Einstein’s Boxes:  
Quantum Mechanical Solution

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**Abstract**

In this paper we present a solution of the Einstein’s boxes paradox by modern Quantum Mechanics in which a notion of density matrix is equivalent to a notion of a quantum state of a system. We use a secondary quantization formalism in the attempt to make a description particularly clear. The aim of this paper is to provide pedagogical help to the students of quantum mechanics.

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1 Introduction

The ”Einstein’s Boxes” thought experiment is originally presented by Einstein in 1927 at Solvay conference to demonstrate the incompleteness in quantum mechanics description of reality. Later it was discussed and modified by Einstein, de Broglie, Schrodinger, Heisenberg, and others, into a simple scenario involving the splitting in half of the wave function in the box.

In his book [1] de Broglie describes it in this way:

Suppose a particle is enclose in a box B with impermeable walls. The associated wave \( \Psi \) is confined to the box and cannot leave it. The usual interpretation assert that the particle is ”potentially” present in the whole box \( B \), with a probability \( |\Psi|^2 \) at each point. Let us suppose that by some process or other, for example, by inserting a partition into the box, the box \( B \) is divided into two separate parts \( B_1 \) and \( B_2 \) and that \( B_1 \) and \( B_2 \) are then transported to two very distant places, for example to Paris and Tokyo. The particle, which has not yet appeared, thus remains potentially present in the assembly of two boxes and its wave function \( \Psi \) consists of two parts, one of which, \( \Psi_1 \), is located in \( B_1 \) and the other, \( \Psi_2 \), in \( B_2 \). The wave function is thus of the form \( \Psi = c_1 \Psi_1 + c_2 \Psi_2 \), where \( |c_1|^2 + |c_2|^2 = 1 \).
The probability laws of wave mechanics now tell us that if an experiment is carried out in box $B_1$ in Paris, which will enable the presence of the particle to be revealed in this box, the probability of this experiment giving a positive result is $|c_1|^2$, whilst the probability of it giving a negative result is $|c_2|^2$. According to the usual interpretation, this would have the following significance: since the particle is present in the assembly of the two boxes prior to the observable, it would be immediately localized in the box $B_1$ in the case of a positive result in Paris. This does not seem to me to be acceptable. The only reasonable interpretation appears to me to be that prior to the observable localization in $B_1$, we know that the particle was in one of the boxes $B_1$ and $B_2$, but we do not know in which one, and the probabilities considered in the usual wave mechanics are the consequence of this partial ignorance. If we show that the particle is in the box $B_1$, it implies simply that it was already there prior to localization...

We might note here how the usual interpretation leads to paradox in the case of experiment with negative result...if nothing is observed, this negative result will signify that the particle is not in box $B_2$ and it is thus in box $B_1$ in Paris. But this can reasonably signify only one thing: the particle was already in Paris in box $B_1$ prior to the drainage experiment made in Tokyo in box $B_2$. Every other interpretation is absurd. How can we imagine that the simple fact of having observed nothing in Tokyo has been able to promote the localization of the particle at a distance of many thousands of miles away?

In paper published recently, T. Norsen [2] presents the history of the problem, several formulation of this thought experiment, analyses and assess it from point of Einstein-Bohr debates, EPR dilemma, and Bell’s theorem. This paper has encouraged us to consider the problem of two boxes in the frame of modern quantum mechanics.

The definition of quantum state of the quantum system was done by von Neumann [3] in 1927. The concept of density matrix’s operator solves EPR-paradox [4] and other paradoxes in Quantum Mechanics.

By means of Einstein’s boxes problem we demonstrate the quantum mechanical approach to description of quantum state of subsystem of complicate quantum system. The paper is organized as follows. In section 2 we recall the von Neumann formalism of Density Matrix operator and discuss the problem of description of the state of subsystems of composite system with the help of reduced and conditional density matrices. In section 3 we formulate the problem in the boxes in secondary quantization formalism. In section 4 we define the quantum states for each box after separation. In section 5 we consider the process of observation of the particle in the box $S_2$ and can see that this process has no influence on the quantum state in the box $S_1$, we also discuss the property of the quantum state of the particle while the boxes are conjugated again. In conclusion we review our consideration and ensure that there is not Einstein’s boxes paradox.

## 2 Density Matrix

### 2.1 Notion of Quantum State

The general definition of quantum state was given by von Neumann [3].

He proposed the following procedure for calculation of average values of physical variables $\hat{F}$:

$$< F > = \text{Tr}(\hat{F}\hat{\rho}).$$
Here operator \( \hat{\rho} \) satisfies three conditions:

1) \( \hat{\rho}^+ = \hat{\rho} \),

2) \( Tr\hat{\rho} = 1 \),

3) \( \forall \psi \in \mathcal{H} \quad <\psi|\hat{\rho}\psi > \geq 0 \).

By the formula for average values von Neumann found out the correspondence between linear operators \( \hat{\rho} \) and states of quantum systems. This formula gives quantum mechanical definition of the notion ”a state of a system”. The operator \( \hat{\rho} \) is called Density Matrix.

Suppose that \( \hat{F} \) is an operator with discrete non-degenerate spectrum. If an observable \( \mathcal{F} \) has a definite value in the state \( \rho \), i.e. a dispersion of \( \mathcal{F} \) in the state \( \hat{\rho} \) equals zero, then the density matrix of this state is a projective operator satisfying the condition

\[
\hat{\rho}^2 = \hat{\rho} = \hat{P}_N, \quad \hat{P}_N = |\Psi_N\rangle\langle \Psi_N|, \quad \langle \Psi_N|\Psi_N \rangle = 1.
\]

The average value of an arbitrary variable in this state is equal to

\[
\langle A \rangle = \langle \Psi_N|\hat{A}\Psi_N \rangle.
\]

It is so-called pure state. If the state is not pure it is known as mixed.

### 2.2 Composite System and Reduced Density Matrix

Suppose the system \( S \) is an unification of two subsystems \( S_1 \) and \( S_2 \):

\[
S = S_1 \cup S_2.
\]

Then the Hilbert space \( \mathcal{H} \) is a direct product of two spaces

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2,
\]

here the space \( \mathcal{H} \) corresponds to the system \( S \) and the spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) correspond to the subsystems \( S_1 \) and \( S_2 \).

Now suppose that a physical variable \( \mathcal{F}^{(1)} \) is connected with subsystem \( S_1 \) only. The average value of this variable in the state \( \rho_{1+2} \) is given by equation

\[
\langle \mathcal{F}^{(1)} \rangle_\rho = Tr(\hat{\mathcal{F}}^{(1)}\hat{\rho}_{1+2}) = Tr_1(\hat{\mathcal{F}}^{(1)}\hat{\rho}_1),
\]

where the operator \( \hat{\rho}_1 \) is defined by the formula

\[
\hat{\rho}_1 = Tr_2\hat{\rho}_{1+2}.
\]

The operator \( \hat{\rho}_1 \) [5] satisfies all the properties of Density Matrix in \( S_1 \). The operator is called Reduced Density Matrix. Thus, the state of the subsystem \( S_1 \) is defined by reduced density matrix. The reduced density matrix for the subsystem \( S_2 \) is defined analogously:

\[
\hat{\rho}_2 = Tr_1\hat{\rho}_{1+2}.
\]

Quantum states \( \rho_1 \) and \( \rho_2 \) of subsystems are defined uniquely by the state \( \rho_{1+2} \) of the composite system.
2.3 Conditional Density Matrix

The average value of a variable $\hat{F}^{(1)}\hat{P}^{(2)}$, where $\hat{P}^{(2)} = |\phi^{(2)}\rangle_2\langle\phi^{(2)}|$, $\langle\phi^{(2)}|\phi^{(2)}\rangle_2 = 1$, in the state $\hat{\rho}$ is equal to

$$\langle F^{(c)} \rangle_\rho = Tr_1(\hat{F}^{(1)}\hat{\rho}_1^c).$$

It is easy to demonstrate that the operator

$$\hat{\rho}_1^c = \frac{1}{p}Tr_2(\hat{\rho}\hat{P}^{(2)})$$

satisfies all the properties of density matrix in the space $\mathcal{H}_1$. Here $p$ is a probability to find a subsystem $S_2$ in the pure state $|\phi^{(2)}\rangle_2$

$$p = Tr_{(1+2)}(\hat{\rho}\hat{P}^{(2)}) = Tr_2(\hat{\rho}_2\hat{P}^{(2)}).$$

So the operator $\hat{\rho}_1^c$ defines some state of the subsystem $S_1$. This state called a Conditional Density Matrix [6].

It is density matrix for the subsystem $S_1$ under condition that the subsystem $S_2$ is selected in the pure state $\hat{P}_2 = |\phi^{(2)}\rangle_2\langle\phi^{(2)}|.$

3 Pure States in Boxes

3.1 One Box

Suppose a particle is in a pure state with the wave function $\Psi(x)$ in the box $S'$. It means that there exists a physical variable $\hat{F}$ with nondegenerate discrete spectrum that has a definite value in this state, i.e. a dispersion of this variable equals zero. We can write $\Psi(x)$ in the form:

$$\Psi(x) = \alpha\Psi_1(x) + \beta\Psi_2(x), \quad \int |\Psi(x)|^2 dx = 1,$$

where

$$|\alpha|^2 + |\beta|^2 = 1,$$

$$\Psi_1(x) = 0(x > 0), \quad 1\langle\Psi|\Psi\rangle_1 = 1,$$

$$\Psi_2(x) = 0(x < 0), \quad 2\langle\Psi|\Psi\rangle_2 = 1.$$

For example, in the box

$$V(x) = \begin{cases} 0 & |x| < a \\ +\infty & |x| > a \end{cases}$$

energy is nondegenerate

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 n^2 \quad n = 1, 2, ...$$

The stationary states with energies $[7]$ $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2 n^2$ have the wave functions

$$\Psi_n(x) = \sqrt{\frac{1}{a}} \sin \left(\frac{\pi n}{2a}(x - a)\right), \quad \int_{-a}^{+a} |\Psi(x)|^2 dx = 1.$$

If $n$ is even then $\Psi(0) = 0$. Suppose a particle is in the state with $n = 2k$. 

3.2 Two Boxes

Now we consider a quantum system of two boxes. The wave functions in separated box have to satisfy a condition $\phi(0) = 0$. The Hilbert space of the system $S = (S_1 U S_2)$ is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where

$$\mathcal{H}_1 = \{ \phi_1(x) : \phi_1(x) = 0, \text{if} \ x > 0, \ \phi_1(-a) = \phi_1(0) = 0 \},$$

$$\mathcal{H}_2 = \{ \phi_2(x) : \phi_2(x) = 0, \text{if} \ x < 0, \ \phi_2(a) = \phi_2(0) = 0 \}.$$

Later we will use secondary quantization formalism:

$$[\hat{\Psi}(x), \hat{\Psi}(y)] = 0 = [\hat{\Psi}^+(x), \hat{\Psi}^+(y)],$$

$$[\hat{\Psi}(x), \hat{\Psi}^+(y)] = \delta(x - y).$$

Vacuum state is

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2, \ \hat{\Psi}(x)|0\rangle = 0.$$

The operators of particles numbers are

$$\hat{N} = \hat{N}_1 + \hat{N}_2,$$

$$\hat{N}_1 = \int_{x<0} \hat{\Psi}^+(x)\hat{\Psi}(x)dx, \ \hat{N}_2 = \int_{x>0} \hat{\Psi}^+(x)\hat{\Psi}(x)dx.$$

One particle quantum mechanics corresponds to the sector with $N = 1$.

The state with wave function $\Psi^{(1)}(x)$ in this representation is

$$|\Psi_1\rangle = \int_{x<0} dx \Psi^{(1)}(x)\hat{\Psi}^+(x)|0\rangle_1 \ast |0\rangle_2 = |\Psi^{(1)}_1\rangle_1 \ast |0\rangle_2$$

and the state with wave function $\Psi^{(2)}(x)$ is

$$|\Psi_2\rangle = |0\rangle_1 \ast \int_{x>0} dx \Psi^{(2)}(x)\hat{\Psi}^+(x)|0\rangle_2 = |0\rangle_1 \ast |\Psi^{(2)}_2\rangle_2.$$ 

If $n$ is even ($n = 2k$) then the wave function $\Psi_n$, satisfies the condition $\Psi(0) = 0$ and belongs to Hilbert space $\mathcal{H}$.

The energy levels of the composite system $S$ are degenerate and there are two basic orthonormal wave functions with the same energy. To define a pure state of the system $S$ with definite energy we have to define additional quantum number. For example, let a parity be this variable:

$$\hat{P}\psi(x) = \psi(-x).$$

Initial wave function $\Psi_n(x)$ is odd

$$\hat{P}\Psi(x) = \Psi(-x) = -\Psi(x).$$

Another wave function with the same energy is even

$$\hat{P}\Psi_+(x) = \Psi_+(-x) = \Psi_+(x) = |\Psi(x)|.$$ 

The variable $\hat{F} = \hat{P}\hat{H}$ with nondegenerate spectrum defines these wave functions uniquely.
Another basis is defined by energy and variable of number of particles in one of the
box, for example, in the box \( S_2 \) \( (\hat{N}_1 = 1 - \hat{N}_2) \):

\[
\begin{align*}
\{N_2 = 0, & E = E_n\} : |\Psi^{(1)}_1\rangle_1 |0\rangle_2, \\
\{N_2 = 1, & E = E_n\} : |0\rangle_1 |\Psi^{(2)}_2\rangle_2.
\end{align*}
\]

The wave function of the state \( \Psi_{n=2k} \) equals

\[
|\Psi_n\rangle = \alpha |\Psi^{(1)}_k\rangle_1 |0\rangle_2 + \beta |0\rangle_1 |\Psi^{(2)}_k\rangle_2, \quad \alpha = -\beta = \frac{1}{\sqrt{2}}.
\]

The average of coordinate in this state is

\[
<x> = \int_{-a}^{+a} |\Psi_n(x)|^2 x dx, \quad \omega(x) = \frac{1}{2} \sin^2 \frac{\pi}{a} k x.
\]

The average of momentum is equal to

\[
<p> = \int_{-a}^{+a} |\tilde{\Psi}_{2k}(p)|^2 p dx, \quad \tilde{\omega}(p) = \frac{2k^2 \pi a}{\hbar} \frac{1}{\left(\frac{p^2 a^2}{\hbar^2} - k^2 \pi^2\right)^2} \left\{ \begin{array}{ll}
\cos^2 pa/\hbar, & \text{if } k \text{ is odd,} \\
\sin^2 pa/\hbar, & \text{if } k \text{ is even.}
\end{array} \right.
\]

Dispersal of energy in this state equals zero.

4 Quantum States of Subsystems

Suppose, that \( \Psi_n(x) \) is an initial wave function. We put a partition at the point \( x = 0 \) carefully.

Hamiltonian of the system is changed. Now there exist two boxes with partition which
were described in subsection 3.2.

The wave function \( \Psi_n(x) \in \mathcal{H} \) does not change. Now it is the composite system of
two boxes in pure state \( |\Psi_n\rangle \). This circumstance defines all quantum properties of the
composite system and its subsystems.

This state has definite energy \( E_n \). The density matrix of this pure state is

\[
\hat{\rho}_n = |\Psi_n\rangle \langle \Psi_n| = |\alpha|^2 |\Psi^{(1)}_1\rangle_1 \langle \Psi^{(1)}_1| \otimes |0\rangle_2 \langle 0| + \\
\alpha \beta^* |\Psi^{(1)}_1\rangle_1 \langle 0| \otimes |0\rangle_2 \langle \Psi^{(2)}_1| + \\
\alpha^* \beta |0\rangle_1 \langle \Psi^{(1)}_1| \otimes |\Psi^{(2)}_2\rangle_2 \langle 0| + |\beta|^2 |0\rangle_1 \langle 0| \otimes |\Psi^{(2)}_2\rangle_2 \langle \Psi^{(2)}_2|.
\]

It is entangled state: whiles the composite system has a wave function, the quantum
states of each of the boxes \( S_1 \) and \( S_2 \) are mixed.

The reduced density matrix of the box \( S_1 \) is

\[
\hat{\rho}_1 = Tr_2 \hat{\rho}_n = |\alpha|^2 |\Psi^{(1)}_1\rangle_1 \langle \Psi^{(1)}_1| + |\beta|^2 |0\rangle_1 \langle 0|.
\]

The box \( S_1 \) is empty with probability \( |\beta|^2 \). There is a particle with wave function \( \Psi^{(1)}_k \) in
the box \( S_1 \) with probability \( |\alpha|^2 \).

The reduced density matrix of the box \( S_2 \) is

\[
\hat{\rho}_2 = Tr_1 \hat{\rho}_n = |\alpha|^2 |0\rangle_2 \langle 0| + |\beta|^2 |\Psi^{(2)}_2\rangle_2 \langle \Psi^{(2)}_2|.
\]
The box $S_2$ is empty with probability $|\alpha|^2$. There is a particle with wave function $\Psi^{(2)}_k$ in the box $S_2$ with probability $|\beta|^2$.

If the particle in $S_2$ is selected in pure state $P_2 = |\phi\rangle\langle\phi|$ (independently how and when it is done) the state of $S_1$ is

$$\hat{\rho}_{1/2}^{(c)} = \frac{1}{p} \langle\phi|\hat{\rho}|\phi\rangle,$$

where $p$ is the probability to find a system $S_2$ in the state $|\phi\rangle$.

For example, if the particle is in the box $S_2$ ($N_2 = 1$) or the energy is equal to $E_k$ (the processes of measurement are different but they select the same pure state $\Psi^{(2)}_k$), then the state of $S_1$ is pure state $|0\rangle_1$, i.e. there is no particle in the box $S_1$ and this event is definite with probability $p_1 = 1$.

For example, if the box $S_2$ is empty ($N_2 = 0$) or the energy is equal to 0, then the state of $S_1$ is pure state $|\Psi^{(1)}_k\rangle_1$ and this event is definite with probability $p_1 = 1$.

In the next section we illustrate these results.

5 Quantum States of Subsystems (continuation)

Now we send the box $S_1$ to Tokyo and the box $S_2$ to Paris.

Suppose that we change our mind and send our boxes back to Moscow. We put away a partition carefully and found that the system $S'$ is in the pure state with the wave function $\Psi_n$.

5.1 Measurement in the Box

If not, suppose, Alice in Paris decided to look into the box $S_2$ and to see if the particle in it or not. We describe this process in manner of von Neumann measurement.

The Alice’s detector (subsystem $S3$) could be in two different positions: if detector did not have registered the particle,

$$|no\rangle = \binom{1}{0},$$

and if it did,

$$|yes\rangle = \binom{0}{1}.$$

At the moment of time $t = 0$ the composite system $\Gamma = S \oplus (S3)$ is in the pure state

$$|\Theta\rangle = |\Psi_n\rangle \otimes \binom{1}{0}.$$

Since vacuum $|0\rangle_2$ is the state without particles in the box $S_2$

$$\hat{N}_2|0\rangle_2 = 0$$

and

$$\hat{N}_2|\Psi^{(2)}\rangle_2 = |\Psi^{(2)}\rangle_2,$$
the Hamiltonian of interaction of the detector and the subsystem $S_2$ could be represented, for example, in the form

$$\hat{H}_I = \gamma \hat{N}_2 \hat{\sigma}_1, \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Unitary evolution of the state of the system $(S \oplus S3)$ leads to

$$|\Theta(t)\rangle = \exp\left(-i \frac{\hat{H}_I t}{\hbar}\right)|\Theta\rangle.$$  

It is well known that

$$e^{-i \frac{\hat{\sigma}_1 t}{\hbar}} = \cos \frac{\gamma t}{\hbar} - i \sin \frac{\gamma t}{\hbar} \hat{\sigma}_1.$$  

Suppose the duration of exposition is $t = \frac{\hbar \gamma}{2}$ then

$$e^{-i \frac{\hat{\sigma}_1 t}{\hbar}} = -i \hat{\sigma}_1.$$  

The wave function of the system $\Gamma$ at the moment $t$ is

$$|\Theta(t)\rangle = \alpha |\Psi^{(1)}\rangle_1 |0\rangle_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \beta |0\rangle_1 |\Psi^{(2)}\rangle_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

The state of the composite system $\Gamma$ rests pure under unitary evolution. The state of the system $S$ has changed. It is mixed now with density matrix:

$$\rho_S = |\alpha|^2 |\Psi^{(1)}\rangle_1 \langle \Psi^{(1)}| |0\rangle_2 \langle 0| + |\beta|^2 |0\rangle_1 \langle 0| |\Psi^{(2)}\rangle_2 \langle \Psi^{(2)}|.$$  

We can see that the quantum state of the subsystem $S_1$ has not changed: it is $\hat{\rho}_1$ again. Thus there is no paradox: the observation of the subsystem $S_2$ in Paris does not change the state of the subsystem $S_1$ in Tokyo. All measurements in the system $S_1$ give the same results before and after observation in the box $S_2$.

### 5.2 Back to One Box

What happens if we send two boxes to Moscow and carefully put away a partition? Suppose we decide to measure the energy of the system $S'$.

With probability $W_{N=2k} = 1/2$ the energy is equal to $E_{N=2k}$. With probability $W_{N=2m} = 0$ the energy is equal to $E_{N=2m, m \neq k}$. With probability $W_{2l+1}$, $l = 0, 1, 2, ...$ the energy equals $E_{(2l+1)}$ where

$$W_{2l+1} = 2 \left( \int_0^1 \sin \pi ky \cos \pi y(l + 1/2) dy \right)^2.$$  

$W_{2l+1} = 0$ if $k$ and $l$ have the same parity.

For example, if $k = 1$ then

$$W_{2l+1} = \frac{2}{\pi^2 \left( (l+1/2)^2 - 1 \right)^2} \sin^2 \frac{\pi(l+1)}{2}.$$  

Distributions of coordinate and momentum do not change.
5.3 Selected States

If Alice decides to select the states such that there is no particle in the box $S_2$ then the quantum state of the composite system $S$ under condition that the detector is found in the state $|\text{yes}\rangle$ is the pure state $|\Psi^{(1)}\rangle_1 \otimes |0\rangle_2$. It is defined by two quantum numbers $N_1 = 1$ and the energy in the system of two separated boxes $E_n$.

The state of the box $S_1$ is the quantum state with conditional density matrix

$$\hat{\rho}_1^c = 2 \langle 0 | \hat{\rho}_S | 0 \rangle_2 \frac{1}{|\alpha|^2}.$$

It is pure state of the particle in the box $S_1$ with wave function $\Psi^{(1)}_k(x)$. This pure state does not change during observation (measurement) in the box $S_2$. We can look into the box $S_1$ before or after looking into $S_2$ but the particle is always in the box $S_1$ if the box $S_2$ is empty. This state is selected under condition that the particle isn’t seen in the box $S_2$.

Let us now put away a partition. The state is not the stationary state of the system: particle in the box $S_1$ $\oplus$ ”empty box” together. The energy distribution is $W_N$. It is not a paradox, it is a quantum logic.

Conclusions

The reduced density matrix and conditional density matrix notions resolve Einstein’s boxes paradox. Quantum state of the composite system defines the quantum states of the subsystems uniquely.

The initial pure state becomes the mixed state during observation. But the observation in the box $S_2$ does not change the state of the box $S_1$.

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