On a subfamily of starlike functions related to hyperbolic cosine function

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Abstract
We introduce and study a new Ma–Minda subclass of starlike functions \(S^*_\psi\), defined as
\[
S^*_\psi := \left\{ f \in A : \frac{zf'(z)}{f(z)} < \cosh \sqrt{z} =: \varphi(z), z \in \mathbb{D} \right\},
\]
associated with an analytic univalent function \(\cosh \sqrt{z}\), where we choose the branch of the square root function so that \(\cosh \sqrt{z} = 1 + z/2! + z^2/4! + \cdots\). We establish certain inclusion relations for \(S^*_\psi\) and deduce sharp \(S^*_\psi\)-radii for certain subclasses of analytic functions.

Keywords Univalent functions · Starlike functions · Radius problems · Hyperbolic Cosine function · Subordination

Mathematics Subject Classification 30C45 · 30C80

1 Introduction
Let \(A_n\) be the class of all analytic functions defined on the open unit disc \(\mathbb{D} := \{z \in \mathbb{C} : |z|<1\}\), with Taylor series representation of the form \(f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\). Let \(A := A_1\). Assume \(S \subset A\) as the class of univalent functions. If \(f(z)\) and \(g(z)\) are analytic functions in \(\mathbb{D}\), then \(f(z)\) is said to be subordinate to \(g(z)\) (\(f \prec g\)), if there exists a self-map \(w(z)\) on \(\mathbb{D}\) such that \(w(0) = 0\) and \(f(z) = g(w(z))\). For instance, if \(g(z)\) is a univalent function in \(\mathbb{D}\), then \(f \prec g\) if

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and only if \( f(0) = g(0) \) and \( f(D) \subset g(D) \). In 1992, Ma and Minda [12] investigated
the following subclasses of \( \mathcal{A} \) using the notion of subordination

\[
\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \phi(z), z \in D \right\}
\]

and

\[
\mathcal{C}(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \phi(z), z \in D \right\}.
\]

In the above defined classes, the expressions \( zf'(z)/f(z) \) and \( 1 + zf''(z)/f'(z) \) are
subordinate to an analytic univalent function \( \phi(z) \) such that \( \phi'(0) > 0 \) and
\( \text{Re}\phi(z) > 0 \) (\( z \in D \)). Furthermore, \( \phi(z) \) is symmetric about the real axis and starlike
with respect to \( \phi(0) = 1 \). Several authors have previously handled the Ma and Minda
classes for various choices of \( \phi(z) \), some are enlisted below (Table 1).

The classes \( \mathcal{S}^*_{e}, \mathcal{S}^*_r, \mathcal{S}^*(q_{\kappa}), \mathcal{S}^*[A,B] \) and \( \mathcal{SS}^*(\beta) \) were widely studied in
[3, 7, 13, 14, 24]. For instance, a number of sufficient conditions in terms of
coefficient estimates for the class \( \mathcal{SS}^*(\beta) \) are studied in [16] and references therein.

For the present study we examine the function \( g_{\sigma}(z) := \cosh \sigma \sqrt{z} \), where \( \sigma \in [-\pi/2, \pi/2] \) and we choose the branch of the square root function so that
\( \cosh \sigma \sqrt{z} = 1 + \sigma^2 z/2! + \sigma^4 z^2/4! + \cdots \). Note that \( g_{\sigma}(z) \) is an analytic univalent
function with \( \text{Re}g_{\sigma}(z) > 0 \) and maps \( D \) onto a convex region. Further it is symmetric
about real axis (i.e. \( g_{\sigma}(z) = g_{\sigma}(\bar{z}) \)) such that \( g_{\sigma}'(0) = \sigma^2/2 > 0 \). Consequently, \( g_{\sigma}(z) \)
is a Ma–Minda type function. In the recent years, cosine and cosine hyperbolic
functions have been investigated, see [2, 4]. Note that \( g(z) = \cosh \sqrt{z}, \phi_1(z) = \cos z \)
and \( \phi_2(z) = \cosh z \) have identical images, however \( \phi_1(z) (\phi_1'(0) < 0) \) and \( \phi_2(z) \) are
non-univalent functions in \( D \), whereas \( g(z) (g'(0) = 1/2 > 0) \) is univalent in \( D \).

Thus the geometry of \( g_{\sigma}(z) \) piqued our interest in formulating the following
definition, by means of subordination.

### Table 1  Ma–Minda starlike classes for special choices of \( \phi(z) \)

| Class \( \mathcal{S}^*(\phi) \) | \( \phi(z) \) | References |
| --- | --- | --- |
| \( \mathcal{S}^*_{e} \) | \( e^z \) | Mendiratta et al. [14] |
| \( \mathcal{S}^*_r \) | \( \sqrt{1 + z} \) | Sokol et al. [26] |
| \( \mathcal{S}^*_q \) | \( z + \sqrt{1 + z^2} \) | Raina et al. [21] |
| \( \mathcal{S}^*_{s} \) | \((1 + sz)^2, -1 \leq s \leq 1\) | Masih et al. [13] |
| \( \mathcal{S}^*(q_{\kappa}) \) | \( \sqrt{1 + \kappa z}, 0 < \kappa \leq 1 \) | Sokol et al. [3] |
| \( \mathcal{SS}^*(\beta) \) | \((1 + z)/(1 - z)^{\beta}, 0 < \beta \leq 1\) | Stankiewicz [27] |
| \( \mathcal{S}^*[A,B] \) | \((1 + Az)/(1 + Bz) \) | Janowski [7] |
| \( \mathcal{S}^*(\beta) \) | \((1 + (1 - 2\beta)z)/(1 - z), 0 \leq \beta < 1\) | Robertson [23] |
Definition 1 Let \( S_{\varepsilon}^* \) be the class of normalized starlike functions, defined as follows:

\[
S_{\varepsilon}^* := \left\{ f \in \mathcal{A} : \frac{zf''(z)}{f'(z)} < \varphi_{\varepsilon}(z) := \cosh \sigma \sqrt{z}, z \in \mathbb{D} \right\} \quad (\sigma \in [-\pi/2, \pi/2] - \{0\}),
\]

where we choose the branch of the square root function so that

\[
cosh \sigma \sqrt{z} = 1 + \frac{\sigma^2 z}{2!} + \frac{\sigma^4 z^2}{4!} + \frac{\sigma^6 z^3}{6!} + \cdots.
\]

The conformal mapping \( \varphi_{\varepsilon} : \mathbb{D} \to \mathbb{C} \), maps the unit disc \( \mathbb{D} \) onto the region

\[
\Omega_{\varphi_{\varepsilon}} := \{ u \in \mathbb{C} : |\log(u + \sqrt{u^2 - 1})|^2 < \sigma^2 \} \quad (\sigma \in [-\pi/2, \pi/2] - \{0\}),
\]

defined on the principle branch of logarithm and square root functions. For each \( \sigma \leq \varphi \), observe that \( \varphi_{\varepsilon}(\mathbb{D}) \subset \varphi_{\varepsilon}(\mathbb{D}) \). Moreover, for each circle \( |z| = r < 1 \),

\[
\begin{align*}
\min_{|z|=r} \text{Re} \varphi_{\varepsilon}(z) &= \min_{|z|=r} |\varphi_{\varepsilon}(z)| = \varphi_{\varepsilon}(\sqrt{-r}) \\
\max_{|z|=r} \text{Re} \varphi_{\varepsilon}(z) &= \max_{|z|=r} |\varphi_{\varepsilon}(z)| = \varphi_{\varepsilon}(\sqrt{r}).
\end{align*}
\]

(1)

Assume \( \varphi_1(z) = \varphi(z) \), therefore we have \( S_{\varepsilon}^* = S_{\varepsilon}^* \). In the present investigation we shall restrict our major workings to a subclass of starlike functions, namely \( S_{\varepsilon}^* \), and deduce radii constants along with some inclusion relations. In terms of integral representation, we have \( f \in S_{\varepsilon}^* \) if and only if

\[
f(z) = z \exp \left( \int_0^z \frac{\hat{\varphi}(t) - 1}{t} \, dt \right)
\]

(2)

where \( \hat{\varphi}(z) < \varphi(z) \). Note that if \( \psi_{\varepsilon}(z) = 1 + z^3/3 + z^2/18 \) and \( \phi_{\varepsilon}(z) = 1 + \sin(z/3) \), then evidently \( \psi_{\varepsilon}(z) \) and \( \phi_{\varepsilon}(z) \) are subordinate to \( \varphi(z) \), so the corresponding functions

\[
f_1(z) = z \exp \left( \frac{z^3 + z^2}{36} \right) \quad \text{and} \quad f_2(z) = ze^{Si(z)}, \text{where} \quad Si(z) = \int_0^z \frac{\sin t}{t} \, dt
\]

lie in \( S_{\varepsilon}^* \). Now using the representation in (2), we obtain different functions, those work as extremal functions for various results. For instance, \( \varphi_{\varepsilon n} \in \mathcal{A} \ (n = 2, 3, 4, \ldots) \), defined as
\[ \varphi_{\varphi}(z) = z \exp \left( \frac{z}{\int_0^z \frac{q(t^{n-1}) - 1}{t} \, dt} \right) = z + \frac{z^n}{2(n-1)} + \frac{z^{2(n-1)}}{48(n-1)} + \cdots, \quad (3) \]

belongs to \( S^*_\varphi \). We denote \( \varphi_{\varphi} := \varphi_{\varphi} \). For completeness of our class \( S^*_\varphi \), we give below a remark using the results of [12, 15].

**Remark 1** For \( f \in S^*_\varphi \) and \( \varphi_{\varphi}(z) \) be as defined in (3), then for \( |z| = r_0 < 1 \), we have

(i) \(-\varphi_{\varphi}(-r_0) \leq |f(z)| \leq \varphi_{\varphi}(r_0) \) (Growth Theorem).

(ii) \( \varphi_{\varphi}'(-r_0) \leq |f'(z)| \leq \varphi_{\varphi}'(r_0) \) (Distortion Theorem).

(iii) \( |\arg(f(z)/z)| \leq \max_{|z|=r_0} \arg(\varphi_{\varphi}(z)/z) \) (Rotation Theorem).

Equality for (i)–(iii) holds for some \( z_0 \neq 0 \) if and only if \( f(z) \) is a rotation of \( \varphi_{\varphi}(z) \). In fact if \( f \in S^*_\varphi \) then either \( f \) is a rotation of \( \varphi_{\varphi}(z) \) or \( f(\mathbb{D}) \supset \{v : |v| \leq -\varphi_{\varphi}(-1) \approx 0.619 \ldots \} \).

Further, from the results in [15] for each \( f \in S^*_\varphi \),

(i) \( |a_2| \leq 1/2 \),

(ii) \( |a_3| \leq 1/4 \),

(iii) \( |a_4| \leq 1/6 \)

and (iv) for any complex constant \( \mu \), \( |a_3 - \mu a_2^2| \leq \frac{1}{4} \max\{1, |\mu - 7/12|\} \).

These estimates are sharp. Equality in (i) holds for the function \( \varphi_{\varphi}(z) \) and \( \tilde{f}(z) = z + z^3/4 \) is an extremal function for (ii) and (iv).

## 2 Properties of hyperbolic cosine function

We begin with a Lemma which demonstrates a maximal disc centered at a point \((c, 0)\) on the real line, that can be subscribed within \( g_\sigma(\mathbb{D}) \).

**Lemma 1** Suppose \( \sigma \neq 0 \), then \( g_\sigma(z) \) satisfies the following inclusion

\[ \{u \in \mathbb{C} : |u - c| < r_{ac} \} \subset g_\sigma(\mathbb{D}) =: \Omega_{g_\sigma}(\sigma) \quad (-\pi/2 \leq \sigma \leq \pi/2), \]

where

\[ r_{ac} = \begin{cases} c - \cos \sigma, & \cos \sigma < c \leq (\cosh \sigma + \cos \sigma)/2 \\ \cosh \sigma - c, & (\cosh \sigma + \cos \sigma)/2 \leq c < \cosh \sigma. \end{cases} \]

**Proof** Let \( \Gamma := g_\sigma(e^{it}), -\pi \leq t \leq \pi \) be the boundary curve of the function \( g_\sigma(z) \). Due to symmetricity of the curve \( \Gamma \) about real-axis, it is enough to consider \( 0 \leq t \leq \pi \). Define a function \( G_c(\tau) \) as follows:

\[ G_c(\tau) := (c - \cosh(\sigma(\cos \tau))) \cos(\sigma(\sin \tau)))^2 + \sinh^2(\sigma(\cos \tau)) \sin^2(\sigma(\sin \tau)), \]

where \( \tau = t/2 \). Observe that \( G_c(\tau) \) (see Fig. 1 for different values of \( c \)) is the square of the distance from point \((c, 0)\) to \( \Gamma \). Now we study the following cases:
Case 1: For \( \cos \sigma < c \leq 1 \), \( G_c(\tau) \) is monotonically decreasing on \([0, \pi/2]\), then

\[
   r_{ac} = \min_{\tau \in [0, \pi/2]} \sqrt{G_c(\tau)} = \sqrt{G_c(\pi/2)} = c - \cos \sigma.
\]

Case 2: When \( 1 \leq c \leq \sigma_0 \), where \( \sigma_0 < (\cosh \sigma + \cos \sigma)/2 \) is a point at which \( G_c(\tau) \) changes its character i.e \( G_c(\tau) \) is monotonically decreasing for \( 1 \leq c \leq \sigma_0 \) and has three critical points \( \{0, \tau_c, \pi/2\} \) for \( \sigma_0 < c \leq (\cosh \sigma + \cos \sigma)/2 \), where \( \tau_c \in (0, \pi/2) \) is the only root of the equation

\[
   2c \tan \tau \cos(\sigma \sin \tau) \sinh(\sigma \cos \tau) + 2 \sin(\sigma \sin \tau) \cosh(\sigma \cos \tau) \\
   = \sin(2\sigma \sin \tau) + \tan \tau \sinh(2\sigma \cos \tau).
\]

(4)

Note that \( \tau_c < \tau_c \) whenever \( c < \hat{c} \). Further

\[ G_c(0) - G_c(\pi/2) = (\cosh \sigma - \cos \sigma)(\cos \sigma + \cosh \sigma - 2c) \geq 0. \]

Therefore this yields

\[
   r_{ac} = \min_{\tau \in [0, \pi/2]} \left\{ \sqrt{G_c(0)}, \sqrt{G_c(\tau_c)}, \sqrt{G_c(\pi/2)} \right\} = \sqrt{G_c(\pi/2)} = c - \cos \sigma.
\]

Case 3: For \( (\cosh \sigma + \cos \sigma)/2 \leq c \leq \sigma_1 \), where \( \sigma_1 < \cosh \sigma \) is a point at which \( G_c(\tau) \) changes its character i.e \( G_c(\tau) \) has three critical points \( \{0, \tau_c, \pi/2\} \), where \( \tau_c \in (0, \pi/2) \) is the only root of Eq. (4) and \( G_c(\tau) \) is an increasing function for \( \sigma_1 < \sigma < \cosh \sigma \). Infact \( G_c(0) \leq G_c(\pi/2) \). Therefore

\[
   r_{ac} = \min_{\tau \in [0, \pi/2]} \left\{ \sqrt{G_c(0)}, \sqrt{G_c(\tau_c)}, \sqrt{G_c(\pi/2)} \right\} = \sqrt{G_c(0)} = \cosh \sigma - c.
\]

Hence the result follows. \( \square \)

Inclusion results in Lemma 2, follows from equation (1) and Lemma 1.

Lemma 2 For the region \( \Omega_{q_\sigma} := q_\sigma(\mathbb{D}) \), following inclusion relations hold:

(i) \( \{u : |u - (\cosh \sigma + \cos \sigma)/2| < (\cosh \sigma - \cos \sigma)/2\} \subset \Omega_{q_\sigma}. \)

(ii) \( \Omega_{q_\sigma} \subset \{u : \cos \sigma < \Re u < \cosh \sigma\} \) and \( \Omega_{q_\sigma} \subset \{u : \cos \sigma < |u| < \cosh \sigma\}. \)
(iii) \( \Omega_{\sigma} \subseteq \{ u : |\operatorname{Im} u| < l \} \) and \( \Omega_{\sigma} \subseteq \{ u : |u - (\cosh \sigma + \cos \sigma)/2| < l \} \), where \( l = |\operatorname{Im}(\cosh(\sigma e^{i t_0}/2))| \), and \( t_0 \) is the root of the equation

\[
\cos \sigma + \cosh \sigma - 2 \cos(\sigma \sin(t/2)) \cosh(\sigma \cos(t/2)) = 0.
\]

For \( \sigma = 1 \), Lemma 1 leads to the following result for the region \( \Omega_{\sigma} =: \Omega_0 \).

**Theorem 1** The region \( \Omega_0 := \mathcal{Q}(\mathbb{D}) \supset \{ u \in \mathbb{C} : |u - c| < r_c \} \) where

\[
r_c = \begin{cases} 
\cosh 1 - c, & \text{if } 1 < c \leq (\cosh 1 + \cos 1)/2 \\
\cosh 1 - c, & \text{if } (\cosh 1 + \cos 1)/2 \leq c < \cosh 1.
\end{cases}
\]

**Remark 2** Theorem 1 ensures that \( D_c := |u - c| < r_c \), is the maximal disc subscribed in \( \mathcal{Q}(\mathbb{D}) \), when \( c = (\cosh 1 + \cos 1)/2 \) and \( r_c = (\cosh 1 - \cos 1)/2 \). Thus \( D_c \subseteq \mathcal{Q}(\mathbb{D}) \).

For all the subsequent results, we shall assume \( c_0 := \cos 1 \) and \( c_1 := \cosh 1 \).

**Lemma 3** For the region \( \Omega_0 := \mathcal{Q}(\mathbb{D}) \), we have the following inclusion relations:

(i) \( \{ u : |u - (c_0 + c_1)/2| < (c_1 - c_0)/2 \} \subseteq \Omega_0 \).

(ii) \( \Omega_0 \subseteq \{ u : |\arg u| < m \} \), where \( m \approx 0.506053 \approx (0.322163) \pi/2 \approx 28.9947^\circ \).

(iii) \( \Omega_0 \subseteq \{ u : c_0 < \Re u < c_1 \} \) and \( \Omega_0 \subseteq \{ u : c_0 < |u| < c_1 \} \).

(iv) \( \Omega_0 \subseteq \{ u : |\operatorname{Im} u| < l \} \) and \( \Omega_0 \subseteq \{ u : |u - (c_0 + c_1)/2| < l \} \), where \( l = |\operatorname{Im}(\cosh(e^{i t_0}/2))| \) and \( t_0 \) is the solution of the equation

\[
c_0 + c_1 - 2 \cos(\sin(t/2)) \cosh(\cos(t/2)) = 0.
\]

**Proof** We can obtain (i), (iii)–(iv) from equations in (1), Remark 2 and Lemma 2 (for \( \sigma = 1 \)). For part (ii) let \( \Gamma := \partial(\mathcal{Q}(z)) = \mathcal{Q}(e^{it}) \), \(-\pi \leq t \leq \pi\), represents the boundary curve of \( \mathcal{Q}(z) \). Assume that

\[
\Re e^{it} = \cos(\sin(t/2)) \cosh(\cos(t/2)) =: X(t)
\]

and

\[
\Im e^{it} = \sin(\sin(t/2)) \sinh(\cos(t/2)) =: Y(t).
\]

Consider
\[ |\log f(z)| < \max_{|z|=1} |\log f(z)| = \max_{t \in [-\pi, \pi]} |\arg f(e^{it})| = \max_{t \in [-\pi, \pi]} \tan^{-1}(Y(t)/X(t)) = \max_{t \in [-\pi, \pi]} \tan^{-1}(\tan(\sin(t/2)) \tanh(\cos(t/2))) =: m(t) \]

Observe that \( \tan^{-1} x \) is a monotonically increasing real valued function. Therefore it is enough to obtain the maximum of \( m(t) \). The roots of
\[
m'(t) = 0.5(\cos(t/2) \tanh(\cos(t/2)) \sec^2(\sin(t/2)) - \sin(\sin(t/2)) \text{sech}^2(\cos(t/2))) = 0
\]
are \( t_1 \approx -1.91672 \) and \( t_2 \approx 1.91672 \). As \( t_1 < t_2 \), therefore maximum of \( m(t) \) is attained at \( t_2 \). Hence the inclusion in (ii) follows.

In Theorem 2 and Corollary 1, we prove inclusion results pertaining to various classes along with the classes \( ST_p(\gamma) \), \( S_{hpl}(s) \), \( k-ST \) and \( M(\beta) \) [1, 8, 9, 28] defined below:

\[
M(\beta) := \left\{ f \in A : \frac{zf'(z)}{f(z)} < \frac{1 + (2\beta - 1)z}{1 + z}, \beta > 1 \right\},
\]

\[
ST_p(\gamma) := \left\{ f \in A : \text{Re} \frac{zf'(z)}{f(z)} + \gamma > \left| \frac{zf'(z)}{f(z)} - \gamma \right|, \gamma > 0 \right\},
\]

\[
S_{hpl}(s) := \left\{ f \in A : \frac{zf'(z)}{f(z)} < (1 - z)^{-s} = e^{-s \log(1-z)}, 0 < s \leq 1 \right\},
\]

\[
k-ST := \left\{ f \in A : \text{Re} \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, k \geq 0 \right\}.
\]

**Theorem 2** Let \( f \in S_{\theta_{\sigma}}^* \) then for each \( \sigma \in [-\pi/2, \pi/2] \) \( \setminus \{0\} \), following inclusions hold:

(i) \( S_{\theta_{\sigma}}^* \subset S^*(\zeta) \), where \( \zeta = \cos \sigma \).

(ii) \( S_{\theta_{\sigma}}^* \subset M(\beta) \), where \( \beta = \cosh \sigma \).

(iii) \( S_{\kappa}^* \subset S_{\theta_{\sigma}}^* \), whenever \( \kappa \leq 1 - \cos^2 \sigma \).

(iv) \( k-ST \subset S_{\theta_{\sigma}}^* \), whenever \( k \geq \cosh \sigma/(\cosh \sigma - 1) \).

(v) \( S_{\theta_{\sigma}}^* \subset S_{hpl}^*(s) \), whenever \( \log(\sec \sigma)/\log 2 \leq s \leq 1 \), \( \sigma \in [-\pi/3, \pi/3] \setminus \{0\} \).

(vi) \( S_{\theta_{\sigma}}^* \subset S_L^*(s) \), whenever \( 1 - \sqrt{\cos \sigma} \leq s \leq \frac{1}{\sqrt{2}} \).

**Proof** Observe that, in Eq. (1), when \( r \) tends to 1\(^{-}\), sharp bounds on real part and modulus of \( \varrho_{\sigma}(z) \) are obtained. Consequently, due to Lemma 2 the inclusions in (i) and (ii) are true for the class \( S_{\theta_{\sigma}}^* \). We know that \( q_{\kappa}(z) = \sqrt{1 + \kappa z} \) where \( 0 < \kappa \leq 1 \), is associated with the region \( |u^2 - 1| < \kappa \). Therefore part (iii) can be easily established as \( q_{\kappa}(\mathbb{D}) \) lies in \( \varOmega_{\theta_{\sigma}} \), if and only if, \( \sqrt{1 - \kappa} \geq \cosh \sigma \), which implies \( \kappa \leq 1 - \cos^2 \sigma \). For part (iv), let \( \Gamma_k = \{ u \in \mathbb{C} : \text{Re} u > k |u - 1| \} \), where \( k \geq 0 \). When \( k > 1 \), the set \( \Gamma_k \) represents the interior of an ellipse.
where \( x_1 = k^2/(k^2 - 1) \), \( a_1 = k/(k^2 - 1) \) and \( b_1 = 1/\sqrt{k^2 - 1} \). For \( \gamma_k \) to lie in \( \Omega_{q_x} \) we must have \( x_1 + a_1 \leq \cosh \sigma \), which gives a sufficient condition for \( \gamma_k \) to lie in \( \Omega_{q_x} \), this leads us to the required condition. From [8] we know that \( \text{Re}(1 - z)^{-s} > 2^{-s} \). Therefore for (v) to hold true \( 2^{-s} \leq \cos \sigma \), which gives \( \log(\sec \sigma) / \log 2 \leq s \leq 1 \), provided \(-\pi/3 \leq \sigma \leq \pi/3 \). Furthermore, it was demonstrated in [13], that \( L_S(\mathbb{D}) \supset \{ u : |u - 1| < 1 - (1 - s)^2 \} \), where \( 0 < s \leq 1/\sqrt{2} \). Thus for (vi) to hold true we must have \( 1 - (1 - s)^2 \geq 1 - \cos \sigma \). Thus \( S_{q_x}^* \subset S_L^*(s) \) for each \( s \geq 1 - \sqrt{\cos \sigma} \). ☐

In the following Corollary we prove inclusion results for the class \( S_{q_x}^* \).

**Corollary 1** For each function \( f \in S_{q_x}^* \) the following inclusions hold:

- (i) \( S_{q_x}^* \subset S^*(c_0) \).
- (ii) \( S_{q_x}^* \subset M(c_1) \).
- (iii) \( S_{q_x}^* \subset SS^*(\beta) \), where \( \beta \approx 0.322163 \).
- (iv) \( S_{q_x}^* \subset S_{q_x}^* \), whenever \( \kappa \leq 1 - c_0^2 \).
- (v) \( k - ST \subset S_{q_x}^* \), whenever \( k \geq c_1/(c_1 - 1) \).
- (vi) \( S_{q_x}^* \subset S_{hp}(s) \), whenever \(- \log c_0 / \log 2 \leq s \leq 1 \).

**Legend**

- \( g_1 : g(z) = \cosh \sqrt{z} \)
- \( g_2 : \text{Re} u = c_0 \)
- \( g_3 : |\arg u| = \beta \pi / 2 \)
- \( \arg \lambda_1 = - \arg \lambda_2 = \beta \pi / 2 \)
- \( \beta \approx 0.322163 \)
- \( g_4 : \text{Re} u = c_1 \)
- \( g_5 : \sqrt{1 + (1 - c_0^2)}z \)
- \( g_6 : \text{Re} u = \frac{c_1}{c_1 - 1} |u - 1| \)
- \( g_7 : \frac{(\text{Re} u - \frac{c_0 + c_1}{2})^2}{(c_1 - c_0) \frac{c_1^2}{2}} + \frac{(\text{Im} u)^2}{(c_2)^2} \),
- \( c_2 = 0.65 \)
- \( g_8 : \text{Re} u + \gamma = |u - \gamma| \),
- \( \gamma \approx 0.0654238 \)
- \( g_9 : \frac{1}{(1 - z)^s}, s_0 = \frac{\log c_0^2}{\log 2} \)

**Fig. 2** Inclusion graphs in context of Corollary 1 associated with \( g(z) \).
(vii) $S_\sigma^* \subset S_L^*(s)$, whenever $1 - \sqrt{c_0} s \leq \frac{1}{\sqrt{z}}$.
(viii) $S_\sigma^* \subset ST_p(\gamma)$, whenever $\gamma \geq \gamma_0 \approx 0.0654238$.

**Proof** Clearly parts (i)–(iii) and (iv)–(vii) can be obtained as a result of Theorem 2 for $\sigma = 1$. Part (iii) is true due to Lemma 3, for the class $S_\sigma^*$ (see Fig. 2). For (viii) in order to show $S_\sigma^* \subset ST_p(\gamma)$, we must have $|u - \gamma| - Reu < \gamma$, where $u(z) = \cosh \sqrt{z}$. For $z = e^{i\tau}$ we have

$$H(\tau) := \frac{\sin^2(\sin \tau) \sinh^2(\cos \tau)}{4 \cos(\sin \tau) \cosh(\cos \tau)} < \gamma,$$

where $\tau = t/2$. Clearly $H'(\tau)$ vanishes on $\{0, \tilde{\tau}, \pi/2\}$, with $\tau = \tilde{\tau} \approx 0.832934$ as the only root of the equation

$$\tan(\sin \tau) \tanh(\cos \tau)((\cos \tau(\cos(2 \sin \tau) + 3) \sin(\cos \tau) \sec(\sin \tau)) - \sin \tau \sin(\sin \tau)(\cosh(2 \cos \tau) + 3) \sech(\cos \tau)) = 0$$

in $(0, \pi/2)$. Therefore $\max_{\tau \in [0, \pi/2]} H(\tau) = H(\tilde{\tau}) \approx 0.0654238$. Observe that $ST_p(\gamma_1) \subset ST_p(\gamma_2)$ whenever $\gamma_1 < \gamma_2$. This leads to the required inclusion relation. \[ \square \]

**Remark 3** Figure 2 displays various inclusion relations related to the region $\Omega_\sigma := \Omega_{\sigma_1}$. A vertical ellipse enclosing the region $\Omega_\sigma$ is $(x - x_2)^2/a_2^2 + y^2/b_2^2 = 1$, where $x_2 = c_0/2$, $a_2 = c_1/2$ and $c_2 \geq \max \text{Im}q(z)$. For visual purposes we illustrate this ellipse $(g_7)$ for $c_2 = 0.65$. Figure 2 depicts the sharpness of inclusion results in Corollary 1.

Let $\mathcal{P}_n(z)$ denote the class of functions $p(z)$ of the type $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ such that $Re(p(z)) > \alpha$ $(0 \leq \alpha < 1)$. Clearly the class $\mathcal{P}_n(z) \subset \mathcal{P}_n$ and assume $\mathcal{P}_n := \mathcal{P}_n(0)$. If a function $p(z)$ of the form $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ satisfies $p(z) \prec (1 + Az)/(1 + Bz)$, for $A \neq B$ and $|B| \leq 1$, then $p \in \mathcal{P}_n[A, B]$. We state a few lemmas in connection with these classes.

**Lemma 4** [22] If $p \in \mathcal{P}_n[A, B]$, then for $|z| = r$

$$|p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}}| \leq |A - B|r^n \frac{1}{1 - B^2 r^{2n}}.$$  

Particularly, if $p \in \mathcal{P}_n(\alpha)$, then

$$|p(z) - \frac{1 + (1 - 2\alpha) r^{2n}}{1 - r^{2n}}| \leq \frac{2(1 - \alpha)r^n}{1 - r^{2n}}.$$
Lemma 5  \[25\] If \( p \in \mathcal{P}_n'(z) \), then for \(|z| = r\)

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - x nr^2)}{(1 - r^2)(1 + (1 - 2x)r^2)}.
\]

Theorem 3 Let \( p(z) = (1 + Az)/(1 + Bz) \), where \(-1 < B < A \leq 1\), then \( p(z) \prec \cosh \sqrt{z} \) if and only if

\[
A \leq \begin{cases} 
1 - (1 - B)c_0 & \text{if } 2(1 - AB) \leq (c_0 + c_1)(1 - B^2) \\
(1 + B)c_1 - 1 & \text{if } 2(1 - AB) \geq (c_0 + c_1)(1 - B^2).
\end{cases}
\]

(6)

Proof Lemma 4 shows that the \( p(z) = (1 + Az)/(1 + Bz) \), maps \( \mathbb{D} \) onto the disc

\[
\left| p(z) - \frac{1 - AB}{1 - B^2} \right| \leq \frac{A - B}{1 - B^2}, \quad -1 < B < A \leq 1.
\]

By Theorem 1, \( p(z) \prec \cosh \sqrt{z} \) if and only if the above disc lies within \( \Omega_q \). Conditions in (6) gives \((1 + A) \leq (1 + B)c_1\), provided \(2(1 - AB) \geq (c_0 + c_1)(1 - B^2)\) holds. In fact \((A - B)/(1 - B^2) \leq c_1 - (1 - AB)/(1 - B^2)\) leads to \((A - B)/(1 - B^2) \leq c_1 - c\) provided \(2c \geq c_0 + c_1\) where \(c = (1 - AB)/(1 - B^2)\). Also from (6), \((1 - A) \geq (1 - B)c_0\) whenever \(2(1 - AB) \leq (c_0 + c_1)(1 - B^2)\). Equivalently, \((A - B)/(1 - B^2) \leq c - c_0\) whenever \(2c \leq c_0 + c_1\). Thus \(p(z)\) lies in \(|u - c| < r_c\), where \(r_c\) is given by (5).

Corollary 2 If conditions on \(A, B\) are as given in Theorem 3, then \(S^*_n[A, B] \subset S^*_q\).

3 Radius problems

Radius problems have been an active area of research in geometric function theory. Some of the pioneering work in this direction have been discussed by several authors, see [3, 14, 22, 24]. For further development on radius problems of analytic functions, readers may refer to [5, 10, 17, 19]. Motivated by the aforestated work, we derive radius results for the following classes

\[
S^*_n(q) = \left\{ f \in \mathcal{A}_n : \frac{zf'(z)}{f(z)} < \cosh \sqrt{z} =: q(z) \right\},
\]

\[
S^*_n[A, B] = \left\{ f \in \mathcal{A}_n : \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \right\}
\]

and
On a subfamily of starlike functions... 2053

\[ M_n(\beta) = \left\{ f \in A_n : \frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\beta)z}{1 - z}, \beta > 1 \right\}. \]

In the sequel we apply lemmas stated in Sect. 2, to obtain sharp \( S^*_n(\rho) \)-radius, \( S^*_n[A, B] \)-radius and \( M_n(\beta) \)-radius for the class \( S^*_\rho \). The following theorem is obtained from Lemma 3 and equations in (1).

**Theorem 4** The class \( S^*_\rho \subset M(\beta) \) for \( |z| < r_\beta \), where

\[ r_\beta = \left\{ \begin{array}{ll} r(\beta), & 1 < \beta < c_1 \\ 1, & \beta \geq c_1. \end{array} \right. \]

and \( r(\beta) \in (0, 1) \) is the smallest root of the equation \( \cosh \sqrt{r} = \beta \). Equality holds when \( f(z) = \phi_\rho(z) \).

**Theorem 5** Suppose \( f \in S^*_\rho \), then \( f(z) \) is starlike of order \( \zeta \), in \( |z| < r_\zeta \), where \( r_\zeta < 1 \) is the least positive root of the equation \( \cos \sqrt{r} = \zeta \). This radius result is sharp.

**Proof** As \( f \in S^*_\rho \), then we have \( zf'(z) = f(z) \cosh \sqrt{w(z)} \), where \( w(z) \) is a Schwarz function with \( w(0) = 0 \) such that for \( -\pi \leq t \leq \pi \), \( w(z) = \text{Re}^t \). For each \( R = |w(z)| \leq |z| = r < 1 \), we have \( \cos \sqrt{R} \geq \cos \sqrt{r} \), and as a result of equations in (1)

\[ \text{Re} \frac{zf'(z)}{f(z)} \geq \min_{|z|=r} \text{Reg}(w(z)) = \cos \sqrt{r} \geq \zeta. \]

If \( s(r, \zeta) := \cos \sqrt{r} - \zeta \), then there exist \( r_{\zeta_0} < r_{\zeta_1} \) such that \( s(r_{\zeta_0}, \zeta) > 0 \) and \( s(r_{\zeta_1}, \zeta) < 0 \), holds. Thus a least positive root \( r_\zeta \) for the equation \( s(r, \zeta) = 0 \), will serve the purpose. In particular, at \( z_0 = -r \), we have \( \text{Re}(z_0f'(z_0)/f(z_0)) = \cos \sqrt{r} = \zeta \), then function \( f(z) = \phi_\rho(z) \) is the extremal function.

On replacing \( \phi(z) = (1 + (1 - 2x)z)/(1 - z) \) in the definition \( C(\phi) \) we get the well-known class of convex functions of order \( \alpha \) \( (0 \leq \alpha < 1) \), denoted by \( C(\alpha) \). For \( \alpha = 0 \), it reduces to the well-known class of convex functions \( C \). In Theorem 6, we establish radius of convexity of order \( \alpha \) for the class \( S^*_\rho \).

**Theorem 6** Let \( f \in S^*_\rho \), then \( f \in C(\alpha) \), where \( \alpha \in [0, 1] \), provided \( |z| \leq r_0 \), where \( r_0 \in [0, 1) \) is the least positive root of the equation

\[ 2(1 - r^2) \cos \sqrt{r} - \sqrt{r} \tan \sqrt{r} = \alpha. \]

**Proof** As \( f \in S^*_\rho \), there exists a Schwarz function \( w(z) \) such that \( w(0) = 0 \) and

\[ \frac{zf'(z)}{f(z)} = \cosh \sqrt{w(z)}. \] (7)

On logarithmically differentiating (7) and applying triangle inequality, we deduce
\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \text{Re} \left( \frac{zf'(z)}{f(z)} + \frac{zw'(z) \tanh \sqrt{w(z)}}{2\sqrt{w(z)}} \right) \]

\[ \geq \cos \sqrt{r} - |z| |w'(z)| \left| \frac{\tanh \sqrt{w(z)}}{2\sqrt{w(z)}} \right| \quad (|z| = r < 1). \tag{8} \]

Further Schwarz Pick Lemma, yields

\[ -|z| |w'(z)| \left| \frac{\tanh \sqrt{w(z)}}{\sqrt{w(z)}} \right| \geq - |z| \frac{1 - |w(z)|^2}{1 - |z|^2} \left| \frac{\tanh \sqrt{w(z)}}{\sqrt{w(z)}} \right|. \tag{9} \]

Assume \( w(z) = Re^{it}, t \in [-\pi, \pi] \) where \( R \leq r \), then inequality (9) yields

\[ \text{Re} \left( \frac{zw'(z) \tanh \sqrt{w(z)}}{2\sqrt{w(z)}} \right) \leq \sqrt{r} \tan \sqrt{r} \frac{2(1 - r^2)}{2(1 - r^2)}. \tag{10} \]

Thus from inequalities (8) and (10) we conclude that

\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \cos \sqrt{r} - \frac{\sqrt{r} \tan \sqrt{r}}{2(1 - r^2)}. \]

Hence the least positive root of the equation of \( 2(1 - r^2) \cos \sqrt{r} - \sqrt{r} \tan \sqrt{r} = \pi \) will serve the purpose. \( \square \)

**Theorem 7** For \(-1 \leq B < A \leq 1\), suppose \( f \in S_n^*[A, B] \), then the sharp \( S_n^*(q) \)–radius is given by

(i) \( \mathcal{R}_{S_n^*(q)}(S_n^*[A, B]) = \min \{ 1; (1 - c_0)/(A - Bc_0)^{1/n} \} =: \mathcal{R}_0 \), where \( 0 \leq B < A \leq 1 \).

(ii) \( \mathcal{R}_{S_n^*(q)}(S_n^*[A, B]) = \begin{cases} \mathcal{R}_0, & \mathcal{R}_0 \leq \mathcal{R}_1 \\ \mathcal{R}_2, & \mathcal{R}_0 > \mathcal{R}_1 \end{cases} \), where \(-1 \leq B < 0 < A \leq 1\).

where

\[ \mathcal{R}_1 = \left( \frac{c_0 - 2}{B(c_0 B - 2A)} \right)^{1/2n}, \quad \mathcal{R}_2 = \min \left\{ 1; \left( \frac{c_1 - 1}{A - Bc_1} \right)^{1/n} \right\}. \]

**Proof** As \( f \in S_n^*[A, B] \), then \( p(z) = zf'(z)/f(z) \) lies in the disc \(|p(z) - c| < R\), where

\[ c = \frac{1 - AB r^{2n}}{1 - B^2 r^{2n}} \quad \text{and} \quad R = \frac{(A - B)r^{2n}}{1 - B^2 r^{2n}}. \]

If \( B \geq 0 \), then \( c \leq 1 \). For \( f(z) \) to lie in \( S_n^*(q) \), Theorem 1 and Lemma 4 yields
\[
\frac{(A - B)r^n}{1 - B^2r^{2n}} \leq \frac{1 - AB^2r^{2n}}{1 - B^2r^{2n}} - c_0.
\]

The above inequality gives \( r \leq R_0 \). Equality here holds for \( \tilde{f}(z) \) of the form
\[
\tilde{f}(z) = \begin{cases}
(1 + Bz^n)^{(A-B)/nB}, & B \neq 0 \\
\exp(Az^n), & B = 0.
\end{cases}
\] (11)

Further, if \(-1 \leq B < 0 < A \leq 1 \) and \( R_0 \leq R_1 \), then \( c \leq (c_0 + c_1)/2 \) if and only if \( r \leq R_1 \). Therefore, for \( 0 \leq r \leq R_0 \), we deduce that \( c \leq (c_0 + c_1)/2 \). In fact, due to Theorem 1 for each \( f \in S_0(\phi) \), we have \( (A - B)r^n/(1 - B^2r^{2n}) \leq c - c_0 \), equivalently \( r \leq R_0 \). Furthermore, assume that \( R_0 > R_1 \). Then \( c \geq (c_0 + c_1)/2 \) if and only if \( r \geq R_1 \). In particular for \( r \geq R_0 \), we have \( c \geq (c_0 + c_1)/2 \). Thus by Theorem 1, for each \( f \in S_0(\phi) \), the inequality \( (A - B)r^n/(1 - B^2r^{2n}) \geq c_1 - c \) is equivalent to \( r \leq R_2 \). The function \( \tilde{f}(z) \) given in (11) works as the extremal function.

**Theorem 8** Let \( \beta > 1 \), then the sharp \( S_0(\phi) \)-radius for the class \( \mathcal{M}(\beta) \), is given by
\[
R_{S_0(\phi)}(\mathcal{M}(\beta)) = \left(\frac{1 - c_0}{2\beta - (1 + c_0)}\right)^{1/n}.
\]

**Proof** As \( f \in \mathcal{M}(\beta) \), then \( zf'(z)/f(z) < (1 + (1 - 2\beta)z)/(1 - z) \). Clearly, for each \( \beta > 1 \), \( (1 + (1 - 2\beta)r^{2n})/(1 - r^{2n}) \leq 1 \). Further by Lemma 4, we get
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(\beta - 1)r^n}{1 - r^{2n}}.
\]

On applying Theorem 1, we have
\[
\frac{2(\beta - 1)r^n}{1 - r^{2n}} \leq \frac{1 + (1 - 2\beta)r^{2n}}{1 - r^{2n}} - c_0
\]
or equivalently \( r^{2n}((1 - 2\beta) + c_0) - 2(\beta - 1)r^n + 1 - c_0 \geq 0 \), which gives \( r \leq R_{S_0(\phi)}(\mathcal{M}(\beta)) \). The required extremal function is \( \tilde{f}(z) = z/(1 - z)^{2(1-\beta)/n} \). \( \square \)

Recently, Lecko et al. [11] investigated the expressions \( \text{Re}(1 - z^2)f(z)/z > 0 \) and \( \text{Re}(1 - z)^2f(z)/z > 0 \), involving the starlike functions \( z/(1 - z^2) \) and \( z/(1 - z)^2 \). In 2019, Cho et al. [6] estimated radii constants for classes characterised by the ratio of two analytic functions \( f(z) \) and \( g(z) \) with certain conditions on \( g(z) \), namely \( \text{Reg}(z)/z > x \) for \( x = 0 \) or \( 1/2 \), such that \( \text{Re}(f(z)/g(z)) > 0 \). Motivated by these classes, here below we define some subclasses of \( \mathcal{A}_n \),
\[
\mathcal{F}_1(\beta) := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ and } \text{Re} \left( g(z) \right) > \beta, g \in \mathcal{A}_n \right\} \quad (\beta \in \{0, 1/2\})
\]

and

\[ Springer \]
\[ \mathcal{F}_2 := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ and } g \in \mathcal{A}_n \text{ is convex} \right\}. \]

**Definition 2** Let \(-1 \leq A \leq 1\) and \(g \in \mathcal{A}_n\), then for each \(n = 1, 2, \ldots\), \(\mathcal{F}_3 \subset \mathcal{A}_n\), be defined as:
\[ \mathcal{F}_3 := \left\{ f \in \mathcal{A}_n : \text{Re} \left( \frac{f(z)}{g(z)} > 0 \right) \text{ and } \text{Re} \left( \frac{(1 - z^n)^{(1+A)/n}}{z} g(z) > 0 \right) \right\}. \]

**Remark 4** The functions \(\tilde{f}(z) = z(1 + (1 - 2\beta)z^n)/(1 - z^n)\) defined on \(\mathbb{D}\) satisfy \(|(\tilde{f}(z)/\tilde{g}(z)) - 1| = |z^n| < 1\) and \(\text{Re}(\tilde{g}(z)/z = \text{Re}(1 + (1 - 2\beta)z^n)/(1 - z^n) > \beta\). Therefore \(\tilde{f} \in \mathcal{F}_1(\beta)\), where \(\beta \in \{0, 1/2\}\). If \(\tilde{f}(z) = z(1 + z^n)/(1 - z^n)^{1/n}\) and \(\tilde{g}(z) = z/(1 - z^n)^{1/n}\), then \(\tilde{f} \in \mathcal{F}_2\). Similarly when \(\tilde{f}(z) = z(1 + z^n)^{(1)+n} z^n)^{(1+A)/n}\) and \(\tilde{g}(z) = z(1 + z^n)/(1 - z^n)^{(1+A)/n}\), then \(\tilde{f} \in \mathcal{F}_3\). Therefore the class \(\mathcal{F}_3\) is non-empty.

**Theorem 9** The sharp \(S^*_n(q)\)-radii for the classes \(\mathcal{F}_1(0)\), \(\mathcal{F}_1(1/2)\) and \(\mathcal{F}_2\), are respectively given by

(i) \[ \mathcal{R}_{S_n^*(q)}(\mathcal{F}_1(0)) = \left( \frac{\sqrt{9n^2 - 4(c_0 - 1)(1 + n - c_0) - 3n}}{2(1 + n - c_0)} \right)^{1/n}. \]

(ii) \[ \mathcal{R}_{S_n^*(q)}(\mathcal{F}_1(1/2)) = \left( \frac{1 - c_0}{2n - (c_0 - 1)} \right)^{1/n}. \]

(iii) \[ \mathcal{R}_{S_n^*(q)}(\mathcal{F}_2) = \left( \frac{\sqrt{1 + n(n + 6) + 4c_0(c_0 - (1 + n)) - (1 + n)}}{2(n - c_0)} \right)^{1/n}. \]

**Proof** Assume \(f(z)/g(z) = p_1(z)\) and \(g(z)/z = p_2(z)\), where \(f(z)\) and \(g(z)\) are analytic functions in \(\mathbb{D}\).

(i) As \(f \in \mathcal{F}_1(0)\), then \(p_2 \in \mathcal{P}_n(0)\). We know that \(|p_1(z) - 1| < 1\) holds if \(\text{Re}(1/p_1(z)) > 1/2\) and vice-versa. Assume \(f(z) = z p_1(z) p_2(z)\). Now using the expressions of \(p_1(z)\), \(p_2(z)\) and by applying Theorem 1 and Lemma 5 we have
\[ \left| \frac{zf''(z)}{f(z)} - 1 \right| = \left| \frac{zp_2'(z)}{p_2(z)} - \frac{zp_1'(z)}{p_1(z)} \right| \leq \frac{(3 + r^m)mr^m}{1 - r^m} \leq 1 - c_0. \]

The above inequality leads to \(r^m(n + 1 - c_0) + 3mr^m - 1 + c_0 \leq 0\), provided \(r \leq \mathcal{R}_{S_n^*(q)}(\mathcal{F}_1(0))\). The functions \(\tilde{f}(z) = z(1 + z^n)/(1 - z^n)^{2}\) and \(\tilde{g}(z) = z(1 + z^n)/(1 - z^n)\) at \(z_0 = \mathcal{R}_{S_n^*(q)}(\mathcal{F}_1(0)) e^{\pi i/n}\) gives
\[
\frac{z_0 \tilde{f}'(z_0)}{\tilde{f}(z_0)} - 1 = \frac{(3 + z_0^2)n z_0^n}{1 - z_0^{2n}} = 1 - c_0.
\]

Thus \(\tilde{f}\) is the extremal function.

(ii) Let \(f \in \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_1}(1/2))\), then \(1/p_1, p_2 \in \mathcal{P}_n(1/2)\). Proceeding as in (i), on applying Theorem 1 and Lemma 5 we get

\[ \frac{|z f'(z)|}{f(z)} - 1 \leq \frac{2n r^n}{1 - r^n} \leq 1 - c_0. \]

This holds true whenever \(r \leq \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_1}(1/2))\). For sharpness, consider \(\tilde{f}(z) = z\) and \(\tilde{g}(z) = z/(1 - z^n)\), then at \(z_0 = \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_1}(1/2)) e^{i\pi/n}\), we get

\[ \frac{z_0 \tilde{f}'(z_0)}{\tilde{f}(z_0)} - 1 = \frac{2n z_0^n}{1 - z_0^n} = 1 - c_0. \]

(iii) Let \(f(z)/g(z) = p(z)\) be a function defined in \(D\). As \(f \in \overline{\mathcal{H}_2}\), then \(|1/p(z) - 1| < 1\) if and only if \(\Re p(z) > 1/2\). As \(g \in A_n\) is convex, then due to Marx-Strohhäcker theorem, \(g \in S^*_n(1/2), (S^*_n(1/2) = \{f \in A_n : \Re f(z)/f(z) > 1/2\})\). Therefore due to Lemma 4,

\[ \frac{|z g'(z)|}{g(z)} - 1 \leq \frac{r^n}{1 - r^{2n}}. \]

On logarithmically differentiating \(f(z)\) and applying Theorem 1, we get

\[ \frac{|z f'(z)|}{f(z)} - \frac{1}{1 - r^{2n}} = \frac{|z g'(z) - z p'(z) - 1 - z^n|}{g(z) - p(z)} \leq \frac{nr^{2n} + (1 + n)r^n}{1 - r^{2n}} \leq \frac{1}{1 - r^{2n}} - c_0, \]

which leads to \(r^{2n}(n - c_0) + r^n(1 + n) - 1 + c_0 \leq 0\), provided \(r \leq \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_2})\). The functions \(\tilde{f}(z) = z(1 + z^n)/(1 - z^n)^{1/n}\) and \(\tilde{g}(z) = z/(1 - z^n)^{1/n}\) at \(z_0 = \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_2}) e^{i\pi/n}\) gives \(|z_0 \tilde{f}'(z_0)/\tilde{f}(z_0)| = c_0\). Hence the result is sharp.

\[ \square \]

**Theorem 10** Let \(r \in [0, 1]\), then the sharp \(S^*_n(\vartheta)\) –radius for the class \(\overline{\mathcal{H}_3}\) is given by

\[ \mathcal{R}_{S^*_n(\vartheta)}(\overline{\mathcal{H}_3}) = \begin{cases} \mathcal{R}_0, & r \leq \mathcal{R}_0, \\ \mathcal{R}_1, & r \geq \mathcal{R}_0, \end{cases} \]

where
\[ \mathcal{R}_0 = \begin{cases} 
\left( \frac{1 + A + 4n + \sqrt{(1 + A + 4n)^2 - 4(1 - c_0)(A + c_0)}}{2(A + c_0)} \right)^{1/n} & \text{if } -1 \leq A < -c_0, \\
\left( \frac{1 - c_0}{1 + 4n - c_0} \right)^{1/n} & \text{if } A = -c_0, \\
\left( \frac{1 + A + 4n - \sqrt{(1 + A + 4n)^2 - 4(1 - c_0)(A + c_0)}}{2(A + c_0)} \right)^{1/n} & \text{if } -c_0 < A \leq 1, 
\end{cases} \]

and
\[ \mathcal{R}_1 = \left( \frac{\sqrt{(1 + A + 4n)^2 + 4(A + c_1)(c_1 - 1) - (1 + A + 4n)}}{2(A + c_1)} \right)^{1/n}. \]

**Proof** Let \( f \in \mathcal{A}_3 \), then \( \text{Re}(f(z)/g(z)) > 0 \) and \( \text{Re}((1 - z^n)^{(1+A)/n} g(z)/z) > 0 \), where \( g \in \mathcal{A}_n \). Define \( g(z)/f(z) = p_1(z) \) and \( (1 - z^n)^{(1+A)/n} g(z)/z = p_2(z) \), where \( p_1(z) \) and \( p_2(z) \) are analytic in \( \mathbb{D} \). Since \( A < 1 \), then for \( |z| = r < 1 \), the inequality \( (1 + Ar^{2n}) \geq 1 - r^{2n} \), holds true. Further on logarithmically differentiating \( zp_1(z)p_2(z)(1 - z^n)^{-(1+A)/n} = f(z) \), we get
\[ \frac{zf'(z)}{f(z)} = \frac{1 + Az^n}{1 - z^n} + \frac{zp'_1(z)}{p_1(z)} + \frac{zp'_2(z)}{p_2(z)}. \]

Due to Lemmas 4–5, for \( |z| = r \), we infer
\[ \left| \frac{zf'(z)}{f(z)} - \frac{1 + Ar^{2n}}{1 - r^{2n}} \right| \leq \frac{4nr^n}{1 - r^{2n}} + \frac{(1 + A)r^n}{1 - r^{2n}}. \tag{12} \]

Assume \( c = (1 + Ar^{2n})/(1 - r^{2n}) \). Then \( c \leq (c_0 + c_1)/2 \) leads to \( r \leq \mathcal{R} \) and vice-versa, where \( \mathcal{R} = ((c_0 - 2)/(2A + c_0))^{1/2n} \). Algebraically, for each \( n = 1, 2, 3, \ldots \), it can be observed that, for the given range of \( A \), we have \( \mathcal{R}_0 < \mathcal{R}_1 < \mathcal{R} \). In particular, if \( r \leq \mathcal{R}_0 \), then \( c \leq (c_0 + c_1)/2 \). Further due to Theorem 1, inequality (12) gives
\[ \frac{4nr^n}{1 - r^{2n}} + \frac{(1 + A)r^n}{1 - r^{2n}} \leq \frac{1 + Ar^{2n}}{1 - r^{2n}} - c_0, \]
whenever \( r \leq \mathcal{R}_0 \). Moreover if \( c \geq (c_0 + c_1)/2 \), then \( r \geq \mathcal{R}_0 \). In fact, when \( r \geq \mathcal{R}_0 \), then we have \( c \geq (c_0 + c_1)/2 \). Now inequality (12) together with Theorem 1 yields
\[ \frac{4nr^n}{1 - r^{2n}} + \frac{(1 + A)r^n}{1 - r^{2n}} \leq c_1 - \frac{1 + Ar^{2n}}{1 - r^{2n}}, \]
provided \( r \leq \mathcal{R}_1 \). Thus the following functions, mentioned in Remark 4
serve as the extremal function for both the cases.

4 Certain estimates for the class $S_q^*$

In this section certain sufficient conditions for the class $S_q^*$ are established.

**Theorem 11** Let $f \in \mathcal{A}$, then $f \in S_q^*$ if and only if

$$\frac{1}{z} \left( f(z) \ast \frac{z - k z^2}{(1 - z)^2} \right) \neq 0 \quad (13)$$

where $k = \cosh e^{i/2} / (\cosh e^{i/2} - 1)$ for $t \in [-\pi, \pi]$. Moreover, $f \in S_q^*$ if and only if

$$1 - \sum_{n=2}^{\infty} \frac{(n - \cosh e^{i/2}) a_n}{\cosh e^{i/2} - 1} z^{n-1} \neq 0. \quad (14)$$

**Proof** Since $f \in S_q^*$, then $zf'(z)/f(z) = \cosh \sqrt{w(z)}$, where $w(z)$ is a Schwarz function with $w(0) = 0$. Equivalently for $w(z) = e^{it}, -\pi \leq t \leq \pi$, we have

$$\frac{zf'(z)}{f(z)} \neq \cosh e^{i/2} \Leftrightarrow zf'(z) - (\cosh e^{i/2})f(z) \neq 0 \quad \text{for} \quad t \in [-\pi, \pi],$$

Eventually it leads to $zf'(z) - k(zf'(z) - f(z)) \neq 0$. Thus through simple computations (13) can be established. The condition in (14) can be deduced using (13). □

**Corollary 3** Let $f \in \mathcal{A}$ satisfy the following

$$\sum_{n=2}^{\infty} \left| \frac{n - \cosh e^{i/2}}{\cosh e^{i/2} - 1} \right| \left| a_n \right| < 1, \quad (15)$$

then $f \in S_q^*$.

**Proof** Consider the following inequality with $k = \cosh e^{i/2} / (\cosh e^{i/2} - 1)$,

$$\left| 1 - \sum_{n=2}^{\infty} (n(k - 1) - k)a_n z^{n-1} \right| \geq 1 - \sum_{n=2}^{\infty} |(n(k - 1) - k)||a_n|.$$
\[ 1 - \sum_{n=2}^{\infty} (n(k - 1) - k) a_n z^{n-1} > 0, \]

Hence due to Theorem 11 we conclude that \( f \in S^*_0 \).

**Theorem 12** Let \( f \in S^*_0 \) then the following inequality holds

\[ c_1^2 - 1 \geq \sum_{k=2}^{\infty} (k^2 - c_1^2) |a_k|^2. \]

**Proof** Since \( f \in S^*_0 \), then \( zf'(z) = \cosh(\sqrt{w(z)}) f(z) \), for a Schwarz function \( w(z) \) with \( w(0) = 0 \). For \( 0 \leq |z| = r < 1 \), we get the following

\[
2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k} = \int_0^{2\pi} |re^{i\theta} f'(re^{i\theta})|^2 \, d\theta \\
= \int_0^{2\pi} \left| \cosh\left(\sqrt{w(re^{i\theta})}\right) f(re^{i\theta}) \right|^2 \, d\theta \\
\leq \int_0^{2\pi} \cosh^2\left(\sqrt{|w(re^{i\theta})|}\right) |f(re^{i\theta})|^2 \, d\theta \\
\leq \int_0^{2\pi} (\cosh^2 r) |f(re^{i\theta})|^2 \, d\theta \\
= 2\pi (\cosh^2 r) \sum_{k=1}^{\infty} |a_k|^2 r^{2k}.
\]

Thus when \( r \) tends to \( 1^- \), we at once obtain the required inequality.

**Example 1** Let \( f \in A \), then following functions are members of \( S^*_0 \).

1. \( f(z) = z + a_n z^n \in S^*_0 \), provided \( |a_n| \leq (1 - c_0)/(n - c_0), \ n \in \mathbb{N} - \{1\} \).
2. \( f(z) = z/(1 - Az)^2 \in S^*_0 \), provided \( |A| \leq (c_1 - 1)/(c_1 + 1) \).
3. \( f(z) = z/(1 - Az) \in S^*_0 \), provided \( |A| \leq (c_1 - 1)/c_1 \).
4. \( f(z) = ze^{Az} \in S^*_0 \), provided \( |A| \leq 1 - c_0 \).

**Proof** For part (i) we require that \( zf'(z)/f(z) = (1 + na_n z^{n-1})/(1 + a_n z^{n-1}) \) must lie in the disc \( \{ u : |u - c| < r_c \} \subseteq \partial(\mathbb{D}) \), centered at \( c \), where \( r_c \) is defined in (5). It is a known fact that the function \( f(z) = z + a_n z^n \) is univalent if and only if \( |a_n| \leq 1/n \). Thus \( c = (1 - n|a_n|^2)/(1 - |a_n|^2) \leq 1 \). If \( u = (1 + na_n z^{n-1})/(1 + a_n z^{n-1}) \) and \( r_c = (1 - n|a_n|^2)/(1 - |a_n|^2) - c_0 \), then due to Theorem 1,
\[
\frac{(n - 1)|a_n|}{1 - |a_n|^2} \leq \frac{1 - n|a_n|^2}{1 - |a_n|^2} - c_0.
\]

The proofs of (ii)–(iv) are much akin to (i), therefore it is skipped.

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References

1. Ali, R.M., K.G. Subramanian, V. Ravichandran, and O.P. Ahuja. 2008. Neighborhoods of starlike and convex functions associated with parabola. Journal of Inequalities and Applications 346279: 9.
2. Alotaibi, A., M. Arif, M.A. Alghamdi, and S. Hussain. 2020. Starlikeness associated with cosine hyperbolic function. Mathematics 8 (7): 1118. https://doi.org/10.3390/math8071118.
3. Aouf, M.K., J. Dziok, and J. Sokół. 2011. On a subclass of strongly starlike functions. Applied Mathematics Letters 24 (1): 27–32.
4. Bano, K., and M. Raza. 2021. Starlike functions associated with cosine functions. Bulletin of the Iranian Mathematical Society 47 (5): 1513–1532.
5. Baricz, Á., M. Obradović, and S. Ponnusamy. 2013. The radius of univalence of the reciprocal of a product of two analytic functions. Journal of Analysis 21: 1–19.
6. Cho, N.E., V. Kumar, S.S. Kumar, and V. Ravichandran. 2019. Radius problems for starlike functions associated with the sine function. Bulletin of the Iranian Mathematical Society 45 (1): 213–232.
7. Janowski, W. 1973. Some extremal problems for certain families of analytic functions. I. Annales Polonici Mathematici 28 (3): 297–326.
8. Kanas, S., V.S. Masih, and A. Ebadian. 2019. Relations of a planar domains bounded by hyperbolas with families of holomorphic functions. Journal of Inequalities and Applications 2019 (1): 1–14.
9. Kanas, S., and A. Wiśniowska. 2000. Conic domains and starlike functions. Revue Roumaine de Mathématiques Pures et Appliquées 45 (4): 647–658.
10. Kumar, S., and S.K. Sahoo. 2021. Radius of convexity for integral operators involving Hornich operations. Journal of Mathematical Analysis and Applications 502 (2): 21.
11. Lecko, A., and Y.J. Sim. 2019. Coefficient problems in the subclasses of close-to-star functions. Results in Mathematics 74 (3): 104.
12. Ma, W.C., and D. Minda. 1992. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis (Tianjin). Conf. Proc. Lecture Notes Anal., 157–169. Cambridge, MA: Int. Press.
13. Masih, V.S., and S. Kanas. 2020. Subclasses of starlike and convex functions associated with the Limaçon Domain. Symmetry 12: 942.
14. Mendorratta, R., S. Nagpal, and V. Ravichandran. 2015. On a subclass of strongly starlike functions associated with exponential function. Bulletin of the Malaysian Mathematical Science Society 38 (1): 365–386.
15. Mundalia, M., and S.K. Shannugam. 2020. Coefficient bounds for a unified class of holomorphic functions. In Mathematical analysis. I. Approximation theory, Springer Proc. Math. Stat., vol. 306, 197–210. Singapore: Springer.
16. Nezhmetdinov, I.R., and S. Ponnusamy. 2005. New coefficient conditions for the starlikeness of analytic functions and their applications. Houston Journal of Mathematics 31 (2): 587–604.
17. Ponnusamy, S., and S.K. Sahoo. 2006. Study of some subclasses of univalent functions and their radius properties. *Kodai Mathematical Journal* 29 (3): 391–405.
18. Ponnusamy, S., S.K. Sahoo, and N.L. Sharma. 2016. Maximal area integral problem for certain class of univalent analytic functions. *Mediterranean Journal of Mathematics* 13 (2): 607–623.
19. Ponnusamy, S., S.K. Sahoo, and T. Sugawa. 2014. Radius problems associated with pre-Schwarzian and Schwarzian derivatives. *Analysis (Berlin)* 34 (2): 163–171.
20. Ponnusamy, S., N.L. Sharma, and K.-J. Wirths. 2020. Logarithmic coefficients problems in families related to starlike and convex functions. *Journal of the Australian Mathematical Society* 109 (2): 230–249.
21. Raina, R.K., and J. Sokół. 2015. Some properties related to a certain class of starlike functions. *Comptes Rendus Mathematique Academie des Sciences Paris* 353 (11): 973–978.
22. Ravichandran, V., F. Rønning, and T.N. Shanmugam. 1997. Radius of convexity and radius of starlikeness for some classes of analytic functions. *Complex Variables, Theory and Application* 33 (1–4): 265–280.
23. Robertson, M.I.S. 1936. On the theory of univalent functions. *Annals of Mathematics* 37 (2): 374–408.
24. Saliu, A., K.I. Noor, S. Hussain, and M. Darus. 2021. Some results for the family of univalent functions related with limaçon domain. *AIMS Mathematics* 6 (4): 3410–3431.
25. Shah, G.M. 1972. On the univalence of some analytic functions. *Pacific Journal of Mathematics* 43: 239–250.
26. Sokół, J., and J. Stankiewicz. 1996. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Naukowe Politechniki Rzeszowskiej Matematyka* 19: 101–105.
27. Stankiewicz, J. 1971. Quelques problèmes extrémaux dans les classes des fonctions α-angulairement étoilées. *Annales Universitatis Mariae Curie-Skłodowska Section A* 20 (1966): 59–75.
28. Uralegaddi, B.A., M.D. Ganigi, and S.M. Sarangi. 1994. Univalent functions with positive coefficients. *Tamkang Journal of Mathematics* 25 (3): 225–230.

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