Long-Range k-core Percolation

Cristian F Moukarzel$^1$ and T Sokolowski$^2$

$^1$ CINVESTAV del IPN, Depto. de Física Aplicada, 97310 Mérida, Yucatán, México.
$^2$ Fachrichtung Theoretische Physik, Universität des Saarlandes, 66041 Saarbrücken, Germany.
E-mail: cristian@mda.cinvestav.mx

Abstract. Bootstrap (or k-core) Percolation with $k = 3$ is studied numerically on two-dimensional lattices with long-range links whose lengths $r_{ij}$ are distributed according to $P(r) \sim r^{-\alpha}$. By varying the decay exponent $\alpha$ the topology of these networks can be made to range from two-dimensional short-range networks to $\infty$-dimensional random graphs. The 3-core transition is found to be of first-order character with a divergent correlation length for $\alpha < 2.75$ and of second order for larger $\alpha$. Whenever the transition is first-order an associated critical corona is found to exist. The correlation length exponent $\nu$ defined from the corona correlation length above the first order transition is estimated as $\nu \approx 1/2$ for $\alpha = 0$, and only shows a weak $\alpha$ dependence for $\alpha \leq 2.50$. The second-order transition at large $\alpha$ is found to be in the universality class of two-dimensional Percolation.

1. Introduction

k-core [1] (or Bootstrap Percolation [2]) generalizes Percolation [3] by requiring that a site have $k$ or more neighbors in a cluster in order to be connected to it. Equivalently the k-core (or ‘core’ for simplicity) is the maximal subset of a graph with minimal coordination $k$. The core identifies a subset of ‘strongly connected sites’, a concept that finds many applications in Network Analysis, including Biological Evolution [4], Dynamics of Avalanches [5, 6, 7], Community Identification [8], and Modelling of Neuron Dynamics [9], among many others.

In mean-field the k-core undergoes, for all $k > 2$, a first-order critical transition when the average coordination $\gamma$ reaches a critical value $\gamma_c$. At this transition the density $P_\infty$ of sites in the largest core cluster (the order parameter) has a power-law singularity on top of a discontinuity, of the form

$$P_\infty = \begin{cases} 
0 & \text{for } \gamma < \gamma_c \\
P_c + a(\gamma - \gamma_c)^{1/2} & \text{for } \gamma > \gamma_c 
\end{cases}$$

In two and three dimensions however, numerical results and analytic arguments indicate that, depending on $k$ and on the lattice coordination, k-core has either a trivial transition at $p = 1$ or a critical point at a non trivial $p_c$ but in the universality class of Percolation [10, 11, 12, 13]. In four dimensions k-core it is believed to have a non-trivial ($p_c \neq 1$) but non-critical first-order transition [14].

Recently it has been shown [15, 16] that a critical correlation-length can be defined for k-core in terms of the corona, which is the subset of the core with exactly $k$ neighbors in the core.
Analytical calculations in mean-field show that the corona correlation length diverges at $\gamma_c^+$ in the usual power-law fashion $\xi \sim (\gamma - \gamma_c)^\nu$, with a thermal exponent $\nu$ that is either 1/2 or 1/4. Schwartz, Liu and Chayes (SLC) [15] pointed out a number of similarities between k-core and the Jamming transition [17] of repulsive spheres in finite dimensions, and suggested that k-core is a simplified model for Jamming. At the Jamming transition of frictionless spheres, the average coordination $z$ jumps discontinuously from zero to the isostatic [18] value $z_c = 2d$ at the Jamming density $\phi_c$, and behaves as

$$z = z_c + a (\phi - \phi_c)^\beta, \quad \text{for } \phi \geq \phi_c,$$

where $\beta$ has been estimated numerically to be close to 1/2 in 2 and 3 dimensions [17]. While packings of stiff spheres are isostatic [19, 18] with $z = z_c$ in any dimension, compressed packings of deformable spheres develop an excess number of contacts $\delta z = z - z_c$ and become hyperstatic. Wyart [20] has argued that for weakly hyperstatic packings a length scale $l^* \sim (\delta z)^{-1}$ can be defined such that the system behaves essentially as an isostatic one below $l^*$. This length scale $l^*$ would therefore diverge as $(\delta \phi)^{-1/2}$ on approach to the unjamming transition. Additionally, numerical simulations [21] near the unjamming transition find a dynamical length that diverges with a thermal exponent which is near 1/4.

SLC also remark that the k-core connectivity condition is satisfied in a jammed packing, since a sphere in $d$ dimensions must be in contact with at least $d + 1$ other spheres in order to be itself locally jammed. Based on these similarities, SLC propose that k-core is a good description of the Jamming transition. Difficulties with this interpretation include the fact that k-core is only first-order in MF but not in two and three dimensions, while the Jamming transition is first order in two and three dimensions.

The aim of the present work is to investigate how the critical behavior of k-core changes from second-order to first-order critical, as space dimensionality increases. As an alternative to studying lattices of various dimensions, which is time-consuming and only allows for small linear lengths in large dimensions, in this work k-core Percolation with $k = 3$ is studied numerically on two-dimensional networks with random long-range (LR) links. By adequately choosing the length-distribution of LR links, networks can be built that have an effective dimension between two and $\infty$ [22, 23, 24, 25].

2. LR networks

Our LR networks are built by taking a 2-dimensional square array of $N = L^2$ points, and adding to it a total of $\gamma N/2$ randomly located links, where $\gamma > 0$ is a parameter specifying the average coordination. A second parameter $\alpha \geq 0$ determines the a priori length-distribution of links through

$$P_{ij} \sim r_{ij}^{-\alpha},$$

where $r_{ij}$ is the Euclidean distance between $i$ and $j$ in the two dimensional array. For $\alpha \to \infty$, only near neighbors are connected, and the resulting network, although being disordered, has a two-dimensional topology if $\gamma$ is above its percolation threshold. For $\alpha \to 0$ all pairs $ij$ are connected with the same probability, and a Random Graph [26] is obtained (See Fig. 1).

2.1. Numerical implementation

In order to implement this link distribution numerically, we first choose a random site $i$ and then a site $j$ in such a way that (3) is satisfied. If $j$ is already connected to $i$ then this link is rejected and a new site $j$ is chosen, otherwise $j$ is connected to $i$.

Consider the following two ways to generate link-length distribution (3) numerically: Method A consists in calculating and storing beforehand, for each relative vector $\{i, j\}$ on the 2D lattice,
Figure 1. Sites in the 3-core (blue squares) for 30 × 30 networks whose LR links have a decay exponent $\alpha$ respectively equal to 0, 2, 4 and 10 (from left to right and from top to bottom). The average coordination is $\gamma = 3.5$ in all cases. Shown in red are the links in the core. Helicoidal boundary conditions were used.

the normalized probability (3) for the given value of $\alpha$, and then choosing one vector $\{i, j\}$ using those probabilities. The choosing step, however, takes time proportional to $N$ for each new link if $\alpha$ is small, making Method A very inefficient in general. Method B consists in first generating a random length $\ell$ between 1 and $L/2$ with probability $P(\ell) \sim \ell^{-(\alpha-d+1)}$ (the extra $\ell^{d-1}$ factor with respect to (3) takes into account the number of lattice points inside a thin shell of radius $\ell$) and then a random direction $\vec{e}$. Site $j$ is chosen to be the one whose position $r_j$ is closest to $\vec{r}_i + \ell \ast \vec{e}$. This alternative method however is not efficient for large values of $\alpha$ and $\gamma$, since very short lengths $\ell$ are generated most of the times, therefore when all near neighbors of $i$ are already connected to $i$, the rejection rate is very large.
We have chosen to implement a hybrid of methods A and B, that circumvents the disadvantages mentioned above. The set of lattice vectors \( \{i,j\} \) is divided into two classes: those respectively shorter and longer than a prespecified length \( \ell_0 \). For all short vectors, the probabilities \( P_{ij} \) are calculated according to (3) and stored, and from them the total weight \( P_s \) of short vectors is obtained:

\[
P_s = \sum_{ij|\ell_{ij}<\ell_0} P_{ij},
\]

where the sum runs over short vectors only. For each new link connection, a random number \( R \) is first generated and if \( R < P_s \), method A is used to choose among the list of short vectors. If \( R > P_s \) then method B is used, i.e. a random length \( \ell > \ell_0 \) is generated, etc. We have found that this hybrid algorithm with a limit length \( \ell_0 = 1.5 \) works efficiently at all values of \( \alpha \) and \( \gamma \) analyzed so far.

2.2. Link dilution

For each fixed \( \alpha \), we determine and store the \( \gamma \)-dependence of the k-core on a given sample, i.e. a realisation of links, by starting with a large coordination and reducing it in small steps (deleting links) until the k-core disappears. Notice however that a network with coordination \( \gamma_0 \) obtained by random link-dilution of a higher-coordination network is not statistically equivalent to one built with coordination \( \gamma_0 \) by the method above. The reason for this is that (3) gives more weight to short links, so that close neighbors are connected first. As \( \gamma \) increases during the building procedure, longer links begin to appear, since most short links are already connected and double connections are not accepted. This shows that the actual link distribution depends not only on \( \alpha \) through (3) but also on \( \gamma \), since networks with large coordination will necessarily have more long links. This effect is stronger the larger \( \alpha \) is. In view of this, and in order to avoid the “irreversibility” or history dependence that random dilution would produce, we chose to implement the dilution process by eliminating links in exactly the opposite order as they were connected. This ensures that a network with coordination \( \gamma \) has the same statistical properties, no matter if it was obtained by adding or diluting links.

3. Numerical results

![Figure 2](image_url)

**Figure 2.** a) The fraction \( f_{\text{core}} \) of sites in the k-core is discontinuous for small \( \alpha \), but non-singular for large \( \alpha \). b) The order parameter \( P_\infty \) shows a singularity at \( \gamma_c \) for all \( \alpha \), being discontinuous for small \( \alpha \). All data are for \( N = 10^7 \) sites.

Early studies [27] of critical phenomena on d-dimensional LR networks suggest that in general, for a system undergoing a *second-order* phase transition, three different regimes can be expected...
as a function of $\alpha$: a MF regime at low $\alpha$ in which classical behaviour is found, a large $\alpha$ limit in which d-dimensional behaviour is expected, and a possible intermediate regime where critical indices depend on $\alpha$.

Bootstrap Percolation on LR networks could potentially display an even more complex behavior,

![Graph showing $P_\infty/f_{\text{core}}$ as a function of coordination number $\gamma$.](image)

**Figure 3.** The ratio $P_\infty/f_{\text{core}}$ indicates the relative importance of the largest core cluster, and clearly displays the character (first-order or second-order) of the transition. All data are for $N = 10^7$ sites.

since in addition to the above mentioned different critical regimes, a crossover from first-order to second-order phase transition occurs as $\alpha$ is increased. In this work we concentrate on understanding the crossover from first-order to second-order transition that happens at a critical decay rate $\alpha_0$.

We have analyzed k=3 Bootstrap Percolation on networks with $\alpha = 0$ to $\alpha = 10$, which enables us to identify two main regimes: for $\alpha < 2.75$ the 3-core transition is first-order critical, as on Random Graphs [16], while for $\alpha > 2.75$ there is a second-order percolation transition. In order to extract critical indices from finite-size scaling (FSS) we have done measurements on networks made of $N = 10^3, 10^4, 10^5, 10^6$ and $10^7$ sites. A sample average was done in each case over $10^9/N$ independent realisations of LR links. For each value of $\alpha$ we measured a number of observables that are relevant for the characterization of the k-core transition occurring when the average coordination $\gamma$ is varied. We report here mainly our results regarding the behavior of the k-core order parameter and corona correlation-length near $\gamma_c(\alpha)$.

### 3.1. Size of the core and order parameter

The fraction $f_{\text{core}}$ of sites in the k-core is shown in Fig. 2a. For small $\alpha$ this quantity suffers a discontinuity as a function of the coordination $\gamma$, but behaves regularly for large $\alpha$. Because of the
tree-like topology of these LR networks for small \( \alpha \), there are no (small) loops and therefore no finite k-core clusters. This implies that the k-core transition happens discontinuously instead of progressively nucleating from the union of many smaller clusters as in a second-order transition. The order parameter \( P_\infty \), which is the fraction of sites in the largest k-core cluster, is shown in Fig. 2b. These data indicate that the k-core transition is discontinuous for small \( \alpha \) and becomes continuous for large \( \alpha \). Further results to be described later are consistent with the fact that the critical decay exponent separating first- and second-order transition regimes is located around \( \alpha_0 \approx 2.75 \). In addition, in a certain \( \alpha \) interval it appears possible that two different transitions, one of second order and the other of first order, exist.

The ratio \( P_\infty / f_{\text{core}} \) (Fig. 3) is particularly interesting. This ratio goes to \( \approx 1 \) above the transition for all values of \( \alpha \), i.e. for first-order as well as for second-order transitions, indicating that the core becomes essentially a single connected cluster. The manner in which this quantity grows, however, depends on the type of transition, i.e. on \( \alpha \). It does so discontinuously for \( \alpha < 2.75 \), and continuously for \( \alpha > 2.75 \). This defines the critical decay rate for the crossover from first-order to second-order transition as roughly \( \alpha_0 \approx 2.75 \).

3.2. The Corona

The corona is defined \([15]\) as the subset of the k-core that has exactly k neighbors in the k-core, and plays an important role for the critical properties of the discontinuous k-core transition \([15, 16]\). The total fraction of corona sites \( f_{\text{corona}} \) jumps discontinuously at the first-order k-core transition for \( \alpha < 2.75 \) (Fig. 4a). The corona comprises a finite fraction of the k-core for all values of the coordination \( \gamma \) for which a k-core exists. As seen in Fig. 4a, the corona ceases to behave singularly as soon as the k-core transition becomes continuous for large \( \alpha \).

3.3. Corona correlation length

A correlation length \( \xi_{\text{corona}} \) can be defined \([15, 16]\) in terms of the decay rate of the number \( N_{\text{crn}}(\ell) \) of corona sites \( \ell \) steps away from a randomly chosen k-core site via a path of corona sites.

\[
\xi_{\text{corona}}^{-1} = \lim_{\ell \to \infty} \frac{1}{\ell} \log N_{\text{crn}}(\ell),
\]

Figure 4. a) The fraction of sites with exactly three neighbors in the core, or corona, is responsible for the critical properties at the first-order transition \([15, 16]\). b) The average chemical extension of the corona, or corona correlation length, is singular at the first-order transition, but remains finite for all \( \gamma \) for large \( \alpha \). Data are for \( N = 10^7 \) sites.
It has been shown that, for random graphs [16] (\(\alpha = 0\) in our networks) or on Bethe lattices [15], this length diverges above the k-core transition as 
\[ \xi_{\text{corona}} \sim |\gamma - \gamma_c|^{-\nu} \]
However there seems to be a discrepancy as to whether \(\nu = 1/2\) [16] or \(\nu = 1/4\) [15].

Alternatively, it is possible to define \(\xi_{\text{corona}}^{(n)}\) from the \(n\)-th moment of \(N_{\text{crn}}(\ell)\) as [28, 29]:
\[ \xi_{\text{corona}}^{(n)} = \left( \frac{\sum_{\ell} \ell^n N_{\text{crn}}(\ell)}{\sum_{\ell} N_{\text{crn}}(\ell)} \right)^{1/n}, \tag{6} \]
where often the second moment \(n = 2\) is used. The \(\gamma\)-dependence of \(\xi_{\text{corona}}^{(2)}\) is shown in Fig. 4b for several values of \(\alpha\). It is seen that \(\xi_{\text{corona}}^{(2)}\) is discontinuous at \(\gamma_c\) when \(\alpha < 2.75\).

If the \(\ell\)-dependence of \(N_{\text{crn}}(\ell)\) is purely exponential, then (6) and (5) should give equivalent results for the estimation of the critical exponent \(\nu\), since in that case \(\xi_{\text{corona}}^{(n)} \propto \hat{\xi} \forall n\). However, if one more generally assumes that
\[ N_{\text{crn}}(\ell) = \ell^{-\theta} e^{-\ell/\hat{\xi}}, \tag{7} \]
then
a) If \(\theta < 1\):
\[ \xi_{\text{corona}}^{(n)} = \hat{\xi}, \Rightarrow \nu_n = n \ \forall n. \tag{8} \]
b) If \(1 < \theta < 2\) on the other hand:
\[ \xi_{\text{corona}}^{(n)} \sim (\hat{\xi})^{1-\frac{\theta-1}{n}}, \Rightarrow \nu_n = \nu \left( 1 - \frac{\theta - 1}{n} \right). \tag{9} \]

Fig. 5 displays \(\xi_{\text{corona}}^{(n)}\), \(n = 1, 2, 4, 8\) as obtained from the sample-averaged layer-mass distribution of corona clusters \(N_{\text{crn}}(\ell)\), as a function of coordination number \(\gamma\) near \(\gamma_c\) for two representative values of \(\alpha\) in the first-order regime. For \(\alpha > 2.75\), on the other hand, \(\xi_{\text{corona}}\) does not seem to grow with \(N\) at any \(\gamma_c\), indicating that the corona is composed of finite clusters when the k-core transition is of second order.

For each \(\xi_{\text{corona}}, n = 1, 2, 4, 8\) as obtained from the sample-averaged layer-mass distribution of corona clusters \(N_{\text{crn}}(\ell)\), a function of coordination number \(\gamma\) near \(\gamma_c\) for two representative values of \(\alpha\) in the first-order regime. For \(\alpha > 2.75\), on the other hand, \(\xi_{\text{corona}}\) does not seem to grow with \(N\) at any \(\gamma_c\), indicating that the corona is composed of finite clusters when the k-core transition is of second order.

For each \(\xi_{\text{corona}},\) the critical exponent \(\nu_n\) can be estimated \(^1\) by fitting the \(\gamma\)-dependence of \(\xi\) above \(\gamma_c\) as \(\xi_{\text{corona}} \sim (\gamma - \gamma_c)^{-\nu_n}\). There is a clear \(n\)-dependence in the estimated critical

---

\(^1\) The superscript \(^1\) indicates a footnote, which is not included in the text. This notation is used to denote additional explanatory notes or references that are not included in the main text.
Figure 6. The critical exponent $\nu_n(\alpha)$ obtained from fits of the $\gamma$-dependence of $\xi^{(n)}$ near $\gamma_c$ in Fig. 5 is fitted for their $n$-dependence according to (9). The estimates for $\nu$ and $\theta$ that result are in all three cases consistent with $\nu = 1/2$ and $\theta \approx 3/2$.

The critical exponent $\nu_n$, which is not consistent with (8). This suggests that $\theta > 1$ in (7). In order to check if this is the case, the exponents $\nu_n$ calculated from the fits in Fig. 5 (and an additional value $\alpha = 2.0$) were fitted for their $n$-dependence using (9), which would hold if $1 < \theta < 2$. The results are shown in Fig. 6 for $\alpha = 0$, 2, and 2.5. All three cases are roughly consistent with $\nu \approx 1/2$ and $\theta \approx 3/2$.

The fact that $\theta > 1$ and therefore that $\xi^n$ do not scale with the same critical exponent is rather unusual, so it is convenient to provide independent evidence of this fact. By fitting $N_{\text{crn}}(\ell)$ using (7) near $\gamma_c$ (Fig. 7a) a power-law decay exponent $\theta = 1.46 \pm 0.05$ is obtained, which is consistent with the result derived from the analysis of the $\gamma$-dependence of the moments of $N_{\text{crn}}(\ell)$. The plot at the center (Fig. 7b) shows that, after subtracting the exponential decay, the remaining $\ell$-dependence is very accurately described by a power law $\ell^{-\theta}$.

However, notice that analytical results [16] for k-core in the $\alpha = 0$ case indicate that $\theta = 0$ should be observed instead. On the other hand, in MF percolation models, the connectivity function $N_{\text{crn}}(\ell)$ usually decays exponentially [30], i.e. $\theta = 0$. Our numerical finding $\theta \approx 3/2$ in (7) is therefore rather puzzling and deserves further study.

3.4. The second-order Percolation transition at large $\alpha$

For large values of the decay exponent $\alpha$, the topology of our networks becomes two-dimensional and therefore we should recover the known results for Bootstrap Percolation in two dimensions.

1 Admittedly, only very roughly, since $\gamma_c$ has to be estimated as well.
Figure 7. a) The number $N_{crn}(\ell)$ (red crosses) of corona sites $\ell$ steps away from a given k-core site is described acceptably well by (7), as this plot shows. In this plot data for $\alpha = 0$ and $\gamma = 3.3505$ are shown. The green line is a fit of the form (7), which results in $\theta = 1.46 \pm 0.05$ and $\xi \approx 10^2$. b) The same data as in a), except for the fact that the exponential dependence $e^{-\ell/\xi}$ has been subtracted, showing that the remaining $\ell$-dependence is to a very good approximation a power-law decay $\ell^{-\theta}$.

Figure 8. Critical scaling exponents, at the second-order transition, for the mass $S$ of a finite core cluster (green dots) and for $\partial \log P_\infty / \partial \gamma$ (red dots) for various values of $\alpha$. These quantities scale respectively as $L^{\gamma/\nu}$ and $L^{1/\nu}$ at $\gamma_c$. Dotted lines indicate the expected (2d Percolation) values of these scaling exponents when $\alpha$ is large.
i.e. critical indices in the universality class of usual Percolation. This is verified in this section for the critical indices $\nu$ and $\gamma$.

The correlation length exponent $\nu$ can be estimated, for a second-order transition, from FSS of $\partial \log P_\infty / \partial \gamma$, which at $\gamma_c$ scales as $L^{1/\nu}$ [31]. For large $\alpha$ one expects to recover $\nu = 4/3$ of 2d Percolation [3]. Similarly, the average mass $S$ of a finite connected cluster (disregarding the largest cluster) behaves at the 2d Percolation transition as $S_L(\gamma_c) \sim L^{\hat{\gamma} / \nu}$, where $\hat{\gamma} = 43/18$ is the susceptibility exponent [3].

FSS exponents for $S$ (green circles) and $\partial \log P_\infty / \partial \gamma$ (red circles) are shown in Fig. 8 for various values of the decay exponent $\alpha$. Dashed lines indicate the expected values from 2d Percolation. For $\alpha > 4.0$ both scaling indices rapidly approach their expected 2d Percolation values, which is consistent with the notion [32] that k-core is in the universality class of usual Percolation whenever it has a second-order transition.

These scaling exponents decrease strongly around $\alpha \approx 2.75$ which is consistent with other numerical evidence indicating that the second-order transition no longer exists for $\alpha < 2.75$.

3.5. Two transitions ?

![Figure 9](image_url)

**Figure 9.** Data for $N = 10^7$ (symbols) and $N = 10^6$ (thin lines) help identify which of the maxima grow with $N$, in each of these four plots.

Data for $\partial P_\infty / \partial \gamma$ in Fig. 9a show two distinct maxima for each $\alpha$ in the range $2.7 \leq \alpha \leq 3.0$. In order for a peak in this quantity to be associated with a phase transition, it must grow with increasing $N$. The leftmost maximum in each curve on this plot indicates the incipient
second-order transition. This identification is supported by the fact that the k-core correlation length $\xi_{\text{core}}$, given by the average gyration radius of a finite k-core cluster [3], diverges (Fig. 9d) around the same value of the coordination $\gamma$. Therefore a second-order transition with a diverging correlation length seems to exist for all $\alpha \geq 2.7$. An additional criterion supporting this conclusion is the fact that $\partial(P_\infty/f_{\text{core}})/\partial \gamma$, which more clearly indicates the second-order transition, also shows growing peaks at the same locations, as Fig. 9c indicates.

The rightmost peaks in Fig. 9a, on the other hand, indicate the location of the potential first-order transition. It is seen that these peaks grow only slowly with $N$ for $\alpha \leq 2.8$, perhaps not at all for $\alpha = 2.9$, and definitely do not grow with $N$ for $\alpha = 3.0$. Looking at $\xi_{\text{corona}}$ in Fig. 9b a similar conclusion is reached. The corresponding peaks in $\xi_{\text{corona}}$ grow slowly with $N$ for $\alpha = 2.8$ and possibly 2.9 but not for $\alpha = 3.0$. This evidence would locate the upper limit of the first-order transition regime somewhere between $\alpha = 2.7$ and $\alpha = 3.0$. Notice that our results are compatible with the existence of a small $\alpha$-region around $\alpha = 2.70$ where two phase transitions (one of second order and the other of first order) might occur as a function of $\gamma$. Also interesting is the fact that the location of the second-order transition has a strong $\alpha$-dependence, while that of the first-order transition only a weak one.

4. Discussion

We have numerically studied 3-core Percolation on two-dimensional lattices with long-range links whose length distribution is $P(l) \sim l^{-\alpha}$. For $\alpha < \alpha_0 \approx 2.75$, a first-order critical transition is found, in accordance with previously known results in mean-field[33, 1, 34]. The corona, that is the subset with exactly three neighbors in the core, is critical at this first-order transition. A correlation length defined from the average chemical extension of the corona is found to diverge with a critical exponent $\nu \approx 1/2$. The decay of connectivity with chemical distance in the corona is found to be well described by (7) near $\gamma_c$, with $\theta \approx 3/2$, in slight discrepancy with recent analytical results [16] predicting $\theta = 0$.

For $\alpha > \alpha_0$ on the other hand we observe a second-order transition. Critical indices measured for this transition are consistent with known percolation indices when $\alpha$ is large.

Despite the relatively large system sizes analyzed, the limit $\alpha_0$ between the first- and second-order regimes could not be determined very precisely, and is estimated to be $\alpha_0 = 2.75 \pm 0.25$. There might as well be a small region around $\alpha = 2.7$ in which two transitions happen: a second order transition followed by a first-order one at a larger $\gamma$. However this could well be an artifact due to finite size effects.

Acknowledgments

Financial support from CONACYT, Mexico, under project 48783-F is acknowledged. TS kindly thanks the ‘Studienstiftung des deutschen Volkes’ for their financial support of his stay at CINVESTAV. CFM wishes to thank the “Computational Physics for Engineering Materials” group of ETH Zürich, for their hospitality during his stay.

[1] Pittel B, Spencer J and Wormald N 1996 J. Comb. Theory Ser. B 67 111–151
[2] Leath P L, Chalupa J and Reich G R 1979 Bull Amer Phys Soc 24 362–362
[3] Stauffer D and Aharony A 1994 Introduction to Percolation Theory 2nd ed (Bristol: Taylor and Francis)
[4] Klimke P, Thurner S and Hanel R 2009 J. Theor. Biol. 256 142–146
[5] Farrow C, Duxbury P M and Moukarzel C F 2005 Physical Review E 72
[6] Farrow C L, Shukla P and Duxbury P M 2007 J. Phys. A-Math. Theor. 40 F581–F587
[7] Iwata M and Sasa S I 2009 J. Phys. A-Math. Theor. 42 075005
[8] Saito K, Yamada T and Kazama K 2008 IEICE Trans. Fundam. Electron. Commun. Comput. Sci. E91A 3304–3311
[9] Tlusty T and Eckmann J P 2009 J. Phys. A-Math. Theor. 42 205004
[10] Vanenter A C D 1987 *J. Stat. Phys.* **48** 943–945
[11] Aizenman M and Lebowitz J L 1988 *J. Phys. A-Math. Gen.* **21** 3801–3813
[12] Schonmann R H 1990 *J. Stat. Phys.* **58** 1239–1244
[13] Chaves C M and Koiller B 1995 *Physica A* **218** 271–278
[14] Parisi G and Rizzo T 2008 *Phys. Rev. E* **78** 022101
[15] Schwartz J M, Liu A J and Chayes L Q 2006 *Europhys. Lett.* **73** 560–566
[16] Goltsev A V, Dorogovtsev S N and Mendes J F F 2006 *Physical Review E* **73**
[17] O’Hern C S, Silbert L E, Liu A J and Nagel S R 2003 *Phys. Rev. E* **68** 011306
[18] Moukarzel C F 1998 *Phys. Rev. Lett.* **81** 1634–1637
[19] Moukarzel C F 2005 *Physica A* **356** 157–161
[20] Xu N, Wyart M, Liu A J and Nagel S R 2007 *Phys. Rev. Lett.* **98** 175502
[21] Silbert L E, Liu A J and Nagel S R 2005 *Phys. Rev. Lett.* **95** 098301
[22] Kleinberg J M 2000 *Nature* **406** 845–845
[23] Jespersen S and Blumen A 2000 *Phys. Rev. E* **62** 6270–6274
[24] Moukarzel C F and de Menezes M A 2002 *Phys. Rev. E* **65** 056709
[25] Moukarzel C F 2005 *Physica A* **356** 157–161
[26] Bollobás B 2001 *Random Graphs* 2nd ed (Cambridge: Cambridge University Press)
[27] Fisher M E, Nickel B G and Ma S 1972 *Phys. Rev. Lett.* **29** 917–&
[28] Nguyen B G 1987 *J. Stat. Phys.* **46** 517–523
[29] Hara T 1990 *Probab. Theory Relat. Field* **86** 337–385
[30] Bunde A and Havlin S (eds) 1996 *Fractals and Disordered Systems* 2nd ed (Heidelberg: Springer Verlag)
[31] Mailhout A, Plumer M L and Caille A 1994 *Phys. Rev. B* **50** 6854–6859
[32] Branco N S and Silva C J 1999 *Int. J. Mod. Phys. C* **10** 921–930
[33] Chalupa J, Leath P L and Reich G R 1979 *J Phys-C-Solid State Phys* **12** L31–L35
[34] Moukarzel C, Duxbury P M and Leath P L 1997 *Phys. Rev. E* **55** 5800–5811