Weak Cyclic Monotonicity and Existence of Solutions of Differential Inclusions

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Abstract

The notion of weak cyclic monotonicity of set-valued maps generalizing the cyclic monotonicity is introduced. The existence of solutions of differential inclusions with compact, upper semi-continuous, not necessarily convex right-hand sides in $\mathbb{R}^n$ is extended from cyclic monotone to weakly cyclic monotone right-hand sides.

Key words: differential inclusion, nonconvex right-hand side, existence of solutions, weakly monotone map, cyclic monotone map

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1 Introduction

We investigate the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \in I = [0, T],$$

where the set-valued mapping $F$ is upper semi-continuous, with non-empty compact, not necessarily convex values in $\mathbb{R}^n$.

To mention some relevant results on existence of solutions of such inclusions, we should first remind the case of upper semi-continuous maps with convex compact values (cp. [4, Chapter 2]). In particular, the classical existence theorem for right-hand side which is the negation of a maximal monotone map (see e.g. [1, Sec. 3.2, Theorem 1]). Maximal monotone set-valued maps are almost everywhere single-valued [22, 13], and it is easy to see that at the points where they are not single-valued, they are upper semi-continuous with convex values. Other important existence results for differential inclusions with non-convex right-hand sides are the results of Filippov [11] in finite dimensions and De Blasi, Pianigiani [3] in Banach spaces for continuous $F$. The continuity may replace the monotonicity and convexity, as is shown there.

An important existence result for upper semi-continuous maps, with not necessarily convex values, is established in [2], where the condition of cyclic monotonicity is imposed. We remind that the values of a cyclic monotone map are subsets of the values of the subdifferential of a proper convex function [20]. Generally, cyclic monotonicity is stronger than just monotonicity, but in $\mathbb{R}^1$ they are equivalent.

In [14] the notion of colliding on the set of discontinuities of $F$ is investigated and geometric conditions to avoid it or to escape of this set are studied.

Existence of solutions is established in [9] under a weak componentwise monotonicity condition and a standard growth condition. This componentwise monotonicity condition is equivalent to the

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strengthened one-sided Lipschitz (SOSL) condition \cite{16} with constant zero imposed on the negation of \( F, -F(\cdot) \). In one dimension the latter condition is equivalent to the one-sided Lipschitz (OSL) property in the sense of \cite{5} of \(-F\) with constant zero. The proof is constructive and uses a limiting procedure for the Euler polygons. The method of Euler polygonal approximations is a widely used powerful tool for deriving existence, approximation, well-posedness and stability properties of solutions of differential equations and inclusions (see e.g. \cite{7, 5, 24}), as well as of variational and control problems \cite{8, 18, 19}.

A typical proof of the existence of solutions in the upper-semicontinuous case uses the Arzela-Ascoli theorem applied to the polygonal solutions or Mazur’s weak closure theorem for the velocities. The limit of the Euler approximants is then a solution of the inclusion with convexified right-hand side. In \cite{9} the existence of solutions without convexity is achieved deriving monotonicity in the time of each coordinate of the velocities from the weak componentwise monotonicity condition. The Helly’s selection principle (see e.g. \cite{17, Chap. 10}) is then applied to obtain strong compactness of the set of velocities.

In this paper we formulate a condition, called here “weak cyclic monotonicity”, which generalizes the cyclic monotonicity and we prove the existence of solutions under this condition. In the proof we construct a cyclic monotone subinclusion of the given inclusion and apply the existence theorem of \cite{2}. Let us note that in general the weak cyclic monotonicity condition is stronger than the weak monotonicity condition. The question whether one can prove existence for weakly monotone maps (maps having OSL negation with constant zero) in more than one dimension is still open.

## 2 Weak cyclic monotonicity

First we introduce some notation. For notions used in the paper, but not explicitly defined here we refer the reader to \cite{4, 20, 1}.

Let \( v \in \mathbb{R}^n \). We denote by \(|v|\) the Euclidean norm of the vector \( v \). Denote by \( \mathbb{B} \) the unit ball in \( \mathbb{R}^n \). For a bounded set \( A \subset \mathbb{R}^n \), we denote \( \|A\| = \sup\{\|a\| : a \in A\} \). The support function of the set \( A \) in direction \( v \) is denoted by \( \delta^*(v, A) = \sup_{a \in A}(v, a) \). The classical convex subdifferential of a convex function \( f \) at the point \( x \) is denoted by \( \partial f(x) \).

We use the fact that the set-valued map \( F \) from \( \mathbb{R}^n \) to a compact set in \( \mathbb{R}^n \) is upper semicontinuous (USC) iff its graph is closed \cite[Section 1.1]{1}.

We recall that the map \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is monotone if for every \( x, y \in \mathbb{R}^n, v_x \in F(x), v_y \in F(y) \),

\[
\langle x - y, v_x - v_y \rangle \geq 0.
\]

We call \( F \) weakly monotone if for every \( x, y \in \mathbb{R}^n \) and every \( v_x \in F(x) \) there exists \( v_y \in F(y) \) such that (2) holds.

We also recall that the set-valued function \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is called cyclic monotone if for every natural \( m \), every \( x_0, x_1, \ldots, x_m \in \mathbb{R}^n \), and every \( v_i \in F(x_i), \ i = 0, \ldots, m \),

\[
\langle x_m - x_0, v_m \rangle \geq \sum_{i=1}^{m} \langle x_i - x_{i-1}, v_{i-1} \rangle.
\]

It is well-known that monotone maps are almost everywhere single-valued (in the sense of measure \cite{22} and in Bair category sense \cite{13}). The weaker property we define below may hold for everywhere non-single-valued maps.

In the next definitions and in all considerations below we fix the point \( x_0 \) to be the one of (1) and also fix an arbitrary \( v_0 \in F(x_0) \).

**Definition 1.** The finite sequence \( \{(x_i, v_i)\}_{i=0}^k \subset \text{Graph}(F) \) is a cyclic monotone sequence (CM sequence) if (3) holds for every \( m = 1, 2, \ldots, k \). The one-term sequence \( \{(x_0, v_0)\} \subset \text{Graph}(F) \) is a CM sequence by definition.
It is clear from this definition that if one cuts several last terms of the CM sequence \(\{(x_i, v_i)\}_{i=0}^k\), the shorter sequence \(\{(x_i, v_i)\}_{i=0}^m\) for \(m \leq k\) is also CM.

Now we give a recursive definition of a weakly cyclic monotone map based on cyclic monotone sequences on its graph.

**Definition 2.** The set-valued map \(F : \mathbb{R}^n \to \mathbb{R}^n\) is called weakly cyclic monotone (WCM) if for every natural \(m\), any CM sequence \(\{(x_i, v_i)\}_{i=0}^{m-1} \subset \text{Graph}(F)\) and every \(x_m \in \mathbb{R}^n\) there exists \(v_m \in F(x_m)\) such that the sequence \(\{(x_i, v_i)\}_{i=0}^m\) is a CM sequence.

In other words, \(F\) is weakly cyclic monotone if every CM sequence may be continued on the graph of \(F\), preserving its cyclic monotonicity.

Clearly, the weak cyclic monotonicity generalizes the cyclic monotonicity. On the other side, the weak cyclic monotonicity of set-valued maps is stronger than the weak monotonicity defined above.

From Definition 2 it is easy to see that the following condition implies weak cyclic monotonicity, but is weaker than cyclic monotonicity:

\[
\delta^*(x_m - x_0, F(x_m)) \geq \sum_{i=1}^{m} \delta^*(x_i - x_{i-1}, F(x_{i-1})),
\]

for every finite sequence \(\{x_i\}_{i=0}^m\).

**Remark 1.** Given a CM sequence \(\{(x_i, v_i)\}_{i=0}^{m-1}\) and \(x_m \in \mathbb{R}^n\), if one can choose \(v_m \in F(x_m)\) satisfying

\[
\langle x_m - x_0, v_m - v_{m-1} \rangle \geq 0,
\]

then the sequence \(\{(x_i, v_i)\}_{i=0}^m\) is CM.

Indeed, (5) may be written as

\[
\langle x_m - x_0, v_m \rangle \geq \langle x_m - x_{m-1}, v_{m-1} \rangle + \langle x_{m-1} - x_0, v_{m-1} \rangle.
\]

From this, one can prove (3) using a simple inductive argument.

**Example 1.** Here we give a simple example of a WCM multifunction which is not cyclic monotone.

Given a compact set \(A\), the constant map \(F : \mathbb{R}^n \to \mathbb{R}^n\) defined by \(F(x) = A, x \in \mathbb{R}^n\), is WCM.

Indeed, for a given CM sequence \(\{(x_i, v_i)\}_{i=0}^{m-1}\) and for given \(x_m \in \mathbb{R}^n\) one can choose \(v_m = v_{m-1} \in A\) so as to fulfill (5). If \(A\) is not a singleton, then \(F\) is not cyclic monotone since cyclic monotone (and all monotone) maps are a.e. single-valued, as we have mentioned above.

### 3 The Main Result

We impose the following assumptions in order to prove the existence of solutions:

- **A1.** \(F : \mathbb{R}^n \to \mathbb{R}^n\) has compact, nonempty values and is upper semicontinuous (has closed graph).

- **A2.** The map \(F\) is weakly cyclic monotone.

The main result of this paper is:

**Theorem 1.** Under the conditions A1, A2 the differential inclusion (1) has a solution.

To proof the theorem, we use the following result proved in [2].

**Theorem 2.** If \(F : \mathbb{R}^n \to \mathbb{R}^n\) has non-empty compact images and is upper semicontinuous and cyclic monotone, then the inclusion (1) has a solution.

We reduce our considerations to the last theorem by the following claim.
Theorem 3. If $F : \mathbb{R}^n \to \mathbb{R}^n$, satisfies A1,A2, then for every fixed $x_0 \in \mathbb{R}^n$, $v_0 \in F(x_0)$ the following map is upper semicontinuous and cyclic monotone, with nonempty compact images:

$$G(x) = \left\{ v \in F(x) : \langle x - x_0, v \rangle \geq \sup \left\{ \langle x - x_m, v_m \rangle + \sum_{i=1}^{m} \langle x_i - x_{i-1}, v_{i-1} \rangle \right\} \right\}, \quad (6)$$

where the supremum is taken on all natural $m$ and all CM sequences $\{(x_i, v_i)\}_{i=0}^{m}$.

Proof. Let $x_0, v_0 \in F(x_0)$ be fixed as above. For a given a finite CM sequence $S = \{(x_i, v_i)\}_{i=0}^{m}$ and for $x \in \mathbb{R}^n$ we denote

$$g(S, x) = \langle x - x_m, v_m \rangle + \sum_{i=1}^{m} \langle x_i - x_{i-1}, v_{i-1} \rangle.$$ 

Define the function

$$g(x) = \sup_S g(S, x), \quad (7)$$

where the supremum is taken on all $m \in \mathbb{N}$ and all finite CM sequences $S = \{(x_i, v_i)\}_{i=0}^{m} \subseteq \text{Graph}(G)$. Clearly, $g$ is convex, as a supremum of linear functions. Further,

$$g(x_0) = \sup_S \left\{ \langle x_0 - x_m, v_m \rangle + \sum_{i=1}^{m} \langle x_i - x_{i-1}, v_{i-1} \rangle \right\},$$

and using (3) we get that $g(x_0) \leq 0$. Taking $m = 0$, we easily see that $g(x_0) = 0$, which implies that $g$ is proper.

Next, we prove that $G(x)$ is non-empty for every $x \in \mathbb{R}^n$. Let us note that in the above notation

$$G(x) = \{ v \in F(x) : \langle x - x_0, v \rangle \geq g(x) \}.$$ 

Given a finite CM sequence $S = \{(x_i, v_i)\}_{i=0}^{m}$ and setting $x_{m+1} = x$, by A2 there is $v_{m+1} \in F(x)$, which we denote by $v(S, x)$, such that

$$\langle x - x_0, v(S, x) \rangle \geq g(S, x).$$

Since $g(x)$ is a supremum, we can choose a sequence $\{S^k\}_{k=1}^{\infty}$ such that

$$g(x) = \lim_{k \to \infty} g(S^k, x) \leq \liminf_{k \to \infty} \langle x - x_0, v(S^k, x) \rangle.$$ 

Since $F(x)$ is compact, the limit of every convergent subsequence of $\{v(S^k, x)\}_{k=1}^{\infty}$ is in $F(x)$. Denoting a limit of a converging subsequence of $\{v(S^k, x)\}_{k=1}^{\infty}$ by $v(x) \in F(x)$, we get that $g(x) \leq \langle x - x_0, v(x) \rangle$, thus $v(x) \in G(x)$. Thus, $G(x) \neq \emptyset$ for any $x \in \mathbb{R}^n$. Clearly, $G(x)$ is compact as a closed subset of the compact $F(x)$.

To prove that $G$ is upper semicontinuous, we show that its graph is closed. Take the sequences $x^k \to x^\infty$ and $v^k \to v^\infty$ with $v^k \in G(x^k)$. Passing to the limit for $k \to \infty$ in the inequalities of (3) and taking in account that $F$ has closed graph, we get that $v^\infty \in G(x^\infty)$, hence $G$ is USC.

To show that $G$ is cyclic monotone, we use the criterion that a map $G$ is cyclic monotone if and only if $G(x)$ is a subset of the subdifferential of a proper convex function [20]. Our argument is similar to the one in Theorem 24.8 in [20].

We now prove that $G(x) \subseteq \partial g(x)$ for any point $x \in \mathbb{R}^n$. For this, it is sufficient to show that for all $(x, v) \in \text{Graph}(G)$ and for any $y \in \mathbb{R}^n$ and $\alpha < g(x)$,

$$g(y) - \alpha > \langle y - x, v \rangle.$$
Indeed, since $g(x)$ is a supremum and $\alpha < g(x)$, there is a finite CM sequence $S = \{(x_i, v_i)\}_{i=0}^{k}$, such that

$$\alpha < g(S, x) = \langle x - x_k, v_k \rangle + \sum_{i=1}^{k} \langle x_i - x_{i-1}, v_{i-1} \rangle.$$  \hspace{1cm} (8)

Now, adding the point $x_{k+1} = x, v_{k+1} = v \in G(x)$ to the sequence $S$, we obtain the CM sequence $\hat{S} = \{(x_i, v_i)\}_{i=0}^{k+1}$, and by the definition of $g$,

$$g(y) \geq g(\hat{S}, y) = \langle y - x, v \rangle + \langle x - x_k, v_k \rangle + \sum_{i=1}^{k} \langle x_i - x_{i-1}, v_{i-1} \rangle.$$  

By (8),

$$g(y) > \langle y - x, v \rangle + \alpha,$$

which completes the proof. \hfill \Box

We obtain Theorem 1 applying Theorem 2 for the inclusion

$$\dot{x}(t) \in G(x(t)), \hspace{0.5cm} x(0) = x_0, \hspace{0.5cm} t \in I = [0, T],$$

with $G(x)$ defined in Theorem 3.

\textbf{Remark 2.} As we mentioned above, in one dimension the weak monotonicity coincides with the weak componentwise monotonicity for which the existence of solutions is proved in [9]. We conjecture that there exist solutions for weakly monotone right-hand sides also in higher dimensions.

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