Lower bounds for tails of sums of independent symmetric random variables

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Abstract

The approach of Kleitman (1970) and Kanter (1976) to multivariate concentration function inequalities is generalized in order to obtain for deviation probabilities of sums of independent symmetric random variables a lower bound depending only on deviation probabilities of the terms of the sum. This bound is optimal up to discretization effects, improves on a result of Nagaev (2001), and complements the comparison theorems of Birnbaum (1948) and Pruss (1997). Birnbaum’s theorem for unimodal random variables is extended to the lattice case.

1 Introduction

For deviation probabilities $\mathbb{P}(|S| > t)$ of sums $S = \sum_{i=1}^{n} X_i$ of independent, real-valued, and symmetrically distributed random variables $X_i$, Nagaev (2001, Theorem 1, in different notation) obtained the lower bound

$$\mathbb{P}(|S| > t) \geq \sum_{k > t/h} 2^{-k} \mathcal{B}_p(\{k\}) \quad (t \in [0, nh])$$

where $h \in ]0, \infty[$ is a free parameter and

$$\mathcal{B}_p := \bigstar_{i=1}^n \mathcal{B}_{p_i}$$

is the convolution of the Bernoulli distributions $\mathcal{B}_{p_i} = (1 - p_i)\delta_0 + p_i\delta_1$ with success probabilities $p_i := \mathbb{P}(|X_i| \geq h)$. Nagaev also provided analytically more tractable lower bounds for the right hand side of (1) and showed that the resulting inequalities for $\mathbb{P}(|S| > t)$ effectively complement other bounds depending on second and third absolute moments of the random variables $X_i$.

The main purpose of the present note is to provide as Theorem 2.4 below a generalization of Kanter’s (1976) concentration function inequality for sums of independent and symmetric random vectors, which yields as Corollary 2.6 below in particular the following improvement of (1), under the same assumptions as above:

$$\mathbb{P}(|S| > t) \geq \sum_{k > t/h} \left(1 - 2^{-k} F_k(\lfloor \frac{t}{h} + 1 \rfloor)\right) \mathcal{B}_p(\{k\}) \quad (t \in [0, nh])$$

Here and below, we use the standard notations $\lfloor x \rfloor := \max \{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil := -\lfloor -x \rfloor$, and write

$$F_n(m) := \max_{r \in \mathbb{Z}} \sum_{i=r}^{r+m-1} \binom{n}{i} \quad (n, m \in \mathbb{N}_0)$$
for the sum of the $m$ largest binomial coefficients of order $n$. For $t \in [(n-1)h, nh[$, the inequalities in (11) and (13) are identical, while for $t \in [0, (n-1)h[$ and $B_p(\{n-1\}) > 0$, inequality (13) is strictly sharper than (11). Moreover, as follows from the proof of Corollary 2.6, inequality (3) is optimal up to discretization effects, in the sense that, subject to the stated assumptions, the right hand side of (3) is the greatest lower bound for $P(|S| > t) + \frac{1}{2}P(|S| = t)$ for every $t = mh$ with $m \in \{1, \ldots, m\}$.

The rest of this note is structured as follows. Section 2 develops the Kleitman-Kanter approach to multivariate concentration function inequalities. A specialization to the one-dimensional case, namely Corollary 2.5, immediately yields the above-mentioned Corollary 2.6 improving Nagaev’s result. Section 3 reformulates Corollary 2.6 as a comparison theorem, stated together with related results of Pruss (1997) and Birnbaum (1948). The latter is generalized to the lattice case. Historical remarks are collected in Section 4.

2 A generalized Kanter inequality

Let $\| \cdot \|$ be a seminorm on an $\mathbb{R}$-vector-space $E$ and let $| \cdot |$ denote the usual absolute value on $\mathbb{R}$. We write $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

2.1 Lemma. Let $a \in E$, $m \in \mathbb{N}$ and $C_1, \ldots, C_m \subset E$ with

(5) $x, y \in C_j \Rightarrow \|x - y\| < \|a\|

for each $j \in \{1, \ldots, m\}$. Then for some $r \in \{1, \ldots, m\}$ the translate $C_r - a$ is disjoint from $\bigcup_{j=1}^m C_j$.

Proof. We may assume that $D := \bigcup_{j=1}^m C_j \neq \emptyset$ and $\|a\| > 0$. In the special case $E = \mathbb{R}$, $\| \cdot \| = | \cdot |$ and $a > 0$, we choose $r$ such that $\min D = \min C_r$ if $\min D$ exists, and $\text{inf} D = \text{inf} C_r$ otherwise. In the general case we apply the Hahn-Banach theorem (compare e.g. Rudin (1991), Theorem 3.3 and its Corollary) to yield a linear functional $\ell$ on $E$ with $\ell(a) = \|a\|$ and $|\ell(x)| \leq \|x\|$ for every $x \in E$, so that the special case applied to $\ell(a) > 0$ and $\ell(C_1), \ldots, \ell(C_m) \subset \mathbb{R}$ yields the claim. \hfill $\square$

2.2 Lemma. For $n, m \in \mathbb{N}_0$, we have $F_n(m) = \sum_{i=r}^s \binom{n}{i}$ with $r = r_{n,m} := \lfloor (n-m+1)/2 \rfloor$ and $s = r + m - 1$, and also with $\lfloor (n-m+1)/2 \rfloor$ in place of $r_{n,m}$. Further,

(6) $F_n(m) = F_{n-1}(m-1) + F_{n-1}(m+1) \quad (n, m \in \mathbb{N})$

and $n \mapsto 2^{-n} F_n(m)$ is for every $m \in \mathbb{N}_0$ a decreasing function.

Proof. The claim up to (6) follows easily from the symmetry, monotonicity and recursion properties of the binomial coefficients. The last claim follows, since the right hand side of (6) is $\leq 2 F_{n-1}(m)$. \hfill $\square$

We write $|A|$ for the cardinality of a set $A$.

2.3 Theorem (essentially Kleitman’s (1970) Theorem I). Let $n, m \in \mathbb{N}$, $a_1, \ldots, a_n \in E$, and $C_1, \ldots, C_m \subset E$ with

(7) $x, y \in C_j \Rightarrow \|x - y\| < \min_{i=1}^n \|a_i\|

for each $j \in \{1, \ldots, m\}$. Then

(8) $\sharp \{ I \subset \{1, \ldots, n\} : \sum_{i \in I} a_i \in \bigcup_{j=1}^m C_j \} \leq F_n(m)$. 

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with equality for \( E = \mathbb{R}, \| \cdot \| = | \cdot |, a_1 = \ldots = a_n = 1, \) and \( C_j = \{(n - m + 1)/2 \} + j - 1 \).

**Proof:** We consider more generally \( n, m \in \mathbb{N}_0 \) and let \( G_n(m) \) denote the supremum of the left hand side of (8) subject to the stated assumptions on \( a_1, \ldots, a_n \) and \( C_1, \ldots, C_m \). Then
\[
G_n(0) = 0 = F_n(0) \quad (n \in \mathbb{N}_0)
\]
(10)
\[
G_0(m) = 1 = F_0(m) \quad (m \in \mathbb{N})
\]

Let \( n, m \in \mathbb{N} \). Given \( a_1, \ldots, a_n \in \mathbb{E} \) and \( C_1, \ldots, C_m \in \mathbb{E} \) with (7), let \( a := a_n \) and choose \( r \) according to Lemma 2.1. Then the left hand side of (8) is
\[
\tilde{A} := \{ \varepsilon \in \{0, 1\}^n : \sum_{i=1}^n \varepsilon_i a_i \in \bigcup_{j=1}^m C_j \} = A_1 \times \{0\} \cup A_2 \times \{1\} \cup A_3 \times \{1\}
\]

where
\[
A_1 := \{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in \bigcup_{j=1}^m C_j \}
\]
\[
A_2 := \{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in C_r - a_n \}
\]
\[
A_3 := \{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in \bigcup_{j \neq r} (C_j - a_n) \}
\]

with \( A_1 \cap A_2 = \emptyset \) and thus
\[
\| \tilde{A} \| \leq \| A_1 \| + \| A_2 \| + \| A_3 \| = \| A_1 \cup A_2 \| + \| A_3 \| \leq G_{n-1}(m+1) + G_{n-1}(m-1)
\]

Hence we have
\[
G_n(m) \leq G_{n-1}(m-1) + G_{n-1}(m+1) \quad (n, m \in \mathbb{N})
\]

Now (3), (9), (10) and (11) together imply \( G_n(k) \leq F_n(m) \) for all \( n, m \in \mathbb{N}_0 \), as was to be shown. The claim about equality is obvious. \( \square \)

We call a random vector \( X \) symmetric if it has the same law as \( -X \). We recall the definitions (2) and (4) and put

\[
Q_p := \prod_{i=1}^n \left( (1 - p_i) \delta_0 + \frac{p_i}{2} (\delta_1 - \delta_0) \right) \quad (p \in [0, 1]^n)
\]

2.4 **Theorem (Kanter’s (1976) Lemma 4.2 generalized).** Let \( h \in ]0, \infty[ , n, m \in \mathbb{N}, \) and \( p \in [0, 1]^n \). Then the supremum of

\[
\mathbb{P} \left( \sum_{i=1}^n X_i \in \bigcup_{j=1}^m C_j \right)
\]

taken over all measurable \( \mathbb{R} \)-vector spaces \( E \), measurable seminorms \( \| \cdot \| \) on \( E \), measurable sets \( C_1, \ldots, C_m \subset E \) with
\[
x, y \in C_j \implies \| x - y \| < 2h
\]

for every \( j \in \{1, \ldots, m\} \), and all independent and symmetric \( E \)-valued random vectors \( X_i \) with
\[
\mathbb{P}(\| X_i \| < h) \leq 1 - p_i \quad (i = 1, \ldots, n)
\]
is attained for $E = \mathbb{R}$, $\| \cdot \| = | \cdot |$, $C_j = \{0, h\} + (2j - m - 1)h$, and the $X_i$ symmetric $\mathbb{R}$-valued with $\mathbb{P}(X_i = 0) = 1 - p_i = 1 - \mathbb{P}(|X_i| = h)$. The value of the supremum is

$$Q_p([-m + 1, m]) = \sum_{k=0}^{n} 2^{-k} F_k(m) B_p(\{k\})$$

**Remark.** Analytically convenient and sharp upper bounds for the quantity in (12) in the special case $m = 1$ are provided by Kanter (1976, Lemma 4.3) and by Mattner & Roos (2006). It is an open problem to prove analogous bounds for $m \geq 2$.

**Proof.** We may assume $h = 1$. Let $n$ etc. up to the $X_i$ be as stated and let us put $\pi_i := 1 - \mathbb{P}(|X_i| < 1)$. We may assume that $X_i = (1 - B_i) Y_i + B_i R_i Z_i$ with $B_1, \ldots, B_n, Y_1, \ldots, Y_n, Z_1, \ldots, Z_n, R_1, \ldots, R_n$ independent, $B_i \sim B_p$, $Y_i \sim \mathbb{P}(X_i \in \cdot \|X_i\| < 1) :=$ the conditional distribution of $X_i$ given $\|X_i\| < 1$, $Z_i \sim \mathbb{P}(X_i \in \cdot \|X_i\| \geq 1)$, and $\mathbb{P}(R_i = -1) = \mathbb{P}(R_i = 1) = 1/2$. Then, with $B := (B_1, \ldots, B_n)$, with $Q$ denoting the law of $(Y, Z) := (Y_1, \ldots, Y_n, Z_1, \ldots, Z_n)$, and with $|b| := \sum_{i=1}^{n} b_i$, we have

$$\mathbb{P}\left( \sum_{i=1}^{n} X_i \in \bigcup_{j=1}^{m} C_j \right) = \sum_{b \in \{0,1\}^n} \mathbb{P}(B = b) \int \mathbb{P}\left( \sum_{i=1}^{n} b_i R_i + \frac{1}{2} z_i \in \bigcup_{j=1}^{m} \frac{1}{2}(C_j + \sum_{i=1}^{n} (b_i z_i - (1 - b_i) y_i)) \right) dQ(y, z) \leq 2^{-|b| F_{\mathbb{R}}(m)}$$

(14) $\leq$ R.H.S. (12) with $\pi$ instead of $p$

(15) $\leq$ R.H.S. (12)

Here the inequality in (13), and hence (14), follows from Theorem 2.3 with those $z_i$ with $b_i = 1$ playing the role of the $a_i$, and with $\frac{1}{2}(C_j + \sum_{i=1}^{n} (b_i z_i - (1 - b_i) y_i))$ in place of $C_j$. Inequality (15) is true since $N_0 \ni k \mapsto 2^{-k} F_k(m)$ is decreasing by Lemma 2.2 and $[0,1]^n \ni p \mapsto B_p$ is increasing with respect to the coordinatewise order on $[0,1]^n$ and the usual stochastic order. In the special case $E = \mathbb{R}$ etc. as stated, we have $Y_i \sim \delta_0$ and may replace the distribution of $Z_i$ by $\delta_1$ in deriving (13), etc. as stated, and hence get equality everywhere. \hfill $\square$

2.5 Corollary. Let $0 < h \leq H < \infty$ with $m := \lfloor H/h \rfloor < H/h + 1/2$, $n \in \mathbb{N}$, and $p \in [0,1]^n$. Then the supremum of

$$\mathbb{P}\left( \sum_{i=1}^{n} X_i \in ] - H, H] + a \right)$$

taken over all independent and symmetric $\mathbb{R}$-valued random variables $X_i$ with

$$\mathbb{P}(|X_i| < h) \leq 1 - p_i \quad (i = 1, \ldots, n)$$

and all $a \in \mathbb{R}$, is attained for $\mathbb{P}(X_i = 0) = 1 - p_i = 1 - \mathbb{P}(|X_i| = h)$ and $a = mh - H$. The value of the supremum is given in (12).

**Proof.** Given $h, H, m, n, p, X_i$ and $a$ as above, we have

$$\mathbb{P}\left( \sum_{i=1}^{n} X_i \in ] - mh, mh] + b \right) \leq \mathbb{P}\left( \sum_{i=1}^{n} X_i \in C_j \right) \leq \text{R.H.S. (12)}$$

$$= \mathbb{P}\left( \sum_{i=1}^{n} X_i \in ] - mh, mh] + b \right) \leq \text{R.H.S. (12)}$$
with \( b := a \) and \( C_j := ]-h,h] + (2j - m - 1)h + b \), using Theorem 2.4 with \( E = \mathbb{R} \) and \( || \cdot || = | \cdot | \).

On the other hand, if \( \mathbb{P}(X_i = 0) = 1 - p_i = 1 - \mathbb{P}(|X_i| = h) \) and \( a = mh - H \), and if we let \( b := 0 \) instead of \( b := a \), then we can replace the two inequalities in the above calculation by equalities, as the assumption \( m < H/h + 1/2 \) yields \(-mh \leq -H + a < -(m - 1)h\).

\[ \square \]

2.6 Corollary. Let \( S = \sum^n_{i=1} X_i \) with independent and symmetric \( \mathbb{R} \)-valued random variables \( X_i \) and let \( h \in ]0, \infty[ \). Then (3) holds with \( p_i := \mathbb{P}(|X_i| \geq h) \) for \( i = 1, \ldots, n \).

Proof. For \( t > 0 \), we apply Corollary 2.5 with \( a = 0 \) and \( H = mh \) with \( m := \lceil t/h \rceil + 1 \) to get

\[ \mathbb{P}(|S| \leq t) \leq \mathbb{P}(S \in ]mh, mh[) \leq \text{R.H.S. eqn.} 12 \]

Inequality (3) follows by taking complements, since \( F_k(m) = 2^k \) for \( k \leq m - 1 \).

\[ \square \]

3 Comparison theorems

For \( \mathbb{R} \)-valued random variables \( U \) and \( V \), we write \( U \geq_{st} V \) if \( U \) is stochastically larger than \( V \), that is, if \( \mathbb{P}(U \geq t) \geq \mathbb{P}(V \geq t) \) for every \( t \in \mathbb{R} \). A specialization of Corollary 2.5 can be viewed as one of three results yielding at least almost a stochastic ordering \( |S| \geq_{st} |T| \) for sums \( S, T \) of independent symmetric random variables assuming a corresponding ordering of their terms, the other two results being theorems of Pruss (1997) and Birnbaum (1948). It therefore appears natural to summarize these results here, and to use this opportunity to extend Birnbaum’s theorem to the lattice case.

Let us agree on the following unimodality definitions for laws \( P \) on \( \mathbb{R} \). We call \( P \) unimodal on \( \mathbb{R} \), if \( P \) is unimodal in the usual sense that, for some \( x_0 \in \mathbb{R} \), the distribution function of \( P \) is convex on \( ]-\infty, x_0]\) and concave on \([x_0, \infty[ \). For \( a \in \mathbb{R} \) and \( h \in ]0, \infty[ \), we call \( P \) unimodal on \( h\mathbb{Z} + a \), if \( P(h\mathbb{Z} + a) = 1 \) and if there is a \( k_0 \in \mathbb{Z} \) such that \( k \mapsto P(hk + a) \) is increasing on \( \{ k \in \mathbb{Z} : k \leq k_0 \} \) and decreasing on \( \{ k \in \mathbb{Z} : k \geq k_0 \} \). For \( h \in ]0, \infty[ \), we call \( P \) unimodal with span \( h \), if either \( h = 0 \) and \( P \) is unimodal on \( \mathbb{R} \), or \( h > 0 \) and \( P \) is unimodal on \( h\mathbb{Z} + a \) for some \( a \in \mathbb{R} \). As usual, we attribute any property just defined to a random variable \( X \) if its distribution enjoys it.

3.1 Theorem. Let \( n \in \mathbb{N} \) and let \( X_1, \ldots, X_n \) as well as \( Y_1, \ldots, Y_n \) be independent and symmetrically distributed \( \mathbb{R} \)-valued random variables with sums \( S = \sum^n_{i=1} X_i \) and \( T = \sum^n_{i=1} Y_i \) and with

\[ |X_i| \geq_{st} |Y_i| \quad (i = 1, \ldots, n) \]

(a) (Pruss (1997)) Then

\[ \mathbb{P}(|S| \geq t) \geq \frac{1}{2} \mathbb{P}(|T| \geq t) \quad (t > 0) \]

(b) If \( h \in ]0, \infty[ \) and \( \mathbb{P}(Y_i \in [-h, 0, h]) = 1 \) for \( i = 1, \ldots, n \), then

\[ \mathbb{P}(|S| > mh) + \frac{1}{2} \mathbb{P}(|S| = mh) \geq \mathbb{P}(|T| > mh) + \frac{1}{2} \mathbb{P}(|T| = mh) \quad (m \in \mathbb{N}) \]

(c) (Birnbaum (1948) generalized) Let \( h \in ]0, \infty[ \) and \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) be unimodal with span \( h \). In case of \( h > 0 \) assume further for each \( i \in \{1, \ldots, n\} \) that \( X_i, Y_i \) are both \( h\mathbb{Z} \)-valued or both \( h(\mathbb{Z} + \frac{1}{2}) \)-valued. Then \( |S| \geq_{st} |T| \).
See Berger (1997, Theorem 1.1) for a further related comparison theorem.

**Example.** Let \( n = 2, X_1, X_2, Y_1 \sim \frac{1}{2} (\delta_{-1} + \delta_1), \) and \( Y_2 = 0. \) Then \( |X_i| \geq_{st} |Y_i| \) for \( i = 1, 2. \) Since \( \mathbb{P}(|S| \geq 1) = \frac{1}{2} \) and \( \mathbb{P}(|T| \geq 1) = 1, \) it follows that the constant \( \frac{1}{2} \) in Pruss’ theorem is best possible. As each of the four random variables is unimodal with span 2, it also follows that the second sentence in part (c) can not be omitted. Further, in this example, \( \mathbb{P}(Y_i \in (-1, 0, 1)) = 1 \) for \( i = 1, 2 \) but \( \mathbb{P}(S > 0) + \frac{1}{2} \mathbb{P}(S = 0) = \frac{3}{4} \geq 1 = \mathbb{P}(T > 0) + \frac{1}{2} \mathbb{P}(T = 0), \) showing that in (19) we may not replace \( N \) by \( N_0. \)

**Proof.** (a) See Pruss (1997).

(b) Here (18) is equivalent to (17) with \( p_i = \mathbb{P}(|Y_i| = h), \) so that Corollary 2.5 with \( H = mh \) and \( a = 0 \) yields (19).

(c) Induction based on Lemmas 3.2 and 3.3 given below. In the step from \( n - 1 \) to \( n, \) we may assume that \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) to be independent, and conclude that

\[
\left| \sum_{i=1}^{n} X_i \right| \geq_{st} \left| \sum_{i=1}^{n-1} X_i + Y_n \right| \geq_{st} \left| \sum_{i=1}^{n} Y_i \right|
\]

by applying Lemma 3.2 first to \( U_1 := \sum_{i=1}^{n-1} X_i, V_1 := X_n, W_1 := Y_n \) and then to \( U_2 := Y_n, V_2 := \sum_{i=1}^{n-1} X_i, W_2 := \sum_{i=1}^{n-1} Y_i, \) observing that by Lemma 3.3 the sum \( U_1 \) is symmetric and unimodal with span \( h, \) and that in case of \( h > 0 \) the sums \( V_2, W_2 \) are both \( h\mathbb{Z}\)-valued or both \( h(\mathbb{Z} + \frac{1}{2})\)-valued.

3.2 **Lemma.** Let \( U, V, W \) be symmetrically distributed \( \mathbb{R}\)-valued random variables with \( U, V, W \) independent, \( |V| \geq_{st} |W|. \) Let \( h \in [0, \infty[ \) with \( U \) unimodal with span \( h. \) In case of \( h > 0 \) let further \( V, W \) be both \( h\mathbb{Z}\)-valued or both \( h(\mathbb{Z} + \frac{1}{2})\)-valued. Then \( |U + V| \geq_{st} |U + W|. \)

**Proof.** We may assume that \( h \in \{0, 1\}. \) In case of \( h = 0 \) we put \( A := B := [0, \infty[, \) while for \( h = 1 \) we let \( A, B \in \{N_0, N_0 + \frac{1}{2}\} \) with \( \mathbb{P}(|U| \in A) = \mathbb{P}(|V| \in B) = 1. \) Then for \( t \in A + B := \{a + b : a \in A, b \in B\} \) and denoting by \( P_U \) etc. the laws of the random variables occurring as subscripts, we have

\[
\mathbb{P}(|U + V| \leq t) = \int_B P_U([v - t, v + t]) \, dP_V(v) \leq \int_B P_U([v - t, v + t]) \, dP_W(v) = \mathbb{P}(|U + W| \leq t)
\]

since in each case the function \( B \ni v \mapsto P_U([v - t, v + t]) \) is decreasing. As \( \mathbb{P}(|U + V| \in A + B) = 1, \) this proves \( |U + V| \geq_{st} |U + W|. \)

3.3 **Lemma (Wintner).** Let \( X \) and \( Y \) be independent \( \mathbb{R}\)-valued random variables and let \( h \in [0, \infty[. \) If \( X \) and \( Y \) are symmetric and unimodal with span \( h, \) then so is \( X + Y. \)

**Proof.** Obvious by writing the laws of \( X \) and \( Y \) as mixtures of uniform distributions on symmetric intervals in \( \mathbb{R} \) or \( h\mathbb{Z} \) or \( h(\mathbb{Z} + \frac{1}{2}). \) See Dharmadhikari & Joag-Dev (1988, pp. 13 and 109) for the cases where \( h = 0 \) or \( X \) and \( Y \) are both symmetric unimodal on \( h\mathbb{Z} \). The remaining three cases are analogous.
4 Historical notes

Theorem 2.3 in the Hilbert space case, and assuming the sets $C_j$ to be slightly smaller than necessary, was proved by Kleitman (1970), generalizing several earlier results and in particular the one-dimensional case due to Erdős (1945, Theorems 1 and 3). Jones (1978, page 4, footnote 7) observed that Kleitman’s result and proof extends to general (semi-)normed spaces. Meanwhile, Kanter (1976, Lemma 4.1) proved a weaker result, assuming in particular symmetry of the sets $C_j$. The present proof of Theorem 2.3 is just a slightly refined rewrite of Kleitman’s proof and Jones’ footnote.

Kanter (1976) essentially stated and proved Theorem 2.4 for $m = 1$ and $C_1$ symmetric. Le Cam (1986, pp. 408-409) adopted Kanter’s approach. Theorem 3.1(c) in the case of $h = 0$ and without atoms at zero is due to Birnbaum (1948). Bickel & Lehmann (1976) and Shaked & Shantikumar (1994, page 78) allowed atoms at zero in their statements, but apparently not in their proofs. Sherman (1955) extended Birnbaum’s result to the absolutely continuous multivariate case. Dharmadhikari & Joag-Dev (1988, p. 164) gave an elegant development of Sherman’s theorem, dispensing with unnecessary continuity assumptions. They also essentially stated without proof Theorem 3.1(c) for $h > 0$ in the case where all random variables are $h\mathbb{Z}$-valued.

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