On the theorem converse to Jordan’s curve theorem.

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Abstract

Theorem converse to Jordan’s curve theorem says that if a compact set $K$ has two complementary domains in $\mathbb{R}^2$, from each of which it is at every point accessible, it is a simple closed curve.

We show that the requirement of this theorem that all points of $K$ were accessible from both complementary domains is surplus and prove one generalization of this theorem.

Introduction.

Approximately three years ago author was collided with the following question: what are the sufficient conditions for a compact set in the plane to be homeomorphic to the two-dimensional disk? In addition, it was desirable to express supposed answer in the terms of local properties of the frontier complementary domains.

In this terms it is possible instead of considering a compact set to pass to the investigation of its frontier. Then the previous question could be restated in the form: what are the sufficient conditions for a compact set in the plane to be a simple closed curve?

Famous Jordan’s curve theorem states that a simple closed curve in the plane has two complementary domains, of each of which it is the complete frontier. But the important conditions given in this theorem are not sufficient. The counterexample is so called polish curve (see the slight modification of it below, example 1.2).
Shoenflies theorem says that an arbitrary homeomorphism of a simple closed curve in the plane onto the unit circle could be extended to a homeomorphism of the plane onto itself. Every point of the unit circle is accessible from the both complementary domains. Hence an arbitrary point of a simple closed curve must be accessible from the components of a complement. The conditions of a sort are found to be the sufficient for a compact set in the plane to be a Jordan curve.

So, the following theorem holds true: if a closed set has two complementary domains in $\mathbb{R}^2$, from each of which it is at every point accessible, it is a simple closed curve. The author’s papers [P1] and [P2] was devoted to the proof of this theorem (it was formulated there in rather different terms).

But the result given in [P1] and [P2] was turned out to be not new. It is known as theorem converse to Jordan’s curve theorem (see [K], [N]). Kuratovsky in his book [K] gives somewhat stronger statement: if for a closed set could be found two complementary domains in $\mathbb{R}^2$, from each of which it is at every point accessible, it is a simple closed curve. But the analysis of Newman’s proof given in [N] allows to conclude that it is suitable for the more general case stated by Kuratovsky.

Author asks the natural question whether it is possible to weaken the requirements of theorem converse to Jordan’s curve theorem.

This problem appears to be nontrivial. For example the set of points on the polish curve accessible from the both complementary domains is open and dense (in particular it is massive). But this set appears to be ”badly disposed” in it.

Let $K$ be a compact set in the plane the complement of which has two components $W_1$ and $W_2$. It proves out to be useful to consider separately the sets $A_1$, $A_2$ of points of $K$, accessible from $W_1$ and $W_2$, respectively. And if the both of them are ”well disposed” in $K$ then $K$ is a simple closed curve.

It is interesting that if a set $K$ is connected and its subsets $K \setminus A_1$ and $K \setminus A_2$ are zero-dimensional then the sets $A_1$ and $A_2$ are ”well disposed” in $K$ and $K$ is a simple closed curve.

It appears that the concept of ”well disposed” subset is connected with one topological property of the space $K$ (with the topology induced from the plane).

Namely, let $(X, \mathcal{T})$ be a topological space. Denote by $\mathcal{LC}(X)$ the weakest from topologies on $X$ with the following property: for each opened subset $W$ of $(X, \mathcal{T})$ every connected component of $W$ is opened in the topology $\mathcal{LC}(X)$.

In these terms the sets $A_1$ and $A_2$ will be ”well disposed” in $K$ if $A_1$ and $A_2$ are dense in $K$ in the topology $\mathcal{LC}(K)$.

In what follows we will define the concept of ”well disposed” subset in a
compact set in the plane and will give one generalization of theorem converse to the Jordan’s curve theorem.

Finally, I wish to express my deep gratitude to Yu. B. Zelinskiy and V. V. Sharko for statement of a problem and formulation of corrolary 1.1, and to M. A. Pankov for a number of valuable remarks.

1 Statement of main results.

First we give some notations and definitions being used in what follows.

Let $F$ be a set in the plane. We shall denote the frontier of $F$ by $\partial F$ and the closure of $F$ by $\overline{F}$.

Let $x \in \mathbb{R}^2$ and $\varepsilon > 0$. Denote

$$U_{\varepsilon}(x) = \{y \in \mathbb{R}^2 \mid \rho(x,y) < \varepsilon\}.$$ 

Let $U$ be an open subset of the plane. Three following definitions (see [N]) are useful to study the local properties of $\overline{U}$.

Definition 1.1 A simple continuous curve $\varphi : I \to \mathbb{R}^2$ is called an end-cut of $U$ in a point $x \in \overline{U}$, if $\varphi(0) = x$ and $\varphi((0,1)) \subset U$.

Definition 1.2 Call a point $x \in \overline{U}$ accessible from $U$, if there exists an end-cut of $U$ in the point $x$.

Definition 1.3 A cross cut of the domain $U$ is a simple continuous curve $\psi : I \to \mathbb{R}^2$ such that $\psi(0), \psi(1) \in \overline{U}$ and $\psi((0,1)) \subset U$.

And now we are ready to begin the main account.

1.1 Definition and elementary properties of $d$-sets.

Let $K$ be a compact subset of the plane dividing it into two connected domains. Let us name such set two-sided.

In what follows we shall denote complementary domains of a two-sided set $K$ by $W_1$ and $W_2$. We shall also denote by $A_i$ a set of points of $K$, accessible from $W_i$, $i = 1, 2$.

Designation 1.1 Let $F$ be a set in $\mathbb{R}^2$. Let $x \in F$ and $U$ be a neighborhood of $x$ (not necessarily open). Designate by $F(U,x)$ the connected component of $F \cap U$ which contains $x$. 
Proposition 1.1 Let $F$ be a subset of the plane. Let $U_1, U_2$ be two neighborhoods of point $x \in F$ and $U_2 \subset U_1$.

Then $F(U_2, x) \subseteq F(U_1, x)$.

\(\nabla\) The connected component of a point $x$ in $F \cap U_1$ is by definition the maximal connected subset of $F \cap U_1$, which contains $x$.

The connected set $F(U_2, x)$ lies in $F \cap U_1$ and contains $x$. Therefore, $F(U_2, x) \subseteq F(U_1, x)$. \(\triangle\)

Definition 1.4 Let $F$ be a set in the plane. We shall say that a subset $R$ of $F$ is sufficiently dense in $F$ if for all $x \in F$ and every opened neighborhood $U$ of $x$ a set $F(U, x) \cap R$ is not empty.

Remark 1.1 It is easy to see that definition of sufficiently dense subset could be reformulated in the following way: a subset $R$ of a set $F$ in the plane is sufficiently dense in $F$ if for any opened set $U$ every connected component of a set $F \cap U$ contains a point from $R$.

Definition 1.5 Two-sided set $K$ is called $d$-set if both $A_1$ and $A_2$ are sufficiently dense in $K$.

Let us deduce some simple properties of $d$-sets.

Proposition 1.2 Let $K$ be a $d$-set.

For any $x \in K$ and its open neighborhood $U$ the sets $A_1$ and $A_2$ are dense in $K(U, x)$.

Specifically, the sets $A_1$ and $A_2$ are dense in $K$.

\(\nabla\) Let $y \in K(U, x)$ and $V$ be any open neighborhood of $y$. Designate the set $V \cap U$ by $V_0$.

From property $d$ follows that there exist $y_1, y_2 \in K(V_0, y)$, such that $y_i \in A_i, i = 1, 2$.

But $K(V_0, y) \subseteq K(U, y) = K(U, x)$. Therefore, $y_1, y_2 \in K(U, x) \cap V$. \(\triangle\)

Proposition 1.3 Let $K$ be a two-sided set.

If the sets $A_1$ and $A_2$ are dense in $K$, then $K$ is the common boundary of it’s complementary domains, that is

$$K = \partial W_1 = \partial W_2.$$  \hspace{1cm} (1.1)

In particular, any $d$-set complies with the relation 1.1.
Obviously, $\mathcal{F}W_i \subseteq K$, $i = 1, 2$.

On the other hand, $A_i \subseteq \mathcal{F}W_i$, $i = 1, 2$. So $K = \overline{A_i} \subseteq \mathcal{F}W_i$, $i = 1, 2$, since the sets $\mathcal{F}W_i$ are closed by the definition. △

For convenience of further account we shall give following

**Definition 1.6** Let $K$ be a two-sided set. If $A_1$ and $A_2$ are dense in $K$ we shall say that $K$ is the simple set.

It follows from proposition 1.2 that $d$-sets are simple.

**Proposition 1.4** Each simple set is connected.

This statement is a direct corollary of the previous statement and the following theorem: *if the domains $W_1$ and $W_2$ in the plane do not meet, but $\mathcal{F}W_1 \subseteq \mathcal{F}W_2$, then $\mathcal{F}W_1$ is connected* (see [N], theorem V.14.1). △

### 1.2 $d$-sets and theorem, inverse to Jordan’s curve theorem.

The famous Jordan’s curve theorem says, that a simple closed curve in the plane has two complementary domains, of each of which it is the complete frontier.

There is a natural question: under what conditions two-sided set $K$ in the plane will be a Jordan curve?

Answer to this question gives the theorem, converse to Jordan’s curve theorem, which states that a two-sided set, all points of which are accessible from both components of the complement, is a simple closed curve (see [N], theorem VI.16.1).

The requirement of this theorem that all points of $K$ were accessible from both complementary domains turns out to be surplus.

How it is possible to weaken this condition?

Trivial direct check shows, that for any domain in $\mathbb{R}^2$ points accessible from it are dense in the frontier of this domain. Therefore, taking account of proposition 1.3 it seems natural to require that sets $A_i$ of points, accessible from the complementary domains $W_i$, $i = 1, 2$, were dense in $K$. This requirement could not be weakened, as shows the following example.

**Example 1.1**

\[
K = \{\text{the unit circle in } \mathbb{R}^2\} + \{\text{an isolated point in } W_1\} + \{\text{an isolated point in } W_2\}.
\]
But even if the sets $A_1$ and $A_2$ are dense in $K$ it is not sufficient for $K$ to be a Jordan curve.

**Example 1.2** Consider a union $K$ of following sets

\[
K_1 = \{0\} \times [-1,1],
K_2 = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,1], y = \sin\frac{\pi}{x} - \sin\pi x\},
K_3 = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,1], y = \sin\frac{\pi}{x} + \sin\pi x\}.
\]

It could be shown that $K$ is the simple set and each point of the curve $K_2 \cup K_3$ is accessible from both components of complement.

However, the set $K$ is not arcwise connected (components of it’s linear connectivity are $K_1$ and $K_2 \cup K_3$).

Also $K$ is not $d$-set. If $\varepsilon < 1$ then

\[
K(U_\varepsilon(0), 0) = \{0\} \times (-\varepsilon, \varepsilon)
\]

for circular neighborhood $U_\varepsilon(0)$, and this set does not contain any point accessible from the limited complementary domain of $K$.

This example can be used as an argument for the benefit of introduction of $d$-sets.

Let $F$ be a set in the plane. Since the topology on the plane has a denumerable base of opened sets, we can find a denumerable subset dense in $F$.

Hence, natural question arises: what is minimal cardinality of sufficiently dense subsets of $F$?

The following example shows that there exists a simple set in $\mathbb{R}^2$ such that every its sufficiently dense subset is of cardinality *continuum*.

**Example 1.3** (See [K], section 48.I, examp. 4) Let $C_0$ be the Cantor set situated at the axes $\xi_1$ of the plane $(\xi_1, \xi_2)$ and $C_1$ be the same set disposed on the line $\xi_2 = 1$. Connect every point of $C_0$ with the appropriate point in $C_1$ by vertical interval. Add to $C_0$ the adjacent intervals of length $1/3$, $1/3^3$, ..., and to $C_1$ add the adjacent intervals of length $1/3^2$, $1/3^4$, ... . We receive the continuum $K_0$ with a connected complement.
Let
\[ K_1 = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \mid (\xi_1 - 1/2)^2 + (\xi_2 - 1)^2 = 1/4, \; \xi_2 \geq 1 \right\}. \]

The arc \( K_1 \) is the cross-cut of \( K_0 \), so the continuum
\[ K = K_0 \cup K_1 \]
is two-sided (see lemma 2.3). It is not difficult to show that the set \( K \) complies with the equality (1.1), hence \( K \) is the simple set. In fact, \( K \) is not a \( d \)-set.

It could be verified that for an opened set
\[ V = \mathbb{R} \times (0,1) \]
every connected component of \( V \cap K = C \times (0,1) \) is \( \{c\} \times (0,1) \), where \( c \in C \) is a certain element of the Cantor set \( C \).

So, the set \( V \cap K \) has a \textit{continuum} connected components and from remark 1.1 it follows that every sufficiently dense subset of \( K \) must have cardinality \textit{continuum}.

The main aim of the further considerations is to prove following theorem and its corollary.

\textbf{Theorem 1.1} Every \( d \)-set in the plane is a simple closed curve.

\textbf{Corollary 1.1} Let \( K \) be a connected two-sided subset of the plane. If sets \( K \setminus A_1 \) and \( K \setminus A_2 \) are zero-dimensional then \( K \) is a simple closed curve.
1.3 Proof of corollary 1.1.

Corollary 1.1 is a consequence of theorem 1.1 and two following lemmas.

**Lemma 1.1** Let $F$ be a compact connected subset of the plane. Let $X$ be a closed neighborhood of a point $x \in F$, such that $F \setminus X \neq \emptyset$.

Then $F(X,x) \cap \mathcal{F}X \neq \emptyset$.

$\nabla$ Fix $y \in F \setminus X$. It is known (see [N], theorem IV.5.1), that for every $\varepsilon > 0$ points $x$ and $y$ can be connected with the $\varepsilon$-chain in the set $F$, i.e. final sequence

$$x = z_1, z_2, \ldots, z_k = y$$

of points could be found in $F$ to comply with the condition

$$\rho(z_i, z_{i+1}) < \varepsilon, \quad i = 1, \ldots, k - 1.$$ 

For every $n \in \mathbb{N}$ select $1/n$-chain in $F$

$$x^n_1, \ldots, x^n_{k(n)}, \quad n \in \mathbb{N},$$

connecting points $x$ and $y$.

Since $y \notin X$, for every $n \in \mathbb{N}$ in the sequence $\{x^n_i\}_{i=1}^{k(n)}$ could be found an element which does not belong to $X$. Therefore, for all $n \in \mathbb{N}$ indexes $j(n) \in \{1, \ldots, k(n) - 1\}$ are defined to meet the conditions

(i) $x^n_i \in X, \ i = 1, \ldots, j(n);$

(ii) $x^n_{j(n)+1} \notin X.$

The set $F \cap X$ is compact, being the closed subset of the compact set $F$. Therefore sequence $\{x^n_{j(n)}\}_{n \in \mathbb{N}}$ has a limit point $x_0 \in X \cap F$. Without loss of generality we shall suppose that $x^n_{j(n)} \to x_0$ (otherwise, it is always possible to pass to a subsequence).

Since $x^n_{j(n)+1} \notin X$ and $\rho(x^n_{j(n)}, x^n_{j(n)+1}) \to 0$ then $\rho(x^n_{j(n)}, \mathbb{R}^2 \setminus X) \to 0$. Therefore, $\rho(x_0, \mathbb{R}^2 \setminus X) = 0$ and $x_0 \in \mathcal{F}X$.

On the other hand, for every $\varepsilon > 0$ there exist $n_1 \in \mathbb{N}$ and $n > n_1$, such that $1/n_1 < \varepsilon$ and $\rho(x^n_{j(n)}, x_0) < \varepsilon$. Then points $x$ and $x_0$ could be connected in $F \cap X$ by the $\varepsilon$-chain

$$x = x^n_1, x^n_2, \ldots, x^n_{j(n)}, x_0.$$ 

Consequently, (see [N], theorem IV.5.4) points $x$ and $x_0$ belong to one connected component of the set $F \cap X$. $\triangle$
Lemma 1.2 Let $K$ be a connected two-sided subset of the plane. If sets $B_1 = K \setminus A_1$ and $B_2 = K \setminus A_2$ are zero-dimensional then $K$ is $d$-set.

$\nabla$ Fix $x \in K$ and open neighborhood $U$ of $x$. Let us show that $K(U, x) \cap A_1 \neq \emptyset$.

The set $K$ divides the plane into two connected components. Therefore it consists more than of one point. Take $\varepsilon > 0$ such that $U_{2\varepsilon}(x) \subset U$ and $K \setminus U_{2\varepsilon}(x) \neq \emptyset$.

According to lemma 1.1 there exists $y \in K(U_{\varepsilon}(x)) \cap \mathcal{F}U_{\varepsilon}(x)$. From proposition 1.1 it follows that $y \in K(U, x)$. Besides, $y \in K \setminus U_{\varepsilon}(x)$.

The set $B_1$ is zero-dimensional. Therefore the space $B_1$ admits a base of open sets $\{V_\alpha\}_{\alpha \in \Lambda}$ such that $\mathcal{F}V_\alpha = \emptyset$ in space $B_1$ (in relative topology) for all $\alpha \in \Lambda$.

Let $x \in B_1$. Then $x \in V_{\beta}$ and $V_{\beta} \subset U_{\varepsilon}(x)$ for some $\beta \in \Lambda$.

$V_\beta$ is an open-closed subset of space $B_1$. If $V = B_1 \setminus V_\beta \neq \emptyset$, the sets $V$ and $V_\beta$ will form a partition of space $B_1$.

Assume that $K(U, x) \subset B_1$. Since $y \notin U_{\varepsilon}(x)$ then $y \in V$ and $V \neq \emptyset$.

Therefore, the sets $V_\beta \cap K(U, x)$ and $V \cap K(U, x)$ form a partition of the set $K(U, x)$ contrary to its connectedness.

So $K(U, x) \cap A_1 \neq \emptyset$. The relation $K(U, x) \cap A_2 \neq \emptyset$ is proved similarly.

By virtue of arbitrariness in the choice of a point $x \in K$ and its neighborhood $U$ we conclude that $K$ is $d$-set. $\triangle$

1.4 On an accessibility of a point of a simple subset of the plane from a component of its complement.

It is easy to see that theorem 1.1 is direct corollary of theorem, inverse to Jordan’s curve theorem, and the following statement.

Theorem 1.2 Let $K$ be a simple subset of the plane. If for some $x \in K$ and every open neighborhood $U$ of $x$ the set $K(U, x) \cap A_1$ is not empty, the point $x$ is accessible from $W_1$.

Plan of the proof of theorem 1.2.

If it is known that the point $x$ is accessible from $W_1$, theorem holds true. Therefore we shall assume further that $(K(U, x) \cap A_1) \setminus \{x\} \neq \emptyset$ for every open neighborhood $U$ of $x$.

Fix $x_1 \in (K(U_1(x), x) \cap A_1) \setminus \{x\}$. 
Suppose for some \( n \in \mathbb{N} \) there are already selected \( n \) different points \( x_1, \ldots, x_n \) such that \( x_i \in K(U_{1/i}(x), x) \cap A_1 \) and \( x_i \neq x \), \( i = 1, \ldots, n \). Let

\[
\varepsilon_1 = \min_{i \in \{1, \ldots, n\}} \rho(x_i, x), \quad \varepsilon = \min(\varepsilon_1, \frac{1}{n + 1}).
\]

Select \( x_{n+1} \in (K(U_{\varepsilon_1}(x), x) \cap A_1) \setminus \{x\} \). Proposition 1.1 implies the inclusion \( x_{n+1} \in K(U_{1/(n+1)}(x), x) \).

By application of this procedure consequently for all \( n \in \mathbb{N} \) we shall receive the sequence \( \{x_n\}_{n \in \mathbb{N}} \), with pairwise different elements which complies with the condition

\[
x_n \in (K(U_{1/n}(x), x) \cap A_1) \setminus \{x\}, \quad n \in \mathbb{N}.
\]

We fix for every \( n \in \mathbb{N} \) an end-cut \( \alpha_n : I \to \mathbb{R}^2 \) of the domain \( W_1 \) in the point \( x_n \) (that is simple continuous curve, for which \( \alpha_n(0) = x_n \) and \( \alpha_n(0, 1] \subset W_1 \)).

Following statement which will be proved in the next section holds true.

**Lemma 1.3** There exists a family of cross-cuts \( \{\beta_n : I \to \mathbb{R}^2\}_{n=2}^\infty \) of the domain \( W_1 \) which complies with the following properties

(i) a cross-cut \( \beta_n \) connects points \( x_n \) and \( x_{n+1} \), \( n \geq 2 \);

(ii) \( \beta_n(I) \subset U_{1/(n-1)}(x) \), \( n \geq 2 \);

(iii) for every \( n \geq 2 \) exists \( \tau_n > 0 \), such that \( \alpha_2([0, \tau_2]) \subset \beta_2(I) \) and \( \alpha_n([0, \tau_n]) \subset (\beta_{n-1}(I) \cap \beta_n(I)), n \geq 3 \).

Let \( \{\beta_n\}_{n=2}^\infty \) and \( \{\tau_n\}_{n=2}^\infty \) be collections of cross-cuts and parameter values from lemma 1.3. From this lemma follows that for every \( n \geq 2 \)

\[
\alpha_n(\tau_n) \in U_{1/(n-1)}(x)
\]

and the points \( \alpha_n(\tau_n) \) and \( \alpha_{n+1}(\tau_{n+1}) \) are contained in the same connected component of \( U_{1/(n-1)}(x) \cap W_1 \). Therefore points \( \alpha_n(\tau_n) \) and \( \alpha_{n+1}(\tau_{n+1}) \) can be connected in the set \( U_{1/(n-1)}(x) \cap W_1 \) by a simple polygonal line \( J_n \) with the final number of links (see [N], theorem V.6.3). We designate

\[
Q_n = \bigcup_{k=2}^n J_k, \quad A = \bigcup_{n=2}^\infty Q_n = \bigcup_{n=2}^\infty J_n.
\]

Obviously, the set \( Q_n \) is connected and closed for every \( n \geq 2 \). Therefore the set \( A \) is connected as the union of the connected sets with a common point \( \alpha_2(\tau_2) \).
Since $A \setminus Q_n \subset U_{1/(n-1)}(x)$, $n \geq 2$, then $\overline{A} = A \cup \{x\}$. Being a closure of the connected set, $\overline{A}$ is connected and on a construction $\overline{A} \setminus \{x\} \subset W_1$.

The further proof is based on the following statement (see [N], theorem VI.14.3).

**Lemma 1.4** Let $a \in \mathbb{R}^2$ and $A$ be the union of a sequence of segments, $x_ny_n$, such that $x_n \to a$, $y_n \to a$. Then if $a$ is connected in $\overline{A}$ to a point $b$ ($b \neq a$), there is a simple arc in $\overline{A}$ with end-points $a$ and $b$.

From this lemma follows that there is a simple continuous curve $\alpha : I \to \overline{A}$ connecting points $x$ and $\alpha_2(\tau_2)$. Since $\overline{A} \setminus \{x\} \subset W_1$, the arc $\alpha$ is an end-cut of the domain $W_1$ in the point $x$. □

So it suffices to prove lemma 1.3 for completion of the proof of theorem 1.2.

## 2 Cross-cuts of complementary domains of simple sets in the plane.

Lemma 1.3 from the previous section is based on one local property of simple sets which will be studied now.

**Theorem 2.1** Let $K$ be a simple set, $x \in K$. Let for some $\varepsilon > 0$ points $y_1, y_2 \in K(U_\varepsilon(x), x)$, $y_1 \neq y_2$, be accessible from a complementary domain $W_1$ of $K$. Let

$$\alpha_i : I \to \mathbb{R}^2$$

be an end-cut of $W_1$ in the point $y_i$, $i = 1, 2$.

For any $\varepsilon_1 > \varepsilon$ there exists cross-cut

$$\beta : I \to \mathbb{R}^2$$

of $W_1$ with end-points $y_1$, $y_2$, such that

(i) $\beta(I) \subset U_{\varepsilon_1}(x)$;

(ii) $\alpha_i([0, \tau_i]) \subset \beta(I)$, $i = 1, 2$, for some $\tau_1, \tau_2 \in (0, 1)$.

**Proof of theorem 2.1** will be decomposed into several steps.

Suppose $\varepsilon_1 > \varepsilon$ is given. Fix $\varepsilon_2 \in (\varepsilon, \varepsilon_1)$.

1. There exists a cross-cut $\beta_0 : I \to \mathbb{R}^2$ of $W_1$ which meets condition (ii) of theorem.
Since \( y_1, y_2 \in U_{\epsilon_2}(x) \), there exists \( \tau_0 \in (0,1) \) meeting the relations
\[
\alpha_i([0, \tau_0]) \subset U_{\epsilon_2}(x), \quad i = 1, 2, \text{ and } \alpha_1([0, \tau_0]) \cap \alpha_2([0, \tau_0]) = \emptyset.
\]
Points \( z_1 = \alpha_1(\tau_0), z_2 = \alpha_2(\tau_0) \) are contained in the open connected set \( W_1 \). It is known that any domain in \( \mathbb{R}^2 \) is arcwise connected. Therefore, there exists simple arc
\[
\gamma : I \to W_1
\]
connecting points \( z_1 \) and \( z_2 \).

Designate
\[
\tau_i = \min \{ \tau \mid \alpha_i(\tau) \in \gamma(I) \}, \quad i = 1, 2.
\]
Nonempty closed sets \( \{0\} = \alpha_i^{-1}(y_i) \) and \( \alpha_i^{-1}(\gamma(I)) \) do not intersect, hence \( \tau_i \in (0, \tau_0], i = 1, 2 \).

Values \( t_i \in [0,1] \) are uniquely defined, such that \( \alpha_i(\tau_i) = \gamma(t_i), i = 1, 2 \). Consider a simple arc
\[
\beta_0 : I \to \mathbb{R}^2,
\]
\[
\beta_0(t) = \begin{cases}
\alpha_1(4t\tau_1), & t \in [0,1/4], \\
\gamma(2(t-1/4)t_2 + 2(3/4-t)t_1), & t \in [1/4,3/4], \\
\alpha_2(4(1-t)\tau_2), & t \in [3/4,1].
\end{cases}
\]
This curve will be the desired cross-cut.

If, besides, \( \beta_0(I) \subset U_{\epsilon_1}(x) \), then theorem is proved. Therefore, we shall suppose further that
\[
\beta_0(I) \setminus U_{\epsilon_1}(x) \neq \emptyset.
\]

2. We shall examine one special case. Let \( K \subset \overline{U_{\epsilon}(x)} \). It is clear that \( K \subset U_{\epsilon_1}(x) \) then.

The following statement (see [N], theorem V.9.3) holds true.

**Lemma 2.1** If a closed set \( F \neq \mathbb{R}^2 \) is contained in a domain \( D \) and \( D_1, D_2, \ldots \) are the components of \( \mathbb{R}^2 \setminus F \), the components of \( D \setminus F \) are \( D \cap D_1, D \cap D_2, \ldots \).

From this statement follows, that points \( z_1 = \alpha_1(\tau_0), z_2 = \alpha_2(\tau_0) \) are contained in the same connected component of the opened set \( U_{\epsilon_1}(x) \setminus K \) (namely, in \( W_1 \cap U_{\epsilon_1}(x) \)). Therefore, there exists a simple arc
\[
\gamma : I \to W_1 \cap U_{\epsilon_1}(x),
\]
with the end-points \( z_1 \) and \( z_2 \).

Repeating argument of the previous step we conclude that there exists a cross-cut
\[
\beta_0 : I \to U_{\epsilon_1}(x)
\]
3. The set $K(U(x), x) \cup \beta_0(I)$ is connected and divides the plane into two connected domains $V_1$ and $V_2$, one of which is contained in $W_1$.

First we prove following

Lemma 2.2 Let $K$ be a simple set. For any proper closed subset $\bar{K} \subseteq K$ it’s complement $\mathbb{R}^2 \setminus \bar{K}$ is connected.

\[ \nabla \quad \text{Let } y \in K \setminus \bar{K}. \text{ There exists } \varepsilon(y) > 0, \text{ such that } U_{\varepsilon(y)}(y) \cap \bar{K} = \emptyset. \]

According to proposition 1.3, we can find $z_1, z_2 \in U_{\varepsilon(y)}(y)$, such that $z_i(y) \in W_i$, $i = 1, 2$.

Designate by $J_i(y)$ the segment with end-points $y$ and $z_i(y)$, $i = 1, 2$. Obviously, $J_i(y) \cap \bar{K} = \emptyset$, $i = 1, 2$. The set

\[ Q(y) = W_1 \cup W_2 \cup J_1(y) \cup J_2(y) \]

is connected and $Q(y) \cap \bar{K} = \emptyset$.

We fix a point $z \in W_1$. Since $z \in Q(y)$ for every $y \in K \setminus \bar{K}$, the set

\[ Q = \bigcup_{y \in K \setminus \bar{K}} Q(y) = \mathbb{R}^2 \setminus \bar{K} \]

is connected. $\Delta$

Being the union of the connected sets which have a common point $y_1 = \beta_0(0)$, the set $K(U_{\varepsilon}(x), x) \cup \beta_0(I)$ is connected.

Under the definition $K(U_{\varepsilon}(x), x)$ is the connected component of a set $U_{\varepsilon}(x) \cap K$. Therefore, it is closed in $U_{\varepsilon}(x) \cap K$. Since $U_{\varepsilon}(x) \cap K$ is closed in $\mathbb{R}^2$, $K(U_{\varepsilon}(x), x)$ is closed in $\mathbb{R}^2$.

Applying lemma 2.2, we conclude that the complement

\[ Q = \mathbb{R}^2 \setminus K(U_{\varepsilon}(x), x) \]

is connected.

Obviously, the curve $\beta_0$ is the cross-cut of $Q$.

The following statement holds true (see [N], theorem V.2.7).
Lemma 2.3 If both the end-points of a cross-cut $L$ in a domain $D$ of $\mathbb{R}^2$ are on the same component of $\mathbb{R}^2 \setminus D$, $D \setminus L$ has two components, and $L$ is contained in the frontier of both.

This lemma implies that the set

$$Q \setminus \beta_0(I) = \mathbb{R}^2 \setminus (K(U_c(x), x) \cup \beta_0(I))$$

has two components $V_1$ and $V_2$, and that

$$\beta_0(I) \subset (\mathcal{F}V_1 \cap \mathcal{F}V_2).$$

(2.1)

Since

$$W_2 \cap \left(K(U_c(x), x) \cup \beta_0(I)\right) = \emptyset,$$

(2.2)

then $(V_1 \cup V_2) \cap W_2 \neq \emptyset$. Without loss of generality, we can assume that $W_2 \cap V_2 \neq \emptyset$.

$W_2$ is connected, therefore from the relation (2.2) it follows that $W_2 \subset V_2$. Otherwise we should receive a partition $W_2 \cap V_1, W_2 \cap V_2$ of $W_2$.

Relation (1.1) implies an equality $W_1 = \mathbb{R}^2 \setminus \overline{W_2}$. On the other hand, $\overline{W_2} \subset \overline{V_2} \subseteq \mathbb{R}^2 \setminus V_1$. Therefore,

$$V_1 \subseteq \mathbb{R}^2 \setminus \overline{V_2} \subseteq \mathbb{R}^2 \setminus \overline{W_2} = W_1.$$

4. Now we shall describe a structure which will allow us to construct a cross-cut $\beta$ of $Q \cap U_{c_1}(x)$, contained in a set $\beta_0(I) \cup V_1$.

It is known, that for any simple continuous curve in the plane there exists a homeomorphism of $\mathbb{R}^2$ onto itself, mapping this curve onto a segment (see [N], paragraph VI.17). Fix homeomorphism

$$f_0 : \mathbb{R}^2 \to \mathbb{R}^2$$

which maps the curve $\beta_0$ onto a segment $[-2, 2] \times \{0\}$.

Easy to see that it is possible to select such homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$, that the composition

$$f = (\varphi \times id) \circ f_0 : \mathbb{R}^2 \to \mathbb{R}^2$$

will satisfy to the following requirements:

$$f(y_1) = f \circ \beta_0(0) = (-2, 0), \quad f(y_2) = f \circ \beta_0(1) = (2, 0);$$
Therefore, \( f \circ \beta_0(1/4) = f \circ \alpha_1(\tau_1) = (-1, 0), f \circ \beta_0(3/4) = f \circ \alpha_2(\tau_2) = (1, 0) \).

Under these conditions it is obvious (see step 1) that
\[
\begin{align*}
    f \circ \beta_0([0, 1/4]) &= f \circ \alpha_1([0, \tau_1]) = [-2, -1] \times \{0\} \subset f(U_{\varepsilon_2}(x)), \\
    f \circ \beta_0([3/4, 1]) &= f \circ \alpha_2([0, \tau_2]) = [1, 2] \times \{0\} \subset f(U_{\varepsilon_2}(x)).
\end{align*}
\]

Since \( f \) is the homeomorphism and the set \( U_{\varepsilon_2}(x) \) is open in \( \mathbb{R}^2 \), there exists \( \delta_1 > 0 \) meeting the relations
\[
U_{\delta_1}(f \circ \beta_0(1/4)) \subset f(U_{\varepsilon_2}(x)) \quad \text{and} \quad U_{\delta_1}(f \circ \beta_0(3/4)) \subset f(U_{\varepsilon_2}(x)).
\]

Fix also \( \delta_2 > 0 \), such that
\[
U_{\delta_2}([-1, 1] \times \{0\}) \cap f(K(\overline{U_{\varepsilon_2}(x)}), x)) = \emptyset.
\]

We can do it, since \( f(K(\overline{U_{\varepsilon_2}(x)}), x)) \) and \( f \circ \beta_0([1/4, 3/4]) \) are disjoint compact sets.

Let \( \delta = \min(\delta_1, \delta_2) \).

Consider the following subsets of \( U_\delta([-1, 1] \times \{0\}) \):
\[
\begin{aligned}
P_1 &= (-1) \times [0, \delta/2] \cup ([-1, 1] \times \{\delta/2\}) \cup \{1\} \times [0, \delta/2], \\
P_2 &= (-1) \times [-\delta/2, 0] \cup ([-1, 1] \times \{-\delta/2\}) \cup \{1\} \times [-\delta/2, 0].
\end{aligned}
\]

Let us set on these two polygonal lines a parametrization \( \mu_i : I \to P_i, i = 1, 2 \), converting them to simple continuous curves, such that
\[
\begin{align*}
    \mu_1(0) &= \mu_2(0) = f \circ \beta_0(1/4) = (-1, 0), \\
    \mu_1(1) &= \mu_2(1) = f \circ \beta_0(3/4) = (1, 0).
\end{align*}
\]

Consider in addition a segment \( J \subset U_\delta((-1, 0)) \) with the end-points \((-1 - \delta/2, 0) \in f \circ \alpha_1([0, \tau_1]) \) and \((-1, \delta/2) \in \mu_1(I) \).

Let \( \mu : S^1 \to \mathbb{R}^2 \) be a simple closed curve, defined by a relation
\[
\mu(t) = \begin{cases} 
\mu_1(2t), & t \in [0, 1/2], \\
\mu_2(2 - 2t), & t \in [1/2, 1]. 
\end{cases}
\]

The curve \( \mu \) divides the plane into two domains. Designate a limited component of the complement \( \mathbb{R}^2 \setminus \mu(S^1) \) by \( V_{in} \) and unlimited component by \( V_{o} \). Obviously,
\[
V_{in} = (-1, 1) \times (-\delta/2, \delta/2) \subset U_\delta([-1, 1] \times \{0\}).
\]

Therefore, \( f(K(\overline{U_{\varepsilon_2}(x)}), x)) \subset V_0 \).
Segment $[-1, 1] \times \{0\} = f \circ \beta_0([1/4, 3/4])$ divides a rectangle $V_{in}$ into two domains. Designate by $V_{in}^+$ the domain contained in the upper half-plane and the other domain by $V_{in}^-.$

By a construction $V_{in} \cap f(K(U_\varepsilon(x), x)) = \emptyset.$ Therefore, according to relation (2.1), $V_{in} \cap f(3V_1) = V_{in} \cap f(3V_2) = f \circ \beta((1/4, 3/4)) = (-1, 1) \times \{0\}.$ Consequently, one from domains $V_{in}^+, V_{in}^-$ is contained in $f(V_1)$, the other is a subset of $f(V_2)$.

We shall suppose that $V_{in}^+ \in f(V_1).$ Otherwise we can replace $f$ by $i \circ f,$ where $i : \mathbb{R}^2 \to \mathbb{R}^2,$ $i : (x_1, x_2) \mapsto (x_1, -x_2),$ is an inversion relative to the first coordinate axes.

Since
\[
\mu_i(I) \cap f \left( K(U_\varepsilon(x), x) \cup \beta_0(I) \right) = \mu_i(0) \cup \mu_i(1), \quad i = 1, 2,
\]
then
\[
\mu_1((0, 1)) \subset f(V_1), \quad \mu_2((0, 1)) \subset f(V_2).
\]

5. Consider a simple closed curve
\[
\gamma : S^1 \to \mathbb{R}^2, \\
\gamma(t) = x + \varepsilon_2 \exp(2\pi it),
\]
being the boundary of a neighborhood $U_{\varepsilon_2}(x)$.

Let us designate
\[
\theta = f^{-1} \circ \mu : \mathbb{R}^2 \to \mathbb{R}^2.
\]

The following statement (see [N], example VI.16) holds true.

**Lemma 2.4 (KerékJáró’s Theorem)** If two simple closed curves, $J_1$ and $J_2$, have more than one common point, all the residual domains of $J_1 \cup J_2$ are Jordan domains.

Show that the curves $\gamma$ and $\theta$ meet the condition of this lemma.

Since $\beta_0(I) \setminus U_{\varepsilon_1}(x) \neq \emptyset$ (see step 1) and $\varepsilon_2 < \varepsilon_1$ then $\beta_0(I) \setminus U_{\varepsilon_2}(x) \neq \emptyset.$ On a construction, $\beta_0(I) \setminus U_{\varepsilon_2}(x) \subset \beta_0((1/4, 3/4)) \subset f^{-1}(V_{in}).$

Fix $z_0 \in \beta_0(I) \setminus U_{\varepsilon_2}(x).$ The boundary of the domain $f^{-1}(V_{in})$ is a Jordan curve $\theta.$ Therefore $\theta(S^1) \setminus U_{\varepsilon_2}(x) \neq \emptyset.$ Really, the set $f^{-1}(V_0)$ is not limited, hence there exists $z'_0 \in f^{-1}(V_0) \cap (\mathbb{R}^2 \setminus U_{\varepsilon_2}(x)).$ A set $\mathbb{R}^2 \setminus U_{\varepsilon_2}(x)$ is arcwise connected, so the simple continuous curve $\psi : I \to \mathbb{R}^2 \setminus U_{\varepsilon_2}(x)$ could be found to connect points $z_0$ and $z'_0.$ Domains $f^{-1}(V_0)$ and $f^{-1}(V_{in})$ are disjoint, $z_0 \in f^{-1}(V_{in}),$ $z'_0 \in f^{-1}(V_0),$ and $\theta(S^1) = \mathcal{F}f^{-1}(V_{in}) = \mathcal{F}f^{-1}(V_0).$ Therefore, $\psi(I) \cap \theta(S^1) \neq \emptyset.$
Let \( z_1 \) be contained in \( \psi(I) \cap \theta(S^1) \subset \theta(S^1) \setminus \overline{U_{\varepsilon}(x)} \).

On the other hand, \( z_2 = \beta_0(1/4) = \theta(0) \in U_{\varepsilon}(x) \) on a construction.

Points \( z_1 \) and \( z_2 \) divide Jordan curve \( \theta \) into two simple arcs with common end-points

\[
\theta_i : I \to \mathbb{R}^2, \quad i = 1, 2,
\]
such that

\[
\begin{align*}
\theta_i(0) &= z_1 \in U_{\varepsilon}(x), & \theta_i(1) &= z_2 \in \mathbb{R}^2 \setminus \overline{U_{\varepsilon}(x)}, & i &= 1, 2; \\
\theta_1(I) \cap \theta_2(I) &= \{z_1\} \cup \{z_2\}.
\end{align*}
\]

It is not difficult to see, that \( \theta_i \cap \mathcal{F}U_{\varepsilon}(x) \neq \emptyset, i = 1, 2 \). Consequently, the intersection \( \gamma(S^1) \cap \theta(S^1) \) contains not less than two points and for the curves \( \gamma \) and \( \theta = f^{-1} \circ \mu \) lemma 2.4 is applicable.

6. Let \( V_K \) be a complementary domain of \( \theta(S^1) \cup \gamma(S^1) \) containing \( K(U_{\varepsilon}(x), x) \).

Let \( \eta : S^1 \to \mathbb{R}^2 \)

be a simple closed curve bounding the domain \( V_K \).

On the construction a connected set

\[
K_0 = \overline{K(U_{\varepsilon}(x), x) \cup \beta_0([0, 1/4) \cup (3/4, 1])} = \overline{K(U_{\varepsilon}(x), x) \cup \alpha_1([0, \tau_1) \cup \alpha_2([0, \tau_2))}
\]

is contained in \( V_K \). Therefore, \( \beta_0(1/4), \beta_0(3/4) \in \mathcal{F}V_K = \eta(S^1) \). Similarly,

\[
(K_0 \cup f^{-1}(J)) \setminus f^{-1}((-1, \delta/2))
\]

is the subset of \( V_K \), hence \( x_0 = f^{-1}((-1, \delta/2)) \in \mathcal{F}V_K = \eta(S^1) \). Since \( x_0 \in f^{-1} \circ \mu_1((0, 1)) \subset V_1 \), then \( V_1 \cap \mathcal{F}V_K \neq \emptyset \).

The curve \( \eta \) is divided by points \( \beta_0(1/4) \) and \( \beta_0(3/4) \) into two arcs

\[
\eta_i : I \to \mathbb{R}^2, \quad i = 1, 2,
\]
such that

\[
\begin{align*}
\eta_i(0) &= \beta_0(1/4), & \eta_i(1) &= \beta_0(3/4), & i &= 1, 2, \\
\eta_1(I) \cap \eta_2(I) &= \{\beta_0(1/4)\} \cup \{\beta_0(3/4)\}.
\end{align*}
\]

Let \( \eta_i \) be that from these two curves, which contains the point \( x_0 \).

7. We shall show, that \( \eta_1((0, 1)) \subset V_1 \).

The set \( V_1 \) is connected, so it will be enough to prove that \( \eta_1((0, 1)) \cap \mathcal{F}V_1 = \emptyset \). We shall check a somewhat stronger relation

\[
\eta_1((0, 1)) \cap \overline{K(U_{\varepsilon}(x)) \cup \beta_0(I)} = \emptyset.
\]
Since $K_0 \subset V_K$, then $\eta_1(I) \cap K_0 = \emptyset$ and

$$\eta_1((0, 1)) \cap \left(K(\overline{U_\varepsilon(x)}) \cup \beta_0([0, 1/4] \cup [3/4, 1])\right) = \emptyset.$$  

On the other hand, the sets $K(\overline{U_\varepsilon(x)}, x)$ and $\beta_0((1/4, 3/4))$ are contained in the different components of $\mathbb{R}^2 \setminus \theta(S^1)$, namely,

$$K(\overline{U_\varepsilon(x)}, x) \subset V_K \subset f^{-1}(V_o), \quad \beta_0((1/4, 3/4)) \subset f^{-1}(V_{in}).$$

Therefore,

$$\eta_1((0, 1)) \cap \beta_0((1/4, 3/4)) \subset V_K \cap f^{-1}(V_{in}) \subset f^{-1}(V_o) \cap f^{-1}(V_{in}) = \emptyset.$$  

8. Recall that $V_1 \subset W_1$ and arcs $\alpha_1$, $\alpha_2$ are end-cuts of the domain $W_1$ in the points $y_1$ and $y_2$, respectively. Since

$$\alpha_i([0, \tau_i)) \subset V_K \subset U_{\varepsilon_2}(x), \quad i = 1, 2,$$

$$\eta(I) \subset \mathcal{F}V_K,$$

$$\alpha_1(\tau_1) = \eta_1(0), \quad \alpha_2(\tau_2) = \eta_1(1),$$

then a simple arc

$$\beta : I \rightarrow \mathbb{R}^2,$$

$$\beta(t) = \begin{cases} 
\alpha_1(4t \tau_1) = \beta_0(t), & t \in [0, 1/4], \\
\eta(t/2 + 1/4), & t \in [1/4, 3/4], \\
\alpha_2(4(1 - t) \tau_2) = \beta_0(t), & t \in [3/4, 1]
\end{cases}$$

is the cross-cut of the domain $W_1$, which complies with all the requirements of theorem 2.1. □

Proof of lemma 1.3. From proposition 1.1 follows, that

$$K(U_{1/(n+1)}(x), x) \subset K(U_{1/n}(x), x) \subset K(\overline{U_1(x)}, x)$$

for all $n \in \mathbb{N}$. Therefore $x_n, x_{n+1} \in K(\overline{U_1(x)}, x), n \in \mathbb{N}$.

Apply now for all $n \geq 2$ theorem 2.1 to the values $\varepsilon = 1/n$, $\varepsilon_1 = 1/(n-1)$, points $y_1 = x_n$, $y_2 = x_{n+1}$ and end-cuts $\alpha_n$, $\alpha_{n+1}$. We receive a sequence $\{\beta_n\}_{n=2}^{\infty}$ of cross-cuts of the domain $W_1$ and two sequences of positive numbers $\{\tau_n\}_{n=2}^{\infty}$, $\{\tau'_n\}_{n=2}^{\infty}$, connected by relations

(i) $\beta_n(I) \subset U_{1/(n-1)}(x), n \geq 2;$

(ii) $(\alpha_n([0, \tau'_n]) \cup \alpha_{n+1}([0, \tau''_n])) \subset \beta_n(I), n \geq 2.$

Assume $\tau_2 = \tau'_2$, $\tau_n = \min(\tau'_n, \tau''_{n-1})$, $n \geq 3$. Then the sequences $\{\beta_n\}$ and $\{\tau_n\}$ satisfy to lemma 1.3. △
3 Concluding remarks.

Theorems 1.1 and 1.2 could be reformulated in an other way. In order to do that we shall consider different topology on a two-sided subset of the plane.

**Definition 3.1** Let \((X, \mathcal{T})\) be a topological space. Let \(\{U_\alpha\}_{\alpha \in A}\) be a base of opened sets in \(X\).

For every \(\alpha \in A\) let \(\{V_{\alpha\beta}\}_{\beta \in B(\alpha)}\) be the family of all connected components of the set \(U_\alpha\).

The topology induced by the family \(\{V_{\alpha\beta}\}_{\alpha \in A, \beta \in B(\alpha)}\) will be designated by \(\mathcal{LC}(X, \mathcal{T})\) (or by \(\mathcal{LC}(X)\) if the original topology on \(X\) is clear from context).

**Remark 3.1** This topology could be informally characterized by the following property: for each opened subset \(W\) of \((X, \mathcal{T})\) every connected component of \(W\) is opened in the topology \(\mathcal{LC}(X, \mathcal{T})\).

Moreover, as it could be seen from definition 3.1 \(\mathcal{LC}(X, \mathcal{T})\) is the weakest topology on \(X\) meeting this property.

So, this remark could be used as the alternative definition of the topology \(\mathcal{LC}(X, \mathcal{T})\).

**Remark 3.2** It could be easily verified that the space \((X, \mathcal{T})\) is locally connected if and only if the topologies \(\mathcal{T}\) and \(\mathcal{LC}(X, \mathcal{T})\) coincide.

Using definition 3.1 theorems 1.1 and 1.2 could be reformulated in the following form

**Theorem 3.1** Let \(K\) be a two-sided subset of the plane. If the sets \(A_1\) and \(A_2\) are dense in \(K\) in the topology \(\mathcal{LC}(K)\) then \(K\) is the simple closed curve.

**Theorem 3.2** Let \(K\) be a simple subset of the plane. Then the sets \(A_1\) and \(A_2\) are closed in the topology \(\mathcal{LC}(K)\).

For a special class of spaces the topology \(\mathcal{LC}(X, \mathcal{T})\) is mentioned in [K], chapter 6, section 49.VII (see also [M]).

Namely, let \((X, \rho)\) be a metric space with the following property: for every \(a, b \in X\) there exists a connected subset \(A \subset X\) such that \(a, b \in A\) and

\[
\text{diam } A = \sup_{x, y \in A} \rho(x, y) < \infty.
\]

Then so called relative distance

\[
\rho_r : X \times X \to \mathbb{R}_+,
\]

\[
\rho_r(a, b) = \inf\{\text{diam } A \mid a, b \in A \text{ and } A \text{ is connected}\}
\]

is well defined and appears to be a metric.
Proposition 3.1 Topology induced by the distance function $\rho_r$ coincides with the topology $\mathcal{LC}(X)$.

$\nabla$ Fix $x \in X$. For $\varepsilon > 0$ designate by $X(U_\varepsilon(x), x)$ the connected component of $x$ in the set $U_\varepsilon(x)$.

Then in accord with definition 3.1 the family $\{X(U_\varepsilon(x), x)\}_{\varepsilon > 0}$ forms a base of opened neighborhoods of the point $x$ in the topology $\mathcal{LC}(X)$.

On the other hand let us denote

$$V_\varepsilon^r(x) = \{y \in X \mid \rho_r(x, y) < \varepsilon\}$$

for every $\varepsilon > 0$.

Then by easy direct verification we receive

$$V_{\varepsilon/3}^r(x) \subseteq X(U_\varepsilon(x), x) \subseteq V_{3\varepsilon}^r(x)$$

for every $\varepsilon > 0$.

Q. E. D. $\triangle$

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