Homoclinic chaos and energy condition violation

J. Mark Heinzle\textsuperscript{1}, Niklas Röhr\textsuperscript{2} and Claes Uggla\textsuperscript{2}

\textsuperscript{1}Institute for Theoretical Physics, University of Vienna, A-1090 Vienna, Austria and
\textsuperscript{2}Department of Physics, University of Karlstad, S-651 88 Karlstad, Sweden

(Dated: July 31, 2006)

In this letter we discuss the connection between so-called homoclinic chaos and the violation of energy conditions in locally rotationally symmetric Bianchi type IX models, where the matter is assumed to be non-tilted dust and a positive cosmological constant. We show that homoclinic chaos in these models is an artifact of unphysical assumptions: it requires that there exist solutions with positive matter energy density \( \rho > 0 \) that evolve through the singularity and beyond as solutions with negative matter energy density \( \rho < 0 \). Homoclinic chaos is absent when it is assumed that the dust particles always retain their positive mass. In addition, we discuss more general models: for solutions that are not locally rotationally symmetric we demonstrate that the construction of extensions through the singularity, which is required for homoclinic chaos, is not possible in general.

PACS numbers: 04.20.-q, 98.80.Jk, 98.80.Cq, 04.20.Ha

The dynamics of locally rotationally symmetric (LRS) and diagonal Bianchi type IX models with a perfect fluid, usually chosen to be dust, and a positive cosmological constant has been discussed recently in a series of articles \textsuperscript{3,4,5,6}. It was claimed that Einstein’s static universe is a chaotic scatterer so that “homoclinic chaos” is generated in the models. In another article \textsuperscript{7}, LRS type IX models with general equations of state (comprising dust and a positive cosmological constant as a special case) were re-investigated by employing a scale-invariant formulation. This permitted the use of rigorous dynamical systems methods which led to results that excluded the possibility of homoclinic chaos; the dynamics of the models was found to be predictable. In spite of these rigorous results, claims about homoclinic chaos persist in the literature. The purpose of this letter is to resolve these contradictions.

In the first part of this letter we show that homoclinic chaos in LRS Bianchi type IX models is an artifact of unphysical assumptions: for the models to exhibit homoclinic chaos solutions have to be extended through the singularity where it must be assumed that the mass of the dust particles changes sign. The assumption of standard singularity where it must be assumed that the mass of the dust particles always retain their positive mass. In addition, we discuss more general models: for solutions that are not locally rotationally symmetric we demonstrate that the construction of extensions through the singularity, which is required for homoclinic chaos, is not possible in general.

The system \textsuperscript{5} admits Einstein’s static universe as a solution, represented by the fixed point \( E_A = p_B = 1, A = B = 1/\sqrt{4A}, E_0 = 1/\sqrt{4A} \). In \textsuperscript{8}, the field equations were reformulated in terms of scale-invariant variables in order to obtain a regular system. This allowed the use of rigorous dynamical systems methods and thus resulted in a comprehensive description of the global dynamics; in particular, the dynamics of models initially close to \( E \) was found to be predictable and non-chaotic. Since the results are completely invariant and do not rely on the particular choice of variables, the findings of \textsuperscript{8} refute earlier claims about “homoclinic chaos” \textsuperscript{3,4,6}.

The claims about homoclinic chaos rely on the assertion that there exist infinitely many solutions of \textsuperscript{8} that evolve from an initial state close to \( E \), leave any neighborhood of \( E \), and return to a final state close to \( E \) again. It is argued that this behavior is the basis of chaos in the dynamics because of the chaotic distribution of “homoclinic” initial data points among initial data close to \( E \). In the following we demonstrate that the asserted “homo-
clinic solutions” can only exist under unphysical assumptions. For pedagogical reasons, in our considerations we focus on the most recent publication of [1, 2, 3, 4], i.e., [4]; in particular we adopt the notation and terminology.

The analysis of [4] is crucially based on the observation that the system [4] is regular for all $A$ including $A = 0$. This prompts the authors to not restrict the domain of the variable $A$ to $A > 0$ (“truncation at $A = 0$ is bound to a loss of information”), but to consider the regions $A < 0$ and $A = 0$ as a part of phase space (“for the Hamiltonian dynamics [the region $A < 0$] constitutes an essential part of the phase space”).

As a first comment, let us note that the regularity of the equations [4] at $A = 0$ is inseparably connected with the assumption of LRS symmetry. Generalizing [1] to the full diagonal type IX case with spatial metric $\sqrt{4\Lambda}$, the assumption of LRS symmetry. Generalizing (1) to the full diagonal type IX case with spatial metric $A^2(\omega^1)^2 + B^2(\omega^2)^2 + C^2(\omega^3)^2$ yields

$$\frac{1}{8} \left[ \frac{A^2}{BC} \left( \frac{p_A}{C} + \frac{p_B}{A} + \frac{p_C}{B} \right) \right] + \frac{1}{4} \left[ \frac{p_{AB}}{C} + \frac{p_{AC}}{B} + \frac{p_{BC}}{A} \right]$$

for the kinetic term of the Hamiltonian. (For $B = C = p_C$, the kinetic term of [4] is recovered by noting that the new momentum $p_B$ differs from the original one by a factor of 2.) We observe that the non-LRS Hamiltonian and the resulting equations are singular at $A = 0$ (and $B = 0, C = 0$). Any deviation from LRS thus invalidates the statement that “the regions of phase space $A > 0$ and $A < 0$ join smoothly in $A = 0$.” This will be further discussed below.

In general relativity, for LRS line elements of the form [11], one usually makes the restriction $A > 0$ (and $B > 0$); in particular, Hamiltonian approaches typically use exponential representations of the scale factors, e.g., $A = e^\alpha$, see, e.g., [5, 6, 7, 8, 9, 10]. It is tacitly, and correctly, assumed that the restriction to positive scale factors is no loss of generality. This is due to the existence of discrete symmetries: the system [10] is invariant under $(A, p_A, t) \rightarrow -(A, p_A, t)$ and $(B, p_B, t) \rightarrow -(B, p_B, t)$ (where the latter reflects the invariance of the non-LRS system under the simultaneous transformations $(B, p_B, t) \rightarrow -(B, p_B, t)$ and $(C, p_C, t) \rightarrow -(C, p_C, t)$). As a consequence, solutions with $A < 0$ arise from solutions with $A > 0$ via this symmetry map; in this context it is important to note that $E_0 \rightarrow -E_0$ and that $\rho$ is invariant under the transformations.

The phase space considered in [4] is the union of the regions $A > 0$ and $A < 0$, “smoothly joined on $A = 0$”; in particular, $A = 0$ is not regarded as a singularity from a Hamiltonian point of view. The authors give numerical evidence for the existence of an infinite set of solutions with initial data close to the fixed point $E$ (i.e., with $A > 0$) that pass through $A = 0$, enter the region $A < 0$, re-enter $A > 0$ through $A = 0$, and eventually enter a neighborhood of $E$ again. It is the existence of these “homoclinic solutions” that is argued to cause “homoclinic chaos” in the model. For the homoclinic solutions $E_0 = \text{const} > 0$ holds; in fact, $E_0$ is close to $1/\sqrt{4\Lambda}$, which is the value characterizing Einstein’s universe $E$.

The positivity of $E_0$ has far reaching consequences: from $E_0 = 2AB^2\rho$ we obtain

$$\rho = \frac{E_0}{2AB^2},$$

which implies that $\rho < 0$, if $A < 0$ (since $E_0 > 0$). Therefore models described by solutions with $A < 0$ violate the (weak/null/strong/dominant) energy condition; indeed, since the matter content is dust, $\rho < 0$ implies that the dust particles have negative inertial and gravitational mass. The homoclinic solutions described in [4] evolve between the regions $A > 0$ and $A < 0$, hence $\rho > 0$ changes to $\rho < 0$ (and back) during the evolution. However, the change of the density of the dust particles from positive to negative is not continuous: when $A \searrow 0$ we find $\rho \rightarrow \infty$, when $A \nearrow 0$ we have $\rho \rightarrow -\infty$. (This is not immediate from [4], but follows from the fact that the spatial volume density $AB^2$ converges to zero as the singularity is approached.) Hence extending a solution from $A > 0$ via $A = 0$ to $A < 0$ (and back) corresponds to joining $\rho = +\infty$ and $\rho = -\infty$ (and again $-\infty$ with $+\infty$).

In [4] the authors emphasize that it is essential to incorporate the regions $A < 0$ and $A = 0$ into the analysis, since the “dynamical phenomena that occur in $A < 0$ are present in $A > 0$ transported there via the Hamiltonian flow”. Naturally, there exist numerous examples in physics where the consideration of unphysical states is used as an abstract mathematical tool that makes possible or facilitates the analysis of physical states. Here the case is totally different. The system [4] is an autonomous system, which makes the flow a local phenomenon. In particular the phase space picture in the region $A > 0$ depends exclusively on $A > 0$; it is a basic principle that (changes of) the flow in $A < 0$ cannot affect the phase space picture in $A > 0$.

Though irrelevant, the assertion that the system [4] in the region $A < 0$ is associated with complex dynamics (see, e.g., Figs. 17 and 18 in [4]) is likely to be correct, but considering that we are dealing with dust particles with negative masses this should perhaps not be too surprising. Note that the occurrence of negative masses is not primarily due to $A < 0$, but due to the fact that $A < 0$ while $E_0$ is kept positive. Recall in this context that application of the symmetry map $(A, E_0) \rightarrow -(A, E_0)$, cf. the discussion above, would leave the dynamics (and the positivity of $\rho$) unchanged.

We conclude that there exist two attitudes one can

---

1 We also adopt the term “homoclinic solutions” here, although this is a deviation from standard dynamical systems nomenclature.
take to homoclinic chaos in LRS type IX models with dust and a positive cosmological constant.

(i) A conservative attitude, which is the prevailing opinion of the classical general relativity community: this is also the view we hold. Restricting the phase space to positive scale factors is no loss of generality; \( A = 0 \) represents a singularity of the model, since \( \rho \to \infty \) when \( A \to 0 \). Extending a cosmological model as a solution with \( A < 0 \) (and \( E_0 > 0 \)) amounts to joining \( \rho = +\infty \) with \( \rho = -\infty \), and it must be assumed that the dust particles replace their originally positive mass by negative mass. Any such extension is unreasonable from the point of view of physics, and irrelevant for the description of physical states from the point of view of mathematics.

(ii) An unconventional attitude. The phase space is given by the union of \( A > 0, A = 0, \) and \( A < 0; A = 0 \) is not a singularity in the sense that the evolution of cosmological models does not stop. There exists a mechanism that transports the model (which is characterized by \( E_0 > 0 \) and thus has positive mass originally) through \( A = 0 \ (\rho = \pm \infty) \) to a model with negative mass, and back. The constructed spacetimes are of the form \( \mathbb{R} \times M \), where the manifold \( M \) is Riemannian except for at a discrete set of times, where the metric on \( M \) is degenerate. At times of degeneracy, the density of the dust particles changes sign from \( \rho > 0 \) via \( \rho = +\infty \) and \( \rho = -\infty \) to \( \rho < 0 \) (or reversed).

If one takes view (i), it can be proved, see [3], that no homoclinic chaos exists and that the dynamics is predictable. The results in [3] are completely invariant and do not rely on the particular scale-invariant formulation; however, the results do rely on the assumption that the dust particles retain their positive mass.\(^{2}\)

View (ii) is associated with severe problems. First, we note that although formally, i.e., from a purely mathematical point of view, the extension of the scale factor \( A \) through \( A = 0 \) is feasible, this extension relies crucially on the assumption of LRS symmetry. When LRS symmetry is broken (as it certainly is for generic models) it is far from clear whether an extension of generic solutions through the singularity exists. Let us elaborate on this issue.

In the LRS case, the regularity of (2) and (3) in \( A \) allowed [4] to also regularize the problem in \( B \) by a change of the time variable that only involved \( B \). In this formulation, \( B = 0 \) was an invariant subset, so that orbits could not pass through \( B = 0 \); in contrast, the set \( A = 0 \) was not invariant, which was exploited to extend orbits beyond the singularity \( A = 0 \). In the full diagonal type IX case the equations are singular at \( A = 0, B = 0, \) and \( C = 0 \). It is possible to regularize the equations by a change of the time variable (in a spirit similar to the \( B \) regularization in [4]). However, this re-parametrization must necessarily involve all three scale factor due to permutation symmetry, e.g., one could choose a lapse proportional to \( ABC \). As a consequence, \( A = 0, B = 0, \) and \( C = 0 \) become invariant subsets, which prevents solutions from changing the sign of the scale factors (and thus from changing the sign of \( \rho \)). Hence, in the non-LRS case, an extension of solutions through the singularity lacks a mathematical foundation.

That LRS models are of a special nature is well known. In particular, singularities that occur in the LRS case are very different from those occurring in the diagonal case. In the LRS case solutions generically approach the singularity along a so-called Taub asymptote. This suggests that the singularity may be a weak null singularity, see [12] and references therein, and that is possible to extend the spacetime through the singularity in a \( C^0 \) manner to the Minkowski spacetime. It is thus not inconceivable that the metric can be extended, but it is unclear whether such a construction bears any relation to the extension of the scale factor \( A \) critically discussed above. The problem of whether the extension of \( A \) leads to an extension of the metric is not discussed in [4]; clearly, the extension of \( A \) is completely irrelevant unless it leads to an extension of the metric.

In the diagonal (non-LRS) case, on the other hand, the situation is completely different: solutions generically approach the singularity in a Mixmaster fashion and the singularity is a spacelike scalar curvature singularity [13]; no extension of the metric is possible. Since the Mixmaster behavior is typical for generic cosmological singularities, see e.g. [14] and references therein, these considerations seem to rule out the type of manipulations and arguments presented in [4] for generic models. It appears that the special LRS case is quite misleading when it comes to spacetimes and phase space extensions.

Even in the LRS case, not all solutions possess extensions of the kind discussed above; a Taub asymptote is a prerequisite. This is because for such solutions \( B \not\to 0 \) when the singularity is approached; solutions with \( B \to 0 \), e.g. solutions with an isotropic singularity, are inextendible. This creates a link to the related question of why it is possible for unphysical solutions \( \rho < 0 \) to expand from a singularity, and then recollapse to a singularity at \( A = 0 \) (and subsequently re-enter the physical part of the phase space). At first this is quite surprising since

\(^{2}\) The essence of view (i) pervades all theories that model phenomena with differential equations: although the equations might be well-defined (and regular) for a range of a variable that is greater than its physical range, the physicality condition prohibits the extension of solutions. Examples abound not only in physics; for instance, in population models in mathematical biology [11], solutions will never be extended to negative numbers of individuals of a species.
dust particles with negative mass generate anti-gravity, and one would thus expect such solutions to expand to a state infinite dilution, $\rho \to 0$, which is confirmed by numerics for typical solutions. However, the solutions found numerically in [4] have Taub asymptotes and are thus vacuum dominated with extreme shear; since shear represents gravity generating gravity, these solutions can recollapse to the singularity and enter the physical region again. Note, however, that this scenario is only correct for solutions with Taub asymptotes with initial data such that the shear dominates during a sufficiently long time.

Finally, let us point out that, irrespective of the mathematical considerations, at present there does not exist any physical theory that could explain the traversing of the singularity in the way implicitly suggested by adherents of view (ii). Note that certain speculative mechanisms like “quantum tunneling of the wave function of the universe” (see [15]) are insufficient, since they lack an explanation for the metamorphosis of the dust particles from positive to negative mass. In addition, in view of the discussion above, such a mechanism is a serious candidate only if it can explain the behavior of generic models and not merely the exceptional LRS type IX models.

It seems to us that the burden of making a convincing case for homoclinic chaos in Bianchi type IX models lies with adherents to view (ii); considering the above, this is a formidable challenge indeed.

* Electronic address: Mark.Heinzle@aei.mpg.de
† Electronic address: niklas.roehr@kau.se
‡ Electronic address: Claes.Uggla@kau.se

Note added.

After submission of this letter an erratum to [4] was published, see [16]. In this small note the authors state that the physical interpretation on the existence of homoclinic chaos has to be corrected, but the authors do not give any explanation for this statement (see, however, the first part of the present letter); furthermore, it is not discussed that the corrected interpretation also affects the status of homoclinic chaos in non-LRS models (see, however, the second part of the present letter).