Recursive Estimation of Dynamic RSS Fields Based on Crowdsourcing and Gaussian Processes

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Abstract—In this paper, we address the estimation of a time-varying spatial field of received signal strength (RSS) by relying on measurements from randomly placed and not very accurate sensors. We employ a radio propagation model where the path loss exponent and the transmitted power are unknown with Gaussian priors whose hyper-parameters are estimated by applying the empirical Bayes method. We consider the locations of the sensors to be imperfectly known, which entails that they represent another source of error in the model. The propagation model includes shadowing which is considered to be a zero-mean Gaussian process where the correlation of attenuation between two spatial points is quantified by an exponential function of the distance between the points. The location of the transmitter is also unknown and estimated from the data with a weighted centroid approach. We propose to estimate time-varying RSS fields by a recursive Bayesian method and crowdsourcing. The method is based on Gaussian processes (GP), and it produces the joint distribution of the spatial field. Further, it summarizes all the acquired information by keeping the size of the needed memory bounded. We also present the Bayesian Cramér-Rao bound (BCRB) of the estimated parameters. Finally, we illustrate the performance of our method with experimental results on synthetic data sets.

Index Terms—sensor networks, Bayesian estimation, spectrum sensing, RSS, Gaussian processes for regression, time-varying fields, crowdsourcing, Cramér-Rao bound.

I. INTRODUCTION

SPECTRUM sensing has gained significant interest for research due to the rapid growth of wireless communication systems. Its main function is to map the distribution of radio frequency (RF) signals within a specific area, detecting intruders in a particular spectrum band and/or free channels that are not being used by any user [1], [2]. For this purpose, spectrum sensing relies on measurements from sensors. Current techniques for spectrum management require these measurements to be quite accurate, i.e., the techniques need an expensive infrastructure where sensors are sparsely and strategically deployed. These sensors provide precise measurements of received power and their positions are perfectly known. Due to the cost of this setting, approaches based on measurements obtained by expensive equipment do not scale well and cannot be extended to large areas. An alternative and appealing solution is to exploit crowdsourcing, where many sensors with low-quality acquire much less accurate measurements [3], [4]. For example, one can use not very accurate measurements obtained by a large number of smart phones, and yet can create a more accurate spectrum map than that obtained by a few sparsely located expensive sensors.

Most of the current spectrum monitoring techniques rely on received signal strength (RSS) measurements. Given the RSS values at some known locations, the goal of spectrum sensing is to estimate the field of RSS at any location within an area of interest. Spectral monitoring is not the only application where RSS measurements play a central role. Others include indoor localization [5], [6], tracking [7], distance estimation [8] and distributed asynchronous regression [9].

Methods that use RSS measurements for making inference are based on radio propagation path loss models. These models depend on different parameters including the locations of the sensors, the path loss exponent, the transmitted power, and shadowing. The more these parameters fit the reality, the more accurate the model is. Furthermore, due to the dynamic nature of the signal propagation, the learning of the model parameters that describe the propagation should allow for their adaptation.

There are two groups of papers where the propagation loss model is used. In one group, the authors consider the path loss exponent to be known and static [1], [10]–[13]. In the other, the exponent is unknown, is possibly dynamic, and is estimated [14]–[16]. Often, when indoor environments are studied, different values of the path loss exponent are assumed and estimated, e.g., with nonlinear least-squares techniques. The Bayesian methods allow for taking into account the estimation error of the exponent while estimating other more important unknowns or in making decisions.

Shadowing is another important notion in these studies. It has been commonly modeled by log-normal distributions. Usually, no spatial correlation due to shadowing fading has been assumed and i.i.d. log-normal distributions with the same variance have been used [10], [12], [14]–[16]. However, it is well-known that shadowing effects at different locations can change significantly due to varying propagation conditions. For this reason, the authors of [11] address the correlation between RSS measurements and propose to estimate RSS at specific locations as the value of nearby sensors. In [1], [2], [13], [17], [18], a full covariance matrix is introduced to model the spatially correlated shadowing. While in [13] this matrix is known, in [1], [2], [17], [18] a multivariate zero-mean Gaussian is used to approximate it. In [1], [2], [17], an
exponentially distributed coefficient of correlation is used.

Most of the approaches proposed in the literature require the transmitted power to be available at a central unit where all the measurements are processed [1], [2], [13], [15], [16], [18]. However, the transmitted power may be unavailable and may change with time [12]. In this situation, one can estimate the transmitted power [19], [20] or eliminate the dependence of the propagation model on it by computing RSS differences between sensors [19], [21], [22].

We point out that the transmitter and sensor locations also affect the radio propagation model. Approaches in the literature assume that these locations are perfectly known or, if estimated locations are available, they are considered as if they are true. To the best of our knowledge, there are no models that introduce errors in distances between transmitters and receivers. Further, one may have an increasing number of measurements at different locations at each time instant, and this can increase considerably the complexity and size of the system [23]. In our paper, we address measurements by sensors whose locations are only approximately known, and we consider a scenario with a large number of measurements that is due to crowdsourcing.

The variability in the RSS and/or the parameters that affect these measurements are rarely discussed. In [18], an adaptive kernel-based approach is proposed, but there is no discussion on the variations of RSS in practice. In [24], some results on RSS measurements are reported for a WLAN with time, while in [6] some variations over a few seconds and days are reported to emphasize the need for updating the fingerprint for localization purposes. In both cases, indoor WiFi signals are considered.

In this paper, we propose a GP-based approach, where the uncertainties in the positions of the sensors and the transmitter as well as the correlation due to shadowing effects are included in the model. We deal with an unknown path loss exponent and transmitted power by modeling them as Gaussian random variables whose hyper-parameters are learned according to the empirical Bayesian approach [25]. Preliminary results of this approach for static fields can be found in [26]. We cope with the temporal changes of the RSS measurements from some sensors by using a recursive technique where the solution is updated with the arrival of new data. This solution also allows for adaptation to accommodate changes in both propagation and movement of sensors, as showed in [27] with a synthetic dataset.

As in previous approaches [26], [27], we fix the locations where estimates of RSS are needed. These locations are represented as nodes of a predefined grid. The computational complexity is fixed and determined by the number of nodes. This differs from other approaches, e.g., in [23], where new available measurements are included sequentially in the system, thereby increasing its size and complexity. Furthermore, in our work we estimate the field at the nodes serve as priors for the field estimates at the next time instant when new measurements are acquired. In this way, we make the approach scalable. To account for the time-variations of the field, we include a forgetting factor in the method which determines the relevance of the previous and current data. We also find the BCRB of our estimates, where we rely on some recent results on bounds for GPs [28]. Finally, we demonstrate the performance of the method on synthetic datasets. We compare the performance with previous static approaches based on GPs [26] and interpolation techniques [4].

The paper is organized as follows. We introduce the notation and the model in Section II. In Section III, we explain how to estimate the location of the transmitter and how to find the Gaussian distribution of the path loss exponent and transmitted power. We develop the GP implementation for RSS estimation at static fields in Section IV. The BCRB is presented in Section V, and the method for estimating time-varying fields in Section VI. Finally, we provide simulation results in Section VII and concluding remarks in Section VIII.

## II. SYSTEM MODEL

In this section we propose a model for the measurements of the sensors. This model will be used to generate synthetic data and to propose an estimation algorithm. We focus on the estimation of RSS at \( M \) fixed grid nodes with perfect known locations given by \( \mathbf{X}_g = [x_{g1}, \ldots, x_{gM}]^\top \in \mathbb{R}^{M \times 2} \). We denote this estimated field at instant \( t \) as \( \mathbf{f}_g^t \in \mathbb{R}^M \). We will use the superscript \( [t] \) to indicate that the variable is a function of time. At time \( t \), we consider that \( N \) random low-cost sensors\(^1\) appear within the area at locations \( \mathbf{X}^t = [\mathbf{x}_1^t, \ldots, \mathbf{x}_N^t]^\top \in \mathbb{R}^{N \times 2} \). We denote the distance between sensors \( i \) and \( j \) at time \( t \) by \( d_{ij}^t \). The distance between them is computed from their locations \( \mathbf{x}_i \) and \( \mathbf{x}_j \) by

\[
d_{ij}^t = \sqrt{\left(\mathbf{x}_i^t - \mathbf{x}_j^t\right)^\top \left(\mathbf{x}_i^t - \mathbf{x}_j^t\right)}.
\]

Given a single transmitter with equivalent isotropically radiated power (EIRP) \( P^t \) (in dBm) at location \( \mathbf{x}_0 \in \mathbb{R}^{2 \times 1} \), the distances between the sensors and the grid nodes to the transmitter are denoted by \( \mathbf{d}^t = [d_1^t, \ldots, d_M^t]^\top \) and \( \mathbf{d}_g = [d_{g1}, \ldots, d_{gM}]^\top \), respectively, and they are defined analogously to (1). All the distances are measured in meters. The sensors also report their measurements of RSS (in dBm), and they are denoted by \( \mathbf{z}^t = [z_1^t, \ldots, z_N^t]^\top \). Assuming a log-normal path loss model, we express the RSS as

\[
\mathbf{z}^t = \mathbf{1}P^t - 10\alpha^t \log_{10}(\mathbf{d}^t) + \mathbf{v}^t + \mathbf{w}^t,
\]

where \( \mathbf{1} \) is an \( N \times 1 \) vector of 1’s, \( \alpha^t \) is the path loss exponent (in dB) at time \( t \), \( \mathbf{v}^t \) is an attenuation due to shadowing effects and modeled according to

\[
\mathbf{v}^t \sim \mathcal{N}\left(\mathbf{v}^t, 0, \Sigma_v^t\right),
\]

where the notation \( \mathcal{N}\left(\mathbf{v}^t, 0, \Sigma_v^t\right) \) signifies that \( \mathbf{v}^t \) has a Gaussian distribution with a mean vector \( \mathbf{0} \) and a covariance matrix \( \Sigma_v^t \). The covariance matrix \( \Sigma_v^t \) is comprised of elements given by

\[
\text{Cov}(v_i^t, v_j^t) = \sigma_v^2 \exp\left(-d_{ij}^t / D_{\text{corr}}\right),
\]

\(^1\)Note that we have removed the superscript \( t \) from \( N \) for easier reading.
where $D_{\text{corr}}$ is a parameter that models the correlation in the measurements, and $w^{[t]}$ is some unrelated additive noise,
\begin{equation}
    w^{[t]} \sim \mathcal{N} \left( w^{[t]} ; 0, \sigma_w^2 I_N \right).
\end{equation}

We reiterate that the sensors do not perfectly know their locations (or distances) and instead they only have the estimates of the distances. More specifically, based on the estimates of their positions, $\hat{X}^{[t]} = [\hat{x}_1^{[t]} \ldots \hat{x}_N^{[t]}]$, one can obtain the estimates of their distances from the transmitter, $\hat{d}^{[t]} = [\hat{d}_1^{[t]} \ldots \hat{d}_N^{[t]}]$ (see below). We model the estimated distance between the $i$th sensor and the transmitter according to
\begin{equation}
    \hat{d}_i^{[t]} = d_i^{[t]} + \epsilon_d \sim \mathcal{N} \left( d_i^{[t]}, d_i^{[t]}^2, \sigma_d^2 \right),
\end{equation}
where $\sigma_d^2$ is known. To reflect the uncertainty in $d_i^{[t]}$, we modify (2) to
\begin{equation}
    z^{[t]} = 1 P^{[t]} - \hat{q}^{[t]} \alpha^{[t]} + u^{[t]} + v^{[t]} + w^{[t]},
\end{equation}
where $\hat{q}^{[t]} = \left[ 10 \log_{10}(\hat{d}_1^{[t]}) \ldots 10 \log_{10}(\hat{d}_N^{[t]}) \right]^T$, and $u^{[t]}$ is the error that reflects the imprecisely known location of the sensors,
\begin{equation}
    u^{[t]} \sim \mathcal{N} \left( u^{[t]} ; 0, (\rho^2_0)^2 \hat{D}^{[t]} \right),
\end{equation}
with $\rho^2_0 = 10 \alpha^2 \sigma_{\log_{10}(\epsilon)}$ (see the Appendix; $\rho^2_0$ is expressed in $\text{dB}$) and $\hat{D}^{[t]} = \text{diag} \left\{ 1/d_1^{[t]} \ldots 1/d_N^{[t]} \right\}$.

In the rest of the paper, we assume that both $P^{[t]}$ and $\alpha^{[t]}$ are unknown, where the EIRP can change with time and the path loss exponent is constant. We consider that both variables are Gaussian distributed as explained in Section III. We also assume that the location of the transmitter, $\mathbf{x}_0$, is unknown and static. We let the remaining parameters to be known with values $\sigma_d = 13.2 \text{mD}$ or $\rho_u = 200 \text{mD}$ and $\sigma_w = \sqrt{7} \text{dB}$. Given $z^{[t]}$, $\hat{X}^{[t]}$, $\hat{d}^{[t]}$, the model in (7), and all the assumptions, we need to estimate the RSS, $f^{[t]}$, at the $M$ grid locations, $\mathbf{X}_g$.

Note that the model in (7) can be simplified by ignoring the estimation errors of the distances. In that case, we remove $u^{[t]}$ from the equation and work with
\begin{equation}
    z^{[t]} = 1 P^{[t]} - 10 \alpha^{[t]} \log_{10}(\hat{d}^{[t]}) + v^{[t]} + w^{[t]},
\end{equation}
In Subsection VII-B, we will compare the performances of the models based on (7) and (10), respectively.

### A. Errors of sensor locations

In Fig. 1, we plotted the measured power at distance $d_i$ from the transmitter as described by $-10 \alpha \log_{10} d_i$ for $\alpha = 3.5$ and distances from 0 to 250 m when the distance is known correctly (red solid line). The dotted blue curve shows the measurements according to the model when the incorrect distance has an error of $+10 \text{m}$, i.e., $\hat{d}_i = d_i + 10 \text{m}$. We also included two more curves, $-10 \alpha \log_{10} \hat{d}_i \pm \rho_u/\hat{d}_i$, where $\rho_u = 200$. It can be observed that the error of the model at small distances is quite high. The curves suggest how the incorrect estimate of the sensor locations would affect the estimates of the remaining unknowns by using the sensor measurements.

![Fig. 1: The measured powers according to the model in (7) when the distance is known correctly (solid red) or with errors of $+10 \text{m}$ (dotted blue) and with $\pm$ one standard deviation of the error (dashed black) when $\rho_u = 200 \text{mD}$ ($\sigma_d = 13.16 \text{m}$) and $\alpha = 3.5 \text{dB}$.

### III. ESTIMATION OF THE UNKNOWN PARAMETERS

In the model given by (7), we consider that the location of the transmitter, $\mathbf{x}_0$, the path loss exponent, $\alpha^{[t]}$, and the EIRP, $P^{[t]}$, are unknown. In the following, we explain how we find the estimates of these parameters.

#### A. Estimation of the transmitter location

We estimate the location of the transmitter from the measurements of RSS at the reporting sensors, $z^{[t]}$, by the weighted centroid approach [29] according to
\begin{equation}
    \hat{X}_0^{[t]} = \frac{1}{N} \sum_{i=1}^{N} w_i^{[t]} \hat{x}_i^{[t]},
\end{equation}
where $w_i^{[t]} = 10^{-z_i^{[t]}/10}$.

Thus,
\begin{equation}
    \hat{d}_i = |\hat{x}_i - \hat{x}_0|, \quad i = 1, 2, \ldots, N,
\end{equation}
\begin{equation}
    \hat{d}_i = |x_i - \hat{x}_0|, \quad i = 1, 2, \ldots, M.
\end{equation}

We note that the error in estimating $\hat{X}_0$ propagates in the estimates of the distances defined in (13) and (14).

#### B. Bayesian estimation of $\alpha^{[t]}$ and $P^{[t]}$

We assume that both variables, $\alpha^{[t]}$ and $P^{[t]}$, are independent and Gaussian distributed, i.e.,
\begin{equation}
    p(P^{[t]}|\theta_P) \sim \mathcal{N} \left( P^{[t]} ; \mu_P^{[t]}, \sigma_P^{2[t]} \right),
\end{equation}
\begin{equation}
    p(\alpha^{[t]}|\theta_\alpha) \sim \mathcal{N} \left( \alpha^{[t]} ; \mu_\alpha^{[t]}, \sigma_\alpha^{2[t]} \right),
\end{equation}
where the hyper-parameters $\theta^{[t]} = [\mu_\alpha^{[t]}, \sigma_\alpha^{[t]}, \mu_P^{[t]}, \sigma_P^{[t]}]$ have to be estimated from the current set of measurements, $z^{[t]}$. To ease the reading, we remove the superscript $[t]$ in the remaining of this section.
The joint posterior distribution of $\alpha$ and $P$ given a set of measurements, $z$, from sensors with given positions, $\hat{X}$, can be computed by using the Bayes’ rule, or

$$ p(\alpha, P | z, \hat{X}, \theta) = \frac{p(z | \alpha, P, \hat{X}) p(\alpha | \theta) p(P | \theta)}{p(z | \hat{X}, \theta)}, \quad (17) $$

where $p(z | \alpha, P, \hat{X})$ is a Gaussian distribution of $z$, $p(z | \alpha, P, \hat{X}) = N \left( z; \mu_z | \alpha, P, \Sigma_z | \alpha, P \right)$, with

$$ \mu_z | \alpha, P = 1P - \hat{q}_\alpha, $$

$$ \Sigma_z | \alpha, P = \sigma^2_\alpha \hat{D} + \Sigma_\alpha + \sigma^2_\theta I_N. \quad (18) $$

The marginalized distribution of the measurements, $p(z | \hat{X}, \theta)$, can be computed by

$$ p(z | \hat{X}, \theta) = \int p(z | \alpha, P, \hat{X}) p(\alpha | \theta) p(P | \theta) d\alpha dP, \quad (20) $$

which is also a Gaussian distribution [30], $p(z | \hat{X}, \theta) \sim N \left( z; \mu_z, \Sigma_z \right)$, with

$$ \mu_z = 1\mu_p - \hat{q}_\mu_z, $$

$$ \Sigma_z = \Sigma_z | \alpha, P + \sigma^2_\alpha \hat{Q} + \sigma^2_\theta J_N, \quad (21) $$

where $\hat{Q} = \hat{q}_\alpha \hat{q}_\alpha^T$ and $J_N$ is an $N \times N$ matrix with all elements equal to one.

We can obtain estimates of the hyper-parameters in (15) and (16) from the data by applying the empirical Bayes method [25], [26]. More specifically, we approximate $\mu_z$ with $z$ and solve the system of equations in (21) to obtain the estimated hyper-parameters of the mean as the following optimization problem:

$$ \begin{align*}
[\hat{\mu}_p] &= \min_{\mu_p \in \mu_\alpha} \left\| [1 - \hat{q}] [\mu_p \mu_\alpha] - z \right\|^2 \quad \text{subject to } \mu_\alpha \geq 2. \quad (23)
\end{align*} $$

We recall that the sensors close to the transmitter introduce higher errors in the estimation of $\alpha$ and $P$ than the sensors that are farther away, as already explained in Subsection II-A. For this reason, we used a weighting scheme to account for this in the sum of the squares in (23) as follows:

$$ \begin{align*}
[\hat{\mu}_p] &= \min_{\mu_p \in \mu_\alpha} \sum_{i=1}^N \left( \frac{\mu_p - q_i \mu_\alpha - z_i}{\sigma_\alpha d_i} \right)^2 \quad \text{subject to } \mu_\alpha \geq 2 \\
&= \min_{\mu_p \in \mu_\alpha} \left\| \sqrt{\hat{D}}^{-1} \left[ \mu_p \mu_\alpha \right] - \sqrt{\hat{D}} \hat{q} \right\|^2 \quad \text{subject to } \mu_\alpha \geq 2. \quad (24)
\end{align*} $$

Next, we approximate $\Sigma_z$ with $(z - \hat{\mu}_z)(z - \hat{\mu}_z)^T$, where $\hat{\mu}_z = 1\mu_p - \hat{q}_\mu_z$, and obtain the estimates of the hyper-parameters of the variance from (22) by solving

$$ \begin{align*}
[\hat{\sigma}^2_\alpha] &= \min_{\sigma^2_\alpha \in \sigma^2_\alpha} \left\| [1 b] [\sigma^2_\alpha \sigma^2_\theta] - a \right\|^2 \quad \text{subject to } \sigma^2_\alpha, \sigma^2_\theta \geq 0, \quad (25)
\end{align*} $$

where $a$ and $b$ are vectors formed by the diagonal elements of $A = (z - \hat{\mu}_z)(z - \hat{\mu}_z)^T - \Sigma_z | \alpha, P$ and $\hat{Q}$, respectively. Note that the estimation of the variances in (25) requires knowledge of $\Sigma_\alpha$, i.e., the values of the parameters $D_{corr}$ and $\sigma^2_\theta$. When these parameters are not available, the previous variances can be learned from the GP, as proposed in the next section.

IV. GP FOR REGRESSION IN STATIC FIELDS

In this section, we present an approach to estimate the spatial distribution of RSS based on a Gaussian process for regression (GPR). We have a set of RSS measurements $z[t]$, obtained at locations $\hat{X}[t]$, and we want to estimate the RSS levels $f[t]$ at a set of given nodes. Note that for now we do not use any information from previous time instants. An equivalent formulation of (7) is given by

$$ \begin{align*}
\begin{bmatrix} z[t] \end{bmatrix} &= \begin{bmatrix} f(\hat{X}[t]) + n[t] \end{bmatrix},
\end{bmatrix} \quad (26)
\end{align*} $$

where $n[t]$ is an additive Gaussian noise vector defined by

$$ \begin{align*}
n[t] &= u[t] + w[t] \\
&\sim N \left( n[t]; 0, \Sigma_n = \sigma^2_n I_N + \rho^2_s \hat{D} \right), \quad (27)
\end{align*} $$

and the function $f(\hat{X}[t]) = 1P[t] - q[t] \mu_\alpha[t] + v[t]$ is modeled as a GP, i.e.,

$$ \begin{align*}
\begin{bmatrix} \hat{X}[t] \end{bmatrix} &\sim \mathcal{GP} \left( \begin{bmatrix} f(\hat{X}[t]) ; m_X[t], K_X[t] \end{bmatrix} \right), \\
\end{bmatrix} \quad (28)
\end{align*} $$

whose mean and covariance functions are given by

$$ \begin{align*}
m_X[t] &= 1\mu_p[t] - \hat{q}[t] \mu_\alpha[t], \\
K_X[t] &= K(\hat{X}[t], \hat{X}[t]), \\
\end{bmatrix} \quad (29) $$

where $K(\hat{X}[t], \hat{X}[t])$ is an $N \times N$ matrix with elements $k(\hat{X}[t], \hat{X}[t])$ determined by a specific kernel, defined in Subsection IV-A, and $\sigma^2[\theta]$ are $[\hat{\mu}_p[t], \hat{\sigma}_p[t], \hat{\sigma}_\alpha[t], \hat{\hat{X}}[t]]$ are the estimated hyper-parameters of the GP computed as described in Section III. Note that we have not adopted zero-mean functions, as commonly done [31], because this assumption would violate the model from (7). As already discussed in Section III, the hyper-parameters $\sigma^2[\theta]$ and $D_{corr}$ can be computed if the values of the parameters $\sigma^2_n$ and $D_{corr}$ are known. If $\sigma^2_n$ and $D_{corr}$ are not known, we propose to estimate them as parameters of the kernel which will be learned from the GP (see Subsection IV-A).

Given a set of training points, $(z[t], \hat{X}[t])$, we want to estimate the spatial field, $f[t]$, at a set of test points, $X_g$, whose prior is distributed according to

$$ \begin{align*}
\begin{bmatrix} p(\hat{f}[t] | \theta[t]) \end{bmatrix} &\sim \mathcal{N} \left( \begin{bmatrix} f[t] ; m_{X_g}[t], K_{X_g}[t] \end{bmatrix} \right), \\
\end{bmatrix} \quad (31)
\end{align*} $$

where

$$ \begin{align*}
m_{X_g}[t] &= 1\hat{\mu}_p[t] - q_\theta \hat{q}_\mu[t], \\
K_{X_g}[t] &= K(\hat{X}_g, \hat{X}_g), \\
q_\theta &= [10 \log_{10}(\hat{d}_g_1) \ldots 10 \log_{10}(\hat{d}_g_u)]^T. \quad (33)
\end{align*} $$

We note that in the last equation we have a hat symbol above $d_g$, because even though we know the exact locations of the nodes where we estimate the RSS, we do not know the exact location of the transmitter.

The joint distribution of the training outputs, $z[t]$, and the test outputs, $f[t]$, that fits the model and priors above is

$$ \begin{align*}
\begin{bmatrix} z[t] \end{bmatrix} &\sim \mathcal{N} \left( \begin{bmatrix} z[t] ; m_X[t], m_{X_g}[t] \end{bmatrix} ; \\
K_X[t] + \Sigma_n[t] &K_{X_g}[t] \end{bmatrix} \right), \quad (35)
\end{align*} $$

where

$$ \begin{align*}
\begin{bmatrix} f[t] \end{bmatrix} &\sim \mathcal{N} \left( \begin{bmatrix} f[t] ; m_X[t], m_{X_g}[t] \end{bmatrix} ; \\
K_X[t] &K_{X_g}[t] \end{bmatrix} \right).
\end{align*} $$


The prediction of the RSS at the desired nodes is presented by their posterior distribution, \( f_g^{[t]} \). This distribution is obtained by conditioning on the observations, \( z^{[t]} \), estimated locations of the sensors \( \hat{X}^{[t]} \), and the estimated hyper-parameters, \( \hat{\theta}^{[t]} \), i.e.,
\[
p\left(f^{[t]}_g | X_g, \hat{X}^{[t]}, z^{[t]}, \hat{\theta}^{[t]}\right) \sim \mathcal{N}\left(f^{[t]}_g | \mu_{g}^{[t]}, \Sigma_{g}^{[t]}\right),
\]
where
\[
\mu_{g}^{[t]} = m_{X}^{[t]} + K_{X}^{[t]}(C_{\hat{X}}^{[t]})^{-1}(z^{[t]} - m_{\hat{X}}^{[t]}),
\]
\[
\Sigma_{g}^{[t]} = K_{X}^{[t]} - K_{X}^{[t]}(C_{\hat{X}}^{[t]})^{-1}K_{X}^{[t] \top},
\]
\[
C_{\hat{X}}^{[t]} = K_{X}^{[t]} + \Sigma_{g}^{[t]}.
\]
We refer to this solution as static GP-based approach (sGP), and it is summarized in Algorithm 1.

**Algorithm 1** Static GP-based approach (sGP) formulation

Given a specific time instant \( t \) and the RSS measured by sensors \( \{z^{[t]}\} \) at positions \( \{X^{[t]}\} \),

1. Obtain the estimates of the hyper-parameters,
\[
\hat{\theta}^{[t]} = \left[ \hat{\mu}_{a}^{[t]}, \hat{\sigma}_{a}^{2}, \hat{\mu}_{P}^{[t]}, \hat{\sigma}_{P}^{2}, \hat{\tilde{z}}_{g}^{[t]} \right],
\]
by solving (23) and (25).

2. Compute the posterior distribution at the grid nodes
\[
p\left(f^{[t]}_d | X_d, \hat{X}^{[t]}, z^{[t]}, \hat{\theta}^{[t]}\right) \text{ in (36).}
\]

\[\text{A. The kernel and its hyper-parameters}\]

In implementing the GPR, we need to select the kernel of the GPR which measures the similarity between points and is used to predict values of RSS at the nodes of interest from the measured RSS. We chose to work with a combination of different kernels given by
\[
k(x_i, x_j) = \sigma_k^2 \exp\left(-\frac{\sqrt{(x_i - x_j)^\top(x_i - x_j)} + \sigma_k^2 q(x_i)q(x_j) + \sigma_k^2}{2l^2}\right),
\]
where \( q(x_i) = 10 \log_{10} \sqrt{(x_i - \hat{X}_0)(x_i - \hat{X}_0)^\top} \). The parameters \( \sigma_k, l, \sigma_a, \) and \( \sigma_P \) are obtained by the GP by minimizing the log marginal likelihood.

**V. BAYESIAN CRAMÉR-RAO BOUND**

In this section we focus on a particular node of the grid, say \( \hat{X}_g^{[t]} \). The GP provides us with a Bayesian estimate of the true RSS at this grid node, \( \hat{f}_g^{[t]} \),
\[
\hat{f}_g^{[t]} \sim \mathcal{N}\left(\hat{f}_g^{[t]} | \mu_{g}^{[t]}, \sigma_{g}^{2[t]}\right),
\]
i.e., our GP approach obtains the mean, \( \mu_{g}^{[t]} \), by (37) and the variance \( \sigma_{g}^{2[t]} \) by (38). We want to find the lower bound of the mean squared-error (MSE) of \( \hat{f}_g^{[t]} \), \( \mathbb{E}_{\hat{f}, f_g}[(f_g^{[t]} - \hat{f}_g^{[t]})^2] \).

We obtain this bound by computing the BCRB [32] according to [28]
\[
\mathbb{E}_{\hat{f}_g, f_g}[\left((f_g^{[t]} - \hat{f}_g^{[t]})^2\right)] \geq \mathbb{E}_{\hat{f}_g, f_g}\left(\frac{\partial}{\partial \hat{f}_g} \ln p(z^{[t]}, f_g^{[t]})\right)^2 \Sigma_{g}^{[t]}^{-1} = \sigma_{g}^{2[t]},
\]
We note that apart from the estimated variable, \( f_g^{[t]} \), we have three more parameters, \( \alpha^{[t]}, P^{[t]} \) and \( x_0 \), that introduce errors in the estimation. In [28] the authors develop a hybrid Cramér-Rao bound (HCRB) for GPR with deterministic hyper-parameters. The authors conclude that a term must be added to the variance of the GPR in (42), as shown below. This term is a function of the derivative with respect to the hyper-parameters of the mean of the GPR. The result is obtained under the assumption that the estimates of the hyper-parameters are unbiased, or at least asymptotically unbiased.

We can cast our solution as a GPR where the deterministic hyper-parameters of the mean are \( \mu_{P,\alpha,x_0}^{[t]} = [\hat{\mu}_P^{[t]}, \hat{\mu}_\alpha^{[t]}, \hat{\tilde{X}}_0^{[t]}] \). Hence we may apply, in a straightforward way, the result in [28] to develop a HCRB where the hyper-parameters \( \hat{\mu}_P^{[t]}, \hat{\mu}_\alpha^{[t]} \) and \( \hat{X}_0 \) are considered as deterministic. The expression for the HCRB is
\[
\mathbb{E}_{\hat{f}_g, f_g}[\left((f_g^{[t]} - \hat{f}_g^{[t]})^2\right)] \geq \sigma_{g}^{2[t]} + g_{g, x_0}^{[t]} M_{g, x_0}^{-1} g_{g, x_0}^{[t]},
\]
where
\[
g_{g, x_0}^{[t]} = \frac{\partial}{\partial \mu_{P,\alpha,x_0}} \left(m_{x_0}^{[t]} - m_{X}^{[t]}(C_{\tilde{X}}^{[t]})^{-1}k_{g}^{[t,\tilde{X}}_{x_0,\tilde{X}}\right),
\]
\[
M_{g, x_0}^{[t]} = \frac{\partial m_{x_0}^{[t] \top}}{\partial \mu_{P,\alpha,x_0}} \left(C_{\tilde{X}}^{[t]}\right)^{-1} \frac{\partial m_{\tilde{X}}^{[t] \top}}{\partial \mu_{P,\alpha,x_0}}.
\]
We point out that the derivatives are with respect to the hyper-parameters \( \mu_{\alpha}^{[t]} \) and \( \mu_P^{[t]} \) and not with respect to \( \alpha^{[t]}, P^{[t]} \). We can show that
\[
g_{g, x_0}^{[t]} = \left[\begin{array}{c}
-10 \log_{10}(d_{g_0}) \\
\frac{c}{\sigma_{g_0}}(\tilde{X}_0 - x_{g_0})
\end{array}\right]
\]
\[
- \left[1 - \tilde{q}_g^{[t]} A\right]^\top \left(C_{\tilde{X}}^{[t]}\right)^{-1}k_{g}^{[t,\tilde{X}}_{x_0,\tilde{X}} + \left[\begin{array}{c}
0 \\
0
\end{array}\right] + \frac{m_{g}^{[t] \top} (C_{\tilde{X}}^{[t]})^{-1} A_1 \left(C_{\tilde{X}}^{[t]}\right)^{-1} k_{g}^{[t,\tilde{X}}_{x_0,\tilde{X}} \\
\frac{m_{g}^{[t] \top} (C_{\tilde{X}}^{[t]})^{-1} A_2 \left(C_{\tilde{X}}^{[t]}\right)^{-1} k_{g}^{[t,\tilde{X}}_{x_0,\tilde{X}}}
\right],
\]
\[
M_{g, x_0}^{[t]} = \left[1 - \tilde{q}_g^{[t]} A\right]^\top C_{\tilde{X}}^{[t]}^{-1} \left[1 - \tilde{q}_g^{[t]} A\right],
\]
where
\[
c = -10 \log_{10}(c^{[t]}),
\]
\[
A = c \hat{D}(J_{N \times 2} \text{diag}(\tilde{X}_0) - \hat{X}) \in \mathbb{R}^{N \times 2},
\]
\[
A_1 = -2 \sigma_{g_0}^2 \text{diag}(\hat{D}^2(1\tilde{X}_0(1) - \tilde{X}(1, 1))) \in \mathbb{R}^{N \times N},
\]
\[
A_2 = -2 \sigma_{g_0}^2 \text{diag}(\hat{D}^2(1\tilde{X}_0(2) - \tilde{X}(1, 2))) \in \mathbb{R}^{N \times N}.
\]
and $\hat{x}_0(j)$ is the $j$th element of the vector $\hat{x}_0$ and $\hat{X}(:,j)$ represents the $j$th column of the matrix $\hat{X}$.

At this point, we emphasize that our GPR has hyper-parameters, $p(t)$ and $P(t)$, with priors that depend on their own hyper-parameters, $\theta(t) = [\mu_0^p, \sigma_0^p, \mu_0^P, \sigma_0^P, \hat{x}_0]$. We are using the results in [28] for the latter hyper-parameters. Therefore, we are somehow assuming that the variance of the GPR already includes the uncertainties of $\alpha(t)$ and $P(t)$ by averaging in a Bayesian way. In turn, the HCRB provides the overall effect of $\mu_0^p$ and $\mu_0^P$ on the MSE of the RSS.

VI. GAUSSIAN PROCESSES FOR TIME-VARYING FIELDS

We also consider the possibility that the RSS field may vary with time. For example, the EIRP, the orientation of the antennas, the objects around the sensors, among others, may change with time, and they may affect the RSS at the locations of the sensors. Also, some sensors may become unavailable, or they can move and change their positions. One possibility to address this is to simply use the approach from the previous section only on the current data and ignore the rest. An alternative is to include all the past information but enforcing reduced influence of older data on the estimates.

As explained in Section II, the main interest here is not finding how the field evolves at every position with time, which is already dealt by other approaches [23]. Instead, we just want to estimate the RSS at the grid positions, i.e., at a set of predefined locations. We propose to update the estimates at these locations with information from the new observations. We compute the posterior distribution at the grid nodes, which is then used as a prior of the RSS for the next instant time. In order to emphasize the information in the more recent samples and “forget” the information from old ones, we inject a forgetting factor, $0 < \lambda \leq 1$. As a result, our solution is a linear combination of past and current information weighted by $\lambda$ and $1 - \lambda$, i.e.,

\[
p(\hat{f}_g^t | X_g, \hat{X}^{1:t}, z^{1:t}) \sim \mathcal{N}(\mu_g^t, \Sigma_g^t),
\]

where

\[
\mu_g^t = m_{X_g}^t + (1 - \lambda) \mu_{prior}^t + \lambda \mu_{post}^t,
\]

\[
\Sigma_g^t = K_{X_g}^t \left( K_{X_g}^t + \Sigma_{n}^t \right)^{-1} \left( z^t - m_{X_g}^t \right),
\]

and

\[
\mu_{post}^t = K_{X_g}^t X \left( K_X^t + \Sigma_{n}^t \right)^{-1} \left( z^t - m_X^t \right),
\]

\[
\Sigma_{post}^t = K_{X_g}^t X \left( K_X^t + \Sigma_{n}^t \right)^{-1} K_{X_g}^t X^\top,
\]

\[
\mu_{prior}^t = \mu_{g}^{t-1} - m_{X_g}^{t-1},
\]

\[
\Sigma_{prior}^t = K_{X_g}^t \left( t-1 \right) - \Sigma_g^{t-1}.
\]

Due to the dynamic nature of the scenario, a new estimation of $\alpha$ and $P$ is computed at every time instant following (23). Note that we are ensuring that the covariance matrix in (54) remains positive definite due to the linear combination weighted by $\lambda$ and $1 - \lambda$. Our approach is summarized in Algorithm 2, which we refer to as recursive Gaussian process (rGP)-based algorithm.

Algorithm 2 Recursive GP (rGP) formulation

Initialization. Compute the posterior probability at the grid nodes for $t = 0$ by executing Algorithm 1 and set $\mu_{prior}^0 = \mu_g^0, \Sigma_{prior}^0 = \Sigma_g^0$.

for $t = 1, \ldots, T$ do

Compute the posterior distribution at the grid nodes at time instant $t$, i.e., $p(\hat{f}_g^t | X_g, \hat{X}^{1:t}, z^{1:t})$ as in (52).

end for

Note that when $\lambda = 1$, we forget all the prior knowledge about the previous data and compute the mean and covariance matrix of the GP with just the current data, yielding the approach developed in Section IV [26].

VII. EXPERIMENTAL RESULTS

In this section we include results that illustrate the performance of the proposed approach. First, we explain the setup.

A. Simulated scenario

The setting is similar to the one used in [26]. We simulated an area of 500 m $\times$ 500 m where a single transmitter is placed at the center of the area. We considered a fixed and uniform grid with $M = 1088$ nodes, where we want to estimate the RSS. We randomly placed $N = 218$ sensors within the area, and their RSS measurements were generated according to (7) with $d_{0}^{[t]}$ replaced by $d_{0}^{[t]}$ and with $u_{t}$ removed, $\alpha_{t} = 3.5$ and $P_{t} = -10$ dBm. The rest of the parameters were set to $\sigma_{w} = \sqrt{7}$ dB, $\sigma_{v} = \sqrt{10}$ dB, $\rho_{w} = 200$ mdB, and $D_{cor}$ = 50 m. We assumed that the values of the parameters $\sigma_{v}$ and $D_{cor}$ are not available and that they would be estimated from the GP.

One example of this scenario is depicted in Fig. 2, where we represent the grid nodes with red squares, the transmitter with a green triangle and the sensors with circles. The different values of RSS at the sensors are represented with a scale of colors where yellow means the highest and blue the lowest values.

Fig. 2: A graphical representation of the space of interest with the locations of the transmitter, nodes, and sensors.
To quantify the error of the performance, we used the mean squared error (MSE). The metric was computed for each time instant as

$$\text{MSE}^{[t]} = \frac{1}{M} \sum_i \left( f_y^{[t]} - f_y^{[t]} \right)^2, \quad \forall i = 1 \cdots M, \quad (59)$$

where $f_y^t$ is the true value of the RSS at the $i$th node and $\hat{f}_y^t$ is its estimate.

### B. Error in the location

In this subsection we show the performance of our GP algorithm for static fields considering the three different cases explained in Section II, i.e.,

1) Case 1: Using the true sensor locations and the model in (2).
2) Case 2: Using the sensor locations with errors and accounting for the errors as in (7).
3) Case 3: Using the sensor locations with errors as if they were accurate and following (10).

In Fig. 3 we show the MSEs for these three cases. Clearly, when there is no error in the location (Case 1), we obtain the lowest error. On the other hand, when considering locations with errors (Cases 2 and 3), the performance is just slightly deteriorated in comparison to Case 1. Specifically, our proposal of introducing one additional source of error to model the error of the user location (Case 2) improves the traditional approach of ignoring this source of error (Case 3).

### C. Static GP

In this subsection we analyze the performance of the sGP approach for static fields. One example of this scenario is depicted in Fig. 2.

We first estimated the path loss exponent and the transmitter power following (23) and (25). To show the robustness of our method for estimating the mean of the path loss exponent and transmitted power, we introduced a high error in the location of the closest sensor to the transmitter. The obtained values of the means are shown in Table I, and they are close to the true values.

| $\hat{\mu}_\alpha$ (dB) | $\hat{\mu}_P$ (dBm) |
|------------------------|---------------------|
| 3.3                    | -14.57              |

**TABLE I:** Estimates of the path loss exponent and transmitted power.

We also include Fig. 4 where we compare the estimated and true RSS along different distances from the transmitter. The figure also shows the noisy measurements of the RSS of the sensors (plotted with circles).

The results show that the estimated values (red dashed line) are approximately the same as the true ones (black solid line), and in agreement with the results from Table I. Note that the closest measurement to the transmitter (whose RSS is around -50 dBm) did not affect negatively the estimates of $\alpha$ and $P$ because we of the precaution we took with (24).

The results of the RSS field estimates are shown in Fig. 5. We represent the RSS measurements of the sensors with red circles and the estimated mean of the posterior distribution of the GP over the coverage area with blue solid surface. The graph shows that the GP smoothly approximates the RSS field.

![RSS Field Estimate](image)

**Fig. 5:** Mean of the estimation (solid) and observations ($\circ$).

Finally, in Table II we present the true and estimated values...
of the hyperparameters of the kernel.

| $\sigma_k$ | l |
|----------|---|
| True value | 3.16 | 5 |
| Estimated value | 3.42 | 5.88 |

TABLE II: True and estimated hyperparameters of the kernel.

D. Recursive GP

Here we present some results with the recursive GP. In Fig. 6, we display results from two different scenarios: 1) an intermittent setting where 20% of the sensors are unavailable at each time instant and 2) a setting where the sensors are moving from their previous positions. The figure provides plots of MSEs for different shadowing variances and at two time instants ($t = 1$ and $t = 10$). The results were averaged over 100 different experiments and compared to the ordinary Kriging with detrending (OKD) technique [4], [33], applied at $t = 1$. As expected, the recursive approach gets better estimates with time. Also as expected, the error of the rGP method with intermittent data is slightly higher than with data of moving sensors. Note that when we use intermittent data at $t = 1$, the performance is not as good as when we use data from moving sensors because there are less available measurements.

![Fig. 6: Mean square error for $\rho_u = 200$ dBm and for $t = 1$ (solid) and $t = 10$ (dotted).](image)

Finally, in Fig. 7, we illustrate the time variability of the RSS at a specific location when dynamic transmitter powers are considered. We changed the value of $P^{[t]}$ two times during the observation interval so that the RSS varied as shown by the green dashed line. The blue solid line represents the mean of the estimated posterior obtained from the GP following (53), and the gray shadowed error bars are bands around the mean whose widths are equal to three times the square root of HCRB deviation in (43). The applied value of $\lambda$ was 0.5.

![Fig. 7: Mean of the estimation (blue solid) and $\pm 3 \times \sqrt{\text{HCRB}}$ (filled), given by (43), and true RSS (green dashed) along time.](image)

**APPENDIX**

**MODEL OF ERROR DUE TO LOCATION ESTIMATES**

We start by taking the logarithm of (6)

$$
\log_{10}\left(\hat{d}_i^{[t]}\right) = \log_{10}\left(d_i^{[t]} + \epsilon_d\right) = \log_{10}\left(d_i^{[t]}\left(1 + \frac{\epsilon_d}{d_i^{[t]}}\right)\right)
$$

$$
= \log_{10}\left(d_i^{[t]}\right) + \log_{10}\left(1 + \frac{\epsilon_d}{d_i^{[t]}}, \right) \quad (60)
$$

Let $x = \epsilon_d/d_i^{[t]}$, which is a variable that takes small values. Then we expand $f(x) = \log_{10}(1 + x)$ by Taylor expansion around $x = 0$ and get

$$
f(x) = \log_{10}(1 + x)|_{x=0} + \frac{\log_{10}(e)}{1 + x}|_{x=0}x + R
$$

$$
= \log_{10}(e)x + R = \frac{\log_{10}(e)\epsilon_d}{d_i^{[t]}} + R, \quad (61)
$$

where $R$ is the residual. Thus,

$$
\log_{10}\left(\hat{d}_i^{[t]}\right) \approx \log_{10}\left(d_i^{[t]}\right) + \frac{\log_{10}(e)\epsilon_d}{d_i^{[t]}}, \quad (62)
$$

**VIII. CONCLUSIONS**

In this paper, we use crowdsourcing to solve the problem of RSS estimation of a possibly time-varying field. We rely on measurements provided by different and inexpensive sensors that are randomly placed within an area of interest. We developed a Bayesian framework based on GP and a model with several unknown parameters. The unknown path loss exponent and transmitted power were modeled as Gaussian variables. The hyper-parameters of their Gaussian distributions were estimated from the data. We also assumed that the user locations were not perfectly known, which introduced an additional source of error in the model. Further, the location of the transmitter was unknown too, and it was estimated from the data, which had its own error. We also addressed the problem where the field may vary with time. In our solution we used a forgetting factor which determines the relevance of previous and current data. In all our solutions, the needed memory of the algorithm is fixed and a function of the number of nodes, and it is independent of the number of sensors. Finally, we derived the HCRB of the estimated parameters, and we showed the performance of our approach with experimental results on synthetic datasets.
Replacing (63) in (2), we obtain that
\[ u_\rho \sim \mathcal{N}(\mu_\rho; 0, \sigma^2_\rho). \]  
If we assume that \( d_{i[t]} \gg \epsilon_d \) and \( \epsilon_d \sim \mathcal{N}(\epsilon_d; 0, \sigma^2_d) \), we can write
\[ u_\epsilon \sim \mathcal{N}(u_\epsilon; 0, \frac{(10\alpha \epsilon_{10}(\epsilon_{10})^2}{d_{i[t]}^2} \sigma^2_d). \]
Thus, for the standard deviation of \( u_\epsilon \), we have
\[ \sigma_{\epsilon[t]} = \rho_{\epsilon[t]}/d_{i[t]}, \]
where
\[ \rho_{\epsilon[t]} = 10\alpha \epsilon_{10}(\epsilon_{10})^2 \epsilon_{10}(\epsilon_{10}), \]
which is what we have in (9).

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