \) a maximal \( \mathfrak{o} \)-order \( \mathfrak{O} \), and fix for any \( v \not\in \text{Ram}(B) \) an \( F_v \)-isomorphism (2.1) with \( \delta_v(\mathfrak{O}_v) = M_2(\mathfrak{o}) \).

In this subsection, we shall choose for each place \( v \) a fixed subspace \( \sigma_v \) of \( \pi'_v \) and some \( g_v \in G_v \) such that \( \delta_v(g_v) \in K_v \) for all but finitely many places.

- We denote by \( S_1 \) the set of places of one of the following cases:
  (0) \( v \in \text{Ram}(B) \). We choose \( \sigma_v = \pi'_v, g_v = 1 \).
  (1) \( v \not\in \text{Ram}(B), F_v = \mathbb{R}, E_v = \mathbb{C} \). We choose \( \sigma_v = C_{v,0,\Omega_v} \) defined in Definition 2.13 \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_0 \) defined in (2.20).
(2) \( \nu \notin \text{Ram}(B), E_v \) non-split, \( \nu < \infty \), \( \pi_v \) ramified non-supercuspidal. We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\), \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \max(0, c(\Omega_v) - c(\pi_v)) \) defined in \((2.10)\).

(3) \( \nu \notin \text{Ram}(B), E_v \) non-split, \( \nu < \infty \), \( \pi_v \) supercuspidal and either \( c(\Omega_v) \leq c(\pi_v) \) if \( E_v/F_v \) is unramified or \( c(\Omega_v) < c(\pi_v) \) if \( E_v/F_v \) is ramified. We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\), \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \max(0, c(\pi_v)) \) defined in \((2.10)\).

Then Lemma \[3.1\] for case (0) resp. the second part of Lemma \[3.2\] for case (1) resp. Lemma \[3.16\] for case (2) resp. Lemma \[3.18\] (1) for case (3) gives (with implicit constant which can be taken as \(2/3\))

\[
\tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v) \gg 1.
\]

- We denote by \( S_2 \) the set of places of one of the following cases:

  (1) \( \nu \notin \text{Ram}(B), E_v \) split, \( \nu \mid \infty \). We choose \( \sigma_v \) to be the \( K_v \)-subrepresentation of \( \pi'_v \) of lowest weight. Choose any \( \nu_0 \in \sigma_v \) and \( r \in F_v \) be as in the first part of Lemma \[3.7\]. We choose then \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \) defined in \((2.5)\).

  (2) \( \nu \notin \text{Ram}(B), E_v \) split, \( \nu < \infty \) and \( \pi_v \) non-spherical or \( c(\Omega_v) > 0 \). We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\) and \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \pi_v \) defined in \((2.18)\).

  (3) \( \nu \notin \text{Ram}(B), E_v \) non-split, \( \nu < \infty \), \( \pi_v \) supercuspidal and either \( c(\Omega_v) > c(\pi_v) \) if \( E_v/F_v \) is unramified or \( c(\Omega_v) \geq c(\pi_v) \) if \( E_v/F_v \) is ramified. We choose, \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\), \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \pi_v \) defined in \((2.16)\).

Then the first part of Lemma \[3.2\] for case (1) resp. the first part of Lemma \[3.12\] for case (2) and Lemma \[3.18\] (2) for case (3) gives

\[
\tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v) \gg_{\pi_v} C(\Omega_v)^{-\frac{1}{2}}.
\]

- We denote by \( S_3 \) the set of places \( \nu \notin \text{Ram}(B), E_v \) non-split, \( \nu < \infty \), \( \pi_v \) spherical and \( c(\Omega_v) > 0 \). We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\) and \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \pi_v \) defined in \((2.16)\). Then Lemma \[3.3\] gives

\[
\tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v) = C(\Omega_v)^{-\frac{1}{2}} \frac{\zeta_v(2)L(1, \eta_v)}{e(E_v/F_v)L(1, \pi_v, \text{Ad})}.
\]

- We denote by \( S_4 = S_{4,s} \cup S_{4,n} \) defined by:

  (1) \( S_{4,s} : \nu \notin \text{Ram}(B), E_v \) split, \( \nu < \infty \), \( \pi_v \) spherical and \( c(\Omega_v) = 0 \). We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\) and \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \pi_v \) defined in \((2.18)\). Note if the conductor of \( \iota_v \) is already 0, we have \( \delta_v(g_v) \in K_v \).

  (2) \( S_{4,n} : \nu \notin \text{Ram}(B), E_v \) non-split, \( \nu < \infty \), \( \pi_v \) spherical and \( c(\Omega_v) = 0 \). We choose \( \sigma_v = \sigma_0(\pi'_v) \) defined before \((2.7)\) and \( g_v \in G(F_v) \) such that \( g_v^{-1} \iota_v g_v = \iota_v \pi_v \) defined in \((2.16)\). Note if the conductor of \( \iota_v \) is already 0, we have \( \delta_v(g_v) \in K_v \).

Then the second part of Lemma \[3.12\] for case (1) resp. the second part of Lemma \[3.14\] gives

\[
\tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v) = \frac{\zeta_v(2)L(1/2, \pi_v, \text{E} \otimes \Omega_v)}{L(1, \eta_v)\zeta_v(2)L(1/2, \pi_v, \text{E} \otimes \Omega_v)}.
\]

**Remark 4.1.** Note that \( \sigma_v \) is \( G(F_v) \)-stable at \( \nu \in \text{Ram}(B) \), and \( K_v \) (or more precisely \( K_v^{-1}(K_v) \) if \( K_v \) refers to the standard maximal compact subgroup of \( GL_2(F_v) \))-stable at \( \nu \notin \text{Ram}(B) \). Note also that \( g_v \) depends on \( \Omega_v \) but \( \sigma_v \) does not.

Combining all the above bounds gives

\[
\prod_{v < \infty} \tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v) \prod_{v < \infty} \frac{L(1, \eta_v)L(1, \pi_v, \text{Ad})}{\zeta_v(2)L(1/2, \pi_v, \text{E} \otimes \Omega_v)} \tilde{\alpha}_v(g_v, \sigma_v; \Omega_v, \iota_v)
\]

\[
\gg_{\pi} \prod_{v \in S_{4,s} \cup S_{4,n} \cup S_{2} \cup S_{3}} \frac{L(1, \eta_v)L(1, \pi_v, \text{Ad})}{\zeta_v(2)L(1/2, \pi_v, \text{E} \otimes \Omega_v)} \prod_{v \in S_{2} \cup S_{3}} C(\Omega_v)^{-\frac{1}{2}} \prod_{v \in S_{3}} \frac{L(1, \eta_v)^2}{e(E_v/F_v)L(1/2, \pi_v, \text{E} \otimes \Omega_v)}
\]

\[
\gg_{\pi} 2^{-|\text{Ram}(B) \cap \mathbb{S}|} \prod_{v \in S_{2} \cup S_{3}} C(\Omega_v)^{-\frac{1}{2}}.
\]

Taking \((2.7)\) into account, we have proved
Lemma 4.2. For the choice of $\sigma = \otimes_v \sigma_v$ and $g = (g_v)_v \in G(\mathbb{A})$ above, we have
\[
\hat{a}(g, \sigma; \Omega; \iota \iota) \gg_{\mathcal{F}, \pi} D(E)^{-\frac{1}{2}} \prod_{v \in S} C(\Omega_v)^{-\frac{1}{2}} \left| \frac{L(1/2, \pi_E \otimes \Omega)}{L(1, \eta)^2 L(1, 1, \pi, \mathbb{A})} \right|.
\]

4.2. Global Estimation: Amplification. Let $E$ be some positive real number to be optimized later and define a set of places
\[ I_E = \{ v : E \leq q_v \leq 2E, v \in S_{4, \pi} \} \]

and the cardinality of $I_E$. Recall that in the definition of $\iota_0$ at a place $v \in S_{4, \pi}$ we fixed an identification $E_v \simeq F_v \times F_v$. Under this identification we observe that for any $\phi \in \sigma_v, v \in I_E$
\[
\tilde{\ell}(g, \phi; \iota, \iota) = \int_{Z(A)T(F) \setminus T(A)} g \phi(t_{\iota_v}((\varpi_v, 1))) \Omega(t_{\iota_v}((\varpi_v, 1))) \, dt
\]
\[
= \Omega_v((\varpi_v, 1)) \int_{Z(A)T(F) \setminus T(A)} g g_v^{-1} \iota_v((\varpi_v, 1)) g \phi(t_{\iota_v}((\varpi_v, 1))) \Omega(t_{\iota_v}((\varpi_v, 1))) \, dt
\]
\[
= \iota_v((\varpi_v, 1)) \tilde{\ell}(g_{\iota_v}^{-1}(a(\varpi_v)), \phi; \iota, \iota).
\]

Denoting by $B(\sigma)$ an orthogonal basis of $\sigma$, we get by Cauchy-Schwarz inequality
\[
\hat{a}(g, \sigma; \Omega, \iota) = \sum_{\varphi \in B(\sigma)} \frac{|\tilde{\ell}(g, \varphi; \iota, \iota)|^2}{\lVert \varphi \rVert^2_{X(\mathcal{B})}}
\]
\[
= \sum_{\varphi \in B(\sigma)} \frac{1}{\lVert \varphi \rVert^2_{X(\mathcal{B})}} \left| \frac{1}{M_E} \sum_{v \in I_E} \Omega((\varpi_v, 1)) \tilde{\ell}(g_{\iota_v}^{-1}(a(\varpi_v)), \varphi; \iota, \iota) \right|^2
\]
\[
\leq \sum_{\varphi \in B(\sigma)} \frac{1}{\lVert \varphi \rVert^2_{X(\mathcal{B})}} \tilde{\ell} \left( \frac{1}{M_E} \sum_{v \in I_E} \Omega((\varpi_v, 1)) g_{\iota_v}^{-1}(a(\varpi_v)), \varphi \right)^2 1, \iota
\]
\[
= \frac{M_E}{E} \sum_{v_1, v_2 \in I_E} \Omega((\varpi_{v_1}, \varpi_{v_2}^{-1}, 1)) \tilde{\ell}(g, \Phi(v_1, v_2); 1, \iota),
\]
where we have denoted
\[
\Phi(v_1, v_2)(x) = \sum_{\varphi \in B(\sigma)} \lVert \varphi \rVert^2_{X(\mathcal{B})} \delta_{\iota_v}^{-1}(a(\varpi_v)) \cdot \delta_{\iota_v}^{-1}(a(\varpi_v)) \cdot \varphi(x), \forall x \in X(\mathcal{B}),
\]

which satisfies:

(0) $\Phi(v_1, v_2)$ is invariant by $Z(\mathbb{A})$;
(1) At $v \in \text{Ram}(\mathcal{B})$, $\Phi(v_1, v_2)$ is invariant by $G(F_v)$;
(2) At $v \notin \{ v_1, v_2 \}$, $\Phi(v_1, v_2)$ is invariant by $K_v$;
(3) At $v = v_1 \neq v_2$ or $v = v_2 \neq v_1$, $\Phi(v_1, v_2)$ is invariant by $K_v \cap \delta_{\iota_v}^{-1}(a(\varpi_v))K_v \delta_{\iota_v}^{-1}(a(\varpi_v))^{-1}$;
(4) At $v = v_1 = v_2$, $\Phi(v_1, v_2)$ is invariant by $\delta_{\iota_v}^{-1}(a(\varpi_v))K_v \delta_{\iota_v}^{-1}(a(\varpi_v))^{-1}$.

Remark 4.3. If we write
\[
K(v_1, v_2) = \prod_{v \in \text{Ram}(\mathcal{B})} G(F_v) \prod_{v \notin \text{Ram}(\mathcal{B})} K_v \prod_{v = v_1 \neq v_2 \text{ or } v = v_2 \neq v_1} K_v \cap \delta_{\iota_v}^{-1}(a(\varpi_v))K_v \delta_{\iota_v}^{-1}(a(\varpi_v))^{-1}
\]
\[
\cdot \prod_{v = v_1 = v_2} \delta_{\iota_v}^{-1}(a(\varpi_v))K_v \delta_{\iota_v}^{-1}(a(\varpi_v))^{-1},
\]
then the group
\[
\Gamma(v_1, v_2) = G(F) \cap K(v_1, v_2)
\]
is the group of invertible elements of some \( \sigma \)-order of \( B \) which has at most prime level at \( v_1 \) and \( v_2 \). It is in fact a lattice of \( \prod_{v_1 v_2} G(F_v) \) and

\[
X(B)/K(v_1, v_2) \cong \Gamma(v_1, v_2) \backslash \prod_{v \mid \infty} G(F_v)/K_v,
\]
on which live the “classical Hilbert modular forms”.

We spectrally decompose \( \Phi(v_1, v_2) \). Note that if \( \text{Ram}(B) \neq \emptyset \) then \( X(B) \) is compact and there is no issue of convergence, while if \( \text{Ram}(B) = \emptyset \) then the normal convergence is established in [44 Theorem 2.16]. We can apply \( \tilde{\ell}(\cdot; 1; \ell) \) to (recall \( \text{Vol}(X(B)) = 2 \))

\[
\Phi(v_1, v_2) = \sum_{x} \frac{\langle \Phi(v_1, v_2), \tilde{x} \rangle_{X(B)} }{2} \tilde{x} + \sum_{\varphi \in \text{cusp}(B)} \sum_{\phi \in B(e)} \frac{\langle \Phi(v_1, v_2), \phi \rangle_{X(B)} }{\| \phi \|^2_{X(B)}} \frac{\langle \Phi(v_1, v_2), E(s, f) \rangle_{X(M_2), \text{hyp}} }{\| E(s, f) \|^2_{\text{Eis}}} E(s, f) \frac{ds}{4\pi i}.
\]

and get

\[
\tilde{\ell}(g, \Phi(v_1, v_2); 1, \ell) = S_1(v_1, v_2) + S_{\text{cusp}}(v_1, v_2) + 1_{\text{Ram}(B) = \emptyset} S_{\text{Eis}}(v_1, v_2),
\]

where

\[
S_1(v_1, v_2) = \frac{1}{2} \sum_{x} \langle \Phi(v_1, v_2), \tilde{x} \rangle_{X(B)} \chi(\det g) \tilde{\ell}(\tilde{x}; 1, \ell),
\]

\[
S_{\text{cusp}}(v_1, v_2) = \sum_{\varphi \in \text{cusp}(B)} \sum_{\phi \in B(e)} \frac{\langle \Phi(v_1, v_2), \phi \rangle_{X(B)} }{\| \phi \|^2_{X(B)}} \frac{\langle \Phi(v_1, v_2), E(s, f) \rangle_{X(M_2), \text{hyp}} }{\| E(s, f) \|^2_{\text{Eis}}} \tilde{\ell}(g, \phi; 1, \ell),
\]

\[
S_{\text{Eis}}(v_1, v_2) = \sum_{\xi \in \mathcal{F} \setminus \chi(1)} \int_{-i\infty}^{i\infty} \sum_{f \in B(\pi_0, \zeta)} \frac{\langle \Phi(v_1, v_2), E(s, f) \rangle_{X(M_2), \text{hyp}} }{\| E(s, f) \|^2_{\text{Eis}}} \tilde{\ell}(g, E(s, f); 1, \ell) \frac{ds}{4\pi i};
\]

and the terms with non zero contribution are only possibly over:

(1) \( \chi \): Hecke characters of \( F \) such that \( \chi(\det K(v_1, v_2)) = 1 \), i.e.

- \( \chi_v = 1 \) for \( v \in \text{Ram}(B), v < \infty \);
- \( \chi_v |_{\mathbb{Q}_v} = 1 \) for \( v \in \text{Ram}(B), v \mid \infty \);
- \( \chi_v \) is unramified for \( v \notin \text{Ram}(B), v < \infty \).

We also recall that \( \tilde{\chi} = \chi \circ \text{det} \).

(2) \( g \): cuspidal representations such that

- \( \varrho_v = 1 \) for \( v \in \text{Ram}(B) \);
- \( \varrho_v \) is spherical at \( v \notin \text{Ram}(B) \cup \{ v_1, v_2 \} \) resp. at \( v = v_1 = v_2 \), and \( \varphi_v \) is the spherical vector resp. spherical vector with respect to \( \delta_v^{-1}(a(\varpi_v)/K_v) \delta_v^{-1}(a(\varpi_v))^{-1} \);
- \( c(\varrho_v) \leq 1 \) at \( v = v_1 \neq v_2 \) or \( v = v_2 \neq v_1 \) and \( \varphi_v \) is invariant by \( K_v \cap \delta_v^{-1}(a(\varpi_v)/K_v) \delta_v^{-1}(a(\varpi_v))^{-1} \).

(3) \( \xi \): Characters of \( \mathbb{F}^\times \setminus \mathbb{A}(1) \) unramified everywhere at \( v < \infty \), i.e. characters of the generalized ideal class group of \( F \). Recall \( \pi_{\alpha} = \pi(\xi, \xi^{-1}) \).

- At \( v \notin \{ v_1, v_2 \} \) resp. \( v = v_1 = v_2 \), \( f_v \) is the spherical vector resp. spherical vector with respect to \( \delta_v^{-1}(a(\varpi_v)/K_v) \delta_v^{-1}(a(\varpi_v))^{-1} \);
- At \( v = v_1 \neq v_2 \) or \( v = v_2 \neq v_1 \), \( f_v \) is invariant by \( K_v \cap \delta_v^{-1}(a(\varpi_v)/K_v) \delta_v^{-1}(a(\varpi_v))^{-1} \).

The inner product \( \langle \cdot, \cdot \rangle_{X(M_2), \text{hyp}} \) is the one obtained from the hyperbolic measure on \( \text{GL}_2(\mathbb{A}) \) (c.f. Section 2.3).

In this way, we can write

\[
\tilde{\alpha}(g, \sigma; \Omega, \ell) = \Sigma_1 + \Sigma_{\text{cusp}} + 1_{\text{Ram}(B) = \emptyset} \Sigma_{\text{Eis}},
\]
\[ \Sigma_1 = \frac{1}{M_E} \sum_{v_1, v_2 \in I_G} \Omega((\varpi_{v_1} \varpi_{v_2}^{-1}, 1)) \chi_1(v_1, v_2), \]

(4.3)

\[ \Sigma_{\text{cusp}} = \frac{1}{M_E} \sum_{v_1, v_2 \in I_G} \Omega((\varpi_{v_1} \varpi_{v_2}^{-1}, 1)) \chi_{\text{cusp}}(v_1, v_2), \]

(4.4)

\[ \Sigma_{\text{Eis}} = \frac{1}{M_E} \sum_{v_1, v_2 \in I_G} \Omega((\varpi_{v_1} \varpi_{v_2}^{-1}, 1)) \chi_{\text{Eis}}(v_1, v_2). \]

(4.5)

### 4.3. Global Estimation: One-dimensional Part.

First we note that

\[ \ell(\tilde{\chi}; 1, t) = \int_{\mathcal{Z}(\Lambda)T(F) \setminus T(\Lambda)} \chi(\det t)[dt] = \int_{\mathbf{A}^* \mathbf{E}^* \setminus \mathbf{A}^*_E} \chi(\mathbf{Nr}_{E}(t))[dt] = 1_{\{1, \eta\}}, \]

hence at most two terms are non zero in \( S_1(v_1, v_2) \) thus in \( \Sigma_1 \).

**Remark 4.4.** The term \( \chi = \eta \) gives non zero contribution only if the following restricted conditions are satisfied:

- \( \forall v \in \text{Ram}(B) \) implies \( v \mid \infty \);
- \( \text{E}_v/F_v \) is everywhere not ramified for \( v < \infty \);
- \( \pi \simeq \pi \otimes \eta \).

Next we note that by regrouping

\[ \Sigma_1 = \frac{1}{2} \sum_{\varphi \in \mathcal{B}(\sigma)} \| \varphi \|_{X(B)}^2 \sum_{\chi \in \{1, \eta\}} \left( \left| \frac{1}{M_E} \sum_{v \in I_G} \Omega((\varpi_v, 1)) g \delta_v^{-1}(a(\varpi_v)) \varphi \right| X(B) \right)^2, \]

hence

\[ |\Sigma_1| \leq \sum_{\varphi \in \mathcal{B}(\sigma)} \| \varphi \|_{X(B)}^2 \left( \left| \frac{1}{M_E} \sum_{v \in I_G} \Omega((\varpi_v, 1)) g \delta_v^{-1}(a(\varpi_v)) \varphi \right| X(B) \right)^2, \]

\[ = \frac{1}{M_E^2} \sum_{\varphi \in \mathcal{B}(\sigma)} \| \varphi \|_{X(B)}^2 \sum_{v_1, v_2 \in I_G} \Omega((\varpi_{v_1} \varpi_{v_2}^{-1}, 1)) \langle \delta_v^{-1}(a(\varpi_v)), \varphi, \delta_{v_2}^{-1}(a(\varpi_{v_2})) \varphi \rangle_{X(B)} \]

\[ = \frac{1}{M_E} \sum_{\varphi \in \mathcal{B}(\sigma)} \| \varphi \|_{X(B)}^2 \sum_{v_1, v_2 \in I_G} \Omega((\varpi_{v_1} \varpi_{v_2}^{-1}, 1)) \prod_{v \in \{v_1, v_2\}} \frac{|\langle \delta_v^{-1}(a(\varpi_v)), \varphi, \delta_{v_2}^{-1}(a(\varpi_{v_2})) \varphi \rangle_v|}{\| \varphi \|_{X(B)}^2}, \]

\[ \leq \frac{\| B(\sigma) \|}{M_E} + \frac{1}{M_E} \sum_{\varphi \in \mathcal{B}(\sigma)} \sum_{v_1, v_2 \in I_G} \sum_{v \in \{v_1, v_2\}} \frac{|\langle \delta_v^{-1}(a(\varpi_v)), \varphi, \delta_{v_2}^{-1}(a(\varpi_{v_2})) \varphi \rangle_{v_1} \varphi_{v_2}|}{\| \varphi \|_{X(B)}^2}, \]

where we have chosen local inner products on \( \pi_v \) such that on \( \pi \in L^2(G_B, \omega) \)

\[ \| \varphi \|_{X(B)} = \prod_v \| \varphi \|_{v}, \text{ for } \varphi \simeq \otimes' \varphi_v \text{ under } \pi \simeq \otimes' \pi_v. \]

By our choice of \( \sigma, \varphi_v \) is the spherical vector in \( \pi_v \) for \( v \in \{v_1, v_2\} \). Hence we can calculate the local inner products in the Kirillov model using the Macdonald’s formula [4, Theorem 4.6.5], i.e. (c.f. [44, Corollary 5.18 (2) & (3)])

\[ \frac{|\langle \varphi_v, \delta_v^{-1}(a(\varpi_v)), \varphi \rangle_{v}|}{\| \varphi \|_{v}^2} = \frac{q_v^{-\frac{1}{2}}}{1 + q_v^{-1} |\text{tr}_v|}, \text{ for } v \in S_{4,s}, \]
where $\text{tr}_v = \alpha_v + \beta_v$ is the sum of the Satake parameters $\alpha_v, \beta_v$ of $\pi_v$. Applying [11] Lemma 6.1 to $\pi$ and $\pi \otimes \eta$ we get for any $\epsilon > 0$

\[(4.6) \quad |\Sigma| \leq \frac{|B(\sigma)|}{M_E} + \frac{|B(\sigma)|}{M_E} \sum_{v_1, v_2 \in I_E} \frac{\tilde{q}_{v_1}^2}{1 + q_{v_1}^{-1}} \frac{\tilde{q}_{v_2}^2}{1 + q_{v_2}^{-1}} |\text{tr}_v| |\text{tr}_{v_2}| \ll_{F, \pi, \epsilon} M_E^{-1} + E^{-1+\epsilon}.
\]

### 4.4. Global Estimation: Cuspidal Part

We apply (2.6) to $\phi \in \mathfrak{g}$ appearing in $S_{\text{cusp}}(v_1, v_2)$ to see

\[\frac{\tilde{\ell}(g, \phi; 1, t)}{\|\phi\|^2_{X(B)}} \leq \frac{D(F)^{1/2} \|\text{Ram}(E/F)\| \Lambda(1/2, \theta_E)}{8D(E)^{1/2} \Lambda(1, \eta)^2 \Lambda(1, \theta, \text{Ad})} \cdot \prod_{v \in \mathfrak{p}_F} \frac{L(1, \eta_v) L(1, \theta_v, \text{Ad})}{\zeta_v(2) L(1/2, \theta_{E,v})} \alpha_v(g_v, \phi_v; 1, t_v).
\]

We estimate the local terms place by place in sections 4.4.1 to 4.4.10 to get

\[\frac{\tilde{\ell}(g, \phi; 1, t)}{\|\phi\|^2_{X(B)}} \ll_{F, \pi, \epsilon} 2^{\|\text{Ram}(E/F)\|} D(E)^{-\frac{t}{2}} C(\Omega)^{-1-2\phi + \epsilon} \lambda_{\phi, \infty}^{1+\epsilon} L(1/2, \theta_E) L(1, \theta, \text{Ad}) \]

where $\lambda_{\phi, \infty} = \lambda_{\phi, \infty}$ is the eigenvalue of the Casimir element $\Delta_\infty = \Delta_{B, \infty}$ of $\prod_{v \in \mathfrak{p}_F} Z \setminus G(F_v)$ on $\mathfrak{g}$. We can thus proceed as in [11] (6.16) to see, for $l$ sufficiently large

\[|S_{\text{cusp}}(v_1, v_2)| \leq \left( \sum_{\mathfrak{g}} \sum_{\phi \in B(\mathfrak{g})} \frac{|(\Delta_{B, \infty}^{1+1} \Phi(v_1, v_2), \phi)_{X(B)}|^2}{\|\phi\|^2_{X(B)}} \right)^{\frac{1}{2}} \left( \sum_{\mathfrak{g}} \sum_{\phi \in B(\mathfrak{g})} \frac{\tilde{\ell}(g, \phi; 1, t)^2}{\|\phi\|^2_{X(B)}} \right)^{\frac{1}{2}}
\]

\[\ll_{F, \pi, \epsilon} 2^{\|\text{Ram}(E/F)\|} D(E)^{-\frac{t}{2}} C(\Omega)^{-1-2\phi + \epsilon} |L(1, \eta)|^{-1} |\Delta_{B, \infty}^{1+1} \Phi(v_1, v_2)|_{X(B)}
\]

\[\ll_{F, \pi, \epsilon} 2^{\|\text{Ram}(E/F)\|} D(E)^{-\frac{t}{2}} C(\Omega)^{-1-2\phi + \epsilon} |L(1, \eta)|^{-1} E^{1+B + \epsilon},
\]

where the last step uses the following theorem:

**Theorem 4.5.** Fix $B$ and $n_v \in \{0, 1\}$ for each $v < \infty, v \not\in \text{Ram}(B)$ such that $n_v = 1$ for zero or two $v$. Let $E \subset B$ vary over quadratic extension of $F$. Let $\phi$ run over the cuspidal representations of $G(\mathbb{A}) \simeq G_B(\mathbb{A})$ with trivial central character, with the following properties:

- $\phi_v = 1$ at $v \in \text{Ram}(B)$;
- $\phi_v$ is spherical at $v \mid \infty, v \not\in \text{Ram}(B)$;
- the local (logarithmic) conductor at each finite place $v < \infty, v \not\in \text{Ram}(B)$ satisfies $c(\phi_v) \leq n_v$.

Then we have for any $\epsilon > 0$, $B = 3/4, \delta = 1/8$ and sufficiently large $A > 0$

\[(4.7) \quad \sum_{\mathfrak{g}} \frac{L(1/2, \theta_E)}{L(1, \theta, \text{Ad})} \lambda_{\phi, \infty}^{-A} \ll_{F, B, \epsilon} (ED(E))^{1/2} E^{1+B} D(E)^{\frac{t}{2} - \delta},
\]

where we have written

- $E = \prod_{v < \infty} q_v^{n_v}$ and $q_v$ is the cardinality of the residue field of $F_v$.
- $\lambda_{\phi, \infty}$ is the eigenvalue of the Casimir element of $\prod_{v \mid \infty, v \not\in \text{Ram}(B)} Z \setminus G(F_v)$ on $\phi$.
- $\phi_E$ is the base change representation of $\phi$ to $G(\mathbb{A}) \simeq \text{GL}_2(\mathbb{A}_E)$. 

Remark 4.6. If we view Theorem 4.3 as a generalization of [24] Theorem 1.1, the average bound Theorem 4.5 is a generalization of [24] Corollary 6.7 to the new framework.

Remark 4.7. The ideal/optimal bound in Theorem 4.3 should give \( B = 0, \delta = 1/2 \) as is the case for [24] Corollary 6.7. This is the Lindelöf Hypothesis on average and seems to be too far from reach using current technologies, since otherwise one would deduce Lindelöf Hypothesis for \( L(1/2, \chi \otimes \eta) \) with respect to \( D(E) = C(\eta) \) by noting that \( L(1/2, \phi_E) = L(1/2, \chi \otimes \eta) \), where \( \eta \in \mathbb{F}/F \) is the quadratic Hecke character associated with the quadratic extension \( E/F \), is non-negative (c.f. [20]).

Remark 4.8. For possible ways to establish Theorem 4.3 as well as known special cases please see Section 5. We will give a complete proof of Theorem 4.3 in a future paper [29].

Hence by (4.3) we get

\[
|\mathbb{S}_{\text{cusp}}| \leq \frac{1}{M_E^2} \sum_{v_1, v_2 \in F_E} |S_{\text{cusp}}(v_1, v_2)| \ll_{F, \eta, \nu} 2^{\frac{|\text{Ram}(E/F)|}{2}} D(E)^{-2} C(\Omega)^{-\frac{1}{1-2\delta} + \epsilon} |L(1, \eta)|^{-1} E^{1+B+\epsilon}.
\]

4.4.1. At \( v \in \text{Ram}(B) \), \( \phi_v = 1 \), hence

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) = \int_{Z \setminus T(F_v)} \tilde{dt} = \text{Vol}(F_v^x \setminus E_v^x, \tilde{dt}) = 1.
\]

4.4.2. At \( v \notin \text{Ram}(B) \), \( F_v = \mathbb{R}, E_v = \mathbb{C} \). \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_0 \) defined in (2.14), hence

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) = \tilde{\alpha}_v(\phi_v; 1, \nu_0) = \int_{\mathbb{R}^x \setminus \mathbb{C}^x} \tilde{dt} = 1.
\]

4.4.3. At \( v \notin \text{Ram}(B), E_v \) nonsplit, \( v < \infty, \pi_v \) ramified non-supercuspidal. \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_0 \) defined in (2.16), hence by Lemma 3.14

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) \ll \left\{ \begin{array}{ll} (c(\Omega_v) - c(\pi_v)) q_v^{-\frac{1}{2\delta}} & \text{if } c(\Omega_v) > c(\pi_v) \\
\zeta_v(2) L(1/2, \phi_{E_v, v}) & \text{if } c(\Omega_v) \leq c(\pi_v). \end{array} \right.
\]

4.4.4. At \( v \notin \text{Ram}(B), E_v \) nonsplit, \( v < \infty, \pi_v \) supercuspidal and either \( c(\Omega_v) \leq c(\pi_v) \) if \( E_v/F_v \) is unramified or \( c(\Omega_v) < c(\pi_v) \) if \( E_v/F_v \) is ramified. \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_0 \) defined in (2.16), hence by Lemma 3.14

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) = \frac{\zeta_v(2) L(1/2, \phi_{E_v, v})}{L(1, \eta_v) L(1, \nu_v, Ad)}.
\]

4.4.5. At \( v \notin \text{Ram}(B), E_v \) split, \( v | \infty \). \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_v \) for some \( r \in F_v \) satisfying \( |r|_v \in [C(\Omega_v), 2C(\Omega_v)] \) and for \( \nu_v \) defined in (2.28), hence by [21] Corollary 6.9

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) \leq C(\theta, \epsilon)^2 \lambda_v^{1+\epsilon} C(\Omega_v)^{-\frac{1}{5\delta} + \epsilon},
\]

where \( \lambda_v \) is the eigenvalue of the Casimir element of \( G(F_v) \) on the space \( \phi_v \).

4.4.6. At \( v \notin \text{Ram}(B), E_v \) split, \( v < \infty \) and \( \pi_v \) non-spherical or \( c(\Omega_v) > 0 \). \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_v \) defined in (2.31), hence by [21] Corollary 6.9

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) \leq C(\theta, \epsilon)^2 C(\Omega_v)^{-\frac{1}{5\delta} + \epsilon} \frac{\zeta_v(2) L(1/2, \phi_{E_v, v})}{L(1, \eta_v) L(1, \nu_v, Ad)}.
\]

4.4.7. At \( v \notin \text{Ram}(B), E_v \) nonsplit, \( v < \infty, \pi_v \) supercuspidal and either \( c(\Omega_v) > c(\pi_v) \) if \( E_v/F_v \) is unramified or \( c(\Omega_v) \geq c(\pi_v) \) if \( E_v/F_v \) is ramified. \( \phi_v \) is a spherical vector in \( \phi_v \) and \( g_v^{-1} \nu_v g_v = \nu_v \) defined in (2.16), hence by Lemma 3.14

\[
\tilde{\alpha}_v(g_v, \phi_v; 1, \nu_v) \ll \left\{ \begin{array}{ll} (c(\Omega_v) - c(\pi_v)) q_v^{-\frac{1}{2\delta}} & \text{if } c(\Omega_v) > c(\pi_v) \\
\zeta_v(2) L(1/2, \phi_{E_v, v}) & \text{if } c(\Omega_v) \geq c(\pi_v). \end{array} \right.
\]
4.4.8. At $v \notin \text{Ram}(B), E_v$ nonsplit, $v < \infty, \pi_v$ spherical and $c(\Omega_v) > 0$. Equivalently, $v \in S_4$. $\phi_v$ is a spherical vector in $\varrho_v$ and $g_v^{-1}t_vg_v = t_v(\Omega_v)$ defined in (2.10), hence by Lemma 3.13
\[ \alpha_v(g_v, \phi_v; 1, t_v) \ll c(\Omega_v) \varrho_v^{(1-2\theta)c(\Omega_v)}. \]

4.4.9. At $v \notin \text{Ram}(B), E_v$ nonsplit, $v < \infty, \pi_v$ spherical and $c(\Omega_v) = 0$ or at $v \notin \text{Ram}(B), E_v$ split, $v < \infty, \pi_v$ spherical and $c(\Omega_v) = 0$ but $v \notin \{v_1, v_2\}$. Equivalently, $v \in S_4 - \{v_1, v_2\}$. $\phi_v$ is a spherical vector in $\varrho_v$ and $g_v^{-1}t_vg_v = t_0$ defined in (2.10), hence by Lemma 3.13
\[ \alpha_v(g_v, \phi_v; 1, t_v) = \frac{\zeta_v(2)L(1/2, \varrho_{E_v})}{L(1, \eta_v)L(1, g_v, Ad)}. \]

4.4.10. At $v \in \{v_1, v_2\}$. $c(\varrho_v) \leq 1$ and the level of $\phi_v$ is at most 1. Also $g_v^{-1}t_vg_v = t_0$ defined in (2.10). We choose $\phi_v \in B(\varrho_v)$ as the one for [44, Lemma 6.12] and get
\[ \alpha_v(g_v, \phi_v; 1, t_v) \leq C(\theta, \epsilon)^2 \frac{\zeta_v(2)L(1/2, \varrho_{E_v})}{L(1, \eta_v)L(1, g_v, Ad)}. \]

4.5. Global Estimation: Continuous Part. We apply an analogue of (2.6) in the case of Eisenstein series to $f \in \pi_\alpha, \xi$ appearing in $S_{\text{Eis}}(v_1, v_2)$ to see
\[ \frac{\|\hat{\ell}(g, E(s, f); 1, t)\|^2}{\|E(s, f)\|^2_{\text{Eis}}} = \frac{2^{\text{Ram}(E/F)}|\Lambda(\frac{1}{2} + s, \xi_E)\Lambda(\frac{1}{2} - s, \xi_E^{-1})|}{4D(E)\Lambda(1, \eta)^2\Lambda(1 + 2s, \xi^2)\Lambda(1 - 2s, \xi^{-2})} \cdot \prod_{v \in V_p} \frac{\zeta_v(1)L(1, \eta_v)L(1 + 2s, \xi_v^2)\Lambda(1 - 2s, \xi_v^{-2})}{\zeta_v(2)L(\frac{1}{2} + s, \xi_{E_v})}\alpha_v(g_v, s, f; 1, t_v), \]

We estimate the local terms place by place as in the cuspidal case to get for $s \in \mathbb{R}$
\[ \frac{\|\hat{\ell}(g, E(s, f); 1, t)\|^2}{\|E(s, f)\|^2_{\text{Eis}}} \ll \frac{1}{2} D(E)^{-\frac{1}{2}} C(\Omega)^{-\frac{1}{2} + \epsilon} \lambda_{s, \infty}^{\frac{1}{2} + \epsilon} \frac{L(1/2 + s, \xi_E)}{L(1, \eta)L(1 + 2s, \xi^2)} \]
where $\lambda_{s, \infty} = \lambda_{s, \infty}$ is the eigenvalue of the Casimir element $\Delta_{\infty} = \Delta_{M_2, \infty}$ of $\prod_{v \mid \infty} Z \cdot \text{GL}_2(F_v)$ on $\pi_s, \xi$. We can thus proceed as before to see for $l$ sufficiently large
\[ |S_{\text{Eis}}(v_1, v_2)| \leq \left( \sum_{\xi} \left( \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\hat{\ell}(g, E(s, f); 1, t)|^2}{\|E(s, f)\|^2_{\text{Eis}}} ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \ll_{F, \pi, \epsilon} 2^{\text{Ram}(E/F)} D(E)^{-\frac{1}{2}} C(\Omega)^{-\frac{1}{2} + \epsilon} L(1, \eta)^{-1} \lambda_{s, \infty}^{\frac{1}{2} + \epsilon} \frac{L(1/2 + s, \xi_E)}{L(1, \eta)L(1 + 2s, \xi^2)} \frac{\alpha_v(g_v, s, f; 1, t_v)}{\alpha_v(g_v, s, f; 1, t_v)} \]

Theorem 4.9. Let $\xi$ run over the Hecke characters of $F^\times \setminus \mathbb{A}^{(1)}$ unramified everywhere at $v < \infty$. Then we have for any $\epsilon > 0$, $\delta' = (1 - 2\theta)/8$ and sufficiently large $A > 0$
\[ \sum_{\xi} \int_{-\infty}^{\infty} \frac{L(\frac{1}{2} + s, \xi_E)^2}{L(1 + 2s, \xi)} ds \ll_{F, \epsilon} \frac{A^4}{4\pi^4} \frac{\lambda_{s, \infty}}{\lambda_{s, \infty}^{A, \infty}} \ll_{F, \epsilon} D(E)^{\frac{1}{2} - \delta' + \epsilon}, \]
where we have written:
• $\xi_E$ is the base change of $\xi$ to $\mathbb{A}^E_{\mathbb{F}}$, i.e. $\xi_E = \xi \otimes \mathbb{F}$.  
• $\lambda_{\xi,s,\infty}$ is the eigenvalue of the Casimir element of $\prod_{v \mid \infty} Z \backslash GL_2(F_v)$ on $\pi_{s,\xi} = \pi(\xi \cdot |^s, \xi^{-1} \cdot |^{-s})$.

Remark 4.10. We have similarly $L(s, \xi_E) = L(s, \xi) L(s, \xi_E)$. Hence Theorem 4.9 is a subconvex bound with respect to $\eta$ on average over $\xi$ and $s \in \mathbb{R}$.

Remark 4.11. For possible ways to establish Theorem 4.9 as well as known special cases please see Section 3. In particular Theorem 4.9 is implied by our future paper [43].

Hence by (4.5) we get

\[ |\Sigma_{Eis}| \leq \frac{1}{M_E} \sum_{v_1, v_2 \in J_E} |S_{Eis}(v_1, v_2)| \ll_{F, \pi, r} 2^{B_{\text{Ram}(E/F)} + 1} D(E)^{-\frac{5}{4}} C(\Omega)^{-\frac{1}{2}} |L(1, \eta)|^{-1}. \]

4.6. Finalising the Estimation. Inserting (4.6), (4.8) and (4.10) into (4.2) and taking Lemma 4.2 into account we get

\[
\left| \frac{L(1, \pi_E \otimes \Omega)}{L(1, \pi, \text{Ad})} \right| D(E)^{-\frac{5}{4}} C(\Omega)^{-\frac{1}{2}} \ll_{F, \pi, r} |\Sigma_1| + |\Sigma_{\text{cusp}}| + 1_{\text{Ram}(B) = 0} |\Sigma_{Eis}|
\]

\[
\ll_{F, \pi, r} |L(1, \eta)| (M_E^{-1} + E^{-1+\varepsilon}) + D(E)^{-\frac{5}{4}} C(\Omega)^{-\frac{1}{2}} E^{1-B+\varepsilon}
\]

\[
+ 1_{\text{Ram}(B) = 0} D(E)^{-\frac{5}{4}} C(\Omega)^{-\frac{1}{2}} + \varepsilon.
\]

At this stage, we would like to choose $E$ as in [44], i.e.

\[ E = D(E)^{\frac{d}{2(2-B-\varepsilon)}} C(\Omega)^{\frac{1-2B}{2(2-B-\varepsilon)}}. \]

However, our $M_E$ here is essentially the number of splitting finite places $v$ such that $q_v \in [E, 2E]$, not the same as in [44]. Consequently for small $E$ the magnitude of $M_E$ can not be as large as $E / \log E$. This is where we need the assumption (1.3) in Theorem 1.1. The optimal choice under (1.3) then becomes

\[ E = D(E)^{\frac{d}{2(2-B-\varepsilon)}} C(\Omega)^{\frac{1-2B}{2(2-B-\varepsilon)}} \]

and the restriction

\[ E = D(E)^{\frac{d}{2(2-B-\varepsilon)}} C(\Omega)^{\frac{1-2B}{2(2-B-\varepsilon)}} \geq D(E)^{\delta} \iff C(\Omega) \geq D(E)^{\frac{4d(2-B-\varepsilon)}{1-2B}}. \]

5. Average Bounds

In this section, we discuss on possible ways to establish Theorem 4.5 and Theorem 4.9.

5.1. Apply Individual Bounds. Since $L(s, \varphi_E) = L(s, \varphi) L(s, \varphi \otimes \eta)$ resp. $L(s, \xi_E) = L(s, \xi) L(s, \xi \otimes \eta)$, one can apply individual bounds of $L(1/2, \varphi \otimes \eta)$ resp. $L(1/2, \xi \otimes \eta)$ subconvex with respect to $C(\eta) = D(E)$ and polynomial with respect to $C(\varphi)$ resp. $C(\xi)$. In particular, an improvement of [44] as [42] resp. a generalization of [44] as [43] can give a version of Theorem 4.5 resp. the full version of Theorem 4.9. However, we do not expect [42] to give the full version of Theorem 4.5. The detailed discussion on the full version of both theorems is as follows.

Although over a general number field such results are rare, over $\mathbb{Q}$ they have been much studied. We begin with Theorem 4.9. In this case, the only possible $\xi$ is the trivial character. We can apply the convex bound for $\zeta(1/2 + s)$, lower bound for $\zeta(1 + 2s)$ like [35] (3.6.3)] and the effective Burgess bound $L(1/2 + s, \eta) \ll_{\varepsilon} |s| D(E)^{3/16 + \varepsilon}$ to see that even $\delta = 1/8$ is allowable. For Theorem 4.5 the bound $|L(1, \varphi, \text{Ad})| \gg_{F, \pi} C(\varphi)^{-\varepsilon}$ is valid [16] even for general number fields [2] Lemma 3. The convex bound $|L(1/2, \varphi)| \ll_{\varepsilon} C(\varphi)^{1/4 \varepsilon}$ and the bound $|L(1/2, \varphi \otimes \eta)| \ll_{\varepsilon} C(\varphi)^{1/2 + \varepsilon} D(E)^{3/8 + \varepsilon}$ [11] Theorem 2 with addendum [3] imply that $B = 3/4, \delta = 1/8$ is allowable.

It should be possible to generalize the above mentioned results to a general number field. Since these results are obtained by amplification with explicit Kuznetsov formula, it is expected that the possible generalization uses some explicit relative trace formulas for unipotent subgroups (c.f. [22] [23]).
5.2. Apply Bounds for Quadratic Twists. Although we have powerful results as \[1\], the idea of applying individual bounds does not seem to be the most natural way. As mentioned in Remark \[1.3\] a more natural way would be to apply something specific for quadratic twists. Taking Theorem \[1.3\] for example, we could apply Hölder’s inequality to see

\[
\sum_{\mathfrak{e}} \frac{L(\frac{1}{2}, g_{\mathfrak{e}})}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \leq \left( \sum_{\mathfrak{e}} \frac{\lambda_{\mathfrak{e}, \infty}^{-A}}{L(1, \varrho, Ad)} \right)^{\frac{1}{4}} \left( \sum_{\mathfrak{e}} \frac{|L(\frac{1}{2}, g_{\mathfrak{e}})|^4}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \right)^{\frac{1}{4}} \left( \sum_{\mathfrak{e}} \frac{|L(\frac{1}{2}, g \otimes \eta)|^3}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \right)^{\frac{1}{4}}.
\]

By Weyl’s law (c.f. [14 Theorem 2.23]), we have

\[
\sum_{\mathfrak{e}} \frac{\lambda_{\mathfrak{e}, \infty}^{-A}}{L(1, \varrho, Ad)} \ll_{\varepsilon} E^{1+\varepsilon}.
\]

By a general result on the fourth moment (c.f. [14 Theorem 6.6]), we have

\[
\sum_{\mathfrak{e}} \frac{|L(\frac{1}{2}, g)|^4}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \ll_{\varepsilon} E^{1+\varepsilon}.
\]

By the famous Weyl-type bound for cubic moments [9] generalized in [14], we have, over \(F = \mathbb{Q}\) and for “\(E = D(E)\)” (actually a condition stronger than this),

\[
\sum_{\mathfrak{e}} \frac{|L(\frac{1}{2}, g \otimes \eta)|^3}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \ll_{\varepsilon} D(E)^{1+\varepsilon}.
\]

By positivity, we then get for “\(E \leq D(E)\)” a strong version of Theorem \[4.5\] over \(\mathbb{Q}\) as

\[
\sum_{\mathfrak{e}} \frac{L(\frac{1}{2}, g)}{L(1, \varrho, Ad)} \lambda_{\mathfrak{e}, \infty}^{-A} \ll_{\varepsilon} E^{\frac{3}{2}+\varepsilon} D(E)^{1+\varepsilon}.
\]

In conclusion, another possibility to obtain Theorem \[4.5\] and \[4.9\] would be via the generalization of \[9, 45\] to the number field case and to get rid of the condition “\(E \leq D(E)\)” in a suitable way. It would be reasonable to expect this method to give bounds even better than what is stated in Theorem \[4.5\] and \[4.9\].

5.3. Apply Relative Trace Formula. We might also expect to get Theorem \[4.5\] and Theorem \[4.9\] by the relative trace formulas on \(G = G_B\) or \(GL_2\) as in \[15\]. Since the calculation is a similar and simpler version of \[11\], we omit the details.

To simplify the discussion, we take a very special case by assuming that \(Ram(B)\) contains all the infinite places of \(F\). We write \(B = B_\infty\) as in (2.2) and recall \(u\) in (2.3), hence \(G = G_\varepsilon\). We specify the maximal order \(\mathfrak{O}\) by specifying it at every place \(v < \infty\):

\begin{itemize}
  \item \(v \notin Ram(B)\): \(B_v\) is a division quaternion algebra over \(F_v\) which has a unique maximal \(\mathfrak{q}_v\)-order \(\mathfrak{O}_v\).
  \item \(v \in Ram(B)\): there is some \(\zeta \in E_v^\times\) such that \(\varepsilon = \text{Nr}_{F_v/F_\infty}(\zeta)\). Recall \(\beta\) defined in (2.10) or (2.17). We let (recall \(\delta_v\) in (2.11))
  \[
  \mathfrak{O}_v = \mathfrak{O}_v e + \mathcal{O}_v, e = (\beta - \bar{\beta})^{-1}(1 - \zeta^{-1}u);
  \]
  \[
  \delta_v : \beta \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \zeta^{-1}u \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
  \]
\end{itemize}

We leave the reader to check that the above defined \(\mathfrak{O}_v\) specifies a unique \(\mathfrak{O}\) (c.f. [11 Section 4]). We take a test function \(f = \otimes'_v f_v \in C_c^\infty(Z\setminus G(\mathbb{A}))\) as follows:

\begin{itemize}
  \item \(v \in Ram(B)\): take \(f_v = 1\).
  \item \(v \notin Ram(B)\) \(\cup \{v_1, v_2\}\): take \(f_v = 1_{Z(F_v)\mathfrak{O}_v^\times}\).
  \item \(v = v_1 = v_2\): take \(f_v = 1_{Z(F_v)\delta_v^{-1}(a(v_\mathfrak{q}_v))\mathfrak{O}_v^\times \delta_v^{-1}(a(v_\mathfrak{q}_v))^{-1}}\).
  \item \(v = v_1 \neq v_2\) or \(v = v_2 \neq v_1\): take \(f_v = 1_{Z(F_v)\delta_v^{-1}(a(v_\mathfrak{q}_v))\mathfrak{O}_v^\times \delta_v^{-1}(a(v_\mathfrak{q}_v))^{-1} \cap Z(F_v)\mathfrak{O}_v^\times}.
\end{itemize}
The operator, with $dg$ chosen to be the standard Tamagawa measure,

$$R(f) : L^2(G, 1) \rightarrow L^2(G, 1), (R(f)\varphi)(x) = \int_{Z \setminus G} f(y)\varphi(xy)dg, \forall x \in Z \setminus G$$

is Hilbert-Schmidt and of trace class with kernel function admitting the spectral decomposition

$$K_f(x, y) = \sum_{\gamma \in Z \setminus G} f(x^{-1}\gamma y)$$

$$= \frac{1}{2} \sum_{\xi} \hat{\xi}(x)\hat{\xi}(y) \int_{Z \setminus G} f(y)\hat{\xi}(g)dg + \sum_{\varphi \in \mathcal{B}(\mathcal{E})} \sum_{\phi \in \mathcal{B}(\mathcal{E})} \frac{(R(\phi)\varphi)(x)\overline{\varphi(y)}}{||\varphi||^2_{X(B)}}$$

We integrate $K_f(x, y)$ along $(Z(\mathcal{A})T(\mathcal{F}))^2$ to get

$$I(f) = \int_{(Z(\mathcal{A})T(\mathcal{F}))^2} K_f(t_1, t_2)dt_1dt_2$$

where $T$ is the $\mathcal{F}$-torus

$$T(\mathcal{F}) = \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & \bar{a} \end{array} \right) : a \in \mathbb{E} \right\}$$

which receives the measure $dt$ as we defined for $T$.

On the one hand, the spectral side is

$$I(f) = \sum_{\xi} I(f; \xi) + \sum_{\varphi} I(f; \varphi)$$

for which we have, writing $q_1 = q_{v_1}$ and $q_2 = q_{v_2}$

- $I(f; \xi) = 0$ unless $\xi = 1$ is the trivial character and
  $$I(f; 1) = \begin{cases} 2\Lambda(1, \eta)^2c_{\mathcal{B}} & \text{if } v_1 \neq v_2 \\ 2\Lambda(1, \eta)c_{\mathcal{B}} & \text{if } v_1 = v_2 \end{cases}$$

  where
  $$c_{\mathcal{B}} = \text{Vol}\left( \prod_{v \in \text{Ram}(\mathcal{B})} Z \setminus G(\mathcal{F}_v) \prod_{v \notin \text{Ram}(\mathcal{B})} Z(\mathcal{F}_v) \setminus Z(\mathcal{F}_v)K_{v, \mathcal{F}_v, dg} \right).$$

- $I(f; \varphi) \neq 0$ only if $\varphi$ appears in the spectral decomposition

$$I(f; \varphi) = \begin{cases} \frac{c_{\mathcal{B}}D(\mathcal{F})\frac{1}{2}L(\text{Ram}(\mathcal{E})/\mathcal{F})}{2(q_1 + 1)(q_2 + 1)\text{D}(\mathcal{E})^{\frac{1}{2}}} \cdot \frac{L(\text{Ram}(\mathcal{B}))(1, \eta, \text{Ad})}{L(\text{Ram}(\mathcal{B}))(1, \eta, \text{Ad})} \cdot \prod_{v \in \text{Ram}(\mathcal{B})} \frac{L(1, \eta_v)}{\zeta_v(2)} & \text{if } v_1 \neq v_2 \\ \frac{c_{\mathcal{B}}D(\mathcal{F})\frac{1}{2}L(\text{Ram}(\mathcal{E})/\mathcal{F})}{2\text{D}(\mathcal{E})^{\frac{1}{2}}} \cdot \frac{L(\text{Ram}(\mathcal{B}))(1, \eta, \text{Ad})}{L(\text{Ram}(\mathcal{B}))(1, \eta, \text{Ad})} \cdot \prod_{v \in \text{Ram}(\mathcal{B})} \frac{L(1, \eta_v)}{\zeta_v(2)} & \text{if } v_1 = v_2. \end{cases}$$

On the other hand, the geometric side is

$$I(f) = \sum_{\zeta \in \text{cNr}(\mathcal{E})} I(\zeta, f) + 2\Lambda(1, \eta)\{I(0, f) + I(\infty, f)\}$$

for which we have

- $I(\infty, f) = \int_{Z \setminus T(\mathcal{A})} f(tu)dt = 0$.
- $I(0, f) = \int_{Z \setminus T(\mathcal{A})} f(t)dt = \frac{D(\mathcal{F})\frac{1}{2}L(\text{Ram}(\mathcal{E})/\mathcal{F})}{D(\mathcal{E})^{\frac{1}{2}}}$.  

\[ \text{(11)} \]
• The function $I(\zeta, f)$, which is calculated locally, should be viewed as the tensor product of locally defined functions on $\mathbb{A}^\times$ with support in $\varepsilon N_{E/F}(\mathbb{A}_F)^	imes$:

$$I(\zeta, f) = \frac{D(F)^{2[\text{Ram}(E/F)]}}{D(E)} \prod_{v \in \text{Ram}(B)} \frac{1_{\zeta \notin \varepsilon N_{E/F}(E_v)^	imes}}{|v(\zeta)|} \prod_{v \notin \text{Ram}(B)} \frac{1_{\zeta \in \varepsilon N_{E/F}(E_v)^	imes}}{|v(\zeta)|} \prod_{E_v/F_v \text{ ramified}} \frac{1}{2^{|v(\zeta)| \leq 0}} \prod_{E_v/F_v \text{ split; } v \notin \{v_1, v_2\}} \frac{1}{|v(\zeta)|}$$

Hence we get

$$\sum_{\zeta} I(\zeta; g) = \sum_{\zeta} I(\zeta, f) + 2\Lambda(1, \eta)I(0, f) - I(f; 1)$$

where as $E$ varies (i.e. as $D(E) \to \infty$) the sum $\sum_{\zeta} I(\zeta, f)$ should admit the main term $I(f; 1)$, the second main term $-2\Lambda(1, \eta)I(0, f)$ and we need to bound the error term to get a bound of the left side. It is not difficult to see

$$\int_{\mathbb{F}^\times \setminus \mathbb{A}^\times} I(\zeta, f) \frac{d\zeta}{|1 - \zeta|^2} = I(f; 1)$$

by the decomposition of the Tamagawa measure of $G_x$ with respect to the action of $T_x \times T_x$.

Unfortunately, we don’t know how to proceed to bound the error term so far.

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