Caffarelli-Kohn-Nirenberg inequalities on Besov and Triebel-Lizorkin-type spaces

Douadi Drihem *

March 14, 2023

Abstract

We present some Caffarelli-Kohn-Nirenberg-type inequalities on Herz-type Besov-Triebel-Lizorkin spaces, Besov-Morrey and Triebel-Lizorkin-Morrey spaces. More precisely, we investigate the inequalities

\[ \| f \|_{\dot{K}^{\alpha_1, r}_{\alpha, \sigma}, 1} \leq c \| f \|_{\dot{K}^{\alpha_3, 1}_{\alpha_1, \delta_1} A_{\beta}}^{1-\theta} \]  

and

\[ \| f \|_{E^{s, p, 1}} \leq c \| f \|_{\dot{M}^s} \| f \|_{N^{p, \beta, v}}, \]

with some appropriate assumptions on the parameters, where \( \dot{K}^{\alpha_1, r}_{\alpha, \sigma} \) is the Herz-type Bessel potential spaces, which are just the Sobolev spaces if \( \alpha_1 = 0, 1 < r = v < \infty \) and \( \sigma \in \mathbb{N}_0 \), and \( \dot{K}^{\alpha_3, 1}_{\alpha_1, \delta_1} A_{\beta} \) are Besov or Triebel-Lizorkin spaces if \( \alpha_3 = 0 \) and \( \delta_1 = p \). The usual Littlewood-Paley technique, Sobolev and Franke embeddings are the main tools of this paper. Some remarks on Hardy-Sobolev inequalities are given.

MSC 2010: 46B70, 46E35.

Key Words and Phrases: Besov spaces, Triebel-Lizorkin spaces, Morrey spaces, Herz spaces, Caffarelli-Kohn-Nirenberg inequalities.

1 Introduction

Major results in harmonic analysis and partial differential equations invoke some inequalities. Some examples can be mentioned such as: Caffarelli, Kohn and Nirenberg in [4]. They proved the following useful inequality:

\[ \| x^\gamma f \|_\tau \leq c \| x^{\beta} f \|_q \| x^{\alpha} \nabla f \|_p^{1-\theta}, \quad f \in \mathcal{D}(\mathbb{R}^n), \]  

(1)

where \( 1 \leq p, q < \infty, \tau > 0, 0 \leq \theta \leq 1, \alpha, \beta, \gamma \in \mathbb{R} \) satisfy some suitable conditions. This inequality plays an important role in theory of PDE’s, which extended to fractional Sobolev spaces by [27]. This estimate can be rewritten in the following form:

\[ \| f \|_{\dot{K}^{\gamma, r}_{\alpha, \sigma}} \leq c \| f \|_{\dot{K}^{\beta, 1}_{\alpha, \sigma}} \| \nabla f \|_{\dot{K}^{\beta, 1}_{\alpha, \sigma}}^{1-\theta}, \quad f \in \mathcal{D}(\mathbb{R}^n), \]

* M’sila University, Department of Mathematics, Laboratory of Functional Analysis and Geometry of Spaces, P.O. Box 166, M’sila 28000, Algeria, e-mail: douadidr@yahoo.fr
where $K^{α,p}_q$ is the Herz space, see Definition 1 below. These function spaces play an important role in Harmonic Analysis. After they have been introduced in [17], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [1], in the semilinear parabolic equations [11], in the summability of Fourier transforms [15], and in the Cauchy problem for Navier-Stokes equations [38]. For important and latest results on Herz spaces, we refer the reader to the papers [29], [45] and to the monograph [20].

Again (1) with $α = β = γ = 0$, is just

$$
\|f\|_{M^s_p} \leq c\|f\|_{M^{q}_{p}}^θ\|\nabla f\|_{M^{q}_{p}}^{1-θ}, \quad f \in \mathcal{D}(\mathbb{R}^n).
$$

where $M^s_p$, $1 ≤ u ≤ p < ∞$ is the Morrey space. The main purpose of this paper is to present more general version of such inequalities. More precisely, we extend this estimate to Herz-type Besov-Triebel-Lizorkin spaces, called $K^{α,p}_q B^s_β$ and $K^{α,p}_q F^s_β$, which generalize the usual Besov and Triebel-Lizorkin spaces. We mean that

$$
\dot{K}^{α,p}_q B^s_β = B^s_ν, \quad \dot{K}^{α,p}_q F^s_β = F^s_ν.
$$

In addition $K^{α,p}_q F^0_2$ are just the Herz spaces $K^{α,p}_q$ when $1 < p, q < ∞$ and $-\frac{2}{q} < α < n(1 - \frac{1}{q})$. In the same manner, we extend these inequalities to Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Our approach based on the Littlewood-Paley technique of Triebel [37] and some results obtained by the author in [6, 7, 8].

The structure of this paper needs some notation. As usual, $\mathbb{R}^n$ denotes the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For any $u > 0$, $k ∈ \mathbb{Z}$ we set

$$
C(u) = \{x ∈ \mathbb{R}^n : \frac{u}{2} < |x| ≤ u\} \quad \text{and} \quad C_k = C(2^k). \quad \chi_k, \quad \text{for} \quad k ∈ \mathbb{Z}, \quad \text{denote the characteristic function of the set} \quad C_k.
$$

The expression $f ≈ g$ means that $C g ≤ f ≤ c g$ for some independent constants $c, C$ and non-negative functions $f$ and $g$.

For any measurable subset $Ω ⊆ \mathbb{R}^n$ the Lebesgue space $L^p(Ω), 0 < p ≤ ∞$ consists of all measurable functions for which

$$
\|f\|_{L^p(Ω)} = \left(\int_Ω |f(x)|^p \, dx\right)^{1/p} < ∞, 0 < p < ∞
$$

and

$$
\|f\|_{L^∞(Ω)} = \text{ess-sup}_{x ∈ Ω} |f(x)| < ∞.
$$

If $Ω = \mathbb{R}^n$, then we put $L^p(\mathbb{R}^n) = L^p$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$. The symbol $S(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$ and we denote by $S'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f ∈ S(\mathbb{R}^n)$ by

$$
\mathcal{F}(f)(ξ) = (2π)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdotξ} f(x) \, dx, \quad ξ ∈ \mathbb{R}^n.
$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $S'(\mathbb{R}^n)$ in the usual way. The Hardy-Littlewood maximal operator $M$ is defined
on $L^1_{\text{loc}}$ by
\[ \mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n \]
and $\mathcal{M}_\tau f = (\mathcal{M}|f|^\tau)^{1/\tau}$, $0 < \tau < \infty$.

Given two quasi-Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of $X$ in $Y$ is continuous. We use $c$ as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

## 2 Function spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_{q}^{\alpha,p}$.

**Definition 1** Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_{q}^{\alpha,p}$ is defined by
\[ \dot{K}_{q}^{\alpha,p} = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{K}_{q}^{\alpha,p}} < \infty \}, \]
where
\[ \| f \|_{\dot{K}_{q}^{\alpha,p}} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| f \chi_k \|_{q}^p \right)^{1/p} \]
with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_{q}^{\alpha,p}$ are quasi-Banach spaces and if $\min(p,q) \geq 1$ then $\dot{K}_{q}^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ the space $\dot{K}_{p}^{0,p}$ coincides with the Lebesgue space $L^p$. In addition
\[ \dot{K}_{p}^{\alpha,p} = L^p(\mathbb{R}^n, |\cdot|^\alpha), \quad \text{(Lebesgue space equipped with power weight)}, \]
where
\[ \| f \|_{L^p(\mathbb{R}^n, |\cdot|^\alpha)} = \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha \, dx \right)^{1/p}. \]

Notice that
\[ \dot{K}_{q}^{\alpha,p} \hookrightarrow S'(\mathbb{R}^n) \]
for any $\alpha < n(1 - \frac{1}{p})$, $1 \leq p, q \leq \infty$ or $\alpha = n(1 - \frac{1}{q})$, $p = 1$ and $1 \leq q \leq \infty$. We mean that,
\[ T_{f}(\varphi) = \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx, \quad \varphi \in S(\mathbb{R}^n), f \in \dot{K}_{q}^{\alpha,p} \]
generates a distribution $T_{f} \in S'(\mathbb{R}^n)$. A detailed discussion of the properties of these spaces may be found in [16, 19, 22], and references therein.

The following lemma is the $\dot{K}_{q}^{\alpha,p}$-version of the Plancherel-Polya-Nikolskij inequality.

**Lemma 1** Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, \tau, q, r \leq \infty$. We suppose that $\alpha_1 + \frac{\alpha_1}{q} > 0, 0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$. Then there exists a positive constant $c > 0$ independent of $R$ such that for all $f \in \dot{K}_{q}^{\alpha_2,s} \cap S'(\mathbb{R}^n)$ with $\text{supp} \, Ff \subset \{ \xi : |\xi| \leq R \}$, we have
\[ \| f \|_{\dot{K}_{q}^{\alpha_1,r}} \leq c \, R^n \left( \begin{array}{c} n \alpha_2 - n \alpha_1 \delta \end{array} \right) \| f \|_{\dot{K}_{q}^{\alpha_2,s}}, \]
where
\[ \delta = \left\{ \begin{array}{ll} r, & \text{if } \alpha_2 = \alpha_1, \\
\tau, & \text{if } \alpha_2 > \alpha_1. \end{array} \right. \]
Lemma 2 Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, \tau, q, r \leq \infty$. We suppose that $\alpha_1 + \frac{n}{s} > 0$, $0 < s \leq q \leq \infty$ and $\alpha_2 \geq \alpha_1 + \frac{n}{s} - \frac{n}{q}$. Then there exists a positive constant $c$ independent of $R$ such that for all $f \in K_q^{\alpha_2, \delta} \cap S^r(\mathbb{R}^n)$ with $\text{supp } \mathcal{F} f \subset \{ \xi : |\xi| \leq R \}$, we have
\[
\| f \|_{K_q^{\alpha_1, r}} \leq c R^{\frac{n}{q} - \frac{n}{s} + \alpha_2 - \alpha_1} \| f \|_{K_q^{\alpha_2, \delta}},
\]
where
\[
\delta = \begin{cases} 
    r, & \text{if } \alpha_2 = \alpha_1 + \frac{n}{s} - \frac{n}{q}, \\
    \tau, & \text{if } \alpha_2 > \alpha_1 + \frac{n}{s} - \frac{n}{q}.
\end{cases}
\]

The proof of these inequalities is given in [3], Lemmas 3.10 and 3.14. Let $1 < q < \infty$ and $0 < p \leq \infty$. If $f$ is a locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, then
\[
\| \mathcal{M} f \|_{K_q^{\alpha, p}} \leq c \| f \|_{K_q^{\alpha, p}},
\]
see [19]. We need the following lemma, which is basically a consequence of Hardy’s inequality in the sequence Lebesgue space $\ell^q$.

Lemma 3 Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive real numbers, such that
\[
\| \{\varepsilon_k\}_{k \in \mathbb{N}_0} \|_{\ell^q} = I < \infty.
\]
Then the sequences $\{\delta_k : \delta_k = \sum_{j \leq k} a^{k-j} \varepsilon_j\}_{k \in \mathbb{N}_0}$ and $\{\eta_k : \eta_k = \sum_{j \geq k} a^{j-k} \varepsilon_j\}_{k \in \mathbb{N}_0}$ belong to $\ell^q$, and
\[
\| \{\delta_k\}_{k \in \mathbb{N}_0} \|_{\ell^q} + \| \{\eta_k\}_{k \in \mathbb{N}_0} \|_{\ell^q} \leq c I,
\]
with $c > 0$ only depending on $a$ and $q$.

Some of our results of this paper are based on the following result, see Tang and Yang [33].

Lemma 4 Let $1 < \beta < \infty, 1 < q < \infty$ and $0 < p \leq \infty$. If $\{f_j\}_{j=0}^\infty$ is a sequence of locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, then
\[
\left\| \left( \sum_{j=0}^\infty (\mathcal{M} f_j)^\beta \right)^{1/\beta} \right\|_{K_q^{\alpha, p}} \leq \left\| \left( \sum_{j=0}^\infty |f_j|^\beta \right)^{1/\beta} \right\|_{K_q^{\alpha, p}}.
\]

Now, we present the Fourier analytical definition of Herz-type Besov and Triebel-Lizorkin spaces and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let $\varphi_0$ be a function in $S(\mathbb{R}^n)$ satisfying $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| \geq \frac{3}{2}$. We put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{1-j}x)$ for $j = 1, 2, 3, \ldots$. 

Remark 1 We would like to mention that Lemma 2 improves the classical Plancherel-Polya-Nikolskij inequality by taking $\alpha_1 = \alpha_2 = 0, r = s$ and using the embedding $\ell^q \hookrightarrow \ell^s$. In the previous lemma we have not treated the case $s < q$. The next lemma gives a positive answer.
Then \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) is a resolution of unity, \( \sum_{j=0}^{\infty} \varphi_j(x) = 1 \) for all \( x \in \mathbb{R}^n \). Thus we obtain the Littlewood-Paley decomposition

\[
f = \sum_{j=0}^{\infty} F^{-1} \varphi_j * f
\]

of all \( f \in S'(\mathbb{R}^n) \) (convergence in \( S'(\mathbb{R}^n) \)). We are now in a position to state the definition of Herz-type Besov and Triebel-Lizorkin spaces.

**Definition 2** Let \( \alpha, s \in \mathbb{R}, 0 < p, q \leq \infty \) and \( 0 < \beta \leq \infty \).

(i) The Herz-type Besov space \( K^{\alpha,p}_s B^{\beta}_\beta \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\| f \|_{K^{\alpha,p}_s B^{\beta}_\beta} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \| F^{-1} \varphi_j * f \|_{K^{\alpha,p}_q}^\beta \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).

(ii) Let \( 0 < p, q < \infty \). The Herz-type Triebel-Lizorkin space \( K^{\alpha,p}_q F^s_{p,\beta} \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\| f \|_{K^{\alpha,p}_q F^s_{p,\beta}} = \left( \sum_{j=0}^{\infty} 2^{js\beta} | F^{-1} \varphi_j * f |_{p}^\beta \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).

**Remark 2** Let \( s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty \) and \( \alpha > -\frac{n}{q} \). The spaces \( K^{\alpha,p}_s B^{\beta}_\beta \) and \( K^{\alpha,p}_q F^s_{p,\beta} \) are independent of the particular choice of the smooth dyadic resolution of unity \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) (in the sense of equivalent quasi-norms). In particular \( K^{\alpha,p}_s B^{\beta}_\beta \) and \( K^{\alpha,p}_q F^s_{p,\beta} \) are quasi-Banach spaces and if \( p, q, \beta \geq 1 \), then they are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [5, 6, 7, 8, 9, 39, 40, 42].

Now we give the definitions of the spaces \( B^{s}_{p,\beta} \) and \( F^{s}_{p,\beta} \).

**Definition 3** (i) Let \( s \in \mathbb{R} \) and \( 0 < p, \beta \leq \infty \). The Besov space \( B^{s}_{p,\beta} \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\| f \|_{B^{s}_{p,\beta}} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \| F^{-1} \varphi_j * f \|_{p}^\beta \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).

(ii) Let \( s \in \mathbb{R}, 0 < p < \infty \) and \( 0 < \beta \leq \infty \). The Triebel-Lizorkin space \( F^{s}_{p,\beta} \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\| f \|_{F^{s}_{p,\beta}} = \left( \sum_{j=0}^{\infty} 2^{js\beta} | F^{-1} \varphi_j * f |_{p}^\beta \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).
The theory of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$ has been developed in detail in [35, 36] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < \beta \leq \infty$,

$$K_p^{0,p}B_{p,\beta} = B_{p,\beta}^s \quad \text{and} \quad K_p^{0,p}F_{p,\beta} = F_{p,\beta}^s.$$ 

Let $w$ denote a positive, locally integrable function and $0 < p < \infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f\|_{L^p(\mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$ 

For $q \in [1, \infty)$ we denote by $\mathcal{A}_q$ the Muckenhoupt class of weights, and $\mathcal{A}_\infty = \bigcup_{q \geq 1} \mathcal{A}_q$. We refer to [12] for the general properties of these classes. Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Besov spaces $B_{p,\beta}^s(\mathbb{R}^n, w)$ to be the set of all distributions $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s(\mathbb{R}^n, w)} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \varphi_j * f\|_{L^p(\mathbb{R}^n, w)}^{\beta} \right)^{1/\beta}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel-Lizorkin spaces $F_{p,\beta}^s(\mathbb{R}^n, w)$ to be the set of all distributions $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,\beta}^s(\mathbb{R}^n, w)} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \varphi_j * f\|_{L^p(\mathbb{R}^n, w)}^{\beta} \right)^{1/\beta}$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required.

The spaces $B_{p,\beta}^s(\mathbb{R}^n, w) = B_{p,\beta}^s(w)$ and $F_{p,\beta}^s(\mathbb{R}^n, w) = F_{p,\beta}^s(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, \beta$ weighted spaces). Let $\varphi \in \mathcal{S}^\prime(\mathbb{R}^n)$ such that $\varphi \circ K(x) \equiv x$ for a positive, locally integrable function and $0 < p < \infty$. We obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces. Let $\mathcal{N}$ be a power weight, i.e., $\mathcal{N}(\xi) = |\xi|^\gamma$ with $\gamma > -n$. Then we have

$$B_{p,\beta}^s(w) = K_p^{\mathcal{N}} B_{p,\beta}^s \quad \text{and} \quad F_{p,\beta}^s(w) = K_p^{\mathcal{N}} F_{p,\beta}^s,$$

in the sense of equivalent quasi-norms.

**Definition 4** (i) Let $1 < q < \infty$, $0 < p < \infty$, $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$ and $s \in \mathbb{R}$. Then the Herz-type Bessel potential space $k_{q,s}^{\alpha,p}$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{k_{q,s}^{\alpha,p}} = \left\| \left( 1 + |\xi|^2 \right)^{\frac{1}{2}} f \right\|_{k_{q}^{\alpha,p}} < \infty.$$ 

(ii) Let $1 < q < \infty$, $0 < p < \infty$, $-\frac{2}{q} < \alpha < n(1 - \frac{1}{q})$ and $m \in \mathbb{N}$. The homogeneous Herz-type Sobolev space $W_{q,m}^{\alpha,p}$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{W_{q,m}^{\alpha,p}} = \sum_{|\beta| \leq m} \left\| \frac{\partial^\beta f}{\partial^\beta x} \right\|_{k_{q,s}^{\alpha,p}} < \infty,$$

where the derivatives must be understood in the sense of distribution.
In the following, we will present the connection between the Herz-type Triebel-Lizorkin spaces and the Herz-type Bessel potential spaces; see [21, 39]. Let $1 < q < \infty$, $1 < p < \infty$ and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$. If $s \in \mathbb{R}$, then
\[ \dot{K}_q^{\alpha,p} F^s_2 = \dot{K}_q^{\alpha,p} \] (3)
with equivalent norms. If $s = m \in \mathbb{N}$, then
\[ \dot{K}_q^{\alpha,p} F^m_2 = W^{\alpha,p}_q \] (4)
with equivalent norms. In particular
\[ \dot{K}_p^{\alpha,p} F^m_2 = W^p_m \] (Sobolev spaces)
and
\[ \dot{K}_q^{\alpha,p} F^0_2 = \dot{K}_q^{\alpha,p} \] (5)
with equivalent norms. Let $0 < \theta < 1$,
\[ \alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{\beta} = \frac{1 - \theta}{\beta_0} + \frac{\theta}{\beta_1} \]
and
\[ s = (1 - \theta)s_0 + \theta s_1. \]
For simplicity, in what follows, we use $\dot{K}_q^{\alpha,p} A^s_\beta$ to denote either $\dot{K}_q^{\alpha,p} B^s_\beta$ or $\dot{K}_q^{\alpha,p} F^s_\beta$. As an immediate consequence of Hölder’s inequality we have the so-called interpolation inequalities:
\[ \|f\|_{\dot{K}_q^{\alpha,p} A^s_\beta} \leq \|f\|_{\dot{K}_q^{\alpha_0,p_0} A^{s_0}_{\beta_0}}^{1-\theta} \|f\|_{\dot{K}_q^{\alpha_1,p_1} A^{s_1}_{\beta_1}}^\theta \] (6)
holds for all $f \in \dot{K}_q^{\alpha_0,p_0} A^{s_0}_{\beta_0} \cap \dot{K}_q^{\alpha_1,p_1} A^{s_1}_{\beta_1}$.
We collect some embeddings on these functions spaces as obtained in [6]-[7]. First we have elementary embeddings within these spaces. Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{q}$. Then
\[ \dot{K}_q^{\alpha,p} B^s_{\min(\beta,p,q)} \hookrightarrow \dot{K}_q^{\alpha,p} F^s_\beta \hookrightarrow \dot{K}_q^{\alpha,p} B^s_{\max(\beta,p,q)}. \] (7)

**Theorem 1** Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r, \beta \leq \infty, \alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that
\[ s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2. \]
Let $0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$ or $0 < s \leq q \leq \infty$ and
\[ \alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}. \] (8)

(i) We have the embedding
\[ \dot{K}_q^{\alpha_2,\theta} B^{s_2}_\beta \hookrightarrow \dot{K}_q^{\alpha_1,\theta} B^{s_1}_\beta, \]
where
\[ \theta = \begin{cases} r, & \text{if } \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 = \alpha_1, q \leq s, \\ p, & \text{if } \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}, s \leq q \text{ or } \alpha_2 > \alpha_1, q \leq s. \end{cases} \]
(ii) Let $0 < q, s < \infty$. The embedding
\[ \dot{K}_q^{\alpha_2, r} F_s^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1, p} F_{s_1}^p \]
holds if $0 < r \leq p < \infty$, where
\[ \theta = \begin{cases} \beta, & \text{if } 0 < s \leq q < \infty \text{ and } \alpha_2 + \frac{n}{q} = \alpha_1 + \frac{n}{s}; \\ \infty, & \text{otherwise.} \end{cases} \]

We now present an immediate consequence of the Sobolev embeddings, which called Hardy-Sobolev inequalities.

**Corollary 1** Let $1 < q \leq s < \infty$, $1 < q < n$ and $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in \dot{W}^1_q$
\[ \int_{\mathbb{R}^n} \left( \frac{|f(x)|^s}{|x|^{-\alpha}} \right) dx \leq c \left( \sum_{|\beta| = 1} \left\| \frac{\partial^\beta f}{\partial x^\beta} \right\|_{K_q^\alpha} \right)^s \leq c \left( \sum_{|\beta| = 1} \left\| \frac{\partial^\beta f}{\partial x^\beta} \right\|_{q} \right)^s. \]

Now we recall the Franke embedding, see [9].

**Theorem 2** Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p, q < \infty, 0 < \theta \leq \infty, \alpha_1 > -\frac{n}{s} \text{ and } \alpha_2 > -\frac{n}{q}$. We suppose that
\[ s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2. \]

Let
\[ 0 < q < s < \infty \quad \text{and} \quad \alpha_2 \geq \alpha_1, \]
or
\[ 0 < s \leq q < \infty \quad \text{and} \quad \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}. \]

Then
\[ \dot{K}_q^{\alpha_2, p} B_{s_2}^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1, p} F_{s_1}^s. \]

**Corollary 2** Let $1 < q \leq s < \infty$ with $1 < q < n$. Let $\alpha = \frac{n}{q} - \frac{n}{s} - 1$. There is a constant $c > 0$ such that for all $f \in B_{q, s}^1$
\[ \int_{\mathbb{R}^n} \left( \frac{|f(x)|^s}{|x|^{-\alpha}} \right) dx \leq c \left\| f \right\|^s_{K_q^{\alpha} B_1^1} \leq c \left\| f \right\|^s_{B_{q, s}^1}. \]

**Remark 3** We would like to mention that in Theorem 1 and Theorem 2 the assumptions $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$, (8) and $0 < r \leq p < \infty$ are necessary, see [6, 7, 9].

Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a resolution of unity. For any $a > 0$, $f \in S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote, Peetre maximal function,
\[ (\mathcal{F}^{-1} \varphi_j)^{*, a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\mathcal{F}^{-1} \varphi_j \ast f(y)|}{(1 + 2^j |x - y|)^a}, \quad j \in \mathbb{N}_0. \]

We now present a fundamental characterization of the above spaces, which plays an essential role in this paper, see [41, Theorem 1].
Theorem 3 Let \( s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \leq \infty \), and \( \alpha > -\frac{n}{q} \). Let \( a > \frac{n}{\min\left(\frac{n}{\beta}, \frac{n}{q}\right)} \).

Then

\[
\|f\|_{\dot{K}^{\alpha,p}_q B^s_\beta}^* = \left( \sum_{j=0}^{\infty} 2^{js\beta} \| (\mathcal{F}^{-1} \varphi_j)^* a f \|_{\dot{K}^{\alpha,p}_q}^{\beta} \right)^{1/\beta},
\]

is an equivalent quasi-norm in \( \dot{K}^{\alpha,p}_q B^s_\beta \). Let \( a > \frac{n}{\min\left(\frac{n}{\beta}, \frac{n}{q}\right)} \).

Then

\[
\|f\|_{\dot{K}^{\alpha,p}_q F^s_\beta}^* = \left( \sum_{j=0}^{\infty} 2^{js\beta} (\mathcal{F}^{-1} \varphi_j)^* a f \right)^{1/\beta},
\]

is an equivalent quasi-norm in \( \dot{K}^{\alpha,p}_q F^s_\beta \).

Let \( 0 < p, q \leq \infty \). For later use we introduce the following abbreviations:

\[
\sigma_q = n \max\left(\frac{1}{q} - 1, 0\right) \quad \text{and} \quad \sigma_{p,q} = n \max\left(\frac{1}{p} - 1, \frac{1}{q} - 1, 0\right).
\]

In the next we shall interpret \( L^1_{\text{loc}} \) as the set of regular distributions.

Theorem 4 Let \( 0 < p, q, \beta \leq \infty, \alpha > -\frac{n}{q}, \alpha_0 = n - \frac{n}{q} \), and \( s > \max(\sigma_q, \alpha - \alpha_0) \). Then

\[
\dot{K}^{\alpha,p}_q A^s_\beta \hookrightarrow L^1_{\text{loc}},
\]

where \( 0 < p, q < \infty \) in the case of Herz-type Triebel-Lizorkin spaces.

Proof. Let \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) be a smooth dyadic resolution of unity. We set

\[
\varrho_k = \sum_{j=0}^{k} \mathcal{F}^{-1} \varphi_j * f, \quad k \in \mathbb{N}_0.
\]

For technical reasons, we split the proof into two steps.

Step 1. We consider the case \( 1 \leq q \leq \infty \). In order to prove we additionally do it into the four Substeps 1.1, 1.2, 1.3 and 1.4.

Substep 1.1. \(-\frac{n}{q} < \alpha < \alpha_0\). Since \( s > 0 \) and \( \dot{K}^{\alpha,p}_q \hookrightarrow \dot{K}^{\alpha,\max(1,p)}_q \), we have

\[
\sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j * f \|_{\dot{K}^{\alpha,\max(1,p)}_q} \lesssim \| f \|_{\dot{K}^{\alpha,p}_q A^s_\beta}.
\]

Then, the sequence \( \{ \varrho_k \}_{k \in \mathbb{N}_0} \) converges to \( g \in \dot{K}^{\alpha,\max(1,p)}_q \). Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). Write

\[
\langle f - g, \varphi \rangle = \langle f - \varrho_N, \varphi \rangle + \langle g - \varrho_N, \varphi \rangle, \quad N \in \mathbb{N}_0.
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{S}(\mathbb{R}^n) \). Clearly, the first term tends to zero as \( N \to \infty \), while by Hölder’s inequality there exists a constant \( C > 0 \) independent of \( N \) such that

\[
|\langle g - \varrho_N, \varphi \rangle| \leq C \| g - \varrho_N \|_{\dot{K}^{\alpha,\max(1,p)}_q}.
\]
which tends to zero as $N \to \infty$. From this and $K_q^{\alpha, \text{max}(1,p)} \hookrightarrow L^1_{\text{loc}}$, because of $\alpha < \alpha_0$, we deduce the desired result. In addition, we have

$$\dot{K}_q^{\alpha,p} A^s_{\beta} \hookrightarrow \dot{K}_q^{\alpha,\text{max}(1,p)}.$$  

**Substep 1.2.** $\alpha \geq \alpha_0$ and $1 < q \leq \infty$. Let $1 < q_1 < \infty$ be such that

$$s > \alpha + \frac{n}{q} - \frac{n}{q_1}.$$  

We distinguish two cases:

- $q_1 = q$. By Theorem \(\text{II}'(i)\), we obtain

$$\dot{K}_q^{\alpha,p} B^s_{\beta} \hookrightarrow \dot{K}_q^{0,q} B^{s-\alpha}_{\beta} = B^{s-\alpha}_{q_1,\beta} \hookrightarrow L^1_{\text{loc}},$$  

where the last embedding follows by the fact that

$$B^{s-\alpha}_{q_1,\beta} \hookrightarrow L^q,$$  

(9)

because of $s - \alpha > 0$. The Herz-type Triebel-Lizorkin case follows by the second embeddings of (7).

- $1 < q_1 < \alpha$ or $1 < q < q_1 < \infty$. If we assume the first possibility then Theorem \(\text{II}'(i)\) and Substep 1.1 yield

$$\dot{K}_q^{\alpha,p} B^s_{\beta} \hookrightarrow \dot{K}_q^{0,q} B^{s-\alpha}_{\beta} \rightarrow L^1_{\text{loc}},$$  

since $\alpha + \frac{n}{q} > \frac{n}{q_1}$. The latter possibility follows again by Theorem \(\text{II}'(i)\). Indeed, we have

$$\dot{K}_q^{\alpha,p} B^s_{\beta} \hookrightarrow \dot{K}_q^{0,q} B^{s+\alpha_0-\alpha}_{\beta} \rightarrow \dot{K}_q^{0,q_1} B^{s-\alpha}_{\beta} \rightarrow L^1_{\text{loc}},$$  

where the last embedding follows by the fact that

$$B^{s-\alpha}_{q_1,\beta} \hookrightarrow L^{q_1}.$$  

(10)

Therefore from (7) we obtain the desired embeddings.

**Substep 1.3.** $q = 1$ and $\alpha > 0$. We have

$$\dot{K}_1^{\alpha,p} B^s_{\beta} \hookrightarrow \dot{K}_1^{0,1} B^{s-\alpha}_{\beta} = B^{s-\alpha}_{1,\beta} \hookrightarrow L^1,$$  

since $s > \alpha$.

**Substep 1.4.** $q = 1$ and $\alpha = 0$. Let $\alpha_3$ be a real number such that $\max(-n, -s) < \alpha_3 < 0$. From Theorem \(\text{II}\) we get

$$\dot{K}_1^{0,p} A^s_{\beta} \hookrightarrow \dot{K}_1^{\alpha_3,p} A^{s+\alpha_3}_{\beta}.$$  

We have

$$\sum_{k=0}^{\infty} \| F^{-1} \varphi_k * f \|_{\dot{K}_1^{\alpha_3,\text{max}(1,p)}} \lesssim \| f \|_{\dot{K}_1^{\alpha_3,p} A^{s+\alpha_3}_{\beta}} \lesssim \| f \|_{\dot{K}_1^{0,p} A^s_{\beta}},$$  

since $\alpha_3 + s > 0$. Using the same type of arguments as in Substep 1.1 it is easy to see that

$$\dot{K}_1^{\alpha_3,p} A^{s+\alpha_3}_{\beta} \hookrightarrow \dot{K}_1^{\alpha_3,\text{max}(1,p)} \hookrightarrow L^1_{\text{loc}}.$$
Step 2. We consider the case $0 < q < 1$.

Substep 2.1. $-\frac{n}{q} < \alpha < 0$. By Lemma 1 we obtain

$$\sum_{j=0}^{\infty} \|\mathcal{F}^{-1} \varphi_j * f\|_{K^{\alpha}_{1, \text{max}(1,p)}} \lesssim \sum_{j=0}^{\infty} 2^{j(\frac{n}{q}-n)} \|\mathcal{F}^{-1} \varphi_j * f\|_{K^{\alpha}_{q,p}} \lesssim \|f\|_{K^{\alpha}_{q,p}A^{s}_\beta},$$

since $s > \frac{n}{q} - n$. The desired embedding follows by the fact that $\hat{K}^{\alpha, \text{max}(1,p)}_{q} \hookrightarrow L^1_{\text{loc}}$ and the arguments in Substep 1.1. In addition

$$\hat{K}^{\alpha,p}_{q}A^{s}_\beta \hookrightarrow \hat{K}^{\alpha, \text{max}(1,p)}_{1}.$$  \hspace{1cm} (11)

Substep 2.2. $\alpha \geq 0$. Let $\alpha_4$ be a real number such that $\max(-n, -s + \frac{n}{q} - n + \alpha) < \alpha_4 < 0$. From Theorem 1, we get

$$\hat{K}^{\alpha,p}_{q}A^{s}_\beta \hookrightarrow \hat{K}^{\alpha_4, \text{max}(1,p)}_{1}A^{s-\frac{n}{q}+\alpha_4-n}_\beta \hookrightarrow \hat{K}^{\alpha, \text{max}(1,p)}_{1}A^{s-\frac{n}{q}+\alpha_4}_\beta \hookrightarrow \hat{K}^{\alpha, \text{max}(1,p)}_{q}A^{s-\frac{n}{q}+\alpha_4}_\beta.$$

As in Substep 1.4, we easily obtain that

$$\hat{K}^{\alpha,p}_{q}A^{s}_\beta \hookrightarrow L^1_{\text{loc}}.$$

Therefore, under the hypothesis of this theorem, every $f \in \hat{K}^{\alpha,p}_{q}A^{s}_\beta$ is a regular distribution. This finishes the proof.

Let $f$ be an arbitrary function on $\mathbb{R}^n$ and $x, h \in \mathbb{R}^n$. Then

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^{M+1} f(x) = \Delta_h(\Delta_h^M f)(x), \quad M \in \mathbb{N}.$$

These are the well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^j C_j^M f(x + (M-j)h), \quad x \in \mathbb{R}^n,$$

where $C_j^M$ are the binomial coefficients. By ball means of differences we mean the quantity

$$d^M_t f(x) = t^{-n} \int_{|h| \leq t} |\Delta_h^M f(x)| dh = \int_{B} |\Delta_h^M f(x)| dh, \quad x \in \mathbb{R}^n.$$

Here $B = \{y \in \mathbb{R}^n : |h| \leq 1\}$ is the unit ball of $\mathbb{R}^n$ and $t > 0$ is a real number. We set

$$\|f\|_{K^{\alpha,p}_{q}B^{s}_\beta} = \|f\|_{K^{\alpha}_{q,p}} + \left( \int_0^\infty t^{-s\beta} \|d^M_t f\|_{K^{\alpha}_{q,p}} dt \right)^{1/\beta}$$

and

$$\|f\|_{K^{\alpha,p}_{q}F^{s}_\beta} = \|f\|_{K^{\alpha}_{q,p}} + \left( \int_0^\infty t^{-s\beta} (d^M_t f)^{\beta} dt \right)^{1/\beta}.$$  

The following theorem play a central role in our paper.
Theorem 5  Let $0 < p, q, \beta \leq \infty$, $\alpha > -\frac{n}{q}$, $\alpha_0 = n - \frac{n}{q}$ and $M \in \mathbb{N} \setminus \{0\}$.

(i) Assume that
$$\max(\sigma_q, \alpha - \alpha_0) < s < M.$$ Then $\|\cdot\|_{K_q^{\alpha,p}B_\beta^s}$ is an equivalent quasi-norm on $\dot{K}_q^{\alpha,p}B_\beta^s$.

(ii) Let $0 < p < \infty$ and $0 < q < \infty$. Assume that
$$\max(\sigma_q, \alpha - \alpha_0) < s < M.$$ Then $\|\cdot\|_{K_q^{\alpha,p}F_\beta^s}$ is an equivalent quasi-norm on $\dot{K}_q^{\alpha,p}F_\beta^s$.

Proof. For ease of presentation, we split the proof into three steps.

Step 1. We will prove that
$$\|f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_s^{\beta}}$$
for all $f \in \dot{K}_q^{\alpha,p}A_s^{\beta}$. We employ the same notations as in Theorem 4. Recall that
$$\varrho_k = \sum_{j=0}^k \mathcal{F}^{-1}\varphi_j * f, \quad k \in \mathbb{N}_0.$$ Obviously $\{\varrho_k\}_{k \in \mathbb{N}_0}$ converges to $f$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\{\varrho_k\}_{k \in \mathbb{N}_0} \subset \dot{K}_q^{\alpha,p}$ for any $0 < p, q \leq \infty$ and any $\alpha > -\frac{n}{q}$. Furthermore, $\{\varrho_k\}_{k \in \mathbb{N}_0}$ is a Cauchy sequences in $\dot{K}_q^{\alpha,p}$ and hence it converges to a function $g \in \dot{K}_q^{\alpha,p}$, and
$$\|g\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_s^{\beta}}.$$ Let us prove that $g = f$ a.e. We will do this into four cases.

Case 1. $-\frac{n}{q} < \alpha < \alpha_0$ and $1 \leq q \leq \infty$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We write
$$\langle f - g, \varphi \rangle = \langle f - \varrho_N, \varphi \rangle + \langle g - \varrho_N, \varphi \rangle, \quad N \in \mathbb{N}_0.$$ Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Clearly, the first term tends to zero as $N \to \infty$, while by Hölder’s inequality there exists a constant $C > 0$ independent of $N$ such that
$$|\langle g - \varrho_N, \varphi \rangle| \leq C\|g - \varrho_N\|_{\dot{K}_q^{\alpha,\max(1,p)}}$$
which tends to zero as $N \to \infty$. Then, with the help of Substep 1.1 of the proof of Theorem 4 we have $g = f$ almost everywhere.

Case 2. $\alpha \geq \alpha_0$ and $1 < q \leq \infty$. Let $1 < q_1 < \infty$ be as in Theorem 4. From (9) and (10), we derive in this case, that every $f \in \dot{K}_q^{\alpha,p}A_s^{\beta}$ is a regular distribution, $\{\varrho_k\}_{k \in \mathbb{N}_0}$ converges to $f$ in $L^{q_1}$ and
$$\|f\|_{L^{q_1}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}A_s^{\beta}}.$$ Indeed, from the embeddings (10) and since $f \in B_{q_1,\beta}^{\frac{\alpha - n}{q} + s}$, it follows that $\{\varrho_k\}_{k \in \mathbb{N}_0}$ converges to a function $h \in L^{q_1}$. Similarly as in Case 1, we conclude that $f = h$ a.e. It remains to prove that $g = f$ a.e. We have
$$\|f - g\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f - \varrho_k\|_{\dot{K}_q^{\alpha,p}} + \|g - \varrho_k\|_{\dot{K}_q^{\alpha,p}}, \quad k \in \mathbb{N}_0.$$
and
\[ \|f - q_k\|_{K_q^{\alpha,p}}^s \leq \sum_{j=k+1}^{\infty} \|\mathcal{F}^{-1} \varphi_j \ast f\|_{K_q^{\alpha,p}}^\sigma \leq \|f\|_{K_q^{\alpha,p} A_\beta}^\sigma \sum_{j=k+1}^{\infty} 2^{-j\sigma}, \]
where \( \sigma = \min(1, p, q) \). Letting \( k \) tends to infinity, we get \( g = f \) a.e. For the latter case \( 1 < q_1 < q \leq \infty \), we have
\[ \dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_q^{0,\max(1,p)} A_\beta^{s-\alpha - \frac{n}{q} + \frac{\alpha}{q_1}}. \]
As in Case 1, \( \{q_k\}_{k \in \mathbb{N}_0} \) converges to a function \( h \in \dot{K}_q^{0,\max(1,p)} \). Then again, similarly to the arguments in Case 1 it is easy to check that \( f = h \) a.e. Therefore, we can conclude that \( g = f \) a.e.

**Case 3.** \( q = 1 \) and \( \alpha \geq 0 \).

*Subcase 3.1.*** \( q = 1 \) and \( \alpha > 0 \). We have
\[ \dot{K}_1^{\alpha,p} B_\beta^s \hookrightarrow L^1, \]
since \( s > \alpha \), see Theorem 4. Substep 1.3. Now one can continue as in Case 2.

*Subcase 3.2.*** \( q = 1 \) and \( \alpha = 0 \). Let \( \alpha_3 \) be a real number such that \( \max(-n, -s) < \alpha_3 < 0 \). From Theorem 1, we get
\[ \dot{K}_1^{0,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha_3,p} A_\beta^{s+\alpha_3}. \]
We have
\[ \sum_{k=0}^{\infty} \|\mathcal{F}^{-1} \varphi_k \ast f\|_{K_1^{\alpha_3,\max(1,p)}} \lesssim \|f\|_{K_1^{\alpha_3,p} A_\beta^{s+\alpha_3}} \lesssim \|f\|_{K_1^{0,p} A_\beta^s}, \]
since \( \alpha_3 + s > 0 \). Hence the sequence \( \{q_k\}_{k \in \mathbb{N}_0} \) converges to \( f \) in \( \dot{K}_1^{\alpha_3,\max(1,p)} \), see Case 1. As in Case 2, we obtain \( g = f \) a.e.

**Case 4.** \( 0 < q < 1 \).

*Subcase 4.1.*** \( -\frac{n}{q} < \alpha < 0 \). From the embedding (11) and the fact that \( s > \frac{n}{q} - n \), the sequence \( \{q_k\}_{k \in \mathbb{N}_0} \) converge to \( f \) in \( \dot{K}_1^{\alpha,\max(1,p)} \). As above we prove that \( g = f \) a.e.

*Subcase 4.2.*** \( \alpha \geq 0 \). Recall that
\[ \dot{K}_q^{\alpha,p} A_\beta^s \hookrightarrow \dot{K}_1^{\alpha_4,\max(1,p)} A_\beta^{s-\frac{n}{q} + n - \alpha + \alpha_4}, \]
see Substep 2.2 of the proof of Theorem 1. As in Subcase 3.2 the sequence \( \{q_k\}_{k \in \mathbb{N}_0} \) converges to \( f \) in \( \dot{K}_1^{\alpha_4,\max(1,p)} \). The same arguments above one can conclude that: \( g = f \) a.e.

**Step 2.** In this step we prove that
\[ \|f\|_{K_q^{\alpha,p} F_\beta^s}^s = \left( \int_0^\infty t^{-s\beta} (d_t^M f)_{\beta} \frac{dt}{t} \right)^{1/\beta} \|f\|_{K_q^{\alpha,p} F_\beta^s} \lesssim \|f\|_{K_q^{\alpha,p} F_\beta^s}, \quad f \in \dot{K}_q^{\alpha,p} F_\beta^s. \]
Thus, we need to prove that
\[ \left( \sum_{k=-\infty}^{\infty} 2^{sk\beta} |d_{2^{-k}} f|_{\beta}^3 \right)^{1/3} \|f\|_{K_q^{\alpha,p}} \]
does not exceed \( c \| f \|_{\dot{K}_{q}^{\alpha,p} F_{\beta}^{s}} \). In order to prove we additionally do it into the two Substeps 2.1 and 2.2. The estimate for the space \( \dot{K}_{q}^{\alpha,p} B_{\beta}^{s} \) is similar.

**Substep 2.1.** We will estimate
\[
\left\| \left( \sum_{k=0}^{\infty} 2^{sk\beta} |d_{2^{-k}}^{M} f|^{\beta} \right)^{1/\beta} \right\|_{\dot{K}_{q}^{\alpha,p}}.
\]

Let \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) be a smooth dyadic resolution of unity. Obviously we need to estimate
\[
\left\{ 2^{ks} \sum_{j=0}^{k} d_{2^{-k}}^{M} (\mathcal{F}^{-1} \varphi_j * f) \right\}_{k \in \mathbb{N}_0}
\]
and
\[
\left\{ 2^{ks} \sum_{j=k+1}^{\infty} d_{2^{-k}}^{M} (\mathcal{F}^{-1} \varphi_j * f) \right\}_{k \in \mathbb{N}_0}.
\]

Recall that
\[
d_{2^{-k}}^{M} (\mathcal{F}^{-1} \varphi_j * f) \lesssim 2^{(j-k)M} (\mathcal{F}^{-1} \varphi_j)^{*,a} f (x)
\]
if \( a > 0, 0 \leq j \leq k, k \in \mathbb{N}_0 \) and \( x \in \mathbb{R}^n \), see, e.g., [11], where the implicit constant is independent of \( j, k \) and \( x \). We choose \( a > \frac{n}{\min(q,\beta)} \). Since \( s < M \), (12) in \( \ell^\beta \)-quasi-norm does not exceed
\[
\left( \sum_{j=0}^{\infty} 2^{js\beta} ((\mathcal{F}^{-1} \varphi_j)^{*,a} f)^{\beta} \right)^{1/\beta}.
\]

By Theorem 3 the \( \dot{K}_{q}^{\alpha,p} \)-quasi-norm of (14) is bounded by \( c \| f \|_{\dot{K}_{q}^{\alpha,p} F_{\beta}^{s}} \).

Now, we estimate (13). We can distinguish two cases as follows:

- **Case 1.** \( \min(q,\beta) \leq 1 \). If \( -\frac{n}{q} < \alpha < n(1 - \frac{1}{q}) \), then \( s > \frac{n}{\min(q,\beta)} - n \). We choose
\[
\max \left( 0, 1 - \frac{s \min(q,\beta)}{n} \right) < \lambda < \min(q,\beta),
\]
which is possible because of
\[
s > \frac{n}{\min(q,\beta)} - n = \frac{n}{\min(q,\beta)} \left( 1 - \min(q,\beta) \right).
\]

Let \( \frac{n}{\min(q,\beta)} < a < \frac{s}{1-\lambda} \). Then \( s > a(1 - \lambda) \). Now, assume that \( \alpha \geq n(1 - \frac{1}{q}) \). Therefore
\[
s > \max \left( \frac{n}{\min(q,\beta)} - n, \frac{n}{q} + \alpha - n \right).
\]

If \( \min(q,\beta) \leq \frac{n}{\frac{q}{q+\alpha}} \), then we choose \( \lambda \) as in (15). If \( \min(q,\beta) > \frac{n}{\frac{q}{q+\alpha}} \), then we choose
\[
\max \left( 0, 1 - \frac{s \min(q,\beta)}{\frac{n}{q} + \alpha} \right) < \lambda < \frac{n}{\frac{n}{q} + \alpha}.
\]
which is possible because of
\[ s > \frac{n}{q} + \alpha - n = \left( \frac{n}{q} + \alpha \right) \left( 1 - \frac{n}{q} + \alpha \right). \]

In that case, we choose \( \frac{n}{q} + \alpha < \frac{s}{2\lambda} \). We set
\[ J_{2,k}(f) = 2^{ks} \sum_{j=k+1}^{\infty} d^M_{2,k}(\mathcal{F}^{-1} \phi_j \ast f), \quad k \in \mathbb{N}_0. \]

Recalling the definition of \( d^M_{2,k}(\phi_j \ast f) \), we have
\[ d^M_{2,k}(\mathcal{F}^{-1} \phi_j \ast f) = \int_B |\Delta^M_{2,kh}(\mathcal{F}^{-1} \phi_j \ast f)| dh \]
\[ \leq \int_B |\Delta^M_{2,kh}(\mathcal{F}^{-1} \phi_j \ast f)|^\lambda dh \sup_{h \in B} |\Delta^M_{2,kh}(\mathcal{F}^{-1} \phi_j \ast f)|^{1-\lambda}. \] (17)

Observe that
\[ |\mathcal{F}^{-1} \phi_j \ast f(x + (M - i)2^{-kh})| \leq c2^{(j-k)\alpha} \phi_j^{*,a} f(x), \quad |h| \leq 1 \] (18)
and
\[ \int_B |\mathcal{F}^{-1} \phi_j \ast f(x + (M - i)2^{-kh})|^\lambda dh \leq c\mathcal{M}(|\mathcal{F}^{-1} \phi_j \ast f|^\lambda)(x). \]
(19)

if \( j > k, i \in \{0, ..., M\} \) and \( x \in \mathbb{R}^n \). Therefore
\[ d^M_{2,k}(\mathcal{F}^{-1} \phi_j \ast f) \leq c2^{(j-k)\alpha(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1} \phi_j \ast f|^\lambda) \]
for any \( j > k \), where the positive constant \( c \) is independent of \( j \) and \( k \). Hence
\[ J_{2,k}(f) \leq c2^{ks} \sum_{j=k+1}^{\infty} 2^{(j-k)\alpha(1-\lambda)}(\phi_j^{*,a} f)^{1-\lambda}\mathcal{M}(|\mathcal{F}^{-1} \phi_j \ast f|^\lambda). \]

Using Lemma 3 we obtain that (13) in \( \ell^\beta \)-quasi-norm can be estimated from above by
\[ c\left( \sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^{(1-\lambda)\beta}\left( \mathcal{M}(|\mathcal{F}^{-1} \phi_j \ast f|^\lambda) \right)^{\beta/\lambda} \right)^{1/\beta} \]
\[ \lesssim \left( \sum_{j=0}^{\infty} 2^{js\beta}(\phi_j^{*,a} f)^\beta \left( \sum_{j=0}^{\infty} 2^{js\beta}\left( \mathcal{M}(|\mathcal{F}^{-1} \phi_j \ast f|^\lambda) \right)^{\beta/\lambda} \right)^{\lambda/\beta} \right)^{1/\beta}. \]

Applying the \( \hat{K}_q^{\alpha,p} \)-quasi-norm and using Hölder’s inequality we obtain that
\[ \left\| \left( \sum_{j=0}^{\infty} (J_{2,k}(f))^\beta \right)^{1/\beta} \right\|_{\hat{K}_q^{\alpha,p}} \]
is bounded by
\[
\| f \|_{K_q^{\alpha,p}A^s_\beta} \lesssim \| f \|_{K_q^{\alpha,p}F^s_\beta},
\]
where we have used Lemma 4 and Theorem 3.

- **Case 2.** min(q, β) > 1. Assume that \( \alpha \geq n(1 - \frac{1}{q}) \). Then we choose \( \lambda \) as in (16) and \( \frac{n}{q} + \alpha < a < \frac{n}{1 - \frac{1}{q}} \). If \(-\frac{n}{q} < a < n(1 - \frac{1}{q})\), then we choose \( \lambda = 1 \). The desired estimate can be done in the same manner as in Case 1.

**Substep 2.2.** We will estimate
\[
\| f \|_{K_q^{\alpha,p}A^s_\beta}.
\]

We employ the same notations as in Substep 1.1. Define
\[
H_{k,2}(f)(x) = \int_B \sum_{j=0}^{\infty} \Delta_{2^{k-j}} F^{-1} \varphi_j f(x) dz, \quad k \leq 0, x \in \mathbb{R}^n.
\]

As in the estimation of \( J_{2,k} \), we obtain that
\[
H_{2,k}(f) \lesssim 2^{-k\frac{\lambda}{\lambda - 1}} \sup_{\mathbb{R}^n} \left( (2^{j\lambda} F^{-1} \varphi_j f)^{\frac{1}{\lambda}} \right)^{1-\frac{1}{\lambda}} \mathcal{M} (2^{2j\lambda} F^{-1} \varphi_j f)^{\frac{1}{\lambda}}
\]
and this yields that
\[
\left( \sum_{k=-\infty}^{-1} 2^{sk\lambda} H_{k,2} f \right)^{\frac{1}{\lambda}} \lesssim \sup_{\mathbb{R}^n} \left( (2^{j\lambda} F^{-1} \varphi_j f)^{\frac{1}{\lambda}} \right)^{1-\frac{1}{\lambda}} \mathcal{M} (2^{2j\lambda} F^{-1} \varphi_j f)^{\frac{1}{\lambda}}.
\]

By the same arguments as used in Substep 2.1 we obtain the desired estimate.

**Step 3.** Let \( f \in K_q^{\alpha,p}A^s_\beta \). We will prove that
\[
\| f \|_{K_q^{\alpha,p}A^s_\beta} \lesssim \| f \|_{K_q^{\alpha,p}A^s_\beta}^{*}.
\]

As the proof for \( K_q^{\alpha,p}A^s_\beta \) is similar, we only consider \( K_q^{\alpha,p}F^s_\beta \). Let \( \Psi \) be a function in \( \mathcal{S}(\mathbb{R}^n) \) satisfying \( \Psi(x) = 1 \) for \( |x| \leq 1 \) and \( \Psi(x) = 0 \) for \( |x| \geq \frac{3}{2} \), and in addition radial symmetric. We make use of an observation made by Nikol’skij [28] (see also [31] and [35] Section 3.3.2). We put
\[
\psi(x) = (-1)^{M+1} \sum_{i=0}^{M-1} (-1)^i C_i^M \Psi(x (M - i)).
\]
The function $\psi$ satisfies $\psi(x) = 1$ for $|x| \leq \frac{1}{4}$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{4}$. Then, taking $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi\left(\frac{x}{2}\right) - \psi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$ for $j = 2, 3, \ldots$, we obtain that $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity. This yields that

$$\left\| \left( \sum_{j=0}^{\infty} 2^{jn\beta} |\mathcal{F}^{-1}\varphi_j * f|^\beta \right)^{1/\beta} \right\|_{K^q_{a,p}}$$

is a quasi-norm equivalent in $\dot{K}^a_{q,p}$. Let us prove that the last expression is bounded by

$$C \left\| f \right\|_{K^a_{q,p}F^s_{p}}.$$  \hspace{1cm} (20)

We observe that

$$\mathcal{F}^{-1} \varphi_0 * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \mathcal{F}^{-1} \Psi(z) \Delta_{-\frac{1}{2}} f(x) dz + f(x) \int_{\mathbb{R}^n} \mathcal{F}^{-1} \Psi(z) dz$$

Moreover, it holds for $x \in \mathbb{R}^n$ and $j = 1, 2, \ldots$

$$\mathcal{F}^{-1} \varphi_j * f(x) = (-1)^{M+1} \int_{\mathbb{R}^n} \Delta_{2^{-j}y} f(x) \tilde{\Psi}(y) dy,$$

with $\tilde{\Psi} = \mathcal{F}^{-1} \Psi - 2^{-n} \mathcal{F}^{-1} \Psi(\cdot/2)$. Now, for $j \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^n} |\Delta_{2^{-j}y} f(x)\tilde{\Psi}(y)| dy$$

$$= \int_{|y| \leq 1} |\Delta_{2^{-j}y} f(x)\tilde{\Psi}(y)| dy + \int_{|y| > 1} |\Delta_{2^{-j}y} f(x)\tilde{\Psi}(y)| dy. \hspace{1cm} (21)$$

Thus, we need only to estimate the second term of (21). We write

$$2^{sj} \int_{|y| > 1} |\Delta_{2^{-j}y} f(x)\tilde{\Psi}(y)| dy$$

$$= 2^{sj} \sum_{k=0}^{\infty} \int_{2^k < |y| \leq 2^{k+1}} |\Delta_{2^{-j}y} f(x)\tilde{\Psi}(y)| dv$$

$$\leq c 2^{sj} \sum_{k=0}^{\infty} 2^{n-j-Nk} \int_{2^k < |h| \leq 2^{k+1}} |\Delta_{h}^M f(x)| dh \hspace{1cm} (22)$$

where $N > 0$ is at our disposal and we have used the properties of the function $\tilde{\Psi}$, $|\tilde{\Psi}(x)| \leq c(1+|x|)^{-N}$, for any $x \in \mathbb{R}^n$ and any $N > 0$. Without lost of generality, we may assume $1 \leq \beta \leq \infty$. Now, the right-hand side of (22) in $\ell_\beta$-norm is bounded by

$$c \sum_{k=0}^{\infty} 2^{-Nk} \left( \sum_{j=0}^{\infty} 2^{s(j+n)} \left( \int_{|h| \leq 2^{k-j+1}} |\Delta_{h}^M f(x)| dh \right)^\beta \right)^{1/\beta}. \hspace{1cm} (23)$$

After a change of variable $j - k - 1 = v$, we estimate (23) by

$$c \sum_{k=0}^{\infty} 2^{(s+n-N)k} \left( \sum_{v=-k-1}^{\infty} 2^{sv\beta} (d_{2^{-v}}^M f(x))^{\beta} \right)^{1/\beta} \lesssim \left( \sum_{v=-\infty}^{\infty} 2^{sv\beta} (d_{2^{-v}}^M f(x))^{\beta} \right)^{1/\beta},$$
where we choose \( N > n + s \). Taking the \( \dot{K}_q^{α,p} \)-quasi-norm we obtain the desired estimate \((20)\). The proof is complete.

We would like to mention that

\[
\| f(λ) \|_{\dot{K}_q^{α,p}B_β^s} \approx λ^{−\frac{n}{q}}\| f \|_{\dot{K}_q^{α,p}} + λ^{s−α−\frac{n}{q}}\left( \int_0^∞ t^{−sβ}\|\dot{M}f\|_{\dot{K}_q^{α,p}}^β \frac{dt}{t} \right)^{\frac{1}{β}}
\]  

(24)

and

\[
\| f(λ) \|_{\dot{K}_q^{α,p}F_β^s} \approx λ^{−\frac{n}{q}}\| f \|_{\dot{K}_q^{α,p}} + λ^{s−α−\frac{n}{q}}\left( \int_0^∞ t^{−sβ}(\dot{M}f)^β \frac{dt}{t} \right)^{\frac{1}{β}}\| \dot{K}_q^{α,p}
\]

for any \( λ > 0, 0 < p ≤ ∞, 0 < q ≤ ∞, α > −\frac{n}{q} \), \( \max(σ_q, α − α_0) < s < M \) \( (0 < p, q < ∞ \) and \( \max(σ_q, β, α − α_0) < s < M \) in the \( KF \)-case) and \( M ∈ \mathbb{N} \).

Let \( φ^j(x) = φ_0(2^{−j}x) − φ_0(2^{−j}x) \) for \( j ∈ \mathbb{Z} \) and \( x ∈ \mathbb{R}^n \). In view of \((39)\) we have the following equivalent norm of \( \dot{K}_q^{α,p} \). Let \( 1 < p, q < ∞ \) and \( −\frac{n}{q} < α < n − \frac{n}{q} \). Then

\[
\left\| \left( \sum_{j = −∞}^∞ |F^{-1} φ^j * f|^2 \right)^{1/2} \right\|_{\dot{K}_q^{α,p}} \approx \| f \|_{\dot{K}_q^{α,p}},
\]

holds for all \( f ∈ \dot{K}_q^{α,p} \).

Let \( s ∈ \mathbb{R}, 0 < p, q < ∞, 0 < β ≤ ∞ \) and \( α > −\frac{n}{q} \). We set

\[
\| f \|_{\dot{K}_q^{α,p}B_β^s} = \left( \sum_{j = −∞}^∞ 2^{jsβ}\| F^{-1} φ^j * f \|_{\dot{K}_q^{α,p}}^β \right)^{1/β}
\]

and

\[
\| f \|_{\dot{K}_q^{α,p}F_β^s} = \left( \sum_{j = −∞}^∞ 2^{jsβ}|F^{-1} φ^j * f|^{β} \right)^{1/β} \| \dot{K}_q^{α,p}.\]

**Proposition 1** Let \( s > \max(σ_q, α − n + \frac{n}{q}), 0 < p, q < ∞, 0 < β ≤ ∞ \) and \( α > −\frac{n}{q} \).

(i) Let \( s > \max(σ_q, α − n + \frac{n}{q}) \) and \( f ∈ \dot{K}_q^{α,p}B_β^s \). Then

\[
\| f \|_{\dot{K}_q^{α,p}B_β^s} \approx \| f \|_{\dot{K}_q^{α,p}} + \| f \|_{\dot{K}_q^{α,p}B_β^s},
\]

(ii) Let \( s > \max(σ_q, β, α − n + \frac{n}{q}) \) and \( f ∈ \dot{K}_q^{α,p}F_β^s \). Then

\[
\| f \|_{\dot{K}_q^{α,p}F_β^s} \approx \| f \|_{\dot{K}_q^{α,p}} + \| f \|_{\dot{K}_q^{α,p}F_β^s}.\]

**Proof.** As the proof for (i) is similar, we only consider (ii). We use the following Marschall’s inequality which given in \((23)\) Proposition 1.5], see also \((10)\). Let \( A > 0, R ≥ 1 \). Let \( b ∈ D(\mathbb{R}^n) \) and a function \( g ∈ C^∞(\mathbb{R}^n) \) such that

\[
supp Fg ⊆ \{ ξ ∈ \mathbb{R}^n : |ξ| ≤ AR \} \quad \text{and} \quad \text{supp} b ⊆ \{ ξ ∈ \mathbb{R}^n : |ξ| ≤ A \}.
\]

Then

\[
|F^{-1} b * g(x)| ≤ c(AR)^2−n \| b \|_{B_1^{-n}} M_t(g)(x)
\]
for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where $c$ is independent of $A$, $R$, $x$, $b$, $j$ and $g$. Here $\dot{B}_{1,t}^{\beta}$ denotes the homogeneous Besov spaces. We have

$$\mathcal{F}^{-1}\varphi^j \ast f = \mathcal{F}^{-1}\varphi^j \ast \mathcal{F}^{-1}\varphi_0 \ast f, \quad -j \in \mathbb{N}. $$

Therefore,

$$|\mathcal{F}^{-1}\varphi^j \ast f(x)| \leq c\|\varphi^j\|_{\dot{B}_{1,t}^{\beta}} \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 \ast f)(x) \leq c2^{j(n-\frac{\beta}{q})} \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 \ast f)(x), \quad x \in \mathbb{R}^n,$$

where the positive constant $c$ is independent of $j$ and $x$. If we choose $\frac{n}{s+n} < t < \min(1, q, \beta, \frac{n}{\alpha+q})$ then

$$\left( \sum_{j=-\infty}^{-1} 2^{js\beta} |\mathcal{F}^{-1}\varphi^j \ast f|^\beta \right)^{1/\beta} \lesssim \mathcal{M}_t(\mathcal{F}^{-1}\varphi_0 \ast f).$$

Taking the $\dot{K}_q^{\alpha,p}$-quasi-norm and using (2) we obtain

$$\left\| \left( \sum_{j=-\infty}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\varphi^j \ast f|^\beta \right)^{1/\beta} \right\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s}.$$

Because of $s > \max(\sigma_q, \alpha - n + \frac{n}{q})$ the series $\sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi^j \ast f$ converges not only in $\mathcal{S}'(\mathbb{R}^n)$ but almost everywhere in $\mathbb{R}^n$. Then

$$\|f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|\mathcal{F}^{-1}\varphi_0 \ast f\|_{\dot{K}_q^{\alpha,p}} + \left( \sum_{j=1}^{\infty} \|\mathcal{F}^{-1}\varphi^j \ast f\|_{\dot{K}_q^{\alpha,p}}^{\min(1,p,q)} \right)^{1/\min(1,p,q)}.$$

Therefore $\|f\|_{\dot{K}_q^{\alpha,p}} + \|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s}$ can be estimated from above by $c\|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s}$. Obviously

$$\mathcal{F}^{-1}\varphi_0 \ast f = \sum_{j=0}^{N} \mathcal{F}^{-1}\varphi^j \ast f - \sum_{j=1}^{N} \mathcal{F}^{-1}\varphi^j \ast f = g_N + h_N, \quad N \in \mathbb{N}. $$

We have

$$\|h_N\|_{\dot{K}_q^{\alpha,p}} \leq \left( \sum_{j=1}^{\infty} \|\mathcal{F}^{-1}\varphi^j \ast f\|_{\dot{K}_q^{\alpha,p}}^{\min(1,p,q)} \right)^{1/\min(1,p,q)}, \quad N \in \mathbb{N}. $$

By Lebesgue’s dominated convergence theorem, it follows that $\|g_N - f\|_{\dot{K}_q^{\alpha,p}}$ tends to zero as $N$ tends to infinity. Therefore $\|\mathcal{F}^{-1}\varphi_0 \ast f\|_{\dot{K}_q^{\alpha,p}}$ can be estimated from above by the quasi-norm

$$c\|f\|_{\dot{K}_q^{\alpha,p}} + c\|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s}.$$

The proof is complete.

**Proposition 2** Let $s > 0, 1 < p, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$. Let

$$\mathcal{S}_0(\mathbb{R}^n) = \{ f \in \mathcal{S}(\mathbb{R}^n) : \text{supp}\mathcal{F}f \subset \mathbb{R}^n \setminus \{0\} \}.$$

Then $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{K}_q^{\alpha,p}$. 
Proof. Let \( \varphi_0 = \varphi \) be as above. As in \([37]\) it is sufficient to approximate \( f \in \mathcal{S}(\mathbb{R}^n) \) in \( W_{q,k}^{\alpha,p}, k \in \mathbb{N} \), by functions belonging to \( \mathcal{S}_0(\mathbb{R}^n) \). We have

\[
|D^\alpha F^{-1}(\varphi(2^j \cdot) Ff)| = 2^{-jn}|\tilde{\varphi}_j * D^\alpha f| \leq 2^{-jn} M(\tilde{\varphi}_j),
\]

where \( \tilde{\varphi}_j = F^{-1} \varphi(2^{-j} \cdot), j \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \). From (2) we obtain

\[
\|D^\alpha F^{-1}(\varphi(2^j \cdot) Ff)\|_{K_{q,p}^\alpha} \leq c 2^{-jn} \|\tilde{\varphi}_j\|_{K_{q,p}^\alpha} \leq c 2^{j(\frac{n}{q} - n + \alpha)},
\]

where the positive constant \( c \) is independent of \( j \). Since \( \alpha < n - \frac{n}{q} \), we obtain that \( f - F^{-1}(\varphi(2^j \cdot) Ff) \) approximate \( f \in \mathcal{S}(\mathbb{R}^n) \) in \( \dot{W}_{q,k}^{\alpha,p}, k \in \mathbb{N} \). The proof of the proposition is complete.

**Proposition 3** Let \( s > 0, 1 < p, q < \infty \) and \(-\frac{n}{q} < \alpha < n - \frac{n}{q}\). Let \( f \in \dot{W}_{q,s}^{\alpha,p} \). Then

\[
\|f\|_{\dot{W}_{q,s}^{\alpha,p}} \approx \|f\|_{K_{q,p}^\alpha} + \|(-\Delta)^{\frac{s}{2}} f\|_{K_{q,p}^\alpha},
\]

where

\[
(-\Delta)^{\frac{s}{2}} f = F^{-1}((|\xi|^s F f).
\]

**Proof.** Let \( f \in \mathcal{S}_0(\mathbb{R}^n) \). We apply Marschall’s inequality to \( g_j = F^{-1}(\varphi^j |x|^s F f), j \in \mathbb{Z} \) and \( b_j(x) = 2^{js}|x|^{-s} \psi_j(x), j \in \mathbb{Z}, x \in \mathbb{R}^n \) where

\[
\varphi^j(x) = \varphi_0(2^{-j} x) - \varphi_0(2^{1-j} x), \quad \psi_j = \varphi^{j-1} + \varphi^j + \varphi^{j+1}, \quad j \in \mathbb{Z}, x \in \mathbb{R}^n.
\]

Then

\[
|F^{-1} b_j * g_j(x)| \leq c \|b_j\|_{B^1_{p,1}} M(F^{-1}(\varphi^j |\xi|^s F f))(x) \leq c M(F^{-1}(\varphi^j |\xi|^s F f))(x)
\]

for any \( j \in \mathbb{Z} \) and any \( x \in \mathbb{R}^n \), where \( c \) is independent of \( j \). Let \( j \in \mathbb{Z} \). In view of the fact that

\[
F^{-1} \varphi^j * f = F^{-1}(\varphi^j F f) = 2^{-js} F^{-1}(2^{js} |\xi|^{-s} \psi_j |x|^s \varphi^j F f) = 2^{-js} F^{-1}(b_j |\xi|^s \varphi^j F f),
\]

by Lemma 4 and (23) we obtain

\[
\left\| \left( \sum_{j=-\infty}^{\infty} 2^{js} |F^{-1} \varphi^j * f|^2 \right)^{\frac{1}{2}} \right\|_{K_{q,p}^\alpha} \lesssim \left\| \left( \sum_{j=-\infty}^{\infty} |F^{-1}(\varphi^j |\xi|^s F f)|^2 \right)^{\frac{1}{2}} \right\|_{K_{q,p}^\alpha}
\]

\[
\lesssim \|F^{-1}(\varphi^j |\xi|^s F f)\|_{K_{q,p}^\alpha}.
\]

The same arguments can be used to prove the opposite inequality in view of the fact that

\[
F^{-1}(\varphi^j |\xi|^s F f) = F^{-1}(2^{-js} \psi_j |\xi|^s 2^{js} \varphi^j F f) = F^{-1}(b_j 2^{js} \varphi^j F f), \quad j \in \mathbb{Z}.
\]

The rest follows by Propositions 1 and 2. The proof is complete.

**Definition 5** Let \( 0 < u \leq p < \infty \). The Morrey space \( M^p_u \) is defined to be the set of all \( u \)-locally Lebesgue-integrable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{M^p_u} = \sup |B|^{\frac{1}{p} - \frac{1}{u}} \|f \chi_B\|_u < \infty,
\]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \).
Remark 4 The Morrey spaces $M_u^p$ are quasi-Banach spaces, Banach spaces for $u \geq 1$, were introduced by Morrey to study some PDE’s, see [26]. One can easily seen that $M_p^p = L^p$ and that for $0 < u < v \leq p < \infty$,

$$M_u^p \hookrightarrow M_v^p.$$ 

The Sobolev Morrey spaces are defined as follows.

Definition 6 Let $1 < u \leq p < \infty$ and $m = 1, 2, \ldots$. The Sobolev Morrey space $M_u^{m,p}$ is defined to be the set of all $u$-locally Lebesgue-integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{M_u^{m,p}} = \|f\|_{M_u^p} + \sum_{|\alpha| \leq m} \|D^\alpha f\|_{M_u^p} < \infty.$$ 

Let now recall the definition of Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a resolution of unity, see Section 2.

Definition 7 Let $s \in \mathbb{R}$, $0 < u \leq p < \infty$ and $0 < q \leq \infty$. The Besov-Morrey space $N_{p,q,u}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{N_{p,q,u}^s} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| F^{-1} \varphi_j * f \right\|_{M_u^p}^q \right)^{1/q} < \infty.$$ 

In the limiting case $q = \infty$ the usual modification is required. The Triebel-Lizorkin-Morrey space $E_{p,q,u}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{E_{p,q,u}^s} = \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} \left| F^{-1} \varphi_j * f \right|^q \right)^{1/q} \right\|_{M_u^p} < \infty.$$ 

In the limiting case $q = \infty$ the usual modification is required.

We have

$$E_{p,2,u}^m = M_u^{m,p}, \quad m \in \mathbb{N}, \quad 1 < u \leq p < \infty,$$

with equivalent norms, see [32] Theorem 3.1]. In particular, we have that

$$E_{p,2,u}^0 = M_u^p, \quad 1 < u \leq p < \infty,$$

also in the sense of with equivalent norms, see [24] Proposition 4.1].

Theorem 6 Let $s_i \in \mathbb{R}$, $0 < q_i \leq \infty$, $0 < u_i \leq p_i < \infty$, $i = 1, 2$. There is a continuous embedding

$$E_{p_1,q_1,u_1}^{s_1} \hookrightarrow E_{p_2,q_2,u_2}^{s_2}$$

if, and only if,

$$p_1 \leq p_2 \quad \text{and} \quad \frac{u_2}{p_2} \leq \frac{u_1}{p_1}$$

and

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{or} \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2} \quad \text{and} \quad p_1 \neq p_2.$$ 

For the proof of these Sobolev embeddings, see [13] Theorem 3.1].

Remark 5 A detailed study of Besov-Morrey and Triebel-Lizorkin-Morrey spaces including their history and properties can be found in [13, 24, 25, 32, 44] and references therein.
3 Caffarelli-Kohn-Nirenberg inequalities

As mentioned in the introduction, Caffarelli-Kohn-Nirenberg inequalities play a crucial role to study regularity and integrability for solutions of nonlinear partial differential equations, see [14, 43]. The main aim of this section is to extend these inequalities to more general function spaces. Let \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) be a resolution of unity and

\[
Q_J f = \sum_{j=0}^{J} \mathcal{F}^{-1} \varphi_j \ast f, \quad J \in \mathbb{N}, \ f \in \mathcal{S}'(\mathbb{R}^n).
\]

3.1 CKN inequalities in Herz-type Besov and Triebel-Lizorkin spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in \( \dot{K}^{\alpha,p}_q A^r_\beta \) spaces. The main results of this section based on the following proposition.

Proposition 4 Let \( \alpha_1, \alpha_2 \in \mathbb{R}, \sigma \geq 0, 1 < r, v < \infty, 0 < \tau, u \leq \infty \) and

\[
-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}.
\]

(i) Assume that \( 1 < u \leq v < \infty \) and \( \alpha_2 \geq \alpha_1 \). Then for all \( f \in \dot{K}^{\alpha_2,\delta}_u \cap \mathcal{S}'(\mathbb{R}^n) \) and all \( J \in \mathbb{N} \),

\[
\|Q_J f\|_{\dot{K}^{\alpha_1,\sigma}_v A^r_\beta} \leq c 2^{J(\frac{\tau}{\sigma} - \frac{\tau}{r} + \alpha_2 - \alpha_1 + \sigma)} \|f\|_{\dot{K}^{\alpha_2,\delta}_u},
\]

where

\[
\delta = \begin{cases} r, & \text{if } \alpha_2 = \alpha_1, \\ \tau, & \text{if } \alpha_2 > \alpha_1 \end{cases}
\]

and the positive constant \( c \) is independent of \( J \).

(ii) Assume that \( 1 < v \leq u < \infty \) and \( \alpha_2 \geq \alpha_1 + \frac{n}{v} - \frac{n}{u} \). Then for all \( f \in \dot{K}^{\alpha_2,\delta}_u \cap \mathcal{S}'(\mathbb{R}^n) \) and all \( J \in \mathbb{N} \), (27) holds where the positive constant \( c \) is independent of \( J \) and

\[
\delta = \begin{cases} r, & \text{if } \alpha_2 = \alpha_1 + \frac{n}{v} - \frac{n}{u}, \\ \tau, & \text{if } \alpha_2 > \alpha_1 + \frac{n}{v} - \frac{n}{u}. \end{cases}
\]

Proof. By similarity, we only give the proof for (i). Let \( \sigma = \theta m + (1 - \theta)0, \alpha \in \mathbb{N}^n \) with \( 0 < \theta < 1 \) and \( |\alpha| \leq m \). From (3) we have

\[
\|Q_J f\|_{\dot{K}^{\alpha_1,\sigma}_v A^r_2} \leq \|Q_J f\|_{\dot{K}^{1-\theta,\sigma}_v A^r_2} \|Q_J f\|_{\dot{K}^{\theta,\sigma}_v A^r_2}.
\]

Observe that

\[
\dot{K}^{\alpha_1,\sigma}_v A^r_2 = \dot{I}^{\alpha_1,\sigma}_{v,\sigma}, \ \dot{K}^{\alpha_1,\sigma}_v A^m_2 = \dot{W}^{\alpha_1,\sigma}_{v,m}, \ \text{and} \ \dot{K}^{\alpha_1,\sigma}_v A^0_2 = \dot{K}^{\alpha_1,\sigma}_v,
\]

see (3), (4) and (5). It follows that

\[
\|Q_J f\|_{\dot{K}^{\alpha_1,\sigma}_v} \leq \|Q_J f\|_{\dot{K}^{1-\theta,\sigma}_v} \|Q_J f\|_{\dot{W}^{\alpha_1,\sigma}_{v,m}},
\]

where the positive constant \( c \) is independent of \( J \). Observe that

\[
Q_J f = 2^{jn} \mathcal{F}^{-1} \varphi_0(2^J \cdot) \ast f.
\]
Therefore,
\[ D^\alpha(Q_j f) = 2^{J(\alpha+n)} \omega_f \ast f = 2^{J(\alpha)} \tilde{Q}_j f, \quad |\alpha| \leq m \]
with \( \omega_f(x) = D^\alpha(F^{-1}\varphi_0)(2^J x), \quad x \in \mathbb{R}^n \). Recall that
\[ \|Q_j f\| \lesssim \mathcal{M}(f). \]

Applying Lemma 1 and the estimate (2), we obtain
\[ \|D^\alpha(Q_j f)\|_{\dot{K}_{u_1,r}^{\alpha_1,r}} \leq c2^{J(\frac{n}{u}+\alpha_2-\alpha_1+|\alpha|)} \|\tilde{Q}_j f\|_{\dot{K}_{u_2}^{\alpha_2,\delta}} \leq c2^{J(\frac{n}{u}+\alpha_2-\alpha_1+m)} \|f\|_{\dot{K}_{u_3}^{\alpha_2,\delta}} \]
for any \( |\alpha| \leq m \). This finish the proof.

**Remark 6** With \( \alpha_1 = \alpha_2 = 0 \) and \( r = v \) the estimation (27) can be rewritten as
\[ \|Q_j f\|_{H^\varphi} \leq c2^{J(\frac{n}{u}+\sigma)} \|f\|_{\dot{K}_{u}^{\alpha,v}} \leq c2^{J(\frac{n}{u}+\sigma)} \|f\|_{u}, \]
because of \( 1 < u \leq v < \infty \) which has been proved by Triebel in [37, Proposition 4.5].

Now we are in position to state the main results of this section.

**Theorem 7** Let \( 0 < p, \tau, \beta, \varphi < \infty, \ 1 < r, v, u < \infty, \sigma \geq 0, \)
\[ -\frac{n}{v} < \alpha_1 < \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < \frac{n}{u}, \quad \alpha_3 > \frac{n}{u}, \quad v \geq \max(p,u), \quad 0 < \sigma < 1, \]
and
\[ s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0 \]
\[ \sigma - \frac{n}{v} = -(1-\theta)\frac{n}{u} + \theta \left( s - \frac{n}{p} \right) + \alpha_1 - \left( (1-\theta)\alpha_2 + \theta \alpha_3 \right), \quad 0 < \theta < 1. \]

Assume that \( s > \sigma_{p,\beta} \) in the \( KF \)-case.

(i) Let \( \alpha_1 \leq \alpha_2 \leq \alpha_3 \). There is a constant \( c > 0 \) such that for all \( f \in \dot{K}_{u}^{\alpha_2,\delta} \cap \dot{K}_{p}^{\alpha_3,\delta_1} B_{\beta} \),
\[ \|f\|_{\dot{K}_{u}^{\alpha_1,\varphi}} \leq c\|f\|_{\dot{K}_{p}^{\alpha_2,\beta}} \|f\|_{\dot{K}_{p}^{\alpha_3,\delta_1} B_{\beta}} \]
with
\[ \delta = \begin{cases} \tau, & \text{if } \alpha_2 = \alpha_1, \\ r, & \text{if } \alpha_2 > \alpha_1. \end{cases} \]
and
\[ \delta_1 = \begin{cases} r, & \text{if } \alpha_3 = \alpha_1, \\ \varphi, & \text{if } \alpha_3 > \alpha_1. \end{cases} \]

(ii) Let \( \frac{1}{r} \leq (1-\theta)\frac{n}{u} + \theta \frac{n}{p} \) and
\[ \alpha_1 = (1-\theta)\alpha_2 + \theta \alpha_3. \]
There is a constant \( c > 0 \) such that for all \( f \in \dot{K}_{u}^{\alpha_2,\varphi} F_{\infty}^0 \cap \dot{K}_{p}^{\alpha_3,\delta} A_{\infty}^s \),
\[ \|f\|_{\dot{K}_{u}^{\alpha,\varphi}} \leq c \|f\|_{\dot{K}_{p}^{\alpha_2,\varphi} F_{\infty}^0} \|f\|_{\dot{K}_{p}^{\alpha_3,\delta} A_{\infty}^s}. \]
Proof.
Proof of (i). For technical reasons, we split the proof into two steps.
Step 1. We consider the case $p \leq u$. Let

$$ f = \sum_{j=0}^{\infty} F^{-1} \varphi_j * f, \quad f \in \mathcal{S}'(\mathbb{R}^n). $$

Then it follows that

$$ f = \sum_{j=0}^{J} F^{-1} \varphi_j * f + \sum_{j=J+1}^{\infty} F^{-1} \varphi_j * f $$

$$ = Q_j f + \sum_{j=J+1}^{\infty} F^{-1} \varphi_j * f, \quad J \in \mathbb{N}. $$

Hence

$$ \| f \|_{K_v^{1, \sigma}} \leq \| Q_j f \|_{K_v^{1, \sigma}} + \| \sum_{j=J+1}^{\infty} F^{-1} \varphi_j * f \|_{K_v^{1, \sigma}}. $$

Using Proposition 4, it follows that

$$ \| Q_j f \|_{K_v^{1, \sigma}} \lesssim 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 + \sigma)} \| f \|_{K_v^{0, \delta_1}}. $$

From the embedding

$$ \dot{K}_v^{\alpha_1, \sigma} B_1^\sigma \hookrightarrow \dot{K}_v^{1, \sigma}, $$

see (7), the last norm in (32) can be estimated by

$$ c \sum_{j=J+1}^{\infty} 2^{j \sigma} \| F^{-1} \varphi_j * f \|_{K_v^{1, \sigma}} \lesssim \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 + \sigma)} \| F^{-1} \varphi_j * f \|_{K_v^{0, \delta_1}} $$

$$ \lesssim 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 + s + \sigma)} \| f \|_{K_v^{0, \delta_1} B_1^\delta}. $$

by Lemma 1 where the last estimate follows by (29). Plug (33) and (35) into (32) we obtain

$$ \| f \|_{K_v^{1, \sigma}} \lesssim 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 + \sigma)} \| f \|_{K_v^{0, \delta_1} B_1^\delta} + 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 - s + \sigma)} \| f \|_{K_v^{0, \delta_1} B_1^\delta} $$

$$ + 2^{j(\frac{n}{p} - \frac{n}{v} + \alpha_2 - \alpha_1 + s)} \left( \| f \|_{K_v^{0, \delta_1} B_1^\delta} + 2^{j(\frac{n}{p} - \frac{n}{v} - s - \alpha_2 + \alpha_3)} \| f \|_{K_v^{0, \delta_1} B_1^\delta} \right), $$

with some positive constant $c$ independent of $J$. Again from, Lemma 1 it follows that

$$ \dot{K}_v^{\alpha_3, \delta_1} B_1^\delta \hookrightarrow \dot{K}_u^{0, \delta}, $$

since $s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > 0$. We choose $J \in \mathbb{N}$ such that

$$ 2^{j(\frac{n}{p} - \frac{n}{v} - s - \alpha_2 + \alpha_3)} \approx \| f \|_{K_v^{0, \delta_1} B_1^\delta} \| f \|_{K_v^{0, \delta_1} B_1^\delta}^{-1}. $$

We obtain

$$ \| f \|_{K_v^{1, \sigma}} \lesssim \| f \|_{K_v^{0, \delta_1} B_1^\delta}^{-1} \| f \|_{K_v^{0, \delta_1} B_1^\delta}. $$
By (29) one has \( s > \max \left( \sigma_p, \alpha_3 - n + \frac{n}{p} \right) \) and by the fact that \(-\frac{n}{u} < \alpha_2 < n - \frac{n}{u}\),

\[
\sigma > \max \left( 0, \alpha_1 + \frac{n}{v} - n \right)
\]

and Theorem [5] or Proposition [1] can be used. Therefore

\[
\|f\|_{\hat{K}^{\alpha_1, r}_u \hat{F}^2} \lesssim \|f\|_{\hat{K}^{\alpha_1, r}_u}
\]

and

\[
\|f\|_{\hat{K}^{\alpha_1, r}_u \hat{F}^2} \lesssim \|f\|^{1-\theta}_{\hat{K}^{\alpha_2, \delta}_u} \left( \|f\|_{\hat{K}^{\alpha_3, \delta_1}_p} + \|f\|_{\hat{K}^{\alpha_3, \delta_1}_p} \right)^\theta.
\]

In this estimate replace \( f \) by \( f(\lambda \cdot) \) we obtain

\[
\|f\|_{\hat{K}^{\alpha_1, r}_u \hat{F}^2} \lesssim \|f\|^{1-\theta}_{\hat{K}^{\alpha_2, \delta}_u} \left( \lambda^{-s} \|f\|_{\hat{K}^{\alpha_3, \delta_1}_p} + \|f\|_{\hat{K}^{\alpha_3, \delta_1}_p} \right)^\theta.
\]

Taking \( \lambda \) large enough we obtain (31) but with \( p \leq u \).

**Step 2.** We consider the case \( u < p \). Taking \( \lambda > 0 \) large enough such that

\[
\frac{\|f(\lambda \cdot)\|_{\hat{K}^{\alpha_2, \delta}_u}}{\|f(\lambda \cdot)\|_{\hat{K}^{\alpha_3, \delta_1}_p}} \leq 1,
\]

which is possible because of \( s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > 0 \), see (24). As in Step 1, with \( f(\lambda \cdot) \) in place of \( f \) and (37) in place of (36), we obtain the desired estimate. The proof of (i) is complete.

**Proof of (ii).** Observe that

\[
\frac{n}{v_1} = \frac{n}{v} + \theta s - \sigma = (1 - \theta) \frac{n}{u} + \theta \frac{n}{p}
\]

and \( \frac{\sigma}{s} \leq \theta < 1 \). Therefore

\[
\hat{K}^{\alpha_1, r}_u F_{\infty}^{\theta s} \hookrightarrow \hat{K}^{\alpha_1, r}_{v_1, \sigma},
\]

see Theorems [1]. From (3), (5) and (6), we obtain

\[
\|f\|_{\hat{K}^{\alpha_1, r}_u F_{\infty}^{\theta s}} \leq \|f\|^{1-\theta}_{\hat{K}^{\alpha_2, \delta}_u F_{\infty}^{\theta s}} \|f\|_{\hat{K}^{\alpha_3, \delta_1}_p F_{\infty}^{\theta s}}.
\]

We have

\[
\hat{K}^{\alpha_3, p}_{\infty} A_{\beta} \hookrightarrow \hat{K}^{\alpha_3, p}_{\infty} F_{\infty}^{\theta s}.
\]

This finishes the proof of (ii). The proof is complete.

**Remark 7** (i) Taking \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) and \( r = v \) we obtain

\[
\|f\|_{H^\sigma} \leq c \|f\|^{1-\theta}_{\hat{K}^{0, v}_u} \|f\|_{\hat{K}^{0, v}_u B^\sigma_\beta}
\]

\[
\leq c \|f\|_{u}^{1-\theta} \|f\|_{B^\sigma_\beta}^\theta
\]

for all \( f \in L_u \cap B^\sigma_\beta \), because of \( L_u \hookrightarrow \hat{K}^{0, v}_u \) and \( \hat{B}^\sigma_\beta = \hat{K}^{0, p}_u \hat{B}^\sigma_{v, \beta} \hookrightarrow \hat{K}^{0, v}_u \hat{B}^\sigma_{\beta}, \) which has been proved by Triebel in [37, Theorem 4.6].
(ii) Under the hypothesis of Theorem 7(ii), with \( 0 < p < \frac{n}{\alpha} \) and \( \frac{1}{r} \leq (1 - \theta) \frac{n}{\alpha} + \theta \left( \frac{n}{p} - s + \frac{\sigma}{\theta} \right) \), we have

\[
\| f \|_{K_{v,u}^{\alpha,\tau}} \leq C \| f \|_{K_{v,u}^{\alpha,2,\tau}} \| f \|_{K_p^{\alpha_3,1,\tau}} A_{\kappa}^s
\]

for all \( f \in K_{u}^{\alpha,2,\tau} F_2^{0} \cap K_p^{\alpha_3,1,\tau} A_{\kappa}^s \), where

\[
\kappa = \begin{cases} 
\frac{1}{p-s+\theta}, & \text{if } A = B, \\
\infty, & \text{if } A = F.
\end{cases}
\]

Indeed, observe that

\[
\frac{n}{v} = (1 - \theta) \frac{n}{u} + \theta \left( \frac{n}{p} - s + \frac{\sigma}{\theta} \right) = (1 - \theta) \frac{n}{u} + \theta \frac{n}{u_1}
\]

and \( \frac{\sigma}{\theta} - s \leq 0 \). Therefore, from (3), (5) and (6), we obtain

\[
\| f \|_{K_{v,u}^{\alpha,\tau}} \leq \| f \|_{K_{u}^{\alpha_3,1,\tau} F_2^{0}} \| f \|_{K_p^{\alpha_3,1,\tau} A_{\kappa}^s}.
\]

The result follows by

\[
K_{p}^{\alpha_3,1,\tau} A_{\kappa}^s \hookrightarrow K_{u}^{\alpha,2,\tau} F_2^{0},
\]

see Theorems 1 and 2.

**Theorem 8** Let \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, 0 < p, \tau, \beta, \theta \leq \infty, 1 < r, v, u < \infty, \)

\[
s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > -\frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0
\]

and

\[
\frac{n}{v} = (1 - \theta) \frac{n}{u} + \theta \left( \frac{n}{p} - s \right) - \alpha_1 + (1 - \theta) \alpha_2 + \theta \alpha_3, \quad 0 < \theta < 1.
\]

Assume that \( 0 < p, \tau < \infty \) and \( s > \sigma_{p,\beta} \) in the \( KF \)-case.

Let \( \delta \) and \( \delta_1 \) be as in Theorem 7(i). Let \( \alpha_1 \leq \alpha_2 \leq \alpha_3, v \geq \max(u,p), \alpha_1 > -\frac{n}{v}, -\frac{n}{u} < \alpha_2 < n - \frac{n}{u} \) and \( \alpha_3 > -\frac{n}{u} \). We have

\[
\| f \|_{K_{v,u}^{\alpha_1,\tau}} \lesssim \| f \|_{K_{v,u}^{\alpha_2,\tau}} \| f \|_{K_p^{\alpha_3,\delta,\delta_1} A_{\beta}^s},
\]

holds for all \( f \in K_{u}^{\alpha_2,\delta} \cap K_p^{\alpha_3,\delta_1} A_{\beta}^s \).

**Proof.** We employ the same notation and conventions as in Theorem 7. As in Proposition 4

\[
\| Q_j f \|_{K_{v,u}^{\alpha_1,\tau}} \lesssim 2^{J \left( \frac{n}{p} - \frac{n}{u} + \alpha_2 - \alpha_1 \right)} \| f \|_{K_{v,u}^{\alpha_2,\delta}}, \quad J \in \mathbb{N}.
\]

Therefore,

\[
\| f \|_{K_{v,u}^{\alpha_1,\tau}} \lesssim 2^{J \left( \frac{n}{p} - \frac{n}{u} + \alpha_2 - \alpha_1 \right)} \| f \|_{K_{v,u}^{\alpha_2,\delta}} + \sum_{j=J+1}^{\infty} \| F^{-1} \varphi_j \ast f \|_{K_{v,u}^{\alpha_1,\tau}}, \quad J \in \mathbb{N}.
\]

Repeating the same arguments of Theorem 7 we obtain the desired estimate.
Remark 8 Under the same hypothesis of Theorem 8 with $1 < p < \infty, -\frac{n}{p} < \alpha_3 < n - \frac{n}{p}, r = v$ and $\beta = 2,$ we obtain

$$
\| | \cdot |^{\alpha_1} f \|_v \lesssim \| f \|^{1-\theta}_{K^{\alpha_2,v}_u} \| f \|^{\theta}_{K^{\alpha_3,v}_p}
$$

$$
\lesssim \| | \cdot |^{\alpha_2} f \|^{1-\theta}_u \| f \|^{\theta}_{K^{\alpha_3,v}_p}
$$

$$
\lesssim \| | \cdot |^{\alpha_2} f \|^{1-\theta}_u \| f \|^{\theta}_{\tilde{K}^{\alpha_3,v}}
$$

for all $f \in L^u(\mathbb{R}^n, | \cdot |^{\alpha_2}) \cap \tilde{K}^{\alpha_3,v}_u,$ because of

$$
K^{\alpha_2,u}_u \hookrightarrow K^{\alpha_2,v}_u \quad \text{and} \quad \tilde{K}^{\alpha_3,v}_p \hookrightarrow \tilde{K}^{\alpha_3,v}_u.
$$

In particular if $s = m \in \mathbb{N},$ then we obtain

$$
\| | \cdot |^{\alpha_1} f \|_v \lesssim \| f \|^{{1-\theta}}_{K^{\alpha_2,v}_u} \left( \sum_{|\beta| \leq m} \| \partial^\beta f \|_{K^{\alpha_3,v}_p} \right)^{\theta}
$$

$$
\lesssim \| | \cdot |^{\alpha_2} f \|^{{1-\theta}}_u \left( \sum_{|\beta| \leq m} \| \cdot \|^{\alpha_3} \partial^\beta f \|_{K^{\alpha_3,v}_p} \right)^{\theta}
$$

for all $f \in L^u(\mathbb{R}^n, | \cdot |^{\alpha_2}) \cap W^m_p(\mathbb{R}^n, | \cdot |^{\alpha_3}).$ As in \[37, Theorem 4.6\] replace $f(\lambda)$ with $\lambda > 0,$ the sum $\sum_{|\beta| \leq m} \cdots$ can be replaced by $\sum_{0<|\beta|\leq m} \cdots.$

From Proposition 3 and Theorem 7/(i) we obtain the following statement.

**Theorem 9** Let $1 < p, p < \infty, 0 < \tau \leq \infty, 1 < r, v, u < \infty, \sigma \geq 0,$ \((28), (29)\) and \((30)\) with $\alpha_3 < n - \frac{n}{p}.$ Let $\alpha_1 \leq \alpha_2 \leq \alpha_3.$ There is a constant $c > 0$ such that for all $f \in \tilde{K}^{\alpha_2,\delta} \cap \tilde{K}^{\alpha_3,\delta_1},$

$$
\| (-\Delta^\delta f) \|_{K^{\alpha_1,\tau}} \leq c \| f \|^{{1-\theta}}_{K^{\alpha_2,\delta}} \| (-\Delta^\delta f) \|^{\theta}_{K^{\alpha_3,\delta_1}}
$$

with

$$
\delta = \left\{ \begin{array}{ll}
\tau, & \text{if } \alpha_2 = \alpha_1, \\
\theta, & \text{if } \alpha_2 > \alpha_1,
\end{array} \right. \quad \text{and} \quad \delta_1 = \left\{ \begin{array}{ll}
\tau, & \text{if } \alpha_3 = \alpha_1, \\
\theta, & \text{if } \alpha_3 > \alpha_1.
\end{array} \right.
$$

In the next we study the case when $p \leq v < u$ in Theorem 7.

**Theorem 10** Let $0 < p, \tau < \infty, 0 < \beta, \kappa \leq \infty, 1 < r, v < \infty, \sigma \geq 0, 1 < u < \infty,$

$$
-\frac{n}{u} < \alpha_1 < n - \frac{n}{u}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p},
$$

$$
s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0
$$

and

$$
\sigma - \frac{n}{v} = -(1 - \theta)\frac{n}{u} + \theta \left( s - \frac{n}{p} \right) + \alpha_1 - ((1 - \theta)\alpha_2 + \theta\alpha_3), \quad 0 < \theta < 1.
$$
(i) Let \( p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u} \) and \( \alpha_3 = \alpha_2 \). There is a constant \( c > 0 \) such that for all \( f \in K_{u}^{\alpha_2, \tau} \cap K_{p}^{\alpha_3, \tau} F_{\beta}^{s} \),

\[
\| f \|_{k_{v,a}^{\alpha_1, r}} \leq c \| f \|_{K_{u}^{\alpha_2, \tau}}^{1-\theta} \| f \|_{K_{p}^{\alpha_3, \tau} F_{\beta}^{s}}^{\theta},
\]

(39)

(ii) Let \( p \leq v < u, \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{u} \) and \( \alpha_3 > \alpha_2 \). There is a constant \( c > 0 \) such that (39) holds for all \( f \in K_{u}^{\alpha_2, \tau} \cap K_{p}^{\alpha_3, \alpha} F_{\beta}^{s} \) with \( K_{p}^{\alpha_3, \alpha} F_{\beta}^{s} \) in place of \( K_{p}^{\alpha_3, \tau} F_{\beta}^{s} \).

**Proof.** Recall that, as in Theorem 7, one has the estimate

\[
\| f \|_{k_{v,a}^{\alpha_1, r}} \leq \| Qf \|_{k_{v,a}^{\alpha_1, r}} + \| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f \|_{k_{v,a}^{\alpha_1, r}}, \quad J \in \mathbb{N}.
\]

From Proposition 4/ (ii),

\[
\| Qf \|_{k_{v,a}^{\alpha_1, r}} \leq c 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)} \| f \|_{K_{u}^{\alpha_2, \tau}},
\]

which is possible since

\[
\frac{n}{v} + \alpha_1 - \alpha_2 \leq \frac{n}{u} < \frac{n}{v}.
\]

Using again the embedding (34) and Lemma 1 we get

\[
\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f \|_{k_{v,a}^{\alpha_1, r}} \leq \sum_{j=J+1}^{\infty} 2^{j\sigma} \| \mathcal{F}^{-1} \varphi_j * f \|_{k_{v,a}^{\alpha_1, r}} \leq \sum_{j=J+1}^{\infty} 2^{j(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1+\sigma)} \| \mathcal{F}^{-1} \varphi_j * f \|_{k_{p}^{\alpha_3, \vartheta}},
\]

where

\[
\vartheta = \begin{cases} 
\tau, & \text{if } \alpha_3 = \alpha_2, \\
\kappa, & \text{if } \alpha_3 > \alpha_2.
\end{cases}
\]

Therefore, \( \| f \|_{k_{v,a}^{\alpha_1, r}} \) can be estimated by

\[
c 2^{J(\frac{n}{u}-\frac{n}{v}+\alpha_2-\alpha_1+\sigma)} \| f \|_{K_{u}^{\alpha_2, \tau}} + 2^{J(\frac{n}{p}-\frac{n}{v}+\alpha_3-\alpha_1-s+\vartheta)} \| f \|_{k_{p}^{\alpha_3, \vartheta} F_{\beta}^{s}}
\]

\[
= c 2^{J(\frac{n}{p}-\frac{n}{u}+\alpha_2-\alpha_1+\sigma)} \left( \| f \|_{K_{u}^{\alpha_2, \tau}} + 2^{J(\frac{n}{p}-\frac{n}{v}-s-\alpha_2+\alpha_3)} \| f \|_{k_{p}^{\alpha_3, \vartheta} F_{\beta}^{s}} \right),
\]

where the positive constant \( c > 0 \) is independent of \( J \). Observe that

\[
K_{p}^{\alpha_3, \vartheta} F_{\beta}^{s} \hookrightarrow K_{u}^{\alpha_2, \tau},
\]

since \( s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > 0 \). We choose \( J \in \mathbb{N} \) such that

\[
2^{J(\frac{n}{p}-\frac{n}{u}-s-\alpha_2+\alpha_3)} \approx \| f \|_{K_{u}^{\alpha_2, \tau}} \| f \|_{k_{p}^{\alpha_3, \vartheta} F_{\beta}^{s}}^{-1},
\]

we obtain the desired estimate. The proof is complete. As in Theorem 8 combined with Theorem 10 we obtain the following conclusion.
Theorem 11 Under the hypothesis of Theorem 10 with $\alpha_1 > -\frac{n}{v}$ and $\sigma = 0$, we have the estimates with $\hat{K}^{\alpha_1, r}_v$ replaced by $\hat{K}^{\alpha_1, r}_{v, \sigma}$.

Finally we study the case of $v \leq \min(p, u)$.

Theorem 12 Let $1 < r < \infty, 0 < p, \beta, \tau \leq \infty, 1 < v \leq \min(p, u), \alpha_2 - \alpha_1 > \frac{n}{v} - \frac{n}{\max(p, u)}, \alpha_3 \geq \alpha_2, \sigma \geq 0$,

$$-\frac{n}{v} < \alpha_1 < n - \frac{n}{v}, \quad -\frac{n}{u} < \alpha_2 < n - \frac{n}{u}, \quad \alpha_3 > -\frac{n}{p}$$

and

$$s - \frac{n}{p} + \frac{n}{u} + \alpha_2 - \alpha_3 > \sigma - \frac{n}{v} + \alpha_2 - \alpha_1 + \frac{n}{u} > 0.$$

Assume that $0 < p, \tau < \infty$ and $s > \sigma_{p, \beta}$ in the $\hat{K}F$-case. There is a constant $c > 0$ such that for all $f \in \hat{K}^{\alpha_2, \tau}_u \cap \hat{K}^{\alpha_3, \tau}_p A^s_{\beta}$,

$$\|f\|_{\hat{K}^{\alpha_1, r}_v} \leq c \|f\|_{\hat{K}^{\alpha_2, \tau}_u}^{1-\theta} \|f\|_{\hat{K}^{\alpha_3, \tau}_p A^s_{\beta}}^\theta$$

with

$$\sigma - \frac{n}{v} = -(1 - \theta)\frac{n}{u} + \theta \left(s - \frac{n}{p}\right) + \alpha_1 - \left((1 - \theta)\alpha_2 + \theta \alpha_3\right).$$

Proof. By similarity, we only consider $\hat{K}^{\alpha_3, \tau}_p B^s_{\beta}$. We split the proof into two steps.

Step 1. We consider the case $p \leq u$. We employ the same notation as in Theorem 7.

In view of Theorem 10 we need only to estimate

$$\left\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f \right\|_{\hat{K}^{\alpha_1, r}_v}, \quad J \in \mathbb{N}.$$

Using the embedding (34) and Lemma 2 we obtain

$$\left\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f \right\|_{\hat{K}^{\alpha_1, r}_v} \lesssim \sum_{j=J+1}^{\infty} 2^{j\sigma} \left\| \mathcal{F}^{-1} \varphi_j * f \right\|_{\hat{K}^{\alpha_1, r}_v} \lesssim \sum_{j=J+1}^{\infty} 2^{j\sigma} \left\| \mathcal{F}^{-1} \varphi_j * f \right\|_{\hat{K}^{\alpha_1, r}_v}$$

with is possible since

$$\frac{n}{v} + \alpha_1 - \alpha_2 < \frac{n}{p} \leq \frac{n}{v}.$$

Repeating the same arguments of Theorem 7 we obtain the desired estimate.

Step 2. We consider the case $u < p$. Using a combination of the arguments used in the corresponding step of the proof of Theorem 7 and those used in the first step above, we arrive at the desired estimate.

Similarly we obtain the following conclusion.

Theorem 13 Under the hypothesis of Theorem 12 with $\sigma = 0$, we have

$$\|f\|_{\hat{K}^{\alpha_1, r}_v} \lesssim \|f\|_{\hat{K}^{\alpha_2, \tau}_u}^{1-\theta} \|f\|_{\hat{K}^{\alpha_3, \tau}_p A^s_{\beta}}^\theta$$

for all $f \in \hat{K}^{\alpha_2, \tau}_u \cap \hat{K}^{\alpha_2, \tau}_p A^s_{\beta}$.

Remark 9 Under the same hypothesis of Theorems 11 and 13 with $r = v, \sigma = 0, \tau = \max(u, p)$ and $\beta = 2$, we improve Caffarelli-Kohn-Nirenberg inequality (1) in some sense.
3.2 CKN inequalities in Besov-Morrey and Triebel-Lizorkin-Morrey spaces

In this section, we investigate the Caffarelli, Kohn and Nirenberg inequalities in $E_{p,q,u}^s$ and $N_{p,q,u}^s$ spaces. The main results of this section based on the following Lemma.

**Lemma 5** Let $1 < u \leq p < \infty, 1 < s \leq q < \infty$ and $R > 0$.

(i) Assume that $1 \leq v \leq u$. There exists a constant $c > 0$ independent of $R$ such that for all $f \in M_{s,v}^q \cap M_s^q$ with supp $\mathcal{F}f \subset \{ \xi : |\xi| \leq R \}$, we have

$$\| f \|_{M_u^s} \leq cR^\frac{n}{s} \| f \|_{M_{u}^q} \| f \|_{M_{\infty}^{\frac{n}{p}}}.$$

(ii) Assume that $\frac{u}{p} \leq \frac{u}{q}$ and $q \leq p$. There exists a constant $c > 0$ independent of $R$ such that for all $f \in M_s^q$ with supp $\mathcal{F}f \subset \{ \xi : |\xi| \leq R \}$, we have

$$\| f \|_{M_u^s} \leq cR^\frac{n}{s} \| f \|_{M_{u}^q}.$$

**Proof.**

We split the proof in two steps.

**Step 1.** We will prove (i). Let $B$ be a ball of $\mathbb{R}^n$. Write

$$\| |B|^{\frac{1}{p} - \frac{1}{u}} f \chi_B \|_u^u = u \int_0^{\infty} t^{u-1} \{ x \in B : |f(x)| |B|^{\frac{1}{p} - \frac{1}{u}} > t \} dt < \infty.$$

We have

$$|f(x)| \leq cR^\frac{n}{s} \| f \|_{M_{u}^q}, \quad x \in \mathbb{R}^n,$$

see [30] Proposition 2.1] where $c > 0$ independent of $R$. Let $p_0 = \frac{u}{u}$. Clearly

$$|f(x)| = |f(x)|^{p_0} |f(x)|^{1 - p_0} \leq |f(x)|^{p_0} (R^\frac{n}{s} \| f \|_{M_{u}^q})^{1 - p_0} = c |f(x)|^{p_0} d^{1 - p_0},$$

which yields that

$$\| |B|^{\frac{1}{p} - \frac{1}{u}} f \chi_B \|_u^u \leq u \int_0^{\infty} t^{u-1} \{ x \in B : |f(x)| |B|^{\frac{1}{p_0} - \frac{1}{u}} > cd^{1 - \frac{1}{p_0}} \} dt = cd^{u-v} \int_0^{\infty} \lambda^{v-1} \{ x \in B : |f(x)| |B|^{\frac{1}{p_0} - \frac{1}{u}} > \lambda \} d\lambda,$$

after the change the variable $\lambda^{p_0} d - p_0 d^{1 - p_0} = t$. The last expression is clearly bounded by

$$cd^{u-v} \| f \|_{M_{u}^p}^{\frac{v}{p}} \leq cR^{\frac{n}{s}} \| f \|_{M_{u}^q}^{\frac{v}{q}} \| f \|_{M_{\infty}^{\frac{n}{p}}}^{u-v}.$$

**Step 2.** We will prove (ii). If $p = q$, then $u \leq s$ and the estimate follows by the Hölder inequality. Assume that $q < p$ and we choose $v > 0$ such that max$(1, \frac{vu}{p}) < v \leq u < \frac{vu}{q}$.

By Step 1, we only need to estimate $R^{\frac{n}{q} - \frac{n}{v}} \| f \|_{M_{u}^q}^{\frac{v}{q}}$. Write

$$R^{\frac{n}{q} - \frac{n}{v}} \| f \|_{M_{u}^q}^{\frac{v}{q}} = R^{\frac{n}{q} - \frac{n}{v}} \| R^{\frac{n}{p} - \frac{n}{q}} f \|_{M_{u}^p}^{\frac{v}{q}}.$$
Let \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) be a resolution of unity. Observe that

\[
\mathcal{F}^{-1} \varphi_j * f = 0 \quad \text{if} \quad R < 2^{j-1}, \quad j \in \mathbb{N}_0.
\]

This observation together with (26) yield

\[
\| R^{\frac{nu}{pv}-\frac{n}{q}} \|_{\mathcal{M}^{p,v}} \approx \left( \sum_{j \in \mathbb{N}_0, 2^{j-1} \leq R} R^{\frac{nu}{pv}-\frac{n}{q}} |\mathcal{F}^{-1} \varphi_j * f|^q \right)^{1/q} \|_{\mathcal{M}^{p,v}},
\]

which follows by the Sobolev embedding, see Theorem 6,

\[
M^q_s = \mathcal{E}^0_{q,2,s} \hookrightarrow \mathcal{E}^{\frac{nu}{pv}-\frac{n}{q}}_{p,2,v},
\]

since

\[
\frac{n}{q} = \frac{nu}{pv} - \frac{n}{q} - \frac{nu}{pv}, \quad q < \frac{vp}{u} \quad \text{and} \quad \frac{u}{p} \leq \frac{s}{q}.
\]

The lemma is proved.

**Proposition 5** Let \( 1 < u \leq p < \infty, 1 < q < \infty \) and \( s > 0 \).

(i) Let \( f \in \mathcal{N}_{p,q,u}^s \). Then

\[
\| f \|_{\mathcal{N}_{p,q,u}^s} \approx \| f \|_{\mathcal{M}^p_u} + \| f \|_{\mathcal{N}_{p,q,u}^{\hat{s}}},
\]

where

\[
\| f \|_{\mathcal{N}_{p,q,u}^{\hat{s}}} = \left( \sum_{j=-\infty}^{\infty} 2^{qjs} |\mathcal{F}^{-1} \varphi_j * f|^q \right)^{1/q} \|_{\mathcal{M}^p_u}.
\]

(ii) Let \( f \in \mathcal{E}_{p,q,u}^s \). Then

\[
\| f \|_{\mathcal{E}_{p,q,u}^s} \approx \| f \|_{\mathcal{M}^p_u} + \| f \|_{\mathcal{E}_{p,q,u}^{\hat{s}}},
\]

where

\[
\| f \|_{\mathcal{E}_{p,q,u}^{\hat{s}}} = \left( \sum_{j=-\infty}^{\infty} 2^{qjs} |\mathcal{F}^{-1} \varphi_j * f|^q \right)^{1/q} \|_{\mathcal{M}^p_u}.
\]

**Proof.** By similarity, we prove only (ii). We have as in the proof of Proposition 1 that

\[
\| f \|_{\mathcal{E}_{p,q,u}^{\hat{s}}} \lesssim \| f \|_{\mathcal{E}_{p,q,u}^s}.
\]

The only difference with the proof of Proposition 1 consists in the fact that we use [34, Lemma 2.5]. Since \( s > 0 \) we observe

\[
\| f \|_{\mathcal{M}^p_u} \approx \| f \|_{\mathcal{E}_{p,q,u}^0} \lesssim \| f \|_{\mathcal{E}_{p,q,u}^{\hat{s}}}.
\]

Now we prove the opposite inequality. Obviously \( \| \mathcal{F}^{-1} \varphi_0 * f \|_{\mathcal{M}^p_u} \) can be estimated from above by \( \| f \|_{\mathcal{M}^p_u} \), which completes the proof.
**Theorem 14** Let $1 < u \leq p < \infty$ and $1 < v \leq q < \infty$. Assume that $\frac{u}{p} \leq \frac{v}{q}$, $q \leq p$ and $\sigma \geq 0$. Then for all $f \in M^q_u$ and all $J \in \mathbb{N}$,

$$
\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \leq c2^{jn(\frac{1}{q} - \frac{1}{p}) + \sigma}\|f\|_{M^q_u},
$$

where the positive constant $c$ is independent of $J$.

**Proof.** Let $\sigma = \theta m + (1 - \theta)0$, $\alpha \in \mathbb{N}$ with $0 < \theta < 1$ and $|\alpha| \leq m$. We have

$$
\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \leq \|Q_J f\|_{\mathcal{E}^0_{p,2,u}}^{1-\theta}\|Q_J f\|_{\mathcal{E}^m_{p,2,u}}^\theta.
$$

Observe that

$$
\mathcal{E}^m_{p,2,u} = M^{m,p}_u \quad \text{and} \quad \mathcal{E}^0_{p,2,u} = M^p_u,
$$

which yield that

$$
\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \leq \|Q_J f\|_{M^m_u}^{1-\theta}\|Q_J f\|_{M^m_u}^\theta,
$$

where the positive constant $c$ is independent of $J$. Lemma 5 yields that

$$
\|D^\alpha (Q_J f)\|_{M^p_u} \lesssim 2^{jn(\frac{1}{q} - \frac{1}{p}) + |\alpha|}\|f\|_{M^q_u}.
$$

Therefore,

$$
\|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} \lesssim 2^{jn(\frac{1}{q} - \frac{1}{p}) + \sigma}\|f\|_{M^q_u}.
$$

This finish the proof.

Now we are in position to state the main result of this section.

**Theorem 15** Let $1 < u \leq p < \infty$, $1 < \mu \leq \delta < \infty$, $1 < \beta < \infty$, $\sigma \geq 0$ and $1 < v \leq q < \infty$. Assume that

$$
\frac{u}{p} \leq \frac{\mu}{\delta} \leq \frac{v}{q}, \quad s > 0 \quad \text{and} \quad p \geq \delta \geq q.
$$

Let

$$
\frac{s - n}{q} > \frac{\sigma - n}{p} \quad \text{and} \quad \frac{\sigma - n}{p} = -(1 - \theta)\frac{n}{\delta} + \theta\left(\frac{s - n}{q}\right), \quad 0 < \theta < 1.
$$

Then

$$
\|f\|_{\mathcal{E}^\sigma_{p,2,u}} \lesssim \|f\|_{M^\mu_u}^{1-\theta}\|f\|_{\mathcal{N}^\sigma_{q,\beta,v}}^\theta, \quad \sigma > 0 \tag{42}
$$

and

$$
\|f\|_{M^\mu_u} \lesssim \|f\|_{M^\mu_u}^{1-\theta}\|f\|_{\mathcal{N}^\sigma_{q,\beta,v}}^\theta \tag{43}
$$

for all $f \in M^\mu_u \cap \mathcal{N}^\sigma_{q,\beta,v}$.

**Proof.** We have

$$
f = Q_J f + \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f, \quad J \in \mathbb{N}.
$$

Hence

$$
\|f\|_{\mathcal{E}^\sigma_{p,2,u}} \leq \|Q_J f\|_{\mathcal{E}^\sigma_{p,2,u}} + \left\| \sum_{j=J+1}^{\infty} \mathcal{F}^{-1} \varphi_j * f \right\|_{\mathcal{E}^\sigma_{p,2,u}}. \tag{44}
$$
Using Theorem 14 it follows that
\[ \| Q_j f \|_{L^s_{p,2,u}} \lesssim 2^{J_n(\frac{1}{q} - \frac{1}{p}) + \sigma J} \| f \|_{M^\delta_\mu}. \]

From the embedding \( N^\sigma_{p,1,u} \hookrightarrow N^\sigma_{p,\min(2,u),u} \hookrightarrow E_{p,2,u}^\sigma \) and Lemma 5 the last term in (44) can be estimated by
\[ c \sum_{j=J+1}^{\infty} 2^j \| \mathcal{F}^{-1} \varphi_j * f \|_{M^\delta_\mu} \lesssim \sum_{j=J+1}^{\infty} 2^{j_n(\frac{1}{q} - \frac{1}{p}) + j \sigma} \| \mathcal{F}^{-1} \varphi_j * f \|_{M^\delta_\mu} \]
\[ \lesssim 2^{J(\frac{n}{q} - \frac{n}{p} + \sigma - s)} \| f \|_{N^s_{q,\infty,v}}. \]

since \( s - \frac{n}{q} > \sigma - \frac{n}{p} \). Therefore,
\[ \| f \|_{E^\sigma_{p,2,u}} \leq c 2^{J(\frac{1}{q} - \frac{1}{p}) + \sigma J} \| f \|_{M^\delta_\mu} + 2^{J(\frac{1}{q} - \frac{1}{p} + \sigma - s)} \| f \|_{N^s_{q,\infty,v}} \]
\[ = c 2^{J(\frac{1}{q} - \frac{1}{p}) + \sigma J} \left( \| f \|_{M^\delta_\mu} + 2^{J(\frac{1}{q} - \frac{1}{p} - s)} \| f \|_{N^s_{q,\infty,v}} \right), \]

where the positive constant \( c \) is independent of \( J \). We wish to choose \( J \in \mathbb{N} \) such that
\[ \| f \|_{M^\delta_\mu} \approx 2^{J(\frac{1}{q} - \frac{1}{p} - s)} \| f \|_{N^s_{q,\infty,v}}, \]
which is possible since \( N^s_{q,\infty,v} \hookrightarrow M^\delta_\mu \). Indeed, from Theorem 6 and (26), we get
\[ N^s_{q,\infty,v} \hookrightarrow E^s_{q,\infty,v} \hookrightarrow E^\sigma_{p,2,u} = M^\delta_\mu, \]
beacuse of \( s - \frac{n}{q} > \frac{n}{p} \). Thus
\[ \| f \|_{E^\sigma_{p,2,u}} \lesssim \| f \|_{M^\delta_\mu} \| f \|_{N^s_{q,\infty,v}}^{\theta}. \]

Using (10) and (11) we arrive at the inequality
\[ \| f \|_{E^\sigma_{p,2,u}} \lesssim \| f \|_{M^\delta_\mu} \left( \| f \|_{M^\delta_\mu} + \| f \|_{N^s_{q,\infty,v}} \right)^\theta. \]

In this estimate replace \( f \) by \( f(\lambda \cdot) \) and using (10) to obtain
\[ \| f \|_{E^\sigma_{p,2,u}} \lesssim \| f \|_{M^\delta_\mu} \left( \lambda^{-s} \| f \|_{M^\delta_\mu} + \| f \|_{N^s_{q,\infty,v}} \right)^\theta. \]

Taking \( \lambda \) large enough we obtain (12) \( \Rightarrow \) (13).

**Acknowledgments**

We thank the referee for carefully reading the paper and for making several useful suggestions and comments.

This work is found by the General Direction of Higher Education and Training under Grant No. C00L03UN280120220004 and by The General Directorate of Scientific Research and Technological Development, Algeria.
References

[1] A. Baernstein II, E.T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*. Mem. Amer. Math. Soc. 53, no. 318, 1985.

[2] H.Q. Bui, *Weighted Besov and Triebel spaces: interpolation by the real method*. Hiroshima Math. J. 12 (1982), 581–605.

[3] H.Q. Bui, *Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures*. J. Funct. Anal. 55, (1984) 39–62.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights*. Compos. Math. 53 (1984), 259–275.

[5] A. Djeriou, D. Drihem, *On the continuity of pseudo-differential operators on multiplier spaces associated to Herz-type Triebel-Lizorkin spaces*, Mediterr. J. Math. (2019) 16: 153. https://doi.org/10.1007/s00009-019-1418-7.

[6] D. Drihem, *Embeddings properties on Herz-type Besov and Triebel-Lizorkin spaces*. Math. Ineq and Appl. 16 (2) (2013), 439–460.

[7] D. Drihem, *Sobolev embeddings for Herz-type Triebel-Lizorkin spaces*, Function Spaces and Inequalities. P. Jain, H.-J. Schmeisser (ed.). Springer Proceedings in Mathematics and Statistics. Springer, 2017.

[8] D. Drihem, *Complex interpolation of Herz-type Triebel-Lizorkin spaces*. Math. Nachr. 291 (13) (2018), 2008–2023.

[9] D. Drihem, *Jawerth-Franke embeddings of Herz-type Besov and Triebel-Lizorkin spaces*. Funct. Approx. Comment. Math. 61 (2) (2019), 207–226.

[10] D. Drihem, H. Hebbache, *Continuity of non-regular pseudodifferential operators on variable Triebel-Lizorkin spaces*. Annales Polonici Mathematici. 122 (2019), 233–248.

[11] D. Drihem, *Semilinear parabolic equations in Herz spaces*. Appl. Anal. (2022), https://doi.org/10.1080/00036811.2022.2047948.

[12] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted norm inequalities and related topics*. In: North-Holland Mathematics Studies, Vol. 116, North-Holland, Amsterdam, 1985.

[13] D.D. Haroske, L. Skrzypczak, *On Sobolev and Franke–Jawerth embeddings of smoothness Morrey spaces*. Rev Mat Complut. 27 (2014), 541–573.

[14] V. Felli, M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*. J. Diff. Equations. 191 (1) (2003), 121–142.

[15] H.G. Feichtinger, F. Weisz, *Herz spaces and summability of Fourier transforms*. Math. Nachr. 281 (3) (2008), 309–324.
[16] E. Hernandez, D. Yang, Interpolation of Herz-type Hardy spaces. Illinois J. Math. 42 (1998), 564–581.

[17] C. Herz, Lipschitz spaces and Bernstein’s theorem on absolutely convergent Fourier transforms. J. Math. Mech. 18 (1968), 283–324.

[18] M. Izuki, Y. Sawano, Atomic decomposition for weighted Besov and Triebel-Lizorkin spaces. Math. Nachr. 285, (2012) 103–126.

[19] X. Li, D. Yang, Boundedness of some sublinear operators on Herz spaces. Illinois J. Math. 40 (1996), 484–501.

[20] Y. Li, D. Yang, L. Huang, Real-variable Theory of Hardy Spaces Associated with Generalized Herz Spaces of Rafeiro and Samko. Lecture Notes in Mathematics 2320, Springer, Cham, 2022.

[21] S. Lu, D. Yang, Herz-type Sobolev and Bessel potential spaces and their applications. Sci. in China (Ser. A). 40 (1997), 113–129.

[22] S. Lu, D. Yang, G. Hu, Herz type spaces and their applications. Beijing: Science Press, 2008.

[23] J. Marschall, Weighted parabolic Triebel spaces of product type. Fourier multipliers and pseudo-differential operators. Forum. Math. 3 (1991), 479-511.

[24] A.L. Mazzucato, Decomposition of Besov-Morrey spaces. In: Harmonic Analysis at Mount Holyoke 2001, Contemporary Math. 320 (2003), 279- 294.

[25] A.L. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE. Trans. Am. Math. Soc. 355 (4) (2003), 1297–1364.

[26] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. 43 (1938), 126–166.

[27] H-M, Nguyen, M. Squassina, Fractional Caffarelli–Kohn–Nirenberg inequalities. J. Funct. Anal. 274 (2018), 2661–2672.

[28] S.M. Nikol’skij, Approximation of function of several variables and imbedding Theorem. Springer, Berlin, Germany, 1975.

[29] H. Rafeiro, S. Samko, Herz spaces meet Morrey type spaces and complementary Morrey type spaces. J. Fourier Anal. Appl. 26 (2020), Paper No. 74, 14 pp.

[30] Y. Sawano, S. Sugano, H. Tanaka, Identification of the image of Morrey spaces by the fractional integral operators. Proc. A. Razmadze Math. Inst. 149 (2009), 87–93.

[31] W. Sickel, On pointwise multipliers for $F^{s}_{p,q}(\mathbb{R}^n)$ in case $\sigma_{p,q} < s < n/p$. Ann. Mat. Pura. Appl. 176(1)(1999), 209–250.

[32] W. Sickel, Smoothness spaces related to Morrey spaces – a survey. I. Eurasian Math. J. 3 (3) (2012), 110–149.
[33] L. Tang, D. Yang, *Boundedness of vector-valued operators on weighted Herz*. Approx. Th. Appl. 16 (2000), 58–70.

[34] L. Tang, J. Xu, *Some properties of Morrey type Besov-Triebel spaces*. Math. Nachr. 278 (2005), 904–917.

[35] H. Triebel, *Theory of function spaces*. Birkhäuser, Basel, 1983.

[36] H. Triebel, *Theory of function spaces II*. Birkhäuser, Basel, 1992.

[37] H. Triebel. *Local function spaces, heat and Navier-Stokes equations*. European Math. Soc. Publishing House, Zürich, 2013.

[38] Y. Tsutsui, *The Navier-Stokes equations and weak Herz spaces*, Adv. Differential Equations. 16 (2011), 1049–1085.

[39] J. Xu, D. Yang, *Applications of Herz-type Triebel-Lizorkin spaces*. Acta. Math. Sci (Ser. B). 23 (2003), 328–338.

[40] J. Xu, D. Yang, *Herz-type Triebel-Lizorkin spaces, I*. Acta Math. Sci (English Ed.). 21 (3) (2005), 643–654.

[41] J. Xu, *Equivalent norms of Herz type Besov and Triebel-Lizorkin spaces*. J Funct Spaces Appl 3 (2005), 17–31.

[42] J. Xu, *Decompositions of non-homogeneous Herz-type Besov and Triebel-Lizorkin spaces*. Sci. China. Math. 47(2) (2014), 315–331.

[43] B. Xuan, *The solvability of quasilinear Brezis–Nirenberg-type problems with singular weights*. Nonlinear Anal., Theory Methods Appl. 62 (2005), 703–725.

[44] W. Yuan, W. Sickel, D. Yang, *Morrey and Campanato Meet Besov, Lizorkin and Triebel*. Lecture Notes in Mathematics, vol. 2005, Springer-Verlag, Berlin 2010.

[45] Y. Zhao, D. Yang, Y. Zhang, *Mixed-norm Herz spaces and their applications in related Hardy spaces*. Anal. Appl. (Singap.) (2022) (to appear).