Heavy Quark Potential in a static and strong homogeneous magnetic field

Mujeeb Hasan\textsuperscript{1}, Bhaswar Chatterjee\textsuperscript{2} and Binoy Krishna Patra\textsuperscript{3}

\textit{Department of Physics, Indian Institute of Technology Roorkee, India, 247 667}

Abstract

We have investigated the properties of quarkonia in a thermal QCD medium in presence of strong magnetic field. For that purpose, we first derive the one-loop gluon self-energy in strong magnetic field for both massless and massive flavors and its static limit gives the Debye screening mass, which in the sequel gives the the effective gluon propagator. Interestingly, in strong magnetic field, the Debye mass has been found to be affected mainly by the magnetic field only, compared to the other scales prevalent in the thermal medium. With the abovementioned ingredients, the potential between heavy quark ($Q$) and anti-quark ($\bar{Q}$) is obtained in a hot QCD medium in the presence of strong magnetic field by correcting both short and long range components of the potential in vacuum. It is found that the long range part of the quarkonium potential is affected much by magnetic field as compared to the short range part. This observation facilitates us to estimate the magnetic field beyond which the potential will be too weak to bind $Q\bar{Q}$ together. For example, the $J/\psi$ is dissociated at $eB \sim 10 \ m_{\pi}^2$ and $\Upsilon$ is dissociated at $eB \sim 100 \ m_{\pi}^2$ whereas its excited states, $\psi'$ and $\Upsilon'$ are dissociated at smaller magnetic field $eB = m_{\pi}^2$, $13m_{\pi}^2$, respectively.

PACS: 12.39.-x, 11.10.St, 12.38.Mh, 12.39.Pn

Keywords: Quantum Chromodynamics, Schwinger proper-time method, Debye mass, strong magnetic field, string tension, dielectric permittivity, Heavy quark potential.

1 Introduction

Lattice gauge theory at very high temperatures and/or baryon densities predicts an interesting window onto the properties of Quantum Chromodynamics (QCD) in guise of a new phase, Quark-gluon Plasma (QGP), which pervaded the early universe, and may be present in the core of neutron stars. To realize this predicted phase, current experimental program of ultra relativistic heavy ion collisions (URHIC) have been designed at different colliders with different center of mass energies, \textit{viz.} Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) at $\sqrt{s}= 200$ GeV per nucleon in Au + Au collisions and Large Hadron Collider (LHC) at European Organization for Nuclear Research (CERN) at $\sqrt{s}= 2.76$ TeV per nucleon in Pb + Pb collisions. Recent analysis suggests that the events of URHIC should be analyzed by incorporating the effect of magnetic field because an intensely strong magnetic field, perpendicular to the reaction plane, is expected to be produced at very early stages of collisions when the event is off-central.
Depending on the centrality, the strength of the magnetic field may reach between $m^2_7$ ($\simeq 10^{18}$ Gauss) at RHIC [6, 7] to 10 $m^2_7$ at LHC [7]. At extreme cases it may reach values of 50 $m^2_7$. A very strong magnetic field ($\sim 10^{21}$ Gauss) was also produced in the early universe during the electroweak phase transition due to the gradients in Higgs field. Ultimately such strong magnetic field might significantly affect the production of particle and its dynamics at very early stage of the collisions. Since magnetic field induces an anisotropy to the momentum of the affected particles, we might expect it to affect the anisotropic flow of the particles.

Heavy quark pairs ($Q\bar{Q}$) are also produced during the very early stages of the collisions when the field strength is expected to be very strong. For example, charm-anti charm ($c\bar{c}$) pairs are produced at a typical time of $t_{c\bar{c}}(\simeq 1/2m_c) \simeq 0.1$ fm whereas the magnetic field is expected to be very strong typically up to $t_B \simeq 0.2$ fm [8]. Moreover, it has been suggested that depending on the conductivity of the medium, the magnetic field produced may still remain substantial for significantly longer time [9]. Thus it becomes reasonable to assume that charmonium production may get significantly influenced by the magnetic field.

The same argument applies to the bottomonium production. A large number of studies on the in-medium properties of $Q\bar{Q}$ bound states has been carried out using the phenomenological potential models [10], where the effects of the medium are encoded in a temperature dependent potential with non perturbative inputs from the lattice simulations. The derivation of such models was however not established from the underlying theory, QCD. However, lattice calculations of free energies and other quantities [11] obtained from the correlation functions of Polyakov loop are often taken as input for the potential. Although these quantities have been thought to be related to the color-singlet and color-octet heavy quark potentials at finite temperature, a precise answer is still missing [12]. Recently quarkonia at finite temperature has been studied by taking the advantage of the hierarchies between the non-relativistic scales associated with quarkonia and the thermal energy scales characterizing the system through the effective field theories, viz. NRQCD, pNRQCD etc [13]. The in-medium modifications of the quarkonium states can be studied from the first principle of QCD by the spectral functions [14] but the reconstruction of the spectral function from the lattice meson correlators turns out to be very difficult.

Recently one of us explored the properties of quarkonia states in a hot medium by correcting both the perturbative and non perturbative terms of the $Q\bar{Q}$ potential through the dielectric function in real-time formalism [15] in both isotropic as well as anisotropic hot QCD medium, arising at the very early stages of the collisions [16]. As mentioned earlier, magnetic field is also produced at the early stages of the collisions thus it becomes worthwhile to examine the effects of magnetic field on the properties of quarkonia bound states in the above framework and subsequently on its yields. For that purpose we will first calculate the gluon self-energy in a background (heat bath) medium in the presence of a static homogeneous magnetic field, and then calculate the interquark potential between $Q$ and $\bar{Q}$ by the static limit of the effective gluon propagator to see the effects of magnetic field on the quarkonium states in a thermal medium.

Our work is arranged in the following way: In subsection 2.1 we will discuss the quark propagator at finite temperature in strong magnetic field approximation (SMFA). In subsection 2.2 we will calculate the
gluon self energy at finite temperature in presence of strong magnetic field. In next subsection 2.3 we will compute the screening mass in SMFA by taking the static limit of gluon-self energy. In Section 3, we will obtain the potential from the static limit of the effective propagator and explore how the properties of quarkonia could be affected by the presence of strong magnetic field. Finally we will conclude in Section 4.

2 Quarkonia in a magnetic field

Quantum mechanically both the quarkonium and heavy meson spectra have been analyzed in both the three-dimensional harmonic potential and Cornell potential with an additional spin-spin interaction term [17, 18]. Moreover the possible anisotropies emerging in the static quark-anti quark potential is also explored in [17]. Schwinger first obtained the fermionic propagator in coordinate space [19] and then Tsai has obtained the same in momentum space and used it to calculate the vacuum polarization in magnetic field [20]. The vacuum polarization tensor has also been obtained in a gauge invariant manner for both strong and weak magnetic field limit [21, 22, 23]. We are now going to extend these propagators to QCD to calculate the gluon self-energy, which will in turn help to study the properties of quarkonia quantum field theoretically in the presence of strong magnetic field.

2.1 Fermionic propagator in presence of magnetic field

2.1.1 Vacuum in a static and homogeneous magnetic field:

For the sake of simplicity, we assume the magnetic field to be constant and homogeneous. We also assume the magnetic field to be along z-direction and of magnitude $B$. Such a magnetic field can be obtained from a vector potential $A_\mu = (0, 0, Bx, 0)$. The choice of vector potential is not unique as the same magnetic field can also be obtained from a symmetric potential given by $A_\mu = (0, -\frac{By}{2}, \frac{Bx}{2}, 0)$. Using the proper-time method formulated by Schwinger, the fermion propagator in such a magnetic field can be written in the coordinate space as [19]

$$S(x, x') = \phi(x, x') \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} S(p) , \tag{1}$$

where the the phase factor, $\phi(x, x')$ is given by

$$\phi(x, x') = e^{ie\int_{x'}^{x} A_\xi d\xi} , \tag{2}$$

which becomes unity for a closed fermion loop with two fermion lines, i.e, $\phi(x, x') = 1$ [24]. However, the same propagator was first calculated by [20, 24] in the momentum space as

$$iS(p) = \int_0^\infty \frac{ds}{eB\cos(s)} e^{-is[m^2-p_0^2+\frac{m(s)}{2}p_1^2]} \left[\left(\cos(s) + \gamma_1 \gamma_2 \sin(s)\right) (m + \gamma \cdot p_\parallel - \frac{\gamma \cdot p_\perp}{\cos(s)})\right] . \tag{3}$$

The propagator (3) in the momentum space can also be expressed in a more convenient way using the
associated Laguerre polynomials \((L_n)\)

\[
iS(p) = \sum_n -id_n(\alpha)D + d'_n(\alpha)\bar{D} + i\gamma \cdot p_\perp \frac{p_0^2 - p_\perp^2}{p_\perp^2} + i\gamma \cdot p_\perp \frac{m^2 - p_\perp^2}{p_\perp^2},
\]

where the following quantities are defined as [24]

\[
D = (m + \gamma \cdot p_\parallel) + \gamma \cdot p_\perp \frac{m^2 - p_\perp^2}{p_\perp^2},
\]

\[
\bar{D} = \gamma_1 \gamma_2 (m + \gamma \cdot p_\parallel),
\]

\[
d_n(\alpha) = (-1)^n e^{-\alpha} C_n(2\alpha),
\]

\[
C_n(2\alpha) = L_n(2\alpha) - L_{n-1}(2\alpha),
\]

\[
d'_n(\alpha) = \frac{\partial d_n}{\partial \alpha},
\]

\[
p_\perp^2 = m^2 - p_\parallel^2,
\]

\[
\alpha = \frac{p_\perp^2}{eB},
\]

\[
p_\parallel^2 = p_0^2 - p_z^2,
\]

\[
p_\perp^2 = p_x^2 + p_y^2.
\]

The order of the Laguerre polynomial also corresponds to the number of energy eigenvalues in a magnetic field, known as Landau levels. In strong magnetic field approximation (SMFA) \((eB >> \text{pole masses (}m^2\text{)})\) the particles occupy the lowest Landau level (LLL) \((n = 0)\) only, thus, in SMFA, the fermion propagator in eq.(4) reduces to the following form

\[
iS_0(p) = \frac{(1 + \gamma^0 \gamma^3 \gamma^5)(\gamma^0 p_0 - \gamma^3 p_z + m)}{p_0^2 - m^2 + i\epsilon} e^{\frac{p_0^2}{m}},
\]

where \(m\) and \(q\) are the mass and electric charge of the fermion, respectively.

### 2.1.2 Heat Bath in a strong homogeneous magnetic field

In thermal medium, the system in weak coupling regime possesses additional scales, viz. \(T, gT\) etc., in addition to the masses of partons (quarks). So at finite temperature, SMFA implies that both conditions \(eB >> T^2\) and \(eB >> m^2\) are to be satisfied. In real-time formalism the fermion propagator can be expressed as

\[
S(p) = \left( \begin{array}{cc} S_0(p)\bar{N}_2^2 + S_0(p)\bar{N}_1^2 & -S_0(p)\bar{N}_1\bar{N}_2 + S_0(p)\bar{N}_0\bar{N}_2 \\ S_0(p)\bar{N}_1\bar{N}_2 - S_0(p)\bar{N}_1\bar{N}_2 & -S_0(p)\bar{N}_2^2 - S_0(p)\bar{N}_2^2 \end{array} \right),
\]

where

\[
\bar{N}_1(p_0) = \sqrt{n_+} \theta(p_0) + \sqrt{n_-} \theta(-p_0),
\]

\[
\bar{N}_2(p_0) = \sqrt{1 - n_+} \theta(p_0) + \sqrt{1 - n_-} \theta(-p_0),
\]

\[
n_\pm(\omega) = \frac{1}{e^{\beta(\omega - m)} + 1},
\]

and \(S_0(p)\) is given by eq.(5).
For calculating the gluon self energy in an equilibrium medium, we need only the “11”-component of the matrix propagator expressed in eq. (6)

\[ S_{11}(p) = S_0(p) \tilde{N}_2^2 + S_0^*(p) \tilde{N}_2^2. \]  

(10)

For zero chemical potential, i.e., \( \mu = 0 \), \( n_+ = n_- = n \), the “11”-component of the fermion propagator becomes

\[ iS_{11}(p) = \left[ \frac{1}{p_0^2 - m^2 + i\epsilon} + 2\pi i n_+ \delta(p_0^2 - m^2) \right] \left( 1 + \gamma_0 \gamma_3 \gamma_5 (\gamma_0 p_0 - \gamma_3 p_z + m) e^{-\frac{p_0^2}{m^2}} \right), \]  

(11)

where the distribution function is given by

\[ n_p(p_0) = \frac{1}{e^{\beta |p_0|} + 1}. \]

The above description for fermionic propagator can be easily generalized to quarks of \( f \)-th flavor with which we are going to calculate the gluon self-energy.

### 2.2 Gluon self-energy in a hot QCD medium in presence of strong magnetic field

For system in equilibrium, we need only the “11”-component of the gluon self-energy matrix, which is given by

\[ \Pi^{\mu\nu}(k) = \frac{i g^2}{2} \sum_f \int \frac{d^4p}{(2\pi)^4} \text{tr}[\gamma^\mu S_{11}(p)\gamma^\nu S_{11}(q)] \]

\[ = \frac{i g^2}{2} \sum_f \int \frac{d^4p}{(2\pi)^4} \text{tr}\left[ (\gamma^\mu (1 + \gamma_0 \gamma_3 \gamma_5) (\gamma_0 p_0 - \gamma_3 p_z + m_f) \gamma^\nu (1 + \gamma_0 \gamma_3 \gamma_5) (\gamma_0 q_0 - \gamma_3 q_z + m_f) \right] \]

\[ \left\{ \frac{1}{p_0^2 - m_f^2 + i\epsilon} + 2\pi i n_+ \delta(p_0^2 - m_f^2) \right\} \left\{ \frac{1}{q_0^2 - m_f^2 + i\epsilon} + 2\pi i n_+ \delta(q_0^2 - m_f^2) \right\} e^{-\frac{p_0^2}{m^2}} e^{-\frac{q_0^2}{m^2}}, \]  

(12)

where the factor 1/2 arises due to trace in color-space and the trace due to \( \gamma \) matrices is given by

\[ L^{\mu\nu} = 8 \left[ p_0^\mu q_0^\nu + p_0^\nu q_0^\mu - g^{\mu\nu}((p.q)_0 - m_f^2) \right]. \]  

(13)

Separating the momentum integration into longitudinal (\( \parallel \)) and transverse (\( \perp \)) components with respect to the magnetic field, the gluon self-energy can be factorized into \( \parallel \) and \( \perp \) components of momentum integration

\[ \Pi^{\mu\nu}(k) = \sum_f \Pi^{\mu\nu}_{\parallel}(k_{\parallel}) A_f(k_{\perp}), \]  

(14)

where the transverse component is given by

\[ A_f(k_{\perp}) = \int dp_z dp_0 e^{-\frac{p_z^2}{m_f^2}} e^{-\frac{p_0^2}{m^2}} \]

\[ = \frac{\pi |q_f| B}{2} e^{-\frac{q_f^2}{2m^2}}. \]  

(15)
2.2.1 Vacuum contribution (T = 0, 4-dimensional)

Using the identity \( T \) is now reduced from 4 to 2. This dimensional reduction in fact removes the divergences usually encountered in vacuum without magnetic field except the fact that the dimension of the momentum integration in LLL approximation, the dependence of self-energy on the magnetic field is fully encapsulated in the transverse component whereas the longitudinal part carries no dependence on the magnetic field. We will now calculate the longitudinal component of the self-energy by decomposing eq. (12) into vacuum and thermal parts:

\[
\Pi_{\parallel}^{\mu\nu} = (\Pi_{\parallel}^{\mu\nu})_V + (\Pi_{\parallel}^{\mu\nu})_n + (\Pi_{\parallel}^{\mu\nu})_{n^2},
\]

where \((\Pi_{\parallel}^{\mu\nu})_V\) is the vacuum part, \((\Pi_{\parallel}^{\mu\nu})_n\) and \((\Pi_{\parallel}^{\mu\nu})_{n^2}\) are the thermal contributions due to single and double distribution functions, respectively. They are explicitly given by

\[
(\Pi_{\parallel}^{\mu\nu})_V = \frac{ig^2}{2(2\pi)^4} \int dp_0 dp_2 \frac{L^{\mu\nu}}{2\pi} \left\{ \frac{1}{(q_0^2 - m_f^2 + i\epsilon)(p_0^2 - m_f^2 + i\epsilon)} \right\}, \tag{17}
\]

\[
(\Pi_{\parallel}^{\mu\nu})_n = \frac{ig^2(2\pi i)}{2(2\pi)} \int dp_0 dp_2 \frac{L^{\mu\nu}}{2\pi} \left\{ \frac{n_\mu n_\nu \delta(p_0^2 - m_f^2)}{(q_0^2 - m_f^2 + i\epsilon)(p_0^2 - m_f^2 + i\epsilon)} \right\}, \tag{18}
\]

\[
(\Pi_{\parallel}^{\mu\nu})_{n^2} = \frac{ig^2}{2(2\pi)^4} \int dp_0 dp_2 \frac{L^{\mu\nu}}{2\pi} \left\{ (-4\pi^2)n_\mu n_\nu \delta(p_0^2 - m_f^2)\delta(q_0^2 - m_f^2) \right\}. \tag{19}
\]

We will now calculate the vacuum term for the gluon self-energy.

2.2.1 Vacuum contribution (T = 0, eB \(\neq 0\))

The vacuum term in strong magnetic field can be calculated easily as it is similar to the calculation of self-energy in vacuum without magnetic field except the fact that the dimension of the momentum integration is now reduced from 4 to 2. This dimensional reduction in fact removes the divergences usually encountered in 4-dimension, thus we do not need any regularization any more. Using the identity

\[
\frac{1}{x \mp i\epsilon} = P\left(\frac{1}{x}\right) \pm i\pi \delta(x), \tag{20}
\]

the real part of the vacuum term in the gluon-self energy has been calculated as

\[
\Re \Pi^{\mu\nu}(k) \mid_V = (g^{\mu\nu}_{\parallel} - \frac{k^\mu k^\nu}{k_\parallel^2}) \Pi(k^2), \tag{21}
\]

where the form factor, \(\Pi(k^2)\) is given by

\[
\Pi(k^2) = \frac{g^2}{4\pi^2} \sum_f |q_f B| e^{-\frac{k^2}{24\pi^2\alpha_s}} \left[ \frac{2m_f^2}{k_\parallel^2} \left(1 - \frac{4m_f^2}{k_\parallel^2}\right)^{-1/2} \left\{ \ln \frac{1 - \left(1 - \frac{4m_f^2}{k_\parallel^2}\right)^{1/2}}{1 + \left(1 + \frac{4m_f^2}{k_\parallel^2}\right)^{1/2} + i\pi} - 1 \right\} \right]. \tag{22}
\]

Therefore the “00”-component (\(\mu = \nu = 0\)) of the real part of vacuum term of the gluon self-energy (using the metric \(g^{\mu\nu}_{\parallel} = \text{diag}(1, 0, 0, -1)\) is given by

\[
\Re \Pi^{00}(k) \mid_V = -\frac{k_\parallel^2}{k_\parallel^2} \Pi(k^2).
\]
In the limit of massless quarks \((m_f = 0)\), the gluon self-energy due to vacuum term in the static limit \((k_0 = 0, \vec{k} \to 0)\) is given by the scale available to the magnetic field only in SMFA

\[
\Re\Pi^{00}(k_0 = 0, \vec{k} \to 0) \big|_V = \frac{g^2}{4\pi^2} \sum_f |q_f B| = 0.
\] (23)

For the physical quark masses \((m_f \neq 0)\), the vacuum term in the static limit \((k_0 = 0, \vec{k} \to 0)\) vanishes

\[
\Re\Pi^{00}(k_0 = 0, \vec{k} \to 0) \big|_V = 0.
\] (24)

### 2.2.2 Medium contribution

The (thermal) medium contribution to the gluon self-energy contains two terms: the first one \((18)\) involves single distribution function and the second one \((19)\) involves the product of two distribution functions. We will first consider the medium contribution due to the single distribution function only. Using the property of Dirac delta function, the gluon self-energy in eq.\((18)\) is reduced to

\[
(\Pi^{\mu\nu})_{\parallel} = -\frac{g^2}{2(2\pi)^3} \int dp_0 dp_z L^{\mu\nu} \left[ n_\rho(p_0) \left\{ \frac{\delta(p_0 - \omega_\rho) + \delta(p_0 + \omega_\rho)}{(q_0^2 - q_\rho^2 - m_\rho^2 + i\epsilon)(2\omega_\rho)} \right\} + n_\eta(q_0) \left\{ \frac{\delta(q_0 - \omega_\eta) + \delta(q_0 + \omega_\eta)}{(p_0^2 - p_\eta^2 - m_\eta^2 + i\epsilon)(2\omega_\eta)} \right\} \right] .
\] (25)

Taking \(\mu = \nu = 0\), the real part of “00”-component of \((\Pi^{\mu\nu})_{\parallel}\) becomes

\[
\Re\Pi^{00}(k_0, k_z) \big|_n = -\frac{g^2}{2(2\pi)^3} \int dp_0 dp_z L^{00} \left[ n_\rho(p_0) \left\{ \frac{\delta(p_0 - \omega_\rho) + \delta(p_0 + \omega_\rho)}{(q_0^2 - q_\rho^2 - m_\rho^2 + i\epsilon)(2\omega_\rho)} \right\} + n_\eta(q_0) \left\{ \frac{\delta(q_0 - \omega_\eta) + \delta(q_0 + \omega_\eta)}{(p_0^2 - p_\eta^2 - m_\eta^2 + i\epsilon)(2\omega_\eta)} \right\} \right] ,
\] (26)

where the “00” component of \(L^{\mu\nu}\) is

\[
L^{00} = 8[p_0 q_0 + p_z q_z + m_\rho^2],
\] (27)

and the other notations are

\[
\omega_\rho = \sqrt{p_\rho^2 + m_\rho^2}, \quad \omega_\eta = \sqrt{(p_z - k_z)^2 + m_\eta^2}.
\]

After performing the \(p_0\) integration we get from eq.\((26)\)

\[
\Re\Pi^{00}(k_0, k_z) \big|_n = \frac{g^2}{4(2\pi)^3} \int dp_z \left[ \frac{L^{00}_1 n_\rho^+}{\omega_\rho[(\omega_\rho - k_0)^2 - \omega_\eta^2]} + \frac{L^{00}_2 n_\rho^-}{\omega_\rho[(\omega_\rho + k_0)^2 - \omega_\eta^2]} + \frac{L^{00}_3 n_\eta^+}{\omega_\eta[(\omega_\eta + k_0)^2 - \omega_\rho^2]} + \frac{L^{00}_4 n_\eta^-}{\omega_\eta[(\omega_\eta - k_0)^2 - \omega_\rho^2]} \right] ,
\] (28)
where we have defined

\[ L_{1}^{00} = L_{1}^{00}(p_0 = \omega_p) = 8(2\omega_p^2 - \omega_p p_z k_z), \]
\[ L_2^{00} = L_2^{00}(p_0 = -\omega_p) = 8(2\omega_p^2 + \omega_p p_0 - p_z k_z), \]
\[ L_3^{00} = L_3^{00}(p_0 = \omega_q + k_0) = 8(2\omega_p^2 + \omega_q k_0 - 3p_z k_z + k_z^2), \]
\[ L_4^{00} = L_4^{00}(p_0 = -\omega_q + k_0) = 8(2\omega_p^2 - \omega_q k_0 - 3p_z k_z + k_z^2), \]

and

\[ n_{p}^{+} = n_{p}(p_0 = \omega_p), \]
\[ n_{p}^{-} = n_{p}(p_0 = -\omega_p), \]
\[ n_{q}^{+} = n_{q}(p_0 = \omega_q + k_0), \]
\[ n_{q}^{-} = n_{q}(p_0 = -\omega_q + k_0). \]

In the limit of massless quarks \((m_f = 0)\), the gluon self-energy in eq.\((28)\) gets simplified into

\[ \Re \Pi_{\perp}^{00}(k_0, k_z) \big|_n = \frac{g^2}{2(2\pi)^3} \left[ \frac{k_z^2}{k_0^2 - k_z^2} + \ln(2) - \ln(1 + e^{\frac{k_z}{T}}) \right]. \tag{29} \]

Using eq.\((14)\) and multiplying the transverse component, \(A(k_{\perp})\) from eq.\((15)\), the contribution to the real part of self-energy from the component having single distribution function becomes

\[ \Re \Pi_{\perp}^{00}(k_0, k_x, k_y, k_z) \big|_n = \frac{g^2}{4\pi^2} \sum_f |q_f| B e^{-\left(\frac{k_z^2 + k_z^2}{2}\right)} \left[ \frac{k_z^2}{k_0^2 - k_z^2} + \frac{k_z T}{k_0^2 - k_z^2} \ln(2) - \frac{k_z T}{k_0^2 - k_z^2} \ln(1 + e^{\frac{k_z}{T}}) \right], \tag{30} \]

which, in the static limit \((k_0 = 0, \vec{k} \to 0)\) becomes

\[ \Re \Pi_{\perp}^{00}(k_0 = 0, \vec{k} \to 0) \big|_n = -\frac{g^2}{4\pi^2} \sum_f |q_f| B + \frac{g^2}{8\pi^2} \sum_f |q_f| B. \tag{31} \]

However, for the physical quark masses \((m_f \neq 0)\), the self-energy in eq.\((26)\) reduces to, by putting \(k_0 = 0\)

\[ \Re \Pi_{\perp}^{00}(k_0 = 0, k_z) \big|_n = -\frac{g^2}{2(2\pi)^3} \int dp_z I_n, \tag{32} \]

where the integrand, \(I_n\), is given by

\[ I_n = \frac{8p_z n_p}{\omega_p k_z} - \frac{8(p_z - k_z) n_q}{\omega_q k_z} + \frac{16m_f^2 n_p}{\omega_p k_z(2p_z - k_z)} - \frac{16m_f^2 n_q}{\omega_q k_z(2p_z - k_z)}, \tag{33} \]

and the distribution functions are given by

\[ n_p = \frac{1}{e^{\frac{\beta |\omega_p|}{T}} + 1}, \quad n_q = \frac{1}{e^{\frac{\beta |\omega_q|}{T}} + 1}. \]

Further taking the \(k_z \to 0\) limit, the integrand, \(I_n\) is simplified into

\[ I_n = -\frac{8}{T} n_p (1 - n_p). \]
Thus for the physical quark masses ($m_f \neq 0$), the contribution to the gluon self-energy having single distribution function in the static limit reduces to

$$\text{Re} \Pi^{00}(k_0 = 0, \vec{k} \to 0) \big|_{n} = \frac{g^2}{4\pi^2 T} \sum_f |q_f B| \int_0^\infty dp_z \frac{e^{\beta \omega_p}}{(1 + e^{\beta \omega_p})^2}. \quad (34)$$

Finally the medium contribution to the gluon self-energy involving the product of two distribution functions given in eq.(19) does not contribute to the real-part of the gluon self-energy, i.e.

$$\text{Re} \Pi^{00}(k_0 = 0, \vec{k} \to 0) \big|_{n^2} = 0. \quad (35)$$

We have thus so far evaluated the vacuum as well as medium contribution to one-loop gluon self energy, therefore we add them up to obtain the real-part of one-loop gluon self-energy in static limit for massless quarks

$$\text{Re} \Pi^{00}(k_0 = 0, \vec{k} \to 0) = \frac{g^2}{8\pi^2} \sum_f |q_f| B, \quad (36)$$

and for the physical quark masses ($m_f \neq 0$)

$$\text{Re} \Pi^{00}(k_0 = 0, \vec{k} \to 0) = \frac{g^2}{4\pi^2 T} \sum_f |q_f| B \int_0^\infty dp_z \frac{e^{\beta \omega_p}}{(1 + e^{\beta \omega_p})^2}. \quad (37)$$

### 2.3 Debye screening mass in strong magnetic field:

The Debye screening manifests in the collective oscillation of the medium via the dispersion relation and is obtained by the static limit of the longitudinal part (“00” component) of gluon self-energy, i.e.

$$m_D^2 = \text{Re} \Pi^{00}(k_0 = 0, \vec{k} \to 0). \quad (38)$$

Therefore, eq.(36) gives the very simple form for the square of the Debye mass for massless quarks, which is already derived in Refs.[25, 26]

$$m_D^2 = \frac{g^2}{8\pi^2} \sum_f |q_f| B. \quad (39)$$

It shows that $m_D^2$ increases linearly with the magnetic field and is independent of temperature, thus the collective behavior of the medium gets strongly affected by the presence of strong magnetic field. However, for physical quark masses, the Debye mass is given by from eq.(37)

$$m_D^2 = m_D^2(m_f = 0) \times \frac{2}{T} \int_0^\infty dp_z \frac{e^{\beta \omega_p}}{(1 + e^{\beta \omega_p})^2}, \quad (40)$$

which depends on both magnetic field and temperature. However, $m_D^2$ depends strongly on the magnetic field and depends very slowly on the temperature and becomes temperature-independent beyond a certain temperature. For example, for $eB = 5m_\pi^2$ and $50m_\pi^2$, $m_D^2$ becomes temperature independent beyond $T=180$ MeV and 280 MeV, respectively. Thus, to see the variation of the Debye masses with the strong magnetic field, we have numerically calculated $m_D^2$ as a function of $eB$ (in units of $m_\pi^2$) for the temperature range
T=150 - 550 MeV in Fig.1-a and noticed that for smaller temperature (150 MeV), $m_D^2$ is almost linearly increasing with $eB$ and for higher temperatures (350-500 MeV), $m_D^2$ deviates slightly from the linearly increasing trend and increases very little at higher $eB$’s.

In strong magnetic field approximation ($eB \gg T^2$ and $eB \gg m^2$), the medium with massless quarks possesses only one scale available related to the magnetic field ($eB$) so by the dimensional arguments the square of the Debye mass is linear in $eB$ whereas for the the medium with physical quark masses, even in SMFA there is a weak competition between the dominant scale, $eB$ and much weaker scales, mass ($m$) and temperature ($T$) (rather their ratio, $m/T$) in the form of Boltzmann damping factor ($\exp(-m/T)$) as in eq.(40). This is seen in Fig.1-b, where a comparison of Debye masses with and without incorporating the quarks masses is made.

To see the temperature dependence of the Debye mass explicitly we have plotted $m_D$ with the temperature directly with increasing values of $eB = 5m^2, 25m^2$ and $50m^2$ in Fig.2, where a decreasing trend is found in the region of smaller temperature and higher magnetic field.

![Fig. 1-a](image1.png)  ![Fig. 1-b](image2.png)

Figure 1: **Left panel:** Separation is seen only between low and high T at high eB. **Right panel:** High eB can distinguish b/w massless and massive fermions (quarks).

### 3 Heavy quark potential in a hot QCD medium:

The derivation of potential between a heavy quark $Q$ and its anti-quark ($\bar{Q}$) either from EFT (pNRQCD) or from first principle QCD may not be plausible because the hierarchy of non relativistic scales and thermal scales assumed in weak coupling EFT calculations may not be satisfied and the adequate quality of the data is not available in the present lattice correlator studies, respectively, so one may use the potential model to circumvent the problem.

Since the mass of the heavy quark ($m_Q$) is very large, so the requirement: $m_Q \gg \sqrt{eB} \gg \Lambda_{QCD}$ and $T \ll m_Q$ is satisfied for the description of the interactions between a pair of heavy quark and anti-quark at finite temperature in the presence of magnetic field in terms of quantum mechanical potential. Thus we
can obtain the medium-modification to the vacuum potential by correcting both its short and long-distance part with a dielectric function $\epsilon(k)$ as

$$V(r, T) = \int \frac{d^3k}{(2\pi)^{3/2}} (e^{ikr} - 1) \frac{V(k)}{\epsilon(k)}, \quad (41)$$

where we have subtracted a $r$-independent term (to renormalize the heavy quark free energy) which is the perturbative free energy of quarkonium at infinite separation [27]. The dielectric function is related to the “$00$”-component of effective gluon propagator in static limit as

$$\frac{1}{\epsilon(k)} = -\lim_{k_0=0} k^2 D_{00}(k_0, k), \quad (42)$$

and $V(k)$ is the Fourier transform (FT) of the Cornell potential. To obtain the FT of the potential, we regulate both terms with the same screening scale. However in the framework of Debye-Hückel theory, Digal et al. [28] employed different screening functions, $f_c$ and $f_s$ for the Coulomb and string terms, respectively, to obtain the free energy.  

At present, we regulate both terms by multiplying with an exponential damping factor and is switched off after the FT is evaluated. This has been implemented by assuming $r$- as distribution ($r \to r \exp(-\gamma r)$). The FT of the linear part $-\sigma r \exp(-\gamma r)$ is

$$-\frac{i}{k\sqrt{2\pi}} \left( \frac{2}{(\gamma - ik)^3} - \frac{2}{(\gamma + ik)^3} \right). \quad (43)$$

After putting $\gamma = 0$, we obtain the FT of the linear term $\sigma r$ as,

$$\tilde{\sigma r} = -\frac{4\sigma}{k^4\sqrt{2\pi}}. \quad (44)$$

The FT of the Coulomb piece is straightforward and is given by

$$V_C(k) = -\sqrt{\frac{2}{2\pi}} \frac{\alpha_s}{k^2}, \quad (45)$$

\footnote{In another calculation, different scales for the Coulomb and linear pieces were also employed in [29, 30] to include non-perturbative effects in the free energy beyond the deconfinement temperature through a dimension-two gluon condensate.}
thus the FT of the full Cornell potential becomes

\[ V(k) = -\sqrt{\frac{2}{\pi}} \frac{\alpha}{k^2} - \frac{4\sigma}{\sqrt{2\pi}k^4}. \]  

(46)

The “00”-component of effective gluon propagator in static limit has been obtained with the help of “00”-component of one-loop gluon self energy. We have already calculated the “00” component of one-loop gluon self energy in presence of strong magnetic field at finite temperature in eq.(37), hence the “00”-component of effective gluon propagator in static limit is given by

\[ D_{01}^{00}(0,k) = \frac{-1}{k^2 + m_D^2}. \]  

(47)

Therefore the real part of the static potential can be obtained by substituting the dielectric permittivity \( \epsilon(k) \) from eq.(42) and the Fourier transformation from eq.(46) into the definition of the potential (41)

\[ V(r;T,B) = V_C(r;T,B) + V_S(r;T,B), \]  

(48)

where the Coulombic and string term of the potential are given by (with the dimensionless quantity \( \hat{r} = r m_D \))

\[ V_C(r;T,B) = -\alpha m_D \left( e^{-\hat{r}} + 1 \right), \]  

(49)

\[ V_S(r;T,B) = \frac{2\sigma}{m_D} \left[ \frac{e^{-\hat{r}} - 1}{\hat{r}} + 1 \right], \]  

(50)

respectively. It is thus evident that the medium dependence in the potential enters through the Debye mass, which in turn depends on both temperature and magnetic field for physical quark masses and depends only on magnetic field for massless quarks. This gives a characteristic dependence of the potential on both temperature and magnetic field. The nonlocal terms in the potential insures \( V(r, T) \) to reduce to the Cornell potential in \( T \to 0 \) limit. However, such terms could also arise naturally from the basic computations of real time static potential in hot QCD [32] and from the real and imaginary time correlators in a thermal QCD medium [33]. These terms in the potential are needed in computing the masses of the quarkonium states and to compare the results with the lattice studies. It is equally important while comparing our effective potential with the free energy in lattice studies.

Since we are exploring the effect of medium on the potential between \( Q \) and \( \bar{Q} \) in strong magnetic field approximation so we probe it by varying the strength of magnetic field \( (eB) \) from \( 5m_D^2 \) to \( 50m_D^2 \) (in Figure 3-a) at a temperature \( T=150 \) MeV. It is found that as the strength of the magnetic field increases the potential becomes stronger. To see the competition between the magnetic field and temperature, we have calculated the potential in Figure 3-b in a hotter medium \( (T=300 \) MeV). As we have seen earlier in Figure 1-a that the (square) Debye mass increases very little, so the screening of the potential becomes stronger compared to Figure 3-a.

Usually potential model studies are limited to the medium-modification of the perturbative part of the potential only where it is assumed that the string-tension vanishes abruptly at the deconfinement
point [34, 35]. Since the phase transition in QCD for physical quark masses is found to be a crossover [36], so the string tension may not vanish at the deconfinement temperature. This issue, usually overlooked in the literature where only a screened Coulomb potential was assumed above $T_c$ and the linear term was neglected, is certainly worth for an investigation. To see the effect of the linear term on the potential, in addition to the Coulomb term, we have plotted the potential (in Fig.4-a) with ($\sigma \neq 0$) and without string term ($\sigma = 0$) in a magnetic field $eB = 5m^2_\pi$. As we know already in vacuum ($T=0$), the inclusion of the linear term makes the potential in short-distance interaction less attractive and in long-distance interaction the linear term makes the potential more repulsive, compared to the Coulomb term alone. However, the medium modification causes the linear term attractive and overall the medium modifications to both Coulomb and string term makes the potential more attractive (seen in Figure 4-a) compared to the vacuum potential.

To see the effect of the scale (Debye mass) at which the screening takes place on both the linear and Coulombic term we have calculated the potential at a larger magnetic field, $eB = 50m^2_\pi$ ($m_D(eB = 50m^2_\pi) = 0.3$ $m_D(eB = 5m^2_\pi)$) in Fig. 4-b, where we found that the increase of the scale (screening mass) makes the linear term less attractive, compared to the lower scale ($eB$). To understand the observations in Figure 4, we have probed the range of interactions, viz. short-range ($r = 0.2$ fm), intermediate ($r = 0.5$ fm) and long-range ($r = 1$ fm) interactions of $Q\bar{Q}$ potential as a function of magnetic field ($eB$) in figure 5 and found that only the long range interaction ($r = 1$ fm) has been affected noticeably. Overall observation is that as the strength of the magnetic field increases the long range QCD force becomes more and more short range, thus implying that magnetic field facilitates early dissolution of $Q\bar{Q}$ states.

### 3.1 Dissociation of Heavy Quarkonia in magnetic field

In this section, we shall discuss the dissociation of charmonium and bottomonium states due to an external strong magnetic field in a hot QCD medium. The concept of dissociation temperature becomes irrelevant here because the scale at which the collective oscillations develop depends only on the magnetic field albeit
we are considering a hot QCD medium because in strong magnetic field approximation \( (eB \gg T^2) \), the scale at which the collective oscillation sets in is associated with magnetic field only because \( eB \) is the most dominant scale in strong magnetic field limit (if the partons are assumed massless), not the thermally generated scales. This in turn makes the potential to depend only on the magnetic field through the dependence of Debye mass on the magnetic field. Thus, it makes sense here to discuss the dissociation of quarkonium states due to the magnetic field only as far as SMFA is valid.

As we know that in the presence of medium, the potential between a heavy quark \( Q \) and its anti-quark \( \bar{Q} \) will be screened, as a result if the screening is strong enough, the potential becomes too weak to form the resonance. Thus we can argue that the quarkonium states will be dissolved in a medium if the Debye screening radius, \( r_D (= \frac{1}{m_D}) \) in a given medium is smaller than the bound state radius of a particular resonance state then the medium inhibits the formation of the particular resonance and \( Q \) and \( \bar{Q} \) will be dissolved into the medium. Since the screening mass in strong magnetic field increases with the magnetic field therefore the (critical) magnetic field at which the \( QQ \) potential becomes too feeble to hold \( QQ \) together becomes smaller for the excited states. We can thus estimate the lower limit of critical magnetic
field for various charmonium and bottomonium states by the criteria: \( \sqrt{\langle r^2 \rangle} = r_D(B_i) \), i.e., for magnetic field larger than \( B_i \), the \( i \)-th quarkonium states cease to exist. For example, \( J/\psi \) will be dissociated at \( eB = 14m_\psi^2 \) and its excited state, \( \psi' \) is dissociated at smaller magnetic field \( eB = m_\psi^2 \) whereas \( \Upsilon \) will be dissociated at \( eB = 130m_\Upsilon^2 \) and \( \Upsilon' \) is dissociated at smaller magnetic field \( eB = 13m_\Upsilon^2 \).

To understand the in-medium properties of the quarkonium states quantitatively, one need to solve the Schrödinger equation with the medium-modified potential, \( V(r; B, T) \). There are some numerical methods to solve the Schrödinger equation either in partial differential form (time-dependent) or eigen value form (time-independent) by the finite difference time domain method (FDTD) or matrix method, respectively. In the later method, the stationary Schrödinger equation can be solved in a matrix form through a discrete basis, instead of the continuous real-space position basis spanned by the states \( |\rightarrow x\rangle \). Here the confining potential \( V \) is subdivided into \( N \) discrete wells with potentials \( V_1, V_2, ..., V_{N+2} \) such that for \( i \)-th boundary potential, \( V = V_i \) for \( x_{i-1} < x < x_i; \ i = 2, 3, ..., (N + 1) \). Therefore for the existence of a bound state, there must be exponentially decaying wave function in the region \( x > x_{N+1} \) as \( x \to \infty \) and has the form:

\[
\Psi_{N+2}(x) = P_x \exp[-\gamma_{N+2}(x - x_{N+1})] + Q_x \exp[\gamma_{N+2}(x - x_{N+1})],
\]

where, \( P_x = \frac{1}{2}(A_{N+2} - B_{N+2}) \), \( Q_x = \frac{1}{2}(A_{N+2} + B_{N+2}) \) and, \( \gamma_{N+2} = \sqrt{2\mu(V_{N+2} - E)} \). The eigenvalues can be obtained by identifying the zeros of \( Q_x \). Using this method, we have found that \( J/\psi \) and \( \Upsilon \) is dissociated at \( eB = 5m_\psi^2 \) and \( eB = 50m_\Upsilon^2 \), respectively.

Though the dissociation magnetic fields, obtained from the two different methods apparently look different, its easy to see that qualitatively they are similar. Using both methods, we found that the dissociation magnetic field for \( \Upsilon \) is roughly an order of magnitude greater than the dissociation magnetic field for \( J/\psi \). Even though their absolute value obtained from the two different methods differ, they lie in the same ball park, which is \( \sim 10 \ m_\psi^2 \) for \( J/\psi \) and \( \sim 100 \ m_\psi^2 \) for \( \Upsilon \).

4 Conclusions

In this article, we have explored the effects of strong and homogeneous magnetic field on the properties of quarkonium states. For that purpose we have derived the potential between a heavy quark and its antiquark by the medium corrections to both Coulomb and linear term of \( Q\bar{Q} \) potential at \( T=0 \), unlike the medium correction to the Coulomb term alone. Although the medium considered is thermal but due to strong magnetic field approximation, all other scales present in the thermal medium becomes irrelevant as the scale related to magnetic field dominates over other. This is exactly what happens in the collective oscillation of the medium in the form of Debye mass. In fact, the Debye mass becomes completely independent of temperature for massless quarks and depends very weakly on temperature for massive quarks. However, beyond a certain temperature, the dependence is so weak that it is almost insignificant. As a result the heavy quark potential mainly depends on the magnetic field with a very feeble dependence on the temperature. This is expected as the effect of the medium on the potential enters through the Debye mass.
In particular the long distance part of the potential gets significantly affected, whereas the short distance part is mildly affected.

We have then studied the dissociation of quarkonium states in a medium. Since the potential in SMFA depends mainly only on the magnetic field thus we have discussed the dissociation of quarkonium states due to the magnetic field only. We have estimated the critical value of magnetic field beyond which the resonance does not form in two methods. The first one gives a lower limit of critical magnetic field for both charmonium and bottomonium states at which the Debye screening radius becomes smaller than the bound state radius of a particular resonance state. The other one comes from the consideration of the binding energies of a specific state obtained from the energy eigenvalues of the Schrödinger equation. In brief, $J/\psi$ is dissociated at $eB \sim 10 \, m_\pi^2$ and $\Upsilon$ is dissociated at $eB \sim 100 \, m_\pi^2$.

5 Acknowledgments

We are thankful to Aritra Bandyopadhyay for a fruitful discussion during this work. Bhaswar is thankful to the Ministry of to Human Resource Development, Government of India for the financial assistance.

References

[1] I. A. Shovkovy, Lect. Notes Phys. 871, 13 (2013).
[2] M. D’Elia, Lect. Notes Phys. 871, 181 (2013).
[3] K. Fukushima, Lect. Notes Phys. 871, 241 (2013).
[4] N. Muller, J. A. Bonnet, and C. S. Fisher, Phys. Rev. D 89, 094023 (2014).
[5] V. A. Miransky and I. A. Shovkovy, Phys. Rep. 576, 1-209 (2015).
[6] D. Kharzeev, L. McLerran, and H. Warringa, Nucl. Phys. A 803, 227 (2008).
[7] V. Skokov, A. Illarionov, and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009).
[8] L. McLerran and V. Skokov, Nucl. Phys. A 929, 184 (2014).
[9] K. Tuchin, Phys. Rev. C 82, 034904 (2010).
[10] F. Karsch, M. T. Mehr and H. Satz, Z. Phys. C 37, 617 (1988).
[11] L. D. McLerran and B. Svetitsky, Phys. Rev. D 24, 450 (1981).
[12] O. Philipsen, Nucl. Phys. A 820, 33C (2009)
[13] N. Brambilla, J. Ghiglieri, A. Vairo and P. Petreczky, Phys. Rev. D 78, 014017 (2008)
[14] W. M. Alberico, A. Beraudo, A. De Pace, A. Molinari, Phys. Rev. D 77, 017502 (2008).
[15] L. Thakur, U. Kakade and B. K. Patra, Phys. Rev. D 89, 094020 (2014).
[16] U. Kakade, B. K. Patra and L. Thakur, Int. J. Mod. Phys. A 30, 155043 (2015).
[17] C. Bonati, M. D’Elia and A. Rucci, Phys. Rev. D 92, 054014 (2015).
[18] J. Alford and M. Strickland, Phys. Rev. D 88, 105017 (2013).
[19] J. Schwinger, Phys. Rev. 82, 664 (1951).
[20] Wu-yang Tsai, Phys. Rev. D 10, 2699 (1974).
[21] H. P. Rojas and A. E. Shabad, Ann. Phys. (N.Y.) 121, 432 (1979).
[22] Avijit K. Ganguly, Sushan Konar, Palash B. Pal, Phys. Rev. D 60, 105014 (1999).
[23] Juan Carlo D’Olive, Jose F. Nieves and Sarira Sahu, Phys. Rev. D 67, 025018 (2003).
[24] T. Chyi et. al, Phys. Rev. D 62, 105014 (2000).
[25] K. Fukushima, K. Hattori, H-U. Yee and Y. Yin, Phys. Rev. D 93, 074028 (2016).
[26] A. Bandyopadhyay, C. A. Islam and M. G. Mustafa, Phys. Rev. D 94, 114034 (2016).
[27] A. Dumitru, Y. Guo, and M. Strickland, Phys. Rev. D 79, 114003 (2009).
[28] S. Digal, O. Kaczmarek, F. Karsch and H. Satz, Eur. Phys. J. C 43, 71 (2005).
[29] E. Megias, E. Ruiz Arriola, and L. L. Salcedo, Indian J. Phys. 85, 1191 (2011).
[30] E. Megias, E. Ruiz Arriola, and L. L. Salcedo, Phys. Rev. D 75, 105019 (2007).
[31] P. Petreczky, Eur. Phys. J. C 43, 51 (2005).
[32] Mikko Laine, O. Philipsen, Marcus Tassler, and Paul Romatschke, JHEP 03 (2007) 054.
[33] A. Beraudo, J.P. Blaizot, and C. Ratti, Nucl. Phys. A806, 312 (2008).
[34] H. Satz, Jour. Phys. G R25, (2006).
[35] A. Mocsy and P. Petreczky, Phys. Rev. D 77, 014501 (2008).
[36] F. Karsch, J. Phys. Conf. Ser. 46, 122 (2006).