Order and minimality of some topological groups

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Abstract

A Hausdorff topological group is called minimal if it does not admit a strictly coarser Hausdorff group topology. This paper mostly deals with the topological group $H_+(X)$ of order-preserving homeomorphisms of a compact LOTS $X$. We provide a sufficient condition on $X$ under which $H_+(X)$ is minimal. This condition is satisfied, for example, by: the unit interval, the ordered square, the extended long line and the circle (endowed with its cyclic order). In fact, these groups are even $a$-minimal, meaning that the topology on $G$ is the smallest Hausdorff group topology on $G$. The technique in this article is mainly based on works of Gamarnik [21] and Gartside-Glyn [22].

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1. Introduction

A Hausdorff topological group $G$ is minimal ([15], [39]) if it does not admit a strictly coarser Hausdorff group topology or, equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding.

All topological spaces are assumed to be Hausdorff and completely regular. Let $X$ be a compact topological space. Denote by $H(X)$ the group of homeomorphisms of $X$, endowed with the compact-open topology $\tau_{co}$. In this setting $H(X)$ is a topological group and the natural action $H(X) \times X \to X$ is continuous.

Clearly, every compact topological group is minimal. The groups $\mathbb{R}$ and $\mathbb{Z}$, on the other hand, are not minimal. Moreover, Stephenson showed in [39] that a LCA group is minimal if and only if it is compact. Nontrivial examples of
minimal groups include $\mathbb{Q}/\mathbb{Z}$ (see [39]) and $S(X)$, the symmetric group of an infinite set (with the pointwise topology). The latter was proved by Gaughan [23] and (independently) by Dierolf and Schwanengel [9]. For more information on minimal groups we refer to the surveys [8], [10] and the book [11].

The following is a question of Stoyanov (cited in [2], for example):

**Question 1.1.** Is it true that for every compact homogeneous spaces $X$ the group $H(X)$ is minimal?

One important positive example of such a space is the Cantor cube $2^\omega$. Indeed, in [21] Gamarnik proved that $H(2^\omega)$ is minimal. Recently van Mill ([31]) provided a counterexample to Question 1.1 proving that for the $n$-dimensional Menger universal continuum $X$, where $n > 0$, the group $H(X)$ is not minimal.

It is well known that the Hilbert cube $[0,1]^\omega$ is a homogeneous compact space as well. The following question of Uspenskij [41] remains unanswered: is the group $H([0,1]^\omega)$ minimal?

**Definition 1.2.**

1. A topological group $G$ is $a$-minimal if its topology is the smallest possible Hausdorff group topology on $G$.

2. A compact space $X$ is $M$-compact ($aM$-compact) if the topological group $H(X)$ is minimal (respectively, $a$-minimal).

3. A compact ordered space $X$ is $M_+$-compact ($aM_+$-compact) if the topological group $H_+(X)$ of all order-preserving homeomorphisms of $X$ is minimal (respectively, $a$-minimal).

Several questions naturally arise at this point:

**Question 1.3.**

1. Which (notable) compact spaces are $M$-compact? $aM$-compact?

2. Which compact ordered spaces are $M_+$-compact? $aM_+$-compact?

The two point compactification of $\mathbb{Z}$ gives a compact LOTS $X$ such that $H_+(X)$ is not minimal (Example 5.1). Thus not every compact LOTS is $M_+$-compact.

Clearly, every $a$-minimal group is minimal. It is well known that $(\mathbb{Z}, \tau_p)$ with its $p$-adic topology is a minimal topological group. Since such topologies are incomparable for different $p$’s, it follows that $(\mathbb{Z}, \tau_p)$ is not $a$-minimal.

Following are a few results related to the question of $M$-compactness:

1. (Gaughan, [23]) 1-point compactification $X^*$ of a discrete set $X$ is $M$-compact since $H(X^*)$ is precisely $S(X)$;

2. (Banakh-Guran-Protasov [3]) Every subgroup of $S(X)$ that contains $S_\omega(X)$ (permutations of finite support) is $a$-minimal (answers a question of Dikranjan [28]);
3. (Gamarnik, [21]) $[0,1]^n$ is $M$-compact if and only if $n = 1$;

4. (Gartside and Glyn [22]) $[0,1]$ and $S^1$ are $aM$-compact;

5. [21] Cantor cube $2^\omega$ is $M$-compact;

6. (Uspenskij, [40]) Every $h$-homogeneous compact space is $M$-compact (note that this is a generalization of (3));

7. (van Mill [31]) $n$-dimensional Menger universal continuum $X$, where $n > 0$, is not $M$-compact (answers Stoyanov’s question [1.1]).

The concept of $a$-minimal groups is in fact an intrinsic algebraic property of an abstract group $G$ (underlying a given topological group). $a$-minimality is interesting for several reasons. It is strongly related to some fundamental topics like Markov’s and Zariski’s topologies. For additional information about $a$-minimality (and minimality) see the recent survey [10]. For Markov’s topology see [13, 3, 14].

One of the main ideas in Gamarnik’s paper [21] was to explore quasibounded actions (defined in [29]) of full homeomorphism groups. In the present paper we found additional new possibilities of this approach, where the groups are some subgroups of $H(X)$ preserving a given structure on $X$. Among others: group structure or an order. For instance, if $X$ is a compact group then the semidirect product $X \rtimes \text{Aut}(X)$ is minimal (Theorem 3.11).

Below we deal with the groups $H^+(X)$. Given an ordered compact space $X$, we are interested in the group $H^+(X)$ of order-preserving homeomorphisms. For a compact space $X$ the group $H(X)$ is complete (with respect to the two-sided uniformity) and therefore $H^+(X)$ is also complete (as a closed subgroup of a complete group).

In certain cases the minimality of $H(X)$ can be deduced from the minimality of $H^+(X)$, as the following two lemmas show.

**Lemma 1.4.** Let $X$ be a compact LOTS such that $H^+(X)$ is minimal. If $H^+(X)$ is a co-compact subgroup of $H(X)$, then $H(X)$ is minimal.

This lemma is a corollary of Lemma [20]. Co-compactness of $H^+(X)$ in $H(X)$ means that the coset space $H(X)/H^+(X)$ is compact.

**Lemma 1.5.** [11, Theorem 4.3] If $H$ is complete then the semidirect product $H \rtimes_\alpha G$ is minimal for minimal groups $H$ and $G$.

From this lemma we can conclude that if the group $H^+(X)$ is minimal, then $H^+(X) \rtimes G$ is also minimal for a minimal topological group $G$. So, for example, from the minimality of $H^+[0,1]$ we can deduce the minimality of $H[0,1]$ since $H[0,1] = H^+[0,1] \rtimes \mathbb{Z}_2$. However, in general, it is unclear how to infer the minimality of $X$ from the minimality of $X \rtimes \mathbb{Z}_2$. For instance, in [10, Example 4.7] it is shown that there exists a non-minimal group $X$ such that $X \rtimes \mathbb{Z}_2$ is minimal.
Extending some ideas of Gamarnik [21] and Gartside-Glyn [22] to linearly ordered spaces we give some new results about minimality of groups \( H_+(X) \) (see Theorems 3.4 and 4.6). The following linearly ordered spaces \( X \) are compact. The groups \( H_+(X) \) (and, in fact, also \( H(X) \)) are \( \alpha \)-minimal.

1. \([0, 1]\);
2. the ordered square \( I^2 \);
3. the extended long line \( L^\ast \);
4. the ordinal space \([0, \kappa]\);
5. \( S^1 \) (note that in this case we work with a cyclic order).

2. Preliminaries

In what follows, every compact topological space will be considered as a uniform space with respect to its natural (unique) uniformity.

**Lemma 2.1.** Let \( X \) be a compact space and let \( G \) be a subgroup of \( H(X) \). If \( \tau \) is a Hausdorff group topology on \( G \) such that the action \( G \times X \to X \) is continuous, then \( \tau_{co} \subseteq \tau \).

For a topological group \((G, \gamma)\) and its subgroup \( H \) denote by \( \gamma/H \) the natural quotient topology on the coset space \( G/H \).

**Lemma 2.2.** (Merson’s Lemma) Let \((G, \gamma)\) be a not necessarily Hausdorff topological group and \( H \) be a not necessarily closed subgroup of \( G \). If \( \gamma_1 \subseteq \gamma \) is a coarser group topology on \( G \) such that \( \gamma_1|H = \gamma|H \) and \( \gamma_1/H = \gamma/H \), then \( \gamma_1 = \gamma \).

**Lemma 2.3.** Let \( H \) be a co-compact complete subgroup of a topological group \( G \). If \( H \) is minimal then \( G \) is minimal too.

**Proof.** Denote by \( \tau \) the given topology on \( G \), and let \( \gamma \subseteq \tau \) be a coarser Hausdorff group topology. Since \( H \) is minimal, we know that \( \gamma|H = \tau|H \). Furthermore, \( H \) is \( \gamma \)-closed in \( G \) because \( H \) is complete. Since \((G/H, \gamma/H)\) is Hausdorff and \((G/H, \tau/H)\) is compact we have \( \gamma/H = \tau/H \). Thus, by Merson’s Lemma 2.2 we conclude that \( \gamma = \tau \).

2.1. Ordered topological spaces

Most of the definitions in this subsection can be found in [33]. Let \( E \) be a set. A binary relation \( \leq \) is a preorder if it is reflexive and transitive. An order (or partial order) is an antisymmetric preorder. A linear order (or total order) is an order that satisfies in addition the totality axiom: for all \( a, b \in E \) either \( a \leq b \) or \( b \leq a \). A preordered (ordered or linearly ordered) set is a set equipped with a preorder (respectively, order or linear order). For two ordered sets \((X, \leq)\) and \((Y, \preceq)\) a function \( f : X \to Y \) is order-preserving if \( x \leq y \) implies \( f(x) \preceq f(y) \).
For a set $X$ equipped with a linear order $\leq$, the order topology (or interval topology) $\tau_\leq$ on $X$ is generated by the subbase that consists of open intervals $(a, b) = \{ x \in X : a < x < b \}$, $(a, \to) = \{ x \in X : b < x \}$. A linearly ordered topological space (or LOTS) is a triple $(X, \tau_\leq, \leq)$ where $\leq$ is a linear order on $X$ and $\tau_\leq$ is the order topology on $X$. For every pair $a < b$ in $X$ the definition of the intervals $(a, b), [a, b]$ is understood.

**Remark 2.4.** Let $X$ be a LOTS and $a < b$ is a given pair of elements in $X$. Let $h \in H^+_{\leq}[a, b]$. Then for the natural extension $\hat{h} : X \to X$ with $\hat{h}(x) = x$ for every $x \in X \setminus (a, b) = (\leftarrow, a] \cup [b, \to)$ we have $\hat{h} \in H^+(X)$.

**Definition 2.5.** (Nachbin[33]) A topological ordered (preordered) space is a triple $(X, \leq, \tau)$ where $X$ is a set, $\leq$ is an order (preorder) on $X$, $\tau$ is a topology on $X$ and the graph of the order (preorder) $Gr(\leq) = \{(x, y) : x \leq y\}$ is closed in $X \times X$. In particular, a compact ordered space is a topological ordered space that is compact.

**Remark 2.6.** Every compact Hausdorff space is a compact partially ordered space with respect to the trivial partial order (equality) because the closedness of the diagonal in $X \times X$ is exactly the Hausdorff property of $X$.

A subset $Y \subseteq X$ is said to be decreasing if $x \leq y \in Y$ implies $x \in Y$. Similarly one defines an increasing subset.

**Lemma 2.7.** [33, Prop. 1] Let $(X, \leq)$ be a partially ordered set and let $\tau$ be a topology on $X$. The following conditions are equivalent:

1. $(X, \leq, \tau)$ is a topological ordered space (that is, $Gr(\leq)$ is $\tau$-closed in $X \times X$);
2. if $x \leq y$ is false then there exist: an increasing neighborhood $W$ of $x$ and a decreasing neighborhood $V$ of $y$ such that $V \cap W = \emptyset$.

**Corollary 2.8.** Let $(X, \leq)$ be a LOTS. If $u_1 < u_2$ are two points in $X$, then there exist disjoint $\tau_\leq$-open neighborhoods $O_1$ and $O_2$ in $X$ of $u_1$ and $u_2$ respectively, such that $O_1 < O_2$, meaning that $x < y$ for every $(y, x) \in O_2 \times O_1$. In particular, the graph of $\leq$ is closed in $(X, \tau_\leq) \times (X, \tau_\leq)$.

**Corollary 2.9.** Any (compact) LOTS is a (compact) ordered space in the sense of Nachbin. Conversely, for every compact ordered space $(X, \tau, \leq)$, where $\leq$ is a linear order, $\tau$ is necessarily the interval topology of $\leq$.

**Proof.** The first part is obvious by Corollary 2.8 and Lemma 2.7. For the second part observe that the $\tau$-closeness of the linear order $\leq$ in $X \times X$ implies that the subbase intervals $(a, \to), (\leftarrow, b)$ (with $a, b \in X$) are $\tau$-open. Hence $\tau_\leq \subseteq \tau$. Since $\tau_\leq$ is a Hausdorff topology and $\tau$ is a compact topology we conclude that $\tau_\leq = \tau$. 

Let $X$ be a partially ordered set. Recall that a subset $A \subseteq X$ is called convex if $[a, b] := \{ x \in X : a \leq x \leq b \} \subseteq A$ for every $a, b \in A$. 


Definition 2.10. Let $(X,\leq)$ be a partially ordered set and let $\mu$ be a uniformity on $X$. Then $\mu$ is uniformly convex if for all $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that $\delta \subseteq \varepsilon$ and $\delta(x)$ is convex for all $x \in X$.

Remark 2.11. Note that if $\delta(x)$ is convex, then so is $(\delta \cap \delta^{-1})(x)$. Thus $\delta$ can be chosen to be symmetric, and it will be so chosen throughout this paper.

Lemma 2.12. Let $\delta \in \mu$ be a (symmetric) entourage such that $\delta(x)$ is convex for all $x \in X$. If $(a,b) \in \delta$ and $a \leq c \leq d \leq b$ then $(c,d) \in \delta$. 

Proof. If $(a,b) \in \delta$ then $a,b \in \delta(b)$. From the assumptions that $a \leq c \leq b$ and $\delta(b)$ is convex it follows that $c \in \delta(b)$ or (equivalently) $b \in \delta(c)$. From $c,b \in \delta(c)$ and $c \leq d \leq b$ it follows that $d \in \delta(c)$ (by convexity of $\delta(c)$) and finally $(c,d) \in \delta$. □

Lemma 2.13. Let $X$ be a compact LOTS. Then the natural unique uniformity $\mu$ on $X$ is uniformly convex.

Proof. This follows by combining some well known results. For instance, [35, Prop. 3.3] and the fact that every compact LOTS is a uniform ordered space in the sense of Nachbin [33, Prop. 13]. □

2.2. Limit points and ultrafilters

All definitions and results of this subsection can be found for example in [4]. Let $X$ be a topological space and $\mathcal{J}$ a filter on $X$. A point $x \in X$ is said to be a limit point of $\mathcal{J}$ if $\mathcal{J}$ is finer than the neighborhood filter $\mathcal{N}_x$ of $x$. We also say that $\mathcal{J}$ is convergent to $x$. The point $x$ is called a limit point of a filter base $\mathcal{B}$ on $X$, if the filter whose base is $\mathcal{B}$ converges to $x$. Let $f$ be a mapping from a set $X$ to a topological space $Y$, and let $\mathcal{J}$ be a filter on $X$. A point $y \in Y$ is a limit point of $f$ with respect to the filter $\mathcal{J}$ if $y$ is a limit point of the filter base $f(\mathcal{J})$.

Proposition 2.14. [4]

1. If $\mathcal{B}$ is an ultrafilter base on a set $X$ and if $f$ is a mapping into a set $Y$, then $f(\mathcal{B})$ is an ultrafilter base on $Y$.

2. Let $f$ be a mapping from a set $X$ into a topological space $Y$, and let $\mathcal{J}$ be a filter on $X$. A point $y \in Y$ is a limit point of $f$ with respect to the filter $\mathcal{J}$ if and only if, $f^{-1}(V) \in \mathcal{J}$ for each neighborhood $V$ of $y$ in $Y$.

3. If $X$ is a compact Hausdorff space, then every ultrafilter on $X$ converges to a unique point.

We can sum these propositions as follows:

Corollary 2.15. Let $\mathcal{J}$ be an ultrafilter on a set $E$ and let $f$ be a mapping from $E$ to a compact space $X$. Then there exists a unique point $\bar{x} \in X$ such that each neighborhood $O$ of $\bar{x}$ satisfies

$$f^{-1}(O) \in \mathcal{J}.$$ 

That is, $\bar{x}$ is the limit point of $f$ with respect to $\mathcal{J}$.
2.3. $\pi$-uniform actions

Let $\pi: G \times X \to X$ be an action of a topological group $G$ on a space $X$. We define two maps:

1. $g$-translation: $\pi^g: X \to X$, $\pi^g(x) = gx$;
2. $x$-orbit: $\pi_x: G \to X$, $\pi_x(g) = gx$.

For a topological group $(G, \tau)$ we denote by $e$ the identity element of $G$ and by $N_{g_0}(\tau)$ the local base of $G$ at $g_0$.

**Definition 2.16.** \cite{30, 29} Let $\pi: G \times X \to X$ be an action of a topological group $(G, \tau)$ on a Hausdorff uniform space $(X, \mu)$. The uniformity (or, the action) is said to be:

1. $\pi$-uniform at $e$ or quasibounded if for every $\varepsilon \in \mu$ there exist $\delta \in \mu$ and $U \in N_e(\tau)$ such that $(gx, gy) \in \varepsilon$ for every $(x, y) \in \delta$ and $g \in U$.
2. $\pi$-uniform if for every $g_0 \in G$ and for every $\varepsilon \in \mu$ there exist $\delta \in \mu$ and $U \in N_{g_0}(\tau)$ such that $(gx, gy) \in \varepsilon$ for every $(x, y) \in \delta$ and $g \in U$.

The notion of a $\pi$-uniform action, defined in \cite{30, 29}, was originally used to study compactifications of $G$-spaces. Later it was employed by Gamarnik \cite{21} to prove that for a compact space $X$ the compact-open topology on $H(X)$ is minimal within the class of $\pi$-uniform topologies. More applications of $\pi$-uniformity can be found in \cite{27} and in \cite{6}.

**Definition 2.17.** Let $X$ be a compact space and $G$ a subgroup of $H(X)$. A Hausdorff group topology $\tau$ on $G$ is said to be $\pi$-uniform if the natural action $(G, \tau) \times X \to X$ is $\pi$-uniform with respect to the unique compatible uniformity on $X$.

For a topological group $X$ denote by $\mu_l, \mu_r, \mu_{l\lor r}$ the left, right and two-sided uniform structures on $X$, respectively.

**Lemma 2.18.** Let $G$ be a topological group and $X$ is a uniform space.

1. If $\pi: G \times X \to X$ is a $\pi$-uniform action and all orbit maps $\pi_x: G \to X$ are continuous, then $\pi$ is continuous;
2. If $X$ is compact and $P \leq H(X)$ is a topological subgroup, then the action $\pi: P \times X \to X$ is $\pi$-uniform if and only if it is $\pi$-uniform at the identity.
3. If $X$ is a topological group and $\pi: G \times X \to X$ is an action by continuous automorphisms, then the action is $\pi$-uniform with respect to $\mu \in \{\mu_l, \mu_r, \mu_{l\lor r}\}$ if and only if $\pi$ is continuous at $(e, e_X)$. 

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3. Minimality properties of $H_+(X)$

Our main objective in this section is to prove that under certain conditions the compact-open topology on $H_+(X)$ is minimal. We prove here a generalized version of Gamarnik’s result for a (partially) ordered setting. In the particular case of the trivial partial order (equivalence relation, see Remark 2.6) we obtain the original result of Gamarnik [21, Prop. 2.1].

**Theorem 3.1.** Let $(X, \tau_\rho)$ be a compact partially ordered space and $H_+(X)$ is the group of all order-preserving homeomorphisms of $X$. Then the compact-open topology $\tau_{co}$ is minimal within the class of $\pi$-uniform topologies on $H_+(X)$.

**Proof.** Assuming the contrary suppose that there exists a $\pi$-uniform group topology $\tau$ on $H_+(X)$ such that $\tau_{co} \subsetneq \tau$. Let $\pi: H_+(X) \times X \to X$ be the natural action of $H_+(X)$ on $X$. If all orbit maps are continuous, then, by Lemma 2.18 (1), $\pi$ is continuous and, by Lemma 2.4 $\tau_{co} \subseteq \tau$. So we can assume that there exists an orbit map that is not continuous (at the identity). That is, there exists $x_0 \in X$ such that $\pi_{x_0}: H_+(X) \to X$ is not continuous at $e \in H_+(X)$. Thus, denoting by $\mu$ the natural uniformity on $X$, there exists $\varepsilon_0 \in \mu$ such that for all $U \in N_e(\tau)$ there exists $g_U \in U$ for which $\left((g_U x_0, x_0) \notin \varepsilon_0, \right.$ (3.1)\left.)

For a given $U \in N_e(\tau)$ define $F(U) = \{V \in N_e(\tau) : V \subseteq U\}$. Denote by $F$ the filter on the set $N_e(\tau)$ generated by the filter base $\{F(U)\}_{U \in N_e(\tau)}$. Since every filter is contained in an ultrafilter, choose an ultrafilter $J$ on $N_e(\tau)$ that contains $F$.

For each $x \in X$ define a map $f_x: N_e(\tau) \to X$ by $f_x(U) = g_U x$ for $g_U$ that satisfies (3.1). Let $\bar{x}$ be the limit point of $f_x$ with respect to the ultrafilter $J$ given by Corollary 2.15. Now define the following transformation

$$ h: X \to X, \quad h(x) = \bar{x}. $$ (3.2)

In the rest of the proof we show that $h$ is a nontrivial order-preserving homeomorphism that belongs to every neighborhood of the identity element in $(H_+(X), \tau)$, in contradiction to $\tau$ being a Hausdorff group topology.

**Claim 3.2.** The map $h$ defined by (3.2) is a nontrivial homeomorphism in $H_+(X)$.

**Proof.** We break the proof into five steps.

Step 1. In order to prove that $h$ is one-to-one, suppose to the contrary that there exist $x, y, z \in X$ such that $h(x) = h(y) = z$ and $x \neq y$. Choose an entourage $\varepsilon \in \mu$ such that $(x, y) \notin \varepsilon$. The action is $\pi$-uniform at the identity, and thus there exist $U_\varepsilon \in N_e(\tau)$ and $\delta_\varepsilon \in \mu$ such that $(g x, g y) \notin \varepsilon$ for every $(x, y) \in \delta_\varepsilon$ and $g \in U_\varepsilon$. Choose a symmetric $\delta \in \mu$ satisfying $\delta^2 \subseteq \delta_\varepsilon$. 
In particular, if \( A(\pi) \) and, similarly, \( A(\eta) \),

Since \( F \) is \( \pi \)-uniform, and in particular for \( \eta \), we have:

\[
A(x) = \{ U \in N_{\epsilon}(\tau) : (g_U x, z) \in \delta \} \in \mathcal{J}.
\]

Similarly,

\[
A(y) = \{ U \in N_{\epsilon}(\tau) : (g_U y, z) \in \delta \} \in \mathcal{J}.
\]

Also, since \( F(U^{-1}) \in \mathcal{J} \), the intersection \( A(x) \cap A(y) \cap F(U^{-1}) \) is not empty. If \( U_0 \in A(x) \cap A(y) \cap F(U^{-1}) \) and \( g_{U_0} \in U_0 \), then \( g_{U_0} \in U_{\epsilon}^{-1} \) (and thus \( g_{U_0}^{-1} \in U_{\epsilon} \)),

\[
(x, y) \in \mathcal{J}, \quad (g_{U_0} x, g_{U_0} y) \in \mathcal{J}.
\]

And since \( U \) is \( \pi \)-uniform, we have \( (g_{U_0}^{-1} g_{U_0} x, g_{U_0}^{-1} g_{U_0} y) = (x, y) \in \mathcal{J} \), and this contradicts the choice of \( \epsilon \). Therefore \( h \) is one-to-one.

**Step 2.** To prove that \( h \) is onto, for a given \( y \in X \) we will find \( x \in X \) such that \( h(x) = y \). Consider the map \( N_{\epsilon}(\tau) \to X \) given by \( U \mapsto g_U^{-1} y \). Let \( x \) be the limit point of this map with respect to the ultrafilter \( \mathcal{J} \). To show that \( h(x) = y \), we will show that \( y \) is the limit point of \( f_x \) with respect to \( \mathcal{J} \). Let \( \epsilon \in \mu \) be an arbitrary entourage and choose \( U_\epsilon, \delta_\epsilon \) from the definition of \( \pi \)-uniform topology.

Since \( x \) is defined as the limit point of \( g_U^{-1} y \), we know that

\[
A(x) = \{ U \in N_{\epsilon}(\tau) : (g_U^{-1} y, x) \in \delta_\epsilon \} \in \mathcal{J}.
\]

And since \( F(U_\epsilon) \) is also an element of \( \mathcal{J} \), the intersection \( A(x) \cap F(U_\epsilon) \) is not empty. Let \( U \in A(x) \cap F(U_\epsilon) \). Then for \( g_U \in U \subseteq U_\epsilon \) and \( g_U^{-1} y, x \in \delta_\epsilon \) we have \( (g_U y, x) \in \mathcal{J} \) (by the choice of \( U_\epsilon, \delta_\epsilon \)). This last condition is satisfied by all \( U \in A(x) \cap F(U_\epsilon) \) and, therefore, \( \{ U \in N_{\epsilon}(\tau) : (g_U y, x) \in \mathcal{J} \} \in \mathcal{J} \) (since \( A(x) \cap F(U_\epsilon) \in \mathcal{J} \) and \( A(x) \cap F(U_\epsilon) \subseteq \{ U \in N_{\epsilon}(\tau) : (g_U y, x) \in \mathcal{J} \} \)). This holds for all \( \epsilon \in \mu \), which proves that \( y \) is the limit point of \( f_x = g_U y \) with respect to \( \mathcal{J} \). And that, in turn, proves that \( h(x) = y \) and therefore \( h \) is onto.

**Step 3.** In order to prove that \( h \) is (uniformly) continuous we will show that for every \( \epsilon \in \mu \) there exists \( \delta \in \mu \) such that \( (h(x), h(y)) \in \mathcal{J} \) for all \( (x, y) \in \mathcal{J} \). Let \( \epsilon_0 \in \mu \) and choose \( \epsilon \in \mu \) such that \( \epsilon^3 \subseteq \epsilon_0 \). Choose \( \delta_\epsilon, U_\epsilon \) from the Definition 2.16 of the \( \pi \)-uniformity. We will show that if \( (x, y) \in \delta_\epsilon \) then \( (h(x), h(y)) \in \epsilon_0 \). Let \( (x, y) \in \delta_\epsilon \) and suppose to the contrary that \( (h(x), h(y)) \notin \epsilon_0 \). This means that if \( t_1, t_2 \) satisfy \( (h(x), t_1) \in \epsilon \), \( (h(y), t_2) \in \epsilon \), then

\[
(t_1, t_2) \notin \epsilon.
\]

(3.3)

Since \( h(x) \) is the limit point of \( g_U y \), \( A(x) = \{ U \in N_{\epsilon}(\tau) : (g_U y, h(y)) \in \epsilon \} \in \mathcal{J} \) and, similarly, \( A(y) = \{ U \in N_{\epsilon}(\tau) : (g_U y, h(y)) \in \epsilon \} \in \mathcal{J} \). Also, since \( F(U_\epsilon) \in \mathcal{J} \), the intersection \( A(x) \cap A(y) \cap F(U_\epsilon) \) is not empty. Let \( V \in A(x) \cap A(y) \cap F(U_\epsilon) \).

In particular, \( (g_V x, h(x)) \in \epsilon \) and \( (g_V y, h(y)) \in \epsilon \). Next, from (3.3) it follows that \( (g_V x, g_V y) \notin \epsilon \). But \( V \subseteq U_\epsilon \), so \( g_V \in U_\epsilon \), and recall that \( (x, y) \in \delta_\epsilon \). So by definition of \( \pi \)-uniformity we get the desired contradiction.

**Step 4.** To see that \( h \) is not trivial, recall that from (3.3) we have \( x_0 \in X \) and \( \epsilon_0 \in \mu \) such that \( (g_U x_0, x_0) \notin \epsilon_0 \) for every \( U \in N_{\epsilon}(\tau) \). This implies that \( h(x_0) \neq x_0 \).
Step 5. Finally, we show that \( h \) is order-preserving. Suppose to the contrary that there exists a pair \( a, b \in X \) such that \( a \leq b \) and \( (a, b) \notin \text{Gr}(\leq) \). By Lemma 2.7, one can find two disjoint and open neighbourhoods \( W, V \) of \( a, b \) respectively, such that \( W \) is increasing and \( V \) is decreasing.

By the definition of \( h \) the following sets are members of \( \mathcal{J} \):
\[
A_1 = \{ U \in N_\varepsilon(\tau) : g_U a \in W \},
A_2 = \{ U \in N_\varepsilon(\tau) : g_U b \in V \}.
\]

Since \( \mathcal{J} \) is a filter the intersection \( A_1 \cap A_2 \) is not empty. Choose \( U_0 \) in the intersection. Thus we have an order-preserving homeomorphism \( g_{U_0} \) such that \( g_{U_0} a \in W \) and \( g_{U_0} b \in V \).

Since \( a \leq b \) and \( g_{U_0} \) is order-preserving, we have \( g_{U_0} a \leq g_{U_0} b \in V \). Since \( V \) is decreasing we have \( g_{U_0} a \in V \). Then \( g_{U_0} a \in W \cap V \). This means that \( W \) and \( V \) are not disjoint, a contradiction.

\[\square\]

The following claim shows that \( \tau \) is not Hausdorff.

**Claim 3.3.** For every \( U \in N_\varepsilon(\tau) \), \( h \in U \).

**Proof.** For \( g \in H_+(X) \) and \( \varepsilon \in \mu \) define
\[
\tilde{\varepsilon}(g) = \{ f \in H_+(X) : (g(x), f(x)) \in \varepsilon \text{ for all } x \in X \}.
\]

It can be easily verified that \( \{ \tilde{\varepsilon}(g) \}_{g \in H_+(X)} \) is a local base of neighborhoods for every point \( g \in H_+(X) \) with respect to the compact-open topology \( \tau_{co} \) on \( H_+(X) \).

In order to prove the statement, it suffices to show that \( h \in [\tilde{\varepsilon}^3(\varepsilon)]^{-1}U_0 \) for every \( U_0 \in N_\varepsilon(\tau) \) and for every \( \varepsilon \in \mu \). Indeed, for each \( U \in N_\varepsilon(\tau) \) we can find \( U_0 \in N_\varepsilon(\tau) \) such that \( U_0^2 \subseteq U \) and \( U_0^{-1} = U_0 \). But \( \tau \subseteq \tau_{co} \), and \( \{ \tilde{\varepsilon}(g) \}_{g \in H_+(X)} \) is a local base at \( e \), thus there exists \( \varepsilon \in \mu \) with \( \tilde{\varepsilon}^3(\varepsilon) \subseteq U_0 \). Therefore \( [\tilde{\varepsilon}^3(\varepsilon)]^{-1}U_0 \subseteq U_0^{-1}U_0 \subseteq U \).

Let \( \varepsilon \in \mu \) and \( U_0 \in N_\varepsilon(\tau) \). Choose \( \delta_\varepsilon \in \mu \) and \( U_\varepsilon \in N_\varepsilon(\tau) \) from the definition of \( \pi \)-uniform topology. For \( x \in X \) define \( A(x) = \{ U \in N_\varepsilon(\tau) : (g_U h^{-1}(x), x) \in \varepsilon \} \).

Since \( h(h^{-1}(x)) = x \), from the definition of \( h \) we have \( A(x) \in \mathcal{J} \) (indeed, \( x = h(h^{-1}(x)) = h^{-1}(x) \), \( x \) is the limit point of the map \( f_{h^{-1}(x)} : N_\varepsilon(\tau) \to X \) defined by \( f_{h^{-1}(x)}(U) = g_U h^{-1}(x) \)). Since \( h \) (and thus \( h^{-1} \)) is uniformly continuous, we can choose \( \alpha \in \mu \) such that \( \alpha \subseteq \varepsilon \) and
\[
(t_1, t_2) \in \alpha \Rightarrow (h^{-1}(t_1), h^{-1}(t_2)) \in \delta_\varepsilon.
\]

(3.4)

Since \( X \) is compact, we can find a finite subset \( \{ x_1, x_2, ..., x_n \} \subseteq X \) such that for every \( x \in X \) there exists \( 1 \leq i \leq n \) for which \( (x, x_i) \in \alpha \). Let
\[
U \in \left( \bigcap_{i=1}^n A(x_i) \right) \cap F(U_\varepsilon \cap U_0).
\]

(3.5)
For every $x \in X$ there exists $i$ such that $(x, x_i) \in \alpha$ and from (3.4) we have $(h^{-1}(x), h^{-1}(x_i)) \in \delta_i$. Since $U \subseteq U_\varepsilon$, by the choice of $U_\varepsilon$ and $\delta_i$ we have $(g_U h^{-1}(x), g_U h^{-1}(x_i)) \in \varepsilon$. Since $U \subseteq A(x_i)$, it follows that $(g_U h^{-1}(x), x_i) \in \varepsilon$. Recalling that $(x, x_i) \in \alpha \subseteq \varepsilon$ we obtain $(g_U h^{-1}(x), x_i) \in \varepsilon^3$, and therefore $g_U h^{-1} \in \varepsilon^3(\varepsilon)$. But since $g_U \in \varepsilon U_0$, we get $h \in \varepsilon^3(\varepsilon) \varepsilon^3(\varepsilon) U_0$, and this completes the proof.

Claims 3.2 and 3.3 complete the proof of Theorem 3.1.

3.1. First application

Theorem 3.1 cannot be strengthened to state that every compact LOTS is $M_\pi$-compact as Example 3.1 demonstrates. Nevertheless, we show that under certain natural conditions (of "local movability") on a compact connected LOTS $X$, the group $H_+(X)$ is minimal.

**Theorem 3.4.** Let $(X, \tau_\varepsilon)$ be a compact connected LOTS that satisfies the following conditions:

(A) for every pair of elements $a < b$ in $X$ the group $H_+[a, b]$ is nontrivial.

Then the group $H_+(X)$ is minimal (that is, $X$ is $M_\pi$-compact).

**Proof.** Suppose to the contrary that there exists a Hausdorff group topology $\tau$ on $H_+(X)$ such that $\tau \not\subseteq \tau_\varepsilon$. From Theorem 3.1 we know that $\tau$ is not $\pi$-uniform and, from Lemma 2.1, that it is not $\pi$-uniform at the identity. That is

$$(\exists \varepsilon \in \mu)(\forall \varepsilon \in \mu)(\forall U \in N_\varepsilon(\tau))(\exists (x_\alpha, y_\alpha) \in \delta)(\exists g_\alpha \in U) : (g_\alpha x_\alpha, g_\alpha y_\alpha) \notin \varepsilon. \tag{3.6}$$

We define two maps $\psi_1, \psi_2: \mu \times N_\varepsilon(\tau) \to X$ as follows:

$$\psi_1: \alpha = (\delta, U) \mapsto g_\alpha x_\alpha$$

$$\psi_2: \alpha = (\delta, U) \mapsto g_\alpha y_\alpha.$$  

For $\alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau)$ define

$$F(\alpha) = \{(\delta', U') \in \mu \times N_\varepsilon(\tau) : \delta' \subseteq \delta, U' \subseteq U\}.$$  

One can verify that $\{F(\alpha)\}_{\alpha \in \mu \times N_\varepsilon(\tau)}$ is a filter base. Denote by $\mathcal{F}$ the ultrafilter on $\mu \times N_\varepsilon(\tau)$ that contains $\{F(\alpha)\}_{\alpha \in \mu \times N_\varepsilon(\tau)}$. Denote by $x_0$ and $y_0$ the limit points of the maps $\psi_1$ and $\psi_2$ respectively, with respect to the ultrafilter $\mathcal{F}$.

**Claim 3.5.** If $\delta \in \mu$ is a symmetric entourage such that $\delta^3 \subseteq \varepsilon$ (for the $\varepsilon$ given in (3.6)), then $(x_0, y_0) \notin \delta$.

**Proof.** Suppose to the contrary that $(x_0, y_0) \in \delta$. Since $x_0$ is a limit point of $\psi_1$ with respect to the ultrafilter $\mathcal{F}$ and $\delta(x_0)$ is a neighborhood of $x_0$ we have:

$$A_\delta(x_0) = \{\alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau) : \psi_1(\alpha) \in \delta(x_0)\} = \{\alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau) : (g_\alpha x_\alpha, x_0) \in \delta\} \in \mathcal{F}.$$
Similarly for $y_0$ we have

$$A_\delta(y_0) = \{ \alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau) : (g_\alpha y_0, y_0) \in \delta \} \in \mathcal{J}.$$ 

Since $\mathcal{J}$ is a filter the intersection $A_\delta(x_0) \cap A_\delta(y_0)$ is not empty and thus there exists $\alpha_1 = (\delta_1, U_1) \in \mu \times N_\varepsilon(\tau)$ such that $(g_{\alpha_1} x_{\alpha_1}, x_0) \in \delta$ and $(g_{\alpha_1} y_{\alpha_1}, y_0) \in \delta$ for $g_{\alpha_1} \in U_1$, $(x_{\alpha_1}, y_{\alpha_1}) \in \delta_1$. Then

$$(g_{\alpha_1} x_{\alpha_1}, g_{\alpha_1} y_{\alpha_1}) = (g_{\alpha_1} x_{\alpha_1} x_0)(x_0, y_0)(y_0, g_{\alpha_1} y_{\alpha_1}) \in \delta^3 \subseteq \varepsilon.$$ 

This contradicts \([3,6]\). \(\square\)

Now, since $(x_0, y_0) \notin \delta$ we know that $x_0 \neq y_0$, so we assume without loss of generality that $x_0 < y_0$. Since $X$ is connected it is densely ordered, thus we can find $t_1, t_2 \in X$ such that $x_0 < t_1 < t_2 < y_0$. According to assumption () there exists a nontrivial order-preserving homeomorphism $[t_1, t_2] \to [t_1, t_2]$. Also, by Remark 2.1 there exists $e \neq \psi \in \mathcal{H}^e(X)$ such that $\psi$ is trivial on $X \setminus (t_1, t_2)$. The following claim shows that $\tau$ is not Hausdorff.

**Claim 3.6.** $\psi \in U$ for all $U \in N_\varepsilon(\tau)$.

**Proof.** It is sufficient to show that for all $U_0 \in N_\varepsilon(\tau)$ and for all $e \in \mu$ we have $\psi \in U_0 O_0 U_0^{-1}$ where $O_e = \{ g \in \mathcal{H}^e(X) : (gx, x) \in e \ \forall x \in X \}$. Let $e \in \mu$, $U_0 \in N_\varepsilon(\tau)$. Since $x_0$ is the limit point of $\psi_1$ under the ultrafilter $\mathcal{J}$ and $(e, t_1)$ is a neighborhood of $x_0$, we have

$$A_{t_1}(x_0) = \{ \alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau) : g_\alpha x_0 < t_1 \} \in \mathcal{J}.$$ 

Similarly for $y_0$ and the neighborhood $(t_2, -)$ of $y_0$ we have:

$$A_{t_2}(y_0) = \{ \alpha = (\delta, U) \in \mu \times N_\varepsilon(\tau) : g_\alpha y_0 > t_2 \} \in \mathcal{J}.$$ 

Since $X$ is a compact LOTS, according to Lemma \([2,13]\) we know that for the $\varepsilon$ given above there exists a symmetric $\delta_0 \in \mu$ such that $\delta_0 \subseteq \varepsilon$ and for all $x \in X$ the set $\delta_0(x)$ is convex. Set $\alpha_0 = (\delta_0, U_0)$. Since $F(\alpha_0) \in \mathcal{J}$, there exists $\hat{\alpha} = (\hat{\delta}, \hat{U}) \in (\alpha_0) \cap A_{t_1}(x_0) \cap A_{t_2}(y_0)$. Thus we have $(x_{\hat{\alpha}}, y_{\hat{\alpha}}) \in \delta \subseteq \delta_0 \subseteq \varepsilon$ and $g_{\hat{\alpha}} \in \hat{U} \subseteq U_0$ such that $g_{\hat{\alpha}} x_{\hat{\alpha}} < t_1$ and $g_{\hat{\alpha}} y_{\hat{\alpha}} > t_2$. In particular, $x_{\hat{\alpha}} \neq y_{\hat{\alpha}}$.

We will show that $\psi \in U_0 O_e U_0^{-1}$ by showing that $g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}} \in O_e$, that is $(g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}} t, t) \in e$ for all $t \in X$. Let $(a, b)$ denote the interval $(x_{\hat{\alpha}}, y_{\hat{\alpha}})$ in case $x_{\hat{\alpha}} < y_{\hat{\alpha}}$, and $(y_{\hat{\alpha}}, x_{\hat{\alpha}})$ otherwise. We break the proof into two cases.

**Case 1:** $t \notin (a, b)$. In this case we have either $t \leq x_{\hat{\alpha}}$ or $t \geq y_{\hat{\alpha}}$. Since $g_{\hat{\alpha}}$ is order-preserving we have either $g_{\hat{\alpha}} t \leq g_{\hat{\alpha}} x_{\hat{\alpha}}$ or $g_{\hat{\alpha}} t \geq g_{\hat{\alpha}} y_{\hat{\alpha}}$, respectively. Recall that $g_{\hat{\alpha}} x_{\hat{\alpha}} < t_1$ and $g_{\hat{\alpha}} y_{\hat{\alpha}} > t_2$ which means that $g_{\hat{\alpha}} t < t_1$ or $g_{\hat{\alpha}} t > t_2$. Thus $g_{\hat{\alpha}} t \in X \setminus [t_1, t_2]$ and, since $X \setminus [t_1, t_2] \subseteq X \setminus (t_1, t_2)$, we have $\psi g_{\hat{\alpha}} t = g_{\hat{\alpha}} t$ and thus $g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}} t = t$ (since $\psi$ is trivial on $X \setminus (t_1, t_2)$). Therefore $(g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}} t, t) \in e$ for every $t \notin (a, b)$.

**Case 2:** $t \in (a, b)$. Since $g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}}$ is trivial outside of $(a, b)$ (as shown in Case 1), we know that $g_{\hat{\alpha}}^{-1} \psi g_{\hat{\alpha}} (a, b) \subseteq (a, b)$. Recall that $(a, b) \in \delta_0$ and, since
of $O$, completing the proof is to show that the homeomorphism $x$, according to Lemma 2.12 we have $(g_a^{-1} \psi g_a, t, t) \in J_0$. Since $\delta_0 \subseteq \varepsilon$, we have $(g_a^{-1} \psi g_a, t, t) \in \varepsilon$ for all $t \in (a, b)$.

Summing up the two cases we see that $(g_a^{-1} \psi g_a, t, t) \in \varepsilon$ for all $t \in X$ and, since $g_a \in U_0$, we conclude that $\psi \in U_0 Q J_0^{-1}$. This completes the proof. ☐

3.2. Minimality of $G \times \text{Aut}(G)$

The main goal of this subsection is to prove that for every compact topological group $G$ the natural semidirect product $G \times \text{Aut}(G)$ is a minimal topological group (Theorem 3.11).

Remark 3.7. Observe that if $\text{Aut}(G)$ is minimal then $G \times \text{Aut}(G)$ is minimal by Theorem 3.1. So, in view of Theorem 3.11 it is important to note that there are compact groups $G$ such that $\text{Aut}(G)$ is not minimal. Indeed, as it was explained in [10, Section 5], one may take $G = (\mathbb{Q}, \text{discrete})^\wedge$ the Pontryagin dual of the discrete group $\mathbb{Q}$ of the rational numbers.

First of all we extend Theorem 3.1 in some algebraic setting. Let $w : K \times K \to K$ be a binary operation on a compact space $K$. Denote by $\text{Aut}(K)$ the group of continuous automorphisms of the structure $(K, w)$. If $K$ is a compact ordered space then we denote by $\text{Aut}_+(K)$ the group of automorphisms of $(K, w, \leq)$. Note that $\text{Aut}_+(K) = \text{Aut}(K) \cap H_+(K)$.

Theorem 3.8. The compact-open topology on $\text{Aut}_+(K)$ is minimal within the class of $\pi$-uniform topologies.

Theorem 3.11 proves that the compact-open topology on $H_+(K)$ is minimal within the class of $\pi$-uniform topologies. The same can be said about the group of automorphisms $\text{Aut}_+(K)$. The only additional ingredient needed for completing the proof is to show that the homeomorphism $h$ is an algebraic automorphism of $(K, w)$. In fact, since $h$ is 1-1 and onto it is enough to show that $h$ is a homomorphism, that is, $h(xy) = h(x) h(y)$, where we write $w(x, y) := x y$ for $x, y \in K$.

Lemma 3.9. The function $h : K \to K$ defined in the proof of Theorem 3.1 is a homomorphism.

Proof. Let $x, y \in K$. We need to show that $h(xy) = h(x) h(y)$, that is $\overline{x y} = \overline{x} \overline{y}$. As before, $\overline{x y}$ is the limit point (with respect to the ultrafilter $J$) of the function $f_{xy} : X \to K$ defined by $U \to g_U(xy)$. Under analogous notation $\overline{x}$ is the limit of $f_x$ and $\overline{y}$ is the limit of $f_y$. Since $g$ is now an automorphism, we have $g_U(xy) = g_U(x) g_U(y)$ and therefore $f_{xy} = f_x f_y$. Thus we have to show that $\overline{x y}$ is the limit point of $f_{xy}$ with respect to $J$. That is, for every neighborhood $O$ of $\overline{x y}$ the set $\{ U \in N_c(\tau) : f_{xy}(U) \in O(x y) \}$ is in $J$. Note that since the multiplication function is continuous, for each $O(\overline{x y})$ there exist neighborhoods $M(\overline{x})$ and $N(\overline{y})$ such that $M(\overline{x}) N(\overline{y}) \subseteq O(\overline{x y})$. Therefore we get the following chain of inclusions: $\{ U \in N_c(\tau) : f_{xy}(U) \in O(\overline{x y}) \} \supseteq \{ U \in N_c(\tau) : f_{xy}(U) \in O(\overline{x y}) \}$
**Theorem 3.10.** Let \((M, \gamma)\) be a topological group such that \(M\) is algebraically a semidirect product \(M = X \times G\), where \(X\) is a compact subgroup. Then the corresponding action

\[
\alpha : (G, \gamma/X) \times (X, \gamma|_X) \to (X, \gamma|_X)
\]

is continuous at \((e_G, e_X)\).

**Proof.** Let \(pr : M \to G = M/X\), \((x, g) \mapsto g\), denote the canonical projection. Algebraically \(M/X = \{X \times \{g\}\}_{g \in G}\), which allows us to identify \(G\) with \(M/X\), and thus the topological group \((G, \gamma/X)\) is well defined.

To show that \(\alpha\) is continuous at \((e_G, e_X)\) let \(O \subseteq \gamma/X \subseteq X\) be a neighborhood of \(e_X\). We will find neighborhoods \(P\) of \(e_G\) (in \((G, \gamma/X)\)) and \(U\) of \(e_X\) (in \((X, \gamma|_X)\)) such that \(\alpha(P \times U) \subseteq O\).

Since \(X\) is a compact group, there exists a neighborhood \(O_1\) of \(e_X\) such that for all \(x \in X\) we have \(x^{-1}O_1x \subseteq O\). The restriction \(M \times X \to X\), \((a, x) \mapsto axa^{-1}\) of the conjugation \(M \times M \to M\) in the topological group \((M, \gamma)\) is (well-defined, because \(X\) is a normal subgroup of \(M\)) continuous at \((e_M, e_X)\). Therefore, for \(O_1\) there exist a neighborhood \(U\) of \(e_X\) (in \((X, \gamma|_X)\)) and a neighborhood \(V\) of \(e_M\) (in \((M, \gamma)\)) such that \(vUv^{-1} \subseteq O_1\) for all \(v \in V\).

We claim that \(P = pr(V)\) and \(U\) satisfy the needed conditions above. That is, we want to show that \(\alpha(g, z) := g(z) \in O\) for all \((g, z) \in pr(V) \times U\). Indeed, if \(g \in pr(V)\) there exists \(x \in X\) such that \((x, g) \in V\), and recall that \(z \in U\) is in fact \((z, e_G)\). We know that \(vzv^{-1} \in O_1\). Therefore,

\[
vzv^{-1} = (x, g)(z, e_G)(x, g)^{-1} = (xg(z), g)(g^{-1}(x^{-1}), g^{-1}) = (xg(z)g^{-1}(x^{-1}), e_G) = xg(z)x^{-1}, e_G = xg(z)x^{-1} \in O_1.
\]

Thus \(\alpha(g, z) \in x^{-1}O_1x \subseteq O\), which completes the proof.

**Theorem 3.11.** If \(G\) is a compact topological group, then \(G \rtimes \text{Aut}(G)\) is a minimal group.

**Proof.** Let \(\tau\) be the given topology on \(G\), and \(\tau_{co}\) the compact-open topology on \(\text{Aut}(G)\). Denote by \(\gamma\) the product topology on \(G \times \text{Aut}(G)\). Assume that \(\gamma_1 \subseteq \gamma\) is a coarser Hausdorff group topology on \(G \times \text{Aut}(G)\). Since \(G\) is compact we have \(\gamma_1|_G = \gamma|_G = \tau\).

The action \(\alpha : (\text{Aut}(G), \gamma_1|_G) \times (G, \gamma_1|_G) \to (G, \gamma_1|_G)\) is continuous at the identity \((id_G, e_G)\) by Theorem 8.10. Furthermore, \(\gamma_1/G\) is a Hausdorff topology on \(\text{Aut}(G)\) since \(G\) is a compact (hence, closed) subgroup of the Hausdorff group \((G \rtimes \text{Aut}(G), \gamma_1)\). Therefore \(\gamma_1/G\) is an \(\alpha\)-uniform topology on \(\text{Aut}(G)\) (Lemma 2.18 (3) and Definition 2.11).
Since $\gamma_1/G \subseteq \gamma/G = \tau_{co}$ and $\tau_{co}$ is minimal within the class of $\alpha$-uniform topologies on $\text{Aut}(G)$ (Theorem 3.8 for the trivial partial order), we have $\gamma_1/G = \gamma/G$. Finally, using Merson’s Lemma 2.2 we deduce that $\gamma_1 = \gamma$ and that concludes the proof.

It seems to be unknown if $G \triangleright P$ is minimal for any compact group $G$ and every (closed) subgroup $P$ of $\text{Aut}(G)$. For other results and questions about $\text{Aut}(G)$ in the context of minimality see [10].

4. Order-preserving homeomorphisms and $\alpha$-minimality

First we recall the following result of Gartside and Glyn:

Theorem 4.1. [22] For any metric one dimensional manifold (with or without boundary) $M$, the compact-open topology on the full homeomorphism group $H(M)$ is the unique minimum group topology on $H(M)$.

The one dimensional compact manifolds, up to homeomorphism, are the closed interval $[0,1]$ and the circle. So, this result can be reformulated in the following way.

Theorem 4.2. [22] $H([0,1])$ and $H(S^1)$ are $\alpha$-minimal groups.

Using some results of Nachbin we extend the ideas of [22] to compact connected linearly ordered spaces (Theorem 4.6). In fact, one can also replace the linear order with the cyclic order.

For the purposes of this section we fix the following notations. Let $(X, \tau_c)$ be a compact LOTS with its unique compatible uniform structure $\mu$ and define
\[ s = \min X, t = \max X. \]
For every $f \in C(X)$ and $\varepsilon > 0$ define
\[ U_{f,\varepsilon} := \{(x,y) \in X \times X : |f(x) - f(y)| \leq \varepsilon\}. \]

Lemma 4.3. (Nachbin [33]) Denote by $C_+(X,[0,1])$ the set of all continuous order-preserving maps $f:X \to [0,1]$. The family $\{U_{f,\varepsilon} : f \in C_+(X,[0,1]), \varepsilon > 0\}$ is a subbase of the uniformity $\mu$ for every compact LOTS $X$.

Proof. For every compact space $X$ and a point-separating family $F$ of functions $X \to [0,1]$, the corresponding weak uniformity $\mu_F$ on $X$ is just the natural unique compatible uniformity $\mu$ on $X$. The corresponding uniform subbase is the family $\{U_{f,\varepsilon} : f \in F, \varepsilon > 0\}$ of entourages.

On the other hand, by a fundamental result of Nachbin [33], p. 48 and 113], for every compact LOTS $X$ the set $F := C_+(X,[0,1])$ of all order-preserving continuous maps on $X$ separates the points.

Definition 4.4. Let $\alpha \in \mu$ be an entourage. We say that a finite chain $A := \{c_0, c_1, \ldots, c_n\}$ in $X$ is an $\alpha$-connected net if:
1. \[ s = c_0 \leq c_1 \leq \cdots \leq c_n = t; \]
2. \((x, y) \in \alpha\) for every \(x, y \in [c_i, c_{i+1}]\).

Notation: \(A \in \Gamma(\alpha)\).

Note that \((x, y) \in \alpha^2\) for every \(x \in [c_k, c_{k+1}]\) and \(y \in [c_{k+1}, c_{k+2}]\).

Lemma 4.5. Let \((X, \tau_\leq)\) be a compact LOTS with its unique compatible uniform structure \(\mu\). The following are equivalent:

1. \(X\) is connected;
2. for every \(\alpha \in \mu\) there exists an \(\alpha\)-connected net.

Proof. \((1) \Rightarrow (2)\)

In the setting of Definition 4.4 every finite chain which contains an \(\alpha\)-connected net is also an \(\alpha\)-connected net. It follows that it is enough to verify the definition for entourages from any given uniform subbase of \(\mu\). So, in our case, by Lemma 4.3 it is enough to check that there exists an \(\alpha\)-connected net for every \(\alpha = U_{f, \varepsilon}\). So we have to show that \(\Gamma(U_{f, \varepsilon})\) is nonempty for every \(f \in C_\ast(X, [0, 1])\) and every \(\varepsilon > 0\).

Since \(X\) is connected and compact the continuous image \(f(X) \subseteq [0, 1]\) is a closed subinterval, say \(f(X) = [u, v]\).

Fix \(n \in \mathbb{N}\) large enough such that \(\frac{v-u}{n} \leq \varepsilon\). For every natural \(i\) with \(0 < i < n\) choose \(c_i \in X\) with \(f(c_i) = \frac{(i-1)u + iv}{n}\) and \(c_0 = s, c_n = t\). Then \(A = \{c_0, c_1, \ldots, c_n\} \in \Gamma(U_{f, \varepsilon})\) for every \(U_{f, \varepsilon}\). Indeed, since \(f\) is order-preserving, for every \(x, y \in X\) with \(x, y \in [c_i, c_{i+1}]\) we have
\[
\frac{v-u}{n} \leq \varepsilon,
\]
so \(|f(x) - f(y)| \leq \frac{v-u}{n} \leq \varepsilon\), and therefore \((x, y) \in \alpha = U_{f, \varepsilon}\).

\((2) \Rightarrow (1)\)

Let \(X = X_1 \cup X_2\) be a partition of \(X\) into two nonempty clopen subsets. There exists \(\alpha \in \mu\) such that \((x, y) \notin \alpha\) for every \(x \in X_1, y \in X_2\). Then \(\Gamma(\alpha)\) is empty.

\[\square\]

Theorem 4.6. Let \((X, \tau_\leq)\) be a compact connected LOTS that satisfies the following condition:

\((B)\) for every pair of elements \(a < b\) in \(X\) the group \(H_+[a, b]\) is nonabelian.

Then the group \(H_+(X)\) is \(a\)-minimal (that is, \(X\) is \(a\)M-\(a\)compact).

Proof. We modify the arguments of [22] and use Lemma 4.5.

For every interval \((a, b) \subseteq X\) (with \(a < b\)) the group \(H_+[a, b]\) is nonabelian. Taking into account Remark 2.4 choose \(p, q \in H_+(X)\) such that \(pq \neq qp\) and \(p(x) = q(x) = x\) for every \(x \notin (a, b)\).
Define
\[
T(a, b) := \{ g \in H_+(X) : g p g^{-1} \text{ does not commute with } q \}. \tag{4.1}
\]

**Claim 1.** \(T(a, b)\) is an open neighborhood of \(e\) in every Hausdorff group topology on \(H_+(X)\).

Clearly, \(e \in T(a, b)\). For every Hausdorff group topology \(\nu\) on \(H_+(X)\) the set \(T(a, b)\) is the inverse image of \(\nu = \{ e \}\) under the continuous map
\[
(H_+(X), \nu) \to (H_+(X), \nu), \quad g \mapsto (gp g^{-1}) q (gp g^{-1})^{-1} q^{-1}.
\]

**Claim 2.** For every \(g \in T(a, b)\) there exists \(x \in (a, b)\) such that \(g(x) \in (a, b)\).

Assuming the contrary, there exists \(g \in T(a, b)\) such that \(g(a, b) \cap (a, b) = \emptyset\).
Equivalently, \((a, b) \cap g^{-1}(a, b) = \emptyset\). Hence, \(g^{-1}(x) \notin (a, b)\) for every \(x \in (a, b)\).
By the choice of \(p\) we have \(p g^{-1}(x) = g^{-1}(x)\) and so \(g p g^{-1}(x) = x\) for every \(x \in (a, b)\).
On the other hand, \(g(x) = x\) for every \(x \in X \setminus (a, b)\) (by the choice of \(q\)).
It follows that \(g p g^{-1}\) and \(q\) commute, which contradicts the definition of \(T(a, b)\) in \(4.1\).

Let \(\tau\) be the collection of all finite intersections of \(T(a, b)\)'s. Then \(\tau \subseteq \nu\) for every Hausdorff group topology \(\nu\) on \(H_+(X)\).

**Claim 3.** Every open neighbourhood \(U\) of \(e\) in \(H_+(X)\), with the usual compact-open topology, contains an element \(T\) from \(\tau\).

We verify Claim 3. Let \(\mu\) be the unique compatible uniformity on \(X\). A basic neighbourhood of \(e\) has the form:
\[
O_\varepsilon := \{ g \in H_+(X) : (g(x), x) \in \varepsilon \quad \forall \ x \in X \},
\]
where \(\varepsilon \in \mu\). Choose a symmetric entourage \(\varepsilon_1 \in \mu\) such that \(\varepsilon_1^2 \subseteq \varepsilon\). For \(\varepsilon_1\) by Lemma \[4.3\] choose an \(\varepsilon_1\)-connected net \(\{c_0, c_1, \ldots, c_n\}\) of \(X\).
By condition (B) and Equation \[4.1\] we have the corresponding \(T(c_i, c_{i+1}) \subseteq H_+(X)\).
Define
\[
T := \bigcap_{i=0}^{n-1} T(c_i, c_{i+1}).
\]
Since \(\tau \subseteq \nu\) and \(T \in \tau\) now it is enough to show:

**Claim 4.** \(T \subseteq O_\varepsilon\).

Assuming the contrary let \(h \in T\) but \(h \notin O_\varepsilon\). Then there exists \(x \in X\) such that \((h(x), x) \notin \varepsilon\). Pick minimal \(i\) such that \(x \in [c_i, c_{i+1}]\). Then by a remark after Definition \[4.3\] we have \((x, y) \in \varepsilon_i^2 \subseteq \varepsilon\) for every \(y \in [c_{i-1}, c_{i+2}]\). Hence,
\[
h(x) \in X \setminus [c_{i-1}, c_{i+1}] = [c_0, c_{i-1}] \cup (c_{i+2}, c_n]. \tag{4.2}
\]
From Claim 3 choose \(x_0\) such that
\[
x_0, h(x_0) \in (c_i, c_{i+1}). \tag{4.3}
\]
It is easy to see that $h$ is not order-preserving. Indeed, we have the following two cases (of course, $x \neq x_0$ by equations 4.2 and 4.3):

1. $x_0 < x$. Since $h \in H_+(X)$ we have $h(x_0) < h(x)$. It follows from 4.2 that $h(x) \in (c_{i+2}, c_n]$. In particular, $i < n - 2$. From Claim 3, there exists $x_1$ such that $x_1, h(x_1) \in (c_{i+1}, c_{i+2})$. Thus, $h(x_1) < h(x)$, while $x < x_1$.

2. $x_0 > x$. Since $h \in H_+(X)$ we have $h(x) < h(x_0)$. It follows from 4.2 that $h(x) \in [c_0, c_{i-1})$. In particular, $i > 2$. From Claim 3, there exists $x_1$ such that $x_1, h(x_1) \in (c_{i-1}, c_i)$. Thus, $h(x) < h(x_1)$, while $x_1 < x$.

Note that condition (B) of Theorem 4.6 implies condition (A) of Theorem 3.4. On the other hand the conclusion of 4.6 is stronger.

Remark 4.7. One may show that not only $H_+(X)$ but also $H(X)$ in Theorem 4.6 is $a$-minimal. Indeed, observe that any autohomeomorphism $f \in H(X)$ is either order-preserving or order-reversing. So, it is easy to modify the proof of Theorem 4.6.

5. Examples

5.1. Not every compact LOTS is $M_+^+$-compact

The following example shows that $H_+(X)$ is not necessarily minimal.

Example 5.1. Denote by $Z^*$ the two-point compactification of $Z$. One can easily verify that $H_+(Z^*)$ is a discrete copy of $Z$ and thus not minimal. That is, the compact LOTS $Z^*$ is not $M_+^+$-compact.

5.2. The Ordinal Space

This example shows that the conditions of Theorems 3.4 and 4.6 are not necessary. For every ordinal number $\kappa$ the space $[0, \kappa]$ is a compact LOTS. This space is scattered and hence not connected for every $\kappa > 0$. Nonetheless, one can show that $H_+[0, \kappa]$ is trivial (hence $a$-minimal). We start by noting that $[0, \kappa]$ is certainly a well-ordered set.

Lemma 5.2. [7, Corollary 4.1.9] If two well-ordered sets $A$ and $B$ are order-isomorphic, then the isomorphism is unique.

It follows from Lemma 5.2 that the identity is the only order-preserving automorphism of a well-ordered set.

Corollary 5.3. For every well-ordered compact LOTS $X$ (e.g., for the ordinal space $X = [0, \kappa]$) we have $H_+(X) = \{e\}$ (thus $X$ is $aM_+^+$-compact).
5.3. The Unit Interval

\[ [0, 1] \] clearly satisfies the conditions of Theorem 4.6. Thus the group \( H_{+}[0, 1] \) is \( a \)-minimal.

5.4. The Ordered Square

Let \( I = [0, 1] \) and define the lexicographic order on \( I \times I \). Then \( I^2 = (I \times I, \tau) \), the unit square with the order topology, is a compact and not metrizable space. We show that it satisfies the conditions of Theorem 4.6. It is clearly connected. As to the second condition, let \( U = ((a_1, b_1), (a_2, b_2)) \in I^2 \) be an open interval. If \( a_1 = a_2 \) then \( U \) is homeomorphic to \( (0, 1) \subseteq \mathbb{R} \). Otherwise, if \( a_1 < a_2 \), \( U \) contains an interval homeomorphic to \( (0, 1) \subseteq \mathbb{R} \) (for example \( ((a_2, 0), (a_2, b_2)) \)). Thus it is sufficient to work with intervals of the form \( U = ((a, b_1), (a, b_2)) \). The interval \( U \) is homeomorphic and, in fact, order isomorphic to \( (0, 1) \subseteq \mathbb{R} \). Since the interval \( [0, 1] \) satisfies the second condition of Theorem 4.6 so does \( I^2 \). So, \( H_{+}(I^2) \) is \( a \)-minimal and \( I^2 \) is an \( aM \)-compact space. Moreover, by Remark 4.7 \( H(I^2) \) is \( a \)-minimal and \( I^2 \) is an \( aM \)-compact space.

5.5. The Extended Long Line

Let \( L = [0, \omega_1) \times [0, 1) \) where \( \omega_1 \) is the least uncountable ordinal. Considering \( L \) with the lexicographic order, the set \( L \) with the topology induced by this order is called the long line. Let \( L^* = L \cup \{ \omega_1 \} \) and extend the ordering on \( L \) to \( L^* \) by letting \( a < \omega_1 \) for all \( a \in L \). The space \( L^* \) with the order topology is a compact space called the extended long line. In fact, \( L^* \) is the one point compactification of \( L \).

Several properties of this space can be found in [26], [32] and [37]. The extended long line satisfies the conditions of Theorem 4.6. Indeed, it is well known that \( L^* \) is a compact connected LOTS. Also, \( L \) (the long line) is locally homeomorphic (by an order-preserving homeomorphism) to the interval \( (0, 1) \). In case the interval in question is of the form \( [a, \omega_1] \), we can verify condition (B) for a subinterval \( (a, b) \) of \( [a, \omega_1] \). So, \( H_{+}(L^*) \) is \( a \)-minimal and \( L^* \) is an \( aM \)-compact space. Moreover, by Remark 4.7 \( H(L^*) \) is \( a \)-minimal and \( L^* \) is an \( aM \)-compact space.

5.6. The case of \( S^1 \)

Denote by \( H_{+}(S^1) \) the Polish group of all orientation preserving homeomorphisms of the circle \( S^1 \). This group, as well as the groups \( H_{+}(X) \), play a major role in many important research lines. See, for example, [24, 34, 25].

The arguments of Theorem 4.6 (or, of [22, Theorem 1]) can be easily modified for the circle \( S^1 \).

**Theorem 5.4.** The group \( H_{+}(S^1) \) is \( a \)-minimal.
Very likely, the same remains true also for any connected cyclically ordered compact space.

A Hausdorff topological group is totally minimal if every Hausdorff quotient is minimal \[12\]. Every minimal algebraically (or, at least, topologically) simple minimal group is totally minimal. \(H_+ (\mathbb{S}^1)\) is algebraically simple as can be seen (for example) in \[38\] \[24\]. Although the group \(H_+ [0, 1]\) is not algebraically simple, it is topologically simple. Indeed, by \[19\] Theorem 14, \(H_+ [0, 1]\) has exactly five normal subgroups: \(\{e\}, H_+ [0, 1], Q, Q_0, Q := Q_0 \cap Q_1\). It is easy to see that \(Q\) is dense in \(H_+ [0, 1]\). This yields that \(H_+ [0, 1]\) is topologically simple.

**Corollary 5.5.** \(H_+ (\mathbb{S}^1)\) and \(H_+ [0, 1]\) are totally minimal groups.

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