Phase space methods: independence of subspaces

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Abstract. The concept of independence of subspaces is introduced in the context of quasi-probability distributions in phase space, for quantum systems with finite-dimensional Hilbert space. It is shown that due to the non-distributivity of the lattice of subspaces, there are various levels of independence, from pairwise independence up to (full) independence.

1. Introduction
We consider quantum systems with finite Hilbert space $H(d)$, and study quasi-probability distributions $R(i)$ related to projectors in subspaces $H_1, \ldots, H_n$ of $H(d)$. They are generalizations of the various functions within the phase space formalism. Special cases are the $Q$-function, the probability distribution in position space, etc. In this context we study the concept of independence of the subspaces $H_1, \ldots, H_n$. The present work complements our previous work [1]. The intention here is to provide a physical exposition with many examples, and without the ‘distraction’ of formal proofs to the various statements.

Linear independence (or for simplicity independence) is a prerequisite for fundamental concepts like dimension and basis. The subject of matroids[2, 3] provides a deep approach, and defines independence through some axioms. von Neumann[4, 5] studied independence within the general framework of the continuous geometries.

In this paper we show that there are various levels of independence for the subspaces $\{H_1, \ldots, H_n\}$. This is intimately related to the non-distributive nature of the lattice of subspaces. Distributivity is a very fundamental property in Boolean (classical) logic, which is formalized with set theory. In Quantum Mechanics distributivity does not hold, and this is related to non-commutativity. Concepts which are trivially equivalent in a distributive structure, might become inequivalent in a non-distributive structure. Quantum theory is usually described through non-commutativity, and in this paper it is described through non-distributivity.

In section II we discuss briefly for later use, finite quantum systems with $d$-dimensional Hilbert space $H(d)$, and the corresponding modular orthocomplemented lattice of subspaces $\mathcal{L}(d)$. In section III we introduce within set theory, the concept of independence. In set theory distributivity holds, and
consequently there is a single concept of independence. We also show that in Kolmogorov’s probability theory, the law of total probability (which is used to define marginals of probability distributions) is based on distributivity. In non-distributive structures the law of total probability does not hold, and the definition of marginals of probability distributions becomes problematic.

In section IV we introduce quasi-probability distributions in phase space, related to the subspaces $H_1, \ldots, H_n$ of $H(d)$. In section V we introduce various levels of independence, and define the degree of independence. We conclude in section VI with a discussion of our results.

2. Preliminaries

2.1. Finite quantum systems

We consider a quantum system with variables in $\mathbb{Z}(d)$ (the integers modulo $d$), described by a $d$-dimensional Hilbert space $H(d)$[6, 7]. We also consider two orthonormal bases, ‘position states’ and ‘momentum states’ which are related through a Fourier transform:

$$|P; \beta\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha} \omega(\alpha \beta) |X; \alpha\rangle; \quad \omega(\alpha) = \exp \left( \frac{2\pi \alpha}{d} \right); \quad \alpha, \beta \in \mathbb{Z}(d).$$

The $X, P$ in the notation are not variables, but they simply indicate position and momentum states, correspondingly. Displacement operators in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space, are defined as

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega(-2^{-1} \alpha \beta); \quad Z = \sum_{m} |X; m\rangle \langle X; m|; \quad X = \sum_{m} |X; m + 1\rangle \langle X; m|$$

The formalism is slightly different in the cases of odd and even $d$, and here we consider odd $d$ so that the factor $2^{-1}$ exists in $\mathbb{Z}(d)$.

Acting with $D(\alpha, \beta)$ on a normalized fiducial vector $|f\rangle$ we get the following $d^2$ ‘coherent states’[6, 7]

$$|C; \alpha, \beta\rangle = D(\alpha, \beta)|f\rangle; \quad \alpha, \beta \in \mathbb{Z}(d).$$

The $C$ in the notation indicates coherent states. For a ‘generic’ fiducial vector, any $d$ of the $d^2$ coherent states are linearly independent.

The coherent states obey the resolution of the identity:

$$\frac{1}{d} \sum_{\alpha, \beta} |C; \alpha, \beta\rangle \langle C; \alpha, \beta| = 1.$$  

2.2. The modular orthocomplemented lattice of subspaces $\mathcal{L}(d)$

The Birkhoff-von Neumann lattice of the closed subspaces of the Hilbert space, has been studied extensively in the literature [8, 9, 10]. In the case of finite dimensional Hilbert spaces considered in this paper, it is a modular orthocomplemented lattice which we denote as $\mathcal{L}(d)$.

We consider the set of subspaces of $H(d)$, and define the conjunction (logical AND) and disjunction (logical OR) [11, 12]:

$$H_1 \land H_2 = H_1 \cap H_2; \quad H_1 \lor H_2 = \text{span}(H_1 \cup H_2).$$
The logical OR is not just the union, but it contains superpositions of states in the two subspaces. Boolean algebra (classical logic) is formalized with set theory and the logical OR is the union of two sets. In the modular orthocomplemented lattice $\mathcal{L}(d)$ (quantum logic of finite quantum systems) the logical OR is more than the union of two subspaces, because it contains all superpositions.

In every lattice the following distributivity inequalities hold ($\prec$ indicates subspace):

$$ (H_1 \land H_2) \lor H_0 \prec (H_1 \lor H_0) \land (H_2 \lor H_0) $$
$$ (H_1 \lor H_2) \land H_0 \succ (H_1 \land H_0) \lor (H_2 \land H_0). $$

(6)

In a distributive lattice they become equalities.

The lattice $\mathcal{L}(d)$ is modular but non-distributive. The modularity property is

$$ H_1 \prec H_3 \rightarrow H_1 \lor (H_2 \land H_3) = (H_1 \lor H_2) \land H_3. $$

(7)

Modularity is a weak version of distributivity, and is related to independence. Birkhoff [11] discussed the link between modular lattices and matroids (which introduce independence in an abstract way). The lattice $\mathcal{L}(d)$ is non-distributive, but some of its sublattices might be distributive.

The orthocomplement of $H_1$ (logical NOT operation) is unique, and is another subspace which we denote as $H_1^\perp$, with the properties

$$ H_1 \land H_1^\perp = \emptyset; \quad H_1 \lor H_1^\perp = \mathcal{I} = H(d); \quad (H_1^\perp)^\perp = H_1 $$
$$ (H_1 \land H_2)^\perp = H_1^\perp \lor H_2^\perp; \quad (H_1 \lor H_2)^\perp = H_1^\perp \land H_2^\perp $$
$$ \dim(H_1) + \dim(H_1^\perp) = d. $$

(8)

3. Independence in set theory and Kolmogorov probability theory

Let $\Omega$ be a finite set. The powerset $2^\Omega$ contains all its subsets. In it we define the conjunction (logical AND), disjunction (logical OR), and negation (logical NOT), as the intersection, union and complement:

$$ A_1 \land A_2 = A_1 \cap A_2; \quad A_1 \lor A_2 = A_1 \cup A_2; \quad \neg A = \Omega \setminus A. $$

(9)

Then $2^\Omega$ with these operations, is a Boolean algebra.

The subsets $A_1, \ldots, A_n$ of $\Omega$ are independent, if

$$ (A_1 \lor \ldots \lor A_{i-1} \lor A_{i+1} \lor \ldots \lor A_n) \land A_i = \emptyset, $$

(10)

for all $i = 1, \ldots, n$. The subsets $A_1, \ldots, A_n$ of $\Omega$ are pairwise independent or pairwise disjoint, if

$$ A_i \land A_j = \emptyset, $$

(11)

for all $i, j$. Using the distributivity property

$$ (A_1 \land A_2) \lor A_3 = (A_1 \lor A_3) \land (A_2 \lor A_3); \quad (A_1 \lor A_2) \land A_3 = (A_1 \land A_3) \lor (A_2 \land A_3), $$

(12)

we prove that in set theory independence is equivalent to pairwise independence.
The marginals of joint probability distributions are based on the law of total probability in Kolmogorov’s probability theory. Let \( \Omega \) be a set of alternatives, \( B_1, ..., B_n \) a partition of the set \( \Omega \) and \( A \subseteq \Omega \). Then

\[
p(A) = \sum_i p(A \cap B_i).
\]  

(13)

Distributivity is essential for the validity of this. In non-distributive structures (like quantum mechanics) this might not hold.

4. Quasi-probability distributions related to the subspaces \( H_1, ..., H_n \)

We consider a set \( \{ H_1, ..., H_n \} \) of \( n \geq 2 \) proper subspaces of \( H(d) \). Let \( \rho \) be a density matrix, and

\[
R(i) = \text{Tr}[\rho \Pi(H_i)] \geq 0; \quad i = 1, ..., n
\]  

(14)

For a given \( i \), \( R(i) \) is the probability that the measurement \( \Pi(H_i) \) will give the outcome ‘yes’. The set of all \( R(i) \) is not in general a probability distribution, but it is a quasi-probability distribution.

Later, we will use the quasi-probability distribution:

\[
\tilde{R}(i) = \text{Tr}[\rho \Pi(J_i^\perp \wedge H_i)] \]  

(15)

where

\[
J_i = \bigvee_{j \neq i} H_j = H_1 \lor ... \lor H_{i-1} \lor H_{i+1} \lor ... \lor H_n
\]

\[
J_i^\perp = \bigwedge_{j \neq i} H_j^\perp = H_1^\perp \land ... \land H_{i-1}^\perp \land H_{i+1}^\perp \land ... \land H_n^\perp.
\]  

(16)

\( J_i^\perp \wedge H_i \) involves the part of the space \( H_i \) which overlaps with all \( H_j^\perp \), and therefore it does not overlap with any of the \( H_j \), for \( j \neq i \). The \( \tilde{R}(i) \) is the probability that a measurement \( \Pi(J_i^\perp \wedge H_i) \) on a system with density matrix \( \rho \), will give ‘yes’. In this case the state belongs to \( H_i \) and it also belongs to all \( H_j^\perp \), with \( j \neq i \).

It is easily seen that \( 0 \leq \tilde{R}(i) \leq R(i) \). The \( \Pi(J_i^\perp \wedge H_i) \) commutes with all \( \Pi(H_j) \):

\[
\Pi(H_j), \Pi(J_i^\perp \wedge H_i) = 0
\]  

(17)

**Example 4.1.** We consider the \( d \) one-dimensional subspaces \( H(X; \alpha) \) that contain the position spaces \( |X; \alpha \rangle \). We get:

\[
R(\alpha) = \langle X; \alpha | \rho | X; \alpha \rangle; \quad \sum_\alpha R(\alpha) = 1; \quad \alpha \in \mathbb{Z}(d).
\]  

(18)

This is the probability distribution in the position space.

**Example 4.2.** We consider the \( d^2 \) one-dimensional subspaces \( H(C; \alpha, \beta) \) that contain the coherent states \( |C; \alpha, \beta \rangle \). Then

\[
R(\alpha, \beta) = \langle C; \alpha, \beta | \rho | C; \alpha, \beta \rangle; \quad \frac{1}{d} \sum_{\alpha, \beta} R(\alpha, \beta) = 1; \quad \alpha, \beta \in \mathbb{Z}(d).
\]  

(19)

\( R(\alpha, \beta) \) is the \( Q \)-function in the \( \mathbb{Z}(d) \times \mathbb{Z}(d) \) phase space.
5. Levels of independence

The subspaces $H_1, ..., H_n$ of $H(d)$ are independent, if for all $i = 1, ..., n,$

$$J_i \land H_i = \mathcal{O} \quad (20)$$

A state cannot belong to both $H_i$ AND to $J_i$. This is equivalent to the statement that the subspaces $H_1, ..., H_n$ are independent, if any $n$ vectors $|v_1\rangle \in H_1, ..., |v_n\rangle \in H_n$ (one vector from each of the subspaces $H_i$), are independent:

$$\lambda_1|v_1\rangle + ... + \lambda_n|v_n\rangle = 0 \rightarrow \lambda_1 = ... = \lambda_n = 0. \quad (21)$$

We can prove that if the set $\{H_1, ..., H_n\}$ contains independent subspaces, then the subspaces in any subset (with cardinality at least 2) are also independent. Also if the subspaces $H_1, ..., H_n$ of $H(d)$ are independent, then

$$\dim(H_1) + ... + \dim(H_n) = \dim(H_1 \lor ... \lor H_n) \leq d. \quad (22)$$

A weaker (due to the non-distributivity of the lattice) concept is pairwise independence. The subspaces $H_1, ..., H_n$ are pairwise independent, if

$$H_i \land H_j = \mathcal{O} \text{ for all } i, j.$$ 

Only for subspaces within a distributive sublattice of $L(d)$, independence is equivalent to pairwise independence. An example of this, is when the $\Pi(H_1), ..., \pi(H_n)$ commute with each other.

Between (full) independence and pairwise independence there are many intermediate concepts which we quantify with the degree of independence. Let $\rho$ be a density matrix. The matrix for the degree of independence $A$, and the degree of independence $\eta(\rho)$, are given by

$$A = \frac{1}{n} \sum_i [\Pi(H_i) - \Pi(J_i^\perp \land H_i)]; \quad \eta(\rho) = \frac{1}{n} \sum_i [R(i) - \tilde{R}(i)] = \text{Tr}(\rho A). \quad (23)$$

$A$ is a $d \times d$ positive semidefinite matrix (as a sum of the projectors $\Pi(H_i) - \Pi(J_i^\perp \land H_i)$).

We note that:

- If $J_i^\perp \land H_i = H_i$ for all $i$, then $A = 0$ and $\eta(\rho) = 0$. The $\{H_1, ..., H_n\}$ are (fully) independent. This is the strongest form of independence.
- If $J_i^\perp \land H_i = \mathcal{O}$ for all $i$, then

$$A = \frac{1}{n} \sum_i \Pi(H_i); \quad \eta(\rho) = \frac{1}{n} \sum_i R(i) \quad (24)$$

The $\{H_1, ..., H_n\}$ are pairwise independent. This is the weakest form of independence.
- Between these two extreme cases, the $\{H_1, ..., H_n\}$ are partially independent. For a given $\rho$, $\eta(\rho)$ takes values in the interval

$$0 \leq \eta(\rho) \leq \frac{1}{n} \sum_i R(i) \quad (25)$$
In $H(d)$ (with fixed $d$), we consider various sets of subspaces $S_1 = \{H_1, ..., H_n\}$, $S_2 = \{H'_1, ..., H'_m\}$, etc, with matrices for the degree of independence $A_1$, $A_2$, etc. We define a partial preorder among them, as follows. The set of subspaces $S_1$ is more independent than $S_2$ (we denote this as $S_1 \triangleright S_2$), if $A_1 - A_2$ is a negative semidefinite matrix (denoted as $A_1 - A_2 \leq 0$). In this case $\eta_1(\rho) \leq \eta_2(\rho)$ for all density matrices $\rho$.

It is easily seen that $\triangleright$ is a partial preorder. The properties of reflexivity ($S_1 \triangleright S_1$) and transitivity (if $S_1 \triangleright S_2$ and $S_2 \triangleright S_3$ then $S_1 \triangleright S_3$), hold. But antisymmetry does not hold (if $S_1 \triangleright S_2$ and $S_2 \triangleright S_1$, it does not follow that $S_1 = S_2$).

5.1. Examples

**Example 5.1.** We consider the $d$ one-dimensional subspaces $H(X; \alpha)$ that contain the position spaces $|X; \alpha\rangle$. In this case

$$J(X; \alpha) = \bigvee_{\beta \neq \alpha} H(X; \beta); \quad [J(X; \alpha)]^\perp = H(X; \alpha).$$

and

$$A = 0; \quad \eta(\rho) = 0; \quad \tilde{R}(\alpha) = R(\alpha) = \langle X; \alpha | \rho | X; \alpha \rangle.$$  

Therefore these subspaces are independent. This is the highest level of independence.

**Example 5.2.** We consider the $d^2$ one-dimensional subspaces $H(C; \alpha, \beta)$ that contain the coherent states $|C; \alpha, \beta\rangle$. Then These subspaces are pairwise independent. We have explained earlier that any $d$ of the $d^2$ coherent states are linearly independent, and therefore

$$J(C; \alpha_0, \beta_0) = \bigvee_{\alpha \neq \alpha_0, \beta \neq \beta_0} H(C; \alpha, \beta) = H(d); \quad [J(C; \alpha_0, \beta_0)]^\perp = \emptyset$$

and

$$A = \frac{1}{d^2} \sum \Pi[H(C; \alpha, \beta)] = \frac{1}{d} 1; \quad \eta(\rho) = \frac{1}{d}; \quad \tilde{R}(\alpha, \beta) = 0.$$  

Therefore these subspaces are pairwise independent. This is the lowest level of independence.

**Example 5.3.** In $H(8)$ we consider the following two-dimensional subspaces:

$$H_1 = \begin{pmatrix} \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}; \quad H_2 = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}; \quad H_3 = \begin{pmatrix} \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b \end{pmatrix} \end{pmatrix}; \quad H_4 = \begin{pmatrix} \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ b \\ b \end{pmatrix} \end{pmatrix}. \quad (30)$$
Here we give a generic vector within these subspaces, which depends on two variables because the subspaces are two-dimensional. Then we calculate the subspaces \( J_i \):

\[
J_1 = H_2 \vee H_3 \vee H_4 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}; \quad J_2 = H_1 \vee H_3 \vee H_4 = \begin{pmatrix}
a \\
b \\
0 \\
0 \\
c \\
0
\end{pmatrix}; \quad J_3 = H_1 \vee H_2 \vee H_4 = \begin{pmatrix}
a \\
b \\
c \\
-\frac{a}{2} \\
\frac{c}{2} \\
0
\end{pmatrix}; \quad J_4 = H_1 \vee H_2 \vee H_3 = \begin{pmatrix}
a \\
b \\
c \\
0 \\
0 \\
f
\end{pmatrix}.
\] (31)

\( J_1, J_3, J_4 \) are six-dimensional subspaces, and the vectors depend on six variables. \( J_2 \) is a five-dimensional subspace, and the vector depends on five variables.

We also consider their orthocomplements which are the subspaces:

\[
J_1^\perp = \begin{pmatrix}
a \\
0 \\
0 \\
0 \\
0 \\
-\frac{a}{2}
\end{pmatrix}; \quad J_2^\perp = \begin{pmatrix}
a \\
b \\
0 \\
0 \\
c \\
0
\end{pmatrix}; \quad J_3^\perp = \begin{pmatrix}
a \\
b \\
c \\
0 \\
0 \\
0
\end{pmatrix}; \quad J_4^\perp = \begin{pmatrix}
a \\
b \\
c \\
0 \\
0 \\
0
\end{pmatrix}
\] (32)

\( J_1^\perp, J_3^\perp, J_4^\perp \), are two-dimensional spaces and \( J_2^\perp \) is three-dimensional. From this follows that

\[
J_1^\perp \wedge H_1 = \begin{pmatrix}
a \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}; \quad J_2^\perp \wedge H_2 = \begin{pmatrix}
a \\
b \\
0 \\
0 \\
0 \\
0
\end{pmatrix}; \quad J_3^\perp \wedge H_3 = \mathcal{O}; \quad J_4^\perp \wedge H_4 = \mathcal{O}.
\] (33)
The corresponding projectors are calculated as follows. Let $a_1, \ldots, a_k$ be $k$ independent vectors, and $M$ the $d \times k$ matrix $(a_1, \ldots, a_k)$ which has as columns these vectors. The projector to the space spanned by these $k$ vectors is

$$\Pi = M(M^\dagger M)^{-1}M^\dagger.$$  \hspace{1cm} (34)

Using this we calculated the matrix for the degree of independence:

$$A = \frac{1}{24} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 3 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 6 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 5 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 9 \end{pmatrix}.$$  \hspace{1cm} (35)

We also consider the state

$$|s\rangle = \frac{1}{\sqrt{60}}([X; 0] + 2[X; 1] + 3[X; 2] + [X; 4] + 5[X; 5] + 2[X; 6] + 4[X; 7])$$  \hspace{1cm} (36)

and the corresponding density matrix $\rho = |s\rangle \langle s|$. In this case the degree of independence is

$$\eta(\rho) = \text{Tr}(\rho A) = 0.285.$$  \hspace{1cm} (37)

Also

$$R(1) = 0.083; \quad R(2) = 0.225; \quad R(3) = 0.341; \quad R(4) = 0.658$$

$$\tilde{R}(1) = 0.016; \quad \tilde{R}(2) = 0.150; \quad \tilde{R}(3) = 0; \quad \tilde{R}(4) = 0.$$  \hspace{1cm} (38)

6. Discussion

Given $n$ subspaces $H_1, \ldots, H_n$ of $H(d)$, we have considered the quasi-probability distribution $R(i)$ of Eq.(14). Both the probability distribution in the position space (Eq.(18)), and the $Q$-function (Eq.(19)), are special cases of this more general distribution. We have also introduced the quasi-probability distribution $\tilde{R}(i)$ of Eq.(15).

In this general context, we have studied the concept of independence. We have shown that there are many levels of independence, from pairwise independence up to (full) independence. This is intimately related to the fact that quantum theory is a non-distributive structure.

Classical physics and Boolean logic, are formalized with set theory, which is a distributive structure. In this case

- various approaches to independence are equivalent to each other
- the law of total probability in Eq.(13) holds. It is used in defining marginals of probability distributions.
Quantum physics and quantum logic are formalized with the Birkhoff-von Neumann lattice of subspaces, and distributivity is replaced by the weaker property of modularity (in systems with finite-dimensional Hilbert space). We have shown in [1] and in this paper, that consequences of this are:

- that there are many levels of independence. They are quantified with the matrix for the degree of independence $A$, and the degree of independence $\eta(\rho)$ in Eq.(23), which is related to the difference $R(i) - \tilde{R}(i)$.
- that the law of total probability does not hold in general.

The work studies quantum theory from the angle of non-distributivity, which can play a complementary role to non-commutativity.

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