Optimal systems of subalgebras for a nonlinear Black-Scholes equation

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Abstract

The main object of our study is a four dimensional Lie algebra which describes the symmetry properties of a nonlinear Black-Scholes model. This model implements a feedback effect which is typical for an illiquid market. The structure of the Lie algebra depends on one parameter, i.e. we have to do with a one-parametric family of algebras. We provide a classification of these algebras using Patera-Winternitz method. Optimal systems of one-, two- and three- dimensional subalgebras are described for the family of symmetry algebras of the nonlinear Black-Scholes equation. The optimal systems give us the possibility to describe a complete set of invariant solutions to the equation.

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1 Introduction

In [2] Frey and Patie examined the feedback effect of the option replication strategy of the large trader on the asset price process. They obtained a new model by the introduction of a liquidity coefficient which depends on the current stock price. The feedback-effect described leads to a nonlinear version of the Black-Scholes partial differential equation,

\[ u_t + \frac{\sigma^2 S^2}{2} \frac{u_{SS}}{(1 - \rho S \lambda(S) u_{SS})^2} = 0, \]

with \( S \in [0, \infty), \quad t \in [0, T] \). As usual, \( S \) denotes the price of the underlying asset and \( u(S, t) \) denotes the hedge-cost of the claim with a payoff \( h(S) \) which will be defined later. The hedge-cost is different from the price of the derivatives product in illiquid markets. In the sequel \( t \) is the time variable, \( \sigma \) defines the volatility of the underlying asset. The Lie group analysis of the equation (1) was provided in [1]. By using this method the Lie point symmetries, the Lie symmetry algebras and groups to the corresponding equations were found; see for details [1] and [4, 6, 3] for a general introduction to the methodology.
Theorem 1.1 (Bordag, [1]). The differential equation (1) with an arbitrary function \(\lambda(S)\) possesses a trivial three dimensional Lie algebra \(\text{Diff}_\Delta(M)\) spanned by infinitesimal generators

\[
v_1 = \frac{\partial}{\partial t}, \quad v_2 = S \frac{\partial}{\partial u}, \quad v_3 = \frac{\partial}{\partial u}.
\]

Only for the special form of the function \(\lambda(S) \equiv \omega S^k\), where \(\omega, k \in \mathbb{R}\) equation (1) admits a non-trivial four dimensional Lie algebra \(L\) spanned by generators

\[
v_1 = \frac{\partial}{\partial t}, \quad v_2 = S \frac{\partial}{\partial u}, \quad v_3 = \frac{\partial}{\partial u}, \quad v_4 = S \frac{\partial}{\partial S} + (1 - k)u \frac{\partial}{\partial u}
\]

with commutator relations

\[
[v_1, v_2] = [v_1, v_3] = [v_1, v_4] = [v_2, v_3] = 0, \\
[v_2, v_4] = -kv_2, \quad [v_3, v_4] = (1 - k)v_3.
\]

In the paper [9] authors try unsuccesses to construct an optimal system of subalgebras for the symmetry algebra (2). The authors used the method suggested in the series of well known papers by P. Winternitz and Co [7, 8] where all three and four-dimensional Lie algebras where classified.

The investigation in [9] contains some misprints and mistakes which demand corrigendum. In that paper the structure of the optimal system of subalgebras [9] does not contain some one-dimensional algebras. On the other sides the classification does not depend on the parameter \(k\) from which the structure of the algebra (2) deeply depends. To be able to construct correct families of invariant solutions we need a correct optimal system of subalgebras.

In our paper we present the correct optimal system of one-, two-, three- dimensional systems of subalgebras.

2 Classification of the algebra \(L\)

Let us consider the four-dimensional Lie algebra \(L\) (2) with the commutator relations (3). To classify this algebra we use the classification method which was introduce by J. Patera and P. Winternitz in [7]. Further we use the notations of this paper.

As we noticed before the structure of the algebra \(L\) depends on the one real-valued parameter \(k\). As it was remarked in [1] the algebra posses a two-dimensional center by \(k = 0\) and \(k = 1\).

Case \(k = 0\). In this case the generators of the algebra \(L\) take the form

\[
v_1 = \frac{\partial}{\partial t}, \quad v_2 = x \frac{\partial}{\partial u}, \quad v_3 = \frac{\partial}{\partial u}, \quad v_4 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.
\]
with the following commutator Table 1

|   | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $v_1$ | 0 | 0 | 0 | 0 |
| $v_2$ | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 0 | $v_3$ |
| $v_4$ | 0 | 0 | $-v_3$ | 0 |

Table 1: The commutator table of the algebra $L$ in case $k = 0$

Let us introduce an algebra $L'_{4,1}$ with operators

$$e_1 = -v_4, \ e_2 = v_3, \ e_3 = v_1, \ e_4 = v_2.$$  \hspace{1cm} (5)

Then $L'_{4,1} = A_2 \oplus 2A_1$ with one nontrivial commutator relation $[e_1, e_2] = e_2$.

**Case $k = 1$.** This case leads to the algebra $L_{4,2}$ with generators

$$v_1 = \frac{\partial}{\partial t}, \ v_2 = x \frac{\partial}{\partial u}, \ v_3 = \frac{\partial}{\partial u}, \ v_4 = x \frac{\partial}{\partial x}.$$  \hspace{1cm} (6)

which is isomorphic to the algebra $L_{4,2}$ spanned on generators

$$e_1 = v_4, \ e_2 = v_2, \ e_3 = v_1, \ e_4 = v_3.$$  \hspace{1cm} (7)

In this case algebra $L_{4,2}$ has the decomposition $A_2 \oplus 2A_1$ with one nontrivial commutator relations $[e_1, e_2] = e_2$.

**Case $k > \frac{1}{2}, k \neq 1$.** In this case the algebra $L$ is isomorphic to the algebra $L'_{4,3} = A_{3,5}^\alpha \oplus A_1$ with the commutator relations $[e_1, e_3] = e_1, \ [e_2, e_3] = \alpha e_2$, where

$$e_1 = v_2, \ e_2 = v_3, \ e_3 = -\frac{1}{k}v_4, \ e_4 = v_1$$  \hspace{1cm} (8)

and $\alpha = \frac{k-1}{k}$.

**Case $k < \frac{1}{2}, k \neq 0$.** This case leads us to the algebra $L$ which is isomorphic to $L_{4,4} = A_{3,5}^\alpha \oplus A_1$ with the commutator relations $[e_1, e_3] = e_1, \ [e_2, e_3] = \alpha e_2$, where

$$e_1 = v_3, \ e_2 = v_2, \ e_3 = \frac{1}{1-k}v_4, \ e_4 = v_1$$  \hspace{1cm} (9)

and $\alpha = \frac{k}{k-1}$.
Case $k = \frac{1}{2}$. In the last case the algebra $L$ has generators of the following type

$$v_1 = \frac{\partial}{\partial t}, \ v_2 = x \frac{\partial}{\partial u}, \ v_3 = x \frac{\partial}{\partial x}, \ v_4 = x \frac{\partial}{\partial x} + \frac{1}{2} u \frac{\partial}{\partial u},$$  \hspace{1cm} (10)

and the commutator Table 2

| $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|-------|-------|-------|-------|
| $v_1$ | 0     | 0     | 0     |
| $v_2$ | 0     | 0     | 0     | $-\frac{1}{2}v_2$ |
| $v_3$ | 0     | 0     | 0     | $\frac{1}{2}v_3$ |
| $v_4$ | 0     | $\frac{1}{2}v_2$ | $-\frac{1}{2}v_3$ | 0 |

Table 2: The commutator table of the algebra $L$ in case $k = \frac{1}{2}$ and is isomorphic to the algebra $L'_{4,5} = A_{3,4} \oplus A_1$ where generators are denoted by

$$e_1 = v_3, \ e_2 = v_2, \ e_3 = 2v_4, \ e_4 = v_1.$$  \hspace{1cm} (11)

The nontrivial commutator relations are $[e_1, e_3] = e_1, \ [e_2, e_3] = -e_2$.

3 An optimal system of subalgebras

The main goal of this paper is to find a correct optimal system of subalgebras for the Lie algebra $L$ [2]. The procedure was described by Pattera & Winternitz in [7]. In the paper all three and four-dimensional algebras were classified and the optimal systems for these algebras were listed. We repeat this algorithm for the algebra $L$ first in general case where $k \neq 0, \frac{1}{2}, 1$. In those two cases (8), (9) the Lie algebra $L$ is isomorphic to the algebra $A_{3,5}^\alpha \oplus A_1$ with following commutator Table 3

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|-------|-------|-------|-------|
| $e_1$ | 0     | 0     | $e_1$ | 0     |
| $e_2$ | 0     | 0     | $\alpha e_2$ | 0     |
| $e_3$ | $-e_1$ | $-\alpha e_2$ | 0     | 0     |
| $e_4$ | 0     | 0     | 0     | 0     |

Table 3: The commutator table of the algebra $L$ where $0 < |\alpha| < 1$.

In the general case $L$ has one central element and can be represent as a direct sum of one- and three-dimensional Lie algebras

$$L = \{e_4\} \oplus L_3.$$  \hspace{1cm} (12)
where $e_4$ is the central element of the algebra $L$ and $L_3 = L \setminus \{e_4\}$.

The representation (12) simplify the procedure of construction of an optimal system of subalgebras.

We start with construction of the corresponding system of subalgebras for both algebras in (12) and then complete the study with the investigation of non-splitting extensions. We follow the paper [7] and describe a solution of this problem in a step-by-step method introduced by the authors.

Step 1. We find all subalgebras of $\{e_4\}$, it means we have $\{0\}$ and $\{e_4\}$.

Step 2. We have to classify all subalgebras of $L_3$ (12) under conjugation which is defined by an interior isomorphism of this algebra. This isomorphism can be presented by the adjoint representation.

**Definition 3.1** (Olver, [6]). Let $G$ be a Lie group. For each $g \in G$, group conjugation $K_g(h) = ghg^{-1}$, $h \in G$, determines a diffeomorphism on $G$. Moreover, $K_g \circ K_g' = K_{gg'}$, $K_e = 1_G$, so $K_g$ determines a global group action of $G$ on itself, with each conjugacy map $K_g$ being a group homomorphism: $K_g(hh') = K_g(h)K_g(h')$ etc. The differential $dK_g : TG|_h \rightarrow *TG|_{K_g(h)}$ is readily seen to preserve the right-invariance of vector fields, and hence determines a linear map on the Lie algebra of $G$, called the adjoint representation:

$$Ad\ g(v) = dK_g(v)$$

The simplest way to find the adjoint representation is the Lie series

$$Ad(\exp(\varepsilon v))w = w - \varepsilon[v, w] + \frac{\varepsilon^2}{2!}[v, [v, w]] - \ldots$$

The adjoint representation table for the algebra $L$ (12) is rather simple and has a form given in Table 4

| $Ad$   | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|--------|-------|-------|-------|-------|
| $e_1$  | $e_1$ | $e_2$ | $e_3 - \varepsilon e_1$ | $e_4$ |
| $e_2$  | $e_1$ | $e_2$ | $e_3 - \alpha \varepsilon e_2$ | $e_4$ |
| $e_3$  | $e^\varepsilon e_1$ | $e^{\alpha \varepsilon} e_2$ | $e_3$ | $e_4$ |
| $e_4$  | $e_1$ | $e_2$ | $e_3$ | $e_4$ |

Table 4: The adjoint representation table of the algebra $L = A_{3.5}^0 \oplus A_1$ with $(i, j)$-th entry indicate $Ad(\exp(\varepsilon e_i)) e_j$ element.

By using the adjoint representation (14) we classify all subalgebras of $L_3$ (12) under conjugacy.
One-dimensional subalgebras. Firstly we consider one-dimensional subalgebras of the general type
\[ A = \{ ae_1 + be_2 + ce_3 \}, \]
where \( a, b, c \) are arbitrary constants. If \( c \neq 0 \) then we can use the first and the second lines of the Table 4. We obtain
\[ Ad(\exp(\xi e_1 + \zeta e_2))A = (a - c\xi)e_1 + (b - ca\zeta)e_2 + ce_3, \]
choosing \( \xi = \frac{a}{c}, \zeta = \frac{b}{ca} \) we prove that \( A \) is isomorphic to \( e_3 \).

If \( c = 0 \) in (15) we have three cases to study. If \( a \neq 0, b = 0 \) then \( A \) is isomorphic to \( e_1 \). If \( a = 0, b \neq 0 \) then \( A \) is isomorphic to \( e_2 \). The last case we obtain if \( ab \neq 0 \) then we can use the third line of the Table 4 and obtain after an action of the adjoint representation
\[ Ad(\exp(\xi e_3))A = ae^\xi e_1 + be^{a\xi}e_2. \]

Using the scaling on \( ae^\xi \) and choosing \( \xi = \frac{1}{a} \log \left| \frac{a}{b} \right| \) we prove that \( A \) is isomorphic to the algebra generated by \( \{ e_1 \pm e_2 \} \).

Collecting all previous results we obtain that the optimal system of one-dimensional subalgebras of \( L_3 \) contains following subalgebras
\[ \{0\}, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1 \pm e_2\}. \]

Two-dimensional subalgebras. Let us consider now two-dimensional subalgebras of \( L_3 \). Let \( B \) be one of the one-dimensional subalgebras (17) and \( A = \{ ae_1 + be_2 + ce_3 \} \). For a subalgebra \( M = B + A \) we demand that \( [A, B] \subset M \).

Let \( B = \{ e_1 \} \) then without loss of generality we can represent \( A \) in the form \( \{ ae_2 + be_3 \} \). Let \( b \neq 0 \), by using of the second line of the adjoint representation Table 4 we prove that \( A \) is isomorphic to \( \{ e_3 \} \). If \( b = 0 \) then \( A = \{ e_2 \} \). In this case we obtain two subalgebras
\[ \{e_1, e_2\}, \{e_1, e_3\}, \]
which are non-isomorphic to each other. In the same way we obtain subalgebras \( \{ e_1, e_2 \}, \{ e_2, e_3 \} \) in case \( B = e_2 \).

Let \( B = \{ e_3 \} \) then without loss of generality we can represent \( A = \{ ae_1 + be_2 \} \). Let us check the commutator relations
\[ [ae_1 + be_2, e_3] = ae_1 + abe_2. \]

We see that \( A + B \) is an algebra just under condition \( ab = 0 \). On this way we obtain the two subalgebras
\[ \{e_1, e_3\}, \{e_2, e_3\} \]
In case \( B = \{ e_1 \pm e_2 \} \) we choose \( A = \{ e_3 \} \). Then \( [e_1 \pm e_2, e_3] = e_1 \pm \alpha e_2 \) is not an algebra.
The optimal system of the one- and two-dimensional subalgebras of $L_3$ contains following subalgebras

$$\{0\}, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1 \pm e_2\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}. \quad (20)$$

Step 3. We have to find all splitting extensions of the algebra $\{e_4\}$. To do this we have to find all subalgebras $N_a$ of $L_3$ such that

$$[e_4, N_a] \subseteq N_a \quad (21)$$

and classify all such subalgebras under $\text{Nor}_L e_4$.

**Definition 3.2** (Ovsyannikov, [5]). Let $N$ be a subalgebra of the Lie algebra $L$. By the normalizer $\text{Nor}_L N$ of $N$ in $L$ we mean the maximal subalgebra of $L$ containing $N$ in which $N$ is an ideal, i.e.

$$\text{Nor}_L N = \{y \in L : [y, x] \in N \, \forall x \in N\}. \quad (22)$$

As soon as $e_4$ is a central element and $\text{Nor}_L e_4 = L$ any adjoint representation does not affect on $e_4$ and $N_a$ is any subalgebra of $L_3$. This step is trivial and we obtain the subalgebras of the type $\{e_4, S\}$ where $S$ running through all subalgebras (20).

Step 4. We have to find all subalgebras of type

$$\left\{ e_4 + \sum_i a_i e_i, N_a \right\}, \quad (23)$$

where $N_a$ is a subalgebra of $L_3$ such that $\text{Nor}_L N_a$ is not contained in $L_3$, $a_i \in \mathbb{R}$ are not all equal to zero and the generators $e_4 + \sum_i a_i e_i$ are not conjugate to $e_4$. Since $e_4$ is a central element of $L$ all of those conditions are satisfied. Let $N_a$ running through the list of algebras (17) and let $A = e_4 + \sum_i a_i e_i$. Let first $N_a = \{0\}$. This case is trivial because $e_4$ is the central element and the procedure was described on the Step 2. We obtain four subalgebras

$$\{ae_1 + e_4\}, \{ae_2 + e_4\}, \{ae_3 + e_4\}, \{a(e_1 \pm e_2) + e_4\}, \quad a \neq 0 \quad (24)$$

If we scale by $\frac{1}{a}$ all of these subalgebras [24] we see that for the two first subalgebras we can use adjoint representation generated by $e_1$ and $e_2$ to reduce these subalgebras to simplest one. We obtain from Table 4 following two subalgebras

$$\{e_1 + be^{-\varepsilon} e_4\}, \{e_2 + be^{-\alpha \varepsilon} e_4\}, \quad (25)$$
where $b = \frac{1}{a} \neq 0$. Choosing $\varepsilon = \log |\frac{1}{b}|$ in the first case and $\varepsilon = \frac{1}{a} \log |\frac{1}{b}|$ in the second one, we finally obtain the following list of one dimensional non-splitting extensions

$$\{ e_1 \pm e_4 \}, \{ e_2 \pm e_4 \}, \{ e_3 + ae_4 \}, \{ e_1 \pm e_2 + ae_4 \}, a \neq 0. \quad (26)$$

Let us now consider two-dimensional non-splitting extensions. To simplify these procedures we use as $N_a$ the subalgebras of the list (20). We notice that under action of the adjoint representation the general form of $e_4 + \sum a_i e_i$ is hold.

Let $N_a$ be equal to $\{ e_1 \}$ then without loss of generality we can represent $A$ as $\{ e_4 + a_2 e_2 + a_3 e_3 \}$. If $a_3$ is not equal to zero we can use the second line of the adjoint representation Table 4 and reduce the algebra $A$ to $e_4 + ae_3$. We obtain the following subalgebra

$$\{ e_3 + ae_4, e_1 \}, a \neq 0. \quad (27)$$

In the case $a_3 = 0$ we rewrite $A = \{ ae_4 + e_2 \}$ and use the third line of the adjoint representation Table 4 to obtain

$$\{ e^{\alpha \varepsilon} e_2 + ae_4, e^{\varepsilon} e_1 \} \quad (28)$$

or

$$\{ e_2 + ae^{-\alpha \varepsilon} e_4, e^{(1-\alpha)\varepsilon} e_1 \}, a \neq 0. \quad (29)$$

By choosing $\varepsilon = \frac{1}{a^2} \log |a|$ and scaling the second generator of the algebra above by the corresponding constant we obtain the following algebra

$$\{ e_2 \pm e_4, e_1 \}. \quad (30)$$

The same procedure in the case $N_a = e_2$ lead us to the non-isomorphic subalgebras

$$\{ e_1 \pm e_4, e_2 \}, \{ e_3 + ae_4, e_2 \}, a \neq 0. \quad (31)$$

Let us consider the case $N_a = \{ e_3 \}$, then we can choose $A = \{ e_4 + a_1 e_1 + a_2 e_2 \}$. Note that $A + N_a$ is an algebra just in case $a_1 a_2 = 0$. Those subalgebras were considered in the previous cases. Let $N_a = \{ e_1 \pm e_2 \}$ then $\{ N_a, e_4 + a_1 e_1 + a_2 e_2 + a_3 e_3 \}$ is an algebra only if $a_3 = 0$. Without loss of generality we can represent $A$ as an algebra generated by $\{ ae_4 + e_1 \}$ then by using the third line of the Table 4 we see that algebras $\{ A, N_a \}$ are isomorphic to the following algebra

$$\{ e_1 \pm e_4, e_1 + ae_2 \}. \quad (32)$$

where $a \in \mathbb{R}$. Note that the case $a = 0$ we consider on the third step. Finally we obtain the following subalgebra

$$\{ e_1 \pm e_4, ae_1 + e_2 \}. \quad (33)$$
where $a \neq 0$.

Now we consider case $N_a = \{e_1, e_2\}$ here we can represent $A$ as $\{ae_4 + e_3\}$ and obtain the following three dimensional subalgebra

$$\{e_3 + ae_4, e_1, e_2\},$$

where $a \neq 0$. It is easy to see that the other choices of $N_a$ do not provide any other non similar subalgebras.

We obtain the following list of the optimal system of subalgebras to the algebra $L^{(2)}$

$$\{0\}, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1 \pm e_2\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_4\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1 \pm e_4\}, \{e_1, e_4\}, \{e_2 \pm e_4\}, \{e_3 \pm e_2 + ae_4\}, \{e_1 \pm e_2 + ae_4\}, \{e_3 + ae_4, e_1\}, \{e_2 \pm e_4, e_1\}, \{e_1 \pm e_4, e_2\}, \{e_2 + ae_4, e_2\}, \{e_1 \pm e_4, ae_1 + e_2\}, \{e_3 + ae_4, e_1, e_2\}.$$

Finally we obtain the complete optimal system of subalgebras of Lie algebra $L$ (see Table 5).

| Dimension | Subalgebras |
|-----------|-------------|
| 1         | $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4\}$, $\{e_1 \pm e_2\}$, $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_2, e_3\}$, $\{e_4\}$, $\{e_1, e_4\}$, $\{e_2, e_4\}$, $\{e_3, e_4\}$, $\{e_1 \pm e_4\}$, $\{e_1, e_4\}$, $\{e_2 \pm e_4\}$, $\{e_3 \pm e_2 + ae_4\}, \{e_1 \pm e_2 + ae_4\}$ |
| 2         | $\{e_1, e_2\}$, $\{e_1, e_4\}$, $\{e_2, e_4\}$, $\{e_3, e_4\}$, $\{e_1 \pm e_2 + ae_4\}$, $\{e_1, e_4\}$, $\{e_2, e_4\}$, $\{e_3, e_4\}$, $\{e_1 \pm e_4\}$, $\{e_1, e_4\}$, $\{e_2 \pm e_4\}$, $\{e_3 \pm e_2 + ae_4\}$ |
| 3         | $\{e_1, e_2, e_4\}$, $\{e_1, e_3, e_4\}$, $\{e_2, e_3, e_4\}$, $\{e_1, e_2, e_3 + ae_4\}$ |

Table 5: The optimal system of subalgebras of the algebra $L^{(2)}$ in case $k \neq 0, \frac{1}{2}, 1$, were $a \in \mathbb{R}$, $\epsilon = \pm 1$. 

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We remark now that in case $k = \frac{1}{2}$ the structure of the algebra $L$ is the same as in the case above hence the optimal system of subalgebras is the same. For $k = 0$ or $k = 1$ we obtain the following system of subalgebras by the similar procedure.

| Dimension | Subalgebras |
|-----------|-------------|
| 1         | $\{e_2\}$, $\{e_3 \cos \varphi + e_4 \sin \varphi\}$, $\{e_1 + a(e_3 \cos \varphi + e_4 \sin \varphi)\}$, $\{e_2 + \epsilon(e_3 \cos \varphi + e_4 \sin \varphi)\}$ |
| 2         | $\{e_1 + a(e_3 \cos \varphi + e_4 \sin \varphi), e_2\}$, $\{e_3, e_4\}$, $\{e_1 + a(e_3 \cos \varphi + e_4 \sin \varphi), e_3 \sin \varphi - e_4 \cos \varphi\}$, $\{e_2 + \epsilon(e_3 \cos \varphi + e_4 \sin \varphi), e_3 \sin \varphi - e_4 \cos \varphi\}$, $\{e_2, e_3 \sin \varphi - e_4 \cos \varphi\}$ |
| 3         | $\{e_1 + a(e_3 \cos \varphi + e_4 \sin \varphi), e_3 \sin \varphi - e_4 \cos \varphi\}$, $\{e_1, e_3, e_4\}$, $\{e_2, e_3, e_4\}$, $\{e_1 + a(e_3 \cos \varphi + e_4 \sin \varphi), e_3 \sin \varphi - e_4 \cos \varphi, e_2\}$ |

Table 6: The optimal system of subalgebras of the algebra $L$ in case $k = 0$ or $k = 1$ where $a \in \mathbb{R}$, $\epsilon = \pm 1$, $0 \leq \varphi \leq \Pi$. 

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4 Conclusion

In this chapter we return to our original algebra $L(2)$ and introduce the optimal system of subalgebras Table 5 and Table 6 in original generators.

| Parameter | Generators | Optimal System of subalgebras |
|-----------|------------|-------------------------------|
| $k = 0$   | $v_1 = \frac{\partial}{\partial x}, v_2 = x \frac{\partial}{\partial x}, v_3 = \frac{\partial}{\partial y}, v_4 = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial u}$ | $\{v_2\}, \{v_1 \cos \varphi + v_2 \sin \varphi\}, \{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_2 + \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_1 \cos \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_2, v_3, v_4\}$, $\{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi$, $\{v_3 + \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi$, $\{v_3, v_1 \sin \varphi - v_2 \cos \varphi\}$, $\{v_4, v_1, v_2\}$, $\{v_3, v_1, v_2\}$, $\{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi, \{v_3\}$. |
| $k = 1$   | $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = \frac{\partial}{\partial y}, v_4 = x \frac{\partial}{\partial y}$ | $\{v_2\}, \{v_1 \cos \varphi + v_2 \sin \varphi\}, \{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_2 + \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_1 \cos \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $\{v_2, v_3, v_4\}$, $\{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi$, $\{v_3 + \epsilon(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi$, $\{v_3, v_1 \sin \varphi - v_2 \cos \varphi\}$, $\{v_4, v_1, v_2\}$, $\{v_3, v_1, v_2\}$, $\{v_4 + a(v_1 \cos \varphi + v_2 \sin \varphi)\}$, $v_1 \sin \varphi - v_2 \cos \varphi, v_2, v_4 \sin \varphi - v_3 \cos \varphi, v_2, v_4 \sin \varphi - v_3 \cos \varphi\}$. |
| $k < \frac{1}{2}, k \neq 0$ | $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = \frac{\partial}{\partial y}, v_4 = x \frac{\partial}{\partial y} + (1-k)y \frac{\partial}{\partial u}$ | $\{v_2\}, \{v_1\}, \{v_3 + av_2\}$, $\{v_1 \cos \varphi + v_2 \sin \varphi\}, \{v_4 + av_1\}, \{v_4 + av_1\}$, $\{v_3 + \epsilon v_2 + av_1\}$, $\{v_3, v_2\}$, $\{v_3, v_1\}$, $\{v_2, v_1\}$, $\{v_4, v_1\}$, $\{v_4 + av_1\}$, $\{v_2 + v_1, v_2\}$, $\{v_1 + av_1, av_3 + v_2\}$, $\{v_4 + av_1, v_3\}$, $\{v_4 + av_1, v_2\}, \{v_3 + v_2, v_3\}, \{v_3, v_4, v_1\}, \{v_2, v_4, v_1\}, \{v_4, v_2, v_4 + av_1\}$, $\{v_3, v_2, v_4 + av_1\}$. |
| $k = \frac{1}{2}$ | $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = \frac{\partial}{\partial y}, v_4 = x \frac{\partial}{\partial y} + \frac{1}{2} (1 - k) y \frac{\partial}{\partial u}$ | $\{v_2\}, \{v_1\}, \{v_3 + av_2\}$, $\{v_1 \cos \varphi + v_2 \sin \varphi\}, \{v_4 + av_1\}, \{v_3 + \epsilon v_2 + av_1\}$, $\{v_3, v_2\}$, $\{v_3, v_1\}$, $\{v_2, v_1\}$, $\{v_4, v_1\}$, $\{v_4 + av_1\}$, $\{v_2 + v_1, v_2\}$, $\{v_1 + av_1, av_3 + v_2\}$, $\{v_4 + av_1, v_3\}$, $\{v_4 + av_1, v_2\}, \{v_3 + v_2, v_3\}, \{v_3, v_4, v_1\}, \{v_2, v_4, v_1\}, \{v_4, v_2, v_4 + av_1\}$. |
| $k > \frac{1}{2}, k \neq 1$ | $v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = \frac{\partial}{\partial y}, v_4 = x \frac{\partial}{\partial y} + (1-k)y \frac{\partial}{\partial u}$ | $\{v_2\}, \{v_1\}, \{v_3 + av_2\}$, $\{v_1 \cos \varphi + v_2 \sin \varphi\}, \{v_4 + av_1\}$, $\{v_2 + \epsilon v_2 + av_1\}$, $\{v_2, v_3\}$, $\{v_2, v_1\}$, $\{v_3, v_1\}$, $\{v_4, v_1\}$, $\{v_2 + v_3, v_1\}$, $\{v_2, \var_3, v_2\}$, $\{v_2, v_1\}$, $\{v_3, v_1\}$, $\{v_4, v_1\}$, $\{v_2 + v_3, v_1\}$, $\{v_2, v_3, v_2\}$, $\{v_2 + v_3, v_2\}$, $\{v_2, v_1\}$, $\{v_3, v_1\}$, $\{v_4, v_1\}$, $\{v_2 + v_3, v_1\}$, $\{v_2, v_3, v_2\}$. |

Table 7: The optimal system of subalgebras of the algebra $L(2)$ with $a \in \mathbb{R}$, $\epsilon = \pm 1$, $0 \leq \varphi \leq \Pi$

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