C*-algebras associated with reversible extensions of logistic maps

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Abstract. The construction of reversible extensions of dynamical systems presented in a previous paper by the author and A. V. Lebedev is enhanced, so that it applies to arbitrary mappings (not necessarily with open range). It is based on calculating the maximal ideal space of C*-algebras that extends endomorphisms to partial automorphisms via partial isometric representations, and involves a new set of ‘parameters’ (the role of parameters is played by chosen sets or ideals).

As model examples, we give a thorough description of reversible extensions of logistic maps and a classification of systems associated with compression of unitaries generating homeomorphisms of the circle.

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§ 1. Introduction

A general C*-method of construction of reversible extensions of irreversible dynamical systems was developed in [1], where in particular a complete description of the maximal ideal spaces of the arising C*-algebras was given. These algebras naturally spring out in the spectral analysis of weighted shift operators and transfer operators (see, in particular, [2] and [3]) and, as is shown in [1], their maximal ideal spaces are tightly related to inverse (projective) limits of dynamical systems.

As a C*-algebraic basis of their construction the authors of [1] explored the leading concept of [4] — an algebra whose elements play the role of Fourier coefficients in partial isometric extensions of C*-algebras. Namely, in [4] there was studied the C*-algebra $C^*(\mathcal{A}, U)$ generated by a *-algebra $\mathcal{A} \subset L(H)$, $1 \in \mathcal{A}$, and an operator $U \in L(H)$ under the assumption that $\mathcal{A}$ has the following three properties:

\begin{align*}
\mathcal{A} \ni a & \rightarrow \delta(a) := UaU^* \in \mathcal{A}, \\
\mathcal{A} \ni a & \rightarrow \delta^*(a) := U^*aU \in \mathcal{A}, \\
Ua = \delta(a)U, & \quad a \in \mathcal{A}.
\end{align*}

In this event $\mathcal{A}$ is called the coefficient algebra of $C^*(\mathcal{A}, U)$, cf. [4], Proposition 2.4.

One has to stress that such objects are the major structural elements of the most
successful crossed product constructions, like the ones developed by Cuntz and Krieger [5], [6], Paschke [7], Murphy [8], Exel [9] and others. In general, conditions (1.1)–(1.3) imply that $\delta(\cdot) = U(\cdot)U^*$ is an endomorphism of $\mathcal{A}$ (then $U$ is necessarily a partial symmetry) and $\delta_*(\cdot) = U(\cdot)U^*$ is a unique nondegenerate transfer operator for $\delta: \mathcal{A} \to \mathcal{A}$ (in the sense of [10]), see [11]. The general cross-product based on relations (1.1)–(1.3) is developed in [12], [13]. As is shown in [14], it could be viewed as the crossed product by a Hilbert bimodule [15], and hence it is one of the fundamental models of relative Cuntz-Pimsner algebras [16], $C^*$-algebras associated with $C^*$-correspondences [17], [18] and Doplicher-R Roberts algebras [19], see [20].

In this article we develop the $C^*$-formalism of [1] and apply it to a series of classical dynamical systems, in order to get the description of maximal ideal spaces of $C^*$-algebras associated with their reversible extensions. We recall that the starting point of [1] was a commutative unital $C^*$-subalgebra $\mathcal{A} \subseteq L(H)$ and an endomorphism $\delta: \mathcal{A} \to \mathcal{A}$ such that

$$\mathcal{A} \ni a \to \delta(a) := UaU^* \in \mathcal{A}, \quad U^*U \in \mathcal{A},$$

(1.4) for a certain $U \in L(H)$. Then (1.1) and (1.3) hold, and the $C^*$-algebra

$$\mathcal{B} = C^* \left( \bigcup_{n=0}^{\infty} U^* \mathcal{A} U^n \right)$$

generated by $\bigcup_{n=0}^{\infty} U^* \mathcal{A} U^n$ is the smallest (still commutative) coefficient $C^*$-algebra of $C^*(\mathcal{A}, U)$ such that $\mathcal{A} \subseteq \mathcal{B}$. The passage from $\mathcal{A}$ to $\mathcal{B}$ corresponds to passage from irreversible to reversible dynamics. Namely, endomorphisms $\delta: \mathcal{A} \to \mathcal{A}$ and $\delta: \mathcal{B} \to \mathcal{B}$ are given, via Gelfand transform, by partial dynamical systems $(M, \alpha)$ and $(\tilde{M}, \tilde{\alpha})$, where $(\tilde{M}, \tilde{\alpha})$ may be viewed as a universal reversible extension of $(M, \alpha)$ (we will make the latter statement precise in Theorem 3.14). The authors of [1] gave a complete description of $(\tilde{M}, \tilde{\alpha})$ in terms of $(M, \alpha)$ and noticed that $(\tilde{M}, \tilde{\alpha})$ contains, as a subsystem, the inverse limit of $(M, \alpha)$. This indicates that the structure of the $C^*$-algebra $\mathcal{B}$ is related to hyperbolic attractors, [21]–[23] (such as solenoids or horseshoes of Smale); irreversible continua [24] (the most known are the Brouwer-Janiszewski-Knaster continuum, or Knaster’s pseudoarc); and systems associated with classical substitution tilings [25] (these include tilings of Penrose, Amman, Fibonacci, Morse etc.).

The results of [1], however, have one drawback. The only seemingly technical assumption $U^*U \in \mathcal{A}$ implies that the image of $\alpha$ is necessarily open, which in turn excludes many important examples. As the first step in the present paper we eliminate this inconvenience. The key hint on how to overcome the mentioned drawback is given in [1], Remark 3.7. Namely, one has to pass from the $C^*$-algebra $\mathcal{A}$ to the $C^*$-algebra $\mathcal{A}_+ := C^*(\mathcal{A}, U^*U)$ generated by $\mathcal{A}$ and the projection $U^*U$ and apply the $C^*$-method of the reversible extension construction to the dynamical system generated on $\mathcal{A}_+$. In §§2 and 3 we provide the corresponding analysis and as a result obtain a description of the extended system $(\tilde{M}, \tilde{\alpha})$ under the conditions (1.1), (1.3) which are weaker than (1.4). As we show in Theorem 3.8, these axioms embrace all endomorphisms of $\mathcal{A}$, and thereby all partial dynamical systems $(M, \alpha)$. The principal novelty here is uncovering the fact that $(\tilde{M}, \tilde{\alpha})$ depends
not only on \((M, \alpha)\) but also on a certain set of parameters \(Y \subset X\) (or ideals in \(\mathcal{A}\)). This new observation has a number of interesting consequences. For instance, we get nontrivial results implementing our method to (already) reversible dynamical systems, such as homeomorphisms of the circle. Nevertheless, our primary example and one of our main goals is a depictive presentation of \(C^*\)-algebras associated with reversible extensions of the family of \(\text{logistic maps} \ \alpha_\lambda : [0, 1] \to [0, 1]:\)

\[
\alpha_\lambda(x) = 4\lambda x(1 - x), \quad 0 < \lambda \leq 1.
\]  

(1.5)

In the process of portraying the maximal ideal spaces of arising \(C^*\)-algebras, apart from the developed formalism, we take advantage of the results concerning inverse limits of logistic maps [26], [27]. In particular, we discuss in detail how the extended systems are influenced by such phenomena as \(\text{bifurcations or chaos.}\)

The paper is organized as follows. In \(\S\) 2 we introduce notation and generalize or adapt from [1] the basic concepts required for presentation of the results of the article. Here we also discuss a general \(C^*\)-method of extending partial dynamical systems. The main result of [1], description of maximal ideal spaces of \(C^*\)-algebras corresponding to reversible extensions of \(C^*\)-dynamical systems, is refined in \(\S\) 3, where apart from giving a purely topological definition we characterize such systems as universal objects. Section 4 is devoted to presentation of reversible extensions of the logistic family, and finally, in \(\S\) 5 we classify the \(C^*\)-algebras associated with homeomorphisms of the circle via their rotation numbers.

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\(\S\) 2. Preliminaries. Endomorphisms of commutative \(C^*\)-algebras, dynamical systems and their extensions

Throughout the article we let \(\mathcal{A}\) be a unital commutative \(C^*\)-algebra. By using the Gelfand transform we assume the identification \(\mathcal{A} = C(M),\) where \(M = M(\mathcal{A})\) is the \(\text{maximal ideal space} \) (also called \(\text{spectrum}\)) of the algebra \(\mathcal{A}\).

2.1. Endomorphisms and partial dynamical systems. It is well known (see, for example, [1], Theorem 2.2) that every endomorphism \(\delta : \mathcal{A} \to \mathcal{A}\) is of the form

\[
\delta(a) = \begin{cases} 
  a(\alpha(x)), & x \in \Delta, \\
  0, & x \notin \Delta,
\end{cases}
\]

where \(\alpha : \Delta \to M\) is a continuous mapping defined on a closed and open (briefly clopen) subset \(\Delta \subset M\). Namely, treating points of \(M\) as functionals on \(\mathcal{A}\) we have

\[
\Delta = \{x \in M : x(\delta(1)) \neq 0\} \quad \text{and} \quad \alpha = \delta^*|_\Delta,
\]

where \(\delta^*\) is the dual operator to \(\delta : \mathcal{A} \to \mathcal{A}\). Therefore we will refer to \(\alpha : \Delta \to M\) as a \(\text{mapping dual to endomorphism} \ \delta\).

To start with we describe the objects related to algebras and endomorphisms that are necessary for our further presentation.

For every subset \(I \subset \mathcal{A}\) the set

\[
hull(I) := \{x \in M : x(I) = 0\}
\]
is a closed subset of $M$, and if $I$ is an ideal in $\mathcal{A}$, then $I = C_{\text{hull}(I)}(M)$, where $C_K(M)$ stands for the set of continuous functions on $M$ vanishing on $K \subset M$. Plainly, in the above notation we have

$$\text{hull}(\ker \delta) = \alpha(\Delta).$$

Since $\Delta$ is closed and $\alpha$ is continuous it follows that $\alpha(\Delta)$ is closed as well, and evidently $\alpha(\Delta)$ is open iff the characteristic function of $\alpha(\Delta)$ belongs to $C(M)$ (in which case it is a unit in $\ker \delta = C_{\alpha(\Delta)}(M)$). Accordingly, we arrive at

**Proposition 2.1.** The image of the mapping $\alpha$ dual to an endomorphism $\delta: \mathcal{A} \rightarrow \mathcal{A}$ is open if and only if the kernel of $\delta$ is unital.

The annihilator of an ideal $I$ in $\mathcal{A}$ is the set

$$I^\perp := \{ a \in \mathcal{A} : aI = \{0\} \}.$$ 

Clearly, $I^\perp$ is an ideal and it could be equivalently defined as the largest ideal in $\mathcal{A}$ such that $I \cap I^\perp = \{0\}$. In particular, if the kernel of $\delta: \mathcal{A} \rightarrow \mathcal{A}$ is unital, then $\mathcal{A}$ admits decomposition into the following direct sum of ideals

$$\mathcal{A} = \ker \delta \oplus (\ker \delta)^\perp, \quad (2.1)$$

and $\delta$ yields an isomorphism between the ideal $(\ker \delta)^\perp$ and the subalgebra $\delta(\mathcal{A}) \subset \mathcal{A}$.

**Definition 2.2.** An endomorphism $\delta: \mathcal{A} \rightarrow \mathcal{A}$ with unital kernel $\ker \delta$ and the image $\delta(\mathcal{A})$, which is an ideal in $\mathcal{A}$ will be called a partial automorphism of $\mathcal{A}$.

Since $\delta(\mathcal{A})$ is an ideal in $\mathcal{A}$ iff $\delta(\mathcal{A}) = \delta(1)\mathcal{A}$, it follows that $\delta$ is a partial automorphism of $\mathcal{A}$ iff the algebra $\mathcal{A}$ admits two decompositions into a direct sum of ideals

$$\mathcal{A} = \ker \delta \oplus (\ker \delta)^\perp, \quad \mathcal{A} = \delta(\mathcal{A}) \oplus \delta(\mathcal{A})^\perp.$$ 

If this is the case, then we denote by $\delta_*: \delta(\mathcal{A}) \rightarrow (\ker \delta)^\perp$ the inverse to the isomorphism $\delta: (\ker \delta)^\perp \rightarrow \delta(\mathcal{A})$ and prolong $\delta_*$ onto $\mathcal{A}$ by putting $\delta_*|_{\delta(\mathcal{A})^\perp} \equiv 0$. Clearly, $\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ is a partial automorphism and

$$\delta \circ \delta_* \circ \delta = \delta, \quad \delta_* \circ \delta \circ \delta_* = \delta_*.$$ \quad (2.2)

Hence $\delta_*$ is the so-called generalized inverse to $\delta$.

The next proposition shows a deeper relation between the objects introduced above.

**Proposition 2.3.** Endomorphism $\delta: \mathcal{A} \rightarrow \mathcal{A}$ is a partial automorphism if and only if its dual map $\alpha: \Delta \rightarrow \alpha(\Delta)$ is a homeomorphism and $\alpha(\Delta)$ is clopen in $M$.

Moreover, if $\delta$ is a partial automorphism, then there is a unique partial automorphism $\delta_*: \mathcal{A} \rightarrow \mathcal{A}$ which is a generalized inverse for $\delta$, and it is given by

$$\delta_*(a) = \begin{cases} a(\alpha^{-1}(x)), & x \in \alpha(\Delta), \\ 0, & x \notin \alpha(\Delta), \end{cases} \quad a \in \mathcal{A} = C(M).$$
Proof. If \( \delta : \mathcal{A} \rightarrow \mathcal{A} \) is a partial automorphism, then \( \alpha(\Delta) \) is clopen (by Proposition 2.1) and \( \alpha : \Delta \rightarrow \alpha(\Delta) \) is a homeomorphism since it is a mapping dual to the isomorphism \( \delta : (\ker \delta) \perp \rightarrow \delta(\mathcal{A}) \). The converse implication is straightforward. Suppose now that \( \delta \) and \( \delta_* \) are partial automorphisms satisfying (2.2). Then from \( \delta = \delta \circ \delta_* \circ \delta \) we get that \( \ker \delta_* \cap \delta(\mathcal{A}) = \{0\} \); similarly, from \( \delta(\mathcal{A}) \subset (\ker \delta^*)_\perp \) and \( \delta_* = \delta_* \circ \delta \circ \delta_* \) we get that \( \delta_*(\delta(\mathcal{A})) = \delta_* (\mathcal{A}) \). But since \( \delta_* \) is a partial automorphism, the latter relation gives \( (\ker \delta^*)_\perp \subset \delta(\mathcal{A}) \) and therefore \( (\ker \delta^*)_\perp = \delta(\mathcal{A}) \). By symmetry we also have \( (\ker \delta^*)_\perp \subset \delta(\mathcal{A}) \) and thus it follows that \( \delta_* : \delta(\mathcal{A}) \rightarrow (\ker \delta)^\perp \) coincides with the inverse for \( \delta : (\ker \delta)^\perp \rightarrow \delta(\mathcal{A}) \). In particular, \( \delta_* \) is uniquely determined by \( \delta \) and its dual mapping is \( \alpha^{-1} \).

The above consideration makes it natural to adopt the following definitions, cf. [1], Definitions 2.4 and 2.6.

**Definition 2.4.** By a (partial) dynamical system we will mean a triple \( (M, \Delta, \alpha) \), where \( M \) is a compact Hausdorff space, \( \Delta \) a clopen subset of \( M \), and \( \alpha : \Delta \rightarrow M \) a continuous map. Unless a misunderstanding can arise, we will simply write \( (M, \alpha) \).

**Definition 2.5.** We will say that a partial dynamical system \( (M, \Delta, \alpha) \) is reversible if \( \alpha(\Delta) \) is an open subset of \( M \) and the map \( \alpha : \Delta \rightarrow \alpha(\Delta) \) is a homeomorphism (so that the triple \( (M, \alpha(\Delta), \alpha^{-1}) \) is also a partial dynamical system).

**2.2. Extensions of partial dynamical systems and endomorphisms.** The main concept of the \( C^* \)-method of construction of reversible extensions of irreversible dynamical systems is given in [1]. The present and the next §§ 2.3, 2.4 and 3 are devoted to a description of the main structural blocks of this construction. Moreover we also give a refinement of the construction that is necessary for a complete analysis of the partial dynamical systems under investigation in the paper, and establish universality and minimality of natural reversible extensions (Theorem 3.14).

**Definition 2.6.** Let \( (M_\alpha, \Delta_\alpha, \alpha) \) and \( (M_\beta, \Delta_\beta, \beta) \) be partial dynamical systems. We say that a surjective continuous map \( \Psi : M_\beta \rightarrow M_\alpha \) is a semiconjugacy (or a factor map) if and only if the following conditions hold:

\[
\Psi^{-1}(\Delta_\alpha) = \Delta_\beta, \quad (2.3)
\]
\[
\alpha(\Psi(x)) = \Psi(\beta(x)), \quad x \in \Delta_\beta. \quad (2.4)
\]

If there is a semiconjugacy from \( (M_\beta, \beta) \) to \( (M_\alpha, \alpha) \) we say that \( (M_\beta, \beta) \) is an extension of \( (M_\alpha, \alpha) \) and \( (M_\alpha, \alpha) \) is a factor of \( (M_\beta, \beta) \). If additionally the system \( (M_\beta, \beta) \) is reversible we call it a reversible extension of \( (M_\alpha, \alpha) \). If the factor map is one-to-one, then its inverse is also a factor map, and we say that \( (M_\beta, \beta) \) and \( (M_\alpha, \alpha) \) are conjugated, or equivalent.

The next proposition clarifies the role of the objects introduced in the above definition and shows that the class of partial dynamical systems with factor maps as morphisms form a category dual to the category of endomorphisms of unital \( C^* \)-algebras, where the role of morphisms is played by unital monomorphisms that conjugate endomorphisms.
Proposition 2.7. Let \( \delta : \mathcal{A} \to \mathcal{A} \) and \( \gamma : \mathcal{B} \to \mathcal{B} \) be endomorphisms of unital commutative \( \mathcal{C}^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \), and let \( (M(\mathcal{A}), \alpha) \) and \( (M(\mathcal{B}), \beta) \) be the corresponding dual partial dynamical systems. Endomorphism \( \gamma \) extends \( \delta \) in the sense that there exists a unital monomorphism \( T : \mathcal{A} \to \mathcal{B} \) such that

\[
T \circ \delta = \gamma \circ T \tag{2.5}
\]

if and only if the partial dynamical system \( (M(\mathcal{B}), \beta) \) is an extension of the system \( (M(\mathcal{A}), \alpha) \). Furthermore, the factor map \( \Psi : M(\mathcal{B}) \to M(\mathcal{A}) \) is the dual map to the unital monomorphism \( T \) satisfying (2.5): \( T(a) = a \circ \Psi \), \( a \in \mathcal{A} = C(M(\mathcal{A})) \).

Proof. A mapping dual to a unital monomorphism \( T : \mathcal{A} \to \mathcal{B} \) maps \( M(\mathcal{B}) \) onto \( M(\mathcal{A}) \) and hence \( T \) is an operator of composition with a surjection \( \Psi : M(\mathcal{B}) \to M(\mathcal{A}) \) where \( \Psi = T^*|_{M(\mathcal{B})} \).

Using these formulas one checks that (2.5) holds iff \( \Psi \) satisfies (2.3) and (2.4).

Remark 2.8. We have to stress that the extension described in Definition 2.6 is a slightly different (weaker) notion than the corresponding one described in [1], Definition 2.7. Namely, note that (2.3) is equivalent to two relations

\[
\Psi(\Delta_\beta) = \Delta_\alpha, \quad \Psi(M_\beta \setminus \Delta_\beta) = M_\alpha \setminus \Delta_\alpha, \tag{2.6}
\]

and (2.4) implies that

\[
\Psi(M_\beta \setminus \beta(\Delta_\beta)) \supset M_\alpha \setminus \alpha(\Delta_\alpha). \tag{2.7}
\]

However, unlike [1], we allow \( \alpha(\Delta_\alpha) \) not to be open (this is vital for the applications considered in this article). Moreover, as the further part of the article will show, the principal situation of interest is the case when \( \beta(\Delta_\beta) \) is open; thus we cannot require to have equality in (2.6) (as in [1], Definition 2.7).

Continuing the above remark one may consider the following consequence of (2.6):

\[
\Psi(M_\beta \setminus \beta(\Delta_\beta)) \supset M_\alpha \setminus \alpha(\Delta_\alpha). \tag{2.7}
\]

In particular, as will be shown in the article, by refining (fixing) the left hand part of this inclusion one can obtain various extensions naturally arising in the analysis of dynamical systems.

Definition 2.9. Let \( (M_\alpha, \Delta_\alpha, \alpha) \) and \( (M_\beta, \Delta_\beta, \beta) \) be partial dynamical systems and let \( Y \subset M_\alpha \). If there is a semiconjugacy \( \Psi : M_\beta \to M_\alpha \) such that

\[
\Psi(M_\beta \setminus \beta(\Delta_\beta)) = Y, \tag{2.8}
\]

then we say that \( (M_\beta, \Delta_\beta, \beta) \) is an extension of \( (M_\alpha, \Delta_\alpha, \alpha) \) associated with \( Y \).
Remark 2.10. Plainly, the set $Y$ satisfying (2.8) is necessarily closed and contains $M_\alpha \setminus \alpha(\Delta_\alpha)$. Conversely, for any such set $Y$ one easily constructs an extension of $(M_\alpha, \Delta_\alpha, \alpha)$ associated with $Y$, see for instance Fig. 1.

Let us describe now the relation between extensions of dynamical systems associated with $Y$ and extensions of endomorphisms associated with ideals.

Suppose that $\delta: \mathcal{A} \to \mathcal{A}$ and $\gamma: \mathcal{B} \to \mathcal{B}$ are endomorphisms conjugated by $T$, as in Proposition 2.7. Then (2.5) implies the relations

$$\ker \delta = T^{-1}(\ker \gamma), \quad (\ker \delta)^\perp \supset T^{-1}((\ker \gamma)^\perp),$$

where by taking hulls the former yields (2.3) and the latter gives (2.7). In particular,

$$\text{hull}((\ker \delta)^\perp) = M_\alpha \setminus \alpha(\Delta_\alpha), \quad \text{hull}(T^{-1}((\ker \gamma)^\perp)) = \Psi(M_\beta \setminus \beta(\Delta_\beta)),$$

where $\Delta_\alpha$ and $\Delta_\beta$ are the domains of $\alpha$ and $\beta$, respectively, and $\Psi$ is the dual mapping to $T$. Accordingly, we arrive at

**Theorem 2.11.** Let $J$ be an ideal in $\mathcal{A}$ and set $Y = \text{hull}(J)$. Under the notation of Proposition 2.7, the system $(M(\mathcal{B}), \beta)$ is an extension of the system $(M(\mathcal{A}), \alpha)$ associated with $Y$ if and only if there exists a unital monomorphism $T: \mathcal{A} \to \mathcal{B}$ such that

$$T \circ \delta = \gamma \circ T, \quad T^{-1}((\ker \gamma)^\perp) = J. \quad (2.10)$$

If this is the case, then $J \subset (\ker \delta)^\perp$ (equivalently $Y \supset M(\mathcal{A}) \setminus \alpha(\Delta_\alpha)$), so that endomorphism $\gamma$ extending $\delta$ ‘shrinks the annihilator of its kernel’ to $J$.

Remark 2.12. Assuming the identification $\mathcal{A} \subset \mathcal{B}$ ($\mathcal{A} \cong T(\mathcal{A}) \subset \mathcal{B}$) we have $J = (\ker \gamma)^\perp \cap \mathcal{A}$ and relations (2.9) tell that $\gamma$ may enlarge the kernel of $\delta$ but only outside $\mathcal{A}$, that is $(\ker \gamma) \cap \mathcal{A} = \ker \delta$. In other words, the only reason why $J$ may be strictly smaller than $(\ker \delta)^\perp$ is that $\ker \gamma \setminus \mathcal{A}$ is nonempty. Thus it seems natural to look for extensions where $J = (\ker \delta)^\perp$ (equivalently $Y = M(\mathcal{A}) \setminus \alpha(\Delta_\alpha)$), cf. [1], Definition 2.7 and Remark 2.8.

### 2.3. $C^*$-dynamical systems and partial dynamical systems.

Partial dynamical systems are naturally associated with $C^*$-dynamical systems and representations of the latter ones in turn are defined by actions of partial isometries in Hilbert spaces. Discussion of this relationship is the theme of the present subsection.

**Definition 2.13.** By a (concrete) $C^*$-dynamical system we mean a pair $(\mathcal{A}, U)$, where $\mathcal{A}$ is a commutative $C^*$-subalgebra of the algebra $L(H)$ of bounded operators in a Hilbert space $H$, $1 \in \mathcal{A}$, and $U \in L(H)$ is a partial isometry such that

$$U\mathcal{A}U^* \subset \mathcal{A}, \quad U^*U \in \mathcal{A}', \quad (2.11)$$

where $\mathcal{A}'$ is a commutator of $\mathcal{A}$ in $L(H)$.

Plainly, relations (2.11) imply that

$$\delta(a) := UaU^*, \quad a \in \mathcal{A}, \quad (2.12)$$

is an endomorphism of $\mathcal{A}$, and we will say that $\delta: \mathcal{A} \to \mathcal{A}$ is an endomorphism generated by $U$. Similarly, if $(M, \Delta, \alpha)$ is the partial dynamical system dual to $\delta$ we will say that $(M, \Delta, \alpha)$ is a partial dynamical system generated by $U$. 
Remark 2.14. The above definition extends [1], Definition 2.11, see relations (1.4). In particular, relations (2.11) and the assumption that \( U \) is a partial isometry are equivalent to (1.1) and (1.3), cf. [4], Proposition 2.2.

Remark 2.15. We will show (in Theorem 3.8) that for an arbitrary endomorphism \( \delta: \mathcal{A} \to \mathcal{A} \) the \( C^* \)-algebra \( \mathcal{A} \) may be identified with an algebra of operators acting in a certain Hilbert space \( H \) in such a way that \( \delta \) is generated by an operator \( U \in L(H) \). Hence every abstract \( C^* \)-dynamical system \((\mathcal{A}, \delta)\) can be represented by a concrete one.

As is indicated by Theorem 2.11, in order to investigate extensions of an endomorphism generated by \( U \in L(H) \) we need to identify the annihilator of its kernel in terms of \((\mathcal{A}, U)\). To this end, let us consider the sets

\[
(1 - U^*U)\mathcal{A} \cap \mathcal{A} = \{ a \in \mathcal{A} : U^*Ua = 0 \}, \quad U^*U \mathcal{A} \cap \mathcal{A} = \{ a \in \mathcal{A} : U^*Ua = a \},
\]

which are (mutually orthogonal) ideals in \( \mathcal{A} \). The next proposition shows the role of these sets in the description of endomorphisms.

Proposition 2.16. Let \((\mathcal{A}, U)\) be a \( C^* \)-dynamical system, \( \delta: \mathcal{A} \to \mathcal{A} \) an endomorphism generated by \( U \), \((M, \Delta, \alpha)\) a partial dynamical system dual to \( \delta: \mathcal{A} \to \mathcal{A} \) and \( Y = \text{hull}(U^*U \mathcal{A} \cap \mathcal{A}) \). Then

\[
\ker \delta = (1 - U^*U)\mathcal{A} \cap \mathcal{A} \quad \text{and} \quad (\ker \delta)^\perp \subset U^*U \mathcal{A} \cap \mathcal{A},
\]

that is, \( Y \supset M \setminus \alpha(\Delta) \). If additionally \( U^*U \in \mathcal{A} \), then

\[
\ker \delta = (1 - U^*U)\mathcal{A} \quad \text{and} \quad (\ker \delta)^\perp = U^*U \mathcal{A},
\]

that is, \( Y = M \setminus \alpha(\Delta) \). In particular

i) if \( U^*U \in \mathcal{A} \), then \( \ker \delta \) is unital (\( \alpha(\Delta) \) is clopen);

ii) if \( U \) is an isometry, then \( \delta: \mathcal{A} \to \mathcal{A} \) is a monomorphism (\( \alpha(\Delta) = M \));

iii) if \( U \) is unitary, then \( \delta: \mathcal{A} \to \mathcal{A} \) is a unital monomorphism (\( \Delta = M \) and \( \alpha: M \to M \) is surjective).

Proof. To see that \((1 - U^*U)\mathcal{A} \cap \mathcal{A}\) coincides with \( \ker \delta \) let \( a \in \mathcal{A} \) and note that

\[
U^*Ua = 0 \implies \delta(a) = UaU^* = U(U^*Ua)U^* = 0,
\]

\[
U^*Ua \neq 0 \implies U^*\delta(a)U = U^*UaU^*U = U^*Ua \neq 0 \implies \delta(a) \neq 0.
\]

Now, since \( \ker \delta \cap (U^*U \mathcal{A} \cap \mathcal{A}) = \{ 0 \} \) we have \( U^*U \mathcal{A} \cap \mathcal{A} \subset (\ker \delta)^\perp \), and if \( U^*U \in \mathcal{A} \), then the projection \((1 - U^*U)\) is the unit for \( \ker \delta = (1 - U^*U)\mathcal{A} \), and consequently \( U^*U \) is the unit for \((\ker \delta)^\perp \). Thus the remaining part of the proposition is straightforward; cf. [1], Proposition 2.3.

In the next Lemma 2.18 and Theorem 2.19 we give a description of \((\ker \delta)^\perp \) and obtain a restraint for \( U^*U \) in terms of the objects related to central carriers of elements in von Neumann algebras (for completeness of presentation we include in Lemma 2.18 the known properties i) and ii) of carriers).

Definition 2.17. By a carrier of a \( C^* \)-subalgebra \( K \subset L(H) \) we mean the orthogonal projection \( Q \in L(H) \) onto the subspace \( KH \subset H \).
Lemma 2.18. Let $Q \in L(H)$ be a carrier of an ideal $I$ in a (not necessarily commutative) $C^*$-algebra $\mathcal{A} \subset L(H)$. Then

i) $Q = \operatorname{s-lim}_{\lambda \in \Lambda} \mu_{\lambda}$, where $\{\mu_{\lambda}\}_{\lambda \in \Lambda}$ is an approximate unit for $I$ (the limit is taken in the strong operator topology);

ii) $Q \in \mathcal{A}'$;

iii) $I \subset Q \mathcal{A} \cap \mathcal{A}$ and $I^\perp = (1 - Q) \mathcal{A} \cap \mathcal{A}$.

Proof. By Vigier’s Theorem the limit $Q := \operatorname{s-lim}_{\lambda \in \Lambda} \mu_{\lambda}$ exists. For $a \in \mathcal{A}$ and $h \in H$ we have

$$Qah = \lim_{\lambda \in \Lambda} \mu_{\lambda} ah = \lim_{\lambda \in \Lambda} \mu_{\lambda} \left( \lim_{\lambda' \in \Lambda} a \mu_{\lambda'} \right) h = QaQh$$

and similarly

$$aQh = \lim_{\lambda \in \Lambda} \mu_{\lambda} ah = \lim_{\lambda \in \Lambda} \mu_{\lambda} \left( \lim_{\lambda' \in \Lambda} a \mu_{\lambda'} \right) h = QaQh.$$

Using these relations one deduces that $Q$ is a carrier of $I$ and $Q \in \mathcal{A}'$. Clearly, $I \subset Q \mathcal{A} \cap \mathcal{A}$ and $I^\perp \subset (1 - Q) \mathcal{A} \cap \mathcal{A}$. To see that $I^\perp \supset (1 - Q) \mathcal{A} \cap \mathcal{A}$ let $a \notin \mathcal{A} \setminus I^\perp$. Then there is $b \in I$ and $h \in H$ such that $abh \neq 0$. Hence $Qabh = abh \neq 0$ and therefore $Qa \neq 0$, which is equivalent to $(1 - Q)a \neq a$.

By item iii) in the above lemma we see that complements of carriers are ‘born’ to deal with annihilators of ideals.

Theorem 2.19. Under the notation of Proposition 2.16, let $P = 1 - Q \in L(H)$ be the complement of the carrier $Q$ of the ideal $(1 - U^*U) \mathcal{A} \cap \mathcal{A} = \ker \delta$ in $\mathcal{A}$. Then

$$U^*U \leq P, \quad P \in \mathcal{A}', \quad (\ker \delta)^\perp = P \mathcal{A} \cap \mathcal{A}.$$ 

Moreover, if $(\ker \delta)^\perp = U^*U \mathcal{A} \cap \mathcal{A}$ (equivalently $Y = \overline{M \setminus \alpha(\Delta)}$), which automatically holds when $U^*U = P$, then the implications in items i)–iii) in Proposition 2.16 are in fact equivalences.

Proof. By Lemma 2.18, $P \in \mathcal{A}'$ and $(\ker \delta)^\perp = P \mathcal{A} \cap \mathcal{A}$. By definition of $Q$, $Q \leq 1 - U^*U$ and thus $U^*U \leq P$. Now suppose that $(\ker \delta)^\perp = U^*U \mathcal{A} \cap \mathcal{A}$; by Proposition 2.16 condition $U^*U \in \mathcal{A}$ gives even more, namely, $(\ker \delta)^\perp = U^*U \mathcal{A}$.

We show the converses to implications in items i)–iii) of Proposition 2.16.

i) If $\ker \delta$ is unital, then $Q$ is the unit for $\ker \delta$ and consequently $P$ is the unit for $(\ker \delta)^\perp = U^*U \mathcal{A} \cap \mathcal{A}$. This implies that $U^*U = P \in \mathcal{A}$.

ii) If $\ker \delta = \{0\}$, then $\mathcal{A} = (\ker \delta)^\perp = U^*U \mathcal{A} \cap \mathcal{A}$, that is, $\mathcal{A} = U^*U \mathcal{A}$ and therefore $U$ is an isometry.

iii) It suffices to combine items i) and ii).

The main theme of [1] is a description of the $C^*$-method of construction of reversible $C^*$-dynamical systems, that is, systems such that not only $(\mathcal{A}, U)$ but also $(\mathcal{A}, U^*)$ is a $C^*$-dynamical system, and thus both of the mappings $\delta(a) := UaU^*$, $\delta_*(a) := U^*aU$, $a \in \mathcal{A}$, are endomorphisms of $\mathcal{A}$. We adopt an equivalent version of [1], Definition 2.15.
Definition 2.20. By a reversible $C^*$-dynamical system we mean a pair $(\mathcal{A}, U)$ where $\mathcal{A} \subset L(H)$ is commutative, $1 \in \mathcal{A}$ and $U \in L(H)$ is a partial isometry such that

$$U\mathcal{A}U^* \subset \mathcal{A}, \quad U^*\mathcal{A} \subset \mathcal{A}.$$ \hspace{1cm} (2.13)

Clearly, relations (2.13) are equivalent to the condition that both $(\mathcal{A}, U)$ and $(\mathcal{A}, U^*)$ are $C^*$-dynamical systems.

Proposition 2.21. If $(\mathcal{A}, U)$ is a reversible $C^*$-dynamical system, then endomorphisms $\delta: \mathcal{A} \to \mathcal{A}$ and $\delta_*: \mathcal{A} \to \mathcal{A}$ generated by $U$ and $U^*$ are mutually generalized inverse partial automorphisms and

$$(\ker \delta)^\perp = \delta_*(\mathcal{A}) = U^*U\mathcal{A}, \quad \delta(\mathcal{A}) = (\ker \delta_*)^\perp = UU^*\mathcal{A}.$$ 

In particular, the partial dynamical system $(M(\mathcal{A}), \alpha)$ generated by $U$ is reversible.

Proof. By Proposition 2.16 and the symmetry between $\delta$ and $\delta_*$ it suffices to show that $\delta(\mathcal{A}) = UU^*\mathcal{A}$, which follows because

$$\delta(\mathcal{A}) = U\mathcal{A}U^* = UU^*U\mathcal{A}U^* \subset UU^*\mathcal{A},$$

$$UU^*\mathcal{A} = UU^*\mathcal{A}UU^* = \delta(\delta_*(\mathcal{A})) \subset \delta(\mathcal{A}).$$

2.4. $C^*$-method of extending partial dynamical systems. Given a concrete $C^*$-dynamical system $(\mathcal{A}, U)$ we have at our disposal two mappings that are defined on the whole of the $C^*$-algebra $L(H)$:

$$\delta(a) := Ua^*, \quad \delta_*(a) := aU, \quad a \in L(H).$$ \hspace{1cm} (2.14)

Moreover, $\delta$ restricted to $\mathcal{A}$ is an endomorphism, and $\delta_*$ restricted to $\mathcal{A}$ is an endomorphism iff $(\mathcal{A}, U)$ is reversible. Obviously, $\delta$ (and similarly $\delta_*$) may define endomorphisms on many different subalgebras of $L(H)$, and if additionally such a subalgebra, say $\mathcal{B}$, contains $\mathcal{A}$, then it yields a natural extension $\delta: \mathcal{B} \to \mathcal{B}$ of the initial endomorphism $\delta: \mathcal{A} \to \mathcal{A}$, cf. [1], Definition 2.17. This in turn can be rewritten in the language of partial dynamical systems.

Theorem 2.22. Let $(\mathcal{A}, U)$ and $(\mathcal{B}, U)$ be $C^*$-dynamical systems and $(M(\mathcal{A}), \alpha)$ and $(M(\mathcal{B}), \beta)$ partial dynamical systems generated by $U$ on the maximal ideal spaces of $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\mathcal{A} \subset \mathcal{B}$, then $(M(\mathcal{B}), \beta)$ is an extension of $(M(\mathcal{A}), \alpha)$ associated with the set

$$Y = \text{hull}((\ker \delta|_\mathcal{B})^\perp \cap \mathcal{A}) = \text{hull}(P\mathcal{B} \cap \mathcal{A}),$$

where $(\ker \delta|_\mathcal{B})^\perp$ is the annihilator of the kernel of $\delta: \mathcal{B} \to \mathcal{B}$ and $P$ is the complement of the carrier of $(1 - U^*U)\mathcal{B} \cap \mathcal{B}$. Moreover,

i) $U^*U \in \mathcal{B} \implies Y = \text{hull}(U^*U\mathcal{A} \cap \mathcal{A}),$

ii) $U^*U \in \mathcal{A} \implies Y = M(\mathcal{A}) \setminus \alpha(\Delta).$
Proof. Since $\delta : \mathcal{B} \to \mathcal{B}$ is an extension of $\delta : \mathcal{A} \to \mathcal{A}$ in the sense of Proposition 2.7, where $T = \text{id}$, the first part of the assertion follows from Theorem 2.11 and Proposition 2.19. If additionally $U^*U \in \mathcal{B}$, then in view of Proposition 2.16 we have

$$(\ker \delta|_{\mathcal{B}}) ^\perp \cap \mathcal{A} = U^*U \mathcal{B} \cap \mathcal{A} = U^*U \mathcal{A} \cap \mathcal{A}.$$ 

Similarly, if $U^*U \in \mathcal{A}$, then

$$(\ker \delta|_{\mathcal{B}}) ^\perp \cap \mathcal{A} = U^*U \mathcal{A} \cap \mathcal{A} = U^*U A \cap \mathcal{A}.$$ 

cf. Proposition 2.1

Using the above method one can always obtain a reversible extension which (in the context of coefficient algebras) was first noticed in [4], cf. [1].

Theorem 2.23. Suppose that, under the notation of Theorem 2.22 $\mathcal{A} \subset \mathcal{B}$ and $(\mathcal{B}, U)$ is reversible. Then $(M(\mathcal{B}), \beta)$ is a reversible extension of $(M(\mathcal{A}), \alpha)$ associated with the set

$$Y = \text{hull}(U^*U \mathcal{A} \cap \mathcal{A}).$$

Moreover, for any $C^*$-dynamical system $(\mathcal{A}, U)$ there exists a minimal reversible $C^*$-dynamical system $(\mathcal{B}, U)$ such that $\mathcal{A} \subset \mathcal{B}$. Namely, the $C^*$-algebra

$$\mathcal{B} = C^* \left( \bigcup_{n=0}^{\infty} U^* \mathcal{A} U^n \right),$$

generated by $\bigcup_{n=0}^{\infty} U^* \mathcal{A} U^n$ is commutative, and it is the smallest $C^*$-algebra that contains $\mathcal{A}$ and satisfies (2.13).

Proof. The first part of the assertion follows from Proposition 2.21 and Theorem 2.22 (since $U^*U = U^*1U \in \mathcal{B}$). Commutativity of the algebra $\mathcal{B}$ given by (2.15) and the property that both $\delta : \mathcal{B} \to \mathcal{B}$ and $\delta_* : \mathcal{B} \to \mathcal{B}$ are endomorphisms were established in [4], Proposition 4.1. The remaining part is straightforward.

§ 3. Natural reversible extensions of dynamical systems

In this section we give a complete purely topological description of partial dynamical systems corresponding to minimal reversible extensions of $C^*$-dynamical systems introduced in Theorem 2.23. Additionally, we characterize such systems as universal objects and discuss their relation to the notion of inverse limit, which therefore we recall now.

Definition 3.1. If $\alpha : M \to M$ is a continuous mapping of a topological space $M$, then the inverse limit of the inverse sequence $M \xleftarrow{\alpha} M \xleftarrow{\alpha} M \xleftarrow{\alpha} \cdots$ is the topological space of the form

$$\lim_{\leftarrow} (M, \alpha) := \left\{ (x_0, x_1, \ldots) \in \prod_{n \in \mathbb{N}} M : \alpha(x_{n+1}) = x_n, n \in \mathbb{N} \right\},$$

equipped with the product topology inherited from $\prod_{n \in \mathbb{N}} M$. Furthermore, on the space $\lim_{\leftarrow} (M, \alpha)$ we have a naturally defined homeomorphism

$$\tilde{\alpha}(x_0, x_1, \ldots) = (\alpha(x_0), x_0, x_1, \ldots), \quad (x_0, x_1, \ldots) \in \lim_{\leftarrow} (M, \alpha),$$
called a homeomorphism induced by the mapping $\alpha : M \to M$. 
3.1. $C^*$-dynamical approach. Throughout this section we let $(\mathcal{A}, U)$ be a $C^*$-dynamical system, $\delta$ and $\delta_*$ the mappings given by (2.14), and $\mathcal{B}$ the $C^*$-algebra from Theorem 2.23. We denote by $(M, \Delta, \alpha)$ the partial dynamical system dual to $\delta: \mathcal{A} \to \mathcal{A}$ and by $(\tilde{M}, \tilde{\Delta}, \tilde{\alpha})$ the reversible partial dynamical system dual to $\delta: \mathcal{B} \to \mathcal{B}$:

$$\mathcal{A} \cong C(M), \quad \mathcal{B} = C^*\left(\bigcup_{n=0}^{\infty} U^n \mathcal{A} U^n\right) \cong C(\tilde{M}).$$

The dynamical system $(\tilde{M}, \tilde{\Delta}, \tilde{\alpha})$ plays the principal role in the paper. As was noted in [1], Remark 3.7, to obtain description of $(f_M, e^{\Delta}, e^{\alpha})$ in terms of $(M, \Delta, \alpha)$ one has to apply the main result of [1] to the $C^*$-algebra $\mathcal{A}^+ := C^*(\mathcal{A}, U^*U)$ generated by $\mathcal{A}$ and the projection $U^*U$. Therefore we need to analyse the algebra $\mathcal{A}^+$ and the partial dynamical system dual to endomorphism $\delta: \mathcal{A}^+ \to \mathcal{A}^+$. Hereafter we proceed to the discussion of these objects.

3.1.1. Adjoining the projection $U^*U$ to the algebra $\mathcal{A}$. Plainly, the $C^*$-algebra $\mathcal{A}^+$ is the direct sum of ideals

$$\mathcal{A}^+ = U^*U \mathcal{A} \oplus (1 - U^*U) \mathcal{A},$$

where $(1 - U^*U) \mathcal{A} = \ker(\delta|_{\mathcal{A}^+})$ is the kernel of $\delta: \mathcal{A}^+ \to \mathcal{A}^+$, see Proposition 2.16. As a simple consequence, cf. for instance [28], Lemma 10.1.6, we get

**Proposition 3.2.** Algebra $\mathcal{A}^+$ is isomorphic to the direct sum of quotient algebras

$$\mathcal{A}^+ \cong \mathcal{A} / \ker(\delta|_{\mathcal{A}}) \oplus \mathcal{A} / (U^*U \mathcal{A} \cap \mathcal{A}),$$

where $\ker(\delta|_{\mathcal{A}}) = (1 - U^*U) \mathcal{A} \cap \mathcal{A}$ is the kernel of $\delta: \mathcal{A} \to \mathcal{A}$ and $U^*U \mathcal{A} \cap \mathcal{A}$ is an ideal contained in the annihilator $\ker(\delta|_{\mathcal{A}})^\perp$ of $\ker(\delta|_{\mathcal{A}})$. Under the isomorphism (3.2) endomorphism $\delta: \mathcal{A}^+ \to \mathcal{A}^+$ takes the form

$$a + \ker(\delta|_{\mathcal{A}}) \oplus b + (U^*U \mathcal{A} \cap \mathcal{A}) \mapsto \delta(a) + \ker(\delta|_{\mathcal{A}}) \oplus \delta(a) + (U^*U \mathcal{A} \cap \mathcal{A}).$$

**Remark 3.3.** The term $(1 - U^*U) \mathcal{A} \cong \mathcal{A} / (U^*U \mathcal{A} \cap \mathcal{A})$ is a unital $C^*$-algebra which coincides with the kernel of $\delta: \mathcal{A} \to \mathcal{A}$. Thus one may interpret adjoining the projection $U^*U$ to the algebra $\mathcal{A}$ as a unitization of the kernel of $\delta: \mathcal{A} \to \mathcal{A}$ (equivalently, compactification of the complement of the image of $\alpha$).

Let $(M_+, \Delta_+, \alpha_+)$ denote the partial dynamical system generated by $U$ on the maximal ideal space of $\mathcal{A}_+$. In view of Theorem 2.22, $(M_+, \alpha_+)$ is an extension of $(M, \alpha)$ associated with the set

$$Y = \text{hull}(U^*U \mathcal{A} \cap \mathcal{A}).$$

By passing in Proposition 3.2 to duals we get the complete description of the partial dynamical system $(M_+, \Delta_+, \alpha_+)$. 
Figure 1. (a) Partial dynamical system \((M, \alpha)\) generated by \(U\) on \(\mathcal{A}\); (b) partial dynamical system \((M_+, \alpha_+)\) generated by \(U\) on \(\mathcal{A}_+\).

**Proposition 3.4.** Under the above notation \(Y\) is the closed set containing \(M \setminus \alpha(\Delta)\), and the spectrum \(M_+\) of the algebra \(\mathcal{A}_+\) is presented as the following (topological) direct sum:

\[ M_+ = \alpha(\Delta) \sqcup Y. \]

The partial mapping \(\alpha_+\) dual to \(\delta:\ \mathcal{A}_+ \to \mathcal{A}_+\), is defined on the set

\[ \Delta_+ = (\alpha(\Delta) \cap \Delta) \sqcup (Y \cap \Delta) \]

and attains values in the first summand of \(M_+ = \alpha(\Delta) \sqcup Y\) acting by the formula

\[ \alpha_+(x) = \alpha(x), \quad x \in \Delta_+, \]

see Fig. 1. In particular, \(U^*U \in \mathcal{A}\), that is, \(\mathcal{A} = \mathcal{A}_+\) if and only if \(Y = M \setminus \alpha(\Delta)\), and then \(\alpha(\Delta)\) is clopen.

**Proof.** Since the maximal ideal spaces of the quotient algebras \(\mathcal{A}/\ker(\delta|_{\mathcal{A}})\) and \(\mathcal{A}/(U^*U\mathcal{A} \cap \mathcal{A})\) may be identified with the sets \(\text{hull}(\ker(\delta|_{\mathcal{A}})) = \alpha(\Delta)\) and \(\text{hull}(U^*U\mathcal{A} \cap \mathcal{A}) = Y\), the first part of the assertion follows from Proposition 3.2. For the second part apply Theorem 2.19.

Pictorially speaking, the space \(M_+\) arises from \((M, \alpha)\) by ‘cutting out’ \(\alpha(\Delta)\) and ‘replacing’ the set \(M \setminus \alpha(\Delta)\) with the closed set \(Y\) that contains \(M \setminus \alpha(\Delta)\). The image of \(\alpha_+\) is the set \(\alpha(\Delta)\) that has been ‘cut out’ and thus may be identified with the image of \(\alpha\). The domain of \(\alpha_+\) is enlarged with respect to the domain of \(\alpha\) by the set \(Y \cap (\Delta \cap \alpha(\Delta))\).

Now, as we have obtained the description of the system \((M_+, \Delta_+, \alpha_+)\) we pass to the main object of this subsection — the system \((\tilde{M}, \tilde{\Delta}, \tilde{\alpha})\).

### 3.1.2. Description of the system \((\tilde{M}, \tilde{\alpha})\) dual to \(\delta:\ \mathcal{B} \to \mathcal{B}\).

Let \(\tilde{x} \in \tilde{M}\) be a multiplicative linear functional on \(\mathcal{B}\). We associate with it a sequence of functionals on \(\mathcal{A}\) given by the formula

\[ x_n(a) := \tilde{x}(\delta_n^*(a)), \quad a \in \mathcal{A}, \quad n \in \mathbb{N}. \]

Since \(\delta_n:\ \mathcal{B} \to \mathcal{B}\) is an endomorphism, the functionals \(x_n:\ \mathcal{A} \to \mathbb{C}\) are multiplicative linear, and thus

\[ x_n \in M \quad \text{or} \quad x_n \equiv 0. \]
Moreover, since $\mathcal{B} = C^*(\bigcup_{n=0}^{\infty} \delta_n(\mathcal{A}))$ the sequence $(x_0, x_1, \ldots)$ determines $\overline{x}$ uniquely. Consequently we have the injective mapping
\[ \overline{x} \mapsto (x_0, x_1, \ldots, x_n, \ldots), \tag{3.3} \]
which embeds the space $\widetilde{M}$ into the Cartesian product $\prod_{n \in \mathbb{N}} (M \cup \{0\})$ of a countable number of copies of the space $M \cup \{0\}$.

The next theorem is a refinement of the main result of [1] on the objects under consideration in this paper.

**Theorem 3.5** (description of the reversible system $(\widetilde{M}, \widetilde{\alpha})$). The maximal ideal space of the algebra $\mathcal{B}$ may be identified via the mapping (3.3) with the following topological space
\[ \widetilde{M} = \bigcup_{N=0}^{\infty} M_N \cup M_\infty, \]
where the spaces
\[ M_N = \{ (x_0, x_1, \ldots, x_N, 0, \ldots) : x_n \in \Delta, \alpha(x_n) = x_{n-1}, n = 1, \ldots, N, x_N \in Y \}, \]
\[ M_\infty = \{ (x_0, x_1, \ldots) : x_n \in \Delta, \alpha(x_n) = x_{n-1}, n \geq 1 \} \]
are equipped with the product topology inherited from $\prod_{n \in \mathbb{N}} (M \cup \{0\})$, where $\{0\}$ is a clopen singleton, and $Y$ is a closed set containing $M \setminus \Delta$ (namely, $Y = \text{hull}(U^*U \mathcal{A} \cap \mathcal{A})$). Furthermore the partial homeomorphisms dual to partial automorphisms $\delta, \delta^*: \mathcal{B} \to \mathcal{B}$ are defined respectively on the clopen sets
\[ \widetilde{\Delta} = \{ (x_0, x_1, \ldots) \in \widetilde{M} : x_0 \in \Delta \}, \quad \widetilde{\alpha}(\widetilde{\Delta}) = \{ (x_0, x_1, \ldots) \in \widetilde{M} : x_1 \neq 0 \} \]
and act according to the formulae
\[ \widetilde{\alpha}(x_0, x_1, \ldots) = (\alpha(x_0), x_0, x_1, \ldots), \quad \widetilde{\alpha}^{-1}(x_0, x_1, \ldots) = (x_1, x_2, \ldots). \tag{3.4} \]

**Proof.** It suffices to apply [1], Theorems 3.5 and 4.1 to the $C^*$-dynamical system $(\mathcal{A}, U)$ and then use Proposition 3.4; cf. also [1], Remark 3.7.

**Remark 3.6.** The mapping $\Phi: \widetilde{M} \to M$ dual to the inclusion $\mathcal{A} \subset \mathcal{B}$ is given by the formula
\[ \Phi(x_0, x_1, \ldots) = x_0. \tag{3.5} \]
It is a factor map establishing that $(\widetilde{M}, \widetilde{\alpha})$ is a reversible extension of $(M, \alpha)$ associated with $Y$.

The next result shows that in the situations when $U$ belongs to a series of commonly exploited classes of operators the structure of $(\widetilde{M}, \widetilde{\alpha})$ becomes more transparent.

**Theorem 3.7.** Under the notation of Theorem 3.5, we have $M \setminus \alpha(\Delta) \subset Y$ and
i) if $U^*U \in \mathcal{A}$, then $Y = M \setminus \alpha(\Delta)$ and hence a sequence $(x_0, x_1, \ldots, x_N, 0, \ldots)$$ \]
is an element of $M_N$ iff
\[ x_N \notin \alpha(\Delta) \quad (\text{that is, } x_N \text{ does not have a preimage}) \]
and $x_n \in \Delta, \alpha(x_n) = x_{n-1}$ for $n = 1, \ldots, N$;
ii) if $U$ is an isometry, then $\alpha: \Delta \to M$ is a surjection, and thus

$$\tilde{M} = M_\infty;$$

iii) if $U$ is unitary, then $\alpha: M \to M$ is a surjection,

$$\tilde{M} = \lim_{\leftarrow}(M, \alpha)$$

and $\tilde{\alpha}: \tilde{M} \to \tilde{M}$ is a homeomorphism induced by $\alpha: M \to M$.

Proof. In view of Theorem 3.5 item i) follows from Proposition 3.4, whereas items ii), iii) follow by Proposition 2.16.

3.2. Construction of operators generating an arbitrary partial dynamical system $(M, \alpha)$. Let $(M, \Delta, \alpha)$ be a partial dynamical system and let $Y \subset M$ be a closed set containing $M \setminus \alpha(\Delta)$. Theorem 3.5 leads to a natural question: does there exist a $C^*$-dynamical system $(\mathcal{A}, U)$ generating the partial dynamical system $(M, \Delta, \alpha)$ and such that its reversible extension is associated with $Y$. In other words, whether all the objects described in Theorem 3.5 are realizable. The answer is: yes. Namely, following [1], 4.2 we use the explicit description of the reversible extension $(\tilde{M}, \tilde{\alpha})$ associated with $Y$ to give a simple construction of operators generating $(M, \alpha)$.

Let

$$(Uf)(\bar{x}) = \begin{cases} f(\tilde{\alpha}(\bar{x})), & \bar{x} \in \bar{\Delta}, \\ 0, & \bar{x} \notin \bar{\Delta}, \end{cases}$$

act in the Hilbert space $H = \ell^2(\tilde{M})$ (which may be treated as $L^2_\mu(\tilde{M})$ where $\mu$ is the counting measure), and let $\mathcal{A} \subset L(H)$ be the algebra consisting of operators of multiplication by functions from $C(\tilde{M})$ dependent only on the zeroth coordinate:

$$\mathcal{A} = \{a \in C(\tilde{M}) : a(\bar{x}) = a(x_0), \text{ where } \bar{x} = (x_0, x_1, \ldots) \in \tilde{M}\}.$$ 

Clearly, $\mathcal{A} \cong C(M)$ and $U$ generates on $M$ the partial mapping $\alpha$. The projection $U^*U$ is the operator of multiplication by the characteristic function of $\tilde{\alpha}(\bar{\Delta})$, and thus $U^*U \mathcal{A} \cap \mathcal{A} = \{a \in \mathcal{A} : U^*Ua = a\} \cong C_Y(M)$. Accordingly, we get

**Theorem 3.8.** Let $(M, \alpha)$ be an arbitrary partial dynamical system and let $Y$ be an arbitrary closed set containing $M \setminus \alpha(\Delta)$. Then there is a Hilbert space $H$, a $C^*$-algebra $\mathcal{A} \subset L(H)$ whose spectrum is homeomorphic to $M$ and a partial isometry $U \in L(H)$ such that

i) on the spectrum of $\mathcal{A}$ operator $U$ generates the partial mapping $\alpha$,

ii) on the spectrum of

$$\mathcal{B} := C^*\left(\bigcup_{n \in \mathbb{N}} U^{*n} \mathcal{A} U^n\right)$$

operator $U$ generates the reversible extension $(\tilde{M}, \tilde{\alpha})$ of $(M, \alpha)$ associated with $Y$.

**Remark 3.9.** It follows from [29], Example 1.6 (see [30], [20]), that concrete $C^*$-dynamical systems generating a fixed ‘abstract’ endomorphism $\delta: \mathcal{A} \to \mathcal{A}$ are
Remark 3.11. In the case when Definition 3.10.

Let \((M, \alpha)\) be a partial dynamical system and \(Y\) a closed subset of \(M\) containing \(M \setminus \alpha(\Delta)\). The partial dynamical system \((\tilde{M}, \tilde{\alpha})\) described in Theorem 3.5 will be called the natural reversible extension of \((M, \alpha)\) associated with \(Y\).

Remark 3.12. The system \((\tilde{M}, \tilde{\alpha})\) has an advantage over the system \((\lim\downarrow (M, \alpha), \tilde{\alpha})\). Namely, \((\tilde{M}, \tilde{\alpha})\) is always an extension of \((M, \alpha)\) whereas \((\lim\downarrow (M, \alpha), \tilde{\alpha})\) may degenerate. The next example illustrates this observation.

Example 3.13. Let \(\alpha: M \to M\) be a constant map with the only value being a non-isolated point \(p \in M\), see Fig. 2, (a). The only closed set containing \(M \setminus \alpha(\Delta) = M \setminus \{p\}\), is \(Y = M\). For this set the system \((M_+, \alpha_+)\) described in Proposition 3.4 arises from \((M, \alpha)\) by adjoining a clopen copy of the singleton \(\{p\}\), see Fig. 2, (b). The space \(\tilde{M}\) is a countable family of copies of \(M\) compactified with a single point \(M_\infty = \{p\}\), Fig. 2, (c). Finally, the space \(\lim\downarrow (M, \alpha), \tilde{\alpha}\) degenerates into a single point, Fig. 2, (d).

A hard piece of evidence that \((\tilde{M}, \tilde{\alpha})\) is an appropriate reversible counterpart of \((M, \alpha)\) is its identification as a universal minimal object that we now present.

Theorem 3.14 (universality of natural reversible extension). Let \((M, \Delta, \alpha)\) be a partial dynamical system, \((\tilde{M}, \tilde{\Delta}, \tilde{\alpha})\) its natural reversible extension associated with a set \(Y \subset M\) and \(\Phi\) the corresponding factor map (3.5).

i) If \((M_\beta, \Delta_\beta, \beta)\) is a reversible extension of \((M, \alpha)\) associated with \(Y\) and \(\Psi\) is the corresponding semiconjugacy, then there is a unique semiconjugacy \(\tilde{\Psi}\)
from \((M_\beta, \beta)\) to \((\widetilde{M}, \alpha)\) such that \(\Psi = \Phi \circ \tilde{\Psi}\), that is, the diagram

\[
\begin{array}{ccc}
(M_\beta, \beta) & \xrightarrow{\tilde{\Psi}} & (\widetilde{M}, \alpha) \\
\downarrow{\Psi} & & \downarrow{\Phi} \\
(M, \alpha) & & \\
\end{array}
\]

commutes, and \(\tilde{\Psi}(M_\beta \setminus \beta(\Delta_\beta)) = \widetilde{M} \setminus \alpha(\Delta)\), so that \((M_\beta, \beta)\) is an extension of \((\widetilde{M}, \alpha)\) associated with \(\widetilde{M} \setminus \alpha(\Delta)\).

ii) If a partial dynamical system \((M_\beta, \beta)\) possesses the property of \((\widetilde{M}, \alpha)\) described in item i), then \((M_\beta, \beta)\) and \((\widetilde{M}, \alpha)\) are equivalent and equivalence is established by means of \(\tilde{\Psi}\).

Proof. i) Denote by \(M^\beta_N, N \in \mathbb{N}\), the set of points that have exactly \(N\) preimages under \(\beta\), that is, \(y \in M^\beta_N\) iff \(y, \beta^{-1}(y), \ldots, \beta^{-(N-1)}(y) \in \beta(\Delta_\beta)\) and \(\beta^{-N}(y) \notin \beta(\Delta_\beta)\). Similarly, we denote by \(M^\beta_\infty\) the set of points that have infinitely many preimages under \(\beta\). We put

\[
M^\beta_N \ni y \mapsto \tilde{\Psi}(y) := (\Psi(y), \Psi(\beta^{-1}(y)), \ldots, \Psi(\beta^{-N}(y)), 0, \ldots) \in M_N, \\
M^\beta_\infty \ni y \mapsto \tilde{\Psi}(y) := (\Psi(y), \Psi(\beta^{-1}(y)), \ldots, \Psi(\beta^{-n}(y)), \ldots) \in M_\infty.
\]

By (2.4) and (2.8) this yields a well defined mapping \(\tilde{\Psi} : M_\beta \to \widetilde{M}\). It is continuous, since for an open set \(\widetilde{U} \subset \widetilde{M}\) of the form

\[
\widetilde{U} = \{(x_0, x_1, \ldots) \in \widetilde{M} : x_n \in U\}, \quad \text{where } U \text{ is open in } M,
\]
the set $\tilde{\Psi}^{-1}(\tilde{U}) = \beta^n(\Psi^{-1}(U))$ is also open. Condition (2.8) implies that $\tilde{\Psi}$ maps $M^\beta$ onto $M_N$. In particular,

$$\tilde{\Psi}(M_\beta \setminus \beta(\Delta_\beta)) = \tilde{\Psi}(M^\beta_0) = M_0 = \tilde{M} \setminus \tilde{\alpha}(\Delta).$$

To show that $\tilde{\Psi}$ maps $M^\beta$ onto $M_\infty$ we fix $\tilde{x} = (x_0, x_1, \ldots) \in M_\infty$ and put

$$D_n = \tilde{\Psi}^{-1}\left(\{\tilde{y} = (y_0, y_1, \ldots) \in \tilde{M} : y_n = x_n\}\right), \quad n \in \mathbb{N}.$$ 

It is evident that $\{D_n\}_{n \in \mathbb{N}}$ forms a decreasing sequence of nonempty compact sets, and therefore $\bigcap_{n \in \mathbb{N}} D_n \neq \emptyset$. Taking $y \in \bigcap_{n \in \mathbb{N}} D_n$ we have $\tilde{\Psi}(y) = \tilde{x}$, which proves surjectivity of $\tilde{\Psi}(y)$. Relations $\tilde{\Psi}^{-1}(\Delta) = \Delta_\beta$ and $\Psi = \Phi \circ \tilde{\Psi}$ are straightforward.

For the uniqueness of $\tilde{\Psi}$ we note that reversibility of the systems $(M_\beta, \beta)$, $(\tilde{M}, \tilde{\alpha})$ and the equality $\tilde{\Psi}(M_\beta \setminus \beta(\Delta_\beta)) = \tilde{M} \setminus \tilde{\alpha}(\Delta)$ imply that a semiconjugacy $\tilde{\Psi}$ from $(M_\beta, \beta)$ to $(\tilde{M}, \tilde{\alpha})$ is automatically a semiconjugacy from $(M_\beta, \beta^{-1})$ to $(\tilde{M}, \tilde{\alpha}^{-1})$. This together with the relation $\Psi = \Phi \circ \tilde{\Psi}$ gives a family of relations $\Psi \circ \beta^{-n} = \Phi \circ \tilde{\alpha}^{-n} \circ \tilde{\Psi}, \ n \in \mathbb{N}$, understood in the sense that not only functions but also their natural domains are equal. This forces $\tilde{\Psi}$ to act according to formulae which we used as a definition in the first part of the proof.

ii) We have two semiconjugacies $\Phi: M_\beta \to \tilde{M}$ and $\tilde{\Psi}: \tilde{M} \to M_\beta$ which by the argument from item i) (with the same convention concerning domains) satisfy

$$\Psi \circ \beta^{-n} = \Phi \circ \tilde{\alpha}^{-n} \circ \tilde{\Psi}, \quad \Phi \circ \tilde{\alpha}^{-n} = \Psi \circ \beta^{-n} \circ \tilde{\Phi}, \quad n \in \mathbb{N}.$$ 

Thus $(\Phi \circ \tilde{\alpha}^{-n}) \circ (\tilde{\Psi} \circ \tilde{\Phi}) = \Phi \circ \tilde{\alpha}^{-n}$ and from the form $\tilde{M}$ it follows that $\tilde{\Psi} \circ \tilde{\Phi} = \text{id}$. Therefore $\tilde{\Phi}$ is a homeomorphism which yields the desired equivalence.

Identifying $\mathcal{A} = C(M)$ with the subalgebra $\{a \circ \Phi : a \in C(M)\}$ of $\tilde{\mathcal{A}} := C(\tilde{M})$, denoting by $\delta$ and $\tilde{\delta}$ endomorphisms corresponding to $\alpha$ and $\tilde{\alpha}$, respectively, and putting $J = C_Y(M)$ one may interpret the above result in terms of endomorphisms as follows.

**Theorem 3.15.** Every partial automorphism $\gamma: \mathcal{B} \to \mathcal{B}$ that extends $\delta: \mathcal{A} \to \mathcal{A}$ in such a way that $(\ker \gamma)^\perp \cap \mathcal{A} = J$, automatically extends the partial automorphism $\tilde{\delta}: \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ in such a way that $(\ker \gamma)^\perp \cap \tilde{\mathcal{A}} = (\ker \tilde{\delta})^\perp$. Moreover, this property characterizes the pair $(\mathcal{A}, \delta)$ (up to isomorphisms conjugating endomorphisms).

### § 4. Reversible extensions of logistic maps

Now as the necessary preparatory work has been implemented and the required $C^*$-objects have been described we pass to a presentation of calculation of concrete examples of reversible extensions of dynamical systems.

In this section, we conduct a thorough analysis of reversible extensions of the family of logistic maps $\{\alpha_\lambda\}_{\lambda \in (0, 1]}$, where by a *logistic map with parameter* $\lambda \in (0, 1]$ we mean a quadratic map $\alpha_\lambda: [0, 1] \to [0, 1]$ given by (1.5); cf. Fig. 3, (a). For better illustration of our $C^*$-method we define the operators generating $\alpha_\lambda$, $\lambda \in (0, 1]$, in a concrete fashion, even though we already know that such operators always exist,
see Theorem 3.8. To this end, we let $H = L^2(\mathbb{R})$ and consider the $C^*$-algebra $\mathcal{A}$ consisting of operators of multiplication by periodic functions $a(t)$ of period 1, continuous on $[0,1)$, and possessing a limit at 1 from below, that is:

$$a(t + 1) = a(t), \quad a|_{[0,1)} \in C([0,1)) \quad \text{and there exists } \lim_{t \to 1^-} a(t). \quad (4.1)$$

Plainly, $\mathcal{A}$ is isomorphic to $C([0,1])$ and we will identify its spectrum $M$ with the unit interval $[0,1]$:

$$M = [0,1].$$

We fix $\lambda \in (0,1]$ and define a piecewise continuous function $\gamma_\lambda : \mathbb{R} \to \mathbb{R}$, where

$$\gamma_\lambda(x) = \begin{cases} 4\lambda t(1-t) + 2k, & t \in [k, k + \frac{1}{2}], \\ 4\lambda t(1-t) + 2k + 1, & t \in [k + \frac{1}{2}, k + 1), \end{cases} \quad k \in \mathbb{Z}. \quad (4.2)$$

The graph of $\gamma_\lambda$ arises from the graph of $\alpha_\lambda$ by a ‘propagation’ of the halves of the parabola $y = \alpha_\lambda(x)$ on $\mathbb{R}^2$, so that one obtains a graph of an injective mapping on $\mathbb{R}$, see Fig. 3, (b). The mapping $\gamma_\lambda$ is injective, and it is bijective iff $\lambda = 1$.

![Figure 3](image.png)

Figure 3. (a) Graph of the logistic map $\alpha_\lambda : [0,1] \to [0,1]$; (b) graph of the injective map $\gamma_\lambda : \mathbb{R} \to \mathbb{R}$ arising from $\alpha_\lambda$.

We let $U_\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a normalized operator of composition with $\gamma_\lambda$:

$$(U_\lambda f)(t) = \sqrt{|\gamma_\lambda'(t)|} f(\gamma_\lambda(t)) = 2\sqrt{|\lambda(1-2t)|} f(\gamma_\lambda(t)), \quad f \in L^2(\mathbb{R}), \quad (4.3)$$

so that $U_\lambda$ is a coisometry where the adjoint isometry is given by the formula

$$(U_\lambda^* f)(t) = \begin{cases} \sqrt{|(\gamma_\lambda^{-1})'(t)|} f(\gamma_\lambda^{-1}(t)), & t \in \gamma_\lambda(\mathbb{R}), \\ 0, & t \notin \gamma_\lambda(\mathbb{R}), \end{cases} \quad f \in L^2(\mathbb{R}).$$

In particular, $U_\lambda$ is unitary if and only if $\lambda = 1$. 
**Proposition 4.1.** For each \( \lambda \in (0, 1] \) the operator \( U_\lambda \) generates on the spectrum of the C*-algebra \( \mathcal{A} \) the logistic map \( \alpha_\lambda \). Moreover the hull of the ideal

\[
J = (U_\lambda^*U_\lambda)\mathcal{A} \cap \mathcal{A} = \{ a \in \mathcal{A} : U_\lambda^*U_\lambda a = a \}
\]

is the smallest possible, cf. Proposition 2.16, that is,

\[
Y = \text{hull}(J) = \overline{M \setminus \alpha_\lambda(M)} = \begin{cases} [\lambda, 1], & \lambda < 1, \\ \emptyset, & \lambda = 1. \end{cases}
\]

**Proof.** Since \( U_\lambda^*U_\lambda \) is the operator of multiplication by the characteristic function \( \chi_{\gamma_\lambda(\mathbb{R})} \) of the set \( \gamma_\lambda(\mathbb{R}) \), we have \( U_\lambda^*U_\lambda \in \mathcal{A}' \) and the hull of \( J \) is the set \( Y = \text{hull}(\{ a \in \mathcal{A} : \chi_{\gamma_\lambda(\mathbb{R})}a = a \}) = [\lambda, 1] \) for \( \lambda < 1 \) and \( Y = \emptyset \) for \( \lambda = 1 \). For any operator \( a \) of multiplication by \( a(t) \), \( U_\lambda aU_\lambda^* \) is an operator of multiplication by

\[
(U_\lambda aU_\lambda^*)(t) = a(\gamma_\lambda(t)) = a(\{\gamma_\lambda(t)\}) = a(\alpha_\lambda(\{t\}))
\]

where \( \{t\} \in [0, 1) \) denotes the fractional part of a number \( t \in \mathbb{R} \). Hence one sees that \( U_\lambda \) generates on \([0, 1]\) the mapping \( \alpha_\lambda \).

In view of the above the first of the following operations

\[
\delta_\lambda(a) := U_\lambda aU_\lambda^*, \quad \delta_{*, \lambda}(a) := U_\lambda^* aU_\lambda, \quad a \in L(L^2(\mathbb{R}))
\]

preserves the algebra \( \mathcal{A} \) and the logistic map \( \alpha_\lambda \) is its dual map. In particular, we may adopt the identifications:

\[
\mathcal{A} = C([0, 1]), \quad \delta_\lambda(a) = a \circ \alpha_\lambda, \quad a \in \mathcal{A}.
\]

On the other hand, for \( a \in \mathcal{A} \), \( \delta_{*, \lambda}(a) \) is the operator of multiplication by the function

\[
\delta_{*, \lambda}(a)(t) = (U_\lambda^* aU_\lambda)(t) = \begin{cases} a(\gamma_\lambda^{-1}(t)), & x \in \gamma_\lambda(\mathbb{R}), \\ 0, & x \notin \gamma_\lambda(\mathbb{R}), \end{cases}
\]

which is periodic but in general its period is two, not one. Therefore the mappings \( \delta_{*, \lambda}, \lambda \in (0, 1] \), do not preserve the algebra \( \mathcal{A} \) and the C*-algebras

\[
B_\lambda := C^*(\bigcup_{n \in \mathbb{N}} U_\lambda^n \mathcal{A} U_\lambda^n)
\]

are essentially bigger than \( \mathcal{A} \).

**Theorem 4.2.** Let \((\tilde{M}_\lambda, \tilde{\alpha}_\lambda), \lambda \in (0, 1] \), be the reversible extension of \([0, 1], \alpha_\lambda \) associated with the set \( Y \) where \( Y = [\lambda, 1] \) if \( \lambda < 1 \) and \( Y = \emptyset \) if \( \lambda = 1 \). The algebra \( B_\lambda \) may be identified with \( C(\tilde{M}_\lambda) \) and then the partial automorphism \( \delta_\lambda : B_\lambda \to B_\lambda \) becomes the operator of composition with the homeomorphism \( \tilde{\alpha}_\lambda : \tilde{M}_\lambda \to \tilde{\alpha}_\lambda(\tilde{\Delta}) \):

\[
B_\lambda = C(\tilde{M}_\lambda), \quad \delta_\lambda(a) = a \circ \tilde{\alpha}_\lambda, \quad a \in B_\lambda.
\]

In particular operator \( U_\lambda \) generates on \( \tilde{M}_\lambda \) the partial homeomorphism \( \tilde{\alpha}_\lambda \).

This follows from Proposition 4.1, Theorem 3.5 and Definition 3.10.
The description of the extended $C^*$-dynamical systems $(\mathcal{B}_\lambda, \delta_\lambda)$ reduces to the description of the family of reversible topological systems

$$\{(\tilde{M}_\lambda, \tilde{\alpha}_\lambda)\}_{\lambda \in (0, 1]}.$$ 

We recall, cf. Remark 3.6, that the mapping $\Phi: \tilde{M}_\lambda \to [0, 1]$ dual to the embedding $\mathcal{A} \subset \mathcal{B}_\lambda$ is a surjection such that the diagram

$$\begin{array}{ccc}
\tilde{M}_\lambda & \xrightarrow{\tilde{\alpha}_\lambda} & \tilde{M}_\lambda \\
\Phi \downarrow & & \Phi \\
[0, 1] & \xrightarrow{\alpha_\lambda} & [0, 1]
\end{array}$$

commutes. We stress that a change of the parameter value $\lambda \in (0, 1]$ does not only influence the dynamics of $\tilde{\alpha}_\lambda$ but also the topology of the space $\tilde{M}_\lambda$. The following notions of continuum theory will be indispensable in our analysis.

**Definition 4.3** (see, for instance, [24]). By a continuum we mean a connected and compact metric space. A continuum will be called

i) nondegenerate if it is not a singleton,

ii) reducible if it may be presented as the sum of two of its proper subcontinua,

iii) irreducible if it is a nondegenerate continuum which is not reducible,

iv) a snake-like or arc-like continuum if it is homeomorphic to an inverse limit of an inverse sequence with bonding maps which are continuous maps of an interval, cf. [24], 12.19.

For all $\lambda \in (0, 1]$, the spectrum $\tilde{M}_\lambda$ of $\mathcal{B}_\lambda$ contains the snake-like continuum $M_\infty = \lim_{\leftarrow}([0, 1], \alpha_\lambda)$, and it is a general dynamical principle, discovered by Barge and Martin [27]; one should expect this continuum to be irreducible, see Theorem 4.6 below.

4.1. General structure of the extended systems $(\tilde{M}_\lambda, \tilde{\alpha}_\lambda)$. Let $\lambda < 1$.

By Remark 3.11 the space $\tilde{M}_\lambda$ consists of the set $M_\infty$, which is the inverse limit $\lim_{\leftarrow}(M, \alpha_\lambda)$ and a countable family of arcs $M_N$, that is, sets homeomorphic to a closed interval $M_N \cong [\lambda, 1]$.

**Theorem 4.4.** For $\lambda < 1$ the maximal ideal space $\tilde{M}_\lambda$ of the algebra $\mathcal{B}_\lambda$ consists of the snake-like continuum $M_\infty = \lim_{\leftarrow}(M, \alpha_\lambda)$ and a sequence of arcs $M_N$ such that

$$\lim_{n \to \infty} M_N = M_\infty,$$

where the limit is taken in the Hausdorff metric. In particular,

$$\tilde{M}_\lambda = \bigcup_{n\in\mathbb{N}} M_N \cup M_\infty = \bigcup_{n\in\mathbb{N}} M_N.$$ 

The mapping $\tilde{\alpha}_\lambda$ generated by $U_\lambda$ on $\tilde{M}_\lambda$, carries homeomorphically arc $M_N$ onto arc $M_{N+1}$, and on the continuum $M_\infty = \lim_{\leftarrow}(M, \alpha_\lambda)$ it coincides with the homeomorphism induced by $\alpha_\lambda$ (Definition 3.1).
Proof. We only need to show the equality \( \lim_{n \to \infty} M_N = M_\infty \) which (see for instance [24], Theorem 4.11) is equivalent to the following two inclusions:

\[
\limsup M_N = \{ \bar{x} \in \bar{M}_\lambda : \forall \text{ open } U \ni \bar{x} \\
\quad \forall k \in \mathbb{N} \exists N > k \ U \cap M_N \neq \emptyset \} \subset M_\infty,
\]

\[
M_\infty \subset \liminf M_N = \{ \bar{x} \in \bar{M}_\lambda : \forall \text{ open } U \ni \bar{x} \\
\quad \exists k \in \mathbb{N} \forall N > k \ U \cap M_N \neq \emptyset \}.\]

If we assume that \( \bar{x} \in \limsup M_N \) and \( \bar{x} \in M_{N_0} \) for certain \( N_0 \in \mathbb{N} \), then taking in the definition of \( \limsup M_N \), \( U = M_{N_0} \) and \( k = N_0 \) we arrive at a contradiction. This proves the first inclusion. To prove the second one, take \( \bar{x} = (x_0, \ldots, x_{N_0}, \ldots) \in M_\infty \) and an open neighbourhood of \( \bar{x} \) of the form

\[
U = \{ \bar{y} = (y_0, \ldots, y_{N_0}, \ldots) \in \bar{M}_\lambda : y_{N_0} \in (x_{N_0} - \varepsilon, x_{N_0} + \varepsilon) \}.
\]

Note that there exists \( n_0 \in \mathbb{N} \) such that \( \alpha^{n_0}_\lambda ([\lambda, 1]) = [0, \lambda] \). Indeed, if \( \lambda \leq \frac{1}{2} \), then \( \alpha_\lambda([\lambda, 1]) = [0, \lambda] \), and if \( \lambda > \frac{1}{2} \), then for certain \( n_0 \) \( \alpha^{n_0}_\lambda(\lambda) > \frac{1}{2} \), and therefore \( \alpha^{n_0}_\lambda([\lambda, 1]) = [0, \lambda] \). Putting \( k = N_0 + n_0 \) we have \( U \cap M_N \neq \emptyset \) for every \( N > k \) and thus \( \bar{x} \in \liminf M_N \).

The above theorem implies that once we have at our disposal a description of \((M_\infty, \tilde{\alpha}_\lambda)\), then we may easily describe the whole system \((\bar{M}_\lambda, \lambda, \tilde{\alpha}_\lambda)\): it suffices to adjoin to \( M_\infty \) the sequence of arcs \( \{ M_N \}_{N \in \mathbb{N}} \) converging to \( M_\infty \) and prolong \( \lambda, \tilde{\alpha}_\lambda \) so that it shifts homeomorphically arcs \( M_N \) towards \( M_\infty \). The importance of this comment lies in that a great deal of facts concerning the systems of \((\liminf(M, \alpha_\lambda), \tilde{\alpha}_\lambda)\) type are known, see [26]. Thus we may use them to achieve our goal.

4.1.1. The extended system for \( \lambda = 1 \) (B-J-K continuum). For \( \lambda = 1 \) the mapping \( \alpha_1 \) is a surjection and the space \( \bar{M}_1 \) coincides with the inverse limit \( \liminf(M, \alpha_1) \) of the full logistic map \( \alpha_1(x) = 4x(1-x) \), cf. Theorem 3.7, iv). Hence \( \bar{M}_1 \) is one of the most famous irreducible continua called the B-J-K continuum in honour of Brouwer, Janiszewski and Knaster [26, [24, [31]. One may graphically depict \( \bar{M}_1 \) by joining the points of the Cantor set with semicircles in a manner presented on Fig. 4, (a).

The logistic map \( \alpha_1(x) = 4x(1-x) \) is topologically conjugate to the tent map \( \alpha_T(x) = 1 - |2x - 1| \) and the B-J-K continuum is usually considered [24, 31] as the inverse limit \( \liminf(M, \alpha_T) \). Then there is a natural parametrization of the composant of the point \((0, 0, 0, \ldots)\) of the B-J-K continuum by nonnegative real numbers [31], see Fig. 4, (b). Within this parametrization the induced homeomorphisms \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_T \) fulfill the following formulae

\[
\tilde{\alpha}_1(t) = \begin{cases} 
2k + \alpha(\{t\}), & t \in \left[k, k + \frac{1}{2}\right), \\
2(k + 1) - \alpha(\{t\}), & t \in \left[k + \frac{1}{2}, k + 1\right),
\end{cases} \quad \tilde{\alpha}_T(t) = 2t, \quad (4.5)
\]

where \( \{t\} \) denotes the fractional part of the number \( t \) (obviously, the systems \((\bar{M}_1, \tilde{\alpha}_1)\) and \((\bar{M}_1, \tilde{\alpha}_T)\) are topologically conjugate). These comments give a central idea about the dynamics of the system \((\bar{M}_\lambda, \lambda, \tilde{\alpha}_\lambda)\) for \( \lambda = 1 \).
Theorem 4.5. Algebra $\mathcal{B}_1$ may be identified with the algebra $C(\widetilde{M}_1)$ of continuous functions on the B-J-K continuum $\widetilde{M}_1$, Fig. 4, (a). Then the automorphism $\delta_1: \mathcal{B}_1 \to \mathcal{B}_1$ becomes the operator of composition with the homeomorphism $\tilde{\alpha}_1: \tilde{M}_1 \to \tilde{M}_1$, which within the parametrization presented on Fig. 4, (b) assumes the form (4.5).

Furthermore irreducibility of $\tilde{M}_1$ expresses the following property of the algebra $\mathcal{B}_1$: if $I_1$, $I_2$ are ideals in $\mathcal{B}_1$ such that $\mathcal{B}_1/I_i$, for $i = 1, 2$, does not contain nontrivial idempotents, then

$$I_1 \cap I_2 = \{0\} \implies I_1 = \{0\} \text{ or } I_2 = \{0\}.$$  

Proof. The first part of the statement follows from Theorem 4.2. To show the second part notice that for $i = 1, 2$ we have

$$I_i = C_{Y_i}(\tilde{M}_1), \quad \text{where } Y_i = \text{hull}(I_i) \subset \tilde{M}_1.$$  

The algebra $C(\tilde{M}_1)/I_i \cong C(Y_i)$ does not contain nontrivial idempotents if and only if $Y_i$ is connected, that is, if $Y_i$ is a subcontinuum of $\tilde{M}_1$. Since $I_1 \cap I_2 = C_{Y_1 \cup Y_2}(\tilde{M}_1)$, the condition $I_1 \cap I_2 = \{0\}$ is equivalent to the equality $Y_1 \cup Y_2 = \tilde{M}_1$ and therefore the asserted property of $\mathcal{B}_1$ is equivalent to the irreducibility of continuum $\tilde{M}_1$.

4.1.2. Feigenbaum limit $\lambda_\infty$. Let

$$\lambda_1 = \frac{3}{4}, \quad \lambda_2 = \frac{1 + \sqrt{6}}{4} \approx 0.862, \quad \lambda_3, \ldots$$  

be the sequence of the parameter values $\lambda$ corresponding to the first cascade of period-doubling bifurcation of the system $([0, 1], \alpha_\lambda)$. This is an increasing sequence and the value

$$\lambda_\infty = \lim_{n \to \infty} \lambda_n \approx 0.89249$$  

is called the Feigenbaum limit [26], [32], [23]. The interval of parameters splits into two parts $(0, \lambda_\infty)$ and $(\lambda_\infty, 1]$ which correspond respectively to regular and chaotic behaviour of the systems $([0, 1], \alpha_\lambda)$, see Fig. 5. This finds a splendid reflection in the structure of the algebra $\mathcal{B}_\lambda$. 

Figure 4. Brouwer-Janiszewski-Knaster continuum.
Theorem 4.6. Let $\lambda \in (0, 1]$. The following conditions are equivalent.

i) $\lambda > \lambda_\infty$;

ii) spectrum $\tilde{M}_\lambda$ of $\mathcal{B}_\lambda$ contains an irreducible continuum;

iii) There exists a proper ideal $J$ in a maximal ideal in the $C^*$-algebra $\mathcal{B}_\lambda$ with the property that for every two ideals $I_1, I_2$ in $\mathcal{B} := \mathcal{B}_\lambda/J$ such that $\mathcal{B}/I_1$, $\mathcal{B}/I_2$ do not contain nontrivial idempotents we have

$$I_1 \cap I_2 = \{0\} \implies I_1 = \{0\} \text{ or } I_2 = \{0\}.$$  

Proof. Equivalence of i) and ii) follows from Theorem 4.4 and [26], Theorems 3, 4 and 7. In the proof of Theorem 4.5 we have shown that the condition in item iii) is equivalent to irreducibility of the continua. Moreover, the statement ‘$J$ is the proper ideal in a maximal ideal in $\mathcal{B}_\lambda$’ means that ‘the set $Z = \text{hull}(J)$ contains more than one point’. This explains the equivalence of ii) and iii).

4.2. The first cascade of bifurcation: $\lambda \in (0, \lambda_\infty]$. For $\lambda < \lambda_\infty$ the dynamics of $([0,1], \alpha_\lambda)$ is completely understood: $\alpha_\lambda$ has exactly one stable orbit which is periodic with period $2^n$, $n \in \mathbb{N}$, exactly one repelling periodic orbit with period $2^k$, for each $k = 0, \ldots, n-1$, and at most two repelling fixed points. As one increases $\lambda$ from 0 to $\lambda_\infty$ the number $n$ gradually increases — the system undergoes a period-doubling bifurcation. In particular the period of the stable orbit of $\alpha_\lambda$, for $\lambda \in (0, \lambda_\infty)$, increases according to Sharkovskii’s order [22], [23]:

$$1 < 2 < 4 < \cdots < 2^n < \cdots$$

$$\cdots < 2^m(2n + 1) < \cdots < 2^m \cdot 7 < 2^m \cdot 5 < 2^m \cdot 3 < \cdots$$

$$\cdots < 2(2n + 1) < \cdots < 14 < 10 < 6 < \cdots$$

$$\cdots < (2n + 1) < \cdots < 7 < 5 < 3.$$  

(4.6)

Let us now imagine that we slowly move the parameter $\lambda$ from 0 to $\lambda_\infty$, and simultaneously observe the maximal ideal space

$$\tilde{M}_\lambda = \bigcup_{n \in \mathbb{N}} M_N \cup M_\infty$$

of the algebra $\mathcal{B}_\lambda$. Let us also assume that we watch the change in $\tilde{M}_\lambda$ from the point of view of the initial space $M = [0, 1] —$ looking at a point $\vec{x} = (x_0, x_1, \ldots) \in \tilde{M}_\lambda$.
we read off $x_0 \in [0, 1]$. Such an approach will allow us to understand in detail how the change of the parameter $\lambda$ affects the topology of the space $\widetilde{M}_\lambda$. In other words we will build an image of $\tilde{M}_\lambda$ with the help of the factor map

$$\Phi(x_0, x_1, \ldots) = x_0, \quad (x_0, x_1, \ldots) \in \tilde{M}_\lambda,$$

cf. Remark 3.6. Since however $\Phi$ is not injective, it may ‘wind’ a piece of an arc $M_N$ several times onto an interval. Thus in order to get a homeomorphic image of $\tilde{M}_\lambda$ one needs to modify $\Phi$ by ‘adjoining’ copies of the corresponding arcs. In this manner one constructs a homeomorphism from $\tilde{M}_\lambda$ onto a certain subset of the plane, cf. [33]. Here, we restrict ourselves to a discussion of the results of that procedure, the trace of which will be seen on pictures of $\tilde{M}_\lambda$ where a point $\tilde{x} \in \tilde{M}_\lambda$ is labelled by its zeroth coordinate $\Phi(\tilde{x}) \in \tilde{M}$.

The crucial role in this enterprise is played by the orbit \(\{q_n\}_{n \in \mathbb{N}}\) of the critical point of the mapping $\alpha_\lambda$. We set

$$q_n := \alpha_\lambda^n \left( \frac{1}{2} \right), \quad n \in \mathbb{N}.$$

We start with parameter $\lambda$ taking values slightly greater than zero (smaller than $\frac{1}{4}$), see Fig. 6, (a). The only periodic point of the mapping $\alpha_\lambda$ is the stable fixed point 0. In particular, we have:

$$q_1 > q_2 > q_3 > \cdots, \quad \lim_{n \to \infty} q_n = 0.$$ 

Accordingly, the space $\widetilde{M}_\lambda$ consists of a singleton $M_\infty$ and a sequence of arcs $M_N$ converging to $M_\infty$, Fig. 7, (a). As we increase the parameter $\lambda$ the length of intervals $[0, q_N]$ increases. Finally, when $\lambda$ surpasses $\frac{1}{4}$ the fixed point 0 loses its stability transferring it onto a newly born fixed point $\omega_{1,1} > 0$:

$$q_1 > q_2 > q_3 > \cdots, \quad \lim_{n \to \infty} q_n = \omega_{1,1} > 0,$$

Fig. 6, (b). The set $M_\infty$ becomes an arc corresponding to the interval $[0, \omega_{1,1}]$, and it grows as $\lambda$ increases, Fig. 7, (b).
Figure 7. (a) Dynamics of \((\tilde{M}_\lambda, \tilde{\alpha}_\lambda)\) where \(0 < \lambda \leq \frac{1}{4}\); (b) \(\frac{1}{4} < \lambda \leq \frac{1}{2}\).

Figure 8. (a) Graph of \(\alpha_\lambda\) for \(\frac{1}{2} < \lambda \leq \frac{3}{4}\); (b) for \(\frac{3}{4} < \lambda \leq \frac{1+\sqrt{6}}{4}\).

When \(\lambda\) reaches the value \(\frac{1}{2}\) all the intervals \([0, q_n], n \in \mathbb{N}, [0, \omega_{1,1}]\) become equal to \([0, \frac{1}{2}]\) and the space \(\tilde{M}_\lambda\) assumes the shape of a ‘regular ladder’ (in which every step has the same length). As we pass \(\lambda = \frac{1}{2}\) the orbit of the critical point loses monotonicity:

\[q_1 > q_3 > q_5 > \cdots > \omega_{1,1} > \cdots > q_4 > q_2, \quad \lim_{n \to \infty} q_n = \omega_{1,1},\]

Fig. 8, (a). Consequently, each arc \(M_N, N \in \mathbb{N} \cup \{\infty\}\), develops a ‘curl’ at one of its end-points: for each \(1 < N < \infty\) the arc \(M_N\) has \(N - 1\) bendings, whereas \(M_\infty\) is bent infinitely many times, Fig. 9, (a).

When \(\lambda\) exceeds the value \(\lambda_1 = \frac{3}{4}\) the first bifurcation occurs — the fixed point \(\omega_{1,1}\) gives birth to a new stable periodic orbit \(\{\omega_{2,1}, \omega_{2,2}\}\), Fig. 8, (b). Then

\[q_1 > q_3 > \cdots > \omega_{2,1} > \omega_{1,1} > \omega_{2,2} > \cdots > q_4 > q_2, \quad \lim_{n \to \infty} q_{2n+i} = \omega_{2,i}, \quad i = 1, 2.\]
This implies that the attracting fixed point in $M_\infty$ grows into an arc corresponding to the interval $[\omega_{2,1}, \omega_{2,2}]$, see Fig. 9, (b). In other words, $M_\infty$ becomes a sin$(\frac{1}{2})$-continuum.

For $\lambda$ lying approximately in the middle of the interval $(\frac{3}{4}, 1 + \frac{\sqrt{6}}{4}] = (\lambda_1, \lambda_2]$ the orbit $\{\omega_{2,1}, \omega_{2,2}\}$ is superstable (it coincides with the critical point orbit). Lengths of the ‘adjoint’ intervals become equal: $|q_{k+1} - q_k| = \omega_{2,1} - \omega_{2,1}$, $k > 0$, and the space $\tilde{M}_\lambda$ assumes a regular shape. Afterwards, as we pass $\lambda = \frac{1}{2}$ the orbit of the critical point converges to the stable orbit in a more complicated manner:

$q_1 > q_5 > \cdots > \omega_{2,1} > \cdots > q_7 > q_3, \quad q_4 > q_8 > \cdots > \omega_{2,2} > \cdots > q_6 > q_2$.

At the level of the subspace $M_\infty \subset \tilde{M}_\lambda$, see Fig. 10, it causes perturbations around the periodic points; the arcs of the limit bar begin to curl around their end-points.

When $\lambda$ exceeds the value $\lambda_2 = \frac{1+\sqrt{6}}{4}$ the second bifurcation occurs. The orbit $\{\omega_{2,1}, \omega_{2,2}\}$ becomes repelling and transfers its stability onto a newly born periodic orbit $\{\omega_{4,1}, \omega_{4,2}, \omega_{4,3}, \omega_{4,4}\}$ of period 4. Consequently, periodic points of $M_\infty$ grow into arcs corresponding to intervals $[\omega_{4,1}, \omega_{4,3}]$ and $[\omega_{4,2}, \omega_{4,4}]$.

The maximal ideal space $M_\lambda$ of algebra $B_\lambda$ for $\lambda \in (\lambda_2, \lambda_3]$ consists of the snake-like continuum $M_\infty$ presented on Fig. 11 and a sequence of arcs $\{M_N\}_{N \in \mathbb{N}}$ converging to $M_\infty$ in the Hausdorff metric.

This process continues: as $\lambda$ approaches approximately the middle of the interval $(\lambda_2, \lambda_3]$ the orbit $\{\omega_{4,1}, \omega_{4,2}, \omega_{4,3}, \omega_{4,4}\}$ becomes superstable and the space $M_\infty$ assumes a regular shape. Afterwards continuum $M_\infty$ develops curls around the points corresponding to the orbit $\{\omega_{4,1}, \omega_{4,2}, \omega_{4,3}, \omega_{4,4}\}$ until we pass $\lambda = \lambda_3$, where each of these points grows into an arc. And so on, and so forth, cf. Fig. 12.

In order to state the result formally we extend the sequence $\lambda_1 = \frac{3}{4}$, $\lambda_2$, $\lambda_3$, $\ldots$, putting $\lambda_{-1} = 0$ and $\lambda_0 = \frac{1}{4}$. By a ray we mean a topological space homeomorphic to $(0, 1]$, and an end-point of a topological space $M$ is a point $p \in M$ every neighbourhood $U$ of which contains an open neighbourhood $V$ such that the boundary
Figure 10. Snake-like continuum $M_\infty \subset \tilde{M}_\lambda$ for $\lambda$, approaching the second bifurcation.

Figure 11. Snake-like continuum $M_\infty \subset \tilde{M}_\lambda$ after the second bifurcation (immersed into the 3-dimensional space $Oxyz$).
of $V$ is a singleton, cf. [24]. The subspace $M_\infty = \lim\leftarrow (M, \alpha_\lambda) \subset \tilde{M}_\lambda$ for $\lambda < \lambda_\infty$ is given by the following recurrent description.

**Theorem 4.7** ([26], Theorem 3). If $\lambda \in (\lambda_n, \lambda_{n+1}]$, $n \in \mathbb{N}$, then the continuum $\lim\leftarrow (M, \alpha_\lambda)$ is the closure of a ray $R$ such that $\lim\leftarrow (M, \alpha_\lambda) \setminus R$ is the union of two copies of $\lim\leftarrow (M, \alpha_\lambda')$, where $\lambda' \in (\lambda_{n-1}, \lambda_n]$, intersecting in a common end-point.

Now we are ready to give a full description of the system $(f_{M_\lambda}, \alpha_\lambda)$ for $\lambda$ lying in the interval $(0, \lambda_\infty) = S_{\infty} = \bigcup_{n=-1}^{\infty} (\lambda_n, \lambda_{n+1}]$. If $\lambda \in (0, \frac{3}{4}]$, such a system is presented in Figs. 7 and 9.

**Theorem 4.8.** Let $U_\lambda$ be the operator given by (4.3) and $\mathcal{B}_\lambda$ the $C^*$-algebra given by (4.4). If $\lambda \in (\lambda_n, \lambda_{n+1}]$, $n > 0$, then

i) the maximal ideal space $\tilde{M}_\lambda$ of the algebra $\mathcal{B}_\lambda$ consists of a snake-like continuum $M_\infty$ and a sequence of arcs $\{M_N\}_{N \in \mathbb{N}}$ converging to $M_\infty$, where

$$M_\infty = R \cup (R_{1,1} \cup R_{1,2}) \cup \cdots \cup (R_{n-1,1} \cup \cdots \cup R_{n-1,2n-1}) \cup (I_1 \cup \cdots \cup I_{2n-1})$$

is the sum of $2^n - 1$ rays $R$, $R_{k,i}$, $k = 1, \ldots, n-1$, $i = 1, \ldots, 2^k$, and $2^{n-1}$ arcs $I_i$, $i = 1, \ldots, 2^{n-1}$, cf. Fig. 12. The closure of $R$ gives $M_\infty$ and

$$\overline{R_{k,i}} = \bigcup_{j=0}^{n-k-1} \bigcup_{l=0}^{2^j-1} R_{k+j,i+1,2^j} \cup \bigcup_{l=0}^{2^{n-k-1}-1} I_{i+1,2^k},$$

where $k = 1, \ldots, n-1$, $i = 1, \ldots, 2^k$;

ii) the partial homeomorphism $\tilde{\alpha}_\lambda$ generated by $U_\lambda$ on $\tilde{M}_\lambda$ carries $M_N$ onto $M_{N+1}$, $N \in \mathbb{N}$, and $\tilde{\alpha}_\lambda: M_\infty \to M_\infty$ is a homeomorphism that preserves $R$, permutes cyclically arcs $I_i$ and the rays $R_{k,i}$ (for each fixed $k = 1, \ldots, n-1$):

$$\tilde{\alpha}_\lambda(I_1) = I_2, \ldots, \tilde{\alpha}_\lambda(I_{2^n}) = I_1, \quad \tilde{\alpha}_\lambda(R_{k,1}) = R_{k,2}, \ldots, \tilde{\alpha}_\lambda(R_{k,2^k}) = R_{k,1}.$$

Figure 12. Subspace $M_\infty \subset \tilde{M}_\lambda$ after the fourth bifurcation (schematic presentation).
Moreover, all the rays \( R, R_{k,i} \) and arcs \( I_i \) are pairwise disjoint except for the following cases:

\[
R_{k,i} \cap R_{k,2^{k-1}+i} = \{ \vec{\omega}_{2k-1,i} \}, \quad k = 1, \ldots, n-1, \ i = 1, \ldots, 2^{k-1},
\]

which form periodic orbits \( \{ \vec{\omega}_{2k-1,1}, \ldots, \vec{\omega}_{2k,2k} \} \) with period \( 2^k \), \( k = 1, \ldots, n-2 \). Midpoints of arcs \( I_i \) form a periodic orbit \( \{ \vec{\omega}_{2n-1,1}, \ldots, \vec{\omega}_{2n-1,2n-1} \} \) with period \( 2^{n-1} \), and end-points of arcs \( I_i \) form a periodic orbit \( \{ \vec{\omega}_{2n,1}, \ldots, \vec{\omega}_{2n,2n} \} \) with period \( 2^n \).

**Proof.** For the description of the space \( \tilde{M}_\lambda \) apply Theorems 4.4 and 4.7. The dynamics of \( \tilde{\alpha}_\lambda \) on \( M_\infty \) may be deduced from the proof of [26], Theorem 3; see also [26], 6.

The infinite sequence of period-doubling bifurcations leaves the following imprint on the structure of \( \tilde{M}_\lambda \) for \( \lambda \) attaining the Feigenbaum limit.

**Theorem 4.9.** For \( \lambda = \lambda_\infty \) the maximal ideal space \( \tilde{M}_\lambda \) of the algebra \( \mathcal{R}_\lambda \) possesses the property that every nondegenerate subcontinuum of \( \tilde{M}_\lambda \) is reducible. Moreover, \( \tilde{M}_\lambda \) contains only three topologically different nondegenerate subcontinua: arcs, copies of the space \( M_\infty \) and sums of two copies of \( M_\infty \) intersecting in the common end-point.

**Proof.** Apply Theorem 4.4 and [26], Theorem 7.

**4.3. The parameter values corresponding to chaotic dynamics.** For \( \lambda > \lambda_\infty \) the dynamics of the system \( ([0,1], \alpha_\lambda) \) is chaotic and the available knowledge concerning mappings \( \{ \alpha_\lambda \}_{\lambda > \lambda_\infty} \) is far from being complete. However we are able to present a number of results that shed much light onto the structure of the considered systems. For example, we already know that \( \tilde{M}_\lambda \) contains irreducible continua (Theorem 4.6) and for \( \lambda = 1 \), \( \tilde{M}_\lambda \) is a B-J-K continuum. We start with the case when B-J-K continua are the only irreducible subcontinua of \( \tilde{M}_\lambda \).

**4.3.1. Cascade of B-J-K continuum doubling.** Let us consider the sequence \( \mu_0 = 1, \mu_1, \mu_2, \ldots \) of parameter values, in which the bifurcation diagram splits into two ‘copies’ of itself, Fig. 5. It is a decreasing sequence converging to \( \lambda_\infty \), and formally \( \mu_n \) could be defined as the solution of the following equation

\[
\alpha_\lambda^{2^n}(\lambda) = \text{the largest fixed point of } \alpha_\lambda^{2^{n-1}},
\]

cf. [26], [32]. The sequence \( \{ \mu_n \}_{n \in \mathbb{N}} \) admits an inductive procedure surprisingly similar to the one presented in Theorem 4.7.

**Theorem 4.10 ([26], Theorem 6).** For \( n > 0 \) \( \lim(M, \alpha_{\mu_n}) \) is the closure of a ray \( R \) such that \( \lim(M, \alpha_{\mu_n}) \setminus R \) is the union of two copies of the space \( \lim(M, \alpha_{\mu_{n-1}}) \) intersecting in a common end-point.

Theorem 4.10 together with Theorems 4.4, 4.5 says that for \( \lambda = \mu_1 \) the space \( \tilde{M}_\lambda \) consists of a sequence of arcs \( \{ M_N \}_{N \in \mathbb{N}} \) converging to the snake-like continuum \( M_\infty \) which is the union of two copies \( B_1, B_2 \) of a B-J-K continuum and a ray \( R \), see Fig. 13, (a). Partial homeomorphism \( \tilde{\alpha}_\lambda \) transforms \( M_N \) onto \( M_{N+1}, N \in \mathbb{N} \), and the dynamics of \( \tilde{\alpha}_\lambda \) on \( M_\infty \) is as follows: the end-point of the ray \( R \) is a fixed point, the remaining points of \( R \) slide on \( R \) toward the continua \( B_1, B_2 \). Points
from $B_1$ are carried onto $B_2$ and vice versa. The intersection $B_1 \cap B_2$ consists of a fixed point.

Analogously, for $\lambda = \mu_2$, see Fig. 13, (b), $M\infty \subset \tilde{M}\lambda$ consists of four copies $B_1, B_2, B_3, B_4$ of a B-J-K continuum and three arcs $R, R_1, R_2$. The end-point of $R$ is a fixed point, and the remaining points of $R$ move toward arcs $R_1, R_2$. Points from $R_1$ are carried onto $R_2$ and vice versa. The intersection $R_1 \cap R_2$ consists of a fixed point. Continua $B_i$ are cyclically permuted:

$$\tilde{\alpha}_\lambda(B_1) = B_2, \quad \tilde{\alpha}_\lambda(B_2) = B_3, \quad \tilde{\alpha}_\lambda(B_3) = B_4, \quad \tilde{\alpha}_\lambda(B_4) = B_1,$$

and $(B_1 \cap B_3) \cup (B_2 \cap B_4)$ constitutes a periodic orbit of period 2.

In general, for $\lambda = \mu_n$ we have the description of $(\tilde{M}\lambda, \tilde{\alpha}_\lambda)$ which differs from the one presented in Theorem 4.8 only in that the arcs $I_i$ are replaced with B-J-K continua, cf. Fig. 14.

**Theorem 4.11.** Let $U\lambda$ be the operator given by (4.3) and $\mathcal{B}\lambda$ the $C^*$-algebra given by (4.4). If $\lambda = \mu_n$, $n \in \mathbb{N}_+$, then

i) the maximal ideal space $\tilde{M}\lambda$ of algebra $\mathcal{B}\lambda$ consists of a snake-like continuum $M\infty$ and a sequence of arcs $\{M_N\}_{N \in \mathbb{N}}$ converging to $M\infty$, where

$$M\infty = R \cup (R_{1,1} \cup R_{1,2}) \cup \cdots \cup (R_{n-1,1} \cup \cdots \cup R_{n-1,2^{n-1}}) \cup (B_1 \cup \cdots \cup B_{2^n})$$

is the sum of $2^n - 1$ rays $R, R_{k,i}, k = 1, \ldots, n-1, i = 1, \ldots, 2^k$, and $2^n$ B-J-K continua $B_i, i = 1, \ldots, 2^n$, cf. Fig. 14. The closure of $R$ gives $M\infty$ and

$$\overline{R_{k,i}} = \bigcup_{j=0}^{n-k-1} \bigcup_{l=0}^{2^j-1} R_{k+j,i+l \cdot 2^k} \cup \bigcup_{l=0}^{2^{n-k}-1} B_{i+l \cdot 2^k}, k = 1, \ldots, n-1, i = 1, \ldots, 2^k;$$
ii) the partial homeomorphism \( \tilde{\alpha}_\lambda \) generated by \( U_\lambda \) on \( \widetilde{M}_\lambda \) carries \( M_N \) onto \( M_{N+1} \), \( N \in \mathbb{N} \), and \( \tilde{\alpha}_\lambda \colon M_\infty \to M_\infty \) is a homeomorphism that preserves \( R \), permutes cyclically continua \( B_i \) and the rays \( R_{k,i} \) (for fixed \( k = 1, \ldots, n-1 \)):

\[
\tilde{\alpha}_\lambda(B_1) = B_2, \ldots, \tilde{\alpha}_\lambda(B_{2^n}) = B_1, \quad \tilde{\alpha}_\lambda(R_{k,1}) = R_{k,2}, \ldots, \tilde{\alpha}_\lambda(R_{k,2^k}) = R_{k,1}.
\]

Moreover, all the rays \( R \), \( R_{k,i} \) and continua \( B_i \) are pairwise disjoint except for the following intersections:

\[
R_{k,i} \cap R_{k,2^{k-1}+i} = \{ \tilde{\omega}_{2^{k-1},i} \}, \quad i = 1, \ldots, 2^{k-1},
\]

\[
B_i \cap B_{2^n-1+i} = \{ \tilde{\omega}_{2^n-1,i} \}, \quad i = 1, \ldots, 2^{n-1},
\]

which form periodic orbits \( \{ \tilde{\omega}_{2^k,1}, \ldots, \tilde{\omega}_{2^k,2^k} \} \) with period \( 2^k \), \( k = 1, \ldots, n-1 \).

**Proof.** This follows from Theorems 4.4, 4.5 and inductively applied Theorem 4.10; see also the proof of [26], Theorem 6.

### 4.4. Windows of stable periodic orbits of odd period

Let \( (\eta_n, \nu_n) \), \( n > 0 \), be the interval of parameter values for \( \lambda \) where \( \alpha_\lambda \) has its first stable orbit of period \( 2n+1 \). The sequences \( \eta_n, \nu_n \) converge decreasingly to \( \mu_1 \), see Fig. 15. Significantly, passing with \( \lambda \) from \( (\eta_n, \nu_n) \) to \( (\eta_{n+1}, \nu_{n+1}) \) the period of the stable periodic orbit of \( \alpha_\lambda \) decreases according to Sharkovskii’s order (see (4.6)).

**Theorem 4.12.** Let \( U_\lambda \) be the operator given by (4.3) and \( \mathcal{B}_\lambda \) the \( C^* \)-algebra given by (4.4). If \( \lambda \in (\eta_n, \nu_n) \), \( n > 0 \), then

i) the maximal ideal space \( \widetilde{M}_\lambda \) of algebra \( \mathcal{B}_\lambda \) consists of a snake-like continuum \( M_\infty \) and a sequence of arcs \( \{ M_N \}_{N \in \mathbb{N}} \) converging to \( M_\infty \), and

\[
M_\infty = R \cup C_{2n+1},
\]
where $R$ is a ray, $C_{2n+1}$ is an irreducible continuum with exactly $2n + 1$ end-points and whose only proper nondegenerate subcontinua are arcs. Furthermore, $C_{2n+1} \cap R = \emptyset$ and $\overline{R} = R \cup C_{2n+1}$;

ii) the partial homeomorphism $\tilde{\alpha}_\lambda$ generated by $U_\lambda$ on $\tilde{M}_\lambda$ carries $M_N$ onto $M_{N+1}$ for $N \in \mathbb{N}$ and preserves both $R$ and $C_{2n+1}$. The end-point of $R$ is a fixed point and the remaining points of $R$ move toward $C_{2n+1}$. The end-points of $C_{2n+1}$ form a periodic orbit with period $2n + 1$.

By Theorem 4.4 it suffices to inductively apply [26], Theorem 8.

For $\lambda \in (\eta_1, \nu_1]$ the continuum $C_3 \subset \tilde{M}_\lambda$ is considered to be the simplest example of an irreducible continuum, cf. [24]. It may be obtained as an attractor of a continuous injective map $T: \Omega \to \Omega$ defined on a compact subset $\Omega = A \cup B \cup C$ of $\mathbb{R}^2$ which acts according to Fig. 16. Actually, the subsystem $(C_3, \tilde{\alpha}_\lambda)$ of $(\tilde{M}_\lambda, \tilde{\alpha}_\lambda)$ is topologically conjugate to the system $(\Lambda, T)$, where $\Lambda = \bigcap_{n \in \mathbb{N}} T^n(\Omega)$.

4.4.1. Cascades of bifurcations that follow the windows of stability. After each of the intervals $(\eta_n, \nu_n]$, cf. Fig. 15, there occurs a cascade of period-doubling bifurcations. For instance, increasing the parameter value $\lambda$ in the window $(\eta_1, \nu_1]$ of the stable orbit of period three, we observe that the continuum $C_3 \subset \tilde{M}_\lambda$ curls around its end-points $\{\tilde{\omega}_{3,1}, \tilde{\omega}_{3,2}, \tilde{\omega}_{3,3}\}$, Fig. 17, (a). When we pass $\lambda = \nu_1$ end-points of $C_3$ grow into arcs: continuum $C_3$ turns into irreducible continuum $C_6$ with 6 end-points, Fig. 17, (b). Afterwards continuum $C_6$ begins to curl around its
end-points which finally grow into 6 arcs whose end-points are end-points of an irreducible continuum $C_{12}$. And so on, and so forth. In particular, after four such bifurcations we get a continuum $C_{64}$ which arises from continuum $C_3$ by replacing each of its end-points with a copy of the continuum presented on Fig. 12.

Similar phenomena occur after every window of stability $(\eta_n, \nu_n]$, $n > 0$. In order to get a formal description, for each $n > 0$, we denote by $\lambda(n)$ the sequence of parameter values $\lambda$ that correspond to the cascade of period-doubling bifurcations of the stable orbit appearing immediately after $\lambda_0 := \eta_n$.

**Theorem 4.13.** Let $U_\lambda$ be the operator given by (4.3) and $\mathcal{B}_\lambda$ the $\mathcal{C}^*$-algebra given by (4.4). If $\lambda \in (\lambda_m^{(n)}, \lambda_{m+1}^{(n)})$ for $n > 0$, $m > 0$, then

1) the maximal ideal space $\tilde{M}_\lambda$ of algebra $\mathcal{B}_\lambda$ consists of the snake-like continuum $M_\infty$ and the sequence of arcs $\{M_N\}_{N \in \mathbb{N}}$, converging to $M_\infty$, where

$$M_\infty = R \cup C_{2m}(2n+1)$$

is the union of a ray $R$ and an irreducible continuum $C_{2m}(2n+1)$ with exactly $2^m(2n+1)$ end-points. Furthermore $\overline{R} = R \cup C_{2m}(2n+1)$ and

$$C_{2m}(2n+1) = C_{2n+1} \cup \bigcup_{k=0}^{m-1} 2^k(2n+1) \cup \bigcup_{i=1}^{2^{m-1}(2n+1)} R_{k,i} \cup \bigcup_{i=1}^{I_i}$$

is the union of $(2^m - 1)(2n + 1)$ rays $R_{k,i}$ and $2^{m-1}(2n + 1)$ arcs $I_i$. The closure of $C_{2n+1}$ coincides with $C_{2m}(2n+1)$ and

$$\overline{R_{k,i}} = \bigcup_{j=0}^{m-k-1} 2^{j-1} \bigcup_{i=0}^{2^{2m-k-1}} R_{k+i, j+i+1.2^k(2n+1)} \bigcup_{l=0}^{2^{m-k-1}} I_{i+l.2^k(2n+1)}$$

for $i = 1, \ldots, 2^k(2n+1)$, $k = 0, \ldots, m - 1$;

2) the partial homeomorphism $\tilde{\alpha}_\lambda$ generated by $U_\lambda$ on $\tilde{M}_\lambda$ carries $M_N$ onto $M_{N+1}$, $N \in \mathbb{N}$, and $\tilde{\alpha}_\lambda$; $M_\infty \to M_\infty$ is a homeomorphism that preserves $R$ and $C_{2n+1}$, permutes cyclically the arcs $I_i$, and (for each fixed $k=0, \ldots, m - 1$) the rays $R_{k,i}$.
Moreover, all the rays $R, R_{k,i}$, all the arcs $I_i$ and the continuum $C_{2n+1}$ are pairwise disjoint except the following intersections:

\[ C_{2n+1} \cap R_{0,i}, \quad i = 1, \ldots, 2n + 1, \]
\[ R_{k,i} \cap R_{k,i+2^{k-1}(2n+1)}, \quad i = 1, \ldots, 2^{k-1}(2n+1), \quad k = 1, \ldots, m - 1. \]

The sets \( \{ \omega_{2k(2n+1)}, \ldots, \omega_{2k(2n+1)+2^{k(2n+1)}} \} \) form periodic orbits with periods \( 2^k(2n+1) \) for \( k = 0, \ldots, m - 2 \). The mid-points of the arcs $I_i$ form a periodic orbit with period \( 2^{m-1}(2n+1) \) and their end-points form a periodic orbit with period \( 2^m(2n+1) \).

In view of Theorem 4.4 it suffices to repeat the argument from the proof of Theorem 8 in [26], see also remarks preceding Theorem 9 in [26], as well as the introduction to [26], 6.

§ 5. Reversible extensions of homeomorphisms of a circle

The characteristic feature of the $C^*$-method developed in this paper is that it leads from irreversible dynamics to reversible dynamics. Therefore, it may seem surprising that applying it to (already) reversible systems one may also get non-trivial results. Clearly, all the interesting phenomena arising in this case are related to the freedom of choice of the set $Y$, equivalently the ideal $J$, see §3.1.1. As we show below such considerations arise naturally in investigation of compressions of unitary operators. We start with the general structure of dynamical systems that we will here deal with.

**Proposition 5.1.** If $\alpha: M \to M$ is a homeomorphism and $(\widetilde{M}, \tilde{\alpha})$ is a reversible extension of $(M, \alpha)$ associated with a set $Y \subset M$, then $\widetilde{M}$ may be treated as a closed subset

\[ \widetilde{M} = \bigcup_{N \in \mathbb{N}} M_N \cup M_\infty \subset \mathbb{N} \times M \]

of the product space, where $\mathbb{N} = \mathbb{N} \cup \{ \infty \}$ is the one-point compactification of the discrete space $\mathbb{N}$,

\[ M_\infty = \{ \infty \} \times M, \quad M_N = \{ N \} \times \alpha^N(Y), \quad N \in \mathbb{N}, \]

and the partial homeomorphism $\tilde{\alpha}: \widetilde{M} \to \widetilde{M}$ acts according to the formulae

\[ \tilde{\alpha}(N, x) = (N + 1, \alpha(x)), \quad \tilde{\alpha}(\infty, x) = (\infty, \alpha(x)). \]

**Proof.** We define a homeomorphism $\Psi$ of

\[ \widetilde{M} = \bigcup_{N \in \mathbb{N}} M_N \cup M_\infty \]

onto a closed subset $\mathbb{N} \times M$, using the factor map $\Phi: \widetilde{M} \to M$, see (3.5), by the formulae

\[ \Psi(\tilde{x}) := (x, \Phi(\tilde{x})), \quad \tilde{x} \in M_\infty, \quad \Psi(\tilde{x}) := (N, \Phi(\tilde{x})), \quad \tilde{x} \in M_N, \quad N \in \mathbb{N}. \]
Since $\alpha$ is a homeomorphism, one readily sees that

$$\Psi: \bigcup_{N \in \mathbb{N}} M_N \cup M_\infty \to \bigcup_{N \in \mathbb{N}} \{N\} \times \alpha^N(Y) \cup \{\infty\} \times M$$

is a homeomorphism, and identifying $\widetilde{M}$ with $\Psi(\widetilde{M})$ the assertion follows.

### 5.1. Compression of unitaries generating homeomorphisms of a circle.

Let $\alpha: S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle and let $\gamma: \mathbb{R} \to \mathbb{R}$ be its lift to $\mathbb{R}$, that is, a continuous mapping satisfying

$$\alpha(e^{2\pi it}) = e^{2\pi i \gamma(t)}, \quad t \in [0, 1].$$

We recall that $\gamma$ is an increasing homeomorphism such that $\gamma(t + 1) = \gamma(t) + 1$, $t \in \mathbb{R}$, determined by $\alpha$ up to a translation by an integer constant. We define a unitary operator $U \in L(H)$ on the space $H = L^2(\mathbb{R})$ by the formula

$$(Uf)(t) = |\gamma'(t)| f(\gamma(t)),$$

which (by monotonicity of $\gamma$) makes sense for almost all $t$ in $\mathbb{R}$. In particular, $(U^*f)(t) = |(\gamma^{-1})'(t)| f(\gamma^{-1}(t))$. We let $A \subset L(H)$ be an algebra of operators of multiplication by continuous periodic functions with period 1: $A \cong C(S^1)$. Clearly,

$$UAU^* \subset A, \quad U^*AU \subset A,$$

and we have the following result.

**Proposition 5.2.** The operators $U$ and $U^*$ generate on the maximal ideal space of $A$ (identified with $S^1$) the systems $(S^1, \alpha)$ and $(S^1, \alpha^{-1})$, respectively.

Let us now consider compressions of the introduced objects in the space $H := L^2([0, \infty))$ naturally treated as a subspace of $H = L^2(\mathbb{R})$. Namely, we denote by $P: \mathbb{H} \to H$ the projection from $\mathbb{H} = L^2(\mathbb{R})$ by the formula

$$U := PU^*P, \quad A := PA^*P.$$ Then the algebra $A \subset L(H)$ is isomorphic to $C(S^1)$ and $U \in L(H)$ is a partial isometry such that $U^* = PU^*P \in L(H)$.

**Proposition 5.3.** With the above notation the following possibilities may occur.

i) If $\gamma(0) > 0$, then $U$ is a noninvertible coisometry in $L(H)$,

$$U \mathcal{A} U^* \subset \mathcal{A}, \quad U^* \mathcal{A} U \nsubseteq \mathcal{A},$$

operator $U$ generates on the spectrum of $\mathcal{A}$ the system $(S^1, \alpha)$ and

$$\text{hull}(U^*U \mathcal{A} \cap \mathcal{A}) = \{e^{2\pi it} : t \in [0, \gamma(0)]\}.$$

ii) If $\gamma(0) = 0$, then $U$ is unitary,

$$U \mathcal{A} U^* \subset \mathcal{A}, \quad U^* \mathcal{A} U \subset \mathcal{A},$$

$U$ and $U^*$ generate respectively the systems $(S^1, \alpha)$ and $(S^1, \alpha^{-1})$. 

iii) If $\gamma(0) < 0$, then $U$ is a noninvertible isometry

$$U^*\mathcal{A}U \subset \mathcal{A}, \quad \mathcal{A}U^* \not\subset \mathcal{A},$$

operator $U^*$ generates on the spectrum of $\mathcal{A}$ the system $(S^1, \alpha^{-1})$ and

$$\text{hull}(U^*U\mathcal{A} \cap \mathcal{A}) = \{e^{2\pi it} : t \in [0, \gamma^{-1}(0)]\}.$$

**Proof.** If $\gamma(0) > 0$, then

$$(Uf)(t) = \sqrt{|\gamma'(t)|} f(\gamma(t)), \quad (U^*f)(t) = \begin{cases} \sqrt{|(\gamma^{-1})'(t)|} f(\gamma^{-1}(t)), & t \in [\gamma(0), \infty), \\ 0, & t \in [0, \gamma(0)). \end{cases}$$

In particular, the projection $U^*U$ is the operator of multiplication by the characteristic function of $[\gamma(0), \infty)$ and thereby $\text{hull}(U^*U\mathcal{A} \cap \mathcal{A}) = \{e^{2\pi it} : t \in [0, \gamma(0)]\}$.

Item iii) is obtained by reversing the roles of $\gamma$ and $\gamma^{-1}$ in item i) and item ii) is straightforward.

We see that in a process of compression one may lose one of the relations (5.1). However, according to our results one may always retrieve what is lost by passing to a bigger algebra. To fix attention let us from now on assume that

$$\gamma(0) > 0$$

(the case when $\gamma(0) < 0$ is completely analogous). We put

$$B = C^*(\bigcup_{n=0}^{\infty} U^*U^n),$$

which by Theorem 2.23 is the smallest $C^*$-algebra containing $\mathcal{A}$ such that

$$UBU^* \subset B, \quad U^*B \subset B.$$

**Theorem 5.4.** The spectrum of $B$ assumes one of the forms:

i) if $\gamma(0) \geq 1$, then $\widetilde{M} = \mathbb{N} \times S^1$, see Fig. 18, (a);

ii) if $\gamma(0) \in (0, 1)$, then $\widetilde{M} = \bigcup_{N \in \mathbb{N}} M_N \cup M_\infty \subset \mathbb{N} \times S^1$, where $M_\infty = \{\infty\} \times S^1$ is a circle and the sets $M_N$, $N \in \mathbb{N}$, are arcs:

$$M_N = \{N\} \times [\alpha^N(1), \alpha^{N+1}(1)],$$

![Figure 18. Spectrum of $B$ related to a homeomorphism of the circle.](image)
where \([\alpha^N(1), \alpha^{N+1}(1)]\) stands for an arc on \(S^1\) with the origin \(\alpha^N(1)\) and ending \(\alpha^{N+1}(1)\), see Fig. 18, (b).

Under the identification \(\mathcal{B} = C(M)\) the operator \(U\) generates on \(\tilde{\mathcal{M}}\) the mapping \(\tilde{\alpha}\) that acts according to the formulae

\[
\tilde{\alpha}(N, x) = (N + 1, \alpha(x)), \quad \tilde{\alpha}(\infty, x) = (\infty, \alpha(x)).
\]

**Proof.** By Proposition 5.3 and Theorem 3.5 the system \((\tilde{\mathcal{M}}, \tilde{\alpha})\) is the reversible extension of \((S^1, \alpha)\) associated with the set \(Y = \{e^{2\pi it} : t \in [0, \gamma(0)]\}\), which either is a circle \(S^1\), when \(\gamma(0) \geq 1\), or an arc with the origin \(1 = e^{2\pi i0}\) and ending \(\alpha(1) = e^{2\pi i\gamma(0)}\). Thus it suffices to apply Proposition 5.1.

It follows that the algebra \(\mathcal{B}\) depends not only on the homeomorphism \(\alpha\) but also on the choice of the lift \(\gamma\). In particular, if \(\gamma(0) \geq 1\), then independently of \(\alpha\)

\[
\mathcal{B} \cong C(\mathbb{N} \times S^1).
\]

If however \(\gamma(0) \in (0, 1)\), then the structure of \(\mathcal{B}\) is uniquely determined by the orbit of the point \(1 \in S^1\). We will discuss this issue in detail below.

### 5.2. Classification of the extended algebras via rotation numbers.

We recall [22], [23] that if \(\gamma : \mathbb{R} \to \mathbb{R}\) is a lift of a homeomorphism \(\alpha : S^1 \to S^1\), then the limit \(\lim_{n \to \infty} \gamma(t)_n\) always exists, does not depend on \(t \in \mathbb{R}\) and its fractional part

\[
\tau(\alpha) := \left\{ \lim_{n \to \infty} \frac{\gamma(t)_n}{n} \right\} \in [0, 1)
\]

depends only on \(\alpha\). The quantity \(\tau(\alpha)\) is called the rotation number of \(\alpha\). Its role is explained by the following theorem due to Poincaré, cf. [22], [23]. The symbol \(\Theta_{\tau}, \tau \in [0, 1)\), stands for the rotation by \(2\pi\tau\): \(\Theta_{\tau}(z) = z \cdot e^{2\pi i\tau}, z \in S^1\).

**Theorem 5.5** (Poincaré classification). Let \(\alpha : S^1 \to S^1\) be an orientation-preserving homeomorphism of the circle.

1) If \(\tau(\alpha) = \frac{m}{n}\), where \(m\) and \(n\) are coprime, then all periodic points for \(\alpha\) are of period \(n\) (and there exists at least one such point).

2) If \(\tau(\alpha) \notin \mathbb{Q}\), then \(\alpha\) does not possess periodic points and the set \(\Omega(\alpha)\) consisting of the accumulation points of an arbitrary orbit \(\{\alpha(x)\}_{x \in \mathbb{Z}}\) does not depend on the choice of \(x \in S^1\). There are two possible subcases

a) when \(\Omega(\alpha) = S^1\), that is, when \(\alpha\) is topologically transitive, then the system \((S^1, \alpha)\) is topologically conjugated to \((S^1, \Theta_{\tau(\alpha)})\);

b) when \(\alpha\) is not topologically transitive, then \(\Omega(\alpha)\) is a perfect nowhere-dense subset of \(S^1\) and there exists a continuous surjection \(\phi : \Omega(\alpha) \to S^1\), which is a semiconjugacy from \((\Omega(\alpha), \alpha)\) onto \((S^1, \Theta_{\tau(\alpha)})\).

We use the above theorem to classify the algebras \(B\) described in item ii) of Theorem 5.4, that is, we assume throughout this subsection that the lift \(\gamma : \mathbb{R} \to \mathbb{R}\) satisfies

\[
0 < \gamma(0) < 1.
\]

Such a lift always exists, provided that \(1 \in S^1\) is not a fixed point of \(\alpha : S^1 \to S^1\). Since \(\gamma\) is uniquely determined by \(\alpha\), so also are the operator \(U\) and the algebra.
\[ B \cong C(\tilde{M}) \text{ defined in the previous subsection. Thus, it makes sense to adopt the following notation:} \]
\[ \mathcal{B}_\alpha := B, \quad \tilde{M}_\alpha := \tilde{M}. \]

The space \( M_\alpha \) consists of a circle \( M_\infty \) and a sequence of arcs \( \{M_N\}, N \in \mathbb{N} \).

**Theorem 5.6.** In the situation under consideration the following cases may occur:

1) if \( \tau(\alpha) = \frac{m}{n} \), where \( m \) and \( n \) are coprime, then the limit points of the end-points of arcs \( \{M_N\}_{N \in \mathbb{N}} \) form a subset of \( M_\infty \) with cardinality \( n \), and
\[ \mathcal{B}_\alpha \cong \mathcal{B}_{\Theta_{m/n}}; \]

2) if \( \tau(\alpha) \notin \mathbb{Q} \), then the two subcases are possible:
   a) \( \alpha \) is topologically transitive, and then \( M_\infty \) is the set of limit points of the end-points of arcs \( \{M_N\}_{N \in \mathbb{N}} \) and
\[ \mathcal{B}_\alpha \cong \mathcal{B}_{\Theta_{\tau(\alpha)}}; \]
   b) \( \alpha \) is not topologically transitive, and then the set of limit points of the end-points of arcs \( \{M_N\}_{N \in \mathbb{N}} \) form a perfect nowhere dense subset of \( M_\infty \).

In particular, \( \mathcal{B}_\alpha \neq \mathcal{B}_{\Theta_{\tau}}, \quad \tau \in [0, 1) \).

**Proof.** The set of limit points of the end-points of arcs \( \{M_N\}_{N \in \mathbb{N}} \) coincides with the set \( \{\infty\} \times \Omega(\alpha) \subset M_\infty \), where \( \Omega(\alpha) \) is the set of limit points of the orbit \( \{\alpha^N(1)\}_{N \in \mathbb{N}} \). By Theorem 5.5 we only need to consider the cases listed below.

1) The set \( \Omega(\alpha) \) consists of \( n \) points that form a periodic orbit of \( \alpha \). More precisely, there are \( n \) points \( x_0, x_1, \ldots, x_{n-1} \in S^1 \), enumerated according to the orientation and such that
\[ \lim_{N \to \infty} \alpha^{Nn+k}(1) = x_{km}, \quad k = 0, \ldots, m-1, \]
cf. [22]. Thus it follows that
\[ \lim_{N \to \infty} M_{Nn+k} = \{\infty\} \times [x_{km} \mod n, x_{(k+1)m} \mod n], \quad k = 0, \ldots, n-1, \]
that is, the sequence of arcs \( \{M_{Nm+k}\}_{N \in \mathbb{N}} \) converges in Hausdorff metric to the arc on \( M_\infty \) with the origin \( \infty, x_{km} \mod (mod m) \) and ending \( \infty, x_{(k+1)m} \mod (mod m) \). Let \( \varphi: \tilde{M}_\alpha \to \tilde{M}_{\Theta_{m/n}} \) be the mapping that acts ‘linearly’ according to the scheme
\[ \{\infty\} \times [x_k, x_{k+1} \mod n] \overset{\varphi}{\mapsto} \{\infty\} \times \left[ \exp\left(2\pi i \frac{k}{n}\right), \exp\left(2\pi i \frac{k+1}{n}\right) \right], \]
\[ k = 0, \ldots, n-1, \]
\[ \{N\} \times [\alpha^N(1), \alpha^{N+1}(1)] \overset{\varphi}{\mapsto} \{N\} \times \left[ \exp\left(2\pi i \frac{N}{n}\right), \exp\left(2\pi i \frac{N+1}{n}\right) \right], \quad N \in \mathbb{N}. \]

It is evident that \( \varphi \) is a homeomorphism and hence the algebras \( \mathcal{B}_\alpha = C(\tilde{M}_\alpha) \) and \( \mathcal{B}_{\Theta_{m/n}} = C(\tilde{M}_{\Theta_{m/n}}) \) are isomorphic.
2a) We have \( \{\infty\} \times \Omega(\alpha) = \{\infty\} \times S^1 = M_\infty \) and there exists a homeomorphism \( \varphi: S^1 \to S^1 \) such that

\[
\begin{array}{c}
S^1 \xrightarrow{\alpha} S^1 \\
\varphi \downarrow \quad \varphi \\
S^1 \xrightarrow{\Theta_{\tau}(\varphi)} S^1
\end{array}
\]

is commutative. Furthermore, \( \varphi \) may be arranged so that \( \varphi(1) = 1 \), see the proof of Theorem 7.1.9 in [22]. It follows that the mapping \( \text{id} \times \varphi: \hat{M}_\alpha \to \hat{M}_{\Theta_{\tau}(\varphi)} \):

\[
(id \times \varphi)(N, x) = (N, \varphi(x)), \quad (N, x) \in M_N, \quad N \in \mathbb{N},
\]

is a homeomorphism. Hence \( B_\alpha \cong B_{\Theta_{\tau}(\varphi)} \).

2b) Since \( \{\infty\} \times \Omega(\alpha) \) is a perfect nowhere-dense subset of \( M_\infty \) the space \( \hat{M}_\alpha \) is not homeomorphic to any of the subspaces \( \hat{M}_{\Theta_{\tau}} \), \( \tau \in [0, 1) \). Equivalently, \( B_\alpha \neq B_{\Theta_{\tau}} \) for all \( \tau \in [0, 1) \).

If \( \varphi \) is a topological conjugacy between \( (S^1, \alpha) \) and \( (S^1, \beta) \), then either \( \tau(\alpha) = \tau(\beta) \) (when \( \varphi \) is orientation preserving) or \( \tau(\alpha) + \tau(\beta) = 1 \) (when \( \varphi \) changes the orientation), so the rotation number is ‘almost an invariant’ for homeomorphisms of the circle. For the algebras \( B_\alpha \) the rotation number is an invariant \textit{sensu stricto}. If \( \varphi \) is a topological conjugacy between \( (S^1, \alpha) \) and \( (S^1, \beta) \), then either \( \tau(\alpha) = \tau(\beta) \) (when \( \varphi \) is orientation preserving) or \( \tau(\alpha) + \tau(\beta) = 1 \) (when \( \varphi \) changes the orientation), so the rotation number is ‘almost an invariant’ for homeomorphisms of the circle. For the algebras \( B_\alpha \) the rotation number is an invariant \textit{sensu stricto}.

**Theorem 5.7.** If algebras \( B_\alpha \) and \( B_\beta \) are isomorphic, then \( \tau(\alpha) = \tau(\beta) \).

**Proof.** Suppose that \( B_\alpha \) and \( B_\beta \) are isomorphic. There exists a homeomorphism \( \varphi: \hat{M}_\alpha \to \hat{M}_\beta \), where \( \hat{M}_\alpha = \bigcup_{N \in \mathbb{N}} M_N \) and \( \hat{M}_\beta = \bigcup_{N \in \mathbb{N}} M'_N \) are maximal ideal spaces of \( B_\alpha \) and \( B_\beta \), respectively. Clearly, \( \varphi \) carries the arcs \( \{M_N\}_{N \in \mathbb{N}} \) onto arcs \( \{M'_N\}_{N \in \mathbb{N}} \), the circle \( M_\infty \) onto circle \( M'_\infty \) and the set \( \Omega(\alpha) \subset M_\infty \) of limit points of end-points of the arcs \( \{M_N\}_{N \in \mathbb{N}} \) onto the set \( \Omega(\beta) \subset M'_\infty \) of limit points of end-points of the arcs \( \{M'_N\}_{N \in \mathbb{N}} \). For the simplicity of notation we adopt the identification \( M_\infty = M'_\infty = S^1 \). We claim that \( \varphi \) establishes conjugacy between the systems \( (\Omega(\alpha), \alpha) \) and \( (\Omega(\beta), \beta) \) or \( (\Omega(\alpha), \alpha) \) and \( (\Omega(\beta), \beta^{-1}) \) depending on whether \( \varphi: M_\infty \to M'_\infty \) preserves or changes the orientation. Once we prove this, the standard argument gives us that either \( \tau(\alpha) = \tau(\beta) \) or \( \tau(\alpha) + \tau(\beta) = 1 \), where in the latter case we get \( \tau(\alpha) = \tau(\beta) \) since \( \tau(\beta^{-1}) = 1 - \tau(\beta) \).

To prove our claim we fix a sequence \( \{\alpha^{N_k}(1)\}_{k \in \mathbb{N}} \) converging to an arbitrarily chosen point \( x_0 \in \Omega(\alpha) \). Then the sequence \( \{\alpha^{N_k+1}(1)\}_{k \in \mathbb{N}} \) converges to \( \alpha(x_0) \), and the sequence of arcs \( \{M_{N_k}\}_{N \in \mathbb{N}} \) converges (in the Hausdorff metric) to the arc \( [x_0, \alpha(x_0)] \). In the case when \( \varphi: M_\infty \to M'_\infty \) preserves the orientation, almost all arcs from the sequence \( \{M_{N_k}\}_{N \in \mathbb{N}} \) are mapped in accordance with (the natural) orientation onto almost all arcs of the sequence \( \{\varphi(M_{N_k})\}_{N \in \mathbb{N}} \). Hence

\[
[\varphi(x_0), \varphi(\alpha(x_0))] = \varphi([x_0, \alpha(x_0)]) = \varphi(\lim_{k \to \infty} M_{N_k}) = [\varphi(x_0), \beta(\varphi(x_0))].
\]

Thus \( \varphi(\alpha(x_0)) = \beta(\varphi(x_0)) \) and consequently \( \varphi \) conjugates the systems \( (\Omega(\alpha), \alpha) \) and \( (\Omega(\beta), \beta) \). In the case when \( \varphi: M_\infty \to M'_\infty \) changes the orientation, arguing
similarly as above one gets
\[ [\varphi(\alpha(x_0)), \varphi(x_0)] = \varphi([x_0, \alpha(x_0)]) = \lim_{k \to \infty} \varphi(M_{N_k}) = [\beta^{-1}(\varphi(x_0)), \varphi(x_0)]. \]
Hence \( \varphi(\alpha(x_0)) = \beta^{-1}(\varphi(x_0)) \) and consequently \( \varphi \) conjugates the systems \( (\Omega(\alpha), \alpha) \) and \( (\Omega(\beta), \beta^{-1}) \). This proves our claim.

Applying the classical result of Denjoy [34], [22], [23], which states that every diffeomorphism of the circle with finite variation and irrational rotation number is topologically transitive, we get that in the class of algebras \( B_\alpha \) associated with such diffeomorphisms the rotation number is not only an invariant but actually a numerical equivalent.

Theorem 5.8. If one of the homeomorphisms \( \alpha, \beta \) is a diffeomorphism with finite variation, then the algebras \( B_\alpha \) and \( B_\beta \) are isomorphic if and only if \( \tau(\alpha) = \tau(\beta) \).

Proof. Apply Theorems 5.6, 5.7 and Denjoy’s Theorem.

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