Almost Kenmotsu metric as Ricci-Yamabe soliton

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Abstract

The object of the present paper is to characterize two classes of almost Kenmotsu manifolds admitting Ricci-Yamabe soliton. It is shown that a \((k, \mu)\)'-almost Kenmotsu manifold admitting a Ricci-Yamabe soliton or gradient Ricci-Yamabe soliton is locally isometric to the Riemannian product \(\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n\). For the later case, the potential vector field is pointwise collinear with the Reeb vector field. Also, a \((k, \mu)\)-almost Kenmotsu manifold admitting certain Ricci-Yamabe soliton with the curvature property \(Q \cdot P = 0\) is locally isometric to the hyperbolic space \(\mathbb{H}^{2n+1}(-1)\) and the non-existence of the curvature property \(Q \cdot R = 0\) is proved.

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1. Introduction

In 1982, the concept of Ricci flow was introduced by Hamilton [11]. The Ricci flow is an evolution equation for metrics on a Riemannian manifold \((M^n, g)\) given by

\[
\frac{\partial g}{\partial t} = -2S,
\]

where \(g\) is the Riemannian metric and \(S\) denotes the \((0, 2)\)-symmetric Ricci tensor. A self similar solution of the Ricci flow is called Ricci soliton [12] and is defined as

\[
\mathcal{L}_V g + 2S = 2\lambda g
\]

where \(\mathcal{L}_V\) denotes the Lie derivative along the vector field \(V\) and \(\lambda\) is a constant.

The notion of Yamabe flow was proposed by Hamilton [13] in 1989, which is defined on a Riemannian manifold \((M^n, g)\) as

\[
\frac{\partial g}{\partial t} = -rg,
\]
where \( r \) is the scalar curvature of the manifold. In dimension 2, we know that 
\[ S = \frac{r}{2} g \] 
and hence, the Ricci flow and Yamabe flow are equivalent. For \( n > 2 \), 
these two notions are not same as the Yamabe flow preserves the conformal 
class of metrics but the Ricci flow does not in general. A Yamabe soliton on a 
Riemannian manifold \((M^n, g)\) is a triplet \((g, V, \lambda)\) satisfying

\[
\frac{1}{2} \mathcal{L}_V g = (\lambda - r)g,
\]

where \( \lambda \) is a constant. The Ricci soliton and Yamabe soliton are said to be 
shrinking, steady or expanding according as \( \lambda \) is positive, zero or negative re-
spectively.

In 2019, Güler and Crasmareanu [10] introduced a scalar combination of the 
Ricci flow and the Yamabe flow under the name of Ricci-Yamabe map. Let 
\((M^n, g)\) be a Riemannian manifold, \(T^2_2(M)\) be the linear space of its \((0,2)-
symmetric tensor fields and \( \text{Riem}(M) \subset T^2_2(M)\) be the infinite space of its 
Riemannian metrics. In [10], the authors gives the following two definitions.

**Definition 1.1.** ([10]) A Riemannian flow on \( M \) is a smooth map:

\[
g : I \subseteq \mathbb{R} \rightarrow \text{Riem}(M),
\]

where \( I \) is a given open interval.

**Definition 1.2.** ([10]) The map \( \text{RY}^{(\alpha, \beta, g)} : I \rightarrow T^2_2(M) \) given by:

\[
\text{RY}^{(\alpha, \beta, g)} = \frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t) g(t)
\]
is called the \((\alpha, \beta)\)-Ricci-Yamabe map of the Riemannian flow \((M, g)\). If

\[
\text{RY}^{(\alpha, \beta, g)} \equiv 0,
\]

then \( g(\cdot) \) will be called an \((\alpha, \beta)\)-Ricci-Yamabe flow.

The \((\alpha, \beta)\)-Ricci-Yamabe map can be a Riemannian or semi-Riemannian or 
singular Riemannian flow due to the signs of \( \alpha \) and \( \beta \). The freedom of choosing 
the sign of \( \alpha \) and \( \beta \) is very useful in geometry and relativity. Recently, a bi-
metric approach of the spacetime geometry appears in [1] and [3]. The notion 
of \((\alpha, \beta)\)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton from the Ricci-
Yamabe flow can be defined as follows:

**Definition 1.3.** A Riemannian manifold \((M^n, g), n > 2\) is said to admit a 
Ricci-Yamabe soliton (in short, \( \text{RYS} \)) \((g, V, \lambda, \alpha, \beta)\) if

\[
\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r)g,
\] (1.1)
Ricci-Yamabe soliton

where \( \lambda, \alpha, \beta \in \mathbb{R} \). If \( V \) is gradient of some smooth function \( f \) on \( M \), then the above notion is called gradient Ricci-Yamabe soliton (in short, GRYS) and then \((1.1)\) reduces to

\[
\nabla^2 f + \alpha S = (\lambda - \frac{1}{2} \beta r)g,
\]

(1.2)

where \( \nabla^2 f \) is the Hessian of \( f \).

The RYS (or GRYS) is said to be expanding, steady or shrinking according as \( \lambda < 0, \lambda = 0 \) or \( \lambda > 0 \) respectively. A RYS (or GRYS) is called an almost RYS (or GRYS) if \( \alpha, \beta \) and \( \lambda \) are smooth functions on \( M \). The above notion generalizes a large class of soliton like equations. A RYS (or GRYS) is said to be a

- Ricci soliton (or gradient Ricci soliton) (see [12]) if \( \alpha = 1, \beta = 0 \).
- Yamabe soliton (or gradient Yamabe soliton) ([13] if \( \alpha = 0, \beta = 1 \).
- Einstein soliton (or gradient Einstein soliton) ([5]) if \( \alpha = 1, \beta = -1 \).
- \( \rho \)-Einstein soliton (or gradient \( \rho \)-Einstein soliton) ([6]) if \( \alpha = 1, \beta = -2\rho \).

We say that the RYS or GRYS is proper if \( \alpha \neq 0, 1 \).

In 2016, Wang [16] proved that a \((k,\mu)'\)-almost Kenmotsu manifold admitting gradient Ricci soliton is locally isometric to \( \mathbb{H}^{n+1}(-4) \times \mathbb{R}^n \). In [15], the authors proved the same result for a gradient \( \rho \)-Einstein soliton. Thus a natural question is

**Question 1.4.** Does the above result is true for a \((2n+1)\)-dimensional \((k,\mu)'\)-almost Kenmotsu manifold admitting RYS or GRYS?

We will answer this question affirmatively. Moreover, we have studied this notion on \((k,\mu)\)-almost Kenmotsu manifolds with some curvature properties. The paper is organized as follows: In section 2, we give some preliminaries on almost Kenmotsu manifolds. Section 3 deals with \((k,\mu)\)-almost Kenmotsu manifolds admitting RYS and GRYS. Section 4 contains some curvature properties of \((k,\mu)\)-almost Kenmotsu manifolds admitting proper RYS.

2. Preliminaries

A \((2n+1)\)-dimensional almost contact metric manifold \( M \) is a smooth manifold together with a structure \((\phi, \xi, \eta, g)\) satisfying

\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0
\]

(2.1)
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \] (2.2)

for any vector fields \( X, Y \) on \( M \), where \( \phi \) is a \((1, 1)\)-tensor field, \( \xi \) is a unit vector field, \( \eta \) is a 1-form defined by \( \eta(X) = g(X, \xi) \) and \( g \) is the Riemannian metric. Using (2.2), we can easily see that \( \phi \) is skew symmetric, that is,

\[ g(\phi X, Y) = -g(X, \phi Y). \] (2.3)

The fundamental 2-form \( \Phi \) on an almost contact metric manifold is defined by \( \Phi(X, Y) = g(X, \phi Y) \) for any vector fields \( X, Y \) on \( M \). An almost contact metric manifold such that \( \eta \) is closed and \( d\Phi = 2\eta \wedge \Phi \) is called almost Kenmotsu manifold (see [9], [14]). Let us denote the distribution orthogonal to \( \xi \) by \( \mathcal{D} \) and defined by \( \mathcal{D} = \text{Ker}(\eta) = \text{Im}(\phi) \). In an almost Kenmotsu manifold, since \( \eta \) is closed, \( \mathcal{D} \) is an integrable distribution.

Let \( M \) be a \((2n + 1)\)-dimensional almost Kenmotsu manifold. We denote by \( h = \frac{1}{2} \mathcal{L}_\xi \phi \) and \( l = R(\cdot, \xi)\xi \) on \( M \). The tensor fields \( l \) and \( h \) are symmetric operators and satisfy the following relations [14]:

\[ h\xi = 0, \ l\xi = 0, \tr(h) = 0, \tr(h\phi) = 0, \ h\phi + \phi h = 0, \] (4.4)

\[ \nabla_X \xi = X - \eta(X)\xi - \phi hX \Rightarrow \nabla_X \xi = 0, \] (5.5)

\[ \phi l\phi - l = 2(h^2 - \phi^2), \] (6.6)

\[ R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \] (7.7)

for any vector fields \( X, Y \) on \( M \). The \((1, 1)\)-type symmetric tensor field \( h' = h\circ \phi \) is anti-commuting with \( \phi \) and \( h'\xi = 0 \). Also it is clear that ([9], [17])

\[ h = 0 \Leftrightarrow h' = 0, \ h'^2 = (k + 1)\phi^2 \Leftrightarrow h^2 = (k + 1)\phi^2. \] (8.8)

The notion of \((k, \mu)\)-nullity distribution on a contact metric manifold \( M \) was introduced by Blair et. al. [2], which is defined for any \( p \in M \) and \( k, \mu \in \mathbb{R} \) as follows:

\[ N_p(k, \mu) = \{ Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \]

\[ + \mu[g(Y, Z)hX - g(X, Z)hY] \} \] (9.9)

for any vector fields \( X, Y \in T_p(M) \), where \( T_p(M) \) denotes the tangent space of \( M \) at any point \( p \in M \) and \( R \) is the Riemann curvature tensor. In [9], Dileo and Pastore introduced the notion of \((k, \mu)'\)-nullity distribution, on an almost Kenmotsu manifold \((M, \phi, \xi, \eta, g)\), which is defined for any \( p \in M \) and \( k, \mu \in \mathbb{R} \) as follows:

\[ N_p(k, \mu)' = \{ Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \]

\[ + \mu[g(Y, Z)h'X - g(X, Z)h'Y] \}. \] (10.10)

for any vector fields \( X, Y \in T_p(M) \). For further details on almost Kenmotsu manifolds, we refer the reader to go through the references ([7]-[9], [16]-[18]).
3. \((k, \mu)^{\prime}\)-almost Kenmotsu manifolds

Let \(X \in \mathcal{D}\) be the eigen vector of \(h^\prime\) corresponding to the eigen value \(\delta\). Then from (2.8), it is clear that \(\delta^2 = -(k + 1)\), a constant. Therefore \(k \leq -1\) and \(\delta = \pm \sqrt{-k - 1}\). We denote by \([\delta]^\prime\) and \([-\delta]^\prime\) the corresponding eigen spaces related to the non-zero eigen value \(\delta\) and \(-\delta\) of \(h^\prime\), respectively. In [9], it is proved that in a \((2n+1)\)-dimensional \((k, \mu)^{\prime}\)-almost Kenmotsu manifold \(M\) with \(h^\prime \neq 0, k < -1, \mu = -2\) and \(\text{Spec}(h^\prime) = \{0, \delta, -\delta\}\), with \(0\) as simple eigen value and \(\delta = \sqrt{-k - 1}\). In [18], Wang and Liu proved that for a \((2n+1)\)-dimensional \((k, \mu)^{\prime}\)-almost Kenmotsu manifold \(M\) with \(h^\prime \neq 0\), the Ricci operator \(Q\) of \(M\) is given by

\[
QX = -2nX + 2n(k + 1)\eta(X)\xi - 2nh^\prime X. \tag{3.1}
\]

Moreover, the scalar curvature of \(M\) is \(r = 2n(k - 2n)\). From (2.10), we have

\[
R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] - 2[\eta(Y)h^\prime X - \eta(X)h^\prime Y], \tag{3.2}
\]

where \(k, \mu \in \mathbb{R}\). Also from (3.2), we get

\[
R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] - 2[g(h^\prime X,Y)\xi - \eta(Y)h^\prime X]. \tag{3.3}
\]

Again from (3.1), we have

\[
S(Y, \xi) = g(QY, \xi) = 2nk\eta(Y). \tag{3.4}
\]

Using (2.5), we obtain

\[
(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h^\prime X,Y). \tag{3.5}
\]

3.1 Ricci-Yamabe soliton

We now consider the notion of RYS in the framework \((k, \mu)^{\prime}\)-almost Kenmotsu manifolds. To prove our first theorem regarding RYS, we need the following lemmas:

**Lemma 3.1.** ([8]) In a \((k, \mu)^{\prime}\)-almost Kenmotsu manifold \(M^{2n+1}\) with \(h^\prime \neq 0\), the following relation holds

\[
(\nabla_ZS)(X,Y) - (\nabla_XS)(Y,Z) - (\nabla_YS)(X,Z)
= -4n(k + 2)g(h^\prime X,Y)\eta(Z).
\]

**Lemma 3.2.** ([8]) In a \((k, \mu)^{\prime}\)-almost Kenmotsu manifold \(M^{2n+1}\), \((\mathcal{L}_X h^\prime)Y = 0\) for any \(X, Y \in [\delta]^\prime\) or \(X, Y \in [-\delta]^\prime\), where \(\text{Spec}(h^\prime) = \{0, \delta, -\delta\}\).
Theorem 3.3. If \((g, V, \lambda, \alpha, \beta)\) be a RYS on a \((2n + 1)\)-dimensional 
\((k, \mu)'\)-almost Kenmotsu manifold \(M\), then the manifold \(M\) is locally isometric to 
\(\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n\), provided \(2\lambda - \beta r \neq 4nk\alpha\).

Proof. From (1.1) we have

\[
(L_V g)(X, Y) + 2\alpha S(X, Y) = [2\lambda - \beta r]g(X, Y). \tag{3.6}
\]

Differentiating the foregoing equation covariantly along any vector field \(Z\), we obtain

\[
(\nabla_Z L_V g)(X, Y) = -2\alpha(\nabla_Z S)(X, Y). \tag{3.7}
\]

Due to Yano \[19\], we have the following commutation formula

\[
(L_V \nabla_X g - \nabla_X L_V g - [V, X]g)(Y, Z) = -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y). \tag{3.8}
\]

Since \(\nabla g = 0\), then the above relation becomes

\[
(\nabla_X L_V g)(Y, Z) = g((L_V \nabla)(X, Y), Z) + g((L_V \nabla)(X, Z), Y). \tag{3.9}
\]

Since \(L_V \nabla\) is symmetric, then it follows from (3.8) that

\[
g((L_V \nabla)(X, Y), Z) = \frac{1}{2}(\nabla_X L_V g)(Y, Z) + \frac{1}{2}(\nabla_Y L_V g)(X, Z) - \frac{1}{2}(\nabla_Z L_V g)(X, Y). \tag{3.10}
\]

Using (3.7) in (3.9), we have

\[
g((L_V \nabla)(X, Y), Z) = \alpha[(\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)]. \tag{3.11}
\]

Using lemma 3.1 in (3.10) yields

\[
g((L_V \nabla)(X, Y), Z) = -4n\alpha(k + 2)g(h'X, Y)\eta(Z),
\]

which implies

\[
(L_V \nabla)(X, Y) = -4n\alpha(k + 2)g(h'X, Y)\xi. \tag{3.12}
\]

Putting \(Y = \xi\) in (3.11), we get \((L_V \nabla)(X, \xi) = 0\) and this implies \(\nabla_Y (L_V \nabla)(X, \xi) = 0\). Therefore,

\[
(\nabla_Y L_V \nabla)(X, \xi) + (L_V \nabla)(\nabla_Y X, \xi) + (L_V \nabla)(X, \nabla_Y \xi) = 0. \tag{3.13}
\]

Using \((L_V \nabla)(X, \xi) = 0\), (3.11) and (2.5) in (3.12), we obtain

\[
(\nabla_Y L_V \nabla)(X, \xi) = 4n\alpha(k + 2)[g(h'X, Y) + g(h^2X, Y)]\xi.
\]
Now, it is well known that (see \cite{19})
\[(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z),\]
We now use (3.13) in the foregoing equation to obtain
\[
(\mathcal{L}_V R)(X, \xi)\xi = (\nabla_X \mathcal{L}_V \nabla)(\xi, \xi) - (\nabla_\xi \mathcal{L}_V \nabla)(X, \xi) = 0. \tag{3.14}
\]
Now, setting \(Y = \xi\) in (3.6) and using (3.4) we have
\[
(\mathcal{L}_V g)(X, \xi) = [2\lambda - \beta r - 4nk\alpha]\eta(X), \tag{3.15}
\]
which implies
\[
(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = [2\lambda - \beta r - 4nk\alpha]\eta(X). \tag{3.16}
\]
Putting \(X = \xi\) in (3.16), we can easily obtain
\[
\eta(\mathcal{L}_V \xi) = -\frac{1}{2}[2\lambda - \beta r - 4nk\alpha]. \tag{3.17}
\]
From (3.2), we can write
\[
R(X, \xi)\xi = k(X - \eta(X)\xi) - 2h'X. \tag{3.18}
\]
Now, using (3.16)-(3.18) and (3.2)-(3.3), we obtain
\[
(\mathcal{L}_V R)(X, \xi)\xi = \mathcal{L}_V R(X, \xi)\xi - R(\mathcal{L}_V X, \xi)\xi - R(X, \mathcal{L}_V \xi)\xi - R(X, \xi)\mathcal{L}_V \xi
= k[2\lambda - \beta r - 4nk\alpha](X - \eta(X)\xi) - 2(\mathcal{L}_V h')X
- 2[2\lambda - \beta r - 4nk\alpha]h'X - 2\eta(X)h'(\mathcal{L}_V \xi)
- 2g(h'X, \mathcal{L}_V \xi). \tag{3.19}
\]
Equating (3.14) and (3.19) and then taking inner product with \(Y\), we get
\[
k[2\lambda - \beta r - 4nk\alpha](g(X, Y) - \eta(X)\eta(Y))
- 2g((\mathcal{L}_V h')X, Y) - 2[2\lambda - \beta r - 4nk\alpha]g(h'X, Y)
- 2\eta(X)g(h'(\mathcal{L}_V \xi), Y) - 2g(h'X, \mathcal{L}_V \xi)\eta(Y) = 0. \tag{3.20}
\]
Replacing \(X\) by \(\phi X\) in the foregoing equation, we get
\[
k[2\lambda - \beta r - 4nk\alpha]g(\phi X, Y) - 2g((\mathcal{L}_V h')\phi X, Y)
- 2[2\lambda - \beta r - 4nk\alpha]g(h'\phi X, Y) = 0. \tag{3.21}
\]
Let \(X \in [-\delta]'\) and \(V \in [\delta]'\), then \(\phi X \in [\delta]'\). Then from (3.21), we have
\[
(k - 2\delta)[2\lambda - \beta r - 4nk\alpha]g(\phi X, Y) - 2g((\mathcal{L}_V h')\phi X, Y) = 0. \tag{3.22}
\]
Since, $V$, $\phi X \in [\delta]'$, using lemma 3.2, we have $(\mathcal{L}_V h')\phi X = 0$. Therefore, equation (3.22) reduces to

$$(k - 2\delta)[2\lambda - \beta r - 4nk\alpha]g(\phi X, Y) = 0,$$

which implies $k = 2\delta$, since by hypothesis $2\lambda - \beta r \neq 4nk\alpha$.

Now $k = 2\delta$ and $\delta^2 = -(k + 1)$ together implies $\delta = -1$ and hence $k = -2$. Then from Proposition 4.2 of [9], we have $\alpha = -1$ and hence, $k = -2$. Then from proposition 4.2 of [9], we have $R(X_\delta, Y_\delta)Z_\delta = 0$ and

$$R(X_\delta, Y_\delta)Z_\delta = -4\{g(Y_\delta, Z_\delta)X_\delta - g(X_\delta, Z_\delta)Y_\delta\},$$

for any $X_\delta$, $Y_\delta$, $Z_\delta \in [\delta]'$ and $X_\delta$, $Y_\delta$, $Z_\delta \in [-\delta]'$. Also noticing $\mu = -2$ it follows from proposition 4.3 of [9] that $K(X, \xi) = -4$ for any $X \in [-\delta]'$ and $K(X, \xi) = 0$ for any $X \in [\delta]'$. Again from proposition 4.3 of [9] we see that $K(X, Y) = -4$ for any $X, Y \in [-\delta]'$ and $K(X, Y) = 0$ for any $X, Y \in [\delta]'$. As shown in [9] that the distribution $[\xi] \oplus [\delta]'$ is integrable with totally geodesic leaves and the distribution $[-\delta]'$ is integrable with totally umbilical leaves by $H = -(1 - \delta)\xi$, where $H$ is the mean curvature tensor field for the leaves of $[-\delta]'$ immersed in $M$. Here $\delta = -1$, then the two orthogonal distributions $[\xi] \oplus [\delta]'$ and $[-\delta]'$ are both integrable with totally geodesic leaves immersed in $M$. Then we can say that $M$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

**Remark 3.4.** If $2\lambda - \beta r - 4nk\alpha = 0$ and since $r = 2n(k - 2n)$, then $\lambda = n\beta(k - 2n) + 2nk\alpha$. Since $k < -1$, if $\alpha, \beta > 0$, then $\lambda < 0$ and hence, the RYS is expanding. If $\alpha, \beta < 0$, then $\lambda > 0$ and the RYS is shrinking. If $n\beta(k - 2n) + 2nk\alpha = -2nk\alpha$, then $\lambda = 0$ and the RYS is steady.

### 3.2 Gradient Ricci-Yamabe soliton

We now consider the notion of GRYs in the framework of $(k, \mu)'$-almost Kenmotsu manifolds and extend the preceding theorem 3.3 by considering $V$ as a gradient vector field. In this regard, the following theorem is proved.

**Theorem 3.5.** If $(g, V, \lambda, \alpha, \beta)$ be a GRYs on a $(2n+1)$-dimensional $(k, \mu)'$-almost Kenmotsu manifold $M$, then the manifold $M$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ or $V$ is pointwise collinear with the characteristic vector field $\xi$.

**Proof.** Let $V$ be the gradient of a non-zero smooth function $f : M \to \mathbb{R}$, that is, $V = Df$, where $D$ is the gradient operator. Then from (1.2), we can write

$$\nabla_X Df = (\lambda - \frac{1}{2} \beta r)X - \alpha Q X.$$  \hspace{1cm} (3.23)
Differentiating this covariantly along any vector field \( Y \), we obtain

\[
\nabla_Y \nabla_X Df = (\lambda - \frac{1}{2} \beta r) \nabla_Y X - \alpha \nabla_Y QX. \tag{3.24}
\]

Interchanging \( X \) and \( Y \) in (3.24) yields

\[
\nabla_X \nabla_Y Df = (\lambda - \frac{1}{2} \beta r) \nabla_X Y - \alpha \nabla_X QY. \tag{3.25}
\]

From (3.23), we get

\[
\nabla_{[X,Y]} Df = (\lambda - \frac{1}{2} \beta r)(\nabla_X Y - \nabla_Y X) - \alpha Q(\nabla_X Y - \nabla_Y X). \tag{3.26}
\]

It is well known that

\[
R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df
\]
Substituting (3.24)-(3.26) in the foregoing equation yields

\[
R(X,Y)Df = \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y]. \tag{3.27}
\]

With the help of (3.1), we obtain

\[
(\nabla_Y Q)X = \nabla_Y QX - Q(\nabla_Y X)
\]
\[
= 2n(k + 1)(g(X,Y) - \eta(X)\eta(Y)) + g(h'X,Y))\xi
+2n(k + 1)\eta(X)(Y - \eta(Y)\xi - \phi hY) + 2n g(h'Y + h'^2 Y, X)\xi
+2n\eta(X)(h'Y + h'^2 Y).
\]

Interchanging \( X \) and \( Y \) in the preceding equation we will obtain \( (\nabla_X Q)Y \).
Now, substituting \( (\nabla_Y Q)X \) and \( (\nabla_X Q)Y \) in (3.27) and using (2.8), we obtain

\[
R(X,Y)Df = 2n\alpha(k + 2)[\eta(X)h'Y - \eta(Y)h'X]. \tag{3.28}
\]

Putting \( X = \xi \) in (3.28) and then taking inner product with \( X \) yields

\[
g(R(\xi,Y)Df,X) = 2n\alpha(k + 2)g(h'Y,X). \tag{3.29}
\]

Again, using (3.3), we have

\[
g(R(\xi,Y)Df,X) = -g(R(\xi,Y)X, Df)
= -kg(X,Y)(\xi f) + k\eta(X)(Y f)
+2g(h'X,Y)(\xi f) - 2\eta(X)((h'Y)f). \tag{3.30}
\]

Equating (3.29) and (3.30) and then antisymmetrizing, we get

\[
kg(X)(Y f) - k\eta(Y)(X f) - 2\eta(X)((h'Y)f) + 2\eta(Y)((h'X)f) = 0.
\]
Replacing $X$ by $\xi$ in the above equation, we have
\[ k(Yf) - k(\xi f)\eta(Y) - 2(h'Y)f = 0, \]
which implies
\[ k[Df - (\xi f)\xi] - 2h'(Df) = 0. \] (3.31)

Operating $h'$ on (3.31) and using (2.8), we obtain
\[ h'(Df) = -\frac{2(k + 1)}{k}[Df - (\xi f)\xi]. \] (3.32)

Substituting (3.32) in (3.31), we get
\[ (k + 2)^2[Df - (\xi f)\xi] = 0, \]
which implies either $k = -2$ or $Df = (\xi f)\xi$.
If $k = -2$, then by the same argument as earlier, the manifold $M$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.
If $V = Df = (\xi f)\xi$, then $V$ is pointwise collinear with $\xi$. This completes the proof. \(\square\)

To obtain some consequences of the above theorem, we need the following definition:

**Definition 3.6.** A vector field $V$ on an almost contact metric manifold $M$ is said to be an infinitesimal contact transformation if $\mathcal{L}_V \eta = \psi \eta$ for some smooth function $\psi$ on $M$. In particular, if $\psi = 0$, then $V$ is called a strict infinitesimal contact transformation.

**Corollary 3.7.** If $(g, V, \lambda, \alpha, \beta)$ be a GRYS on a $(2n+1)$-dimensional $(k, \mu)'$-almost Kenmotsu manifold $M$ with $k \neq -2$, then

1. The potential vector field $V$ is a constant multiple of $\xi$.
2. $V$ is a strict infinitesimal contact transformation.
3. $V$ leaves $h'$ invariant.

**Proof.** Since $k \neq -2$, then from theorem 3.5, we have $V = (\xi f)\xi = b\xi$, where $(\xi f) = b$ is some smooth function on $M$. Then using (2.5), we can easily obtain
\[ (\mathcal{L}_{b\xi}g)(X,Y) = (Xb)\eta(Y) + (Yb)\eta(X) + 2b[g(X,Y) - \eta(X)\eta(Y) - g(\phi hX,Y)]. \] (3.33)
Substituting (3.33) in (3.6), we obtain
\[
(Xb)\eta(Y) + (Yb)\eta(X) + 2b[g(X,Y) - \eta(X)\eta(Y) - g(\phi hX,Y)]
= (2\lambda - \beta r)g(X,Y) - 2\alpha S(X,Y).
\]
(3.34)

Putting \(X = Y = \xi\) in (3.34) and using (3.4), we obtain
\[
2(\xi b) = 2\lambda - \beta r - 4nk\alpha.
\]
(3.35)

Let \(\{e_i\}\) be an orthonormal basis of the tangent space at each point of \(M\). Substituting \(X = Y = e_i\) in (3.34) and then summing over \(i\), we obtain
\[
2(\xi b) = (2\lambda - \beta r)(2n + 1) - 2\alpha r - 4nb.
\]
(3.36)

Since \(\alpha, \beta, \lambda\) and \(r\) is constant, then equating (3.35) and (3.36), we can easily see that \(b\) is constant. This proves (1).

Since \(b\) is constant, then from (3.35), we have
\[
2\lambda - \beta r = 4nk\alpha.
\]
(3.37)

Again since \(b\) is constant and \(V = b\xi\), then \(\mathcal{L}_V\xi = 0\). Using (3.37) in (3.15), we get \((\mathcal{L}_V g)(X,\xi) = 0\), which implies \((\mathcal{L}_V \eta)X = 0\) for any vector field \(X\). This proves (2).

Now using \(\mathcal{L}_V \xi = 0\) and (3.37) in (3.20), we get \((\mathcal{L}_V h')(X = 0\) for any vector field \(X\), which means \(V\) leaves \(h'\) invariant. This proves (3).

Remark 3.8. It is known that for a smooth tensor field \(T\), \(\mathcal{L}_X T = 0\) if and only if \(\phi_t\) is a symmetric transformation for \(T\), where \(\{\phi_t : t \in \mathbb{R}\}\) is the 1-parameter group of diffeomorphisms corresponding to the vector field \(X\) on a manifold (see [4]). Since \(h'\) is a smooth tensor field of type \((1,1)\) on \(M\), then \(\mathcal{L}_V h' = 0\) if and only if \(\psi_t\) is a symmetric transformation for \(h'\), where \(\{\psi_t : t \in \mathbb{R}\}\) is the 1-parameter group of diffeomorphisms corresponding to the vector field \(V\).

4. \((k, \mu)\)-almost Kenmotsu manifolds

In this section, we study RYS on \((k, \mu)\)-almost Kenmotsu manifolds with some curvature properties. From (2.9), we have
\[
R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].
\]
(4.1)

In [9], Dileo and Pastore proved that for a \((k, \mu)\)-almost Kenmotsu manifold, \(k = -1\) and \(h = 0\). Hence, from (4.1), we get the followings:
\[
R(X,Y)\xi = \eta(X)Y - \eta(Y)X.
\]
(4.2)
$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X. \quad (4.3)$

$S(X, \xi) = -2n\eta(X) \quad \text{and} \quad Q\xi = -2n\xi. \quad (4.4)$

Also from (2.5), we get

$$\nabla_X\xi = X - \eta(X)\xi. \quad (4.5)$$

Again in [9], it is proved that in an almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution, the sectional curvature $K(X, \xi) = -1$. From this, we get $r = -2n(2n + 1)$.

**Lemma 4.1.** If $(g, \xi, \lambda, \alpha, \beta)$ be a proper RYS on a $(2n + 1)$-dimensional $(k, \mu)$-almost Kenmotsu manifold $M$, then the manifold $M$ is $\eta$-Einstein.

**Proof.** Considering $V = \xi$ in (1.1), then we get

$$(\mathcal{L}_\xi g)(X, Y) + 2\alpha S(X, Y) = (2\lambda - \beta r)g(X, Y). \quad (4.6)$$

Now,

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X).$$

Using (4.5) in the foregoing equation yields

$$(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)]. \quad (4.7)$$

Substituting (4.7) in (4.6) and using the properness of the RYS and $r = -2n(2n + 1)$, we obtain

$$S(X, Y) = \frac{1}{\alpha}[\lambda + n\beta(2n + 1) - 1]g(X, Y) + \frac{1}{\alpha}\eta(X)\eta(Y). \quad (4.8)$$

This proves that the manifold is $\eta$-Einstein. \qed

Now from (4.4), we have

$$S(\xi, \xi) = -2n.$$

Again from (4.8), we get

$$S(\xi, \xi) = \frac{1}{\alpha}[\lambda + n\beta(2n + 1) - 1] + \frac{1}{\alpha}.$$

Equating these two values of $S(\xi, \xi)$, we obtain

$$\lambda + n\beta(2n + 1) = -2n\alpha. \quad (4.9)$$

Using (4.9) in (4.8), we have

$$S(X, Y) = -\frac{1}{\alpha}(2n\alpha + 1)g(X, Y) + \frac{1}{\alpha}\eta(X)\eta(Y), \quad (4.10)$$

which implies

$$QX = -\frac{1}{\alpha}(2n\alpha + 1)X + \frac{1}{\alpha}\eta(X)\xi. \quad (4.11)$$
4.1 Proper RYS and the curvature condition $Q \cdot P = 0$

We now aim to study a proper RYS on a $(2n + 1)$-dimensional $(k, \mu)$-almost Kenmotsu manifold admitting the curvature property $Q \cdot P = 0$, where $P$ is the projective curvature tensor defined for a $(2n + 1)$-dimensional Riemannian manifold as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]. \quad (4.12)$$

**Theorem 4.2.** If $(g, \xi, \lambda, \alpha, \beta)$ is a proper RYS on a $(2n + 1)$-dimensional $(k, \mu)$-almost Kenmotsu manifold $M$ satisfying the curvature property $Q \cdot P = 0$, then the manifold $M$ is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

**Proof.** Let us suppose that the curvature property $Q \cdot P = 0$ holds on $M$. Then for any vector fields $X, Y, Z$ on $M$, we have

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0.$$  

Using (4.11) in the foregoing equation yields

$$\frac{1}{\alpha}[\eta(P(X, Y)Z)\xi - \eta(X)P(\xi, Y)Z - \eta(Y)P(X, \xi)Z$$

$$- \eta(Z)P(X, Y)\xi + 2(2n\alpha + 1)P(X, Y)Z] = 0. \quad (4.13)$$

Now, with the help of (4.2)-(4.4), we calculate the followings:

$$P(\xi, Y)Z = -g(Y, Z)\xi - \frac{1}{2n}S(Y, Z)\xi. \quad (4.14)$$

$$P(X, \xi)Z = g(X, Z)\xi + \frac{1}{2n}S(X, Z)\xi. \quad (4.15)$$

$$P(X, Y)\xi = 0. \quad (4.16)$$

$$\eta(P(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z)$$

$$- \frac{1}{2n}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \quad (4.17)$$

Substituting (4.14)-(4.17) in (4.13) yields

$$2(2n\alpha + 1)P(X, Y)Z = 0,$$

which implies either $2n\alpha + 1 = 0$ or $P(X, Y)Z = 0$.

If $2n\alpha + 1 = 0$, then from (4.10), we get

$$S(X, Y) = \frac{1}{\alpha}\eta(X)\eta(Y) = -2n\eta(X)\eta(Y),$$
which implies $r = -2n$, a contradiction to the fact that $r = -2n(2n + 1)$. Hence, $P(X, Y)Z = 0$ and therefore, (4.12) implies

$$R(X, Y)Z = \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y].$$

(4.18)

Taking inner product of (4.18) with $W$ and then contracting $Y$ and $Z$ yields

$$S(X, W) = -2ng(X, W).$$

(4.19)

Using (4.19) in (4.18), we obtain

$$R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y].$$

This proves that $M$ is locally isometric to $\mathbb{H}^{2n+1}(-1)$. □

### 4.2 Proper RYS and non-existence of $Q \cdot R = 0$

The next theorem is concerned about the non-existence of the curvature property $Q \cdot R = 0$ on $(k, \mu)$-almost Kenmotsu manifolds admitting proper RYS.

**Theorem 4.3.** If $M$ be a $(2n+1)$-dimensional $(k, \mu)$-almost Kenmotsu manifold admitting a proper RYS $(g, \xi, \lambda, \alpha, \beta)$, then the curvature property $Q \cdot R = 0$ does not hold on $M$.

**Proof.** Let the curvature property $Q \cdot R = 0$ holds on $M$. Then for any vector fields $X$, $Y$ and $Z$ on $M$, we have

$$Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ = 0.$$  

(4.20)

Using (4.11) in (4.20), we get

$$2(2n\alpha + 1)R(X, Y)Z + \eta(R(X, Y)Z)\xi - \eta(X)R(\xi, Y)Z$$

$$-\eta(Y)R(X, \xi)Z - \eta(Z)R(X, Y)\xi = 0.$$  

(4.21)

With the help of (4.2), we obtain

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X).$$  

(4.22)

Using (4.3) and (4.22) in (4.21) yields

$$(2n\alpha + 1)R(X, Y)Z = \eta(Z)[\eta(X)Y - \eta(Y)X],$$  

(4.23)

which implies

$$R(X, Y)\xi = \frac{1}{2n\alpha + 1}[\eta(X)Y - \eta(Y)X].$$  

(4.24)

Comparing (4.2) and (4.24), we get $\frac{1}{2n\alpha + 1} = 1$, which implies $\alpha = 0$, a contradiction to the fact that the RYS is proper. This completes the proof. □
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