Contributing of the Sixth Order Effective Chiral Lagrangian to the $\pi K$ Scattering at Large $N_c$

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Abstract

Using the method of asymptotic sum rules we estimated the size of $O(m_s p^4)$ and $O(m_s^2 p^2)$ corrections to $\pi K$ scattering amplitude in large $N_c$ limit. These corrections arise from the sixth order effective chiral lagrangian (EChL). Our method enables us to estimate the corresponding terms of the sixth order EChL in leading order of $1/N_c$ expansion in model independent way. We found that the corrections numerically are suppressed in spite of naive expectation of 30–35%. Our estimation gives the value of these corrections about 5–10%.
The technique of the Effective Chiral Lagrangians (EChL) provides us with a systematic way of low-energy expansion of correlators of different colourless currents in the QCD [1, 3]. The information about large distance behaviour of the Quantum Chromodynamics is hidden in a set of coupling constants which is finite if we restrict ourselves by finite order in momentum expansion.

In the lowest order of momentum expansion $O(p^2)$ the interactions of (pseudo)goldstone mesons (pions, kaons and eta mesons) are described by the famous Weinberg lagrangian [1, 2]:

$$L^{(2)} = \frac{F_0^2}{4} tr(\partial_\mu U^\dagger \partial_\mu U) + \frac{B_0^2}{4} tr(\chi),$$

(1)

where $\chi = 2B_0(\hat{m}U + U^\dagger \hat{m})$, $\hat{m} = \text{diag}(0, 0, m_s)$ is a quark mass matrix and $F_0$ and $B_0$ are low-energy coupling constants contained an information about long-distance behaviour of the QCD. The latter can be either extracted from an experiment or calculated in the framework of some models of strong interactions.

In the next $O(p^4)$ order the interactions of the (pseudo)goldstone mesons are described by the following EChL [4, 5]

$$L^{(4)} = L_1 \left[ tr(\partial_\mu U^\dagger \partial_\mu U) \right]^2 + L_2 tr(\partial_\mu U^\dagger \partial_\nu U) \cdot tr(\partial_\mu U^\dagger \partial_\nu U) +$$

$$+ L_3 tr(\partial_\mu U^\dagger \partial_\nu U \partial_\rho U^\dagger \partial_\rho U) + L_4 tr(\partial_\mu U^\dagger \partial_\nu U) \cdot tr(\chi) +$$

$$+ L_5 tr(\partial_\mu U^\dagger \partial_\nu U \chi) + L_6 tr(\chi)^2 + L_7 tr((\hat{m}U^\dagger + U\hat{m})^2) + L_8 tr(\chi^2) + L_9 tr(\chi^2).$$

(2)

Here new eight coupling constants $L_{1,8}$ are appeared. These coupling constants enter the mass splitting in the (pseudo)goldstone octet, $d$-wave $\pi\pi$ and $\pi K$ scattering, etc. They have been calculated by integration of non-topological axial anomaly in QCD [6, 7, 8] and in the instanton liquid model of QCD vacuum [9]. Throughout the paper we shall neglect the masses of light $u$ and $d$ quarks and we shall work in the leading order of $1/N_c$ expansion, so we can put $L_4 = L_6 = 2L_1 - L_2 = 0$ [5]. Hence in this order the (pseudo)goldstone meson interactions in the fourth order are described by six universal parameters of the EChL—$F_0$, $B_0$, $L_2$, $L_3$, $L_5$ and $L_8$.

Using the EChL eqs.(1) and (2) one can calculate, say, $\pi K$ scattering amplitude to the orders $O(p_\pi^2)$, $O(m_s p_\pi^2)$ and $O(m_s^2)$, where $p_\pi$ is pion momentum. The corrections of the form $O(m_s p_\pi^4)$, $O(m_s^2 p_\pi^2)$ and $O(m_s^3)$ due to
higher order EChL naively are expected to be of order $O(m_K^2/m_\rho^2)$ i.e. about 30–35%. To calculate the size of these corrections one needs to know the sixth order EChL. Unfortunately the knowledge of the higher order EChL is rather poor. In this paper we calculate the size of these corrections to $\pi K$ scattering amplitude of the orders $O(p_\pi^4)$ and $O(m_\pi^2p_\pi^2)$, using the method of asymptotic sum rules [10].

Method of asymptotic sum rules for the (pseudo) Goldstone elastic scattering in large $N_c$ limit was suggested in [10]. In a sense, it is based on the generalization of Weinberg’s asymptotic restrictions for chiral amplitudes [11] for the case of nonzero (pseudo) Goldstone mass. This method allows to express the parameters of the effective chiral lagrangian through the masses and widths of the admissible resonances and put a dynamical restrictions on resonance spectrum at large $N_c$ [10]. We apply this method to particular case of $\pi\pi$ and $\pi K$ elastic processes.

Using this method we find that the corrections to $\pi K$ scattering amplitudes of the form $O(m_\pi p_\pi^4)$ and $O(m_\pi^2p_\pi^2)$ ($p_\pi$ is soft pion momentum) arising from the sixth order EChL numerically are suppressed in spite of naive expectation of 30–35%. Our estimation gives the value of these corrections about 5–10%, so one can use EChL (2) for, say, calculation of d-wave $\pi K$ scattering amplitude with good accuracy.

2. Let us consider the elastic $\pi\pi$–scattering process

$$\pi_a(k_1) + \pi_b(k_2) \to \pi_c(k_3) + \pi_d(k_4).$$

($a, b, c, d = 1, 2, 3$ are the isotopic indices and $k_1, ..., k_4$ — pion momenta.) Its amplitude $M^{abcd}$ can be written in the form:

$$M^{abcd} = \delta^{ab}\delta^{cd}A + \delta^{ac}\delta^{bd}B + \delta^{ad}\delta^{bc}C,$$  \hspace{1cm} (3)

where $A, B, C$ are the scalar functions of Mandelstam variables $s, t, u$:

$$s = (k_1 + k_2)^2, \quad t = (k_1 - k_3)^2, \quad u = (k_1 - k_4)^2,$$ \hspace{1cm} (4)

obeying the Bose–symmetry requirements:

$$A(s, t, u) = A(s, u, t),$$

$$B(s, t, u) = A(t, s, u),$$ \hspace{1cm} (5)

$$C(s, t, u) = A(u, t, s).$$
The amplitude of the $\pi K$ scattering process
\[ \pi_a(k_1) + K_\alpha(k_2) \to \pi_b(k_3) + K_\beta(k_4). \] (6)
can be expressed in terms of two (iso)scalar functions $A_+(\nu, t)$ and $A_-(\nu, t)$ by
\[ M_{\alpha\beta}^{ab} = \delta^{ab}\delta_{\alpha\beta}A_+(\nu, t) + i\epsilon^{abc}\sigma^c_{\alpha\beta}A_-(\nu, t), \] (7)
where invariant variable $\nu = s - u$ is expressed via Mandelstam variables eq. (4). Near threshold of the reactions one can expand the (iso)scalar amplitudes $A(s, t)$, $A_+(\nu, t)$ and $A_-(\nu, t)$ in power series of pion momenta:
\[ A(s, t) = \sum_{i,j} a_{ij} s^i t^j, \] (8)
\[ A_+(\nu, t) = a_0(m_s) + a_1(m_s) \cdot t + a_2(m_s) \cdot t^2 + a_3(m_s) \cdot \nu^2 t + \ldots, \] (9)
\[ A_-(\nu, t) = b_1(m_s) \cdot \nu + b_2(m_s) \cdot \nu t + \ldots \] (10)
Non-analitic parts of the amplitudes (like $E^4 \log(E)$) are suppressed by additional $1/N_c$. Parameters of the near threshold expansion of the $\pi K$ scattering amplitude depend on strange quark mass.

From the Effective Chiral Lagrangian eq. (4),eq. (2) one gets an expression for the low-energy parameters of the $\pi\pi$ scattering amplitude and for expansion in $m_s$ of the corresponding parameters of the $\pi K$ scattering amplitudes in terms of universal parameters of the fourth order EChL $F_0$, $B_0$, $L_2$ and $L_3$:
\[ a_{00} = 0; \quad a_{10} = \frac{1}{F_0}; \quad a_{01} = 0; \]
\[ a_{20} = \frac{4(2L_2 + L_3)}{F_0^4}; \quad a_{11} = a_{02} = \frac{8L_2}{F_0^4}, \]
\[ b_1(m_s) = -\frac{1}{F_0^3}, \] (11)
(It easy to prove that the corrections due to the non-zero strange quark mass to $b_1$ should vanish by chiral Ward identity i.e. $\frac{d}{dm_s}b_1(m_s) \equiv 0$)
\[ a_3(m_s) = \frac{8(L_3 + 4L_2)}{F_0^4} \cdot (1 + m_s \xi_1) + O(m_s^2). \] (12)
\[ L_5, L_7 \text{ and } L_8 \text{ do not enter explicitely into these quantities. Dependent on them appears in one–loop order i.e. in next order of the } 1/N_c \text{ expansion} \]
Here we introduce the quantity $\xi_1$ which determines the values of $m_s$-corrections to the low-energy parameters of $\pi K$ amplitude of order $O(p^4)$. This quantity can be calculated in terms of parameters of the sixth order EChL, unfortunately the EChL to these order is not known. In next section, using method of asymptotic sum rules [10], we shall express these quantities via parameters (masses and widths) of $\pi\pi$ and $\pi K$ resonances and show that $m_s\xi_1 \sim 0.1$ what justified an applicability of $m_s$-expansion for the $\pi K$ amplitude of order $O(p^4)$ (applicability of the ECL (1,2) for description of low-energy $\pi K$ elastic scattering with an accuracy of 10%).

3. In this section we derive sum rules (SR) relating low-energy coefficients $a_{ij}$ (for $\pi\pi$ amplitude), $a_i(m_s)$ and $b_i(m_s)$ (for $\pi K$ amplitude) to parameters (masses and widths) of $\pi\pi$ and $\pi K$ resonances in the leading order of the $1/N_c$ expansion. To do this we use the Weinberg’s method of the asymptotic restrictions [1] generalized for the case of nonzero mass of (pseudo)goldstone bosons and for the case of scattering on nonzero angle [10]. Idea of the method is very simple – in the large $N_c$ limit we can represent $\pi K$ and $\pi\pi$ elastic scattering amplitudes as an infinite sum of resonance pole contributions, in this limit the amplitude is free of other singularities. Then one assumes that the asymptotic behaviour of the amplitude at $s \to \infty$ and fixed $t$ is not worse than the one in Regge’s theory. This assumption should be fulfilled in the large $N_c$ limit to ensure the renormalizability of the underlying theory - QCD. Using the resonance form of the amplitude one can express coefficients of near threshold expansion of the amplitudes through the masses and widths of $\pi\pi$ and $\pi K$ resonances.

$$\frac{1}{F_0^2} = \sum V_1(J) \frac{1}{M_t^4} + \sum V_0(J) \frac{1}{M_0^4} ; \quad (13)$$

$$\frac{4L_2}{F_0^4} = \sum V_1(J) \frac{1}{M_t^6} ; \quad (14)$$

$$\frac{4(2L_2 + L_3)}{F_0^4} = -\sum V_1(J) \frac{1}{M_t^6} + \sum V_0(J) \frac{1}{M_0^6} , \quad (15)$$

where $M_I(0)$ ($I=1,2$) are masses of the $\pi\pi$ resonances with isospin $I$ and constants $V_I(J)$ can be related to the decay widths of corresponding resonances with isospin $I$ and spin $J$:

$$V_1(J) = 8\pi(2J + 1) \cdot \frac{M_t^2}{k_\pi} \Gamma(R \to \pi\pi) ,$$

$$V_0(J) = \frac{4\pi(2J + 1)}{k_\pi} \Gamma(R \to \pi\pi) ,$$

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$$V_0(J) = \frac{4\pi(2J + 1)}{k_\pi} \Gamma(R \to \pi\pi) .$$
\[ V_0(J) = \frac{2}{3} \cdot 8\pi(2J + 1) \cdot \frac{M_R^2}{k_{\pi}} \Gamma(R \rightarrow \pi\pi), \]

\( k_{\pi} \) being the pion CMS momentum.

\[ b_1(m_s) = \frac{1}{F_0^2} = \sum \frac{4R(m_s)}{(M_R(m_s)^2 - m_K^2)^2}, \quad (16) \]

\[ a_3(m_s) = \frac{8(L_3 + 4L_2)}{F_0^4} \cdot (1 + m_s\xi_1) + O(m_s^2) = \sum \frac{8R(m_s)}{(M_R(m_s)^2 - m_K^2)^3}, \quad (17) \]

\( R(m_s) \) and \( M_R(m_s) \) are residues and masses of the \( \pi K \)-resonances.

\[ R(m_s) = \frac{1}{3} \cdot 8\pi(2j + 1) \cdot \frac{M_R}{k} \Gamma(R \rightarrow \pi K), \quad (18) \]

\[ k = \frac{1}{2M_R} \sqrt{(M_R^2 - (m_K + m_\pi)^2)(M_R^2 - (m_K - m_\pi)^2)}. \quad (19) \]

When \( m_s \rightarrow 0 \) one gets:

\[ R(m_s) = \frac{1}{4} V_I \cdot (1 + m_s\Delta + O(m_s^2)), \quad (20) \]

(here \( I = 0 \) for even spin resonances, \( I = 1 \) for odd spin resonances)

\[ M_R(m_s)^2 = M_R(0)^2 \cdot (1 + m_s\delta + O(m_s^2)). \quad (21) \]

The quantities \( \delta \) and \( \Delta \) are related to the mass and widths splitting inside resonance nonet. For example, for well established \( \rho \)-meson octet one has:

\[ m_s\Delta \approx \frac{4R(K^* \rightarrow \pi K) - V_1(\rho \rightarrow \pi\pi)}{V_1(\rho \rightarrow \pi\pi)} = -0.25, \quad (22) \]

\[ m_s\delta \approx \frac{M_{K^*}^2 - M_\rho^2}{M_\rho^2} = 0.35. \quad (23) \]

Expanding the eqs. (16,17) in powers of \( m_s \) one can express \( m_s\xi_1 \) via resonance parameters. From eq. (16) one gets:

\[ 0 = \frac{\partial b_1(m_s)}{\partial m_s} = \langle m_s\Delta - 2 \cdot (m_s\delta - \frac{m_K^2}{M_R(0)^2}) \rangle_2. \quad (24) \]
where we introduce notations

$$\langle F \rangle_k = \frac{\sum_{I=0,1} \sum_{\text{res}} \frac{V_I}{M_{R(0)^2}} F}{\sum_{I=0,1} \sum_{\text{res}} \frac{V_I}{M_{R(0)^2}}}$$  \hspace{1cm} (25)$$

for any quantity $F$ depending on resonance parameters.

Naively one can expect that relative corrections due to nonzero strange quark mass to $b_1(m_s)$ must be of order $m_s \delta \approx (M_{K^*}^2/M_{\rho}^2 - 1)$ (e.i. $\sim 0.35$ for realistic resonance spectrum) but the chiral Ward identities require that (see eq. (16)) resonance spectrum must be adjusted to provide exact cancelation of different corrections in any order of $m_s$-expansion each of those is not small. Expanding rhs of the sum rules (16,17) in power of $m_s$ one can express $\xi_1$ in terms of resonances parameters:

$$m_s \xi_1 = <m_s \Delta - 3 \cdot (m_s \delta - \frac{m_K^2}{M_{R(0)^2}}) >_3.$$  \hspace{1cm} (26)$$

In the previous (second) order we found complete cancellations among different corrections of order $m_s$, so we could, in principle, expect similar cancellations in considered (fourth) order.

To estimate numerically $m_s \xi_1$ we use two different methods. In the first one we use the typical parameters of the $\rho$-meson nonet to estimate $m_s \xi_1$ by eq. (26). The result is:

$$m_s \xi_1 \approx \frac{4R(K^* \to \pi K) - V_1(\rho \to \pi\pi)}{V_1(\rho \to \pi\pi)} - 3 \cdot \frac{M_{K^*}^2 - (M_{\rho}^2 + m_K^2)}{M_{\rho}^2} =$$

$$= -0.25 + 3 \cdot 0.06 = -0.07.$$  \hspace{1cm} (27)$$

In the second method we calculate $b_1(m_s)$ and $a_3(m_s)$ directly by eqs. (16,17) using the realistic resonance spectrum [13] (results are presented in Table 1 for $\pi\pi$ SR and Table 2 for $\pi K$ SR) ) and $L_2$, $L_3$ and $F_0^2$ by eqs. (13-15), and then extract $\xi_1$ using:

$$m_s \cdot \xi_1 \approx \frac{a_3(m_s) \cdot F_0^4 - 8(L_3 + 4L_2)}{8(L_3 + 4L_2)}.$$  \hspace{1cm} (28)$$

From the Tables 1. and 2. one can conclude that $m_s \cdot \xi_1 = -0.085 \pm 0.005$ what confirm our previous estimation (27). Also we see that the parameter
$b_1(m_s)$ does not depend on $m_s$ with good accuracy, what is required by chiral Ward identities. Numerically both $\pi\pi$ and $\pi K$ sum rules (15, 16) give $F_0^{(exp)}/F_0^{(SR)} = 0.83 \pm 0.1$. From the latter number one can conclude that next to leading $1/N_c$ corrections to $F_0$ have a size around 30% and negative sign.

4. The applicability of the fourth order effective chiral lagrangian (1,2) for the calculation $\pi K$ scattering amplitudes up to order $O(p^4)$ with 5-10% accuracy is checked at least in large $N_c$ limit. The $\pi K$ scattering amplitude calculated with use of fourth order EChL has a relative corrections of order $m_s$ arising from the sixth order EChL. Naively we would expect that the corrections are of order $m_k^2/m_\rho^2 \sim 30 - 35\%$ (the same as for mass splittings in $SU(3)$ meson multiplets), but, at least in the leading order of $1/N_c$ expansion, these corrections are anomalously small. Roughly speaking the actual behaviour of the corrections with $m_s$ is not $\sim m_k^2/m_\rho^2$, but $\sim m_k^2 - (m_\rho^2 + m_K^2)$. Though both these quantities are proportional to $m_s$ the latter has additional numerical suppression. Simultaneously we found that resonance spectrum has to satisfy an equation (24) to provide chiral Ward identities. We checked that this equation is satisfied with quite good accuracy on the known resonance spectrum [13]. We think our sum rules (15,16,17) for parameters of fourth order EChL and (24) for resonance spectrum are the good consistency check of different models of QCD at large number of colours.

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\footnote{An attempt to explain the observed difference by missing resonance(s) is not likelihood because in this case one has to add resonance simultaneously in $\pi\pi$ and $\pi K$ channels and hence we would have a light strange resonance what seems to be excluded by data.}
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\[ 4L_2 + L_3 = \frac{a_3(m_s=0)F_0^2}{8} F_\pi^2/F_0^2 \]

Table 1: Contributions of known $\pi\pi$ resonances to sum rules eqs.(13,15)

| Resonance | spin | $4L_2 + L_3 = \frac{a_3(m_s=0)F_0^2}{8}$ | $F_\pi^2/F_0^2$ |
|-----------|------|------------------------------------------|------------------|
| $f_0(975)$ | 0    | $(1.98 \pm 0.44) \times 10^{-5}$        | 0.01             |
| $f_0(1400)$ | 0    | $(3 \pm 2) \times 10^{-5}$              | 0.03             |
| $f_0(1590)$ | 0    | $(2 \pm 2) \times 10^{-6}$              | 0.002            |
| $f_2(1275)$ | 2    | $(1.51 \pm 0.075) \times 10^{-4}$       | 0.1              |
| $f_2(1525)'$ | 2    | $\sim 0$                                | $\sim 0.$        |
| $f_2(1720)$ | 2    | $\sim 0$                                | 0.002            |
| $f_4(2050)$ | 4    | $\sim 0$                                | 0.001            |
| $f_6(2510)$ | 6    | $(1.2 \pm 0.6) \times 10^{-6}$          | 0.001            |
| $\rho(770)$ | 1    | $1.66 \times 10^{-3}$                   | 0.51             |
| $\rho(1450)$ | 1    | $(1.2 \pm 0.2) \times 10^{-4}$          | 0.04             |
| $\rho(1700)$ | 1    | $(2 \pm 3.) \times 10^{-6}$             | 0.003            |
| $\rho_3(1690)$ | 3    | $(2.5 \pm 0.4) \times 10^{-6}$          | 0.03             |
| $\rho_5(2400)$ | 5    | $(2.7 \pm 1.4) \times 10^{-6}$          | 0.003            |
| **Sum**    | -    | $(2.0 \pm 0.1) \times 10^{-3}$          | $\sim 0.7$      |

Table 2: Contributions of known $\pi K$ resonances to sum rules eqs.(16,17)

| Resonance | spin | $\frac{a_1(m_s)F_0^2}{8}$ | $b_1(m_s)F_\pi^2 = F_\pi^2/F_0^2$ |
|-----------|------|----------------------------|-----------------------------------|
| $K^*(1430)$ | 0    | $(9.5 \pm 2.0) \times 10^{-5}$ | 0.08                             |
| $K^*(891)$  | 1    | $(1.53 \pm 0.02) \times 10^{-3}$ | 0.39                             |
| $K^*(1370)$ | 1    | $(1. \pm 1.) \times 10^{-5}$   | 0.01                             |
| $K^*(1400)$ | 1    | $(1. \pm 1.) \times 10^{-6}$   | 0.001                            |
| $K^*(1680)$ | 1    | $(7. \pm 6.) \times 10^{-5}$   | 0.086                            |
| $K^*(1430)$ | 2    | $(8.8 \pm 0.9) \times 10^{-5}$ | 0.073                            |
| $K^*(1780)$ | 3    | $(2.2 \pm 0.4) \times 10^{-5}$ | 0.03                             |
| $K^*(2045)$ | 4    | $(7.9 \pm 2.3) \times 10^{-6}$ | 0.014                            |
| **Sum**    | -    | $(1.83 \pm 0.1) \times 10^{-3}$ | $\sim 0.7$                      |