INVARINACE OF 0-CURRENTS UNDER DIFFUSIONS.

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ABSTRACT. We study two actions of a stochastic flow \( \varphi_t \) on the space of 0–currents \( T \) of a differentiable manifold \( M \). In particular, we give conditions on a current \( T \) to be invariant under these actions. Also, we apply our results to the case of 0–currents that come from harmonic measures associated to certain type of foliations on homogeneous spaces.

1. INTRODUCTION.

In dynamical systems some evolution problems can be related to the evolution of geometrical currents over a differentiable manifold. For example, consider the following situation: let \( D \) be a domain in \( \mathbb{R}^n \) and suppose that we want to study the evolution of the volume of \( D \) under a flow \( \varphi_t : \mathbb{R}^n \to \mathbb{R}^n \). This problem can be studied from different points of views. In [10], Kinateder and McDonald used the fact that \( D_t = \varphi_t(D) \) is a stochastic process on a Frechet manifold, and developed a stochastic calculus for that setup to study the behavior of the volume \( \text{vol}(D_t) \).

Another approach to the same problem, can be given by a simple use of the change of variables formula. In fact, we observe that \( \text{vol}(D_t) = \int_{D_t} dx = \int_D \varphi_t^* dx \).

Therefore, the study of the evolution of \( \text{vol}(D_t) \) turns into the study of the evolution of the geometric current \( T_t \) induced by the stochastic \( n \)-form \( \varphi_t^* dx \) on \( \mathbb{R}^n \). The main advantage of making this is that we change a structure of a Frechet manifold by a vector space. Then, our object of study is a stochastic process \( T_t \) on a vector space.

In this work we study the natural class of stochastic processes on the space of currents which are induced by stochastic processes in the base space. Our main set up is the following: let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, let \( \varphi_t \) be a stochastic flow on a differentiable manifold \( M \), and let \( C^\infty(M) \) be the set of smooth real-valued functions \( f : M \to \mathbb{R} \) with compact supports. We denote by \( C^0(M) \) to the space of 0–currents, i.e. the space of linear functionals \( T : C^\infty(M) \to \mathbb{R} \). The flow \( \varphi_t \) acts naturally on \( C^0(M) \) by \( \varphi_t^* T(f) = T(f \circ \varphi_t) \), for every \( f \in C^\infty(M) \). We observe that \( C^0(M) \) is an infinite dimensional vector space and that \( \varphi_t^* T \) is a random variable on \( C^0(M) \). Thus, it is possible to take means and we are able to define another action on the space of currents by \( (\varphi_t * T)(f) = \mathbb{E}[\varphi_t^* T(f)] \).

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Inspired in the definition above we say that $\varphi_t \ast T$ is the mean of the random variable $\varphi_t^* T$.

A current $T$ will be called invariant under $\varphi_t$ if

$$\varphi_t^* T = T,$$

and invariant in mean by $\varphi_t$ if

$$(\varphi_t \ast T)(f) = T(f), \quad \forall f \in C^\infty(M).$$

When the current $T$ is defined in terms of a borel measure $\mu$ on $M$, i.e.

$$T(f) = \int_M f \, d\mu,$$

for every $f \in C^\infty(M)$, the definition of invariance under $\varphi_t$ coincides with the usual definition of the measure $\mu$ being invariant under $\varphi_t$. Also, for the case of borel measures, the notion of invariance in mean is equivalent to the invariance of $\mu$ under the heat semigroup generated by $\varphi_t$ (see e.g. Klieman [11]).

It is easy to see that an invariant current is invariant in mean. In fact, for an invariant current $T$ we have

$$E[\varphi_t^* T(f)] = E[T(f)] = T(f).$$

But the converse is hard to happen. We will give conditions that characterize the currents that are invariant and invariant in mean under diffusions. In particular, we will study the case of $0-$currents given by volume forms as a natural consequence of its theory and, as an application, we will specialize on the particular case of harmonic measures of foliations over homogeneous spaces.

The article is organized as follows: in Section 2, we study the action of a diffusion over a current $T$ and obtain conditions to guarantee the different types of invariance. In Section 3, we work in the specific case of currents defined by $n-$forms. Particularly, we study the problem discussed above but now on the set up of domains in Riemannian manifolds. Finally, in Section 4, we apply the above results to the case of foliations on a homogeneous space.

2. INVARINCE UNDER STOCHASTIC FLOW GENERATED BY SDE.

Here, we start our study of the actions over a current $T$ of a diffusion defined by a SDE on $M$. In particular, we obtain conditions to guarantee the different types of invariance. Our main tools are the convergence theorems of de Rham (see [6] pg 68) and the Itô formula for the flow of a SDE given by Kunita in [12, pg 263].

Let $(M, g)$ be a compact Riemannian manifold. Consider a stochastic flow $\varphi_t$ generated by a Stratonovich SDE on a manifold $M$:

$$dx_t = \sum_{i=0}^{m} X_i(x_t) \circ dB_t^i,$$

$$x_0 = x,$$

where $B^0_t = t$, $(B^1_t, \ldots, B^m_t)$ is a Brownian motion in $\mathbb{R}^m$ constructed on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $X_0, X_1, \ldots, X_m$ are smooth vector fields over $M$. We assume that there exist a stochastic solution flow of diffeomorphisms $\varphi_t : \mathbb{R} \times M \times \Omega \to M$. In particular, when $M$ is compact the flow solution always exists, see e.g. Elworthy [7] or Ikeda and Watanabe [9] among others.
Proposition 1. Let $T$ be a $0$–current. Then
\[
T \left( \int_0^t f(\varphi_t) \circ dB_t \right) = \int_0^t T(f(\varphi_t)) \circ dB_t.
\]

Proof. We first observe that $\int_0^t f(\varphi_t) \circ dB_t$, is a random function on $M$. By a theorem of de Rham (see [6] pg 68) we have that there are smooth $n$–forms $\{\nu_n\}_{n \in \mathbb{N}}$ such that the currents $T_n$, defined by
\[
T_n(f) = \int_M f \nu_n,
\]
satisfy $T_n(f) \to T(f)$ for every $f \in C^\infty(M)$. Since
\[
T_n \left( \int_0^t f(\varphi_t) \circ dB_t \right) = \int_0^t T_n(f(\varphi_t)) \circ dB_t
\]
for each $n$, by Fubini’s theorem, we have that
\[
T \left( \int_0^t f(\varphi_t) \circ dB_t \right) = \lim_{n \to \infty} T_n \left( \int_0^t f(\varphi_t) \circ dB_t \right) = \lim_{n \to \infty} \int_0^t T_n(f(\varphi_t)) \circ dB_t = \int_0^t T(f(\varphi_t)) \circ dB_t.
\]

In a similar way, we obtain
\[
T \left( \int_0^t f(\varphi_t) \, dB_t \right) = \int_0^t T(f(\varphi_t)) \, dB_t,
\]
and
\[
T \left( \int_0^t f(\varphi_t) \, dt \right) = \int_0^t T(f(\varphi_t)) \, dt.
\]

Given a vector field $X$ over $M$ and a $0$–current $T$ the derivative $XT$ is a $0$–current such that
\[
XT(f) = -T(Xf),
\]
cf. de Rham [6] pg 46.

Theorem 1. A $0$–current $T$ is invariant under $\varphi_t$ if and only if
\[
X_i T = 0,
\]
for all $i = 0, \ldots, m$. A $0$–current $T$ is invariant in mean under $\varphi_t$ if and only if
\[
\left( X_0 - \frac{1}{2} \sum_{i=1}^m X_i^2 \right) T = 0.
\]
Proof. By Itô formula for \( \varphi_t^{-1} \) (see [12 pg 263]), we obtain that
\[
\varphi_t^{-1*}T(f) = T(f \circ \varphi_t^{-1}) = T(f - \sum_{i=1}^{m} \int_0^t X_i(f \circ \varphi_s^{-1}) dB^i_s - \int_0^t X_0(f \circ \varphi_s^{-1}) ds + \int_0^t \frac{1}{2} \sum_{i=1}^m X_i^2(f \circ \varphi_s^{-1}) ds).
\]

Thus, if \( \varphi_t^*(T(f)) = T(f) \) then
\[
\varphi_t^{-1*}T(f) = T(f),
\]
and therefore \( X_i T = 0 \) for \( i = 0, \ldots, m \). On the other hand, if \( T(X,f) = 0 \) then \( \varphi_t^{-1*}T(f) = T(f) \), and therefore \( \varphi_t^*(T(f)) = T(f) \) for every \( f \).

To see the other case, we observe that
\[
E[\varphi_t^{-1*}T(f)] = T(f) + \int_0^t E\left[ T\left( -X_0(f \circ \varphi_s^{-1}) + \frac{1}{2} \sum_{i=1}^m X_i^2(f \circ \varphi_s^{-1}) \right) \right] ds.
\]

Thus, if \( (X_0 - \frac{1}{2} \sum_{i=1}^m X_i^2) T = 0 \) we obtain that \( E[\varphi_t^{-1*}T(f)] = T(f) \). Then,
\[
E[\varphi_t^*(T(f))] = E[T(f \circ \varphi_t)] = E[E[\varphi_t^{-1*}T(f \circ \varphi_t)]] = T(f).
\]

Analogously, if \( E[\varphi_t^*(T(f))] = T(f) \) then \( E[\varphi_t^{-1*}T(f)] = T(f) \), and therefore
\[
\left( X_0 - \frac{1}{2} \sum_{i=1}^m X_i^2 \right) T = 0.
\]

\( \square \)

Remark 1. Although we are dealing here with 0—currents it is easy to see that the same kind of calculations can be done for \( k—currents \) with \( 0 < k \leq n \), since the main tools used here, the de Rham theorems and Kunita’s formulas, remain valid for differential (\( n-k \))-forms, with Lie derivatives, instead of the natural vector field derivative on functions. In particular, Ustunel in [14] obtained the formula above, for the case of \( n—currents \) in \( \mathbb{R}^n \), on the space of distributions on \( \mathbb{R}^n \).

3. Currents defined by a \( n \)-forms.

General results on 0—currents can be difficult to be obtained if we have no information about its structure. When a 0—current is defined by an \( n—form \) we can apply the differential calculus to study it. In this section we will see that the invariance of a 0—current defined by an \( n—form \) can be characterized in terms of usual differential operators.
Consider, over an orientable compact Riemannian \( n \)-manifold \( (M, g) \), the volume form \( \mu_g \) defined by the metric \( g \). It is well known that any other \( n \)-form \( \nu \) can be written as \( \nu = f \mu_g \) for a smooth function \( f \).

The volume form allows us to introduce the concept of divergence of a vector field and Jacobian of a transformation in the following way. The divergence of the vector field \( X \) with respect to \( \mu \) is the function \( \text{div}_\mu(X) \) such that

\[
L_X \mu = (\text{div}_\mu(X)) \mu.
\]

The Jacobian of a diffeomorphism \( \phi \) is the smooth function \( J_{\mu} \phi \) such that

\[
\phi^* \mu = J_{\mu} \phi \mu.
\]

In particular, when we consider the diffeomorphism \( \varphi_t \) arising from the stochastic case, we observe that

\[
\varphi_t^{-1} T_{\mu_g} = \int_M f \varphi_t^{-1} \mu_g
= \int_M f \varphi_t^* \mu_g
= \int_M f J\varphi_t \mu_g
= T_{\mu_g}(J\varphi_t f).
\]

So, the evolution of \( \varphi_t^{-1} T_{\mu_g} \) can be studied through the Jacobian of \( \varphi_t \). In particular, if \( J\varphi_t = 1 \) then the current \( T_{\mu_g} \) is invariant under \( \varphi_t \).

It is well known (see [12], Theorem 2.1 and Theorem 4.2) that the following formula holds:

\[
\varphi_t^* \omega - \omega = \sum_{i=0}^{m} \int_0^t \varphi_s^* L_{X_i} \omega \circ dB_s^i.
\]

Lemma 1. The Jacobian \( J_{\mu} \varphi_t(x) \) satisfy the formula

\[
J_{\mu} \varphi_t(x) = 1 + \sum_{i=0}^{m} \int_0^t \text{div}_{X_i}(\varphi_s(x)) (J_{\mu}(\varphi_s(x)) \circ dB_s^i).
\]

Proof.

\[
J_{\mu} \varphi_t \mu_g - \mu_g = \varphi_t^* \mu_g - \mu_g
= \sum_{i=0}^{m} \int_0^t \varphi_s^* L_{X_i} \mu_g \circ dB_s^i
= \sum_{i=0}^{m} \int_0^t \varphi_s^* (\text{div}_{\mu_g}(X_i) \mu_g) \circ dB_s^i
= \sum_{i=0}^{m} \int_0^t \text{div}_{\mu_g}(X_i) \circ \varphi_s \varphi_s^* \mu_g \circ dB_s^i
= \sum_{i=0}^{m} \int_0^t \text{div}_{\mu_g}(X_i) \circ \varphi_s J_{\mu}(\varphi_s \mu_g) \circ dB_s^i
= \left( \sum_{i=0}^{m} \int_0^t \text{div}_{\mu_g}(X_i) \circ \varphi_s J_{\mu}(\varphi_s \circ dB_s^i) \right) \mu_g.
\]
Therefore,
\[ J_{\mu_g}\varphi_t - 1 = \left( \sum_{i=0}^{m} \int_0^t \text{div}_{\mu_g}(X_i) \circ \varphi_s \ J_{\mu_g}\varphi_s \circ dB_s \right). \]

Thus, we see that if \( \text{div}_{\mu_g}(X_i) = 0 \) then \( T_{\mu_g} \) is invariant under \( \varphi_t \). This result can be generalized as in the following theorem.

**Theorem 2.** Let \( T_\nu \) be a 0–current defined by a \( n \)–form \( \nu = f_{\mu_g} \). Then \( T_\nu \) is invariant under \( \varphi_t \) if and only if
\[ \text{div}_{\mu_g}(fX_i) = 0, \]
for \( i = 0, ..., m \).

**Proof.** As we saw before, \( T_\nu \) is invariant under \( \varphi_t \) if and only if \( X_iT_\nu = 0 \) for \( i = 0, ..., m \). For any smooth function \( h \), we have that
\[ X_iT_\nu(h) = -T_\nu(X_ih) \]
\[ = -\int_M X_i h \nu \]
\[ = \int_M h L_{X_i}\nu, \]
where we used Stokes’s theorem and Cartan’s formula for \( L_{X_i} \). Since,
\[ \text{div}_\nu(X_i) = \text{div}_{\mu_g}(X_i) + \frac{X_i f}{f}, \]
\[ \text{div}_{\mu_g}(fX_i) = f \text{div}_{\mu_g}(X_i) + X_i f, \]
we obtain that
\[ X_i T_\nu(h) = \int_M h L_{X_i}\nu \]
\[ = \int_M h \text{div}_\nu(X_i) \nu \]
\[ = \int_M h \left( f \text{div}_{\mu_g}(X_i) + X_i f \right) \nu \]
\[ = \int_M h \text{div}_{\mu_g}(fX_i) \nu. \]

Thus, \( X_iT_\nu = 0 \) iff \( \text{div}_{\mu_g}(fX_i) = 0 \). \( \square \)

**Corollary 1.** The current \( T_{\mu_g} \) is invariant under \( \varphi_t \) if and only if \( \text{div}_{\mu_g}(X_i) = 0 \), for \( i = 0, ..., m \).

**Example 1.** Let \( M^{2n} \) be a smooth manifold endowed with a symplectic form \( w \), and \( h_i : M \to \mathbb{R}, i = 0,1,...,k \), be a smooth functions. Let \( X_{h_i} \) the Hamiltonian vector field associated to \( h_i \), for each \( i = 0,1,...,k \), i.e. the unique vector field that \( i_{X_{h_i}}w = dh_i \). Supposing that \( M \) is compact, or at least that each \( X_{h_i} \) is complete, let \( \phi_t : M \to M \) be the one-parameter family of diffeomorphisms generated by the equation
\[ dx_t = X_{h_0}(x_t) \ dt + \sum_{i=1}^{k} X_{h_i}(x_t) \circ dB_t. \]
The Liouville measure $\mu$ (or symplectic measure) of $M$ is given by

$$\mu = \frac{w^n}{n!}.$$ 

Notice that

$$L_{X_b} \mu = \frac{1}{n!} \sum_{j=1}^{n} w \wedge \cdots \wedge L_{X_{b_j}} w \wedge \cdots \wedge w.$$ 

Since

$$L_{X_{b_j}} w = d(i_{X_{b_j}} w) = ddh = 0,$$

we have that $L_{X_{b_j}} \mu = 0$, and thus $\text{div}_\mu(X_{b_i}) = 0$ for $i = 0, 1, ..., k$. In this way, the stochastic flow $\phi_t$ preserves $\mu$ almost surely for all $t$.

**Example 2.** Let $(M, g)$ be a Riemannian manifold that admit a basis of orthonormal vector fields $\{X_1, \ldots, X_n\}$ such that

$$[X_i, X_j] = \sum_{l=1}^{n} a_{ij}^l X_l.$$ 

Consider the flow $\phi_t$ associated to the SDE

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dB_t.$$ 

Then, the volume measure $\mu$ is invariant if $\text{div}_\mu(X_i) = 0$ for each $i = 1 \ldots n$ or equivalently, if $\sum_{j=1}^{n} a_{ji}^j = 0$ for each $i = 1 \ldots n$. In fact, we observe that

$$\text{div}_\mu(X_i) = \sum_{j=1}^{n} (\nabla_{X_j} X_i, X_j)$$

$$= \sum_{j=1}^{n} \frac{1}{2} ([X_j, X_i], X_j)$$

$$= \frac{1}{2} \sum_{j=1}^{n} (a_{ji}^k, X_k, X_j)$$

$$= \frac{1}{2} \sum_{j=1}^{n} a_{ji}^j.$$ 

In the same way, we can study when a current $T_\nu$ is invariant in mean under $\varphi_t$.

**Theorem 3.** Let $\nu$ and $\mu_g$ as above. The current $T_\nu$ is invariant in mean under $\varphi_t$ if

$$\text{div}(f X_0) + \frac{1}{2} \sum_{i=1}^{n} (X_i + \text{div}_{\mu_g}(X_i))(\text{div}_{\mu_g}(f X_i)) = 0.$$ 

**Proof.** Let $h$ be a smooth function. We have that

$$T_\nu \left( X_0 h - \frac{1}{2} \sum_{i=1}^{n} X_i^2 h \right) = \int_M X_0 h \nu - \frac{1}{2} \sum_{i=1}^{n} \int_M X_i^2 h \nu.$$
For the second integral above, we observe that

\[
\int_M X^2 h \nu = \int_M h L_X^2 \nu \\
= \int_M h[X^2 f + 2X f \text{div}_{\mu g}(X) + f(X(\text{div}_{\mu g}(X))) + (\text{div}_{\mu g}(X))^2] \mu_g \\
= \int_M h(X + \text{div}_{\mu g}(X))(\text{div}_{\mu g}(f X)) \mu_g.
\]

Therefore, \(T_{\nu}(X_0 h - 1/2 \sum_{i=1}^n X_i^2 h) = 0\) if

\[-\text{div}_{\mu g}(f X_0) - 1/2 \sum_{i=1}^n (X_i + \text{div}_{\mu g}(X_i))(\text{div}_{\mu g}(f X_i)) = 0.\]

\(\square\)

4. **Application to foliations on Homogeneous manifold.**

A measure \(\mu\) on a Riemannian manifold \((M, g)\) define a 0–current \(T_\mu\) by

\[T_\mu(f) = \int_M f \mu,\]

for every \(f \in C^\infty(M)\). In this section, we study the specific case when the measure \(\mu\) is related to the geometric structure given by a foliation \(F\) over \(M\) associated with the stochastic flow defined by a SDE that respect this structure. In particular, we relate the invariance with the well known concepts of holonomy invariance and harmonic measures.

There are two particular measures associated to a foliation \(F\) on a Riemannian manifold \(M\). The first one is related to a dynamical system on transverse direction of \(F\) called the holonomy pseudogroup. The invariant measures for such a dynamical system, when they exist, are called holonomy invariant measures (see, for example, [3] or [5]). When the foliation is oriented, these measures are characterized in terms of currents as a positive current \(\psi\) such that

\[\psi(\text{div}_L(X)) = 0,\]

for every vector field \(X\) tangent to the leaf (see Candel [2] or Connes [5]). Here \(\text{div}_L\) is the divergent operator in the leaf direction.

An alternative way of associating a measure to a foliation \(F\) is via the foliated Brownian motion (see Garnett [8]). This is a stochastic process whose infinitesimal generator is given by the Laplace operator in the leaf direction \(\Delta_L\). The invariant measure associated to this stochastic process is called harmonic measure and is characterized in the following way: a measure \(\mu\) is harmonic if

\[\int_M \Delta_L(f) \, d\mu = 0,\]

for each smooth function \(f\). Harmonic measures can be described in terms of currents as a positive current \(\psi\) such that \(\psi(\Delta_L f) = 0\) for each smooth function \(f\).

It is interesting to observe that holonomy invariant measures produce harmonic measures, i.e. if \(\psi\) is the current associated to a holonomy invariant measure, then

\[\psi(\Delta_L(f)) = 0,\]
for every smooth function $f$ (see Candel [2], Garnett [8]). Although the converse not always happens.

In our setup, we can interpret harmonic measures as follows. Let $\varphi_t$ the stochastic flow associated to the Brownian motion, then the current $\psi$ associated to the harmonic measure is invariant in mean under $\varphi_t$. We want to find conditions, for the case of a particular kind of foliations on homogeneous spaces, such that the current $\psi$ is also invariant under $\varphi_t$.

Let $G$ be a Lie group and $K$ be a closed subgroup of $G$ with cofinite volume, i.e. $M \cong G/K$ is a compact homogeneous manifold with an invariant metric $\langle \ , \rangle$. Since $M$ is compact, there is a probability invariant measure $\nu$, such that

$$\int_M f(gx) \nu(dx) = \int_M f(x) \nu(dx),$$

for every $f \in C_c(M)$ (continuous real-valued functions with compact support). See e.g. Abbaspour and Moskowitz [1, Chap. II, p.106] among others.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ with orthonormal basis $\{v_1, \ldots, v_r\}$ and consider the integrable subbundle $E$ of $M$ by $\{V^*_1, \ldots, V^*_r\}$ defining a foliation $F$ on $M$. Here, we used the fact that each $v \in \mathfrak{g}$ induce a vector field $V^*_v$ on $M$ by the formula

$$V^*_v(gK) = \frac{d}{dt} \bigg|_{t=0} \exp(tv) \cdot (gK).$$

Let $\nabla$ the Levi-Civita connection on $M$ and $\nabla^E$ the connection induced on $E$. With this notation, the divergence operator in the leaf direction and the Laplace operator in the leaf direction are given by

$$\text{div}_L(X) = \sum_{i=1}^r g(\nabla^E_{V^*_i}X, V^*_i),$$

for every vector field $X$ tangent to the foliation, and

$$\Delta^E = \sum_{i=1}^r V^*_i V^*_i - \nabla^E_{V^*_i}V^*_i.$$

**Lemma 2.** We have that

$$\nabla^E_{V^*_i}V^*_i = -\sum_{k=1}^r c^i_{jk} V^*_k,$$

where the $c^i_{ij}$ are the structure constants in terms of basis $\{V^*_1, \ldots, V^*_r\}$ satisfying

$$[V^*_i, V^*_j] = \sum_{l=1}^r c^i_{lj} V^*_l.$$

**Proof.** We have by Levi-Civita connection that

$$2 < \nabla^E_{V^*_i}V^*_j, V^*_k> = 2 < \nabla^E_{V^*_i}V^*_j, V^*_k>$$

$$= < [V^*_i, V^*_j], V^*_k> - < [V^*_j, V^*_k], V^*_i> - < [V^*_i, V^*_k], V^*_j>$$

$$= (c^i_{kj} - c^j_{ik} - c^i_{jk}).$$
Therefore,
\[ \nabla^E V_i^* = - \sum_{k=1}^{r} c_{ik} V_k^* . \]

**Lemma 3.**
\[ \int_M (V^* f)(x) \nu(dx) = 0 \text{ for all } f \in C^\infty(M). \]

**Proof.** Since
\[ (V^* f)(x) = \frac{d}{dt} \bigg|_{t=0} f(\exp(tv)(x)), \]
we have that
\[ \int_M (V^* f)(x) \nu(dx) = \frac{d}{dt} \bigg|_{t=0} \int_M f(\exp(tv)(x)) \nu(dx) \]
\[ = \frac{d}{dt} \bigg|_{t=0} \int_M f(x) \nu(dx) \]
\[ = 0. \]

**Theorem 4.** The measure \( \nu \) is a harmonic measure for the induced foliation.

**Proof.** Let \( f \in C^\infty(M) \). Then,
\[ \int_M \Delta_E f(x) \nu(dx) = \sum_{i=1}^{r} \int_M \nabla^E V_i^* f(x) \nu(dx) \]
\[ = - \sum_{i,k=1}^{r} c_{ik} \int_M V_k^* f(x) \nu(dx) \]
\[ = 0. \]

**Remark 2.** Due to the formula above and the Dynkin formula, we can describe the foliated Brownian motion as the solution \( \varphi_t \) of the following SDE
\[ dx_t = \frac{1}{2} \sum_{k=1}^{r} c_{ik} V_k^* (x_t) dt + \sum_{i=1}^{r} V_i^* (x_t) dB^i_t, \]
where each vector field \( V_k^* \) is defined on compact manifold \( M \). Thus, the theorem above can be seen as
\[ \mathbb{E}[\varphi_t^* T_\nu] = T_\nu . \]

**Theorem 5.** The measure \( \nu \) is totally invariant if and only if \( \text{Tr}_h \text{ad}(v_i) = 0 \) for all \( v_i \in \mathfrak{h} \) (where \( \text{Tr}_h \) denotes the trace operator with respect the subspace \( \mathfrak{h} \)).

Moreover, if \( \text{Tr}_h \text{ad}(v_i) = 0 \) for all \( v_i \in \mathfrak{h} \) then, a measure \( \mu \) on \( M \) is totally invariant if and only if \( T_\mu \) is invariant under the flow of the foliated Brownian motion \( \varphi_t \).
Invariance of 0-currents under diffusions.

Proof. Firstly, we show that $\nu$ is holonomy invariant. Let $X^* = \sum_{i=1}^r f_i V_i^*$, with each $f_i \in C^\infty(M)$. Then,

$$T_\nu(\text{div}_E(X^*)) = \int_M \text{div}_E(X^*)(x) \nu(dx) = \sum_{i=1}^r \int_M (V_i^* f_i)(x) \nu(dx) + \sum_{i=1}^r \int_M (f_i \text{div}_E(V_i^*))(x) \nu(dx).$$

From definition of divergence, we have that

$$\text{div}_E(V_i^*) = \sum_{j=1}^r (\nabla^{\text{E}}_{V^*_j} V^*_i, V^*_j)$$

$$= \sum_{j=1}^r (\nabla V^*_j V^*_i, V^*_j)$$

$$= - \sum_{j=1}^r (V^*_i, \nabla V^*_j V^*_j)$$

$$= \sum_{j=1}^r c_{ji}^{j}$$

$$= \text{Tr}_h(ad(v_i)).$$

Thus,

$$\int_M \text{div}_E(X^*)(x) \nu(dx) = \sum_{i,j=1}^r \int_M \text{Tr}_h(ad(v_i)) f_i(x) \nu(dx).$$

Where the first part follows.

For the second part, assume that $\text{Tr}_h(ad(v_i)) = 0$. Let $f$ be a smooth function, then by the invariance of $T_\mu$, we obtain that

$$V_i^* T_\mu(f) = - \int_M V_i^* f \, d\mu = 0$$

Repeating the calculations above with $\mu$ in the place of $\nu$, we obtain the desired. \hfill \Box

Corollary 2. If $\mathfrak{h}$ is a semisimple Lie algebra then $\nu$ is totally invariant.

Remark 3. We observe that the same result is valid if $G/K$ is replaced by a smooth manifold $M$ with a Lie group transitive action $G \times M \to M$ that foliates $M$, an invariant measure $\nu$ and an invariant metric $\langle \, , \rangle$ on $M$.

Corollary 3. If $G$ is nilpotent then $\nu$ is totally invariant.

Proof. If $G$ is nilpotent, then any subalgebra of $\mathfrak{g}$ is nilpotent. Therefore, $\text{Tr}_h ad(v_i) = 0$ for all $v_i \in \mathfrak{h}$. \hfill \Box

Example 3. Consider a compact Riemannian homogeneous 3-dimensional manifold $G/H$, where the associated Lie algebra $\mathfrak{g}$ of $G$ is equipped with an orthonormal basis $\{X, Y, Z\}$ satisfying

$$[X, Y] = 2Y; \quad [X, Z] = -2Z; \quad [Y, Z] = X.$$
This Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. And, for the Lie subalgebra $\mathfrak{h}$ of closed subgroup $H$ we consider the vector space generated by $\{X, Y\}$. Here $\mathcal{F}$ is the foliation induced by $\mathfrak{h}$. We have that $\text{ad}(X)$ and $\text{ad}(Y)$ with respect to the basis $\{X, Y\}$ of $\mathfrak{h}$ are given by

$$[\text{ad}(X)]_\mathfrak{h} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}; 
[\text{ad}(Y)]_\mathfrak{h} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}. $$

Since $\text{Tr}_\mathfrak{h}(\text{ad}(X)) = 2$ the volume measure defined by the orthonormal basis is not totally invariant.

**Example 4.** For a foliation on a compact manifold $M$ that is obtained by a subalgebra of the Lie algebra of the transitive group $G$ acting on $M$, the volume measure is harmonic and also holonomy invariant by the theorem above.

In particular this is the case of a foliation on a Heisenberg manifold, when viewed as a circle bundle over the $2$–torus, where the leaves are the fibers of the bundle.

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