A generalized figure of merit for qubit readout

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Many promising approaches to fault-tolerant quantum computation require repeated quantum nondemolition (QND) readout of binary observables such as quantum bits (qubits). A commonly used figure of merit for readout performance is the error rate for binary assignment in a single repetition. However, it is known that this figure of merit is insufficient. Indeed, real-world readout outcomes are typically analog instead of binary. Binary assignment therefore discards important information on the level of confidence in the analog outcomes. Here, a generalized figure of merit that fully captures the information contained in the analog readout outcomes is proposed. This figure of merit is the Chernoff information associated with the statistics of the analog readout outcomes in one repetition. Unlike the single-repetition error rate, the Chernoff information uniquely determines the cumulative error rate for arbitrary readout noise. Importantly, this universal description persists for the small number of repetitions and non-QND imperfections relevant to real experiments. It follows that arbitrary non-Gaussian readout noise common in experiments can be replaced by effective Gaussian noise with the same Chernoff information. This correspondence leads to a simple universal expression to estimate the cumulative error rate of a QND readout. In addition, the Chernoff information is used to rigorously quantify the amount of information discarded by analog-to-binary conversion. These results provide a unified framework for qubit readout and should facilitate optimization and engineering of near-term quantum devices across all platforms.

I. INTRODUCTION

The ability to readout binary quantum observables such as quantum bits (qubits) is an important desideratum for quantum information processing [1]. In particular, it is often highly desirable that the readout have high fidelity and be quantum nondemolition (QND). For instance, many promising fault-tolerant architectures for scalable fault-tolerant quantum computation require that stabilizer parities be repeatedly read out during the computation [2–13]. For fault tolerance to be achieved in these architectures, it is crucial that the readout fidelity be above the threshold of the error-correcting code [14, 15]. Moreover, it is necessary that the readout be QND so that the code is projected onto the eigenstate corresponding to the observed stabilizer values. QND readouts have the important advantage that repeated readouts leave the system state unchanged. Therefore, each repetition provides additional information on the system state. As a result, the readout fidelity increases exponentially with the number of repetitions. This property has been exploited to improve the readout fidelity of quantum bits (qubits) for a variety of implementations including trapped ion qubits [16, 17], solid-state spin qubits [18–31], and superconducting qubits [32]. The same temporal correlations in the outcomes of consecutive QND readouts can be used to correct stabilizer readout errors in quantum error-correcting codes [33–35].

A seemingly natural figure of merit for the performance of repetitive QND readout is the probability ϵ of a readout error occurring in a single repetition. Here, each repetition is assigned a binary outcome, with ϵ being the probability of an incorrect assignment. The readout errors are then corrected by performing a majority vote on the binary outcomes. The cumulative readout error rate eN after N repetitions is simply proportional to the probability that an error has occurred in more than half of the repetitions, eN ∝ ϵN/2. Therefore, it appears that the cumulative readout error rate is fully determined by the single-repetition error rate ϵ. However, a typical real-world readout does not only have two outcomes. Rather, the readout outcomes are commonly analog (see Fig. 1) and need not even be scalar. For instance, the single-repetition readout outcome could be a continuous electrical voltage or current [36–57], a non-binary photon count at a photodetector [21, 58–63], or a collection of such outcomes. If each individual repetition is assigned a binary outcome, information on the level of confidence in each analog readout outcome is discarded. Such analog-to-binary conversion is known as “hard decoding”. It was shown that taking into account the additional information contained in the distribution of analog readout outcomes, or “soft decoding”, can significantly reduce eN compared to hard decoding [31, 64–67]. It follows that two repetitive QND readouts characterized by the same value of ϵ can yield different values of eN. This suggests that ϵ is not a universal descriptor of readout performance [68]. Moreover, it was shown that the existence of a soft-decoding advantage is highly dependent on the details of the often highly non-Gaussian distributions of analog readout outcomes. Heuristic arguments have been put forward to predict when an advantage exists in common cases [31, 64], but a unified and economical description that fully captures the performance of repetitive QND readout for all outcome distributions is highly desirable.

The present work introduces a figure of merit that fully
captures the cumulative error rate \( \epsilon_N \) of the repetitive QND readout of binary observables with an arbitrary distribution of analog readout outcomes. That figure of merit is the asymptotic rate of decrease of \( \ln \epsilon_N \) with the number of repetitions \( N \). In the classical theory of hypothesis testing, this quantity is known as the Chernoff information for the discrimination of two probability distributions [69, 70]. Like the single-repetition readout error rate \( \epsilon \), the Chernoff information can be obtained solely from the statistics of readout outcomes in a single repetition. In fact, it is shown that it is closely related to \( \epsilon \) when the readout outcomes are binary. Unlike \( \epsilon \), however, the Chernoff information does not discard information associated with the level of confidence in each analog readout outcome. Therefore, the Chernoff information enables a universal description of repetitive QND readout, in the sense that all outcome distributions with the same Chernoff information have the same cumulative readout fidelity. Moreover, this universality persists in the non-asymptotic regime and in the presence of non-QND imperfections. Therefore, theoretical analysis of repetitive QND readout may be restricted to Gaussian noise without loss of generality. This leads to a simple and universal expression for the cumulative error rate of a QND readout that remains accurate for small \( N \). Finally, the Chernoff information is used to predict the soft decoding advantage in cases of practical importance without having to resort to time-consuming simulations [30, 31, 64, 65, 67]. The present work paves the way for a generalized understanding of the readout of quantum observables and should facilitate the engineering of high-fidelity QND readout in near-term quantum devices on all platforms.

II. REPETITIVE QUANTUM NONDEMOLITION READOUT

A. Quantum nondemolition readout

Consider a quantum observable \( A \) with only two distinct eigenvalues \( a = +1 \) and \( a = -1 \). These eigenvalues correspond to two orthogonal eigenspaces with projectors \( \Pi_+ \) and \( \Pi_- \), respectively. The observable \( A \) could be, e.g., the Pauli observable \( Z \) of a qubit or a parity-check observable in an error-correcting code. If the system is prepared in an eigenstate of \( \Pi_\pm \), an ideal QND readout of \( A \) yields the outcome \( a = \pm 1 \) with certainty. Moreover, the post-readout state remains identical to the prepared eigenstate. However, a real-world QND readout is subject to noise that introduces uncertainty in the value of \( a \). In general, it is therefore not possible to know with certainty in which eigenspace the system was prepared after a single readout. Fortunately, the QND property guarantees that every subsequent readout yields the same outcome as in the first readout. Thus, repeated readouts allow to “average out” the noise and to measure the observable \( A \) to arbitrary accuracy.

B. Single repetition

First consider a single-repetition of the QND readout. In general, the noisy readout yields an outcome \( \mathcal{O} \) that depends on the eigenvalue \( a \). More precisely, the statistics of the readout outcomes are described by the probability distribution \( P_\pm(\mathcal{O}) \) for observing \( \mathcal{O} \) if \( a = \pm 1 \). Here, the distributions \( P_\pm(\mathcal{O}) \) can take any form. For instance, the outcome \( \mathcal{O} \) could be a discrete random variable, a continuous random variable, or a multidimensional set of random variables. Example distributions \( P_\pm(\mathcal{O}) \) are shown in Fig. 1.

The most commonly used figure of merit for readout performance in a single-repetition is the error rate \( \epsilon \), defined as the average probability of assigning the observed outcome to the incorrect eigenvalue. The value of \( \epsilon \) depends on the rule chosen to assign an eigenvalue \( a \) to each outcome \( \mathcal{O} \). The assignment rule that minimizes \( \epsilon \) is obtained by comparing the posterior probabilities \( P_\mathcal{O}(\pm) \) of \( a = \pm 1 \) given the observed outcome \( \mathcal{O} \) [71]. The ratio of these two probabilities is

\[
P_\mathcal{O}(+) = \frac{P_+(\mathcal{O}) P(+) + P_-(\mathcal{O}) P(-)}{P_+(\mathcal{O}) P(+) + P_-(\mathcal{O}) P(-)},
\]

Here, \( P(\pm) \) are the \textit{a priori} probabilities for the eigenvalues \( a = \pm 1 \). When the ratio in Eq. (1) is larger (smaller) than 1, the outcome \( a = +1 \) (\( a = -1 \)) is assigned. In the following, it is assumed that the two eigenvalues are equally likely \textit{a priori}, \( P(+) = P(-) \). This leads to a definition of \( \epsilon \) that is agnostic about the value of \( a \). Moreover, this case is common and desirable because it maximizes the information extracted by readout. Setting \( P(+) = P(-) \) and introducing the log-likelihood ratio

\[
\lambda(\mathcal{O}) = \ln \frac{P_+(\mathcal{O})}{P_-(\mathcal{O})},
\]
it follows from Eq. (1) that $a = +1$ [$a = -1$] is assigned when $\lambda(\mathcal{O}) > 0$ [$\lambda(\mathcal{O}) < 0$]. If $\lambda(\mathcal{O}) = 0$, the eigenvalue is assigned at random. The log-likelihood ratio, Eq. (2), is central to hypothesis testing. It should be interpreted as the observer’s level of confidence in the assignment given the observed outcome $\mathcal{O}$. The log-likelihood ratio is depicted in Fig. 1 along with the distributions $P_{\pm}(\mathcal{O})$.

It is instructive to introduce the conditioned single-repetition error rates $\epsilon_{\pm}$, defined as the probabilities of incorrectly assigning the eigenvalue conditioned on a state with eigenvalue $a = \pm 1$ being prepared. The average single-repetition error rate is then the average over the two possible eigenvalues, $\epsilon = (\epsilon_+ + \epsilon_-)/2$. Under the above assignment rule, the conditioned single-repetition error rates are simply

$$
\epsilon_+ = \int_{-\infty}^{0} d\lambda P_+(\lambda), \quad \epsilon_- = \int_{0}^{\infty} d\lambda P_-(\lambda).
$$

Here, $P_\pm(\lambda)$ are the probability distributions for $\lambda$ conditioned on $a = \pm 1$. In terms of the distributions $P_\pm(\mathcal{O})$, $\epsilon_+ [\epsilon_-]$ is the area under the distribution $P_+(\mathcal{O}) [P_-(\mathcal{O})]$ in the regions where $\lambda(\mathcal{O}) < 0$ [$\lambda(\mathcal{O}) > 0$]. This is represented by the shaded areas in Fig. 1. Because $\epsilon_+$ and $\epsilon_-$ are respectively integrals of $P_+(\mathcal{O})$ and $P_-(\mathcal{O})$ only, the error rate $\epsilon$ cannot contain information on the relative value of $P_+(\mathcal{O})$ and $P_-(\mathcal{O})$. Therefore, important information contained in the functional form of log-likelihood ratio $\lambda(\mathcal{O})$ is discarded.

### C. Multiple repetitions

That the single-repetition error rate discards information is most readily seen by redoing the above analysis for repeated QND readout of the binary observable $A$. Repeated readout yields a string of outcomes $\mathcal{O}_N = \{\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{N-1}\}$. Due to the QND nature of the readout, all outcomes are independently sampled from the same distribution $P_\pm(\mathcal{O})$ when the observable is prepared in the eigenspace of $\Pi_\pm$. Accordingly, the joint distribution of the readout outcomes conditioned on the eigenvalue $a = \pm 1$ is $P_\pm(\mathcal{O}_N) = \prod_{k=0}^{N-1} P_\pm(\mathcal{O}_k)$. The cumulative log-likelihood ratio for the entire string of outcomes $\mathcal{O}_N$ is thus

$$
l_N = \ln \frac{P_+(\mathcal{O}_N)}{P_-(\mathcal{O}_N)} = \sum_{k=0}^{N-1} \lambda(\mathcal{O}_k).
$$

As before, the eigenvalue $a = +1$ [$a = -1$] is assigned when $l_N > 0$ [$l_N < 0$]. Equation (4) shows that in the general case, each outcome must be weighed by $\lambda(\mathcal{O}_k)$ in order to perform optimal assignment. Therefore, discarding information contained in $\lambda(\mathcal{O})$ in each repetition is necessarily suboptimal.

Analogously to Eq. (3), the cumulative conditioned error rates $\epsilon_{\pm,N}$, defined as the probability of incorrectly identifying the eigenvalue $a$ given the string $\mathcal{O}_N$, are introduced:

$$
\epsilon_{+,N} = \int_{-\infty}^{0} dl_N P_+(l_N), \quad \epsilon_{-,N} = \int_{0}^{\infty} dl_N P_-(l_N).
$$

Here, $P_\pm(l_N)$ are the probability distributions for $l_N$ conditioned on $a = \pm 1$. The corresponding average cumulative error rate is then $\epsilon_N = (\epsilon_{+,N} + \epsilon_{-,N})/2$. Because the repetitions are independent, $\epsilon_N$ is expected to decrease exponentially as $N$ grows, $\epsilon_N \sim \exp(-CN)$ for some constant $C$. It will be argued that the constant $C$, which will be shown to equal the Chernoff information, is the appropriate figure of merit for repetitive QND readout.

### III. A GENERALIZED FIGURE OF MERIT: THE CHERNOFF INFORMATION

#### A. Large deviation theory

The cumulative log-likelihood ratio $l_N$, Eq. (4), is the sum of the independent and identically distributed (i.i.d.) variables $\lambda(\mathcal{O}_k)$. According to the central limit theorem, the distributions $P_{\pm}(l_N)$ therefore asymptotically converge to Gaussians. Thus, one might hope to evaluate the cumulative error rates, Eq. (5), using the cumulative distribution function (CDF) of a Gaussian distribution. As was first noted by Cramér, however, this approach produces wildly inaccurate results [73]. Indeed, according to the Berry-Esseen theorem [74, 75], the CDF of $P_{\pm}(l_N)$ converges only polynomially to a Gaussian as $N$ increases. Meanwhile, the error rates $\epsilon_{\pm,N}$ decrease exponentially with $N$. Therefore, the relative accuracy in $\epsilon_{\pm,N}$ explodes as $N \to \infty$ and the central-limit theorem fails. This problem is solved by the theory of large deviations developed by Cramér [73] and Sanov [76], and applied to hypothesis testing by Chernoff and Hoeffding [69, 70]. The theory is summarized in Ref. [77]. The result is that

$$
\ln \epsilon_N \sim -CN \quad \text{as} \quad N \to \infty,
$$

where

$$
C = -\inf_{t \in [0,1]} \ln \left[ \int d\mathcal{O} P_+(\mathcal{O})^{1-t} P_-(\mathcal{O})^t \right].
$$

Here, “$\sim$” denotes asymptotic equality. The quantity $C$ is known as the Chernoff information [78]. It is a symmetric distance measure between the distributions $P_+(\mathcal{O})$ and $P_-(\mathcal{O})$ and can be interpreted as a rate of information gain per repetition. Like the single-repetition readout error rate $\epsilon$, the Chernoff information depends only on the statistics of readout outcomes in a single repetition. Unlike $\epsilon$, however, it depends on both $P_+(\mathcal{O})$ and $P_-(\mathcal{O})$. Consequently, the Chernoff information encodes information contained in the relative level of confidence
$\lambda(O)$ in each readout outcome. As a result, readout outcome distributions $P_{\pm}(O)$ with the same single-repetition error rates $\epsilon_{\pm}$ do not necessarily have the same Chernoff information.

**B. Universality**

The power of using the Chernoff information as a figure of merit for readout of binary observables is that all readout outcome distributions $P_{\pm}(O)$ with the same Chernoff information, no matter their shape, give the same asymptotic behavior for $\ln \epsilon_N$ as $N \to \infty$. Here, it is argued that such universal behavior holds even in the non-asymptotic regime $N \gtrsim 1$. As discussed in Sec. IV B, the Chernoff information for Gaussian noise with signal-to-noise ratio $r$ is simply given by $C = r/2$. This suggests an interpretation of the Chernoff information as an effective Gaussian signal-to-noise ratio. More precisely, it suggests that as far as the cumulative error rate $e_N$ is concerned, non-Gaussian noise may be replaced by an effective Gaussian noise with signal-to-noise ratio $2C$. In the case of Gaussian noise, however, the central-limit theorem for $l_N$ holds exactly for all $N$. Consequently, an exact expression for $e_N$ can be obtained for all $N$, namely, $e_N = \text{erfc} \left( \sqrt{rN/2} \right)/2$ [79]. This naturally leads to the conjecture that the same relationship holds for arbitrary noise by setting $r = 2C$:

$$e_N = \frac{1}{2} \text{erfc} \left( \sqrt{CN} \right). \quad (8)$$

It can be shown from simple counter-examples at $N = 1$ that the ansatz of Eq. (8) is not exact for finite $N$. Nevertheless, its approximate validity was assessed by performing numerical Monte Carlo simulations [77] for a variety of non-Gaussian outcome distributions $P_{\pm}(O)$. The results are shown in Fig. 2. It is found that Eq. (8) captures $\ln \epsilon_N$ extremely well for all $N \gtrsim 1$ and for all the considered noise models. These include Gaussian noise ubiquitous in quantum readouts relying on electronic and homodyne detection, Poissonian noise ubiquitous in quantum readouts relying on fluorescence detection, Cauchy noise with fat polynomial tails, and the heavily bimodal and non-Gaussian readout noise observed empirically in Ref. [31]. The inference from these results is that Eq. (8) can be used to accurately estimate the cumulative error rate $e_N$ for arbitrary readout noise and finite $N$, obviating the need for time-consuming simulations [30, 31, 64, 65, 67] that are specialized to the noise model. This universal non-asymptotic behavior should greatly facilitate readout engineering by reducing the analysis of the great variety of noise models discussed in the literature to the calculation of the Chernoff information.

In practice, the error rate of a QND readout is typically limited by non-QND imperfections. Most commonly, the observable $A$ eventually relaxes with small probability in each repetition due to coupling to the environment. It is important to verify that universality persists in this more realistic scenario. Additional Monte-Carlo simulations are performed in Ref. [77] to show that this is indeed the case.

**IV. HARD AND SOFT DECODING**

**A. Soft decoding advantage**

The Chernoff information can also be used to quantify the information lost by converting analog readout outcomes to binary values. To do this, the Chernoff information $C$ for analog outcomes is compared to the Chernoff information $C_b$ for the corresponding binarized outcomes. This is reflected in the soft decoding advantage

$$\rho = \frac{C}{C_b}. \quad (9)$$

If $\rho = 1$, no information is discarded by binarizing readout outcomes. If $\rho > 1$, however, significant amount of information has been lost. Inspection of Eq. (6) shows that binarizing readout outcomes reduces the order of magnitude of $\epsilon_N$ by a factor $\rho$. Equivalently, the number of readouts required to achieve a desired value of $\epsilon_N$ is $\rho$ times larger with hard decoding than with soft decoding. Due to the persistence of universality at small $N$ discussed in Sec. III B, the asymptotic soft decoding advantage is expected to also persist in the non-asymptotic limit. Indeed, it was verified that Eq. (9) accurately predicts the soft decoding advantage observed for small $N$ in Ref. [31]. Note that there exists a general analytical expression for $C_b$ [77]. In the important limit $\epsilon_{\pm} \to 0$, it takes the form

$$C_b \sim \left[ \frac{1}{\ln(\epsilon_+^{-1})} + \frac{1}{\ln(\epsilon_-^{-1})} \right]^{-1}. \quad (10)$$

Similar to the average single-repetition error rate, $\epsilon = (\epsilon_+ + \epsilon_-)/2$, $C_b$ is a monotonic function of both $\epsilon_+$ and $\epsilon_-$. This makes $C_b$ an appropriate substitute for $\epsilon$ to quantify the performance of a single repetition.

To illustrate the usefulness of the Chernoff information in characterizing readout, the soft decoding advantage, Eq. (9), is now calculated for two examples of interest.

**B. Example 1: Gaussian distributed readout outcomes**

Assume that the readout of the eigenvalues $a = \pm 1$ is subject to additive Gaussian noise, such that the distributions of analog readout outcomes are $P_{\pm}(O) = \sqrt{r/2\pi} \exp \left[ -r (O \mp 1)^2 / 2 \right]$. Here, $r$ is the (power) signal-to-noise ratio. Gaussian noise is ubiquitous in real experiments. For instance, electronic noise in the readout
of semiconductor spin qubits [37, 39, 42, 44, 45, 47, 49–52, 56] as well as quantum noise in the readout of superconducting qubits [41, 43, 54, 57] are well modelled by additive Gaussian noise. An application of Eq. (7) gives $C = r/2$ for all $r$. The single-repetition error rates corresponding to these distributions are $\epsilon_\pm = \text{erfc}(\sqrt{r/2})/2$. Using this expression, it is possible to show that $C_b \approx r/\pi$ for $r \ll 1$ and $C_b \approx r/4$ for $r \gg 1$ [77]. Thus, the soft decoding advantage varies smoothly from

$$\rho = \frac{\pi}{2} \text{ for } r \ll 1 \text{ to } \rho = 2 \text{ for } r \gg 1 \ . \quad (11)$$

Therefore, hard decoding of Gaussian distributions leads to loss of information for all signal-to-noise ratios. In particular, the number of repetitions required to reach a desired error rate is always at least $\pi/2 \approx 1.57$ times larger if the analog outcomes are binarized. The result for $r \gg 1$ is consistent with the analysis of Ref. [64] and with known results from the classical theory of soft-decision decoding [80, 81].

C. Example 2: Gaussian distributed readout outcomes with conversion errors

In the presence of Gaussian readout noise, the eigenvalues $a = \pm 1$ are ideally each converted to Gaussian distributions with means $\pm 1$. In practice, however, imperfections in the readout scheme may lead to conversion errors. As a result, the distributions $P_\pm(\mathcal{O})$ often resemble mixtures of Gaussian distributions [31, 37, 39, 41, 45, 49–51]. Such imperfections can be modelled with the distributions $P_\pm(\mathcal{O}) = (1-\eta) \sqrt{r/2\pi} \exp[-r(\mathcal{O} \mp 1)^2/2] + \eta \sqrt{r/2\pi} \exp[-r(\mathcal{O} \pm 1)^2/2]$. Here, $\eta$ is the rate of conversion errors. Expressions for $C$ and $C_b$ for these distributions are given in Ref. [77]. The resulting soft decoding advantage $\rho$ is shown in Fig. 3 as a function of $\eta$ and of the rate of errors due to Gaussian noise, $\epsilon_G \equiv \text{erfc}(\sqrt{r/2})/2$. There is a clear transition from a region of parameter space where $\rho = 1$ when conversion...
errors dominate to a region where $\rho > 1$ when Gaussian errors dominate. This agrees with the heuristic conclusions of Refs. [31, 64]. Note, however, that while previous work had to resort to time-consuming Monte Carlo simulations to quantify soft decoding advantage of non-Gaussian distributions [30, 31, 64, 65, 67], the present approach enables accurate prediction of $\rho$ by computing a single integral. This makes it much easier to explore the parameter space to engineer and optimize readout.

V. CONCLUSION

In conclusion, a generalized figure of merit for the repetitive QND readout of a binary quantum observable $A$ was introduced. This figure of merit is the Chernoff information associated with the analog distributions of readout outcomes for each eigenvalue of $A$ [see Eq. (7)]. When the readout outcomes are binary, the Chernoff information is closely related to the commonly used single-repetition error rate. Contrary to the single-repetition error rate, however, the Chernoff information is a universal figure of merit: all noise models with the same Chernoff information yield the same functional form for the cumulative error rate. It was shown that this persists to a very good approximation for small number of repetitions and non-QND imperfections relevant to real-world experiments. It follows that arbitrary readout noise can be replaced by effective Gaussian noise without loss of generality. Finally, the Chernoff information was used to quantify the amount of information discarded by binarizing readout outcomes in each repetition, and simple results were derived analytically for experimentally relevant readout models. The results presented here provide a unified description of repetitive QND readout and should greatly facilitate the standardization, optimization, and engineering of quantum readout across all experimental platforms.

There are several possible avenues for future research. Firstly, it would be interesting to generalize the figure of merit presented here for noise that is non-stationary or correlated between repetitions, two features that are likely to appear in real experiments. Moreover, a rigorous justification of the universal behavior described by Eq. (8) for finite repetition number $N$ is highly desirable. Such a justification could potentially be obtained through the relationship between probability theory and the renormalization group [82]. In this approach, the readout outcomes $O_k$ are thought of as classical degrees of freedom on an $N$-site lattice. Finite size corrections could then be obtained by analyzing the renormalization group flow around the fixed point at $N = \infty$. Such an approach could also potentially provide generalizations of Eq. (8) in the presence of non-QND transitions by interpreting transition probabilities as weak interaction parameters between the degrees of freedom $O_k$.

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[1] D. P. DiVincenzo, in Scalable Quantum Computers: Paving the Way to Realization, edited by S. L. Braunstein, H.-K. Lo, and P. Kok (Wiley-VCH, Berlin, Germany, 2001) Chap. 1, pp. 1–13.
[2] A. G. Fowler, A. M. Stephens, and P. Groszkowski, Phys. Rev. A 80, 052312 (2009).
[3] L. Sun, A. Petrenko, Z. Lehtas, B. Vlastakis, G. Kirchmair, K. Sliwa, A. Narla, M. Hatridge, S. Shankar, J. Blumoff, L. Frunzio, M. Mirrahimi, H. M. Devoret, and R. J. Schoelkopf, Nature 511, 444 (2014).
[4] J. Kelly, R. Barends, A. Fowler, A. Megrant, E. Jeffrey, T. White, D. Sank, J. Mutus, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, I.-C. Hoi, C. Neill, O. P. J. J. C. Quintana, P. Roushan, A. Vainsencher, J. Wenner, A. N. Cleland, and J. M. Martinis, Nature 519, 66 (2015).
[5] J. Cramer, N. Kalb, M. A. Rol, B. Hensen, M. S. Blok, M. Markham, D. J. Twitchen, R. Hanson, and T. H. Taminiau, Nat. Commun. 7, 11526 (2016).
[6] N. Ofek, A. Petrenko, R. Heeres, P. Reinhold, Z. Lehtas, B. Vlastakis, Y. Liu, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, H. M. Devoret, and R. J. Schoelkopf, Nature 536, 441 (2016).
[7] S. Rosenblum, P. Reinhold, M. Mirrahimi, L. Jiang, L. Frunzio, and R. J. Schoelkopf, Science 361, 266 (2018).
[8] V. Negnevitsky, M. Marinelli, K. K. Mehta, H.-Y. Lo, C. Flügmann, and J. P. Home, Nature 563, 527 (2018).
[9] L. Hu, Y. Ma, W. Cai, X. Mu, Y. Xu, W. Wang, Y. Wu, H. Wang, Y. P. Song, and C.-L. Zou, Nat. Phys. 15, 503 (2019).
[10] D. Rist, L. C. G. Govia, B. Donovan, S. D. Fullek, W. D. Kalfus, M. Brink, N. T. Bronn, and T. A. Ohki, 1911.12280.
[11] C. K. Andersen, A. Remm, S. Lazar, S. Krinner, J. Heinsso, J.-C. Besse, M. Gabureac, A. Wallraff, and C. Eichler, npj Quantum Inf. 5, 69 (2019).
[12] C. K. Andersen, A. Remm, S. Lazar, S. Krinner, N. Lacroix, G. J. Norris, M. Gabureac, C. Eichler, and A. Wallraff, Nat. Phys. 16, 875 (2020).
[13] C. C. Bultink, T. E. O’Brien, R. Vollmer, N. Muthusubramanian, M. W. Beekman, M. A. Rol, X. Fu, B. Tarasiński, V. Ostroukh, B. Varbanov, A. Bruno, and L. DiCarlo, Sci. Adv. 6, eaay3050 (2020).
[14] E. Knill, R. Laflamme, and W. Zurek, arXiv:quant-ph/9610011 (1996), quant-ph/9610011.
[15] D. Aharonov, Phys. Rev. A 62, 062311 (2000).
[16] T. Schaetz, M. D. Barrett, D. Leibfried, J. Britton,
J. Chiaverini, W. M. Itano, J. D. Jost, E. Knill, C. Langer, and D. J. Wineland, Phys. Rev. Lett. 94, 010501 (2005).

[17] D. B. Hume, T. Rosenband, and D. J. Wineland, Phys. Rev. Lett. 99, 120502 (2007).

[18] T. Mennier, I. T. Vink, L. H. W. van Beveren, F. H. L. Koppens, H. P. Tranitz, W. Wegscheider, L. P. Kouwenhoven, and L. M. K. Vandersypen, Phys. Rev. B 74, 195303 (2006).

[19] L. Jiang, J. S. Hodges, J. R. Maze, P. Maurer, J. M. Taylor, D. G. Cory, P. R. Hemmer, R. L. Walsworth, A. Yacoby, A. S. Zibrov, and M. D. Lukin, Science 326, 267 (2009).

[20] P. Neumann, J. Beck, M. Steiner, F. Rempp, H. Federer, P. R. Hemmer, J. Wrachtrup, and F. Jelezko, Science 329, 542 (2010).

[21] L. Robledo, L. Childress, H. Bernien, B. Hensen, P. F. Alkemade, and R. Hanson, Nature 477, 574 (2011).

[22] G. Waldherr, J. Beck, M. Steiner, P. Neumann, A. Gali, T. Frauenheim, F. Jelezko, and J. Wrachtrup, Phys. Rev. Lett. 106, 157601 (2011).

[23] P. C. Maurer, G. Kucsko, C. Latta, L. Jiang, N. Y. Yao, S. D. Bennett, F. Pastawski, D. Hunger, N. Chislock, M. Markham, D. J. Twitchen, J. I. Cirac, and M. D. Lukin, Science 332, 1283 (2012).

[24] A. Dreyer, P. Spinicelli, J. R. Maze, J.-F. Roch, and V. Jacques, Phys. Rev. Lett. 110, 060502 (2013).

[25] J. J. Pla, K. Y. Tan, J. P. Dehollain, W. H. Lim, J. J. Morton, F. A. Zwanenburg, D. N. Jamieson, A. S. Dzurak, and A. Morelo, Nature 496, 334 (2013).

[26] I. Lovchinsky, A. O. Sushkov, E. Urbach, N. P. de Leon, S. Choi, K. De Greve, R. Evans, R. Gertner, E. Bersin, C. Müller, L. McGuinness, F. Jelezko, R. L. Walsworth, H. Park, and M. D. Lukin, Science 351, 836 (2016).

[27] J. M. Boss, K. S. Cujia, J. Zopes, and C. L. Degen, Science 356, 837 (2017).

[28] S. Schmitt, T. Gefen, F. M. Stner, T. Urdampilleta, M. Atatüre, npj Quantum Inf. 5, 13 (2019).

[29] T. Nakajima, A. Noiri, J. Yoneda, M. R. Delbecq, P. Stano, S. Amaha, J. Yoneda, A. Noiri, K. Kawasaki, K. Takeda, G. Allison, A. Ludwig, A. D. Wieck, D. Loss, and S. Tarucha, Phys. Rev. Lett. 119, 017701 (2017).

[30] P. Pakkiam, A. V. Timofeev, M. G. House, M. R. Hogg, T. Kobayashi, M. Koch, S. Rogge, and M. Y. Simmons, Phys. Rev. X 8, 041032 (2018).

[31] L. Vukušić, J. Kukučka, H. Watzinger, F. Schäffler, and G. Katsaros, Nano Letters 18, 7141 (2018).

[32] P. Harvey-Collard, B. D’Anjou, M. Rudolph, N. T. Jacobson, J. Dominguez, G. A. Ten Eyck, J. R. Wendt, T. Pluym, M. P. Lilly, W. A. Coish, M. Pioro-Ladrière, and M. S. Carroll, Phys. Rev. X 8, 021046 (2018).

[33] A. Opremcak, I. V. Pechenezhskiy, C. Howington, B. G. Christophersen, M. A. Beck, E. Leonard, J. Stuttle, C. Wilen, K. N. Nesterov, G. J. Ribeill, T. Thorbeck, F. Schlenker, M. G. Vavilov, B. L. Plourde, and R. McDermott, Science 361, 1239 (2018).

[34] A. West, B. Hensen, A. Jouan, T. Tanttu, C.-H. Yang, A. Rossi, M. F. Gonzalez-Zalba, F. Hudson, A. Morello, D. J. Reilly, and A. S. Dzurak, Nat. Nanotech. 14, 437 (2019).

[35] D. J. Urdampilleta, Matias und Niegemann, E. Chanrion, B. Jadot, C. Spence, C. Mortemousque, Pierre-André und Bäuerle, L. Hutin, B. Bertrand, S. Barraud, R. Maupin, M. Sanquer, X. Jelh, S. De Franceschi, M. Vinet, and T. Mennier, Nat. Nanotech. 14, 737 (2019).

[36] G. Zheng, N. Samkhraie, M. L. Noordam, N. Kalhor, D. Brousse, A. Sammak, G. Scappucci, and L. M. K. Vandersypen, Nat. Nanotech. 14, 742 (2019).

[37] D. Keith, S. K. Gorman, L. Kranz, Y. He, J. G. Keizer, M. A. Broome, and M. Y. Simmons, New J. Phys. 21, 063011 (2019).

[38] D. Keith, M. G. House, M. B. Donnelly, T. F. Watson, B. Weber, and M. Y. Simmons, Phys. Rev. X 9, 041003 (2019).

[39] A. Opremcak, C. H. Liu, C. Wilen, K. Okubo, B. G. Christophersen, D. Sank, T. C. White, A. Vainsencher, M. Giustina, A. Meigrant, B. Burkett, B. L. T. Plourde, and R. McDermott, 2008.02346.

[40] J. Ebel, T. Joas, M. Schalk, A. Angerer, J. Majer, and
F. Reinhard, 2003.07562.

[56] E. J. Connors, J. J. Nelson, and J. M. Nichol, Phys. Rev. Applied 13, 024019 (2020).

[57] E. I. Rosenthal, C. M. F. Schneider, M. Malnou, Z. Zhao, F. Leditzky, B. J. Chapman, W. Wustmann, X. Ma, D. A. Falken, M. F. Zanner, L. R. Vale, G. C. Hilton, J. Gao, G. Smith, G. Kirchmair, and K. W. Lehnert, 2008.03805.

[58] A. H. Myerson, D. J. Szwer, S. C. Webster, D. T. C. Allcock, M. J. Curtis, G. Imreh, J. A. Sherman, D. N. Stacey, A. M. Steane, and D. M. Lucas, Phys. Rev. Lett. 100, 200502 (2008).

[59] R. Gehr, J. Volz, G. Dubois, T. Steinmetz, Y. Colombe, B. L. Lev, R. Long, J. Estève, and J. Reichel, Phys. Rev. Lett. 104, 203602 (2010).

[60] T. P. Harty, D. T. C. Allcock, C. J. Ballance, L. Guidoni, H. A. Janacek, N. M. Linke, D. N. Stacey, A. M. Steane, and D. M. Lucas, Phys. Rev. Lett. 113, 220501 (2014).

[61] B. J. Shields, Q. P. Unterreithmeier, N. P. de Leon, H. Park, and M. D. Lukin, Phys. Rev. Lett. 114, 136402 (2015).

[62] B. D’Anjou, L. Kuret, L. Childress, and W. A. Coish, Phys. Rev. X 6, 011017 (2016).

[63] S. L. Todaro, V. B. Verma, K. C. McCormick, D. T. C. Allcock, R. P. Mirin, D. J. Wineland, S. W. Nam, A. C. Wilson, D. Leibfried, and D. H. Slichter, 2008.00065.

[64] B. D’Anjou and W. A. Coish, Phys. Rev. Lett. 113, 230402 (2014).

[65] H. Dinani, D. W. Berry, R. Gonzalez, J. R. Maze, and C. Bonato, Phys. Rev. B 99, 125413 (2019).

[66] J. H. Xu, A. X. Chen, W. Yang, and G. R. Jin, Phys. Rev. A 100, 063839 (2019).

[67] G. Liu, M. Chen, Y.-X. Liu, D. Layden, and P. Cappellaro, Mach. Learn.: Sci. Technol. 1, 015003 (2020).

[68] The importance of soft decoding was also recognized in the context of continuous-variable quantum error correction [83–86], continuous-variable quantum communication [87], and quantum parameter estimation [64, 66, 88].

[69] H. Chernoff, Ann. Math. Stat. 23, 493 (1952).

[70] W. Hoeffding, Ann. Math. Stat. 36, 396 (1965).

[71] S. M. Kay, Fundamentals of Statistical Signal Processing, Vol. II: Detection Theory, Vol. II (Prentice Hall, New Jersey, U.S.A., 1998).

[72] Or hypervolume in the case of multidimensional O.
Supplemental material for “A generalized figure of merit for qubit readout”

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S1. LARGE DEVIATION THEORY

Let $\bar{s} = (1/N) \sum_{k=1}^{N} s_k$ be the sample mean of $N$ i.i.d. variables $s_k$. In its simplest form, the main result of large deviation theory is that the CDF of $\bar{s}$ asymptotically satisfies [1]

$$\ln P(\bar{s} > x) \sim -NI(x), \quad (S1)$$

where the so-called rate function $I(x)$ is independent of $N$. Here, “$\sim$” denotes asymptotic equality as $N \to \infty$. Large deviation theory provides an explicit expression for the function $I(x)$:

$$I(x) = -\inf_t [K(t) - xt]. \quad (S2)$$

Here, $K(t)$ is the cumulant generating function of the distribution $P(s_k)$:

$$K(t) = \ln \left[ \int ds P(s)e^{st} \right]. \quad (S3)$$

Applying Eqs. (S1), (S2), and (S3) to the conditioned cumulative error rates [Eq. (5) of the main text] yields

$$\ln e_{\pm,N} \sim -C_{\pm} N, \quad (S4)$$

where

$$C_{\pm} = -\inf_{t\in[0,1]} \ln \left[ \int d\lambda P_\pm(\lambda)e^{t\lambda} \right]. \quad (S5)$$

Here, “$\sim$” denotes asymptotic equality. Note that the infimum occurs in the interval $t \in [0,1]$. This is because the cumulant generating functions $K_{\pm}(t) = \ln \left[ \int d\lambda P_{\pm}(\lambda)e^{t\lambda} \right]$ are convex and vanish at $t = 0$ and $t = 1$. Therefore, the infimum must occur between $t = 0$ and $t = 1$. Finally, the integration variable is changed back to the readout outcome $O$ by setting $\lambda = \ln \frac{P_+(O)}{P_-(O)}$ in Eq. (S5). This gives

$$C_+ = C_- = C = -\inf_{t\in[0,1]} \ln \left[ \int dO P_+(O)^{1-t}P_-(O)^t \right], \quad (S6)$$

which is the Chernoff information introduced in Eq. (7) of the main text.
S2. CHERNOFF INFORMATION FOR BINARY READOUT OUTCOMES

A hard decoding strategy converts each analog outcome \( O \) to binary outcomes \( \pm \) in each repetition. The conditioned probabilities for the binary outcomes are

\[
P_+(+) = 1 - \epsilon_+, \quad P_-(+) = \epsilon_-
\]

\[
P_+(-) = \epsilon_+, \quad P_-(-) = 1 - \epsilon_-
\]

Here, \( \epsilon_\pm \) are the conditioned single-repetition error rates defined in Eq. (3) of the main text. The Chernoff information \( C_b \) for binary outcomes is then obtained by substituting Eq. (S7) into Eq. (S6). The optimization over \( t \) can be performed exactly. The result is

\[
C_b = -\ln \left[ (1 - \epsilon_+)^{1-t^*} \epsilon_+^{t^*} + \epsilon_-^{1-t^*} (1 - \epsilon_-)^{t^*} \right],
\]

where

\[
t^* = \frac{\ln \left( \frac{1-\epsilon_+}{\epsilon_+} - \epsilon_+^{1-t^*} \right)}{\ln \left( \frac{1-\epsilon_+}{\epsilon_+} \right)}, \quad 1-t^* = \frac{\ln \left( \frac{1-\epsilon_-}{\epsilon_-} \right)}{\ln \left( \frac{1-\epsilon_-}{\epsilon_-} \right)}.
\]

(S9)

Simple expressions for \( C_b \) can be obtained in two limits of practical relevance. When \( \epsilon_\pm \to 0 \), Eq. (S9) takes the form

\[
t^* \sim \frac{\ln(\epsilon_-^{-1})}{\ln(\epsilon_+^{-1}) + \ln(\epsilon_-^{-1})}, \quad 1-t^* \sim \frac{\ln(\epsilon_-^{-1})}{\ln(\epsilon_+^{-1}) + \ln(\epsilon_-^{-1})}.
\]

(S10)

Substituting these expressions back into Eq. (S8) gives Eq. (10) of the main text,

\[
C_b \sim \left[ \frac{1}{\ln(\epsilon_+^{-1})} + \frac{1}{\ln(\epsilon_-^{-1})} \right]^{-1} \quad \text{as} \quad \epsilon_\pm \to 0.
\]

(S11)

Another case of interest is the symmetric case, \( \epsilon_+ = \epsilon_- \). In this case, \( t^* = 1/2 \) and Eq. (S8) reduces to

\[
C_b = \ln \left[ \frac{1}{\sqrt{4\epsilon(1-\epsilon)}} \right].
\]

(S12)

S3. EXAMPLES OF SOFT DECODING ADVANTAGE

1. Gaussian noise

Consider a readout that converts the eigenvalues \( a = \pm 1 \) to readout signals of amplitude \( \pm 1 \). Moreover, assume that these signals are hidden by Gaussian noise with (power) signal-
to-noise ratio $r$. The resulting distributions of analog readout outcomes are

$$P_{\pm}(O) = \sqrt{\frac{r}{2\pi}} \exp \left[-\frac{r(O \mp 1)^2}{2}\right].$$

(S13)

The Chernoff information is obtained from Eq. (S6). By symmetry of the distributions, the infimum occurs at $t^* = 1/2$. Performing the Gaussian integral yields

$$C = \frac{r}{2}.$$  

(S14)

Next suppose that the analog outcomes are binarized in each repetition. By symmetry, it is clear that $\epsilon_+ = \epsilon_- = \epsilon$. The average single-repetition error rate for Gaussian noise is simply

$$\epsilon_G = \int_{-\infty}^{0} dO \ P_+(O) = \int_{0}^{\infty} dO \ P_-(O) = \frac{1}{2} \text{erfc} \left(\sqrt{\frac{r}{2}}\right).$$

(S15)

The Chernoff information for binary outcomes, Eq. (S12), is then

$$C_b = \ln \left[\frac{1}{\sqrt{4\epsilon_G(1 - \epsilon_G)}}\right].$$

(S16)

Expanding this expression in the two extreme limits $r \ll 1$ and $r \gg 1$ gives

$$C_b = \begin{cases} \frac{r}{\pi} - \left(\frac{\pi-3}{3\pi^2}\right) r^2 + O(r^3) & \text{for } r \ll 1, \\ \frac{r}{4} + \frac{1}{4} \ln \left(\frac{\pi r}{8}\right) + O \left(\frac{1}{r}\right) & \text{for } r \gg 1. \end{cases}$$

(S17)

Comparing Eqs. (S14) and (S17) gives the soft decoding advantage $\rho = C/C_b$ in both limits:

$$\rho = \begin{cases} \frac{\pi}{2} + \left(\frac{\pi-3}{6}\right) r + O(r^2) & \text{for } r \ll 1, \\ 2 - 2 \left(\frac{1}{r} \ln \left(\frac{\pi r}{8}\right)\right) + O \left(\frac{1}{r^2}\right) & \text{for } r \gg 1. \end{cases}$$

(S18)

The expression for $r \gg 1$ is the one obtained in Ref. [2] by other means.

2. Gaussian noise with conversion errors

As in Sec. S31, consider a readout that attempts to convert the eigenvalues $a = \pm 1$ to readout signals $\pm 1$ hidden by Gaussian noise. However, now suppose that the conversion fails with probability $\eta$. The resulting probability distributions of analog readout outcomes are mixtures of Gaussian distributions:

$$P_{\pm}(O) = (1 - \eta) \sqrt{\frac{r}{2\pi}} \exp \left[-\frac{r(O \mp 1)^2}{2}\right] + \eta \sqrt{\frac{r}{2\pi}} \exp \left[-\frac{r(O \pm 1)^2}{2}\right].$$

(S19)
The Chernoff information is obtained from Eq. (S6). By symmetry of the distributions, the infimum occurs at $t^* = 1/2$. Rearranging the integral gives

$$C = \frac{r}{2} - \ln \left[ \int_{-\infty}^{\infty} dx \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \sqrt{1 + 4\eta(1 - \eta) \sinh^2(\sqrt{r}x)} \right].$$  \hspace{1cm} (S20)

The second term gives a correction to Eq. (S14) due to the finite rate of conversion errors $\eta$. The symmetry of the distributions also means that $\epsilon_+ = \epsilon_- = \epsilon$. The average single-repetition error rate in the presence of conversion errors is

$$\epsilon_\eta = (1 - \eta)\epsilon_G + \eta(1 - \epsilon_G),$$  \hspace{1cm} (S21)

where $\epsilon_G = \text{erfc}\left(\sqrt{r/2}\right)/2$ as before. The Chernoff information for binary outcomes is then obtained from Eq. (S12):

$$C_b = \ln \left[ \frac{1}{\sqrt{4\epsilon_\eta(1 - \epsilon_\eta)}} \right].$$  \hspace{1cm} (S22)

The soft decoding advantage $\rho = C/C_b$ can be calculated from Eqs. (S20) and (S22) by performing a simple integral. The result is plotted as a function of $\epsilon_G$ and $\eta$ in Eq. (3) of the main text.

**S4. MONTE CARLO SIMULATIONS**

**1. Hidden Markov model**

The results stated in Sec. III B of the main text are obtained by simulating the cumulative readout error rate $e_N$ for arbitrary distributions of analog readout outcomes and in the presence of non-QND imperfections. In the presence of non-QND imperfections, the eigenvalue $a$ may change from one repetition to the next. Therefore, the distribution of analog readout outcomes $P_{a_N}(O_N)$ is now conditioned on the eigenvalue $a_N$ realized in repetition $N$. In addition, it is assumed that non-QND transitions are Markovian. Therefore, the dynamics are fully characterized by the probability $P_{a_N}(a_{N+1})$ to transition to eigenvalue $a_{N+1}$ given the previous eigenvalue $a_N$ \[3\]. Processes described by the distributions $P_{a_N}(O)$ and $P_{a_N}(a_{N+1})$ define a hidden Markov model \[4\]. Such models can be simulated efficiently as described below.
2. Sampling

A large number $M = 10^6$ of independent strings of readout outcomes $O_N = \{O_0, O_1, \ldots, O_{N-1}\}$ is sampled for both initial eigenvalues $a_0 = \pm 1$. This is done with the following algorithm:

1. Set the initial eigenvalue $a_0$ and set $k = 0$;
2. Repeat the following until $k = N$:
   (a) Sample the readout outcome $O_k$ from the distribution $P_{a_k}(O_k)$;
   (b) Sample the next eigenvalue $a_{k+1}$ according to the distribution $P_{a_k}(a_{k+1})$;
   (c) Increase $k$ by 1.

3. Decoding

Decoding in performed by determining which initial eigenvalue $a_0$ most likely generated the sampled data. As discussed in the main text, this is done by calculating the log-likelihood ratio
\[
l_N = \ln \left[ \frac{P_{a_0=+1}(O_N)}{P_{a_0=-1}(O_N)} \right],
\] (S23)
where $P_{a_0}(O)$ are the probabilities of obtaining the string $O_N$ conditioned on the initial eigenvalue $a_0$. The probabilities $P_{a_0}(O_N)$ are also known as the likelihoods for the eigenvalues $a_0$. If $l_N > 0$, $a_0 = +1$ is assigned. If $l_N < 0$, $a_0 = -1$ is assigned. If $l_N = 0$, the value of $a_0$ is assigned at random. If the assigned value of $a_0$ differs from the true value, an error has occurred. For each true value $a_0 = \pm 1$, the number of errors $E_{\pm,N}$ is divided by the number of simulations $M$ to yield an estimate $e_{\pm,N} \approx E_{\pm,N}/M$ of the cumulative error rate. The statistical uncertainty in that estimate is $\delta e_{\pm,N} \approx \sqrt{e_{\pm,N}(1-e_{\pm,N})/M}$. Moreover, the statistical uncertainty in the average error rate $e_N = (e_{+,N} + e_{-,N})/2$ is $\delta e_N \approx \sqrt{\delta e_{+,N}^2 + \delta e_{-,N}^2}/2$.

4. Calculation of the likelihood

The likelihoods $P_{a_0}(O_k)$ can be efficiently calculated for all substrings $O_k$, $k \leq N$, using the procedure described below. In what follows, the dependence on $a_0$ is omitted to simplify
notation. The likelihood can be decomposed as

$$P(O_k) = \sum_{a_k} \ell_k(a_k),$$

(S24)

where

$$\ell_k(a_k) \equiv P(O_k, a_k).$$

(S25)

The advantage of this decomposition is that $\ell_k(a_k)$ can be calculated iteratively using the theory of hidden Markov models. Let $\ell_k$ be a column vector with elements $\ell_k(a_k)$ in the basis $\{a_k = +1, a_k = -1\}$. This vector obeys the recurrence relation [5, 6]

$$\ell_{k+1} = V_k \cdot \ell_k.$$  

(S26)

Here, $V_k$ is a matrix with elements that depend on $O_k$:

$$V_k(a_{k+1}, a_k) = P_{a_k}(O_k) P_{a_k}(a_{k+1}).$$  

(S27)

The probability of the string $O_k$ occurring is then

$$P(O_k) = \text{Tr} [\ell_k].$$  

(S28)

Here, the trace of a vector is defined as the sum of its elements. Note that the above recurrence automatically yields the likelihood for all $k \leq N$ after $N$ iterations. The initial state is set to $\ell_0 = (1, 0)^T$ to calculate the likelihood for $a_0 = +1$ and to to $\ell_0 = (0, 1)^T$ to calculate the likelihood for $a_0 = -1$.

S5. UNIVERSALITY IN THE PRESENCE OF NON-QND IMPERFECTIONS

Consider the common case where the observable eigenvalue relaxes from $a_k = +1$ to $a_{k+1} = -1$ at rate $\Gamma$. For this relaxation process, the transition probabilities $P_{a_k}(a_{k+1})$ in a repetition of duration $\Delta t$ are

$$P_{a_k=+1}(a_{k+1} = +1) \approx 1 - p, \quad P_{a_k=+1}(a_{k+1} = +1) = 0$$

$$P_{a_k=+1}(a_{k+1} = -1) \approx p, \quad P_{a_k=+1}(a_{k+1} = -1) = 1,$$

(S29)

where $p = \Gamma \Delta t \ll 1$ is the transition probability. In the main text, it is argued that arbitrary noise with Chernoff information $C$ can be modelled by effective Gaussian noise with signal-to-noise ratio $r = 2C$. For Gaussian noise, it is known that $e_N$ is a function of $rN$ and
FIG. S1. (a) Monte Carlo simulations of the cumulative error rate $e_N$ for (b) Gaussian noise, (c) Cauchy noise, (d) Poissonian noise, and (e-f) the analog and binary noise observed in Ref. [6]. In (b-f), the blue distribution corresponds to $a = +1$ and the red distribution corresponds to $a = -1$. The cumulative error rate is a universal function of $CN$ and $C/p$ for all simulated noise models. The solid black line is obtained from Eq. (8) of the main text. The details of the simulations are discussed in Sec. S4.

$r/p$ only [7]. Therefore, $e_N$ should be a universal function of $CN$ and $C/p$ regardless of the details of the noise. It is verified that this is indeed the case by performing Monte Carlo simulations using the procedure of Sec. S4. The results are shown in Fig. S1. All noise models with the same value of $C/p$ collapse on the same curve when plotted as a function of $CN$. Therefore, the cumulative error rate $e_N$ may simply be tabulated for various values $CN$ and $C/p$ by assuming Gaussian noise. The cumulative error rate for arbitrary non-Gaussian noise can then simply be read off from the Gaussian results.

We note that small deviations from universality were observed in the minimum value of $e_N$ when that minimum is reached at small values of $N$. This occurs because early relaxation interacts with the small non-asymptotic corrections to the Gaussian result, Eq. (8) of the main text. In the case of Gaussian noise, it is known that the minimum occurs after $N_m \propto \ln(C/p)/C$ repetitions [7]. As $N_m$ increases, deviations from universality disappear. Since $N_m \gg 1$ is the only situation where repetitive QND readout is useful, the saturation
value remains universal in most cases of practical interest.

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, 2005).

[2] B. D’Anjou and W. A. Coish, *Phys. Rev. Lett.* **113**, 230402 (2014).

[3] Here, the state space is two-dimensional, $a = \pm 1$. In general, however, the measured binary observable $A$ may be embedded in a larger state space with more complex dynamics.

[4] W. Zucchini, I. L. MacDonald, and R. Langrock, *Hidden Markov Models for Time Series* (Apple Academic Press Inc., 2016).

[5] T. Nakajima, A. Noiri, J. Yoneda, M. R. Delbecq, P. Stano, T. Otsuka, K. Takeda, S. Amaha, G. Allison, K. Kawasaki, A. Ludwig, A. D. Wieck, D. Loss, and S. Tarucha, *Nat. Nanotech.* **14**, 555 (2019).

[6] X. Xue, B. D’Anjou, T. F. Watson, D. R. Ward, D. E. Savage, M. G. Lagally, M. Friesen, S. N. Coppersmith, M. A. Eriksson, W. A. Coish, and L. M. K. Vandersypen, *Phys. Rev. X* **10**, 021006 (2020).

[7] J. Gambetta, W. A. Braff, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, *Phys. Rev. A* **76**, 012325 (2007).