Factorization methods for noncommutative KP
and Toda hierarchy

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Abstract

We show that the solution space of the noncommutative KP hierarchy is the same
as that of the commutative KP hierarchy owing to the Birkhoff decomposition of
groups over the noncommutative algebra. The noncommutative Toda hierarchy is
introduced. We derive the bilinear identities for the Baker–Akhiezer functions and
calculate the $N$-soliton solutions of the noncommutative Toda hierarchy.

1 Noncommutative KP and Toda hierarchy

Recently, the classical and quantum field theories over the noncommutative (NC)
space-times have been extensively studied. The NC gauge theory has had a great success in
the string theory. In particular, the NC deformation of the ADHM construction of the
(anti)SDYM equation [9] and the Nahm construction [1] were shown. It seems to imply
the significance of the NC deformations of the integrable systems. Some authors have
discussed the NC integrable systems on lower dimension [2, 3, 6, 13, 17] (see also [5, 15, 16]),
especially the NC KP hierarchy. In this letter, we study the moduli space of the NC KP
hierarchy throughout the Birkhoff decomposition of a certain formal group over a NC
algebra. We also introduce the NC Toda hierarchy and derive the bilinear identities and
the $N$-soliton solutions.

Let $(\hat{t}_1, \hat{t}_2, \cdots)$ be coordinates of NC plane $\mathbb{R}^{2\infty}$ which satisfy $[\hat{t}_n, \hat{t}_m] = i\theta_{nm}$, and $\mathcal{A}_\theta$
a set of functions on it. The deformation parameters $\theta_{nm} \in \mathbb{R}$ are non-zero constants
and we assume that a matrix $(\theta_{nm})$ is invertible. By an orthogonal change of coordinates
as $[\hat{t}_{n-1}, \hat{t}_n] = i\theta_n$ ($n \geq 1$), the algebra is realized as operators over the Fock space
$\mathcal{F} = \oplus_{n_1, n_2, \cdots} \mathbb{C} |n_1 n_2 \cdots \rangle \langle n_1 n_2 \cdots |$ ($n_i = 0, 1, \cdots$).

The NC KP hierarchy of the operator form is defined as follows [2, 3, 17]. We consider
an operator valued monic pseudo differential operator\footnote{The variable $x$ needs not to be an operator on $\mathcal{F}$, since the Lax equations give $\frac{\partial \hat{w}_n}{\partial x} = \frac{\partial \hat{w}_n}{\partial t_n}$ for the Lax operator $L = \sum_{n \geq -1} \hat{u}_n \hat{x}^{-n}$.} (PDO)

$$\hat{W} = 1_{\mathcal{F}} + \sum_{n \geq 1} \hat{w}_n(x, \hat{t}) \hat{x}^{-n},$$

(1)

where $1_{\mathcal{F}}$ is an identity of $\mathcal{A}_\theta$. The coefficients of $\hat{W}$ are operators

$$\hat{w}_n(x, \hat{t}) = \sum_{l_i, m_i \geq 0} w_n^{(l, m)}(x) |l_1 l_2 \cdots \rangle \langle m_1 m_2 \cdots |,$$

(2)
where \( w^{(l,m)}_n \) are \( \mathbb{C} \)-valued functions of \( x \). The Sato–Wilson equations of the NC KP hierarchy are given by

\[
\frac{\partial}{\partial t_n} \hat{W} = -(\hat{W} \partial^2_x \hat{W}^{-1})_+ \hat{W} \quad (n \geq 1),
\]

where\(^2\) \( \frac{\partial}{\partial t_n} := -i \sum_m \theta^{-1}_{nm} [\hat{t}_m, \cdot] \). For a PDO \( P \), \((P)_+\) denotes a part of a differential operator of \( P \) and \((P)_-\) denotes \( P - (P)_+\).

Here we introduce the NC Toda hierarchy. We add coordinates \((\hat{s}_1, \hat{s}_2, \cdots)\) which satisfy \([\hat{s}_n, \hat{s}_m] = i \tilde{\theta}_{nm}, [\hat{s}_n, \hat{t}_m] = 0\) and \( A_{\hat{s}, \hat{t}} \) denotes a set of functions on it. This algebra is realized over a tensor product of two Fock spaces \( \mathcal{F} \otimes \mathcal{F} \). Let \( \hat{W}^{(+)} \) (resp. \( \hat{W}^{(-)} \)) be an upper (resp. lower) triangular \( \mathbb{Z} \times \mathbb{Z} \) matrix whose components are valued in \( A_{\hat{s}, \hat{t}} \). All diagonal components of the matrix \( \hat{W}^{(-)} \) are an identity \( 1_{\mathcal{F} \otimes \mathcal{F}} \) of \( A_{\hat{s}, \hat{t}} \). We define the Sato–Wilson equations of the NC Toda hierarchy as

\[
\begin{align*}
\frac{\partial \hat{W}^{(\pm)}_{-n}}{\partial t_n} &= \mp (\hat{W}^{(-)}_{-n} \hat{W}^{(-1)}_{-1})_{\pm} \hat{W}^{(\pm)}, \\
\frac{\partial \hat{W}^{(\pm)}_{-n}}{\partial s_n} &= \mp (\hat{W}^{(+)}_{-n} \hat{W}^{(-1)}_{-1})_{\pm} \hat{W}^{(\pm)},
\end{align*}
\]

for \( n \geq 1 \), where \((n, m)\) component of \( \Lambda \) is \( \delta_{n+1,m} \) and \( \frac{\partial}{\partial s_n} := -i \sum_m \tilde{\theta}^{-1}_{nm} [\hat{s}_m, \cdot] \). For a matrix \( P \), \((P)_+\) denotes the upper triangular part of \( P \) and \((P)_-\) denotes \( P - (P)_+\).

Note that these equations give the Lax equations and the Zakharov–Shabat (ZS) equations as usual \([14]\). We define \( \hat{b}_n, \hat{c}_n \in A_{\hat{s}, \hat{t}} \) as \( B := (\hat{W}^{(-)} \Lambda \hat{W}^{(-1)})_+ = \Lambda + \sum_{n \in \mathbb{Z}} \hat{b}_n E_{nn} \) and \( C := (\hat{W}^{(+)})^{-1} \Lambda^{-1} \hat{W}^{(+) - 1})_- = \sum_{n \in \mathbb{Z}} \hat{c}_n E_{nn} \) where \((k, l)\) component of \( E_{nn} \) is \( \delta_{km} \delta_{ln} \). The ZS equation \( \frac{\partial B}{\partial s_1} - \frac{\partial C}{\partial t_1} + [B, C] = 0 \) gives the NC Toda equations

\[
\frac{\partial \hat{b}_n}{\partial s_1} = -\hat{c}_{n+1} + \hat{c}_n, \quad \frac{\partial \hat{c}_n}{\partial t_1} = \hat{b}_n \hat{c}_n - \hat{c}_n \hat{b}_{n-1},
\]

for \( n \in \mathbb{Z} \). In the commutative case we can introduce functions \( \phi_n \) such that \( \hat{b}_n = -\frac{\partial}{\partial t_1} \phi_n \) and \( \hat{c}_n = e^{\phi_{n-1} - \phi_n} \). However, in the NC case we cannot define such functions.

# 2 A solution space of the NC KP hierarchy

First we consider the NC KP hierarchy. As in the commutative case, the Sato–Wilson equations \((3)\) are equivalent to the Birkhoff decomposition of a certain formal group \( G \). The integrable hierarchy throughout the factorization problems \( G = G_- G_+ \) of a group \( G \) over a NC algebra \( R \) was studied in \([8, 15]\), for example (see also \([7]\)). In our case \( G, G_\pm \) correspond to the formal groups \( \exp G, \exp G_\pm \) in \([15]\) over the NC plane \( \mathbb{R}^{2\infty} \) instead of \( \mathbb{R}^2 \) (see remark \(1)\).

Let \( \hat{W}_0 \in G_- \) be an ‘initial value’ of \( \hat{W} \) which is a monic PDO and satisfies \( \frac{\partial}{\partial t_n} \hat{W}_0 = 0 \) for \( n \geq 1 \). The factorization of \( e^\hat{t} \hat{W}_0^{-1} \)

\[
e^\hat{t} \hat{W}_0^{-1} = \hat{W}_0^{-1} \cdot \hat{W}^{(+)} \quad (\hat{W} \in G_-, \hat{W}^{(+) \in G_+}),
\]

\(^2\)This notation is different from \( \frac{\partial}{\partial t_n} = \hat{t}_n = -i \sum_m \theta^{-1}_{nm} \hat{t}_m \) which is often used in NC gauge theory.
where \( \hat{\xi} = \sum_{n \geq 1} \hat{t}_n \theta^n \) gives the Sato–Wilson equations (3) as in the commutative case [7, 8, 14]. Conversely, we assume that \( \hat{W} \) satisfies the Sato–Wilson equations, the factorization

\[
e^{-\hat{\xi}} \hat{W}^{-1} = \hat{M}^{(-)} \hat{M}^{(+)} \quad (\hat{M}^{(\pm)} \in G_{\pm}),
\]

gives equations

\[
-\hat{M}^{(+)} (\hat{W} \partial_x \hat{W}^{-1}) \hat{M}^{(+)} = -\frac{\partial \hat{M}^{(-)}}{\partial \hat{t}_n} \hat{M}^{(-)} + \frac{\partial \hat{M}^{(+)}}{\partial \hat{t}_n} \hat{M}^{(+)}^{-1},
\]

for \( n \geq 1 \). Taking a part of \((\cdot)_{-}\) of both sides, we obtain \( \frac{\partial}{\partial \hat{t}_n} \hat{M}^{(-)} = 0 \). Since \( e^{\hat{\xi}} \hat{M}^{(-)} = \hat{W}^{-1} \hat{M}^{(+)}^{-1} \) gives the factorization of \( e^{\hat{\xi}} \hat{M}^{(-)} \) by the uniqueness of the Birkhoff decomposition, we obtain equation (6).

In the noncommutative case, \( \{\hat{t}_n\} \) are operators and the fact \( \frac{\partial}{\partial \hat{t}_n} \hat{W}_0 = 0 \) has a quite different meaning from that of the commutative case. Since the derivative of \( \hat{t}_n \) is defined as \( \frac{\partial}{\partial \hat{t}_n} = -i \sum_m \theta^{1+}_{nm} [\hat{t}_m, \cdot] \), it means that \( \hat{W}_0 \) commutes with all \( \hat{t}_n \) for \( n \geq 1 \). Such an operator must be proportional to the identity of \( A_\theta \), and \( \hat{W}_0 \) must have a form\(^3\)

\[
\hat{W}_0 = 1_F \left( 1 + \sum_{n \geq 1} w_n^{(0)}(x) \partial_x^{-n} \right).
\]

Therefore \( \hat{W}_0 \) defines a point of the solution space of the commutative KP hierarchy (so-called the Sato Grassmanian [12] and so on). Since the operator \( \hat{W} \) corresponds to \( \hat{W}_0 \) uniquely through the action of operator \( e^{\hat{\xi}} \), we conclude that, in this sense, the solution space of NC KP hierarchy is the same as the commutative one. This fact is valid for the multicomponent case and their reductions. In general, for a NC hierarchy obtained by the Birkhoff decomposition, such as the NC Toda hierarchy, the solution space is same with the space of the initial values.

**Remark 1** In [8], they studied the group \( G = \hat{E}^\times \) which is a set of PDOs \( P \) whose coefficients are valued in \( R[[t]] \) with time parameters \( t = \{t_1, t_2, \cdots\} \) under the conditions \( P|_{t=0} \). In the case of the NC KP hierarchy, \( \{\hat{t}_n\} \) are elements of \( R = A_\theta \) and we cannot adopt their results to our case directly. Considering the Neumann series (see the proof of Theorem 3.2 [8]), we obtain the Birkhoff decomposition of \( G \) as formal series \( \mathbb{C}[[\hat{t}]] = \mathbb{C}[[\hat{t}_1, \hat{t}_2, \cdots]] \). The problem is whether components are well-defined under the re-arrangement of generators \( \{\hat{t}_n\} \). However, substituting \( \text{val}_t(\theta_{nm}) := n + m \) by the relations \( [\hat{t}_n, \hat{t}_m] = i\theta_{nm} \), we find that components are well-defined as a formal power series of \( A_\theta[\theta_{nm}] \) from theorem 3.2 [8]. Note that we need the assumption that the LHS of (7) is an element of \( \hat{E}^\times \) [8] with replacing operators \( \hat{t}_n \) to time variables \( t_n \) for its factorization.

**Remark 2** In the limit \( \theta_{nm} \to 0 \), the NC KP hierarchy is reduced to the commutative one and \( \hat{w}_n \) to

\[
w_n(t, x) = \lim_{\theta \to 0} \sum_{l, m \geq 0} w_n^{(l, m)}(x) f_{(l, m)}(t, x, \theta),
\]
with commutative times \( t = (t_1, t_2, \cdots) \), where \( f_{(l,m)} \) are functions obtained by the Wigner–Weyl transformation (see for e.g., [4]) of \( |l_1 l_2 \cdots \rangle \langle m_1 m_2 \cdots | \). The point of the solution space where \( w_n \) are defined is same as that of \( \hat{w}_n \).

**Remark 3** In [2], they derive the differential equations of the deformation parameters \( \theta_{nm} \), and remark that those equations allow us to construct the solutions of the NC KP hierarchy as a formal Taylor series of \( \theta_{nm} \) from the solutions of the commutative KP hierarchy.

### 3 Noncommutative Toda hierarchy

In this section, we define the Baker–Akhiezer (BA) functions for the NC Toda hierarchy, and show that they satisfy the bilinear identities. We also calculate the \( N \)-soliton solutions.

#### 3.1 Bilinear identities

Let \( \hat{w}_\pm (s), \hat{w}_\pm^*(s) \in \mathcal{A}_{\hat{t}, \hat{s}} \) be operators which are coefficients of \( \hat{W}^{(\pm)} \) and \( \hat{W}^{(\pm)-1} \)

\[
\hat{W}^{(\pm)} = \sum_{n \geq 0} \text{diag}(\hat{w}_{\pm n}(s)) \Lambda^{\pm n}, \quad \hat{W}^{(\pm)-1} = \sum_{n \geq 0} \Lambda^{\pm n} \text{diag}(\hat{w}_{\pm n}^*(s)) \tag{11}
\]

where \( \text{diag}(\hat{w}_{\pm n}(s)) = \text{diag}(\cdots, \hat{w}_{\pm n}(1), \hat{w}_{\pm n}(0), \cdots) \) and so on. We introduce the operator valued Baker–Akhiezer functions \( \hat{w}_\pm, \hat{w}^*_\pm \)

\[
\hat{w}_-(s, \lambda) = \left( \sum_{n \geq 0} \hat{w}_{-n}(s) \lambda^{-n+s} \right) e^{\hat{\xi}_-}, \quad \hat{w}_+(s, \lambda) = \left( \sum_{n \geq 0} \hat{w}_{+n}(s) \lambda^{-n+s} \right) e^{\hat{\xi}_+},
\]

\[
\hat{w}^*_-(s, \lambda) = e^{-\hat{\xi}_-} \left( \sum_{n \geq 0} \hat{w}^*_{-n}(s) \lambda^{-n-s} \right), \quad \hat{w}^*_+(s, \lambda) = e^{-\hat{\xi}_+} \left( \sum_{n \geq 0} \hat{w}^*_{+n}(s) \lambda^{-n-s} \right), \tag{12}
\]

where \( \hat{\xi}_- := e^{\sum_{n \geq 1} \hat{t}_n \lambda^n} \) and \( \hat{\xi}_+ := e^{\sum_{n \geq 1} \hat{s}_n \lambda^n} \). Considering the same arguments with the commutative case [14], we obtain the bilinear identities as follows. By the definition of \( \hat{w}_\pm, \hat{w}^*_\pm \) (11), the BA functions satisfy

\[
\oint \frac{d\lambda}{2\pi i} \hat{w}_-(s', \lambda) \hat{w}_+^*(s, \lambda) = \oint \frac{d\lambda}{2\pi i} \hat{w}_+(s', \lambda) \hat{w}_-^*(s, \lambda) (= 1_{F \otimes F} \delta_{s', s-1}). \tag{13}
\]

Since \( \hat{w}_\pm \) and \( \hat{w}_- \) satisfy the same linear differential equations of \( \hat{t}_n \) and \( \hat{s}_n \),

\[
\frac{\partial \hat{w}_\pm}{\partial \hat{t}_n} = (\hat{W}^{(\pm)} \Lambda^n \hat{W}^{(\pm)-1})_+ \hat{w}_\pm, \quad \frac{\partial \hat{w}_\pm}{\partial \hat{s}_n} = (\hat{W}^{(\pm)} \Lambda^{-n} \hat{W}^{(\pm)-1})_- \hat{w}_\pm, \tag{14}
\]

there exist an operator \( \hat{B}_{1J} \) such that \( \partial^j_\hat{t} \partial^j_\hat{s} \hat{w}_\pm = \hat{B}_{1J} \hat{w}_\pm \) where \( \partial^j_\hat{t} = \frac{\partial}{\partial \hat{t}_1} \cdots \frac{\partial}{\partial \hat{t}_i} \) and so on. We multiply \( \hat{B}_{1J} \) on both sides of (13) with the identification \( \Lambda \) with \( e^{\hat{J}} \) for \( \hat{w}_\pm(s', \lambda) \) [14], and obtain the bilinear identities

\[
\oint \frac{d\lambda}{2\pi i} \partial^j_\hat{t} \partial^j_\hat{s} \hat{w}_-(s', \lambda) \cdot \hat{w}_-^*(s, \lambda) = \oint \frac{d\lambda}{2\pi i} \partial^j_\hat{t} \partial^j_\hat{s} \hat{w}_+(s', \lambda) \cdot \hat{w}_+^*(s, \lambda), \tag{15}
\]
for any $I, J$. For $\partial^I_t \partial^J_s = \frac{\partial}{\partial t^n}$ (resp. $\partial^I_t \partial^J_s = \frac{\partial}{\partial s^n}$), equations (15) are equal to $(\hat{W}(-) e^{n \sum \hat{\theta} \hat{W}(+)^{-1}} + \delta_{s', s-1})$ (resp. $(\hat{W}(+) e^{-n \sum \hat{\theta} \hat{W}(+)^{-1}} + \delta_{s', s-1})$) and the bilinear identities are equivalent to the Sato–Wilson equations (4) as in the commutative case. Note that we cannot introduce the notion of the $\tau$-functions [12, 14] by the same reason with the NC KP hierarchy [2]. If we set $s = s'$, the RHS of (15) is zero and these identities correspond to those of the NC KP hierarchy [2]. This fact means that if we restrict the Fock space $F \otimes F$ to $F$ on which $\hat{t}$ acts, the BA functions of the NC Toda hierarchy for fixed $s$ are also those of the NC KP hierarchy.

### 3.2 $N$-soliton solutions

In this section we calculate $N$-soliton solutions for the NC Toda hierarchy considering the factorization problem. We put $\hat{V}(-) := \hat{W}(-) e^{-\sum \nabla_{\Lambda}^{-n}}$ and $\hat{V}(+) := \hat{W}(+) e^{-\sum \nabla_{\Lambda}^n}$. Then, the Sato–Wilson equations (4) are equivalent to the factorization [14]

$$
\hat{V}(-1) \cdot \hat{V}(+) = e^{\hat{\xi}} \hat{V}_0^{-1} e^{-\hat{\xi}},
$$

where $\hat{\xi} := \exp(\sum \nabla_{\Lambda}^n + \nabla_{\Lambda}^n)$. We choose the initial value for the $N$-soliton solutions (cf (9))

$$
\hat{V}_0^{-1} = 1_{F \otimes F} \left( I + \epsilon \sum_{i=1}^N a_i \sum_{m,n \in \mathbb{Z}} p_i^m q_i^{-n} E_{mn} \right),
$$

where $a_i, p_i, q_i$ are positive constant parameters such that $q_N < \cdots < q_1 < p_1 < \cdots < p_N$, $\epsilon$ is a formal parameter, and $I := \sum_{n \in \mathbb{Z}} E_{nn}$. We put $\hat{V}(-) = 1_{F \otimes F} I + \hat{Z}$, $\hat{Z} = (\hat{z}_{ij})$ where $\hat{z}_{nm} \in A_{\hat{\theta}, \hat{\theta}}$ is zero for $n \leq m$. Considering equation (16) we have

$$
(\cdots, \hat{z}_{s, s-2}, \hat{z}_{s, s-1}) \left( 1_{F \otimes F} I + \epsilon \sum_{i=1}^N a_i e^{\hat{\eta}(p_i)} p_i q_i^{-\hat{\eta}(q_i)} \right) = -\epsilon \sum_{j=1}^N a_j p_j^s e^{\hat{\eta}(p_j)} q_j e^{-\hat{\eta}(q_j)},
$$

where $p_i := \hat{t}(\cdots, p_i^{s-2}, p_i^{s-1})$, $q_i := (\cdots, q_i^{-s+2}, q_i^{-s+1})$ and $\hat{\eta}(p) := \sum_{n \geq 1} (\hat{t}_n p^n + \hat{s}_n p^{-n})$. In the commutative case, the Cramer formula solves these equations and solutions are expressed by the $\tau$-function. In the NC case we cannot adopt such a method. We solve this equation as a formal series\(^4\) of $\epsilon$,

$$
(\cdots, \hat{z}_{s, s-2}, \hat{z}_{s, s-1}) = -\epsilon \sum_{j=1}^N a_j p_j^s e^{\hat{\eta}(p_j)} q_j e^{-\hat{\eta}(q_j)} \sum_{k \geq 0} \left( -\epsilon \sum_{i=1}^N a_i e^{\hat{\eta}(p_i)} p_i q_i^{-\hat{\eta}(q_i)} \right)^k
$$

$$
= \sum_{k \geq 0} (-\epsilon)^{k+1} \sum_{1 \leq j, i_1, \cdots, i_k \leq N} \frac{p_j^s \phi_j \phi_{i_1} \cdots \phi_{i_k}}{p_i - q_j} \frac{q_{i_1}^{-s+1} p_{i_1} q_{i_2}^{-s+1} p_{i_2}^{s+1} \cdots q_{i_k}^{-s+1} p_{i_k}^{s+1}}{p_k - q_{i_k-1}} q_{i_k},
$$

\(^4\)Since it reduces to the soliton solutions of the NC KP hierarchy and depends only on $\hat{t}_n p^n + \hat{s}_n p^{-n}$ ($p \in \{p_i, q_i\}$) for $\hat{t}_n, \hat{s}_n$, we find that this series is well-defined as an element of $A_{\hat{\theta}, \hat{\theta}}[[\theta_{nm}, \theta_{nm}]]$ without $\epsilon$. 
where \( \hat{\phi}_i := a_i e^{\hat{\eta}(p_i)} e^{-\hat{\eta}(q_i)} \). Here we used \( qp = \frac{q^{s+1}p^s}{p-q} \). Thus we obtain the \( N \)-soliton solution of the NC Toda hierarchy,

\[
\hat{z}_{nm} = -\sum_{k \geq 0} \epsilon^{k+1} \sum_{1 \leq i_0, i_1, \ldots, i_k \leq N} q_{ik}^{-m+n-1} \frac{(p_{i_0} \cdots p_{i_k})^n}{(q_{i_0} \cdots q_{i_k})^{n-1}} \times \frac{\hat{\phi}_{i_0} \hat{\phi}_{i_1} \cdots \hat{\phi}_{i_k}}{(q_{i_0} - p_{i_1})(q_{i_1} - p_{i_2}) \cdots (q_{i_{k-1}} - p_{i_k})},
\]

for \( n > m \). As noted in previous section, if we restrict the Fock space \( \mathcal{F} \otimes \mathcal{F} \) to \( \mathcal{F} \) on which \( \hat{t}_n \) act, the BA functions of the NC Toda hierarchy are reduced to those of the NC KP hierarchy. In fact, the solution with a change of parameters \( a_i \rightarrow a_i/p_i \)

\[
\hat{z}_{10} = \sum_{k \geq 0} \epsilon^{k+1} \sum_{1 \leq i_0, \ldots, i_k \leq N} \frac{\hat{\phi}_{i_0} \cdots \hat{\phi}_{i_k}}{(q_{i_0} - p_{i_1})(q_{i_1} - p_{i_2}) \cdots (q_{i_{k-1}} - p_{i_k})},
\]

(21)

 corresponds the \( N \)-soliton solutions of the NC KP hierarchy obtained in [2, 11] using the trace method [10].

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