Elliptically Symmetric Lenses and Violation of Burke’s Theorem

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ABSTRACT

We show that the outside equation of a bounded elliptically symmetric lens (ESL) exhibits a pseudo-caustic that arises from a branch cut. A pseudo-caustic is a curve in the source plane across which the number of images changes by one. The inside lens equation of a bounded ESL is free of a pseudo-caustic. Thus the total parity of the images of a point source lensed by a bounded elliptically symmetric mass is not an invariant in violation of the Burke’s theorem. A smooth mass density function does not guarantee the validity of the Burke’s theorem.

Pseudo-caustics of various lens equations are discussed. In the Appendix, Bourassa and Kantowski’s deflection angle formula for an elliptically symmetric lens is reproduced using the Schwarz function of the ellipse for an easy access; the outside and inside lens equations of an arbitrary set of truncated circularly or elliptically symmetric lenses, represented as points, sticks, and disks, are presented as a reasonable approximation of the realistic galaxy or cluster lenses.

One may consider smooth density functions that are not bounded but fall sufficiently fast asymptotically to preserve the total parity invariance. Any bounded function may be sufficiently closely approximated by an unbounded smooth function obtained by truncating its Fourier integral at a high frequency mode. Whether to use a bounded function or an unbounded smooth function for an ESL lens mass density, whereby whether to observe the total parity invariance or not, incurs philosophical questions. For example, is it sensible to insist that the elliptical symmetry of an elliptic lens galaxy be valid in the entire sky? How a pseudo-caustic close to or intersecting with a caustic must be withered away during a smoothing process and what it means will be investigated in a separate work.

Subject headings: gravitational lensing
1. Introduction

Burke’s theorem (Burke 1981) is known as odd number theorem (MacKenzie 1985; Schneider et al. 1992), and is made of two pieces of claims. For a bounded mass distribution lens, 1) the total parity of the images is an invariant and 2) the number of images of a source sufficiently far away from the lens system is one. Here we will only focus on the invariance of the total parity.

Thirty five years ago, Bourassa and Kantowski (1975) found that the deflection angle of an elliptically symmetric lens (ESL) can be expressed as an integral over the interval [0, 1]. In our notations described in the Appendix, the normalized lens equation of an ESL is written as

\[ \omega = z - \int_0^{t_{\text{cut}}} \tilde{\sigma}(t) t dt \left( \frac{1}{(z^2 - c^2t^2)^{1/2}} \right) \]  

where \( \omega \) and \( z \) are the positions of the source and an image, \( c \) is the focal length of the mass distribution ellipse, and \( t \) is the elliptic radial parameter such that the position variable of the mass distribution is given by \( \xi = t(a \sin \theta + b \cos \theta) \) : \( t \in [0, 1] \) where \( a \) and \( b \) are the semi-major and semi-minor axes of the mass ellipse. The mass in the mass ellipse is used for the normalization of the lens equation. \( \tilde{\sigma}(t) \) is the projected mass density function re-normalized such that

\[ 1 = \int_0^1 \tilde{\sigma}(t) t dt . \]  

If the image position \( z \) is inside the mass ellipse, then \( t_{\text{cut}} = \eta \) where \( \eta \) is the elliptic radial parameter of \( z \): \( z = \eta(a \sin \theta + b \cos \theta) \), and if the image position \( z \) is outside the mass ellipse, then \( t_{\text{cut}} = 1 \).

A pseudo-caustic refers to a curve in the source plane across which the number of images changes by one and was considered a curious phenomenon (Evans and Wilkinson 1998). The appearance of a pseudo-caustic is an indicator that the total parity of the images is not an invariant in violation of the Burke’s theorem. The integrand of eq. (1) includes a square root function, and we will see that that is the source of the pseudo-caustic of an ESL outside lens equation.

We adopted the terminology pseudo-caustic because 1) a pseudo-caustic curve can form cusps in conjunction with open caustic curves and 2) we can continue to use the term “caustic domains” to describe the domains of the source plane with definite number of images that are defined not only by caustics but also by pseudo-caustics.

In section 2 we discuss the pseudo-caustics of various lens equations. The outside equation of a generic elliptically symmetric lens is shown to have a pseudo-caustic that arises from a branch cut. The inside equation of an elliptically symmetric finite density lens
does not have a pseudo-caustic. Thus a bounded elliptically symmetric lens violates the invariance of the total parity of the images; the arguments of smooth density functions for Burke’s theorem are invalidated. In section 3 we give a summary and raise a possibility to approximate a bounded mass density by a smooth function with infinite extension to preserve the invariance of the total parity of the images. We leave the details to a separate work. It is a natural wonder if other types of singularities than point singularities and branch cuts may be to be found in gravitational lensing. In the Appendix, we reproduce the deflection angle formula of Bourassa and Kantowski (1975) using the Schwarz function of the ellipse (Fassnacht et al. 2007; Khavinson and Lundberg 2009) and derive various lens equations used in section 2; we also spell out the inside and outside lens equations of an arbitrary set of circularly symmetric lenses and elliptically symmetric lenses as a reasonable model for galaxy and cluster lenses; they are represented as points, sticks, and disks.

2. Pseudo-caustics of Elliptically Symmetric Lens Equations

We start with examining the singular isothermal lens in which the pseudo-caustic arises from a point mass density singularity where the Burke’s vector field is ill defined and move on to discussing the pseudo-caustics of various elliptically symmetric lenses (ESL). We will see that the outside lens equation of any bounded ESL (mass density vanishes outside a certain finite radius) has a pseudo-caustic that arises from the double-valuedness of the c-cut that is from a branch cut. The inside equation of a bounded ESL does not have a pseudo-caustic. Thus a bounded ESL violates the invariance of the total parity.

2.1. SIS or Inside a Cylindrically Truncated Isothermal Sphere

The “curious behavior” of a pseudo-caustic was noted first in an analysis of the isothermal sphere of an infinite extension, whose projected mass density is inversely proportional to the radius (referred to as SIS: singular isothermal sphere), and was attributed to the density singularity at the center (Kovner 1987). The nature of the pseudo-caustic can be readily read off from the lens equation.

\[ \omega = z - \frac{\eta}{z} \]  

(3)

where \( \eta \) is the radial parameter of \( z \): \( z = \eta ae^{i\theta} \) and \( a \) is a reference radius the mass within which is used to normalize the lens equation. Eq. (3) can be obtained by using the Newton’s theorem of a spherical symmetric mass or by calculating the deflection angle as in the Appendix.
If we consider a monopole lens, for which $\eta$ should be replaced by 1, an $\epsilon$-circle around the singularity at the origin is mapped to a large circle at infinity and the lens equation is a double-covering mapping. In other words, the image plane is divided into two regions by the critical curve, which is mapped to the origin of the source plane, and each of the two regions is mapped to the entire source plane.

Now $z = 0$ is a singularity of eq.(3) where infinitely many finite-size vectors clash. Speaking in terms of the vector field considered by Burke (1981), the vector field is not well defined at $z = 0$. If we cut out an $\epsilon$-hole, the vector field behaves well in the neighborhood of the punctured origin, and the $\epsilon$-boundary around the singularity is mapped to a circle of radius $1/a$ in the source plane. The $z$-plane outside the critical curve covers the whole $\omega$-plane, but the punctured disk inside the critical curve covers only the disk of radius $1/a$ centered at the origin of the $\omega$-plane. In other words, the disk of radius $1/a$ of the $\omega$-plane has two images and the rest of the $\omega$-plane has only one image; the number of images changes as the source crosses the circle of radius $1/a$ or the pseudo-caustic. As the source crosses from inside the disk of radius $1/a$, one of the two images “disappears into the $\epsilon$-hole.”

In summary, the pseudo-caustic is an indicator that the lensing equation is not a full double covering but a “one and half” covering of the source plane and that the total parity of the images is not an invariant.

If we truncate the isothermal sphere cylindrically such that the truncated mass is infinitely elongated along the line of sight, the projected mass density is inversely proportional to the radius, and the inside lens equation is given by the same equation as eq.(3) where $a$ is the radius of the mass disk. The reason why the lens equations are the same is that only the mass inside the probing circle affects the two-dimensional gravitational field at the probing point $z$ as is well known as the Newton’s theorem. The deflection angle is essentially, apart from some constants including the distance factor, the two-dimensional gravitational field.

### 2.2. Constant Density ESL Inside Equation

The deflection angle of the ESL can be expressed as an elementary function when the density is constant (Fassnacht et al. 2007). The (normalized) lens equation inside the mass is a linear equation

$$\omega = \left( 1 - \frac{1}{ab} \right) z + \left( \frac{a - b}{ab a + b} \right) \bar{z}$$

where $a$ and $b$ are the semi-major and semi-minor axes of the mass ellipse. The lens equation is well defined everywhere, and there is no pseudo-caustic. The equation has always
one solution and needs to be screened by hand or by studying the mass boundary and its secondary curves (map the mass boundary by the lens equation into a curve in the source plane and study the pre-image curves in the image plane that are mapped to the curve in the source plane; one of the pre-image curves is the mass boundary) if the solution is inside the mass and hence is an actual image. \( \omega = 0 \) has a solution at \( z = 0 \), and \( \omega = \infty \) has a solution at \( z = \infty \). Therefore a sufficiently large \( \omega \) will fail to form an image inside the mass where the equation is valid.

The same equation applies to an infinite mass constant density ESL with an infinite extension, which is due to the well known Newton's theorem for the gravitational field of an elliptically symmetric mass.

### 2.3. Arcsine Lens Inside Equation

The deflection angle of the ESL can be expressed as an arcsine function when the projected mass density is inversely proportional to the elliptic radius [Khavinson and Lundberg 2009]. The mass model is commonly referred to as a SIE (singular isothermal ellipsoid) since it was named as such in [Kormann et al. (1994)](#) even though it was neither proved nor demonstrated to be an isothermal state. We shall call it arcsine lens. The inside lens equation of an arcsine lens is given as follows as can be found in the Appendix.

\[
\omega = z - \frac{1}{c} \arcsin \left( \frac{c\eta}{z} \right)
\]

(5)

where \( c \) is the focal length and \( \eta \) is the elliptic radial parameter of \( z \).

As in the case of the SIS, the elliptic version eq.(5) is ill-defined at \( z = 0 \), and an \( \epsilon \)-circle around the singularity is mapped to a finite loop in the source plane as \( \epsilon \rightarrow 0 \). The arcsine lens inside equation has a pseudo-caustic. The curve is a simple loop given by

\[
w = -\frac{1}{c} \left( a \cos \theta - ib \sin \theta \right).
\]

(6)

The pseudo-caustic loop has the same sense with the \( \epsilon \)-circle but has its phase lagged by \( \pi \).

[Kormann et al. (1994)](#) called the pseudo-caustic curve a cut following the convention used by [Kovner (1987)](#) and observed that the pseudo-caustic can intersect with the caustic curve. The caustic is a symmetric astroid and is either enclosed inside or intersect with the pseudo-caustic curve.

In order to calculate the critical curve and caustic curve, it is convenient to use the elliptic coordinates: \( z = \eta(\cos \theta + ib \sin \theta) \) because the deflection angle is independent of the
elliptic radius. The partial derivative with respect to \(z\), \(\partial_z\), can be expressed in terms of the partial derivatives in elliptic coordinates, 
\[
\partial \equiv \partial_z = \frac{a \sin \theta + ib \cos \theta}{2iab} \partial_y + \frac{a \cos \theta - ib \cos \theta}{2\eta ab} \partial_\theta,
\]
and the Jacobian matrix components of the inside lens equation (5) are
\[
\partial \omega = 1 - \frac{1}{2\eta ab}; \quad \bar{\partial} \omega = \frac{1}{2\eta ab} \frac{z}{\bar{z}}
\]
where \(\bar{\partial}\) is the complex conjugate of \(\partial\). The Jacobian determinant is
\[
J = |\partial \omega|^2 - |\bar{\partial} \omega|^2 = 1 - \frac{1}{\eta ab},
\]
and on the critical curve
\[
\eta = \frac{1}{ab}.
\]
The critical curve is an ellipse.
\[
z = \frac{1}{ab}(a \cos \theta + ib \sin \theta).
\]
The corresponding caustic is a quadroid (or astroid) with the four cusps on the real and imaginary axes (or symmetry axes). The precusps can be found using the same method as in [Rhie and Bennett (2009)]. Since \(\partial \omega\) is real, the precusp condition \(0 = \partial_z J\) becomes
\[
0 = z\partial J - \bar{z}\bar{\partial} J.
\]
The equation leads to \(\sin 2\theta = 0\), hence the precusps are at \(\theta = 0, \pi/2, \pi,\) and \(3\pi/2\). The precusps are mapped to cusps on the positive real, negative imaginary, negative real, and positive imaginary axis respectively.
\[
\Delta \omega_{\text{real}} \equiv \omega_0 - \omega_\pi = \frac{2}{b} - \frac{2}{c} \arcsin(\frac{c}{a})
\]
\[
\Delta \omega_{\text{imag}} \equiv \omega_{\pi/2} - \omega_{3\pi/2} = \frac{2i}{a} - \frac{2}{c} \arcsin(\frac{ic}{b})
\]
\(\Delta \omega_{\text{real}} > 0\) and \(\Im \Delta \omega_{\text{imag}} < 0\), and hence the caustic curve has the opposite sense with the critical curve.

Figure 1 shows the cases of the caustic and pseudo-caustic. The number of the images of the caustic zones are indicated and clearly demonstrate that the total parity of the images is not an invariant. There is no case where the pseudo-caustic is enclosed inside the caustic curve because the cusps on the imaginary axes are always inside the pseudo-caustic. The points of the pseudo-caustic and caustic on the positive imaginary axis (semi-minor axis of the mass ellipse) are from \(\theta = 3\pi/2\), and it is easy to see that \(\Im (\omega_{\text{pseudo}}(3\pi/2) - \omega_{\text{caustic}}(3\pi/2)) = 1/a > 0\).
2.4. Truncated Constant Density ESL Outside Equation

We have discussed a few cases in which a pseudo-caustic arises because of a point singularity where Burke’s vector field is not well defined and around where the vector values are not all infinite. In this section we will introduce a line singularity where Burke’s vector field is not well defined and around where the vector values are not all infinite.

Consider truncating the constant density ESL at \( t = 1 \), and the lens equation valid outside the mass is given by

\[
\omega = z - \frac{2}{c^2} (\bar{z} - \sqrt{\bar{z}^2 - c^2}) .
\] (15)

\( z = \pm c \) are the branch points of the square root function and we need to place branch cuts to establish a convention where the Riemann sheets are sewed up. Mathematically a branch cut can be placed wherever we are pleased to put it as far as it is connected to the branch point in concern. Physically, however, we are constrained because two images in an \( \epsilon \)-neighborhood cannot come from two widely separated sources unless there is a physical reason. Placing the branch cuts randomly will induce such an unphysical behavior. The outside lens equation \([15]\) depends only on the focal length \( c \) and is valid for an infinite family of confocal lenses for which the image zone can be everywhere except for an \( \epsilon \)-ellipse around the line segment connecting the two focal points or the two branch points. Thus physically the line segment connecting the two focal points is the only option as the branch cut. The square root function should be understood as

\[
\sqrt{\bar{z}^2 - c^2} = (r_+ r_-)^{1/2} e^{i(\theta_+ + \theta_-)/2} \] where \( z \pm c = r_\pm e^{i\theta_{\pm}} \).

Across the branch cut, the (complex) deflection angle changes the phase by \( \pi \). In other words, Burke’s vector field is not well defined on the branch cut because it is double-valued. We can cut along the branch cut to remedy it. The boundary of the cut is mapped to a finite loop by the lens equation. The finite loop is the pseudo-caustic, and across which a negative image disappears into or appears from the branch cut.

2.5. Arcsine Lens Outside Equation

If we truncate the arcsine lens at \( t = 1 \), the outside lens equation is given by

\[
\omega = z - \frac{1}{c} \arcsin \left( \frac{c}{\bar{z}} \right) .
\] (16)

The deflection angle is double-valued on the line segment connecting the two focal points. Call it \( c \)-cut. Cut the image plane along the \( c \)-cut, and the boundary of the \( c \)-cut is mapped to a semi-infinite loop whose extension is finite in the direction of the \( c \)-cut. The semi-infinite loop is the pseudo-caustic.
Bergweiler and Eremenko (2010) (BE10 herefrom) studied the number of images of the lens equation (16) using a converted equation and found that the number of images can be anywhere from 1 to 6. Since the functional BE10 used is different from that of (16), the caustic planes in BE10 should be reinterpreted. It turns out that the transformation is simple: 

$$\omega' = -c\omega$$

where $$\omega'$$ is the source plane variable of the associated equation, and the morphology of its caustic plane can be read almost as if it were the caustic plane of the original equation (16) if one overlooks the absolute value of the coordinates and takes into account of the reversing of the parities. The pseudo-caustic in a caustic plane in BE10 arises from the image plane boundary (thus the converted equation cannot be the lens equation of a realistic lens), and it corresponds to the pseudo-caustic that arises from the c-cut that is essentially from a branch cut.

Figure 2 shows the caustic domains for $$c = 1.65$$. The critical curve bifurcates at $$c = \sqrt{2}$$, and at $$c = 1.65$$ it is made of three loops which enclose the origin and two focal points respectively. The caustic curves are open and closed by the pseudo-caustic curve. The transition from the domain 1/0 to the domain 2/0 is by an increase of a positive image as is indicated in the notations, and the reason for the relevance of a positive image is that the segment of the c-cut is outside the critical curves. One of the four “positive” segments of the pseudo-caustic is indicated by the symbol ⊕.

### 2.6. Arbitrary Bounded Mass Lenses

We have examined the pseudo-caustics of a few algebraic lens equations. For an arbitrary elliptically symmetric lens, the equation is an integro-algebraic equation. However the existence or non-existence of the pseudo-caustic arising from the c-cut can be easily discerned by looking at the behavior of the c-cut at the origin, $$z = 0$$. Since the deflection angle is continuous on the ϵ-boundary of the c-cut, double-valuedness at $$z = 0$$ is an indicator that the c-cut is mapped to a finite loop, i.e., a pseudo-caustic. The ESL outside lens equation is in general, as can be found in the Appendix,

$$\omega = z - \int_{0}^{1} \frac{\bar{\sigma}(t)dt}{(\bar{z}^2 - c^2t^2)^{1/2}}$$  \hspace{1cm} (17)

where $$c$$ is the focal length of the mass boundary ellipse. At $$z = 0 \pm i0$$,

$$DAngle = \int_{0}^{1} \frac{\pm i}{c} \bar{\sigma}(t)dt$$  \hspace{1cm} (18)

It is double-valued and finite. At the intersections of the ϵ-boundary of the branch cut and the real line, $$z = \pm(a\eta + i0)$$, the deflection angle is single-valued and finite. Therefore the pseudo-caustic is a finite loop.
The ESL inside lens equation, also found in the Appendix, is

\[
\omega = z - \int_0^\eta \frac{\tilde{\sigma}(t)tdt}{(\bar{z}^2 - c^2t^2)^{1/2}}
\]

where \(c\) is the focal length of the mass boundary ellipse and \(\eta\) is elliptic radial parameter of \(z\). \(z = 0 \pm i\zeta\) are mapped to the single value \(DAngle = 0\) because the integral upper bound \(\eta = 0\). Any other point \(z\) on the branch cut is also single-valued because it is always outside the branch cut connecting the branch points \(\pm c\eta\). Thus the ESL inside lens equation of an arbitrary bounded mass has no pseudo-caustic.

The examples of the bounded ESL lenses discussed in the previous subsections have sharp cutoffs at \(t = 1\). However, the class of bounded density functions include smooth functions. For example, a family of bounded smooth functions can be constructed as follows (Eremenko 2010).

\[
\sigma(t) \propto e^{-\alpha^2/(1-t)^2} : \quad t < 1
\]

\[
0 : \quad t \geq 1
\]

Thus smooth bounded density functions are not immune from pseudo-caustics and contradict the smooth density function arguments for the Burke’s theorem.

3. Summary and Discussion

Burke’s theorem invokes Poincaré-Hopf index theorem of the vector field on the compactified complex plane as a sphere. The index of a loop is determined only by the zeros of the vector field enclosed in the loop. The behavior of Burke’s vector field is determined by the behavior of the deflection angle where the latter is an integral of a mass density function with the kernel \((\bar{z} - \bar{\xi})^{-1}\) where \(z\) is the probing position and \(\xi\) is the position variable of the density function. If the vector field is continuous everywhere, then the index of the total number of the images can be accounted for by drawing a large loop at infinity. The large loop at infinity encloses one positive (unmagnified) image at infinity when the source is at infinity, and hence the index of the large loop is 1 and Burke’s theorem follows.

However, the deflection angle integral is not necessarily continuous. The best known singularity is that of a monopole or a monopole-equivalent (due to the Newton’s theorem), and it introduces a hole in the sphere of the complex plane. In other words, the underlying manifold is not compact, and strictly, the Poincaré-Hopf index theorem does not apply. However, a pole respects the total parity invariance, and the hole due to the monopole singularity only violates the odd-numberedness of the Burke’s theorem; the asymptotic number
of the images is determined by the zero at infinity and the number of the poles due to the monopole lens positions; the total parity of the $n$-point lens is $1 - n$.

The cylindrically truncated isothermal lens is another example of the singularities and has been known for causing a pseudo-caustic violating the total parity invariance. The mass density singularity at the origin is responsible for the pseudo-caustic, and it has been argued that Burke’s theorem should hold if the mass density function is smooth.

Here we have shown another type of singularity that is common in bounded elliptically symmetric lenses; the new type of singularity comes from a branch cut and a smooth ESL mass density function is not immune from it. In conclusion, a smooth mass density function is not a guarantee for the validity of the Burke’s theorem.

Morse theory studies (MacKenzie 1985; Petters 1992) had not turned up the branch cut singularities that we have found in the bounded elliptically symmetric lenses that are physically perfectly reasonable. It may be the time to reexamine the Morse theory in relation to gravitational lensing or null geodesics.

Point masses are the foundation of the Galactic microlensing experiments including searches for microlensing planets. The point masses preserve the total parity invariance but do not respect the odd number theorem because an odd number of point masses produces an even number of images asymptotically. Here we found a line singularity of bounded elliptic masses that is related to a branch cut and causes a pseudo-caustic. A pseudo-caustic violates the total parity invariance. A bounded elliptically symmetric mass is a reasonable model for galaxy lenses or cluster lenses. Are there other types of singularities yet to be discovered in gravitational lensing?

One can consider a smooth density function that is unbounded but asymptotically falls reasonably fast to avoid a pseudo-caustic and an infinite mass. For example, a bounded mass density function that reasonably fits a real lens may be reasonably approximated by truncating the large frequency modes of its Fourier integral. The truncated Fourier integral should be infinitely differentiable and infinitely extended. Then does the real lens respect or violate the invariance of the total parity? If we insist on the preservation of the total parity invariance, we need to assume that the lens mass is infinitely extended. However small a density it might be at a large distance, we need to assume the elliptic symmetry. Is it sensible to do so? One practical approach may be to investigate the behavior of the pseudo-caustics of bounded masses that are close to or intersect with their caustics in the process of increasing the differentiability. It will be explored in a separate work.
A. SIS or Cylindrically Truncated Isothermal Lens

If we consider a self-gravitating isothermal sphere of gas mass in thermal equilibrium, the algebraically known solution of the Lane-Emden equation, the so-called SIS (singular isothermal sphere), has the density profile that is proportional to the inverse radius square. The total mass is infinite because of the indefinite extension of the constant mass shells. In practice, the tidal interaction with the neighboring systems are likely to define the edges and mass breaks of the self-gravitating objects. If we truncate the SIS at a certain (3-d) radius, say \( r = a \), and integrate for the projected mass of the lens, the 2-d mass density function has the form

\[
\Sigma(t) \propto \frac{1}{t} \arctan \left( \frac{(1 - t^2)^{1/2}}{t} \right) ; \quad t \leq 1
\]  

where the position variable is expressed in terms of the circular coordinates as \( \xi = ta(\cos \theta + is \sin \theta) \) and the mass truncation is made at \( t = 1 \). In order to find the lens equation, we can use the Newton’s theorem that the gravitational field of a circularly symmetric mass is determined by the mass inside the circle of the radius of the probing point placed at the center, and hence we need to integrate the density function in eq. (A1) from the center to an arbitrary \( t \) less than 1. The integration doesn’t relent to a nice manageable algebraic function. So we cheat a bit (maybe a lot) as is customary and ignore the arctangent factor to obtain an easy-to-handle density function \( \Sigma(t) \propto 1/t \) and truncate it at \( t = 1 \). In other words, the mass being considered is the infinite isothermal sphere cut into an infinite cylinder that is infinitely long along the line of sight. Thus the (2-d) mass distribution may be best referred to as a cylindrically truncated isothermal lens (CTIL).

The gravitational lens equation is given by

\[
\omega = z - 4GD \int \frac{\Sigma(t)}{\bar{z} - \bar{\xi}} d^2\xi
\]  

(\text{A2})

\( \omega \) and \( z \) are variables for a source and its image in the lens plane at the distance of the lens and \( D \) is the reduced distance: \( 1/D = 1/D_1 + D_2 \) where \( D_1 \) and \( D_2 \) are the distances from the lens to the observer and the source. The area element \( d^2\xi = d\xi_1 d\xi_2 = (-2i)^{-1} d\xi d\bar{\xi} \).

If the mass of the lens is \( M \), introduce the normalized density function \( \sigma(t) = \Sigma(t)/M \) and renormalize the position variables by the Einstein ring radius of the mass \( M \), \( R_E = (4GMD)^{1/2} \), so that the unit distance of the lens plane is given by \( R_E \). The lens equation \( \text{(A2)} \) is rewritten as

\[
\omega = z - \int \frac{\sigma(t)}{\bar{z} - \bar{\xi}} d^2\xi .
\]  

(\text{A3})

Since \( \sigma(t) \) depends only on \( t \), we can manipulate the deflection angle integral as follows. If \( \partial D_t \) denotes the circle of radius \( ta \) and \( D_t \) is the circular disk inside the boundary circle
\[ \partial D_t, \text{ then the area of the annulus } D_{t+dt} - D_t \text{ is of the order of } dt \text{ and } \sigma(t) \text{ can be considered constant in the annulus.} \]

\[ \left[ \int_{D_{t+dt}} - \int_{D_t} \right] \frac{\sigma(t) d^2 \xi}{(\bar{z} - \xi)} = \sigma(t) \left[ \int_{D_{t+dt}} - \int_{D_t} \right] \frac{d^2 \xi}{\bar{z} - \xi} = \sigma(t) \frac{\partial}{\partial t} \left[ \int_{D_t} \frac{d^2 \xi}{\bar{z} - \xi} \right]. \] (A4)

Therefore the deflection angle DAngle is given by

\[ DAngle = \int_0^1 dt \sigma(t) \frac{\partial}{\partial t} \left[ \int_{D_t} \frac{d^2 \xi}{\bar{z} - \xi} \right]. \] (A5)

We can calculate the integral inside the square bracket in eq(A5), which we shall label D\_t\_Integral, using the Schwarz function of the circle and three facts are useful to know: the Stoke’s theorem or Green’s theorem in two dimensions, the Cauchy integral of an analytic function, and the Schwarz function of the circle. In this case of the circular mass boundary, it is simpler to apply the Newton’s theorem, but we will develop the machinery that can be also used for an elliptic mass boundary.

The Stoke’s theorem is well known in arbitrary dimensions because of their physical relevance in electrodynamics, fluid dynamics, etc. See for example Jackson (1975). In two dimensions,

\[ \int_{\partial \Omega} u dx + v dy = \int_{\Omega} (\partial_x v - \partial_y u) dxdy, \] (A6)

in which it is implicit that \( u \) and \( v \) are differentiable and the differentials are continuous, and which can be rewritten in terms of the exterior differentiation \( d \).

\[ \int_{\partial \Omega} u dx + v dy = \int_{\Omega} d(udx + vdy) \] (A7)

(Sometimes \( d\wedge \) is used instead of \( d \) in order to make the exterior differentiation more explicit.) If \( u \) and \( v \) are \( x \) and \( y \) components of vector \( \vec{w} \), it can be written in the vector form.

\[ \int_{\partial \Omega} \vec{w} \cdot d\vec{l} = \int_{\Omega} \nabla \times \vec{w} dxdy. \] (A8)

The real space Stoke’s theorem can be readily translated into the complex form in the complex plane using eq.(A7), and it states (Gong and Gong 2007) that if \( w = w_1 dz + w_2 d\bar{z} \) is an exterior differential one form in a domain \( \Omega \) and \( \partial \Omega \) is the boundary of \( \Omega \), where \( w_1(z, \bar{z}) \) and \( w_2(z, \bar{z}) \) are differentiable once and the differentials are continuous, then

\[ \int_{\partial \Omega} w = \int_{\Omega} dw \] (A9)
The orientation of the boundary curve \( \partial \Omega \) is assumed to be counterclockwise as usual. If one walks along the boundary, the left hand points to the domain \( \Omega \). Walking along a loop with the right hand in is the negative of the left hand in.)

The Cauchy integral of \( f(\xi) \) is defined as (Titchmarsh 1991)

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi
\]

(A10)

where \( f(\xi) \) is analytic inside and on a simple closed curve \( \Gamma \). The integral is equal to \( f(z) \) if \( z \) is inside \( \Gamma \) and the equality is called Cauchy’s integral formula; it is equal to 0 if \( z \) is outside \( \Gamma \) because the integrand is analytic (Cauchy’s theorem). The derivative of \( f(z) \) can be expressed in terms of the integral of \( f(\xi) \) and a kernel.

\[
f'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^2} d\xi
\]

(A11)

The Schwarz reflection symmetry principle (Needham 2004) states that given a smooth enough arc in the complex plane, there is an analytic function (in a domain that includes the arc) that maps the arc into its complex conjugate. The analytic function is called the Schwarz function of the arc and commonly denoted \( S(\xi) \). The complex conjugate \( \overline{S(\xi)} \) reflects the domain with respect to the arc. On the arc, \( \xi = S(\overline{\xi}) \), or \( \overline{\xi} = S(\xi) \). Here the relevance is that the Schwarz function of the circle is nothing but \( S(\xi) = t^2a^2/z \) because on the circle \( \xi \overline{\xi} = t^2a^2 \) where a position variable in the circular coordinate is written as \( \xi = ta(\cos \theta + i \sin \theta) \).

Now we are ready to integrate the deflection angle over the unit circular disk \( D \).

**A.1. CTIL Outside Equation**

If the probing point \( z \) is outside the mass disk, the integrand of the \( D_t \text{Integral} \) does not have a singularity and is analytic. The complex conjugate of the \( D_t \text{Integral} \) is

\[
\overline{D_t \text{Integral}} = \int_{D_t} \frac{d^2 \xi}{z - \xi} = \frac{1}{2i} \int_{D_t} \frac{d\xi d\overline{\xi}}{z - \xi} = \frac{1}{2i} \int_{D_t} \frac{d}{d\xi} \left( \overline{\xi} d\xi \right)
\]

(A12)

\[
= \frac{1}{2i} \int_{\partial D_t} \overline{\xi} d\xi = \frac{1}{2i} \int_{\partial D_t} \frac{t^2a^2 d\xi}{\overline{\xi}(z - \xi)} = \frac{\pi t^2a^2}{z}
\]

(A13)

where \( z \) is outside the boundary \( \partial D_t \). Therefore the deflection angle in eq.(A5) is

\[
DAngle = \int_{0}^{1} dt \sigma(t) \frac{d}{dt} \left( \frac{\pi t^2a^2}{z} \right) = \int_{0}^{1} 2\pi a^2 \sigma(t) t dt \frac{1}{z} = \int_{0}^{1} \sigma(t) t dt \frac{1}{z}
\]

(A14)
where
\[
1 = \int_0^1 \bar{\sigma}(t) t \, dt ,
\]  
and hence the outside lens equation is given by the monopole lens equation.
\[
\omega = z - \frac{1}{\bar{z}}.
\]

The outside lens equation of a circularly symmetric lens is independent of the mass density function.

### A.2. CTIL Inside Equation

If the probing point \( z \) is inside the mass disk \( D \), we need to mind the singularity at \( \bar{\xi} = \bar{z} \) in \( D \)Integral. If we cut out from the mass disk \( D \) a small disk of radius \( \epsilon \) centered at \( z \), \( D(z, \epsilon) \), then we can apply the Green's theorem to the \( D \)Integral over \( D - D(z, \epsilon) \).

\[
\int_{D(z, \epsilon)} \frac{d^2\xi}{z - \xi} = \int_{\partial D(z, \epsilon)} \frac{\bar{\xi} d\xi}{z - \xi} .
\]  

Now the surface integral over \( D(z, \epsilon) \) on the LHS vanishes,
\[
\int_{D(z, \epsilon)} \frac{d^2\xi}{z - \xi} = \int_0^{2\pi} \int_0^\epsilon \frac{rdrd\theta}{re^{i\theta}} = \epsilon \int_0^{2\pi} \frac{d\theta}{e^{i\theta}} = 0 ,
\]  

hence the \( D \)Integral is given by the RHS of eq.(A17).
\[
\frac{1}{2i} \left[ \int_{\partial D_t} - \int_{\partial D(z, \epsilon)} \right] \frac{\bar{\xi} d\xi}{z - \xi} .
\]  

The first contour integral over \( \partial D_t \) vanishes if \( z \) is inside \( D_t \) because the integrand is analytic outside the contour; if \( z \) is outside the contour, the integral picks up the contribution from the pole at \( \xi = 0 \) inside the contour or the pole \( \xi = z \) outside the contour which result in the same value.
\[
\frac{1}{2i} \int_{\partial D_t} \frac{\bar{\xi} d\xi}{z - \xi} = \frac{1}{2i} \int_{\partial D_t} \frac{t^2a^2d\xi}{z - \xi} = \pi \frac{t^2a^2}{z} : \quad \eta \geq t .
\]  

where \( z = \eta a(\cos \theta + i \sin \theta) \). The second contour integral over \( \partial D(z, \epsilon) \) can be calculated using the Schwarz function of the circle: \( (\bar{\xi} - \bar{z})(\xi - z) = \epsilon^2 \).
\[
\frac{1}{2i} \int_{\partial D(z, \epsilon)} \frac{\bar{\xi} d\xi}{z - \xi} = \frac{1}{2i} \int_{\partial D(z, \epsilon)} \left( \frac{z}{z - \xi} - \frac{\epsilon^2}{(\xi - z)^2} \right) d\xi = -\pi \bar{z} : \quad \eta < t
\]  

(A21)
The contribution comes from the first term only. The second term vanishes because the derivative of the analytic function $\epsilon^2$ (constant) is zero. Since the result is independent of $t$, it does not contribute to the deflection angle integral. Thus the deflection angle at $z$ inside the mass disk is given as follows.

$$DAngle = \int_0^\eta \tilde{\sigma}(t) t \, dt \cdot \bar{z}.$$  \hspace{1cm} (A22)

For a CTIL, $\tilde{\sigma}(t) = 1/t$, and the inside lens equation is given by

$$\omega = z - \frac{\eta}{\bar{z}}.$$  \hspace{1cm} (A23)

**B. Elliptically Symmetric Lenses**

For galaxy lenses that have elliptic shapes and light distributions, one can consider a class of mass distributions whose projected densities depend on only the elliptical radius. For example, $\Sigma(t) \propto 1/t$ where the position variable $\xi = t(a \sin \theta + ib \cos \theta)$. It is currently unknown if a singular elliptic mass distribution with infinite or finite extension is exactly or closely related to a self-gravitating gas mass in thermal equilibrium. However one can suspect that it might be the case at least for small ellipticities because of the existence of the circularly symmetric system SIS and the fact that an angular momentum tends to add oblateness to the system.

Consider the family of ellipses parameterized by $t(\geq 0)$ that fills the two-dimensional space.

$$\xi = t(a \cos \theta + ib \sin \theta), \quad a \geq b$$  \hspace{1cm} (B1)

The curve $t = constant$ is an ellipse that is converted to the familiar equation for the ellipse in real coordinates $\xi_1$ and $\xi_2$ where $\xi = \xi_1 + i\xi_2$.

$$\frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} = t^2, \quad t \geq 0$$  \hspace{1cm} (B2)

Assign a projected mass density profile truncated at $t = 1$, and the deflection angle of an elliptically symmetric lens (ESL) is given as follows similarly to eq.(A5).

$$DAngle = \int_0^1 dt \sigma(t) \frac{\partial}{\partial t} \left[ \int_{\Omega_t} d^2\xi \right] \cdot \frac{\bar{z}}{\bar{\xi} - \xi}$$  \hspace{1cm} (B3)

where $\Omega_t$ is the ellipse with the elliptic radial parameter $t$.

In order to calculate the integral in the square bracket of eq.(B3), which we label $\Omega_t$Integral, we need to calculate the Schwarz function of the ellipse. Note that an analytic
function is uniquely determined in a domain if it is known in an arc in the domain. Thus it suffices to find the Schwarz function on the ellipse where it is satisfied that \( \bar{z} = S(z) \). Since we know the equation of the ellipse in real coordinates, eq.(B2), we only need to substitute \( x = (z + \bar{z})/2 \) and \( y = (z - \bar{z})/(2i) \) and solve \( \bar{z} \) in terms of \( z \).

\[
\bar{z} = \frac{a^2 + b^2}{c^2} z \pm \frac{2ab}{c^2} \left( z^2 - c^2 t^2 \right)^{1/2}
\]  

(B4)

where \( c^2 = a^2 - b^2 \) and \( z = \pm ct \) are the foci of the ellipse. The square root function is intrinsically a double-valued function and here one branch is chosen so that it is a univalent analytic function (except at the branch points \( z = \pm ct \)). If \( z \pm ct = r_+e^{i\theta_2} \), we choose

\[
\left( z^2 - c^2 t^2 \right)^{1/2} = \left( r_+ - r_- \right)^{1/2} e^{i(\theta_2 + \theta_1)/2}
\]

Then the Schwarz function of the ellipse is given by

\[
S(z) = \frac{a^2 + b^2}{c^2} z - \frac{2ab}{c^2} \left( z^2 - c^2 t^2 \right)^{1/2}.
\]  

(B5)

The other branch is not a solution because the function does not return \( \bar{z} \). (The readers are encouraged to check numerically.)

The Schwarz function can be decomposed into two parts: \( S(z) = S_1(z) + S_2(z) \) where

\[
S_1(z) = \frac{a-b}{a+b} z ; \quad S_2(z) = \frac{2ab}{c^2} \left( z - \left( z^2 - c^2 t^2 \right)^{1/2} \right).
\]  

(B6)

\( S_1(z) \) diverges at infinity and \( S_2(z) \) is singular (branch points) at the focii. Thus \( S_1(z) \) is analytic inside the ellipse \( \partial \Omega_t \) and \( S_2(z) \) is analytic outside \( \partial \Omega_t \). (The decomposition is not essential for the integration even though one can say it is convenient. Handling \( S(z) \) directly without the decomposition by manipulating the pole at \( z = \infty \) by an inverse transformation is also instructive.)

Be reminded from the previous section that the singularity at \( \xi = z \) when the probing point \( z \) is inside the mass disk \( D_t \) does not contribute to the deflection angle. Thus, the only difference between the outside lens equation and inside lens equation is the range of the integration in \( t \) in eq.(B3). Now assuming that \( \eta > t \), the \( \Omega_t \) integral can be calculated using the machineries of Stoke’s theorem, Cauchy integral, and the Schwarz function of the ellipse.

\[
\int_{\Omega_t} \frac{d^2 \xi}{z - \xi} = \frac{1}{2i} \int_{\Omega_t} \frac{d\xi d\bar{\xi}}{z - \xi} = \frac{1}{2i} \int_{\Omega_t} \frac{d\xi d\bar{\xi}}{z - \xi} = \frac{1}{2i} \int_{\partial \Omega_t} \frac{d\bar{\xi} d\xi}{z - \xi}
\]

\[
= \frac{1}{2i} \int_{\partial \Omega_t} \frac{S_1(\xi) + S_2(\xi)}{z - \xi} d\xi = \pi S_2(z) = \frac{2\pi ab}{c^2} \left( z - \left( z^2 - c^2 t^2 \right)^{1/2} \right)
\]

Thus the ESL outside lens equation is

\[
\omega = z - \int_0^1 \frac{\tilde{\sigma}(t) t dt}{(z^2 - c^2 t^2)^{1/2}}.
\]  

(B7)
and the ESL inside lens equation is

$$\omega = z - \int_0^\eta \frac{\tilde{\sigma}(t)tdt}{(\tilde{z}^2 - c^2\eta^2)^{1/2}}$$  \hspace{1cm} \text{(B8)}$$

where $c$ is the focal length of the mass boundary ellipse and $\eta$ is elliptic radial parameter of $z$.

### B.1. Inverse Elliptic Radial Density: $\tilde{\sigma}(t) = 1/t$

If the mass density is proportional to the inverse of the elliptic radius, $\tilde{\sigma}(t) = 1/t$, then the outside and inside lens equations are

$$\omega = z - \frac{1}{c} \arcsin \left( \frac{c}{\tilde{z}} \right) : \quad \text{z is outside the lens mass} \hspace{1cm} \text{(B9)}$$

and

$$\omega = z - \frac{1}{c} \arcsin \left( \frac{c\eta}{\tilde{z}} \right) : \quad \text{z is inside the lens mass, } z \neq 0 \hspace{1cm} \text{(B10)}$$

where $\eta$ is the elliptic radius of $z$: $z = \eta(a \cos \theta + ib \sin \theta)$. We call the ESL lens with $\tilde{\sigma}(t) = 1/t$ an arcsin lens named after the arcsin functions of the deflection angles.

### B.2. Constant Density: $\tilde{\sigma}(t) = 2$

If the projected mass density is constant, $\tilde{\sigma}(t) = 2$. The outside equation is

$$\omega = z - \frac{2}{c^2}(\tilde{z} - (\tilde{z}^2 - c^2)^{1/2})$$  \hspace{1cm} \text{(B11)}$$

and the inside equation is

$$\omega = z - \frac{2}{c^2}(\tilde{z} - (\tilde{z}^2 - c^2\eta^2)^{1/2}) = z - \frac{1}{ab} \left( z - \frac{a - b}{a + b} \tilde{z} \right)$$  \hspace{1cm} \text{(B12)}$$

where the last equality holds because $\tilde{z} = S(z)$ on the $\eta$-ellipse.

### C. Points and Sticks and Disks

A set of lens masses that are either spherically symmetric or elliptically symmetric can be represented by a set of points and sticks for the images outside the lensing masses. If we
normalize the lens equation by the total mass, then the normalized outside lens equation is given by

\[
\omega = z - \sum_j \frac{\epsilon_{cj}}{z - x_{cj}} - \sum_k \int_0^1 \frac{\epsilon_{ek} \bar{\sigma}(t) t dt}{((z - \bar{x}_{ek})^2 - e^{i2\theta_k} c_k^2 t^2)^{1/2}}; \quad (C1)
\]

\[
\sum_j \epsilon_{cj} + \sum_k \epsilon_{ek} = 1. \quad (C2)
\]

\(\epsilon_{cj}\) and \(x_{cj}\) are the fractional mass and position of the center of mass of the \(j\)-th point (circularly symmetric mass); \(\epsilon_{ek}\), \(x_{ek}\), \(c_k\) and \(\theta_k\) are the fractional mass, position of the center of mass, focal length and direction angle of the \(k\)-th stick (elliptically symmetric mass). The focal points of the \(k\)-th elliptic mass are at \(z = x_{ek} \pm e^{i\theta_k} c_k\).

For an image inside the disk of a circularly symmetric lens \(j_0\), the lens equation is obtained by replacing the \(j_0\)-th point mass lens deflection angle in eq. (C1) by its inside deflection angle.

\[
\frac{\epsilon_{cj_0}}{z - \bar{x}_{cj_0}} \Rightarrow \frac{\epsilon_{cj}\eta[z - x_{cj_0}]}{z - \bar{x}_{cj}} \quad (C3)
\]

where \(\eta[z - x_{cj_0}]\) denotes the elliptic radial parameter of \(z - x_{cj_0}\).

For an image inside the disk of an elliptically symmetric lens \(k_0\), the lens equation is obtained by replacing the \(k_0\)-th elliptical mass lens deflection angle in eq. (C1) by its inside deflection angle by changing the upper bound of the integral from 1 to \(\eta[z - c_{ek_0}]\).

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Fig. 1.— There are two morphologies of the caustic domains of the Arcsin lens inside equation: 1) the caustic is enclosed inside the pseudo-caustic: the images are of 1/0, 1/1, and 2/2 where $m$ and $n$ of $m/n$ are the numbers of positive and negative images. 2) the caustic and pseudo-caustic intersect: the images are of 1/0, 1/1, 2/1, and 2/2. The caustics are cuspy and the pseudo-caustics are smooth. The numbers in the caustic domains indicate the total number of images of each caustic domain. The total parity is 0 or 1 for both cases.
Fig. 2.— The caustic domains of the arcsine lens outside equation with $c = 1.65$. The red is the (open) caustic and the blue is the pseudo-caustic. At the joining points, the caustic and pseudo-caustic are tangential. $m/n$ stands for #(positive images)/#(negative images). The total parity is 0, 1, or 2. $\oplus$ indicates that the image crossing the segment of the pseudo-caustic and its symmetric counterparts is a positive image. For the rest of the pseudo-caustic, the image appearing or disappearing is a negative image.