Multiple vortices for a self-dual $CP(1)$
Maxwell-Chern-Simons model

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Abstract
We prove the existence of at least two doubly periodic vortex solutions for a self-dual $CP(1)$ Maxwell-Chern-Simons model. To this end we analyze a system of two elliptic equations with exponential nonlinearities. Such a system is shown to be equivalent to a fourth-order elliptic equation admitting a variational structure.

Key Words: nonlinear elliptic system, nonlinear fourth-order elliptic equation, Chern-Simons vortex theory

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0 Introduction
The vortex solutions for the self-dual $CP(1)$ Maxwell-Chern-Simons model introduced in [14] (see also the monographs [9, 12, 22]) are described by a system of two elliptic equations with exponential nonlinearities defined on a two-dimensional Riemannian manifold. Such a system (henceforth, the “$CP(1)$ system”) was considered in [7], where among other results the authors prove the existence of one doubly periodic solution by super/sub methods. On the other hand, formal arguments from physics as well as certain analogies with the $U(1)$ Maxwell-Chern-Simons model [5, 19] and with the $CP(1)$ “pure” Chern-Simons model [6, 13] suggest that solutions to the $CP(1)$ system should be multiple. In the special case of single-signed negative vortex points, a second solution for the $CP(1)$ system was exhibited in [18]. The method employed in [18] is not directly applicable to the general case, due to the singularities produced by the positive vortex points. Our aim in this note is to prove multiplicity of solutions for the $CP(1)$ system in the general case of vortex points of either sign. In fact, we shall prove multiplicity for an abstract system of nonlinear elliptic

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equations which includes the $CP(1)$ system as a special case, thus emphasizing some essential features of the $CP(1)$ system which ensure the multiplicity of vortex solutions.

For the sake of simplicity we define our equations on the flat 2-torus $M = \mathbb{R}^2/\mathbb{Z}^2$, although it will be clear that corresponding results hold true on general compact Riemannian 2-manifolds. We fix $p_1, \ldots, p_m \in M$ the “positive vortex points” and $q_1, \ldots, q_n \in M$ the “negative vortex points”. The $CP(1)$ system as introduced in [14] and analyzed in [7] is given by

$$
\Delta \tilde{u} = 2q \left( -N + S - \frac{1 - e^\tilde{u}}{1 + e^\tilde{u}} \right) - 4\pi \sum_{j=1}^{m} \delta_{p_j} + 4\pi \sum_{k=1}^{n} \delta_{q_k} \quad \text{on } M
$$

$$
\Delta N = -\kappa^2 q^2 \left( -N + S - \frac{1 - e^\tilde{u}}{1 + e^\tilde{u}} \right) + q \frac{4e\tilde{u}}{(1 + e\tilde{u})^2} N \quad \text{on } M
$$

where the couple $(\tilde{u}, N)$ is the unknown variable, $q, \kappa > 0$ and $S \in \mathbb{R}$ are constants and $\delta_{p_j}, \delta_{q_k}$ are the Dirac measures centered at $p_j, q_k$. Setting $v = N - S, s = -S, \lambda = 2/\kappa, \varepsilon = 1/(\kappa q)$, the above system takes the form:

(1) $-\Delta \tilde{u} = \varepsilon^{-1} \lambda (v - f(\tilde{u})) + 4\pi \sum_{j=1}^{m} \delta_{p_j} - 4\pi \sum_{k=1}^{n} \delta_{q_k} \quad \text{on } M$

(2) $-\Delta v = \varepsilon^{-1} \left[ \lambda f'(\tilde{u}) \delta(\tilde{u} - s) - \varepsilon^{-1} (v - f(\tilde{u})) \right] \quad \text{on } M,$

where $f : [0, +\infty) \to \mathbb{R}$ is defined by $f(t) = (t - 1)/(t + 1)$. In the special case $m = 0$ system (1)–(2) was introduced in [17] with the aim of providing a unified framework for the results in [5, 16, 19] and in [7]. A multiplicity result for (1)–(2) when $m = 0$ was obtained in [18]. Our main result concerns the multiplicity of solutions for system (1)–(2) in the case $m > 0$ under the following

**Assumptions on $f$:**

(f0) $f : [0, +\infty) \to \mathbb{R}$ smooth and $f'(t) > 0 \ \forall t > 0$;

(f1) $f(0) < s < \sup_{t > 0} f(t) < +\infty$;

(f2) $\sup_{t > 0} t^4 |f''(t)| < +\infty$.

For later use, we note that assumptions (f0)–(f1)–(f2) imply that there exists $f_\infty > s$ such that

$$
(3) \quad \sup_{t > 0} \left[ t |f(t) - f_\infty| + t^2 f'(t) + t^3 |f''(t)| + t^4 |f'''(t)| \right] < +\infty.
$$

Clearly, $f$ defined by $f(t) = (t - 1)/(t + 1)$ satisfies (f0)–(f1)–(f2) for every $-1 < s < 1$. We restrict our attention to the case $m > n$. It will be clear that the case $m < n$ may be treated analogously, while the case $m = n$ requires an altogether different method and will not be considered here. Our main result is the following

**Theorem 0.1.** Let $m > n$ and suppose that $f$ satisfies assumptions (f0)–(f1)–(f2). Then there exists $\lambda_0 > 0$ with the property that for every fixed $\lambda \geq \lambda_0$ there exists $\varepsilon_\lambda > 0$ such that for each $0 < \varepsilon \leq \varepsilon_\lambda$ system (1)–(2) admits at least two solutions.
The remaining part of this note is devoted to the proof of Theorem 0.1. In Section 1 we prove that system (1)–(2) is equivalent to the following nonlinear elliptic equation of the fourth order:

\begin{equation}
\epsilon^2 \Delta^2 u - \Delta u = -\epsilon \lambda \left[ f'' \left( e^{\sigma + u} \right) e^{\sigma + u} + f' \left( e^{\sigma + u} \right) \right] e^{\sigma + u} |\nabla (\sigma + u)|^2 \\
+ 2\epsilon \lambda \Delta f \left( e^{\sigma + u} \right) + \lambda^2 f' \left( e^{\sigma + u} \right) e^{\sigma + u} (s - f \left( e^{\sigma + u} \right)) + 4\pi (m - n) \quad \text{on } M,
\end{equation}

where \( \sigma \) is the Green function uniquely defined by

\[-\Delta \sigma = 4\pi \sum_{j=1}^{m} \delta_{p_j} - 4\pi \sum_{k=1}^{n} \delta_{q_k} - 4\pi (m - n), \int_M \sigma = 0 \text{ (note that } |M| = 1) \]. By formally setting \( \epsilon = 0 \) in (4) we obtain the “limit” equation

\begin{equation}
-\Delta u = \lambda^2 f' \left( e^{\sigma + u} \right) e^{\sigma + u} (s - f \left( e^{\sigma + u} \right)) + 4\pi (m - n) \quad \text{on } M.
\end{equation}

For \( f(t) = (t-1)/(t+1) \) equation (5) describes the vortex solutions for the \( CP(1) \) Chern-Simons model introduced in [13] and analyzed in [6]. When \( f(t) = t \) and \( s = 1 \), equation (5) describes the vortex solutions for the \( U(1) \) Chern-Simons model introduced in [10, 11], which has received considerable attention by analysts in recent years, see [4, 8, 15, 21] and the references therein. In turn, solutions to (4) correspond to critical points in the Sobolev space \( H^2(M) \) for the functional \( I_\epsilon \) defined by

\[
I_\epsilon(u) = \frac{\epsilon^2}{2} \int (\Delta u)^2 + \frac{1}{2} \int |\nabla u|^2 + \epsilon \lambda \int f' \left( e^{\sigma + u} \right) e^{\sigma + u} |\nabla (\sigma + u)|^2 \\
+ \frac{\lambda^2}{2} \int (f \left( e^{\sigma + u} \right) - s)^2 - 4\pi (m - n) \int u.
\]

The two desired solutions for (1)–(2) will be obtained as a local minimum and a mountain pass for \( I_\epsilon \) (in the sense of Ambrosetti and Rabinowitz [1]). The main issue will be to produce a local minimum for \( I_\epsilon \). To this end, in Section 2 we first construct a supersolution \( \bar{u} \) for equation (5). By adapting to the fourth order equation (4) the constrained minimization technique for second order equations in Brezis and Nirenberg [3] (see also Tarantello [21]), we set

\[ A = \{ u \in H^2(M), \ u \leq \bar{u} \ \text{a.e. on } M \} \]

and we consider \( u_\epsilon \in A \) satisfying

\[
I_\epsilon(u_\epsilon) = \min_A I_\epsilon.
\]

Then \( u_\epsilon \) is a subsolution for (4). By an accurate analysis we show that for small values of \( \epsilon \) we have in fact the strict inequality \( u_\epsilon < \bar{u} \) everywhere on \( M \). Consequently, \( u_\epsilon \) is an internal minimum point for \( I_\epsilon \) on \( A \) in the sense of \( H^2 \), and thus it yields a local minimum for \( I_\epsilon \). On the other hand we have \( I_\epsilon(c) \to -\infty \) on constant functions \( c \to +\infty \). Consequently, \( I_\epsilon \) has a mountain pass geometry. In Section 3 we prove the Palais-Smale condition for \( I_\epsilon \). At this point, the classical mountain pass theorem in [1] concludes the proof of Theorem 0.1. The Appendix contains some simple technical facts which are repeatedly used throughout the proofs.

**Notation.** Henceforth, unless otherwise specified, all equations are defined on \( M \), all integrals are taken over \( M \) with respect to the Lebesgue measure and all functional spaces are defined on \( M \) in the usual way. In particular, we denote by \( L^p \), \( 1 \leq p \leq +\infty \), the Lebesgue spaces and by \( H^k \), \( k \geq 1 \) the Sobolev spaces. We denote by \( C > 0 \) a general constant, independent of certain parameters that will be specified in the sequel, and whose actual value may vary from line to line.
1 Preliminaries

In this section we show that system (1)–(2) admits a variational structure. We set \( A = 4\pi(m - n) > 0 \). Following a technique introduced by Taubes for self-dual models (see [12]), we denote by \( \sigma \) the Green’s function uniquely defined by

\[
- \Delta \sigma = 4\pi \sum_{j=1}^{m} \delta_{p_j} - 4\pi \sum_{k=1}^{n} \delta_{q_k} - A
\]

(recall that \( |M| = 1 \)). Setting \( \tilde{u} = \sigma + u \) system (1)–(2) takes the form

\[
\begin{align*}
- \Delta u &= \varepsilon^{-1} \lambda (v - f(e^{\sigma + u})) + A \\
- \Delta v &= \varepsilon^{-1} \left[ \lambda f'(e^{\sigma + u}) e^{\sigma + u} (s - v) - \varepsilon^{-1} (v - f(e^{\sigma + u})) \right].
\end{align*}
\]

In turn, system (6)–(7) is equivalent to a fourth order equation. We note that equation (7) may be written in the equivalent form:

\[
- \Delta v + \varepsilon^{-2} (1 + \varepsilon \lambda f'(e^{\sigma + u}) e^{\sigma + u}) v = \varepsilon^{-2} [f(e^{\sigma + u}) + \varepsilon \lambda f'(e^{\sigma + u}) e^{\sigma + u}].
\]

By uniqueness for equation (8) for every fixed \( u \), if \( v \in L^1 \) is a distributional solution for (8), then it is in fact \( H^1 \). We first show:

**Lemma 1.1.** Suppose \((u, v) \in H^1 \times H^1\) is a weak solution for system (6)–(7). Then \((u, v)\) is a classical solution.

**Proof.** Throughout this proof, we denote by \( \alpha > 0 \) a general Hölder exponent.

By (8), \( v \in C^\alpha \). Then by (6), \( u \in C^{1, \alpha} \). By Lemma 4.2 and (3), \( f(e^{\sigma + u}) \) and \( f'(e^{\sigma + u}) e^{\sigma + u} \) are Lipschitz continuous. Therefore, by (8) \( v \in C^{2, \alpha} \). In turn, by (6) \( u \in C^{2, \alpha} \) and in particular \((u, v)\) is a classical solution. \(\square\)

**Lemma 1.2.** The couple \((u, v) \in H^1 \times H^1\) is a weak solution for system (6)–(7) if and only if \( u \in H^2 \) is a weak solution for the fourth order equation

\[
\varepsilon^2 \Delta^2 u - \Delta u = -\varepsilon \lambda \left[ f''(e^{\sigma + u}) e^{\sigma + u} + f'(e^{\sigma + u}) \right] e^{\sigma + u} |\nabla (\sigma + u)|^2 + 2\varepsilon \lambda \Delta f(e^{\sigma + u}) + \lambda^2 f'(e^{\sigma + u}) e^{\sigma + u} (s - f(e^{\sigma + u})) + A,
\]

and \( v \) is defined by

\[
v = -\varepsilon \lambda^{-1} \Delta u - \varepsilon \lambda^{-1} A + f(e^{\sigma + u}).
\]

**Proof.** Suppose \((u, v) \in H^1 \times H^1\) is a weak solution for (6)–(7). Then by Lemma 1.1 we have in particular \( u \in H^2 \). Solving (6) for \( v \), we obtain (10). Inserting the expression for \( v \) as given by (10) into (7), we find that \( u \) is a distributional solution for the equation

\[
\varepsilon^2 \Delta^2 u - \Delta u = \varepsilon \lambda \Delta f(e^{\sigma + u}) + \varepsilon \lambda f'(e^{\sigma + u}) e^{\sigma + u} (\Delta u + A) + \lambda^2 f'(e^{\sigma + u}) e^{\sigma + u} (s - f(e^{\sigma + u})) + A.
\]
On the other hand, by the identities (46) and (47) in the Appendix we have, in the sense of distributions:

\[
\Delta f(e^{\sigma+u}) + f'(e^{\sigma+u}) e^{\sigma+u} (\Delta u + A) = \Delta f(e^{\sigma+u}) + f'(e^{\sigma+u}) e^{\sigma+u} \Delta(\sigma + u) = 2\Delta f(e^{\sigma+u}) - \{ f''(e^{\sigma+u}) e^{\sigma+u} + f'(e^{\sigma+u}) \} e^{\sigma+u} |\nabla(\sigma + u)|^2.
\]

Inserting into (11), we conclude that \( u \in H^2 \) satisfies (9).

Conversely, suppose \( u \in H^2 \) is a weak solution for (9). Then \( v \) defined by (10) belongs to \( L^2 \), and thus it is a distributional solution for (7). By uniqueness of solutions to (8) for fixed \( u \), we conclude that \( v \in H^1 \).

Equation (9) admits a variational formulation, as stated in the following

**Lemma 1.3.** \( u \in H^2 \) is a weak solution for (9) if and only if it is a critical point for the \( C^1 \) functional \( I_\varepsilon \) defined on \( H^2 \) by:

\[
I_\varepsilon(u) = \varepsilon^2 \int (\Delta u)^2 + \frac{1}{2} \int |\nabla u|^2 + \varepsilon \lambda \int f'(e^{\sigma+u}) e^{\sigma+u} |\nabla(\sigma + u)|^2 + \frac{\lambda^2}{2} \int (f(e^{\sigma+u}) - s)^2 - A \int u.
\]

**Proof.** By Lemma 4.1 and properties of \( f \) as in (3), \( I_\varepsilon \) is well-defined and \( C^1 \) on \( H^2 \). We compute, for any \( \phi \in H^2 \):

\[
\frac{d}{dt} \bigg|_{t=0} \int f'(e^{\sigma+u+t\phi}) e^{\sigma+u+t\phi} |\nabla(\sigma + u + t\phi)|^2 = \int \left[ f''(e^{\sigma+u}) e^{\sigma+u} + f'(e^{\sigma+u}) \right] e^{\sigma+u} |\nabla(\sigma + u)|^2 \phi + 2 \int f'(e^{\sigma+u}) e^{\sigma+u} \nabla(\sigma + u) \cdot \nabla \phi.
\]

Consequently,

\[
\langle I_\varepsilon'(u), \phi \rangle = \varepsilon^2 \int \Delta u \Delta \phi + \int \nabla u \cdot \nabla \phi + \varepsilon \lambda \int \left[ f''(e^{\sigma+u}) e^{\sigma+u} + f'(e^{\sigma+u}) \right] e^{\sigma+u} |\nabla(\sigma + u)|^2 \phi + 2 \varepsilon \lambda \int f'(e^{\sigma+u}) e^{\sigma+u} \nabla(\sigma + u) \cdot \nabla \phi + \lambda^2 \int f'(e^{\sigma+u}) e^{\sigma+u} (f(e^{\sigma+u}) - s) \phi - A \int \phi.
\]

Since

\[
\int f'(e^{\sigma+u}) e^{\sigma+u} \nabla(\sigma + u) \cdot \nabla \phi = \int \nabla f(e^{\sigma+u}) \cdot \nabla \phi = -\int \Delta f(e^{\sigma+u}) \phi,
\]

it follows that critical points of \( I_\varepsilon \) correspond to solutions for (9), as asserted. \( \square \)
2 A local minimum

Our aim in this section is to prove the existence of a local minimum for the functional $I_\varepsilon$, as stated in the following

**Proposition 2.1.** There exists $\lambda_0 > 0$ with the property that for every fixed $\lambda \geq \lambda_0$ there exists $\varepsilon_\lambda > 0$, such that for any $0 < \varepsilon \leq \varepsilon_\lambda$ there exists a solution $u_\varepsilon$ to (9) corresponding to a local minimum for the functional $I_\varepsilon$.

Throughout this section, we denote by $C > 0$ a general constant independent of $\varepsilon > 0$. Following an idea in [6], we first construct a supersolution for the “limit” equation

$$-\Delta u = \lambda^2 f'(e^{\sigma+u}) e^{\sigma+u} (s - f(e^{\sigma+u})) + A,$$

which is formally obtained from (9) by setting $\varepsilon = 0$.

**Lemma 2.1.** There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ equation (16) admits a (distributional) supersolution $\bar{u}$.

**Proof.** We fix $\rho > 0$ such that

$$\bigcup_{j=1}^m B_\rho(p_j) \cap \bigcup_{j=1}^m B_\rho(p_l) = \emptyset$$

and

$$\sum_{j=1}^m |B_\rho(p_j)| < \frac{1}{2}.$$

We denote by $g$ a smooth cutoff function satisfying

$$g(x) = \begin{cases} 
1, & \text{if } x \in \bigcup_{j=1}^m B_\rho(p_j) \\
0, & \text{if } x \in M \setminus \bigcup_{j=1}^m B_{2\rho}(p_j)
\end{cases}$$

and $0 \leq g(x) \leq 1$ for all $x \in M$. Let $u^*$ be the function uniquely defined by

$$-\Delta u^* = A - 4\pi m + 8\pi m \left( g - \int g \right) + 4\pi \sum_{k=1}^n \delta_{q_k},$$

$$\int u^* = 0.$$

We define

$$\bar{u} = u^* + \tilde{C},$$

with $\tilde{C} > 0$ sufficiently large so that

$$f(e^{\sigma + \bar{u}}) - s > c_0$$

on $M$

for some $c_0 > 0$. Such a $\tilde{C}$ exists by (f1) since $\sigma + u^*$ is bounded below on $M$. We claim that for all $\lambda$ sufficiently large, $\bar{u}$ is a supersolution for (16). Indeed, if $x \in \bigcup_{j=1}^m B_\rho(p_j)$, then $g(x) = 1$ and in view of (17)

$$-\Delta \bar{u} \geq A - 4\pi m + 8\pi m \left( 1 - \sum_{j=1}^m |B_\rho(p_j)| \right) + 4\pi \sum_{j=1}^m \delta_{q_k} \geq A$$

$$\geq \lambda^2 f'(e^{\sigma+\bar{u}}) e^{\sigma+\bar{u}} (s - f(e^{\sigma+\bar{u}})) + A.$$
On the other hand, if \( x \in M \setminus \bigcup_{j=1}^{m} B_{\rho}(p_j) \), we have:

\[-\Delta \bar{u} \geq A - 4\pi m - 8\pi m \int g + 4\pi \sum_{k=1}^{n} \delta_{y_k} \geq A - 12\pi m.\]

Let us check that on \( M \setminus \bigcup_{j=1}^{m} B_{\rho}(p_j) \) we have

\[A - 12\pi m \geq \lambda u \epsilon (s - f (e^{\sigma + \bar{u}})) + A\]

for all \( \lambda \) sufficiently large. Indeed, we can choose \( C_1 > 0 \) such that

\[C_1^{-1} \leq e^{\sigma + \bar{u}} \leq C_1 \text{ on } M \setminus \bigcup_{j=1}^{m} B_{\rho}(p_j).\]

Therefore, for \( \lambda \) large, \( \bar{u} \) is a subsolution for (16) in \( M \setminus \bigcup_{j=1}^{m} B_{\rho}(p_j) \).

Henceforth, we fix \( \lambda \geq \lambda_0 \). We note that solutions to (16) correspond to critical points in \( H^1 \) for the functional \( I_0 \) defined by

\[I_0(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{\lambda^2}{2} \int (f(e^{\sigma + u}) - s)^2 - A \int u,\]

for \( u \in H^1 \). We define

\[A = \{ u \in H^2 : u \leq \bar{u} \}.\]

Then \( A \) is a convex closed subset of \( H^2 \) and consequently there exists \( u_\epsilon \in H^2 \) satisfying

\[I_\epsilon(u_\epsilon) = \min_{A} I_\epsilon.\]

Since \( u_\epsilon - \phi \in A \) for every \( \phi \in H^2, \phi \geq 0 \) we have \( I_\epsilon(u_\epsilon - \phi) \geq I_\epsilon(u_\epsilon) \) for every \( \phi \in H^2, \phi \geq 0 \). Therefore, \( u_\epsilon \) is a weak subsolution for (9), i.e., it satisfies

\[\varepsilon^2 \Delta^2 u_\epsilon - \Delta u_\epsilon \leq -\varepsilon \lambda \left[ f''(e^{\sigma + u_\epsilon}) e^{\sigma + u_\epsilon} + f'(e^{\sigma + u_\epsilon}) \right] e^{\sigma + u_\epsilon} |\nabla(\sigma + u_\epsilon)|^2 + 2\varepsilon \lambda \Delta f(e^{\sigma + u_\epsilon}) + \lambda^2 f'(e^{\sigma + u_\epsilon}) e^{\sigma + u_\epsilon} (s - f (e^{\sigma + u_\epsilon})) + A.\]

in the weak sense. The main step towards proving Proposition 2.1 will be to prove the strict inequality \( u_\epsilon < \bar{u} \) on \( M \), see Lemma 2.4 below. We begin by establishing:

**Lemma 2.2.** There exists a subsolution \( u_0 \in H^1 \) for equation (16) such that \( u_\epsilon \to u_0 \) weakly in \( H^1 \), strongly in \( L^p \) for every \( p \geq 1 \) and a.e. on \( M \). Furthermore, \( u_0 < u \).
Proof. We denote by \( \mu \in A \) the constant function defined by
\[
\mu(x) = \min_M \bar{u} \quad \text{for all } x \in M.
\]
Then
\[
I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(\mu) \leq C.
\]
Since we also have \( \int u_\varepsilon \leq \int \bar{u} \leq C \), we readily derive the estimates
\[
\varepsilon \| \Delta u_\varepsilon \|^2 + \| \nabla u_\varepsilon \|^2 + \varepsilon \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} |\nabla (\sigma + u_\varepsilon)|^2 + \left| \int u_\varepsilon \right| \leq C
\]
In particular, we have \( \| u_\varepsilon \|_{H^1} \leq C \). Therefore, by Sobolev embeddings there exists \( u_0 \in H^1 \) such that up to subsequences \( u_\varepsilon \rightharpoonup u_0 \) weakly in \( H^1 \), strongly in \( L^p \) for every \( p \geq 1 \) and a.e. on \( M \). In particular, \( u_0 \leq \bar{u} \) on \( M \). Taking limits in (18), we find that \( u_0 \) is a subsolution for (16). Now the strong maximum principle yields \( u_0 < \bar{u} \) on \( M \).

Now we can strengthen the convergences stated in Lemma 2.2.

Lemma 2.3. The following limits hold:

(i) \( \lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) = \inf_A I_0 = I_0(u_0) \)

(ii) \( \lim_{\varepsilon \to 0} \varepsilon \| \Delta u_\varepsilon \| = 0 \)

(iii) \( \lim_{\varepsilon \to 0} \varepsilon \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} |\nabla (\sigma + u_\varepsilon)|^2 = 0. \)

Furthermore, \( u_0 \) is in fact a solution for (16).

Proof. Proof of (i). The functional \( I_\varepsilon \) may be written in the form
\[
I_\varepsilon(u) = \frac{\varepsilon^2}{2} \| \Delta u \|^2 + \varepsilon \lambda \int f'(e^{\sigma+u})e^{\sigma+u} |\nabla (\sigma + u)|^2 + I_0(u)
\]
for every \( u \in H^2 \). Consequently
\[
I_\varepsilon(u_\varepsilon) = \inf_A I_\varepsilon \geq \inf_A I_0,
\]
and therefore
\[
\liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \geq \inf_A I_0.
\]

In order to prove that
\[
\limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \leq \inf_A I_\varepsilon,
\]
we observe that for any \( \eta > 0 \) we can select \( u_\eta \in A \) such that
\[
I_0(u_\eta) \leq \inf_A I_0 + \eta.
\]

Then we have
\[
I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(u_\eta) \leq I_0(u_\eta) + o_\varepsilon(1)
\leq \inf_A I_0 + \eta + o_\varepsilon(1).
\]
Therefore
\[(23) \limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \leq \inf_\mathcal{A} I_0 + \eta, \]
and since \(\eta\) can be chosen arbitrarily small we obtain (22). From (21) and (22) we obtain (i).

Proof of (ii)–(iii). Since \(u_\varepsilon \rightharpoonup u_0\) weakly in \(H^1\), we have
\[
\liminf_{\varepsilon \to 0} I_0(u_\varepsilon) \geq I_0(u_0) = \inf_\mathcal{A} I_0.
\]
Therefore,
\[
\inf_\mathcal{A} I_0 = \lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \left\{ \frac{\varepsilon^2}{2} \|\Delta u_\varepsilon\|^2 + \varepsilon \lambda \int f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} |\nabla (\sigma + u_\varepsilon)|^2 + I_0(u_\varepsilon) \right\} \geq I_0(u_0) \geq \inf_\mathcal{A} I_0.
\]
Hence, (ii) and (iii) are established. By (i) we obtain that \(u_\varepsilon \to u_0\) strongly in \(H^1\) and \(I_0(u_0) = \inf_\mathcal{A} I_0\). Since we also have \(u_0 < \bar{u}\) (see Lemma 2.2), we have that \(u_0\) belongs to the interior of \(\mathcal{A}\) in the \(C^0\)-topology. In particular, \(u_0\) is a local minimum for \(I_0\) in the \(C^1\)-topology. By the Brezis and Nirenberg argument in [3], \(u_0\) is a local minimum for \(I_0\) in the \(H^1\)-topology and thus it is in fact a solution for (16).

Now we are ready to prove the following crucial strict inequality:

**Lemma 2.4.** For every fixed \(\lambda \geq \lambda_0\) there exists \(\varepsilon_\lambda > 0\) such that for every \(0 < \varepsilon < \varepsilon_\lambda\) there holds
\[(24) u_\varepsilon < \bar{u} \text{ on } M.\]

**Proof.** We denote
\[a(u) = -\left[ f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u}) \right] e^{\sigma+u} |\nabla (\sigma + u)|^2 + 2\Delta f(e^{\sigma+u})\]
for all \(u \in H^2\) and
\[
F_\varepsilon = \varepsilon \lambda a(u_\varepsilon) + \lambda^2 f'(e^{\sigma+u_\varepsilon}) e^{\sigma+u_\varepsilon} (s - f(e^{\sigma+u_\varepsilon})) + A,
\]
\[
F_0 = \lambda^2 f'(e^{\sigma+u_0}) e^{\sigma+u_0} (s - f(e^{\sigma+u_0})) + A.
\]
Then (18) may be written in the form:
\[
\varepsilon^2 \Delta^2 u_\varepsilon - \Delta u_\varepsilon \leq F_\varepsilon.
\]
Now we exploit the decomposition \(\varepsilon^2 \Delta^2 - \Delta = (-\varepsilon^2 \Delta + 1)(-\Delta)\). Let \(G_\varepsilon\) be the Green function for the operator \(-\varepsilon^2 \Delta + 1\) on \(M\). In what follows we shall repeatedly use the properties of \(G_\varepsilon\) established in Lemma 4.4 in the Appendix. Since \(G_\varepsilon > 0\) on \(M\), from the above inequality we derive
\[(25) -\Delta u_\varepsilon \leq G_\varepsilon * F_\varepsilon.\]
Claim: There exists $1 < q < 2$ such that

(26) $\|F_\varepsilon - F_0\|_q \to 0$ as $\varepsilon \to 0$.

Proof of (26). We only show that $\varepsilon \|a(u_\varepsilon)\|_q \to 0$ as $\varepsilon \to 0$, since the remaining estimates follow by compactness arguments in a straightforward manner. By identity (47) in the Appendix we may write

$$a(u_\varepsilon) = [f''(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} + f'(e^{\sigma + u_\varepsilon})]e^{\sigma + u_\varepsilon} |\nabla(\sigma + u_\varepsilon)|^2$$

$$+ 2f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta(\sigma + u_\varepsilon).$$

Therefore, in view of (3) it suffices to show that as $\varepsilon \to 0$

(27) $\varepsilon \| [f''(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} + f'(e^{\sigma + u_\varepsilon})]e^{\sigma + u_\varepsilon} |\nabla\sigma|^2 \|_q \to 0$

(28) $\varepsilon \| |\nabla u_\varepsilon|^2 \|_q \to 0$

(29) $\varepsilon \| f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta(\sigma + u_\varepsilon) \|_q \to 0$.

To see (27), note that by Lemma 4.1 in the Appendix and properties of $f$ as stated in (3),

$$[f''(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} + f'(e^{\sigma + u_\varepsilon})]e^{\sigma + u_\varepsilon} |\nabla\sigma|^2 \leq C (1 + e^{u_\varepsilon} + e^{-u_\varepsilon}).$$

By (19) and the Moser-Trudinger inequality (see, e.g., Aubin [2]), we have $\|e^{u_\varepsilon}\|_q \leq Ce^{|u_\varepsilon|\|\nabla u_\varepsilon\|_2^2} \leq C$. Similarly, we obtain $\|e^{-u_\varepsilon}\|_q \leq C$. Therefore, (27) is established. To see (28), let $1 < q < \alpha < 2$. Then by Hölder’s inequality and (19),

$$\int |\nabla u_\varepsilon|^2 \leq \int |\nabla u_\varepsilon|^q |\nabla u_\varepsilon|^{2q-\alpha}$$

$$\leq \left( \int |\nabla u_\varepsilon|^2 \right)^{\alpha/2} \left( \int |\nabla u_\varepsilon|^{2(2q-\alpha)/(2-\alpha)} \right)^{(2-\alpha)/2}$$

$$\leq C \|\nabla u_\varepsilon\|^{2q-\alpha}_{2(2q-\alpha)/(2-\alpha)}.$$

Consequently, in view of Lemma 2.3–(ii):

$$\varepsilon \| |\nabla u_\varepsilon|^2 \|_q \leq C \varepsilon \|\nabla u_\varepsilon\|^{(2q-\alpha)/q}_{2(2q-\alpha)/(2-\alpha)}$$

$$\leq C \varepsilon^{1-(2q-\alpha)/q} (\varepsilon \|\nabla u_\varepsilon\|_{2(2q-\alpha)/(2-\alpha)}^{(2q-\alpha)/q})$$

$$\leq C \varepsilon^{1-(2q-\alpha)/q} \|\varepsilon \Delta u_\varepsilon\|^{(2q-\alpha)/q}_{2(2q-\alpha)/(2-\alpha)} \to 0,$$

and (28) follows. Finally, by identity (46) in the Appendix and (3) we have

$$\varepsilon \| f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta(\sigma + u_\varepsilon) \|_q$$

$$= \varepsilon \| A f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} + f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta u_\varepsilon \|_q$$

$$\leq \varepsilon \| A f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \|_q + \varepsilon \| f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta u_\varepsilon \|_q$$

$$= \varepsilon \| f'(e^{\sigma + u_\varepsilon})e^{\sigma + u_\varepsilon} \Delta u_\varepsilon \|_q + o_\varepsilon(1).$$
By the Hölder inequality, Lemma 2.3-(ii) and (3) we have

$$\varepsilon \| f'(e^{\sigma + u} \Delta u) \| \leq C \varepsilon \| \Delta u \| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

which yields (29). We conclude that $$\varepsilon \| a(u) \| \rightarrow 0$$, as asserted, and the desired claim (26) follows.

By (26) and properties of $$G\varepsilon$$ as in Lemma 4.4 we derive

$$\| G\varepsilon * F\varepsilon - F0 \| \leq \| G\varepsilon * (F\varepsilon - F0) \| + \| G\varepsilon * F0 - F0 \| \leq \| F\varepsilon - F0 \| + \varepsilon^2 \| \Delta F0 \| \rightarrow 0$$

as $$\varepsilon \rightarrow 0$$. We define $$w\varepsilon$$ as the unique solution for

$$(-\Delta + 1) w\varepsilon = G\varepsilon * F\varepsilon + u\varepsilon.$$

Then in view of (25) we have

$$(-\Delta + 1)(u\varepsilon - w\varepsilon) \leq 0$$

and therefore by the maximum principle

$$u\varepsilon \leq w\varepsilon, \text{ on } M.$$

Since $$u0$$ satisfies (16), we have

$$-\Delta u0 = F0.$$

Consequently,

$$(-\Delta + 1)(w\varepsilon - u0) = G\varepsilon * F\varepsilon - F0 + u\varepsilon - u0,$$

and therefore (30), Lemma 2.2 and standard elliptic estimates yield

$$\| w\varepsilon - u0 \| \leq C \left( \| G\varepsilon * F\varepsilon - F0 \| + \| u\varepsilon - u0 \| \right) \rightarrow 0.$$

In particular, $$w\varepsilon$$ converges uniformly to $$u0$$. Taking into account that $$u\varepsilon \leq w\varepsilon$$ and $$u0 < \bar{u}$$ on $$M$$, we conclude that for all $$\varepsilon > 0$$ sufficiently small we have the desired strict inequality $$u\varepsilon < \bar{u}$$.

Now we can provide the

Proof of Proposition 2.1. Let $$\lambda0 > 0$$ as in Lemma 2.1 and for every fixed $$\lambda \geq \lambda0$$ let $$\varepsilon\lambda > 0$$ as in Lemma 2.4. Then by Lemma 2.4 the function $$u\varepsilon$$ defined by

$$I\varepsilon(u\varepsilon) = \min_{\mathcal{A}} I\varepsilon$$

satisfies the strict inequality $$u\varepsilon < \bar{u}$$ for every $$0 < \varepsilon < \varepsilon\lambda$$. In particular, by the Sobolev embedding $$\| u \| \leq C \| u \|$$ for all $$u \in H^2$$, for every $$\varepsilon > 0$$ sufficiently small there exists an $$H^2$$-neighborhood of $$u\varepsilon$$ entirely contained in $$\mathcal{A}$$. Therefore, for such values of $$\varepsilon$$, $$u\varepsilon$$ belongs to the interior of $$\mathcal{A}$$ in the sense of $$H^2$$. It follows that $$u\varepsilon$$ is a critical point for $$I\varepsilon$$ corresponding to a local minimum, as asserted. \(\square\)
3 The Palais-Smale condition

In this section we prove the Palais-Smale condition for $I_{\varepsilon}$ for every fixed $\varepsilon, \lambda > 0$.

**Proposition 3.1.** For every fixed $\varepsilon, \lambda > 0$ the functional $I_{\varepsilon}$ satisfies the Palais-Smale condition.

We denote by $(u_j), u_j \in H^2, j = 1, 2, 3, \ldots$ a Palais-Smale sequence for the functional $I_{\varepsilon}$. That is, $(u_j)$ satisfies:

(31) \quad $I_{\varepsilon}(u_j) \to \alpha \in \mathbb{R},$

(32) \quad $\|I'_{\varepsilon}(u_j)\|_{H^{-1}} \to 0$

as $j \to +\infty$. We have to show that $(u_j)$ admits a subsequence strongly convergent in $H^2$. By compactness, it suffices to show that $(u_j)$ is bounded in $H^2$. It will be useful to decompose $u_j$ in the following way

$$u_j = u_j' + c_j, \text{ where } \int u_j' = 0 \text{ and } c_j \in \mathbb{R}.$$ 

Then condition (31) is equivalent to

(33) \quad $I_{\varepsilon}(u_j) = \frac{\varepsilon^2}{2} \int (\Delta u_j)^2 + \frac{1}{2} \int \|\nabla u_j\|^2 + \varepsilon \lambda \int f'(e^{\sigma + u_j}) e^{\sigma + u_j} |\nabla (\sigma + u_j)|^2$

$$+ \frac{\lambda^2}{2} \int (f(e^{\sigma + u_j}) - s)^2 - Ac_j \to \alpha$$

and (32) implies (see (14))

(34) \quad $\diamondsuit(1) \|\Delta u_j\|_2 = \langle I'_{\varepsilon}(u_j), u_j' \rangle = \varepsilon^2 \int (\Delta u_j)^2 + \int \|\nabla u_j\|^2$

$$+ \varepsilon \lambda \int \left[ f''(e^{\sigma + u_j}) e^{\sigma + u_j} + f'(e^{\sigma + u_j}) \right] e^{\sigma + u_j} |\nabla (\sigma + u_j)|^2 u_j'$$

$$+ 2\varepsilon \lambda \int f'(e^{\sigma + u_j}) e^{\sigma + u_j} \nabla (\sigma + u_j) \cdot \nabla u_j$$

$$+ \lambda^2 \int f'(e^{\sigma + u_j}) e^{\sigma + u_j} (f(e^{\sigma + u_j}) - s) u_j'$$

It is readily checked that $c_j \geq -C$ for some $C > 0$. Indeed, by (33) we have

(35) \quad $-Ac_j \leq \frac{\varepsilon^2}{2} \int (\Delta u_j)^2 - Ac_j \leq I_{\varepsilon}(u_j) \leq C.$

Furthermore, if either $c_j \leq C$ or $\|\Delta u_j\|_2 \leq C$, then $u_j$ is bounded in $H^2$. Indeed, if $c_j \leq C$ then we readily obtain from (35) that $\|\Delta u_j\|_2 \leq C$. Suppose $\|\Delta u_j\|_2 \leq C$. Then by Sobolev embeddings we also have $\int |\nabla u_j|^2 \leq C$ and $\|u_j'\|_\infty \leq C$. We have

$$\int f'(e^{\sigma + u_j}) e^{\sigma + u_j} |\nabla (\sigma + u_j)|^2$$

$$= \int \nabla f(e^{\sigma + u_j}) \cdot \nabla (\sigma + u_j) = - \int f(e^{\sigma + u_j}) \Delta (\sigma + u_j)$$

$$= -A \int f(e^{\sigma + u_j}) + 4\pi mf_\infty - 4\pi nf(0) - \int f(e^{\sigma + u_j}) \Delta u_j$$

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and therefore by (3)

\[ \int f'(e^{\sigma+u_j})e^{\sigma+u_j} |\nabla (\sigma + u_j)|^2 \leq C(1 + \|\Delta u_j\|_2) \leq C. \]

On the other hand the term \( \int (f(e^{\sigma+u_j}) - s)^2 \) is bounded. Therefore we derive from (33) that

\[ \alpha + \circ_j(1) = I_\varepsilon(u_j) \leq -Ac_j + C \]

and consequently \( c_j \leq C \). In view of the above remarks, henceforth we assume that

(36) \[ \|\Delta u_j\|_2 \to +\infty \text{ and } c_j \to +\infty \text{ as } j \to +\infty. \]

By (35) and assumption (36) we then have

(37) \[ \|\Delta u_j\|_2 \leq Cc_j^{1/2}. \]

The following identity will be useful.

**Lemma 3.1.** For all \( u \in H^2 \) the following identity holds:

\[
\int \left[ f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u}) \right] |\nabla (\sigma + u)|^2 u
+ 2 \int f'(e^{\sigma+u})e^{\sigma+u} \nabla (\sigma + u) \cdot \nabla u
= \int f'(e^{\sigma+u})e^{\sigma+u} \nabla (\sigma + u) \cdot \nabla u - \int f'(e^{\sigma+u})e^{\sigma+u} \Delta (\sigma + u) u.
\]

**Proof.** Integrating by parts we have

(38) \[
\int \left[ f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u}) \right] |\nabla (\sigma + u)|^2 u
= \int \nabla \left[ f'(e^{\sigma+u})e^{\sigma+u} \right] \cdot \nabla (\sigma + u) u
= - \int f'(e^{\sigma+u})e^{\sigma+u} \Delta (\sigma + u) u - \int f'(e^{\sigma+u})e^{\sigma+u} \nabla (\sigma + u) \cdot \nabla u.
\]

The asserted identity follows. \( \square \)

Now we can provide the
Proof of Proposition 3.1. By (34) and Lemma 3.1 we have
\[o_j(1)\|\Delta u_j\|_2\]
\[\geq \varepsilon^2 \|\Delta u_j\|^2 + \varepsilon \lambda \int f''(e^{\sigma^+u_j})e^{\sigma^+u_j} + f'(e^{\sigma^+u_j})|\nabla (\sigma + u_j)|^2 u_j' + 2\varepsilon \lambda \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} - f'(e^{\sigma^+u_j}) \nabla (\sigma + u_j) \cdot \nabla u_j + \lambda^2 \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j' = \varepsilon^2 \|\Delta u_j\|^2 + \varepsilon \lambda \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \nabla (\sigma + u_j) \cdot \nabla u_j - \varepsilon \lambda \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j' + \lambda^2 \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j'.\]
and therefore, since \(\int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} |\nabla u_j|^2 \geq 0,\)
\[o_j(1)\|\Delta u_j\|_2 \geq \varepsilon^2 \|\Delta u_j\|^2 + \varepsilon \lambda \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \nabla \sigma \cdot \nabla u_j - \varepsilon \lambda \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j' + \lambda^2 \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j'.\]
By properties of \(f\) and the Sobolev embedding \(\|u_j'\|_\infty \leq C \|\Delta u_j\|_2\) we have
\[\int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} (f(e^{\sigma^+u_j}) - s) u_j' \leq C \|\Delta u_j\|_2.\]
By the Hölder inequality and Sobolev embeddings we have
\[\left| \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \nabla \sigma \cdot \nabla u_j \right| \leq C \|\nabla \sigma\|_p \|\nabla u_j\|_{p'} \leq C \|\Delta u_j\|_2,\]
for any \(1 \leq p < 2\). By (46) in the Appendix and Sobolev embeddings we have
\[\left| \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \Delta \sigma u_j \right| = A \left| \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} u_j' \right| \leq C \|\Delta u_j\|_2.\]
Finally, we claim that there exists \(\bar{j}\) such that for all \(j \geq \bar{j}\)
\[\left| \int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \Delta u_j u_j \right| \leq C_2 \|\Delta u_j\|^2_2.\]
To prove (43), we write for \(\rho > 0\)
\[\int f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \Delta u_j u_j = \int_{B_\rho(q_j)} f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \Delta u_j u_j' + \int_{M \setminus \bigcup_{\rho=1}^{\bar{j}} B_\rho(q_j)} f'(e^{\sigma^+u_j})e^{\sigma^+u_j} \Delta u_j u_j'.\]
Therefore, by properties of \(f\), the Cauchy-Schwarz inequality and Sobolev embeddings, we estimate

\[
\left| \int_{\bigcup_{h=1}^{n} B_{\rho}(q_{j})} f'(e^{\sigma+u_{j}}) e^{\sigma+u_{j}} \Delta u_{j} u'_{j} \right| \\
\leq C \left( \int_{\bigcup_{h=1}^{n} B_{\rho}(q_{j})} (\Delta u_{j})^2 \right)^{1/2} \left( \int_{\bigcup_{h=1}^{n} B_{\rho}(q_{j})} u'^{2} \right)^{1/2} \\
\leq C \| \Delta u_{j} \|_{2} \| u'_{j} \|_{\infty} \left( \sum_{h=1}^{n} |B_{\rho}(q_{j})| \right) \leq C \rho \| \Delta u_{j} \|_{2}^{2}.
\]

Therefore, we may choose \(\rho > 0\) such that

\[
\left( \int_{\bigcup_{h=1}^{n} B_{\rho}(q_{j})} f'(e^{\sigma+u_{j}}) e^{\sigma+u_{j}} \Delta u_{j} u'_{j} \right) \leq \frac{\varepsilon^{2}}{4} \| \Delta u_{j} \|_{2}^{2}.
\]

We define

\[
e_{0} = \min_{M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})} e^{\sigma} > 0.
\]

By (36), (37) and the embedding \(\| u'_{j} \|_{\infty} \leq C \| \Delta u_{j} \|_{2}\) we have

\[
\min_{M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})} e^{\sigma+u_{j}} \geq \min_{M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})} e^{\sigma-u_{j}'} \geq e_{0} e^{-C \sqrt{\sigma+c_{j}}} > e_{0} e^{c_{j}/2} \rightarrow +\infty \text{ as } j \rightarrow +\infty.
\]

Therefore, by properties of \(f\) and since \(c_{j} \rightarrow \infty\), for every \(\mu > 0\) there exists \(j_{\mu} \in \mathbb{N}\) such that if \(j \geq j_{\mu}\) then \(f'(e^{\sigma+u_{j}}) e^{\sigma+u_{j}} \leq \mu\) on \(M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})\). We conclude that for \(j \geq j_{\mu}\) we have

\[
\left| \int_{M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})} f'(e^{\sigma+u_{j}}) e^{\sigma+u_{j}} \Delta u_{j} u'_{j} \right| \leq C \mu \| \Delta u_{j} \|_{2}^{2}.
\]

We choose \(\mu > 0\) such that

\[
\left| \int_{M \setminus \bigcup_{h=1}^{n} B_{\rho}(q_{j})} f'(e^{\sigma+u_{j}}) e^{\sigma+u_{j}} \Delta u_{j} u'_{j} \right| \leq \frac{\varepsilon^{2}}{4} \| \Delta u_{j} \|_{2}^{2}.
\]

Now (44) and (45) yield (43) with \(j = j_{\mu}\).

Now we can conclude the proof of Proposition 3.1. Indeed, inserting the estimates (40)–(41)–(42)–(43) into (39) we obtain

\[
\varepsilon^{2} \| \Delta u_{j} \|_{2}^{2} \leq C \| \Delta u_{j} \|_{2} + \frac{\varepsilon^{2}}{2} \| \Delta u_{j} \|_{2}^{2}
\]

and consequently we derive that \(\| \Delta u_{j} \|_{2} \leq C\). This is a contradiction since we have assumed (36).

Now we can finally prove our main result:
Proof of Theorem 0.1. By Proposition 2.1, there exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ fixed, there exists $\varepsilon_\lambda > 0$ such that for every $0 < \varepsilon \leq \varepsilon_\lambda$ the functional $I_\varepsilon$ admits a critical point corresponding to a local minimum. By Proposition 3.1, $I_\varepsilon$ satisfies the Palais-Smale condition. If $u_\varepsilon$ is not a strict local minimum, it is known that $I_\varepsilon$ has a continuum of critical points (see, e.g., [21]). In particular, $I_\varepsilon$ has at least two critical points. If $u_\varepsilon$ is a strict local minimum, we note that on constant functions $c \to +\infty$ we have $L_\varepsilon(c) \to -\infty$. Therefore $I_\varepsilon$ admits a mountain pass structure in the sense of Ambrosetti and Rabinowitz [1]. Hence by the mountain pass theorem [1] we obtain the existence of a second critical point for $I_\varepsilon$. In either case, we conclude that the fourth order equation (9) admits at least two solutions. By the equivalences as stated in Lemma 1.1 and in Lemma 1.2, system (6)–(7) admits at least two solutions, as asserted.

4 Appendix

We collect in this Appendix the proofs of some simple properties which have been repeatedly used throughout this note. Recall that $\sigma$ is defined as the unique distributional solution for $-\Delta \sigma = 4\pi \sum_{j=1}^m \delta_{p_j} - 4\pi \sum_{k=1}^m \delta_{q_k}$, $f \sigma = 0$. Therefore, there exist smooth functions $\gamma_j$, $\theta_k$ and $\rho > 0$ such that $\sigma(x) = \gamma_j(x) + \log |x - p_j|^2$ in $B_\rho(p_j)$ and $\sigma(x) = \theta_k(x) + \log |x - q_k|^2$ in $B_\rho(q_k)$.

Lemma 4.1. Suppose $\phi : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying

$$|\phi(t)| \leq C_\phi \min\{t, t^{-1}\}$$

for some $C_\phi > 0$. Then there exists $\overline{C_\phi} > 0$ depending on $\phi$ only such that for all measurable functions $u$ we have

$$\phi(e^{\sigma+u})|\nabla \sigma|^2 \leq \overline{C_\phi}(1 + e^u + e^{-u}).$$

Proof. In $M \setminus \bigcup_{j=1}^m B_\rho(p_j) \setminus \bigcup_{k=1}^m B_\rho(q_k)$ we have

$$\phi(e^{\sigma+u})|\nabla \sigma|^2 \leq C_\phi \sup_{M \setminus \bigcup_{j=1}^m B_\rho(p_j) \setminus \bigcup_{k=1}^m B_\rho(q_k)} |\nabla \sigma|^2.$$

In $B_\rho(p_j)$ we have

$$\phi(e^{\sigma+u})|\nabla \sigma|^2 \leq C_\phi e^{-(\sigma+u)}|\nabla \sigma|^2 \leq C_\phi \sup_{B_\rho(p_j)} (e^{-\sigma}|\nabla \sigma|^2) e^{-u}.$$

In $B_\rho(q_k)$ we have

$$\phi(e^{\sigma+u})|\nabla \sigma|^2 \leq C_\phi e^{\sigma+u}|\nabla \sigma|^2 \leq C_\phi \sup_{B_\rho(q_k)} (e^{\sigma}|\nabla \sigma|^2) e^u.$$

Now the asserted estimate follows.

Lemma 4.2. If $u \in C^1$ and $\phi : [0, +\infty) \to \mathbb{R}$ is a smooth function satisfying

$$|\phi'(t)| \leq C_\phi \min\{t^{-1/2}, t^{-3/2}\}$$

for some $C_\phi > 0$, then $\phi(e^{\sigma+u})$ is Lipschitz continuous on $M$ (with Lipschitz constant depending on $u$).
Proof. We need only check the claim near the vortex points \( p_j, q_k \). By the mean value theorem we have, for \( x, y \in B_\rho(p_j), x, y \neq p_j \):

\[
\phi(e^{(\sigma+u)(x)}) - \phi(e^{(\sigma+u)(y)}) = \phi'(e^{(\sigma+u)(y+\theta(x-y))})e^{(\sigma+u)(y+\theta(x-y))}(y+\theta(x-y)) \cdot (x-y)
\]

for some \( 0 \leq \theta \leq 1 \). By properties of \( \sigma \) and \( \phi \),

\[
|\phi(e^{(\sigma+u)(x)}) - \phi(e^{(\sigma+u)(y)})| \\
\leq C_\phi|e^{-(\sigma+u)(y+\theta(x-y))}|^{1/2}|\nabla(\sigma + u)(y + \theta(x-y))| |x-y| \\
\leq C_\sigma e^{\|u\|_\infty/2}|y + \theta(x-y) - p_j| (1 + |y + \theta(x-y) - p_j|^{-1} + \|\nabla u\|_\infty) |x-y| \\
\leq C_\sigma e^{\|u\|_\infty/2}(1 + \|\nabla u\|_\infty) |x-y|.
\]

A similar argument yields Lipschitz continuity near the \( q_k \)'s, and the statement follows.

Lemma 4.3. For any \( u \in H^2 \) the following identities hold, in the sense of distributions:

(46) \[
f' (e^{\sigma+u}) e^{\sigma+u} \Delta \sigma = Af' (e^{\sigma+u}) e^{\sigma+u}
\]

and

(47) \[
\Delta f (e^{\sigma+u}) = \{ f'' (e^{\sigma+u}) e^{\sigma+u} + f' (e^{\sigma+u}) \} e^{\sigma+u} |\nabla(\sigma + u)|^2 + f' (e^{\sigma+u}) e^{\sigma+u} \Delta(\sigma + u).
\]

Proof. By (3) the function \( f' (e^{\sigma+u}) e^{\sigma+u} \) may be extended by continuity to the whole of \( M \) by setting it equal to 0 at \( p_j, q_k \). In view of the definition of \( \sigma \) we obtain (46).

Since \( u \in H^2 \), (47) holds pointwise almost everywhere on \( M \). By Lemma 4.1, the right hand side of (47) belongs to \( L^1 \), and it is absolutely continuous in the \( x \)-variable, for almost every fixed \( y \). Therefore, (47) holds in the sense of distributions.

Finally, we prove some properties for the Green function \( G_\varepsilon \) for the operator \(-\varepsilon^2 \Delta + 1 \) on \( M \).

Lemma 4.4. Let \( G_\varepsilon = G_\varepsilon(x,y) \) be the Green function defined by

\[
(-\varepsilon^2 \Delta + 1)G_\varepsilon = \delta_y \text{ on } M.
\]

Then

i) \( G_\varepsilon > 0 \) on \( M \times M \) and for every fixed \( y \in M \) we have \( G_\varepsilon \to \delta_y \) as \( \varepsilon \to 0 \), weakly in the sense of measures;

ii) \( \|G_\varepsilon * h\|_q \leq \|h\|_q \) for all \( 1 \leq q \leq +\infty \);

iii) If \( \Delta h \in L^q \) for some \( q \geq 1 \) then \( \|G_\varepsilon * h - h\|_q \leq \varepsilon^2 \|\Delta h\|_q \).
Proof. Proof of (i). Note that since $-\varepsilon^2\Delta + 1$ is coercive, $G_\varepsilon$ is well defined (e.g., by Stampacchia’s duality argument [20]). By the maximum principle, $G_\varepsilon > 0$ on $M \times M$. Integrating over $M$ with respect to $x$, we have $\int G_\varepsilon(x, y) \, dx = \int |G_\varepsilon(x, y)| \, dx = 1$ and therefore there exists a Radon measure $\mu$ such that $G_\varepsilon(\cdot, y) \to \mu$ as $\varepsilon \to 0$, weakly in the sense of measures. For $\varphi \in C^\infty$ we compute:

$$\varphi(y) = \varepsilon^2 \int G_\varepsilon(x, y)(-\Delta \varphi)(x) \, dx + \int G(x, y)\varphi(x) \, dx \to \int \varphi \, d\mu$$

as $\varepsilon \to 0$. By density of $C^\infty$ in $C$, we conclude that $\mu = \delta_y$. Proof of (ii). For $q = 1$, we have:

$$\|G_\varepsilon * h\|_1 = \int |(G_\varepsilon * h)(x)| \, dx \leq \int |h(y)| \int G_\varepsilon(x, y) \, dx = \|h\|_1.$$

For $q = \infty$ we have, for any $x \in M$:

$$|G_\varepsilon * h(x)| \leq \|h\|_\infty \int G_\varepsilon(x, y) \, dy = \|h\|_\infty \int G_\varepsilon(x, y) \, dx = \|h\|_\infty,$$

and therefore $\|G_\varepsilon * h\|_\infty \leq \|h\|_\infty$. The general case follows by interpolation.

Proof of (iii). Suppose $1 < q < +\infty$. Let $U_\varepsilon = G_\varepsilon * h$. Then we can write

$$-\varepsilon^2 \Delta(U_\varepsilon - h) + (U_\varepsilon - h) = \varepsilon^2 \Delta h.$$

Multiplying by $|U_\varepsilon - h|^{q-2}(U_\varepsilon - h)$ and integrating, we obtain

$$\varepsilon^2 (q-1) \int |U_\varepsilon - h|^{q-2}|\nabla(U_\varepsilon - h)|^2 + \int |U_\varepsilon - h|^q$$

$$= \varepsilon^2 \int \Delta h |U_\varepsilon - h|^{q-2}(U_\varepsilon - h).$$

By positivity of the first term above and Hölder’s inequality,

$$\int |U_\varepsilon - h|^q \leq \varepsilon^2 \int |\Delta h||U_\varepsilon - h|^{q-1} \leq \varepsilon^2 \|\Delta h\|_q \|U_\varepsilon - h\|^{q-1}.q^{-1}.$$

Hence $\|U_\varepsilon - h\|_q \leq \varepsilon^2 \|\Delta h\|_q$ and (iii) follows recalling the definition of $U_\varepsilon$ in the case $1 < q < +\infty$. Taking limits for $q \to 1$ and $q \to +\infty$, we obtain the general case.

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