The leading hadronic contribution to \((g-2)\) of the muon: The chiral behavior using the mixed representation method

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Overview
What is the leading order anomalous magnetic moment of the muon $a_{\mu}^{HLO}$ and what precision can be reached from lattice calculations?

The central quantity $a_{\mu}^{HLO}$ is accessible from the lattice by computing the hadronic vacuum polarization (HVP) function $\Pi(Q^2)$

$$a_{\mu}^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int dQ^2 K_E(Q^2, m_\mu) \left(\Pi(Q^2) - \Pi(0)\right)$$  \hspace{1cm} (1)

In the following and the talks by V. Gülpers, H. Horch and G. Herdoiza, we study different methods of obtaining $\left(\Pi(Q^2) - \Pi(0)\right)$ and discuss the uncertainties arising from their respective systematics, as well as the disconnected diagrams.
Hadronic vacuum polarization

In phenomenology the hadronic vacuum polarization can be computed via

\[
\left( \Pi(Q^2) - \Pi(0) \right) = \frac{Q^2}{3} \int_{0}^{\infty} ds \frac{R(s)}{s(s + Q^2)}
\]

where \( R(s) \propto \sigma(e^+e^- \rightarrow \text{hadrons}) \)
On the lattice both sides of (2) can be used to compute the HVP

- **Lhs:**

\[
\left( \Pi(Q^2) - \Pi(0) \right) = ... 
\]

Extract the \( \Pi(Q^2 > Q^2_{\text{latt,min}}(L,a)) \) by noting

\[
\Pi_{\mu\nu}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2) 
\]

where \( \Pi_{\mu\nu}(Q) \) is given in terms of the vector meson current-current correlator \( \langle j_\mu(x) j_\nu(0) \rangle \)

\[
\Pi_{\mu\nu}(Q) \equiv \int d^4x \, e^{iQ \cdot x} \langle j_\mu(x) j_\nu(0) \rangle 
\]

- We refer to this approach as the "standard method",

[0212018], [0608011], [1103.4818], [1011.5793]
On the lattice both sides of (2) can be used to compute the HVP

**Rhs:**

\[
\ldots = \frac{Q^2}{3} \int_0^\infty ds \frac{R(s)}{s(s + Q^2)}
\]

Extract the *difference* \(\Pi(Q^2) - \Pi(0)\) by noting

\[
R(s) = 12\pi^2 \rho(s)
\]

where \(\rho(s)\) is the spectral function of the vector meson current-current correlator \(\langle j_\mu(x)j_\nu(0)\rangle\)

\[
G(x_0, \vec{k}) \equiv \int d^3x \, e^{i\vec{k}\vec{x}} \langle J_\mu(x_0, \vec{x})J_\nu(0)\rangle = \int_0^\infty ds \sqrt{s}\rho(s)K(s, x_0).
\]

One finds:

\[
\Pi(Q^2) - \Pi(0) = \int_0^\infty dx_0 \, G(x_0) \left[ x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2}Qx_0\right) \right]
\]

- We refer to this approach as the "mixed representation method",

[1305.5878], [1306.2532]
Caveats of the current methods

\[
\left( \Pi(Q^2) - \Pi(0) \right) = \frac{Q^2}{3} \int_0^\infty ds \frac{R(s)}{s(s + Q^2)}
\]

\[
\Pi(Q^2 > Q_{min}^2) - \Pi(Q^2 \to 0) = \ldots \quad \ldots = \int_0^\infty dx_0 G(x_0) K(x_0, Q^2)
\]

**Lhs:** Forming in the standard method \( \left( \Pi(Q^2) - \Pi(0) \right) \) and \( a_{HLO}^\mu \) note ...

- ... lattice data is not available at \( \Pi(0) \)
- ... extrapolation from \( Q^2 = 0 \) to \( Q_{min}^2 = \min(Q_{latt}^2(L, a)) \) is required
- ... \( a_{HLO}^\mu \) depends crucially on precise data/interpolation at low \( Q^2 \)

**Rhs:** Integrating using the mixed rep. method for \( \left( \Pi(Q^2) - \Pi(0) \right) \) and \( a_{HLO}^\mu \) note ...

- ... the correlator has to be known for all times \( t \to \infty \)
- ... lattice data has to extrapolated to its asymptotic behavior
- ... \( a_{HLO}^\mu \) depends crucially on precise knowledge of the correlator/spectrum
Towards a precision determination of $a_{\mu}^{HLO}$

Both the standard and the mixed rep. method rely on the same data and ultimately process/display equivalent information.

However, what is low $Q^2 \rightarrow 0$ in one is large Euclidean times $t \rightarrow \infty$ in the other. Can we use this to our advantage?

Ad-/disadvantages of the standard method → talks at this conference.

In the mixed rep. method the key observable, $G(x_0, \vec{k} = 0)$, ...

- ... has a well established machinery to study signal/noise behavior and finite size/mass/lattice spacing effects.
- ... can draw on a large body of experience/methods to systematically improve the results.
- ... enables a systematic study of the spectrum of QCD.
- ... opens the possibility of a straight forward inclusion of the disconnected diagrams (see talk by V. Gülpers).
What advantages can we expect to exploit?

- The results for \( G(x_0, \kappa = 0) \) can be extracted at runtime from a program computing \( \Pi(Q^2) \) at negligible cost.

- The difference of the standard method and mixed rep. method can be monitored

\[
\Pi_{STD}(Q^2) - \left( \Pi(Q^2) - \Pi(0) \right)_{MRM} = \Pi(0) \quad (3)
\]

iff both analysis are in fact equivalent,

- difference should be approx. constant
- gives a measure of \( \Pi(0) \)

- The different systematics should become visible in quantities like (3)
Towards chiral behaviour of $a_{\mu}^{HLO}$

- We explore the chiral behaviour of $a_{\mu}^{HLO}$ at fixed lattice spacing $a = 0.063$ and $\beta = 5.30$

| lattice     | $L$ [fm] | $m_\pi$ [MeV] | $m_\pi L$ | $N_{meas}(N_{conf})$ | Label |
|-------------|----------|---------------|-----------|-----------------------|-------|
| $64 \times 32^3$ | 2.0      | 451           | 4.7       | 4000(1000)            | E5    |
| $96 \times 48^3$ | 3.0      | 324           | 5.0       | 1200(300)             | F6    |
| $96 \times 48^3$ | 3.0      | 277           | 4.2       | 1000(250)             | F7    |
| $128 \times 64^3$ | 4.0      | 190           | 4.0       | 820(205)              | G8    |

- All ensembles were generated within the CLS effort with two flavors of $O(a)$ improved Wilson-Clover fermions
- Correlation functions for strange and charm (not shown here) quark masses are available as quenched, valence observables
Standard method

To do:

- Extrapolation to $Q^2 = 0$, via Padé-fit \([1112.2894], [1205.3695], [1309.2153]\).

- Integration of $\Pi_{\text{fit}}(Q^2) - \Pi_{\text{fit}}(0)$ to obtain $a_{\mu}^{HLO}$.
Mixed rep. method

**To do:**

- Asymptotic extrapolation, via single exponential-fit
  
- Integration of $G(x_0)$ for 
  \[
  \left( \Pi(Q^2) - \Pi(0) \right)
  \]

- or direct integration of $G(x_0)$ to obtain $a^{HLO}_\mu$
The extrapolation of \( G(x_0) \) depends on the low mass spectrum

- We assume a single ground-state exponential contributes beyond \( x_0 \simeq 1.2 \text{fm} \) (approx. \( x_0/a \simeq 18 - 20 \))
- Further contributions cannot be excluded/included from the current data
- We use smeared-smeared interpolating operators in the light case
Comparing both methods

The difference \( \Pi_{STD}(Q^2) - \left( \Pi(Q^2) - \Pi(0) \right)_{MRM} \) shows an almost constant behavior.

In principle \( \Pi(0) \) can be extracted from the difference ...

... here we will extract \( \Pi(0) \) for the standard method from Padé-fit
The anomalous magnetic moment of the muon for $N_f = 2$
The anomalous magnetic moment of the muon for $N_f = 2 + 1q$
We explored the chiral behavior of $a^{HLO}_\mu$
We extended our analysis of the standard method to $m_\pi = 190\text{MeV}$
We included the mixed rep. method and compared systematically
The standard method and the MRM are seen to be highly compatible.

The MRM gives a new handle to study the systematic effects and their induced errors on $a^{HLO}_\mu$.

In the future, we will give an estimate of $a^{HLO}_\mu$ at the physical point including also disconnected diagrams (see talk by V. Gülpers).