Charmless $B \to PPP$ Decays: the Fully-Symmetric Final State

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Abstract

In charmless $B \to PPP$ decays, where $P$ is a pseudoscalar meson, there are six possibilities for the symmetry of the final state. In this paper, for $P = \pi, K$, we examine the properties of the fully-symmetric final state. We present expressions for all 32 $B \to PPP$ decay amplitudes as a function of both SU(3) reduced matrix elements and diagrams, demonstrating the equivalence of diagrams and SU(3). We also give 25 relations among the amplitudes in the SU(3) limit, as well as those that appear when the diagrams $E/A/PA$ are neglected. In the SU(3) limit, one has the equalities $\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{FS} = A(B^+ \to K^+K^-K^-)_{FS}$ and $\sqrt{2}A(B^+ \to \pi^+K^+K^-)_{FS} = A(B^+ \to \pi^+\pi^+\pi^-)_{FS}$, where FS denotes the fully-symmetric final state. These provide good tests of the standard model that can be carried out now by the LHCb Collaboration.

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I INTRODUCTION

Over the past several years, a certain amount of attention, both theoretical and experimental, has focused on charmless, three-body $B \to PPP$ decays ($P$ is a pseudoscalar meson). First, in Ref. [1] the diagrammatic method was proposed to describe $B \to PPP$ decays. Using this, it was shown that clean information about weak Cabibbo-Kobayashi-Maskawa (CKM) phases can be extracted from three-body decays [2, 3]. This idea was applied in Ref. [4] to obtain the weak phase $\gamma$ from the BABAR measurements of the Dalitz plots for some $B \to K\pi\pi$ and $B \to K K \bar{K}$ decays. Second, the LHCb collaboration recently reported nonzero measurements of CP asymmetries for the $\Delta S = 1$ decays $B^+ \to K^+\pi^+\pi^-$ and $B^+ \to K^+K^+K^-$ [5], as well as for the $\Delta S = 0$ decays $B^+ \to \pi^+\pi^+\pi^-$ and $B^+ \to \pi^+K^+K^-\bar{K}$ [6]. Furthermore, considerably larger CP asymmetries were measured in these decays for localized regions of phase space. That is, the CP-asymmetry measurements are momentum dependent. Theoretical analyses of these results can be found in Refs. [7, 8, 9, 10].

For charmless $B \to PPP$ decays, under flavor SU(3) the three final-state particles are treated as identical, so that the six permutations of these particles must be considered. There are thus six possibilities for the final state: a totally symmetric state, a totally antisymmetric state, or one of four mixed states. The fully-symmetric state is particularly intriguing for the following reason. The state that is fully symmetric under permutations of the particle indices is, by definition, also fully symmetric under permutations of the particle momenta. As such, this state does not receive contributions from spin-1 resonances because the decay products of such resonances are necessarily in a state antisymmetric in the particle momenta. This means that all effects generated by spin-1 resonances, such as direct CP asymmetries, SU(3) breaking, etc., are absent in the fully-symmetric $PPP$ state.

In this paper we examine in detail the properties of the fully-symmetric final state in $B \to PPP$. For simplicity, we assume that the final-state particles are all $\pi$’s or $K$’s. We first establish which SU(3) reduced matrix elements contribute to the fully-symmetric state, and write all $B \to PPP$ decay amplitudes as a function of these matrix elements. It turns out that there are seven independent combinations of matrix elements. However, there are a total of 32 decays – 16 $b \to s$ and 16 $b \to d$ transitions. There are therefore 25 relations among the amplitudes in the SU(3) limit. A number of these are subject to experimental tests. If the relations are found not to hold, this would indicate SU(3) breaking, or possibly even new physics.

We also write all $B \to PPP$ decay amplitudes as a function of diagrams. A comparison of the two expressions for the amplitudes allows us to write the SU(3) matrix elements as a function of diagrams, which establishes the equivalence of diagrams and SU(3), as was also done for $B \to PP$ decays [11]. As in two-body decays, three of the diagrams – $E$, $A$ and $PA$ – involve interactions of the spectator quark, and are expected to be quite a bit smaller than the other diagrams. In the limit in which $E$, $A$ and $PA$ are neglected, the number of combinations of matrix elements is reduced from seven to five. This implies additional relations among the amplitudes, and several of these can also be tested experimentally.

All experimental tests require that the fully-symmetric final state be probed. For any
$B \to PPP$ decay, this can be done using an isobar analysis of the Dalitz plot. This was discussed in Ref. [1], and we review the method later in the paper. But the point is that, for any three-body decay for which a Dalitz plot has been measured, one can extract the fully-symmetric amplitude.

One experimental test that is quite compelling, and which can be performed now, is related to the above LHCb measurements of CP asymmetries. In the SU(3) limit, one has the equalities
\[ \sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{FS} = A(B^+ \to K^+K^+K^-)_{FS} \]
\[ \sqrt{2}A(B^+ \to \pi^+K^+K^-)_{FS} = A(B^+ \to \pi^+\pi^+\pi^-)_{FS} \] (FS stands for fully symmetric). This says that the amplitudes for the fully-symmetric state of the two decays in each equality are predicted by the standard model (SM) to be equal. Furthermore, these equalities are momentum dependent, so that one should find the same values for the amplitudes at all points in the Dalitz plot. Now, LHCb has already measured the Dalitz plots for these decays. By performing an isobar analysis, the fully-symmetric amplitudes can be constructed. These relations can then be examined, providing a test of the SM.

In Sec. II, we express the fully-symmetric amplitudes for all $B \to PPP$ decays in terms of SU(3) reduced matrix elements. We present the relations among these amplitudes in Sec. III. In Sec. IV, we express the fully-symmetric $B \to PPP$ decay amplitudes in terms of diagrams, and demonstrate the equivalence of SU(3) and diagrams. We discuss the experimental tests of the amplitude relations in Sec. V, with an emphasis on the relations involving $B^+ \to K^+\pi^+\pi^-$ and $B^+ \to K^+K^+K^-$, and $B^+ \to \pi^+\pi^+\pi^-$ and $B^+ \to \pi^+K^+K^-$. We conclude in Sec. VI.

II AMPLITUDES & REDUCED MATRIX ELEMENTS

In this section we perform SU(3) Wigner-Eckart decompositions of the $B \to PPP$ decay amplitudes. Each element of SU(3) can be represented by $|\mathbf{r}YI_3\rangle$, where $\mathbf{r}$ is the irreducible representation (irrep), $Y$ is the hypercharge, and $I$ and $I_3$ stand for isospin and its third component, respectively. Note that, in general, Lie algebras are not associative, so that the order of multiplication is important. Here we take products from left to right.

In order to construct products of SU(3) states we use the SU(3) isoscalar factors from Refs. [12] [13], along with SU(2) Clebsch-Gordan coefficients (CG’s). We have checked that the results match with those SU(3) CG’s that are listed in Ref. [14].

There are 16 $b \to s$ and 16 $b \to d$ charmless three-body $B \to PPP$ decays, where $P = \pi$ or $K$. Under flavor SU(3), all three final-state particles belong to the same multiplet (octet of SU(3)), and hence they can be treated as identical, so that the six possible permutations of these particles must be considered. Here we focus on the fully-symmetric final state, which has dimension 120. This can be decomposed into irreps of SU(3) as follows:
\[ (8 \times 8 \times 8)_{FS} = 64 + 27_{FS} + 10_{FS} + 10^*_{FS} + 8_{FS} + 1 \]
where
\[ 27_{FS} = \sqrt{\frac{8}{13}}27_{8 \times 27} + \sqrt{\frac{7}{13}}27_{8 \times 8} \]
\[ 10_{FS} = \sqrt{\frac{2}{5}} 10_{8 \times 27} + \sqrt{\frac{3}{5}} 10_{8 \times 8}, \]
\[ 10^*_{FS} = \sqrt{\frac{2}{5}} 10^*_{8 \times 27} - \sqrt{\frac{3}{5}} 10^*_{8 \times 8}, \]
\[ s_{FS} = \frac{3}{2\sqrt{5}} s_{8 \times 27} - \sqrt{\frac{2}{15}} s_{8 \times 8} + \sqrt{\frac{5}{12}} s_{8 \times 1}. \] (2)

In the first equation above, note that \( 27_{8 \times 27} \) is symmetric under the interchange of \( 8 \) and \( 27 \). This distinguishes this irrep from \( 27' \) which also appears in the Wigner-Eckart decomposition of \( 8 \times 27 \), but is antisymmetric under the interchange of \( 8 \) and \( 27 \). Similarly, in the fourth equation, \( 8_{8 \times 8} \) is symmetric under the interchange of two \( 8 \)'s, distinguishing it from the other \( 8 \) in the product \( 8 \times 8 \) that is antisymmetric.

### A SU(3) assignments of pseudoscalar mesons

We begin by representing the light-quark states \( (u, d \text{ and } s) \) in flavor SU(3). The three quarks transform as the fundamental \( (3) \) of SU(3). The antiquarks transform as the \( 3^* \) of SU(3). We have assigned the following representations to the quarks and antiquarks:

\[
\begin{align*}
|u\rangle &= \begin{cases} 1 & 1 \\ 1 & 2 \\ 2 & 2 \\ \end{cases}, \\
|\bar{u}\rangle &= \begin{cases} 3^* & 1 \\ 3^* & 2 \\ 2^* & 2 \\ \end{cases}, \\
|d\rangle &= \begin{cases} 1 & 1 \\ 3 & 2 \\ 2 & 1 \\ \end{cases}, \\
|\bar{d}\rangle &= \begin{cases} 3^* & 1 \\ 3^* & 2 \\ 2^* & 2 \\ \end{cases}, \\
|s\rangle &= \begin{cases} 1 & 0 \\ 2 & 0 \\ \end{cases}, \\
|\bar{s}\rangle &= \begin{cases} 3^* & 0 \\ 3^* & 0 \\ \end{cases}.
\end{align*}
\] (3)

Using this convention we find that the \( \pi \)'s and \( K \)'s can be represented as follows:

\[
\begin{align*}
|\pi^+\rangle &= |u\rangle |\bar{d}\rangle = |8011\rangle, \\
|\pi^-\rangle &= -|d\rangle |\bar{u}\rangle = |801-1\rangle, \\
|\pi^0\rangle &= \frac{|d\rangle |\bar{d}\rangle - |u\rangle |\bar{u}\rangle}{\sqrt{2}} = |8010\rangle, \\
|K^+\rangle &= |u\rangle |\bar{s}\rangle = |811\frac{1}{2}\rangle, \\
|K^0\rangle &= |d\rangle |\bar{s}\rangle = |81\frac{1}{2}-1\rangle, \\
|K^-\rangle &= -|s\rangle |\bar{u}\rangle = |8-1\frac{1}{2}-1\rangle.
\end{align*}
\] (4)

### B Fully-symmetric three-body final states

The first step is to construct normalized three-body states within flavor SU(3) representing fully-symmetric \( P_1P_2P_3 \) final states. There are three cases. In the first, all three particles in the final state are distinct from one another (e.g., \( \pi^0\pi^+\pi^- \)). We begin by constructing states that are symmetrized over the first two particles. We then add all three combinations...
symmetrized in this way to obtain the fully-symmetric state. In what follows the state is symmetrized over particles that are included within parentheses.

\[
\langle (P_1 P_2) P_3 \rangle = \frac{1}{\sqrt{2}} [\langle P_1 \rangle \langle P_2 \rangle \langle P_3 \rangle + \langle P_2 \rangle \langle P_1 \rangle \langle P_3 \rangle],
\]

\[
\langle (P_1 P_2 P_3) \rangle_{FS} = \frac{1}{\sqrt{3}} [\langle (P_1 \langle P_2 \rangle) \langle P_3 \rangle \rangle + \langle (P_2 \langle P_3 \rangle) \langle P_1 \rangle \rangle + \langle (P_3 \langle P_1 \rangle) \langle P_2 \rangle \rangle].
\]

In the second case, two of the three particles are identical but distinct from the third (e.g., \(\pi^0 \pi^0 \pi^+\)). The three-particle state in which the first two particles are identical is automatically symmetric in the first two particles. Note that this state cannot be obtained from the first of Eq. (5) above by simply setting \(P_1 = P_2\). In writing Eq. (5) above we have used the fact that the states \(\langle P_1 \rangle\) and \(\langle P_2 \rangle\) are orthogonal to one another, which is no longer true if they represent the same particle. Keeping this in mind, we construct the fully-symmetric three-particle state by adding only distinct combinations symmetrized over the first two particles.

\[
\langle (PPP) \rangle_{FS} = \langle P \rangle \langle P \rangle \langle P \rangle.
\]

(6)

The final case is where all three particles are identical (e.g., \(\pi^0 \pi^0 \pi^0\)). In this case the state obtained by multiplying three single-particle states is automatically symmetric over all three particles. Once again note here that, in order to construct a state that is appropriately normalized, it is not possible to simply set \(P_1 = P_2 = P_3\) in the cases discussed above.

\[
\langle (PPP) \rangle_{FS} = \langle P \rangle \langle P \rangle \langle P \rangle.
\]

(7)

C Three-body \(\bar{b} \to \bar{s}q\bar{q}\) and \(\bar{b} \to \bar{d}q\bar{q}\) transitions using flavor SU(3)

The Hamiltonian for three-body \(B\) decays follows from the underlying quark-level transitions \(\bar{b} \to \bar{s}q\bar{q}\) and \(\bar{b} \to \bar{d}q\bar{q}\), where \(q\) is an up-type quark \((u, c, t)\). However, the unitarity of the CKM matrix, given as

\[
\sum_{q=u,c,t} V_{qb}^* V_{qs} = 0, \quad \sum_{q=u,c,t} V_{qb}^* V_{qd} = 0,
\]

(8)

allows us to trade one of the up-type quarks for the other two. Here we choose to replace the \(t\)-quark operators and retain only the \(c\)-quark and \(u\)-quark operators. Thus the weak-interaction Hamiltonian is composed of four types of operators, namely \(\bar{b} \to \bar{s}c\bar{c}\), \(\bar{b} \to \bar{d}c\bar{c}\), \(\bar{b} \to \bar{s}u\bar{u}\), and \(\bar{b} \to \bar{d}u\bar{u}\). The flavor-SU(3) representations of these operators are dictated by
the final-state light quarks, since the heavy $c$, $b$ and $t$ quarks are flavor-SU(3) singlets. The transition operators are given as follows:

$$
O_{b \to \bar{s}c\bar{c}} = V_{ub}^* V_{cs} B_{(2,0,0)}^{(3^*)}, \quad O_{b \to dc\bar{c}} = V_{ub}^* V_{cd} B_{(-1,1,1,1)}^{(3^*)},
$$

$$
O_{b \to s\bar{u}\bar{u}} = V_{ub}^* V_{us} \left\{ A_{(2,0,0)}^{(3^*)} + R_{(2,1,0)}^{(6)} + \sqrt{6} P_{(2,1,0)}^{(15^*)} + \sqrt{3} P_{(2,0,0)}^{(15^*)} \right\},
$$

$$
O_{b \to du\bar{u}} = V_{ub}^* V_{ud} \left\{ A_{(-1,1,1,1)}^{(3^*)} - R_{(-1,1,1,1)}^{(6)} + \sqrt{8} P_{(-1,1,1,1)}^{(15^*)} + P_{(-1,1,1,1)}^{(15^*)} \right\}, \quad (9)
$$

where we have used the notation $O_{(Y,I,I_3)}^{(r)}$ to represent each flavor-SU(3) operator ($O = \{A, B, R, P\}$). We have taken the names and relative signs between these operators from Ref. [15]. The weak-interaction Hamiltonian that governs charmless $B$ decays is then simply a sum of these four operators:

$$
\mathcal{H} = O_{b \to \bar{s}c\bar{c}} + O_{b \to dc\bar{c}} + O_{b \to s\bar{u}\bar{u}} + O_{b \to du\bar{u}}. \quad (10)
$$

The above Hamiltonian governs the decay of the flavor-SU(3) triplet of $B$-mesons [$B^3 = (B_s^+, B_s^0, B_s^0)$], whose components have the same flavor-SU(3) representations as their corresponding light quarks. The fully-symmetric three-body decay amplitude for the process $B \to P_1 P_2 P_3$ can now be constructed easily as follows:

$$
A_{FS}(p_1, p_2, p_3) = \text{FS} \left\langle (P_1 P_2 P_3) \right| \mathcal{H} \left| B^3 \right\rangle, \quad (11)
$$

where $p_i$ represents the momentum of the final-state particle $P_i$.

### III AMPLITUDE RELATIONS

The 32 charmless three-body $B$ decay amplitudes (16 $\bar{b} \to \bar{s}$ and 16 $\bar{b} \to \bar{d}$) can all be written in terms of nine matrix elements (here we have suppressed the $Y, I, I_3$ indices of the operators):

$$
B_1^{(fs)} \equiv \text{FS} \left\langle 1 \right| B^{(3^*)} \left| 3 \right\rangle, \quad B^{(fs)} \equiv \text{FS} \left\langle 8 \right| B^{(3^*)} \left| 3 \right\rangle, \quad A_1^{(fs)} \equiv \text{FS} \left\langle 1 \right| A^{(3^*)} \left| 3 \right\rangle, \quad A^{(fs)} \equiv \text{FS} \left\langle 8 \right| A^{(3^*)} \left| 3 \right\rangle, \quad R_8^{(fs)} \equiv \text{FS} \left\langle 8 \right| R^{(6)} \left| 3 \right\rangle, \quad P_{10}^{(fs)} \equiv \text{FS} \left\langle 10 \right| P^{(15^*)} \left| 3 \right\rangle, \\
\bar{P}_{10}^{(fs)} \equiv \text{FS} \left\langle 10^* \right| P^{(15^*)} \left| 3 \right\rangle, \quad P_8^{(fs)} \equiv \text{FS} \left\langle 8 \right| P^{(15^*)} \left| 3 \right\rangle, \quad P_{27}^{(fs)} \equiv \text{FS} \left\langle 27 \right| P^{(15^*)} \left| 3 \right\rangle. \quad (12)
$$
The decomposition of all 32 amplitudes in terms of these matrix elements is given in Tables I and II.

Note that there are only seven combinations of matrix elements in the amplitudes since $B^{(fs)}$ and $A^{(fs)}$, as well as $B_1^{(fs)}$ and $A_1^{(fs)}$, always appear together:

$$\lambda_c^{(q)} B^{(fs)} + \lambda_u^{(q)} A^{(fs)} , \quad \lambda_c^{(q)} B_1^{(fs)} + \lambda_u^{(q)} A_1^{(fs)},$$

(13)

with $\lambda_p^{(q)} = V_{pb}^* V_{pq}$; $q = d, s$; $p = u, c$.

Given that there are 32 decay amplitudes, all expressed in terms of seven combinations of matrix elements, there must be 25 independent relations among the amplitudes. What are they?

This question is addressed as follows. The 16 $\bar{b} \to \bar{s}$ decay amplitudes are expressed in terms of the seven combinations of matrix elements, so there must be nine relations among the amplitudes. Some of the amplitude relations are determined by considering isospin alone. The remaining relations can be found by expanding the symmetry to full SU(3). This procedure is then applied to the 16 $b \to d$ decays, giving an additional nine amplitude relations. Finally, one can relate certain $\bar{b} \to \bar{s}$ and $\bar{b} \to \bar{d}$ decay amplitudes using U spin. There are eight such relations, of which seven are independent of the above 18 relations. This makes a total of 25 independent relations.

A.  $\bar{b} \to \bar{s}$ Decays

The 16 $\bar{b} \to \bar{s}$ decays (see Table I) include $B \to K\pi\pi$ (6 decays), $B \to KK\bar{K}$ (4 decays), $B_0^s \to \pi K\bar{K}$ (4 decays), and $B_0^0 \to \pi\pi\pi$ (2 decays). We begin by applying isospin alone. All initial- and final-state particles in the decays are eigenstates of isospin, and the weak Hamiltonian has $\Delta I = 0$ or 1. One can relate the individual amplitudes using the Wigner-Eckart theorem.

The isospin relations among the $\bar{b} \to \bar{s}$ and $B \to PPP$ amplitudes are (some of these are given in Ref. [1]):

1. $B \to K\pi\pi$:

$$A(B^+ \to K^0\pi^+\pi^0)_{FS} = -A(B_0^0 \to K^+\pi^0\pi^-)_{FS},$$

$$\sqrt{2} A(B^+ \to K^0\pi^+\pi^0)_{FS} = A(B_0^0 \to K^0\pi^+\pi^-)_{FS} + \sqrt{2} A(B_d^0 \to K^0\pi^0\pi^0)_{FS},$$

$$\sqrt{2} A(B_0^0 \to K^+\pi^0\pi^-)_{FS} = A(B^+ \to K^+\pi^+\pi^-)_{FS} + \sqrt{2} A(B^+ \to K^+\pi^0\pi^0)_{FS},$$

(14)

2. $B \to KK\bar{K}$:

$$A(B^+ \to K^+K^+K^-)_{FS} + \sqrt{2} A(B^+ \to K^+K^0\bar{K}^0)_{FS} =$$

$$\sqrt{2} A(B_d^0 \to K^0K^+K^-)_{FS} + A(B_d^0 \to K^0K^0\bar{K}^0)_{FS},$$

(15)
Table I: Amplitudes for $\Delta S = 1$ $B$-meson decays to fully-symmetric $PPP$ states as a function of the SU(3) matrix elements.

| Decay Amplitude | $V_{cb}^*V_{cs}$ | $V_{ub}^*V_{us}$ |
|-----------------|------------------|------------------|
|                 | $B_1^{(fs)}$     | $B^{(fs)}$       | $A_1^{(fs)}$ | $A^{(fs)}$ | $R_8^{(fs)}$ | $R_{10}^{(fs)}$ | $P_8^{(fs)}$ | $P_{10}^{(fs)}$ | $P_2^{(fs)}$ |
| $A(B^+ \rightarrow K^+\pi^+\pi^-)$ | $0$ | $0$ | $0$ | $0$ | $1$ | $3$ | $2$ | $3$ | $0$ | $-3\sqrt{3}$ |
| $\sqrt{2}A(B^+ \rightarrow K^+\pi^0\pi^0)$ | $0$ | $-\frac{1}{\sqrt{2}}$ | $0$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $\frac{18\sqrt{7}}{5\sqrt{7}}$ |
| $\sqrt{2}A(B^+ \rightarrow K^0\pi^+\pi^0)$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $3$ | $2$ | $3$ | $0$ | $-3\sqrt{3}$ |
| $A(B^0 \rightarrow K^0\pi^+\pi^-)$ | $0$ | $\frac{1}{\sqrt{5}}$ | $0$ | $\frac{1}{\sqrt{5}}$ | $-\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $\frac{2\sqrt{2}}{3}$ | $-\frac{9\sqrt{2}}{5\sqrt{7}}$ |
| $\sqrt{2}A(B^0 \rightarrow K^0\pi^0\pi^0)$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $-\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $-\frac{2\sqrt{2}}{3}$ | $-\frac{6\sqrt{2}}{5\sqrt{7}}$ |
| $\sqrt{2}A(B^0 \rightarrow K^+\pi^0\pi^-)$ | $0$ | $0$ | $0$ | $0$ | $-\frac{1}{\sqrt{3}}$ | $0$ | $0$ | $3\sqrt{3}$ | $0$ | $\frac{3\sqrt{3}}{\sqrt{7}}$ |
| $A(B^+ \rightarrow K^+K^0\bar{K}^0)$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $-\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $-\frac{12\sqrt{2}}{5\sqrt{7}}$ |
| $\frac{1}{\sqrt{2}}A(B^+ \rightarrow K^+K^+K^-)$ | $0$ | $\frac{1}{\sqrt{5}}$ | $0$ | $\frac{1}{\sqrt{5}}$ | $\frac{1}{\sqrt{15}}$ | $-\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $-\frac{3\sqrt{3}}{5\sqrt{7}}$ |
| $A(B^0 \rightarrow K^0K^+K^-)$ | $0$ | $\frac{1}{\sqrt{5}}$ | $0$ | $\frac{1}{\sqrt{5}}$ | $-\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $-\frac{9\sqrt{3}}{5\sqrt{7}}$ |
| $\frac{1}{\sqrt{2}}A(B^0 \rightarrow K^0K^0\bar{K}^0)$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $0$ | $-\frac{1}{\sqrt{5}}$ | $\frac{1}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{3}{5}$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $-\frac{6\sqrt{3}}{5\sqrt{7}}$ |
| $\sqrt{2}A(B_s^0 \rightarrow \pi^0K^+K^-)$ | $\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $\frac{2}{6\sqrt{3}}$ | $\frac{1}{5}$ | $\frac{1}{3\sqrt{2}}$ | $\frac{51}{10\sqrt{14}}$ |
| $\sqrt{2}A(B_s^0 \rightarrow \pi^0K^0\bar{K}^0)$ | $\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $-\frac{2}{6\sqrt{3}}$ | $\frac{1}{5}$ | $\frac{1}{3\sqrt{2}}$ | $\frac{3\sqrt{7}}{10\sqrt{2}}$ |
| $A(B_s^0 \rightarrow \pi^-K^+\bar{K}^0)$ | $-\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $-\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $0$ | $\frac{1}{2\sqrt{3}}$ | $0$ | $-\frac{\sqrt{2}}{3}$ | $-\frac{3}{2\sqrt{14}}$ |
| $A(B_s^0 \rightarrow \pi^+K^-\bar{K}^0)$ | $-\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $-\frac{\sqrt{3}}{2\sqrt{10}}$ | $0$ | $0$ | $\frac{1}{2\sqrt{3}}$ | $0$ | $\frac{1}{\sqrt{2}}$ | $-\frac{3}{2\sqrt{14}}$ |
| $\frac{2}{\sqrt{3}}A(B_s^0 \rightarrow \pi^0\pi^0\pi^0)$ | $0$ | $0$ | $0$ | $0$ | $-\frac{2}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{4}{5}$ | $\frac{\sqrt{2}}{3}$ | $\frac{6\sqrt{2}}{5\sqrt{7}}$ |
| $\sqrt{2}A(B_s^0 \rightarrow \pi^0\pi^+\pi^-)$ | $0$ | $0$ | $0$ | $0$ | $\frac{2}{\sqrt{15}}$ | $\frac{1}{3\sqrt{3}}$ | $\frac{4}{5}$ | $-\frac{\sqrt{2}}{3}$ | $-\frac{6\sqrt{2}}{5\sqrt{7}}$ |
Table II: Amplitudes for $\Delta S = 0$ $B$-meson decays to fully-symmetric $PPP$ states as a function of the SU(3) matrix elements.

| Decay Amplitude | $V_{cb}^\ast V_{cd}$ | $V_{ub}^\ast V_{ud}$ |
|-----------------|---------------------|---------------------|
| $A(B^+ \to \pi^+ K^0 \overline{K}^0)$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ |
| $A(B^+ \to \pi^+ K^+ K^-)$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ |
| $\sqrt{2}A(B^+ \to \pi^0 K^+ \overline{K}^0)$ | $0$ | $0$ |
| $\sqrt{2}A(B^0 \to \pi^0 K^0 \overline{K}^0)$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ |
| $\sqrt{2}A(B^0 \to \pi^0 K^+ K^-)$ | $0$ | $0$ |
| $A(B^0 \to \pi^+ K^0 K^-)$ | $-\frac{1}{\sqrt{3}}$ | $-\frac{1}{\sqrt{3}}$ |
| $A(B^0 \to \pi^- K^+ \overline{K}^0)$ | $0$ | $0$ |
| $\sqrt{2}A(B^+ \to \pi^+ \pi^0 \pi^0)$ | $0$ | $0$ |
| $\frac{1}{\sqrt{2}}A(B^+ \to \pi^+ \pi^+ \pi^-)$ | $0$ | $0$ |
| $\frac{2}{\sqrt{3}}A(B^0 \to \pi^0 \pi^0 \pi^0)$ | $0$ | $0$ |
| $\sqrt{2}A(B^0 \to \pi^0 \pi^+ \pi^-)$ | $0$ | $0$ |
| $\sqrt{2}A(B_s^0 \to \overline{K}^0 \pi^0 \pi^0)$ | $0$ | $0$ |
| $A(B_s^0 \to K^0 \pi^+ \pi^-)$ | $0$ | $0$ |
| $\sqrt{2}A(B_s^0 \to K^- \pi^+ \pi^0)$ | $0$ | $0$ |
| $\frac{1}{\sqrt{2}}A(B_s^0 \to \overline{K}^0 K^0 \overline{K}^0)$ | $0$ | $0$ |
| $A(B_s^0 \to \overline{K}^0 K^+ K^-)$ | $0$ | $0$ |
3. $B_s^0 \rightarrow \pi K \bar{K}$:

$$\sqrt{2}A(B_s^0 \rightarrow \pi^0 K^+ K^-)_{FS} + \sqrt{2}A(B_s^0 \rightarrow \pi^0 K^0 \bar{K}^0)_{FS} + A(B_s^0 \rightarrow \pi^- K^+ \bar{K}^0)_{FS} + A(B_s^0 \rightarrow \pi^+ K^- K^0)_{FS} = 0 \ . \quad (16)$$

4. $B_s^0 \rightarrow \pi \pi \pi$:

$$\frac{2}{\sqrt{3}}A(B_s^0 \rightarrow \pi^0 \pi^0 \pi^0)_{FS} = -\sqrt{2}A(B_s^0 \rightarrow \pi^0 \pi^+ \pi^-)_{FS} \ . \quad (17)$$

This makes a total of six isospin relations.

Now, under isospin, the $K\pi\pi$ and $KK\bar{K}$ matrix elements (for example) are unrelated. However, they are equal under full SU(3). Indeed, under SU(3) all $\bar{b} \rightarrow \bar{d}$ matrix elements are equal to their corresponding $\bar{b} \rightarrow \bar{d}$ matrix elements. When one applies full SU(3), one additional relation is

$$\sqrt{2}A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{FS} = A(B^+ \rightarrow K^+ K^+ K^-)_{FS} \ . \quad (18)$$

In fact, in Ref. [16], Gronau and Rosner showed that U spin implies that

$$A(K^+ \pi^+ \pi^-)_{p_1p_2p_3} + A(K^+ \pi^+ \pi^-)_{p_2p_1p_3} = A(K^+ K^+ K^-)_{p_1p_2p_3} \ . \quad (19)$$

If one applies this to the fully-symmetric final state, the two terms on the left-hand side are equal. If one also takes into account the different normalizations for $(K^+ \pi^+ \pi^-)_{FS}$ and $(K^+ K^+ K^-)_{FS}$, one gets

$$\frac{2}{\sqrt{6}}A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{FS} = \frac{1}{\sqrt{3}}A(B^+ \rightarrow K^+ K^+ K^-)_{FS} \implies \sqrt{2}A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{FS} = A(B^+ \rightarrow K^+ K^+ K^-)_{FS} \ , \quad (20)$$

as above. Thus, one does not really need full SU(3) to obtain Eq. (18) – only the U-spin SU(2) subgroup is needed.

The remaining two relations are not associated with any symmetry. They involve a number of amplitudes associated with different decays. As they are not particularly interesting, we do not present them here.

**B $\bar{b} \rightarrow \bar{d}$ Decays**

The 16 $\bar{b} \rightarrow \bar{d}$ decays (see Table [1] include $B \rightarrow \pi K \bar{K}$ (7 decays), $B \rightarrow \pi \pi \pi$ (4 decays), $B_s^0 \rightarrow K \pi \pi$ (3 decays), and $B_s^0 \rightarrow K K \bar{K}$ (2 decays). Again, all initial- and final-state particles in the decays are eigenstates of isospin, and the weak Hamiltonian has $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$. Using the Wigner-Eckart theorem, the isospin relations among the $\bar{b} \rightarrow \bar{d}$ $B \rightarrow PPP$ amplitudes are (some of these are given in Ref. [1]):
1. $B \to \pi K\bar{K}$:
\[
\sqrt{2}A(B^0 \to \pi^0 K^+ K^-)_{FS} + A(B^0 \to \pi^+ K^0 K^-)_{FS} - A(B^+ \to \pi^+ K^+ K^-)_{FS} \\
+ \sqrt{2}A(B^0 \to \pi^0 K^0 \bar{K}^0)_{FS} + A(B^0 \to \pi^- K^+ \bar{K}^0)_{FS} \\
- A(B^+ \to \pi^+ K^0 \bar{K}^0)_{FS} - \sqrt{2}A(B^+ \to \pi^0 K^+ K^-)_{FS} = 0.
\] (21)

2. $B \to \pi\pi\pi$:
\[
\sqrt{2}A(B^0 \to \pi^0 \pi^0 \pi^0)_{FS} = -\sqrt{3}A(B^0 \to \pi^+ \pi^0 \pi^-)_{FS}, \\
2A(B^+ \to \pi^+ \pi^0 \pi^0)_{FS} = -A(B^+ \to \pi^- \pi^+ \pi^+)_{FS}.
\] (22)

3. $B_s^0 \to K\pi\pi$:
\[
\sqrt{2}A(B_s^0 \to K^0 \pi^0 \pi^0)_{FS} + A(B_s^0 \to K^0 \pi^+ \pi^-)_{FS} + \sqrt{2}A(B_s^0 \to K^- \pi^0 \pi^0)_{FS} = 0.
\] (23)

There are no relations involving $B_s^0 \to KKK$ amplitudes. This makes a total of four isospin relations.

As with $\bar{b} \to \bar{s}$ decays, one can find the additional five relations by applying full SU(3), in which case the $\bar{b} \to \bar{d}$ matrix elements are the same for all decays. One of these relations,
\[
A(B^+ \to \pi^+ \pi^+ \pi^-)_{FS} + \sqrt{2}A(B^+ \to \pi^+ K^0 \bar{K}^0)_{FS} = \\
\sqrt{2}A(B_s^0 \to \bar{K}^0 \pi^+ \pi^-)_{FS} + A(B_s^0 \to K^0 K^0 \bar{K}^0)_{FS}.
\] (24)

follows by applying U-spin reflection ($d \leftrightarrow s$) to the $\bar{b} \to \bar{s}$ relation of Eq. (15). Another relation,
\[
\sqrt{2}A(B^+ \to \pi^+ K^+ K^-)_{FS} = A(B^+ \to \pi^+ \pi^+ \pi^-)_{FS},
\] (25)

is due to full U spin \[16\], as in Eq. (18).

As was the case for $\bar{b} \to \bar{s}$ decays, the remaining three relations are not associated with any symmetry. They involve numerous amplitudes associated with different decays. As they are not particularly interesting, we do not present them here.

C U Spin

For each $\bar{b} \to \bar{s}$ decay in Table I whose final state does not involve any $\pi^0$’s, there is a corresponding $\bar{b} \to \bar{d}$ decay in Table II related by U-spin reflection ($d \leftrightarrow s$). The eight pairs are

1. $A(B^0 \to K^0 K^0 K^0)$ and $A(B_s^0 \to \bar{K}^0 K^0 K^0)$,
2. $A(B^0 \to K^0 \pi^+ \pi^-)$ and $A(B_s^0 \to K^+ K^0 K^0)$,
3. \(A(B^0 \to K^+K^0K^-)\) and \(A(B^0_s \to K^0\pi^+\pi^-)\),
4. \(A(B^0 \to \pi^-K^+\bar{K}^0)\) and \(A(B^0 \to \pi^+K^0K^-)\),
5. \(A(B^0_s \to \pi^+K^0K^-)\) and \(A(B^0 \to \pi^-K^+\bar{K}^0)\),
6. \(A(B^+ \to K^+K^+K^-)\) and \(A(B^+ \to \pi^+\pi^+\pi^-)\),
7. \(A(B^+ \to K^+\pi^+\pi^-)\) and \(A(B^+ \to \pi^+K^+K^-)\),
8. \(A(B^+ \to K^+K^0\bar{K}^0)\) and \(A(B^+ \to \pi^+K^0\bar{K}^0)\).

The first (second) decay is \(\bar{b} \to \bar{s}\) (\(\bar{b} \to \bar{d}\)). This shows that there is a U-spin dependence between Eqs. (15) and (24).

In each pair of amplitudes, terms multiplying \(V^*_{cb}V_{cs}\) and \(V^*_{ub}V_{us}\) in one process equal terms multiplying \(V^*_{cb}V_{cd}\) and \(V^*_{ub}V_{ud}\) in the other [17]. (See Tables I and II.) Thus, one can write relations among the \(\bar{b} \to \bar{s}\) and \(\bar{b} \to \bar{d}\) decay amplitudes involving CKM matrix elements. However, it was shown in Refs. [16, 17, 18] that a more experimentally-useful relation between U-spin pairs is the following:

\[
\frac{A_s}{A_d} \frac{B_s}{B_d} = -1 ,
\]  

where

\[
B_d = |A(\bar{b} \to \bar{d})|^2 + |A(b \to d)|^2 ,
B_s = |A(\bar{b} \to \bar{s})|^2 + |A(b \to s)|^2 ,
A_d = \frac{|A(\bar{b} \to \bar{d})|^2 - |A(b \to d)|^2}{|A(\bar{b} \to \bar{d})|^2 + |A(b \to d)|^2} ,
A_s = \frac{|A(\bar{b} \to \bar{s})|^2 - |A(b \to s)|^2}{|A(\bar{b} \to \bar{s})|^2 + |A(b \to s)|^2} .
\]  

(27)

\(B_d\) and \(B_s\) are related to the CP-averaged \(\bar{b} \to \bar{d}\) and \(\bar{b} \to \bar{s}\) decay rates, while \(A_d\) and \(A_s\) are direct CP asymmetries. The CP-conjugate amplitude \(A(\bar{b} \to \bar{q})\) is obtained from \(A(b \to q)\) by changing the signs of the weak phases. (Since we have U-spin reflections, these relations hold for all final symmetry states.)

There are eight U-spin relations of this kind. Along with the nine \(\bar{b} \to \bar{s}\) and nine \(\bar{b} \to \bar{d}\) decay amplitude relations, of which one pair [Eqs. (15) and (24)] is related by U-spin reflection, this makes a total of 25 independent relations. This is consistent with 32 decay amplitudes all expressed as a function of seven combinations of SU(3) matrix elements.
IV AMPLITUDES AND DIAGRAMS

A \( B \to PP \) decays

In Ref. [11], it was argued that two-body \( B \to PP \) decays can be described by six diagrams: the color-favored and color-suppressed tree amplitudes \( T \) and \( C \), the gluonic-penguin amplitude \( P \), the exchange amplitude \( E \), the annihilation amplitude \( A \), and the penguin-annihilation amplitude \( PA \). Now, \( P \) receives contributions from the internal quarks \( t, c \) and \( u \):

\[
P = V_{tb}V_{td}P_t + V_{cb}V_{cd}P_c + V_{ub}V_{ud}P_u \\
= V_{ct}^*P_{ct} + V_{ut}^*P_{ut},
\]

where \( P_t \equiv P_c - P_t, P_{ut} \equiv P_u - P_t, \) and the unitarity of the CKM matrix [Eq. (8)] has been used in the second line. Note that two-body diagrams are generally (re)defined to absorb the CKM matrix elements that multiply them. That is, \( T, P_{ut} \), etc. include \( V_{ub}^*V_{ud} \), and \( P_{ct}, P_{ct} \) include \( V_{ct}^*V_{cd} \).

In \( B \to PP \) decays there are seven matrix elements. \( \{1\}_u, \{8\}_u, \{82\}, \{83\}, \) and \( \{27\} \) are all multiplied by \( V_{ub}^*V_{ud} \), while \( \{1\}_c \) and \( \{8\}_c \) are multiplied by \( V_{cb}^*V_{cd} \). For \( \Delta C = 0, \Delta S = 0 \) decays, the following relations between matrix elements and diagrams were established [11]:

\[
V_{ub}^*V_{ud} \{1\}_u = 2\sqrt{3} [PA_{ut} + \frac{2}{3}P_{ut} + \frac{2}{3}E - \frac{1}{12}(C + \frac{1}{4}f)] \\
V_{ub}^*V_{ud} \{8\}_u = -\frac{5}{3} \left[ P_{ut} + \frac{3}{8}(T + A) - \frac{3}{8}(C + E) \right], \\
V_{ub}^*V_{ud} \{82\} = \frac{\sqrt{5}}{4} (C + A - T - E), \\
V_{ub}^*V_{ud} \{83\} = -\frac{1}{8\sqrt{3}} (T + C) - \frac{5}{8\sqrt{3}} (A + E), \\
V_{ub}^*V_{ud} \{27\} = -\frac{1}{2\sqrt{3}} [T + C], \\
V_{cb}^*V_{cd} \{1\}_c = 2\sqrt{3} \left[ PA_{ct} + \frac{2}{3}P_{ct} \right], \\
V_{cb}^*V_{cd} \{8\}_c = -\frac{5}{3} P_{ct}.
\]

A corresponding set of relations exists for the diagrams corresponding to strangeness-changing \( \Delta C = 0 \) decays. This shows that the description of decay amplitudes in terms of diagrams is equivalent to an SU(3) description.

Above, the equivalence of diagrams and SU(3) is shown for \( B \to PP \) decays. However, there are two diagrams that have not been included: the color-favored and color-suppressed electroweak-penguin (EWP) amplitudes \( P_{EW} \) and \( P_{EW}^C \) [19]. If they are added, the relations...
between matrix elements and diagrams are modified, but this does not change the fact that diagrams and SU(3) are equivalent. Indeed, the addition of EWP’s only has the effect of redefining the diagrams:

\[
\begin{align*}
T & \rightarrow T + P_{EW}^C, \\
C & \rightarrow C + P_{EW}, \\
(P_{ut} + P_{ct}) & \rightarrow (P_{ut} + P_{ct}) - \frac{1}{3}P_{EW}^C.
\end{align*}
\]

(30)

Although this shows that, in fact, EWP’s are not independent, it does not indicate how EWP’s are related to the other diagrams. The actual $B \rightarrow PP$ EWP-tree relations were found later, in Refs. [20, 21].

## B $B \rightarrow PPP$ decays

In Ref. [1], it was argued that $B \rightarrow PPP$ decays can also be described by diagrams, similar to those of $B \rightarrow PP$. For the three-body analogues of $T$, $C$ and $P$, one has to “pop” a quark pair from the vacuum. The subscript “1” (“2”) is added if the popped quark pair is between two non-spectator final-state quarks (two final-state quarks including the spectator). For the $P$-type diagrams, it turns out that only the combination $\tilde{P} \equiv P_1 + P_2$ appears in amplitudes. For the three-body analogues of $E$, $A$ and $PA$, the spectator quark interacts with the $\bar{b}$, and one has two popped quark pairs. Here there is only one of each type of diagram, so there are a total of eight diagrams: $T_{1,2}$, $C_{1,2}$, $\tilde{P}$, $E$, $A$, $PA$. Furthermore, for each of $\tilde{P}$ and $PA$, we allow for two contributions, giving $\tilde{P}_{ut}$, $\tilde{P}_{ct}$, $PA_{ut}$ and $PA_{ct}$, where $\tilde{P}_{ut} \equiv \tilde{P}_u - \tilde{P}_t$, and similarly for the other diagrams. For certain decays, the diagrams may have a popped $s\bar{s}$ quark pair. Under SU(3), these are equal to the same diagrams with a popped $d\bar{d}$ or $u\bar{u}$. As in $B \rightarrow PP$ decays, we define the diagrams to absorb the CKM matrix elements that multiply them. The decomposition of all 32 amplitudes in terms of diagrams is given in Tables III and IV.

For the $\bar{b} \rightarrow \bar{d}$ decays there are eight diagrams that include $V_{ub}^*V_{ud}$ and two that include $V_{cb}^*V_{cd}$. Similarly, there are seven matrix elements proportional to $V_{ub}^*V_{ud}$ and two proportional to $V_{cb}^*V_{cd}$. By analyzing all 16 $\bar{b} \rightarrow \bar{d}$ decays, one finds that the relations between these are

\[
\begin{align*}
V_{ub}^*V_{ud} A_1^{(fs)} &= \frac{\sqrt{5}}{2\sqrt{6}} (8\tilde{P}_{ut} + 8PA_{ut} + 3T_1 + 3T_2 - C_1 - C_2 + 8E), \\
V_{ub}^*V_{ud} A^{(fs)} &= \frac{\sqrt{5}}{8} (-8\tilde{P}_{ut} - T_1 - 3T_2 - 5C_1 + C_2 + E - 3A), \\
V_{ub}^*V_{ud} R_8^{(fs)} &= \frac{\sqrt{5}}{4\sqrt{3}} (T_1 - T_2 - C_1 + C_2 + 3E - 3A), \\
V_{ub}^*V_{ud} R_{10}^{(fs)} &= \frac{\sqrt{3}}{2} (-T_1 + T_2 + C_1 - C_2),
\end{align*}
\]

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Table III: Amplitudes for $\Delta S = 1$ $B$-meson decays to fully-symmetric $PPP$ states as a function of the three-body diagrams ($\bar{b} \to \bar{s}$ diagrams are written with primes).

| Decay                                      | $V_{cb}^*V_{cs}$ | $V_{ub}^*V_{us}$ |
|--------------------------------------------|------------------|------------------|
| $A(B^+ \to K^+\pi^+\pi^-)$                | $\tilde{P}_{ct}'$ $PA_{ct}'$ | $\tilde{P}_{ut}'$ $C_1'$ $C_2'$ $T_1'$ $T_2'$ $E'$ $A'$ $PA_{ut}'$ |
| $\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)$      | 1 0              | 1 0              |
| $\sqrt{2}A(B^+ \to K^0\pi^+\pi^-)$      | 0 0              | 0 0              |
| $A(B^0 \to K^0\pi^+\pi^-)$               | $\sqrt{2}A(B^0 \to K^0\pi^0\pi^0)$ | $\sqrt{2}A(B^0 \to K^+\pi^0\pi^-)$ |
| $A(B^0 \to K^+K^0\bar{K}^0)$             | 1 0              | 1 0              |
| $\frac{1}{\sqrt{2}}A(B^+ \to K^+K^+K^-)$ | $\sqrt{2}A(B^0 \to K^+K^-\bar{K}^0)$ | $\sqrt{2}A(B^0 \to K^\pi^-K^+\bar{K}^0)$ |
| $\sqrt{2}A(B^0 \to K^\pi^-K^+\bar{K}^0)$ | $\sqrt{2}A(B^0 \to K^\pi^-K^+\bar{K}^0)$ | $\sqrt{2}A(B^0 \to K^\pi^-K^+\bar{K}^0)$ |
| $A(B^0 \to K^0\pi^+\pi^-)$               | $\sqrt{2}A(B^0 \to K^0\pi^0\pi^0)$ | $\sqrt{2}A(B^0 \to K^0\pi^0\pi^0)$ |
| $\sqrt{2}A(B^0 \to \pi^0K^+\bar{K}^0)$  | $\sqrt{2}A(B^0 \to \pi^0K^0\bar{K}^0)$ | $\sqrt{2}A(B^0 \to \pi^0K^0\bar{K}^0)$ |
| $\frac{2}{\sqrt{3}}A(B^0 \to \pi^0\pi^0\pi^0)$ | $\frac{2}{\sqrt{3}}A(B^0 \to \pi^0\pi^0\pi^0)$ | $\frac{2}{\sqrt{3}}A(B^0 \to \pi^0\pi^0\pi^0)$ |
Table IV: Amplitudes for $\Delta S = 0$ $B$-meson decays to fully-symmetric $PPP$ states as a function of the three-body diagrams.

| Decay | $V_{cb}^*V_{cd}$ | $V_{ub}^*V_{ud}$ |
|-------|-----------------|-----------------|
|       | $\tilde{P}_{ct}$ $PA_{ct}$ | $\tilde{P}_{ut}$ $C_1$ $C_2$ $T_1$ $T_2$ $E$ $A$ $PA_{ut}$ |
| $A(B^+ \to \pi^+ K^0 \bar{K}^0)$ | 1 0 | 1 0 0 0 0 0 1 0 |
| $A(B^+ \to \pi^+ K^+ K^-)$ | -1 0 | -1 -1 0 0 -1 0 -1 0 |
| $\sqrt{2} A(B^+ \to \pi^0 K^+ \bar{K}^0)$ | 0 0 | 0 0 -1 -1 0 0 0 0 |
| $\sqrt{2} A(B^0 \to \pi^0 K^0 K^0)$ | 2 1 | 2 0 -1 0 0 0 0 1 |
| $\sqrt{2} A(B^0 \to \pi^0 K^+ K^-)$ | 0 1 | 0 -1 0 0 0 0 2 0 1 |
| $A(B^0 \to \pi^+ K^0 K^-)$ | -1 -1 | -1 0 0 0 -1 -1 0 -1 |
| $A(B^0 \to \pi^- K^+ K^-)$ | -1 -1 | -1 0 0 -1 0 -1 0 -1 |
| $\sqrt{2} A(B^+ \to \pi^+ \pi^0 \pi^0)$ | 1 0 | 1 1 0 0 1 0 1 0 |
| $\frac{1}{\sqrt{2}} A(B^+ \to \pi^+ \pi^+ \pi^-)$ | -1 0 | -1 -1 0 0 -1 0 -1 0 |
| $\frac{2}{\sqrt{3}} A(B^0 \to \pi^0 \pi^0 \pi^0)$ | 1 0 | 1 1 -1 0 0 -1 0 0 |
| $\sqrt{2} A(B^0 \to \pi^0 \pi^- \pi^-)$ | -1 0 | -1 -1 1 0 0 1 0 0 |
| $\sqrt{2} A(B^0_s \to \bar{K}^0 \pi^0 \pi^0)$ | 1 0 | 1 1 -1 0 0 0 0 0 0 |
| $A(B^0_s \to \bar{K}^0 \pi^+ \pi^-)$ | -1 0 | -1 -1 0 0 -1 0 0 0 |
| $\sqrt{2} A(B^0_s \to K^- \pi^+ \pi^-)$ | 0 0 | 0 0 1 0 1 0 0 0 0 |
| $\frac{1}{\sqrt{2}} A(B^0_s \to \bar{K}^0 K^{0}\pi^0)$ | 1 0 | 1 0 0 0 0 0 0 0 0 |
| $A(B^0_s \to \bar{K}^0 K^+ K^-)$ | -1 0 | -1 -1 0 -1 0 0 0 0 0 |
A corresponding set of relations exists for the diagrams corresponding to $\bar{b} \rightarrow \bar{s}$ decays. This demonstrates the equivalence of diagrams and SU(3) for the fully-symmetric $PPP$ state.

Now, there are also four EWP diagrams that can contribute to the amplitudes: $P_{EW1,2}$ and $P_{EW1,2}^C$. However, as Eq. (31) shows, in $B \rightarrow PPP$ the equivalence of diagrams and SU(3) holds without EWP’s (as was the case for $B \rightarrow PP$). The EWP diagrams are therefore not independent. Their addition only has the effect of redefining the diagrams:

\[ T_i \rightarrow T_i + P_{EWi}^C, \]
\[ C_i \rightarrow C_i + P_{EWi}, \]
\[ (\tilde{P}_{ut} + \tilde{P}_{ct}) \rightarrow (\tilde{P}_{ut} + \tilde{P}_{ct}) - \frac{2}{3} P_{EW1} - \frac{1}{3} (P_{EW1}^C + P_{EW2}^C), \]
\[ (PA_{ut} + PA_{ct}) \rightarrow (PA_{ut} + PA_{ct}) + \frac{2}{3} P_{EW1}, \]

where $i = 1, 2$. The above shows that EWP contributions do not introduce new independent SU(3) amplitudes, but does not indicate how EWP’s are related to the other diagrams. The exact EWP-tree relations for the fully-symmetric state were given in Ref. [2]. Taking $c_1/c_2 = c_9/c_{10}$ (which holds to about 5%), the simplified form is

\[ P_{EWi} = \kappa T_i, \quad P_{EWi}^C = \kappa C_i, \]

where

\[ \kappa \equiv \frac{3 |\lambda_i^{(d)}| c_9 + c_{10}}{2 |\lambda_i^{(d)}| c_1 + c_2}. \]

C Neglect of $E/A/PA$

The utility of expressing the $B \rightarrow PPP$ amplitudes in terms of diagrams is that it is relatively easy to ascertain which diagrams contribute to a given decay amplitude, while it is much more difficult to do this for the SU(3)-matrix elements. However, the great advantage of using diagrams is that it provides dynamical input. In particular, the diagrams $E$, $A$ and $PA$ all involve the interaction of the spectator quark. As such, they are expected to
be considerably smaller than the $T_i$, $C_i$ and $\tilde{P}$, and can therefore be neglected, to a first approximation.

In $B \to PPP$ decays, if $E/A/PA$ are neglected, one has

\[
\begin{align*}
R_{10}^{(fs)} &= -\frac{6}{\sqrt{5}} R_8^{(fs)}, \\
P_{27}^{(fs)} &= \frac{1}{18} \sqrt{\frac{7}{2}} (12 P_8^{(fs)} - 5\sqrt{2} P_{10}^{(fs)}), \\
B_1^{(fs)} &= -2\sqrt{\frac{2}{3}} B^{(fs)}.
\end{align*}
\]

The first two relations above reduce the number of combinations of reduced matrix elements in SU(3) from seven to five. However, because $B_1^{(fs)}$ and $B^{(fs)}$ always appear with $A_1^{(fs)}$ and $A^{(fs)}$, respectively [Eq. (13)], the third relation does not lead to a further change in the number of combinations of reduced matrix elements. Given that the 16 $\bar{b} \to \bar{s}$ decay amplitudes are now expressed in terms of five combinations of matrix elements, there must be 11 independent relations among the amplitudes. That is, there must be two additional relations, and similarly for the $\bar{b} \to \bar{d}$ amplitudes. What are they?

For $\bar{b} \to \bar{s}$, the additional relations are as follows:

1. The $KK\bar{K}$ relation of Eq. (15) is split into two relations:

\[
\begin{align*}
A(B^+ \to K^+K^+K^-)_{FS} &= \sqrt{2} A(B^0 \to K^+K^0K^-)_{FS}, \\
\sqrt{2} A(B^+ \to K^+K^0\bar{K}^0)_{FS} &= A(B^0 \to K^0K^0\bar{K}^0)_{FS}.
\end{align*}
\]

2. $A(B^0_s \to \pi^0\pi^+\pi^-)_{FS} = 0$, i.e., the two decays $B^0_s \to \pi^0\pi^+\pi^-$ and $B^0_s \to \pi^0\pi^0\pi^0$ are pure $E'$.

For $\bar{b} \to \bar{d}$, the additional relations are as follows:

1. $A(B^0_s \to \overline{K}^0\pi^+\pi^-)_{FS} = A(B^+ \to \pi^+K^+K^-)_{FS}$.

2. $A(B^+ \to \pi^+\pi^0\pi^0)_{FS} + A(B^0 \to \pi^0\pi^+\pi^-)_{FS} = A(B^0_s \to K^-\pi^+\pi^0)_{FS}$.

The above relations are not due to group theory alone – there is dynamical input.

The remaining five relations among the $B \to PPP$ amplitudes can be found among the eight U-spin relations described in Sec. IIIC. In fact, for the case where $E/A/PA$ are neglected, most of the $B \to PPP$ amplitude relations can be cast in terms of U-spin relations. The full list, including the true U-spin pairs, is [18]

1. $(B^0 \to K^+K^-K^0, B^+ \to K^+K^+K^0, B^+ \to K^+\pi^+\pi^-)$ and $(B^+ \to \pi^+K^-K^+, B^+ \to \pi^+\pi^0\pi^0, B^+ \to \pi^+\pi^+\pi^-, B^0_s \to \bar{K}^0\pi^+\pi^-)$,

2. $(B^+ \to K^+K^0\bar{K}^0, B^0 \to K^0K^0\bar{K}^0)$ and $(B^+ \to \pi^+K^0\bar{K}^0, B^0_s \to \bar{K}^0\bar{K}^0K^0)$,
3. $B^0 \rightarrow K^0\pi^+\pi^-$ and $B^0_s \rightarrow \overline{K}^0K^+K^-$,
4. $B^0_s \rightarrow \pi^-K^+\overline{K}^0$ and $B^0 \rightarrow \pi^+K^0K^-$,
5. $B^0_s \rightarrow \pi^+K^-K^0$ and $B^0 \rightarrow \pi^-K^+\overline{K}^0$,
6. $(B^+ \rightarrow K^0\pi^+\pi^0, B^0 \rightarrow K^+\pi^-\pi^0)$ and $B^+ \rightarrow K^+\overline{K}^0\pi^0$,
7. $B^0 \rightarrow K^0\pi^0\pi^0$ and $(B^0 \rightarrow \pi^0\pi^0\pi^0, B^0 \rightarrow \pi^0\pi^+\pi^-, B^0_s \rightarrow \overline{K}^0\pi^0\pi^0)$.

The decays in the first (second) parentheses are $\bar{b} \rightarrow \bar{s}$ ($\bar{b} \rightarrow \bar{d}$) transitions.

V EXPERIMENTAL TESTS

We have expressed all decay amplitudes in terms of matrix elements, and shown the equivalence with the diagrammatic description. However, the real goal is to produce predictions that can be tested experimentally. Some of the amplitude relations we have presented relate more than two decay processes and hence involve relative strong phases among amplitudes. Since such strong phases are difficult to measure, these relations are hard to verify experimentally. Thus, the most interesting of them are equalities between two amplitudes. Although there can be a relative strong phase between the two amplitudes, such a phase does not affect the relationship between the magnitudes, which are much easier to measure. In this section we focus on such “two-amplitude equalities” and discuss how they can be tested.

A Measuring the PPP fully-symmetric amplitude

When considering any experimental tests, the first question is: how can the fully-symmetric final state be probed? The method for doing this was discussed in Ref. [1]; it proceeds as follows.

For the decay $B \rightarrow P_1P_2P_3$, one defines the three Mandelstam variables $s_{ij} \equiv (p_i + p_j)^2$, where $p_i$ is the momentum of each $P_i$. These are not independent, but obey $s_{12} + s_{13} + s_{23} = m_{P_1}^2 + m_{P_2}^2 + m_{P_3}^2$. The $B \rightarrow P_1P_2P_3$ Dalitz plot is given in terms of two Mandelstam variables, say $s_{12}$ and $s_{13}$. Now, one can obtain the decay amplitude $\mathcal{M}(s_{12}, s_{13})$ describing this Dalitz plot. To do this, one uses an isobar model. Here the amplitude is expressed as the sum of a non-resonant and several intermediate resonant contributions:

$$\mathcal{M}(s_{12}, s_{13}) = N_{DP} \sum_j c_j e^{i\theta_j} F_j(s_{12}, s_{13}),$$

where the index $j$ runs over all contributions. Each contribution is expressed in terms of isobar coefficients $c_j$ (magnitude) and $\theta_j$ (phase), and a dynamical wave function $F_j$. $N_{DP}$ is a normalization constant. The $F_j$ take different forms depending on the contribution. The $c_j$ and $\theta_j$ are extracted from a fit to the Dalitz-plot event distribution.
Now, given $M(s_{12}, s_{13})$, one can construct the fully-symmetric amplitude. It is given simply by
\[
M_{\text{FS}} = \frac{1}{\sqrt{6}} \left[ M(s_{12}, s_{13}) + M(s_{13}, s_{12}) + M(s_{12}, s_{23}) + M(s_{23}, s_{12}) + M(s_{23}, s_{13}) + M(s_{13}, s_{23}) \right].
\] (38)
Thus, for any three-body decay for which a Dalitz plot has been measured, one can extract the fully-symmetric amplitude.

B SU(3) amplitude equalities

As noted above, two-amplitude equalities are the most interesting since they are subject to direct experimental tests. There are four such equalities under isospin symmetry, given in Eqs. (14), (17) and (22). These isospin relations may be useful as a test of the SM, constraining new-physics models in which the effective Hamiltonian involves strangeness-conserving $\Delta I = 5/2$ operators or strangeness-changing $\Delta I = 2$ operators.

If one restricts attention to final states involving only charged kaons and pions, there are two equalities due to U spin; they relate two $b \to \bar{s}$ decays, and two $b \to \bar{d}$ decays. They are given in Eqs. (18) and (25), and repeated for convenience below:
\[
\sqrt{2} A(B^+ \to K^+\pi^+\pi^-)_{\text{FS}} = A(B^+ \to K^+K^+K^-)_{\text{FS}},
\]
\[
\sqrt{2} A(B^+ \to \pi^+K^+K^-)_{\text{FS}} = A(B^+ \to \pi^+\pi^+\pi^-)_{\text{FS}}.
\] (39)
Consider the first equality, the “$K\pi\pi$-$KK\bar{K}$ relation.” Since we are not interested in the overall phase of the amplitudes, the relation can be written as
\[
\frac{|A(B^+ \to K^+K^+K^-)_{\text{FS}}|}{\sqrt{2}|A(B^+ \to K^+\pi^+\pi^-)_{\text{FS}}|} = 1,
\] (40)
and note that it also holds for $B^-$ decays. The key point is that this relation holds at every point in the Dalitz plot (though we can only use one sixth of the Dalitz plot due to the fact that the amplitudes are fully symmetric). Thus, this ratio should be measured for each Dalitz-plot point, and then one should average over all points. This is extremely important, as it has the effect of reducing the errors. (As always, correlations among the different points must be taken into account in calculating the error.)

Of course, SU(3)-breaking effects can lead to a violation of the $K\pi\pi$-$KK\bar{K}$ relation. One can see such effects, for instance, in the different boundaries of the Dalitz plots for $B \to \pi\pi\pi$, $K\pi\pi$, $\pi K\bar{K}$, and $KK\bar{K}$, as shown in Fig. II. In order to treat SU(3) breaking rigorously, one should introduce into the $B \to PPP$ matrix elements a spurion mass term $M_{brk} \sim (2s\bar{s} - u\bar{u} - d\bar{d})$, behaving as an SU(3) octet. When one does this, one finds several SU(3)-breaking terms. However, as far as the $K\pi\pi$-$KK\bar{K}$ relation is concerned, the overall effect is to modify the two-amplitude equality. In the presence of SU(3) breaking, we have
\[
A(B^+ \to K^+K^+K^-)_{\text{FS}} = X \sqrt{2} A(B^+ \to K^+\pi^+\pi^-)_{\text{FS}},
\] (41)
where $X$ is the SU(3)-breaking factor. Note that $X$ does not take the same group-theoretical form at each point in the Dalitz plot. It is simply a complex number that affects the equality. Eq. (40) is now modified:

$$\frac{|A(B^+ \to K^+ K^+ K^-)_{FS}|}{\sqrt{2}|A(B^+ \to K^+ \pi^+ \pi^-)_{FS}|} = |X|.$$ (42)

Now, as is always the case, the size of SU(3) breaking is unknown (though it is typically $\lesssim 25\%$). It is logically possible that $|X| - 1 > 0$ at each point in the Dalitz plot, perhaps very much larger. However, given that $X$ depends in a complicated way on all the SU(3)-breaking terms, it seems more likely that $|X| - 1 > 0$ at some points, and $|X| - 1 < 0$ at others. In this situation, averaging over all Dalitz-plot points will reduce the effect of SU(3) breaking. Of course, there is no guarantee that this occurs for the $K\pi\pi$-$KK\bar{K}$ relation, but whether or not it happens will be determined experimentally. And SU(3) breaking could be smaller
simply due to the fact that all spin-1 resonances, and the associated SU(3) breaking, are absent from the fully-symmetric amplitudes.

Indeed, this type of effect was seen in Ref. [4], where some $B \to K\pi\pi$ and $B \to KK\bar{K}$ decays were analyzed. There, the approximation was made that, to leading order, the SU(3) breaking in all diagrams is equal, so that $A(B \to KK\bar{K}) = \alpha_{SU(3)}A(B \to K\pi\pi)$, where $\alpha_{SU(3)}$ measures the amount of SU(3) breaking. Averaged over the entire Dalitz plot, it was found that $|\alpha_{SU(3)}| = 0.97 \pm 0.05$, i.e., an SU(3) breaking of $\sim 5\%$ was found. Perhaps something like this happens with the $K\pi\pi-KK\bar{K}$ relation.

Now, LHCb has measured [5]

\[
A_{CP}(B^+ \to K^+\pi^+\pi^-) = +0.032 \pm 0.008(\text{stat}) \pm 0.004(\text{syst}) \pm 0.007(J/\psi K^+) ,
\]

\[
A_{CP}(B^+ \to K^+K^-) = -0.043 \pm 0.009(\text{stat}) \pm 0.003(\text{syst}) \pm 0.007(J/\psi K^+) ,
\]

and [6]

\[
A_{CP}(B^+ \to \pi^+K^-K^-) = -0.141 \pm 0.040(\text{stat}) \pm 0.018(\text{syst}) \pm 0.007(J/\psi K^+) ,
\]

\[
A_{CP}(B^+ \to \pi^+\pi^+\pi^-) = +0.117 \pm 0.021(\text{stat}) \pm 0.009(\text{syst}) \pm 0.007(J/\psi K^+) ,
\]

with larger asymmetries, with the same signs as the above, observed in localized regions of phase space. As with all measurements, one wants to compare these results with the predictions of the SM.

Using measured values of corresponding branching ratios [22], relative signs and relative magnitudes of the above asymmetries were shown to be in modest and reasonable agreement with two U-spin predictions [7],

\[
\frac{A_{CP}(B^+ \to \pi^+K^-K^-)}{A_{CP}(B^+ \to \pi^+\pi^-)} = \frac{\mathcal{B}(B^+ \to K^+\pi^-)}{\mathcal{B}(B^+ \to \pi^+K^-)} ,
\]

\[
\frac{A_{CP}(B^+ \to \pi^+\pi^-)}{A_{CP}(B^+ \to K^+K^-)} = \frac{\mathcal{B}(B^+ \to K^+K^-)}{\mathcal{B}(B^+ \to \pi^+\pi^-)} .
\]

On the other hand, SM predictions for the asymmetries themselves are much less certain because direct CP asymmetries involve unknown strong phases.

The $K\pi\pi-KK\bar{K}$ relation [Eq. (40)] and a similar $\pi K\bar{K}-\pi\pi\pi$ relation provide clean tests of the SM within the flavor SU(3) approximation. They say that, for the fully-symmetric states, the amplitudes of the two pairs of decays, whatever values they take, are equal at all points of the Dalitz plot. As we have argued, SU(3) breaking in these two equalities is expected to be reduced when averaged over the entire Dalitz plot. In order to test these two relations, LHCb must extract the fully-symmetric amplitudes for the four decays. As noted above, this requires performing an isobar analysis of the Dalitz plots of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+K^-K^-$, $B^+ \to \pi^+K^+K^-$ and $B^+ \to \pi^+\pi^+\pi^-$. However, LHCb already has these Dalitz-plot data, so it is straightforward to test the SM using the $K\pi\pi-KK\bar{K}$ and $\pi K\bar{K}-\pi\pi\pi$ relations. This can be done now, and we strongly encourage LHCb to carry out these analyses.
Finally, in Sec. IIIC it was noted that there are eight pairs of $\bar{b} \to \bar{s}$ and $\bar{b} \to \bar{d}$ three-body decays that are related by U spin. As pointed out in Refs. [17, 18], the decay rates and direct CP asymmetries for each U-spin pair satisfy Eq. (26). Two examples, Eqs. (45) and (46), have already been tested by the asymmetries of Eqs. (43) and (44), but for the unsymmetrized final states. One thing we will add is that the prediction of Eq. (26) holds at each point of the Dalitz plot. Thus, for the fully-symmetric final states of the U-spin pairs, one should average over all points. That is, in the presence of U-spin breaking, Eq. (26) is modified to be

$$-\frac{A_s}{A_d} \frac{B_s}{B_d} = Y,$$

(47)

where $Y$ is the U-spin-breaking factor. It is a real number that can take different values at different points in the Dalitz plot. If $Y > 1$ at some points, and $Y < 1$ at others – and this will be determined experimentally – then averaging over all Dalitz-plot points will reduce the effect of U-spin breaking, as well as the statistical error.

C Neglect of $E/A/PA$

When $E/A/PA$ diagrams are neglected, one finds some new two-amplitude equalities. These provide a good test of the assumption:

$$A(B^+ \to K^+ K^+ K^-)_{FS} = \sqrt{2} A(B^0 \to K^+ K^0 K^-)_{FS},$$

$$\sqrt{2} A(B^+ \to K^+ K^0 K^0)_{FS} = A(B^0 \to K^0 K^0 K^0)_{FS},$$

$$A(B^0_s \to \pi^0 \pi^+ \pi^-)_{FS} = 0,$$

$$A(B^0_s \to \overline{K}^0 \pi^+ \pi^-)_{FS} = A(B^+ \to \pi^+ K^+ K^-)_{FS}.$$  

(48)

The BaBar collaboration has studied the decays $B^+ \to K^+ K^+ K^-$ and $B^0 \to K^+ K^0 K^-$, as well as $B^+ \to K^+ K^0 \overline{K}^0$ and $B^0 \to K^0 \overline{K}^0 \overline{K}^0$ [23]. BaBar data can be used to construct the fully-symmetric states in these processes to test the first two of the above relationships.

VI CONCLUSIONS

In charmless $B \to PPP$ decays, flavor SU(3) treats the three final-state particles as identical. As there are six permutations of these particles, there are six possibilities for the final state: a totally symmetric state, a totally antisymmetric state, or one of four mixed states. In this paper, we examine the properties of the fully-symmetric final state for the case where the final-state particles are all $\pi$’s or $K$’s.

We begin by writing all $B \to PPP$ decay amplitudes as a function of the SU(3) reduced matrix elements. There are seven independent combinations of these matrix elements. On the other hand, there are 16 $\bar{b} \to \bar{s}$ and 16 $\bar{b} \to \bar{d}$ decays, for a total of 32. We work out the 25 relations among the amplitudes in the SU(3) limit. Several of these can be tested experimentally.
We also present all $B \to PPP$ decay amplitudes as a function of diagrams. By comparing the two expressions for the amplitudes, we are able to write the matrix elements as a function of diagrams, demonstrating the equivalence of diagrams and SU(3). One of the advantage of using diagrams is that it provides dynamical input. In particular, three of the diagrams – $E$, $A$ and $PA$ – all involve the interaction of the spectator quark, and are expected to be considerably smaller than the other diagrams. If $E/A/PA$ are neglected, there are additional relations among the amplitudes, some of which can also be tested experimentally.

One relation that provides a good test of the SM is the following. In the SU(3) limit, one has the equality between two $\bar{b} \to \bar{s}$ decay amplitudes: $\sqrt{2} A(B^+ \to K^+ \pi^+ \pi^-)_{FS} = A(B^+ \to K^+ K^+ K^-)_{FS}$. That is, the amplitudes for the fully-symmetric state of these two decays are predicted to be equal at each point in the Dalitz plot. Now, LHCb has already measured the Dalitz plots for these decays. An isobar analysis of the Dalitz plots allows the fully-symmetric amplitudes to be constructed, so that this equality, and the SM, can be tested. It is important to average over all Dalitz-plot points. This reduces the statistical error, and possibly even the effect of SU(3) breaking. A similar analysis can be done for the equality between two $\bar{b} \to \bar{d}$ decay amplitudes: $\sqrt{2} A(B^+ \to \pi^+ K^+ K^-)_{FS} = A(B^+ \to \pi^+ \pi^+ \pi^-)_{FS}$. These tests can be done now; it is hoped that LHCb will carry out these analyses.

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