Abstract. We study the well-posedness of compressible vortex sheets and entropy waves in two-dimensional steady supersonic Euler flows over Lipschitz walls with BV incoming flows. Both the Lipschitz wall of BV tangential angle function and the BV incoming flow perturb a background strong vortex sheet/entropy wave. In particular, when the total variation of the incoming flow perturbation around the background strong vortex sheet/entropy wave is small, we prove that the two-dimensional steady supersonic Euler flows containing a strong vortex sheet/entropy wave past the Lipschitz wall are $L^1$-stable. The weak waves are reflected after the nonlinear waves interact with the strong vortex sheet/entropy wave and the wall boundary. Using the wave-front tracking method, the existence of solutions in BV over the Lipschitz walls is first shown, when the total variation of the incoming flow perturbation around the background strong vortex sheet/entropy wave is suitably small. Then we establish the $L^1$-stability of the solutions with respect to the incoming flows. To achieve this, a Lyapunov functional, equivalent to the $L^1$-distance between two solutions containing the strong vortex sheets/entropy waves, is carefully constructed to include the nonlinear waves generated by both the wall boundary and the incoming flow. This Lyapunov functional is then proved to decrease in the flow direction, leading to the $L^1$-stability of the solutions. Furthermore, the uniqueness of these solutions extends to a larger class of viscosity solutions.

1. Introduction

We study the well-posedness of compressible vortex sheets and entropy waves in two-dimensional (2-D) steady supersonic Euler flows over Lipschitz walls with BV incoming flows. The inviscid compressible flows are governed by the 2-D steady Euler system:

\[
\begin{aligned}
(pu)_x + (pv)_y &= 0, \\
(pu^2 + p)_x + (p(uv))_y &= 0, \\
(puv)_x + (p(u^2 + p))_y &= 0, \\
(pu(E + \frac{p}{\rho}))_x + (pv(E + \frac{p}{\rho}))_y &= 0,
\end{aligned}
\]

(1.1)

with $u = (u, v)$, $p$, $\rho$, and $E$ representing the fluid velocity, scalar pressure, density, and total energy, respectively. Furthermore, the total energy $E$ is explicitly given by

\[
E = \frac{1}{2}|u|^2 + e,
\]

where the internal energy $e$ can be written as a function of $(p, \rho)$ defined through the thermodynamical relations: $e = e(p, \rho)$. The other two thermodynamic variables are entropy $S$ and temperature $T$. In the case of an ideal gas, pressure $p$ and internal energy $e$ can be expressed as

\[
p = R\rho T, \quad e = c_v T
\]

Date: April 27, 2022.

2010 Mathematics Subject Classification. 35B35, 35B40, 76J20, 35L65, 35B05, 85A05.

Key words and phrases. Full Euler equations, entropy waves, compressible vortex sheets, $L^1$-stability, steady flows, supersonic Euler flow, Riemann solutions, Lipschitz wall, BV perturbation, Glimm’s functional, nonlinear interaction, global existence, uniqueness.
with the adiabatic exponent $\gamma = 1 + \frac{R}{c_v} > 1$. In particular, in terms of density $\rho$ and entropy $S$, they have the form:

$$p(\rho, S) = \kappa \rho^{\gamma} e^{S/c_v}, \quad e(\rho, S) = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v} = \frac{R}{\gamma - 1} T(\rho, S),$$

where $R$, $c_v$, and $\kappa$ are all positive constants.

When entropy $S$ is constant, the flow becomes isentropic, which is governed by the first three equations in (1.1) with the pressure-density relation as $p = p(\rho) = \frac{R}{\gamma} \rho^{\gamma}$ for $\gamma > 1$. The limiting case $\gamma = 1$ corresponds to the isothermal flow.

Define $$c := \sqrt{p(\rho, S)}$$
as the sonic speed. Then, for polytropic gases, the sonic speed is $c = \sqrt{\gamma p/\rho}$. The type of flow is classified by the Mach number $M := \frac{u}{c} = \frac{\sqrt{\nabla e}}{e^{S/c_v}}$. When $M > 1$, system (1.1) governs a supersonic flow (i.e., $|u| > c$), which has all real eigenvalues and is hyperbolic. For $M < 1$, system (1.1) governs a subsonic flow (i.e., $|u| < c$), which has complex eigenvalues and is mixed-composite elliptic-hyperbolic. A sonic state corresponds $M = 1$.

We are interested in the $L^1$-stability of compressible vortex sheets and entropy waves in steady supersonic flow over the Lipschitz walls under the BV perturbations of the incoming flow (see Fig. 1.1). Multidimensional (M-D) steady supersonic Euler flows are important in many physical applications (cf. Courant-Friedrichs [15]). In particular, when the upstream flow is a uniform steady flow above the plane wall in $x < 0$ all the time, the downstream flow above a Lipschitz wall in $x > 0$ is governed by a steady Euler flow after a sufficiently long time. Moreover, compressible vortex sheets and entropy waves occur ubiquitously in nature and are fundamental waves. Furthermore, since steady Euler flows are large-time asymptotic states and may be global attractors of the corresponding unsteady compressible Euler flows, it is important to establish the existence of steady Euler flows and understand their qualitative properties, which are still widely open. In particular, the uniqueness and stability of compressible vortex sheet/entropy wave solutions in a class of physical entropy solutions for steady supersonic flow has been a longstanding open problem. On the other hand, compressible vortex sheets and entropy waves may be formulated as characteristic free boundaries, and the stability problem can also be formulated as a free boundary problem (cf. [7, 9, 10]), whose solution is a direct corollary of the well-posedness results in $BV$ established in this paper, for which the regularity of the free boundary in $\mathbb{R}^2$ is Lipschitz (with bounded total variation of the tangential angle function) but not in $C^1$ in general.

The stability of contact discontinuities for the Cauchy problem for 1-D strictly hyperbolic systems under a $BV$ perturbation has been studied by Sablé-Tougeron [30] and Corli-Sablé-Tougeron [10]. In particular, when a weak wave interacts with the boundary of strip $\{(t, x) : t \geq 0, -1 < x < 1\}$, the reflection coefficients for the reflected waves (similar to $K_{11}$ in below) are required to be less than one, which is the stability condition for the mixed problem in the strip in the earlier works; see, e.g., Sablé-Tougeron [30]. The nonlinear structural stability with the local-in time existence of 2-D compressible vortex sheet solutions was first established for the Mach number $M > \sqrt{2}$ in Coulombel-Secchi [13 14], while the 2-D compressible vortex sheets are not stable in general even locally when $M < \sqrt{2}$; also see [10] for further results for compressible vortex sheets. Moreover, multiple wild solutions for the Cauchy problem of the compressible Euler equations have been constructed; see [12] [22] and the references cited therein for both the isentropic and full Euler cases. Thus, it is fundamental to understand further the underlying physics of the stability/instability of compressible vortex sheets and entropy waves and their interactions with other nonlinear waves, even for the large Mach number cases. In particular, it is important to understand whether strong steady compressible vortex sheets and entropy waves are $L^1$-stable in the class of entropy solutions in $BV$ for steady supersonic flow for any Mach number $M > 1$, different from the time-dependent case in [13 14] as the results of this paper have indicated. We hope that the analysis and results in this paper will inspire further physics-based modeling and analysis of the interactions between the two types of strong waves and other nonlinear waves, and possibly light a path forward to
further understanding of the global existence and nonlinear stability of vortex sheet/entropy wave solutions for the 2-D compressible Euler equations in gas dynamics.

Working with the full Euler system \([1.1]\) and a uniform upstream flow containing one straight strong vortex sheet/entropy wave, Chen-Zhang-Zhu \([11]\) first established the global existence of supersonic Euler flows in \(BV\) with a strong vortex sheet/entropy wave under the \(BV\) perturbation of the Lipschitz wall by using the Glimm scheme. The essential difference between system \([1.1]\) (as analyzed in \([11]\) and §2–§7 below) and strictly hyperbolic systems as considered in \([16, 30]\) is that two of the four characteristic eigenvalues coincide and have two corresponding linearly independent eigenvectors that determine precisely the compressible vortex sheet and entropy wave, so that two independent parameters are required to describe them, respectively.

Consider the following vector functions of the physical variables \(U = (u, p, \rho)\):

\[
W(U) = (\rho u, \rho u^2 + p, \rho u v, p u (h + \frac{u^2 + v^2}{2}))^\top,
\]

\[
H(U) = (\rho v, \rho u v, \rho v^2 + p, \rho v (h + \frac{u^2 + v^2}{2}))^\top,
\]

with \(h = \frac{\gamma p}{(\gamma - 1) \rho}\). Then the steady Euler equations in \([1.1]\) can be expressed in the following conservative form:

\[
W(U)_x + H(U)_y = 0. \tag{1.4}
\]

In this paper, for completeness, we first show, via the wave-front tracking method, the existence of solutions containing strong vortex sheets/entropy waves is established. As corollaries of these results, the estimates on the uniformly Lipschitz semigroup of entropy solutions generated by the wave-front tracking approximations are obtained, and the uniqueness of solutions containing strong vortex sheets/entropy waves is established in a larger class of solutions, i.e., the class of viscosity solutions. More precisely, we focus mainly on the problem in domain \(\Omega\) over the Lipschitz wall for the supersonic Euler flows \(U\) governed by system \([1.4]\), given that the corresponding problem for the isentropic system is simpler to analyze; see Fig. 1.1. The boundary and initial data are given as follows:

(i) The Lipschitz function \(g \in \text{Lip}(\mathbb{R}_+; \mathbb{R})\) satisfies

\[
g(0) = g'(0+) = 0, \quad \lim_{x \to 0} \arctan(g'(x+)) = 0, \quad g' \in \text{BV}(\mathbb{R}_+; \mathbb{R}),
\]

and

\[
\text{TV}(g'(\cdot)) < \varepsilon \quad \text{for some constant } \varepsilon > 0.
\]

Denote \(\Omega := \{(x, y) : y > g(x), x \geq 0\}, \Gamma := \{(x, y) : y = g(x), x \geq 0\},\) and \(n(x\pm) = \frac{-g'(x\pm)}{\sqrt{(g'(x\pm))^2 + 1}}\) as the outer normal vectors to \(\Gamma\) at the respective points \(x\pm\) (cf. Fig. 1.1).

(ii) The incoming flow \(\bar{U}(y) := U_0^b(y) + \tilde{U}_0(y)\) at \(x = 0\) is composed of two parts:

(a) The upstream flow \(U_0^b(y)\) consists of one straight vortex sheet/entropy wave \(y = y_0^b > 0\) and two constant vectors \(U_0^b(y) = U_-\) when \(0 < y < y_0^b\) and \(U_0^b(y) = U_+\) when \(y > y_0^b > 0\) such that

\[
v_- = v_+ = 0, \quad u_+ = c_+, > 0,
\]

where \(c_\pm = \sqrt{\gamma p_\pm / \rho_\pm}\) is the sonic speed of state \(U_\pm\).

(b) The \(BV\) perturbation \(\tilde{U}_0(y) = (\bar{u}_0, \bar{p}_0, \bar{\rho}_0)(y) \in (L^1 \cap \text{BV})(\mathbb{R}; \mathbb{R}^4)\) at \(x = 0\) with \(\text{TV}(\tilde{U}_0) \ll 1\).

Then we consider the following initial-boundary value problem for system \([1.4]\):

**Boundary Condition:** \[
\mathbf{u} \cdot \mathbf{n}|_\Gamma = 0, \tag{1.5}
\]

**Cauchy Condition:** \[
U|_{x=0} = \bar{U}(y) = U_0^b(y) + \tilde{U}_0(y). \tag{1.6}
\]
Definition 1.1 (Admissible entropy solutions). A BV function $U = U(x, y)$ is said to be an entropy solution of the initial-boundary value problem (1.4)–(1.6) if and only if the following conditions hold:

(i) $U$ is a weak solution of (1.4) and satisfies (1.5)–(1.6) in the trace sense;

(ii) $U$ satisfies the steady entropy Clausius inequality:

\[
(\rho u S)_x + (\rho v S)_y \geq 0
\]

in the distributional sense in $\Omega$ including the Lipschitz wall boundary.

To solve the initial-boundary value problem (1.4)–(1.6), in this paper, we develop suitable methods to deal with the difficulties caused by the nonstrict hyperbolicity of the system and the Lipschitz wall boundary, in comparison with the previous results for the Cauchy problem of strictly hyperbolic systems of conservation laws. For supersonic Euler flow with a strong shock-front emanating from the wedge vertex, Chen-Li [8] worked out the issue for a Lipschitz wedge boundary. We now discuss some main differences in our work here from the Cauchy problem and the resulting key difficulties. We remark that, in the case of the Cauchy problem concerning only weak waves, the decrease of the Lyapunov functional and the $L^1$–stability of the solutions were obtained through the cancellation of distances on both sides of the waves. In the presence of a strong shock, for the $L^1$–stability of solutions of the Cauchy problem for strictly hyperbolic systems of conservation laws, the Lyapunov functional was identified to decrease by employing the strength of the strong shock to control the strengths of weak waves of the other families (e.g., see Lewicka-Trivisa [27]). In contrast with our Lipschitz wall problem, which is an initial-boundary value problem, there is no such cancellation as only one-side is possible near the boundary. Furthermore, no strong vortex sheets/entropy waves (characteristic discontinuities) nor strong shocks are present to handle the strength of the weak waves of the other families, and the terms in the estimates for the first and fourth families carry different signs. As such, it is difficult to say whether the functional can be made to decrease for our case of strong vortex sheets and entropy waves with multiplicity of eigenvalues. One of the key steps resolving this difficulty is to use the physical feature of the boundary condition that the flow of two solutions near the boundary must run in parallel (also see [8]). This observation helps us obtain additional quantitative relations near the boundary. Then, applying suitable weights and adjustments in the coefficients of the Lyapunov functional and using the cancellation between the different families, the functional is proved to decrease in the flow direction.

The rest of the paper is organized as follows: In §2, we recall some fundamental properties of the 2-D steady Euler system (1.1), i.e., (1.3), and discuss related nonlinear waves and wave interaction estimates. In §3, the wave-front tracking algorithm in the presence of strong vortex sheets/entropy waves is discussed, the suitable interaction potential $Q$ including the effect of the Lipschitz wall is constructed, and the existence of entropy solutions in $BV$ is established for the initial-boundary

Figure 1.1. Stability of the compressible vortex sheet/entropy wave in supersonic flow
value problem. In §4, we construct a Lyapunov functional (equivalent to the $L^1$-distance between two entropy solutions $U$ and $V$) to include the nonlinear waves produced by the wall boundary vertices. Then, in §5, the monotone decrease of the Lyapunov functional is established in the flow direction, leading to the $L^1$-stability of the solutions containing the strong vortex sheets/entropy waves. In §6, we employ the the estimates established in §3–§5 to obtain the existence of a Lipschitz semigroup of solutions generated by a wave-front tracking approximation, as well as some estimates on the uniformly Lipschitz semigroup produced by the limit of wave-front tracking approximations. Moreover, the uniqueness of solutions with strong vortex sheets/entropy waves is obtained in the larger class of viscosity solutions in §7.

2. Steady Full Euler Equations: Nonlinear Waves and Wave Interactions

In this section, we first present some basic properties of the steady Euler system \( \text{(1.1)} \), i.e., \( \text{(1.4)} \), and then discuss nonlinear waves and related interaction estimates, which will be employed in the subsequent development.

Notice that, when \( U(x, y) \in C^1 \), system \( \text{(1.4)} \) is equivalent to

\[
\nabla_U W(U) U_x + \nabla_U H(U) U_y = 0. \tag{2.1}
\]

Then the roots of the fourth degree polynomial

\[
det(\lambda \nabla_U W(U) - \nabla_U H(U)) \tag{2.2}
\]

are the eigenvalues of \( \text{(1.4)} \); that is, the solutions of the equation:

\[
(v - \lambda u)^2((v - \lambda u)^2 - c^2(1 + \lambda^2)) = 0, \tag{2.3}
\]

where \( c = \sqrt{\gamma p/\rho} \) is the sonic speed. For supersonic flows (i.e., \( |u| > c \)), system \( \text{(1.4)} \) is hyperbolic. Specifically, when \( u > c \), system \( \text{(1.4)} \) has four real eigenvalues in the \( x \)-direction:

\[
\lambda_j = \frac{uv + (-1)^j c \sqrt{u^2 + v^2 - c^2}}{u^2 - c^2} \quad \text{for } j = 1, 4;
\]

\[
\lambda_k = \frac{v}{u} \quad \text{for } k = 2, 3,
\]

with the four corresponding linearly independent eigenvectors given by

\[
r_j = \kappa_j(-\lambda_j, 1, \rho(\lambda_j u - v), \frac{\rho(\lambda_j u - v)}{c^2})^T \quad \text{for } j = 1, 4, \tag{2.4}
\]

\[
r_2 = (u, v, 0, 0)^T, \quad r_3 = (0, 0, 0, \rho)^T, \tag{2.5}
\]

where \( \kappa_j \) the re-normalization factors such that \( r_j \cdot \nabla \lambda_j = 1 \), given that the \( j \)-th-characteristic fields are genuinely nonlinear, \( j = 1, 4 \). The second and third linearly degenerate characteristic fields satisfy \( r_k \cdot \nabla \lambda_k = 0 \), \( k = 2, 3 \), which correspond to vortex sheets and entropy waves, respectively.

The wave curves in the phase space for \( \text{(1.4)} \) are determined by the Rankine-Hugoniot jump conditions:

\[
s[W(U)] = [H(U)], \tag{2.6}
\]

where \( s \) is the propagation speed of the discontinuity.

There are two different waves associated with the linearly degenerate families \( \lambda_k = \frac{u_0}{u_0}, k = 2, 3 \), with the corresponding linearly independent right eigenvectors \( r_k, k = 2, 3 \), in \( \text{(2.5)} \).

**Vortex sheets:**

\[
C_2(U_0) : \quad s = \frac{v}{u} = \frac{v_0}{u_0}, \quad p = p_0, \quad S = S_0, \quad u^2 + v^2 \neq u_0^2 + v_0^2. \tag{2.7}
\]

**Entropy waves:**

\[
C_3(U_0) : \quad s = \frac{v}{u} = \frac{v_0}{u_0}, \quad p = p_0, \quad (u, v) = (u_0, v_0), \quad S \neq S_0. \tag{2.8}
\]

The vortex sheet and the entropy wave above match as a single characteristic discontinuity in the physical \( (x, y) \)-plane, two independent parameters are needed to describe them in the phase space.
$U = (u, p, \rho) = (u, v, p, \rho)$ since there are two linearly independent eigenvectors corresponding to the repeated eigenvalues $\lambda_2 = \lambda_3 = \frac{\gamma}{u}$ of the linearly degenerate characteristics fields.

The nonlinear waves associated with $\lambda_j, j = 1, 4$, are shock waves and rarefaction waves. The propagation speeds of the shock waves are

$$s_j := \frac{u_0 v_0 + (-1)^j c_0 \sqrt{u_0^2 + v_0^2 - c_0^2}}{u_0^2 - c_0^2} \quad \text{for } j = 1, 4,$$

where $c_0^2 = \frac{\gamma}{\rho_0} \frac{\rho}{\rho_0}$ and $b_0 = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \frac{\rho}{\rho_0}$. Substituting $s_j$ into (2.10), the $j$-Hugoniot curves $S_j(U_0)$ through state $U_0$ are

$$S_j(U_0) : \quad [p] = \frac{c_0^2}{b_0} [\rho], \quad [u] = -s_j [v], \quad \rho_0(s_j u_0 - v_0) [v] = [p] \quad \text{for } j = 1, 4.$$

Written as $S_j^+(U_0), j = 1, 4$, the half curves of $S_j(U_0)$ for $\rho > \rho_0$ in the phase space are said to be the shock curves on which any state forms a shock with the below state $U_0$ in the $(x, y)$–plane respecting the entropy condition (1.7). Furthermore, for each $j = 1$ or $j = 4$, curves $S_j^+(U_0)$ and $R_j^-(U_0)$ at state $U_0$ have the same curvature.

If $U$ is a piecewise smooth solution, then any of the following conditions below is equivalent to the entropy inequality (1.7) in Definition 1.1 for a shock wave (see also (11)):

(i) The physical entropy condition: The density increases across the shock in the flow direction,

$$\rho_{\text{back}} < \rho_{\text{front}}. \quad (2.9)$$

(ii) The Lax entropy condition: On the $j$th-shock, the shock speed $s_j$ satisfies

$$\lambda_j(\text{back}) < s_j < \lambda_j(\text{front}) \quad \text{for } j = 1, 4,$$

$$s_1 < \lambda_{2,3}(\text{back}), \quad \lambda_{2,3}(\text{front}) < s_4. \quad (2.10)$$

The rarefaction wave curves $R_j^-(U_0)$ through state $U_0$ in the state space are given by

$$R_j^- : \quad dp = c_0^2 dp, \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v)dv = dp \quad \text{for } \rho < \rho_0, \quad j = 1, 4. \quad (2.11)$$

We now discuss several essential properties of the nonlinear waves and related wave interaction estimates in Lemmas 2.1–2.7 below. These facts will be used in the subsequent development; see also Chen-Zhang-Zhu (11) for further details.

2.1. Riemann Problems and Riemann Solutions

We focus on the related Riemann problems and their solutions in this section, which serve as the building blocks for the front tracking algorithm for the initial-boundary value problem (1.4)–(1.6).

Lateral Riemann problem. Consider the following lateral Riemann problem with boundary $\Gamma$:

$$\begin{align*}
U \big|_{x=x_0} &= U^0, \\
U \big|_{\Gamma} &= 0.
\end{align*} \quad (2.13)$$

It has been observed in (15) that, if the angle between the flow direction of the constant front-state $U^0$ and the wall at a boundary vertex is smaller than $\pi$ and larger than the extreme angle determined by the incoming flow state and $\gamma > 1$, then a unique 4-shock is generated, separating the front-state from the supersonic back-state. If the angle between the flow direction of the front-state and the wall at a boundary vertex is larger than $\pi$ and less than the extreme angle, then a 4-rarefaction wave is produced, emanating from the vertex. These waves are easily seen through the shock polar analysis (cf. (11) (15)). This signifies that, when the angle between the flow direction of the front-state and the wall at a boundary vertex is close to $\pi$, the lateral Riemann problem can be uniquely solved. For further details, see (11) (15).

In particular, the background solution $U = U_0^b$ is the unique entropy solution of problem (2.13) with $U = U_0^b$ and $g \equiv 0$, consisting of two constant states $U_- = (u_-, 0, p_-, \rho_-)$ and $U_+ = (u_+, 0, p_+, \rho_+)$, satisfying $u_+ > c_+ > 0$ in subdomains $\Omega_+$ and $\Omega_-$ of $\Omega$ separated by
the straight vortex sheet/entropy wave. The principal aim of this paper is to establish the $L^1$ well-posedness for problem \((1.4)–(1.6)\) for the solutions near the background solution \(U_0^b\) containing the strong vortex sheet/entropy wave \(\{U_-, U_+\}\).

Riemann problem involving only weak waves. Consider the Riemann problem:

\[
\begin{aligned}
(1.4), \\
U|_{x=x_0} = U = \begin{cases} 
U_a & \text{for } y > y_0, \\
U_b & \text{for } y < y_0,
\end{cases}
\end{aligned}
\]

with the constant states \(U_a\) and \(U_b\) denoting the above state and below state with respect to line \(y = y_0\), respectively. Then there exists \(\varepsilon > 0\) so that, for any states \(U_b\) and \(U_a\) in the neighborhood \(O_\varepsilon(U_+)\) of \(U_+\), or \(U_b\) and \(U_a\) in the neighborhood \(O_\varepsilon(U_-)\) of \(U_-\), the Riemann problem \((2.14)\) has a unique admissible solution consisting of at most four waves of shocks, rarefaction waves, a vortex sheet, and an entropy wave.

Riemann problem involving the strong vortex sheets/entropy waves. From now on, the notation \(\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) will be used to write \(U_a = \Phi (\alpha_1, \alpha_2, \alpha_3, \alpha_1; U_b)\) as the solution of the Riemann problem, where \(\Phi \in C^2\), and \(\alpha_j\) is the strength of the \(j\)-wave (measuring the jump across the wave). For any waves with \(U_b \in O_\varepsilon(U_-)\) and \(U_a \in O_\varepsilon(U_+)\), we also use \(\{U_b, U_a\} = (0, \sigma_2, \sigma_3, 0)\) to denote the strong vortex sheet/entropy wave that connects \(U_b\) and \(U_a\) with strength \((\sigma_2, \sigma_3)\).

That is,

\[
U_m = \Phi_2 (\sigma_2; U_b) := (u_b e^{\sigma_2}, v_b e^{\sigma_2}, p_b, \rho_b), \quad U_a = \Phi_3 (\sigma_3; U_m) := (u_m, v_m, p_m, \rho_m e^{\sigma_3}).
\]

In particular, for the background solution \(U_0^b\), \(\{U_-, U_+\} = (0, \sigma_2, \sigma_3, 0)\):

\[
U_+ = (u_+, 0, p_+, \rho_+) = (u_- e^{\sigma_2}, 0, p_-, \rho_- e^{\sigma_3}).
\]

We write \(G(\sigma_2, \sigma_3; U_b) := \Phi_3 (\sigma_3; \Phi_2 (\sigma_2; U_b))\) for any \(U_b \in O_\varepsilon(U_-)\). Then we have

**Lemma 2.1.** The vector function \(G(\sigma_2, \sigma_3; U_b)\) satisfies

\[
G_{\sigma_2} (\sigma_3, \sigma_2; U_b) = (u_b e^{\sigma_2}, v_b e^{\sigma_2}, 0, 0), \quad G_{\sigma_3} (\sigma_3, \sigma_2; U_b) = (0, 0, 0, \rho_b e^{\sigma_3}),
\]

and

\[
\nabla_U G(\sigma_3, \sigma_2; U_b) = \text{diag}(e^{\sigma_2}, e^{\sigma_2}, 1, e^{\sigma_3}).
\]

Furthermore, for the background plane vortex sheet and entropy wave with the below state \(U_- = (u_-, 0, p_-, \rho_-)\), above state \(U_+ = (u_+, 0, p_+, \rho_+)\), and strength \((\sigma_2, \sigma_3)\),

\[
\det(\mathbf{r}_4(U_+), G_{\sigma_3} (\sigma_3, \sigma_2; U_-), G_{\sigma_2} (\sigma_2, \sigma_2; U_-), \nabla_U G(\sigma_3, \sigma_2; U_-) \cdot \mathbf{r}_1(U_-)) > 0.
\]

These can be easily obtained from direct calculations, which are thus omitted. The properties in \((2.15)–(2.17)\) above play a fundamental role in achieving the necessary estimates for the strengths of reflected weak waves in the interaction between the strong vortex sheet/entropy wave and weak waves (see the proofs for Lemmas 2.4–2.7).

2.2. Estimates for Wave Interactions and Reflections. In the following, several essential estimates for wave interactions and reflections are provided. For their proofs and related details, see [III].

**Estimates for weak wave interactions.** For the weak wave interaction away from both the strong vortex sheet/entropy wave and the wall boundary in subdomains \(\Omega_+\) or \(\Omega_-\), we have the following estimate:

**Lemma 2.2.** Assume that \(U_b, U_m, U_a \in O_\varepsilon(U_+)\), or \(U_b, U_m, U_a \in O_\varepsilon(U_-)\), are three states with \(\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) and \(\{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, \beta_4)\). Then \(\{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)\) with

\[
\gamma_i = \alpha_i + \beta_i + O(1)\Delta(\alpha, \beta),
\]

\(i = 1, 2, 3, 4\),
where $\Delta(\alpha, \beta) = (|\alpha_4| + |\alpha_3| + |\alpha_2|)|\beta_1| + |\alpha_4|(|\beta_3| + |\beta_2|) + \sum_{j=1,4} \Delta_j(\alpha, \beta)$ and

$$\Delta_j(\alpha, \beta) = \begin{cases} 0, & \alpha_j \geq 0, \beta_j \geq 0, \\ \alpha_j ||\beta_j||, & \text{otherwise.} \end{cases}$$ (2.19)

Estimates on the boundary perturbation of weak waves and the reflection of weak waves on the boundary. We write $\{C_l(a_l,b_l)\}_{l=0}^{\infty}$ for points $\{(a_l,b_l)\}_{l=0}^{\infty}$ in the $(x,y)$-plane with $0 < a_l < a_{l+1}$.

Define

$$\begin{align*}
\theta_{l+1} &= \arctan \left( \frac{b_{l+1}-b_{l}}{a_{l+1}-a_{l}} \right), \quad \theta_{l} = \theta_{l+1} - \theta_{l-1}, \quad \theta_{-1,0} = 0, \\
\Omega_{l+1} &= \{(x, y) : x \in [a_l, a_{l+1}], y > b_l + (x-a_l)\tan(\theta_{l+1})\}, \\
\Gamma_{l+1} &= \{(x, y) : x \in (a_l, a_{l+1}), y = b_{l+1} + (x-a_l)\tan(\theta_{l+1})\},
\end{align*}$$

and the outer normal vector to $\Gamma_{l}$:

$$n_{l+1} = \frac{(b_{l+1} - b_{l}, a_{l+1} - a_{l})}{\sqrt{(b_{l+1} - b_{l})^2 + (a_{l+1} - a_{l})^2}} = (\sin(\theta_{l+1}), -\cos(\theta_{l+1})).$$ (2.21)

With the constant state $U_\infty$ consider the following lateral Riemann problem:

$$\begin{align*}
\begin{cases}
(2.1) & \text{in } \Omega_{l+1}, \\
U|_{x=a_l} = U_\infty, \\
u \cdot n_{l+1}|_{\Gamma_{l+1}} = 0.
\end{cases}
\end{align*}$$ (2.22)

Lemma 2.3. Suppose $\{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, 0)$ and $\{U_l, U_m\} = (0, 0, 0, \alpha_4)$ with $u_l \cdot n_l|_{\Gamma_l} = 0$. Then there exists a unique solution $U_{l+1}$ of problem (2.22) such that $\{U_{l+1}, U_a\} = (0, 0, 0, \delta_4)$ and $u_{l+1} \cdot n_{l+1}|_{\Gamma_{l+1}} = 0$. Moreover,

$$\delta_4 = \alpha_4 + K_{b_1}\beta_1 + K_{b_2}\beta_2 + K_{b_3}\beta_3 + K_{b_0}\theta_l,$$

where $K_{b_1}, K_{b_2}, K_{b_3},$ and $K_{b_0}$ are $C^2$-functions of $\beta_3, \beta_2, \beta_1, \alpha_4, \theta_{l+1},$ and $U_a$ satisfying

$$K_{b_i}|_{\{\theta_l=\alpha_4=\beta_1=\beta_2=\beta_3=0, U_a=U_{-}\} = 1, \quad K_{b_i}|_{\{\theta_l=\alpha_4=\beta_1=\beta_2=\beta_3=0, U_a=U_{-}\} = 0 \quad \text{for } i = 2, 3,$$ (2.24)

and $K_{b_0}$ is bounded. In particular, $K_{b_0} < 0$ at the origin.

This lemma has two purposes. The first is to estimate the weak waves generated by the vertices on the Lipschitz wall boundary. This boundedness will be used to control the boundary perturbation; see (3.2) below in the construction of the wave interaction potential $Q(x)$. The second is to estimate the strength of the reflected wave $\delta_4$ with respect to the incident wave $\alpha_1$. Property (2.24) of the coefficients will play an important role in controlling the reflected waves.

Estimates for the interactions between the strong vortex sheet/entropy wave and weak waves from below. Estimate (2.24) below plays a key role in ensuring the $L^1$-stability of entropy solutions, especially for the existence of constants $w^k$ and $w^k_\infty$ in Lemma 5.1 (see below). This estimate also ensures the existence of $K^* \in (K_{11}, 1)$ in the construction of the wave interaction potential $Q(x)$ in (3.2).

Lemma 2.4. Let $U_b, U_m \in O_\varepsilon(U_-)$ and $U_a \in O_\varepsilon(U_+)$ with

$$\{U_b, U_m\} = (0, \alpha_2, \alpha_3, \alpha_4), \quad \{U_m, U_a\} = (\beta_1, \sigma_2, \sigma_3, 0).$$

Then there exists a unique $(\delta_1, \sigma_2', \sigma_3', \delta_4)$ such that the Riemann problem (2.14) admits an admissible solution that consists of a weak $1-$wave of strength $\delta_1$, a strong vortex sheet/entropy wave of strength $(\sigma_2', \sigma_3')$, and a weak $4-$wave of strength $\delta_4$:

$$\{U_b, U_a\} = (\delta_1, \sigma_2', \sigma_3', \delta_4).$$
so that
\[\delta_1 = \beta_1 + K_{11}\alpha_4 + O(1)\Delta', \quad \delta_4 = K_{14}\alpha_4 + O(1)\Delta',\]
\[\sigma'_2 = \sigma_2 + \alpha_2 + K_{12}\alpha_4 + O(1)\Delta', \quad \sigma'_3 = \sigma_3 + \alpha_3 + K_{13}\alpha_4 + O(1)\Delta',\]
\[|K_{11}\{\alpha_4=\alpha_3=\alpha_2=0,\sigma_2=\sigma_3=\sigma_3=\sigma_30\}| = \frac{|\lambda_4(U_+)|e^{2\sigma_2+\sigma_3} - \lambda_4(U_-)|}{|\lambda_4(U_+)e^{2\sigma_2+\sigma_3} + \lambda_4(U_-)|} < 1, \quad (2.25)\]
where \(\sum_j |K_{1j}| \) is bounded, and \(\Delta' = |\beta_1|(\alpha_2 + \alpha_3)|.\)

**Lemma 2.5.** The coefficient, \(|K_{14}|\{\alpha_4=\alpha_3=\alpha_2=0,\sigma_2=\sigma_3=\sigma_3=\sigma_30\},\) in the strength \(\delta_4\) of a weak 4-wave in Lemma 2.4 remains bounded away from zero.

**Proof.** By Lemma 2.4, we can find a unique solution \((\delta_1, \sigma'_2, \sigma'_3, \delta_4)\) as a \(C^2\)-function of \(\alpha_2, \alpha_3, \alpha_4, \beta_1, \sigma_2, \sigma_3,\) and \(U_b\) to
\[\Phi_4(\delta_4; G(\sigma'_3, \sigma'_2; \Phi_1(\delta_1; U_b))) = G(\sigma_3, \sigma_2; \Phi_1(\beta_1; \Phi(\alpha_4, \alpha_3, \alpha_2, 0; U_b))). \quad (2.26)\]
That is,
\[\sigma'_i = \sigma'_i(\alpha_2, \alpha_3, \alpha_4, \beta_1, \sigma_2, \sigma_3) \quad \text{for} \ i = 2, 3, \quad \delta_j = \delta_j(\alpha_2, \alpha_3, \alpha_4, \beta_1, \sigma_2, \sigma_3) \quad \text{for} \ j = 1, 4,\]
where we have omitted \(U_b\) for simplicity. Moreover, from (11), we have
\[K_{1j} = \int_0^1 \partial_{\alpha_4} \delta_j(\alpha_2, \alpha_3, \theta \alpha_4, \beta_1, \sigma_2, \sigma_3) \, d\theta \quad \text{for} \ j = 1, 4.\]
Differentiate (2.25) with respect to \(\alpha_4,\) and let \(\beta_1 = \alpha_4 = \alpha_3 = \alpha_2 = 0, \sigma_2 = \sigma_20,\) and \(\sigma_3 = \sigma_30.\)

We obtain
\[\nabla_U G(\sigma_30, \sigma_20; U_-) \cdot \mathbf{r}_4(U_-) = \partial_{\alpha_4} \delta_4 \mathbf{r}_4(U_+) + \partial_{\alpha_4} \sigma'_3 G(\sigma_30, \sigma_20; U_-) + \partial_{\alpha_4} \Theta_2 G(\sigma_30, \sigma_30; U_-) + \partial_{\alpha_4} \delta_1 \nabla_U G(\sigma_30, \sigma_20; U_-) \cdot \mathbf{r}_1(U_-).\]

By Lemma 2.1, we have
\[|\partial_{\alpha_4} \delta_4| = \frac{|\det(\nabla_U G(\sigma_30, \sigma_20; U_-) \cdot \mathbf{r}_4(U_-), G(\sigma_30, \sigma_20; U_-), G(\sigma_30, \sigma_30; U_-), \nabla_U G(\sigma_30, \sigma_20; U_-) \cdot \mathbf{r}_1(U_-))|}{|\det(\nabla_U G(U_+), G(\sigma_30, \sigma_20; U_-), G(\sigma_30, \sigma_30; U_-), \nabla_U G(\sigma_30, \sigma_20; U_-) \cdot \mathbf{r}_1(U_-))|} \]
\[\geq \frac{\kappa_4(U_+)^2 \rho^2 u^2 e^{2\sigma_3+\sigma_30}(\lambda_4(U_+) - \lambda_1(U_-))}{\kappa_4(U_-)^2 \rho^2 u^2 e^{2\sigma_3+\sigma_30}(\lambda_4(U_+) - \lambda_1(U_-))} \]
\[= \frac{2k_4(U_+)^2 \lambda_4(U_-) e^{2\sigma_3+\sigma_30} + \lambda_4(U_-)}{\lambda_4(U_+) e^{2\sigma_3+\sigma_30} + \lambda_4(U_-)} > 0.\]

This completes the proof.

**Estimates for the interactions between the strong vortex sheet/entropy wave and weak waves from above. We have**

**Lemma 2.6.** Let \(U_b \in O_\varepsilon(U_-)\) and \(U_m, U_a \in O_\varepsilon(U_+)\) with
\[\{U_b, U_m\} = (0, \sigma_2, \sigma_3, \alpha_4), \quad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, 0).\]

Then there exists a unique \((\delta_1, \sigma'_2, \sigma'_3, \delta_4)\) such that the Riemann problem (2.14) admits an admissible solution that consists of a weak 1-wave of strength \(\delta_1,\) a strong vortex sheet/entropy wave of strength \((\sigma'_2, \sigma'_3),\) and a weak 4-wave of strength \(\delta_4);\)
\[\{U_b, U_a\} = (\delta_1, \sigma'_2, \sigma'_3, \delta_4),\]
so that
\[
\delta_1 = K_{21}\beta_1 + O(1)\Delta'', \quad \sigma'_2 = \sigma_2 + \beta_2 + K_{22}\beta_1 + O(1)\Delta'', \\
\sigma'_3 = \sigma_3 + \beta_3 + K_{23}\beta_1 + O(1)\Delta'', \quad \delta_4 = \alpha_4 + K_{24}\beta_1 + O(1)\Delta'',
\]
where \(\sum_{j=1}^4 |K_{2j}|\) is bounded, and \(\Delta'' = |\alpha_4|(|\beta_2| + |\beta_3|)\).

The constant, \(K_{21}\), here is used in the definition of weighted strength \(b_i\) of weak waves in (3.1).

**Lemma 2.7.** The coefficient, \(|K_{21}|\), is bounded away from zero, while the reflection coefficient \(|K_{24}|\) remains bounded by \(1\).

**Proof.** By Lemma 2.6, we can find a unique solution \((\delta_1, \sigma'_2, \sigma'_3, \delta_4)\) as a \(C^2\)-function of \(\alpha_2, \alpha_3, \alpha_4, \beta_1, \sigma_2, \sigma_3, \) and \(U_b\) to
\[
\Phi_4(\delta_1; G(\sigma'_3, \sigma'_2; \Phi_1(\delta_1; U_b))) = \Phi(0, \beta_3, \beta_2, \beta_1; \Phi_4(\alpha_4; G(\sigma_3, \sigma_2; U_b))). \tag{2.27}
\]
That is,
\[
\sigma'_i = \sigma'_i(\beta_1, \beta_2, \beta_3, \alpha_4, \sigma_2, \sigma_3) \quad \text{for} \quad i = 2, 3, \quad \delta_j = \delta_j(\beta_1, \beta_2, \beta_3, \alpha_4, \sigma_2, \sigma_3) \quad \text{for} \quad j = 1, 4,
\]
where we have omitted \(U_b\) for simplicity. Moreover, from [11], we have
\[
K_{2j} = \int_0^1 \partial_\beta_i \partial_\sigma_i (\theta, \beta_1, \beta_2, \beta_3, \alpha_4, \sigma_2, \sigma_3) \, d\theta \quad \text{for} \quad j = 1, 4.
\]

Similarly, differentiate (2.27) with respect to \(\beta_1\), and let \(\alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0, \sigma_2 = \sigma_20, \text{and} \sigma_3 = \sigma_30\). Then we obtain
\[
r_1(U_+) = \partial_{\beta_1} \delta_4 \, r_4(U_+) + \partial_\beta_2 \sigma'_2 \, G_{\sigma_3}(\sigma_30, \sigma_20; U) + \partial_\beta_3 \, \sigma_1' \, G_{\sigma_2}(\sigma_30, \sigma_20; U) + \partial_{\beta_1} \, \delta_1 \, \nabla_{\sigma} G(\sigma_30, \sigma_20; U) \cdot r_1(U_-).
\]

By Lemma 2.1, we have
\[
|\partial_{\beta_1} \delta_4| = \left| \frac{\det(r_1(U_+), G_{\sigma_1}(\sigma_30, \sigma_20; U_+), G_{\sigma_2}(\sigma_30, \sigma_20; U_+), r_1(U_+))}{\det(r_1(U_+), G_{\sigma_3}(\sigma_30, \sigma_20; U_-), G_{\sigma_2}(\sigma_30, \sigma_20; U_-), \nabla_{\sigma} G(\sigma_30, \sigma_20; U_-) \cdot r_1(U_-))} \right| > 0.
\]

However, for the reflection coefficient \(|K_{24}|\), we have
\[
|\partial_{\beta_3} \delta_4| = \left| \frac{\det(r_1(U_+), G_{\sigma_1}(\sigma_30, \sigma_20; U_+), G_{\sigma_2}(\sigma_30, \sigma_20; U_+), \nabla_{\sigma} G(\sigma_30, \sigma_20; U_-) \cdot r_1(U_-))}{\det(r_1(U_+), G_{\sigma_3}(\sigma_30, \sigma_20; U_-), G_{\sigma_2}(\sigma_30, \sigma_20; U_-), \nabla_{\sigma} G(\sigma_30, \sigma_20; U_-) \cdot r_1(U_-))} \right| < 1,
\]
where \(|K_{24}|\) is not necessarily bounded away from zero, but is less than one.

3. THE WAVE-FRONT TRACKING ALGORITHM AND GLOBAL EXISTENCE OF ENTROPY SOLUTIONS

We first start with a brief description of the wave-front tracking method to be employed throughout in §4–§7, and then establish the existence of entropy solutions when the perturbation of the incoming flow has small total variation at \(x = 0\).

The main scheme in the wave-front tracking method is to construct approximate solutions within a class of piecewise constant functions. We first approximate the initial data by a piecewise constant vector function. Then we solve the resulting Riemann problems exactly, with the exception of the rarefaction waves that are replaced by the rarefaction fans with many small wave-fronts of equal strengths. The outgoing fronts are continued up to the first time when two waves collide and a
new Riemann problem is solved. In this process, one has to modify the algorithm and introduce a simplified Riemann solver in order to keep the number of wave-fronts finite for all \( x \geq 0 \) in the flow direction. See Bressan [2, 1] and Baiti-Jenssen [1] for related references.

3.1. The Riemann Solvers.

As indicated in §2, the solution of the Riemann problem \( \{U_b, U_a\} \) is a self-similar solution given by at most five states separated by shocks, vortex sheet/entropy wave, or rarefaction waves. To connect state \( U_a \) to \( U_b \), there exist \( C^2 \)-curves \( \eta \to \varphi(\eta)(U) \) with parametrization (which is equivalent to the arc length and consistent with renormalization \( r_j \cdot \nabla \lambda_j = 1, j = 1, 4 \)) such that

\[
U_b = \varphi(\eta)(U_a) := \Upsilon_1(\eta_1) \circ \cdots \circ \Upsilon_1(\eta_1)(U_a)
\]

for some \( \eta = (\eta_1, \ldots, \eta_4) \), and \( U_j = \Upsilon_j(\eta_j) \circ \cdots \circ \Upsilon_1(\eta_1)(U_a) \) for \( j = 1, 2, 3 \).

Next, we describe the construction of front tracking approximations for the initial-boundary value problem \( \{1.4\} \). Denote \( \vartheta > 0 \) as the initial approximation parameter. Then the given initial data function \( \Upsilon \) is first approximated by a sequence of piecewise constant functions \( \Upsilon^\vartheta \) in the \( L^1 \)-norm, and the wall boundary is also approximated as described in [2, 20] in §2 with

\[
a_t = l \Delta x, \quad b_t = g(l \Delta x) \quad \text{for some } \Delta x > 0.
\]

For fixed \( \vartheta > 0 \), denote \( Z_\vartheta \) as the set of the total number of jumps in the approximate initial data functions \( \Upsilon^\vartheta \) and the tangential angle function of the wall boundary. Let \( \delta_\vartheta > 0 \) be a parameter so that a rarefaction wave is replaced by a step function whose steps are no further apart than \( \delta_\vartheta \). The discontinuity between two steps is set to propagate with a speed equal to the Rankine-Hugoniot speed of the jump connecting the states corresponding to the two steps. At any time, the simplified Riemann solver (defined below) is employed with constant \( \lambda > 0 \) (as the speed of the generated non-physical wave) which is strictly greater than all the wave speeds of system (2.1). The strength of the non-physical wave is the error generated when the simplified Riemann solver is applied.

**Accurate Riemann solver.** The accurate Riemann solver (ARS) is the exact solution of the Riemann problem, with the exception that every rarefaction wave \( \{w, R_j(w)(\alpha)\}, j = 1, 4 \), is divided into equal parts and replaced by a piecewise constant rarefaction fan of several new wave-fronts of equal strength.

**Simplified Riemann solver.** When only very weak waves are involved, the simplified Riemann solver (SRS) here is the same as the one described in [1, 3]. That is, all new weak waves are put together in a single *non-physical front* with positive speed larger than all the characteristic speeds. In the case of a weak wave interacting with the strong vortex sheet/entropy wave, the purpose of SRS is to ignore the strength of the weak wave, while preserving the strength of the strong vortex sheet/entropy wave, and to place the error in the non-physical wave in the following manner:

**Case 1.** A weak wave \( \{U_-, U_1\} \) collides with the strong vortex sheet/entropy wave \( \{U_1, U_+\} \) from below. The Riemann problem \( \{U_-, U_+\} \) is solved as follows:

\[
\begin{align*}
U_- & \quad \text{for } \frac{\chi}{2} < \chi(U_1, U_+), \\
U_2 & \quad \text{for } \chi(U_1, U_+) < \frac{\chi}{2} < \lambda, \\
U_+ & \quad \text{for } \frac{\chi}{2} > \lambda,
\end{align*}
\]

with \( \chi(U_1, U_+) \) as the speed of the strong vortex sheet/entropy wave, and state \( U_2 \) is solved in a way that \( \{U_-, U_2\} \) is the strong vortex sheet/entropy wave starting from \( U_- \) and \( \chi(U_1, U_+) = \chi(U_-, U_2) \). Hence, (SRS) keeps the same strength of the strong vortex sheet/entropy wave, and the error appears in the non-physical fronts.
Case 2. A weak wave \( \{U_2, U_+\} \) collides with the strong vortex sheet/entropy wave \( \{U-, U_2\} \) from above. The Riemann problem \( \{U-, U_+\} \) is solved as follows:

\[
\begin{cases}
U_- & \text{for } \frac{y}{x} < \chi(U_-, U_2), \\
U_2 & \text{for } \chi(U_-, U_2) < \frac{y}{x} < \hat{\lambda}, \\
U_+ & \text{for } \frac{y}{x} > \hat{\lambda},
\end{cases}
\]

with \( \chi(U_-, U_2) \) denoting the speed of the strong vortex sheet/entropy wave.

3.2. Construction of Wave Front Tracking Approximations

Given \( \vartheta > 0 \), the corresponding front tracking approximate solution \( U^q(x, y) \) is constructed as follows: At \( x = 0 \), all the Riemann problems in \( U^q \) are solved by using the accurate Riemann solver. Furthermore, we can change the speed of one of the incoming fronts so that, at any time \( x > 0 \), there is at most one collision involving only two incoming fronts. This adjustment of speed can be chosen arbitrarily small. Let \( \omega_0 \) be a fixed small parameter with \( \omega_0 \to 0 \) as \( \vartheta \to 0 \), which will be determined later. For convenience, subscript \( j \) in \( \alpha_j \) will be dropped henceforward, and we will write \( \alpha_j \) as \( \alpha \) when no ambiguity arises and employ the same notation \( \alpha \) as a wave and its strength as before; the same applies for \( \beta \).

Case 1. Two weak waves with strengths \( \alpha \) and \( \beta \) interact at some \( x > 0 \). The Riemann problem produced by this collision is solved in the following way:

- If \( |\alpha \beta| > \omega_0 \) and the two waves are physical, then the accurate Riemann solver is employed.
- If \( |\alpha \beta| < \omega_0 \) and the two waves are physical, or there is a non-physical wave, then the simplified Riemann solver is employed.

Case 2. A weak wave \( \alpha \) interacts with the strong vortex sheet/entropy wave and one weak wave at some \( x > 0 \). The Riemann problem produced by this collision is solved in the following way:

- If \( |\alpha| > \omega_0 \) and the weak wave is physical, then the accurate Riemann solver is applied.
- If \( |\alpha| < \omega_0 \) and the weak wave is physical, or this wave is non-physical, then the simplified Riemann solver is applied.

Case 3. The flow perturbation due to the Lipschitz wall boundary.

- When the change of angle of the boundary is larger than \( \omega_0 \) and the weak wave is physical, then the accurate Riemann solver is employed to solve the lateral Riemann problem.
- If the change of angle of the boundary is less than \( \omega_0 \), then this perturbation is ignored.

Case 4. The physical wave collides with the boundary. The accurate Riemann solver is employed to solve the lateral Riemann problem.

3.3. Glimm’s Functional and Wave Interaction Potential

The goal in this subsection is to construct the suitable Glimm-type functional and the associated wave interaction potential \( Q \) for the initial-boundary value problem \( \text{14} \text{– 16}. \) This involves a careful combination of the additional nonlinear waves generated from the wall boundary vertices.

**Definition 3.1 (Approaching waves).**

(i) Two weak fronts \( \alpha \) and \( \beta \), located at points \( y_\alpha < y_\beta \) and of the characteristic families \( j_\alpha, j_\beta \in \{1, \ldots, 4\} \), respectively, are said to be approaching each other if the following two conditions are concurrently satisfied:

- \( y_\alpha \) and \( y_\beta \) are both in one of the two intervals into which \( \mathbb{R} \) is partitioned by the location of the strong vortex sheet/entropy wave. That is, both waves are either in \( \Omega_- \) or \( \Omega_+ \);
- Either \( j_\alpha > j_\beta \) or else \( j_\alpha = j_\beta \) and at least one of them is a shock.

In this case, we write \( (\alpha, \beta) \in A \).

(ii) A weak wave \( \alpha \) of the characteristic family \( j_\alpha \) is said to be approaching the strong vortex sheet/entropy wave if either \( \alpha \in \Omega_- \) and \( j_\alpha = 4 \), or \( \alpha \in \Omega_+ \) and \( j_\alpha = 1 \). We then write \( \alpha \in A_{\alpha/\epsilon} \).
(iii) A weak wave $\alpha$ of the characteristic family $j_{\alpha}$ is said to be approaching the boundary if $\alpha \in \Omega_-$ and $j_{\alpha} = 1$. We then write $\alpha \in \mathcal{A}_b$.

Define the total (weighted) strength of weak waves in $U^\vartheta(x, \cdot)$ as
\[
\mathcal{V}(x) = \sum_{\alpha} |b_{\alpha}|,
\]
where, for a weak wave $\alpha$ of the $j$-family, its weighted strength is defined as
\[
b_{\alpha} = \begin{cases} 
k_+ \alpha & \text{if } \alpha \in \Omega_+ \text{ and } j_{\alpha} = 1, \\
\alpha & \text{if } \alpha \in \Omega_-,
\end{cases}
\]
where $k_+ = \frac{2K_{\vartheta}}{\bar{V}^\vartheta}$, and coefficient $K_{21}$ is given as in Lemma 2.6.

Next, the wave interaction potential $\mathcal{Q}(x)$ is defined as
\[
\mathcal{Q}(x) = C^* \sum_{(\alpha, \beta) \in \mathcal{A}} |b_{\alpha} b_{\beta}| + K^* \sum_{\alpha \in \mathcal{A}_{\vartheta/e}} |b_{\alpha}| + \sum_{\beta \in \mathcal{A}_b} |b_{\beta}| + \bar{K}_{b_0} \sum_{\alpha > x} |\theta_{\alpha}|
\]
\[
= \mathcal{Q}_A + \mathcal{Q}_{\vartheta/e} + \mathcal{Q}_b + \mathcal{Q}_{\vartheta},
\]
where $K^* \in (K_{11}, 1)$ and $K_{b_0} > K_{b_0}$, while $C^*$ is a constant to be specified later. To control the total variation of the new waves produced by the boundary vertices, $\mathcal{Q}_{\vartheta}$ in our wave interaction potential $\mathcal{Q}(x)$ is an added term, compared to that for the Cauchy problem.

The Glimm-type functional $\mathcal{G}$ is defined as follows:
\[
\mathcal{G}(x) = \mathcal{V}(x) + K \mathcal{Q}(x) + |U^\vartheta(x) - U_0^+| + |U_\vartheta(x) - U_0^-|,
\]
where states $U_\vartheta(x)$ and $U^\vartheta(x)$ are the below state and the above state of the strong vortex sheet/entropy wave at time $x$ respectively, $U_0^-$ and $U_0^+$ are the below and above state of the strong vortex sheet/entropy wave at $x = 0$ respectively, and $K$ is a large positive constant to be determined later.

Notice that $\mathcal{V}$, $\mathcal{Q}$, and $\mathcal{G}$ remain unchanged between any pair of subsequent interaction times. However, we will show that, across an interaction time $x$, both $\mathcal{Q}$ and $\mathcal{G}$ decrease.

**Lemma 3.1.** Assume that $TV(U_0(\cdot)) + TV(g' (\cdot))$ is sufficiently small. Then $\mathcal{V}(x)$ remains sufficiently small for all $x > 0$, and $TV(U^\vartheta(x, \cdot))$ has a uniform bound for any $\vartheta > 0$.

**Proof.** With the Glimm-type functional $\mathcal{G}$, consider
\[
\Delta \mathcal{G}(x) = \mathcal{G}(x^+) - \mathcal{G}(x^-),
\]
where $x^-$ and $x^+$ denote the times before and after the interaction time $x > 0$, respectively.

**Case 1.** Two weak waves $\alpha$ and $\beta$ collide. States $U^\vartheta(x)$ and $U_\vartheta(x)$ do not alter across this interaction time $x > 0$, so that
\[
\Delta \mathcal{G}(x) = \mathcal{V}(x^+) - \mathcal{V}(x^-) + K (\mathcal{Q}(x^+) - \mathcal{Q}(x^-))
\]
\[
\leq B_1 |b_{\alpha} b_{\beta}| - K ((C^* - B_0) |b_{\alpha} b_{\beta}| - C^* |b_{\alpha} b_{\beta}| \mathcal{V}(x^-)),
\]
where $B_0$ and $B_1$ are constants independent of $\vartheta$.

**Case 2.** A weak wave $\alpha$ of the 1-family interacts with the boundary.
\[
\Delta \mathcal{G}(x) = K_{b_1} \alpha - \alpha - K ((1 - K^* K_{b_1}) \alpha - C^* K_{b_1} \mathcal{V}(x^-) \alpha),
\]
where $K^* K_{b_1} < 1$.

**Case 3.** A new 4-wave $\alpha$ produced by the Lipschitz wall boundary.
\[
\Delta \mathcal{G}(x) = K_{b_0} \theta_1 - K ((\bar{K}_{b_0} - K^* K_{b_0}) \theta_1 - C^* K_{b_0} \theta_1 \mathcal{V}(x^-)),
\]
where $K_{b_0} < \bar{K}_{b_0}$ is large.

In the following two cases, states $U_\vartheta(x)$ and $U^\vartheta(x)$ change across this interaction time $x > 0$. 

Case 4. A weak wave $\alpha$ of the 4-family collides with the strong vortex sheet/entropy wave from below.

\[
\Delta G(x) \leq V(x^+) - V(x^-) + |U^o(x^+) - U^o(x^-)| + |U_\alpha(x^+) - U_\alpha(x^-)| + K(Q(x^+) - Q(x^-)) \\
\leq (K_{11} + K_{14} + C^*)\alpha - \alpha - K\left(\alpha - C^*(K_{11} + K_{14})V(x^-)\alpha\right).
\]

Case 5. A weak wave $\alpha$ of the 1-family collides with the strong vortex sheet/entropy wave from above.

\[
\Delta G(x) \leq (K_{21} + K_{24} + C^*)\alpha - b_\alpha - K\left((K^*b_\alpha - K_{21}\alpha) - C^*(K_{21} + K_{24})V(x^-)\alpha\right).
\]

In the cases above, $K_{11} < K^* < 1$, $K^*b_\alpha \geq 2K_{21}\alpha$ in connection with weight $k_+$, and the constant $C^* > \mathcal{B}_0 > 0$ is large.

Next, we establish that the total (weighted) strength of waves in $U^\theta(x, \cdot)$ remains sufficiently small for all $x > 0$ if it is sufficiently small at $x = 0$. More precisely,

\[V(x) \ll 1 \quad \text{for all } x > 0.\]

This can be proved as follows:

(i) $x_1 > 0$ is the first interaction time. Given that $V(x^-_1) = V(0) \leq TV(U_0(\cdot)) \ll 1$ and $\sum_{l=0}^{\infty} \theta_l \leq TV(g(\cdot)) \ll 1$ in Cases 1–5 above, we conclude that, for $K$ sufficiently large and $\omega_\theta$ sufficiently small,

\[\Delta G(x_1) \leq 0, \quad \text{i.e.,} \quad G(x^+_1) \leq G(x^-_1) = G(0).\]

Therefore,

\[V(x^+_1) \leq G(x^+_1) \leq G(0) \leq V(0) + K\mathcal{Q}(0) = V(0) + K\left(C^*V^2(0) + V(0) + K\sum_{l=0}^{\infty} \theta_l\right) \leq C\left(V(0) + \sum_{l=0}^{\infty} \theta_l\right) \ll 1.
\]

(ii) $V(x^-_m) \ll 1$ and $G(x^+_m) \leq G(x^-_m)$ for any $m < n$. Then, for the next interaction time $x_n$, similar to Case (i), we also conclude

\[\Delta G(x_n) \leq 0, \quad \text{i.e.,} \quad G(x^+_n) \leq G(x^-_n) = G(x^+_n-1).\]

Therefore, all together, we obtain

\[V(x^+_1) + |U^o(x^+_1) - U_0^+| + |U_\alpha(x^+_n) - U^-_n| \leq G(x^+_n) \leq G(x^-_m) = G(x^+_n-1) \leq \ldots \leq G(0) = V(0) + K\mathcal{Q}(0) = V(0) + K\left(C^*V^2(0) + V(0) + K\sum_{l=0}^{\infty} \theta_l\right) \leq C\left(V(0) + \sum_{l=0}^{\infty} \theta_l\right) \ll 1.
\]

This implies that $V(x) \ll 1$ for all $x > 0$, since $C$ is independent of $x$.

Furthermore, the total variation of $U^\theta(x, \cdot)$ is uniformly bounded:

\[TV\{U^\theta(x, \cdot)\} \approx V(x)|U^o(x) - U_0^+| + |U_\alpha(x) - U^-_0| + |\sigma_{20}| + |\sigma_{30}| = \mathcal{O}(1). \quad (3.4)
\]

This completes the proof.
In order to define the front tracking approximate solution \( U^0(x, \cdot) \) for any \( x > 0 \), along with a uniform bound on the total variation, we also need to have a finite number of wave-fronts in \( U^0(x, \cdot) \). This is given by the following lemma.

**Lemma 3.2.** For any fixed \( \vartheta > 0 \) small enough, the number of wave-fronts in \( U^0(x, y) \) is finite and the approximate solutions \( U^0(x, y) \) are defined for all \( x > 0 \). Moreover, for any \( x > 0 \), the total strength of all the non-physical waves is of order \( O(1) (\vartheta + \omega) \).

**Proof.** We first note that the total interaction potential \( Q(x) \) remains unchanged when there is no interaction and decreases across an interaction time \( x > 0 \), as discussed in Cases 1–5 in Lemma 3.1. Furthermore, from Cases 1–5 and the subsequent analysis above, we have concluded that \( V(x) \ll 1 \). Thus, we can fix some number \( \nu \in (0, 1) \) such that

\[
\Delta Q(x) = Q(x^+) - Q(x^-) \leq \begin{cases} 
-\nu |b_\alpha b_\beta| & \text{if both waves } \alpha \text{ and } \beta \text{ are weak,} \\
-\nu |b_\alpha| & \text{if the weak wave } \alpha \text{ hits the strong vortex sheet/entropy wave,} \\
-\nu |\theta| & \text{if the angle of the boundary changes.}
\end{cases}
\] (3.5)

Now, following an argument similar to the one given in [1, 2], we reach the following conclusions: Note that initially \( Q(0) \) is bounded and \( Q \) decreases thereafter for each case. Moreover, in the case where the interaction potential between the incoming waves or the change of angle of the boundary is larger than \( \omega \), \( Q \) decreases by at least \( \nu \omega \) in these interactions, as implied by the bounds given in [3–5]. Following the wave-front tracking method in our problem, new physical waves can only be produced by such interactions. Furthermore, when the weak wave \( \alpha \) of 1-family collides with the wall boundary, we have solved the lateral Riemann problem and shown that, after this interaction, there is only a reflected wave of 4-family with the reflection coefficient 1. Thus, before and after this interaction, the number of the waves stays the same, which implies that the number of the waves is finite. Finally, because the non-physical waves are generated only when the physical waves collide, we can also conclude that the number of non-physical wave fronts is finite; if two waves can only collide once, the number of interactions is also finite. Consequently, it follows that the approximate solutions \( U^0(x, \cdot) \) are defined for all \( x > 0 \). The similar argument allows us to conclude that the total strength of all the non-physical wave fronts at any \( x \) is of order \( O(1) (\vartheta + \omega) \). This completes the proof.

Following the line of arguments as in [1, 2] for the wave-front tracking algorithm and Lemma 3.1 above, we conclude this section with the following theorem for the global existence of entropy solutions of the initial-boundary value problem \([1.4]–[1.6] \).

**Theorem 3.1.** Suppose that \( \text{TV}(\tilde{U}_0(\cdot)) + \text{TV}(\tilde{g}(\cdot)) \) is suitably small. Then, for the initial-boundary value problem \([1.4]–[1.6] \), there exists a global entropy solution in BV satisfying the steady Clausius entropy inequality \([1.7] \).

4. **The Lyapunov Functional for the \( L^1 \)-Distance between Two Solutions**

To show that the wave-front tracking approximations, constructed for the existence analysis in §3, converge to a unique limit, we estimate the distance between any two \( \theta \)-approximate \( U \) and \( V \) of problem \([1.4]–[1.6] \). To this end, we develop the Lyapunov functional \( \Phi(U, V) \), equivalent to the \( L^1 \)-distance:

\[
C^{-1} \|U(x, \cdot) - V(x, \cdot)\|_{L^1} \leq \Phi(U, V) \leq C \|U(x, \cdot) - V(x, \cdot)\|_{L^1},
\]

and prove that \( \Phi(U, V) \) is almost decreasing:

\[
\Phi(U(x_2, \cdot), V(x_2, \cdot)) - \Phi(U(x_1, \cdot), V(x_1, \cdot)) \leq C \vartheta (x_2 - x_1) \quad \text{for all } x_2 > x_1 > 0,
\]

for some constant \( C > 0 \). Here \( U \) and \( V \) are two approximate solutions constructed via the wave-front tracking method, and the small approximation parameter \( \vartheta \) is responsible for controlling the subsequent errors:
• Errors in the approximation of the initial data and the boundary.
• Errors in the speeds of shocks, vortex sheets, entropy waves, and rarefaction fronts.
• The total strength of non-physical fronts.
• The maximum strength of rarefaction fronts.

Along the line of arguments presented in [6, 27, 29], with time $x$ fixed, at each $y$, one connects state $U(y)$ with $V(y)$ in the state space by going along the Hugoniot curves $S_1, C_2, C_3,$ and $S_4$. Depending on the location of the strong vortex sheet/entropy wave in $U(y)$ and $V(y)$, the distance between $U(y)$ and $V(y)$ is estimated along discontinuity waves in possibly different directions, determining the strength of the $j$-Hugoniot wave, $h_j(y)$, in the following way:

- If $U(y)$ and $V(y)$ are both in $\Omega_-$ and $\Omega_+$, then it begins at state $U(y)$ and moves along the Hugoniot curves to reach state $V(y)$.
- If $U(y)$ is in $\Omega_-$ and $V(y)$ is in $\Omega_+$, then it begins at state $U(y)$ and moves along the Hugoniot curves to reach state $V(y)$.
- If $V(y)$ is in $\Omega_-$ and $U(y)$ is in $\Omega_+$, then it begins at state $V(y)$ and moves along the Hugoniot curves to reach state $U(y)$.

Define the $L^1$–weighted strengths of the waves in the solution of the Riemann problem $\{U(y), V(y)\}$ or $\{V(y), U(y)\}$ as follows:

$$q_j(y) = \begin{cases} w^b_j h_j(y) & \text{whenever } U(y) \text{ and } V(y) \text{ are both in } \Omega_-, \\ w^m_j h_j(y) & \text{whenever } U(y) \text{ and } V(y) \text{ are both in different domains,} \\ w^a_j h_j(y) & \text{whenever } U(y) \text{ and } V(y) \text{ are both in } \Omega_+, \end{cases}$$

(4.1)

with constants $w^b_j$, $w^m_j$, and $w^a_j$ above to be specified later on, based on the estimates of wave interactions and reflections in Lemmas 2.2–2.7.

We define the following Lyapunov functional:

$$\Phi(U, V) = \sum_{j=1}^{4} \int_{y(x)}^{\infty} |q_j(y)| W_j(y) \, dy,$$  

(4.2)

where the weights are given by

$$W_j(y) = 1 + \mathcal{K}_1 A_j(y) + \mathcal{K}_2 \left( Q(U) + Q(V) \right)$$

(4.3)

with constants $\mathcal{K}_1$ and $\mathcal{K}_2$ to be determined later. Here $Q$ denotes the total wave interaction potential incorporating the boundary effect as defined in (3.2), and $A_j(y)$ denotes the total strength of waves in $U$ and $V$, which approach the $j$-wave $q_j(y)$, defined in the following manner (for $y$ where there is no jump in $U$ or $V$):

$$A_j(y) = F_j(y) + G_j(y) + \begin{cases} H_j(y) & \text{if } j \text{-wave } q_j(y) \text{ is small and the } j \text{-field is genuinely nonlinear,} \\ 0 & \text{if } j = 2, 3, \text{ and } q_j(y) = B \text{ is large.} \end{cases}$$  

(4.4)

We first define the following global weights $G_j$:

| $G_j(y)$ | $U, V$ are both in $\Omega_-$ | $U, V$ are in distinct regions | $U, V$ are both in $\Omega_+$ |
|----------|-------------------------------|-------------------------------|-------------------------------|
| $G_1(y)$ | 4B                            | 2B                            | 4B                           |
| $G_{2,3}(y)$ | 0                            | 0                             | 0                            |
| $G_4(y)$ | 4B                            | 2B                            | 2B                           |
The summands in (4.4) are defined as follows:

\[
F_j(y) = \sum_{\alpha \in \mathcal{J} \setminus \mathcal{SC}} |\alpha| + \sum_{y_a < y, j < k_\alpha \leq 4} |\alpha|,
\]

\[
H_j(y) = \begin{cases} 
\left(\sum_{\alpha \in \mathcal{J}(U) \setminus \mathcal{SC}, y_a < y, k_\alpha = j} + \sum_{\alpha \in \mathcal{J}(V) \setminus \mathcal{SC}, y_a > y, k_\alpha = j}\right) |\alpha| & \text{if } q_j(y) < 0, \\
\left(\sum_{\alpha \in \mathcal{J}(U) \setminus \mathcal{SC}, y_a < y, k_\alpha = j} + \sum_{\alpha \in \mathcal{J}(V) \setminus \mathcal{SC}, y_a > y, k_\alpha = j}\right) |\alpha| & \text{if } q_j(y) > 0,
\end{cases}
\]

where, at each \(x\), \(\alpha\) stands for the (non-weighted) strength of wave \(\alpha \in \mathcal{J}\), located at point \(y_a\) and belonging to the characteristic family \(k_\alpha\): \(\mathcal{J} = \mathcal{J}(U) \cup \mathcal{J}(V)\) and \(\mathcal{SC} = \mathcal{SC}(U) \cup \mathcal{SC}(V)\) are the set of all the waves (in \(U\) and \(V\)) and the set of all the strong characteristic discontinuities (in \(U\) and \(V\)), respectively.

Under the assumption that \(TV(\tilde{U}_0(\cdot)) + TV(\tilde{V}_0(\cdot)) + TV(g(\cdot))\) is small enough with \(U(x, \cdot), V(x, \cdot) \in \text{BV} \cap L^1\), one concludes

\[
\mathcal{M}^{-1} \left\| U(x, \cdot) - V(x, \cdot) \right\|_{L^1} \leq \sum_{j=1}^{4} \int_{y(x)}^{\infty} |q_j(y)| \, dy \leq \mathcal{M} \left\| U(x, \cdot) - V(x, \cdot) \right\|_{L^1},
\]

\[
1 \leq W_j(y) \leq \mathcal{M}, \quad j = 1, \ldots, 4,
\]

where constant \(\mathcal{M}\) is independent of \(\vartheta\) and time \(x\). Here we define the strength of any large wave of the 2-characteristic or 3-characteristic family to equal to some fixed number \(B\) (larger than all the strengths of the small waves), and the terms “small” and “large” refer to the waves that connect the states in the same or in the distinct domains \(\Omega^-\) and \(\Omega^+\), respectively.

Consequently, we have

\[
C^{-1} \left\| U(x, \cdot) - V(x, \cdot) \right\|_{L^1} \leq \Phi(U, V) \leq C \left\| U(x, \cdot) - V(x, \cdot) \right\|_{L^1} \tag{4.5}
\]

for any \(x \geq 0\) with constant \(C > 0\) depending only on the quantities independent of \(x\): the strength of the strong vortex sheet/entropy wave and \(TV(\tilde{U}_0(\cdot)) + TV(\tilde{V}_0(\cdot)) + TV(g(\cdot))\).

5. The \(L^1\)-Stability Estimates

In this section, we establish the \(L^1\)-stability estimates.

5.1. Evolution of the Lyapunov Functional \(\Phi\) in the Flow Direction \(x > 0\).

For each \(j = 1, \ldots, 4\), \(\lambda_j(y)\) is the speed of the \(j\)-wave \(q_j(y)\) (along the Hugoniot curve in the phase space). Then, at a time \(x > 0\) that is not the interaction time of the waves in either \(U(x) = U(x, \cdot)\) or \(V(x) = V(x, \cdot)\), an explicit computation gives

\[
\frac{d}{dx} \Phi(U(x), V(x)) = \sum_{\alpha \in \mathcal{J}} \sum_{j=1}^{4} (q_j(y_{\alpha})|W_j(y_{\alpha})| - q_j(y_{\alpha})|W_j(y_{\alpha})|) \, \hat{y}_{\alpha} - \sum_{j=1}^{4} q_j(b)|W_j(b)| \, \hat{y}_b
\]

\[
= \sum_{\alpha \in \mathcal{J}} \sum_{j=1}^{4} (q_j(y_{\alpha})|W_j(y_{\alpha}) - \lambda_j(y_{\alpha})|y_{\alpha} - \lambda_j(y_{\alpha})| + q_j(y_{\alpha})|W_j(y_{\alpha}) - \lambda_j(y_{\alpha})|y_{\alpha} - \lambda_j(y_{\alpha})|)
\]

\[
+ \sum_{j=1}^{4} q_j(b)|W_j(b)| - \hat{y}_b + \lambda_j(b), \tag{5.1}
\]

where \(\hat{y}_\alpha\) denotes the speed of the Hugoniot wave \(\alpha \in \mathcal{J}\), \(b = g(x)^+\) stands for the points close to the boundary, and \(\hat{y}_b\) is the slope of the boundary.
Then (5.1) can be written as
\[
\frac{d}{dx} \Phi(U(x), V(x)) = \sum_{\alpha \in J} \sum_{j=1}^{4} E_{\alpha,j} + \sum_{j=1}^{4} E_{b,j},
\]
(5.2)
where
\[
E_{\alpha,j} = |q_{j}^{\pm}|W_{j}^{\pm}(\lambda_{j}^{\pm} - \hat{y}_{\alpha}) - |q_{j}^{\pm}|W_{j}^{\pm}(\lambda_{j}^{\pm} - \hat{y}_{\alpha}),
\]
(5.3)
\[
E_{b,j} = |q_{j}(b)|W_{j}(b)(-\hat{y}_{b} + \lambda_{j}(b),
\]
(5.4)
with \(q_{j}^{\pm} = q_{j}(y_{\alpha}^{\pm})\), \(W_{j}^{\pm} = W_{j}(y_{\alpha}^{\pm})\), and \(\lambda_{j}^{\pm} = \lambda_{j}(y_{\alpha}^{\pm})\).

Our central aim in §5.2 below is to prove the bounds:
\[
\sum_{j=1}^{4} E_{\alpha,j} \leq O(1)\vartheta|\alpha| \quad \text{when } \alpha \text{ is a weak wave in } J,
\]
(5.5)
\[
\sum_{j=1}^{4} E_{\alpha,j} \leq O(1)|\alpha| \quad \text{when } \alpha \text{ is a non-physical wave in } J,
\]
(5.6)
\[
\sum_{j=1}^{4} E_{\alpha,j} \leq O(1)B\vartheta \quad \text{when } \alpha \text{ is a strong vortex sheet/entropy wave in } J,
\]
(5.7)
\[
\sum_{j=1}^{4} E_{b,j} \leq 0 \quad \text{near the boundary},
\]
(5.8)
where the quantities denoted by the Landau symbol \(O(1)\) are independent of constants \(K_1\) and \(K_2\).

With these bounds (5.5)–(5.8) together with the uniform bound on the total strengths of waves (3.4), we obtain
\[
\frac{d}{dx} \Phi(U(x), V(x)) \leq O(1)\vartheta.
\]
(5.9)
Integration of (5.9) over the interval \([0, x]\) yields
\[
\Phi(U(x), V(x)) \leq \Phi(U(0), V(0)) + O(1)\vartheta x.
\]
(5.10)

We remark that, at each interaction time \(x\) when two fronts of \(U\) or two fronts of \(V\) interact, by the Glimm interaction estimates, all the weight functions \(W_{j}(y)\) decrease, if constant \(K_2\) in the Lyapunov functional is taken to be sufficiently large. Furthermore, due to the self-similar property of the Riemann solutions, \(\Phi\) decreases at this time.

### 5.2. Estimates for Bounds (5.5)–(5.8)

We now establish bounds (5.5)–(5.8), particularly (5.7)–(5.8), when \(\alpha\) is a strong vortex sheet/entropy wave in \(J\) and near the Lipschitz wall boundary, respectively.

For the case that the weak wave \(\alpha \in J := J(U) \cup J(V)\) and the non-physical waves in \(J\), which appears when \(U\) and \(V\) are both in \(\Omega_-\) or \(\Omega_+\), estimates (5.5)–(5.6) are shown similarly based on the arguments in Bressan-Liu-Yang [6], provided that \(|J| \leq |\sigma_2| + |\sigma_3|\) is sufficiently small and \(K_1\) is sufficiently large. In what follows, we focus only on the other two cases, namely (5.7)–(5.8).

**Case 1. The first strong vortex sheet/entropy wave \(\alpha\) in \(U\) or \(V\) is crossed.** Using Lemma 2.4, we have the estimates:
\[
h_{1}^{+} = h_{1}^{-} + K_{11} h_{4}^{-},
\]
(5.11)
\[
h_{4}^{+} = K_{11} h_{4}^{-}.
\]
(5.12)
Moreover, the essential estimate \(|K_{11}| < 1\) given in Lemma 2.4 ensures the existence of desired weights \(w_{1}^{b}\) and \(w_{4}^{b}\) in the following way.
Lemma 5.1. There exist \( w_1^b, w_4^b \), and \( \gamma_b \) such that

\[
\frac{w_1^b}{w_1^b} < 1, 
\frac{w_4^b}{w_4^b} < 1, 
\lambda_1^+ - \lambda_{2,3} \bigg| K_{11} < \gamma_b < 1. 
\]

With Lemma 5.1, we estimate \( E_j \) for \( j = 1, \ldots, 4 \), starting with \( E_1 \): By (5.11) and (5.14),

\[
E_1 = |q_1^-|((\lambda_1^- - \dot{y}_a)(W_1^+ - W_1^-) + W_1^+ (|q_1^-|((\lambda_1^+ - \dot{y}_a) - |q_1^-|((\lambda_1^- - \dot{y}_a))) \\
\leq 2B \lambda_1^ b|h|^1|\lambda_1^- - \dot{y}_a| + W_1^+(|q_1^-|((\lambda_1^+ - \dot{y}_a) - |q_1^-|((\lambda_1^- - \dot{y}_a))) \\
\leq 2B \lambda_1^ b|h|^1|\lambda_1^- - \dot{y}_a| + W_1^+(|q_1^-|((\lambda_1^+ - \dot{y}_a) - |q_1^-|((\lambda_1^- - \dot{y}_a))) \\
\leq (K_1 A_{W_1} + \tilde{\kappa})|q_1^-|((\lambda_1^- - \dot{y}_a) - (K_1 A_{W_1} + \tilde{\kappa})|q_1^-|((\lambda_1^- - \dot{y}_a)), 
\]

where

\[
\tilde{\kappa} := 1 + K_2(Q(U) + Q(V)) > 0, 
\]

\( W_1^+ = W_1(y_a^+) = 2B \lambda_1^ b K_1 + K_1 A_{W_1} + \tilde{\kappa}, \ A_{W_1} = F_1(y_a^+) + H_1(y_a^+) \) here is the total strength of all the weak waves in \( U \) and \( V \) which approach the 1-wave \( q_1^+ = q_1(y_a^+) \), and \( 2B \lambda_1^ b \) is from weight \( G_1(y_a^+) \).

For \( j = 2, 3 \), \( W_j^+ = W_j^- \) so that (5.11) reduces to

\[
E_j = W_j^- (|q_j^-|((\lambda_j^+ - \dot{y}_a) - |q_j^-|((\lambda_j^- - \dot{y}_a))) \leq O(1)B \left( \theta + \sum_{i=1,4} |q_i^-| \right), 
\]

where \( k \notin \{j, 1, 4\} \).

For \( j = 4 \),

\[
E_4 = |q_4^-|((\lambda_4^- - \dot{y}_a)(W_4^+ - W_4^-) + W_4^+ (|q_4^-|((\lambda_4^+ - \dot{y}_a) - |q_4^-|((\lambda_4^- - \dot{y}_a))) \\
\leq 2B \lambda_4^ b|q_4^-|((\lambda_4^+ - \dot{y}_a) + (2B \lambda_4^ b + K_1 A_{W_4} + \tilde{\kappa})|q_4^+|((\lambda_4^- - \dot{y}_a) \\
- (2B \lambda_4^ b + K_1 A_{W_4} + \tilde{\kappa})|q_4^-|((\lambda_4^- - \dot{y}_a), 
\]

where \( W_4^+ = W_4(y_a^+) = 2B \lambda_4^ b K_1 + K_1 A_{W_4} + \tilde{\kappa} \) with constant \( \tilde{\kappa} \) determined by (5.13), \( A_{W_4} = F_2(y_a^+) + H_2(y_a^+) \) is the total strength of all the weak waves in \( U \) and \( V \) which approach the 4-wave \( q_4^+ = q_4(y_a^+) \), and \( 2B \lambda_4^ b \) is from weight \( G_4(y_a^+) \).

For the weighted \( L^1 \)-strength \( q_j(y) \) in (4.11), we choose \( w_1^b \) to be small enough relative to \( w_{11}^b \), \( w_4^b \) large enough relative to \( w_{11}^b \), and \( K_1 \) large enough and the total variation of \( U \) and \( V \) small enough.
Then we use (5.11)–(5.12) to obtain
\[
\sum_{j=1}^{4} E_j \leq 2BK_1 \left( w_1^h|h_1^+||\lambda_1^- - \hat{y}_a | + \gamma_0 w_4^h|h_4^-|(|\lambda_4^- - \hat{y}_a | - w_1^m|h_1^+||\lambda_1^+ - \hat{y}_a |) + (K_1A_{W_1^+} + \hat{K})|q_1^-||\lambda_1^- - \hat{y}_a | - (K_1A_{W_1^+} + \hat{K})|q_1^+||\lambda_1^+ - \hat{y}_a | \right) \\
+ 2BK_1 w_1^h|h_1^-||\lambda_1^- - \hat{y}_a | - 2BK_1|q_4^-||\lambda_4^- - \hat{y}_a | \\
+ (2BK_1 + K_1A_{W_4^+} + \hat{K}) w_1^m|K_1h_4^-||\lambda_1^+ - \hat{y}_a | \\
- (2BK_1 + K_1A_{W_4^+} + \hat{K})|q_4^-||\lambda_4^- - \hat{y}_a | \\
+ O(1)B \left( \theta + \sum_{i=1,4} |q_i^-| \right) \\
\leq O(1)B\theta.
\]

Case 2. The weak wave $\alpha$ between the two strong vortex sheets/entropy waves in $U$ and $V$ is crossed.

For $j = 1$, we have
\[
E_1 = |q_1^+||W_1^+ - W_1^-|(|\lambda_1^- - \hat{y}_a | + W_1^-|(|\lambda_1^+ - \hat{y}_a | - |q_1^-|(|\lambda_1^- - \hat{y}_a |)) \\
\leq\begin{cases} 
-K_1|q_1^-|\|\alpha\||\lambda_1^- - \hat{y}_a | + O(1)(BK_1 + 1)(|q_1^+| - |q_1^-|\|\alpha\| + O(1)|\alpha|) & \text{when } k_\alpha = 2, 3, 4, \\
O(1)(BK_1 + 1)(|q_1^-| - |q_1^-|\|\alpha\| + O(1)|\alpha|) & \text{when } k_\alpha = 1.
\end{cases}
\]

For $j = 2, 3$, we have
\[
E_j = B \left( (W_j^+ - W_j^-)(|\lambda_j^+ - \hat{y}_a | + W_j^-||\lambda_j^- - \hat{y}_a |) \right) \\
\leq\begin{cases} 
-B \left( K_1|\alpha||\lambda_j^- - \hat{y}_a | - O(1)|\alpha| \right) & \text{when } k_\alpha = 1, 4, \\
BK_1|\alpha|\|\theta + O(1)|h_4^-| \right) & \text{when } k_\alpha = 2, 3.
\end{cases}
\]

For $j = 4$, we have
\[
E_4 = |q_4^+||W_4^+ - W_4^-|(|\lambda_4^- - \hat{y}_a | + W_4^-||\lambda_4^+ - \hat{y}_a | - |q_4^-|(|\lambda_4^- - \hat{y}_a |)) \\
\leq\begin{cases} 
-K_1|q_4^-|\|\alpha\||\lambda_4^- - \hat{y}_a | + (2BK_1 + O(1))(|q_4^+| - |q_4^-|)(|\lambda_4^- - \hat{y}_a | + |q_4^-|(|\lambda_4^+ - \lambda_4^-)) & \text{when } k_\alpha = 1, 2, 3, \\
O(1)(BK_1 + 1)(|q_4^-| - |q_4^-|\|\alpha\| + O(1)|\alpha|) & \text{when } k_\alpha = 4.
\end{cases}
\]

Then we obtain that, when $k_\alpha = 1, 4$,
\[
\sum_{j=1}^{4} E_j \leq - BK_1(|\lambda_2^+ - \hat{y}_a | + |\lambda_3^+ - \hat{y}_a |)|\alpha| + O(1)(1 + B)|\alpha| \\
+ 2BK_1 O(1)(|q_1^-| - |q_1^-|\|\alpha\| + |q_4^-| - |q_4^-|\|\alpha\|) + (|q_1^-|\|\alpha\| + |q_4^-|\|\alpha\|);
and, when \( k_\alpha = 2, 3, \)
\[
\sum_{j=1}^{4} E_j \leq -K_4|\alpha|(\|q_1^+\|_{L^1}^\perp - \dot{y}_\alpha) + \|q_1^-\|_{L^1}^\perp - \dot{y}_\alpha| + BK_4|\alpha|(\dot{\theta} + O(1)|h_\perp^4|) + O(1)(BK_4 + 1)(w_{4m} + w_{4n})|h_\perp^4| |\alpha|.
\]
Choosing \( w_{jm}, j = 1, 4, \) and \( B \) small enough, and \( K_4 \) large enough, we conclude
\[
\sum_{j=1}^{4} E_j \leq O(1)\dot{\theta}.
\]

**Case 3. The second strong vortex sheet/entropy wave \( \alpha \) in \( U \) or \( V \) is closed.** For this case, by Lemma 2.6, we have
\[
\begin{align*}
h_1^- &= K_{21}h_0^+, \\
h_4^- &= h_4^+ + K_{24}h_0^+.
\end{align*}
\] (5.18) (5.19)

Moreover, the essential estimate \( |K_{24}| < 1 \) in Lemma 2.6 ensures the existence of desired weights \( w_4 \) and \( w_4^\alpha \) in the following manner.

**Lemma 5.2.** There exist \( w_4^\alpha, w_4 \), and \( \gamma_\alpha \) such that
\[
\begin{align*}
\frac{w_4^\alpha}{w_4} \frac{\lambda_1^+ - \lambda_{2,3}}{\lambda_1^+ - \lambda_{2,3}} |K_{24}| \leq \gamma_\alpha < 1.
\end{align*}
\] (5.20)

With Lemma 5.2, we estimate \( E_j \) for \( j = 1, \ldots, 4 \) as follows: By (5.18),
\[
\begin{align*}
E_1 &= |q_1^-|(\lambda_1^+ - \dot{y}_\alpha)(W_1^+ - W_1^-) + W_1^+ \left(|q_1^+|(\lambda_1^+ - \dot{y}_\alpha) - |q_1^-|(\lambda_1^+ - \dot{y}_\alpha)\right) \\
&\quad + w_{4m}K_4|\alpha|(|\lambda_1^+ - \dot{y}_\alpha| + (4BK_4 + K_4A_{W_0^+} + \hat{\mathcal{K}})|q_1^+|(\lambda_1^+ - \dot{y}_\alpha) \\
&\quad + w_{4n}K_4|\alpha|(|\lambda_1^+ - \dot{y}_\alpha| + (4BK_4 + K_4A_{W_0^+} + \hat{\mathcal{K}})|q_1^+|(\lambda_1^+ - \dot{y}_\alpha) \\
&\quad - w_{4n}h_4^+|\alpha|(|\lambda_1^+ - \dot{y}_\alpha| - 2BK_4w_{4n}|h_\perp^2| |\lambda_1^+ - \dot{y}_\alpha|,
\end{align*}
\]
where \( W_1^+ = W_1(y_\alpha^+) = 4BK_4 + K_4A_{W_0^+} + \hat{\mathcal{K}} \) with \( \hat{\mathcal{K}} \) determined by (5.15), \( A_{W_0^+} = F_1(y_\alpha^+)^+ + H_1(y_\alpha^+) \) here is the total strength of all the weak waves in \( U \) and \( V \) which approach the 1-wave \( q_1^+ = q_1(q_\alpha^+) \), and \( 4BK_4 \) is from weight \( G_1(y_\alpha^+) \).

For \( j = 2, 3, W_2^+ = W_3^- \) so that (5.1) reduces to
\[
E_j = W_j^+\left(|q_1^+|(\lambda_1^+ - \dot{y}_\alpha) - |q_1^-|(\lambda_1^+ - \dot{y}_\alpha)\right) \leq O(1)B\left(\dot{\theta} + \sum_{i=1,4} |q_i^+|\right).
\]

By (5.19) and (5.20), we have
\[
\begin{align*}
E_4 &= |q_4^-|(\lambda_4^- - \dot{y}_\alpha)(W_4^+ - W_4^-) + W_4^+ \left(|q_4^+|(\lambda_4^+ - \dot{y}_\alpha) - |q_4^-|(\lambda_4^+ - \dot{y}_\alpha)\right) \\
&\quad = W_4^+ \left(|q_4^+|(\lambda_4^+ - \dot{y}_\alpha) - |q_4^-|(\lambda_4^- - \dot{y}_\alpha)\right) \\
&\quad \leq W_4^+ \left(w_{4m}h_4^-|\lambda_4^+ - \dot{y}_\alpha| + w_{4n}K_4|\alpha|(|\lambda_4^+ - \dot{y}_\alpha| - w_{4n}h_4^-|\lambda_4^- - \dot{y}_\alpha|) \\
&\quad \leq 2BK_4 \left(w_{4m}h_4^-|\lambda_4^+ - \dot{y}_\alpha| + \gamma_\alpha w_{4n}|\alpha|(|\lambda_4^+ - \dot{y}_\alpha| - w_{4n}h_4^-|\lambda_4^- - \dot{y}_\alpha|) \\
&\quad + (K_4A_{W_0^+} + \hat{\mathcal{K}})|q_4^+|(\lambda_4^- - \dot{y}_\alpha) - (K_4A_{W_0^+} + \hat{\mathcal{K}})|q_4^-|(\lambda_4^- - \dot{y}_\alpha),
\end{align*}
\]
where \( W_4^+ = W_4(y_\alpha^+) = 2K_4B + K_4A_{W_0^+} + \hat{\mathcal{K}} \) with \( \hat{\mathcal{K}} > 0 \) determined by (5.15), \( A_{W_0^+} = F_4(y_\alpha^+) + H_4(y_\alpha^+) \) here is the total strength of all the weak waves in \( U \) and \( V \) which approach the 4-wave \( q_4^+ = q_4(y_\alpha^+) \), and \( 2BK_4 \) is from weight \( G_4(y_\alpha^+) \).
For the weighted $L^1$–strength $q_i(y)$ in (4.1), when $w_i^0$ is small enough relative to $w_i^m$, $w_1^0$ is large enough relative to $w_1^m$, $K_1$ is large enough, applying (5.18)–(5.19), suitably small total variation of $U$ and $V$ yields

$$
\sum_{j=1}^{4} E_j \leq 2BK_1 \left( w_4^0 |\lambda_4^+ - y_\alpha| + \gamma_\alpha w_4^0 |\lambda_4^+ - y_\alpha| - w_4^0 |\lambda_4^- - y_\alpha| \right)
+ (K_1 A_{W_4^+} + \hat{K}) |q_4^+| (\lambda_4^+ - y_\alpha) - (K_1 A_{W_4^+} + \hat{K}) |q_4^-| (\lambda_4^- - y_\alpha)
- 2BK_1 |q_1^-| |\lambda_4^- - y_\alpha| - 2BK_1 |q_1^+| |\lambda_4^+ - y_\alpha|
+ (4BK_1 + K_1 A_{W_4^+} + \hat{K}) w_1^0 |K_{21} h_1^+||\lambda_4^- - y_\alpha|
- (2BK_1 + K_1 A_{W_4^+} + \hat{K}) w_1^0 |h_1^+||\lambda_4^+-y_\alpha|
+ O(1)B \left( \varrho + \sum_{i=1,4} |q_i^+| \right)
= -2(1 - \gamma_\alpha)BK_1 w_4^0 |h_1^+||\lambda_4^- - y_\alpha|
+ (K_1 A_{W_4^+} + \hat{K})(w_1^0 |K_{21} h_1^+||\lambda_4^- - y_\alpha| - w_1^0 |h_1^+||\lambda_4^+ - y_\alpha|)
+ 2BK_1 w_4^0 |h_1^+||\lambda_4^+ - y_\alpha| - 2BK_1 w_4^0 |h_1^+||\lambda_4^- - y_\alpha|
+ 2BK_1 (w_1^0 |K_{21} h_1^+| + 2w_1^0 |K_{21} h_1^+|) |\lambda_4^- - y_\alpha| - 2BK_1 w_1^0 |h_1^+| |\lambda_4^+ - y_\alpha|
+ (K_1 A_{W_4^+} + \hat{K}) w_4^0 |h_4^+|(\lambda_4^- - y_\alpha) - (K_1 A_{W_4^+} + \hat{K}) w_4^0 |h_4^+|(\lambda_4^- - y_\alpha)
+ O(1)B \left( \varrho + \sum_{i=1,4} |q_i^+| \right)
\leq 0,
$$

which implies (5.20).

**Case 4. Close to the Lipschitz wall boundary.** This case differs from the Cauchy problem. Here we will use the particular property of the boundary condition (1.4): The flows of $U$ and $V$ are tangent to the Lipschitz wall, which implies that they must be parallel with each other along the boundary. Then a piecewise constant weak solution is constructed only along the Hugoniot curves tangent to the Lipschitz wall, which implies that they must be parallel with each other along the boundary.

**Lemma 5.3.** Let $U(b) = (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho})$ and $V(b) = (\tilde{u}, \tilde{v}, \tilde{p}, \tilde{\rho})$ be two states in a small neighborhood $O_{\varepsilon}(U_{-})$ of $U_{-}$ satisfying $\frac{\partial U}{\partial \varepsilon} = \frac{\partial V}{\partial \varepsilon} = \tilde{z}_b$ and $\tilde{v}, \tilde{\varphi} \approx 0$. Denote by $h_j(b)$ as the strength of the $j^{th}$-shock in the Riemann problem determined by $U(b)$ and $V(b)$, and denote by $\lambda_j$ as the corresponding $j^{th}$-characteristic speed. Then

$$
|\lambda_j - \tilde{z}_b| \sim |h_1(b)| \quad \text{for } j = 2, 3,
$$

(5.21)

$$
|h_4(b)| \leq |h_1(b)| + O(1)|h_2(b)||\lambda_2 - \tilde{z}_b| + |h_1(b)||O(1)|\tilde{z}_b|,
$$

(5.22)

$$
|h_1(b)| = \tilde{O}(1)|h_4(b)| \quad \text{with } \frac{1}{2} < \tilde{O}(1) < \frac{3}{2},
$$

(5.23)

where $\tilde{z}_b$ is the slope of the Lipschitz wall.

**Proof.** We prove this by analyzing the two cases.

**Case 1.** $h_1(b) = 0$ and $h_4(b) = 0$ that corresponds to the case $\tilde{p} = \tilde{\rho}$. Starting at state $\tilde{U}_b$, we move along the Hugoniot curves of the second and third families to reach $\tilde{V}_b$. Note that these two families are the contact Hugoniot curves, so that $\lambda_2$ and $\lambda_3$ are constant along the Hugoniot curves. Given that $\lambda_{2,3} = \frac{q_2}{p}, \frac{q_2}{p} = (1, \frac{3}{\gamma}, 0, 0)^T$, and $\frac{q_3}{p} = (0, 0, 0, \rho)^T$, $\frac{q_3}{p}$ remains unchanged as the initial value $\frac{q(U)}{u(U)}$, i.e., $\tilde{z}_b$ in this process by the boundary condition (1.5). Therefore, we conclude that


$\lambda_{2,3} = \dot{z}_b$, equivalently,

$$\dot{z}_b - \lambda_{2,3} = 0.$$

**Case 2.** $h_1(b) \neq 0$ that corresponds to $p \neq \bar{p}$. Starting at state $U(b)$, we move along the 1-Hugoniot curve to reach $U_1$, then possibly move along the 2-contact Hugoniot curve to reach $U_2$, the 3-Hugoniot curve to reach $U_3$, and the 4-Hugoniot curve to reach $V(b)$.

To make clear some essential relations among the strengths: $h_1(b), h_2(b), h_3(b),$ and $h_4(b)$, we project $(u,v,p,\rho)$ onto the $(u,v)$–plane. Denote $\mathbf{r}_{1|u}$ as the projection of $\mathbf{r}_1$ onto the $u$-axis, $\mathbf{r}_{2|u,v}$ as the projection of $\mathbf{r}_2$ onto the $(u,v)$–plane, etc. Then, at the background state $U_-$,

$$\mathbf{r}_{1}|_u = -\mathbf{r}_4|_u, \quad \mathbf{r}_1|_v = \mathbf{r}_4|_v, \quad \mathbf{r}_1|_{(p,\rho)} = -\mathbf{r}_4|_{(p,\rho)}, \quad \mathbf{r}_2 = \mathbf{r}_2|_{(u,v)}, \quad \mathbf{r}_3|_{(u,v)} = 0.$$

We first note that $h_4(b) \neq 0$. Given that $\mathbf{r}_1|_{(u,v)} = k_1(-\lambda_1,1)^T$ along with finite characteristic speeds $\lambda_1$ and $\dot{z}_b \approx 0$, then $\dot{z}_b < -\frac{\lambda_1}{\lambda_2}$ near state $U_-$ Thus, we can conclude that, in the $(u,v)$–plane, $\frac{\partial}{\partial u}$ along the 1-curve is always larger than $\dot{z}_b$. This implies that $\frac{v(U_1)}{u(U_1)} \neq \frac{v(U_2)}{u(U_2)}$. Meanwhile,

$$\frac{v(U_1)}{u(U_1)} = \frac{v(U_2)}{u(U_2)} = \frac{v(U_3)}{u(U_3)} = \frac{v(U_4)}{u(U_4)}.$$ 

Therefore, we conclude that there is some distance along the 4-Hugoniot curve to reach $V(b)$ so that $h_4 \neq 0$.

Next, we present an essential estimate to bound $|h_4|$ more precisely in terms of $|h_4|$. To that end, define the signed length of $(U_1 - U_b)|_{(u,v)}$ and $(V_b - U_3)|_{(u,v)}$ by $d_1$ and $d_4$ on the $(u,v)$–plane:

$$d_1 = \begin{cases} \| (U_1 - U_b) \|_{(u,v)} & \text{if } h_1 > 0, \\ -\| (U_1 - U_b) \|_{(u,v)} & \text{if } h_1 < 0, \end{cases}$$

and

$$d_4 = \begin{cases} \| (V_b - U_3) \|_{(u,v)} & \text{if } h_4 > 0, \\ -\| (V_b - U_3) \|_{(u,v)} & \text{if } h_4 < 0, \end{cases}$$

Note that

$$|\lambda_2 - \dot{z}_b| = \mathcal{O}(1)|d_1| = \mathcal{O}(1)|h_1(b)|.$$ 

Since $\lambda_2 = \frac{v(U_1)}{u(U_1)} = \frac{v(U_2)}{u(U_2)} = \lambda_3$, we can similarly conclude

$$|\lambda_3 - \dot{z}_b| = \mathcal{O}(1)|d_4| = \mathcal{O}(1)|h_1(b)|,$$

by using the following projections on the $(u,v)$–plane:

$$\mathbf{r}_1|_u = -\mathbf{r}_4|_u, \quad \mathbf{r}_1|_v = \mathbf{r}_4|_v, \quad \mathbf{r}_2 = \mathbf{r}_2|_{(u,v)}, \quad \mathbf{r}_3|_{(u,v)} = 0.$$

Moreover, we note that

$$-d_4 = \mathcal{O}(1)h_2(b)(\lambda_2 - \dot{z}_b) + \tilde{d},$$

where $\tilde{d} \cos \varphi_1 = d_1 \cos \varphi_2, \varphi_1$ denotes the angle between $(1, \dot{z}_b)$ and $\mathbf{r}_4|_{(u,v)}$, $\varphi_2$ denotes the angle between $\mathbf{r}_1|_{(u,v)}$ and $(1, \dot{z}_b)$, $\varphi_1 = \varphi_2 + 2\alpha$ for $\alpha = \arctan(\dot{z}_b)$, and

$$\tilde{d} = d_1 \frac{\cos \varphi_2}{\cos \varphi_1} = d_1 \frac{\cos(\varphi_1 - 2\alpha)}{\cos \varphi_1} = d_1 \frac{\cos \varphi_1 \cos(2\alpha) + \sin \varphi_1 \sin(2\alpha)}{\cos \varphi_1} = d_1 (\cos(2\alpha) + 2\mathcal{O}(1)\sin(2\alpha)) = d_1 (1 + \mathcal{O}(1)\alpha) = d_1 (1 + \mathcal{O}(1)\dot{z}_b),$$

so that

$$-d_4 = \mathcal{O}(1)h_2(b)(\lambda_2 - \dot{z}_b) + d_4 (1 + \mathcal{O}(1)\dot{z}_b).$$

At $U_-$, $\mathbf{r}_1|_{(u,p,\rho)} = -\mathbf{r}_4|_{(u,p,\rho)}$ and $\mathbf{r}_1|_v = \mathbf{r}_4|_v$, which implies

$$\frac{d_1}{h_1} = \frac{d_4}{h_4}.$$
Thus, we obtain the following key estimate:

$$- h_4(b) = O(1)h_2(b)(\lambda_2 - \hat{z}_b) + h_1(b)(1 + O(1)\hat{z}_b).$$

(5.24)

Estimate (5.24) now implies

$$|h_4(b)| \leq |h_1(b)| + O(1)|h_2(b)||\lambda_2 - \hat{z}_b| + |h_1(b)||O(1)|\hat{z}_b|$$

$$\leq |h_1(b)| + O(1)(|h_2(b)| + |\hat{z}_b|)|h_1(b)|,$$

which yields

$$|h_1(b)| = O(1)|h_4(b)|$$

with $$\frac{1}{2} < O(1) < \frac{3}{2},$$

given that $$|h_2(b)| + |\hat{z}_b|$$ is always small enough. This is guaranteed by the sufficiently small total variation of the initial perturbation $$\tilde{U}_0$$ and the boundary perturbation. This completes the proof.

Notice that the requirement $\frac{v}{u} = \frac{\hat{v}}{\hat{u}} = \hat{z}_b$ in Lemma 5.3 is just the boundary condition (1.5) because $\hat{z}_b$ here is the slope of the Lipschitz wall.

Applying Lemma 5.3 now yields

$$E_{b,1} = |q_1(b)|W_1(b)(-\hat{z}_b + \lambda_1)$$

$$= -4BK_1w^b_j|h_1(b)||\lambda_1| + O(1)|h_1(b)|$$

$$= -4BK_1w^b_j|h_1(b)||\lambda_1| + O(1)|h_4(b)|,$$

$$E_{b,j} = |q_j(b)|W_j(b)(-\hat{z}_b + \lambda_j) = O(1)w^b_j|h_j(b)|(-\hat{z}_b + \lambda_j) = O(1)h_1(b) = O(1)h_4(b) \quad \text{for } j = 2, 3,$$

$$E_{b,4} = |q_4(b)|W_4(b)(-\hat{z}_b + \lambda_4)$$

$$= 4BK_1w^b_4|h_4(b)||\lambda_1| + O(1)|h_4(b)|$$

$$\leq 4BK_1\lambda_1|w^b_4(h_1(b) + O(1)|h_2(b)||\lambda_2 - \hat{z}_b| + O(1)|h_1(b)||\hat{z}_b|) + O(1)|h_4(b)|.$$

Using Lemma 5.1, we can choose $w^b_1$ and $w^b_4$ such that

$$w^b_4 < w^b_1.$$

Then, with the total variation of the incoming flow perturbation and the boundary perturbation small enough and $K_1$ large enough, we have

$$\sum_{j=1}^{4} E_{b,j} = 4BK_1(w^b_j - w^b_j)|h_1(b)||\lambda_1| + O(1)BK_1|\lambda_1|w^b_j(1 + O(1)|h_2(b)||\hat{z}_b|)|h_1(b)| + O(1)|h_4(b)|$$

$$\leq O(1)4BK_1(w^b_j - w^b_j)|h_4(b)||\lambda_1| + O(1)BK_1|\lambda_1|w^b_j(1 + O(1)|h_2(b)||\hat{z}_b|)|h_1(b)| + O(1)|h_4(b)| \leq 0,$$

provided that $|h_2(b)| + |\hat{z}_b|$ is sufficiently small. This is guaranteed since the total variation of the incoming flow perturbation and the boundary perturbation are sufficiently small.

6. Existence of A Semigroup of Solutions

As a corollary of the essential estimates in §3–§5, we can now establish both the existence of semigroup $S$ generated by the wave-front tracking method and the Lipschitz continuity of $S$.

Lemma 6.1. If $TV(\tilde{U}_0(\cdot)) + TV(g'(\cdot))$ is small enough, then the map:

$$(\tilde{U}(\cdot), x) \mapsto U^0(x, \cdot) := S^\theta_x(\tilde{U}(\cdot))$$

produced by the wave-front tracking algorithm is a uniformly Lipschitz continuous semigroup satisfying the properties:

(i) $S^\theta_x \tilde{U} = \tilde{U}$, and $S^\theta_x w^\theta_{x_2} \tilde{U} = S^\theta_{x_1 + x_2} \tilde{U}$ for all $x_1, x_2 \geq 0$;

(ii) $\|S^\theta_x \tilde{U} - S^\theta_x \tilde{V}\|_{L^1} \leq C\|\tilde{U} - \tilde{V}\|_{L^1} + C\theta x$ for all $x \geq 0$.
Proof. Since $S^0$ is generated by the wave-front tracking algorithm, property (i) is immediate. Next, property (ii) is proved as follows: Take a pair of front tracking $\vartheta$-approximate solutions $U^\vartheta$ and $V^\vartheta$ of problem (1.4)–(1.6) with $\overline{U}(\cdot)$ and $\overline{V}(\cdot)$ as the initial data, respectively. Using (4.5) and (5.10), at any $x \geq 0$, we have
\[
\|U^\vartheta(x) - V^\vartheta(x)\|_{L^1} \leq C \Phi(U^\vartheta(x), V^\vartheta(x)) \leq C \Phi(U^\vartheta(0), V^\vartheta(0)) + C \partial x \leq C \|\overline{U} - \overline{V}\|_{L^1} + C \partial x.
\]
(6.1)
Therefore, the $\vartheta$-semigroup is Lipschitz continuous.

For a given $\nu > 0$, we define the domain:
\[
\mathcal{D} = \text{cl}\left\{ U : \mathbb{R} \mapsto \mathbb{R}^4 \mid \exists \text{ one point } y^i \in \mathbb{R} \text{ and } U_+ \text{ such that } U - \overline{U} \in L^1(\mathbb{R}; \mathbb{R}^4) \text{ and } \text{TV}(U - \overline{U}) \leq \nu \right\}.
\]
Given a solution $U(x, y)$ to the initial-boundary value problem (1.4)–(1.6), we note that, if $U^\varepsilon(x) := U(x, y) \in \mathcal{D}$ at any fixed $x \geq 0$, then $y^i > g(0) = 0$ at $x = 0$ and $y^i > g(x)$ for $x > 0$ as a strong vortex sheet/entropy wave is present.

The semigroup $S$ generated by the wave-front tracking algorithm is provided by the following theorem.

**Theorem 6.1.** If $\text{TV}(\tilde{U}_0(\cdot)) + \text{TV}(g(\cdot))$ is small enough, then $S^0$ produced by the wave-front tracking algorithm is a Cauchy sequence in the $L^1$–norm, so that $S^0$ converges to a unique limit $S$ satisfying that $S_x(\overline{U}) = \lim_{\vartheta \to 0} S_x^0(\overline{U})$ for any $x > 0$. Then the map $S : [0, \infty) \times \mathcal{D} \mapsto \mathcal{D}$ is a uniformly Lipschitz semigroup in $L^1$. In particular, the entropy solution to the initial-boundary value problem (1.4)–(1.6) constructed by the wave-front tracking algorithm is unique and $L^1$–stable.

Based on the essential estimates in §3–§5, Theorem 6.1 can be proved in the same way as the argument given in [3]. Also see Chen-Li [8].

7. **Uniqueness of Entropy Solutions in the Class of Viscosity Solutions**

In this section, as an immediate consequence of the estimates obtained in §4–§6, we find that the semigroup $S$ produced by the wave-front tracking method is the only standard Riemann semigroup (SRS) in the sense of Definition 7.1 given below. In other words, the semigroup defined by the wave-front tracking method is the canonical trajectory of the standard Riemann semigroup (SRS). This yields the uniqueness of entropy solutions in a broader class of viscosity solutions as introduced by Bressan in [4]. Furthermore, it coincides with the semigroup trajectory generated by the wave-front tracking method.

the initial-boundary value problem (1.4)–(1.6), is said to have a standard Riemann semigroup if, for some small $\nu_0$, there exist both a continuous mapping $\mathcal{R} : [0, \infty) \times \mathcal{D} \mapsto \mathcal{D}$ and a constant $L$ satisfying the following properties:

(i) **Semigroup property**: $\mathcal{R}_{0\overline{U}} = \overline{U}$ and $\mathcal{R}_{x_1} \mathcal{R}_{x_2} \overline{U} = \mathcal{R}_{x_1 + x_2} \overline{U}$;

(ii) **Lipschitz continuity**: $\|\mathcal{R}_x \overline{U} - \mathcal{R}_x \overline{V}\|_{L^1} \leq L \|\overline{U} - \overline{V}\|_{L^1}$;

(iii) **Consistency with the Riemann solver**: Given piecewise constant initial data $\overline{U} \in \mathcal{D}$, then, for all $x \in [0, \nu_0]$, $U(x, \cdot) = \mathcal{R}_x \overline{U}$ coincides with the solution of problem (1.4)–(1.6) obtained by piecing together the standard Riemann solutions and the lateral Riemann solutions.

Following the argument in [4], we employ the estimates obtained in §4–§6 to conclude

**Theorem 7.1.** Suppose that problem (1.4)–(1.6) has a standard Riemann semigroup $\mathcal{R} : [0, \infty) \times \mathcal{D} \mapsto \mathcal{D}$. Consider the semigroup $S$ produced by the wave-front tracking method: $S_x(\overline{U}) = \lim_{\vartheta \to 0} S_x^0(\overline{U})$. 


Assume $\overline{U} \in D$. Then, for all $x > 0$, $R_x \overline{U} = S_x \overline{U}$. Furthermore, a continuous map $U : [0, X] \mapsto D$ is a viscosity solution of problem (1.4)–(1.6) defined in [3] if and only if

$$U(x, \cdot) = R_x \overline{U} \quad \text{for any } x \in [0, T].$$

(7.1)

In particular, a continuous map $U : [0, X] \mapsto D$ is a viscosity solution if and only if

$$U(x, \cdot) = S_x \overline{U} \quad \text{for any } x \in [0, T].$$

(7.2)

The proof here follows a similar argument to the one presented in [3]. The only difference is the strong vortex sheet/entropy wave in our problem. Nonetheless, one can proceed with the proof by considering the convergence of the wave-front tracking method which is shown in §3.

**Remark 7.1.** In the simpler cases of the isentropic or isothermal Euler flow, as well as the potential flow, as far as the $L^1$–stability problem is concerned, we realize the same results as those for the full Euler system (1.1).

**Acknowledgements:** The authors would like to thank Yun Pu for his helpful suggestions. The research of Gui-Qiang G. Chen was supported in part by the UK Engineering and Physical Sciences Research Council Awards EP/L015811/1, EP/V008854/1, and EP/V051121/1. The research of Vaibhav Kukreja was supported in part by the National Science Foundation under Grants DMS-0935967 and DMS-0807551, the UK EPSRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1).

**References**

[1] P. Baiti and K. Jenssen, On the front-tracking algorithm, J. Math. Anal. Appl. 217 (1998), 395–404.

[2] A. Bressan, Global solutions of systems of conservation laws by wave-front tracking, J. Math. Anal. Appl. 170 (1992), 414–432.

[3] A. Bressan, The unique limit of the Glimm scheme, Arch. Ration. Mech. Anal. 130 (1995), 205–230.

[4] A. Bressan, Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem, Oxford University Press: Oxford, 2000.

[5] A. Bressan and R. M. Colombo, The semigroup of $2 \times 2$ conservation laws, Indiana Univ. Math. J. 44 (1995), 677–725.

[6] A. Bressan, T.-P. Liu, and T. Yang, $L^1$ stability estimates for $n \times n$ conservation laws, Arch. Ration. Mech. Anal. 149 (1999), 1–22.

[7] G.-Q. Chen and M. Feldman, Mathematics of Shock Reflection-Diffraction and von Neumann’s Conjecture, Research Monograph, Annals of Mathematics Studies, 197, Princeton University Press: Princeton, 2018.

[8] G.-Q. Chen and T.-H. Li, Well-posedness for two-dimensional steady supersonic Euler flows past a Lipschitz wedge, J. Diff. Equ. 244 (2008), 1521–1550.

[9] G.-Q. Chen, H. Shahgholian, and J.-V. Vázquez, Free boundary problems: The forefront of current and future developments, In: Free Boundary Problems and Related Topics. Theme Volume: Phil. Trans. R. Soc. A. 373 (2015), 20140285.

[10] G.-Q. Chen and Y.-G. Wang, Characteristic discontinuities and free boundary problems for hyperbolic conservation laws. In: Nonlinear Partial Differential Equations, 53–81, Abel Symp. 7, Springer, Heidelberg, 2012.

[11] G.-Q. Chen, Y.-Q. Zhang, and D.-W. Zhu, Stability of compressible vortex sheets in steady supersonic Euler flows over Lipschitz walls, SIAM J. Math. Anal. 38 (2007), 1660–1693.

[12] E. Chiodaroli, C. De Lellis, and O. Kreml, Global ill-posedness of the isentropic system of gas dynamics, Comm. Pure Appl. Math. 68 (2015), 1157–1190.

[13] J. F. Coulombel and P. Secchi, The stability of compressible vortex sheets in two space dimensions, Indiana Univ. Math. J. 53 (2004), 941–1012.

[14] J. F. Coulombel and P. Secchi, Nonlinear compressible vortex sheets in two space dimensions, Ann. Sci. Ec. Norm. Super. 41 (2008), 85–139.

[15] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience: New York, 1948.

[16] A. Corli and M. Sablé-Tougeron, Stability of contact discontinuities under perturbations of bounded variation, Rend. Sem. Mat. Univ. Padova, 97 (1997), 35–60.

[17] C. M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, J. Math. Anal. Appl. 38 (1972), 33–41.

[18] C. M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Second Edition, Springer-Verlag: Berlin, 2005.
[19] R. J. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of conservation laws, J. Diff. Equ. 20 (1976), 187–212.
[20] J. Glimm, Solution in the large for nonlinear systems of conservation laws, Comm. Pure Appl. Math. 18 (1965), 697–715.

[21] H. Holden and N. Risebro, Front Tracking for Hyperbolic Conservation Laws, Springer-Verlag: New York, 2002.

[22] C. Klingenberg, O. Kreml, V. Mácha, and S. Markfelder, Shocks make the Riemann problem for the full Euler system in multiple space dimensions ill-posed, Nonlinearity, 33 (2020), 6517–6540.

[23] P. D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves. CBMS- RCSAM, No. 11. SIAM: Philadelphia, Pa., 1973.

[24] Ph. LeFloch, Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves, Birkhäuser-Verlag: Basel, 2002.

[25] M. Lewicka, $L^1$ stability of patterns of non-interacting large shock waves, Indiana Univ. Math. J. 49 (2000), 1515–1537.

[26] M. Lewicka, Stability conditions for patterns of noninteracting large shock waves, SIAM J. Math. Anal. 32 (2001), 1094–1116.

[27] M. Lewicka and K. Trivisa, On the $L^1$ well posedness of systems of conservation laws near solutions containing two large shocks, J. Diff. Equ. 179 (2002), 133–177.

[28] T.-P. Liu, The deterministic version of the Glimm scheme, Commun. Math. Phys. 57 (1977), 135–148.

[29] T.-P. Liu and T. Yang, Well-posedness theory for hyperbolic conservation laws, Comm. Pure Appl. Math. 52 (1999), 1553–1586.

[30] M. Sablé-Tougeron, Méthode de Glimm et problème mixte, Ann. Inst. H. Poincaré Anal. Nonlinéaire, 10 (1993), 423–443.

Gui-Qiang G. Chen, Mathematical Institute, University of Oxford, Oxford, OX1 3LB, UK
Email address: chengq@maths.ox.ac.uk

Vaibhav Kukreja, Moshman Research, Portland, OR 97219, USA; Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
Email address: vaibhavk@moshmanresearch.com