Taming the heavy-tailed features by shrinkage and clipping

Ziwei Zhu
Department of Operations Research and Financial Engineering
Princeton University

Abstract

In this paper, we consider the generalized linear models (GLM) with heavy-tailed features and corruptions. Besides clipping the response, we propose to shrink the feature vector by its $\ell_4$-norm under the low dimensional regime and clip each entry of the feature vector in the high-dimensional regime. Under bounded fourth moment assumptions, we show that the MLE based on shrunk or clipped data enjoys nearly the minimax optimal rate with exponential deviation bound. Simulations demonstrate significant improvement in statistical performance by feature shrinkage and clipping in linear regression with heavy-tailed noise and logistic regression with noisy labels. We also apply shrinkage to deep features of MNIST images and find that classifiers trained by shrunk deep features are fairly robust to noisy labels: it achieves 0.9% testing error in the presence of 40% mislabeled data.

1 Introduction

Heavy-tailed data abound in modern data analytics. For instance, financial log-returns and macroeconomic variables usually exhibit heavy tails (Cont (2001)). In genomic study, microarray data are always wildly fluctuated (Liu et al. (2003), Purdom et al. (2005)). In deep learning, features learned by deep neural nets are generated via highly nonlinear transformation of the original data and thus have no guarantee of exponential-tailed distribution. These real-world cases contradict the common sub-Gaussian or sub-exponential conditions in statistics literature. A series of questions thus arise: with heavy-tailed data, can we still achieve good statistical properties of the previous standard estimators or testing statistics? If not, is there a solution to overcome heavy-tailed corruptions and achieve equally well statistical performance as with exponential-tailed data?

To answer these questions, perhaps the easiest statistical problem to start with is the mean estimation problem. It turns out surprisingly, as first pointed out by Catoni (2012), that the empirical mean is far from optimal when data has only a few finite moments. In Catoni (2012), the author instead proposed a novel M-estimator for the population mean
and revealed its sub-Gaussian concentration behavior around the true mean under merely bounded second moment assumptions. The correspondent score function is constructed to grow in a logarithmic fashion when deviation is large, thereby being insensitive to outliers and delivering a robust M-estimator. Later on [Devroye et al. (2016), Hsu and Sabato (2016) and Minsker (2017+)] established a similar sub-Gaussian concentration property for the median-of-means estimator ([Nemirovsky et al. (1982)]). Particularly, Hsu and Sabato (2016) and Minsker (2017+) consider the median-of-means approach under general metric spaces.

Beyond the mean estimation problem, robust loss minimization and the median-of-means approach are proved to be successful under a great variety of problem setups with heavy-tailed data, e.g., covariance matrix or general matrix estimation ([Minsker (2017+); Fan et al. (2017)]), empirical loss minimization ([Brownlees et al. (2015); Hsu and Sabato (2016)]), low-dimensional regression and high-dimensional sparse linear regression ([Fan et al. (2017); Sun et al. (2017)]), low-rank matrix recovery ([Fan et al. (2017)]) and so forth.

Despite heated research on statistics with heavy-tailed data, very few has studied the effect of heavy-tailed features or designs in general regression problems. It still remains unclear whether widely spread features or designs are blessings or curses to statistical efficiency. This motivates us to consider both heavy-tailed designs and responses under a new variant of the generalized linear models (GLM) called corrupted GLM (CGLM), which imposes extra random corruptions on the response from the traditional GLM. The introduction of this additional corruption is to address the limited expression capacity of the traditional GLM, which can only generate exponential-tailed responses. The CGLM enjoys much broader model capacity and embraces a myriad of important real-world problems, e.g., linear regression with heavy-tailed noise, classification with mislabeled data, etc.

One main message of our paper is that heavy-tailed features will drastically aggravate the induced corruptions in the CGLM and downgrade the statistical efficiency of the standard approach. To shed light on this point, in Figure 1 we contrast the performance of the standard MLE under the logistic regression model when the features are light-tailed and

(a) Gaussian Features  (b) Student’s $t_2$ Features (c) Shrunken Student’s $t_2$ Features

Figure 1: Logistic Regression with 10% mislabeled data based on different features
heavy-tailed. As we can observe, when data points are wildly spread as illustrated in Panel (b), the boundary derived from MLE deviates far from the true boundary. When we have Gaussian features, however, Panel (a) shows nearly perfect alignment between the MLE boundary and the true boundary. The reason is that outliers, especially those mislabeled, have huge influence on the log-likelihood and can thus easily drive the MLE boundary away from the truth.

To tame the wild behavior of heavy-tailed features, we propose to appropriately shrink or clip the features before calculating the M-estimator. Given feature vectors \( \{ \mathbf{x}_i \in \mathbb{R}^d \}_{i=1}^n \), a threshold value \( \tau \) and certain norm \( \| \cdot \| \), the shrunk features \( \{ \tilde{x}_i^s \}_{i=1}^n \) are defined as:

\[
\tilde{x}_i^s = \min(\|\mathbf{x}_i\|, \tau) \cdot \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|}.
\]

In other words, we restrict \( \|\tilde{x}_i^s\| \) below the level \( \tau \). The clipped features \( \{ \tilde{x}_i^c \}_{i=1}^n \) are defined as

\[
\tilde{x}_{ij}^c = \min(|x_{ij}|, \tau) \cdot \frac{x_{ij}}{|x_{ij}|}, \text{ for any } 1 \leq j \leq d.
\]

For ease of notations, we will drop the superscript "s" or "c" unless it is not clear whether shrinkage and clipping is applied. We will illustrate both theoretically and numerically that this data shrinkage and clipping trade little bias for dramatic variance reduction such that the correspondent MLE achieves (nearly) the minimax optimal statistical rate. Panels (b) and (c) of Figure 1 compare the performance of MLE under Logistic regression based on original heavy-tailed features and shrunk features. As we can clearly see, after feature shrinkage, the new MLE boundary becomes much more aligned with the true boundary than the original one. The advantage of feature shrinkage is tremendous here, since it drags back the mislabeled outliers and significantly mitigates their perturbation on the log-likelihood. Note that similar ideas have been explored to overcome adversarial corruptions on features. For example, [Chen et al. (2013)] uses the trimmed inner product to robustify standard high-dimensional regression toolkits and establishes strong statistical guarantees while allowing certain fraction of samples to be arbitrarily corrupted. [Feng et al. (2014)] proposes to throw out samples with large magnitude to avoid adversarial feature corruptions in logistic regression and binary classification problems. The major difference between our work and theirs is that (1) we do not assume any corruption on the features; all the corruptions here are imposed on responses; (2) our features are heavy-tailed per se, while in [Chen et al. (2013)] and [Feng et al. (2014)] the genuine designs are sub-Gaussian. As clearly seen, our focus is heavy-tailedness, rather than corruptions, of features in regression problems.

We organize our paper as follows. In Section 2, we introduce in detail the model we work with, i.e., the corrupted GLM. In Section 3, we elucidate the specific feature shrinkage and clipping methods and present our main theoretical results. Under the low-dimensional regime, we prove that the MLE based on \( \ell_4 \)-norm shrunk features and clipped responses enjoys the same optimal statistical error rate as the standard MLE with sub-Gaussian features and response up to a \( \sqrt{\log n} \) factor. For high-dimensional models, we show that the \( \ell_1 \)-regularized MLE based on appropriately clipped feature and responses achieve exactly the minimax optimal rate. One technical contribution that is worth emphasis is that we provide rigorous
justification of the (restricted) strong convexity of the negative likelihood based on shrunk or clipped features. In Section 4, we illustrate the numerical superiority of our proposed estimators over the standard MLE under both low-dimensional and high-dimensional regimes. We investigate two important problem setups: the linear regression with heavy-tailed noise and binary logistic regression with mislabeled data. Besides, we implement our methods on the MNIST dataset and discover that shrinkage on deep features improves the prediction power of logistic classifiers when noisy labels occur.

2 Problem setup

In this paper we consider the corrupted generalized linear model (CGLM). Recall the definition of the standard GLM with the canonical link. Suppose we have $n$ samples $\{(y_i, x_i)\}_{i=1}^n$, where $y_i$ is the response and $x_i$ is the feature vector. Under the GLM with the canonical link, the response follows the distribution

$$f_n(y; X, \beta^*) = \prod_{i=1}^n f(y_i; \eta_i^*) = \prod_{i=1}^n \left\{ c(y_i) \exp \left[ \frac{y_i \eta_i^* - b(\eta_i^*)}{\phi} \right] \right\}, \quad (2.1)$$

where $y = (y_1, \ldots, y_n)^T$, $X = (x_1, \ldots, x_n)^T$, $\beta^* \in \mathbb{R}^d$ is a regression vector, $\eta_i^* = x_i^T \beta^*$ and $\phi > 0$ is the dispersion parameter. The negative log-likelihood corresponding to (2.1) is given, up to an affine transformation, by

$$\ell_n(\beta) = \frac{1}{n} \sum_{i=1}^n -y_i x_i^T \beta + b(x_i^T \beta) = \frac{1}{n} \sum_{i=1}^n -y_i \eta_i + b(\eta_i) = \frac{1}{n} \sum_{i=1}^n \ell_i(\beta), \quad (2.2)$$

and the gradient and Hessian of $\ell_n(\beta)$ are respectively

$$\nabla \ell_n(\beta) = -\frac{1}{n} \sum_{i=1}^n (y_i - b'(x_i^T \beta^*)) x_i \quad (2.3)$$

and

$$\nabla^2 \ell_n(\beta) = \frac{1}{n} \sum_{i=1}^n b''(x_i^T \beta^*) x_i x_i^T, \quad (2.4)$$

For ease of notations, we write the empirical hessian $\nabla^2 \ell_n(\beta)$ as $H_n(\beta)$ and $\mathbb{E}(b''(x_i^T \beta) x_i x_i^T)$ as $H(\beta)$.

Now we consider further an extra random corruption on the response $y_i$. Suppose we can only observe corrupted responses $z_i = y_i + \epsilon_i$, where $\epsilon_i$ is a random noise. We emphasize here that the introduction of $\epsilon_i$ significantly improves the flexibility of the original GLM. The new model now embraces many more real-world problems with complex structures, e.g., the linear regression model with heavy-tailed noise, the logistic regression with mislabeled samples and so forth.
To handle the heavy-tailed features and corruptions, we propose to shrink or clip the data \( \{(z_i, x_i)\}_{i=1}^n \) first and feed them to the log-likelihood (2.2) to calculate MLE. More rigorously, our negative log-likelihood is evaluated on the shrunk data \( \{\tilde{z}_i, \tilde{x}_i\}_{i=1}^n \) as follows.

\[
\tilde{\ell}_n(\beta) := \frac{1}{n} \sum_{i=1}^{n} -\tilde{z}_i \tilde{x}_i^T \beta + b(\tilde{x}_i^T \beta).
\] (2.5)

We denote the hessian matrix of \( \tilde{\ell}_n(\beta) \) by \( \tilde{H}_n(\beta) \) and its population version \( \mathbb{E} \tilde{H}_n(\beta) \) by \( \tilde{H}(\beta) \).

In the next section, we will elucidate the specific shrinkage and clipping methods in both the low-dimensional and high-dimensional regimes and explicitly derive the statistical error rates of the MLE based on \( \tilde{\ell}_n(\beta) \).

3 Main results

3.1 Notation

We collect all the notations here for convenience of presentation. We use regular letters for scalars, bold regular letters for vectors and bold capital letters for matrices. Denote the \( d \)-dimensional Euclidean unit sphere by \( S^{d-1} \). Denote the Euclidean and \( \ell_1 \)-norm ball with the center \( \beta^* \) and radius \( r \) by \( B_2(\beta^*, r) \) and \( B_1(\beta^*, r) \) respectively. We write the set \( \{1, \cdots, d\} \) as \([d]\). For two scalar sequences \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \), we say \( a_n \asymp b_n \) if there exist two universal constants \( C_1 \) and \( C_2 \) such that \( C_1 b_n \leq a_n \leq C_2 b_n \) for all \( n \geq 1 \). We use \( \|v\|_2 \), \( \|v\|_1 \) and \( \|v\|_4 \) to denote the Euclidean norm, \( \ell_1 \) norm and \( \ell_4 \)-norm of \( v \) respectively. Particularly, recall that \( \|x_i\|_4 := \left( \sum_{j=1}^d x_{ij}^4 \right)^{\frac{1}{4}} \). We use \( \|A\|_{op} \) and \( \|A\|_{\max} \) to denote the operator norm and elementwise max-norm of \( A \) respectively and use \( \lambda_{\min}(A) \) to denote the minimum eigenvalue of \( A \). For any \( \beta^* \in \mathbb{R}^d \) and any differential map \( f : \mathbb{R}^d \to \mathbb{R} \), define the first-order Taylor remainder of \( f(\beta) \) at \( \beta = \beta^* \) to be

\[
\delta f(\beta; \beta^*) := f(\beta) - f(\beta^*) - \nabla f(\beta^*)^T (\beta - \beta^*).
\]

From now on, we refer to some quantities as constants if they are independent of the sample size \( n \), the dimension \( d \), and the sparsity \( s \) of \( \beta^* \) in the high-dimensional regime.

3.2 Low-dimensional regime

The standard MLE estimator is defined as \( \hat{\beta} := \arg\min_{\beta \in \mathbb{R}^d} \ell_n(\beta) \), where \( \ell_n(\cdot) \) is characterized as in (2.2). It is widely known that under genuine GLM with bounded features, \( \hat{\beta} \) enjoys \( \sqrt{d/n} \)-consistency to the true parameter \( \beta^* \) in terms of the Euclidean norm. However, when the feature vectors have only bounded moments, there is no guarantee of \( \sqrt{d/n} \)-consistency any more, let alone further perturbation \( \epsilon_i \) on the response. To reduce the disruption due to
heavy-tailed data, we apply the following $\ell_4$-norm shrinkage to feature vectors:

$$\tilde{x}_i := \min\left(\frac{\|x_i\|_4}{\|x_i\|_4}, \tau_1\right) \cdot x_i$$

and also clipping to the response:

$$\tilde{z}_i := \min(|z_i|, \tau_2)$$

before MLE calculation, where $\tau_1$ and $\tau_2$ are predetermined thresholds. Clipping on the response is natural; when $|z_i|$ is abnormally large, clipping reduces its magnitude to prevent potential wild corruptions by $\epsilon_i$. Here we explain more on why we shrink features in terms of the $\ell_4$-norm rather than other norms. The $\ell_4$-norm shrinkage has been proven to be successful in low-dimensional covariance estimation in Fan et al. (2017). Theorem 6 therein shows that when data have only bounded fourth moment, the $\ell_4$-norm shrinkage sample covariance enjoys operator-norm convergence rate of order $O_p\left(\sqrt{d \log d/n}\right)$ to the population covariance matrix. This inspires us to apply similar $\ell_4$-norm shrinkage to heavy-tailed features to ensure that the empirical hessian $\tilde{H}_n(\beta)$ is close to its population version $H(\beta)$ and thus well-behaved. We shall rigorize this argument in Lemma 1 later.

After data shrinkage and clipping, we minimize the negative log-likelihood based on the new data $\{\tilde{z}_i, \tilde{x}_i\}_{i=1}^n$ with respect to $\beta$ to derive the M-estimator. Specifically, define

$$\tilde{\ell}_n(\beta) := \frac{1}{n} \sum_{i=1}^n -\tilde{z}_i \tilde{x}_i^T \beta + b(\tilde{x}_i^T \beta)$$

and we choose $\tilde{\beta} := \arg\min_{\beta \in \mathbb{R}^d} \tilde{\ell}_n(\beta)$ to estimate $\beta^*$. In the sequel, we will show that $\|\tilde{\beta} - \beta^*\|_2 = O_p\left(\sqrt{d \log n/n}\right)$ with exponential exception probability. One crucial step in this statistical error analysis is to establish a uniform strong convexity of $\tilde{\ell}(\beta)$ over $\beta \in B_2(\beta^*, r)$ up to some small tolerance term, where $r > 0$ is small. The following lemma rigorously justifies this point.

**Lemma 1.** Suppose the following conditions hold: (1) $b''(x_i^T \beta^*) \leq M < \infty$ always holds and $\forall \omega > 0$, $\exists m(\omega) > 0$ such that $b'(\eta) \geq m(\omega) > 0$ for $|\eta| \leq \omega$; (2) $\mathbb{E}x_i = 0$, $\lambda_{\min}(\mathbb{E}x_i x_i^T) \geq \kappa_0 > 0$ and $\mathbb{E}(\mathbf{v}^T x_i)^4 \leq R < \infty$ for all $\mathbf{v} \in S^{d-1}$; (3) $\|\beta^*\|_2 \leq L < \infty$. Choose the shrinkage threshold $\tau_1 \asymp (n/\log n)^{\frac{1}{4}}$. For any $0 < r < 1$ and $t > 0$, as long as $\sqrt{d \log n/n}$ is sufficiently small, it holds with probability at least $1 - 2\exp(-t)$ that for all $\Delta \in \mathbb{R}^d$ such that $\|\Delta\|_2 \leq r$,

$$\delta \tilde{\ell}_n(\beta^* + \Delta; \beta^*) \geq \kappa \|\Delta\|_2^2 - C\tau^2 \left(\sqrt{\frac{t}{n}} + \sqrt{\frac{d}{n}}\right),$$

where $\kappa$ and $C$ are some constants.
Remark 1. Here we explain the conditions for Lemma 1. Condition (1) assumes that the response from the GLM has bounded variance and is non-degenerate when $\eta$ is bounded. Note here that we do not assume a uniform lower bound of $b''(\eta)$. $m(\omega)$ is allowed to decay to zero as $\omega \to \infty$. Condition (2) says that the population covariance matrix of the design vector $x_i$ is positive definite and $x_i$ has bounded fourth moment. Condition (3) is natural; it holds if we have $\text{var}(\omega^T \beta^*) < \infty$ and $\lambda_{\min}(\mathbb{E} x_i x_i^T) \geq \kappa_0 > 0$. Note that the ordinary least square (OLS) estimator has been shown to enjoy consistency under similar bounded fourth moment conditions (Hsu et al. (2012), Audibert et al. (2011), Oliveira (2016)). Theorem 1 later establishes similar results for CGLM that embraces a much broader family of loss functions.

Remark 2. When deriving the statistical rate of $\hat{\beta}$ in Theorem 1 later, we will let the radius of the local neighborhood $r$ decay to zero so that the tolerance term $r^2(\sqrt{d/n} + \sqrt{d/n})$ is negligible.

Given Lemma 1, we are now ready to derive the statistical error rate $\|\hat{\beta} - \beta^*\|_2$.

Theorem 1. Suppose the conditions of Lemma 1 hold. We further assume that (1) $\mathbb{E} z_i^4 \leq M_1 < \infty$; (2) $\|\mathbb{E} [\epsilon_i x_i]\|_2 \leq M_2 \sqrt{d/n}$ for some constant $M_2$. Choose $\tau_1, \tau_2 \propto (n/\log n)^{\frac{1}{2}}$. There exists a constant $C > 0$ such that for any $\xi > 1$,

$$
\mathbb{P}(\|\tilde{\beta} - \beta^*\|_2 \geq C\xi \sqrt{\frac{d\log n}{n}}) \leq 3n^{1-\xi}.
$$

Remark 3. Condition 1 requires merely bounded fourth moments of the response from CGLM. Condition 2 requires the additional corruption to be nearly uncorrelated with the design, which is trivially satisfied if $\mathbb{E} (\epsilon_i | x_i) = 0$.

Sometimes the covariance between $\epsilon_i$ and $x_i$ does not vanish as $n$ and $d$ grow. For example, in binary logistic regression with a random corruption $\epsilon_i$, suppose

$$
\mathbb{P}(\epsilon_i = -1 | y_i = 1) = p, \mathbb{P}(\epsilon_i = 0 | y_i = 1) = 1 - p,
\mathbb{P}(\epsilon_i = 1 | y_i = 0) = p, \mathbb{P}(\epsilon_i = 0 | y_i = 0) = 1 - p.
$$

(3.1)

where $p < 0.5$. In other words, we flip the genuine label $y_i$ with probability $p$. Then we have

$$
\mathbb{E} \epsilon_i x_i = \mathbb{E}(\epsilon_i x_i \cdot 1_{\{y_i=0\}}) + \mathbb{E}(\epsilon_i x_i \cdot 1_{\{y_i=1\}}) = p \mathbb{E}(x_i (1_{\{y_i=0\}} - 1_{\{y_i=1\}})) = 2p \mathbb{E}(x_i \cdot 1_{\{y_i=0\}}).
$$

The last equality holds because we assume $\mathbb{E} x_i = 0$. Therefore, $\mathbb{E} \epsilon_i x_i \propto p$ and if $p$ does not decay, neither will $\mathbb{E} \epsilon_i x_i$. Natarajan et al. (2013) solves this noisy label problem through minimizing weighted negative log-likelihood

$$
\tilde{\beta}^w := \arg\min_{\beta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \ell''(x_i, z_i; \beta) = \arg\min_{\beta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \frac{(1 - p) \ell(x_i, z_i; \beta) - p \cdot \ell(x_i, 1 - z_i; \beta)}{1 - 2p}.
$$

(3.2)
Lemma 1 therein shows that \( \mathbb{E}_i \ell^w(x_i, z_i) = \ell(x_i, y_i) \). This implies that when the sample size is sufficiently large, minimizing the weighted negative log-likelihood above is similar to minimizing the negative log-likelihood with true labels. In the presence of heavy-tailed features, however, we propose to replace \( x_i \) with the \( \ell_4 \)-norm shrunk feature \( \tilde{x}_i \), i.e., we recruit

\[
\tilde{\beta}^w := \arg\min_{\beta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \ell^w(\tilde{x}_i, z_i; \beta) = \frac{1}{n} \sum_{i=1}^{n} \frac{(1 - p)\ell(\tilde{x}_i, z_i; \beta) - p \cdot \ell(\tilde{x}_i, 1 - z_i; \beta)}{1 - 2p}.
\]

(3.3)

to estimate the regression vector \( \beta^* \). The following corollary establishes the statistical error rate of \( \tilde{\beta}^w \) with exponential deviation bound.

**Corollary 1.** Under the logistic regression with a random corruption \( \epsilon_i \) satisfying (3.1), choose \( \tau_1 \approx (n / \log n)^{1/4} \). Under the conditions of Lemma 1, it holds for some constant \( C \) and any \( \xi > 1 \) such that as long as \( \sqrt{d \log d/n} \) is sufficiently small,

\[
P(\|\tilde{\beta}^w - \beta^*\|_2 \geq C\xi \sqrt{d \log n / n}) \leq 2n^{1-\xi}.
\]

**Remark 4.** Here we do not need to truncate the response by \( \tau_2 \), because in logistic regression the response is always bounded.

### 3.3 High-dimensional regime

In this section, we consider the regime where the dimension \( d \) grows much faster than the sample size \( n \). Recall that the standard \( \ell_1 \)-regularized MLE of the regression vector \( \beta^* \) under the GLM is

\[
\hat{\beta} := \arg\min_{\beta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} (-y_i x_i^T \beta + b(x_i^T \beta)) + \lambda \|\beta\|_1,
\]

(3.4)

where \( (y_i, x_i) \) comes from the genuine GLM and \( \lambda > 0 \) is the tuning parameter. Negahban et al. (2012) shows that \( \|\hat{\beta} - \beta^*\|_2 = O_p(\sqrt{s \log d/n}) \) under GLM when \( \{x_i\}_{i=1}^{n} \) are sub-Gaussian. However, in the presence of heavy-tailed features \( x_i \) and corruptions \( \epsilon_i \), the statistical accuracy of \( \hat{\beta} \) might deteriorate if we directly evaluate the log-likelihood (3.4) on \( \{(z_i, x_i)\}_{i=1}^{n} \). In this section, we aim to develop a robust \( \ell_1 \)-regularized MLE for the regression vector \( \beta^* \). Define the clipped feature vector \( \tilde{x}_i \) such that for any \( j \in [d] \),

\[
\tilde{x}_{ij} := \min(|x_{ij}|, \tau_1) x_{ij} / |x_{ij}|
\]

and clipped response

\[
\tilde{z}_i = \min(|z_i|, \tau_2) z_i / |z_i|.
\]

We propose to evaluate the negative log-likelihood on the truncated data.

\[
\tilde{\ell}_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} -\tilde{z}_i \tilde{x}_i^T \beta + b(\tilde{x}_i^T \beta).
\]
Again we denote the hessian matrix of $\tilde{\ell}_n(\beta)$ by $\tilde{H}_n(\beta)$ and $\mathbb{E}(\tilde{H}_n(\beta))$ by $\tilde{H}(\beta)$. We study the following $\mathcal{M}$-estimator as the robust estimator of $\beta^*$.

$$\tilde{\beta} = \arg \min_{\beta \in \mathbb{R}^d} \tilde{\ell}_n(\beta) + \lambda \|\beta\|_1.$$

For $S \subset [d]$ and $|S| = s$, define the restricted cone $C(S) := \{v \in \mathbb{R}^d : \|v_{S^c}\|_1 \leq 3\|v_S\|_1\}$. By Lemma 1 in Negahban et al. (2012), when $\lambda > 2\|\nabla \tilde{\ell}_n(\beta)\|_{\max}$, $\beta - \beta^* \in C(S)$, which is a crucial property that gives rise to statistical consistency of $\tilde{\beta}$ under high-dimensional regimes. Therefore, in the following we first present a lemma that characterizes the order of $\|\nabla_{\beta} \tilde{\ell}_n(\beta^*)\|_{\max}$.

**Lemma 2.** Under the following conditions: (1) $b'(\mathbf{x}_i^T \beta^*) \leq M < \infty$ always holds and $\forall \omega > 0$, $\exists m(\omega) > 0$ such that $b'(\eta) \geq m(\omega) > 0$ for $|\eta| \leq \omega$; (2) $\mathbb{E}x_{ij} = 0$, $\mathbb{E}x_{ij}^2 x_{ik}^2 \leq R < \infty$ for all $1 \leq j, k \leq d$; (3) $\mathbb{E}x_{ij}^4 \leq M_1$ and $\mathbb{E}x_{ij}^4' \leq M_1'$; (4) $\|\beta^*\|_1 \leq L < \infty$; (5) $|\mathbb{E}x_{ij}x_{ij}| \leq M_2/\sqrt{n}$ for some universal constant $M_2 < \infty$ and all $1 \leq j \leq d$. With $\tau_1, \tau_2 \asymp (n/\log d)^{\frac{1}{4}}$, it holds for some constant $C$ and all $\xi > 1$ that

$$\mathbb{P}(\|\nabla \tilde{\ell}(\beta^*)\|_{\max} \geq C\xi \sqrt{\log d \over n}) \leq 2d^{1-\xi}.$$

**Remark 5.** In this lemma we show that $\|\nabla \tilde{\ell}_n(\beta^*)\|_{\max} = O_\mathbb{P}(\sqrt{\log d/n})$. In the sequel we will choose $\lambda \asymp \sqrt{\log d/n}$ to achieve the minimax optimal rate for $\beta$.

Another requirement for statistical guarantee of $\tilde{\beta}$ is the restricted strong convexity (RSC) of $\tilde{\ell}_n$, which is first formulated in Negahban et al. (2012). RSC ensures that $\tilde{\ell}_n(\beta)$ is “not too flat”, so that if $|\tilde{\ell}_n(\beta) - \tilde{\ell}_n(\beta^*)|$ is small, then $\beta$ and $\beta^*$ are close. In high-dimensional sparse linear regression, RSC is implied by the restricted eigenvalue (RE) condition (Bickel et al. (2009), van de Geer (2007), etc.), a widely studied and acknowledged condition for statistical error analysis of the Lasso estimator.

Unlike the quadratic loss in linear regression, the negative log-likelihood $\tilde{\ell}_n(\beta)$ has its hessian matrix $\tilde{H}_n(\beta)$ depend on $\beta$, which creates technical difficulty of verifying its RSC. Generally speaking, RSC condition does not hold uniformly over all $\beta \in \mathbb{R}^d$. This motivates us to study localized RSC (LRSC), i.e., RSC with $\beta$ constrained within a small neighborhood of $\beta^*$. This idea was first explored by Fan et al. (2017+) and Sun et al. (2017). Specifically, we say a loss function $\mathcal{L}(\beta)$ satisfies LRSC($\beta^*, r, S, \kappa, \tau_\mathcal{L}$) if for all $\Delta \in C(S) \cap B_2(0, r)$,

$$\delta \mathcal{L}(\beta^* + \Delta; \beta^*) \geq \kappa \|\Delta\|_2^2 - \tau_\mathcal{L},$$

where $\tau_\mathcal{L}$ is a small tolerance term. Later we will see that this localized version of RSC suffices for establishing the statistical rate of $\tilde{\beta}$. The following lemma verifies the LRSC of $\tilde{\ell}_n(\beta)$.

**Lemma 3.** Suppose the conditions of Lemma 2 hold. Let $S$ be the true support of $\beta^*$ with $|S| = s$. Assume that for any $v \in \mathbb{R}^d$ such that $v \in C(S)$ and $\|v\|_2 = 1$, $0 < \kappa_0 \leq \ldots$
Figure 2: High dimensional sparse linear regression with light-tailed features (left) and heavy-tailed features (right)

\( T^T E(x_i x_i^T) v \leq \kappa_1 < \infty \). Set \( \tau_1 \asymp (n / \log d)^{\frac{1}{4}} \). For any \( 0 < r < 1 \) and \( t > 0 \), as long as \( s^2 \log d / n \) is sufficiently small, it holds with probability at least \( 1 - 2 \exp(-t) \) that for all \( \Delta \in \mathbb{R}^d \) such that \( \|\Delta\|_2 \leq r \) and \( \Delta \in \mathcal{C}(S) \),

\[
\delta \tilde{\ell}_n(\beta^* + \Delta; \beta^*) \geq \kappa \|\Delta\|_2^2 - C_0 r^2 \left( \sqrt{\frac{t}{n}} + \sqrt{\frac{s \log d}{n}} \right),
\]

where \( \kappa \) and \( C_0 \) are some constants.

**Remark 6.** This lemma is the high-dimensional analogue of Lemma 1. Similarly, in the sequel we will let \( r \) converge to zero when analyzing the statistical rate of \( \tilde{\beta} \) so that the tolerance term \( r^2 (\sqrt{t/n} + \sqrt{s \log d / n}) \) is negligible.

The lemma above together with Lemma 2 underpins the statistical guarantee of \( \tilde{\beta} \) as follows.

**Theorem 2.** Under the assumptions of Lemma 2 and 3, choose \( \lambda = 2 C \xi \sqrt{\log d / n} \) and \( \tau_1, \tau_2 \asymp (n / \log d)^{\frac{1}{2}} \), where \( \xi \) and \( C \) are the same constants as in Lemma 2. It holds for some constant \( C_1 \) that

\[
P \left( \|\tilde{\beta} - \beta^*\|_2 \geq C_1 \xi \sqrt{\frac{s \log d}{n}} \right) \leq 4d^{1-\xi}.
\]

### 4 Numerical study

#### 4.1 High-dimensional sparse linear regression

We first consider the high-dimensional sparse linear model \( y_i = x_i^T \beta^* + \epsilon_i \). We set \( d = 1000, n = 100, 200, 500, 1000, 5000, 10000 \) and \( \beta^* = (1, 1, 1, 1, 0, \cdots, 0)^T \). Recall that in the
high-dimensional regime, we propose elementwise clipping on both the heavy-tailed features and responses. In Figure 2, we compare estimation error of the $\ell_1$-regularized least squares estimators based on clipped data and original data under standard Gaussian features and $t_{4.1}$ features respectively. All feature vectors $\{x_i\}_{i=1}^n$ are i.i.d. generated and within each $x_i$, all dimensions $\{x_{ij}\}_{j=1}^d$ are i.i.d. $\{\epsilon_i\}_{i=1}^n$ are i.i.d. noises that are independent to the features and we adjust the magnitude of the noise such that $SD(\epsilon_i) = 5$ regardless of the distribution it conforms to. The clipping threshold levels $\tau_1, \tau_2$ and the regularization parameter $\lambda$ are selected by cross-validation. The plot is based on 1,000 independent Monte Carlo simulations. From Figure 2, we first observe that under both light-tailed and heavy-tailed features, the heavier tail $\epsilon_i$ has, the more data clipping improves the statistical accuracy. More importantly, the benefit from data clipping is much more significant in the presence of heavy-tailed features, which justifies our conjecture and theories.

4.2 Logistic regression with mislabeled data

In this subsection we consider the logistic regression with mislabeled data as characterized by (3.1). We minimize the weighted negative log-likelihood to derive $\hat{\beta}^w$ and $\tilde{\beta}^w$ as described in (3.2) and (3.3) to estimate the regression vector $\beta^*$ and compare their performances. All the samples are independently generated and all dimensions of features are independent to each other as well. The tuning parameters $\lambda$ and $\tau_1$ are chosen based on cross validation. We investigate both the low-dimensional and high-dimensional regimes.

- In the low-dimensional regime, we set $d = 10$, $n = 100, 200, 500, 1000, 2000, 5000, 10000$, $\beta^* = (0.5, \cdots, 0.5, -0.5, \cdots, -0.5)$, $p = 0.1$. The left panel of Figure 3 compares $\|\hat{\beta}^w - \beta^*\|_2$ and $\|\tilde{\beta}^w - \beta^*\|_2$ under $t_{2.1}, t_{4.1}$ and Gaussian features. We can observe that $\hat{\beta}^w$ significantly outperforms $\tilde{\beta}^w$ under $t_{2.1}$ and $t_{4.1}$ features, and they perform equally.
well when features are Gaussian. This perfectly matches our intuition and supports our theories.

- In the high-dimensional regime, we apply elementwise clipping to \( x_i \) to derive \( \tilde{\beta}^w \). We set \( d = 100, n = 50, 100, 250, 500, 1000, 2500, 5000, \beta^* = (1, 1, -1, 0, \cdots, 0) \), \( p = 0.1 \).

As shown in the right panel of Figure 3, \( \tilde{\beta}^w \) enjoys sharper statistical accuracy than \( \hat{\beta}^w \) under all the three feature distributions. The outstanding performance of \( \tilde{\beta}^w \) under the Gaussian feature scenario is particularly surprising. We conjecture that feature clipping here downsizes \( \| \nabla \tilde{\ell}_n^w (\beta^*) \|_{\text{max}} \) and thus leads to more effective regularization.

4.3 Experiments on MNIST

We extract deep features of all images of the digits 4 and 9 in the MNIST dataset through a pre-trained convolutional neural network, which has 0.8% testing error in recognizing 0 to 9. Readers can refer to [Deep MNIST Tutorial] by Google for the details of the architecture of the neural nets. We aim to use the extracted deep features of images to classify 4’s and 9’s with artificial mislabelling. We randomly flip the true labels with probability \( p \) and minimize the weighted negative log-likelihood \( \tilde{\ell}_n^w (\beta) = (1/n) \sum_{i=1}^{n} \ell^w (\tilde{x}_i, z_i; \beta) \) as characterized in (3.3) to estimate the regression vector. We repeat the procedure for 100 times and compare the average performance of the resulting MLE based on original deep features \( \{x_i\}_{i=1}^{n} \) and shrunk features \( \{x_i/\|x_i\|_4 \cdot \min(\|x_i\|_4, 0.7)\} \). The result is presented in Table 1. We discover that feature shrinkage robustifies the MLE so well such that the performance is insensitive to the mislabelling probability.

| Mislabel Prob | Original Features | Shrunk Features |
|---------------|------------------|----------------|
| 0%            | 0.5%             | 0.8%           |
| 10%           | 1.0%             | 0.8%           |
| 20%           | 1.4%             | 0.8%           |
| 40%           | 3.1%             | 0.9%           |

Table 1: Testing classification error using original deep features and shrunk deep features

References

AUDIBERT, J.-Y., CATONI, O. ET AL. (2011). Robust linear least squares regression. The Annals of Statistics 39 2766–2794.

BICKEL, P. J., RITOV, Y. and TSYBAKOV, A. B. (2009). Simultaneous analysis of lasso and dantzig selector. The Annals of Statistics 37 1705–1732.

*https://www.tensorflow.org/get_started/mnist/pros
Boucheron, S., Lugosi, G. and Massart, P. (2013). Concentration inequalities: A nonasymptotic theory of independence. OUP Oxford.

Brownlees, C., Joly, E. and Lugosi, G. (2015). Empirical risk minimization for heavy-tailed losses. The Annals of Statistics 43 2507–2536.

Catoni, O. (2012). Challenging the empirical mean and empirical variance: a deviation study. Annales de l’Institut Henri Poincaré 48 1148–1185.

Chen, Y., Caramanis, C. and Mannor, S. (2013). Robust sparse regression under adversarial corruption. In International Conference on Machine Learning.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance 1.

Devroye, L., Lerasle, M., Lugosi, G., Oliveira, R. I. et al. (2016). Sub-gaussian mean estimators. The Annals of Statistics 44 2695–2725.

Fan, J., Li, Q. and Wang, Y. (2017). Robust estimation of high-dimensional mean regression. Journal of Royal Statistical Society, Series B 79 247–265.

Fan, J., Liu, H., Sun, Q. and Zhang, T. (2017+). I-lamm for sparse learning: Simultaneous control of algorithmic complexity and statistical error. The Annals of Statistics, to appear.

Fan, J., Wang, W. and Zhu, Z. (2017). A shrinkage principle for heavy-tailed data: High-dimensional robust low-rank matrix recovery. arXiv preprint arXiv:1603.08315.

Feng, J., Xu, H., Mannor, S. and Yan, S. (2014). Robust logistic regression and classification. In Advances in Neural Information Processing Systems.

Hsu, D., Kakade, S. M. and Zhang, T. (2012). Random design analysis of ridge regression. In Conference on Learning Theory.

Hsu, D. and Sabato, S. (2016). Loss minimization and parameter estimation with heavy tails. Journal of Machine Learning Research 17 1–40.

Ledoux, M. and Talagrand, M. (2013). Probability in Banach Spaces: isoperimetry and processes. Springer Science & Business Media.

Liu, L., Hawkins, D. M., Ghosh, S. and Young, S. S. (2003). Robust singular value decomposition analysis of microarray data. Proceedings of the National Academy of Sciences 100 13167–13172.

Massart, P. (2000). About the constants in talagrand’s concentration inequalities for empirical processes. The Annals of Probability 28 863–884.

Minsker, S. (2017+). Sub-gaussian estimators of the mean of a random matrix with heavy-tailed entries. The Annals of Statistics, to appear.
Natarajan, N., Dhillon, I. S., Ravikumar, P. K. and Tewari, A. (2013). Learning with noisy labels. In Advances in Neural Information Processing Systems.

Negahban, S., Yu, B., Wainwright, M. J. and Ravikumar, P. K. (2012). A unified framework for high-dimensional analysis of \( m \)-estimators with decomposable regularizers. Statistical Science 24 538–577.

Nemirovsky, A.-S., Yudin, D.-B. and Dawson, E.-R. (1982). Problem complexity and method efficiency in optimization. SIAM Review 27 264–265.

Oliveira, R. I. (2016). The lower tail of random quadratic forms with applications to ordinary least squares. Probability Theory and Related Fields 166 1175–1194.

Purdom, E., Holmes, S. P. et al. (2005). Error distribution for gene expression data. Statistical Applications in Genetics and Molecular Biology 4 1070.

Sun, Q., Zhou, W. and Fan, J. (2017). Adaptive huber regression: Optimality and phase transition. Preprint.

van de Geer, S. (2007). The deterministic lasso. Seminar für Statistik, Eidgenössische Technische Hochschule (ETH) Zürich.

Vershynin, R. (2010). Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027.

5 Technical lemmas, propositions and proofs

Lemma 4. Suppose \( \mathbb{E}(v^T x_i)^4 \leq R \) for any \( v \in S^{d-1} \). Define the \( \ell_4 \)-norm shrunk samples

\[
\tilde{x}_i := \min \left( \frac{\|x_i\|_4, \tau}{\|x_i\|_4} \right) \cdot x_i,
\]

where \( \tau \) is a threshold value. Then we have the following:

1. \( \| \tilde{x}_i \tilde{x}_i^T - \mathbb{E} \tilde{x}_i \tilde{x}_i^T \|_{op} \leq \| \tilde{x}_i \|_2^2 + \sqrt{R} \leq \sqrt{d\tau^2} + \sqrt{R} \);

2. \( \| \mathbb{E}((\tilde{x}_i \tilde{x}_i^T - \mathbb{E} \tilde{x}_i \tilde{x}_i^T)(\tilde{x}_j \tilde{x}_j^T - \mathbb{E} \tilde{x}_j \tilde{x}_j^T)) \|_{op} \leq R(d + 1) \);

3. For all \( \xi > 0 \), \( \mathbb{P} \left( \| \hat{\Sigma}_n(\tau) - \Sigma \|_{op} \geq \xi \sqrt{\frac{Rd \log n}{n}} \right) \leq n^{1-C\xi} \), where \( \tau \approx (nR/(\log n))^{1/4} \) and \( C \) is a universal constant.

Proof. This result is from Fan et al. (2017). For convenience of adapting the lemma to other settings, we present its proof here. Notice that

\[
\| \tilde{x}_i \tilde{x}_i^T - \mathbb{E} \tilde{x}_i \tilde{x}_i^T \|_{op} \leq \| \tilde{x}_i \tilde{x}_i^T \|_{op} + \| \mathbb{E} \tilde{x}_i \tilde{x}_i^T \|_{op} = \| \tilde{x}_i \|_2^2 + \sqrt{R} \leq \sqrt{d\tau^2} + \sqrt{R}.
\]
Also for any \( v \in S^{d-1} \), we have
\[
\mathbb{E}(v^T \tilde{x}_i \tilde{x}_i^T \tilde{x}_i^T v) = \mathbb{E}(\| \tilde{x}_i \|^2_2 (v^T \tilde{x}_i)^2) \leq \mathbb{E}(\| \tilde{x}_i \|^2_2 (v^T x_i)^2) = \sum_{j=1}^{d} \mathbb{E}(x_{ij}^2) \leq \sum_{j=1}^{d} \sqrt{\mathbb{E}(x_{ij}^4)} \mathbb{E}(v^T x_i)^4 \leq Rd
\]

Then it follows that \( \| \mathbb{E}(\tilde{x}_i \tilde{x}_i^T \tilde{x}_i^T \tilde{x}_i^T) \|_{op} \leq Rd \). Since \( \| (\mathbb{E}(\tilde{x}_i \tilde{x}_i^T) + \mathbb{E}((\tilde{x}_i \tilde{x}_i^T))^T) \|_{op} \leq R \),
\[
\| \mathbb{E}((\tilde{x}_i \tilde{x}_i^T - \mathbb{E}(\tilde{x}_i \tilde{x}_i^T))^T) \|_{op} \leq R(d + 1).
\] (5.2)

By the matrix Bernstein’s inequality (Theorem 5.29 in Vershynin (2010)), we have for some constant \( c_1 \),
\[
\mathbb{P} \left( \| \frac{1}{n} \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^T - \mathbb{E}(x_i x_i^T) \|_{op} > t \right) \leq 2d \exp \left( -c_1 \left( \frac{nt^2}{R(d + 1)} \right)^{\frac{1}{2}} \right).
\] (5.3)

For any \( v \in S^{d-1} \), it holds that
\[
\mathbb{E}(v^T(x_i x_i^T)v \cdot 1_{\|x_i\|_2 \geq \tau}) \leq \sqrt{\mathbb{E}(v^T x_i)^4} P(\|x_i\|_2 > \tau) \leq \sqrt{\frac{R^2d}{\tau^4}} = \frac{R \sqrt{d}}{\tau^2}.
\] (5.4)

Therefore we have
\[
\| \mathbb{E}(x_i x_i^T - \tilde{x}_i \tilde{x}_i^T) \|_{op} \leq R \sqrt{d}/\tau^2.
\] (5.5)

Choose \( \tau \approx (nR/d \log d)^{\frac{1}{2}} \) and substitute \( t \) with \( \xi R \sqrt{d \log n/n} \). Then we reach the final conclusion by combining the concentration bound and bias bound.

\[\square\]

**Proof of Lemma [7]**. Define a contraction function
\[
\phi(x; \theta) = x^2 \cdot 1_{\{x \leq \theta\}} + (x - 2\theta)^2 \cdot 1_{\{\theta < x \leq 2\theta\}} + (x + 2\theta)^2 \cdot 1_{\{-2\theta \leq x < -\theta\}}.
\]

One can verify that \( \phi(x; \theta) \leq x^2 \) for any \( \theta \). This contraction function was used in a preliminary version of Negahban et al. (2012) to establish the RSC of negative log-likelihood. Given any \( \Delta \in B_2(0, r) \), by the Taylor expansion, we can find \( v \in (0, 1) \) such that
\[
\delta \tilde{\ell}_n(\beta^* + \Delta; \beta^*) = \tilde{\ell}_n(\beta^* + \Delta) - \tilde{\ell}_n(\beta^*) - \nabla \tilde{\ell}_n(\beta^*)^T \Delta = \frac{1}{2} \Delta^T \tilde{H}_n(\beta^* + v \Delta) \Delta
\]
\[
= \frac{1}{2n} \sum_{i=1}^{n} b''(x_i^T(\beta^* + v \Delta)) (\Delta^T x_i)^2 \geq \frac{1}{2n} \sum_{i=1}^{n} b''(x_i^T(\beta^* + \Delta)) \phi(\Delta^T x_i; \alpha_1 r) \cdot 1_{\{\|x_i\|_2 \leq \alpha_2\}}
\]
\[
\geq \frac{m(\omega)}{2n} \sum_{i=1}^{n} \phi(\Delta^T x_i; \alpha_1 r) \cdot 1_{\{\|x_i\|_2 \leq \alpha_2\}},
\] (5.6)
where we choose \( \omega = \alpha_1 + \alpha_2 > \alpha_1 r + \alpha_2 \) so that the last inequality holds by Condition (1).

For ease of notation, let \( A_i := \{ |\Delta^T \tilde{x}_i| \leq \alpha_1 r \} \) and \( B_i := \{ |\beta^T \tilde{x}_i| \leq \alpha_2 \}. \) We have

\[
\mathbb{E}[\phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i}] \geq \mathbb{E}[(\Delta^T \tilde{x}_i)^2 \cdot 1_{A_i \cap B_i}]
\geq \Delta^T \mathbb{E}[x_i x_i^T \cdot 1_{A_i \cap B_i}] \Delta - \Delta^T \mathbb{E}[(x_i x_i^T - \tilde{x}_i \tilde{x}_i^T) \cdot 1_{A_i \cap B_i}] \Delta
\geq \Delta^T \mathbb{E}[x_i x_i^T \cdot 1_{A_i \cap B_i}] \Delta - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta
\geq \Delta^T \mathbb{E}[x_i x_i^T] \Delta - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta
\geq \kappa_0 \|\Delta\|^2_2 - \sqrt{\mathbb{E}(\Delta^T x_i)^4 \cdot (\mathbb{P}(A_i^c) + \mathbb{P}(B_i^c))} - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta.
\]

By the Markov Inequality,

\[
\mathbb{P}(A_i^c) \leq \frac{\mathbb{E}(\Delta^T \tilde{x}_i)^4}{\alpha_1^4 r^4} \leq \frac{R^4}{\alpha_1^4} \quad \text{and} \quad \mathbb{P}(B_i^c) \leq \frac{\mathbb{E}(\beta^T \tilde{x}_i)^4}{\alpha_2^4} \leq \frac{R^4 \|\beta\|^4}{\alpha_2^4} \leq \frac{RL^4}{\alpha_2^4}.
\]

Besides, according to (5.5),

\[
\Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta \leq \frac{R \sqrt{d} \|\Delta\|^2_2}{\tau_1^2} \leq C_1 R \|\Delta\|^2_2 \sqrt{\frac{d \log d}{n}},
\]

where \( C_1 \) is certain constant. Therefore, for sufficiently large \( \alpha_1, \alpha_2, n \) and \( d \),

\[
\mathbb{E}[\phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i}] \geq \frac{\kappa_0}{2} \|\Delta\|^2_2. \tag{5.7}
\]

For notational convenience, define \( Z_i := \phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i} = \phi(\Delta^T \tilde{x}_i \cdot 1_{B_i}; \alpha_1 r) \) and \( \Gamma_r := \sup_{\|\Delta\|_2 \leq r} \|n^{-1} \sum_{i=1}^n Z_i - \mathbb{E}Z_i\|. \) Then an application of Massart’s inequality (Massart (2000)) delivers that

\[
\mathbb{P}\left( |\Gamma_r - \mathbb{E}\Gamma_r| \geq \alpha_1^2 r^2 \sqrt{\frac{\tau}{n}} \right) \leq 2 \exp \left( - \frac{t}{8} \right). \tag{5.8}
\]

The remaining job is to derive the order of \( \mathbb{E}\Gamma_r \). Note that \( |\phi(x_1; \theta) - \phi(x_2; \theta)| \leq 2\theta |x_1 - x_2| \) for any \( x_1, x_2 \in \mathbb{R} \). By the symmetrization argument and then Ledoux-Talagrand contraction inequality (see Ledoux and Talagrand (2013), p. 112), for a sequence of i.i.d. Rademacher variables \( \{\gamma_i\}_{i=1}^n \),

\[
\mathbb{E}\Gamma_r \leq 2 \mathbb{E} \sup_{\|\Delta\|_2 \leq r} \left| \frac{1}{n} \sum_{i=1}^n \gamma_i Z_i \right| \leq 8 \alpha_1 r \cdot \mathbb{E} \sup_{\|\Delta\|_2 \leq r} \left| \left( \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{|\beta^T \tilde{x}_i| \leq \alpha_2\}} \right) \Delta \right|
\leq 8 \alpha_1 r^2 \cdot \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{|\beta^T \tilde{x}_i| \leq \alpha_2\}} \right|_2 \leq 8 \alpha_1 r^2 \cdot \sqrt{\mathbb{E} \left| \left( \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{|\beta^T \tilde{x}_i| \leq \alpha_2\}} \right) \right|^2} \leq 8 \alpha_1 r^2 \cdot \sqrt{\frac{1}{n} \sum_{i=1}^n \mathbb{E} \|\tilde{x}_i\|^2_2} \leq 8 \alpha_1 r^2 R^2 \cdot \sqrt{\frac{d}{n}}.
\]
Combining the above inequality with (5.6), (5.7) and (5.8) yields that for any $t > 0$, with probability at least $1 - 2 \exp(-t)$, for all $\Delta \in \mathbb{R}^d$ such that $\|\Delta\|_2 \leq r$,

$$
\delta \tilde{e}_n(\beta; \beta^*) \geq \frac{mn\kappa_0}{4} \|\Delta\|_2^2 - \alpha_1^2 \sqrt{\frac{8t}{n}} \cdot r^2 - 8\alpha_1 R \sqrt{\frac{d}{n}} \cdot r^2.
$$

\[ \square \]

**Proof of Theorem 4** Construct an intermediate estimator $\tilde{\beta}_\eta$ between $\tilde{\beta}$ and $\beta^*$:

$$
\tilde{\beta}_\eta = \beta^* + \eta(\tilde{\beta} - \beta^*),
$$

where $\eta = 1$ if $\|\tilde{\beta} - \beta^*\|_2 \leq r$ and $\eta = r/\|\tilde{\beta} - \beta^*\|_2$ if $\|\tilde{\beta} - \beta^*\|_2 > r$. Write $\tilde{\beta}_\eta - \beta^*$ as $\tilde{\Delta}_\eta$. By Lemma 1, it holds with probability at least $1 - 2 \exp(-t)$ that

$$
\kappa \|\tilde{\Delta}_\eta\|_2^2 - C r^2 \left( \sqrt{\frac{t}{n}} + \sqrt{\frac{d}{n}} \right) \leq \delta \tilde{e}_n(\tilde{\beta}_\eta; \beta^*) \leq -\nabla \tilde{e}_n(\beta^*)^T \tilde{\Delta}_\eta \leq \|\nabla \tilde{e}_n(\beta^*)\|_2 \cdot \|\tilde{\Delta}_\eta\|_2,
$$

which further implies that

$$
\|\tilde{\Delta}_\eta\|_2 \leq \frac{3\|\nabla \tilde{e}_n(\beta^*)\|_2}{\kappa} + \sqrt{\frac{3c_1 r^2}{\kappa}} \cdot \left( \frac{t}{n} \right)^{\frac{1}{4}} + \sqrt{\frac{3c_2 r^2}{\kappa}} \cdot \left( \frac{d}{n} \right)^{\frac{1}{4}}. \tag{5.9}
$$

Now we derive the rate of $\|\nabla \tilde{e}_n(\beta^*)\|_2$.

$$
\nabla \tilde{e}_n(\beta^*) = \frac{1}{n} \sum_{i=1}^{n} (\tilde{z}_i - b(\tilde{x}_i^T \beta^*)) \tilde{x}_i
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \tilde{z}_i \tilde{x}_i - \mathbb{E} \tilde{z}_i \tilde{x}_i + \mathbb{E} (\tilde{z}_i - b(\tilde{x}_i^T \beta^*)) \tilde{x}_i + \frac{1}{n} \sum_{i=1}^{n} b(\tilde{x}_i^T \beta^*) \tilde{x}_i - \mathbb{E} (b(\tilde{x}_i^T \beta^*) \tilde{x}_i).
$$

(5.10)

where $\tilde{x}_i$ is between $x_i$ and $\tilde{x}_i$ by the mean value theorem. In the following we will bound $T_1, T_2$ and $T_3$ respectively.

**Bound for $T_1$:** Define the Hermitian dilation matrix

$$
\tilde{Z}_i := \tilde{z}_i \cdot \begin{pmatrix} 0 & \tilde{x}_i^T \\ \tilde{x}_i & 0 \end{pmatrix}
$$

Note that

$$
\|\mathbb{E}\tilde{Z}_i^2\|_{op} = \|E \begin{pmatrix} \tilde{z}_i^2 & \tilde{x}_i^T \tilde{x}_i \\ 0 & \tilde{x}_i \tilde{x}_i^T \end{pmatrix} \|_{op} = \max(\mathbb{E}(\tilde{z}_i^2 \tilde{x}_i^T \tilde{x}_i), \|\mathbb{E}(\tilde{z}_i^2 \tilde{x}_i \tilde{x}_i^T)\|_{op})
$$

For any $j \in [d]$,

$$
E(\tilde{z}_i^2 \cdot \tilde{x}_{ij}) \leq \sqrt{E z_i^4 \cdot E x_{ij}^4} \leq \sqrt{M_1 R},
$$

17
so \( \mathbb{E}[\tilde{z}_i^2 \cdot \tilde{x}_i^T \tilde{x}_i] \leq d \sqrt{M_1 R} \). In addition, for any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = 1 \),

\[
\mathbb{E}(\tilde{z}_i^2 (\mathbf{v}^T \tilde{x}_i)^2) \leq \sqrt{M_1 R}.
\]

We thus have \( \|\mathbb{E}\tilde{Z}_i\|_{op} \leq d \sqrt{M_1 R} \). In addition, \( \|\mathbb{E}\tilde{Z}_i\|_{op} = \mathbb{E}(\tilde{z}_i \cdot \|\tilde{x}_i\|_2) \leq \sqrt{\mathbb{E}z_i^2 \mathbb{E}\|\tilde{x}_i\|_2^2} \leq \sqrt{d(M_1 R)^{\frac{1}{2}}} \), which further implies that \( \|\mathbb{E}(\tilde{Z}_i - \mathbb{E}\tilde{Z}_i)^2\|_{op} \leq (d + 1) \sqrt{M_1 R} \). Also notice that since \( \|\tilde{x}_i\|_4 \leq \tau_1 \) and \( \tilde{z}_i \leq \tau_2 \), \( \|\tilde{Z}_i\|_{op} \leq \frac{1}{2} d^{\frac{1}{4}} \cdot \tau_1 \tau_2 \). By the matrix Bernstein’s inequality,

\[
P\left(\left\| \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i - \mathbb{E}\tilde{Z}_i \right\|_{op} \geq t \right) \leq d \cdot \exp\left(-c_1 \min\left(\frac{nt^2}{(d + 1)\sqrt{M_1 R}}, \frac{2nt}{d^{\frac{1}{4}} \tau_1 \tau_2}\right)\right).
\]

Given that \( \|T_1\|_2 = 2\|n^{-1} \sum_{i=1}^n \tilde{Z}_i - \mathbb{E}\tilde{Z}_i\|_{op} \), it thus holds that

\[
P\left(\|T_1\|_2 \geq 2t \right) \leq d \cdot \exp\left(-c_1 \min\left(\frac{nt^2}{(d + 1)\sqrt{M_1 R}}, \frac{2nt}{d^{\frac{1}{4}} \tau_1 \tau_2}\right)\right).
\]

**Bound for \( T_2 \):** We decompose \( T_2 \) as follows:

\[
\|T_2\|_2 \leq \left(\sum_{i=1}^n \tilde{Z}_i - z_i \right) \|\tilde{x}_i\|_2 + \left(\sum_{i=1}^n z_i - y_i \right) \|\tilde{x}_i\|_2 + \left(\sum_{i=1}^n y_i - b'(x_i^T \beta^*) \right) \|\tilde{x}_i\|_2
\]

\[
+ \left(\sum_{i=1}^n b'(x_i^T \beta^*) - b'(x_i^T \beta^*) \right) \|\tilde{x}_i\|_2.
\]

Now we work on \( \{T_2\}^i_{i=1} \) one by one. For any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = 1 \),

\[
\mathbb{E}(\tilde{z}_i \cdot z_i) (\mathbf{v}^T \tilde{x}_i) \leq \mathbb{E}(\|z_i\|_2^2 \mathbf{v}^T x_i) \cdot 1_{\{|z_i| > \tau_2\}} \leq \sqrt{\mathbb{E}(z_i^2 (\mathbf{v}^T x_i)^2)} \cdot \mathbb{P}(|z_i| > \tau_2).
\]

Thus we have \( \|T_{2i}\|_2 \leq M_1 \frac{R^{\frac{1}{2}}}{\tau_2^2} \). Again, for any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = 1 \), since \( \|\mathbb{E}\epsilon_i x_i\|_2 \leq M_2 \sqrt{d/n} \),

\[
\mathbb{E}[\epsilon_i (\tilde{x}_i^T \mathbf{v})] = \mathbb{E}[\epsilon_i (\tilde{x}_i^T \tilde{x}_i) \mathbf{v})] + \mathbb{E}[\epsilon_i (\tilde{x}_i^T \mathbf{v})] \leq \mathbb{E}[\epsilon_i (\tilde{x}_i^T \mathbf{v})] \cdot 1_{\{||x_i|\|_4 \geq \gamma_1\}} + M_2 \sqrt{\frac{d}{n}}
\]

\[
\leq \sqrt{\mathbb{E}(\epsilon_i (x_i^T \mathbf{v})^2)} \cdot \mathbb{P}(|\|x_i\|_4 \geq \gamma_1) + M_2 \sqrt{\frac{d}{n}}
\]

\[
\leq (M_1 R)^{\frac{1}{2}} \cdot \frac{\sqrt{d \tau_2}}{\tau_1^2} + M_2 \sqrt{\frac{d}{n}}.
\]

Therefore \( \|T_{22}\|_2 \leq (M_2 R)^{\frac{1}{2}} \sqrt{dR/\tau_2^2} + M_2 \sqrt{d/n} \). For \( T_{23} \), since \( \mathbb{E}[y_i - b'(x_i^T \beta^*) | x_i] = 0 \), \( T_{23} = 0 \). Finally we bound \( T_{24} \). For any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = 1 \),

\[
\|T_{24}\|_2 \leq M \mathbb{E}(\beta^T (x_i - \tilde{x}_i)) (\mathbf{v}^T \tilde{x}_i) \leq M \mathbb{E}[(\beta^T x_i)(\mathbf{v}^T x_i) \cdot 1_{\{|\|x_i\|_4 \geq \gamma_1\}}]
\]

\[
\leq M \sqrt{\mathbb{E}(\beta^T x_i)^2 (\mathbf{v}^T x_i)^2} \cdot P(|\|x_i\|_4 \geq \gamma_1) \leq M L \sqrt{dR/\tau_1^2}.
\]
To summarize here, we have
\[
\|T_2\|_2 \leq (M_1 R)^\frac{1}{2} \left( \sqrt{\frac{M_1}{\tau_2}} + \sqrt{d R} \right) + M_2 \frac{d}{\tau_1} + M_2 \frac{d}{n}. \tag{5.12}
\]

**Bound for** $T_3$: We apply a similar proof strategy as in the bound for $T_1$. Define the following Hermitian dilation matrix:
\[
\tilde{X}_i := b' (\tilde{x}_i^T \beta^*) \cdot \left( \begin{array}{c} 0 \\ \tilde{x}_i \\ 0 \end{array} \right).
\]
First,
\[
\|\mathbb{E} \tilde{X}_i^2\|_{op} = \max(\|\mathbb{E}(b' (\tilde{x}_i^T \beta^*) \tilde{x}_i^T \tilde{x}_i)\|, \|\mathbb{E} b' (\tilde{x}_i^T \beta^*)^2 \tilde{x}_i \tilde{x}_i^T\|_{op}).
\]
Write $|b'(1)|$ as $b_1$. For any $j \in [d]$,
\[
\mathbb{E}(b' (\tilde{x}_i^T \beta^*)^2 \cdot \tilde{x}_i^2) \leq \mathbb{E}((b_1 + M(\tilde{x}_i^T \beta^* - 1))^2 \tilde{x}_i^2) \leq 2 \mathbb{E}((b_1 + M)^2 (\tilde{x}_i^T \beta^*)^2) \tilde{x}_i^2 \leq 2M^2 R \|\beta^*\|^2 (d+\sqrt{d}) V =: V,
\]
so $\mathbb{E}(b' (\tilde{x}_i^T \beta^*)^2 \cdot \tilde{x}_i^T \tilde{x}_i) \leq dV$. In addition, for any $v \in \mathbb{R}^d$ such that $\|v\|_2 = 1$,
\[
\mathbb{E}(b' (\tilde{x}_i^T \beta^*)^2 (v^T \tilde{x}_i)^2) \leq \mathbb{E}((b_1 + M(\tilde{x}_i^T \beta^* - 1))^2 (v^T \tilde{x}_i)^2) \leq V.
\]
We thus have $\|\mathbb{E} \tilde{X}_i^2\|_{op} \leq dV$. In addition, $\|\mathbb{E} \tilde{X}_i\|_{op} = \mathbb{E}(b' (\tilde{x}_i^T \beta^*) \cdot \tilde{x}_i\|_2) \leq \sqrt{\mathbb{E}b' (\tilde{x}_i^T \beta^*)^2} \mathbb{E}\|\tilde{x}_i\|_2 \leq \sqrt{\mathbb{E}(\tilde{x}_i - \mathbb{E} \tilde{x}_i)^2}) \leq (d+\sqrt{d}) V$. Also notice that $\|\tilde{X}_i\|_{op} \leq ((b_1 + M) + M\|\beta^*\|_2 \cdot d^2 \tau_1) d^2 \tau_1$. By the matrix Bernstein’s inequality,
\[
\mathbb{P}\left(\left\| \frac{1}{n} \sum_{i=1}^n \tilde{X}_i - \mathbb{E} \tilde{X}_i \right\|_{op} \geq t \right) \leq d \cdot \exp\left(-c_1 \min\left(\frac{nt^2}{(d+\sqrt{d})V}, \frac{nt}{(b_1 + M + M\|\beta^*\|_2 \cdot d^2 \tau_1) d^2 \tau_1}\right)\right).
\]
Given that $\|T_3\|_2 = 2\|\frac{1}{n} \sum_{i=1}^n \tilde{X}_i - \mathbb{E} \tilde{X}_i\|_{op}$, it thus holds that
\[
\mathbb{P}\left(\|T_3\|_2 \geq 2t \right) \leq d \cdot \exp\left(-c_1 \min\left(\frac{nt^2}{(d+\sqrt{d})V}, \frac{nt}{(b_1 + M + M\|\beta^*\|_2 \cdot d^2 \tau_1) d^2 \tau_1}\right)\right). \tag{5.13}
\]
Finally, choose $\tau_1, \tau_2 \approx (n/ \log n)^\frac{1}{4}$. Combining (5.11), (5.12) and (5.13) delivers that for some constant $C_1$ any $\xi > 1$,
\[
\mathbb{P}(\|\nabla \tilde{\ell}_n (\beta^*)\|_2 \geq C_1 \xi \sqrt{\frac{d \log n}{n}}) \leq n^{1-\xi}. \tag{5.14}
\]
Choose $t = \xi \log n$ and let $r$ be larger than the RHS of (5.9). When $d/n$ is sufficiently small and $n$ is sufficiently large, we can obtain that
\[
r \geq C_2 \xi \sqrt{\frac{d \log n}{n}} =: r_0,
\]

19
where $C_2$ is a constant. Choose $r = r_0$. Then by (5.9), $\|\Delta\|_2 \leq r_0$ and thus $\Delta = \tilde{\Delta}$. Finally, we reach the conclusion that

$$\mathbb{P}\left(\|\tilde{\Delta}\|_2 \geq C_2 \xi \sqrt{\frac{d \log n}{n}}\right) \leq n^{1-\xi} + 2n^{-\xi} \leq 3n^{1-\xi}.$$ 

\[\square\]

Proof of Corollary 1. The proof strategy is nearly the same as that for deriving Theorem 1 so we provide a roadmap here and do not dive into great details. For ease of notation, write $n^{-1} \sum_{i=1}^{n} \ell^w(\tilde{x}_i, z_i; \beta)$ as $\ell_n^w(\beta)$ and denote the hessian matrix of $\ell_n^w(\beta)$ by $H_n^w(\beta)$. Since $\tilde{H}_n^w(\beta) = \nabla^2 \ell_n^w(\beta) = \tilde{H}_n(\beta)$, we can directly obtain the uniform strong convexity of $\tilde{H}_n^w(\beta)$ from Lemma 1. In addition,

$$\nabla_{\beta} \ell_n^w(\beta^*) = \frac{1-p}{1-2p} \cdot \frac{1}{n} \sum_{i=1}^{n} (b'(\tilde{x}_i^T \beta^*) - z_i)\tilde{x}_i - \frac{p}{1-2p} \cdot \frac{1}{n} \sum_{i=1}^{n} (b'(\tilde{x}_i^T \beta^*) - (1 - z_i))\tilde{x}_i$$

$$= \frac{1-p}{1-2p} (T_1 - \mathbb{E}T_1) - \frac{p}{1-2p} (T_2 - \mathbb{E}T_2) + \frac{1-p}{1-2p} \mathbb{E}T_1 - \frac{p}{1-2p} \mathbb{E}T_2$$

$$= \frac{1-p}{1-2p} (T_1 - \mathbb{E}T_1) - \frac{p}{1-2p} (T_2 - \mathbb{E}T_2) + \mathbb{E}(b'(\tilde{x}_i^T \beta^*) - y_i)\tilde{x}_i.$$

Since $|b'(\tilde{x}_i^T \beta^*) - z_i| \leq 1$ and $|b'(\tilde{x}_i^T \beta^*) - (1 - z_i)| \leq 1$, following the bound for $T_1$ in Theorem 1 we will obtain

$$\mathbb{P}(\|\frac{1-p}{1-2p} (T_1 - \mathbb{E}T_1) - \frac{p}{1-2p} (T_2 - \mathbb{E}T_2)\|_2 \geq c_1 \xi \sqrt{\frac{d \log n}{n}}) \leq n^{1-\xi},$$

where $c_1 > 0$ depends on $R$ and $p$ and $\xi > 1$. In addition, following the bound for $T_{23}$ and $T_{24}$ in Theorem 1, we shall obtain

$$\|\mathbb{E}(b'(\tilde{x}_i^T \beta^*) - y_i)\tilde{x}_i\|_2 \leq M_2 L \frac{\sqrt{dR}}{\tau_1^2} \leq c_2 M_2 \sqrt{\frac{dR \log n}{n}}.$$ 

where $c_2 > 0$ is a constant. Therefore, for some constant $c_3$ depending on $R, p, M_2, R$, we have

$$\mathbb{P}(\|\nabla_{\beta} \ell_n^w(\beta^*)\|_2 \geq c_3 \xi \sqrt{\frac{d \log n}{n}}) \leq n^{1-\xi}. $$

Combining this with the uniform strong convexity of $\tilde{H}_n^w(\beta)$ delivers the final conclusion. \[\square\]
Proof of Lemma 2. According to (2.3), \([\nabla \tilde{\ell}((\beta^*)^\top)] = (b'((\tilde{\beta}^*)^\top) - \tilde{z}_i)\tilde{x}_{ij}\). Then we have
\[
\frac{1}{n} \sum_{i=1}^{n} (b'((\tilde{x}_i^T \beta^*)) - \tilde{z}_i)\tilde{x}_{ij} \leq \frac{1}{n} \sum_{i=1}^{n} b'((\tilde{x}_i^T \beta^*))\tilde{x}_{ij} - \mathbb{E}b'((\tilde{x}_i^T \beta^*))\tilde{x}_{ij} + \mathbb{E}(b'((\tilde{x}_i^T \beta^*)) - \tilde{z}_i)\tilde{x}_{ij} + \frac{1}{n} \sum_{i=1}^{n} \tilde{z}_i\tilde{x}_{ij} - \mathbb{E}\tilde{z}_i\tilde{x}_{ij}.
\]

We start with the upper bound of \(T_1\). By the Mean Value Theorem, for any \(i \in [n]\), there exists \(\xi_i\) between 1 and \(\tilde{x}_i^T \beta^*\) such that \(b'((\tilde{x}_i^T \beta^*)) = b'(1) + b''(\xi_i) \cdot (\tilde{x}_i^T \beta^* - 1)\). Therefore we have
\[
T_1 \leq \frac{1}{n} \sum_{i=1}^{n} b'(1)\tilde{x}_{ij} - \mathbb{E}(b'(1)\tilde{x}_{ij}) + \frac{1}{n} \sum_{i=1}^{n} b''(\xi_i)\tilde{x}_{ij}(\tilde{x}_i^T \beta^* - 1) - \mathbb{E}(b''(\xi_i)\tilde{x}_{ij}(\tilde{x}_i^T \beta^* - 1))
\leq \frac{1}{n} \sum_{i=1}^{n} b'(1)\tilde{x}_{ij} - \mathbb{E}(b'(1)\tilde{x}_{ij}) + \sum_{k=1}^{d} |\beta_k^*| \cdot \frac{1}{n} \sum_{i=1}^{n} b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik} - \mathbb{E}b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik}
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} b''(\xi_i)\tilde{x}_{ij} - \mathbb{E}(b''(\xi_i)\tilde{x}_{ij})
\]

Since \(\text{var}(\tilde{x}_{ij}) \leq \sqrt{R}\) and \(|\tilde{x}_{ij}| \leq \tau_1\), an application of Bernstein’s inequality (Theorem 2.10 in Boucheron et al. (2013)) yields that
\[
\mathbb{P}(\left|\frac{1}{n} \sum_{i=1}^{n} b'(1)\tilde{x}_{ij} - \mathbb{E}(b'(1)\tilde{x}_{ij})\right| \geq |b'(1)|\left(\sqrt{\frac{\sqrt{R} \cdot 2t}{n}} + \frac{c_1 \tau_1 t}{n}\right)) \leq 2 \exp(-t),
\]
where \(c_1 > 0\) is some universal constant. In addition, \(b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik} \leq M\tau_1^2\) and \(\text{var}(b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik}) \leq \mathbb{E}(b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik})^2 \leq M^2 R\). Again by Bernstein’s inequality,
\[
\mathbb{P}(\left|\frac{1}{n} \sum_{i=1}^{n} b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik} - \mathbb{E}(b''(\xi_i)\tilde{x}_{ij}\tilde{x}_{ik})\right| \geq \sqrt{\frac{2M^2 RT}{n}} + \frac{c_1 M\tau_1^2 t}{n}) \leq 2 \exp(-t).
\]
Similarly,
\[
\mathbb{P}(\left|\frac{1}{n} \sum_{i=1}^{n} b''(\xi_i)\tilde{x}_{ij} - \mathbb{E}(b''(\xi_i)\tilde{x}_{ij})\right| \geq \sqrt{\frac{M^2 \sqrt{R} t}{n}} + \frac{M\tau_1 t}{n}) \leq 2 \exp(-t).
\]
Combining the above three inequalities delivers that
\[
\mathbb{P}(T_1 \geq |b'(1)|\left(\sqrt{\frac{\sqrt{R} \cdot 2t}{n}} + \frac{c_1 \tau_1 t}{n} + \sqrt{\frac{2M^2 RT}{n}} + \frac{c_1 M\tau_1^2 t}{n} + \sqrt{\frac{M^2 \sqrt{R} t}{n}} + \frac{M\tau_1 t}{n}\right)
\leq 6 \exp(-t).
\]
\[(5.15)\]
Now we bound $T_2$.

$$T_2 = \mathbb{E}[(z_i - \tilde{z}_i)\tilde{x}_{ij}] + \mathbb{E}\epsilon_i\tilde{x}_{ij} + \mathbb{E}[(b'(x_i^T\beta^*) - b'(\tilde{x}_i^T\beta^*))\tilde{x}_{ij}]$$

$$\leq \mathbb{E}[|z_i\tilde{x}_{ij} - \tilde{z}_i\tilde{x}_{ij}| \cdot 1_{\{|z_i|\geq \tau_2\}}] + \mathbb{E}(\epsilon_i x_{ij}) + \mathbb{E}(x_{ij} - \tilde{x}_{ij}) + M \cdot \sum_{k=1}^{d} |\beta_k^*| \cdot \mathbb{E}|\tilde{x}_{ik}(\tilde{x}_{ij} - x_{ij})|$$

$$\leq (M_1R)^{\frac{1}{2}} \cdot \frac{\sqrt{M_1}}{\tau_2^2} + \frac{M_3}{\sqrt{n}} + \frac{(M_1R)^{\frac{1}{2}}}{\tau_1^2} + MM_2 \cdot \frac{\sqrt{R}}{\tau_1^2}. \tag{5.16}$$

Finally we bound $T_3$. Note that $|\tilde{z}_i\tilde{x}_{ij}| \leq \tau_1\tau_2$, $\text{var}(\tilde{x}_{ij}\tilde{z}_i) \leq \mathbb{E}|\tilde{z}_i\tilde{x}_{ij}|^2 \leq \sqrt{M_1R}$. According to the Bernstein’s inequality,

$$\mathbb{P}(|T_3| \geq \sqrt{2t\sqrt{M_1R}} \cdot \frac{c_1\tau_1\tau_2 t}{n}) \leq 2 \exp(-t). \tag{5.17}$$

Choose $\tau_1, \tau_2 \approx (n/\log d)^{\frac{1}{4}}$. Combining (5.15), (5.16) and (5.17) delivers that for some constant $C_1 > 0$ that depends on $M, R, \{M_i\}_{i=1}^{3}, b(1)$ and any $\xi > 1$,

$$\mathbb{P}(|\nabla_{\beta^*}(\bar{\ell})|_j \geq C_1\xi \sqrt{\frac{\log d}{n}}) \leq 2d^{-\xi}. \tag{5.18}$$

Then by the union bound for all $j \in [d]$, it holds that

$$\mathbb{P}(\max_{j \in [d]}|\nabla_{\beta^*}(\bar{\ell})|_j \geq C_1\xi \sqrt{\frac{\log d}{n}}) \leq 2d^{1-\xi}. \tag{5.18}$$

Proof of Lemma 3. The proof strategy is quite similar to that for Lemma 1, except that we need to take advantage of the restricted cone $C(S)$ that $\Delta$ lies in. First of all, for any $1 \leq j, k \leq d$,

$$|\mathbb{E}(\tilde{x}_{ij}\tilde{x}_{ik} - x_{ij}x_{ik})| \leq \sqrt{\mathbb{E}(x_{ij}x_{ik})^2 \cdot (\mathbb{P}(|x_{ij}| \geq \tau_1) + \mathbb{P}(|x_{ik}| \geq \tau_1))} \leq \frac{\sqrt{2R}}{\tau_1^2}. \tag{5.18}$$

We thus have

$$\|\mathbb{E}[x_i x_i^T - \bar{x}_i\bar{x}_i^T]\|_{\text{max}} \leq \frac{\sqrt{2R}}{\tau_1^2} \leq CR \sqrt{\frac{2\log d}{n}}, \tag{5.18}$$

where $C > 0$ is some constant. Again, define a contraction function

$$\phi(x; \theta) = x^2 \cdot 1_{\{|x| \leq \theta\}} + (x - 2\theta)^2 \cdot 1_{\{|x| < 2\theta\}} + (x + 2\theta)^2 \cdot 1_{\{|x| < 2\theta\}}.$$
Given any $\Delta \in B_2(0, r) \cap C(S)$, by the Taylor expansion, we can find $v \in (0, 1)$ such that
\begin{align*}
\delta l_n(\beta^* + \Delta; \beta^*) &= \tilde{l}_n(\beta^* + \Delta) - \tilde{l}_n(\beta^*) - \nabla \tilde{l}_n(\beta^*)^T \Delta = \frac{1}{2} \Delta^T \tilde{H}_n(\beta^* + v \Delta) \Delta \\
&= \frac{1}{2n} \sum_{i=1}^{n} b''(\tilde{x}_i^T (\beta^* + v \Delta)) (\Delta^T \tilde{x}_i) \geq \frac{1}{2n} \sum_{i=1}^{n} b''(\tilde{x}_i^T (\beta^* + v \Delta)) \phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{\{|\beta^* \tilde{x}_i| \leq \alpha_2\}} \\
&\geq \frac{m(\omega)}{2n} \sum_{i=1}^{n} \phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{\{|\beta^* \tilde{x}_i| \leq \alpha_2\}},
\end{align*}
(5.19)
where we choose $\omega = \alpha_1 + \alpha_2 > \alpha_1 r + \alpha_2$ so that the last inequality holds by Condition (1).

For ease of notation, let $A_i := \{|\Delta^T \tilde{x}_i| \leq \alpha_1 r\}$ and $B_i := \{|\beta^* \tilde{x}_i| \leq \alpha_2\}$. We have
\begin{align*}
\mathbb{E}[\phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i}] &\geq \mathbb{E}[(\Delta^T \tilde{x}_i)^2 \cdot 1_{A_i \cap B_i}] \\
&\geq \Delta^T \mathbb{E}[x_i x_i^T \cdot 1_{A_i \cap B_i}] \Delta - \Delta^T \mathbb{E}[(x_i x_i^T - \tilde{x}_i \tilde{x}_i^T) \cdot 1_{A_i \cap B_i}] \Delta \\
&\geq \Delta^T \mathbb{E}[x_i x_i^T \cdot 1_{A_i \cap B_i}] \Delta - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta \\
&\geq \Delta^T \mathbb{E}[x_i x_i^T] \Delta - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta \\
&\geq \kappa_0 \|\Delta\|^2_2 - \sqrt{\mathbb{E}(\Delta^T x_i)^2 \cdot (\mathbb{P}(A^c_i) + \mathbb{P}(B^c_i))} - \Delta^T \mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T] \Delta \\
&\geq \kappa_0 \|\Delta\|^2_2 - \sqrt{\mathbb{E}(\Delta^T x_i)^2 \cdot (\mathbb{P}(A^c_i) + \mathbb{P}(B^c_i))} \cdot \|\Delta\|^2_2 - \|\mathbb{E}[x_i x_i^T - \tilde{x}_i \tilde{x}_i^T]\|_\text{max} \cdot \|\Delta\|^2_2
\end{align*}
By the Markov Inequality and (5.18),
\begin{align*}
\mathbb{P}(A^c_i) &\leq \frac{\mathbb{E}(\Delta^T \tilde{x}_i)^2}{\alpha_1^2 r^2} \leq \frac{\mathbb{E}(\Delta^T x_i)^2 + \Delta^T \mathbb{E}(\tilde{x}_i \tilde{x}_i^T - x_i x_i^T) \Delta}{\alpha_1^2 r^2} \\
&\leq \frac{\sqrt{R} \|\Delta\|^2_2 + CR s \|\Delta\|^2_2 \sqrt{2 \log d/n}}{\alpha_1^2 r^2} \leq \frac{\sqrt{R} + CR s \sqrt{\log d/n}}{\alpha_1^2 r^2},
\end{align*}
and
\begin{align*}
\mathbb{P}(B^c_i) &\leq \frac{\mathbb{E}(\beta^T \tilde{x}_i)^2}{\alpha_2^2} \leq \frac{\mathbb{E}(\beta^T x_i)^2 + \beta^T \mathbb{E}(\tilde{x}_i \tilde{x}_i^T - x_i x_i^T) \beta^*}{\alpha_2^2} \\
&\leq \frac{\sqrt{R} \|\beta\|^2_2 + CR s \|\beta\|^2_2 \sqrt{2 \log d/n}}{\alpha_2^2} \leq \frac{\sqrt{R} L^2 + CR L^2 s \sqrt{2 \log d/n}}{\alpha_2^2}.
\end{align*}
Overall, as long as $\alpha_1, \alpha_2$ are sufficiently large and $s \sqrt{\log d/n}$ is not large,
\begin{align*}
\mathbb{E}[\phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i}] &\geq \frac{\kappa_0}{2} \|\Delta\|^2_2.
\end{align*}
(5.20)

For notational convenience, define $Z_i := \phi(\Delta^T \tilde{x}_i; \alpha_1 r) \cdot 1_{B_i} = \phi(\Delta^T \tilde{x}_i \cdot 1_{B_i}; \alpha_1 r)$ and $\Gamma_r := \sup_{\|\Delta\|_2 \leq r, \Delta \in C(S)} \frac{n-1}{n} \sum_{i=1}^{n} Z_i - \mathbb{E}Z_i$. Then an application of Massart’s inequality (Massart (2000)) delivers that
\begin{align*}
\mathbb{P}\left(\|\Gamma_r - \mathbb{E}\Gamma_r\| \geq \alpha_1 r^2 \sqrt{\frac{t}{n}}\right) &\leq 2 \exp\left(-\frac{t}{8}\right).
\end{align*}
(5.21)
The remaining job is to derive the order of $\mathbb{E} \Gamma_r$. By the symmetrization argument and Ledoux-Talagrand contraction inequality, for a sequence of i.i.d. Rademacher variables $\{\gamma_i\}_{i=1}^n$,

$$\mathbb{E} \Gamma_r \leq 2 \mathbb{E} \sup_{\|\Delta\|_2 \leq r, \Delta \in \mathcal{C}(S)} \left| \frac{1}{n} \sum_{i=1}^n \gamma_i Z_i \right| \leq 8 \alpha_1 r \cdot \mathbb{E} \sup_{\|\Delta\|_2 \leq r, \Delta \in \mathcal{C}(S)} \left| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{\|\beta^* \tilde{x}_i\| \leq \alpha_2\}} \cdot \Delta \right|
$$

$$\leq 8 \alpha_1 \sqrt{s} r \cdot \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{\|\beta^* \tilde{x}_i\| \leq \alpha_2\}} \right|_{\max}.$$

For any $1 \leq j \leq d$, by Bernstein inequality,

$$\mathbb{P}\left( \left| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_{ij} \cdot 1_{\{\|\beta^* \tilde{x}_i\| \leq \alpha_2\}} \right| \right) \geq \sqrt{\frac{2 \sqrt{R t}}{n} + \frac{C_1 t}{n} \leq 2 \exp(-t),$$

where $C_1$ is some constant. By the union bound, we can deduce that for some constant $C_2$,

$$\mathbb{P}\left( \left\| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{\|\beta^* \tilde{x}_i\| \leq \alpha_2\}} \right\|_{\max} \geq C_2 \sqrt{\frac{t \log d}{n}} \right) \leq 2d^{1-t},$$

which further implies that

$$\mathbb{E} \Gamma_r \leq 8 \alpha_1 \sqrt{s} r \cdot \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n \gamma_i \tilde{x}_i \cdot 1_{\{\|\beta^* \tilde{x}_i\| \leq \alpha_2\}} \right\|_{\max} \leq 8C_3 \alpha_1 r^2 \sqrt{\frac{s \log d}{n}}.$$

for some constant $C_3$. Combining the above inequality with (5.19), (5.20) and (5.21) yields that for any $t > 0$, with probability at least $1 - 2 \exp(-t)$,

$$\delta \tilde{\ell}_n(\beta; \beta^*) \geq \frac{m \kappa_0}{4} \|\Delta\|^2 - \alpha_2^2 r^2 \sqrt{\frac{8t}{n} - 8C_3 \alpha_1 r^2 \sqrt{\frac{s \log d}{n}}}. \quad \square$$

Proof of Theorem 3. According to Lemma 1 in [Negahban et al. (2012)], as long as $\lambda \geq 2 \|\nabla \tilde{\ell}_n(\beta)\|_{\max}$, $\tilde{\Delta} \in \mathcal{C}(S)$. We construct an intermediate estimator $\tilde{\beta}_\eta$ between $\tilde{\beta}$ and $\beta^*$:

$$\tilde{\beta}_\eta = \beta^* + \eta(\beta - \beta^*),$$

where $\eta = 1$ if $\|\tilde{\beta} - \beta^*\|_2 \leq r$ and $\eta = r/\|\tilde{\beta} - \beta^*\|_2$ if $\|\tilde{\beta} - \beta^*\|_2 > r$. Choose $\lambda = 2C \xi \sqrt{\log d \cdot n}$, where $C$ and $\xi$ are the same as in Lemma 2. By Lemmas 2 and 3 it holds with probability at least $1 - 2 \exp(-t)$,

$$\kappa \|\tilde{\Delta}_\eta\|_2^2 - C_0 r^2 \left( \frac{t}{n} + \sqrt{\frac{s \log d}{n}} \right) \leq \delta \tilde{\ell}_n(\beta^*; \beta^*) \leq -\nabla \tilde{\ell}_n(\beta^*)^T \tilde{\Delta}_\eta \leq \|\nabla \tilde{\ell}_n(\beta^*)\|_{\max} \cdot \|\tilde{\Delta}_\eta\|_1 \leq 4 \|\nabla \tilde{\ell}_n(\beta^*)\|_{\max} \cdot \|[\tilde{\Delta}_\eta]_S\|_1 \leq 4 \sqrt{s} \|\nabla \tilde{\ell}_n(\beta^*)\|_{\max} \cdot \|\tilde{\Delta}_\eta\|_2. \quad (5.22)$$
Some algebra delivers that

\[ \| \hat{\Delta}_n \|_2 \leq \frac{4\sqrt{s}\|\nabla \tilde{l}_n(\beta^*)\|_{\text{max}}}{\kappa} + r \sqrt{\frac{C_0}{\kappa}} \left( \sqrt{\frac{t}{n}} + \sqrt{\frac{s \log d}{n}} \right). \]  

(5.23)

Choose \( t = \xi \log d \) above. Let \( r \) be greater than the RHS of the inequality above. For sufficiently sufficiently small \( s \log d/n \), we have \( r \geq 5\sqrt{s} \| \nabla \tilde{l}_n(\beta^*) \|_{\text{max}}/\kappa \). Define \( r_0 := 5\sqrt{s} \| \nabla \tilde{l}_n(\beta^*) \|_{\text{max}}/\kappa \) and choose \( r = r_0 \). Therefore, \( \| \hat{\Delta}_n \|_2 \leq r \) and \( \hat{\Delta}_n = \hat{\Delta} \). By Lemma 2 we reach the conclusion.