Convergence of Structured Quadratic Forms With Application to Theoretical Performances of Adaptive Filters in Low Rank Gaussian Context

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Abstract—This paper addresses the problem of deriving the asymptotic performance of adaptive Low Rank (LR) filters used in target detection embedded in a disturbance composed of a LR Gaussian noise plus a white Gaussian noise. In this context, we use the Signal to Interference to Noise Ratio (SINR) loss as performance measure which is a function of the estimated projector onto the LR noise subspace. However, although the SINR loss can be determined through Monte-Carlo simulations or real data, this process remains quite time consuming. Thus, this paper proposes to predict the SINR loss behavior in order to not depend on the data anymore and be quicker. To derive this theoretical result, previous works used a restrictive hypothesis assuming that the target is orthogonal to the LR noise. In this paper, we propose to derive this theoretical performance by relaxing this hypothesis and using Random Matrix Theory (RMT) tools. These tools will be used to present the convergences of simple quadratic forms and perform new RMT convergences of structured quadratic forms and SINR loss in the large dimensional regime, i.e., the size and the number of the data tend to infinity at the same rate. We show through simulations the interest of our approach compared to the previous works when the restrictive hypothesis is no longer verified.

Index Terms—Low Rank SINR loss, Random Matrix Theory, Adaptive Filtering, Quadratic Forms convergence, Spiked model

I. INTRODUCTION

In array processing, the covariance matrix \( R \) of the data is widely involved for main applications as filtering [1], [2], radar/sonar detection [3] or localization [4], [5]. However, when the disturbance in the data is composed of the sum of a Low Rank (LR) correlated noise and a White Gaussian Noise (WGN), the covariance matrix is often replaced by the projector onto the LR noise subspace \( \Pi_c \) [6]–[9]. In practice, the projector onto the LR noise subspace (and the covariance matrix) is generally unknown and an estimate is consequently required to perform the different processing. This estimation procedure is based on the so-called secondary data assumed to be independent and to share the same distribution. Then, the true projector is replaced by the estimated one in order to obtain an adaptive processing. An important issue is then to derive the theoretical performances of the adaptive processing as a function of the number of secondary data \( K \). The processing based on the covariance matrix has been widely studied and led to many theoretical results in filtering [1] and detection [10]–[13]. For example, for classical adaptive processing, \( K = 2m \) secondary data (where \( m \) is the data size) are required to ensure good performance of the adaptive filtering, i.e. a 3dB loss of the output Signal to Interference plus Noise Ratio (SINR) compared to optimal filtering [1]. For LR processing, some results have been obtained especially in filtering [6], [14]–[16] and localization [17]. Similarly, in LR filtering, the number \( K \) of secondary data required to ensure good performance of the adaptive filtering is equal to \( 2r \) (where \( r \) is the rank of the LR noise subspace) [6], [14].

These last results are obtained from the theoretical study of the Signal to SINR loss. More precisely, in [14], [16], the derivation of the theoretical results is based on the hypothesis that the steering vector is orthogonal to the LR noise subspace. Nevertheless, even if the result seems to be close to the simulated one when the hypothesis is no longer valid anymore [18], it is impossible with traditional techniques of [14], [16] to obtain a theoretical performance as a function of the distance between the steering vector and the LR noise subspace. Since, in practice, this dependence is essential to predict the performance of the adaptive filtering, we propose in this paper to derive the theoretical SINR loss, for a disturbance composed of a LR noise and a WGN, as a function of \( K \) and the distance between the steering vector and the LR noise subspace. The proposed approach is based on the study of the SINR loss structure.

The SINR loss (resp. LR SINR loss) is composed of a simple Quadratic Form (QF) in the numerator, \( s^H_1 \hat{R}^{-1} s_2 \) (resp. \( s^H_1 \hat{\Pi}_c^\perp s_2 \)) and a structured QF in the denominator \( s^H_1 \hat{R}^{-1} \hat{R} \hat{R}^{-1} s_2 \) (resp. \( s^H_1 \hat{\Pi}_c^\perp \hat{R} \hat{R}^{-1} \hat{\Pi}_c^\perp s_2 \)). These recent years, the simple QFs (numerator) have been broadly studied [19]–[22] using Random Matrix Theory (RMT) tools contrary to structured QFs (denominator). RMT tools have also been used in array processing to improve the MUSIC algorithm [23], [24] and in matched subspace detection [25], [26] where the rank \( r \) is unknown. The principle is to examine the spectral behavior of \( \hat{R} \) by RMT to obtain their convergences, performances and asymptotic distribution when \( K \) tends to infinity and when both the data size \( m \) and \( K \) tend to infinity at the same ratio, i.e. \( m/K \to c \in [0, +\infty) \), for different models of \( \hat{R} \) of the observed data as in [19], [20], [23], [22] and [21]. Therefore, inspired by these works, we propose in this paper to study the convergences of the structured QFs \( s^H_1 \hat{\Pi}_c^\perp \hat{R} \hat{R}^{-1} \hat{\Pi}_c^\perp s_2 \) (resp. \( s^H_1 \hat{\Pi}_c^\perp \hat{R} \hat{R}^{-1} \hat{\Pi}_c^\perp s_2 \)): when \( K \to \infty \) with a fixed \( m \)
and when 2) \(m, K \to \infty\) at the same ratio under the most appropriated model for our data and with the rank assumed to be known. From [27], [28], the spiked model has proved to be the more appropriated one to our knowledge. This model, introduced by [29] (also studied in [30], [31] from an eigenvector point of view) considers that the multiplicity of the eigenvalues corresponding to the signal (the LR noise for us) is fixed for all \(m\) and leads to the SPIKE-MUSIC estimator [32] of \(s^H \hat{\Pi} s_2\). Then, the new results are validated through numerical simulations. From these new theoretical convergences, the paper derives the convergence of the SINR loss for both adaptive filters (the classical and the LR one). The new theoretical SINR losses depend on the number of secondary data \(K\) but also on the distance between the steering vector and the LR noise subspace. This work is partially related to those of [33], [34] and [35] which uses the RMT tools to derive the theoretical SINR loss in a full rank context (previously defined as classical).

Finally, these theoretical SINR losses are validated in a jamming application context where the purpose is to detect a target thanks to a Uniform Linear Antenna (ULA) composed of \(m\) sensors despite the presence of jamming. The response of the jamming is composed of signals similar to the target response. This problem is very similar to the well-known Space Time Adaptive Processing (STAP) introduced in [2]. Results show the interest of our approach with respect to other theoretical results [6], [14]–[16] in particular when the target is close to the jammer.

The paper is organized as follows. Section II presents the received data model, the adaptive filters and the corresponding SINR losses. Section III summarizes the existing studies on the simple QFs \(s^H \hat{\Pi} s_2\) and \(s^H \hat{\Pi} s_2\), and exposes the covariance matrix model, the spiked model. Section IV gives the theoretical contribution the paper with the convergences of the structured QFs \(s^H \hat{\Pi} c \hat{\Pi}^c s_2\) and \(s^H \hat{\Pi} c \hat{\Pi}^c s_2\) and the SINR losses. The results are finally applied on a jamming application in Section V.

**Notations:** The following conventions are adopted. An italic letter stands for a scalar quantity, boldface lowercase (uppercase) characters stand for vectors (matrices) and \((\cdot)^H\) stands for the conjugate transpose. \(I_N\) is the \(N \times N\) identity matrix, \(\text{tr}()\) denotes the trace operator and \(\text{diag}(\cdot)\) denotes the diagonalization operator such as \((\text{diag}(a))_{i,i} = (a)_i\) and \((\text{diag}(a))_{i,j} = 0\) if \(i \neq j\). \# \{\cdot\} denotes the cardinality of the set \(\{a, b\}\) is the set defined by \(\{x \in \mathbb{Z} : a \leq x \leq b, \forall (a, b) \in \mathbb{Z}^2\}\). \(\mathcal{D}_{a \times N}\) is a \(n \times N\) matrix full of 0. The abbreviations iid and a.s. stem for independent and identically distributed and almost surely respectively.

## II. PROBLEM STATEMENT

The aim of the problem is to filter the received observation vector \(x \in \mathbb{C}^{m \times 1}\) in order to whiten the noise without mitigating an eventual complex signal of interest \(d\) (typically a target in radar processing). In this paper, \(d\) will be a target response and is equal to \(\alpha a(\Theta)\) where \(\alpha\) is an unknown complex deterministic parameter (generally corresponding to the target amplitude), \(a(\Theta)\) is the steering vector and \(\Theta\) is an unknown deterministic vector containing the different parameters of the target (e.g. the localization, the velocity, the Angle of Arrival (AoA), etc.). In the remainder of the article, in order to simplify the notations, \(\Theta\) will be omitted of the steering vector which will simply be denoted as \(a\). If necessary, the original notation will be taken.

This section will first introduces the data model. Then, the filters, adaptive filters and the corresponding SINR loss, the quantity characterizing their performances, will be defined.

### A. Data model

The observation vector can be written as:

\[
x = d + c + b
\]

(1)

where \(c + b\) is the noise that has to be whitened. \(b \in \mathbb{C}^{m \times 1} \sim \mathcal{CN}(0, \sigma^2 I_m)\) is an Additive WGN (AWGN) and \(c\) is a LR Gaussian noise. \(c \in \mathbb{C}^{m \times 1}\) modeled by a zero-mean complex Gaussian vector with a normalized covariance matrix \(C (\text{tr}(C) = m)\), i.e. \(c \sim \mathcal{CN}(0, C)\). Consequently, the covariance matrix of the noise \(c + b\) can be written as:

\[
R = C + \sigma^2 I_m \in \mathbb{C}^{m \times m}.
\]

Moreover, considering a LR Gaussian noise, one has rank \((C) = r < m\) and hence, the eigendecomposition of \(C\) is:

\[
C = \sum_{i=1}^{m} \gamma_i u_i u_i^H
\]

(2)

where \(\gamma_i\) and \(u_i\), \(i \in [1:r]\) are respectively the non-zero eigenvalues and the associated eigenvectors of \(C\), unknown in practice. This leads to:

\[
R = \sum_{i=1}^{m} \lambda_i u_i u_i^H
\]

(3)

where \(\lambda_i\) and \(u_i\), \(i \in [1,m]\) are respectively the eigenvalues and the associated eigenvectors of \(R\) with \(\lambda_1 = \gamma_1 + \sigma^2 > \cdots > \lambda_r = \gamma_r + \sigma^2 > \lambda_{r+1} = \cdots = \lambda_m = \sigma^2\). Then, the projector onto the LR Gaussian noise subspace \(\Pi_c\) and the projector onto the orthogonal subspace to the LR Gaussian noise subspace \(\Pi_c^\perp\) are defined as follows:

\[
\begin{align*}
\Pi_c &= \sum_{i=1}^{r} u_i u_i^H \\
\Pi_c^\perp &= I_m - \Pi_c = \sum_{i=r+1}^{m} u_i u_i^H
\end{align*}
\]

(4)

However, in practice, the covariance matrix \(R\) of the noise is unknown. Consequently, it is traditionally estimated with the Sample Covariance Matrix (SCM) which is computed from \(K\) iid secondary data \(x_k = c_k + b_k, k \in [1,K]\), and can be written as:

\[
\hat{R} = \frac{1}{K} \sum_{k=1}^{K} x_k x_k^H = \sum_{i=1}^{m} \hat{\lambda}_i \hat{u}_i \hat{u}_i^H
\]

(5)

where \(\hat{\lambda}_i\) and \(\hat{u}_i\), \(i \in [1,m]\) are respectively the eigenvalues and the eigenvectors of \(\hat{R}\) with \(\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_m \sim \mathcal{CN}(0, \sigma^2 I_m)\) and \(b_k \sim \mathcal{CN}(0, \sigma^2 I_m)\). Finally, the traditional estimated projectors estimated from the SCM are:

\[
\begin{align*}
\hat{\Pi}_c &= \sum_{i=1}^{r} \hat{u}_i \hat{u}_i^H \\
\hat{\Pi}_c^\perp &= I_m - \hat{\Pi}_c = \sum_{i=r+1}^{m} \hat{u}_i \hat{u}_i^H
\end{align*}
\]

(6)
C. SINR Loss

which is defined by [6]:

\[ w_{opt} = R^{-1}a \] (7)

Since, in practice, the covariance matrix \( R \) of the noise is unknown, the estimated optimal filter or adaptive filter (sub-optimal) is:

\[ \hat{w} = \hat{R}^{-1}a \] (8)

In the case where one would benefit of the LR structure of the noise, one should use the optimal LR filter, based on the fact that \( \Pi^\perp_c \) is the best rank \( r \) approximation of \( R^{-1} \), which is defined by [6]:

\[ w_{LR} = \Pi^\perp_c a \] (9)

As, in practice, the projector is not known and is estimated from the SCM, the estimated optimal filter or adaptive filter (sub-optimal) is:

\[ \hat{w}_{LR} = \hat{\Pi}^\perp_c a \] (10)

C. SINR Loss

Then, we define the SINR Loss. In order to characterize the performance of the estimated filters, the SINR loss compares the SINR at the output of the filter to the maximum SINR:

\[ \rho = \frac{SINR_{out}}{SINR_{max}} = \frac{|\hat{w}^H d|^2}{(\hat{w}^H \hat{R} \hat{w})(d^H \hat{R}^{-1}d)} \] (11)

\[ = \frac{|a^H \hat{R}^{-1}a|^2}{(a^H \hat{R}^{-1} \hat{R}^{-1}a)(a^H R^{-1}a)} \] (12)

If \( \hat{w} = w_{opt} \), the SINR loss is maximum and is equal to 1. When we consider the LR structure of the noise, the theoretical SINR loss can be written as:

\[ \rho_{LR} = \frac{|\hat{w}_{LR}^H d|^2}{(\hat{w}_{LR}^H \hat{R} \hat{w}_{LR})(d^H \hat{R}^{-1}d)} \] (13)

\[ = \frac{|a^H \hat{R}^{-1}a|^2}{(a^H \hat{R}^{-1} \hat{R}^{-1}a)(a^H R^{-1}a)} \] (14)

Finally, the SINR loss corresponding to the adaptive filter in Eq. (10) is defined from Eq. (14) as:

\[ \hat{\rho}_{LR} = \rho_{LR} |\hat{\Pi}^\perp_c = \hat{\Pi}^\perp_c \] (15)

Since we are interested in the performance of the filters, we would like to obtain the theoretical behavior of the SINR losses. Some asymptotic studies on the SINR loss in LR Gaussian context have already been done [14], [16]. In [14], [16], the theoretical result is derived by using the assumption that the LR noise is orthogonal to the steering vector and, in this case, [16] obtained an approximation of the expectation of the SINR loss \( \hat{\rho}_{LR} \). However, this assumption is not always verified, not very relevant and is a restrictive hypothesis in real cases. We consequently propose to relax it and study the convergence of the SINR loss using RMT tools through the study of the nominators and denominators. Indeed, one can already note that the numerators are simple QFs whose

convergences were widely considered in RMT. However, the denominators contain more elaborated QFs which were not tackled in RMT yet and will be the object of Sec.IV.

III. RANDOM MATRIX THEORY TOOLS

This section is dedicated to the introduction of classical results from the RMT for the study of the convergence of QFs. This theory and the convergences are based on the behavior of the eigenvalues of the SCM when \( m, K \to \infty \) at the same rate, i.e. \( m/K \to c \in ]0, +\infty[ \). In order to simplify the notations, we will abusively note \( c = m/K \).

The useful tools for the study of the eigenvalues behavior and the assumptions to the different convergences will be first presented. Secondly, the section will expose the data model, the spiked model [22]. Finally, the useful convergences of simple QFs (\( s_1^H \hat{R}^{-1}s_2, s_1^H \hat{I}s_2 \)) will be introduced.

A. Preliminaries

The asymptotic behavior of the eigenvalues when \( m, K \to \infty \) at the same rate is described through the convergence of their associated empirical Cumulative Distribution Function (CDF) \( \hat{F}_m(x) \) or their empirical Probability Density Function (PDF) \( f_m(x) \)\(^1\). The asymptotic PDF \( f_m(x) \) will allow us to characterize the studied data model. The empirical CDF of the sample eigenvalues of \( \hat{R} \) can be defined as:

\[ \hat{F}_m(x) = \frac{1}{m} \# \{ k : \hat{\lambda}_k \leq x \} \] (16)

However, in practice, the asymptotic characterization of \( \hat{F}_m(x) \) is too hard. Consequently, one prefers to study the convergence of the Stieltjes transform (\( ST[ \cdot ] \)) of \( \hat{F}_m(x) \):

\[ \hat{b}_m(z) = ST[\hat{F}_m(x)] = \int_{\mathbb{R}} \frac{1}{x-z} d\hat{F}_m(x) \] (17)

\[ = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{\lambda_i-z} = \frac{1}{m} tr\left( (\hat{R} - zI_m)^{-1} \right) \] (18)

with \( z \in \mathbb{C}^+ \), \( \mathbb{C}^+ \) is the set of non-negative real numbers. \( \hat{b}_m(z) \) is almost surely converges to \( b_m(z) \). It is interesting to note that the PDF can thus be retrieve from the Stieltjes transform of its CDF:

\[ f_m(x) = \lim_{\Im[z] \to +\infty} \frac{1}{\pi} \Im \left[ b_m(z) \right] \] (19)

with \( x \in \mathbb{R} \). In an other manner, the characterization of \( \hat{f}_m(x) \) (resp. \( f_m(x) \)) can be obtained from \( \hat{b}_m(z) \) (resp. \( b_m(z) \)). Then, to prove the convergences, we assume the following standard hypotheses.

(As1) \( R \) has uniformly bounded spectral norm \( \forall m \in \mathbb{N}^* \), i.e. \( \forall i \in \{1, m\} \), \( \lambda_i < \infty \).

(As2) The vectors \( s_1, s_2 \in \mathbb{C}^{m \times 1} \) used in the QFs (here \( a(\Theta) \) and \( x \)) have uniformly bounded Euclidean norm \( \forall m \in \mathbb{N}^* \).

(As3) Let \( Y \in \mathbb{C}^{m \times K} \) having iid entries \( y_{ij} \) with zero mean and unit variance, absolutely continuous and with \( E[|y_{ij}|^2] < \infty \).

\(^1\)One can show that under (As1As3As5) described later, \( \hat{f}_m(x) \) a.s. converges towards a nonrandom PDF \( f(x) \) with a compact support.
(As4) Let $Y \in \mathbb{C}^{m \times K}$ defined as in (As3), then its distribution is invariant by left multiplication by a deterministic unitary matrix. Moreover, the eigenvalues empirical PDF of $\frac{1}{K}YY^H$ a.s. converges to the Marčenko-Pastur distribution [36] with support $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$.

(As5) The maximum (resp. minimum) eigenvalue of $\frac{1}{K}YY^H$ a.s. tends to $(1 + \sqrt{c})^2$ (resp. $(1 - \sqrt{c})^2$).

B. Covariance matrix models and convergence of eigenvalues

We first expose the considered data model and, then, the eigenvalues behavior of the SCM. The SCM can be written as $\hat{R} = \frac{1}{K}XX^H$ with:

$$X = R^{1/2}Y = (I_m + P)^{1/2}Y$$

with $X = [x_1, \ldots, x_K]$. The iid entries of $Y$ follow the $CN(0,1)$ distribution according to our data model in Sec.II. The complex normal distribution being a particular case of such distributions defined in (As3), the $Y$ entries consequently verify it. Thus, the forthcoming convergences hold in the more general case defined by (As3), $R^{1/2}$ is the $m \times m$ Hermitian positive definite square root of the true covariance matrix. The matrix $P$ is the rank $r$ perturbation matrix and can be eigendecomposed as $P = U\Omega U^H = \sum_{i=1}^{\tilde{M}} \omega_i U_i U_i^H$ with:

$$\Omega = \begin{bmatrix} \omega_1 I_{K_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \omega_{\tilde{M}} I_{K_{\tilde{M}}}
\end{bmatrix}$$

with $U = [U_1 \cdots U_{\tilde{M}}]$ and $\tilde{M}$ the number of distinct eigenvalues of $R$. Moreover, $U_i \in \mathbb{C}^{m \times K_i}$, where $K_i$ is the multiplicity of $\omega_i$. Hence, the covariance matrix (3) can be rewritten as:

$$R = \sum_{i=1}^{\tilde{M}} \lambda_i U_i U_i^H$$

where $\lambda_i$, of multiplicity $K_i$, and $U_i$ are the eigenvalues and the associated subspaces (concatenation of the $K_i$ eigenvectors associated to $\lambda_i$) of $R$ respectively, with $\lambda_1 = 1 + \omega_1 > \cdots > \lambda_{\tilde{M}} = 1 + \omega_{\tilde{M}} > 0$ and $\sum_{i=1}^{\tilde{M}} K_i = m$. The properties of the spiked model are the following:

- $\exists n \in [1, \tilde{M}]$ such that $\omega_n = 0$.
- The multiplicity $K_i$ is fixed $\forall i \in [1, \tilde{M}] \setminus n$ and does not increase with $m$, i.e. $K_i/m \to 0^+$, $\forall i \in [1, \tilde{M}] \setminus n$.

Consequently, we have $\text{rank}(\Omega) = \sum_{i=1}^{\tilde{M}} K_i = r$ and $K_n = m - r$. In other words, the model specifies that only a few eigenvalues are non-unit (and do not contribute to the noise unit-eigenvalues) and fixed. Consequently, $\lambda_n = 1$ is the eigenvalue of $R$ corresponding to the white noise and the others correspond to the rank $r$ perturbation.

In our case (see Sec.II), we recall that the covariance matrix $R$ can be written as in Eq.(3) and Eq.(22). More specifically, the noise component $b$ corresponds to the white noise and its eigenvalue is $\lambda_{\tilde{M}} = 1$ as, for simplicity purposes, we set $\sigma^2 = 1$. The $r$ eigenvalues of the LR noise component $c$ are strictly higher than 1. Thus, $M = r + 1$, $\lambda_1 = 1 + \omega_1 > \cdots > \lambda_{M-1} = 1 + \omega_{M-1} > \lambda_{\tilde{M}} = 1$, $K_i = 1$ is the multiplicity of $\lambda_i$, $\forall i \in [1, r]$, and $K_{\tilde{M}} = m - r$ is the multiplicity of $\lambda_{\tilde{M}}$.

$$R = \lambda_{\tilde{M}} U_{\tilde{M}} U_{\tilde{M}}^H + \sum_{i=1}^{\tilde{M}-1} \lambda_i U_i U_i^H = U_{r+1} U_{r+1}^H + \sum_{i=1}^{r} \lambda_i u_i u_i^H$$

This model leads to a specific asymptotic eigenvalues PDF of $R$ as detailed hereafter. The convergence of the eigenvalues is achieved through the convergence of the Stieltjes transform of the eigenvalues CDF. The asymptotic eigenvalue behavior of $R$ for the spiked model was introduced by Johnstone [29] and its eigenvalue behavior was studied in [37]. In order to derive it, [37] exploited the specific expression given in Eq.(20). Then, [22] introduced the final assumption (separation condition) under which the following convergences are given.

(As6.S) The eigenvalues of $P$ satisfy the separation condition, i.e. $|\omega_i| > \sqrt{c}$ for all $i \in [1, \tilde{M}] \setminus n$ ($i \in [1, r]$ in our case).

Thus, under (As1-As5, As6.S), we have:

$$\tilde{f}_m(x) \to f(x)$$

where $f(x)$ is the Marčenko-Pastur law:

$$f(x) = \begin{cases} (1 - \frac{1}{c}), & \text{if } x = 0 \text{ and } c > 1 \\ \frac{1}{2\pi c x} \sqrt{(\lambda_- - x)(x - \lambda_+)} , & \text{if } x \in [\lambda_-, \lambda_+] \\ 0, & \text{otherwise} \end{cases}$$

with $\lambda_- = (1 - \sqrt{c})^2$ and $\lambda_+ = (1 + \sqrt{c})^2$. However, it is essential to note that, for all $i \in [1, \tilde{M}] \setminus n$:

$$\hat{\lambda}_j \in \mathcal{M}_i \xrightarrow{m,K \to \infty} \tau_i = 1 + \omega_i + \frac{1 + \omega_i}{\omega_i}$$

where $\mathcal{M}_i$ is the set indexes corresponding to the $j$-th eigenvalue of $R$ (for example $\mathcal{M}_{r+1} = \{r+1, \ldots, m\}$ for $\lambda_{r+1}$). Two representations of $\tilde{f}_m(x)$ for two different $c$ and a sufficient large $m$ are shown on Fig. 1 when the eigenvalues of $R$ are 1, 2, 3, and 7 with the same multiplicity, where the eigenvalue 1 is the noise eigenvalue. One can observe that say (As6.S) is verified is equivalent to say that $\tau_{n-1} > \lambda_+$ and $\tau_{n+1} < \lambda_-$. In other words, all the sample eigenvalues corresponding to the non-unit eigenvalues of $R$, converge to a value $\tau_i$ which is outside the support of the Marčenko-Pastur law (‘asymptotic’ PDF of the ‘unit’ sample eigenvalues). As an illustration, one can notice that, in Fig. 1, for $\tilde{f}_m(x)$ plotted for $c = 0.1$, the separation condition is verified ($\omega_1 = 6$, $\omega_2 = 2$ and $\omega_3 = 1$ are greater than $\sqrt{c} = 0.316$) and the three non-unit eigenvalues are represented on the PDF and outside the support of the Marčenko-Pastur law by their respective limits $\tau_1 = 7.116$, $\tau_2 = 3.15$ and $\tau_3 = 2.2$. On the contrary, for $\tilde{f}_m(x)$ plotted for $c = 1.5$, only the two greatest eigenvalues are represented on the PDF by their respective limits $\tau_1 = 8.75$ and $\tau_2 = 5.25$ while the separation condition is not verified for the eigenvalue $\lambda_3 = 2$ ($\omega_3 = 1 < \sqrt{c} = 1.223$). In this case, the sample eigenvalues corresponding to the eigenvalue $\lambda_3 = 2$ belongs to the Marčenko-Pastur law.
C. Convergence of simple quadratic forms

Here, we compare the convergence of two QFs in two convergence regimes: when $K \to \infty$ with a fixed $m$ and when $m, K \to \infty$ at the same rate.

We first present the useful convergences of simple QFs function of $\hat{R}$. It is well known that, due to the strong law of large numbers, when $K \to \infty$ with a fixed $m$, $\hat{R} \to R$ a.s. [38]. Thus,

$$s_1^H \hat{R}^{-1} s_2 \xrightarrow{a.s. \ K \to \infty \ m \to \infty} s_1^H R^{-1} s_2$$

(27)

Moreover, when $m, K \to \infty$ at the same rate [19], [39]:

$$s_1^H \hat{R}^{-1} s_2 \xrightarrow{a.s. \ m, K \to \infty} (1 - c)^{-1} s_1^H R^{-1} s_2$$

(28)

The useful convergences of simple QFs function of $\hat{\Pi}^\perp_c$ are then presented. As $\hat{R} \to R$ a.s. when $K \to \infty$ with a fixed $m$, $\hat{\Pi}^\perp_c \to \Pi^\perp_c$ a.s. [19] in the same convergence regime. Thus:

$$s_1^H \hat{\Pi}^\perp_c s_2 \xrightarrow{a.s. \ K \to \infty \ m \to \infty} s_1^H \Pi^\perp_c s_2$$

(29)

For the convergences in the large dimensional regime ($m, K \to \infty$ at the same rate), the convergences are presented under (As1-As5) and the separation condition As6.S. [22] showed that, $\forall i \in [1, M - 1]$

$$s_1^H \hat{U}_i^H \hat{U}_i s_2 \xrightarrow{a.s. \ m, K \to \infty} \frac{1 - c\omega_i^{-2}}{1 + c\omega_i^{-2}} s_1^H U_i U_i^H s_2$$

(30)

with $\omega_i = \lambda_i - 1$. $\lambda_i$ is the $i$-th distinct eigenvalue of $R$. Let $\chi_i = \frac{1}{1 + c\omega_i^{-2}}$. Thus, using the following relationship,

$$\hat{\Pi}^\perp_c = I_m - \sum_{i=1}^{M-1} \hat{U}_i \hat{U}_i^H = I_m - \sum_{i=1}^{r} \hat{u}_i \hat{u}_i^H$$

(31)

one can deduce that with the spiked model and in the large dimensional regime:

$$s_1^H \hat{\Pi}^\perp_c s_2 \xrightarrow{m, K \to \infty} s_1^H \Pi^\perp_c s_2$$

(32)

with $\Pi^\perp_c = \sum_{i=1}^{m} \psi_i u_i u_i^H$ and

$$\psi_i = \begin{cases} 1, & \text{if } i > r \\ 1 - \chi_i, & \text{if } i \leq r \end{cases}$$

(33)

Consequently, $s_1^H \hat{R}^{-1} s_2$ is consistent in the two convergence regimes and, although $s_1^H \hat{\Pi}^\perp_c s_2$ is consistent when $K \to \infty$ with a fixed $m$, it is no more consistent under the regime of interest i.e. when both $m, K \to \infty$ at the same rate.

IV. NEW CONVERGENCE RESULTS

A. Convergence of structured quadratic forms

In this section, the convergence of the structured QF function of $\hat{\Pi}^\perp_c$ is analyzed and results to Proposition 1.

**Proposition 1**: Let $B$ be a $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all $m$. Then, under (As1-As5, As6.S) and the spiked model,

$$s_1^H \hat{\Pi}^\perp_c B \hat{\Pi}^\perp_c s_2 \xrightarrow{m, K \to \infty} s_1^H \Pi^\perp_c B \Pi^\perp_c s_2$$

(34)

where $\Pi^\perp_c = \sum_{i=1}^{m} \psi_i u_i u_i^H$ with $\psi_i$ defined by Eq.(33).

**Proof**: See Appendix.

Moreover, one can remark that if $B = R$, where $R$ is the covariance matrix as defined in Eq.(5), the following convergence holds:

$$s_1^H \hat{\Pi}^\perp_c B \hat{\Pi}^\perp_c s_2 \xrightarrow{m, K \to \infty} s_1^H \Pi^\perp_c B \Pi^\perp_c s_2$$

(35)

A visualization of the convergence of Eq.(34) in terms of Mean Squared Error (MSE) can be found in Fig. 2 when $m, K \to \infty$ at a fixed ratio. It is compared to the MSE corresponding to the following convergence when $K \to \infty$ with a fixed $m$:

$$s_1^H \hat{\Pi}^\perp_c B \hat{\Pi}^\perp_c s_2 \xrightarrow{K \to \infty} s_1^H \Pi^\perp_c B \Pi^\perp_c s_2$$

(36)

B. Convergence of SINR losses

Now, we provide the convergences of the estimated SINR loss using the convergences previously presented and the following convergence. We recall that, as $\hat{R} \to R$ a.s. when $K \to \infty$ with a fixed $m$, one has:

$$s_1^H \hat{R}^{-1} \hat{R}^{-1} s_2 \xrightarrow{m, K \to \infty} s_1^H R^{-1} s_2$$

(37)

Hence, when $K \to \infty$ with a fixed $m$ and using Eq.(27), Eq.(37) and the continuous mapping theorem [38]:

$$\hat{\rho} \xrightarrow{K \to \infty} \frac{|a^H R^{-1} a|^2}{(a^H R^{-1} a)(a^H R^{-1} a)} = 1$$

(38)
And, under (As1-As5), when \( m, K \to \infty \) at the same rate, from [33], we have:

\[
\hat{\rho}_{\text{LR}} \overset{\text{a.s.}}{\longrightarrow}_{m,K\to\infty} \frac{(1-c)|a^H R^{-1} a|^2}{(a^H R^{-1} a)(a^H R^{-1} a)} = 1 - c
\]  

Thus, the estimated SINR loss \( \hat{\rho} \) is consistent when \( K \to \infty \) with \( m \) fixed and when \( m, K \to \infty \) at the same rate, up to an additive constant \( c \). Consequently, RMT cannot help us to improve the estimation of the theoretical SINR loss.

For the SINR loss corresponding to the adaptive LR filters, when \( K \to \infty \) with a fixed \( m \), using Eq.(29), Eq.(36) and the continuous mapping theorem theorem, we have:

\[
\hat{\rho}_{\text{LR}} \overset{\text{a.s.}}{\longrightarrow}_{m\to\infty} \rho_{\text{LR}}
\] 

where \( \rho_{\text{LR}} \) is defined by Eq.(14). When \( m, K \to \infty \) at the same rate, we obtain the following convergence:

\[
\hat{\rho}_{\text{LR}} \overset{\text{a.s.}}{\longrightarrow}_{m,K\to\infty} \rho_{\text{LR}}^{(S)} = \rho_{\text{LR}}|\bar{\Pi} = \bar{\Pi}^a \neq \rho_{\text{LR}}
\] 

where Eq.(32), Proposition 1 and the continuous mapping theorem were used to prove Eq.(41). One can observe that, although the traditional estimator of \( \rho_{\text{LR}} \) is consistent when \( K \to \infty \) with a fixed \( m \), it is no more consistent when \( m, K \to \infty \) at the same rate. It is also important to underline that the new convergence result leads to a more precise approximation of \( \hat{\rho}_{\text{LR}} \) than previous works [16]. Indeed, [16] proposes an approximation dependent on \( K \) and the approximation proposed here depends on \( K \) (and of course on \( c \)) as well as on the parameter \( \Theta \).

V. SIMULATIONS

A. Parameters

As an illustration of the interest of the application of RMT in filtering, the jamming application is chosen. The purpose of this application is to detect a target thanks to a ULA composed of \( m \) sensors despite the presence of jamming. The response of the jamming, \( c \) is composed of signals similar to the target response. In this section, except for the convergences when \( m, K \to \infty \) at the same rate \( c \), we choose \( m = 100 \) in order to have a large number for the data dimension. Even if, in some basic array processing applications, this number could seem significant, it actually became standard in many applications such as STAP [2], MIMO applications [40], [41], MIMO-STAP [40], etc. Here, \( \Theta = \theta \) where \( \theta \) is the AoA. The jamming is composed of three synthetic targets with AoA -20°, 0° and 20° and wavelength \( l_0 = 0.667m \). Thus, the jamming (LR noise) has a rank \( r = 3 \). Then, the AWGN \( b \) power is \( \sigma^2 = 1 \). Finally, the theoretical covariance matrix of the total noise can be written as \( R = \frac{JNR}{\text{tr}(\Lambda)} \text{U} \Lambda \text{U}^H + \sigma^2 \text{I}_m \) with \( \Lambda = \text{diag}([6,2,1]) \) and where \( JNR \) is the jamming to noise ratio. \( \frac{JNR}{\text{tr}(\Lambda)} \) is fixed at 10dB except for Fig. 4.

In order to validate the \( \text{spiked} \) model as covariance matrix model, we visualize a zoom of the experimental PDF of the eigenvalues of our data without target in Fig. 3 over 5\( \times \)10\( ^3 \) Monte-Carlo iterations. We observe a Marčenko-Pastur law around 1 (eigenvalues of the white noise) and Gaussian distributions for the eigenvalues of the jamming, which is consistent to the CLT for the \( \text{spiked} \) model proved in [22]. The \( \text{spiked} \) model is consequently relevant for our data model.

Moreover, in order to verified that the \( \text{spiked} \) model is realistic in terms of \textit{separation condition}, Fig. 4 shows \( \omega_r - \sqrt{c} \) as a function of \( \frac{JNR}{\text{tr}(\Lambda)} \) in dB. This figure will be the same for all \( m \) and \( K \) at a fixed ratio. We recall that, in order to satisfy the \textit{separation condition}, one should have \( \omega_r - \sqrt{c} > 0 \). Consequently, we gladly observe that it is satisfied for \( \frac{JNR}{\text{tr}(\Lambda)} > 4 \text{dB} \) for the majority of \( c \) even \( c > 2 \). Indeed, in practice, if the \( \frac{JNR}{\text{tr}(\Lambda)} \) is lower, the jamming will not have any effects on the performance.

B. Performance of filters

We now observe the performances of filters through the SINR loss. We are first interested in the validation of the convergence of \( \hat{\rho}_{\text{LR}} \) in Eq.(41) as \( m, K \to \infty \) at the same rate. This convergence is validated and presented in Fig. 5 in
terms of MSE over \(10^3\) realizations with \(c = 3\) for an AoA of the target (\(\theta = 50^\circ\)) and an AoA of the jamming (\(\theta = 20^\circ\)).

Fig. 5 shows the visualizations of Eq.(14) (blue line with stars), Eq.(15) (blue dashed line), the right side of the convergence in Eq.(41) (green line with circles) and the traditional estimation of \(E[\hat{\rho}_{LR}]\) (black line) as a function of \(K\) with \(J_N R_{tr}(\Lambda) = 10\) dB, \(m = 100\) and \(\theta = 20.5^\circ\).

VI. CONCLUSION

In this paper, we proposed new results in random matrix theory with a specific covariance matrix model fitted to our data model: the spiked model. Based on this, we studied the convergence of the traditional estimators of the SINR loss in their full rank and low rank version when the number of secondary data \(K \to \infty\) with a fixed data dimension \(m\) and when \(m, K \to \infty\) at the same rate \(c = m/K\). We observed that the full rank version is consistent in the two regimes. However, the low rank version is consistent when \(K \to \infty\) with a fixed \(m\) but is not consistent when \(m, K \to \infty\) at the same rate \(c\). Finally, we applied these results to a jamming application. We first observed that the experimental probability density function of the eigenvalue of the covariance matrix of jamming data is relevant with the probability density function...
of the spiked model. Then, we validated the convergence of the SINR loss in its low rank version and we observed that random matrix theory and more precisely the spiked model better evaluate the asymptotic performances of the low rank SINR loss corresponding to the adaptive LR filter, especially when the steering vector parameter is close to the jamming one and contrary to previous works. Moreover, it permits to predict the steering vector parameter value corresponding to the performance break.

VII. APPENDIX

The proof is decomposed as follows. We first develop the structured QF as a sum of simple QFs and base structured QF (Subsec. VII-A). In a second time, we formulate the base structured QF as a complex integral (Subsec. VII-B) and split it into several integrals (Subsec. VII-C). Then, we determine the deterministic complex integral equivalent of the base structured QF (Subsec. VII-D) and its formal expression (Subsec. VII-E). Finally, we use this result to determine the convergence of the structured QF in the large dimensional regime (Subsec. VII-F). The regime of convergences in the Appendix, if not precised, is $m, K \rightarrow \infty$ at a fixed ratio $c$.

A. Development of the structured QF

Let $s_1$ and $s_2$ be two deterministic complex vectors and $B$ be an $m \times m$ deterministic complex matrix with uniformly bounded spectral norm for all $m$. In order to obtain the convergence of the structured QF $s^H_1 \Pi^+ \Pi^+ B \Pi^+ s_2$, one can rewrite, using the notations of Eq.(23) and the spiked model, $\Pi^+ = \Pi^+_{r+1} = U^H \Pi^+_{r+1} = I_m - U^H \Pi^+_{r+1} = I_m - \Pi^+_{r+1}$ where $\Pi^+_{r+1} = [\hat{u}_r, \ldots, \hat{u}_m]$, $\Pi_i = \hat{u}_i \hat{u}_i^H$, $\forall i \in \{1, r\}$ and $\hat{u}_i$ are the eigenvectors of the SCM. We recall that $r$ is fixed for all $m$, i.e. $r/m \rightarrow 0^+$. Thus, one can develop the structured QF as:

$$s^H_1 \Pi^+ \Pi^+ B \Pi^+ s_2 = s^H_1 \left( I_m - \sum_{i=1}^r \hat{u}_i \hat{u}_i^H \right) B \left( I_m - \sum_{i=1}^r \hat{u}_i \hat{u}_i^H \right) s_2$$

$$= s^H_1 B s_2 - s^H_1 \sum_{i=1}^r \hat{u}_i B s_2 - s^H_1 \sum_{i=1}^r \hat{u}_i B \sum_{i=1}^r \hat{u}_i s_2$$

$$= s^H_1 B s_2 - \sum_{i=1}^r s^H_1 \Pi^+ B \sum_{i=1}^r \hat{u}_i s_2$$

$$= \sum_{j=1}^r s^H_1 \Pi^+ B \Pi^+ s_2 + \sum_{j_1 \neq j_2} s^H_1 \Pi^+ B \Pi^+ s_2 (42)$$

B. Formulation of the base structured QF as a complex integral

Remark that Eq.(44) is a sum of simple QFs and base structured QFs, we first focus on the convergence of the base structured QF $\hat{\eta}(j_1, j_2) = s^H_1 \Pi^+ B \Pi^+ s_2$, $\{j_1, j_2\} \in \{1, r\}^2$. Let us now formulate the base structured QF as a complex integral.

**Proposition 2:** Let $B$ be an $m \times m$ deterministic complex matrix with a uniformly bounded spectral norm for all $m$. Then, under (As1-As5, As6.S) and the spiked model, $\forall j_1, j_2 \in [1, r + 1]$, if $\hat{\eta}(j_1, j_2) = s^H_1 \Pi^+ B \Pi^+ s_2$,

$$\hat{\eta}(j_1, j_2) = \frac{1}{(2\pi)^2} \int_{C_{j_1}} \int_{C_{j_2}} s^H_1 (\hat{R} - z_1 I_m)^{-1} \times B (\hat{R} - z_2 I_m)^{-1} s_2 dz_1 dz_2 (45)$$

**Proof:** If $j_1 \neq j_2$, it can be easily shown that $\hat{\eta}(j_1, j_2)$ can be expressed as the following Cauchy integral:

$$A = \frac{1}{(2\pi)^2} \int_{C_{j_1}} \int_{C_{j_2}} s^H_1 G(z_1) B (\hat{R} - z_2 I_m)^{-1} s_2 dz_1 dz_2 (46)$$

where $C_{j_1}$ in a negatively oriented contour encompassing the eigenvalues of $\hat{R}$ corresponding to the $j_1$-th eigenvalue of $R$ and $z_1$ and $z_2$ are independent variables. Indeed, let $G(z_k) = (\hat{R} - z_k I_m)^{-1} = (\frac{1}{2} XX^H - z_k I_m)^{-1}$ with $k \in \{1, 2\}$. Thus:

$$A = \frac{1}{(2\pi)^2} \int_{C_{j_1}} \int_{C_{j_2}} s^H_1 G(z_1) B \sum_{n=1}^m \lambda_n u_n u_n^H s_2 (47)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \int_{C_{j_2}} \sum_{n=1}^m s^H_1 G(z_1) B u_n u_n^H s_2 (48)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \int_{C_{j_2}} \sum_{n=1}^m \frac{1}{2\pi i} f_n^{(2)}(z_2) dz_2 dz_1 (49)$$

where $f_n^{(2)}(z_2) = s^H_1 G(z_1) B u_n u_n^H s_2$. From the expression $f_n^{(2)}(z_2)$, one observe that $f_n^{(2)}(z_2)$ has a single simple pole $\lambda_n$ which is encompassed by $C_{j_2}$ for the indexes $n \in M_{j_2}$ where $M_{j_2}$ is the set of indexes corresponding to the $j_2$-th eigenvalue of $R$. Consequently, from complex analysis:

$$A = \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \frac{1}{2\pi i} \int_{C_{j_2}} \frac{s^H_1 G(z_1) B u_n u_n^H s_2}{\lambda_n - z_2} dz_2 dz_1 (50)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \frac{1}{2\pi i} \frac{f_n^{(2)}(z_2) dz_2 dz_1}{\lambda_n - z_2} (51)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \left[ - \operatorname{Res}\left(f_n^{(2)}(z_2), \lambda_n\right) \right] dz_1 (52)$$

where $\operatorname{Res}\left(f_n^{(2)}(z_2), \lambda_n\right)$ is the residue of $f_n^{(2)}(z_2)$ at $\lambda_n$. Thus, using the residue theorem and residue calculus:

$$A = \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \left[ - \lim_{z_2 \to \lambda_n} \frac{f_n^{(2)}(z_2)}{\lambda_n - z_2} \right] dz_1 (53)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \lim_{z_2 \to \lambda_n} \frac{s^H_1 G(z_1) B u_n u_n^H s_2}{\lambda_n - z_2} dz_1 (54)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \lim_{z_2 \to \lambda_n} \left( s^H_1 G(z_1) B u_n u_n^H s_2 \right) dz_1 (55)$$

$$= \frac{1}{(2\pi)^2} \int_{C_{j_1}} \sum_{n \in M_{j_2}} \lim_{z_2 \to \lambda_n} \left( s^H_1 G(z_1) B u_n u_n^H s_2 \right) dz_1 (56)$$
\[ A = \frac{1}{2\pi i} \oint_{C_{j_1}} s_n^H G(z_1) B \sum_{n \in M_{j_1}} \hat{u}_n u_n^H s_d dz_1 \]  
\[ = \frac{1}{2\pi i} \oint_{C_{j_1}} s_n^H (\hat{R} - z_1 I_m)^{-1} B \hat{p}_{j_2} s_d dz_1 \]  
\[ = \frac{1}{2\pi i} \oint_{C_{j_1}} s_n^H \left( \sum_{n=1}^m \lambda_n \hat{u}_n u_n^H - z_1 I_m \right)^{-1} B \hat{p}_{j_2} s_d dz_1 \]  
\[ = \frac{1}{2\pi i} \oint_{C_{j_1}} s_n^H \frac{1}{\sum_{n=1}^m \lambda_n \hat{u}_n u_n^H - z_1 I_m} B \hat{p}_{j_2} s_d dz_1 \]  
\[ = \sum_{n=1}^m \frac{1}{2\pi i} \oint_{C_{j_1}} f_n^{(1)}(z_1) dz_1 \]  
where \( f_n^{(1)}(z_1) = s_n^H u_n \hat{u}_n^H B \hat{p}_{j_2} s_d \). Similarly, \( f_n^{(1)}(z_1) \) has a single simple pole \( \lambda_n \) which is encompassed by \( C_{j_1} \) for the indexes \( n \in M_{j_1} \). Thus:

\[ A = \sum_{n \in M_{j_1}} \frac{1}{2\pi i} \oint_{C_{j_1}} f_n^{(1)}(z_1) dz_1 \]  
\[ = - \sum_{n \in M_{j_1}} \text{Res} \left( f_n^{(1)}(z_1), \lambda_n \right) \]  
\[ = - \sum_{n \in M_{j_1}} \text{Res} \left( f_n^{(1)}(z_1), \lambda_n \right) \]  
\[ = \sum_{n \in M_{j_1}} \text{Res} \left( f_n^{(1)}(z_1), \lambda_n \right) \]  
\[ = \sum_{n \in M_{j_1}} \lim_{z_1 \to \lambda_n} \left( \lambda_n - z_1 \right) \left( s_n^H u_n \hat{u}_n^H B \hat{p}_{j_2} s_d \right) \]  
\[ = \sum_{n \in M_{j_1}} \lim_{z_1 \to \lambda_n} \left( \lambda_n - z_1 \right) \left( s_n^H u_n \hat{u}_n^H B \hat{p}_{j_2} s_d \right) \]  
\[ = \sum_{n \in M_{j_1}} \lim_{z_1 \to \lambda_n} \left( \lambda_n - z_1 \right) \left( s_n^H u_n \hat{u}_n^H B \hat{p}_{j_2} s_d \right) \]  
\[ = s_n^H \sum_{n \in M_{j_1}} \hat{u}_n u_n^H B \hat{p}_{j_2} s_d = s_n^H \hat{P}_{j_1} B \hat{p}_{j_2} s_d \]  

Consequently, \( \hat{\eta}(j_1, j_2) = A \) for \( j_1 \neq j_2 \).

Then, if \( j_1 = j_2 = j \) and using the same arguments as previously, one has:

\[ s_n^H \hat{P}_{j} B \hat{p}_{j} s_d = \frac{1}{2\pi i} \oint_{C_{j}} s_n^H \sum_{n=1}^m \hat{u}_n \hat{u}_n^H B \hat{u}_n u_n^H s_d dz_1 \]  

However, the remaining of the proof is based on the fact that the resolvent \( G(z) \) of the SCM can be found in the complex integral, which is not the case in the previous equation. Consequently, noticing that:

\[ g(\hat{P}_{j}) = \frac{1}{2\pi i} \oint_{C_{j}} \sum_{n=1}^m g(\hat{P}_{j}) \frac{1}{\lambda_n - z} dz \]  

where \( g(\cdot) \) is a functional, Eq.(68) is equivalent to Eq.(45). As a consequence, \( \forall j_1, j_2 \in [1, r+1] \):

\[ \hat{\eta}(j_1, j_2) = \frac{1}{2\pi i} \oint_{C_{j_1}} s_n^H (\hat{R} - z_1 I_m)^{-1} \times B \left( \hat{R} - z_2 I_m \right)^{-1} s_d dz_1 dz_2 \]  

C. Development of the complex integral

Next, one want to split the previous line integral into several line integrals where some of them will tend to 0. Thus, from [22], with \( k \in [1, 2] \), one can write:

\[ (\hat{R} - z_k I_m)^{-1} = (I_m + P)^{-1/2} (Q(z_k) - z_k Q(z_k)) U \]  
\[ \times \tilde{\hat{H}}(z_k)^{-1} \Omega (I_r + \Omega)^{-1} U^H Q(z_k) \]  
\[ \times (I_m + P)^{-1/2} \]  

with

\[ Q(z_k) = (\frac{1}{2} YY^H - z_k I_m)^{-1} \]  
\[ \tilde{\hat{H}}(z_k) = I_m + z_k \Omega (I_m + \Omega)^{-1} U^H Q(z_k) U \]  

Then, replacing \( (\hat{R} - z_k I_m)^{-1} \) by Eq.(71) in Eq.(45) and developing the obtained result, one obtains:

\[ \hat{\eta}(j_1, j_2) = \frac{1}{(2\pi i)^2} \oint_{C_{j_1}} \oint_{C_{j_2}} s_n^H E(z_1) B E(z_2) s_d dz_1 dz_2 \]  
\[ - \frac{1}{(2\pi i)^2} \oint_{C_{j_1}} \oint_{C_{j_2}} \hat{\hat{H}}(z_1) \hat{\hat{C}}(z_2) B \times E(z_1) s_d dz_1 dz_2 \]  
\[ - \frac{1}{(2\pi i)^2} \oint_{C_{j_1}} \oint_{C_{j_2}} \hat{\hat{H}}(z_1) \hat{\hat{C}}(z_2) B \times E(z_1) s_d dz_1 dz_2 \]  
\[ + \frac{1}{(2\pi i)^2} \oint_{C_{j_1}} \oint_{C_{j_2}} \hat{\hat{H}}(z_1) \hat{\hat{C}}(z_2) B \times E(z_1) s_d dz_1 dz_2 \]  
\[ = D_1 - D_2 - D_3 + D_4 \]  

D. Determination of the deterministic complex integral equivalent

The convergence of the terms \( D_1 \) to \( D_4 \) has now to be studied. Some of them will tend to 0 and the remainder of the terms will tend to a deterministic integral equivalent.

**Proposition 3**: Let \( B \) be a \( m \times m \) deterministic complex matrix with a uniformly bounded spectral norm for all \( m \). Then, under (As1-As5, As6,S) and the spiked model, \( \forall j_1, j_2 \in [1, r+1] \), \( \hat{\eta}(j_1, j_2) \rightarrow 0 \) with

\[ \eta(j_1, j_2) = \frac{1}{(2\pi i)^2} \oint_{C_{j_1}} \oint_{C_{j_2}} e_n^H (z_1) H(z_1)^{-1} C_2(z_1) \times B C_2^H(z_2) H(z_2)^{-1} e_2(z_2) dz_1 dz_2 \]  

where \( \gamma_j^- \) is a deterministic negatively oriented circle only enclosing \( \tau_j \) (cf. Eq.(26)) and

\[ H(z) = I_m + z \tilde{b}_m(z) \Omega (I_m + \Omega)^{-1} \]  
\[ e_n^H(z) = zb_n(z) s_n^H (I_m + P)^{-1/2} U \]  
\[ C_2(z) = \tilde{b}_m(z) \Omega (I_m + \Omega)^{-1} U^H (I_m + P)^{-1/2} \]  
\[ C_2^H(z) = z \tilde{b}_m(z) (I_m + P)^{-1/2} U \]  

**Proof**: We first recall that we are interested in the indexes \( j_1, j_2 \in [1, r] \). Then, the function \( E(z) \) in \( D_1, D_2 \) and \( D_3 \) can be rewritten as:

\[ E(z) = (\hat{R} - z(I_m + P))^{-1} = \sum_{n=1}^m \hat{u}_n u_n^H \]  

Thus, \( E(z_1) \) (resp. \( E(z_2) \)) has a single simple pole \( \tilde{\lambda}_n \) when \( \omega_n \neq 0 \), i.e. \( n \in [1, r] \) (As5, As6,S) are verified and
Thus, from Eq. (89), one obtains:
\[
\hat{H}(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \Omega(I_m + \Omega)^{-1}
\]
(95)
\[
\hat{e}_1^H(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \hat{e}_2^H(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \hat{e}_2^H(z)
\]
(96)
\[
\hat{C}_2(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \Omega(I_m + \Omega)^{-1} \hat{H}^H(I_m + P)^{-1/2}
\]
(97)
\[
\hat{C}_2^H(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \Omega(I_m + \Omega)^{-1} \hat{H}^H(I_m + P)^{-1/2}
\]
(98)
\[
\hat{e}_2(z) = \frac{\hat{b}_m(z)}{\hat{e}_m(z)} \hat{e}_2(z)
\]
(99)
As a result, \( \hat{\eta}(j_1, j_2) - \eta(j_1, j_2) \to 0 \) with
\[
\eta(j_1, j_2) = \frac{1}{2\pi i} \oint_{C(j_2)} \oint_{C(j_1)} \hat{e}_1^H(z) \hat{H}(z)^{-1} \hat{C}_2(z)
\]
\[ \times BC_H(z) \hat{H}(z)^{-1} \hat{e}_2(z) dz_1 dz_2 \]
(100)
where \( \gamma_j \) is a deterministic negatively oriented circle only enclosing \( \tau_j \) (cf. Eq. (26)).

E. Determination of the expression of the deterministic equivalent

Let us now find the expression of the deterministic equivalent \( \eta(j_1, j_2) \) as a function of the eigenvalues and eigenvectors of the covariance matrix \( R \).

**Proposition 4**: Let \( B \) be an \( m \times m \) deterministic complex matrix with a uniformly bounded spectral norm for all \( m \). Then, under (As1-As5, As6.S) and the spiked model,
\[
\eta(j_1, j_2) = \chi_{j_1} \chi_{j_2} s_{j_1} \Pi_{j_1} B \Pi_{j_2} s_{j_2}
\]
(101)
with \( \chi_j = \frac{1}{1+\omega_j^2} \) and \( \{j_1, j_2\} \in [1, r]^2 \).

**Proof**: We first rewrite Eq. (100) as:
\[
\eta(j_1, j_2) = \frac{1}{2\pi i} \oint_{C(j_2)} \oint_{C(j_1)} \hat{e}_1^H(z_1) \hat{H}(z_1)^{-1} \hat{C}_2(z_1) B \hat{H}(z_2)^{-1} \hat{e}_2(z_2) dz_1 dz_2
\]
(102)
with
\[
g = \frac{1}{2\pi} \oint_{C(j_1)} \phi_1^H(z_1) \hat{H}(z_1)^{-1} c_2(z_1) dz_1
\]
(103)
in order to determine \( g \) in a first time.

We recall that, in our case, \( \omega_1 > \cdots > \omega_r > \omega_{r+1} = 0 \). After an eigendecomposition of \( \phi_1^H(z_1) \) and \( c_2(z_1) \) and, noticing from [22] that:
\[
\hat{H}(z)^{-1} = \text{diag}
\]
\[ \left( \frac{1}{1+\omega_1 b_m(z) \tau_{j_{r+1}}^*}, \cdots, \frac{1}{1+\omega_r b_m(z) \tau_{j_{r+1}}^*} \right) \]
(104)
with
\[
\mathcal{I} = \left[ \Omega_{k_1+\ldots+k_{r-1}}, \mathbf{I}_{k_1} \right] \Theta_{k_1+\ldots+k_{r-1}} \mathcal{I} \]
(105)
one obtains:
\[
e_{j_1}^H(z_1) \hat{H}(z_1)^{-1} c_2(z_1) = s_{j_1} \sum_{l=1}^{r+1} \frac{\omega_l}{(1+\omega_l b_m(z_{j_1})) \tau_{j_{r+1}}^*} \]
(106)
Thus,
\[
g = \frac{1}{2\pi i} \oint_{C(j_1)} \oint_{C(j_2)} s_{j_1} \sum_{l=1}^{r+1} \frac{\omega_l}{(1+\omega_l b_m(z_{j_1})) \tau_{j_{r+1}}^*} \]
(107)
\[
\frac{1}{1+\omega_l b_m(z_{j_1}) \tau_{j_{r+1}}^*} \]
(108)
\[
= \frac{1}{2\pi i} \oint_{C(j_1)} s_{j_1} \sum_{l=1}^{r+1} \frac{\omega_l}{(1+\omega_l b_m(z_{j_1})) \tau_{j_{r+1}}^*} \]
(109)
\[
= \frac{1}{2\pi i} \oint_{C(j_1)} s_{j_1} \sum_{l=1}^{r+1} \frac{\omega_l}{(1+\omega_l b_m(z_{j_1})) \tau_{j_{r+1}}^*} \]
(110)
From [22], \( \frac{1+\omega_j}{\omega_j} + z_1 b_m(z_{j_1}) = 0 \) only for \( z_1 = \tau_{j_1} \) and \( z_1 b_m(z_{j_1}) \neq 0, j_1 \in [1, r] \). Hence, \( \frac{1+\omega_j}{\omega_j} + z_1 b_m(z_{j_1}) \) has a single simple pole at \( \tau_{j_1} \), \( j_1 \in [1, r] \). As a consequence, with \( h(z) = z b_m(z) \)
\[
g = \frac{1}{2\pi i} \oint_{C(j_1)} \oint_{C(j_2)} s_{j_1} \sum_{l=1}^{r+1} \frac{\omega_l}{(1+\omega_l b_m(z_{j_1})) \tau_{j_{r+1}}^*} \]
(111)
\[
\frac{1}{1+\omega_j} \]
Thus, similarly as with \( g \), one deduces that:
\[
\tilde{g} = \frac{1}{2\pi} \int_{\gamma_{2}} C^{H}_{2}(z_{2})H(z_{2})^{-1}e_{2}(z_{2})dz_{2}
\]
(118)

As a result:
\[
\eta(j_{1}, j_{2}) = \xi(j_{1})\xi(j_{2})s_{H}^{H}\Pi_{j_{1}}B_{j_{1}}s_{2}
\]
(120)

Finally, the last step consists in expressing \( \xi(j_{2}) \) as a function of \( \omega_{j} \). Using Corollary 2 from [22], one expresses \( \xi(j_{2}) \) as:
\[
\xi(j_{2}) = \frac{1 - cm_{0}^{-2}}{1 + cm_{0}^{-2}}
\]
(122)

As a consequence,
\[
\eta(j_{1}, j_{2}) \overset{m_{K} \to \infty}{\longrightarrow} \eta(j_{1}, j_{2}) = \chi_{j_{1}}\chi_{j_{2}}s_{H}^{H}\Pi_{j_{1}}B_{j_{1}}s_{2}
\]
(123)

with \( \{j_{1}, j_{2}\} \in [1, r]^{2} \).

F. Convergence of the structured QF

From the development of the structured QF, we recall that the convergences of the simple QFs \( s_{H}^{H}\Pi_{i}B_{s_{2}} \) and \( s_{H}^{H}\Pi_{i}B_{s_{2}} \), \( \forall i \in [1, r] \), can be easily determined from [22]:
\[
s_{H}^{H}\Pi_{i}B_{s_{2}} \overset{m_{K} \to \infty}{\longrightarrow} \chi_{s_{H}^{H}\Pi_{i}B_{s_{2}}}
\]
(124)
\[
s_{H}^{H}\Pi_{i}B_{s_{2}} \overset{m_{K} \to \infty}{\longrightarrow} \chi_{s_{H}^{H}\Pi_{i}B_{s_{2}}}
\]
(125)

where \( \chi_{i} \) is defined as in Section III.C.

Then, also using Eq.(123) in Eq.(44), one obtains:
\[
s_{H}^{H}\Pi_{i}B_{s_{2}} \overset{m_{K} \to \infty}{\longrightarrow} \chi_{s_{H}^{H}\Pi_{i}B_{s_{2}}}
\]
(126)

or equivalently
\[
s_{H}^{H}\Pi_{i}B_{s_{2}} \overset{m_{K} \to \infty}{\longrightarrow} \chi_{s_{H}^{H}\Pi_{i}B_{s_{2}}}
\]
(127)

\[
s_{H}^{H}\Pi_{i}B_{s_{2}} \overset{m_{K} \to \infty}{\longrightarrow} \chi_{s_{H}^{H}\Pi_{i}B_{s_{2}}}
\]
(128)

with \( \tilde{\Pi}_{c,s_{2}} = \sum_{i=1}^{m} \psi_{i}u_{i}a_{i}^{H} \) and
\[
\psi_{i} = \begin{cases} 1, & \text{if } i > r \ 
1 - \chi_{i}, & \text{if } i \leq r \end{cases}
\]
(129)
[26] N. Asendorf and R. Nadakuditi, “The performance of a matched subspace detector that uses subspaces estimated from finite, noisy, training data,” IEEE Trans. on Sig. Proc., vol. 61, no. 8, pp. 1972 – 1985, April 2013.
[27] A. Combernoux, F. Pascal, G. Ginolhac, and M. Lesturgie, “Performances of low rank detectors based on random matrix theory with application to stap,” RADAR, Oct. 2014.
[28] ——, “Asymptotic performance of the low rank adaptive normalized matched filter in a large dimensional regime,” ICASSP, Apr. 2015, accepted.
[29] I. Johnstone, “On the distribution of the largest principal component,” The Annals of Statistics, vol. 29, no. 2, pp. 295 – 327, 2001.
[30] F. Benaych-Georges and R. Nadakuditi, “The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices,” Adv. in Math., vol. 227, no. 1, pp. 494 – 521, 2011.
[31] D. Paul, “Asymptotics of sample eigenstructure for a large dimensional spiked covariance model,” Statistica Sinica, vol. 17, no. 4, pp. 1617 – 1642, 2007.
[32] W. Hachem, P. Louhaton, X. Mestre, J. Najim, and P. Vallet, “A subspace estimator of fixed rank perturbations of large random matrices,” Journal of Multivariate Analysis, vol. 114, 2013.
[33] B. Tang, J. Tang, and Y. Peng, “Performance of knowledge aided space time adaptive processing,” IET Radar Sonar Navig., vol. 5, no. 3, pp. 331 – 340, 2010.
[34] ——, “Clutter nulling performance of SMI in amplitude heterogeneous clutter environments,” IEEE Trans. on Aero. and Elec. Syst., vol. 49, no. 2, pp. 1366 – 1373, April 2013.
[35] J. Yu, F. Rubio, and M. McKay, “Performance analysis of minimum variance asset allocation with high frequency data,” ICASSP, pp. 6496 – 6500, May 2013.
[36] V. Marčenko and L. Pastur, “Distributions of eigenvalues for somme set of random matrices,” Math USSR-Sbornik, vol. 1, no. 4, pp. 457 – 483, April 1967.
[37] J. Baik and J. W. Silverstein, “Eigenvalues of large sample covariance matrices of spiked population models,” Journal of Multivariate Analysis, vol. 97, pp. 1643 – 1697, 2006.
[38] P. Billingsley, Probability and Measure, 3rd ed. New York, NY: Wiley, 1995.
[39] V. Girko, An Introduction to Statistical Analysis of Random Arrays. VSP International Science Publishers, 1998, ch. 14 - Ten years of general statistical analysis, http://www.general-statistical-analysis.girko.freewebspace.com/chapter14.pdf.
[40] J. Li and P. Stoica, MIMO Radar Signal Processing, 1st ed. Wiley, 2009.
[41] D. Tse and P. Viswanath, Fundamentals of Wireless Communication, 1st ed. Cambridge University Press, 2005.