COMPLEX HOROSPHERICAL TRANSFORM ON REAL SPHERE

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Abstract. We define a new integral transform on the real sphere which is invariant relative to the orthogonal group and similar to the horospherical Radon transform for the hyperbolic space. This transform involves complex geometry associated with the sphere.

In integral geometry on hyperbolic spaces and other noncompact symmetric spaces there are 2 versions of the Radon transform: geodesic and horospherical [GGG03]. The horospherical Radon transform has more essential connections with the harmonic analysis on these spaces than its geodesic analogue. The geodesic version of the Radon transform is well known on the sphere. It is the famous Minkowsky-Funk transform of integration along the big subspheres [GGG03]. Let us recall that this transform was discovered earlier than the Radon transform. We want in this note to construct the analogue of the horospherical transform on the sphere. At first glance it looks strange since there are no horospheres on the sphere, but we will show that such a transform exists if in the geometrical background we replace real horospheres by complex ones. We will follow the idea of [Gi00] which was developed for some pseudohyperbolic spaces, but, as we will see, gives an interesting construction on the sphere also. Usually, in integral geometry we integrate functions (or other analytic objects) along some submanifolds. In other words we integrate along a manifold δ-functions with supports on these manifolds. The idea is to replace these δ-functions with Cauchy kernels with singularities on similar complex submanifolds, and then instead of real submanifolds, we are interested in corresponding complex submanifolds which have no real points. Such complex horospheres exist for the real sphere and we can define the horospherical transform. Such applications of the complex geometry to analysis on real manifolds can give significant advantages; they are very much in the spirit of the classical geometry of XIX century.

Let \( S = S^{n-1} \) be the \((n-1)\)-dimensional sphere in \( \mathbb{R}^n \):

\[
\Delta(x) = x_1^2 + \cdots + x_n^2 = 1
\]

and \( \mathbb{C}S = \mathbb{C}S^{n-1} \) be its complexification in \( \mathbb{C}^n \): \( \Delta(z) = 1, z = x + iy \). Let \( \Xi \) be the cone \( \Delta(\zeta) = 0, \zeta = \xi + i\eta \neq 0 \). Let us consider \( \mathbb{C}_z^n \) and \( \mathbb{C}_\zeta^n \) as dual relative to the form

\[
\zeta \cdot z = \zeta_1 z_1 + \cdots + \zeta_n z_n.
\]
We will call complex horospheres sections $\mathcal{H}(\zeta)$ of the complex sphere $\mathbb{C}S$ by the isotropic hyperplane:

$$\mathcal{H}(\zeta) = \{ z \in \mathbb{C}S; \zeta \cdot z = 1 \}, \quad \zeta \in \Xi.$$  

We will also sometimes parameterize horospheres by homogeneous coordinates $(\zeta, p)$:

$$\mathcal{H}(\zeta, p) = \{ z \in \mathbb{C}S; \zeta \cdot z = p \}, \quad \zeta \in \Xi, p \in \mathbb{C}, p \neq 0,$$

where $\mathcal{H}(\lambda \zeta, \lambda p) = \mathcal{H}(\zeta, p), \lambda \neq 0$. We do not consider the degenerate horospheres with $p = 0$. The cone $\Xi$ in real coordinates is defined by the conditions

$$\Delta(\xi) = \Delta(\eta), \quad \xi \cdot \eta = 0.$$

**Lemma.** The horosphere $\mathcal{H}(\zeta)$ does not intersect the real sphere $S$ if and only if

$$0 < \Delta(\xi) = \Delta(\eta) < 1.$$

It is a simple direct computation in which it is convenient to use the rotation invariancy. Let us denote the domain in the cone $\Xi$ defined in Lemma, through $\Xi^+$. In homogeneous coordinates this condition is $0 < \Delta(\xi) < |p|^2$. $\Xi^+$ is a Stein submanifold in $\Xi$.

The boundary $\partial \Xi^+$ admits a natural fibering over $S$ on $(n-2)$-dimensional spheres

$$\gamma(x) = \{ \zeta = x + i\eta, \Delta(\eta) = 1, x \cdot \eta = 0 \}, x \in S.$$  

The corresponding horospheres intersect $S$ in one point $x$.

In this note we will assume that the functions $f \in C^\infty(S)$. Of course, it is possible to define the horospherical transform under very weak conditions and it would be interesting to consider different Paley-Wiener theorems for this transform. We will need some notations for differential forms. Let us denote through $[a_1, \ldots, a_n]$ the determinant of the matrix with the columns $a_1, \ldots, a_n$ some of which can be 1-forms. We expand such determinants from left to right and use the exterior product for the multiplication of 1-forms. Such a determinant with identical columns can differ from zero: $[dx_1, \ldots, dx_n] = n! dx_1 \wedge \cdots \wedge dx_n$. We will write $a^{(k)}$ if a column $a$ repeats $k$ times.

We will use the interior product of forms $\phi \lhd \psi$. It is a form $\alpha$ such that $\phi \wedge \alpha = \psi$; its restriction on a submanifold where $\phi = 0$ is uniquely defined. If $\phi = df$ where $f$ is a function then $df \lhd \psi$ up to a multiplicative constant is the residue of $\psi/f$ on $\{ f = 0 \}$.

Let us define the horospherical transform as

$$\hat{f}(\zeta, p) = \int_S \frac{f(x)}{\zeta \cdot x - p} [x, dx^{(n-1)}], \quad \zeta \in \Xi^+,$$

where $[x, dx^{(n-1)}] = 2d(\Delta(x)) [dx^{(n)}] = (n-1)! \sum_{1 \leq f \leq n} (-1)^{(j-1)} x_j \wedge i_{i \neq j} dx_i$ is the invariant measure on $S$.

We have

$$\hat{f}(\lambda \zeta, \lambda p) = \lambda^{-1} \hat{f}(\zeta, p).$$

Let $\hat{f}(\zeta) = \hat{f}(\zeta, 1)$. Apparently, the transform is well defined and $\hat{f}(\zeta)$ is the holomorphic function in $\Xi^+$. Also under our conditions on $f$ boundary values of $\hat{f}$ are well defined. Our aim is to find an inversion formula for the horospherical transform $f \rightarrow \hat{f}$. 


Theorem. There is an inversion formula

\[ f(x) = \frac{n-1}{2(2\pi)^{n-1}} \int_{\gamma_1(x)} \mathcal{L}_p f(x + i\eta, p)|_{p=1}[x, \eta, d\eta^{(n-2)}], \]

\[ \mathcal{L}_p = \frac{n-2}{p} \frac{\partial^{(n-3)}}{\partial p^{(n-3)}} - 2 \frac{\partial^{(n-2)}}{\partial p^{(n-2)}}. \]

Here we take the boundary values of \( \hat{f} \) on \( \partial \Xi \) and integrate along cycles \( \gamma_1(x) \).

The proof of this theorem follows the ideas of [Gi95, Gi00] (see also Ch. 5 in [GGG03]). We will construct the decomposition on plane waves on the sphere. Namely, let us consider the differential form

\[ \text{intersection of } \zeta \text{ on } S \]

\[ \Delta(\zeta, u, x) = \frac{f(u)}{(\zeta \cdot (u - x))^{n-1}} [u, du^{(n-1)}] \wedge [u + x, \zeta, d\zeta^{(n-2)}], \quad u \in S, \zeta \in \mathbb{C}^n \setminus \{0\}. \]

Here \( x \in S \) is a fixed point. It is the analogue for the sphere \( S \) of the form in decomposition on plane waves in \( \mathbb{R}^n \). It is important that we permit complex \( \zeta \).

The crucial point is that this form is closed. It is the direct consequence of a technical lemma [Gi95, GGG03]:

\[ d(a(\zeta), \zeta, d\zeta^{(n-2)}) = -\frac{1}{n-2} \sum_{1 \leq j \leq n} \frac{\partial a_j(\zeta)}{\partial \zeta_j}[\zeta, d\zeta^{(n-1)}]. \]

The application of this formula for the column-function \( a(\zeta) = \frac{\zeta \cdot u + x}{(\zeta \cdot (u - x))^{n-1}} \) shows the closeness of \( \kappa \) on \( \zeta \) (since \( \Delta(u) = \Delta(x) = 1 \)); it is closed on \( u \) since \( [u, du^{(n-1)}] \) has the maximal degree on \( S \).

The form \( \kappa \) is the source of inversion formulas. We integrate it on \( \Gamma(x) = S \times \gamma(x) \) where \( \gamma(x) \) is a cycle in \( \mathbb{C}^n \). Let us now define \( \hat{f}(\zeta, p) \) for any \( (\zeta, p) \) such that the intersection of \( S \) by the hyperplane \( \zeta \cdot z = p \) is empty (we temporarily removed the condition \( \Delta(\zeta) = 0 \)). If \( (\zeta, p = \zeta \cdot x) \) in the integral satisfy these conditions, then the integral makes sense and we can interpret the integrand as a differential operator of \( \hat{f}(\zeta, p) \).

Let us start from the cycle \( \gamma_0(x) \) of real \( \zeta = \xi \) such that \( \xi \cdot x = 0 \). We will pick up the cycle such that it intersects each ray of 0 in one point (we can take as \( \gamma_0(x) \) the intersection of the sphere \( \Delta(\xi) = 1 \) by the hyperplane \( \xi \cdot x = 0 \)). Of course the integral does not depend on the specifics of such a choice. Let us restrict \( \kappa \) on the corresponding cycle \( \Gamma_0(x) \) and we will interpret it as the form

\[ \kappa_x[f]|_{\Gamma_0} = \frac{f(u)}{(\zeta \cdot u - x - i0)^{n-1}} [u, du^{(n-1)}] \wedge [x + u, \xi, d\xi^{(n-2)}]. \]

Here for the regularization we use the distribution \( (\zeta \cdot u - i0)^{-(n-1)} \). Let us present this form as the sum of 2 forms \( \kappa_1 + \kappa_2 \) corresponding to \( x, u \) in \( x + u \).

If \( f \) is even on \( S \) then we have

\[ \int_{\Gamma_0(x)} \kappa_1 = \frac{2(2\pi)^{n-1}}{(n-1)!} f(x). \]
It is just the inversion of the Minkowski-Funk transform (integration of even functions on \( S \) along sections by \( \xi \cdot u = 0 \)). This inversion formula is the specialization of the inversion of the projective Radon transform \([GGG03]\) (if one were to present the projective space by pairs of antipodal points) and equivalent to the inversion of the affine Radon transform.

This integral is equal to zero for odd functions; the corresponding integral of \( \kappa_2 \) is equal to zero for even functions, but will reproduce the odd ones. To see it (using the rotation invariancy) it is sufficient to consider one point \( x = (1, 0, \ldots, 0) \). Then on \( \gamma_0(x) \) we have \( \xi_1 = 0 \) and the integral of \( \kappa_2 \) for an odd function \( f \) coincides with the integral of \( \kappa_1 \) for the even function \( u_1 f(u) \). As the result, we have for all functions on \( S \)

\[
\int_{\Gamma_0(x)} \kappa_x[f] = \frac{2(2\pi i)^{n-1}}{(n-1)!} f(x).
\]

Now we want to integrate \( \kappa \) on the cycle \( \Gamma_1(x) \) corresponding to the cycle \( \gamma_1(x) \) of horospheres passing through \( x \) (see above cycles on \( \partial\Xi_+ \)). First, we remark that the form \( \kappa \) with \( \Delta(\zeta) = 0 \) (corresponding to horospheres) can be transformed in

\[
\tilde{\kappa}_x[f] = \frac{f(u)(\zeta \cdot (u + x))}{(\zeta \cdot x)(\zeta \cdot (u - x))^{n-1}} [u, du^{(n-1)}] \land [x, \zeta, d\zeta^{(n-2)}], \quad \Delta(\zeta) = 0.
\]

To see it let us recall that on the cone \( \Xi \) if \( \lambda \cdot \zeta \neq 0 \) then the form

\[
\frac{[\lambda, \zeta, d\zeta^{(n-2)}]}{\lambda \cdot \zeta}
\]

is independent of \( \lambda \). This is simple to see by transforming the determinant using the condition \( \Delta(\zeta) = 0 \), but the reason for this a phenomenon is that this form, up to a constant factor, is the residue of the form \( [\zeta, d\zeta^{(n-1)}]/\Delta(\zeta) \) on \( \Xi \) for any \( \lambda \). A special case of this independence gives

\[
[x + u, \zeta, d\zeta^{(n-2)}] = \frac{\zeta \cdot (x + u)}{\zeta \cdot x} [x, \zeta, d\zeta^{(n-2)}]
\]

and we transformed the form \( \kappa \) in \( \tilde{\kappa} \). This transformation is important, since we want maximally eliminate \( u \) out of the integrand. Now \( u \) participates only trough \( p = \zeta \cdot u \).

The direct computation gives that

\[
\frac{1}{\zeta \cdot x} \int_{\mathcal{S}} f(u)(\zeta \cdot (u + x))^{n-1} [u, du^{(n-1)}] = \frac{(-1)^{n-3}}{(n-2)!} \mathcal{L}_p \hat{f}(\zeta \cdot x)
\]

and since on the cycle \( \gamma_1(x) \) we have \( \zeta = x + i\eta, x \cdot \eta = 0, \zeta \cdot x = 1 \), we obtain

\[
\int_{\Gamma_1(x)} \kappa_x[f] = i^{n-1} \int_{\gamma_1(x)} \mathcal{L}_p \hat{f}(x + i\eta, 1) [x, \eta, d\eta^{n-2}].
\]

To finish the proof of Theorem it is sufficient to prove that the inversion formula for Radon’s cycle \( \Gamma_0(x) \) holds also for the horospherical cycle \( \Gamma_1(x) \). Since the form
κ is closed it is sufficient to construct a homotopy of the cycle γ₀(κ) to the cycle γ₁(κ) through such ζ that the section of CS by the hyperplane ζ · z = ζ · x − iε has no real points.

It is very simple to construct such a deformation explicitly. Using the invariancy it is sufficient to consider only one point x. Let x = (1, 0 . . . , 0). Then for ξ ∈ γ₀(κ) we have ξ₁ = 0; let ξ = (0, ˜ξ). Consider cycles

\[ γ_δ = \{ζ(˜ξ, δ) = (iδ \sqrt{Δ(˜ξ)}, ˜ξ)\}, \quad 0 ≤ δ ≤ 1. \]

We have γ₀ = γ₀(κ); γ₁ = γ₁(κ) for this x. Then all complex hyperplanes ζ(˜ξ) · z = p(ξ, δ) = iδ \sqrt{Δ(ξ)} intersect the real sphere S only on the point x and if one were to replace this p on p(ξ, δ) + iε, ε > 0 then the hyperplane will have no intersections at all. As a result, we can define \( f(ζ(ξ, δ), p(ξ, δ)), 0 ≤ δ ≤ 1 \) as the boundary values when ε → 0. So we constructed the desirable deformation and Theorem is proved.

The horospherical transform \( f → ˆf \) commutes with rotations (the group SO(n)).

The domain Ξ is not invariant relative to this action, but the action of the circle \( \mathbb{T} = \{|ζ| = 1\} \) preserves \( Ξ_+ \). The decomposition of holomorphic functions on \( Ξ_+ \) in Fourier series on T corresponds to the decomposition on subspaces of homogeneous polynomials and it is the decomposition on irreducible representations of SO(n):

**Proposition.** The horospherical transform \( f → ˆf \) commutes with the actions of \( SO(n) \) and intertwines subspaces of spherical polynomials on S and homogeneous polynomials on \( Ξ \).

This representation of spherical polynomials through homogeneous polynomials on the complex cone Ξ goes back to Maxwell’s formulas for spherical polynomials [B53], p.251. This inverse horospherical transform is in a sense the generating function for Maxwell’s formula. Several components of this construction admit generalizations. If one were to replace S by any surface \( \{Φ(x) = 1\} \) where Φ is a polynomial and \( Φ(u) − Φ(x) = \sum(u_j − x_j)φ_j(u, x) \) then the form

\[ \kappa_x[f] = \frac{f(u)}{(ζ · (u − x))^{n−1}}[u, du^{(n−1)}] ∧ [φ(u, x), ζ, dζ^{(n−2)}] \]

is closed, but it is unclear how to build interesting cycles different from Radon’s cycle. For some examples with non definite quadratic forms cf. in [Gi00].

The most natural development of this example is the construction of horospherical transform on arbitrary compact symmetric spaces using complex horospheres which is connected with harmonic analysis on such spaces through Fourier series. This will be the subject of our future publication.

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