Measurement incompatibility does not give rise to Bell violation in general

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Abstract
In the case of a pair of two-outcome measurements, incompatibility is equivalent to Bell nonlocality. Indeed, any pair of incompatible two-outcome measurements can violate the Clauser–Horne–Shimony–Holt Bell inequality, which has been proven by Wolf et al (2009 Phys. Rev. Lett. 103 230402). In the case of more than two measurements the equivalence between incompatibility and Bell nonlocality is still an open problem, though partial results have recently been obtained. Here we show that the equivalence breaks for a special choice of three measurements. In particular, we present a set of three incompatible two-outcome measurements, such that if Alice measures this set, independent of the set of measurements chosen by Bob and the state shared by them, the resulting statistics cannot violate any Bell inequality. On the other hand, complementing the above result, we exhibit a set of \( N \) measurements for any \( N \geq 2 \) that is \( N-1 \)-wise compatible, nevertheless it gives rise to Bell violation.

1. Introduction
Correlations resulting from incompatible local measurements on an entangled quantum state can violate Bell inequalities [1, 2]. However, Bell violation is not possible if either the measurements are compatible or the shared state is unentangled. In this respect, one may ask whether (i) all entangled states lead to Bell violation. This turns out not to be true for projective measurements [3] and for the general case of positive-operator-valued-measure (POVM) measurements as well [4] (see also [5, 6] for more recent results). Similarly, one may ask whether (ii) all incompatible measurements lead to Bell violation. Specifically, the question is whether for any given set of incompatible measurements performed by Alice, one can always find a shared entangled state and a set of measurements for Bob, such that the resulting statistics will lead to Bell inequality violation.

This holds true in the case of any number of incompatible projective measurements [7], and for a pair of dichotomic measurements as well [8]. However, in the case of more than two non-projective dichotomic measurements (or in the case of two non-dichotomic measurements) the problem is still open. Though, there is recent progress toward this aim. For example, a strong link between incompatibility of measurements and Einstein–Podolsky–Rosen steering [9, 10], a phenomenon in between entanglement and Bell nonlocality, has been established [11–13].

In this paper, we present a set of three incompatible dichotomic measurements, such that if Alice uses this triple, independent of the set of measurements chosen by Bob and the state shared by them, the resulting statistics cannot violate any Bell inequality. This result remains valid for Bell inequalities with arbitrary number of settings and outcomes on Bob’s side, including the general case that Bob is allowed to carry out arbitrary POVM measurements. Note that the case where Bob’s settings are restricted to projective measurements have been settled recently [13].

In addition, and complementary to the above results, we present a set of \( N \) measurements, such that any \( N-1 \) measurements out of this set are compatible. However, we show that using this set of \( N \) measurements on
one side, and another set of \( N \) measurements on the other side along with a suitable shared state between them leads to violation of a Bell inequality. This result holds true for any number of \( N > 2 \) settings.

The paper is structured as follows. In section 2, we start by defining the setup and we fix notation. Section 3 is devoted to the detailed proof of our main result. To do so, we simplify the problem in section 3.1 by showing that given Alice’s specific set of three measurements, it is sufficient to deal with pure two-qubit states in the Schmidt form \( |\psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle \) along with Bob’s real-valued ternary-outcome POVMs. Then, depending on the value of the parameter \( \theta \), we will split the proof into two parts. The case of small \( \theta \) values are considered in section 3.2, whereas the case of large \( \theta > \theta^* \) values are treated in section 3.3. Then in section 4, complementing the above results, we exhibit \( N \) measurements for any \( N > 2 \) that are \((N-1)\)-wise compatible, however they give rise to Bell violation. The paper ends with conclusion in section 5.

2. Setup

A general quantum measurement is represented by a set of positive definite operators \( \{M_a\} \), \( M_a \geq 0 \) that sum to the identity, \( \sum_a M_a = I \). We consider the following set of three dichotomic qubit POVMs, so-called trine measurements (labeled by \( x = 0, 1, 2 \)):

\[
M_{a|x}^0 = \frac{1}{2}(I + (-1)^x \eta \tilde{a}_x \cdot \hat{\sigma}),
\]

where \( a \) labels the two possible outcomes \( \{0, 1\} \), and the vector \( \hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) stand for the three Pauli matrices \( X, Y, \) and \( Z \), respectively. Above \( \eta \) is a parameter between zero and one. In case of \( \eta = 1 \), the measurement is projective, and in case of \( \eta = 0 \), the measurement is the identity. The three Bloch vectors of Alice’s measurements are chosen as

\[
\tilde{a}_x = \cos(2\pi/3)\tilde{e}_1 + \sin(2\pi/3)\tilde{e}_3
\]

for \( x = 0, 1, 2 \). That is, the three measurement directions \( \tilde{a}_x \), \( (x = 0, 1, 2) \) point toward the vertices of a regular triangle on the real plane (see figure 1).

Let us now define what we mean by incompatibility of a given set of \( n \) measurements. We say that Alice’s set of measurements \( \{M_{a|x}\} \), \( x = (1, \ldots, n) \) is \( n \)-wise jointly measurable [14, 15], if there exists a \( 2^n \)-outcome parent measurement with POVM elements \( M_a \), such that each outcome corresponds to a bit string \( a = (a_1, a_2, \ldots, a_n) \) such that

\[
M_{a|x} = \sum_{a \setminus a_x} M_a,
\]

where the notation \( a \setminus a_x \) stands for an \((n-1)\) bit string formed of all the bits of \( a \) except for \( a_x \).

If the set \( \{M_{a|x}\} \) is not \( n \)-wise jointly measurable, the set is said to be incompatible. Specifically, the measurements given by equations (1) and (2) are known to be pairwise jointly measurable below \( \eta_2 = \sqrt{3} - 1 \approx 0.7321 \) and triplywise jointly measurable below \( \eta_3 = 2/3 \) [16–18]. Hence, there is a range

\[
\eta_2 \lesssim 0.7321, \quad \eta_3 \lesssim \frac{2}{3}
\]
\[
2/3 < \eta < \sqrt{3} - 1, \text{ where the set forms a so-called hollow triangle [12]. In this range, the set of three POVMs is pairwise jointly measurable, but not triplywise jointly measurable, hence the three measurements are incompatible.}
\]

Let us now fix \(\eta^* = 0.67\). According to the above, the set \(\{M_{\alpha|x}\}\) defines a hollow triangle. In this notes, we show that there is no Bell inequality which can be violated if Alice measures this set. Namely, we show that the probability distribution
\[
p(ab|xy) = \text{Tr}(\rho M_{\alpha|x}^\eta \otimes M_{\beta|y}), \quad x = 0, 1, 2, \ a = 0, 1
\]
is local for any state \(\rho\) shared by Alice and Bob and arbitrary measurements \(\{M_{\beta|y}\}\) (including an arbitrary number of settings \(y\) and outcomes \(b\) for Bob). Note that a probability distribution \(p(ab|xy)\) is local if and only if it admits a decomposition of the form
\[
p(ab|xy) = \sum_\lambda p(\lambda)p_A(\lambda|a, x)p_B(\lambda|b, y, \lambda),
\]
where \(\lambda\) is a shared variable and \(p(\lambda)\) defines weights summing up to 1, whereas \(p_A\) and \(p_B\) define Alice and Bob’s respective local response functions. The construction of such a local hidden variable (LHV) model will prove our assertion that measurement incompatibility does not imply Bell nonlocality in general. Below we present the detailed proof, which starts with a slight simplification of the problem.

### 3. Proof

#### 3.1. Simplification

First, instead of a general mixed state \(\rho\) in equation (4) we can consider pure states without loss of generality [13]. This is due to the convexity of the set of local correlations and the fact that \(\rho\) depends linearly on the probabilities \(p(ab|xy)\) in equation (5). Next, since Alice’s measurements (1) act on a qubit, the shared state takes the general form of two-qubit pure states
\[
|\psi\rangle = U_A \otimes U_B(\cos \theta |00\rangle + \sin \theta |11\rangle),
\]
where \(\theta \in [0, \pi/4]\) and \(U_A, U_B\) are arbitrary (unitary) qubit rotations. On the other hand, Bob’s set of measurements \(\{M_{\beta|y}\}\) are qubit POVMs (with possibly infinite number of inputs \(y\) and outputs \(b\) for Bob). Furthermore, instead of generic qubit \(U_A\) and \(U_B\) unitaries we can choose \(U_A = O(\varphi)\) and \(U_B = I\) in the state (6), where \(O(\varphi)\) is given by a planar rotation
\[
O(\varphi) = \cos \varphi |0\rangle \langle 0| + |1\rangle \langle 1| + \sin \varphi (-|0\rangle \langle 1| + |1\rangle \langle 0|)
\]
and we can further assume that Bob’s measurements \(\{M_{\beta|y}\}\) are real valued. The corresponding proofs are deferred to appendix A. In addition, since any extremal real-valued qubit POVM has at most three outcomes [19], this entails that it suffices to consider Bob’s real-valued measurements with at most three outcomes (that is, \(b \in \{0, 1, 2\}\)).

Due to the above simplifications, the proof boils down to show that the probability distribution
\[
p(ab|xy) = \text{Tr}(\rho(\theta, \varphi)M_{\alpha|x}^\eta \otimes M_{\beta|y}), \quad x = 0, 1, 2, \ a = 0, 1,
\]
where \(\eta^* = 0.67\), admits a LHV model in the form (5), where the two-parameter family of states
\[
\rho(\theta, \varphi) = |\psi(\theta, \varphi)\rangle \langle \psi(\theta, \varphi)|
\]
is as follows
\[
|\psi(\theta, \varphi)\rangle = O_A(\varphi) \otimes I(\cos \theta |00\rangle + \sin \theta |11\rangle),
\]
and the set \(\{M_{\beta|y}\}\) consists of an arbitrary number of real valued qubit measurements \(y\) with ternary outcomes \(b = 0, 1, 2\).

As we stated in the introduction, the proof will be split into two parts, the case of small values \(\theta < \theta^*\), and the case of large values \(\theta^* < \theta \leq \pi/4\), where the threshold \(\theta^*\) appears to be
\[
\theta^* = \frac{1}{2} \arcsin \sqrt{\frac{100}{67}} \approx 0.2279.
\]

Let us first start with the case of small \(\theta\) values.

#### 3.2. Small \(\theta\) values

In this regime the proof is fully analytical. Let us consider the two Pauli measurements \(\sigma_1 = X\) and \(\sigma_3 = Z\) with respective projectors
where \( a \in \{0, 1\} \). We next consider the noisy trine measurements defined by the formulas (1) and (2), where the three shrunk vectors \( \eta \hat{\mu}_x \), \( x = 0, 1, 2 \) point toward the vertices of an equilateral triangle (see figure 1). It is a simple exercise to show that the shrunk vectors are inside the square spanned by the unit vectors \( \pm \hat{e}_1 \) and \( \pm \hat{e}_2 \) if 
\[
\eta \leq \eta_2 = \sqrt{3} - 1 \approx 0.7321.
\]
Therefore the noisy trine measurements (1) and (2) for \( \eta \leq \eta_2 \) can be expressed as convex combinations of the two Pauli measurements \( X \) and \( Z \). In other words, given an input choice (one of the noisy trine measurements), one can translate it into choosing one of the two Pauli measurements \( X \) and \( Z \) along with some randomness [20].

Similarly, if we have noisy Pauli measurements
\[
P_{a|0}^p = (1 + (-1)^a \nu X)/2, \\
P_{a|1}^p = (1 + (-1)^a \nu Z)/2,
\]
where \( a \in \{0, 1\} \), the trine measurements (1) and (2) can be simulated up to a visibility of \( \eta = \nu \eta_2 \) with measurements (14).

Suppose now that the distribution
\[
p(ab|xy) = \text{Tr} \left( \rho(\theta, \varphi) P_{a|x}^{rv} \otimes M_{b|y}^{rv} \right)
\]
where \( \rho(\theta, \varphi) \) is the state of the system, \( P_{a|x}^{rv} \) is the projector on the outcome corresponding to the input \( x \) and the measurement \( a \) and \( M_{b|y}^{rv} \) is an arbitrary set of qubit measurements on Bob’s side. Then the simulability of the trine measurements with the noisy Paulis (14) above entails that the distribution
\[
p(ab|xy) = \text{Tr} \left( \rho(\theta, \varphi) M_{a|x}^{rv} \otimes M_{b|y}^{rv} \right),
\]
where \( M_{a|x}^{rv} \) are the trine measurements (1) and (2) with a visibility of \( \nu \eta_2 = \nu (\sqrt{3} - 1) \). Indeed, if the distribution \( p(ab|xy) \) in equation (16) was nonlocal, i.e. there existed a Bell inequality violated by \( p(ab|xy) \) the use of measurements (14) in equation (16) would give at least the same Bell violation due to the above simulability results of measurements and the linearity of the trace rule. This is a contradiction, hence the distribution (16) has to admit a LHV model.

Let us now invoke [21], where it has been proven that the Clauser–Horne–Shimony–Holt (CHSH) inequality [22] is the only inequivalent Bell inequality in the bipartite scenario, where Alice has two dichotomic settings and Bob has any number of settings \( y \) with arbitrary number of outcomes \( b \). Therefore, a probability distribution \( p(ab|xy) \) where \( a, x = 0, 1, b, y \) are possibly infinite, admits a LHV model if and only if \( p(ab|xy) \) does not violate (any of the versions of) the CHSH inequality. Put together with the above simulability result, if the probability distribution (15) does not give rise to Bell–CHSH-violation, it implies that the probability distribution (16) admits a LHV model.

Then it is enough to check the range of parameters \( (\theta, \varphi, \nu) \) for which the distribution (15) does not give rise to CHSH violation. Due to the Horodecki criterion [23], a pure two-qubit state (10) has a maximal CHSH violation of \( 2\sqrt{1 + \sin^2 2\theta} \), which value can be attained with the Pauli measurements (12) (in some rotated bases on Alice’s side). Note that this violation is independent of the angle \( \varphi \). Also, for the noisy Paulis (14) with visibility \( \nu \), the maximum CHSH value becomes \( 2\nu \sqrt{1 + \sin^2 2\theta} \). Since the local bound of the CHSH inequality is 2, we get the criterion
\[
\nu \leq \nu^* = \frac{1}{\sqrt{1 + \sin^2 2\theta}}
\]
to have a local model for the distribution (15) using a two-qubit pure state (10) independently of the set of measurements chosen by Bob.

Putting all the above results together, the trine measurements (1) and (2) with a visibility of \( \eta = \nu \eta_2 \), where the state is defined by (10) and Bob has arbitrary measurements, gives a local distribution \( p(ab|xy) \). Above, \( \nu^* \) is given by (17) and \( \eta_2 \) is given by (13). Suppose, we want a LHV model for \( \eta = \eta^* = 0.67 \), then the critical \( \theta^* \) below which the distribution \( p(ab|xy) \) is local is given by the solution of the equation \( \eta^* = 67/100 = \nu^* \eta_2 \). This value is \( \theta^* \approx 0.2279 \) (rad), and the exact value is given by formula (11).

### 3.3. Large \( \theta \) values

For the region \( \theta^* \leq \theta \leq \pi/4 \) we use a different approach. Recall that our task is to show that the probability distribution (8) with \( \nu^* = 0.67 \) admits a LHV model (5). The pure state \( \rho(\theta, \varphi) \) is defined by equation (10), where we now focus on the range \( \theta^* < \theta \leq \pi/4 \) and \( 0 \leq \varphi \leq 2\pi \), where \( \theta^* \) is given by equation (11). On the other hand, Bob’s set of measurements \( \{M_{b|y}\} \) consists of an arbitrary number of real valued qubit
measurements $y$ with ternary outcomes each (that is we have $b \in \{0, 1, 2\}$ for each setting $y$). Our procedure is based on discretizing the set $\theta \in [\theta^a, \pi/4]$. Note that a similar procedure has been carried out in [11, 24].

In particular, we give a linear program in section 3.3.1 which lowerbounds the value of $\eta$ considering any fixed state $\rho(\theta, \varphi)$ in equations (9) and (10), for which a LHV model exists. Defining a fine enough grid for $\theta$ and $\varphi$, and taking the minimum $\eta$ over the grid points allow us to lowerbound $\eta$ globally for this particular grid. Then, in section 3.3.2 the continuous case will be considered. In particular, starting from a finite set $\{(\theta_i, \varphi_i), i = 1, \ldots, n\}$, which gives us a LHV model for $\eta(\theta_i, \varphi_i)$, we provide a LHV model for $\eta = 0.67$ for a continuous values of $(\theta_i, \varphi_i)$. The treatment of this continuous case is based on the method presented in [13].

3.3.1. Finite grid

In order to lowerbound $\eta$ for any given pair of angles $(\theta, \varphi)$, we first discretize Bob’s POVM measurements using the method presented in [25] (see appendix A of this reference for the case of general POVM measurements). Instead of considering an infinite continuous set, we take a finite number of POVM elements $\{M_{b|y}\}$. Given this finite set of POVM elements, one can simulate a continuous set of (noisy) measurements for some $\eta_b$.

\[
M_b^{\eta_b} = \eta_b M_b + (1 - \eta_b) \text{Tr}(M_b \zeta_b) \mathbb{1},
\]

where $\{M_b, (b = 0, 1, 2)\}$ is an arbitrary three-outcome POVM on the real plane, and $\zeta_b$ is some fixed qubit state. The above simulation means that $M_b^{\eta_b}$ can always be written as a convex combination of the finite number of POVM elements $\{M_{b|y}\}$. In particular, we pick a finite set consisting of 9 binary-outcome and 4 ternary-outcome measurements. The binary-outcome measurements

\[
M_{b|y} = \frac{1 + (-1)^y \bar{u}_y \cdot \hat{\sigma}}{2}, \quad b = 0, 1
\]

are defined by the Bloch vectors

\[
\bar{u}_y = \cos(y\pi/9)\hat{e}_1 + \sin(y\pi/9)\hat{e}_3,
\]

where $y = 0, 1, \ldots, 8$. On the other hand, the ternary-outcome measurements $M_{b|y}$, $y = (9, 10, 11, 12)$ are defined by the three POVM elements as follows

\[
M_{b|y} = (1 + \bar{v}_y \cdot \hat{\sigma})/3,
\]

\[
M_{1|y} = (1 + \bar{v}_y \cdot \hat{\sigma})/3,
\]

\[
M_{2|y} = (1 + \bar{v}_y \cdot \hat{\sigma})/3,
\]

where the respective Bloch vectors are

\[
\bar{v}_y = \cos(y\pi/2)\hat{e}_1 + \sin(y\pi/2)\hat{e}_3,
\]

\[
\bar{v}_y = \cos(y\pi/2 + 2\pi/3)\hat{e}_1 + \sin(y\pi/2 + 2\pi/3)\hat{e}_3,
\]

\[
\bar{v}_y = \cos(y\pi/2 + 4\pi/3)\hat{e}_1 + \sin(y\pi/2 + 4\pi/3)\hat{e}_3
\]

for $y = 9, 10, 11, 12$. In addition, we also include the three degenerate measurements and the six different outcome relabellings of each POVM $M_{b|y}$, $b = 0, 1, 2$, for all $y = 0, 1, \ldots, 12$ in the finite set, where the binary-outcome measurements are embedded into the space of three-outcome POVM elements. This amounts to $3 + 6 \times 13 = 81$ POVMs, which define a polytope with 81 vertices, whose facets can be determined using a polytope software. Let us define $\zeta_b$ through $\alpha$ as follows

\[
\zeta_b = \alpha|0\rangle \langle 0| + (1 - \alpha)\mathbb{1}/2.
\]

We choose two distinct $\alpha$ values, $\alpha = 0$ and $\alpha = 5/6$. Following the method in the appendix of [25] and running the program cdd [26], we get the threshold values $\eta_b = 0.9268$ for $\alpha = 0$ and $\eta_b = 0.8900$ for $\alpha = 5/6$. Therefore, we can express Bob’s (noisy) measurements $M_b^{\eta_b}$ in equation (18) by the above $\eta_b$ values as a convex combination of the 81 POVMs above.

We are now ready to use the trick of [25, 27] to simulate a distribution $p(ab|xy)$ coming from a continuous set of Bob’s measurements $M_b$ using a finite set $\{M_{b|y}\}$. The optimization problem below is a modified version of protocol 2 in [25]:

\[
\max \quad \eta
\]

subject to

\[
\text{Tr}(M_a^{\eta_a} \otimes M_{y|y} \chi) = \sum_\lambda P_\lambda D_\lambda
\]

\[
\sum_\lambda P_\lambda = 1, \quad P_\lambda \geq 0, \quad \forall \lambda, \quad \forall a, b, x, y
\]

\[
\rho(\theta, \varphi) = \eta_b \chi + (1 - \eta_b) \chi_b \otimes \zeta_b
\]

The input to this program are $\eta_b$ and $M_{b|y}$ from equations (19) and (21), $\rho(\theta, \varphi)$ in equation (9), and the deterministic strategies $D_\lambda$. On the other hand, the optimization variables are $P_\lambda$ and $\chi$. This is not a linear
program yet, however notice that $\chi_A = \rho_A \equiv \text{Tr}_B \rho(\theta, \varphi)$ and $\chi$ can be expressed from the last line of the problem (24) as

$$\chi = \frac{1}{\eta} \rho(\theta, \varphi) + \frac{\eta - 1}{\eta} \rho_A \otimes \zeta_B.$$  

(25)

This allows us to obtain the following linear program:

$$\begin{align*}
\max_{\eta} & \quad \eta \\
\text{subject to} & \quad \text{Tr}(M^\eta_{a_j} \otimes M_{b_j} \chi) = \sum_{\lambda} p_{\lambda} D_{\lambda} \\
& \quad \sum_{\lambda} p_{\lambda} = 1, \quad p_{\lambda} \geq 0 \quad \forall \lambda, \quad \forall a, b, x, y,
\end{align*}$$  

(26)

where the input $\chi$ and $M_{b_j}$ come from equation (25) and equations (19) and (21), respectively, and the optimization variables are $p_{\lambda}$. Note that we can further write $p_{\lambda} = O(\varphi)\text{Tr}_B \rho(\theta, 0) O(\varphi)$. Calling the solver Mosek [28] either with $\alpha = 0$ or $\alpha = 5/6$, it takes about 7 s to solve the linear program (24) and return $\eta$ in our standard desktop PC for a fixed value of $\rho(\theta, \varphi)$. Let us denote $\eta = \max \{\eta(\alpha = 0), \eta(\alpha = 5/6)\}$ for a given pair $(\theta, \varphi)$. The above program allows us to evaluate $\eta$ for any fixed $(\theta, \varphi)$. Our goal is to prove that $\eta$ is above the threshold $\eta = 0.67$ in the whole interval $\theta^* \leq \theta \leq \pi/4$ and $0 \leq \varphi \leq 2\pi$. We cover this continuous case in the next subsection. To this end, we resort to the technique proposed in [13].

3.3.2. Continuous case

We first minimized $\eta$ in the two variables $\theta^* \leq \theta \leq \pi/4$ and $0 \leq \varphi \leq 2\pi$ using the heuristic search Amoeba [29], and obtained the minimum $\eta = 0.6808$ by the variables $\theta = \pi/4$, $\varphi \approx 0.1192$ and $\alpha = 0$. This gives a strong numerical evidence that $\eta \geq 0.67$ for the continuous case as well.

We next prove this result in a semi-analytical way. To this end, we closely follow the method introduced in [13]. Suppose we have a state $\rho(\theta, \varphi)$ in equation (9) for $\theta = \theta_i$, $\varphi = \varphi_j$, and $\eta$ in Alice’s measurements (1), such that the distribution (8) admits a LHV model. Then we also have a LHV model for a state (with the same measurements of Alice) which is a convex mixture of our state $\rho$ and a separable state

$$p \rho(\theta_i, \varphi_j) + (1 - p) \sigma,$$

(27)

where $0 \leq p \leq 1$ and $\sigma$ denotes a separable state. Let $\rho_0 = \text{Tr}_A \rho(\theta, \varphi)$. Therefore, if we can write

$$\nu \rho(\theta, \varphi) + (1 - \nu) \frac{1}{2} \otimes \rho_B = p \rho(\theta, \varphi) + (1 - p) \sigma$$

(28)

for some weight $p$ and separable state $\sigma$, then the distribution (8) admits a LHV model for the state $\rho(\theta, \varphi)$ and for Alice’s trine measurements $M^\eta_{a_j}$ in equation (1). Let us note that in order to get the above equation, we also passed an amount of $(1 - \nu)$ noise from Alice’s measurements to the state. We expect to find such a decomposition in (28) which in the limits $\theta \to \theta_i$, $\varphi \to \varphi_j$ gives us the value of $\nu$ close to 1. Recall that we obtained $\eta > 0.6808$ over all $(\theta, \varphi)$ using a heuristic search. Hence, if we can make a fine enough grid of the $(\theta_i, \varphi_j)$ values with $\eta \geq 0.6808$ for all grid points, we expect to have $\eta(\theta, \varphi) = \nu \eta(\theta_i, \varphi_j) > 0.67$ for the continuous case $(\theta, \varphi)$. Note also that due to symmetries it is enough to consider the regime $\varphi \in [0, \pi/6]$ and $\theta \in [0.2279, \pi/4]$.

We have to discuss two separate cases according to the movement from the coordinate $(\theta_i, \varphi_j)$ to the two orthogonal directions. In the case of both directions, we start from a pair $(\theta_i, \varphi_j)$ and a fixed $\alpha_i$, either 0 or 5/6, and call the linear program (26) to compute $\eta$. Then we find analytical formulas which allow us to obtain $\eta(\theta_i, \varphi_j)$ in the case of $\theta = \theta_i - \delta \theta$, and $\eta(\theta_i, \varphi)$ in the case of $\varphi = \varphi_j + \delta \varphi$. The respective formulas are as follows:

$$\eta(\theta_i, \varphi_j) = \frac{\eta(\theta_i, \varphi_j)}{\cot \theta \tan \theta_i (1 + \eta(\theta_i, \varphi_j)) - \eta(\theta_i, \varphi_j)}$$

(29)

and

$$\eta(\theta_i, \varphi) = \frac{1 - 2 \sqrt{2} \sqrt{1 - \cos(2\delta \varphi)}}{8 \cos(2\delta \varphi) - 7} \eta(\theta_i, \varphi_j),$$

(30)

where the proofs are given in appendices B and C. These formulas give us a method to tackle the continuous case $(\theta, \varphi)$ given the values of $\eta$ for a finite grid $\{(\theta_i, \varphi_j)\}$.

Given these formulas, we first find a lower bound on $\eta(\theta_i)$ $\equiv \min_{\varphi} \eta(\theta_i, \varphi)$ for a fixed $\theta_i$ value, where optimization is carried out over all $\varphi$. We use equation (30) and set $\delta \varphi = \varphi - \varphi_j = 0.1$ degree to obtain a lower bound of $\eta(\theta_i, \varphi) = \nu \eta(\theta_i, \varphi_j)$, where $\nu = 0.993067$. Therefore, in order to get a lower bound for a given angle
\[ \theta_i \] and all \( \varphi \) we have to compute

\[ \eta(\theta_i) = 0.993\,067 \times \min_j \{ \eta(\theta_i, \varphi_j) \}, \]

where the angles \( \varphi_j \) scan the discrete set \( \varphi_j = \{0, 0.1, 0.2, \ldots, 29.8, 29.9, 30\} \) degrees consisting of 301 different angles. This method provides us with the bound \( \eta(\theta_i) \) valid for a fixed \( \theta_i \) and any values of \( \varphi \). Note that it takes 7 s for our computer to solve the linear program for \( \eta \) in a single instance of \( (\theta_i, \varphi_j) \), hence the overall time to compute \( \eta(\theta_i) \) is \( 301 \times 7 \) s, that is, roughly half an hour.

Having the above bound \( \eta(\theta_i) \) for a fixed \( \theta_i \), we can compute the lower bound \( \eta(\theta) \) for any \( 0 < \theta < \theta_i \) by using the formula:

\[ \eta(\theta) = \frac{\eta(\theta_i)}{\cot \theta \tan(1 + \eta(\theta_i)) - \eta(\theta_i)}. \]

In this way, we get \( \eta \) valid for a continuous set of \( \theta \) and \( \varphi \) values. The proof of the above formula is based on the fact that formula (32) for any fixed \( 0 < \theta < \theta_i \) is a monotonic (increasing) function of \( \eta \). Then, for any fixed \( 0 < \theta < \theta_i \), we have

\[ \min_{\varphi} \eta(\theta, \varphi) \geq \min_{\varphi} \frac{\eta(\theta_i, \varphi)}{\cot \theta \tan(1 + \eta(\theta_i, \varphi)) - \eta(\theta_i, \varphi)}, \]

which is further lower bounded by equation (32) due to the above mentioned monotonic property.

The actual numerical treatment for \( \alpha = 0 \) in equation (31) proceeds as follows:

1. Set \( i = 0 \) and \( \theta_0 = \pi/4 \).
2. Compute \( \eta(\theta_i) \) in equation (31).
3. Compute \( \theta < \theta_i \) for which \( \eta(\theta) = 0.67 \) using formula (32) and identify \( \theta_{i+1} = \theta \).
4. Set \( i = i + 1 \) and go back to step 2 while \( (\theta_i - \theta_{i+1}) > \epsilon \), where \( \epsilon \) is a small number, say, \( 10^{-4} \).

We do the same computation by choosing \( \alpha = 5/6 \) and \( \theta_0 = 0.6 \) in the first step of the algorithm above. The results are visualized in figure 2 (the diamonds stand for \( \alpha = 0 \) and the empty circles are for \( \alpha = 5/6 \)). Let us stress that in the region in between two consecutive markers the above analytical lower bounds guarantee that \( \eta \) cannot drop below 0.67. On the other hand, the solid curve corresponds to the analytical lower bound. As we see, the three curves cover all the range \( 0 \leq \theta \leq \pi/4 \), which completes the proof.

4. Bell violation with \((N - 1)\)-wise compatible measurements

In this section, we further explore the link between joint measurability and Bell violations. It has been proven in [12] that there exists a specific pairwise jointly measurable set of \( N = 3 \) dichotomic POVMs which give rise to the violation of the \( I_{3322} \) three-setting two-outcome Bell inequality [30]. Below we generalize this result to any
$N > 3$. In particular, we present $N$ observables, which are $(N - 1)$-wise jointly measurable, and give rise to violation of an $N$-setting Bell inequality.

To this end, we use the construction from [31]. Namely, it has been proven there that there exist a pure quantum state $\rho$ acting on $\mathbb{C}^N \otimes \mathbb{C}^N$ and specific two-outcome projective measurements $M_{a|x}$ and $M_{b|y}$, $a, b = 0, 1, x, y = 1, \ldots, N$ (defined by equations (3)–(5) in [31]), giving rise to the probability distribution

$$p(00|xy) = \eta \text{Tr} (\rho M_{0|x} \otimes M_{0|y}),$$
$$p_a (0|x) = \eta \text{Tr} (\rho M_{a|x} \otimes \mathbb{1}),$$
$$p_b (0|y) = \text{Tr} (\rho \otimes M_{0|y}),$$

(34)

which has been shown to violate the $N$-setting $I_{NN22}$ inequality for the parameter range $\eta \geq 1/(N - 1)$. Note that we switched Alice and Bob with respect to the notation in [31]. It is also noted that $p_a (0|x) = \sum_{y=1}^{N} p(0b|xy)$ and $p_b (0|y) = \sum_{x=1}^{N} p(a0|xy)$ stand for Alice’s and Bob’s respective marginal distributions. The $N$-setting Bell inequality $I_{NN22}$ was discovered by Collins and Gisin [30], for which the $I_{322}$ inequality is the first member $N = 3$.

Let us now pass the finite $\eta$ value in equation (34) to the measurements by defining the following POVM elements for Alice:

$$M_{a=0|x} = \rho M_{a=0|x},$$
$$M_{a=1|x} = 1 - \rho M_{a=0|x},$$

(35)

for $x = (1, \ldots, N)$. Indeed, with these lossy measurements we have

$$p(ab|xy) = \text{Tr} (\rho M_{a|x} \otimes M_{b|y}),$$

(36)

which gives the same statistics as equation (34) violating the $N$-setting $I_{NN22}$ inequality for $\eta = 1/(N - 1)$. However, if we pick any $(N - 1)$ measurements from the set defined by the POVM elements (35) above, they turn out to be $(N - 1)$-wise jointly measurable for the parameter $\eta = 1/(N - 1)$. This is analogous to the one presented in appendix E of [32], and is as follows.

Let us consider $n$ lossy two-outcome measurements. We start with arbitrary two outcomes measurements $M_{a|x}$, where $a = 0, 1$ and $x = (1, \ldots, n)$. Then the lossy sets are constructed as follows

$$M_{a|x} = \eta M_{a|x},$$
$$M_{b|x} = 1 - \eta M_{a|x},$$

(37)

Clearly, these measurements define valid POVM elements for all $x$. It is proven below that any such set of $n$ measurements is in fact jointly measurable in case of $\eta \leq 1/n$. Let us consider a parent POVM $\{M_a\}$ with $2^n$ elements, where $a$ is a length $n$ binary string. Let all the POVM elements $M_a$ vanish except the ones corresponding to the strings $01 \ldots 11, 10 \ldots 11, \ldots, 11 \ldots 01, 11 \ldots 10$ (that is, when the string contains a single 0), and $11 \ldots 11$ (that is, all digits are 1). In these cases, we have the following elements:

$$M_{01 \ldots 11} = (1/n) M_{01},$$
$$M_{10 \ldots 11} = (1/n) M_{02},$$
$$\vdots$$
$$M_{11 \ldots 01} = (1/n) M_{0n-1},$$
$$M_{11 \ldots 10} = (1/n) M_{0n},$$
$$M_{11 \ldots 11} = (1/n) \sum_{x=1}^{n} M_{0|x},$$

(38)

If we consider a parent POVM defined by equation (3), we indeed recover the measurements appearing in equation (37) with $\eta = 1/n$. Using this result, we let $n = (N - 1)$, and identify any $N - 1$ measurements in the set (35) by the parameter $\eta = 1/(N - 1)$ with the set (37). This proves that the set of $N$ specific measurements defined by equation (35) are $(N - 1)$-wise jointly measurable in the case of $\eta \leq 1/(N - 1)$.

5. Conclusion

We investigated the link between Bell nonlocality and incompatibility of measurements and proved that there exists a set of three incompatible dichotomic qubit measurements which never give rise to Bell nonlocality. We recall that this is the simplest situation in which the two notions may differ, since for a pair of dichotomic measurements it has been proved by Wolf et al [8] that measurement incompatibility entails violation of Bell inequalities. Recently, the case of more than two dichotomic measurements has been addressed. Importantly, Quintino et al [13] constructed a LHV model for a set of incompatible qubit measurements. The present study can be considered as a generalization of [13] in different aspects: on one hand, Bob’s two outcome settings have...
been generalized to measurement settings with arbitrary outcomes. On the other, Alice’s set of measurements could be decreased from an infinite number to the minimum number of three settings. Note also a more recent work [33] obtaining related results.

Moving away from the bipartite case, we can ask the following question. Does there exist a set of incompatible measurements such that if Alice measures this set independently of the set of measurements chosen by Bob and Charlie and the three-party state shared by them, the resulting statistics is not genuinely tripartite nonlocal (in the sense of not able to violate any Svetlichny–type inequality [34–36])? This question can be considered as a generalization of the two-party case to more parties.

Finally, we presented a set of \( N \) suitably chosen measurements in dimension \( N \), which are \((N - 1)\)-wise jointly measurable, such that they provide a Bell violation. It remains an open problem if such a set of \( N \) measurements can be found in the case of minimal dimension 2.

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Appendix A. Real-valued unitaries

We prove that one can choose \( U_A = O(\varphi) \) and \( U_B = I \) in the state (6) without the loss of generality, where \( O(\varphi) \) is the planar rotation (7) and Bob’s qubit measurements \( \{M_{b|y}\} \) are real valued.

Suppose that the distribution \( p(ab|xy) \) in equation (4) is local for all \( \{M_{b|y}\} \) real valued, however, it lies outside the local set (i.e. nonlocal) for \( \{M_{0|y}\} \) complex valued. Let us denote this nonlocal distribution by \( p'(ab|xy) \). We next show that this situation cannot occur. Hence this is a proof by contradiction.

Since the LHV set (5) is convex, the nonlocal distribution \( p'(ab|xy) \) implies that there must exist a hyperplane with associated (real-valued) Bell coefficients \( c_{ab|xy} \), such that

\[
\beta \equiv \sum_{a,b,x,y} c_{ab|xy} p'(ab|xy) > \max \sum_{a,b,x,y} c_{ab|xy} p(ab|xy),
\]

where maximization is over all \( p(ab|xy) \) within the LHV set. However, as we will show the value of \( \beta \) in equation (A1) can also be attained with \( U_A = O(\varphi) \) and \( U_B = I \) and real valued qubit measurements \( M_{0|y} \) for Bob. Hence, there exists some nonlocal distribution \( p(ab|xy) \) where the set \( \{M_{b|y}\} \) is real-valued, which is a contradiction.

We now show that \( \beta \) can be attained using \( U_A = O(\varphi) \) and \( U_B = I \) and a real valued set \( \{M_{b|y}\} \). To this end, let us denote

\[
\sigma_{b|y} = \text{Tr}_B(\rho I \otimes M_{b|y}),
\]

and let

\[
F_{b|y} = \sum_{a,x} c_{ab|xy} M_{a|x}^y.
\]

With these, we have \( \beta = \sum_{b,y} \text{Tr}(F_{b|y} \sigma_{b|y}) \). Since \( M_{a|x}^y \) is real valued, \( F_{b|y} \) are symmetric matrices. Then, by redefining \( \sigma_{b|y} \) as \( (\sigma_{b|y} + \sigma_{b|y}^*)/2 \), we get a real-valued assemblage, which provides the same \( \beta \) value in equation (A1). Due to the GHJW construction [37, 38], any such real valued no-signaling qubit assemblage \( \{\sigma_{b|y}\} \) has a quantum realization with a state \( \rho = |\psi\rangle \langle \psi| \) in the form

\[
|\psi\rangle = \sum_{i=0,1} \sqrt{\lambda_i} (O_\varphi (\varphi) |i\rangle)|i\rangle,
\]

where \( \lambda_i \) are positive Schmidt coefficients and \( O_\varphi \) is the orthogonal qubit matrix defined by (7). These can be obtained through the diagonalization \( \sigma_{b|y} = \sum_{k} \sigma_{b|y} = O_\varphi (\varphi) \sum_{\lambda_i} \lambda_i |i\rangle \langle i| O_\varphi (\varphi) \). On the other hand, Bob’s measurements \( M_{b|y} \) can be written in the form

\[
M_{b|y} = \sum_{i=0,1} \sum_{j=0,1} \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle j| \sigma_{b|y} |i\rangle \langle i|,
\]

which define valid real-valued qubit POVM elements (as they are readily positive and sum up to the identity).
Appendix B. Computation of $\eta(\theta, \varphi)$

We have the special case of equation (28), where $\varphi_j$ is fixed:

$$v\rho(\theta, \varphi) + (1 - v)\frac{1}{2} \otimes \text{Tr}_A \rho(\theta, \varphi) = p\rho(\theta, \varphi) + (1 - p)\sigma.$$  \hspace{1cm} (B1)

Then we have $\eta(\theta, \varphi) = v\eta(\theta, \varphi)$. First let us observe that we can rotate Alice’s system by an angle $-\varphi_j$, such that we get the same $v$ in the un-rotated system. Then it is enough to determine $v$ and $p$ in the decomposition (B1) when $\varphi_j = 0$.

Our goal is to get a good lower bound on $v$ in function of $\delta\theta = \theta_i - \theta > 0$. Following similar steps as in the derivation carried out in [13], that is constraining that $\sigma$ is a diagonal matrix in equation (B1), and demanding the positivity of the diagonal elements of $\sigma$, we get the following upper bound formulas for $v$:

$$v = \cot \theta \tan \theta (1 + \eta(\theta, \varphi)) - \eta(\theta, \varphi).$$  \hspace{1cm} (B3)

where $\eta$ above is expressed by the angles $(\theta_i, \varphi)$ and we also assume that $\theta_i \leq \theta$. It turns out that the smallest value corresponds to the last line, which is the most constraining, hence we can take

$$\eta(\theta, \varphi) = \frac{\eta(\theta_i, \varphi)}{\cot \theta \tan \theta (1 + \eta(\theta, \varphi)) - \eta(\theta, \varphi)}.$$  \hspace{1cm} (B3)

It is noted that in the other case of $\theta \geq \theta_i$, the most constraining relation in equations (B2) corresponds to the first line.

Appendix C. Computation of $\eta(\theta_i, \varphi)$

We have the special case of equation (28), when $\theta_i$ is fixed:

$$v\rho(\theta_i, \varphi) + (1 - v)\frac{1}{2} \otimes \text{Tr}_A \rho(\theta_i, \varphi) = p\rho(\theta_i, \varphi) + (1 - p)\sigma.$$  \hspace{1cm} (C1)

Then we have $\eta(\theta_i, \varphi) = v\eta(\theta_i, \varphi)$. We can rotate Alice’s system by an angle $-\varphi_j$, such that we get the same $v$ in the un-rotated system. Then it is enough to determine $v$ and $p$ in the decomposition

$$v\rho(\theta_i, \varphi) + (1 - v)\frac{1}{2} \otimes \text{Tr}_A \rho(\theta_i, \varphi) = p\rho(\theta_i, \varphi) + (1 - p)\sigma.$$  \hspace{1cm} (C2)

We wish to get a good lower bound on $v$ in the function of $\delta\varphi = \varphi - \varphi_j > 0$ for fixed $\theta_i$.

To this end, we prove that we can take $p = v$ in equation (C2) above, where $v$ is given by

$$v(\delta\varphi) = \frac{1 - 2\sqrt{2} \sqrt{1 - \cos(2\delta\varphi)}}{8 \cos(2\delta\varphi) - 7},$$  \hspace{1cm} (C3)

which results in $\sigma$ separable. Indeed, if we rearrange equation (C2) for $\sigma$, it will take the form

$$\sigma = \frac{v}{1 - v}(\rho(\theta_i, \delta\varphi) - \rho(\theta_i, 0)) + \frac{1}{2} \otimes \text{Tr}_A \rho(\theta_i, \delta\varphi).$$  \hspace{1cm} (C4)

If we insert $v$ from (C3) into (C4), one can see that $\sigma$ is a valid two-qubit separable state. This can be checked by first noting that $\text{PT}(\sigma) = \sigma$ (for arbitrary $v$), where $\text{PT}$ denotes partial transposition [39, 40] with respect to system $B$. On the other hand, $\sigma$ is a valid density matrix. Readily, $\text{Tr} \sigma = 1$ and all its eigenvalues turn out to be positive

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_{3,4} = \frac{3 \pm \sqrt{5 + 4 \cos(4\theta_i)}}{8}$$  \hspace{1cm} (C5)

for any $\theta_i$. Then the relation $\eta(\theta_i, \varphi) = v(\varphi - \varphi_j)\eta(\theta_i, \varphi_j)$ follows, where $v$ is given by equation (C3).
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