Characterizations of monadic NIP

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OVERVIEW

1. Monadic stability
2. Monadic NIP (Shelah)
3. Characterizations
4. Hereditary classes
5. Questions
**Definition and Examples**

**Definition**

A theory $T$ is *monadically stable/NIP* if any expansion of $T$ by arbitrarily many unary predicates remains stable/NIP.

- Analysis of monadically stable theories is due to Baldwin and Shelah [1].
- Refining equivalence relations and mutually algebraic theories are monadically stable
- DLO and various tree-like theories are monadically NIP.
- Essentially anything with a non-unary function is not monadically NIP, e.g. vector spaces.
Theorem (Baldwin-Shelah [1])

The following are equivalent.

1. $T$ is monadically stable.
2. $T$ is stable and monadically NIP.
3. $T$ is stable and does not admit coding.
4. Models of $T$ admit a nice decomposition into trees of countable models.
5. $T$ is stable and if $B \downarrow_D C$, then for any $a$, $aB \downarrow_D C$ or $B \downarrow_D aC$. 
Tree decompositions

Definition

A tree decomposition of $M$ is a collection of countable submodels of $M$, indexed by a tree, such that

1. $\bigcup M_i = M$
2. If $i < j$ then $M_i \subset M_j$.
3. The children of a model $M_i$ are independent over $M_i$.

- Example: An equivalence relation with $\kappa$ classes of size $\lambda$.

- The children of $M_i$ form a congruence over $M_i$. 

(NON-)FORKING

• Recall: $T$ is stable and if $A \downarrow_D B$, then for any $c$, $cA \downarrow_D B$ or $A \downarrow_D Bc$.

• Equivalently, forking is trivial (i.e. if $A \not\fork C B$, then $a \not\fork C b$ for some $a \in A, b \in B$) and transitive on singletons.

• So forking defines an equivalence relation on singletons.

• Can use this equivalence relation to build the tree decomposition. (Or can use first characterization to iteratively extend by one point).

• Given a $a, b, c$ failing this property, take Morley sequences in $a$ and $b$ and automorphic images of $c$ to get coding, as in vector spaces. ($c_{ij}$ behaves non-generically over $a_i b_j$.)
**Shelah’s Theorem**

- Soon afterward, Shelah analyzed monadic NIP [4].
- Concerned with structure theory, since non-structure was clear in his setting.

**Definition**

Let $A imp_{\text{fs}_M} B$ mean that $tp(A/MB)$ is finitely satisfiable in $M$. Let $A imp_{\text{fs}_{M\subseteq C}} B$ mean that $tp(A/CB)$ is finitely satisfiable in $M$. A theory $T$ has the f.s.-dichotomy if given $A imp_{\text{fs}_M} B$, then for any $c$, $cA imp_{\text{fs}_M} B$ or $A imp_{\text{fs}_M} Bc$.

**Theorem ([4])**

*If $T$ does not have the f.s.-dichotomy, then $T$ admits a pre-coding configuration, and so is not monadically NIP.*

*If $T$ has the f.s.-dichotomy, then models of $T$ admit a nice linear decomposition into substructures.*
THE f.s.-DICHOTOMY

- Recall: given $A \upharpoonright_M^{fs} B$, then for any $c$, $cA \upharpoonright_M^{fs} B$ or $A \upharpoonright_M^{fs} Bc$.
- Implies dependence is trivial* and transitive on singletons.
- *: If $A \downarrow_M^{fs} B$, then $A \downarrow_M^{fs} b$ for some $b \in B$.
  If $C \supset M$ is large (i.e. realizes all types over $M$), and
  $A \upharpoonright_M^{fs} \downarrow C B$, then $a \upharpoonright_M^{fs} \downarrow C B$ for some $a \in A$.
- So if we work over a large $C \supset M$, dependence gives a quasi-order.
- Why do we need $C$? Stationarity: If $p \in S(C)$ is fin. sat. in $M$,
  then for any $D \supset C$ there is a unique extension $p$ over $D$ that
  is fin. sat. in $M$. (No assumption of f.s.-dichotomy.)
**M-f.s. Sequences**

**Definition**

Given a model $M$, $(a_i : i \in I)$ is an $M$-f.s. sequence if $a_i \downarrow_M^{fs} \{a < i\}$.

- Similar to Morley sequences. If also indiscernible, then a special case of Morley sequences.

**Theorem (No assumption of f.s.-dichotomy)**

Given an indiscernible sequence $I \subset \mathcal{C}$, we can find some model $M$ so that $I$ is an $M$-f.s. sequence.

Furthermore, we can find large $C \supset M$ so that $I$ remains indiscernible and $M$-f.s. over $C$.

- Finite satisfiability and $M$-f.s. sequences seem like useful notions in arbitrary theories.
If the f.s.-dichotomy fails, we want a failure of monadic NIP.

Given $\bar{a}, \bar{b}, c, M$ failing the f.s.-dichotomy, extend $\bar{a}b$ to an $M$-f.s. sequence over a large $C \supset M$.

By automorphisms, for $i < j$ find $c_{ij}$ so $\text{tp}(\bar{a}bc) = \text{tp}(\bar{a}_i \bar{b}_jc_{ij})$ (so $c_{ij}$ is non-generic over $\bar{a}_i \bar{b}_j$) but is reasonably generic over the rest of the sequence.

This gives a pre-coding configuration as below.

**Definition**

A pre-coding configuration is an indiscernible sequence $(\bar{d}_i : i \in I)$ and formula $\phi(\bar{x}, \bar{y}, z)$ such that for every $s < t$, there is $c_{st}$ satisfying the following.

1. $\models \phi(\bar{d}_s, \bar{d}_t, c_{st})$
2. $\not\models \phi(\bar{d}_u, \bar{d}_t, c_{st})$ for $u < s$
3. $\not\models \phi(\bar{d}_s, \bar{d}_v, c_{st})$ for $t < v$
Non-structure contd.

- After Ramsey’s theorem, combinatorial arguments give coding in a unary expansion.
- The unary expansion is used to “recover the rows” $\bar{d}_i$ from the first element, so the tuples can be replaced by singletons.
- Shelah’s unary expansion is non-explicit.
LINEAR DECOMPOSITIONS

Definition

A linear decomposition of $M$ is a partition $M = \bigsqcup_i A_i$ and a model $N$ (not necessarily in $M$) such that $(A_i : i \in I)$ is an $N$-f.s. sequence.

- From the f.s.-dichotomy, we can extend partial linear decompositions one point at a time.
- Example: DLO

- Somewhat like one step of the tree decomposition, although the parts are ordered.
- Linear decompositions give an order-congruence over any large $C \supset N.$
Main theorem

Theorem (B, Laskowski)

The following are equivalent.

1. $T$ is monadically NIP.
2. $T$ does not admit coding in a unary expansion.
3. $T$ does not admit a pre-coding configuration.
4. $T$ has the f.s.-dichotomy.
5. Partial linear decompositions of models of $T$ extend to full linear decompositions.
6. $T$ is dp-minimal and indiscernible trivial.

- From Shelah’s results, we still need $(5) \Rightarrow (1)$, and to show the equivalence with $(6)$.
- We also redo the non-structure part of Shelah’s proof more carefully to get our result about finite structures.
FROM DECOMPOSITIONS TO MONADIC NIP

- Given an indiscernible sequence $\mathcal{I} = (a_i : i \in I)$, we consider a partition of $\mathfrak{C}$ with each $a_i$ in a different part.
- We choose a finite subset of that partition, and count the number of types realized over it.
- If $T$ has IP, then by taking $\mathcal{I}$ sufficiently long and shattered, we must realize unboundedly many types.
- If $T$ can extend $\mathcal{I}$ to a linear decomposition over $M$, then doing so will realize few types ($\beth_2(\aleph_0)$).
- This uses that each part is finitely satisfiable in $M$, so few types in each part, and the parts form an order-congruence.
- So few quantifier-free types realized in any monadic expansion of $T$.
- But we can bound the number of types realized in terms of the number of q.f.-types realized (by applying $\beth_{\omega+1}$).
- This type-counting seems similar to linear clique width (cf [2]).
INDISCERNIBLES

Definition

$T$ is *dp-minimal* if whenever $\mathcal{I}$ is dense indiscernible, then $\mathcal{I}$ splits into at most three parts indiscernible over a parameter $c$, with one part initial, one a singleton, and one terminal.

$T$ is *indiscernible trivial* if whenever $\mathcal{I}$ is indiscernible over each $a \in A$, then $\mathcal{I}$ is indiscernible over $A$.

- Thanks to Pierre Simon for suggesting this characterization.
- Example: DLO

- Fairly easy that if have these properties, then can’t have a pre-coding configuration.
- If $T$ is monadically NIP, linear decompositions show its models “look like DLO”.
A DIVIDING LINE

- Monadic NIP should be a dividing line for several properties of hereditary classes.
- Should provide a general setting for decompositions as in structural graph theory.
- For example, see recent work on twin-width and ordered graph classes, where it coincides with monadic NIP [5].
- Also see work on sparse graph classes, started by Nešetřil and Ossona de Mendez.
“Structurally $P$” closes $P$ under definability in unary expansions.
HOMOGENEOUS STRUCTURES

Definition

Given a structure $M$, the growth rate of $M$ is a function $\varphi_M(n)$ counting the (unlabeled) isomorphism types of $n$-substructures.

We add monadic NIP to a question of Macpherson.

Conjecture

Let $M$ be a homogeneous $\omega$-categorical structure. The following are equivalent.

1. $M$ is monadically NIP.
2. The growth rate of $M$ is at most exponential.
3. Age($M$) is well-quasi-ordered by embeddability, i.e. there is no infinite antichain.

We prove non-structure results: (2) $\implies$ (1) and a weak form of (3) $\implies$ (1), just assuming QE.
Theorem (B, Laskowski)

Suppose $M$ has QE and is not monadically NIP. Then

1. the growth rate of $M$ is at least $(n/k)!$ for some $k \in \mathbb{N}$
2. there is some expansion $M^*$ of $M$ by $\ell$ unary predicates with $\text{Age}(M^*)$ not well-quasi-ordered

- No uniform bounds on $k$, $\ell$. 
CODING FINITE GRAPHS

- We want to encode bipartite graphs with $n$ edges and $n$ vertices in each part in $O(n)$-substructures of a unary expansion of $M$.
- By our characterization, if $M$ is not monadically NIP, it admits a pre-coding configuration.
- Shelah showed how to then code bipartite graphs in an unspecified unary expansion.
- You only need to name the “columns” of the pre-coding configuration, which lets you recover the “rows” from any element [2].
- If $\psi(x, y, z)$ witnesses coding, we want to ensure $\psi$ behaves the same in our finite structures as in $\mathcal{C}$.
- We keep track of which elements are needed to witness (the failure of) quantifiers in $\psi$ so we can include them in our finite structures.
Questions

Question

Can we give uniform bounds on $k$ and $\ell$ in the last theorem? In particular, can we get rid of $\ell$?
Can the linear decomposition be refined to a tree decomposition? [1]

Question

Can the quantifier-elimination for mutually algebraic theories be generalized to monadic stability?

Question

Is there a tree-decomposition for monadically stable structures more suited to finite combinatorics?
Does monadic stability imply low VC-density, i.e. $vc(\phi(\bar{x}; \bar{y})) = |\bar{x}|$?
Is a hereditary graph class monadically stable iff it is definable in a unary expansion of a nowhere-dense class? [3]
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