LOW-SLOPE LEFSCHETZ FIBRATIONS

ADALET CENGEL, MUSTAFA KORKMAZ

Abstract. For $g \geq 3$, we construct genus-$g$ Lefschetz fibrations over the two-sphere whose slopes are arbitrarily close to 2. The total spaces of the Lefschetz fibrations can be chosen to be minimal and simply connected. It is also shown that the infimum and the supremum of slopes all Lefschetz fibrations are not realized as slopes.

1. Introduction

By the work of Donaldson [4], every closed symplectic 4-manifold, perhaps after blowing up, admits a Lefschetz fibration over the two-sphere. Conversely, Gompf [8] showed that the total space of genus-$g$ Lefschetz fibration is a symplectic 4-manifold if $g \geq 2$. For a given product of positive Dehn twists representing the identity of the mapping class group of a closed oriented surface of genus $g$, one can explicitly construct a genus-$g$ Lefschetz fibration over the two-sphere $S^2$. Conversely, every Lefschetz fibration with base $S^2$ gives a positive factorization of the identity, unique up to Hurwitz moves and global conjugation. This gives a combinatorial way to study symplectic 4-manifolds.

Let $f : X \to S^2$ be a (nontrivial) relatively minimal genus-$g$ Lefschetz fibration, where $X$ is a closed oriented smooth 4-manifold. Let $c_1^2(X)$ and $\chi_h(X)$ denote the first Chern number and the holomorphic Euler characteristic of $X$, respectively. The slope of $f$ is defined as the quotient

$$\lambda_f = \frac{c_1^2(X) + 8(g-1)}{\chi_h(X) + (g-1)}.$$ 

Xiao [19] proved that relatively minimal holomorphic genus-$g$ Lefschetz fibrations over a genus-$k$ curve satisfy the slope inequality $4 - 4/g \leq \lambda_f$. Hain conjectured that for $g \geq 2$ every relatively minimal genus-$g$ Lefschetz fibration over $S^2$ satisfies the slope inequality (cf. [5, Conjecture 4.12] and [1, Question 5.10]). In [12] and [13], several examples of Lefschetz fibrations violating this conjecture are constructed. In all of these examples, the slopes of the Lefschetz fibrations are close to $4 - 4/g$ for large $g$. We refer reader to [13] for a history of the slope conjecture.

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The purpose of this paper is to construct genus-$g$ Lefschetz fibrations with base $S^2$ having slope arbitrarily close to 2 for every $g \geq 3$. All of the slopes we get are greater than 2. Moreover, the total spaces of these Lefschetz fibrations can be chosen to be simply connected and minimal. We also show that the infimum and the supremum of the slopes of all Lefschetz fibrations cannot be realized as the slope of any Lefschetz fibration. See Section 4.

Here is the idea of the proof of our result. Consider a Lefschetz fibration $f_0$ of genus $g \geq 3$ with the base $S^2$. Its monodromy contains $C^r$ for some Dehn twist $C$ about a nonseparating simple closed curve and for $r \geq 1$, as was shown by Smith in [1, 15] that the monodromy cannot be contained in the Torelli group. By taking a twisted fiber sum of the Lefschetz fibration with itself carefully, one can guarantee that the resulting monodromy contains a product $(DE)^r$, where $D$ and $E$ are positive Dehn twists about two disjoint curves cobounding a subsurface of genus $h$ in a regular fiber for any chosen $1 \leq h \leq g - 1$. Thus each $DE$ can be replaced by a product of $(2h + 1)(2h + 2)$ Dehn twists about nonseparating curves by using the odd chain relation. These two operations can be applied the resulting Lefschetz fibration repeatedly to get Lefschetz fibration $f_n$ for every $n$. It turns out that the slopes of $f_n$ converge to $4h/(h + 1)$, which is independent of the initial choice $f_0$. See Theorem 4.1.

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$$4 - \frac{4}{g} - \frac{4(g - 2)}{g(g + 2)}$$

was constructed for each $g \geq 3$. We would like to thank İnanç Baykur, Noriyuki Hamada and Naoyuki Monden for their comments and their interests in this work.

2. Preliminaries and Notations

This section gives the necessary background and the known results used in our proofs.

2.1. Mapping Class Groups. Let $\Sigma_{g,k}$ be a compact connected oriented surface of genus $g$ with $k \geq 0$ boundary components, and let $\text{Mod}(\Sigma_{g,k})$ be the mapping class group of $\Sigma_{g,k}$, the group consisting of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,k} \to \Sigma_{g,k}$ fixing all points on the boundary. Isotopies are also assumed to be the identity on the boundary. In this paper, we have $k \leq 1$. If $k = 0$, we drop it from the notation and write $\Sigma_g$ and $\text{Mod}(\Sigma_g)$.

In this article, we always denote simple closed curves by the lowercase letters, and the positive (right) Dehn twist about them by the corresponding capital letters. If $f$ and $h$ are two diffeomorphisms of $\Sigma_{g,k}$, the composition $fh$ means that $h$ is applied first. The conjugation $fhf^{-1}$ is denoted by $h^f$. All diffeomorphisms and curves are considered up to isotopy.
Throughout the paper, we fix the surface $\Sigma_{g,1}$ in Figure 1 and the curves on it. The closed surface $\Sigma_g$ is the surface obtained from $\Sigma_{g,1}$ by gluing a disk along the boundary component. A curve on $\Sigma_{g,1}$ gives rise to a curve on $\Sigma_g$; these two curves are denoted by the same letter.

Dehn twists are basic mapping classes. A product of Dehn twists representing the identity in $\text{Mod}(\Sigma_{g,k})$ is called a relator. We first remind the following relations among Dehn twists. For the proofs the reader is referred to [7, 10].

- **Conjugation relation**: If $f \in \text{Mod}(\Sigma_{g,k})$ and if $c$ and $d$ are two simple closed curves on $\Sigma_{g,k}$ with $f(c) = d$, then $Cf = D$.

- **Commutativity relation**: If $c$ and $d$ are two disjoint simple closed curves on $\Sigma_{g,k}$, then $CD = DC$.

- **Braid relation**: If $c$ intersects $d$ transversely at one point, then $CDC = DCD$.

- **Lantern relation**: Let $a_1, a_2, a_3, a_4$ be four disjoint simple closed curves bounding a subsurface $\Sigma$ homeomorphic to a sphere with four boundary components. Then there are three simple closed curves $x, y, z$ on $\Sigma$ such that the Dehn twists about them satisfy the lantern relation

$$A_1 A_2 A_3 A_4 = XYZ.$$  

The relator

$$L = Z^{-1} Y^{-1} X^{-1} A_1 A_2 A_3 A_4$$  

is a lantern relator.

- **Chain relation**: A chain $(a_1, a_2, \ldots, a_n)$ of length $n$ is an ordered $n$-tuple of simple closed curves on $\Sigma_{g,k}$ satisfying

  (i) $a_i$ intersects $a_{i+1}$ transversely at one point if $1 \leq i \leq n - 1$, and
  (ii) $a_i$ and $a_j$ are disjoint if $|i - j| > 1$.

If the length of the chain is odd, say $n = 2h + 1$, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_{2h+1}$ is a genus-$h$ subsurface of $\Sigma_{g,k}$ with two boundary components. If $d$ and $e$ are two simple closed curves parallel to these two boundary components, then, in $\text{Mod}(\Sigma_{g,k})$, we have the relation

$$(A_1 A_2 \cdots A_{2h+1})^{2h+2} = DE,$$

called an odd chain relation [18]. In this case,

$$L_{2h+1} = (A_1 A_2 \cdots A_{2h+1})^{2h+2} D^{-1} E^{-1}$$

is an odd chain relator.

If the length of the chain is even, say $n = 2h$, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_{2h}$ is a genus-$h$ subsurface of $\Sigma_{g,k}$ with one
boundary component. If \( d \) is a simple closed curve parallel to this boundary component, then we have the relation
\[
(A_1 A_2 \cdots A_{2h})^{4h+2} = D
\]
in \( \text{Mod}(\Sigma_{g,k}) \), called an even chain relation [18]. We say that
\[
(3) \quad \mathcal{C}_{2h} = (A_1 A_2 \cdots A_{2h})^{4h+2} D^{-1}
\]
is an even chain relator.

\begin{itemize}
\item **Hyperelliptic relation:** Suppose that the surface is closed, i.e., \( k = 0 \). The Dehn twists about the curves of the maximal chain \((c_1, \ldots, c_{2g+1})\) on \( \Sigma_g \) satisfy the relation
\[
(C_1 C_2 \cdots C_{2g} C_{2g+1} C_{2g} \cdots C_2 C_1)^2 = 1,
\]
called the hyperelliptic relation. The product
\[
(4) \quad h_g = (C_1 C_2 \cdots C_{2g} C_{2g+1} C_{2g} \cdots C_2 C_1)^2.
\]
is called the hyperelliptic relator.
\end{itemize}

2.2. Lefschetz fibrations. Let \( S^2 \) denote the 2-sphere and let \( X \) be a closed connected oriented smooth 4-manifold. A Lefschetz fibration on \( X \) is a smooth surjective map \( f : X \to S^2 \) with finite set of critical points \( P = \{p_1, \ldots, p_n\} \) such that around each critical point \( p_i \) and critical value \( f(p_i) \), there are orientation-preserving complex coordinate charts on which \( f \) is of the form \( f(z_1, z_2) = z_1^2 + z_2^2 \). (In general, the base of a Lefschetz fibration can be a closed orientable surface, but we only consider those with the base \( S^2 \).) We may assume that each singular fiber \( f^{-1}(f(p_i)) \) contains only one singular point, which can be achieved by a small perturbation of the fibration. Throughout the paper, we assume that Lefschetz fibrations are nontrivial and relatively minimal, i.e. there is at least one singular fiber and no fiber contains a \((-1)\)-sphere.

A regular fiber of \( f \) is a closed connected orientable surface \( \Sigma_g \) of genus \( g \). The number \( g \) is called the genus of the Lefschetz fibration. A singular fiber \( f^{-1}(f(p_i)) \) is obtained from a regular fiber by collapsing a simple closed curve \( a_i \) on \( \Sigma_g \), called a vanishing cycle, to a point. The diffeomorphism

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The surface \( \Sigma_{g,1} \) and the curves on it. The closed surface \( \Sigma_g \) is obtained from \( \Sigma_{g,1} \) by gluing a 2-disk along \( \delta \).}
\end{figure}
type of a regular neighborhood of a singular fiber is determined by that vanishing cycle.

Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration. For a fixed regular value $b_0$, let us fix an identification of the regular fiber $f^{-1}(b_0)$ with the surface $\Sigma_g$. The Lefschetz fibration $f$ is then determined by a factorization

$$A_1 A_2 \cdots A_n = 1,$$

called the monodromy of the Lefschetz fibration, of the identity into positive Dehn twists in the mapping class group $\text{Mod}(\Sigma_g)$. The monodromy of $f$ is unique up to a sequence of Hurwitz moves and a global conjugation by a diffeomorphism. Recall that a Hurwitz move is

$$A_1 A_2 \cdots A_i A_{i+1} \cdots A_n \sim A_1 A_2 \cdots A_{i+1} A_i \cdots A_n,$$

or

$$A_1 A_2 \cdots A_i A_{i+1} \cdots A_n \sim A_1 A_2 \cdots A_{i+1} A_i^{-1} \cdots A_n.$$

A section of a Lefschetz fibration $f : X \to \mathbb{S}^2$ is a map $s : \mathbb{S}^2 \to X$ such that the composition $fs$ is the identity map of $\mathbb{S}^2$.

It follows from from the standard theory of Lefschetz fibrations that if $f : X \to \mathbb{S}^2$ is a Lefschetz fibration with a regular fiber $\Sigma_g$ and with the monodromy $A_1 A_2 \cdots A_n = 1$, the fundamental group $\pi_1(X)$ of $X$ is isomorphic to a quotient of the group

$$\Gamma_f = \pi_1(\Sigma_g)/\langle a_1, a_2, \ldots, a_n \rangle,$$

$\pi_1(\Sigma_g)$ divided by the normal closure of the vanishing cycles (cf. [8]). Moreover, if $f$ has a section, then the groups $\pi_1(X)$ and $\Gamma_f$ are isomorphic.

For $i = 1, 2$, let $f_i : X_i \to \mathbb{S}^2$ be a Lefschetz fibration of genus $g$ with a regular fiber $\Sigma_g$ and with the monodromy $W_i$. After removing tubular neighborhoods $\nu \Sigma_g$ of $\Sigma_g$ from each $X_i$, we identify the boundaries of $X_1 - \nu \Sigma_g$ and $X_2 - \nu \Sigma_g$ via a fiber-preserving orientation-reversing diffeomorphism $\psi$. The (twisted) fiber sum of $X_1$ and $X_2$ is defined as

$$f_1^\psi f_2 : X_1^\psi X_2 \to \mathbb{S}^2,$$

where $X_1^\psi X_2 = (X_1 - \nu \Sigma_g) \cup_{\psi} (X_2 - \nu \Sigma_g)$. If $\psi$ is unimportant, then we drop it from the notation. The monodromy of the new Lefschetz fibration is $W_1 W_2^\psi$.

**Theorem 2.1** ([17, 2]). Let $g \geq 2$. A fiber sum of any two Lefschetz fibrations of genus $g$ is minimal.

A Lefschetz fibration $f$ is called holomorphic if $X$ is a complex surface and $f$ is a holomorphic map for a suitable complex structure on $\mathbb{S}^2$.

The Euler characteristic of a Lefschetz fibration $f : X \to \mathbb{S}^2$ can be computed as $\epsilon(X) = 4 - 4g + n$, where $n$ is the number of singular fibers of $f$. Another invariant of $f$ is the signature $\sigma(X)$ of $X$. There are various techniques to compute the signature: see [14, 15, 6, 5, 11]. Endo and Nagami [5] showed that the signature of a Lefschetz fibration can be calculated by
using the signatures of relations contained in its monodromy. We use this method to calculate the signatures of Lefschetz fibrations we construct.

Let $K_f^2 = c_1^2(X) + 8(g - 1)$ and $\chi_f = \chi_h(X) + (g - 1)$, where $c_1^2(X)$ is the first Chern number and $\chi_h(X) = (\sigma(X) + e(X))/4$ is the holomorphic Euler characteristic of $X$. The slope $\lambda_f$ of $f$ is defined as the quotient

$$\lambda_f = \frac{K_f^2}{\chi_f}.$$ 

It follows from Lemma 3.2 in [16] that $K_f^2 \geq 4g - 4$. Since the 4-manifold $X$ is symplectic [8], $b_2^+(X) \geq 1$. The standard handle decomposition of $X$ has $n + 2$ two-handles. By Corollary 1.3 in [15], at least one of the vanishing cycles is nonseparating. Thus, $1 \leq b_2^+(X) + b_2^-(X) \leq n + 1$. It follows now that $\sigma(X) = b_2^-(X) - b_2^+(X) \geq 1 - n$, i.e., $\sigma(X) + n \geq 1$. From the equality

$$\chi_h = \frac{\sigma(X) + e(X)}{4} = \frac{\sigma(X) + 4 - 4g + n}{4} = \frac{\sigma(X) + n}{4} + 1 - g,$$

we conclude that $\chi_f \geq 1$.

2.3. Signatures of relations. The signature of a Lefschetz fibration can be computed from the signatures of the relations involved in its monodromy.

**Theorem 2.2 ([5, Theorem 4.2]).** Let $f : X \to \mathbb{S}^2$ be a Lefschetz fibration of genus $g$ with the monodromy $A_1A_2 \cdots A_n = 1$. Then the signature of $X$ is

$$\sigma(X) = I_g(A_1A_2 \cdots A_n).$$

Here, $I_g$ is an integer-valued function on the set of relators of the mapping class group of $\Sigma_g$, whose definition and properties can be found in [5].

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ be simple closed curves on $\Sigma_g$. Suppose that the Dehn twists about them satisfy the relation

$$A_1A_2 \cdots A_n = B_1B_2 \cdots B_m.$$ 

For a positive relator $W = U \cdot A_1A_2 \cdots A_n \cdot V$, a new positive relator

$$W' = U \cdot B_1B_2 \cdots B_m \cdot V$$

is obtained by replacing the product $A_1A_2 \cdots A_n$ in $W$ with $B_1B_2 \cdots B_m$. If $R = B_1B_2 \cdots B_mA_n^{-1} \cdots A_2^{-1}A_1^{-1}$, we say that $W'$ is obtained from $W$ by an $R$-substitution. In this case, we also say that the Lefschetz fibration with monodromy $W'$ is obtained from the Lefschetz fibration with monodromy $W$.

**Theorem 2.3 ([5, Theorem 4.3]).** Let $f : X \to \mathbb{S}^2$ and $f' : X' \to \mathbb{S}^2$ be two genus-$g$ Lefschetz fibrations. Suppose that $f'$ is obtained from $f$ by an $R$-substitution. Then the signatures of $X$ and $X'$ satisfy the equality

$$\sigma(X') = \sigma(X) + I_g(R).$$

The signatures of the relations we need are given below. For the proof, see [5, Section 3].
• **Lantern relator:** The signature of the lantern relator (1) is

\[ I_g(\mathcal{L}) = -1. \]

• **Odd chain relator:** The signature of the odd chain relator (2) is

\[ I_g(\mathcal{C}_{2h+1}) = -2h(h+2). \]

• **Hyperelliptic relator:** The signature of the hyperelliptic relator (4) is

\[ I_g(hg) = -4(g+1). \]

We note that if \( C \) and \( D \) commute (resp. satisfy the braid relation), then the signature of the relator \( CDC^{-1}D^{-1} \) (resp. \( CDC^{-1}C^{-1}D^{-1} \)) is 0. Also, the signature of the even chain relator (3) is

\[ I_g(\mathcal{C}_{2h}) = -4h(h+1) + 1. \]

3. Fiber sum, substitutions and slope

In this section, we determine the slope of the fiber sum of two Lefschetz fibrations. We also investigate how the slope changes under Lantern substitutions and \( \mathcal{C}_{2h+1} \)-substitutions. We remind that all Lefschetz fibrations we consider are nontrivial and relatively minimal, i.e. there is at least one singular fiber and no fiber contains a \((-1)\)-sphere.

**Lemma 3.1.** Let \( g \geq 2 \) and let \( f_1 : X_1 \to \mathbb{S}^2 \) and \( f_2 : X_2 \to \mathbb{S}^2 \) be two Lefschetz fibrations of genus \( g \). For any fiber sum \( f : X_1 \sharp X_2 \to \mathbb{S}^2 \) of \( f_1 \) and \( f_2 \), we have

(i) \( K_f^2 = K_{f_1}^2 + K_{f_2}^2 \) and

(ii) \( \chi_f = \chi_{f_1} + \chi_{f_2} \).

In particular, \( \lambda_f = (K_{f_1}^2 + K_{f_2}^2)/(\chi_{f_1} + \chi_{f_2}) \).

**Proof.** The Euler characteristic and the signature of the total space of \( f \) are

\[ e(X_1 \sharp X_2) = e(X_1) + e(X_2) + 4(g-1) \]

and

\[ \sigma(X_1 \sharp X_2) = \sigma(X_1) + \sigma(X_2), \]

respectively. From these two equalities, we compute

\[
K_f^2 = c_1^2(X_1 \sharp X_2) + 8(g-1) \\
= 3\sigma(X_1 \sharp X_2) + 2e(X_1 \sharp X_2) + 8(g-1) \\
= 3\sigma(X_1) + 2e(X_1) + 8(g-1) + 3\sigma(X_2) + 2e(X_2) + 8(g-1) \\
= K_{f_1}^2 + K_{f_2}^2,
\]

and

\[
\chi_f = \frac{\sigma(X_1 \sharp X_2) + e(X_1 \sharp X_2)}{4} + g - 1 \\
= \frac{\sigma(X_1) + \sigma(X_2) + e(X_1) + e(X_2) + 4(g-1)}{4} + g - 1 \\
= \chi_{f_1} + \chi_{f_2}.
\]

This proves the lemma. \( \square \)
Corollary 3.2. Let \( g \geq 2 \) and let \( f_1 : X_1 \to \mathbb{S}^2 \) and \( f_2 : X_2 \to \mathbb{S}^2 \) be two Lefschetz fibrations of genus \( g \) having the same slope \( \lambda \). Then the slope of any fiber sum of \( f_1 \) and \( f_2 \) is also \( \lambda \).

Lemma 3.3. Let \( g \geq 3 \). Suppose that a genus-\( g \) Lefschetz fibration \( f_2 : X_2 \to \mathbb{S}^2 \) is obtained from \( f_1 : X_1 \to \mathbb{S}^2 \) by a lantern substitution. Then \( \lambda_{f_2} < \lambda_{f_1} \).

Proof. Let \( n_1 \) and \( n_2 \) be the numbers of singular fibers of \( f_1 \) and \( f_2 \), respectively. Since \( f_2 \) is obtained from \( f_1 \) by a lantern substitution \( \mathcal{L} = Z^{-1}Y^{-1}X^{-1}A_1A_2A_3A_4 \), the signatures of \( f_1 \) and \( f_2 \) satisfies
\[
\sigma(X_2) = \sigma(X_1) + I_g(\mathcal{L}) = \sigma(X_1) - 1,
\]
and \( n_2 = n_1 + 1 \), so that
\[
e(X_2) = 4 - 4g + n_2 = 4 - 4g + n_1 + 1 = e(X_1) + 1.
\]
Thus,
\[
K_{f_2}^2 = K_{f_1}^2 - 1 \quad \text{and} \quad \chi_{f_2} = \chi_{f_1}.
\]
The lemma now follows.

Lemma 3.4. Let \( g \geq 3 \). Suppose that a genus-\( g \) Lefschetz fibration \( f_2 : X_2 \to \mathbb{S}^2 \) is obtained from \( f_1 : X_1 \to \mathbb{S}^2 \) by a \( \mathcal{C}_{2h+1} \)-substitution. Then
(i) \( K_{f_2}^2 = K_{f_1}^2 + 2h^2 \) and
(ii) \( \chi_{f_2} = \chi_{f_1} + \frac{1}{2}(h^2 + h) \).

Proof. Let \( n_1 \) and \( n_2 \) be the number of singular fibers of \( f_1 \) and \( f_2 \), respectively. Since \( f_2 \) is obtained from \( f_1 \) by a \( \mathcal{C}_{2h+1} \)-substitution,
\[
n_2 = n_1 - 2 + (2h + 1)(2h + 2) = n_1 + 4h^2 + 6h.
\]
Therefore,
\[
e(X_2) = 4 - 4g + n_2 = e(X_1) + 4h^2 + 6h
\]
and
\[
\sigma(X_2) = \sigma(X_1) + I_g(\mathcal{C}_{2h+1}) = \sigma(X_1) - 2h^2 - 4h.
\]
The lemma follows from these two equalities.

Corollary 3.5. Let \( g \geq 3 \). Suppose that a genus-\( g \) Lefschetz fibration \( f : X \to \mathbb{S}^2 \) is obtained from a fiber sum of the Lefschetz fibrations \( f_1 : X_1 \to \mathbb{S}^2 \) and \( f_2 : X_2 \to \mathbb{S}^2 \) followed by \( \mathcal{C}_{2h+1} \)-substitutions applied \( r \) times. Then
(i) \( K_f^2 = K_{f_1}^2 + K_{f_2}^2 + 2rh^2 \), and
(ii) \( \chi_f = \chi_{f_1} + \chi_{f_2} + \frac{1}{2}r(h^2 + h) \).
4. Main results

In this section we prove our main theorems. We construct a sequence of Lefschetz fibrations of genus \( g \geq 3 \) such that their slopes converge to \( \frac{4h}{n+1} \) for any \( 1 \leq h \leq g-1 \). Taking \( h = 1 \) gives our main theorem. The Lefschetz fibrations can be chosen to be simply connected and minimal. We then show that the infimum and the supremum of the slopes of all Lefschetz fibrations are not realized as the slope.

**Theorem 4.1.** Let \( g \geq 3, \ r \geq 1 \) and \( 1 \leq h \leq g-1 \) be integers and let \( f_0 : X_0 \to S^2 \) be a genus-\( g \) Lefschetz fibration with the monodromy \( VC^r \), where \( C \) is a positive Dehn twist about a nonseparating curve and \( V \) is a product of positive Dehn twists. For each integer \( n \geq 1 \), there is a genus-\( g \) Lefschetz fibration \( f_n : X_n \to S^2 \) such that

\[
\begin{align*}
(i) \quad & K_{f_n}^2 = 2^n K_{f_0}^2 + 2^n r [(h+1)^n - 1] h, \quad \text{and} \\
(ii) \quad & \chi_{f_n} = 2^n \chi_{f_0} + 2^{n-2} r [(h+1)^n - 1] (h+1).
\end{align*}
\]

In particular, the slope of \( f_n \) is

\[
\lambda_{f_n} = \frac{4K_{f_0}^2 + 4r [(h+1)^n - 1] h}{4\chi_{f_0} + r [(h+1)^n - 1] (h+1)},
\]

so that its limit is \( \frac{4h}{n+1} \).

**Proof.** Consider a regular fiber of \( f_0 \) and identify it with the surface \( \Sigma_g \). By a global conjugation of the monodromy, we may assume that \( \lambda = \lambda_1 \). Let us set \( V_0 = V, r_0 = r \) and \( W_0 = V_0 C_1^{r_0} \). Let \( \phi_1 \) and \( \phi_2 \) be two diffeomorphisms of the surface \( \Sigma_g \) such that \( \phi_1(c_1) = d_{h+1} \) and \( \phi_2(c_1) = e_{h+1} \). Note that the curves \( d_{h+1} \) and \( e_{h+1} \) are isotopic to the two boundary components of a regular neighborhood of \( c_1 \cup c_2 \cup \cdots \cup c_{2h+1} \) on \( \Sigma_g \), so that

\( D_{h+1} E_{h+1} = (C_1 C_2 \cdots C_{2h+1})^{2(h+1)} \).

Suppose that for an integer \( i \geq 0 \), \( f_i : X_i \to S^2 \) is a Lefschetz fibration with a regular fiber \( \Sigma_g \) and with the monodromy factorization \( W_i = V_i C_1^{r_i} \), where \( V_i \) is a product of positive Dehn twists. Then the product \( W_i^{\phi_1} W_i^{\phi_2} \) may be written as

\[
W_i^{\phi_1} W_i^{\phi_2} = V_i^{\phi_1} V_i^{D_{h+1}^{r_i} \phi_2} (D_{h+1} E_{h+1})^{r_i}.
\]

By trading each \( D_{h+1} E_{h+1} \) with \( (C_1 C_2 \cdots C_{2h+1})^{2h+2} \), we get a new factorization \( W_{i+1} \) of the identity

\[
W_{i+1} = V_i^{\phi_1} V_i^{D_{h+1}^{r_i} \phi_2} (C_1 C_2 \cdots C_{2h+1})^{(2h+2)r_i} = V_{i+1} C_1^{r_{i+1}},
\]

where \( r_{i+1} = 2(h+1)r_i \). Let \( f_{i+1} : X_{i+1} \to S^2 \) be the Lefschetz fibration with the monodromy \( W_{i+1} \).
Notice that $f_{i+1}$ is obtained from $f_i$ by taking a twisted fiber sum with itself followed by $C_{2h+1}$-substitutions performed $r_i$ times. By Corollary 3.5, we have

\[ K^2_{f_{i+1}} = 2K^2_{f_i} + 2r_i h^2, \text{ and} \]
\[ \chi_{f_{i+1}} = 2\chi_{f_i} + \frac{1}{2}r_i (h^2 + h). \]

It follows that

\[ K^2_{f_n} = 2^n K^2_{f_0} + \left( \sum_{i=0}^{n-1} 2^n r_i \right) h^2 \]
\[ = 2^n K^2_{f_0} + \left( \sum_{i=0}^{n-1} 2^n (h+1)^i r_0 \right) h^2 \]
\[ = 2^n K^2_{f_0} + 2^n r [(h+1)^n - 1] h. \]

By a similar computation we get

\[ \chi_{f_n} = 2^n \chi_{f_0} + 2^{n-2} r [(h+1)^n - 1] (h+1). \]

It is easy to check that the sequence $\lambda_{f_n}$ is decreasing if $\lambda_{f_0} > \frac{4h}{h+1}$ and is increasing if $\lambda_{f_0} < \frac{4h}{h+1}$. In either case, its limit is $\frac{4h}{h+1}$.

This completes the proof of the theorem.

Corollary 4.2. For each $g \geq 3$, there is a Lefschetz fibration $f$ whose slope is greater than 2 but arbitrarily close to 2.

**Proof.** Choose any Lefschetz fibration $f_0$ whose slope is greater than 2, and apply Theorem 4.1 for $h = 1$.

Theorem 4.3. Let $g \geq 3$. For every integer $n \geq 0$, there is a genus-$g$ Lefschetz fibration $F_n : Y_n \to S^2$ such that

(i) the slope of $F_n$ satisfies $2 < \lambda_{F_n} < 2 + \frac{4g-8}{2n}$, and

(ii) the 4-manifold $Y_n$ is minimal and simply connected.

**Proof.** In the mapping class group of the surface $\Sigma_{g,1}$ in Figure 1, it can easily be checked that the Dehn twist $\Delta$ about a curve parallel to boundary component $\delta$ may be written as

\[ \Delta = (C_1 C_2 \cdots C_{2g} D_g E_g C_{2g} \cdots C_2 C_1)^2. \]

Gluing a disc along $\delta$ gives rise to a surjective homomorphism $\text{Mod}(\Sigma_{g,1}) \to \text{Mod}(\Sigma_g)$. The relation in (6) gives the hyperelliptic relation

\[ W_0 = (C_1 C_2 \cdots C_{2g} C_{2g+1} C_{2g} \cdots C_2 C_1)^2 = 1 \]

in $\text{Mod}(\Sigma_g)$. Recall that a curve on $\Sigma_{g,1}$ and its image on $\Sigma_g$ are denoted by the same letter.

Let $f_0$ be the Lefschetz fibration with the monodromy $W_0$ and let $X_0$ be the total space of $f_0$. The Euler characteristic of $X_0$ is $e(X_0) = 4g+8$. Since
$W_0$ is the hyperelliptic relator, the signature of $X_0$ is $\sigma(X_0) = -4g - 4$, so that
\[ K_{f_0}^2 = 4g - 4 \]
and
\[ \chi_{f_0} = g. \]

Let $\phi_1$ and $\phi_2$ be two diffeomorphisms of the surface $\Sigma_g$ defined as
\[ \phi_1 = UD_2C_1U \text{ and } \phi_2 = UE_2C_1U, \]
so that $\phi_1(c_1) = d_2$ and $\phi_2(c_1) = e_2$. By taking $r = h = 1$, for each $n \geq 1$, Theorem 4.1 gives a genus-$g$ Lefschetz fibration $f_n : X_n \rightarrow S^2$ whose slope is
\[ \lambda_{f_n} = \frac{4K_{f_0}^2 + 4(2^n - 1)}{4\chi_{f_0} + (2^n - 1)2} = \frac{2^{n+1} + 8g - 10}{2^n + 2g - 1}. \]

It is easy to check that $2 < \lambda_{f_n} < 2 + \frac{4g-8}{2^n}$.

Since the diffeomorphisms $\phi_1$ and $\phi_2$ of $\Sigma_g$ fix the simple closed curves $c_3, \ldots, c_{2g+1}$, for every $n \geq 1$, the monodromy $W_n$ contains the Dehn twists $C_1, C_2, C_3, \ldots, C_8, C_9, \ldots, C_{2g+1}$ (cf. the equality (5)).

Now let $\psi$ be a self-diffeomorphism of $\Sigma_g$ such that $\psi(c_1) = c_4$ and $\psi(c_2) = c_5$. The word $W_n W_n^\psi$ is then a factorization of the identity into positive Dehn twists containing $C_j$ for all $1 \leq j \leq 2g+1$. Let $F_n : Y_n \rightarrow S^2$ be the Lefschetz fibration with the monodromy $W_n W_n^\psi$, so that it is a fiber sum of $f_n$ with itself. By Corollary 3.2, the slope of $F_n$ is equal to $\lambda_{f_n}$.

Since $\lambda_{F_n}$ is equal to $\lambda_{f_n}$, it satisfies the inequalities in (i). Since the curves $c_1, c_2, \ldots, c_{2g+1}$ are among vanishing cycles of $F_n$, the 4-manifold $Y_n$ is simply connected. Moreover, the manifold $Y_n$ is minimal by Theorem 2.1.

This finishes the proof of the theorem.

**Remark 4.4.** It is natural to ask whether one may use other substitutions in the proofs of Theorems 4.1 and 4.3. The number of Dehn twist $C_1$ appearing in the monodromy $W_n$ in above proofs grows exponentially with $n$. For this reason, it seems that one cannot use the lantern relation instead of the odd chain relation. Even chain relation cannot be used either because it involves a separating Dehn twist. We also note that in Theorem 4.3, each Lefschetz fibration $F_n$ has a section.

**Theorem 4.5.** Let $g \geq 3$. For every genus-$g$ Lefschetz fibration $f$, there is another Lefschetz fibration whose slope is less (resp. greater) than $\lambda_f$. In particular, there is no Lefschetz fibration whose slope is equal to the infimum (resp. supremum) of slopes of all Lefschetz fibrations.

**Proof.** Let $\Sigma_g$ be a regular fiber of $f$. Since $g \geq 3$, there are nonseparating simple closed curves $a_1, a_2, a_3, a_4, x, y, z$ on $\Sigma_g$ such that the Dehn twist about them satisfy the lantern relation $A_1A_2A_3A_4 = XYZ$.

The monodromy $W$ of $f$ contains at least one Dehn twist $C$ about a nonseparating curve $c$. Choose three diffeomorphisms $\varphi_1, \varphi_2, \varphi_3$ of $\Sigma_g$ such that $\varphi_1(c) = x$, $\varphi_2(c) = y$ and $\varphi_3(c) = z$, so that $C^{\varphi_1} = X$, $C^{\varphi_2} = Y$
and $C^{ϕ_3} = Z$. By applying Hurwitz moves, it is easy to see that $W'' = W^{ϕ_1}W^{ϕ_2}W^{ϕ_3}$ contains the factor $XYZ$, so that a lantern substitution can be applied. The slope of the Lefschetz fibration $f'$ with monodromy $W'$ is equal to $\lambda_f$, as it is a fiber sum of three copies of $f$. By Lemma 3.3, the slope of the Lefschetz fibration obtained from $f'$ by a lantern substitution has slope less that $\lambda_f$.

In the same vein, by taking the fiber sum of four copies of $f$ if necessary, we may assume that the monodromy of contains the product $A_1A_2A_3A_4$ so that inverse of the lantern substitution may be applied to get a Lefschetz fibration with larger slope.

The second statement follows from the first.

5. Final remarks

As mentioned in [13], slopes of Lefschetz fibrations are bounded. For each $g \geq 2$, let us define the functions $m_λ(g)$ and $M_λ(g)$ to be, respectively, the infimum and the supremum of the set $\{λ_f : f$ is a genus-$g$ Lefschetz fibration\}.

From the discussions in Section 2.2, we have $m_λ(g) > 0$.

Ozbagci [14] proved that $c_1^2(X) \leq 10χ_h + 2g - 2$ for every genus-$g$ Lefschetz fibration $f : X \to S^2$. This inequality is equivalent to $λ_f \leq 10$, so that $M_λ(g) \leq 10$. But then we have $λ_f < 10$ by Theorem 4.5, so that $c_1^2(X) < 10χ_h + 2g - 2$.

Since every genus-2 Lefschetz fibration is hyperelliptic, $m_λ(2) = 2$. Moreover, this number is the slope of every genus-2 Lefschetz fibration without separating vanishing cycles (c.f. [13]). Contrary to the $g = 2$ case, Theorem 4.5 says that for every Lefschetz fibration $f$ of genus $g \geq 3$, we have $m_λ(g) < λ_f < M_λ(g)$.

It might be interesting to determine the numbers $m_λ(g)$ and $M_λ(g)$. Baykur and Hamada suspect that $λ_f \geq 2$ for all genus-$g$ Lefschetz fibrations. In the light of this and of the results of this paper, we conjecture that $m_λ(g) = 2$ for all $g \geq 2$.

References

[1] J. Amaros, F. Bogomolov, L. Katzarkov, and T. Pantev Symplectic Lefschetz fibrations with arbitrary fundamental groups, J. Differential Geom. 54 (2000), no.3, 489–545, With an appendix by Ivan Smith.
[2] R. I. Baykur, Minimality and fiber sum decompositions of Lefschetz fibrations. Proc. Amer. Math. Soc. 144 (2016), no. 5, 2275–2284.
[3] R. I. Baykur, N. Hamada, Private communication.
[4] S. K. Donaldson, Lefschetz pencils on symplectic manifolds, J. Differential Geom., 53 (1999), no.2, 205–236.
[5] H. Endo and S. Nagami, Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations, Trans. Amer. Math. Soc. 357 (2005), no.8, 3179–3199.
[6] H. Endo, Meyer’s signature cocycle and hyperelliptic fibrations, Math. Ann. 316 (2000), no.2, 237–257.
[7] B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol.49, Princeton University Press, 2012.
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[8] R. E. Gompf and A. I. Stipsicz, 4-Manifolds and Kirby Calculus, Graduate Studies in Mathematics, vol.20. Amer. Math. Soc., Providence, RI, 1999.

[9] A. Kas, On the handlebody decomposition associated to a Lefschetz fibration, Pacific J. Math. 89 (1980), no.1, 89–104.

[10] M. Korkmaz, Minimal generating sets for the mapping class group of a surface, Handbook of Teichmüller theory. Volume III, IRMA Lect. Math Theor. Phys., vol.17, Eur. Math. Soc., Zurich, 2012, pp.441–463.

[11] Y. Matsumoto, Lefschetz fibrations of genus two—a topological approach, Topology and Teichmüller spaces (Katinkulta,1995), World Sci.Publ., River Edge, NJ, 1996, pp. 123–148.

[12] H. Miyachi and H. Shiga, Holonomies and the slope inequality of Lefschetz fibrations, Proc. Amer. Math. Soc. 139:4 (2011), 1299–1307.

[13] N. Monden, Lefschetz fibrations with small slope Pacific J. Math. 267 (2014), no.1, 243–256.

[14] B. Ozbagci, Signatures of Lefschetz fibrations, Pacific J.Math. 202 (2002), no.1, 99–118.

[15] I. Smith, Lefschetz fibrations and the Hodge bundle, Geom.Topol. 3 (1999), 211–233.

[16] A. Stipsicz, On the number of vanishing cycles in Lefschetz fibrations, Math. Res. Lett. 6 (1999), no. 3–4, 449–456.

[17] M. Usher, Minimality and symplectic sums, Int. Math. Res. Not. 2006, Art. ID 49857.

[18] B. Wajnryb, An elementary approach to the mapping class group of a surface, Geom.Topol. 3 (1999), 405–466.

[19] G. Xiao, Fibered algebraic surfaces with low slope, Math.Ann. 276 (1987), no.3, 449–466.

Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey
Email address: adalet.cengel@gmail.com
Email address: korkmaz@metu.edu.tr