Density profiles of the exclusive queuing process

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Received 3 October 2012
Accepted 12 November 2012
Published 6 December 2012

Online at stacks.iop.org/JSTAT/2012/P12004
doi:10.1088/1742-5468/2012/12/P12004

Abstract. The exclusive queuing process (EQP) incorporates the exclusion principle into classic queuing models. It is characterized by, in addition to the entrance probability \(\alpha\) and exit probability \(\beta\), a third parameter: the hopping probability \(p\). The EQP can be interpreted as an exclusion process of variable system length. Its phase diagram in the parameter space \((\alpha, \beta)\) is divided into a convergent phase and a divergent phase by a critical line which consists of a curved part and a straight part. Here we extend previous studies of this phase diagram. We identify subphases in the divergent phase, which can be distinguished by means of the shape of the density profile, and determine the velocity of the system length growth. This is done for EQPs with different update rules (parallel, backward sequential and continuous time). We also investigate the dynamics of the system length and the number of customers on the critical line. They are diffusive or subdiffusive with non-universal exponents that also depend on the update rules.

Keywords: phase diagrams (theory), driven diffusive systems (theory), stochastic particle dynamics (theory), traffic models
1. Introduction

Queuing theory is one of the most important topics in the field of operations research [1]–[3]. It has a broad spectrum of applications ranging from telecommunications to traffic engineering and supply chains. One of the simplest queuing processes is the so-called M/M/1 model, where customers enter the system with probability α and leave the system with probability β at one server. The current state of the M/M/1 queuing process is completely specified by the number of customers. The system converges to a stationary state with a finite number of customers when α < β, whereas the number of waiting customers diverges for α > β.

A feature which seems to be important for pedestrian queues and other traffic applications is the excluded-volume effect: pedestrians can proceed only when there is enough space in front of them [4]. This is seen e.g. in queues at the check-in at airports where passengers have to move the luggage when moving forward. However, standard queuing models like the M/M/1 model neglect the excluded-volume effect, and do not have a spatial structure. Then the length $L$ of the system is given by the number of
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Figure 1. Exclusive queuing process.

waiting customers \( N \) (if customers have unit length) and the density is constant in space \( (\rho = N/L = 1) \).

The ‘exclusive queuing process’ (EQP) was introduced in [5, 6] to investigate how the excluded volume affects queues. It is obtained by modifying the input procedure of the one-dimensional totally asymmetric simple exclusion process (TASEP) with the ordinary ‘open boundary condition’. Customers are injected always at the end of the queue, called the left end in the following, i.e. behind the last customer waiting (figure 1). This is in contrast to the usual open TASEP case, where the input always happens at the same site, irrespective of the occupation of the other sites. The output is not changed compared to the usual open TASEP one: in both models customers are extracted at the right end (the server) which is fixed.

The EQP is not the only variant of the TASEP on a dynamic lattice. An earlier example is the dynamically extending exclusion process (DEEP) introduced in [7]–[10] as a model for fungal growth. In contrast to the EQP, the DEEP has no mechanism for reducing the system length and therefore the length of the system is always diverging. Other possible biological applications are length regulation of microtubules [11, 12] and bacterial flagellar growth [13].

The state space of the EQP is the set of configurations of the customers. It is more precisely given as

\[
S = \{ \tau = 1\tau_{L-1} \cdots \tau_1 | L \in \mathbb{Z}_{\geq 0}, \tau_j = 0, 1 \},
\]

where \( \tau = \emptyset, 1 \) for \( L = 0, 1 \), respectively. The state \( \tau = 1\tau_{L-1} \cdots \tau_1 \in S \setminus \{ \emptyset \} \) corresponds to the customer configuration where each site \( j \) is occupied or empty according to \( \tau_j = 1 \) or 0, and \( L \) defines the length of the system. The symbol \( \emptyset \) corresponds to the state ‘no customer in the system’. We denote the number of customers \( \#(\tau_j = 1) \) by \( N \) for a given state \( \tau = 1\tau_{L-1} \cdots \tau_1 \). Each customer enters the system, hops and leaves the system as

\[
\begin{align*}
\emptyset & \rightarrow 1 \quad \text{with probability } \alpha, \\
1 \cdots & \rightarrow 11 \cdots \quad \text{with probability } \alpha, \\
\cdots 10 \cdots & \rightarrow \cdots 01 \cdots \quad \text{with probability } p, \\
\cdots 1 & \rightarrow \cdots 0 \quad \text{with probability } \beta.
\end{align*}
\]

The local update rules (2) are not sufficient for specifying the dynamics fully. In addition, the sequence in which the rules are applied to the sites or customers needs to be defined. Here we consider two discrete-time update rules (parallel and backward sequential) and a continuous-time dynamics where the parameters \( \alpha, \beta \) and \( p \) are transition rates (not probabilities).

In [5, 14], exact stationary states for the continuous-time case and the parallel-update case were constructed as matrix product states based on the known forms for
the corresponding TASEPs with a fixed length \([15, 16]\). The phase boundary between the convergent and divergent phases was found to be modified compared to the classical M/M/1 queue case. In particular, for the convergence to the stationary states, the injection rate (or probability) \(\alpha\) cannot be bigger than the maximal current of the TASEP, i.e. ‘the queue itself is a bottleneck’, as well as the server.

In \([17]\), the phase diagram was analyzed in more detail. The convergent and divergent phases are both further subdivided into two subphases analogous to the maximal-current and high-density phases of the TASEP. Furthermore time-dependent properties were investigated. However the asymptotic form of the velocity for the growth of \(L\) in the divergent phase was left as an open problem. In this paper we will clarify this point, giving density profiles with the help of Monte Carlo simulations. (In the case where the customer hopping is deterministic \((p = 1)\), an exact ‘dynamical state’ in matrix product form exists which enables us to rigorously derive the behavior of \(\langle L_t \rangle\) and \(\langle N_t \rangle\) \([18]\).)

This paper is organized as follows. In section 2, we define the EQPs with the three update rules (parallel, backward and continuous time) in more detail. We determine the phase diagram for each update rule from the behavior of the system length (convergent versus divergent) and the form of the outflow (the current of customers through the right end) \([17]\). The outflow \(J_{\text{out}}\) depends on \(\beta\) and \(p\) when \(\beta < \beta_c\), and only on \(p\) when \(\beta > \beta_c\) with a critical point \(\beta_c\). The system converges to a stationary state for \(\alpha < J_{\text{out}}\) and diverges for \(\alpha > J_{\text{out}}\). The growth velocity of the number of customers is given by \(\alpha - J_{\text{out}}\). Thus at this stage the phase diagram is divided into four phases. The critical line \(\alpha = J_{\text{out}}\) consists of a curve \((\beta < \beta_c)\) and a straight line \((\beta > \beta_c)\). We also discuss the relations among the three EQPs and some special or limiting cases including the classical queuing processes.

In section 3, on the basis of simulation results we characterize further subphases of the divergent phases according to the shapes of the density profiles; plateau, plateau–slope, plateau–slope–plateau, slope and slope–plateau. They are obtained from a ‘rarefaction wave’ with a cutoff. We determine the form of the velocity \(V\) of growth of the system length \(\langle L_t \rangle \simeq V t\) as well.

In section 4, we investigate the parallel and backward EQPs on the critical line. We find that the behaviors of the system length \(\langle L_t \rangle \sim t^{\gamma_L}\) and the number of customers \(\langle N_t \rangle \sim t^{\gamma_N}\) are diffusive, \(\gamma_X = \frac{1}{2}\), or subdiffusive, \(\gamma_X < \frac{1}{2}\) \((X = L, N)\), according to whether \(\beta < \beta_c\) or \(\beta > \beta_c\). Furthermore \(\gamma_X\) surprisingly depends on the update rules on the straight line \((\beta > \beta_c)\); the exponents seem to be \(\frac{1}{4}\) for the parallel EQP, and depend on \(p\) for the backward EQP.

Finally, we give some conclusions in section 5. In two appendices we provide explicit analytical results for the special case of deterministic hopping \((p = 1)\) and simulation results for the backward and continuous-time updates.

2. Update rules

The TASEP is a prototypical model of stochastic interacting particle systems. It has been studied intensively in the past few decades both from the viewpoint of nonequilibrium statistical physics \([19]–[21], [4]\) and from that of mathematics (see e.g. \([22]\)). Like for the TASEP with the ordinary open boundary condition \([23]\) one can study the EQP with different update rules. In the next two subsections we define two discrete-time EQPs and
Density profiles of the exclusive queuing process determine their phase diagrams. These are divided into four phases according to their asymptotic lengths (convergent versus divergent) and the parameter dependence of the outflows. Then we consider special cases and limits including the continuous-time EQP.

2.1. Parallel update

In [6, 14, 17, 18], the EQP with the parallel-update rule, which we call parallel EQP for short, was investigated. In the parallel dynamics all sites are updated simultaneously, e.g. as

\[
\begin{align*}
101011 \rightarrow 1011010 & \quad \text{with probability } \alpha \times p \times (1-p) \times \beta, \\
110111 \rightarrow 110111 & \quad \text{with probability } (1-\alpha) \times p \times (1-\beta), \\
\emptyset \rightarrow \emptyset & \quad \text{with probability } 1-\alpha.
\end{align*}
\]

The current–density relation for the TASEP with parallel update and periodic boundary conditions is given by [24, 25]

\[ J_{\parallel}(\rho) = \frac{1 - \sqrt{1 - 4p\rho(1-\rho)}}{2}, \]

which is also true for the ordinary open boundary condition [16]. The ‘critical line’ that separates the parameter space into divergent and convergent phases is given by

\[ \alpha = J_{\parallel}(\rho_{\parallel}) = \begin{cases} 
\frac{\beta(p - \beta)}{p - \beta^2} & (0 < \beta \leq \beta_c), \\
\frac{1 - \sqrt{1 - p}}{2} & (\beta_c < \beta \leq 1),
\end{cases} \]

where \( \rho_{\parallel} = \begin{cases} 
\frac{p - \beta}{p - \beta^2} & (0 < \beta \leq \beta_c), \\
\frac{1}{2} & (\beta_c < \beta \leq 1),
\end{cases} \) (5)

with \( \beta_c = 1 - \sqrt{1 - p} \). When \( \alpha < J_{\parallel}(\rho_{\parallel}) \), the system converges to a stationary state which has a matrix product form [14]. The number of customers \( \langle N_t \rangle \) decreases linearly in time as

\[ \langle N_t \rangle \sim (\alpha - J_{\parallel}(\rho_{\parallel}))t + \langle N_0 \rangle \] (6)

while \( t \lesssim \langle N_0 \rangle / (J_{\parallel}(\rho_{\parallel}) - \alpha) \) starting from a sufficiently large \( \langle N_0 \rangle \) at time \( t = 0 \). The quantity \( J_{\parallel}(\rho_{\parallel}) \) is actually the customer current through the right end, i.e. the outflow. The system length exhibits a similar behavior:

\[ \langle L_t \rangle \sim \frac{\alpha - J_{\parallel}(\rho_{\parallel})}{\rho_{\parallel}}t + \langle L_0 \rangle, \]

where the density profile is almost flat with the bulk density \( \rho_{\parallel} \). In view of the form (5), we call the region \( \alpha < J_{\parallel}(\rho_{\parallel}) \) with \( \beta > \beta_c \) the ‘maximal-current-convergent (MC-C) phase’, and \( \alpha < J_{\parallel}(\rho_{\parallel}) \) with \( \beta < \beta_c \) the ‘high-density-convergent (HD-C) phase’.

When \( \alpha > J_{\parallel}(\rho_{\parallel}) \), the system does not have a stationary state, and \( \langle N_t \rangle \) and \( \langle L_t \rangle \) diverge linearly in time. For \( \langle N_t \rangle \), the form (6) is valid and we have the asymptotic
behavior

\[ \langle N_t \rangle \simeq (\alpha - J_{\parallel}(\rho_{\parallel})) t \quad (t \to \infty). \quad (8) \]

In view of the form (5), we call the region \( \alpha > J_{\parallel}(\rho_{\parallel}) \) with \( \beta > \beta_c \) the ‘maximal-current-divergent (MC-D) phase’, and \( \alpha > J_{\parallel}(\rho_{\parallel}) \) with \( \beta < \beta_c \) the ‘high-density-divergent (HD-D) phase’. On the other hand, the form (7) is not always valid in the divergent phase (see equation (23) below). The main purpose of this paper is to determine the velocity \( V_{\parallel} \) for \( \langle L_t \rangle \simeq V_{\parallel} t \) as well as the density profile in the divergent phase.

2.2. Backward-sequential update

We now consider the discrete-time EQP with backward-sequential update (backward EQP): first a customer arrives with probability \( \alpha \), and the customer at the right end is extracted with probability \( \beta \) (if it exists). Then starting from the rightmost customer and going sequentially to the left up to the leftmost customer, we move each customer forward with probability \( p \) if possible. For example,

\[
\begin{align*}
101011 & \rightarrow 1011001 & \text{with probability } \alpha \times \beta \times p \times (1-p) \times p \times (1-p), \\
111011 & \rightarrow 11111 & \text{with probability } (1-\alpha) \times (1-\beta) \times p \times p \times p, \\
\emptyset & \rightarrow \emptyset & \text{with probability } (1-\alpha) + \alpha \times \beta. \\
\end{align*}
\]

(9)

In the first example of transitions (9), the customer on the second site can move to the rightmost site, thanks to the backward update. On the other hand, this customer cannot move in the parallel case; see the first example of equation (3).

The current–density relation for the backward-sequential update\(^3\) TASEP is [23]

\[ J_\leftarrow(\rho) = \frac{\rho\rho(1-\rho)}{1-p \rho}. \quad (10) \]

Simulation results imply that the critical line separating the parameter space of the backward EQP into the convergent and divergent phases is given by

\[
\alpha = J_\leftarrow(\rho_\leftarrow) = \begin{cases} \\
\frac{\beta(p-\beta)}{p(1-\beta)} & (0 < \beta \leq \beta_c), \\
\frac{(1-\sqrt{1-p})^2}{p} & (\beta_c < \beta < 1), \\
\end{cases}
\]

where \( \rho_\leftarrow = \begin{cases} \\
\frac{p-\beta}{p(1-\beta)} & (0 < \beta \leq \beta_c), \\
\frac{1-\sqrt{1-p}}{p} & (\beta_c < \beta < 1), \\
\end{cases} \)

with \( \beta_c = 1 - \sqrt{1-p} \) as in the parallel case. The phase diagram is qualitatively similar to that of the parallel EQP (5); see figure 2. In the special case \( \beta = 1 \) the system always has the stationary state \( P(\emptyset) = 1 \) and \( P(\text{otherwise}) = 0 \). Therefore we set \( 0 < \beta < 1 \) in the following.

\(^3\) Note that the sitewise- and customerwise-sequential (particlewise-sequential) updates [23] are identical here.
When $\alpha < J_-(\rho_-)$, we expect the system to converge to a stationary state, and the number of customers $\langle N_t \rangle$ decreases linearly in time as

$$\langle N_t \rangle \sim (\alpha - J_-(\rho_-)) t + \langle N_0 \rangle \quad (12)$$

while $t \lesssim \langle N_0 \rangle / (J_-(\rho_-) - \alpha)$ starting from a sufficiently large $\langle N_0 \rangle$ at time $t = 0$. The system length also exhibits a similar behavior:

$$\langle L_t \rangle \sim \frac{\alpha - J_-(\rho_-)}{\rho_-} t + \langle L_0 \rangle. \quad (13)$$

When $\alpha > J_-(\rho_-)$, we expect the system not to have a stationary state and $\langle N_t \rangle$ and $\langle L_t \rangle$ to diverge linearly in time. For $\langle N_t \rangle$, the form (12) is valid and we have the asymptotic behavior

$$\langle N_t \rangle \approx (\alpha - J_-(\rho_-)) t \quad (t \to \infty). \quad (14)$$

On the other hand, for the divergence of the length, the form (13) is not always valid (see equation (29) below).
2.3. Limits and special cases

The discrete-time EQPs have several known models as special cases or limits. The following diagram illustrates the relations between the various models:

- Parallel EQP $\Delta s \to 0 \quad \Rightarrow \quad \text{Continuous EQP} \quad \Delta s \to 0 \quad \Rightarrow \quad \text{Backward EQP}$
- Rule 184 CA with stochastic boundaries

Here $\Delta s$ is the length of the discrete-time step. Formally, we replace the probabilities $\alpha$, $\beta$, and $p$, and time $t$ by $\alpha \Delta s + o(\Delta s)$, $\beta \Delta s + o(\Delta s)$, $\Delta s + o(\Delta s)$, and $t/\Delta s$, respectively. Then we take the continuous-time limit $\Delta s \to 0$.

The continuous-time limits of both discrete-time EQPs yield the continuous-time EQP studied in [5]. The current–density relation for the continuous-time case is simply

$$J_{\text{cont}}(\rho) = p\rho(1 - \rho), \quad (16)$$

and the critical line is given as

$$\alpha = J_{\text{cont}}(\rho_{\text{cont}}) = \begin{cases} \frac{\beta(p - \beta)}{p} & (0 < \beta \leq p/2), \\ \frac{p}{4} & (p/2 < \beta < 1), \end{cases}$$

where $\rho_{\text{cont}} = \begin{cases} 1 - \frac{\beta}{p} & (0 < \beta \leq p/2), \\ \frac{1}{2} & (\beta > p/2). \end{cases} \quad (17)$

The time-dependent properties in the continuous-time case are obtained simply by replacing $J_{\parallel}$ and $\rho_{\parallel}$ by $J_{\text{cont}}$ and $\rho_{\text{cont}}$ in equation (6)–(8). For example the velocity of the number of customers in the divergent phase is given as

$$\lim_{t \to \infty} \frac{\langle N_t \rangle}{t} = \alpha - J_{\text{cont}}(\rho_{\text{cont}}). \quad (18)$$

The deterministic hopping cases ($p = 1$) of the discrete-time EQPs correspond to two different processes. The bulk dynamics of the parallel EQP with $p = 1$ corresponds to the rule 184 cellular automaton (CA) with stochastic boundaries. It is still an EQP although the customer hopping is deterministic [6]. The MC-D and MC-C phases vanish from the phase diagram (figure 2). This case has an exact ‘dynamical state’ in a matrix product form which enables us to derive asymptotic behaviors of the system length, the number of customers and the density profile in the limit $t \to \infty$ [18]. Explicit analytical results are given in appendix A.

The backward EQP with $p = 1$ is equivalent to the discrete-time M/M/1 queuing process which is no longer an EQP as e.g. the state $\underbrace{1 \cdots 1}_{N}$ changes to $\underbrace{1 \cdots 1}_{N-1}$ when the customer at the server gets service. Thus no empty site appears between the leftmost customer and the server, i.e. we have always $N_t = L_t$, if the system starts from the empty chain. Also in this case, the MC-D and MC-C phases vanish from the phase diagram.
Figure 3. A schematic picture for the density profile in the divergent phase, where $x$ is the rescaled position $j/t$. According to the injection probability (rate) $\alpha$, the rarefaction wave is ‘cut’ by the leftmost customer ($x = V$) and the server ($x = 0$).

(figure 2). In the limit $t \to \infty$, the system shows different behavior, depending on the phase. Again, explicit analytical results are given in appendix A.

The continuous-time M/M/1 queuing process is recovered by the continuous-time limits of the rule 184 case and the discrete-time M/M/1 queue. It is also obtained by the $p \to \infty$ limit of the continuous-time EQP.

3. Subphases in the divergent phase

We consider the TASEP on an infinite chain with the initial densities $\rho_{\text{right}}$ (at sites $j < 0$) and $\rho_{\text{left}}$ (at sites $j \geq 0$), where $\rho_{\text{left}} > \rho_{\text{right}}$. When the current $J$ from left to right is given by a function of the density $\rho$, the rescaled density profile $\rho(x = j/t)$ is well described by

$$
\rho(x) \simeq \begin{cases} 
\rho_{\text{right}} & (x < f(\rho_{\text{right}})), \\
 f^{-1}(x) & (f(\rho_{\text{left}}) > x > f(\rho_{\text{right}})), \\
 \rho_{\text{left}} & (x > f(\rho_{\text{left}}))
\end{cases}
$$

with $f(\rho) = -dJ/d\rho$ [26]. We will see that, for the EQPs, the density profiles in the divergent phase are obtained by cutting this rarefaction wave as in figure 3.

3.1. The parallel case

From the current–density relation (4) for the parallel-update TASEP we have

$$
f_{\parallel}(\rho) = -\frac{dJ_{\parallel}}{d\rho} = \frac{p(2\rho - 1)}{\sqrt{1 - 4pp(1 - \rho)}}, \quad f_{\parallel}^{-1}(x) = \frac{1}{2} + \frac{x}{2} \sqrt{\frac{1 - p}{p(p - x^2)}}.
$$

We assume that the (rescaled) density profile $\rho_{x,t,t}$ (at site $xt$ and time $t$) has the form

$$
\rho_{x,t,t} \simeq \begin{cases} 
0 & (x > V, 0 > x), \\
\rho(x) & (V > x > 0),
\end{cases}
$$

where $\rho(x)$ is given by (19) with $\rho_{\text{right}} = \rho_{\parallel}$ and $\rho_{\text{left}} = 1$. This assumption is supported by simulation results. Here $V$ is the velocity of the system length $\langle L_t \rangle \simeq Vt$.
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Under assumption (21) we have
\[
t(\alpha - J^{\text{out}}) \simeq t \int_0^V \rho(x) \, dx,
\]
where the two sides are different expressions for the number of customers. Inserting \( J^{\text{out}} = J^\parallel(\rho^\parallel) \) (see equation (5)) into equation (22), we find the velocity
\[
V = V^\parallel = \left\{ \begin{array}{ll}
\frac{\alpha - J(\rho^\parallel)}{\rho^\parallel} &= \frac{\alpha p - \beta^2}{p - \beta} - \beta \quad \text{(I)}, \\
2p\alpha - p + 2\sqrt{p\alpha(1-p)(1-\alpha)} &\quad \text{(II)}, \\
\alpha &\quad \text{(III)},
\end{array} \right.
\]
where
\[
\text{I: } 0 < V^\parallel \leq f^\parallel(\rho^\parallel) \quad \text{i.e. } \frac{\beta(p - \beta)}{p - \beta^2} < \alpha \leq \frac{(p - \beta)^2}{p - 2p\beta + \beta^2}, \\
\text{II: } f^\parallel(\rho^\parallel) < V^\parallel \leq f^\parallel(1) \quad \text{i.e. Max} \left\{ \frac{(p - \beta)^2}{p - 2p\beta + \beta^2}, \frac{1 - \sqrt{1-p}}{2} \right\} < \alpha \leq p, \\
\text{III: } f^\parallel(1) \geq V^\parallel \quad \text{i.e. } p < \alpha \leq 1.
\]
Combining this and the form of \( \rho^\parallel \) given in equation (5), we obtain five subphases in the divergent phase. In each phase the rescaled density \( \rho_{x,t} \) has a different form (figure 4):
\[
\text{HD-D-I: } \rho_{x,t} \simeq \left\{ \begin{array}{ll}
\rho^\text{right} & (V > x > 0), \\
0 & (x > V),
\end{array} \right.
\]
\[
\text{HD-D-II: } \rho_{x,t} \simeq \left\{ \begin{array}{ll}
\rho^\text{right} & (v_1 > x > 0), \\
f^{-1}(x) & (V > x > v_1), \\
0 & (x > V),
\end{array} \right.
\]
\[
\text{MC-D-II: } \rho_{x,t} \simeq \left\{ \begin{array}{ll}
f^{-1}(x) & (V > x > 0), \\
0 & (x > V),
\end{array} \right.
\]
\[
\text{HD-D-III: } \rho_{x,t} \simeq \left\{ \begin{array}{ll}
\rho^\text{right} & (v_1 > x > 0), \\
f^{-1}(x) & (v_2 > x > v_1), \\
1 & (V > x > v_2), \\
0 & (x > V),
\end{array} \right.
\]
\[
\text{MC-D-III: } \rho_{x,t} \simeq \left\{ \begin{array}{ll}
f^{-1}(x) & (v_2 > x > 0), \\
1 & (V > x > v_2), \\
0 & (x > V),
\end{array} \right.
\]
where
\[
\rho^\text{right} = \rho^\parallel, \quad v_1 = f^\parallel(\rho^\parallel) = \frac{p(p - 2\beta + \beta^2)}{p - 2p\beta + \beta^2}, \quad v_2 = f^\parallel(1) = p,
\]
\[
f^{-1}(x) = f^{-1}_\parallel(x).
\]
The HD-D phase is divided into three phases: (I) plateau, (II) plateau–slope and (III) plateau–slope–plateau. On the other hand, the MC-D phase is divided into only two phases: (II) slope and (III) slope–plateau. The plateau near the exit does not appear.
Figure 4. The subphases of the EQPs: the parallel EQP (top left), the continuous-time EQP (top right), the backward EQP with $\frac{1}{2} \leq p < 1$ (bottom left) and the backward EQP with $0 < p < \frac{1}{2}$ (bottom right).

Figures 5 and 6 show simulation results for the velocities and the density profiles, respectively, with parameters

$$(\alpha, \beta, p) = \begin{cases} 
(0.4, 0.3, 0.84) & \text{HD-D-I (blue)}, \\
(0.75, 0.3, 0.84) & \text{HD-D-I (red)}, \\
(0.9, 0.3, 0.84) & \text{HD-D-III (purple)}, \\
(0.55, 0.8, 0.84) & \text{MC-D-II (orange)}, \\
(0.9, 0.8, 0.84) & \text{MC-D-III (green)}. 
\end{cases} \quad (27)$$

We see that in the special case $p = 1$ (rule 184 CA), the subphases except HD-D-I vanish from the divergent phase. Indeed the one-plateau density profile and the velocity with $p = 1$ agree with the rigorous results (see appendix A, equation (A.1)).

3.2. The backward case

From the current–density relation (10) for the TASEP with the backward update we have

$$f_-(x) := \frac{dJ_0}{d\rho} = -\frac{p(1 - 2\rho + pp^2)}{(1 - pp)^2}, \quad f^{-1}_-(x) = \frac{1}{p} - \frac{1}{p} \sqrt{\frac{1}{1 - p}} \frac{1}{1 + x}. \quad (28)$$

As in the parallel-update case, we assume equation (21) with $\rho(x)$ as in (19), $\rho_{\text{right}} = \rho_-$ and $\rho_{\text{left}} = 1$. From equation (22), we find that the velocity $V_-$ for the system length
Figure 5. The growth velocities of the system length and the number of customers for the parallel EQP. The simulation data were obtained by averaging for $10^4$ samples. We see that they agree with the lines corresponding to equations (23) and (8). The parameters are chosen as in equations (27).

Figure 6. Rescaled density profiles $\rho_{jt}$ of the parallel EQP ($t = 10^4$). The parameters are chosen as in equation (27). The simulation data were obtained by averaging for $10^4$ samples.

$\langle L_t \rangle \simeq V_\rightarrow t$ is given by

$$V_\rightarrow = \begin{cases} \frac{\alpha - J(\rho_\rightarrow)}{\rho_\rightarrow} = \frac{p(1 - \beta)}{p - \beta} - \beta & (I), \\ 2\sqrt{p(1 - p)} \alpha - p(1 - \alpha) & (II), \\ \alpha & (III), \end{cases}$$

where

I: $0 < V_\rightarrow \leq f_\rightarrow(\rho_\rightarrow)$ i.e. $\frac{\beta(p - \beta)}{p(1 - \beta)} < \alpha \leq \frac{(p - \beta)^2}{p(1 - p)}$,

II: $f_\rightarrow(\rho_\rightarrow) < V_\rightarrow \leq f_\rightarrow(1)$ i.e. $\text{Max} \left( \frac{(p - \beta)^2}{p(1 - p)}, \frac{(1 - \sqrt{1 - p})^2}{p} \right) < \alpha \leq \frac{p}{1 - p}$, (30)

III: $f_\rightarrow(1) > V_\rightarrow$ i.e. $\frac{p}{1 - p} < \alpha \leq 1$.

When $\frac{1}{2} \leq p < 1$, the case D-III vanishes and the divergent phase is divided into three phases (figure 4). On the other hand, when $0 < p < \frac{1}{2}$, the structure of the subphases is qualitatively similar to that for the parallel case. The density profiles are given by

doi:10.1088/1742-5468/2012/12/P12004
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\begin{equation}
\rho_{\text{right}} = \rho_-, \quad v_1 = f_- (\rho_-) = \frac{p - 2\beta + \beta^2}{1 - p}, \\
v_2 = f_- (1) = \frac{p}{1 - p}, \quad f^{-1} (x) = f_-(x). \tag{31}
\end{equation}

Simulation results for the velocities and density profiles are presented in appendix B. They agree with the forms (29) and (25), (31).

We see that in the special case \( p = 1 \) (i.e. the discrete-time M/M/1), the subphases except HD-D-I vanish from the divergent phase. Indeed the one-plateau density profile and the velocity with \( p = 1 \) agree with the rigorous (A.4).

### 3.3. The continuous-time case

From the current–density relation (16) for the continuous-time TASEP, we have

\begin{equation}
f_{\text{cont}} (\rho) := - \frac{dJ_{\text{cont}}}{d\rho} = p(2\rho - 1). \tag{32}
\end{equation}

The velocity \( V_{\text{cont}} \) of the system length, the subphases and the density profiles can be obtained following the same procedure as for the parallel and backward EQPs, or simply by taking the continuous-time limits of the results for the two discrete cases:

\begin{equation}
V_{\text{cont}} = \begin{cases}
\frac{\rho_{\text{cont}} - J (\rho_{\text{cont}})}{\rho_{\text{cont}}} = \frac{p}{p - \beta} \alpha - \beta & (\text{I}), \\
2\sqrt{p\alpha - p} & (\text{II}), \\
\alpha & (\text{III}),
\end{cases} \tag{33}
\end{equation}

where

I : \quad 0 < V_{\text{cont}} \leq f_{\text{cont}} (\rho_{\text{cont}}) \quad \text{i.e.} \quad \frac{\beta(p - \beta)}{p} < \alpha \leq \frac{(p - \beta)^2}{p}, \tag{34}

II : \quad f_{\text{cont}} (\rho_{\text{cont}}) < V_{\text{cont}} \leq f_{\text{cont}} (1) \quad \text{i.e.} \quad \text{Max} \left( \frac{(p - \beta)^2}{p}, \frac{p}{4} \right) < \alpha \leq p, \tag{35}

III : \quad f_{\text{cont}} (1) > V_{\text{cont}} \quad \text{i.e.} \quad p < \alpha. \tag{36}

The density profiles are given by equation (25) with

\begin{equation}
\rho_{\text{right}} = \rho_{\text{cont}}, \quad v_1 = f_{\text{cont}} (\rho_{\text{cont}}) = p - 2\beta, \\
v_2 = f_{\text{cont}} (1) = p, \quad f^{-1} (x) = f^{-1}_{\text{cont}} (x) = \frac{1}{2} + \frac{x}{2p}. \tag{37}
\end{equation}

Simulation results are given in appendix B, which agree with the forms (33) and (25), (37).

We see that in the limiting case \( p \to \infty \) (with \( \alpha \) and \( \beta \) fixed), i.e. the continuous-time M/M/1 queuing process, the subphases except HD-D-I (one plateau) vanish from the divergent phase. The density of the plateau and the velocity simply become 1 and \( \alpha - \beta \), respectively.
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Figure 7. The exponents $\gamma_X$ for the system length $X = L$ and the number of customers $N$ on the critical lines of the parallel ($p = 0.84$, $\beta_c = 0.6$, left) and backward ($p = 0.36$, $\beta_c = 0.2$ and $p = 0.19$, $\beta_c = 0.1$, right) EQPs. The markers □ (orange) and × (purple) correspond to $\gamma_L$ and $\gamma_N$, respectively, for the parallel case with $p = 0.84$ and the backward case with $p = 0.36$, and ○ (blue) and + (red) correspond to $\gamma_L$ and $\gamma_N$, respectively, for the backward case with $p = 0.19$.

4. On the critical line

Since $\langle N_t \rangle$ and $\langle L_t \rangle$ converge to stationary values in the convergent phase, and diverge proportionally to $t$ in the divergent phase, we expect them to behave as

$$\langle X_t \rangle \sim t^{\gamma_X}, \quad 0 \leq \gamma_X \leq 1, \quad \text{for } X = L, N. \quad (38)$$

Under this assumption we have

$$\frac{\ln X_t - \ln X_{t/b}}{\ln b} \rightarrow \gamma_X \quad (t \rightarrow \infty), \quad (39)$$

for $X = L, N$. After verifying that the growth behavior is indeed well described by power laws of the form (38), we estimate the exponents $\gamma_X$ by applying (39) to simulated samples with $b = 10$ and $t \leq 5 \times 10^5$. The number of samples for this estimation for each parameter set is basically $10^4$, but $10^6$ or $5 \times 10^6$ samples were used for the backward EQP in the region $0.28 \leq \beta < 0.8$ and $\beta > 0.8$, respectively, because there fluctuations of $L$ and $N$ are very large. The results shown in figure 7 are consistent with the expectation $\gamma_L = \gamma_N$ everywhere on the critical line. This is supported by the observation that the total density $\rho_{\text{tot}} = \langle N_t \rangle / \langle L_t \rangle$ reaches quickly an almost stationary value, which implies that $\gamma_L = \gamma_N$. More detailed results will be presented in a future publication.

The critical lines of the EQPs consist of two parts: a curve and a straight line (figure 2). On the curved part, the simulation results indicate

$$\langle N_t \rangle, \langle L_t \rangle = O(\sqrt{t}). \quad (40)$$

The behavior of $\langle L_t \rangle$ and $\langle N_t \rangle$ on the straight part of the critical line is not so clear, although diffusive behavior can be excluded. As figure 7 indicates, the exponents are smaller than on the curved part, i.e. $\gamma_L = \gamma_N < 1/2$. For the parallel case, $\gamma = 1/4$ (with large corrections near $\beta_c$) cannot be excluded, but for the backward case, the exponents seem to depend on the value of $p$. For example, the exponents for $p = 0.19$ appear to be larger than those for $p = 0.36$; see the right graph of figure 7. Our simulation results are

doi:10.1088/1742-5468/2012/12/P12004
not sufficient to determine conclusively the dependence of $\gamma$ on the parameters, e.g. how it varies with $\beta$ near $\beta_c$.

5. Conclusion

We have continued our studies of the exclusive queuing process (EQP) which extends the classical M/M/1 queuing process by incorporating the excluded-volume effect. We have compared the behavior of the model with different update rules (parallel, backward sequential, continuous time).

The phase diagram of the EQP turns out to be rather rich. Here we have shown that the divergent phase is subdivided into up to five different subphases according to the parameter dependence of the current and the density profiles. The MC-D phase has two different subphases (the slope and slope–plateau phases), and the HD-D phase has three different subphases (the plateau, plateau–slope and plateau–slope–plateau phases). The phase diagrams of different update procedures are qualitatively similar, except for the backward case with $p \geq \frac{1}{2}$ and some limiting cases.

In the divergent phase we have conjectured the analytic form of the density profiles which show good agreement with simulation results. The shapes of the rescaled profiles can be understood in terms of a rarefaction wave that is ‘cut’ at both ends.

On the critical line separating the divergent from the convergent phase the length of the system grows sublinearly. On the basis of simulation results we find diffusive behavior on the curved part of the critical line (i.e. $\beta < \beta_c$) for all updates. In the special case $p = 1$ for the two discrete EQPs, the density profiles can be written in terms of the complementary error function; see equations (A.2) and (A.5). Identifying the density profile on the curved part for the EQPs with general values of $p$ is one of problems that will need to be clarified in the future.

The behavior on the straight part $\beta > \beta_c$ of the critical line depends on the update rule. However, it is subdiffusive ($\gamma < 1/2$) for all cases. We could not clearly determine how the exponent $\gamma$ depends on the parameters, and more simulation data with sufficient accuracy are needed to determine the behavior on the straight part.

Acknowledgments

C Arita is a JSPS fellow for research abroad. The authors thank Kirone Mallick for useful discussions.

Appendix A. Deterministic hopping cases

A.1. Parallel update with $p = 1$

For the deterministic hopping limit $p = 1$ of the parallel update we have previously obtained the exact dynamical state in a matrix product form [18]. This enables us to derive asymptotic behaviors of the system length, the number of customers and the density profile in the limit $t \to \infty$:  

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- Divergent phase ($\alpha > \beta/(1 + \beta)$):
  \[
  \langle L_t \rangle = (\alpha - \beta + \alpha \beta)t + o(t), \quad \langle N_t \rangle = \frac{\alpha - \beta + \alpha \beta}{1 + \beta} t + o(t),
  \]
  \[
  \rho_{xt,t} \to \begin{cases} 
  1/(1 + \beta) & (x < 1), \\
  0 & (x > 1). 
  \end{cases}
  \tag{A.1}
  
- Critical line ($\alpha = \beta/(1 + \beta)$):
  \[
  \langle L_t \rangle = 2 \sqrt{\frac{\beta t}{\pi(1 + \beta)}} + o(\sqrt{t}), \quad \langle N_t \rangle = 2 \sqrt{\frac{\beta t}{\pi(1 + \beta)^3}} + o(\sqrt{t}),
  \]
  \[
  \rho_{x\sqrt{t},t} \to \frac{1}{1 + \beta} \text{erfc} \left( \frac{x}{2} \sqrt{\frac{1 + \beta}{\beta}} \right).
  \tag{A.2}
  
- Convergent phase ($\alpha < \beta/(1 + \beta)$):
  \[
  \langle L_t \rangle \to \frac{\alpha}{\beta - \alpha - \alpha \beta}, \quad \langle N_t \rangle \to \frac{\alpha(1 - \alpha)}{\beta - \alpha - \alpha \beta}, \quad \rho_{jt} \to (1 - \alpha) \left( \frac{\alpha}{(1 - \alpha)\beta} \right)^j,
  \tag{A.3}
  \]
  where erfc is the complementary error function $\text{erfc}(x) = \int_x^\infty e^{-y^2} dy$.

A.2. Backward-sequential update with $p = 1$

The deterministic hopping limit $p = 1$ of the backward update is equivalent to the discrete-time M/M/1 queuing process. In the limit $t \to \infty$, the system shows different behavior, depending on the phase:

- Divergent phase ($\alpha > \beta$):
  \[
  \langle L_t \rangle = \langle N_t \rangle = (\alpha - \beta)t + o(t), \quad \rho_{xt,t} \to \begin{cases} 
  1 & (x < 1), \\
  0 & (x > 1). 
  \end{cases}
  \tag{A.4}
  
- Critical line ($\alpha = \beta$):
  \[
  \langle L_t \rangle = \langle N_t \rangle = 2 \sqrt{\frac{\beta(1 - \beta)}{\pi}} t + o(\sqrt{t}), \quad \rho_{x\sqrt{t},t} \to \text{erfc} \left( \frac{x}{2 \sqrt{\beta(1 - \beta)}} \right).
  \tag{A.5}
  
- Convergent phase ($\alpha < \beta$):
  \[
  \langle L_t \rangle = \langle N_t \rangle \to \frac{\alpha(1 - \beta)}{\beta - \alpha}, \quad \rho_{jt} \to \left( \frac{\alpha(1 - \beta)}{\beta(1 - \alpha)} \right)^j.
  \tag{A.6}
  
Appendix B. Simulation results for backward and continuous-time updates

Here we provide simulation results for the velocities and density profiles for the backward and continuous-time cases, respectively (figures B.1 and B.2). The parameters are chosen

doi:10.1088/1742-5468/2012/12/P12004
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Figure B.1. The growth velocities of the system length and the number of customers for the backward (top) and continuous (bottom) EQPs. The simulation data were obtained by averaging for $10^4$ samples. We see that they agree with the lines corresponding to equations (29), (14) (33) and (18). The parameters are chosen as in (B.1) and (B.2).

Figure B.2. Rescaled density profiles $\rho_{jt}$ of the backward (top, $t = 10^4$) and continuous-time (bottom, $t = 2500$) EQPs. The parameters are chosen as in (B.1) and (B.2). The simulation data were obtained by averaging for $10^4$ samples.
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as

\[
(\alpha, \beta, p) = \begin{cases}
(0.2, 0.1, 0.36) & \text{HD-D-I} \quad \bigcirc \text{ (blue),}
(0.4, 0.1, 0.36) & \text{HD-D-I} \quad \bigtriangleup \text{ (red),}
(0.8, 0.1, 0.36) & \text{HD-D-III} \quad \times \text{ (purple),}
(0.3, 0.6, 0.36) & \text{MC-D-II} \quad \square \text{ (orange),}
(0.8, 0.6, 0.36) & \text{MC-D-III} \quad + \text{ (green)}
\end{cases}
\]

for the backward case, and

\[
(\alpha, \beta, p) = \begin{cases}
(0.35, 0.25, 1) & \text{HD-D-I} \quad \bigcirc \text{ (blue),}
(0.8, 0.25, 1) & \text{HD-D-I} \quad \bigtriangleup \text{ (red),}
(1.2, 0.25, 1) & \text{HD-D-III} \quad \times \text{ (purple),}
(0.6, 1, 1) & \text{MC-D-II} \quad \square \text{ (orange),}
(1.2, 1, 1) & \text{MC-D-III} \quad + \text{ (green)}
\end{cases}
\]

for the continuous case.

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