Strong solutions of the incompressible Navier-Stokes equations in external domains: local existence and uniqueness

M. Tessarotto\textsuperscript{a,b} and C. Cremaschini\textsuperscript{c}

\textit{Department of Mathematics and Informatics, University of Trieste, Italy}
\textit{Consortium of Magneto-fluid-dynamics, University of Trieste, Italy}
\textit{International School for Advanced Studies, SISSA, Trieste, Italy}

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Abstract

In this paper the problem of strong solvability of the incompressible Navier-Stokes equations (INSE) is revisited, with the goal of determining the minimal assumptions for the validity of a local existence and uniqueness theorem for the Navier-Stokes fluid fields (solutions of INSE). Emphasis is placed on fluid fields which, together with suitable derivatives, do not necessarily decay at infinity and hence do not belong to Sobolev spaces. For this purpose a novel approach based on a so-called inverse kinetic theory, recently developed by Tessarotto and Ellero, is adopted. This requires the construction of a suitable kinetic equation, advancing in time a suitably smooth kinetic distribution function and providing exactly, as its moment equations, the complete set of fluid equations. In turn, by proper definition of the kinetic equation, this permits the introduction of the so-called Navier-Stokes dynamical system, i.e., the dynamical system which advances in time self-consistently the Navier-Stokes fluid fields. Investigation of the properties of this dynamical system is crucial for the establishment of an existence and uniqueness theorem for strong solutions of INSE. The new theorem applies both to bounded and unbounded domains and in the presence of generalized boundaries, represented by surfaces, curves or even sets of isolated points. In particular, for unbounded domains, solutions are considered, which do not necessarily vanish at infinity. Basic consequences for the functional setting of classical solutions are analyzed.

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I. INTRODUCTION

The goal of this paper is to investigate a well-known theoretical issue of fluid dynamics. This is concerned with the existence and smoothness of solutions of the fluid equations which characterize an incompressible Newtonian fluid, i.e., the so-called *incompressible Navier-Stokes equations* (INSE). In particular, we will be interested in solutions which are physically realizable, i.e., can be identified with *physical observables*. In the theory of continua these are necessarily described by strong solutions. Several aspects of the mathematical theory of these fluid equations remain unsolved. This occurs, in particular, in the case of:

- **3D external domains**;
- **forcing**;
- **solutions which do not necessarily decay at infinity in configuration space**.

A related problem of mathematical research involves the search of the *classical dynamical system* - here denoted as *Navier-Stokes (NS) dynamical system* - which uniquely advances in time the related fluid fields that characterize the Newtonian fluid, i.e., the fluid velocity \( V(\mathbf{r},t) \) and the fluid pressure \( p(\mathbf{r},t) \). In other words, if the NS dynamical system actually exists, it will permit to cast the complete set of fluid equations in terms of an equivalent (and possibly infinite) set of ordinary differential equations which define the dynamical system itself. From the mathematical viewpoint the reason why such a dynamical system is so important is that, if a theorem of existence and uniqueness can be established for the set of ordinary differential equations which determines the dynamical system, than it is obvious that it will imply the same conclusion also for INSE. For contemporary science the determination of such a dynamical system represents not merely an unsolved intellectual challenge, but a fundamental prerequisite for the proper formulation of all physical theories which are based on the description of these fluids. These involve, for example, the understanding of the related phase-space Lagrangian fluid descriptions and the consistent formulation of turbulence theory and of the related approximate statistical descriptions (i.e., obtained introducing appropriate stochastic models able to reproduce phenomenological data), both essential in fluid dynamics and in applied sciences.
Surprisingly, until recently (see Tessarotto and Ellero, 2000-2005 [1, 2, 3]), the problem has remained unsolved. Its solution, discussed in detail in Refs. [3, 4, 5], is based on the construction of an *inverse kinetic theory (IKT)* for the incompressible NS equations, i.e., a suitable phase-space description which provides the complete set of fluid equations in terms of an appropriate inverse kinetic equation. Such an equation, in particular, can be constructed in such a way to define uniquely a classical dynamical system which advances in time the fluid fields, i.e., so that they result - by construction - solutions of the same fluid equations. For the extension of the theory to the treatment of incompressible thermofluids and quantum fluids see Refs. [6, 7, 8, 9].

In this paper we intend to point out basic consequences which can be drawn from IKT as far as the problem of existence and smoothness of solutions of INSE is concerned.

### A. Strong solutions of the INSE problem

For definiteness, we shall assume that the relevant fluids \( \{ \rho, \mathbf{V}, p \} \), i.e., respectively the mass density, fluid velocity and pressure describing the fluid, are defined point-wise and suitably smooth in an appropriate domain \( \Omega \times I \). Here \( \Omega \) (*fluid domain*) is an open connected subset of the Euclidean space \( \mathbb{R}^3 \), with boundary \( \partial \Omega \) and closure \( \overline{\Omega} \). \( \overline{\Omega} \) is defined as the set where the mass density is a constant \( \rho \equiv \rho_o > 0 \); moreover, \( I \) is either a time interval \( I = [t_0, t_1] \) (with closure \( \overline{I} = [t_0, t_1] \)), or the real axis \( \mathbb{R} \). We assume that the fluid fields are continuous in \( \overline{\Omega} \times \overline{I} \), fulfill suitable initial and boundary conditions respectively at \( t = t_o \) and on the boundary \( \partial \Omega \), while in the open set \( \Omega \times I \) they satisfy the fluid equations:

\[
\frac{D}{Dt} \rho + \rho \nabla \cdot \mathbf{V} = 0, \tag{1}
\]

\[
\rho \frac{D}{Dt} \mathbf{V} + \nabla p - f - \mu \nabla^2 \mathbf{V} = 0, \tag{2}
\]

\[
\nabla \cdot \mathbf{V} = 0, \tag{3}
\]

\[
\rho = \rho_o, \tag{4}
\]

which are subject to the inequalities

\[
p > 0, \tag{5}
\]

\[
\rho > 0. \tag{6}
\]
Here the notation is standard. Thus, $\rho_0$ and $\mu > 0$ are respectively the constant mass density and fluid viscosity and $f(r,t)$ the volume force density. Moreover, $\frac{D}{Dt} \equiv \frac{D}{Dt}V + V \cdot \nabla$ is the convective derivative, Eq. (1) denotes the so-called continuity equation, while Eqs. (2), (3) and (4) are respectively the Navier-Stokes (NS), isochoricity and incompressibility equations. Finally, the inequalities (5) and (6) represent the so-called physical realizability conditions of the fluid, which must be prescribed in order that $p$ and $\rho$ are physical observables.

The complete set of equations (1)-(4) - subject to the inequalities (5) and (6) - are denoted as incompressible Navier-Stokes equations (INSE) and its solutions NS fluid fields. In the following it is assumed that: 1) $\{\rho, V, p\}$ are strong solutions of the INSE problem which are suitably smooth so that Eqs. (12) and (13) are identically satisfied in the whole set $\Omega \times I$; 2) the solutions of INSE are assumed to satisfy suitable initial-(Dirichlet-)boundary value problem (INSE problem). These are defined as follows:

A) fluid initial conditions: the initial conditions for the fluid fields $A(r,t) = \{\rho, V, p\}(r,t)$ are defined imposing

$$A(r,t) = A_o(r), \tag{7}$$

where $t_o \in I$, and $A_o(r) \equiv \{\rho_o, V_o(r), p_o(r)\}$ satisfy respectively the condition of isochoricity and the Poisson equation:

$$\nabla \cdot V_o = 0, \tag{8}$$

$$\nabla^2 p_o = -\nabla \cdot [V_o \nabla V_o] - \nabla \cdot f(r,t_o). \tag{9}$$

B) fluid boundary conditions: $\partial \Omega$ is considered for greater generality as a moving boundary. In particular, for all points $r_W(t) \in \partial \Omega$ let us assume that their velocity

$$V_w(r_W(t),t) \equiv \frac{d}{dt}r_W(t) \tag{10}$$

is a suitably smooth function of time defined in $I$. Then, the boundary conditions are obtained, in both cases, imposing the Dirichlet boundary conditions

$$\lim_{r \in \Omega \rightarrow r_W \in \partial \Omega} A(r,t) = A_W(r_W,t), \tag{11}$$

where $A_W(r_W,t) \equiv \{\rho_w, V_w, p_w\}(r_W,t)$.

In particular, we remark that:
by taking the divergence of the NS equation (2) and respectively its scalar product with \( \mathbf{V} \), it follows the Poisson equation for the fluid pressure \( p \)

\[
\nabla^2 p = -\rho_o \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f} \tag{12}
\]

and the energy equation

\[
\rho \frac{D}{Dt} \frac{V^2}{2} + \mathbf{V} \cdot [\nabla p - \mathbf{f} - \mu \nabla^2 \mathbf{V}] = 0. \tag{13}
\]

since conventionally the domain of vacuum is defined as the subdomain (of \( \mathbb{R}^3 \)) in which both \( p \) and \( \rho \) vanish identically, the inequalities (5) and (6) provide the physical requirements in order that the domain \( \Omega \) is non-empty;

the boundary of the fluid domain, \( \partial \Omega \), may generally include so-called free boundaries where the fluid pressure \( p(r,t) \) locally vanishes;

in case \( \Omega \) is unbounded and \( \partial \Omega \) (or a subset of \( \partial \Omega \)) is an improper surface of \( \mathbb{R}^3 \), the fluid fields \( A(r,t) \) need not decay necessarily at infinity. This means that \( A(r_W,t) \) may not vanish at infinity. As a consequence, in such a case it follows that \( \mathbf{V}(r,t) \) shall not be required to be \( L^2(\Omega) \).

**B. Motivations and historical background**

The modern status of mathematical theory of NS equations is largely due to the pioneering work of Leray in the years 1933-34 \[10, 11, 12\] and Hopf in 1950-51 \[13\] who reformulated the NS PDE’s in terms of a suitable set of integral equations and introduced the concept of weak solutions for NS equations. These solutions have played, since their introduction, a major role in the mathematical research dealing with NS equations. Indeed, they are the only solutions which, so far, have been proven to exist for all times and without restrictions on the initial data, apart the requirement of a suitable functional setting for the fluid fields (for a review see Galdi, 2002 \[37\]). However, the fundamental problem of a unique global weak solution and, conversely, the possibility that uniqueness holds only locally due to the appearance of a local "bifurcation" phenomenon (the so-called Leray conjecture \[23\]) still remain open issues.
On the other hand, the distinction between strong (or *classical*) and weak solutions has important relevance also for the modelling of real fluids in the framework of continuous mechanics. It should be pointed out that the existence of weak solutions, rather than the classical ones, for NS equations may be also potentially an indication of the failure of the mathematical model based on the incompressible Navier-Stokes equations. In fact, it may involve the violation of the subsidiary conditions indicated above, in particular since a weak solution is generally not defined everywhere in $\Omega \times I$ (Doering and Gibbon, 1997 [31]).

An important feature which characterizes weak solutions is the way in which they are usually manufactured, i.e., their existence and uniqueness is established. This is obtained replacing the NS PDE’s by an infinite set of ODE’s and by constructing explicitly their solutions, which furnish a sequence of successive approximations, the so-called *Galerkin approximates* ([31]). The existence and uniqueness theorem for weak solutions of NS equations is achieved by demonstrating that the Galerkin approximates converge in a suitably weak sense to the solutions of the original PDE’s (which is another way of justifying the name given to these solutions).

In its traditional approach [12, 13, 16, 18, 19, 20, 21, 26] the treatment of the existence and uniqueness problem in the local sense, i.e., for a finite time interval $I$, requires a suitable functional setting which depends, in particular, on the choice of the boundary conditions. A necessary condition for the fluid fields to be strong solutions of INSE is that they are one-sided continuous on boundaries which separate the configuration space. Moreover, in the open domain $\Omega$, since all fluid equations [i.e., Eqs. (1)–(4) as well as (12) and (13)] must be defined by functions which are at least continuous, i.e., $C^{(0,0)}(\Omega \times I)$, the fluid fields and the volume force density $f(\mathbf{r},t)$ which satisfy INSE must evidently have the *native functional setting* (NF setting). These conditions involve both the fluid fields $\{\rho, \mathbf{V}, p\}$ and the volume force $f(\mathbf{r},t)$, which are required to satisfy at least

\[
\left\{ \begin{array}{l}
\{\rho, \mathbf{V}, p\} \in C^{(0)}(\Omega \times I), \\
\mathbf{V}(\mathbf{r},t) \in C^{(3,1)}(\Omega \times I), \\
p(\mathbf{r},t) \in C^{(2,0)}(\Omega \times I), \\
f(\mathbf{r},t) \in C^{(1,0)}(\Omega \times I),
\end{array} \right. \tag{14}
\]
as well as their initial and boundary conditions, which imply also

\[
\begin{align*}
V_o(r) & \in C^3(\Omega), \\
p_o(r) & \in C^2(\Omega), \\
f(r,t) & \in C^1(\Omega),
\end{align*}
\]

and

\[
\begin{align*}
V_W(r_W(t),t) & \in C^1(I), \\
p_W(r_W(t),t) & \in C^1(\Omega),
\end{align*}
\]

Indeed, in validity of (14) [and (15), (16)], by invoking equations (2)-(3) and imposing in the whole set \(\overline{\Omega}\) the initial condition

\[
\rho(r,t_o) = \rho_o > 0,
\]

it is obvious that Poisson equation Eq. (12) implies the incompressibility and the isochoricity conditions (3) and (4), as well as the energy equation (13) and vice versa. Hence, imposing Eq. (17), INSE can be reduced to the equivalent set given by Eqs. (1)- (4) and the physical realizability conditions (5), (6). Here \(\overline{\Omega} = \Omega \cup \partial \Omega\) is the closure of \(\Omega\) and \(C^{(i,j)}(\Omega \times I) \equiv C^{(i)}(\Omega) \cap C^{(j)}(I)\).

The customary approach to weak solutions is, instead, based on the requirement that the fluid fields, together with the data, belong to appropriate Sobolev spaces [19, 20, 24, 25], endowed with a suitable scalar product, usually defined by the integral on the configuration space \(\Omega\)

\[
(V(r,t), V_1(r,t)) \equiv \int_{\Omega} d\mathbf{r} V(r,t) \cdot V_1(r,t),
\]

being \(V(r,t), V_1(r,t)\) two independent velocity fields, and with norm

\[
\|V(r,t)\|_1^2 \equiv \int_{\Omega} d\mathbf{r} \sum_{\alpha=0,1} (D^\alpha V,D^\alpha V).
\]

The latter definition involves, besides the velocity field itself \(V = D^0 V\), its gradient \(\nabla V(r,t) \equiv D^1 V\). If the domain \(\Omega\) is bounded and the fluid fields belong to the NF setting both the scalar product and the norm can be trivially defined. However, this is not generally true in the case of an external (i.e., unbounded) domain \(\Omega\), unless the velocity field \(V(r,t)\) vanishes for \(|\mathbf{r}| \to \infty\). This is usually denoted Sobolev-space functional setting (SB setting).
C. Open problems

Referring to the 3D case the theory of existence and uniqueness of solutions is not yet complete in several aspects.

In particular regarding existence, a major open question is whether strong solutions exist for all times (global strong solutions), i.e., the fluid fields $A(r,t)$ remain at all times in the same (NF) functional class, or cease to exist in a proper sense, i.e., they develop spontaneously singularities which violate, at least locally, the Navier-Stokes equations. In both cases the answer would be of useful: in the first case to establish a global existence and uniqueness theorem with far-reaching consequences; in the second, to understand the nature of ”singularities” or ”irregularities” (i.e., for example, discontinuities arising in suitably higher-order derivatives of the fluid fields or violations of suitable bounds in appropriate functional spaces) possibly produced by the spontaneous evolution and decay of turbulence in a finite or infinite time interval [23].

Another issue of fundamental importance in its own right is the problem of uniqueness of strong solutions, which are assumed to exist and to belong to a suitable functional class (for example NF or SB). Several authors have investigated the role of functional settings and supplementary regularity assumptions to be satisfied by the fluid fields and/or the data. In this regard, of fundamental importance is, in particular, the result obtained by Ladyzhenskaya [19], who was able to prove the uniqueness of the fluid fields $A(r,t)$, assumed to exist as strong solutions belonging to the SB setting, by imposing the additional regularity constraint that the velocity field remains bounded in $L^4(\Omega)$. An analogous result was obtained later for strong solutions in $L^p(\Omega)$, with $p \geq 2$, (Kato and Ponce, 1984 [26]; Sohr and von Wahl, 1986 [28]; Deuring and von Wahl, 1995 [30]) and has been addressed by several authors also for weak solutions (see, for example, Iftimie, 1999 [32]; Koch and Tataru, 2001 [35], Galdi, 2002 [37]). However, these results fail, in general, in the case of an external domain $\Omega$, for the same reason indicated above. Therefore, a fundamental problem, as yet unanswered, both for weak and strong solutions, is the search of the minimal functional setting for the existence of local and respectively global solutions in exterior domains (Galdi and Maremonti, 1986 [27]) and which do not decay at infinity in $\Omega$. Additional issues concern both uniqueness (J.Kato, 2003 [38]) and existence (Giga et al.,1999 [33]; Giga et al. [34]).
D. Goal of the paper

Goal of the paper is to address the problem of strong solvability of the incompressible Navier-Stokes equations (INSE) in $\Omega \times I$. $\Omega$ is here identified with a subset of the Euclidean space $\mathbb{R}^3$, formed at most by a finite number of disjoint, open and connected subsets $\Omega_i$ of non-vanishing, and possibly non-finite measure (fluid subdomains); as a particular case, $\Omega$ ($\Omega_i$) can also be identified with an unbounded subdomain of $\mathbb{R}^3$ (external domain). Moreover, $I$ is subset of $R$, assumed either bounded or unbounded. In the following we intend to seek the minimal functional setting for the well-posedness of local strong solutions which, together with suitable derivatives of the fluid fields, in the case in which $\Omega$ is an external domain, do not necessarily decay at infinity in $\Omega$ (nondecaying strong solutions) and hence do not belong to Sobolev spaces. More precisely, this involves the search of appropriate minimal functional classes for the NS fluid fields $V(r,t), p(r,t)$, assuring the validity of a local existence and uniqueness theorem for the initial-boundary value problem for NS equations.

The treatment here developed is based on the IKT earlier developed by Tessarotto and Ellero [3, 4, 5], which relies on the construction of a suitable kinetic equation and providing exactly, as its moments equations, the required set of fluid equations. The IKT here adopted is shown to provide the basis for a local existence and uniqueness theorem for strong solutions of the NS equations.

Key features of the present treatment are as follows:

1) The fluid domain $\Omega$ is a three-dimensional subset of $\mathbb{R}^3$ which is not necessarily bounded, i.e., it can be an external domain.

2) In the case in which $\Omega$ is unbounded, the fluid fields, and in particular the fluid velocity $V(r,t)$, are not required to be $L^p(\Omega)$ in the fluid domain $\Omega$, with $p \geq 2$. Hence, the fluid fields $A(r,t) = \{\rho, V, p\}$, together with their derivatives $\frac{\partial}{\partial t} A(r,t), \nabla A(r,t), \nabla^2 A(r,t)$ and the volume force density $f(r,t)$, are permitted to be nondecaying, i.e., to admit non-vanishing asymptotic values in $\Omega$. Thus, denoting $\hat{e}_r$ the unit vector $\hat{e}_r = r/r$ and $B \equiv \{A(\hat{e}_r, t), \frac{\partial}{\partial t} A, \nabla A, \nabla^2 A, f\}$, it must be such that

$$\begin{cases} 
\lim_{r \in \Omega, |r| \to \infty} B(r,t) = B_\infty(\hat{e}_r, t), \\
0 \leq |B_\infty(\hat{e}_r, t)| < \infty ,
\end{cases}$$

(20)

where the limits $B_\infty \equiv \{A_\infty(\hat{e}_r, t), \frac{\partial}{\partial t} A_\infty, \nabla A_\infty, \nabla^2 A_\infty, f_\infty\}$ are allowed to be non-zero.
3) The fluid pressure \( p(\mathbf{r}, t) \) is assumed non-negative in the closure domain \( \overline{\Omega} \), i.e., \( p \geq 0 \), thus permitting the existence of *vacuum regions*, i.e., subdomains of \( \mathbb{R}^3 \) in which \( \rho \) and \( p \) vanish identically, and corresponding *free boundaries*, i.e., the parts of (the border of) \( \overline{\Omega} \) on which the fluid pressure vanishes identically.

4) The boundary set \( \partial \Omega \) may generally be formed by piece-wise regular surfaces, curves, as well as isolated points of \( \Omega \subseteq \mathbb{R}^3 \).

5) In validity of assumptions 1)-3), a local existence and uniqueness theorem for strong solutions of the initial-boundary value problem for INSE is found by assuming that the fluid fields \( \{\rho, \mathbf{V}, p\} \) and the volume force density acting on the fluid in \( \Omega \times I \) satisfy stronger requirements than those provided by the NF setting and \( \text{(5, 6)} \) which are given by the following assumptions

\[
\begin{align*}
\{\rho, \mathbf{V}, p, \Pi, Q, f(\mathbf{r}, t)\} &\in C^{(0)}(\overline{\Omega} \times I), \\
\mathbf{V}(\mathbf{r}, t) &\in C^{(3,2)}(\Omega \times I), \\
\Pi, Q, p(\mathbf{r}, t) &\in C^{(2,2)}(\Omega \times I), \\
f(\mathbf{r}, t) &\in C^{(2,0)}(\Omega \times I),
\end{align*}
\]

while

\[
\begin{align*}
\mathbf{V}_o(\mathbf{r}) &\in C^{(3)}(\Omega), \\
\Pi, Q, p_o(\mathbf{r}) &\in C^{(2)}(\Omega), \\
f(\mathbf{r}, t_o) &\in C^{(2)}(\Omega).
\end{align*}
\]

Moreover, the velocity \( \mathbf{V}_w(\mathbf{r}_W(t), t) \) of each point, \( \mathbf{r}_W(t) \), of the boundary \( \partial \Omega \) is assumed at least of class

\[
\mathbf{V}_w(\mathbf{r}_W(t), t) \in C^{(2)}(I).
\]

Here \( \Pi(\mathbf{r}, t), Q(\mathbf{r}, t) \) and \( \Pi_o(\mathbf{r}), Q_o(\mathbf{r}) \) denote auxiliary fluid fields to be suitably defined. Furthermore, in the case in which \( \Omega \) is unbounded, the fluid fields \( A(\mathbf{r}, t) = \{\rho, \mathbf{V}, p, \Pi, Q\} \) and the volume force density \( f(\mathbf{r}, t) \), together with their derivatives \( \frac{\partial}{\partial t} A(\mathbf{r}, t), \nabla A(\mathbf{r}, t), \nabla^2 A(\mathbf{r}, t) \), are permitted to be non-decaying, i.e., to admit non-vanishing asymptotic values in \( \Omega \). Thus, denoting \( \hat{e}_r \) the unit vector \( \hat{e}_r = \mathbf{r}/r \) and \( B \equiv \{A(\hat{e}_r), \frac{\partial}{\partial t} A, \nabla A, \nabla^2 A, f\} \), it must be

\[
\begin{align*}
\lim_{r \in \Omega, |r| \to \infty} B(\mathbf{r}, t) &= B_{\infty}(\hat{e}_r, t), \\
0 &\leq |B_{\infty}(\hat{e}_r, t)| < \infty,
\end{align*}
\]

where the limits \( B_{\infty} \equiv \{A_{\infty}(\hat{e}_r, t), \frac{\partial}{\partial t} A_{\infty}, \nabla A_{\infty}, \nabla^2 A_{\infty}, f_{\infty}\} \) are allowed to be non-zero. Assumptions \( \text{(21)-(24)} \) are denoted as *minimal functional setting* (MF setting).
E. Scheme of presentation

The plan of the paper is as follows. First (in Sec.2), the IKT developed by Tessarotto and Ellero [3, 4, 5] is recalled and its basic assumptions are pointed out. Basic consequences of the kinetic theory are analyzed in the subsequent sections 3 and 4. In particular, in Sec.3 the Navier-Stokes dynamical system, defined by the integral curves of the kinetic equation, is introduced and its conditions of existence, uniqueness and regularity are investigated. In Sec.4, the problem of existence, uniqueness and regularity of the kinetic distribution function, for suitable initial and boundary conditions is analyzed, with particular reference to Maxwellian solutions. This permits us to obtain a theorem of existence and uniqueness of strong solutions of INSE in the MF setting.

II. CONSTRUCTION OF IKT FOR INSE

A remarkable aspect of fluid dynamics is related to the construction of inverse kinetic theories (IKT) for hydrodynamic equations in which the fluid fields are identified with suitable moments of an appropriate kinetic probability distribution. Recently the topic has been the subject of theoretical investigations on the incompressible Navier-Stokes (NS) equations (INSE) [3, 4, 5]. The importance of the IKT-approach goes beyond the academic interest. In fact, fluid equations represent usually a mixture of hyperbolic and elliptic PDE’s, which are extremely hard to study both analytically and numerically. As such, their investigation represents a challenge both for mathematical analysis and for computational fluid dynamics. For this reason in the past alternative approaches, based on asymptotic kinetic theories, have been devised which permit to advance in time the fluid fields, to be determined in terms of suitable moments of an appropriate kinetic distribution function. These methods, which approximate the fluid equations only in an asymptotic sense, are based on the introduction of suitably modified (fluid) equations which permit to advance in time the fluid fields only in an approximate sense. In particular, typically, their modified fluid equations actually describe weakly-compressible fluids. The discovery of IKT [1] provides, however, a new starting point for the theoretical and numerical investigation of hydrodynamic equations, since it does not require any modification of the exact fluid equations. In particular it holds for strong solutions, and permits to advance in time exactly the fluid fields by means of a
suitable kinetic distribution function \( f(x,t) \). Here \( x \) is the state vector \( x = (r, v) \), where respectively \( r \) and \( v \) denote the corresponding "configuration" and "velocity" vectors, and \( \Gamma \) is the phase-space spanned by \( x \). In the following we shall assume that \( \Gamma \) is a phase-space of dimension 6. This is achieved introducing a phase-space classical dynamical system (here denoted \textit{NS dynamical system})

\[
x_o \rightarrow x(t) = T_{t,t_0}x_o,
\]

which uniquely advances in time the fluid fields by means of an appropriate evolution operator \( T_{t,t_0} \). This is assumed to be generated by a suitably smooth vector field \( X(x,t) \),

\[
\frac{d}{dt}x = X(x,t),
\]

\[
x(t_o) = x_o,
\]

where \( x = (r, v) \in \Gamma = \Omega \times \mathbb{R}^3 \) is a suitable state vector, \( r \) and \( v \) denoting suitable position and velocity vectors spanning respectively the configuration and velocity spaces \( \Omega \) and \( \mathbb{R}^3 \), and finally the set of points \( x(t) \) (for \( t \in I \)) defines the (phase-space) Lagrangian trajectory of the NS dynamical system \( \textit{(NS Lagrangian trajectory)} \). Therefore, introducing the corresponding pdf (probability distribution function) \( f(x,t) \geq 0 \), it fulfills necessarily in \( \Gamma \) the differential Liouville equation

\[
Lf(x,t) = 0,
\]

where \( L \) is the Liouville streaming operator

\[
Lf = \frac{\partial}{\partial t} f + \frac{\partial}{\partial x} \cdot \{X(x,t) f\}.
\]

Eq. (28) can be interpreted as a Lagrangian \textit{inverse kinetic equation}. The corresponding equivalent \textit{Lagrangian form} reads

\[
J(x(t),t)f(x(t),t) = f(x_o,t_o) \equiv f_o(x_o),
\]

where \( f(x(t),t) \) is the Lagrangian representation of the pdf, \( x(t) \) is the solution of the initial-value problem (26)-(27), \( f_o(x_o) \) is a suitably smooth initial pdf and

\[
J(x(t),t) = \left| \frac{\partial x(t)}{\partial x_o} \right|
\]

is the Jacobian of the map \( x_o \rightarrow x(t) \).
The vector field $X(x,t)$ is in principle completely arbitrary. Therefore it can be defined in such a way that the inverse kinetic equation (28) satisfies an appropriate set of constraint equations and in particular so that it admits as a particular solution the local Maxwellian distribution

$$f_M(x,t) = \frac{\rho_o}{\pi^{3/2}v_{th}^3} \exp \left\{ -\frac{u^2}{v_{th}^2} \right\}, \quad (32)$$

where $u \equiv v - V(r,t)$ is the relative velocity and $v_{th}(r,t)$ denotes the thermal velocity defined in terms of the kinetic scalar pressure $p_1(r,t)$ (see below):

$$v_{th} \equiv \sqrt{\frac{2p_1(r,t)}{\rho_o}}. \quad (33)$$

As proven in Ref. [3], thanks to the arbitrariness in the definition of the vector field $f(x,t)$, and in the definition of the velocity moments of $f$, this permits us to construct an inverse kinetic theory for INSE. Hence, it follows in particular that:

1. For prescribed initial pdf $f_o(x_o)$ the time-evolved pdf $f(x(t),t)$, solution of the inverse kinetic equation (28), is uniquely determined by the NS dynamical system (25). Thus, if the NS dynamical system, solution of the initial-value-problem (26)-(27), exists then necessarily $f(x(t),t)$ defined by Eq.(30) is a solution of the inverse kinetic equation (28);

2. For a prescribed choice of the vector field $X(x,t)$, Eq.(28) is an inverse kinetic equation for INSE, i.e., the fluid equations which define INSE - namely Eqs.(1)-(4) - are provided by a suitable set of velocity-moment equations of (28). Therefore, if $f(x,t)$ is solution of the inverse kinetic equation (28), necessarily a suitable subset of its velocity moments coincide with the fluid fields $\{\rho, V, p\}$, which are solution of INSE. As a consequence, the corresponding velocity moments of Eq.(28) must coincide identically with INSE.

A. Assumptions of IKT

The IKT approach for INSE and the corresponding NS dynamical system can be obtained in a straightforward way following the approach of Refs. [2, 3]. For this purpose, let us require that:
• The pdf \( f(x,t) \) is summable in velocity space, in the sense that the velocity moments
\[
F_G(x, t) = \int_V d^3v G(x, t) f(x, t)
\] (34)
exist in the closure domain \( \overline{\Omega} \times I \) and result suitably smooth in \( \Omega \times I \) at least for the weight functions \( G(x, t) = 1, v, E \equiv \frac{1}{3} u^2, uu, Eu \), where \( u \equiv v - V(r, t) \) denotes the relative velocity (with respect to the fluid velocity).

• The fluid fields \( \{\rho, V, p\} \) coincide with the moments:
\[
\begin{cases}
\rho(r, t) = \int_V d^3v f(x, t) = \rho_o > 0, \\
V(r, t) = \frac{1}{\rho} \int_V d^3v v f(x, t), \\
p(r, t) = p_1(r, t) - P_o, \\
p_1(r, t) = \int_V d^3v \frac{1}{3} u^2 f(x, t),
\end{cases}
\] (35)

being \( P_o \) a positive constant and \( p_1(r, t) \) denoting the kinetic pressure which is defined so that
\[
p_1(r, t) \geq P_o,
\] (36)

and hence it results \( p(r, t) > 0 \) in \( \Omega \times I \). In addition we introduce the higher-order moments of the pdf:
\[
Q = \int_V d^3v uf, 
\] (37)
\[
\Pi = \int_V d^3v uu f.
\] (38)

This implies that, by construction, the physical realizability conditions (5) and (6) are identically satisfied. Furthermore, one can prove that INSE are satisfied by suitably selecting the vector field \( X(x, t) \), i.e., by requiring that in the domain \( \Omega \times I \) the moment equations
\[
\int_V d^3v G(x, t) Lf(x, t) = 0,
\] (39)
corresponding to the first three moments, i.e., \( G(x, t) = 1, v, \frac{1}{3} u^2 \), coincide with the equations of INSE.

• Let us impose suitable kinetic initial conditions on the pdf \( f(x,t) \). The initial conditions are manifestly of the form
\[
f(x, t_0) = f_o(x),
\] (40)
where the initial pdf $f_o(x)$ is assumed to satisfy the initial conditions (7), (8), (9) for the fluid fields, which requires

$$\begin{align*}
\rho_o &= \int_V d^3v f_o(x), \\
V_o(r) &= \frac{1}{\rho_o} \int_V d^3v v f_o(x), \\
p_o(r) &= \int_V d^3v \frac{1}{3} u^2 f_o(x) - P_o,
\end{align*}$$

where $u = v - V_o(r)$ and $P_o > 0$ is an arbitrary initial constant. Instead, the initial moments

$$\begin{align*}
Q_o &= \int_V d^3v u E f_o(x), \\
\Pi_o &= \int_V d^3v uu f_o(x),
\end{align*}$$

are arbitrary, so that - for example - it is always possible to impose that they vanish identically in $\Omega$ [by suitably defining the initial kinetic distribution function $f_o(x)$].

B. Additional assumptions - Extension of the NS Lagrangian trajectories on $\partial \Omega$

An arbitrary Lagrangian trajectory defined by the NS dynamical system can generally reach the boundary $\partial \Omega$, so that the solution of the initial value problem (26)-(27) $x(t) = \{r(t), v(t)\}$ generally may not be defined in the whole existence domain of the NS fluid fields ($I$). This requires its extension on the boundary $\partial \Omega$. The result is achieved by defining suitable boundary conditions for the NS dynamical system, which must apply for an arbitrary phase-space trajectory $x(t)$ of the same dynamical system. Certain restrictions must be placed on the possible motion of the boundary. This is due both to the assumed regularity of the fluid fields [i.e., the settings (21) and (22)] and the no-slip conditions to be imposed on the fluid velocity due to the Dirichlet boundary conditions (11). In particular, we shall require that:

- If $r_W$ is an arbitrary point of $\partial \Omega$, its velocity, defined as $V_W(t) \equiv V_W(r_W(t), t) = \frac{d}{dt}r_W(t)$, is by assumption a smooth real function of time in the sense (23).

Nevertheless, the precise nature of $\partial \Omega$ (i.e., if it is a surface, curve or an isolated point, remains in principle largely arbitrary. Thus, for example, $\partial \Omega$ may be assumed as formed by piece-wise surfaces or curves, as well as isolated points, of $\mathbb{R}^3$. The definition of the
boundary conditions for the NS dynamical system can, in fact, be achieved in all such cases.

For this purpose, let us consider an arbitrary Lagrangian trajectory \( x(t) \equiv \{ r(t), v(t) \} \) which at time \( t = t_c \) reaches the boundary \( \partial \Omega \) at the position \( r_W(t_c) \) with nonvanishing relative velocity, i.e., is such that
\[
\begin{align*}
  r(t_c) &= r_W(t_c), \\
  \lim_{t \to t_c^-} |v(t) - V_W(t)| &> 0.
\end{align*}
\] (44) (45)

Then, introducing the unit vector \( n_w(t_c) \) defined so that
\[
\begin{align*}
  n_w(t_c) &= \lim_{t \to t_c^-} \frac{r_W(t) - r(t)}{|r_W(t) - r(t)|},
\end{align*}
\] (46)

let us denote by \( x^{(-)}(t_c) \) and \( x^{(+)}(t_c) \), respectively, the incoming and outgoing Lagrangian trajectories, which are defined as:
\[
\begin{align*}
  x^{(\pm)}(t_c) &= \lim_{t \to t_c^{(\pm)}} x(t).
\end{align*}
\] (47)

The boundary conditions for the NS dynamical system are obtained by imposing the bounce-back boundary conditions
\[
\begin{align*}
  r^{(+)}(t_c) &= r^{(-)}(t_c), \\
  v^{(+)}(t_c) - V_W(t_c) &= -[v^{(-)}(t_c) - V_W(t_c)].
\end{align*}
\] (48) (49)

C. Kinetic boundary conditions

To complete the set of assumptions required by IKT, appropriate kinetic boundary conditions must be defined for \( f(x, t) \). Consistent with Eqs.(48) and (49), they are achieved requiring that \( f(x, t) \) satisfies on \( \partial \Gamma \) the following constraints (A-C):

- A) kinetic bounce-back condition: this is obtained by imposing the conservation of probability density at \( t_c \), i.e.,
\[
\begin{align*}
  f^{(+)}(r_W(t_c), v^{(+)}(t_c), t_c) &= f^{(-)}(r_W(t_c), 2V_W(t) - v^{(-)}(t), t_c),
\end{align*}
\] (50)

where \( V_W(t_c) \equiv V_W(r_W(t_c), t_c) \) denotes again the velocity of the boundary \( \partial \Omega \) at the position \( r_W(t_c) \);
B) *first fluid constraint:* it is provided by the requirement that the first moment of the pdf yields the mass density, i.e., there results
\[
\rho = \int_V d^3v f(r_W(t_c), v, t_c). \tag{51}
\]

C) *second fluid constraint:* it requires that, consistent with the Dirichlet boundary condition for the fluid pressure [provided by Eqs.(11)], on \(\partial \Omega\) the kinetic pressure satisfies the constraint
\[
p_1(r_W, t) - P_o = p_W(r_W, t), \tag{52}
\]
where \(p_W(r_W, t)\) and \(V_W(r_W, t_c)\) are prescribed in accordance with Eqs.(11) and \(p_1(r_W, t)\), due to the position \(35\), reads
\[
p_1(r_W, t_c) = \int_V d^3v \frac{1}{3} \{v - V_W(r_W(t_c), t_c)\}^2 f(r_W(t_c), v, t_c). \tag{53}
\]

D. Determination of the NS vector field

Provided the fluid fields \(\{\rho, V, P, \Pi, Q\}\) and the volume force density \(f(r, V, t)\) are continuous in \(\Omega \times I\) and suitably smooth, the vector field \(X(x,t)\) is found \([2, 3]\) to be of the form \(X(x,t) \equiv \{v, F\}\). Here the vector field \(F \equiv F(x,t; f)\) can be written as
\[
F(x,t; f) = F_0(x,t; f) + F_1(x,t; f), \tag{54}
\]
where the two vector fields \(F_0(x,t; f)\) and \(F_1(x,t; f)\), which depend functionally on the kinetic distribution \(f(x,t)\) (via the moments \(\{\rho, V, P, \Pi, Q\}\)), are defined respectively as:
\[
F_0(x,t; f) = \frac{1}{\rho} \nabla \cdot \Pi - \nabla p_1(r,t) + f + u \cdot \nabla V + \nu \nabla^2 V, \tag{55}
\]
\[
F_1(x,t; f) = \frac{1}{2} u \left\{ \frac{\partial}{\partial t} \ln p_1 - \frac{1}{p_1} V \cdot \left[ \rho \frac{\partial}{\partial t} V + \rho V \cdot \nabla V - f - \mu \nabla^2 V \right] + \frac{1}{p_1} \nabla \cdot Q - \frac{1}{2p_1} \left[ \nabla \Pi \right] \cdot Q \right\} + \frac{v_{th}^2}{2p_1} \frac{u^2}{v_{th}^2 - 3/2}. \tag{56}
\]
Then the following theorem (proven in Ref. [3]) has the flavor of:

**THM. 1 - IKT for INSE**

Let us assume that:

1) the fluid fields \( \{ \rho, V, p, \Pi, Q \} \) and the volume force \( f(r,t) \) belong to the functional class defined by the MF setting \((21)-(24)\);

2) the pdf \( f(x,t) \) is strictly positive and is a particular solution of IKE \([\text{Eq. (28)}]\) in \( \Omega \times I \);

3) the velocity moments of \( f(x,t) \) are defined by Eqs. \((35), (42) \) and \((43)\);

4) the pdf \( f(x,t) \) satisfies the initial conditions \((40) \) and \((41)\);

5) the pdf \( f(x,t) \) satisfies on \( \partial \Omega \) the kinetic boundary conditions \((50), (51) \) and \((52)\).

Then it follows that

\[ T_{11} \] \( \{ \rho, V, p \} \) are solutions of INSE in \( \Omega \times I \);

\[ T_{12} \] if \( f(x,t) \) satisfies in the whole phase-space \( \Gamma = \Omega \times I \) the initial conditions \( f(x,t_0) = f_M(x,t_0) \), and on \( \Omega \delta \times I \) the boundary conditions

\[ f^+(r_W(t_c),v,t) = f^-(r_W, 2V_W(t) - v,t), \quad (57) \]

then it follows that \( f(x,t) \) is solution of IKE \([\text{Eq. (28)}]\) in the whole set \( \Gamma = \Omega \times I \).

\[ T_{13} \] if \( f(x,t) \) is a particular solution of IKE, then \( \{ \rho, V, p \} \) are solutions of INSE in \( \Omega \times I \).

**III. THE NAVIER-STOKES DYNAMICAL SYSTEM: A THEOREM OF LOCAL EXISTENCE AND UNIQUENESS**

Let us now analyze the existence, uniqueness and regularity of the IKT. In the following sections we intend to address in particular the well-posedness of the initial-value problem \((26)-(27)\) which defines the Navier-Stokes dynamical system.

It is immediate to prove that under suitable smoothness assumptions on \( \{ \rho, V, p, \Pi, Q \}_{(r,t)} \), the problem \((26)-(27)\) admits a local existence and uniqueness theorem and therefore defines a dynamical system \( S_{NS} \), to be denoted as *Navier-Stokes dynamical system*.

**THM. 2 - Local existence, uniqueness and regularity of the NS dynamical system**
Let us require that the fluid fields \( \{ \rho, \mathbf{v}, p, \Pi, Q \} \) and the volume force density \( f(r,t) \) are bounded in \( \bar{\Omega} \times I \) and belong to the MF setting (21)-(24).

Then it follows that:

a) the vector field \( \mathbf{X}(x,t) = [\mathbf{v}, \mathbf{F}(x,t)] \) is of class \( C^{(1,1)}(\Gamma \times I) \);

b) the solution of the problem (20)-(27) exists locally and is unique:

\[
\mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t) \equiv \{ \mathbf{r}(t), \mathbf{v}(t) \} \equiv \{ \chi_r(\mathbf{x}_o, t_o, t), \chi_v(\mathbf{x}_o, t_o, t) \}.
\]

Moreover:

c) \( \mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t) \) and its inverse, \( \mathbf{x}_o = \chi(\mathbf{x}(t), t, t_o) \) are at least:

\[
\chi(\mathbf{x}_o, t_o, t), \chi(\mathbf{x}(t), t, t_o) \in C^{(2)}(\Gamma \times I \times I);
\] (58)

d) the Jacobian \( J(\mathbf{x}(t), t) \) of the flow

\[
\mathbf{x}_o \rightarrow \mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t)
\] (59)

is at least of class

\[
J(\mathbf{x}(t), t) \in C^{(1)}(\Gamma \times I)
\] (60)

and for all \( \mathbf{x}_o \in \Gamma \) and for all \( t_o, t \in I \) is non-vanishing and finite, i.e.,

\[
J(\mathbf{x}(t), t) \neq 0, \infty.
\] (61)

Proof - In validity of the regularity assumption the vector field \( \mathbf{X}(x,t) \) is manifestly of class \( C^{(1,1)}(\Gamma \times I) \). Hence, thanks to the fundamental theorem of existence and uniqueness (see for example, Coddington and Levinson, 1955 [14], Hirsch and Smale, 1974 [22]) the regularity conditions (58) are implied. The additional result (60) follows directly from Eq. (31) and the definition given above for the vector field \( \mathbf{F}(x,t) \) [Eq. (54)]. Moreover, thanks to Liouville theorem the Jacobian \( J(\mathbf{x}(t), t) \) does not vanish or diverge for all \( \mathbf{x}_o \in \Gamma \) and for all \( t_o, t \in I \). Q.E.D.

An immediate consequence of THM.2 is the following

**COROLLARY 1 of THM.2 - Minimal functional setting of the fluid fields**

If the fluid fields obey the MF setting (21)-(24) there follows in particular that:
a) the NS vector field $\mathbf{X}(\mathbf{x},t) = [\mathbf{v}, \mathbf{F}(\mathbf{x},t)]$ results of class $C^{(1,\infty,1)}(\Omega \times V \times I)$;

b) the problem (26)-(27) admits one and only one solution $\mathbf{x}(t)$ of class $C^{(1,2,2)}(\Gamma \times I \times I)$;

c) the Jacobian $J(\mathbf{x}(t),t)$ of the phase-flow is at least $C^{(1,2)}(\Gamma \times I)$.

**Proof** - The result follows from THM's 1 and 2 and the assumption of the MF setting for the fluid fields. Q.E.D.

From THM.2 there follows the obvious further corollary:

**COROLLARY 2 of THM.2** - Extension of the solution $\mathbf{x}(t)$ on $\partial\Omega$

The solution $\mathbf{x}(t) = \chi(\mathbf{x}_o,t_o,t)$ of the initial value problem Eq.(26)-(27), prolonged on $\partial\Omega$ by means of the bounce-back boundary condition (50), exists and is unique.

**Proof** - The result is an immediate consequence of THM.1, the assumption of validity of the MF setting and the boundary conditions previously introduced (see Sec.2.3) for the NS dynamical system. Q.E.D.

**IV. THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE KINETIC PDF AND INSE: EXISTENCE, UNIQUENESS AND REGULARITY**

The previous theorem of local existence and uniqueness for the NS dynamical system, and its extension given by the 2nd Corollary of THM.2, can now be used to obtain a local theorem of existence, uniqueness and regularity for the kinetic distribution function $f(\mathbf{x},t)$. Due to its arbitrariness, it is always possible to limit ourselves to the investigation of initial conditions of the form

$$f(\mathbf{x},t_o) = f_M(\mathbf{x},t_o). \quad (62)$$

In view of the positions (54),(55),(56) one can prove (see below) that this implies identically $f(\mathbf{x},t) \equiv f_M(\mathbf{x},t)$. We stress that a more general result holds for an arbitrary (but suitably smooth and summable) distribution $f(\mathbf{x},t)$ (see related discussion in Ref.[3]). The choice (62) warrants summability and existence of all the required velocity moments and in addition is consistent with the physical requirement set by the PEM (principle of entropy maximization; see Refs.[6,7]). Then, the following theorem holds:

**THM. 3 - Local existence and uniqueness of the kinetic pdf**

*Let us require that:*
1) the initial kinetic pdf $f_0(x)$ coincides with the Maxwellian kinetic pdf $f_M(x,t_o)$;
2) the fluid fields $\{\rho, V, p\}_{(r,t)}$ and the volume force density $f(r,t)$ belong to the MF setting defined by Eqs. (21)-(24);
3) the fluid fields satisfy the fluid initial and boundary conditions (7)-(9) and (11);
4) for all $(x,t) \in \Gamma \times I$ the kinetic pdf $f(x,t)$, if it exists, satisfies the bounce-back condition and the conditions on pressure (50), (51) and (52);

It follows that in the domain $\Gamma \times I$:

a) $f_M(x,t)$ is a particular solution of IKE [Eq.(28)];
b) $f_M(x,t)$ is differentiable;
c) $f_M(x,t)$ is summable;
d) the velocity moments of $f_M(x,t)$, corresponding to $G(x,t) = 1, v.E \equiv \frac{1}{3}u^2$, coincide with the fluid fields

$$\{\rho, V, p_1 \equiv p + P_o\}_{(r,t)};$$

(63)

and moreover:
e) $f_M(x,t)$ is defined and summable also on the boundary set $\partial \Gamma \times I$, where $\partial \Gamma = \partial \Omega \times V$.

Proof - The proof of a) follows by direct substitution of the position $f(x,t) = f_M(x,t)$ in Eq.(28). In addition, Eq.(30), shows that this solution necessarily corresponds only to the initial condition (62), i.e., there results necessarily in $\Gamma \times I$:

$$f_M(x(t), t) = \frac{f_M(x_o, t_o)}{J(x(t), t)}.$$  

(64)

b) Thanks to assumptions 1)-4), $f_M(x,t)$ is manifestly differentiable in $\Gamma \times I$. c) Similarly, $f_M(x,t)$ is manifestly summable and d) its moments corresponding to $G(x,t) = 1, v.E \equiv \frac{1}{3}u^2$ are by definition $\{\rho, V, p_1 \equiv p + P_o\}_{(r,t)}$. d) Finally, thanks to Corollary 2 of THM.2, $f_M(x,t)$ is also manifestly defined on the boundary $\partial \Omega$. Q.E.D.

THM.3 already contains in itself the basic ingredients required to reach the theorem of existence and uniqueness for INSE, which can be written as:

**THM. 4 - Local existence and uniqueness of the NS fluid fields**

In validity of THM.3 it follows that in the domain $\Omega \times I$, the fluid fields $\{\rho, V, p\}_{(r,t)}$ are necessarily strong solutions of the initial-boundary value problem of INSE.

Proof - In fact, thanks to THM.1 and 3 it follows that $f_M(x,t)$ is a particular solution of IKE [Eq.(28)] if and only if $\{\rho, V, p\}_{(r,t)}$ are necessarily strong solutions of the initial-
boundary value problem of INSE which belong to the MF setting. As a consequence the
moments of $f_M(x,t), \{\rho, V, p_1\}_{(r,t)}$ obey, by construction, INSE and the associated initial and
boundary conditions, set by prescribing $f_M(x,t_0)$ and respectively Eqs.(50), (51) and (52).
Q.E.D.

We remark that, in the case $\Omega$ is an unbounded domain, thanks to assumptions (24), in
principle no restriction is placed on the asymptotic behavior of the fluid fields for $|r| \to \infty$
at time $t$. Hence, in contrast to the customary approach [19, 20, 24, 25], these solutions do
not necessarily belong to Sobolev spaces.

THM’s. 3 and 4 contain the main contribution of the paper and the basic new results
regarding the existence, uniqueness and regularity of nondecaying strong solutions of INSE.
Thus, provided the fluid fields $\{\rho, V, p\}_{(r,t)}$ satisfy the assumptions of the MF setting in
the set $\Omega \times I$, together with the initial and boundary conditions, defined respectively by
Eqs.(7)-(9) and (11), the solution of INSE exists and is unique.

It is possible to show that the regularity assumptions on the fluid pressure are minimal
in the context of the present approach. For example, if the fluid pressure is assumed to
be unbounded from below, the inverse kinetic approach manifestly fails, since it cannot be
related to the kinetic pressure in this case. In fact, the Maxwellian distribution is defined
and results summable in velocity space (in the sense indicated above) only if the kinetic
pressure is strictly positive $p_1(r,t) > 0$. Moreover, if the partial time derivative $\frac{\partial p}{\partial t}$ is assumed
unbounded or discontinuous, the NS dynamical system cannot be defined any more, since
the same NS vector field ceases to be at least continuous.

The present result generalizes also the treatment of uniqueness of nondecaying strong
solutions recently given by J. Kato [38]. The following remark is relevant in this context.
First, there is a one-to-one correspondence between fluid fields and the Maxwellian distri-
bution $f_M(x,t)$, while the uniqueness of $f_M(x,t)$ and of the phase-flow (43) imply each
other. Both are determined, in turn, by the solution of the initial-value problem (26)-(27)
and therefore by the value of the Jacobian $J(x(t),t)$. Therefore, the uniqueness of strong
solutions is intrinsically related to their existence. In fact, $J(x(t),t)$ is defined and finite if
and only if the fluid fields belong to the MF setting, and in particular the kinetic pressure
$p_1(r,t)$ results strictly positive, while - at the same time - the fluid fields are finite, i.e.,
$p_1(r,t) < \infty$ and $|V(r,t)| < \infty$. 

22
V. DISCUSSION AND CONCLUSIONS

In this paper the problem of existence of strong solutions of the initial-boundary value problem of INSE has been addressed in the context of the IKT earlier developed \[1, 2, 3, 4, 5\]. The striking new features of present approach is that the fluid fields, and in particular the fluid velocity, do not necessarily decay at infinity in \( \Omega \), hence the solution is not embedded in Sobolev spaces.

The proof of existence, uniqueness and regularity of strong solutions of INSE is reached in two steps. The first one is based on the introduction of the NS vector fields \( \mathbf{X}(\mathbf{x},t) \), suitably related to the fluid fields \( \{ \rho, \mathbf{V}, p \} \), and by establishing the conditions of existence of its associated dynamical system, the NS dynamical system. The second step consists in the introduction of a suitable inverse kinetic equation and a related initial-boundary value problem, for which a theorem of existence, uniqueness and regularity can be reached.

As a consequence, INSE are found to be expressed in terms of suitable velocity moments of the kinetic equation, whereas the initial and boundary conditions for the fluid fields are satisfied identically by proper definition of the initial and boundary conditions for the kinetic distribution function.

A discussion on consequences and applications of IKT in fluid dynamics can be found in Refs.\[6, 7\] and \[39\].

Here we mention that by means of the IKT here adopted, the fluid pressure \( p(\mathbf{r},t) \) can be advanced in time self-consistently without solving explicitly the Poisson equation, since it can be determined directly as a moment of the kinetic distribution function by advancing in time the initial kinetic distribution function \( f(\mathbf{x},t_0) \) in terms of NS dynamical system \( (25) \) and IKE \( (28) \). As a fundamental result, an exact pressure evolution equation, advancing in time self-consistently the fluid pressure, can actually be achieved \[40\].

This feature is potentially relevant for the development of numerical solution methods for INSE based on the discretization of the kinetic distribution in phase space and yields a possible alternative to direct solution methods based on the solution of Poisson equation for the fluid pressure \( p \) \[3\] (see also for example, \[41, 42, 43, 44\]).
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