A Simpler Scaling Algorithm for Weighted Matching in General Graphs

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Abstract

We present a new scaling approach for the maximum weight perfect matching problem in general graphs, with running time $O((m + n \log n)\sqrt{n \log(nN)})$, where $n$, $m$, $N$ denote the number of vertices, number of edges, and largest magnitude of integral costs. Comparing with the complicated long-standing algorithm by [Gabow and Tarjan 1991] of running time $O(m\sqrt{n \log n \log(nN)})$, our algorithm not only has a better time bound when $m = \omega(n \sqrt{\log n})$, but is also dramatically simpler to describe and analyze. Our algorithm also matches the time bound $O(m\sqrt{n \log(nN)})$ of maximum weight perfect matching for bipartite graphs [Gabow and Tarjan 1989] when $m = \Omega(n \log n)$.

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1 Introduction

Graph matching is one of the most fundamental problems in combinatorial optimization. A matching \( M \) in a graph \( G \) is a set of edges without common vertices. A vertex associated with an edge in the matching is called matched, otherwise it is free. A matching in which all vertices are matched is called a perfect matching. In a weighted graph, the maximum weight matching (MWM) is a matching maximizing the sum of weights of matching edges, and the maximum weight perfect matching (MWPM) is a prefect matching with maximum weight, which is equivalent to the minimum cost perfect matching. The MWM and MWPM are reducible to each other.\[6\]. The unweighted version of maximum matching is called maximum cardinality matching (MCM). There are also results reducing MWM to \( N \) instances of MCM, see \[20, 26, 19\].

Although the maximum matching problem has been studied for decades, the computational complexity of finding an optimal matching remains quite open. In 1965, Edmonds presented elegant polynomial time algorithms for MCM \[8\] and MWPM \[7\] in general graphs. Early implementations of Edmonds’s algorithm required \( O(n^3) \) time \[11, 22\] using elementary data structures. Following the approach of Hopcroft and Karp’s MCM algorithm for bipartite graphs \[18\], Micali and Vazirani \[23\] presented an MCM algorithm for general graphs running in \( O(m\sqrt{n}) \) time. (See Vazirani’s recent report \[30\].)

For maximum weight (perfect) matching, the implementation of the Hungarian algorithm \[21\] using Fibonacci heaps \[10\] runs in \( O(mn + n^2\log n) \) time in bipartite graphs, a bound that is matched in general graphs by Gabow \[13\] using more complex data structures. Faster algorithms are known when the edge weights are nonnegative integers in \([-N, \ldots, N]\), where a word RAM model is assumed, with \( \log(\max\{N, n\}) \)-bit words. Gabow and Tarjan \[15, 16\] gave bit-scaling algorithms for MWPM running in \( O(m\sqrt{n}\log(nN)) \) time in bipartite graphs and \( O(m\sqrt{n}\log n\log(nN)) \) time in general graphs.\[3\] However, Gabow and Tarjan’s scaling algorithm on general graphs is very complicated and difficult to analyze. In this paper we give a much simpler approach which is potentially applicable in real applications.

Extending \[24\], Sankowski \[28\] gave an \( O(Nn^\omega) \) \[1\]MWPM algorithm for bipartite graphs. Recently, Huang and Kavitha \[19\] and Cygan, Gabow and Sankowski \[3\] obtained a similar time bound for MWM and MWPM in general graphs, respectively. There are also results for approximate maximum weight matching in linear time, see \[4, 27, 6\].

The MWPM problem in general graphs has many applications, such as Christofide’s approximate algorithm for travelling salesman problem \[1\], the exact algorithm for the Chinese postman problem \[39\], and Gabow and Sankowski’s new algorithm for single source shortest path problem \[14\].

Our Result We first present a new scaling approach for the MWPM problem in general graphs which runs in \( O((m + n\log n)\sqrt{n}\log(nN)) \) time. Without loss of generality, we assume that the edge weights are nonnegative integers in the range \([0, \ldots, N]\). As we know, the most difficult part for scaling approaches in general graphs is how to deal with the blossoms formed in the previous scale. We treat blossoms by their sizes. If a root blossom has at most \( \sqrt{n} \) vertices, we can dissolve it with an Edmonds’ search with time \( O((m(B) + n(B)\log n(B))n(B)) \) by \[13\], where \( m(B) \) and \( n(B) \) are the numbers of the edges and vertices in the blossom.
For large blossoms with more than \( \sqrt{n} \) vertices, we need to bound their total \( z \)-value to linear in order to bound the running time after dissolving them directly, since dissolving one blossom \( B \) directly will lead a matching edge to have a \( z(B)/2 \) violation of the tightness condition. However, if in every scale we only run the dual adjustment step \( \Omega(\sqrt{n}) \) times and leaves \( O(\sqrt{n}) \) vertices free, it is easy to the bound the total \( z \)-value of large blossoms to \( O(n) \). For every free vertex at the end of each scale, we can make it matched by adding an artificial edge and an artificial vertex. The artificial edges and vertices simplify our analysis as we can get a perfect matching after every scale. We can make them free again after the last scale by deleting artificial edges and vertices. Then we can run the Edmonds’ search from these vertices to find the real maximum weight perfect matching.

**Organization** Section 2 introduces basic notations of matching and blossoms. Section 3 presents our main algorithm. In Section 3.1, we define the dual variables and relaxed complementary slackness, and in Section 3.2, we give the description of two versions of Edmonds’ search. The main procedure will be described in Section 3.3 while we will prove its correctness in Section 3.4 and analyze its running time in Section 3.5.

### 2 Definitions and Basics

The input is a graph \( G = (V, E, \hat{w}) \) where \( |V| = n, |E| = m \), and \( \hat{w} \) is an integer in the range \([0, \cdots, \mathbb{N}]\). A matching is a set of vertex-disjoint edges. A vertex is free if it is not adjacent to an \( M \) edge. An alternating path is a path whose edges alternate between \( M \) and \( E \setminus M \). An alternating path \( P \) is augmenting if there is one more non-matching edges than matching edges in \( P \), that is, both ends of \( P \) are free vertices. We define the weight of a matching \( M \) w.r.t. edge weights \( w \) to be \( w(M) = \sum_{e \in M} w(e) \).

As in [15, 16], we change the original graph weight to \( \bar{w} \) where \( \bar{w}(e) = (n + 1)\hat{w}(e) \) for all edge \( e \in E \). So all edge weights \( \bar{w}(e) \) are integer multiples of \( (n + 1) \), and we transform them to binary representation with \( \lceil \log((n + 1)\mathbb{N}) \rceil \) bits. There are \( \lceil \log((n + 1)\mathbb{N}) \rceil \) scales. If we have found a perfect matching \( M \) such that \( \bar{w}(M) \geq \bar{w}(M^*) - n \) for a maximum weight perfect matching \( M^* \), then \( M \) is also an MWPM w.r.t. \( \bar{w} \) as well as \( \hat{w} \).

For simplicity, in each scale we initialize every \( y \)-value to be zero by changing edge weights, so we need the following lemma.

**Lemma 2.1.** For any vertex \( u \in V \), decreasing the weights of all edges \( (u, v) \) adjacent to \( u \) by the same amount will not change the MWPM.

**Proof.** Since the weights of all perfect matchings decrease by the same amount, the lemma holds.\[\square\]

### 2.1 Blossoms

Each scale of the algorithm maintains a dynamic set \( \Omega \) of nested blossoms as in Edmond’s algorithm [7, 8]. Blossoms are formed inductively as follows. If \( v \in V \) then the set \( \{v\} \) is a trivial blossom. An odd length sequence \( (A_0, A_1, \ldots, A_\ell) \) forms a nontrivial blossom \( B = \bigcup_i A_i \) if the \( \{A_i\} \) are blossoms and there is a sequence of edges \( e_0, \ldots, e_\ell \) where \( e_i \in A_i \times A_{i+1} \) (modulo \( \ell + 1 \)) and \( e_i \in M \) if and only if \( i \) is odd, that is, \( A_0 \) is incident to unmatched edges \( e_0, e_\ell \). The base of blossom \( B \) is the base of \( A_0 \); the base of a trivial blossom is its only vertex. The set of blossom edges
$E_B$ are \{$e_0, \ldots, e_\ell$\} and those used in the formation of $A_0, \ldots, A_\ell$. The set $E(B) = E \cap (B \times B)$ may, of course, include many non-blossom edges. A short proof by induction shows that $|B|$ is odd and that the base of $B$ is the only unmatched vertex in the subgraph induced by $B$. The vertex set $V(B)$ is the set of vertices in blossom $B$ and all of its descendants. We also let $n(B) = |V(B)|$ and $m(B) = |E(B)|$.

The set $\Omega$ of active blossoms is represented by rooted trees in our algorithm, where leaves represent vertices and internal nodes represent nontrivial blossoms. A root blossom is one not contained in any other blossom. The children of an internal node representing a blossom $B$ are ordered by the odd cycle that formed $B$, where the child containing the base of $B$ is ordered first. It is often possible to treat blossoms as if they were single vertices. \[8, 7\] The contracted graph $G/\Omega$ is obtained by contracting all root blossoms and removing the edges in those blossoms. To dissolve a root blossom $B$ means to delete its node in the blossom forest and, in the contracted graph, to replace $B$ with individual vertices $A_0, \ldots, A_\ell$. In our algorithm, we may dissolve blossoms with positive $z$-values, so we need to increase $y(v)$ by $z(B)/2$ for all $v \in V(B)$. The $y$-values and $z$-values are dual variables defined on vertices and blossoms. (See Section 3.1.)

2.2 Running time for Edmond’s algorithm

The first polynomial algorithm for weighted matching is given by Edmonds \[7\]. The best implementation of Edmonds’ algorithm takes $O(n(m + n \log n))$ time \[13\], where the time to find each augmenting path is $O(m + n \log n)$. When the weighted lengths of all augmenting paths are $O(m)$, we can improve it to $O(m)$. \[16\]

3 The Maximum Weight Perfect Matching Algorithm

3.1 Dual variables

We maintain two dual variables $y : V \to \mathbb{Z}$ and $z : \Omega \to \mathbb{N}$ that satisfy Property \[1\] with respect to the current matching $M$. For $(u, v) \in E$, let $yz(u, v) = y(u) + y(v) + \sum_{B \in \Omega, (u, v) \in E(B)} z(B)$.

**Property 1.** (Relaxed Complementary Slackness) Similar to \[16\],

1. $z(B)$ is a nonnegative even integer for all $B \in \Omega$, and $y(u)$ is an integer for all vertex $u$;
2. All root blossoms $B$ have $z(B) > 0$. (Non-root blossoms may have zero $z$-values.)
3. $yz(e) \geq w(e) - 2$ for all edge $e = (u, v) \in E$.
4. $yz(e) \leq w(e)$ when $e = (u, v)$ is a current matching edge or a blossom edge.

The next lemma shows that a perfect matching satisfying Property \[1\] will be a good approximation of the maximum weight perfect matching.

**Lemma 3.1.** Let $M$ be a perfect matching and $\Omega$ be the set of blossoms. If Property \[2\] is satisfied for $M, \Omega$ and some $y, z$-values, then $w(M) \geq w(M^*) - n$ where $M^*$ is a maximum weight perfect matching in $G$.

**Proof.** Since there are exactly $|B|/2$ edges of $M$ included in a blossom $B \in \Omega$, so

$$w(M) \geq \sum_{e \in M} yz(e) = \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B)|M \cap E(B)| = \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B)|B|/2$$
Since the number of edges of any matching included in \( B \) is at most \( \lfloor |B|/2 \rfloor \), so for the maximum weight perfect matching \( M^* \), we have:

\[
w(M^*) \leq \sum_{u \in V(M^*)} y(u) + \sum_{e \in M^*} \sum_{B \in \Omega, e \in E(B)} z(B) + 2|M^*|
\]

\[
= \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B)|M^* \cap E(B)| + n
\]

\[
\leq \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B)\lfloor |B|/2 \rfloor + n
\]

We have \( w(M) \geq w(M^*) - n \), thus proving the claim. ■

3.2 Edmonds’ Search

Suppose in a graph \( G \) in which all edge weights are even, we have a matching \( M \) and \( y, z, \Omega \) satisfying Property 1. If \( M \) is not perfect, we can run the Edmonds’ search [7] on the set of eligible edges to augment \( M \). Next, we will define two searching procedures \( \text{SearchOne}(F) \) and \( \text{SearchTwo}(F) \) as shown in Figure [1] where \( F \) is a set of free vertices with the same \( y \)-value. The only difference of these two procedures is the definitions of eligible edges. (See Definitions 3.2 and 3.3.) Let \( E_{\text{elig}} \) be the set of eligible edges and let \( G_{\text{elig}} = (V, E_{\text{elig}})/\Omega \) be the unweighted graph obtained by discarding ineligible edges and contracting root blossoms.

**Definition 3.2.** An edge \( e \) is eligible in \( \text{SearchOne} \) if at least one of the following holds.

1. \( e \in E_B \) for some \( B \in \Omega \).
2. \( e \notin M \) and \( yz(e) = w(e) - 2 \).
3. \( e \in M \) and \( yz(e) = w(e) \).

**Definition 3.3.** An edge \( e \) is eligible in \( \text{SearchTwo} \) if it is eligible by Definition 3.2 or it was a blossom edge just after the last time we augmented \( M \).

Therefore, in \( \text{SearchTwo} \), \( G_{\text{elig}} \cup \Omega \) will keep growing until we find an augmenting path. The running time per every augmenting path is \( O(m + n \log n) \) from [13], where the \( n \log n \) term comes from the priority queue used to find new eligible edges. However, in \( \text{SearchOne} \), if we can bound the weighted length of augmenting paths to be \( O(m) \), the time for each iteration is \( O(m) \). [10]

Here “one iteration” means one “Augmentation, Blossom Shrinking, Dual Adjustment” step. By Lemma 3.6 in \( \text{SearchOne} \), only one “Augmentation and Blossom Shrinking” step is needed before Dual Adjustment. We conclude these to:

**Lemma 3.4.** The running time of one iteration in \( \text{SearchOne}(F) \) is \( O(m) \) if we can bound the weighted length of augmenting paths to be \( O(m) \).

**Lemma 3.5.** If \( F \) does not change, \( \text{SearchTwo}(F) \) takes \( O(m + n \log n) \) time for any number of iterations before an augmenting path is discovered.

The following two lemmas about the Edmonds’ search come directly from [5]. They show the correctness of these procedures.
• **Augmentation:**
  From $F$, find a maximal set $\Psi$ of augmenting paths in $G_{elig}$ and set $M \leftarrow M \oplus (\bigcup_{P \in \Psi} P)$. Update $G_{elig}$.

• **Blossom Shrinking:**
  Let $V_{out} \subseteq V(G_{elig})$ be the vertices (that is, root blossoms) reachable from free vertices in $F$ by even-length alternating paths; let $\Omega'$ be a maximal set of (nested) blossoms on $V_{out}$. (That is, if $(u, v) \in E(G_{elig}) \setminus M$ and $u, v \in V_{out}$, then $u$ and $v$ must be in a common blossom.) Let $V_{in} \subseteq V(G_{elig}) \setminus V_{out}$ be those vertices reachable from those free vertices by odd-length alternating paths. Set $z(B) \leftarrow 0$ for $B \in \Omega'$ and set $\Omega \leftarrow \Omega \cup \Omega'$. Update $G_{elig}$.

• **Dual Adjustment:**
  Let $\hat{V}_{in}, \hat{V}_{out} \subseteq V$ be original vertices represented by vertices in $V_{in}$ and $V_{out}$. The $y$- and $z$-values for some vertices and root blossoms are adjusted:

  $y(u) \leftarrow y(u) - 1$, for all $u \in \hat{V}_{out}$.  
  $y(u) \leftarrow y(u) + 1$, for all $u \in \hat{V}_{in}$.  
  $z(B) \leftarrow z(B) + 2$, if $B \in \Omega$ is a root blossom with $B \subseteq \hat{V}_{out}$.  
  $z(B) \leftarrow z(B) - 2$, if $B \in \Omega$ is a root blossom with $B \subseteq \hat{V}_{in}$.

After dual adjustments some root blossoms may have zero $z$-values. Dissolve such blossoms (remove them from $\Omega$) as long as they exist. Note that non-root blossoms are allowed to have zero $z$-values. Update $G_{elig}$ by the new $\Omega$.

Figure 1: The Edmonds’ search $SearchOne(F)$ and $SearchTwo(F)$, where $F$ is a set of free vertices with the same $y$-value.
Lemma 3.6. In SearchOne, after the Augmentation and Blossom Shrinking steps $G_{\text{elig}}$ contains no augmenting path from free vertices in $F$, nor is there a path from a free vertex in $F$ to a blossom.

Proof. Suppose there is an augmenting path $P$ from $F$ in $G_{\text{elig}}$ after augmenting along paths in $\Psi$. Since $\Psi$ is maximal, $P$ must intersect some $P' \in \Psi$ at a vertex $v$. However, after the Augmentation step every edge in $P'$ will become ineligible, so the matching edge $(v, v') \in M$ is no longer in $G_{\text{elig}}$, contradicting the fact that $P$ consists of eligible edges. Since $\Omega'$ is maximal, there can be no blossom reachable from a free vertex of $F$ in $G_{\text{elig}}$ after the Blossom Shrinking step. ■

Lemma 3.7. In both SearchOne and SearchTwo, after the Dual Adjustment step, all edges will not offend Property 1 if all edge weights are even.

Proof. Property 1(1) is obviously maintained. Property 1(2) is also maintained since all the new root blossoms discovered in the Blossom Shrinking step are in $V_{\text{out}}$ and will have positive $z$-values after adjustment. Furthermore, each root blossom whose $z$-value drops to zero is removed.

After the blossom is formed, we can see $yz(e)$ for all blossom edges will not change until the blossom is dissolved, so each blossom edge $e$ has $yz(e) = w(e)$ or $yz(e) = w(e) - 2$. Thus, in SearchTwo, the eligible edges still have $yz(e) = w(e)$ or $yz(e) = w(e) - 2$.

Since the Dual Adjustment is only from the free vertices with the same $y$-value, and $w(e)$ is even for every $e$, the $y$-values of all vertices in $\hat{V}_{\text{in}} \cup \hat{V}_{\text{out}}$ must have the same parity. Recall that an eligible edge $e$ must have $yz(e) = w(e)$ or $w(e) - 2$. Let $e = (u, v)$ be an edge, and suppose first that both $u, v$ are in $\hat{V}_{\text{in}} \cup \hat{V}_{\text{out}}$. If $u, v \in B \in \Omega$ then $yz(e)$ is unchanged, preserving the property, so we can assume that $u$ and $v$ are in different root blossoms. If $e \notin M$ is ineligible then, due to parity, $yz(e) \geq w(e)$ before adjustment and $yz(e) \geq w(e) - 2$ afterward. If $e \notin M$ is eligible then at least one $u, v$ is in $\hat{V}_{\text{out}}$ (otherwise another blossom or augmenting path would have been formed), so $yz(e)$ cannot be reduced. If $e \in M$ then it must be eligible, so $u \in \hat{V}_{\text{in}}$ and $v \in \hat{V}_{\text{out}}$ and $yz(e)$ is unchanged. Now suppose $u$, but not $v$, is in $\hat{V}_{\text{in}} \cup \hat{V}_{\text{out}}$. If $e \notin M$ is eligible then $u \in \hat{V}_{\text{in}}$ and $yz(e)$ will increase. If it is ineligible, $yz(e) \geq w(e) - 1$ before adjustment and $yz(e) \geq w(e) - 2$ afterward. If $e \in M$ then it must be ineligible, so $u \in \hat{V}_{\text{in}}, yz(e) \leq w(e) - 1$ before adjustment and $yz(e) \leq w(e)$ afterward. ■

3.3 The main procedure

Now we are ready to describe the main procedure of our algorithm. The difficult part for scaling algorithms in general graphs is how to treat blossoms of previous scale. In this algorithm, we dissolve all blossoms immediately when coming to a new scale. However, we treat blossoms differently by their sizes. For “large” blossoms, we bound their $z$-values so the total violation of Property 1 is only $O(n)$. For “small” blossoms, we can run the Edmonds’ search inside them first. The large and small blossoms are defined as:

Definition 3.8. We say a blossom $B$ is “large” if its number of vertices $n(B) > \sqrt{n}$, otherwise, it is “small”.

To deal with the “remaining” free vertices after each scale, we make it matched by adding an artificial edge and an artificial vertex. Note that artificial edges and vertices cannot be in blossoms.

Let $w', y', z', M', \Omega'$ be the edge weights, $y$-value, $z$-value, the matching and blossom set in the previous scale. In the first scale, $w', y', z' = 0$ and $M', \Omega' = \emptyset$. We can see they satisfy Property 1. Note that $z$ is a function both on $\Omega$ and $\Omega'$. 7
For scales $i = 1, \cdots, \lceil \log((n+1)N) \rceil$, run the following steps.

- **Initialization** If $i = 1$, $G_1 \leftarrow G$ with $y' = 0$, otherwise $G_i \leftarrow G_{i-1}$. Note that we will keep the artificial edges and vertices of $G_{i-1}$, so that we can use the perfect matching $M'$ in $G_{i-1}$ to analyze the algorithm. (See Lemma 3.13)

  1. For each edge $e$, $w(e) \leftarrow 2(w'(e) + (the \ i-th \ bit \ of \ \overline{w}(e)))$. For each vertex $v$, $y(v) \leftarrow 2y'(v) + 3$. For each blossom $B' \in \Omega'$, $z(B') = 2z'(B')$.
  2. If a root blossom $B' \in \Omega'$ is large, dissolve $B'$ by setting $y(v) \leftarrow y(v) + z(B')/2$ for each $v \in B'$. Recursively run this step until all root blossoms in $\Omega'$ are small.
  3. Set $w(u, v) \leftarrow w(u, v) - y(u) - y(v)$ for each edge $(u, v)$, and set $y(u) \leftarrow 0$ for each vertex $u$.
  4. Dissolve all small blossom $B' \in \Omega'$ by setting $y(v) \leftarrow y(v) + z(B')/2$ for each $v \in B'$.
  5. Set $M \leftarrow \emptyset$ and $\Omega \leftarrow \emptyset$.

- **Search** Denote the largest $y$-value of free vertices by $Y$ and the set of those free vertices with largest $y$-value $Y$ by $F$. There are two steps of the search:
  * If $Y > 0$, repeatedly run $SearchTwo(F)$ until $Y = 0$.
  * Then repeatedly run $SearchOne(F)$ until $Y = -2\lceil \sqrt{n} \rceil$ or we have found a perfect matching.

- **Perfection** After $Y$ reaches $-2\lceil \sqrt{n} \rceil$, for each free vertex $u$ that are not artificial, we add an artificial vertex $v$ with $y(v) = 2\lceil \sqrt{n} \rceil$ and an artificial edge $(u, v)$ with $w(u, v) = 0$ in the graph $G_i$. Make the artificial edge a matching edge, and also delete all artificial free vertices with their edges in $G_i$. Then we get a perfect matching in the current graph $G_i$.

- **Finalization** After the final scale, we erase all the artificial vertices and edges in the final graph $G_{\lceil \log((n+1)N) \rceil}$ to return to the original graph $G$, then run $SearchTwo(\{v\})$ for every free vertex $v$ (vertex previously matched with an artificial edge) until we get a perfect matching.

### 3.4 Correctness

During the analysis, we let $yz(u, v) = y(u) + y(v) + \sum_{B' \in \Omega', (u,v) \in E(B')} z(B') + \sum_{B \in \Omega, (u,v) \in E(B)} z(B)$. However, since $\Omega'$ becomes empty after Step 4, only one of $\Omega'$ and $\Omega$ can be non-empty at any time.

**Lemma 3.9.** For all edge $(u, v) \in G_i$,

- After Step 1 of Initialization, $w(e) \leq yz(e)$, and for matching edges $e \in M'$, $w(e) \geq yz(e) - 6$. In the first scale, $w(e) \geq yz(e) - 6$ for all edge $e$.

- After Step 3 of Initialization, $w(e) \leq \sum_{B' \in \Omega', (u,v) \in E(B')} z(B')$.

- After Initialization, $w(u,v) \leq 2 \min\{y(u), y(v)\}$ thus $w(u,v) \leq y(u) + y(v)$, so Property 7 holds.

- $w(e)$ is always an even number.

**Proof.** Since $y' (e) \geq w'(e) - 2$ for $yz'(u, v) = y'(u) + y'(v) + \sum_{B' \in \Omega', (u,v) \in E(B')} z'(B')$, after Step 1, $yz(e) = 2y'z'(e) + 6 \geq 2w'(e) + 2 \geq w(e)$. For matching edges $e$, $yz(e) = 2y'z'(e) + 6 \leq 2w'(e) + 6 \leq w(e) + 6$. In the first scale, $yz(e) = 6$ and $w(e) = 0$ or 2 for all edge $e$. 
Steps 2 and 4 will increase some \(y\)-values but will maintain \(w(e) \leq yz(e)\). After Step 3, \(w(u, v)\) will decrease by \(y(u) + y(v)\), so \(w(u, v) \leq \sum_{B' \in \Omega', (u, v) \in E(B')} z(B')\). After Step 4, \(y(u) = \sum_{B' \in \Omega', (u, v) \in E(B')} z(B')/2 \geq \sum_{B' \in \Omega', (u, v) \in E(B')} z(B')/2\) for any edge \((u, v)\), so \(w(u, v) \leq 2y(u)\). Similarly \(w(u, v) \leq 2y(v)\).

From Property 1 in the previous scale, after Step 1 \(y\)-values are odd and \(z\)-values are multiples of 4, so \(y\)-values remain odd after Step 2. Since initially \(w(e)\) is even, \(w(e)\) remains even in the whole procedure. ■

We can also easily obtain the basic observation:

**Property 2.** After the largest \(y\)-value of free vertices decreases to zero, all free vertices will have the same \(y\)-value \(Y\).

Every call of SearchOne or SearchTwo maintains the properties that free vertices in \(F\) have the same \(y\)-value, and all edge weights are even. Thus Lemma 3.9 and Lemma 3.7 apply.

**Lemma 3.10.** (Correctness) The algorithm will return the maximum weight perfect matching of \(G\).

**Proof.** After Finalization, by Lemma 2.1, we can change the weight function to \(w(e) = 2\bar{w}(e)\), so a maximum weight perfect matching \(M^*\) for \(w\) will also be a MWPM for \(\bar{w}\). By Lemma 3.1, \(w(M) \geq w(M^*) - n\), so \(\bar{w}(M) \geq \bar{w}(M^*) - n/2\). Since the \(\bar{w}(e)\) are all multiples of \((n + 1)\), \(\bar{w}(M) = \bar{w}(M^*)\). ■

### 3.5 Running time

Next, we analyze the running time.

**Lemma 3.11.** In the SearchTwo step with \(Y > 0\), we only need to consider the edges within small blossoms of previous scale. The total time needed for the SearchTwo steps with \(Y > 0\) in one scale is \(O((m + n \log n) \sqrt{n})\).

**Proof.** First, we prove that the \(y\)-value of matched vertices cannot be smaller than \(Y\) by induction. After Initialization, since \(M = \emptyset\), it is true. Assume it is true just before a Dual Adjustment step. After this Dual Adjustment step, \(Y\) decrease by one, and for any matched vertex \(u\), \(y(u)\) can at most decrease by one, so \(y(u) \geq Y\) after Dual Adjustment. If there is a new eligible edge \((u, v)\) connecting to a free vertex \(v\), \((u, v)\) must be non-matching, so \(y(u) + y(v) = w(u, v) - 2\). If \(y(v) < Y\), the Search never reached it before, so \(y(v)\) is unchanged after Initialization. By Lemma 3.9, \(w(u, v) \leq 2y(v)\) so \(y(u) + y(v) + 2 \leq 2y(v)\) and \(y(u) \leq y(v) - 2 < y(v) < Y\), comes to a contradiction. So \(y(v) \geq Y\). Thus all vertices comes into \(G_{elig}\) have \(y\)-values at least \(Y\), proving the statement.

In the SearchTwo steps with \(Y > 0\), we only need to consider the edges within small blossoms of previous scale since other edges have non-positive weights. For each such small root blossom \(B'\), \(n(B') \leq \sqrt{n}\). If we only consider the SearchTwo procedure inside \(B'\), by Lemma 3.5, each augmenting path will take \(O(m(B') + n(B') \log n(B'))\) time. We may also need to restart SearchTwo(\(F\)) when new free vertices with smaller initial \(y\)-values comes into \(F\), but this will only happen \(O(n(B'))\) times within one blossom \(B'\). Thus the time needed for SearchTwo inside \(B'\) will be \(O((m(B') + n(B') \log n(B'))n(B'))\). Thus, the total time for \(Y > 0\) is \(O((m + n \log n) \sqrt{n})\).

**Lemma 3.12.** The sum of \(z\)-values of large blossoms at the end of a scale is less than \(4n\).
Proof. By Lemma 3.11, the large blossoms can only form after $Y = 0$. Every dual-adjustment step can only increase the $z$-values of at most $\lfloor \sqrt{n} \rfloor$ large blossoms, and the amount is at most 2. (The artificial edges cannot be in a blossom.) Since there are at most $2 \lfloor \sqrt{n} \rfloor$ dual-adjustment steps after $Y = 0$, the lemma holds.

Lemma 3.13. After the Search step, the number of free vertices is at most $6\sqrt{n}$.

Proof. Consider the perfect matching $M'$ obtained in the previous scale with blossoms set $\Omega'$. For the first scale, we let $M'$ be any perfect matching and $\Omega = \emptyset$. When we dissolve a blossom $B' \in \Omega'$, there will be one matching edge connecting to its base such that $w(u, v) - yz(u, v)$ decrease by $z(B')/2 = z'(B')$. Let $K = \sum_{|B'| > \sqrt{n}, B' \in \Omega'} z'(B')$ at the beginning of one scale, then $K < 4n$ by Lemma 3.12.

By Lemma 3.9 after Step 3 of Initialization,

$$w(M') \geq -6|M'| - K + \sum_{|B'| \leq \sqrt{n}, B' \in \Omega'} 2z'(B') \lfloor |B'|/2 \rfloor \geq -10n + \sum_{|B'| \leq \sqrt{n}, B' \in \Omega'} 2z'(B') \lfloor |B'|/2 \rfloor \leq w(M') + 10n$$

Note that $|M'| \leq n$ since the number of artificial vertices is at most $n$. For any other matching $M''$, after Step 3 of Initialization, by Lemma 3.9

$$w(M'') \leq \sum_{e \in M'' B' \in \Omega', |B'| \leq \sqrt{n}, e \in E(B')} 2z'(B') \leq \sum_{|B'| \leq \sqrt{n}, B' \in \Omega'} 2z'(B') \lfloor |B'|/2 \rfloor \leq w(M') + 10n$$

When $Y$ reaches $Y' = -2\lfloor \sqrt{n} \rfloor$, let the number of free vertices be $f = |F|$, the current matching be $M$, and the current blossom set be $\Omega$. From Property 1 and Property 2

$$w(M) \geq \sum_{u \in M} y(u) + \sum_{B \in \Omega} z(B) \lfloor |B|/2 \rfloor \quad f \cdot Y' + w(M) \geq \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B) \lfloor |B|/2 \rfloor$$

and,

$$w(M') \leq \sum_{u \in V} y(u) + \sum_{e \in M', B \in \Omega, e \in E(B)} z(B) + 2|M'| \leq \sum_{u \in V} y(u) + \sum_{B \in \Omega} z(B) \lfloor |B|/2 \rfloor + 2n \leq f \cdot Y' + w(M) + 2n \leq f \cdot Y' + w(M') + 12n$$

The last line is from the equation $w(M'') \leq w(M') + 10n$ for any matching $M''$. So $(-2\sqrt{n})f \geq -12n$, and $f \leq 6\sqrt{n}$. ■
Theorem 3.14. The running time for this algorithm is $O((m + n \log n)\sqrt{n}\log(nN))$.

Proof. By Lemma 3.11, the time needed for $Y > 0$ in each scale is $O((m + n \log n)\sqrt{n})$. After $Y = 0$, since $Y$ will only change by $O(\sqrt{n})$, the weights of augmenting paths relative to the $yz$-values of $Y = 0$ will be $O(\sqrt{n})$. By Lemma 3.4, each iteration of SearchOne takes $O(m)$ time, so the time needed after $Y = 0$ in each scale is $O(m\sqrt{n})$. Thus, the time will be $O((m + n \log n)\sqrt{n})$ in each scale. For the Finalization step, by Lemma 3.13, at most $6\sqrt{n} \lceil \log((n + 1)N) \rceil$ free vertices emerge after we remove artificial edges. Since we have changed the scales of weights for many times, we cannot bound the weight length of augmenting paths to $O(m)$, so we cannot use SearchOne. By Lemma 3.5, every augmenting path needs $O(m + n \log n)$ time in SearchTwo, so the time needed is $O((m + n \log n)\sqrt{n}\log(nN))$. □

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