Coinduction in Uniform
Foundations for Corecursive Proof Search with Horn Clauses

Henning Basold\textsuperscript{1}, Ekaterina Komendantskaya\textsuperscript{2}, and Yue Li\textsuperscript{3}

\textsuperscript{1}CNRS, ENS Lyon (email henning.basold@ens-lyon.fr, orcid 0000-0001-7610-8331)
\textsuperscript{2}Heriot-Watt University (email ek19@hw.ac.uk)
\textsuperscript{3}Heriot-Watt University (email yl55@hw.ac.uk, orcid 0000-0003-0545-0732)

November 2018

We establish proof-theoretic, constructive and coalgebraic foundations for proof search in coinductive Horn clause theories. Operational semantics of coinductive Horn clause resolution is cast in terms of coinductive uniform proofs; its constructive content is exposed via soundness relative to an intuitionistic first-order logic with recursion controlled by the later modality; and soundness of both proof systems is proven relative to a novel coalgebraic description of complete Herbrand models.

1 Introduction

Horn clause logic is a Turing complete and constructive fragment of first-order logic, that plays a central role in verification \cite{21}, automated theorem proving \cite{52, 57, 53} and type inference. Examples of the latter can be traced from the Hindley-Milner type inference algorithm \cite{55, 74}, to more recent uses of Horn clauses in Haskell type classes \cite{51, 25} and in refinement types \cite{42, 27}. Its popularity can be attributed to well-understood fixed point semantics and an efficient semi-decidable resolution procedure for automated proof search.

According to the standard fixed point semantics \cite{78, 52}, given a set $P$ of Horn clauses, the least Herbrand model for $P$ is the set of all (finite) ground atomic formulae inductively entailed by $P$. For example, the two clauses below define the set of natural numbers in the least Herbrand model.

\begin{align*}
\kappa_{\text{nato}} & : \text{nat } 0 \\
\kappa_{\text{nats}} & : \forall x. \text{nat } x \rightarrow \text{nat } (s \, x)
\end{align*}

Formally, the least Herbrand model for the above two clauses is the set of ground atomic formulae obtained by taking a (forward) closure of the above two clauses. The model for $\text{nat}$ is given by $\mathcal{N} = \{ \text{nat } 0, \text{nat } (s \, 0), \text{nat } (s \, (s \, 0)), \ldots \}$. 

\textsuperscript{*}This work is supported by the European Research Council (ERC) under the EU’s Horizon 2020 programme (CoVeCe, grant agreement No. 678157)

\textsuperscript{†}This work is supported by EPSRC research grant EP/N014758/1.

\textbf{arXiv:1811.07644v3 [cs.LO] 15 Mar 2022}
We can also view Horn clauses coinductively. The greatest complete Herbrand model for a set \( P \) of Horn clauses is the largest set of finite and infinite ground atomic formulae coinductively entailed by \( P \). For example, the greatest complete Herbrand model for the above two clauses is the set

\[
\mathcal{N}^\infty = \mathcal{N} \cup \{\text{nat} \ (s \ (s \ldots))\},
\]

obtained by taking a backward closure of the above two inference rules on the set of all finite and infinite ground atomic formulae. The greatest Herbrand model is the largest set of finite ground atomic formulae coinductively entailed by \( P \). In our example, it would be given by \( \mathcal{N} \) already. Finally, one can also consider the least complete Herbrand model, which interprets entailment inductively but over potentially infinite terms. In the case of \( \text{nat} \), this interpretation does not differ from \( \mathcal{N} \). However, finite paths in coinductive structures like transition systems, for example, require such semantics.

The need for coinductive semantics of Horn clauses arises in several scenarios: the Horn clause theory may explicitly define a coinductive data structure or a coinductive relation. However, it may also happen that a Horn clause theory, which is not explicitly intended as coinductive, nevertheless gives rise to infinite inference by resolution and has an interesting coinductive model. This commonly happens in type inference. We will illustrate all these cases by means of examples.

**Horn clause theories as coinductive data type declarations**

The following clause defines, together with \( \kappa_{\text{nat}0} \) and \( \kappa_{\text{nats}} \), the type of streams over natural numbers.

\[
\kappa_{\text{stream}} : \forall xy. \text{nat} \ x \land \text{stream} \ y \rightarrow \text{stream} \ (\text{scons} \ x \ y)
\]

This Horn clause does not have a meaningful inductive, i.e. least fixed point, model. The greatest Herbrand model of the clauses is given by

\[
\mathcal{S} = \mathcal{N}^\infty \cup \{\text{stream}(\text{scons} \ x_0 \ (\text{scons} \ x_1 \ldots )) \mid \text{nat} \ x_0, \text{nat} \ x_1, \ldots \in \mathcal{N}^\infty\}
\]

In trying to prove, for example, the goal \( \text{stream} \ x \), a goal-directed proof search may try to find a substitution for \( x \) that will make \( \text{stream} \ x \) valid relative to the coinductive model of this set of clauses. This search by resolution may proceed by means of an infinite reduction \( \text{stream} \ x \xrightarrow{\kappa_{\text{stream}}[\text{scons} \ y \ x' / x']} \text{nat} \ y \land \text{stream} \ x' \xrightarrow{\kappa_{\text{nat}0}[0 / y]} \text{stream} \ x' \xrightarrow{\kappa_{\text{stream}}[\text{scons} \ y' / x']} \ldots \), thereby generating a stream \( Z \) of zeros via composition of the computed substitutions: \( Z = (\text{scons} \ 0 \ x') (\text{scons} \ 0 \ x'' / x') \ldots \). Above, we annotated each resolution step with the label of the clause it resolves against and the computed substitution. A method to compute an answer for this infinite sequence of reductions was given by Gupta et al. [40] and Simon et al. [70]: the underlined loop gives rise to the circular unifier \( x = \text{scons} \ 0 \ x \) that corresponds to the infinite term \( Z \). It is proven that, if a loop and a corresponding circular unifier are detected, they provide an answer that is sound relative to the greatest complete Herbrand model of the clauses. This approach is known under the name of CoLP.

**Horn Clause Theories in Type Inference**

Below clauses give the typing rules of the simply typed \( \lambda \)-calculus, and may be used for type inference or type checking:

\[
\begin{align*}
\kappa_{\text{f1}} : & \forall x \Gamma a. \ \text{var} \ x \land \text{find} \ \Gamma \ x \ a \rightarrow \text{typed} \ \Gamma \ x \ a \\
\kappa_{\text{f2}} : & \forall x \Gamma a \ m \ b. \ \text{typed} \ [x : a] \Gamma \ m \ b \rightarrow \text{typed} \ \Gamma \ (\lambda \ x \ m) \ (a \rightarrow b) \\
\kappa_{\text{f3}} : & \forall \Gamma a \ m \ n \ b. \ \text{typed} \ \Gamma \ m \ (a \rightarrow b) \land \text{typed} \ \Gamma \ n \ a \rightarrow \text{typed} \ \Gamma \ (\text{app} \ m \ n) \ b
\end{align*}
\]
It is well known that the $Y$-combinator is not typable in the simply-typed $\lambda$-calculus and, in particular, self-application $\lambda x.x$ is not typable either. However, by switching off the occurs-check in Prolog or by allowing circular unifiers in CoLP [40, 70], we can resolve the goal “typed \(\lambda x (\text{app} \ x \ x) \ a\)” and would compute the circular substitution: $a = b \rightarrow c, b = b \rightarrow c$ suggesting that an infinite, or circular, type may be able to type this $\lambda$-term. A similar trick would provide a typing for the $Y$-combinator. Thus, a coinductive interpretation of the above Horn clauses yields a theory of infinite types, while an inductive interpretation corresponds to the standard type system of the simply typed $\lambda$-calculus.

### Table 1: Examples of greatest (complete) Herbrand models for Horn clauses $\gamma_1$, $\gamma_2$, $\gamma_3$. The signatures are $\{a\}$ for the clause $\gamma_1$ and $\{a, f\}$ for the others.

| Horn clauses | $\gamma_1 : \forall x. \ p x \rightarrow p x$ | $\gamma_2 : \forall x. \ p(f \ x) \rightarrow p x$ | $\gamma_3 : \forall x. \ p x \rightarrow p(f \ x)$ |
|--------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| Greatest Herbrand model: | $\{p \ a\}$ | $\{p(a), p(f \ a), p(f(f \ a)), \ldots\}$ | $\emptyset$ |
| Greatest complete Herbrand model: | $\{p \ a\}$ | $\{p(a), p(f \ a), p(f(f \ a)), \ldots\}$ | $\{p(f(f \ \ldots))\}$ |
| CoLP substitution for query $p \ a$ | $id$ | fails | fails |
| CoLP substitution for query $p \ x$ | $id$ | $x = f \ x$ | $x = f \ x$ |

Horn Clause Theories in Type Class Inference  

Haskell type class inference does not require circular unifiers but may require a cyclic resolution inference [51, 35]. Consider, for example, the following mutually defined data structures in Haskell.

```haskell
data OddList a = OCons a (EvenList a)
data EvenList a = Nil | ECons a (OddList a)
```

This type declaration gives rise to the following equality class instance declarations, where we leave the, here irrelevant, body out.

```haskell
instance (Eq a, Eq (EvenList a)) => Eq (OddList a) where
    instance (Eq a, Eq (OddList a)) => Eq (EvenList a) where
```

The above two type class instance declarations have the shape of Horn clauses. Since the two declarations mutually refer to each other, an instance inference for, e.g., $\text{Eq} (\text{OddList} \ \text{Int})$ will give rise to an infinite resolution that alternates between the subgoals $\text{Eq} (\text{OddList} \ \text{Int})$ and $\text{Eq} (\text{EvenList} \ \text{Int})$. The solution to terminate the computation as soon as the cycle is detected [51], and this method has been shown sound relative to the greatest Herbrand models in [34]. We will demonstrate this later in the proof systems proposed in this paper.

The diversity of these coinductive examples in the existing literature shows that there is a practical demand for coinductive methods in Horn clause logic, but it also shows that no unifying proof-theoretic approach exists to allow for a generic use of these methods. This causes several problems.

**Problem 1.** The existing proof-theoretic coinductive interpretations of cycle and loop detection are unclear, incomplete and not uniform.

To see this, consider Tab. 1 which exemplifies three kinds of circular phenomena in Horn clauses: The clause $\gamma_1$ is the easiest case. Its coinductive models are given by the finite set $\{p \ a\}$. On the other extreme is the clause $\gamma_3$ that, just like $\kappa_{\text{stream}}$, admits only an infinite formula in its coinductive model. The intermediate case is $\gamma_2$, which could be interpreted by an infinite set of finite formulae in its greatest Herbrand model, or may admit an infinite formula in its greatest complete Herbrand model.
Examples like $\gamma_1$ appear in Haskell type class resolution [51], and examples like $\gamma_2$ in its experimental extensions [35]. Cycle detection would only cover computations for $\gamma_1$, whereas $\gamma_2$, $\gamma_3$ require some form of loop detection. However, CoLP’s loop detection gives confusing results here. It correctly fails to infer $p\ a$ from $\gamma_3$ (no unier for subgoals $p\ a$ and $p\ (f\ a)$ exists), but incorrectly fails to infer $p\ a$ from $\gamma_2$ (also failing to unify $p\ a$ and $p\ (f\ a)$). The latter failure is misleading bearing in mind that $p\ a$ is in fact in the coinductive model of $\gamma_2$. Vice versa, if we interpret the CoLP answer $x = f\ x$ as a declaration of an infinite term $(f\ f\ \ldots)$ in the model, then CoLP’s answer for $\gamma_3$ and $p\ x$ is exactly correct, however the same answer is badly incomplete for the query involving $p\ x$ and $\gamma_2$, because $\gamma_2$ in fact admits other, finite, formulae in its models. And in some applications, e.g. in Haskell type class inference, a finite formula would be the only acceptable answer for any query to $\gamma_2$.

This set of examples shows that loop detection is too coarse a tool to give an operational semantics to a diversity of coinductive models.

**Problem 2. Constructive interpretation of coinductive proofs in Horn clause logic is unclear.** Horn clause logic is known to be a constructive fragment of FOL. Some applications of Horn clauses rely on this property in a crucial way. For example, inference in Haskell type class resolution is constructive: when a certain formula $F$ is inferred, the Haskell compiler in fact constructs a proof term that inhabits $F$ seen as type. In our earlier example $\text{Eq}$ (OddList $\text{Int}$) of the Haskell type classes, Haskell in fact captures the cycle by a fixpoint term $t$ and proves that $t$ inhabits the type $\text{Eq}$ (OddList $\text{Int}$). Although we know from [34] that these computations are sound relative to greatest Herbrand models of Horn clauses, the results of [34] do not extend to Horn clauses like $\gamma_3$ or $\kappa_{\text{stream}}$, or generally to Horn clauses modelled by the greatest complete Herbrand models. This shows that there is not just a need for coinductive proofs in Horn clause logic, but constructive coinductive proofs.

**Problem 3. Incompleteness of circular unification for irregular coinductive data structures.** Table 1 already showed some issues with incompleteness of circular unification. A more famous consequence of it is the failure of circular unification to capture irregular terms. This is illustrated by the following Horn clause, which defines the infinite stream of successive natural numbers.

$$ \kappa_{\text{from}} : \forall x\ y. \text{from} (s\ x)\ y \to \text{from} (s\ (s\ cons\ x\ y)) $$

The reductions for $\text{from} 0\ y$ consist only of irregular (non-unifiable) formulae:

$$ \text{from} 0\ y \kappa_{\text{from}:[s\ cons\ 0\ y'/y]} \leadsto \text{from} (s\ 0)\ y' \kappa_{\text{from}:[s\ cons\ (s\ 0)\ y''/y']} \leadsto \ldots $$

The composition of the computed substitutions would suggest as answer an infinite term that is given by $\text{from} 0\ (s\ (s\ (s\ 0)\ \ldots))$. However, circular unification no longer helps to compute this answer, and CoLP fails. Thus, there is a need for more general operational semantics that allows irregular coinductive structures.

**A New Theory of Coinductive Proof Search in Horn Clause Logic**

In this paper, we aim to give a principled and general theory that resolves the three problems above. This theory establishes a constructive foundation for coinductive resolution and allows us to give proof-theoretic characterisations of the approaches that have been proposed throughout the literature.

To solve Problem 1, we follow the footsteps of the uniform proofs by Miller et al. [53,54], who gave a general proof-theoretic account of resolution in first-order Horn clause logic (fohc) and three

---

1We follow the standard terminology of [75] and say that two formulae $F$ and $G$ form a cycle if $F = G$, and a loop if $F[\theta] = G[\theta]$ for some (possibly circular) unifier $\theta$. 

---
extensions: first-order hereditary Harrop clauses (fohh), higher-order Horn clauses (hohc), and higher-order hereditary Harrop clauses (hohh). In Sec. 3, we extend uniform proofs with a general coinduction proof principle. The resulting framework is called coinductive uniform proofs (CUP). We show how the coinductive extensions of the four logics of Miller et al., which we name co-fohc, co-fohh, co-hohc and co-hohh, give a precise proof-theoretic characterisation to the different kinds of coinduction described in the literature. For example, coinductive proofs involving the clauses \( \gamma_1 \) and \( \gamma_2 \) belong to co-fohc and co-fohh, respectively. However, proofs involving clauses like \( \gamma_3 \) or \( \kappa_{\text{stream}} \) require in addition fixed point terms to express infinite data. These extensions are denoted by co-fohc\(_{\text{fix}}\), co-fohh\(_{\text{fix}}\), co-hohc\(_{\text{fix}}\) and co-hohh\(_{\text{fix}}\). Sec. 3 shows that this yields the cube in Fig. 1, where the arrows show the increase in logical strength. The invariant search for regular infinite objects done in CoLP is fully described by the logic co-fohc\(_{\text{fix}}\), including proofs for clauses like \( \gamma_3 \) and \( \kappa_{\text{stream}} \). An important consequence is that CUP is complete for \( \gamma_1 \), \( \gamma_2 \), and \( \gamma_3 \), e.g. \( p a \) is provable from \( \gamma_2 \) in CUP, but not in CoLP.

In tackling Problem 3, we will find that the irregular proofs, such as those for \( \kappa_{\text{from}} \), can be given in co-hohh\(_{\text{fix}}\). The stream of successive numbers can be defined as a higher-order fixed point term \( s_\text{fr} = \text{fix } f. \lambda x. \text{scons } x (f (s x)) \), and the proposition \( \forall x. \text{from } x (s_\text{fr} x) \) is provable in co-hohh\(_{\text{fix}}\). This requires the use of higher-order syntax, fixed point terms and the goals of universal shape, which become available in the syntax of Hereditary Harrop logic.

In order to solve Problem 2 and to expose the constructive nature of the resulting proof systems, we present in Sec. 4 a coinductive extension of first-order intuitionistic logic and its sequent calculus. This extension (iFOL\(_{\pi}\)) is based on the so-called later modality (or Löb modality) known from provability logic [15, 72], type theory [58, 7] and domain theory [19]. However, our way of using the later modality to control recursion in first-order proofs is new and builds on [12, 13]. In the same section we also show that CUP is sound relative to iFOL\(_{\pi}\), which gives us a handle on the constructive content of CUP. This yields, among other consequences, a constructive interpretation of CoLP proofs.

Section 5 is dedicated to showing soundness of both coinductive proof systems relative to complete Herbrand models [52]. The construction of these models is carried out by using coalgebras and category theory. This frees us from having to use topological methods and will simplify future extensions of the theory to, e.g., encompass typed logic programming. It also makes it possible to give original and constructive proofs of soundness for both CUP and iFOL\(_{\pi}\) in Section 5. We finish the paper with discussion of related and future work.

**Originality of the contribution**

The results of this paper give a comprehensive characterisation of coinductive Horn clause theories from the point of view of proof search (by expressing coinductive proof search and resolution as
coinductive uniform proofs), constructive proof theory (via a translation into an intuitionistic sequent calculus), and coalgebraic semantics (via coinductive Herbrand models and constructive soundness results). Several of the presented results have never appeared before: the coinductive extension of uniform proofs; characterisation of coinductive properties of Horn clause theories in higher-order logic with and without fixed point operators; coalgebraic and fibrational view on complete Herbrand models; and soundness of an intuitionistic logic with later modality relative to complete Herbrand models.

2 Preliminaries: Terms and Formulae

In this section, we set up notation and terminology for the rest of the paper. Most of it is standard, and blends together the notation used in [53] and [10].

Definition 1. We define the sets $T$ of types and $P$ of proposition types by the following grammars, where $\iota$ and $\omicron$ are the base type and base proposition type, respectively.

\[
\begin{align*}
T &::= \iota \mid \sigma \rightarrow \tau \\
\Pi &::= \omicron \mid \sigma \rightarrow \rho \\
\end{align*}
\]

We adapt the usual convention that $\rightarrow$ binds to the right.

Definition 2. A term signature $\Sigma$ is a set of pairs $c : \tau$, where $\tau \in T$, and a predicate signature is a set $\Pi$ of pairs $p : \rho$ with $\rho \in \Pi$. The elements in $\Sigma$ and $\Pi$ are called term symbols and predicate symbols, respectively. Given term and predicate signatures $\Sigma$ and $\Pi$, we refer to the pair $(\Sigma, \Pi)$ as a signature. Let $\text{Var}$ be a countable set of variables, the elements of which we denote by $x, y, \ldots$. We call a finite list $\Gamma$ of pairs $x : \tau$ of variables and types a context. The set $\Lambda_\Sigma$ of (well-typed) terms over $\Sigma$ is the collection of all $M$ with $\Gamma \vdash M : \tau$ for some context $\Gamma$ and type $\tau \in T$, where $\Gamma \vdash M : \tau$ is defined inductively in Fig. 2. A term is called closed if $\Gamma \vdash M : \tau$, otherwise it is called open. Finally, we let $\Lambda_\Sigma^c$ denote the set of all terms $M$ that do not involve $x$.

Definition 3. Let $(\Sigma, \Pi)$ be a signature. We say that $\varphi$ is a (first-order) formula in context $\Gamma$, if $\Gamma \vdash \varphi$ is inductively derivable from the rules in Fig. 3.

Definition 4. The reduction relation $\rightarrow$ on terms in $\Lambda_\Sigma$ is given as the compatible closure (reduction under applications and binders) of $\beta$- and fix-reduction:

\[
\begin{align*}
(\lambda x. M)N &\rightarrow M[N/x] \\
\text{fix } x. M &\rightarrow M[\text{fix } x. M/x]
\end{align*}
\]
We denote the reflexive, transitive closure of \( \rightarrow \) by \( \rightarrow^* \). Two terms \( M \) and \( N \) are called **convertible**, if \( M \equiv N \), where \( \equiv \) is the equivalence closure of \( \rightarrow \). Conversion of terms extends to formulae in the obvious way: if \( M_k \equiv M'_k \) for \( k = 1, \ldots, n \), then \( pM_1 \cdots M_n \equiv pM'_1 \cdots M'_n \).

We will use in the following that the above calculus features subject reduction and confluence, cf. [61]: if \( \Gamma \vdash M : \tau \) and \( M \equiv N \), then \( \Gamma \vdash N : \tau \); and \( M \equiv N \) iff there is a term \( P \), such that \( M \rightarrow^* P \) and \( N \rightarrow^* P \).

The order of a type \( \tau \in T \) is given as usual by \( \text{ord}(i) = 0 \) and \( \text{ord}(\sigma \rightarrow \tau) = \max\{\text{ord}(\sigma) + 1, \text{ord}(\tau)\} \).

If \( \text{ord}(\tau) \leq 1 \), then the arity of \( \tau \) is given by \( \text{ar}(i) = 0 \) and \( \text{ar}(i \rightarrow \tau) = \text{ar}(\tau) + 1 \). A signature \( \Sigma \) is called **first-order**, if for all \( f : \tau \in \Sigma \) we have \( \text{ord}(\tau) \leq 1 \). We let the arity of \( f \) then be \( \text{ar}(\tau) \) and denote it by \( \text{ar}(f) \).

**Definition 5.** The set of **guarded base terms** over a first-order signature \( \Sigma \) is given by the following type-driven rules.

\[
\begin{array}{ccc}
\text{ord}(\tau) \leq 1 & f : \tau \in \Sigma & \Gamma \vdash f : \tau \\
\Gamma \vdash x : \tau & \Gamma, x : \tau, y_1 : \iota, \ldots, y_{\text{ar}(\tau)} : \iota \vdash \Gamma \vdash M_i : \iota & 1 \leq i \leq \text{ar}(f) \\
& \Gamma \vdash \text{fix} \ x. \Lambda \ y \ : \ f \ M : \tau
\end{array}
\]

General **guarded terms** are terms \( M \), such that all fix-subterms are guarded base terms, which means that they are generated by the following grammar.

\[
G ::= M \ (\text{with } \text{ord}(M : \tau) \text{ for some type } \tau) \mid c \in \Sigma \mid x \in \text{Var} \mid G \ G \mid \lambda x. \ G
\]

Finally, \( M \) is a **first-order** term over \( \Sigma \) with \( \Gamma \vdash M : \tau \) if \( \text{ord}(\tau) \leq 1 \) and the types of all variables occurring in \( \Gamma \) are of order \( 0 \). We denote the set of guarded first-order terms \( M \) with \( \Gamma \vdash M : \iota \) by \( \Lambda^{G,1}_\Sigma(\Gamma) \) and the set of guarded terms in \( \Gamma \) by \( \Lambda^G_\Sigma(\Gamma) \). If \( \Gamma \) is empty, we just write \( \Lambda^{G,1}_\Sigma \) and \( \Lambda^G_\Sigma \), respectively.

Note that an important aspect of guarded terms is that no free variable occurs under a fix-operator. **Guarded base terms** should be seen as specific fixed point terms that we will be able to unfold into potentially infinite trees. **Guarded terms** close guarded base terms under operations of the simply typed \( \lambda \)-calculus.

**Example 6.** Let us provide a few examples that illustrate (first-order) guarded terms. We use the first-order signature \( \Sigma = \{\text{scons} : i \rightarrow i \rightarrow i, s : i \rightarrow i, 0 : i\} \).

1. Let \( \text{sf} = \text{fix } f. \lambda x . \text{scons } (f \ (s \ x)) \) be the function that computes the streams of numerals starting at the given argument. It is easy to show that \( \text{ord}(\text{sf}) = 0 \rightarrow i \) and so \( \text{sf} \in \Lambda^{G,1}_\Sigma \).

2. For the same signature \( \Sigma \) we also have \( x : i \rightarrow i \rightarrow i \). Thus \( x \in \Lambda^{G,1}_\Sigma(x : i) \) and \( s \ x \in \Lambda^{G,1}_\Sigma(x : i) \).

3. We have \( x : i \rightarrow i \rightarrow 0 : i \), but \( x(0) \notin \Lambda^{G,1}_\Sigma(x : i \rightarrow i) \).

The purpose of guarded terms is that these are productive, that is, we can reduce them to a term that either has a function symbol at the root or is just a variable. In other words, guarded terms have head normal forms: We say that a term \( M \) is in **head normal form**, if \( M = f \overline{N} \) for some \( f \in \Sigma \) or if \( M = x \) for some variable \( x \). The following lemma is a technical result that is needed to show in Lem. 8 that all guarded terms have a head normal form.
Lemma 7. Let $M$ and $N$ be guarded base terms with $\Gamma, x : \sigma \vdash y : \tau$ and $\Gamma \vdash g : \sigma$. Then $M \left[ N/x \right]$ is a guarded base term with $\Gamma \vdash g M \left[ N/x \right] : \tau$.

Proof. Let $M$ and $N$ be as above. We proceed by induction on the derivation that $M$ is guarded to show that $M \left[ N/x \right]$ is guarded as well.

- Suppose $M = y$ for some variable. If $y = x$, then $M \left[ N/x \right] = N$ and $\tau = \sigma$. Thus, we have $\Gamma \vdash g M \left[ N/x \right] : \tau$.

- The case for signature symbols is immediate, as for $f \in \Sigma$ we have $f \left[ N/x \right] = f$.

- Suppose $\Gamma, x : \sigma \vdash f \left( M \right) : \tau$. By the IH, we have $\Gamma \vdash f \left( M \left[ N/x \right] \right) : \gamma$ and $\Gamma \vdash g \left( N \right) : \gamma$. Thus, we obtain $\Gamma \vdash g \left( M_1 \right) : \tau$.

- Finally, assume that $\Gamma, x : \sigma \vdash g \left( f M \right) : \tau$. Then by IH, we have

$$\Gamma, z : \tau, y_1 : t_1, \ldots, y_{\text{ar}(\tau)} : t_1 \vdash g \left( M_1 \left[ N/x \right] \right) : t$$

and so $\Gamma \vdash g \left( \text{fix } z. \lambda y. f M \right) \left[ N/x \right] : \tau$. \hfill \Box

Lemma 8. If $M$ is a first-order guarded term with $M \in \Lambda^G_{\Sigma} \left( \Gamma \right)$, then $M$ reduces to a unique head normal form. This means that either (i) there is a unique $f \in \Sigma$ and terms $N_1, \ldots, N_{\text{ar}(f)}$ with $\Gamma \vdash g \left( N_k \right) : t$ and $M \rightarrow f \left( N \right)$, and for all $L$ if $M \rightarrow L$, then $N \equiv L$; or (ii) $M \rightarrow x$ for some $x : t \in \Gamma$.

Proof. The term $M$ with $\Gamma \vdash g M : t$ can have either of the following three shapes:

1. $x$, where $x : t \in \Gamma$
2. $f \left( N \right)$ with $\Gamma \vdash g \left( N_k \right) : t$, or
3. $(\text{fix } x. \lambda y. f M) \left( N \right)$ with $\Gamma, x : \tau, y_1 : t_1, \ldots, y_{\text{ar}(\tau)} : t_1 \vdash g \left( M_k \right) : t$ for $k = 1, \ldots, \text{ar}(f)$ and $\Gamma \vdash g \left( N_i \right) : t$ for $i = 1, \ldots, \text{ar}(\tau)$,

because variables can only occur in argument position due to the order restriction of the types in $\Gamma$. In the first two cases we are done immediately. For the third case, we let $P = \text{fix } x. \lambda y. f M$ and then find that

$$P \rightarrow f \left( M \left[ P/x, N/y \right] \right)$$

Lemma 7 gives us now that each $M_l \left[ P/x, N/y \right]$ is guarded. Finally, if $M \rightarrow L$, then $N \equiv L$ by confluence of the reduction relation. \hfill \Box

In Lem. 8, we have shown that guarded base terms are stable under substitution, that is, substituting a guarded base term into another results into a guarded base term. The following lemma shows that the same is true for guarded terms. This result is necessary to define substitution for formulae over guarded terms, see Def. 10

Lemma 9. Let $M \in \Lambda^G_{\Sigma} \left( \Gamma, x \right)$ and $N \in \Lambda^G_{\Sigma} \left( \Gamma \right)$ be guarded terms with $\Gamma, x : \tau \vdash M : \sigma$ and $\Gamma \vdash N : \tau$. Then $M \left[ N/x \right] \in \Lambda^G_{\Sigma} \left( \Gamma \right)$ and $\Gamma \vdash M \left[ N/x \right] : \sigma$.

Proof. By an easy induction on $M$. \hfill \Box
We end this section by introducing the notion of an atom and refinements thereof. This will enable us to define the different logics and thereby to analyse the strength of coinduction hypotheses, which we promised in the introduction.

**Definition 10.** A formula $\varphi$ of the shape $\top$ or $p \cdot M_1 \cdots M_n$ is an atom and a

- first-order atom, if $p$ and all the terms $M_i$ are first-order;
- guarded atom, if all terms $M_i$ are guarded; and
- simple atom, if all terms $M_i$ are non-recursive, that is, are in $\Lambda_\omega$.

First-order, guarded and simple atoms are denoted by $\text{At}_1$, $\text{At}_\omega$ and $\text{At}_\omega^\omega$. We denote conjunctions of these predicates by $\text{At}_1^\omega = \text{At}_1 \cap \text{At}_\omega$ and $\text{At}_1^\omega = \text{At}_1 \cap \text{At}_\omega^\omega$.

Note that the restriction for $\text{At}_\omega$ only applies to fixed point terms. Hence, any formula that contains terms without fix is already in $\text{At}_1^\omega$ and $\text{At}_1^\omega \cap \text{At}_\omega^\omega = \text{At}_\omega^\omega$. Since these notions are rather subtle, we give a few examples

**Example 11.** We list three examples of first-order atoms.

1. For $x : i$ we have $\text{stream} \ x \in \text{At}_1$, but there are also “garbage” formulae like “$\text{stream} \ (\text{fix} \ x. \ x)$” in $\text{At}_1$. Examples of atoms that are not first-order are $p \cdot M$, where $p : (i \to i) \to o$ or $x : i \to i + M : \tau$.

2. Our running example “from 0 (fr 0)” is a first-order guarded atom in $\text{At}_1^\omega$.

3. The formulae in $\text{At}_1^\omega$ may not contain recursion and higher-order features. However, the atoms of Horn clauses in a logic program fit in here.

### 3 Coinductive Uniform Proofs

This section introduces the eight logics of the coinductive uniform proof framework announced and motivated in the introduction. The major difference of uniform proofs with, say, a sequent calculus is the “uniformity” property, which means that the choice of the application of each proof rule is deterministic and all proofs are in normal form (cut free). This subsumes the operational semantics of resolution, in which the proof search is always goal directed. Hence, the main challenge, that we set out to solve in this section, is to extend the uniform proof framework with coinduction, while preserving this valuable operational property.

We begin by introducing the different goal formulae and definite clauses that determine the logics that were presented in the cube for coinductive uniform proofs in the introduction. These clauses and formulae correspond directly to those of the original work on uniform proofs [53] with the only difference being that we need to distinguish atoms with and without fixed point terms. The general idea is that goal formulae (G-formulae) occur on the right of a sequent, thus are the goal to be proved. Definite clauses (D-formulae), on the other hand, are selected from the context as assumptions. This will become clear once we introduce the proof system for coinductive uniform proofs.

**Definition 12.** Let $D_i$ be generated by the following grammar with $i \in \{1, \omega\}$.

$$D_i ::= \text{At}_i^\omega \mid G \to D \mid D \land D \mid \forall x : \tau. D$$
We say that $\varphi$ will therefore admit coinduction goals of different strength. This ability to explicitly use stronger

The sets of definite clauses $(D\text{-}\text{formulae})$ and goals $(G\text{-}\text{formulae})$ of the four logics $co$-$fohc$, $co$-$fohh$, $co$-$hohc$, $co$-$hohh$ are the well-formed formulae of the corresponding shapes defined in Tab. 2. For the variations $co$-$fohh_{fix}$ etc. of these logics with fixed point terms, we replace upper index “$s$” with “$g$” everywhere in Tab. 2. A $D$-formula of the shape $\forall \vec{x}. A_1 \land \cdots \land A_n \rightarrow A_0$ is called $H$-formula or Horn clause if $A_k \in A^f_1$, and $H^g$-formula if $A_k \in A^g_1$. Finally, a logic program (or program) $P$ is a set of $H$-formulae. Note that any set of $D$-formulae in $fohc$ can be transformed into an intuitionistically equivalent set of $H$-formulae.

We are now ready to introduce the coinductive uniform proofs. Such proofs are composed of two parts: an outer coinduction that has to be at the root of a proof tree, and the usual the usual uniform proofs by Miller et al. [54]. The latter are restated in Fig. 4. Of special notice is the rule $co$-$fix$ in Fig. 5. Our proof system mimics the typical recursion with a guard condition found in coinductive programs and proofs [3, 5, 18, 30, 38]. This guardedness condition is formalised by applying the guarding modality $\langle \_ \rangle$ on the formula being proven by coinduction and the proof rules that allow us to distribute the guard over certain logical connectives, see Fig. 5. The guarding modality may be discharged only if the guarded goal was resolved against a clause in the initial program or any hypothesis, except for the coinduction hypotheses. This is reflected in the rule $de$-$cod$ (\_), where we may only pick a clause from $P$, and is in contrast to the rule $de$-$cod$, in which we can pick any hypothesis. The proof may only terminate with the initial step if the goal is no longer guarded.

Note that the $co$-$fix$ rule introduces a goal as a new hypothesis. Hence, we have to require that this goal is also a definite clause. Since coinduction hypotheses play such an important role, they deserve a separate definition.

\textbf{Definition 13.} Given a language $L$ from Tab. 2, a formula $\varphi$ is a coinduction goal of $L$ if $\varphi$ simultaneously is a $D$- and a $G$-formula of $L$.

Note that the coinduction goals of $co$-$fohc$ and $co$-$fohh$ can be transformed into equivalent $H$- or $H^g$-formulae, since any coinduction goal is a $D$-formula.

Let us now formally introduce the coinductive uniform proof system.

\textbf{Definition 14.} Let $P$ and $\Delta$ be finite sets of, respectively, definite clauses and coinduction goals, over the signature $\Sigma$, and suppose that $G$ is a goal and $\varphi$ is a coinduction goal. A \textit{sequent} is either a uniform provability sequent of the form $\Sigma; P; \Delta \Rightarrow G$ or $\Sigma; P; \Delta \Rightarrow A$ as defined in Fig. 4, or it is a coinductive uniform provability sequent of the form $\Sigma; P \vdash \varphi$ as defined in Fig. 5. Let $L$ be a language from Tab. 2. We say that $\varphi$ is coinductively provable in $L$, if $P$ is a set of $D$-formulae in $L$, $\varphi$ is a coinduction goal in $L$ and $\Sigma; P \vdash \varphi$ holds.

The logics we have introduced impose different syntactic restrictions on $D$- and $G$-formulae, and will therefore admit coinduction goals of different strength. This ability to explicitly use stronger

| Definite Clauses | Goals |
|------------------|-------|
| $co$-$fohc$      | $D_1$  | $G \equiv A_1^f \ | G \land G \ | G \lor G \ | \exists x : r. G$ |
| $co$-$hohc$      | $D_{co}$ | $G \equiv A_1^f \ | G \land G \ | G \lor G \ | \exists x : r. G$ |
| $co$-$fohh$      | $D_1$  | $G \equiv A_1^g \ | G \land G \ | G \lor G \ | \exists x : r. G \ | D \rightarrow G \ | \forall x : r. G$ |
| $co$-$hohh$      | $D_{co}$ | $G \equiv A_1^g \ | G \land G \ | G \lor G \ | \exists x : r. G \ | D \rightarrow G \ | \forall x : r. G$ |

Table 2: \textit{D}- and \textit{G}-formulae for coinductive uniform proofs.
coinduction hypotheses within a goal-directed search was missing in CoLP, for example. And it allows us to account for different coinductive properties of Horn clauses as described in the introduction. We finish this section by illustrating this strengthening.

The first example is one for the logic co-fohc, in which we illustrate the framework on the problem of type class resolution.

**Example 15.** Let us restate the Haskell type class inference problem discussed in the introduction in terms of Horn clauses:

\[ \kappa_1 : \text{eq} \ i \]

\[ \kappa_{\text{odd}} : \forall x. \text{eq} \ x \land \text{eq} \ (\text{even} \ x) \rightarrow \text{eq} \ (\text{odd} \ x) \]

\[ \kappa_{\text{even}} : \forall x. \text{eq} \ x \land \text{eq} \ (\text{odd} \ x) \rightarrow \text{eq} \ (\text{even} \ x) \]

To prove \text{eq} \ (\text{odd} \ i) for this set of Horn clauses, it is sufficient to use this formula directly as coinduction hypothesis, as shown in Fig. 6. Note that this formula is indeed a coinduction goal of
co-fohc, hence we find ourselves in the simplest scenario of coinductive proof search. In Tab. 1, \( \gamma_1 \) is a representative for this kind of coinductive proofs with simplest atomic goals.

It was pointed out in [35] that Haskell’s type class inference can also give rise to irregular corecursion. Such cases may require the more general coinduction hypothesis (e.g. universal and/or implicative) of co-fohh or co-hohh. The below set of Horn clauses is a simplified representation of a problem given in [35]:

\[
\begin{align*}
\kappa_1 : \ & \text{eq} \ i \\
\kappa_2 : \ & \forall x. (\text{eq} \ x) \land \text{eq} \ (s \ (g \ x)) \rightarrow \text{eq} \ (s \ x) \\
\kappa_3 : \ & \forall x. \text{eq} \ x \rightarrow \text{eq} \ (g \ x)
\end{align*}
\]

Trying to prove \( \text{eq} \ (s \ i) \) by using \( \text{eq} \ (s \ i) \) directly as a coinduction hypothesis is deemed to fail, as the coinductive proof search is irregular and this coinduction hypothesis would not be applicable in any guarded context. But it is possible to prove \( \text{eq} \ (s \ i) \) as a corollary of another theorem: \( \forall x. (\text{eq} \ x) \rightarrow \text{eq} \ (s \ x) \). Using this formula as coinduction hypothesis leads to a successful proof, which we omit here. From this more general goal, we can derive the original goal by instantiating the quantifier with \( i \) and eliminating the implication with \( \kappa_i \). This second derivation is sound with respect to the models, as we show in Thm. 39.

We encounter \( \gamma_2 \) from Tab. 1 in a similar situation: To prove \( p \ a \), we first have to prove \( \forall x. p \ x \) in co-fohh, and then obtain \( p \ a \) as a corollary by appealing to Thm. 39. The next example shows that we can cover all cases in Tab. 1 by providing a proof in co-hohh that involves irregular recursive terms.

**Example 16.** Recall the clause \( \forall x. y. \text{from} \ (s \ x) \ y \rightarrow \text{from} \ x \ (s \text{cons} \ x \ y) \) that we named \( \kappa_{\text{from}} \) in the introduction. Proving \( \exists y. \text{from} \ 0 \ y \) is again not possible directly. Instead, we can use the term \( s_{fr} = \text{fix} \ f. \lambda x. \text{scons} \ x \ (f \ (s \ x)) \) from Ex. 6 and prove \( \forall x. \text{from} \ x \ (s_{fr} \ x) \) coinductively, as shown in Fig. 7. This formula gives a coinduction hypothesis of sufficient generality. Note that the correct coinduction hypothesis now requires the fixed point definition of an infinite stream of successive
Figure 7: The co-hohhfix proof for \( \forall x. \text{from} \ (s_{fr} x) \). Note that the last step of the leftmost branch involves \( \text{from} \ (s \ (\text{scons} \ (s_{fr} \ x))) \equiv \text{from} \ (s_{fr} \ x) \).

There are examples of coinductive proofs that require a fixed point definition of an infinite stream, but do not require the syntax of higher-order terms or hereditary Harrop formulae. Such proofs can be performed in the co-fohcfix logic. A good example is a proof that the stream of zeros satisfies the Horn clause theory defining the predicate \( \text{stream} \) in the introduction. The goal (\( \text{stream} \ s_0 \)), with \( s_0 = \text{fix} x. \text{scons} \ 0 \ x \) can be proven directly by coinduction. Similarly, one can type self-application with the infinite type \( a = \text{fix} t. t \to b \) for some given type \( b \). The proof for typed \( [x : a] \ (\text{app} \ x \ x) \ b \) is then in co-fohcfix. Finally, the clause \( y s \) is also in this group. More generally, circular unifiers obtained from CoLP’s [40] loop detection yield immediately guarded fixed point terms, and thus CoLP corresponds to coinductive proofs in the logic co-fohcfix. A general discussion of Horn clause theories that describe infinite objects was given in [47], where the above logic programs were identified as being productive.

4 Coinductive Uniform Proofs and Intuitionistic Logic

In the last section, we introduced the framework of coinductive uniform proofs, which gives an operational account to proofs for coinductively interpreted logic programs. Having this framework at hand, we need to position it in the existing ecosystem of logical systems. The goal of this section is to prove that coinductive uniform proofs are in fact constructive. We show this by first introducing an extension of intuitionistic first-order logic that allows us to deal with recursive proofs for coinductive predicates. Afterwards, we show that coinductive uniform proofs are sound relative to this logic by means of a proof tree translation. The model-theoretic soundness proofs for both logics will be provided in Section 5.

We begin by introducing an extension of intuitionistic first-order logic with the so-called later modality, written \( \triangleright \). This modality is the essential ingredient that allows us to equip proofs with a controlled form of recursion. The later modality stems originally from provability logic, which characterises transitive, well-founded Kripke frames [29, 73], and thus allows one to carry out induction without an explicit induction scheme [15]. Later, the later modality was picked up by the type-theoretic community to control recursion in coinductive programming [7, 8, 20, 56, 58], mostly with the intent
to replace syntactic guardedness checks for coinductive definitions by type-based checks of well-definedness.

Formally, the logic iFOL\sb{\textbullet} is given by the following definition.

**Definition 17.** The formulae of iFOL\sb{\textbullet} are given by Def. 3 and the rule:

\[ \Gamma \vdash \phi \]

Conversion extends to these formulae in the obvious way. Let \( \phi \) be a formula and \( \Delta \) a sequence of formulae in iFOL\sb{\textbullet}. We say \( \phi \) is **provable in context** \( \Gamma \) **under the assumptions** \( \Delta \) in iFOL\sb{\textbullet}, if \( \Gamma \vdash \Delta \vdash \phi \) holds. The **provability relation** \( \vdash \) is thereby given inductively by the rules in Fig. 8 and Fig. 9.

The rules in Fig. 8 are the usual rules for intuitionistic first-order logic and should come at no surprise. More interesting are the rules in Fig. 9, where the rule (Löb) introduces recursion into the proof system. Furthermore, the rule (Mon) allows us to to distribute the later modality over implication, and consequently over conjunction and universal quantification. This is essential in the translation in Thm. 20 below. Finally, the rule (Next) gives us the possibility to proceed without any recursion, if necessary.

Note that so far it is not possible to use the assumption \( \text{\textbullet} \phi \) introduced in the (Löb)-rule. The idea is that the formulae of a logic program provide us the obligations that we have to prove, possibly by recursion, in order to prove a coinductive predicate. This is cast in the following definition.

**Definition 18.** Given an \( H^{\text{\textbullet}} \)-formula \( \phi \) of the shape \( \forall \bar{x}. (A_1 \land \cdots \land A_n) \rightarrow \psi \), we define its **guarding** \( \overline{\phi} \) to be \( \forall \bar{x}. (\text{\textbullet} A_1 \land \cdots \land \text{\textbullet} A_n) \rightarrow \psi \). For a logic program \( P \), we define its guarding \( \overline{P} \) by guarding each formula in \( P \).
The following admissible rules are easily derivable in the logic \(i\text{FOL}_\leq\) and are essential in showing soundness of \(co-hoh\text{h}_{fix}\) with respect to \(i\text{FOL}_\leq\).

**Lemma 19.**
\[
\begin{align*}
\Gamma \vdash \varphi \quad \varphi \equiv \psi & \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi} \\
\Gamma \vdash \varphi & \quad \frac{\Delta \vdash \varphi \quad x : \tau \notin \Gamma}{\Gamma, x : \tau \vdash \Delta \vdash \varphi} \quad \text{(Weak)} \\
\Gamma \vdash \varphi_1 \rightarrow \varphi_2 \vdash \psi & \quad \frac{\Gamma \vdash \varphi_1 \rightarrow \varphi_2 \vdash \psi}{\Gamma \vdash (\varphi_1 \rightarrow \varphi_2) \vdash \psi} \quad \text{(Mon)} \\
\Gamma \vdash \Delta \vdash (\varphi \land \psi) & \quad \frac{\Gamma \vdash \Delta \vdash \varphi \land \psi}{\Gamma \vdash \Delta \vdash (\varphi \land \psi)} \quad \text{(-Pres-\land_r)} \\
\Gamma \vdash \Delta \vdash (\forall x : \tau. \varphi) & \quad \frac{\Gamma \vdash \Delta \vdash \forall x : \tau. \varphi \vdash \psi}{\Gamma \vdash \Delta \vdash (\forall x : \tau. \varphi) \vdash \psi} \quad \text{(-Pres-\forall_r)} \\
\Gamma \vdash \Delta \vdash \varphi \land \psi & \quad \frac{\Gamma \vdash \Delta \vdash \varphi \land \psi}{\Gamma \vdash \Delta \vdash (\varphi \land \psi)} \\
\Gamma \vdash \Delta \vdash (\forall x : \tau. \varphi) & \quad \frac{\Gamma \vdash \Delta \vdash \forall x : \tau. \varphi \vdash \psi}{\Gamma \vdash \Delta \vdash (\forall x : \tau. \varphi) \vdash \psi} \quad \text{(-Pres-\forall_l)}
\end{align*}
\]

**Proof.**
- Preservation of well-formed formulae under conversion follows by induction on formulae from type preservation of reductions.
- The well-formedness of \(\varphi\) follows from provability by induction on \(\varphi\).
- Weakening is also given by induction on \(\varphi\).
- The other rules follow from implication introduction and elimination, and monotonicity of \(\vdash\). \(\square\)

The translation given in Def. 18 of a logic program into formulae that admit recursion corresponds unfolding a coinductive predicate, cf. [13]. We show now how to transform a coinductive uniform proof tree into a proof tree in \(i\text{FOL}_\leq\), such that the recursion and guarding mechanisms in both logics match up.

**Theorem 20.** If \(P\) is a logic program over a first-order signature \(\Sigma\) and the sequent \(\Sigma ; P \iff \varphi\) is provable in \(co-hoh\text{h}_{fix}\), then \(P \vdash \varphi\) is provable in \(i\text{FOL}_\leq\).

To prove this theorem, one uses that each coinductive uniform proof tree starts with an initial tree that has an application of the \(\text{co-fix}\)-rule at the root and that eliminates the guard by using the rules in Fig. 5. At the leaves of this tree, one finds proof trees that proceed only by means of the rules in Fig. 5. The initial tree is then translated into a proof tree in \(i\text{FOL}_\leq\) that starts with an application of the \(\text{Löb}\)-rule, which corresponds to the \(\text{co-fix}\)-rule, and that simultaneously transforms the coinduction hypothesis and applies introduction rules for conjunctions etc. This ensures that we can match the coinduction hypothesis with the guarded formulae of the program \(P\).

**Proof.** We provide a sketch of the proof. First, we note that the coinduction goal in \(co-hoh\text{h}_{fix}\) is given by the following grammar.

\[
\text{CG} ::= \text{At}_{\sigma}^\mathcal{G} \mid \text{CG} \rightarrow \text{CG} \mid \text{CG} \land \text{CG} \mid \forall x : \tau. \text{CG}
\]
Thus, a coinduction goal is the restriction of FOL to implication, conjunction and universal quantification. Note that such a coinduction goal is intuitionistically equivalent to a conjunction of Horn-clauses. Assume that we are given a uniform proof tree $T$. We translate this tree into a proof tree $T'$ in \( \text{iFOL}_\land \). The proof proceeds in the following steps.

1. The first step of a proof tree $T$ starting in $\Sigma; P \leftrightarrow \varphi$ must be an application of the co-fix rule to a proof tree $T_1$ ending in $\Sigma; P; \varphi \rightarrow \langle \varphi \rangle$. This step can be directly translated into an application of the Löb rule. Hence, if $T'_1$ is the translation of $T_1$ with conclusion $\overline{P}, \triangleright \varphi \dashv \vdash \varphi$, then $T'$ is given by applying (Löb) to $T'_1$, thereby obtaining a proof tree ending in the desired sequent $\overline{P} \vdash \varphi$.

2. The next step must then be either $\forall R()$, $\land R()$, $\rightarrow R()$ or $\text{DECIDE}()$. To prove this by induction on the proof tree, we need to define coinduction goal contexts. These are contexts $\varphi[-]$ with a hole $[-]$, such that plugging an atom from $\text{At}_0\gamma$ into the hole yields a coinduction goal. More generally, we will need contexts with multiple holes $[-]$, that are indexed from 0 to $n$ for some $n \in \mathbb{N}$. Formally, such contexts are given by the following grammar.

$$H ::= [-]_i \mid [-]_i \rightarrow H \mid \forall x : \tau.H$$

$$C ::= H \mid C \land C$$

Let $C$ be a context, we write $C[\overline{\varphi}]$ for the formula that arises by replacing the holes $[-]_i$ by $\varphi_i$. Note that this may result in binding of free variables in $\varphi_1$ and $\varphi_2$.

We prove by induction on proof trees that for any context $\Gamma$, any set of formulae $P$, any context $C$ and any proof for $\Sigma, \Gamma; P; C[\varphi], \Delta \rightarrow \langle \varphi \rangle$ that there is a proof for $\Gamma \mid \overline{P}, C[\triangleright \varphi], \Delta \vdash \varphi$. The translation for this step follows then by taking $\Gamma$ and $\Delta$ to be empty and $C$ to be $[-]$.

In the $\forall R()$ case, we get a proof tree for $\Sigma, \Gamma; P; C[\forall x. \varphi], \Delta \rightarrow \langle \forall x. \varphi \rangle$ that has the sequent $\Sigma, \Gamma, x; P; C[\forall x. \varphi], \Delta \rightarrow \langle \varphi \rangle$ as its only premise. By putting $C' = C[\forall x. [-]]$, this premise can be written as $\Sigma, \Gamma, x; P; C'[\varphi], \Delta \rightarrow \langle \varphi \rangle$ from which we obtain by induction a proof tree for $\Gamma, x \mid \overline{P}, C'[\triangleright \varphi], \Delta \vdash \varphi$. Using the derived rule (\triangleright\text{-Pres}\triangleright\forall_i) from Lem. 19 and by the rule (V-I), we thus obtain a proof tree for the sequent $\Gamma \mid \overline{P}, C[\triangleright (\forall x. \varphi)] \vdash \forall x. \varphi$.

For the cases $\land R()$ and $\rightarrow R()$, one proceeds similarly as for $\forall R()$ by appealing to the fact that $\triangleright$ preserves conjunction and implication, respectively. The only things to be taken care of are the multi-contexts in the conjunction and the extension of $\Delta$ in the implication case. Finally, the $\text{DECIDE}()$ rule is dealt with in the next step.

3. For an application of either of the decide rules, there are generally two cases to consider: either the clause $D$ is selected from $P \cup \Delta$ by $\text{DECIDE}()$ or $\text{DECIDE}$, or $\text{DECIDE}$ selects $C[\overline{A}]$. In both cases, we proceed by induction to analyse of the proof tree for $\Sigma, \Gamma; P; C[\overline{A}], \Delta \Rightarrow B$.

Define $H ::= C[\triangleright \overline{A}]$. We then obtain the following cases from the fact that $D$ and $H$ are Horn-clauses with the later modality in specific places.

a) $D \in P$ is selected. Then the proof tree in \( \text{iFOL}_\land \) will have at its root $\Gamma \mid \overline{P}, H, \Delta \vdash B$ and at its leaves sequents of the form $\Gamma \mid \overline{P}, H, \Delta \vdash \triangleright C$ for some atoms $C$.

b) $C[\overline{A}]$ is selected. Then the resulting proof tree in \( \text{iFOL}_\land \) will have at its root $\Gamma \mid \overline{P}, H, \Delta \vdash \triangleright A_k$ for some $k$, and as its leaves sequents of the form $\Gamma \mid \overline{P}, H, \Delta \vdash \triangleright A_i$ for some $i$.

Our goal is now to combine such proof trees. The only mismatch might occur when we have a proof tree that has $\Gamma \mid \overline{P}, H \vdash B$ as root (first case) that has to be attached to a leaf of another
\[
\frac{x | \Delta \vdash \forall x. \forall t. \text{from } (s \ x) \ t \to \text{from } x \ (\text{scons } x \ t)}{(\text{Proj})}
\]
\[
\frac{x | \Delta \vdash \text{from } (s \ x) \ (s_{\text{fr}} \ x) \to \text{from } x \ (\text{scons } c \ (s_{\text{fr}} \ x))}{(\text{Conv})}
\]
\[
\frac{x | \Delta \vdash \text{from } x \ (\text{scons } x \ (s_{\text{fr}} \ x))}{(\text{V-E})}
\]
\[
\frac{x | \Delta \vdash \text{from } x \ (s_{\text{fr}} \ x)}{(\text{V-1})}
\]
\[
\frac{P, \forall x. \text{from } x \ (s_{\text{fr}} \ x) + \forall x. \text{from } x \ (s_{\text{fr}} \ x)}{(\text{Löb})}
\]
\[
\frac{P, \forall x. \text{from } x \ (s_{\text{fr}} \ x) + \forall x. \text{from } x \ (s_{\text{fr}} \ x)}{(\text{Pres-} \forall)}
\]
\[
\frac{P, \forall x. \text{from } x \ (s_{\text{fr}} \ x) + \forall x. \text{from } x \ (s_{\text{fr}} \ x)}{(\text{Next})}
\]
\[
\frac{P, \forall x. \text{from } x \ (s_{\text{fr}} \ x) + \forall x. \text{from } x \ (s_{\text{fr}} \ x)}{(\text{E})}
\]

Figure 10: \(iFOL\) proof for Example 21. \(\Delta\) abbreviates \(\overline{P}, \forall x. \text{from } x \ (s_{\text{fr}} \ x)\).

proof tree (from either case), which will be of the form \(\Gamma | \overline{P}, H + \Rightarrow C\) for some atom \(C\). Since this match arises from a uniform proof, we have that \(C = B\). Hence, we can combine these two trees by appealing to the (Next) rule:

\[
\frac{\Gamma | \overline{P}, H \vdash B}{\Gamma | \overline{P}, H + \Rightarrow B}
\]

In all the other cases, the trees can be combined directly. \(\Box\)

Example 21. Recall the following clause from the introduction.

\[
\forall x. \forall t. \text{from } (s \ x) \ t \to \text{from } x \ (\text{scons } x \ t) \tag{1}
\]

In Ex. [16] we provided the CUP proof for \(\forall x. \text{from } x \ (s_{\text{fr}} \ x)\). In this example, we show how that proof is translated in a proof in \(iFOL\).

The guarding of clause \(1\) is given by the clause \(2\).

\[
\forall x. \forall t. \Rightarrow (\text{from } (s \ x) \ t) \to \text{from } x \ (\text{scons } x \ t) \tag{2}
\]

To save space, when we build a proof in \(iFOL\) using (V-I), (V-E) or (Conv), etc., we may omit the condition branch, which is \(x : \tau \notin \Gamma, \Gamma + M : \tau \text{ or } \psi \equiv \psi'\) respectively, if and only if we know that the condition holds.

Now let \(\overline{P}\) denote the singleton set of clause \(2\). In Fig. 10 we display the \(iFOL\) proofs for \(\overline{P} + \forall x. \text{from } x \ (s_{\text{fr}} \ x)\) that arises from the CUP proof.

The results of this section show that it is irrelevant whether the guarding modality is used on the right (CUP-style) or on the left (\(iFOL\)-style), as the former can be translated into the latter. However, CUP uses the guarding on the right to preserve proof uniformity, whereas \(iFOL\) extends a general sequent calculus. Thus, to obtain the reverse translation, we would have to have an admissible cut rule.
in CUP. The main ingredient to such a cut rule is the ability to prove several coinductive statements simultaneously. This is possible in CUP by proving the conjunction of these statements. Unfortunately, we cannot eliminate such a conjunction into one of its components, since this would require non-deterministic guessing in the proof construction, which in turn breaks uniformity. Thus, we leave a solution of this problem for future work.

5 Herbrand Models and Soundness

In Sec. 4 we showed that coinductive uniform proofs are sound relative to the intuitionistic logic $\text{iFOL}_I$. This gives us a handle on the constructive nature of coinductive uniform proofs. Since $\text{iFOL}_I$ is a non-standard logic, we still need to provide semantics for that logic. We do this by interpreting in Sec. 5.4 the formulae of $\text{iFOL}_I$ over the well-known (complete) Herbrand models and prove the soundness of the accompanying proof system with respect to these models. Although we obtain soundness of coinductive uniform proofs over Herbrand models from this, this proof is indirect and does not give a lot of information about the models captured by the different calculi co-fohc etc. For this reason, we will give in Sec. 5.3 a direct soundness proof for coinductive uniform proofs. We also obtain coinduction invariants from this proof for each of the calculi, which allows us to describe their proof strength.

5.1 Coinductive Herbrand Models and Semantics of Terms

Before we come to the soundness proofs, we introduce in this section (complete) Herbrand models by using the terminology of final coalgebras. We then utilise this description to give operational and denotational semantics to guarded terms. These semantics show that guarded terms allow the description and computation of potentially infinite trees.

The coalgebraic approach has been proven very successful both in logic and programming [1, 76, 77]. We will only require very little category theoretical vocabulary and assume that the reader is familiar with the category $\text{Set}$ of sets and functions, and functors, see for example [11, 24, 50]. The terminology of algebras and coalgebras [4, 46, 65, 66] is given by the following definition.

**Definition 22.** A coalgebra for a functor $F: \text{Set} \to \text{Set}$ is a map $c: X \to FX$. Given coalgebras $d: Y \to FY$ and $c: X \to FX$, we say that a map $h: Y \to X$ is a homomorphism $d \to c$ if $Fh \circ d = c \circ h$. We call a coalgebra $c: X \to FX$ final, if for every coalgebra $d$ there is a unique homomorphism $h: d \to c$. We will refer to $h$ as the coinductive extension of $d$.

The idea of (complete) Herbrand models is that a set of Horn clauses determines for each predicate symbol a set of potentially infinite terms. Such terms are (potentially infinite) trees, whose nodes are labelled by function symbols and whose branching is given by the arity of these function symbols. To be able to deal with open terms, we will allow such trees to have leaves labelled by variables. Such trees are a final coalgebra for a functor determined by the signature.

**Definition 23.** Let $\Sigma$ be first-order signature. The extension of a first-order signature $\Sigma$ is a (polynomial) functor $\llbracket \Sigma \rrbracket: \text{Set} \to \text{Set}$ given by

$$\llbracket \Sigma \rrbracket(X) = \bigsqcup_{f \in \Sigma} X^{\text{ar}(f)},$$

where $\text{ar}: \Sigma \to \mathbb{N}$ is defined in Sec. 2 and $X^n$ is the $n$-fold product of $X$. We define for a set $V$ a functor $\llbracket \Sigma \rrbracket + V: \text{Set} \to \text{Set}$ by $(\llbracket \Sigma \rrbracket + V)(X) = \llbracket \Sigma \rrbracket(X) + V$, where $+$ is the coproduct (disjoint union) in $\text{Set}$. 18
To make sense of the following definition, we note that we can view $\Pi$ as a signature and we thus obtain its extension $[\Pi]$. Moreover, we note that the final coalgebra of $[\Sigma] + V$ exists because $[\Sigma]$ is a polynomial functor.

**Definition 24.** Let $\Sigma$ be a first-order signature. The coterms over $\Sigma$ are given by a final coalgebra $\text{root}_V : \Sigma^\omega(V) \to \Sigma^\omega(V) + V$. For brevity, we denote the coterms with no variables, i.e. $\Sigma^\omega(\emptyset)$, by root: $\Sigma^\omega \to \Sigma^\omega(\Sigma^\omega)$, and call it the (complete) Herbrand base and its elements ground coterms. Finally, we let the (complete) Herbrand base $B^\omega$ be the set $[\Pi](\Sigma^\omega)$.

The construction $\Sigma^\omega(V)$ gives rise to a functor $\Sigma^\omega : \text{Set} \to \text{Set}$, called the free completely iterative monad [5]. If there is no ambiguity, we will drop the injections $\kappa_i$ when describing elements of $\Sigma^\omega(V)$. Note that $\Sigma^\omega(V)$ is final with property that for every $s \in \Sigma^\omega(V)$ either there are $f \in \Sigma$ and $T \in (\Sigma^\omega(V))^{\text{sort}(f)}$ with $\text{root}_V(s) = f(T)$, or there is $x \in V$ with $\text{root}_V(s) = x$. Finally allows us to specify unique maps into $\Sigma^\omega(V)$ by giving a coalgebra $X \to \Sigma^\omega(X) + V$. In particular, one can define for each $\theta : V \to \Sigma^\omega$ the substitution $t[\theta]$ of variables in the coterm $t$ by $\theta$ as the coinductive extension of the following coalgebra.

$$\Sigma^\omega(V) \xrightarrow{\text{root}_V} \Sigma^\omega(\Sigma^\omega(V)) + V \xrightarrow{\text{id} \circ \text{root}_V} \Sigma^\omega(\Sigma^\omega(V))$$

Now that we have set up the basic terminology of coalgebras, we can give semantics to guarded terms from Def. [5]. The idea is that guarded terms guarantee that we can always compute with them so far that we find a function symbol in head position, see Lem. [8]. This function symbol determines then the label and branching of a node in the tree generated by a guarded term. If the computation reaches a constant or a variable, then we stop creating the tree at the present branch. This idea is captured by the following lemma.

**Lemma 25.** There is a map $\llbracket - \rrbracket_1 : \Lambda^{G,1}_\Sigma(\Gamma) \to \Sigma^\omega(\Gamma)$ that is unique with

1. if $M \equiv N$, then $\llbracket M \rrbracket_1 = \llbracket N \rrbracket_1$, and
2. for all $M$, if $M \rightarrow f \overrightarrow{\text{N}}$ then $\text{root}_V(\llbracket M \rrbracket_1) = f(\llbracket N \rrbracket_1)$, and if $M \rightarrow x$ then $\text{root}_V(\llbracket M \rrbracket_1) = x$.

**Proof.** We define a coalgebra $c : \Lambda^{G,1}_\Sigma(\Gamma)/\equiv \to \Sigma\llbracket \Lambda^{G,1}_\Sigma(\Gamma)/\equiv \rrbracket + \Gamma$ on the quotient of guarded terms by convertibility as follows.

$$c[M] = \begin{cases} f[\overrightarrow{N}], & \text{if } M \rightarrow f \overrightarrow{\text{N}} \\ x, & \text{if } M \rightarrow x \end{cases}$$

This is a well-defined map by Lem. [8]. By finality of $\Sigma^\omega(\Gamma)$, we obtain a unique homomorphism $h : \Lambda^{G,1}_\Sigma(\Gamma)/\equiv \to \Sigma^\omega(\Gamma)$. This allows us to define $\llbracket - \rrbracket_1 = h \circ [-]$, which gives us immediately for $M \equiv N$ that $\llbracket M \rrbracket_1 = h[M] = h[N] = \llbracket N \rrbracket_1$. Moreover, we have

$$\text{root}(\llbracket M \rrbracket_1) = \text{root}(h[M])$$

$$= (\llbracket \Sigma \rrbracket(h) + \text{id})(c[M])$$

$$= \begin{cases} \llbracket \Sigma \rrbracket(h)(f[\overrightarrow{N}]) = f[h[N]] = f[\llbracket N \rrbracket_1], & \text{if } M \rightarrow f \overrightarrow{\text{N}} \\ \text{id}(x) = x, & \text{if } M \rightarrow x \end{cases}$$

Finally, assume that we are given a map $k : \Lambda^{G,1}_\Sigma(\Gamma) \to \Sigma^\omega(\Gamma)$ with the above two properties. The first allows us to lift $k$ to a map $k' : \Lambda^{G,1}_\Sigma(\Gamma)/\equiv \to \Sigma^\omega(\Gamma)$ with $k' \circ [-] = k$. Due to the second property we know that $k'$ is then a coalgebra homomorphism and by finality $k' = h$. Hence, we obtain from $\llbracket - \rrbracket_1 = h \circ [-] = k' \circ [-] = k$ that $\llbracket - \rrbracket_1$ is unique. $\square$
5.2 Interpretation of Basic Intuitionistic First-Order Formulae

In this section, we give an interpretation of the formulae in Def. 3, in which we restrict ourselves to guarded terms. This interpretation will be relative to models in the complete Herbrand universe. Since we later extend these models to Kripke models to be able to handle the later modality, we formulate these models already now in the language of fibrations [16, 45].

**Definition 27.** Let \( p : E \to B \) be a functor. Given an object \( I \in B \), the fibre \( E_I \) above \( I \) is the category of objects \( A \in E \) with \( p(A) = I \) and morphisms \( f : A \to B \) with \( p(f) = \text{id}_I \). The functor \( p \) is a (split) fibration if for every morphism \( u : I \to J \) in \( B \) there is a functor \( u^* : E_J \to E_I \), such that \( \text{id}_I^* = \text{Id}_{E_I} \) and \( (v \circ u)^* = u^* \circ v^* \). We call \( u^* \) the reindexing along \( u \).

To give an interpretation of formulae, consider the following category \( \text{Pred} \).

\[
\text{Pred} = \left\{ \begin{array}{l}
\text{objects:} \quad (X, P) \text{ with } X \in \text{Set} \text{ and } P \subseteq X \\
\text{morphisms:} \quad f : (X, P) \to (Y, Q) \text{ is a map } f : X \to Y \text{ with } f(P) \subseteq Q
\end{array} \right.
\]

The functor \( \mathbb{P} : \text{Pred} \to \text{Set} \) with \( \mathbb{P}(X, P) = X \) and \( \mathbb{P}(f) = f \) is a split fibration, see [45], where the reindexing functor for \( f : X \to Y \) is given by taking preimages: \( f^*(Q) = f^{-1}(Q) \). Note that each fibre \( \text{Pred}_X \) is isomorphic to the complete lattice of predicates over \( X \) ordered by set inclusion. Thus, we refer to this fibration as the predicate fibration.

Let us now expose the logical structure of the predicate fibration. This will allow us to conveniently interpret first-order formulae over this fibration, but it comes at the cost of having to introduce a good amount of category theoretical language. However, doing so will pay off in Sec. 5.4, where we will construct another fibration out of the predicate fibration. We can then use category theoretical results to show that this new fibration admits the same logical structure and allows the interpretation of the later modality.

The first notion we need is that of fibred products, coproducts and exponents, which will allow us to interpret conjunction, disjunction and implication.

**Definition 28.** A fibration \( p : E \to B \) has **fibred finite products** \( (1, \times) \), if each fibre \( E_I \) has finite products \( (1_I, \times_I) \) and these are preserved by reindexing: for all \( f : I \to J \), we have \( f^*(1_J) = 1_I \) and \( f^*(A \times_J B) = f^*(A) \times_I f^*(B) \). Fibred finite coproducts and exponents are defined analogously.

The fibration \( \mathbb{P} \) is a so-called first-order fibration, which allows us to interpret first-order logic, see [45 Def. 4.2.1].

**Definition 29.** A fibration \( p : E \to B \) is a first-order fibration if

- \( B \) has finite products and the fibres of \( p \) are preorders,

- \( p \) has fibred finite products \( (\top, \land) \) and coproducts \( (\bot, \lor) \) that distribute;

**Example 26.** Recall \( sfr \) from Ex. 6 and note that \( sfr \ 0 \to \text{scons} \ 0 \ (sfr \ (s \ 0)) \). Hence, we have \( \text{root}(\text{sfr} \ 0 \ 1) = \text{scons} \ 0 \ 1 \ (sfr \ (s \ 0)) \ 1 \). If we continue unfolding \( \text{sfr} \ 0 \ 1 \), then we obtain the infinite tree \( \text{scons} \ 0 \ (s \ 0) \to \text{scons} \ (s \ (s \ 0)) \to \cdots \).
• $p$ has fibred exponents $\rightarrow$; and

• $p$ has existential and universal quantifiers $\exists I, J \mapsto \pi_{I,J} \mapsto \forall I, J$ for all projections $\pi_{I,J} : I \times J \rightarrow I$.

A first-order $\lambda$-fibration is a first-order fibration with Cartesian closed base $B$.

The fibration $P : \text{Pred} \rightarrow \text{Set}$ is a first-order $\lambda$-fibration, as all its fibres are posets and $\text{Set}$ is Cartesian closed; $P$ has fibred finite products $(\top, \cap)$, given by $\top_X = X$ and intersection; fibred distributive coproducts $(\emptyset, \cup)$; fibred exponents $\Rightarrow$, given by $(P \Rightarrow Q) = \{ t \mid \text{if } t \in P, \text{then } \overline{t} \in Q \}$; and universal and existential quantifiers given for $P \in \text{Pred}_{X,Y}$ by

\[
\forall X,Y P = \{ x \in X \mid \forall y \in Y. (x, y) \in P \} \quad \exists X,Y P = \{ x \in X \mid \exists y \in Y. (x, y) \in P \}.
\]

The purpose of first-order fibrations is to capture the essentials of first-order logic, while the $\lambda$-part takes care of higher-order features of the term language. In the following, we interpret types, contexts, guarded terms and formulae in the fibration $P : \text{Pred} \rightarrow \text{Set}$: We define for types $\tau$ and context $\Gamma$ sets $\llbracket \tau \rrbracket$ and $\llbracket \Gamma \rrbracket$; for guarded terms $M$ with $\Gamma \vdash M : \tau$ we define a map $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$ in $\text{Set}$; and for a formula $\Gamma \vdash \phi$ we give a predicate $\llbracket \phi \rrbracket \in P_{\text{Pred}}[\Gamma]$.

Remark. It should be noted that we give in the following an interpretation over concrete fibrations with their base over $\text{Set}$. However, the interpretations could also be given over general first-order $\lambda$-fibrations $p : E \rightarrow B$. The main issue is to get an interpretation of guarded terms over a final coalgebra for $[\Sigma]$ in a general category $B$. Currently, this interpretation crucially requires the category of sets as base category, see Lem. 25.

The semantics of types and contexts are given inductively in the Cartesian closed category $\text{Set}$, where the base type $\iota$ is interpreted as coterms, as follows.

\[
\llbracket \iota \rrbracket = \Sigma^\omega \\
\llbracket \tau \rightarrow \sigma \rrbracket = \llbracket \sigma \rrbracket^{\llbracket \tau \rrbracket} \\
\llbracket \emptyset \rrbracket = 1 \\
\llbracket \Gamma, x: \tau \rrbracket = \llbracket \Gamma \rrbracket \times \llbracket \tau \rrbracket
\]

We note that a coterm $t \in \Sigma^\omega(V)$ can be seen as a map $(\Sigma^\omega)^V \rightarrow \Sigma^\omega$ by applying a substitution in $(\Sigma^\omega)^V$ to $t : \sigma \mapsto t[\sigma]$. In particular, the semantics of a guarded first-order term $M \in \Lambda^G_{\Sigma}(\Gamma)$ is equivalently a map $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \Sigma^\omega$. We can now extend this map inductively to $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$ for all guarded terms $M \in \Lambda^G_{\Sigma}(\Gamma)$ with $\Gamma \vdash M : \tau$ by

\[
\llbracket M \rrbracket(y)(\overline{t}) = \llbracket M \rrbracket(x)[(\overline{x} \mapsto \overline{t})] \\
\llbracket c \rrbracket(y)(\overline{t}) = c \overline{t} \\
\llbracket x \rrbracket(y) = y(x) \\
\llbracket M N \rrbracket(y) = \llbracket M \rrbracket(y) \llbracket N \rrbracket(y) \\
\llbracket \lambda x. M \rrbracket(y)(t) = \llbracket M \rrbracket(y[x \mapsto t])
\]

Lemma 30. The mapping $\llbracket - \rrbracket$ is a well-defined function from guarded terms to functions, such that $\Gamma \vdash M : \tau$ implies $\llbracket M \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$.

Proof. Immediate by induction on $M$. \hfill \square

Since $P : \text{Pred} \rightarrow \text{Set}$ is a first-order fibration, we can interpret inductively all logical connectives of the formulae from Def. 3 in this fibration. The only case that is missing is the base case of predicate symbols. Their interpretation will be given over a Herbrand model that is constructed as the largest fixed point of an operator over all predicate interpretations in the Herbrand base. Both the operator and the fixed point are the subjects of the following definition.
Definition 31. We let the set of interpretations $\mathcal{I}$ be the powerset $\mathcal{P}(\mathcal{B}^\omega)$ of the complete Herbrand base. For $I \in \mathcal{I}$ and $p \in \Pi$, we denote by $I|_p$ the interpretation of $p$ in $I$ (the fibre of $I$ above $p$)

$$I|_p = \{ \vec{t} \in (\Sigma^\omega)^{ar(p)} \mid p(\vec{t}) \in I \}.$$ 

Given a set $P$ of $H^\omega$-formulae, we define a monotone map $\Phi_P : \mathcal{I} \rightarrow \mathcal{I}$ by

$$\Phi_P(I) = \{ \psi[\theta] \mid (\forall \vec{x}). \land_{k=1}^n \varphi_k \rightarrow \psi \in P, \theta : |\vec{x}| \rightarrow \Sigma^\omega, \forall k. \|\varphi_k\|_I[\theta] \in I \},$$

where $\|[-]_I[\theta]$ is the extension of semantics and substitution from coterms to the Herbrand base by functoriality of $\|\Pi\|$. The (complete) Herbrand model $M_P$ of $P$ is the largest fixed point of $\Phi_P$, which exists because $\mathcal{I}$ is a complete lattice.

Remark. Note that if $P$ is a set of Horn clauses (logic program), then the definition of the operator $\Phi_P$ in Def. 31 just becomes

$$\Phi_P(I) = \{ \psi[\theta] \mid (\forall \vec{x}). \land_{k=1}^n \varphi_k \rightarrow \psi \in P, \theta : |\vec{x}| \rightarrow \Sigma^\omega, \forall k. \varphi_k[\theta] \in I \},$$

as we do not have to unfold fixed point terms. Thus, in most cases, except in the proof of Thm. 39, we will drop the semantic brackets $\|[-]_I\|_I$.

Given a formula $\varphi$ with $\Gamma \vdash \varphi$ that contains only guarded terms, we define the semantics of $\varphi$ in $\text{Pred}$ from an interpretation $I \in \mathcal{I}$ inductively as follows.

$$\|\Gamma \vdash p \ M\|_I = \left(\|M\|_I\right)^.(I|_p)$$

$$\|\Gamma \vdash \tau\|_I = \tau^.(\Gamma|_I)$$

$$\|\Gamma \vdash \varphi \land \psi\|_I = \|\Gamma \vdash \varphi\|_I \land \|\Gamma \vdash \psi\|_I$$

$$\|\Gamma \vdash Q x : \tau. \varphi\|_I = \Omega^.(\Gamma|_I, \Gamma|_I) \|\Gamma, x : \tau \vdash \varphi\|_I$$

$Q \in \{\forall, \exists\}$

\(\square \in \{\land, \lor, \rightarrow\}\)

Lemma 32. The mapping $\|[-]_I\|$ is a well-defined function from formulae to predicates, such that $\Gamma \vdash \varphi$ implies $\|\varphi\|_I \subseteq \|\Gamma\|_I$ or, equivalently, $\|\varphi\|_I \in \text{Pred}|_{\Gamma|_I}$.

Proof. Immediate by induction on $\varphi$. \(\square\)

Let us demonstrate the interpretation of formulae on an example.

Example 33. Recall the formula $\forall x. y. \text{from} (s \ x) \ y \rightarrow \text{from} x (s \text{cons} \ x \ y)$, which we introduced as clause $\kappa_{\text{from}0}$. We spell out the interpretation of this formula. Note that root($\|s \ x\|_I$) = $s \ x$ = $s \ x$. Abusing notation, we write $s \ u$ for $\|s \ x\|_I [u/x]$, and analogously for the terms $y$, $x$ and $s \text{cons} \ x \ y$. We then have

$$\|\text{from} (s \ x) \ y\|_I = \left(\|s \ x\|_I, \|y\|_I\right)^.(I|_{\text{from}})$$

$$= \{(u, v) \in (\Sigma^\omega)^2 \mid (s \ u, v) \in I|_{\text{from}}\}$$

Using similar calculations for the other terms in the clause $\kappa_{\text{from}0}$, we obtain

$$\|\kappa_{\text{from}0}\|_I = \|\forall x. y. \text{from} (s \ x) \ y \rightarrow \text{from} x (s \text{cons} \ x \ y)\|_I$$

$$= \forall_1 \forall_2 (\text{from} (s \ x) \ y \rightarrow \text{from} x (s \text{cons} \ x \ y)|_I)$$

$$= \{* \mid \forall u, v. \text{if} (s \ u, v) \in I|_{\text{from}}, \text{then} (u, s \text{cons} \ u \ v) \in I|_{\text{from}}\}$$

As expected, we thus have $\|\kappa_{\text{from}0}\|_I = \{*\}$ if $I$ validates $\kappa_{\text{from}0}$.

This concludes the semantics of types, terms and formulae. We now turn to show that coinductive uniform proofs are sound for this interpretation.
5.3 Soundness of Coinductive Uniform Proofs for Herbrand Models

In this section, we give a direct proof of soundness for the coinductive uniform proof system from Sec. 5. Later, we will obtain another soundness result by combining the proof translation from Thm. 20 with the soundness of iFOL (Thm. 44 and 47). The purpose of giving a direct soundness proof for uniform proofs is that it allows the extraction of a coinduction invariant, see Lem. 37.

The main idea is as follows. Given a formula \( \varphi \) and a uniform proof \( \pi \) for \( \Sigma; P \iff \varphi \), we construct an interpretation \( I \in I \) that validates \( \varphi \), i.e., \( \models \varphi \), and that is contained in the complete Herbrand model \( M_P \). Combining these two facts, we obtain that \( \models \varphi \) \( M_P = \top \), and thus the soundness of uniform proofs.

To show that the constructed interpretation \( I \) is contained in \( M_P \), we use the usual coinduction proof principle:

**Definition 34.** An invariant for \( K \in I \) is a set \( I \in I \), such that \( K \subseteq I \) and \( I \) is a \( \Phi_P \)-invariant, that is, \( K \subseteq \Phi_P(I) \). If \( t \in B_\infty \), we also say that \( I \) is an invariant for \( t \), if it is an invariant for \( \{t\} \).

Since \( M_P \) is the largest fixed point of \( \Phi_P \), we immediately have that, if \( K \) has an invariant, then \( K \subseteq M_P \), see also [49].

In the remainder of this section, we will often have to refer to substitutions by coterms and their composition. The following definition will make dealing with this easier by organising these substitutions into a (Kleisli-)category. These notations are derived from the monad \( (\Sigma^\infty, \eta, \mu) \) with \( \eta \): \( \text{Id} \rightarrow \Sigma^\infty \) and \( \mu \): \( \Sigma^\infty \Sigma^\infty \rightarrow \Sigma^\infty \), cf. [3]:

**Definition 35.** A (Kleisli-)substitution \( \theta \) from \( V \) to \( W \), written \( \theta : V \rightarrow W \), is a map \( V \rightarrow \Sigma^\infty(W) \). Composition of \( \theta : V \leftrightarrow W \) and \( \delta : U \leftrightarrow V \) is given by

\[
\theta \circ \delta = U \xrightarrow{\delta} \Sigma^\infty(V) \xrightarrow{\Sigma^\infty(\theta)} \Sigma^\infty(\Sigma^\infty(W)) \xrightarrow{\text{inv}} \Sigma^\infty(W).
\]

In what follows, we extract, for any instance of a formula \( \varphi \), an explicit invariant from a proof \( \pi \) of \( \Sigma; P \iff \varphi \), which then yields the soundness of \( \mu \). More precisely, let \( \varphi \) be an \( H^\theta \)-formula with \( \varphi = \forall \vec{x}.A_1 \land \cdots \land A_n \rightarrow A_0 \) and let \( X \) be the set of variables in \( \vec{x} \). Given a substitution \( \theta : X \leftrightarrow \emptyset \), we need to show that if for all \( 1 \leq k \leq n \) we have \( \models A_k \{\theta\} \in M_P \), then \( \models A_0 \{\theta\} \in M_P \). The remainder of this section is devoted to constructing an invariant for \( \models A_0 \{\theta\} \).

We note that a uniform proof \( \pi \) for \( \Sigma; P \iff \varphi \) starts with

\[
\begin{align*}
\overset{\cdots}{\vdots} \\
\text{\( c : i, \Sigma; P; \varphi, h p \rightarrow \langle A_0[\vec{c}/\vec{x}] \rangle \) } \\
\text{\( \xrightarrow{\text{R}} \) } \\
\text{\( \vdash \) : \( i \notin \Sigma \) } \\
\text{\( \forall \vec{R} \) } \\
\end{align*}
\]

where \( \varphi = \forall \vec{x}.A_1 \land \cdots \land A_n \rightarrow A_0 \), \( h p = (A_1 \land \cdots \land A_n)[\vec{c}/\vec{x}] \), and the eigenvariables \( \vec{c} \) are all distinct. Let \( C \) the set of variables in \( \vec{c} \) and \( \Sigma \) the signature \( \vec{c} : i, \Sigma \). For brevity, we define \( A_k^C = A_k[\vec{c}/\vec{x}] \). Note that from here, the further proof of the given goal will only be based on the signature \( \Sigma^C \), that is, no new eigenvariables will be introduced higher in the proof. Thus, we can focus on \( A_0^C \) in our construction of an invariant: Given a substitution \( \theta_0 : C \leftrightarrow \emptyset \), we need to construct an invariant for \( \models A_0^C \{\theta_0\} \), given that we already have an invariant for the assumptions \( \models A_k^C \{\theta_0\} \) with \( 1 \leq k \leq n \).

We need to refer to the levels of the proof \( \pi \), which is the distance from the root sequent \( \Sigma; P \iff \varphi \). For example, the above proof tree displays levels 0 to 3 of the proof \( \pi \).
From here, the proof of \( \langle A_0[\overline{c}/\overline{x}] \rangle \) can only proceed by applying the rule \( \text{DECEIVE(\_)} \), with a chosen clause \( \kappa \) from \( P \). If \( \kappa \) is of the form \( \forall \gamma \cdot A \), then we define \( I = \{ \llbracket A \rrbracket_1[\theta] \mid \theta \text{ is a substitution} \} \). It is straightforward to show that this is an invariant. Since \( A_0[\overline{c}/\overline{x}] \equiv A[\gamma] \) for some substitution \( \gamma \), we can find a substitution \( \theta : \gamma \rightarrow \emptyset \), such that \( \{ A_0^C \} _1[\theta_0] = \{ A \} _1[\theta] \) (put \( \theta(y_i) = \{ y(y_i) \} _1[\theta_0] \)). Thus, we have that \( \{ A_0^C \} _1[\theta_0] \in I \), which shows that \( \{ A_0^C \} _1[\theta_0] \in \mathcal{M}_p \).

Having considered this simple case, we will now analyse the case when a chosen \( \kappa \) is of the form \( \forall \gamma \cdot \psi \rightarrow \psi' \), and \( A_0^C \equiv \psi'[\gamma] \) for some substitution \( \gamma \). In this case, applications of \( \text{DECEIVE(\_)} \) and \( \rightarrow L \) and \( \text{INITIAL} \) will eventually deliver the subgoal:

\[
\Sigma^C ; P; \varphi, h_\psi \Rightarrow \psi[\gamma]
\]

We will refer to this stage in the proof \( \pi \) as the \textit{level} \( l \) in \( \pi \). We now consider the rest of the proof tree above level \( l \), i.e. we consider the proof for \( \psi[\gamma] \). This is where the non-trivial part of the invariant construction will be obtained. In general, \( \psi[\gamma] \) will be given by \( A'_1 \land \ldots \land A'_m \), and the rule \( \land R \) will require subproofs for each of \( A'_i \). Let us consider a sub-proof of \( \pi \) for an arbitrary such \( A'_i \).

The proof can only proceed here by applying the rule \( \text{DECEIVE(\_)} \), in which case there are three options: to choose a program clause from \( P \), or choose \( h_\psi \) or \( \varphi \). Only the latter case is interesting for the invariant construction, as this is where the coinductive goal \( \varphi \) is instantiated, giving rise to a substitution that we will use in the invariant construction:

\[
\Sigma^C ; P; \varphi, h_\psi \Rightarrow A'_i \quad \forall L
\]

\[
\Sigma^C ; P; \varphi, h_\psi \Rightarrow A'_i \quad \text{DECEIVE}
\]

Generally, \( \varphi \) may be used with different substitutions multiple times within the proof tree \( \pi \). However, \( \varphi \) itself is uniquely determined by the only use of the rule \( \text{co-fix} \) in the root of \( \pi \). The above fragment of the proof for \( A'_i \) gives rise to a substitution \( \rho_1 = [\overline{N}/\overline{x}] \) that we can extend to a substitution \( \theta_1 : C \rightarrow C \) by defining \( \theta_1(c_j) = \llbracket \rho_1(c_j) \rrbracket \), where \( c_j \) is the eigenvariable that was initially substituted for \( x_j \). Since \( \varphi \) is the goal of the coinductive proof \( \pi \), we are lead to use \( \theta_1 \) and its iterations in the invariant that will prove coinduction goal \( \varphi \).

The notions in the following definition will allow us to easily organise and iterate the substitutions that occur in a uniform proof. Recall that in general, \( \varphi \) can be used \( n \) times in the proof \( \pi \), giving rise to \( n \) substitutions \( \theta_1, \ldots, \theta_n \). The following abstract definition is motivated by this observation.

\begin{definition}
Let \( S \) be a set with \( S = \{ 1, \ldots, n \} \) for some \( n \in \mathbb{N} \). We call the set \( S^* \) of lists over \( S \) the set of \textit{substitution identifiers}. Suppose that we have substitutions \( \theta_0 : V \rightarrow \emptyset \) and \( \theta_k : V \rightarrow V \) for each \( k \in S \). Then we can define a map \( \Theta : S^* \rightarrow (\Sigma^C)^V \), which turns each substitution identifier into a substitution, by iteration from the right:

\[
\Theta(\epsilon) = \theta_0 \quad \text{and} \quad \Theta(w : k) = \Theta(w) \otimes \theta_k
\]

Coming back to the analysis of the proof \( \pi \), we assign to each substitution \( \rho_i = [\overline{N}/\overline{x}] \) with \( n \geq 1 \), which arises from a use of \( \varphi \) in the proof tree \( \pi \), a substitution \( \theta_i : C \rightarrow C \) by \( \theta_i(c_j) = \llbracket \rho_i(c_j) \rrbracket \). Note that each \( N_j \) in \( \rho_i \) has only variables from \( C \), that is, \( \Sigma^C \vdash N_j : \tau \). We call \( \theta_i \) an \textit{agent} of \( \pi \).

We let \( D \subseteq A \) be the set of atoms that are proven in \( \pi \):

\[
D = \{ A \mid \Sigma^C; P; \Delta \Rightarrow \langle A \rangle \text{ or } \Sigma^C; P; \Delta \Rightarrow A \text{ appears in } \pi \}
\]
From the agents and atoms in π we extract an invariant for the goal formula. In the following lemma we take $S = \{1, \ldots, n\}$ to be the set of identifiers for the $n$ uses of $\varphi$ in the given proof $\pi$.

**Lemma 37.** Suppose that $\varphi$ is an $H^q$-formula of the form $\forall \overline{x}. A_1 \land \cdots \land A_n \rightarrow A_0$ and that there is a proof $\pi; P \leftrightarrow \varphi$. Let $D$ be the proven atoms in $\pi$, $\theta_1, \ldots, \theta_n$ be the agents of $\pi$ and $\theta_0$: $C \leftrightarrow \emptyset$ some initial substitution. Define $A_k^C = A_k[\overline{c}/\overline{x}]$ and suppose further that $I_1$ is an invariant for \{$A_k^C[\Theta(e)] \mid 1 \leq k \leq n$\}. If we put

$$I_2 = \bigcup_{w \in S^*} \llbracket D \rrbracket_1[\Theta(w)]$$

then $I = I_1 \cup I_2$ is an invariant for $\llbracket A_0^C \rrbracket_1[\Theta(e)]$.

**Proof.** This proof refers to the notation and the construction of the proof $\pi$ as given above.

We first need show that $\llbracket A_0^C \rrbracket_1[\Theta(e)] \in I_2$, which follows trivially from the fact that $\llbracket A_0^C \rrbracket_1[\Theta(e)] \in I_2$ by construction of $I_2$. It remains to show that $I \subseteq \Phi_P(I)$. We consider two cases: either $y \in I_1$ or $y \in I_2$. If the former, then $y \in \Phi_P(I)$ by definition of $I_1$ as an invariant and monotonicity of $\Phi_P$.

Consider the case when $y \in I_2$, i.e. when $y \in \llbracket B \rrbracket_1[\Theta(w)]$ for some $w$, and therefore $y = \llbracket B \rrbracket_1[\Theta(w)]$. We have to show that $y \in \Phi_P(I)$, that is, we have to show that there is a clause $\forall \overline{y}. \wedge_{k=1}^m B_k \rightarrow \psi'$ and a substitution $\theta: \llbracket \overline{y} \rrbracket \mapsto \emptyset$ such that $y = \llbracket \psi' \rrbracket_1[\theta]$ and for all $1 \leq k \leq m$ we have $\llbracket B_k \rrbracket_1[\theta] \in I$. We show that by case analysis on the proof $B$ and induction on $w$.

As discussed in the outline of proof $\pi$, only the rules `decide` or `decide()` are applicable to an atomic goal, and there are 3 possibilities of choosing a formula via these: it may be a program clause from $P$, the hypothesis $hp$ or $\varphi$. When one of these options is taken in a proof, we will say $B$ is `resolved against` a clause from $P$, $hp$ or $\varphi$, respectively. Moreover, choosing an atomic clause in $P$ gives us the base case, for which the proof has been given already. The remaining cases are:

a) Suppose $B$ is resolved against a clause $\forall \overline{y}. \wedge_{k=1}^m B_k \rightarrow \psi'$ in $P$, that is, we have $\psi'[y] \equiv B$ for some substitution $\gamma$. Note that if we define $\theta = \Theta(w) \odot \gamma$, then

$$y = \llbracket B \rrbracket_1[\Theta(w)] = \llbracket \psi'[y] \rrbracket_1[\Theta(w)] = \llbracket \psi' \rrbracket_1[\Theta(w) \odot \gamma] = \llbracket \psi' \rrbracket_1[\theta].$$

Since for all $1 \leq k \leq m$ the atom $B_k[y]$ must have a proof somewhere in $\pi$, we have $B_k[y] \in D$. Thus, also $\llbracket B_k \rrbracket_1[\theta] \in I_2$ and so $y \in \Phi_P(I)$ with the initial program clause and the substitution $\theta$.

b) If $B$ is resolved against $\varphi$, then this can only occur above the level $l$ in the proof tree $\pi$. By the already given schematic analysis of $\pi$, we also know that this case requires that $B \equiv A_0[y_r]$ for some $y_r$, and moreover this substitution is already incorporated in the construction of $\Theta$, as $y_r$ gives rise to an agent $\theta_r$. Thus, we have

$$y = \llbracket B \rrbracket_1[\Theta(w)] = \llbracket A_0[y_r] \rrbracket_1[\Theta(w)] = \llbracket A_k^C \rrbracket_1[\Theta(w) \odot \theta_r] = \llbracket A_k^C \rrbracket_1[\Theta(w : r)].$$

The latter equality follows from the definition of $\Theta(w : r)$ and of substitution composition. Note that $A_0^C$ was initially, below level $l$, resolved against a program clause $\forall \overline{y}. \wedge_{k=1}^m \psi_k \rightarrow \psi'$ with $A_0^C \equiv \psi'[y]$ for some substitution $\gamma$. Thus, by putting $\theta = \Theta(w : r) \odot \gamma$, we further have

$$y = \llbracket \psi' \rrbracket_1[\Theta(w : r)] = \llbracket \psi' \rrbracket_1[\Theta(w : r) \odot \gamma] = \llbracket \psi' \rrbracket_1[\theta].$$

Since each $\psi_k[y]$ is an atom in $\pi$, we also have that $\psi_k[y] \in D$ and thus $\llbracket \psi_k[y] \rrbracket_1[\Theta(w : r)] \in I_2$. This yields in turn that $\llbracket \psi_k[y] \rrbracket_1[\theta] \in I_2$. Putting this all together, we have that $y \in \Phi_P(I)$ by using the initial program clause and the substitution $\theta$.
we note that $\Phi$

Theorem 39. This induction and case analysis shows that for any $M$ we have $M \Phi$ largest map on $\Phi$.

Proof. conservative extension of $\Phi$ follows easily by coinduction.

Finally, we show that extending logic programs with coinductively proven lemmas is sound. This once we have Lem. 37 the following soundness theorem is easily proven.

Theorem 38. If $\varphi$ is an $H^q$-formula and $\Sigma; P \iff \varphi$, then $\llbracket \varphi \rrbracket_{M_P} = \top$.

Proof. We construct an invariant $I$ for any instance of $\Sigma; P \iff \varphi$, as per Lem. 37. Since $I \subseteq \Phi_P(I)$, we obtain $\llbracket \varphi \rrbracket_{M_P} = \top$.

This follows easily by coinduction.

Theorem 39. Let $\varphi$ be an $H^q$-formula. Then $\Sigma; P \iff \varphi$ implies $M_{P \cup \{\varphi\}} = M_P$, that is, $P \cup \{\varphi\}$ is a conservative extension of $P$ with respect to the Herbrand model.

Proof. Suppose $\varphi$ is an $H^q$-formula of the shape $\forall x. \psi_1 \rightarrow \psi_2$. Let $M = M_P$ and $M' = M_{P \cup \{\varphi\}}$. First, we note that $\Phi_P = \Phi_P \cup \Phi_\varphi$, where $\cup$ is the point-wise union. This gives us immediately that $M \subseteq \Phi_P(M) \subseteq \Phi_P(M')$ and thus $M \subseteq M'$ by coinduction. For the other direction, that is $M' \subseteq M$ one uses soundness and coinduction as follows. We have

$$M' = \Phi_P(M')$$

(by definition of $M'$)

$$= \Phi_P(M') \cup \{\llbracket \psi_2 \rrbracket_{\Theta} : \Theta \rightarrow \Sigma^\omega, \llbracket \psi_1 \rrbracket_{\Theta} \in M'\}$$

(by definition of $\Phi_P$)

$$\subseteq \Phi_P(M') \cup \{\llbracket \psi_2 \rrbracket_{\Theta} : \Theta \rightarrow \Sigma^\omega, \llbracket \psi_1 \rrbracket_{\Theta} \in M\}$$

(by assumption)

$$\subseteq \Phi_P(M') \cup M$$

(by $\Sigma; P \iff \varphi$ and Thm. 38)

Now we use the soundness of a so-called up-to-technique. Specifically, let $F$ be the monotone map on $I$ given by $F(I) = I \cup M$. Then $F$ is $\Phi_P$-compatible, that is, $F \circ \Phi_P \subseteq \Phi_P \circ F$ because $M$ is the largest $\Phi_P$ fixed point. It follows for every $I \in I$ that whenever $I \subseteq \Phi_P(F(I))$ then $I \subseteq M$. By the above calculation, we have that $M' \subseteq \Phi_P(F(M'))$. Thus, $M' \subseteq M$ as we wanted to show. Altogether, this gives us that $M_P = M_{P \cup \{\varphi\}}$.

I removed the condition now.
As a corollary we obtain that, if there is a proof for $\Sigma; P \leftrightarrow \varphi$, then a proof for $\Sigma; P, \varphi \leftrightarrow \psi$ is sound with respect to $\mathcal{M}_P$. Indeed, by Thm. 39 we have that $\mathcal{M}_P = \mathcal{M}_{P \cup \{ \varphi \}}$ and by Thm. 38 that $\Sigma; P, \varphi \leftrightarrow \psi$ is sound with respect to $\mathcal{M}_{P \cup \{ \varphi \}}$. Thus, the proof of $\Sigma; P, \varphi \leftrightarrow \psi$ is also sound with respect to $\mathcal{M}_P$. We use this property implicitly in our running examples, and refer the reader to [14][49][48] for proofs, further examples and discussion.

### 5.4 Soundness of $\text{IFOL}_\Box$ over Herbrand Models

In this section, we demonstrate how the logic $\text{IFOL}_\Box$ can be interpreted over Herbrand models. Recall that we obtained a fixed point model from the monotone map $\Phi_P$ on interpretations. In what follows, it is crucial that we construct the greatest fixed point of $\Phi_P$ by iteration, c.f. [6][31][79]: Let $\text{Ord}$ be the class of all ordinals equipped with their (well-founded) order. We denote by $\text{Ord}^{op}$ the class of ordinals with their reversed order and define a monotone function $\overline{\Phi}_P : \text{Ord}^{op} \rightarrow I$, where we write the argument ordinal in the subscript, by

$$
(\overline{\Phi}_P)_\alpha = \bigcap_{\beta < \alpha} \Phi_P(\overline{\Phi}_P) .
$$

Note that this definition is well-defined because $<$ is well-founded and because $\Phi_P$ is monotone, see [13]. Since $I$ is a complete lattice, there is an ordinal $\alpha$ such that $\overline{\Phi}_P = \Phi_P(\overline{\Phi}_P)_\alpha$, at which point $\overline{\Phi}_P$ is the largest fixed point $\mathcal{M}_P$ of $\Phi_P$. In what follows, we will utilise this construction to give semantics to $\text{IFOL}_\Box$.

The fibration $\overline{\mathcal{F}} : \text{Pred} \rightarrow \text{Set}$ gives rise to another fibration as follows. We let $\overline{\text{Pred}}$ be the category of functors (monotone maps) with fixed predicate domain:

$$\overline{\text{Pred}} = \left\{ \begin{array}{ll}
\text{objects:} & u : \text{Ord}^{op} \rightarrow \text{Pred}, \text{ such that } \overline{\mathcal{F}} \circ u \text{ is constant} \\
\text{morphisms:} & u \rightarrow v \text{ are natural transformations } f : u \Rightarrow v , \\
& \text{ such that } \overline{\mathcal{F}} f : \overline{\mathcal{F}} \circ u \Rightarrow \overline{\mathcal{F}} \circ v \text{ is the identity} 
\end{array} \right\}
$$

The fibration $\overline{\mathcal{F}} : \text{Pred} \rightarrow \text{Set}$ is defined by evaluation at any ordinal (here 0), i.e. by $\overline{\mathcal{F}}(u) = \overline{\mathcal{F}}(u(0))$ and $\overline{\mathcal{F}}(f) = (\overline{\mathcal{F}})f_0$, and reindexing along $f : X \rightarrow Y$ by applying the reindexing of $\overline{\mathcal{F}}$ point-wise, i.e. by $f^*(u)_{\alpha} = f^*(u)_{\alpha}$.

Note that there is a (full) embedding $K : \text{Pred} \rightarrow \overline{\text{Pred}}$ that is given by $K(X, P) = (X, \overline{\mathcal{F}})$ with $\overline{\mathcal{F}}_{\alpha} = P$. One can show [13] that $\overline{\mathcal{F}}$ is again a first-order fibration and that it models the later modality, as in the following theorem.

**Theorem 40.** The fibration $\overline{\mathcal{F}}$ is a first-order fibration. If necessary, we denote the first-order connectives by $\uparrow, \wedge$ etc. to distinguish them from those in $\text{Pred}$. Otherwise, we drop the dots. Finite (co)products and quantifiers are given point-wise, while for $X \in \text{Set}$ and $u, v \in \overline{\text{Pred}}_X$, exponents are given by

$$(v \Rightarrow u)_{\alpha} = \bigcap_{\beta \leq \alpha} (\alpha_{\beta} \Rightarrow u_{\beta}) .$$

There is a fibred functor $\uparrow : \overline{\text{Pred}} \rightarrow \overline{\text{Pred}}$ with $\overline{\mathcal{F}} \circ \uparrow = \overline{\mathcal{F}}$ given on objects by

$$(\uparrow u)_{\alpha} = \bigcap_{\beta < \alpha} u_{\beta}$$

and a natural transformation next: $\text{Id} \Rightarrow \uparrow$ from the identity functor to $\uparrow$. The functor $\uparrow$ preserves reindexing, products, exponents and universal quantification: $\uparrow(f^*u) = f^*(\uparrow u)$, $\uparrow(u \land v) = \uparrow u \land \uparrow v$, $\uparrow(u^v) \rightarrow (\uparrow u)^v$, $\uparrow(\exists u) = \exists(\uparrow u)$. Finally, for all $X \in \text{Set}$ and $u \in \overline{\text{Pred}}_X$, there is löb: $(\uparrow u \Rightarrow u) \rightarrow u$ in $\overline{\text{Pred}}_X$. 

27
Intuitively, the later modality shifts a given sequence by one position and concatenates it with the terminal object. This can be seen if we have a description ordinals through successor and limit ordinals. Given $\sigma \in \text{Pred}_X$, we can visualise the beginning of $\sigma$ and $I \sigma$ as follows.

$\sigma:$

$\sigma_0 \supseteq \sigma_1 \supseteq \sigma_2 \supseteq \sigma_3 \supseteq \sigma_4 \supseteq \cdots$

$\Gamma \vdash \sigma$

$I \sigma:$

$X \supseteq \sigma_0 \supseteq \sigma_1 \supseteq \sigma_2 \supseteq \sigma_3 \supseteq \cdots$

Using the above theorem, we can extend the interpretation of formulae to $iFOL_{\diamond}$ as follows. Let $u: \text{Ord}^{op} \rightarrow I$ be a descending sequence of interpretations. As before, we define the restriction of $u$ to a predicate symbol $p \in \Pi$ by $(u|p|_\alpha) = u_\alpha|_p = \{ \overline{r} \mid p(\overline{r}) \in u_\alpha \}$. The semantics of formulae in $iFOL_{\diamond}$ as objects in $\text{Pred}$ is given by the following iterative definition.

$\Gamma \vdash \exists x \cdot \tau \rightarrow \overline{M}_u = \Gamma \vdash \tau \rightarrow \overline{M}

\Gamma \vdash \phi \rightarrow \psi \rightarrow \Gamma \vdash \phi \rightarrow \psi

\Gamma \vdash Q \cdot x : \tau \rightarrow \phi \rightarrow \Gamma \vdash \phi

\Gamma \vdash \Gamma \rightarrow \phi \rightarrow \phi

The following lemma is the analogue of Lem. 32 for the interpretation of formulae without the later modality.

**Lemma 41.** The mapping $\Gamma \vdash \phi \rightarrow \psi$ is a well-defined map from formulae in $iFOL_{\diamond}$ to sequences of predicates, such that $\Gamma \vdash \phi$ implies $\Gamma \vdash \phi \rightarrow \psi \rightarrow \psi$. Proof. Immediate by induction on $\phi$. □

**Lemma 42.** All rules of $iFOL_{\diamond}$ are sound with respect to the interpretation $\Gamma \vdash \phi \rightarrow \psi$ of formulae in $\text{Pred}$, that is, if $\Gamma \vdash \Delta \rightarrow \phi$ and $\Gamma \vdash \phi \rightarrow \psi$, then $\Gamma \vdash \Delta \rightarrow \psi$. In particular, $\Gamma \vdash \phi$ implies $\Gamma \vdash \phi \rightarrow \psi \rightarrow \psi$. Proof. The soundness for the rules of first-order logic in Fig. 8 is standard for the given interpretation over a first-order fibration as in Thm. 40. Soundness of the rules for the rules of the later modality in Fig. 9 follows from the existence of the morphisms next and l"ob, and functoriality of $\Gamma \rightarrow$ that were proved in Thm. 40. □

The following lemma shows that the guarding of a set of formulae is valid in the chain model that they generate.

**Lemma 43.** If $\phi$ is an $H$-formula in $P$, then $\Gamma \vdash \phi \rightarrow \phi$. Proof. Let $\phi$ be an $H$-formula in $P$ of shape $\Gamma \vdash \phi \rightarrow \phi$ and let $\Gamma$ be the context $\Gamma$. Our goal is to show that $\Gamma \vdash \phi \rightarrow \psi = \psi$. First, we have by definition of the semantics for all
\( \alpha \in \text{Ord} \) that

\[
\bigwedge_{i=1}^{n} \bullet (p_i \bar{M}_i) \rightarrow q \bar{N} \alpha \\
= \bigcap_{\beta < \alpha} \left( \bigwedge_{i=1}^{n} \bullet (p_i \bar{M}_i) \right)_{\beta} \Rightarrow \left[ q \bar{N} \right]_{\beta}
= \bigcap_{\beta < \alpha} \left( \bigcap_{i=1}^{n} \bigwedge_{\gamma < \beta} (p_i \bar{M}_i)_{\gamma} \right) \Rightarrow q \bar{N} \beta
= \bigcap_{\beta < \alpha} \left\{ \sigma \in \left[ \Gamma \right] \mid (\forall i. \forall Y < \beta. \bar{M}_i[\sigma] \in \Phi_{\beta}(p)) \right\}
= \bigcap_{\beta < \alpha} \left\{ \sigma \in \left[ \Gamma \right] \mid (\forall i. \forall Y < \beta. \bar{M}_i[\sigma] \in \Phi_{\beta}(p)) \right\}
= \left( \forall Y < \beta. \bar{N}[\sigma] \in \Phi_{\beta}(\bar{M}_i(q)) \right)
\]

We intend to show now that this set is equal to \( \top_{\left[ \Gamma \right]} \). Let \( \sigma \in \left[ \Gamma \right] \), such that \( \forall i. \forall Y < \beta. \bar{M}_i[\sigma] \in \Phi_{\beta}(\bar{M}_i(p)) \). We have to show that \( \forall Y < \beta. \bar{N}[\sigma] \in \Phi_{\beta}(\bar{M}_i(q)) \). To this end, suppose \( \gamma < \beta \). Then \( \forall i. \bar{M}_i[\sigma] \in \Phi_{\beta}(\bar{M}_i(p)) \) by assumption. By definition of \( \Phi_{\beta} \) we obtain \( \bar{N}[\sigma] \in \Phi_{\beta}(\bar{M}_i(q)) \) as required. Hence, \( \left[ \bigwedge_{i=1}^{n} \bullet (p_i \bar{M}_i) \rightarrow q \bar{N} \right] = \top_{\left[ \Gamma \right]} \). But then \( \left[ \bar{M}_i \right]_{\Phi_{\beta}} = \forall \gamma \top_{\left[ \Gamma \right]} = \top \). \( \square \)

Combining this with soundness from Lem. 42 we obtain that provability in \( iFOL_\bullet \) relative to a logic program \( P \) is sound for the model of \( P \).

**Theorem 44.** For all logic programs \( P \), if \( \Gamma \vdash \varphi \) then \( \left[ \varphi \right]_{\Phi_{\beta}} = \top \).

**Proof.** Combine Lem. 42 and Lem. 43 \( \square \)

The final result of this section is to show that the descending chain model, which we used to interpret formulae of \( iFOL_\bullet \), is sound and complete for the fixed point model, which we used to interpret the formulae of coinductive uniform proofs. This will be proved in Thm. 47 below. The easiest way to prove this result is by establishing a functor \( \overline{\text{Pred}} \rightarrow \text{Pred} \) that maps the chain \( \overline{\Phi_{\beta}} \) to the model \( M_P \), and that preserves and reflects truth of first-order formulae (Prop. 16). We will phrase the preservation of truth of first-order formulae by a functor by appealing to the following notion of fibrations maps, cf. [45 Def.4.3.1].

**Definition 45.** Let \( p : E \rightarrow B \) and \( q : D \rightarrow A \) be fibrations. A **fibration map** \( p \rightarrow q \) is a pair \( (F : E \rightarrow D, G : B \rightarrow A) \) of functors, s.t. \( q \circ F = G \circ p \) and \( F \) preserves Cartesian morphisms: if \( f : X \rightarrow Y \) in \( E \) is Cartesian over \( p(f) \), then \( F(f) \) is Cartesian over \( G(p(f)) \).

Let us now construct a first-order \( \lambda \)-fibration map \( \overline{\text{Pred}} \rightarrow \text{Pred} \). We note that since every fibre of the predicate fibration is a complete lattice, for every chain \( u \in \overline{\text{Pred}}_X \) there exists an ordinal \( \alpha \) at which \( u \) stabilises. This means that there is a limit \( \lim u \) of \( u \) in \( \text{Pred}_X \), which is the largest subset of \( X \), such that \( \forall \alpha. \lim u \subseteq u_\alpha \). This allows us to define a map \( L : \overline{\text{Pred}} \rightarrow \text{Pred} \) by

\[
L(X, u) = (X, \lim u) \\
L(f : (X, u) \rightarrow (Y, v)) = f.
\]
In the following proposition, we show that $L$ gives us the ability to express first-order properties of limits equivalently through their approximating chains. This, in turn, provides soundness and completeness for the interpretation of the logic $\text{iFOL}_*$ over descending chains with respect to the largest Herbrand model.

**Proposition 46.** The functor $L: \text{Pred} \to \text{Pred}$, as defined above, is a map of fibrations and preserves fibred (co)products, and existential and universal quantification. Furthermore, $L$ is right-adjoint to the embedding $K: \text{Pred} \to \text{Pred}$. Finally, for each $p \in \Pi$ and $u \in \text{Pred}_{2^p}$, we have $L(u|^p) = L(u)$.\text{Pred}_{2^p}$.

**Proof.** First, we show that if $f: (X, u) \to (Y, v)$, then $f$ is indeed a morphism $(X, \lim u) \to (Y, \lim v)$. This means that we have to show that $f(\lim u) \subseteq \lim v$. By the limit property, it suffices to show for all $\alpha \in \text{Ord}$ that $f(\lim u) \subseteq v_\alpha$:

$$f(\lim u) \subseteq f(u_\alpha) \subseteq v_\alpha \quad \lim u \subseteq u_\alpha \quad \text{image of } f \text{ monotone} \quad f \text{ is morphism } (X, u) \to (Y, v)$$

That $L$ preserves identities and composition is evident, as is the preservation if indices: $\pi = \pi \circ L$.

Next, we show that Cartesian morphisms are preserved as well. Let $f: (X, u) \to (Y, v)$ be Cartesian in $\text{Pred}$, and suppose we are given $g$ and $h$ as in the lower triangle in the following diagram in $\text{Set}$ and $(Z, P)$ in $\text{Pred}$.

![Diagram](image_url)

We have to show that $h$ is a morphism $(Z, P) \to (X, \lim u)$ in $\text{Pred}$. To that end, we define a constant chain $w: \text{Ord} \to \text{Pred}_P$ by $w_\alpha = P$. Note that $\lim w = P$, thus $L(Z, w) = (Z, P)$. Moreover, for all $\alpha \in \text{Ord}$ we have that $g(w_\alpha) = g(Z) \subseteq \lim v$. Thus, $g(w_\alpha) \subseteq v_\alpha$ and $g$ is a morphism in $\text{Pred}$. Since $f$ is Cartesian, we obtain that $h$ is a morphism $(Z, w) \to (X, u)$ in $\text{Pred}$, that is, for all $\alpha$, $h(P) = h(w_\alpha) \subseteq u_\alpha$. This gives us in turn that $h(P) \subseteq \lim u$, which means that $h$ is a morphism $(Z, P) \to (Y, \lim v)$ in $\text{Pred}$.

Showing that $L$ preserves coproducts and existential quantifiers is somewhat nasty, while products and universal quantification are straightforward. First, we prove that conjunction is preserved, that is, we want to prove that $\lim(u \lor v) = \lim u \lor \lim v$. We note now that, because $u$ and $v$ are descending, that there are ordinals $\alpha, \beta, \gamma$ such that $\lim(u \lor v) = (u \lor v)_\gamma$ and $\lim u \lor \lim v = u_\alpha \lor v_\beta$. Let now $\gamma' = \alpha \cup \beta \cup \gamma$ be the least upper bound of these ordinals. Then we have by the above assumptions that

$$\lim(u \lor v) = (u \lor v)_{\gamma'}$$

Descending chains

$$= u_{\gamma'} \lor v_{\gamma'}$$

Point-wise def. of $\lor$

$$= u_\alpha \lor v_\beta$$

Descending chains

$$= \lim u \lor \lim v.$$
Thus, \( L((X, u) \lor (X, v)) = L(X, u) \lor L(X, v) \) as desired.

Similarly, to prove \( \lim(\exists_{X, Y} \sigma) = \exists_{X, Y}(\lim \sigma) \), we let \( \beta \) be such that \( \lim \sigma = \sigma_{\beta} \). The inclusion \( \exists_{X, Y}(\lim \sigma) \subseteq L(\exists_{X, Y} \sigma) \) is, as usual, unconditionally true. For the other direction, we have

\[
x \in L(\exists_{X, Y} \sigma) \iff x \in \lim\{x \in X \mid \exists y \in Y. (x, y) \in \sigma_x\}
\]

\[
\iff \forall \alpha. \exists y. (x, y) \in \sigma_{\alpha}
\]

\[
\iff \exists y. \forall \alpha. (x, y) \in \sigma_{\alpha}
\]

\( \sigma \) descending and stable at \( \beta \)

\[
\iff x \in \exists_{X, Y}(\lim \sigma).
\]

This proves that also existential quantification is preserved by \( L \).

Finally, to show that there is an adjunction \( K \dashv L \), we have to show for all \((X, P) \in \text{Pred} \) and \((Y, u) \in \text{Pred} \) that there is a natural isomorphism \( \text{Hom}_{\text{Pred}}((X, P), (Y, \lim u)) \cong \text{Hom}_{\text{Pred}}((X, \overline{P}), (Y, u)) \). This boils down to showing that for any map \( f : X \to Y \) we have \( f(P) \subseteq \lim u \iff \forall \alpha. f(P) \subseteq u_x \). In turn, this is immediately given by the limit property of \( \lim u \).

We get from Prop. 46 soundness of \( \overline{\Phi_P} \) for Herbrand models. More precisely, if \( \varphi \) is a goal formula that has only implication-free formulas on the left of an implication (first-order goal), then its interpretation in the coinductive Herbrand model is true if its interpretation over the chain approximation of the Herbrand model is true.

**Theorem 47.** If \( \varphi \) is a first-order goal and \( \| \varphi \|_{\overline{\Phi_P}} = \top \), then \( \| \varphi \|_{\overline{M_P}} = \top \).

**Proof.** First, we show for an implication-free D-formula \( \psi \) that

\[
L(\| \psi \|_{\overline{\Phi_P}}) = \| \psi \|_{\overline{M_P}} \tag{3}
\]

by induction on \( \psi \) and using Prop. 46 as follows. For atoms, we have that

\[
L(\| p \|_{\overline{\Phi_P}}) = \| \overline{M}^*(\overline{\Phi_P}|_p) \|
\]

\[
= \| \overline{M}^*(\overline{\Phi_P}|_p) \|
\]

\[
= \| \overline{M}^*(\overline{\Phi_P}) \|_{\overline{M_P}}
\]

\( \overline{M_P} \) is limit of \( \overline{\Phi_P} \)

The cases for universal quantification and conjunction are given by using that \( L \) preserves these connectives (again Prop. 46). From this, we obtain for a first-order goal \( \varphi \) that \( L(\| \varphi \|_{\overline{\Phi_P}}) \subseteq \| \varphi \|_{\overline{M_P}} \) by induction on \( \varphi \) and using again Prop. 46.

To show that the semantics over \( \text{Pred} \) and \( \overline{\text{Pred}} \) coincide, that is, that we have the following correspondence.

\[
\| \varphi \|_{\overline{\Phi_P}} = \top \quad \| \varphi \|_{\overline{M_P}} = \top
\]

Since any predicate is included in the maximal predicate \( \top \), it suffices to show that there is a correspondence as in

\[
\top \to \| \varphi \|_{\overline{\Phi_P}}
\]

\( \top \to \| \varphi \|_{\overline{M_P}} \)
Note that $\dagger$ is given by the embedding $K(\tau)$. Using Prop. [46] and [3] we obtain the desired correspondence as follows.

$$
\dagger = K(\tau) \rightarrow \phi \equiv \phi_{\Psi} \quad \text{in Pred}
$$

$$
\tau \rightarrow L(\phi \equiv \phi_{\Psi}) \quad \text{in Pred}
$$

This concludes the proof of soundness for first-order goals with respect to the Herbrand model.  

6 Conclusion, Related Work and the Future

In this paper, we provided a comprehensive theory of resolution in coinductive Horn-clause theories and coinductive logic programs. This theory comprises of a uniform proof system that features a form of guarded recursion and that provides operational semantics for proofs of coinductive predicates. Further, we showed how to translate proofs in this system into proofs for an extension of intuitionistic FOL with guarded recursion, and we provided sound semantics for both proof systems in terms of coinductive Herbrand models. The Herbrand models and semantics were thereby presented in a modern style that utilises coalgebras and fibrations to provide a conceptual view on the semantics.

Related Work. It may be surprising that automated proof search for coinductive predicates in first-order logic does not have a coherent and comprehensive theory, even after three decades [3, 60], despite all the attention that it received as programming [2, 28, 41, 43] and proof [32, 33, 37, 38, 44, 59, 65, 66, 67, 68] method. The work that comes close to algorithmic proof search is the system CIRC [64], but it cannot handle general coinductive predicates and corecursive programming. Inductive and coinductive data types are also being added to SMT solvers [23, 63]. However, both CIRC and SMT solving are inherently based on classical logic and are therefore not suited to situations where proof objects are relevant, like programming, type class inference or (dependent) type theory. Moreover, the proposed solutions, just like those in [40, 70] can only deal with regular data, while our approach also works for irregular data, as we saw in the from-example.

This paper subsumes Haskell type class inference [51, 35] and exposes that the inference presented in those papers corresponds to coinductive proofs in co-fohc and co-hohh. Given that the proof systems proposed in this paper are constructive and that uniform proofs provide proofs (type inhabitants) in normal form, we could give a propositions-as-types interpretation to all eight coinductive uniform proof systems. This was done for co-fohc and co-hohh in [35], but we leave the remaining cube from the introduction for future work.

Future Work. There are several directions that we wish to pursue in the future. First, we know that CUP is incomplete for the presented models, as it is intuitionistic and it lacks an admissible cut rule. The first can be solved by moving to Kripke/Beth-models, as done by Clouston and Goré [29] for the propositional part of iFOL. However, the admissible cut rule is more delicate. To obtain such a rule one has to be able to prove several propositions simultaneously by coinduction, as discussed at the end of Sec. [4]. In general, completeness of recursive proof systems depends largely on the theory they are applied to, see [71] and [17]. However, techniques from cyclic proof systems [25, 69] may help. We also aim to extend our ideas to other situations like higher-order Horn clauses [42, 27] and interactive proof assistants [39, 9, 30, 22], typed logic programming, and logic programming that mix inductive and coinductive predicates.
Acknowledgements.  We would like to thank Damien Pous and the anonymous reviewers for their valuable feedback.

References

[1] M. Abbott, T. Altenkirch, and N. Ghani. Containers: Constructing strictly positive types. TCS, 342(1):3–27, Sept. 2005. ISSN 0304-3975. doi:10.1016/j.tcs.2005.06.002

[2] A. Abel, B. Pientka, D. Thibodeau, and A. Setzer. Copatterns: programming infinite structures by observations. In POPL’13, pages 27–38, 2013. doi:10.1145/2429069.2429075

[3] P. Aczel. Non-well-founded sets. Center for the Study of Language and Information, Stanford University, 1988.

[4] P. Aczel. Algebras and Coalgebras. In Algebraic and Coalgebraic Methods in the Mathematics of Program Construction. Springer, 2002.

[5] P. Aczel, J. Adámek, S. Milius, and J. Velebil. Infinite trees and completely iterative theories: A coalgebraic view. TCS, 300(1–3):1–45, May 2003. ISSN 0304-3975. doi:10.1016/S0304-3975(02)00728-4

[6] J. Adámek. On final coalgebras of continuous functors. Theor. Comput. Sci., 294(1/2):3–29, 2003. doi:10.1016/S0304-3975(01)00240-7

[7] A. W. Appel, P.-A. Melliès, C. D. Richards, and J. Vouillon. A very modal model of a modern, major, general type system. In POPL, pages 109–122. ACM, 2007. doi:10.1145/1190216.1190235

[8] R. Atkey and C. McBride. Productive coprogramming with guarded recursion. In ICFP, pages 197–208. ACM, 2013. doi:10.1145/2500365.2500597

[9] D. Baelde, K. Chaudhuri, A. Gacek, D. Miller, G. Nadathur, A. Tiu, and Y. Wang. Abella: A system for reasoning about relational specifications. J. Formalized Reasoning, 7(2):1–89, 2014. doi:10.6092/issn.1972-5787/4650

[10] H. Barendregt, W. Dekkers, and R. Statman. Lambda Calculus with Types. Cambridge University Press, Cambridge; New York, July 2013. ISBN 978-0-521-76614-2.

[11] M. Barr and C. Wells. Category Theory for Computing Science. Prentice Hall International Series in Computer Science. Prentice Hall, 2nd edition, 1995. ISBN 978-0-13-323809-9. URL http://www.tac.mta.ca/tac/reprints/articles/22/tr22abs.html

[12] H. Basold. Mixed Inductive-Coinductive Reasoning: Types, Programs and Logic. PhD thesis, Radboud University Nijmegen, 2018. URL http://hdl.handle.net/2066/190323

[13] H. Basold. Breaking the Loop: Recursive Proofs for Coinductive Predicates in Fibrations. ArXiv e-prints, Feb. 2018. URL https://arxiv.org/abs/1802.07143

[14] H. Basold, E. Komendantskaya, and Y. Li. Coinduction in uniform: Foundations for corecursive proof search with horn clauses. extended version of this paper. CoRR, abs/1811.07644, 2018. URL http://arxiv.org/abs/1811.07644
[15] L. D. Beklemishev. Parameter Free Induction and Provably Total Computable Functions. TCS, 224 (1-2):13–33, 1999. doi:10.1016/S0304-3975(98)00305-3

[16] J. Bénabou. Fibered categories and the foundations of naive category theory. Journal of Symbolic Logic, 50(1):10–37, 1985. doi:10.2307/2273784

[17] S. Berardi and M. Tatsuta. Classical System of Martin-Löf’s Inductive Definitions Is Not Equivalent to Cyclic Proof System. In J. Esparza and A. S. Murawski, editors, Proc. of FOSSACS 2017, volume 10203 of LNCS, pages 301–317, 2017. doi:10.1007/978-3-662-54458-7_18

[18] L. Birkedal and R. E. Møgelberg. Intensional Type Theory with Guarded Recursive Types qua Fixed Points on Universes. In LICS, pages 213–222. IEEE Computer Society, 2013. doi:10.1109/LICS.2013.27

[19] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First Steps in Synthetic Guarded Domain Theory: Step-Indexing in the Topos of Trees. In Proc. of LICS’11, pages 55–64. IEEE Computer Society, 2011. doi:10.1109/LICS.2011.16

[20] A. Bizjak, H. B. Grathwohl, R. Clouston, R. E. Møgelberg, and L. Birkedal. Guarded Dependent Type Theory with Coinductive Types. In FoSSaCS, volume 9634 of LNCS, pages 20–35. Springer, 2016. URL https://arxiv.org/abs/1601.01586

[21] N. Bjørner, A. Gurfinkel, K. L. McMillan, and A. Rybalchenko. Horn clause solvers for program verification. In Fields of Logic and Computation II, volume 9300 of LNCS, pages 24–51, 2015. doi:10.1007/978-3-319-23534-9_2

[22] J. C. Blanchette, F. Meier, A. Popescu, and D. Traytel. Foundational nonuniform (co)datatypes for higher-order logic. In LICS’17, pages 1–12. IEEE Computer Society, 2017. doi:10.1109/LICS.2017.8005071

[23] J. C. Blanchette, N. Peltier, and S. Robillard. Superposition with datatypes and codatatypes. In IJCAR 2018, LNCS, 2018. doi:10.1007/978-3-319-94205-6_25

[24] F. Borceux. Handbook of Categorical Algebra: Volume 1, Basic Category Theory. Cambridge University Press, Apr. 2008. ISBN 978-0-521-06119-3.

[25] G. Bottu, G. Karachalios, T. Schrijvers, B. C. d. S. Oliveira, and P. Wadler. Quantified class constraints. In Haskell Symp., pages 148–161. ACM, 2017. doi:10.1145/3122955.3122967

[26] J. Brotherston and A. Simpson. Sequent calculi for induction and infinite descent. J. Log. Comput., 21(6):1177–1216, 2011. doi:10.1093/logcom/exq052

[27] T. C. Burn, C. L. Ong, and S. J. Ramsay. Higher-order constrained horn clauses for verification. PACMPL, 2(POPL):11:1–11:28, 2018. doi:10.1145/3158099

[28] V. Capretta. General Recursion via Coinductive Types. Logical Methods in Computer Science, 1(2), July 2005. ISSN 18605974. doi:10.2168/LMCS-1(2:1)2005

[29] R. Clouston and R. Goré. Sequent Calculus in the Topos of Trees. In A. M. Pitts, editor, Proc. of FoSSaCS 2015, volume 9034 of LNCS, pages 133–147. Springer, 2015. doi:10.1007/978-3-662-46678-0_9
[30] T. Coquand. Infinite objects in type theory. In *TYPES’93*, volume 806, pages 62–78, 1994.

[31] P. Cousot and R. Cousot. Constructive versions of Tarski’s fixed point theorems. *Pacific J. Math.*, 82 (1):43–57, 1979. URL http://projecteuclid.org/euclid.pjm/1102785059

[32] C. Dax, M. Hofmann, and M. Lange. A Proof System for the Linear Time $\mu$-Calculus. In S. Arun-Kumar and N. Garg, editors, *Proceedings of FSTTCS 2006*, volume 4337 of LNCS, pages 273–284. Springer, 2006. doi:10.1007/11944836_26.

[33] J. Endrullis, H. H. Hansen, D. Hendriks, A. Polonsky, and A. Silva. A coinductive framework for infinitary rewriting and equational reasoning. In *RTA’15*, pages 143–159, 2015. doi:10.4230/LIPIcs.RTA.2015.143

[34] F. Farka, E. Komendantskaya, and K. Hammond. Coinductive soundness of corecursive type class resolution. In *Post-proc. of LOPSTR’16*, volume 10184 of LNCS, pages 312–326. Springer, 2017. doi:10.1007/978-3-319-319-4_18

[35] P. Fu, E. Komendantskaya, T. Schrijvers, and A. Pond. Proof relevant corecursive resolution. In *FLOPS’16*, pages 126–143. Springer, 2016. doi:10.1007/978-3-319-29604-3_9

[36] N. Gambino and J. Kock. Polynomial functors and polynomial monads. *Math. Proc. Cambridge Phil. Soc.*, 154(01):153–192, Jan. 2013. ISSN 0305-0041, 1469-8064. doi:10.1017/S0305004112000394

[37] J. Giesl, C. Aschersmann, M. Brockschmidt, F. Emmes, F. Frohn, C. Fuhs, J. Hensel, C. Otto, M. Plücker, P. Schneider-Kamp, T. Ströder, S. Swiderski, and R. Thiemann. Analyzing program termination and complexity automatically with AProVE. *J. Autom. Reasoning*, 58(1):3–31, 2017. doi:10.1007/s10817-016-9388-y.

[38] E. Giménez. Structural recursive definitions in type theory. In *ICALP’98*, pages 397–408, 1998. doi:10.1007/BFb0055070.

[39] P. L. group on Agda. Agda Documentation. Technical report, Chalmers and Gothenburg University, 2015. URL http://wiki.portal.chalmers.se/agda/ Version 2.4.2.5.

[40] G. Gupta, A. Bansal, R. Min, L. Simon, and A. Mallya. Coinductive logic programming and its applications. In V. Dahl and I. Niemelä, editors, *Proc. of ICLP’07*, volume 4670 of LNCS, pages 27–44. Springer, 2007. doi:10.1007/978-3-540-74610-2_4.

[41] T. Hagino. A typed lambda calculus with categorical type constructors. In *Category Theory in Computer Science*, Lecture Notes in Computer Science, pages 140–157. Springer, 1987. doi:10.1007/3-540-18508-9_24

[42] K. Hashimoto and H. Unno. Refinement type inference via horn constraint optimization. In *SAS*, volume 9291 of LNCS, pages 199–216, 2015. doi:10.1007/978-3-662-48288-9_12

[43] B. T. Howard. Inductive, Coinductive, and Pointed Types. In R. Harper and R. L. Wexelblat, editors, *Proc. of ICFP’96*, pages 102–109. ACM, 1996. doi:10.1145/232627.232640.

[44] C.-K. Hur, G. Neis, D. Dreyer, and V. Vafeiadis. The Power of Parameterization in Coinductive Proof. In *Proc. of POPL’13*, POPL ’13, pages 193–206. ACM, 2013. ISBN 978-1-4503-1832-7. doi:10.1145/2429069.2429093
[45] B. Jacobs. *Categorical Logic and Type Theory*. Number 141 in Studies in Logic and the Foundations of Mathematics. North Holland, Amsterdam, 1999.

[46] B. Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Number 59 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2016. ISBN 978-1-107-17789-5. URL [http://www.cs.ru.nl/B.Jacobs/CLG/JacobsCoalgebraIntro.pdf](http://www.cs.ru.nl/B.Jacobs/CLG/JacobsCoalgebraIntro.pdf)

[47] E. Komendantskaya and Y. Li. Productive corecursion in logic programming. *J. TPLP (ICLP’17 post-proc.)*, 17(5-6):906–923, 2017. doi:10.1017/S147106841700028X

[48] E. Komendantskaya and Y. Li. Towards coinductive theory exploration in horn clause logic: Position paper. In T. Kahsai and G. Vidal, editors, *Proceedings 5th Workshop on Horn Clauses for Verification and Synthesis, HCVS 2018, Oxford, UK, 13th July 2018.*, volume 278, pages 27–33, 2018. doi:10.4204/EPTCS.278.5

[49] E. Komendantskaya and Y. Li. Coinductive uniform proofs. *CoRR*, abs/1903.07371, 2018. URL [https://arxiv.org/abs/1903.07371](https://arxiv.org/abs/1903.07371)

[50] J. Lambek and P. J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, Mar. 1988. ISBN 978-0-521-35653-4.

[51] R. Lämmel and S. L. Peyton Jones. Scrap your boilerplate with class: Extensible generic functions. In *ICFP’05*, pages 204–215. ACM, 2005. doi:10.1145/1086365.1086391

[52] J. W. Lloyd. *Foundations of Logic Programming, 2nd Edition*. Springer, 1987. ISBN 3-540-18199-7.

[53] D. Miller and G. Nadathur. *Programming with Higher-order logic*. Cambridge University Press, 2012.

[54] D. Miller, G. Nadathur, F. Pfenning, and A. Scedrov. Uniform Proofs as a Foundation for Logic Programming. *Ann. Pure Appl. Logic*, 51(1-2):125–157, 1991. doi:10.1016/0168-0072(91)90068-W

[55] R. Milner. A theory of type polymorphism in programming. *J. Comput. Syst. Sci.*, 17(3):348–375, 1978. doi:10.1016/0022-0000(78)90014-4

[56] R. E. Møgelberg. A type theory for productive coprogramming via guarded recursion. In *CSL-LICS*, pages 71:1–71:10. ACM, 2014. doi:10.1145/2603088.2603132

[57] G. Nadathur and D. J. Mitchell. Teyjus - A compiler and abstract machine based implementation of lambda-prolog. In *CADE-16*, volume 1632 of LNCS, pages 287–291. Springer, 1999. doi:10.1007/3-540-48660-7_25

[58] H. Nakano. A Modality for Recursion. In *LICS*, pages 255–266. IEEE Computer Society, 2000. doi:10.1109/LICS.2000.855774

[59] D. Niwinski and I. Walukiewicz. Games for the $\mu$-Calculus. *TCS*, 163(1&2):99–116, 1996. doi:10.1016/0304-3975(95)00136-0

[60] D. M. R. Park. Concurrency and Automata on Infinite Sequences. In P. Deussen, editor, *Proceedings of TCS’81*, volume 104 of LNCS, pages 167–183. Springer, 1981. doi:10.1007/BFb0017309
[61] G. D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5(3):223–255, Dec. 1977. ISSN 0304-3975. doi:[10.1016/0304-3975(77)90044-5]

[62] D. Pous. Complete Lattices and Up-To Techniques. In Z. Shao, editor, *APLAS ’07*, volume 4807 of *LNCS*, pages 351–366. Springer, 2007. doi:[10.1007/978-3-540-76637-7_24]

[63] A. Reynolds and V. Kuncak. Induction for SMT solvers. In *VMCAI ’15*, volume 8931 of *LNCS*, pages 80–98, 2015. doi:[10.1007/978-3-662-46081-8_5]

[64] G. Roşu and D. Lucanu. Circular Coinduction: A Proof Theoretical Foundation. In *CALCO*, volume 5728 of *LNCS*, pages 127–144. Springer, 2009. doi:[10.1007/978-3-642-03741-2_10]

[65] J. Rutten. Universal Coalgebra: A Theory of Systems. *TCS*, 249(1):3–80, 2000. ISSN 0304-3975. doi:[10.1016/S0304-3975(00)00056-6]

[66] D. Sangiorgi. *Introduction to Bisimulation and Coinduction*. Cambridge University Press, New York, NY, USA, 2011. ISBN 978-1-107-00363-7.

[67] L. Santocanale. A Calculus of Circular Proofs and Its Categorical Semantics. In *FoSSaCS*, pages 357–371, 2002. doi:[10.1007/3-540-45931-6_25]

[68] L. Santocanale. $\mu$-Bicomplete Categories and Parity Games. *RAIRO - ITA*, 36(2):195–227, 2002. URL [https://doi.org/10.1051/ita:2002010](https://doi.org/10.1051/ita:2002010)

[69] D. S. Shamkanov. Circular proofs for the Gödel-Löb provability logic. *Mathematical Notes*, 96(3):575–585, Sept. 2014. ISSN 1573-8876. doi:[10.1134/S0001434614090326]

[70] L. Simion, A. Bansal, A. Mallya, and G. Gupta. Co-logic programming: Extending logic programming with coinduction. In L. Arge, C. Cachin, T. Jurdziński, and A. Tarlecki, editors, *Automata, Languages and Programming*, volume 4596 of *LNCS*, pages 472–483. Springer, 2007. doi:[10.1007/978-3-540-73420-8_42]

[71] A. Simpson. Cyclic Arithmetic is Equivalent to Peano Arithmetic. In *Proceedings of FoSSaCS’17*, LNCS, 2017. doi:[10.1007/978-3-662-54458-7_17]

[72] C. Smoryński. *Self-Reference and Modal Logic*. Universitext. Springer-Verlag, 1985. ISBN 0-387-96209-3.

[73] R. M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25(3):287–304, 1976. ISSN 1565-8511. doi:[10.1007/BF02757006]

[74] M. Sulzmann and P. J. Stuckey. HM(X) type inference is CLP(X) solving. *J. Funct. Program.*, 18(2):251–283, 2008. doi:[10.1017/S0956796807006569]

[75] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

[76] D. A. Turner. Elementary Strong Functional Programming. In P. H. Hartel and M. J. Plasmeijer, editors, *Proceedings of FPLE ’95*, volume 1022 of *LNCS*, pages 1–13. Springer, 1995. doi:[10.1007/3-540-60675-0_35]

[77] B. van den Berg and F. de Marchi. Non-well-founded trees in categories. *Annals of Pure and Applied Logic*, 146(1):40–59, Apr. 2007. ISSN 0168-0072. doi:[10.1016/j.apal.2006.12.001]
[78] M. van Emden and R. Kowalski. The semantics of predicate logic as a programming language. *Journal of the Assoc. for Comp. Mach.*, 23:733–742, 1976.

[79] J. Worrell. On the final sequence of a finitary set functor. *Theor. Comput. Sci.*, 338(1-3):184–199, 2005. doi[10.1016/j.tcs.2004.12.009]