Collision of two general geodesic particles around a Kerr-Newman black hole

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Abstract

We study collision of two general geodesic particles around the Kerr-Newman (KN) black hole and get the center-of-mass (CM) energy of the non-marginally and marginally bound critical particles in the direct collision and LSO collision scenarios. We find the constraint conditions that arbitrarily high CM energy can be obtained for the near-horizon collision of two general geodesic particles in the extremal KN black hole, and note that the charge decreases the value of the latitude in which arbitrarily high CM energy can occurs. We also interpret why the high-velocity collision belt centers to the equator with the increase of the charge.

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I. INTRODUCTION

Banados, Silk and West (BSW) [1] showed recently that the extremal Kerr black hole surrounded by relic dark matter density spikes could be regarded as a Planck-energy-scale collider, which might allow us to explore ultra high energy collisions and astrophysical phenomena, such as the gamma ray bursts and the active galactic nuclei. At the same time, several authors [2–4] pointed out that the ultra-energetic collisions cannot occur near the black hole in nature due to the astrophysical limitation. For the kerr black hole, to circumvent the fine-tuning problem to obtain the CM energy for the collision, different scenarios, e.g. multiple scattering [5, 6], the innermost stable circular orbit (ISCO) [7] and the last stable orbit (LSO) [8, 9], was proposed by several author. In Refs. [10–12], the author elucidated the universal property of acceleration of particles for rotating black holes and try to give a general explanation of this BSW mechanism. The similar BSW mechanism has also been found in other kinds of black holes and special spacetime, such as stringy black hole [13], Kerr-Newman black holes [14] and Kaluza-Klein Black Hole [15], Kerr-Taub-NUT spacetime [16], and naked singularity [17–19]. The BSW mechanism also stimulated some implications concerning the effects of gravity generated by colliding particles [20], the emergent flux from particle collision near the Kerr black holes [21], and the numerical estimation of the escaping flux of massless particles created in collisions around the Kerr black hole [22].

Recently, Harada [8] generalized the analysis of the CM energy of two colliding particles to general geodesic massive and massless particles in the Kerr black hole. They showed that, in the direct collision and LSO collision scenarios, the collision with an arbitrarily high CM energy can occur near the horizon of maximally rotating black holes not only at the equator but also on a belt centered at the equator. In this paper, we will extend Harada’s work to the KN black hole. Besides the rotation parameter $a$, the KN black hole has another parameter, the charge $q$. Thus, we will demonstrate what effects of the charge on the CM energy for the particles in the near-horizon collision of the general geodesic particles. And we also will give an interpretation why the high-velocity collision belt centered at the equator with the increase of the charge.

This paper is organized as follows. In Sec. II, we briefly review general geodesic particles in the KN spacetime. In Sec. III, we obtain an expression for the CM energy of two general
geodesic particles at any spacetime point and then obtain a general formula for the near-horizon collision. We also discuss the collision with an arbitrarily high CM energy in the direct collision and LSO collision scenarios, and see what effects of the charge on the CM energy for the particles in the near-horizon collision of the general geodesic particles. Sec. IV is devoted to a brief summary. We use the units $c = G = 1$ throughout the paper.

II. GENERAL GEODESIC ORBITS IN KERR-NEWMAN SPACETIME

The metric of the KN spacetime in the Boyer-Lindquist coordinates can be expressed as

$$ds^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[ a dt - (r^2 + a^2) d\phi \right]^2$$

$$- \frac{\Delta}{\rho^2} \left[ dt - a \sin^2 \theta d\phi \right]^2,$$

where $\Delta = r^2 - 2r + a^2 + q^2$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, and $M$, $a$ and $q$ are the mass, the rotation parameter and the electric charge. The event horizon of the KN black hole is given by $r_H = M + \sqrt{M^2 - a^2 - q^2}$, and the extremal case corresponds to the condition $a^2 + q^2 = M^2$. The angular velocity of the KN black hole is

$$\Omega_H = \frac{a}{(r_H^2 + a^2)} = \frac{a}{(2M^2 - q^2 + 2M \sqrt{M^2 - q^2 - a^2})}. \quad (2.2)$$

The nonvanishing contravariant components $g^{\mu\nu}$ of the metric are

$$g^{tt} = -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta}, \quad g^{t\phi} = g^{\phi t} = -\frac{2Mar - q^2}{\rho^2 \Delta},$$

$$g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\theta \theta} = \frac{1}{\rho^2}, \quad g^{\phi \phi} = \frac{\Delta - a^2 \sin^2 \theta}{\Delta \rho^2 \sin^2 \theta}. \quad (2.3)$$

The general geodesic motion of massive particles in the KN spacetime was analyzed in Refs. [23, 24]. So we here briefly review general geodesic particles in the KN spacetime. The Hamiltonian for the geodesic motion is given by

$$\mathcal{H}[x^\alpha, p_\beta] = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu,$$

where $p_\mu$ is the conjugate momentum to $x^\mu$. Let $S = S(\lambda, x^\alpha)$ be the action as a function of the parameter $\lambda$ and coordinates $x^\alpha$, the conjugate momentum $p_\alpha$ is described by $p_\alpha = \frac{\partial S}{\partial x^\alpha}$. Then the corresponding Hamilton-Jacobi equation is

$$- \frac{\partial S}{\partial \lambda} = \mathcal{H} \left[ x^\alpha, \frac{\partial S}{\partial x^\alpha} \right] = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}. \quad (2.4)$$
For the KN black hole, the action can be expressed as
\[
S = \frac{1}{2} m^2 \lambda - Et + L\phi + \sigma_r \int^r \frac{\sqrt{R}}{\Delta} + \sigma_\theta \int^\theta \sqrt{\Theta},
\]
(2.5)
where the constants \( m \), \( E \) and \( L \) are the rest mass, energy and angular momentum of the particle. The sign functions \( \sigma_r = 1(-1) \) and \( \sigma_\theta = 1(-1) \) correspond to the outgoing (ingoing) geodesics. From Eq. (2.4) we can obtain
\[
\rho^2 \frac{dt}{d\lambda} = -a(aE \sin^2 \theta - L) + \frac{(r^2 + a^2)P}{\Delta},
\]
(2.6)
\[
\rho^2 \frac{dr}{d\lambda} = \sigma_r \sqrt{R},
\]
(2.7)
\[
\rho^2 \frac{d\theta}{d\lambda} = \sigma_\theta \sqrt{\Theta},
\]
(2.8)
\[
\rho^2 \frac{d\phi}{d\lambda} = - \left( aE - \frac{L}{\sin^2 \theta} \right) + \frac{aP}{\Delta}.
\]
(2.9)
with
\[
\Theta = \Theta(\theta) = Q - \cos^2 \theta \left[ a^2 (m^2 - E^2) + \frac{L^2}{\sin^2 \theta} \right],
\]
\[
R = R(r) = P(r)^2 - \Delta(r) \left[ m^2 r^2 + (L - aE)^2 + Q \right],
\]
\[
P = P(r) = (r^2 + a^2)E - aL.
\]
(2.10)
where \( Q \) is the Carter constant [23]. Then the radial equation for the timelike particle moving along geodesics is
\[
\frac{1}{2} u^r u^r + V_{\text{eff}}(r) = 0,
\]
(2.11)
where the effective potential is defined by
\[
V_{\text{eff}}(r) = - \frac{R(r)^2}{2\rho^4}.
\]
(2.12)
The circular orbit of the particle can be found by the conditions
\[
V_{\text{eff}}(r) = 0, \quad \frac{dV_{\text{eff}}(r)}{dr} = 0.
\]
(2.13)
Because we are interested in causal geodesics, we also need to impose the condition \( \frac{dt}{d\lambda} > 0 \).
As \( r \to r_H \) for the timelike particle, this condition reduces to
\[
E \geq \frac{aL}{2(a^2 + r_H^2)} = \Omega_H L,
\]
which shows us that the angular momentum must be equal to or smaller than the critical value \( L_c \equiv \Omega_H^{-1} E \).
III. CM ENERGY OF TWO COLLIDING GENERAL GEODESIC PARTICLES IN KN SPACETIME

In this section, we will study the CM energy for the collision of two particles moving along general geodesic in the KN spacetime and the high-velocity collision belts on the extremal KN black hole.

A. CM energy of two colliding particles in KN spacetime

Let us now consider two uncharged colliding particles with rest masses $m_1$ and $m_2$. We assume that two particles 1 and 2 are located at the same spacetime point with the four momenta $p_{(i)}^a = m_{(i)} u_{(i)}^a$. The CM energy $E_{cm}$ of the two particles is shown by

$$E_{cm}^2 = m_1^2 + m_2^2 - 2g^{ab} p_{(1)a} p_{(2)b}. \quad (3.1)$$

With the help of Eqs. (2.3) and (3.1), the CM energy of two colliding general geodesic particles in the KN spacetime is

$$E_{cm}^2 = m_1^2 + m_2^2 + \frac{2}{\rho^2} \left[ \frac{P_1 P_2 - \sigma_1 \sqrt{R_1} \sigma_2 \sqrt{R_2}}{\Delta} - \frac{(L_1 - a \sin^2 \theta E_1)(L_2 - a \sin^2 \theta E_2)}{\sin^2 \theta} \right]. \quad (3.2)$$

Now we will investigate the properties of the CM energy as the radius $r$ approaches to the horizon $r_H$ of the non-extremal black hole. $\sigma_1$ and $\sigma_2$ have the same sign on the horizon $r = r_H$. Note that both denominator and the numerator of the fraction $\frac{P_1 P_2 - \sigma_1 \sqrt{R_1} \sigma_2 \sqrt{R_2}}{\Delta}$ on the right-hand side of Eq. (3.2) vanishes at $r_H$. Using l’Hospital’s rule and taking into account $r_H^2 - 2r_H - a^2 - Q^2 = 0$, the CM energy of two general geodesic particles in the near-horizon limit can be expressed as

$$E_{cm}^2(r_H) = m_1^2 + m_2^2 + \frac{1}{\rho_{\gamma H}^2} \left[ (m_1^2 r_H^2 + K_1) \frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} + (m_2^2 r_H^2 + K_2) \frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2} \right]
- \frac{2(L_1 - a \sin^2 \theta E_1)(L_2 - a \sin^2 \theta E_2)}{\sin^2 \theta} - 2\sigma_1 \sqrt{\Theta_1} \sigma_2 \sqrt{\Theta_2}, \quad (3.3)$$

where $K_i = Q_i + (L_i - a E_i)^2$ for the particles $i$. We can now find that the necessary condition to obtain an arbitrarily high CM energy is that $E_i - \Omega_H L_i = 0$, i.e., either of the two particles must possess the critical angular momentum $L_{i c} = \frac{E_i}{\Omega_H}$. 

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For the colliding particles with the same rest mass \( m_0 \) moving on the equatorial plane, Eq. (3.3) reduces to

\[
E_{\text{cm}}^2(r_H) = 2m_0^2 + \frac{1}{r_H^2} \left\{ \left[ m_0^2 r_H^2 + (L_1 - aE_1)^2 \right] \frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} - \left[ m_0^2 r_H^2 + (L_2 - aE_2)^2 \right] \frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2} - 2(L_1 - aE_1)(L_2 - aE_2) \right\}.
\]

(3.4)

If we further assume that the colliding particles have the same energy \( E_1 = E_2 \), Eq. (3.4) becomes

\[
\frac{E_{\text{cm}}(r_H)}{2m_0} = \sqrt{1 + \frac{(L_1 - L_2)^2}{(L_1 - L_c)(L_2 - L_c) 4a}}.
\]

which coincides with the result in Ref. [14].

**B. The high-velocity collision belts in extremal KN black hole**

We are now in the position to study the collision of two particles with an arbitrarily high CM energy. In Ref. [8], T. Harada and M. Kimura divide the collision scenario into four types according to the effective potentials for the critical particles: The first type is the direct collision with the conditions \( R'(r_H) = 0, R(r_H) = 0, \) and \( R''(r_H) > 0; \) the second one is LSO [9] collision with the conditions \( R'(r_H) = 0, R(r_H) = 0, \) and \( R''(r_H) = 0; \) the third one is multiple scattering with the conditions \( R'(r_H) = 0, R(r_H) = 0, \) and \( R''(r_H) < 0; \) and the fourth one is also multiple scattering but with the condition that \( R'(r_H) = 0 \) and \( R''(r_H) < 0. \) Here we take Harada-Kimura’s classification and concentrate our attention on the direct collision and LSO collision scenarios.

For the critical particles defined by its angular momentum \( L_c = \frac{E}{\Omega_H} \), from Eq. (2.10), we can easily find that the condition \( R(r_H) = 0 \) is satisfied by both the nonextremal and extremal black holes. However, for the first derivative, \( R'(r_H) = -2(r_H - 1)(2m^2 r_H^2 + K) \), the condition \( R'(r_H) = 0 \) holds only for the extremal KN spacetime, and the condition \( R'(r_H) < 0 \) is true for the nonextremal KN black hole which shows that direct collision and LSO collision scenarios for the critical particles in the nonextremal KN black hole do not exist. Therefore, we only consider the critical particles in the extremal KN black hole. Using Eq. (2.10), \( R''(r_H) \) becomes

\[
R''(r_H) = 2 \left[ \left( 4 - \frac{M^2}{a^2} \right) E^2 - m^2 \right] M^2 - Q.
\]

(3.5)
Then, $R''(r_H) \geq 0$ shows that
\[
\left(4 - \frac{M^2}{a^2}\right)E^2 - m^2 \right) \geq Q. \tag{3.6}
\]

From Eq. (2.10), we find the following condition
\[
\cos^2 \theta \left[ a^2(m^2 - E^2) + \frac{(M^2 + a^2)^2 E^2}{a^2 \sin^2 \theta} \right] \leq Q \leq \left(4 - \frac{M^2}{a^2}\right)E^2 - m^2 \right) M^2, \tag{3.7}
\]
where $L_c = \frac{(M^2 + a^2)E}{a}$ was used. Using Eq. (3.7) and taking $M^2 = q^2 + a^2$ for the extremal black hole, the following condition must be satisfied
\[
(m^2 - E^2) \sin^4 \theta + \left( (7E^2 - 2m^2)(a^2 + q^2) + a^2 E^2 \right) \sin^2 \theta - \frac{(q^2 + 2a^2)^2 E^2}{a^2} \geq 0. \tag{3.8}
\]

For the marginally bound orbit $m^2 = E^2$, we can find
\[
\sin \theta \geq \sqrt{\frac{(q^2 + 2a^2)^2}{a^2(6a^2 + 5q^2)}}. \tag{3.9}
\]

The result confirms that charge $q$ of the black hole indeed influences on the angle of the collision of two general critical particles. We obtain $\sin \theta \geq \sqrt{\frac{2}{3}}$ when $q = 0$, which coincides with the Kerr case [8]. If we set $M = 1$, the right hand of the inequality (3.9) becomes \[
\frac{(2-q^2)^2}{(1-q^2)(6-q^2)}, \]
which monotonically increases with the charge $q$. Therefore we can get the maximum charge $q$ when $\sin \theta = 1$, i.e., $q = \sqrt{\frac{2}{3}}$ (the corresponding $a$ is $\frac{1}{\sqrt{3}}$). From the Fig. 1, we find that the highest latitude $(\frac{\pi}{2} - \theta)$ decreases with the increase of the charge $q$.

For the bound ($m^2 - E^2 > 0$) and the unbound ($m^2 - E^2 < 0$) particles, we can find that $\theta$ must satisfy
\[
\sin \theta \geq \sqrt{-B + \sqrt{B^2 - 4AC}} \over 2A, \tag{3.10}
\]
where $A = m^2 - E^2, \; B = (7E^2 - 2m^2)(a^2 + q^2) + a^2 E^2, \; C = -\frac{(q^2 + 2a^2)^2 E^2}{a^2}$. The highest absolute value of the latitude is shown in Fig. 2 as a function of the specific energy of the particle. We find that the charge $q$ decreases the highest value of the latitude in which the arbitrarily high CM energy can occurs. Thus, the highest value of the latitude $\alpha$ of both the bound and unbound critical particles is given as
\[
\alpha(E, m) = \arccos \sqrt{-B + \sqrt{B^2 - 4AC}} \over 2A. \tag{3.11}
\]
FIG. 1: The variation of the highest value of the latitude for the critical particles with charge $q$ of the extremal KN black hole in the marginally bound orbit. Here we set $M = 1$.

FIG. 2: The variation of the highest value of the latitude for the critical particles with the energy of the particles in the non marginally bound orbit with different $q$ and $a$. Here we set $M = 1$ and $m = 0.2$. 
FIG. 3: The variation of the highest value of the latitude for the critical massless particles with charge $q$ of the extremal KN black hole in the non marginally bound orbit. Here we take $M = 1$.

In the limit $E \to \infty$, Eq. (3.10) gives

$$\sin \theta \geq \sqrt{\frac{(8a^2 + 7q^2) - \sqrt{a^2(8a^2 + 7q^2)^2 - 4(2a^2 + q^2)^2}}{2}}.$$  \hspace{1cm} (3.12)

Then we can get the constraint on the value of the rotation parameter $a$ and charge $q$: $1 \geq a \geq \frac{1}{2}$, $0 \leq q \leq \frac{\sqrt{3}}{2}$. For a massless particle with critical angular momentum, we can also get the same result. In Fig. 3, we also find that with the increase of the charge $q$ the highest value of the latitude for the critical massless particles decreases.

When the charge $q \to 0$, the inequality (3.12) reduces to the result in Ref. 8

$$\sin \theta \geq \sqrt{3} - 1.$$  \hspace{1cm} (3.13)

It is interesting to note that the constraint conditions for the charge $q$ and rotation parameter $a$ are $0 \leq q \leq \frac{\sqrt{2}}{3}$, $1 \geq a \geq \frac{1}{3}$ for marginally bound particles, which are different from $0 \leq q \leq \frac{\sqrt{3}}{2}$, $1 \geq a \geq \frac{1}{2}$ for non-marginally bound critical particles.

C. Interpretation of constrains to value of $a$ and $q$ in extremal KN black hole

Now we try to give an interpretation of the constrains on the value of rotation parameter $a$ and the charge $q$. According to the Penrose mechanism, in general, the BSW mechanism
allows rotational energy of a rotating black hole to be extracted by scattered particles escaping from the ergosphere to infinity, and the ergosphere will become thinner as the energy extracted from the black hole. Thus, the thickness of the ergosphere plays an important role in the process of obtaining high CM energy. The fact shows us that the constrains on parameters $a$ and $q$ to obtain arbitrarily high CM energy corresponds to the constrains on the thickness of the ergosphere.

The infinite redshift surface of the KN black hole is

$$r^\infty_\pm = M \pm \sqrt{M^2 - q^2 - a^2 \cos^2 \theta}.$$ 

The ergosphere is the region bounded by the event horizon $r_H$ and the outer stationary limit surface $r^\infty_+$. The thickness of the ergosphere for the extremal KN black hole is

$$H \equiv r^\infty_+ - r_H = a \sin \theta. \tag{3.14}$$

Using Eqs. (3.8) and (3.10), we can get the minimum thickness of the ergosphere for the marginally bound and massless particle with the critical angular momentum to obtain arbitrary high CM energy. The minimum thickness of the ergosphere is shown in Fig. 4.

![Figure 4: The variation of the minimum thickness of the ergosphere $H_{min}$ with charge $q$ for the marginally bound critical particle (left) and massless critical particle (right) for the extremal KN black hole.](image)

From the Fig. 4 we find that the minimum thickness of the ergosphere satisfies $H_{min} \geq \frac{1}{\sqrt{3}}$ for marginally bound particles. However, it satisfies $H_{min} \geq \frac{1}{2}$ for the massless particles.
FIG. 5: the variation of the minimum thickness of the ergosphere with the specific energy vs the charge $q$ and rotation parameters $a$ of the extremal KN black hole in the non marginally bound orbit, we set $M = 1, m = 0.2$.

For the non marginally bound particle, the variation of $H_{min}$ as the function of specific energy with different charge $q$ is shown in Fig. 5, which shows that the minimum thickness of the ergosphere decreases with increase of the charge $q$.

In fact, when the extremal KN black hole carries more the charge $q$, the rotation energy is more less. The collision with an arbitrarily high CM energy must have enough rotation energy. Thus, the high-velocity collision belt centers at the equator with the increase of the charge $q$ because the equator has the maximum thickness of the ergosphere $H_{max} = a$ and the rotation energy. When the charge $q \rightarrow \frac{\sqrt{3}}{2}$ for the non marginally bound particle, the high-velocity collision only occurs at the equator where the maximum thickness of the ergosphere $H_{max} = a$ which coincides with the minimum thickness $H_{min}$. Above interpretation is also true for the marginally bound particle. Therefore, we argue that the constrains on the value of rotation parameter $a$ and the charge $q$ in order to obtain arbitrarily high CM energy is corresponding to the constrains on the thickness of the ergosphere.
IV. SUMMARY

We studied collision of two general geodesic particles and get the formula for the CM energy of the non-marginally and marginally bound critical particles in the direct collision and LSO collision scenarios. Our study showed that arbitrarily high CM energy can be obtained for the near-horizon collision of two general geodesic particles in the extremal KN black hole under the two conditions: (1) either of the impingement particles has the critical angular momentum $L_c = \frac{E}{\Omega_H}$; and (2) the charge $q$ and rotation parameter $a$ satisfy $0 \leq q \leq \sqrt{\frac{3}{2}}$, $1 \geq a \geq \sqrt{\frac{1}{3}}$ for marginally bound particles, and $0 \leq q \leq \frac{\sqrt{3}}{2}$, $1 \geq a \geq \frac{1}{2}$ for non-marginally bound critical particles. We find that the presence of the charge $q$ will decrease the value of the latitude in which arbitrarily high CM energy can occurs. Finally, we present an interpretation why the high-velocity collision belt centers to the equator with the increase of the charge $q$, i.e, the constrains on the value of $a$ and $q$ in order to obtain arbitrarily high CM energy is corresponding to the constrain on the thickness of the ergosphere.

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