Arbitrary Overlap Constraints in Graph Packing Problems

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Abstract. In earlier versions of the community discovering problem, the overlap between communities was restricted by a simple count upper-bound \cite{17,11,18}. In this paper, we introduce the \(\Pi\)-Packing with \(\alpha()\)-Overlap problem to allow for more complex constraints in the overlap region than those previously studied. Let \(\mathcal{V}\) be all possible subsets of vertices of \(V(G)\) each of size at most \(r\), and \(\alpha : \mathcal{V} \times \mathcal{V} \to \{0, 1\}\) be a function. The \(\Pi\)-Packing with \(\alpha()\)-Overlap problem seeks at least \(k\) induced subgraphs in a graph \(G\) subject to: (i) each subgraph has at most \(r\) vertices and obeys a property \(\Pi\), and (ii) for any pair \(H_i, H_j\), with \(i \neq j\), \(\alpha(H_i, H_j) = 0\) (i.e., \(H_i, H_j\) do not conflict). We also consider a variant that arises in clustering applications: each subgraph of a solution must contain a set of vertices from a given collection of sets \(\mathcal{C}\), and no pair of subgraphs may share vertices from the sets of \(\mathcal{C}\). In addition, we propose similar formulations for packing hypergraphs. We give an \(O(r^k k^{(r+1)k} \alpha(k) c n^c)\) algorithm for our problems where \(k\) is the parameter and \(c\) and \(r\) are constants, provided that: i) \(\Pi\) is computable in polynomial time in \(n\) and ii) the function \(\alpha()\) satisfies specific conditions. Specifically, \(\alpha()\) is hereditary, applicable only to overlapping subgraphs, and computable in polynomial time in \(n\). Motivated by practical applications we give several examples of \(\alpha()\) functions which meet those conditions.

1 Introduction

Many complex systems arising in the real world can be represented by networks, e.g. social and biological networks. In these networks, a node represents an entity, and an edge represents a relationship between two entities. A community arises in a network when two or more entities have common interests. In this way, members of a community tend to share several properties. Extracting the communities in a network is known as the community discovering problem \cite{10}.

In practice communities may overlap by sharing one or more of their members \cite{10,12}. In \cite{18}, the \(\mathcal{H}\)-Packing with \(t\)-Overlap was proposed as an abstraction for the community discovering problem. The goal is to find \(k\) subgraphs in a given graph \(G\) (the network) where each subgraph (a community) should be isomorphic to a graph \(H \in \mathcal{H}\) where \(\mathcal{H}\) is a family of graphs (the community models). Every pair of subgraphs in the solution should not overlap by more than \(t\) vertices (shared members).
However, in some cases the type of overlap that is allowed may be more complex. For example, it has been observed in [23] that overlapping regions are denser than the rest of the community. Also, in [11] it is suggested that overlapping regions should contain nodes which have a relationship with all the communities they belong to. Moreover, in [24] only boundaries nodes can happen in the overlapping regions. Motivated by this, we generalize the \( \mathcal{H} \)-Packing with \( t \)-Overlap to restrict the pairwise overlap by a function \( \alpha() \) rather than by an upper-bound \( t \). We also consider other communities models besides a family \( \mathcal{H} \).

The scope of community definitions is vast, see [10]. Thus, we define the much more general problem of \( \Pi \)-Packing with \( \alpha() \)-Overlap.

**The \( \Pi \)-Packing with \( \alpha() \)-Overlap problem**

*Input:* A graph \( G \) and a non-negative integer \( k \).

*Parameter:* \( k \)

*Question:* Does \( G \) contain a \((k, \alpha)\)-\( \Pi \)-packing, i.e., a set of at least \( k \) induced subgraphs \( K = \{H_1, \ldots, H_k\} \) subject to the following conditions: i. each \( H_i \) has at most \( r \) vertices and obeys the property \( \Pi \), and ii. for any pair \( H_i, H_j \), with \( i \neq j \), \( \alpha(H_i, H_j) = 0 \)?

We also propose a similar generalization for the problem of packing sets with pairwise overlap that we call the \( r \)-Set Packing with \( \alpha() \)-Overlap problem.

**The \( r \)-Set Packing with \( \alpha() \)-Overlap problem**

*Input:* A collection \( S \) each of size at most \( r \), drawn from a universe \( \mathcal{U} \), and a non-negative integer \( k \).

*Parameter:* \( k \)

*Question:* Does \( S \) contain a \((k, \alpha())\)-set packing, i.e., at least \( k \) sets \( K = \{S_1, \ldots, S_k\} \) where for each pair \( S_i, S_j \) \((i \neq j)\) \( \alpha(S_i, S_j) = 0 \)?

Some of our generalized problems are NP-complete; this follows from the NP-complete \( \mathcal{H} \)-Packing and \( r \)-Set Packing problems. Our goal is to achieve fixed-parameter (or FPT) algorithms which are algorithms that provide a solution in \( f(k) \cdot n^{O(1)} \) running time, where \( f \) is some arbitrary computable function depending only on the parameter \( k \). In all our problems, \( k \) (the size of the solution) is the parameter, \( r \) is a fixed constant, and \( n \) denotes the order of the graph or the number of elements in the universe (depending on the problem).

**Related Work.** H. Fernau et al., [8] provide an \( O(r^r k^{r-1}) \) kernel for the \( \mathcal{H} \)-Packing and \( r \)-Set Packing with \( t \)-Overlap problems. In addition, an \( O(r^k k^{(r-1)} k^{t-1} + 2 n^r) \) algorithm for these problems can be found in [15]. A \( 2 (r k - r) \) kernel when \( \mathcal{H} = \{K_r\} \) and \( t = r - 2 \) is given in [15].

The \( \mathcal{H} \)-Packing problem has an \( O(k^{r-1}) \) kernel, where \( H \) is an arbitrary graph on \( r \) vertices. Kernelization algorithms when \( H \) is a prescribed graph can be found in [7,9,13,19]. The \( r \)-Set Packing problem has an \( O(r^r k^{r-1}) \) kernel [1].
The community discovering problem is studied with a variety of approaches in [17, 12, 24, 2, 4, 18], and comprehensive surveys are [10, 22].

**Our Results.** In this work, we introduce the $r$-Set Packing and $\Pi$-Set Packing with $\alpha()$-Overlap problems as more universal versions for the problem of packing graphs and sets subject to overlap constraints modeled by a function $\alpha()$. Our generalizations capture a much broader range of potential real life applications.

We show in Section 3 that the $r$-Set Packing with $\alpha()$ Overlap problem is fixed-parameter tractable when $\alpha()$ meets specific requirements ($\alpha()$ is well-conditioned, see Definition 1). Our FPT-algorithm generalizes our previous algorithm [15]. Previously, we considered only a specific type a conflict between a pair of sets: overlap larger than $t$. In our extended algorithm, we will consider the more general $\alpha$-conflicts. To solve the $\Pi$-Packing with $\alpha()$-Overlap problem, we reduce it to its set version. This allows us to achieve an algorithm with $O(r^k k^{r+1} n^\epsilon)$ running time, provided that $\alpha()$ is well-conditioned and $\Pi$ is verifiable in polynomial time.

In Section 4, we give specific examples of well-conditioned $\alpha()$ functions, some motivated by practical applications while others by theoretical considerations. Specifically, a well-conditioned $\alpha()$ can restrict (but it is not limited to): i) the size of the overlap, ii) the weight in the overlap region, (assuming as input a weighted graph), iii) the pattern in the overlap region, i.e. the induced subgraph in the overlap should be isomorphic to a graph in $\mathcal{F}$, where $\mathcal{F}$ is a graph class that is hereditary, iv) that all overlapping vertices must satisfy a specific property $\xi$, v) that the overlap region should have a specific density, and finally, v) the maximum distance between any pair of vertices in the overlap.

Lastly, we study the PCH-$r$-Set Packing with $\alpha()$-Overlap problem in Section 5. In this setting, every set in the solution must contain a specific set of elements from a given collection of sets $\mathcal{C}$. This problem remain fixed-parameter tractable if $|\mathcal{C}| = O(g(k))$ for some computable function $g$ dependent on $k$ and independent of $n$.

## 2 Preliminaries

Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a universe of elements and $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of sets, where $S_i \subseteq \mathcal{U}$. We will use the letters $u, s, S$ in combination with subindices to refer to elements in $\mathcal{U}$, sets of elements of $\mathcal{U}$, and members of $\mathcal{S}$, respectively. Notice that we will identify a subset of elements of $\mathcal{U}$ (that is not necessarily a member of $\mathcal{S}$) using a lower-case $s$ with a subindex, while we restrict the use of upper-case letters to identify members of $\mathcal{S}$.

For $S' \subseteq S$, $\text{val}(S')$ denotes the union of all members of $S'$. We say that a subset of elements $s$ is contained in a set $S$, if $s \subseteq S$. In addition, let $\mathcal{S}(s)$ be the collection of all sets in $\mathcal{S}$ that contain $s$. That is, $s \subseteq S$ for each $S \in \mathcal{S}(s)$ and $s \not\subseteq S'$ for each $S' \not\in (\mathcal{S}\setminus \mathcal{S}(s))$. For any two sets $S, S' \in \mathcal{S}$, $|S \cap S'|$ is the overlap size while $\{S \cap S'\}$ is the overlap region.
Definition 1. Let $\mathcal{U}$ be all possible subsets of elements of $\mathcal{U}$ each of size at most $r$, and $\alpha : \mathcal{U} \times \mathcal{U} \to \{0, 1\}$ be a function. A pair of sets $s_i, s_j \in \mathcal{U}$ \alpha-conflict if $\alpha(s_i, s_j) = 1$ else they do not \alpha-conflict. If $\alpha()$ satisfies the following requirements, we say $\alpha()$ is well-conditioned.

i) $\alpha()$ is hereditary. Specifically, if $s_i$ and $s_j$ do not $\alpha$-conflict ($\alpha(s_i, s_j) = 0$), $\alpha(s_i', s_j') = 0$ for any pair of subsets $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$.

ii) If $s_i$ and $s_j$ $\alpha$-conflict ($\alpha(s_i, s_j) = 1$), $|s_i \cap s_j| \geq 1$. Furthermore, for any pair of subsets $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ with $\alpha(s_i', s_j') = 0$, $((s_i \cap s_j) \setminus (s_i' \cap s_j')) \neq \emptyset$.

The elements in $s_i \cap s_j$ are referred to as the conflicting elements.

iii) $\alpha$ is computable in polynomial time in $n$.

A maximal $\alpha()$-set packing $\mathcal{M} \subseteq \mathcal{S}$ is a maximal collection of sets from $\mathcal{S}$ such that for each pair of sets $S_i, S_j \in \mathcal{M}$ ($i \neq j$) $\alpha(S_i, S_j) = 0$, and for each $S \in \mathcal{S}\setminus \mathcal{M}$, $S$ $\alpha$-conflicts with some $S' \in \mathcal{M}$, i.e., $\alpha(S, S') = 1$.

All graphs in this paper are undirected and simple, unless otherwise stated. For a graph $G$, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. $|V(G)|$ is the order of the graph. For a set of vertices $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by $S$ in $G$. The distance (shortest path) between two vertices $u$ and $v$ is denoted as $\text{dist}_G(u, v)$. We use the letter $n$ to denote both $|\mathcal{U}|$ and $|V(G)|$.

3 Packing Problems with Well-Conditioned Overlap

We start by developing an FPT-algorithm for the $r$-Set Packing with $\alpha()$-Overlap problem. After that, we provide a solution for $\Pi$-Packing with $\alpha()$-Overlap by reducing it to the set version. Our FPT-algorithm assumes that the function $\alpha()$ is well-conditioned.

3.1 An FPT Algorithm for the $r$-Set Packing with $\alpha()$-Overlap

The next lemmas state important observations of a maximal $\alpha()$-set packing and are key components in the correctness of our algorithm.

Lemma 1. Let $\mathcal{M}$ be a maximal $\alpha()$-set packing. If $|\mathcal{M}| \geq k$, then $\mathcal{M}$ is a $(k, \alpha())$-set packing.

Proof. Assume otherwise that $\mathcal{M}$ is not a $(k, \alpha())$-set packing. This would be only possible if there is at least one pair of sets $S_i, S_j$ in $\mathcal{M}$ for which $\alpha(S_i, S_j) = 1$ but in that case $\mathcal{M}$ would not be a maximal $\alpha()$-set packing.

Lemma 2. Given an instance $(\mathcal{U}, \mathcal{S}, k)$ of $r$-Set Packing with $\alpha()$-Overlap, where $\alpha()$ is well-conditioned, let $\mathcal{K}$ and $\mathcal{M}$ be a $(k, \alpha())$-set packing and a maximal $\alpha()$-set packing, respectively. For each $S^* \in \mathcal{K}$, $S^*$ shares at least one element with at least one $S \in \mathcal{M}$.
Proof. If $S^* \in \mathcal{M}$, the lemma simply follows. Assume by contradiction that there is a set $S^* \in \mathcal{K}$ such that $S^* \notin \mathcal{M}$ and there is no set $S \in \mathcal{M}$ $\alpha$-conflicting with $S^*$. However, we could add $S^*$ to $\mathcal{M}$, contradicting its maximality. Thus, there exists at least one $S \in \mathcal{M}$ $\alpha$-conflicting with $S^*$. Since $\alpha$ is well-conditioned, by Definition \ref{def:weighted_overlap} (ii) $|S \cap S^*| \geq 1$. 

Our Bounded Search Tree algorithm (abbreviated as BST-$\alpha$-algorithm) for $r$-Set Packing with $\alpha$-Overlap has three main components: Initialization, Greedy, and Branching. We start by computing a maximal $\alpha$-set packing $\mathcal{M}$ of $\mathcal{S}$. If $|\mathcal{M}| \geq k$ then $\mathcal{M}$ is a $(k, \alpha)$-set packing and the BST-$\alpha$-algorithm stops (Lemma \ref{lem:maximal}). Otherwise, we create a search tree $T$ where at each node $i$, there is a collection of sets $Q^i = \{s^i_1, \ldots, s^i_k\}$ with $s^i_j \subseteq S$ for some $S \in \mathcal{S}$. The goal is to complete $Q^i$ to a solution, if possible. That is, to find $k$ sets $K = \{S_1, \ldots, S_k\}$ of $\mathcal{S}$, such that $s^i_j \subseteq S_j$ for $1 \leq j \leq k$ and $K$ is a $(k, \alpha)$-set packing.

The children of the root of $T$ are created according to a procedure called Initialization. After that for each node $i$ of $T$, a routine called Greedy will attempt to complete $Q^i$ to $(k, \alpha)$-set packing. If Greedy succeeds then the BST-$\alpha$-algorithm stops. Otherwise, the next step is to create children of the node $i$ using the procedure Branching. The BST-$\alpha$-algorithm will repeat Greedy in these children. Eventually, the BST-$\alpha$-algorithm either finds a solution at one of the leaves of the tree or determines that it is not possible to find one.

We next explain the three main components of the BST-$\alpha$-algorithm individually. Let us start with the Initialization routine. By Lemma \ref{lem:initialization} if there is a solution $\mathcal{K} = \{S^*_1, \ldots, S^*_k\}$ each $S^*_j$ contains at least one element of $\text{val}(\mathcal{M})$. Notice that each element of $\text{val}(\mathcal{M})$ could be in at most $k$ sets of $\mathcal{K}$. Thus, we create a set $\mathcal{M}_k$ that contains $k$ copies of each element in $\text{val}(\mathcal{M})$. That is, per each element $u \in \text{val}(\mathcal{M})$ there are $k$ copies $u_1, u_2, \ldots, u_k$ in $\mathcal{M}_k$ and $|\mathcal{M}_k| = k|\text{val}(\mathcal{M})|$. The root will have a child $i$ for each possible combination of $k$ elements from $\mathcal{M}_k$. A set of $Q^1$ is initialized with one element of that combination. For example, if the combination is $\{u_1, u_2, u_k, a_1, b_1\}$, $Q^1 = \{\{u_1\}, \{u_2\}, \{u_k\}, \{a_1\}, \{b_1\}\}$. After that, we remove the indices from the elements in $Q^1$, e.g., $Q^1 = \{\{u\}, \{u\}, \{u\}, \{a\}, \{b\}\}$.

At each node $i$, the Greedy routine returns a collection of sets $Q^{gr}$. Initially, $Q^{gr} = \emptyset$ and $j = 1$. At iteration $j$, Greedy searches for a set $S$ that contains $s^i_j \in Q^i$ (the $j$th set of $Q^i$) subject to two conditions (**): (1) $S$ is not already in $Q^{gr}$ and (2) $S$ does not $\alpha$-conflict with any set in $Q^{gr}$ (i.e., $\alpha(S, S') = 0$ for each $S' \in Q^{gr}$). If such set $S$ exists, Greedy adds $S$ to $Q^{gr}$, i.e., $Q^{gr} = Q^{gr} \cup S$ and continues with iteration $j = j+1$. Otherwise, Greedy stops executing and returns $Q^{gr}$. If $|Q^{gr}| = k$, then $Q^{gr}$ is a $(k, \alpha)$-set packing and the BST-$\alpha$-algorithm stops. If $Q^i$ cannot be completed into a solution (Lemma \ref{lem:initialization}), Greedy returns $Q^{gr} = \infty$. Greedy searches for the set $S$ in the collection $S(s^i_j, Q^i) \subseteq S(s^i_j)$ which is obtained as follows: add a set $S' \in S(s^i_j, Q^i)$ to $S(s^i_j, Q^i)$, if $S'$ does not $\alpha$-conflict with any set in $(Q^i \setminus s^i_j)$ and $S'$ is distinct of each set in $(Q^i \setminus s^i_j)$.

The Branching procedure executes every time that Greedy does not return a $(k, \alpha)$-set packing but $Q^i$ could be completed into one. That is, $Q^{gr} \neq \emptyset$. 


\( |Q^\text{gr}| < k \). Let \( j = |Q^\text{gr}| + 1 \) and \( s^j_i \) be the \( j \)-th set in \( Q^i \). Greedy stopped at \( j \) because each set \( S \in S(s^j_i, Q^1) \) either it was already contained in \( Q^\text{gr} \), or it \( \alpha \)-conflicts with at least one set in \( Q^\text{gr} \) (see **). We will use the conflicting elements between \( S(s^j_i, Q^1) \) and \( Q^\text{gr} \) to create children of the node \( i \). Let \( I^* \) be the set of those conflicting elements. Branching creates a child \( l \) of the node \( i \) for each element \( u_l \in I^* \). The collection \( Q^l \) of child \( l \) is the same as the collection \( Q^1 \) of its parent \( i \) with the update of the set \( s^j_i \) as \( s^j_i \cup u_l \), i.e., \( Q^l = \{s^j_1, \ldots, s^j_{j-1}, s^j_j \cup u_l, s^j_{j+1}, \ldots, s^j_k\} \). The set \( I^* \) is obtained as \( I^* = I^* \cup (S(s^j_i, Q^1) \cap S') \) for each pair \( S \in S(s^j_i, Q^1) \) and \( S' \in Q^\text{gr} \) that \( \alpha \)-conflict (\( \alpha(S, S') = 1 \)) or that \( S = S' \). The pseudocode of all these routines is detailed in the Appendix.

**Correctness.** With the next series of lemmas we establish the correctness of the \( \text{BST-}\alpha()\)-algorithm for any well-conditioned function \( \alpha() \).

A collection \( Q^1 = \{s^j_1, \ldots, s^j_i, \ldots, s^j_k\} \) is a partial-solution of a \((k, \alpha())\)-set packing \( K = \{S^*_1, \ldots, S^*_j, \ldots, S^*_k\} \) if and only if \( s^j_i \subseteq S^*_j \), for \( 1 \leq j \leq k \). The next lemma states the correctness of the Initialization routine and it follows because we created a node for each selection of \( k \) elements from \( M_k \), i.e., \( \binom{M^*}{k} \).

**Lemma 3.** If there exists at least one \((k, \alpha())\)-set packing of \( S \), at least one of the children of the root will have a partial-solution.

**Proof.** By Lemma \( \mathbb{P} \) every set in \( K \) contains at least one element of \( \text{val}(M) \). It is possible that the same element be in at most \( k \) different sets of \( K \). Therefore, we replicated \( k \) times each element in \( \text{val}(M) \) collected in \( M_k \). Since we created a node for each selection of \( k \) elements from \( M_k \), i.e., \( \binom{M^*}{k} \), the lemma follows. \( \square \)

The next lemma states that the \( \text{BST-}\alpha()\)-algorithm correctly stops attempting to propagate a collection \( Q^1 \). Due to the (i) property of a well-conditioned \( \alpha() \), we can immediately discard a collection \( Q^1 \), if it has a pair of sets that \( \alpha \)-conflicts. In addition, the collection \( S(s^j_i, Q^1) \) contains all sets from \( S(s^j_i) \) that are not \( \alpha \)-conflicting with any set in \( Q^1 \) (excluding \( s^j_i \)). So again, if \( Q^1 \) is a partial-solution, due to the (i) property, \( S(s^j_i, Q^1) \) cannot be empty.

**Lemma 4.** Assuming \( \alpha() \) is well-conditioned, \( Q^1 \) is not a partial solution either: i. if there is a pair of distinct sets in \( Q^1 \) that \( \alpha \)-conflict, or ii. if for some \( s^j_i \in Q^1 \) \( S(s^j_i, Q^1) = \emptyset \).

**Proof.** (i) Suppose otherwise that \( Q^1 \) is a partial-solution, but \( s^j_j, s^j_i \) \( \alpha \)-conflict. Since \( Q^1 \) is a partial-solution, \( s^j_j \subseteq S^*_j \) and \( s^j_i \subseteq S^*_i \) where \( S^*_j, S^*_i \in K \) and \( K \) is a \((k, \alpha())\)-set packing.

The pair \( S^*_j, S^*_i \) does not \( \alpha \)-conflict, otherwise, \( K \) would not be a solution. However, \( \alpha() \) is hereditary, \( s^j_j \subseteq S^*_j \), and \( s^j_i \subseteq S^*_i \), thus, \( s^j_j \) and \( s^j_i \) do not \( \alpha \)-conflict either.

(ii) To prove the second part of the lemma, we will prove the next stronger claim.
Claim 1. If \( Q^i = \{ s_i^1, \ldots, s_i^j, \ldots, s_i^k \} \) is a partial-solution then \( S_j^* \in S(s_j^i, Q^i) \) for each \( 1 \leq j \leq k \).

Proof. Assume by contradiction that \( Q^i \) is a partial-solution but \( S_j^* \notin S(s_j^i, Q^i, \alpha) \) for some \( j \).

If \( Q^i \) is a partial-solution, \( s_j^i \subseteq S_j^* \subseteq K \) and \( ((Q^i \setminus s_j^i) \cup S_j^*) \) is a partial-solution as well. The set \( S_j^* \in S(s_j^i) \) and \( S(s_j^i, Q^i) \subseteq S(s_j^i) \) (see Algorithm 4 for the computation of \( S(s_j^i, Q^i) \)). The only way that \( S_j^* \) would not be in \( S(s_j^i, Q^i) \) is if there is at least one set \( S \) in \( (Q^i \setminus s_j^i) \) that \( \alpha \)-conflicts with \( S_j^* \) or if \( S_j^* \) is equal to a set in \( (Q^i \setminus s_j^i) \) but then \( ((Q^i \setminus s_j^i) \cup S_j^*) \) would not be a partial-solution a contradiction to \((i)\).

Branching creates at least one child whose collection is a partial-solution, if the collection of the parent is a partial-solution as well. Recall that \( I^* \) is computed when Greedy stopped its execution at some \( j \leq k \), i.e., it could not add a set that contains \( s_j^i \) to \( Q^{gr} \). If \( Q^i \) is a partial-solution, \( s_j^i \subseteq S_j^* \) and \( S_j^* \subseteq K \). Given property (ii) for a well-conditioned \( \alpha() \), \( S_j^* \) must be intersecting in at least one element with at least one set in \( Q^{gr} \). Therefore, at least one element of \( S_j^* \) will be in \( I^* \).

Lemma 5. If \( Q^i = \{ s_i^1, \ldots, s_i^j, \ldots, s_i^k \} \) is a partial-solution then there exists at least one \( u_i \in I^* \) such that \( Q^i = \{ s_i^1, \ldots, s_i^j \cup u_i, \ldots, s_i^k \} \) is a partial-solution.

Proof. Assume to the contrary that \( Q^i = \{ s_i^1, \ldots, s_i^j, \ldots, s_i^k \} \) is a partial-solution but that there exists no element \( u_i \in I^* \) such that \( Q^i = \{ s_i^1, \ldots, s_i^j \cup u_i, \ldots, s_i^k \} \) is a partial-solution. This can only be possible if \((S_j^* \setminus s_j^i) \cap I^* = \emptyset\).

First, given that \( Q^i \) is a partial-solution \( S_j^* \in S(s_j^i, Q^i, \alpha) \) (Claim 1). In addition, either \( S_j^* \) is already in \( Q^{gr} \) (i.e., it is equal to some set \( S' \in Q^{gr} \)) or \( S_j^* \) must be \( \alpha \)-conflicting with at least one set \( S' \in Q^{gr} \); otherwise, \( S_j^* \) would have been selected by Greedy. Any of these situations implies that \(|S_j^* \cap S'| \geq 1\) (Definition 1(ii)). By the computation of \( I^* \) (Algorithm 4), \( S_j^* \cap S' \subseteq I^* \).

Now it remains to show, that at least one element of \( S_j^* \cap S' \) is in \( S_j^* \setminus s_j^i \). This will guarantee that the set \( s_j^i \) will be increased by one element at the next level of the tree. This immediately follows if \( S_j^* = S' \).

Thus, we will show it for the case that \( S_j^* \neq S' \) but \( S_j^* \alpha \)-conflicts with \( S' \). Suppose that \((S_j^* \cap S') \cap (S_j^* \setminus s_j^i) = \emptyset\) by contradiction. Recall that \( S' \) contains some set \( s_h^i \) of \( Q^i \) (for some \( h \leq j \)). Furthermore, \( S' \in S(s_h^i, Q^i, \alpha) \); otherwise \( S' \) would not have been selected by Greedy.

If \( S' \) is \( \alpha \)-conflicting with \( S_j^* \) but \( S' \) is not \( \alpha \)-conflicting with \( s_j^i \) (otherwise \( S' \) would not have been in \( S(s_h^i, Q^i, \alpha) \)), then \((S' \cap (S_j^* \setminus s_j^i)) \neq \emptyset \) by property (ii) in Definition 1.

\[ \square \]

Theorem 1. The BST-\( \alpha() \)-algorithm finds a \((k, \alpha())\)-set packing of \( S \), if \( S \) has at least one and \( \alpha() \) is well-conditioned.
Furthermore, we require that $\alpha$ is well-conditioned. The universe $U$ we are not asking to compute the largest subgraph of $G$ rather only verifying whether a specific induced subgraph of at most $\Pi$ collection of all induced $r$-subgraphs of $G$ is verifiable in polynomial time in $n$ where $n = |V(G)|$.

Examples of properties $\Pi$ that could represent communities are the following. Let $S$ be an induced subgraph of $G$ with at most $r$ vertices. $S$ is a community, if it has a density of at least $t$ ($|E(S)| \geq t$) and the number of edges connecting $S$ to rest of the network is at most a specific value $17$. $S$ is a community, if every vertex in $S$ is adjacent to at least $|V(S)| - c$ vertices in $S$ (for some constant $c$). Observe that with our property $\Pi$, we still can use a family of graphs $\mathcal{H}$ to represent a community as in the $\mathcal{H}$-Packing with $t$-Overlap problem. In that case, $\Pi$ would correspond to the condition that $S$ is a community if $S$ is isomorphic to a graph $H$ in $\mathcal{H}$.

In the $\Pi$-Packing with $\alpha()$-Overlap problem, we regulate the pairwise overlap with a function $\alpha : \mathcal{V}^r \times \mathcal{V}^r \rightarrow \{0, 1\}$ where $\mathcal{V}^r$ is the collection of all possible subsets of vertices of $V(G)$ each of size at most $r$. We say that two subgraphs $H_i$ and $H_j$ are $\alpha$-conflict if $\alpha(H_i, H_j) = 1$. Abusing the terminology, we extend the definition of a well-conditioned $\alpha()$ (Definition 1) to consider subsets of vertices as well. This implies that $\mathcal{U}^r = \mathcal{V}^r$, $s_i = V(H_i)$ and $s_j = V(H_j)$ in Definition 1.

To provide a solution for the $\Pi$-Packing with $\alpha()$-Overlap problem, we will basically follow the approach of reducing this problem to the set version, i.e., to the $r$-Set Packing with $\alpha()$-Overlap problem. To this end, we first compute the collection of all induced $\Pi$-subgraphs of $G$, and we collect them in $\mathcal{P}_G$. This is done by naively testing all sets of at most $r$ vertices from $G$. We highlight that we are not asking to compute the largest subgraph of $G$ that follows $\Pi$, but rather only verifying whether a specific induced subgraph of at most $r$ vertices satisfies $\Pi$ or not. In this way, $|\mathcal{P}_G| = O(n^r)$.

Next, we construct an instance of $r$-Set Packing with $\alpha()$-Overlap as follows. The universe $\mathcal{U}$ equals $V(G)$ and there is a set $S = V(H)$ in $\mathcal{S}$ for each $H \in \mathcal{P}_G$. Furthermore, we require that $\alpha()$ be well-conditioned.

**Theorem 2.** The $r$-Set Packing with $\alpha()$-Overlap problem can be solved in $O(n^r)$ time, when $\alpha()$ is well-conditioned.

### 3.2 The $\Pi$-Packing with $\alpha()$-Overlap problem

The $\Pi$-Packing with $\alpha()$-Overlap problem generalizes the $\mathcal{H}$-Packing with $t$-Overlap problem 15 by including other community definitions in addition to prescribed graphs and by allowing more complex overlap restrictions.

We will represent a community through a graph property $\Pi$. Intuitively, if a subgraph $H$ of order at most $r$ has the property $\Pi$ (called a $\Pi$-subgraph), $H$ is a community. To obtain an FPT algorithm, we require however that $\Pi$ be verifiable in polynomial time in $n$ where $n = |V(G)|$. The number of children of the root is given by $\binom{|\mathcal{M}_h|}{k} \leq (\binom{k(r(k-1))}{k}) = O(rk^2k)$ and the height of the tree is at most $(r-1)k$. The number of children of each node at level $h$ is equivalent to the size of $I^*$ at each level $h$. The number of elements in $\text{val}(Q^*)$ is at most $r(k-1)$, thus, $|I^*| \leq r(k-1)$. Therefore, the size of the tree is given by: $(\binom{k(r(k-1))}{k}) \prod_{h=1}^{(r-1)k} r(k-1)$ which is $O(rk^k(k+1)^k)$. In addition, $\alpha()$ is computable in $O(n^c)$ for some constant $c$ (property (iii), Definition 1).

$\Pi$-Set Packing with $\alpha()$-Overlap problem generalizes the $\Pi$-Packing with $t$-Overlap problem. In that case, $\Pi$ is a community if $S$ is a community, if every vertex in $S$ is adjacent to at least $|V(S)| - c$ vertices in $S$ (for some constant $c$). To provide a solution for the $\Pi$-Packing with $\alpha()$-Overlap problem, we will basically follow the approach of reducing this problem to the set version, i.e., to the $r$-Set Packing with $\alpha()$-Overlap problem.
Theorem 3. In the next section, we provide several examples of functions that are well-conditioned and \( \Pi \) is polynomial time verifiable.

4 Well-Conditioned Overlap Constraints

In the next section, we provide several examples of functions that are well-conditioned. That is, they satisfy the conditions in Definition 1. In the first section, we focus on functions concerning the \( r \)-Set Packing problem that by our discussion on Section 3.2 could be used to restrict the overlap for graph version as well. After that in Section 4.2 we provide functions that consider graph properties.

4.1 Restricting the Overlap Between Sets

Weighted Overlap. Let us assume that each \( u_i \in U \) has associated a non-negative weight \( w(u_i) \). We could restrict the overlap region by its weight. The function \( \alpha \)-Weight returns no-conflict if \( w(s_i \cap s_j) = (\sum_{u \in (s_i \cap s_j)} w(u)) \leq w_t \) where \( w_t \geq 0 \) is a constant, else returns \( \alpha \)-conflict.

Notice that we could use \( \alpha \)-Weight\((s_i, s_j)\) to upper-bound the overlap size by a constant \( t \). To this end, we make \( w(u) = 1 \) for each \( u \in U \), and \( w_t = t \).

Lemma 7. The function \( \alpha \)-Weight is well-conditioned.

Proof. (i) \( \alpha \)-Weight is hereditary. For any pair of sets \( s_i, s_j \) with \( w(s_i \cap s_j) \leq w_t \) \( (\alpha(s_i, s_j) = 0) \), there is no pair of subsets \( s_i' \subseteq s_i, s_j' \subseteq s_j \) with \( w(s_i' \cap s_j') > w_t \) \( (\alpha(s_i, s_j) = 1) \). For the sake of contradiction, suppose otherwise. Notice that \( (s_i' \cap s_j') \subseteq (s_i \cap s_j) \). Thus, if \( w(s_i \cap s_j) \leq w_t \) but \( w(s_i' \cap s_j') > w_t \) then there must be some elements in \( (s_i \cap s_j) \backslash (s_i' \cap s_j') \) with negative weights, a contradiction.

(ii) If \( \alpha(s_i, s_j) = 1 \) then \( w(s_i \cap s_j) > w_t \) and \( (s_i \cap s_j) \neq \emptyset \). Let \( s_i' \subseteq s_i \) and \( s_j' \subseteq s_j \) be any pair of subsets with \( \alpha(s_i', s_j') = 0 \), (i.e., \( w(s_i' \cap s_j') \leq w_t \)). Since \( w(s_i \cap s_j) > w_t \), \( w(s_i \cap s_j) - w(s_i' \cap s_j') > 0 \). Therefore, \( ((s_i \cap s_j) \backslash (s_i' \cap s_j')) \neq \emptyset \).

(iii) Finally, we can determine in \( O(r) \) time, if \( w(s_i \cap s_j) > w_t \). \( \square \)
We could also restrict the overlap region by both its size and its weight. This combined restriction is a well-conditioned function as well.

**Measures Overlap.** A measure of a set $S$ is a function $\mu$ that satisfies (i) $\mu(S) \geq 0$, (ii) $\mu(S) = 0$ if $S = \emptyset$, and (iii) for any collection of pairwise disjoint subsets $S_1, \ldots, S_l$ of $S$, $\mu(\bigcup_{i=1}^{l} S_i) = \sum_{i=1}^{l} \mu(S_i)$. The last property implies that for any $S' \subseteq S$, $\mu(S') \leq \mu(S)$. Let $\mu$ be a measure on each set $\{s_i \cap s_j\}$ that is computable in polynomial time. The function $\alpha$-Measure$(s_i, s_j)$ returns no-conflict if $\mu(s_i \cap s_j) \leq t$ (where $t \geq 0$ is a constant) otherwise returns $\alpha$-conflict.

**Lemma 8.** The function $\alpha$-Measure is well-conditioned.

**Proof.** (i) $\alpha$-Measure is hereditary. Assume by contradiction that $\mu(s_i \cap s_j) \leq t$ but there is pair of subsets $s_i' \subseteq s_i, s_j' \subseteq s_j$ with $\mu(s_i' \cap s_j') > t$. Let $S = (s_i' \cap s_j')$. First, $S \neq \emptyset$, otherwise, $\mu(S) = 0$ and since $t \geq 0$ there would be a contradiction. Second $S \subseteq (s_i \cap s_j)$, thus by the additive property of $\mu$, $\mu(S) \leq \mu(s_i \cap s_j)$. Since $\mu(s_i \cap s_j) \leq t$, the claim holds.

(ii) If $\mu(s_i \cap s_j) > t$ then $|s_i \cap s_j| \geq 1$. This follows because $\mu(\emptyset) = 0$ and $t \geq 0$. Let $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ be a pair of subsets with $\alpha(s_i' \cap s_j') = 0$, (i.e., $\mu(s_i' \cap s_j') \leq t$). Note that at most one $s_i' = s_i$ or $s_j' = s_j$; otherwise $\alpha(s_i' \cap s_j') = 1$. Since $\mu(s_i \cap s_j) > t$, $\mu(s_i \cap s_j) = \mu(s_i' \cap s_j') > 0$. In this way, $(|s_i \cap s_j|) \neq \emptyset$.

(iii) The function $\mu$ is computed in polynomial time; thus, we can verify in constant time whether $\mu(s_i \cap s_j) > t$ or not. \qed

**Metric Overlap.** Let us assume that $U$ is a metric space. That is, there is a metric or a distance function that defines a distance between each pair of elements $u, v$ of $U$, subject to the following conditions: $dist_U(u, v) \geq 0$, $dist_U(u, v) = 0$ if $(u = v)$, $dist_U(u, v) = dist_U(v, u)$ and $\parallel dist_U(u, w) - dist_U(v, w) \parallel \leq dist_U(u, v)$. For a constant $d_i > 0$, we define the function $\alpha$-Metric$(s_i, s_j)$ which returns no-conflict if $|s_i \cap s_j| \leq 1$ or $dist_U(u, v) \leq d_i$ for each pair $u, v (u \neq v)$ in $s_i \cap s_j$ else returns $\alpha$-conflict.

**Lemma 9.** The function $\alpha$-Metric is well-conditioned.

**Proof.** (i) $\alpha$-Metric is hereditary. For any pair of sets $s_i, s_j$ that do not $\alpha$-conflict (i.e., $\alpha(s_i, s_j) = 0$), there is no pair of subsets $s_i' \subseteq s_i, s_j' \subseteq s_j$ with $\alpha(s_i' \cap s_j') = 1$. Assume the opposite by contradiction. $|s_i' \cap s_j'| \geq 1$, otherwise $s_i'$ and $s_j'$ would not $\alpha$-conflict. Observe that $(s_i' \cap s_j') \subseteq (s_i \cap s_j)$. In addition, since $U$ is a metric-space and $\alpha(s_i, s_j) = 0$, $dist_U(u, v) \leq t$ for each pair $u, v (u \neq v)$ in $(s_i \cap s_j)$. Given that we are using $dist_U$ and not $dist_{U \cap s_j'}$, there is no pair of elements in $(s_i' \cap s_j')$ with $dist_U(u, v) > t$.

(ii) Since $d_i > 0$, for any pair $s_i, s_j$ with $\alpha(s_i, s_j) = 1$, $|s_i \cap s_j| > 1$. Let $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ be a pair of subsets with $\alpha(s_i', s_j') = 0$. Note that at most one $s_i' = s_i$ or $s_j' = s_j$; otherwise $\alpha(s_i', s_j') = 1$. $dist_U(u, v) \leq t$ for each pair $u, v (u \neq v)$ in $(s_i' \cap s_j')$. Since $\alpha(s_i, s_j) = 1$ but $\alpha(s_i', s_j') = 0$ then it must exist at least one element $u$ in $(s_i \cap s_j) \setminus (s_i' \cap s_j')$ such that $dist_U(u, v) > t$ for some $v$ in $(s_i \cap s_j)$. In this way, $(|s_i \cap s_j|) \neq \emptyset$.

(iii) Assuming as input a metric space $U$, $\alpha$-Metric is verified in $O(n^2)$ time. \qed
4.2 Restricting the Overlap Between Subgraphs

**Prescribed Pattern.** It has been observed in social networks that the overlap region is often more densely connected than the rest of the community [23]. Inspired by this, we will allow pairwise-overlap in a \((k, \alpha())\)-II-packing if the overlap region has a specific pattern, for example, it’s a clique. More precisely, we say that a pair of subgraphs \(H_i, H_j\) do not \(\alpha\)-conflict if \(G[V(H_i) \cap V(H_j)]\) is isomorphic to a graph \(F\) in a class \(\mathcal{F}\). To define a well-conditioned \(\alpha()\), \(\mathcal{F}\) is a graph class that is hereditary (i.e., it is closed under taking induced subgraphs).

To preserve our FPT results, any graph in \(\mathcal{F}\) should be polynomial time verifiable. Examples of \(\mathcal{F}\) are cliques, planar and chordal graphs. Indeed this applies to any graph class that is closed under minors, since this is hereditary and by the Robertson-Seymour theorem the graph is polynomially testable by checking for the forbidden minors [20]. We define the function \(\alpha\)-Pattern\((s_i, s_j)\) that returns no-conflict if \(|s_i \cap s_j| = 0\) or if \(G[s_i \cap s_j]\) is isomorphic to a graph \(F\) in \(\mathcal{F}\); otherwise, it returns \(\alpha\)-conflict.

**Lemma 10.** The function \(\alpha\)-Pattern is well-conditioned.

**Proof.** (i) \(\alpha\)-Pattern is hereditary. Assume by contradiction that there is pair \(s_i, s_j\) with \(\alpha(s_i, s_j) = 0\) but there is a pair of subsets \(s_i' \subseteq s_i\) and \(s_j' \subseteq s_j\) with \(\alpha(s_i', s_j') = 1\). If \(\alpha(s_i', s_j') = 1\) this implies that \(G[s_i' \cap s_j']\) is not isomorphic to a graph \(F\) in \(\mathcal{F}\). Notice that \((s_i' \cap s_j') \subseteq (s_i \cap s_j)\). In addition, \((G[s_i \cap s_j])\) is isomorphic to a graph \(F \in \mathcal{F}\) (otherwise, \(\alpha(s_i, s_j) = 1\)). Since \(\mathcal{F}\) is a graph class that is hereditary, \(G[s_i' \cap s_j']\) is also isomorphic to \(F\) and \(s_i'\) and \(s_j'\) do not \(\alpha\)-conflict.

(ii) It follows by definition of \(\alpha\)-Pattern\((s_i, s_j)\) that for any pair \(s_i, s_j\) that \(\alpha\)-conflict (i.e., \(\alpha(s_i, s_j) = 1\)) \(|s_i \cap s_j| \geq 1\). Let \(s_i' \subseteq s_i\) and \(s_j' \subseteq s_j\) be a pair of non-empty subsets where \(\alpha(s_i', s_j') = 0\). This implies that \(G[s_i' \cap s_j']\) is isomorphic to a graph \(F \in \mathcal{F}\). Recall that \((s_i' \cap s_j') \subseteq (s_i \cap s_j)\). Since \(G[s_i \cap s_j]\) is not isomorphic to a graph in \(\mathcal{F}\) but \(G[s_i' \cap s_j']\) is, and \(\mathcal{F}\) is closed under taking induced subgraphs, \((s_i \cap s_j) \setminus (s_i' \cap s_j')\) \(\neq \emptyset\).

(iii) Finally, \(\alpha\)-Pattern is computed in polynomial time as it is a constraint of the class \(\mathcal{F}\).

**Distance.** In [24], overlapping nodes occur only in the boundary regions of overlapping communities in sensor networks. Motivated by this, we consider in the overlap region nodes that are “closer” to each other. In this way, two subgraphs \(H_i, H_j\) do not \(\alpha\)-conflict if the distance in \(G\) between any pair of vertices \(u, v\) in \(V(H_i) \cap V(H_j)\) is at most a constant \(d_i > 0\), i.e., \(dist_G(u, v) \leq d_i\). Recall that a subgraph \(H_i\) is represented by a set \(S_i = V(H_i)\) in \(\mathcal{S}\). Since the graph distance is a metric on \(V(G)\), we use the function \(\alpha\)-Metric defined previously (Lemma 9).

**Lemma 11.** The function \(\alpha\)-Distance is an \(\alpha\)-Condition.

Note that we are using \(dist_G(u, v) \leq d_i\) instead of \(dist_G|_{S_i \cap S_j}(u, v) \leq d_i\). The second one is not an hereditary property and thus not well-conditioned.
Property. There are several vertex properties that are relevant to the analysis of real networks: vertex strength [3,16], vertex weight [14], and disparity [16], among others. Hence, we suggest considering only overlapping nodes that present the same property $\xi$ (or properties). We assume however that the properties values for each vertex are given as part of the input. We define $\alpha$-Property($s_i, s_j$) which simply returns no-conflict either if $|s_i \cap s_j| = 0$ or if each element $u$ in $\{s_i \cap s_j\}$ satisfies $\xi$. Otherwise, it returns $\alpha$-conflict.

Lemma 12. The function $\alpha$-Property is well-conditioned.

Proof. (i) $\alpha$ is hereditary. Assume by contradiction that there is pair $s_i, s_j$ with $\alpha(s_i, s_j) = 0$ but there is a pair of subsets $\alpha(s_i', s_j') = 1$ where $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$. If $\alpha(s_i, s_j) = 0$ every element in $s_i \cap s_j$ satisfies the property $\xi$. Since $(s_i' \cap s_j') \subseteq (s_i \cap s_j)$, every element in $s_i' \cap s_j'$ satisfies $\xi$ as well.

(ii) By definition of $\alpha$-Property any pair of disjoint sets do not $\alpha$-conflict. Let $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ be a pair of subsets with $\alpha(s_i', s_j') = 0$. If $\alpha(s_i, s_j) = 1$ but $\alpha(s_i', s_j') = 0$ there must exist at least one element in $((s_i \cap s_j \setminus (s_i' \cap s_j'))$ that does not follow $\xi$ and therefore $((s_i \cap s_j \setminus (s_i' \cap s_j')) \neq \emptyset$.

(iii) The property $\xi$ for each element of $U$ is given as part of the input. Thus, we can verify in constant time whether $s_i, s_j \alpha$-conflict or not. \hfill $\square$

Dense Overlap. We design another $\alpha$ function to model the behavior that the overlap region is densely connected. To that end, we define $\alpha$-DenseOverlap($s_i, s_j$) that returns no-conflict if $|s_i \cap s_j| = 0$ or $|E(G[s_i \cap s_j])| \geq \frac{O'(O-1)}{2} - c$, where $O = |s_i \cap s_j|$ and $c \geq 0$ is a constant; otherwise, it returns $\alpha$-conflict.

Lemma 13. The function $\alpha$-DenseOverlap is well-conditioned.

Proof. (i) $\alpha$-DenseOverlap is hereditary. Assume by contradiction that there is pair of sets $s_i, s_j$ with $\alpha(s_i, s_j) = 0$ but there is a pair of subsets $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ with $\alpha(s_i', s_j') = 1$. Notice that $(s_i' \cap s_j') \neq \emptyset;\text{ otherwise, }\alpha(s_i', s_j') = 0$. Therefore, if $\alpha(s_i', s_j') = 1$ then $|E(G[s_i' \cap s_j'])| < \frac{O'(O-1)}{2} - c$, where $O' = |s_i' \cap s_j'|$. However since $G[s_i' \cap s_j']$ is an induced subgraph of $G[s_i \cap s_j]$, then $|E(G[s_i \cap s_j])| < \frac{O'(O-1)}{2} - c$, a contradiction.

(ii) If $\alpha(s_i, s_j) = 1, |s_i \cap s_j| \geq 1$. Furthermore, for any pair of subsets $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ with $\alpha(s_i', s_j') = 0 (s_i \cap s_j) \setminus (s_i' \cap s_j') \neq \emptyset$. Assume otherwise by contradiction. If $\alpha(s_i', s_j') = 0$ then $|E(G[s_i' \cap s_j'])| \geq \frac{O'(O-1)}{2} - c$, where $O' = |s_i' \cap s_j'|$. Thus, if $(s_i \cap s_j) \setminus (s_i' \cap s_j')$ would be the empty set, then $|E(G[s_i \cap s_j])| \geq \frac{O'(O-1)}{2} - c$, where $O = |s_i \cap s_j|$, a contradiction to our assumption that $\alpha(s_i, s_j) = 1$.

(iii) We can verify in polynomial time this condition.

Density. We could ask that the subgraph induced by the overlapping vertices has both at most $t$ vertices and $c$ edges. To that end, the function $\alpha$-Density returns no-conflict if $|s_i \cap s_j| = 0$ or $(|s_i \cap s_j| \leq t$ and $|E(G[s_i \cap s_j])| \leq c)$, where $c \geq 0$, else returns $\alpha$-conflict.
Lemma 14. The function $\alpha$-Density is well-conditioned.

Proof. (i) $\alpha$-Density is hereditary. Assume by contradiction that there is pair $s_i, s_j$ with $\alpha(s_i, s_j) = 0$, but there is a pair of subsets $\alpha(s_i', s_j') = 1$ where $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$. Since $\alpha(s_i, s_j) = 0$, both $(s_i \cap s_j) \leq t$ and $E(G[s_i \cap s_j]) \leq c$. For any pair of sets $s_i, s_j$ with $\alpha(s_i, s_j) = 0$, $|s_i \cap s_j| \leq t$. Thus, there cannot be a pair of subsets $s_i' \subseteq s_i, s_j' \subseteq s_j$ with $|s_i' \cap s_j'| > t$. Given that $(s_i' \cap s_j') \subseteq (s_i \cap s_j)$, $|E(G[s_i' \cap s_j'])| \leq |E(G[s_i \cap s_j])| \leq c$.

(ii) For any pair $s_i, s_j$ that $\alpha$-conflict (i.e., $\alpha(s_i, s_j) = 1$) $|s_i \cap s_j| \geq 1$. In addition, for any pair of subsets $s_i' \subseteq s_i$ and $s_j' \subseteq s_j$ with $\alpha(s_i', s_j') = 0$ both $|s_i' \cap s_j'| \leq t$ and $E(G[s_i' \cap s_j']) \leq c$. Since $\alpha(s_i, s_j) = 1$ either $|s_i \cap s_j| > t$ or $E(G[s_i \cap s_j]) > c$. In the first case, since $\alpha(s_i', s_j') = 0$, $|s_i' \cap s_j'| \leq t$. Therefore, $((s_i \cap s_j) \backslash (s_i' \cap s_j')) \neq \emptyset$. For the second case, $|E(G[s_i \cap s_j])| - |E(G[s_i' \cap s_j'])| > 0$. Therefore, $|(s_i \cap s_j) \backslash (s_i' \cap s_j')| \geq 1$.

(iii) In $O(r)$ time, we can verify if $|E(G[s_i \cap s_j])| \leq c$. □

5 Predetermined Cluster Heads

The problem of discovering communities in networks has been tackled with clustering algorithms as well [13]. Many of these algorithms consider as part of the input a collection of sets of vertices $C = \{C_1, \ldots, C_l\}$ where each set $C_i \subset V(G)$ is called a cluster head. The objective is to find a set of communities in $G$ where each community contains exactly one cluster head. In addition, communities should not share members of the cluster heads [21,14,15,24].

Motivated by this, we introduce the PCH-$r$-Set Packing with $\alpha()$-Overlap problem, where PCH stands for Predetermined Clusters Heads. The input of this problem is as before a universe $U$, a collection $S$, an integer $k$, but now it also has a collection of sets $C = \{C_1, \ldots, C_l\}$ where $C_i \subset U$. The goal there is to find a $(k, \alpha())$-set packing (PCH), i.e., a set of at least $k$ sets $K = \{S_1, \ldots, S_k\}$ subject to the following conditions: each $S_i$ contains at least one set of $C$; for any pair $S_i$, $S_j$ with $i \neq j$, $(S_i \cap S_j) \cap \text{val}(C) = \emptyset$, and $S_i$, $S_j$ do not $\alpha$-conflict. Recall that a $\Pi$-subgraph (or a community) is represented by a set in $S$ (Section 3.2). Thus, this problem translates into a PCH variation for our $\Pi$-Packing problem as well.

To solve the $r$-Set Packing with $\alpha$-Overlap problem (PCH), we need to do two modifications to the BST-algorithm described in Section 2.

First, we redefine the routine that creates the children of the root of the search tree, and we call it Initialization (PCH). By Lemma 2 a maximal solution $M$ is used to determine the children of the root. In the (PCH)-variation, we no longer compute $M$ but rather we use $C$ to compute those children. That is, the root will have a child $i$ for each possible combination of $(C_i)$. Recall that a node $i$ has a collection $Q^i = \{s_1^i, \ldots, s_k^i\}$. Each set of $Q^i$ is initialized with set of that combination.

Lemma 15. If there exists at least one $(k, \alpha())$-set packing (PCH) of $S$, at least one of the children of the root will have a partial-solution.
Proof. It follows by the explicit condition that each set in a \((k, \alpha())\)-set packing (PCH) should contain at least one set from \(C\) and because the routine \textbf{Initialization} (PCH) tries all possible selections of size \(k\) from \(C\) to create the children of the root.

Second, we redefine the \(\alpha\) function of the \textbf{BST-algorithm} as \(\alpha\)-PCH. This new function returns \(\alpha\)-\textit{conflict} if \(((s_i \cap s_j) \cap \text{val}(C)) \neq \emptyset\); otherwise executes the original \(\alpha()\) function and returns \(\alpha(s_i, s_j)\).

\textbf{Lemma 16.} If the function \(\alpha()\) is well conditioned, the function \(\alpha\)-PCH is also well-conditioned.

\textit{Proof.} (i) \(\alpha\)-PCH is hereditary. Assume that \(\alpha\)-PCH\((s_i, s_j) = 0\), and there is pair of subsets \(s'_i \subseteq s_i\) and \(s'_j \subseteq s_j\) with \(\alpha\)-PCH\((s'_i, s'_j) = 1\). Since \(\alpha\) is well conditioned, this is only possible if \(((s'_i \cap s'_j) \cap \text{val}(C)) \neq \emptyset\). However, \((s'_i \cap s'_j) \subseteq (s_i \cup s_j)\) and by our assumption \(((s_i \cap s_j) \cap \text{val}(C)) = \emptyset\), a contradiction.

(ii) If \(\alpha\)-PCH\((s_i, s_j) = 1\), \(|s_i \cap s_j| \geq 1\). Assume otherwise by contradiction. Since \(\alpha\) is well-conditioned, this is only possible if the extra condition in \(\alpha\)-PCH returns \(\alpha\)-\textit{conflict} when \(\{s_i \cap s_j\} = \emptyset\). However, in that case there can not be an intersection with the set \(\text{val}(C)\), and \(\alpha\)-PCH\((s_i, s_j) = 0\) instead. It remains to show that for any pair of subsets \(s'_i \subseteq s_i\) and \(s'_j \subseteq s_j\) with \(\alpha\)-PCH\((s'_i, s'_j) = 0\), \((s_i \cap s_j) \setminus (s'_i \cup s'_j) \neq \emptyset\). Assume otherwise by contradiction, but in that case again there cannot be an intersection with \(\text{val}(C)\) and \(\alpha\)-PCH\((s_i, s_j) = 0\).

(iii) Since \(\alpha\) is well-conditioned, and it takes \(O(r)\) time to verify the extra condition in \(\alpha\)-PCH, \(\alpha\)-PCH is verified in polynomial time. \(\square\)

The above two modifications guarantee that the \textbf{BST Algorithm} will find a \((k, \alpha())\)-Set Packing (PCH) if \(S\) has at least one. Given that each set in \(S\) has size at most \(r\), we can immediately discard any set in \(C\) of size more than \(r\). In this way, each set in \(C\) is be upper-bounded by a constant \(c\), \(1 \leq c \leq r - 1\). To maintain our running time, the size of \(C\) should be \(O(g(k))\), where \(g\) is a computable function dependent only on \(k\) and possibly \(r\) but independent of \(n\). Hence, we can state:

\textbf{Theorem 4.} If \(\alpha()\) is well-conditioned, the \(\text{PCH}\)-r-Set Packing with \(\alpha()\)-Overlap problem is solved in \(O((g(k))^k(rk)^{(r-1)k} n^r)\) time, where \(|C| = g(k)|\).

We could also omit the condition that clusters cannot share members of the cluster heads as in \([6]\). In that case, we do not need to redefine the function \(\alpha()\).

6 Conclusion

We have proposed a more general framework for the problem of finding overlapping communities where the pairwise overlap meets a constraint function \(\alpha()\). This framework captures much more realistic settings of the community discovering problem and can lead to interesting questions on its own. We have also
shown that our problems are fixed-parameter tractable when the overlap constraint $\alpha()$ is subject to a set of rather general conditions (Definition 1). In addition, we have given several $\alpha()$ functions that meet those conditions.

There are several interesting paths remaining to explore. It would be interesting to provide a fixed-parameter algorithm for our problems for functions other than those as in Definition 1. For example, when the overlap is bounded by a percentage of the sizes of the communities or when the overlap size has a lower-bound instead of an upper-bound. In addition, a natural step would be to obtain kernelization algorithms for our problems.

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7 Appendix

7.1 Pseudocode

Algorithm 1 BST $\alpha()$-Algorithm

1: Compute a maximal ($\alpha()$)-set packing $\mathcal{M}$
2: if $|\mathcal{M}| \geq k$ then Return $\mathcal{M}$ end if
3: $T$=Initialization($\mathcal{M}$)
4: for each node $i$ of $T$ do
5: Let $\mathcal{Q}^i$ be the collection of sets at node $i$
6: $\mathcal{Q}^{gr} = \text{Greedy}(\mathcal{Q}^i)$
7: if $\mathcal{Q}^{gr} \neq \infty$ then
8: if $|\mathcal{Q}^{gr}| = k$ then Return $\mathcal{Q}^{gr}$ end if
9: Branching($T$, node $i$, $\mathcal{Q}^i$, $\mathcal{Q}^{gr}$)
10: end if
11: end for

Algorithm 2 Initialization($\mathcal{M}$)

1: Replicate $k$ times each element $u \in \text{val}(\mathcal{M})$ and identify them as $u_1, \ldots, u_k$.
2: Let $\mathcal{M}_k$ be the enlarged set $\text{val}(\mathcal{M})$
3: $i = 0$, $T = \text{null}$
4: while $i < |(\mathcal{M}_k)_k|$ do
5: Let $\mathcal{Q}^i = \{s^i_1, \ldots, s^i_k\}$ be the ith combination of $\binom{\mathcal{M}_k}{k}$
6: CreateNode($T$, root, node $i$, $\mathcal{Q}^i$)
7: $i = i + 1$
8: end while
9: Return $T$
Algorithm 3 Greedy($Q^i$)

1: $Q^{gr} = \infty$
2: //Check if $Q^i$ could not be a partial solution
3: if there is no pair $s^i_j, s^i_g$ in $Q^i$ ($f \neq g$) with $\alpha(s^i_j, s^i_g) = 1$ then
4: $Q^{gr} = \emptyset$; $j = 0$
5: repeat
6: Let $s^i_j$ be the $j$th set of $Q^i$
7: if $S(s^i_j, Q^i, \alpha) = \emptyset$ then
8: $Q^{gr} = \infty$
9: else
10: //Choose arbitrarily a set $S$ from $S(s^i_j, Q^i, \alpha)$ such that
11: //S does not $\alpha$-conflict with any set in $Q^{gr}$
12: //and S is not already in $Q^{gr}$
13: $f = 0$
14: while $f < |S(s^i_j, Q^i, \alpha)|$ do
15: Let $S_f$ be the $f$-th set in $S(s^i_j, Q^i, \alpha)$
16: $Conflicts = 0$
17: for each $S' \in Q^{gr}$ do
18: if $(\alpha(S_f, S') == 1) \text{ OR } (S_f == S')$ then
19: $Conflicts = Conflicts + 1$
20: end if
21: end for
22: if $Conflicts == 0$ then $S = S_f$; $f = |S(s^i_j, Q^i, \alpha)| + 1$ end if
23: end while
24: //Add the set $S$ to $Q^{gr}$
25: if such set $S$ does not exist then
26: $j = k + 1$
27: else
28: $Q^{gr} = Q^{gr} \cup S$
29: end if
30: $j = j + 1$
31: end if
32: until $(j \geq k) \text{ OR } (Q^{gr} = \infty)$
33: end if
34: Return $Q^{gr}$
Algorithm 4 Branching($T$, node $i$, $Q^i$, $Q^{gr}$)

1: Let $s^i_j$ be the first set of $Q^i$ not completed by Greedy, i.e.,
2: $j = |Q^{gr}| + 1$ and $s^i_j = Q^i[j]$
3: $I^* = \emptyset$
4: for each $S \in S(s^i_j, Q^i, \alpha)$ do
5:   for each $S' \in Q^{gr}$ do
6:     if $\alpha(S, S') = 1$ or $(S == S')$ then
7:       $I^* = I^* \cup ((S \setminus s^i_j) \cap S')$
8:   end for
9: end for
10: $l = 0$
11: while $l \leq |I^*|$ do
12:   Let $u_l$ be the $l$th element of $I^*$
13:   $Q^l = \{s^i_1, s^i_2, \ldots, s^i_j \cup u_l, \ldots, s^i_k\}$
14:   CreateNode($T$, node $i$, node $l$, $Q^l$)
15:   $l = l + 1$
16: end while

Algorithm 5 Compute $S(s^i_j, Q^i, \alpha)$

1: $l = 0$, $S(s^i_j, Q^i, \alpha) = \emptyset$
2: while $l < |S(s^i_j)|$ do
3:   Let $S_l$ be the $l$-th set in $S(s^i_j)$
4:   $f = 0$, $conflicts = 0$
5:   while $f < |Q^i|$ do
6:     if $f \neq j$ then
7:       if $\alpha(s^i_f, S_l) == 1$ or $(s^i_f == S_l)$ then
8:         $conflicts = conflicts + 1$
9:       end if
10:   end if
11:   $f = f + 1$
12: end while
13: if $conflicts == 0$ then $S(s^i_j, Q^i, \alpha) = S(s^i_j, Q^i, \alpha) \cup S_l$
14: $l = l + 1$
15: end while
16: Return $S(s^i_j, Q^i, \alpha)$
Algorithm 6 Initialization (PCH)(C)

1: $i = 0$, $T = null$
2: while $i < \binom{|C|}{k}$ do
3:   Let $\{C_1^i, \ldots, C_k^i\}$ be the $i$th combination of $\binom{C}{k}$
4:   Make $Q^i = \{s_1^i, \ldots, s_k^i\}$ equal to $\{C_1^i, \ldots, C_k^i\}$, i.e, $s_j^i = C_j^i$
5:   CreateNode($T$, root, node $i$, $Q^i$)
6:   $i = i + 1$
7: end while
8: Return $T$