Özer Talo, Feyzi Başar

CERTAIN SPACES OF SEQUENCES OF FUZZY NUMBERS DEFINED BY A MODULUS FUNCTION

Abstract. The main purpose of the present paper is to introduce the spaces $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ of sequences of fuzzy numbers defined by a modulus function. Furthermore, some inclusion theorems related to these sets are given and shown that $\ell_\infty(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ are solid.

1. Preliminaries, background and notation

Let $W$ be the set of all closed bounded intervals $A$ of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. The operations addition and multiplication by a real number on $W$ are defined, as follows:

$$A + B := \{a + b : a \in A, b \in B\} := [\underline{A} + B, \overline{A} + B],$$

$$\alpha A := \{\alpha a : a \in A\}.$$  

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$  

Define the relation $d$ on $W$ by

$$d(A, B) := \max\{|A - B|, |\overline{A} - \overline{B}|\}.$$  

Then it can easily be observed that $d$ is a metric on $W$ (cf. Diamond and Kloeden [3]) and $(W, d)$ is a complete metric space, (cf. Nanda [8]).

A fuzzy number is a fuzzy set on the real axis, i.e. a mapping $u : \mathbb{R} \to [0, 1]$ associating with each real number $t$ its grade of membership $u(t)$ which satisfies the following four conditions.

(i) $u$ is normal, i.e. there exists an $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$.

2000 Mathematics Subject Classification: Primary 46S40; Secondary 40A05.

Key words and phrases: sequence of fuzzy numbers, modulus function, solidity and completeness of a sequence space and inclusion relations.
(ii) $u$ is convex, i.e. $u(\lambda t_1 + (1-\lambda)t_2) \geq \min\{u(t_1), u(t_2)\}$ for all $t_1, t_2 \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) $u$ is upper semi-continuous.

(iv) The set $[u]_0 := \{t \in \mathbb{R} : u(t) > 0\}$ is compact, (cf. Zadeh [15]).

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $L(\mathbb{R})$ and called it as the space of fuzzy numbers. $\alpha$-level set $[u]_\alpha$ of $u \in L(\mathbb{R})$ is defined by

$[u]_\alpha := \begin{cases} 
\{ t \in \mathbb{R} : u(t) \geq \alpha \}, & (0 < \alpha \leq 1) \\
\{ t \in \mathbb{R} : u(t) > \alpha \}, & (\alpha = 0) 
\end{cases}$

which is a convex set in $\mathbb{R}$ if and only if $u$ is convex, where $\{ t \in \mathbb{R} : u(t) > \alpha \}$ denotes the closure of the set $\{ t \in \mathbb{R} : u(t) > \alpha \}$ in the usual topology of $\mathbb{R}$. The set $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0,1]$. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $r$ defined by $r(t) := \begin{cases} 
1, & (t = r) \\
0, & (t \neq r) 
\end{cases}$.

The operations addition and scalar multiplication are defined on $L(\mathbb{R})$ by

$[u + v]_\alpha := [u]_\alpha + [v]_\alpha$ and $[au]_\alpha := a[u]_\alpha$,

for each $0 \leq \alpha \leq 1$.

Define the map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$\overline{d}(u, v) := \sup_{0 \leq \alpha \leq 1} d([u]_\alpha, [v]_\alpha)$.

For $u, v \in L(\mathbb{R})$, define $u \leq v$ if and only if $[u]_\alpha \leq [v]_\alpha$ for any $\alpha \in [0,1]$. It is known that $(L(\mathbb{R}), \overline{d})$ is a complete metric space, (cf. [8]).

Following Matloka [5], we give some definitions below, which are needed in the text:

**Definition 1.1.** A sequence $x = (x_k)$ of fuzzy numbers is a function $x$ from the set $\mathbb{N} = \{0, 1, 2, \ldots \}$ into $L(\mathbb{R})$. The fuzzy number $x_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called the $k^{th}$ term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

**Definition 1.2.** A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number $l$, if for every $\varepsilon > 0$ there exists a positive integer $n_0$ such that

$\overline{d}(x_k, l) < \varepsilon$ for all $k > n_0$.

By $c(F)$ and $c_0(F)$, we denote the set of all convergent sequences and the set of all sequences converging to $\overline{0}$ of fuzzy numbers, respectively.
**Definition 1.3.** A sequence \( x = (x_k) \) of fuzzy numbers is said to be Cauchy if for every \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that

\[
\overline{d}(x_k, x_m) < \varepsilon \quad \text{for all} \quad k, m > n_0.
\]

By \( C(F) \), we denote the set of all Cauchy sequences of fuzzy numbers.

**Definition 1.4.** A sequence \( x = (x_k) \) of fuzzy numbers is said to be bounded if the set of fuzzy numbers consisting of the terms of the sequence \( (x_k) \) is a bounded set. That is to say that a sequence \( x = (x_k) \) of fuzzy numbers is said to be bounded if there exist two fuzzy numbers \( l \) and \( u \) such that \( l \leq x_n \leq u \) for any \( n \in \mathbb{N} \). By \( \ell_\infty(F) \), we denote the set of all bounded sequences of fuzzy numbers.

It is not hard to see that \( c_0(F) \subset c(F) \subset C(F) \subset \ell_\infty(F) \subset w(F) \).

**Definition 1.5.** Let \( x = (x_k) \) be a sequence of fuzzy numbers. Then the expression \( \sum_{k=0}^{\infty} x_k \) is called a series of fuzzy numbers. Denote \( s_n = \sum_{k=0}^{n} x_k \) for all \( n \in \mathbb{N} \), if the sequence \( (s_n) \) converges to a fuzzy number \( s \) then we say that the series \( \sum_{k=0}^{\infty} x_k \) of fuzzy numbers converges to \( s \) and write \( \sum_{k=0}^{\infty} x_k = s \). We say otherwise the series of fuzzy numbers is divergent.

The notion of modulus function was introduced by Nakano [7], as follows;

**Definition 1.6.** A function \( f : [0, \infty) \to [0, \infty) \) is called a modulus if the following conditions hold:

(a) \( f(x) = 0 \) if and only if \( x = 0 \).

(b) \( f(x + y) \leq f(x) + f(y) \) for all \( x, y \geq 0 \).

(c) \( f \) is increasing.

(d) \( f \) is continuous from the right at \( 0 \).

Hence, \( f \) is continuous on the interval \([0, \infty)\).

Now, we may give the concept of solidity of a space of sequences of fuzzy numbers defined by Sarma [11].

**Definition 1.7.** A set \( \lambda \) of sequences of fuzzy numbers is said to be solid (or normal) if \((y_k) \in \lambda \) whenever \( \overline{d}(y_k, 0) \leq \overline{d}(x_k, 0) \) for all \( k \in \mathbb{N} \), for some \((x_k) \in \lambda \).

Zadeh introduced the concept of fuzzy sets and define the fuzzy set operations, in his significant article [15]. Subsequently several authors discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Especially, El Naschie [9] studied the E infinity theory which has very important applications in quantum particle physics. In [8], it was shown that \( c(F) \) and \( \ell_\infty(F) \) are
complete metric spaces with the Haussdorff metric $D_\infty$ defined by

$$D_\infty(x, y) := \sup_{k \in \mathbb{N}} d(x_k, y_k),$$

where $x = (x_k), y = (y_k)$ are the elements of the spaces $c(F)$ or $\ell_\infty(F)$. Of course, $c_0(F)$ is also a complete metric space with respect to the Haussdorff metric $D_\infty$. Further, Nanda [8] has introduced and proved that the space $\ell_p(F)$ of all absolutely $p$-summable sequences of fuzzy numbers defined by

$$\ell_p(F) := \left\{ x = (x_k) \in w(F) : \sum_k \left[ d(x_k, \theta) \right]^p < \infty \right\}$$

is a complete metric space with the Haussdorff metric $D_p$ defined by

$$D_p(x, y) := \left\{ \sum_k \left[ d(x_k, y_k) \right]^p \right\}^{1/p},$$

where $x = (x_k), y = (y_k)$ are in the space $\ell_p(F)$ and $\theta := (\bar{0}, \bar{0}, \ldots, \bar{0}, \ldots)$. Nuray and Savaş [10] have recently shown that the space $\ell(p, F)$ of sequences of fuzzy numbers

$$\ell(p, F) := \left\{ x = (x_k) \in w(F) : \sum_k \left[ d(x_k, \theta) \right]^{p_k} < \infty \right\}$$

is a complete metric space with the metric $\varrho$ defined by

$$\varrho(x, y) := \left\{ \sum_k \left[ d(x_k, y_k) \right]^{p_k} \right\}^{1/M},$$

where $p = (p_k)$ is a bounded sequence of strictly positive real numbers and $M = \max\{1, \sup_{k \in \mathbb{N}} p_k\}$, and $x = (x_k), y = (y_k)$ are the points of the space $\ell(p, F)$. Mursaleen and Başarır [6] have introduced some new spaces of sequences of fuzzy numbers generated by a non-negative regular matrix $A$ some of which reduced to the Maddox spaces $\ell_\infty(p, F), c(p, F), c_0(p, F)$ and $\ell(p, F)$ of sequences of fuzzy numbers for the special cases of that matrix $A$. Altın, Et and Çolak [1] have recently introduced the concepts of lacunary statistical convergence and lacunary strongly convergence of generalized difference sequences of fuzzy numbers, and gave some relations related to these concepts. Talo and Başar [12] have extended the main results of Başar and Altay [2] to the fuzzy numbers.

In [13], Talo and Başar have recently studied the corresponding sets $\ell_\infty(F), c(F), c_0(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers to the classical spaces $\ell_\infty, c, c_0$ and $\ell_p$ of sequences with real or complex terms. After determining the $\alpha$, $\beta$- and $\gamma$-duals of the sets $\ell_\infty(F), c(F), c_0(F)$ and $\ell_p(F)$, they characterize some classes of matrix transformations between the
Spaces of sequences of fuzzy numbers

2. The spaces $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ of sequences of fuzzy numbers defined by a modulus function

Let $f$ be a modulus function. We introduce the sets $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ of sequences of fuzzy numbers defined by a modulus function by

$$\ell_\infty(F, f) := \left\{ x = (x_k) \in w(F) : \sup_{k \in \mathbb{N}} f[\overline{d}(x_k, 0)] < \infty \right\},$$

$$c(F, f) := \left\{ x = (x_k) \in w(F) : \lim_{k \to \infty} f[\overline{d}(x_k, l)] = 0 \text{ for some } l \in L(\mathbb{R}) \right\},$$

$$c_0(F, f) := \left\{ x = (x_k) \in w(F) : \lim_{k \to \infty} f[\overline{d}(x_k, 0)] = 0 \right\}$$

and

$$\ell_p(F, f, s) := \left\{ x = (x_k) \in w(F) : \sum_k \frac{\{f[\overline{d}(x_k, 0)]\}^p}{k^s} < \infty \right\}, \quad (s \geq 0).$$

Now, we may begin with the following theorem which is essential in the text:

**Theorem 2.1.** The sets $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ of sequences of fuzzy numbers defined by a modulus function are closed under the coordinatewise addition and scalar multiplication.

**Proof.** Since it is not hard to show that the sets $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ are closed with respect to the coordinatewise addition and scalar multiplication, we omit the detail. \[\square\]
**Theorem 2.2.** The sets $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ of sequences of fuzzy numbers defined by a modulus function are complete metric spaces with respect to the metrics $D_\infty$ and $D_p$ defined by

$$D_\infty(x, y) := \sup_{k \in \mathbb{N}} f[d(x_k, y_k)],$$

and

$$D_p(x, y) := \left\{ \sum_k \frac{f[d(x_k, y_k)]^p}{k^s} \right\}^{1/p},$$

respectively; where $x = (x_k)$, $y = (y_k)$ are the elements of the sets $\ell_\infty(F, f)$, $c(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$.

**Proof.** Since the proof is analogue for the spaces $\ell_\infty(F, f)$, $c(F, f)$ and $\ell_p(F, f, s)$, we consider only the space $c_0(F, f)$. One can easily establish that $D_\infty$ defines a metric on $c_0(F, f)$ which is a routine verification, so we leave it to the reader.

It remains to prove the completeness of the space $c_0(F, f)$. Let $\{x^i\}$ be any Cauchy sequence in the space $c_0(F, f)$, where $x^i := \{x_{0}^{(i)}, x_{1}^{(i)}, x_{2}^{(i)}, \ldots\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$D_\infty(x^i, x^j) = \sup_{k \in \mathbb{N}} f\left[ d\left( x_k^{(i)}, x_k^{(j)} \right) \right] < \varepsilon$$

for all $i, j \geq n_0(\varepsilon)$. We obtain for each fixed $k \in \mathbb{N}$ from (2.1) that

$$f\left[ d\left( x_k^{(i)}, x_k^{(j)} \right) \right] < \varepsilon$$

for every $i, j \geq n_0(\varepsilon)$. (2.2) means that

$$\lim_{i,j \to \infty} f\left[ d\left( x_k^{(i)}, x_k^{(j)} \right) \right] = 0.$$  

Since $f$ is continuous we have from (2.3) that

$$f\left[ \lim_{i,j \to \infty} d\left( x_k^{(i)}, x_k^{(j)} \right) \right] = 0.$$  

Therefore, since $f$ is a modulus function one can derive by (2.4) that

$$\lim_{i,j \to \infty} d\left( x_k^{(i)}, x_k^{(j)} \right) = 0$$

which means that $\{x_k^{(i)}\}$ is a Cauchy sequence in $L(\mathbb{R})$ for every fixed $k \in \mathbb{N}$. Since $L(\mathbb{R})$ is complete, it converges, say $x_k^{(i)} \to x_k$ as $i \to \infty$. Using these infinitely many limits, we define the sequence $x = (x_0, x_1, x_2, \ldots)$. Let us pass to limit firstly as $j \to \infty$ and nextly taking supremum over $k \in \mathbb{N}$ in (2.2) to obtain
\[ \overline{D}_\infty(x^i, x) \leq \frac{\varepsilon}{2}. \]

Since \( x^i = \{ x_k^{(i)} \} \in c_0(F, f) \) for each \( i \in \mathbb{N} \), there exists \( k_0(\varepsilon) \in \mathbb{N} \) such that
\[ f \left[ \overline{d} \left( x_k^{(i)}, 0 \right) \right] \leq \frac{\varepsilon}{2} \]
for every \( k \geq k_0(\varepsilon) \) and for each fixed \( i \in \mathbb{N} \). Therefore, since
\[ f \left[ \overline{d}(x_k, 0) \right] < f \left[ \overline{d} \left( x_k^{(i)}, x_k \right) \right] + f \left[ \overline{d} \left( x_k^{(i)}, 0 \right) \right] \]
holds by triangle inequality for all \( i, k \in \mathbb{N} \) for fixed \( i \geq n_0(\varepsilon) \) we have
\[ f \left[ \overline{d}(x_k, 0) \right] \leq \varepsilon \text{ for all } k \geq k_0(\varepsilon). \]

This shows that \( x \in c_0(F, f) \). Since \( \{ x^i \} \) was an arbitrary Cauchy sequence, the space \( c_0(F, f) \) is complete.

This step concludes the proof. \( \blacksquare \)

**Theorem 2.3.** Let \( \lambda(F, f) \) denotes the anyone of the spaces \( \ell_\infty(F, f) \), \( c(F, f) \), \( c_0(F, f) \) and \( f_1, f_2 \) be two modulus functions. Then, the following inclusion relations hold:

(a) \( \lambda(F, f_1) \cap \lambda(F, f_2) \subseteq \lambda(F, f_1 + f_2) \).
(b) \( \lambda(F, f_1) \subseteq \lambda(F, f_2 \circ f_1) \).
(c) If \( f_1(t) \leq f_2(t) \) for all \( t \in [0, \infty) \), \( \lambda(F, f_2) \subseteq \lambda(F, f_1) \).

**Proof.** We give the proof for \( \lambda = c_0 \). Since the proof can also be given for \( \lambda = c \) or \( \lambda = \ell_\infty \), we leave the detail to the reader.

(a) Let \( x = (x_k) \in \lambda(F, f_1) \cap \lambda(F, f_2) \). Since
\[ (f_1 + f_2)[\overline{d}(x_k, 0)] = f_1[\overline{d}(x_k, 0)] + f_2[\overline{d}(x_k, 0)], \]
one can see by passing to limit as \( k \to \infty \) and taking supremum over \( k \in \mathbb{N} \) from (2.6) that \( x \in \lambda(F, f_1 + f_2) \), where \( \lambda \in \{ \ell_\infty, c, c_0 \} \).

(b) Let \( x = (x_k) \in \lambda(F, f_1) \). Since \( f_2 \) is continuous, there exists an \( \eta > 0 \) such that \( f_2(\eta) = \varepsilon \) for all \( \varepsilon > 0 \). Since \( x = (x_k) \in \lambda(F, f_1) \) there exists an \( k_0 \in \mathbb{N} \) such that \( f_1[\overline{d}(x_k, 0)] \leq \eta \) for all \( k \geq k_0 \). Therefore, one can derive by applying \( f_2 \) that \( f_2 \{ f_1[\overline{d}(x_k, 0)] \} < f_2(\eta) = \varepsilon \), i.e. \( x \in \lambda(F, f_2 \circ f_1) \).

(c) Since \( f_1(t) \leq f_2(t) \) for all \( t \in [0, \infty) \), we have \( f_1[\overline{d}(x_k, 0)] \leq f_2[\overline{d}(x_k, 0)] \). This leads us to the consequence that \( x \in \lambda(F, f_2) \) which implies that \( x \in \lambda(F, f_1) \), as expected. \( \blacksquare \)

**Lemma 2.4.** (cf. [4]) Let \( f_1 \) and \( f_2 \) be two modulus functions, and \( 0 < \delta < 1 \). If \( f_1(t) > \delta \), then
\[ (f_2 \circ f_1)(t) \leq \frac{2f_2(1)}{\delta} f_1(t) \]
holds for all \( t \in [0, \infty) \).
**Theorem 2.5.** Let $f_1$, $f_2$ be two modulus functions. Then, the following inclusion relations hold:

(a) If $s > 1$, \( \ell_p(F, f_1, s) \cap \ell_p(F, f_2, s) \subseteq \ell_p(F, f_1 + f_2, s) \).

(b) If $s > 1$, \( \ell_p(F, f_1, s) \subseteq \ell_p(F, f_2 \circ f_1, s) \).

(c) If \( \limsup_{t \to \infty} |f_1(t)/f_2(t)| < \infty \), \( \ell_p(F, f_2, s) \subseteq \ell_p(F, f_1, s) \).

(d) If $s_1 < s_2$, \( \ell_p(F, f_1, s_1) \subseteq \ell_p(F, f_1, s_2) \).

**Proof.**

(a) \( \{(f_1 + f_2)[\bar{d}(x_k, \bar{0})]\}^p \leq 2^{p-1} \{[f_1[\bar{d}(x_k, \bar{0})]]^p + [f_2[\bar{d}(x_k, \bar{0})]]^p\} \) holds which yields us by taking summation over \( k \in \mathbb{N} \) that \( x \in \ell_p(F, f_1 + f_2, s) \), as desired.

(b) Since \( f_2 \) is continuous from the right at 0, there is \( \delta \) with \( 0 < \delta < 1 \) such that \( f_2(t) < \varepsilon \) for all \( \varepsilon > 0 \) whenever \( 0 \leq t \leq \delta \). Define the sets \( N_1 \) and \( N_2 \) by

\[
N_1 = \{ k \in N : f_1[\bar{d}(x_k, \bar{0})] \leq \delta \},
\]

\[
N_2 = \{ k \in N : f_1[\bar{d}(x_k, \bar{0})] > \delta \}.
\]

Then, we obtain from Lemma 2.4 for \( f_1[\bar{d}(x_k, \bar{0})] > \delta \) that

\[
(f_2 \circ f_1)[\bar{d}(x_k, \bar{0})] \leq \frac{2f_2(1)}{\delta} f_1[\bar{d}(x_k, \bar{0})].
\]

Therefore, we derive for \( x = (x_k) \in \ell_p(F, f_1, s) \) with \( s > 1 \) that

\[
\sum_k \frac{\{(f_2 \circ f_1)[\bar{d}(x_k, \bar{0})]\}^p}{k^s} = \sum_{k \in N_1} \frac{\{(f_2 \circ f_1)[\bar{d}(x_k, \bar{0})]\}^p}{k^s} + \sum_{k \in N_2} \frac{\{(f_2 \circ f_1)[\bar{d}(x_k, \bar{0})]\}^p}{k^s} \leq \sum_{k \in N_1} \frac{\varepsilon^p}{k^s} + \sum_{k \in N_2} \frac{\left(\frac{2f_2(1)}{\delta} f_1[\bar{d}(x_k, \bar{0})]\right)^p}{k^s} \leq \varepsilon^p \sum_{k \in N_1} \frac{1}{k^s} + \left[\frac{2f_2(1)}{\delta}\right]^p \sum_{k \in N_2} \frac{\{f_2[\bar{d}(x_k, \bar{0})]\}^p}{k^s} < \infty.
\]

Hence, \( x = (x_k) \in \ell_p(F, f_2 \circ f_1, s) \).

(c) Suppose that \( \limsup_{t \to \infty} |f_1(t)/f_2(t)| < \infty \). Then, there is a number \( M > 0 \) such that \( |f_1(t)/f_2(t)| \leq M \) for all \( t \in (0, \infty) \). Since \( \bar{d}(x_k, \bar{0}) \geq 0 \) for all \( k \in \mathbb{N} \) and for all \( x = (x_k) \in \ell_p(F, f_2, s) \), we have \( f_1[\bar{d}(x_k, \bar{0})] \leq M f_2[\bar{d}(x_k, \bar{0})] \) which leads us

\[
\sum_k \frac{\{f_1[\bar{d}(x_k, \bar{0})]\}^p}{k^s} \leq \sum_k \frac{\{M f_2[\bar{d}(x_k, \bar{0})]\}^p}{k^s} = M^p \sum_k \frac{\{f_2[\bar{d}(x_k, \bar{0})]\}^p}{k^s} < \infty.
\]

Thus \( x = (x_k) \in \ell_p(F, f_1, s) \), as desired.
(d) Let $s_1 \leq s_2$. Since $0 < k^{-1} \leq 1$ for all $k \in \mathbb{N}$, it is immediate that $k^{-s_2} \leq k^{-s_1}$. Then, one can see that
\[
\sum_k \frac{1}{k^{s_2}} \{f_1[d(x_k, 0)]\}^p \leq \sum_k \frac{1}{k^{s_1}} \{f_1[d(x_k, 0)]\}^p < \infty
\]
holds, for all $x = (x_k) \in \ell_p(F, f_1, s_1)$. This means that $x = (x_k) \in \ell_p(F, f_1, s_2)$, which completes the proof.

**Corollary 2.6.** Define the sets $\ell_p(F, s)$ and $\ell_p(F, f)$ by
\[
\ell_p(F, s) := \left\{ x = (x_k) \in w(F) : \sum_k \frac{1}{k^s} [d(x_k, 0)]^p < \infty \right\}; \quad (s \geq 0),
\]
\[
\ell_p(F, f) := \left\{ x = (x_k) \in w(F) : \sum_k \{f[d(x_k, 0)]\}^p < \infty \right\}.
\]

Then, we have
(a) If $s > 1$, $\ell_p(F, s) \subseteq \ell_p(F, f, s)$.
(b) $\ell_p(F, f) \subseteq \ell_p(F, f, s)$.

**Proof.** (a) This follows from the Theorem 2.5 (b) with $f_1(t) = t$ and $f_2 = f$.

(b) This is immediate by taking $s_1 = 0$, $s_2 = s$ and $f_1 = f$, from the Theorem 2.5 (d).

Now, we can give the next theorem.

**Theorem 2.7.** Let $s > 1$. Then, the following relations hold:
(a) $\ell_\infty(F) \subseteq \ell_p(F, f, s)$.
(b) If $f$ is bounded, then $\ell_p(F, f, s) = w(F)$.

**Proof.** (a) Let $x = (x_k) \in \ell_\infty(F)$. Then, there is a number $M > 0$ such that $d(x_k, 0) \leq M$ for all $k \in \mathbb{N}$. Since $f$ is continuous and increasing, there is a number $N > 0$ such that $f[d(x_k, 0)] \leq f(M) \leq N$. Therefore, we get for $s > 1$ that
\[
\sum_k \frac{\{f[d(x_k, 0)]\}^p}{k^s} \leq N^p \sum_k \frac{1}{k^s} < \infty.
\]
Hence, $x = (x_k) \in \ell_p(F, f, s)$.

(b) Suppose that $f$ is bounded. Then, one can find a number $N > 0$ such that $f(t) \leq N$ for all $t \in [0, \infty)$. Thus, for $x = (x_k) \in w(F)$ we have
\[
\sum_k \frac{\{f[d(x_k, 0)]\}^p}{k^s} \leq N^p \sum_k \frac{1}{k^s} < \infty
\]
which says that the inclusion $w(F) \subseteq \ell_p(F, f, s)$ holds. Since the inclusion $\ell_p(F, f, s) \subseteq w(F)$ also holds we conclude that $w(F) = \ell_p(F, f, s)$, as desired.
Prior to giving the final theorem of the paper on the solidity of the spaces $\ell_\infty(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$, we state and prove the lemma concerning the solidity of the spaces $\ell_\infty(F)$, $c_0(F)$ and $\ell_p(F)$.

**Lemma 2.8.** The spaces $\ell_\infty(F)$, $c_0(F)$ and $\ell_p(F)$ are solid.

**Proof.** Let $\lambda(F)$ denotes the anyone of the spaces $\ell_\infty(F)$, $c_0(F)$ and $\ell_p(F)$. Suppose that

$$d(y_k, 0) \leq d(x_k, 0) \quad (2.9)$$

holds, for some $x = (x_k) \in \lambda(F)$. Therefore, one can easily see by (2.9) that

$$\begin{align*}
\sup_{k \in \mathbb{N}} d(y_k, 0) &\leq \sup_{k \in \mathbb{N}} d(x_k, 0) < \infty \\
\lim_{k \to \infty} d(y_k, 0) &\leq \lim_{k \to \infty} d(x_k, 0) = 0 \\
\sum_{k} [d(y_k, 0)]^p &\leq \sum_{k} [d(x_k, 0)]^p < \infty.
\end{align*}$$

The above inequalities yield the desired consequence that $y = (y_k) \in \lambda(F)$.

This completes the proof of the lemma.

**Theorem 2.9.** The spaces $\ell_\infty(F, f)$, $c_0(F, f)$ and $\ell_p(F, f, s)$ are solid.

**Proof.** This is immediate by Lemma 2.8, since the modulus function $f$ is increasing.

**Acknowledgement.** We would like to thank the anonymous reviewer for his/her careful reading and making some useful comments on earlier version of this paper which improved the presentation and its readability.

**References**

[1] Y. Altın, M. Et, R. Çolak, *Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers*, Comput. Math. Appl. 52 (2006), 1011–1020.

[2] F. Başar, B. Altay, *On the space of sequences of $p$-bounded variation and related matrix mappings*, Ukrainian Math. J. 55 (1) (2003), 136–147.

[3] P. Diamond, P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets Syst. 35 (1990), 241–249.

[4] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc. 100 (1986), 161–166.

[5] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL 28 (1986), 28–37.

[6] M. Mursaleen, M. Başarir, *On some new sequence spaces of fuzzy numbers*, Indian J. Pure Appl. Math. 34 (9) (2003), 1351–1357.

[7] H. Nakano, *Concave modulars*, J. Math. Soc. Japan 5 (1953), 29–49.

[8] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets Syst. 33 (1989), 123–126.

[9] M. S. El Naschie, *A review of $\varepsilon^{(\infty)}$ theory and the mass spectrum of high energy particle physics*, Chaos, Solitons & Fractals 19 (1) (2004), 209–236.
[10] F. Nuray, E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca 45 (3) (1995), 269–273.

[11] B. Sarma, *On a class of sequences of fuzzy numbers defined by modulus function*, Internat. J. Sci. Technol. 2 (1) (2007), 25–28.

[12] Ö. Talo, F. Başar, *On the space $b_{v_p}(F)$ of sequences of $p$-bounded variation of fuzzy numbers*, Acta Math. Sin. Eng. 24 (7) (2008), 1205–1212.

[13] Ö. Talo, F. Başar, *Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations*, Comput. Math. Appl. 58 (2009), 717–733.

[14] Ö. Talo, F. Başar, *Quasilinearity of the classical sets of sequences of the fuzzy numbers and some applications*, Taiwanese J. Math. 13 (2009), to appear.

[15] L. A. Zadeh, *Fuzzy sets*, Inform. & Control 8 (1965), 338–353.

Ö. Talo
ANADOLU LİSESI MATEMATİK ÖĞRETMENİ
44170-MALATYA, TÜRKİYE
E-mail: ozertalo@hotmail.com

F. Başar
FATİH ÜNİVERSİTESİ, FEN-EDEBIYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ
BÜYÜKÇEKMECE KAMPÜSÜ
34500-İSTANBUL, TÜRKİYE
E-mail: fbasar@fatih.edu.tr, feyzibasar@gmail.com

Received March 2, 2007; revised version November 19, 2009.