Bayesian Nonparametric Mixtures of Exponential Random Graph Models for Ensembles of Networks

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January 21, 2022

Abstract

Ensembles of networks arise in various fields where multiple independent networks are observed on the same set of nodes, for example, a collection of brain networks constructed on the same brain regions for different individuals. However, there are few models that describe both the variations and characteristics of networks in an ensemble at the same time. In this paper, we propose to model the ensemble of networks using a Dirichlet Process Mixture of Exponential Random Graph Models (DPM-ERGMs), which divides the ensemble into different clusters and models each cluster of networks using a separate Exponential Random Graph Model (ERGM). By employing a Dirichlet process mixture, the number of clusters can be determined automatically and changed adaptively with the data provided. Moreover, in order to perform full Bayesian inference for DPM-ERGMs, we employ the intermediate importance sampling technique inside the Metropolis-within-slice sampling scheme, which addressed the problem of sampling from the intractable ERGMs on an infinite sample space. We also demonstrate the performance of DPM-ERGMs with both simulated and real datasets.

Keywords: Dirichlet process; Importance sampling; Metropolis Hastings; Slice sampling

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1 Introduction

Networks, as representations of relational data, are widely used in various scientific fields, such as sociology, neuroscience and biology. They provide valuable insight in understanding the diverse processes behind the complex dependent interactions among different objects. With the recent development of technology, ensembles of networks are increasingly available, which stand for multiple observations obtained on the same or similar set of nodes across different subjects or time points. Examples of ensembles of networks include a collection of brain networks from a number of participants (Simpson et al., 2013), social networks across different schools (Sweet et al., 2019), and among others. There are high demands for developing the methodology to identify the characteristics that are common or unique across individuals by taking advantage of the wealth of data presented in an ensemble.

The statistical modeling of ensembles of networks has also been motivated by the accessibility of abundant network data. Some researchers treat networks in an ensemble as replicates or duplicates of a true underlying network. Durante et al. (2017) extended the latent space models using a Bayesian nonparametric approach to infer common network patterns of all networks. The differences across networks are ignored in this way as they assume that all networks within an ensemble have the same structure. In contrast, some authors argue that networks from an ensemble vary from subject to subject. Paul and Chen (2020) developed a random effect stochastic block model, where the individual variations from the mean community structure of the population are considered in the model. Similarly, Arroyo et al. (2021) introduced a common subspace independent-edge multiple random graph model that includes both the common invariant submatrix for modeling the shared latent structures and an individual score matrix for describing the individual difference.

Within an ensemble, some networks share common structures, while others exhibit distinct features. Group representation is a powerful tool to capture the similarities and differences of network structures in the same ensemble. Durante and Dunson (2018) introduced a Bayesian method to test the differences between two given groups of networks. Lehmann and White (2021) developed a multilevel network model to compare networks from different groups. In most cases, the underlying group structure is unknown and it is therefore necessary to develop a methodology that identifies the group membership and compares groups of networks simultaneously. Signorelli and Wit (2020) introduced a model-based clustering method based on mixtures of generalized linear models for populations of networks. Yin et al. (2020) proposed a finite mixture of exponential random graph models to model the ensemble of networks using the pseudo likelihood method. However, for both models, the number of clusters need to be determined in advance. Also, the generalized linear models and the pseudo likelihood method assume the edges within a network are independent, which is not practical in real datasets.

In this paper, we propose the Dirichlet Process Mixtures of Exponential Random Graph Models (DPM-ERGMs) for ensembles of networks. The Dirichlet process mixture model uses the Dirichlet process as a prior over an infinite mixture model, where the number of mixtures can grow adaptively with the data. This enables the model to determine the group structure of the ensemble automatically, in other words, to compare different
networks without prior knowledge of the number of clusters. Moreover, the Dirichlet process provides a large sample space and tractable posterior distributions, facilitating inference on the infinite sample space (Ferguson, 1973). On the other hand, the Exponential Random Graph Model (ERGM), a versatile network model, is employed to model networks for its ability to represent various types of topological features. Thus, DPM-ERGMs are capable of determining the group structure and describing the group characteristics of an ensemble simultaneously.

There are two challenges in performing Bayesian inference for DPM-ERGMs: the infinite sample space of Dirichlet process mixtures and the intractability of ERGM likelihood. The slice sampling algorithm (Walker, 2007) provides a way to sample from the posterior distribution of Dirichlet process mixture models. To sample from the infinite sample space, we borrow the idea of slice sampling and introduce a latent variable for the model, which helps us to find a finite set of components required to produce the correct Markov chain. Then the inference can be performed by sampling from the full conditional distributions of all variables on a finite space. However, the slice sampling algorithm was designed for the Dirichlet process mixtures of normal distributions, where the sampling methods related to the normal distribution are widely available. In DPM-ERGMs, sampling from posterior distributions of ERGM parameters and membership variables is challenging due to the intractable ERGM likelihood.

One way to sample from the posterior distributions of ERGMs is to use Metropolis Hastings algorithms. Standard Metropolis Hastings algorithms are not applicable since the acceptance probability depends on the intractable normalizing constants. To address this issue, Caimo and Friel (2011) applied the exchange algorithm (Murray et al., 2006), where a perfect sampler is employed to facilitate the Metropolis Hastings algorithm, avoiding the calculation of the intractable normalizing constant. As the perfect sampler from the ERGM is unavailable in most cases, a sample from the MCMC method is used in practice. Liang and Jin (2013) developed a Monte Carlo Metropolis Hastings (MCMH) algorithm to sample from the intractable posterior distributions. The algorithm is implemented by approximating the unknown normalizing constant ratio in the acceptance probability using a Monte Carlo estimate and is proved to converge to the desired target distribution. The exchange algorithm can be seen as a special case of the MCMH algorithm. However, most of the literature on ERGMs only deals with the single network situation. In DPM-ERGMs, networks from the same group are multiple samples from the same ERGM distribution. This requires the Bayesian inference to have the ability of incorporating multiple network samples.

To sample from the posterior distributions of DPM-ERGMs, we develop a Metropolis-within-slice sampling algorithm that employs Metropolis Hastings inside the slice sampling algorithm. Specifically, we extend the MCMH algorithm to a Multi-network MCMH (MM-CMH) algorithm in order to update the ERGM parameters that represent multiple networks from the same group. An importance sampling estimator with intermediate values is used in MM-CMH to approximate the normalizing constant ratio in the acceptance probability to ensure the accuracy of the estimation. In this way, the characteristics of the whole group can be captured by pooling information across networks. Besides, posterior samples of membership variables also suffer from the intractability issue. We express the membership
variable distributions in such a way that a ratio of normalizing constants is obtained, and employ an intermediate importance sampling estimator to approximate the constructed ratio. We refer to the combined algorithm as Intermediate Importance Metropolis-within-Slice (IIMS) sampling algorithm. The IIMS sampling algorithm allows the full Bayesian inference to be performed based on the true likelihood, and is capable of modeling complex dependency structures beyond the pairwise interactions. Moreover, we can replace the true likelihood with the pseudo likelihood function in the Metropolis-within-slice scheme to achieve a faster, approximate computation. We will illustrate both methods in detail later.

The rest of paper is organized as follows. In Section 2, we describe how the DPM-ERGMs are formulated. Section 3 provides the sampling methodology. Section 4 presents the simulation studies. We summarize the paper in Section 5.

2 Model Formulation

2.1 Exponential Random Graph Models

ERGMs describe the generating process of networks through exponential family distributions with summary statistics showing various connecting patterns as explanatory variables. A network with $n$ nodes is typically represented by a random adjacency matrix $Y \in \{0, 1\}^{n \times n}$, where $Y_{ij} = 1$ indicates an edge between nodes $i$ and $j$, and $Y_{ij} = 0$ otherwise. The realization of $Y$ is denoted by $y$ while the set of all possible outcomes of $Y$ is denoted by $\mathcal{Y}$. The covariate information regarding the nodal or network attribute that affects the connections are denoted by $X \in \mathcal{X}$. The network structures of interest are expressed using a summary statistics vector, $S(y, X) : \mathcal{Y} \times \mathcal{X} \rightarrow \mathbb{R}^d$. It represents the characteristics of the network, such as the number of edges, triangles, etc, which are crucial to the formation and dissolution of networks. The general ERGM has the following form,

$$P(Y = y | \theta, X) = \frac{\exp\{\theta^\top S(y, X)\}}{k(\theta)},$$

where $\theta \in \mathbb{R}^d$ is the vector of model parameters, and $S(y, X)$ is the summary statistics (Morris et al., 2008). The normalizing constant $k(\theta) = \sum_{y \in \mathcal{Y}} \exp\{\theta^\top S(y, X)\}$ is the sum over all potential graphs in the sample space, which is usually intractable except for very small networks. Given a realization of network $y$, the aim of statistical inference is to find which value of $\theta$ provides best description for the data under ERGM framework. The intractability of the normalizing constant is a strong barrier to the estimation of ERGMs as the likelihood function can only be specified up to a parameter dependent constant.

Bayesian inference is a natural choice for ERGMs since it allows uncertainty on model parameters. The posterior distribution of ERGMs is

$$f(\theta | y, X) = \frac{\pi(\theta)P(Y = y | \theta, X)}{P(Y = y | X)},$$

where $\pi(\theta)$ is the prior, $P(Y = y | X) = \int_{\mathbb{R}^d} \pi(\theta)P(Y = y | \theta, X)d\theta$. The standard MCMC algorithm is not suitable since the acceptance probability as shown in (3) to move from $\theta$
to the new proposal $\theta'$ requires evaluation of the intractable constants $k(\theta)$ and $k(\theta')$ at each step of the algorithm

$$\frac{\pi(\theta') h(\theta|\theta')}{\pi(\theta) h(\theta'|\theta)} \cdot \frac{\exp{\theta'^T S(y_i, X_i)}}{\exp{\theta^T S(y_i, X)}} \cdot \frac{k(\theta)}{k(\theta')},$$

(3)

Here, $h(\cdot)$ stands for the proposal distribution. MCMH algorithm (Liang and Jin, 2013) samples from the posterior ERGMs by using an importance sampling estimator to approximate $k(\theta)/k(\theta')$ in the Metropolis Hastings algorithm.

### 2.2 Dirichlet Process Mixtures of ERGMs

Ensembles of networks include multiple network observations. In addition to the complex structures within each network, one may also be interested in studying the variations across different networks. Mixture models are a natural approach to describe such a population as they can detect and characterize the subpopulations that share common structures and distinguish networks that are different automatically. In particular, the infinite mixture model is applied here because the corresponding model complexity is adjusted to the data. Here, we propose to model the ensemble of networks through an infinite mixture of ERGMs, each component of which represents a cluster (subpopulation) of networks that share common structures using a cluster-specific ERGM.

An ensemble with $N$ networks is denoted by $\{Y_i\}_{i=1}^{N}$, and the corresponding covariate information is $\{X_i\}_{i=1}^{N}$. In such an ensemble, the single network $Y_i$ is represented using an infinite mixture of ERGMs as follows

$$P_{w,\theta}(Y_i = y_i | X_i) = \sum_{j=1}^{\infty} w_j \frac{\exp{\theta_j^T S(y_i, X_i)}}{k(\theta_j)},$$

(4)

where $j$ is the cluster label, $w_j$ is the mixing proportion, $\theta_j$ is the cluster specified parameter vector, $S(y_i, X_i)$ is the summary statistics of network $y_i$, and $k(\theta_j) = \sum_{y \in Y} \exp{\theta_j^T S(y, X)}$ is the normalizing constant. Without requiring a fixed number of clusters in advance, the infinite mixture model is able to determine the number of clusters adaptively with the data provided.

The likelihood of the ensemble of networks can be expressed as

$$P_{w,\theta}(\{Y_i = y_i\}_{i=1}^{N} | \{X_i\}_{i=1}^{N}) = \prod_{i=1}^{N} \sum_{j=1}^{\infty} w_j \frac{\exp{\theta_j^T S(y_i, X_i)}}{k(\theta_j)},$$

or

$$P_{\theta}(\{Y_i = y_i\}_{i=1}^{N} | \{X_i, Z_i = k_i\}_{i=1}^{N}) = \prod_{i=1}^{N} \frac{\exp{\theta_{k_i}^T S(y_i, X_i)}}{k(\theta_{k_i})}.$$

where $Z = (Z_1, Z_2, \ldots, Z_N)$ is a latent variable to indicate the membership of each network, e.g. $Z_i = k_i$ if $y_i$ belongs to cluster $k_i$. It is informative to consider an infinite mixture model
especially when it is not appropriate to have a limit on the number of groups. However, the inference of this model is challenging because the intractable normalizing constant has to be evaluated in the infinite sample space.

To perform Bayesian inference on the proposed infinite mixture of ERGMs, we adopt a Dirichlet process prior DP(β, H) (Ferguson, 1973), which is arguably the most commonly used Bayesian nonparametric prior. Under the constructive definition, also known as the stick-breaking representation (Sethuraman, 1994), the mixing proportion \( w \) is constructed using a stick-breaking procedure with an auxiliary variable \( v \). A sequence of independent and identically distributed auxiliary variables \( v_1, v_2, \ldots \) are sampled from a prior distribution Beta(1, \( \beta \)), and the mixing proportions are set as \( w_1 = v_1, w_j = v_j \prod_{l=1}^{j-1}(1 - v_l) \) (for \( j > 1 \)). The membership indicator variable \( Z \) follows a multinomial distribution Mult(\( w \)) with probability \( w = (w_1, w_2, \ldots) \). For the prior of ERGM parameter \( \theta_j \), we use a multivariate Gaussian distribution \( \mathcal{N}(\mu_0, \Sigma_0) \). Given the membership \( Z_i = k_i \), the network \( Y_i \) is modeled by an ERGM with parameter \( \theta_{k_i} \). In the remaining of this paper, we will use Dirichlet Process Mixtures of Exponential Random Graph Models (DPM-ERGMs) with the following form,

\[
\begin{align*}
v_j &\sim \text{Beta}(1, \beta) \\
w_1 &= v_1, w_j = v_j \prod_{l=1}^{j-1}(1 - v_l) \\
z_i|w &\sim \text{Mult}(w) \\
\theta_j|\mu_0, \Sigma_0 &\sim \mathcal{N}(\mu_0, \Sigma_0) \\
y_i|Z_i = k_i, \theta &\sim P_{\theta_{k_i}}(Y_i = y_i | X_i).
\end{align*}
\]

Here, \( P_{\theta_{k_i}}(Y_i = y_i | X_i) = \exp\{\theta_{k_i}^T S(y_i, X_i)\}/k(\theta_{k_i}) \) is the ERGM with parameter \( \theta_{k_i} \).

### 3 Posterior Computation

The statistical inference for the proposed model is very challenging due to the infinite number of mixture components and the intractable ERGM likelihood. In this section, we first develop a Metropolis-within-slice sampling algorithm to address the issue of sampling from the infinite sample space of DPM-ERGMs. Then, we provide details of the algorithms based on a true and pseudo likelihood approach separately.

The slice sampling algorithm (Walker, 2007; Kalli et al., 2011) provides a way to sample from the infinite mixture components. Similar to the slice sampling, we first introduce a latent variable \( u \) to our proposed model to identify the exact number of components that are required to produce a valid Markov chain with the correct stationary distributions. The joint density of \((y, u)\) is written as

\[
P_{w,\theta}(Y = y, u | X, \xi) = \sum_{j : \xi_j > u} \frac{w_j}{\xi_j} P_{\theta_j}(Y = y | X).
\]

Compared with the original density (4), there are only finite numbers of \( j \) satisfying \( w_j > u \). In other words, the inference can be performed by sampling from the finite set \( \{j : \xi_j > u\} \),
which simplifies the problem dramatically. $\xi$ is a deterministic decreasing sequence used to address the update of $u$. See Kalli et al. (2011) for details and choices of $\xi$.

Furthermore, with indicator variable $Z$, the joint density can be expressed as

$$P_{w,\theta}(Y = y, u, Z = k | X, \xi) = \frac{w_k}{\xi_k} 1(u < \xi_k) P_{\theta_k}(Y = y | X).$$

Hence, the likelihood for the ensemble $\{Y_i\}_{i=1}^N$ with latent variable $u$ and sequence $\xi$ is

$$l_{w,\theta}(\{Y_i = y_i, Z_i = k_i, u_i\}_{i=1}^N | \{X_i\}_{i=1}^N, \xi) = \prod_{i=1}^N \frac{w_k}{\xi_k} 1(u_i < \xi_k) P_{\theta_k}(Y_i = y_i | X_i).$$  \(6\)

With the prior distribution specified in \(5\), the full conditional distributions of all variables $(u, w, \theta, Z)$ are available. The Metropolis-within-slice sampling scheme is performed by sampling $(u, w, \theta, Z)$ from their full conditional distributions in turn. In particular, as the direct sampling from ERGMs is not possible, Metropolis Hastings algorithm is used to assist the sampling of $\theta$.

### 3.1 True likelihood based IIMS Algorithm

In order to overcome the intractability issue and perform accurate estimation to the original model, we propose to employ the intermediate importance sampling technique in the Metropolis-within-slice sampling scheme, and name this algorithm as IIMS algorithm. The sampling procedures of the true likelihood based IIMS algorithm are listed as follows.

**Step 1.** Sample $u_i$ from a uniform distribution,

$$u_i \sim U(0, \xi_{k_i}) \quad (i = 1, 2, \ldots, N),$$  \(7\)

where $k_i$ is the current allocation of network $y_i$.

**Step 2.** Sample $v_j$ from a beta posterior distribution,

$$v_j \sim \text{Beta}(1 + a_j, \beta + b_j) \quad (j = 1, 2, \ldots, K^*).$$  \(8\)

Here, $a_j = \sum_{i=1}^N 1(k_i = j)$ denotes the number of networks in group $j$ and $b_j = \sum_{i=1}^N 1(k_i > j)$ corresponds to the number of networks in the groups whose label are bigger than $j$. $K^*$ denotes the current number of clusters.

Update $w_j$ with

$$w_1 = v_1, w_j = v_j \prod_{l=1}^{j-1} (1 - v_l) \quad (j = 2, \ldots, K^*).$$  \(9\)

**Step 3.** Sample $\theta_j$ ($j = 1, 2, \ldots, K^*$) using the MMCMH algorithm with the following procedures,

(1) Draw $\theta'_j$ from a proposal distribution $h(\cdot | \theta_j)$.

(2) Simulate $m_2$ networks from each intermediate distribution with parameter $\theta^{im}_r$ ($r = 0, 1, \ldots, m_1$) individually and store the network statistics using $S(z^s_r)$ ($s = 1, 2, \ldots, m_2$), where $\theta^{im}_r$ ($r = 1, 2, \ldots, m_1$) are $m_1$ intermediate values between $\theta^{im}_0 = \theta_j$ and $\theta^{im}_{m_1+1} = \theta'_j$.
(3) Estimate the normalizing constant ratio \( k(\theta'_{j})/k(\theta_{j}) \) with an intermediate importance sampling estimator

\[
\gamma = \prod_{r=0}^{m_1} \frac{1}{m_2} \sum_{s=1}^{m_2} \exp\{((\theta'^{im}_{r+1} - \theta^{im}_{r})^\top S(z^s_r))\}. \tag{10}
\]

(4) Accept \( \theta'_{j} \) with probability

\[
\alpha = \min\left(1, \frac{\pi(\theta'_{j})h(\theta'_{j}|\theta'_{j}) \exp\{((\theta'_{j} - \theta_{j})^\top \sum_{z_{i}=j} S(y_{i}, X_{i}))\}}{\pi(\theta_{j})h(\theta'_{j}|\theta_{j}) \gamma \sum_{i} 1(z_{i}=j)} \right). \tag{11}
\]

\( \pi(\theta_{j}) \) is the prior distribution.

**Step 4.** Sample \( Z_{i} \) from a multinomial distribution with probability proportional to a normalizing constant dependent ratio,

\[
P(Z_{i} = k_{i} | \cdots) \propto 1(\xi_{k_{i}} > u_{i}) \frac{w_{k_{i}}}{\xi_{k_{i}}} \cdot \exp\{\theta'^{\top} S(y_{i}, X_{i})\} \cdot \frac{k(\theta_{c})}{k(\theta_{k_{i}})} \quad (i = 1, 2, \ldots, N). \tag{12}
\]

Here, \( k(\theta_{c}) \) is multiplied to construct a computable normalizing constant ratio and the normalizing constant ratios for different groups \( k(\theta_{c})/k(\theta_{k_{i}}) \) \((k_{i} = 1, \ldots, K^{*})\) are approximated using an intermediate importance sampling estimator as in (10).

Remark: in **Step 3**, we use a MMCMH algorithm to sample \( \theta_{j} \) from the posterior ERGMs with multiple networks. Next, we will explain how the MMCMH algorithm is developed in Section 3.1.1. Also, we will show the construction of formula (12) in Section 3.1.2.

### 3.1.1 Sample \( \theta \)

The posterior distribution of group parameter \( \theta_{j} \) is proportional to the product of prior \( \pi(\theta_{j}) \) and the joint likelihood of the networks in group \( j \), which is

\[
f(\theta_{j}| \cdots) \propto \pi(\theta_{j}) \prod_{Z_{i}=j} \frac{\exp\{\theta'^{\top} S(y_{i}, X_{i})\}}{k(\theta_{j})}. \tag{13}
\]

Sampling from such a posterior distribution is challenging as it depends on the product of multiple intractable likelihood functions. MCMH algorithm (Liang and Jin, 2013) was designed to sample from the posterior ERGM of a single network. Here, we extend the MCMH algorithm to a MMCMH algorithm for the multiple network case.

In MCMH algorithm, \( k(\theta'_{j})/k(\theta_{j}) \) is approximated with an importance sampling estimator

\[
\frac{1}{m_2} \sum_{s=1}^{m_2} \exp\{((\theta'_{j} - \theta_{j})^\top S(z^s))\}, \tag{14}
\]

with \( z^s (s = 1, 2, \ldots, m_2) \) denoting a sequence of \( m_2 \) independent auxiliary networks sampled from the ERGM with parameter \( \theta_{j} \). However, the importance sampling estimate will be incorrect if \( \theta'_{j} \) and \( \theta_{j} \) are not close enough (Neal, 2005). This obstacle can be overcome
by introducing intermediate distributions between \( \theta'_j \) and \( \theta_j \). Specifically, we interpolate \( m_1 \) values, \( \theta_{r+1}^{im} \) (\( r = 1, 2, \ldots, m_1 \)), so that \( \theta_{r+1}^{im} \) and \( \theta_{r+1}^{im} \) are close enough, and factorize the normalizing constant ratio using intermediate values,

\[
\frac{k(\theta'_j)}{k(\theta_j)} = \prod_{r=0}^{m_1} \frac{k(\theta_{r+1}^{im})}{k(\theta_{r+1}^{im})} = \frac{k(\theta_{r+1}^{im}) k(\theta_{r+1}^{im}) \cdots k(\theta_{m_1+1}^{im})}{k(\theta_{m_1+1}^{im})},
\]

where \( \theta_0^{im} = \theta_j \) and \( \theta_{m_1+1}^{im} = \theta'_j \). Then, each factor \( k(\theta_{r+1}^{im})/k(\theta_{r}^{im}) \) are estimated using importance sampling estimator.

Therefore, the intermediate importance sampling estimator to \( k(\theta'_j)/k(\theta_j) \) is written as

\[
\gamma = \prod_{r=0}^{m_1} \frac{1}{m_2} \sum_{s=1}^{m_2} \exp\{((\theta_{r+1}^{im} - \theta_{r}^{im}) \top S(z^s_r))\}.
\]

where \( z^s_r \) (\( s = 1, 2, \ldots, m_2 \)) is a sequence of \( m_2 \) independent networks sampled from the ERGM with parameter \( \theta_{r}^{im} \).

To sample from \( \{13\} \) using MMCMH algorithm, we propose \( \theta'_j \) from \( h(\cdot | \theta_j) \), and accept \( \theta'_j \) with probability

\[
\frac{\pi(\theta'_j) h(\theta_j | \theta'_j) \exp\{((\theta'_j - \theta_j) \top \sum_{z_i=j} S(y_i, X_i))\}}{\pi(\theta_j) h(\theta'_j | \theta_j) \gamma \sum_i 1(z_i=j)}.
\]

With the approximation to the normalizing constant ratio available, the acceptance ratio is calculable and thus the posterior sampling is feasible. Compared with importance sampling, the use of intermediate values increases the quality of estimation by introducing intermediate distributions. Similar techniques like annealed importance sampling and linked importance sampling \([\text{Neal, 2005}]\) can be used as well.

### 3.1.2 Sample \( Z \)

The full conditional distribution of \( Z_i \) is

\[
P(Z_i = k_i | \cdots) \propto 1(\xi_{k_i} > u_i) \frac{w_{k_i}}{\xi_{k_i}} \cdot \frac{\exp\{\theta_{k_i}^\top S(y_i, X_i)\}}{k(\theta_{k_i})}.
\]

The ratio on the right hand side depends on an intractable normalizing constant \( k(\theta_{k_i}) \), which makes the direct sampling infeasible. Unlike the acceptance probability, there is no normalizing constant ratio involved in the posterior membership probability. However, if we can construct a normalizing constant ratio in the posterior membership probability, we will be able to borrow the strength of intermediate importance sampling to allocate the network samples. To do so, we multiply a constant \( k(\theta_c) \) to each term of the posterior probability vector and obtain

\[
P(Z_i = k_i | \cdots) \propto 1(\xi_{k_i} > u_i) \frac{w_{k_i}}{\xi_{k_i}} \cdot \exp\{\theta_{k_i}^\top S(y_i, X_i)\} \frac{k(\theta_c)}{k(\theta_{k_i})}.
\]
where the constructed normalizing constant ratios $k(\theta_c)/k(\theta_{k_i})$ ($k_i = 1, 2, \ldots, K^*$) using intermediate importance sampling estimation as shown in (10). Thus, the posterior probability ratios will not change and sampling can be performed.

The choice of $\theta_c$ is important to the accuracy of the intermediate importance sampling estimation. The estimation will be incorrect if the parameters to be compared, $\theta_c$ and $\theta_j$, are not close enough. As each group has a unique $\theta_j$, it is impossible to find one $\theta_c$ close to all $\theta_j$ at the same time. Simple importance sampling is not applicable here and multiple intermediate values must be used to ensure the quality of estimation.

### 3.2 Pseudo likelihood based PMS Algorithm

In addition to the true likelihood approach in 3.1, we also propose a fast estimation method based on the pseudo likelihood (Strauss and Ikeda, 1990), which is an approximation to the true likelihood. To be specific, the algorithm is developed by employing a pseudo likelihood approximation in the Metropolis-within-slice sampling algorithm. We name this pseudo likelihood based algorithm as PMS algorithm. In the PMS algorithm, $(u, w)$ are sampled in the same way as in the IIMS algorithm, and $(\theta, Z)$ are updated with pseudo likelihood replacement.

The pseudo likelihood method approximates the true likelihood using the product of conditional probabilities of all edges in a network,

$$PL_\theta(Y = y | X) = \prod_{r \neq s} P(y_{rs} = 1 | y_{-rs}, X)^{y_{rs}} \left(1 - P(y_{rs} = 1 | y_{-rs}, X)\right)^{1 - y_{rs}},$$

where $y_{-rs} = \{y_{kl}, (k, l) \neq (r, s)\}$ denotes all the dyads of the graph excluding $y_{rs}$. Here, $y_{rs}$ is described using Bernoulli distribution with probability defined by change statistics, $\Delta S_{rs} = S(y_{rs} = 1, y_{-rs}, X) - S(y_{rs} = 0, y_{-rs}, X)$, which indicates the changes of $y_{rs}$ on the summary statistics,

$$P(y_{rs} = 1 | y_{-rs}, X) = \frac{\exp(\theta^T \Delta S_{rs})}{1 + \exp(\theta^T \Delta S_{rs})}.$$

If we replace the true likelihood with use pseudo likelihood, then the acceptance ratio for sampling $\theta_j$ using Metropolis Hastings algorithm is

$$\frac{\pi(\theta'_j) h(\theta_j | \theta'_j) \prod_{i = j}^{\xi_j} PL_{\theta'_j} (Y_i = y_i | X) \prod_{i = j}^{\xi_j} PL_{\theta_j} (Y_i = y_i | X)}{\pi(\theta_j) h(\theta'_j | \theta_j) \prod_{i = j}^{\xi_j} PL_{\theta'_j} (Y_i = y_i | X) \prod_{i = j}^{\xi_j} PL_{\theta_j} (Y_i = y_i | X)},$$

and the posterior probability of cluster membership $Z_i$ is proportional to

$$1(\xi_j > u_i) \frac{w_j}{\xi_j} \cdot PL_{\theta_j} (Y_i = y_i).$$

Thus, the sampling of $\theta, Z$ is possible with the pseudo likelihood replacement.

PMS algorithm is faster than IIMS algorithm, but it is less accurate. The major issue is that it may underestimate the endogenous network formation process, since pseudo likelihood only uses local information within a whole graph (van Duijn et al., 2009). Moreover, when the model is near-degenerate, posterior samples from pseudo likelihood method may fall into the degenerate region (Caimo and Friel, 2011).
4 Empirical Results

In this section, we illustrate the performance of the proposed DPM-ERGMs through a synthetic and a real ensemble. The network samples from the given ERGM distribution are generated using R package ergm (Hunter et al., 2008).

4.1 Synthetic Networks

An ensemble of $N = 40$ undirected networks are generated from a mixture model with $K = 2$ groups. Two statistics are used to describe the networks, the number of edges $S_1(y) = \sum_{i<j} y_{ij}$ to reflect on the network density and the number of triangles $S_2(y) = \sum_{i<j<k} y_{ij}y_{jk}y_{ik}$ to represent the transitivity. The mixing proportion is $w_{true} = (0.5, 0.5)$. The network size is $n = 30$. The ERGM parameters for group 1 are selected as $\theta^1_{true} = (-3, 0.9)$, which has low density and high transitivity parameter, meaning that some edges are generated because of endogenous formation process. The second group parameter is $\theta^2_{true} = (-1, 0)$, representing Bernoulli networks which have independent edges.

![Figure 1: Clustering results of IIMS algorithm (left), and PMS algorithm (right).](image)

We applied both the IIMS and PMS algorithm to the synthetic ensemble. The prior of variable $v$ is a beta distribution $\text{Beta}(1, 0.1)$. The prior of ERGM parameters $\theta$ is selected to be a multivariate normal distribution $\mathcal{N}(\mu_0, \Sigma_0)$ with $\mu_0 = (-3, 0)$, $\Sigma_0 = 4^2 I_2$, where $I_2$ is a two dimension diagonal matrix. The proposal distribution is $\mathcal{N}(0, \Sigma_p)$, $\Sigma_p = 0.05^2 I_2$. For sequence $\xi_1, \xi_2, \ldots$, we use an exponential decreasing sequence, $\xi_i = e^{-i}$. $K_i$, the number of components that satisfies $\{j : \xi_j > u_i\}$, is also the smallest integer that satisfies $\{e^{-K_i} > u_i\}$, thus $K_i = \lceil -\log(u_i) \rceil$. We start with all networks in one group with initial value $\theta_0 = (-2, 0)$ and choose $m_1 = 2, m_2 = 10$ in the MMCMH step and $m_1 = 5, m_2 = 10$ in the sampling of membership variable. More details on the choices of $m_1, m_2$ can be found in the appendix.

The simulation is run for 12000 iterations with 2000 iterations as burn in. The clustering results are shown in Figure 1. Both IIMS and PMS algorithms are able to detect the true group memberships of all networks correctly. The acceptance ratio is 0.60 for group 1, 0.27 for group 2 using IIMS algorithm. The acceptance ratio is 0.62 for group 1, 0.18 for group...
2 using PMS algorithm. The posterior density plots are displayed in Figure 2. As we can see, the triangle estimator of group 1 from IIMS is smaller than PMS. This confirms the finding of van Duijn et al. (2009) that pseudo likelihood method tends to underestimate the endogenous network formation process. For Bernoulli networks in group 2, the pseudo likelihood method underestimates the parameter variance and provides a narrower interval. This is consistent with the finding of Bouranis et al. (2017).

Figure 2: Density plots of ERGM parameters after 2000 burn in.

Figure 3: Density plots of simulated network statistics generated from true likelihood estimation and pseudo likelihood estimation. The left two plots are simulated from 200 different posterior samples, while the right two are sampled from posterior mean.

In order to further assess the quality of estimation, we simulate networks based on the estimation. Specifically, we firstly simulate 200 networks, each from one of the 200 different posterior samples obtained after 2000 burn in and 50 thinning, then we simulate another 200
networks from the posterior mean. The simulated network statistics from true likelihood is shown in the first row of Figure 3 with statistics from the posterior samples on the left side and statistics from the posterior mean on the right side. The observed network statistics is covered well by the simulated network statistics, indicating that the estimator is a good fit to the data. However, in the second row of Figure 3 there are significant amount of full graphs (graphs with 435 edges) simulated from the pseudo estimation, because posterior samples from PMS method have degenerate parameter values.

In this simulation, we applied both the IIMS and PMS algorithm to the synthetic network ensemble. IIMS algorithm provided accurate estimation to the model. PMS algorithm clustered all the network samples correctly, but the estimated model for group 1 failed to generate networks resembling the observed graphs.

4.2 Krackhardt’s Advice Networks

We next apply the proposed DPM-ERGMs to an advice network ensemble. David Krackhardt (Krackhardt, 1987) studied a sequence of 21 networks about 21 employees in a high-tech machine manufacturing firm. The networks are constructed based on the data collected from a survey on the query “Who does X go to for advice and help with work?” Everyone is asked not only the advice relationship of themselves but also other people. Therefore, a collection of 21 perception networks \( y_i (i = 1, 2, \ldots, 21) \) is built where every network represents an individual’s perspective about the advice relationships among the 21 individuals. \( y_{rs,i} = 1 \) indicates that in the opinion of individual \( i, r \) asks help from \( s \). The covariate information of each individual is represented by a vector \( X \). The original paper focuses on exploring the differences of perception networks through node centrality scores to measure the importance of the nodes. Here, we are interested in learning the differences and similarities of the perception networks using the mixture of ERGMs. In this way, the generating mechanism of the perception networks can be analyzed. This helps us to better understand the perception network relationships. For the structure statistics, we choose the following:

- \( S^1(y_i) = \sum_{r \neq s} y_{rs,i} \), the total number of edges in the network. This reflects on the communication strength.
- \( S^2(y_i) = \sum_{r \neq s} y_{rs,i} 1(X_r = X_s) \), the total number of connections between individuals in the same level. The positive coefficient indicates that people tend to ask for help from people of the same level, while the negative coefficient means that more help is sought from others in a different level.
- \( S^3(y_i) = e^\phi \sum_{k=1}^{n-2} (1-(1-e^{-\phi})^k) DP_k(y_i), \phi = 0.25 \), geometrically weighted dyad-wise shared partner, GWDSP, a good representation for local clustering property, where \( DP_k(y_i) \) represents the number of dyads with \( k \) shared partners in the network \( y_i \).

We apply the IIMS algorithm to the advice network ensemble. The hyperparameter are specified as follows. A multivariate Gaussian distribution with mean \( \mu_0 = (-3, 0, 0) \) and covariance \( \Sigma_0 = 4^2 I_3 \) is chosen as the prior distribution for ERGM parameters. The proposal variance in the MMCMH algorithm is set as \( \Sigma_q = 0.05^2 I_3 \). A beta prior Beta(1, 0.1)
Figure 4: Clustering results of advice network ensemble using IIMS sampling algorithm. Left: the number of clusters at every iteration. Right: the frequency of allocating to each group after 50,000 burn in.

is used for the mixing proportion. $\theta_0 = (-2, 0, 0)$ is the initial value for ERGM parameter. In the intermediate importance sampling procedure, we use $m_1 = 2$ intermediate distributions and $m_2 = 10$ auxiliary networks for MMCMH algorithm and $m_1 = 5, m_2 = 10$ in the allocation step.

Figure 5: Density plots of group parameters after 50,000 burn for advice network ensemble.

The number of clusters at each iteration and the allocating frequency of each network from the IIMS algorithm are shown at Figure 4. We can see that 4 groups are clustered with networks 15, 20 in the first group, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 14, 18, 19, 21 in the second group, 6, 13, 16, 17 in the third group, and network 1 in the fourth group. The acceptance probability in the MMCMH algorithm for 4 groups are 0.43, 0.16, 0.49, 0.38 respectively. To learn about the characteristics of each group, we display the posterior density plots from IIMS algorithm in Figure 5. Group 1 has the smallest coefficient for edges but the biggest for GWDSP. This means that networks 15 and 20 have strong local clustering property, which is consistent with the fact that networks 15 and 20 have hub structures where fewer nodes have most of the connections. The advice relationships they nominate are centered around themselves. Group 2 has a big coefficient for edges and negative coefficient for level effect, indicating that networks are dense in this group and there are more advice between employees of different levels than of same levels. Group 3
has the smallest negative level effect, meaning that the advice relationships they observed are most across employees of different levels. Network 1 individually forms group 4. The level effect of network 1 is around 0, suggesting that individual level does not play a big role in network 1.

Our results are supported by the findings of Krackhardt (1987). Next, we compare our results with the centrality calculated in Krackhardt (1987). Betweenness centrality reflects on the influence of a node has over the flow of information. Group 1 consists of networks 15 and 20, which have unique performances on betweenness centrality. The betweenness centrality of nodes 15, 20 is 81.15 and 65.35, which are much bigger than the rest of nodes. Both of them mentioned a lot of advice relationships they are involved in. This is consistent with our finding of local clustering phenomenon implied by high GWDSP coefficient. The networks in group 3 are distinct from the rest of individuals in terms of low indegree and betweenness centrality. The indegree of individuals 6, 13, 16, 17 is all 0, indicating that they are not asked for advice by anybody. Also, the betweenness centrality of them is 0, 0.2, 0.11, 0.28, smaller than the rest of nodes in the locally aggregated networks. Moreover, employee 1 has high indegree centrality 18, but low betweenness centrality 2.81. It is asked advice often, but rarely asks advice from other people. Of all the 18 edges individual 1 claimed, only 1 relationship is confirmed by others. The specialty of individual 1 explains why the network 1 formed a group of its own.

![Figure 6: Density plots of network statistics based on networks simulated from posterior mean. The vertical lines stand for the value of structure statistics of observed networks.](image)

Posterior assessments can be done by comparing the observed network statistics with simulated network statistics sampled from ERGM with estimation as parameters. Specifically, we generate 500 networks using the posterior mean as parameters and draw the density plots of the simulated network statistics in Figure 6. As we can see, the simulated network statistics are close to the observed network statistics, suggesting that IIMS algorithm fits the data well. Note that network 1 located on the right end of the plot is far from other networks regarding the number of total edges and the number of edges within the same level. This is another reason that we think network 1 is better to be in a separate group.

Next, we apply PMS algorithm to the advice ensemble. After 100,000 iterations, 6 stable groups are detected, as shown at Figure 7. The networks in groups 1, 3, 4 from PMS algorithm are the same as from IIMS algorithm. The group 2 from IIMS algorithm
is divided further into 3 groups, where networks 2, 4, 5, 8, 9, 10, 14, 19, 21 form the new second group, 3, 7, 12, 18 make the new fifth group, and 11 is in the sixth group. The acceptance probability of the MMCMH algorithm for 6 groups are 0.36, 0.23, 0.40, 0.36, 0.29, 0.50 respectively.

Furthermore, we calculate the distance between observed network statistics and simulated network statistics as follows,

\[
\sum_{i:Z_i=k} (S(y_i) - \frac{1}{500} \sum_{l=1}^{500} S(z_{ik}^l))^2 \quad (k = 1, 2, \ldots, K),
\]

where \(S(y_i)\) represents the summary statistics of observed network \(y_i\), and \(S(z_{ik}^l)\) stands for the summary statistics of simulated networks from ERGM with group parameter \(\theta_k\). The results of are shown in Table 1. Comparing both results, the estimation for groups 1, 3, 4 from IIMS is more accurate, especially that the IIMS estimation of group 4 is much better than the PMS estimation. To get more details, we show the density plots of the simulated network statistics on Figure 8. Simulated network statistics from IIMS are centered around the observed statistics on the top row, while simulated statistics from PMS are distant from the observed statistics on the second row. This is because the model for group 4 is near-degenerate. For a near-degenerate model, the underlying parameter values are close to a degenerate region, which increases the difficulty for estimation. This
can happen quite often when we fit a ERGM with complicated statistics to real datasets. The pseudo likelihood method does not work for the near-degenerate model (Caimo and Friel 2011). In this case, we can only use true likelihood method. For the 14 networks in group 2, the total distance is smaller for PMS method. This is understandable because the IIMS method fits all these 14 networks with one model, while the PMS method fits these networks with 3 models.

Figure 8: Simulated network statistics from IIMS estimation for network 1 (or group 4) is in the first row. Simulated network statistics from PMS estimation for network 1 (or group 4) is in the second row.

In this simulation, we applied the IIMS algorithm to the advice ensemble and found 4 meaningful clusters. Although pseudo likelihood based methods managed to divide the ensemble into reasonable clusters, they failed to represent the features of networks because they are not suitable for estimating the near-degenerate model in this example. More simulation results can also be found in the appendix.

5 Discussion

In this paper, we proposed to model the ensemble of networks using a Dirichlet process mixture of ERGMs. Through such a framework, the subpopulations consisting of similar networks can be detected and compared automatically without requiring a fixed number of clusters in advance. On the other hand, multiple networks with similar characteristics are described by the same ERGM, namely, the cluster-specific ERGM, which is better than a single network ERGM, because information from all networks in the same cluster are gathered together on the cluster-specific ERGMs. Moreover, we also developed a novel IIMS sampling algorithm for the full Bayesian inference of the DPM-ERGMs in order to
capture the higher order interactions within a network.

The full Bayesian inference of ERGMs is known to be time consuming as generating networks from desired ERGMs requires a long run of Markov chain using MCMC technique. We provided a PMS sampling algorithm as a fast approximation method which can be used for pre-analysis of the dataset. However, as we mentioned before, PMS algorithm can not capture the higher order interactions within the network and can fail estimation when the model is near-degenerate. For a more accurate estimation, IIMS sampling algorithm is recommended.

Acknowledgments

Sa Ren was supported by the Graduate Teaching Assistant scholarship from University of Kent. The authors report there are no competing interests to declare.

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Appendix

1.1 Intermediate Importance Sampling

Here, we use simulation studies to show how the number of intermediate distributions $m_1$ and the number of auxiliary networks $m_2$ affect the normalizing constant ratio approximation with varying distances between compared parameters.

![Figure 9](image-url)

Figure 9: Estimation of normalizing constant ratios when the compared parameters are far from each other. Three plots correspond to three repetitions of the parameters.

We first show that how the estimation changes with different values of $m_1$ and $m_2$ when the compared parameters are distant. To do so, we sample two parameters $\theta_1, \theta_2$ independently from the prior $\mathcal{N}(\mu_0, \Sigma_0)$ and estimate $k(\theta_2)/k(\theta_1)$ with different values of $m_1, m_2$. The results of three repetitions are shown in the three plots of Figure 9 separately. In each plot, lines with different colors correspond to different numbers of intermediate values $m_1$ and x-axis represents different numbers of auxiliary variables $m_2$. As is shown, the line of $m_1 = 0$ is far from the other lines, meaning that the estimation is incorrect and intermediate distributions have to be used to get a good estimation.

![Figure 10](image-url)

Figure 10: Estimation of normalizing constant ratios when the compared parameters are close to each other. Three plots correspond to three repetitions of the parameters.

In the second simulation, we show how $m_1, m_2$ affect the intermediate importance sampling estimation when the compared parameters are close. Here, we generate a sample
\( \theta_1 \) from the prior \( \mathcal{N}(\mu_0, \Sigma_0) \), and propose \( \theta_2 \) from a normal distribution \( \mathcal{N}(\theta_1, \Sigma_p) \). Then we estimate normalizing constant ratio \( k(\theta_2)/k(\theta_1) \) with different \( m_1, m_2 \), and show the estimation in Figure 10. As we can see, all lines merge together with increasing \( m_1 \) and \( m_2 \), indicating that the intermediate importance sampling estimation is consistent. The simple importance sampling estimation (the line with \( m_1 = 0 \)) has big variations, and intermediate importance sampling estimators (lines with \( m_1 > 0 \)) are more stable.

We recommend \( m_1 = 2, m_2 = 10 \) for MMCMH algorithm and \( m_1 = 5, m_2 = 10 \) for the posterior membership sampling as initial values, and similar techniques can be applied to choose \( m_1, m_2 \) in the specific dataset.

### 1.2 International Trade Networks

We also apply the proposed DPM-ERGMs to a world trade network ensemble. The ensemble of trade networks is observed on 60 countries \((n = 60)\) over the period 2001-2016 \((N = 16)\), denoted as \( y_i (i = 1, 2, \ldots, 16) \). The networks are built based on the annual import data between every two countries from the UN Comtrade website\(^1\). The trade amount was collected in constant 2010 US dollars. A directed edge exists from node \( r \) to \( s \), \( y_{rs,i} = 1 \), if the import amount from country \( r \) to \( s \) is more than 3 billion dollars at year \( i \). The geographic distance between countries, represented by a matrix \( X \), is an important factor in analyzing trade relationship. Here, we treat distance as edge covariate and explore its influence on the trade ensemble. The distance between countries is calculated using the coordinate of the capital city, downloaded from CEPII database\(^2\).

In this application, we choose four statistics to explore the ensembles of trade networks from different aspects,

- **\( S^1(y_i) = \sum_{r \neq s} y_{rs,i} \)**, the total number of edges in network \( y_i \). The density of trade networks can reflect the universality of global trade relationship.

- **\( S^2(y_i) = \sum_{r \neq s} y_{rs,i} y_{sr,i} \)**, the total number of mutual edges. The mutual edge in trade networks stands for bilateral trade. It is helping in understanding trade types.

- **\( S^3(y_i) = e^\phi \sum_{k=1}^{n-2} \{1 - (1 - e^{-\phi})^k\} EP_k(y_i), \phi = 0.25 \)**, geometrically weighted edgewise shared partner, GWESP, a representation for transitivity. \( EP_k(y_i) \) is the number of connected pairs that have \( k \) common neighbors.

- **\( S^4(y_i) = \sum_{r \neq s} y_{rs,i} X_{rs} \)**, the effect of the distance covariate. This helps to explore how distance affects the trade network structure.

We ran 100,000 iterations using IIMS sampling algorithm with the first 50,000 iterations as burn in. The hyperparameter and initial values are set as follows, \( \theta_0 = (-2, 0, 0, 0) \) for ERGM parameter initial, \( \mu_0 = (-3, 0, 0, 0), \Sigma_0 = 4^2 I_4 \) for ERGM parameter prior, a diagonal matrix \( \Sigma_q \) with diagonal entries \((0.05^2, 0.02^2, 0.02^2, 0.02^2)\) for the variance of the proposal distribution in MMCMH, Beta(1,0.1) for the sticking breaking prior. In intermediate importance sampling, we choose \( m_1 = 2, m_2 = 10 \) for MMCMH and \( m_1 =

\(^1\)https://comtrade.un.org/
\(^2\)http://www.cepii.fr
Figure 11: Clustering results of trade ensemble using IIMS sampling algorithm. Left: the number of clusters at every iteration. Right: the frequency of allocating to each group after 50,000 burn in.

$m_2 = 10$ for posterior membership sampling. As shown in Figure 11, the ensemble of trade networks is clustered into 2 groups. Group 1 corresponds to networks of earlier years, from 2001 to 2005, and group 2 is formed by networks of later years, between 2006 and 2016. The acceptance ratio is 0.45, 0.30 for two groups separately. The network membership is closely related to the time, which is reasonable as trade networks are collected over time.

Figure 12: Density plots of estimation for groups 1 (red), 2 (blue) in trade ensemble.

The characteristics of each group can be further described using a group-specific ERGM and the comparisons between groups can be performed by comparing the parameters of each
ERGM. The density plots for the posterior samples are shown in the first row of Figure 12. As we can see, group 2 has bigger density parameter than group 1, meaning that the trade relationships are denser. It also has bigger mutuality, which indicates that bilateral trade is more common. More countries prefer to form a mutual trade relationship with their trading partners. The smaller transitivity coefficient of group 2 suggests that the international trade is becoming more universal, although that the local clustering phenomenon still exists, implied by the positive transitivity parameter.

Next, we ran the PMS sampling algorithm 100,000 iterations. The clustering result is displayed in Figure 13, which is similar to the IIMS algorithm. Networks from 2001 to 2004 are in the group 1 and networks between 2005 and 2016 are in the group 2. The acceptance ratio of each group is 0.59, 0.32. The density plots for each group are shown in the second row of Figure 12. Regardless of the similar clustering result, the density plots for ERGM parameter estimation are quite different. Comparing with the IIMS algorithm, PMS provides a narrow and sharp estimation, because pseudo likelihood method underestimates the variance of estimation. Moreover, the coefficient for GWESP term from PMS method is much smaller compared with the IIMS method. This is because the pseudo likelihood method can not capture the dependent structures within a network.

In order to assess the model results, we generate 1000 networks from the estimated model, and plot the simulated network statistics in Figure 14. The black dots stand for networks of group 1 and red dots represent networks of group 2. In the first row of the figure, the density plots of simulated network statistics are close to the observed samples, indicating that IIMS algorithm provides good estimation to the data. However, in the second row, the posterior mode of the first group is far away from the 4 samples in the group, implying that the model is not a good fit to the data. As we mentioned before, networks in group 1 have strong transitivity, which can not be captured by PMS algorithm.

Furthermore, we calculate the distance between observed network statistics and simulated network statistics. Results are shown in Table 2. IIMS method fits the networks in group 1 better than PMS method as the distance 89344 is much smaller than 145566.
Figure 14: Density plots of network statistics, edges, mutual, GWESP and edgecov, based on networks simulated from estimation. The black vertical lines represent networks of group 1 and the red lines stand for networks of group 2.

Figure 15: Clustering results of trade ensemble using PMG sampling algorithm. Left: DIC values for different number of clusters. Right: the frequency of allocating to each group after 50,000 burn in when there are 2 groups.
Table 2: The distance between observed network statistics and simulated network statistics.

| Method | Group 1 | Group 2 |
|--------|---------|---------|
| IIMS   | 80344   | 188624  |
| PMS    | 145566  | 175461  |

For comparison, we also applied the pseudo likelihood Metropolis-within-Gibbs (PMG) sampling algorithm, developed by Yin et al. (2020) for a finite mixture of ERGMs. The prior for the group parameter is the same as it for the infinite method. Without knowing the number of clusters in advance, we fit the model with the number of clusters $K = 1, 2, 3, 4$ in sequence and calculate deviance information criteria (DIC) for each model accordingly. The DIC value with different number of clusters is shown in Figure 15. The best model is the one with the smallest DIC value, meaning that the number of clusters is chosen to be 2 here. Networks between 2001 and 2004 are allocated to the first group and networks between 2005 and 2016 are in the second group. The individual network membership and the posterior density of each group from PMG algorithm are the same as our proposed PMS algorithm.

In this simulation, IIMS algorithm clustered the trade ensemble into 2 groups and fit each group with a different ERGM. PMS algorithm provides similar clustering result to IIMS, but the networks in group 1 are fit poorly because the pseudo likelihood method failed to capture the transitivity of trade networks. The results of PMG algorithm are comparable with PMS algorithm, which guaranteed our method.