Fast calculation of p-values for one-sided Kolmogorov-Smirnov type statistics

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Abstract

A novel method for computing exact p-values of one-sided statistics from the Kolmogorov-Smirnov family is presented. It covers the Higher Criticism statistic, one-sided weighted Kolmogorov-Smirnov statistics, and the one-sided Berk-Jones statistics. In addition to p-values, the method can also be used for power analysis, finding alpha-level thresholds, and the construction of confidence bands for the empirical distribution function.

With its quadratic runtime and numerical stability, the method easily scales to sample sizes in the hundreds of thousands and takes less than a second to run on a sample size of 25,000. This allows practitioners working on large data sets to use exact finite-sample computations instead of approximation schemes.

The method is based on a reduction to the boundary-crossing probability of a pure jump stochastic process. FFT convolutions of two different sizes are then used to efficiently propagate the probabilities of the non-crossing paths. This approach has applications beyond statistics, for example in financial risk modeling.

Keywords: Continuous goodness-of-fit, Higher criticism, Stochastic process, Boundary crossing, Hypothesis testing

1. Introduction

Let $X_1, \ldots, X_n$ be random variables drawn independently from a distribution $F$ and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be their order statistics. In this paper, we present a fast and numerically stable algorithm for computing one-sided non-crossing probabilities of the form

$$\Pr[\forall i : X_{(i)} \leq \beta_i],$$

where $\beta_1, \ldots, \beta_n$ are (arbitrary) upper bounds. This probability may be rewritten as

$$NC\text{PROB}(B_1, \ldots, B_n) := \Pr[\forall i : U_{(i)} \leq B_i | U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0, 1]].$$

where $B_i = F(\beta_i)$ and $U_{(1)} \leq \cdots \leq U_{(n)}$ are the order statistics of a uniform sample in $[0, 1]$. This equivalence follows by expressing $X_i$ using the inverse transformation $X_i = F^{-1}(U_i)$ where...
\( F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \geq u\} \) is the generalized inverse distribution function and then noting that \( F^{-1}(U_{(i)}) \leq \beta_i \) holds if and only if \( U_{(i)} \leq F(\beta_i) = B_i \).

A closely related problem, that is also covered by our algorithm, is the computation of one-sided non-crossing probabilities for the empirical cumulative distribution function (eCDF). Given a function \( b : [0, 1] \rightarrow \mathbb{R} \), this is the probability that \( b(t) \) bounds the empirical CDF from below,

\[
\Pr[\forall t \in [0, 1] : b(t) \leq nF_n(t) \mid U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0, 1]].
\]

(3)

where \( F_n(t) := \frac{1}{n} \sum_{i=1}^{n} 1(U_i \leq t) \) is the eCDF of the sample \( U_1, \ldots, U_n \). The non-crossing probability (3) is equal to the non-crossing probability \( \text{NCPROB}(B_1, \ldots, B_n) \) with upper bounds given by the first integer crossings of the lower boundary function (Gleser, 1985),

\[
B_i = \inf\{t \in [0, 1] : b(t) > i - 1\}.
\]

(4)

Hence, methods for computing the probability (2) can be readily applied to the calculation of non-crossing probabilities for the empirical CDF. See Figure 1 for an illustration. Conversely, given a set of upper bounds \( B_1, \ldots, B_n \), one may construct a step function \( b(t) = \sum_{i=1}^{n} 1(B_i \leq t) \) for which the non-crossing probability (3) is equal to \( \text{NCPROB}(B_1, \ldots, B_n) \). To conclude, the calculation of the probabilities (2) and (3) are two different formulations of the same problem. This equivalence is well-known in the literature and has also been extended to discontinuous distributions (Steck, 1971; Gleser, 1985; Dimitrova et al., 2020b). Since it is fundamental to our algorithm description, we include a concise proof of this equivalence in Appendix A.

1.1. Outline

The main contribution of this paper is a fast and numerically stable \( O(n^2) \) algorithm for computing the \textit{one-sided non-crossing probabilities} (2) and (3). In Section 2 we describe the application of our method to Kolmogorov-Smirnov-type goodness of fit testing and list several other potential applications. In Section 3 we review the existing methods for computing one-sided and two-sided non-crossing probabilities. In sections 4 and 5 we describe the proposed algorithm in detail. In Section 6 we apply our method to the computation of \( p \)-values for a one-sided statistic by Berk and Jones (1979) with sample sizes up to one million, demonstrating state-of-the-art performance. The full source code is linked in Section 6.1.

2. Motivation

The primary motivation for this work is the computation of \( p \)-values and power for a large family of one-sided continuous goodness-of-fit statistics. Examples include the Higher-Criticism
Figure 1: A one-sided lower boundary function $b(t)$ (in yellow) and a non-decreasing step function $f: [0, 1] \rightarrow \{0, 1, \ldots\}$ (in blue) with increments at $U_1, \ldots, U_5 \in [0, 1]$. The empty circles mark the first integer crossings of $b(t)$ as defined in (4). By Lemma 2 (Appendix A), $f(t)$ does not cross $b(t)$ if and only if for all $i$, $f(B_i) \geq i$, or equivalently, that the order statistics $U_{(1)} \leq \cdots \leq U_{(5)}$ all satisfy $U_{(i)} \leq B_i$. The blue circles thus define a layer graph of possible non-crossing transitions for $f(t)$. In this example, $f(t)$ crosses $b(t)$ at $B_4$.

statistic (Donoho and Jin, 2004), one-sided variants of the Kolmogorov-Smirnov statistic (Kolmogorov, 1933; Rényi, 1953; Eicker, 1979; Jaeschke, 1979; Mason and Schuenemeyer, 1983; Jager and Wellner, 2004), variants of the one-sided Berk-Jones statistics (Berk and Jones, 1979; Jager and Wellner, 2005), $\phi$-divergence statistics (Jager and Wellner, 2007), tests based on local-levels (Finner and Gontscharuk, 2018), and gGOF statistics (Zhang et al., 2020). All of these one-sided statistics have the maximum form (or an equivalent minimum form)

$$S := \max_{i=1,\ldots,n} s_i(F(x_{(i)})),$$

where $x_{(1)} \leq \cdots \leq x_{(n)}$ are the order statistics of a sample that, under the null hypothesis, is drawn from a continuous distribution $F$, and $s_1, \ldots, s_n : \mathbb{R} \rightarrow \mathbb{R}$ are either all monotone increasing functions or all monotone decreasing functions. For example, the one-sided Kolmogorov-Smirnov statistics are

$$D_n^+ := \max_{i=1,\ldots,n} \left( \frac{i}{n} - F(x_{(i)}) \right), \quad D_n^- := \max_{i=1,\ldots,n} \left( F(x_{(i)}) - \frac{i-1}{n} \right).$$

Here, $D_n^+$ is a maximum over the monotone decreasing functions $s_i(u) = \frac{i}{n} - u$ and $D_n^-$ is a maximum over monotone increasing functions $s_i(u) = u - \frac{i-1}{n}$.

Another example for a statistic of the form (5) is the Higher Criticism statistic of Donoho
and Jin (2004),

$$\text{HC}_n^* := \sqrt{n} \max_{1 \leq i \leq n} \frac{\frac{X_i}{n} - F(x_{(i)})}{\sqrt{F(x_{(i)})(1 - F(x_{(i)})}.} \tag{7}$$

The $\text{HC}_n^*$ statistic can be viewed as a variant of the one-sided Kolmogorov-Smirnov statistic which takes the maximum standardized deviation of the transformed order statistics $F(x_{(1)}), \ldots, F(x_{(n)})$ from their respective expectations.

Rather than maximizing over standardized deviations, or Z-scores, of the transformed order statistics, one can instead consider the one-sided $p$-value of $F(x_{(i)})$ with respect to the null distribution of uniform order statistics $F(x_{(i)}) \sim \text{Beta}(i, n - i + 1)$, and take the minimum over all such $p$-values. This is the one-sided $M^+_n$ statistic of Berk and Jones (1979), which has the minimum form (analogous to Eq. (5)),

$$M^+_n := \min_{i=1, \ldots, n} s_i(F(x_{(i)})), \tag{8}$$

where $s_i$ are the (monotone-increasing) CDFs of the corresponding Beta distributions,

$$s_i(u) := \text{Pr}[\text{Beta}(i, n - i + 1) < u] = \frac{n!}{(i-1)!(n-i)!} \int_0^u t^{i-1}(1-t)^{n-i} dt. \tag{9}$$

In Section 6 we present an application of our method for computing one-sided non-crossing probabilities to the computation of $p$-values for the $M^+_n$ statistic.

In the next subsections, we describe in detail how the computation of $p$-values and power for one-sided statistics of the general form (5) can be reduced to a calculation of the probability (2), we discuss test statistic distribution inversion for obtaining $\alpha$-level thresholds and mention some applications that involve the non-crossing probability (2).

Remark 1. An alternative to exact computation is the use of asymptotics. For the Higher Criticism, Berk-Jones, and some related statistics, the asymptotic distributions are known (Eicker, 1979; Jaeschke, 1979; Wellner and Koltchinskii, 2003; Moscovich et al., 2016). Unfortunately, the convergence of the null distribution to its limiting form can be exceedingly slow (Gontscharuk et al., 2015), rendering the asymptotics inapplicable. More sophisticated approximations were developed (for example, by Li and Siegmund (2015)), but these are specific to a particular statistic and the quality of their approximation is difficult to analyze. Exact finite-sample computations are generally preferable, provided that they are fast enough to be practical.

2.1. $p$-value and power calculations

Assume that a sample $x_1, \ldots, x_n$ is drawn independently from a continuous distribution $F$ and let $S$ be a statistic of the maximum form (5). Clearly, $S \leq s$ if and only if $s_i(F(x_{(i)})) \leq s$
for all $i$. Since $s_i$ is monotone increasing, this occurs if and only if $F(x_{(i)}) \leq s_i^{-1}(s)$. The distribution of the statistic $S$ under the null hypothesis, that $X_i \overset{i.i.d.}{\sim} F$ is thus

$$\Pr[S \leq s \mid X_i \overset{i.i.d.}{\sim} F] = \Pr[\forall i : F(X_{(i)}) \leq s_i^{-1}(s) \mid X_i \overset{i.i.d.}{\sim} F].$$

(10)

Let $U_i = F(X_i)$, since $F$ is continuous, we have that $U_i \sim U[0, 1]$ and $U_{(i)} = F(X_{(i)})$. This means that (10) can be rewritten as

$$\Pr[\forall i : U_{(i)} \leq s_i^{-1}(s) \mid U_i \overset{i.i.d.}{\sim} U[0, 1]].$$

(11)

Thus the computation of the distribution of a maximum statistic $S$ as defined by Equation (5) reduces to the calculation of the probability (2) with $B_i = s_i^{-1}(s)$. The $p$-value of the statistic $S$ is given by

$$\text{p-value}(s) = \Pr[S \geq s \mid X_i \overset{i.i.d.}{\sim} F] = 1 - \text{NCPROB}(s_1^{-1}(s), \ldots, s_n^{-1}(s)).$$

(12)

Computing the power of such a statistic against a known alternative that $X_1, \ldots, X_n \overset{i.i.d.}{\sim} G$ similarly reduces to Eq. (2), since in that case

$$\Pr[S \geq s \mid X_i \overset{i.i.d.}{\sim} G] = 1 - \Pr[\forall i : s_i(F(X_{(i)})) < s \mid X_i \overset{i.i.d.}{\sim} G]$$

$$= 1 - \Pr[\forall i : G(X_{(i)}) \leq G(F^{-1}(s_i^{-1}(s))) \mid X_i \overset{i.i.d.}{\sim} G]$$

$$= 1 - \Pr[U_{(i)} \leq G(F^{-1}(s_i^{-1}(s))) \mid U_i \overset{i.i.d.}{\sim} U[0, 1]]$$

$$= 1 - \text{NCPROB}(G(F^{-1}(s_1^{-1}(s))), \ldots, G(F^{-1}(s_n^{-1}(s))))).$$

(13) (14) (15) (16)

For a more intricate analysis that considers distributions with discontinuities, see the analyses of Gleser (1985); Dimitrova et al. (2020b).

### 2.2. Computation of $\alpha$-level thresholds

Given a test statistic $S$ of the maximum form in Eq. (5), how can we pick a threshold to obtain an $\alpha$-level test? This is the threshold $s_{n,\alpha}$ that satisfies $\Pr[S \geq s_{n,\alpha} \mid X_i \overset{i.i.d.}{\sim} F] = \alpha$. Since the probability $\Pr[S \geq s]$ is monotone-decreasing in $s$, a common approach is to find $s_{n,\alpha}$ by repeated bisection, thus inverting the cumulative distribution of the statistic $S$ numerically.

If we know that $s_{n,\alpha} \in [a, b]$ then an approximation of $s_{n,\alpha}$ with additive error $< \epsilon$ may be obtained using binary search. This search involves $O(\log((b-a)/\epsilon))$ calculations of probabilities of the form (2). When the range of $s_{n,\alpha}$ is not known in advance, one can use a doubling search (Bentley and Yao, 1976) to obtain an $\epsilon$-approximation with $O(\log(s_{n,\alpha}/\epsilon))$ probability calculations of the form (2).
2.3. Additional applications

Additional applications which involve probabilities of the form (2) and (3) include the construction of confidence bands for empirical distribution functions and Q-Q plots (Owen, 1995; Frey, 2008; Matthews, 2013; Weine et al., 2023), multiple hypothesis testing (Meinshausen and Rice, 2006; Roquain and Villers, 2011; von Schroeder and Dickhaus, 2020; Miecznikowski and Wang, 2023), change-point detection (Worsley, 1986), sequential testing (Dongchun, 1998), financial risk modeling (Dimitrova et al., 2017; Goffard, 2019), genome-wide association studies Sabatti et al. (2009); Barnett et al. (2017); Sun and Lin (2019); Liu et al. (2022), exoplanet detection (Sulis et al., 2017), cryptography (Ding et al., 2018), econometrics (Goldman and Kaplan, 2018), and inventory management (Dimitrova et al., 2020a).

3. Existing methods

In this section, we review leading computational methods for evaluating non-crossing probabilities. We begin with methods for computing one-sided non-crossing probabilities of the form (2) and then proceed to two-sided non-crossing probabilities of the form

\[
\Pr\{\forall i : b_i \leq U_{(i)} \leq B_i | U_1, \ldots, U_n \stackrel{i.i.d.}{\sim} U[0,1]\}. \tag{17}
\]

Note that any algorithm for computing two-sided non-crossing probabilities is, in particular, applicable to the one-sided problem (2) by setting \( b_i = 0 \) for all \( i \).

3.1. One-sided boundaries

Many methods for computing or estimating the one-sided non-crossing probability (2) have been proposed over the years. One approach is to repeatedly generate \( X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} F \) and measure the percentage of times that the inequalities \( X_{(i)} \leq \beta_i \) hold. This Monte-Carlo approach does not yield accurate results and can be slow when the probability of interest is small and the sample size \( n \) is large.

For the exact computation of the non-crossing probability (2), first note that for each set of order statistics with no repetitions \( X_{(1)} < X_{(2)} < \cdots < X_{(n)} \) there are exactly \( n! \) instances of \( (X_1, \ldots, X_n) \) that map to it. Let \( U_i = F(X_i) \). For a continuous \( F \) we have \( U_i \stackrel{i.i.d.}{\sim} U[0,1] \), the density of the random vector \( (U_1, \ldots, U_n) \) is equal to 1 on the unit cube. It follows that the density of the sorted vector of order statistics \( (U_{(1)}, \ldots, U_{(n)}) \) is equal to \( n! \) on the simplex that satisfies \( U_{(1)} < \cdots < U_{(n)} \) and zero elsewhere (we may ignore events of measure zero that
\( U(i) = U(i+1) \). It follows that

\[
\Pr[\forall i : U(i) \leq B_i] = \Pr[\forall i : U(i) < B_i]
\]

\[
= n! \Pr[\forall i : U_i < B_i \text{ and } U_1 < U_2 < \cdots < U_n].
\]

This may be decomposed recursively as

\[
n! \Pr[\forall i : U_i < B_i \text{ and } U_1 < U_2 < \cdots < U_n] = n! \int_0^{B_1} dU_1 \Pr[\forall i = 2, \ldots, n : U_1 < B_i \text{ and } U_1 < U_2 < \cdots < U_n | U_1]
\]

\[
= n! \int_0^{B_1} dU_1 \int_0^{B_2} dU_2 \Pr[\forall i = 3, \ldots, n : U_i < B_i \text{ and } U_2 < U_3 < \cdots < U_n | U_2]
\]

\[
= \cdots = n! \int_0^{B_1} \int_0^{B_2} \int_0^{B_3} \cdots \int_0^{B_n} dU_n.
\]

This recursion was first noted by Wald and Wolfowitz (1939) who demonstrated the symbolic computation of the integral with \( n = 6 \) samples. The integral (23) was analyzed by Durbin (1973) for the case where the bounds \( B_i \) increase linearly, leading to a closed-form expression for the distribution of the one-sided Kolmogorov-Smirnov statistics. More recently, Moscovich et al. (2016) developed a method for the numerical integration of (23) with computational cost \( O(n^2) \). That method was shown to be stable up to \( n \approx 30,000 \) using standard double-precision floating-point numbers.

Many other recursive formulas have been proposed for the calculation of one-sided non-crossing probabilities. Of particular note is the formula in Proposition 3.2 of Denuit et al. (2003). This \( O(n^2) \) recursive formula was first derived by Noé and Vandewiele (1968) and used to tabulate percentage points of standardized one-sided Kolmogorov-Smirnov statistics for sample sizes up to \( n = 100 \). Another \( O(n^2) \) recursive procedure was proposed by Kotel’nikova and Chmaladze (1983). A major limitation of these methods is that they contain sums of large binomial coefficients multiplied by very small numbers, leading to numerical instabilities. Thus, using standard floating-point numbers, the methods become unstable for sample sizes beyond a few hundred (see Section 1 of Khmaladze and Shinjikashvili (2001)). While it is possible to use variable precision floating-point numbers or rational arithmetic to alleviate the loss of numerical accuracy (Brown and Harvey, 2008a,b; von Schroeder and Dickhaus, 2020), this approach incurs heavy runtime penalties compared to the use of numerically stable methods that can use standard floating-point numbers.
3.2. Two-sided boundaries

For the computation of two-sided non-crossing probabilities of the form (17), several methods have been proposed (Epanechnikov, 1968; Steck, 1971; Durbin, 1971; Noé, 1972; Friedrich and Schellhaas, 1998; Khmaladze and Shinjikashvili, 2001; Moscovich and Nadler, 2017). Unfortunately, all of these methods have a high computational cost of $O(n^3)$ with the exception of the following:

- The FFT-based algorithm of Moscovich and Nadler (2017), on which the current paper is based, has a running time of $O(n^2 \log n)$.

- The procedure of Durbin (1971) is based on solving a system of linear equations. While standard solutions are $O(n^3)$, using the Coppersmith-Winograd algorithm or related methods, the theoretical asymptotic runtime is approximately $O(n^{2.373})$. However, such methods involve huge runtime constants and are not practical.

- It was noted by Miecznikowski et al. (2017) that the determinant-based formula of Steck (1971) can be computed in $O(n^2)$ thanks to the Hessenberg form of the matrix. However, due to a rapid loss of numerical accuracy, this approach is difficult to scale to large values of $n$. In a recent paper by Wang and Miecznikowski (2022), the authors compared seven variants of high-precision and rational arithmetic algorithms for computing Steck’s determinant. They demonstrated their approach to the task of computing $p$-values for the one-sided exact Berk-Jones statistic $M_n^+$ (see Eq. (8)). For the largest sample size that they tested ($n = 15,000$) the running time of computing the probability (17) was 40 seconds. In contrast, for the same sample size, the $O(n^2)$ method presented in this paper runs in 0.33 seconds.

4. Technical background

We now describe the methods of Friedrich and Schellhaas (1998); Khmaladze and Shinjikashvili (2001); Moscovich and Nadler (2017) that form the basis of our algorithm. These methods compute the two-sided non-crossing probability (17) given a set of lower and upper boundaries. However, since the focus of this paper is on the one-sided case, our exposition describes these methods in the simpler case of a one-sided boundary, where $b_i = 0$ for all $i$. 

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4.1. Stepwise Binomial propagation

In this subsection, we describe a minor variant of “scheme 1” of Friedrich and Schellhaas (1998), specialized to the one-sided boundary case. Let \( S(i, j) \) be the following probability,

\[
S(i, j) := \Pr\left[nF_n(B_i) = j \text{ and } U(1) \leq B_1, U(2) \leq B_2, \ldots, U(i-1) \leq B_{i-1}\right] \tag{24}
\]

where \( U(1) \leq \cdots \leq U(n) \) are the order statistics of a sample \( U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0, 1] \). Define \( B_0 := 0 \) and \( B_{n+1} := 1 \). (26)

With this notation,

\[
S(0, j) = \Pr[nF_n(0) = j] = \delta_{0,j}. \tag{27}
\]

Our quantity of interest is

\[
S(n + 1, n) = \Pr[nF_n(1) = n \text{ and } \forall i \in \{1, \ldots, n\} : U(i) \leq B_i] \tag{28}
\]

\[
= \Pr[\forall i \in \{1, \ldots, n\} : U(i) \leq B_i] \tag{29}
\]

\[
= \text{NCPROB}(B_1, \ldots, B_n). \tag{30}
\]

We now explain how this quantity is computed using recursion relations. The initial conditions are \( S(0, j) = \delta_{0,j} \). The transition probabilities are given by the following Chapman-Kolmogorov equations,

\[
S(i + 1, j) = \sum_{k=i}^{n} S(i, k) \cdot \Pr[nF_n(B_{i+1}) = j \mid nF_n(B_i) = k]. \tag{31}
\]

Note that the summation is done over \( k \geq i \) to guarantee that we only sum over non-crossing paths for which \( nF_n(B_i) \geq i \), or equivalently that \( U(i) \leq B_i \) (see Figure 1 and Appendix A). The transition probability \( k \to j \) is the probability that exactly \( j - k \) of the points fall in the interval \((B_i, B_{i+1}]\), conditioned on the fact that \( k \) of them fell in the interval \([0, B_i]\). This is given by the following Binomial probability mass function,

\[
\Pr[nF_n(B_{i+1}) = j \mid nF_n(B_i) = k] = \Pr[\text{Binomial}(n-k, p_i) = j-k] \tag{32}
\]

\[
= \binom{n-k}{j-k} p_i^{j-k} (1 - p_i)^{n-j}. \tag{33}
\]

where \( p_i := \Pr[B_i < U \leq B_{i+1} \mid U > B_i] \) with \( U \sim U[0, 1] \). Hence, \( p_i = (B_{i+1} - B_i)/(1 - B_i) \). To compute the non-crossing probability \( S(n + 1, n) \), one can start by setting \( S(1, j) \) for all \( j \) via
Eq. (31), then proceed to compute \( S(2, j) \) for all \( j \), etc. at a total runtime cost of \( O(n^3) \). This procedure, which we dub *Stepwise Binomial propagation*, is illustrated in Figure 1. The filled circles represent elements of \( S(i, j) \) with \( j \geq i \) whereas the hollow circles \( S(i, i-1) \) correspond to paths for which \( nF_n(t) \) crosses the lower boundary at \( B_i \).

4.2. *Stepwise Poisson propagation*

There is a simple connection between the empirical CDF of an i.i.d. sample and a conditioned Poisson process:

**Lemma 1.** Let \( U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0, 1] \) be a sample and let \( F_n(t) = \frac{1}{n} \sum_i 1(U_i \leq t) \) be its empirical CDF. The distribution of the process \( nF_n(t) \) is identical to that of a Poisson process \( \xi_n(t) \) with intensity \( n \), conditioned on \( \xi_n(1) = n \).

For the proof, see Shorack and Wellner (2009, Prop 2.2, Ch. 8). The calculation of the non-crossing probability in Eq. (3) may thus be reduced to the calculation of the non-crossing probability of a Poisson process \( \xi_n \) with intensity \( n \), conditioned on \( \xi_n(1) = n \).

The recursion relations for all \( i = 0, \ldots, n+1 \) and \( j = 0, \ldots, n \) mimic those of \( S(i, j) \),

\[
Q(i, j) := \Pr[\xi_n(B_i) = j \text{ and } \forall \ell \in \{1, 2, \ldots, i-1\} : \xi_n(B_\ell) \geq \ell].
\]  

(34)

The recursion relations for all \( i = 0, \ldots, n+1 \) and \( j = 0, \ldots, n \) mimic those of \( S(i, j) \),

\[
Q(0, j) = \delta_{0,j},
\]

(35)

\[
Q(i+1, j) = \sum_{k=i}^{n} Q(i, k) \cdot \Pr[\xi_n(B_{i+1}) = j | \xi_n(B_i) = k],
\]

(36)

where the transition probabilities are now given by Poisson counts,

\[
\Pr[\xi_n(B_{i+1}) = j | \xi_n(B_i) = k] = \Pr[\text{Pois}(n(B_{i+1} - B_i)) = j - k] = \frac{(n(B_{i+1} - B_i))^{j-k}e^{-n(B_{i+1} - B_i)}}{(j-k)!}.
\]

(38)

As before, in Equation (36) we sum over \( k \geq i \) to guarantee that we only consider the non-crossing paths for which \( \xi_n(B_i) \geq i \). The algorithm based on this recursion, which we dub *stepwise Poisson propagation* proceeds by computing \( Q(1, j) \) for all \( j \), then \( Q(2, j) \) for all \( j \), etc. Finally, by Lemma 1,
S(n + 1, n) = \Pr[nF_n(B_1) \geq 1, \ldots, nF_n(B_n) \geq n] \quad (39)
= \Pr[\xi_n(B_1) \geq 1, \ldots, \xi_n(B_n) \geq n | \xi_n(1) = n] \quad (40)
= \frac{\Pr[\xi_n(B_1) \geq 1, \ldots, \xi_n(B_n) \geq n \text{ and } \xi_n(1) = n]}{\Pr[\xi_n(1) = n]} \quad (41)
= \frac{\Pr[\xi_n(B_1) \geq 1, \ldots, \xi_n(B_n) \geq n \text{ and } \xi_n(B_{n+1}) = n]}{\Pr[\text{Pois}(n) = n]} \quad (42)
= \frac{Q(n + 1, n)}{n^ne^{-n/n!}}. \quad (43)

This method was proposed by Khmaladze and Shinjikashvili (2001) for two-sided boundary crossing probabilities. It has the same \(O(n^3)\) asymptotic running time as the stepwise Binomial propagation of Friedrich and Schellhaas (1998) which we described in Section 4.1.

4.3. Fourier-based stepwise Poisson propagation

In contrast to the Binomial propagation described in Section 4.1, in the stepwise Poisson propagation, the transition probabilities \(\Pr[\xi_n(B_{i+1}) = j | \xi_n(B_i) = k]\) in Eq. (37) do not depend on \(j\) or \(k\) but only on their difference. This is due to the memorylessness property of the Poisson process. As a result, the recurrence (36) has the form of a linear convolution,

\[
Q(i + 1, j) = \sum_{k=i}^{n} Q(i, k) \cdot \Pr[\text{Pois}(\lambda^{(i+1)}) = j - k],
\]

where \(\lambda^{(i+1)} := n(B_{i+1} - B_i)\) is the expected number of jumps of the Poisson process in the interval \((B_{i+1}, B_i]\). Let zero(v, i) denote a copy of the vector \(v\) with the first \(i\) elements set to zero, and let \(Q^{(i)} \in \mathbb{R}^{n+1}\) denote the vector

\[
Q^{(i)} := (Q(i, 0), Q(i, 1), \ldots, Q(i, n)).
\]  

With this notation, the vector \(Q^{(i+1)}\) is given by a truncated linear convolution,

\[
Q^{(i+1)} = \text{zero}(Q^{(i)}, i) \ast \pi^{(i+1)},
\]

where \(\pi^{(i+1)} := (\Pr[\text{Pois}(\lambda^{(i+1)}) = 0], \ldots, \Pr[\text{Pois}(\lambda^{(i+1)}) = n])\) is the Poisson PMF vector. The zeroing operation is done to account for the fact that the summation in Eq. (44) is performed only for \(k \geq i\).

Each of these linear convolutions can be computed efficiently in \(O(n \log n)\) time steps using the fast Fourier transform (FFT) and the circular convolution theorem for discrete signals (Press et al., 1992, Ch. 12, 13). The resulting procedure has a total running time of \(O(n^2 \log n)\) and is
numerically stable for large sample sizes using standard double-precision (64-bit) floating-point numbers (Moscovich and Nadler, 2017).

This stepwise FFT-based procedure can also be used to compute the non-crossing probabilities for non-homogeneous Poisson processes, negative binomial processes, and other types of stochastic jump processes, as well as non-crossing probabilities for discontinuous distributions (Dimitrova et al., 2020a,b).

5. Proposed algorithm

In the previous section, we described how, for any \( i \), one can obtain the non-crossing probabilities vector \( Q^{(i+1)} \), defined in Eq. (45), by computing a truncated linear convolution of \( Q^{(i)} \) and the PMF of a Poisson random variable. Starting from \( Q^{(i)} \) for some \( i \) and repeating this process \( k \) times we obtain \( Q^{(i)} \rightarrow Q^{(i+1)} \rightarrow \ldots \rightarrow Q^{(i+k)} \) in \( O(k n \log n) \) time. In this section, we show how for any \( k \in (\log n, n/\log n) \), it is possible to go directly from \( Q^{(i)} \) to \( Q^{(i+k)} \) using just \( O(k n) \) steps. This makes the total runtime for computing \( Q^{(n+1)} \) be \( \lceil n + 1 \rceil O(k n) = O(n^2) \).

The main idea behind our method is simple. We first compute all the transition probabilities of the Poisson process \( \xi_n \),

\[
Q(i, j) \cdot \Pr[\xi_n(B_{i+k}) = j | \xi_n(B_i) = j],
\]

from all non-zero elements \( Q(i, j) \) at a cost of \( O(n \log n) \) using a single convolution. These transition probabilities include the contributions of non-crossing paths and also the contributions of crossing paths that intersect the lower boundary in the interval \([B_i, B_{i+k})\). All that remains is to subtract the contributions of the crossing paths. Non-crossing paths satisfy \( \xi_n(B_j) \geq j \) for all \( j \). In contrast, a path that crosses the lower boundary inside the interval \([B_i, B_{i+k})\) must satisfy \( \xi_n(B_j) < j \) for at least one index \( j \in \{i, \ldots, i + k - 1\} \). With some careful accounting that we describe in the next section, we can efficiently compute the probability of having a first crossing at each of these points and then subtract their individual contributions from the arrival probabilities in Eq. (47).

**Definition 1.** Let \( f : [0, 1] \rightarrow \{0, 1, 2, \ldots\} \) be a function. For every \( i \in \{0, \ldots, n + 1\} \), we define two logical predicates,

\[
\text{NC}(f, i) := \forall \ell < i : f(B_i) \geq \ell, \quad \text{(no crossing before } B_i) \tag{48}
\]

\[
\text{FC}(f, i) := \text{NC}(f, i) \text{ and } f(B_i) < i \quad \text{(first crossing at } B_i) \tag{49}
\]

\[
= \forall \ell < i : f(B_i) \geq \ell \text{ and } f(B_i) = i - 1. \tag{50}
\]
Proposition 1. Let $k > 0$ be some integer. Given the vector $Q^{(i)}$ as defined in Eq. (45), we can compute the probabilities $\Pr[\xi_n(B_{i+k}) = j \text{ and } NC(\xi_n, i)]$ for all $j$ in $O(n \log n)$ time.

Proof. We first note that for every $\ell \in \{i, \ldots, i+k-1\}$, the probabilities $\Pr[\xi_n(B_{\ell}) = j \text{ and } NC(\xi_n, i)]$ can be computed in $O(n \log n)$ time. As explained in Section 4.3, this convolution can be computed in $O(n \log n)$ steps.

Proposition 2. Given $Q^{(i)}$ the probabilities $\Pr[FC(\xi_n, j) \text{ and } \xi_n(B_{i+k}) = \ell]$ for all values of $\ell \in \{i, \ldots, i+k-1\}$ and $j \in \{i, \ldots, i+k-1\}$ can be computed in $O(k^2 \log k + nk)$ time.

Proof. We first note that for every $j \geq 1$, by Eq. (50),

$$FC(\xi_n, j) = \forall \ell < j - 1 : f(B_{\ell}) \geq \ell \text{ and } f(B_{j-1}) = f(B_j) = j - 1.$$  \hfill (53)

By the chain rule, we have

$$\Pr[FC(\xi_n, j)] = \Pr[\forall \ell < j - 1 : f(B_{\ell}) \geq \ell \text{ and } f(B_{j-1}) = j - 1] \cdot \Pr[f(B_j) = j - 1 | f(B_{j-1}) = j - 1]$$

\begin{equation}
= Q(j - 1, j - 1) \cdot \Pr[\text{Pois}(n(B_j - B_{j-1})) = 0]. \tag{55}
\end{equation}

From the definition of $FC$, if $FC(\xi_n, j)$ then $\xi_n(B_j) = j - 1$, hence by the memorylessness of the Poisson process,

$$\Pr[\xi_n(B_{i+k}) = \ell | FC(\xi_n, j)] = \Pr[\xi_n(B_{i+k}) = \ell | \xi_n(B_j) = j - 1]$$

\begin{equation}
= \Pr[\text{Pois}(n(B_{i+k} - B_j)) = \ell - (j - 1)]. \tag{57}
\end{equation}

Putting it all together, we have

$$\Pr[FC(\xi_n, j) \text{ and } \xi_n(B_{i+k}) = \ell] = \Pr[FC(\xi_n, j)] \cdot \Pr[\xi_n(B_{i+k}) = \ell | FC(\xi_n, j)]$$

\begin{equation}
= Q(j - 1, j - 1) \cdot \Pr[\text{Pois}(n(B_j - B_{j-1})) = 0] \cdot \Pr[\text{Pois}(n(B_{i+k} - B_j)) = \ell - j + 1]. \tag{58}
\end{equation}

Evaluating this probability for all $j \in \{i, \ldots, i+k-1\}$ and $\ell \in \{i+k, \ldots, n\}$ takes a total of $O(nk)$ time. As for the computation of $Q(j - 1, j - 1)$ for all $j \in \{i, \ldots, i+k-1\}$, note that $Q(j, \ell)$ for all $j, \ell \in \{i - 1, \ldots, i + k - 1\}$ is a $k \times k$ sub-array that can be computed in time $O(k^2 \log k)$ using the FFT-based algorithm described in Section 4.2. \hfill \Box
Proposition 3. Given $Q^{(i)}$, one can compute $Q^{(i+k)}$ in $O(n \log n + nk + k^2 \log k)$ time.

Proof. If the predicate NC($f, i$) holds then either NC($f, i + k$) or FC($f, j$) for exactly one of $j \in \{i, \ldots, i + k - 1\}$. Hence for a Poisson process $\xi_n(t)$

$$\Pr[\text{NC}(\xi_n, i)] = \Pr[\text{NC}(\xi_n, i + k)] + \sum_{j=i}^{i+k-1} \Pr[\text{FC}(\xi_n, j)].$$ \hspace{1cm} (59)

This equality holds even when we add the constraint that $\xi_n(B_{i+k}) = \ell$. Adding this constraint and subtracting the sum on the RHS of Eq. (59) from both sides, we get,

$$\Pr[\text{NC}(\xi_n, i + k) \text{ and } \xi_n(B_{i+k}) = \ell] = \Pr[\text{NC}(\xi_n, i) \text{ and } \xi_n(B_{i+k}) = \ell] \hspace{1cm} (\ast)$$

$$- \sum_{j=i}^{i+k-1} \Pr[\text{FC}(\xi_n, j) \text{ and } \xi_n(B_{i+k}) = \ell].$$ \hspace{1cm} (60)

By Proposition 1 the probabilities (\ast) can be computed in time $O(n \log n)$ and by Proposition 2 the probabilities (\ast\ast) are computable in time $O(nk + k^2 \log k)$. Evaluating (60) costs $O(nk)$. The total running time of computing $Q^{(i+k)}$ given $Q^{(i)}$ is therefore $O(n \log n + nk + k^2 \log k)$.

We can now put it all together. Starting from $Q^{(0)} = (1, 0, 0, \ldots, 0)$, we compute $Q^{(k)}$ and then compute $Q^{(2k)}$, $Q^{(3k)}$, etc., until we reach $Q^{(n+1)}$. By Proposition 3, each of these steps takes $O(n \log n + nk + k^2 \log k)$ time. Thus the total running time is

$$\left\lceil \frac{n + 1}{k} \right\rceil O\left(n \log n + nk + k^2 \log k\right) = O\left(\frac{n^2 \log n}{k} + n^2 + nk \log k\right).$$ \hspace{1cm} (61)

For any choice of $k \in (\log n, n/\log n)$, the running time is $O(n^2)$.

6. Benchmarks

In this section, we test the running time and accuracy of our method. The application chosen here is the computation of p-values for the $M_n^+$ one-sided statistic of Berk and Jones (1979) as defined in Eq. (8). Following the work of Donoho and Jin (2004), the Higher Criticism and Berk Jones statistics have attracted renewed interest due to their optimality with respect to various sparse signal detection problems (Hall and Jin, 2010; Arias-Castro et al., 2011; Li and Siegmund, 2015; Arias-Castro et al., 2020; Porter and Stewart, 2020; Zhang et al., 2020; Kipnis, 2022). In particular, the $M_n^+$ and closely related $R_n^+$ statistics of Berk and Jones (1979) have been applied to inference tasks in various domains, including survival analysis, astrophysics, genetics, and social network anomaly detection (Owen, 1995; Sulis et al., 2017; Sun and Lin, 2019; Zhang and Wu, 2022; Matthews, 2013; Cadena et al., 2019).
For each sample size \( n \), we first computed an \( \alpha \)-level threshold \( s_{n,\alpha} \) with the bisection method described in Section 2.2 for \( \alpha = 5\% \). The bounds \( B_i = s_i^{-1}(s) \) were computed using the `betaincinv` function, which computes the inverse of the CDF of a Beta random variable. The probability \( \Pr[M_n^+ < s_{n,\alpha}] \) was calculated using a single NCPROB evaluation as described in Section 2.1. The following methods for computing NCPROB were tested:

- **KS (2001):** the \( O(n^3) \) two-sided algorithm of Khmaladze and Shinjikashvili (2001), described in Section 4.2.
- **MNS (2016):** the \( O(n^2) \) one-sided algorithm of Moscovich et al. (2016) mentioned in Section 3.
- **MN (2017):** the \( O(n^2 \log n) \) two-sided algorithm of Moscovich and Nadler (2017), described in Section 4.3.
- **New:** the \( O(n^2) \) one-sided algorithm described in this paper.

Figure 2 shows the running times for the sample sizes \( n = 5000, 10000, \ldots, 100,000 \) (best out of 3 runs). The Poisson-propagation-based methods KS (2001)/MN (2017)/New all produce the same results in the tested range, with relative errors less than \( 10^{-10} \) using standard double-precision (64-bit) floating-point numbers. In contrast, MNS (2016) is only accurate up to about \( n = 30,000 \). For the sample size \( n = 35,000 \) it produces a relative error of 7\% and for \( n > 50,000 \) it breaks down completely. Therefore, we did not test the running time of MNS (2016) for sample sizes larger than 30,000.

An additional set of large-scale benchmarks is shown in Figure 3. This time, due to the long running times involved, we only performed a single measurement for every data point (rather than taking the best out of 3 runs) and used a fixed threshold for all sample sizes, chosen to be the \( \alpha \)-level threshold for \( n = 100,000 \) with \( \alpha = 5\% \). This figure does not show benchmarks for KS (2001) due to its excessive running time for large sample sizes. In the bottom panel, we show the relative difference between the boundary-crossing probabilities computed using MN (2017) and New. This relative error is small throughout the tested range. See Appendix B for additional details on our implementation and benchmarks.

Remark 2. A different choice of test statistic should yield very similar running times. The chosen test statistic and threshold merely determine the bounds \( B_1, \ldots, B_n \), but the running time does not typically depend on their particular values. One exception is the case where there are multiple repeating bounds (e.g. \( B_1 = B_2 \)), which we optimized for.
Figure 2: Running times for computing the $p$-value of a one-sided goodness-of-fit statistic. The times shown are the best out of three runs. Note the logarithmic y-axis.
Figure 3: Large scale benchmarks for computing the $p$-value of a one-sided goodness-of-fit statistic. (top) Running times of our method vs. Moscovitch and Nadler (2017). Note the logarithmic y-axis. (bottom) Relative numerical errors of the $p$-values computed by the two methods. The maximum relative numerical error is about $10^{-9}$. 
6.1. Code availability

A C++ implementation of the tested methods for computing one-sided and two-sided boundary crossing probabilities of the form (2) and (17) is provided at the following link:

https://github.com/mosco/crossing-probability

This repository also includes a Python language wrapper, as well as code for running the benchmarks and creating the figures in Section 6.

7. Conclusion

Given a set of bounds \( B_1, \ldots, B_n \in [0, 1] \), this paper presents a new \( O(n^2) \) method for the calculation of the non-crossing probability

\[
\text{NCPROB}(B_1, \ldots, B_n) := \Pr[\forall i : U(i) \leq B_i],
\]

where \( U(1) \leq \cdots \leq U(n) \) are the order statistics of a uniform draw in the unit interval. The fast calculation of these probabilities has many applications, in particular for sparse signal detection, goodness-of-fit testing, financial risk modeling, and the construction of one-sided confidence bands for the empirical distribution function.

We have applied our method to the computation of \( p \)-values for a one-sided goodness-of-fit statistic of Berk and Jones (1979) and compared its running time to other leading methods, with sample sizes as large as one million. For all sample sizes, our method is shown to be the fastest one available by a wide margin.

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Appendix A. Reduction of the continuous boundary crossing problem to a discrete set of inequalities

Let \( U_1, \ldots, U_n \sim U[0, 1] \) be a sample of independent uniform variables with order statistics \( U(1) \leq \cdots \leq U(n) \) and empirical cumulative distribution function \( F_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1(U_i \leq t) \). In this appendix, we present the reduction between the non-crossing probability of the empirical cumulative distribution,

\[
\Pr[\forall t \in [0, 1] : b(t) \leq nF_n(t)],
\]

(A.1)
and the simultaneous non-crossing probability of the order statistics,

\[
\text{NCPROB}(B_1, \ldots, B_n) = \Pr[\forall i : U_{(i)} \leq B_i].
\] (A.2)

This reduction is well-known and has also been extended to discontinuous distributions (Steck, 1971; Gleser, 1985; Dimitrova et al., 2020b). Nonetheless, we thought it would benefit the reader to include a concise proof of this basic result, which is at the foundation of the methods described in this paper. First, we show that, rather than considering the entire boundary function \(b(t)\), it suffices to consider its first integer passage times,

\[
B_i := \inf\{t \in [0,1] : b(t) > i - 1\}, \quad i = 1, \ldots, k
\] (A.3)

where \(k\) is the largest integer for which the set \(\{t : b(t) > k - 1\}\) is non-empty. The following lemma holds the key observation that allows one to replace the infinite set of inequality constraints \(\forall t \in [0,1] : b(t) \leq nF_n(t)\) with a finite set of inequalities.

**Lemma 2.** Let \(f : [0,1] \to \{0,1,2,\ldots\}\) be a non-decreasing right-continuous function and let \(b : [0,1] \to \mathbb{R}\) be a function with first integer crossings \(B_1, \ldots, B_k\), then

\[
\forall i : f(B_i) \geq i \iff \forall t : b(t) \leq f(t).
\]

**Proof.** \((\Rightarrow)\) Divide the interval \([0,1]\) into a disjoint union,

\[
[0, B_1) \cup [B_1, B_2) \cup \ldots \cup [B_{k-1}, B_k) \cup [B_k, 1].
\] (A.4)

We now prove that \(b \leq f\) in each of these intervals:

1. \([0, B_1)\): By the definition of \(B_i\), for all \(t < B_1\) we have \(b(t) \leq 0\), and since \(f\) is non-negative it follows that \(b(t) \leq 0 \leq f(t)\).

2. \([B_i, B_{i+1})\): If \(t < B_{i+1}\) then \(b(t) \leq i\). Since we assumed \(f(B_i) \geq i\) it follows that for all \(t \in [B_i, B_{i+1})\) we have \(b(t) \leq i \leq f(B_i) \leq f(t)\), where the last inequality is due to the monotonicity of \(f\).

3. \([B_k, 1]\): \(f(t) \geq f(B_k) \geq k \geq b(t)\). The first inequality follows from the monotonicity of \(f\), the second is an assumption of the lemma, and the last inequality follows from the definition of \(k\).

\((\Leftarrow)\) By the definition of \(B_i\) there is a series of real numbers \(t_j \in [B_i, 1]\) such that \(t_j \to B_i\) and \(b(t_j) > i - 1\). By the assumption \(f(t_j) \geq b(t_j) > i - 1\). Since \(i\) and \(f(t_j)\) are both integers, this
means that \( f(t_j) \geq i \). From the right-continuity of \( f \) we conclude that

\[
  f(B_i) = \lim_{j \to \infty} f(t_j) \geq i.
\] (A.5)

A direct consequence of this lemma is that the probability that the empirical CDF of a uniform sample does not cross a lower boundary is equal to the probability that the order statistics satisfy a set of simultaneous upper bounds.

**Corollary 1.** Let \( U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0,1] \) be a sample with empirical cumulative distribution \( F_n(t) = \frac{1}{n} \sum_i 1(U_i \leq t) \) and let \( b : [0,1] \to \mathbb{R} \) be a function, then

\[
  \Pr[\forall t \in [0,1] : b(t) \leq nF_n(t)] = \Pr[\forall i \in \{1, \ldots, k\} : U_{(i)} \leq B_i],
\]

where \( U_{(1)} \leq \cdots \leq U_{(n)} \) are the order statistics of the sample and \( B_1, \ldots, B_k \) are the first integer crossings of \( b(t) \) as defined in Eq. (A.3).

**Proof.** By Lemma 2, \( b(t) \leq nF_n(t) \) for all \( t \) if and only if \( nF_n(B_i) \geq i \) for all \( i \). By the definition of the empirical CDF, \( nF_n(B_i) \geq i \) if and only if at least \( i \) elements of the sample are at most \( B_i \), in other words, that \( U_{(i)} \leq B_i \). The result follows. \( \square \)

Hence the problem of computing the non-crossing probability (3), is reduced to the probability that the inequalities \( nF_n(B_i) \geq i \) hold at a finite set of times. The reduction can also be made in the other direction, from the calculation of the discrete boundary crossing probability (A.2) to the continuous boundary crossing (A.1).

**Corollary 2** (reduction from the discrete to the continuous problem). Let \( U_1, \ldots, U_n \overset{i.i.d.}{\sim} U[0,1] \) and let \( B_1, \ldots, B_k \) be a set of upper bounds in the discrete boundary crossing probability (2). Define their cumulative function as \( b(t) = \sum_{i=1}^k 1(B_i \leq t) \), then

\[
  \Pr[\forall i \in \{1, \ldots, k\} : U_{(i)} \leq B_i] = \Pr[\forall t : b(t) \leq nF_n(t)],
\] (A.6)

where \( F_n \) is the empirical CDF of \( U_1, \ldots, U_n \).

**Proof.** By the construction of \( b(t) \), for all \( i \), \( B_i = \inf\{t : b(t) > i - 1\} \). This coincides with the definition of \( B_i \) in Eq. (A.3). Equation (A.6) follows. \( \square \)

**Remark 3.** For data from a non-uniform distribution \( X_i \sim F \), we may transform the variables as \( U_i = F(X_i) \). If the distribution \( F \) is continuous then \( U_i \sim U[0,1] \), thus we may directly apply the reductions above to the transformed variables as described in Section 2.1. However,
discontinuous distributions require a more intricate analysis. For the full details of the reduction in the discontinuous case, see Theorem 1 of Gleser (1985) which extends Corollary 1 above. These results were used by Dimitrova et al. (2020b) to compute the distribution of the Kolmogorov-Smirnov statistic when the underlying distribution \( F \) is discontinuous.

**Appendix B. Benchmark and implementation details**

All four methods compared in Figure 2 were implemented in C++, compiled in clang 11.0.3, and tested on a 2019 Intel Core i7-8569U CPU. For computing the fast Fourier transform we used the library FFTW 3.3.8 in single-threaded mode (Frigo and Johnson, 2005).

In the calculation of \( Q(i+1,:) \) according to Eq. (46), we represent the zero elements implicitly, thus reducing the size of the FFT convolutions from \( n+1 \) to \( n+1-i \). This optimization already existed in our previous code for computing two-sided non-crossing probabilities (Moscovich and Nadler, 2017). We added additional optimizations to the two-sided Poisson-propagation algorithms KS (2001) and MN (2017) that specifically handles consecutive lower bounds that satisfy \( b_{i+1} = b_i \) as a special case (see Eq. (17)). This makes the two methods more competitive for the computation of one-sided boundary crossing probabilities. This, in addition to several other technical code optimizations and the improvement in processor speed, resulted in an 8-fold decrease in the running time of MN (2017) in the one-sided boundary case, compared to our previous benchmark (Moscovich and Nadler, 2017).

Our proposed algorithm has a configurable jump size parameter \( k \). The entire range \( k \in [\log n, n/\log n] \) gives asymptotically optimal results of \( O(n^2) \). To get a ballpark estimate for the optimal value of \( k \) we set \( k(x) = x\sqrt{n} \) and minimized the asymptotic runtime in Eq. (61). The resulting minimizer is \( k = \sqrt{2n} \). However, the setting used in the benchmarks was \( k = \sqrt{n} \) as this was empirically found to be faster.

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