Regularity of the $3D$ stationary Hall magnetohydrodynamic equations on the plane

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Abstract

We study the regularity of weak solutions to the $3D$ valued stationary Hall magnetohydrodynamic equations on $\mathbb{R}^2$. We prove that every weak solution is smooth. Furthermore, we prove a Liouville type theorem for the Hall equations.

AMS Subject Classification Number: 35Q35, 35Q85, 76W05

keywords: stationary Hall-MHD equations, regularity, Liouville type theorem

1 Introduction and the main theorems

We study the following $3D$ valued stationry Hall-magnetohydrodynamics (Hall-MHD) system on $\mathbb{R}^2$.

\begin{align}
(v \cdot \nabla)v - \Delta v &= -\nabla p + (\nabla \times B) \times B + f, \\
\nabla \times (B \times v) - \Delta B &= -\nabla \times ((\nabla \times B) \times B) + \nabla \times g, \\
\n\nabla \cdot v &= 0, \quad \nabla \cdot B = 0.
\end{align}

Here, $v = (v^1, v^2, v^3), B = (B^1, B^2, B^3)$, where $v^j = v^j(x_1, x_2), B^j = B^j(x_1, x_2), j = 1, 2, 3$, and $p = p(x_1, x_2), x = (x_1, x_2) \in \mathbb{R}^2$. The vector fields $f$ and $g$ represent the external forces. The system for $B$ obtained from (1.2) by setting $v = 0$ is called the Hall equation. Physically the full time-dependet version of the system (1.1)-(1.3) on $\mathbb{R}^3$ describes the dynamics plasma flows with strong shear of magnetic fields such case as in
the solar flares. We refer [1] and the references therein for the physical backgrounds for the full system, and [2, 3, 4, 5, 6, 7, 8, 9] for recent studies of the mathematical problems of the equations. In particular in [7] it is shown that there exist weak solutions of the time dependent 3D time dependent Hall-MHD system on the plane, having the possible set of space-time singularities, whose Hausdorff dimension is at most two. On the other hand, in [6] it is proved that there exist weak solutions of the full 3D stationary Hall-MHD equations having the possible set of singularities with the Hausdorff dimension at most one. In the case of our system (1.1)-(1.3) of the 3D stationary Hall-MHD on the plane, if we apply the argument of [6], then we could easily deduce that there exist weak solutions having possible set of singularities with the Hausdorff dimension zero. Note that this is still far from the conclusion that the set of singularities is empty. Therefore, the regularity problem of (1.1)-(1.3) in $\mathbb{R}^2$ could be regarded as an interesting critical problem, which is our main subject of study in this paper. One of our main results in this paper is to show the full regularity of any weak solutions to the above system, namely the set of singularities is indeed empty. For the proof of this result we modify the Widman’s hole filling method (cf. [12]) in order to handle the case, where the logarithmically blowing-up coefficient is allowed in the Caccioppoli type inequality. We also prove a Liouville type result for the Hall system, which means that any weak solution the equations (1.2)-(1.3) with $v = \nabla \times g = 0$ having finite Dirichlet integral is zero. Below by $\nabla^\perp$ we denote the orthogonal gradient operator $(-\partial_2, \partial_1)^\top$. We also denote that $v' = (v^1, v^2)$ for given $v = (v^1, v^2, v^3)$. According to $\nabla \cdot B = 0$ we find a potential $\Phi \in W^{2, 2}_{\text{loc}}(\mathbb{R}^2)$ such that

$$
(1.4) \quad (B^1, B^2)^\top = \nabla^\perp \Phi.
$$

Setting $\Psi := B_3$, the equations in (1.2) turn into

$$
(1.5) \quad \Delta \Phi = \nabla^\perp \Psi \cdot \nabla \Phi + h^3,
$$
$$
(1.6) \quad \Delta \Psi = -\nabla^\perp \Delta \Phi \cdot \nabla \Phi + \partial_1 h^2 - \partial_2 h^1,
$$

where

$$
(1.7) \quad \begin{cases}
    h^1 = -\partial_1 \Phi v^3 + \Psi v^2 + g^1,
    
    h^2 = -\partial_2 \Phi v^3 - \Psi v^1 + g^2,
    
    h^3 = \nabla \Phi \cdot v' + g^3 + \text{const}.
\end{cases}
$$

We call the system (1.5) - (1.6) the $\Phi$-$\Psi$-system.

By $\tilde{W}^{1,2}_\sigma(\mathbb{R}^2)$ we denote the space of all $v \in L^2_{\text{loc}}(\mathbb{R}^2)$ with $\nabla v \in L^2(\mathbb{R}^2)$ and $\nabla \cdot v = 0$ almost everywhere in $\mathbb{R}^2$. In addition, by $\tilde{W}^{m,s}_\sigma(\mathbb{R}^2)$ we denote the space of all $\Phi \in W^{m,s}_{\text{loc}}(\mathbb{R}^2)$ with $\partial^\alpha \Phi \in L^2(\mathbb{R}^2)$ for all $|\alpha| = m$.

We introduce the following notion of weak solution to (1.1)-(1.3), and the notion of weak-strong solution to the system (1.6), (1.5).

**Definition 1.1.** Let $f, g \in L^2(\mathbb{R}^2)$. A pair $(v, B) \in \tilde{W}^{1,2}_\sigma(\mathbb{R}^2) \times \tilde{W}^{1,2}_\sigma(\mathbb{R}^2)$ is called a weak solution to (1.1)-(1.3) if the following identities hold for all $\varphi \in C^\infty_\text{c}(\mathbb{R}^2)$, and...
\( \psi \in C_c^\infty(\mathbb{R}^2) \) respectively

\[
\begin{align*}
(1.8) \quad & \int_{\mathbb{R}^2} \nabla v : \nabla \varphi = \int_{\mathbb{R}^2} (v \otimes v) : \nabla \varphi + (\nabla \times B) \times B \cdot \varphi + f \cdot \varphi, \\
(1.9) \quad & \int_{\mathbb{R}^2} \nabla B : \nabla \psi = -\int_{\mathbb{R}^2} ((\nabla \times B) \times B + B \times v - g) \cdot \nabla \psi.
\end{align*}
\]

2. Let \( h \in L^2(\mathbb{R}^2) \). A pair \((\Phi, \Psi) \in \hat{W}^{1,2}(\mathbb{R}^2) \times \hat{W}^{1,2}(\mathbb{R}^2)\) is called a \textit{strong-weak solution} to (1.5), (1.6) if (1.5) is satisfied almost everywhere in \( \mathbb{R}^2 \), and (1.6) is fulfilled in the sense of distributions, i.e. for every \( \varphi \in C_c^\infty(\mathbb{R}^2) \),

\[
\begin{align*}
(1.10) \quad & \int_{\mathbb{R}^2} \nabla \Phi \cdot \nabla \varphi = \int_{\mathbb{R}^2} -\Delta \Phi \nabla \Phi \cdot \nabla \varphi + h' \cdot \nabla \varphi.
\end{align*}
\]

\textbf{Remark 1.2.} Note that if \((v, B) \in \hat{W}^{1,2}_{\sigma}(\mathbb{R}^2) \times \hat{W}^{1,2}_{\sigma}(\mathbb{R}^2)\) is a weak solution to (1.1)–(1.3), then \((\Phi, \Psi) \in \hat{W}^{1,2}(\mathbb{R}^2) \times \hat{W}^{1,2}(\mathbb{R}^2)\) is a weak-strong solution to (1.5), (1.6) with right-hand side \( h \) given according to (1.7). Indeed, noting

\[
(\nabla \times B) \times B = -\frac{1}{2}(\nabla \psi^2, 0)^\top - (\Delta \Phi \nabla \Phi, \nabla \psi \cdot \nabla \Phi),
\]

from (1.9) with \( \psi = (\eta^1, \eta^2, 0)^\top \in C_c^\infty(\mathbb{R}^2) \) we find

\[
\int_{\mathbb{R}^2} \Delta \Phi \text{curl} \eta = \int_{\mathbb{R}^2} \partial_i \nabla \psi \cdot \partial_i \eta
\]

\[
= \int_{\mathbb{R}^2} (\nabla \nabla \psi \cdot \nabla \Phi - (B \times v)^3 + g^3) \text{curl} \eta,
\]

where \( \text{curl} \eta = \partial_1 \eta^2 - \partial_2 \eta^1 \). Whence, (1.5).

To verify (1.6), we insert into (1.9) the test functions \( \psi = (0, 0, \varphi)^\top, \varphi \in C_c^\infty(\mathbb{R}^2) \). This gives

\[
\int_{\mathbb{R}^2} \nabla \varphi \cdot \nabla \varphi = \int_{\mathbb{R}^2} \Delta \Phi \nabla \varphi \cdot \nabla \varphi + (B \times v - g)' \cdot \nabla \varphi,
\]

and therefore (1.10) holds with \( h \) given by (1.7). Accordingly, the pair \((\Phi, \Psi)\) is a strong-weak solution to (1.5), (1.6).

Our first main result is the following regularity theorem for the system (1.1)–(1.3).

\textbf{Theorem 1.3.} Let \((v, B) \in \hat{W}^{1,2}_{\sigma}(\mathbb{R}^2) \times \hat{W}^{1,2}_{\sigma}(\mathbb{R}^2)\) be a weak solution to the steady Hall-MHD system in \( \mathbb{R}^2 \) with \( f, g \in C^\infty(\mathbb{R}^2) \). Then both \( v \) and \( B \) are smooth.
Next, we consider the following stationary Hall system,
\[(1.11) \quad \Delta B = \nabla \times ((\nabla \times B) \times B) \quad \text{in} \quad \mathbb{R}^2,\]
which is obtained from the \(B\) equations of the Hall-MHD system with \(v \equiv 0\). Our second main result is the following Liouville type theorem for the system (1.11).

**Theorem 1.4.** Let \(B\) be a weak solution to (1.11) having the finite Dirichlet integral, i.e. \(\int_{\mathbb{R}^2} |\nabla B|^2 < +\infty\). Then \(B \equiv 0\).

2 A modified hole filling method

**Theorem 2.1.** Let \(f \in \hat{W}^{1,2}(\mathbb{R}^2)\), and let \(\mu \in (0, 1)\). Suppose that for all \(B_r \subset \mathbb{R}^2\), \(0 < r < \frac{1}{2}\) the following inequality holds true
\[(2.1) \quad \int_{B_{r/2}} |\nabla f|^2 \leq c_0 (1 + |(f)_{B_r}|) \int_{B_r \setminus B_{r/2}} |\nabla f|^2 + c_1 r^\mu,\]
where \(c_0, c_1\) are positive constants. Then \(f\) is Hölder continuous.

**Proof:** 1. In view of [10, Lemma 2.4] we see that for all \(0 < r < \frac{1}{2}\),
\[(2.2) \quad 1 + |(f)_{B_r}| \lesssim (\log r^{-1})^{\frac{1}{2}}.\]
According to (2.1) together with (2.2) we find that there exists a constant \(c > 0\) such that for all \(0 < r < \frac{1}{2}\).

\[(2.3) \quad \int_{B_{r/2}} |\nabla f|^2 \leq c (\log r^{-1})^{\frac{1}{2}} \int_{B_r \setminus B_{r/2}} |\nabla f|^2 + cr^\mu.\]

Now in (2.3) filling the hole by adding \(c (\log r^{-1})^{\frac{1}{2}} \int_{B_{r/2}} |\nabla f|^2\) to both sides, we infer
\[\int_{B_{r/2}} |\nabla f|^2 \leq \frac{c (\log r^{-1})^{\frac{1}{2}}}{1 + c (\log r^{-1})^{\frac{1}{2}}} \int_{B_r} |\nabla f|^2 + cr^\mu.\]
Accordingly, we are in a position to apply Lemma A.2 which yields for all \(0 < r < \frac{1}{2}\) and \(\alpha > 1\),
\[(2.4) \quad \int_{B_r} |\nabla f|^2 \lesssim \frac{1}{|\log r^{-1}|^{2\alpha}}.\]

2. Next, applying Lemma A.3, we conclude that
\[\sup_{0 < r < 1} |(f)_{B_r}| \lesssim \zeta(\alpha) < +\infty,\]
where the hidden constant in this inequality is independent of the center of the ball. Thus observing (2.1), we get a constant $c_2 > 0$ such that for all $0 < r < \frac{1}{2}$,

\[(2.5) \quad \int_{B_{r/2}} |\nabla f|^2 \leq c_2 \int_{B_r \setminus B_{r/2}} |\nabla f|^2 + c_1 r^\mu.\]

Now in (2.5) filling the hole, we arrive at

\[(2.6) \quad \int_{B_{r/2}} |\nabla f|^2 \leq \theta \int_{B_r} |\nabla f|^2 + c_1 r^\mu, \quad \text{where} \quad \theta = \frac{c_2}{1 + c_2} < 1.\]

3. Let $0 < \lambda < \min \left\{ -\frac{\log \theta}{\log 2}, \mu \right\}$ arbitrarily chosen but fixed. Thanks to Lemma A.4 we get constant $c_3 > 0$ such that for all $0 < r < \frac{1}{2}$

\[(2.7) \quad \int_{B_{r/2}} |\nabla f|^2 \leq c_3 r^\lambda.\]

Note that $c_3$ depends neither on $r$ nor on the center of the ball.

4. Finally, applying Poincaré’s inequality from (2.7) we conclude that for all $0 < r < \frac{1}{2}$

\[\int_{B_r} |f - (f)_{B_r}|^2 \lesssim r^2 \int_{B_{r/2}} |\nabla f|^2 \leq c_3 r^{2+\lambda}.\]

By Campanato’s theorem (see e.g. [11]) we get the Hölder continuity of $f$. \[\blacksquare\]

## 3 Local energy equality for weak solutions to the $\Phi$-$\Psi$-system

The aim of this section is to show that every weak-strong solution to (1.5), (1.6) satisfies a corresponding local energy equality. We have the following

**Lemma 3.1.** Let $h \in L^2(\mathbb{R}^2)$. Let $(\Phi, \Psi) \in \dot{W}^{2,2}(\mathbb{R}^2) \times \dot{W}^{1,2}(\mathbb{R}^2)$ be a strong-weak solution to the $\Phi$-$\Psi$ system (1.5), (1.6). Then the following energy identity holds true for all $\zeta \in C_c^\infty(\mathbb{R}^2)$, and for all $c \in \mathbb{R}$

\[
\int_{\mathbb{R}^2} ((\Delta \Phi)^2 + |\nabla \Psi|^2) \zeta
\]

\[= - \int_{\mathbb{R}^2} \left( (\Psi - c) \nabla \Psi - (\Psi - c) \Delta \Phi \nabla \Phi \right) \nabla \zeta + \int_{\mathbb{R}^2} h^3 \Delta \Phi \zeta + h' \cdot \nabla \left( (\Psi - c) \zeta \right).\]
Proof: For $\rho > 0$ we define

$$
\gamma(\tau) = \begin{cases} 
\frac{1}{2} \tau^2 & \text{if } |\tau| \leq \rho \\
\rho \left( |\tau| - \frac{r}{2} \right) & \text{if } |\tau| > \rho.
\end{cases}
$$

Clearly, $\gamma \in C^{1,1}(\mathbb{R})$, and

$$
\gamma'(\tau) = \text{sign}(\tau) \min\{|\tau|, \rho\}, \quad \gamma''(\tau) = \chi(-\rho, \rho).
$$

Let $\zeta \in C^\infty_c(\mathbb{R}^2)$ be arbitrarily chosen. By virtue of Sobolev’s embedding theorem we see that $\Phi$ is Hölder continuous, and thus bounded on supp($\zeta$). Without loss of generality we may assume that $\Phi \geq 1$ on supp($\zeta$). Let $\alpha > 0$. We multiply (1.5) by $\alpha \Phi^{\alpha-1} \gamma(\Psi) \zeta$, integrate it over $\mathbb{R}^2$, and integrate by part. This leads to the following identity

$$
\alpha \int_{\mathbb{R}^2} \Delta \Phi \Phi^{\alpha-1} \gamma(\Psi) \zeta = \alpha \int_{\mathbb{R}^2} (\nabla^\perp \Psi \cdot \nabla \Phi + h^3) \Phi^{\alpha-1} \gamma(\Psi) \zeta
$$

(3.2)

$$
= - \int_{\mathbb{R}^2} \Phi^\alpha \gamma(\Psi) \nabla^\perp \Psi \cdot \nabla \zeta + \alpha \int_{\mathbb{R}^2} \Phi^{\alpha-1} \gamma(\Psi) h^3 \zeta.
$$

On the other hand, applying integration by parts, we find

$$
\alpha \int_{\mathbb{R}^2} \Delta \Phi \Phi^{\alpha-1} \gamma(\Psi) \zeta
$$

$$
= -\alpha(\alpha - 1) \int_{\mathbb{R}^2} |\nabla \Phi|^2 \Phi^{\alpha-2} \gamma(\Psi) \zeta - \int_{\mathbb{R}^2} \nabla \Phi \cdot \nabla \gamma(\Psi) \zeta
$$

$$
- \alpha \int_{\mathbb{R}^2} \Phi^{\alpha-1} \gamma(\Psi) \nabla \Phi \cdot \nabla \zeta
$$

$$
= -\alpha(\alpha - 1) \int_{\mathbb{R}^2} |\nabla \Phi|^2 \Phi^{\alpha-2} \gamma(\Psi) \zeta - \int_{\mathbb{R}^2} \nabla (\Phi^\alpha \gamma'(\Psi) \zeta) \cdot \nabla \Psi
$$

$$
+ \int_{\mathbb{R}^2} \Phi^\alpha |\nabla \Psi|^2 \gamma''(\psi) \zeta + \int_{\mathbb{R}^2} \Phi^\alpha \gamma'(\Psi) \nabla \Psi \cdot \nabla \zeta
$$

$$
- \alpha \int_{\mathbb{R}^2} \Phi^{\alpha-1} \gamma(\Psi) \nabla \Phi \cdot \nabla \zeta.
$$

(3.3)

In what follows, we focus on evaluating the second integral on the right-hand side. For this purpose we first replace $\gamma'(\Psi)$ by $\gamma'(\Psi)_\epsilon = \gamma'(\Psi) \ast \eta_\epsilon$, where $\eta_\epsilon$ denotes the usual Friedrich’s mollifying kernel. From (1.10) with $\varphi = \Phi^\alpha \gamma'(\Psi)_\epsilon$ we get

$$
- \int_{\mathbb{R}^2} \nabla (\Phi^\alpha \gamma'(\Psi)_\epsilon \zeta) \cdot \nabla \Psi
$$

(3.4)

$$
= \int_{\mathbb{R}^2} \Delta \Phi \nabla \Phi \cdot \nabla (\Phi^\alpha \gamma'(\Psi)_\epsilon \zeta) - \int_{\mathbb{R}^2} h' \cdot \nabla (\Phi^\alpha \gamma'(\Psi)_\epsilon \zeta) = I_\epsilon + II_\epsilon.
$$
By an elementary calculus we get

\[
I_\varepsilon = \int_{\mathbb{R}^2} \Phi^\alpha \gamma'_\rho(\Psi)_\varepsilon \Delta \Phi \nabla \Phi \cdot \nabla \perp \zeta + \int_{\mathbb{R}^2} \Phi^\alpha \Delta \Phi (\nabla \Phi \cdot \nabla \perp \gamma''_\rho(\Psi))_\varepsilon \zeta
\]

\[
- \int_{\mathbb{R}^2} \Phi^\alpha \Delta \Phi \left[(\nabla \Phi \cdot \nabla \perp \gamma'_\rho(\Psi))_\varepsilon - \nabla \Phi \cdot (\nabla \perp \gamma'_\rho(\Psi))_\varepsilon \right] \zeta.
\]

(3.5)

As it can be checked easily, the first integral on the right-hand side of (3.5) tends to

\[
\int_{\mathbb{R}^2} \Phi^\alpha \gamma'_\rho(\Psi) \Delta \Phi \nabla \Phi \cdot \nabla \perp \zeta \quad \text{as} \quad \varepsilon \to 0,
\]

while the second integral tends to

\[
\int_{\mathbb{R}^2} \Phi^\alpha \Delta \Phi \nabla \Phi \cdot \nabla \perp \gamma''_\rho(\Psi) \zeta \quad \text{as} \quad \varepsilon \to 0,
\]

where we have used the fact that \( \nabla \Phi \cdot \nabla \perp \gamma''_\rho(\Psi) = \Delta \Phi \gamma''_\rho(\Psi) \in L^2(\mathbb{R}^2) \). Finally, appealing to Lemma 3.2 below with \( \psi = \gamma'_\rho(\Psi) \), and \( \phi = \Phi \), we infer that the third integral tends to zero as \( \varepsilon \to 0 \).

Furthermore, the convergence of the integral \( II_\varepsilon \), and the convergence of the integral on the left-hand side of (3.4) can be obtained by using routine arguments, recalling the fact that \( f_\varepsilon \to f \) in \( L^1(\mathbb{R}^2) \) as \( \varepsilon \to 0 \) for any \( L^1 \) function \( f \). This together with (1.5) shows that

\[
- \int_{\mathbb{R}^2} \nabla (\Phi^\alpha \gamma'_\rho(\Psi))_\varepsilon \cdot \nabla \Psi
\]

\[
= - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \nabla (\Phi^\alpha \gamma'_\rho(\Psi))_\varepsilon \cdot \nabla \Psi
\]

\[
= \int_{\mathbb{R}^2} \Phi^\alpha \gamma'_\rho(\Psi) \Delta \Phi \nabla \Phi \cdot \nabla \perp \zeta + \int_{\mathbb{R}^2} \Phi^\alpha ((\Delta \Phi)^2 - h^3 \Delta \Phi) \gamma''_\rho(\Psi) \zeta
\]

\[
- \int_{\mathbb{R}^2} h' \cdot \nabla \perp (\Phi^\alpha \gamma'_\rho(\Psi)) \zeta.
\]

Replacing the second integral on the right-hand side of (3.3) by the identity, we have just
derived, we obtain

\[ \alpha \int_{\mathbb{R}^2} \Delta \Phi \Phi^{\alpha-1} \gamma(\Psi) \zeta \]

\[ = -\alpha(\alpha - 1) \int_{\mathbb{R}^2} |\nabla \Phi|^2 \Phi^{\alpha-2} \gamma(\Psi) \zeta + \int_{\mathbb{R}^2} \Phi^{\alpha} \gamma'(\Psi) \Delta \Phi \nabla \cdot \nabla \zeta \]

\[ + \int_{\mathbb{R}^2} \Phi^{\alpha} ((\Delta \Phi)^2 - h^3 \Delta \Phi) \gamma''(\Psi) \zeta - \int_{\mathbb{R}^2} h' \cdot \nabla \perp (\Phi^{\alpha} \gamma'(\Psi) \zeta) \]

\[ (3.6) + \int_{\mathbb{R}^2} \Phi^{\alpha} |\nabla \Psi|^2 \gamma''(\Psi) \zeta + \int_{\mathbb{R}^2} \Phi^{\alpha} \gamma'(\Psi) \nabla \cdot \nabla \zeta. \]

Combining (3.2) and (3.6), we are led to

\[ - \int_{\mathbb{R}^2} \Phi^{\alpha} \gamma'(\Psi) \nabla \perp \Psi \cdot \nabla \zeta - \alpha \Phi^{\alpha-1} \gamma'(\Psi) h^3 \zeta \]

\[ = -\alpha(\alpha - 1) \int_{\mathbb{R}^2} |\nabla \Phi|^2 \Phi^{\alpha-2} \gamma(\Psi) \zeta + \int_{\mathbb{R}^2} \Phi^{\alpha} \gamma'(\Psi) \Delta \Phi \nabla \cdot \nabla \zeta \]

\[ + \int_{\mathbb{R}^2} \Phi^{\alpha} ((\Delta \Phi)^2 - h^3 \Delta \Phi) \gamma''(\Psi) \zeta - \int_{\mathbb{R}^2} h' \cdot \nabla \perp (\Phi^{\alpha} \gamma'(\Psi) \zeta) \]

\[ (3.7) + \int_{\mathbb{R}^2} \Phi^{\alpha} |\nabla \Psi|^2 \gamma''(\Psi) \zeta + \int_{\mathbb{R}^2} \Phi^{\alpha} \gamma'(\Psi) \nabla \cdot \nabla \zeta. \]

In (3.7), first letting \( \alpha \to 0 \), and afterwards letting \( \rho \to +\infty \), we conclude

\[ - \frac{1}{2} \int_{\mathbb{R}^2} \Psi^2 \nabla \perp \Psi \cdot \nabla \zeta \]

\[ = - \int_{\mathbb{R}^2} \Psi \Delta \Phi \nabla \perp \Phi \cdot \nabla \zeta + \int_{\mathbb{R}^2} ((\Delta \Phi)^2 - h^3 \Delta \Phi) \zeta - \int_{\mathbb{R}^2} h' \cdot \nabla \perp (\Psi \zeta) \]

\[ (3.8) + \int_{\mathbb{R}^2} |\nabla \Psi|^2 \zeta + \int_{\mathbb{R}^2} \Psi \nabla \Psi \cdot \nabla \zeta. \]

Noting that the integral on the left-hand side of (3.8) vanishes, we deduce the local energy identity (3.1) from (3.8) for \( c = 0 \). Since in the discussion above \( \Psi \) can be replaced by \( \Psi - c \) for any \( c \in \mathbb{R} \), we get the assertion of the lemma.

The following lemma we have used in the proof of Lemma 3.1.

**Lemma 3.2.** Let \( \phi \in \hat{W}^{2,2}(\mathbb{R}^2) \), and \( \psi \in \hat{W}^{1,2}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). Then

\[ \nabla \phi \cdot \nabla \perp \psi_{\varepsilon} - (\nabla \phi \cdot \nabla \perp \psi)_{\varepsilon} \to 0 \quad \text{weakly in} \quad L^2(\mathbb{R}^2) \quad \text{as} \quad \varepsilon \to +\infty. \]
Proof: By using the absolutely continuity of the Lebesgue measure we see that
\begin{equation}
\nabla \phi \cdot \nabla \perp \psi_{\varepsilon} - (\nabla \phi \cdot \nabla \perp \psi)_{\varepsilon} \to 0 \quad a.e. \text{ in } \mathbb{R}^2 \text{ as } \varepsilon \to +\infty.
\end{equation}
Thus, it suffices to show that the $L^2$-norm of $\nabla \phi \cdot \nabla \perp \psi_{\varepsilon} - (\nabla \phi \cdot \nabla \perp \psi)_{\varepsilon}$ is bounded independently on $\varepsilon$. To see this, we first calculate for almost everywhere $x \in \mathbb{R}^2$
\begin{equation*}
\nabla \phi(x) \cdot \nabla \perp \psi_{\varepsilon}(x) - (\nabla \phi \cdot \nabla \perp \psi)(x)
= \int_{B_{\varepsilon}} (\nabla \phi(x) - \nabla \phi(x-y)) \cdot \nabla \perp \psi(x-y) \eta_{\varepsilon}(y) dy
\end{equation*}
\begin{equation*}
= - \int_{B_{\varepsilon}} \psi(x-y) \int_{0}^{1} \partial_i \nabla \phi(x-ty) dty_i \cdot \nabla \perp \eta_{\varepsilon}(y) dy.
\end{equation*}
Noting that $|y| |\nabla \perp \eta_{\varepsilon}(y)| \lesssim \varepsilon^{-2}$, and $\psi \in L^\infty(\mathbb{R}^2)$, along with Jensen’s inequality we find
\begin{equation*}
(\nabla \phi(x) \cdot \nabla \perp \psi_{\varepsilon}(x) - (\nabla \phi \cdot \nabla \perp \psi)(x))^2 \lesssim \varepsilon^{-2} \|\psi\|_{L^\infty}^2 \int_{B_{\varepsilon}} \int_{0}^{1} |\nabla^2 \phi(x-ty)|^2 dtdy.
\end{equation*}
Integrating this inequality over $\mathbb{R}^2$, and employing Fubini’s theorem, we obtain
\begin{equation*}
\int_{\mathbb{R}^2} (\nabla \phi \cdot \nabla \perp \psi_{\varepsilon} - (\nabla \phi \cdot \nabla \perp \psi))^2 \lesssim \varepsilon^{-2} \|\psi\|_{L^\infty}^2 \int_{B_{\varepsilon}} \int_{0}^{1} \int_{\mathbb{R}^2} |\nabla^2 \phi(x-ty)|^2 dxdtdy
= \|\psi\|_{L^\infty}^2 \|\nabla^2 \phi\|_{L^2}^2.
\end{equation*}
This completes the proof of the Lemma 3.2.}

4 Proof of Theorem 1.3

We now consider the system (1.6), (1.5) in $\mathbb{R}^2$ with general right-hand side. From Lemma 3.1 we infer that
\begin{equation}
(\Delta \Phi)^2 + |\nabla \Psi|^2 = \nabla \cdot \left( (\Psi - c) \nabla \Psi - (\Psi - c) \Delta \Phi \nabla \perp \Phi \right) + h^3 \Delta \Phi - (\Psi - c)(\partial_1 h^1 + \partial_2 h^2)
\end{equation}
in $\mathbb{R}^2$ in the sense of distributions.

Our aim is to prove the following local regularity result

Theorem 4.1. Let $h \in M^{2,\mu}_{loc}$ for some $\mu > 0$. Let $(\Phi, \Psi) \in \dot{W}^{2,2}(\mathbb{R}^2) \times \dot{W}^{1,2}(\mathbb{R}^2)$ be a strong-weak solution to (1.5), (1.6). Then $\Psi, \partial_i \Phi \in C^\alpha(\mathbb{R}^2)$, $i = 1, 2$.

The proof of Theorem 4.1 is based on Caccioppoli-type inequalities as well as a crucial logarithmic decay estimate. In what follows we make use of the following notion of a suitable cut off function.
**Definition 4.2.** Given balls $B_\rho \subset B_R = B_R(x_0)$, $0 < \rho < R$, a function $\zeta \in C^\infty_c(B_R)$ is said to be a suitable cut off function for this balls if $0 \leq \zeta \leq 1$ in $B_R$, $\zeta \equiv 1$ on $B_\rho$, and

$$|\nabla^2 \zeta| + |\nabla \zeta|^2 \lesssim (R - \rho)^{-2}. $$

In what follows, let $h \in L^2(\mathbb{R}^2)$, and let $(\Psi, \Phi) \in \hat{W}^{1,2}(\mathbb{R}^2) \times \hat{W}^{2,2}(\mathbb{R}^2)$ be a weak-strong solution to (1.6), (1.5). Furthermore, for a measurable set $A \subset \mathbb{R}^2$ with $\text{meas} A > 0$ we write

$$(f)_A = \int_A f = \frac{1}{\text{meas } A} \int_A f, \quad f \in L^1(A).$$

The following lemmas are an immediate consequence of the local energy identity (3.1).

**Proof of Theorem 4.1.** Let $\zeta \in C^\infty_c(B_R)$ be a cut off function suitable for the balls $B_{R/2} \subset B_R$. In (3.1) we replace $\zeta$ by $\zeta^2$ and set $c = \Psi_{B_R \setminus B_{R/2}}$. This yields

$$\int_{B_R} ((\Delta \Phi)^2 + |\nabla \Psi|^2) \zeta^2 \lesssim R^{-1} \int_{B_R \setminus B_{R/2}} |\Psi - (\Psi)_{B_R \setminus B_{R/2}}| |\nabla \Psi| \zeta$$

$$+ R^{-1} \int_{B_R \setminus B_{R/2}} |\Psi - (\Psi)_{B_R \setminus B_{R/2}}| |\Delta \Phi| |\nabla \Phi - (\nabla \Phi)_{B_R \setminus B_{R/2}}| \zeta$$

$$+ R^{-1} |(\nabla \Phi)_{B_R \setminus B_{R/2}}| \int_{B_R \setminus B_{R/2}} |\Psi - (\Psi)_{B_R \setminus B_{R/2}}| |\Delta \Phi| \zeta$$

$$+ R^{-1} \int_{B_R \setminus B_{R/2}} |\Psi - (\Psi)_{B_R \setminus B_{R/2}}| |h'| \zeta$$

$$+ \int_{B_R} |h|^2 |\Delta \Phi| \zeta^2 + |h'| |\nabla \Psi| \zeta^2. $$

(4.2)

Then by the aid of Hölder’s inequality, Young’s inequality, and Sobolev-Poincaré inequality, we deduce from (4.2)

$$\int_{B_R} ((\Delta \Phi)^2 + |\nabla \Psi|^2) \zeta^2 \lesssim \int_{B_R \setminus B_{R/2}} |\nabla \Psi|^2 + \left( \int_{B_R \setminus B_{R/2}} |\nabla^2 \Phi|^2 \right)^2 + |(\nabla \Phi)_{B_R}| \int_{B_R \setminus B_{R/2}} |\nabla \Psi|^2 + |\Delta \Phi|^2$$

$$+ \int_{B_R} |h|^2. $$

(4.3)
Next, we provide the estimate of $\nabla^2 \Phi$ in term of $\Delta \Phi$. In fact, using integration by parts, we easily get
\[
\int_{B_R} |\nabla^2 \Phi|^2 \zeta^2 = \int_{B_R} \partial_i \partial_j \Phi (\partial_i \partial_j \Phi) \zeta^2 \\
= \int_{B_R} |\Delta \Phi|^2 \zeta^2 - 2 \int_{B_R} (\partial_j \Phi - (\partial_j \Phi)_{B_R \setminus B_{R/2}})(\partial_i \partial_j \Phi) \zeta \partial_i \zeta \\
+ 2 \int_{B_R} (\partial_j \Phi - (\partial_j \Phi)_{B_R \setminus B_{R/2}})(\partial_i \partial_j \Phi) \Delta \Phi \zeta \partial_j \zeta.
\]

Applying Cauchy-Schwarz’s inequality along with Young’s inequality and Poincaré’s inequality, we find
\[
\int_{B_{R/2}} |\nabla^2 \Phi|^2 \zeta^2 \lesssim \int_{B_R} |\Delta \Phi|^2 \zeta^2 + R^{-2} \int_{B_R \setminus B_{R/2}} |\nabla \Phi - (\nabla \Phi)_{B_R \setminus B_{R/2}}|^2 \\
\lesssim \int_{B_R} |\Delta \Phi|^2 \zeta^2 + \int_{B_R \setminus B_{R/2}} |\nabla^2 \Phi|^2.
\]

Thus, estimating the first term on the right-hand side of (4.4) by means of (4.3), we are led to
\[
\int_{B_{R/2}} (|\nabla^2 \Phi|^2 + |\nabla \Psi|^2) \\
\lesssim \left[ 1 + |(\nabla \Phi)_{B_R}| + \int_{B_R} |\nabla^2 \Phi|^2 \right] \int_{B_R \setminus B_{R/2}} |\nabla \Psi|^2 + |\nabla^2 \Phi|^2 + \int_{B_R} |h|^2 \\
\lesssim \left[ 1 + |(\nabla \Phi)_{B_R}| \right] \int_{B_R \setminus B_{R/2}} (|\nabla \Psi|^2 + |\nabla^2 \Phi|^2) + R^\mu
\]

Thus, by means of (4.5) we are in a position to apply Theorem 2.1 with $f = (\partial_1 \Phi, \partial_2 \Phi, \Psi)$. Accordingly, $\partial_1 \Phi, \partial_2 \Phi$ and $\Psi$ are Hölder continuous.

Proof of Theorem 1.3 Recalling that $B = (-\partial_2 \Phi, \partial_1, \Psi)$ we obtain the Hölder continuity of $B$. Arguing as in [7], we get the smoothness of $(v, B)$.

5 Proof of Theorem 1.4

Thanks to Theorem 1.3 we already know that $B$ is smooth, and therefore the following energy identity holds true for all $\zeta \in C_0^\infty(\mathbb{R}^2)$ and $\Lambda \in \mathbb{R}^2$
\[
\int_{\mathbb{R}^2} |\nabla B|^2 \zeta = \frac{1}{2} \int_{\mathbb{R}^2} |B - \Lambda|^2 \Delta \zeta + \int_{\mathbb{R}^2} (\nabla \times B) \times B \cdot (B - \Lambda) \times \nabla \zeta.
\]
We define
\[ \mu(r) := r^{-2} \int_{B_r \setminus B_{r/2}} |B| \, dx, \quad r > 0. \]

By using change of coordinates \( x = ry \), we see that
\[ \mu(r) = \int_{B_1 \setminus B_{1/2}} |B(ry)| \, dy, \quad r > 0. \]

By a straightforward arguments we easily get for all \( r > 1 \),
\[ \mu'(r) = \int_{B_1 \setminus B_{1/2}} y_j \frac{\partial_j B(ry) \cdot B(ry)}{|B(ry)|} \, dy \leq r^{-2} \int_{B_r \setminus B_{r/2}} |\nabla B| \, dx \]
\[ \leq \sqrt{\pi} r^{-1} \left( \int_{B_r \setminus B_{r/2}} |\nabla B|^2 \, dx \right)^{1/2} \leq \sqrt{\pi} (\log r)' \left( \int_{\mathbb{R}^2 \setminus B_{1/2}} |\nabla B|^2 \, dx \right)^{1/2}. \]

Thus, the function
\[ r \mapsto \mu(r) - \sqrt{\pi} \log r \left( \int_{\mathbb{R}^2 \setminus B_{1/2}} |\nabla B|^2 \, dx \right)^{1/2} \]
is non increasing on \([1, +\infty)\), which implies for all \( r \geq e \)
\[ (5.2) \quad \mu(r) \leq \mu(1) + \sqrt{\pi} \log r \left( \int_{\mathbb{R}^2 \setminus B_{1/2}} |\nabla B|^2 \, dx \right)^{1/2} \leq C_0 \log(r), \]

where \( C_0 = \mu(1) + \sqrt{\pi} \left( \int_{\mathbb{R}^2 \setminus B_{1/2}} |\nabla B|^2 \, dx \right)^{1/2}. \)

Let \( \zeta \) be a cut-off function for \( B_r \) and \( B_{r/2} \). In (5.1) we replace \( \zeta \) by \( \zeta^2 \), take \( \Lambda = B_{B_r \setminus B_{r/2}} \), and integrate by parts. This gives
\[ \int_{B_r} |\nabla B|^2 \zeta^2 = -2 \int_{B_r \setminus B_{r/2}} \nabla B : (B - B_{B_r \setminus B_{r/2}}) \otimes \zeta \nabla \zeta \]
\[ + \int_{B_r \setminus B_{r/2}} (\nabla \times B) \times B \cdot (B - B_{B_r \setminus B_{r/2}}) \times \zeta \nabla \zeta \]
\[ = I_1 + I_2. \]

In order to estimate the first integral we use Cauchy-Schwarz inequality together with Poincaré’s inequality. This gives \( I_1 = o(1) \). For the estimation of the second integral \( I_2 \)
we first write

\[ I_2 = \int_{B_r \setminus B_{r/2}} (\nabla \times B) \times (B - B_{B_r \setminus B_{r/2}}) \cdot (B - B_{B_r \setminus B_{r/2}}) \times \zeta \nabla \zeta + \int_{B_r \setminus B_{r/2}} (\nabla \times B) \times B_{B_r \setminus B_{r/2}} \cdot (B - B_{B_r \setminus B_{r/2}}) \times \zeta \nabla \zeta = I_{21} + I_{22}. \]

Then applying Cauchy-Schwarz inequality together with Sobolev-Poincaré’s inequality, we get \( I_{21} = O(1) \) as \( r \to +\infty \). Thus, it only remains to estimate \( I_{22} \). By the aid of Cauchy-Schwarz inequality and Poincaré’s inequality along with (5.2) we infer

\[ I_{22} \lesssim \mu(r) \int_{B_r \setminus B_{r/2}} |\nabla B|^2 \leq C_0 \log(r) \int_{B_r \setminus B_{r/2}} |\nabla B|^2. \]

Let \( \varepsilon > 0 \), and \( R_0 > 0 \) be chosen sufficiently large which will be specified below. We now set \( r = 2^k, k \in \mathbb{N} \). Let \( \varepsilon > 0 \) be arbitrarily chosen. As \( |\nabla B|^2 \) is integrable, for every \( m \in \mathbb{N} \) there exists \( k \in \mathbb{N}, k \geq m \) such that

\[ \int_{B_{2k} \setminus B_{2k-1}} |\nabla B|^2 \leq \frac{\varepsilon}{k}. \]

Otherwise, there exists \( m \in \mathbb{N} \) such that the reverse inequality of (5.4) holds for all \( k \geq m \), which leads to the following contradiction

\[ +\infty > \int_{\mathbb{R}^2} |\nabla B|^2 \geq \sum_{k \geq m} \int_{B_{2k} \setminus B_{2k-1}} |\nabla B|^2 \geq \sum_{k \geq m} \frac{\varepsilon}{k} = +\infty. \]

Thus (5.3) with \( r = 2^k \) reads

\[ I_{22} \lesssim C_0 \varepsilon. \]

As \( \varepsilon > 0 \) can be chosen arbitrarily small, we conclude that \( \int_{\mathbb{R}^2} |\nabla B|^2 = 0 \) and therefore \( B = \text{const.} \).

**Acknowledgements** Chae was partially supported by NRF grants 2016R1A2B3011647, while Wolf has been supported by the German Research Foundation (DFG) through the project WO1988/1-1; 612414.

## A Auxiliary Lemmas

By \( \tilde{W}^{m,s}(\mathbb{R}^2) \), \( 1 \leq s < +\infty, m \in \mathbb{N} \), we denote the homogeneous Sobolev space of all \( f \in W^{m,s}_{\text{loc}}(\mathbb{R}^2) \) with \( D^\alpha f_{B_1} = \frac{1}{\text{meas } B_1} \int_{\mathbb{R}^2} D^\alpha f = 0 \) for all \( |\alpha| \leq m - 1 \) such that \( D^\alpha f \in L^s(\mathbb{R}^2) \).
Lemma A.1. Let $B_1, B_2 \in \dot{W}^{1,2}(\mathbb{R}^2)$ with $\partial_1 B_1 + \partial_2 B_2 = 0$ in $\mathbb{R}^2$. Then there exists $\Phi \in \dot{W}^{2,2}(\mathbb{R}^2)$, such that $(B_1, B_2)^\top = \nabla^\perp \Phi$, where $\nabla^\perp \Phi = (\partial_2 \Phi, -\partial_1 \Phi)^\top$.

Proof: We consider the equation

$$-\Delta \Phi = \partial_1 B_2 - \partial_2 B_1 \text{ in } \mathbb{R}^2. \tag{A.1}$$

Let $\Phi \in \dot{W}^{2,2}(\mathbb{R}^2)$ denote the unique weak solution to (A.1). Set $A = \nabla^\perp \Phi$. Then

$$\partial_1 A^2 - \partial_2 A^1 = -\Delta \Phi = \partial_1 B_2 - \partial_2 B_1.$$

Thus, there exists $p \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, such that $(B_1 - A^1, B_2 - A^2) = (\partial_1 p, \partial_2 p)$. Since $\partial_1 (B_1 - A^1) + \partial_2 (A^2 - B_2)$ it follows $\Delta p = 0$. Noting that $\nabla p$ has logarithmic growth at infinity we get $\nabla p = \text{const}$. Eventually, replacing $\Phi$ by $\Phi + Q$, where $Q$ is a polynomial of degree $\leq 1$ we may assume that $(B_i - A_i)_{B_i} = 0$ ($i = 1, 2$), which gives $(B_1, B_2) = (A_1, A_2) = \nabla^\perp \Phi$. ■

Lemma A.2. Let $\phi : [0, 1) \to [0, +\infty)$ be a non decreasing function. Assume there exists $c = \text{const} > 0$ such that for all $0 < r < \frac{1}{2}$

$$\phi(r/2) \leq \frac{c[\log r^{-1}]^{\frac{1}{2}}}{1 + c[\log r^{-1}]^{\frac{1}{2}}} \phi(r) + cr^\mu. \tag{A.2}$$

Then for every $\alpha > 1$ there holds for all $0 < r < \frac{1}{2}$

$$\phi(r) \lesssim \frac{1}{[\log r^{-1}]^\alpha}, \tag{A.3}$$

where the hidden constant in (A.3) depend only on $c, q, \alpha$ and $\mu$.

Proof: Clearly (A.2) with $r = 2^{-k}$ reads

$$\phi(2^{-k}) \leq \frac{ck^\frac{1}{2}}{1 + ck^\frac{1}{2}} \phi(2^{-k+1}) + c2^{-k\mu+\mu}. \tag{A.4}$$

Let $n \in \mathbb{N}, n \geq 2$. Iterating (A.4) from $k = n^2$ to $k = n + 1$, we obtain

$$\phi(2^{-n^2}) \leq \left( \prod_{k=n+1}^{n^2} \frac{ck^\frac{1}{2}}{1 + ck^\frac{1}{2}} \right) \phi(2^n) + \frac{c2^{-\mu}}{1 - 2^{-\mu}}. \tag{A.5}$$

Furthermore, estimating

$$\log \prod_{k=n+1}^{n^2} \frac{ck^\frac{1}{2}}{1 + ck^\frac{1}{2}} = \sum_{k=n+1}^{n^2} \log \left( 1 - \frac{1}{1 + ck^\frac{1}{2}} \right) \leq - \sum_{k=n+1}^{n^2} \frac{1}{1 + ck^\frac{1}{2}} \lesssim -n^{\frac{1}{2}},$$

we find that

$$\prod_{k=n+1}^{n^2} \frac{ck^\frac{1}{2}}{1 + ck^\frac{1}{2}} \lesssim e^{-n^{\frac{1}{2}}}. \tag{A.6}$$
Noting that \( \frac{2^{-n\mu}}{1-2^{-n\mu}} \lesssim e^{-n^{1/2}} \), we deduce from (A.5) along with (A.6) that

(A.7) \( \phi(2^{-n^2}) \lesssim e^{-n^{1/2}} \)

Let \( \alpha > 1 \) be arbitrarily chosen. Clearly, there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \)

\[
2\alpha \log n \leq n^{1/2} \quad \Rightarrow \quad e^{-n^{1/2}} \leq \frac{1}{|\log 2^{n^2}|^\alpha}.
\]

Let \( 0 < r < \frac{1}{2} \). Then there exists unique \( n \in \mathbb{N} \) such that \( 2^{-(n+1)^2} < r \leq 2^{-n^2} \). In particular, \( (n+1)^2 > \frac{\log r^{-1}}{\log 2} \), and (A.7) yields

\[ \phi(r) \leq \phi(2^{-(n+1)^2}) \lesssim \frac{1}{|\log r^{-1}|^\alpha}. \]

Whence, the claim.

Lemma A.3. Let \( f \in \hat{W}^{1,2}(\mathbb{R}^2) \). Suppose there exists \( \alpha > 1 \) and a constant \( c_0 > 0 \) such that for all \( 0 < r < \frac{1}{2} \)

(A.8) \( \left( \int_{B_r} |\nabla f|^2 \right)^{1/2} \leq \frac{c_0}{|\log r^{-1}|^\alpha}. \)

Then

(A.9) \( \sup_{0 < r < 1} |(f)_{B_r}| < +\infty. \)

Proof: We set \( \mu(r) = \int_{B_r} f \). We first claim the following inequality for all \( 0 < r < R < +\infty \).

(A.10) \( |\mu(r)| \leq |\mu(R)| + \log \frac{R}{r} \left( \pi^{-1} \int_{B_R} |\nabla f|^2 \right)^{1/2}. \)

Indeed, since \( \mu(r) = \int_{B_1} f(ry)dy \), we have

\[
\mu'(r) = \int_{B_1} y \cdot \nabla f(ry)dy = \int_{B_r} x \cdot \nabla f dx \geq -\int_{B_r} |\nabla f| \geq -\frac{1}{r} \left( \pi^{-1} \int_{B_R} |\nabla f|^2 \right)^{1/2}.
\]

This implies that

\[
\frac{d}{dr} \left[ \mu(r) + \log \frac{R}{r} \left( \pi^{-1} \int_{B_R} |\nabla f|^2 \right)^{1/2} \right] \geq 0,
\]

which after integration over \( [r, R] \) provides us with

(A.11) \( \mu(r) \leq \mu(R) + \log \frac{R}{r} \left( \pi^{-1} \int_{B_R} |\nabla f|^2 \right)^{1/2}. \)
Similarly, we have
\[ \mu'(r) \leq \frac{1}{r} \left( \pi^{-1} \int_{B_r} |\nabla f|^2 \right)^{1/2}, \]
which leads to the opposite inequality,
\[ (A.12) \quad \mu(r) \geq \mu(R) - \log \frac{R}{r} \left( \pi^{-1} \int_{B_R} |\nabla f|^2 \right)^{1/2}. \]
Combining (A.11) and (A.12), we obtain (A.10) as claimed.

For \( k \in \mathbb{N}, k \geq 2 \) we insert \( r = 2^{-k-1} \) and \( R = 2^{-k} \) in (A.10). This yields
\[ (A.13) \quad |\mu(2^{-k-1})| \leq |\mu(2^{-k})| + \log 2 \frac{c_0 \pi^{-\frac{1}{2}}}{k^{\alpha} \log 2} \leq |\mu(2^{-k})| + \frac{c_0}{k^{\alpha}}. \]
Iterating (A.13) \( n-1 \)-times from \( k = n \) to \( k = 2 \), we arrive at
\[ |\mu(2^{-n-1})| \leq |\mu(2^{-2})| + c_0 \sum_{k=2}^{n} k^{-\alpha} \lesssim \zeta(\alpha). \]
Here \( \zeta \) stands for Riemann’s Zeta-function, and \( \zeta(\alpha) = \sum_{k=1}^{\infty} k^{-\alpha} \).

Now, let \( 0 < r < \frac{1}{2} \) be arbitrarily chosen. There exists unique \( n \in \mathbb{N} \) such that \( 2^{-n-2} < r \leq 2^{-n-1} \). In particular, \( 2n \geq n + 2 \geq \frac{\log r^{-1}}{\log 2} \geq \log r^{-1} \). Thus, the inequality above yields \( |\mu(r)| \lesssim \zeta(\alpha) \). This completes the proof of (A.9). \( \blacksquare \)

**Lemma A.4.** Let \( \phi : [0, 1) \to [0, +\infty) \) be a non decreasing function. Assume there exists a constants \( \theta, \mu \in (0, 1) \) such that for all \( 0 < r < \frac{1}{2} \)
\[ (A.14) \quad \phi(r/2) \leq \theta \phi(r) + cr^\mu. \]
Then for every \( \alpha > 1 \) there holds for all \( 0 < r < \frac{1}{2} \)
\[ (A.15) \quad \begin{cases} \phi(r) \lesssim r^\alpha & \text{if } \theta \neq 2^{-\mu}, \quad \alpha = \min \left\{ -\frac{\log \theta}{\log 2}, 2 \right\}, \\ \phi(r) \lesssim (\log r^{-1}) r^\mu & \text{if } \theta = 2^{-\mu} \end{cases} \]
where the hidden constant in (A.3) depends only on \( c, \theta \) and \( \mu \).

**Proof:** The inequality (A.14) with \( r = 2^{-k} \) reads
\[ (A.16) \quad \phi(2^{-k-1}) \leq \theta \phi(2^{-k}) + c2^{-k\mu}. \]
Iterating (A.16) \( n \)-times, we obtain
\[ (A.17) \quad \phi(2^{-n}) \leq c2^{-n\mu} + c\theta 2^{-(n-1)\mu} + c\theta^2 2^{-(n-2)\mu} + \ldots + c\theta^{n-1} 2^\mu + c\theta^n. \]
In case \( 2^{-\mu} < \theta \) we deduce from (A.17)
\[ (A.18) \quad \phi(2^{-n}) \leq c\theta^n \left[ \left( \frac{2^{-\mu}}{\theta} \right)^n + \left( \frac{2^{-\mu}}{\theta} \right)^{n-1} + \ldots + 1 \right] \leq \frac{c\theta^n}{\theta - 2^{-\mu}}. \]
On the contrary, in case $2^{-\mu} > \theta$ (A.17) yields

\[(A.19) \quad \phi(2^{-n}) \leq c2^{-\mu n} \left[ 1 + \frac{\theta}{2^\mu} + \ldots + \left( \frac{\theta}{2^{-\mu}} \right)^{n-1} + \left( \frac{\theta}{2^{-\mu}} \right)^n \right] \leq \frac{c2^{-\mu n}}{2^{-\mu} - \theta}.
\]

Thus, from (A.18) and (A.19) we deduce that for all $n \in \mathbb{N}$

\[(A.20) \quad \phi(2^{-n}) \lesssim 2^{-n\alpha}, \quad \alpha = \min \left\{ \frac{\log \theta}{\log 2}, \mu \right\}.
\]

This implies (A.15) in case $\theta \neq 2^{-\mu}$. In case $\theta = 2^{-\mu}$ we infer from (A.17)

\[(A.21) \quad \phi(2^{-n}) \lesssim n2^{-n\mu}.
\]

Noting that $n = \frac{\log 2^n}{\log 2}$, we get (A.15) in case $\theta = 2^{-\mu}$.

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