A note on knot Floer homology and fixed points of monodromy

Yi NI
Department of Mathematics, Caltech, MC 253-37
1200 E California Blvd, Pasadena, CA 91125
Email: yini@caltech.edu

Abstract
Using an argument of Baldwin–Hu–Sivek, we prove that if $K$ is a hyperbolic fibered knot with fiber $F$ in a closed, oriented 3–manifold $Y$, and $\hat{HF} K(Y, K, [F], g(F) − 1)$ has rank 1, then the monodromy of $K$ is freely isotopic to a pseudo-Anosov map with no fixed points. In particular, this shows that the monodromy of a hyperbolic L-space knot is freely isotopic to a map with no fixed points.

1 Introduction
Knot Floer homology, defined by Ozsváth–Szabó [21] and Rasmussen [26], is a powerful knot invariant. It contains a lot of information about the topology of the knot. For example, it detects the Seifert genus of a knot $K$ [22], and it determines whether $K$ is fibered [7, 15]. In fact, such information is contained in $\hat{HF}(K, g(K))$, the first nontrivial term of the knot Floer homology with respect to the Alexander grading, which is often referred as the “topmost term” in knot Floer homology.

In recent years, it became clear that $\hat{HF}(K, g(K) − 1)$, which is often called “the second term” or “the next-to-top term” in knot Floer homology, also contains interesting information about the knot. For example, Lipshitz–Ozsváth–Thurston [14] showed that the “second term” of the bordered Floer homology of a surface mapping class completely determines this mapping class. Baldwin and Vela-Vick [2] proved that the second term of the knot Floer homology of a fibered knot is always nontrivial, and Ni [18] proved the same result for knots whose topmost term is supported in a single $\mathbb{Z}/2\mathbb{Z}$-grading.

The main theorem in this paper is also of this type.

Theorem 1.1. Let $Y$ be a closed, oriented 3–manifold, and $K \subset Y$ be a hyperbolic fibered knot with fiber $F$ and monodromy $\varphi$. If

$$\text{rank} \hat{HF}(Y, K, [F], g(F) − 1) = 1,$$
then $\varphi$ is freely isotopic to a pseudo-Anosov map with no fixed points.

Recall that a knot $K \subset S^3$ is an L-space knot if a positive surgery on $K$ is an L-space. The most interesting scenario to apply Theorem 1.1 is when $K \subset S^3$ is a hyperbolic L-space knot. In this case it is well-known that $\text{rank} \widehat{HF}(S^3, K, g(K) - 1) = 1$ [24]. We hope Theorem 1.1 will shed more light on the understanding of L-space knots.

The proof of Theorem 1.1 uses a strategy due to Baldwin–Hu–Sivek [1], who proved Theorem 1.1 when $K$ has the same $\widehat{HF}$ as the cinquefoil $T_{5,2}$. In [1], the authors made use of the zero surgery formula on alternating knots in Heegaard Floer homology due to Ozsváth–Szabó [19]. We basically replace this result with a more general zero surgery formula. For simplicity, we only state here a formula for knots in $S^3$. If we are just interested in proving Theorem 1.1 for knots in $S^3$, this case will suffice. The zero surgery formula used in the proof of Theorem 1.1 is Proposition 3.1.

Given a null-homologous knot $K \subset Y$, let $Y_{p/q}(K)$ be the manifold obtained by $\frac{p}{q}$–surgery on $K$.

**Proposition 1.2.** Let $K \subset S^3$ be a hyperbolic fibered knot with genus $g \geq 3$. Suppose that the monodromy of $K$ is neither left-veering nor right-veering, then

$$\text{rank} \ HF^+(S^3_0(K), g - 2) = \text{rank} \widehat{HF}(S^3, K, g - 1) - 2.$$
**Pseudo-Anosov:** \( \varphi \) is freely isotopic to a pseudo-Anosov map \( \widetilde{\varphi} \). That is, there exist two singular measured foliations \( (F^u, \mu^u), (F^s, \mu^s) \) of \( F \), which are transverse everywhere except at the singular points, such that

\[
\widetilde{\varphi}(F^u, \mu^u) = \lambda(F^u, \mu^u), \quad \widetilde{\varphi}(F^s, \mu^s) = \lambda^{-1}(F^s, \mu^s)
\]

for a fixed real number \( \lambda > 1 \).

**Reducible:** \( \varphi \) is freely isotopic to a reducible map \( \widetilde{\varphi} \). That is, there exists a collection \( C \) of mutually disjoint essential simple closed curves, such that \( \widetilde{\varphi}(C) = C \). Moreover, \( F \setminus C \) can be divided into (possibly disconnected) subsurfaces \( F_1, \ldots, F_m \), such that the mapping class of \( \widetilde{\varphi}|F_i \) is either periodic or pseudo-Anosov. We also require that \( C \) is the minimum collection of curves with this property, and \( C \neq \emptyset \).

Thurston proved that the mapping torus of \( \varphi \) has a complete, finite volume hyperbolic structure in its exterior if and only if the mapping class of \( \varphi \) is pseudo-Anosov.

Let \( \varphi : F \to F \) be a diffeomorphism such that \( \varphi(\partial F) = \text{id}_{\partial F} \). By Thurston’s classification, \( \varphi \) is freely isotopic to a standard representative \( \widetilde{\varphi} \), whose restriction to \( \partial F \) may not be the identity. For each component \( C \) of \( \partial F \), one can define a fractional Dehn twist coefficient (FDTC) \( c(\varphi) \in \mathbb{Q} \), which is the rotation number of \( \widetilde{\varphi} \) on \( C \) compared with \( \varphi \). This concept essentially appeared in work of Gabai [6, Remark 8.7 ii)], and the term FDTC was coined by Honda–Kazez–Matić [10].

Honda–Kazez–Matić [10] also defined right-veering diffeomorphisms.

**Definition 2.1.** Let \( F \) be a compact surface with boundary, \( a, b \subset F \) be two properly embedded arcs with \( a(0) = b(0) = x \). We isotope \( a, b \) with endpoints fixed, so that \( |a \cap b| \) is minimal. We say \( b \) is to the right of \( a \), denoted \( a \leq b \), if either \( b \) is isotopic to \( a \) with endpoints fixed, or \( (b \cap U) \setminus \{x\} \) lies in the “right” component of \( U \setminus a \), where \( U \subset F \) is a small neighborhood of \( x \). See Figure 1.

![Figure 1](image_url)

**Figure 1:** The arc \( b \) is to the right of \( a \).

**Definition 2.2.** Let \( \varphi : F \to F \) be a diffeomorphism that restricts to the identity map on \( \partial F \). Let \( C \) be a component of \( \partial F \). Then \( \varphi \) is right-veering with respect to \( C \) if for every \( x \in C \) and every properly embedded arc \( a \subset F \) with \( x \in a \), the image \( \varphi(a) \) is to the right of \( a \) at \( x \). Similarly, we can define left-veering with respect to \( C \). If \( \varphi \) is right-veering with respect to every component
of $\partial F$, we say $\varphi$ is a right-veering diffeomorphism. Similarly, we can define left-veering diffeomorphisms.

If the mapping class of $\varphi$ is pseudo-Anosov, then $\varphi$ is right-veering with respect to $C$ if and only if $c(\varphi) > 0$ for $C$ [10 Proposition 3.1]. If the mapping class of $\varphi$ is periodic, then $\varphi$ is right-veering with respect to $C$ if $c(\varphi) > 0$ for $C$ [10 Proposition 3.2].

3 Knot Floer homology and the zero surgery formula

In this section, we will use arguments in [16, 18] to prove Propositions 1.2 and 3.1. We assume the readers are reasonably familiar with Heegaard Floer homology.

Proposition 3.1. Let $Y$ be a closed, oriented 3–manifold, and $K \subset Y$ be a hyperbolic fibered knot with fiber $F$. Let $s \in \text{Spin}^c(Y)$ be the underlying Spin$^c$ structure for the open book decomposition of $Y$ with binding $K$ and page $F$. Let $\hat{F} \subset Y_0(K)$ be the closed surface obtained by capping off $\partial F$ with a disk, and let $t_k \in \text{Spin}^c(Y_0(K))$ be the Spin$^c$ structure satisfying

$$t_k|Y \setminus K = s|Y \setminus K, \quad \langle c_1(t_k), \hat{F} \rangle = 2k, \quad k \in \mathbb{Z}.$$ 

Suppose that either $s$ is torsion, or $HF^+(Y, s) = 0$. Suppose also that the monodromy of $K$ is neither left-veering nor right-veering. If

$$\text{rank} HFK(Y, K, [F], g(F) - 1) = 2$$

and $g(F) \geq 3$, then

$$\text{rank} HF^+(Y_0(K), t_{g-2}) = 0.$$ 

Let $Y$ be a closed, oriented 3–manifold, $K \subset Y$ be a fibered knot with a fiber $F$ of genus $g$, and let $s \in \text{Spin}^c(Y)$ be the Spin$^c$ structure of the open book decomposition with binding $K$ and page $F$. By [23], there exists a Heegaard diagram for $(Y, K)$, such that the topmost knot Floer chain complex $\hat{CFK}(Y, K, [F], g)$ has a single generator. Let

$$C = \hat{CFK}^\infty(Y, K, s, [F])$$

be the knot Floer chain complex. Let $\partial$ be the differential on $C$, and let $\partial_0$ be the summand of $\partial$ which preserves the $(i,j)$-grading.

We will consider the chain complex $C\{i < 0, j \geq g - 2\}$, which has the form

$$C(-1, g - 1)$$

(1)

$$C(-2, g - 2)$$

$$C(-1, g - 2).$$
where
\[ C(-i, g - i) \cong \widehat{CFK}(Y, K, s, [F], g) \cong \mathbb{Z}, \text{ for all } i \in \mathbb{Z}, \quad (2) \]
and
\[ C(-1, g - 2) \cong \widehat{CFK}(Y, K, s, [F], g - 1). \]

By (2), \( \partial_0 = 0 \) on \( C(-1, g - 1) \) and \( C(-2, g - 2) \). Since \( \partial^2 = 0 \), we see that
\[ \partial_w \partial_z = 0 \quad (3) \]
and \( \partial^2_{zw} = 0 \), where we use (2) to identify every \( C(-i, g - i) \) with \( \mathbb{Z} \). So we must have
\[ \partial_{zw} = 0. \]

**Lemma 3.2.** If the monodromy of \( K \) is neither left-veering nor right-veering, then the rank of the homology of the mapping cone (1) is
\[ \text{rank} \widehat{HF} \bar{K}(Y, K, s, [F], g - 1) - 2. \]

**Proof.** As we have showed above, the mapping cone (1) becomes the chain complex
\[ \begin{array}{c}
\mathbb{Z} \\
\downarrow \partial_z \\
\mathbb{Z} \\
\downarrow \partial_w \\
C(-1, g - 2).
\end{array} \quad (4) \]

Since the monodromy of \( K \) is neither left-veering nor right-veering, by the argument in the proof of [2, Theorem 1.1], the induced map
\[ (\partial_z)_* : \mathbb{Z} \to H_*(C(-1, g - 2)) \]
is injective, and the induced map
\[ (\partial_w)_* : H_*(C(-1, g - 2)) \to \mathbb{Z} \]
is surjective. Using (3), we see that the rank of the homology of (4) is
\[ \text{rank} H_*(C(-1, g - 2)) - 2 = \text{rank} \widehat{HF} \bar{K}(Y, K, s, [F], g - 1) - 2. \]

**Proof of Proposition 1.2.** By a standard argument, (see, for example, [21, Corollary 4.5],) \( \widehat{HF}^+(S^3_0(K), g - 2) \) is isomorphic to the homology of (1), so our conclusion follows from Lemma 3.2. \( \square \)

As in [25], for any \( k \in \mathbb{Z} \), let
\[ A^+_k = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z} \]
and \( B^+ = C\{i \geq 0\} \cong CF^+(Y, s) \). There are chain maps
\[ v_k^+, h_k^+ : A^+_k \to B^+. \]
Here \( v_k^+ \) is the vertical projection, and \( h_k^+ \) is essentially the horizontal projection.
Proof of Proposition 3.1. It is well known that $HF^+(Y_0(K), t_{g-2})$ is isomorphic to the homology of $MC(v_{g-2}^+ + h_{g-2}^+)$, the mapping cone of 

$$v_{g-2}^+ + h_{g-2}^+: A_k^+ \to B^+.$$ 

In fact, when $s$ is torsion, this result follows from the same argument as in [25, Subsection 4.8]; when $s$ is non-torsion, the formula is proved in [17, Theorem 3.1].

By the exact triangle

$$H_*(A_{g-2}^+) \xrightarrow{(v_{g-2}^+)_*} H_*(B^+) \xrightarrow{\delta} H_*(C\{i < 0, j \geq g-2\})$$

(5)

$C\{i < 0, j \geq g-2\}$ is quasi-isomorphic to $MC(v_{g-2}^+)$, the mapping cone of $v_{g-2}^+$.

By Lemma 3.2, $H_*(MC(v_{g-2}^+); Q) = 0$, so $(v_{g-2}^+)_*$ is an isomorphism over $Q$. If $s$ is torsion, there is an absolute $\mathbb{Q}$-grading on $A_k^+$ and $B^+$. Since $g-2 > 0$, the grading shift of $h_{g-2}^+$ is strictly less than the grading shift of $v_{g-2}^+$. Hence $(v_{g-2}^+)_* + (h_{g-2}^+)_*$ is also an isomorphism over $\mathbb{Q}$, which implies that

$$HF^+(Y_0(K), t_{g-2}; \mathbb{Q}) = 0.$$ 

(6)

If $HF^+(Y, s) = 0$, $H_*(B^+) = 0$, so $H_*(A_{g-2}^+; Q) = 0$ since $(v_{g-2}^+)_*$ is an isomorphism over $\mathbb{Q}$. We again have (6).

4 Proof of the main theorem

Lemma 4.1. There exists a hyperbolic fibered knot $L \subset Z = S^1 \times S^2$ with fiber $G$, such that the monodromy of the fibration is right-veering, and the $\text{Spin}^c$ structure of the open book decomposition with binding $L$ and page $G$ is non-torsion.

Proof. By [5], there exists a non-torsion contact structure $\xi$ on $Z$. Let $(S, h)$ be an open book decomposition supporting $\xi$. By [3], we can stabilize $(S, h)$ many times to get a new open book $(G, \psi)$ with connected binding $L$, such that the monodromy $\psi$ is pseudo-Anosov and right-veering.

Let $L$ be the knot as in Lemma 1.1 and let $L'$ be the $(2n + 1, 2)$-cable of $L$ for a sufficiently large integer $n$. Let $E$ be the fiber of the new fibration of $Z$ with $\partial E = L'$, then

$$E = T \cup G_1 \cup G_2,$$

where $T$ is a genus $n$ surface with 3 boundary components, and $G_1, G_2$ are two copies of $G$. Let $\rho$ be the monodromy of $L'$. Then $\rho|T$ is isotopic to a periodic map of period $4n + 2$, and $\rho$ swaps $G_1, G_2$. 

6
Lemma 4.2. The FDTC of \(\rho\) on \(L'\) is \(\frac{1}{4n+2}\).

Proof. The complement of \(L'\) is the union of a cable space and \(Z \setminus L\). The slope on \(L'\) of the Seifert fiber of the cable space is \(4n+2\). Our conclusion follows from the definition of FDTC.

Proof of Theorem 1.1. The proof of Theorem 1.1 uses a similar argument as [1, Theorem 3.5]. The new input here is to replace [1, Equation (3.3)] with Proposition 3.1.

Since \(\text{rank} \hat{HF}K(Y, K, [F], g(F) - 1) = 1\), it follows from [16, Theorem A.1] that \(\varphi\) is either right-veering or left-veering. Without loss of generality, we assume \(\varphi\) is right-veering.

Let \(L'\) be as in Lemma 4.2. By [9], we have

\[
\hat{HF}K(Z, L', [E], g(E) - 1) \cong \mathbb{Z}.
\]

Consider the connected sum \(K \# L'\), which is a knot in \(Y \# Z\). Let \(g'\) be the genus of the Seifert surface \(F \natural E\). By the Künneth formula,

\[
\text{rank} \hat{HF}K(Y \# Z, K \# L', [F \natural E], g' - 1) = 2.
\]

(7)

The monodromy of \(K \# L'\) is a map \(\sigma\) on \(F \natural E\). By Lemma 4.2, \(\sigma|E\) is left-veering, so \(\sigma\) is neither left-veering nor right-veering.

Let \(s \in \text{Spin}^c(Y \# Z)\) be the \(\text{Spin}^c\) structure of the open book decomposition with binding \(K \# L'\) and page \(F \natural E\). Since the restriction of \(s\) to \(Z \setminus B^3\) is non-torsion, \(HF^+(Y \# Z, s) = 0\) by the adjunction inequality [20].

Now Proposition 3.1 and (7) imply that

\[
\text{rank} \, HF^+(Y \# Z)_0(K \# L', g' - 2) = 0.
\]

(8)

The manifold \((Y \# Z)_0(K \# L')\) is a surface bundle over \(S^1\). Its fiber \(P\) is a closed surface which is the union of \(F\) and \(E\). Let \(\tilde{\sigma}\) be the monodromy, then \(\tilde{\sigma}|F = \varphi\).

The rest of our argument is similar to [11, Theorem 3.5]. Using work of Lee–Taubes [13], Kutzthman–Lee–Taubes [12], Kronheimer–Mrowka [11], one sees that \(HF^+(Y \# Z)_0(K \# L', g' - 2)\) is isomorphic to the symplectic Floer homology \(HF^{2ymp}(P, \tilde{\sigma})\) of \((P, \tilde{\sigma})\). By [5], \(HF^{2ymp}(P, \tilde{\sigma}; \mathbb{Q}) = 0\).

Recall that we assume \(\varphi\) is right-veering. In the terminology of [11], \(\varphi\) has no Type IIIb fixed points. By [3, Theorem 4.16], \(HF^{2ymp}(P, \tilde{\sigma}; \mathbb{Q})\) contains a direct summand which is freely generated by a superset of the fixed points of the pseudo-Anosov representative \(\tilde{\varphi}\) of \(\varphi\). So \(\tilde{\varphi}\) has no fixed points.

Remark 4.3. In [11], the authors computed \(S^3_0(K \# K, g(K \# K) - 2)\) to get the conclusion. This approach will also work in the general case if the underlying \(\text{Spin}^c\) structure of the open book decomposition corresponding to \(K\) is torsion.
References

[1] John A. Baldwin, Ying Hu, and Steven Sivek, Khovanov homology and the cinquefoil (2021), preprint, available at https://arxiv.org/abs/2105.12102.

[2] John Baldwin and David Vela-Vick, A note on the knot Floer homology of fibered knots, Algebr. Geom. Topol. 18 (2018), no. 6, 3669–3690.

[3] Vincent Colin and Ko Honda, Stabilizing the monodromy of an open book decomposition, Geom. Dedicata 132 (2008), 95–103.

[4] Andrew Cotton-Clay, Symplectic Floer homology of area-preserving surface diffeomorphisms, Geom. Topol. 13 (2009), no. 5, 2619–2674.

[5] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds, Invent. Math. 98 (1989), no. 3, 623–637.

[6] David Gabai, Problems in foliations and laminations, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 1–33.

[7] Paolo Ghiggini, Knot Floer homology detects genus-one fibred knots, Amer. J. Math. 130 (2008), no. 5, 1151–1169.

[8] Paolo Ghiggini and Gilberto Spano, in preparation.

[9] Matthew Hedden, On knot Floer homology and cabling, Algebr. Geom. Topol. 5 (2005), 1197–1222.

[10] Ko Honda, William H. Kazez, and Gordana Matić, Right-veering diffeomorphisms of compact surfaces with boundary, Invent. Math. 169 (2007), no. 2, 427–449.

[11] Peter Kronheimer and Tomasz Mrowka, Monopoles and three-manifolds, New Mathematical Monographs, vol. 10, Cambridge University Press, Cambridge, 2007.

[12] Çağatay Kutluhan, Yi-Jen Lee, and Clifford Henry Taubes, HF=HM, I: Heegaard Floer homology and Seiberg-Witten Floer homology, Geom. Topol. 24 (2020), no. 6, 2829–2854.

[13] Yi-Jen Lee and Clifford Henry Taubes, Periodic Floer homology and Seiberg-Witten-Floer cohomology, J. Symplectic Geom. 10 (2012), no. 1, 81–164.

[14] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, A faithful linear-categorical action of the mapping class group of a surface with boundary, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 4, 1279–1307.

[15] Yi Ni, Knot Floer homology detects fibred knots, Invent. Math. 170 (2007), no. 3, 577–608.

[16] Yi Ni, Seifert fibered and reducible surgeries on hyperbolic fibered knots (2020), preprint, available at https://arxiv.org/abs/2007.11774.

[17] Yi Ni, Property G and the 4–genus (2020), preprint, available at https://arxiv.org/abs/2007.03721.

[18] Yi Ni, The next-to-top term in knot Floer homology (2021), preprint, available at https://arxiv.org/abs/2104.14687.

[19] Peter Ozsváth and Zoltán Szabó, Heegaard Floer homology and alternating knots, Geom. Topol. 7 (2003), 225–254.

[20] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and three-manifold invariants: properties and applications, Ann. of Math. (2) 159 (2004), no. 3, 1159–1245.

[21] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), no. 1, 58–116.

[22] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334.

[23] Peter Ozsváth and Zoltán Szabó, Heegaard Floer homology and contact structures, Duke Math. J. 129 (2005), no. 1, 39–61.

[24] Peter Ozsváth and Zoltán Szabó, On knot Floer homology and lens space surgeries, Topology 44 (2005), no. 6, 1281–1300.
[25] Knot Floer homology and integer surgeries, Algebr. Geom. Topol. 8 (2008), no. 1, 101–153.

[26] Jacob Andrew Rasmussen, Floer homology and knot complements, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)--Harvard University.

[27] William P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417–431.