A COMBINATORIAL MAPPING FOR THE HIGHER-DIMENSIONAL
MATRIX-TREE THEOREM

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Abstract. For a natural class of matroids that are equipped with a multiplicity
function, we provide a family of combinatorially meaningful maps from the sandpile
group of a matroid to its bases such that the size of the preimage of every basis is its
multiplicity squared. This generalizes a bijection given by Backman, Baker, and Yuen
and extends work by Duval, Klivans, and Martin.

1. Introduction

The sandpile group of a graph $G$ (denoted $S(G)$) is a subgroup of the free abelian
group on the vertices of $G$ that arises in a variety of contexts, including the field of
chip-firing. For more details, see e.g. [Kli18, Chapter 4].

An important result in chip-firing is the following theorem.

Theorem 1.1 (Sandpile Matrix-Tree Theorem on Graphs [Big99, Theorem 7.3]). Let
$G$ be a graph and $T(G)$ be the set of spanning trees of $G$.

$$|S(G)| = |T(G)|.$$

The result follows from Kirchhoff’s Matrix-Tree Theorem. While this theorem im-
plies the existence of bijections between $S(G)$ and $T(G)$, the standard proof is not
bijective. There has been a great deal of interest in providing combinatorially mean-
ingful bijections between these two sets. See, for example, [MD92], [BW97], [HLM+08],
and [Ber08].

For this paper, instead of working with graphs, we work with representable matroids.
The matroid represented by a matrix $D$, is the pair $(E, B)$ where the ground set $E$ is
the set of columns of $D$ and the bases $B$ are the maximal linearly independent subsets
of $E$ over some field (in this paper, we always use $\mathbb{R}$). We will sometimes write $B(D)$
to denote the bases of the matroid represented by $D$. It is a well known fact that $|B|
is the same for all $B \in B$ and this quantity is called the rank of $(E, B)$.

A regular matroid is a matroid that can be represented by a matrix that is totally
unimodular which means that every minor has determinant 1, 0, or -1.

Every graph has an associated matroid, whose edges correspond to the ground set
and whose spanning trees correspond to the bases. It is well-known that the set of
matroids arising from graphs is strictly contained in the set of regular matroids.

$$\{\text{Graphical Matroids}\} \subsetneq \{\text{Regular Matroids}\} \subsetneq \{\text{Representable Matroids}\} \subsetneq \{\text{Matroids}\}$$

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There is a natural way to define the sandpile group $S(M)$ of a regular matroid $M$ such that $S(M) \cong S(G)$ when $M$ is the matroid of a graph $G$. In this context, the Sandpile Matrix-Tree Theorem also extends naturally.

**Theorem 1.2** (Sandpile Matrix-Tree Theorem on Regular Matroids [Mer99, Theorem 4.6.1]). Let $M$ be a regular matroid and $B(M)$ be the set of bases of $M$.

$$|S(M)| = |B(M)|.$$  

As with the graphical case, the original proof is enumerative. In 2017, Backman, Baker, and Yuen gave a bijective proof of this result [BBY19].

Meanwhile, Duval, Klivans, and Martin explored the non-regular case. They took a more topological approach and defined the sandpile group on simplicial complexes and later cell complexes [DKM09] [DKM13] [DKM15]. This work was also partially inspired by Kalai’s work on complete simplicial complexes [Kal83]. The authors were mainly interested in the matroid represented by the top-dimensional boundary map of the cell complex. Bases of this matrix are what they call *cellular spanning forests*.

Through working with cell complexes, Duval, Klivans, and Martin extended the Matrix-Tree Theorem to any representable matroid. To state our version of their result, we introduce the *multiplicity function* from the theory of *arithmetic* matroids (see [DM13]).

**Definition 1.3.** Let $(E, B)$ be the matroid represented by a matrix $D$ with rank $n$ and let $B \in B$. Let $D_B$ be $D$ restricted to the columns in $B$. The *multiplicity* of $B$, denoted $m(B)$, is the GCD of the magnitudes of the determinants of $n \times n$ minors of $D_B$.

Note that when $D$ has $n$ rows, $m(B)$ is just the magnitude of $\det(D_B)$. The multiplicity function is not purely matroidal: it depends on which representation we choose for $(E, B)$.

Duval, Klivans, and Martin give several candidates for what should be called the sandpile group of a cell complex (or, equivalently, of a matrix $D$). However, when we assume that there is at least one basis with multiplicity 1, these groups are all isomorphic (see Remark 2.9). We define one of these groups, in particular what they call the cutflow group, in Definition 2.8. We will call this group the sandpile group and write it as $S(D)$. In this context, Duval, Klivans, and Martin give the following version of the Matrix-Tree Theorem$^1$:

**Theorem 1.4** (Sandpile Matrix-Tree Theorem on Cell Complexes [DKM15, Theorem 8.1]). Let $D$ be a matrix representing a matroid $(E, B)$ such that for some $B \in B$, $m(B) = 1$.

$$|S(D)| = \sum_{B \in B} m(B)^2.$$  

**Remark 1.5.** Instead of a multiplicity function, Duval, Klivans, and Martin gave this result in terms of the sizes of certain torsion groups. The size of the torsion group associated with a particular basis is exactly its multiplicity (as defined above).

$^1$The theorem in [DKM15] is stronger since it gives the sizes of several groups that may be distinct when we do not assume $D$ has a basis with multiplicity 1.
When $D$ is a totally unimodular representation of a regular matroid, $m(B) = 1$ for all $B \in \mathcal{B}$. Thus, Theorem 1.4 is a strengthening of Theorem 1.2. A natural question to ask is whether we can elevate the relationship between $\mathcal{S}(D)$ and $\mathcal{B}$ to a many-to-one map that is akin to a bijection. In particular, we consider the following:

**Question 1.6** ([Kli18, Exercise 7.10.15]). *Can one define a natural map $f : \mathcal{S}(D) \rightarrow \mathcal{B}(D)$ such that for any $B \in \mathcal{B}(D)$, we have $|f^{-1}(B)| = m(B)^2$?*

The main result of this paper is Theorem 3.22, which answers this question affirmatively by providing a family of explicit maps of this form. These maps depend only on a choice of multiplicity 1 basis and a generic direction we call a *shifting vector*. Our general construction is geometric as in [BBY19]. We associate each basis with a parallelepiped of volume $m(B)^2$. These parallelepipeds do not intersect and their union produces a non-convex polyhedron that tiles $\mathbb{R}^{|E|}$. Using our shifting vector, we associate $m(B)^2$ points of $\mathbb{Z}^{|E|}$ to each parallelepiped. Finally, we show that these points are all distinct in $\mathcal{S}(D)$.

In Section 4, we give an alternate presentation of our map that tiles a lower-dimensional space, making it easier to visualize. In Section 5, we explore the choices of shifting vector more closely and show that they correspond to the chambers of a certain hyperplane arrangement. Furthermore, we associate these chambers with acyclic circuit and cocircuit signatures and show that our map generalizes a map given by Backman, Baker, and Yuen [BBY19]. Finally, in Section 6, we give several open questions.

## 2. The Matrix Sandpile Group

### 2.1. Lattices.

**Definition 2.1.** A *lattice* is a subgroup of a finite-dimensional vector space $V$ that is isomorphic to $\mathbb{Z}^k$ for some $k$.

A *sublattice* of a lattice is a subgroup that is also a lattice. Given any set $S$ of $\mathbb{Z}^k$ vectors, the integer linear combinations of these vectors form a lattice $L$ of dimension at most $k$. We say that $S$ *generates* $L$ and, if the vectors in $S$ are linearly independent, we say that $S$ is an *integral basis* for $L$.

**Remark 2.2.** When working with vector spaces, any maximal linearly independent set of generators is a basis. However, a maximal linearly independent set of generators for a lattice is not always an integral basis of this lattice. For example, the set $\{2, 3\}$ generates $\mathbb{Z}$, but neither $\{2\}$ nor $\{3\}$ is an integral basis for $\mathbb{Z}$.

**Proposition 2.3.** [GR01, Theorem 14.5.3] *If $S$ is a set of $k$ $\mathbb{Z}^k$ vectors that are an integral basis for a lattice $L$, then the group $\mathbb{Z}^k/L$ has size equal to the magnitude of the determinant of the matrix formed by the vectors of $S$.***

### 2.2. Cuts and Flows.

There has been a lot of work regarding cuts and flows of graphs (see [BLHN97] and [GR01]). More recently, Duval, Klivans, and Martin extended these results to cell complexes. All of the following definitions, except for the continuous sandpile group, also appear in [DKM15]. However, we use “sandpile lattice” in place of “cutflow lattice” and “sandpile group” in place of “cutflow group”.

Given any integer matrix $D$, we define the following:

**Definition 2.4.**
The cut space and cut lattice are generated by the rows of $D$. The flow space and flow lattice are generated by the coefficients of integer linear combinations of columns of $D$ that sum to 0. Note that these generators are all elements of $\mathbb{Z}^k$ where $k$ is the number of columns of $D$.

Lemma 2.5 ([DKM15, Proposition 5.1]). For any integer matrix $D$, the spaces $\text{Cut}(D)$ and $\text{Flow}(D)$ are orthogonal complements.

Note that because $\mathcal{C}(D) \subset \text{Cut}(D)$ and $\mathcal{F}(D) \subset \text{Flow}(D)$, this also means that $\mathcal{C}(D)$ and $\mathcal{F}(D)$ are always orthogonal. We also get the following corollary:

Corollary 2.6. Let $D$ be an integer matrix and $\hat{D}$ be a matrix with rows that generate $\text{Flow}(D)$. $\text{Flow}(D) = \text{Cut}(\hat{D})$ and $\text{Flow}(\hat{D}) = \text{Cut}(D)$.

Proof. The first equality follows immediately from the fact that $\text{Cut}(\hat{D})$ is generated by the rows of $\hat{D}$ which also generate $\text{Flow}(D)$ by definition.

For the second equality, by Lemma 2.5, $\text{Flow}(\hat{D})$ is the orthogonal complement of $\text{Cut}(\hat{D})$ which we established is equal to $\text{Flow}(D)$. By a second application of Lemma 2.5, $\text{Flow}(D)$ is the orthogonal complement of $\text{Cut}(D)$. Since the composition of two orthogonal complements is the identity, we conclude that $\text{Flow}(\hat{D}) = \text{Cut}(D)$. □

Definition 2.7. The sandpile lattice of $D$ is the group $\mathcal{C}(D) \oplus \mathcal{F}(D)$.

Definition 2.8. The sandpile group, denoted $\mathcal{S}(D)$ of $D$ is the group $\mathbb{Z}^k/(\mathcal{C}(D) \oplus \mathcal{F}(D))$ where $k$ is the number of columns of $D$.

Remark 2.9. The authors of [DKM15] provide several candidates for a cellular sandpile group: the critical group, the cocritical group, the discriminant group of the cut lattice, the discriminant group of the flow lattice, and the cutflow group. However, they show in Corollary 7.8 that when the multiplicities of the bases of $D$ are relatively prime, these five groups are all isomorphic. We generally make the even stronger assumption that $D$ has basis of multiplicity 1. Our definition corresponds to the cutflow group.

Definition 2.10. The continuous sandpile group, denoted $\tilde{\mathcal{S}}(D)$ is the group $\mathbb{R}^k/(\mathcal{C}(D) \oplus \mathcal{F}(D))$ where $k$ is the number of columns of $D$.

Definition 2.11. A standard representative matrix is a matrix of the form

$$D = \begin{bmatrix} I_n & M \end{bmatrix}$$

where $M$ is any $n \times m$ integer matrix.

We will focus on matroids represented by standard representative matrices because of the following proposition. We do not prove this fully because it is technical and follows from results in [Pag20].

Proposition 2.12. Let $(E, B)$ be a matroid represented by a matrix $D$. $(E, B)$ can be represented by a standard representative matrix $D'$ (after rearranging $E$) such that the multiplicity of bases is also maintained if and only if there is some $B \in B$ such that $m(B) = 1$. 

Proof. The backwards direction is immediate because the first \( n \) columns of a standard representative matrix always correspond to a basis of multiplicity 1.

For the forward direction, we can think of \( D \) as representing an oriented arithmetic matroid and apply the algorithm from the proof of Proposition 8.2 in [Pag20]. \( \square \)

**Definition 2.13.** Given a standard representative matrix \( D \), let \( \hat{D} \) be the \( m \times (n+m) \) matrix:

\[
\hat{D} = \begin{bmatrix} -M^T & I_m \end{bmatrix}
\]

and let \( D \) be the \( (n+m) \times (n+m) \) matrix:

\[
D = \begin{bmatrix} D \hat{D} \\
\hat{D} \end{bmatrix} = \begin{bmatrix} I_n & M \\
-M^T & I_m \end{bmatrix}.
\]

**Remark 2.14.** In [Oxl03, Section 2.2], Oxley introduces the matrix \( \hat{D} \) and shows that it represents the dual of the matroid represented by \( D \).

**Theorem 2.15.**

- The rows of \( D \) are an integral basis for the cut lattice \( C(D) \).
- The rows of \( \hat{D} \) are an integral basis for the flow lattice \( F(D) \).
- The rows of \( D \) are an integral basis for the sandpile lattice \( C(D) \oplus F(D) \).

Proof. The first claim follows from Theorem 6.2 in [DKM15].

For the second claim, by Lemma 2.5, we need to show that the rows of \( \hat{D} \) are an integral basis for the orthogonal complement of \( C(D) \). When we multiply the \( i \)th row of \( D \) by the \( j \)th row of \( \hat{D} \), the only two nonzero terms are the \((i,j)\) entry of \( M \) and the \((j,i)\) entry of \(-M^T\), which sum to 0. Furthermore, the rows of \( D \) and \( \hat{D} \) are each linearly independent because they have the full-rank identity as a minor. This means that the rows of \( \hat{D} \) form an integral basis for some sublattice of \( F(D) \). If this lattice were not all of \( F(D) \), we could get an integer vector from a non-integer linear combination of rows. This is impossible because because of the full-rank identity minor.

The third claim follows immediately from the first two. \( \square \)

Recall that \( C(D) \oplus F(D) \) is the sandpile lattice of \( D \). This implies the following corollary:

**Corollary 2.16.** Let \( D \) be an \( n \times (n+m) \) standard representative matrix. Two vectors \( v_1, v_2 \in \mathbb{Z}^{n+m} \) (resp. \( \mathbb{R}^{n+m} \)) are equivalent as elements of \( S(D) \) (resp. \( \tilde{S}(D) \)) if and only if \( v_1 - v_2 \in \text{Im}_\mathbb{Z}(D^T) \).

We also provide two other sets of integral bases.

**Proposition 2.17.** The rows of the following matrices are each integral bases for \( C(D) \oplus F(D) \):

\[
\begin{bmatrix} I_n & M \\
0 & \hat{D} \hat{D}^T \end{bmatrix} \quad \begin{bmatrix} DD^T & 0 \\
-M^T & I_m \end{bmatrix}
\]

Proof. We know from Theorem 2.15 that \( D \) is an integral basis for \( C(D) \oplus F(D) \). We need to show that we can write any row of the above matrices as an integer linear
combination of rows of $\mathcal{D}$ and that we can write any row of $\mathcal{D}$ as an integer linear combination of rows in either of the above matrices.

To get from $\mathcal{D}$ to the left matrix, we multiply the first $n$ rows by $M^T$ and add this to the last $m$ rows. The equality holds by the observation $DD^T = MM^T + I_n$. To get back to $\mathcal{D}$, we multiply the first $n$ rows by $-M^T$ and add this to the last $m$ rows.

Similarly, To get from $\mathcal{D}$ to the right matrix, we multiply the last $m$ rows by $-M$ and add this to the first $n$ rows. The equality holds by the observation that $\hat{\mathcal{D}}\hat{\mathcal{D}}^T = M^TM + I_m$. To get back to $\mathcal{D}$, we multiply the last $m$ rows by $M$ and add this to the first $n$ rows. □

3. Sandpile to Basis Map

In order to answer Question 1.6, we first construct a tiling of $\mathbb{R}^{n+m}$.

**Definition 3.1.** The *fundamental parallelepiped* of a square matrix $A$ with column vectors $c_1, \ldots, c_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i c_i \mid 0 \leq a_i \leq 1 \right\}.$$

We use the notation $\Pi_\bullet(A)$ to indicate the fundamental parallelepiped of $A$.

**Definition 3.2.** The *half-open fundamental parallelepiped* of a square matrix $A$ with column vectors $c_1, \ldots, c_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i c_i \mid 0 \leq a_i < 1 \right\}.$$

We use the notation $\Pi_\circ(A)$ to indicate the half-open fundamental parallelepiped of $A$.

It is a classical result that the volume of $\Pi_\bullet(A)$ or $\Pi_\circ(A)$ is the magnitude of $\det(A)$.

**Definition 3.3.** For any basis $B$ of $M(D)$, we think of $B$ as a subset of columns of $\mathcal{D}$ (which restricts to a subset of columns of $D$ or $\hat{D}$).

- $P_1(B)$ is the fundamental parallelepiped of $D$ restricted to columns in $B$.
- $P_2(B)$ is the fundamental parallelepiped of $\hat{D}$ restricted to columns not in $B$.
- $P(B)$ is the direct product of $P_1(B)$ and $P_2(B)$.

Note that if $D$ has $n$ rows and $n+m$ columns, then $P_1(B)$ is $n$-dimensional, $P_2(B)$ is $m$-dimensional, and $P(B)$ is $(n+m)$-dimensional.

We can also describe $P(B)$ in the following way: For each column of $\mathcal{D}$, if this column corresponds to an index of $B$, replace the last $m$ entries with 0’s. If this column does not correspond to an index of $B$, replace the first $n$ entries with 0’s. The fundamental parallelepiped of this matrix is $P(B)$. See Example 3.5.

**Lemma 3.4.** For any basis $B$ of $M(D)$, $P_1(B)$ and $P_2(B)$ both have volume $m(B)$ while $P(B)$ has volume $m(B)^2$. 
Proof. $P_1(B)$ has volume $m(B)$ by definition. $P(B)$ is the direct product of $P_1(B)$ and $P_2(B)$, so its volume must be the product of the volumes of $P_1(B)$ and $P_2(B)$. This means that the only thing to check is that the determinant of $D$ restricted to columns in $B$ has the same magnitude as the determinant of $\hat{D}$ restricted to columns not in $B$.

First, we partition $B$ into 2 subsets, $B_a := B \cap [1, n]$ and $B_b := B \cap [n+1, n+m]$. On $D$, each $i \in B_a$ corresponds to $e_i$ and each $j \in B_b$ corresponds to the $(j - n)^{th}$ column of $M$.

For each $i \in B_a$, if we remove the column $e_i$, along with the $i^{th}$ row, the determinant of the resulting submatrix has the same magnitude. We can continue this process for all elements of $B_a$. This shows that the determinant of $D$ restricted to the columns of $B$ has the same magnitude as det$(\hat{M})$, where $\hat{M}$ is $M$ restricted to the columns $j - n$ for $j \in B_b$ and rows not in $B_a$.

The calculation for $\hat{D}$ is very similar. This time, the first $n$ columns correspond to columns of $-M^T$ and the last $m$ correspond to standard basis vectors. We first restrict $-M^T$ to columns that are not in $B_a$. Then, for each $j \in [n+1, n+m] \setminus B_b$, we can remove this column and the $j - n^{th}$ row without changing the magnitude of the determinant. After removing all of these rows, the remaining rows are the elements of $B_b$. Thus, the magnitude of this determinant is the same as the magnitude of the determinant of $-M^T$ restricted to columns not in $B_a$ and the rows $j - n$ for each $j \in B_b$. This is just the transpose of $-\hat{M}$ as defined above, whose determinant has the same magnitude as det$(\hat{M})$.

Example 3.5. Consider the matrix

$$D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix}$$

which is associated with the matrix $D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix}$.

In this example, there are 3 bases of $M(D)$, one for every pair of columns. The associated parallelepipeds are given below:

$$P_1(\{1, 2\}) = \Pi_\bullet \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \quad P_2(\{1, 2\}) = \Pi_\bullet \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \quad P(\{1, 2\}) = \Pi_\bullet \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$$

$$P_1(\{1, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 1 \\ 0 \\ 2 \end{array} \right] \quad P_2(\{1, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \quad P(\{1, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 1 \\ 0 \\ 3 \end{array} \right]$$

$$P_1(\{2, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right] \quad P_2(\{2, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \quad P(\{2, 3\}) = \Pi_\bullet \left[ \begin{array}{c} 0 \\ 0 \\ 3 \end{array} \right]$$

See Figure 1 for a plot of these three parallelepipeds. Notice that they only intersect at their boundaries. We show that this is true in general.

Proposition 3.6. The parallelepipeds $P(B)$ for each basis $B$ of $M(D)$ do not intersect except at their boundaries.
Figure 1. Here is a plot of the three parallelepipeds from Example 3.5 in 3 dimensional space. The square is \( P(\{1, 2\}) \), the smaller of the two remaining parallelepipeds is \( P(\{1, 3\}) \) and the larger is \( P(\{2, 3\}) \). We will see in Corollary 3.10 that the union of these parallelepipeds periodically tiles the plane.

Proof. Let \( c_1, c_2, \ldots, c_{n+m} \) be the columns of \( D \) and \( \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{n+m} \) be the columns of \( \hat{D} \). Let \( B_1 \) and \( B_2 \) be two distinct bases of \( M(D) \).

\( P(B_1) \) and \( P(B_2) \) have intersecting interiors if and only if \( P_1(B_1) \) and \( P_1(B_2) \) have intersecting interiors and \( P_2(B_1) \) and \( P_2(B_2) \) have intersecting interiors. Assume that \( P_1(B_1) \) and \( P_1(B_2) \) have intersecting interiors. Then, for some coefficients \( a_1, \ldots, a_{n+m}, b_1, \ldots, b_{n+m} \) we have the following equality

\[
\sum_{i=1}^{n+m} a_i c_i = \sum_{i=1}^{n+m} b_i c_i
\]

where \( a_i \in (0, 1) \) for \( i \in B_1 \), \( a_i = 0 \) for \( i \notin B_1 \), \( b_i \in (0, 1) \) for \( i \in B_2 \), and \( b_i = 0 \) for \( i \notin B_2 \). If we subtract the second sum from the first and define \( d_i = a_i - b_i \), we get:

\[
\sum_{i=1}^{n+m} d_i c_i = 0.
\]

If \( e_i \) is the \( i^{th} \) standard basis vector, the above equality tells us that:

\[
\sum_{i=1}^{n+m} d_i e_i \in \ker \mathbb{R}(D) = \text{Flow}(D).
\]

Similarly, if \( P_2(B_1) \) and \( P_2(B_2) \) have intersecting interiors then for some set \( \hat{a}_1, \ldots, \hat{a}_{n+m}, \hat{b}_1, \ldots, \hat{b}_{n+m} \), we have the following equality:

\[
\sum_{i=1}^{n+m} \hat{a}_i \hat{c}_i = \sum_{i=1}^{n+m} \hat{b}_i \hat{c}_i
\]

where \( \hat{a}_i \in (0, 1) \) for \( i \notin B_1 \), \( a_i = 0 \) for \( i \in B_1 \), \( b_i \in (0, 1) \) for \( i \notin B_2 \), and \( b_i = 0 \) for \( i \in B_2 \). If we define \( \hat{d}_i = \hat{a}_i - \hat{b}_i \), we get:
\[
\sum_{i=1}^{n+m} \hat{d}_i \hat{c}_i = 0.
\]

It follows that:
\[
\sum_{i=1}^{n+m} \hat{d}_i e_i \in \ker_R(\hat{D}) = \text{Flow}(\hat{D}) = \text{Cut}(D)
\]

where the last equality follows from Corollary 2.6.

By Lemma 2.5, we have:
\[
0 = \left( \sum_{i=1}^{n+m} d_i e_i \right) \cdot \left( \sum_{i=1}^{n+m} \hat{d}_i e_i \right) = \sum_{i=1}^{n+m} d_i \hat{d}_i.
\]

For each \(i\), there are 4 possibilities:

**Case 1** \(i \in B_1 \cap B_2\):
\(\hat{a}_i = \hat{b}_i = 0\) so \(\hat{d}_i = 0\) and \(d_i \cdot \hat{d}_i = 0\).

**Case 2** \(i \in B_1 \setminus B_2\):
\(b_i = 0\), so \(d_i = a_i\) which means \(d_i \in (0, 1)\). \(\hat{a}_i = 0\), so \(\hat{d}_i = -\hat{b}_i\) which means \(\hat{d}_i \in (-1, 0)\). It follows that \(d_i \cdot \hat{d}_i < 0\).

**Case 3** \(i \in B_2 \setminus B_1\):
\(a_i = 0\), so \(d_i = -b_i\) which means \(d_i \in (-1, 0)\). \(\hat{b}_i = 0\), so \(\hat{d}_i = \hat{a}_i\) which means \(\hat{d}_i \in (0, 1)\). It follows that \(d_i \cdot \hat{d}_i < 0\).

**Case 4** \(i \not\in B_1 \cup B_2\):
\(a_i = b_i = 0\) so \(d_i = 0\) and \(d_i \cdot \hat{d}_i = 0\).

\(B_1\) and \(B_2\) are the same size and distinct, so cases 2 and 3 must occur at least once. This means that:
\[
\sum_{i=1}^{n+m} d_i \hat{d}_i < 0.
\]

This is a contradiction. \(\square\)

**Definition 3.7.** \(T(D)\), the *tile associated with \(D\)*, is:
\[
\bigcup_{B \in \mathcal{B}(D)} P(B).
\]

Corollary 3.10 will justify why we call this non-convex polyhedron a *tile*.

The following corollary follows directly from Lemma 3.4 and Proposition 3.6.

**Corollary 3.8.** The volume of \(T(D)\) is equal to
\[ \sum_{B \in \mathcal{B}(D)} m(B)^2. \]

Note that this is also equal to \(|\mathcal{S}(D)|\) by the Sandpile Matrix-Tree Theorem on Cell Complexes (Theorem 1.4).

When considering all of \(T(D)\), we can strengthen Proposition 3.6 to the following:

**Proposition 3.9.** Two distinct points of \(T(D)\) can only be equivalent as elements of \(\tilde{\mathcal{S}}(D)\) if they are both on the boundary of \(T(D)\).

**Proof.** First, we show that two points of \(T(D)\) can only be equivalent as elements of \(\tilde{\mathcal{S}}(D)\) if they are each on the boundary of some \(P(B)\).

Let \(p_1\) and \(p_2\) be interior points of \(P(B_1)\) and \(P(B_2)\) respectively. Using the notation and reasoning from Proposition 3.6, we can write \(p_1 - p_2\) as the vector whose first \(n\) entries are:

\[ \sum_{i=1}^{n+m} d_i c_i \]

and whose last \(m\) entries are:

\[ \sum_{i=1}^{n+m} \hat{d}_i \hat{c}_i. \]

By Corollary 2.16, \(p_1\) and \(p_2\) are equivalent as elements of \(\tilde{\mathcal{S}}(D)\) if and only if:

\[ p_1 - p_2 = D^T(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m})^T \]

for some \((z_1, \ldots, z_{n+m})^T \in \mathbb{Z}^{n+m}\).

Let \(r_i\) be the restriction of the \(i^{th}\) row of \(D\) to the first \(n\) entries and \(\hat{r}_i\) be the restriction of the \(i^{th}\) row of \(\hat{D}\) to the last \(m\) entries. Then, the first \(n\) entries of:

\[ D^T(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m})^T \]

are given by:

\[ \sum_{i=1}^{n+m} z_i r_i \]

and the last \(m\) entries are given by:

\[ \sum_{i=1}^{n+m} z_i \hat{r}_i \]

From the structure of \(D\), \(r_i\) and \(c_i\) as well as \(\hat{r}_i\) and \(\hat{c}_i\) are closely related. In particular, for \(i \in [1, n]\), we have \(r_i = c_i\) and \(\hat{r}_i = -\hat{c}_i\). For \(i \in [n+1, n+m]\), we have \(r_i = -c_i\) and \(\hat{r}_i = \hat{c}_i\).

This means that the first \(n\) entries of:

\[ D^T(z_1, \ldots, z_n, -z_{n+1}, \ldots, -z_{n+m})^T \]
are given by:
\[
\sum_{i=1}^{n+m} z_i c_i
\]
and the last \(m\) entries are given by:
\[
\sum_{i=1}^{n+m} -z_i \hat{c}_i.
\]

\(p_1\) and \(p_2\) are equivalent as elements of \(\tilde{S}(D)\) if and only if we have:
\[
\sum_{i=1}^{n+m} d_i c_i = \sum_{i=1}^{n+m} z_i c_i \iff \sum_{i=1}^{n+m} (d_i - z_i) c_i = 0.
\]
and:
\[
\sum_{i=1}^{n+m} \hat{d}_i \hat{c}_i = \sum_{i=1}^{n+m} -z_i \hat{c}_i \iff \sum_{i=1}^{n+m} (\hat{d}_i + z_i) \hat{c}_i = 0.
\]

By the same logic that we used for Proposition 3.6, the coefficients of the first sum form an element of \(\text{Flow}(D)\) while the coefficients of the second form an element of \(\text{Cut}(D)\). Lemma 2.5 again tells us that their dot product is 0. In other words:
\[
\sum_{i=1}^{n+m} (d_i - z_i) \cdot (\hat{d}_i + z_i) = 0
\]

For each \(i\), there are 4 possibilities:

**Case 1** \(i \in B_1 \cap B_2\):
\(\hat{a}_i = \hat{b}_i = 0\) so \(\hat{d}_i = 0\). \(a_i, b_i \in (0, 1)\) so \(d_i \in (-1, 1)\). \((d_i - z_i) \cdot (\hat{d}_i + z_i) = 0\) if \(z_i = 0\). Otherwise, the two factors have a different sign and the product is negative.

**Case 2** \(i \in B_1 \setminus B_2\):
\(b_i = 0\), so \(d_i = a_i\) which means \(d_i \in (0, 1)\). \(\hat{a}_i = 0\), so \(\hat{d}_i = -\hat{b}_i\) which means \(\hat{d}_i \in (-1, 0)\). If \(z_i > 0\), then \(d_i - z_i < 0\) and \(\hat{d}_i + z_i > 0\). If \(z_i \leq 0\), \(d_i - z_i < 0\) and \(\hat{d}_i + z_i > 0\). In either case, \((d_i - z_i) \cdot (\hat{d}_i + z_i) < 0\).

**Case 3** \(i \in B_2 \setminus B_1\):
\(a_i = 0\), so \(d_i = -b_i\) which means \(d_i \in (-1, 0)\). \(\hat{b}_i = 0\), so \(\hat{d}_i = \hat{a}_i\) which means \(\hat{d}_i \in (0, 1)\). If \(z_i \geq 0\), then \(d_i - z_i < 0\) and \(\hat{d}_i + z_i > 0\). If \(z_i < 0\), \(d_i - z_i < 0\) and \(\hat{d}_i + z_i > 0\). In either case, \((d_i - z_i) \cdot (\hat{d}_i + z_i) < 0\).

**Case 4** \(i \not\in B_1 \cup B_2\):
\(a_i = b_i = 0\) so \(d_i = 0\). \(\hat{a}_i, \hat{b}_i \in (0, 1)\) so \(\hat{d}_i \in (-1, 1)\). \((d_i - z_i) \cdot (\hat{d}_i + z_i) = 0\) if \(z_i = 0\). Otherwise, the two factors have a different sign and the product is negative.
In all four cases the product is negative, unless we are always in case 1 or case 4 and $z_i = 0$ for all $i$. However, if $z_i = 0$ for all $i$, then $p_1 = p_2$. Thus, our claim holds by contradiction.

We showed that two distinct points $p_1$ and $p_2$ of $T(D)$ that are equivalent as elements of $\hat{S}(D)$ must each lie on the boundary of some $P(B)$. We now show by contradiction that they are on both on the boundary of $T(D)$.

Assume that $p_1$ is an interior point of $T(D)$. Since $T(D)$ is the union of non-degenerate parallelepipeds, there is some direction vector $v$ such that for all sufficiently small $\varepsilon > 0$, $p_2 + \varepsilon v$ is in $T(D)$ but not on the boundary of any $P(B)$. If we make $\varepsilon$ small enough, $p_1 + \varepsilon v$ must be in $T(D)$ as well, since $p_1$ is an interior point of $T(D)$ by assumption. This means that $p_1 + \varepsilon v$ and $p_2 + \varepsilon v$ are equivalent as elements of $\hat{S}(D)$. We get a contradiction because both points are in $T(D)$, but $p_2 + \varepsilon v$ is not on the boundary of any $P(B)$. This means that $p_1$ and $p_2$ must both be on the boundary of $T(D)$. \hfill \Box

The next corollary shows that copies of $T(D)$ can be used to periodically tile $\mathbb{R}^{n+m}$.

**Corollary 3.10.** For any $D$, the set of translates $T(D) + D^T(v_1, \ldots, v_{n+m})^T$ for all $(v_1, \ldots, v_{n+m}) \in \mathbb{Z}^{n+m}$ cover all of $\mathbb{R}^{n+m}$ and only intersect at their boundaries.

**Proof.** By Corollary 2.16, for any point $p \in \mathbb{R}^{n+m}$, the sandpile group equivalent elements are those of the form $p + D^T(v_1, \ldots, v_{n+m})$ for all $(v_1, \ldots, v_{n+m}) \in \mathbb{Z}^{n+m}$. Since these are exactly the translates of $T(D)$, the condition that the translates do not intersect except at their boundaries follows directly from Proposition 3.9.

We also have to show that the translates cover all of $\mathbb{R}^{n+m}$ given that they do not overlap except at their boundaries. We first note that $\Pi_\circ(D^T)$ must tile $\mathbb{R}^{n+m}$ under the same translation because for every $p \in \mathbb{R}^{n+m}$, there is a unique solution to $(D^T)r = p$ (in particular $r = (D^T)^{-1}p$). We can map each point of $T(D)$ to a point in $\Pi_\circ(D^T)$ by translating it by an integer combination of columns of $D^T$. Let $t$ be this piecewise translation from $T(D) \rightarrow \Pi_\circ(D^T)$. Each translation preserves the volume of the region we transform and the only overlap is from the boundary of $T(D)$, which is a 0 volume set. It follows that the volume of the image of $t$ is equal to the volume of $T(D)$. Since $\Pi_\circ(D^T)$ has the same volume as $T(D)$, the set of points that are not in the image of $t$ must have volume 0.

Let $p$ be a point of $\Pi_\circ(D^T)$ that is not in the image of $t$. The preimage of $p$ is the collection of points in the same equivalence class with respect to $\hat{S}(D)$. By assumption, none of these points are in $T(D)$. Since $T(D)$ is closed, this means that none of these points are limit points of $T(D)$ either, so there is a neighborhood of $p$ that is also not in the image of $t$. However, this neighborhood must have positive volume, which is a contradiction. \hfill \Box

**Remark 3.11.** Our tile construction can be applied to any square matrix of the form

$$
\mathcal{D}' = \begin{bmatrix} D' \\
\tilde{D}' \end{bmatrix}.
$$

Proposition 3.6 holds as long as $D'$ and $\tilde{D}'$ generate orthogonal spaces. However, Proposition 3.9 and Corollary 3.10 are more restrictive because Proposition 3.9 relies on the relationship between rows and columns of $D'$. To satisfy these results, we need
Above are 9 copies of $\mathcal{T}(D)$ for $D = [1 \ 3]$. The dashed lines indicate the boundary between the parallelepipeds that make up $T(D)$ while the solid lines indicate the boundary of translates of $T(D)$. We get a similar pattern whenever $n = m = 1$.

$D' = [S_1 \ M]$ and $\hat{D}' = [-M^T \ S_2]$, where $S_1$ and $S_2$ are symmetric (and $D'$ and $\hat{D}'$ still generate orthogonal spaces). Most of the results in this paper (except those in Section 4) generalize naturally to matrices of this form.

**Example 3.12.** The simplest case is when $n = m = 1$ Here, $\mathcal{D}$ is of the form:

$$\mathcal{D} = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$$

for some integer $k$. When $k = 3$, we get the pattern in Figure 2.

**Remark 3.13.** Because our tiling is of $(n + m)$-dimensional space, it is difficult to present more complicated examples. However, in Section 4, we show that we can take an $n$-dimensional or $m$-dimensional slice of our tiling and get many of the same results. This will allow us to present more interesting tilings of 2-dimensional space (see Figure 7).

In order to define a function that answers Question 1.6, we will need $T(D)$ and an appropriate $\mathbb{R}^{n+m}$ direction vector.

**Definition 3.14.** A shifting vector $w = (w_1, \ldots, w_{n+m})$ is a vector in $\mathbb{R}^{n+m}$ that is not in the affine span of a facet of $P(B)$ for any $B \in \mathcal{B}(D)$.

**Remark 3.15.** In Section 5, we will show that choosing a shifting vector is equivalent to choosing *acyclic circuit and cocircuit signatures*.

**Definition 3.16.**

- For any $r \in \mathbb{R}^{n+m}$, $r$ is a *w-representative* of $\mathcal{S}(D)$ if $r + \varepsilon w \in T(D)$ for all sufficiently small $\varepsilon > 0$. If $r + \varepsilon w \in P(B)$, we say that $r$ is *associated* with $B$.
- For any $z \in \mathbb{Z}^{n+m}$, $z$ is a *w-representative* of $\hat{\mathcal{S}}(D)$ if $z + \varepsilon w \in T(D)$ for all sufficiently small $\varepsilon > 0$. If $z + \varepsilon w \in P(B)$, we say that $z$ is *associated* with $B$. 

![Figure 2](image-url)
Lemma 3.17. Each $w$-representative of $\tilde{S}(D)$ or $S(D)$ is associated with exactly one $B \in B(D)$.

Proof. Since $w$-representatives of $S(D)$ are also $w$-representatives of $\tilde{S}(D)$, it suffices to prove the result for $S(D)$. Because $T(D) = \bigcup P(B)$, we know that $r + \varepsilon w \in P(B)$ for some $B \in B(D)$. Since, $w$ is not in the affine span of any facet of $P(B)$, $r + \varepsilon w$ must be in the interior of $P(B)$. By Proposition 3.6, this is true for a unique $B$. □

Proposition 3.18. For any shifting vector $w$, there is exactly one $w$-representative in $\mathbb{R}^{n+m}$ for each equivalence class of $\tilde{S}(D)$ and exactly one $w$-representative in $\mathbb{Z}^{n+m}$ for each equivalence class of $S(D)$.

Proof. The second result is a direct corollary of the first (and could also be proven with an enumerative argument). By Corollary 3.10, every point $r \in \mathbb{R}^{n+m}$ lies on some translation of $T(D)$ by an integer linear combination of the rows of $D$. We can translate this point to a point on $T(D)$ without changing the equivalence class with respect to $\tilde{S}(D)$. If $r$ maps to an interior point $r'$ of $T(D)$, then by Proposition 3.9, this is the unique point on $T(D)$ that is equivalent to $r$. Furthermore, since $r'$ is in the interior of $T(D)$, $r'$ is always a $w$-representative of $\tilde{S}(D)$ regardless of $w$.

If $r$ maps to a boundary point of $T(D)$, then by Proposition 3.9, any point of $T(D)$ that is in the same $\tilde{S}(D)$ equivalence class must also lie on the boundary of $T(D)$. Label these points as $\{r_1, \ldots, r_k\}$. We need to show that exactly one of these points is a $w$-representative.

By the condition that $w$ is not in the affine span of any facet of $T(D)$, for all sufficiently small $\varepsilon > 0$, $r_i + \varepsilon w$ must not lie on the boundary of $T(D)$ for any $i$. If $r_i + \varepsilon w$ and $r_j + \varepsilon w$ are both in $T(D)$ for $i \neq j$, then these are two distinct points in the interior of $T(D)$ that are equivalent as elements of $\tilde{S}(D)$. This is impossible by Proposition 3.9.

We have shown uniqueness, so we just need existence. Because $w$ is not in the affine span of any facet of $T(D)$, we can choose $\varepsilon > 0$ so that all points between $r$ and $r + w\varepsilon$ map to interior points of $T(D)$. Let $\hat{r}$ be the point mapped to by $r + w\varepsilon$. Then, $\hat{r} - \varepsilon w$ must be equivalent to $r$ with respect to $\tilde{S}(D)$. By our condition on $\varepsilon$, we see that this point is a $w$-representative. □

Proposition 3.19. For any shifting vector $w$, and for any $B \in B(D)$, there are exactly $m(B)^2$ $w$-representatives of $S(D)$ that are associated with $B$.

To prove this result, we apply the following lemma from Ehrhart Theory:

Lemma 3.20 ([BR07, Lemma 9.2]). For any integer matrix $A$, the number of integer points in the half-open fundamental parallelepiped $\Pi_\varepsilon(A)$ is equal to its volume (the magnitude of $\det(A)$).

Proof of Proposition 3.19. The points $z$ of $P(B)$ can all be uniquely written as:

$$z = \sum_{i=1}^{n+m} a_i c_i$$

where each $a_i \in [0, 1]$ and each $c_i$ is a vector of length $n + m$. Since the $c_i$ are linearly independent (otherwise $P(B)$ would not be a basis), we can write any vector $w$ uniquely as:
\[ w = \sum_{i=1}^{n+m} b_i c_i \]

where we do not place a restriction on the \( b_i \). If \( w \) is a shifting vector, then the condition that \( w \) is not in the affine span of any facet is equivalent to the condition that \( b_i \neq 0 \) for all \( i \). We have:

\[ z + \varepsilon w = \sum_{i=1}^{n+m} (a_i + \varepsilon b_i) c_i. \]

From here, we see that \( z \) is a \( w \)-representative of \( \mathcal{S}(D) \) that satisfies the conditions of the proposition if and only if \( a_i \in (0, 1] \) for \( b_i < 0 \) and \( a_i \in [0, 1) \) for \( b_i > 0 \). This region is the integer translation of a half-open fundamental parallelepiped with volume equal to the volume of \( P(B) \). By Lemma 3.20, the number of integer points in this region is equal to this volume, and the integer translation does not change the number of integer points. Finally, by Lemma 3.4, the volume is \( m(B)^2 \), completing the proof. \( \square \)

We now define a function \( \tilde{f}_w \) from \( \tilde{\mathcal{S}}(D) \to \mathcal{B}(D) \) given a shifting vector \( w \). For any \( s \in \tilde{\mathcal{S}}(D) \), we first take the \( w \)-representative of \( s \) (which is unique by Proposition 3.18). Then, we let \( \tilde{f}_w(s) = B \) where \( B \) is the basis associated with this \( w \)-representative (which is unique by Lemma 3.17).

**Definition 3.21.** \( f_w \) is \( \tilde{f}_w \) (as defined above) but with its domain restricted to \( \mathcal{S}(D) \).

The following theorem is the main result of this paper and provides a family of functions that satisfy Question 1.6.

**Theorem 3.22.** For any \( B \in \mathcal{B}(D) \), we have \( |f_w^{-1}(B)| = m(B)^2 \).

**Proof.** We showed in Propositions 3.18 and 3.6 that \( f_w \) is a well-defined map from \( \mathcal{S}(D) \) to \( \mathcal{B}(D) \). The fact that \( |f_w^{-1}(B)| = m(B)^2 \) is a corollary of Proposition 3.19. \( \square \)

**Example 3.23.** Consider the matrix and associated tile from Example 3.5. One can show that \( w = (1, 1, 1) \) satisfies the requirements of a shifting vector. There are 14 different \( w \)-representatives of \( \mathcal{S}(D) \) given in the list below:

\[
\{(0, 0, 0), (0, 0, -1), (1, 0, -1), (1, 1, -1), (2, 1, -1), (2, 2, -1), (0, 0, -2), \\
(1, 0, -2), (1, 1, -2), (2, 1, -2), (2, 2, -2), (0, 0, -3), (1, 1, -3), (2, 2, -3)\}.
\]

Furthermore, we have:

\[
\begin{align*}
\tilde{f}_w^{-1}(\{1, 2\}) &= \{(0, 0, 0)\}.
\tilde{f}_w^{-1}(\{1, 3\}) &= \{(1, 0, -1), (2, 1, -1), (1, 0, -2), (2, 1, -2)\}.
\tilde{f}_w^{-1}(\{2, 3\}) &= \{(0, 0, -1), (1, 1, -1), (2, 2, -1), (0, 0, -2), (1, 1, -2), \\
& (2, 2, -2), (0, 0, -3), (1, 1, -3), (2, 2, -3)\}.
\end{align*}
\]
where \( w \)-representatives are shorthand for “the equivalence class of \( S(D) \) containing this \( w \)-representative”. We can confirm that \( f_w \) satisfies the conditions of Question 1.6 by noting that:

\[
\begin{align*}
|f_w^{-1}(\{1, 2\})| &= 1 = m(\{1, 2\})^2. \\
|f_w^{-1}(\{1, 3\})| &= 4 = m(\{1, 3\})^2. \\
|f_w^{-1}(\{2, 3\})| &= 9 = m(\{2, 3\})^2.
\end{align*}
\]

If we use a different shifting vector, some of our representatives may change. For example, for \( w' = (-1, 2, -2) \), we have:

\[
\begin{align*}
f_w^{-1}(\{1, 2\}) &= \{(1, 0, 1)\}. \\
f_w^{-1}(\{1, 3\}) &= \{(1, 0, 0), (2, 1, 0), (1, 0, -1), (2, 1, -1)\}. \\
f_w^{-1}(\{2, 3\}) &= \{(3, 2, 0), (1, 1, 0), (2, 2, 0), (3, 2, -1), (1, 1, -1), \\
&\quad (2, 2, -1), (3, 2, -2), (1, 1, -2), (2, 2, -2)\}.
\end{align*}
\]

Note that interior points of \( P(B) \) are always associated with \( B \), but boundary points depend on the shifting vector.

### 4. Lower-Dimensional Representatives

In Section 3, we showed how to construct a tiling of \( \mathbb{R}^{n+m} \) (see Theorem 3.10) and how to use this tiling (and a shifting vector) to produce a set of representatives for \( S(D) \) (see Theorem 3.22). In this section, we show how to use the tiling of \( \mathbb{R}^{n+m} \) to produce a tiling of \( \mathbb{R}^{n} \) or \( \mathbb{R}^{m} \) that also (given a shifting vector) produces a set of representatives of \( S(D) \). The representatives associated with the tiling of \( \mathbb{R}^{n} \) all have zero in their last \( m \) entries while the representatives associated with the tiling of \( \mathbb{R}^{m} \) all have zero in their first \( n \) entries. However, if we use the same shifting vector, the sandpile equivalence classes mapped to each basis stay constant.

One benefit of this alternate construction is that it is often easier to work in lower dimensional space. In particular, we are now able to provide a wide variety of tilings of \( \mathbb{R}^{2} \) (see Figure 7). With our original map, all tilings of \( \mathbb{R}^{2} \) were similar to the one given in Example 3.12.

The main tool we use in this section is the following lemma.

**Lemma 4.1.** Let \( D \) be the standard representative matrix

\[
D = [I_n \ M]
\]

and let \( z = (z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_m)^T \in \mathbb{Z}^{n+m}. \) \( z \) is equivalent, with respect to \( S(D) \), to the vector whose first \( n \) entries are:

\[
(z_1, \ldots, z_n)^T + M^T(\tilde{z}_1, \ldots, \tilde{z}_m)^T
\]

and whose last \( m \) entries are zero.
The desired vectors are equal to 

\[(\hat{z}_1, \ldots, \hat{z}_m)^T - M(z_1, \ldots, z_n)^T.\]

**Proof.** The desired vectors are equal to 

\[z - D^T(0, \ldots, 0, \hat{z}_1, \ldots, \hat{z}_m)^T\]

and

\[z - D^T(z_1, \ldots, z_n, 0, \ldots, 0)^T\]

respectively. The lemma follows from Corollary 2.16. □

Recall from Definition 3.3 that for any \(B \in \mathcal{B}(D)\), we have parallelepipeds \(P_1(B)\), \(P_2(B)\), and \(P(B)\), where \(P(B)\) is the direct product of \(P_1(B)\) and \(P_2(B)\). Let \(w \in \mathbb{R}^{n+m}\) be a shifting vector of \(D\) (see Definition 3.14). In Definition 3.16, we defined the \(w\)-representatives associated with \(B\) to be the integer vectors \(z \in \mathbb{Z}^{n+m}\) such that for all sufficiently small \(\varepsilon > 0\), \(z + \varepsilon w \in P(B)\).

For this section, instead of thinking of \(w\)-representatives as a single \(\mathbb{Z}^{n+m}\) vector, we will think of them as a pair of vectors, one corresponding to the first \(n\) entries and one corresponding to the last \(m\) entries.

**Definition 4.2.** Given a basis \(B \in \mathcal{B}(D)\) and a shifting vector \(w = (w_1, \ldots, w_n, \hat{w}_1, \ldots, \hat{w}_m)\), we say that a vector \((z_1, \ldots, z_n) \in \mathbb{Z}^n\) is associated with \(P_1(B)\) if \((z_1, \ldots, z_n) + \varepsilon(w_1, \ldots, w_n) \in P_1(B)\) for all sufficiently small \(\varepsilon > 0\). We say that a vector \((\hat{z}_1, \ldots, \hat{z}_m) \in \mathbb{Z}^m\) is associated with \(P_2(B)\) if \((\hat{z}_1, \ldots, \hat{z}_m) + \varepsilon(\hat{w}_1, \ldots, \hat{w}_m) \in P_2(B)\) for all sufficiently small \(\varepsilon > 0\).

**Lemma 4.3.** A point \(z = (z_1, \ldots, z_n, \hat{z}_1, \ldots, \hat{z}_m)\) is a \(w\)-representative of \(S(D)\) associated with \(B\) if and only if \((z_1, \ldots, z_n)\) is associated with \(P_1(B)\) and \((\hat{z}_1, \ldots, \hat{z}_m)\) is associated with \(P_2(B)\).

**Proof.** By definition, a point is in \(P(B)\) if and only if it is in \(P_1(B)\) when restricted to the first \(n\) coordinates and \(P_2(B)\) when restricted to the last \(m\) coordinates. The lemma follows from the fact that \(z + \varepsilon w\) is \((z_1, \ldots, z_n) + \varepsilon(w_1, \ldots, w_n)\) when restricted to the first \(n\) coordinates and \((\hat{z}_1, \ldots, \hat{z}_m) + \varepsilon(\hat{w}_1, \ldots, \hat{w}_m)\) when restricted to the last \(m\) coordinates. □

By a slight adjustment of Proposition 3.19, one can show that there are \(m(B)\) integer vectors associated with \(P_1(B)\) and \(m(B)\) integer vectors associated with \(P_2(B)\). We now show how to construct an \(n\)-dimensional tile and an \(m\)-dimensional tile. For both constructions, we use a standard representative matrix \(D\) and a shifting vector \(w = (w_1, \ldots, w_n, \hat{w}_1, \ldots, \hat{w}_m)\).

**Definition 4.4.**

\[T'(D) := \bigcup_{B \in \mathcal{B}} \left( \bigcup_{v \in \mathbb{Z}^m \text{ associated with } P_2(B)} (P_1(B) + MT_v^T) \right)\]

\(T'(D)\) is made up of \(m(B)\) parallelepipeds for each \(B \in \mathcal{B}(D)\) and depends on \((\hat{w}_1, \ldots, \hat{w}_m)\) but not \((w_1, \ldots, w_n)\). Figure 3 gives an example of \(T'(D)\).
\textbf{Definition 4.5.}

\[ T''(D) := \bigcup_{B \in \mathcal{B}} \left( \bigcup_{v \in \mathbb{Z}^n} \text{associated with } P_1(B) (P_2(B) - M v^T) \right) \]

\( T''(D) \) is made up of \( m(B) \) parallelepipeds for each \( B \in \mathcal{B}(D) \) and depends on \((w_1, \ldots, w_m) \) but not \((\hat{w}_1, \ldots, \hat{w}_n) \). Figure 5 gives an example of \( T''(D) \).

The following theorem says that \( T'(D) \) and \( T''(D) \) have many similar properties to \( T(D) \). This is the main result of this section.

\textbf{Theorem 4.6.}

- The parallelepipeds that make up \( T'(D) \) only intersect at their boundaries.
- The parallelepipeds that make up \( T''(D) \) only intersect at their boundaries.
- The set of translates \( T'(D) + D D^T (v_1, \ldots, v_n)^T \) for all \((v_1, \ldots, v_n) \in \mathbb{Z}^n \) cover all of \( \mathbb{R}^n \) and only intersect at their boundaries.
- The set of translates \( T''(D) + \hat{D} \hat{D}^T (\hat{v}_1, \ldots, \hat{v}_m)^T \) for all \((\hat{v}_1, \ldots, \hat{v}_m) \in \mathbb{Z}^m \) cover all of \( \mathbb{R}^m \) and only intersect at their boundaries.
- For each \( B \in \mathcal{B}(D) \), there are exactly \( m(B)^2 \) integer points \((z_1, \ldots, z_n) \) of \( T'(D) \) such that for all sufficiently small \( \varepsilon > 0 \), \((z_1, \ldots, z_n) + \varepsilon (w_1, \ldots, w_n) \) is in one of the translates of \( P_1(B) \) that make up \( T'(D) \).
- For each \( B \in \mathcal{B}(D) \), there are exactly \( m(B)^2 \) integer points \((\hat{z}_1, \ldots, \hat{z}_m) \) of \( T''(D) \) such that for all sufficiently small \( \varepsilon > 0 \), \((\hat{z}_1, \ldots, \hat{z}_m) + \varepsilon (\hat{w}_1, \ldots, \hat{w}_m) \) is in one of the translates of \( P_2(B) \) that make up \( T''(D) \).

\textbf{Proof.} The general strategy for every part of this proof is to apply Lemma 4.1 to results from Section 3 about \( T(D) \).

The first 2 parts follow from Proposition 3.6 and Lemma 4.1.

For the next 2 parts, Proposition 2.17 implies that two \( \mathbb{R}^{n+m} \) vectors that end with \( m \) zeros are equivalent iff the difference of their first \( n \) entries is in \( \text{Im}_{\mathbb{Z}}(DD^T) \). Similarly, two \( \mathbb{R}^{n+m} \) vectors that begin with \( n \) zeros are equivalent iff the difference of their first \( m \) entries is in \( \text{Im}_{\mathbb{Z}}(\hat{D} \hat{D}^T) \). The results follow from this observation as well as Corollary 3.10 and Lemma 4.1.

Finally, for the last 2 parts, the integer points we obtain are exactly the \( w \)-representatives of \( T(D) \) translated by Lemma 4.1 so that either the first \( n \) or last \( m \) coordinates are 0. Thus, we can just apply Theorem 3.22. \( \square \)

\textbf{Example 4.7.} Consider the matrix

\[ D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \]

which is associated to the matrix \( \mathcal{D} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix} \).

In Example 3.5, we gave a perspective drawing for the 3-dimensional \( T(D) \). In Example 3.23, we gave the set of \( w \)-representatives when \( w = (1, 1, 1) \). Here, we will show how to construct \( T'(D) \) and \( T''(D) \) and find a set of \( w \)-representatives for these lower-dimensional tiles.

To construct \( T'(D) \), we first look at \( P_2(B) \) for each \( B \in \mathcal{B}(D) \). Because \( m = 1 \), these are intervals.
Figure 3. This is $T'(D)$ for $\tilde{w} = (1)$. It is made up of 1 parallelogram of area 1 corresponding to $\{1, 2\}$, 2 parallelograms of area 2 corresponding to $\{1, 3\}$, and 3 parallelograms of area 3 corresponding to $\{2, 3\}$. Note that the origin is the bottom left corner of the square in the upper right.

$$P_2(\{1, 2\}) = [0, 1].$$
$$P_2(\{1, 3\}) = [−2, 0].$$
$$P_2(\{2, 3\}) = [−3, 0].$$

Then, for each $B \in B(D)$, we find the set of integer points that are mapped into $P_2(B)$ by the shifting vector $(1)$ (the last $m$ entries of $w$). For $P_2(\{1, 2\})$, this is $\{(0)\}$. For $P_2(\{1, 3\})$, this is $\{−2, (−1)\}$. For $P_2(\{2, 3\})$, this is $\{−3, (−2), (−1)\}$. Then, we multiply each of these by $(3, 2)^T$ and shift $P_1(B)$ by these amounts. The resulting tile is given in Figure 3.

Finally, to find a set of representatives for $S(D)$, we take all of points $(z_1, z_2) \in \mathbb{Z}^2$ such that for all sufficiently small $\varepsilon > 0$, $(z_1, z_2) + \varepsilon(1, 1) \in T'(D)$ (where the shifting vector $(1, 1)$ is from the first two elements of $w$).

Let $f_{w'}$ be the map that sends $S(D) \to B(D)$ by mapping the lattice points in Figure 4 to bases associated to the parallelograms they are shifted into. We get the following set of representatives for $S(D)$:

$$f_{w'}^{-1}(\{1, 2\}) = \{(0, 0, 0)\}.$$  
$$f_{w'}^{-1}(\{1, 3\}) = \{−2, −2, 0\}, \{−1, −1, 0\}, \{−5, −4, 0\}, \{−4, −3, 0\}.$$  
$$f_{w'}^{-1}(\{2, 3\}) = \{−3, −2, 0\}, \{−2, −1, 0\}, \{−1, 0, 0\}, \{−6, −4, 0\}, \{−5, −3, 0\},$$  
$$\quad \{−4, −2, 0\}, \{−9, −6, 0\}, \{−8, −5, 0\}, \{−7, −4, 0\}.$$
We show which integer points map into $T'(D)$ by the shifting vector $(1, 1)$. The color of the point corresponds to which basis the point is mapped to. As expected from Theorem 3.22, there is 1 point mapped to $\{1, 2\}$, 4 points mapped to $\{1, 3\}$, and 9 points mapped to $\{2, 3\}$. If we append 0 to each of these points, we get a set of representatives for $S(D)$.

Note that these are the same representatives that we get as if we apply the first part of Lemma 4.1 to the representatives we obtained in Example 3.23 with the same shifting vector.

We can also find a set of representatives by using the tiling $T''(D)$ of $\mathbb{R}$. For each $B \in \mathcal{B}(D)$, we find the set of lattice points that are mapped into $P_1(B)$ by the shifting vector $(1, 1)$.

For $P_1(\{1, 2\})$ this is $\{(0, 0)\}$.
For $P_1(\{1, 3\})$ these are $\{(1, 0), (2, 1)\}$.
For $P_1(\{2, 3\})$ these are $\{(0, 0), (1, 1), (2, 2)\}$.

Then, we multiply each of these points by $(-3, -2)$ and shift $P_2(B)$ by these amounts. This gives the following collection of intervals that form $T''(D)$ (where the different intervals are separated by dashed lines):

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{This is $T''(D)$ for $w'' = (1, 1)$. It is made up of 1 interval of length 1 corresponding to $\{1, 2\}$, 2 intervals of length 2 corresponding to $\{1, 3\}$, and 3 intervals of length 3 corresponding to $\{2, 3\}$. Note that the origin is the leftmost point of the rightmost interval.}
\end{figure}

Finally, to find a set of representatives for $S(D)$, we take all points $z$ such that for all sufficiently small $\varepsilon > 0$, $z + \varepsilon(1) \in T''(D)$.
Let $f''_w$ be the map that sends $S(D) \to B(D)$ by mapping the lattice points in Figure 6 to bases associated to the intervals they are shifted into. We get the following set of representatives for $S(D)$:

- $f''_w^{-1}([1, 2]) = \{(0, 0, 0)\}.$
- $f''_w^{-1}([1, 3]) = \{(0, 0, -10), (0, 0, -9), (0, 0, -5), (0, 0, -4)\}.$
- $f''_w^{-1}([2, 3]) = \{(0, 0, -13), (0, 0, -12), (0, 0, -11), (0, 0, -8), (0, 0, -7), (0, 0, -6), (0, 0, -3), (0, 0, -2), (0, 0, -1)\}.$

Note that these are the same representatives that we get as if we apply the second part of Lemma 4.1 to the representatives we obtained in Example 3.23 with the same shifting vector.

Figure 7 gives some examples of tiles in $\mathbb{R}^2$ computed using Sage. On the left is the tile with different colors indicating different bases and on the right is 9 copies of the tile to show how the tiling works.

5. Connection to the Backman-Baker-Yuen Bijections

In [BBY19], Backman, Baker, and Yuen construct a family of bijections between the bases and sandpile group equivalence classes of a regular matroid. These bijections depend on a choice of basis and a choice of acyclic circuit and cocircuit signatures (which we will define below). Furthermore, for an arbitrary integer matrix $D$, The authors give a family of bijections between $B(D)$ and a subset of $S(D)$ of size $|B(D)|$.

The maps we defined in Definition 3.21 are from $S(D)$ to $B(D)$ and depend on a choice of multiplicity 1 basis and shifting vector. In this section, we show that the collection of shifting vectors that give a specific map correspond canonically to a choice of acyclic circuit and cocircuit signature. Furthermore, we show that for a proper choice of shifting vector, and the same choice of distinguished basis, our maps generalize the Backman-Baker-Yuen bijections.

5.1. Shifting Vectors and Hyperplane Arrangements. Let $D$ be a rank $n$ standard representative matrix. For this section, we think of $B \in B(D)$ as a subset of columns of $D$ instead of their indices.
Figure 7. Above are 3 examples of tiles that we obtain by applying Lemma 4.1 to the map from Section 3.
Let $S$ be a linearly independent subset of $n - 1$ columns of $D$. These vectors define a hyperplane in $\mathbb{R}^n$ that we call $H(S)$. Let $\mathcal{H}(D)$ be the central hyperplane arrangement made up of all such $H(S)$. For a reference on hyperplane arrangements, see e.g. [BLVS+99].

For each $B \in \mathcal{B}(D)$, recall the parallelepiped $P_1(B)$ from Definition 3.3. One way to describe $P_1(B)$ is as the region bounded by the following set of $2n$ hyperplanes:

$$\{H(B \setminus c) \mid c \in B\} \cup \{H(B \setminus c) + c \mid c \in B\}$$

where adding a vector to a hyperplane indicates translating all points on the hyperplane by this amount.

**Definition 5.1.** For some $B \in \mathcal{B}(D)$. Let $\phi_B$ map the pair $(z, c)$ to $\{0, 1, 2\}$ where $z \in \mathbb{Z}^n \cap P_1(B)$ and $c \in B$.

$$\phi_B(z, c) = \begin{cases} 1 & \text{if } z \in H(B \setminus c) \\ 2 & \text{if } z \in (H(B \setminus c) + c) \\ 0 & \text{if } z \notin H(B \setminus c) \cup (H(B \setminus c) + c). \end{cases}$$

This map is well-defined since a point cannot lie in two parallel hyperplanes.

**Definition 5.2.** A corner point of $P_1(B)$ is a $z \in P_1(B)$ such that $\phi_B(z, c) \neq 0$ for all $c \in B$.

**Lemma 5.3.** There are exactly $2^n$ corner points (one for each element of $\{1, 2\}^n$) and they are all in $\mathbb{Z}^n$.

**Proof.** Label the columns associated with $B$ as $(c_1, \ldots, c_n)$. For each $\xi \in \{1, 2\}^n$ there is exactly one point $z$ such that $\phi_B(z, c_i)$ is the $i^{th}$ entry of $\xi$. This point is explicitly given by:

$$z = \sum_{\{i \mid \text{the } i^{th} \text{ entry of } \xi \text{ is } 2\}} c_i.$$ 

Since each $c_i \in \mathbb{Z}^n$, this point $z$ is also in $\mathbb{Z}^n$. \hfill \Box

Let $w = (w_1, \ldots, w_n, \hat{w}_1, \ldots, \hat{w}_m)$ be a shifting vector. As in Definition 4.2, we will think of this as a pair of vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For now, we focus on $(w_1, \ldots, w_n)$.

**Proposition 5.4.** A vector $w \in \mathbb{R}^n$ is the first part of a shifting vector if and only if it does not lie on any hyperplane in $\mathcal{H}(D)$. Furthermore, if $w_1, w_2 \in \mathbb{R}^{n+m}$ have the same last $m$ entries, $f_w$ and $f_{w'}$ are equivalent maps if and only if the first $n$ entries of $w$ and the first $n$ entries of $w'$ lie in the same chamber of $\mathcal{H}(D)$.

**Proof.** The first claim holds because the affine spans of the facets of $P_1(B)$ over all $B \in \mathcal{B}(D)$ are exactly the hyperplanes in $\mathcal{H}(D)$. For the second claim, let $w = (w_1, \ldots, w_n)$ be the first $n$ entries of a shifting vector. For some $B \in \mathcal{B}(D)$ consider an arbitrary $z$ in $\mathbb{Z}^n \cap P_1(B)$. $z$ is associated with $B$ (in the sense of Definition 4.2) if and only if:

- For every $c \in B$ where $\phi(z, c) = 1$, $w$ is on the same side of $H(B \setminus c)$ as $c$.
- For every $c \in B$ where $\phi(z, c) = 2$, $w$ is on the opposite side of $H(B \setminus c)$ as $c$. 


This description depends only on which chamber \( w \) is in, which proves the if portion. For the only if portion, for each shifting vector \( w \) and \( B \in \mathcal{B}(D) \), there is a unique corner point associated with \( B \). This corner point will be different if \( w \) is in a different chamber of:

\[
\bigcup_{c \in B} H(B \setminus c).
\]

The proposition follows because we can write every \( H \in \mathcal{H}(D) \) as \( H(B \setminus c) \) for some \( B \in \mathcal{B}(D) \) and \( c \in B \).

**Definition 5.5.** \( \mathcal{H}(\hat{D}) \) is the hyperplane arrangement made up of linearly independent subsets of \( m-1 \) columns of \( \hat{D} \).

We can use very similar logic when working with \( P_2(B) \) and the last \( m \) entries of a shifting vector. In particular, we have the following proposition.

**Proposition 5.6.** Let \( w = (w_1, \ldots, w_n, \hat{w}_1, \ldots, \hat{w}_m) \) and \( w' = (w'_1, \ldots, w'_n, \hat{w}'_1, \ldots, \hat{w}'_m) \) be two different shifting vectors. \( f_w \) and \( f_{w'} \) are equivalent maps if and only if \( (w_1, \ldots, w_n) \) lies in the same chamber of \( \mathcal{H}(D) \) as \( (w'_1, \ldots, w'_n) \) and \( (\hat{w}_1, \ldots, \hat{w}_n) \) lies in the same chamber of \( \mathcal{H}(\hat{D}) \).

Thus, the number of different maps that we get from \( S(D) \to \mathcal{B}(D) \) (given a choice of distinguished basis) is equal to the number of chambers of \( \mathcal{H}(D) \) multiplied by the number of chambers of \( \mathcal{H}(\hat{D}) \). This quantity is known to be purely matroidal (not depending on basis multiplicities) and can be calculated using Zaslavsky’s Theorem (see [Zas75]).

### 5.2. Acyclic Signatures

The map given by Backman, Baker, and Yuen is defined in terms of **acyclic circuit and cocircuit signatures**. We will see that a choice of acyclic circuit or cocircuit signature corresponds to a choice of chamber in an arrangement of \((n+m-1)\)-dimensional hyperplanes in \( \mathbb{R}^{n+m} \). Furthermore, we show that the chambers in the arrangement corresponding to acyclic cocircuit (resp. circuit) signature are in natural bijection with chambers of \( \mathcal{H}(D) \) (resp. \( \mathcal{H}(\hat{D}) \)). In other words, choosing a shifting vector is equivalent to choosing a pair of acyclic signatures.

Let \( (E, \mathcal{B}) \) be a matroid. A **cocircuit** is a minimal subset of \( E \) that has nontrivial intersection with every \( B \in \mathcal{B} \).

**Lemma 5.7 ([DKM15, Lemma 4.4])**. Let \( (E, \mathcal{B}) \) be a rank \( n \) matroid represented by a matrix \( D \) and let \( C \) be a cocircuit of this matroid. The set:

\[
\{0\} \cup \{v \in \text{Cut}(D) \mid \text{supp}(v) = C\}
\]

is a 1-dimensional subspace of \( \text{Cut}(D) \) (where \( \text{supp}(v) \) is the support of \( v \)).

This subspace is normal to a unique central hyperplane in \( \mathbb{R}^{|E|} \). We call this hyperplane \( H_C \). Let \( \mathcal{H}'(D) \) be the **central hyperplane arrangement** made up of \( H_C \) for all cocircuits \( C \).

**Definition 5.8.** An **acyclic cocircuit signature** is a choice of chamber of \( \mathcal{H}'(D) \).

\(^2\text{Cocircuits can also be equivalently defined as circuits of the dual matroid.}\)}
This isn’t the same definition as the one given in [BBY19], but it is equivalent.

We will show that \( \mathcal{H}(D) \) and \( \mathcal{H}'(D) \) are closely related. Every \( H_C \) has a normal vector in \( \text{Cut}(D) \). This means that if we restrict \( \mathcal{H}'(D) \) to \( \text{Cut}(D) \) instead of all of \( \mathbb{R}^{|E|} \), we still get the same number of chambers. Because the rows of \( D \) are full rank, they are a basis for \( \text{Cut}(D) \). Thus, multiplication by \( D \) is a bijection between \( \mathbb{R}^n \) and \( \text{Cut}(D) \).

**Lemma 5.9.** A set \( S \subseteq E \) is a maximal rank \( n - 1 \) subset of \( E \) if and only if \( E \setminus S \) is a cocircuit.

**Proof.** \( \text{rk}(S) = n \) if and only if \( S \) contains a basis. \( S \) contains a basis if and only if there is some \( B \in \mathcal{B} \) such that \( B \cap (E \setminus S) = \emptyset \). It follows that \( E \setminus S \) contains a cocircuit if and only if \( \text{rk}(S) < n \). When we maximize the size of \( S \), we minimize the size of \( E \setminus S \) and get a cocircuit. \( \square \)

**Proposition 5.10.** Let \( S \) be a maximal rank \( n - 1 \) subset of \( E \) and \( H(S) \) be the hyperplane defined in Definition 5.5. Multiplication by \( D \) sends \( H(S) \) to \( H_{E \setminus S} \cap \text{Cut}(D) \).

**Proof.** Let \( v \) be a normal vector to \( H(S) \). By definition, all of the columns corresponding to \( S \) are orthogonal to \( v \). Thus, \( \text{supp}(D \cdot v) \subset (E \setminus S) \). By Lemma 5.9, \( E \setminus S \) is a cocircuit, and \( v \) is mapped to a normal vector to \( H_{E \setminus S} \). The proposition follows because \( D \) is an linear map with trivial kernel. \( \square \)

We now have a natural way to associate chambers of \( \mathcal{H}(D) \) with chambers of \( \mathcal{H}'(D) \). If we are given a chamber of \( \mathcal{H}(D) \), we multiply any vector in this chamber by \( D \) to get a chamber of \( \mathcal{H}'(D) \). Alternatively, if we are given a chamber of \( \mathcal{H}'(D) \), we choose a vector \( v \) in this chamber that is also in \( \text{Cut}(D) \). There is a unique vector \( v' \) such that \( D \cdot v' = v \) and this vector gives a chamber of \( \mathcal{H}(D) \).

If we replace \( D \) with \( \hat{D} \) and cocircuit with circuit, we get the exact same results. Thus, we can associate each chamber of \( \mathcal{H}(\hat{D}) \) with an acyclic circuit signature. This means that choosing a shifting vector is equivalent to choosing acyclic circuit and cocircuit signatures.

### 5.3. Comparison to the Backman-Baker-Yuen Map.

Let \( D \) be a standard representative matrix and \( (E, \mathcal{B}) \) be the matroid it represents. Given acyclic circuit and cocircuit signatures, Backman, Baker, and Yuen give a bijection between \( \mathcal{B}(D) \) and a size \( |\mathcal{B}(D)| \) subset of \( \mathcal{S}(D) \) [BBY19]. Given the same information, we give a map from \( \mathcal{S}(D) \) to \( \mathcal{B}(D) \). We will show that the \( s \in \mathcal{S}(D) \) that Backman, Baker, and Yuen associate with a given \( B \in \mathcal{B}(D) \) is also associated with \( B \) in our map. Furthermore, we show that this equivalence class contains the unique corner point associated with \( B \) (see Definition 5.2).

Choose a shifting vector \( w = (w_1, \ldots, w_n, \hat{w}_1, \ldots, \hat{w}_n) \). For each set \( S \) of \( n - 1 \) linearly independent columns of \( D \), let \( n_S \) be a normal vector to the hyperplane \( H(S) \) whose dot product with \( (w_1, \ldots, w_n) \) is positive. For each set \( \hat{S} \) of \( m - 1 \) linearly independent columns of \( \hat{D} \), let \( \hat{n}_S \) be a normal vector to the hyperplane \( H(\hat{S}) \) whose dot product with \( (\hat{w}_1, \ldots, \hat{w}_n) \) is positive. Label the columns of \( D \) as \( (c_1, \ldots, c_{n+m}) \) and the columns of \( \hat{D} \) as \( (\hat{c}_1, \ldots, \hat{c}_{n+m}) \).

Consider any \( B \in \mathcal{B}(D) \) and let \( \hat{B} \) be the columns of \( \hat{D} \) that make up \( P_2(B) \). We remarked in the proof of Proposition 5.4 that there is a unique corner point associated with \( B \). This point, which we will call \( p(B,w) \) has its first \( n \) entries given by:
\[ \sum_{\{c \in B | c \cdot n_{B,\backslash} < 0\}} c \]

and its last \( m \) entries given by:

\[ \sum_{\{\hat{c} \in \hat{B} | \hat{c} \cdot \hat{n}_{\hat{B},\backslash} < 0\}} \hat{c}. \]

Let \( p^*(B,w) \) be the point in \( \{0,1\}^{n+m} \) such that for each \( i \) in \([n+m]\):

- If \( i \in B \), the \( i^{th} \) entry of \( p^*(B,w) \) is 1 if \( c_i \cdot n_{B,\setminus} < 0 \) and 0 otherwise.
- If \( i \not\in B \), the \( i^{th} \) entry of \( p^*(B,w) \) is 1 if \( \hat{c}_i \cdot \hat{n}_{\hat{B},\setminus} < 0 \) and 0 otherwise.

**Lemma 5.11.** \( p(B,w) \) is in the same sandpile equivalence class as \( p^*(B,w) \). This means that every corner point is in the same sandpile equivalence class as a vector in \( \{0,1\}^{n+m} \).

**Proof.** For \( i > n \), if \( c_i \in B \), we add the \((i - n)^{th}\) row of \( \hat{D} \) to \( p(B,w) \). For \( i \leq n \), if \( c_i \not\in B \) (or equivalently if \( \hat{c}_i \not\in \hat{B} \)), we add the \( i^{th} \) row of \( D \) to \( p(B,w) \). Using the relationship between rows and columns of \( D \), one can show directly that this procedure gives the point \( p^*(B,w) \).

\( \square \)

**Example 5.12.** Consider the matrix

\[ D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \]

which we also explored in Examples 3.5, 3.23, and 4.7. Label the rows of \( D \) as \( c_1, c_2, \) and \( c_3 \). \( \mathcal{H}(D) \) is made up of the lines \( y = 0 \) (corresponding to \( H(c_1) \)), \( x = 0 \) (corresponding to \( H(c_2) \)), and \( 2x - 3y = 0 \) (corresponding to \( H(c_3) \)). Label the columns of \( \hat{D} \) as \( \hat{c}_1, \hat{c}_2, \) and \( \hat{c}_3 \). \( \mathcal{H}(\hat{D}) \) contains a single hyperplane, \( H(\emptyset) \), which is the origin. If we choose the shifting vector \( w = (1,1,1) \), then we can choose \( n_{c_1} = (0,1), n_{c_2} = (1,0), n_{c_3} = (-2,3), \) and \( \hat{n}_0 = (1) \).

Let \( B_1 = \{c_1,c_2\}, B_2 = \{c_1,c_3\}, \) and \( B_3 = \{c_2,c_3\} \). Then, we have:

- \( p(B_1,w) = (0,0,0) \)
- \( p(B_2,w) = (1,0,-2) \)
- \( p(B_3,w) = (0,0,-3) \)

- \( p^*(B_1,w) = (0,0,0) \)
- \( p^*(B_2,w) = (1,1,0) \)
- \( p^*(B_3,w) = (1,0,0) \).

**Theorem 5.13.** Let \( w \) be a shifting vector. Given the acyclic circuit and cocircuit signatures associated with \( w \), the Backman-Baker-Yuen bijection from \([BBY19]\) maps each basis \( B \) to the sandpile group equivalence class containing \( p^*(B,-w) \).

Justification for this theorem can be found in \([BBY19]\) or \([Yue18]\). We could replace \( p^*(B,-w) \) with \( p(B,-w) \) since these are in the same equivalence class, but \( p^*(B,-w) \) is more naturally obtained from the Backman-Baker-Yuen map. Note that the fact that the this theorem depends on \(-w\) instead of \(w\) is not a meaningful difference, it comes from an arbitrary choice of direction in the definition of each map.
6. Further Questions

One prevalent assumption we made in this paper was the existence of a basis with multiplicity 1. This allowed us to work with standard representative matrices (see Theorem 2.12) and gave us a natural choice for integral basis of the cut and flow lattices. The weaker assumption that the multiplicities are relatively prime helped justify our definition of the sandpile group of cell complexes (see Remark 2.9).

**Question 6.1.** Is there a way to extend the map from Section 3 to work with matrices that do not have a basis with multiplicity 1? What if we still assume that the multiplicities are relatively prime?

The main purpose of our map was to associate each equivalence class of the sandpile group to a basis. However, in constructing this map, we also give a representative for each equivalence class. In particular, this is the set of \( w \)-representatives.

**Question 6.2.** What are some properties of the \( w \)-representatives that we get from different choices of distinguished basis or shifting vector? Are they generalizations of any known sets of representatives of the graphical sandpile group (such as superstable or critical configurations)?

Finally, while this paper does not require us to work with graphical or regular matroids, we do require a specific matrix representative in order to define the sandpile group.

**Question 6.3.** Is there a reasonable way to define the sandpile group of a non-representable matroid?

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