UNITARILY INVARIANT NORM INEQUALITIES FOR OPERATORS

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Abstract. We present several operator and norm inequalities for Hilbert space operators. In particular, we prove that if $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$, then

$$|||A_1A_2^* + A_2A_3^* + \cdots + A_nA_1^*||| \leq \sum_{i=1}^{n} ||A_iA_i^*||,$$

for all unitarily invariant norms.

We also show that if $A_1, A_2, A_3, A_4$ are projections in $\mathcal{B}(\mathcal{H})$, then

$$\left\| \left( \sum_{i=1}^{4} (-1)^{i+1} A_i \right) \oplus 0 \oplus 0 \oplus 0 \right\| \leq ||| (A_1 + |A_3A_1|) \oplus (A_2 + |A_4A_2|) \oplus (A_3 + |A_1A_3|) \oplus (A_4 + |A_2A_4|)|||$$

for any unitarily invariant norm.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ stand for the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and let $I$ denote the identity operator. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\| = \sup\{\|Ax\| : \|x\| = 1 \}$ denote the usual operator norm of $A$ and $|A| = (A^*A)^{1/2}$ be the absolute value of $A$. For $1 \leq p < \infty$, the Schatten $p$-norm of a compact operator $A$ is defined by $\|A\|_p = (\text{tr}|A|^p)^{1/p}$, where tr is the usual trace functional. If $A$ and $B$ are operators in $\mathcal{B}(\mathcal{H})$ we use $A \oplus B$ to denote the $2 \times 2$ operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathcal{H} \oplus \mathcal{H}$. One can show that

$$\|A \oplus B\| = \max(\|A\|, \|B\|)$$

$$\|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{1/p}$$

An operator $A \in \mathcal{B}(\mathcal{H})$ is positive and write $A \geq 0$ if $\langle A(x), x \rangle \geq 0$ for all $x \in \mathcal{H}$. we say $A \leq B$ whenever $B - A \geq 0$.

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We consider the wide class of unitarily invariant norms $||| \cdot |||$. Each of these norms is defined on an ideal in $\mathcal{B}(\mathcal{H})$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator $T$ belongs to the norm ideal associated with $||| \cdot |||$. Each unitarily invariant norm $||| \cdot |||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators $T$ in the norm ideal associated with $||| \cdot |||$ and for all unitary operators $U$ and $V$ in $\mathcal{B}(\mathcal{H})$.

The following are easily follows from the basic properties of unitarily invariant norms

\[
|||A \oplus A^*||| = |||A \oplus A|||, \tag{1.3}
\]

\[
|||A \oplus B||| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| \tag{1.4}
\]

\[
|||AA^*||| = |||A^*A||| \tag{1.5}
\]

for all operators $A, B \in \mathcal{B}(\mathcal{H})$. For the general theory of unitarily invariant norms, we refer the reader to Bhatia and Simon [1, 11].

It follows from the Fan dominance principle (see [7]) that the following three inequalities for all unitarily invariant norms are equivalence:

\[
|||A||| \leq |||B|||, \tag{1.6}
\]

\[
|||A \oplus 0||| \leq |||B \oplus 0|||, \tag{1.7}
\]

\[
|||A \oplus A||| \leq |||B \oplus B|||. \tag{1.8}
\]

It has been shown by Kittaneh [9] that if $A_1, A_2, B_1, B_2, X, \text{and} Y$ are operators in $\mathcal{B}(\mathcal{H})$, then

\[
2|||(A_1XA_2^* + B_1YB_2^*) \oplus 0||| \leq \left\| \begin{bmatrix} A_1^*A_1X + XA_2^*A_2 & A_1^*B_1Y + XA_2^*B_2 \\ B_1^*A_1X + YB_2^*A_2 & B_1^*B_1Y + YB_2^*B_2 \end{bmatrix} \right\| \tag{1.9}
\]

for all unitarily invariant norms.

It has been shown by Bhatia and Kittaneh [2] that if $A$ and $B$ are operators in $\mathcal{B}(\mathcal{H})$, then

\[
|||A^*B + B^*A||| \leq |||A^*A + B^*B|||, \tag{1.10}
\]

for all unitarily invariant norms. Kittaneh [10] proved that if $A$ and $B$ are positive operators in $\mathcal{B}(\mathcal{H})$, then

\[
|||(A + B) \oplus 0||| \leq \left\| \left( A + |B^{1/2}A^{1/2}| \right) \oplus \left( B + |A^{1/2}B^{1/2}| \right) \right\| \tag{1.11}
\]

for any unitarily invariant norm.

It was shown by Fong [4] that if $A \in M_n(C)$, then

\[
||AA^* - A^*A|| \leq ||A||^2, \tag{1.12}
\]

and it was shown by Kittaneh [8] that

\[
||AA^* + A^*A|| \leq ||A||^2 + ||A||^2. \tag{1.13}
\]
In this paper we establish some operator and norm inequalities. We generalize inequalities (1.9) and (1.10) and present a norm inequality analogue to (1.11). Based on our main result, we provide new proofs of inequalities (1.12) and (1.13).

2. Main results

To achieve our main result we need the following lemma.

**Lemma 2.1.** [3, Theorem 1] If $A$, $B$ and $X$ are operators in $B(\mathcal{H})$, then

$$2|||AXB^*||| \leq |||A^*AX + XB^*B|||$$

(2.1)

for any unitarily invariant norm.

The main result below is an extension of [9, Theorem 2.2]. We will prove it by an approach different from [9, Theorem 2.2].

**Theorem 2.2.** Let $A_i, B_i, X_i \in B(\mathcal{H})$ for $i, j = 1, 2, \ldots, n$. Then

$$2|||\sum_{i=1}^{n} (A_iX_iB_i^*) \oplus 0 \oplus \ldots \oplus 0|||$$

$$\leq \left|\begin{bmatrix}
A_1^*A_1X_1 + X_1B_1^*B_1 & A_1^*A_2X_2 + X_1B_1^*B_2 & \cdots & A_1^*A_nX_n + X_1B_1^*B_n \\
A_2^*A_1X_1 + X_2B_2^*B_1 & A_2^*A_2X_2 + X_2B_2^*B_2 & \cdots & A_2^*A_nX_n + X_2B_2^*B_n \\
\vdots & \vdots & \ddots & \vdots \\
A_n^*A_1X_1 + X_nB_n^*B_1 & A_n^*A_2X_2 + X_nB_n^*B_2 & \cdots & A_n^*A_nX_n + X_nB_n^*B_n
\end{bmatrix}\right|$$

for all unitarily invariant norms.

**Proof.** Consider the following operators on $\bigoplus_{i=1}^{n}\mathcal{H}$

$$A = \begin{bmatrix}
A_1 & A_2 & \cdots & A_n \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
B_1 & B_2 & \cdots & B_n \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}$$

and

$$X = \begin{bmatrix}
X_1 & 0 & \cdots & 0 \\
0 & X_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_n
\end{bmatrix}.$$
Corollary 2.3. Let $A_1, A_2, \ldots, A_n \in \mathbb{B}(\mathcal{H})$. Then

$$\||| A_1 A_2^* + A_2 A_3^* + \cdots + A_n A_1^* ||| \leq \left\| \sum_{i=1}^{n} A_i A_i^* \right\|,$$

for all unitarily invariant norms. In particular,

$$\| A_1 A_2^* + A_2 A_3^* + \cdots + A_n A_1^* \|_p \leq \left\| \sum_{i=1}^{n} A_i A_i^* \right\|_p \quad \text{for} \quad 1 \leq p \leq \infty.$$
Proof. Letting $B_i = A_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $B_n = A_1$ and $X_i = I$ in Theorem 2.2, we get

$$2|||A_1 A_2^* + A_2 A_3^* + \cdots + A_n A_1^* \oplus 0 \cdot \cdot \cdot \oplus 0|||
\leq \begin{bmatrix} A_1^* A_1 & A_1^* A_2 & \cdots & A_1^* A_n \\ A_2^* A_1 & A_2^* A_2 & \cdots & A_2^* A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^* A_1 & A_n^* A_2 & \cdots & A_n^* A_n \end{bmatrix}
+ \begin{bmatrix} A_2^* A_2 & A_2^* A_3 & \cdots & A_2^* A_1 \\ A_3^* A_2 & A_3^* A_3 & \cdots & A_3^* A_1 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^* A_2 & A_1^* A_3 & \cdots & A_1^* A_1 \end{bmatrix}
= \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
+ \begin{bmatrix} A_2^* & A_3^* & \cdots & A_1^* \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
= \begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ 0 & A_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^* \end{bmatrix}
+ \begin{bmatrix} A_2^* & 0 & \cdots & 0 \\ 0 & A_3^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_1^* \end{bmatrix}
= 2 \left( \sum_{i=1}^{n} A_i A_i^* \right) \oplus 0 \cdot \cdot \cdot \oplus 0
$$

By the equivalence of inequalities (1.6) and (1.7) we have

$$|||A_1 A_2^* + A_2 A_3^* + \cdots + A_n A_1^*||| \leq \left\| \sum_{i=1}^{n} A_i A_i^* \right\|
$$

for all unitarily invariant norms. \qed
To establish the next result we need the following Lemma. The lemma is a basic triangle inequality comparing, in unitarily invariant norms, the sum of two normal operators to the sum of their absolute values.

**Lemma 2.4.** [5] If $A$ and $B$ are normal operators in $B(\mathcal{H})$, then

$$|||A + B||| \leq |||A|| + ||B|||.$$  \hspace{1cm} (2.2)

for all unitarily invariant norms.

**Corollary 2.5.** Let $A_1, A_2, A_3, A_4$ be projections in $B(\mathcal{H})$. Then

$$||\left(\sum_{i=1}^{4}(-1)^{i+1}A_i\right) \oplus 0 \oplus 0 \oplus 0||$$

$$\leq |||(A_1 + |A_3A_1|) \oplus (A_2 + |A_4A_2|) \oplus (A_3 + |A_1A_3|) \oplus (A_4 + |A_2A_4|)|||$$  \hspace{1cm} (2.3)

for all unitarily invariant norms. In particular,

$$\left\|\sum_{i=1}^{4}(-1)^{i+1}A_i\right\|$$

$$\leq \max\{\|A_1 + |A_3A_1|\|, \|A_2 + |A_4A_2|\|, \|A_3 + |A_1A_3|\|, \|A_4 + |A_2A_4|\|\}$$

and

$$\left\|\sum_{i=1}^{4}(-1)^{i+1}A_i\right\|_p$$

$$\leq \left(\|A_1 + |A_3A_1|\|_p^p + \|A_2 + |A_4A_2|\|_p^p + \|A_3 + |A_1A_3|\|_p^p + \|A_4 + |A_2A_4|\|_p^p\right)^{1/p}$$

\hspace{1cm} (1 \leq p < \infty).
Proof. Letting \( n = 4 \), and replacing \( A_i \) and \( B_i \) by \( A_i \), and \( X_i = (-1)^{i+1}I \) for \( i = 1, 2, 3, 4 \) in Theorem 2.2, we get

\[
\begin{bmatrix}
\sum_{i=1}^{4} (-1)^{i+1} A_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\leq
\begin{bmatrix}
A_1 & 0 & A_1 A_3 & 0 \\
0 & -A_2 & 0 & -A_2 A_4 \\
A_3 A_1 & 0 & A_3 & 0 \\
0 & -A_4 A_2 & 0 & -A_4
\end{bmatrix}
\leq
\begin{bmatrix}
A_1 & 0 & A_1 A_3 & 0 \\
0 & A_2 & 0 & A_2 A_4 \\
A_3 A_1 & 0 & A_3 & 0 \\
0 & A_4 A_2 & 0 & A_4
\end{bmatrix}
= \begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & A_1 A_3 & 0 \\
0 & 0 & 0 & A_2 A_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\leq \begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix}
+ \begin{bmatrix}
|A_3 A_1| & 0 & 0 & 0 \\
0 & |A_4 A_2| & 0 & 0 \\
0 & 0 & |A_1 A_3| & 0 \\
0 & 0 & 0 & |A_2 A_4|
\end{bmatrix}
= \begin{bmatrix}
|A_3 A_1| & 0 & 0 & 0 \\
0 & |A_4 A_2| & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix}
\leq \begin{bmatrix}
A_1 + |A_3 A_1| & 0 & 0 & 0 \\
0 & A_2 + |A_4 A_2| & 0 & 0 \\
0 & 0 & A_3 + |A_1 A_3| & 0 \\
0 & 0 & 0 & A_4 + |A_2 A_4|
\end{bmatrix}.
\]

This proves inequality (2.3).
The rest inequalities follow from (2.3), (1.1) and (1.2). \(\square\)

Corollary 2.6. Let \( A_1, A_2, \ldots, A_n \) be positive operators in \( \mathbb{B}(\mathcal{H}) \). Then

\[
\|A_1 + A_2 + \cdots + A_n\| \leq \max \{ \|A_i + (n-1)A_i\| : i = 1, 2, \ldots, n \}. \quad (2.4)
\]
Proof. First we show that

\[
\begin{bmatrix}
A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\
A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n
\end{bmatrix}
\]

It is enough to show that

\[
\left[ A_1 + (n-1)\|A_1\| \quad 0 \quad \cdots \quad 0 \\
0 \quad A_2 + (n-1)\|A_2\| \quad \cdots \quad 0 \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
0 \quad 0 \quad \cdots \quad A_n + (n-1)\|A_n\| \right].
\]

To see this, we note that

\[
nC = \begin{bmatrix}
(n-1)A_1^{1/2} & -A_1^{1/2} & \cdots & -A_1^{1/2} \\
-A_2^{1/2} & (n-1)A_2^{1/2} & \cdots & -A_2^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
-A_n^{1/2} & -A_n^{1/2} & \cdots & (n-1)A_n^{1/2}
\end{bmatrix}
\times
\begin{bmatrix}
(n-1)A_1^{1/2} & -A_1^{1/2} & \cdots & -A_1^{1/2} \\
-A_2^{1/2} & (n-1)A_2^{1/2} & \cdots & -A_2^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
-A_n^{1/2} & -A_n^{1/2} & \cdots & (n-1)A_n^{1/2}
\end{bmatrix} \geq 0.
\]

Next, by letting \(X_i = I\) and replacing both \(A_i\) and \(B_i\) by \(A_i^{1/2}\) in Theorem 2.2, we obtain

\[
\|A_1 + A_2 + \cdots + A_n\| \leq \left\| \begin{bmatrix}
A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\
A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\
\vdots & \vdots & \ddots & \vdots \\
A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n
\end{bmatrix} \right\|
\]
\[\begin{bmatrix}
A_1 + (n-1)\|A_1\| & 0 & \cdots & 0 \\
0 & A_2 + (n-1)\|A_2\| & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_n + (n-1)\|A_n\|
\end{bmatrix}\]

Hence
\[\|A_1 + A_2 + \cdots + A_n\| \leq \max\{\|A_i + (n-1)\|A_i\|\} : i = 1, 2, \ldots, n\]. \qed

**Corollary 2.7.** Let \(A\) and \(B\) be normal operators in \(\mathcal{B}(\mathcal{H})\), then
\[\|A + B\| \leq \max\{\|A\| + \|A\|, \|B\| + \|B\|\}\].

**Proof.** Letting \(n = 2\), \(A_1 = |A|\), \(A_2 = |B|\) in (2.4), therefore
\[\|A + B\| \leq \|\|A\| + |B|\|\] (by (2.2))
\[\leq \max\{\|A\| + \|A\|, \|B\| + \|B\|\}\]. \qed

To establish the next result we need the following lemma.

**Lemma 2.8.** [6, Theorem 1.1] If \(A, B, C\) and \(D\) are operators in \(\mathcal{B}(\mathcal{H})\), then
\[\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\| \leq \|\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\|\]. \tag{2.6}

**Corollary 2.9.** Let \(A \in \mathcal{B}(\mathcal{H})\). Then
\[\|AA^* + A^*A\| \leq \|A^2\| + \|A\|^2\] \tag{2.7}

and
\[\|AA^* - A^*A\| \leq \|A\|^2.\] \tag{2.8}

**Proof.** Letting \(n = 2\), \(A_1 = B_1 = A\), \(A_2 = B_2 = A^*\) and \(X_1 = X_2 = I\) in Theorem 2.2 to get
\[\|AA^* + A^*A\| \leq \|\begin{bmatrix} A^*A & A^* \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
This prove inequality (2.7). To prove inequality (2.8), letting \(n = 2\), \(A_1 = B_1 = A\), \(A_2 = B_2 = A^*\) and \(X_1 = I = -X_2\) in Theorem 2.2, we get

\[
\|AA^* - A^*A\| \leq \left\| \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix} \right\|
\leq \left\| \begin{bmatrix} \|A^*A\| & 0 \\ 0 & \|AA^*\| \end{bmatrix} \right\| \quad \text{(by inequality (2.2))}

= \|A\|^2.
\]

□

References

[1] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
[2] R. Bhatia and F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl. 11 (1990), 272–277.
[3] R. Bhatia and C. Davis, More matrix forms the arithmetic-geometric mean inequality, SIAM J. Matrix Anal. Appl. 287 (1990), 719–725.
[4] C.K. Fong, Norm estimates related to self-commutators, Linear Algebra Appl. 74 (1986), 151–156.
[5] R.A. Horn and X. Zhan, Inequalities for C-S seminorms and Lieb functions, Linear Algebra Appl. 291 (1999), 103–113.
[6] J.C. Hou and H.K. Du, Norm inequalities of positive operator matrices, Integral Equations Operator Theory. 22 (1995), 281-294.
[7] F. Kittaneh, A note on the arithmetic-geometric mean inequality for matrices, Linear Algebra Appl. 171 (1992), 1-8.
[8] F. Kittaneh, Commutator inequalities associated with the polar decomposition, Proc. Amer. Math. Soc. 130 (2002) 1279–1283.
[9] F. Kittaneh, Norm inequalities for sums of positive operators, J. Operator Theory. 48 (2002), 95–103.
[10] F. Kittaneh, Norm inequalities for sums of positive operators.II, Positivity 10 (2006), 251–260.
[11] B. Simon, Trace Ideals and their Applications, Cambridge University Press, Cambridge, 1979.

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