Abstract

The $S = 1/2$ XXZ spin chain with the staggered XY anisotropy

$$H = J \sum_n (S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + \Delta S^z_n S^z_{n+1}) - \delta \sum_n (-1)^n (S^x_n S^x_{n+1} - S^y_n S^y_{n+1})$$

is shown to possess gapless, Luttinger-liquid-like phases in a wide range of its parameters: the XY-like phase and spin nematic phases, the latter characterized by a two-spin order parameter breaking translational and spin rotation symmetries. In the simplest, exactly solvable case $\Delta = 0$, the spectrum remains gapless at arbitrary $J$ and $\delta$ and is described by two massless Majorana (real) fermions with different velocities $v_{\pm} = |J \pm \delta|$. At $|\delta| < J$ the staggered XY anisotropy does not influence the ground state of the system (XY phase). At $|\delta| > J$, due to level crossing, a spin nematic state is realized, with $\uparrow\uparrow\downarrow\downarrow$ and $\uparrow\downarrow\downarrow\uparrow$ local symmetry of the $xx$ and $yy$ spin correlations. The spin correlation functions are calculated and the effect of thermally induced spin nematic ordering in the XY phase ("order from disorder") is discussed.
The role of a finite $\Delta$ is studied in the limiting cases $|\delta| \ll J$ and $|\delta| \gg J$, using bosonization method. On the basis of the derived self-dual field-theoretical model, similar to the one recently proposed to treat two weakly coupled Luttinger chains, a confinement regime is described, in which fermionic excitations fail to be observable due to Luttinger-liquid effects. Kosterlitz-Thouless transitions to massive phases, driven by the $\delta$--anisotropy, are discussed.

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There has been considerable recent interest in the $S = 1/2$ quantum spin chains with alternating ferromagnetic and antiferromagnetic bonds\cite{1,2}:

$$H = \sum_j [(S_{2j} \cdot S_{2j+1}) - \beta (S_{2j-1} \cdot S_{2j})]. \quad (1)$$

The model (1) continuously connects a gapful phase of decoupled singlets ($\beta = 0$) with the limit of the $S = 1$ chain ($\beta \to \infty$) and so far was mostly considered in the context of the Haldane gap problem\cite{3}.

In this paper, we would like to draw attention to another, in a sense, opposite aspect of the problem of bond-alternating spin chains. In the rotationally (SU(2)) invariant $S = 1/2$ chain the formation of a massive phase appears as a result of bond alternation ($\beta \neq -1$). On the other hand, as follows from the (Bethe ansatz) exact solution of the translationally invariant XYZ model\cite{4}, an Ising-like mass gap is always present in the excitation spectrum as long as continuous spin rotation symmetry remains fully broken. One then may ask: Are the requirements of translational and spin rotation symmetries robust for realization of a gapless, Luttinger-liquid low-energy behavior of $S = 1/2$ spin chains? Can the bond alternation and full breakdown of spin rotational symmetry, each separately generating a gap, result in a gapless (critical) regime when acting together? As shown below (Sec.2), among several unitarily equivalent possibilities, the XXZ spin chain with the staggered XY anisotropy

$$H = H_{XXZ} + H_\delta$$

$$= J \sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z) - \delta \sum_n (-1)^n (S_n^x S_{n+1}^x - S_n^y S_{n+1}^y). \quad (2)$$

is a nontrivial example of such a system exhibiting gapless phases in a wide range of its parameters with rather interesting properties.

In Sec.3 we first consider the simplest, exactly solvable case of the XY bond-alternating chain ($\Delta = 0$). The Jordan-Wigner transformation reduces it to a 1d system of spinless
fermions with a half-filled band and a Cooper pairing with the total momentum equal $\pi$. This model is also equivalent to two decoupled quantum Ising chains (i.e. 1d Ising models in transverse magnetic fields) at criticality. Therefore, for any ratio $\delta/J$, the excitation spectrum is gapless, and elementary excitations are represented by two Majorana (real) fermions propagating with different velocities $v_{\pm} = |J \pm \delta|a$. At $|\delta| < J$ the staggered XY anisotropy does not influence the ground state of the system but reveals at the level of excitations, leading to splitting of the two gapless branches of the spectrum. This affects the time dependence of the spin correlation functions and thermodynamics.

At $|\delta| > J$, due to level crossing, the XY ground state is changed by another gapless phase - a spin nematic (SN) state, characterized by broken translational and spin rotational U(1) symmetry, but preserved time reversal invariance. In this phase, the local symmetry of the $xx$ and $yy$ spin correlations is changed by the $\uparrow\uparrow\downarrow\downarrow$ and $\uparrow\downarrow\uparrow\uparrow$ structures, respectively. This reflects the sign-alternating character of exchange couplings between neighboring spins at $|\delta| > J$ (see Eq.(2)). The new spin correlations (following the same power-law decay with distance as that in the XY phase) induce a two-spin SN ordering of the system described by the order parameter $<\text{vac}|S_n^x S_{n+1}^x - S_n^y S_{n+1}^y|\text{vac}> \sim (-1)^n \text{sgn } \delta$. We also find a manifestation of Villain’s ”order from disorder” - the effect of thermally induced SN ordering of the spins in the XY-like phase ($|\delta| < J$).

In Sec.4 we extend consideration to the case of a finite $\Delta$ to study effects of a weak staggered XY anisotropy ($|\delta| \ll J$) in a Luttinger spin liquid. Using bosonization method, we show that the low-energy properties of the system (2) are described by a self-dual field-theoretical model, similar to the one recently proposed to describe two weakly coupled Luttinger chains. In the range $-1/\sqrt{2} < \Delta < 1$, the off-diagonal perturbation $H_\delta$, tending to split the velocities of fermionic excitations, is irrelevant due to the infrared catastrophe that eliminates single-fermion states from the low-energy part of the spectrum. This behavior is an example of Anderson’s confinement which has been extensively discussed recently in connection with the two-chain problem. In this regime, the system maintains basic properties of the gapless Luttinger liquid, slightly modified by thermally induced SN order.
At $\Delta < -1/\sqrt{2}$, a translationally invariant, two-axis anisotropy, generated in the second order in $\delta$, becomes a relevant perturbation, and the system undergoes a Kosterlitz-Thouless transition to a Neel-ordered state with polarization axis $\hat{x}$ or $\hat{y}$.

By the $\delta \leftrightarrow J$ interchange symmetry, the above picture also holds in the opposite limiting case $|\delta| \gg J$. Here one finds Luttinger-liquid-like SN phases and Kosterlitz-Thouless transitions to phases with $\uparrow\uparrow\downarrow\downarrow$ ordering of the spins, either along $\hat{x}$- or $\hat{y}$-direction.

**II. CHOICE OF THE MODEL**

Consider a bond-alternating $S = 1/2$ quantum XYZ spin chain given by the Hamiltonian

$$H = \sum_{n=1}^{N} \sum_{\alpha=x,y,z} [J_{\alpha} + (-1)^{n} J'_{\alpha}] S_{\alpha}^{\alpha} S_{\alpha+1}^{\alpha}, \tag{3}$$

The model can be mapped onto two coupled quantum Ising chains by a nonlocal unitary transformation, previously used by Kohmoto, den Nijs and Kadanoff in their study of the 2D Ashkin-Teller model. This transformation is a special combination of spin rotation and duality transformation. Its basic steps are: i) a global $\pi/2$ spin rotation, $\sigma_{y}^{\alpha} \rightarrow \sigma_{z}^{\alpha}$, $\sigma_{z}^{\alpha} \rightarrow -\sigma_{y}^{\alpha}$; ii) duality transformation $\sigma_{n}^{x} \rightarrow \sigma_{n-1/2}^{z} \sigma_{n+1/2}^{z}$, $\sigma_{n}^{z} \sigma_{n+1}^{z} \rightarrow \sigma_{n+1/2}^{x}$, where $\{n + 1/2\}$ are the dual lattice sites; iii) relabeling the sites, $j \rightarrow \frac{1}{2}(j + \frac{1}{4})$, and introduction of two kinds of the Pauli matrices, $\sigma_{j}^{\alpha}$ and $\tau_{j+1/2}^{\alpha}$, defined on the two sublattices of the dual lattice; iv) duality transformation of the $\tau$-spins only. As a result, spin operators $S_{n}^{\alpha}$ of the staggered XYZ chain (2) are represented in terms of $\sigma$ and $\tau$ matrices as products of order and disorder operators:

$$2S_{2j}^{x} = \sigma_{j}^{x} \tau_{j+1/2}^{x}, \quad 2S_{2j+1}^{x} = \sigma_{j+1}^{x} \tau_{j+1/2}^{x},$$

$$2S_{2j}^{y} = \sigma_{j+1/2}^{x} \tau_{j}^{x}, \quad 2S_{2j+1}^{y} = \sigma_{j+1/2}^{z} \tau_{j+1}^{z},$$

$$2S_{2j}^{z} = i \sigma_{j}^{z} \sigma_{j+1/2}^{x} \tau_{j+1}^{x}, \quad 2S_{2j+1}^{z} = i \sigma_{j+1/2}^{z} \sigma_{j+1}^{x} \tau_{j+1/2}^{x},$$

where

$$\sigma_{j+1/2}^{x} = \prod_{l=j+1}^{N/2} \sigma_{l}^{x}, \quad \tau_{j+1/2}^{z} = \prod_{l=1}^{j} \tau_{l}^{z}.$$
The transformed Hamiltonian

\[ H = H_\sigma + H_\tau + H_{\sigma\tau} \]

\[
= \frac{N/2}{\sum_{j=1}^{N/2}} \left[ (A_x \sigma_j^x \sigma_{j+1}^x + B_y \sigma_j^y) + (A_y \tau_j^z \tau_{j+1}^z + B_x \tau_j^x) - (A_z \sigma_j^z \sigma_{j+1}^z + 1 + B_z \sigma_j^x \tau_{j+1}^x) \right] \quad (5)
\]
describes two coupled quantum Ising chains. The constants \(A_\alpha = (1/4)(J_\alpha + J'_\alpha), \ B_\alpha = (1/4)(J_\alpha - J'_\alpha)\). Following Ref.10, it can be shown that this Hamiltonian determines the transfer matrix of the two-dimensional asymmetric Ashkin-Teller (AT) in the highly anisotropic, the so-called \(\tau\)-continuum limit. The asymmetric AT model represents two different 2D Ising models coupled by a four-spin interaction.

To sort out those cases when the model (5) may show a critical (gapless) behavior, we shall treat the interchain \(\sigma\tau\) coupling as a weak perturbation and then consider conditions when at least one of the two quantum Ising chains is critical. There are three such possibilities.

1) \(H_\sigma\) is critical, while \(H_\tau\) is not (or vice versa), implying that \(A_x = \pm B_y, A_y \neq \pm B_x\). Using the Majorana fermion representation for each chain, one finds one massless (\(H_\sigma\)) and one massive (\(H_\tau\)) branch in the excitation spectrum. The interchain interaction \(H_{\sigma\tau}\) is irrelevant; its role is exhausted by renormalization of the mass and critical temperature (or group velocity of the massless fermion). In this case, the critical behavior of the system belongs to the 2d Ising universality class.

2) \(H_\sigma\) and \(H_\tau\) are identical and critical; e.g. \(A_x = B_y = A_y = B_x\). If at the same time \(A_z = B_z\), the model (5) becomes equivalent to the exactly solved \(U(1)\)-symmetric, translationally invariant XXZ spin chain,\(^{12}\) related to the six-vertex model.\(^{13}\) The interchain coupling \(H_{\sigma\tau}\) is marginal: the system shows a Luttinger-liquid critical behaviour with continuously varying (coupling-dependent) critical exponents.

3) \(H_\sigma\) and \(H_\tau\) are both critical but different. This case occurs when

\[
A_x = \pm B_y, \ A_y = \pm B_x, \ |A_x| \neq |A_y| \quad (6)
\]
and corresponds to two massless Majorana fields with different velocities. This is a new, yet not considered possibility.
Conditions (3) give four possible distributions of the exchange couplings \( A_x, A_y, B_x \) and \( B_y \) on lattice bonds, each distribution breaking both U(1) spin rotational symmetry and translational invariance (below we indicate only the \( XY \)-part of the corresponding Hamiltonians):

\[
H_1 = 2(A + B) \sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + 2(A - B) \sum_n (-1)^n (S_n^x S_{n+1}^x - S_n^y S_{n+1}^y),
\]

\[
H_2 = 2(A - B) \sum_n (-1)^n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + 2(A + B) \sum_n (S_n^x S_{n+1}^x - S_n^y S_{n+1}^y),
\]

\[
H_3 = 2 \sum_n [A - (-1)^n B] (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + 2 \sum_n [B + (-1)^n A] (S_n^x S_{n+1}^x - S_n^y S_{n+1}^y),
\]

\[
H_4 = 2 \sum_n [B + (-1)^n A] (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + 2 \sum_n [A - (-1)^n B] (S_n^x S_{n+1}^x - S_n^y S_{n+1}^y).
\]

All these cases are related to each other by nonlocal unitary transformations. Transforming \( S_n^y \rightarrow (-1)^n S_n^y \), \( S_n^z \rightarrow (-1)^n S_n^z \), one finds that

\[
H_2(A, B; J_z, J'_z) \leftrightarrow H_1(A, B; -J_z, -J'_z), \quad H_4(A, B; J_z, J'_z) \leftrightarrow H_3(A, B; -J_z, -J'_z).
\]

On the other hand, under \( S_n^y \rightarrow -S_n^y \), \( S_n^z \rightarrow -S_n^z \) only at \( n = 4j + 1 \) and \( n = 4j + 2 \),

\[
H_3(A, B; J_z, J'_z) \leftrightarrow H_1(A, B; -J_z, -J'_z).
\]

In what follows, we shall concentrate on \( H_1 \). Choosing then \( A_z = B_z \), we arrive at the XXZ spin chain with XY bond-alternating (staggered) anisotropy, given by Eq.(2). It is equivalent to the following Hamiltonian describing two critical but different quantum Ising chains coupled by a self-dual interchain interaction:

\[
H = \frac{1}{4} \sum_{j=1}^{N/2} [(J - \delta)(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x) + (J + \delta)(\tau_j^z \tau_{j+1}^z + \tau_j^x) - J \Delta(\sigma_j^z \sigma_{j+1}^z \tau_j^z \tau_{j+1}^z + \tau_j^x \tau_j^x)]. \tag{7}
\]

The model (2) has the following symmetry properties which will be used below:

1) Under a global \( \pi/2 \) rotation \( S_n^x \rightarrow S_n^y, \quad S_n^y \rightarrow -S_n^x \)

\[
H(J, \delta, J_z) \rightarrow H(J, -\delta, J_z). \tag{8}
\]
2) Under a staggered transformation, $S^x_n \rightarrow (-1)^n S^x_n$, $S^y_n \rightarrow (-1)^{n+1} S^y_n$

$$H(J, \delta, J_z) \rightarrow H(-J, \delta, J_z).$$

(9)

3) A nonlocal transformation to a $\pi/2$-twisted reference frame,

$$S^x_n \rightarrow -S^x_n \quad (n = 4j + 1, 4j + 2),$$

$$S^y_n \rightarrow -S^y_n \quad (n = 4j + 2, 4j + 3),$$

$$S^z_n \rightarrow (-1)^n S^z_n$$

(10)

leads to the important symmetry under interchange of $J$ and $\delta$:

$$H(J, \delta, J_z) \rightarrow H(\delta, J, -J_z).$$

(11)

III. XY CHAIN WITH BOND-ALTERNATING ANISOTROPY

In this section we set $J_z = 0$ and consider properties of the XY chain with a bond-alternating anisotropy:

$$H = H_0 + H_\delta = \sum_{n=1}^{N} [J(S^x_n S^x_{n+1} + S^y_n S^y_{n+1}) - \delta(-1)^n(S^x_n S^x_{n+1} - S^y_n S^y_{n+1})].$$

(12)

Due to the symmetry property (9), we can choose without loss of generality $J > 0$. The Jordan-Wigner (JW) transformation

$$S^+_n = a^+_n \exp(-i\pi \sum_{j=1}^{n-1} a^+_j a_j), \quad S^z_n = a^+_n a_n :\equiv a^+_n a_n - \frac{1}{2}$$

(13)

represents Hamiltonian (12) as a quadratic form of spinless fermion operators

$$H = \frac{1}{2} \sum_n [J(a^+_n a_{n+1} + a^+_n a_n) - \delta(-1)^n(a^+_n a_{n+1} + a_{n+1} a_n)],$$

(14)

desccribing a tight-binding half-filled band and a Cooper pairing of the fermions on neighboring sites with a sign-alternating amplitude. The latter circumstance leads to commutativity of $H_0$ and $H_\delta$ which, in turn, indicates the absence of a gap in the spectrum at arbitrary
values of $J$ and $\delta$. This is in agreement with the equivalent representation of model (2) in terms of two decoupled quantum Ising chains at criticality, as given by Eq.(4) at $\Delta = 0$. Representation (14) implies that the excitation spectrum consists of two branches which, in the continuum limit, correspond to two massless Majorana fields with group velocities $v_\pm = |J \pm \delta|$ (we set the lattice constant $a = 1$).

To understand the effect of the staggered XY anisotropy on the ground state properties of the model (14), we rewrite the Hamiltonian in momentum representation

$$H = H_0 + H_\delta = \sum_{|k|<\pi} \left[ \epsilon(k)a_k^+a_k + \frac{1}{2}\Delta(k)(a_k^+a_{\pi-k}^+ + h.c.) \right],$$  \hspace{1cm} (15)

where

$$\epsilon(k) = -\epsilon(k + \pi) = J \cos k, \quad \Delta(k) = -\Delta(k + \pi) = \delta \cos k.$$  

We see that $H_\delta$ describes Cooper pairing of JW fermions with total momentum equal $\pi$. Mapping Hamiltonian (15) onto reduced Brillouin zone, $|q| < \pi/2$

$$H = \sum_{|q|<\pi/2} \left[ \epsilon(q)(a_q^+a_q - a_{\pi-q}^+a_{\pi-q}) + \Delta(q)(a_q^+a_{\pi-q} + a_{\pi-q}a_q) \right],$$

and introducing a Nambu 2-spinor

$$\psi_q = \begin{pmatrix} a_q \\ a_{\pi-q}^+ \end{pmatrix},$$

one rewrites $H$ in the form

$$H = E_0 + \sum_{|q|<\pi/2} \psi_q^+ \left[ \epsilon(q) + \Delta(q)\tau^1 \right] \psi_q,$$  \hspace{1cm} (16)

where $\tau^\alpha$ are the Pauli matrices, and

$$E_0 = -\sum_q \epsilon(q) = -\frac{NJ}{\pi}$$  \hspace{1cm} (17)

is the ground state energy of $H_0$.

Eq.(10) explains the origin of commutativity of $H_0$ and $H_\delta$. The Hamiltonian of the "pure" XY model, $H_0$, with global U(1) spin rotation symmetry in real space, posseses a
local SU(2) pseudospin symmetry in momentum space. Generators of this SU(2) group are:

\[ J_\alpha^q = \frac{1}{2} \bar{\psi}_q^+ r^\alpha \psi_q \quad (\alpha = x, y, z). \]  

For a given \( q \), \( \Delta(q) \) then appears as a "magnetic" field oriented along the \( x \)-axis in the local pseudospin space.

Diagonalization of \( H \) reduces to a \( \pi/2 \)-rotation \( \psi_q \rightarrow \exp[-(\pi/2)J_\alpha^q] \psi_q \), under which

\[ H = E_0 + H_{exc}, \]

\[ H_{exc} = \sum_{|q|<\pi/2} [E_+(q)\alpha_q^+\alpha_q + E_-(q)\beta_q^+\beta_q], \]  

where new fermion operators

\[ \alpha_q = \frac{1}{\sqrt{2}}(a_q + a_{\pi-q}^+), \quad \beta_q = \frac{1}{\sqrt{2}}(-a_q + a_{\pi-q}^+) \]  

and

\[ E_{\pm}(q) = (J \pm \delta) \cos q \]  

At \( |\delta| < J E_{\pm}(q) \geq 0 \) at all \( |q| < \pi/2 \). Therefore in Eq.(19) \( H_{exc} \) is positive definite and represents the excitation energy of the system, \( E_{\pm}(q) \) thus being energies of the single-fermion excitations. The ground state \( |vac> \) is defined via relations \( \alpha_q|vac> = \beta_q|vac> = 0 \) and coincides with that of \( H_0 \). The local pseudospin SU(2) symmetry is unbroken: \( J_q^\alpha|vac> = 0 \).

Thus, at \( |\delta| < J \), the alternating XY anisotropy has no effect on the ground state of the system. This can be understood from the following simple arguments. Start with the ground state of the pure XY model \( H_0 \),

\[ |vac> = \prod_{\pi/2<|k|<\pi} a_k^+|0> \]

and then consider \( H_\delta \) as a perturbation. Note that if a single-fermion state \( |k> \) is occupied, the state \( |\pi-k> \) is necessarily empty, or vice versa. As a result, the Cooper pairing with
total momentum $\pi$ cannot occur in the ground state, if $\delta$ is small enough; so the ground state is that of the XY model. This means that there is no off-diagonal long-range order (ODLRO) in the ground state

\[
\langle \text{vac}|a_{k}^{+}a_{-k}^{+}|\text{vac}\rangle = 0, \\
\langle \text{vac}|S_{n}^{x}S_{n+1}^{x} - S_{n}^{y}S_{n+1}^{y}|\text{vac}\rangle = 0, \quad (|\delta| < J),
\]

(22)
implying that spin rotational U(1) symmetry and translational invariance remain intact. However, perturbation $H_{\delta}$ reveals at the level of excitations. Consider one-particle and one-hole excitations $|q>_{p} = a_{q}^{+}|\text{vac}\rangle$, $|\pi - q>_{h} = a_{\pi - q}|\text{vac}\rangle$ with $|q| < \pi/2$. At $\delta = 0$ these states are degenerate: $E_{p}(q) = E_{h}(\pi - q) = J \cos q \geq 0 \ (J > 0)$. The degeneracy is removed by $H_{\delta}$, but only at $q \neq \pm k_{F}$, since the pairing amplitude $\Delta(q)$ has nodes at the two Fermi points. Therefore, a gap does not open; instead the doubly degenerate spectrum splits into two gapless branches with different Fermi velocities.

The above picture persists at all $|\delta| < J$. However, at $|\delta| > J$ one of the two excitation energies (21) becomes negative, and the ground state changes. Since $[H_{0}, H_{\delta}] = 0$, this simply occurs due to level crossing: one of the former excited states becomes the ground state. As a result, the symmetry of the vacuum is changed. Choosing, e.g., $\delta > J > 0$, and making a particle-hole transformation $\beta_{q} \rightarrow \bar{\beta}_{q}^{+}$, one finds that the ground state is now determined by $\alpha_{q}|\text{vac}\rangle = \bar{\beta}_{q}|\text{vac}\rangle = 0$, the ground state energy and Hamiltonian of excitations being changed, respectively, by

\[
E_{0} = -\sum_{q} \Delta(q) = -\frac{N|\delta|}{\pi},
\]

(23)

\[
H_{\text{exc}} = \sum_{|q|<\pi} [E_{+}(q)\alpha_{q}^{+}\alpha_{q} + E_{-}(q)\bar{\beta}_{q}^{+}\bar{\beta}_{q}]
\]

(24)

with

\[
E_{\pm}(q) = (\delta \pm J) \cos q \geq 0.
\]

(25)

Notice that, at $|\delta| > J$, $H_{0}$ does not contribute to the ground state energy:
The local pseudospin SU(2) symmetry is now broken down to U(1), implying that the ground state is characterized by a nonzero ODLRO which breaks global U(1) and translational symmetries:

$$\langle \text{vac}|S^x_n S^x_{n+1} + S^y_n S^y_{n+1}|\text{vac}\rangle = 0.$$  \hspace{1cm} (26)

where \(\text{vac}\) is the vacuum state.

Using the two-chain representation (4) of the spin operators, one finds that, at all values of \(J\) and \(\delta\), the local magnetization of the model (12) vanishes:

$$\langle S^\alpha_n \rangle = 0.$$  \hspace{1cm} (27)

This simply follows from the fact that, since the \(\sigma\) and \(\tau\) quantum Ising chains are decoupled and self-dual (critical), the \(Z_2 \times Z_2\) symmetry related to transformations \(\sigma^z_j \rightarrow -\sigma^z_j, \quad \tau^z_j \rightarrow -\tau^z_j\) remains unbroken. Absence of local magnetization means that time reversal symmetry is always preserved. Preservation of time reversal symmetry together with breakdown of spin rotational U(1) symmetry allows to conclude that, at \(|\delta| > J\), the ground state represents a spin-nematic (SN) phase, with the order parameter given by symmetric tensor (27). The SN state is doubly \((Z_2)\) degenerate; the two SN phases at \(\delta > J\) and \(\delta < -J\) transform to each other under transformation (8).

Let us now consider spin correlations in the ground state of the XY-like (\(|\delta| < J\)) and SN (\(|\delta| > J\)) phases. According to the two-chain representation (4) of the Hamiltonian and multiplicative \(\sigma\tau\) form (4) of the spin operators, the correlation functions factorize into products of independent contributions of each quantum Ising chain. At large space-time separations, the \(\sigma\) and \((\tau)\) contributions are determined by gapless excitations, described in terms of (right- and left-moving) Majorana fermions with velocities \(v_+\) and \(v_-\), respectively.

We shall first consider the case \(|\delta| < J\). Using the Majorana fermion representation of the quantum Ising chain (4) and passing to a continuum limit, one can show that the longitudinal
(zz) spin correlation function is bilinear in the Majorana single-fermion Green’s functions:
\[
<S^z_n(t)S^z_m(0)> \approx -\frac{1}{4\pi^2} \left( \frac{1}{x + v_+ t} - \frac{(-1)^{n-m}}{x - v_+ t} \right) \left( \frac{1}{x + v_- t} - \frac{(-1)^{n-m}}{x - v_- t} \right) 
\]
\[
= -\frac{1}{2\pi^2} \frac{[1 - (-1)^{n-m}]x^2 + [1 + (-1)^{n-m}]v_+ v_- t^2}{(x^2 - v_+^2 t^2)(x^2 - v_-^2 t^2)},
\]
where \( x = n - m \). This result can also be obtained by using the JW representation (13) for \( S^z_n \) together with Bogoliubov transformation (20) to quasiparticle operators \( \alpha_q \) and \( \beta_q \). The transverse spin correlation function is a product of two Ising correlation functions:
\[
<S^x_n(t)S^x_m(0)> = <S^y_n(t)S^y_m(0)> \sim -(-1)^{n-m} \frac{1}{(x^2 - v_+^2 t^2)^{1/8}} \frac{1}{(x^2 - v_-^2 t^2)^{1/8}}.
\]
At equal times, formulas (28) and (29) coincide with known results for the “pure” XY model (\( \delta = 0 \)).

Now we consider the case \( |\delta| > J \). For simplicity, we shall restrict ourselves by equal time correlations. As follows from the \( \delta \leftrightarrow J \) symmetry transformations (10), (11), in the SN phases the antiferromagnetic zz-correlations (28) transform to the ferromagnetic ones:
\[
<S^z_n S^z_m>_{vac} = \frac{1}{2\pi^2} \frac{[1 - (-1)^{n-m}]}{|n - m|^2}
\]
In the XY-like phase, the power-law decay (28) of the antiferromagnetic zz-correlations leads to the well known singular response of the system to a staggered magnetic field along the \( z \)-axis, logarithmically divergent at \( T \to 0 \). (In terms of the JW fermions, this property appears as a logarithmic charge-density-wave instability of a noninteracting 1D Fermi system). Similarly, in the SN phases, ferromagnetic zz-correlations (30) give rise to a logarithmically divergent response to a homogeneous magnetic field. The ferromagnetic and antiferromagnetic susceptibilities are finite at \( |\delta| < J \) and \( |\delta| > J \), respectively,
\[
\chi_F \sim \frac{1}{J} \ln \left( \frac{J + \delta}{J - \delta} \right), \quad (|\delta| < J),
\]
\[
\chi_{AF} \sim \frac{1}{J} \ln \left( \frac{\delta + J}{\delta - J} \right), \quad (|\delta| > J).
\]
At the boundaries $\delta = \pm J$ between the XY-like and SN phases, the model shows both ferromagnetic and antiferromagnetic instabilities.

Applying the $\delta \leftrightarrow J$ transformations (11) to (29), we find that, at $|\delta| > J$, the transverse spin correlations show the same power-law decay as in the XY-like phase, but are characterized by a different type of local ordering:

$$<S^x_{2j}S^x_{2j+2l+1}> \sim \frac{(-1)^l}{|2l+1|^{1/2}}; \quad <S^x_{2j}S^x_{2j+2l+1}> \sim \frac{(-1)^{l+1}}{|2l+1|^{1/2}}. \quad (31)$$

$$<S^y_{2j}S^y_{2j+2l+1}> \sim \frac{(-1)^l}{|2l|^{1/2}}; \quad <S^y_{2j}S^y_{2j+2l+1}> \sim \frac{(-1)^{l+1}}{|2l+1|^{1/2}}. \quad (32)$$

Eqs. (31) and (32) describe $\uparrow\uparrow\downarrow\downarrow$ and $\uparrow\downarrow\downarrow\uparrow$ periodic structures for the $xx$ and $yy$ spin correlations, respectively. The origin of these correlations is easily seen from (12). At $\delta > J$ the $x$-components of neighboring spins are coupled ferromagnetically on even bonds $<2j, 2j+1>$ and antiferromagnetically on odd bonds $<2j - 1, 2j>$, whereas for the $y$-components the picture is opposite (at $\delta < -J$ one has simply to interchange $x$- and $y$-components).

The transverse spin correlations (31), (32) locally describe a classical ground state of Hamiltonian $H_\delta$. Such a state represents a periodic structure, with a unit cell consisting of four spins $S_m = \hat{x} \cos \varphi_m + \hat{y} \sin \varphi_m$, $m = 1, 2, 3, 4$, where angles $\varphi_m$ are given by $\varphi_1 = \varphi$, $\varphi_2 = -\varphi$, $\varphi_3 = \pi + \varphi$, $\varphi_4 = -\varphi$ at $\delta > 0$, and $\varphi_1 = \varphi$, $\varphi_2 = -\varphi$, $\varphi_3 = \pi + \varphi$, $\varphi_4 = -\varphi$ at $\delta < 0$, $\varphi$ being an arbitrary angle. These configurations can be viewed as two interpenetrating antiferromagnetic sublattices whose staggered magnetizations transform to each other under reflection about the $y$-axis ($\delta > 0$), or $x$-axis ($\delta < 0$), as required by the symmetry of $H_\delta$. Apparently, these states can be obtained from a classical Neel ground state of the XY model $H_0$ by means of transformations (11) and (8). Furthermore, the nonlocal transformation (11) maps the continuous symmetry of $H_0$ under uniform U(1) spin rotations onto the symmetry of $H_\delta$ under continuous rotations of the two sublattices by the same angle but in opposite directions. As a result, the classical ground state of $H_\delta$ is continuously degenerate under varying the angle $\varphi$. 

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In the quantum case of interest, the corresponding gapless "phason" mode destroys single-spin long-range order precisely in the same way as it occurs in the XY model. However, despite the loss of local magnetization, the survival of transverse correlations \( (31), (32) \) gives rise to SN ordering of the system, described by two-spin tensor \( (27) \). This is similar to the picture described by Chandra, Coleman and Larkin\(^{17} \) for 2d frustrated antiferromagnets, where short-range correlations with local twist structure lead to the formation of a SN state. In the latter case, the order parameter is a parity-breaking antisymmetric tensor \( T(x, x') = < S(x) \times S(x') > \), corresponding to the p-type SN phase.\(^{14} \) The difference between this and our cases is the absence of local twist structure in the ground state of our model at \( |\delta| > J \), \( (T(x, x') = 0) \). Instead we have the above described local spin structure generated by the staggered XY anisotropy, which results in a SN order parameter \( (27) \) with a symmetric tensor form.

To conclude this section, let us briefly discuss the role of finite temperature. As it was already pointed out, the staggered XY anisotropy does not affect the ground state of the model at \( \delta < J \), but reveals in splitting the excitation spectrum in two gapless branches with different Fermi velocities. One of interesting manifestations of this splitting is the appearance of the temperature-induced SN long-range order in the XY phase. Using the diagonalized form \( (19) \) of the Hamiltonian, one finds that

\[
< S^x_{2j} S^x_{2j+1} > = < S^y_{2j-1} S^y_{2j} > = -\frac{1}{2\pi} + \frac{1}{J - \delta} \frac{E_-}{N},
\]

\[
< S^x_{2j-1} S^x_{2j} > = < S^y_{2j} S^y_{2j+1} > = -\frac{1}{2\pi} + \frac{1}{J + \delta} \frac{E_+}{N}.
\]

Here \( E_{\pm} \) are the average thermal energies of the \( \alpha \)- and \( \beta \)-quasiparticles. At low temperatures, \( T \ll J \pm \delta \), \( E_{\pm}/N = (\pi/24)(T^2/(J \pm \delta)) \). Assuming for simplicity that \( \delta \ll J \) one finds

\[
< S^x_{2j} S^x_{2j+1} > = < S^y_{2j-1} S^y_{2j} > = \left( -\frac{1}{2\pi} + \frac{\pi T^2}{24J^2} \right) + \frac{\pi}{12} \left( \frac{\delta}{J} \right) \left( \frac{T}{J} \right)^2,
\]

\[
< S^x_{2j-1} S^x_{2j} > = < S^y_{2j} S^y_{2j+1} > = \left( -\frac{1}{2\pi} + \frac{\pi T^2}{24J^2} \right) - \frac{\pi}{12} \left( \frac{\delta}{J} \right) \left( \frac{T}{J} \right)^2.
\]
The first term in the right-hand sides of these equations describes antiferromagnetic correlations in the XY model, slightly suppressed by thermal fluctuations. The second terms with opposite signs represent a combined effect of the temperature and XY bond-alternating anisotropy. These terms have the symmetry of the above described transverse spin correlations in the ground state of the SN phase and give rise to a finite value of the SN order parameter at $|\delta| < J$

$$< S^x_n S^x_{n+1} - S^y_n S^y_{n+1} > = (-1)^n \left( \frac{\pi \delta}{6J} \right) \left( \frac{T}{J} \right)^2$$ (33)

The phenomenon we are dealing with here is an example of Villain’s ”order from disorder”. Its manifestations in 2d frustrated magnets have been recently described in Refs.17,18. The $\delta$-anisotropy cannot influence the ground state at $\delta < J$; however, it lowers thermal excitation energy of the system by splitting the spectrum and repopulating the excited quasiparticles. This leads to the generation of new spin correlations, absent in the ground state, indicating the onset of SN order from thermal disorder.

On increasing $|\delta|$, one of the two modes becomes very soft and nearly completely populated, when $|J - |\delta|| \ll T \ll J + |\delta|$. In this temperature range, the $xx$ spin correlations on even (odd) bonds and $yy$ correlations on odd (even) bonds get suppressed, and the amplitude of the SN order parameter is close to $1/2\pi$. At $|\delta| \to J$, the SN long-range order eventually penetrates into the ground state.

**IV. XXZ CHAIN WITH A WEAK BOND-ALTERNATING XY ANISOTROPY**

In this section, we extend the above considerations to the model (2) with a finite $J_z$ and study effects of a weak bond-alternating XY anisotropy in the disordered gapless phase of the XXZ chain, assuming that $|\delta| \ll J$, $|\Delta| < 1$. The question we address here is: How off-diagonal perturbation $H_\delta$, tending to split velocities of fermionic excitations, reveals in a Luttinger spin liquid, where orthogonality catastrophe suppresses single-fermion states in the low-energy part of the spectrum. Notice that the $\delta \leftrightarrow J$ symmetry allows to translate results of this section to the case of very strong $\delta$-anisotropy, $|\delta| \gg J$. 

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Being interested in the infrared properties of the model, we pass to a continuum description developed for the XXZ spin chain by Luther and Peschel. Linearizing the spectrum of the JW fermions in the vicinity of two Fermi points $\pm k_F = \pm \pi/2$ and introducing two Fermi fields corresponding to right-moving ($\psi_1$) and left-moving ($\psi_2$) particles

$$a_n \rightarrow (-i)^n \psi_1(x) + i^n \psi_2(x),$$

one makes use of Abelian bosonization:

$$\psi_{1,2}^+(x)\psi_{1,2}(x) : = \frac{1}{\sqrt{\pi}} \partial_x \varphi_{1,2}(x)$$

$$\psi_{1,2}(x) \rightarrow \frac{1}{\sqrt{2\pi \alpha}} \exp[\pm i\sqrt{4\pi} \varphi_{1,2}(x)],$$

$$\varphi_{1,2}(x) = \frac{1}{2} \left( \varphi(x) \mp \int^x \Pi(y) \right),$$

to describe the low-energy spin excitations in terms of a scalar field theory

$$H_{XXZ} = \frac{1}{2} v_F \int dx \left[ : \Pi^2(x) : + (1 + 4\Delta/\pi) : (\partial_x \varphi(x))^2 : \right] + H_U.$$  

Here $\varphi(x)$ and $\Pi(x)$ are the scalar field and its conjugate momentum, respectively. The term

$$H_U \sim \Delta \int dx \cos(\sqrt{16\pi}\varphi(x))$$

originates from Umklapp scattering of the JW fermions. At $\Delta > 1$, these processes drive the system to a strong-coupling, massive Neel phase, but are irrelevant in the disordered phase, $|\Delta| < 1$. Using continuum representation (34), the staggered XY anisotropy $H_\delta$ transforms to a pairing term

$$H_\delta = -\frac{i\delta}{2} \int dx \left[ : \psi_1^+(x)\psi_1^+(x+a) : - : \psi_2^+(x)\psi_2^+(x+a) : \right] - h.c.]$$

Bosonizing (38) by means of (35) and performing canonical transformation of the field and momentum

$$\varphi(x) \rightarrow \frac{\beta}{\sqrt{4\pi}} \phi(x), \quad \Pi(x) \rightarrow \frac{\tilde{\beta}}{\sqrt{4\pi}} P(x), \quad \beta \tilde{\beta} = 4\pi,$$
one arrives at the following self-dual field-theoretical model

\[ H = \int dx \left[ \frac{u}{2} \left( : P^2(x) : + : (\partial_x \phi(x))^2 : \right) + \frac{\delta}{\pi \alpha} \cos \beta \phi(x) \cos \tilde{\beta} \tilde{\phi}(x) \right]. \] (40)

Here \( u \) is the renormalized velocity, \( \alpha \) is a cutoff parameter, and \( \tilde{\phi}(x) \) is a field dual to \( \phi(x) \), defined as \( \partial_x \tilde{\phi}(x) = P(x) \). The correct parametrization of \( \beta \) is provided by the Bethe-ansatz solution of the XXZ model

\[ \frac{\beta^2}{2\pi} = \left( 1 - \frac{\theta}{\pi} \right)^{-1}, \quad \cos \theta = \Delta \] (41)

Amazingly, the model (40) is similar to the one recently proposed to describe two weakly coupled spinless Luttinger chains. The difference between the two models stems from different values of the conformal spin \( S_c = \beta \tilde{\beta}/2\pi \) of the perturbation: \( S_c = 1 \) for the two-chain model, and \( S_c = 2 \) for the present one. As a result, the two-chain and present models belong to different universality classes. It was shown in Ref.8 that the self-dual theory (40) can be equivalently reformulated in terms of a 2d Coulomb gas of charge-monopole composites. The latter is given by the grand partition function:

\[ \frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \int \prod_{j=1}^{2n} d^2 x_j \frac{d^2 \mathbf{x}_j}{\alpha^2} \sum_{\sigma, \tilde{\sigma}} \exp \left( 2K \sum_{i<j} \sigma_i \sigma_j l_{ij} + 2\tilde{K} \sum_{i<j} \tilde{\sigma}_i \tilde{\sigma}_j l_{ij} + 2i \sum_{i \neq j} \sigma_i \tilde{\sigma}_j \varphi_{ij} \right) \] (42)

where \( Z_0 \) is the partition function of the unperturbed, Gaussian part of the model. Eq.(12) describes a 2d system of classical particles with coordinates \( \mathbf{x}_j = (u\tau_j, x_j) \), \( 0 < \tau_j < 1/T \), \( z = \delta \alpha / \pi u \) being the fugacity. Each particle carries an ”electric” charge \( \sigma_j = \pm 1 \) and ”magnetic” charge, or monopole, \( \tilde{\sigma}_j = \pm 1 \). The four-component system of charge-monopole composites is neutral with respect to each kind of charges. The charges and monopoles interact with logarithmic potential \( l_{ij} = \ln(|\mathbf{x}_i - \mathbf{x}_j|/\alpha) \) amongst themselves, and there is also a statistical Aharonov-Bohm phase \( \varphi_{ij} = \arctan((x_i - x_j)/u(\tau_i - \tau_j)) \) which couples the charges and monopoles belonging to different composites. The parameters \( K = \beta^2 / 4\pi \) and \( \tilde{K} = 1/K \) are naturally interpreted as inverse dimensionless temperatures for the charges and monopoles, respectively.

The perturbation in (40) has critical dimension \( D = K + \tilde{K} \geq 2 \). So, in the XY-like, gapless phase of the XXZ chain, the staggered XY anisotropy is marginal at the XY
point $\Delta = 0$ and irrelevant at $\Delta \neq 0$. The renormalized amplitude $\delta_{\text{eff}}(T)$, and hence the difference between the velocities of two Majorana fermionic excitations, scales down to zero according to the power law $\delta_{\text{eff}}(T) \sim \delta(\alpha T/u)^{D-2}$. This is analogous to the confinement regime in the two-chain model\,[9], in which Luttinger-liquid effects suppress single-particle interchain hopping in the infrared limit\,[8,23].

At finite temperatures, thermally induced SN ordering will take place. However, in the Luttinger spin-liquid regime ($\Delta \neq 0$), the SN order parameter increases with the temperature more slowly than in a noninteracting Fermi gas, Eq.\,(33). Using Coulomb gas representation \,[12] and calculating the second-order correction to the free energy of the XXZ model, one finds that

$$(-1)^n < S_n^x S_{n+1}^x - S_n^y S_{n+1}^y > = -\frac{1}{L} \frac{\partial F}{\partial \delta} \sim \left( \frac{\delta \alpha}{u} \right) \left( \frac{T \alpha}{u} \right)^{2D-2} \quad (43)$$

The nonuniversal power-law temperature dependence of the SN order parameter reflects the well-known infrared catastrophe in a 1D Luttinger liquid that eliminates single-fermion states from the low-energy part of the spectrum. The r.h.s. of \,(43) should then be understood as $\sim T^2[N_{\text{LL}}(T)]^2$, where $N_{\text{LL}}(\omega) \simeq N_0(|\omega| \alpha/u)^{D-2}$ is the single-fermion density of states vanishing in the zero-energy limit.

The suppression of the XX anisotropy in the lowest order in $\delta$ does not really mean that $H_\delta$ is a totally irrelevant perturbation. The usual criterium of relevance, based on the comparison of the critical dimension of a perturbation with space-time dimension 2, is not applicable here, since the perturbation has a nonzero conformal spin\,[25]. This is seen from the Coulomb gas representation \,[12]. The suppression of the XX staggered anisotropy in the first order in $\delta$ originates from pairing of the charge-monopole composites with zero total "electric" and "magnetic" charges. On the other hand, binding of composites in pairs with total "magnetic" or "electric" charge $\pm 2$ gives rise to two operators

$$O(x) = \cos 2\beta \phi(x), \quad \tilde{O}(x) = \cos 2\tilde{\beta} \phi(x) \quad (44)$$

which are absent in the original model \,[10], but are generated upon renormalization in the
effective Hamiltonian in the second order in $\delta$. The operators (44) have zero conformal spin; so their relevance can be established by the usual criterium.

Clearly, $O(x)$ is the Umklapp operator (see (37)), relevant only at $|\Delta| > 1$. $\tilde{O}(x)$ is a new operator whose critical dimension is $4\tilde{K}$. It becomes marginal at the point $\tilde{K} = 1/2$, or equivalently, $\Delta = -\frac{1}{\sqrt{2}}$. Therefore, in the region $-\frac{1}{\sqrt{2}} < \Delta < 1$, where both operators $O(x)$ and $\tilde{O}(x)$ are irrelevant, the model (40) preserves basic features of the gapless Luttinger-liquid phase of the XXZ chain, slightly modified by thermally induced SN or dering (43). This is a pure realization of Anderson’s confinement, i.e. stability of the Luttinger liquid against single-fermion off-diagonal perturbations, as opposed to the case of two coupled Luttinger chains, where the generation of two-particle interchain correlations drives inevitably the system away from the Luttinger-liquid fixed point at arbitrary in-chain interaction (see, e.g. Ref.24 and references therein).

The point $\Delta = -\frac{1}{\sqrt{2}}$ is expected to be a Kosterlitz-Thouless transition point. The nature of the operator $\tilde{O}(x)$ and, accordingly, the type of ordering in a massive phase which occurs at $\Delta < -\frac{1}{\sqrt{2}}$ can be elucidated by means of the effective sine-Gordon model written in terms of the dual field $\tilde{\phi}$

$$H_{eff} = \int dx \left[ \frac{n'}{2} \left( \tilde{P}^2(x) + (\partial_x \tilde{\phi}(x))^2 \right) - \frac{m}{\alpha} \cos 2\beta \tilde{\phi}(x) \right],$$

(45)

together with the continuum representation of spin operators in terms of scalar fields $\phi$ and $\tilde{\phi}$:

$$S_n^z \rightarrow \frac{\beta}{2\pi} \partial_x \phi(x) + \lambda (-1)^n \sin \beta \phi(x),$$

(46)

$$S_n^x \rightarrow \mu (-1)^n \cos \frac{1}{2} \beta \phi(x), \quad S_n^y \rightarrow \mu (-1)^n \sin \frac{1}{2} \beta \phi(x)$$

(47)

In the above equations, $m$ is a positive parameter proportional to $\delta^2 a/u$, and $\lambda$ and $\mu$ are numerical constants. As follows from (45), the onset of a strong-coupling regime in the model (40) results in ordering of the dual field, $\tilde{\phi}$.

Let us look at the symmetry properties of the model (9) under translations and spin rotations about the $\hat{z}$-axis. The spin operators in (10) and (11) are invariant under $\phi \rightarrow \phi + ...$
$\phi + 2\pi/\beta, \tilde{\phi} \rightarrow \tilde{\phi} + 4\pi/\tilde{\beta}$. This is the general symmetry of any spin model. Under a translation by one lattice spacing ($T_a$), $\phi \rightarrow \phi + \pi/\beta, \tilde{\phi} \rightarrow \tilde{\phi} + 2\pi/\tilde{\beta}$. Under a U(1) rotation about the $\hat{z}$-axis by an angle $\alpha$ ($U_\alpha$), $\phi$ does not change, while $\tilde{\phi} \rightarrow \tilde{\phi} + 2\alpha/\tilde{\beta}$. Using these transformations of the fields $\phi$ and $\tilde{\phi}$, one can check that the perturbation term in the original continuum model (40) changes its sign under $T_a$ and $U_{\pi/2}$, thus reflecting breakdown of translational symmetry (bond alternation) and the $\pi/2$ spin rotation symmetry (XY-anisotropy). We then notice that, in the effective model (45), each of these two symmetries is recovered; however the operator $\cos 2\tilde{\beta}\tilde{\phi}$ changes its sign under $U_{\pi/4}$. We therefore conclude that the effective perturbation, generated in the second order in $\delta$, represents a translationally invariant, two-axis ($Z_4$) anisotropy. Its lattice version is given by a four-spin term

$$\cos 2\tilde{\beta}\tilde{\phi} \leftrightarrow (S_{n}^{x}S_{n+1}^{x} - S_{n}^{y}S_{n+1}^{y})(S_{n+2}^{x}S_{n+3}^{x} - S_{n+2}^{y}S_{n+3}^{y})$$

$$-(S_{n}^{x}S_{n+1}^{y} + S_{n}^{y}S_{n+1}^{x})(S_{n+2}^{x}S_{n+3}^{y} + S_{n+2}^{y}S_{n+3}^{x})$$

In the strong-coupling phase ($-1 < \Delta < -1/\sqrt{2}$) the $Z_4$-symmetry is broken. The field $\tilde{\phi}$ acquires a nonzero vacuum expectation value, determined by one of the four degenerate minima $\tilde{\phi} = 0, \pi/\tilde{\beta}, 2\pi/\tilde{\beta}, 3\pi/\tilde{\beta}$ of the ”potential” $-m\cos 2\tilde{\beta}\tilde{\phi}$ (within the main periodicity interval $4\pi/\tilde{\beta}$). As follows from (47), this leads to a Neel ordering of the spins, either along the $x$-axis,

$$(-1)^n < S_{n}^{x} > \sim \pm \mu, \quad (-1)^n < S_{n}^{y} >= 0 \quad (48)$$

at $\tilde{\phi} = 0, 2\pi/\tilde{\beta}$, or along the $y$-axis,

$$(-1)^n < S_{n}^{x} >= 0, \quad (-1)^n < S_{n}^{y} >\sim \pm \mu \quad (49)$$

at $\tilde{\phi} = \pi/\tilde{\beta}, 3\pi/\tilde{\beta}$.

As we already mentioned, the results obtained for a weak bond-alternating anisotropy can be applied to the opposite limiting case $|\delta| \gg J$, using the $\delta \leftrightarrow J$ symmetry (11), (10). In the region $-1 < \Delta' < 1/\sqrt{2}$, where $\Delta' = J_z/|\delta|$, we find two gapless SN phases (for each sign of $\delta$), with low-temperature properties of a Luttinger liquid. Applying nonlocal
transformations (10) to Eqs.(48) and (49), we find that at \( \Delta' = 1/\sqrt{2} \) a Kosterlitz-Thouless transition takes place to an ordered phase, characterized by the \( \uparrow\uparrow\downarrow\downarrow \) periodic long-range alignment of the spins, either along the \( x \)- or \( y \)-direction in spin space.

V. CONCLUSIONS

We have shown in this paper that, in the \( S = 1/2 \) quantum spin chains, the gapless, Luttinger spin-liquid state with a power-law decay of the correlation functions is not exhausted by systems possessing translational and spin rotation symmetries. Specifically, this type of low-energy behavior also characterizes the XXZ spin chain with the staggered XY anisotropy, exhibiting XY-like and spin-nematic phases with a gapless excitation spectrum. The model possesses a number of interesting properties. We have shown that massless Majorana fermions with different velocities, being elementary excitations in the noninteracting case of the XY bond-alternating chain, fail to be observable (confinement) in the Luttinger-liquid regime, when interaction caused by a finite \( zz \)-anisotropy \( \Delta \) is switched on. Another interesting feature of the model are order from disorder and Kosterlitz-Thouless transitions to massive ordered phases, driven by the \( \delta \)-anisotropy.

Although our description was restricted by limiting cases \( |\delta| \ll J \) and \( |\delta| \gg J \), it is clear that the gapless XY-like and SN phases occupy large domains in the parameter space of the model. However, to understand the phase diagram of the system in more detail and, in particular, describe a transition from the XY-like phase to the SN phases on changing \( \delta \) with \( \Delta \) kept finite, the region \( |\delta| \sim J \) should be considered. This region is not accessible by bosonization method. As follows from the two-chain representation (7) of the original model (2), in this region one has to consider an interacting system of “light” and “heavy” Majorana fermions with strongly different velocities \( v_+ \gg v_- \). This problem resembles the Kondo-lattice one and deserves a special analysis. We hope to return to this and related questions in the future.
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A particle-hole transformation $a_{n-q} \leftrightarrow a_{n-q}^\dagger$ maps this subspace onto the orthogonal one, with basis vectors $|10> \text{ and } |01>$. The transformed operators $\hat{J}_\alpha^\sigma$ generate one more SU(2) algebra. So, the total local symmetry group of $H_0$ is $SU(2) \otimes SU(2)$.

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