The Klein bottle in its classical shape: 
a further step towards a good parametrization

Gregorio Franzoni

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Abstract
Together with the Möbius strip, the Klein bottle is one of the intriguing objects in the 
universe of geometry, sometimes appearing in non-mathematical contexts too. Until now, 
several parametrizations of it as a surface immersed in ordinary three-space have been found, 
some of which are very elegant and lead to nice and well understandable shapes. Nevertheless, these shapes are quite different from the object imagined by F. Klein in the late 19th century: a tube which passes through itself with the two ends glued together. Parametrizations for this version of the Klein bottle do exist, but they are not fully satisfactory for some reasons. We discuss some of the existing representations and propose two new immersions of the Klein bottle in $\mathbb{R}^3$, which are intended to be a step towards a canonical expression of this surface in the shape imagined by its first discoverer.

1 Introduction

The Klein bottle is a topological object that can be defined as $K = \mathbb{R}^2 / \sim$, where $\sim$ is the equivalence relation $(u, v) \sim (u', v') \iff (u', v') = (u + 2k\pi, (-1)^k v + 2h\pi)$. It is a well known fact that $K$ is a genus 2 non-orientable closed surface and that it is non-embeddable in $\mathbb{R}^3$. It is possible, however, to immerse it in $\mathbb{R}^3$, that is, to map it in $\mathbb{R}^3$ obtaining an image with no singular points. To give an immersion of $K$ in $\mathbb{R}^3$ it suffices to define, on the fundamental square $[0, 2\pi] \times [0, 2\pi]$, an immersion that passes to the quotient with respect to the relation $\sim$. Among the existing known immersions of $K$ in $\mathbb{R}^3$, we recall the two following:

\begin{align}
\text{Kb1}(u, v) : \quad & \begin{cases} 
x = (a + \cos(\frac{u}{2}) \sin v - \sin(\frac{u}{2}) \sin(2v) \cos u \\
y = (a + \cos(\frac{u}{2}) \sin v - \sin(\frac{u}{2}) \sin(2v)) \sin u \\
z = \sin(\frac{u}{2}) \sin v + \cos(\frac{u}{2}) \sin(2v)
\end{cases} \\
\text{Kb2}(u, v) : \quad & \begin{cases} 
x = \cos 2u \sin v/(1 - (\sin u \cos u + \sin 2u \sin v)/\sqrt{2}) \\
y = (\sin 2u \sin v - \sin u \cos v)/\sqrt{2}(1 - (\sin u \cos u + \sin 2u \sin v)/\sqrt{2}) \\
z = \cos u \cos v/(1 - (\sin u \cos u + \sin 2u \sin v)/\sqrt{2})
\end{cases}
\end{align}
The first one can be obtained by moving a lemniscate around a circle while it rotates, in its plane, half a turn around its center (see, for example, [1, 2, 9]). The second one can be seen as a one-parameter family of circles in space and its image is also the zero set of a polynomial of degree eight (see [1]). Moreover, in the work of H. B. Lawson [8], it arises as particular case as the image, by stereographic projection, of a minimal surface on $S^3$. The two versions of the Klein bottle described above could be considered canonical for the simplicity of their geometrical construction and of their formulas, and for the clearness of the resulting shapes.

But if we think of the bottle as it was conceived by F. Klein in 1882 ([7, §23], see also [6, pp. 308–310]): tucking a rubber hose, making it to penetrate to itself, and then smoothly gluing the two ends together, they are far from such a visual idea. On the other hand, known formulas that lead to the bottle in its original shape are quite complicated and don’t have the elegance of (1) and (2). In the following we will recall two definitions of the Klein bottle in its first shape, then we will propose a further step towards what we would like to call a canonical parametrization for it as a self-penetrating tube.
2 Two classical looking definitions of the Klein bottle

In this paragraph we recall two parametrizations of the bottle due to S. Dickson [5] and to M. Trott [10] respectively. Both surfaces are realized as tubes around plane curves, according to the following scheme: let $\alpha(t) = (x(t), y(t)), t \in [a, b]$ be a curve lying on the $xy$ plane satisfying $\|\alpha'(t)\| \neq 0$, $k = (0, 0, 1)$ the $z$-axis unit vector, $T = \frac{\alpha'}{\|\alpha'\|}$ the unit tangent vector field of $\alpha(t)$ and $J$ the canonical complex structure of the $xy$ plane, i.e. the linear map from $\mathbb{R}^2$ to itself defined by $J(v_1, v_2) = (-v_2, v_1)$. Then the couple of unit vectors $(J(T), k)$ is always orthogonal to $\alpha'(t)$ along $\alpha(t)$ and can be used to construct a tube around $\alpha(t)$ as follows:

$\text{Tube}(t, \theta) = \alpha(t) + r(t)(\cos \theta \ J(T) + \sin \theta \ k)$

$(t, \theta) \in [a, b] \times [0, 2\pi]$ \hspace{1cm} (3)

where the scalar continuous function $r(t)$ represents the radius of the tube.

Dickson's bottle is built up by that scheme, except for the choice of the moving pair of vector fields, which is not orthogonal to the curve. Here is its parametrization

$$\begin{align*}
Kb3(u, v) &= \begin{cases}
  x = \begin{cases}
    6 \cos u (1 + \sin u) + 4(1 - \frac{1}{2} \cos u) \cos u \cos v & \text{for} & 0 \leq u \leq \pi \\
    6 \cos u (1 + \sin u) + 4(1 - \frac{1}{2} \cos u) \cos (v + \pi) & \text{for} & \pi < u \leq 2\pi
  \end{cases} \\
  y = \begin{cases}
    16 \sin u + 4(1 - \frac{1}{2} \cos u) \sin u \cos v & \text{for} & 0 \leq u \leq \pi \\
    16 \sin u & \text{for} & \pi < u \leq 2\pi
  \end{cases} \\
  z = 4(1 - \frac{1}{2} \cos u) \sin v
\end{cases}
\end{align*} \hspace{1cm} (4)
Parametrization (4) defines two distinct tubes, the first one being built up on a frame which moves along the central curve remaining parallel to the \(xz\) plane, the second one connecting the two ends of the first tube through a rotation of \(\pi\) of the moving frame. The union of these two parts turns out to be a well-looking object (Fig. 4), which renders properly the idea outlined by F. Klein.

![Figure 4: Klein bottle according to S. Dickson’s definition; on the left, the central curve.](image)

Note that the curve implemented in this construction

\[
\alpha(t) : \begin{cases} 
x = 6 \cos t(1 + \sin t) \\
y = 16 \sin t
\end{cases}
\]  

is a *piriform*, a well-known curve (see for example [3] or [4]) whose parametrization is

\[
\text{piriform}(t) : \begin{cases} 
x = a(1 + \sin t) \\
y = b \cos t(1 + \sin t)
\end{cases}
\]  

It can be easily proven, and also guessed by looking at Figure 4, that (4) does not define an immersion, as the two tubes do not meet tangentwise along the common boundaries. And, of course, it would be better to get the whole surface as the image of a single parametrization, with no use of inequalities as in (4). This has been done by M. Trott, who follows closely the scheme defined by (3) and puts some constraints on the central curve (that he calls *directrix*) and on the radius, that can be resumed in the following:

\[
i) \quad \alpha(a) = \alpha(b) \\
ii) \quad \alpha'(a) = -\alpha'(b) \\
iii) \quad r(a) = r(b) \\
iv) \quad r'(a) = r'(b) = \pm\infty
\]  

Conditions i), ii) and iii) mean that the two ends of the tube must be coincident, while iv)
means that they must meet tangentwise. The curve and the radius he uses are

\[ \beta(t) = \left( \frac{1}{t^4 + 1}, \frac{t^2 + t + 1}{t^4 + 1} \right) \quad t \in (-\infty, +\infty) \]

\[ r(t) = \frac{84t^4 + 56t^3 + 21t^2 + 21t + 24}{672(1 + t^4)} \]

and the resulting image is shown in Figure 5. Equations (8) define an immersion, but the resulting shape is somehow edgy, because of the choice of a directrix whose curvature has a non-smooth behavior. Moreover, as \( t \) ranges on an open interval, when one tries to plot the surface there is a missing strip in correspondence of a neighborhood of the cusp (Trott uses \( t \in [-20, +20] \) in his plots).

Figure 5: Klein bottle according to M. Trott’s definition; on the left, the central curve.

3 A new description

Starting from the two constructions described in the previous paragraph, it is natural trying to construct a new surface by taking the best features from both: the beautiful and symmetric directrix of the first one and the rigorous geometric scheme of the second one. In order to use the piriform as a directrix for our tube, we re-parametrize it to make it start and end at the cusp:

\[ \gamma(t) = \begin{cases} 
\alpha(1 - \cos t) \\
\beta \sin t(1 - \cos t) \\
t \in (0, 2\pi)
\end{cases} \]

A suitable radius, which satisfies iii) and iv) of (7), is for example

\[ r(t) = c - d(t - \pi) \sqrt{t(2\pi - t)} \]
Parameters $c$ and $d$ affect respectively the radius of the whole tube and the difference between its minimum and maximum value. The resulting plot, with $(a, b, c, d) = (20, 8, \frac{13}{5}, \frac{3}{2})$ and $(t, \theta) \in (0, 2\pi) \times [0, 2\pi]$ is shown in Figure 6.

Figure 6: The piriform curve and a tube around it: an immersion of the Klein bottle in $\mathbb{R}^3$.

4 Some remarks

Although, in our opinion, the described result is rather satisfactory, there are some facts we want to point out. First, the parametrization of our surface, extensively written, has a long and complicated expression. Secondly, similarly to what happens with Trott’s parametrization, the image of the immersion fails to be closed because it misses a circle at the cusp of the directrix, as $\|\gamma'\|$ vanishes at $t = 0$ and $t = \pi$, while the used scheme needs $\|\gamma'\|$ to be nonzero everywhere. A way to get rid of this issue is using one half of the Dumbbell curve (see 4) as a directrix. This is a famous sextic curve which also has the following parametrization:

$$\text{dumbbell}(t) : \begin{cases} x = \sin t \\ y = \sin^2 t \cos t \end{cases} \quad t \in [0, 2\pi]$$

If $t$ ranges in $I = [0, \pi]$, one obtains a curve that satisfies the first three conditions of 7 and whose tangent vector is well-defined for all $t \in I$, so it is possible to use a closed rectangle as a domain for the immersion, obtaining a closed image. By using a stretched Dumbbell curve
\[ \alpha(t) = (5 \sin t, 2 \sin^2 t \cos t) \] as directrix and \[ r(t) = \frac{1}{2} - \frac{1}{30} (2t - \pi) \sqrt{2t(2\pi - 2t)} \] as radius function, with \( t \in [0, \pi] \), we obtain another example of immersion of the Klein bottle, more suitable to be plotted with a computer (see Figure 8).

Figure 8: Some views of the Klein bottle as a tube around Dumbbell curve. On the right, a high definition plot.

## 5 Conclusion

We defined two immersions of the Klein bottle in \( \mathbb{R}^3 \) in the shape outlined by F. Klein in 1882, with a reasonably good appearance. The mathematical expression of both is still too complicated and far from the elegance of versions like (1) and (2). The present work is intended to be a further step towards an immersion of this famous surface in \( \mathbb{R}^3 \) which we would like to call “canonical” from both a mathematical and an aesthetic point of view.
References

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