We show that for all very special quaternionic manifolds a different $N = 1$ reduction exists, defining a Kähler Geometry which is “dual” to the original very special Kähler geometry with metric $G_{ab} = -\partial_a \partial_b \ln V$ ($V = \frac{1}{6} d_{abc} \lambda^a \lambda^b \lambda^c$). The dual metric $g^{ab} = V^{-2}(G^{-1})^{ab}$ is Kähler and it also defines a flat potential as the original metric. Such geometries and some of their extensions find applications in Type IIB compactifications on Calabi–Yau orientifolds.
1 Isometries of dual quaternionic manifolds

One of the basic constructions in dealing with the low energy effective Lagrangians of Type IIA and Type IIB superstrings is the so called $c$–map [1], which associates to any Special Kähler manifold of complex dimension $n$ a “dual” quaternionic manifold of quaternionic dimension $n_H = n + 1$. In particular it was shown [2] that “dual” quaternionic manifolds always have at least $2n + 4$ isometries: one scale isometry $\epsilon_0$ and $2n + 3$ shift isometries $\beta_I, \alpha_I, \epsilon_+$ ($I = 0, \cdots, n$), whose generators close a Heisenberg algebra [3]:

$$[\beta^I, \epsilon^+] = [\alpha_I, \epsilon^+] = 0; \quad [\beta^I, \alpha_J] = \delta^I_J \epsilon^+; \quad [\epsilon^0, \alpha_I] = \frac{1}{2} \alpha_I; \quad [\epsilon^0, \beta^I] = \frac{1}{2} \beta^I; \quad [\epsilon^0, \epsilon^+] = \epsilon^+$$

The corresponding generators can be written according to their $\epsilon^0$ weight as [4, 5, 6, 7]:

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} + \mathcal{V}_1. \quad (1.2)$$

However it was shown in [6, 7] that when the Special Kähler manifold has some isometries, then some “hidden symmetries” are generated in the $c$–map spaces which are classified by $\mathcal{V}_{-1}, \mathcal{V}_{-\frac{1}{2}}$, with

$$\dim(\mathcal{V}_{-1}) \leq 1; \quad \dim(\mathcal{V}_{-\frac{1}{2}}) \leq 2n + 2. \quad (1.3)$$

In particular, for a generic very special geometry, with a cubic polynomial prepotential

$$F(z) = \frac{1}{48} d_{abc} z^a z^b z^c \quad (1.4)$$

with generic $d_{abc}$, with no additional isometries, it was shown that:

$$\dim(\mathcal{V}_{-1}) = 0; \quad \dim(\mathcal{V}_{-\frac{1}{2}}) = 1; \quad \dim(\mathcal{V}_0) = n + 2. \quad (1.5)$$

Since the isometries of a generic very Special Geometry of dimension $n$ are $n + 1$, the dual manifold has then $3n + 6$ isometries, where the $n + 2$ additional isometries lie, $n + 1$ in $\mathcal{V}_0$, denoted by $\omega_I, (I = 0, \cdots, n)$, and one $\hat{\beta}_0$ in $\mathcal{V}_{-\frac{1}{2}}$. For symmetric spaces the upper bound in equation (1.3) is saturated so that $\dim G_Q = \dim G_{SK} + 4n + 7$ where $G_{SK}$ and $G_Q$ are the isometry groups of the Special Kähler and Quaternionic spaces respectively.

2 The very Special $\sigma$–model Lagrangian and its $N = 1$ reduction

The quaternionic “dual” $\sigma$–model for a generic Special Geometry was derived in [2] by dimensional reduction of a $N = 2$ Special Geometry to three dimensions. By adapting the
conventions of \[2\] to those of \[6\] and \[8\] we call the special coordinates \(z^a\) as \(z^a = x^a + iy^a\) and define:

\[
V = \frac{1}{6}(\kappa y y y) \equiv \frac{1}{6}\kappa \quad (\kappa y y y) = d_{abc}y^a y^b y^c
\]

\[
\kappa_a = d_{abc}y^b y^c ; \quad \kappa_{ab} = d_{abc}y^c
\]

The \(2n + 4\) additional coordinates are denoted by \(\zeta^I \equiv (\zeta^0, \zeta^a), \tilde{\zeta}^I \equiv (\tilde{\zeta}^0, \tilde{\zeta}_a), D, \tilde{\Phi}\).

The \(\alpha^I, \beta_I\) isometries act as shifts on the \(2n + 2\) coordinates \(\zeta^I, \tilde{\zeta}^I\):

\[
\delta \zeta^I = \alpha^I ; \quad \delta \tilde{\zeta}^I = \beta_I
\]

while the \(\omega^a\) shift isometries of the special geometry, \(\delta x^a = \omega^a\), act as duality rotations on the \(\zeta^I, \tilde{\zeta}^I\) symplectic vector:

\[
\delta \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & -A^T \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix}
\]

with

\[
A = \begin{pmatrix} 0 & 0 \\ \omega^a & 0 \end{pmatrix} ; \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 3d_{abc} \omega^c \end{pmatrix}
\]

On the other hand the \(\beta_0\) isometry rotates \(\zeta^a\) into \(x^a\) so that the \(x^a, \tilde{\zeta}_a\) variables are related by quaternionic isometries. It is immediate to see that the full \(\sigma\)–model Lagrangian \[2, 6, 8\] is invariant under the following parity operation \(\Omega\):

\[
y^a \to y^a ; \quad \tilde{\zeta}_a \to \tilde{\zeta}_a ; \quad \zeta^0 \to \zeta^0 ; \quad D \to D ;
\]

\[
x^a \to -x^a ; \quad \zeta^a \to -\zeta^a ; \quad \tilde{\zeta}^0 \to -\tilde{\zeta}^0 ; \quad \tilde{\Phi} \to -\tilde{\Phi}
\]

so that, restricting to the plus-parity sector is a consistent truncation, giving rise to the following Lagrangian for \(2n + 2\) (real) variables:

\[
(V^{-1})^{-1}L = -\frac{1}{4}(\partial_\mu D)^2 - \frac{1}{4}G_{ab} \partial_\mu y^a \partial_\mu y^b - \frac{1}{8} e^{2D} V (\partial_\mu \zeta)^2
\]

\[
-2e^{2D} V^{-1} (G^{-1})^{ab} \partial_\mu \tilde{\zeta}_a \partial_\mu \tilde{\zeta}_b
\]

where \(G_{ab} = -\partial_a \partial_b \log V\). By a change of variables we can decouple the \((D, \zeta^0)\) fields from the rest as follows: define two new variables \((\Phi, \lambda^a)\):

\[
V(y) e^{2D} = e^{2\Phi} ; \quad y^a = \lambda^a e^{\frac{\Phi}{2}}
\]

Thus it follows that \(V(\lambda)e^{2D} = e^\Phi\) and the Lagrangian becomes:

\[
(V^{-1})^{-1}L = -\frac{1}{4}(\partial_\mu \Phi)^2 - \frac{1}{8} e^{2\Phi} (\partial_\mu \zeta^0)^2 - \frac{1}{4}G_{ab} \partial_\mu \lambda^a \partial_\mu \lambda^b
\]

\[
- \frac{1}{4} (\partial_\mu \log V(\lambda))^2 - 2V(\lambda)^{-2} (G^{-1})^{ab} \partial_\mu \tilde{\zeta}_a \partial_\mu \tilde{\zeta}_b
\]
the \((\Phi, \zeta^0)\) part defines a SU\((1,1)/U(1)\) \(\sigma\)-model.

The coefficient of the two terms in the \(\partial_\mu \lambda^a \partial^\mu \lambda^b\) part combine into \(-\frac{3}{2} \left(\frac{\kappa_{ab}}{\kappa} - 3 \kappa_a \kappa_b / \kappa^2\right)\).

We now define a new variable \(t_a = \frac{1}{2} \kappa_{ab} \lambda^b\) such that \(d \lambda^b = (\kappa^{-1})^{ba} t_a\) we obtain that

\[
\begin{align*}
g^{ab} = -6 \left(\frac{\kappa_{cd}}{\kappa} - 3 \frac{\kappa_{c} \kappa_{d}}{\kappa^2}\right) (\kappa^{-1})^{ac}(\kappa^{-1})^{bd} = -\frac{6}{\kappa^2} \left[(\kappa^{-1})^{ab} - 3 \lambda^a \lambda^b\right] = \frac{36}{\kappa^2} (G^{-1})^{ab}
\end{align*}
\]

Therefore in the \((t_a, \tilde{\zeta}_a)\) variables we finally get

\[
\begin{align*}
(\sqrt{-g})^{-1} \mathcal{L} &= \frac{1}{4} (\partial_\mu \Phi)^2 - \frac{1}{8} e^{2\Phi} (\partial_\mu \zeta^0)^2 - \frac{1}{4} g^{ab} \partial_\mu t_a \partial^\mu t_b \\
&\quad - 2 g^{ab} \partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b
\end{align*}
\]

Therefore by defining the complex variables

\[
\eta_a = t_a + 2 \sqrt{2} i \tilde{\zeta}_a
\]

we get for the \(2n\)-dimensional \(\sigma\)-model:

\[
\begin{align*}
- \frac{1}{4} g(\Re \eta)^{ab} (\partial_\mu \Re \eta_a \partial^\mu \Re \eta_b + \partial_\mu \Im \eta_a \partial^\mu \Im \eta_b) &= - \frac{1}{4} g^{ab} \partial_\mu \eta_a \partial^\mu \bar{\eta}_b
\end{align*}
\]

The previous Lagrangian is Kähler provided

\[
g(t)^{ab} = \frac{\partial^2 \hat{K}}{\partial t_a \partial t_b}.
\]

This condition is achieved by setting \(\hat{K} = -2 \log V(\lambda)\). Indeed

\[
\begin{align*}
\frac{\partial}{\partial t_a} \log V &= (\kappa^{-1})^{ac} \frac{\partial}{\partial \lambda^c} \log V = \frac{3 \lambda^a}{\kappa} \\
\frac{\partial^2}{\partial t_a \partial t_b} \log V &= 3 \left[\frac{(\kappa^{-1})^{ab}}{\kappa} - 3 \lambda^a \lambda^b / \kappa^2\right] = -\frac{1}{2} \times \frac{36}{\kappa^2} (G^{-1})^{ab}
\end{align*}
\]

### 3 Isometries of the \(N = 1\) reduction

The \(\sigma\)-model isometries of the c–map, using the notations of [7] are parametrized by

\[
\epsilon^+, \epsilon^0, \alpha^I, \beta_I, \omega^a, \omega^0, \hat{\beta}_0.
\]

The \(N = 1\) reduction projects out \(\epsilon^+, \alpha^a, \beta_0, \omega^a\), so the remaining isometries are \(n + 4\), namely:

\[
\beta_a, \omega^0, \epsilon^0, \alpha^0, \hat{\beta}_0.
\]

Three of the latter generate a SL\((2,\mathbb{R})\) symmetry (otherwise absent in generic dual quaternionic manifolds), the others generate a shift symmetry in \(\Im \eta_a\) and a scale symmetry in the
The dual manifold has the same isometries of the original Special Kähler. Since the \( \tilde{\eta}_a \) variables are related to the \( x^a \) variables by quaternionic isometries, the two manifolds need in fact not be distinct. Even though the \( \tilde{\eta}_a \) variables are related to the \( x^a \) variables by quaternionic isometries, the two manifolds are in general distinct. However, in the particular case of homogeneous–symmetric spaces \([9]\), it turns out that the dual manifold coincide with the original one. The proof of this statement will be given elsewhere.

4 Connection with Calabi Yau orientifolds

The \( \sigma \)-map was originally studied in relation to the Type II \( A \to B \) mirror map in Calabi–Yau compactifications. In Calabi Yau orientifolds of Type II B strings with D–branes present, the bulk Lagrangian is obtained combining a world–sheet parity with a manifold parity which, for generic spaces \([10]\), is precisely doing the truncation we have encountered in this note.

For certain Calabi Yau manifolds more generic orientifoldings are possible where the set of special coordinates \( z^A \) is separated in two parts with opposite parity, \( z^A_{\pm} (n_+ + n_- = n) \) such that \([11]\)

\[
\begin{align*}
y_{\pm} & \to \pm y_{\pm} \\
x_{\pm} & \to \mp x_{\pm}
\end{align*}
\]

and then consequently

\[
\begin{align*}
\zeta_{\pm} & \to \mp \zeta_{\pm} ; \quad \zeta^0 \to \zeta_0 \\
\tilde{\zeta}_{\pm} & \to \pm \tilde{\zeta}_{\pm} ; \quad \tilde{\zeta}_0 \to -\tilde{\zeta}_0
\end{align*}
\]

However in this case one must demand

\[
d_{++} = d_{---} = 0
\]

in order for the \( N = 1 \) reduction to be consistent \([12]\).

In this case the \( \sigma \)-model Lagrangian acquires more terms and can be symbolically written as:

\[
(\sqrt{|g|})^{-1} \mathcal{L} = -(\partial D)^2 - \frac{1}{4} G_{++} (\partial y_+) - \frac{1}{4} G_{--} (\partial x_-)^2 - \frac{1}{8} e^{2D} V (\partial \zeta^0)^2 - \frac{1}{8} e^{2D} V G_{--} (x_- \partial \zeta^0 - \partial \zeta_-)^2 - 2e^{2D} V^{-1} (G^{-1})^{++} (\partial \tilde{\zeta}_+ + \frac{1}{8} d_{++} x_- \partial \zeta^0 - \frac{1}{4} d_{---} x_- \partial \zeta_-)^2
\]
where for the sake of simplicity space–time indices have been suppressed from partial derivatives and contraction over them is understood. In (4.4) \( G^{++} \) is as before since \( d^{+++} \neq 0 \), \( G^{+-} = 0 \) and \( G^{-+} = -6 (d^{+-} y_+)/(d^{++} y_+ y_+ y_+) \).

The total set of coordinates are: \( y_+, x_-, \zeta_-, \tilde{\zeta}_+ \) and \( (\Phi, \zeta^0) \). Since in this case some of the \( y \) coordinates, namely \( y_- \), have been replaced by \( x_- \), the new variables define a Kähler manifold of complex dimension \( n + 1 \) certainly distinct from the original one.

There is an \( N = 4 \) analogue of this dual \( N = 1 \) geometries if we consider different embeddings of \( N = 4 \) supergravity into \( N = 8 \). This corresponds to Type II B on \( T^6/\mathbb{Z}_2 \) orientifold with \( D3 \) or \( D9 \) branes (Type I string) or Heterotic string on \( T^6 \). In all these cases the bulk sector corresponds to \( [\text{SO}(6,6)/\text{SO}(6) \times \text{SO}(6)] \times [\text{SU}(1,1)/\text{U}(1)] \sigma \)-model but the 15 axions in \( \text{SO}(6,6)/\text{SO}(6) \times \text{SO}(6) \) are coming from \( C_4, C_2, B_2 \) [13, 14, 15, 16, 17, 18, 19].

Also cases in which a further splitting appears are realized if the orientifold projection acts differently on \( T^{p-3} \times T^{9-p} \) \((p = 3, 5, 7, 9)\). This is the analogue of the \( y_\pm, x_\pm \) splitting [11]. In all these cases the dual manifolds coincide, as predicted by \( N = 4 \) supergravity.

### 5 Properties of the dual Special Kähler spaces and no–scale structure.

The dual Kähler space, obtained by a \( N = 1 \) truncation of the \( (c–map) \) very special quaternionic space has a metric that satisfies a “duality” relation with the original very special Kähler space:

\[
g_{ab}^D = \frac{1}{V^2} (G^{-1})^{ab} \tag{5.1}
\]

Moreover it can be shown that its affine connection is simply related to the affine connection of original Kähler space:

\[
\Gamma_d^{bc} = \frac{1}{V} (G^{-1})^{ca} \Gamma^b_{ad}. \tag{5.2}
\]

Actually in the one–dimensional case the two connections coincide.

These dual spaces are also no–scale [20, 21, 22]. Indeed it is sufficient to prove that

\[
\frac{\partial \hat{K}}{\partial \Re \eta_a} (g^{-1})_{ab} \frac{\partial \hat{K}}{\partial \Re \eta_b} = 3. \tag{5.3}
\]

But this is indeed the case since

\[
\lambda^a G_{ab} \lambda^b = 3. \tag{5.4}
\]

From a Type II B perspective, this was anticipated in [23]
6 Concluding remarks.

In this note we have shown that for an arbitrary very special geometry, through the c–map, it is possible to construct a “dual” Kähler geometry which has a dual metric, it is Kähler and it provides a dual no–scale potential. Recently such constructions have found applications in Calabi Yau orientifolds [24, 11] but the procedure considered here is intrinsic to the four dimensional context.

We have not shown that the final Lagrangian is supersymmetric but, using the reduction techniques of [12], it can be shown that this is indeed the case. It is reassuring that the SL(2,R) symmetry, related to the Type II B interpretation, comes out in a pure four dimensional context, thanks to the results of [6 7].

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