Diffusive-Ballistic Transition in Random Polymers with Drift and Repulsive Long-Range Interactions

L.M. Cioletti\textsuperscript{a,} \textsuperscript{1}, C.C.Y. Dorea\textsuperscript{a} and S. Vasconcelos da Silva\textsuperscript{b}

\textsuperscript{a} Universidade de Brasilia, 70910-900 Brasilia-DF, Brazil
\textsuperscript{b} Universidade Federal de Goias, 74001-970 Goiania-GO, Brazil

Abstract

In this note phase transition issues are addressed for random polymers on $\mathbb{Z}^2$ with long-range self-repulsive interactions. It is shown that, in the absence of drift and with power law interactions, the polymer exhibits transition from diffusive to a ballistic behavior. When non-null drifts are added and positive translation invariant interactions are considered, the polymer presents a ballistic behavior. Our results complement some previous studies on the matter and we also derive a Central Limit Theorem for the model.

MSC: 82B20, 82B41, 82B26.
Keywords: self-repelling random polymers; Ising model; long-range interactions; diffusive-ballistic phase transition; CLT.

1 Introduction

Random polymers can be modelled as connected subsets of $\mathbb{Z}^2$. More precisely, a $N$-th step polymer $S$ is an element of $\mathbb{W}_N$ given by

$$\mathbb{W}_N := \{ S = (S_0, S_1, \ldots, S_N) : S_i \in \mathbb{Z}^2, S_0 = 0 \text{ and } \|S_{i+1} - S_i\| = 1 \},$$

being $\| \cdot \|$ the $\ell^1$ norm. Under Gibbs measure setting at inverse temperature $\beta > 0$ and Hamiltonian $\mathbb{H}_N$ we can write the probability

$$\mathbb{P}_N^{\beta, h}(S) = \frac{\exp[-\beta \mathbb{H}_N(S)]}{\mathbb{Z}_N^\beta(h)}, \quad \mathbb{H}_N(S) = -\sum_{1 \leq i < j \leq N} V_{ij} \langle X_i, X_j \rangle + \langle h, S_N \rangle, \quad (1)$$

where $X_i = S_i - S_{i-1}$ stands for the $i$-th random step, $V_{ij}$ are the prescribed interactions, $\langle \cdot, \cdot \rangle$ denotes the usual inner product, $h \in \mathbb{R}^2$ is the fixed drift vector and $\mathbb{Z}_N^\beta(h)$ is the partition function.

\textsuperscript{1}Corresponding author : leandro.mat@gmail.com
Co-authors : changdorea@unb.br ; simone@ufg.br
Research partially supported by CNPq and FEMAT.
Caracciolo et al. ([3], 1993) introduced a self-repelling random polymer model with Hamiltonian on $\mathbb{W}_N$ given by $H_N(S) = g_0 \sum_{0 \leq i < j \leq N} V_{ij} \delta_{S_i, S_j}$, where $g_0 > 0$ and the interactions $V_{ij} = |i - j|^{-\alpha}$. Their model interpolates between the lattice Edwards model ($\alpha = 0$) and ordinary SRW ($\alpha = \infty$). Moreover, it was conjectured that for dimension $1 \leq d \leq 4$ there exists a strictly positive exponent $\gamma = \gamma(d, \alpha)$ such that the mean square end-to-end distance satisfies the asymptotics

$$E_{P_N}[\|S_N\|^2] = \sum_{S \in \mathbb{W}_N} \|S_N\|^2 P_N(S) \sim c N^\gamma,$$

where the Gibbs measure $P_N$ is given by the Hamiltonian $H_N$. In [10], the Hamiltonian $\tilde{H}_N(S) = -\sum_{0 \leq i < j \leq N} |i - j|^{-\alpha} \|S_i - S_j\|^2$, where $3 < \alpha \leq 4$, was considered. They proved the existence of positive constants $\beta_1$ and $\beta_2$ that led to phase transition from diffusive regime ($\beta < \beta_1$) to a ballistic one ($\beta > \beta_2$). However, it was left unknown what undergoes when $\beta \in [\beta_1, \beta_2]$. As usual, the different diffusive regimes are classified according to the asymptotic behavior of the mean square displacement and for our model (1) it reduces in determining $\gamma > 0$ for which the following limit exists, is positive and finite

$$\lim_{N \to \infty} \frac{1}{N^\gamma} E_{P_N^{\beta,h}}[\|S_N\|^2] = \lim_{N \to \infty} \frac{1}{N^\gamma} \sum_{S \in \mathbb{W}_N} \|S_N\|^2 P_N^{\beta,h}(S). \quad (2)$$

We say that the polymer model is **diffusive** if $\gamma = 1$, **superdiffusive** if $1 < \gamma < 2$ and **ballistic** if $\gamma = 2$.

Our main motivation is to build a self-repelling random polymer model for which we can derive a genuine diffusive-ballistic phase transition, i.e. the existence of a unique positive constant $\beta_c$ separating the model into two regimes. In this note, assuming zero drift and $V_{ij} = |i - j|^{-\alpha}$ with $1 < \alpha \leq 2$, we prove (Theorem 3) that there exists a unique positive number $\beta_c$ (the critical temperature of a related one dimensional Ising model) such that the model is diffusive for $\beta < \beta_c$ and ballistic for $\beta > \beta_c$. On the other hand, considering non-null drift and positive, translation invariant and regular interactions, we conclude from Theorem 2 that for all $\beta \in (0, \infty)$ the model is ballistic.

The Lemma 1 is an essential tool in this work. Its proof is similar in spirit to the one introduced for the Potts model by M. Suzuki [13] in 1967. It consists in decoupling the steps of the polymer as two independent Ising random variables. The background idea is the same applied when looking at SRW in lattice $\mathbb{Z}^2$.

In 1983 Newman [9] proved a CLT for block random variables satisfying the FKG inequalities under finite susceptibility hypothesis. In Section 3 we investigate the validity of the CLT for our model. Here assuming non zero drifts and consequently infinite susceptibility, we prove (Theorem 4) a CLT for the projections of suitably normalized displacements. This is obtained by using both the Lee-Yang circle theorem and the $C^2$-regularity condition from Wu Liming [8].


2 Mean Square Displacement and Phase Transition

For the volume $\Lambda_N = \{1, 2, \ldots, N\}$ consider the one dimensional Ising model with free boundary conditions defined by the Hamiltonian

$$
H_{\Lambda_N} (\sigma) = - \sum_{1 \leq i < j \leq N} V_{ij} \sigma_i \sigma_j - \sum_{i=1}^{N} h \sigma_i,
$$

where $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N := \Sigma_N$, $V_{ij} \in \mathbb{R}$ are the coupling constants and $h \in \mathbb{R}$ is an external field. To simplify notation for a given a real-valued function $f : \Sigma_N \to \mathbb{R}$ write

$$
\langle f \rangle_{\Lambda_N}^{\beta,h} = \mathbb{E}_{P_{\Lambda_N}^{\beta,h}}[f] \quad \text{with} \quad P_{\Lambda_N}^{\beta,h}(\sigma) = \frac{1}{Z_{\Lambda_N}^{\beta}(h)} \exp\left(-\beta H_{\Lambda_N}(\sigma)\right)
$$

where $Z_{\Lambda_N}^{\beta}(h)$ is the partition function.

Lemma 1. For $e_1 = (1, 0)$ and $e_2 = (0, 1)$ define $h_1 = \langle h, e_1 - e_2 \rangle$ and $h_2 = \langle h, e_1 + e_2 \rangle$. Then

$$
\mathbb{E}_{P_{\Lambda_N}^{\beta,h}}[\|S_N\|^2] = \frac{1}{2} \sum_{i,j=1}^{N} \left[ \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,h_1} + \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,h_2} \right].
$$

Proof. The proof follows closely the ideas from [2, 10]. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation $TX_i = (\sigma_i e_1 + \tilde{\sigma}_i e_2)/\sqrt{2}$, with $\sigma_i, \tilde{\sigma}_i \in \{-1, 1\}$. A simple computation shows that

$$
P_{\Lambda_N}^{\beta,h}(S) = P_{\Lambda_N}^{\beta,h_1}(\sigma) P_{\Lambda_N}^{\beta,h_2}(\tilde{\sigma}) \quad \text{and} \quad \|S_N\|^2 = \frac{1}{2} \sum_{i,j=1}^{N} (\sigma_i \sigma_j + \tilde{\sigma}_i \tilde{\sigma}_j).
$$

$\square$

Theorem 2. Suppose that $V_{ij}$ is positive and translation invariant, i.e. $V_{ij} = V(|i - j|) > 0$ for all $i \neq j$. If $h \in \mathbb{R}^2$ is such that $h_1$ and $h_2$ satisfy $h_1 h_2 > 0$ and $\beta > 0$, then for some constant $C(\beta, h) > 0$ we have

$$
C(\beta, h) \leq \frac{\mathbb{E}_{P_{\Lambda_N}^{\beta,h}}[\|S_N\|^2]}{N^2}.
$$

Proof. Let $k \in \mathbb{R}$. Since $V_{ij} > 0$ for $i \neq j$ we get from the second Griffiths inequality,

$$
\langle \sigma_i \rangle_{\Lambda_N}^{\beta,k,nn} \langle \sigma_j \rangle_{\Lambda_N}^{\beta,k,nn} \leq \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,k,nn} \leq \langle \sigma_i \rangle_{\Lambda_N}^{\beta,k} \langle \sigma_j \rangle_{\Lambda_N}^{\beta,k}.$$

3
where the left hand side expected values are taken with respect to the Gibbs measure of the nearest neighbours Ising model on $\Lambda_N$ with free boundary conditions and Hamiltonian given by $H_{\Lambda_N}(\sigma) = -\sum_{n=1}^{N-1} V(1)\sigma_n\sigma_{n+1} - k \sum_{n=1}^{N} \sigma_n$. Using monotonicity with respect to the volume and classical transfer matrix computation (see [5, p. 107]) we have for any $i \in \Lambda_N$

$$\langle \sigma_i \rangle_{\Lambda_N}^{2,\beta} \to \frac{\sinh \beta k}{\sqrt{\sinh^2(\beta k) + e^{-4V(1)}}},$$

Theorem 3. Let $1 < \alpha \leq 2$, $h = 0$ and $V_{ij} = |i - j|^{-\alpha}$ for $i \neq j$. Then there exist a constant $\beta_c \in (0, \infty)$ and positive numbers $m_+(\beta)$ and $K(\beta)$ such that

$$\frac{1}{2} m_+^2(\beta) \leq \frac{1}{N^2} \mathbb{E}_{\mathbb{P}_{N}^{2,\beta,0}}[||S_N||^2] \leq 1,$$

if $\beta > \beta_c$ (3)

and

$$1 \leq \frac{1}{N} \mathbb{E}_{\mathbb{P}_{N}^{2,\beta,0}}[||S_N||^2] \leq K(\beta),$$

if $0 < \beta < \beta_c$. (4)

Proof. For $1 < \alpha \leq 2$ the existence of a critical $\beta_c \in (0, \infty)$ for the long range Ising model with coupling $V_{ij}$ is shown in [4, 6]. In this case, we have spontaneous magnetization $m_+(\beta) > 0$ for all $\beta > \beta_c$ and the two-point function with free boundary condition satisfies (cf. [7]),

$$\langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta} = \frac{1}{2} \left[ \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0,+} + \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0,-} \right] \geq m_+^2(\beta) \geq m_+^2(\beta_c).$$

Using the same type of arguments as in Theorem 2 we have for large $N$

$$\mathbb{E}_{\mathbb{P}_{N}^{2,\beta,0}}[||S_N||^2] = \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta} \geq \frac{1}{2} \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta} \geq \frac{1}{2} m_+^2(\beta) N^2.$$

To prove (4) one needs lower and upper bounds for $\langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0}$. From the monotonicity with respect to the volume and [1], if $\beta < \beta_c$ there are constants $0 < C(\beta) \leq C'(\beta) < \infty$ such that $\langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0}$ with free boundary condition satisfies

$$\frac{C(\beta)}{|i - j|^\alpha} \leq \langle \sigma_i \sigma_j \rangle_{\Lambda_N}^{\beta,0} \leq \frac{C'(\beta)}{|i - j|^\alpha},$$

where $C(\beta) \equiv (\beta \tanh \beta_c)/\beta_c$. The uniformity of the lower bound is a simple application of FKG inequality. Using Lemma 1 we have

$$N + \sum_{1 \leq i < j \leq N} \frac{2C(\beta)}{|i - j|^\alpha} \leq \mathbb{E}_{\mathbb{P}_{N}^{2,\beta,0}}[||S_N||^2] \leq N + \sum_{1 \leq i < j \leq N} \frac{2C'(\beta)}{|i - j|^\alpha}.$$

Inequality (4) follows by observing that $\sum_{1 \leq i < j \leq N} \frac{1}{|i - j|^\alpha} = O(N)$. \qed


3 Central Limit Theorem

To derive a CLT for (1) we make use of Theorem 1.2 and Theorem 3.1 from [8]. It is required that a $C^2$-regularity condition to be satisfied. We say that a sequence of probability measures $\{\mu_N\}$ satisfies the $C^2$-regularity condition if for $Y_N$ with probability measure $\mu$ the following limit exists

$$\Psi(t) = \lim_{N \to \infty} \Psi_N(t) = \lim_{N \to \infty} \frac{1}{N} \ln E_{\mu_N}[\exp(tNY_N)].$$

Moreover, for some neighborhood $[-\delta, \delta]$ of zero we have $\Psi(\cdot) < \infty$ and

$$\Psi''_N(t) \xrightarrow{N \to \infty} \Psi''(t) \text{ uniformly on } [-\delta, \delta].$$

Under these hypotheses $Y_N$ is asymptotically Gaussian.

For our polymer model take $v \in \mathbb{R}^2$ fixed and consider the empirical field projection

$$L_N = \frac{1}{N} \langle S_N, v \rangle = \frac{1}{N} \sum_{j=1}^{N} \langle X_j, v \rangle.$$ 

Set $\mu_N = \mathbb{P}_{\beta, h}^{\beta, h}$ and define the pressure functional by

$$\Psi_{\beta, h, v}(t) = \lim_{N \to \infty} \Psi_{\beta, h, v}^N(t) = \lim_{N \to \infty} \frac{1}{N} \ln E_{\mathbb{P}_{\beta, h}^{\beta, h}}[\exp(\beta t N L_N)].$$

**Theorem 4.** Assume that the interactions are translation invariant and summable, that is, $V_{ij} = V(|i - j|) > 0$ and $\sum_{i \in \mathbb{Z}} V(i) < \infty$. For $h \in \mathbb{R}^2$ with $h_1 h_2 \neq 0$ and any fixed $v \in \mathbb{R}^2$ we have

$$\frac{1}{\sqrt{N}} \left[ \beta \langle S_N, v \rangle - N \mathbb{E}_{\mathbb{P}_{\beta, h}^{\beta, h}}[\beta \langle S_N, v \rangle] \right] \xrightarrow{D} N \left( 0, \frac{\partial^2}{\partial t^2} \Psi_{\beta, h, v}(0) \right)$$

where “$\xrightarrow{D}$” stands for convergence in distribution.

**Proof.** Under the hypotheses, the existence of the limit $\Psi_{\beta, h, v}(\cdot)$ is proved in [12]. To complete the $C^2$-regularity verification take complex number $z \in \mathbb{C}$ and express $\Psi_{\beta, h, v}^N(z)$ in terms of partition functions of one-dimensional Ising model. As in Lemma 1 write $Z_{\beta}^N(h) = Z_{\beta/2}^{\beta/2}(h_1) Z_{\beta/2}^{\beta/2}(h_2)$. Using the principal-value logarithm identities

$$\ln(zw) = \ln z + \ln w + 2\pi i \mathcal{K} (\ln z + \ln w)$$

$$\ln(z/w) = \ln z - \ln w + 2\pi i \mathcal{K} (\ln z - \ln w)$$

5
where $\mathcal{K}(x+iy) = -\sum_{n \geq -1} n I((2n-1)\pi < y \leq (2n+1)\pi)$ with $I(\cdot)$ being the indicator function, we have for $v_1 = \langle v, e_1 - e_2 \rangle$ and $v_2 = \langle v, e_1 + e_2 \rangle$

$$
\Psi_{\beta,h,v}^N(z) = \frac{1}{N} \left[ \ln Z_{\beta,N}^\beta(h+zv) - \ln Z_{\beta,N}^\beta(h) + 2\pi i \mathcal{K} \left( \ln Z_{\beta,N}^\beta(h+zv) - \ln Z_{\beta,N}^\beta(h) \right) \right]
$$

$$
= \frac{1}{N} \ln Z_{\Lambda_N}^{\beta/2}(h_1 + zv_1) + \frac{1}{N} \ln Z_{\Lambda_N}^{\beta/2}(h_2 + zv_2)
$$

$$
+ \frac{2\pi i}{N} \mathcal{K} \left( \ln Z_{\Lambda_N}^{\beta/2}(h_1 + zv_1) + \ln Z_{\Lambda_N}^{\beta/2}(h_2 + zv_2) \right)
$$

$$
- \frac{1}{N} \ln Z_{\Lambda_N}^{\beta/2}(h_1) - \frac{1}{N} \ln Z_{\Lambda_N}^{\beta/2}(h_2) + \frac{2\pi i}{N} \mathcal{K} \left( \ln Z_{N}^\beta(h + zv) \right)
$$

By assuming that $\Re(h_i + zv_i) \neq 0$ and $h_1 h_2 \neq 0$ it follows from Lee-Yang Theorem’s and standard arguments from [11, p. 111] that

$$
\Psi_{\beta,h,v}^N(z) \to \Psi_{\beta,h,v}(z), \text{ locally uniformly in } z.
$$

Also, it follows that the derivatives of $\Psi_{\beta,h,v}^N(z)$ converge uniformly on the compact subsets of $\mathbb{C}$. Hence the $C^2$-regularity condition is satisfied. Since

$$
\frac{\partial^2}{\partial t^2} \Psi_{\beta,h,v}^N(t) \bigg|_{t=0} = \frac{1}{N} E_{p_{\beta,h}^N} \left[ \beta \langle S_N, v \rangle - N E_{p_{\beta,h}^N} [\beta \langle S_N, v \rangle] \right]^2 \to \frac{\partial^2}{\partial t^2} \Psi_{\beta,h,v}(0)
$$

we conclude the proof using Theorem 3.1 from [8].

**Remark 1.** We emphasize that in the above theorem we proved more than $C^2$-regularity condition. In fact, we proved that the pressure is analytic. Another way to obtain the $C^2$-regularity condition for our polymer model is to apply both FKG and GHS inequalities, see [8, p. 426].

4 Concluding Remarks

The random polymer model considered here interpolates between the SRW (infinite temperature) and a deterministic straight line (zero temperature). At very high temperatures this random polymer should be recurrent and transience would occur at very low temperatures, so we expect a recurrence-transience phase transition. It would be interesting to prove the existence of such phase transition and also to determine the critical temperature that separates these two regimes.

**Acknowledgments.** We are grateful to L.R. Fontes for his many valuable comments and careful reading of this manuscript.
References

[1] M. Aizenman, J. T. Chayes, L. Chayes, C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x−y|^2$ Ising and Potts models, J. Statist. Phys. 50, 1-40 (1988)

[2] P. Buttà, A. Procacci, B. Scoppola. Kac polymers, J. Statist. Phys. 119, no. 3-4, 643-658 (2005)

[3] S. Caracciolo, G. Parisi, A. Pelissetto. Random Walks with short-range interaction and mean field behaviour, J. Statist. Phys., 77, no. 3-4, 519-543 (1994)

[4] F. J. Dyson. Existence of a phase-transition in a one-dimensional Ising ferromagnet. Comm. Math. Phys. 12, no. 2, 91-107 (1969)

[5] Richard S. Ellis. Entropy, Large Deviation and Statistical Mechanics. Springer. (2005)

[6] J. Fröhlich, T. Spencer. The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy. Comm. Math. Phys. 84, no. 1, 87-101 (1982)

[7] J. L. Lebowitz. Coexistence of Phases in Ising Ferromagnets, J. Statist. Phys., 16, no. 6, 463-476 (1977)

[8] Liming, Wu. Moderate Deviations of Dependent Random Variables Related to CLT, Ann. Probab., 23, no. 1, 420-445 (1995).

[9] C. M. Newman. A General Central Limit Theorem for FKG Systems, Comm. Math. Phys. 91, 75-80 (1983)

[10] A. Procacci, R. Sanchis and B. Scoppola. Diffusive-Ballistic Transition in Random Walks with Long-Range Self-Repulsion, Lett. Math. Phys. 83, 181-187 (2008)

[11] D. Ruelle. Statistical Mechanics - Rigorous Results, World Scientific, Imperial College Press. (2007).

[12] D. Ruelle. Thermodynamic Formalism, Addison-Wesley, Reading, MA. (1978).

[13] M. Suzuki. Solution of Potts Model for Phase Transition. Prog. Theor. Phys. 37 (1967)