On an equation by primes with one Linnik prime

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Abstract

Let \( \lfloor \cdot \rfloor \) be the floor function. In this paper, we prove that when \( 1 < c < \frac{16559}{15276} \), then every sufficiently large positive integer \( N \) can be represented in the form

\[
N = [p_1^c] + [p_2^c] + [p_3^c],
\]

where \( p_1, p_2, p_3 \) are primes, such that \( p_1 = x^2 + y^2 + 1 \).

Keywords: Diophantine equation · Prime · Exponential sum · Asymptotic formula

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1 Introduction and main result

A turning point in analytic number theory is 1937, when Vinogradov [20] proved the ternary Goldbach problem. He showed that every sufficiently large odd integer \( N \) can be represented in the form

\[
N = p_1 + p_2 + p_3,
\]

where \( p_1, p_2, p_3 \) are prime numbers.

Helfgott [10] recently showed that this is true for all odd \( N \geq 7 \).

The enormous consequences of Vinogradov’s [21] method for estimating exponential sums over primes find objective expression in the hundreds of articles on diophantine equations and inequalities by primes. Source of detailed proof of Vinogradov’s theorem, beginning with an historical perspective along with an overview of essential lemmas and theorems, can be found in monograph of Rassias [18]. For almost a century, Vinogradov’s three primes theorem has been proved many times with prime numbers of a special form. Recent interesting results in this regard are for example [5], [15], [16], [17]. As an analogue of the ternary Goldbach problem, in 1995, Laporta and Tolev [13] investigated the
diophantine equation

\[ [p_1^c] + [p_2^c] + [p_3^c] = N, \]  

(1)

where \( p_1, p_2, p_3 \) are primes, \( c > 1 \) and \( N \) is positive integer. For \( 1 < c < \frac{17}{16} \) and \( \varepsilon > 0 \) they proved that for the sum

\[ R(N) = \sum_{[p_1^c] + [p_2^c] + [p_3^c] = N} \log p_1 \log p_2 \log p_3 \]

the asymptotic formula

\[ R(N) = \frac{\Gamma^3 \left( 1 + \frac{1}{c} \right)}{\Gamma \left( \frac{3}{c} \right)} N^{\frac{2}{c} - 1} + O \left( N^{\frac{3}{c} - 1} \exp \left( - \frac{1}{3} \log N \right) \right) \]

holds.

Afterwards the result of Laporta and Tolev was sharpened by Kumchev and Nedeva \[12\] to \( 1 < c < \frac{12}{11} \), by Zhai and Cao \[23\] to \( 1 < c < \frac{258}{235} \), by Cai \[3\] to \( 1 < c < \frac{137}{119} \), by Zhang and Li \[24\] to \( 1 < c < \frac{2581}{3106} \) and this is the best result up to now.

On the other hand in 1960 Linnik \[14\] showed that there exist infinitely many prime numbers of the form \( p = x^2 + y^2 + 1 \), where \( x \) and \( y \) are integers. More precisely he proved the asymptotic formula

\[ \sum_{p \leq X} r(p - 1) = \pi \prod_{p > 2} \left( 1 + \frac{\chi_4(p)}{p(p - 1)} \right) \frac{X}{\log X} + O \left( \frac{X(\log \log X)^7}{(\log X)^{1+\theta_0}} \right), \]

where \( r(k) \) is the number of solutions of the equation \( k = x^2 + y^2 \) in integers, \( \chi_4(k) \) is the non-principal character modulo 4 and

\[ \theta_0 = \frac{1}{2} - \frac{1}{4} \varepsilon \log 2 = 0.0289... \]

(2)

Recently the author \[7\] showed that for any fixed \( 1 < c < \frac{427}{400} \), every sufficiently large positive number \( N \) and a small constant \( \varepsilon > 0 \), the diophantine inequality

\[ |p_1^c + p_2^c + p_3^c - N| < \varepsilon \]

has a solution in primes \( p_1, p_2, p_3 \), such that \( p_1 = x^2 + y^2 + 1 \).

Given the last result, it is natural to expect that the diophantine equation \[11\] has a solution in primes \( p_1, p_2, p_3 \), such that \( p_1 = x^2 + y^2 + 1 \). Let \( N \) is a sufficiently large positive integer and

\[ X = N^{\frac{1}{c}}. \]

(3)
Define
\[ \Gamma = \sum_{\substack{X/2 < p_1, p_2, p_3 \leq X \\ |p_1^2| + |p_2^2| + |p_3^2| = N}} r(p_1 - 1) \log p_1 \log p_2 \log p_3. \] (4)

We establish the following theorem.

**Theorem 1.** Let \( 1 < c < \frac{16559}{15276} \). Then for every sufficiently large positive integer \( N \) the asymptotic formula
\[
\Gamma = \pi \prod_p \left( 1 + \frac{\chi_4(p)}{p(p - 1)} \right) \frac{\Gamma^3 \left( 1 + \frac{1}{c} \right)}{\Gamma \left( 1 + \frac{3}{c} \right)} \left( 1 - \frac{1}{2^{3-c}} \right) N^{\frac{2}{c}-1} + O \left( \frac{N^{\frac{2}{c}-1}(\log \log N)^5}{(\log N)^{\theta_0}} \right) \] (5)
holds. Here \( \theta_0 \) is defined by (2).

In addition we have the following challenge for the future.

**Conjecture 1.** There exists \( c_0 > 1 \) such that for any fixed \( 1 < c < c_0 \), and every sufficiently large positive integer \( N \), the diophantine equation
\[
[p_1^2] + [p_2^2] + [p_3^2] = N,
\]
has a solution in prime numbers \( p_1, p_2, p_3 \), such that \( p_1 = x_1^2 + y_1^2 + 1 \), \( p_2 = x_2^2 + y_2^2 + 1 \), \( p_3 = x_3^2 + y_3^2 + 1 \).

## 2 Notations

Let \( N \) be a sufficiently large positive integer. By \( \varepsilon \) we denote an arbitrary small positive number, not the same in all appearances. The letter \( p \) with or without subscript will always denote prime number. The notation \( m \sim M \) means that \( m \) runs through the interval \((M/2, M]\). As usual \( \varphi(n) \) is Euler’s function and \( \Lambda(n) \) is von Mangoldt’s function. We shall use the convention that a congruence, \( m \equiv n \pmod{d} \) will be written as \( m \equiv n \pmod{d} \). Moreover \( e(y) = e^{2\pi i y} \). As usual \( \lfloor t \rfloor \), \( \{t\} \) and \( \|t\| \) denote the integer part of \( t \), the fractional part of \( t \) and the distance from \( t \) to the nearest integer, respectively. We recall that \( t = \lfloor t \rfloor + \{t\} \) and \( \|t\| = \min(\{t\}, 1 - \{t\}) \). We denote by \( r(k) \) the number of solutions of the equation \( k = x^2 + y^2 \) in integers. The symbol \( \chi_4(k) \) will mean the non-principal character modulo 4. Throughout this paper unless something else is said, we suppose that \( 1 < c < \frac{16559}{15276} \).
Denote
\[ D = \frac{X^{\frac{3}{2}}}{(\log N)^{\frac{6.4 + \varepsilon}{3}}} , \quad A > 3 ; \]  
\[ \Delta = X^{\frac{1}{4} - e} ; \]  
\[ H = X^{\frac{1283}{10276}} ; \]  
\[ S_{l,d,J}(t) = \sum_{\substack{p \in J \\ p = l(d)}} e(t[p^c]) \log p ; \]  
\[ S(t) = S_{1,1,(X/2,X]}(t) ; \]  
\[ S_l(t) = \sum_{\substack{p \in J \\ p = l(d)}} e(tp^c) \log p ; \]  
\[ I_J(t) = \int_{J} e(ty^c) \, dy ; \]  
\[ I(t) = I_{(X/2,X]}(t) ; \]  
\[ E(y, t, d, a) = \sum_{\mu y < n \leq \gamma} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c) \, dx , \]
where \( 0 < \mu < 1 . \)

3 Preliminary lemmas

Lemma 1. For any complex numbers \( a(l) \) we have

\[ \left| \sum_{L < l \leq 2L} a(l) \right|^2 \leq \left( 1 + \frac{L}{Q} \right) \sum_{|q| \leq Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{L < l, l+q \leq 2L} a(l+q)\overline{a(l)} , \]

where \( Q \geq 1 . \)

Proof. See ([9], Lemma 5). \( \square \)

Lemma 2. Let \( |f^{(m)}(u)| \asymp YX^{1-m} \) for \( 1 \leq X < u < X_0 \leq 2X \) and \( m \geq 1 . \) Then

\[ \left| \sum_{X < n \leq X_0} e(f(n)) \right| \ll Y^x X^\lambda + Y^{-1} , \]

where \( (x, \lambda) \) is any exponent pair.
Proof. See ([8], Ch. 3).

**Lemma 3.** Let \( x, y \in \mathbb{R} \) and \( H \geq 3 \). Then the formula
\[
e(-x \{ y \}) = \sum_{|h| \leq H} c_h(x)e(hy) + O \left( \min \left( 1, \frac{1}{H \| y \|} \right) \right)
\]
holds. Here
\[
c_h(x) = \frac{1 - e(-x)}{2\pi i(h + x)}.
\]
Proof. See ([2], Lemma 12).

**Lemma 4.** Let \( 3 < U < V < Z < X \) and suppose that \( Z - \frac{1}{2} \in \mathbb{N} \), \( X \gg Z^2U \), \( Z \gg U^2 \), \( V^3 \gg X \). Assume further that \( F(n) \) is a complex valued function such that \( |F(n)| \leq 1 \). Then the sum
\[
\sum_{n \sim X} \Lambda(n)F(n)
\]
can be decomposed into \( O \left( \log^{10}X \right) \) sums, each of which is either of Type I
\[
\sum_{m \sim M} a(m) \sum_{l \sim L} F(ml),
\]
where
\[
L \gg Z, \quad LM \asymp X, \quad |a(m)| \ll m^\epsilon,
\]
or of Type II
\[
\sum_{m \sim M} a(m) \sum_{l \sim L} b(l)F(ml),
\]
where
\[
U \ll L \ll V, \quad LM \asymp X, \quad |a(m)| \ll m^\epsilon, \quad |b(l)| \ll l^\epsilon.
\]
Proof. See ([9], Lemma 3).

**Lemma 5.** Let \( 1 < c < 3 \), \( c \neq 2 \) and \( |t| \leq \Delta \). Then the asymptotic formula
\[
\sum_{X/2 < p \leq X} e(tp^c) \log p = \int_{X/2}^X e(ty^c) \frac{dy}{y} + \mathcal{O} \left( \frac{X}{(\log X)^{\frac{1}{2}}} \right)
\]
holds.
Proof. See ([19], Lemma 14).
Lemma 6. Let $1 < c < 3$, $c \neq 2$, $|t| \leq \Delta$ and $A > 0$ be fixed. Then the inequality

$$\sum_{d \leq \sqrt{X/(\log X)}^6} \max_{y \leq X} \max_{(a,d)=1} |E(y,t,d,a)| \ll \frac{X}{\log A}$$

holds. Here $\Delta$ and $E(y,t,d,a)$ are denoted by (7) and (15).

Proof. See (7, Lemma 18).

Lemma 7. For the sum denoted by (10) and the integral denoted by (14) we have

(i) $\int_{-\Delta}^{\Delta} |S(t)|^2 dt \ll X^{2-c} \log^2 X$,

(ii) $\int_{-\Delta}^{1} |I(t)|^2 dt \ll X^{2-c} \log X$,

(iii) $\int_{0}^{1} |S(t)|^2 dt \ll X \log X$.

Proof. It follows from the arguments used in (19, Lemma 7).

Lemma 8. For the sum denoted by (9) we have

$$\int_{-\Delta}^{\Delta} |S_{l,d,J}(t)|^2 dt \ll \frac{X^{2-c} \log^3 X}{d^2}.$$ 

Proof. It follows by the arguments used in (4, Lemma 6 (i)).

Lemma 9. Let $\alpha$, $\beta$ be real numbers such that

$$\alpha \beta (\alpha - 1)(\beta - 1) \neq 0.$$ 

Set

$$\Sigma_I = \sum_{m \sim M} a(m) \sum_{l \in I_m} e \left( F \frac{m^{\alpha} l^{\beta}}{M^{\alpha} L^{\beta}} \right),$$

where

$$F > 0, \quad M \geq 1, \quad L \geq 1, \quad |a_m| \ll 1$$

and $I_m$ is a subinterval of $[L/2, L]$. Then for any exponent pair $(\kappa, \lambda)$, we have

$$\Sigma_I \ll \left( F \frac{1}{\kappa+\lambda} + M \frac{1}{\kappa+\lambda} L^{\frac{1}{\kappa+\lambda}} + M^{\frac{1}{\kappa+\lambda}} L + M L^{\frac{1}{\kappa+\lambda}} + F^{-1} M L \right) \log (2 + F M L).$$
Proof. See ([22], Theorem 2). 

The next two lemmas are due to C. Hooley.

**Lemma 10.** For any constant \( \omega > 0 \) we have

\[
\left( \sum_{p \leq X} \left| \sum_{d \mid p-1, \sqrt{X} \leq d < \sqrt{X} \log X} \chi_4(d) \right|^2 \right)^{\frac{1}{2}} \ll \frac{X \log \log X}{\log X} ,
\]

where the constant in Vinogradov’s symbol depends on \( \omega > 0 \).

**Lemma 11.** Suppose that \( \omega > 0 \) is a constant and let \( F_\omega(X) \) be the number of primes \( p \leq X \) such that \( p-1 \) has a divisor in the interval \( (\sqrt{X} \log X)^{-\omega}, \sqrt{X} \log X)^{\omega} \). Then

\[
F_\omega(X) \ll \frac{X \log \log X}{(\log X)^{1+2\theta_0}},
\]

where \( \theta_0 \) is defined by (2) and the constant in Vinogradov’s symbol depends only on \( \omega > 0 \).

The proofs of very similar results are available in ([11], Ch.5).

### 4 Outline of the proof

From (11) and well-known identity

\[
r(n) = 4 \sum_{d \mid n} \chi_4(d)\]

we obtain

\[
\Gamma = 4(\Gamma_1 + \Gamma_2 + \Gamma_3),
\]

where

\[
\Gamma_1 = \sum_{X \leq p \leq X} \left( \sum_{d \mid p-1} \chi_4(d) \right) \log p_1 \log p_2 \log p_3 ,
\]

\[
\Gamma_2 = \sum_{X \leq p \leq X} \left( \sum_{D < d \leq X} \chi_4(d) \right) \log p_1 \log p_2 \log p_3 ,
\]

\[
\Gamma_3 = \sum_{X \leq p \leq X} \left( \sum_{d \geq X/D} \chi_4(d) \right) \log p_1 \log p_2 \log p_3 .
\]
In order to estimate $\Gamma_1$ and $\Gamma_3$ we have to consider the sum
\[
I_{l,d,J}(N) = \sum_{X/2 < p_2, p_3 \leq X \atop [p_1] + [p_2] + [p_3] = N \atop p_1 \equiv l \pmod d} \log p_1 \log p_2 \log p_3,
\]
where $d$ and $l$ are coprime natural numbers, and $J \subset (X/2, X]$-interval. If $J = (X/2, X]$ then we write for simplicity $I_{l,d}(N)$. Clearly
\[
I_{l,d,J}(N) = \int_{-\Delta}^{1-\Delta} S_{l,d,J}(t) S^2(t) e(-tN) \, dt - \int_{-\Delta}^{1-\Delta} S_{l,d,J}(t) S^2(t) e(-tN) \, dt + \int_{1-\Delta}^{1-\Delta} S_{l,d,J}(t) S^2(t) e(-tN) \, dt
\]
\[
= I^{(1)}_{l,d,J}(N) + I^{(2)}_{l,d,J}(N),
\]
where
\[
I^{(1)}_{l,d,J}(N) = \int_{-\Delta}^{1-\Delta} S_{l,d,J}(t) S^2(t) e(-tN) \, dt,
\]
\[
I^{(2)}_{l,d,J}(N) = \int_{1-\Delta}^{1-\Delta} S_{l,d,J}(t) S^2(t) e(-tN) \, dt.
\]

We shall estimate $I^{(1)}_{l,d,J}(N)$, $\Gamma_3$, $\Gamma_2$ and $\Gamma_1$, respectively, in the sections 5, 6, 7 and 8. In section 9 we shall finalize the proof of Theorem 1.

5  Asymptotic formula for $I^{(1)}_{l,d,J}(N)$

Using (9), (11) and $|t| \leq \Delta$ we write
\[
S_{l,d,J}(t) = \sum_{p \in J \atop p \equiv l \pmod d} e(tp^c + \mathcal{O}(|t|)) \log p = \sum_{p \in J \atop p \equiv l \pmod d} e(tp^c) (1 + \mathcal{O}(|t|)) \log p
\]
\[
= \mathcal{O} \left( \frac{\Delta X \log X}{d} \right).
\]
Put
\[ S_1 = S(t), \quad S_2 = S_{l,d,J}(t), \quad I_1 = I(t), \quad I_2 = \frac{I_J(t)}{\varphi(d)}. \] (25) (26) (27) (28)

We use the identity
\[ S_1^2 S_2 = I_1^2 I_2 + (S_2 - I_2) I_1^2 + S_2(S_1 - I_1) I_1 + S_1 S_2(S_1 - I_1). \] (29)

Define
\[ \Phi_{\Delta,J}(X, d) = \frac{1}{\varphi(d)} \int_{-\Delta}^{\Delta} I_J(t) e(-Nt) dt. \] (30)

From (7) – (14), (22), (24) – (30), Lemma 5, Lemma 7, Lemma 8 and Cauchy’s inequality it follows
\[ I_{t,d,J}(X) - \Phi_{\Delta,J}(X, d) \approx \left( \max_{|t| \leq \Delta} \left| S_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| + \frac{\Delta X \log X}{d} \right) \int_{-\Delta}^{\Delta} |I(t)|^2 dt \]
\[ + \left( \frac{X}{e(\log X)^{\frac{3}{4}}} + \Delta X \right) \left( \int_{-\Delta}^{\Delta} |S_{l,d,J}(t)|^2 dt \right) \left( \int_{-\Delta}^{\Delta} |I(t)|^2 dt \right)^{\frac{1}{2}} \]
\[ + \left( \frac{X}{e(\log X)^{\frac{3}{4}}} + \Delta X \right) \left( \int_{-\Delta}^{\Delta} |S(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_{-\Delta}^{\Delta} |S_{l,d,J}(t)|^2 dt \right)^{\frac{1}{2}} \]
\[ \approx X^{2-c} \log X \max_{|t| \leq \Delta} \left| S_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| + \frac{X^{3-c}}{de(\log X)^{\frac{3}{4}}}. \] (31)
Put
\[
\Phi_J(X, d) = \frac{1}{\varphi(d)} \int_{-\infty}^{\infty} I^2(t)I_J(t)e(-Nt) \, dt .
\]  

Using (13), (14), (30), (32) and the estimations
\[
I_J(t) \ll \min \left( X, \frac{X^{1-c}}{|t|} \right), \quad I(t) \ll \min \left( X, \frac{X^{1-c}}{|t|} \right)
\]
we deduce
\[
\Phi_{\Delta,J}(X, d) - \Phi_J(X, d) \ll \frac{1}{\varphi(d)} \int_{\Delta} |I(t)|^2 |I_J(t)| \, dt \ll \frac{X^{3-3c}}{\varphi(d) t^3} \ll \frac{X^{3-3c}}{\varphi(d) \Delta^2}
\]
and therefore
\[
\Phi_{\Delta,J}(X, d) = \Phi_J(X, d) + O \left( \frac{X^{3-3c}}{\varphi(d) \Delta^2} \right).
\]

Finally (7), (31), (34) and the identity
\[
I_{l,d,J}(X) = \Phi_J(X, d) + O \left( X^{2-c} (\log X)^{\max} \left| \mathcal{S}_{l,d,J}(t) - \frac{I_J(t)}{\varphi(d)} \right| \right) + O \left( \frac{X^{3-c}}{d e (\log X)^{10}} \right).
\]

We are now in a good position to estimate the sum $\Gamma_3$.

6 Upper bound of $\Gamma_3$

Consider the sum $\Gamma_3$.

Since
\[
\sum_{d \mid p_1 - 1} \sum_{d \leq X/D} \chi_4(d) = \sum_{m \leq (p_1 - 1) D/X} \chi_4 \left( \frac{p_1 - 1}{m} \right) = \sum_{j = \pm 1} \chi_4(j) \sum_{m \leq (p_1 - 1) D/X} 1
\]
then from (19) and (20) we get
\[
\Gamma_3 = \sum_{m < D} \sum_{j = 1} \chi_4(j) I_{1+ jm, \Delta m; J_m} (N),
\]
where \( J_m = \left( \max\{1 + mX/D, X/2\}, X \right) \). The last formula and (21) imply

\[
\Gamma_3 = \Gamma_3^{(1)} + \Gamma_3^{(2)},
\]

where

\[
\Gamma_3^{(i)} = \sum_{m < D} \sum_{j = \pm 1} \chi_4(j) I_{1+jm,4m,J_m}(N), \quad i = 1, 2.
\]

### 6.1 Estimation of \( \Gamma_3^{(1)} \)

From (35) and (37) we obtain

\[
\Gamma_3^{(1)} = \Gamma^* + \mathcal{O}\left(X^{2-c}(\log X)\Sigma_1\right) + \mathcal{O}\left(\frac{X^{3-c}}{e^{(\log X)\Sigma_2}}\right),
\]

where

\[
\Gamma^* = \sum_{m < D} \Phi_f(X, 4m) \sum_{j = \pm 1} \chi_4(j),
\]

\[
\Sigma_1 = \sum_{m < D} \max_{|t| \leq \Delta} \left| S_{1+jm,4m;J_m}(t) - \frac{I_J(t)}{\varphi(4m)} \right|,
\]

\[
\Sigma_2 = \sum_{m < D} \frac{1}{4m}.
\]

From the properties of \( \chi(k) \) we have that

\[
\Gamma^* = 0.
\]

By (6), (9), (13), (40) and Lemma 6 we find

\[
\Sigma_1 \ll \frac{X}{\log^4 X}.
\]

It is well known that

\[
\Sigma_2 \ll \log X.
\]

Bearing in mind (38), (42), (43) and (44) we deduce

\[
\Gamma_3^{(1)} \ll \frac{X^{3-c}}{\log X}.
\]
6.2 Estimation of $\Gamma_3^{(2)}$

Now we consider $\Gamma_3^{(2)}$. The formulas (23) and (37) give us

$$\Gamma_3^{(2)} = \int_{\Delta} 1 - \Delta S_2(t)K(t)e(-Nt) dt,$$

(46)

where

$$K(t) = \sum_{m<D} \sum_{j=\pm 1} \chi_4(j)S_{1+jm,4m;J_m}(t).$$

(47)

Lemma 12. For the sum denoted by (47) we have

$$\int_0^1 |K(t)|^2 dt \ll X \log^6 X.$$

Proof. See ([6], Lemma 22).

Lemma 13. Assume that

$$\Delta \leq |t| \leq 1 - \Delta, \quad |a(m)| \ll m^\varepsilon, \quad LM \asymp X, \quad L \gg X^{\frac{1}{2}}$$

(48)

and $c_h(t)$ denote complex numbers such that $|c_h(t)| \ll (1 + |h|)^{-1}$.

Set

$$S_I = \sum_{|h| \leq H} c_h(t) \sum_{m-M} a(m) \sum_{l-L} e((h + t)m^\varepsilon l^\varepsilon).$$

Then

$$S_I \ll X^{\frac{12993}{10270} + \varepsilon}.$$

Proof. We have

$$S_I \ll X^\varepsilon \max_{|\eta| \in (\Delta, H+1)} \sum_{m-M} \left| \sum_{l-L} e(\eta m^\varepsilon l^\varepsilon) \right|.$$

(49)

We first consider the case when

$$M \ll X^{\frac{12620}{34371}}.$$

(50)
From (47), (48), (49), (50) and Lemma 2 with the exponent pair \((\frac{2}{40}, \frac{33}{40})\) it follows

\[
S_I \ll X^\epsilon \max_{|\eta| \in (\Delta, H+1)} \sum_{m \sim M} \left( \frac{|\eta|X^\epsilon L^{-1}}{|\eta|} \right)^{\frac{2}{40}} L^{\frac{33}{40}} + \frac{1}{|\eta|X^\epsilon L^{-1}}
\]

\[
\ll X^\epsilon \max_{|\eta| \in (\Delta, H+1)} \left( |\eta|^{\frac{2}{40}} X^{\frac{9}{40}} M L^{\frac{33}{40}} + \frac{LM}{|\eta|X^\epsilon} \right)
\]

\[
\ll X^\epsilon \left( H^{\frac{11}{50}} X^{\frac{2c+41}{40}} M^{\frac{9}{40}} + \Delta^{-1} X^{1-\epsilon} \right)
\]

\[
\ll X^{13993_{15276}+\epsilon}. \quad (51)
\]

Next we consider the case when

\[
X^{\frac{12520}{15276}} \ll M \ll X^\frac{5}{7}. \quad (52)
\]

Using (49), (52) and Lemma 9 with the exponent pair \((\frac{7}{11}, \frac{4}{7})\) we deduce

\[
S_I \ll X^\epsilon \max_{|\eta| \in (\Delta, H+1)} \left( |\eta|X^\epsilon \right)^{\frac{11}{50}} M^{\frac{7}{50}} L^{\frac{41}{50}} + M^{\frac{1}{2}} L + ML^{\frac{1}{2}} + |\eta|^{-1} X^{-c} LM
\]

\[
\ll X^\epsilon \left( H^{\frac{7}{11}} M^{\frac{7}{50}} X^{\frac{11c+41}{50}} + XM^{-\frac{1}{2}} + X^{\frac{1}{2}} M^{\frac{1}{2}} + \Delta^{-1} X^{1-\epsilon} \right)
\]

\[
\ll X^{13993_{15276}+\epsilon}. \quad (53)
\]

Bearing in mind (51) and (53) we establish the statement in the lemma. \(\square\)

**Lemma 14.** Assume that

\[
\Delta \leq |t| \leq 1 - \Delta, \quad |a(m)| \ll m^\epsilon, \quad |b(l)| \ll l^\epsilon, \quad LM \asymp X, \quad X^{\frac{4}{7}} \ll L \ll X^{\frac{4}{5}} \quad (54)
\]

and \(c_h(t)\) denote complex numbers such that \(|c_h(t)| \ll (1 + |h|)^{-1}\).

Set

\[
S_{II} = \sum_{|h| \leq H} c_h(t) \sum_{m \sim M} a(m) \sum_{l \sim L} b(l) e((h + t)ml^\epsilon).
\]

Then

\[
S_{II} \ll X^{13993_{15276}+\epsilon}.
\]

**Proof.** Using Cauchy’s inequality and Lemma 1 with \(Q = X^{\frac{3839}{15276}}\) we obtain

\[
|S_{II}| \ll X^\epsilon \sum_{|h| \leq H} |c_h(t)| \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{l \sim L} \left| \sum_{m \sim M} e(f(l, m, q)) \right| \right)^{\frac{1}{2}}, \quad (55)
\]
where \( f_h(l, m, q) = (h + t)m^c((l + q)^c - l^c) \). Now (8), (54), (55) and Lemma 2 with the exponent pair 

\[
(x, \lambda) = BABABA^2BA^3BA^2B(0, 1) = \left( \frac{214}{845}, \frac{199}{338} \right)
\]

imply

\[
S_{II} \ll X^{\varepsilon} \sum_{|h| \leq H} \left| c_h(t) \right| \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q \leq Q} \sum_{l \sim L} \left( |h + t| q X^{c-1} \right)^{\frac{214}{845} M^{\frac{199}{338}} + \frac{1}{|h + t| q X^{c-1}}} \right)^{\frac{1}{2}}
\]

\[
\ll X^{\varepsilon} \sum_{|h| \leq H} \left| c_h(t) \right| \left( \frac{X^2}{Q} + \frac{X}{Q} \left( H^{\frac{214}{845} X^{\frac{214(c-1)}{845}}} M^{\frac{199}{338} Q^{\frac{1059}{845}}} L + \Delta^{-1} X^{1-c} L \log Q \right) \right)^{\frac{1}{2}}
\]

\[
\ll X^{\frac{13993}{15276} + \varepsilon} \sum_{|h| \leq H} \left| c_h(t) \right| \ll X^{\frac{13993}{15276} + \varepsilon} \sum_{|h| \leq H} \frac{1}{1 + |h|} \ll X^{\frac{13993}{15276} + \varepsilon},
\]

which proves the statement in the lemma.

\[\square\]

**Lemma 15.** Let \( \Delta \leq |t| \leq 1 - \Delta \). Then for the exponential sum denoted by (10) we have

\[
S(t) \ll X^{\frac{13993}{15276} + \varepsilon}.
\]

**Proof.** In order to prove the lemma we will use the formula

\[
S(t) = S^*(t) + \mathcal{O}(X^{\frac{1}{2}}), \tag{56}
\]

where

\[
S^*(t) = \sum_{X/2 < n \leq X} \Lambda(n) e(t[n^c]). \tag{57}
\]

By (8), (57) and Lemma 3 with \( x = t \) and \( y = n^c \) we get

\[
S^*(t) = \sum_{X/2 < n \leq X} \Lambda(n) e(tn^c - t\{n^c\}) = \sum_{X/2 < n \leq X} \Lambda(n) e(tn^c) e(-t\{n^c\})
\]

\[
= \sum_{X/2 < n \leq X} \Lambda(n) e(tn^c) \left( \sum_{|h| \leq H} c_h(t)e(hn^c) + \mathcal{O} \left( \min \left( 1, \frac{1}{H\|n^c\|} \right) \right) \right)
\]

\[
= \sum_{|h| \leq H} c_h(t) \sum_{X/2 < n \leq X} \Lambda(n) e((h + t)n^c) + \mathcal{O} \left( (\log X) \sum_{X/2 < n \leq X} \min \left( 1, \frac{1}{H\|n^c\|} \right) \right)
\]

\[
= S_0^*(t) + \mathcal{O} \left( (\log X) \sum_{X/2 < n \leq X} \min \left( 1, \frac{1}{H\|n^c\|} \right) \right), \tag{58}
\]

14
where

\[ S_0^*(t) = \sum_{|h| \leq H} c_h(t) \sum_{X/2 < n \leq X} \Lambda(n)e((h + t)n^c). \] (59)

Arguing as in (3, Lemma 3.3) we find

\[ \sum_{X/2 < n \leq X} \min\left(1, \frac{1}{H||n^c||}\right) \ll X^\varepsilon \left(H^{-1}X + H^{1/2}X^{2}\right). \] (60)

Let

\[ U = X^{1/3}, \quad V = X^{1/3}, \quad Z = [X^{2/3}] + \frac{1}{2}. \]

According to Lemma 3, the sum \( S_0^*(t) \) can be decomposed into \( O\left(\log^{10} X\right) \) sums, each of which is either of Type I

\[ \sum_{|h| \leq H} c_h(t) \sum_{m \sim M} a(m) \sum_{l \sim L} e((h + t)m^c l^c), \]

where

\[ L \gg Z, \quad LM = X, \quad |a(m)| \ll m^\varepsilon, \]

or of Type II

\[ \sum_{|h| \leq H} c_h(t) \sum_{m \sim M} a(m) \sum_{l \sim L} b(l)e((h + t)m^c l^c), \]

where

\[ U \ll L \ll V, \quad LM = X, \quad |a(m)| \ll m^\varepsilon, \quad |b(l)| \ll l^\varepsilon. \]

Using (59), Lemma 13 and Lemma 14 we obtain

\[ S_0^*(t) \ll X^{\frac{13993}{15276} + \varepsilon}. \] (61)

Taking into account (8), (56), (58), (60) and (61) we establish the statement in the lemma. \( \square \)

Bearing in mind (16), Cauchy’s inequality, Lemma 7, Lemma 12 and Lemma 13 we deduce

\[ \Gamma_3^{(2)} \ll \max_{\Delta \leq t \leq 1-\Delta} |S(t)| \left(\int_{\Delta}^{1-\Delta} |S(t)|^2 \, dt\right)^{1/2} \left(\int_{\Delta}^{1-\Delta} |K(t)|^2 \, dt\right)^{1/2} \]

\[ \ll \max_{\Delta \leq t \leq 1-\Delta} |S(t)| \left(\int_{0}^{1} |S(t)|^2 \, dt\right)^{1/2} \left(\int_{0}^{1} |K(t)|^2 \, dt\right)^{1/2} \]

\[ \ll X^{\frac{2969}{15276} + \varepsilon} \ll \frac{X^{3-c}}{\log X}. \] (62)
6.3 Estimation of $\Gamma_3$

Summarizing (36), (45) and (62) we obtain

$$\Gamma_3 \ll \frac{X^{3-c}}{\log X}. \quad (63)$$

7 Upper bound of $\Gamma_2$

In this section we need a lemma that gives us information about the upper bound of the number of solutions of the binary equation corresponding to (1).

Lemma 16. Let $1 < c < 3$, $c \neq 2$ and $N_0$ is a sufficiently large positive integer. Then for the number of solutions $B_0(N_0)$ of the diophantine equation

$$[p_1^c] + [p_2^c] = N_0 \quad (64)$$

in prime numbers $p_1, p_2 \in \left( N_0^{\frac{1}{c}} / 2, N_0^{\frac{1}{c}} \right]$ we have that

$$B_0(N_0) \ll \frac{N_0^{\frac{2}{c}-1}}{\log^2 N_0}. \quad (65)$$

Proof. Define

$$B(X_0) = \sum_{X_0^{\frac{1}{c}} \leq p_1, p_2 \leq X_0} \frac{\log p_1 \log p_2}{[p_1^c] + [p_2^c] = N_0} \quad (65)$$

where

$$X_0 = N_0^{\frac{1}{c}}. \quad (66)$$

By (65) we write

$$B(X_0) = \int_{-\Delta_0}^{1-\Delta_0} S_0^2(t) e(-N_0 t) \, dt$$

$$= B_1(X_0) + B_2(X_0), \quad (67)$$

where
\[ S_0(t) = \sum_{X_0/2 < p \leq X_0} e(t[p^c]) \log p , \]  
\( (68) \)

\[ \overline{S}_0(t) = \sum_{X_0/2 < p \leq X_0} e(tp^c) \log p , \]  
\( (69) \)

\[ \Delta_0 = \frac{(\log X_0)^{A_0}}{X_0^{\Delta_0}}, \quad A_0 > 10 , \]  
\( (70) \)

\[ B_1(X_0) = \int_{-\Delta_0}^{1-\Delta_0} S_0^2(t)e(-N_0t) \, dt , \]  
\( (71) \)

\[ B_2(X_0) = \int_{\Delta_0}^{1} S_0^2(t)e(-N_0t) \, dt . \]  
\( (72) \)

First we estimate \( B_1(X_0) \). Put

\[ I_0(t) = \int_{X_0/2}^{X_0} e(ty^c) \, dy , \]  
\( (73) \)

\[ \Psi_{\Delta_0}(X_0) = \int_{-\Delta_0}^{\Delta_0} I_0^2(t)e(-N_0t) \, dt , \]  
\( (74) \)

\[ \Psi(X_0) = \int_{-\infty}^{\infty} I_0^2(t)e(-N_0t) \, dt . \]  
\( (75) \)

Using \( (33), (73) \) and \( (75) \) we obtain

\[ \Psi(X_0) = \int_{-X_0^{-c}}^{X_0^{-c}} I_0^2(t)e(-N_0t) \, dt + \int_{|t| > X_0^{-c}} I_0^2(t)e(-N_0t) \, dt , \]

\[ \ll \int_{-X_0^{-c}}^{X_0^{-c}} X_0^2 \, dt + \int_{X_0^{-c}}^{\infty} \left( \frac{X_0^{1-c}}{t} \right)^2 \, dt , \]

\[ \ll X_0^2\Delta_0^{-c} . \]  
\( (76) \)

On the other hand \( (24), (68) - (71), (74) \), Lemma 5 and the trivial estimations

\[ S_0(t) \ll X_0 , \quad I_0(t) \ll X_0 \]  
\( (77) \)
imply

\[ B_1(X_0) - \Psi_{\Delta_0}(X_0) \ll \int_{-\Delta_0}^{\Delta_0} |S_0^2(t) - I_0^2(t)| \, dt \]

\[ \ll \int_{-\Delta_0}^{\Delta_0} \left| S_0(t) - I_0(t) \right| \left( \left| S_0(t) \right| + \left| I_0(t) \right| \right) \, dt \]

\[ \ll \left( \max_{|t| \leq \Delta_0} \left| S_0(t) - I_0(t) \right| + \Delta_0X_0 \right) \left( \int_{-\Delta_0}^{\Delta_0} |S_0(t)| \, dt + \int_{-\Delta_0}^{\Delta_0} |I_0(t)| \, dt \right) \]

\[ \ll \left( \frac{X_0}{e^{(\log X_0)^\frac{1}{2}}} + \Delta_0X_0 \right) \Delta_0X_0 \]

\[ \ll \frac{X_0^{2-c}}{e^{(\log X_0)^\frac{1}{2}}}. \]  

(78)

From (33), (70), (74) and (75) it follows

\[ |\Psi(X_0) - \Psi_{\Delta_0}(X_0)| \ll \int_{\Delta_0}^{\infty} |I_0(t)|^2 \, dt \ll \frac{1}{X_0^{2(c-1)}} \int_{\Delta_0}^{\infty} \frac{dt}{t^2} \]

\[ \ll \frac{1}{X_0^{2(c-1)} \Delta_0} \ll \frac{X_0^{2-c}}{\log X_0}. \]  

(79)

Now (76), (78) and (79) and the identity

\[ B_1(X_0) = B_1(X_0) - \Psi_{\Delta_0}(X_0) + \Psi_{\Delta_0}(X_0) - \Psi(X_0) + \Psi(X_0) \]

give us

\[ B_1(X_0) \ll X_0^{2-c}. \]  

(80)

Further we estimate \( B_2(X_0) \). By (66), (72), (77) and partial integration we deduce

\[ B_2(X_0) = -\frac{1}{2\pi i} \int_{\Delta_0}^{1-\Delta_0} \frac{S_0^2(t)}{N_0} \, d e(-N_0t) \]

\[ = -\frac{S_0^2(t)e(-N_0t)}{2\pi i N_0} \bigg|_{\Delta_0}^{1-\Delta_0} + \frac{1}{2\pi i N_0} \int_{\Delta_0}^{1-\Delta_0} e(-N_0t) \, d \left( S_0^2(t) \right) \]

\[ \ll X_0^{2-c} + X_0^{-c} |\Omega|, \]  

(81)
where

\[ \Omega = \int_{\Delta_0}^{1-\Delta_0} e(-N_0 t) d\left( S_0^2(t) \right). \]  

(82)

Next we consider \( \Omega \). Put

\[ \Gamma : z = f(t) = S_0^2(t), \quad \Delta_0 \leq t \leq 1 - \Delta_0. \]  

(83)

Now (82) and (83) imply

\[ \Omega = \int_{\Gamma} e\left( -N_0 f^{-1}(z) \right) dz. \]  

(84)

Using (77), (83) and that the integral (84) is independent of path we derive

\[ \Omega = \int_{\Gamma} e\left( -N_0 f^{-1}(z) \right) dz \ll \int_{\Gamma} |dz| \ll |f(\Delta_0)| + |f(1 - \Delta_0)| \ll X_0^2, \]  

(85)

where \( \Gamma \) is the line segment connecting the points \( f(\Delta_0) \) and \( f(1 - \Delta_0) \). Bearing in mind (81) and (85) we find

\[ B_2(X_0) \ll X_0^{2-c}. \]  

(86)

Summarizing (67), (80) and (86) we obtain

\[ B(X_0) \ll X_0^{2-c}. \]  

(87)

Taking into account (65), (66) and (87), for the number of solutions \( B_0(N_0) \) of the diophantine equation (64) we get

\[ B_0(N_0) \ll \frac{N_0^{2-1}}{\log^2 N_0}. \]

The lemma is proved.

We are now ready to estimate the sum \( \Gamma_2 \). We denote by \( \mathcal{F}(X) \) the set of all primes \( X/2 < p \leq X \) such that \( p - 1 \) has a divisor belongs to the interval \( (D, X/D) \). The inequality \( xy \leq x^2 + y^2 \) and (15) give us

\[ \Gamma_2^2 \ll (\log X)^6 \sum_{X/2 < p_1, \ldots, p_6 \leq X} \left| \sum_{d|p_1-1} \chi_4(d) \right| \left| \sum_{t|p_2-1} \chi_4(t) \right| \left| \sum_{t|p_3-1} \chi_4(t) \right| \left| \sum_{t|p_4-1} \chi_4(t) \right| \left| \sum_{t|p_5-1} \chi_4(t) \right| \left| \sum_{t|p_6-1} \chi_4(t) \right| \left| \sum_{t|p_6-1} \chi_4(t) \right|^{2}. \]
The summands in the last sum for which \( p_1 = p_4 \) can be estimated with \( \mathcal{O}(X^{3+\varepsilon}) \).

Thus

\[
\Gamma_2^2 \ll (\log X)^6 \Sigma_0 + X^{3+\varepsilon},
\]

where

\[
\Sigma_0 = \sum_{X/2 < p_1 \leq X} \left| \sum_{d|p_1-1} \chi_4(d) \right|^2 \sum_{\substack{X/2 < p_4 \leq X \atop p_4 \in \mathbb{P}(X) \atop p_4 \neq p_1}} \sum_{\substack{X/2 < p_2, p_3, p_5, p_6 \leq X \atop |p_1| + |p_2| + |p_4| = N \atop |p_3| + |p_5| + |p_6| = N}} 1.
\]

(88)

Now (89) and Lemma 16 yield

\[
\Sigma_0 \ll \frac{X^{4-2c}}{\log X} \Sigma_0' \Sigma_0'',
\]

(90)

where

\[
\Sigma_0' = \sum_{X/2 < p \leq X} \left| \sum_{d|p-1} \chi_4(d) \right|^2, \quad \Sigma_0'' = \sum_{\substack{X/2 < p \leq X \atop p \in \mathbb{P}(X)}} 1.
\]

Applying Lemma 10 we obtain

\[
\Sigma_0' \ll \frac{X (\log \log X)^7}{\log X}.
\]

(91)

Using Lemma 11 we get

\[
\Sigma_0'' \ll \frac{X (\log \log X)^{3}}{(\log X)^{1+2\theta_0}},
\]

(92)

where \( \theta_0 \) is denoted by (2).

Finally (88), (90), (91) and (92) imply

\[
\Gamma_2 \ll \frac{X^{3-c}(\log \log X)^5}{(\log X)^{\theta_0}}.
\]

(93)

8 Asymptotic formula for \( \Gamma_1 \)

Consider the sum \( \Gamma_1 \). From (17), (20) and (21) we deduce

\[
\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)},
\]

(94)

where

\[
\Gamma_1^{(i)} = \sum_{d \leq D} \chi_4(d) I_{1,d}(N), \quad i = 1, 2.
\]

(95)
8.1 Estimation of $\Gamma_1^{(1)}$

First we consider $\Gamma_1^{(1)}$. Using formula (35) for $J = (X/2, X]$, (95) and treating the reminder term by the same way as for $\Gamma_3^{(1)}$ we find

$$\Gamma_1^{(1)} = \Phi(X) \sum_{d \leq D} \frac{\chi_4(d)}{\varphi(d)} + \mathcal{O}\left(\frac{X^{3-c}}{\log X}\right),$$  

where

$$\Phi(X) = \int_{-\infty}^{\infty} I^3(t)e(-Nt) \, dt .$$  

(97)

Lemma 17. For the integral denoted by (97) the asymptotic formula

$$\Phi(X) = \frac{\Gamma^3\left(1 + \frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} \left(1 - \frac{1}{2^{3-c}}\right) X^{3-c} + \mathcal{O}\left(\frac{X^{3-c}}{e^{(\log X)^{\frac{1}{6}}}}\right)$$

holds.

Proof. From (97) write

$$\Phi(X) = \Theta_1 - \Omega_S + \Omega_S + \Theta_2 ,$$  

(98)

where

$$\Theta_1 = \int_{-\Delta}^{\Delta} I^3(t)e(-Nt) \, dt ,$$  

(99)

$$\Theta_2 = \int_{|t|>\Delta} I^3(t)e(-Nt) \, dt ,$$  

(100)

$$\Omega_S = \int_{-\Delta}^{\Delta} S^3(t)e(-Nt) \, dt .$$  

(101)
By (7), (10), (11), (24), (99), (101), Lemma 5 and Lemma 7 we obtain

\[ \Theta_1 - \Omega_S \ll \int_{-\Delta}^{\Delta} |S^3(t) - I^3(t)| \, dt \]
\[ \ll \int_{-\Delta}^{\Delta} |S(t) - I(t)| \left( |S^2(t) - I^2(t)| \right) \, dt \]
\[ \ll \left( \max_{|t| \leq \Delta} |S(t) - I(t)| + \Delta X \right) \left( \int_{-\Delta}^{\Delta} |S(t)|^2 \, dt + \int_{-\Delta}^{\Delta} |I(t)|^2 \, dt \right) \]
\[ \ll \left( \frac{X}{e^{(\log X) \frac{3}{4}}} + \Delta X \right) X^{2-c} \log^2 X \]
\[ \ll \frac{X^{3-c}}{e^{(\log X) \frac{3}{4}}} \cdot \tag{102} \]

Arguing as in [13] we get

\[ \Omega_S = \frac{\Gamma^3 \left( 1 + \frac{1}{c} \right)}{\Gamma \left( \frac{3}{4} \right)} \left( 1 - \frac{1}{2^{3-c}} \right) X^{3-c} \mathcal{O} \left( \frac{X^{3-c}}{e^{(\log X) \frac{3}{4}-c}} \right) \cdot \tag{103} \]

From (7), (33) and (100) it follows

\[ \Theta_2 \ll \int_{\Delta}^{\infty} |I(t)|^3 \, dt \ll \frac{1}{X^{3(c-1)}} \int_{\Delta}^{\infty} \frac{dt}{t^3} \]
\[ \ll \frac{1}{X^{3(c-1)} \Delta^2} \ll X^{3-c-c} \cdot \tag{104} \]

Now the lemma follows from (98), (102), (103) and (104). \( \square \)

According to [6] we have

\[ \sum_{d \leq D} \chi_4(d) \varphi(d) = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi_4(p)}{p(p-1)} \right) + \mathcal{O} \left( X^{-1/20} \right) \cdot \tag{105} \]

From (96) and (105) we obtain

\[ \Gamma^{(1)}_1 = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi_4(p)}{p(p-1)} \right) \Phi(X) + \mathcal{O} \left( \frac{X^{3-c}}{\log X} \right) + \mathcal{O} \left( \Phi(X) X^{-1/20} \right) \cdot \tag{106} \]

Now (106) and Lemma 17 yield

\[ \Gamma^{(1)}_1 = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi_4(p)}{p(p-1)} \right) \frac{\Gamma^3 \left( 1 + \frac{1}{c} \right)}{\Gamma \left( \frac{3}{4} \right)} \left( 1 - \frac{1}{2^{3-c}} \right) X^{3-c} + \mathcal{O} \left( \frac{X^{3-c}}{\log X} \right) \cdot \tag{107} \]
8.2 Estimation of $\Gamma_1^{(2)}$

Arguing as in the estimation of $\Gamma_3^{(2)}$ we get

$$\Gamma_1^{(2)} \ll \frac{X^{3-c}}{\log X}. \quad (108)$$

8.3 Estimation of $\Gamma_1$

Summarizing (94), (107) and (108) we deduce

$$\Gamma_1 = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)} \right) \frac{\Gamma^3 \left(1 + \frac{1}{2} \right)}{\Gamma \left(\frac{3}{2} \right)} \left(1 - \frac{1}{2^{3-c}} \right) X^{3-c} + O \left(\frac{X^{3-c}}{\log X} \right). \quad (109)$$

9 Proof of the Theorem

Bearing in mind (3), (16), (63), (93) and (109) we establish asymptotic formula (5).

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