Slice Energy and Theories of Gravitation

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Abstract

We review recent work on the use of the slice energy concept in generalized theories of gravitation. We focus on two special features in these theories, namely, the energy exchange between the matter component and the scalar field generated by the conformal transformation to the Einstein frame of such theories and the issue of the physical equivalence of different conformal frame representations. We show that all such conformally-related, generalized theories of gravitation allow for the slice energy to be invariably defined and its fundamental properties be insensitive to conformal transformations.
1 Introduction

Higher-order and scalar-tensor theories of gravity are currently very popular as mechanisms providing alternative ways to explain the observed cosmic acceleration, cf. [1, 2] and Refs. therein. Although different these theories share two important common characteristics: Firstly they can all be formulated in different conformal frames and secondly they all require for their proper formulation in at least one of the conformally-related frames the existence of scalar fields. These two characteristics turn out to be closely related: the conformal transformation that relates two different conformal representations of a theory is usually defined through the introduction of a scalar field. Further the existence of different conformal frames poses nontrivial relations between the scalar fields present in them, which would otherwise have no connection. It is therefore important to be able to state clearly such relationships: What is the precise relation between the scalar fields present in two different, conformally-related frames? Are two frame representations of the same theory mathematically and/or physically equivalent?

In this paper we first analyze these questions from the viewpoint of a geometric quantity, the energy of fields on a slice in spacetime (slice energy). In the next Section we review the basic properties of this quantity and derive an important equation which describes how the field energy is transported from slice to slice in spacetime. In Section 3, we analyze the behaviour of the slice energy in different theories of gravity and compare our findings with that known in general relativity. This comparison shows that slice energy is a kind of ‘universal invariant’ in metric theories of gravitation may be used to clarify the relations between the different forms of scalar fields appearing in such theories.

The conformal technique further means that we may view the $f(R)$-vacuum theory as a unified theory of gravitation (described in the Einstein frame by the metric $\tilde{g}$) and the scalar field $\phi$, that is as a theory uniting general relativity and the (lagrangian) theory of the scalar field. In the Jordan frame only one single geometric object appears, the ‘metric’ $g$, and the conformal transformation then serves as a tool to ‘fragment’ $g$ into its two pieces in the Einstein frame, namely the gravitational field $\tilde{g}$ and the scalar field
\( \phi \). In higher-order gravity we may describe the different pieces of information involved in the conformal transformation in the following way: \textit{Gravity}, the field \( \tilde{g} \) (the field \( g \) contains more than pure gravity); \textit{dark energy}, the scalar field \( \phi \); \textit{dark matter}, the “matter” fields \( \psi \) which couple non-minimally to gravity and to \( \phi \); \textit{ordinary matter}, the conformal transform of the matter component, \( \tilde{\psi} \), which couples minimally to gravity but is not coupled to \( \phi \). We show in the last Section how this interaction, induced in the Einstein frame by adding, as an example, a dust cloud in the Jordan frame and conformally transforming, leads to an exchange of energy between the this dust cloud and the ‘unobserved’ component—the \( \phi \)—field and vice versa. This in turn translates to an interplay between the matter component in the original Jordan frame (dark matter) and the scalar field \( \phi \) (dark energy).

2 \hspace{1em} \textbf{Slice energy and transport}

Consider a time-oriented spacetime \((V, g)\) with \( V = M \times \mathbb{R} \), where \( M \) is a smooth manifold of dimension \( n \), \( g \) a spacetime metric and the spatial slices \( M_t (= M \times \{t\}) \) are spacelike submanifolds endowed with the time-dependent spatial metric \( g_t \). (In the following, Greek indices run from 0 to \( n \), while Latin indices run from 1 to \( n \). We also assume that the metric signature is \((+ - \cdots -)\).) On \((V, g)\) we consider a family of matterfields denoted collectively as \( \psi \), assume that the field \( \psi \) arises from a lagrangian density which we denote by \( L \) and denote the stress tensor of the field \( \psi \) by \( T(\psi) \).

For \( X \) any causal vectorfield of \( V \) we define the \textit{energy-momentum vector} \( P \) of a stress tensor \( T \) relative to \( X \) to be

\[
P^\beta = X_\alpha T^{\alpha\beta}.
\]

(2.1)

The \textit{energy on} \( M_t \) \textit{with respect to} \( X \) is defined by the integral (when it exists)

\[
E_t = \int_{M_t} P^\alpha n_\alpha d\mu_t,
\]

(2.2)

where \( n \) is the unit normal to \( M_t \) and \( d\mu_t \) is the volume element with respect to the spatial metric \( g_t \). We call \( P^\alpha n_\alpha \) the \textit{energy density}. Assuming that \( X \) and \( T \) are smooth,
we find
\[ \nabla_{\alpha} P^{\alpha} = \frac{1}{2} T^{\alpha\beta} (\nabla_{\alpha} X_{\beta} + \nabla_{\beta} X_{\alpha}) + X_{\beta} \nabla_{\alpha} T^{\alpha\beta}. \] (2.3)

Thus, if \( \mathcal{K} \subset \mathcal{V} \) is a compact domain with smooth boundary \( \partial \mathcal{K} \), it follows from Stokes’ theorem that
\[ \int_{\mathcal{K}} \nabla_{\alpha} P^{\alpha} d\mu = \int_{\partial \mathcal{K}} P^{\alpha} n_{\alpha} d\sigma, \] (2.4)

where \( d\mu \) is the volume element of \( \mathcal{V} \) and \( d\sigma \) that of \( \partial \mathcal{K} \), and so we find
\[ \int_{\partial \mathcal{K}} P^{\alpha} n_{\alpha} d\sigma = \frac{1}{2} \int_{\mathcal{K}} T^{\alpha\beta} (\nabla_{\alpha} X_{\beta} + \nabla_{\beta} X_{\alpha}) d\mu + \int_{\mathcal{K}} X_{\beta} \nabla_{\alpha} T^{\alpha\beta} d\mu. \] (2.5)

Hence, when \( \mathcal{M} \) is compact or the field falls off appropriately at infinity, on the spacetime slab \( \mathcal{D} = \Sigma \times [t_0, t_1], \Sigma \subset \mathcal{M} \), and with \( T \) having support on \( \mathcal{D} \) we have the following relation for the energies on the two end-slices
\[ E_{t_1} - E_{t_0} = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta} (\nabla_{\alpha} X_{\beta} + \nabla_{\beta} X_{\alpha}) d\mu + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} X_{\beta} \nabla_{\alpha} T^{\alpha\beta} d\mu. \] (2.6)

Thus we have shown the following result (cf. [3], p. 87-88).

**Theorem 1** When \( X \) is a Killing vectorfield and the field is conserved, i.e., \( \nabla_{\alpha} T^{\alpha\beta} = 0 \), we have
\[ E_{t_1} = E_{t_0}. \] (2.7)

This means that, when the energy-momentum tensor of a field is conserved, the same is true for its slice energy relative to a Killing vectorfield as a function of time.

Below, we pay particular attention to the case for which the field is a matter field \( \psi \) interacting with a scalar field \( \phi \) with potential \( V(\phi) \). We take the scalar field lagrangian density to be
\[ L = -\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi + V(\phi). \] (2.8)

Then the energy-momentum tensor of \( \phi \) is
\[ T^{\alpha\beta}(\phi) = \partial^{\alpha} \phi \partial^{\beta} \phi - \frac{1}{2} g^{\alpha\beta} (\partial^{\lambda} \phi \partial_{\lambda} \phi - 2V(\phi)), \] (2.9)

and before we proceed further we note the following important result (see [1] for a proof).
Theorem 2  The energy density $P^\alpha n_\alpha$ of the scalar field $\phi$ with potential $V(\phi)$ is positive when $V(\phi) > 0$.

As noted already, we shall be interested in the case of a matter field $\psi$ interacting with a scalar field $\phi$ with potential $V(\phi)$, especially in a conformally equivalent frame. The following notation for conformally related quantities is used: Let $g$ and $\tilde{g}$ be two conformal metrics, $\tilde{g} = \Omega^2 g$ on the manifold $\mathcal{V}$. This means that in two orthonormal moving frames, $\theta^\alpha$ and $\tilde{\theta}^\alpha$, the two conformal metrics satisfy

$$\tilde{g} = \eta_{\alpha\beta} \tilde{\theta}^\alpha \tilde{\theta}^\beta, \quad g = \eta_{\alpha\beta} \theta^\alpha \theta^\beta$$

and

$$\tilde{\theta}^\alpha = \Omega \theta^\alpha,$$

(2.10)

with $\eta_{\alpha\beta} = \text{diag}(+,-\cdots,-)$ being the flat metric. Setting $\Omega^2 = e^\phi$ we see that $\tilde{\theta}^\alpha = e^{\phi/2} \theta^\alpha$ and obviously $\tilde{\theta}_\alpha = e^{-\phi/2} \theta_\alpha$. The same rules are true for any 1-form or vectorfield on $\mathcal{V}$. We take $\tilde{T}^{\alpha\beta}(\phi)$ and $\tilde{T}^{\alpha\beta}(\tilde{\psi})$ to denote the stress tensors of the two fields $\phi$ and $\tilde{\psi}$ and assume that their sum is conserved

$$\tilde{\nabla}_\alpha \left( \tilde{T}^{\alpha\beta}(\phi) + \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right) = 0,$$

(2.11)

but the two components are not conserved separately, that is

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\phi) \neq 0,$$

(2.12)

and

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) \neq 0,$$

(2.13)

unless the conservation equations $\nabla_\alpha T^{\alpha\beta}(\psi) = 0$ for the field $\psi$ in the original frame are conformally invariant. This implies that there must be a nontrivial $\phi - \tilde{\psi}$ interaction between the matter field $\tilde{\psi}$ and the $\phi$-field and an associated exchange of energy between $\phi$ and $\tilde{\psi}$. Writing Eq. (2.6) for the scalar field $\phi$ and substituting for the last term in the right-hand-side from Eq. (2.11) we arrive at the general energy transport equation in the conformally related frame

$$E_{t_1}(\phi) - E_{t_0}(\phi) = \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{X}_\beta + \tilde{\nabla}_\beta \tilde{X}_\alpha) d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu},$$

(2.14)

with $d\tilde{\mu}$ being the volume element of $\tilde{g}$. 4
3 Field equations and slice energy conservation

We are interested below in a comparison of the conservation properties of slice energy of certain fields in general relativity, higher-order gravity theories and scalar-tensor theories of gravitation. In general relativity we take the field equations to be of the form

\[ G_{\alpha\beta} = T_{\alpha\beta}(\phi) + T_{\alpha\beta}(\psi), \]  

where \( G_{\alpha\beta} \) is the Einstein tensor, \( \phi \) is a scalar field with stress tensor given by Eq. (2.9), \( T(\psi) \) represents the stress tensor of a field \( \psi \), and we assume the conservation identities \( \nabla_\alpha T^{\alpha\beta}(\phi) = 0 \) and \( \nabla_\alpha T^{\alpha\beta}(\psi) = 0 \). In higher-order gravity theories we consider the Jordan-frame equations

\[ L_{\alpha\beta} \equiv f'R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}f - \nabla_\alpha \nabla_\beta f' + g_{\alpha\beta} \Box_g f' = T_{\alpha\beta}(\psi), \]  

which, because \( \nabla_\alpha L^{\alpha\beta} = 0 \), imply the conservation identities \( \nabla_\alpha T^{\alpha\beta}(\psi) = 0 \). The Einstein frame representation of this theory is

\[ \tilde{G}_{\alpha\beta} = T_{\alpha\beta}(\phi) + \tilde{T}_{\alpha\beta}(\tilde{\psi}), \]  

where \( \phi = \ln f' \) and \( T_{\alpha\beta}(\phi) \) is of the form \( \nabla_\alpha \nu^{\alpha\beta} + \nu_{\alpha\beta} \Box_g f' \), cf. [4]. Here we have the situation introduced in the previous Section wherein the whole tensor in the right-hand-side is conserved,

\[ \tilde{\nabla}_\alpha \left( \tilde{T}^{\alpha\beta}(\phi) + \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right) = 0, \]  

but the two components are not conserved separately, that is

\[ \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\phi) \neq 0, \quad \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) \neq 0. \]  

The field \( \phi \) appearing both in general relativity and in (the Einstein frame representation of) higher-order gravity theories is in certain contexts responsible for the existence of an inflationary period. For concreteness we call it the inflaton and distinguish it from
a scalar field, say $\xi$, that may appear directly in the Jordan frame equations (3.2) in addition to the matterfield $\psi$.

Lastly we take the defining equations of our scalar-tensor theory to be the Brans-Dicke (BD) ones, with $\chi$ denoting the BD scalar field (everything we do below is valid if, instead of the BD theory assumed here only for brevity, we consider the most general scalar-tensor action having couplings of the form $h(\chi)$, where $h$ is any differentiable function of the field $\chi$),

$$S_{\alpha\beta} \equiv \chi G_{\alpha\beta} = T_{\alpha\beta}(\chi) + T_{\alpha\beta}(\psi).$$

(3.6)

The novel feature of this equation is the requirement that, if in accordance with the equivalence principle we assume that

$$\nabla_\alpha T^{\alpha\beta}(\psi) = 0,$$

(3.7)

only, then, because $\nabla_\alpha G^{\alpha\beta} = 0$ we find

$$\nabla_\alpha S^{\alpha\beta} = \nabla_\alpha T^{\alpha\beta}(\chi).$$

(3.8)

Here $T^{\alpha\beta}(\chi)$ is not given by (2.9) but by a different, more complicated, expression (cf. [5], pp. 159-60). For definiteness below we call the field $\chi$ the dilaton to distinguish it from the other scalar fields appearing in the $f(R)$ Eqs. (3.2), (3.3) and in general relativity, Eq. (3.1). Many currently popular string theories appear as special cases of the scalar-tensor equations.

Starting from the general transport equation, Eq. (2.14), we derive relations showing the dependence of the total slice energy of the system on the special features of each one of the three theories given by Eqs. (3.1), (3.2) and (3.3) and (3.6). Writing Eq. (2.14) for the scalar field $\phi$ and substituting from Eqs. (3.1) and the conservation identity for the terms $T(\phi)$ and $X \nabla T$ respectively, we find

$$E_{t_1}(\phi) - E_{t_0}(\phi) = \int_{t_0}^{t_1} \int_{M_t} [G^{\alpha\beta} - T^{\alpha\beta}(\psi)] \nabla_\alpha X_\beta d\mu$$

$$= \int_{t_0}^{t_1} \int_{M_t} G^{\alpha\beta} \nabla_\alpha X_\beta d\mu - \int_{t_0}^{t_1} \int_{M_t} T^{\alpha\beta}(\psi) \nabla_\alpha X_\beta d\mu.$$

(3.9)
Using Stokes’ theorem, the last term is just
\[
\int_{t_0}^{t_1} \int_{M_t} T^\alpha{}^\beta(\psi) \nabla_\alpha X_\beta d\mu = \int_{M_{t_1}} P^\alpha{}_n d\mu_{t_1} - \int_{M_{t_0}} P^\alpha{}_n d\mu_{t_0} = E_{t_1}(\psi) - E_{t_0}(\psi),
\] (3.10)
and so, setting \(E_t(\phi + \psi) = E_t(\phi) + E_t(\psi)\), we find that in general relativity the total slice energy of a system comprised of the field \(\phi\) and a matter field \(\psi\) depends on the Einstein tensor as follows:
\[
E_{t_1}(\phi + \psi) - E_{t_0}(\phi + \psi) = \int_{t_0}^{t_1} \int_{M_t} G^\alpha{}^\beta \nabla_\alpha X_\beta d\mu. \tag{3.11}
\]
Further, since \(\nabla_\alpha G^\alpha{}^\beta = 0\), integrating by parts and using Stokes theorem we have
\[
\int_{t_0}^{t_1} \int_{M_t} G^\alpha{}^\beta \nabla_\alpha X_\beta d\mu = \int_{M_{t_1}} G^\alpha{}^\beta X_\alpha N_\beta d\mu_{t_1} - \int_{M_{t_0}} G^\alpha{}^\beta X_\alpha N_\beta d\mu_{t_0}, \tag{3.12}
\]
where \(N\) is the unit normal to the slices. Using this form we have the following result.

**Theorem 3** The total slice energy of the system comprised of the scalar field \(\phi\) and a matterfield \(\psi\) satisfying the Einstein equations (3.1), is given by
\[
E_{t_1}(\phi + \psi) - E_{t_0}(\phi + \psi) = \int_{M_{t_1}} G^\alpha{}^\beta X_\alpha N_\beta d\mu_{t_1} - \int_{M_{t_0}} G^\alpha{}^\beta X_\alpha N_\beta d\mu_{t_0}. \tag{3.13}
\]
In particular, when \(X\) is a Killing field of the metric \(g\), the total slice energy of the system is conserved.

The terms of the form \(\int_{M_t} G^\alpha{}^\beta X_\alpha N_\beta d\mu_t\) represent a gravitational flux through the slice \(M_t\). When \(X\) is a Killing field, the right hand side of Eq. (3.11) is zero and we have an integral conservation law given by the equality of the two terms in the right hand side of Eq. (3.13) and this agrees with the corresponding result originally given in [6], Chap. VI, and proved there via a different route.

The situation in higher-order gravity is in fact, despite the different conservation
laws, similar. In the Einstein frame we have

\[
E_{t_1}(\phi) - E_{t_0}(\phi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \left[ \tilde{G}^{\alpha\beta} - \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right] \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}
\]

\[
= \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{G}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\tilde{\psi}) \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu}
\]

\[
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}.
\] (3.14)

Using Stokes’ theorem, the middle term is

\[
\int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\tilde{\psi}) \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} = \int_{\mathcal{M}_{t_1}} \tilde{P}^{\alpha} \tilde{n}_\alpha d\tilde{\mu}_t - \int_{\mathcal{M}_{t_0}} \tilde{P}^{\alpha} \tilde{n}_\alpha d\tilde{\mu}_0
\]

\[
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}
\]

\[
= E_{t_1}(\tilde{\psi}) - E_{t_0}(\tilde{\psi}) - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}.
\] (3.15)

and so, setting \(E_t(\phi + \tilde{\psi}) = E_t(\phi) + E_t(\tilde{\psi})\), we find that in higher-order gravity, because of the marvelous fact that the terms of the general form \(\int \tilde{X} \tilde{\nabla} \tilde{T}(\tilde{\psi})\) which were absent in general relativity now precisely cancel each other, the total slice energy of a system composed of the field \(\phi\) and a matter field \(\tilde{\psi}\) in the Einstein frame depends on the Einstein tensor in the same way as before:

\[
E_{t_1}(\phi + \tilde{\psi}) - E_{t_0}(\phi + \tilde{\psi}) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{G}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu}.
\] (3.16)

Hence we arrive at the following result.

**Theorem 4** The total slice energy of the system composed of the scalar field \(\phi\) and a matterfield \(\tilde{\psi}\) satisfying the Einstein equations (3.3), is given by

\[
E_{t_1}(\phi + \tilde{\psi}) - E_{t_0}(\phi + \tilde{\psi}) = \int_{\mathcal{M}_{t_1}} \tilde{G}^{\alpha\beta} \tilde{X}_\alpha \tilde{N}_\beta d\tilde{\mu}_t - \int_{\mathcal{M}_{t_0}} \tilde{G}^{\alpha\beta} \tilde{X}_\alpha \tilde{N}_\beta d\tilde{\mu}_0.
\] (3.17)

In particular, when \(X\) is a Killing field of the metric \(\tilde{g}\), the total slice energy of the system is conserved.
Note that if we have a scalar field $\xi$ in addition to the matter field $\psi$ present in the original Jordan frame of the higher order gravity theory, then we obtain a result similar to that in general relativity but with $L_{\alpha\beta}$ in place of the Einstein tensor, namely,

$$E_{t_1}(\xi + \psi) - E_{t_0}(\xi + \psi) = \int_{M_{t_1}} L^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{M_{t_0}} L^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \quad (3.18)$$

Then terms of the form $\int_{M_t} L^{\alpha\beta} X_\alpha N_\beta d\mu$ represent a higher-order gravitational flux through the slice $M_t$. When $X$ is a Killing field, we again have an integral conservation law as before.

We now move to the analysis of the scalar-tensor theory (3.6). In this case Eq. (2.14) becomes

$$E_{t_1}(\chi) - E_{t_0}(\chi) = \int_{t_0}^{t_1} \int_{M_t} \nabla_\alpha S^{\alpha\beta} d\mu + \int_{t_0}^{t_1} \int_{M_t} T^{\alpha\beta}(\chi) \nabla_{(\alpha} X_{\beta)} d\mu, \quad (3.19)$$

and after some algebra we find that

$$E_{t_1}(\chi) - E_{t_0}(\chi) = \int_{M_{t_1}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_1} - \int_{M_{t_0}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_0} - \int_{t_0}^{t_1} \int_{M_t} T^{\alpha\beta}(\psi) \nabla_{(\alpha} X_{\beta)}. \quad (3.20)$$

Since $\nabla_\alpha S^{\alpha\beta} = G^{\alpha\beta} \nabla_\alpha \phi$, the first two terms can be expressed more simply as follows

$$\int_{M_{t_1}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_1} - \int_{M_{t_0}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_0} = \int_{t_0}^{t_1} \int_{M_t} S^{\alpha\beta} \nabla_\alpha X_\beta + \int_{t_0}^{t_1} \int_{M_t} G^{\alpha\beta} X_\beta \partial_\alpha \chi. \quad (3.21)$$

Therefore using Eq. (3.10) we find that

$$E_{t_1}(\chi + \psi) - E_{t_0}(\chi + \psi) = \int_{t_0}^{t_1} \int_{M_t} \chi G^{\alpha\beta} \nabla_\alpha X_\beta + \int_{t_0}^{t_1} \int_{M_t} G^{\alpha\beta} X_\beta \partial_\alpha \chi$$

$$= \int_{t_0}^{t_1} \int_{M_t} G^{\alpha\beta}(\chi \nabla_\alpha X_\beta + X_\beta \partial_\alpha \chi)$$

$$= \int_{t_0}^{t_1} \int_{M_t} G^{\alpha\beta} \nabla_\alpha (\chi X_\beta)$$

and we are led to the following result.
Theorem 5 The total slice energy of the dilaton-matter system satisfying the scalar-tensor equations (3.7) is given by

\[ E_{t_1}(\chi + \psi) - E_{t_0}(\chi + \psi) = \int_{M_{t_1}} S^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{M_{t_0}} S^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \] (3.22)

In conclusion we have found the different forms that slice energy takes in various classes of generalized theories of gravitation which include higher-order gravity theories and scalar-tensor ones. These forms may be described symbolically as follows:

\[ E_{t_1}(\lambda + \psi) - E_{t_0}(\lambda + \psi) = \int_{M_{t_1}} \Lambda^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{M_{t_0}} \Lambda^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \] (3.23)

Here \( \lambda \) denotes either an inflaton field, \( \phi \), which couples to matter in general relativity or in the Einstein frame in higher-order gravity, the scalar field \( \xi \) which may appear in the Jordan frame of higher-order gravity, or the dilaton \( \chi \) in scalar-tensor theory, while \( \Lambda \) is a gravitational operator defining the left-hand-sides of the associated field equations, that is, \( \Lambda = G^{\alpha\beta}, \tilde{G}^{\alpha\beta}, L^{\alpha\beta} \) or \( S^{\alpha\beta} \) respectively. As a last remark we note that one might think that since all these theories have equations of motion of the form

\[ \Lambda^{\alpha\beta} = T^{\alpha\beta}_{\text{total}}, \] (3.24)

and since

\[ E_{\text{total}} = \int_{M_t} d\mu_t X^\alpha T_{\alpha\beta} n^\beta, \] (3.25)

then

\[ E_{\text{total}} = \int_{M_t} d\mu_t X^\alpha \Lambda_{\alpha\beta} n^\beta, \] (3.26)

and so Eq. (3.23) follows trivially from Eq. (3.26). What is wrong with this argument? The basic point here is that it is not obvious that the slice energy will necessarily satisfy a linear law of the form \( E_{\text{total}} = E(\phi) + E(\psi) \). In fact, that this is the only (currently feasible) way to do it follows from Eq. (3.9) using Eq. (3.10) as we acknowledge it immediately after that equation. Usually one has only one field in general relativity and so the equivalent of Eq. (3.11) is obvious. To say that Eq. (3.11) is valid when more
than one fields are present, one has to assume that slice energy will satisfy the above linear law. So the results in this Section clear up this point and prepare the necessary ground for the more complicated case of interacting (i.e., directly coupled) fields.

A closely related issue is the crucial fact that, in the different theories we consider, not only one typically has a number of fields in the RHSs of the equations of motion but each of these fields is not separately conserved. This gives an added complication to the effect that in the equations involving the slice energy one has the slice energy of one field appearing in the LHS and the matter tensor of another field on the RHS, and there is no conservation law in the RHS as that field is not separately conserved. Of course one may guess that the same behavior as that previously found for general relativity will be valid here. But then this is clearly not a proof. The analysis in this Section clears up this situation completely in the more general case of higher order and scalar tensor theories. The field equations do not say that slice energy is an additive quantity, only assume it. This is an important point clarified in this analysis.

4 Dust clouds in higher order gravity

We now study the interaction and associated energy exchange between $\phi$ and $\tilde{\psi}$ more closely by assuming that $\psi$ is the simplest form of a fluid, a dust cloud on $(V, g)$ with 4-velocity $V_{\alpha}$ and stress tensor

$$T_{\alpha\beta, \text{dust}} = \rho V_{\alpha} V_{\beta},$$

satisfying the $f(R)$-dust equations in the Jordan frame, namely

$$f'R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f - \nabla_{\alpha} \nabla_{\beta} f' + g_{\alpha\beta} \Box f' = \rho V_{\alpha} V_{\beta},$$

with

$$\nabla_{\alpha}(\rho V^{\alpha} V^{\beta}) = 0.$$  (4.3)

After the conformal transformation we find

$$\tilde{G}_{\alpha\beta} = \tilde{T}_{\alpha\beta}(\phi) + \tilde{\rho} \tilde{V}_{\alpha} \tilde{V}_{\beta},$$

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with $\tilde{T}_{\alpha\beta}(\phi)$ given by (2.3) with tildes where appropriate, and

$$\tilde{V}_a = e^{-\phi/2} V_a, \quad \tilde{\rho} = e^{\phi/2} \rho, \quad s \in \mathbb{R}.$$  \hspace{1cm} (4.5)

(We have set $\tilde{\rho} = \Omega^* \rho$ and since $\Omega^2 = e^\phi$, $\Omega^* = e^{\phi/2}$.) What is the field equation satisfied by the scalar field $\phi$? From Eq. (4.4) the divergence of the stress tensor of $\phi$ is minus that of the dust, but

$$\tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha \tilde{V}^\beta) = \nabla_a (\tilde{\rho} \tilde{V}^a \tilde{V}^\beta) + A^\alpha_{\alpha\gamma} \tilde{\rho} \tilde{V}^\gamma \tilde{V}^\beta + A^\beta_{\alpha\gamma} \tilde{\rho} \tilde{V}^\alpha \tilde{V}^\gamma,$$  \hspace{1cm} (4.6)

where

$$A^\alpha_{\beta\gamma} = \frac{1}{2} \left( \delta^\alpha_{\beta} \partial_{\gamma} \phi + \delta^\alpha_{\gamma} \partial_{\beta} \phi - g_{\beta\gamma} g^{\alpha\delta} \partial_{\delta} \phi \right).$$  \hspace{1cm} (4.7)

From these equations and Eq. (4.3) we deduce the modified scalar field equation in the form

$$\partial^\beta \phi (\tilde{\square} \phi + V') + \frac{s + 5}{2} \tilde{\rho} \tilde{V}^\alpha \tilde{V}_\alpha + \frac{1}{2} \tilde{\rho} \partial^\beta \phi = 0.$$  \hspace{1cm} (4.8)

Another way to derive the scalar field equation is as follows. Since

$$\tilde{\nabla}_\alpha \tilde{T}_{\alpha\beta}^{\text{dust}} = \tilde{\nabla}_a (\tilde{\rho} \tilde{V}^a \tilde{V}^\beta) = \tilde{V}^\beta \tilde{\nabla}_a (\tilde{\rho} \tilde{V}^a) + \tilde{\rho} (\tilde{\nabla}_a \tilde{V}^\beta) \tilde{V}^a,$$  \hspace{1cm} (4.9)

multiplying (4.9) by $\tilde{V}_\beta$ we have

$$\tilde{V}_\beta \tilde{\nabla}_a \tilde{T}_{\alpha\beta}^{\text{dust}} = \tilde{\nabla}_a (\tilde{\rho} \tilde{V}^a) \tilde{V}_\beta \tilde{V}^\beta + \tilde{\rho} (\tilde{\nabla}_a \tilde{V}^\beta) \tilde{V}^\alpha \tilde{V}_\beta$$  \hspace{1cm} (4.10)

and, since $\tilde{V}_\beta \tilde{V}^\beta = 1$, we obtain

$$\tilde{\nabla}_a \tilde{T}_{\alpha\beta}^{\text{dust}} = \tilde{V}^\beta \tilde{\nabla}_a (\tilde{\rho} \tilde{V}^a) + \tilde{\rho} (\tilde{\nabla}_a \tilde{V}^\beta) \tilde{V}^a.$$  \hspace{1cm} (4.11)

After some algebra we arrive at the equation of motion for the scalar field $\phi$ in the Einstein frame, namely

$$\partial^\beta \phi (\tilde{\square} \phi + V') + \tilde{V}^\beta \tilde{\nabla}_a (\tilde{\rho} \tilde{V}^a) + \tilde{\rho} \tilde{V}^\alpha (\tilde{\nabla}_a \tilde{V}^\beta) = 0.$$  \hspace{1cm} (4.12)

Recalling that dust matter follows geodesics on the original Jordan frame, $V^\alpha \nabla_\alpha V^\beta = 0$, we find that the last two terms in this equation equal to the last two terms in Eq. (4.8) and so we conclude that Eq. (4.12) provides an equivalent form of Eq. (4.8).
Note that in the very special case where
\[ \tilde{V}_\beta = \partial_\beta \phi, \]  
(4.13)
which implies some sort of ‘alignment’ between the dust and the scalar field, the scalar field equation (4.8) becomes
\[ \tilde{\square} \phi + V' + \frac{s + 4}{2} \tilde{\rho} = 0. \]  
(4.14)
This is now easily compared to the more commonly used scalar field equation, but we have to bear in mind that it has been obtained from (4.8) by imposing the serious restriction (4.13).

We now study the behaviour of the total slice energy of the system comprised of \( \phi \) and the dust component. We prove that the total slice energy is conserved only when spacetime is stationary. We choose \( V = X \) so that
\[ P^\alpha n_\alpha = X_\beta n_\alpha \rho V^\alpha V^\beta = \rho V^\alpha n_\alpha. \]  
(4.15)
Hence, from Eq. (4.11), applying Stokes’ theorem we obtain
\[ \int_K \tilde{\nabla}_\alpha (\tilde{\rho} \tilde{V}^\alpha) d\tilde{\mu} = \int_{\partial K} \tilde{\rho} \tilde{V}^\alpha \tilde{n}_\alpha d\tilde{\sigma}, \]  
(4.16)
Therefore Eq. (2.14), after some manipulation becomes
\[ E_t(\phi) + E_t(\text{dust}) = E_0(\phi) + E_0(\text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu}, \]  
(4.17)
where by definition and Eq. (4.15), for any \( t \),
\[ E_t(\text{dust}) = \int_{\mathcal{M}_t} \rho V^\alpha n_\alpha d\mu_t. \]  
(4.18)
We see that the last term in Eq. (4.17) can be zero only when \( V \) is a Killing vectorfield. We therefore arrive at the following result about the total slice energy with respect to the fluid itself.
Theorem 6 The total slice energy with respect to the timelike vectorfield $\tilde{V}$, tangent to the dust timelines, of the scalar field-dust system satisfying the field equations (4.2), satisfies

$$E_t(\phi + \text{dust}) = E_0(\phi + \text{dust}) + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) d\tilde{\mu}. \quad (4.19)$$

In particular the slice energy of the scalar field-dust system is conserved when $\tilde{V}$ is a Killing vectorfield of $\tilde{g}$.

We also conclude that the property of the conservation of slice energy for dust is a conformal invariant. However, when $V$ is not a Killing vectorfield, we see that there is a nontrivial contribution to the slice energy coming from the stress tensor of the scalar field generated by the conformal transformation. Note that this contribution is also nonzero even in the special case that Eq. (4.13) is assumed for in that case the first term in $\tilde{T}^{\alpha\beta}(\phi)(\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha)$ is zero because $\tilde{V}^\alpha \tilde{V}^\beta (\tilde{\nabla}_\alpha \tilde{V}_\beta + \tilde{\nabla}_\beta \tilde{V}_\alpha) = 0$, but the whole combination is still not zero as there are additional terms coming from the contributions of the other terms in Eq. (4.9) (unless the fluid satisfies an extra condition – see [1]).

This analysis can be extended to the more general case of a perfect fluid interacting with a scalar field, cf. [1].

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