Supersymmetry of the Nonstationary Schrödinger Equation and Time-Dependent Exactly Solvable Quantum Models

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Abstract — New exactly solvable time-dependent quantum models are obtained with the help of the supersymmetric extension of the nonstationary Schrödinger equation.

1. INTRODUCTION

An essential ingredient of the conventional supersymmetric quantum mechanics (for reviews see [1]) is the well known Darboux transformation [2] for the stationary Schrödinger equation. This transformation permits us to construct new exactly solvable stationary potentials from known ones. Similar constructions may be developed for the time-dependent Schrödinger equation [3].

Our approach to Darboux transformation is based on a general notion of transformation operator introduced by Delsart [4]. In terms of this notion Darboux [2] studied differential first order transformation operators for the Sturm-Liouville problem. This is the reason in our opinion to call every differential transformation operator Darboux transformation operator. Different approach to Darboux transformation is exposed in the book [5]. It is worthwhile mentioning that our approach in contrast to that of Ref. [5] leads to real potential differences. This property is crucial for constructing the supersymmetric extension of the nonstationary Schrödinger equation.

2. FORMALISM

In this section we review briefly the basic constructions leading to the supersymmetry of the nonstationary Schrödinger equation established in Ref. [3].

Consider two time-dependent Schrödinger equations

\( (i\partial_t - h_0) \psi(x,t) = 0, \quad h_0 = -\partial_x^2 + V_0(x,t), \) \hspace{1cm} (1)

\( (i\partial_t - h_1) \varphi(x,t) = 0, \quad h_1 = -\partial_x^2 + V_1(x,t). \) \hspace{1cm} (2)

We assume the potential \( V_0(x,t) \) and the solutions of Eq. (1), called the initial Schrödinger equation, to be known. By definition transformation operator denoted by \( L \) transforms solutions \( \psi(x,t) \) into solutions \( \varphi(x,t) = L\psi(x,t) \). It is obvious

\footnote{Talk presented by B.F. Samsonov at the VIII International Conference on "Symmetry Methods in Physics", Dubna, Russia, 28 July – 2 August, 1997.}
that this condition is fulfilled if \( L \) participates in the following intertwining relation:
\[
L (i \partial_t - h_0) = (i \partial_t - h_1) L.
\]
In the simplest case of a first order differential operator \( L \) this equation can readily be solved with respect to operator \( L \) and potential difference \( A(x,t) \)
\[
L = L_1(t)[-u_x(x,t)/u(x,t) + \partial_x], \quad L_1(t) = \exp \left[ 2 \int dt \Im (\log u)_{xx} \right],
\]
\[
A(x,t) = V_1(x,t) - V_0(x,t) = -[\log |u(x,t)|^2]_{xx}.
\]
Note that the operator \( L \) and the new potential \( V_1(x,t) \) are completely defined by a function \( u(x,t) \) called the transformation function. This function is a particular solution to the initial Schrödinger equation (1) subject to the condition \( \log u \, u_{xxx} = 0 \) called the reality condition of the new potential.

Operator \( L^+ = -L_1(t)[\pi_x(x,t)/\pi(x,t) + \partial_x] \) which is Laplace adjoint to \( L \) realizes the transformation in the inverse direction, i.e., the transformation from the solutions of Eq. (2) to the ones of Eq. (1). The product \( L^+ L \) is a symmetry operator for Eq. (1) and \( LL^+ \) is the similar one for Eq. (3).

With the help of the transformation operators \( L \) and \( L^+ \) we build up the time-dependent nilpotent supercharge operators
\[
Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}
\] (3)
which commute with the Schrödinger super-operator \( i \partial_t - H \), where \( H = \text{diag}\{h_0, h_1\} \) is the time-dependent super-Hamiltonian and \( I \) is the unit \( 2 \times 2 \) matrix. In general, the super-Hamiltonian is not integral of motion for the quantum system guided by the matrix Schrödinger equation
\[
(i \partial_t - H) \Psi(x,t) = 0.
\] (4)
Two-component function \( \Psi(x,t) \) belongs to the linear space defined over complex number field and spanned by the basis \( \Psi_+ = \psi e_+, \Psi_- = L\psi e_- \) where \( e_+ = (1,0)^T \) and \( e_- = (0,1)^T \). The sign "T" stands for the transposition.

The operators (3) are integrals of motion for Eq. (1). Using the symmetry operators \( L^+ L \) and \( LL^+ \) we may construct other integral of motion for this equation: \( S = \text{diag} \{L^+ L, LL^+\} \). The operators \( Q, Q^+ \) and \( S \) realize the well-known superalgebra \( sl(1/1) \)
\[
[Q, S] = [Q^+, S] = 0, \quad \{Q, Q^+\} = S - \alpha I
\]
where instead of the Hamiltonian we see other symmetry operator. In general, the operators \( S, Q, \) and \( Q^+ \) depend on time and consequently we have obtained the time-dependent superalgebra.
3. HARMONIC OSCILLATOR WITH A TIME-VARYING FREQUENCY

Consider the Hamiltonian

\[ h_0 = -\partial^2_x + \omega^2(t)x^2. \tag{5} \]

Variety of potentials we may obtain by the technique described above depends on variety of solutions of the initial Schrödinger equation suitable for use as transformation functions. A wide class of solutions may be found with the help of \textit{R-separation of variables} method \cite{6} based on the orbits structure of the symmetry algebra with respect to the adjoint representation of corresponding group symmetry. Symmetry algebra of the Schrödinger equation with Hamiltonian (5) is well-known Schrödinger algebra \textit{G} \textsubscript{2} \cite{6}. The following representation of this algebra is suitable for our purpose

\[ K_1 = a - a^+, \quad K_- = -i(a + a^+), \quad K_0 = i, \]

\[ K_{-2} = -i(a + a^+)^2, \quad K_2 = -i(a - a^+)^2, \quad K^0 = -2 \left[ a^2 - (a^+)^2 \right], \]

\[ a = \varepsilon \partial_x - i\dot{\varepsilon}x/2, \quad a^+ = -\bar{\varepsilon} \partial_x + i\bar{\varepsilon}x/2, \quad aa^+ - a^+a = 1/4 \]

where \( \varepsilon = \varepsilon(t) \) is a (complex) solution to the classical equation of motion for the oscillator \( \ddot{\varepsilon}(t) + 4\omega^2(t)\varepsilon(t) = 0 \).

Five orbits are known for this algebra which give four nonequivalent solutions to the Schrödinger equation in R-separated variables with respect to transformations from the Schrödinger group. Below is a summary of all suitable transformation functions and corresponding potentials.

1). Two orbits with the representatives \( J_1 = K_1 \) and \( J_1 = K_2 \) :

\[ u(x,t) = u_{\lambda} + u_{\bar{\lambda}} = \gamma^{-1/2} \cosh \left( \frac{\nu x}{8\gamma} + \mu \frac{\delta}{32\gamma} \right) \]

\[ \times \exp \left[ \frac{i\nu^2\dot{\gamma}}{4\gamma} - \frac{i\mu x}{8\gamma} + i(\nu^2 - \mu^2) \frac{\delta}{64\gamma} \right], \quad \lambda = -\mu - i\nu, \quad L_1(t) = \gamma = (\varepsilon + \bar{\varepsilon})/2, \]

\[ V_1(x,t) = \omega^2(t)x^2 - \frac{\nu^2}{32\gamma^2} \cosh^{-2} \left( \frac{\nu x}{8\gamma} + \mu \frac{\delta}{32\gamma} \right). \]

2). The orbit with representative \( J_2 = K_2 - K_1 \) :

\[ \psi_{\lambda}(x,t) = \delta^{-1/2} \exp \left( \frac{i\nu^2 \dot{\delta}}{4\delta} - ix \frac{\gamma}{2\delta} + i \frac{\gamma^2}{6\delta^3} + i\lambda \frac{\gamma}{\delta} \right) \]

\[ \times Q \left( 2^{-1/2} \left( \frac{x}{\delta} - \frac{\gamma^2}{2\delta^2} \right) - 2^{2/3} \lambda \right) \]

where \( \gamma = \varepsilon + \bar{\varepsilon}, \quad i\delta = \varepsilon - \bar{\varepsilon}, \lambda \) is a separation constant, and \( Q(z) \) is an \textit{Airy} function defined by the equation: \( Q''(z) = zQ(z). \) An exactly solvable potential is expressed
in this case through the Airy function $Ai(z)$. It is an easy exercise to show that a
regular on full real axis potential may be obtained with the help of the second order
Darboux transformation operator with transformation functions $\psi_\lambda$ and $\psi_\chi$.

3). The orbit with the representative $J_3 = K_2 - K_{-2}$ gives several classes of
potentials. First, we may choose solutions which form a discrete basis in the Hilbert
space of states $[7]$ as transformation functions

$$u_n(x, t) = N_n \gamma^{-1/4} \left( \frac{\gamma}{\varepsilon} \right)^{n/2+1/4} \exp \left( \frac{2i\gamma - 1}{16\gamma} x^2 \right) He_n \left( \frac{x}{2\sqrt{\gamma}} \right), \ \gamma = \varepsilon \varepsilon$$

where $He_n(z) = 2^{-n/2}H_n(z/\sqrt{2})$ are the Hermite polynomials. The second order
Darboux transformation with transformation functions $u_n$ and $u_{n+1}$ produces po-
tentials

$$V_n(x, t) = \omega^2(t)x^2 - \frac{1}{2\gamma} \left[ \frac{J''_n(z)}{J_n(z)} - \left( \frac{J'_n(z)}{J_n(z)} \right)^2 - 1 \right], \ z = x/(2\sqrt{\gamma}),$$

$$J_n(z) = \sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(k+1)} He_k^2(z) = kJ_{k-1}(z) + He_k^2(z),$$

Second, we may use a general solution to the quantization equation for the operator $J_3$

$$u(x, t) = \varepsilon^{-1/2} \exp \left( \frac{2i\gamma + 1}{16\gamma} x^2 \right) \left[ C + \text{erf} \left( \frac{x}{2\sqrt{2\gamma}} \right) \right].$$

This leads to potentials which in the case of $\omega(t) = \text{const}$ reduce to the well-known
isospectral potentials.

$$V(x, t) = \omega^2(t)x^2 - \frac{1}{4\gamma} \left[ 1 - 2zQ^{-1}(z)e^{-z^2/2} - 2Q^{-2}(z)e^{-z^2} \right],$$

$$Q(z) = \sqrt{\frac{\pi}{2}} \left[ C + \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right], \ z = \frac{x}{2\sqrt{\gamma}}, \ |C| > 1.$$

Other cases are similar to these described in [3] for the free particle Schrödinger
equation so that be omitted here.

4. TIME DEPENDENT SINGULAR OSCILLATOR

Consider now the following Hamiltonian:

$$h_0 = -\partial_x^2 + \omega^2(t)x^2 + gx^{-2}.$$ 

Symmetry algebra of the Schrödinger equation with this Hamiltonian is $su(1.1) \sim \text{sl}(2, \mathbb{R})$. We use the following representation for this algebra:

$$K_+ = 2 \left( (a^+)^2 - \varepsilon^2 gx^{-2} \right), \ K_- = 2 \left( a^2 - \varepsilon^2 gx^{-2} \right),$$
Consider solutions of the Schrödinger equation which are eigenstates of $K_0$: $K_0 \varphi_\lambda(x, t) = \lambda \varphi_\lambda(x, t)$. When $\lambda = n + k, n = 0, 1, 2, \ldots$ we have a discrete basis of the Hilbert space

$$
\varphi_n(x, t) = 2^{1/2 - 3k} \sqrt{\frac{n!}{\Gamma (n + 2k)}} \gamma^{-k} \left( \frac{\varepsilon}{\gamma} \right)^{n+k} x^{2k-1/2} \exp \left[ i \frac{x^2 \gamma}{8 \gamma} \right] L_n^{2k-1} \left( \frac{x^2}{8 \gamma} \right), \ k = \frac{1}{2} + \frac{1}{4} \sqrt{1 + 4g}, \gamma = \varepsilon \varepsilon.
$$

To construct spontaneously broken supersymmetric model we need transformation functions $u(x, t)$ such that neither $u(x, t)$ nor $u^{-1}(x, t)$ are not from the Hilbert space and $u(x, t)$ is nodeless for all real values of $t$ and $x > 0$. These conditions are fulfilled for the functions

$$
u_p(x, t) = \gamma^{-k} \left( \frac{\varepsilon}{\gamma} \right)^{-p-k} x^{2k-1/2} \exp \left[ i \frac{x^2 \gamma}{8 \gamma} + \frac{x^2}{16 \gamma} \right] L_p^{2k-1} \left( -\frac{x^2}{8 \gamma} \right),$$

$$K_0 u_p(x, t) = -(p + k) u_p(x, t).$$

These transformation functions create the following exactly solvable family of potential differences $A(x, t) = \omega^2(t)x^2 + gx^{-2} - V_1(x, t)$:

$$A(x, t) = A_p(x, t) = \frac{1}{4 \gamma} - \frac{4k - 1}{x^2} - \frac{1}{8} \left( \frac{xL^2_{p-1}(z)}{\gamma L^2_{p-1}(z)} \right)^2 + \frac{x^2 L_{p-2}^{2k+1}(z) + 4 \gamma L_{p-1}^{2k}(z)}{8 \gamma^2 L_{p-1}^{2k-1}(z)}, \ z = -\frac{x^2}{8 \gamma}.$$}

To construct a model with exact supersymmetry we need transformation functions $u(x, t)$ such that $u^{-1}(x, t)$ is square integrable on semiaxis $x \geq 0$ and satisfies the zero boundary condition at the origin for all values of $t$. The following solution of the Schrödinger equation may be chosen in this case:

$$u_p(x, t) = \gamma^{-k} \left( \frac{\varepsilon}{\gamma} \right)^{k-p-1} x^{3/2 - 2k} \exp \left[ i \frac{x^2 \gamma}{8 \gamma} + \frac{x^2}{16 \gamma} \right] L_p^{1-2k} \left( -\frac{x^2}{8 \gamma} \right),$$

$$K_0 u_p(x, t) = (k - p - 1) u_p(x, t).$$

It is not difficult to establish the possible values of $p$. If $p$ is even it may takes the values $p < 2k - 1$ and $p = [2k] + 1, [2k] + 3, \ldots$. For odd $p$ values we may use only $p = [2k], [2k] + 2, \ldots$, where $[2k] \equiv \text{entire}(2k)$. For regular potential differences we obtain

$$A_p(x, t) = \frac{1}{4 \gamma} + \frac{4k - 3}{x^2} - \frac{1}{2} \left( \frac{xL^2_{p-1}(z)}{2 \gamma L^1_{p-1}(z)} \right)^2 + \frac{x^2 L^3_{p-2}(z) + 4 \gamma L^2_{p-1}(z)}{8 \gamma^2 L^1_{p-1}(z)}.$$
5. CONCLUSION

The supersymmetry of the time-dependent Schrödinger equation based on the nonstationary Darboux transformation is very useful to obtain a wide class of exactly solvable nonstationary quantum models. With the help of the Darboux transformation operator we may obtain solutions for transformed equations. In particular, if we know the coherent states for the initial system then by applying to them the Darboux transformation operator we obtain coherent states for transformed quantum system \[8\]. The coherent states are known \[7\] for the systems considered here. Next step is to obtain and investigate coherent states for the transformed systems. Corresponding results will be presented elsewhere.

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