Photons with half-integral spin as q-Fermions

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Abstract

The recently discovered 'light (photons) with half-integral spin' is interpreted as q-Fermions proposed by us in 1991, as these q-Fermions satisfy q-deformed anti-commutation relations (pertaining to spin half) and have the property that more than one q-Fermion can occupy a given quantum state. In this article, in view of the recent discovery, we recall the construction of q-Fermions and give the statistical properties of q-Fermion gas, based on our preprint in 1992.

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Recently, Ballantine, Donegan and Eastham [1] reported the discovery of 'light (photons) with half-integral spin'. This new form of light is interpreted as q-Fermions proposed by Parthasarathy and Viswanathan [2] in 1991. The q-Fermions satisfy q-deformed anti-commutation relations pertaining to spin half and have the novel property that more than one q-Fermion can occupy a given state. In the limit $q \to 1$, we recover ordinary fermions. With $q \neq 1$, the q-Fermion system could be the right description of the newly discovered light with half-integral spin. We [3] have given the q-deformed statistics of q-Fermions, obtaining the number density, chemical potential and the specific heat $C_V$. In view of the recent discovery, we give the statistical properties of q-Fermion gas, essentially taken from the preprint [3].

The result in [1] shows, in two dimensions photons can have a half-integer total angular momentum, comprising of unequal mixture of spin and orbital contributions and demonstrate the half-integer quantization of this total angular momentum. They involve beams of light propagating in a particular direction when the full rotational symmetry is not present. The restricted symmetry leads to a new form of total angular momentum which has a half-integer; that is 'fermionic' spectrum. This half-integer spectrum shows that for light, reduced dimensionality allows for new form of quantization. The half-integer quantization, which they demonstrate through noise measurement, implies fermionic exchange statistics. A description of such photons could be provided by the q-Fermions and q-Fermion statistics. In our preprint [3], we have demonstrated that q-deformed version is not merely a mathematical construct but perhaps effectively takes into account the interactions. These interactions are contact interactions and either the surroundings or the system itself generate these interactions. It is in this sense that the restricted symmetry is effectively taken into account by the q-deformation. While the q-deformed Boson description maintains the commutation relations though q-deformed, to describe half-integer total angular momentum of the new photons, q-Fermion description is appropriate. Nevertheless, this q-Fermion describing the new photon cannot be a fundamental particle, but particle with interactions included, as if it is a quasi-particle.

**q-Fermion Algebra - A brief Review**

A q-deformed non-trivial (in the sense that there is another version of
q-fermion algebra [4] which can be transformed to ordinary fermion algebra and hence trivial) fermion algebra proposed by us [2] is given by

\[ ff^\dagger + \sqrt{q} f^\dagger f = q^{-\frac{N}{2}}, \]

\[ [N, f] = -f, \]

\[ [N, f^\dagger] = f^\dagger, \]

\[ f^2 \neq 0 ; \ (f^\dagger)^2 \neq 0, \]

where \( N \) is the q-Fermion number operator \( \neq f^\dagger f \). The orthonormal \( n \) q-Fermion state is defined by

\[ |n\rangle_F^q = \frac{1}{([n]_q^F)^{\frac{1}{2}}} (f^\dagger)^n |0\rangle, \]

where the vacuum \( |0\rangle \) is defined as \( f|0\rangle = 0 \) and

\[ [n]_q^F = [n]_q^F [n-1]_q^F \ldots [2]_q^F [1]_q^F, \]

\[ [n]_q^F = q^{\frac{(n-1)}{2}} \sum_{k=0}^{(n-1)} (-q)^k = \frac{\sqrt{q}}{1+q} (q^{-\frac{N}{2}} - (-1)^n q^\frac{N}{2}). \]

This algebra describes q-fermions such that any number of q-fermions can occupy a given state for \( 0 < q < 1 \). Here \( q \) is taken to be real. When \( q = 1 \), all states other than the vacuum and one particle state collapse, thereby recovering Pauli principle. Subsequently, Viswanathan, Parthasarathy and Jagannathan [5] have constructed coherent states for q-fermions. In it, we used transformed q-fermion operator as \( f = q^{-\frac{N}{4}} F, \ f^\dagger = F^\dagger q^{-\frac{N}{4}}, \) so that we find equivalent algebra

\[ FF^\dagger + qF^\dagger F = 1, \]

\[ FF^\dagger - F^\dagger F = (-q)^N, \]

\[ [N, F] = -F, \]

\[ [N, F^\dagger] = F^\dagger, \]

\[ F^2 \neq 0 ; \ (F^\dagger)^2 \neq 0, \]

from which we see

\[ FF^\dagger = [N + 1]_F; \ F^\dagger F = [N]_F, \]
where
\[ [n]^f = \frac{1 - (-q)^n}{1 + q}. \] (6)

The superscript \( 'f' \) is used to indicate that we are dealing with q-fermion. Here \( N \) is the number operator, \( \neq F^\dagger F \). The Fock space \(|n\rangle\) can be constructed as
\[
F|n\rangle = ([n]^f)^\frac{1}{2}|n - 1\rangle, \\
F^\dagger|n\rangle = ([n + 1]^f)^\frac{1}{2}|n + 1\rangle.
\] (7)

As before, the vacuum is defined by \( F|0\rangle = 0 \). For \( q \neq 1 \), more than one q-Fermion can occupy a given state and only when \( q = 1 \), all states other than \(|0\rangle \) and \(|1\rangle \) collapse, recovering Pauli principle. The above q-fermion algebra (either in terms of \( f \) or \( F \)) is non-trivial as it cannot be transformed to ordinary fermion algebra. Using (6), we find
\[
[n + 1]^f + q[n]^f = 1.
\] (8)

**Many q-Fermion system - q-Fermion Gas**

As a model Hamiltonian for many q-fermion system, we take
\[
H = \sum_k (E_k - \mu)N_k, \] (9)
where \( E_k \) is the kinetic energy for q-fermion of momentum \( k \). We assume that q-fermion operators with different momenta commute. We define the thermal average of \( F^\dagger F \) as
\[
\langle F^\dagger_k F_k \rangle = \frac{Tr(\exp(-\beta H)F^\dagger_k F_k)}{Tr \exp(-\beta H)}, \] (10)
where $\beta = \frac{1}{K T}$, $K$ the Boltzmann constant. Using the cyclic property of the trace and (4), it follows

$$\langle F_k^\dagger F_k \rangle = \exp(-\beta (E_k - \mu)) \langle F_k F_k^\dagger \rangle.$$  \hspace{1cm} (11)

Using (8), we find

$$\langle [N_k]^f \rangle = \frac{1}{\exp(\beta(E_k - \mu)) + q},$$  \hspace{1cm} (12)

which is the distribution function for $q$-fermions. It is noted that when $T \to 0$, for $E_k > \mu$, $\langle [N_k]^f \rangle \to 0$; for $E_k < \mu$, $\langle [N_k]^f \rangle \to \frac{1}{q}$ and for $E_k = \mu$, $\langle [N_k]^f \rangle \to \frac{1}{1+q}$. So upto $E_k = \mu_0 < \mu$, the levels are filled and empty when $E_k > \mu$. Further from (5) and (11), we see that

$$\frac{[N_k]^f}{[N_k + 1]^f} = e^{-\beta(E_k - \mu)},$$  \hspace{1cm} (13)

From (6), in the limit $q \to 1$, $[n] = n$, $[n + 1] = 1 - n$, $n = 0, 1$, and so (13) gives the familiar Fermi-Dirac distribution when $q = 1$.

The distribution function (12) for many $q$-Fermion system derived by us in Ref.3 (in 1992) has been the subject of study by others, Narayana Swamy [6] in 2006, Algin and Senay [7] in 2012, Algin, Irk and Topcu [8] in 2015. The expression (12) can be solved for $N_k$ as

$$N_k = \frac{1}{|\ell n q|} |\ell n \left( \frac{|e^{\eta_k} - 1|}{e^{\eta_k} + q} \right)|,$$  \hspace{1cm} (14)

where $\eta_k = \beta (E_k - \mu)$, agreeing with [6], [7].

The Hamiltonian (9) allows us to evaluate the number density $\rho$ and the internal energy $u$ for the $q$-Fermion gas, by going from discrete sum to integral. Following Feynman [9], we consider

$$I = \int_0^\infty \frac{g(E) dE}{\exp(\beta(E - \mu)) + q},$$  \hspace{1cm} (15)
where \( g(E) = c\sqrt{E} \) for calculating the number density \( \rho \) and \( g(E) = cE\sqrt{E} \) for calculating the internal energy, \( c \) a constant. Splitting the integral as

\[
I = \frac{1}{q} \int_{0}^{\mu} g(E) dE - \frac{1}{q} \int_{0}^{\mu} \frac{g(E) dE}{1 + q \exp(-\beta(E - \mu))} + \int_{\mu}^{\infty} \frac{g(E) dE}{\exp(\beta(E - \mu)) + q},
\]

which can be easily verified, setting \( x = -\beta(E - \mu) \) in the second integral and \( x = \beta(E - \mu) \) in the third integral, we find

\[
I = \frac{1}{q} \int_{0}^{\mu} g(E) dE - \frac{1}{q\beta} \int_{0}^{\mu^3} \frac{g(\mu - \frac{x}{\beta}) dx}{1 + qe^x} + \frac{1}{\beta} \int_{0}^{\infty} \frac{g(\mu + \frac{x}{\beta}) dx}{e^x + q},
\]

For low enough temperatures \( g(\mu \pm \frac{x}{\beta}) \approx g(\mu) \pm \frac{x}{\beta} g'(\mu) \) and \( \int_{0}^{\mu^3} \to \int_{0}^{\infty} \). Then

\[
I = \frac{1}{q} \int_{0}^{\mu} g(E) dE - \frac{g(\mu)}{q\beta} \int_{0}^{\infty} \frac{dx}{1 + qe^x} + \frac{g'(\mu)}{q\beta^2} \int_{0}^{\infty} \frac{xdx}{1 + qe^x} + \frac{g(\mu)}{\beta} \int_{0}^{\infty} \frac{dx}{e^x + q} + \frac{g'(\mu)}{\beta^2} \int_{0}^{\infty} \frac{xdx}{e^x + q}.
\]

Now

\[
\int_{0}^{\infty} \frac{dx}{1 + qe^x} = \ln \left( \frac{1 + q}{q} \right) ; \int_{0}^{\infty} \frac{dx}{e^x + q} = \frac{1}{q} \ln(1 + q).
\]

Collecting \( g'(\mu) \) terms, we have

\[
\frac{g'(\mu)}{\beta^2} \left( \frac{1}{q} \int_{0}^{\infty} \frac{xdx}{1 + qe^x} + \int_{0}^{\infty} \frac{xdx}{e^x + q} \right),
\]

which can be evaluated as

\[
\frac{g'(\mu)}{q\beta^2} \left( \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right).
\]

Thus,

\[
I = \frac{1}{q} \int_{0}^{\mu} g(E) dE + \frac{g(\mu)}{q\beta} \ln q + \frac{g'(\mu)}{q\beta^2} \left( \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right).
\]
For finding the number density $\rho$, set $g(E) = c\sqrt{E}$ and then we find

$$\rho = \frac{2c}{3q} \mu^\frac{3}{2} + \frac{c}{q\beta} \ln q + \frac{c}{2q\beta^2 \sqrt{\mu}} \left( \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right). \quad (23)$$

From this, we see

$$\rho_{T=0} = \frac{2c}{3q} \mu_0^\frac{3}{2}. \quad (24)$$

Equating $\rho_{T=0} = \rho_{T\neq0}$ as required by number conservation, we find

$$\mu^\frac{3}{2} = \mu_0^\frac{3}{2} \left( 1 + \frac{3}{2\mu_0^2} \ln q - \frac{3}{4\beta^2\mu^2} \left[ \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right] \right)^{-1}. \quad (25)$$

Here we approximate $\mu$ by $\mu_0$ in the $(......)^{-1}$ and then

$$\mu \simeq \mu_0 \left( 1 - \frac{1}{\mu_0^2} \ln q - \frac{\pi^2}{12\mu_0^2\beta^2} + \frac{1}{\mu_0^2\beta^2}(\ln q)^2 \right). \quad (26)$$

The internal energy of q-fermion gas is evaluated by taking $g(E) = cE \sqrt{E}$ and we find

$$u = \frac{2c}{5q} \mu^\frac{5}{2} + \frac{c}{q\beta} \mu^\frac{3}{2} \ln q + \frac{3c}{2q\beta^2 \sqrt{\mu}} \left[ \frac{\pi^2}{6} + \frac{1}{2}(\ln q)^2 \right]. \quad (27)$$

Using the expression for $\mu$ above and after some steps, the internal energy becomes

$$u = u_0 + \gamma T^2, \quad (28)$$

where

$$u_0 = \frac{2c}{5q} \mu_0^\frac{5}{2},$$

$$\gamma = \frac{c\sqrt{\mu_0} K^2}{q} \frac{\pi^2}{6} + \frac{c\sqrt{\mu_0} K^2}{q} (\ln q)^2,$$

$$= \frac{c\sqrt{\mu_0} K^2 \pi^2}{6} \left( \frac{1}{q} + \frac{6}{\pi^2 q}(\ln q)^2 \right). \quad (29)$$
The pre-factor in $\gamma$ is the Feynman’s value and the correction is multiplicative parenthesis. Since $C_V = \frac{\partial u}{\partial T}$, we find for q-fermion gas

$$C_V = C^\text{Feynman}_V \left( \frac{1}{q} + \frac{6}{\pi^2 q} (\ln q)^2 \right).$$

(30)

In the limit $q = 1$, we recover Feynman’s value. The results in (23), (25), (26), (27) and (30) were derived by us in our preprint [3].

By measuring $C_V$ for q-fermion gas, $q$ can be determined which can then be used to find $\rho$. These expressions can be applied to the ‘newly discovered photon with half-integral spin’, if the $\rho$, $C_V$ are measured for this system. Further, the expression for $N_k$ in (14) can be used to plot the q-deformed statistical distribution function for q-Fermion gas for various $T$ as a function of $\beta(E - \mu)$ for values of $q < 1$. This plot is available from Fig.1 of [7] and can be used for the distribution function of the ‘new light with half-integer spin’.

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