New quasi-exactly solvable class of generalized isotonic oscillators

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Abstract
We introduce a new family of quasi-exactly solvable (QES) generalized isotonic oscillators which are based on the Hermite exceptional orthogonal polynomials. We obtain exact closed-form expressions for the energies and wavefunctions as well as the allowed potential parameters for the first two members of the family using the Bethe ansatz method. Numerical calculations of the energies reveal that member potentials have multiple QES eigenstates and the number of states for higher members are parameter dependent.

Keywords: quasi-exactly solvable systems, Bethe ansatz, isotonic oscillators
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(Some figures may appear in colour only in the online journal)

1. Introduction

Due to their significance in aspects of exact solvability in quantum mechanics, exceptional orthogonal polynomials (EOPs) have attracted a wide range of interests [1, 12]. Unlike the classical orthogonal polynomials, EOPs start with polynomials of degree one or higher. One essential characteristic of the EOPs is that they form complete orthogonal set with respect to some positive definite measure. Recent attention has been on the construction of new one-dimensional quantum systems involving EOPs. Such constructions are most often discussed in connection with supersymmetric quantum mechanics [1] and multi-dimensional super-integrable systems with higher-order integrals of motion [6, 8]. Moreover, it has been shown that exactly solvable quantum systems related to EOPs also allow different type of ladder operators with infinite sequences or multiplet states [7, 8]. In addition, some very interesting
and unusual properties of superintegrable systems which relate to the EOPs have been discussed very recently [8].

An interesting feature of quantum mechanical models based on certain EOPs is that the corresponding Schrödinger equation can be reduced to differential equations which possess exact, partially, algebraically solvable spectra. Such systems are said to be quasi-exactly solvable (QES). Thus a quantum mechanical system is said to be QES if only a finite number of eigenvalues and corresponding eigenvectors can be obtained exactly through algebraic means [13–15]. A fundamental characteristic of QES systems is that the coefficients of the power series solutions to the underlying differential equations satisfy three-terms or more recursion relations, in contrast to the two-term recursions for exactly solvable cases. The complexity of higher order recursion relations makes it difficult to get power series solutions of such systems. In some cases, however, one can terminate the infinite series at certain power by imposing certain constraints on the system parameters. By so doing, analytic (polynomial) solutions to the systems can be obtained under certain constraints on the potential parameters.

Although, a full classification of QES potentials based on exceptional polynomial subspaces of codimension one has been provided [16]. However, the functional Bethe ansatz method [15, 17, 18] (also see appendix for generalization) has proven very effective in obtaining exact closed-form polynomial solutions to many QES quantum mechanical models of codimensions one and greater [19–21]. Thus, the aim of this paper is to obtain exact solutions by means of the Bethe ansatz method, for a class of quantum models based on the Hermite EOP which are of codimension two and greater.

The work is organized as follows. In section 2, we transform the Schrödinger equation for a family of quantum systems into a QES differential equation. The general, closed form expressions for the energies, wavefunctions and the allowed parameters for the first two members of the family are derived in sections 3 and 4 respectively. We conclude the work in section 5, with summary and some remarks.

2. The models and the underlying differential equation

We consider a family of quantum systems with potentials

\[
V_m(r) = \frac{1}{2} \sum_{k=1}^{m-1} A_k r^{2k} + g \left( \frac{H_m}{H_m} - \left( \frac{H_m}{H_m} \right)^2 \right),
\]

where \( m = 2, 4, 6, \ldots \) is an even integer, \( r \in (0, \infty) \), \( A_k \) represent potential parameters, \( g \) is a constant and \( H_m \) is the Hermite EOP of degree \( m \) with all its coefficients positive, defined by

\[
H_m(r) = m! \sum_{p=0}^{\infty} \frac{2^p (2p)!}{p! (m-2p)!}, \quad m = 2, 4, 6, \ldots
\]

and related to the ordinary Hermite polynomial \( H_m(r) \) by

\[
H_m(r) = (-i)^m H_m(i r).
\]

In recent papers [22–24], cases of the first member potential \( (m = 2) \)

\[
V_2(x) = \frac{a^2 x^2}{2} + g a^2 \left( \frac{x^2 - a^2}{x^2 + a^2} \right),
\]

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where $a$ is a positive parameter have been studied. Particularly in [22], the model was shown to be exactly solvable in the case of $g_a = 2$ and $\omega a^2 = 1/2$, with a general solution

$$
\begin{align*}
\Psi_n(x) &= \frac{P_n(x)}{(2x^2 + 1)} e^{-x^2/2}, \\
E_n &= -\frac{3}{2} + n, \quad n = 0, 3, 4, 5, \ldots, 
\end{align*}
$$

(2.5)

where $\Psi_n(x)$ is the wavefunction and $E_n$ is the energy and the polynomial $P_n(x)$ relates to the Hermite polynomial $H_n(x)$ as follows

$$
P_n(x) = \begin{cases} 
1 & \text{for } n = 0, \\
H_n(x) + 4nH_{n-2}(x) + 4(n - 3)H_{n-4}(x) & \text{for } n = 3, 4, 5, \ldots, 
\end{cases}
$$

(2.6)

and is a particular case of exceptional Hermite polynomial, corresponding to a state-adding Darboux transformation at level $-3$, or equivalently, to a two-step state-deleting Darboux transformation at levels 1 and 2 [10]. In what follows, we show that the isotonic oscillator is a member of a solvable class of EOP potentials (2.1), of which we shall obtain the solutions of the first two members using the Beth ansatz method.

The Schrödinger equation corresponding to the potential equation (2.1) is given by ($\hbar = \mu = 1$)

$$
\begin{align*}
- \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + \sum_{k=1}^{m-1} A_k r^{2k} + g \left( \frac{H_m}{H_m} - \left( \frac{H_m}{H_m} \right)^2 \right) \Psi(r) &= 2E\Psi(r), \\
\end{align*}
$$

(2.7)

where $\ell = -1, 0, 1, \ldots, E$ is the energy eigenvalue and $\Psi(r)$ is the wavefunction. After a brief inspection of the differential equation, we use the transformation

$$
\Psi(r) = r^{\ell+1} H_m e^{w(r)} f(r), \quad w(r) = \sum_{p=1}^{m} B_p r^{2p}, \\
\nu = \frac{1}{2} \left[ 1 - \sqrt{1 - 4g} \right], \quad m = 2, 4, \ldots,
$$

(2.8)

to reduce equation (2.7) to

$$
\begin{align*}
f'''(r) + 2 \left[ \frac{\ell + 1}{r} + \frac{H_m}{H_m} + w'(r) \right] f'(r) \\
+ 2\nu \left( w'(r) + \frac{\ell + 1}{r} \right) H_m + (\nu - g) \frac{H_m}{H_m} \\
+ 2w'(r) \left( w'(r)^2 + w''(r) \right) - \sum_{k=1}^{m-1} A_k r^{2k} \right] f(r) &= -2E f(r),
\end{align*}
$$

(2.9)

where $B_p$ are some constants related to $A_k$. 

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3. The $H_2$ model: quantum isotonic oscillator

For $m = 2$, we have the polynomial $H_2 = 4r^2 + 2$, with the potential

$$V_2(r) = \frac{1}{2}a r^2 - 2g \frac{2r^2 - 1}{(2r^2 + 1)^2}.$$  

(3.1)

This is the quantum isotonic oscillator studied in [19, 25, 26]. The corresponding equation (2.9) for potential $V_2(r)$ reads

$$f''(r) + \left[ \frac{\ell + 1}{r} + \frac{4\nu r}{2r^2 + 1} + 2ar \right] f'(r) + \frac{4}{2r^2 + 1} \left[ 2\nu \left( 2ar^2 + \ell + 1 \right) + \nu - g \right] + a \left( \ell + \frac{3}{2} \right) f(r) - 2Ef(r),$$  

(3.2)

where by comparison with equation (2.9), we have $w(r) = ar^2$. Thus if we take $\alpha = -\frac{\sqrt{\omega}}{2}$ and introduce the new variable $z = r^2$, we have

$$2z(2z + 1)f''(z) + \left[ -4\sqrt{\omega}z^2 + (4\ell - 2\sqrt{\omega} + 8\nu + 3)z + 2\ell + 3 \right] f'(z) + \left[ (2E - \sqrt{\omega}(4\nu + 2\ell + 3))z + 4\nu(\ell + 1) - \frac{\sqrt{\omega}}{2} \left( \ell + \frac{3}{2} \right) + \frac{E}{2} - 2g \right] f(z) = 0.  \quad (3.3)$$

Equation (3.3) is QES and therefore possesses polynomial solutions of degree $n \geq 0$, which we write in the form

$$f(z) = \prod_{i=1}^{n} (z - z_i), \quad f(z) \equiv 1 \quad \text{for} \quad n = 0, \quad (3.4)$$

where $\{z_i\}$ are the roots of the polynomial to be determined. To solve equation (3.3), we apply the functional Bethe ansatz method. Substituting (3.4) into (3.3), we obtain the energies and wavefunction

$$E_n = \sqrt{\omega} \left( 2n + \ell + \frac{5}{2} - \sqrt{1 - 4g} \right) = \sqrt{\omega} \left( 2n + 2\nu + \ell + \frac{3}{2} \right),$$

$$\Psi_n(r) \sim r^{n+1}(2r^2 + 1)^{\nu/2} e^{-\sqrt{\omega} \nu^2} \prod_{i=1}^{n} (r^2 - z_i) \quad (3.5)$$

subject to the constraint

$$\nu^2 + \nu \left( 2\ell + \frac{\sqrt{\omega}}{2} + 1 \right) = 2\sqrt{\omega} \sum_{i=1}^{n} z_i - n \left( 2n + 2\ell + 4\nu - \frac{\sqrt{\omega}}{2} - \frac{1}{2} \right)$$

(3.6)

with $\{z_i\}$ satisfying the equations

$$\sum_{j \neq i}^{n} \frac{2}{z_i - z_j} = \frac{4\sqrt{\omega} z_i^2 - (4\ell - 2\sqrt{\omega} + 8\nu + 3)z_i - 2\ell - 3}{2z_i(2z_i + 1)}, \quad i = 1, 2, ..., n. \quad (3.7)$$
As examples, we shall obtain the solutions for \( n = 0, 1, 2 \). For \( n = 0 \), we have

\[
E_0 = \sqrt{\omega} \left( 2\nu + \ell + \frac{3}{2} \right),
\]

\[
\Psi_0(r) \sim r^{\ell+1} \left( 2r^2 + 1 \right)^{\nu} e^{-\frac{\sqrt{\omega}}{2} r^2},
\] (3.8)

subject to the constraint

\[
\nu = 0 \quad (g = 0) \quad \text{or} \quad \nu = - \left( 2\ell + \frac{\sqrt{\omega}}{2} + 1 \right).
\] (3.9)

Similarly for \( n = 1 \), we have the energy and wavefunction

\[
E_1 = \sqrt{\omega} \left( 2\nu + \ell + \frac{7}{2} \right),
\]

\[
\Psi_1(r) \sim r^{\ell+1} \left( 2r^2 + 1 \right)^{\nu} e^{-\frac{\sqrt{\omega}}{2} r^2} \left( r^2 - z_1 \right),
\] (3.10)

subject to the constraint

\[
\nu^2 + \nu \left( 2\ell + \frac{\sqrt{\omega}}{2} + 3 \right) = 2\sqrt{\omega} z_1 - \left( 2\ell - \frac{\sqrt{\omega}}{2} + \frac{3}{2} \right),
\] (3.11)

where \( z_1 \) satisfies

\[
4\sqrt{\omega} z_1^2 - (4\ell - 2\sqrt{\omega} + 8\nu + 3)z_1 - 2\ell - 3 = 0.
\] (3.12)

Equations (3.11) and (3.12) give the condition

\[
\nu^2 + \nu \left( 2\ell + \frac{\sqrt{\omega}}{2} + 3 \right) = -\left( \ell - \frac{3}{4} \pm \frac{1}{4} \sqrt{(4\ell - 2\sqrt{\omega} + 8\nu + 3)^2 + 16(2\ell + 3)\sqrt{\omega}} \right).
\] (3.13)

Also, for \( n = 2 \), we have the solutions

\[
E_2 = \sqrt{\omega} \left( 2\nu + \ell + \frac{9}{2} \right),
\]

\[
\Psi_2(r) \sim r^{\ell+1} \left( 2r^2 + 1 \right)^{\nu} e^{-\frac{\sqrt{\omega}}{2} r^2} \left( r^2 - z_1 \right) \left( r^2 - z_2 \right),
\] (3.14)

with the constraint

\[
\nu^2 + \nu \left( 2\ell + \frac{\sqrt{\omega}}{2} + 9 \right) = 2\sqrt{\omega} (z_1 + z_2) - \left( 2\ell - \frac{\sqrt{\omega}}{2} + \frac{7}{2} \right),
\] (3.15)

where \( z_1, z_2 \) satisfy

\[
\frac{2}{z_1 - z_2} = \frac{4\sqrt{\omega} z_1^2 - (4\ell - 2\sqrt{\omega} + 8\nu + 3)z_1 - 2\ell - 3}{2z_1(2z_1 + 1)},
\]

\[
\frac{2}{z_2 - z_1} = \frac{4\sqrt{\omega} z_2^2 - (4\ell - 2\sqrt{\omega} + 8\nu + 3)z_2 - 2\ell - 3}{2z_2(2z_2 + 1)}.
\] (3.16)

We now note the following: for physically valid solutions, \( \nu \leq \frac{1}{2} \) and for numerical evaluation of the energies \( E_0 \) and \( E_1 \), \( \omega \) can take any value. However, for the second excited state, one must carefully select the value of \( \omega \) as this determines the validity of the solution.
Table 1. $E_{1, \ell}$ levels for $\mathcal{H}_2$ model with $\omega = 0.1$. The ‘$+$’ and ‘$-$’ superscripts represent the possible roots of equation (3.13).

| $\ell$ | $\nu^-$ | $E_{1,\ell}^{--}$ | $\nu^+$ | $E_{1,\ell}^{+-}$ | $\nu^{--}$ | $E_{1,\ell}^{-+}$ | $\nu^{--}$ | $E_{1,\ell}^{++}$ | $\nu^{--}$ | $E_{1,\ell}^{++}$ |
|-------|--------|-----------------|--------|-----------------|--------|-----------------|--------|-----------------|--------|-----------------|
| 1     | -3.049 84 | -0.505 87 | 0.076 58 | 1.471 46 | -0.676 050 | 0.995 45 | -6.666 90 | -2.793 51 |
| 2     | -5.098 83 | -1.485 53 | 0.043 69 | 1.766 88 | -0.719 447 | 1.284 23 | -8.541 64 | -3.662 95 |
| 3     | -7.117 23 | -2.445 85 | 0.029 61 | 2.074 20 | -0.764 277 | 1.572 11 | -10.464 3 | -4.562 74 |
| 4     | -9.126 91 | -3.400 65 | 0.022 11 | 2.385 69 | -0.799 033 | 1.866 36 | -12.412 4 | -5.478 58 |
| 5     | -11.1329 | -4.353 11 | 0.017 54 | 2.699 03 | -0.825 598 | 2.165 78 | -14.375 3 | -6.403 80 |
| 6     | -13.1369 | -5.304 36 | 0.014 49 | 3.013 33 | -0.846 266 | 2.468 94 | -16.347 5 | -7.334 92 |
| 7     | -15.1399 | -6.254 90 | 0.012 33 | 3.328 19 | -0.862 70 | 2.774 77 | -18.326 0 | -8.269 98 |
| 8     | -17.1421 | -7.204 99 | 0.010 71 | 3.643 39 | -0.876 036 | 3.082 57 | -20.308 8 | -9.207 80 |
| 9     | -19.1438 | -8.154 77 | 0.009 46 | 3.958 83 | -0.887 055 | 3.391 82 | -22.294 8 | -10.147 6 |
| 10    | -21.1452 | -9.104 34 | 0.008 47 | 4.274 43 | -0.896 301 | 3.702 20 | -24.283 2 | -11.089 0 |
Figure 1. Potential plots and corresponding wavefunctions (inserts) for \( H_2 \) model with \((\omega, n, \ell) = (0.1, 1, 1)\).

Table 2. Non-degenerate \( E_{2,\ell} \) levels for \( H_2 \) model with \( \omega = 0.1 \)

| \( \ell \) | \( \nu_a \) | \( E_a \) | \( \nu_b \) | \( E_b \) | \( \nu_c \) | \( E_c \) |
|---|---|---|---|---|---|---|
| 1 | 10.6691 | -5.00847 | -7.39674 | -2.93886 | 0.22493 | 1.88151 |
| 2 | -12.5857 | -5.90443 | -9.34834 | -3.85693 | 0.18139 | 2.17020 |
| 3 | -14.5252 | -6.81480 | -11.3161 | -4.78522 | 0.15198 | 2.46783 |
| 4 | -16.4793 | -7.73449 | -13.2931 | -5.71938 | 0.13073 | 2.77062 |
| 5 | -18.4435 | -8.66052 | -15.2760 | -6.65721 | 0.11466 | 3.07668 |
| 6 | -20.4148 | -9.59104 | -17.2627 | -7.59747 | 0.10208 | 3.38495 |
| 7 | -22.3912 | -10.5248 | -19.2520 | -8.53944 | 0.09198 | 3.69479 |
| 8 | -24.3716 | -11.4611 | -21.2434 | -9.48264 | 0.08368 | 4.00577 |
| 9 | -26.3550 | -12.3993 | -23.2362 | -10.4268 | 0.07675 | 4.31762 |
| 10 | -28.3408 | -13.3390 | -25.2301 | -11.3716 | 0.07088 | 4.63013 |
Thus in tables 1 and 2, by carefully selecting \( \omega = 0.1 \), we give some numerical values of the energies and the allowed potential parameter for \( \ell = 0 \ldots 10 \), for \( n = 1 \) and 2 respectively. Moreover as shown in figures 1 to 3, there is a relationship between the potential structure and the value of the parameters.

4. The \( \mathcal{H}_4 \) model: deformed isotonic oscillator

Similarly, for \( m = 4 \) we have the polynomial \( \mathcal{H}_4 = 16r^4 + 48r^2 + 12 \), such that the corresponding potential reads

\[
V_4(r) = \frac{1}{2} \left[ pr^6 + pr^4 + kr^2 + g \left( \frac{96r^6 + 336r^4 - 2r^3 + 216r^2 - 3r + 36}{2(4r^4 + 12r^2 + 3)^2} \right) \right],
\]

(4.1)

with the corresponding equation

\[
f''(r) + 2 \left[ \frac{\ell + 1}{r} + v - \frac{48r^2 + 24}{4r^4 + 12r^2 + 3} + 4\delta r^3 + 2\beta r \right] f'(r) \\
+ 16 \left[ \nu \left( 2r^2 + 3 \right) \left( 4\delta r^3 + 2\beta r^2 + \ell + 1 \right) + \left( 3r^2 + 2 \right) (v - g) \right] \\
+ 4 \left( \beta + 2\gamma r^2 \right) \left( \ell + 1 \right) + \left( 4\delta r^3 + 2\beta r \right)^2 \\
+ 2\beta + 12\delta r - \rho r^5 - \rho r^4 - kr^2 \right] f(r) = -2Ef(r). 
\]

(4.2)

By comparison with equation (2.9), we have \( w(r) = \delta r^4 + \beta r^2 \). Thus, if we choose \( \delta = -\frac{\sqrt{v}}{4} \) and \( \beta = -\frac{\rho}{4\sqrt{v}} \), and introduce the new variable \( z = r^2 \), we have

\[
\left( 4z^3 + 12z^2 + 3z \right) f''(z) + \left[ -4\sqrt{v} z^4 - \left( 12\sqrt{v} + \frac{2\rho}{\sqrt{v}} \right) z^3 \\
+ \left( 6 - 3\sqrt{v} + 4\ell + 16v - \frac{6\rho}{\sqrt{v}} \right) z^2 \\
+ 3 \left( 6 + 4\ell + 8v - \frac{\rho}{2\sqrt{v}} \right) z + 3 \left( \ell + \frac{3}{2} \right) \right] f'(z) \\
+ \left[ \left( \frac{\rho^2}{4v} - \sqrt{v} (5 + 2\ell + 8v) - \kappa \right) z^3 \\
- 2E - 3\sqrt{v} (2\ell + 4v + 5) - \frac{\rho}{\sqrt{v}} \left( \ell + 4v + \frac{3}{2} \right) - \frac{3\rho^2}{4v} - 3\kappa \right] z^2 \\
+ \left( 6E - 12g + \left( 4v - \frac{3\sqrt{v}}{4} \right) (2\ell + 5) - \frac{3\rho}{4\sqrt{v}} \left( 6 + 4\ell + 8v + \frac{3\rho^2}{4} \right) - \frac{3\kappa}{4} \right] z \\
+ \frac{3E}{2} + 6v (2\ell + 3) - 6g - \frac{3\rho}{4\sqrt{v}} \left( \ell + \frac{3}{2} \right) \right] f(z) = 0.
\]

(4.3)
If we seek the solution of the form (3.4), we have the energy and the wavefunction

\[
E_n = 2 \sqrt{\gamma} \sum_{i=1}^{n} z_i + 3 \sqrt{\gamma} \left(2n + 2\nu + \ell + \frac{5}{2}\right) \\
+ \frac{\rho}{\sqrt{\gamma}} \left(n + 2\nu + \frac{\ell}{2} + \frac{3}{4}\right) + \frac{3\rho^2}{8\gamma} + \frac{3\kappa}{2} \\
\Psi_n(r) \sim r^{\ell+1} \left(4r^4 + 12r^2 + 3\right)^\nu e^{-\frac{\sqrt{\gamma}}{4}r^2 - \frac{\rho}{4\sqrt{\gamma}}} \prod_{i=1}^{n} \left(r^2 - z_i\right), \tag{4.4}
\]

subject to the constraints

\[
\kappa = \frac{\rho^2}{4\gamma} - \sqrt{\gamma} (4n + 2\ell + 8\nu + 5), \\
-2 \sqrt{\gamma} \sum_{i=1}^{n} z_i^3 - \left(6 \sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}}\right) \sum_{i=1}^{n} z_i^2 + \left(4n + 2\ell + 8\nu - \frac{3\rho}{\sqrt{\gamma}} - \frac{3\sqrt{\gamma}}{2} - 1\right) \sum_{i=1}^{n} z_i \\
+ n \left(6\ell + 12\nu - \frac{3\rho}{4\sqrt{\gamma}} + 15\right) + \frac{3E_n}{4} + 3\nu (2\ell + \nu + 2) = 0, \\
6E_n - 12\nu (1 - \nu) - \frac{3\kappa}{4} \\
= 4 \sqrt{\gamma} \sum_{i=1}^{n} z_i^2 + \left(6 \sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}}\right) \sum_{i=1}^{n} z_i - n \left(4n + 4\ell + 16\nu - 3\sqrt{\gamma} + 2\right) \\
+ \frac{\rho}{\sqrt{\gamma}} \left(6n + 6\nu + 2\ell - \frac{3\rho^2}{16}\right) = (2\ell + 5) \left(4\nu - \frac{3\sqrt{\gamma}}{4}\right), \tag{4.5}
\]

with the roots \(\{z_i\}\) satisfying the equations

\[
\sum_{j \neq i}^{n} \frac{2}{z_i - z_j} \\
= \left[-4 \sqrt{\gamma} z_i^4 - \left(12 \sqrt{\gamma} + \frac{2\rho}{\sqrt{\gamma}}\right) z_i^3 + \left(6 - 3 \sqrt{\gamma} + 4\ell + 16\nu - \frac{6\rho}{\sqrt{\gamma}}\right) z_i^2 \right] \\
+ \left[3 \left(6 + 4\ell + 8\nu - \frac{\rho}{2\sqrt{\gamma}}\right) z_i + 3 \left(\ell + \frac{3}{2}\right) \right] \\
= \frac{z_i \left(4z_i^2 + 12z_i + 3\right)}{z_i \left(4z_i^2 + 12z_i + 3\right)}, \tag{4.6}
\]

\(i = 1, 2, \ldots, n\).

We now obtain the solutions corresponding to \(n = 0, 1, 2\). For \(n = 0\), we have the solutions

\[
E_0 = 3 \sqrt{\gamma} \left(2\nu + \ell + \frac{5}{2}\right) + \frac{\rho}{\sqrt{\gamma}} \left(2\nu + \frac{\ell}{2} + \frac{3}{4}\right) + \frac{3\rho^2}{8\gamma} + \frac{3\kappa}{2} \\
\Psi_0(r) \sim r^{\ell+1} \left(4r^4 + 12r^2 + 3\right)^\nu e^{-\frac{\sqrt{\gamma}}{4}r^2 - \frac{\rho}{4\sqrt{\gamma}}} \prod_{i=1}^{n} \left(r^2 - z_i\right), \tag{4.7}
\]
subject to the constraints

\[ \kappa = \frac{\rho^2}{4\gamma} - \sqrt{\gamma} (2\ell + 8\nu + 5), \]

\[ \frac{3E_0}{4} + 3\nu (2\ell + \nu + 2) = 0, \]

\[ 6E_0 - 12\nu (1 - \nu) - \frac{3\kappa}{4} = \rho \left( 6\nu + 2\ell - \frac{3\rho^2}{16} \right) - (2\ell + 5) \left( 4\nu - \frac{3\sqrt{\gamma}}{4} \right). \quad (4.8) \]

Similar to the \( H_2 \) model, the \( n = 0 \) solution has two levels for each \( \ell \). The first level corresponds to \( \nu = 0 \), while the numerical values for the second level are given in table 3.

Similarly for \( n = 1 \), we have the solutions

\[ E_1 = 2\sqrt{\gamma}z_1 + 3\sqrt{\gamma} \left( 2\nu + \ell + \frac{9}{2} \right) + \frac{\rho}{\sqrt{\gamma}} \left( 2\nu + \ell + \frac{7}{4} \right) + \frac{3\rho^2}{8\gamma} + \frac{3\kappa}{2} \]

\[ \Psi(r) \sim \rho^{\ell+1} (4r^4 + 12r^2 + 3)^{\nu/4} e^{-\sqrt{\gamma}r^2 - \rho^2/x^2} \left( r^2 - z_1 \right). \quad (4.9) \]

subject to the constraints

\[ \kappa = \frac{\rho^2}{4\gamma} - \sqrt{\gamma} (2\ell + 8\nu + 9) \]

\[ -2\sqrt{\gamma}z_1^3 + \left( 6\sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}} \right)z_1^2 \]

\[ + \left( 2\ell + 8\nu - \frac{3\rho^2}{\sqrt{\gamma}} - \frac{3\sqrt{\gamma}}{2} + 3 \right)z_1 \]

\[ + \left( 6\ell + 12\nu - \frac{3\rho^2}{4\sqrt{\gamma}} + 15 \right) + \frac{3E_1}{4} + 3\nu (2\ell + \nu + 2) = 0, \]

\[ 6E_1 - 12\nu (1 - \nu) - \frac{3\kappa}{4} = \]

\[ 4\sqrt{\gamma}z_1^3 + \left( 6\sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}} \right)z_1 - (4\ell + 16\nu - 3\sqrt{\gamma} + 6) \]

\[ + \frac{\rho}{\sqrt{\gamma}} \left( 6 + 6\nu + 2\ell - \frac{3\rho^2}{16} \right) - (2\ell + 5) \left( 4\nu - \frac{3\sqrt{\gamma}}{4} \right) \quad (4.10) \]

with the roots \( z_1 \) satisfying the equation

\[ -4\sqrt{\gamma}z_1^4 - \left( 12\sqrt{\gamma} + \frac{2\rho}{\sqrt{\gamma}} \right)z_1^3 + \left( 6 - 3\sqrt{\gamma} + 4\ell + 16\nu - \frac{6\rho}{\gamma} \right)z_1^2 \]

\[ + 3 \left( 6 + 4\ell + 8\nu - \frac{\rho}{2\sqrt{\gamma}} \right)z_1 + 3 \left( \ell + \frac{3}{2} \right) = 0 \quad (4.11) \]

By solving equations (4.10) and (4.11) simultaneously using the numerical Mathematica function NSolve, which is program to give a more complete set of solutions for multivariate nonlinear algebraic equations, we obtain the allowed values of the potential.
Figure 2. Potential plots and corresponding wavefunctions (inserts) for $H_2$ model with $(\omega, n, \ell) = (0.1, 2, 1)$.

Table 3. $E_{0,\ell}$ levels and allowed parameters $\nu$ and $\rho$ for $H_4$ model for $\gamma = 0.1$.

| $\ell$ | $\rho$  | $\kappa$ | $\nu$  | $E_{0,\ell}$ |
|--------|---------|-----------|--------|--------------|
| 1      | -0.407 08 | -1.878 83 | 0.031 43 | -0.506 83 |
| 2      | -0.464 11 | -2.432 04 | 0.049 21 | -1.190 70 |
| 3      | -0.516 25 | -2.968 66 | 0.061 82 | -1.993 46 |
| 4      | -0.566 59 | -3.491 53 | 0.072 38 | -2.916 24 |
| 5      | -0.616 08 | -4.001 81 | 0.081 94 | -3.959 76 |
| 6      | -0.665 14 | -4.499 84 | 0.090 92 | -5.124 48 |
| 7      | -0.713 97 | -4.985 79 | 0.099 55 | -6.410 77 |
| 8      | -0.762 67 | -5.459 73 | 0.107 95 | -7.818 94 |
| 9      | -0.811 30 | -5.921 66 | 0.116 19 | -9.349 26 |
| 10     | -0.859 91 | -6.371 58 | 0.124 32 | -11.0020 |
parameters. We observe that as we change the parameter $\gamma$ from 1 to 100, the total number of solutions (real and complex) changes while a change in the angular momentum $\ell$ alters the number of real solutions for any given $\gamma$. By plotting the energy values on the plane of the potential parameters ($\gamma - \nu - \rho$), as depicted in figure 4 where the dots represent the number of solutions with real energies, for different values of the parameters, one can easily see the multiplicity and distribution in the energy eigenvalues (and the corresponding wavefunction) for any given eigenstate.

To our knowledge, such behaviour has not been pointed out before within the context of (quasi-) exact solvability of quantum systems. Although such behaviour may appear rather uncommon within the context of QES systems, however, we note that this is merely a generalization of [23], where the authors showed that the complete spectrum for the $H_1$ model is only obtainable for specific values of the potential parameters ($\omega = 1/2$ and $g = 2$). Explicitly, we have demonstrated that the eigenstates for higher members of the family have multiple QES sectors which are parameter dependent. Moreover, one other interesting

Figure 3. Potential plots and corresponding wavefunctions (inserts) for $H_1$ model with $(\gamma, n, \ell) = (0.1, 0)$ and $\ell = 1, 5, 10$. 
characteristic of the solution is that for some points \((\gamma, n, \ell)\), take for instance \((1, 1, 1)\), the solutions space becomes entirely complex (in terms of the energy and the allowed potential parameters). Thus, at such points, the Hamiltonian cease to be hermitian but still QES.

Finally for \(n = 2\), we have the energy and wavefunction

\[
E_2 = 2\sqrt{\gamma}(z_1 + z_2) + 3\sqrt{\gamma}\left(2\nu + \ell + \frac{13}{2}\right) + \frac{\rho}{\sqrt{\gamma}}\left(2\nu + \frac{\ell}{2} + \frac{11}{4}\right) + \frac{3\rho^2}{8\gamma} + \frac{3\kappa}{2}
\]

\[
\Psi_1(r) \sim r^{\ell+1}\left(4r^4 + 12r^2 + 3\right)e^{-\frac{\sqrt{\gamma}}{4}r^4 - \frac{\rho^2}{4\sqrt{\gamma}}\left(r^2 - z_1\right)\left(r^2 - z_2\right)}
\]

\[(4.12)\]
subject to the constraints

\[ \kappa = \frac{\rho^2}{4\sqrt{\gamma}} - \sqrt{\gamma}(2\ell + 8\nu + 13) \]

\[ -2\sqrt{\gamma}(z_1^3 + z_2^3) - \left(6\sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}}\right)(z_1^2 + z_2^2) \]

\[ + \left(2\ell + 8\nu - \frac{3\rho}{\sqrt{\gamma}} - \frac{3\sqrt{\gamma}}{2} + 7\right)(z_1 + z_2) \]

\[ + 2\left(6\ell + 12\nu - \frac{3\rho}{4\sqrt{\gamma}} + 15\right) + \frac{3E_2}{4} + 3\nu(2\ell + \nu + 2) = 0, \]

\[ 6E_2 - 12\nu(1 - \nu) - \frac{3\kappa}{4} = \]

\[ 4\sqrt{\gamma}(z_1^2 + z_2^2) + \left(6\sqrt{\gamma} + \frac{\rho}{\sqrt{\gamma}}\right)(z_1 + z_2) - 2(4\ell + 16\nu - 3\sqrt{\gamma} + 10) \]

\[ + \frac{\rho}{\sqrt{\gamma}} \left(12 + 6\nu + 2\ell - \frac{3\rho^2}{16}\right) - (2\ell + 5)\left(4\nu - \frac{3\sqrt{\gamma}}{4}\right), \]

(4.13)

with the roots \( z_i, z_2 \) satisfying the equations

\[ \sum_{j \neq i}^2 \frac{2}{z_i - z_j} \]

\[ -4\sqrt{\gamma}z_i^4 - \left(12\sqrt{\gamma} + \frac{2\rho}{\sqrt{\gamma}}\right)z_i^3 + \left(6 - 3\sqrt{\gamma} + 4\ell + 16\nu - \frac{6\rho}{\sqrt{\gamma}}\right)z_i^2 \]

\[ + 3\left(6 + 4\ell + 8\nu - \frac{\rho}{2\sqrt{\gamma}}\right)z_i + 3\left(\ell + \frac{3}{2}\right) \]

\[ = \frac{z_i\left(4z_i^2 + 12z_i + 3\right)}{z_i}, \]

(4.14)

5. Conclusions

In summary, we have discussed the bound-state solutions to a family of isotonic oscillators based on the Hermite EOPs. We showed that the corresponding Schrödinger equation for the first two members of the family is reducible to QES differential equations. Using the Bethe ansatz approach, we systematically obtained the exact closed-form energies, wavefunctions and allowed potential parameters for these member oscillators.

We pointed out some interesting properties exhibited by these oscillators. In addition, extensive numerical computations reveal that member potentials have multiple QES eigenstates and the number of states for higher members are parameter dependent. It is pertinent to note that though our method gives general quasi-exact solutions for these models for all allowed values of the potential parameters, however, not all of them yield a physical solutions.
It would be interesting to extend the present work to the QES models which are based on type I, II or III Laguerre EOPs and two-step extensions of harmonic oscillator related to $X_{m_p,m_q}$. Hermite EOPs. Research along this path is underway, and the results will be reported elsewhere.

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Appendix. Generalized Bethe ansatz method

Here we give a description of the Bethe ansatz method [18]. We consider the second order differential equation

$$\left[ P(z) \frac{d^2}{dz^2} + Q(z) \frac{d}{dz} + W(z) \right] S(z) = 0, \quad (A.1)$$

where $P(z)$, $Q(z)$ and $W(z)$ are polynomials of degree ($t < s$)

$$P(z) = \sum_{k=0}^{t} p_k z^k, \quad Q(z) = \sum_{k=0}^{s} q_k z^k, \quad W(z) = \sum_{k=0}^{s} w_k z^k, \quad (A.2)$$

$p_k$, $q_k$ and $w_k$ are constants. If we seek a polynomial solution of the form

$$S(z) = \prod_{i=1}^{n} (z - z_i), \quad S(z) \equiv 1 \text{ for } n = 0, \quad (A.3)$$

then equation (A.1) becomes

$$\sum_{k=0}^{r} p_k z^k \sum_{i=1}^{n} \frac{1}{z - z_i} \sum_{j \neq i}^{n} \frac{2}{z_j - z_i} + \sum_{k=0}^{s} q_k z^k \sum_{i=1}^{n} \frac{1}{z - z_i} + \sum_{k=0}^{s} w_k z^k = -w_0, \quad (A.4)$$

where $\{z_i\}$ are distinct roots of the polynomial solution. The right-hand side of this equation is a constant, while the left-hand side (lhs) is a meromorphic function with simple poles $z = z_i$ and singularity at $z = \infty$. The residue at the simple pole $z = z_i$ are given as

$$\text{Res } (-w_0)_{z=z_i} = \sum_{k=0}^{r} p_k z^k \sum_{j \neq i}^{n} \frac{2}{z_j - z_i} + \sum_{k=0}^{s} q_k z^k, \quad (A.5)$$

such that

$$\sum_{k=0}^{r} p_k \sum_{i=1}^{n} \left( \frac{z_i - z_j}{z - z_i} \right)^i \sum_{j \neq i}^{n} \frac{2}{z_j - z_i} + \sum_{k=0}^{s} q_k \sum_{i=1}^{n} \left( \frac{z_i - z_j}{z - z_i} \right)^i \left( \frac{z_j - z_k}{z - z_j} \right) + \sum_{k=1}^{s} w_k z^k = -w_0 - \sum_{i=1}^{n} \text{Res } (-w_0)_{z=z_i} \frac{1}{z - z_i}, \quad (A.6)$$
If we define for $\nu \in \mathbb{Z}_+$,

$$
\mathcal{M}^\nu [z, z_i] = \frac{z^\nu - z_i^\nu}{z - z_i} = z^{\nu - 1} + z^{\nu - 2} z_j + \cdots + z_i z_j^{\nu - 2} + z_j^{\nu - 1} \quad (A.7)
$$

and

$$
S[j; j] = \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{z_i^\nu}{z - z_i} = \sum_{i=1}^{n} \sum_{j \neq i}^{n} z_i^{\nu - 1} + z_i^{\nu - 2} z_j + \cdots + z_i z_j^{\nu - 2} + z_j^{\nu - 1} \quad (A.8)
$$

then equation (A.6) reduces to

$$
2 \sum_{k=1}^{r} p_k S[\mathcal{M}^\nu [z, z_i]; z_j] + \sum_{k=1}^{s} q_k \sum_{i=1}^{n} \mathcal{M}^\nu [z, z_i]
+ \sum_{k=1}^{t} w_k z^k = -w_0 - \sum_{i=1}^{n} \text{Res} ( -w_0 z_i z_j ) / (z - z_i).
$$

(A.9)

For this equation to be valid, the right-hand side must also be a constant. By Liouville’s theorem, we demand that the coefficients of the powers of $z$ as well as the residues at the simple poles of the right-hand side be zero. As a result, one can evaluate the three terms on the lhs of equation (A.9) for some $k$, such that the sums of all possible coefficients of $z^k$ are equated to zero,

$$
2 p_k S[\mathcal{M}^\nu [z, z_i], z_j] + q_k \sum_{i=1}^{n} \mathcal{M}^\nu [z, z_i] + w_k z^k + w_0 = 0, \quad (A.10)
$$

where $t = 1, \ldots, s = 1$ (necessary for quasi-exact solutions), and the roots $\{z_i\}$ satisfy the Bethe ansatz equations

$$
\sum_{k=0}^{r} p_k z^k \sum_{j \neq i}^{n} \frac{2}{z_j - z_i} + \sum_{k=0}^{s} q_k z^k = 0. \quad (A.11)
$$

Finally, we note that if $P(z)$, $Q(z)$ and $W(z)$ are of degrees 4, 3 and 2 respectively, then the above procedure reduces to that discussed in [18].

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