On spectral gap properties and extreme value theory for multivariate affine stochastic recursions

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Abstract
We consider a general multivariate affine stochastic recursion and the associated Markov chain on $\mathbb{R}^d$. We assume a natural geometric condition which implies existence of an unbounded stationary solution and we show that the large values of the associated stationary process follow extreme value properties of classical type, with a non trivial extremal index. The proof is based on a spectral gap property for the action of the corresponding Markov operator on spaces of regular functions with slow growth, and on the clustering properties of large values in the recursion.

Keywords : Spectral gap, Extreme value, Affine random recursion, Limit theorem, Point process, Cluster index.

1 Introduction

Let $V = \mathbb{R}^d$ be the $d$-dimensional Euclidean space and let $\lambda$ be a probability on the affine group $A$ of $V$. Let $(A_n, B_n)$ be a sequence of $A$-valued i.i.d. random variables distributed according to $\lambda$ and let us consider the affine stochastic recursion on $V$ defined by

$$X_n = A_n X_{n-1} + B_n$$

for $n \in \mathbb{N}$. We denote by $P$ the corresponding Markov kernel on $V$ and by $\mathbb{P}$ the product measure $\lambda \otimes N$ on $A^\mathbb{N}$. Our geometric hypothesis $(H)$ on $\lambda$ implies that $P$ has a unique invariant probability $\rho$ on $V$ and the support of $\rho$ is unbounded. In our situation (see [13]), the quantity $\rho_\lambda(|v| > t)$ is asymptotic ($t \to \infty$) to $a^{-1} c t^{-\alpha}$ with $\alpha > 0$, $c > 0$. Furthermore, the measure $\rho$ is multivariate regularly varying, a basic property for the development of extreme value theory, i.e. for the study of exceptionally large values of $X_k(1 \leq k \leq n)$ for $n$ large (see [28]). In such a situation of weak dependence, spectral gap properties of operators associated to $P$ play also an important role via a multiple mixing condition described in [7] for the case of step functions. We observe that the same idea was used in the proofs of limit theorems for the largest coefficient in the continued fraction expansion of a real number uniformly distributed in the interval $[0, 1]$ (see [25], [31]), as well as in the proofs of limit theorems for $\sum_{k=1}^{n} X_k$ (see [15]). In the context of geometric ergodicity, assuming a density condition on the law of $B_n$, partial results were obtained in [21]. Here
we go further in this direction replacing geometric ergodicity by condition \((H)\). We observe that, if \(\lambda\) is singular with respect to Haar measure on \(A\), then the operator \(P\) on \(V\) is not \(\nu\)-geometrically ergodic in general, hence the classical framework of \([22]\) for asymptotics of Markov chains is not convenient in our setting. However condition \((H)\) implies that the operator \(P\) has a spectral gap property in the spaces of Hölder functions with polynomial growth considered below, a fact which allow us to deduce convergence with exponential speed on Hölder functions. A typical example of this situation occurs if the support of \(\lambda\) is finite and generates a dense subsemigroup of the affine group \(A\).

If \(Z_+\) is the set of non negative integers, we denote by \(\mathbb{P}_\rho\) the Markov probability on \(V^{Z_+}\) defined by the kernel \(P\) and the initial probability \(\rho\). In this paper we establish spectral gap properties for the action of \(P\) on Hölder functions and we deduce fundamental extreme value statements for the point processes defined by the \(\mathbb{P}_\rho\)-stationary sequence \((X_k)_{k \geq 0}\). Our results are based on the fact that the general conditions of multiple mixing and anticlustering used in extreme value theory of stationary processes (see \([7]\)) are valid for affine stochastic recursions, under condition \((H)\). We observe that, in the context of Lipchitz functions, the above mixing property is a consequence of the spectral gap properties studied below; it turns out that the use of advanced point process theory allows us to extend this mixing property to the classical context of continuous functions. We note that, if \(d > 1\), the set of probabilities \(\lambda\) on the affine group \(A\), which satisfy condition \((H)\), is open in a natural weak topology; hence in this sense, hypothesis \((H)\) is generic. Then, our framework allow us to develop extreme value theory for a large class of natural examples, including the so-called GARCH process as a very special case (see \([10]\)).

In order to sketch our results, we recall that Fréchet’s law \(\Phi^\omega\) with positive parameters \(\alpha, a\) is the probability on \(\mathbb{R}_+\) defined by the distribution function \(\Phi^\alpha_\omega([0, t]) = \exp(-at^{-\alpha})\).

We denote by \(\mu\) the projection of \(\lambda\) on the linear group \(G = GL(V)\). Also, we consider the associated stochastic linear recursion \(Y_n = A_nY_{n-1}\), we denote by \(Q\) the corresponding Markov kernel on \(V \setminus \{0\}\) and by \(Q = \mu^\otimes\mathbb{N}\) the product measure on \(G^\mathbb{N}\); we write \(S_n = A_n \cdots A_1\) for the product of random matrices \(A_k(1 \leq k \leq n)\). Extending previous work of H. Kesten (see \([19]\)), a basic result proved in \([13]\) under condition \((H)\) is that for some \(\alpha > 0\), the probability \(\rho\) is \(\alpha\)-homogeneous at infinity, hence \(\rho\) has an asymptotic tail measure \(\Lambda \neq 0\) which is a \(\alpha\)-homogeneous \(Q\)-invariant Radon measure on \(V \setminus \{0\}\). The multivariate regular variation of \(\rho\) is a direct consequence of this fact. Also, it follows that, if \(B_t \subset V\) is the ball of radius \(t > 0\) centered at \(0 \in V\) and \(B'_t = V \setminus B_t\), then we have \(\Lambda(B'_t) = \alpha^{-1}ct^{-\alpha}\) with \(c > 0\). In particular, \(\Lambda(B'_t)\) is finite and the projection of \(\rho\) on \(\mathbb{R}_+\), given by the norm map \(v \to |v|\) has the same asymptotic tail as \(\Phi^\alpha_\omega\). If \(\alpha_n = (cn)^{1/\alpha}\), it follows that the mean number of exceedances of \(u_n\) by \(|X_k| (1 \leq k \leq n)\) converges to one. It will appear below that \(u_n\) is an estimate of \(\text{sup}\{\Delta_k \geq 0; 1 \leq k \leq n\}\).

Then, one of our main results is the convergence in law of the normalized maximum of the sequence \(|X_1|, |X_2|, \ldots, |X_n|\) towards Fréchet’s law \(\Phi^\theta_\alpha\) with \(\theta \in ]0, 1[\). A closely related point process result is the weak convergence (see \([28]\)) of the time exceedances process
\[ N_t^n = \sum_{k=1}^n \varepsilon_{n-1} k 1\{|X_k|>u_n\} \]

towards a compound Poisson process with intensity \( \theta \) and cluster probabilities depending on the renewal point process \( \pi^v_n = \sum_{i=0}^\infty \varepsilon s_i(\omega) v \) and on the \( Q \)-invariant measure \( \Lambda \). The significance of the relation \( \theta < 1 \) is that, in our situation, values of the sequence \( \{X_k\}_{k \geq 0} \) and is in contrast with the well known situation of positive i.i.d. random variables with tail also given by \( \Phi^c_\alpha \), where the property \( \theta = 1 \) is satisfied. If Euclidean norm is replaced by another norm, the value of \( \theta \) in the new setting is changed but the condition \( \theta \in [0, 1] \) remains valid. For affine stochastic recursions in dimension one, if \( A_n, B_n \) are positive and condition \( (H) \) is satisfied, convergence to Fréchet’s law and \( \theta \in [0, 1] \) was proved in [17]. We observe that our result is the natural multivariate extension of this fact. Also, if \( d \geq 1 \), assuming technical conditions on the random walk \( X_n \) and density for the law of \( B_n \) with respect to Lebesgue measure, the two above convergences were shown in [21]. Here our proofs use the tools of point processes theory and a remarkable formula (see [2]) for the Laplace functional of a cluster process \( C = \sum_{j>0} \varepsilon Z_j \) on \( V \setminus \{0\} \), depending only on \( \mu, \Lambda \), which describes locally the large values of \( \{X_n\}_{n \geq 0} \). As a consequence of Fréchet’s law and in the spirit of [26], we obtain a logarithm law for affine random walk.

To go further, we consider the linear random walk \( Y_n^v = S_n(\omega) v \) on \( V \setminus \{0\} \), we observe that condition \( (H) \) implies \( \lim_{n \to \infty} S_n(\omega) v = 0 \), \( \mathbb{Q} \)-a.e and we denote by \( \mathbb{Q}_\Lambda \) the Markov measure on \( (V \setminus \{0\})^{Z_+} \) defined by the kernel \( \mathbb{Q} \) and the \( Q \)-invariant initial measure \( \Lambda \). We show below the weak convergence to a limit process \( N \) of the sequence of space-time exceedances processes

\[ N_n = \sum_{i=1}^n \varepsilon_{n-1, i, u_n^{-1} X_i} \]
on \([0, 1] \times (V \setminus \{0\})\). In restriction to \([0, 1] \times B'_\delta \), with \( \delta > 0 \), \( N \) can be expressed in terms of \( C \) and of a Poisson component on \([0, 1]\) with intensity \( \theta \delta^{-\alpha} \); the expression of \( C \) involves the renewal point process \( \pi^v_n \) and \( \mathbb{Q}_\Lambda \). Using the framework and the results of ([2], [3], [7]), we describe a few consequences of this convergence. In particular we consider also, as in ([7], [8]), the convergence of the normalized partial sums \( \sum_{i=1}^n X_i \) towards stable laws, if \( 0 < \alpha < 2 \), in the framework of extreme value theory. Also, as observed in [7], this convergence is closely connected to the convergence of the sequence of space exceedences point processes on \( V \)

\[ N_n^s = \sum_{i=1}^n \varepsilon_{u_n^{-1} X_i} \]
towards a certain infinitely divisible point process \( N^s \). Here the Laplace functional of \( N^s \) can be expressed in terms of \( \Lambda \) and \( \pi^v_n \). In these studies we follow closely the approaches previously developed in ([2], [3], [7]) in the context of extreme value theory for general stationary processes, in particular we make
use of the concepts of tail and cluster processes introduced in [2]. This allow us to prove explicit extreme values properties for affine random walks, under condition (H), and to recover, in a natural setting, the characteristic functions of the above \( \alpha \)-stable laws, as described in [15] if \( d = 1 \) and in ([5], [11]) if \( d > 1 \), completing thereby the results of ([1], [2], [7]). For self containment reasons we have developed anew a few arguments of ([2], [3]) in our situation. We refer to [4] for information on products of random matrices and to ([14], [16]) for short surveys of the above results.

2 The tail process and the cluster process

In this paper, we will always assume that \( \lambda \) satisfies condition (H) explained below.

2.1 Homogeneity at infinity of the stationary measure

We recall condition (H) from [13], for the probability \( \lambda \) on the affine group of \( V \). A semigroup \( T \) of \( GL(V) = G \) is said to satisfy i-p if

a) \( T \) has no invariant finite family of subspaces

b) \( T \) contains an element with a dominant eigenvalue which is real and unique.

Condition i-p implies that the action of \( T \) on the projective space of \( V \) is proximal; heuristically speaking this means that, \( T \) contracts asymptotically two arbitrary given directions to a single one, hence the situation could be compared to a 1-dimensional one. Condition i-p for \( T \) is valid if and only if it is valid for the group which is the Zariski closure of \( T \). Hence it is valid in particular if \( T \) is Zariski dense in \( G \) (see [27]) ; also it is valid for \( T^{-1} \) if and only if it is valid for \( T \). Below we will denote by \( T \) the closed subsemigroup generated by \( \text{supp}(\mu) \), the support of \( \mu \).

For \( g \in G \) we write \( \gamma(g) = \sup(\|g\|, \|g^{-1}\|) \) and we assume \( \int \log \gamma(g) d\mu(g) < \infty \). For \( s \geq 0 \) we write \( \log k(s) = \lim_{n \to \infty} -\frac{1}{n} \log \int |g|^s d\mu^n(g) \) where \( \mu^n \) denotes the \( n \)-th convolution power of \( \mu \) and we write \( L(\mu) \) for the dominant Lyapunov exponent of the product \( S_n(\omega) = A_n \cdots A_1 \) of random matrices \( A_k (1 \leq k \leq n) \) i.e. \( L(\mu) = \lim_{n \to \infty} -\frac{1}{n} \int \log |g| d\mu^n(g) = k'(0) \). We denote by \( r(g) \) the spectral radius of \( g \in G \). We say that \( T \) is non arithmetic if \( r(T) \) contains two elements with irrational ratio. Condition (H) is the following :

1) \( \text{supp}(\lambda) \) has no fixed point in \( V \).
2) There exists \( \alpha > 0 \) such that \( k(\alpha) = \lim_{n \to \infty} \mathbb{E}(|S_n|^\alpha)^{1/n} = 1 \).
3) There exists \( \varepsilon > 0 \) with \( \mathbb{E}(|A|^\alpha \gamma^\varepsilon(A) + |B|^{\alpha + \varepsilon}) < \infty \).
4) If \( d > 1 \), \( T \) satisfies i-p and if \( d = 1 \), \( T \) is non arithmetic.

The above conditions imply in particular that \( L(\mu) < 0 \), \( k(s) \) is analytic, \( k(s) < 1 \) for \( s \in [0, \alpha[ \) and there exists a unique stationary probability \( \rho \) for \( \lambda \) acting by convolution on \( V \); the support of \( \rho \) is unbounded. Property 1 guarantees that \( \rho \) has no atom and says that the action of \( \text{supp}(\lambda) \) is not conjugate to a linear action. Property 2 is responsible for the
α-homogeneity at infinity of ρ described below; if k(s) is finite on |0, ∞| and there exists g ∈ T with r(g) > 1, then Property 2 is satisfied. Also if d > 1, condition i-p is basic for renewal theory of the random walk $S_n(\omega)v$ and it implies that T is non arithmetic.

In the appendix we will show that condition (H) is open in the weak topology of probabilities on the affine group, defined by convergence of moments and of values on continuous compactly supported functions.

Below, we use the decomposition of $V \setminus \{0\} = S^{d-1} \times \mathbb{R}_{>0}$ in polar coordinates, where $S^{d-1}$ is the unit sphere of V. We consider also the Radon measure $\ell^\alpha$ on $\mathbb{R}_{>0}$ ($\alpha > 0$) given by $\ell^\alpha(dt) = t^{-\alpha-1}dt$. We recall (see [19]) that, if $(A_n,B_n)_{n \in \mathbb{N}}$ is an i.i.d sequence of A-valued random variables with law λ and $L_\mu < 0$, then ρ is the law of the $\mathbb{P} - a.e$ convergent series $X = \sum_{0 \leq n} A_1 \cdots A_k B_{k+1}$. We observe that a family $h_t(t \in \mathbb{R}^*)$ of automorphisms of the group A is given by $h_t(a,b) = (a, tb)$. Then it follows that the stationary probability for $h_t(\lambda)$ is $t.\rho$, where $t.\rho$ denotes the push forward of $\rho$ under the dilation $v \to tv$.

**Theorem 2.1** (see [13], Theorem C)

Assume that λ satisfies condition (H). Then the operator P has a unique stationary probability $\rho$, the support of $\rho$ is unbounded and we have the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \to 0^+} t^{-\alpha}(t.\rho) = \Lambda = c(\sigma^\alpha \otimes \ell^\alpha)$$

where $c > 0$ and $\sigma^\alpha$ is a probability on $S^{d-1}$. Furthermore $\Lambda$ is a $Q$-invariant Radon measure on $V \setminus \{0\}$.

We observe that, for $d > 1$, if $supp(\lambda)$ is compact, generates a Zariski dense subgroup of A (see [21]), $L(\mu) < 0$, and T is unbounded, then condition (H) is satisfied. For $d = 1$, if $supp(\lambda)$ is compact, the hypothesis of ([17], Theorem 1.1) is equivalent to condition (H).

The existence of Λ stated in the theorem implies multivariate regular variation of $\rho$. If the convergence stated in the theorem is valid we say that $\rho$ is homogeneous at infinity; below we will make essential use of this property.

Under condition (H), Λ gives zero mass to any submanifold and $\sigma^\alpha$ has positive dimension. We observe that, if the sequence $(A_n,B_n)_{n \in \mathbb{N}}$ is replaced by $(A_n,tB_n)_{n \in \mathbb{N}}$ with $t \in \mathbb{R}^*$, then the asymptotic tail measure is replaced by $t.\Lambda$, in particular the constant $c$ is replaced by $|t|^\alpha c$. We observe that, as shown in [13], the $Q$-invariant Radon measure $\Lambda$ is extremal or can be decomposed in two extremal measures. Hence, if the action of T on $S^{d-1}$ has a unique minimal subset, then $\Lambda$ is symmetric and the shift invariant measure $\mathbb{Q}_\Lambda$ on $(V \setminus \{0\})^{\mathbb{Z}^+}$ is ergodic. Otherwise $\mathbb{Q}_\Lambda$ decomposes into two ergodic measures. Hence $\Lambda$ depends only of $\mu$, up to one or two coefficients.

The following is a classical consequence of vague convergence.

**Corollary 2.2** Let f be a non negative $\Lambda$-integrable Borel function on $V \setminus \{0\}$ which has a $\Lambda$-negligible discontinuity set. Then we have $\lim_{t \to 0^+} t^{-\alpha}(t.\rho)(f) = \Lambda(f)$. 

5
2.2 The tail process

We denote $\Omega = G^N$, $\hat{\Omega} = G^Z$, and we endow $\Omega$ (resp $\hat{\Omega}$) with the product probability $Q = \mu^\otimes N$ (resp $\hat{Q} = \mu^\otimes Z$).

We define the $G$-valued cocycle $S_n(\omega)$ where $\omega = (A_k)_{k \in \mathbb{Z}} \in \hat{\Omega}$, $n \in \mathbb{Z}$ by:

$$S_n(\omega) = A_n \cdots A_1 \text{ for } n > 0, \quad S_n(\omega) = A_n^{-1} \cdots A_1^{-1} \text{ for } n < 0, \quad S_0(\omega) = \text{Id}$$

We consider also the random walk $Y_n^v = S_n(\omega)v$ on $V \setminus \{0\}$, starting from $v \neq 0$ and we denote by $Q_A$ (resp $\hat{Q}_A$) the Markov measure on $(V \setminus \{0\})^Z$ (resp $(V \setminus \{0\})^Z$) for the random walk $Y_n^v$ with initial measure $\Lambda$. These measures are invariant under the shift $\tau$ on $(V \setminus \{0\})^Z$ (resp $(V \setminus \{0\})^Z$).

If we denote by $\sigma$ the shift on $\hat{\Omega}$ and by $\hat{\sigma}$ the extended shift on $\hat{\Omega} \times (V \setminus \{0\})$ defined by $\hat{\sigma}(\omega, v) = (\sigma \omega, A_0v)$, then $\hat{Q} \otimes \Lambda$ is $\hat{\sigma}$-invariant and $\hat{Q}_A$ is simply the projection of $\hat{Q} \otimes \Lambda$ on $(V \setminus \{0\})^Z$, under the map $((\omega, v) \mapsto (S_k(\omega)v)_{k \in \mathbb{Z}})$.

The normalized restriction of $\Lambda$ to $B'$ is denoted $\Lambda_1$, hence $\Lambda_1(B') = \tau^{-\alpha}$ if $t > 1$ and we write $Q_{A_1} = c^{-1} \alpha (1_{B'} \circ \pi)Q_A$, $\hat{Q}_{A_1} = c^{-1} \alpha (1_{B'} \circ \pi)\hat{Q}_A$, where $\pi$ denotes the projection on $V \setminus \{0\}$. We note that the probability $Q_{A_1}$ (resp $\hat{Q}_{A_1}$) extends to $V^Z$ (resp $\hat{V}^Z$) and its extension will be still denoted $Q_{A_1}$ (resp $\hat{Q}_{A_1}$).

We consider the probability $\rho$, the shift $\tau$ on $V^Z$ (resp $\hat{V}^Z$) the shift-invariant Markov measure $P_\rho$ (resp $\hat{P}_\rho$) on $V^Z$ (resp $\hat{V}^Z$), where $\rho$ is the law of $X_0$ and $P_\rho$ is the projection of $\hat{P}_\rho$ on $V^Z$. Since $\rho(\{0\}) = 0$, we can replace $V$ by $V \setminus \{0\}$ when working under $P_\rho$. For $0 < j \leq i$ we write $S_j^\rho = A_i \cdots A_j$ and $S_{i+1}^{\rho} = I$. Expectation with respect to $P$ or $Q$, $\hat{Q}$ will be simply denoted by the symbol $E$. If expectation is taken with respect to a Markov measure with initial measure $\nu$ we will write $E_{\nu}$. For a family $Y_j(j \in \mathbb{Z})$ of $V$-valued random variables and $k, \ell \in \mathbb{Z}$, it holds:

$$E_{\nu}[\sup_{k \leq \ell} |X_k| > t] = \hat{Q}_{A_1} \{ M_{1,\infty}(Y) \leq 1 \}$$

In particular we have:

$$\lim_{n \to \infty} \lim_{t \to \infty} P_\rho \left( \sup_{1 \leq k \leq n} |X_k| \leq t/|X_0| > t \right) = \hat{Q}_{A_1} \{ M_{1,\infty}(Y) \leq 1 \} := \theta \in [0,1]$$

**Proof:** a) We observe that, since for any $i \geq 0 X_i = S_i X_0 + \sum_{j=1}^i S_{j+1}^i B_j$ and $\lim_{t \to \infty} \sum_{j=1}^p S_{j+1}^i B_j = \ldots$
0, \mathbb{P}_\rho - a.e., the random vectors \((t^{-1}X_i)_{0 \leq i \leq p+q}\) and \((t^{-1}S_iX_0)_{0 \leq i \leq p+q}\) have the same asymptotic behaviour in \(\mathbb{P}_\rho\)-law, conditionally on \(|X_0| > t\). Also by stationarity of \(\widehat{\mathbb{P}}_\rho\), for \(f\) continuous and bounded on \(V^{p+q+1}\) we have

\[
\mathbb{E}_\rho \{ f(t^{-1}X_{-q}, \ldots, t^{-1}X_p) / |X_0| > t \} = \mathbb{E}_\rho \{ f(t^{-1}X_0, t^{-1}X_1, \ldots, t^{-1}X_{p+q}) / |X_q| > t \}.
\]

From above, using Corollary 2.2, \(\Lambda \{ |x| = 1 \} = 0\), and the formula \(\Lambda \{ |x| > 1 \} = \alpha^{-1}c\) we see that the right hand side converges to:

\[
\alpha^{-1} \int \mathbb{E} \{ f(x, S_1x, \ldots, S_{p+q}x) \} 1_{\{|S_qx| > 1\}} d\Lambda(x).
\]

Hence, using the definition of \(\mathcal{Q}_A\), we get the weak convergence of the process \((Y'_i)_{i \in \mathbb{Z}}\) to \((Y_i)_{i \in \mathbb{Z}}\) as stated in a). Since for any \(x \in V\) we have \(\lim_{t \to \infty} S_t x = 0 \mathbb{Q} - a.e., the formula \(Y_i = S_i Y_0\) gives \(\lim_{i \to \infty} Y_i = 0\), \(\mathcal{Q}_A_1 - a.e.\). If \(\mathcal{Q}_A \{ \lim_{i \to \infty} |Y_i| > 0 \} \neq 0\) then, for some \(\varepsilon, \varepsilon' > 0\) and a sequence \(i = i_k \to -\infty\) we have \(\mathcal{Q}_A \{ |Y_{i_k}| > \varepsilon \} > \varepsilon'\); since \(\mathcal{Q}_A\) is \(\tau\)-invariant we get \(\lim_{i \to -\infty} \mathcal{Q}_A \{ |Y_{i-k}| > \varepsilon \} > \varepsilon'\). Since \(\Lambda(B_x) < \infty\) and \(\lim_{n \to \infty} S_n x = 0 \mathbb{Q} - a.e., this gives the required contradiction.

b) In view of a) and Corollary 2.2, since the discontinuity sets of the functions \(1_{[0,1]}(M^n_t(Y))\) and \(1_{[1,\infty]}(Y_0)\) on \(V^n\) are \(\mathcal{Q}_A_1\)-negligible, we have \(\lim_{t \to \infty} \mathbb{P}_\rho \{ \sup_{1 \leq k \leq n} |t^{-1}X_k| \leq 1/t^{-1} |X_0| > 1 \} = \mathcal{Q}_A_1 \{ M^n_1(Y) \leq 1 \} \).

Hence \(\theta = \lim_{n \to \infty} \lim_{t \to \infty} \mathbb{P}_\rho \{ \sup_{1 \leq k \leq n} |X_k| \leq t^{-1} |X_0| > t \} = \mathcal{Q}_A_1 \{ \sup_{k \geq 1} \mathcal{Q}_A \{ M^{-1}_{\infty}(Y) > 1 \} \}
\]

We write \(\mathcal{Q}_A_1 \{ M^{-1}_{\infty}(Y) \leq 1 \} = 1 - \mathcal{Q}_A_1 \{ M^{-1}_{\infty}(Y) > 1 \}\) and we define the random time \(T\) by \(T = \inf \{ k \geq 1; |Y_{-k}| > 1 \}\) if there exists \(k \geq 1\) with \(|Y_{-k}| > 1\); if such a \(k\) do not exist we take \(T = \infty\). We have by definition of \(T\):

\[
\mathcal{Q}_A_1 \{ M^{-1}_{\infty}(Y) > 1 \} = \sum_{k=1}^{\infty} \mathcal{Q}_A_1 \{ T = k \}
\]

Using stationarity of \(\mathcal{Q}_A\), the definition of \(\mathcal{Q}_A_1\) and a) we get

\[
\mathcal{Q}_A_1 \{ T = k \} = \mathcal{Q}_A \{ |Y_{-1}| \leq 1, |Y_{-2}| \leq 1, \ldots, |Y_{-k+1}| \leq 1; |Y_{-k}| > 1 \}.
\]

The formula \(\mathcal{Q}_A_1 \{ M^{-1}_{\infty}(Y) \leq 1 \} = \mathcal{Q}_A_1 \{ M^{-1}_{\infty}(Y) \leq 1 \}\) follows.

The formula \(\theta = \mathcal{Q}_A \{ M_{\infty}(Y) \leq 1 \}\) and the form of \(Y_i(i \geq 0)\) given in a) imply \(\theta = \mathbb{E} \{ \sup_{1 \leq i \leq 1} d\Lambda_1(x) \} \leq 1\). The condition \(\theta = 1\) would imply for any \(i \geq 1; |S_ix| \leq 1\)

\(\mathcal{Q} \otimes \Lambda_1 - a.e.,\) hence \(\text{supp}(S\Lambda_1) \subset \{ x \in V; |x| \leq 1 \}\). This would contradict the fact that \(\text{supp}\Lambda_1\) is unbounded, hence we have \(\theta < 1\). The inequality \(\theta > 0\) is obtained in Proposition 2.5 below. For a direct proof using standard arguments in ergodic theory see [3] and appendix. □

We note that the process \((Y_j)_{j \in \mathbb{Z}}\) is not stationary. However this process can be viewed as a simplified version of the stationary process \((X_j)_{j \in \mathbb{Z}}\); for example the property \(\lim_{|j| \to \infty} |Y_j| = 0\)
\( \hat{Q}_{\Lambda_1} - a.e \) is an analogue of the weak convergence of \( X_j(|j| \to \infty) \) to the probability \( \rho \).

### 2.3 Anticlustering property

We are going to show that the set of large values of \( X_k(1 \leq k \leq n) \) consists of localized elementary clusters with a few values. An important sufficient condition for localization (see [7]) is proved in Proposition 2.4 below and will allow us to show the existence of a cluster process as defined in [2]. It is called anticlustering and is used in section 4 to decompose the set of values of \( X_k(1 \leq k \leq n) \) into successive quasi-independent blocks. For \( k \leq \ell \) in \( \mathbb{Z} \) we write

\[
M^\ell_k = \sup_{0 \leq i \leq \ell} |X_i|, \quad R^k = \sum_{i=k}^{\ell} \hat{P}_\rho(|X_i| > u_n/|X_0| > u_n) = \mathbb{E}_\rho(\sum_{i=1}^{\ell} \mathbb{1}_{|u_n,|\infty}(|X_i|)|X_0| > u_n),
\]

where we have used stationarity of \( X_k \) in the last equality. Hence it suffices to show the finiteness of such expectations, in the limit.

The following is based on the homogeneity at infinity of \( \rho \), the inequality \( 0 < k(s) < 1 \) if \( 0 < s < \alpha \), and it will imply the finiteness of such expectations, in the limit.

### Proposition 2.4

Assume \( r_n \leq [n^s] \) with \( 0 < s < 1 \), \( \lim_{n \to \infty} r_n = \infty \). Then \( \lim_{n \to \infty} \lim_{m \to \infty} R^m_{r_n} = 0 \). In particular \( \lim_{n \to \infty} \lim_{m \to \infty} \hat{P}_\rho(\sup(M^m_{r_n}, M_{r_n}^m) > u_n/|X_0| > u_n) = 0 \), hence the random walk \( X_n \) satisfies anticlustering. For \( \theta_n \) defined by \( \theta_n^{-1} = \mathbb{E}_\rho(\sum_{i=1}^{\ell} \mathbb{1}_{|u_n,|\infty}(|X_i|)/M_{r_n} > u_n) \) we have

\[
\liminf_{n \to \infty} \theta_n > 0, \text{ and } \theta_n \leq 1.
\]

**Proof:** We observe that:

\[
\hat{P}_\rho(M^m_{r_n} > u_n/|X_0| > u_n) \leq R^m_{r_n}, \quad \hat{P}_\rho(M_{r_n}^m > u_n/|X_0| > u_n) \leq R_{r_n}^m
\]

where we have used stationarity of \( X_k \) in the last equality. Hence it suffices to show

\[
\lim_{n \to \infty} \lim_{m \to \infty} R^m_{r_n} = 0.
\]

For \( i \geq 0 \) we have \( X_i = S_i X_0 + \sum_{j=1}^{i} S_{j+1} B_j \) where \( S_i, X_0 \) are independent, as well as \( X_0, \sum_{j=1}^{i} S_{j+1} B_j \). We write \( I_n^i = \hat{P}_\rho(|X_i| > u_n/|X_0| > u_n) \),

\[
J_n^i = \hat{P}_\rho(|S_i X_0| > 2^{-1} u_n/|X_0| > u_n), \quad K_n^i = \hat{P}_\rho(\sum_{j=1}^{i} |S_{j+1} B_j| > 2^{-1} u_n/|X_0| > u_n),
\]

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hence \( R^n_n = \sum_{i=m}^{n} I^i_n \leq \sum_{i=m}^{n} J^i_n + \sum_{i=m}^{n} K^i_n. \)

We are going to show \( \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=m}^{n} J^i_n = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=m}^{n} K^i_n = 0. \)

We apply Chebyshev's inequality to the \( \chi \)-moments of \( X_n \) with \( \chi \in [0, \alpha[. \) We have:

\[
J^i_n \leq (2u_n^{-1})^\chi \mathbb{E}_\rho(|S_iX_0|^{\chi}/|X_0| > u_n) \leq (2u_n^{-1})^\chi \mathbb{E}(|S_i|^{\chi}) \rho(|X_0|^{\chi}/|X_0| > u_n),
\]

where independence of \( S_i \) and \( X_0 \) have been used in the last formula. Since the law of \( X_0 \) is \( \alpha \)-homogeneous at infinity we have:

\[
\lim_{x \to \infty} x^{-\chi} \mathbb{E}_\rho(|X_0|^{\chi}/|X_0| > x) = \alpha(\alpha - \chi)^{-1}, \quad \limsup_{n \to \infty} J^i_n \leq 2\alpha(\alpha - \chi)^{-1} \mathbb{E}(|S_i|^{\chi}).
\]

Also, using independence of \( X_0 \) and \( \sum_{j=1}^{i} |S^i_{j+1} B_j| \):

\[
K^i_n = \mathbb{P}\{ \sum_{j=1}^{i} |S^i_{j+1} B_j| > 2^{-1} u_n \} \leq (2u_n^{-1})^\chi \mathbb{E}(\sum_{j=1}^{i} |S^i_{j+1} B_j|) \leq (2u_n^{-1})^\chi \mathbb{E}(\sum_{j=1}^{\infty} |S^i_{j+1} B_j|) \chi.
\]

It follows \( \limsup_{m \to \infty} \sum_{i=m}^{n} J^i_n \leq 2\alpha(\alpha - \chi)^{-1} \mathbb{E}(\sum_{i=m}^{\infty} |S_i|^{\chi}). \)

From [13] we know that, since \( 0 < \chi < \alpha \), we have \( k(\chi) < 1 \), hence \( \mathbb{E}(|S_i|^{\chi}) \) decreases exponentially fast to zero; hence the series \( \mathbb{E}(\sum_{i=m}^{\infty} |S_i|^{\chi}) \) converges and \( \lim_{m \to \infty} \mathbb{E}(\sum_{i=m}^{\infty} |S_i|^{\chi}) = 0, \)

\[
\lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=m}^{n} J^i_n = 0.
\]

From above we know that \( R_0 = \sum_{i=1}^{\infty} |S^i_{j+1} B_j| \) has finite \( \chi \)-moment if \( \chi < \alpha \). Then by Chebyshev’s inequality:

\[
\lim_{m \to \infty} \mathbb{P}\{ M^{r_n}_{m+1} > u_n/|X_1| > u_n \} = 0.
\]

By definition of \( M_{r_n} \):

\[
\mathbb{P}\{ M_{r_n} > u_n \} \geq m^{-1} \mathbb{P}\{ |X_{km+1}| > u_n, M_{(k+1)m+1} \leq u_n \},
\]

hence using stationarity:

\[
\mathbb{P}\{ M_{r_n} > u_n \} \geq m^{-1} \mathbb{P}\{ |X_1| > u_n, M^{r_n}_{m+1} \leq u_n \}.
\]

By definition of \( \theta_n \) this can be rewritten as:

\[
\theta_n \geq m^{-1}(1 - \mathbb{P}\{ M^{r_n}_{m+1} > u_n/|X_1| > u_n \}).
\]

Then for \( n \) and \( m \) large, since from above the right hand side is close to \( m^{-1} \), we have \( \theta_n \geq (2m)^{-1} \), hence \( \liminf_{n \to \infty} \theta_n > 0 \). By definition of \( \theta_n^{-1} \) we have \( \theta_n^{-1} \geq 1 \), hence \( \theta_n \leq 1 \)
2.4 The cluster process

In general, for a stationary $V$-valued point process the properties of anticlustering and positivity of the extremal index $\theta$ for a sequence $r_n = o(n)$ with $\lim_{n \to \infty} r_n = \infty$, stated in Proposition 2.4, imply the existence of the cluster process (see [2]). For self containment reasons we give in Proposition 2.5 below a proof of this fact, using arguments of [2]; this gives us also the convergence of $\theta_n$ defined in Proposition 2.4 to $\theta$. For later use we include also in the statement the formula of ([2], Theorem 4.3) giving the Laplace functional for the cluster process restricted to $B'_1$. We recall that the Laplace functional of a random measure $\nu = (\nu_x)_{x \in E}$ on a locally compact separable metric space $E$ endowed with a probability $m$ is given by

$$\psi_\nu(f) = \int e^{x} - \nu_x(f) dm(x)$$

where $f$ is continuous and compactly supported. We recall that weak convergence of a sequence of point processes is equivalent to convergence of their Laplace functionals.

We denote by $r_n$ a sequence as above and we consider the sequence of point processes $C_n = \sum_{i=1}^n \varepsilon_{u_i^{-1}X_i}$ on $V \setminus \{0\}$ under $\mathbb{P}_\rho$ and conditionally on $M_{r_n} = M_{r_n}^\theta > u_n$. Using the tail process $(Y_n)_{n \in \mathbb{Z}}$ defined in Proposition 2.3 above, we show that $C_n$ converges weakly to the point process $C$; $C$ is a basic quantity for the asymptotics of $X_n$ and is called the cluster process of $X_n$. As shown in Proposition 2.5 below, the law of $C$ depends only of $\mu, \Lambda$.

We denote by $\pi_v^\omega$ the renewal point process of the random walk $S_n(\omega)v$ on $V \setminus \{0\}$, given by $\pi_v^\omega = \sum_{i=0}^\infty \varepsilon_{S_i(\omega)v}$. For $v$ fixed, the mean measure of the point process $\pi_v^\omega$ is the potential measure $\sum_{i=0}^\infty Q^i(v,\cdot)$ of the Markov kernel $Q$; if $L(\mu) < 0$ the asymptotics ($|v| \to \infty$) of this Radon measure are described in [13]. The formula below for the Laplace functional of $C$ involves the renewal point process $\pi_v^\omega$ of the linear random walk $S_n(\omega)v$.

**Proposition 2.5** Under $\mathbb{P}_\rho$, the sequence of point processes $C_n$ converges weakly to a point process $C = \sum_{1}^{\infty} \varepsilon_{Z_i}$. The law of the point process $C$ is equal to the $\widehat{Q}_{\lambda_1}$—law of the point process $\sum_{j \in \mathbb{Z}} \varepsilon_{S_jx}$ conditional on $\sup_{i \leq -1} |S_jx| \leq 1$. In particular we have

$$\widehat{Q}_{\lambda_1} \left\{ \lim_{i \to \infty} |Z_0| = 0 \right\} = 1, \quad \widehat{Q}_{\lambda_1} \left\{ \sup_{i \geq 0} |Z_i| \geq 1 \right\} = 1.$$

Furthermore the sequence $\theta_n$ defined in Proposition 2.4 converges to the positive number $\theta = Q_{\lambda_1} \{M_1^\infty(Y) \leq 1\}$ and we have $\theta^{-1} = E_{\lambda_1} (\sum_{j \geq 0} 1_{B'_1(Z_j)}) < \infty$.

If $\text{supp}(f) \subset B'_1$ the Laplace functional of $C$ on $f$ is given by

$$1 - \theta^{-1} E_{\lambda_1} [\exp(f(v) - 1)exp - \pi_v^\omega(f)].$$

**Proof:** We recall that convergence of Laplace functionals implies weak convergence of the corresponding point processes. Let $f$ be a non-negative and continuous function on $V \setminus \{0\}$ which is compactly supported, hence $f(x) = 0$ if $|x| \leq \delta$ with $\delta > 0$. 

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We write for \( k \leq \ell \) with \( k, \ell \in \mathbb{Z} \cup \{ \pm \infty \} \), \( M_k^\ell(Y) = \sup_{t \leq j \leq \ell} |Y_j| \) with \( Y_j = S_j Y_0 \). For \( k, \ell, f \) as above we write \( C_k^\ell = \exp - \sum_k f(u_n^{-1} X_j) \), \( C_k^\ell(Y) = \exp - \sum_k f(Y_j) \) and we observe that \( C_k^\ell \leq 1 \). We fix \( m > 0 \) and we take \( n \) so large that the sequence \( r_n \) of the above proposition satisfies \( r_n > 2m + 1 \). When convenient we write \( r_n = r \), hence \( E_{\rho}(C_1^r; M_1^r > u_n) = \sum_1^r E_{\rho}(C_1^r; M_1^r \leq u_n < X_j) \). We observe that, for \( r - m \geq j > m + 1 \), we have \( C_1^r = C_{j-m}^{j+m} \) except if \( \sup(M_1^{r-m}, M_{j+m+1}^r) > u_n \). We are going to compare \( E_{\rho}(C_1^r; M_1^r > u_n) \) and \( (r - 2m)E_{\rho}(C_{m}^{m}; M_{-m-1}^r \leq u_n < |X_0|) \) using those \( j \)'s which satisfy \( m + 1 < j \leq r - m \) and we denote by \( \Delta_{n,m} \) their difference.

If we write
\[
\Delta_{n,m}(j) = E_{\rho}(C_1^r; M_1^{j-1} \leq u_n < |X_j|) - E_{\rho}(C_{j-m}^{j+m}; M_{j-m-1}^{j-1} \leq u_n < |X_j|)
\]
then we have, since \( C_k^\ell \leq 1 \)
\[
|\Delta_{n,m}| \leq \sum_{m+1}^{r-m} |\Delta_{n,m}(j)| + 2m \mathbb{P}_\rho(|X_0| > u_n).
\]

Using stationarity of \( X_n \) and the above observation we have
\[
|\Delta_{n,m}(j)| \leq \mathbb{P}(\sup(M_{-r-m}^r, M_{m+1}^r) > u_n \delta; |X_0| > u_n),
\]

hence, using stationarity and the formula \( \theta_n = (r_n \mathbb{P}_\rho(|X_0| > u_n))^{-1} \mathbb{P}_\rho(|M_1^r| > u_n) \)
\[
|\theta_n E_{\rho}(C_1^r; M_1^r > u_n) - r^{-1}(r - 2m)E_{\rho}(C_{m}^{m}; M_{-m-1}^{r} \leq u_n/|X_0| > u_n)| \leq \mathbb{P}_\rho(\sup(M_{-r-m}^r, M_{m+1}^r) > u_n \delta/|X_0| > u_n) + 2r^{-1} m.
\]

Using Proposition 2.3, we see that the discontinuity set of the function \( r_{-\infty,-1}(M_{-m-1}^r(Y)) \)
is \( \widehat{Q}_{\Lambda_1} \)-negligible hence, using again Proposition 2.3,
\[
\lim_{n \to \infty} \mathbb{P}_\rho(\sup(M_{-r-m}^r, M_{m+1}^r) > u_n/|X_0| > u_n) = E_{\Lambda_1}(C_{m}^{m}(Y); M_{-m-1}^r(Y) \leq 1).
\]

Also \( \lim_{n \to \infty} r_n^{-1}(r_n - 2m) = 1 \) since \( \lim_{n \to \infty} r_n = \infty \). We observe that by definition of \( \theta_n \) and \( C_1^r \leq 1 \) we have \( \theta_n E_{\rho}(C_1^r; M_1^r > u_n) \leq \theta_n \leq 1 \), hence we can consider convergent subsequences \( \theta_{n_k} \) with \( \lim_{k \to \infty} \theta_{n_k} \in [0,1] \). The anticlustering property of \( X_n \) implies that the limiting values \( (n \to \infty) \) of \( \mathbb{P}_\rho(\sup(M_{-r-m}^r, M_{m+1}^r) > u_n/|X_0| > u_n) \) are bounded by \( \varepsilon_m > 0 \) with \( \lim_{m \to \infty} \varepsilon_m = 0 \). Then the above inequality implies with \( r = r_{n_k} \)
\[
\lim_{k \to \infty} \left| \theta_{n_k} E_{\rho}(C_1^r; M_1^r > u_n) - E_{\Lambda_1}(C_{m}^{m}(Y); M_{-m-1}^r(Y) \leq 1) \right| \leq \varepsilon_{m}.
\]

Since \( E_{\Lambda_1}(\{C_{m}^{m}(Y); M_{-m-1}^r(Y) \leq 1\}) = E_{\Lambda_1}(exp - \sum_{-\infty} f(Y_j); M_{-m-1}^r(Y) \leq 1) = I \)
we have \( \lim_{k \to \infty} \mathbb{P}_\rho(C_1^r; M_1^r > u_n) = I \), hence the limit of \( \theta_n E_{\rho}(C_1^r; M_1^r > u_n) \) exists and is equal to \( I \). In particular with \( f = 0 \) and using Proposition 2.3 we get
\[
\lim_{n \to \infty} \theta_n = \widehat{Q}_{\Lambda_1}(M_{-m-1}^r(Y) \leq 1) = \theta. \]

From above and Proposition 2.4 : \( \theta = \lim_{n \to \infty} \theta_n = \liminf_{n \to \infty} \theta_n > 0 \)

Then we get \( \lim_{n \to \infty} E_{\rho}(C_1^r; T_1^n > u_n) = \theta^{-1} I = \widehat{E}_{\Lambda_1}(exp - \sum_{-\infty} f(Y_j)/M_{-m-1}^r(Y) \leq 1) \) hence the first assertion, using Proposition 2.3. The expression of \((Z_j)_{j \in \mathbb{N}} \) in terms of \((Y_n)_{n \in \mathbb{Z}} \)
and the relation \( \lim_{|n|\to\infty} Y_n = 0 \), \( \mathbb{Q}_{A_1} \) a.e stated in Proposition 2.3 gives
\[
\mathbb{Q}_{A_1} \{ \lim_{i\to\infty} Z_i = 0 \} = 1.
\]
Since the discontinuity set of \( 1_{B_1^1} \) is \( \Lambda_1 \)-negligible, using the weak convergence of \( C_n \) to \( C \),
the continuous mapping theorem (see [28]) and the convergence of \( \theta_n^{-1} \) to \( \theta^{-1} \), we get the formula \( \theta^{-1} = E_{A_1}(\sum_1^{\infty} 1_{B_1^1}(Z_j)) \). The last formula is proved in ([2], Theorem 4.1). □

3 A spectral gap property and multiple mixing

We denote \( X_k^x(k \in \mathbb{N}) \) the affine random walk on \( V \) governed by \( \lambda \), starting from \( x \in V \) and we write \( P\varphi(x) = \int \varphi(hx) d\lambda(h) = E(\varphi(X_1^x)) \).
In this section we use a spectral gap property for a family of operators associated to the process \( X_k(1 \leq k \leq n) \), in order to show the quasi-independence of its successive blocks of length \( r_n \), where \( r_n \) is defined in subsection 2.3.

3.1 Spectral gap property

It was proved in ([11], Theorem 1) that, given a probability \( \lambda \) on \( A \) which satisfies condition \( (H) \), the corresponding convolution operator \( P \) on \( V \) satisfies a ”Doeblin-Fortet” inequality (see [15]) for suitable Banach spaces \( C_\chi \) and \( \mathcal{H}_{\chi,\varepsilon,\kappa} \) defined below. In particular, it will be essential here to use that the operator \( P \) on \( \mathcal{H}_{\chi,\varepsilon,\kappa} \) is the direct sum of a 1-dimensional projection \( \pi \) and a contraction \( U \) where \( \pi \) and \( U \) commute, hence we give also a short proof of this fact below. In order to obtain the relevant multiple mixing property, we show a global Doeblin-Fortet inequality for a family of operators closely related to \( P \). For \( \chi,\kappa \geq 0 \), we consider the weights \( \omega,\eta \) on \( V \) defined by \( \omega(x) = (1+|x|)^{-\chi}, \eta(x) = (1+|x|)^{-\kappa} \). The space \( C_\chi \) is the space of continuous function \( \varphi \) on \( V \) such that \( \varphi(x)\omega(x) \) is bounded and we write \( |\varphi|_\chi = \sup_{x \in V} |\varphi(x)|\omega(x) \).

For \( \varepsilon \in [0,1] \) we write :
\[
[\varphi]_{\varepsilon,\kappa} = \sup_{x \neq y} |x-y|^{-\varepsilon}\eta(x)\eta(y)|\varphi(x) - \varphi(y)|, \quad ||\varphi|| = |\varphi|_\chi + [\varphi]_{\varepsilon,\kappa},
\]
and we denote by \( \mathcal{H}_{\chi,\varepsilon,\kappa} \) the space of functions \( \varphi \) on \( V \) such that \( ||\varphi|| < \infty \). We observe that \( C_\chi \) and \( \mathcal{H}_{\chi,\varepsilon,\kappa} \) are Banach spaces with respect to the norms \( |.|_\chi \) and \( ||.|| \) defined above. Also \( \mathcal{H}_{\chi,\varepsilon,\kappa} \subset C_\chi \) with compact injection if \( \kappa + \varepsilon < \chi \). We observe that the operator \( P \) acts continuously on \( C_\chi \) and \( \mathcal{H}_{\chi,\varepsilon,\kappa} \). For a Lipschitz function \( f \) on \( V \) with non negative real part we define the Fourier-Laplace operator \( P^f \) by \( P^f\varphi(x) = P(\varphi exp(-f)) \). In [11], spectral gap properties for Fourier operators were studied for \( f(v) = i \neq x, v >, x \in V \).

Here the calculations are analogous but \( f \) will be Lipchitz and bounded. We observe that for functions \( f_k(1 \leq k \leq n) \) and \( \varphi \) as above we have :
\[
P^f_1 P^f_2 \ldots P^f_n \varphi(x) = E\{\varphi(X_n^x)exp - \sum_{k=1}^{n} f_k(X_k^x)\}\}
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Also we note that, for $f$ bounded, with $k(f) = \sup_{x \neq y} |x - y|^{-1} |f(x) - f(y)|$

$$|x - y|^{-\varepsilon} |f(x) - f(y)| \leq \inf_{x \neq y} (2|f|_{\infty}|x - y|^{-\varepsilon}, k(f)|x - y|^{1-\varepsilon}) \leq 2|f|_{\infty} + k(f) := k_1(f),$$

For $u, v$ with non-negative real parts we have $|\exp(-u) - \exp(-v)| \leq |u - v|$. In particular, for $f$ as above, $|\exp(-f(x) - \exp(-f(y))| \leq k_1(f)|x - y|^{\varepsilon}$.

It follows that multiplication by $\exp(-f)$ acts continuously on $C_\chi, H_{\chi,\varepsilon,\kappa}$, hence $P^f$ is a bounded operator on $C_\chi$ and $H_{\chi,\varepsilon,\kappa}$. For $m, \gamma > 0$ we denote by $O(m, \gamma)$ the set of operators $P^f$ such that $|f|_{\infty} \leq m$ and $k(f) \leq \gamma$, hence $k_1(f) \leq 2m + \gamma$. For $p \in \mathbb{N}$ let $O^p(m, \gamma)$ be the set of products of $p$ elements in $O(m, \gamma)$ and $\hat{O}(m, \gamma) = \cup_{p>0} O^p(m, \gamma)$. We will endow $\hat{O}(m, \gamma)$ with the natural norm from $\text{End}(H_{\chi,\varepsilon,\kappa})$. Then we have the

**Theorem 3.1** With the above notations and $0 \leq \chi < 2\kappa < 2\varepsilon < \alpha$, there exists $C(m, \gamma) \geq 1$ such that for any $Q \in O(m, \gamma)$ the norm $\|Q\|$ of $Q$ on $H_{\chi,\varepsilon,\kappa}$ is bounded by $C(m, \gamma)$. Furthermore there exists $r \in [0, 1[, \ p \in \mathbb{N}, \ D > 0$ such that for any $Q \in O^p(m, \gamma), \ \varphi \in H_{\chi,\varepsilon,\kappa}$:

$$\|Q\varphi\| \leq r\|\varphi\| + D\|\varphi\|_{\chi}.$$

In particular $\hat{O}(m, \gamma)$ is a bounded subset of $\text{End}(H_{\chi,\varepsilon,\kappa})$ and $C(m, \gamma), r, D$ depend only of $m, \gamma$.

The proof depends on the two lemmas given below, and of calculations analogous to those of [11] for Fourier operators.

**Lemma 3.2** $\hat{O}(m, \gamma)$ is a bounded subset of $\text{End}(C_\chi)$.

**Proof:** Since $\Re(f) \geq 0$ we have for $Q \in O^\ell(m, \gamma)$ with $\ell \in \mathbb{N}$, $\varphi \in C_\chi : |Q\varphi|_{\chi} \leq |P^\ell|\varphi||_{\chi}$, hence it suffices to show that the set $\{P^\ell; \ell \in \mathbb{N}\}$ is bounded in $\text{End}(C_\chi)$. We have for $\varphi \geq 0$, with $A = P^\ell$ :

$$\omega(x)A\varphi(x) = \omega(x)\mathbb{E}(\varphi(X^*_\ell)) \leq |\varphi|_{\chi}\mathbb{E}[\omega(x)\omega^{-1}(X^*_\ell)].$$

If $\chi \leq 1$, using independence and the expression of $X^*_\ell$ we get

$$\omega(x)A\varphi(x) \leq |\varphi|_{\chi}(1 + \mathbb{E}(|S_1|^{\chi}) + \sum_{1 \leq \ell \leq \ell} \mathbb{E}(|S_{\ell+1}|^{\chi} | \mathbb{E}(|B_k|^{\chi})),$$

hence $\sup_{x \in V} \omega(x)A\varphi(x) \leq |\varphi|_{\chi}(1 + \sup_{\ell \geq 1} \mathbb{E}(|S_1|^{\chi}) + \mathbb{E}(|B_1|^{\chi})\sum_{1 \leq \ell \leq \ell} \mathbb{E}(|S_{\ell+1}|^{\chi})).$

Since $\chi < \alpha$, we have $\lim_{\ell \to \infty} \mathbb{E}(|S_\ell|^{\chi})^{1/\ell} = k(\chi) < 1$, hence $\sup_{x \in V} \omega(x)A\varphi(x)$ is bounded by $C_\chi|\varphi|_{\chi}$ with $C_\chi < \infty$.

If $\chi > 1$, we use Minkowski inequality in $L^\chi$ and write:

$$\omega(x)A\varphi(x) \leq |\varphi|_{\chi}(1 + \mathbb{E}(|S_1|^{\chi})^{1/\chi} + \sum_{1 \leq \ell \leq \ell} \mathbb{E}(|S_{\ell+1}|^{\chi})^{1/\chi} \mathbb{E}(|B_k|^{\chi})^{1/\chi})$$

As above we get

$$\sup_{x \in V} \omega(x)A\varphi(x) \leq C_\chi|\varphi|_{\chi}$$

with $C_\chi < \infty$. □
Lemma 3.3 a) For $\beta \in [0, \alpha[$ we have $\sup E(|X_n^0|^\beta) < \infty$.

b) For $\beta_1, \beta > 0$ and $\beta + \beta_1 < \alpha$, we have $\lim_{n \to \infty} (E(|S_n|^{\beta_1}|X_n^0|^\beta))^{1/n} < 1$.

c) If $\chi + \varepsilon < \alpha$ the quantity $\tilde{C}_n = \mathbb{E}\{\sum |S_i|^\varepsilon (1 + |S_n| + |X_n^0|)^{\chi}\}$ is bounded. Furthermore, if $2\kappa + \varepsilon < \alpha$, $\tilde{D}_n = E\{|S_n|^\varepsilon (1 + |S_n| + |X_n^0|)^{2\kappa}\}$ satisfies $\lim_{n \to \infty} (\tilde{D}_n)^{1/n} < 1$.

Proof: a) We write $|X_n^0|^\beta = |\sum \frac{n}{1} S_{kn+1} B_k|^\beta$. If $\beta \leq 1$ we get:

$$E(|X_n^0|^\beta) \leq \frac{n}{1} E(|S_{kn+1}|^\beta) E(|B_k|^\beta) = E(|B_1|^\beta) \sum \frac{n}{0} E(|S_j|^\beta).$$

Since $\lim_{j \to \infty} (E(|S_j|^\beta))^{1/j} < 1$ if $\beta < \alpha$ we get $\sup_{n \geq 0} E(|X_n^0|^\beta) \leq E(|B_1|^\beta) \sum \frac{n}{0} E(|S_j|^\beta) < \infty$.

If $\beta > 1$, we use Minkowski’s inequality in $\mathbb{L}^\beta$ as in the proof of Lemma 1.

b) Using Hölder’s inequality we have

$$E(|S_n|^{\beta_1}|X_n^0|^{\beta_1}) \leq E(|S_n|^{\beta_1+\beta_1})^{\beta_1/\beta_1+\beta_1} E(|X_n^0|^{\beta_1+\beta_1})^{\beta_1/\beta_1+\beta_1},$$

hence the result follows from a) and the fact that $\lim_{n \to \infty} E(|S_n|^{\beta_1+\beta_1})^{1/n} < 1$ since $\beta + \beta_1 < \alpha$.

c) The assertions follows from easy estimations as in b) and the conditions $\chi + \varepsilon < \alpha$, $2\kappa + \varepsilon < \alpha$. □

Proof of Theorem 3.1 We start with a basic observation. For $n > 0$ we have $X_n^x = h_n \cdots h_1 x = S_n x + \sum \frac{n}{1} S_{kn+1} B_k$, hence $|X_n^x - X_n^y| = |S_n(x - y)| \leq |S_n||x - y|$. It follows for $k(f) \leq \gamma$, $x$ and $y$ in $V$:

$$|f(X_n^x) - f(X_n^y)| \leq \gamma |S_n||x - y|.$$

We write $A = T_1 T_2 \cdots T_n$ with $T_i = P^{f_i} \in O(m, \gamma) 1 \leq i \leq n$. We have using Markov property,

$$A f(x) - A f(y) = I_n(x, y) + J_n(x, y) \text{ with }$$

$$I_n(x, y) = E\{|\sum \frac{n}{1} f_i(X_i^x) - \sum \frac{n}{1} f_i(X_i^y)| f(X_n^x)\}$$

$$J_n(x, y) = E\{|\sum \frac{n}{1} f_i(X_i^y)(f(X_n^x) - f(X_n^y))| \}$$

Since $Re(f) \geq 0$ we have:

$$|\sum \frac{n}{1} f_i(X_i^x) - \sum \frac{n}{1} f_i(X_i^y)| \leq \sum \frac{n}{1} |f_i(X_i^x) - f_i(X_i^y)| \leq (2m + \gamma) \sum \frac{n}{1} |X_i^x - X_i^y|.$$

The basic observation gives:

$$I_n(x, y) \leq (2m + \gamma)|f|_{\chi} |x - y|^\varepsilon C_n(x) \text{ with } C_n(x) = E\{|\sum \frac{n}{1} |S_i|^\varepsilon (1 + |X_n^x|)\}$$

$$J_n(x, y) \leq E\{|f(X_n^x) - f(X_n^y)| \} \leq |f|_{\varepsilon, \kappa} |x - y| D_n(x, y),$$

with $D_n(x, y) = E\{|S_n|^\varepsilon (1 + |X_n^x|)^\kappa (1 + |X_n^y|)^\kappa\}$. Using symmetry of $|A f(x) - A f(y)|$, $\chi \leq 2\varepsilon$ and $|X_n^x| \leq |S_n||x| + |X_n^0|$, we get $|A f|_{\varepsilon, \kappa} \leq (2m + \gamma)|f|_{\chi} C_n + |f|_{\varepsilon, \kappa} D_n$ where $C_n, D_n$ are as in Lemma 3.3.
Using Lemma 3.3 we can choose p ∈ N such that r = \bar{D}_p < 1, hence for A ∈ O^p(m, γ),
[Aφ]_{ε,κ} ≤ k_1(f)\tilde{C}_p|φ|_x + r[φ]_{ε,κ}.

Using Lemma 3.2 we see that there exists C_χ ≥ 1 such that |Aφ|_x ≤ C_χ|φ|_x for A ∈ \bar{O}(m, γ), φ ∈ C_χ. Then for A ∈ O^p(m, γ), φ ∈ H_{χ, ε, κ} and p as above:

\|Aφ\| ≤ r||φ|| + (C_χ + 2m + γ)\tilde{C}_p|φ|_x = r||φ|| + D|φ|_x with D > 0.

For the last assertion, assume A ∈ O^0(m, γ) and write n = pn_1 + n_0 with n_1 ∈ N, 0 ≤ n_0 < p. We have A = Q_1 · · · Q_{n_1}R_1 · · · R_{n_0} with Q_i ∈ O^0(m, γ) (1 ≤ i ≤ p) and R_j ∈ O(m, γ) (0 ≤ j ≤ n_0), hence \|R_j\| ≤ C_χ(m, γ). Finally we get

\|Aφ\| ≤ (C_χ(m, γ))^{n_0} \left[r^{n_1}\|φ\| + D|φ|_x(r^{n_1-1} + C_χ^\frac{n_1-2}{0} r^k)\right],

\|A\| ≤ C_χ(m, γ)^p \left[1 + D(1 + C_χ(1 - r)^{-1})\right] := C(m, γ), which gives the result. □

For χ ∈ [0, α[ we consider the function W^χ on V defined by W^χ(x) = |x|^χ. In Proposition 3.4 below we show that, due to the inequality 0 < k(χ) < 1 for χ ∈ [0, α[, P satisfies a drift condition (see [22]) with respect to W^χ. The same inequality implies also a spectral gap property in the Banach space \mathcal{H}_{χ, κ} considered in Proposition 3.4 below. For reader’s convenience we recall the Doeblin-Fortet spectral gap theorem (see [18]).

Let (B, |.|) be a Banach space, (L, ||.|) another Banach space with a continuous injection L → B. Let P be a bounded operator on B, which preserves L and satisfies the following conditions

1) The sequence of operator norms |P^n| in is bounded.
2) The injection L → B is compact.
3) There exists an integer k and r ∈ [0, 1[, D > 0 such that for any v in L :

\|P^k v\| ≤ r\|v\| + D|v|

4) If v_n ∈ L is a sequence and v ∈ B are such that \|v_n\| ≤ 1 and \lim_{n→∞} |v - v_n| = 0, there v ∈ L and \|v\| ≤ 1

Then in restriction to L, P is the commuting direct sum of a finite dimensional operator π with unimodular spectral values and a bounded operator U with spectral radius r(U) < 1. We observe that, frequently the norm ||.| on L is given as a sum of a semi-norm [.|] and the norm |.|; then the inequality in condition 3 can be replaced by

\|P^k v\| ≤ r\|v\| + D|v|

such an inequality is called Doeblin-Fortet’s inequality.

Our substitute for the strong mixing property (see [29]) uses regularity of functions and is the following.

Proposition 3.4 For any β ∈ [0, 1] there exists ℓ ∈ N and b ≥ 0 such that P^\ell W^χ ≤ βW^χ + b for n ≥ ℓ. In particular the sequence of norms |P^n|_x is bounded. Furthermore, if 0 < κ + ε < χ < 2κ < 2κ + ε < α, the injection of \mathcal{H}_{χ, κ} into \mathcal{C}_χ is compact and on \mathcal{H}_{χ, κ, κ}, the Markov operator P satisfies the direct sum decomposition

P = ρ ⊗ 1 + U

where r(U) < 1 and U(ρ ⊗ 1) = (ρ ⊗ 1)U = 0
If $\alpha = 1$ and $0 < \varepsilon < \chi < 1$, the same result is valid.

Proof: We verify successively the four above conditions. First we observe that for any $x \in V$,

$$|X_n - X_0^n| \leq |S_n| |x|, \quad |X_n^x| \leq |X_0^n| + |S_n| |x|.$$  

If $\chi \leq 1$, it follows

$$E(|X_n^x|) \leq E(|X_0^n|) + E(|S_n| x) |x|.$$  

Using the expression of $X_0^n$ and independence we get $E(|X_0^n|) \leq E(|B_1| x) \sum_0^{\infty} E(S_k |x|)$. Since $\chi < \alpha$, we have $E(|X_0^n|) \leq b < \infty$. On the other hand we have $\lim_{n \to \infty} (E(|S_n| x))/n = k(\chi) < 1$, hence for some $\varepsilon > 0$, $k(\chi) + \varepsilon < 1$, and for $n \geq \ell$, $|S_n| \leq \beta' \leq (k(\chi) + \varepsilon)^n$. It follows, for $n \geq \ell$:

$$P^nW^{\chi}(x) = E(|X_n^x|) \leq \beta'W^{\chi}(x) + b$$  

If $\chi > 1$ we use Minkowski inequality, hence:

$$E(|X_n^x|) \leq 2^{\chi}(E(|X_0^n|) + E(|S_n| x)) |x|$$  

As above, using $k(\chi) + \varepsilon < 1$ and $n \geq \ell$ we get

$$E(|X_n^x|) \leq 2^{\chi}b + 2^{\chi}(k(\chi) + \varepsilon)^n |x|^\chi, \quad P^nW^{\chi} \leq \beta''W^{\chi} + b'$$  

with $\beta'' < b' < \infty$. We take $\beta = \beta'$ or $\beta''$ depending on $\chi \leq 1$ or $\chi > 1$. This allow us now to show that $|P^n|_\chi$ is bounded. We observe that $|\varphi(x)| \leq (1 + W(x))^{\chi}|\varphi|_\chi$, hence the positivity of $P$ and $P1 = 1$ implies for $n \in \mathbb{N}$,

$$|P^n\varphi|(x) \leq |\varphi|_\chi P^n(2^\chi + 2^\chi P^nW^{\chi}(x)) = |\varphi|_\chi(2^\chi + 2^\chi P^nW^{\chi}(x)).$$

From above we get

$$|P^n\varphi|(x) \leq |\varphi|_\chi[2^\chi + 2^\chi(b + \beta W^{\chi}(x))].$$

Then the definition of $|P^n|_\chi$ gives $|P^n|_\chi \leq 2^\chi(1 + b + \beta)$, hence the boundedness of $|P^n|_\chi$.

In order to show that if $\kappa + \varepsilon < \chi$, the injection of $B_0 = H_{\kappa,\varepsilon,\chi}$ in $B = C_\chi$ is compact, we use Ascoli argument and consider a large ball $B_t$ with $t > 0$. We consider $\varphi_n \in H_{\kappa,\varepsilon,\chi}$ with $||\varphi_n|| < 1$. The definition on $||\varphi_n||$ implies for any $x, y \in B_t$

$$|\varphi_n(x)| \leq (1 + t)^\chi, |\varphi_n(x) - \varphi_n(y)| \leq (1 + t)^{2\chi}|x - y|^\varepsilon.$$  

Hence, the restrictions of $\varphi_n$ to $B_t$ are equicontinuous and we can find a convergent subsequence $\varphi_{n_k}$. Using the diagonal procedure and a sequence $t_i$ with $\lim_i t_i = \infty$, we get a convergent subsequence $\varphi_{n_j} \in H_{\kappa,\varepsilon,\chi}$ with limit a continuous function $\varphi$ on $V$. From above we have $|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^{\kappa}|x|^\varepsilon$. Hence for some $A, B > 0$, since $\kappa + \varepsilon < \chi$

$$|\varphi_{n_j}(x) - \varphi_{n_j}(0)| \leq (1 + |x|)^{\kappa + \varepsilon}, |\varphi(x)| \leq A + B(1 + |x|)^\chi.$$  

It follows that $\varphi \in C_\chi$. The above inequalities for $\varphi_{n_j}$ imply

$$|(\varphi_{n_j}(x) - \varphi_{n_j}(0)) - (\varphi(x) - \varphi(0))| \leq 2(1 + |x|)^{\kappa + \varepsilon}.$$  

Then the convergence of $\varphi_{n_j}$ to $\varphi$, implies with $\varepsilon_{n_j} = |\varphi_{n_j}(0) - \varphi(0)|$,
\begin{align*}
|\varphi_{n_j}(x) - \varphi(x)| &\leq \varepsilon_{n_j} + 2(1 + |x|)^{\kappa + \varepsilon}(1 + |x|)^{-\chi} |\varphi_{n_j}(x) - \varphi(x)| \\
&\leq \varepsilon_{n_j} + 2(1 + |x|)^{\kappa + \varepsilon - \chi}
\end{align*}

with \( \lim_{j \to \infty} \varepsilon_{n_j} = 0 \). Also for \( t \) sufficiently large, and \( |x| \geq t \), since \( \kappa + \varepsilon < \chi \) we have \( (1 + |x|)^{\kappa + \varepsilon - \chi} \leq \varepsilon_{n_j} \). Furthermore, the uniform convergence of \( \varphi_{n_j} \) to \( \varphi \) on \( B_t \) implies \( \lim_{j \to \infty} (\sup\{|\varphi_{n_j}(x) - \varphi(x)| ; |x| \leq t\}) = 0 \). The convergence of \( |\varphi_{n_j} - \varphi|_\chi \) to zero follows.

The convergence of \( \varphi_{n_j}(x) \) to \( \varphi(x) \) for any \( x \in V \) and the definition of \( \|\varphi_{n_j}\| \), implies \( \|\varphi\| \leq \lim_{j \to \infty} \|\varphi_{n_j}\| \leq 1 \), hence \( \varphi \in L \), hence condition 4 is satisfied.

With \( f = 0 \) in Theorem 3.1 we have \( P^f = P \). In particular there exists \( k > 0 \) such that \( \|P^k \varphi\| \leq r \|\varphi\| + D \|\varphi\|_\chi \) if \( \varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa} \). Hence from \[18\], we know that the above conditions imply that \( P \) is the direct sum of a finite rank operator and a bounded operator \( U \) which satisfies \( r(U) < 1 \). Now it suffices to show that the equation \( P \varphi = z \varphi \) with \( |z| = 1 \), \( \varphi \in \mathcal{H}_{\chi, \varepsilon, \kappa} \) implies that \( \varphi \) is constant and \( z = 1 \). From the convergence in law of \( X^t \) to \( P \) we know that for any \( x \in V \), the sequence of measures \( P^n(x, \cdot) \) converges weakly to \( \rho \). Also we have \( |\varphi| \in \mathcal{H}_{\chi, \varepsilon, \kappa} \) and the sequence \( n^{-1} \sum P^k |\varphi| \) converges to \( \rho(|\varphi|) \).

Since \( |\varphi(x)| = |z^n \varphi(x)| \leq P^n(x, |\varphi|) \) we get \( |\varphi(x)| \leq \rho(|\varphi|) \), hence \( |\varphi| \) is bounded. Since \( z^n \varphi(x) = \mathbb{E}(\varphi(X^n_d)) \) and \( X^n_d \) converges in law to \( \rho \), we get \( \lim_{n \to \infty} z^n \varphi(x) = \rho(\varphi) \). This implies \( z = 1 \) and \( \varphi(x) = \rho(\varphi) \) for any \( x \in V \).

For the last assertion, in view of the above, we have only to verify the contraction condition. We write \([\varphi]_\varepsilon = \sup_{x \neq y} |x - y|^{-\varepsilon} |\varphi(x) - \varphi(y)|\). Then we have

\[ 
\mathbb{E}(|\varphi(X^n_d) - \varphi(X^n_d)|) \leq [\varphi]_\varepsilon |X^n_d - X^n_d|^{\varepsilon} \leq [\varphi]_\varepsilon |x - y|^{\varepsilon} \mathbb{E}(|S^n|^{\varepsilon}).
\]

Since \( \varepsilon < \alpha \), we have \( 0 < k(\varepsilon) < r < 1 \) for some \( r \), hence \( [P^n \varphi]_\varepsilon \leq r [\varphi]_\varepsilon \) for \( n \) large. \( \square \)

### 3.2 A mixing property with speed for the system \((V^{z^+}, \tau, P_\rho)\)

In general, if the law of \( B_n \) has no density with respect to Lebesgue measure, the operator \( P \) on \( L^2(\rho) \) don't satisfy spectral gap properties hence the stationary process is not strongly mixing in the sense of \[29\], but Proposition 3.4 above shows that it is still ergodic. Then, using Theorem 3.1 and Proposition 3.4, it is shown below that the system \((V^{z^+}, \tau, P_\rho)\) satisfies a multiple mixing condition with respect to Lipchitz functions. For a very general framework covering mixing conditions with respect to regular functions, see \[20\]. For a study of extreme value properties for random walks on some classes of homogeneous spaces, using \( L^2 \)-spectral gap methods, we refer to \[20\]. Since, using Proposition 2.4, the stationary process \((X_n)_{n \in \mathbb{N}}\) satisfies also anticlustering, we see below that extreme value theory can be developed for \((X_n)_{n \in \mathbb{N}}\) following the arguments of \((\mathbb{2} , \mathbb{3})\) which were developed under conditions \( \mathcal{A}(u_n), \mathcal{A}'(u_n) \), using continuous functions.

However it turns out that the mixing property \( \mathcal{A}(u_n) \) of \( \mathbb{2} \) for continuous functions can be proved, as a consequence of the corresponding convergences involving Lipchitz functions.
Let $f$ be a bounded continuous function with non-negative real part on $[0, 1] \times (V \setminus \{0\})$. Let $r_n$ be an integer valued sequence with $\lim_{n \to \infty} r_n = r$, $r_n = o(n)$ and $k_n = [\frac{1}{r_n}]$. For $0 \leq i \leq n$, $0 \leq j \leq n$, $x \in V \setminus \{0\}$, $\omega \in V^{\mathbb{Z}^+}$ we write:

$$\mathcal{T}_n(f) = f(\frac{x}{r_n}, \frac{u}{r_n}), f_{i,n}(\omega) = \mathcal{T}_n(X_i), f_{i,n}^j(\omega) = \mathcal{T}_n^j(X_i).$$

In view of heavy notations, in some formulae we will write $r_n = r$, $k_n = k$, $\ell_n = \ell$. For $f$ Lipschitz we denote by $\lambda(f)$ the Lipschitz constant of $f$, and assume $\text{supp}(f) \subset [0, 1] \times B_\delta^c$ with $\delta > 0$. We consider below the quantity $\mathbb{E}_\rho(\exp - \sum_{i=1}^n f_{i,n})$ which is the Laplace functional of the point process $\frac{n}{i=1} \epsilon_{u_n} X_i$. For its analysis we use the classical Bernstein method of gaps, i.e. we decompose the interval $[1, n]$ into large subintervals separated by smaller but still large ones.

**Proposition 3.5** Let $f$ be a compactly supported Lipschitz function on $[0, 1] \times (V \setminus \{0\})$ with $\text{Re} f \geq 0$. Assume that the sequence $r_n \in \mathbb{N}$ satisfies $r_n = o(n)$, $\lim_{n \to \infty} (\log n)^{-1} r_n = \infty$ and write $|f| = m, \lambda(f) = \gamma, \text{supp}(f) \subset [0, 1] \times B_\delta^c, \delta > 0$. Then, with the above notations there exists $C(\delta, m, \gamma) < \infty$ such that,

$$I_n(f) := |\mathbb{E}_\rho(\exp(-\sum_{i=1}^n f_{i,n})) - \prod_{j=1}^k \mathbb{E}_\rho(\exp(-\sum_{j=1}^{j_{r_n}} f_{i,n}^j))| \leq C(\delta, m, \gamma) \sup(r_n^{-1}, n^{-1} r_n).$$

In particular with $r_n = [n^{1/2}]$ we get $\sup(n^{-1} r_n^{-1}) \leq 2n^{-1/2}$

**Proof**: We write $[0, n] = [0, k_n r_n] \cup [k_n r_n, n]$, we decompose the interval $[0, k_n r_n]$ into $k_n$ intervals $J_j = [j r_n, (j+1) r_n]$ and we distinguish in $J_j$ the subinterval of length $\ell_n$ $J_j' = [(j+1) r_n - \ell_n, (j+1) r_n]$; the large integer $\ell_n$ will be specified below.

We write for $f$ fixed, $I(n) = |\mathbb{E}_\rho(\exp - \sum_{i=1}^n f_{i,n}) - \prod_{j=1}^k \mathbb{E}_\rho(\exp - \sum_{i=1}^{j_{r_n}} f_{i,n}^j)|$.

Then the triangular inequality gives $I(n) \leq I_1(n) + I_2(n) + I_3(n) + I_4(n)$ with

$$I_1(n) = |\mathbb{E}_\rho(\exp - \sum_{i=1}^n f_{i,n})|$$

$$I_2(n) = |\mathbb{E}_\rho(\exp - \sum_{i=1}^{kr_n} f_{i,n}) - \prod_{j=1}^k \mathbb{E}_\rho(\exp - \sum_{i=1}^{j_{r_n}} f_{i,n}^j)|$$

$$I_3(n) = |\mathbb{E}_\rho(\exp - \sum_{i=1}^{\ell_n} f_{i,n}) - \prod_{j=1}^k \mathbb{E}_\rho(\exp - \sum_{i=1}^{\ell_n} f_{i,n}^j)|$$

$$I_4(n) = \prod_{j=1}^k \mathbb{E}_\rho(\exp - \sum_{i=1}^{r_n-j_{r_n}} f_{i,n}^j) - \prod_{j=1}^k \mathbb{E}_\rho(\exp - \sum_{i=1}^{r_n} f_{i,n}^j)|$$

where stationarity of $\mathbb{P}_\rho$ has been used in the expressions of $I_3(n), I_4(n)$. The quantities $I_1, I_2, I_4$ are boundary terms; their estimation below is based only on the fact that $r_n$ (resp. $\ell_n$) is small with respect to $n$ (resp. $r_n$), the form of $u_n$, and $f$ has non negative real part. On the other hand estimation of $I_3$ depends on Theorem 3.1 and Proposition 3.4.
Using the inequality $|\exp(-x) - \exp(-y)| \leq |x - y|$ for $x, y$ with non negative real parts we get $I_1(n) \leq \sum_{k=1}^{n} \mathbb{E}_{\rho}(f_{i,n})$. Let $\delta > 0$ be as above such that $f(t, x) = 0$ for $t \in [0, 1]$, $|x| < \delta$, and observe that $n - kr < r$. Then the above bound for $I_1(n)$ gives:

$$I_1(n) \leq r_n |f|_{\infty} \mathbb{P}_{\rho}(u_n^{-1}|X_1| \geq \delta).$$

Since $\lim_{n \to \infty} n^{-1} r_n = 0$, the definition of $u_n$ and Theorem 2.1 gives $\lim_{n \to \infty} I_1(n) = 0$. Also $I_1(n)$ is bounded by $n^{-1} r_n$, up to a coefficient depending only on $m, \delta$. For $I_2(n)$, a similar argument involving each interval $J_j$ and the subinterval $J'_j$ gives:

$$I_2(n) \leq k_n \ell_n |f|_{\infty} \mathbb{P}_{\rho}(u_n^{-1}|X_1| \geq \delta).$$

Using $k_n r_n \leq n$ we get $\lim_{n \to \infty} n^{-1} k_n \ell_n \leq \lim_{n \to \infty} n^{-1} \ell_n$ i.e $\lim_{n \to \infty} I_2(n) = 0$ if $\lim_{n \to \infty} r^{-1}_n \ell_n = 0$.

Also we can bound $I_2(n)$ by $r_n^{-1} \ell_n$, up to a coefficient depending only on $m, \delta$.

For $I_4(n)$, we use the inequality $|\prod_{1}^{n} z_j - \prod_{1}^{n} w_j| \leq \sum |z_j - w_j|$ if $|z_j|$ and $|w_j|$ are less than 1.

Hence:

$$I_4(n) \leq \sum_{j=1}^{k} |\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n}) - \mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n})| \leq |f|_{\infty} k_n \ell_n \mathbb{P}_{\rho}(|X_1| > \delta u_n)$$

As above we get $\lim_{n \to \infty} I_4(n) = 0$ if $\lim_{n \to \infty} r^{-1}_n \ell_n = 0$, and a bound for $I_4(n)$ of the same form as for $I_2(n)$.

The estimation of $I_3(n)$ is more delicate and depends on Lemma 3.6 below. We begin with the inequality: $I_3(n) \leq D(n) + I_5(n) + I_6(n - r, n)$ where

$$D(n) = |\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} \sum_{j=1}^{r-\ell} f_{i,n}) - \mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} \sum_{j=1}^{r-\ell} f_{i,n})|,$$

$$I_5(n) = |(\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n})\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n}) - \mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n})\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n})|,$$

$$I_6(n - r) = |\mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n}) - \prod_{j=2}^{k} \mathbb{E}_{\rho}(\exp - \sum_{j=1}^{k} f_{i,n})|.$$
the operators \( P_{i,n} \) belong to \( O(m, \gamma) \subset End \mathcal{H}_{X, \varepsilon, \kappa} \). With the above notations, the products of operators \( P_{i,n} \) belong to \( \tilde{O}(m, \gamma) \). Also, using Proposition 3.4 we know that on \( \mathcal{H}_{X, \varepsilon, \kappa} \) we can write \( P = \rho \otimes 1 + U \) where \( U \) has spectral radius \( r(U) \) less then 1 and \( U \) commutes with the projection \( \rho \otimes 1 \). We note also that for \( f \) as above and \( \psi \in \mathcal{H}_{X, \varepsilon, \kappa} \) we have:

\[
|\rho(P^j \psi)| \leq \rho(P |\psi|) = \rho(|\psi|) \leq \|\psi\|
\]

Then Lemma 3.6 below implies the convergence of \( D(n) \) to zero with speed.

Now, in order to prove the proposition, we are left to show \( \lim_{n \to \infty} I_3(n) = 0 \). We iterate \( k_n \) times the inequality : \( I_3(n) \leq D(n) + I_5(n) + I_3(n - r_n) \). We get, using Lemma 3.6 :

\[
I_3(n) \leq I_3(n - r_n) + C'(f)(n^{-2} r_n^2 + r_1^f n(U)) \leq C'(f)(k_n r_1^f n(U) + n^{-1} r_n),
\]

with \( C'(f) \geq 1 \), depending on \( m, \gamma \). Since \( r_n = o(n) \), it remains to choose \( \ell_n \) such that \( \ell_n = o(r_n) \) with \( \lim_{n \to \infty} k_n r_1^f n(U) = 0 \). These conditions can be written as:

\[
\lim_{n \to \infty} r_n^{-1} \ell_n = 0, \quad \lim_{n \to \infty} r_n^{-1} m r_1^f n(U) = 0.
\]

The choice of \( \ell_n \) with the above properties is possible since:

\( r_1(U) < 1, \lim_{n \to \infty} n^{-1} r_n = 0 \) and \( \lim_{n \to \infty} (\log n)^{-1} r_n = \infty \).

One can take \( \ell_n < r_n \) with \( (\log n)^{-1} \ell_n = \infty \). The above estimations of \( I_1, I_2, I_3, I_4, I_5 \) give bounds by \( \sup(n^{-1} r_n, r_n^{-1}) \), up to a coefficient depending on \( \delta, m, \gamma \).

**Lemma 3.6** There exists positive numbers \( C_1(U), \ r_1(U) \in [r(U), 1[ \) and \( C(f) \) depending only of \( m, \gamma \) such that, for \( n \in \mathbb{N} \) and \( \ell_n < r_n \), \( D(n) \), as above :

\[
D(n) = |\rho(P_1 n \cdots P_{r - \ell_n} U^{\ell_n} \psi_n)| \leq C_1(U) C(f)(r_1(U))^{\ell_n}.
\]

**Proof:** We observe that Markov’s property implies \( \mathbb{E}(e^{-f(X_i^k)} | g(\omega)) = P^f(\mathbb{E}(g(\omega))) \) where \( f \) is as above, \( g(\omega) \) is a function depending only of \( \omega \) through the random variables \( X_i^k (k \geq 2) \) and \( \mathbb{E}(g(\omega)) \) is a function of \( x \). We apply this property to \( \mathcal{H}_{X, \varepsilon, \kappa} \) with \( f = f_{i,n} \) \( (1 \leq i \leq r - \ell) \) or \( f = 0, g = \psi_n \) as above, hence writing \( P^f = \rho \otimes 1 + U^{\ell} \) and

\[
D(n) = |\rho(P_1 n \cdots P_{r - \ell_n} U^{\ell_n} \psi_n) - \rho(P_1 n \cdots P_{r - \ell_n} U^{\ell_n} \rho(\psi_n))| = |\rho(P_1 n \cdots P_{r - \ell_n} U^{\ell_n} \psi_n)|,
\]

Proposition 3.4 implies the existence of \( C_1(U) < \infty, r_1(U) \in [r(U), 1[ \) with

\[
\|U^{\ell_n} \psi_n\| \leq C_1(U) r_1(U) \|\psi_n\|.
\]

On the other hand, since \( \psi_n \) is of the form \( \psi_n = A 1 \) with \( A \in \tilde{O}(m, \gamma) \), we have, using Theorem 3.1, \( \|\psi_n\| \leq C(f) \) with \( C(f) \) depending on \( m, \gamma \). It follows \( D(n) \leq C_1(U) C(f)(r_1(U))^{\ell_n} \).

**□**

4 Asymptotics of exceedances processes

4.1 Statements of results

Let \( E \) be a complete separable metric space which is locally compact, \( M_+(E) \) the space of positive Radon measures on \( E \), \( M_p(E) \) its subspace of point measures, \( C_c^\circ(E) \) (resp \( \mathcal{L}_+^c(E) \)) the space of non negative and compactly supported continuous (resp Lipschitz) functions. Then it is well known that the vague topology on \( M_+(E) \) is given by a metric
and then $M_+(E)$ becomes a complete separable metric space. Furthermore this metric is constructed (see [28] Lemma 3.11, Proposition 3.17) using a countable family $(h_i)_{i \in I}$ of functions in $L^c_+(E)$ and $M_+(E)$ is a closed subset of $M_+(E)$. It follows that, in various situations with respect to weak convergence of random measures, $C_+(E)$ can be replaced by $L^c_+(E)$.

Below, assuming condition $(H)$, we describe the asymptotics of the space-time exceedance process $N_n = \sum_{i=1}^{n} \varepsilon_{(n^{-1}i, u^{-1}X_i)}$ under the probability $\mathbb{P}_\rho$ and we state a few corollaries. The results are formally analogous to results for stationary processes proved in ([2], [3]) under general conditions. Here however, corresponding conditions have been proved in sections 2, 3 for the affine random walk $X_n$; hence the results described below are new for affine random walks.

It is convenient to express the Laplace formula below in terms of the renewal point process $\pi_\omega^v = \sum_{i \geq 0} \mathcal{E}_S_{n_i}(\omega)\delta Z_{ij}$ of the linear random walk $S_{n_i}(\omega)$.

We denote by $\sum_{i \geq 0} \mathcal{E}_{T^i_\delta}$ the homogeneous point Poisson process on $[0, 1]$ with intensity $p(\delta) = \theta \delta^{-\alpha}$ and by $\sum_{j > 0} \mathcal{E}_{Z_{ij}}$ an i.i.d collection of copies of the cluster process $C = \sum_{j > 0} \varepsilon Z_j$ described in Proposition 2.5, independent of $\sum_{i \geq 0} \mathcal{E}_{T^i_\delta}$. Since we have $|X_n^x - X_n^y| \leq |S_n||x - y|$ and $\lim_{n \to \infty} |S_n| = 0$, $\mathbb{P}$ a.e it is possible to replace $\mathbb{P}_\rho$ by $\mathbb{P}$ and $X_n$ by $X_n^x$ with $x$ fixed, in the statements. We give the corresponding proof for the logarithm law only.

**Theorem 4.1** The sequence of normalized space-time point processes $N_n = \sum_{i=1}^{n} \varepsilon_{(n^{-1}i, u^{-1}X_i)}$ on the space $[0, 1] \times (V \setminus \{0\})$ converges weakly to a point process $N$. For any $\delta > 0$, the law of the restriction of $N$ to $[0, 1] \times B'_\delta$ is the same as the law of the point process on $[0, 1] \times B'_\delta$ given by:

$$\sum_{i \geq 0} \sum_{j > 0} \mathcal{E}_{(T^i_\delta, \delta Z_{ij})} 1\{|Z_{ij}| > 1\}.$$

If $\eta$ denotes the law of $N$ and $f \in C_c^c([0, 1] \times B'_\delta)$, then $\log \psi_\eta(f)$ is equal to

$$\theta \delta^{-\alpha} \int_{0}^{1} \mathbb{E}_1(1 - \exp - \sum_{j > 0} f(t, \delta Z_j)) dt = c^{-1} \int_{0}^{1} \mathbb{E}_1[(\exp f_t(v) - 1)\exp - \pi_\omega^v(f_t)] dt$$

where $f_t(x) = f(t, x)$

Assuming the mixing and anticlustering conditions for continuous functions, this statement was proved in [3]. Here we will use Propositions 2.5, 3.4 and point process theory.

Now as a consequence of Theorem 4.1, the mixing property stated in Proposition 3.5 for Lipchitz functions can be extended to compactly supported continuous functions. Then, in particular, the mixing condition $A(u_n)$ of [2] is valid here and the basic conditions of extreme value theory (see [8]) are satisfied in our context.
Corollary 4.2 With the notation of Proposition 3.5, assume \( f \) is a continuous compactly supported function on \([0, 1] \times (V \setminus \{0\})\). Then we have the convergence \( \lim_{n \to \infty} I_n(f) = 0 \).

Since the space exceedances process \( N^s_n = \sum_{i=1}^{n} \varepsilon u_n^{-1} X_i \) is the projection of \( N_n \) on \( V \setminus \{0\} \) we have the

Corollary 4.3 The normalized space exceedance process \( N^s_n \) converges weakly to a point process \( N^s \). The law of the restriction of \( N^s \) to \( B_1' \) is the same as the law of the point process

\[
\mathcal{Q}^s = \sum_{i=0}^{\infty} \varepsilon \delta Z_i 1_{\{|Z_i| > 1\}}
\]

where \( T^s \) is a Poisson random variable with mean \( p(\delta) = \theta \delta^{-\alpha} \).

The Laplace functional of \( N^s \) is given by

\[
\exp c^{-1} \mathbb{E}_\Lambda \left[ (\exp f(v) - 1) \exp - \pi^w_f(f) \right].
\]

Assuming the mixing and anticlustering conditions for continuous functions, this statement was proved in [2], using the formula for Laplace functionals in Proposition 2.5.

We consider the \( N \)-valued random variable \( \zeta = \pi^w(B_1') \) and we write \( \theta_k = Q_{\Lambda_1} \{ \zeta = k \} \) for \( k \geq 1 \); in particular we have \( \theta_1 = \theta \), \( \theta_k \geq \theta_{k+1} \).

Corollary 4.4 The sequence of normalized time exceedances process \( N^t_n = \sum_{i=1}^{n} \varepsilon u_n^{-1} 1_{\{|X_i| > u_n\}} \)

converges weakly \((n \to \infty)\) to the homogeneous compound Poisson process \( N^t \) on \([0, 1]\) with intensity \( \theta \), and cluster probabilities \( \nu_k(k \geq 1) \) where \( \nu_k = \theta^{-1}(\theta_k - \theta_{k+1}) \).

Under special hypotheses, including density of the law of \( B_n \) with respect to Lebesgue measure, this statement was proved in [21].

Fréchet’s law for \( M^x_n = \sup\{X^x_k; 1 \leq k \leq n\} \) is a simple consequence of Corollary 4.4 as follows.

Corollary 4.5 For any \( x \in V \) and \( t > 0 \) we have the convergence in law of \( u_n^{-1} M^x_n \) to Fréchet’s law \( \Phi_\alpha \),

\[
\lim_{n \to \infty} \mathbb{P}\{u_n^{-1} M^x_n < t\} = \exp - \theta t^{-\alpha} = \Phi_\alpha^\theta([0, t])
\]

with \( \theta = Q_{\Lambda_1} \{ \sup_{n \geq 1} |S_n(\omega)v| \leq 1 \} \). Furthermore the normalized law of the entrance time \( \tau^x_\alpha \) of \( |X^x_n| \) in \([t, \infty]\) converges to the exponential law with parameter \( c \theta \), i.e

\[
\lim_{t \to \infty} \mathbb{P}\{t^{-\alpha} \tau^x_\alpha > k\} = \exp - c \theta k.
\]
It was observed in [26] that Sullivan’s logarithm law for excursions of geodesics around the cusps of hyperbolic manifolds (see [30]), in the case of the modular surface, is a consequence of Fréchet’s law for the continuous fraction expansion of a real number uniformly distributed in [0,1](see [25]). For more detailed extreme value properties in the context of pointwise convergence, we refer to ([10], p 168-179). Here, in this vein, we have the following logarithm law.

**Corollary 4.6** For any \( x \in V \), we have the \( \mathbb{P} - a.e \) convergence

\[
\limsup_{n \to \infty} \frac{\log |X^n_x|}{\log n} = \lim_{n \to \infty} \frac{\log M^n_x}{\log n} = \frac{1}{\alpha}.
\]

We observe that a logarithm law and a modified Fréchet law have been obtained in [20] for random walks on some homogeneous spaces of arithmetic character, using \( L^2 \)-spectral gap methods.

### 4.2 Proofs of point process convergences

The proof of Theorem 4.1 will follow of three lemmas. We denote by \( (X_{k,j})_{k \in \mathbb{N}} \) an i.i.d sequence of copies of the process \( (X_j)_{j \in \mathbb{N}} \) and we write

\[
\tilde{N}_{k,n} = r_n \sum_{j=1}^n \delta_{(n^{-1}kr_n,u^{-1}_nX_{k,j})}, \quad \tilde{N}_n = \sum_{k=1}^{k_n} \tilde{N}_{k,n},
\]

where \( r_n, k_n \) are as in section 2. For \( k_n > 0 \) we denote by \( \mathbb{E}^{(k_n)}_\rho \) the expectation corresponding to the product probability of \( k_n \) copies of \( \mathbb{P}_\rho \).

If \( f \) is a non negative and compactly supported Lipchitz function on \([0,1] \times V \setminus \{0\}\), we have, using independence:

\[
\mathbb{E}^{(k_n)}_\rho (\exp - \tilde{N}_n(f)) = \prod_{k=1}^{k_n} \mathbb{E}_\rho (\exp - \sum_{j=1}^{r_n} f(n^{-1}k,u^{-1}_nX_{k,j})).
\]

This relation and the multiple mixing property in Proposition 3.5 show that, on functions \( f \) as above, the asymptotic behaviour of the Laplace functionals of \( N_n \) under \( \mathbb{E}_\rho \), and \( \tilde{N}_n \) under \( \mathbb{E}^{(k_n)}_\rho \), are the same. We begin by considering the convergence of \( \mathbb{E}^{(k_n)}_\rho (\exp - \tilde{N}_n(f)) \).

Lemma 4.7 below is a general statement giving the weak convergence of a sequence of random measures, using only the convergence of the values of the Laplace functionals on Lipchitz functions.

Lemmas 4.8, 4.9 are reformulations of part of the proof of Theorem 2.3 in [3], which was considered in a general setting.

**Lemma 4.7** Let \( E \) be a separable metric space endowed with a probability \( m \) and assume \( E \) to be locally compact. Let \( \nu_n \) be a sequence of random measures on \( E \) and, for \( f \) non negative Lipchitz and compactly supported, assume that the sequence of Laplace functionals \( \psi_{\nu_n}(f) \) converges to \( \psi(f) \) and \( \psi(sf) \) is continuous at \( s = 0 \), then the sequence \( \nu_n \) converges
weakly. A random measure $\nu = (\nu_x)_{x \in E}$ on $(E, m)$ is well defined by the values of its Laplace functionals $\psi(f) = \int \exp - \nu_x(f) dm(x)$ with $f$ as above.

Proof: We begin by the last assertion and we use the family of Lipschitz functions $(h_i)_{i \in I}$ considered in the above subsection. If the random measures $\nu, \nu'$ satisfy $\psi(f) = \psi_{\nu'}(f)$ for any $f \in \mathcal{L}_+^c(E)$ and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are non-negative numbers then we have $\psi_{\nu}(\sum_{i=1}^{p} \lambda_i h_i) = \psi_{\nu'}(\sum_{i=1}^{p} \lambda_i h_i)$. It follows that the random vectors $(\nu(h_1), \ldots, \nu(h_p))$ and $(\nu'(h_1), \ldots, \nu'(h_p))$ have the same Laplace transforms, hence the same laws. Furthermore, for rational numbers $r_j < r_j'$ the finite intersections of sets of the form $\{\mu \in M_+(E), \mu(h_i) \in [r_j, r_j']\}$ form a countable basic $B$ of open subsets in $M_+(E)$ stable under finite intersection, hence a $\pi$-system (see [28]). Then from above, $\nu, \nu'$ are equal on $B$; since the $\sigma$-field generated by $B$ coincide with the Borel field, one has $\nu = \nu'$.

We observe that, if a sequence of random measures $\nu_n$ is such that for any $f \in \mathcal{L}_+^c(E)$ the sequence of real random variables $\nu_n(f)$ is tight, then the sequence $\nu_n$ itself is tight. This follows for a corresponding result in [28] for $f \in \mathcal{C}_+^c(E)$ since any such $f$ is dominated by an element of $\mathcal{L}_+^c(E)$.

Assuming the convergence of $\psi_{\nu_n}(f)$ to $\psi_{\nu}(f)$ for any $f \in \mathcal{L}_+^c(E)$ and the continuity at $s = 0$ of $\psi_{\nu}(sf)$, we get that $\psi_{\nu}(sf)$ is the Laplace transform of the real random variable $\nu(f)$, and the convergence of the sequence $\nu_n(f)$ to $\nu(f)$ for any $f \in \mathcal{L}_+^c(E)$. From above and the continuity hypothesis of $\psi(sf)$ at $s = 0$, we get that the sequence $\nu_n$ is tight. If $\nu_n$ is a subsequence converging weakly to the random measure $\nu$ we have $\lim_{n \to \infty} \psi_{\nu_n}(f) = \lim_{j \to \infty} \psi_{\nu_{n_j}}(f)$ for any $f \in \mathcal{L}_+^c(E)$. Since such a limit is independent of the subsequence, we get from above that two possible weak limits of random measures are equal. Hence the sequence $\nu_n$ converges weakly to $\nu$. □

**Lemma 4.8** Let $f$ be a non negative and compactly supported continuous function on $[0, 1] \times B'_\delta$ and let $\sum_{j \geq 0} \varepsilon_{Z_j}$ be the cluster process for the affine random walk $(X_k)_{k \in \mathbb{N}}$. Then :

a) $\lim_{n \to \infty} \left[ \log \mathbb{E}_\rho^{(k_n)}(\exp - \bar{N}_n(f)) + \frac{k_n}{1} (1 - \mathbb{E}_\rho(\exp - \bar{N}_k(f))) \right] = 0.$

b) $\lim_{n \to \infty} \frac{k_n}{1} \left( 1 - \mathbb{E}_\rho(\exp - \bar{N}_k(f)) \right) = \theta \delta^{-\alpha} \int_0^1 \mathbb{E}_{\Lambda_1}(1 - \exp - \sum_{j \geq 0} f(t, \delta Z_j)) dt.$

**Lemma 4.9** Let $\sum_{i \geq 0} \varepsilon_{T_i}$ be a homogeneous Poisson process of intensity $p(\delta) > 0$ on $[0, 1]$, which is independent of the sequence of cluster processes $\sum_{j \geq 0} \varepsilon_{Z_{ij}}$.

Then for any non negative and compactly supported continuous function $f$ on $[0, 1] \times B'_\delta$, the Laplace functional of the point process $Q^{\delta} = \sum_{i \geq 0, j \geq 0} \varepsilon_{(T_i, \delta Z_{ij})} f(|Z_{ij}| > 1)$ restricted to $[0, 1] \times B'_\delta$ is equal to :
\[
\psi^\delta(f) = \exp - p(\delta) \int_0^1 \mathbb{E}_{\Lambda_1}(1 - \exp - \sum_{n=1}^{\infty} f(t, \delta Z_n)) dt
\]

**Proof of Theorem 4.1** Let \( f \) be a compactly supported Lipchitz function on \([0, 1] \times B_\delta'\).

Using Proposition 3.5, lemmas 4.8 implies that, on non-negative compactly supported Lipchitz functions on \([0, 1] \times B_\delta'\), the Laplace functionals of \( N_n \) and \( \tilde{N}_n \) have the same limit, namely
\[
\psi^\delta(f) = \exp - p(\delta) \int_0^1 \mathbb{E}_{\Lambda_1}(1 - \exp - \sum_{n=1}^{\infty} f(t, \delta Z_n)) dt.
\]

We observe that, for fixed \( f \) as above, the function \( s \to \psi^\delta(s f) \) is continuous at \( s = 0 \). Since the function \( s \to \psi_n(s f) = \mathbb{E}_{\rho}(\exp - s N_n(f)) \) is the Laplace transform of the non negative random variable \( N_n(f) \), the continuity theorem for Laplace transforms implies that the sequence \( N_n(f) \) converges in law to some random variable. Since this is valid for any \( f \) as above, Lemma 4.7 implies that the sequence of point processes \( N_n \) itself is tight. Since moreover the sequence of Laplace functionals \( \psi_n(f) \) converges to \( \psi^\delta(f) \), Lemma 4.7 implies that there exists a unique point process \( N \) on \([0, 1] \times (V \setminus \{0\})\) such that the sequence \( N_n \) converges weakly to \( N \). As stated in Lemmas 4.8, 4.9 the restriction of \( N \) to \([0, 1] \times B_\delta'\) is given by the point process formula in the theorem. Lemma 4.9 implies that the Laplace functional of \( N \) on the function \( f \in C_\infty([0, 1] \times B_\delta') \) is equal to
\[
\psi^\delta(f) = \exp - \theta \delta^\alpha \int_0^1 \mathbb{E}_{\Lambda_1}[1 - \exp - \sum_{j>0} f(t, \delta Z_j)] dt.
\]

The first part of the formula giving the Laplace functional of \( N \) on \( f \) follows. The second part is a consequence of the last formula in Proposition 2.5 applied to the function \( v \to f(t, \delta v) \) and of the \( \alpha \)-homogeneity of \( \Lambda \). \( \Box \)

**Proof of Theorem 4.2** The first term \( \mathbb{E}_\rho(\exp - \sum_{n=1}^{\infty} f_{i,n}) \) in \( I_n(f) \) is the value of the Laplace functional of \( N_n \) on the continuous function \( f \). Hence Theorem 4.1 implies its convergence to the Laplace functional of \( N \) on \( f \). The same remark is valid for the second term in \( I_n(f) \), if \( N_n \) is replaced by \( \tilde{N}_n \); the limit of \( \tilde{N}_n \) is also \( N \), using Lemma 4.7 and Proposition 3.5. Then for any \( f \) in \( C_\infty([0, 1] \times (V \setminus \{0\})) \) we have \( \lim_{n \to \infty} I_n(f) = \lim_{n \to \infty} [\mathbb{E}_\rho[\exp - N_n(f) - \mathbb{E}_{\rho}(\exp - \tilde{N}_n(f))] = 0 \) \( \Box \)

**Proof of Corollary 4.3** The point process \( N_n^s \) is the projection of \( N_n \) on \( V \setminus \{0\} \). Since \([0, 1] \) is compact and the projection is continuous, the continuous mapping theorem implies the required convergence, using the first part of Theorem 4.1. The formula for the Laplace functional of \( N^s \) is a direct consequence of the second part in Theorem 4.1 applied to a function independent of \( t \). \( \Box \)

**Proof of Corollary 4.4** For \( \varphi \in C_\infty([0, 1]) \) we have \( N_n^t(\varphi) = N_n(\varphi 1_{B_1'}) \). Since the discontinuity set of \( 1_{B_1'} \) is \( \Lambda \)-negligible, Theorem 4.1 gives the convergence of \( N_n^t(\varphi) \) to \( N^t(\varphi) \). With \( f = \varphi \otimes 1_{B_1'} \), the formula for the Laplace functional \( \psi_n(f) \) of \( N \) gives the
Laplace functional $\psi_{n}(\varphi)$ of $N_{\delta}$ in the logarithmic form
\[ \log \psi_{n}(\varphi) = -\theta \int_{0}^{1} \mathbb{E}_{\Lambda_{1}}[1 - \varphi(t)\gamma]dt. \]
The expression of the generating function of the random variable $\gamma = \sum_{j \geq 1} 1_{B_{1}}(Z_{j})$ follows from the last formula in Proposition 2.5:
\[ \sum_{k=1}^{\infty} e^{-sk} \nu_{k} = 1 - (e^{s} - 1)\theta^{-1}\mathbb{E}_{\Lambda_{1}}[\exp\{s \gamma(B_{1}^{\;}')\}]. \]
Hence $\nu_{k} = \theta^{-1}(\theta_{k} - \theta_{k+1})$
In view of Theorem 4.1, the point process $N_{\delta}$ can be written as $N_{\delta} = \sum_{k \geq 0} \gamma_{k} \varepsilon_{T_{k}^{\;\delta}}$, where the random variables $\gamma_{k}$ are i.i.d with the same law as $\gamma$, hence $N_{\delta}$ coincides with the compound Poisson process described in the statement. □

**Proof of Corollary 4.5** Replacing $u_{n}$ by $\delta u_{n}$ ($\delta > 0$) in Corollary 4.4, we see that the point process on $[0, 1]$ given by $N_{n, \delta}^{l} = \sum_{k=1}^{\infty} \varepsilon_{n^{-1}} 1_{\{|X_{n}| > \delta u_{n}\}}$ converges to $N_{\delta}^{l} = \sum_{k \geq 0} \gamma_{k} \varepsilon_{T_{k}^{\;\delta}}$ where $\sum_{k \geq 0} \varepsilon_{T_{k}^{\;\delta}}$ is the Poisson process on $[0, 1]$ with intensity $\theta^{-\alpha}$ and the $\gamma_{k}$ are i.i.d random variables as in the proof of Corollary 4.4. It follows that for any $\delta > 0$,
\[ \lim_{n \to \infty} \mathbb{P}_{\rho}\{N_{n, \delta}^{l}(1) = 0\} = \mathbb{P}_{\rho}\{M_{n} \leq u_{n}\delta\}, \]the convergence of $u_{n}^{-1}M_{n}$ to Fréchet’s law follows.
If $M_{n}$ is replaced by $M_{n}^{\varepsilon}(x \in V)$, the same proof as the one given below for the logarithm law remains valid. The last assertion in the corollary is a direct consequence of Fréchet’s law. □

### 4.3 Proofs of logarithm’s law

The proof of the logarithm’s law is based on Fréchet’s law and depends on two lemmas as follows.

**Lemma 4.10** We have $\mathbb{P}_{\rho} - a.e.$
\[ \limsup_{n \to \infty} \frac{\log|X_{n}|}{\log n} \leq \limsup_{n \to \infty} \frac{\log M_{n}}{\log n} \leq \frac{1}{\alpha}. \]

**Proof**: Let $\varepsilon > 0$, $A_{n}(\varepsilon) = \{|X_{n}| \geq n^{1/\alpha + \varepsilon}\} \subset V_{\alpha, T_{\alpha}}$, $A'_{n}(\varepsilon) = V_{\alpha, T_{\alpha}} \setminus A_{n}(\varepsilon)$. Stationarity of $X_{n}$ implies $\mathbb{P}_{\rho}(A_{n}(\varepsilon)) = \mathbb{P}_{\rho}\{|X_{0}| \leq n^{1/\alpha + \varepsilon}\}$. Since $\lim_{n \to \infty} n^{1+\alpha\varepsilon} \mathbb{E}_{\alpha, T_{\alpha}}(n^{1/\alpha + \varepsilon}) = 1$, we have $\sum_{i=0}^{\infty} \mathbb{P}_{\rho}\{A_{n}(\varepsilon)\} < \infty$. Then Borel-Cantelli’s lemma implies that $\mathbb{P}_{\rho}\{\bigcup_{i=0}^{\infty} A'_{n}(\varepsilon)\} = 1$ i.e $\mathbb{P}_{\rho} - a.e.$ there exists $n_{0}(\omega)$ such that for $n \geq n_{0}(\omega)$, $|X_{n}(\omega)| \leq n^{1/\alpha + \varepsilon}$. Then we deduce that $\mathbb{P}_{\rho} - a.e.$
\[ \limsup_{n \to \infty} \frac{\log M_{n}}{\log n} \leq \frac{1}{\alpha} + \varepsilon. \]Since $\varepsilon$ is arbitrary we get:
\[ \limsup_{n \to \infty} \frac{\log M_{n}}{\log n} \leq \frac{1}{\alpha}. \]□
Lemma 4.11 We have \( P_\rho - a.e : \limsup_{n \to \infty} \frac{\log |X_n|}{\log n} \geq \frac{1}{\alpha}. \)

**Proof:** Let \( \varepsilon \in ]0, 1/\alpha[, B(\varepsilon) = \left\{ \limsup_{n \to \infty} \frac{\log |X_n|}{\log n} \leq \frac{1}{\alpha} - \varepsilon \right\}, B_n(\varepsilon) = \left\{ \sup_{j \geq n} \frac{\log |X_j|}{\log j} \leq \frac{1}{\alpha} - \frac{\varepsilon}{2} \right\}. \)

The sequence \( B_n(\varepsilon) \) is increasing and \( B(\varepsilon) \subset \bigcup_{j=2}^{\infty} B_j(\varepsilon). \) We are going to show \( P_\rho \{ B_n(\varepsilon) \} = 0. \)

For \( p \geq n \geq 2, p \in \mathbb{N}, \) we define \( B_{n,p}(\varepsilon) = \left\{ \sup_{n \leq j \leq p} |X_j| \leq p^{1/\alpha - \varepsilon/2} \right\}, \) hence \( B_n(\varepsilon) \subset B_{n,p}(\varepsilon). \) Using stationarity we get \( P_\rho \{ B_{n,p}(\varepsilon) \} = P_\rho \{ p^{-\varepsilon/2} M_{p-n+1} \leq p^{-\varepsilon/2} \}. \) Also, using Corollary 4.3, we have \( \limsup_{n \to \infty} \{ |P_\rho \{ n^{-1/\alpha} M_n \leq t \} - e^{-\varepsilon \alpha n} \} = 0 \) which implies \( \lim_{p \to \infty} P_\rho \{ B_{n,p}(\varepsilon) \} = 0. \)

Since the function \( p \to B_{n,p}(\varepsilon) \) is decreasing and \( B_n(\varepsilon) = \bigcap_{p \geq 2} B_{n,p}(\varepsilon) \) we have for \( n \geq 2 : \)

\[ \forall \varepsilon > 0, \quad P_\rho \{ B_n(\varepsilon) \} = \lim_{p \to \infty} P_\rho \{ B_{n,p}(\varepsilon) \} = 0, \quad \text{i.e} \quad P_\rho \{ B(\varepsilon) \} = 0. \]

We see that \( P_\rho - a.e, \limsup_{n \to \infty} \frac{\log |X_n|}{\log n} \geq \frac{1}{\alpha} - \varepsilon, \) and, since \( \varepsilon \) is arbitrary we conclude :

\[ \limsup_{n \to \infty} \frac{\log |X_n|}{\log n} \geq \frac{1}{\alpha}. \]

**Proof of Corollary 4.6** From Lemmas 4.10, 4.11 we have \( P_\rho - a.e, \)

\[ \lim_{n \to \infty} \frac{\log M_n}{\log n} = \limsup_{n \to \infty} \frac{\log |X_n|}{\log n} = \frac{1}{\alpha}. \]

Hence the definition of \( M_n \) implies \( \lim_{n \to \infty} \frac{\log M_n}{\log n} = \frac{1}{\alpha}. \)

Hence, for a set of \( \rho \otimes P_\rho \) probability 1 in \( V \times A^\mathbb{N} \) we have

\[ \frac{1}{\alpha} = \limsup_{n \to \infty} \frac{\log |X_n(\omega)|}{\log n} = \lim_{n \to \infty} \frac{\log |X_n(\omega)|}{\log n}, \]

hence for a subsequence \( n_k(\omega), \frac{1}{\alpha} = \lim_{k \to \infty} \frac{\log |X_{n_k}(\omega)|}{\log n_k}. \)

On the other hand we have for any \( x \in V : |X_n - X_n^x| \leq |S_n||X_0 - x| \) and \( P_\rho - a.e : \lim_{n \to \infty} |S_n| = 0. \)

Also \( |\log |X_n| - \log |X_n^x|| \leq |S_n||X_0 - x| \sup(|X_n|^{-1}, |X_n^x|^{-1}), \) hence for any \( x \in V, P_\rho - a.e : \)

\[ \lim_{n \to \infty} |\log |X_n| - \log |X_n^x|| = 0. \]

It follows \( \lim_{k \to \infty} \frac{\log |X_{n_k}|}{\log n_k} = \frac{1}{\alpha}, \) and \( P_\rho - a.e, \limsup_{n \to \infty} \frac{\log |X_n^x|}{\log n} \geq \frac{1}{\alpha}. \)

A similar argument shows that \( P_\rho - a.e, \limsup_{n \to \infty} \frac{\log |X_n^x|}{\log n} \leq \frac{1}{\alpha}. \)

Furthermore, for any \( n \geq 1, x \in V : \)

\[ |M_n^x - M_n| \leq \sup \{|S_k|; 1 \leq k \leq n\}|x - X_0| \]

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where the sequence on the right is $\mathbb{P} - a.e$ bounded. Then, an argument as above shows that, $\mathbb{P} - a.e : \lim_{n \to \infty} \frac{\log M_n}{\log n} = \frac{1}{\alpha}$. □

5 Convergence to stable laws

The convergence to stable laws of the normalized sums $\sum_{i=1}^{n} X_i$ under hypothesis (H) was shown in ([11], [15]) where explicit formulae for the corresponding characteristic functions were given. It was observed there that these formulae involved the asymptotic tail $\Lambda$ of $\rho$, as well as the renewal point process $\pi_\omega^v = \sum_{i=0}^{\infty} \varepsilon S_i(\omega)v$. A similar situation occurred in the dynamical context of [12], where the limiting law was expressed in terms of an induced transformation. We observe that the connection between stable laws for $\sum_{i=1}^{n} X_i$, where $(X_i)_{i \in \mathbb{N}}$ is a stationary process, and point process theory had been already developed in [7] in the context of sample autocorrelation functions. For a recent analysis of the involved properties in this setting see [23]. Here we give new proofs of the results given in ([11], [15]), following the point process approach. In particular we get also a direct proof of the convergence for the related space point process $N_s = \sum_{i=1}^{n} \varepsilon u^{-1} X_i$, via the analysis of Laplace functionals.

5.1 On the space exceedances process

We give here a direct proof of the convergence of $N_s^n$ and we deduce the convergence of the characteristic function for the random variable $N_s^n(f)$, for $f$ compactly supported.

**Theorem 5.1** Let $f$ be a complex valued compactly supported Lipchitz function on $V \setminus \{0\}$ which satisfies $\text{Re}(f) \geq 0$. Then we have

$$\lim_{n \to \infty} \mathbb{E}_\rho(\exp - N_s^n(f)) = \exp c^{-1} \mathbb{E}_\Lambda[(\exp f(v) - 1)\exp - \pi_v^\omega(f)].$$

The proof depends on two lemmas where notations explained above are used. For $i \leq j$ we write $C_n(i, j) = \exp - \sum_{k=i}^{j} f(u_n^{-1} X_k) - 1$, and we note the equality

$$C_n(i, r_n) = \sum_{i=1}^{r_n} [C_n(i, r_n) - C_n(i+1, r_n)]$$

where $C_n(r_n + 1, r_n) = 0$ and $r_n$ is a sequence as in Proposition 2.5. We note also that

$$|C_n(i, j)| \leq 2, \quad |C_n(i, j) - C_n(i+1, j)| \leq 2.$$  

We are going to compare $C_n(1, r_n)$ to $C_{n,k}(1, r_n)$ where

$$C_{n,k}(1, r_n) = \sum_{i=1}^{r_n} [C_n(i, i+k) - C_n(i+1, i+k)],$$

we write $\Delta_{n,k}$ for their difference, $\varepsilon_n = r_n\mathbb{P}_\rho\{|X| > u_n\}$ and $\text{supp}(f) \subset B'_\delta$ with $\delta > 0$.  

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Then we have the

**Lemma 5.2** \( \lim_{k \to \infty} \limsup_{n \to \infty} \varepsilon^{-1}_n \mathbb{E}_\rho(|\Delta_{n,k}|) = 0. \)

**Proof:** We can assume that \( r_n > k. \) We observe that
\[ 1 + C_n(i, r_n) = (1 + C_n(i + 1, r_n)) \exp - f(u^{-1}_n X_i). \]
Also \( C_n(i + 1, r_n) - C_n(i + 1, i + k) = (\exp - \sum_{j=i+1}^{i+k} f(X_j))(\exp - \sum_{j=i+k+1}^{r_n} f(X_j) - 1). \)

Hence we have
\[ \Delta_{n,k} = \sum_{i=1}^{r_n} (\exp - f(u^{-1}_n X_i) - 1)[C_n(i + 1, r_n) - C_n(i + 1, i + k)] = \Delta'_{n,k} + \Delta''_{n,k} \]
with \( \Delta'_{n,k} \) (resp \( \Delta''_{n,k} \)) is the above sum with index \( i \) restricted to \([1, r_n - k]\) (resp \( r_n - k, r_n)\)).

As observed above the expression under \( \Sigma \) is bounded by \( 4 \) and vanishes unless \( |X_i| > \delta u_n \)
for some \( i \in [1, r_n - k] \) and \( M'_{r_n} \) \( M'_{r_n} > \delta u_n. \) Then we get using stationary,
\[ \mathbb{E}_\rho(|\Delta'_{n,k}|) \leq 4r_n \mathbb{P}_\rho\{|X| > \delta u_n, M'_{r_n} > \delta u_n\}. \]

Since the process \((X_k)_{k \in \mathbb{Z}^+}\) satisfies anticlustering, it follows \( \lim_{k \to \infty} \limsup_{n \to \infty} \varepsilon^{-1}_n \mathbb{E}_\rho(|\Delta'_{n,k}|) = 0. \)
Also, stationarity implies
\[ \mathbb{E}_\rho(|\Delta''_{n,k}|) \leq 4 \sum_{i=r_n-k+1}^{r_n} \mathbb{P}_\rho\{|X| > \delta u_n\} = 4k \mathbb{P}_\rho\{|X| > \delta u_n\}. \]

Since \( \rho \) is homogeneous at infinity and \( \lim_{n \to \infty} r_n^{-1} k = 0, \) we get \( \lim_{k \to \infty} \limsup_{n \to \infty} \varepsilon^{-1}_n \mathbb{E}_\rho(|\Delta''_{n,k}|) = 0, \)
hence the required assertion. \( \square \)

**Lemma 5.3** We have the following convergences.

1) For any \( k \geq 1 \) \( \lim_{n \to \infty} k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] = c^{-1} \mathbb{E}_\Lambda[(\exp f(v) - 1) \exp - \sum_{i=0}^{k} f(S_iv)] \)
2) \( \lim_{k \to \infty} \limsup_{n \to \infty} k_n \mathbb{E}_\rho(|\Delta_{n,k}|) = 0 \)

**Proof:** 1) Using stationarity we have
\[ k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] = k_n r_n \mathbb{E}_\rho[C_n(1, k + 1) - C_n(2, k + 1)] = \]
\[ k_n r_n \mathbb{E}_\rho[\exp - \sum_{j=0}^{k} f(u^{-1}_n X_j) - \exp - \sum_{j=1}^{k} f(u^{-1}_n X_j)]. \]

The function \( f^{(k)} \) on \( (V \setminus \{0\})^{k+1} \) given by
\[ f^{(k)}(x_0, x_1, \ldots, x_k) = \exp - \sum_{j=0}^{k} f(x_j) - \exp - \sum_{j=1}^{k} f(x_j) = (\exp f(x_0) - 1) \exp - \sum_{j=0}^{k} f(x_j) \]
is bounded, uniformly continuous on \( (B^k)' \) and \( \lim_{n \to \infty} n^{-1} k_n = 1. \) Hence, the homogeneity at infinity of \( \rho, \) the conditional convergence of \( u^{-1}_n X_j \) to \( S_j X_0 \) and the definition of \( \Lambda \) imply
\[ \lim_{n \to \infty} k_n \mathbb{E}_\rho[C_{n,k}(1, r_n)] = c^{-1} \mathbb{E}_\Lambda[(\exp f(v) - 1) \exp - \sum_{j=0}^{k} f(S_j v)]. \]
2) We have the equality, \( k_n \mathbb{E}_\rho(|\Delta_{n,k}|) = k_n r_n \mathbb{P}_\rho(|X| > u_n) e_n^{-1} \mathbb{E}_\rho(|\Delta_{n,k}|) \).

Then, using Lemma 5.2, the relation \( \lim_{n \to \infty} n^{-1} k_n r_n = 1 \) and the homogeneity at infinity of \( \rho \), assertion 2 follows. \( \Box \)

**Proof of Theorem 5.1** With \( r_n \) as in Proposition 5.3 above, the multiple mixing property gives
\[
\lim_{n \to \infty} [\mathbb{E}_\rho(exp - N_n(f)) - (\mathbb{E}_\rho(1 + C_n(1, r_n)))^k] = 0,
\]
hence it suffices to study the sequence \( (1 + \mathbb{E}_\rho(C_n(1, r_n)))^k \). Since \( \text{Ref} \geq 0 \) and \( \text{supp}(f) \subset B_\delta' \) we have
\[
\mathbb{E}_\rho(|C_n(1, r_n)|) \leq \mathbb{E}_\rho(|1 - exp - \sum_{i=1}^{r_n} f(u_n^{-1} X_i)|) \leq \mathbb{E}_\rho(\sum_{i=1}^{r_n} |f(u_n^{-1} X_i)|) \leq r_n |f|_\infty \mathbb{P}_\rho\{|X_0| > \delta u_n\}.
\]
The last inequality implies the \( L^1 \)-convergence to zero of \( \sum_{i=1}^{r_n} f(u_n^{-1} X_i) \). Then the first in equality gives \( \lim_{n \to \infty} \mathbb{E}_\rho(|C_n(1, r_n)|) = 0 \).

It follows that the behaviour of the sequence \( (1 + \mathbb{E}_\rho(C_n(1, r_n)))^k \) for \( n \) large is determined by the behaviour of \( k_n \mathbb{E}_\rho(C_n(1, r_n)) \). We have for \( k \geq 1, \)
\[
k_n \mathbb{E}_\rho(C_n(1, r_n)) = k_n \mathbb{E}_\rho(C_{n,k}(1, r_n)) + k_n \mathbb{E}_\rho(\Delta_{n,k}).
\]
Since \( \text{supp}(f) \subset B_\delta' \) and \( \lim_{j \to \infty} |S_j v| = 0 \ Q - a.e. \), the series \( \sum_{j=1}^{\infty} f(S_j v) \) converges \( Q - a.e. \).

Since \( \text{Ref} \geq 0 \), it follows \( \lim_{k \to \infty} \mathbb{E}(exp - \sum_{j=1}^{k} f(S_j v)) = exp - \pi'_v(f) \).

Then dominated convergence and Lemma 5.3 imply
\[
\lim_{k \to \infty} \lim_{n \to \infty} k_n \mathbb{E}_\rho(C_{n,k}(1, r_n)) = \mathbb{E}_\Lambda[(exp f(v) - 1)exp - \pi'_v(f)].
\]
This equality and the second assertion in Lemma 5.3 give the result. \( \Box \)

**Corollary 5.4** Let \( m > 0, \delta > 0, \gamma \geq 0, \) and let \( f \) be a \( \mathbb{R}^m \)-valued continuous function on \( V \setminus \{0\} \) which satisfies the conditions
1) \( f \) is locally Lipchitz
2) \( f(v) = 0 \) for \( |v| < \delta \)
3) \( \sup_{v \in V} |v|^{-\gamma} |f(v)| = c_\gamma < \infty \)

Then we have, for any \( u \in \mathbb{R}^m, \)
\[
\lim_{n \to \infty} \mathbb{E}_\rho(exp - i < u, N_n(f) >) = exp c^{-1} \mathbb{E}_\Lambda[(exp i < u, f(v) > - 1)exp - i < u, \pi'_v(f) >]
\]

**Proof:** We consider the random variable \( Y_n = N_n(f) \). For \( a \geq 1, \) let \( \theta_a \) be the function from \( V \setminus \{0\} \) to \([0, 1]\) defined by
\[
\theta_a(v) = 1 \text{ for } |v| \leq a, \quad \theta_a(v) = a + 1 - |v| \text{ for } |v| \in [a, a + 1], \quad \theta_a(v) = 0 \text{ for } |v| \geq a + 1.
\]
Then $\theta_a$ is Lipchitz, hence $f \theta_a$ is Lipchitz and compactly supported. Then the theorem gives,
\[
\lim_{n \to \infty} E_p(\exp i < u, N^a_n(f \theta_a) >) = \exp c^{-1} E \Lambda[(\exp - i < u, f \theta_a(v) > - 1)\exp \pi^a(\exp - i < u, f \theta_a(v) >)]] = \Phi_a(u)
\]
Since $\Lambda(B^\delta_0) < \infty$, the function $u \to \Phi_a(u)$ is continuous on $\mathbb{R}^m$. It follows that the sequence of random variables $Y^a_n = N^a_n(f \theta_a)$ converges in law to the random variable $Y^a_\infty$ which has characteristic function $\Phi_a$. On the other hand we have $\lim_{n \to \infty} \theta_a = 1$, hence by dominated convergence we get,
\[
\lim_{a \to \infty} \Phi_a(u) = \exp c^{-1} E \Lambda[(\exp - i < u, f(v) > - 1)\exp i < u, \pi^a(f) >] = \Phi(u).
\]
We recall that, for $v$ fixed, the series $\sum_{j=0}^\infty f(S_j v)$ converges $\mathbb{Q} - a.e$ to a finite sum, hence the function $u \to \Phi(u)$ is continuous. In other words, $Y^a_\infty$ converges in law ($a \to \infty$) to the random variable $Y$ with characteristic function $\Phi$.
Also for, $z, z' \in C$ with $\text{Re}z \leq 0, \text{Re}z' \leq 0$ we have $|\exp z - \exp z'| \leq |z - z'|$. If we choose $\beta \in [0,1[, \gamma > 0$ such that $\beta \gamma \in [0,\alpha]$, then we have for any $\varepsilon > 0$,
\[
\mathbb{P}_\rho\{|Y^a_n - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} E \rho[(\sum_{j=1}^n (f(u^{-1} X_j)1_{\{|X_j| > au\}})^\beta],
\]
\[
\mathbb{P}_\rho\{|Y^a_n - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} c^m_n E \rho[|u^{-1} X|^\beta 1_{\{|X| > au\}}].
\]
Using Corollary 2.2, with $W(x) = |x|$ it follows $\limsup_{a \to \infty} \mathbb{P}_\rho\{|Y^a_n - Y_n| > \varepsilon\} \leq \varepsilon^{-\beta} c^m_n \Lambda(W^\beta 1_{\{|W| > a\}}).
\]
Since $\varepsilon > 0$ is arbitrary, the convergence in law of $Y_n$ to the random variable $Y$ follows, hence the corollary.

In order to prepare the study of limits for the sums $T_n = \sum_{j=1}^n X_j$ if $0 < \alpha < 2$, we write for $a > 0 : \psi_a(v) = v(1 - \varphi_a(v)$ where
\[
\varphi_a(v) = 1$ if $|v| \leq a, \varphi_a(v) = 2 - a^{-1}|v|$ if $a \leq |v| \leq 2\varepsilon, \varphi_a(v) = 0$ if $|v| > 2a$.
\]
Hence $0 \leq \varphi_a \leq 1_{[0,2a]}$ and $k(\varphi_a) \leq a^{-1}$. Then a consequence of Corollary 5.4 with $m = d, \gamma = 1$ is the following

**Corollary 5.5** The sequence of $V$-valued random variables $N^a_n(\psi_a)$ converges in law to the random variable with characteristic function
\[
\exp c^{-1} E \Lambda[(\exp - i < u, \psi_a(v) > - 1)\exp i < u, \pi^a(\psi_a) >] = \Phi(u).
\]

**5.2 Convergence to stable laws for $T_n = \sum_{i=1}^n X_i$**

In this subsection we write $\psi(v) = v$ and we study the convergence of $N^a_n(\psi) = u^{-1} T_n$ towards a stable law, extending the weak convergence of $N_n$ studied in the above subsection. We need here the last part of the spectral gap result in Proposition 3.4 for the operator $P$. 31
We have the

Theorem 5.6 Let $0 < \alpha < 2$. Then there exists a sequence $d_n$ in $V$ such that the sequence of random variables $n^{-1/\alpha}(T_n - d_n)$ converges in law to a non degenerate stable law.
If $0 < \alpha < 1$, we have $d_n = 0$.
If $1 < \alpha < 2$, we have $d_n = n\mathbb{E}_\rho(X)$
If $\alpha = 1$, we have $d_n = n \mathbb{E}_\rho[X\varphi_1(X)]$.

Explicit expressions for the characteristic functions of the limits are given in the proofs. Non degeneracy of the limit laws are proved in [11] and [15]. For the proofs, we follow the approach of [7] and we need two lemmas corresponding to the cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$.
In the proofs below we use the normalization $u_n = (cn)^{1/\alpha}$ instead of $n^{1/\alpha}$ as in the theorem.

Lemma 5.7 Assume $0 < \alpha < 1$. Then for any $u \in V$ and with the notation of Corollary 5.5, $T^a = N^a(\psi_a)$ converges in law ($a \to 0$) to $T$ with characteristic function given by
$$
\exp \frac{c}{1-\alpha} \mathbb{E}_\Lambda(\exp -i \langle u, v \rangle - 1) \exp i \langle u, \sum_{j=0}^{\infty} S_j v \rangle = \Phi(u).
$$

Also, for any $\delta > 0$ we have
$$
\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}_\rho \left\{ | \sum_{j=1}^{n} u_n^{-1} X_j \varphi_a(u_n^{-1} X_j) | > \delta \right\} = 0.
$$

Proof : Using dominated convergence, the first part follows from Corollary 5.5.

On the other hand, Markov inequality gives
$$
\mathbb{P}_\rho \left\{ | \sum_{j=1}^{n} u_n^{-1} X_j \varphi_a(u_n^{-1} X_j) | > \delta \right\} \leq \delta^{-1} u_n^{-1} \mathbb{E}_\rho(|X|1_{\{|X|<2au_n\}})
$$
The homogeneity at infinity of $\rho$ and Karamata’s lemma (see [28] p.26) gives that the right hand side is equivalent to
$$
\delta^{-1} n^{1-\alpha} (1-\alpha)^{-1} (2au_n)^{-1} \mathbb{P}_\rho(\{|X|>2au_n\}),
$$
i.e to $\delta^{-1} a^{1-\alpha}$ up to a coefficient independent of $n$. Hence the result since $1-\alpha > 0$. □

Lemma 5.8 Assume $1 \leq \alpha < 2$ and write $\overline{\psi}_a(v) = \psi_a(v)$. Then we have the convergence
$$
\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{E}_\rho(|u_n \overline{\psi}_a| - \mathbb{E}_\rho(N_n^s(\overline{\psi}_a))^2) = 0.
$$

Proof : It suffices to show that for any $u \in S^{d-1} :$
$$
\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{E}_\rho(|u, N_n^s(\overline{\psi}_a) > -\mathbb{E}_\rho(<u, N_n^s(\overline{\psi}_a)>)^2) = 0.
$$
We write $f_{a,n}(v) = \overline{\psi}_a(u_n^{-1} v)$, $\overline{\psi}_{a,n} = f_{a,n} - \rho(f_{a,n})$.
Hence $|f_{a,n}(v)| \leq u_n^{-1} |v|1_{\{|v|\leq 2u_n\}}$, $|f_{a,n}| \leq 3u_n^{-1}$. We have the equality
$$
\mathbb{E}_\rho(|u, N_n(\overline{\psi}_a) > -\mathbb{E}_\rho(<u, N_n(\overline{\psi}_a)>)^2) = A_{n,a} + 2B_{n,a}
$$
with
\[ A_{n,a} = nE_\rho(\langle u, \psi_{a,n}(X_0) \rangle^2), \]
\[ B_{n,a} = \sum_{j=1}^{\eta} (n-j)E_\rho(\langle u, \overline{\psi}_{a,n}(X_0) \rangle \langle u, \overline{\psi}_{a,n}(X_j) \rangle). \]

Now the proof splits into two parts a) and b) corresponding to the studies of \( A_{n,a}, B_{n,a} \).

a) We have, using the above estimation of \( f_{a,n} \),
\[ nE_\rho(\langle u, \overline{\psi}_{a,n}(X_0) \rangle^2) \leq nE_\rho(\| f_{a,n}(X_0) \|^2) \leq nE_\rho(u_n^{-2}|X_0|^21_{\{|X_0|<2a_n\}}). \]

Then Karamata’s lemma implies that, for \( n \) large, the right hand side is equivalent to \( n^{1-2\alpha^{-1}}(2a_n)^2((2a)^\alpha n)^{-1}, i.e. to a^{2-\alpha} \). Hence, since \( \alpha \in ]0,2[ \), we get \( \lim_{a \to 0} \lim_{n \to \infty} A_{n,a} = 0 \) uniformly in \( u \in S^{d-1} \).

b) Markov property for the process \( (X_i)_{i \geq 0} \) implies for \( i \geq 1 \),
\[ E_\rho(\langle u, \overline{\psi}_{a,n}(X_0) \rangle \langle u, \overline{\psi}_{a,n}(X_i) \rangle) = E_\rho(\langle u, \overline{\psi}_{a,n}(X_0) \rangle \langle u, P_i \overline{\psi}_{a,n}(X_0) \rangle). \]

First we consider the case \( \alpha \in [1,2] \) and we apply Proposition 3.4 to \( P \) acting on the Banach space \( \mathcal{H} = \mathcal{H}_{\chi,\varepsilon,\kappa} \) with \( \chi \in ]1,\alpha[, \varepsilon = 1 \) and \( \kappa \) chosen according to Proposition 3.4. We observe that for \( h \in \mathcal{H} \) we have \( |h(v)| \leq \|h\|(1+|v|) \). Since \( \overline{\psi}_{a,n} \in \mathcal{H} \), we have
\[ E_\rho(\langle u, \overline{\psi}_{a,n}(X_0) \rangle \langle u, P_i \overline{\psi}_{a,n}(X_0) \rangle) = E_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U_i f_{a,n}(X_0) \rangle), \]
where we have used the decomposition \( P^\alpha = \rho \otimes 1 + U^\alpha \). Schwarz inequality allows us to bound the right hand side by the square root of \( E_\rho(\| f_{a,n}(X_0) \|^2)E_\rho(\| U_i f_{a,n}(X_0) \|^21_{\{|X_0|<2a_n\}}) \).

Since \( |U_i f_{a,n}(v)| \leq (1+|v|)\|U^\alpha\| \| f_{a,n} \| \), the quantity \( E_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U_i f_{a,n}(X_0) \rangle) \) is bounded by \( \|U^\alpha\| \| f_{a,n} \| (\rho(1+|X_0|)^21_{\{|X_0|<2a_n\}})^{1/2} \). Then Karamata’s lemma implies that, up to a coefficient independent of \( n \), the above expression is bounded by \( \|U^\alpha\| \| f_{a,n} \| (n^{-1/2}a^{1-\alpha/2}) (1+n^{\alpha^{-1}-1/2}a^{1-\alpha/2}) \).

Since \( \| f_{a,n} \| \leq n\|u_n^{-1} \), it follows that \( B_{n,a} \), uniformly in \( u \in S^{d-1} \) and up to a coefficient, is bounded by
\[ n(\sum_{i=0}^\infty \|U^\alpha\| n^{-1}[n^{1/2-\alpha^{-1}} a^{1-\alpha/2} + a^{2-\alpha}] = \sum_{i=0}^\infty \|U^\alpha\| [a^{2-\alpha} + a^{1-\alpha/2} n^{1/2-\alpha^{-1}}]. \]

Since \( r(U) < 1 \) we have \( \sum_{i=0}^\infty \|U^\alpha\| < \infty \), hence \( \lim_{n \to \infty} B_{n,a} \) is bounded by \( a^{2-\alpha} \), up to a coefficient independent of \( n \). Since \( 1 < \alpha < 2 \), the lemma follows of the two above convergences.

If \( \alpha = 1 \), we need to use the Banach space \( \mathcal{H}' = \mathcal{H}_{\chi,\varepsilon,\kappa} \) with \( 0 < \varepsilon < \chi < 1, \kappa = 0 \), considered in Proposition 3.4. We use also the inequality \( \|f\|_{a,n} \leq c_1 a^{1-\chi n^{-\varepsilon}} \) with \( c_1 > 0 \), shown below. We note that for \( h \in \mathcal{H}' \), we have \( |h(v)| \leq \|h\|(1+|v|^\varepsilon) \) in particular and up to a constant independent of \( n \), we have
\[ |U_i f_{a,n}(v)| \leq \|U^\alpha\| \| f_{a,n} \|(1+|v|)^\varepsilon \leq \|U^\alpha\|(a^{1-\chi n^{-\varepsilon}})(1+|v|^\varepsilon). \]

Hence we can bound \( E_\rho(\langle u, f_{a,n}(X_0) \rangle \langle u, U_i f_{a,n}(X_0) \rangle) \) by
\[ 2c_1 \|U^\alpha\|(a^{1-\chi n^{-\varepsilon}}) n^{-2} E_\rho(|X_0|^2) 1_{\{|X_0|<2a_n\}}^{1/2} \|E_\rho(1+|X_0|^{2\varepsilon}) 1_{\{|X_0|<2a_n\}}\|^{1/2}, \]
which, if \( \varepsilon > 1/2 \), can be estimated using Karamata’s lemma by
\[ 2c_1 \|U^\alpha\|(a^{1-\chi n^{-\varepsilon}})(an^{-1})^{1/2}(na)^{\varepsilon-1/2} = c_2 a^{1-\chi + \varepsilon} n^{-1}. \]
It follows that $B_{n,a}$ can be estimated by $2c_2(\sum_{i=0}^{\infty} \|U^i\|) a^{1-\chi+\varepsilon}$. Since, using Proposition 3.4 we have $r_1(U) < 1$, it follows if $1/2 < \varepsilon < \chi < 1$, then 

$$\lim_{n \to \infty} \limsup_{a \to 0} B_{n,a} = 0. \quad \square$$

**Proof of Theorem 5.6** For $\alpha \in [0,1]$ the proof follows of Lemma 5.7. We observe that dominated convergence implies the continuity of $\Phi$ at zero, hence $\Phi$ is a characteristic function. From Lemma 5.7 we know that if $Y_n = N^s_n(1)$, $Y^a_n = N^s_n(1 - \varphi_a)$,

1) For any $a > 0$, $Y^a_n$ converges in law ($n \to \infty$) to $T^a$

2) $T^a$ converges in law ($a \to 0$) to $T$

3) For any $\varepsilon > 0$, we have 

$$\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}\{|Y_n - Y^a_n| > \varepsilon\} = 0.$$ 

It follows that the sequence $Y_n = \sum_{i=1}^{n} u^{-1}X_i$ converges in law ($n \to \infty$) to the random variable $T$ with characteristic function $\Phi$.

For $1 < \alpha < 2$, we write

$$Y_n = N^s_n(\psi) - \mathbb{E}_\rho(N^s_n(\psi)(u^{-1}X_j) - \mathbb{E}_\rho(X), Y^a_n = N^s_n(\psi_a) - \mathbb{E}_\rho(N^s_n(\psi_a)),$$

so that $Y^a_n - Y_n = N^s_n(\psi_a) - \mathbb{E}_\rho(N^s_n(\psi_a))$. Then for any $\varepsilon > 0$ Lemma 5.8 gives,

$$\lim_{a \to 0} \limsup_{n \to \infty} \mathbb{P}\{|Y^a_n - Y_n| > \varepsilon\} = 0.$$ 

Furthermore, the sequence $N^s_n(\psi_a)$ converges in law ($n \to \infty$) to $T^a$ and $\mathbb{E}_\rho(N^s_n(\psi_a)) = n\mathbb{E}_\rho[u^{-1}X(1-\varphi_a)(u^{-1}X)]$ converges to the value $b(a)$ of $\Lambda$ on the function $v \to v(1-\varphi_a(v))$, as follows from $\alpha > 1$ and the homogeneity at infinity of $\rho$. Hence the sequence $Y^a_n$ converges in law ($n \to \infty$) to $T^a - b(a) = Y^a$. Finally $Y^a$ converges in law ($a \to 0$) to the random variable $T$ with characteristic function $\Phi$ defined by,

$$\Phi(u) = exp c^{-1}\mathbb{E}_{\Lambda}[(\text{exp } i < u, v > -1 + i < u, v >)\text{exp } i < u, \sum_{j=0}^{\infty} S_j v >]$$

$$+ i\mathbb{E}_{\Lambda}[< u, v > (\text{exp } i < u, \sum_{j=0}^{\infty} S_j v > -1)].$$

This follows of Theorem 5.1, of dominated convergence ($a \to 0$) and of the following inequalities

$$|\text{exp } i < u, \psi_a(v) > -1 + i < u, \psi_a(v) > | \leq \text{Inf} (2 + |u| v, 4|v|^2|u|^2),$$

$$| < u, \psi_a(v) > \mathbb{E}(|\text{exp } i < u, \sum_{j=0}^{\infty} \psi_a(S_j v) > -1)| \leq \text{Inf} (|u| v, 2|u|^2|v|^2 \sum_{j=1}^{\infty} \mathbb{E}(|S_j|)),$$

where $\alpha > 1$ gives $\sum_{j=1}^{\infty} \mathbb{E}(|S_j|) < \infty$. Continuity of $\Phi$ at zero follows also from the above inequalities.

As in [4], we deduce the convergence in law of the sequence $Y_n$ to $T$.

If $\alpha = 1$, we write $Y_n = \sum_{j=1}^{n} u^{-1}X_j - \mathbb{E}_\rho(\varphi_1(u^{-1}X))$ and

$$Y^a_n = \sum_{j=1}^{n} u^{-1}X_j(1 - \varphi_a(u^{-1}X)) - b_n(a) = N^s_n(\psi_a) - b_n(a)$$

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where \( b_n(a) = \mathbb{E}_\rho[X(\varphi_1 - \varphi_a)(n^{-1}X)] \). With the new notations the above inequalities are still valid. The homogeneity at infinity of \( \rho \) gives now
\[
\lim_{n \to \infty} b_n(a) = c^{-1} \mathbb{E}_\Lambda(\varphi(\varphi_1 - \varphi_a)) = b(a).
\]
It follows that the sequence \( Y_n^a \) converges in law \((n \to \infty)\) to the random variable \( T_1^a \) with characteristic function,
\[
\exp c^{-1} \mathbb{E}_\Lambda([\exp i < u, \psi_a(v) > -1] \exp i < u, \pi_v(\psi_a) > -i < u, b(a) >)]
\]
We insert the expression \( i < u, v > (\varphi_1 - \varphi_a(v)) \) with the adequate sign in each of the above factors inside the expectation \( \mathbb{E}_\Lambda \). Then dominated convergence \((a \to 0)\) shows that \( T_1^a - b(a) \) converges in law to the random variable \( T \) with characteristic function
\[
\Phi(u) = \exp c^{-1} \mathbb{E}_\Lambda[A(u, v) + B(u, v)]
\]
with \( A(u, v) = (\exp i < u, v > -1 + i < u, v >) \varphi_1(v), \)
\[
B(u, v) = i < u, v > \varphi_1(v)(\exp i < u, \sum_{j=1}^{\infty} S_j v > -1).
\]
As in ([1], [15]), the stability of the limiting laws follow from the formula for \( \Phi(u) \). If \( 0 < \alpha < 2 \), \( \alpha \neq 1 \) the formula for \( \Phi(u) \) shows that for any \( n \in \mathbb{N} \) we have \( \Phi^n(u) = \Phi(n^{1/\alpha}u) \), hence \( T \) has a stable law of index \( \alpha \).
If \( \alpha = 1 \), we have with \( \gamma_n = c \mathbb{E}_\Lambda[v(\varphi_1(n^{-1}v) - \varphi_1(v))] \), \( \Phi^n(u) = \Phi(n u) \exp -in < u, \gamma_n > \).
This implies that \( T \) follows a stable law with index 1. \( \square \)

6 Appendix

6.1 On positivity of the extremal index

We give a direct proof of the positivity of \( \mathbb{Q}_{\Lambda_1}\{\sup_{n > 0} |S_n(\omega)\varepsilon| < 1\} \) following [15].

**Proposition 6.1** Let \((Y, \tau, m)\) be a dynamical system where \( m \) is a \( \tau \)-invariant probability and let \( f \) be a measurable function on \( Y \). If \( \lim_{n \to \infty} \sum_{k=0}^{n-1} f(\tau^k y) = -\infty \) \( m - a.e. \) then there exists \( c < 0 \) and a subset \( Y_1 \subset Y \) of positive measure such that for every \( y \in Y_1, n \geq 1, \)
\[
T_n(y) = \sum_{k=0}^{n-1} f(\tau^k y) \leq c
\]

**Proof:** Let \( M_n(y) = \sup_{0 \leq k \leq n} T_k(y) \), hence
\[
M_{n+1}(y) = \sup_{0 \leq k \leq n} [f(y), f(y) + M_n(\tau y)] = f(y) + M_n^+(\tau y).
\]
Since \( \lim_{n \to \infty} T_n(y) = -\infty \) \( m - a.e. \), the function \( M_\infty(y) = \lim_{n \to \infty} M_n(y) \) is finite \( m - a.e. \), hence
\[
f(y) = M_\infty^+(y) - M_\infty^-(y) - M_\infty^+(\tau y).
\]
If \( M_\infty = 0 \) \( m - a.e. \) then \( f(y) = M_\infty^+(y) - M_\infty^+(\tau y) \) is a coboundary. By considering the return times to a set on which \( M_\infty^+ \) is bounded from below, we get a contradiction.
with the above coboundary equation. Hence we have, for some $c < 0$, $M_\infty(y) \leq c$ on a set $Y_1$ of positive measure, i.e $M_\infty(y) \leq c < 0$ on $Y_1 \subset X$ with $m(Y_1) > 0$. Hence $T_n(y) \leq M_n(y) \leq M_\infty(y) \leq c$ on $Y_1$ for any $n \geq 1$. □

Using the notations of section 2 we consider the action of $T$ on the unit sphere $(g,x) \rightarrow \frac{gx}{|gx|} = g.x$, we assume that $T$ is strongly irreducible and we denote by $\nu$ a $\mu$-stationary probability on $\mathbb{S}^{d-1}$. We denote by $\mathcal{U}(\text{supp}(\nu))$ the set of non void open subsets of $\text{supp}(\nu)$, hence for any $U \in \mathcal{U}(\text{supp}(\nu))$ we have $\nu(U) > 0$.

**Corollary 6.2** With the above notations, we assume that the semigroup $T$ is strongly irreducible and $L_\mu = \int \log|g|^2 d\mu(g) d\nu(x) < 0$. Then, there exists $\epsilon > 0$, $\Omega_1 \subset \Omega$ with $Q(\Omega_1) > 0$ and a map $\omega \rightarrow U_\epsilon(\omega)$ from $\Omega_1$ to $\mathcal{U}(\text{supp}(\nu))$ such that
\[
\sup\{|S_n(\omega)x| < 1 ; n > 0\} < 1 - \epsilon
\]
for any $x \in U_\epsilon(\omega)$, $\omega \in \Omega_1$.
In particular, we have $Q_{\Lambda_1}\{\sup |S_n(\omega)v| < 1 ; n > 0\} > 0$

**Proof:** We denote $Y = \Omega \times \text{supp}(\nu)$ and we write $y = (\omega,x) \in Y$. We consider the dynamical system $(Y,\tau,m)$ defined by $\tau(\omega,x) = (\sigma\omega, A_1(\omega)x)$. The hypothesis implies with $f(y) = \log|A_1(\omega)x|$, $m(f) < 0$ $T_n(y) = \log|S_n(\omega)x| : \lim_{n \rightarrow \infty} T_n(y) = -\infty, m-a.e.$

Then Proposition 6.1 implies the existence of a set $Y_1 \subset \Omega \times \text{supp}(\nu)$ of positive $Q \otimes \nu$ measure and $\epsilon' > 0$ such that for $n > 0$, $|S_n(\omega)v| < 1 - \epsilon'$. Hence there exists $\Omega_1 \subset \Omega$ with $Q(\Omega_1) > 0$ such that, for $\omega \in \Omega_1$, there exists $S_\omega \subset \mathbb{S}^{d-1}$ with $\nu(S_\omega) > 0$ and $\sup\{|S_n(\omega)v| ; n > 0\} < 1 - \epsilon'$ for $v \in S_\omega$. Since $T$ is strongly irreducible, $\nu$ gives measure zero to any proper subspace (see [13]), hence $S_\omega$ contains $v_1^\omega, \ldots, v_d^\omega$ which are linearly independent, for any $\omega \in \Omega_1$. Then, for any $v = \sum_{i=1}^d \lambda_i v_i^\omega$ with $\sum_{i=1}^d |\lambda_i| < (1 - \epsilon'/2)^{-1}$ we have
\[
|S_n(\omega)v| \leq \sum_{i=1}^d |\lambda_i| |S_i(\omega)v_i^\omega| \leq (1 - \epsilon'/2)^{-1}(1 - \epsilon') < 1 - \epsilon'/2,
\]
for $n > 0$. But the set
\[
U_\omega = \text{supp}(\nu) \cap \{v, \sum_{i=1}^d |\lambda_i| < (1 - \epsilon'/2)^{-1}\}
\]
is an open non void subset of $\text{supp}(\nu)$ since $(1 - \epsilon'/2)^{-1} > 1$ and $v_i^\omega \in U_\omega$, for $i = 1, \ldots, d$.
Hence, for $\omega \in \Omega_1$ and $x \in U_\omega$, $\epsilon = \epsilon'/2$, we have $\sup\{|S_n(\omega)x| ; n > 0\} < 1 - \epsilon$. From [13], we know that the set $\text{supp}(\sigma^\alpha)$ is also the support of a $\mu$-stationary measure $\nu$ and $\Lambda = \sigma^\alpha \otimes \ell^\alpha$ with $c > 0$. Since for any $U \in \mathcal{U}(\text{supp}(\nu))$ the set $\{v = tx ; 1 < t < (1 - \epsilon)^{-1}, x \in U\}$ has positive $\Lambda_1$-measure, we get by definition of $Q_{\Lambda_1}$
\[
Q_{\Lambda_1}\{\sup |S_n(\omega)v| < 1 ; n > 0\} > 0
\]
□

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6.2 Condition \((H)\) is open if \(d > 1\)

We denote by \(T_\mu\) the closed subsemigroup of \(G\) generated by \(\text{supp}(\mu)\), where \(\mu\) is a probability on \(G\). We consider weak topologies for probability measures on \(G\) and on \(A\). We denote by \(M^1(G)\) (resp \(M^1(A)\)) the set of probabilities on \(G\) (resp \(A\)). We denote by \(W(A)\) the weak topology on \(M^1(A)\) defined by the convergence on continuous compactly supported functions as well as of the moments \(\int (\gamma^k(g) + |b|^k(h))d\lambda(h)\) for any \(k \in \mathbb{N}\).

**Theorem 6.3** If \(d > 1\), condition \((H)\) is open in the weak topology \(W(A)\) on \(M^1(A)\).

We will need the Proposition

**Proposition 6.4** Condition \(i-p\) is open for the weak topology on \(M^1(G)\).

**Proof:** Assume \(\mu \in M^1(G)\), satisfies \(i-p\) and let \(\mu_n \in M^1(G)\) be a sequence which converges weakly to \(\mu\). Then \(\text{supp}(\mu_n)\) and \(T_{\mu_n}\) are closed subsets of \(G\) which converges to \(\text{supp}(\mu)\) and \(T_\mu\) respectively. If \(\gamma\) is a proximal element of \(T_\mu\), then by perturbation theory there exists a neighbourhood of \(\gamma\) in \(G\) which consists of proximal elements. Hence there exists \(\gamma_n \in T_{\mu_n}\) which is also proximal.

On the other hand \(T_{\mu_n}\) is irreducible for large \(n\). Otherwise there exists a proper subspace \(W^n \subset V\) with \(T_{\mu_n}(W^n) = W^n\). Let \(W \subset V\) be the limit of a subsequence of \(W^n\). Then, clearly \(T_\mu(W) = W\), which contradicts the irreducibility of \(T_\mu\).

In order to show the strong irreducibility of \(T_{\mu_n}\) for large \(n\), we show the irreducibility of \(Zc_0(T_{\mu_n})\), the connected component of the Zariski closure \(Zc(T_{\mu_n})\) of \(T_{\mu_n}\) (see [23]). Since \(T_{\mu_n}\) is irreducible, the Lie group \(Zc_0(T_{\mu_n})\) is reductive and has finite index in \(Zc(T_{\mu_n})\).

We decompose \(V\) as the direct sum of its isotypic components \(V_i'^{(n)}(1 \leq i \leq p_n)\) under the action of \(Zc_0(T_{\mu_n})\) : \(V = \bigoplus_{i=1}^{p_n} V_i'^{(n)}\). Since \(Zc_0(T_{\mu_n})\) has finite index in \(Zc(T_{\mu_n})\) we can assume, by taking a suitable power, that \(\gamma_n \in Zc_0(T_{\mu_n})\). The uniqueness of the above decomposition of \(V\) and the relation \(\gamma_nv = \lambda_nv, v = \sum_{i=1}^{p_n} v_i, v_i \in V_i'^{(n)}, \) with \(\lambda_n\) a simple dominant eigenvalue of \(\gamma_n\) implies \(\gamma_nv_i = \lambda_nv_i\); hence the proximality of \(\gamma_n\) implies that \(v\) belongs to a unique \(V_i'^{(n)}\), to \(V_i'^{(n)}\) say. Also the irreducibility of \(T_{\mu_n}\) implies that \(T_{\mu_n}\) permutes the subspaces \(V_i'^{(n)}(1 \leq i \leq p_n)\). Since \(V_1'^{(n)}\) is isotypic and \(\gamma_n\) is proximal, the subspace \(V_i'^{(n)}\) is \(T_{\mu_n}\)-irreducible. The same is valid for any \(V_i'^{(n)} = g(V_1'^{(n)})\) since \(g\gamma_ng^{-1}\) is also proximal, for \(g \in T_{\mu_n}\). Assume \(Zc_0(T_{\mu_n})\) is not irreducible for \(n\) large; then it follows that \(p_n \in [1, d]\) and \(r_n = \text{dim} V_i'^{(n)} \in [1, d]\). It follows that we can assume \(p_n = p\) and \(r_n = r\) for \(n\) large. Hence, taking convergent subsequences of \(V_i'^{(n)}(1 \leq i \leq p)\) we obtain proper subspaces \(V_i(1 \leq i \leq p)\) which are permuted by \(T_\mu\); the irreducibility of \(T_\mu\) implies that their sum is \(V\), hence we have \(V = \bigoplus_1^p V_i\), which contradicts the strong irreducibility of \(T_\mu\). Hence \(T_{\mu_n}\) satisfies condition \(i - p\) for \(n\) large. \(\square\)
Proof of Theorem 6.3
Let $\lambda_n \in M^1(A)$ be a sequence which converges to $\lambda \in M^1(A)$ in the weak topology $W(A)$ and let us denote by $\mu_n$ the projection of $\lambda_n$ on $G$. We verify the stability of conditions 1, 2 in $(H)$, since condition 3 follows of the definition of $W(A)$ and condition 4 is a direct consequence of Proposition 6.4.

1) Assume that $\text{supp}(\lambda_n)$ has a fixed point $x_n \in V$ for $n$ large. Since the closed subset $\text{supp}(\lambda_n)$ converges to $\text{supp}(\lambda)$, we can find a convergent subsequence of $x_n$ to a point $x$ in $(V)U(S^{d-1})$, endowed with the visual topology, such that $x$ is $\text{supp}(\lambda)$-invariant. If $x \in V$ we have a contradiction since $\text{supp}(\lambda)$ has no fixed point in $V$. If $x \in S^{d-1}$, we have also a contradiction since the projective action of $\text{supp}(\mu)$ has no fixed point.

2) Using Lemma 6.4, since finiteness of moments for $\mu_n$ is valid, we get that for $\mu_n$ and for any $s \geq 0$, the corresponding operator $P^s$ has a spectral gap on the relevant Hölder space on $S^{d-1}$ (see [13]). The moment condition implies that perturbation theory is valid for the operators $P^s$. Hence the spectral radius $k(s)$ varies continuously. In particular, since we have $k(s) > 1$ for $\mu$ and $s > \alpha$, and $L(\mu) < 0$ the same is valid for $\mu_n$ with $n$ large. Hence there exists $\alpha_n > 0$ close to $\alpha$ such that $k(\alpha_n) = 1$. □

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