COUPLED HAMILTONIAN SYSTEMS WITH EXTENDED AFFINE
WEYL GROUP SYMMETRY OF TYPE $D_3^{(2)}$

YUSUKE SASANO

ABSTRACT. We find a two-parameter family of ordinary differential systems in dimension five with the affine Weyl group symmetry of type $D_3^{(2)}$. We show its symmetry and holomorphy conditions. This is the second example which gave higher order Painlevé type systems of type $D_3^{(2)}$. By obtaining its first integrals of polynomial type, we can obtain a two-parameter family of coupled Hamiltonian systems in dimension four with the polynomial Hamiltonian.

1. INTRODUCTION

In this paper, we find a 2-parameter family of ordinary differential systems in dimension five with the affine Weyl group symmetry of type $D_3^{(2)}$ explicitly given by

\begin{align}
\frac{dx}{dt} &= -(xw - \alpha_2)x + \frac{1}{2}, \\
\frac{dy}{dt} &= (xw + zq - 1)y + \alpha_1 wq, \\
\frac{dz}{dt} &= -(zq - \alpha_0)z - \frac{\eta}{2}, \\
\frac{dw}{dt} &= (xw - zq - \alpha_2)w + yz, \\
\frac{dq}{dt} &= (zq - xw - \alpha_0)q + xy.
\end{align}

Here $x, y, z, w$ and $q$ denote unknown complex variables, and $\alpha_0, \alpha_1, \alpha_2$ are complex parameters satisfying the relation:

\begin{equation}
\alpha_0 + \alpha_1 + \alpha_2 = 1.
\end{equation}

This is the second example which gave higher order Painlevé type systems of type $D_3^{(2)}$.

We also remark that 2-coupled Painlevé II system in dimension four given in the paper [1] admits the affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, s_2$ are determined by the invariant divisors.

On the other hand, the Bäcklund transformations $s_0, s_2$ of this system do not satisfy so except for the transformation $s_1$ (see Theorem 2.2).

We show its symmetry and holomorphy conditions. By obtaining its first integrals of polynomial type, we can obtain a two-parameter family of coupled Hamiltonian systems in

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dimension four with the polynomial Hamiltonian. Each principal part of the Hamiltonian $H$ has its first integral, respectively. Nevertheless, the Hamiltonian $H$ itself is not its first integral.

2. Symmetry and holomorphy conditions

**Theorem 2.1.** Let us consider the following ordinary differential system in the polynomial class:

$$\frac{dx}{dt} = f_1(x, y, z, w, q), \cdots, \frac{dq}{dt} = f_5(x, y, z, w, q) \quad (f_i \in \mathbb{C}(t)[x, y, z, w, q]).$$

We assume that

(A1) $\text{deg}(f_i) = 3$ with respect to $x, y, z, w, q$.

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system $(x_i, y_i, z_i, w_i, q_i) \ (i = 0, 1, 2)$:

0) $x_0 = x, \quad y_0 = y - \frac{2\alpha_0 w}{z} + \frac{\eta w}{z^2}, \quad z_0 = z, \quad w_0 = w, \quad q_0 = q - \frac{2\alpha_0}{z} + \frac{\eta}{z^2},$

1) $x_1 = x + \frac{\alpha_1 q}{y}, \quad y_1 = y, \quad z_1 = z + \frac{\alpha_1 w}{y}, \quad w_1 = w, \quad q_1 = q,$

2) $x_2 = x, \quad y_2 = y - \frac{2\alpha_2 q}{x} - \frac{q}{x^2}, \quad z_2 = z, \quad w_2 = w - \frac{2\alpha_0}{x} - \frac{1}{x^2}, \quad q_2 = q.$

Then such a system coincides with the system (1).

We note that these transition functions satisfy the condition:

$$dx_i \wedge dy_i \wedge dz_i \wedge dw_i \wedge dq_i = dx \wedge dy \wedge dz \wedge dw \wedge dq \quad (i = 0, 1, 2).$$

**Theorem 2.2.** The system (1) admits the affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, s_2$ defined as follows:

with the notation $(*) := (x, y, z, w, q, \eta; \alpha_0, \alpha_1, \alpha_2)$,

$$s_0 : (*) \rightarrow \left( x, y - \frac{2\alpha_0 w}{z} + \frac{\eta w}{z^2}, z, w, q - \frac{2\alpha_0}{z} + \frac{\eta}{z^2}, -\eta; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2 \right),$$

$$s_1 : (*) \rightarrow \left( x + \frac{\alpha_1 q}{y}, y, z + \frac{\alpha_1 w}{y}, w, q, \eta; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1 \right),$$

$$s_2 : (*) \rightarrow \left( -x, y - \frac{2\alpha_2 q}{x} - \frac{q}{x^2}, -z, -w + \frac{2\alpha_0}{x} + \frac{1}{x^2}, -q, -\eta; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2 \right).$$

**Proposition 2.3.** Let us define the following translation operators:

$$T_1 := s_1 s_2 s_1 s_0, \quad T_2 := s_1 T_1 s_1.$$

These translation operators act on parameters $\alpha_i$ as follows:

$$T_1(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0, \alpha_1, \alpha_2) + (-2, 2, 0),$$

$$T_2(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0, \alpha_1, \alpha_2) + (0, -2, 2).$$
3. Particular solution

In this section, we study a solution of the system \( \text{(1)} \) which is written by the use of known functions.

**Proposition 3.1.** The system \( \text{(1)} \) has the following invariant divisor:

| parameter’s relation | invariant divisor |
|----------------------|------------------|
| \( \alpha_1 = 0 \)   | \( y \)          |

Under the condition \( \alpha_1 = 0 \), elimination of \( y \) from the system \( \text{(1)} \) gives

\[
\begin{align*}
\frac{dx}{dt} &= -(xw - \alpha_2)x + \frac{1}{2}, \\
\frac{dz}{dt} &= -(zq - \alpha_0)z - \eta \frac{1}{2}, \\
\frac{dw}{dt} &= (xw - zq - \alpha_2)w, \\
\frac{dq}{dt} &= (zq - xw - \alpha_0)q.
\end{align*}
\]

At first, we find a particular solution \( (x, z, w, q) = (x, z, 0, 0) \), and the system in \( (x, z) \) are given by

\[
\begin{align*}
\frac{dx}{dt} &= \alpha_2 x + \frac{1}{2}, \\
\frac{dz}{dt} &= \alpha_0 z - \eta \frac{1}{2}.
\end{align*}
\]

This system can be solved by

\[
\begin{align*}
x(t) &= C_1 e^{\alpha_2 t} - \frac{1}{2\alpha_2}, \\
z(t) &= C_2 e^{\alpha_0 t} + \frac{\eta}{2\alpha_0},
\end{align*}
\]

where \( C_1, C_2 \) are integral constants.

Next, we find a particular solution \( (x, z, w, q) = (x, z, w, 0) \), and the system in \( (x, z, w) \) are given by

\[
\begin{align*}
\frac{dx}{dt} &= -(xw - \alpha_2)x + \frac{1}{2}, \\
\frac{dz}{dt} &= \alpha_0 z - \eta \frac{1}{2}, \\
\frac{dw}{dt} &= (xw - \alpha_2)w.
\end{align*}
\]

Elimination of \( w \) from the system in the variables \( x, w \) gives

\[
\frac{d^2 x}{dt^2} = \frac{1}{x} \left( \frac{dx}{dt} \right)^2 - \frac{\alpha_2}{2} - \frac{1}{4x}.
\]
This system can be solved by the following two solutions

\begin{align}
(11) \quad x(t) &= e^{-C_1(t+C_2)} \{(e^{C_1(t+C_2)} - \alpha_2)^2 - C_1^2 \} \\
&= \frac{e^{-C_1(t+C_2)} \{(e^{2C_1(t+C_2)}(\alpha_2^2 - C_1^2) - 2\alpha_2 e^{C_1(t+C_2)} + 1)\}}{4C_1^2},
\end{align}

and

\begin{align}
(12) \quad x(t) &= \frac{e^{-C_1(t+C_2)} \{(e^{2C_1(t+C_2)}(\alpha_2^2 - C_1^2) - 2\alpha_2 e^{C_1(t+C_2)} + 1)\}}{4C_1^2},
\end{align}

where \( C_1, C_2 \) are integral constants. Of course, the system in the variable \( z \) can be solved by (8).

4. POLYNOMIAL HAMILTONIAN

For the system (1) let us try to seek its first integrals of polynomial type with respect to \( x, y, z, w, q \).

**Proposition 4.1.** This system (1) has its first integral:

\[
d(y - wq) = -(y - wq).
\]

By solving this equation, we can obtain

\[ y - wq = e^{-t}. \]

By using this, we show that elimination of \( y \) from the system (1) gives a polynomial Hamiltonian system.

**Theorem 4.2.** By the transformations

\[
q_1 = w, \quad p_1 = x, \quad q_2 = e^t q, \quad p_2 = \frac{z}{e^t}, \quad s = \frac{1}{e^t},
\]

elimination of \( y \) from the system (1) gives a 2-parameter family of coupled Hamiltonian systems in dimension four explicitly given by

\[
\begin{cases}
\frac{dq_1}{ds} = \frac{\partial H}{\partial p_1} = -\frac{q_1^2 p_1}{s} + \alpha_2 q_1 - \frac{p_2}{s}, \\
\frac{dp_1}{ds} = -\frac{\partial H}{\partial q_1} = \frac{q_1^2 p_1}{s} - \frac{\alpha_2 p_1}{s} - \frac{1}{2s}, \\
\frac{dq_2}{ds} = \frac{\partial H}{\partial p_2} = -\frac{q_2^2 p_2}{s} - \frac{(\alpha_1 + \alpha_2) q_2}{s} - \frac{p_1}{s}, \\
\frac{dp_2}{ds} = -\frac{\partial H}{\partial q_2} = \frac{q_2^2 p_2}{s} + \frac{(\alpha_1 + \alpha_2) p_2}{s} + \frac{\eta}{2},
\end{cases}
\]

with the polynomial Hamiltonian

\[
H = K_1(q_1, p_1, s; \alpha_2) + K_2(q_2, p_2, s; \alpha_1 + \alpha_2) - \frac{p_1 p_2}{s}
\]

\[
= -\frac{q_1^2 p_1^2}{2s} - 2\alpha_2 q_1 p_1 - q_1 - \frac{q_2^2 p_2^2}{2s} - 2(\alpha_1 + \alpha_2) q_2 p_2 + \eta s q_2 - \frac{p_1 p_2}{s}.
\]
The symbols $K_1(q_1, p_1, s; \alpha)$ and $K_2(q_2, p_2, s; \alpha)$ denote
\[
K_1(q_1, p_1, s; \alpha) = -\frac{q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1}{2s},
\]
\[
K_2(q_2, p_2, s; \alpha) = -\frac{q_2^2 p_2^2 + 2\alpha q_2 p_2 + \eta s q_2}{2s}.
\]
(15)

This Hamiltonian system can be considered as a 1-parameter family of coupled polynomial Hamiltonian systems in dimension four.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to $q_1, p_1, q_2, p_2$. However, we can not find. Of course, the Hamiltonian $H$ is not the first integral.

**Proposition 4.3.** The system $(13)$ is equivalent to the coupled equations:
\[
\begin{align*}
\frac{d^2 y}{ds^2} &= \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} - \frac{\alpha_2}{s^2} - \frac{1}{4s^2 y} - \frac{y^2 w}{s^2}, \\
\frac{d^2 w}{ds^2} &= \frac{1}{w} \left( \frac{dw}{ds} \right)^2 - \frac{1}{s} \frac{dw}{ds} + \frac{\alpha_0 \eta}{2s} - \eta^2 - \frac{yw^2}{s^2},
\end{align*}
\]
(16)

where $y := p_1$ and $w := p_2$.

We study two Hamiltonians $K_1$ and $K_2$ in the principal parts of the Hamiltonian $H$.

At first, we study the Hamiltonian system
\[
\frac{dq_1}{ds} = \frac{\partial K_1}{\partial p_1}, \quad \frac{dp_1}{ds} = -\frac{\partial K_1}{\partial q_1}
\]
(17)

with the polynomial Hamiltonian
\[
K_1 := -\frac{q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1}{2s},
\]
(18)

where setting $q_2 = p_2 = 0$ in the Hamiltonian $H$, we obtain $K_1$.

The system has the first integral $I_1$:
\[
I_1 = q_1^2 p_1^2 - 2\alpha q_1 p_1 - q_1.
\]
(19)

Next, we study the Hamiltonian system
\[
\frac{dq_2}{ds} = \frac{\partial K_2}{\partial p_2}, \quad \frac{dp_2}{ds} = -\frac{\partial K_2}{\partial q_2}
\]
(20)

with the polynomial Hamiltonian
\[
K_2(q_2, p_2, s; \alpha) = -\frac{q_2^2 p_2^2 + 2\alpha q_2 p_2 + \eta s q_2}{2s},
\]
(21)

where setting $q_1 = p_1 = 0$ in the Hamiltonian $H$, we obtain $K_2$.

**Step 1:** We make the change of variables:
\[
x_1 = s q_2, \quad y_1 = \frac{p_2}{s}
\]
(22)
Then, we can obtain the system with the polynomial Hamiltonian:

$$\tilde{K}_2(x_1, y_1, s; \alpha) = -\frac{x_1^2 y_1^2 + 2(\alpha - 1)x_1 y_1 + \eta x_1}{2s}. \tag{23}$$

This system has the first integral $I_2$:

$$I_2 = x_1^2 y_1^2 + 2(\alpha - 1)x_1 y_1 + \eta x_1. \tag{24}$$

We also study its symmetry.

**Theorem 4.4.** The system \([13]\) admits extended affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, s_2, \pi$ defined as follows: with the notation $(*):= (q_1, p_1, q_2, p_2, \eta, s; \alpha_0, \alpha_1, \alpha_2)$:

$$s_0 : (*) \rightarrow \left( q_1, p_1, q_2 - \frac{2\alpha_0}{p_2} + \frac{\eta s}{p_2^2}, p_2, \eta, -s; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2 \right),$$

$$s_1 : (*) \rightarrow \left( q_1, p_1 + \frac{\alpha_1 q_2}{q_1 q_2 + 1}, q_2, p_2 + \frac{\alpha_1 q_1}{q_1 q_2 + 1}, \eta, s; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1 \right),$$

$$s_2 : (*) \rightarrow \left( -q_1 + \frac{2\alpha_2}{p_1} + \frac{1}{p_1^2}, -p_1, -q_2, -p_2, -\eta, -s; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2 \right),$$

$$\pi : (*) \rightarrow \left( -\eta s q_2, -\frac{p_2}{\eta s}, \frac{q_1}{\eta s}, -\eta s p_1, \eta, s; \alpha_2, \alpha_1, \alpha_0 \right),$$

where $\pi$ is its diagram automorphism of Dynkin diagram of type $D_3^{(2)}$.

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