The existence of \( \{p, q\}\)-orientations in edge-connected graphs

Morteza Hasanvand

Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
morteza.hasanvand@alum.sharif.edu

Abstract

In 1976 Frank and Gyárfás gave a necessary and sufficient condition for the existence of an orientation in an arbitrary graph \( G \) such that for each vertex \( v \), the out-degree \( d^+_G(v) \) of it satisfies \( p(v) \leq d^+_G(v) \leq q(v) \), where \( p \) and \( q \) are two integer-valued functions on \( V(G) \) with \( p \leq q \). In this paper, we give a sufficient edge-connectivity condition for the existence of an orientation in \( G \) such that for each vertex \( v \), \( d^+_G(v) \in \{p(v), q(v)\} \), provided that for each vertex \( v \), \( p(v) \leq \frac{1}{2} \Delta_G(v) \leq q(v) \), \( |q(v) - p(v)| \leq k \), and there is \( t(v) \in \{p(v), q(v)\} \) in which \( |E(G)| = \sum_{v \in V(G)} t(v) \). This result is a generalization of a theorem due to Thomassen (2012) on the existence of modulo orientations in highly edge-connected graphs.

Keywords:
Modulo orientation; edge-connectivity; out-degree; spanning tree.

1 Introduction

In this article, graphs have no loops, but multiple edges are allowed, and a general graph may have loops and multiple edges. Let \( G \) be a graph. The vertex set and the edge set of \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. We denote by \( d_G(v) \) the degree of a vertex \( v \) in the graph \( G \). If \( G \) has an orientation, the out-degree and in-degree of \( v \) are denoted by \( d^+_G(v) \) and \( d^-_G(v) \). For a vertex set \( A \) of \( G \) with at least two vertices, the number of edges of \( G \) with exactly one end in \( A \) is denoted by \( d_G(A) \). Also, we denote by \( e_G(A) \) the number of edges with both ends in \( A \). For each vertex, let \( L(v) \) be a set of integers. We denote by \( gap(L(v)) \) the maximum of all \( |a - b| \) taken over of consecutive integers in \( a, b \in L(v) \), and the denote by \( gap(L) \) the maximum of all \( gap(L(v)) \) taken over of all vertices \( v \). An orientation of \( G \) is said to be (i) \( L \)-orientation, if for each vertex \( v \), \( d^+_G(v) \in L(v) \), (ii) \( z \)-defective \( L \)-orientation, if for each vertex \( v \) with \( v \neq z \), \( d^+_G(v) \in L(v) \), (iii) \( (p, q) \)-orientation, if for each vertex \( v \), \( p(v) \leq d^+_G(v) \leq q(v) \), where \( p \) and \( q \) are two integer-valued functions on \( V(G) \). Let \( k \) be a positive integer. The cyclic group of order \( k \) is denoted by \( \mathbb{Z}_k \).

An orientation of \( G \) is said to be \( p \)-orientation, if for each vertex \( v \), \( d^+_G(v) \equiv p(v) \), where \( p : V(G) \rightarrow \mathbb{Z}_k \) is a mapping. A graph \( G \) is called \( m \)-tree-connected, if it contains \( m \) edge-disjoint spanning trees. Note that by
the result of Nash-Williams [9] and Tutte [12] every 2m-edge-connected graph is m-tree-connected. A graph $G$ is said to be $(m, l_0)$-partition-connected, if it can be decomposed into an $m$-tree-connected factor and a factor $F$ having an orientation such that for each $v$, $d_G^+(v) \geq l_0(v)$, where $l_0$ is a nonnegative integer-valued function on $V(G)$. For a graph $G$ with a vertex $z$, we denote by $\chi_z$ the mapping $\chi_z : V(G) \to \{0, 1\}$ such that $\chi(z) = 1$ and $\chi(v) = 0$ for all vertices $v$ with $v \neq z$. Also, we define $\bar{\chi}_z = 1 - \chi_z$. For two edges $xu$ and $uy$ incident with the vertex $u$, lifting of $xu$ and $uy$ is an operation that removes $xu$ and $uy$ and adds a new edge $xy$ (when the purpose is to generate a loopless graph we must not add the next edge $xy$ when $x = y$). Throughout this article, all variables $k$ and $m$ are positive integers.

In 1965 Hakimi introduced the following criterion for the existence of an orientation with a given upper bound on out-degrees.

**Theorem 1.1.([6])** Let $G$ be a graph and let $q$ be an integer-valued function on $V(G)$. Then $G$ has an orientation such that for each $v \in V(G)$, $d_G^+(v) \leq q(v)$, if and only if $e_G(S) \leq \sum_{v \in S} q(v)$ for all $S \subseteq V(G)$.

In 1976 Frank and Gyárfás generalized Hakimi’s result to the following bounded out-degree version.

**Theorem 1.2.([5])** Let $G$ be a graph and let $p$ and $q$ be two integer-valued functions on $V(G)$ with $p \leq q$. Then $G$ has an orientation such that for each $v \in V(G)$, $p(v) \leq d_G^+(v) \leq q(v)$, if and only if for all $S \subseteq V(G)$,

$$
e_G(S) \leq \min\{\sum_{v \in S} q(v) : \sum_{v \in S} (d_G(v) - p(v))\}.$$

In 2012 Thomassen gave a sufficient edge-connectivity condition for the existence of modulo orientations as the following theorem.

**Theorem 1.3.([11])** Let $G$ be a $(2k^2 + k)$-edge-connected graph and let $p : V(G) \to \mathbb{Z}_k$ be a mapping. Then $G$ has a $p$-orientation if and only if $|E(G)| \equiv k \sum_{v \in V(G)} p(v)$.

In this paper, we provide a development for Thomassen’s result by giving a sufficient edge-connectivity for the existence of $\{p, q\}$-orientations as the following theorem.

**Theorem 1.4.** Let $G$ be a $8k^2$-edge-connected graph and let $p$ and $q$ be two integer-valued functions on $V(G)$ in which for each vertex $v$, $p(v) \leq d_G(v)/2 \leq q(v)$ and $|q(v) - p(v)| \leq k$, Then $G$ has an orientation such that for each vertex $v$, $d_G^+(v) \in \{p(v), q(v)\}$ if and only if there is an integer-valued function $t$ on $V(G)$ in which $t(v) \in \{p(v), q(v)\}$ for each vertex $v$, and $|E(G)| = \sum_{v \in V(G)} t(v)$. 

2
2 Edge-connected graphs: \((p, q)\)-orientations and \(\{p, q\}\)-orientations

2.1 \((p, q)\)-orientations

In this section, we are going to derive some corollary of the following reformulation of Hakimi’s Theorem. This version exhibits that why edge-connectivity plays an important role for finding orientations whose out-degrees are far from the half of the corresponding degrees in \(G\).

**Theorem 2.1.** (Hakimi [6]) Let \(G\) be a graph and let \(q\) be an integer-valued function on \(V(G)\). Then \(G\) has an orientation such that for all \(v \in V(G)\), \(d^+_G(v) \leq q(v)\), if and only if for all \(S \subseteq V(G)\),

\[
\sum_{v \in S} (d_G(v) - 2q(v)) \leq d_G(S),
\]

Furthermore, under this condition, \(d^+_G(v) = q(v)\) for all \(v \in V(G)\), if and only if \(|E(G)| = \sum_{v \in V(G)} q(v)\).

**Proof.** Apply Theorem 1.1 and the fact that \(\sum_{v \in S} d_G(v)/2 - d_G(S)/2 = e_G(S)\) for every vertex set \(S\). □

Another immediate consequence of Theorem 2.1 is given in the next corollary.

**Corollary 2.2.** Let \(G\) be a graph and let \(q\) be an integer-valued function on \(V(G)\) satisfying \(|E(G)| \leq \sum_{v \in V(G)} q(v)\). If \(G\) is \(\lambda\)-edge-connected, then it admits an orientation such that for each vertex \(v\), \(d^+_G(v) \leq q(v)\), where

\[
\lambda = \sum_{v \in V(G)} \max\{0, d_G(v) - 2q(v)\}.
\]

Furthermore, under this condition, \(d^+_G(v) = q(v)\) for all \(v \in V(G)\), if and only if \(|E(G)| = \sum_{v \in V(G)} q(v)\).

**Proof.** For every nonempty proper subset \(S\) of \(V(G)\), \(\sum_{v \in S} (d_G(v) - 2q(v)) \leq \sum_{v \in V(G)} \max\{0, d_G(v) - 2q(v)\} = \lambda \leq d_G(S)\). If \(S = V(G)\), then by the assumption, \(\sum_{v \in S} (d_G(v) - 2q(v)) \leq 0 = d_G(S)\). Now, it enough to apply Theorem 2.1. □

The following corollary makes an interesting tool for constructing orientations from a given orientation.

**Corollary 2.3.** Let \(G\) be a graph with an orientation \(D\) and let \(\varepsilon\) be a rational number with \(0 \leq \varepsilon \leq 1\). Then \(G\) admits an orientation \(D_0\) such that for all \(v \in V(G)\), \(d^+_D(v) = \frac{1-\varepsilon}{2} d_G(v) + \varepsilon d^+_D(v)\) if and only if for all \(v \in V(G)\), \(\frac{1-\varepsilon}{2} d_G(v) + \varepsilon d^+_D(v)\) is integer.

**Proof.** By Theorem 2.1, for every vertex \(S\), we have \(d_G(S) \geq \sum_{v \in S} (d_G(v) - 2d^+_D(v)) \geq \sum_{v \in S} (\varepsilon d_G(v) - 2\varepsilon d^+_D(v)) = \sum_{v \in S} \varepsilon (d_G(v) - 2f(v))\), where \(f(v) = \frac{1-\varepsilon}{2} d_G(v) + \varepsilon d^+_D(v)\). If \(f\) is integer-valued, then by Theorem 2.1, one can deduce that there is an orientation \(D_0\) such that for all \(v \in V(G)\), \(d^+_D(v) = f(v)\). This can complete the proof. □
Corollary 2.4. Let $G$ be a graph and let $k$ and $k_0$ be two odd positive integers with $k_0 \leq k$. If $G$ has an orientation $D$ such that for all $v \in V(G)$, $d_D^+(v) - d_G(v)/2 \in \{0, \pm k/2\}$, then it has an orientation $D_0$ such that for all $v \in V(G)$, $d_{D_0}^+(v) - d_G(v)/2 \in \{0, \pm k_0/2\}$.

Proof. Apply Corollary 2.3 with $\varepsilon = k_0/k$. Note that if $d_D^+(v) = d_G(v)/2$, then $(1 - \varepsilon)d_G(v)/2 + \varepsilon d_D^+(v) = d_G(v)/2$, and if $d_D^+(v) = d_G(v)/2 \pm k/2$, then $(1 - \varepsilon)d_G(v)/2 + \varepsilon d_D^+(v) = d_G(v)/2 \pm k/2$. $\Box$

The following theorem is an edge-connected reformulation of Frank and Gyárfás’ Theorem.

Theorem 2.8. (5) Let $G$ be a graph and let $p$ and $q$ be two integer-valued functions on $V(G)$ with $p \leq q$. Then $G$ has an orientation such that for each vertex $v$, $p(v) \leq d_G^+(v) \leq q(v)$, if and only if for all $S \subseteq V(G)$.

$$\max\{\sum_{v \in S} (2p(v) - d_G(v)), \sum_{v \in S} (d_G(v) - 2q(v))\} \leq d_G(S),$$

Proof. Apply Theorem 1.2 and use the fact that $d_G(S) = e_G(S) - \sum_{v \in S} d_G(v)/2$. $\Box$

2.2 Defective $\{p, q\}$-orientations

In order to prove Theorem 1.4, we shall first formulate a weaker version. For this purpose, we need the following two lemmas. The first one guarantees the existence of modulo orientations with bounded out-degrees in edge-connected graphs which is a refinement of the main result in [10].

Lemma 2.6. (7) Let $G$ be a graph, let $n$ be a positive integer, and let $p : V(G) \to \mathbb{Z}_n$ be a mapping satisfying $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is $(3n-3)$-edge-connected, then it has a $p$-orientation such that for each vertex $v$, $|d_G^+(v) - d_G(v)/2| < n$. Furthermore, for an arbitrary vertex $z$, we can have $-x \leq d_G^+(z) - d_G(z)/2 < n-x$, where $x$ is an arbitrary real number $x \in [0, n]$.

Lemma 2.7. (see [4]) Let $G$ be a connected graph with $Q \subseteq V(G)$. If $|Q|$ is even, then $G$ has a spanning forest $F$ such that $Q = \{v \in V(F) : d_F(v) \text{ is odd}\}$.

The following theorem gives a sufficient edge-connectivity for the existence of defective $\{p, q\}$-orientations.

Theorem 2.8. Let $G$ be a graph with $z \in V(G)$, let $k$ be a positive integer, and let $p$ and $q$ be two integer-valued functions on $V(G)$ in which for each vertex $v$, $p(v) \leq d_G(v)/2 \leq q(v)$ and $|q(v) - p(v)| \leq k$. If $G$ is $(\frac{4}{3}k + 1)(k-1)$-tree-connected, then it has an orientation such that for each $v \in V(G) \setminus \{z\}$, $d_G^+(v) \in \{p(v), q(v)\}$.

Furthermore, for the vertex $z$, we can have $-x \leq d_G^+(z) - d_G(z)/2 < k-x$, where $x$ is an arbitrary real number $x \in [0, k]$.
Proof. We may assume that \( k \geq 2 \), as the assertion trivially holds when \( k = 1 \). Since \( G \) is \( m \)-tree-connected, we can decompose \( G \) into \( k - 1 \) spanning trees \( T_2, \ldots, T_k \) and \( k - 1 \) factors \( H_2, \ldots, H_k \) such that every \( H_i \) is \((3i - 3)\)-tree-connected, where \( m = \sum_{2 \leq i \leq k} (3i - 3) + k - 1 \). For each \( i \in \{2, \ldots, k\} \), define
\[
V_i = \{ v \in V(G) \setminus \{z\} : |p(v) - q(v)| = i \},
\]
and \( U_i = V(G) \setminus (V_i \cup \{z\}) \). In addition, by Lemma 2.7, we can take \( F_i \) to be a spanning forest of \( T_i \) such that for each \( v \in U_i \), \( d_{F_i}(v) + d_{H_i}(v) \) is even and for each \( v \in V_i \), \( d_{F_i}(v) + d_{H_i}(v) \) and \( d_G(v) \) have the same parity. Note that the following upper bound of \( G \), \( \sum x_i \), since \( 0 \leq x_n+1 < k \), since \( x_{n+1} = x_n + d_G^+(z) - d_G^-(z)/2 \). Let \( v \in V(G) \setminus \{z\} \) so that \( v \in V_i \) and \( 1 \leq i \leq k \). Therefore,
\[
\begin{align*}
d_G^+(v) &= \sum_{1 \leq j \leq k} d_G^+(v) - d_G^+(v) + \sum_{1 \leq j \leq k, j \neq i} d_G^+(v)/2 = d_G^+(v)/2 - d_G^+(v)/2 \in \{p(v), q(v)\}.
\end{align*}
\]
Furthermore \(-x \leq d_G^+(z) - d_G^-(z)/2 < k - x \), since \( 0 \leq x_{k+1} < k \). Hence the proof is completed. \( \square \)

2.3 \( \{p, q\} \)-orientations

In this section, we shall improve Theorem 2.8 by refining the condition for the vertex \( z \). To do this, we first form the following lemma for working with integer numbers. Note that the following upper bound of \( k(k-1) \) is sharp by setting \((m, n) = (k-1, k)\), \( x_i = k \), and \( y_j = k - 1 \), where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

Lemma 2.9. Let \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_n \) be positive integers and let \( k \) be the maximum of them. If \( \sum_{1 \leq i \leq m} x_i = \sum_{1 \leq j \leq n} y_j \) and for any two integer sets \( I \subseteq \{1, \ldots, m\} \) and \( J \subseteq \{1, \ldots, n\} \) satisfying \(|I| + |J| < m + n\), \( \sum_{i \in I} x_i \neq \sum_{j \in J} y_j \), then
\[
\sum_{1 \leq i \leq m} x_i \leq k(k-1).
\]

Proof. We may assume that \( k \) is the maximum of \( y_1, \ldots, y_n \) and so \( \max_{1 \leq i \leq m} x_i < k \). Let \( I_0 = J_0 = \emptyset \) and \( g(0) = f(0) = 0 \). Let \( s \) be a positive integer. If \( I_1 \cup \cdots \cup I_{s-1} \neq \{1, \ldots, m\} \), then we recursively define
\( I_s \) to be a nonempty subset of \( \{1, \ldots, m\} \setminus (I_1 \cup \cdots \cup I_{s-1}) \) such that
\[
f(s-1) = \sum_{1 \leq t \leq s-2} \sum_{1 \leq j \in J_t} y_j \leq \sum_{1 \leq t \leq s} \sum_{1 \leq j \in J_t} x_i = g(s).
\]

If \( J_1 \cup \cdots \cup J_{s-1} \neq \{1, \ldots, n\} \), then we recursively define \( J_s \) to be a nonempty subset of \( \{1, \ldots, n\} \setminus (J_1 \cup \cdots \cup J_{s-1}) \) such that
\[
g(s) = \sum_{1 \leq t \leq s} \sum_{1 \leq j \in J_t} x_i \leq \sum_{1 \leq t \leq s} \sum_{1 \leq j \in J_t} y_j = f(s).
\]

We consider \( I_s \) and \( J_s \) with the minimum size. These can imply that \( g(1) - f(0) = \min_{1 \leq i \leq m} x_i \leq k - 1 \) and \( g(s) - f(s-1) \leq \min_{i \in I_s} x_i - 1 \leq k - 1 \) when \( s > 1 \) and \( f(s) - g(s) \leq \min_{j \in I_s} y_j - 1 \leq k - 1 \). According to the assumption on summations of \( x_i \) and \( y_j \), we must also have \( f(s-1) \neq g(s) \) and \( g(s) \neq f(s) \).

Define \( g_d(s) = g(s) - f(s-1) \) and \( f_d(s) = f(s) - g(s) \). Assume that \( I_1 \cup \cdots \cup I_q = \{1, \ldots, m\} \) and \( J_1 \cup \cdots \cup J_q = \{1, \ldots, n\} \) so that
\[
0 = g(0) = f(0) < g(1) < f(1) < \cdots < f(q-1) \leq g(q) = \sum_{1 \leq i \leq m} x_i = \sum_{1 \leq j \leq n} y_j = f(q).
\]

Let \( s, s' \in \{1, \ldots, q\} \) with \( s \geq s' \). If \( f_d(s) = f_d(s') \), then
\[
\sum_{s' < t \leq s} \sum_{1 \leq i \in I_t} x_i = g(s) - g(s') = f(s) - f(s') = \sum_{s' < t \leq s} \sum_{1 \leq j \in J_t} y_j.
\]

According to the assumption on summations of \( x_i \) and \( y_j \), we must have \( s = s' \). Note that we consider \( J_q \) to be the empty, and consider \( I_q \) to be the empty set when \( g(q) = f(q-1) \). Similarly, if \( g_d(s) = g_d(s') \), then \( s = s' \). In other words, \( g_d \) and \( f_d \) are injective functions with the restricted domain \( \{1, \ldots, q\} \) and the co-domain \( \{0, 1, \ldots, k-1\} \). Therefore,
\[
\sum_{1 \leq i \leq m} x_i = g(q) = \sum_{1 \leq s \leq q} g_d(s) + \sum_{1 \leq s \leq q-1} f_d(s) \leq 2 \sum_{1 \leq i \leq k-1} i = k(k-1).
\]

Hence the proof is completed. \( \square \)

The following theorem gives a sufficient edge-connectivity for the existence of \( \{p, q\} \)-orientations...

**Theorem 2.10.** Let \( G \) be a \( 4k^2 \)-tree-connected graph and let \( p \) and \( q \) be two integer-valued functions on \( V(G) \) in which for each \( v \in V(G) \), \( p(v) \leq d_G(v)/2 \leq q(v) \) and \( |q(v) - p(v)| \leq k \). Then \( G \) has an orientation such that for each \( v \in V(G) \),
\[
d_G^+(v) \in \{p(v), q(v)\},
\]
if and only if there is an integer-valued function \( t \) on \( V(G) \) in which \( t(v) \in \{p(v), q(v)\} \) for each \( v \in V(G) \), and \( |E(G)| = \sum_{v \in V(G)} t(v) \). Furthermore, for an arbitrary given vertex \( v \), we can have \( d_G^+(v) = t(v) \).

**Proof.** Since every 2-tree-connected graph has a spanning Eulerian subgraph [8], one can decompose \( G \) into a \( 2k^2 \)-tree-connected graph \( G_0 \) and a \( 2k^2 \)-edge-connected Eulerian graph \( H \). By Theorem 2.8, the
graph $G_0$ has an orientation such that for each $v \in V(G) \setminus \{z\}$, $d^+_G(v) = (p(v) - d_H(v)/2, q(v) - d_H(v)/2)$, and $|d^+_{G_0}(z) - d_{G_0}(z)|/2 \leq k$ in which $d^+_{G_0}(z) \geq d_{G_0}(z)/2$ if and only if $t(z) \geq d_G(z)/2$. For each vertex $v$, define $s(v) = t(v) - d^+_G(v) - d_H(v)/2$. According to this definition, for each $v \in V(G) \setminus \{z\}$, $s(v) = 0$ when $d^+_G(v) = t(v) - d_H(v)/2$, and $|s(v)| = |q(v) - p(v)| \leq k$ otherwise. In addition, $|s(z)| = |t(z) - d_G(z)/2 - (d^+_G(z) - d_{G_0}(z)/2)| \leq k$. Let $S$ be a subset of $V(G)$ including $z$ satisfying $\sum_{v \in S} s(v) = 0$. Note that $V(G)$ is a candidate for $S$, since

$$\sum_{v \in V(G)} s(v) = \sum_{v \in V(G)} (t(v) - d^+_G(v) - d_H(v)/2) = |E(G)| - |E(G_0)| - |E(H)| = 0.$$  

Consider $S$ with the minimum $|S|$. Thus for every nonempty proper subset $S_0$ of $S$, $\sum_{v \in S_0} s(v) \neq 0$. Otherwise, $\sum_{v \in S \setminus S_0} s(v) = 0$ which is a contradiction, because either $S_0$ or $S \setminus S_0$ includes $z$. Thus by Lemma 2.9 and the minimal property of $S$, one can conclude that $\sum_{v \in S} |s(v)| \leq 2k(k - 1)$. More precisely, variables $x_i$ in Lemma 2.9 are those positive integers $|s(v)|$ with $s(v) > 0$ and variables $y_j$ are those positive integers $|s(v)|$ with $s(v) < 0$, where $v \in S$. Since $H$ is $2k(k - 1)$-edge-connected, by Corollary 2.2, it has an orientation such that for each $v \in S$, $d^+_H(v) = d_H(v)/2 + s(v)$ and for each $v \in V(G) \setminus S$, $d^+_H(v) = d_H(v)/2$. Note that $\sum_{v \in S} \max\{0, d_H(v) - 2(d_H(v)/2 + s(v))\} = \sum_{v \in S} |s(v)| \leq 2k(k - 1)$. Consider the orientation of $G$ obtained from these orientations. For each vertex $v$,

$$d^+_G(v) = d^+_G(v) + d^+_H(v) = \begin{cases} d^+_G(v) + d_H(v)/2 + s(v) = t(v) \in \{p(v), q(v)\}, & \text{if } v \in S; \\ d^+_G(v) + d_H(v)/2 \in \{p(v), q(v)\}, & \text{otherwise}. \end{cases}$$ 

Hence the theorem holds.

Remark 2.11. We will use the above-mentioned theorem to refine some results in [1, 2] for edge-connected graphs. We will do it in a forthcoming paper.

3 Partition-connected graphs: orientations with sparse lists on out-degrees

In this subsection, we are going to prove the following assertion on the existence of orientations with sparse lists on out-degrees in partition-connected graphs. For dense lists in all graphs, it was investigated by Akbari, Dalirrooyfard, Ehsani, Ozeki, and Sherkati (2020) [3]. Before stating the main result, we need to recall the following lemma from [7].

Lemma 3.1. ([7]) Let $G$ be a general graph with $z \in V(G)$ and let $l_0$ be a nonnegative integer-valued function on $V(G)$. Assume that $z$ is not incident with loops. If $G$ contains an $(m, l_0)$-partition-connected factor $H$ with $d_G(z) \geq 2d_H(z) - 2l_0(z) - 2$, then there are $d_H(z) - l_0(z) - 1$ pair of edges incident with $z$ such that by lifting them the resulting general graph $G_0$ with $V(G_0) = V(G) \setminus \{z\}$ is still $(m, l_0)$-partition-connected.

Now, are we are ready to prove the main result of this section.
Theorem 3.2. Let $G$ be a general graph with $z \in V(G)$ and let $L: V(G) \to 2^{\mathbb{Z}}$ be a mapping satisfying \( \text{gap}(L) \leq k \) and \( \text{gap}(L(z)) = k \). Let $s$, $s_0$, and $l_0$ be three integer-valued functions on $V(G)$ satisfying $s(v) + s_0(v) + \text{gap}(L(v)) < d_G(v)$ and $\max\{s(v), s_0(v)\} \leq l_0(v) + (2k^2 - \text{gap}(L(v)) + 1)\chi_z(v)$ for each vertex $v$. If $G$ is $(2k^2, l_0)$-partition-connected, then it admits a $z$-defective $L$-orientation such that for each vertex $v$,

\[ s(v) \leq d^+_G(v) \leq d_G(v) - s_0(v). \]

Proof. We may assume that $l_0$ is nonnegative and $G$ is loopless. The proof is by induction on $|V(G)|$. For $|V(G)| \leq 2$ the proof is straightforward. So, suppose $|V(G)| \geq 3$. For notational simplicity, let us define $m = 2k^2$. For proving the theorem, we shall consider the following four cases.

Case 1. There is a vertex $u \in V(G) \setminus \{z\}$ with $d_G(u) = 2l_0(u) + 2m - r$ such that $0 < r \leq l_0(u) + m$ and $l_0(u) + m - i \in L(u)$, where $0 \leq i \leq \min\{r, \text{gap}(L(u)) - 1\}$.

By Lemma 3.1, there are $l_0(u) + m - r$ pair of edges incident with $u$ such that by lifting them the resulting general graph $H$ with $V(H) = V(G) \setminus u$ is still $(m, l_0)$-partition-connected. Obviously, $d_R(u) = d_G(u) - 2(l_0(u) + m - r) = r$, where $R$ is the factor of $G$ consisting of all edges incident with $u$ that are not lifted. Since $i \leq r$, the edges of $R$ can be orientated such that $d^+_R(u) + l_0 + m - r \in L(u)$. Define $s'(v) = s(v) - (l_0(u) + m - r)$ and $s'_0(u) = s_0(u) - (l_0(u) + m - r)$. By the assumption, we must have $\max\{s'(v), s'_0(u)\} \leq d_R(u) - \text{gap}(L(u)) - 1$, and $s'(v) + s'_0(u) \leq d_R(u) - \text{gap}(L(u)) - 1$. Therefore, if $d_R(u) \geq \text{gap}(L(u)) - 1$ then the orientation of $R$ can be selected such that $s'(v) \leq d^+_R(u) \leq d_R(u) - s'_0(u)$. If $d_R(u) \leq \text{gap}(L(u)) - 1$, then we must automatically have

\[ s'(v) \leq 0 \leq d^+_R(u) \leq d_R(u) \leq d_R(u) - s'_0(u). \]

Define $L'(v) = \{j - d^+_R(v) : j \in L(v)\}$, where $v \in V(H)$. Obviously, $\max\{s(v) - d^+_R(v), s_0(v) - d^+_R(v)\} \leq l_0(v) + m - (\text{gap}(L(v)) - 1)$ and $s(v) - d^+_R(v) + s_0(v) - d^+_R(v) + \text{gap}(L(v)) - 1 \leq d_H(v)$. Thus by the induction hypothesis, $H$ has a $z$-defective $L'$-orientation such that for each $v \in V(H)$,

\[ s(v) - d^+_R(v) \leq d^+_H(v) \leq d_H(v) - (s_0(v) - d^+_R(v)) = d_G(v) - s_0(v) - d^+_H(v). \]

This orientation induces a $z$-defective $L$-orientation for $G$ such that for each $v \in V(H)$, $d^+_G(v) = d^+_H(v) + d^+_R(v)$, and also $d^+_G(u) = d^+_H(u) + l_0(u) + m - r$. This can complete the proof of Case 1. \hfill \Box

Case 2. $d_G(z) < 2l_0(z) + \text{gap}(L(z)) - 1$.

Since $d_G(z) \geq \text{gap}(L(z)) - 1$, we must have $l_0(z) > 0$ and hence there is an edge $zu$ incident with $z$ such that the graph $G_0$ is $(m, l_0 - \chi_z)$-partition-connected, where $G_0 = G - zu$.

First assume that $s(z) < l_0(z)$. Since $s(z) < l_0(z)$, we must have $s(z) \leq l_0(z) - \chi_z(z)$. Thus by the induction hypothesis, the graph $G_0$ has a $z$-defective $(L - \chi_u)$-orientation such that for each vertex $v$,

\[ s(v) - \chi_u(v) \leq d^+_G_0(v) \leq d_G_0(v) - (s_0(v) - \chi_z(v)). \]

Now, this orientation induces the desired $z$-defective $L$-orientation for $G$ by adding an edge directed from $u$ to $z$. 

8
Now, assume that \( s(z) = l_0(z) \). This implies that \( s_0(z) < l_0(z) \), because \( s(z) + s_0(z) + \text{gap}(L(z)) - 1 \leq d_G(z) \) and \( d_G(z) < 2l_0(z) + \text{gap}(L(z)) - 1 \). Thus by the induction hypothesis, the graph \( G_0 \) has a \( z \)-defective \( (L - \chi_z) \)-orientation such that for each vertex \( v \), \( s(v) - \chi_z(v) \leq d^+_{G_0}(v) \leq d_{G_0}(v) - (s_0(v) - \chi_u(v)) \). Now, this orientation induces the desired \( z \)-defective \( L \)-orientation for \( G \) by adding an edge directed from \( z \) to \( u \). This completes the proof of Case 2. □

Now, by applying Theorem 2.8, the graph \( G \) has an orientation such that \( |d^+_{G_0}(z) - d_G(z)/2| \leq k/2 \) and for all \( v \in V(G) \setminus \{z\} \), \( d^+_{G_0}(z) \in \{p(v), q(v)\} \), where \( p(v) \) and \( q(v) \) are the integers in \( L(v) \) with the smallest \( |q(v) - p(v)| \) such that \( p(v) \leq d_G(v)/2 \leq q(v) \). According to Case 2, \( d_G(z) \geq 2l_0(z) + m - (k - 1) \), which implies that \( s_0(z) \leq l_0(z) \leq d^+_{G_0}(z) \leq d_G(z) - l_0(z) \leq d_G(z) - s_0(z) \). Let \( v \in V(G) \setminus \{z\} \). If \( d_G(v) \geq 2l_0(v) + 2m \), then we must have
\[
s(v) \leq \lfloor d_G(v)/2 \rfloor - (\text{gap}(L(v)) - 1) \leq d^+_G(v) \leq \lfloor d_G(v)/2 \rfloor + (\text{gap}(L(v)) - 1) \leq d_G(v) - s_0(v).
\]
Otherwise, \( d_G(v) = 2l_0(v) + 2m - r \) in which \( 0 < r < \text{gap}(L(v)) - 1 \). According to Case 1, \( \{l_0(v) + m - i : 0 \leq i \leq r\} \cap L(v) = \emptyset \), which implies that
\[
l_0(v) + m - \text{gap}(L(v)) < p(v) \leq d^+_{G_0}(v) \leq q(v) < l_0(v) + m - r + \text{gap}(L(v)) = d_G(v) - (l_0(v) + m - \text{gap}(L(v))),
\]
and so \( s(v) \leq d^+_{G_0}(v) \leq d_G(v) - s_0(v) \). Hence the proof is completed. □

References

[1] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason, Vertex-colouring edge-weightings, Combinatorica 27 (2007) 1–12.

[2] L. Addario-Berry, K. Dalal, and B.A. Reed, Degree constrained subgraphs, Discrete Appl. Math. 156 (2008) 1168–1174.

[3] S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati, Orientations of graphs avoiding given lists on out-degrees, J. Graph Theory 93 (2020) 483–502.

[4] J. Edmonds and E.L. Johnson, Matching, Euler tours and the Chinese postman, Mathematical Programming 5 (1973) 88–124.

[5] A. Frank and A. Gyárfás, How to orient the edges of a graph? in Combinatorics, Coll Math Soc J Bolyai 18 (1976) 353–364.

[6] S.L. Hakimi, On the degrees of the vertices of a directed graph, J. Franklin Inst. 279 (1965) 290–308.

[7] M. Hasanvand, Modulo orientations with bounded out-degrees, arXiv:1702.07039.

[8] F. Jaeger, A note on sub-Eulerian graphs, J. Graph Theory 3 (1979) 91–93.
[9] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.

[10] L.M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang, Nowhere-zero 3-flows and modulo $k$-orientations, J. Combin. Theory Ser. B 103 (2013) 587–598.

[11] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, J. Combin. Theory Ser. B 102 (2012) 521–529.

[12] W.T. Tutte, On the problem of decomposing a graph into $n$ connected factors, J. London Math. Soc. 36 (1961) 221–230.