Multi-Path Low Delay Network Codes

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Abstract—The capability of mobile devices to use multiple interfaces to support a single session is becoming more prevalent. Prime examples include the desire to implement WiFi offloading and the introduction of 5G. Furthermore, an increasing fraction of Internet traffic is becoming delay sensitive. These two trends drive the need to investigate methods that enable communication over multiple parallel heterogeneous networks, while also ensuring that delay constraints are met. This paper approaches these challenges using a multi-path streaming code that uses forward error correction to reduce the in-order delivery delay of packets in networks with poor link quality and transient connectivity. A simple analysis is developed that provides a good approximation of the in-order delivery delay. Furthermore, numerical results help show that the delay penalty of communicating over multiple paths is insignificant when considering the potential throughput gains obtained through the fusion of multiple networks.

I. INTRODUCTION

The increasing availability of wireless devices with multiple radios is driving the push to merge network resources across multiple radio technologies and cellular access nodes. Prime examples include the desire to offload traffic from cellular networks to WiFi networks and the desire to simultaneously utilize both macro and small cells in 5G networks. While the merging of network resources has the potential to drastically increase throughput, packet losses due to congestion, poor link quality, transient network connections, etc. can have serious consequences for meeting users’ quality of service (QoS) requirements. A multi-path streaming code, derived from a low delay code designed for single paths [1], is presented that helps overcome these challenges by reducing the end-to-end, or in-order delivery, delay. Through an analysis of the in-order delivery delay, we further show that merging parallel networks together using network coding enables the summation of individual path throughputs without significant impacts to the overall in-order delivery delay.

The delivery of information in the order it was first transmitted is a requirement for most applications. Unfortunately, packet losses that occur during transmission can cause significant disruptions and delays. As an example, automatic repeat request (ARQ) is one approach to recover from these packet losses. Whenever a packet loss occurs, all packets received after the loss are buffered until ARQ corrects the erasure. This takes on the order of a round-trip time (RTT) or more. If the RTT (more precisely the bandwidth-delay product (BDP)) is small, the disruption in packet delivery is relatively minor. However, if the BDP is large, the added delay necessary to recover from a packet erasure can be detrimental to the QoS of delay sensitive applications.

The remainder of this paper is organized as follows. Section II describes the multi-path streaming code. This includes a discussion regarding the code rate used on each path and the management of the code window used to generate redundancy. Section III describes the system model that is used for the analysis of the in-order delivery delay presented in Section IV. A comparison of the analysis with simulated results is shown in Section V. This section demonstrates that the analysis provides a good estimate of the in-order delivery delay, the use of network coding to merge parallel network paths results in
gains to throughput without impacting the delay, and the trade-off between rate and delay. Finally, the paper is concluded in Section VI.

II. MULTI-PATH STREAMING CODE ALGORITHM

Consider a systematic coding scheme based on random linear network coding (RNLC) \[9\] that allows a server to communicate over parallel paths or networks while helping to reduce the overall in-order delivery delay. Information packets \(p_i, i \in [1, M]\), are injected into each network uncoded. Note that the server has limited knowledge of the packets that will be sent in the future (i.e., it does not have access to the entire file). If an opportunity arises that allows the server to transmit a new packet, it does so without attempting to ensure specific packets arrive at the client in a predetermined order. After a specific number of information packets have been transmitted on any given path, the server generates and transmits a coded packet \(c_i\) on a path of its choosing to help overcome any packet losses that may have occurred.

Define \(l_i\) to be the duration between transmitted coded packets on path \(i \in \mathcal{P}\) (i.e., \(l_i - 1\) information packets are transmitted followed by a single coded packet). This results in a code rate of \(c_i = l_i^{-1}/l\). If a path is idle, the server will transmit either an information packet or coded packet depending on the previously transmitted packets on that specific path. When a coded packet is generated, the information packets used to produce the linear combination are drawn from a dynamically changing code window defined by the 2-tuple \(w = (w_L, w_U)\). This results in the following packet:

\[
c_{n,k} = \sum_{i=w_L}^{w_U} \alpha_{n,k,i} p_i.
\]  

The coefficients \(\alpha_{n,k,i} \in F_q\) are chosen at random and each information packet \(p_i\) is treated as a vector in \(F_q\). All of this is summarized in Algorithm 1 where \(1_i\) is a vector of size \(i\) consisting of all ones. In addition, an example of the generator matrix used to produce the streaming code is provided in Figure 1. In the example, information packets \(p_1\) through \(p_4\) and \(p_5\) through \(p_8\) are transmitted systematically in time-slots 1 through 4 and 6 through 9 respectively. In time-slots 5 and 10, coded packets \(c_{n,1} = \sum_{i=1}^{4} \alpha_{n,1,i} p_i\) and \(c_{n,2} = \sum_{i=3}^{8} \alpha_{n,2,i} p_i\) are transmitted respectively. It is assumed in this example that the server has obtained feedback from the client by time 10 indicating that it successfully received and decoded packets \(p_1\) and \(p_2\). This allows the server to adjust the lower edge of the coding window to exclude the packets during the generation of coded packet \(c_{n,2}\).

![Figure 1: Example generator matrix used to produce the streaming code. The elements of the matrix contain the coefficients used to produce each transmitted packet. The rows show the composition of each transmitted packet and the columns indicate the information packet that must be transmitted.](image)

Algorithm 1: Streaming Multi-Path Code Generation

```
Initialize \(k = 1\) and \(u = 1_{|\mathcal{P}|}

\[\text{while } k \leq M \text{ do}
\]

\[\text{if } u_{n} < l_{n} \text{ then}
\]

\[\begin{align*}
&\text{Transmit packet } p_k \\
&u_{n} \leftarrow u_{n} + 1 \\
&k \leftarrow k + 1
\end{align*}
\]

\[\text{else}
\]

\[\begin{align*}
&\text{Transmit coded packet } c_{n,k} = \sum_{i=w_{L}}^{w_{U}} \alpha_{n,k,i} p_i \\
&u_{n} \leftarrow 1
\end{align*}
\]
```

Before proceeding, it must be noted that Algorithm 1 does not explicitly take advantage of feedback in determining when to inject coded packets into the packet stream. Rather, feedback is only used to estimate the packet erasure probability \(c_i\) on each path \(i \in \mathcal{P}\). This is done in order to simplify the analysis that will be presented later. However, using feedback can only improve the algorithm’s performance; and implemented versions should use feedback intelligently when determining when to inject redundancy to help reduce the delay further.

While Algorithm 1 is fairly simple, two topics jump out that require special consideration. First, the selection of code rates \(c_i\) on every path \(i \in \mathcal{P}\) must be done properly to ensure that the client can decode within a reasonable time period regardless of the observed packet losses. Second, management of the coding window \(w\) must be performed carefully to ensure coded packets add to the knowledge space of the client. These two topics will be addressed by defining the following:

**Definition 1.** A coding policy \(\pi\) determines the code structure and rates used on each path between a server and client.

In other words, the coding policy defines the code rates used to generate coded packets on each path, as well as the algorithm for managing \(w\). There are, in fact, an infinite number of coding policies. However, policies that allow the client to decode with high probability within a reasonable time frame are the only ones of interest. This leads to the next definition:

**Definition 2.** A coding policy that ensures the client will decode with probability equal to 1 is said to be admissible.

Let \(\Pi = \{\pi_1, \pi_2, \pi_3, \ldots\}\) be the set of all admissible coding policies. The code rates and code window management rules for policy \(\pi_i\) will be referred to as the \(|\mathcal{P}|\)-tuple
An admissible coding policy must ensure the client’s capability to decode. One of the most important parts is choosing the appropriate code rate $c(\pi)$. The following theorem helps determine this rate where its proof is provide in the appendix.

**Theorem 3.** An admissible coding policy $\pi$ must satisfy the following constraints:

$$
\sum_{i \in \mathcal{P}} (1 - \epsilon_i) \left((1 - \epsilon_i) - c_i(\pi)\right) r_i > 0, \quad (2)
$$

and

$$
c_i(\pi) \in [0, 1]. \quad (3)
$$

Transmitting coded packets over multiple networks maybe the appropriate strategy in some cases; but in others, it maybe better to only send coded packets over a single network. This leads to the following corollary.

**Corollary 4.** For the case when coded packets are only transmitted over a single path and there exists a $\pi$ such that $c_i(\pi) \in [0, 1]$ and $c_j(\pi) = 1$ for $i, j \in [1, |\mathcal{P}|], i \neq j$, the admissible coding policy $\pi$ must satisfy the following:

$$
c_i(\pi) < 1 - \sum_{k \in \mathcal{P}} \frac{(1 - \epsilon_k) \epsilon_k r_k}{(1 - \epsilon_i) r_i}. \quad (4)
$$

### B. Code Window Management

For admissible coding policies, the code window used to generate coded packets must provide the potential for the client’s knowledge space to increase in the presence of packet losses. There are many ways of accomplishing this goal ranging from schemes that code on a generation-by-generation bases to schemes that code over the entire packet stream. While there is no guarantee that the scheme proposed here is optimal, it does lead to an admissible coding policy.

As a reminder, it is assumed that coded packets are used solely for redundancy. If the path or network is error-free, coded packets will not contribute to the knowledge space of the client. In addition, it is assumed that coding occurs over a packet stream where the server has limited to no knowledge of packets that will be sent in the future. Therefore, any decisions regarding the code window management must be made using information packets that have already been sent and information from feedback that is at least $RTT$ seconds old. Before an algorithm is proposed, the concept of a seen packet from [10] must be established.

**Definition 5.** The client is said to have seen a packet $p_i$ if it has enough information to compute a linear combination of the form $(p_i + q)$ where $q = \sum_{k > i} \alpha_k p_k$ with $\alpha_k \in \mathbb{F}_q$ and $\forall k > i$. Therefore, $q$ is a linear combination involving information packets with indices larger than $i$.

Define seen to be the index of the last seen information packet at the client that is composed of the set of all consecutive seen information packets. It is assumed that the client informs the server of the value of seen through feedback. Once the feedback has been received by the server, seen will be used to set the lower edge of the code window. The upper edge of the code window will be managed based on the index of the last transmitted uncoded information packet. This is summarized in Algorithm 2 which is executed by the server and is agnostic to the path on which any one packet is transmitted.

**Algorithm 2: Code Window Management**

```
Initialize $(w_L, w_U) = (0, 0)$ and $j = 0$
if $p_j$ transmitted uncoded and $i > w_U$ then
    $w_U \leftarrow i$
endif
if Feedback received and seen > $w_L$ then
    $w_L \leftarrow \text{seen}$
```

Since seen is required to be the last seen packet out of the set of consecutive packets, the client will eventually be able to decode given the transfer of enough degrees of freedom. Furthermore, using seen packets to manage the code window helps to decrease the size of the coding/decoding buffers on the server/client respectively.

### III. System Model

A time-slotted model is assumed where a single server-client pair are communicating with each other over multiple parallel networks. Similar to the last section, we denote this set of disjoint networks as $\mathcal{P}$. Data is first placed into information packets $p_1, p_2, \ldots$. These information packets are then used to generate coded packets $c_1, c_2, c_3, \ldots$. Depending on the coding policy, the server chooses to transmit either an information packet or coded packet over one of the network paths. The time it takes to transmit either type of packet is $t_i = \frac{1}{r_i}$ seconds where $r_i$ is the transmission rate in packets/second of network $i \in \mathcal{P}$. Furthermore, it takes each packet $d_i$ seconds to propagate through network $i$ (e.g., $RTT_i = t_i + 2d_i$ on network $i$ assuming that the size of the feedback packet is sufficiently small).

Delayed feedback is available to the server allowing it to estimate each path’s independent and identically distributed (i.i.d.) packet erasure rate $\epsilon_i$ and round-trip time $RTT_i$ (in seconds). However, the server is unable to determine the cause of the packet erasures (e.g., poor network conditions or congestion). Furthermore, the server has knowledge of each network’s transmission rate $r_i$, which can either be determined from feedback obtained from the client or from the size of the server’s congestion window on any specific path (e.g., $r_i = \text{cwnd} / RTT_i$). This feedback can also be used to communicate to the server the number of dofs received by the client. While the following analysis assumes feedback does not contain this information, numerical and simulated results will use feedback to dynamically adjust the code rate depending on the client’s dof deficit.

\[c(\pi) = (c_1(\pi), \ldots, c_{|\mathcal{P}|}(\pi)) \text{ and } W(\pi) \text{ respectively.}\]
IV. Analysis of the In-Order Delivery Delay

Before proceeding, several assumptions are required to simplify the analysis. First, it is assumed that coded packets are only sent over a single network and the code rates conform to Corollary 4. The rate and packet erasure probability of the network used to send coded packets will be referred to as \( r_c \) and \( e_c \), respectively. Second, packets transmitted over faster networks are delayed so that they arrive in-order with packets transmitted over slower networks. For example, assume that packets are transmitted over two disjoint networks with propagation delays \( d_1 \) and \( d_2 \), where \( d_1 < d_2 \). Packets transmitted over network 1 will be delayed an additional \( d_2 - d_1 \) seconds. This assumption affects the analysis by over-estimating the delay since there is a possibility that packets transmitted over the faster networks can be delivered in-order without waiting for a packet from the slower network. However, the number of packets transmitted over the faster networks that can be delivered without packets from the slower networks is relatively small. Third, the coding window used for each coded packet contains all transmitted information packets. This assumption is not necessary if the code window management follows Algorithm 2. However, it does remove any ambiguity regarding the usefulness of a received coded packet.

The in-order delivery delay for the code provided in Algorithm 1 can be determined using a renewal-reward process. Consider a packet transmission process over two disjoint networks with prop-

\[
l_c = \frac{1}{1 - e_c (\pi)}
\]

information packets on each network \( i \in \mathcal{P} \) for every transmitted coded packet. Let \( \alpha_i = (l_c - 1) \frac{r_i}{r_c} \) be the number of lost packets on path \( p_i \) for every transmitted coded packet. Now consider a modified time-slotted model where each time slot has duration \( l_c/r_c \). Define the sequence \( X_1, X_2, \ldots \), where \( X_n = 0, 1, 2, \ldots \) slots, to be the i.i.d. inter-arrival times between decode events with first and second moments \( \mathbb{E}[X] \) and \( \mathbb{E}[X^2] \), respectively. The arrival process is then a sequence of non-decreasing random variables, or arrival epochs, \( 0 \leq S_1 \leq S_2 \leq \cdots \) where \( n \) is the \( n \)th epoch \( S_n = \sum_{i=1}^{n} X_n \).

In order to determine the distribution and moments of \( X_1, X_2, \ldots \), several additional random variables need to be defined. Let the random variable \( Y_{n,i} \), \( i = 1, 2, \ldots \), be the number of lost packets (both information and coded) between \( S_{n-1} + (i - 1) \) and \( S_{n-1} + i \) in the \( n \)th arrival epoch. The exact distribution is the convolution of \( |\mathcal{P}| \) binomial distributions with parameters \( \alpha_i \) and \( \epsilon_i \) for each \( i \in \mathcal{P} \). In order to simplify the analysis, this distribution is approximated by the following

\[
P_{Y_{n,i}}(y_{n,i}) = \frac{\lambda^{y_{n,i}}}{y_{n,i}!} e^{-\lambda}, \quad y_{n,i} = 0, 1, \ldots,
\]

where

\[
\lambda = \epsilon_c + \sum_{i \in \mathcal{P}} \alpha_i \epsilon_i. \tag{8}
\]

If \( Y_{n,1} = 0 \), the number of received packets between \( S_{n-1} \) and \( S_{n-1} + 1 \) is \( 1 + \sum_{i \in \mathcal{P}} \alpha_i \); however, only the \( \sum_{i \in \mathcal{P}} \alpha_i \) packets were necessary to decode (i.e., the coded packet is of no benefit and is dropped). Therefore, \( X_n = 0 \). However if \( Y_{n,1} > 0 \), at least one packet was lost which may prevent delivery. Therefore, the extra \( \text{DOF} \) obtained from coded packets will help correct these erasures and eventually lead to a decode event. As an example, consider the case when \( Y_{n,1} = 1 \). A single packet was lost, but enough \( \text{DOF}s \) were received to decode all of the packets transmitted between \( S_{n-1} \) and \( S_{n-1} + 1 \). Therefore, the inter-arrival time is \( X_n = 1 \). Now consider the case when \( Y_{n,1} = 2 \) and \( Y_{n,2} = 0 \). It is impossible for a renewal to occur between at \( S_{n-1} \) or \( S_{n-1} + 1 \); however a renewal does occur at \( S_{n-1} + 2 \). This results in an inter-arrival time \( X_n = 2 \). Continuing on in this way, it becomes clear that a renewal occurs the first time \( Z = \sum_{i=1}^{Z} Y_{n,i} \leq Z \).

In fact, \( Z \) is a random variable and can be modeled as a \( \text{M}/\text{D}/1 \) queue with a constant service time of one packet per slot and an arrival rate of \( \lambda \) packets per slot. Define \( Y = \sum_{i=1}^{Z} Y_{n,i} \), then \( Z \) conditioned on \( Y \) has the following Borel-Tanner distribution \([11]\):

\[
p_{Z|Y}(z|y) = \frac{y^{z-1} \lambda^z e^{-\lambda y}}{(z-1)!}, \tag{9}
\]

for \( y, z = 1, 2, \ldots \) and \( y = y, y + 1, \ldots \). Equations (7) and (9) can now be used to determine the distribution of \( X_n \) and its first two moments (proof is provided in the appendix).

**Theorem 6.** Let the number of packets lost in a single time-slot be independent and identically distributed according to equation (7). The distribution of the time between decode events for all \( \epsilon_i \) and \( r_i \), \( i \in \mathcal{P} \), that satisfy Corollary 4 is

\[
p_{X_n}(x_n) = \begin{cases} e^{-\lambda} & \text{for } x_n = 0 \\ \lambda e^{-\lambda} \frac{(x_n-1)x_n^2}{2} x_n e^{-x_n} \lambda^{x_n} & \text{for } x_n = 1 \\ 0 & \text{otherwise,} \end{cases}
\]

with first and second moments

\[
\mathbb{E}[X] = \frac{\lambda}{1 - \lambda} e^{-\lambda} \tag{11}
\]

and

\[
\mathbb{E}[X^2] = \left(1 - \frac{\lambda + \lambda^2}{(1 - \lambda)^2}\right) \mathbb{E}[X]. \tag{12}
\]

The first two moments of \( X_n \) can now be used to determine the the renewal-reward process that describes the in-order delivery delay. Before this is done, the following lemma from [12] is needed.
to equation (7) and define the following.

Lemma 7. Let \( \{R(t): t > 0\} \) be a non-negative renewal-reward function for a renewal process with expected inter-renewal time \( \mathbb{E}[X] \) < \( \infty \). If each \( R_n \) is a random variable with \( \mathbb{E}[R_n] < \infty \), then with probability 1,

\[
\lim_{t \to \infty} \frac{1}{t} \int_{t}^{\infty} R(\tau) d\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]}. \tag{13}
\]

Rather than defining the renewal reward function \( R(t) \) and the renewal-reward process using the inter-arrival times \( X_n \), an estimate is considered where the inter-arrival times of this new process are \( W_n = \max(X_n, 1) \) (i.e., \( W_n = 1, 2, \ldots \)). The distribution on \( W_n \) and its first moment are defined in the following.

Corollary 8. Let the number of packets lost in a single time-slot be independent and identically distributed according to equation (2) and define \( W_n = \max(X_n, 1) \). The distribution of inter-arrival times that satisfy Corollary 4 is

\[
p(W_n(w_n)) = \begin{cases} 
  \frac{\lambda e^{-\lambda}(\lambda + 1)}{w_n!} & \text{for } w_n = 1 \\
  \frac{\lambda e^{-\lambda}}{(w_n-1)!} & \text{for } w_n \geq 2 \\
  0 & \text{otherwise,}
\end{cases} \tag{14}
\]

where

\[
\mathbb{E}[W] = \frac{1}{\lambda} \mathbb{E}[X]. \tag{15}
\]

Figure 2 provides a sample function of both renewal processes \( X_n \) and \( W_n \). As a reminder, a renewal is only possible when coded packets (shown using shaded boxes within the figure) are received by the decoder. The reward function that describes the in-order delay precisely is the area under the curve shown for process \( X_n \) (i.e., the blocks with solid borders). However, the reward function \( R(t) \) that describes the delay for process \( W_n \) (i.e., the union of boxes with dotted and solid borders) is the one that is used. The in-order delivery delay can now be determined by combining Lemma 7, Theorem 6, and Corollary 8 (proof is provided in the appendix).

Theorem 9. Consider the coding scheme described by Algorithm 7 where redundant packets are only transmitted on path

\[
\lambda \left( l - 1 \right) \left( 1 - \lambda + \lambda^2 \right) / (2 - \lambda^2). \tag{17}
\]

In the case of the multi-path session, a single path with transmission rate \( r_1 \) and packet erasure rate \( \epsilon_1 \) is used to transmit all of the coded packets in addition to information packets. The second path, which is only used to transmit only information packets, has transmission rate \( r_2 \) and packet erasure rate \( \epsilon_2 \). The figure demonstrates that the approximation
The delivery delay is not very sensitive to changes in rate. This is illustrated by comparing Figure 3 with Figure 4. The ratio of delay penalty was insignificant with respect to the possible packet stream traversing multiple parallel networks. These results were then presented showing that the analysis provides throughput gains obtained by fusing two parallel networks together.

APPENDIX

Proof: (Theorem 3) A path with code rate \( c_i(\pi) \) and transmission rate \( r_i \) packets/seconds results in a coded packet being generated every \( (1 - c_i(\pi))r_i \). Therefore, the expected rate that coded packets arrive at the client on path \( i \) is \( (1 - \epsilon_i)(1 - c_i(\pi))r_i \). Thus, \( \sum_{i \in P} (1 - \epsilon_i)(1 - c_i(\pi))r_i \) total coded packets/second. Now consider the case when each path is treated as a separate session. The code rate on path \( i \) must satisfy \( c_i < 1 - \epsilon_i \) in order to ensure the client’s ability to decode (i.e., the probability of a decoding error \( P_{\text{error}}(E) \to 0 \) as the file size increases for all \( c_i < 1 - \epsilon_i \)). Allowing the code rate to be \( c_i = 1 - \epsilon_i \), the expected rate at which coded packets arrive at the client on path \( i \) is then equal to \( (1 - \epsilon_i)\epsilon_ir_i \) resulting in the sum rate \( \sum_{i \in P} (1 - \epsilon_i)\epsilon_ir_i \). This produces the following bound:

\[
\sum_{i \in P} (1 - \epsilon_i)(1 - c_i(\pi))r_i \geq \sum_{i \in P} (1 - \epsilon_i)\epsilon_i r_i. \tag{18}
\]

Rearranging, we arrive at (2). With regard to (3), it is obvious that \( c_i(\pi) \) must satisfy \( c_i(\pi) \in [0, 1] \).

Proof: (Theorem 6) The inter-arrival time \( X_n \) takes the values of \( x_n = 0 \) if and only if \( y_{n,1} = 0 \) with probability \( e^{-\lambda} \) and \( y_{n,1} = 1 \) with probability \( \lambda e^{-\lambda} \) respectively. For all \( X_n \geq 2 \), we must have \( y_{n,1} \geq 2 \). This results in a decoding error in the first time-slot. Conditioning on \( Y_{n,1} \), we can use equation (9) to find the probability for \( X_n \geq 2 \) by setting \( Z = X_n - 1 \) and \( Y = Y_{n,1} - 1 \):

\[
p_{X_n}(x_n) = \sum_{y_{n,1}=2}^{\infty} p_{Y_{n,1}}(y_{n,1}) p_Z(y_n - 1|y_{n,1} - 1) \tag{19}
= \sum_{y_{n,1}=2}^{\infty} \frac{\lambda y_{n,1} e^{-\lambda}}{y_{n,1}!}, \frac{(x_n - 1)^{y_{n,1} - 1}}{(x_n - y_{n,1})!} \frac{y_{n,1} - 1}{x_n - y_{n,1}} \frac{\lambda x_n - y_{n,1}}{z} e^{-(x_n - 1)\lambda} \tag{20}
= \frac{(x_n - 1)^{x_n - 2}}{(x_n - 2)!} \frac{\lambda x_n - y_{n,1}}{x_n} \frac{\lambda^2 x_n e^{-x_n \lambda}}{x_n - z} \frac{\lambda x_n - y_{n,1}}{x_n} \tag{21}
= \frac{(x_n - 1)^{x_n - 2}}{(x_n - 2)!} \frac{\lambda x_n e^{-x_n \lambda}}{x_n} \tag{22}
\]

To determine the moments of \( X_n \), first note that

\[
\sum_{x_n=2}^{\infty} \frac{(x_n - 1)^{x_n - 2}}{(x_n - 2)!} x_n^k e^{-x_n \lambda} = E[X^k] - \lambda e^{-\lambda} \tag{23}
\]

and

\[
\sum_{x_n=2}^{\infty} \frac{x_n(x_n - 1)^{x_n - 2}}{(x_n - 2)!} \lambda x_n e^{-x_n \lambda} = E[X^2] - 2\lambda e^{-\lambda}. \tag{24}
\]

We can then take the first and second derivatives of

\[
p_{X_n}(x_n) = 1, \tag{25}
\]

VI. Conclusions

A streaming code that uses forward error correction to reduce in-order delivery delay over multiple parallel networks was presented. This included a discussion regarding the requirements for each path’s code rate, in addition to methods to manage the generation of redundancy through the use of a sliding code window. A simple analysis of the code scheme was then developed that approximated the packet losses on each of the paths using a Poisson distribution. Numerical results were then presented showing that the analysis provides a good estimate of the actual in-order delivery delay for a packet stream traversing multiple parallel networks. These results illustrated that the path with the largest packet erasure rate drives the overall in-order delivery delay, and the delay is not very sensitive to changes in transmission rates. Finally, the delay penalty for using multiple paths over a single path was discussed. Numerical results helped show that this delay penalty was insignificant with respect to the possible
Taking the expectation of $R_n$ to be the sum delay of all information packets on path $P$:

$$R_n = \begin{cases} \sum_{k=1}^{W_n+i_c} \frac{e^{-(\lambda+i_c)e^{-\lambda}}}{k!} - 1 \quad \text{for} \quad P = i_c \\ \sum_{k=0}^{\infty} \frac{e^{-\lambda}}{k!} - 1 \quad \text{for} \quad P \neq i_c \end{cases} \tag{27}$$

Substituting $x_n$ for $w_n$ from Corollary 8

$$\mathbb{E}[R_n] = \frac{1}{\sum_{j \in P} \tau_j} \sum_{x_n=0}^{\infty} \sum_{k=1}^{\min(x_n, 1)} r_{x_n} \left[ \frac{x_n}{k} - \frac{x_n}{k-1} \right]$$

$$+ \sum_{i \in P \setminus i_c} r_i \mathbb{E}[W_n](\tau_i) \mathbb{E}[X_n](x_n) \tag{30}$$

Since both $\mathbb{E}[X] < \infty$ and $\mathbb{E}[X^2] < \infty$, the expectation $\mathbb{E}[R_n] < \infty$ and Lemma 7 can be applied. Keeping in mind that every time-slot in the process defined by $W_n$ is divided into $l_c$ smaller time-slots, the expected in-order delivery delay is

$$\mathbb{E}[T] = \lim_{t \to \infty} \frac{1}{t} \int_{t}^{\infty} R(\tau) \, d\tau = \mathbb{E}[R_{\infty}] \tag{32}$$

$$= \mathbb{E}[R_{\infty}] \mathbb{E}[W] \tag{33}$$

$$= \mathbb{E}[X] \sum_{j \in P} r_j \left( \frac{r_c}{l_c e^{-\lambda}} \mathbb{E}[X^2] - \frac{r_c}{l_c} \mathbb{E}[X] \right) \tag{34}$$

$$= \frac{\lambda}{2r_c^2 (1 - \lambda)^2} \sum_{j \in P} r_j \left( r_c^2 (l_c - 1) (1 - \lambda + \lambda^2) \right) \tag{35}$$

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