Feynman diagrams and polylogarithms: shuffles and pentagons

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We summarize the Hopf algebra structure on Feynman diagrams and emphasize the interest in further algebraic structures hidden in Feynman graphs.

1. COMBINATORIAL RENORMALIZATION

There is a universal combinatorial Hopf algebra structure hidden in the process of renormalization. The universality of this structure can be conveniently understood if one considers the UV singularities of Feynman graphs from the viewpoint of configuration spaces. Then, the combinatorial structure of renormalization can be summarized as follows. Each Feynman diagram has various sectors which suffer from short-distance singularities. These sectors are stratified by rooted trees, from which the Hopf algebra structures of are obtained. Figure 1 gives an example. The corresponding Hopf algebra can be formulated on rooted trees or, equivalently, directly on Feynman graphs. Details of the relation to configuration spaces will be described elsewhere. Here, we want to use this representation to motivate further investigation in algebraic relations between Feynman graphs, and report some encouraging first results which will be discussed in much greater detail in future work.

Figure 1 essentially shows how the short distance singularities are located in sectors stratified by rooted trees. This is no surprise as the singularities are constrained to (sub-)diagonals. Along such subdiagonals, one essentially confronts the product of distributions with coinciding support.

This localization of singularities along diagonals conveniently allows to rely on suitable local subtractions, whose combinatorics can be de-

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Figure 1. A Feynman diagram with overlapping divergences. Its short distance (UV) singularities are stratified by two rooted trees as indicated. We can either describe the divergent sectors by rooted trees decorated with primitive diagrams, or in a manner closely related to the notations used by Fulton and MacPherson in the study of configuration spaces. In the latter case, singular sectors appear when either the three vertices \{2,3,4\} approach a diagonal first (with the final overall divergence appearing when vertex 1 approaches the same diagonal) or when the vertices denoted \{1,2,3\} come together first and then approach vertex 4. In all cases, the rooted trees which govern the Hopf algebra structure are the rooted trees with two vertices, here given in bold black lines.
scribed as Hopf algebra operations on the trees of Figure [2]. It can also be directly described in terms of Feynman graphs with coproducts of the form

\[ \Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum_{\gamma \in \Gamma} \gamma \otimes \Gamma/\gamma, \]

where the relation to rooted trees can be read off from the figure. Essentially, the sum over all rooted trees describing the various divergent sectors in the graph replaces the graph when we go from the Hopf algebra of graphs to the Hopf algebra of trees [2].

Here, \( \Gamma/\gamma \) corresponds to a suitable collapse of subgraphs to a point, which in the analytic expressions corresponding to the graphs, furnishes an appropriate insertions of local, polynomial, operators, a process which can be conveniently formulated by considering distributions on the space of external momenta of a graph [6].

Finally, the transition from an unrenormalized graph to a renormalized graph can be summarized in a succinct formula. It reads on graphs as the convolution

\[ S_R \ast \phi(\Gamma) = S_R(\Gamma) + \phi(\Gamma) + \sum_{\gamma \in \Gamma} S_R(\gamma)\phi(\Gamma/\gamma), \]

where \( R : V \to V \) specifies the renormalization scheme, and \( \phi, S_R : H \to V \) are algebra maps which assign to Hopf algebra elements, graphs, analytic expressions in \( V \) corresponding to the Feynman rules (\( \phi \)) and to the counterterm (\( S_R \)). \( S_R \) appears here as a twist of the map \( \phi \circ S \equiv S_{R=\text{id}_V} : H \to V \) given by

\[ S_R(\Gamma) = -R(\phi(\Gamma) + \sum_{\gamma \in \Gamma} S_R(\gamma)\phi(\Gamma/\gamma)). \]

Thus, we obtain a group structure due to the fact that the counit, coproduct and converse of the Hopf algebra provide a unit, a product, and an inverse for characters of the Hopf algebra.

This enables us to describe changes in renormalization schemes via an obvious generalization of Chen’s lemma [3]

\[ S_{R'} \ast \phi = S_{R'} \ast S_R \circ S \ast S_R \ast \phi, \]

and similarly for a change of the character \( \phi \),

\[ S_R \ast \phi' = S_R \ast \phi \ast \phi \circ S \ast \phi'. \]

There is a distinguished renormalization scheme which further enables us to identify the transition from the unrenormalized Green functions to the renormalized ones with the Birkhoff decomposition: the minimal subtraction scheme in dimensional regularization [3, 8]. This allowed Alain Connes and the author to gain further insight in the nature of renormalization: in its very essence, renormalization is about diffeomorphisms of physical observables. If one works this out for the simplest but generic instance of the diffeomorphism of the coupling constant, one confronts two Hopf algebra structures: the Hopf algebra delivered by the composition of diffeomorphisms [8] and the Hopf algebra structure of Feynman graphs, as we can regard the effective coupling constant as a series in graphs via \( g_{\text{eff}}(g) = gZ_1Z_3^{-3/2} \), and one ends up with the fact that this formula delivers a homomorphism of Hopf algebras. Here, \( Z_1 \equiv Z_1(g), Z_3 \equiv Z_3(g) \) are both to be regarded as invertible formal series in \( g \) with \( Z_1(0) = Z_3(0) = 1 \).

Upon the natural action of the rescaling group, this implies the ’t Hooft conditions which ensure the existence of a well-defined \( \beta \)-function, which is a pull-back of the natural one-parameter group of rescalings in the above group [3].

These Hopf algebras can be made into efficient algorithms [3] and to develop routines which possibly start at the level of Wick contractions and completely automate the BPHZ recursions for a given QFT remains as a desirable and achievable challenge for computational physics. Along the same lines, one should hope that in the long run the very fact that this Hopf algebra can be formulated in quite general set-theoretic terms will enable us in the future to succinctly approach asymptotic expansions and stratifications of regions [4] in conceptually satisfying and efficient manners, using appropriate conditions to identify the subgraphs of interest for a particular asymptotic expansion.

Having achieved in such a way a completely satisfactory description of the multiplicative subtraction mechanism which guarantees finiteness.
of renormalized Green functions, one is tempted to ask for more structure which follows from the disclosed Hopf algebra structure. This should lead us eventually to a consideration of the transcendent content of a field theory, towards its polylogarithmic content which plays such a prominent role in QFT as also this conference gives ample evidence.

2. SHUFFLES

It is not difficult to see that the Hopf algebra structure of Feynman diagrams allows to define shuffle algebras \([14]\). To this end, we can utilize the formulation in terms of rooted trees, which allows to write the coproduct as

\[
\Delta \circ B_+^{(x)}(X) = B_+^{(x)}(X) \otimes e + [\text{id} \otimes B_+^{(x)}] \Delta(X).
\]

Corresponding operators exist, due to a theorem in \([13]\), in the formulation in terms of Feynman graphs, and amount to plug the graphs \(X\) in the graph \(x\) in all possible ways. One can even consider operators \(B_+^{(x,y)}(X)\) which correspond to plug in the graph \(X\) at the place \(i\) in the primitive graph \(x\). Here, a place \(i\) can be a specified internal line (if \(X\) is a self-energy graph for that type of line) or vertex (if \(X\) is a vertex correction graph) of \(x\) which is replaced by \(X\).

One can now define a (quasi-)shuffle product \(\Box\) iteratively on graphs (for fixed chosen places say) by defining

\[
B_+^{(x)}(X) \Box B_+^{(y)}(Y) = B_+^{(x)}(X \Box B_+^{(y)}(Y)) + B_+^{(y)}(B_+^{(x)}(X) \Box Y) + B_+^{(x,y)}(X \Box Y).
\]

If a shuffle identity \(\phi(X \Box Y) = \phi(X) \phi(Y)\) holds up to \(n+1\) loops, then the shuffle product is well-defined at \(n+1\) loops. Here, \(C_2\) is a map which assigns to two primitive Feynman graphs a new one \([12]\). Such shuffle algebras are provided by iterated integrals, and hence in particular by polylogarithms as well as in the study of MZVs and Euler sums.

To obtain a well-defined shuffle algebra, one needs commutativity and associativity of this map. It is the latter which, in the case of Yukawa theory, was explicitly constructed and shown to hold up to finite parts \([13]\). The finite violation of associativity was only a function of the grading, the loop-numbers, \(n_x, n_y, n_z\), of the involved primitive graphs,

\[
C_2(x, C_2(y, z)) - C_2(C_2(x, y), z) = (n_x - n_z)C(x, y, z)
\]

with some constant \(C(x, y, z) = F(x)F(y)F(z)\) factorizing with respect to the primitive graphs \(x, y, z\). This ensures the pentagon relation of Figure 3.

It is an open question if at the next order in \((D - 4)\) a pentagon relation still holds or if a higher coherence law is needed. A full understanding of this question will provide valuable insight as to what extent Feynman diagrams might
Figure 3. The pentagon relation. There are two ways to go from $C_2(C_2(x_1, x_2), x_3, x_4) \equiv ((12)3)4$ to $C_2(x_1, C_2(x_2, C_2(x_3, x_4))) \equiv 1(2(34))$. Both should agree up to finite parts. Here, a shift of a pair of brackets is accompanied by the indicated difference in loop numbers which add to the same total in both ways.

generalize the algebraic structures of the polylogarithm. Here is not the place to go into details of the structure of polylogarithm, which is a quite fascinating subject in its own right \[\text{[13]}\], also for a physicist \[\text{[14]}\]. The polylogarithm, as an iterated integral, fulfills shuffle identities. But the polylogarithm also relates to the Grothendieck Teichmüller group, and the remarks in the next section can be regarded as the first cautious steps to investigate similar structure in Feynman graphs.

3. MORE STRUCTURE

Figure \[1\] by itself suggest to investigate more structure. The Lie algebra structure which comes along with the Hopf algebra of Feynman graphs is determined by a Lie bracket $[\Gamma_1, \Gamma_2]$ which sums over all possible ways of plugging a graph $\Gamma_1$ in $\Gamma_2$ and subtracts doing it the other way round \[\text{[6]}\]. Figure \[2\] gives an example. There is a more basic operation involved in this process, the insertion of a graph $\Gamma_1$ at a chosen internal line or vertex of $\Gamma_2$. In the Lie bracket, we involve only the sum over all internal such places, and are thus completely insensitive to structures which depend on these places. There is an obvious operad structure involved when one labels internal lines and vertices, and it is of interest to investigate Feynman graphs with respect to this operad structure. Figure \[3\] gives an example of plugging in a Feynman graph at two different places in a graph. One of the simplest such instances can be obtained by letting $\Gamma_2$ be a one-loop vertex function in massless $\phi^3$-theory in six dimensions at zero momentum transfer, and $\Gamma_1$ be some self-energy graph in this theory. Then, there are two different places

\[[\begin{array}{c} \text{vertex correction} \\ \text{self-energy graph} \end{array}] = 2\]

Figure 4. The bracket of two graphs, a vertex correction and a self-energy graph in this example. The self-energy graph has two internal vertices. Both can be replaced by the vertex correction, which delivers twice the first Feynman graph in the second row. The other way round, any internal line of the vertex correction can be dressed by the self-energy graph, delivering the remaining three graphs of the final row.

\[\begin{array}{c} \text{self-energy at the specified internal line 2 in a vertex correction.} \end{array}\]

Figure 5. Insertion of a self-energy at the specified internal line 2 in a vertex correction.
Figure 6. These two graphs are only distinguished by the places into which subdivergences are plugged, and their difference has only a first order pole. They correspond to different topologies, but have similar renormalization parts. The difference in their topology is apparent when one reads them as chord diagrams, providing different Gauss codes \{1, 1, 2, 2, 3, 3\} and the rational (ladder) code \{1, 2, 3, 3, 2, 1\} \[15\]. Upon evaluation, their difference is typically proportional to \(\zeta(3)/z\).

As \(p\) is finite, plugging a divergent subgraph into it still cannot generate a negative part in the Birkhoff decomposition, while plugging the so constructed graph as a subgraph in a primitive graph delivers a new primitive element. Its evaluation tests out the topological differences between two graphs which have the same substructure with regard to their renormalization parts, as was already observed some time ago [15], with the most striking observation being that this measure of the difference in the topology is proportional to \(\zeta(3)\).

In [6] Alain Connes and the author presented a nice geometrical picture for the Birkhoff decomposition of unrenormalized graphs. In light of the kinship expressed in Figure 1 between singularities in Feynman graphs and the study of generalized functions on configuration spaces, algebraic relations like the above will connect naturally to structures familiar from Grothendieck Teichmüller groups, and we will investigate these connections in the future in a hope to clarify the role which the polylogarithm plays in quantum field theory, in particular also with regard to gauge theories.

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