A simple microstructural explanation of concave price impact

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Abstract

This article describes a simple model of market microstructure which explains a concave price impact. In the proposed model, the local relationship between the order flow and the fundamental price (i.e., the local price impact) is linear, which makes the model dynamically consistent. Nevertheless, the expected impact on midprice from a large sequence of co-directional trades is nonlinear and asymptotically concave. The main practical conclusion of the model is that, throughout a meta-order, the volumes at the best bid and ask prices change (on average) in favor of the executor. This conclusion, in turn, relies on two more concrete predictions of the model, one of which is tested using publicly available market data without the information about meta-orders.

1 Introduction and main results

The term price impact often refers to the fact that a trade tends to move the asset price in its direction. However, this general observation has several more specific interpretations, and, as pointed out in [20], it is important to differentiate between various types of price impact. First, one may consider the local impact: i.e., the expected price change as a function of the volume of a single trade. The latter is only relevant in the markets where large trades occur often (e.g., OTC markets), but is not relevant, e.g., for the most popular stock exchanges. The second type of impact is the meta-order impact: i.e., the expected price change as a function of the volume of a sequence of co-directed trades (such sequence is called a meta-order). This type of impact is more relevant for public exchanges, where most participants willing to buy or sell a large quantity of the asset split this quantity into smaller pieces (child orders) and submit each of them separately. Within the meta-order impact, one can distinguish two sub-types, depending on whether the relative rate of the execution, or its total duration, is fixed.

This paper studies the meta-order impact for a fixed execution rate (also known as the expected price trajectory). There exist various empirical studies that confirm the concavity of meta-order impact (see, e.g., [1, 3, 2, 21, 17, 4]). They show that the expected price change as a function of traded volume is concave (see the left part of Figure 2), and some even claim a specific power (in particular, square-root) dependence of the impact on the volume. These empirical results motivated the search for a sound theoretical explanation of the concavity of price impact. The following four paragraphs describe the main types of explanations available to date.

The first explanation is rather heuristic and has not been documented in the academic literature (to the author’s best knowledge). It explains the concavity of price impact by the predictability of future prices and

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order flow. To understand this argument, notice that the concavity of price impact is equivalent to the statement that the marginal impact of the individual child orders decreases throughout a meta-order. Then, assuming, for example, that the price dynamics have a mean-reverting component, one can argue that the latter will slow down the deviation of the price from its initial level, causing smaller marginal impact of every subsequent trade. Similarly, if the market participants can predict (with some accuracy) the size of a meta-order, then, toward the end of its execution, they become less certain that the execution will continue, hence they limit their predatory actions against the executor (i.e., moving their limit orders or front running), causing smaller marginal impact towards the end of the meta-order. The main flaw of this type of arguments, of course, is their contradiction with the fundamental principles of the efficient market hypothesis. Although, in practice, the latter holds only to a certain extent, typically, the inefficiencies in the market disappear over time, once they become known and are of a significant magnitude. Therefore, given the prolonged history of the empirical studies in this area, it is hard to imagine that the predictability of future prices and order flows would still exist on a level that would make a significant contribution to the price impact.

Another type of explanation is based on game-theoretic models in which a market-maker provides liquidity for the executor. The concavity is explained, for example, via the shape of market-maker’s preferences ([8]) or via the distribution of the sizes of meta-orders ([6]). While such explanation is admissible for OTC markets, it is not clear whether the conclusions of such models extend to the order-driven markets (e.g., exchanges), where multiple heterogeneous liquidity providers with varying inventories (as they trade with other liquidity consumers) compete for clients’ orders.

Additional evidence of the concave (even square-root) price impact curve is obtained via dimensional analysis ([15, 18]). Indeed, if one assumes that the impact depends only on a certain group of factors, then, after few additional invariance assumptions, one can deduce that the square-root is the only function that produces the right unit of measurement for the impact. This is a rather powerful method, but, unfortunately, it sheds very little light on the mechanism that generates the price impact, which is an important part of explaining the phenomenon.

The latent limit order book model (LLOB), described, e.g. in [20, 5], provides an explanation of the concave price impact that is the closest to the one proposed herein. The LLOB model assumes the existence of a large number of potential buyers and sellers whose reservation prices for the asset follow a common stochastic process plus an idiosyncratic Brownian component independent across agents. When the reservation price of a buyer matches that of a seller, they trade and eliminate each other’s orders. The price is defined as the point that separates the reservation prices of the buyers and sellers. In the large-population limit, the resulting distribution of reservation prices has a density that evolves deterministically, according to the heat equation, around the common price component. Then, assuming that a meta-order (of constant rate) can be represented by a source term in the heat equation, the authors of [5] solve the resulting equation explicitly to recover the square-root price impact. Even though the model proposed in this paper starts from a very different setting, the key feature of both models is that the liquidity improves on average throughout the meta-order, which implies that every subsequent child order makes smaller impact. This phenomenon of “improving liquidity” is described in more detail further in this section and in Section [3,1] It is also worth mentioning that, unlike the first two approaches discussed above, the explanation of concave impact provided by the LLOB and by the present model is purely mechanical and is consistent with the efficient market hypothesis: i.e., there is not need to assume any strategic behavior of other market participants, nor is it necessary to assume predictability of future prices and order flow.

The explanation of the concave price impact proposed herein has several advantages. First, as shown in the

1 Of course, this is only true if the initial price is on the right side of the level around which it mean-reverts. This is one of the flaws of this argument.

2 It is worth mentioning that, strictly speaking, the present model only explains asymptotic concavity: i.e., that the marginal impact at the beginning of a long meta-order is higher than at its end. With a slight abuse of terminology, we refer to this property as concavity
remainder of this section, the main results of this paper can be stated in plain language, without appealing to any sophisticated technical arguments. These results are, nevertheless, proven rigorously in Section 2 (and this is where the technical arguments are needed). The model allows for multiple market participants, consuming and providing liquidity, which makes it well suited for order-driven markets. The setting of the model is "bottom-up"; i.e., its inputs have clear economic meaning and can be measured from market data. The proposed explanation does not require the predictability of future prices or order flow and, hence, is consistent with the efficient market hypothesis. Finally, a significant advantage of the proposed explanation is that one of its two most important predictions (which directly imply the concavity of price impact) can be tested on real market data without the information about meta-orders (as the latter is notoriously difficult to obtain).

The remainder of this section is devoted to the description of the main results of the paper. Their precise statements and mathematical proofs are given in Section 2 and their numerical and empirical analysis is presented in Section 3. We measure all prices in ticks and assume that the asset under consideration has constant spread of size one (see the end of Section 3.2 for the discussion of this assumption). We also assume the existence a real-valued process \( X \), which we refer to as the fundamental price, and which has the meaning of a signal predicting the direction of the next trade. Namely, \( X \) is always between the best bid and the best ask prices, and the closer it is to the bid (ask) the more likely it is that the next trade will be a sell (buy). It is clear that the best bid and the best ask prices are given by the roundings \( \lfloor X \rfloor \) and \( \lceil X \rceil \), respectively, and that they change whenever \( X \) hits an integer. We also assume that every trade makes a linear local impact \( \alpha \) on \( X \), with the coefficient \( \alpha \). For example, a buy trade of size \( \delta \) changes \( X \) to \( X + \alpha \delta \). Assuming that, between trades, \( X \) follows a symmetric random walk, we conclude that its global dynamics must have a force, or a drift term, that pushes \( X \) away from the midprice. Indeed, if \( X \) is slightly above the midprice, the next trade is more likely to be a buy, which pushes \( X \) further up and makes the next buy even more likely, and so on. Section 2.2 gives the precise form of this drift and of the dynamics of \( X \). For simplicity, let us focus on the dynamics of \( X \) between two nearest integers – i.e., we consider \( X \mod 1 \) (\( X \) modulo one). As explained above, the process \( X \mod 1 \) is a random walk (on the unit circle) with a drift that pushes it away from 1/2. The stationary distribution of such a process must have a U-shaped density (see the top left part of Figure 1). Thus, before the execution of a meta-order begins, the distribution of the fundamental price modulo one has a U-shaped density. Next, let us analyze what happens toward the end of a meta-order. It is natural to assume that a meta-order, which is a sequence of co-directed trades, introduces an additional drift term in \( X \mod 1 \), of the same sign as the meta-order itself (see equation (13) for the dynamics of the fundamental price during a meta-order). If this additional (constant sign) drift \( \delta \) is very large, it is easy to deduce that the stationary distribution of the resulting process is uniform (think, e.g., of a Brownian motion with a very large drift, run on a circle). It is clear that the wings (i.e., the values at 0 and 1) of a U-shaped density are higher than the wings of a uniform density (the latter are equal to one). Interpolating heuristically between zero additional drift and the infinitely large drift, we conclude that the wings of the stationary density of \( X \mod 1 \) before a sufficiently long meta-order are higher than the wings of its density at the end of this meta-order, for any positive execution rate (compare the top left and the bottom right parts of Figure 1). This heuristic interpolation is made rigorous in Proposition 5 under additional modeling assumptions. Now, the phenomenon of “improving liquidity” becomes clear. Interpreting \( X \) as the microprice (see [19]), one easily sees that the lower wings of the stationary density of \( X \mod 1 \) imply
smaller probability of observing low liquidity (i.e., low volume of limit orders) at the best bid or ask. It only remains to connect the wings of the stationary density of \( X \mod 1 \) to the price impact directly. To this end, notice that the expected impact on the midprice of a buy trade of size \( \delta \), submitted at time \( t = 0 \), is given by

\[
\mathbb{E} \left( \left\lfloor X_0 \mod 1 + \alpha \delta \right\rfloor - 1 \right),
\]

where \( X_0 \mod 1 \) is a random variable whose density is given by the stationary density of \( X \mod 1 \). For \( \delta \downarrow 0 \), it is clear that the leading order of the above expectation is given by the probability that \( X_0 + \alpha \delta > 1 \). The latter, in turn, is proportional to the wings of the stationary density of \( X \mod 1 \). Thus, the expected marginal impact on the midprice is proportional to the wings of the stationary distribution of the fundamental price modulo one. Proposition 3 makes this statement rigorous. As mentioned above, a meta-order introduces an additional drift term in the dynamics of the fundamental price, thus, switching the market into a different regime. In this new regime, \( X \mod 1 \) attains a new stationary density (provided the meta-order lasts long enough), whose wings are lower than the wings of the original stationary density. Repeating the above argument that connects the wings of the stationary density and the marginal impact (see Proposition 4), we conclude that the marginal impact at the end of the meta-order is lower than at the beginning (see Theorem 1), which implies the (asymptotic) concavity of price impact.

Note that the explanation of concave price impact presented above relies only on two predictions. The first one is the U-shape of the (global) stationary distribution of the fundamental price modulo one. The second prediction is that, during a meta-order, the fundamental price obtains an additional drift term of constant sign. While it is impossible to test the second prediction without the meta-order data (although this prediction appears to be self-evident), the first prediction is verified using real market data in Section 3.2.

2 Mathematical analysis of the model

The roots of the proposed model go back to the game-theoretic setting of [10, 9, 11]. However, the specific model proposed herein is natural enough and does not require any additional justification via equilibrium arguments. The core of the model is the assumption that the potential buyers and sellers arrive to the market one by one, having their own reservation prices (i.e., the “fair” prices for the asset, in their view). If a reservation price of an agent is above (below) the current best ask (bid) price, a sell (buy) trade occurs. The reservation prices are not independent across the potential buyers and sellers: they have a common component \( X \) and the idiosyncratic additive part, generated from a (symmetric around zero) distribution with c.d.f. \( F \). It is shown in [11] that, for reasonable values of the model parameters, the agents providing liquidity via limit orders set the equilibrium bid and ask prices exactly at \( \lfloor X \rfloor \) and \( \lceil X \rceil \), respectively, thus, making the model consistent with the setting described in Section 1. The details of the model are presented in the remainder of this section, with the main result (the asymptotic concavity of price impact) stated in Theorem 1. It is worth mentioning that a part of this section contains the description of two models: with finite and infinite trading activity. The former is easier to interpret from the economic (or practical) point of view. The latter provides more tractable expressions for the target quantities. We show that the two are consistent and, ultimately, focus our attention on the infinite activity model.

\footnote{It is important to note that the proposed explanation works even if \( X \) is not the microprice, as long as it has the prescribed connection to the order flow.}
2.1 Finite activity jump-diffusion model

Potential buyers/sellers arrive according to a Poisson process with jump times \( \{S_i\} \) and intensity \( \lambda \). The reservation price of the \( i \)-th buyer/seller is

\[
p^0_{S_i} = \tilde{X}_{S_i} - \xi_i,
\]

where \( \{\xi_i\} \) are i.i.d. random variables, with c.d.f. \( F \), and \( \tilde{X} \) evolves according to

\[
\tilde{X}_t = X_0 + \alpha \sum_{S_i \leq t} \Delta V_{S_i} + \sigma(\tilde{X}_t) \tilde{B}_t \mod 1,
\]

with \( \tilde{B} \) being a Brownian motion independent of \( \{S_i, \xi_i\} \), and with

\[
\Delta V_{S_i} := \delta \mathbf{1}_{\{\xi_i \geq \lfloor X_{S_i} \rfloor - X_{S_i}\}} - \delta \mathbf{1}_{\{\xi_i \leq \lceil X_{S_i} \rceil - X_{S_i}\}}.
\]

Throughout the rest of the paper, we make the following standing assumptions on \( F \) and \( \sigma \).

- \( F \in C^{1+\epsilon}([-1, 1]) \), for some \( \epsilon \in (0, 1) \), and \( F' \) is symmetric around \( x = 0 \) in this range.
- \( \inf_{x \in [0, 1]} (F(x - 1) + F(-x)) > 0 \).
- \( \sigma \) has period one and is symmetric around \( x = 1/2 \).
- \( \inf_{x \in [0, 1]} \sigma(x) > 0 \).
- \( \sigma \in C^{1+\epsilon}([0, 1]) \).

Denote by \( M \) a Poisson random measure with the compensator

\[
\mu(dt, dx) = \lambda dt \otimes (P \circ \xi_i^{-1})(dx) = \lambda dt \otimes dF(x),
\]

independent of \( \tilde{B} \). Then, the fundamental price and the order flow are described by the following system:

\[
\begin{align*}
\tilde{X}_t &= X_0 + \alpha \left( N^+_t - N^-_t \right) + \sigma(\tilde{X}_t) \tilde{B}_t, \\
N^+_t &= \delta \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{x \geq \beta^+(\tilde{X}_u)\}} M(du, dx), \\
N^-_t &= \delta \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{x \leq \beta^-(\tilde{X}_u)\}} M(du, dx),
\end{align*}
\]

(1)

where

\[
\beta^+(x) := \lfloor x \rfloor - x, \quad \beta^-(x) := \lceil x \rceil - x.
\]

Note that

\[
N^+_t = \sum_{S_i \leq t} \Delta V^+_{S_i}, \quad N^-_t = \sum_{S_i \leq t} \Delta V^-_{S_i}.
\]

The input to the model is \( (\alpha, \sigma, F, \lambda, \delta) \).
2.2 Infinite activity diffusion model

For analytic tractability, it is convenient to consider an infinite-activity limit of the model (1), as \( \lambda \to \infty \). In order to avoid the explosion of total order flow, we need to assume that \( \delta \to 0 \), so that \( \lambda \delta \to \gamma \), with some constant \( \gamma > 0 \). For simplicity, we assume that \( \delta = \gamma / \lambda \). Notice that

\[
\begin{align*}
N_t := d(N_t^+ - N_t^-) &= \lambda \delta \left( F(-\beta^+(\tilde{X}_t)) - F(\beta^-(\tilde{X}_t)) \right) \, dt + dZ_t,
\end{align*}
\]

where \( Z \) is a martingale. Then,

\[
\langle Z \rangle_t = \int_0^t \lambda \delta^2 \left( F(-\beta^+(\tilde{X}_s)) + F(\beta^-(\tilde{X}_s)) \right) \, ds \sim \delta \gamma \int_0^t \left( F(-\beta^+(\tilde{X}_s)) + F(\beta^-(\tilde{X}_s)) \right) \, ds \to 0.
\]

Thus, we expect \( \tilde{X} \) to converge to \( X \) which is the solution of

\[
dX_t = \alpha \gamma \left( F(-\beta^+(X_t)) - F(\beta^-(X_t)) \right) \, dt + \sigma(X_t)dB_t,
\]

equipped with the same initial condition as \( \tilde{X} \). We do not make this statement precise, as we will in fact need the convergence of time-changed processes, established in Lemma 1. At this stage it is only important to notice that the drift of \( X \) in (2) is positive in \((n,n+1/2)\), and is negative in \((n+1/2,n+1)\), for any integer \( n \). Hence, this drift pushes \( X \) away from the midprice.

2.3 Expected impact on midprice by a VWAP strategy

2.3.1 Finite activity model

Recall that the process \( (N_t = N_t^+ - N_t^-) \) represents the total order flow in the market. In the finite activity model (1), we have

\[
\begin{align*}
\tilde{X}_t &= X_0 + \alpha N_t + \int_0^t \sigma(\tilde{X}_u) dB_u, \\
N_t &= \delta \int_0^t \int_\mathbb{R} \left( 1_{\{x \geq \beta^+(\tilde{X}_u^-)\}} - 1_{\{x \leq \beta^-(\tilde{X}_u^-)\}} \right) M(du, dx),
\end{align*}
\]

where \( B \) is a Brownian motion and \( M \) is an independent Poisson random measure with the compensator

\[
\mu(dt, dx) = \lambda dt \otimes dF(x).
\]

The total order flow is a sum of the order flows of \( K \) individual market participants (agents):

\[
N = \sum_{j=1}^K N^j.
\]

We assume that the agents follow VWAP-type strategies (this terminology is explained below). Namely, the \( j \)th agent has the order flow

\[
N_j^t = \delta \int_0^t \int_\mathbb{R} \left( 1_{\{x \geq \beta^+(\tilde{X}_u^-)\}} + 1_{\{x \leq \beta^-(\tilde{X}_u^-)\}} \right) \zeta_j(\tilde{X}_u^-, u) M^j(du, dx), \quad j = 1, \ldots, K,
\]
where \( \{M^j\} \) are independent Poisson random measures with the compensators

\[
\mu^j(dt, dx) = \lambda^j dt \otimes dF(x), \quad j = 1, \ldots, K, \quad \sum_{j=1}^K \lambda^j = \lambda,
\]

and \((\omega, x, u) \mapsto \zeta^j(x, u) \in \{\pm 1\}\) is a random field, defined for all \((x, u) \in \mathbb{R} \times \mathbb{R}_+\), s.t.

- \(\{\zeta^j(\cdot, u)\}\) are independent across \(j = 1, \ldots, K\), across \(u \geq 0\), and independent of \(\{M^j\}, \bar{B}\),
- for each \(j = 1, \ldots, K\) and \((x, u) \in \mathbb{R} \times \mathbb{R}_+\), we have

\[
P(\zeta^j(x, u) = 1) = \frac{F(-\beta^+(x))}{F(-\beta^+(x)) + F(\beta^-(x))}.
\]

The random fields \(\{\zeta^j\}\) represent the heterogeneity of agents in terms of their trading styles. For example, at any given moment in time, one agent may buy (i.e., \(\zeta^j = 1\)), while another one may sell (i.e., \(\zeta^j = -1\)). In general, we allow the agents’ trading styles to depend on the fundamental price and on their idiosyncratic sources of randomness. The assumptions we make on \(\{\zeta^j\}\) are not the most general, but they ensure that the model is consistent: i.e., the total order flow \(\sum_{j=1}^K N^j\) satisfies (4), with the Poisson random measure \(M = \sum_{j=1}^K \zeta^j M^j\), having the prescribed compensator. The latter observation follows easily by conditioning on \(\bar{B}\) and applying the standard results on thinning and superposition of compound Poisson processes.

To see why we call the agents’ strategies VWAP-type, recall that VWAP is an execution strategy that prescribes to trade at the rate that is a fixed fraction of the total traded volume rate in the market. This strategy is very popular among practitioners and has been studied extensively in academic literature (see, e.g., [13]). Notice that, in the present model, the total traded volume in the market at time \(t\) is given by

\[
\bar{V}_t = \delta \int_0^t \int_{\mathbb{R}} \left(1_{\{x \geq \beta^+(\hat{X}_{u-})\}} + 1_{\{x \leq \beta^-(\hat{X}_{u-})\}}\right) M(du, dx).
\]

Comparing the above with (6), we conclude that the \(j\)th agent in the proposed model trades with the rate that is \(\lambda^j/\lambda\) fraction of the overall rate, which makes it similar to VWAP. However, unlike the classical VWAP strategy, where the trades are made in the same direction, our agents trade in different directions.

Let us now assume that the first agent aims to compute the expected impact of a sequence of her buy trades (referred to as the execution interval) on the midprice\(^8\). Recalling (6), we conclude that any such execution interval can be characterized by the condition \(\zeta^1 = 1\). In order to compute this expected impact in practice, the agent (i) finds the past execution intervals in a sample of past \(L\) trades, (ii) records the changes in the quoted ask price over each interval, and (iii) computes the sample average of these changes. Mathematically, this process corresponds to computing

\[
\tilde{I}_L(Q, \delta, \lambda, \lambda^1) := \mathbb{E}\left(\left[\hat{X}_{\tau_1} - \left[\hat{X}_{\tau_0}\right]\right] \mid \zeta^1(\cdot, \cdot) = 1, \tau_1 \in [\tau_0, \tau_1]\right),
\]

where the constants \(Q \geq 0\) and \(L > 0\) denote, respectively, the (fixed) total amount of the asset purchased by the first agent over each randomly selected execution interval and the size of the sample (measured in trades of the first agent) from which the execution intervals are selected. The random times \(\tau_0\) and \(\tau_1\) denote, respectively, the start and end times of a randomly chosen execution interval:

\[
\tau_0 := \tilde{U}^1_{\eta+Q}, \quad \tau_1 := \bar{U}^1_{\eta+Q}, \quad \tilde{U}^1 := \left(\tilde{V}^1\right)^{-1}, \quad \eta \sim U(0, L),
\]

\(^8\)We only consider the execution intervals of buy trades, as the case of sell trades is analogous.
\[ \hat{V}_t^1 = \delta \int_0^t \int_{\mathbb{R}} \left( 1_{\{x \geq \beta^+ (\hat{x}_{u-})\}} + 1_{\{x \leq \beta^- (\hat{x}_{u-})\}} \right) M^1(du, dx), \]  

where we choose a right-continuous inverse function and assume that \( \eta \) is independent of everything else. The above choice of \( \tau_0 \) corresponds to the assumption that each trade of the agent is equally likely to be the first trade of an execution interval.

It is easy to see that the conditional distribution of \((\hat{V}^1_{\tau_0+}, -\hat{V}^1_{\tau_0+}, \hat{X}_{\tau_0+})\), given \( \zeta^j(\cdot, t) = 1 \) for \( t \in [\tau_0, \tau_1] \), is the same as the distribution of \((\hat{N}, \hat{X}^\tau_0, \hat{Y}^\tau_0)\), where

\[ \hat{N}^x_t = \delta \int_0^t \int_{\mathbb{R}} \left( 1_{\{z \geq \beta^+ (\hat{y}_{u-})\}} + 1_{\{z \leq \beta^- (\hat{y}_{u-})\}} \right) M(du, dz), \]  
\[ \hat{Y}^x_t = x + \alpha \hat{N}^x_t + \alpha \delta \int_0^t \int_{\mathbb{R}} \left( 1_{\{z \geq \beta^+ (\hat{y}_{u-})\}} - 1_{\{z \leq \beta^- (\hat{y}_{u-})\}} \right) M(du, dz) + \int_0^t \sigma(\hat{y}^x_u) d\hat{W}_u, \]

where \( \hat{W} \) is a Brownian motion, \( \hat{M} \) and \( \hat{\tilde{M}} \) are Poisson random measures with the compensators

\[ \hat{\nu}(dt, dx) = \lambda^1 dt \otimes dF(x), \quad \hat{\nu}(dt, dx) = (\lambda - \lambda^1) dt \otimes dF(x), \]

and all are mutually independent and independent of \((\hat{X}, N^1, \eta)\). Thus, the expected impact on midprice (of a VWAP buy interval) can be rewritten as

\[ \bar{I}_L(Q, \delta, \lambda, \lambda^1) := \mathbb{E} \left( \left[ \hat{Y}^x_Q \right] - \left[ \hat{X}_Q \right] \right), \quad \hat{X} := \hat{X}^{\hat{V}^1}_{(\hat{V}^1)^{-1}}, \quad \hat{Y}^x := \hat{Y}^x_{(\hat{V}^1)^{-1}}, \]  

where \((\hat{X}, \hat{V})\) are defined in [3], [5], [6], [7], and \((\hat{Y}^x, \hat{N}^x)\) are defined in [8], [9].

The process \( \hat{N} \) represents the order flow of the first agent during her (buy) execution intervals. The interpretation of \( \hat{Y} \) is also clear: it is a model for the fundamental price during the (buy) execution intervals of the first agent. Since the order flow is biased upwards in such intervals, the integrand in [8] has a sum of two (mutually exclusive) indicators, and the resulting nondecreasing process \( \hat{N} \) creates an upward trend in the dynamics of \( \hat{Y} \).

### 2.3.2 Infinite activity model

Note that the expected impact on midprice, given by [10], depends only on the distributions of \( \hat{X} \) and \( \hat{Y} \), defined in [10]. Let us fix \( \gamma > 0 \) and \( \theta \in (0, 1] \), and consider the limit of the expected impact:

\[ I_L(Q, \theta) := \lim_{\lambda \to \infty} \bar{I}_L(Q, \gamma/\lambda, \lambda, \theta \lambda), \]

provided it is well defined. The restriction \( \delta = \gamma/\lambda \) is discussed at the beginning of Subsection 2.2. The condition \( \lambda^1 = \theta \lambda \) is equivalent to the assumption that the market participation rate of the first agent is fixed as we vary \( \lambda \).

It turns out that \( \hat{X} \) converges (weakly) to \( \hat{X} \), given by

\[ \hat{X}_t = X_0 + \int_0^t \hat{\mu}_0(\theta, \hat{X}_u) du + \int_0^t \hat{\sigma}(\theta, \hat{X}_u) d\hat{B}_u, \]

with a Brownian motion \( \hat{B} \) and with

\[ \hat{\mu}_0(\theta, y) := \alpha \frac{F(-\beta^+(y)) - F(\beta^-(y))}{\theta (F(-\beta^+(y)) + F(\beta^-(y)))}, \quad \hat{\sigma}(\theta, y) := \sqrt{\theta \gamma (F(-\beta^+(y)) + F(\beta^-(y)))}, \]
while \( \hat{Y}^x \) converges (weakly) to \( \hat{Y}^x \), given by
\[
\hat{Y}_t^x = x + \int_0^t \hat{\mu}_1(\theta, \hat{Y}_u^x) du + \int_0^t \hat{\sigma}(\theta, \hat{Y}_u^x) d\hat{W}_u,
\]
with a Brownian motion \( \hat{W} \) and with
\[
\hat{\mu}_1(\theta, y) := \alpha \frac{2\theta F(\beta^-(y)) + F(-\beta^+(y)) - F(\beta^{-}(y))}{\theta (F(\beta^+(y)) + F(\beta^{-}(y)))}.
\]

To make the above statement precise, we view \( \hat{X}, \hat{Y}^x \) as random elements with values in the Skorokhod space \( D([0, \infty)) \), and \( \hat{X}, \hat{Y}^x \) as random elements with values in \( C([0, \infty)) \). Note also that the laws of the processes \( \hat{X} \) and \( \hat{Y}^x \) are uniquely determined by \( (12) \) and \( (13) \), which can be easily seen by applying the scale function transformation and reducing these SDEs to the ones with no drifts and with Lipschitz diffusion coefficients.

**Lemma 1.** As \( \lambda \to \infty \), for any \( x \in \mathbb{R} \), we have:
\[
\mathbb{P} \circ \hat{X}^{-1} \to \mathbb{P} \circ \hat{X}^{-1}, \quad \mathbb{P} \circ (\hat{Y}^X)^{-1} \to \mathbb{P} \circ (\hat{Y}^X)^{-1},
\]
where the convergence is in weak topology induced by the \( C([0, \infty)) \)-seminorms.

**Proof:**
W.l.o.g., we only prove the convergence of \( \hat{X} \). Recall that the latter is a composition of two processes, \( \hat{X} \) and \( \hat{V}^1 \). First, we prove the \( C \)-tightness of the joint law of \( (\hat{X}, \hat{V}^1) \) over \( \lambda \to \infty \) on a finite time interval \([0, T]\). To prove the latter, it suffices to show (i) that the absolute values of the two processes are bounded in probability, uniformly over \( \lambda \), and (ii) that
\[
\forall \varepsilon > 0, \quad \lim_{\varepsilon' \to 0} \lim_{\lambda \to \infty} \mathbb{P} \left( \sup_{t, s \in [0, T], |t-s| \leq \varepsilon'} |Z_t - Z_s| > \varepsilon \right) = 0,
\]
for \( Z = \hat{X}, \hat{V}^1 \). Both (i) and (ii) follow from Chebyshev’s inequality and the estimate
\[
\mathbb{E} \sup_{u \in [t, s]} |Z_u - Z_s| \leq C \lambda_1 \delta |t - s|,
\]
where \( C \) is a constant and we recall \( \lambda_1 = \theta \lambda \) and \( \delta = \gamma / \lambda \).

Next, we consider any limit point \( \Lambda \) (a probability measure on \( (C([0, T]))^2 \)) of the family \( \{\mathbb{P} \circ (\hat{X}, \hat{V}^1)^{-1}\} \), with the associated sequence \( \{\lambda_n \to \infty\} \). In particular, \( (\hat{X}^n, \hat{V}^{1,n}) \to (\hat{X}, \hat{V}^1) \) in weak topology induced by the \( C \)-norm. Let us describe the dynamics of \( (\hat{X}, \hat{V}^1) \). It is easy to see that, for any \( f \in C_b(\mathbb{R}) \),
\[
\mathbb{E} \sup_{t \in [0, T]} \left| f(\hat{V}_t^1) - \gamma \theta \int_0^t f'(\hat{V}_s^1) \left( F(-\beta^+(\hat{X}_s)) + F(\beta^{-}(\hat{X}_s)) \right) ds \right| = \lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} \left| f(\hat{V}_{t,n}^1) - \gamma \theta \int_0^t f'(\hat{V}_{s,n}^1) \left( F(-\beta^+(\hat{X}_s^n)) + F(\beta^{-}(\hat{X}_s^n)) \right) ds \right| = 0.
\]
Choosing an appropriate sequence of \( f \) approximating the identity, we deduce from the above that
\[
\hat{V}_t^1 = \gamma \theta \int_0^t \left( F(-\beta^+(\hat{X}_s)) + F(\beta^{-}(\hat{X}_s)) \right) ds, \quad t \in [0, T].
\]
Similarly, for any $0 < t < s \leq T$, $f \in C_b(\mathbb{R})$, $0 \leq t_1 < \cdots < t_k \leq t$, $g \in C_b(\mathbb{R}^{2k})$,
\[
\mathbb{E} \left[ g \left( \bar{X}_{t_1}, \bar{V}_{r_1}^{1}, \ldots, \bar{X}_{t_k}, \bar{V}_{r_k}^{1} \right) \left( f(\bar{X}_s) - f(\bar{X}_t) - \alpha \gamma \int_t^s f'(-\beta^+(ar{X}_u)) \left( F(-\beta^+(\bar{X}_u)) - F(-\bar{X}_u) \right) du \right) \right] = \lim_{n \to \infty} \mathbb{E} \left[ g \left( \bar{X}_{t_1}^n, \bar{V}_{r_1}^{1,n}, \ldots, \bar{X}_{t_k}^n, \bar{V}_{r_k}^{1,n} \right) \left( f(\bar{X}_s^n) - f(\bar{X}_t^n) \right) \right] = 0.
\]

From the above, we deduce that the process $M_t := \bar{X}_t - \gamma \int_0^t \left( F(-\beta^+(\bar{X}_u)) - F(-\bar{X}_u) \right) du$, defined on the canonical space $(C([0, T]))^2$, is a continuous martingale under $\Lambda$ and, therefore, is given by a Brownian integral. Using the test function, as in the above, we easily deduce that $d < M \sim \sigma^2(\bar{X}_t)dt$ a.s. under $\Lambda$. Thus we have shown that $\bar{X}$ can be written as
\[
\bar{X}_t = X_0 + \alpha \gamma \int_0^t \left( F(-\beta^+(\bar{X}_u)) - F(-\bar{X}_u) \right) du + \int_0^t \sigma(\bar{X}_u)d\tilde{B}_u, \quad t \in [0, T],
\]
where $\tilde{B}$ is a Brownian motion under $\Lambda$. As the law of $(\bar{X}, \bar{V}^1)$ is uniquely determined by (14) and (15), the convergence of $(\bar{X}_n, \bar{V}^{1,n})$ holds along any sequence $\{\lambda_n \to \infty\}$.

To conclude the proof, we notice that there exists $\varepsilon > 0$, s.t., $\Lambda$-a.s., $\bar{V}^1 \in K^\varepsilon$, with $K^\varepsilon := \{ f \in C([0, T]) : f(t) - f(s) \geq \varepsilon(t-s), \forall 0 \leq s < t \leq T \}$. It is easy to see that the mapping $(f, g) \mapsto f \circ g^{-1}$ is a continuous mapping from $C([0, T]) \times K^\varepsilon$ into $C([0, T])$. Thus, using the Skorokhod’s representation theorem and the portmanteau theorem, we conclude that, along any $\{\lambda_n \to \infty\}$,
\[
\bar{X}_n := \bar{X}^n(\bar{V}^{1,n}) \to \bar{X}(\bar{V}^1) =: \bar{X},
\]
with the convergence being in weak topology induced by the $C$-norm. Using (14) and (15), we easily show that $\bar{X}$ satisfies (12). Recalling that the solution to (12) is unique in law, we complete the proof of the lemma. 

**Remark 1.** It is easy to see from the proof of Lemma 7 that, in the infinite activity model, during an execution interval of the first agent, her order flow is given by
\[
\theta \gamma \int_0^t \left( F(-\beta^+(Y_u)) + F(-\bar{X}_u) \right) du,
\]
where $Y$ represents the dynamics of the fundamental price in such intervals (it is the limit of $\tilde{Y}$). In addition, the total traded volume in the market is given by
\[
\gamma \int_0^t \left( F(-\beta^+(Y_u)) + F(-\bar{X}_u) \right) du.
\]
Thus, in the infinite activity limit, the first agent still uses a VWAP strategy, with the participation rate $\theta$.

In view of Lemma 7 it is natural to expect that $I_L(Q, \gamma, \theta)$, given by (11), can be computed by replacing $(\bar{X}, \tilde{Y})$ by $(\bar{X}, Y)$ in (10).

**Proposition 1.** For any $L > 0$, $Q \geq 0$, and $\theta \in (0, 1]$, the limit in (11) is well defined, and we have
\[
I_L(Q, \theta) = \mathbb{E} \left( \left[ \tilde{Y}^L \bar{X}_F \right] - \left[ \bar{X}_F \right] \right),
\]
with $\eta \sim U(0, L)$ independent of $(\bar{X}, \tilde{Y})$. 


Proof:

We prove the following statement using Lemmas 1 and 2, the portmanteau theorem, and the fact that neither \( \hat{Y}_Q \) nor \( \hat{X}_n \) have atoms.

In the remainder of the paper, we stay in the setting of the infinite activity model.

2.4 Large-sample limit

Recall that \( \eta \sim U(0, L) \), where \( L \) represents the length of the data sample from which the execution intervals are collected. As it is natural to estimate impact over a large sample, we consider

\[
I(Q, \theta) := \lim_{L \to \infty} I_L(Q, \theta),
\]

provided the limit is well defined. Not surprisingly, the large-sample expected impact on midprice turns out to be connected to the stationary distribution of the fundamental price. We begin with the following technical result.

Lemma 2. Let us fix an arbitrary \( \theta > 0 \). Then, there exist unique stationary distributions of \( \hat{X} \mod 1 \) and \( \hat{Y} \mod 1 \), with the densities \( \psi \) and \( \chi \), respectively. These densities are uniquely determined by the following conditions:

\[
\hat{\sigma}^2(\theta, \cdot)\psi \in C^{2+\epsilon}([0, 1]), \quad \frac{1}{2} \partial_x^2 (\hat{\sigma}^2(\theta, x) \psi(x)) - \partial_x (\hat{\mu}_0(\theta, x) \psi(x)) = 0, \quad x \in (0, 1),
\]

\[
\psi(0^+) = \psi(1^-), \quad \int_0^1 \psi(x) = 1,
\]

\[
\hat{\sigma}^2(\theta, \cdot)\chi \in C^{2+\epsilon}([0, 1]), \quad \frac{1}{2} \partial_x^2 (\hat{\sigma}^2(\theta, x) \chi(x)) - \partial_x (\hat{\mu}_1(\theta, x) \chi(x)) = 0, \quad x \in (0, 1),
\]

\[
\chi(0^+) = \chi(1^-), \quad \int_0^1 \chi(x) = 1.
\]

Moreover, for any bounded Borel-measurable function \( G \), we have, for any \( x \in \mathbb{R} \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} G(\hat{X}_t \mod 1) dt = \lim_{T \to \infty} \mathbb{E} G(\hat{X}_T \mod 1) = \int_0^1 G(z) \psi(z) dz,
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} G(\hat{Y}_t^x \mod 1) dt = \lim_{T \to \infty} \mathbb{E} G(\hat{Y}_T^x \mod 1) = \int_0^1 G(z) \chi(z) dz.
\]

Proof:

W.l.o.g. we only consider the case of \( \hat{X} \mod 1 \). First, we notice that the assumptions on \( F \) and \( \sigma \) imply that \( \hat{\mu}_i(\theta, \cdot)/\hat{\sigma}^2(\theta, \cdot) \in C^{1+\epsilon}([0, 1]) \), for \( i = 0, 1 \). Then, for any \( c > 0 \), Theorem 6.5.3 in [14] yields the existence and uniqueness of \( \hat{\sigma}^2 \psi \in C^{2+\epsilon}([0, 1]) \) satisfying the ODE in (19) with the boundary conditions \( \hat{\sigma}^2(\theta, 0^+)\psi(0^+) = \hat{\sigma}^2(\theta, 1^-)\psi(1^-) = c \). Moreover, the maximum principle (or the Feynman-Kac formula) implies that \( \psi > 0 \). Hence, choosing \( c > 0 \) appropriately, we can ensure that \( \int_0^1 \psi(x) = 1 \). Thus, we have shown the existence and uniqueness of the solution to (19).

Note that the length of an individual execution interval (measured in trades of the first agent) may be much smaller than \( L \); i.e., at this stage, we do not make the assumption that the execution intervals are long.
By choosing an arbitrary $f \in C^2([0,1])$, satisfying $f(0) = f(1) = 0$ and $f'(0^+) = f'(1^-)$, applying Itô’s formula to $f \circ (\cdot \mod 1)(X)$, integrating by parts, and using $(\tilde{\sigma}^2 \psi)(0^+) = (\tilde{\sigma}^2 \psi)(1^-)$, along with the ODE \((19)\) and the dominated convergence, we show that
\[
\frac{d}{dt} \int_0^1 E[f(\hat{X}_t \mod 1)] \psi(x) dx \bigg|_{t=0} = 0,
\]  
where $\hat{X}^x$ is the solution to \((12)\) with $X_0 = x$. As follows from Theorem 5.4.20 and Remark 5.4.21 in [12], \(\{\hat{X}^x\}_{x \in \mathbb{R}}\) is a Markov family with the transition denoted by $K(x, A)$. Due to periodicity of the coefficients in \((12)\), we have $K(x + n, A + n) = K(x, A)$. Then, it is easy to see that \(\{\hat{X}^x \mod 1\}_{x \in [0,1]}\) is a Markov family with the transition kernel $\sum_{n=-\infty}^{\infty} K(x, A + n)$. The Markov property and \((21)\) imply that $\psi$ is stationary.

To show uniqueness of the stationary distribution of $\hat{X} \mod 1$, consider any such distribution and use the scale function transformation, along with the continuous differentiability and Gaussian estimates for the fundamental solution of a linear (strictly) parabolic PDE with Lipschitz coefficients (see, e.g., [7]), to conclude that $\hat{X}^x \mod 1$ is stationary. Using Itô’s formula, we show that $\psi$ is a weak solution to the ODE in \((19)\) on $(0,1)$, with the test functions in $C_0^\infty((0,1))$. Using the weak form of the ODE \((19)\), we improve the regularity and conclude that $\tilde{\sigma}^2(\cdot, \cdot) \psi \in C^{2+\epsilon}((0,1))$, and, in turn, that the ODE \((19)\) holds in classical sense. Thus, the first part of the proof yields uniqueness of the stationary distribution.

Finally, to obtain the last statement of the lemma, it is a standard exercise to check that the families of measures
\[
\mathbb{Q}_T(dx) := \frac{1}{T} \int_0^T \mathbb{P}(\hat{X}_t \mod 1 \in dx) dt, \quad \hat{\mathbb{Q}}_T(dx) := \mathbb{P}(\hat{X}_T \mod 1 \in dx), \quad x \in [0,1],
\] parameterized by $T \geq 0$, are tight and that each of their limit points (in weak topology), as $T \to \infty$, is a stationary distribution of $\hat{X} \mod 1$. Since such distribution is unique, we obtain the statement of the lemma for bounded continuous $G$. As the stationary distribution has no atoms, this statement is extended to all bounded Borel-measurable $G$.

\textbf{Proposition 2.} For any $Q \geq 0$ and $\theta > 0$, the limit in \((18)\) is well defined, and we have:
\[
I(Q, \theta) = \int_0^1 \mathbb{E} \left( \left[ \hat{Y}_Q^x \right] - 1 \right) \psi(x) dx,
\]  
where $\psi$ is the density of the stationary distribution of $\hat{X} \mod 1$.

\textbf{Proof:}
Notice that, for any $x \in \mathbb{R}$ and any integer $n$, $\hat{Y}^{x+n} = \hat{Y}^x$. Then, using the independence of $\hat{X}$, $\hat{Y}$ and $\eta$, and the uniform distribution of $\eta$, we have:
\[
I_L(Q) = \mathbb{E} \left( \left[ \hat{Y}_Q^{\hat{X}^x} \right] - \left[ \hat{X}^x \right] \right) = \mathbb{E} \frac{1}{L} \int_0^L \left( \left[ \hat{Y}_Q^{\hat{X}^x \mod 1} \right] - \left[ \hat{X}^x \mod 1 \right] \right) ds = \mathbb{E} \frac{1}{L} \int_0^L G \left( \hat{X}_s \mod 1 \right) ds,
\]  
where
\[
G(x) := \mathbb{E} \left( \left[ \hat{Y}_Q^x \right] - \left[ x \right] \right).
\]  
Using the ergodicity of $X \mod 1$ (see Lemma 2),
\[
\mathbb{E} \frac{1}{L} \int_0^L G \left( \hat{X}_s \mod 1 \right) ds \to \int_0^1 G(x) \psi(x) dx.
\]
2.5 Marginal expected impact

First, we analyze the asymptotic behavior of \( \partial_Q I(Q, \theta) \) as \( Q \downarrow 0 \). Since the drift and volatility of \( \hat{Y}^x \) are bounded and continuous and the volatility is bounded away from zero, as \( Q \to 0 \), we have, uniformly over \( x \in [0, 1] \):

\[
\mathbb{E} \left( \left[ \hat{Y}^x_Q \right] - 1 \right) = \left( \mathbb{P} \left( x + \hat{\mu}_1(\theta, x)Q + \hat{\sigma}(\theta, x)\sqrt{Q} \hat{W}_1 \geq 1 \right) - \mathbb{P} \left( x + \hat{\mu}_1(\theta, x)Q + \hat{\sigma}(\theta, x)\sqrt{Q} \hat{W}_1 \leq 0 \right) \right) (1 + o(1)) = \left( \Phi \left( \frac{x - 1 + \hat{\mu}_1(\theta, x)Q}{\hat{\sigma}(\theta, x)\sqrt{Q}} \right) - \Phi \left( \frac{-x + \hat{\mu}_1(\theta)Q}{\hat{\sigma}(\theta, x)\sqrt{Q}} \right) \right) (1 + o(1)),
\]

where \( \Phi \) is the standard normal c.d.f.. Then, using the continuity of \( \psi \), the conditions \( \psi(0^+) = \psi(1^-) \) and \( \hat{\sigma}(\theta, x) = \hat{\sigma}(\theta, 1 - x) \), for \( x \in (0, 1) \), as well as the mean value theorem and the dominated convergence, we obtain, as \( Q \to 0 \):

\[
\int_0^1 \mathbb{E} \left( \left[ \hat{Y}^x_Q \right] - 1 \right) \psi(x) \, dx = \int_0^1 \left( \Phi \left( \frac{x - 1 + \hat{\mu}_1(\theta, x)Q}{\hat{\sigma}(\theta, x)\sqrt{Q}} \right) - \Phi \left( \frac{x - 1 - \hat{\mu}_1(\theta, 1 - x)Q}{\hat{\sigma}(\theta, x)\sqrt{Q}} \right) \right) \psi(1 - x) \, dx (1 + o(1))
\]

\[
= \sqrt{Q} \int_{-1/\sqrt{Q}}^0 \left[ \Phi \left( \frac{x}{\hat{\sigma}(\theta, 1 + x\sqrt{Q})\sqrt{Q}} + \frac{\hat{\mu}_1(\theta, 1 + x\sqrt{Q})Q}{\hat{\sigma}(\theta, 1 + x\sqrt{Q})} \right) \psi(1 + x\sqrt{Q}) \right. \\
\left. - \Phi \left( \frac{x}{\hat{\sigma}(\theta, 1 + x\sqrt{Q})} - \frac{\hat{\mu}_1(\theta, -x\sqrt{Q})Q}{\hat{\sigma}(\theta, 1 + x\sqrt{Q})} \right) \psi(-x\sqrt{Q}) \right] dx (1 + o(1))
\]

\[
= Q \psi(1^-) \left( \frac{\hat{\mu}_1(\theta, 1^-)}{\hat{\sigma}(\theta, 1^-)} + \frac{\hat{\mu}_1(\theta, 0^+)}{\hat{\sigma}(\theta, 1^-)} \right) \int_{-\infty}^0 \phi \left( \frac{x}{\hat{\sigma}(\theta, 1^-)} \right) dx (1 + o(1))
\]

\[
= Q \psi(1^-) \frac{\hat{\mu}_1(\theta, 1^-) + \hat{\mu}_1(\theta, 0^+)}{2} (1 + o(1)) = Q \alpha \psi(1^-) (1 + o(1)).
\]

Thus, we have proved the following proposition.

**Proposition 3.** For any \( \theta > 0 \),

\[
\partial_Q I(0, \theta) = \alpha \psi(1^-).
\]

Similarly, we can analyze \( \partial_Q I(Q, \theta) \) as \( Q \to \infty \). Notice that, due to the Markov property of \( \hat{Y} \) (see analogous argument for the Markov property of \( \hat{X} \) in the proof of Lemma 2) and the periodicity of the coefficients in (13), we have:

\[
I(Q + \Delta Q, \theta) - I(Q, \theta) = \lim_{L \to \infty} \left( I_L(Q + \Delta Q, \theta) - I_L(Q, \theta) \right)
\]

\[
= \lim_{L \to \infty} \mathbb{E} \left( \left[ \hat{Y}^x_{Q + \Delta Q} \right] - \left[ \hat{Y}^x_Q \right] \right) = \lim_{L \to \infty} \mathbb{E} \left( \left[ \hat{Y}^Z_{Q + \Delta Q} \right] - \left[ \hat{Y}^Z_Q \right] \right) = \lim_{L \to \infty} \mathbb{E} \left( \left[ \hat{Y}^Z_{Q + \Delta Q} \right] - 1 \right),
\]

where \( R^x \sim \hat{Y}^x \mod 1 \) and \( Z \sim R^x_Q \mod 1 \) are independent of \( (\hat{X}, \hat{Y}) \). Repeating the proof of Proposition 3, we obtain

\[
\lim_{L \to \infty} \mathbb{E} \left( \left[ \hat{Y}^Z_{Q + \Delta Q} \right] - 1 \right) = \int_0^1 \mathbb{E} \left( \left[ \hat{Y}^Z_{Q + \Delta Q} \right] - 1 \right) \psi(x) \, dx.
\]
Applying the scale transformation to $\hat{Y}^x$, to eliminate the drift, it is easy to see that the density of $\hat{Y}^x$, denoted $\chi^x_t$, can be written as

$$\chi^x_t(y) = \Gamma(t, x, y) P(y), \quad t > 0, x, y \in \mathbb{R},$$

where $\Gamma(\cdot, x, \cdot) \in C^{1+\varepsilon, 1+\varepsilon}$, with some $\varepsilon \in (0, 1)$, is the fundamental solution of the parabolic PDE associated with the transformed $\hat{Y}^x$, and $P$ is an exponentially bounded Lipschitz-continuous function whose derivative is continuous everywhere except integers, where it has first order discontinuities. A direct computation shows

$$\partial_t \chi^x_t - \frac{1}{2} \partial_y^2 (\sigma^2 \chi^x_t) + \partial_y (\mu_1 \chi^x_t) = 0,$$

(23)

where the equation holds globally in $(t, y) \in (0, \infty) \times \mathbb{R}$ in a weak sense and pointwise (with all derivatives being well defined) everywhere except $(0, \infty) \times \mathbb{Z}$, with the left and the right limits being well defined at every integer $y$. Then, applying the Gaussian estimates for $\Gamma$, it is easy to see that, for any $t > 0$, the distribution of $R^t \sim Y^x_{t \mod 1}$ has density

$$\chi_t^x(y) = \sum_{n \in \mathbb{Z}} \chi_t^x(y + n), \quad y \in [0, 1)$$

(a similar argument was used in the proof of Lemma 2). Due to periodicity of the coefficients, we deduce that $\chi^x$ satisfies (23) in the same sense as $\chi^x_t$.

Repeating the proof of Proposition 3 we obtain, as $\Delta Q \to 0$:

$$\mathbb{E} \left( [\hat{Y}^x_{\Delta Q}] - 1 \right) = \int_0^1 \mathbb{E} \left( [\hat{Y}^x_{\Delta Q}] - 1 \right) \chi^x_Q(y) dy = \Delta Q \alpha \chi^x_Q(1^-) (1 + o(1)),$$

for every $x \in (0, 1)$. Thus, using the dominated convergence theorem, we conclude:

$$\partial_Q I(Q, \theta) = \alpha \int_0^1 \chi^x_Q(1^-) \psi(x) dx.$$ (24)

The following proposition describes the asymptotic behavior of $\partial_Q I$ for large $Q$.

**Proposition 4.** For any $\gamma, \theta > 0$,

$$\lim_{Q \to \infty} \partial_Q I(Q, \theta) = \alpha \chi(1^-),$$

with $\chi$ defined in Lemma 2

**Proof:**

Using Ito’s formula, it is easy to see that $u(t, y) := \int_0^1 \chi^x(y) \psi(x) dx$ is a weak solution to

$$\partial_t u - \frac{1}{2} \partial_y^2 (\sigma^2 u) + \partial_y (\mu_1 u) = 0, \quad y \in (0, 1), \quad u(t, 0) = u(t, 1) = \int_0^1 \chi^x(1^-) \psi(x) dx, \quad u(0, y) = \psi(y),$$

(25)

with $u, \partial_y u \in L^2([0, T] \times [0, 1])$. Recalling that $\chi^x$ satisfies (23) along $(0, \infty) \times \{1^-\}$, we deduce that $v(t, y) := u(t, y) - \int_0^1 \chi^x(1^-) \psi(x) dx$ satisfies (25) with the same initial and with zero boundary conditions. Applying Theorem III.2.1 in [16] to $v$, we conclude that $\|\partial_y v(t, \cdot) = \partial_y u(t, \cdot)\|_{L^2}$ is bounded uniformly over $t \geq 0$. This, in turn, yields uniform continuity of the family $\{u(t, \cdot)\}_{t \geq 0}$. Since the weak limit of this family, as $t \to \infty$ is $\chi$ (see Lemma 2), we conclude that it is also a strong limit in the uniform norm. This, along with (24), completes the proof of the proposition. –
Thus, we have shown that the marginal expected impact on the midprice at the beginning of an execution sequence is proportional to the stationary distribution of \( X \mod 1 \) at \( 1^- \). Similarly, we have shown that the marginal expected impact at the end of a sufficiently long execution sequence is proportional to the stationary distribution of \( Y \mod 1 \) at \( 1^- \). In the next section, we show that, for small \( \theta > 0 \), the former exceeds the latter, which proves the asymptotic concavity of the expected impact curve.

### 2.6 Asymptotic concavity of price impact

In this section, we show that, for small \( \theta \), \( \psi(1^-) > \chi(1^-) \). First, we recall the ODEs (19) and (20) and, multiplying them by \( \theta \), we deduce the existence and uniqueness of function \( (\theta, x) \mapsto f(\theta, x) \), s.t. \( f(\theta, \cdot) \in C^{2+\epsilon}([0, 1]) \) and

\[
\frac{1}{2} \frac{d^2}{dx^2} f(\theta, x) - \frac{d}{dx} \left( (\mu_0(x) + \theta \bar{\mu}_1(x)) f(\theta, x) \right) = 0, \quad f(\theta, 1) = f(\theta, 0), \quad \int_0^1 \frac{f(\theta, x)}{\sigma^2(x)} dx = 1,
\]

where

\[
\sigma(y) := \sqrt{\theta} \sigma(y) = \frac{\sigma(y)}{\sqrt{\gamma(F(y - 1) + F(-y))}}, \quad \mu_0(y) = \alpha \gamma \frac{F(y - 1) - F(-y)}{\sigma^2(y)}, \quad \bar{\mu}_1(y) := 2 \alpha \gamma \frac{F(-y)}{\sigma^2(y)},
\]

and we used

\[
\beta^+(y) = 1 - y, \quad \beta^-(y) = -y, \quad y \in (0, 1).
\]

It is clear that \( \psi = f(0, \cdot)/\sigma^2 \) and \( \chi = f(\theta, \cdot)/\sigma^2 \). Our goal is to show that, for small enough \( \theta > 0 \), we have \( f(\theta, 1) < f(0, 1) \). The next proposition establishes the desired result, but under two additional technical assumptions.

**Assumption 1.** There exists a constant \( \rho \geq 1 \), s.t.

\[
\sigma(x) = \rho \sqrt{\gamma(F(x - 1) + F(-x)), \quad x \in (0, 1).}
\]

Note that all standing assumptions on \( \sigma \) are implied by the above assumption and the properties of \( F \).

**Assumption 2.** The function \( F \) is log-concave in \([-1, 0] \) and \( F' \) is nondecreasing in this range\(^\text{10}\).

**Proposition 5.** For any \( x \in [0, 1] \), there exists \( \partial_0 f(\cdot, x) \in C(\mathbb{R}) \). Moreover, under Assumptions 1 and 2, we have: \( \partial_0 f(0, 1) < 0 \).

**Proof:**

It is easy to see (e.g., using Feynman-Kac formula) that \( f(\theta, x) \) is continuously differentiable in \( \theta \). Then, we differentiate (26) and (27) w.r.t. \( \theta \) to obtain

\[
\frac{1}{2} g_{xx} - (\bar{\mu}_0 + \theta \bar{\mu}_1) g_x - (\bar{\mu}_0' + \theta \bar{\mu}_1') g = \partial_x (\bar{\mu}_1 f(0, \cdot)), \quad g(\theta, 0) = g(\theta, 1),
\]

\[
\int_0^1 \frac{g(\theta, x)}{\sigma^2(x)} dx = 0,
\]

\(^{10}\text{Note that the log-concavity of } F \text{ can be ensured by requiring that the density of the distribution defined by } F \text{ is log-concave.}\)
for $g(\theta, x) := \partial_\theta f(\theta, x)$.

Next, we consider the case $\theta = 0$. The PDE\;\cite{26} and the property $\partial_\theta f(0, 1/2) = \bar{\mu}_0(1/2) = 0$ (which follows from the fact that $f(0, \cdot)$ is symmetric around $x = 1/2$) yield:

$$\partial_\theta f(0, x) = 2\bar{\mu}_0(x)f(0, x), \quad x \in (0, 1).$$

Then,

$$\partial_\theta(\bar{\mu}_1 f(0, x)) = \bar{\mu}_1' f(0, x) + \bar{\mu}_1 \partial_\theta f(0, x) = (\bar{\mu}_1' + 2\bar{\mu}_1 \bar{\mu}_0) f(0, x)$$

$$= \frac{\gamma f(0, x)}{\sigma^4} \left(-2\alpha F''(-x)\sigma^2 - 2\alpha F(-x)\partial_x\sigma^2 + 4\alpha \gamma (F(x - 1) - F(-x))F(-x)\right)$$

$$= \frac{2\alpha \gamma^2 f(0, x)}{\sigma^4} \left(-\rho F'(-x)F(x - 1) - \rho F'(-x)F(-x) - \rho F(-x)F'(x - 1)ight.$$  

$$+ \rho F(-x)F'(-x) + 2F(x - 1)F(-x) - 2F(-x)F(-x))$$

$$\leq \frac{2\alpha \gamma^2 f(0, x)}{\sigma^4} \left(-F'(-x)F(x - 1) - F(-x)F'(x - 1) + 2F(x - 1)F(-x) - 2F(-x)F(-x)\right)$$

$$= \frac{2\alpha \gamma^2 f(0, x)}{\sigma^4} \left(-F'(-x)F(x - 1) + F(-x)F'(x - 1) - 2F(-x)F'(x - 1) + 2F(-x)\int_{-x}^{x-1} F'(z)dz\right).$$

The right hand side of the above is clearly non-positive for $x \in (0, 1/2)$. For $x \in (1/2, 1)$ its negativity is implied by the monotonicity of $F'$ on $\mathbb{R}_+$ and by the log-concavity of $F$:

$$(\log F)'(x - 1) < (\log F)'(-x),$$

$$F(-x)F'(x - 1) < F'(-x)F(x - 1).$$

Recall the ODE for $g(0, \cdot)$:

$$\frac{1}{2}g_{xx} - \bar{\mu}_0 g_x - \bar{\mu}_0' g = \partial_x(\bar{\mu}_1 f(0, \cdot)), \quad g(0, 0) = g(0, 1). \quad (30)$$

The rest of the proof follows from the maximum principle. Indeed, the ODE in\;\cite{26} and the conditions $\partial_\theta(\bar{\mu}_1 f(0, \cdot)) < 0, \bar{\mu}_0' > 0$ imply that $g(0, \cdot)$ cannot have a strictly negative minimum in $(0, 1)$ (otherwise, the ODE cannot be satisfied at the minimum point of $g(0, \cdot)$). Then, if $g(0, 1) \geq 0$, we conclude that $g(0, \cdot) \geq 0$, which contradicts\;\cite{26} (the case $g(0, \cdot) \equiv 0$ is easily excluded, since $\bar{\mu}_1 f(0, \cdot)$ cannot be constant). Thus, we conclude that $g(0, 1) < 0$ and complete the proof of the proposition. \blacksquare

Thus, we have proved the main mathematical result of this paper.

**Theorem 1.** Under Assumptions\;\cite{7} and\;\cite{2} there exists $\varepsilon > 0$, s.t.

$$\partial_Q I(0, \theta) > \lim_{Q \to \infty} \partial_Q I(Q, \theta),$$

for all $\theta \in (0, \varepsilon)$. 

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3 Empirical analysis

3.1 Numerical example

Assume, for simplicity, that $\rho = 1$ and that $F$ is the c.d.f. of a uniform distribution on $[-a, a]$, for some $a > 1$. Then, all assumptions made in Section 2 are satisfied and, for $x \in [0, 1]$,

$$F(x) = \frac{1}{2a}(x + a), \quad F(x - 1) + F(-x) = \frac{2a - 1}{2a}, \quad \sigma^2(x) = \frac{\gamma(2a - 1)}{2a}$$

$$\bar{\sigma}(x) = 1, \quad \bar{\mu}_0(x) = \frac{\alpha}{2a - 1}(2x - 1), \quad \bar{\mu}_1(x) := \frac{2\alpha}{2a - 1}(a - x).$$

The ODE (26) becomes

$$\frac{1}{2} \frac{\partial^2 f(\theta, x)}{\partial x^2} = \frac{\alpha}{2a - 1} \frac{\partial}{\partial x} \left((2x - 1 - \theta + 2\theta a - 1)f(\theta, x)\right) = 0.$$

To find the general solution of this ODE, we solve:

$$u_x - \frac{2\alpha}{2a - 1}(2x(1 - \theta) + 2\theta a - 1)u = C_1,$$

$$u(x) = g(x) \exp\left(\frac{\alpha}{2(2a - 1)(1 - \theta)}(2x(1 - \theta) + 2\theta a - 1)^2\right),$$

$$g'(x) = C_1 \exp\left(-\frac{\alpha}{2(2a - 1)(1 - \theta)}(2x(1 - \theta) + 2\theta a - 1)^2\right),$$

$$g(x) = C_1 \int_{-\infty}^{x} \exp\left(-\frac{2\alpha(1 - \theta)}{2a - 1} \left(y + \frac{2\theta a - 1}{2(1 - \theta)}\right)^2\right) dy + C_2,$$

$$u(x) = \exp\left(\frac{\alpha}{2(2a - 1)(1 - \theta)}(2x(1 - \theta) + 2\theta a - 1)^2\right) \left[C_3 \Phi\left(\frac{\alpha(1 - \theta)}{2a - 1}\left(x + \frac{2\theta a - 1}{2(1 - \theta)}\right)\right) + C_2\right].$$

The boundary conditions yield

$$\exp\left(\frac{\alpha}{2(2a - 1)(1 - \theta)}(2\theta a - 1)^2\right) \left[C_3 \Phi\left(\frac{\alpha(1 - \theta)}{2a - 1}\left(2\theta a - 1\right)\right) + C_2\right]$$

$$= \exp\left(\frac{\alpha}{2(2a - 1)(1 - \theta)}(2(1 - \theta) + 2\theta a - 1)^2\right) \left[C_3 \Phi\left(\frac{\alpha(1 - \theta)}{2a - 1}\left(1 + \frac{2\theta a - 1}{2(1 - \theta)}\right)\right) + C_2\right],$$

$$\frac{\exp\left(\frac{\alpha(1 - 2\theta + 2\theta a)^2}{2(2a - 1)(1 - \theta)}\right)}{\exp\left(\frac{\alpha(1 - 2\theta + 2\theta a)^2}{2(2a - 1)(1 - \theta)}\right)} \Phi\left(\frac{\alpha(1 - \theta)}{2a - 1}\left(1 + \frac{2\theta a - 1}{2(1 - \theta)}\right)\right) - \exp\left(\frac{\alpha(2\theta a - 1)^2}{2(2a - 1)(1 - \theta)}\right) \Phi\left(\frac{\alpha(1 - \theta)}{2a - 1}\left(2\theta a - 1\right)\right).$$

$$C_2 = C_3 \frac{\exp\left(\frac{\alpha(2\theta a - 1)^2}{2(2a - 1)(1 - \theta)}\right) - \exp\left(\frac{\alpha(1 - 2\theta + 2\theta a)^2}{2(2a - 1)(1 - \theta)}\right)}{\exp\left(\frac{\alpha(1 - 2\theta + 2\theta a)^2}{2(2a - 1)(1 - \theta)}\right) - \exp\left(\frac{\alpha(1 - 2\theta + 2\theta a)^2}{2(2a - 1)(1 - \theta)}\right)}.$$
Thus, we have
\[ f(\theta, x) = \frac{u(x)}{\int_0^1 u(y) dy}, \]

\[ u(x) = \exp \left( \frac{\alpha(2\theta(1-\theta) + 2\theta a - 1)^2}{2(2a-1)(1-\theta)} \right) \left[ \exp \left( \frac{\alpha(1-2\theta + 2\theta a)^2}{2(2a-1)(1-\theta)} \right) \Phi \left( \sqrt{\frac{\alpha(1-\theta)}{2a-1}} \frac{1-2\theta + 2\theta a}{1-\theta} \right) - \exp \left( \frac{\alpha(2\theta a - 1)^2}{2(2a-1)(1-\theta)} \right) \Phi \left( \sqrt{\frac{\alpha(1-\theta)}{2a-1}} \frac{2\theta a - 1}{1-\theta} \right) \right] \]
of the executor, \( \bar{Y} \), had entered into its stationary regime before the execution was over. Mathematically, the latter means that, after the execution, the fundamental price run on the business time of the market follows the process

\[
\bar{Y}_t = x + \int_0^t \hat{\mu}_1(\bar{Y}_u) \, du + \hat{W}_t,
\]

where \( \hat{W} \) is a Brownian motion, and

\[
\hat{\mu}_1(y) := \frac{\alpha}{2a - 1} (2(y \mod 1) - 1).
\]

Hence, the price resilience is defined as

\[
R(\bar{V}, \theta) := \int_0^1 \mathbb{E}[\bar{Y}_t] f(\theta, x) \, dx - 1,
\]

where \( \bar{V} \) represents the total traded volume in the market. The right part of Figure 2 shows \( R(\bar{V}, \theta) \) as a function of \( \bar{V} \). Note that, since \( \theta = 0.2 \), the range of values of \( \bar{V} \), in the right part of Figure 2, is chosen to match the range of values of \( Q \), in the left part: indeed, the execution of a meta-order of size \( Q \) via a VWAP strategy with participation rate \( \theta = 0.2 \) will terminate when the total traded volume becomes \( \bar{V} = Q/\theta = 5Q \).

It is clear from the right part of Figure 2, that the price resilience is convex and that the expected midprice does not decay to its initial level, which is consistent with the existing theoretical and empirical findings.

### 3.2 Testing the model predictions on market data

The heuristic argument described in Section 1 shows that the concavity of price impact can be derived from two predictions. The first prediction is that the stationary distribution of the fundamental price modulo tick size, run on the business time, is U-shaped. And the second prediction is that, during the execution of a meta-order, the fundamental price drifts in the direction of the order. Although one cannot test empirically the second assumption without having access to meta-orders, it does not seem that this assumption requires any additional justification beyond common sense. Thus, this subsection tests the first assumption, using publicly available data without any information about the meta-orders themselves.

The experiment presented here uses data from NASDAQ exchange obtained via ITCH protocol, which provides information about every event in the limit order book. An event may be an execution, an addition, or a cancelation of a limit order. The associated prices and volumes are either specified directly or can be recovered from prior events. Using this data, one can reverse engineer the trade volumes and the volumes at the first few levels of the limit order book, at the time of every event. The time interval we use covers November 3–7 of 2014 and includes the tickers CSCO, INTC, LBTYK, LVNTA, MSFT, VOD. We perform the analysis for each ticker separately.

Following the discussion presented in Section 1, we interpret the fundamental price modulo one, \( X \mod 1 \), as the microprice: i.e.,

\[
X_t \mod 1 \approx \frac{V^a_t}{V^b_t + V^e_t},
\]

\(^{11}\)The author thanks S. Jaimungal for providing this data. It is important to note, however, that the data currently posted on S. Jaimungal’s website is different from the one used herein and yields different results. The currently posted sample aggregates co-directed market orders of the same sign if they occur within one millisecond. Such pre-processing makes the sample more convenient for many statistical experiments, but it is not well suited for the present analysis, as it distorts the relationship between the volumes of limit and market orders.
where $V^b$ and $V^a$ are the volumes at the best bid and ask respectively. We split the interval $[0, 1]$ into 10 intervals $J_1, \ldots, J_{10}$ of length 0.1 and use the following approximation for the stationary density:

$$f(0, x) \approx \sum_{S_i} \mathbf{1}_{J_k} \left( \frac{V^a_{S_i}}{V^a_{S_i} + V^b_{S_i}} \right) \frac{|\Delta V_{S_i}|}{\sum_{S_i} |\Delta V_{S_i}|}, \quad x \in J_k, \quad k = 1, \ldots, 10,$$

(31)

where $\{S_i\}$ represent all the events in the limit order book, for the given ticker and the given time interval, for which the bid-ask spread, right before the event, does not exceed two ticks (this restriction is discussed further in this subsection), and $|\Delta V_{S_i}|$ denotes the size of the trade at the event $S_i$ (equal to zero if no trade occurred at this event). The volumes $V^b/a_{S_i}$ are recorded right before the event $S_i$. The motivation for such approximation of $f(0, \cdot)$ comes from the assumption of ergodicity of $\hat{X} \mod 1$ (see Lemma 2).

The estimated stationary density $f(0, \cdot)$ is presented in Figures 3–5, for each ticker in our sample. It is clear that the U-shape property holds for most of the tickers in the sample except LBTYK and LVNTA. There are two possible reasons why the U-shape prediction fails for the latter two tickers. The first, and most straightforward, explanation is the relatively small traded volume of the two stocks. Indeed, the first row of Table 1 shows the total traded volume, in the number of shares, of each ticker over the sample time period, and it indicates that the traded volumes of LBTYK and LVNTA are below the traded volumes of other tickers. However, the volume of LBTYK is very close to that of VOD, and the latter ticker has U-shaped stationary density. This observation motivates the need for a second explanation. As shown in the second row of Table 1, the fraction of order book events at which LBTYK and LVNTA have small (1 or 2 ticks) spreads is significantly lower than that of the other tickers. And it is clear that, for a small-tick stock, whose spread varies across many multiples of the tick size, the proposed model may not be a good description of the dynamics of the fundamental price. Indeed, as the spread size is known to be strongly mean-reverting, the drift and volatility of the fundamental price must, at the very least, depend on the value of the spread itself. In addition, the best bid and ask prices can no longer be assumed to be the roundings of the fundamental price, if the spread varies significantly. The failure of these assumptions is the reason why only the trades occurring at small spreads are included in the computation (31).

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|                | CSCO    | INTC    | LBTYEK | LVNTA   | MSFT    | VOD     |
|----------------|---------|---------|--------|---------|---------|---------|
| Traded volume  | 18,741,841 | 24,034,953 | 3,082,359 | 261,610  | 28,467,003 | 3,384,139 |
| % of events at small spreads | 99.99  | 99.93   | 75.28  | 45.72   | 99.92   | 87.22   |
Figure 1: Stationary density \( f(\theta, \cdot) \), for \( \theta = 0 \) (top left), \( \theta = 0.2 \) (top right), \( \theta = 0.4 \) (bottom left), and \( \theta = 0.6 \). Parameters used are: \( F(x) = 1_{[-a,a]}(x)/(2a) \), \( a = 1.2 \), \( \rho = 1 \), \( \alpha = 10 \).

Figure 2: Left: the expected impact on midprice \( I(Q, \theta) \) (measured in the number of ticks) as a function of executed volume \( Q \). Right: the price resilience \( R(\bar{V}, \theta) \) as a function of total traded volume \( \bar{V} \). Parameters used are: \( F(x) = 1_{[-a,a]}(x)/(2a) \), \( a = 1.2 \), \( \rho = 1 \), \( \alpha = 10 \), \( \theta = 0.2 \).
Figure 3: Estimated stationary density \( f(0, \cdot) \) for CSCO (left) and INTC (right).

Figure 4: Estimated stationary density \( f(0, \cdot) \) for MSFT (left) and VOD (right).

Figure 5: Estimated stationary density \( f(0, \cdot) \) for LBTYK (left) and LVNTA (right).