A DESCRIPTION OF $A_\infty$-WEIGHTS FOR VMO

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ABSTRACT. We present a new characterization of Muckenhoupt $A_\infty$-weights whose logarithm is in $VMO(\mathbb{R})$ in terms of vanishing Carleson measures on $\mathbb{R}^2_+$ and vanishing doubling weights on $\mathbb{R}$. This also gives a novel description of strongly symmetric homeomorphisms on the real line by using a geometric quantity.

1. INTRODUCTION

A locally integrable non-negative measurable function $\omega$ on $\mathbb{R}$ is called a weight. We say that $\omega$ is a doubling weight if there exists a positive constant $\rho$ such that

$$\rho^{-1} \omega(J) \leq \omega(I) \leq \rho \omega(J)$$

for any adjacent bounded intervals $I$ and $J$ of length $|I| = |J|$. Here, $\omega(I) = \int_I \omega(x)dx$. We call the optimal value of such $\rho$ the doubling constant for $\omega$. Moreover, a doubling weight $\omega$ is called vanishing if $\omega(I)/\omega(J) \to 1$ as $|I| = |J| \to 0$.

We say that $\omega$ is a Muckenhoupt $A_\infty$-weight (abbreviated to $A_\infty$-weight) (see [6, C.6]) if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$|E| \leq \delta |I| \Rightarrow \omega(E) \leq \varepsilon \omega(I)$$

whenever $I \subset \mathbb{R}$ is a bounded interval and $E \subset I$ a measurable subset. Naturally an $A_\infty$-weight is doubling. Fefferman and Muckenhoupt [5] gave this a direct computation, and they also provided an example of a function that satisfies the doubling condition but not $A_\infty$.

For a weight function $\omega$ on the real line $\mathbb{R}$, we define a sense-preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = h(0) + \int_0^x \omega(t)dt$. The homeomorphism $h$ is called strongly quasisymmetric if $\omega$ is an $A_\infty$-weight. In particular, $\log \omega \in BMO(\mathbb{R})$, the space of functions of bounded mean oscillation on the real line (see Section 2 for precise definition). This subclass of quasisymmetric homeomorphisms and its Teichmüller space were much investigated (see [1, 2, 4, 14]) because of their great importance in the application to harmonic analysis and elliptic operator theory (see [5, 11, 12, 13]). In particular, it was proved that a sense-preserving homeomorphism $h$ is strongly quasisymmetric if and only if it can be extended to a quasi-conformal homeomorphism of $\mathbb{R}^2_+$ onto itself whose Beltrami coefficient $\mu$ induces a Carleson measure $|\mu(z)|^2y^{-1}dxdy$ on $\mathbb{R}^2_+$. Moreover, a strongly quasisymmetric...
homeomorphism \( h \) is said to be strongly symmetric if the \( A_\infty \)-weight \( \omega \) satisfies that \( \log \omega \in \text{VMO}(\mathbb{R}) \), the space of functions of vanishing mean oscillation on the real line (see Section 2 for precise definition). This class was investigated further in [17] [18]. In particular, it was proved that \( h \) is strongly symmetric if and only if it can be extended to a quasiconformal homeomorphism of \( \mathbb{R}^2_+ \) onto itself whose Beltrami coefficient \( \mu \) induces a vanishing Carleson measure \( |\mu(z)|^2 y^{-1} dxdy \) on \( \mathbb{R}^2_+ \).

Here, a positive measure \( \lambda(x, y) dxdy \) on \( \mathbb{R}^2_+ \) is called a Carleson measure if
\[
\|\lambda\|^{1/2}_{c/2} = \sup_{|I|} \frac{1}{|I|} \int_{I} \int_{I} \lambda(x, y) dxdy < \infty,
\]
where the supremum is taken over all bounded intervals \( I \subset \mathbb{R} \). Furthermore, if a Carleson measure \( \lambda \) satisfies
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{I} \int_{I} \lambda(x, y) dxdy = 0,
\]
we call \( \lambda \) a vanishing Carleson measure.

The purpose of the present paper is to give a new description of strongly symmetric homeomorphisms without using quasiconformal extensions (see Theorem 1 below). Before stating the theorem we need to introduce some notations. For \( z = (x, y) \in \mathbb{R}^2_+ \), let \( I_z = \{ s \mid |s - x| < y/2 \} \), and let \( I_z^+ \) and \( I_z^- \) be the left and right half of the interval \( I_z \), respectively. For a doubling weight \( \omega \) on \( \mathbb{R} \), we introduce a geometric quantity
\[
\frac{2^{-1} \omega(I_z)}{\omega(I_z^+) \omega(I_z^-)^{1/2}},
\]
which is bigger or equal to 1 since it is actually the ratio between the arithmetic mean and the geometric mean of the densities of \( \omega \) over the intervals \( I_z^+ \) and \( I_z^- \), that is \( \omega(I_z^+) / |I_z^+| \) and \( \omega(I_z^-) / |I_z^-| \). Then we define a nonnegative quantity
\[
\eta(z) = \log \frac{2^{-1} \omega(I_z)}{\omega(I_z^+) \omega(I_z^-)^{1/2}}.
\]

In [17], it was shown that the doubling weight \( \omega \) is an \( A_\infty \)-weight if and only if the positive measure \( \eta(z)y^{-1} dxdy \) is a Carleson measure on \( \mathbb{R}^2_+ \). This expresses the close connection between \( A_\infty \)-weights (or strongly quasisymmetric homeomorphisms) and Carleson measures. We consider to what extent one can extend this result to \( A_\infty \)-weights whose logarithm is in \( \text{VMO}(\mathbb{R}) \) (or strongly symmetric homeomorphisms) and prove the following.

**Theorem 1.** For an \( A_\infty \)-weight \( \omega \) on \( \mathbb{R} \), the function \( \log \omega \in \text{VMO}(\mathbb{R}) \) if and only if the Carleson measure \( \eta(z)y^{-1} dxdy \) is vanishing on \( \mathbb{R}^2_+ \) and the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \).

**Remark 1.** The condition that \( \omega \) is an \( A_\infty \)-weight with \( \log \omega \in \text{VMO}(\mathbb{R}) \) implies that the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \), which we may obtain by examining the proof of corresponding claim for the case of unit circle in [15] Lemma 3.3.

**Remark 2.** Let \( \bar{\eta}(z) = |1 - \omega(I_z^+)/\omega(I_z^-)|^2 \). Since the weight \( \omega \) is doubling, it is easy to compute that \( \bar{\eta}(z) \) and \( \eta(z) \) are comparable with comparison constant depending only on doubling constant for \( \omega \). Then, the Carleson measure \( \eta(z)y^{-1} dxdy \) is vanishing on \( \mathbb{R}^2_+ \) if and only if the Carleson measure \( \bar{\eta}(z)y^{-1} dxdy \) is. We consider \( \lambda_\delta(\omega) = \sup_{0 < y \leq \delta} \bar{\eta}(z) \). Then the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \) if and
only if \( \lim_{\delta \to 0^+} \lambda_\delta(\omega) = 0 \). Further, if the rate of convergence of \( \lambda_\delta(\omega) \) satisfies the condition

\[
\int_0^\infty \frac{\lambda_\delta(\omega)}{\delta} d\delta < \infty,
\]

then, by the estimate

\[
\int_{I(x_0, t)} \int_0^t \tilde{\eta}(z) \frac{dx dy}{y} \leq \int_{I(x_0, t)} \int_0^t \lambda_y(\omega) \frac{dx dy}{y} = t \int_0^t \frac{\lambda_y(\omega)}{y} dy,
\]

we can conclude that the measure \( \tilde{\eta}(z) y^{-1} dx dy \) is a vanishing Carleson measure on \( \mathbb{R}^2_+ \).

The paper is structured as follows: in Section 2, we give some basic definitions and results on BMO functions and Muckenhoupt weights which will be used in the proof of Theorem \( \text{I} \). Section 3 is devoted to the proof of Theorem \( \text{I} \).

2. Preliminaries

Let \( I_0 \) be any interval on the real line \( \mathbb{R} \). A locally integrable function \( u \in L^1_{\text{loc}}(I_0) \) is said to have bounded mean oscillation (abbreviated to BMO) if

\[
\|u\|_{\text{BMO}(I_0)} = \sup_{|I|} \frac{1}{|I|} \int_I |u(x) - u_I| dx < \infty,
\]

where the supremum is taken over all bounded intervals \( I \) of \( I_0 \). The set of all BMO functions on \( I_0 \) is denoted by BMO(\( I_0 \)). This is regarded as a Banach space with norm \( \|\cdot\|_{\text{BMO}(I_0)} \) modulo constants since obviously constant functions have norm zero. Moreover, if \( u \) also satisfies the condition

\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_I |u(x) - u_I| dx = 0,
\]

we say \( u \) has vanishing mean oscillation (abbreviated to VMO). The set of all VMO functions on \( I_0 \) is denoted by VMO(\( I_0 \)). The John–Nirenberg inequality for BMO functions (see \([6, \text{VI.2}], [16, \text{IV.1.3}]\)) asserts that there exists two universal positive constants \( C_1 \) and \( C_2 \) such that for any \( u \in \text{BMO}(I_0) \), any bounded interval \( I \) of \( I_0 \), and any \( \lambda > 0 \), it holds that

\[
1 \geq \frac{1}{|I|} \int_I \left| \{ t \in I : |u(t) - u_I| \geq \lambda \} \right| \leq C_1 \exp \left( \frac{-C_2 \lambda}{\|u\|_{\text{BMO}(I_0)}} \right).
\]

(2.1)

We say that the weight \( \omega \) is a Muckenhoupt \( A_p \)-weight (abbreviated to \( A_p \)-weight) for \( p > 1 \) if there exists a constant \( C_p(\omega) \geq 1 \) such that

\[
\left( \frac{1}{|I|} \int_I \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \left( \frac{1}{\omega(x)} \right)^{\frac{1}{p-1}} dx \right)^{p-1} \leq C_p(\omega)
\]

for any bounded interval \( I \subset \mathbb{R} \). We call the optimal value of such \( C_p(\omega) \) the \( A_p \)-constant for \( \omega \). It is known that \( \bigcup_{p>1} A_p = A_\infty \) and \( A_p \subset A_q \) for \( p < q \) (see \([11]\)).

The Jensen inequality implies that

\[
\exp \left( \frac{1}{|I|} \int_I \log \omega(x) dx \right) \leq \frac{1}{|I|} \int_I \omega(x) dx.
\]

(2.3)
Another characterization of $A_\infty$-weights can be given by the reverse Jensen inequality. Namely, $\omega \geq 0$ belongs to the class of $A_\infty$-weights if and only if there exists a constant $C_\infty(\omega) \geq 1$ such that

$$\frac{1}{|I|} \int_I \omega(x)dx \leq C_\infty(\omega) \exp \left( \frac{1}{|I|} \int_I \log \omega(x)dx \right)$$

for every bounded interval $I \subset \mathbb{R}$ (see [8]). We call the optimal value of such $C_\infty(\omega)$ the $A_\infty$-constant for $\omega$.

Sarason gave a characterization of VMO functions by means of $A_2$-weights (see [12, Theorem 2]).

**Proposition 2.** Let $\omega$ be a weight function with $\log \omega \in \text{BMO}(\mathbb{R})$. Then, $\log \omega$ belongs to $\text{VMO}(\mathbb{R})$ if and only if

$$\lim_{|I| \to 0} \left( \frac{1}{|I|} \int_I \omega(x)dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{\omega(x)}dx \right) = 1.$$  

Here, (2.5) may be thought of as a limit $A_2$-condition. Inspired by Proposition 2, Mitsis [10] pushed the analogy between $A_p$-weights and $A_\infty$-weights further by replacing the limit $A_2$-condition with the following so-called asymptotic reverse Jensen inequality (see (2.6) below):

**Proposition 3.** Let $\omega$ be a weight function with $\log \omega \in \text{BMO}(\mathbb{R})$. Then, $\log \omega$ belongs to $\text{VMO}(\mathbb{R})$ if and only if

$$\lim_{|I| \to 0} \left( \frac{1}{|I|} \int_I \omega(x)dx \right) \exp \left( -\frac{1}{|I|} \int_I \log \omega(x)dx \right) = 1.$$  

In other words,

$$\lim_{|I| \to 0} (\log \omega_I - (\log \omega)_I) = 0.$$  

More precisely, Mitsis [10] proved sufficiency with respect to nonatomic measures (covering Lebesgue measure) by performing a dyadic decomposition of the involved interval and omitted the detailed proof of necessity by pointing out it is a standard argument involving the John-Nirenberg inequality for nonatomic measures. Proposition 3 is relevant to the proof of Theorem 1 in the following section. For the completeness of our paper, we complement the proof of necessity and give sufficiency a simple proof involving only elementary measure theoretic considerations.

**Remark 3.** Put it differently, Propositions 2 and 3 imply that $A_2$-condition and $A_\infty$-condition coincide if one restricts to weights which tend to be constant on arbitrarily small intervals.

**Proof of Proposition 3.** Suppose $\log \omega \in \text{VMO}(\mathbb{R})$. By the Jensen inequality, we see

$$1 \leq \left( \frac{1}{|I|} \int_I \omega(x)dx \right) \exp \left( -\frac{1}{|I|} \int_I \log \omega(x)dx \right) \leq \left( \frac{1}{|I|} \int_I \omega(x)dx \right)\left( \frac{1}{|I|} \int_I \frac{1}{\omega(x)}dx \right).$$

Then, obviously, (2.6) follows from (2.5).

Suppose (2.6) holds. To show $u := \log \omega \in \text{VMO}(\mathbb{R})$, we use a strategy of measure theory in [12, Lemma 3]. Let $I$ be a bounded interval in $\mathbb{R}$ such that

$$\left( \frac{1}{|I|} \int_I \omega(x)dx \right) \exp \left( -\frac{1}{|I|} \int_I \log \omega(x)dx \right) = 1 + \varepsilon^3$$
for $0 < \varepsilon < 1/2$. Assuming

$$u_I = \frac{1}{|I|} \int_I \log \omega(x) \, dx = 0,$$

we have

$$\frac{1}{|I|} \int_I \omega(x) \, dx = 1 + \varepsilon^3.$$

Let $F$ be the set where $e^{-\varepsilon} < \omega < e^\varepsilon$ and $E = I - F$. We have

$$(1 + \varepsilon^3)|I| = \int_E (\omega(x) - \log \omega(x)) \, dx + \int_F (\omega(x) - \log \omega(x)) \, dx$$

$$\geq (e^{-\varepsilon} + \varepsilon)|E| + |F|$$

$$\geq (1 + \frac{1}{4}\varepsilon^2)|E| + |F|$$

$$= |I| + \frac{1}{4}\varepsilon^2|E|,$$

which implies $|E| \leq 4\varepsilon|I|$ and $|F| \geq (1 - 4\varepsilon)|I|$. Thus,

$$\int_E \omega(x) \, dx = (1 + \varepsilon^3)|I| - \int_F \omega(x) \, dx$$

$$\leq (1 + \varepsilon)|I| - e^{-\varepsilon}|F|$$

$$\leq (1 + \varepsilon)|I| - (1 - \varepsilon)(1 - 4\varepsilon)|I|$$

$$< 6\varepsilon|I|.$$ 

On the other hand, by (2.9) we have

$$-\int_E \log \omega(x) \, dx = \int_F \log \omega(x) \, dx \leq \varepsilon|F| \leq \varepsilon|I|.$$

Noting that $|\log \omega| < \varepsilon$ on $F$, and $|\log \omega| \leq \omega - \log \omega$ generally, we conclude that

$$\int_I |\log \omega(x)| \, dx = \int_E |\log \omega(x)| \, dx + \int_F |\log \omega(x)| \, dx$$

$$\leq \int_E (\omega(x) - \log \omega(x)) \, dx + \varepsilon|F|$$

$$\leq 6\varepsilon|I| + \varepsilon|I| + \varepsilon|I| = 8\varepsilon|I|.$$ 

Combined with (2.9), this implies that

$$\frac{1}{|I|} \int_I |u(x) - u_I| \, dx < 8\varepsilon$$

for $u = \log \omega$. If $\log \omega$ does not satisfy (2.9), then we write $\log \omega = (\log \omega - a) + a$ with $a = \frac{1}{|I|} \int_I \log \omega(x) \, dx$. Since $\log \omega - a$ satisfies (2.8), and (2.10) holds for $\log \omega - a$, we conclude from (2.10) that

$$\frac{1}{|I|} \int_I |u(x) - u_I| \, dx = \frac{1}{|I|} \int_I |(u(x) - a) - (u - a)_I| \, dx < 8\varepsilon.$$ 

Thus, (2.10) holds for every $u = \log \omega$ which satisfies (2.8). Consequently, (2.6) implies $u = \log \omega \in \text{VMO}(\mathbb{R})$. This completes the proof of Proposition 3.\qed
3. Proof of Theorem 1

In this section, we focus on the proof of Theorem 1.

For any \( x_0 \in \mathbb{R} \) and \( t > 0 \), we set \( I(x_0, t) = \{ x \mid |x - x_0| < t/2 \} \) and set

\[
A(x_0, t) = \frac{1}{t} \int_{I(x_0, t)} \int_{I(x_0, t)} \eta(z) \frac{dx\,dy}{y} = \frac{1}{t} \int_{I(x_0, t)} \int_{0}^{t} \eta(z) \frac{dy\,dx}{y}
\]

It is remarkable that \( \eta(z)/y \) is locally integrable (see [7]), the above equality holds due to Fubini’s theorem.

Considering that the integrand \( \eta(z)/y \) is nonnegative, we divide the integral by \( dy \) over \([0, t]\) into those on dyadic intervals and then by changing the variables, we obtain

\[
\int_{0}^{t} \int_{I(x_0, t)} \eta(z) \frac{dy}{y} = \sum_{k=0}^{\infty} \int_{\frac{k}{2^N}}^{\frac{k+1}{2^N}} \int_{I(x_0, t)} \log \frac{2^{-N} \omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} \frac{dy}{y}
\]

By rearranging the order of the following sum:

\[
\sum_{k=1}^{N} \log \frac{2^{-1} \omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} = \log \frac{2^{-N} \omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} + \sum_{k=1}^{N-1} \log \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})}
\]

and by using further observation on the ratio of the first term:

\[
\frac{2^{-N} \omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} = \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{y} \left( \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} \right)^{\frac{1}{2}},
\]

from (3.1) and (3.2) we see that \( N \)-th partial sum of the series \( A(x_0, t) \) can be written as

\[
\sum_{k=1}^{N} \frac{1}{t} \int_{\frac{k}{2^N}}^{\frac{k+1}{2^N}} \int_{I(x_0, t)} \log \frac{2^{-1} \omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{\omega(x, x + \frac{y}{2^N})} \frac{dx\,dy}{y}
\]

\[
= \frac{1}{t} \int_{I(x_0, t)} \int_{I(x_0, t)} \log \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{y} \frac{dx\,dy}{y}
\]

\[
- \frac{1}{2t} \left( \int_{I(x_0, t)} \int_{I(x_0, t)} \log \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{y} \frac{dx\,dy}{y} + \int_{I(x_0, t)} \int_{I(x_0, t)} \frac{\omega(x, x + \frac{y}{2^N})}{y} \frac{dx\,dy}{y} \right)
\]

\[
+ \sum_{k=1}^{N-1} \frac{1}{t} \int_{\frac{k}{2^N}}^{\frac{k+1}{2^N}} \int_{I(x_0, t)} \log \frac{\omega(x - \frac{y}{2^N}, x + \frac{y}{2^N})}{y} \frac{dx\,dy}{y}
\]

\[
:= A_1(x_0, t) - A_2(x_0, t) + A_3(x_0, t).
\]

Then, we can separate \( A(x_0, t) \) into four parts as follows:

\[
A(x_0, t) = \lim_{N \to \infty} [A_1(x_0, t) - A_2(x_0, t) + A_3(x_0, t)]
\]

\[
(A_1(x_0, t) - \log 2 \log \omega(I(x_0, t))) - \lim_{N \to \infty} [A_2(x_0, t) - \log 2(\log \omega)(I(x_0, t))]
\]
Proof. for every $A$

Here, $\omega_I$ is the average of $\omega$ on the interval $I$ as above: namely, $\omega_I = \frac{1}{|I|} \int_I \omega(x) dx$.

**Claim 1.** If $\omega$ is an $A_\infty$-weight on $\mathbb{R}$, it holds that

$$\lim_{N \to \infty} \widehat{A}_2(x_0, t) = 0.$$ 

**Proof.** We shall apply Lebesgue’s dominated theorem to prove the claim. For any fixed $x_0$, $t$, set $h(x) = \log \omega(x) \chi_{I(x_0, 2t)}$. Then using our hypothesis on $\omega$ that $\omega$ is an $A_\infty$-weight, it readily follows that

$$\left| \log \frac{\omega(x - \frac{y}{2^k}, x)}{\omega(x)} \right| \leq \left| \log \frac{\omega(x - \frac{y}{2^k}, x)}{\omega(x)} \right| - \frac{1}{2^k} \int_{x-\frac{y}{2^k}}^{x} \log \omega(u) du + \frac{1}{2^k} \int_{x-\frac{y}{2^k}}^{x} |\log \omega(u)| du \leq C + Mh(x),$$

where $M$ is Hardy-Littlewood maximal operator. It is not hard to see that $\log \omega \in \text{BMO}(\mathbb{R})$ and hence $h(x) \in L^1(\mathbb{R})$. Therefore, $Mh$ is in weak-$L^1$ and thus locally integrable. From Lebesgue’s dominated theorem and Lebesgue differentiation Theorem, we deduce that

$$\lim_{N \to \infty} \frac{1}{t} \int_0^t \int_{I(x_0, t)} \log \frac{\omega(x - \frac{y}{2^k}, x)}{\omega(x)} \frac{dx dy}{y} = \log 2(\log \omega)_{I(x_0, t)}.$$ 

This concludes the proof of Claim 1. □

We give a simple observation on vanishing doubling weights, which will be used to prove Claims 2 and 3 in the following.

**Lemma 4.** Let the doubling weight $\omega$ be vanishing; namely, for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\tag{3.4} (1 + \varepsilon)^{-1} \leq \frac{\omega(x_0, x_0 + t)}{\omega(x_0 - t, x_0)} \leq 1 + \varepsilon$$

for every $x_0 \in \mathbb{R}$ and for every $t \in (0, \delta)$. Then,

(a) $$\frac{(1 + \varepsilon)^n - 1}{\varepsilon(1 + \varepsilon)^{n-1}} \leq \frac{\omega(x_0 - nt, x_0 + nt)}{\omega(x_0 - t, x_0 + t)} \leq \frac{(1 + \varepsilon)^n - 1}{\varepsilon},$$

(b) $$\frac{x_1, x_1 + t)}{\omega(x_0 - t, x_0)} \leq 1 + \varepsilon$$

are satisfied for every positive integer $n$ and for any $x_1 \in (x_0 - t, x_0)$, respectively.

**Proof.** We now estimate $\omega(x_0 - nt, x_0 + nt)$ from both above and below in terms of $\omega(x_0 - t, x_0 + t)$.

$$\omega(x_0 - nt, x_0 + nt) = \sum_{k=1}^{n} (\omega(x_0 - kt, x_0 - (k-1)t) + \omega(x_0 + (k-1)t, x_0 + kt)) \leq \sum_{k=1}^{n} (1 + \varepsilon)^{k-1} (\omega(x_0 - t, x_0) + \omega(x_0, x_0 + t))$$
uniformly for $x_0 \in \mathbb{R}$. We shall make an estimate on $\hat{A}_1(x_0, t)$:

\begin{equation}
|\hat{A}_1(x_0, t)| \leq \frac{1}{t} \int_{I(x_0, t)} \int_{\frac{t}{2}}^t \left| \log \frac{\omega_I(x, y)}{\omega_I(x_0, t)} \right| \frac{dy dx}{y} \leq \frac{1}{t} \int_{I(x_0, t)} \int_{\frac{t}{2}}^t \left| \log \frac{\omega_I(x, y)}{\omega_I(x_0, t)} \right| \frac{dy dx}{y} + \frac{1}{t} \int_{I(x_0, t)} \int_{\frac{t}{2}}^t \left| \log \frac{\omega_I(x, y)}{\omega_I(x_0, t)} \right| \frac{dy dx}{y}.
\end{equation}

Since $\omega$ is vanishing, for any arbitrarily small $\varepsilon > 0$ there exists some $\delta > 0$ such that (3.4) holds for every $x_0 \in \mathbb{R}$ and for every $t \in (0, \delta)$. We suppose $t \in (0, \delta)$ in the following.

First, we estimate the first term in the last line of (3.5). Set $N = 1/\sqrt{\varepsilon}$ (we may adjust $\varepsilon$ so that $N$ becomes an integer). Then,

\begin{equation}
\int_{\frac{t}{2}}^t \left| \log \frac{\omega_I(x, y)}{\omega_I(x_0, t)} \right| \frac{dy}{y} \leq \sum_{k=1}^N \int_{\frac{t}{2} \left(1 + \frac{k}{N}\right)}^t \left(1 + \frac{k}{N}\right) \left| \log \omega_I(x, y) - \log \omega_I(x_0, t) \right| \frac{dy}{y}.
\end{equation}

We note that as $y \in \left[\frac{t}{2} \left(1 + \frac{k}{N}\right), \frac{t}{2} \left(1 + \frac{k+1}{N}\right)\right]$, the difference value $|\log \omega_I(x, y) - \log \omega_I(x_0, t)|$ is less than the maximum of

\begin{equation}
\max_{1 \leq k \leq N} \left| \log \left( \frac{1}{\frac{t}{2} \left(1 + \frac{k}{N}\right)} \int_{\frac{t}{2} - \frac{k}{N}}^{\frac{t}{2} \left(1 + \frac{k}{N}\right)} \omega_I(u) du \right) - \log \omega_I(x_0, t) \right|.
\end{equation}
and
\[
\max_{1 \leq k \leq N} \left| \log \left( \frac{1}{t} \left(1 + \frac{k}{N} \right) \right) \int_{x - \frac{k}{N}}^{x + \frac{k}{N}} \omega(u) du \right| - \log \omega_{I(x,t)}.
\]

We assume that (3.6) is bigger than (3.7) and continue our computation (the other case can be treated similarly). Combined with
\[
\sum_{k=1}^{N} \int_{\frac{k}{N}}^{\frac{k+1}{N}} \frac{dy}{y} = \log 2 < 1,
\]
it implies
\[
\begin{align*}
\int_{\frac{1}{2}}^{t} \log \frac{\omega_{I(x,y)}}{\omega_{I(x,t)}} \frac{dy}{y} & \leq \max_{1 \leq k \leq N} \left| \log \left( \frac{1}{t} \left(1 + \frac{k}{N} \right) \right) \int_{x - \frac{k}{N}}^{x + \frac{k}{N}} \omega(u) du \right| - \log \omega_{I(x,t)} \\
& \leq \max_{1 \leq k \leq N} \left\{ \log \left( \frac{1}{t} \left(1 + \frac{k}{N} \right) \right) + \left| \log \omega_{I(x,uN+k)} \right| - \log \omega_{I(x,t)} \right\} \\
& \leq \log \left( 1 + \frac{1}{N} \right) + \max_{1 \leq k \leq N} \left| \log \omega_{I(x, \frac{1}{N} \times (N+k))} - \log \omega_{I(x, \frac{1}{N} \times 2N)} \right|.
\end{align*}
\]

By the statement (a) in Lemma 4, we conclude
\[
\begin{align*}
\left| \log \omega_{I(x, \frac{1}{N} \times (N+k))} - \log \omega_{I(x, \frac{1}{N} \times 2N)} \right| & \leq \left| \log \left( \frac{1 + \varepsilon}{(N + k)} \right) \right| - \log \left( \frac{(1 + \varepsilon)^{2N - 1}}{2N} \right) \\
& + \left| \log \left( \frac{1 + \varepsilon}{N} \right) \right| - \log \left( \frac{(1 + \varepsilon)^{N+k-1}}{2N} \right) \\
& \leq 2 \left| \log \left( \frac{1 + \varepsilon}{N} \right) \right| + (3N + k - 2) \log (1 + \varepsilon).
\end{align*}
\]
The monotonicity of \( \frac{a^{x-1}}{x} (a > 1) \) yields that
\[
\int_{\frac{1}{2}}^{t} \log \frac{\omega_{I(x,y)}}{\omega_{I(x,t)}} \frac{dy}{y} \leq \log \left( 1 + \frac{1}{N} \right) + 2 \left| \log \frac{(1 + \varepsilon)^{N+1}}{(1 + \varepsilon)^{2N-1}} \right| + (4N - 2) \log (1 + \varepsilon).
\]

This can be arbitrarily small as \( \varepsilon \) is sufficiently small. Therefore, the first term in the last line of (3.3) tends to 0 uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \).

Next, we consider the second term in the last line of (3.3). By using the statement (b) in Lemma 4, we see that
\[
\begin{align*}
& \frac{1}{t} \int_{I(x_0, t)} \int_{\frac{1}{2}}^{t} \log \frac{\omega_{I(x,t)}}{\omega_{I(x_0,t)}} \frac{dy}{y} dx \\
& \leq \frac{1}{t} \int_{I(x_0, t)} \log \frac{\omega_{I(x,t)}}{\omega_{I(x_0,t)}} dx \times \log 2 \\
& = \frac{1}{t} \int_{I(x_0, t)} \left| \log \frac{\omega(x - \frac{1}{2}, x + \frac{1}{2})}{\omega(x_0 - \frac{1}{2}, x_0 + \frac{1}{2})} \right| dx \times \log 2 \\
& \leq \varepsilon \times \log 2.
\end{align*}
\]
Thus, the second term in the last line of (3.3) is bounded by a constant multiple of \( \varepsilon \). This completes the proof of Claim [2].

**Claim 3.** If the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \), then it holds that

\[
\lim_{t \to 0} \hat{A}_3(x_0, t) = 0
\]

uniformly for \( x_0 \in \mathbb{R} \).

**Proof.** Set

\[
F_k(y) = \int_{I(x_0, t)} \log \frac{\omega(x - \frac{y}{2k}, x + \frac{y}{2k})}{\omega(x - \frac{y}{2k}, x + \frac{y}{2k})} \, dx.
\]

Then,

\[
\hat{A}_3(x_0, t) = \sum_{k=1}^{\infty} \frac{1}{t} \int_{\frac{t}{2}}^{t} F_k(y) \frac{dy}{y}.
\]

Assume that \( \omega \) is vanishing; namely, for any arbitrarily small \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that (3.3) holds for every \( x_0 \in \mathbb{R} \) and for every \( t \in (0, \delta) \). We suppose \( t \in (0, \delta) \) in the following.

We now estimate the ratio \( |F_k(y)|/y\). The expression [3.9] for \( F_k(y) \) is changed in form step by step in the following for convenience of the estimate. We first notice that

\[
2F_k(y) = 2 \int_{I(x_0, t)} \log(\omega(I(x, \frac{y}{2k}))) \, dx - \int_{I(x_0, t)} \log(\omega(x - \frac{y}{2k}, x)) \, dx
\]

By the change of the variables, we make the integrands in the second and third terms same as the first one; namely,

\[
\int_{I(x_0, t)} \log(\omega(x - \frac{y}{2k}, x)) \, dx + \int_{I(x_0, t)} \log(\omega(x, x + \frac{y}{2k})) \, dx
\]

\[
= \int_{x_0 - \frac{y}{2k}}^{x_0 + \frac{y}{2k} + \frac{y}{2k+1}} \log(\omega(I(x, \frac{y}{2k}))) \, dx + \int_{x_0 - \frac{y}{2k}}^{x_0 + \frac{y}{2k} + \frac{y}{2k+1}} \log(\omega(I(x, \frac{y}{2k}))) \, dx.
\]

Then, by rearranging the interval of integration, \( 2F_k(y) \) is divided into four terms:

\[
2F_k(y) = \left( \int_{x_0 - \frac{y}{2k}}^{x_0 - \frac{y}{2k} + \frac{y}{2k+1}} \log(\omega(I(x, \frac{y}{2k}))) \, dx - \int_{x_0 - \frac{y}{2k} - \frac{y}{2k+1}}^{x_0 - \frac{y}{2k}} \log(\omega(I(x, \frac{y}{2k}))) \, dx \right)
\]

\[
+ \left( \int_{x_0 + \frac{y}{2k} - \frac{y}{2k+1}}^{x_0 + \frac{y}{2k}} \log(\omega(I(x, \frac{y}{2k}))) \, dx - \int_{x_0 + \frac{y}{2k} + \frac{y}{2k+1}}^{x_0 + \frac{y}{2k} + \frac{y}{2k+1}} \log(\omega(I(x, \frac{y}{2k}))) \, dx \right) .
\]

Finally, by the change of the variables again, we have

\[
2F_k(y) = \int_{x_0 - \frac{y}{2k} + \frac{y}{2k+1}}^{x_0 + \frac{y}{2k} + \frac{y}{2k+1}} \frac{\omega(I(x, \frac{y}{2k}))}{\omega(x - \frac{y}{2k}, x)} \, dx + \int_{x_0 + \frac{y}{2k} + \frac{y}{2k+1}}^{x_0 + \frac{y}{2k} + \frac{y}{2k+1}} \frac{\omega(x - \frac{y}{2k}, x)}{\omega(I(x, \frac{y}{2k}))} \, dx.
\]

By using the statement (b) in Lemma [4] as above, we have that

\[
(3.11) \quad \left| \log \frac{\omega(I(x, \frac{y}{2k}))}{\omega(x - \frac{y}{2k}, x)} \right| \leq \log(1 + \varepsilon) \leq 4\varepsilon,
\]
which implies that

\[
|F_k(y)| \leq 2^{-(k-1)} \varepsilon.
\]

Consequently, we conclude that

\[
|\widehat{A_3}(x_0, t)| = \sum_{k=1}^{\infty} \frac{t}{2^k} \int_{\frac{t}{2^k}}^{t} |F_k(y)| dy \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon
\]

for any \( x_0 \in \mathbb{R} \) and any \( t \in (0, \delta) \). This completes the proof of Claim 3.

Proof of Theorem 1. Suppose \( \omega \) is an \( A_\infty \)-weight on \( \mathbb{R} \) and \( \log \omega \in \text{VMO}(\mathbb{R}) \). We have \( \lim_{N \to \infty} \widehat{A_2}(x_0, t) = 0 \) by Claim 1 and the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \) by Remark 1. Then \( \widehat{A_1}(x_0, t) \to 0 \), \( \widehat{A_3}(x_0, t) \to 0 \) and \( A_4(x_0, t) \to 0 \) uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \) by Claims 2 and 3 and Proposition 3, respectively. Thus, \( A(x_0, t) \to 0 \) uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \); namely, the Carleson measure \( \eta(z) y^{-1} dxdy \) is vanishing on \( \mathbb{R}^2_+ \).

Now we prove the sufficiency. Suppose \( \omega \) is an \( A_\infty \)-weight on \( \mathbb{R} \), then

\[
\lim_{N \to \infty} \widehat{A_2}(x_0, t) = 0
\]

by Claim 1. Since the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \), we have \( \widehat{A_1}(x_0, t) \to 0 \) and \( \widehat{A_3}(x_0, t) \to 0 \) uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \) by Claims 2 and 3, respectively. Combining with the condition that the Carleson measure \( \eta(z) y^{-1} dxdy \) is vanishing on \( \mathbb{R}^2_+ \), we see \( A_4(x_0, t) \to 0 \) uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \). It follows from Proposition 3 again that \( \log \omega \in \text{VMO}(\mathbb{R}) \). This completes the proof of Theorem 1.

□

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