ON REGULAR SOLUTIONS OF THE 3-D COMPRESSIBLE
ISENTROPIC EULER-BOLTZMANN EQUATIONS WITH VACUUM

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Abstract. In this paper, we discuss the Cauchy problem for the compressible isentropic Euler-Boltzmann equations with vacuum in radiation hydrodynamics. Firstly, we establish the local existence of regular solutions by the fundamental methods in the theory of quasi-linear symmetric hyperbolic systems under some physical assumptions. Then we give the non-global existence of regular solutions caused by the effect of vacuum for polytropic gases with adiabatic exponent $1 < \gamma \leq 3$. Finally, we extend our results to the initial-boundary value problem under some suitable boundary conditions. These blow-up results tell us that the radiation effect cannot prevent the formation of singularities caused by the appearance of vacuum.

1. Introduction

This paper is concerned with the local existence of regular solutions (see Definition 2.1) and the formation of singularities to the Cauchy problem for the isentropic Euler-Boltzmann equations with vacuum arising from the radiation hydrodynamics.

This system appears in various astrophysical contexts [7] and in high-temperature plasma physics [15]. The couplings of fluid field and radiation field involve momentum source and energy source depending on the specific radiation intensity driven by the so-called radiation transfer integro-differential equation [15]. Suppose that the matter is in local thermodynamical equilibrium, the coupled system of Euler-Boltzmann equations for the mass density $\rho(t, x)$, the fluid velocity $u(t, x) = (u_1, u_2, u_3)$, and the specific radiation
intensity $I(v, \Omega, t, x)$ in three-dimensional space reads as 

$$
\begin{aligned}
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I &= A_r, \\
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t \left( \rho u + \frac{1}{c^2} F_r \right) + \nabla \cdot (\rho u \otimes u + P_r) + \nabla p_m &= 0,
\end{aligned}
$$

(1.1)

where $(t, x) \in \mathbb{R}^+ \cap \mathbb{R}^3$, $v \in \mathbb{R}^+$ is the frequency of photons and $\Omega \in S^2$ is the travel direction of photons, here $S^2$ stands for the unit sphere in $\mathbb{R}^3$; $p_m$ is the material pressure satisfying the equation of state

$$
p_m = \rho^\gamma, \quad 1 < \gamma \leq 3,
$$

(1.2)

where $\gamma$ is the adiabatic exponent.

$$
A_r = S - \sigma_a I + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s (v' \rightarrow v, \Omega', \rho) I' - \sigma_s (v \rightarrow v', \Omega, \rho) I \right) d\Omega' dv' \tag{1.3}
$$

is the collision term involving emission, absorption and scattering of energy, where $I = I(v, \Omega, t, x)$, $I' = I(v', \Omega', t, x)$. $S = S(v, t, x, \rho)$ is the rate of energy emission due to spontaneous process; $\sigma_a = \sigma_a (v, t, x, \rho)$ stands for the absorption coefficient that may also depend on the mass density $\rho$; $\sigma_s = \sigma_s (v' \rightarrow v, \Omega', \Omega, \rho)$ is the “differential scattering coefficient” (see [8] or [15]) such that the probability of a photon being scattered from $v'$ to $v$ contained in $dv$, and from $\Omega'$ to $\Omega$ contained in $d\Omega$, and traveling a distance $ds$ is given by $\sigma_s (v' \rightarrow v, \Omega' \cdot \Omega) d\Omega' dv ds$.

For the isentropic flow, the impact of radiation on the dynamical properties of the fluid is described by the following two quantities

$$
F_r = \int_0^\infty \int_{S^2} I(v, \Omega, t, x) \Omega d\Omega dv, \quad P_r = \frac{1}{c} \int_0^\infty \int_{S^2} I(v, \Omega, t, x) \Omega \otimes \Omega d\Omega dv, \tag{1.4}
$$

which are called the radiation flux and the radiation pressure tensor, respectively. The radiation field affects the dynamical properties of the fluid significantly, which makes it difficult to get the estimates of some physical quantities. For example, the material momentum $\int \rho u dx$ of the fluid is not conserved because of the impact coming from the radiation flux $F_r$ and the radiation pressure tensor $P_r$.

For pure compressible hydrodynamics equations without radiation, there have been many results on the local existence of regular solutions and the formation of singularities caused by the appearance of vacuum. The study on the appearance of vacuum in fluid dynamics can be traced back at least to the collected work of von Neumann [14]. He made some remarks on the general hydrodynamical discussion about motions in one dimension.
following Riemann’s theory. Makino-Ukai-Kawashima [13] discussed the Cauchy problem for the compressible Euler equations with both initial density and velocity compactly supported. They established the local existence of the regular solutions and showed that the life span is finite for any non-trivial solution. Liu-Yang [10] first showed that the regular solution of three-dimensional compressible Euler equations with damping will not be global if the initial density has compact support. Xu-Yang [18] established the local existence of smooth solutions to Euler equations with damping under the assumption of physical vacuum boundary condition.

Recently, similar problems for compressible radiation hydrodynamics equations started drawing attention of people. For Euler-Boltzmann equations, when the initial density is away from vacuum, Jiang-Zhong [5] obtained the local existence of $C^1$ solutions for the Cauchy problem. Jiang-Wang [4] showed that some $C^1$ solutions will blow up in finite time regardless of the size of the initial perturbation. For Navier-Stokes-Boltzmann equations, in addition to the local existence of strong solutions with vacuum we obtained in [9], we also established the non-global existence of classical solutions to the Cauchy problem with compactly supported initial density by introducing a new functional which is a linear combination of some mechanical quantities and some radiation quantities in [8], we even studied the case that the viscosity coefficients depend on the mass density $\rho$. Ducomet-Nečasová [2] [3] studied the global weak solutions and their large time behavior for one-dimensional case.

In this paper, we are interested in the isentropic Euler-Boltzmann equations with the occurrence of vacuum. We first prove the local existence of regular solutions to the Cauchy problem, then we studied the formation of singularities caused by the vacuum. Our paper is greatly inspired by the arguments in Makino-Ukai-Kawashima [13], Jiang-Zhong [5] and Xin-Yan [19]. An important technique for symmetrization is adopted from [13], which will be used to prove the local existence of the regular solutions with nonnegative initial density. Due to the complexity of this physical model, we have to make some structure assumptions to the corresponding physical quantities such as $\sigma_a$, $\sigma_s$ and $S$, etc. Then we showed the formation of singularities for Cauchy problem when the initial data contain vacuum in some local domain, which is similar to the assumptions in [19]. Additionally, via the analysis of the finite influence domain, we also get some blow-up results for classical solutions to some initial-boundary value problems. These blow-up results imply that the radiation effect is not strong enough to prevent the formation of singularities caused by
the appearance of vacuum. Similar results have been proved for damped Euler equations \[10\] and Euler-Possion equations \[12\], which are very different from the results obtained in \[16\] \[17\] for the case without vacuum. Some discussion on the relation between this kind of singularities and the formation of shock can be seen in \[10\].

We organize this paper as follows. In section 2, we first reformulate the Cauchy problem for system (1.1) into a simpler form for the case $\sigma_s = 0$, and then we establish the local existence of regular solutions. In section 3, we show that the regular solution obtained in Section 2 will develop singularities in finite time provided that the initial data contain vacuum in some local domain for $\gamma > 1$, and we extend our blow-up result for Cauchy problem to some initial-boundary value problems. Finally, in Section 4, we show the local existence of regular solutions for the case $\sigma_s \neq 0$ under some assumptions to $S$, $\sigma_a$ and $\sigma_s$.

2. Reformulation and Local Existence

2.1. Reformulation.

We only consider the case $\sigma_s = 0$ in this section. For the case $\sigma_s \neq 0$, some corresponding results will be shown in Section 4. We reformulate the Cauchy problem of the compressible Euler-Boltzmann equations (1.1) to a quasi-linear symmetric hyperbolic system, so that we can get the local existence of regular solutions (see Definition 2.1).

From the assumptions of "induced process" and local thermal equilibrium (LTE, see \[8\] \[15\]), $S$ and $\sigma_a$ can be written as

$$
\begin{align*}
S(v, t, x, \rho) &= K_a \mathcal{B}(v) \left(1 + \frac{c^2 I}{2h^3}\right), \\
\sigma_a(v, t, x, \rho) &= K_a \cdot \left(1 + \frac{c^2}{2h^3} \mathcal{B}(v)\right),
\end{align*}
$$

where $\mathcal{B}(v) \in L^2(\mathbb{R}^+)$ is a function of $v$, $h$ is the Planck constant, and

$$
K_a = K_a(v, t, x, \rho) = \rho K_a(v, t, x, \rho) = o(\rho) \geq 0,
$$

where $K_a \in C^\infty$ for $(v, t, x, \rho)$ and $\lim_{\rho \to 0} K_a(v, t, x, \rho) = 0$. More comments on $S(v, t, x, \rho)$ and $\sigma_a(v, t, x, \rho)$ can be seen in \[5\] as well as in \[15\]. So, when $\sigma_s = 0$, the photon transport equation in (1.1) can be written as

$$
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \mathcal{B}(v)).
$$
Then the compressible isentropic Euler-Boltzmann equations (1.1) can be reduced to
\[
\begin{cases}
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \mathcal{B}(v)), \\
\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_m = \frac{1}{c} \int_0^\infty \int_{S^2} K_a \cdot (I - \mathcal{B}(v)) \Omega d\Omega d\nu.
\end{cases}
\] (2.4)

We consider the Cauchy problem with initial data
\[
I|_{t=0} = I_0(v, \Omega, x), \quad (\rho, u)|_{t=0} = (\rho_0(x), u_0(x)),
\] (2.5)

where \(\rho_0(x) \geq 0\).

We first introduce the definition of regular solutions to Cauchy problem (2.4)-(2.5).

**Definition 2.1.** Let \(T > 0\) be a positive constant. A solution \((I, \rho, u)\) to Cauchy problem (2.4)-(2.5) is called a regular solution if
\[
\begin{itemize}
(i) \quad I(v, \Omega, t, x) \in L^2(\mathbb{R}^+ \times S^2; C^1([0, T) \times \mathbb{R}^3)), \quad (\rho(t, x), u(t, x)) \in C^1([0, T) \times \mathbb{R}^3);
(ii) \quad \rho^{\frac{\gamma-1}{2}} \in C^1([0, T) \times \mathbb{R}^3), \quad \rho \geq 0;
(iii) \quad \partial_t u + u \cdot \nabla u = 0 \text{ holds in the exterior of } \text{supp}\rho.
\end{itemize}
\]

**Remark 2.1.**

1. We point out that this definition for regular solutions is almost the same as that of Makio-Ukai-Kawashima [13], in which the local existence of regular solutions is studied for Euler equations with initial data arbitrarily large and \(\inf \rho_0 = 0\). Similar result has been obtained for damped Euler equations in [10]. Moreover, \(\sqrt{\gamma \rho^{\frac{\gamma-1}{2}}}\) is a very important physical quantity called local sound speed in gas dynamics.

2. When \(\rho > 0\), if \((\rho, u)\) has the regularity showed in (i)-(ii), then it naturally holds that
\[
u_t + u \cdot \nabla u + \frac{2\gamma}{\gamma - 1} \rho^{\frac{\gamma-1}{2}} \nabla \rho^{\frac{\gamma-1}{2}} = \frac{1}{c} \int_0^\infty \int_{S^2} K_a \cdot (I - \mathcal{B}(v)) \Omega d\Omega d\nu.
\] (2.6)
Passing to the limit as \(\rho \to 0\), we have
\[
\nu_t + u \cdot \nabla u = \lim_{\rho \to 0} \left( \frac{1}{c} \int_0^\infty \int_{S^2} K_a \cdot (I - \mathcal{B}(v)) \Omega d\Omega d\nu \right) = 0.
\]
So condition (iii) is reasonable at points \((t, x)(t > 0)\) satisfying \(\rho(t, x) = 0\) due to the continuity of \(\rho\) and properties of \(K_a\).

3. We emphasize that condition (iii) is very important to make the velocity \(u\) well defined at vacuum points and to ensure the uniqueness of regular solutions. Without condition (iii), it is very difficult to get enough information about velocity even...
when considering specific cases such as point vacuum or continuous vacuum of one piece.

Now we symmetrize hyperbolic system (2.4). After introducing a new variable
\[ w = p_m^\gamma = \rho^{\frac{\gamma - 1}{2}} \]
system (2.4) can be written as
\[
\begin{align*}
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I &= -K_a \cdot (I - \overline{B}(v)), \\
\partial_t w + u \cdot \nabla w + \frac{\gamma - 1}{2} w \nabla \cdot u &= 0, \\
(\gamma - 1)^2 \left( \partial_t u + u \cdot \nabla u \right) + \frac{\gamma - 1}{2} w \nabla w &= (\gamma - 1)^2 \frac{1}{4\gamma c} \int_0^\infty \int_{S^2} \overline{K}_a \cdot (I - \overline{B}(v)) \Omega d\Omega dv.
\end{align*}
\]
(2.7)

Let \( U = U(t, x) = (w(t, x), u(t, x)) \). Then (2.8) is reduced to the following system of \((I, U)\):
\[
\begin{align*}
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I &= -K_a \cdot (I - \overline{B}(v)), \\
A_0(U) \partial_t U + \sum_{j=1}^3 A_j(U) \partial x_j U &= G(I, U),
\end{align*}
\]
(2.9)
and the initial condition (2.5) turns into
\[
(I, U)|_{t=0} = (I_0(v, \Omega, x), w_0(x), u_0(x)),
\]
(2.10)
where \( w_0(x) = \rho_0(x)^{\frac{\gamma - 1}{2}} \geq 0 \), and
\[
A_0(U) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(\gamma - 1)^2}{4\gamma} I_3 \end{pmatrix}, \quad A_j(U) = \begin{pmatrix} u_j & \frac{\gamma - 1}{2} w \delta_j \\ \frac{\gamma - 1}{2} w \delta_j^\top & \frac{(\gamma - 1)^2}{4\gamma} u_j I_3 \end{pmatrix} \quad (j = 1, 2, 3),
\]
where \( I_3 \) is the \( 3 \times 3 \) unit matrix, \( \delta_j = (\delta_{1j}, \delta_{2j}, \delta_{3j}) \) is the Kronecker symbol satisfying \( \delta_{ij} = 1, \ i = j \) and \( \delta_{ij} = 0 \), otherwise. The source terms \( G = (G_0, G_1, G_2, G_3) \) are
\[
G_0(I, U) = 0, \quad G_j(I, U) = \frac{(\gamma - 1)^2}{4c\gamma} \left( \int_0^\infty \int_{S^2} \overline{K}_a \cdot (I - \overline{B}(v)) \Omega_j d\Omega dv \right) \quad (j = 1, 2, 3).
\]
We note that \( A_j(U)(j = 0, 1, 2, 3) \) are \( C^\infty \) for \( U \) and \( G(I, U) \) are \( C^\infty \) for \( I \) and \( U \). Moreover, \( A_j(U) \) \((j = 1, 2, 3)\) are all symmetric, and \( A_0(U) \) is bounded and positively definite. In fact, we have
\[
\frac{(\gamma - 1)^2}{4\gamma} |\xi|^2 \leq (A_0(U)\xi, \xi) \leq |\xi|^2, \text{ for } \xi \in \mathbb{R}^3.
\]
(2.11)
In order to get the local existence of regular solutions to the Cauchy problem (2.4)-(2.5), it suffices to prove the local existence of classical solutions to the reformulated Cauchy problem (2.9)-(2.10).

2.2. Local existence and uniqueness of classical solutions to (2.9)-(2.10).

We first introduce some notations. Denote by $W^{s,p}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ the ordinary Sobolev spaces, and

$$
\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^n)}, \quad \| \cdot \|_s = \| \cdot \|_{H^s(\mathbb{R}^n)}, \quad \| \cdot \|_{s,T} = \max_{t \in [0,T]} \| \cdot \|_s.
$$

The following well-known estimates for the derivatives of product are useful in the energy estimates for local existence.

**Lemma 2.1.** Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \leq p, q, r \leq +\infty$. For any integer $s \geq 0$, if $f \in W^{s,p}(\mathbb{R}^n)$, $g \in W^{s,q}(\mathbb{R}^n)$, then

$$
\| D^s(fg) \|_{L^r(\mathbb{R}^n)} \leq C_s \left( \| f \|_{L^p(\mathbb{R}^n)} \| D^s g \|_{L^q(\mathbb{R}^n)} + \| g \|_{L^p(\mathbb{R}^n)} \| D^s f \|_{L^q(\mathbb{R}^n)} \right),
$$

and when $s \geq 1$,

$$
\| D^s(fg) - f D^s g \|_{L^r(\mathbb{R}^n)} \leq C_s \left( \| Df \|_{L^p(\mathbb{R}^n)} \| D^{s-1} g \|_{L^q(\mathbb{R}^n)} + \| g \|_{L^p(\mathbb{R}^n)} \| D^s f \|_{L^q(\mathbb{R}^n)} \right),
$$

where $C_s$ is a constant depending on $s$.

**Remark 2.2.** The proof of Lemma 2.1 can be found in [1]. From Sobolev imbedding theorem we know that, if $s > \frac{n}{2}$, then we have

$$
\| f \|_{L^\infty(\mathbb{R}^n)} \leq C_s \| f \|_s,
$$

where $f \in L^\infty \cap H^s$. Then letting $r = 2$ and $p = \infty, q = 2$ in Lemma 2.1, we obtain

$$
\| D^\alpha(fg) \| \leq C_s \| f \|_s \| g \|_s \quad \text{for } s > \frac{n}{2}, \quad f, g \in H^s \text{ and } |\alpha| \leq s.
$$

To obtain the local existence of classical solutions to (2.9)-(2.10), we need some assumptions on the coefficient $K_a$. If there exists a positive constant $M$ such that $\| w^i \|_s \leq M$ ($i = 1, 2$), and we denote $K^i_a = K_a(v, t, x, w^i)$ and $\overline{K}_a = \overline{K}_a(v, t, x, w^i) \geq 0$, then we assume that

$$
\| K_a^i \|_{L^\infty(\mathbb{R}^+; C([0,T];H^s))} + \| \overline{K}_a^i \|_{L^2 \cap L^\infty(\mathbb{R}^+; C([0,T];H^s))} \leq C_s, M \| w^i \|_s,
$$

$$
| K_a(v, t, x, w^1) - \overline{K}_a(v, t, x, w^2) | \leq K(v, t, x) |w^1 - w^2|,
$$

$$
\| K(v, t, x) \|_{L^\infty \cap L^2(\mathbb{R}^+; L^\infty([0,T] \times \mathbb{R}^3))} \leq C_s, M
$$

(2.14)
for any $t \in [0, T]$, where $M$ is a positive constant and $C_{s,M}$ is a positive constant depending only on $s$ and $M$.

Now we give the local existence result.

**Theorem 2.1.** Let $s \geq 3$ be an integer. If the initial data satisfy

$$(I_0, U_0) \in \Phi := \{(I, U)|U(x) \in H^s(\mathbb{R}^3), \ (I(v, \Omega, x) - \overline{B}(v)) \in L^2(\mathbb{R}^+ \times S^2; H^s(\mathbb{R}^3))\},$$

then there exists $T > 0$ such that Cauchy problem (2.9)-(2.10) admits a unique classical solution $(I, U)$ satisfying

$$U(t, x) \in C^1([0, T) \times \mathbb{R}^3),$$

$$I(v, \Omega, t, x) - \overline{B}(v) \in L^2(\mathbb{R}^+ \times S^2; C^1([0, T) \times \mathbb{R}^3)).$$

**Remark 2.3.** The assumptions in Theorem 2.1 for isentropic flows can be satisfied when the absorption coefficient is given by, for example (see [5] or [15]),

$$K_a(v, t, x, \rho) = D_1 \rho \theta^{-\frac{1}{2}} \exp \left(-\frac{D_2}{\theta^{\frac{1}{2}}} \left(\frac{v-v_0}{v_0}\right)^2\right),$$

where $\theta$ is the temperature, $v_0$ is the fixed frequency, $D_i (i = 1, 2)$ are positive constants. For isentropic polytropic gas, we know that $p_m = R \rho \theta = \rho^\gamma$, where $R$ is a positive constant. So we have $w = \rho^{\frac{\gamma - 1}{2}} = \sqrt{R} \theta^{\frac{1}{2}}$, and

$$\lim_{\rho \to 0} \frac{K_a(v, t, x, \rho)}{\rho} = \lim_{\theta \to 0} D_1 \theta^{-\frac{1}{2}} \exp \left(-\frac{D_2}{\theta^{\frac{1}{2}}} \left(\frac{v-v_0}{v_0}\right)^2\right) = 0,$$

$$\lim_{\rho \to +\infty} \frac{K_a(v, t, x, \rho)}{\rho} = \lim_{\theta \to +\infty} D_1 \theta^{-\frac{1}{2}} \exp \left(-\frac{D_2}{\theta^{\frac{1}{2}}} \left(\frac{v-v_0}{v_0}\right)^2\right) = 0.$$

Then in the case (2.15), we have

$$K_a(v, t, w) = D_1 \sqrt{R} w^{\frac{1}{2\gamma-1}} \exp \left(-\frac{D_2 \sqrt{R}}{w} \left(\frac{v-v_0}{v_0}\right)^2\right) = w^{-\frac{2\gamma-1}{2}} K_a(v, t, \cdot, w),$$

$$K_a(v, t, w) = D_1 \sqrt{R} \frac{1}{w} \exp \left(-\frac{D_2 \sqrt{R}}{w} \left(\frac{v-v_0}{v_0}\right)^2\right).$$

When $1 < \gamma \leq 3$, it is easy to verify that the assumptions in Theorem 2.1 are satisfied if $s = 3$.

Now we start proving Theorem 2.1.
Proof. The proof is based on standard energy estimates as well as Banach contraction mapping principle. Let \( j(x) \in C_0^\infty(\mathbb{R}^3) \) be the standard mollifier satisfying

\[
\text{supp} j(x) \subseteq \{ x : |x| \leq 1 \}, \quad \int_{\mathbb{R}^3} j(x) dx = 1, \quad \forall j \geq 0.
\]

Set \( j_\epsilon = \epsilon^{-3} j(\frac{x}{\epsilon}) \) and define \( j_\epsilon u \in C^\infty \) by

\[
j_\epsilon u(x) = \int_{\mathbb{R}^3} j_\epsilon(x - y) u(y) dy.
\]

For \( k = 0, 1, 2, \ldots \), take \( \epsilon_k = 2^{-k} \epsilon_0 \) and

\[
U_0^{(k)} = j_\epsilon U_0^{(0)}(x), \quad I_0^{(k)} = j_\epsilon I_0^{(0)}(v, \Omega, x),
\]

where \( \epsilon_0 \) is to be chosen later. We construct approximate solutions to (2.9)- (2.10) through the following iteration scheme. We take

\[
U^{(0)}(t, x) = U_0^{(0)}(x), \quad I^{(0)}(v, \Omega, t, x) = I_0^{(0)}(v, \Omega, x).
\]

For \( k = 0, 1, \ldots \), we define \( U^{(k+1)}(t, x) \) and \( I^{(k+1)}(v, \Omega, t, x) \) inductively as the solution of the following linearized problem:

\[
\begin{cases}
\frac{1}{c} \partial_t (I^{(k+1)} - \bar{B}(v)) + \Omega \cdot \nabla (I^{(k+1)} - \bar{B}(v)) = -K_a^{(k)} \cdot (I^{(k+1)} - \bar{B}(v)), \\
A_0(U^{(k)}) \partial_t U^{(k+1)} + \sum_{j=1}^{3} A_j(U^{(k)}) \partial x_j U^{(k+1)} = G(I^{(k)}, U^{(k)}), \\
I^{(k+1)}|_{t=0} = I_0^{(k+1)}(v, \Omega, x), \quad U^{(k+1)}|_{t=0} = U_0^{(k+1)}(x),
\end{cases}
\tag{2.18}
\]

where

\[
G_0 = 0, \quad G_j(I^{(k)}, U^{(k)}) = \frac{(\gamma - 1)^2}{4c\gamma} \left( \int_0^\infty \int_{S^2} K_a^{(k)} \cdot (I^{(k)} - \bar{B}(v)) \Omega_j d\Omega d\nu \right),
\]

\[
\bar{K}_a^{(k)} = \bar{K}_a(v, t, x, w^{(k)}), \quad K_a^{(k)} = K_a(v, t, x, w^{(k)}).
\tag{2.19}
\]

It follows immediately that

\[
U^{(k+1)} \in C^\infty([0, T_k] \times \mathbb{R}^3), \quad I^{(k+1)} - \bar{B}(v) \in L^2((0, \infty) \times S^2; C^\infty([0, T_k] \times \mathbb{R}^3))
\tag{2.20}
\]

with \( T_k \) being the largest time of existence for (2.15) where the estimates

\[
\int_0^\infty \int_{S^2} |||I^{(k)} - I_0^{(0)}|||_{s,T_k}^2 d\Omega \leq C \quad \text{and} \quad |||U^{(k)} - U_0^{(0)}|||_{s,T_k} \leq C
\tag{2.21}
\]

are valid for any given constant \( C > 0 \). Hereinafter, \( C \) stands for a generic positive constant.

Of course, in order to get the compactness, we have to guarantee that there exists a \( T > 0 \) such that \( T_k \geq T \) for each \( k \). So the following lemma which gives the uniform estimates of high order norms is very important.
Lemma 2.2 (Boundness of high order norms).

There exist constants $C_1 > 0$ and $T_s > 0$ such that the solutions $(I^{(k)}, U^{(k)})$ $(k = 0, 1, 2, ...)$ to (2.18) satisfy

$$\|U^{(k)} - U^{(0)}\|_{s,T} + \|\partial_t U^{(k)}\|_{s-1,T} + \int_0^\infty \int_{S^2} \|I^{(k)} - I^{(0)}\|^2_{s,T_s} d\omega dv \leq C_1. \quad (2.22)$$

**Proof.** By induction, it is sufficient to prove that (2.22) holds for $(I^{(k+1)}, U^{(k+1)})$ under the assumption that (2.22) holds for $(I^{(k)}, U^{(k)})$. We divide the proof into three steps.

Step 1. The estimate of $\int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(0)}\|^2_{s,T} d\omega dv$.

Let $Q^{(k+1)} = I^{(k+1)} - I^{(0)}$. Then $Q^{(k+1)}$ satisfies

$$\left\{ \begin{align*}
\frac{1}{c} \partial_t Q^{(k+1)} + \Omega \cdot \nabla Q^{(k+1)} &= -K_a^{(k)} Q^{(k+1)} + H^{(k)}, \\
Q^{(k+1)}|_{t=0} &= I^{(k+1)} - I^{(0)},
\end{align*} \right. \quad (2.23)$$

where

$$H^{(k)} = -K_a^{(k)} \cdot (I^{(0)} - \overline{B}(v)) - \Omega \cdot \nabla (I^{(0)} - \overline{B}(v)).$$

Differentiating the equation in (2.23) $\alpha$-times ($|\alpha| \leq s$) with respect to $x$ and multiplying the resulting equation by $D^\alpha Q^{(k+1)}$, we have

$$\frac{1}{2c} \frac{d}{dt} (D^\alpha Q^{(k+1)})^2 + \frac{1}{2} \Omega \cdot \nabla (D^\alpha Q^{(k+1)})^2 = -D^\alpha (K_a^{(k)} Q^{(k+1)}) D^\alpha Q^{(k+1)} + D^\alpha H^{(k)} D^\alpha Q^{(k+1)}. \quad (2.24)$$

Integrating (2.24) with respect to $x$ over $\mathbb{R}^3$ and using the Cauchy’s inequality, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |D^\alpha Q^{(k+1)}|^2 dx \leq C \|D^\alpha Q^{(k+1)}\|^2 + \|D^\alpha (K_a^{(k)} Q^{(k+1)})\|^2 + \|D^\alpha H^{(k)}\|^2 \quad (2.25)$$

$$= : J_1 + J_2 + J_3.$$

According to Lemma 2.1 and (2.13), we get

$$J_2 = \|D^\alpha (K_a^{(k)} Q^{(k+1)})\|^2 \leq C \|K_a^{(k)}\|_s^2 \|Q^{(k+1)}\|_s^2, \quad (2.26)$$

and

$$J_3 = \|D^\alpha H^{(k)}\|^2 = \|D^\alpha (-K_a^{(k)} \cdot (I^{(0)} - \overline{B}(v)) - \Omega \cdot \nabla (I^{(0)} - \overline{B}(v)))\|^2 \leq C \|K_a^{(k)}\|_s^2 \|I^{(0)} - \overline{B}(v)\|^2 + \|I^{(0)} - \overline{B}(v)\|^2_{s+1}. \quad (2.27)$$

Combining (2.14) and (2.25)–(2.27), we arrive at

$$\frac{d}{dt} \|Q^{(k+1)}\|_s^2 \leq C (1 + \|U^{(k)}\|_s) \|Q^{(k+1)}\|_s^2 + \|U^{(k)}\|_s \|I^{(0)} - \overline{B}(v)\|^2 + \|I^{(0)} - \overline{B}(v)\|^2_{s+1}$$

$$\leq C \left( \|Q^{(k+1)}\|_s^2 + \|I^{(0)} - \overline{B}(v)\|^2_s + \|I^{(0)} - \overline{B}(v)\|^2_{s+1} \right).$$

By Gronwall’s inequality we obtain

$$\|Q^{(k+1)}\|^2_{s,T} \leq e^{CT} \left( \|Q^{(k+1)}\|_s^2 + T \|I^{(0)} - \overline{B}(v)\|^2_s + \|I^{(0)} - \overline{B}(v)\|^2_{s+1} \right).$$
It is obvious that
\[ \int_0^\infty \int_{S^2} ||Q^{(k+1)}||^2_{s,T} d\Omega dv \leq e^{CT} \left( \int_0^\infty \int_{S^2} ||Q_0^{(k+1)}||^2_{s,T} d\Omega dv + T \right). \]
Taking \( T = T_1 \) to be small enough, we have
\[ \int_0^\infty \int_{S^2} \|I^{(k+1)} - I_0^{(0)}\|^2_{s,T_1} d\Omega dv \leq CC_1. \] (2.28)

**Step 2.** The estimate of source terms \( \|D^\alpha G(I^{(k)}, U^{(k)})\|, \forall |\alpha| \leq s. \)
Due to Minkowski’s inequality, Holder’s inequality and (2.13), for \( |\alpha| \leq s \), we have
\[
\|D^\alpha G_j(I^{(k)}, U^{(k)})\| \leq C \|D^\alpha \int_0^\infty \int_{S^2} \alpha_k \cdot (I^{(k)} - \mathcal{B}(v))_{\Omega_j} d\Omega dv\|
\leq C \int_0^\infty \int_{S^2} \|\alpha_k\| \|I^{(k)} - \mathcal{B}(v)\|_{s,T} d\Omega dv
\leq C \left( \int_0^\infty \int_{S^2} \|\alpha_k\|^2_{s,T} d\Omega dv + \int_0^\infty \int_{S^2} \|I^{(k)} - \mathcal{B}(v)\|^2_{s,T} d\Omega dv \right),
\]
which implies that
\[ \|G(I^{(k)}, U^{(k)})\| \leq C \left( \int_0^\infty \int_{S^2} \|\alpha_k\|^2_{s,T} d\Omega dv + \int_0^\infty \int_{S^2} \|I^{(k)} - \mathcal{B}(v)\|^2_{s,T} d\Omega dv \right). \] (2.30)
Then from (2.28) and assumptions (2.14), we see that
\[ \|G(I^{(k)}, U^{(k)})\| \leq C \left( \|U^{(k)}\| + \int_0^\infty \int_{S^2} \|I^{(k)} - \mathcal{B}(v)\|^2_{s,T} d\Omega dv \right) \leq C_1. \] (2.31)

**Step 3.** In order to estimate (2.22), define \( M^{(k+1)} = U^{(k+1)} - U_0^{(0)} \), and it is easy to get
\[
\begin{align*}
A_0(U^{(k)}) \partial_t M^{(k+1)} + \sum_{j=1}^3 A_j(U^{(k)}) \partial_{x_j} M^{(k+1)} &= G(I^{(k)}, U^{(k)}) + \mathcal{P}^{(k)},
\end{align*}
\] (2.32)
where
\[ \mathcal{P}^{(k)} = - \sum_{j=1}^3 A_j(U^{(k)}) \partial_{x_j} U_0^{(0)}. \]
With the aid of the steps 1 and 2, it is easy to follow the standard procedure as in [1] and obtain that there exists a time \( T_2 \) such that
\[ \|U^{(k+1)} - U_0^{(0)}\|_{s,T_2} + \|\partial_t U^{(k+1)}\|_{s-1,T_2} \leq C_1. \] (2.33)
Let \( T_s = \min\{T_1, T_2\} \). Then the conclusions in Lemma 2.2 are obtained. \( \square \)

The following lemma implies that the operator associated with \((I^{(k)}, U^{(k)})\) is contracted.
Lemma 2.3. There exist constants $T_* \in [0, T_*)$, $\eta < 1$, $\{\beta_k\} (k = 1, 2, \ldots)$ and $\{\mu_k\} (k = 1, 2, \ldots)$ with $\sum_k |\beta_k| < +\infty$, and $\sum_k |\mu_k| < +\infty$, such that for each $k$

$$
|||U^{(k+1)} - U^{(k)}|||_{0, T_*} + \left( \int_0^\infty \int_{S^2} |||I^{(k+1)} - I^{(k)}|||^2_{0, T_*} d\Omega dv \right)^{\frac{1}{2}}
\leq \eta \left( |||U^{(k)} - U^{(k-1)}|||_{0, T_*} + \left( \int_0^\infty \int_{S^2} |||I^{(k)} - I^{(k-1)}|||^2_{0, T_*} d\Omega dv \right)^{\frac{1}{2}} \right) + \beta_k + \mu_k.
$$

(2.34)

Proof. According to the second equation in (2.18), we have

$$
A_0(U^{(k)})\partial_t(U^{(k+1)} - U^{(k)}) + \sum_{j=1}^3 A_j(U^{(k)})\partial_x_j(U^{(k+1)} - U^{(k)})
= G(I^{(k)}, U^{(k)}) - G(I^{(k-1)}, U^{(k-1)}) + Z^{(k)},
$$

(2.35)

where

$$
Z^{(k)} = -(A_0(U^{(k)}) - A_0(U^{(k-1)}))\partial_t U^{(k)} - \sum_{j=1}^3 (A_j(U^{(k)}) - A_j(U^{(k-1)}))\partial_x_j U^{(k)}.
$$

From Lemma 2.2 and Taylor’s expansion, we easily deduce that, $\forall \tau \in [0, T_*)$,

$$
|||Z^{(k)}|||_{0, \tau} \leq C|||U^{(k)} - U^{(k-1)}|||_{0, \tau}.
$$

(2.36)

According to assumptions (2.14), and by Holder’s inequality and Minkowski’s inequality, we find that

$$
|||G_0(I^{(k)}, U^{(k)}) - G_0(I^{(k-1)}, U^{(k-1)})|| = 0,
$$

(2.37)

and for $j = 1, 2, 3$, $\forall \tau \in [0, T_*)$,

$$
|||G_j(I^{(k)}, U^{(k)}) - G_j(I^{(k-1)}, U^{(k-1)})(v, \Omega, \tau, \cdot)||
\leq C \int_0^\infty \int_{S^2} |||I^{(k-1)} - I^{(k-1)}|||^2_{0, T_*} d\Omega dv
\leq C_{s,M} |||U^{(k)} - U^{(k-1)}||| \int_0^\infty \int_{S^2} |||I^{(k-1)}(v, \Omega, \tau, \cdot) - \overline{B}(v)|||_{L^\infty(\mathbb{R}^3)} |||K(v, \tau, \cdot)|||_{L^\infty(\mathbb{R}^3)} d\Omega dv
+ C \int_0^\infty \int_{S^2} |||K|||_{L^2(\mathbb{R}^3, L^\infty([0,T] \times \mathbb{R}^3))} \left( \int_0^\infty \int_{S^2} |||I^{(k-1)} - I^{(k-1)}|||^2_{0, T_*} d\Omega dv \right)^{\frac{1}{2}}
\leq C_{s,M} |||U^{(k)} - U^{(k-1)}||| |||K|||_{L^2(\mathbb{R}^3, L^\infty([0,T] \times \mathbb{R}^3))} \left( \int_0^\infty \int_{S^2} |||I^{(k)} - I^{(k-1)}|||^2_{0, T_*} d\Omega dv \right)^{\frac{1}{2}}
\leq C \left( |||U^{(k)} - U^{(k-1)}|||_{0, \tau} + \left( \int_0^\infty \int_{S^2} |||I^{(k)} - I^{(k-1)}|||^2_{0, \tau} d\Omega dv \right)^{\frac{1}{2}} \right).
$$

(2.38)
Applying the standard energy estimates to (2.35) and using (2.36)-(2.38), we easily get

\[ \|U^{(k+1)} - U^{(k)}\|_{0,\tau} \leq e^{CT}\|U^{0}_{0} - U^{(k)}\| + e^{CT} \left( \|Z\|_{0,\tau} + \|G(I^{(k)}, U^{(k)}) - G(U^{(k-1)}, I^{(k-1)})\|_{0,\tau} \right). \]  

(2.39)

According to the properties of mollifier, for \( \epsilon_{0} \) small enough, we know that

\[ \|J_{\epsilon}u - u\| \leq C\epsilon\|u\|_{1}, \quad \forall u \in H^{1}, \quad \epsilon \leq \epsilon_{0}. \]

So if we take \( \epsilon_{0} << 1 \), then we have

\[ \|U^{(k+1)}_{0} - U^{(k)}_{0}\| \leq C2^{-k}\|U^{0}_{0}\|. \]  

(2.40)

From (2.39), if we choose \( T_{3} \in [0, T_{*}] \) to be small enough, then it is easy to get

\[ \|U^{(k+1)} - U^{(k)}\|_{0, T_{3}} \leq \eta_{1} \left( \|U^{(k)} - U^{(k-1)}\|_{0, T_{3}} + \left( \int_{0}^{\infty} \int_{\mathbb{S}^{2}} \|I^{(k)} - I^{(k-1)}\|^{2}_{0, T_{3}} d\Omega dv \right)^{1/2} \right) + \beta_{k}, \]

(2.41)

where \( \eta_{1} < \frac{1}{2} \) and \( \sum_{k} |\beta_{k}| < \infty. \)

To bound \( I^{(k+1)} - I^{(k)} \), we use the first equation in (2.18) to see that

\[ \frac{1}{c} \partial_{t} (I^{(k+1)} - I^{(k)}) + \Omega \cdot \nabla (I^{(k+1)} - I^{(k)}) = (\overline{B}(v) - I^{(k)})(K^{(k)} - K^{(k-1)}) - K^{(k)} \cdot (I^{(k+1)} - I^{(k)}). \]  

(2.42)

It is easy to show that, \( \forall \tau \in [0, T_{*}] \),

\[ \frac{d}{dt} \|I^{(k+1)} - I^{(k)}\|^{2} \leq \|I^{(k)} - \overline{B}(v)\|_{s}\|K^{(k)} - K^{(k-1)}\|\|I^{(k+1)} - I^{(k)}\|, \]

(2.43)

where we used the fact that \( K_{a} \geq 0. \) From (2.7), Lemma 2.2 and assumptions (2.14), we have

\[ |K_{a}(v, t, x, w^{(k)}) - K_{a}(v, t, x, w^{(k-1)})| \]

\[ \leq K(v, t, x)|w^{(k-1)} - w^{(k)}| \quad \leq C|w^{(k)} - w^{(k-1)}|. \]  

(2.44)

Then using Young’s inequality, we get

\[ \|I^{(k)} - \overline{B}(v)\|_{s}\|K^{(k)} - K^{(k-1)}\|\|I^{(k+1)} - I^{(k)}\| \]

\[ \leq C\|U^{(k)} - U^{(k-1)}\|^{2}\|I^{(k)} - \overline{B}(v)\|^{2} + \|I^{(k+1)} - I^{(k)}\|^{2}. \]  

(2.45)

Combining (2.43)-(2.45), we have

\[ \int_{0}^{\infty} \int_{\mathbb{S}^{2}} \|I^{(k+1)} - I^{(k)}\|^{2}_{0, \tau} d\Omega dv \]

\[ \leq e^{CT} \left( \int_{0}^{\infty} \int_{\mathbb{S}^{2}} \|I^{(k+1)} - I^{(k)}\|^{2}_{0, \tau} d\Omega dv + C\tau\|U^{(k)} - U^{(k-1)}\|^{2}_{0, \tau} \right). \]  

(2.46)
Similarly to the estimate of $\|U_0^{(k+1)} - U_0^{(k)}\|$, we easily get
$$
\left( \int_0^\infty \int_{S^2} \|I_0^{(k+1)} - I_0^{(k)}\|^2 d\Omega dv \right)^{\frac{1}{2}} \leq C 2^{-k} \left( \int_0^\infty \int_{S^2} \|I_0\|^2 d\Omega dv \right)^{\frac{1}{2}}.
$$
If we choose $T_4 \in [0, T_\ast]$ to be small enough, then we have
$$
\left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2 d\Omega dv \right)^{\frac{1}{2}} \leq \eta_2 \|U^{(k)} - U^{(k-1)}\|_{I_0, T_4 + \mu_k}, \quad (2.47)
$$
where $\eta_2 < \frac{1}{2}$ and $\sum_k |\mu_k| < \infty$. Finally, taking $T_\ast = \min\{T_3, T_4\}$, we obtain Lemma 2.3 by adding (2.41) and (2.47) together.

Now we continue to prove Theorem 2.1.

Lemma 2.3 tells us that
$$
\sum_{k=1}^\infty \|I^{(k+1)} - I^{(k)}\|_{0, T_\ast} + \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{0, T_\ast} d\Omega dv \right)^{\frac{1}{2}} < +\infty,
$$
which implies that
$$
\begin{align*}
\lim_{k \to \infty} \|U^{(k+1)} - U^{(k)}\|_{0, T_\ast} &= 0, \\
\lim_{k \to \infty} \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{0, T_\ast} d\Omega dv \right)^{\frac{1}{2}} &= 0. \quad (2.48)
\end{align*}
$$
In addition, from Lemma 2.2 we know that sequence $\{U^{(k)}(t, \cdot)\} \subset \subset \Phi$ for any fixed $t$ and
$$
\|U^{(k)}\|_{s, T_\ast} + \|\partial_t U^{(k)}\|_{s-1, T_\ast} \leq 2C_1. \quad (2.49)
$$
Then from Sobolev interpolation inequalities, we have
$$
\|U^{(k+1)} - U^{(k)}\|_{s'} \leq C \|U^{(k+1)} - U^{(k)}\|_{1 - \frac{s'}{2}} \|U^{(k+1)} - U^{(k)}\|_{\frac{s'}{2}}. \quad (2.50)
$$
for any $0 < s' < s$. So from (2.49) and (2.50), we get
$$
\|U^{(k+1)} - U^{(k)}\|_{s', T_\ast} \leq C \|U^{(k+1)} - U^{(k)}\|_{1 - \frac{s'}{2}, T_\ast}, \quad \text{for any } 0 < s' < s.
$$
Similarly, using Sobolev interpolation inequalities and Hölder’s inequality, we get
$$
\begin{align*}
\int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{s', T_\ast} d\Omega dv &\leq C \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|_{1 - \frac{s'}{2}}^2 \|I^{(k+1)} - I^{(k)}\|_{\frac{s'}{2}, T_\ast}^2 d\Omega dv \\
&\leq C \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{0, T_\ast} d\Omega dv \right)^{\frac{s-s'}{s}} \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{s, T_\ast} d\Omega dv \right)^{\frac{s'}{s}} \\
&\leq C \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{0, T_\ast} d\Omega dv \right)^{\frac{s-s'}{s}}
\end{align*}
$$
for any $0 < s' < s$. 

\[\Box\]
According to (2.48), we conclude that
\[
\begin{aligned}
\lim_{k \to \infty} \|U^{(k+1)} - U^{(k)}\|_{L^2(T_{**})} &= 0, \\
\lim_{k \to \infty} \left( \int_0^\infty \int_{S^2} \|I^{(k+1)} - I^{(k)}\|^2_{L^2(T_{**})} d\Omega dv \right)^{\frac{1}{2}} &= 0
\end{aligned}
\]  
(2.51)
for any \(0 < s' < s\).

Therefore, if we choose \(s' > \frac{5}{2}\), then from Sobolev embedding theorem, there exists \((I, U)\) such that

\[
U^{(k)} \to U \in C \left( [0, T_{**}]; C^1(\mathbb{R}^3) \right), \quad I^{(k)} \to I \in L^2 \left( \mathbb{R}^+ \times S^2; C([0, T_{**}]; C^1(\mathbb{R}^3)) \right).
\]

Furthermore, from the second equation in (2.18), we have

\[
\partial_t U^{(k+1)} = -A_0^{-1}(U^{(k)}) \sum_{j=1}^3 A_j(U^{(k)}) \partial_{x_j} U^{(k+1)} + A_0^{-1}(I^{(k)}) G(I^{(k)}) U^{(k)},
\]
then \(\partial_t U^{(k+1)} \to \partial_t U \in C \left( [0, T_{**}] \times \mathbb{R}^3 \right)\). Similarly, from the first equation in (2.18), we easily have \(\partial_t I \in L^2 \left( \mathbb{R}^+ \times S^2; C([0, T_{**}] \times \mathbb{R}^3) \right)\). Thus, \((I, U)\) is a classical solution to (2.9)-(2.10).

Finally, we consider the uniqueness of classical solutions. Let \((I, U) = (I, w, u)\) and \((\tilde{I}, \tilde{U}) = (\tilde{I}, \tilde{w}, \tilde{u})\) be two classical solutions to (2.9)-(2.10). From the proof of Lemma 2.3, we have

\[
\begin{aligned}
&\left\{ \begin{aligned}
\frac{1}{c} \partial_t (I - \tilde{I}) + \Omega \cdot \nabla (I - \tilde{I}) \\
&= (\tilde{I} - \overline{B}(v))(K_a(v, t, x, \tilde{w}) - K_a(v, t, x, w)) - K_a(v, t, x, w) \cdot (I - \tilde{I}).
\end{aligned} \right. \\
&\quad + A_0(U) \partial_t (U - \tilde{U}) + \sum_{j=1}^3 A_j(U) \partial_{x_j} (U - \tilde{U}) = G(I, U) - G(\tilde{I}, \tilde{U}) + Z,
\end{aligned}
\]

(2.52)
where

\[
Z = (A_0(\tilde{U}) - A_0(U)) \partial_t \tilde{U} + \sum_{j=1}^3 (A_j(\tilde{U}) - A_j(U)) \partial_{x_j} \tilde{U}.
\]

Similarly to the proof of Lemma 2.3, we can prove that \((I, U) = (\tilde{I}, \tilde{U})\).

The proof of Theorem 2.1 is finished. \(\square\)

**Remark 2.4.** By the standard method in Majda [1], the classical solution obtained in the above theorem also satisfies

\[
U(t, x) = (w(t, x), u(t, x)) \in C^1 \left( [0, T]; H^s(\mathbb{R}^3) \right) \cap C \left( [0, T]; H^{s-1}(\mathbb{R}^3) \right),
\]
\[
I(v, \Omega, t, x) - \overline{B}(v) \in L^2 \left( \mathbb{R}^+ \times S^2; C^1([0, T]; H^s(\mathbb{R}^3)) \right) \cap C([0, T]; H^{s-1}(\mathbb{R}^3)).
\]
Back to the Cauchy problem (2.4)-(2.5), we will give the local existence and uniqueness of regular solutions based on the above results for classical solutions to Cauchy problem (2.9)-(2.10).

2.3. Local existence and uniqueness of regular solutions to (2.4)-(2.5).

In this section, we will give the local existence and uniqueness of regular solutions to the original Cauchy problem (2.4)-(2.5) based on the results obtained in Section 2.2.

Theorem 2.2. Let $s \geq 3$ be an integer. If the initial data satisfy

$$
(I_0, \rho_0, u_0) \in \Psi := \{(I, \rho, u) | \rho(x) \geq 0; (\rho^{\frac{s-1}{2}}, u_x)(x) \in H^s(\mathbb{R}^3),
\quad I(v, \Omega, x) - \overline{B}(v) \in L^2(\mathbb{R}^+ \times S^2; H^s(\mathbb{R}^3)) \},
$$

then there exists a time $T > 0$ such that Cauchy problem (2.4)-(2.5) admits a unique regular solution $(I, \rho, u)$.  

Proof. From Theorem 2.1, we know that there exists $T > 0$ such that Cauchy problem (2.9)-(2.10) has a unique classical solution $(I, U)$ satisfying

$$
U = (w, u) \in C^1 \left([0, T) \times \mathbb{R}^3\right), \quad I \in L^2 \left(\mathbb{R}^+ \times S^2; C^1([0, T) \times \mathbb{R}^3)\right).
$$

(2.53)

According to transformation (2.7), since $\rho(t, x) = w^{\frac{2}{\gamma-1}}$ and $\frac{2}{\gamma-1} \geq 1$ due to $1 < \gamma \leq 3$, it is easy to show that $(\rho, u)(t, x) \in C^1 \left([0, T) \times \mathbb{R}^3\right)$.

Multiplying (2.8) by $\frac{\partial \rho}{\partial w} = \frac{2}{\gamma-1}w^{\frac{4}{\gamma-1}} \in C \left([0, T) \times \mathbb{R}^3\right)$, we get

$$
\partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0,
$$

(2.54)

which is exactly the continuity equation in (2.4). Multiplying (2.8) by $\frac{4\gamma}{(\gamma-1)^2}w^{\frac{4}{\gamma-1}} \in C^1 \left([0, T) \times \mathbb{R}^3\right)$, we get the momentum equations in (2.4):

$$
\rho \partial_t u + \rho u \cdot \nabla u + \nabla p_m = \frac{1}{c} \int_0^\infty \int_{S^2} K_a \cdot (I - \overline{B}(v)) \Omega d\Omega dv.
$$

(2.55)

That is to say, $(I, \rho, u)$ satisfies the Euler-Boltzmann equations classically. Then from the continuity equation, it is easy to get that $\rho$ can be expressed by

$$
\rho(t, x) = \rho_0(X(0, 0, x)) \exp \left(- \int_0^t \text{div}(s, X(s, t, x)) ds\right) \geq 0,
$$

(2.56)

where $X \in C \left([0, T] \times [0, T] \times \mathbb{R}^3\right)$ is the solution of the initial value problem

$$
\left\{
\begin{array}{ll}
\frac{d}{dt}X(t, s, x) = u(t, X(t, s, x)), & 0 \leq t \leq T, \\
X(s, s, x) = x, & 0 \leq s \leq T, \quad x \in \mathbb{R}^3,
\end{array}
\right.
$$

(2.57)

In conclusion, Cauchy problem (2.4)-(2.5) has a unique regular solution $(I, \rho, u)$. □
3. Finite time Blow-up of regular solutions

In this section, we consider the formation of singularities to regular solutions obtained in Section 2.3. We first assume that the initial data (2.5) satisfy the following local vacuum state condition:

**Definition 3.1 (Local vacuum state).**

Let $A_0$ and $B_0$ be two bounded open sets in $\mathbb{R}^3$, $B_0$ is connected and $\overline{A_0} \subset B_0 \subseteq B_{R_0}$, where $R_0$ is a positive constant and $B_{R_0} := \{x \in \mathbb{R}^3 : |x| \leq R_0\}$. If the initial data $(I_0, \rho_0, u_0)$ satisfy

\[
\begin{align*}
\rho_0(x) &= u_0(x) = 0, \quad \forall x \in B_0 - A_0; \quad \int_{A_0} \rho_0(x) \, dx = m_0 > 0, \\
I_0 &\geq \overline{B}(v), \quad \forall (v, \Omega, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{R}^3, \\
I_0 &\equiv \overline{B}(v), \quad \text{if} \quad |x| \geq R_0 \text{ or } x \cdot \Omega \leq 0,
\end{align*}
\]

(3.1)

then we say that the initial data $(I_0, \rho_0, u_0)$ contain local vacuum state.

**Remark 3.1.** $\overline{B}(v)$ is actually a simplification of the Planck function which represents the energy density of black-body radiation. Black-body has the smallest radiation, so condition $I_0 \geq \overline{B}(v)$ is natural. In Theorem 3.1, we will see that the assumption $I_0 \equiv \overline{B}(v)$ for $|x| \geq R_0$ results in the phenomenon that the impact of radiation on the dynamical properties of the fluid vanishes in the far field, then the system serves as the Euler equations as $|x| \to +\infty$.

In order to observe the evolution of $A_0$ and $B_0$, we need the following definition.

**Definition 3.2 (Particle path and flow map).**

Let $x(t; x_0)$ be the particle path starting from $x_0$ at $t = 0$, i.e.,

\[
\frac{d}{dt}x(t; x_0) = u(t, x(t; x_0)), \quad x(0; x_0) = x_0.
\]

(3.2)

Then we denote by $A(t)$, $B(t)$, $(B - A)(t)$ the images of $A_0$, $B_0$, and $B_0 - A_0$, respectively, under the flow map of (3.2), i.e.,

\[
\begin{align*}
A(t) &= \{x(t; x_0) | x_0 \in A_0\}, \quad B(t) = \{x(t; x_0) | x_0 \in B_0\}, \\
(B - A)(t) &= \{x(t; x_0) | x_0 \in (B_0 - A_0)\}.
\end{align*}
\]

It is easy to know that $(B - A)(t)$ is the vacuum domain.

Then we have
Lemma 3.1. Let \((I, \rho, u)\) be the regular solution on \(\mathbb{R}^+ \times S^2 \times [0, T) \times \mathbb{R}^3\) of the Cauchy problem (2.4)-(2.5) satisfying (3.1), then we have

\[
I(v, \Omega, t, x) \geq B(v), \quad (v, \Omega, t, x) \in \mathbb{R}^+ \times S^2 \times [0, T) \times \mathbb{R}^3; \tag{3.3}
\]

\[
I(v, \Omega, t, x) \equiv B(v), \quad \text{if } |x| \geq R_0 + ct \text{ or } x \cdot \Omega \leq 0; \tag{3.4}
\]

\[
B(t) = B_0, \quad A(t) = A_0, \quad \text{for } t \in [0, T). \tag{3.5}
\]

Proof. Firstly, because \(B(v)\) is independent of \(x\) and \(t\), the first equation of system (2.4) can be rewritten as

\[
\frac{1}{c} \partial_t (I - \overline{B}(v)) + \Omega \cdot \nabla (I - \overline{B}(v)) = -K_a \cdot (I - \overline{B}(v)).
\]

We denote by \(y(t; y_0)\) the photon path starting from \(y_0\) at \(t = 0\), i.e.,

\[
\frac{d}{dt} y(t; y_0) = c\Omega, \quad y(0; y_0) = y_0.
\]

Along the photon path, we obtain

\[
(I - \overline{B}(v))(t, y(t; y_0)) = (I_0 - \overline{B}(v))(y_0) \exp \left( \int_0^t -cK_a(v, \tau, y(\tau; y_0), \rho) d\tau \right), \tag{3.6}
\]

where \(y_0 = y - c\Omega t\).

Then from the initial conditions, it is easy to have

\[
I(v, \Omega, t, x) \geq \overline{B}(v), \quad \forall (v, \Omega, t, x) \in \mathbb{R}^+ \times S^2 \times [0, T] \times \mathbb{R}^3.
\]

\[
I(v, \Omega, t, x) \equiv \overline{B}(v), \quad \text{for } |x| \geq R_0 + ct.
\]

If \(I_0(v, \Omega, x) \equiv \overline{B}(v)\) for \(x \cdot \Omega \leq 0\), we can choose any point \((t, x) \in [0, T] \times \mathbb{R}^d\) satisfying \(x \cdot \Omega \leq 0\), then along the photon path

\[
x_0 \cdot \Omega = (x - c\Omega t) \cdot \Omega = x \cdot \Omega - c|\Omega|^2 t \leq x \cdot \Omega \leq 0.
\]

Due to (3.6), it yields

\[
I(t, x, v, \Omega) \equiv \overline{B}(v), \quad \text{for } x \cdot \Omega \leq 0.
\]

Secondly, on the domain \((B - A)(t)\), \(\rho = \overline{K}_a(v, t, x, \rho) \equiv 0\). Due to the momentum equations in (2.8) and the definition of regular solutions, we have

\[
\partial_t u + u \cdot \nabla u = 0, \quad \text{in } (B - A)(t). \tag{3.7}
\]

That is to say, \(u\) is invariant along the particle path. Thus, according to the local vacuum state condition, we have

\[
u(t, x) \equiv 0, \quad \text{in } (B - A)(t).
\]
Using the continuity of \( u(t, x) \), we get
\[
d\frac{d}{dt} x(t; x_0) = u(t, x(t; x_0)) \equiv 0, \quad x_0 \in \partial B_0 \bigcup \partial A_0,
\]
so \( x(t; x_0) \equiv x_0 \). Thus \( B(t) = B_0, \ A(t) = A_0 \).

Now we give the main result of this section, which shows the formation of singularities caused by the appearance of vacuum in some local domain. We first introduce the mass and second moment over \( B(t) \):
\[
m(t) = \int_{B(t)} \rho(t, x) dx \quad \text{(mass)},
\]
\[
M(t) = \int_{B(t)} \rho(t, x)|x|^2 dx \quad \text{(second moment)}.
\]

**Theorem 3.1 (Cauchy problem).**

Assume that \((I, \rho, u)\) is the regular solution on \( \mathbb{R}^+ \times S^2 \times [0, T) \times \mathbb{R}^3 \) of the Cauchy problem (2.4)-(2.5) satisfying (3.1), then it will blow up in finite time, i.e., \( T < +\infty \).

**Proof.** According to Lemma 3.1, we know that \( B(t) = B_0 \). So we easily have
\[
m(t) = m_0, \ \forall \ t \in [0, T).
\]
(3.8)

From the continuity equation and integration by parts, we have
\[
\frac{d}{dt} M(t) = 2 \int_{B_0} x \cdot \rho u dx.
\]
(3.9)

From the momentum equations and integration by parts, we get
\[
\frac{d^2}{dt^2} M(t) = 2 \int_{B_0} (\rho |u|^2 + 3 \rho m) dx + \frac{2}{c} \int_{B_0} \int_0^\infty \int_{S^2} K_a \cdot (I - \overline{B}(v)) x \cdot \Omega d\Omega dv dx
\]
(3.10)

From Lemma 3.1, we know that
\[
\frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} K_a \cdot (I - \overline{B}(v)) x \cdot \Omega d\Omega dv dx
\]
\[
= \frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2 \cap \{x \cdot \Omega \geq 0\}} K_a \cdot (I - \overline{B}(v)) x \cdot \Omega d\Omega dv dx
\]
\[
+ \frac{1}{c} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2 \cap \{x \cdot \Omega < 0\}} K_a \cdot (I - \overline{B}(v)) x \cdot \Omega d\Omega dv dx \geq 0.
\]
(3.11)

It follows from (3.10)-(3.11) that
\[
\frac{d^2}{dt^2} M(t) \geq 2 \int_{B_0} (\rho |u|^2 + 3 \rho m) dx \geq 6 \int_{B_0} \rho m dx.
\]
(3.12)

From Holder’s inequality, we give
\[
m_0 = \int_{B_0} \rho(t, x) dx \leq \left( \int_{|x| \leq R_0} \rho^\gamma(t, x) dx \right)^{\frac{1}{\gamma}} \left( \int_{|x| \leq R_0} dx \right)^{\frac{1}{\gamma}},
\]
(3.13)
where \( \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \). Associated with (3.8), we give
\[
\int_{|x| \leq R_0} p_m(t, x) \, dx \geq m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}.
\] (3.14)
Then (3.12) yields
\[
\frac{d^2}{dt^2} M(t) \geq 6m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}.
\] (3.15)
So, using Taylor’s expansion, we have
\[
M(t) \geq M(0) + M'(0)t + 3m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}t^2.
\] (3.16)
From Lemma 3.1 it is clear that
\[
M(t) \leq m_0 R_0^2, \quad \forall t \in [0, T).
\] (3.17)
Combining (3.16)–(3.17), we have
\[
m_0 R_0^2 \geq M(0) + M'(0)t + 3m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}t^2.
\] (3.18)
Solving this inequality, we get
\[
t \leq \frac{-M'(0) + \sqrt{M'(0)^2 - 12m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}(M(0) - m_0 R_0^2)}}{6m_0^\gamma R_0^{3(1-\gamma)} |B_1|^{1-\gamma}}.
\] (3.19)
In other words, the life span \( T \) must be finite. \( \square \)

**Remark 3.2.** Theorem 3.1 stated that the appearance of vacuum will cause the blow-up of the regular solutions of Euler-Boltzmann equations in finite time. However, this kind of singularity is different from the shock wave which is caused by the compression of fluid. The corresponding results for Euler equations, damped Euler equations, and Euler-Possion equations can be found in [10], [11], [12], [16], [17], etc. Moreover, the result obtained in Theorem 3.1 also improved the conclusion in [10] [12] in the sense that we removed the crucial assumption that the initial mass density is compactly supported.

The similar blow-up estimate can be extended to the initial-boundary problem in a smooth and bounded domain \( \Xi \subset \mathbb{R}^3 \) under some suitable boundary condition. \( \forall T > 0 \), we assume that a solution \( (I, \rho, u) \) of the initial-boundary problem is regular if
\[
(i) I(v, \Omega, t, x) \in L^2(\mathbb{R}^+ \times S^2; C^1([0, T] \times \Xi)), \quad (\rho, u)(t, x) \in C^1([0, T] \times \Xi),
\]
\[
(ii) \rho^{\gamma - 1} \in C^1([0, T] \times \Xi), \quad \rho \geq 0, \quad \text{and} \quad \partial_t u + u \cdot \nabla u = 0
\] (3.20)
holds in the exterior of \( \text{supp}\rho \).
Corollary 3.1 (The initial-boundary value problem).

We assume that $B_0 \subseteq \Xi$ satisfying $\text{dist}(B_0, \Xi) > 0$. Then there is no global regular solution $(I, \rho, u)$ to the initial-boundary value problem for the compressible Euler-Boltzmann equations (2.4) with initial data (2.5) satisfying (3.1) and suitable boundary condition.

Proof. The key point now is that we have to make sure that the flow map $B(t)$ of $B_0$ cannot reach the boundary of $\Xi$. Then we can use the same analysis for Cauchy problem in Lemma 3.1 and Theorem 3.1 to get the corresponding blow-up result for initial-boundary value problem. Let $T$ be the life span of the regular solution $(I, \rho, u)$ and $T^* = \inf\{t \in [0, T) | \text{dist}(B(t), \partial \Xi) = 0, B(t) \subseteq \Xi; \text{dist}(B(\tau), \partial \Xi) > 0, B(\tau) \subseteq \Xi, \forall \tau \in [0, t]\}$

We divide our proof into three steps.

Step 1. We claim that there exists a positive lower bound $\epsilon > 0$ such that $T^* \geq \epsilon$. In fact, without loss of generality, we assume that $T > 1$. Then from the definition of particle path $x(t; x_0)$, we have

$$|x(t; x_0) - x_0| \leq t \|u(t, x)\|_{L^\infty([0,1] \times \Xi)}, \forall t \in [0, 1], x_0 \in \partial B_0. \quad (3.21)$$

If we let $T_5 \in (0, 1]$ be small enough such that

$$T_5 \|u(t, x)\|_{L^\infty([0,1] \times \Xi)} < \text{dist}(B_0, \partial \Xi),$$

then we know that

$$\text{dist}(B(\tau), \partial \Xi) > 0, B(\tau) \subseteq \Xi, \forall \tau \in [0, T_5].$$

So, we get that $T^* \geq \epsilon = T_5 > 0$.

Step 2. We claim that a finite $T^*$ does not exist. In fact, if there exists a finite $T^*$ such that $\epsilon \leq T^* < T$, then due to the definition of $T^*$, we have

$$\text{dist}(B(\tau), \partial \Xi) > 0, B(\tau) \subseteq \Xi, \forall \tau \in [0, T^*).$$

Via the same analysis as in Lemma 3.1, we have

$$B(t) = B_0 \subset \Xi, 0 \leq t < T^*, \quad (3.22)$$

which contradicts with the definition of $T^*$.

Step 3. Now we show that the life span $T$ of regular solutions is finite, i.e., $T < +\infty$. From Step 2, we know that

$$B(t) \subseteq \Xi, \text{dist}(B(t), \partial \Xi) > 0, 0 \leq t < T. \quad (3.23)$$
Using again the same analysis in Lemma 3.1, we have

\[ B(t) = B_0 \subset \subset \Xi, \quad 0 \leq t < T. \] (3.24)

Therefore, we can handle the initial-boundary value problem in the same way as the Cauchy problem. In other words, we can introduce the same functionals \( m(t) \) and \( M(t) \) to prove the finiteness of \( T \) accordingly. We omit the details here. \( \Box \)

4. Local Existence for the case \( \sigma_s \neq 0 \)

In this section, we will give the corresponding local existence of regular solutions for the case \( \sigma_s \neq 0 \), which is similar to the result obtained in Section 2, and we use the same notation as in Section 2. When \( \sigma_s \neq 0 \), if we still consider the assumptions of ‘induced process’ and local thermal equilibrium as in Section 2 for the case \( \sigma_s = 0 \), then the Euler-Boltzmann equations (1.1) become very complicated, for example, the radiation transfer equation in (1.1) reads as

\[
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I = -K_a \cdot (I - \overline{B}(v)) + \Pi,
\]

where

\[
\Pi = \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s I' \left( 1 + \frac{c^2 I'}{2hv'^3} \right) - \sigma_a' I \left( 1 + \frac{c^2 I}{2hv^3} \right) \right) d\Omega' dv',
\]

which is rather complicated and hard to deal with. Therefore, for simplicity, we start from the original Euler-Boltzmann equations (1.1). For this, we need some assumptions. Let

\[
\sigma_s = \rho \overline{\sigma}_s (v' \rightarrow v, \Omega' \cdot \Omega), \quad \sigma_a' = \rho \overline{\sigma}_a' (v \rightarrow v', \Omega \cdot \Omega'),
\]

where \( \overline{\sigma}_s \geq 0 \) and \( \overline{\sigma}_s' \geq 0 \) satisfy

\[
\int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \frac{|v|}{v'} \left| \overline{\sigma}_s \right|^2 d\Omega' dv' \right)^{\lambda_1} d\Omega dv \leq C,
\]

\[
\int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \sigma_a' d\Omega d\Omega' \right)^{\lambda_2} d\Omega dv + \int_0^\infty \int_{S^2} \overline{\sigma}_s' d\Omega d\Omega' \leq C,
\]

where \( \lambda_1 = 1 \) or \( \frac{1}{2} \), and \( \lambda_2 = 1 \) or 2. Let

\[
S = S(v, t, x, \rho) = \rho \overline{S}(v, t, x, \rho) = \rho \overline{\sigma}, \quad \overline{S} \geq 0,
\]

\[
\sigma_a = \sigma_a(v, t, x, \rho) = \rho \overline{\sigma}_a(v, t, x, \rho) = \rho \overline{\sigma}_a, \quad \overline{\sigma}_a \geq 0,
\]

(4.1)
and for any \( \| w(t, \cdot) \|_s \leq M \) (here \( w = \rho^{\frac{n+1}{2}} \), \( M \) is a positive constant), we assume

\[
\| \mathcal{S} \|_{L^1(\mathbb{R}^+; C([0,T]; H^s))} + \| \mathcal{S} \|_{L^2(\mathbb{R}^+; C([0,T]; H^s))} \leq C_{s,M} \| w \|_s,
\]

\[
\| \sigma_a \|_{L^\infty(\mathbb{R}^+; C([0,T]; H^s))} + \| \sigma_a \|_{L^2(\mathbb{R}^+; C([0,T]; H^s))} \leq C_{s,M} \| w \|_s,
\]

\[
\| (\partial_w \mathcal{S}) + (\partial_w \mathcal{S})(v, t, x, w) \|_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^+; L^\infty(\mathbb{R})))} \leq C_{s,M}, \quad \forall t \in [0, T].
\]

**Remark 4.1.** The evaluation of these physical coefficients is a difficult problem in quantum mechanics and their general form is not known. Some similar assumptions and physical examples on \( S \), \( \sigma_a \) and \( \sigma_s \) can be found in Remarks 2.2 and 3.1 of [5] as well as in [15]. When \( \gamma \leq 1 + \frac{2}{s} \), that is \( \frac{2}{\gamma - 1} \geq s \), (4.3) is similar to the assumptions in [5] for the local existence of classical solutions to Cauchy problem (1.1) and (2.5) with initial mass density away from vacuum.

Similarly, in order to change (4.1) into a symmetric hyperbolic system, we introduce the new variable

\[
w = \rho^{\frac{n-1}{2}} = \rho^{\frac{n-1}{2}},
\]

and denote \( U = U(t, x) = (w(t, x), u(t, x))^\top \). Then system (4.1) of the isentropic Euler-Boltzmann equations can be reduced to the following system:

\[
\begin{aligned}
\frac{1}{c} \partial_t I + \Omega \cdot \nabla I &= A_r, \\
A_0(U) \partial_t U + \sum_{j=1}^3 A_j(U) \partial_{x_j} U &= F(I, U),
\end{aligned}
\tag{4.4}
\]

where \( A_0(U) \) and \( A_j(U) \) are defined in Section 2, and

\[
F(I, U) = (F_0, F_1, F_2, F_3), \quad F_0(I, U) = 0,
\]

\[
F_j(I, U) = -\frac{(\gamma - 1)^2}{4c\gamma} \int_0^\infty \int_{S^2} \left( \mathcal{S} - \sigma_a I + \int_0^\infty \int_{S^2} \left( \frac{v'}{v} \sigma_a I' - \sigma_a I \right) d\Omega' dv' \right) \Omega_j d\Omega dv
\]

\( (j = 1, 2, 3) \).

Similarly to Theorem 2.1, in order to get the local existence of the original Cauchy problem (1.1) and (2.5), we need the following key theorem.

**Theorem 4.1.** Let \( s \geq 3 \) be an integer and (4.2)-(4.4) hold. If the initial data satisfy

\[
(I_0, U_0) \in \Psi := \{(I, U)|U(x) \in H^s(\mathbb{R}^3), I(v, \Omega, x) \in L^2(\mathbb{R}^+ \times S^2; H^s(\mathbb{R}^3))\},
\]

then there exists \( T > 0 \) such that the problem (4.4) and (2.10) has a unique classical solution \((I, U)\) satisfying

\[
U \in C^1([0, T) \times \mathbb{R}^3), \quad I \in L^2(\mathbb{R}^+ \times S^2; C^1([0, T) \times \mathbb{R}^3)).
\]
We can follow the same procedure as the proof of Theorem 2.1 to prove Theorem 4.1. For $k = 0, 1, 2, \ldots$, we define $U^{(k+1)}(t, x)$ and $I^{(k+1)}(v, \Omega, t, x)$ inductively as the solution of the following linearized problem:

\[
\begin{aligned}
&\frac{1}{c} \partial_t I^{(k+1)} + \Omega \cdot \nabla I^{(k+1)} + \left( \sigma_a^{(k)} + \int_0^\infty \int_{S^2} \sigma_s^{(k)} d\Omega' dv' \right) I^{(k+1)} = A_v^{(k)}, \\
&A_0(U^{(k)}) \partial_t U^{(k+1)} + \sum_{j=1}^3 A_j(U^{(k)}) \partial_{x_j} U^{(k+1)} = F(I^{(k)}, U^{(k)}), \\
&I^{(k+1)}(0, x, v, \Omega) = I_0^{(k+1)}(x, v, \Omega), \quad U^{(k+1)}(0, x) = U_0^{(k+1)}(x),
\end{aligned}
\]

where

\[
\begin{aligned}
&A_v^{(k)} = S^{(k)} + \int_0^\infty \int_{S^2} \frac{v}{v'} \sigma_s^{(k)} I^{(k)} d\Omega' dv', \quad F_0(I^{(k)}, U^{(k)}) = 0, \\
&F_j(I^{(k)}, U^{(k)}) = -\frac{(\gamma - 1)^2}{4c\gamma} \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s^{(k)} I^{(k)} - \sigma_a^{(k)} I^{(k+1)} \right) \Omega_j d\Omega dv, \\
&\sigma_s^{(k)} = \sigma_s(v' \to v, \Omega' \cdot \Omega, w(k)) = \sigma_s(v \to v', \Omega' \cdot \Omega, w(k)), \\
&S^{(k)} = S(v, t, x, w(k)), \quad \overline{S}^{(k)} = \overline{S}(v, t, x, w(k)), \\
&\sigma_a^{(k)} = \sigma_a(v, t, x, w(k)), \quad \overline{\sigma}_a^{(k)} = \overline{\sigma}_a(v, t, x, w(k)).
\end{aligned}
\]

It follows immediately that

\[
U^{(k+1)} \in C^\infty([0, T_k] \times \mathbb{R}^3), \quad I^{(k+1)} \in L^2(\mathbb{R}^+ \times S^2, C^\infty([0, T_k] \times \mathbb{R}^3)),
\]

where $T_k$ is the largest time of existence for (4.5) such that the estimates

\[
\int_0^\infty \int_{S^2} ||I^{(k)} - I_0^{(0)}||^2_{s,T_k} d\Omega \leq C, \quad ||U^{(k)} - U_0^{(0)}||_{s,T_k} \leq C
\]

are valid for any given constant $C > 0$. Next we give two key lemmas as in Section 2, which imply the compactness of the above-constructed approximate solutions.

**Lemma 4.1 (Boundedness in the high norm).**

There exist constants $C_2 > 0$ and $T_* > 0$ such that the solution $(I^{(k)}, U^{(k)})$ to (4.3) and (4.5) satisfies

\[
|||U^{(k)} - U_0^{(0)}|||_{s,T_*} + |||\partial_t U^{(k)}|||_{s-1,T_*} + \int_0^\infty \int_{S^2} |||I^{(k)} - I_0^{(0)}|||_{s,T_*}^2 d\Omega dv \leq C_2,
\]

for $k = 0, 1, 2, \ldots$.

**Proof.** It is sufficient to prove that (4.9) holds for $(I^{(k+1)}, U^{(k+1)})$ under the assumption that (4.9) holds for $(I^{(k)}, U^{(k)})$. We divide the proof into three steps.
Step 1. The estimate of $\int_0^\infty \int_{S^2} \|I^{(k+1)} - I_0^{(0)}\|^2_s d\Omega dv$. Let $Q^{(k+1)} = I^{(k+1)} - I_0^{(0)}$. Then $Q^{(k+1)}$ satisfies

$$\frac{1}{c} \partial_t Q^{(k+1)} + \Omega \cdot \nabla Q^{(k+1)} + \left(\sigma_a^{(k)} + \int_0^\infty \int_{S^2} \sigma_s^{(k)} d\Omega' dv'\right)Q^{(k+1)} = S^{(k)} + \Theta^{(k)},$$

where $\Theta^{(k)} = -\left(\sigma_a^{(k)} + \int_0^\infty \int_{S^2} \sigma_s^{(k)} d\Omega' dv'\right)I_0^{(0)} - \Omega \cdot \nabla I_0^{(0)} + \int_0^\infty \int_{S^2} \frac{v}{v'} \sigma_s^{(k)} I^{(k)} d\Omega' dv'$.

Differentiating the equations in (4.10) $\alpha$-times ($|\alpha| \leq s$) with respect to $x$, from Young’s inequality we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |D^\alpha Q^{(k+1)}|^2 dx \leq C \|D^\alpha Q^{(k+1)}\|^2 + \|D^\alpha S^{(k)}\|^2 + \|D^\alpha \Theta^{(k)}\|^2$$

$$+ \|D^\alpha (\sigma_a^{(k)} Q^{(k+1)})\|^2 + \int_0^\infty \int_{S^2} \int_{\mathbb{R}^3} D^\alpha (\sigma_s^{(k)} Q^{(k+1)}) D^\alpha Q^{(k+1)} dx d\Omega' dv'$$

$$= : J_1 + J_2 + J_3 + J_4 + J_5.$$

According to Lemma [23], we get

$$J_3 = \|D^\alpha \Theta^{(k)}\|^2 \leq \left\|D^\alpha \left(-\sigma_a^{(k)} I_0^{(0)} - \int_0^\infty \int_{S^2} \sigma_s^{(k)} d\Omega' dv' I_0^{(0)} - \Omega \cdot \nabla I_0^{(0)}\right)\right\|^2$$

$$+ \left\|\int_0^\infty \int_{S^2} D^\alpha \left(\frac{v}{v'} \sigma_s^{(k)} I^{(k)}\right) d\Omega' dv'\right\|^2$$

$$\leq C \|\sigma_a^{(k)}\|^2 \|I_0^{(0)}\|^2_s + \left(1 + \|U^{(k)}\|^2_s\right) \|I_0^{(0)}\|^2_s + \Delta,$$

where

$$\Delta = \left\|\int_0^\infty \int_{S^2} D^\alpha \left(\frac{v}{v'} \sigma_s^{(k)} I^{(k)}\right) d\Omega' dv'\right\|^2 \leq C \left(\int_0^\infty \int_{S^2} \left|\frac{v}{v'} \sigma_s^{(k)}\right| \|I^{(k)}\|_s d\Omega' dv'\right)^2$$

$$\leq C \|U^{(k)}\|_s \int_0^\infty \int_{S^2} \|I^{(k)}\|_s^2 d\Omega dv \cdot \int_0^\infty \int_{S^2} \frac{v^2}{v'^2} \|\sigma_s\|^2 d\Omega' dv'.$$
Combining (4.1)–(4.3) and (4.12)–(4.16), we have
\[
\frac{d}{dt} \|Q^{(k+1)}\|_s^2 \leq C \left( \|D^\alpha S^{(k)}\|^2 + \|Q^{(k+1)}\|_s^2 + \|I_0^{(0)}\|_{s+1}^2 + \int_0^\infty \int_{S^2} \frac{\sigma_s^2}{v'}^2 \|\sigma_s\|^2 d\Omega dv' \right).
\]

Integrating this inequality with respect to \(t\) over \([0, T]\), according to Gronwall’s inequality, we obtain
\[
\int_0^T \int_{S^2} \|Q^{(k+1)}\|_s^2 d\Omega dv \leq Ce^{(C_2+1)T} \left( \int_0^\infty \int_{S^2} \|Q_0^{(k+1)}\|_s^2 d\Omega dv + T \right).
\]

Taking \(T_1\) to be enough small, we arrive at
\[
\int_0^\infty \int_{S^2} \|I^{(k+1)} - I_0^{(0)}\|_s^2 d\Omega dv \leq C_2. \tag{4.17}
\]

**Step 2.** The estimate of source term \(\|D^\alpha F(I^{(k)}, U^{(k)})\|, \forall |\alpha| \leq s\).

Due to Minkowski’s inequality, Holder’s inequality and (2.13), for \(|\alpha| \leq s\), we have
\[
\|D^\alpha F_j(I^{(k)}, U^{(k)})\| 
\leq C \left( \int_0^\infty \int_{S^2} \left( \|S^{(k)} - \sigma_s^{(k)} I^{(k)}\| + \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s I^{(k)} - \sigma_s(I^{(k+1)}) \right) d\Omega dv' \right) \right) \Omega_j d\Omega dv 
\leq C \left( \int_0^\infty \int_{S^2} \left( \|S^{(k)}\|_s + \|\sigma_s^{(k)} I^{(k)}\|_s \right) d\Omega dv + \mathcal{J}_6 + \mathcal{J}_7 \right) 
\leq C \left( \int_0^\infty \int_{S^2} \left( \|S^{(k)}\|_s d\Omega dv + \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left( \|\sigma_s\|_s^2 d\Omega dv' \right) \right) \left( \int_0^\infty \int_{S^2} \|I^{(k)}\|_s^2 d\Omega dv \right)^{1/2} \right) \right) \Omega_j d\Omega dv + \mathcal{J}_6 + \mathcal{J}_7, \tag{4.18}
\]

where
\[
\mathcal{J}_6 = \left( \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s^{(k)} I^{(k)} \right) \Omega_j d\Omega dv' d\Omega dv \right) \|s\|_s^2 d\Omega dv
\]
\[
\leq C \left( \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left( \frac{v}{v'} \sigma_s^2 \|I^{(k)}\|_s^2 d\Omega dv' \right) \right) \right) \frac{1}{2} \Omega_j d\Omega dv
\]
\[
\leq C \left( \int_0^\infty \int_{S^2} \left( \frac{1}{2} \|I^{(k)}\|_s^2 d\Omega dv \right)^{1/2} \right) \left( \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left( \frac{1}{2} \sigma_s^2 \|I^{(k)}\|_s^2 d\Omega dv' \right) \right) \right) \frac{1}{2} \Omega_j d\Omega dv, \tag{4.19}
\]

\[
\mathcal{J}_7 = \left( \int_0^\infty \int_{S^2} \int_0^\infty \int_{S^2} \sigma_s I^{(k+1)} \Omega_j d\Omega dv' d\Omega dv \right) \|s\|_s^2 d\Omega dv
\]
\[
\leq C \left( \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \sigma_s^2 \|I^{(k+1)}\|_s^2 d\Omega dv' \right) \right) \Omega_j d\Omega dv
\]
\[
\leq C \left( \int_0^\infty \int_{S^2} \left( \frac{1}{2} \|I^{(k+1)}\|_s^2 d\Omega dv \right)^{1/2} \right) \left( \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \sigma_s^2 \|I^{(k+1)}\|_s^2 d\Omega dv' \right) \right) \frac{1}{2} \Omega_j d\Omega dv.
\]

Together with assumptions (4.12)–(4.13), we obtain
\[
\|F(I^{(k)}, U^{(k)})\|_s \leq C \left( \|U^{(k)}\|_s + \int_0^\infty \int_{S^2} \left( \|I^{(k)}\|_s^2 + \|I^{(k+1)}\|_s^2 \right) d\Omega dv \right) \leq CC_2. \tag{4.20}
\]
Step 3. In order to estimate (4.9), let \( M^{(k+1)} = U^{(k+1)} - U_0^{(0)} \). It is easy to get
\[
\begin{cases}
A_0(U^{(k)}) \partial_t M^{(k+1)} + \sum_{j=1}^{3} A_j(U^{(k)}) \partial_x_j M^{(k+1)} = F(I^{(k)}, U^{(k)}) + \bar{\Theta}^{(k)}, \\
M^{(k+1)}(0, x) = U_0^{(k+1)}(x) - U_0^{(0)}(x),
\end{cases}
\tag{4.21}
\]
where
\[
\bar{\Theta}^{(k)} = -\sum_{j=1}^{3} A_j(U^{(k)}) \partial_x_j U_0^{(0)}.
\]

With the aid of the steps 1 and 2, we can easily follow the standard procedure as in [1] to show that there is a time \( T_2 \) satisfying
\[
\| U^{(k+1)} - U_0^{(0)} \|_{s,T_2} + \| \partial_t U^{(k+1)} \|_{s-1,T_2} \leq C_2.
\tag{4.22}
\]
Let \( T_* = \min\{T_1, T_2\} \). Then Lemma 4.1 is proved. \( \square \)

Lemma 4.2. There exist constants \( T_{**} \in [0, T_3], \bar{\eta} < 1, \{\bar{\beta}_k\} (k = 1, 2, ...) \) and \( \{\bar{\mu}_k\} (k = 1, 2, ...) \) with \( \sum_k |\bar{\beta}_k| < +\infty \) and \( \sum_k |\bar{\mu}_k| < +\infty \), such that for each \( k \)
\[
\begin{align*}
\| U^{(k+1)} - U^{(k)} \|_{0,T_{**}} + \left( \int_0^{\infty} \int_{S^2} \| I^{(k+1)} - I^{(k)} \|^2_{0,T_{**}} \, d\Omega \, dv \right)^{\frac{1}{2}} \\
\leq \bar{\eta} \left( \| U^{(k)} - U^{(k-1)} \|_{0,T_{**}} + \left( \int_0^{\infty} \int_{S^2} \| I^{(k)} - I^{(k-1)} \|^2_{0,T_{**}} \, d\Omega \, dv \right)^{\frac{1}{2}} \right) + \bar{\beta}_k + \bar{\mu}_k.
\end{align*}
\tag{4.23}
\]

Proof. Similarly to the proof of Lemma 2.3 according to assumptions (4.1)-(4.3), for \( T_3 \) small enough, we have
\[
\begin{align*}
\| U^{(k+1)} - U^{(k)} \|_{0,T_3} \\
\leq \bar{\eta}_1 \left( \| U^{(k)} - U^{(k-1)} \|_{0,T_3} + \left( \int_0^{\infty} \int_{S^2} \| I^{(k)} - I^{(k-1)} \|^2_{0,T_3} \, d\Omega \, dv \right)^{\frac{1}{2}} \right) + \bar{\beta}_k,
\end{align*}
\tag{4.24}
\]
where \( \bar{\eta}_1 < \frac{1}{2} \) and \( \sum_k |\bar{\beta}_k| < +\infty \).

To bound \( I^{(k+1)} - I^{(k)} \), we use equation (2.18) to show that
\[
\begin{align*}
&\frac{1}{c} \partial_t (I^{(k+1)} - I^{(k)}) + \Omega \cdot \nabla (I^{(k+1)} - I^{(k)}) \\
= S^{(k+1)} - S^{(k)} - \sigma_a^{(k)} (I^{(k+1)} - I^{(k)}) - I^{(k)} (\sigma_a^{(k)} - \sigma_a^{(k-1)}) + \bar{J}_8 + \bar{J}_9,
\end{align*}
\tag{4.25}
\]
where
\[
\begin{align*}
\bar{J}_8 &= \int_0^\infty \int_{S^2} \int_{S^2} \frac{v}{v'} \left\{ \sigma_s^{(k)} (I^{(k)} - I^{(k-1)}) + I^{(k-1)} (\sigma_s^{(k)} - \sigma_s^{(k-1)}) \right\} \, d\Omega' \, dv', \\
\bar{J}_9 &= \int_0^\infty \int_{S^2} \left\{ \sigma_s^{(k)} (I^{(k+1)} - I^{(k)}) + I^{(k)} (\sigma_s^{(k)} - \sigma_s^{(k-1)}) \right\} \, d\Omega' \, dv'.
\end{align*}
\]
Similarly to the proof of Lemma 2.3, \( \forall \tau \in [0, \mathcal{T}_*] \), we have
\[
\int_0^\infty \int_{S^2} \| \| (I^{(k+1)} - I^{(k)}) \|^2_{0, \tau} \, d\Omega \, dv \leq e \mathcal{C}_\tau \left( \int_0^\infty \int_{S^2} \| I^{(k+1)} - I_0^{(k)} \|^2 \, d\Omega \, dv \right) + \tau \left( \| U^{(k)} - U^{(k-1)} \|^2_{0, \tau} + \int_0^\infty \int_{S^2} \| I^{(k)} - I^{(k-1)} \|^2_{0, \tau} \, d\Omega \, dv \right).
\]
Choosing \( \mathcal{T}_4 \in [0, \mathcal{T}_*] \) to be small enough, we have
\[
\left( \int_0^\infty \int_{S^2} \| I^{(k+1)} - I^{(k)} \|^2_{0, \mathcal{T}_4} \, d\Omega \, dv \right)^\frac{1}{2} \leq \eta_2 \left( \| U^{(k)} - U^{(k-1)} \|^2_{0, \mathcal{T}_4} + \left( \int_0^\infty \int_{S^2} \| I^{(k)} - I^{(k-1)} \|^2_{0, \mathcal{T}_4} \, d\Omega \, dv \right)^\frac{1}{2} \right) + \mu_k,
\]
where \( \eta_2 < \frac{1}{2} \) and \( \sum_k |\mu_k| < +\infty \).

Finally, taking \( \mathcal{T}_{**} = \min\{ \mathcal{T}_3, \mathcal{T}_4 \} \), Lemma \ref{lem:4.2} is proved by adding (4.24) and (4.26) together.

Based on Lemmas \ref{lem:4.1} and \ref{lem:4.2}, we can prove Theorem \ref{thm:4.1} analogously. We omit the details here.

It turns out that we have the following local existence of regular solutions to the original Cauchy problem (1.1) and (2.5) when \( \sigma_s \neq 0 \).

**Theorem 4.2.** Let \( s \geq 3 \) be an integer and \( \{4.2\}-\{4.3\} \) hold. If the initial data satisfy
\( (I_0, \rho_0, u_0) \in \overline{\mathcal{V}} = \left\{ (I, \rho, u) | \rho \geq 0, (\rho^{\frac{s-1}{2}}, u) \in H^s, I(v, \Omega, x) \in L^2(\mathbb{R}^+ \times S^2; H^s(\mathbb{R}^3)) \right\} \),
then there exists a time \( T > 0 \) such that the Cauchy problem (1.1) and (2.5) has a unique regular solution \( (I, \rho, u) \).

The proof is the same as the corresponding theorem in Section 2, here we omit it.

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