We study the dispersive behaviors of two-particles Schrödinger and wave equations in the Aharonov-Bohm field. In particular, we prove the Strichartz estimates for Schrödinger and wave equations in this setting. The key point is to construct spectral measure of Schrödinger operator with an Aharonov-Bohm type potential in \( \mathbb{R}^4 \). As applications, we finally prove a scattering theory for the nonlinear defocusing subcritical two-particles Schrödinger equation with Aharonov-Bohm potential.

Key Words: Aharonov-Bohm potential, Decay estimates, Strichartz estimate, Scattering theory.

AMS Classification: 42B37, 35Q40, 35Q41.

1. Introduction

In this paper, we continue our program [6, 15, 18] studying the dispersive equations in Aharonov-Bohm field. The Aharonov-Bohm magnetic potential reads

\[
A_B : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2, \quad A_B(x) = \alpha \left( \frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \quad \alpha \in \mathbb{R}, \quad x = (x_1, x_2)
\]

so that the Hamiltonian becomes

\[
H_A = \left( -i\nabla + \alpha \left( \frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right) \right)^2.
\]

The vector potential \( A_B(x) \) generates a \( \delta \)-type magnetic field

\[
B = \nabla \times A_B(x) = 2\pi \alpha \delta,
\]

which was initially studied by Aharonov and Bohm in [1] to show the significance of electromagnetic vector potentials in quantum mechanics. The Aharonov-Bohm effect predicted in [1] is one of the most interesting and intriguing phenomena of quantum physics. This effect occurs when electrons propagate in a domain with a zero magnetic field but with a nonzero vector potential \( A_B \), see [26] and the references therein. In another typical cosmic-string scenarios observed by Alford and Wilczek [2], the fermionic charges can be non-integer multiples of the Higgs charges. As the flux is quantized with respect to the Higgs charge, this will lead to a non-trivial Aharonov-Bohm scattering of these fermions. From the mathematical points, we refer to [11, 12] and references therein for an overview of the spectral theory of this Hamiltonian in Aharonov-Bohm magnetic field. This Hamiltonian model is doubly critical, because of the scaling invariance of the model and the singularities of the potentials, which are not locally integrable. In \( \mathbb{R}^2 \), a generalization of \( H_A \) was considered in [11, 12] in which the dispersive estimates were proved for
Schrödinger equation. Recently, in [15], the authors have proved the Strichartz estimates for wave and Klein-Gordon equations in the Aharonov-Bohm magnetic fields. It worths mentioning that the wave and Klein-Gordon equations are lack of pseudoconformal invariance which plays an important role in [11] for Schrödinger equation, hence we have to use a new method in [15].

In the present paper, we consider the Hamiltonian for magnetic many-particles Schrödinger operator arising in the Hall effect [22, 25]. In particular, we recall the Hamiltonian for magnetic multi-particle Dirichlet forms with Aharonov-Bohm type in [20]. Let \( \vec{x}_j = (x_{j1}, x_{j2}) \in \mathbb{R}^2, j = 1, 2, \ldots, N \) and let

\[
F_j = \alpha \left( -\sum_{k \neq j} \frac{x_{j2} - x_{k2}}{r_{jk}^2} - \sum_{k \neq j} \frac{x_{j1} - x_{k1}}{r_{jk}^2} \right), \quad r_{jk}^2 = \sum_{i=1}^2 (x_{ji} - x_{ki})^2, \tag{1.3}
\]

then the Hamiltonian is given by

\[
H_{A,N} = \sum_{j=1}^N (-i\nabla_j + F_j)^2, \tag{1.4}
\]

which is singular in the set

\[
S = \bigcup_{j \neq k} S_{j,k},
\]

where for all \( j, k \in \{1, 2, \ldots, N\}, j \neq k \),

\[
S_{j,k} = \{ x = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) \in \mathbb{R}^{2N} = \mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 : \vec{x}_j = \vec{x}_k \}. \tag{1.5}
\]

We aim to study the dispersive behaviors of the Schrödinger equation

\[
\begin{cases}
i\partial_t u + H_{A,N} u = 0, \\
u(0,x) = u_0,
\end{cases}
\quad (t,x) \in \mathbb{R} \times \mathbb{R}^{2N} \setminus S,
\quad x \in \mathbb{R}^{2N},
\tag{1.6}
\]

and the wave equation

\[
\begin{cases}
\partial_{tt} u + H_{A,N} u = 0, \\
u(0,x) = f, \quad \partial_t u(0,x) = g,
\end{cases}
\quad (t,x) \in \mathbb{R} \times \mathbb{R}^{2N} \setminus S,
\quad x \in \mathbb{R}^{2N}. \tag{1.7}
\]

The research on the dispersive and Strichartz estimates of the wave and Schrödinger propagators with one single potential has a long history, we refer to [28] and the survey [29] and the references therein. From the physical perspective, it is natural to ask whether similar estimates hold in the presence of interaction potentials. In the \( N \)-particles interaction case, there is much fewer result than single case. It is known that the local-in-time dispersive and Strichartz estimates follow from the kernel in [16] with a large class of potential interactions. Recently, Hong [19] has obtained the global-in-time Strichartz estimates but with “small” interaction potentials. Chong, Grillakis, Machedon and Zhao [11] proved the global-in-time Strichartz estimates for the equation \( (i\partial_t - \Delta_x - \Delta_y + \frac{1}{N} V_N (x - y)) u(t, x, y) = F \) in mixed coordinates norm \( L_t^q(\mathbb{R}; L_x^r L_y^s(\mathbb{R}^n \times \mathbb{R}^n)) \). To the best of our knowledge, in the \( N \)-particles case, there is no result about the decay and Strichartz estimates for [10] and [17]. The two main obstacles to prove the decay estimates arise from the scaling critical Aharonov-Bohm potential and interactions between the many particles. Fortunately, in the special two-particles case \( N = 2 \), which is the simplest case, the Hamiltonian \( H_{A,2} \) can be reduced to a one-particle Hamiltonian \( L_A \) in [2, 4] below but with more singular Aharonov-Bohm type potential in higher
dimensions. Because of this, we can obtain the decay estimates and Strichartz estimates in the two-particles case.

Our main results about the Strichartz estimates are as follows.

**Theorem 1.1.** Let $N = 2$ and let $u(t,x)$ be a solution of Schrödinger equation \([1.10]\), then there exists a constant $C$ such that
\[
\|u(t,x)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C\|u_0\|_{L^q_x(\mathbb{R}^4)},
\]
where
\[
(q,r) \in \Lambda_0^N := \left\{ (q,r) \in [2,\infty] \times [2,\infty) : \frac{2}{q} = 4\left(\frac{1}{2} - \frac{1}{r}\right) \right\}.
\]

**Remark 1.2.** By using dual argument, the endpoint Strichartz estimates \([1.8]\) with $(q,r) = (2,4)$ gives the inhomogeneous Strichartz estimates
\[
\left\| \int_0^t e^{i(s-t)H_{A,2}} F(s) ds \right\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C\|F\|_{L^q_t(L^r_x(\mathbb{R}^4))}.
\]
Then by using the argument in \([21, \text{Remark 8.8}]\), we can obtain the uniform resolvent estimate
\[
\|(H_{A,2} - z)^{-1}\|_{L^q_x(\mathbb{R}^4) \rightarrow L^r_x(\mathbb{R}^4)} \leq C,
\]
where the constant $C$ is independent of $z \in \mathbb{C} \setminus \mathbb{R}^+$. 

**Remark 1.3.** It would be interesting to generalize the result to $N$-particles with $N \geq 3$, but the method of this paper is not available. Compared with the result of \([19]\), here we consider the scaling critical magnetic interaction potentials and the interaction between potentials may be larger. In contrast to the Strichartz estimates of \([19]\), the dispersive and Strichartz estimates \([1.8]\) are available for the norm of $L^q_t(L^r_x(\mathbb{R}^4))$ but not for the mixed norm of $L^q_t(\mathbb{R}; L^r_x(\mathbb{R}^2))L^2(\mathbb{R}^2)$.

**Theorem 1.4.** Let $N = 2$ and let $u(t,x)$ be a solution of wave equation \([1.17]\), then there exists a constant $C$ such that
\[
\|u(t,x)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C\left(\|f\|_{H^s(\mathbb{R}^4)} + \|g\|_{H^{s-1}(\mathbb{R}^4)}\right),
\]
where $s \geq 0$ and
\[
(q,r) \in \Lambda_0^N := \left\{ (q,r) \in [2,\infty] \times [2,\infty) : \frac{2}{q} \leq 3\left(\frac{1}{2} - \frac{1}{r}\right), s = 4\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} \right\}.
\]

By the change of variables
\[
\bar{y}_1 = \frac{x_1 - x_2}{\sqrt{2}}, \quad \bar{y}_2 = \frac{x_1 + x_2}{\sqrt{2}},
\]
and denoting $y = (\bar{y}_1, \bar{y}_2) \in \mathbb{R}^4$ and
\[
v(\bar{y}_1, \bar{y}_2) = u\left(\frac{\bar{y}_1 + \bar{y}_2}{\sqrt{2}}, -\frac{\bar{y}_1 - \bar{y}_2}{\sqrt{2}}\right) \quad \text{or} \quad u(x_1, x_2) = v(\bar{y}_1, \bar{y}_2),
\]
then, we derive
\[
H_{A,2} u(x_1, x_2) = \mathcal{L}_A v(\bar{y}_1, \bar{y}_2),
\]
where
\[
\mathcal{L}_A = \left(-i\nabla_y + A(y)\right)^2, \quad A(y) = \alpha\left(\frac{-y_{12}}{|y_{12}|^2}, \frac{y_{11}}{|y_{11}|^2 + |y_{12}|^2}, \frac{y_{11}}{|y_{11}|^2 + |y_{12}|^2}, 0, 0\right).
\]
Hence,\[\|e^{iH_{A,2}}u_0(x)\|_{L_t^4(\mathbb{R};L_x^s(\mathbb{R}^4))} \simeq \|e^{iL_A}v_0(y)\|_{L_t^4(\mathbb{R};L_x^s(\mathbb{R}^4))}.\] (1.15)

Thus, we can reduce to consider one-particle operator \(L_A\).

As applications, we further study the nonlinear subcritical Schrödinger equation
\[
\begin{cases}
i\partial_t u + H_{A,2}u = |u|^{p-1}u, & (t, x) \in \mathbb{R} \times (\mathbb{R}^4 \setminus S), \\
u(0, x) = u_0 \in H_A^1(\mathbb{R}^4), & x \in \mathbb{R}^4.
\end{cases}
\] (1.16)

By changing variables (1.13), it is equivalent to consider the nonlinear Schrödinger equation with Aharonov-Bohm potential in 4D
\[
\begin{cases}
i\partial_t u + L_A u = |u|^{p-1}u, & (t, x) \in \mathbb{R} \times (\mathbb{R}^4 \setminus \{0, 0, x_3, x_4\}), \\
u(0, x) = u_0 \in H_A^1(\mathbb{R}^4), & x \in \mathbb{R}^4,
\end{cases}
\] (1.17)

where \(u : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}, 2 < p < 3, \) and \(L_A = (-i\nabla_x + A(x))^2\).

We aim to establish the scattering theory for the nonlinear Schrödinger equation (1.17). In particular, when \(A = 0\), there are rich results about scattering theory for the nonlinear Schrödinger equation. For the defocusing and subcritical nonlinear Schrödinger cases, Ginibre-Velo [17] proved the scattering theory in energy space \(H^1(\mathbb{R}^n)\) with \(n \geq 3\) by using the classical Morawetz estimate. Later, in [8], the authors developed a powerful interaction Morawetz estimate and studied the scattering theory for the cubic defocusing Schrödinger equation in a space with regularity less than energy space. The interaction Morawetz estimate reads
\[
\|\nabla |\nabla|^\frac{p-2}{2} u\|^4_{L_t^4} \leq C\|u(0)\|^2_{L_x^4}\|u(t)\|^2_{L_x^\infty H_x^\frac{4}{p}},
\] (1.18)

if \(u\) solves a defocusing nonlinear Schrödinger equation without potentials. The interaction Morawetz estimate (1.18) is so powerful that the authors of [9] proved the scattering theory for energy-critical defocusing Schrödinger and the authors of [30] presented a different proof of the scattering for energy subcritical cases which simplified the argument of [17].

In the proof of the scattering theory of nonlinear Schrödinger equation, the Strichartz estimates and interaction Morawetz estimates are two fundamental tools. For our purpose of studying the long time behavior of solution to (1.17), in spirit of the papers mentioned above, it is natural to establish the Strichartz and the interaction Morawetz estimates for the linear and nonlinear Schrödinger equations (1.17) respectively. However, the Aharonov-Bohm magnetic potential is scaling-critical and the perturbation is non-trivial, so the situation becomes more complicated. In \(\mathbb{R}^4\), the magnetic field is \(B = \text{curl} A\) and the trapping component \(B_r = \frac{\tau}{|r|} \wedge B\) is an obstruction to the dispersion which was observed by [3] [14]. Fortunately, the trapping component \(B_r = 0\) which is generated by the special potential \(A\) in (2.5).

Hence we can obtain the classical Morawetz estimates from the virial identities derived by [14]. While for interaction Morawetz estimates, one needs a positivity of the quantity like \(B(x) \frac{x-y}{|x-y|^2}\) which seems impossible for any \(y\). Colliander et.al in [17] proved the interaction Morawetz estimates for the defocusing nonlinear Schrödinger with magnetic potentials. A key step is to use the method of [14] to bound the quantity involving \(B(x) \frac{x-y}{|x-y|^2}\) when the decay of the magnetic potential \(A\) is fast enough. However, for \(A\) considered here, the decay is not enough to satisfy their assumption [7] [14] (1.38), therefore the method breaks down. The authors in [34]
obtained the interaction Morawetz estimate for Schrödinger with Aharonov-Bohm potential in 2D, but assumed that the solution is radial. In 2D, $A \cdot \nabla f = 0$ if $f$ is a radial function. Hence, at present, we have to study the scattering theory via the classical Morawetz estimate by following the method in Ginibre-Velo \cite{17}.

Now we state our main theorems.

**Definition 1.5.** Let $\dot{H}^1_A(\mathbb{R}^4)$ be the completion of $C_c^\infty(\mathbb{R}^4 \setminus \{0\})$ with respect to the norm

$$ \|f\|_{\dot{H}^1_A(\mathbb{R}^4)} := \left( \int_{\mathbb{R}^4} |\nabla_A f(x)|^2 dx \right)^{\frac{1}{2}}, $$

where $\nabla_A := -i \nabla + A(x)$. Then the inhomogenous space is

$$ \dot{H}^1_A(\mathbb{R}^4) = \dot{H}^1(\mathbb{R}^4) \cap L^2(\mathbb{R}^4). $$

**Theorem 1.6.** Let $u_0(x) \in \dot{H}^1_A(\mathbb{R}^4)$. Then, there exists a unique global solution $u \in C(\mathbb{R}, \dot{H}^1_A(\mathbb{R}^4))$ to \eqref{1.17}. Moreover, the solution $u$ scatters in the sense that there exists $u_{\pm} \in \dot{H}^1_A(\mathbb{R}^4)$ such that

$$ \lim_{t \to \pm \infty} \left\| u(t, \cdot) - e^{it\nabla_A} u_{\pm} \right\|_{\dot{H}^1_A(\mathbb{R}^4)} = 0. $$

As mentioned above, in order to establish a scattering theory, we have to prove Strichartz estimates for our model. Fanelli-Vega \cite{14} obtained the Strichartz estimates for the wave equation with a non-trapping electromagnetic potential with almost Coulomb decay by establishing a family of virial-type identities for $n \geq 3$. However, the method is not applicable to magnetic Schrödinger equation because they only proved the weak dispersive estimate hold in $\dot{H}^\frac{1}{2}$ instead of $L^2$. Later in \cite{3}, they proved the Strichartz estimates for Schrödinger when the potentials are almost critical which do not include the potential \eqref{2.5}. Their method can prove the endpoint Strichartz estimates with derivatives, but can not cover the usual endpoint Strichartz estimates without derivatives, see \cite{3} (1.13), (1.14)].

To achieve the purpose of scattering theory in Theorem \ref{1.6}, we need the chain rules associated with $\mathcal{L}_A$ for differential operators of non-integer order, hence we can derive the desired nonlinear estimates. It is well known that the traditional Littlewood-Paley theory gives the proof of Leibniz (=product) and chain rules for differential operators of non-integer order. For example, if $1 < p < \infty$ and $s > 0$, then

$$ \|fg\|_{H^{s-p} \cap \dot{H}^{p_2} \cap L^p} \lesssim \|f\|_{H^{s-p_1} \cap \dot{H}^{p_3} \cap L^p} \|g\|_{L^{p_2} \cap \dot{H}^{p_4} \cap L^p} + \|f\|_{L^{p_3} \cap \dot{H}^{p_4} \cap L^p} \|g\|_{H^{s-p} \cap \dot{H}^{p_2} \cap L^p} $$

whenever $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. For a textbook presentation of these theorems and original references, see \cite{32}. Our strategy is to prove a boundedness of generalized Riesz transform in $L^p$

$$ \left\| (-\Delta)^\frac{s}{2} \mathcal{L}_A f \right\|_{L^p(\mathbb{R}^4)} \leq C \|f\|_{L^p(\mathbb{R}^4)} $$

and reversed Riesz transform

$$ \left\| \mathcal{L}_A^\frac{s}{2} (-\Delta)^\frac{s}{2} f \right\|_{L^p(\mathbb{R}^4)} \leq C \|f\|_{L^p(\mathbb{R}^4)}. $$

Hence we obtain

$$ \left\| (-\Delta)^\frac{s}{2} f \right\|_{L^p(\mathbb{R}^4)} \simeq \left\| \mathcal{L}_A^\frac{s}{2} f \right\|_{L^p(\mathbb{R}^4)}. $$

To prove these, we need a boundedness of heat kernel.
Theorem 1.7. Let $\mathcal{L}_A$ be in (2.3). Then there exists a constant $C$ such that

$$|e^{-t\mathcal{L}_A(x, y)}| \leq Ct^{-2}e^{-\frac{|x-y|^2}{4R^2}}, \quad t > 0.$$  \hspace{1cm} (1.25)

From the heat kernel estimates and our previous work [23] with Killip, Visan and Miao, one can obtain the equivalence of Sobolev norms

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L^p(R^4)} \sim \|L^A_s f\|_{L^p(R^4)}, \quad 1 < p < \frac{4}{s}$$  \hspace{1cm} (1.26)

therefore, there holds chain rules

$$\|f\|_{H_{s,p}^A(R^4)} \lesssim \|f\|_{H_{s,p_1}^A} \|g\|_{L^{p_2}(R^4)} + \|f\|_{L^{p_3}(R^4)} \|g\|_{H_{s,p_4}^A(R^4)},$$  \hspace{1cm} (1.27)

for $s > 0, 1 < p, p_j < \frac{4}{s}(j = 1, 2, 3, 4)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Our paper is organized as follows. Section 2 presents the representation of the spectral measure associated with $\mathcal{L}_A$. Section 3 is devoted to the proofs of the Strichartz estimates. In Section 4, we establish a scattering theory for the defocusing nonlinear subcritical Schrödinger equation (1.17).

2. The spectral measure

In this section, we first reduce the operator $H_{A,2}$ to an operator $\mathcal{L}_A$ given in (2.3) below by making change of variables. By modifying the method of [15] [18], we prove the representations of Schrödinger and wave propagators, spectral measure associated with the operator $\mathcal{L}_A$.

2.1. The operator. From (1.3) and (1.4) with $N = 2$, we see

$$H_{A,2} = \sum_{j=1}^{2}(-i\nabla_j + F_j)^2,$$

where $\vec{x}_1 = (x_{11}, x_{12}), \vec{x}_2 = (x_{21}, x_{22}) \in \mathbb{R}^2$ and

$$\nabla_{\vec{x}_1} = (\partial_{x_{11}}, \partial_{x_{12}}), \quad F_1 = \alpha \left(-\frac{x_{12} - x_{22}}{|\vec{x}_1 - \vec{x}_2|}, \frac{x_{11} - x_{21}}{|\vec{x}_1 - \vec{x}_2|}\right),$$

$$\nabla_{\vec{x}_2} = (\partial_{x_{21}}, \partial_{x_{22}}), \quad F_2 = \alpha \left(-\frac{x_{22} - x_{12}}{|\vec{x}_1 - \vec{x}_2|}, \frac{x_{21} - x_{11}}{|\vec{x}_1 - \vec{x}_2|}\right).$$

Let

$$\vec{y}_1 = \frac{\vec{x}_1 - \vec{x}_2}{\sqrt{2}}, \quad \vec{y}_2 = \frac{\vec{x}_1 + \vec{x}_2}{\sqrt{2}},$$

then

$$\nabla_{\vec{x}_1} = \frac{1}{\sqrt{2}}(\nabla_{\vec{y}_1} + \nabla_{\vec{y}_2}), \quad \nabla_{\vec{x}_2} = \frac{1}{\sqrt{2}}(\nabla_{\vec{y}_2} - \nabla_{\vec{y}_1}),$$

and

$$H_{A,2} = \left(-i\nabla_{\vec{y}_1} + \alpha \left(-\frac{y_{12}}{|\vec{y}_1|}, \frac{y_{11}}{|\vec{y}_1|^2}\right)\right)^2 - \Delta_{\vec{y}_2}$$

$$= \left(-i(\nabla_{\vec{y}_1}, \nabla_{\vec{y}_2}) + \alpha \left(-\frac{y_{12}}{|\vec{y}_1|^2}, \frac{y_{11}}{|\vec{y}_1|^2}\right)\right)^2.$$
In fact
\[ H_{A,2} = (-i\nabla_{\vec{x}_1} + F_1)^2 + (-i\nabla_{\vec{x}_2} + F_2)^2 \]
\[ = \frac{1}{2} \left( -i\nabla_{\vec{y}_1} - i\nabla_{\vec{y}_2} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 + \frac{3}{2} \left( i\nabla_{\vec{y}_1} - i\nabla_{\vec{y}_2} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 \]
\[ = \frac{1}{2} \left( -i\nabla_{\vec{y}_1} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 + \frac{3}{2} \left( i\nabla_{\vec{y}_1} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 \]
\[ = \frac{1}{2} \left( -i\nabla_{\vec{y}_1} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 - \frac{1}{2} \Delta_{\vec{y}_2} \]
\[ = \frac{1}{2} \left( -i\nabla_{\vec{y}_1} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 \]
\[ = \left( -i\nabla_{\vec{y}_1} + \alpha \left( -\frac{y_{12}}{|y_1|^2}, \frac{y_{11}}{|y_1|^2} \right) \right)^2 - \Delta_{\vec{y}_2} \]

which implies (2.2). Hence, if we denote \( y = (\vec{y}_1, \vec{y}_2) \in \mathbb{R}^4 \) and
\[ v(\vec{y}_1, \vec{y}_2) = u \left( \frac{\vec{y}_1 + \vec{y}_2}{\sqrt{2}}, \frac{\vec{y}_1 - \vec{y}_2}{\sqrt{2}} \right) \quad \text{or} \quad u(\vec{x}_1, \vec{x}_2) = v(\vec{y}_1, \vec{y}_2), \]
then,
\[ H_{A,2}u(\vec{x}_1, \vec{x}_2) = \mathcal{L}_A v(\vec{y}_1, \vec{y}_2), \quad (2.3) \]
with
\[ \mathcal{L}_A = \left( -i\nabla_y + \mathbf{A}(y) \right)^2, \quad \mathbf{A}(y) = \alpha \left( -\frac{y_{12}}{|y_1|^2 + |y_2|^2}, \frac{y_{11}}{|y_1|^2 + |y_2|^2}, 0, 0 \right) \]
which is analogue of (1.11). Both of them are effected in the first two variables by the magnetic potential. Without confusing, from now on, we briefly write \( \mathcal{L}_A \) to consist with (1.11) as follows:
\[ \mathcal{L}_A = \left( -i\nabla_x + \mathbf{A}(x) \right)^2, \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \setminus \{0, 0, x_3, x_4\}, \quad (2.4) \]
with
\[ \mathbf{A}(x) = \alpha \left( -\frac{x_2}{x_1 + x_2}, \frac{x_1}{x_1 + x_2}, x_3, x_4 \right), \quad \alpha \in \mathbb{R}. \quad (2.5) \]
This operator can be regraded as a one-particle model but with more singular Aharonov-Bohm type potential in \( \mathbb{R}^4 \).

We remark that the operator (2.4) with magnetic potential (2.5) was studied in [13] from the Hardy inequality viewpoint and in [14] from the virial identities, see the special model in [14, (1.19)] and [13, (3)]. However, the Strichartz estimates for dispersive equations with potential (2.4) have not been proved yet even though the Strichartz estimates obtained in [14] for a bit faster decaying \( \mathbf{A} \). It is known that the magnetic field \( B = \text{curl} \mathbf{A} = (0, 0, 0, \delta) \) with \( \delta \) denoting the Dirac delta function. Heuristic, since the dispersion obstruction \( B_\tau = \frac{i}{\tau} \wedge B \) (the trapping component of magnetic field arising from \( \mathbf{A} \)) vanishes, the Strichartz estimate should hold.

Therefore we can modify the method of [15] [18] to study the operator (2.4).
2.2. The Schrödinger kernel. In this subsection, we modify the argument of [18], in which we construct the spectral measure for a similar model in \( \mathbb{R}^2 \), to construct the kernel of Schrödinger operator \( e^{it\mathcal{L}_A} \).

**Proposition 2.1.** Let \( x = (r \cos \theta, r \sin \theta, x_3, x_4), \) \( y = (\bar{r} \cos \bar{\theta}, \bar{r} \sin \bar{\theta}, y_3, y_4), \) and let \( K(t; r, \theta, x_3, x_4; \bar{r}, \bar{\theta}, y_3, y_4) \) be the kernel of \( e^{it\mathcal{L}_A} \) with \( \mathcal{L}_A \) being as in (2.4), then there holds

\[
K(t; r, \theta, x_3, x_4; \bar{r}, \bar{\theta}, y_3, y_4) = \frac{1}{4\pi (it)^\frac{3}{2}} e^{-\frac{|x-y|^2}{4it}} A_\alpha(\theta, \bar{\theta}) + \frac{1}{4\pi (it)^\frac{3}{2}} \times e^{-\frac{r^2 + \bar{r}^2 + |x_3-y_3|^2 + |x_4-y_4|^2}{4it}} \int_0^\infty e^{-\frac{r \cosh s}{2it}} B_\alpha(s, \theta, \bar{\theta}) ds,
\]

where \( A_\alpha(\theta, \bar{\theta}) \) and \( B_\alpha(s, \theta, \bar{\theta}) \) are respectively

\[
A_\alpha(\theta, \bar{\theta}) = e^{i\alpha(\theta-\bar{\theta})} \left[ 1_{|0,\pi|}(\theta-\bar{\theta}) + e^{-2\pi i\alpha} 1_{|\pi,2\pi|}(\theta-\bar{\theta}) \right],
\]

and

\[
B_\alpha(s, \theta, \bar{\theta}) = -\left[ \sin(|\alpha|\pi) e^{-|\alpha|s} + \sin(\alpha \pi) \times (e^{-s} - \cos(\theta-\bar{\theta} + \pi)) \sin \alpha s - i \sin(\theta-\bar{\theta} + \pi) \cosh \alpha s \right].
\]

**Proof.** We prove (2.6) by following the idea in [18]. First, we obtain the fundamental solution of Schrödinger operator \( e^{it\mathcal{L}_A} \). In the cylindrical coordinates

\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = x_3, \quad x_4 = x_4,
\]

from (2.5), we write

\[
\mathcal{L}_A = -\Delta + 2iA(x) \cdot \nabla + |A(x)|^2
\]

\[
= -\partial_r^2 - \frac{1}{r} \partial_r + \frac{(i\partial_\theta + \alpha)^2}{r^2} - \partial_{x_3}^2 - \partial_{x_4}^2.
\]

For each \( k \in \mathbb{Z}, \) \( Y_k(\theta) := (2\pi)^{-1/2} e^{-ik\theta} \) satisfies

\[
\begin{cases} 
(i\partial_\theta + \alpha)^2 Y_k(\theta) = (k + \alpha)^2 Y_k(\theta), \quad \text{on } S^1, \\
\int_{S^1} |Y_k(\theta)|^2 d\theta = 1,
\end{cases}
\]

hence, the eigenfunctions \( \{Y_k(\theta)\}_{k \in \mathbb{Z}} \) construct a complete orthonormal basis in \( L^2(S^1) \). Moreover, let \( x' = (x_3, x_4) \in \mathbb{R}^2 \), we can write \( f(x) \in L^2(\mathbb{R}^4) \) in the form of

\[
f(x) = f(r, \theta, x') = \sum_{k \in \mathbb{Z}} a_k(r, x') Y_k(\theta),
\]

where

\[
a_k(r, x') = \int_0^{2\pi} f(r, \theta, x') Y_k(\theta) d\theta.
\]

Therefore, from (2.10), (2.11) and (2.12), we know

\[
\mathcal{L}_A f = \sum_{k \in \mathbb{Z}} \left( A_{k,\alpha} a_k(r, x') - \partial_{x_3}^2 a_k(r, x') - \partial_{x_4}^2 a_k(r, x') \right) Y_k(\theta)
\]

where

\[
A_{k,\alpha} = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{(k + \alpha)^2}{r^2}.
\]
Let $\nu = \nu(k) = |k + \alpha|$, and recall Hankel transform (e.g. see [31]) of $\nu$ order defined by

$$
(H \nu f)(\rho, \theta, x') = \int_{0}^{\infty} J_\nu(rp)f(r, \theta, x')rdr, \ \forall f \in L^2(\mathbb{R}^4),
$$

(2.15)

where the Bessel function

$$
J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2)} \int_{-1}^{1} e^{i\nu s(1 - s^2)^{1/4}}ds, \ \nu > -\frac{1}{2}, \ r > 0.
$$

(2.16)

For our purpose, we recall some properties of the Hankel transform, see [31].

Lemma 2.2 (Hankel transform). Let $\mathcal{H}_\nu$ be the Hankel transform in (2.15) and $A_\nu := -\partial^2_r - \frac{1}{r}\partial_r + \frac{1}{r^2}$. Then

1. $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$,
2. $\mathcal{H}_\nu$ is self-adjoint, i.e. $\mathcal{H}_\nu = \mathcal{H}_\nu^*$,
3. $\mathcal{H}_\nu$ is an $L^2$ isometry, i.e. $\|\mathcal{H}_\nu f\|_{L^2(\mathbb{R}^4)} = \|f\|_{L^2(\mathbb{R}^4)}$,
4. $\mathcal{H}_\nu(A_\nu)(\rho, \theta) = \rho^2(\mathcal{H}_\nu f)(\rho, \theta)$, for $f \in L^2$.

In the cylindrical coordinates (2.9), if the initial data

$$
u_0(x) = f(r, \theta, x') = \sum_{k \in \mathbb{Z}} a_k(r, x')Y_k(\theta),
$$

then $u(t, x) = e^{it\xi \cdot \mathbf{x}}f$ satisfies

$$
u(t, x) = u(t; r, \theta, x') = \sum_{k \in \mathbb{Z}} u_k(t; r, x')Y_k(\theta),
$$

where $u_k(t; r, x')$ solves

$$
\begin{cases}
i\partial_t u_k + A_{\nu, \alpha}u_k - \partial^2_{x_3}u_k - \partial^2_{x_4}u_k = 0, & (t, r, x') \in \mathbb{R} \times (0, \infty) \times \mathbb{R}^2, \\
u_k(0, r, x') = a_k(r, x').
\end{cases}
$$

(2.17)

By taking Fourier transform in variables $x'$ and Hankel transform in variable $r$ and using Lemma 2.2, and let $\xi = (\xi_3, \xi_4) \in \mathbb{R}^2$, we obtain

$$
\begin{cases}
i\partial_t \tilde{u}_k(t; \rho, \xi) + \rho^2 \tilde{u}_k(t; \rho, \xi) + |\xi|^2 \tilde{u}_k(t; \rho, \xi) = 0, \\
\tilde{u}_k(0; \rho, \xi) = \tilde{a}_k(\rho, \xi).
\end{cases}
$$

(2.18)

where

$$
\tilde{u}_k(t; \rho, \xi) = \{H_\nu(k)\tilde{u}_k\}(t; \rho, \xi), \ \tilde{a}_k(\rho, \xi) = \{H_\nu(k)\tilde{a}_k\}(\rho, \xi)
$$

and $\tilde{a}(\xi) = \int_{\mathbb{R}^2} e^{-ix' \cdot \xi}a(x')dx'$. The solution for ordinary differential equation (2.18) is

$$
\tilde{u}_k(t; \rho, \xi) = \tilde{a}_k(\rho, \xi)e^{it(\rho^2 + |\xi|^2)}.
$$

By using Lemma 2.2 again, we obtain

$$
\begin{align*}
u(t, \theta, x') &= \int_{\mathbb{R}^2} \int_{0}^{2\pi} \int_{0}^{\infty} K(t; r, \theta, x'; \tilde{r}, \tilde{\theta}, y')J_{\nu}(\tilde{r})J_{\nu}(r\rho)\rho d\rho d\tilde{\theta}d\tilde{r}dy', \\
K(t; r, \theta, x'; \tilde{r}, \tilde{\theta}, y') &= \sum_{k \in \mathbb{Z}} Y_k(\theta)Y_k(\tilde{\theta})
\end{align*}
$$

(2.19)

where

$$
K(t; r, \theta, x'; \tilde{r}, \tilde{\theta}, y') = \sum_{k \in \mathbb{Z}} Y_k(\theta)Y_k(\tilde{\theta})
$$

$$
\times \int_{\mathbb{R}^2} \int_{0}^{\infty} e^{i(x' - y') \cdot \xi}e^{it(\rho^2 + |\xi|^2)}J_{\nu}(k)J_{\nu}(\tilde{k})\rho d\rho d\xi.
$$
Noting that
\[ \int_{\mathbb{R}^2} e^{-it|\xi|^2} e^{-ix\cdot \xi} d\xi = \frac{\pi}{it} e^{-\frac{|x|^2}{4t^2}} \]
and \( Y_\lambda(\theta) = (2\pi)^{-1/2} e^{-ik\theta} \), we rewrite \( K(t; r, \theta, x'; \bar{r}, \bar{\theta}, y') \) as
\[ K(t; r, \theta, x'; \bar{r}, \bar{\theta}, y') = \frac{1}{2it} e^{-\frac{|x'-y'|^2}{4t^2}} \sum_{k \in \mathbb{Z}} e^{-i(k-\theta)\theta} \int_0^\infty J_{\nu(k)}(r\rho) J_{\nu(k)}(\bar{r}\rho) e^{it\rho^2} \rho d\rho, \]
where the Weber identity \([31]\) is used in the last equality and the term \( I_{\nu(k)}(z) \) \([31]\) is given by
\[ I_{\nu(k)}(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos \nu \cos \theta d\theta = \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \cosh s} e^{-s\nu} ds. \]
Recall \( \nu = |k + \alpha| \), similarly as in \([15, 18]\), we get
\[ \frac{1}{\pi} \sum_{k \in \mathbb{Z}} e^{-i(k-\theta)\theta} \int_0^\pi e^{z \cos \theta} \cos \nu \cos \theta d\theta = \frac{1}{\pi} \times \begin{cases} e^{z \cos \theta} e^{i\nu \cos \theta}, & |\theta - \bar{\theta}| < \pi, \\ e^{z \cos \theta} e^{i\nu \cos \theta - 2\pi}, & \pi < |\theta - \bar{\theta}| < 2\pi, \end{cases} \] (2.21)
and
\[ \frac{1}{\pi} \sum_{k \in \mathbb{Z}} e^{-i(k-\theta)\theta} \sin \nu \int_0^\infty e^{-z \cosh s} e^{-s\nu} ds = \frac{1}{\pi} \int_0^\infty e^{-z \cosh s} \left[ \sin(|\alpha|\pi) e^{-|\alpha|s} + \sin(\alpha \pi) \right. \\
\left. \times \left( \frac{e^{-s} - \cos(\theta - \bar{\theta} + \pi)}{\cosh s - \cos(\theta - \bar{\theta} + \pi)} \right) \sinh \alpha s - i \sin(\theta - \bar{\theta} + \pi) \cosh \alpha s \right] ds. \] (2.22)
By noticing that
\[ (x_1 - y_1)^2 + (x_2 - y_2)^2 = r^2 + \bar{r}^2 - 2\bar{r} \cos(\theta - \bar{\theta}) \]
and collecting \((2.20), (2.22)\) together, we finally prove Proposition 2.1. \( \square \)

Decay estimates follow from the representation of fundamental solution \(2.16\).

**Proposition 2.3.** There exists a constant \( C \) such that
\[ |e^{itL^\lambda}(x, y)| \leq C|t|^{-2}, \quad \forall t \in \mathbb{R} \setminus \{0\}. \] (2.23)

**Proof.** Indeed, from \([15]\), we know
\[ |A_\alpha(\theta, \bar{\theta})| + \int_0^\infty |B_\alpha(s, \theta, \bar{\theta})| \leq C, \] (2.24)
then combining with \(2.16\), dispersive estimates \(2.23\) hold. \( \square \)
2.3. The representation of spectral measure. To prove the disper- ser estimates for wave equation, we need the representation of the spectral measure associated with $\mathcal{L}_A$.

**Proposition 2.4.** Let $x = (r \cos \theta, r \sin \theta, x')$, $y = (\bar{r} \cos \bar{\theta}, \bar{r} \sin \bar{\theta}, y')$ where $x' = (x_3, x_4) \in \mathbb{R}^2$ and $y' = (y_3, y_4) \in \mathbb{R}^2$. Let $dE_{\sqrt{\mathcal{L}_A}}(\lambda; x, y)$ be the spectral measure kernel associated with Schrödinger operator $\mathcal{L}_A$ given by (2.4). Then

$$dE_{\sqrt{\mathcal{L}_A}}(\lambda; x, y) = \frac{\lambda^3}{\pi^2} \sum_{\pm} a_{\pm}(\lambda|x-y|) e^{\pm i\lambda|x-y|} \times A_{\alpha}(\theta, \bar{\theta})$$

$$+ \frac{\lambda^3}{\pi^2} \int_0^\infty \sum_{\pm} a_{\pm}(\lambda|\bar{n}|) e^{\pm i\lambda|\bar{n}|} \times B_{\alpha}(s, \theta, \bar{\theta}) ds,$$

where $a_{\pm}(r)$ satisfies

$$|\dot{a}_{\pm}(r)| \leq C_k (1 + r)^{-\frac{3}{2} - k}, \quad \forall k \geq 0,$$

and $A_{\alpha}(\theta, \bar{\theta})$, $B_{\alpha}(s, \theta, \bar{\theta})$ are as in (2.7) and (2.8) respectively, and

$$\bar{n} = \bar{n}_0 = (r + \bar{r}, \sqrt{2r\bar{r}}(\cosh s - 1), x_3 - y_3, x_4 - y_4).$$

**Proof.** Note that for each $z \in \mathbb{C}$ satisfying $\text{Im} z > 0$, there holds

$$\frac{1}{s - z} = i \int_0^\infty e^{-ist} e^{itz} dt, \quad \forall s \in \mathbb{R}.$$

This together with (2.6) shows

$$(\mathcal{L}_A - (\lambda^2 + i0))^{-1} = \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-i\lambda \mathcal{L}_A} e^{i(\lambda^2 + i0)} dt$$

$$= \lim_{\varepsilon \searrow 0} \int_0^\infty G(t; r, \theta, x_3, x_4; \bar{r}, \bar{\theta}, y_3, y_4) e^{i(\lambda^2 + i0)} dt$$

$$+ i \lim_{\varepsilon \searrow 0} \int_0^\infty D(t; r, \theta, x_3, x_4; \bar{r}, \bar{\theta}, y_3, y_4) e^{i(\lambda^2 + i0)} dt.$$  

(2.29)

As did in (8), noting that

$$\int_{\mathbb{R}^4} e^{-i \xi \cdot x} e^{-i|\xi|^2} d\xi = \frac{\pi}{(it)^2} e^{-\frac{|x|^2}{it^2}},$$

we get for $z = \lambda^2 + i\varepsilon$ with $\varepsilon > 0$

$$\int_0^\infty e^{-\frac{|x-y|^2}{(it)^2}} e^{itx} dt = \frac{1}{\pi} \int_0^\infty \int_{\mathbb{R}^4} e^{-i(x-y) \cdot \xi} e^{-i|\xi|^2} d\xi e^{itx} dt$$

$$= \frac{1}{\pi} \int_{\mathbb{R}^4} e^{-i(x-y) \cdot \xi} \int_0^\infty e^{-it(|\xi|^2 - z)} d\xi d\xi = \frac{1}{i\pi} \int_{\mathbb{R}^4} e^{-i(x-y) \cdot \xi} \frac{e^{-i|\xi|^2}}{|\xi|^2 - z} d\xi,$$

and

$$\int_0^\infty e^{-\frac{r^2 + \bar{r}^2 + |x-y|^2}{(it)^2}} e^{-i\bar{n} \cdot x} dt = \frac{1}{i\pi} \int_{\mathbb{R}^4} e^{-i\bar{n} \cdot \xi} \frac{e^{-i|\xi|^2}}{|\xi|^2 - z} d\xi,$$

where $\bar{n} = (r_1 + r_2, \sqrt{2r_1r_2}(\cosh s - 1), x' - y')$. Then from (2.4), we have

$$i \lim_{\varepsilon \searrow 0} \int_0^\infty G(t; r, \theta, x'; \bar{r}, \bar{\theta}, y') e^{it(\lambda^2 + i0)} dt = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{e^{-i(x-y) \cdot \xi}}{|\xi|^2 - (\lambda^2 + i0)} d\xi \times A_{\alpha}(\theta, \bar{\theta}),$$

(2.31)
and

\[
\begin{align*}
&\lim_{\epsilon \to 0+} \int_{0}^{\infty} e^{-i(r+\epsilon \sqrt{2r(\cosh s-1)})x_3-y_3-y_4} \, d\xi \\
&= \frac{1}{4\pi^2} \int_{0}^{\infty} e^{-i(r+\epsilon \sqrt{2r(\cosh s-1)})x_3-y_3-y_4} \, d\xi \times B_\alpha(s, \theta, \bar{\theta}) \, ds.
\end{align*}
\]

(2.32)

Therefore, collecting (2.29)–(2.32) together and recalling (2.27), we get

\[
(L_\lambda - (\lambda^2 + i0))^{-1} = \frac{1}{4\pi^2} \int_{R^4} \frac{e^{-i(x-y) \cdot \xi}}{|\xi|^2 - (\lambda^2 + i0)} \, d\xi \times A_\alpha(\theta, \bar{\theta}) \\
+ \frac{1}{4\pi^2} \int_{R^4} \frac{e^{-i\bar{\xi} \cdot \xi}}{|\xi|^2 - (\lambda^2 + i0)} \, d\xi \times B_\alpha(s, \theta, \bar{\theta}) \, ds.
\]

(2.33)

Observing that \(A_\alpha = A_\alpha\) and \(B_\alpha = B_\alpha\), then we obtain

\[
(L_\lambda - (\lambda^2 - i0))^{-1} = \frac{1}{4\pi^2} \int_{R^4} \frac{e^{-i(x-y) \cdot \xi}}{|\xi|^2 - (\lambda^2 - i0)} \, d\xi \times A_\alpha(\theta, \bar{\theta}) \\
+ \frac{1}{4\pi^2} \int_{R^4} \frac{e^{-i\bar{\xi} \cdot \xi}}{|\xi|^2 - (\lambda^2 - i0)} \, d\xi \times B_\alpha(s, \theta, \bar{\theta}) \, ds.
\]

(2.34)

By using the Stone’s formula, we deduce that

\[
dE_{\mathcal{T}_\lambda}(\lambda; x, y) = \frac{\lambda}{i\pi} ([L_\lambda - (\lambda^2 + i0)]^{-1} - [L_\lambda - (\lambda^2 - i0)]^{-1}) \, d\lambda.
\]

(2.35)

Combining (2.33)–(2.35), we further write

\[
dE_{\mathcal{T}_\lambda}(\lambda; x, y) = \frac{\lambda}{i\pi} \int_{R^4} \frac{e^{-i(x-y) \cdot \xi}}{|\xi|^2 - (\lambda^2 - i0)} \, d\xi \times A_\alpha(\theta, \bar{\theta}) \\
+ \frac{1}{4\pi^2} \int_{0}^{\infty} \frac{e^{-i\bar{\xi} \cdot \xi}}{|\xi|^2 - (\lambda^2 + i0)} \, d\xi \times B_\alpha(s, \theta, \bar{\theta}) \, ds.
\]

(2.36)

Similarly to (1.6) again, we observe the fact that the Poisson kernel is an approximation to the identity which implies that, for any reasonable function \(m(x)\)

\[
m(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{R} \Im \left( \frac{1}{x - (y + \epsilon i)} \right) m(y) \, dy
\]

(2.37)

then we have

\[
\lim_{\epsilon \to 0^+} \frac{\lambda}{i\pi} \int_{R^4} e^{-i\epsilon x \cdot \xi} \left( \frac{1}{|\xi|^2 - (\lambda^2 + i\epsilon)} - \frac{1}{|\xi|^2 - (\lambda^2 - i\epsilon)} \right) \, d\xi
\]

(2.38)

\[
= \lim_{\epsilon \to 0^+} \frac{\lambda}{\pi} \int_{R^4} e^{-i\epsilon x \cdot \xi} \Im \left( \frac{1}{|\xi|^2 - (\lambda^2 + i\epsilon)} \right) \, d\xi
\]

(2.39)

\[
= \lim_{\epsilon \to 0^+} \frac{\lambda}{\pi} \int_{0}^{\infty} \frac{\epsilon}{(\rho^2 - \lambda^2)^2 + \epsilon^2} \int_{|\omega| = 1} e^{-i\rho x \cdot \omega} d\sigma_\omega \rho^2 \, d\rho
\]

(2.40)

\[
= \lambda^3 \int_{|\omega| = 1} e^{-i\lambda x \cdot \omega} d\sigma_\omega.
\]
On the other hand, from [27, Theorem 1.2.1], we also note that
\[ \int_{S^3} e^{-ix \cdot \omega} d\sigma(\omega) = \sum_\pm a_\pm(|x|) e^{\pm i|x|}, \] (2.38)
where
\[ |\partial_k^k a_\pm(r)| \leq C_k (1 + r)^{-\frac{k}{2} - k}, \quad k \geq 0. \] (2.39)
Therefore we finally obtain representation (2.25).

3. Strichartz estimates

In this section, we prove Theorem 1.1 and Theorem 1.4 about the Strichartz estimates.

3.1. The proof of Theorem 1.1. From the dispersive estimates in Proposition 2.3 and the \( L^2 \)-estimate (obtained from the mass conservation law for Schrödinger equation or the unitary property of \( e^{it\mathcal{L}_A} \)), the abstract method of Keel-Tao [24] directly shows

**Theorem 3.1.** Let \( \mathcal{L}_A \) be in \([24]\) and let \( u(t,x) \) be a solution of Schrödinger equation
\[ \begin{cases}
  i\partial_t u + \mathcal{L}_A u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \{0,0,x_3,x_4\} \\
  u(0,x) = u_0(x), & x \in \mathbb{R}^4.
\end{cases} \] (3.1)
Then there exists a constant \( C \) such that
\[ \|u(t,x)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C \|u_0\|_{L^2(\mathbb{R}^4)}, \] (3.2)
where \((q,r) \in \Lambda^S_0\) defined in \([19]\).

**Remark 3.2.** The Strichartz estimates (3.2) can be extended to the solution of
\[ \begin{cases}
  i\partial_t u + \mathcal{L}_{A,a} u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \{0\}, \\
  u(0,x) = u_0(x), & x \in \mathbb{R}^4,
\end{cases} \] (3.3)
where the electromagnetic operator \( \mathcal{L}_{A,a} = \left(-i\nabla + A(x)\right)^2 + a(\hat{x})|x|^2 \) and \( x \in \mathbb{R}^4 \setminus \{0\}, \hat{x} \in S^3 \) and \( \min_{\hat{x} \in S^3} a(\hat{x}) > -1 \). Indeed, this is a consequence of (3.2) and the local smoothing estimates
\[ \|x^{-1}u(t,x)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C \|u_0\|_{L^2(\mathbb{R}^4)}, \] (3.4)
which is implied by the weighted resolvent estimates in Barceló-Vega-Zubeldía [5]
\[ \sup_{\sigma \in \mathbb{C} \setminus [0,\infty]} \|x^{-1}(\mathcal{L}_{A,a} - \sigma)^{-1}x^{-1}\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C. \] (3.5)

**Remark 3.3.** The method also works for dimension three in which
\[ A(x) = \alpha \left( -\frac{x_2}{|x_1|^2 + |x_2|^2}, \frac{x_1}{|x_1|^2 + |x_2|^2}, 0 \right), \quad \alpha \in \mathbb{R}. \]
In this case, the endpoint Strichartz estimate (3.2) with \((q,r) = (2,6)\) is a generalization of [3] (1.14).
Now we prove Theorem 1.3 by using (3.2). Let \( u(t, \vec{x}_1, \vec{x}_2) \) solve (1.6), then
\[
v(t; \vec{y}_1, \vec{y}_2) = u\left(t; \frac{\vec{y}_1 + \vec{y}_2}{\sqrt{2}}, \frac{\vec{y}_2 - \vec{y}_1}{\sqrt{2}}\right)
\] (3.6)
solves (1.9). Therefore, by (1.15) and (3.2), we obtain for \( (q, r) \in \Lambda_0^5 \) defined in (1.9)
\[
\|u(t, \vec{x}_1, \vec{x}_2)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C\|v(t; \vec{y}_1, \vec{y}_2)\|_{L^q_t(L^r_x(\mathbb{R}^4))}
\]
(3.7)
Thus, we complete the proof of Theorem 1.1.

3.2. The proof of Theorem 1.4. By the same argument of (3.7), to prove Theorem 1.4, it suffices to establish the Strichartz estimates for the wave equation
\[
\begin{align*}
\partial_{tt}u + \mathcal{L}_A u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \{0, 0, x_3, x_4\} \\
u(0, x) &= f(x), \quad \partial_t u(0, x) = g(x) \quad x \in \mathbb{R}^4.
\end{align*}
\] (3.8)
To this goal, we first prove the localized dispersive estimates for wave (3.3).

Proposition 3.4 (Dispersive estimate for wave). Let \( \mathcal{L}_A \) be given in (2.4) and let \( \phi \in C_c^{\infty}([1/2, 2]) \) take value in \([0, 1]\) such that \( 1 = \sum_{k \in \mathbb{Z}} \phi(2^{-k} \lambda) \). Assume \( f = \phi(2^{-k} \sqrt{\mathcal{L}_A}) f \) with \( k \in \mathbb{Z} \), then there exists a constant \( C \) independent of \( t \) and \( k \in \mathbb{Z} \) such that
\[
\|e^{it \sqrt{\mathcal{L}_A}} f\|_{L^\infty(\mathbb{R}^4)} \leq C 2^{\frac{k}{2}} (2^{-k} + |t|)^{-\frac{3}{2}} \|f\|_{L^1(\mathbb{R}^4)}.
\] (3.9)
Proof. With Proposition 2.4 in hand, we use stationary phase argument to prove Proposition 3.4. We write
\[
e^{it \sqrt{\mathcal{L}_A}} f = \int_{\mathbb{R}^4} \int_0^\infty e^{it \lambda} \phi(2^{-k} \lambda) dE_{\sqrt{\mathcal{L}_A}}(\lambda; x, y) f(y) dy.
\]
Now we aim to show kernel estimate
\[
\left| \int_0^\infty e^{it \lambda} \phi(2^{-k} \lambda) dE_{\sqrt{\mathcal{L}_A}}(\lambda; x, y) \right| \leq C 2^{\frac{k}{2}} (2^{-k} + |t|)^{-\frac{3}{2}}.
\] (3.10)
For this purpose, from Proposition 2.4, it suffices to show
\[
\left| \int_0^\infty e^{it \lambda} \phi(2^{-k} \lambda) \lambda^3 a_{\pm}(\lambda |x - y|) e^{\pm i \lambda |x - y|} d\lambda A_\alpha(\theta, \bar{\theta}) \right| \leq C 2^{\frac{k}{2}} (2^{-k} + |t|)^{-\frac{3}{2}},
\] (3.11)
and
\[
\left| \int_0^\infty e^{it \lambda} \phi(2^{-k} \lambda) \lambda^3 \int_0^\infty a_{\pm}(\lambda |\vec{n}|) e^{\pm i \lambda |\vec{n}|} B_\alpha(s, \theta, \bar{\theta}) ds d\lambda \right| \leq C 2^{\frac{k}{2}} (2^{-k} + |t|)^{-\frac{3}{2}},
\] (3.12)
where \( a_{\pm} \) satisfies (2.26), \( \vec{n} \) is in (2.27), and \( A_\alpha(\theta, \bar{\theta}), B_\alpha(s, \theta, \bar{\theta}) \) are in (2.21) and (2.24). Since \( a_{\pm} \) satisfies (2.26), let \( r = |x - y| \) or \( |\vec{n}| \), hence
\[
\left| \partial_N^n [a_{\pm}(\lambda r)] \right| \leq C_N \lambda^{-N} (1 + \lambda r)^{-\frac{3}{2}}, \quad N \geq 0.
\] (3.13)
We first prove (3.11). If $2^k |t \pm x - y| \leq 1$, by (3.13), we directly estimate
\[
\left| \int_0^\infty e^{it\lambda} \phi(2^{-k}\lambda) \lambda^3 a_\pm(\lambda|x - y|) e^{\pm i\lambda|x - y|} d\lambda \right|
\leq C \int_{2^{k-1}}^{2^{k+1}} \lambda^3 (1 + \lambda|x - y|)^{-3/2} d\lambda
\leq C 2^{4k} (1 + 2^k |x - y|)^{-3/2}.
\]
If $2^k |t \pm x - y| \geq 1$, we use (3.13) again and perform $N$-times integration by parts to obtain
\[
\left| \int_0^\infty e^{it\lambda} \phi(2^{-k}\lambda) \lambda^3 a_\pm(\lambda|x - y|) e^{\pm i\lambda|x - y|} d\lambda \right|
\leq C_N |t \pm x - y|^{-N} \int_{2^{k-1}}^{2^{k+1}} \lambda^{3-N} (1 + \lambda|x - y|)^{-3/2} d\lambda
\leq C_N 2^{k(4-N)} |t \pm x - y|^{-N} (1 + 2^k |x - y|)^{-3/2}.
\]
It follows that
\[
\left| \int_0^\infty e^{it\lambda} \phi(2^{-k}\lambda) \lambda^3 a_\pm(\lambda|x - y|) e^{\pm i\lambda|x - y|} d\lambda \right|
\leq C_N 2^{4k} (1 + 2^k |t \pm x - y|)^{-N} (1 + 2^k |x - y|)^{-3/2}.
\]
If $|t| \sim |x - y|$, we see (3.11). Otherwise, we have $|t \pm x - y| \geq c |t|$ for some small constant $c$, choose $N = 1$ and $N = 0$, and then use geometric mean argument to obtain (3.11).

We next prove (3.12). We follow the same lines to obtain
\[
\left| \int_0^\infty e^{it\lambda} \phi(2^{-k}\lambda) \lambda^3 \int_0^\infty a_\pm(\lambda|\vec{s}_y|) e^{\pm i\lambda|\vec{s}_y|} B_\alpha(s, \theta, \tilde{\theta}) ds d\lambda \right|
\leq C 2^{\frac{4k}{2}} (2^{-k} + |t|)^{-\frac{3}{2}} \int_0^\infty |B_\alpha(s, \theta, \tilde{\theta})| ds.
\]
From (2.24), we have
\[
\int_0^\infty |B_\alpha(s, \theta, \tilde{\theta})| ds \leq C,
\]
which implies (3.12). Therefore we prove (3.10), hence (3.9) holds.

We also need the following proposition to achieve our goal.

**Proposition 3.5 (A kernel estimate).** Let $\mathcal{L}_\lambda$ be given in (2.24) and let $\psi \in C_c^\infty([1/2, 2])$ and take value in $[0, 1]$. For $k \in \mathbb{Z}$ and any $K \geq 0$, there exists a constant $C$ independent of $k \in \mathbb{Z}$ such that
\[
\left| \int_0^\infty \psi(2^{-k}\lambda) dE_{\mathcal{L}_\lambda}(\lambda; x, y) \right| \lesssim \frac{2^{4k}}{(1 + 2^k |x - y|)^K}.
\]
Proof. On one hand, by using scaling, we see
\[
\left| \int_0^\infty \psi(2^{-k}\lambda)dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y) \right| = 2^{4k} \left| \int_0^\infty \psi(\lambda)dE_{\sqrt{\bar{E}\lambda}}(\lambda; 2^kx, 2^ky) \right|. \tag{3.18}
\]
On the other hand, by using Proposition 2.4 and integration by parts K times, we have
\[
\left| \int_0^\infty \psi(\lambda)dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y) \right| \lesssim \frac{1}{(1 + |x - y|)^K} + \int_0^\infty \frac{1}{(1 + |\bar{\eta}|)^K} |B_\alpha(s, \theta, \bar{\theta})| ds.
\]
From \(2.27\), we observe that \(|\bar{\eta}|^2 = (r_1 + r_2)^2 + |x_3 - y_3|^2 + |x_4 - y_4|^2 \geq |x - y|^2\). Thus it follows from (3.16) that
\[
\left| \int_0^\infty \psi(\lambda)dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y) \right| \lesssim \frac{1}{(1 + |x - y|)^K}.
\]
This together with (3.18) gives (3.17).

Next we need the Littlewood-Paley square function inequality associated with the operator \(L_A\). This can be done by following the proof of [15, Proposition 2.2], once we show the heat kernel estimate in Theorem 1.7. Even \(L_A\) is slightly different from the operator \(L_{A,0}\) in [15], we can use the argument to prove (2.27). Indeed, if one replaces the variable \(t\) by \(t/i\), by the analytic continuous argument, we can obtain (2.27) from Proposition 2.4. We omit the detail of the proof of Theorem 1.7.

Therefore, by the standard argument deriving Littlewood-Paley theory from the Gaussian boundedness of heat kernel (e.g. see [15]), we have

**Proposition 3.6 (LP square function inequality).** Let \(\{\phi_k\}_{k \in \mathbb{Z}}\) be in Proposition 3.4 and let \(L_A\) be given in (2.4). Then for \(1 < p < \infty\), there exist constants \(c_p\) and \(C_p\) depending on \(p\) such that
\[
c_p \|f\|_{L^p(\mathbb{R}^4)} \leq \left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k(\sqrt{\bar{E}\lambda})f(\lambda)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^4)} \leq C_p \|f\|_{L^p(\mathbb{R}^4)}. \tag{3.19}
\]

For each \(k \in \mathbb{Z}\), let \(\phi_k(\lambda) = \phi(2^{-k}\lambda)\), we define
\[
U_k(t) = \int_0^\infty e^{it\lambda} \phi_k(\lambda)dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y), \tag{3.20}
\]
where \(dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y)\) is the spectral measure in Proposition 2.4. Hence by spectral property (e.g. [21, Lemma 5.3]), we see
\[
U_k(t)U_k^*(s) = \int_0^\infty e^{i(t-s)\lambda} \phi_k(\lambda)^2 dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y).
\]
On one hand, by Proposition 3.4 we have
\[
\|U_k(t)U_k^*(s)\|_{L^1 \rightarrow L^\infty} \lesssim 2^{7k}(2^{-k} + |t - s|)^{-3}. \tag{3.21}
\]
On the other hand, by duality argument, we have
\[
\|U(t)\|_{L^2 \rightarrow L^2} = ||U(t)U_k^*(t)||_{L^2 \rightarrow L^2} \lesssim 1. \tag{3.22}
\]
Indeed, this is implied by
\[
\left| \int_0^\infty (\phi_k(\lambda))^2 dE_{\sqrt{\bar{E}\lambda}}(\lambda; x, y) \right| \lesssim \frac{2^{4k}}{(1 + 2^k|x - y|)^N} \in L^1(\mathbb{R}^4),
\]
which follows from Proposition 3.5.
Then using Keel-Tao’s argument (also see [24]), for \((q, r) \in \Lambda^\omega\), we get
\[
\|U_k(t)f\|_{L^r_t(L^q_x(\mathbb{R}^4))} \lesssim 2^{ks}\|f\|_{L^2(\mathbb{R}^4)}.
\]  
(3.23)

Let \(f_j(x) = \phi(2^{-j} \sqrt{\Lambda})f\), then
\[
e^{it\sqrt{\Lambda}}f(x) = \sum_{k \in \mathbb{Z}} U_k(t)f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} U_k(t)f_j(x).
\]  
(3.24)

By using (3.19) and Minkowski’s inequality, we show that
\[
\|e^{it\sqrt{\Lambda}}f\|_{L^r_t(L^q_x(\mathbb{R}^4))} \lesssim \left(\sum_{j \in \mathbb{Z}} \|U_k(t)f\|_{L^r_t(L^q_x(\mathbb{R}^4))}^2\right)^{1/2}.
\]  
(3.25)

We observe that \(\phi_k(\sqrt{\Lambda})f_j\) vanishes when \(|j - k| \geq 10\), thus
\[
\left(\sum_{j \in \mathbb{Z}} \|\sum_{k \in \mathbb{Z}} U_k(t)f_j\|_{L^r_t(L^q_x(\mathbb{R}^4))}^2\right)^{1/2} \lesssim \left(\sum_{j \in \mathbb{Z}} \|U_k(t)f\|_{L^r_t(L^q_x(\mathbb{R}^4))}^2\right)^{1/2}
\]  
(3.26)

Since
\[
u(t, x) = \frac{e^{it\sqrt{\Lambda}} + e^{-it\sqrt{\Lambda}}}{2}f(x) + \frac{e^{it\sqrt{\Lambda}} - e^{-it\sqrt{\Lambda}}}{2i\sqrt{\Lambda}}g(x),
\]  
(3.27)

we finally prove the Strichartz estimates
\[
\|\nu(t, x)\|_{L^r_t(L^q_x(\mathbb{R}^4))} \lesssim \|f\|_{H^\omega(\mathbb{R}^4)} + \|g\|_{H^\omega(\mathbb{R}^4)}.
\]  
(3.28)

By the equivalence of Sobolev norms (1.26), we finally obtain (1.11). The proof of Theorem 1.4 is now complete.

4. Scattering Theory

We divide this section into three parts. The first part proves the global well-posedness for the solution to the equation (1.17). Then we show the classical Morawetz estimate by using the virial identity in [19, Theorem 1.2]. Finally, we prove Theorem 1.6 by following the idea in [17].

4.1. Global well-posedness. By interpolating (2.22) and mass conservation, we obtain the following dispersive estimates.

**Proposition 4.1** (Dispersive Estimate). Let \(\alpha \in \mathbb{R}\) and \(2 \leq p \leq +\infty\). Then we have the following estimate
\[
\|e^{it\sqrt{\Lambda}}f\|_{L^p(\mathbb{R}^4)} \leq C|t|^{-\sigma - \frac{4}{p}}\|f\|_{L^p(\mathbb{R}^4)}, \quad \forall t \in \mathbb{R} \setminus \{0\},
\]  
(4.1)

for some constant \(C = C(\alpha, p) > 0\) which does not depend on \(t\) and \(f\).

Then Strichartz estimates follow from a direct application of Keel-Tao’s argument [24].
Proposition 4.2 (Strichartz estimate including inhomogeneous term). Let $\mathcal{L}_A$ be in (2.4), $u(t, x)$ be a solution of nonlinear Schrödinger equation

$$
\begin{cases}
i \partial_t u + \mathcal{L}_A u = F(t, x), & (t, x) \in I \times \mathbb{R}^4, \\
u(t_0, x) = u_0, & x \in \mathbb{R}^4,
\end{cases}
$$

(4.2)

then for admissible pairs $(q, r), (\overline{q}, \overline{r}) \in \Lambda^n_0$, there exists a constant $C$ such that

$$
\|u(t, x)\|_{L^q_t(L^r_x(\mathbb{R}^4))} \leq C\|u_0\|_{L^2(\mathbb{R}^4)} + \|F\|_{L^q_t L^r_x(I \times \mathbb{R}^4)},
$$

(4.3)

Remark 4.3. From (4.13) and (1.26), we obtain

$$
\|\nabla_A u\|_{L^q_t L^r_x(I \times \mathbb{R}^4)} \lesssim \|u_0\|_{H^n_x(\mathbb{R}^4)} + \|\nabla_A F\|_{L^q_t L^r_x(I \times \mathbb{R}^4)},
$$

(4.4)

and

$$
\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^4)} \lesssim \|u_0\|_{H^n_x(\mathbb{R}^4)} + \|\nabla F\|_{L^q_t L^r_x(I \times \mathbb{R}^4)}, \quad r < 4.
$$

(4.5)

With dispersive estimates (4.1), Strichartz estimates (4.3),(4.5) and chain rules of non-integer order for operator $\mathcal{L}_A$ in hand, we derive the following local well-posedness theory.

Proposition 4.4 (Local well-posedness theory). Let $u_0 \in H^n_x(\mathbb{R}^4)$. Then there exists $T = T(\|u_0\|_{H^n_x}) > 0$ such that the equation (1.17) with initial data $u_0$ has a unique solution $u$ with

$$
u, \nabla_A u \in C(I; L^2(\mathbb{R}^4)) \cap L^q_t(I; L^r_x(\mathbb{R}^4)), \quad I = [0, T),
$$

(4.6)

with $(q, r) \in \Lambda^n_0$.

Proof. We argue by the standard Banach fixed point argument. To this end, let

$$
X(I) = C(I; L^2(\mathbb{R}^4)) \cap L^{q_0}_t(I; L^{r_0}(\mathbb{R}^4)), \quad (q_0, r_0) = \left(\frac{p+1}{p}, p+1\right),
$$

we consider the map

$$
\Phi : u \mapsto e^{it\mathcal{L}_A} u_0 - i \int_0^t e^{i(t-s)\mathcal{L}_A} \left|u(s)^{p-1} u(s)\right| ds
$$
on the complete metric space

$$
B := \left\{ u, \nabla_A u \in C(I; L^2(\mathbb{R}^4)) \cap L^{q_0}_t(I; L^{r_0}(\mathbb{R}^4)) : \|u\|_{X(I)}, \|\nabla_A u\|_{X(I)} \leq 2C\|u_0\|_{H^n_x}\right\}
$$

and the metric

$$
d(u, v) := \|u - v\|_{L^{q_0}_t(I; L^{r_0}(\mathbb{R}^4))}.
$$

The constant $C$ depends only on the dimension and $\alpha$, and it reflects implicit constants in the Strichartz and Sobolev embedding inequalities. We need to prove that the operator $\Phi$ is well-defined on $B$ and is a contraction map under the metric $d$ for $I$.

Throughout the proof, all spacetime norms will be on $I \times \mathbb{R}^4$. To see that $\Phi$ maps the ball $B$ to itself, we use the Strichartz inequality in Theorem 4.2 and Sobolev
embedding theorem and equivalent norms (1.26) to arrive at

\[ \| \Phi(u) \|_{X(I)} \leq C \| u_0 \|_{L^2} + C \| |u|^{p-1} u \|_{L^\infty(I,L^6)} \]

\[ \leq C \| u_0 \|_{L^2} + CT^{-\frac{2}{q_0}} \| |u|^{p-1} u \|_{L^\infty(I,L^6)} \leq C \| u_0 \|_{L^2} + CT^{-\frac{2}{q_0}} \| \nabla u \|_{L^\infty(I,L^6)} \]

Noting that \( \| u \|_{X(I)} \leq C \| u_0 \|_{H^{\frac{1}{2}}}, \) we see that for \( u \in B, \)

\[ \| \Phi(u) \|_{X(I)} \leq C \| u_0 \|_{L^2} + CT^{-\frac{2}{q_0}} (2C \| u_0 \|_{H^{\frac{1}{2}}})^p. \]

Since \( 1 < p < 3, \)

\[ 1 - \frac{2}{q_0} = 1 - \frac{2(p-1)}{p+1} > 0. \quad (4.7) \]

Therefore, we take \( T \) sufficiently small such that

\[ T^{1-\frac{2}{q_0}} (2C \| u_0 \|_{H^{\frac{1}{2}}})^p \leq \| u_0 \|_{H^{\frac{1}{2}}}, \]

we have \( \| \Phi(u) \|_{X(I)} \leq 2C \| u_0 \|_{H^{\frac{1}{2}}}, \) for \( u \in B. \) Similarly, by Strichartz estimate (1.3), chain rules (1.27), equivalent norms (1.26) and inequality (4.7), we get

\[ \| \nabla \Phi(u) \|_{X(I)} \leq C \| \nabla u_0 \|_{L^2} + C \| \nabla (|u|^{p-1} u) \|_{L^\infty(I,L^6)} \]

\[ \leq C \| \nabla u_0 \|_{L^2} + CT^{-\frac{2}{q_0}} \| |u|^{p-1} u \|_{L^\infty(I,L^6)} \| \nabla u \|_{L^\infty(I,L^6)} \]

\[ \leq C \| u_0 \|_{H^{\frac{1}{2}}} + CT^{-\frac{2}{q_0}} \| \nabla |u|^{p-1} u \|_{L^\infty(I,L^6)} \| \nabla u \|_{L^\infty(I,L^6)} \]

\[ \leq C \| u_0 \|_{L^2} + CT^{-\frac{2}{q_0}} \| \nabla \Phi(u) \|_{X(I)}. \]

This shows that \( \Phi \) maps the ball \( B \) to itself.

Finally, to prove that \( \Phi \) is a contraction, for \( u, v \in B, \) we argue as above

\[ d(\Phi(u), \Phi(v)) \leq C \| |u|^{p-1} u - |v|^{p-1} v \|_{L^\infty(I,L^6)} \]

\[ \leq CT^{-\frac{2}{q_0}} \| u - v \|_{L^\infty(I,L^6)} (\| \nabla u \|_{L^2}^{p-1} + \| \nabla v \|_{L^2}^{p-1}) \]

\[ \leq 2CT^{-\frac{2}{q_0}} (2C \| u_0 \|_{H^{\frac{1}{2}}})^{p-1} d(u, v) \]

by taking \( T \) sufficiently small such that

\[ 2CT^{-\frac{2}{q_0}} (2C \| u_0 \|_{H^{\frac{1}{2}}})^{p-1} \leq \frac{1}{2}. \]

The standard fixed point argument gives a unique solution \( u \) of (1.17) on \( I \times \mathbb{R}^4 \). By Strichartz estimate, we also get the bound

\[ \| u \|_{L^4_tL^4_x(I \times \mathbb{R}^4)} + \| \nabla u \|_{L^4_tL^4_x(I \times \mathbb{R}^4)} \leq 2C \| u_0 \|_{H^{\frac{1}{2}}(\mathbb{R}^4)}, \quad \forall \ (q,r) \in A^S_0, \ r < 4 \]

On the other hand, conservation laws for equation (1.17) hold.
Lemma 4.5 (Conservation laws). If the solution $u$ for equation (1.17) has sufficient decay at infinity and smoothness, it conserves the mass

$$M(u) = \int_{\mathbb{R}^4} |u(t, x)|^2 dx = M(u_0)$$

(4.8)

and energy

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla A u(t)|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^4} |u(t)|^{p+1} dx = E(u_0).$$

(4.9)

Proof. From equation (1.17) and using the integration by parts, we see

$$\frac{d}{dt} M(u(t)) = 2 \Re \int_{\mathbb{R}^4} u_t \bar{u} dx = 2 \Re \int_{\mathbb{R}^4} i (L_A u - |u|^{p-1} u) \bar{u} dx = 0,$$

which implies (4.8).

A similar computation yields

$$\frac{d}{dt} E(u(t)) = \Re \int_{\mathbb{R}^4} \nabla A u u_t \nabla A u + |u|^{p-1} \bar{u} u_t dx = 0,$$

which implies (4.9). □

Using the mass and energy conservation laws, we obtain

$$\|u(t)\|_{H_A^2(\mathbb{R}^4)}^2 \leq M(u_0) + 2E(u_0).$$

(4.10)

Therefore, the global well-posedness follows from the local well-posedness theory and mass and energy conservation.

4.2. Morawetz estimate. In this subsection, we derive the classical Morawetz estimate from the virial identity in Theorem 4.7.

Proposition 4.6 (Morawetz estimate). Assume that $u : I \times \mathbb{R}^4 \to \mathbb{C}$ solves the equation (1.17), then there holds

$$\int_I \int_{\mathbb{R}^4} \frac{|u(t, x)|^{p+1}}{|x|} dx dt \leq C(M(u_0), E(u_0)).$$

(4.11)

Proposition 4.6 follows from Theorem 4.7 with $a(x) = |x|$ below.

Theorem 4.7 (Virial identity). Let $a : \mathbb{R}^4 \to \mathbb{R}$ be a radial, real-valued multiplier, $a(x) = a(|x|)$ and let

$$\Phi_a(t) = \int_{\mathbb{R}^4} a(x)|u|^2 dx.$$

Then for any solution $u$ of magnetic Schrödinger equation (1.17) and initial datum $u_0 \in L^2$, $\nabla A u_0 \in L^2$, the following virial-type identity holds:

$$\frac{d^2}{dt^2} \Phi_a(t) = 4 \int_{\mathbb{R}^4} \nabla A u D^2 a \nabla A u dx + \int_{\mathbb{R}^4} |u|^2 \Delta^2 a dx + \frac{2(p-1)}{p+1} \int_{\mathbb{R}^4} |u|^{p+1} \Delta a dx,$$

(4.12)

where

$$(D^2 a)_{j,k} = \frac{\partial^2}{\partial x_j \partial x_k} a, \quad \Delta^2 a = \Delta (\Delta a),$$

for $j, k = 1, \cdots, 4$ are respectively the Hessian matrix and the bi-Laplacian of $a$. 

Proof. From equation (1.17), we see that
\[ u_t = i(\mathcal{L}_A + |u|^{p-1})u. \]
Then a simple calculation yields
\[ \frac{d\Phi_a(t)}{dt} = i \langle u, [\mathcal{L}_A, a]u \rangle, \]
and
\[ \frac{d^2\Phi_a(t)}{dt^2} = -\langle u, [\mathcal{L}_A, [\mathcal{L}_A, a]]u \rangle - \langle u, [\mathcal{L}_A, a]u \rangle^{p-1}u + \langle |u|^{p-1}, [\mathcal{L}_A, a]u \rangle. \] (4.13)
Consider the first term in (4.13), using the virial identity in [14, Theorem1.2], we arrive at
\[ -\langle u, [\mathcal{L}_A, [\mathcal{L}_A, a]]u \rangle = 4 \int_{\mathbb{R}^4} \nabla A u D^2 a \nabla A u dx = \int_{\mathbb{R}^4} |u|^2 \Delta^2 a dx. \] (4.14)
Recalling that the commutator \([\mathcal{L}_A, a] = 2 \nabla a \cdot \nabla A + \Delta a (\text{see}[14, (2.7)])\), we obtain for the reminder term by using integration by parts
\[ -\langle u, [\mathcal{L}_A, a]u \rangle^{p-1}u + \langle |u|^{p-1}, [\mathcal{L}_A, a]u \rangle = 2 \langle p-1 \rangle \int_{\mathbb{R}^4} |u|^{p+1} \Delta a dx. \] (4.15)
Collecting (4.13)-(4.15) together, we finally conclude (4.12). \(\square\)

4.3. Scattering Theory. In this subsection, we aim to establish a scattering theory by following the idea in [17].

4.3.1. Decay of potential energy. We will utilize the classical Morawetz estimate in Proposition 4.6 and dispersive estimates (4.1) and Strichartz estimates (4.3)-(4.5) that we obtained in the last two subsections to show the decay of potential energy.

Proposition 4.8. Let \( u : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C} \) be the global solution to (1.17). Then, there holds
\[ \lim_{t \to \pm \infty} \| u(t, \cdot) \|_{L^{p+1}(\mathbb{R}^4)} = 0. \] (4.16)
By interpolating with mass and energy, we obtain
\[ \lim_{t \to \pm \infty} \| u(t, \cdot) \|_{L^r(\mathbb{R}^4)} = 0, \quad \forall \ 2 < r < p + 1. \] (4.17)

Proof. We adapt Ginibre-Velo’s method in [17] to show this proposition. For completeness, we give the detailed proof. We only need to show the positive time direction.

We have
\[ \int_{\mathbb{R}^4} |u(t, x)|^{p+1} dx = \int_{|x| \leq t \log t} |u(t, x)|^{p+1} dx + \int_{|x| > t \log t} |u(t, x)|^{p+1} dx. \] (4.18)

Step 1: In this step, we will show
\[ \lim_{t \to \infty} \int_{|x| > t \log t} |u(t, x)|^{p+1} dx = 0. \] (4.19)
To do this, for given $M > 0$, we define the smoothing function
\[ \theta_M(x) = \begin{cases} \frac{|x|}{M} & \text{if } |x| \leq M \\ 1 & \text{if } |x| \leq 2M. \end{cases} \] (4.20)

Then, it is easy to check that
\[ \theta_M \in W^{1,\infty}(\mathbb{R}^4), \quad \| \nabla \theta_M \|_\infty \lesssim \frac{1}{M}, \quad \theta_M u \in C(\mathbb{R}, H^1_A(\mathbb{R}^4)). \]

By (1.17) and a simple computation, we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^4} \theta_M(x)|u(t,x)|^2 \, dx = \text{Re} \int_{\mathbb{R}^4} \theta_M u_t \bar{u} \, dx \\
= \text{Re} \int_{\mathbb{R}^4} \theta_M \bar{u} [i \mathcal{L}_A u + i|u|^{p-1} u] \, dx \\
= - \text{Im} \int_{\mathbb{R}^4} \theta_M \bar{u} \mathcal{L}_A u \, dx \\
= \text{Im} \int_{\mathbb{R}^4} (-i \nabla + A) \theta_M \cdot (i \nabla + A) u \bar{u} \, dx.
\]

Hence
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^4} \theta_M(x)|u(t,x)|^2 \, dx \right| \lesssim \|(-i \nabla + A) \theta_M \|_\infty \cdot \|u\|_{L^2_x} \cdot \|(-i \nabla + A) u\|_{L^2_x} \leq \frac{C}{M}.
\]

Integrating in time, we obtain
\[
\int_{\mathbb{R}^4} \theta_M(x)|u(t,x)|^2 \, dx \leq \frac{C}{M} t + \int_{\mathbb{R}^4} \theta_M(x)|u_0(x)|^2 \, dx, \quad \forall \, t > 0. \quad (4.21)
\]

Taking $M = t \log t$, we have
\[
\int_{|x| > t \log t} |u(t,x)|^2 \, dx \lesssim \frac{1}{\log t} + \int_{\mathbb{R}^4} \theta_M(x)|u_0(x)|^2 \, dx \\
\lesssim \frac{1}{\log t} + \int_{|x| \leq t \log t} \frac{|x|}{t \log t} |u_0(x)|^2 \, dx + \int_{|x| > t \log t} |u_0(x)|^2 \, dx \\
\lesssim \frac{1}{\log t} + \frac{1}{\sqrt{t \log t}} \int_{|x| \leq \sqrt{t \log t}} |u_0(x)|^2 \, dx + \int_{|x| > \sqrt{t \log t}} |u_0(x)|^2 \, dx \to 0 \quad \text{as } t \to \infty.
\]

Since $3 < p + 1 < 4$, so (4.19) follows by interpolating the above inequality with $\|u(t,x)\|_{L^4_x} \lesssim \|u(t,x)\|_{H^1_A} \lesssim 1$.

**Step 2:** By Morawetz estimate (4.11), we derive that for any $\epsilon > 0$, $t > 1$ and $\tau > 0$, there exists $t_0 > \max\{t, 2\tau\}$ such that
\[
\int_{t_0}^{t_0 + \tau} \int_{|x| > s \log s} |u(s,x)|^{p+1} \, dx \, ds < \epsilon. \quad (4.22)
\]
Actually, by Morawetz estimate (4.11), we see that
\[ \int_1^\infty \int_{\mathbb{R}^4} \frac{|u(s,x)|^{p+1}}{|x|} \, dx \, ds > \int_1^\infty \frac{1}{s \log s} \int_{|x| \leq s \log s} |u(s,x)|^{p+1} \, dx \, ds \]
\[ = \sum_{k=0}^\infty \int_{t+2(k+1)\tau}^{t+2k\tau} \frac{1}{s \log s} \int_{|x| \leq s \log s} |u(s,x)|^{p+1} \, dx \, ds \]
\[ \geq \sum_{k=0}^\infty \frac{1}{(t+2(k+1)\tau) \log(t+2(k+1)\tau)} \int_{t+2(k+1)\tau}^{t+2(k+1)\tau} \int_{|x| \leq s \log s} |u(s,x)|^{p+1} \, dx \, ds. \]

We note that
\[ \sum_{k=0}^\infty \frac{1}{(t+2(k+1)\tau) \log(t+2(k+1)\tau)} = +\infty, \]
then there exists \( k > 0 \) such that
\[ \int_{t+2(k+1)\tau}^{t+2k\tau} \int_{|x| \leq s \log s} |u(s,x)|^{p+1} \, dx \, ds \leq \epsilon, \]
therefore, (4.22) holds by taking \( t_0 = t + 2(k+1)\tau \).

**Step 3:** We claim that for any \( \epsilon > 0 \) and \( b > 0 \), there exists \( t_0 > b \) such that
\[ \sup_{s \in [t_0-b,t_0]} \|u(s,\cdot)\|_{L^{p+1}(\mathbb{R}^4)} < \epsilon. \] (4.23)

Taking \( t > \tau > 0 \), we have by Duhamel's formula
\[ u(t) = e^{i\tau \mathcal{L}_A} u_0 - i \int_{t}^{t-\tau} e^{i(t-s)\mathcal{L}_A} (|u|^{p-1}u)(s) \, ds - i \int_{t-\tau}^{t} e^{i(t-s)\mathcal{L}_A} (|u|^{p-1}u)(s) \, ds \]
\[ = v(t) + w(t,\tau) + z(t,\tau). \]

First, by density \( S(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4) \) and the dispersive estimate, we obtain
\[ \lim_{t \to +\infty} \|v(t)\|_{L^{p+1}(\mathbb{R}^4)} = 0. \] (4.24)

**Estimate of** \( w(t,\tau) \): Using dispersive estimate and \( \|u\|_{L^{p+1}_x} \lesssim \|u\|_{H^1} \lesssim 1 \), we get
\[ \|w(t,\tau)\|_{L^p_x} \lesssim \int_{t-\tau}^{t-\tau} (t-s)^{-2} \|u\|_{L^{p+1}_x}^{p+1} \, ds \lesssim \tau^{-1}. \] (4.25)

On the other hand, we have
\[ w(t,\tau) = e^{i\tau \mathcal{L}_A} u(t-\tau) - e^{it \mathcal{L}_A} u_0. \]
This implies
\[ \|w(t,\tau)\|_{L^2_x} \lesssim \|u_0\|_{L^2_x} \lesssim 1. \]

Interpolating this with (4.25) yields that
\[ \|w(t,\tau)\|_{L^{p+1}_x} \leq K \tau^{-\frac{1}{p+1}} \] (4.26)
for some \( K > 0 \).
**Estimate of** $z(t, \tau)$: Using Minkowski’s inequality, the dispersive estimate and Hölder’s inequality and the boundedness of potential energy, one has

\[ \|z(t, \tau)\|_{L^{p+1}_x} \leq \int_{t-\tau}^{t} \left( t - s \right)^{-\frac{3p-1}{3p+1}} \|u\|_{L^{p+1}_{x,t}} \, ds \]

\[ \leq \left( \int_{t-\tau}^{t} \left( t - s \right)^{-\frac{3p-1}{3p+1}} \left( \int_{t-\tau}^{t} \|u\|_{L^{p+1}_{x,t}}^{\frac{p-1}{3p-1}} \, ds \right)^{\frac{3p-1}{p}} \right)^{\frac{p}{3p-1}} \]

\[ \leq L_{\tau} \left( \frac{p-1}{p+1} \right) \left( \int_{t-\tau}^{t} \left( \sup_{s > t-\tau} \int_{|x| > s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} \right) \]

\[ + \int_{t-\tau}^{t} \left( \int_{|x| < s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} \]

for some $L > 0$.

Therefore,

\[ \|u(t)\|_{L^{p+1}_x} \leq \|v(t)\|_{L^{p+1}_x} + K_{\tau}^{-\frac{p-1}{p+1}} \]

\[ + L_{\tau} \left( \frac{p-1}{p+1} \right) \left( \sup_{s > t-\tau} \int_{|x| > s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} \]

\[ + L_{\tau} \left( \frac{p-1}{p+1} \right) \left( \int_{t-\tau}^{t} \left( \int_{|x| < s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} \right) . \]  

(4.27)

For any $\epsilon > 0$, there exists $\tau_\epsilon > b$ such that for any $\tau > \tau_\epsilon$ s.t.

\[ K_{\tau}^{-\frac{p-1}{p+1}} \leq K_{\tau_\epsilon}^{-\frac{p-1}{p+1}} = \frac{\epsilon}{4} . \]  

(4.28)

On the other hand, by Step 1 and (4.24), we deduce that there exists $t_1 > \tau_\epsilon$ such that for any $t \geq t_1$

\[ \|v(t)\|_{L^{p+1}_x} + L_{\tau_\epsilon} \left( \frac{p-1}{p+1} \right) \left( \sup_{s > t-\tau} \int_{|x| > s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} < \frac{\epsilon}{2} . \]  

(4.29)

Furthermore, by Step 2, we derive that there exists $t_0 > \max\{t_1 + b, 2\tau_\epsilon\}$ such that

\[ L_{\tau_\epsilon} \left( \frac{p-1}{p+1} \right) \left( \int_{t_0-2\tau_\epsilon}^{t_0} \int_{|x| < s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} < \frac{\epsilon}{4} . \]  

(4.30)

Noting that for $t \in [t_0 - b, t_0]$ and $\tau_\epsilon > b$, we have

\[ [t - \tau_\epsilon, t] \subset [t - b, t] \subset [t_0 - 2\tau_\epsilon, t_0] . \]

Thus,

\[ L_{\tau_\epsilon} \left( \frac{p-1}{p+1} \right) \left( \int_{t-\tau_\epsilon}^{t} \int_{|x| < s \log s} |u(s, x)|^{p+1} \, dx \right)^{\frac{3p-1}{p}} < \frac{\epsilon}{4} \]  

(4.31)

Plugging (4.28), (4.29) and (4.31) into (4.27), we obtain the claim (4.24).

**Step 4:** Now we turn to show (4.10). It is equivalent to show that for any $\epsilon > 0$, there exists $T_\epsilon > 0$ such that for any $t > T_\epsilon$

\[ \|u(t, x)\|_{L^{p+1}_x} < \epsilon . \]  

(4.32)
From Step 3, we know that for any $t > \tau > 0$
\[
\|u(t, x)\|_{L^{p+1}} \leq \|v(t)\|_{L^{p+1}} + K\tau^{-\frac{p-1}{p+1}} + \|z(t, \tau)\|_{L^{p+1}}. \tag{4.33}
\]
For any $\epsilon > 0$, we take $\tau_\epsilon > 0$ such that
\[
K\tau_\epsilon^{-\frac{p-1}{p+1}} = \frac{\epsilon}{4}. \tag{4.34}
\]
On the other hand, by (4.24), we can choose $t_1 > \tau_\epsilon > 0$ such that
\[
\|v(t)\|_{L^{p+1}} < \frac{\epsilon}{4}, \quad \forall \ t > t_1. \tag{4.35}
\]
Thus,
\[
\|u(t, x)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + \|z(t, \tau_\epsilon)\|_{L^{p+1}}, \quad \forall \ t > t_1. \tag{4.36}
\]
Using the dispersive estimate, we get
\[
\|z(t, \tau_\epsilon)\|_{L^{p+1}} \lesssim \int_{t-\tau_\epsilon}^{t} (t-s)^{-\frac{2(p-1)}{p+1}} \|u(s)\|_{L^{p+1}}^{p} ds \\
\leq M\tau_\epsilon^{-\frac{2(p-1)}{p+1}} \sup_{s \in [t-\tau_\epsilon, t]} \|u(s)\|_{L^{p+1}}^{p}, \tag{4.37}
\]
for some $M > 0$. Hence,
\[
\|u(t, x)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + M\tau_\epsilon^{-\frac{2(p-1)}{p+1}} \sup_{s \in [t-\tau_\epsilon, t]} \|u(s)\|_{L^{p+1}}^{p}, \quad \forall \ t > t_1. \tag{4.38}
\]
On the other hand, applying (4.23) with $b = \tau_\epsilon$, we derive that there exists $t_0 \geq \tau_\epsilon$ such that
\[
\sup_{t \in [t_0 - \tau_\epsilon, t_0]} \|u(t)\|_{L^{p+1}} < \epsilon. \tag{4.39}
\]
Now, we utilize the bootstrap argument to prove (4.32). To do this, we define
\[
t_\epsilon := \sup\{t \geq t_0 : \|u(s)\|_{L^{p+1}} < \epsilon, \quad \forall \ s \in [t_0 - \tau_\epsilon, t]\}. \tag{4.40}
\]
Then, it is equivalent to show that $t_\epsilon = +\infty$. By contradiction, we assume that $t_\epsilon < +\infty$. Then,
\[
\|u(t_\epsilon)\|_{L^{p+1}} = \epsilon. \tag{4.41}
\]
Using (4.36) with $t = t_\epsilon$, we obtain
\[
\epsilon \leq \frac{\epsilon}{2} + M\tau_\epsilon^{-\frac{2(p-1)}{p+1}} \|u(t_\epsilon)\|_{L^{p+1}}^{p} \implies \frac{3-p}{2} \tau_\epsilon^{\frac{3-p}{p+1}} e^{p-1} \geq \frac{1}{2M}. \tag{4.42}
\]
This together with (4.34) implies that
\[
\frac{3-p}{2} \tau_\epsilon^{\frac{3-p}{p+1}} (4K\tau_\epsilon^{-\frac{p-1}{p+1}})^{p-1} \geq \frac{1}{2M} \implies \tau_\epsilon^{\frac{2-p}{p+1}} \leq 2M(4K)^{p-1}
\]
which contradicts with the fact that
\[
\lim_{\epsilon \to 0} \tau_\epsilon = +\infty.
\]
Therefore, we conclude the proof of Proposition 4.8.
4.4. Global space-time bound. In this subsection, we utilize the decay of potential energy in Proposition 4.8 and the continuous argument to show the global space-time bound for the global solution.

**Proposition 4.9.** Let \( u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C} \) be the global solution to (1.17). Then, there holds

\[
\| u(t, x) \|_{L^q_t H^s_x(\mathbb{R} \times \mathbb{R}^d)} < +\infty,
\]

for any \((q, r) \in \Lambda_0^s\). As a consequence, the solution \( u(t, x) \) scatters.

**Proof.** We first utilize Proposition 4.8 and continuous argument to prove

\[
\| u \|_{L^q_t H^s_x([0, \infty), H^{1, r}_x(\mathbb{R}^d))} < +\infty, \quad (\gamma, \rho) = \left( \frac{p+1}{p-1}, p+1 \right).
\]

Using local well-posed theory, we get for any \( T > 0 \),

\[
\| u \|_{L^q_t H^s_x([0, T), H^{1, r}_x(\mathbb{R}^d))} \leq C(T, M(u_0), E(u_0)).
\]

Thus, we only need to show for some \( T > 0 \)

\[
\| u \|_{L^q_t H^s_x([T, \infty), H^{1, r}_x(\mathbb{R}^d))} < +\infty. \tag{4.44}
\]

For \( t > T > 0 \), we have by Duhamel’s formula

\[
u(t) = e^{i(t-T)\mathcal{L}_A}u(T) - i \int_T^t e^{i(t-s)\mathcal{L}_A}(|u|^{p-1}u)(s) \, ds.
\]

By Strichartz’s estimate and Sobolev embedding, we get

\[
\| u \|_{L^q_t H^s_x([T, \infty), H^{1, r}_x(\mathbb{R}^d))} \leq C\| u \|_{H^1_x} + C\| u^{p-1}u \|_{L^q_t H^s_x(\mathbb{R}^d))} \leq C\| u \|_{H^1_x} + C\sup_{t \geq T} \| u(t) \|_{L^q_t H^s_x(\mathbb{R}^d))} \leq C\| u \|_{H^1_x} + C\epsilon(T) \| u \|_{L^q_t H^s_x(\mathbb{R}^d))}.
\]

From Proposition 4.8, we know that

\[
\lim_{T \to \infty} \epsilon(T) = 0.
\]

This together with the continuous argument yields (4.44). And so (4.43) follows. Furthermore, using the estimate

\[
\| u^{p-1}u \|_{L^q_t H^s_x(\mathbb{R}^d))} \leq \| u \|_{L^q_t H^s_x(\mathbb{R}^d))} \| u \|_{H^s_x(\mathbb{R}^d))}
\]

and Strichartz estimate, one can now deduce that \( u \in L^q_t H^s_x(\mathbb{R}^d) \) for any admissible \((q, r) \in \Lambda_0^s\).

\[\square\]

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