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On the recursive properties of one kind hybrid power mean involving two-term exponential sums and Gauss sums

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Abstract: The main purpose of this paper is to study the computational problem of one kind hybrid power mean involving two-term exponential sums and quartic Gauss sums using the analytic method and the properties of the classical Gauss sums, and to prove some interesting fourth-order linear recurrence formulae for this problem. As an application of our result, we can also obtain an exact computational formula for one kind congruence equation mod \( p \), an odd prime.

Keywords: The quartic Gauss sums, Two-term exponential sums, Hybrid power mean, The fourth-order linear recurrence formula

MSC: 11L05, 11L07

1 Introduction

Let \( p \geq 3 \) be an odd prime. For any integer \( m \) with \( (m, p) = 1 \), the quartic Gauss sums \( B(m) = B(m, p) \) is defined as

\[
B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right),
\]

where as usual, \( e(y) = e^{2\pi iy} \).

Recently, some scholars have studied the hybrid power mean problems of various trigonometric sums, and obtained many interesting results. For example, Chen Li and Hu Jiayuan [1] studied the computational problem of the hybrid power mean

\[
S_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^k \cdot \left( \sum_{c=1}^{p-1} e\left(\frac{mc + \bar{c}}{p}\right) \right)^2,
\]

where \( \bar{c} \) denotes the multiplicative inverse of \( c \) mod \( p \). That is, \( c \cdot \bar{c} \equiv 1 \mod p \).

For \( p \equiv 1 \mod 3 \), they used the elementary method to obtain an interesting third-order linear recurrence formula for \( S_k(p) \).

Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

\[
\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^2 \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2,
\]

and proved an exact computational formula for (1).

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Zhang Han and Zhang Wenpeng [3] proved the identity
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^3 + na}{p} \right) \right)^4 = \begin{cases} 
2p^3 - p^2 & \text{if } 3 \mid p - 1, \\
2p^3 - 7p^2 & \text{if } 3 \mid p - 1.
\end{cases}
\]

Other related results can also be found in references [4-13].

In this paper, we will consider the calculating problem of the following hybrid power mean:
\[
V_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) \right)^k \left( \sum_{b=0}^{p-1} e \left( \frac{mb^4 + b}{p} \right) \right)^3,
\]
where \(k \geq 0\) is an integer.

If \(p = 4h + 3\), then from the properties of the Legendre’s symbol \(\mod p\) we have (see [14], formula (30) in Chapter 9)
\[
\sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) = 1 + \sum_{a=1}^{p-1} (1 + \chi_2(a)) e \left( \frac{ma^2}{p} \right) = \sum_{a=0}^{p-1} e \left( \frac{ma^2}{p} \right) = i\chi_2(m) \sqrt{p},
\]
where \(\chi_2 \left( \frac{a}{p} \right)\) denotes the Legendre’s symbol \(\mod p\).

So in this case, the problem we considered in (2) is trivial. If \(p = 4h + 1\), then the situation is more complicated. We will use the analytic method and the properties of classical Gauss sums to study this problem, and prove some new interesting fourth-order linear recurrence formulae for (2) with \(p = 4h + 1\). That is, we will give the following four results.

**Theorem 1.1.** Let \(p\) be a prime with \(p = 24h + 1\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four values are \(V_0(p) = p^2 - 6p\alpha\), \(V_1(p) = p \left( p^2 - 16p - 4\alpha^2 \right)\), \(V_2(p) = p^2 \left( 2p\alpha + 3p - 58\alpha \right)\) and \(V_3(p) = p^2 \left( 7p^2 + 4p\alpha - 92p - 72\alpha^2 \right)\), \(\alpha = \alpha(p) = \frac{1}{p} \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right)\) is an integer, which satisfies the identity (see Theorem 4-11 in [15])
\[
p = \alpha^2 + \beta^2 \equiv \left( \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right) \right)^2 + \left( \sum_{a=1}^{p-1} \left( \frac{a + r\overline{a}}{p} \right) \right)^2,
\]
which \(r\) is any quadratic non-residue \(\mod p\).

**Theorem 1.2.** Let \(p\) be a prime with \(p = 24h + 17\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four values are \(V_0(p) = -p^2 - 6p\alpha\), \(V_1(p) = p \left( p^2 - 18p - 4\alpha^2 \right)\), \(V_2(p) = p^2 \left( 2p\alpha - 3p - 62\alpha \right)\) and \(V_3(p) = p^2 \left( 7p^2 - 4p\alpha - 106p - 72\alpha^2 \right)\).

**Theorem 1.3.** Let \(p\) be a prime with \(p = 24h + 5\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( 9p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four terms are \(V_0(p) = -(p^2 + 6p\alpha)\), \(V_1(p) = -p \left( p^2 - 8p + 4\alpha^2 \right)\), \(V_2(p) = -p^2 \left( 2p\alpha - p - 22\alpha \right)\) and \(V_3(p) = p^3 \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right)\).
Theorem 1.4. Let \( p \) be a prime with \( p = 24h + 13 \). Then for any integer \( k \geq 4 \), we have the fourth-order linear recurrence formula

\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( 9p - 4\alpha^2 \right) V_{k-4}(p),
\]

where the first four terms are \( V_0(p) = p^2 - 6p\alpha \), \( V_1(p) = -p \left( p^2 - 6p + 4\alpha^2 \right) \), \( V_2(p) = -p^2 (2p\alpha + p - 18\alpha) \) and \( V_3(p) = p^2 (5p^2 + 6p\alpha - 18p - 36\alpha^2) \).

From our theorems we may immediately deduce the following:

**Corollary 1.5.** Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^6}{p} \right) \right)^3 \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right)^3 = \begin{cases} p^2 \left( 7p^2 + 4p\alpha - 92p - 72\alpha^2 \right) & \text{if } p = 24h + 1, \\
\frac{p^2}{12} \left( 7p^2 - 4p\alpha - 106p - 72\alpha^2 \right) & \text{if } p = 24h + 17, \\
\frac{p^2}{8} \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right) & \text{if } p = 24h + 5, \\
\frac{p^2}{6} \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right) & \text{if } p = 24h + 13. 
\end{cases}
\]

Note that the estimate \(|\alpha| \leq \sqrt{p}\), from Corollary 1.5 we also have the following:

**Corollary 1.6.** Let \( p \) be a prime with \( p \equiv 1 \mod 8 \), then we have the asymptotic formula

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^6}{p} \right) \right)^3 \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right)^3 = 7p^4 + O \left( p^{\frac{5}{2}} \right).
\]

**Corollary 1.7.** Let \( p \) be a prime with \( p \equiv 5 \mod 8 \), then we have the asymptotic formula

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^6}{p} \right) \right)^3 \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right)^3 = 5p^4 + O \left( p^{\frac{5}{2}} \right).
\]

For any prime \( p \) with \( p \equiv 1 \mod 4 \) and any positive integer \( k \), let \( M_k(p) \) denote the number of the solutions of the congruence equation

\[
x_1^4 + x_2^4 + \cdots + x_k^4 + y_1^4 + y_2^4 + y_3^4 \equiv 0 \mod p, \quad y_1 + y_2 + y_3 \equiv 0 \mod p,
\]

where \( 0 \leq x_i, y_j \leq p - 1, i = 1, 2, \cdots, k, j = 1, 2, 3 \).

Then from our theorems we can give an exact computational formula for \( M_k(p) \). For example, let \( H_s(p) \) denote the number of the congruence equation

\[
x_1^4 + x_2^4 + \cdots + x_s^4 \equiv 0 \mod p, \quad 0 \leq x_i \leq p - 1, \quad i = 1, 2, \cdots s.
\]

Then we have the identity

\[
V_k(p) = \frac{p^2}{p - 1} \cdot M_k(p) - \frac{p}{p - 1} \cdot H_k(p).
\]

Since \( H_k(p) \) has a fourth-order linear recurrence formula (see [8]), so from the above formula and our theorems we can deduce the exact value of \( M_k(p) \).

## 2 Several lemmas

To complete the proofs of our theorems, we need to prove four simple lemmas. Hereafter, we will use many properties of the classical Gauss sums and the fourth-order character mod \( p \), all of which can be found in
books concerning Elementary Number Theory or Analytic Number Theory, such as references [7], [14] or [15]. Some important results related to Gauss sums can also be found in [16] and [17]. These contents will not be repeated here. First we have the following:

**Lemma 2.1.** Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), \( \lambda \) be any fourth-order character \( \mod p \), then we have

\[
\tau^2(\lambda) + \tau^2(\lambda) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left( \frac{a + a}{p} \right) = 2\sqrt{p} \cdot \alpha,
\]

where \( \tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a) e\left( \frac{a}{p} \right) \) denotes the classical Gauss sums, and \( \left( \frac{a}{p} \right) \) is the Legendre’s symbol \( \mod p \).

**Proof.** In fact this is Lemma 2 of [18], so its proof is omitted. \( \square \)

**Lemma 2.2.** Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then for any fourth-order character \( \lambda \mod p \), we have the identity

\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \left( -5p\tau(\lambda) - 2\sqrt{p}\alpha\tau(\lambda) \right) \mod 9 \quad \text{if} \quad p \equiv 1 \mod 8,
\]

\[
\left( -p\tau(\lambda) - 2\sqrt{p}\alpha\tau(\lambda) \right) \mod 9 \quad \text{if} \quad p \equiv 5 \mod 8,
\]

where \( \alpha \) is the same as in Lemma 2.1.

**Proof.** First applying trigonometric identity

\[
\sum_{m=1}^{q} e\left( \frac{nm}{q} \right) = \begin{cases} q \quad \text{if} \quad q \mid n, \\ 0 \quad \text{if} \quad q \nmid n \end{cases}
\]

and note that \( \lambda^4 = \chi_0 \), the principal character \( \mod p \), we have

\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 \\
+ \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right) \\
= \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + b^4 + 1) \left( \sum_{a=0}^{p-1} e\left( \frac{c(a + b + 1)}{p} \right) \right) \\
+ \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + b^4) e\left( \frac{a + b}{p} \right) \\
= \tau(\lambda)p \sum_{a=0}^{p-1} \lambda(a^4 + b^4 + 1) - \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + b^4 + 1) \\
- \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1) \left( \sum_{b=1}^{p-1} e\left( \frac{b(a + 1)}{p} \right) \right). \quad (4)
\]

From (3) we have

\[
\tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1) \left( \sum_{b=1}^{p-1} e\left( \frac{b(a + 1)}{p} \right) \right) = \lambda(2)\tau(\lambda)(p - 1) - \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1) \\
= \lambda(2)\tau(\lambda)p - \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1). \quad (5)
\]

Note that the identity \( \lambda\chi_2 = \lambda \) and

\[
B(m) = \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) = \chi_2(m)\sqrt{p} + \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda). \quad (6)
\]
From (6) we have
\[
\tau(\lambda) \sum_{a=0}^{p-1} \chi(a^4 + 1) = \sum_{b=1}^{p-1} \lambda(b) \sum_{a=0}^{p-1} e \left( \frac{b(a^4 + 1)}{p} \right)
\]
\[
= \sum_{b=1}^{p-1} \lambda(b) \left( \chi_2(b) \sqrt{p} + \chi(b) \tau(\lambda) + \lambda(b) \tau(\lambda) \right) e \left( \frac{b}{p} \right)
\]
\[
= \sqrt{p} \tau(\lambda) - \tau(\lambda) + \sqrt{p} \tau(\lambda) = 2\sqrt{p} \tau(\lambda) - \tau(\lambda).
\] (7)

If \( p \equiv 5 \mod 8 \), then note that \( \lambda(-1) = -1 \) and \( \tau(\lambda) \tau(\lambda) = -p \), applying (6) and Lemma 2.1 we also have
\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi(a^4 + b^4 + 1) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} e \left( \frac{ca^4 + cb^4 + c}{p} \right)
\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e \left( \frac{c}{p} \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ca^4}{p} \right) \right)^2
\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left( \chi_2(b) \sqrt{p} + \chi(b) \tau(\lambda) + \lambda(b) \tau(\lambda) \right)^2
\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left( 2\chi_2(c) \sqrt{p} - \chi_2(c) \sqrt{p} + \chi(\lambda) \right) e \left( \frac{c}{p} \right)
\]
\[
= p + \frac{2\sqrt{p}(\alpha - 1) \tau(\lambda)}{\tau(\lambda)}.
\] (8)

Note that \( \lambda^2 = x_2 = \chi^2 \) and the congruence \( a + b + 1 \equiv 0 \mod p \) implies the congruence \( a^4 + b^4 + 1 \equiv 2(a^2 + a + 1)^2 \mod p \). So we have
\[
\sum_{a+b+1=0 \mod p} \sum_{d=0}^{p-1} \chi(a^4 + b^4 + 1) = \sum_{a=0}^{p-1} \chi \left( 2(a^2 + a + 1)^2 \right)
\]
\[
= \chi(2) \sum_{a=0}^{p-1} \chi_2(a^2 + a + 1) = \chi(2) \sum_{a=0}^{p-1} \chi_2(4a^2 + 4a + 4)
\]
\[
= \chi(2) \sum_{a=0}^{p-1} \chi_2((2a + 1)^2 + 3) = \chi(2) \sum_{a=0}^{p-1} \chi_2(a^2 + 3) = -\chi(2).
\] (9)

Combining (4), (5), (7), (8) and (9) we have the identity
\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right) = -p \tau(\lambda) - 2\sqrt{p} \alpha \tau(\lambda).
\] (10)

If \( p \equiv 1 \mod 8 \), then \( \lambda(-1) = 1 \) and \( \tau(\lambda) \tau(\lambda) = p \), from the method of proving (8) we have
\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi(a^4 + b^4 + 1) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} e \left( \frac{ca^4 + cb^4 + c}{p} \right)
\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e \left( \frac{c}{p} \right) \left( \chi_2(b) \sqrt{p} + \chi(b) \tau(\lambda) + \lambda(b) \tau(\lambda) \right)^2
\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left( 3p + 2\chi_2(c) \sqrt{p} + 2\lambda(c) \sqrt{p} \tau(\lambda) + 2\chi(\lambda) \sqrt{p} \tau(\lambda) \right) e \left( \frac{c}{p} \right)
\]
\[
= 5p + \frac{2\sqrt{p}(\alpha - 1) \tau(\lambda)}{\tau(\lambda)}.
\] (11)

Combining (4), (5), (7), (8) and (11) we have the identity
\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right) ^3 = -5p \tau(\lambda) - 2\sqrt{p} \alpha \tau(\lambda).
\] (12)

Now Lemma 2.2 follows from (10) and (12). 
\( \Box \)
Lemma 2.3. Let $p$ be a prime with $p \equiv 1 \mod 4$, then we have the identity

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} 
p^2 - 6p\alpha & \text{if } p = 24h + 1 \text{ or } p = 24h + 13, \\
-p^2 - 6p\alpha & \text{if } p = 24h + 5 \text{ or } p = 24h + 17. 
\end{cases}
$$

Proof. From (3) we have

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e\left( \frac{a + b + c}{p} \right)
$$

Now we calculate each term in (13). If $p \equiv 5 \mod 8$, then note that $\lambda(-1) = -1$ we have

$$
p \sum_{a=0}^{p-1} 1 = 0. \quad (14)
$$

Applying (6) and Lemma 2.1 we have

$$
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) \right) \left( \sum_{a=0}^{p-1} e\left( \frac{m}{p} \right) \right)
$$

It is clear that the congruences $a^4 + b^4 + 1 \equiv 0 \mod p$ and $a + b + 1 \equiv 0 \mod p$ implies that $ab \equiv 1 \mod p$ and $a^3 \equiv b^3 \equiv 1 \mod p$ with $a \neq b$. So we have

$$
p^2 \sum_{a=2}^{p-1} \sum_{b=2}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) \right) \left( \sum_{a=0}^{p-1} e\left( \frac{m}{p} \right) \right)
$$

Applying (13), (14), (15) and (16) we have the identity

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} 
p^2 - 6p\alpha & \text{if } p = 24h + 13, \\
-p^2 - 6p\alpha & \text{if } p = 24h + 5. 
\end{cases} \quad (17)
$$
If \( p \equiv 1 \pmod{8} \), then we also have
\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = 4p. \tag{18}
\]
\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} e \left( \frac{m(a^4 + b^4 + 1)}{p} \right)
= p^2 + \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(k) + 2\sqrt{\tau(k)} \right) \left( \frac{m}{p} \right)
= p^2 + 2p\alpha - 3p + 2\sqrt{\tau^2(\lambda)} + 2\sqrt{\tau^2(k)} = p^2 - 3p + 6p\alpha. \tag{19}
\]
\[
p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right)
= \left\{ \begin{array}{ll}
p^2 - 6p\alpha & \text{if } p = 24h + 1, \\
0 & \text{if } p = 24h + 17.
\end{array} \right. \tag{20}
\]
Applying (13), (18), (19) and (20) we have
\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3 = \left\{ \begin{array}{ll}
p^2(p - 6) & \text{if } p = 24h + 1, \\
p^2(p - 8) & \text{if } p = 24h + 17, \\
-p^2(p - 4) & \text{if } p = 24h + 13, \\
-p^2(p - 6) & \text{if } p = 24h + 5.
\end{array} \right. \tag{21}
\]
It is clear that Lemma 2.3 follows from (17) and (21).

**Lemma 2.4.** Let \( p \) be a prime with \( p \equiv 1 \pmod{4} \), then we have the identity
\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3 = \left\{ \begin{array}{ll}
p^2(p - 6) & \text{if } p = 24h + 1, \\
p^2(p - 8) & \text{if } p = 24h + 17, \\
-p^2(p - 4) & \text{if } p = 24h + 13, \\
-p^2(p - 6) & \text{if } p = 24h + 5.
\end{array} \right. \tag{22}
\]

**Proof.** From the properties of the Legendre's symbol \( \pmod{p} \) we have
\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3 = \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^2
+ \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right) \left( \sum_{c=0}^{p-1} \left( \frac{mc^4 + c}{p} \right) \right)
= \sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1) \sum_{a=0}^{p-1} \chi_2 \left( \frac{b(a + 1)}{p} \right)
+ \sqrt{p} \sum_{a=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) \sum_{a=0}^{p-1} \chi_2 \left( \frac{c(a + b + 1)}{p} \right)
= -\sqrt{p} + \chi_2(2)p^2 - \sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1) - \sqrt{p} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right)
+ p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right). \tag{22}
\]
From the properties of fourth-order \( \pmod{p} \) and Lemma 2.1 we have
\[
\sum_{a=0}^{p-1} \chi_2(a^4 + 1) = 1 + \sum_{a=1}^{p-1} \chi_2(a + 1) \left( 1 + \lambda(a) + \chi_2(a) + \bar{\tau}(a) \right)
\]
Combining (22), (23), (24) and (25) we have

\[ \frac{1}{\sqrt{p}} \left( \tau^2(\lambda) + \tau^2(\overline{\lambda}) \right) - 1 = 2\alpha - 1. \]  

(23)

Applying Lemmas 2.1–2.4 we also have

\[ \sum_{a+b+1=0 \mod p} \chi_2 \left( a^6 + b^6 + 1 \right) = \sum_{a=0}^{p-1} \chi_2 \left( a^6 + (a+1)^6 + 1 \right) \]

\[ = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( a^6 + 2a^3 + 3a^2 + 2a + 1 \right) = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( (a^2 + a + 1)^2 \right) \]

\[ = \begin{cases} 
\chi_2(2)p & \text{if } p = 12h + 1, \\
\chi_2(2)(p-2) & \text{if } p = 12h + 5.
\end{cases} \]  

(24)

Now we prove our main results. First we prove Theorem 1.1. If \( p = 24h + 1 \), then from Lemmas 2.1, 2.2 and 2.4 we have

\[ \frac{1}{p} \left( \sum_{m=1}^{p-1} \chi_2 \left( ma^4 + a \right) \right)^3 = \begin{cases} 
p^{\frac{1}{3}}(p-6) & \text{if } p = 24h + 1, 

p^{\frac{1}{3}}(p-8) & \text{if } p = 24h + 7, 

-p^{\frac{1}{3}}(p-4) & \text{if } p = 24h + 13, 

-p^{\frac{1}{3}}(p-6) & \text{if } p = 24h + 5.
\end{cases} \]

This proves Lemma 2.4. \( \square \)

3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 1.1. If \( p = 24h + 1 \), then from Lemmas 2.1, 2.2 and 2.4 we have

\[ V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = \sum_{m=1}^{p-1} \left( \chi_2(m)\sqrt{p} + \overline{\chi}(m)\tau(\lambda) + \lambda(m)\tau(\overline{\lambda}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = p^{\frac{1}{3}}(p-6) - 5p^{\frac{1}{3}} - 2\sqrt{p}\alpha \tau^2(\lambda) - 5p^{\frac{1}{3}} - 2\sqrt{p}\alpha \tau^2(\overline{\lambda}) \]

\[ = p \left( p^2 - 16p - 4\alpha^2 \right). \]  

(26)

Applying Lemmas 2.1–2.4 we also have

\[ V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = \sum_{m=1}^{p-1} \left( \chi_2(m)\sqrt{p} + \overline{\chi}(m)\tau(\lambda) + \lambda(m)\tau(\overline{\lambda}) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
Applying (28) and the method of proving (29) we also have

\[
V_3(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^2 \left( 2p^2 + 4p - 2p^2 \right).
\]  

(27)

If \( p = 8h + 1 \), then from (6) we have

\[
B^1(m) = \left( \chi_2(m) \sqrt{p} + \lambda(m) \right)^3
\]

\[
= 7\chi_2(m)p^2 + 4p + 5p(\lambda(m) + \lambda(m)) + 2(\lambda(m) + \lambda(m)) \sqrt{p}.
\]  

(28)

So if \( p = 24h + 1 \), then from (28), Lemmas 2.1–2.4 we have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 7p^3(p - 6) + 4p(\lambda^2) - 5p \lambda^3(\lambda) + 2(\lambda^3) + 2\sqrt{p} \lambda
\]

\[
= 2\sqrt{p} \lambda(\lambda) + 2(\lambda^3) + 2\sqrt{p} \lambda.
\]  

(29)

If \( p = 24h + 17 \), then from Lemmas 2.1–2.4 we have

\[
V_1(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^2(p - 8) - 5p^2 - 2\sqrt{p} \lambda^2 - 5p^2 - 2\sqrt{p} \lambda^2
\]

\[
= p^2(p - 8 + 18p + 4\lambda^2).
\]  

(30)

\[
V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(c) \sqrt{p} + 2\lambda(c) \sqrt{p} \lambda + 2\lambda(c) \sqrt{p} \lambda \right)^3
\]

\[
= p^2(2p^2 - 3p - 62\lambda).
\]  

(31)

Applying (28) and the method of proving (29) we also have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 7p^3(p - 8) - 4p(\lambda^2) + 6p(\lambda^3) - 5p \lambda^3(\lambda) + 2(\lambda^3) + 2\sqrt{p} \lambda
\]

\[
= 2\sqrt{p} \lambda(\lambda) + 2(\lambda^3) + 2\sqrt{p} \lambda.
\]  

(32)

Similarly, if \( p = 24h + 5 \), then we have

\[
V_1(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= -p^2(p - 6) + p^2 - 2\sqrt{p} \lambda^2 + p^2 - 2\sqrt{p} \lambda^2
\]

\[
= -p(p^2 - 8p + 4\lambda^2).
\]  

(33)
If $p = 24h + 5$, then from (6) we have

$$B^3(m) = \left( \chi_2(m) \sqrt{\beta \tau (\lambda)} + \lambda(m) \tau (\tilde{\lambda}) \right) \sqrt{\beta} \alpha.$$  \hfill \text{(34)}

So from (35) and the method of proving (29) we have

$$V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= \sum_{m=1}^{p-1} \left( 2 \chi_2(c) \sqrt{\beta \alpha} - p + 2\lambda(c) \sqrt{\beta \tau (\lambda)} + 2\tilde{\lambda}(c) \sqrt{\beta \tau (\tilde{\lambda})} \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= -p^2 \left( 2p \alpha - p - 22\alpha \right).$$  \hfill \text{(36)}

If $p = 24h + 13$, then (35), Lemmas 2.1–2.4 we have

$$V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= -p^2 \left( p - 4 \right) + p^2 - 2\sqrt{\beta} \alpha \tau^2 (\lambda) + p^2 - 2\sqrt{\beta} \alpha \tau^2 (\tilde{\lambda})$$

$$= -p \left( p^2 - 6p + 4\alpha^2 \right).$$  \hfill \text{(37)}

$$V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= \sum_{m=1}^{p-1} \left( 2 \chi_2(c) \sqrt{\beta \alpha} - p + 2\lambda(c) \sqrt{\beta \tau (\lambda)} + 2\tilde{\lambda}(c) \sqrt{\beta \tau (\tilde{\lambda})} \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= -p^2 \left( 2p \alpha - p - 18\alpha \right).$$  \hfill \text{(38)}

$$V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3$$

$$= 5p^3 \left( p - 4 \right) + 6p\alpha \left( p^2 - 6p\alpha \right) - \tau (\tilde{\lambda}) \left( \tau (\lambda) + 2\sqrt{\beta} \alpha \tau (\tilde{\lambda}) \right)$$

$$-p \tau (\lambda) \left( \tau (\tilde{\lambda}) + 2\sqrt{\beta} \alpha \tau (\lambda) \right) - 2\sqrt{\beta} \alpha \tau (\lambda) \left( \tau (\lambda) + 2\sqrt{\beta} \alpha \tau (\tilde{\lambda}) \right)$$

$$= p^2 \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right).$$  \hfill \text{(39)}
Finally, note that if \( p = 8h + 1 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we have the identity
\[
B^4(m) = 6pB^2(m) + 8\alpha B(m) - p\left(p - 4\alpha^2\right).
\] (40)

For any prime \( p = 24h + 1 \) and integer \( k \geq 4 \), from (26), (27), (29) and (40) we may immediately deduce the fourth-order linear recurrence formula
\[
V_k(p) = \frac{p-1}{2} \sum_{m=1}^{p-1} B^4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right)^3
= \sum_{m=1}^{p-1} B^4(m) \left( 6pB^2(m) + 8\alpha B(m) - p\left(p - 4\alpha^2\right) \right) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right)^3
= 6pV_{k-2}(p) + 8\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right)V_{k-4}(p),
\]
where the first four values \( V_0(p) = p^2 - 6\alpha, V_1(p) = p\left(p^2 - 16p - 4\alpha^2\right), V_2(p) = p^2\left(2p\alpha + 3p - 58\alpha\right) \) and \( V_3(p) = p^2\left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right) \).

This proves Theorem 1.1.

If \( p = 24h + 17 \), then from (30), (31), (32) and (40) we have
\[
V_k(p) = 6pV_{k-2}(p) + 8\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right)V_{k-4}(p),
\]
where the first four values \( V_0(p) = -p^2 - 6\alpha, V_1(p) = p\left(p^2 - 18p - 4\alpha^2\right), V_2(p) = p^2\left(2p\alpha - 3p - 62\alpha\right) \) and \( V_3(p) = p^2\left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right) \).

This proves Theorem 1.2.

If \( p = 8h + 5 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we also have
\[
B^4(m) = -2pB^2(m) + 8\alpha B(m) - p\left(9p - 4\alpha^2\right).
\] (41)

For any prime \( p = 24h + 5 \) and integer \( k \geq 4 \), from (33), (34), (35) and (41) we can deduce the fourth-order linear recurrence formula
\[
V_k(p) = -2pV_{k-2}(p) + 8\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right)V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = -(p^2 + 6\alpha), V_1(p) = -p\left(p^2 - 8p + 4\alpha^2\right), V_2(p) = -p^2\left(2p\alpha - p - 22\alpha\right) \) and \( V_3(p) = p^2\left(5p^2 - 6p\alpha - 28p - 36\alpha^2\right) \).

This proves Theorem 1.3.

If \( p = 24h + 13 \), then from (37), (38), (39) and (41) we also have
\[
V_k(p) = -2pV_{k-2}(p) + 8\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right)V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = p^2 - 6\alpha, V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right), V_2(p) = -p^2\left(2p\alpha + p - 18\alpha\right) \) and \( V_3(p) = p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right) \).

This completes the proofs of our all results.

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