Solving the Nonlinear Monotone Equations by Using a New Line Search Technique

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Abstract. In this work a new technique of line search is proposed to solve the nonlinear monotone equations. For this purpose we combine a modified line search rule with monotone technique. The new proposed algorithm can decrease the CPU time, the number of iterations and the function evaluations and can increase the efficiency of the approach. The global convergence result of the recommended procedures is established under some standard conditions. Preliminary numerical results indicate that the proposed algorithm are interesting and remarkably promising.

Keywords: Nonlinear system of equations. Iterative method. Line search method. Monotone strategy. Global convergence.

1. Introduction

Generally, the nonlinear systems are a family of problems that is close to optimization problems, also its one of the problems that arise in different fields of science and computational geometry, especially in the interpretation of nonlinear partial differential equations, the problem of specified value, etc. There are some cases where many nonlinear equations of some independent variable can be effectively solved. Thus, the roots of systems of nonlinear equations can be found in many applications in applied and numerical mathematics.

Therefore, to solve these non-linear systems, many researchers focused on finding and providing appropriate means and methods for this by proposing some common algorithms to solve this problem.
Nonlinear equations can be considered one of the most important problems with multiple scientific uses as tremolo systems in computer science [1, 2], some sub-problems in generalization and also the first-order necessary condition for the problem of unconstrained convex optimization [3, 4].

One of the indications that there is a close relationship between optimization problems and nonlinear systems of equations is that the fixed points that can be found from the optimization problem are equal in finding the answer to the system of nonlinear equations, so it is appropriate to use unconstrained optimization algorithms to solve this problem.

To solve a system of nonlinear equations, we use one of the two important iterative methods, which are the line search strategy, and the other method is the confidence zone. Emphasis will be placed here on how to search for lines and their frame.

This method is fairly simple so its understanding and application is easy. However, they are ineffective, and have some disadvantage, for example, on the large scale, the work and convergence of the line search method are slow.

So most of researchers used the monotone strategy to address that problem.

Consider the nonlinear system of equations

\[ F(x) = 0 \]  \hspace{1cm} (1)

Where \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous and monotone, i.e.

\[ \langle F(x) - F(y), x - y \rangle \geq 0, \forall x, y \in \mathbb{R}^n. \]

It is possible to convert some of the disparities into monotonous nonlinear equations through the natural map or the fixed point map, but after fulfilling some of the coercive conditions that the objective function has to achieve. There are several algorithms for solving the problem (1), which of the quasi-Newton methods are among the most important of which have made clear progress in theory, and the reason is due to the development in solutions to a large number of problems, which is particularly reflected in the analysis of local convergence [5, 6]. In addition, many researchers have established a global approximation of quasi-Newton methods to unconstrained optimization problems, see [7-10].

In (1986), Griewank suggested a derivative-free line search which is considered the closest approximation to global convergence [11]. In (2000), Li and Fukushima had another point of view by creating and inferring an example showing that searching for a line technique of Griewank contains certain difficulties in some special cases [12]. As a result of their work and by using the non-monotonous line search method, they suggested a Gauss-Newton-based BFGS method to solve nonlinear symmetric equations and a Broydens method to solve nonlinear equations, also they proved these methods converge globally [13]. However, some of the merit functions, such as the quadratic merit function, are used to ensure the global approximation of quasi-Newton method.

The authors suggested many techniques to find the best solution in varicose fields, such as optimization, operation research, reliability, see [14-23], but in this paper, and for solving the nonlinear monotone equations, we used the new algorithm to find the best solution, and it was also proven that they are globally convergence without using the merit function. Compared with the method in [14] and the method in [15], the new method is more efficient.

Now, we will give our algorithm.
2. The New Algorithm (K)

**Step 0.** Choose an initial point \(x_0 \in \mathbb{R}^n\) and constants \(\mu \in (0,1), \rho \in (0,1), \beta \in \left(\frac{1}{2}, 1\right), \sigma \in (0, \frac{1}{2})\), \(m > 0, r > 0\). Let \(k := 0\).

**Step 1.** Compute the search direction \(d_k\) by

\[
d_k = -F(x_k)
\]

Stop if \(d_k = 0\)

**Step 2.** Determine step length \(\alpha_k = \mu^h \beta^k\) such that \(h_k\) is the smallest nonnegative integer \(h\) satisfies

\[
-\langle F(x_k + \mu^h \beta d_k), d_k \rangle \geq \rho \sigma_k \mu^h \beta \| F(x_k + \beta \mu^h d_k) \| \| d_k \|^2
\]

Where \(\sigma_k = \frac{\rho}{1 + \| F(x_k) \|}\)

Let \(z_k = x_k + \alpha_k d_k\).

Stop if \(\| F(z_k) \| = 0\).

**Step 3.** Compute

\[
x_{k+1} = x_k - \frac{\langle F(z_k), x_k - z_k \rangle}{\| F(z_k) \|^2} F(z_k)
\]

Set \(k := k + 1\). Go to Step 1.

**Remark:**

(i) the mapping \(F\) is Lipschitz continuous (LC), satisfies for a positive constant \(L > 0\) that:

\[
\| F(x) - F(y) \| \leq L \| x - y \|, \forall x, y \in \mathbb{R}^n.
\]

(ii) it is clear that \(L + m > m\) so,

\[
\frac{\| F(x_k) \|}{L + m} \leq \| d_k \| \leq \frac{\| F(x_k) \|}{m}
\]

Now we will show that the line search (3) is well-defined in a similar way to [24].

suppose that for some iteration index \(k\) and for any nonnegative integer \(h\), the line search (3) is not satisfied i.e.,

\[
-\langle F(x_k + \mu^h \beta d_k), d_k \rangle < \rho \sigma_k \mu^h \beta \| F(x_k + \beta \mu^h d_k) \| \| d_k \|^2
\]

Now if we take \(\lim_{h \to \infty}\) for two side to (*),

\[
\lim_{h \to \infty} \langle F(x_k + \mu^h \beta d_k), d_k \rangle < \lim_{h \to \infty} \rho \sigma_k \mu^h \beta \| F(x_k + \beta \mu^h d_k) \| \| d_k \|^2
\]

\[
\Rightarrow -\langle F(x_k), d_k \rangle < 0
\]

\[
\Rightarrow -\langle F(z_k - \alpha_k d_k), d_k \rangle < 0
\]

(Since \(x_k = z_k - \alpha_k d_k\))

\[
\Rightarrow -\langle -\alpha_k, F(z_k + d_k), d_k \rangle < 0
\]

\[
\Rightarrow \alpha_k \| F(z_k) \| \| d_k \|^2 < 0
\]

Then we have a contradiction, since it is not possible to have each of \(\alpha_k\), \(\| F(z_k) \|\) and \(\| d_k \|^2\) less than zero, so the line search is well-defined.

3. Convergence property

In this section, to obtain the global convergence of our algorithm then we need the following lemma.

**Lemma 1.** [16]. let \(F\) be monotone and \(x, y \in \mathbb{R}^n\) satisfy \(\langle F(y), x - y \rangle > 0\). Let
\[
x^+ = x - \frac{\langle F(y) , x - y \rangle}{\| F(y) \|^2} F(y)
\]

Then for any \( \bar{x} \in R^n \) such that \( F(\bar{x}) = 0 \), it holds that
\[
\| x^+ - \bar{x} \|^2 \leq \| x - \bar{x} \|^2 - \| x^+ - x \|^2
\]

Now we can state our convergence result by the following theorem similar to [24].

**Theorem 1.** Suppose that \( F \) is (LC) and monotone, and let \( \{x_k\} \) be any sequence generated by Algorithm (K). Also we suppose that the solution set of (1) is nonempty. Then for any \( \bar{x} \) satisfying \( F(\bar{x}) = 0 \), we have
\[
\| x_k - \bar{x} \|^2 \leq \| x_k - x_k+1 \|^2 - \| x_k - x_{k+1} \|^2
\]

In particular, the sequence \( \{x_k\} \) is bounded. Also, its satisfy that either \( \{x_k\} \) is finite and the last iterate is a solution, or the sequence is infinite and
\[
\lim_{k\to\infty} \| x_k+1 - x_k \| = 0
\]

Furthermore, the sequence \( \{x_k\} \) converges to some \( \bar{x} \) such that \( F(\bar{x}) = 0 \).

**Proof.** First, if the algorithm finishes at some iteration \( k \) then:
- either \( d_k = 0 \), so by the positive definiteness of \( B_k \), we get \( F(x_k) = 0 \).
- or \( \| F(z_k) \| = 0 \), in this case \( x_k \) or \( z_k \) will be a solution of (1).

Now suppose that \( d_k \neq 0 \) and \( F(x_k) \neq 0 \) for all \( k \). Then:
\[
\langle F(z_k) , x_k - z_k \rangle = \langle F(z_k) , x_k - x_k - \alpha_k d_k \rangle
\]
\[
= \langle F(z_k) , -\alpha_k d_k \rangle
\]
\[
= -\alpha_k \langle F(x_k) , d_k \rangle
\]
\[
\geq \rho \sigma_k \| F(z_k) \| \alpha_k^2 \| d_k \|^2 > 0
\]

Then,
\[
\langle F(z_k) , x_k - z_k \rangle = -\alpha_k \langle F(z_k) , d_k \rangle \geq \rho \sigma_k \| F(z_k) \| \alpha_k^2 \| d_k \|^2 > 0.
\]

(7)

Let \( \bar{x} \) be any solution of (1) and \( F(\bar{x}) = 0 \). From lemma 1, (4) and (7) we obtain
\[
\| x_k+1 - \bar{x} \|^2 \leq \| x_k - \bar{x} \|^2 - \| x_k+1 - x_k \|^2.
\]

(8)

In particular, the sequence \( \{\| x_k - \bar{x} \|\} \) is decreasing and hence convergent. Consequently, the sequence \( \{x_k\} \) will be bounded, and also we have
\[
\lim_{k\to\infty} \| x_k+1 - x_k \| = 0.
\]

(9)

By (6) It is clear that \( \{d_k\} \) holds to be bounded and so is \( \{z_k\} \).

From (4):
\[
x_k+1 - x_k = -\frac{\langle F(z_k) , x_k - z_k \rangle}{\| F(z_k) \|^2} F(z_k)
\]

Since \( \langle F(z_k) , x_k - z_k \rangle = -\alpha_k \langle F(z_k) , d_k \rangle \) then,
\[
x_k+1 - x_k = \frac{\alpha_k \langle F(z_k) , d_k \rangle}{\| F(z_k) \|^2} F(z_k) \geq \frac{\rho \| F(z_k) \| \alpha_k^2 \| d_k \|^2}{\| F(z_k) \|^2} F(z_k)
\]
\[
= \rho \alpha_k^2 \| d_k \|^2
\]

So,
\[
\| x_k+1 - x_k \| = \frac{\| F(z_k) , x_k - z_k \|}{\| F(z_k) \|} \geq \rho \alpha_k^2 \| d_k \|^2.
\]

(10)
From (9) and (10), we get
\[ \lim_{k \to \infty} \alpha_k \| d_k \| = 0, \quad \lim_{k \in K, \ k \to \infty} \alpha_k \| d_k \| = 0. \tag{11} \]

From (6), we get \( \lim \inf_{k \to \infty} \| F(x_k) \| = 0 \). If \( \lim \inf_{k \to \infty} \| d_k \| = 0 \), then by (11) we get
\[ \lim_{k \to \infty} \alpha_k = 0 \tag{12} \]

Now since \( \{x_k\} \) is bounded and by continuity of \( F \), it is clear that \( \{x_k\} \) has some accumulation point \( \hat{x} \) with \( F(\hat{x}) = 0 \). We also have from (8) that the sequence \( \{\|x_k - \hat{x}\|\} \) converges. Therefore, \( \{x_k\} \) converges to \( \hat{x} \).

(3) gives us
\[ -(F(x_k + \mu^{h_{k-1}}d_k),d_k) < \rho \sigma_k \mu^{h_{k-1}} \beta \| F(x_k + \beta \mu^{h_{k-1}}d_k) \| \| d_k \| \]  

Since \( \{x_k\}, \{d_k\} \) are bounded, so we can choose a subsequence, let \( k \to \infty \) in (13), we obtain
\[ -(F(\hat{x}), \hat{d}) \leq 0 \tag{14} \]

Such that \( \hat{x} \) and \( \hat{d} \) are limits of subsequences that chosen. Otherwise, by (6) and already familiar argument,
\[ -(F(\hat{x}), \hat{d}) > 0 \tag{15} \]

From (14) and (15) we get a contradiction. Hence it is not possible to get that
\[ \lim \inf_{k \to \infty} \| F(x_k) \| > 0. \]

This finishes the proof.

4. Numerical Results
In this section, we compare the performance of the new method (K) discussed earlier with the following algorithms

**PRP**: It is coming from [12].

**BFGS**: It is coming from using the line search in [13] with the direction of this paper.

We wrote all the codes in MATLAB with version R2014a, also the experiments are running on a computer with 4 GB of RAM and CPU 2.30 GHz. The purpose of running the codes is to compare the results of the new algorithm (K) with the algorithms mentioned above.

When \( \|F_k\| \leq 10^{-6} \) or \( \|F(z_k)\| \leq 10^{-8} \) or the total number of iterates exceeds 500000 then all the algorithms will be end. In all of the algorithms, the parameters are specified as follows \( \mu = 0.4, \rho = 0.3, \sigma = 0.25, \epsilon = 10^{-6} \).

The comparison of these methods is based on three things: \( N_i \) (number of iterations), \( N_f \) (number of function evaluations) and the CPU time. Also, the special dimensions to compare these algorithms are limited to 5000 – 50000 for the following initial points:

\[ x_0 = (10,10, ...,10)^T, \]
\[ x_1 = (-10, -10, ..., -10)^T, \]
\[ x_2 = (1,1, ...,1)^T, \]
\[ x_3 = (-1, -1, ..., -1)^T, \]
\[ x_4 = (1, \frac{1}{2}, ..., \frac{1}{2})^T, \]
\[ x_5 = (0,1,0,1, ...,0,1)^T, \]
\[ x_6 = (\frac{1}{2}, \frac{1}{2}, ...,0)^T, \]
\[ x_7 = (1 - \frac{1}{2}, 1 - \frac{1}{2}, ...,0)^T. \]
Numerical results are displayed in tables (4.1, 4.2), the first table contains both of $N_i$ and $N_f$ for all algorithms while the second table contains CPU times of these algorithms.

**Table 4.1:** Numerical results ($N_i, N_f$)

| P. | Dim. | S.P | New | PRP | BFGS |
|----|------|-----|-----|-----|------|
| $P_1$ | 50000 | $x_0$ | 16 | 145 | 188 | 994 | 1255 | 8837 |
| | 50000 | $x_1$ | 16 | 145 | 188 | 994 | 1255 | 8837 |
| | 50000 | $x_2$ | 14 | 101 | 40 | 194 | 148 | 798 |
| | 50000 | $x_3$ | 14 | 101 | 40 | 194 | 148 | 798 |
| | 50000 | $x_4$ | 15 | 81 | 25 | 115 | 15 | 79 |
| | 50000 | $x_5$ | 10 | 46 | 20 | 97 | 27 | 160 |
| | 50000 | $x_6$ | 19 | 136 | 45 | 202 | 49 | 254 |
| | 50000 | $x_7$ | 19 | 136 | 45 | 202 | 49 | 254 |
| $P_2$ | 50000 | $x_0$ | 16 | 145 | 188 | 994 | 1255 | 8837 |
| | 50000 | $x_1$ | 14 | 109 | 196 | 1016 | 1327 | 9350 |
| | 50000 | $x_2$ | 14 | 101 | 40 | 194 | 148 | 798 |
| | 50000 | $x_3$ | 15 | 121 | 46 | 235 | 65 | 375 |
| | 50000 | $x_4$ | 15 | 81 | 71 | 376 | 16 | 89 |
| | 50000 | $x_5$ | 10 | 46 | 20 | 97 | 27 | 160 |
| | 50000 | $x_6$ | 18 | 126 | 50 | 221 | 49 | 254 |
| | 50000 | $x_7$ | 30 | 250 | 50 | 256 | 50 | 267 |
| $P_3$ | 10000 | $x_0$ | 22534 | 135605 | 149031 | 911135 | 409228 | 2866839 |
| | 10000 | $x_1$ | 13699 | 87210 | 36187 | 217508 | 149424 | 1057285 |
| | 10000 | $x_2$ | 61248 | 385783 | 99331 | 521291 | 446334 | 3081533 |
| | 10000 | $x_3$ | 29703 | 149950 | 111908 | 587876 | 241250 | 1466562 |
| | 10000 | $x_4$ | 60325 | 386954 | 94078 | 592519 | 102701 | 586236 |
| | 10000 | $x_5$ | 11159 | 38466 | 14865 | 49971 | 28606 | 133867 |
| | 10000 | $x_6$ | 9873 | 57261 | 20338 | 104094 | 51190 | 307639 |
| | 10000 | $x_7$ | 10121 | 59001 | 20326 | 104018 | 51133 | 307162 |
| $P_4$ | 10000 | $x_0$ | 26 | 263 | 8567 | 106957 | 12677 | 163074 |
| | 10000 | $x_1$ | 26 | 272 | 14175 | 204841 | 19577 | 256721 |
| | 10000 | $x_2$ | 23 | 219 | 421 | 5346 | 5072 | 58466 |
| | 10000 | $x_3$ | 146 | 1471 | 5282 | 60829 | 7269 | 83804 |
| | 10000 | $x_4$ | 21 | 185 | 3599 | 35949 | 4134 | 41272 |
| | 10000 | $x_5$ | 1365 | 14992 | 257 | 1809 | 2328 | 20899 |
| | 10000 | $x_6$ | 536 | 4801 | 6679 | 73228 | 4428 | 49192 |
| | 10000 | $x_7$ | 539 | 4828 | 7036 | 77151 | 4525 | 50314 |
| $P_5$ | 50000 | $x_0$ | 88 | 973 | 62668 | 609170 | 228454 | 2427634 |
| | 50000 | $x_1$ | 67 | 675 | 61618 | 597442 | 225811 | 2396330 |
| | 50000 | $x_2$ | 88 | 973 | 62578 | 608175 | 228235 | 2425065 |
| | 50000 | $x_3$ | 86 | 933 | 62398 | 606160 | 227774 | 2419561 |
| | 50000 | $x_4$ | 92 | 1041 | 62491 | 607212 | 228010 | 2422394 |
| | 50000 | $x_5$ | 91 | 1024 | 62497 | 607267 | 228023 | 2422517 |
| | 50000 | $x_6$ | 91 | 1024 | 62516 | 607479 | 228068 | 2423049 |
| | 50000 | $x_7$ | 88 | 973 | 62549 | 607843 | 228167 | 2424258 |
### Table 4.1: Numerical results (\(N_i, N_f\)) – continued

| P. | Dim. | S.P | New | PRP | BFGS |
|----|------|-----|-----|-----|------|
|    |      |     | \(N_i\) | \(N_f\) | \(N_i\) | \(N_f\) | \(N_i\) | \(N_f\) |
| \(P_6\) | 50000 | \(x_0\) | 16 | 91 | 376 | 1916 | 659 | 4044 |
|      | 50000 | \(x_1\) | 16 | 115 | 378 | 1944 | 2560 | 17934 |
|      | 50000 | \(x_2\) | 14 | 87 | 40 | 169 | 181 | 1010 |
|      | 50000 | \(x_3\) | 16 | 91 | 117 | 510 | 659 | 4044 |
|      | 50000 | \(x_4\) | 15 | 101 | 117 | 510 | 182 | 1023 |
|      | 50000 | \(x_5\) | 14 | 87 | 117 | 510 | 181 | 1010 |
|      | 50000 | \(x_6\) | 14 | 87 | 117 | 510 | 181 | 1010 |
| \(P_7\) | 50000 | \(x_0\) | 10 | 41 | 350 | 1755 | 606 | 3643 |
|      | 50000 | \(x_1\) | 12 | 63 | 401 | 2064 | 684 | 4145 |
|      | 50000 | \(x_2\) | 31 | 65 | 62 | 126 | 62 | 189 |
|      | 50000 | \(x_3\) | 22 | 164 | 26 | 142 | 97 | 538 |
|      | 50000 | \(x_4\) | 49 | 100 | 99 | 200 | 25 | 52 |
|      | 50000 | \(x_5\) | 2584 | 5170 | 5168 | 10338 | 1292 | 2586 |
|      | 50000 | \(x_6\) | 55 | 112 | 110 | 222 | 110 | 333 |
|      | 50000 | \(x_7\) | 55 | 112 | 110 | 222 | 110 | 333 |

### Table 4.2: Numerical results (CPU time)

| P. | Dim. | S.P | New | PRP | BFGS |
|----|------|-----|-----|-----|------|
| \(P_1\) | 50000 | \(x_0\) | 0.5148 | 4.9296 | 40.3730 |
|      | 50000 | \(x_1\) | 0.5148 | 4.9608 | 41.4182 |
|      | 50000 | \(x_2\) | 0.2652 | 0.7332 | 2.5584 |
|      | 50000 | \(x_3\) | 0.3120 | 0.7020 | 2.5896 |
|      | 50000 | \(x_4\) | 0.2652 | 0.4368 | 0.2652 |
|      | 50000 | \(x_5\) | 0.1716 | 0.3744 | 0.5460 |
|      | 50000 | \(x_6\) | 0.4212 | 0.7956 | 0.7176 |
|      | 50000 | \(x_7\) | 0.3744 | 0.8424 | 0.8892 |
| \(P_2\) | 50000 | \(x_0\) | 0.5148 | 4.9452 | 42.6974 |
|      | 50000 | \(x_1\) | 0.4056 | 5.1012 | 44.7410 |
|      | 50000 | \(x_2\) | 0.2964 | 0.6708 | 2.8548 |
|      | 50000 | \(x_3\) | 0.3588 | 0.8736 | 1.2636 |
|      | 50000 | \(x_4\) | 0.2652 | 1.2480 | 0.2808 |
|      | 50000 | \(x_5\) | 0.1560 | 0.3432 | 0.5304 |
|      | 50000 | \(x_6\) | 0.3900 | 0.7956 | 0.7332 |
|      | 50000 | \(x_7\) | 0.7488 | 0.9828 | 0.8580 |
Table 4.2: Numerical results (CPU time) – continued

| P   | Dim. | S.P | New     | PRP     | BFGS    |
|-----|------|-----|---------|---------|---------|
| $P_3$ | 10000 | $x_0$ | 0.5265  | 3.6298  | 1.1398  |
|      | 10000 | $x_1$ | 0.3429  | 0.8903  | 0.4389  |
|      | 10000 | $x_2$ | 1.4979  | 2.0869  | 1.3758  |
|      | 10000 | $x_3$ | 0.5859  | 2.3832  | 0.5998  |
|      | 10000 | $x_4$ | 1.5049  | 2.3747  | 0.2346  |
|      | 10000 | $x_5$ | 0.1506  | 0.2027  | 0.0535  |
|      | 10000 | $x_6$ | 0.2228  | 0.4189  | 0.1233  |
|      | 10000 | $x_7$ | 0.2290  | 0.4198  | 0.1229  |
| $P_4$ | 10000 | $x_0$ | 0.1716  | 0.7960  | 1.0721  |
|      | 10000 | $x_1$ | 0.1716  | 1.5104  | 1.7052  |
|      | 10000 | $x_2$ | 0.1248  | 0.0388  | 0.4040  |
|      | 10000 | $x_3$ | 1.0140  | 0.4524  | 0.5508  |
|      | 10000 | $x_4$ | 0.1248  | 0.2664  | 0.2676  |
|      | 10000 | $x_5$ | 10.3116 | 0.0143  | 0.1396  |
|      | 10000 | $x_6$ | 3.2292  | 0.5561  | 0.3291  |
|      | 10000 | $x_7$ | 3.3384  | 0.5779  | 0.3325  |
| $P_5$ | 5000  | $x_0$ | 0.3900  | 2.6088  | 9.1360  |
|      | 5000  | $x_1$ | 0.2808  | 2.5382  | 8.9972  |
|      | 5000  | $x_2$ | 0.4056  | 2.6067  | 9.0797  |
|      | 5000  | $x_3$ | 0.3744  | 2.5844  | 9.0674  |
|      | 5000  | $x_4$ | 0.3744  | 2.5744  | 9.0594  |
|      | 5000  | $x_5$ | 0.3744  | 2.5384  | 9.0630  |
|      | 5000  | $x_6$ | 0.3744  | 2.5518  | 9.1413  |
|      | 5000  | $x_7$ | 0.3744  | 2.5476  | 9.2811  |
| $P_6$ | 50000 | $x_0$ | 0.5616  | 13.3224 | 0.2694  |
|      | 50000 | $x_1$ | 0.8580  | 13.3380 | 1.1963  |
|      | 50000 | $x_2$ | 0.5616  | 1.2012  | 0.0670  |
|      | 50000 | $x_3$ | 0.5616  | 3.6660  | 0.2751  |
|      | 50000 | $x_4$ | 0.7020  | 3.6972  | 0.0680  |
|      | 50000 | $x_5$ | 0.6552  | 3.6660  | 0.0656  |
|      | 50000 | $x_6$ | 0.5304  | 3.5256  | 0.0641  |
|      | 50000 | $x_7$ | 0.5772  | 3.6348  | 0.0658  |
| $P_7$ | 50000 | $x_0$ | 0.1560  | 8.4396  | 16.9261 |
|      | 50000 | $x_1$ | 0.2340  | 9.9060  | 18.7045 |
|      | 50000 | $x_2$ | 0.2496  | 0.5304  | 0.6708  |
|      | 50000 | $x_3$ | 0.4836  | 0.4836  | 1.7472  |
|      | 50000 | $x_4$ | 0.2808  | 0.9204  | 0.1248  |
|      | 50000 | $x_5$ | 16.8325 | 43.9922 | 8.6424  |
|      | 50000 | $x_6$ | 0.4368  | 0.9984  | 0.9672  |
|      | 50000 | $x_7$ | 0.3744  | 1.0608  | 0.9828  |

From the comparisons of the results we can see the superiority of the new approach compared to other methods for solving the nonlinear systems of monotone equations.
Table (4.1) shows the total of \( N_i \) and \( N_f \) for the three algorithms, while table (4.2) shows the CPU time for the three algorithms to reach the solution. The algorithm (K) solved reached to the solution with less number of iteration and function evaluation and with less CPU time among the three methods. It means that the new algorithm (K) is the best algorithm closing to the performance index.

5. Conclusion
From the numerical results obtained through the comparison technique presented in the tables above of different problems with different initial points and dimensions, it is easy to conclude that the performance of the proposed algorithm (K) is the most efficient and effective in terms of \( N_i, N_f \) and CPU time compared with the two famous algorithms. This can improve the behavior of the new algorithm to solve the nonlinear monotone equations which does not require Jacobian information of the nonlinear equations. The algorithm (K) is able to calculate the best solution of problem (1), also its global convergence has been created without using any merit functions.

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