Morphing Schnyder Drawings of Planar Triangulations

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Abstract
We consider the problem of morphing between two planar drawings of the same triangulated graph, maintaining straight-line planarity. In “How to morph planar graph drawings” (SIAM Journal on Computing) the authors give a morph that consists of $O(n)$ steps where each step is a linear morph that moves each of the $n$ vertices in a straight line at uniform speed. However, their method imitates edge contractions so the grid size of the intermediate drawings is not bounded and the morphs are not good for visualization purposes. Using Schnyder embeddings, we are able to morph in $O(n^2)$ linear morphing steps and improve the grid size to $O(n) \times O(n)$ for a significant class of drawings of triangulations, namely the class of weighted Schnyder drawings. The morphs are visually attractive. Our method involves implementing the basic “flip” operations of Schnyder woods as linear morphs.

Keywords Planar graphs · Computational geometry · Schnyder woods · Graph drawing · Morphing

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A preliminary version of this work was presented in the 22nd International Symposium on Graph Drawing, see [8].
1 Introduction

Given a triangulation on \( n \) vertices and two straight-line planar drawings of it, \( \Gamma \) and \( \Gamma' \), that have the same unbounded face, it is possible to morph from \( \Gamma \) to \( \Gamma' \) while preserving straight-line planarity. This was proved by Cairns in 1944 [13]. Cairns’s proof is algorithmic but requires exponentially many steps, where each step is a linear morph that moves every vertex in a straight line at uniform speed. Floater and Gotsman [21] gave a polynomial time algorithm using Tutte’s graph drawing algorithm [29], but their morph is not composed of linear morphs so the trajectories of the vertices are more complicated, and there are no guarantees on how close vertices and edges may become. Alamdari et al. [1] gave a polynomial time algorithm based on Cairns’s approach that uses \( O(n^2) \) linear morphs, and this has now been improved to \( O(n) \) in [2,3]. The main idea is to contract (or almost contract) edges. With this approach, perturbing vertex positions to prevent coincidence is already challenging, and keeping them on a nice grid seems impossible.

In this paper we propose a new approach to morphing based on Schnyder drawings. We give a planarity-preserving morph that is composed of \( O(n^2) \) linear morphs and for which the vertices of each of the \( O(n^2) \) intermediate drawings are on a \( 6n \times 6n \) grid. Our algorithm works for weighted Schnyder drawings, which are obtained from a Schnyder wood together with an assignment of positive weights to the interior faces. A Schnyder wood (see Sect. 2) is a special type of partition (colouring) and orientation of the edges of a planar triangulation into three rooted directed trees. Schnyder [26,27] used this notion to obtain straight-line planar drawings of triangulations in an \( O(n) \times O(n) \) grid. To do this he defined barycentric coordinates for each vertex in terms of the number of faces in certain regions determined by the Schnyder wood. Dhandapani [14] noted that assigning any positive weights to the faces still gives straight-line planar drawings. We call these weighted Schnyder drawings—they are the drawings on which our morphing algorithm works.

Two weighted Schnyder drawings may differ in weights and in the Schnyder wood. We address these separately: we show that changing weights corresponds to a single planar linear morph; altering the Schnyder wood is the more significant aspect.

The set of Schnyder woods of a given planar triangulation forms a distributive lattice [12,17,25] possibly of exponential size [16]. The basic operation for traversing this lattice is a “flip” that reverses a cyclically oriented triangle and changes colours appropriately. It is known that the flip distance between two Schnyder woods in the lattice is in \( O(n^2) \) (see Sect. 2). Therefore, to morph between two Schnyder drawings in \( O(n^2) \) steps, it suffices to show how a flip can be realized via a constant number of planar linear morphs. We show that flipping a facial triangle corresponds to a single planar linear morph, and that a flip of a separating triangle can be realized by three planar linear morphs.

For examples of our morphs, see Figs. 1 and 2. Further examples and animations can be found in [9]. We note the following properties of our morphs: (1) In each step of our morph, the vertex trajectories are parallel, and furthermore, for a facial flip, the trajectories are parallel to one of the three exterior edges; (2) Intermediate drawings lie on a \( 6n \times 6n \) grid. This implies that in the intermediate drawings, vertices are at least distance 1 apart and that the minimum distance between a vertex and an edge is
proportional to $1/n$. It also implies that face areas are at least $1/2$—this follows from Pick’s theorem that the area of a triangle with vertices on the integer grid is equal to the number of grid points interior to the triangle plus half the number of grid points on the boundary of the triangle. We do not have an analysis of these area and distance measures at intermediate time points of a morph step—see further discussion in the Conclusions.

The above properties make our morphs promising for visualization purposes. For static graphs, the properties of graph drawings that lead to good visualizations are well-studied [10,22]. The basic criteria used by early graph drawing algorithms, such as minimizing edge crossings and maintaining some minimum distance between features, have been evaluated via user studies. For “dynamic” graphs (graphs that change over time), there are some studies, see, for example [5], though none that specifically address morphing, to the best of our knowledge.

Comparing the visualization properties of our approach to those of previous morphing methods, the edge-contraction method of Alamdari et al. [2] is not good for visualization purposes—at the end of the recursion, the whole graph has contracted to a triangle. The method of Floater and Gotsman [21] gives good visualizations, based on experiments and heuristic improvements developed by Shurazhsky and Gotsman [28]. However, their method suffers from the same drawbacks as Tutte’s graph drawing method, namely that vertices and edges may come very close together in intermediate snapshots.

As mentioned above, our method applies to weighted Schnyder drawings. Not all straight-line planar triangulations are weighted Schnyder drawings, but we can recognize those that are in polynomial time (see Sect. 7). The problem of extending our result to all straight-line planar triangulations remains open. In particular, it would suffice to morph an arbitrary straight-line planar triangulation to a weighted Schnyder drawing in a nice way. The first author’s thesis [6] gives one method to do this that is better than general morphing techniques, but maintaining a small grid is still an open problem.
This paper is structured as follows. Section 2 contains the relevant background on Schnyder woods. Section 3 contains the precise statement of our main result, and the general outline of the proof. In Sect. 4 we show that changing face weights corresponds to a linear morph. Flips of facial triangles are handled in Sect. 5 and flips of separating triangles are handled in Sect. 6. In Sect. 7 we point out how the work of Bonichon et al. [11], and Felsner and Zickfeld [20] leads to an algorithm to decide if a planar drawing is a Schnyder drawing with positive weights.
1.1 Definitions and Notation

Consider two drawings $\Gamma$ and $\Gamma'$ of a planar triangulation $T$. A morph between $\Gamma$ and $\Gamma'$ is a continuous family of drawings of $T$, $\{\Gamma_t\}_{t \in [0,1]}$, such that $\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$. We say a face $xyz$ collapses during the morph $\{\Gamma_t\}_{t \in [0,1]}$ if there is $t \in (0,1)$ such that $x$, $y$, and $z$ are collinear in $\Gamma_t$. We call a morph between $\Gamma$ and $\Gamma'$ planar if $\Gamma_t$ is a planar drawing of $T$ for all $t \in [0,1]$. Note that a morph of a triangulation is planar if and only if no face collapses during the morph. We call a morph linear if each vertex moves from its position in $\Gamma_0$ to its position in $\Gamma_1$ along a line segment and at constant speed. Note that each vertex may have a different speed. We denote such a linear morph by $(\Gamma_0, \Gamma_1)$.

Throughout the paper we deal with a planar triangulation $T$ with a distinguished exterior face with vertices $a_1, a_2, a_3$ in clockwise order. The set of interior faces is denoted by $F(T)$. A 3-cycle $C$ whose removal disconnects $T$ is called a separating triangle, and in this case we define $T \mid C$ to be the triangulation induced on the vertices inside $C$ together with $C$ as the exterior face, and we define $T \setminus C$ to be the triangulation obtained from $T$ by deleting the vertices inside $C$ and their incident edges.

2 Schnyder Woods and their Properties

A Schnyder wood of a planar triangulation $T$ with exterior vertices $a_1, a_2, a_3$ is an assignment of directions and colours 1, 2, and 3 to the interior edges of $T$ such that the following two conditions hold (see Fig. 3).

(D1) Each interior vertex has three outgoing edges and they have colours 1, 2, 3 in clockwise order. All incoming edges of colour $i$ appear between the two outgoing edges of colours $i-1$ and $i+1$ (index arithmetic modulo 3).

(D2) At the exterior vertex $a_i$, all the interior edges are incoming and of colour $i$.

The following basic concepts and properties are due to Schnyder [26]. For any Schnyder wood the edges of colour $i$ form a tree $T_i$ rooted at $a_i$. The path from an interior vertex $v$ to $a_i$ in $T_i$ is denoted by $P_i(v)$.
(P) If $T_i^-$ denotes the tree of colour $i$ with all arcs reversed, then $T_i^- \cup T_{i-1} \cup T_{i+1}^-$ contains no directed cycle. This implies that any two outgoing paths from a vertex $v$ have no vertex in common, except for $v$, i.e., $P_i(v) \cap P_{j}(v) = \{v\}$ for $i \neq j$.

The descendants of a vertex $v$ in $T_i$, denoted by $D_i(v)$, are the vertices that have paths to $v$ in $T_i$. Note that $v$ is included in $D_i(v)$. For any interior vertex $v$ the three paths $P_i(v), i = 1, 2, 3$, partition the triangulation into three closed regions $R_i(v), i = 1, 2, 3$, where $R_i(v)$ is bounded by $P_{i+1}(v), P_{i-1}(v)$ and the edge $(a_{i+1}, a_{i-1})$, see Fig. 3 b. Schnyder proved that every triangulation $T$ has a Schnyder wood and that a planar drawing of $T$ can be obtained from coordinates that count faces inside regions:

**Theorem 2.1** (Schnyder [26,27]) Let $T$ be a planar triangulation on $n$ vertices equipped with a Schnyder wood $S$. Consider the map $f : V(T) \rightarrow \mathbb{R}^3$ defined by $f(a_i) = (2n - 5)e_i$, where $e_i$ denotes the $i$-th standard basis vector in $\mathbb{R}^3$, and for each interior vertex $v$, $f(v) = (v_1, v_2, v_3)$, where $v_i$ denotes the number of faces contained inside region $R_i(v)$. Then $f$ defines a straight-line planar drawing.

Note that the drawing specified in Schnyder’s theorem lies in the plane $x + y + z = 2n - 5$ in 3D space. A drawing in the $xy$-plane can be obtained by a projection that maps the three outer vertices to non-collinear points, for example, by dropping the third coordinate. Throughout the paper we will work with the 3D coordinates. Observe that a projection of a straight-line planar morph is a straight-line planar morph.

Dhandapani [14, Thm. 2] noted that Schnyder’s result generalizes to weighted faces. A *weight distribution* $\mathbf{w}$ is a function that assigns a positive weight to each interior face such that the weights sum to $2n - 5$. For any weight distribution the $i$-th coordinate $v_i$ of vertex $v$ is defined as

$$v_i = \sum \{\mathbf{w}(f) : f \in R_i(v)\}. \quad (1)$$

Theorem 2.1 still holds if we use coordinates as defined in (1). We call the resulting straight-line planar drawing the *weighted Schnyder drawing* obtained from $\mathbf{w}$ and $S$.

We will make use of the following properties that follow from (D1), (D2), and (P). Let $T$ be a planar triangulation and let $S$ be a Schnyder wood of $T$.

- **(R1)** For any edge $(u, v) \in E(T)$ and any $w \in V \setminus \{u, v\}$, there is $k \in \{1, 2, 3\}$ so that $u, v \in R_k(w)$. Then $R_k(u) \subseteq R_k(w)$ and $R_k(v) \subseteq R_k(w)$, and consequently, in the corresponding weighted Schnyder drawing we have $u_k, v_k < w_k$.

- **(R2)** Consider a face of $T$ with vertices $x, y, z$ in counterclockwise order. If the three edges form a directed cycle in $S$ and the edge leaving vertex $x$ has colour 1, then:

  (i) the three edges of $T$ have different colours (by Property (P)), and if the cycle is directed counterclockwise the edges have colours 1, 3, 2, and if the cycle is directed clockwise the edges have colours 1, 2, 3 (see Fig. 4),

  (ii) the interiors of $R_1(x), R_2(z)$ and $R_3(y)$ are pairwise disjoint, and

  (iii) $D_1(x) \setminus \{x\}$ is contained in the interior of $R_1(x)$ and similarly for $y$ and $z$. Consequently $D_1(x), D_2(z)$ and $D_3(y)$ are pairwise disjoint.
2.1 Triangle Flips

In this section we describe results of Brehm [12], Ossona de Mendez [25], and Felsner [17] on the flip operation that can be used to convert any Schnyder wood to any other. Let $S$ be a Schnyder wood of a planar triangulation $T$. A flip operates on a cyclically oriented triangle $C$ of $T$. The triangle $C$ may be a face, in which case Properties (R1) and (R2) above apply, or $C$ may be a separating triangle, in which case the properties still apply because of the following result which is proved by counting in-degrees and out-degrees of vertices interior to $C$.

**Lemma 2.2** (Brehm [12, Prop. 3.2.2, p. 35]) If $C$ is a separating triangle, then the restriction of $S$ to the interior edges of $T|_{C}$ is a Schnyder wood of $T|_{C}$, and the restriction of $S$ to the edges of $T \setminus C$ is a Schnyder wood of $T \setminus C$.

Let $C$ be a cyclically oriented triangle in $T$. If the triangle $C$ is oriented counterclockwise, then by Property (R2) and Lemma 2.2, we may assume that $C = xyz$ with edges $(x, y), (y, z), (z, x)$ of colour 1, 3, 2 respectively. By Lemma 2.2 and Property (D1) the edges interior to $C$ that are incident to $x, y$ or $z$ are oriented and coloured as shown in Fig. 3a with $x, y, z$ taking the roles of $a_2, a_1, a_3$, respectively. A clockwise flip of $C$ alters the colours and orientations of $S$ as follows:

1. Edges on the cycle are reversed and colours change from $i$ to $i - 1$. See triangle $xyz$ in Fig. 6.
2. Any interior edge of $T|_{C}$ remains with the same orientation and changes colour from $i$ to $i + 1$. See the edges incident to $b$ in Fig. 8.

Other edges are unchanged. The reverse operation is a *counterclockwise flip*, which Brehm calls a *flop*. Brehm [12, p. 44] proves that a flip yields another Schnyder wood. Consider the graph with a vertex for each Schnyder wood of $T$ and a directed edge $(S, S')$ when $S'$ can be obtained from $S$ by a clockwise flip. This graph forms a distributive lattice [12,17,25]. Ignoring edge directions, the distance between two Schnyder woods in this graph is called their *flip distance*. We now work towards proving Brehm’s result that the flip distance is in fact in $O(n^2)$, where $n$ is the number of vertices of the planar triangulation.
A triangle in a Schnyder wood $S$ is called \textit{flippable} if it is cyclically oriented counterclockwise. \textit{Floppable} triangles are triangles whose edges are cyclically oriented clockwise. Let $C$ be a flippable triangle in $S$ and denote by $S^C$ the Schnyder wood obtained from flipping $C$ in $S$. We say $C_1, C_2, \ldots, C_k$ is a \textit{flip sequence} if $C_1$ is flippable in $S_1 := S$, and $C_i$ is flippable in $S_i$ with $S_{i+1} := S_i^C$, for $2 \leq i \leq k$. If $C_1, \ldots, C_k$ is a flip sequence, then $C_k, \ldots, C_1$ defines a \textit{flop sequence}.

An important property of flips is the following.

\textbf{Lemma 2.3} (Brehm [12, Thm. 1.3.3 and Lem. 1.5.1]) \textit{For any planar triangulation there is a unique Schnyder wood, $L$, with no flippable face.}

The idea is that if two Schnyder woods $S$ and $S'$ differ, then they differ on the orientation of some edge. Because all vertices have outdegree 3, this edge is then part of an oriented cycle that has different orientations in $S$ and $S'$. (Note that following a path of edges of different orientations in $S$ and $S'$ will not lead to any of the three outer vertices because the orientations of their incident edges are fixed.) Suppose the cycle is oriented counterclockwise in $S$. Then by taking chords of the cycle, we obtain a counterclockwise oriented triangle in $S$, which is flippable.

Brehm then proves that a maximal flip sequence always ends at $L$. We will enhance his results to give a bound on the length of such a flip sequence. The analysis begins with the case of 4-connected triangulations. Note that in a 4-connected planar triangulation $T$ the flippable and floppable triangles are precisely the cyclically oriented faces of $T$. We begin with the following easy result.

\textbf{Proposition 2.4} (Brehm [12, Obs. 1.7.4, p. 15]) \textit{Let $T$ be a 4-connected planar triangulation, let $S$ be a Schnyder wood of $T$ and let $f$ be an interior face of $T$. If there is a flip sequence that contains $f$ at least twice, then between any two flips of $f$, all the faces adjacent to $f$ must be flipped.}

We can now derive the following bound, as suggested in [12, p. 15]. We use $d(f, g)$ to denote the distance between faces $f$ and $g$ in the planar dual of a triangulation.

\textbf{Theorem 2.5} \textit{Let $T$ be a 4-connected planar triangulation on $n$ vertices with exterior face $f^*$. The length of any flip sequence $F$ is bounded by the sum of the distances in the planar dual of $T$ to $f^*$, that is,}

$$|F| \leq \left( \sum_{f \in F(T)} d(f, f^*) \right) \in O(n^2).$$

\textit{Furthermore, any maximal flip sequence terminates at $L$, the unique Schnyder wood containing no flippable triangles.}

\textbf{Proof} Observe that any face adjacent to the exterior face cannot be flipped. By Proposition 2.4 it follows that the number of times a face $f$ can be flipped is bounded by the distance from $f$ to the exterior face $f^*$. Therefore the length of any flip sequence is bounded by $\sum_f d(f, f^*)$, which is in $O(n^2)$, as claimed.
Since every flip sequence has bounded length, a flip sequence cannot cycle back to the same Schnyder wood so any maximal flip sequence terminates at a Schnyder wood containing no flippable face, and that Schnyder wood is unique by Lemma 2.3.

In order to extend the above bound to non-4-connected triangulations, we use the following result, which is easy to prove directly.

**Lemma 2.6** (Brehm [12, pp. 36–37]) Consider a planar triangulation $T$ with a Schnyder wood $S$. If $C$ is a separating triangle of $T$ then there is no flippable triangle using edges of both $T|_C$ and $T \setminus C$.

We now prove that the flip distance between any two Schnyder woods of a planar triangulation on $n$ internal vertices is in $O(n^2)$. We have attributed the following result to Brehm [12], but he does not state it as a single result and, in any case, his concern is the finiteness of the sequence, not its asymptotic length. It helps to read Lemma 6 from Miracle et al. [23]$^1$ and Eppstein et al. [15].

**Theorem 2.7** In a planar triangulation with $n$ internal vertices the flip distance between any two Schnyder woods is in $O(n^2)$, and a flip/flop-sequence of that length can be found in linear time per flip/flop.

**Proof** We begin by showing that the length of a maximal flip sequence is in $O(n^2)$. Let $T$ be a planar triangulation with $n$ internal vertices. The proof is by induction on the number of separating triangles in $T$. The base case, when there are no separating triangles and the triangulation is 4-connected, is Theorem 2.5. For the induction step, suppose that $T$ has a separating triangle $C$ with $n_C$ vertices internal to $C$. Then $T|_C$ has $n_C$ internal vertices and $T \setminus C$ has $n - n_C$ internal vertices. Note that $T|_C$ and $T \setminus C$ have fewer separating triangles than $T$.

Consider a flip sequence $F$ starting from a Schnyder wood $S$ of $T$. By Lemma 2.6 the sequence of flips $F$ can be partitioned into a sequence of flips $F|_C$ that apply to $T|_C$, and a sequence of flips $F \setminus C$ that apply to $T \setminus C$. By induction the length of flip sequence $F|_C$ is in $O(n_C^2)$ and the length of flip sequence $F \setminus C$ is in $O((n - n_C)^2)$. Thus the length of $F$ is in $O(n^2)$ (using the fact that $a^2 + b^2 \leq (a + b)^2$).

We now consider the flip distance between two Schnyder woods. Let $S_1$ and $S_2$ be Schnyder woods of $T$. By the above, any maximal flip sequence must terminate at a Schnyder wood containing no flippable triangle, and that Schnyder wood, $L$, is unique by Lemma 2.3. Thus, we may concatenate a maximal flip sequence from $S_1$ to $L$ with the reverse of a maximal flip sequence from $S_2$ to $L$ to obtain a flip/flop sequence of length in $O(n^2)$ from $S_1$ to $S_2$.

Finally we prove the flip/flop sequence can be found in linear time per flip. If Schnyder woods $S$ and $S'$ differ by a flip, we can obtain $S'$ from $S$ by reversing $O(1)$ arcs and by updating the colour of $O(n)$ arcs. The result now follows.

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$^1$ The journal version [24] does not include this lemma.
3 Main Result

Theorem 3.1 Let $T$ be a planar triangulation and let $S$ and $S'$ be two Schnyder woods of $T$. Let $\Gamma$ and $\Gamma'$ be weighted Schnyder drawings of $T$ obtained from $S$ and $S'$ together with some weight distributions. There exists a sequence $\Gamma = \Gamma_0, \ldots, \Gamma_{k+1} = \Gamma'$ of straight-line planar drawings of $T$ such that

- $k$ is in $O(n^2)$,
- the linear morph $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is planar, for each $i$, $0 \leq i \leq k$, and
- the vertices of $\Gamma_i$ lie in a $(6n-15) \times (6n-15)$ grid, for each $i$, $1 \leq i \leq k$.

Furthermore, these drawings can be obtained in $O(n^3)$ time.

We now describe how the results in the upcoming sections prove the theorem. Lemma 4.1 (Sect. 4) proves that if we perform a linear morph between two weighted Schnyder drawings that differ only in their weight distribution then planarity is preserved. Thus, we may take $\Gamma_1$ and $\Gamma_k$ to be the drawings obtained from the uniform weight distribution on $S$ and $S'$ respectively. By Schnyder’s Theorem (Theorem 2.1) these drawings lie on a $(2n-5) \times (2n-5)$ grid and we can interpret this as a drawing on our finer $(6n-15) \times (6n-15)$ grid. (We do not alter the drawing, only the coordinate frame of reference.) By Lemma 2.7 (Sect. 2) there is a sequence of $k$ flips, $k \in O(n^2)$, that converts $S$ to $S'$. Therefore it suffices to show that each flip in the sequence can be realized via a planar morph composed of a constant number of linear morphs. In Theorem 5.5 (Sect. 5) we prove that if we perform a linear morph between two weighted Schnyder drawings that differ only by a flip of a face then planarity is preserved. In Theorem 6.4 (Sect. 6) we prove that if two Schnyder drawings with the same uniform weight distribution differ by a flip of a separating triangle then there is a planar morph between them composed of three linear morphs. The intermediate drawings involve altered weight distributions (here Lemma 4.1 is used again), and lie on a grid of the required size. Putting these results together gives the final sequence $\Gamma_0, \ldots, \Gamma_{k+1}$. All the intermediate drawings lie in a $(6n-15) \times (6n-15)$ grid and each of them can be obtained in $O(n)$ time from the previous one. This completes the proof of Theorem 3.1 modulo the proofs in the following sections.

4 Morphing to Change Weight Distributions

In this section we give the first ingredient of our main result. We consider two weighted Schnyder drawings $\Gamma$ and $\Gamma'$ that differ only in their weight distributions, i.e., $\Gamma$ and $\Gamma'$ are obtained from the same graph with the same Schnyder wood but with different weight distributions. (Recall the definition of weight distributions from Sect. 2.) In the following result we show that the linear morph from $\Gamma$ to $\Gamma'$ preserves planarity.

Lemma 4.1 Let $T$ be a planar triangulation and let $S$ be a Schnyder wood of $T$. Consider two weight distributions $w$ and $w'$ on the faces of $T$, and denote by $\Gamma$ and $\Gamma'$ the weighted Schnyder drawings of $T$ obtained from $w$ and $w'$ respectively. Then the linear morph $\langle \Gamma, \Gamma' \rangle$ is planar.
The trajectory of rectangular shaped vertices is parallel to the exterior edge (a_2, a_3). Similar properties hold for triangular and pentagonal shaped vertices.

**Proof** Consider the family of functions \( \{w'_t\}_{t \in [0,1]} \), defined by \( w'_t(f) = (1-t)w(f) + tw'(f) \). Note that for every \( t \in [0,1] \), the function \( w' \) is a weight distribution since \( w'(f) \) is positive for all \( f \) and \( \sum_f w'(f) = 2n - 5 \). By the results of Dhandapani [14] each \( w' \) yields a planar drawing. This family of drawings defines a planar morph from \( \Gamma \) to \( \Gamma' \). To conclude the proof we only need to show that this morph is linear. The position of vertex \( x \) at time \( t \) is \( x'_t = (x'_t^1, x'_t^2, x'_t^3) \) where \( x'_t \) is \( \sum_{f \in R'_t(x)} w'(f) \). Note that \( x'_t = (1-t)x^0 + tx^1 \), so the result now follows. \( \square \)

## 5 Morphing to Flip a Facial Triangle

In this section we prove that the linear morph from one Schnyder drawing to another one, obtained by flipping a cyclically oriented face and keeping the same weight distribution, preserves planarity. This is Theorem 5.5 below. See Fig. 5. We begin by showing how the regions for each vertex change during such a flip and then we use this to show how the coordinates change.

Let \( S \) and \( S' \) be Schnyder woods of a triangulation \( T \) that differ by a flip of a face \( xyz \) oriented counterclockwise in \( S \) with \( (x,y) \) of colour 1. Let \( (v_1, v_2, v_3) \) and \( (v'_1, v'_2, v'_3) \) be the coordinates of vertex \( v \) in the weighted Schnyder drawings corresponding to \( S \) and \( S' \), respectively, with respect to weight distribution \( w \). For an interior edge \( (p,q) \) of \( T \), let \( \Delta_i(pq) \) be the set of faces in the region bounded by the segment \( pq \) and the paths \( P_i(p) \) and \( P_i(q) \) in \( S \), and define \( \delta_i(pq) \) to be the weight of that region, i.e., \( \delta_i(pq) = \sum_{f \in \Delta_i(pq)} w(f) \). If \( \Delta_i(pq) \) is empty then \( \delta_i(pq) = 0 \). We use notation \( P_i(v), R_i(v), \) and \( D_i(v) \) as defined in Sect. 2 and \( \Delta_i(pq) \) as above and add primes to denote the corresponding structures in \( S' \).

Figure 6 shows how the regions change between the weighted Schnyder drawings corresponding to \( S \) and \( S' \). Observe that the outgoing paths \( P_2(x) \) and \( P_3(x) \) from \( x \) do not change, so region \( R_1(x) \) is unchanged. The outgoing path \( P_1(x) \) changes, so there are some faces, in particular those in \( \Delta_1(xz) = \Delta_1(yz) \cup \{f\} \), that leave \( R_2(x) \) and join \( R'_2(x) \). We capture these properties in the following lemma.

**Lemma 5.1** The following conditions hold:

1. \( R_1(x) = R'_1(x) \), \( R_3(y) = R'_3(y) \) and \( R_2(z) = R'_2(z) \).
Thus the only vertices whose coordinates change are those in $D_1$ only if the paths leaving $1$, and similarly for $y$ and $z$.

3. $D_1(x) = D_1'(x)$, $D_2(z) = D_2'(z)$ and $D_3(y) = D_3'(y)$.

**Proof** We show the first result in each condition since the others can be derived similarly.

1. Note that the only path outgoing from $x$ that changes is $P_1(x)$ (see Fig. 6), that is, $P_2(x) = P_2'(x)$ and $P_3(x) = P_3'(x)$ and therefore $R_1(x) = R_1'(x)$.

2. Observe that $z \in R_2(x)$ so $P_1(z)$ begins inside $R_2(x)$ and joins $P_1(x)$ at some point. See Fig. 6. Furthermore, $P_1(x)$ consists of edge $(x, y)$ and path $P_1(y)$. Similarly, $P_1'(x)$ consists of edge $(x, z)$ and path $P_1(z)$. Thus, $\Delta_1(xz) = \Delta_1(yz) \cup \{f\}$, and these faces lie in $R_2(x)$ and in $R_3'(x)$. There are no other changes to the faces of the regions $R_i(x)$. Therefore $R_2'(x) = R_2(x) \setminus \Delta_1(xz)$ and $R_3'(x) = R_3(x) \cup \Delta_1(xz)$.

3. This is an immediate consequence of 1, since no edges of $S$ in $R_1(x)$ change orientation or colour from $S$ to $S'$.

Next we study the difference between the coordinates of the weighted Schnyder drawings corresponding to $S$ and $S'$. Since the weights do not change, the coordinates of a vertex $v$ change only if its regions change. Furthermore, the regions of $v$ change only if the paths leaving $v$ change, and a path changes only if it uses an edge of $f$. Thus the only vertices whose coordinates change are those in $D_1(x) \cup D_2(z) \cup D_3(y)$. Furthermore, for a vertex in $D_i(x)$ the amount of change is determined by the weights of the faces that switch regions. We make this more precise in the following lemma.

**Lemma 5.2** For each $v \in V(T)$,

$$(v_1', v_2', v_3') = \begin{cases} 
(v_1, v_2, v_3) & \text{if } v \notin D_1(x) \cup D_2(z) \cup D_3(y), \\
(v_1, v_2 - \delta_1(xz), v_3 + \delta_1(xz)) & \text{if } v \in D_1(x), \\
(v_1 + \delta_2(yz), v_2, v_3 - \delta_2(yz)) & \text{if } v \in D_2(z), \\
(v_1 - \delta_3(xy), v_2 + \delta_3(xy), v_3) & \text{if } v \in D_3(y). 
\end{cases}$$
Proof As mentioned above, the only vertices whose coordinates change are those in \( D_1(x) \cup D_2(z) \cup D_3(y) \). Consider a vertex \( v \in D_1(x) \). (The other cases will follow by symmetry.) From Lemma 5.1, we have \( R'_1(x) = R_1(x), R'_2(x) = R_2(x) \setminus \Delta_1(xz), \) and \( R'_3(x) = R_3 \cup \Delta_1(xz) \). Therefore \((v'_1, v'_2, v'_3) = (v_1, v_2 - \delta_1(xz), v_3 + \delta_1(xz))\).  

As we can see from the previous lemma, only vertices in \( D_1(x), D_3(y) \) and \( D_2(z) \) change coordinates. The first coordinate of a vertex \( u \in D_1(x) \) does not change. Thus when morphing linearly between the corresponding Schnyder drawings \( u \) will move in a direction parallel to \( a_2a_3 \) throughout the morph. Similar statements are true for vertices in \( D_3(y) \) and \( D_2(z) \). The following lemma will be useful in proving planarity of the morph when vertices move in the same direction and at constant speeds.

**Lemma 5.3** ([7, Cor. 5]) Let \( M \) be a morph acting on points \( p, q \) and \( r \) such that their motions are along the same direction and at constant speeds. If \( p \) is to the right of the line through \( qr \) at the beginning and the end of the morph \( M \), then \( p \) is to the right of the line through \( qr \) throughout the morph \( M \).

**Proof** Without loss of generality we may assume that the motion is horizontal. Let \( s \) be the point on line \( qr \) that has the same height as \( p \). It suffices to show that the first coordinate of \( s \) is smaller than the first coordinate of \( p \) throughout the morph. Suppose the first coordinate of \( s \) changes from \( s^0 \) to \( s^1 \) and that the first coordinate of \( p \) changes from \( p^0 \) to \( p^1 \). Thus \( s^0 < p^0 \) and \( s^1 < p^1 \). Since \( s \) and \( p \) move at constant speed then at time \( t \) we have

\[
  s' = (1-t)s^1 + ts^0 < (1-t)p^1 + tp^0 = p',
\]

as desired. \( \square \)

We will also use the following basic property of linear morphs.

**Lemma 5.4** Let \( M \) be a linear morph acting on points \( p \) and \( q \). If \( p \) is to the right of (i.e., with larger \( x \)-coordinate than) \( q \) at the beginning and the end of the morph \( M \), then \( p \) is to the right of \( q \) throughout the morph \( M \).

**Proof** This follows immediately from the fact that the projection of the morph \( M \) to the \( x \)-axis is again a linear morph. \( \square \)

We are ready to prove the main result of this section. We express it in terms of a general (non-uniform) weight distribution since we will need that in the next section.

**Theorem 5.5** Let \( S \) be a Schnyder wood of a planar triangulation \( T \) that contains a face \( f \) bounded by a counterclockwise directed triangle \( xyz \), and let \( S' \) be the Schnyder wood obtained from \( S \) by flipping \( f \). Denote by \( \Gamma \) and \( \Gamma' \) the weighted Schnyder drawings obtained from \( S \) and \( S' \) respectively with weight distribution \( w \). Then \( \langle \Gamma, \Gamma' \rangle \) is a planar morph.

**Proof** If a triangle collapses during the morph, then it must be incident to at least one vertex that moves, i.e., a vertex that belongs to \( D_1(x), D_2(y) \) or \( D_3(z) \). Vertices in
First consider a triangle $pqr$ whose vertices move in at most one direction. Without loss of generality, suppose the direction is the one parallel to $a_2a_3$, i.e. horizontally. We prove that $pqr$ does not collapse during the morph. Suppose that in $\Gamma$, without loss of generality, $p$ lies between $q$ and $r$ in its first coordinate, and that $p$ lies to the right of the line through $qr$. This must also be the case in $\Gamma'$ because the points keep their first coordinate and the combinatorial embedding of the triangulation is unique. Therefore by Lemma 5.3, it follows that $p$ lies to the right of $qr$ throughout $\langle \Gamma, \Gamma' \rangle$. In particular this implies that $pqr$ does not collapse during the morph.

Next consider a triangle whose vertices move in two directions. Then, by the same argument, the triangle must include exactly two vertices of $x, y, z$. Without loss of generality, suppose the triangle is $xyz$ where $u \neq y$. Then $u$ must lie in $\Delta_3(xz)$ and is stationary during the morph. Furthermore, by Property (D1), the edge $(u, z)$ must have colour 1 and the edge $(u, x)$ must have colour 2, which implies that $x, z \in R_3(u)$. Similarly, $x, z \in R'_3(u)$. Therefore $x_3, z_3 < u_3$ and $x'_3, z'_3 < u'_3$. By Lemma 5.4, $u$ remains strictly above $x$ and $z$ in their third coordinate throughout the morph and therefore $u$ never lies on the line segment $xz$ during the morph. Similarly, we also conclude that $z$ never lies on the line segment $ux$ during the morph by using Lemma 5.4 and the fact that $u, x \in R_1(z)$ and $u, x \in R'_1(z)$. Finally, by a similar argument, $x$ never lies on the line segment $uz$ during the morph since $u, z \in R_2(x)$ and $u, z \in R'_2(x)$. We conclude that triangle $xz_u$ does not collapse during the morph.

6 Morphing to Flip a Separating Triangle

In this section we prove that there is a planar morph between any two weighted Schnyder drawings that differ by a separating triangle flip (Theorem 6.4 below). Our morph will be composed of three linear morphs.

Throughout this section we let $S$ and $S'$ be Schnyder woods of a planar triangulation $T$ such that $S'$ is obtained from $S$ after flipping a counterclockwise oriented separating triangle $C = xyz$, with $(x, y)$ coloured 1 in $S$. Let $\Gamma$ and $\Gamma'$ be two weighted Schnyder drawings obtained from $S$ and $S'$, respectively, with weight distribution $w$. For the main result of the section, it suffices to consider a uniform weight distribution because we
The linear morph defined by a flip of a separating triangle might not be planar if weights are not distributed appropriately. Here we illustrate the flip of the separating triangle in thick edges using the uniform weight distribution. Snapshots at $t = 0$, and $t = 1$ are at the top. The bottom drawing corresponds to $t = 0.7$; note the edge crossings. The shaded triangle is the one that inverts and the vertex drawn with a larger dot in the top figures is the one that appears in the close-up.

Fig. 7

We now give an outline of the strategy we follow. Morphing linearly from $\Gamma$ to $\Gamma'$ may cause faces inside $C$ to collapse. An example is provided in Fig. 7. However, we can show that there is a “nice” weight distribution that prevents this from happening. Our plan, therefore, is to morph linearly from $\Gamma$ to a drawing $\Gamma'$ with a nice weight distribution, then morph linearly to drawing $\Gamma''$ to effect the separating triangle flip. A final change of weights back to the uniform distribution gives a linear morph from $\Gamma''$ to $\Gamma'$.

This section is structured as follows. First we study how the coordinates change between $\Gamma'$ and $\Gamma''$. Next we show that any face whose vertices are interior to $T|_C$ does
not collapse during a linear morph between $\Gamma$ and $\Gamma'$. We then give a similar result for faces of $T|_C$ that share a vertex or edge with $C$ provided that the weight distribution satisfies certain properties. Finally we prove the main result by showing that there is a weight distribution with the required properties.

Let us begin by examining the coordinates of the vertices. As in Sect. 5 we use $\delta_i(xy)$ to denote the weight inside the region bounded by the edge $(x,y)$, $P_i(x)$ and $P_i(y)$. For a vertex $b \in V(T)$ let $(b_1, b_2, b_3)$ and $(b'_1, b'_2, b'_3)$ denote its coordinates in $\Gamma$ and $\Gamma'$ respectively. By Lemma 2.2 the restriction of $S$ to $T|_C$ is a Schnyder wood. For $b$ an interior vertex of $T|_C$ let $\beta_i$ be the $i$-th coordinate of $b$ in $T|_C$ when considering the restriction of $S$ to $T|_C$ with respect to the weight distribution $w$. By analyzing Fig. 8, we can see that the coordinates for $b$ in $\Gamma'$ are

$$
(b_1, b_2, b_3) = (x_1 + \delta_3(xz) + \beta_1, z_2 + \delta_1(yz) + \beta_2, y_3 + \delta_2(xy) + \beta_3)
$$

$$
= (x_1, z_2, y_3) + (\delta_3(xz), \delta_1(yz), \delta_2(xy)) + (\beta_1, \beta_2, \beta_3).
$$

(2)

We now analyze how the coordinates of vertices change from $\Gamma$ to $\Gamma'$. Note that reading the proof of the lemma first will make it easier to understand the formulas stated in the lemma.

**Lemma 6.1** For each $b \in V(T)$,

$$
(b'_1, b'_2, b'_3) = \begin{cases} 
(b_1, b_2 - \delta_1(xz), b_3 + \delta_1(xz)) & \text{if } b \in D_1(x), \\
(b_1 + \delta_2(yz), b_2, b_3 - \delta_2(yz)) & \text{if } b \in D_2(z), \\
(b_1 - \delta_3(xy), b_2 + \delta_3(xy), b_3) & \text{if } b \in D_3(y), \\
(x_1, z_2, y_3) + (\delta_2(xy), \delta_3(xz), \delta_1(yz)) + (\beta_3, \beta_1, \beta_2) & \text{if } b \in \mathcal{I}, \\
(b_1, b_2, b_3) & \text{otherwise}
\end{cases}
$$

where $\mathcal{I}$ is the set of interior vertices of $T|_C$.

**Proof** Observe that the coordinates of a vertex $b$ change only if its regions change, and its regions change only if an outgoing path from $b$ uses an interior edge of $T|_C$ or an edge of $C$. Therefore the only vertices whose coordinates change are the interior vertices of $T|_C$ or vertices in $D_1(x) \cup D_2(z) \cup D_3(y)$. The part of the result for $b \in D_1(x) \cup D_2(z) \cup D_3(y)$ follows from Lemma 5.2 applied to $T \setminus C$ with the restrictions of the Schnyder woods $S$ and $S'$ to $T \setminus C$ and the weight distribution where the weight of the face $xyz$ is equal to $\sum_{f \in \mathcal{S}(T|_C)} w(f)$ and the weights of the other faces are as specified by $w$.

Now suppose $b$ is an interior vertex of $T|_C$. The coordinates of $b$ in $T|_C$ are given by (2).

Similarly (see Fig. 8) the coordinates for $b$ in $\Gamma'$ are given by

$$
(b'_1, b'_2, b'_3) = (x_1, z_2, y_3) + (\delta_2(xy), \delta_3(xz), \delta_1(yz)) + (\beta'_1, \beta'_2, \beta'_3),
$$

where $\beta'_i$ denotes the $i$-th coordinate of $b$ in $T|_C$. Finally, since the colours of the interior edges of $T|_C$ change from $i$ to $i + 1$ we have $(\beta'_1, \beta'_2, \beta'_3) = (\beta_3, \beta_1, \beta_2)$, which gives the required formula. \qed
We now examine what happens during a linear morph from $\Gamma_1$ to $\Gamma'_1$. We first deal with faces strictly interior to $C$.

**Lemma 6.2** For an arbitrary weight distribution no face formed by interior vertices of $T|_C$ collapses in the morph $(\Gamma_1, \Gamma'_1)$.

**Proof** The idea of the proof is as follows. Consider a face inside $C$ formed by interior vertices $b, c, e$. As above, let $\beta = (\beta_1, \beta_2, \beta_3)$ be the coordinates of $b$ in $T|_C$. Similarly, let $\gamma$ be the coordinates of $c$ in $T|_C$ and let $\epsilon$ be the coordinates of $e$ in $T|_C$. Examining (2) and Lemma 6.1 we see that the coordinates of each of $b, c, e$ in $T|_C$ depend in exactly the same way on the parameters from $T \setminus C$ and differ only in the parameters $\beta, \gamma, \epsilon$. Therefore triangle $bce$ collapses during the morph if and only if it collapses during the linear transformation on $\beta, \gamma, \epsilon$ where we perform a cyclic shift of coordinates, viz., $(\beta_1, \beta_2, \beta_3)$ becomes $(\beta_3, \beta_1, \beta_2)$, etc. No triangle collapses during this transformation because it corresponds to moving each of the three outer vertices $x, y, z$ in a straight line to its clockwise neighbour.

We now give algebraic details. Let $b, c, e \in V(T|_C)$ be interior vertices of $T|_C$ such that $bce$ is an interior face of $T|_C$. We proceed by contradiction by assuming there is a time $r \in (0, 1)$ during the linear morph such that $bce$ collapses, say $b^r$ is in the line segment joining $c^r$ and $e^r$. That is, assume that

$$b^r = (1 - s)e^r + sc^r \quad (3)$$

for some $r \in (0, 1)$ and $s \in [0, 1]$. By (2) and Lemma 6.1, the left hand side of (3) can be written as

$$(x_1, z_2, y_3) + (1 - r)(\delta_3(xz), \delta_1(yz), \delta_2(xy)) + r(\delta_2(xy), \delta_3(xz), \delta_1(yz)) + \beta^r,$$

where $\beta^r = (1 - r)\beta^0 + r\beta^1$ and $\beta^0 = (\beta_1, \beta_2, \beta_3)$ and $\beta^1 = (\beta_3, \beta_1, \beta_2)$. Similar to what we have above, by using (2) and Lemma 6.1, we can rewrite the right hand side of (3) as
where \( \varepsilon^r \) and \( \gamma^r \) are defined analogously to \( \beta^r \). So, (3) is equivalent to

\[
\beta^r - ((1-s)\varepsilon^r + s\gamma^r) = (0, 0, 0).
\]

This can be rewritten as

\[
(1-r)\beta^0 + r\beta^1 - ((1-s)((1-r)\varepsilon^0 + r\varepsilon^1) + s((1-r)\gamma^0 + r\gamma^1)) = (0, 0, 0),
\]

and rearranging terms yields

\[
(1-r)(\beta^0 - ((1-s)\varepsilon^0 + s\gamma^0)) + r(\beta^1 - ((1-s)\varepsilon^1 + s\gamma^1)) = (0, 0, 0).
\]

Plugging in \( \beta^0 = (\beta_1, \beta_2, \beta_3) \) and \( \beta^1 = (\beta_3, \beta_1, \beta_2) \) (and similarly for \( \gamma \) and \( \varepsilon \)), we obtain the following equivalent system of equations.

\[
\begin{align*}
(1-r)(\beta_1 - ((1-s)\varepsilon_1 + s\gamma_1)) + r(\beta_3 - ((1-s)\varepsilon_3 + s\gamma_3)) &= 0, \\
(1-r)(\beta_2 - ((1-s)\varepsilon_2 + s\gamma_2)) + r(\beta_1 - ((1-s)\varepsilon_1 + s\gamma_1)) &= 0, \\
(1-r)(\beta_3 - ((1-s)\varepsilon_3 + s\gamma_3)) + r(\beta_2 - ((1-s)\varepsilon_2 + s\gamma_2)) &= 0.
\end{align*}
\]

To simplify the following arguments, we let

\[
Q_i = \beta_i - ((1-s)\varepsilon_i + s\gamma_i).
\]

So the system of equations now becomes

\[
\begin{align*}
(1-r)Q_1 + rQ_3 &= 0, & (4) \\
(1-r)Q_2 + rQ_1 &= 0, & (5) \\
(1-r)Q_3 + rQ_2 &= 0. & (6)
\end{align*}
\]

Since these coordinates were obtained from a weighted Schnyder drawing, we know there is a \( j \in \{1, 2, 3\} \) so that \( \beta_j > \gamma_j, \varepsilon_j \) by Property (R1). Then \( Q_j > 0 \). Now, we have that

\[
\begin{vmatrix}
(1-r) & 0 & r \\
r & (1-r) & 0 \\
0 & r & (1-r)
\end{vmatrix} = 3r^2 - 3r + 1 > 0.
\]

Therefore it must be the case that \( Q_i = 0, i = 1, 2, 3 \). This contradicts \( Q_j > 0 \), so the result now follows. \( \Box \)

Next we consider the faces interior to \( C \) that share an edge or vertex with \( C \). We show that no such face collapses, provided that the weight distribution \( w \) satisfies \( \delta_1 = \delta_2 = \delta_3 \) where we use \( \delta_1, \delta_2 \) and \( \delta_3 \) to denote \( \delta_1(yz), \delta_2(xy) \) and \( \delta_3(xz) \) respectively.
Lemma 6.3 Let \( w \) be a weight distribution for the interior faces of \( T \) such that \( \delta_1 = \delta_2 = \delta_3 \). No interior face of \( T|_C \) incident to a vertex of \( C \) collapses during \( \langle \Gamma, \Gamma' \rangle \).

**Proof** The idea of the proof is as follows. We will deal separately with the cases where the interior face is incident to the edge \((x, y)\) of \( C \) and where the interior face is only incident to the vertex \( x \) of \( C \). Consider the case of an interior face \( bxy \) incident to edge \((x, y)\). Suppose by contradiction that at time \( r \in [0, 1] \) during the morph the face collapses with \( b' \) lying on segment \( x'y' \), say \( b' = (1 - s)x' + sy' \) for some \( s \in [0, 1] \). We use formula (2) and Lemma 6.1 to re-write this equation. Some further algebraic manipulations (details to follow) show that there is no solution for \( r \). The remaining cases follow by analogous arguments.

We now turn to the details. We may assume, without loss of generality, that the interior face, \( F \), that we are considering is incident to the vertex \( x \). We have two possible cases.

Case 1: The interior face \( F \) is incident to an edge of \( C \).

Case 2: The interior face \( F \) is incident to \( x \) and two interior vertices.

In each case we formulate algebraically the fact that the face \( F \) collapses and then proceed by contradiction.

Case 1: Without loss of generality we may assume that the edge of \( C \) that is incident to face \( F \) is \((x, y)\), so that the face \( F \) is \( bxy \) for some interior vertex \( b \). Assume, by contradiction, that there is a time \( r \in (0, 1) \) during the morph such that \( b, x \) and \( y \) are collinear. Observe that, by Theorem 5.5, triangle \( C \) does not collapse during the morph. Thus, the only way for \( bxy \) to collapse is for vertex \( b \) to lie on the segment \( xy \), that is,

\[
b' = (1 - s)x' + sy' \quad \text{for some} \quad s \in [0, 1]. \tag{7}\]

Since \((b, x)\) has colours 2 and 3 in \( S \) and \( S' \) respectively, it follows that \( x_1^0 < b_1^0 \) and \( x_1^0 = x_1^1 < b_1^1 \). Similarly \( y_3 < b_3^0, b_3^1 \). Therefore for any \( r \in [0, 1] \), the first coordinate of \( b \) is greater than \( x_1 \) and the third coordinate of \( b \) is greater than \( y_3 \). This implies that \( s \in (0, 1) \), as otherwise at least one of these conditions would not hold. Using (2) and Lemma 6.1 we can write \( b' \) as

\[
(x_1, z_2, y_3) + (1 - r)(\delta_3, \delta_1, \delta_2) + r(\delta_2, \delta_3, \delta_1) + (1 - r)(\beta_1, \beta_2, \beta_3) + r(\beta_3, \beta_1, \beta_2).
\]

In order to gather like terms, we will also express \( x'^r \) and \( y'^r \) in terms of \((x_1, z_2, y_3)\). We have \( x'^r = (1 - r)x^0 + rx^1 \) and \( y'^r = (1 - r)y^0 + ry^1 \). Let \( w_C = \sum_{f \in F(T|_C)} w(f) \). Using Fig. 8, we obtain

\[
\begin{align*}
x^0 &= (x_1, z_2, y_3) + (0, \delta_3 + \delta_1 + w_C, \delta_2), \\
x^1 &= (x_1, z_2, y_3) + (0, \delta_3, \delta_2 + \delta_1 + w_C), \\
y^0 &= (x_1, z_2, y_3) + (\delta_2 + \delta_3 + w_C, \delta_1, 0), \\
y^1 &= (x_1, z_2, y_3) + (\delta_2, \delta_1 + \delta_3 + w_C, 0).
\end{align*}
\]
Plugging these all into (7) gives

\[
(x_1, z_2, y_3) + (1 - r)(\delta_3, \delta_1, \delta_2) + r(\delta_2, \delta_3, \delta_1) \\
+ (1 - r)(\beta_1, \beta_2, \beta_3) + r(\beta_3, \beta_1, \beta_2) \\
= (1 - s)\left[(x_1, z_2, y_3) + (1 - r)(0, \delta_3 + \delta_1 + w_C, \delta_2) \\
+ r(0, \delta_3, \delta_2 + \delta_1 + w_C)\right] \\
+ s\left[(x_1, z_2, y_3) + (1 - r)(\delta_2 + \delta_3 + w_C, \delta_1, 0) + r(\delta_2, \delta_1 + \delta_3 + w_C, 0)\right].
\]

By simplifying and analyzing each coordinate separately we obtain the following system of equations where \( \delta = \delta_1 = \delta_2 = \delta_3 \)

\[
(1 - r)(\delta + \beta_1 - s(2\delta + w_C)) + r(\delta + \beta_3 - s\delta) = 0, \quad (8) \\
(1 - r)(\beta_2 - (1 - s)(\delta + w_C)) + r(\beta_1 - s(\delta + w_C)) = 0, \quad (9) \\
(1 - r)(\delta + \beta_3 - (1 - s)\delta) + r(\delta + \beta_2 - (1 - s)(2\delta + w_C)) = 0. \quad (10)
\]

Now, since we have that \( \delta + \beta_3 - s\delta > 0 \) then from (8) we conclude that \( \delta + \beta_1 - s(2\delta + w_C) < 0 \). Therefore

\[
\beta_1 - s(\delta + w_C) < -(1 - s)\delta < 0. \quad (11)
\]

A similar analysis on (10) yields

\[
\beta_2 - (1 - s)(\delta + w_C) < -s\delta < 0. \quad (12)
\]

It now follows from inequalities (11) and (12) that there is no \( r \in [0, 1] \) satisfying (9). Therefore the interior face incident to edge \((x, y)\) does not collapse.

Case 2: Let us now consider the case where the interior face \( F \) is incident to \( x \) and two interior vertices, say \( b \) and \( c \). Let us assume that \( c \in R_1(b) \) and \( b \in R_3(c) \) in \( S \). This implies in particular that \( \gamma_1 < \beta_1 \) and that \( \beta_3 < \gamma_3 \). Suppose by contradiction that the face \( xbc \) collapses at time \( r \).

Suppose first, for a contradiction, that \( b \) lies on the line segment \( xc \) at time \( r \) of the morph. So we have \( b' = (1 - s)x'r + sc' \), with \( r \in (0, 1) \) and \( s \in (0, 1) \). By an analogous argument, it will follow that \( c \) cannot be on the line segment \( xb \). Also, \( x \) cannot be in the line segment \( bc \) because triangle \( C \) does not collapse.

We write the previous equation explicitly by expanding \( b' \) and \( x'r \) as in Case 1, and expanding \( c' \) similar to \( b' \). We obtain

\[
(x_1, z_2, y_3) + (1 - r)(\delta_3, \delta_1, \delta_2) + r(\delta_2, \delta_3, \delta_1) \\
+ (1 - r)(\beta_1, \beta_2, \beta_3) + r(\beta_3, \beta_1, \beta_2) \\
= (1 - s)\left[(x_1, z_2, y_3) + (1 - r)(0, \delta_3 + \delta_1 + w_C, \delta_2) + r(0, \delta_3, \delta_2 + \delta_1 + w_C)\right] \\
+ s\left[(x_1, z_2, y_3) + (1 - r)(\delta_2 + \delta_3 + w_C, \delta_1, 0) + r(\delta_2, \delta_1 + \delta_3 + w_C, 0)\right] \\
+ (1 - r)(\gamma_1, \gamma_2, \gamma_3) + r(\gamma_3, \gamma_1, \gamma_2).
\]
By simplifying and writing equations for each coordinate we obtain the following system of equations where \( \delta = \delta_1 = \delta_2 = \delta_3 \)

\[
\begin{align*}
(1 - r)(\delta + \beta_1 - s(\delta + \gamma_1)) + r(\delta + \beta_3 - s(\delta + \gamma_2)) &= 0, \\
(1 - r)(\beta_2 - (1 - s)(\delta + w_C) - s\gamma_2) + r(\beta_1 - s\gamma_1) &= 0, \\
(1 - r)(\beta_3 - s\gamma_3) + r(\beta_2 - (1 - s)(\delta + w_C) - s\gamma_2) &= 0.
\end{align*}
\] (13) (14) (15)

Now, since \( \beta_1 > \gamma_1 \) from (13) we get that \( \delta + \beta_1 - s(\delta + \gamma_1) > 0 \) and therefore \( \delta + \beta_3 - s(\delta + \gamma_2) < 0 \). From the previous inequality we get \( \beta_3 - s\gamma_3 < -(1 - s)\delta < 0 \). Now, using this in (15) we get that \( \beta_2 - (1 - s)(\delta + w_C) - s\gamma_2 > 0 \). Finally, using the previous inequality in (14) we obtain that \( \beta_1 - s\gamma_1 < 0 \), contradicting our original assumption that \( \beta_1 > \gamma_1 \). So the result follows.

We are now ready to prove the main result of this section.

**Theorem 6.4**  Let \( T \) be a planar triangulation and let \( S \) and \( S' \) be two Schnyder woods of \( T \) such that \( S' \) is obtained from \( S \) by flipping a counterclockwise cyclically oriented separating triangle \( C = xyz \) in \( S \). Let \( \Gamma \) and \( \Gamma' \) be weighted Schnyder drawings obtained from \( S \) and \( S' \), respectively, with uniform weight distribution. Then there exist weighted Schnyder drawings \( \overline{\Gamma} \) and \( \overline{\Gamma}' \) on a \((6n - 15) \times (6n - 15)\) integer grid such that each of the following linear morphs is planar: \( \langle \Gamma, \overline{\Gamma} \rangle \), \( \langle \overline{\Gamma}, \Gamma' \rangle \), and \( \langle \overline{\Gamma}', \Gamma' \rangle \).

**Proof**  Our aim is to define the planar drawings \( \overline{\Gamma} \) and \( \overline{\Gamma}' \). Each one will be realized in a grid that is three times finer than the \((2n - 5) \times (2n - 5)\) grid, i.e., in a \((6n - 15) \times (6n - 15)\) grid with weight distributions that sum to \(6n - 15\). Under this setup, the initial uniform weight distribution \( u \) takes a value of \( 3 \) in each interior face.

Drawings \( \overline{\Gamma} \) and \( \overline{\Gamma}' \) will be the weighted Schnyder drawings obtained from \( S \) and \( S' \) respectively with a new weight distribution \( \overline{w} \). We use \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) to denote the regions \( \Delta_1(zy), \Delta_2(xy) \) and \( \Delta_3(xz) \) respectively, in \( S \). We use \( \delta_i \) and \( \overline{\delta}_i \) to denote the weight of \( \Delta_i \), for each \( i = 1, 2, 3 \), with respect to the uniform weight distribution and the new weight distribution \( \overline{w} \), respectively.

We will define \( \overline{w} \) so that \( \overline{\delta}_1, \overline{\delta}_2, \) and \( \overline{\delta}_3 \) all take on the average value \( \delta := (\delta_1 + \delta_2 + \delta_3)/3 \). Let \( f_i \) be the number of faces in \( \Delta_i \), for \( i = 1, 2, 3 \). Then \( \delta_i = 3f_i \), and \( \delta = f_1 + f_2 + f_3 \). We want the new integer weights of the \( f_i \) faces of \( \Delta_i \) to sum to \( \delta \). This can be accomplished by distributing the weight as equally as possible, i.e., assigning weight \( \left\lfloor \frac{\delta}{f_i} \right\rfloor \) to \( \left\lfloor \delta \mod f_i \right\rfloor \) faces of \( \Delta_i \) and assigning weight \( \left\lfloor \frac{\delta}{f_i} \right\rfloor \) to the remaining faces of \( \Delta_i \). Since \( \delta > f_i \), these weights are all non-zero integers. Observe that \( \overline{\delta}_i = f_i \left\lfloor \frac{\delta}{f_i} \right\rfloor + (\delta \mod f_i) = \delta \).

This completes the description of \( \overline{\Gamma} \) and \( \overline{\Gamma}' \). It remains to show that the three linear morphs are planar. The morphs \( \langle \Gamma, \overline{\Gamma} \rangle \) and \( \langle \overline{\Gamma}', \Gamma' \rangle \) only involve changes to the weight distribution so they are planar by Lemma 4.1. Consider the linear morph \( \langle \overline{\Gamma}, \Gamma' \rangle \). The two drawings differ by a flip of a separating triangle. They have the same weight distribution \( \overline{w} \) which satisfies \( \overline{\delta}_1 = \overline{\delta}_2 = \overline{\delta}_3 \). By Lemmas 6.2 and 6.3 no interior face of \( T|_C \) collapses during the morph. By Theorem 5.5 no face of \( T \setminus C \) collapses during the morph. Thus \( \langle \overline{\Gamma}, \Gamma' \rangle \) defines a planar morph.  \( \square \)
7 Identifying Weighted Schnyder Drawings

In this section we outline a polynomial time algorithm to test if a given straight-line planar drawing $\Gamma$ of triangulation $T$ is a weighted Schnyder drawing. The first step is to identify the Schnyder wood. A result of Bonichon et al. [11] shows that, given a point set $P$ with triangular convex hull, a Schnyder drawing on $P$ is exactly the “half-$\Theta_6$-graph” of $P$ (see [11] for definition of half-$\Theta_6$-graph), which can be computed efficiently. Thus, given the drawing $\Gamma$, we first ignore the edges and compute the half-$\Theta_6$ graph of the points. If this differs from $\Gamma$, we do not have a weighted Schnyder drawing. Otherwise, the half-$\Theta_6$ graph determines the Schnyder wood $S$. We next find the face weights. We claim that there exists a unique assignment of (not necessarily positive) weights $w$ on the faces of $T$ such that $\Gamma$ is the drawing obtained from $S$ and $w$ as described in (1). Furthermore, $w$ can be found in polynomial time by solving a system of linear equations in the $2n - 5$ variables $w(f), f \in \mathcal{F}(T)$. The equations are those from (1). The rows of the coefficient matrix are the characteristic vectors of $R_i(v), i \in \{1, 2, 3\}, v$ an interior vertex of $T$, and the system of equations has a solution because the matrix has rank $2n - 5$. This was proved by Felsner and Zickfeld [20, Thm. 9]. (Note that their theorem is about coplanar orthogonal surfaces; however, their proof considers the exact same set of equations and their Claims 1 and 2 give the needed result.)

We conclude this section by providing an example of a planar triangulation and a drawing of it such that there is no weight distribution (having only positive weights) that realizes it. We claim that the drawing shown in Fig. 9 is such an example. Note that $R_1(x) \subseteq R_1(y)$ in any Schnyder wood $S$, independent of whether $(y, x)$ or $(y, z)$ belongs to $T_3$ in $S$. If weights were non-negative then we would have $x_1 \leq y_1$, which is clearly not the case here. A simpler example (though not 4-connected) may be obtained by contracting $z$ to $x$ and contracting $u$ to $a_1$, i.e., $x$ is adjacent to $a_1, a_2, a_3$ and $y$ is adjacent to $x, a_1, a_2$. 

Fig. 9 A drawing that cannot be realized with positive weights
8 Conclusions and Open Problems

We have made a first step towards morphing straight-line planar graph drawings with a polynomial number of linear morphs on a well-behaved grid. Our method applies to weighted Schnyder drawings. There is hope of extending our method to all straight-line planar triangulations. The first author’s thesis [6] gives partial progress in this direction: an algorithm to morph from any straight-line planar triangulation to a weighted Schnyder drawing in $4(n - 4)$ steps—but not, unfortunately, on a nice grid. This method is simpler than that of Angelini et al. [3] since the idea is to simply contract vertices of degree at most 5 and then uncontract them in reverse order while maintaining a Schnyder drawing. No convexifying routines are needed since there is no target drawing.

It is an open question to extend our result to general (non-triangulated) planar graphs. This might be possible using the extension of Schnyder’s results to 3-connected planar graphs by Felsner [18,19].

Another open question is to analyze face areas and the distances between vertices and edges at time points intermediate to each morphing step. For a flip of a triangular face, we conjecture that these measure are well-behaved because vertices move in such a limited way, namely parallel to the outer edges of the drawing. However, the situation may be worse for a flip of a separating triangle, which involves a morph to alter face weights.

The problem of efficiently morphing planar graph drawings to preserve convexity of faces has been recently solved by Angelini et al. [4]. Another open question is whether we can find a convexity-preserving morph where intermediate drawings lie on a polynomial size grid.

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