Symmetries of the Interacting Boson Model

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I. INTRODUCTION

In 1975 Arima and Iachello proposed a new approach to nuclear collective motion, the interacting boson model or IBM [1]. It quickly became a popular model for the interpretation of nuclear data and acquired the center stage of discussions within the nuclear-structure community in the remainder of the decade and much of the 1980s.

What made and still makes the model so appealing? One of its strengths is that it offers a unified view of several descriptions which until the 1970s existed more or less separately. Nuclei can be viewed as incompressible, charged liquid drops, which vibrate and, if deformed, also rotate [2, 3]. From this picture are derived a variety of charged liquid drops, which vibrate and, if deformed, also less separately. Nuclei can be viewed as incompressible, ural descriptions which until the 1970s existed more or necessarily so for the extended notions of partial dynamical symmetry and quasi dynamical symmetry, which can be beautifully illustrated in the context of the interacting boson model. The main conclusion of the analysis is that dynamical symmetries are scarce while their partial and quasi extensions are ubiquitous.

II. THE INTERACTING BOSON MODEL

A. Basic properties

In the original version of the IBM as applied to even-even nuclei, collective properties of the nucleus are described in terms of a set of interacting $s$ and $d$ bosons carrying the angular momenta $\ell = 0$ and $\ell = 2$, respectively. In the simplest version of the model, referred to as IBM-1, it is assumed that there is only one kind of boson (i.e., no distinction is made between neutron and proton bosons) and that they carry no further intrinsic labels such as spin or isospin. The associated creation and annihilation operators satisfy the standard boson commutation relations

$$[b_{\ell m}, b_{\ell' m'}^\dagger] = \delta_{\ell \ell'} \delta_{m m'},$$
$$[b_{\ell m}, b_{\ell' m'}] = [b_{\ell m}^\dagger, b_{\ell' m'}^\dagger] = 0. \quad (1)$$

The IBM, which makes in particular extensive use of the notion of dynamical symmetry. While the latter might still be unfamiliar as a term, it appears in diverse areas of physics, also in nuclear physics where it received wide-spread attention. Notable examples are Wigner’s supermultiplet model [12], Racah’s pairing model [13], and Elliott’s rotation model [14], and their many extensions that can be formulated in terms of dynamical symmetries [15, 16].

It is not the aim of this contribution to give a comprehensive review of the properties of the IBM but rather to focus on the use of notions of symmetry in the model. It specifically deals with two further generalizations of the concept of dynamical symmetry, which have been developed over the last two decades, namely partial dynamical symmetry and quasi dynamical symmetry. Again, the IBM proved to be instrumental in the development of these novel symmetry notions and the basic ideas behind these extensions can be illustrated beautifully with a simplified IBM Hamiltonian [17]. Before turning to these extended notions of symmetry, a brief review of the IBM is given.
The IBM-1 assumes that low-lying collective states of an even-even nucleus can be described in terms of boson excitations acting upon a vacuum state \(|0\rangle\), which is interpreted as the doubly-closed core of the nucleus under consideration. There are six basic excitations, \(s|0\rangle\) and \(d_m|0\rangle\), \(m = 0, \pm 1, \pm 2\), and the unitary transformations among them generate the Lie algebra \(U(6)\). A different way of expressing the same property is through the construction of the bilinear operators \(b_{lm}^\dagger b_{lm'}\), which generate \(U(6)\) \cite{15}.

As mentioned in the introduction, the bosons are associated with (collective) pairs of nucleons in the valence shell. Because of this interpretation, a collective state of an even-even nucleus with \(2N_b\) valence nucleons is approximated as a state with \(N_b\) bosons. In general, the separate boson numbers \(n_s\) and \(n_d\) are not conserved but the sum \(n_s + n_d = N_b\) is. The Hamiltonian of the IBM-1 can therefore be written in terms of the generators of the Lie algebra \(U(6)\) and acquires the generic form

\[
\hat{H} = E_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \cdots, \tag{2}
\]

where the index refers to the order of the interaction in the generators of \(U(6)\). The first term \(E_0\) is a constant and represents the (negative of the) binding energy of the doubly-closed core. The second term is the one-body part

\[
\hat{H}_1 = \epsilon_s [s|s\rangle|0\rangle + \epsilon_d \sqrt{5}[d|d\rangle|0\rangle
= \epsilon_s s^\dagger \cdot s + \epsilon_d d^\dagger \cdot d \equiv \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d, \tag{3}
\]

where the coupling in angular momentum is shown as an upperscript in round brackets and the dot indicates a scalar product. Furthermore, \(\hat{b}_{lm} \equiv (-)^{l-m} b_{l,-m}\) and the coefficients \(\epsilon_s\) and \(\epsilon_d\) are the single-boson energies in the \(s\) and \(d\) state, respectively. The third term in the Hamiltonian \cite{2} is the two-body interaction,

\[
\hat{H}_2 = \sum_{\ell_1 \leq \ell_2, \ell'_1 \leq \ell'_2, L} \tilde{v}_{\ell_1 \ell_2 \ell'_1 \ell'_2}^{(L)} [b_{\ell_1 \ell_2}^\dagger (L) \cdot [b_{\ell'_1 \ell'_2} (L) \cdot [b_{\ell_1 \ell_2} (L)] \cdot [b_{\ell'_1 \ell'_2} (L)]^{\dagger}. \tag{4}
\]

where the coefficients \(\tilde{v}_{\ell_1 \ell_2 \ell'_1 \ell'_2}^{(L)}\) are related to the interaction matrix elements between normalized two-boson states,

\[
\tilde{v}_{\ell_1 \ell_2 \ell'_1 \ell'_2}^{(L)} = (-)^L \frac{\langle \ell_1 \ell_2; L | \hat{H}_2 | \ell'_1 \ell'_2; L \rangle}{\sqrt{(1 + \delta_{\ell_1 \ell_2})(1 + \delta_{\ell'_1 \ell'_2})}}. \tag{5}
\]

The bosons are symmetrically coupled and allowed two-boson states are: \(s^2\) with angular momentum \(L = 0\), \(sd\) with \(L = 2\), and \(d^2\) with \(L = 0, 2, 4\). This leads to seven independent two-body interactions: three for \(L = 0\), three for \(L = 2\), and one for \(L = 4\).

This analysis can be extended to higher-order interactions. The number of possible interactions at each order \(n\) is summarized in Table I up to \(n = 3\). Some of these interactions contribute to the binding energy but do not influence the excitation spectrum of a nucleus, as indicated with ‘BE’ in the table. The remaining interactions, listed under ‘\(E_2\)’, affect also the relative energies of the eigenstates.

### B. Geometric interpretation

Before entering the discussion of symmetries, a brief discussion of the geometric interpretation is in order, which can be obtained by means of coherent (or intrinsic) states \cite{18–20}. For the IBM-1 the coherent state is of the form

\[
|N_b; \alpha_\mu \rangle \propto \Gamma(\alpha_\mu)^{N_b}|0\rangle, \tag{6}
\]

where \(\alpha_\mu\) are five complex variables in the expression

\[
\Gamma(\alpha_\mu) = s^\dagger + \sum_{\mu, -2}^{+2} \alpha_\mu d^\dagger_\mu. \tag{7}
\]

The \(\alpha_\mu\) have the interpretation of quadrupole shape variables and their associated conjugate momenta, analogous to those introduced in the droplet model of the nucleus \cite{21}. The real part of the \(\alpha_\mu\) can be related to three Euler angles \(\{\theta, \psi, \phi\}\), which define the orientation of an intrinsic frame of reference, and two variables, \(\beta\) and \(\gamma\), that parametrize the intrinsic shape of the nuclear surface. In terms of the latter variables the state \(|N_b; \alpha_\mu \rangle\) is rewritten as

\[
\Gamma(\beta, \gamma) = s^\dagger + \beta \left[ \cos \gamma d^\dagger_0 + \sin \gamma \sqrt{\frac{1}{2}} (d^\dagger_{-2} + d^\dagger_{+2}) \right]. \tag{8}
\]

The calculation of the expectation value of an operator in the coherent state \(|N_b; \alpha_\mu \rangle\) leads to a function of \(N_b\), \(\beta\), and \(\gamma\). The IBM-1 Hamiltonian \cite{2} can be converted in this way into a total-energy surface \(E(\beta, \gamma; N_b, \epsilon, \tilde{v}, \ldots)\), where \(\epsilon, \tilde{v}, \ldots\) is a short-hand notation for the complete set of parameters in the Hamiltonian.

The study of the energy surface \(E(\beta, \gamma; N_b, \epsilon, \tilde{v})\) has shown us the properties of the IBM-1 in two important ways. First, it was instrumental in showing that the three symmetry limits of the model, to be discussed below, have counterparts that are also known from the geometric model of the nucleus \cite{21}. Establishing the correspondence between the IBM and the

| Table I: Number of \(n\)-body interactions in IBM-1. |
|---|---|---|
| Order \(n\) | Number of interactions \(\sum\) \(\text{type BE}^+ \) \(\text{type } E_2\) |
| \(n = 0\) | 1 | 1 | 0 |
| \(n = 1\) | 2 | 1 | 1 |
| \(n = 2\) | 7 | 2 | 5 |
| \(n = 3\) | 17 | 7 | 10 |

See text for explanation.
geometric model was, in fact, one of the major achievements in the early days of the model [13,20]. Secondly, the energy surface was studied from the point of view of catastrophe theory [22], with the single-boson energies $\epsilon$ and boson-boson interactions $\tilde{v}$ viewed as control parameters that determine the minima, saddle points etc. of $E(\beta, \gamma; N_b, \epsilon, \tilde{v})$. This problem was worked out for the most general IBM-1 Hamiltonian with up to two-body interactions [22] and also in the context of the classical Landau theory of phase transitions [24, 25]. It has given rise in recent years to a flurry of activity, which can be characterized as the study of quantum phase transitions in nuclei (see, e.g., the review [24]).

C. Dynamical symmetries

The numerical solution of the eigenvalue problem associated with the IBM-1 Hamiltonian [2] can be obtained in all cases of interest, that is, for values of $N_b$ corresponding to numbers of valence pairs occurring in nuclei and for up to three-body interactions between the bosons. In addition, the interacting-boson problem can be solved analytically for certain boson energies and boson-boson interactions $\tilde{v}$, viewed as control parameters in the Hamiltonian. The underlying reason is that the Hamiltonian in that case can be written as a sum of commuting operators and that, as a consequence, the associated quantum numbers are conserved. The three limits can therefore be summarized with a chain of nested algebras and their associated quantum numbers. For the IBM-1 they are

$$U(6) \supset U(5) \supset SO(5) \supset SO(3) \supset SO(2)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[N_b] \quad \lambda d \quad \nu L \quad \lambda \sigma$$

$$U(6) \supset SU_+(3) \supset SO(3) \supset SO(2)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[N_b] \quad (\lambda \pm, \mu \pm) \quad KL \quad \lambda \sigma$$

$$U(6) \supset SO(6) \supset SO(5) \supset SO(3) \supset SO(2)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$[N_b] \quad \sigma \quad \nu L \quad \lambda \sigma$$

The $N_b$ bosons, which can be in an $s$ or a $d$ state, must transform symmetrically under $U(6)$, as indicated with the square brackets $[N_b]$. The allowed values for the labels of the subalgebras appearing in the lattice [9] then follow from standard group-theoretical reduction rules [15]. The quantum numbers for $SU_-(3)$ and $SU_+(3)$ are not identical but are obtained from each other under the interchange $\lambda \leftrightarrow \mu$, equivalent to a particle-hole transformation. In the following only the $SU_-(3)$ labels are needed, henceforth denoted for simplicity as $(\lambda, \mu)$. Note also the presence of the additional labels $K$ and $\nu_\Delta$, which are needed to distinguish repeated angular momenta $L$ in a single $SO(5)$ or $SU(3)$ representation.

The preceding discussion defines the concept of a dynamical symmetry, which has received particular attention in the context of the IBM [32]. However, even a simplified IBM-1 Hamiltonian reserves many further surprises when it comes to symmetries, as will be shown in Sects. III and IV.

D. Graphical illustration

The property of dynamical symmetry can be displayed in a graphical fashion. To explain the procedure, consider the IBM-1 Hamiltonian, not in its full complexity of Eq. (2), but a simplified version of it, known as the Hamiltonian of the extended consistent-$Q$ formalism (ECQF) [33, 34], which reads

$$\hat{H}_{ECQF} = \omega \left[ (1 - \xi) \hat{n}_d - \frac{\xi}{4N_b} \hat{Q}^\chi \cdot \hat{Q}^\chi \right],$$

(11)

where $\hat{n}_d$ is an operator that counts the number of $d$ bosons and $\hat{Q}_\chi^\chi$ is the quadrupole operator of the model containing a parameter $\chi$,

$$Q_\chi^\chi = [s^\dagger \hat{d} + d^\dagger s]^{(2)} + \chi [d^\dagger \hat{d}]^{(2)}.$$  

(12)

The eigenfunctions of the ECQF Hamiltonian do not depend on the overall scale $\omega$ but only on $\xi$ and $\chi$, which are therefore the structural parameters of the problem. The parameter $\xi$ ranges from 0, where $\hat{H}_{ECQF}$ reduces to $\hat{n}_d$, the linear Casimir operator of $U(5)$, to 1, where it reduces to the quadrupole term $\hat{Q}^\chi \cdot \hat{Q}^\chi$. The latter is a combination of quadratic Casimir operators of $SU_\pm(3)$.
and SO(3) for \( \chi = \pm \sqrt{7}/2 \) while for \( \chi = 0 \) it is (up to a constant) the quadratic Casimir operator of SO(6). A convenient range of the parameters is therefore \( 0 \leq \xi \leq 1 \) and \(-\sqrt{7}/2 \leq \chi \leq +\sqrt{7}/2\), which allows to attain the U(5), SU\(_+\)(3), and SO(6) dynamical symmetries. The parameter space of the ECQF Hamiltonian can be represented on a so-called Casten triangle \([35]\), with each point corresponding to a given \((\xi, \chi)\).

The symmetry properties of a given Hamiltonian can be probed with use of a property called ‘wave-function entropy’ \([36]\). For any eigenstate \(|k\rangle\) of the Hamiltonian that can be expanded in a basis \(\{|i\rangle, i = 1, \ldots, D\}\) with components \(\alpha_i^k\),

\[
|k\rangle = \sum_{i=1}^{D} \alpha_i^k |i\rangle,
\]

the wave-function entropy is defined as

\[
-\sum_{i=1}^{D} (\alpha_i^k)^2 \ln(\alpha_i^k)^2.
\]

The wave-function entropy of a set \(S\) of eigenstates of the Hamiltonian is defined as the sum

\[
-\frac{1}{|S|} \sum_{k \in S} \left( \sum_{i=1}^{D} (\alpha_i^k)^2 \ln(\alpha_i^k)^2 \right),
\]

where \(|S|\) is the cardinality of the set \(S\), that is, the number of eigenstates considered in the set, such that the quantity \((15)\) represents the average wave-function entropy per eigenstate. It is clear from the definition that wave-function entropy depends on the basis \(|i\rangle\), which in the IBM-1 can be taken as U(5), SU\(_+\)(3), or SO(6). The property of interest here is that a vanishing wave-function entropy \((15)\) implies a dynamical symmetry. For example, all eigenstates \(|k\rangle\) of an SO(6) Hamiltonian have vanishing wave-function entropy in the SO(6) basis: for each eigenstate one component \(\alpha_i^k\) equals 1 and all others are 0. However, the same SO(6) Hamiltonian has a non-zero wave-function entropy in the U(5) or SU\(_+\)(3) basis, where the SO(6) eigenstates have a fragmented structure. The extent of this fragmentation is measured by the wave-function entropy—the higher it is, the more fragmentation occurs. The maximal value of the wave-function entropy is obtained if, in a given basis of dimension \(D\), the eigenstate is completely fragmented with equal components \(\pm D^{-1/2}\). The wave-function entropy in that case reaches the value of \(\ln D\).

Figure 1 shows the wave-function entropy, on a scale from 0 to its maximum value \(\ln D\), in the three different bases U(5), SU\(_-\)(3), and SO(6) for all eigenstates of the ECQF Hamiltonian \([11]\) with angular momentum \(L = 0\) and boson number \(N_b = 15\). As argued above, wave-function entropy can be considered as a measure of dynamical symmetry and vanishes when all quantum numbers of the basis are conserved for all eigenstates. Therefore, a blue region (low wave-function entropy) is found around the vertex that corresponds to the basis used to compute the wave-function entropy. It is seen that the wave-function entropy in the bases U(5) and SO(6) is reflection symmetric with respect to the axis U(5)–SO(6).

Following a similar line of argument, it is not necessary to show the wave-function entropy in the SU\(_+\)(3) basis since the resulting plot is the reflection-symmetric version of the one obtained in the SU\(_-\)(3) basis.

The preceding results can be conveniently summarized in a single Fig. 2 which shows the lowest value of the wave-function entropy, as calculated in one of the four possible bases, U(5), SU\(_-\)(3), SU\(_+\)(3), or SO(6). This corresponds to overlaying the three plots of Fig. 1 and taking the minimum value at each \((\xi, \chi)\) point, with the added requirement that also the reflection-symmetric version of the middle plot in Fig. 1 is considered to account for the wave-function entropy in the SU\(_+\)(3) basis. In the appreciation of Fig. 1 it should be remembered...
III. PARTIAL DYNAMICAL SYMMETRIES

The results of Fig. 2 are obtained with the expression (15) where the set $S$ is defined as the eigenstates with angular momentum $L = 0$ and boson number $N_b = 15$. Similar results are obtained if the sum is taken over all eigenstates but for different choices of $L$ and $N_b$. However, one is usually interested only in eigenstates at low energy and it makes therefore sense to restrict the set $S$ to such states. In addition, it may be that some quantum numbers of a dynamical-symmetry classification are broken while others are conserved. Symmetry characteristics of this kind can be studied by restricting the sum in Eq. (15) to a subset of eigenstates of the Hamiltonian and/or by decomposing the eigenstates onto subspaces characterized by a single label (instead of all labels of a dynamical-symmetry chain). Such restricted symmetries are known collectively as partial dynamical symmetries, of which there are three different types. In the first, PDS-1 [37, 38], only eigenstates in a restricted set $S$ retain all quantum numbers. In the second type, PDS-2 [39, 40], all eigenstates of the IBM-1 Hamiltonian conserve a single label of one of the classifications [40]. To render the definition of the associated wave-function entropy more explicit in this case, one decomposes each eigenstate onto subspaces spanned by the representations of single subalgebra $G$ of $U(6)$, leading to the expansion

$$|k\rangle = \sum_{j=1}^{d} \sum_m a_{jm}^k |jm\rangle,$$

where the first sum runs over the $d$ different representations of $G$ while the second enumerates the basis states that span this representation. With the definition of the coefficients

$$(\beta_j^k)^2 = \sum_m (a_{jm}^k)^2,$$}

the relevant wave-function entropy of a set $S$ of eigenstates can be written as

$$-\frac{1}{|S|} \sum_{k \in S} \left( \sum_{j=1}^{d} (\beta_j^k)^2 \ln(\beta_j^k)^2 \right),$$

which, by a similar argument as above, has a maximum value of $\ln d$. Finally, the third type of partial dynamical symmetry, PDS-3 [41], combines the two properties and thus concerns a subset of eigenstates, which is analyzed with respect to a single label.

Algorithms exist for the construction of Hamiltonians with the required symmetry properties, PDS-$i$, and can be found in the review [42].

As before, the concept of partial dynamical symmetry can be illustrated graphically with the wave-function entropy of the ECQF Hamiltonian [41]. Figures 3 to 6 shows the results of nine different calculations, varying the set $S$ of eigenstates, the choice of the label $[n_d, (\lambda, \mu), \sigma]$, and the basis $[U(5), SU_-(3), SU_+(3), SO(6)]$, always for angular momentum $L = 0$ and boson number $N_b = 15$. On the left-hand panel of each figure is plotted the wave-function entropy of the $0^+_1$ eigenstate, that is, for $S = \{|0^+_1\rangle\}$, decomposed in the three different bases, $U(5)$, $SU_-(3)$, and $SO(6)$. In the middle panel the wave-function entropy is summed over all $0^+$ eigenstates but the components $\beta_j^k$ correspond to the decomposition onto subspaces that are characterized by a single label, $n_d, (\lambda, \mu), \sigma$, as in Eq. (18). The wave-function entropy of the $0^+_1$ ground state with respect to a single label is shown on the right-hand panel of Figs. 3 to 5. The figures therefore illustrate graphically the three types of partial dynamical symmetry PDS-$i$. A remarkable result is found as regards the conservation of the SO(6) label $\sigma$ namely, the existence of an entire band of ECQF Hamiltonians with close to exact SO(6) symmetry in the ground state [43], see the left-hand panel of Fig. 5.

A partial dynamical symmetry can also be defined with respect to the SO(5) label $v$—associated with $d$-boson seniority. Given its single-label character, it concerns either PDS-2 or PDS-3. The top panel in Fig. 6 shows that the conservation of the SO(5) label is exact for the entire $U(5)$–SO(6) transitional Hamiltonian, as is known...
FIG. 3: Illustration of the three partial dynamical symmetries of the type PDS-1 in the IBM-1. The plots show the wave-function entropy of the $0^+_1$ eigenstate with respect to all labels of the $U(5)$, $SU_-(3)$, and $SO(6)$ limits for the ECQF Hamiltonian with boson number $N_b = 15$.

FIG. 4: Illustration of the three partial dynamical symmetries of the type PDS-2 in the IBM-1. The plots show the average wave-function entropy of all $0^+_1$ eigenstates with respect to a single label (as indicated) of the $U(5)$, $SU_-(3)$, and $SO(6)$ limits for the ECQF Hamiltonian with boson number $N_b = 15$.

FIG. 5: Illustration of the three partial dynamical symmetries of the type PDS-3 in the IBM-1. The plots show the wave-function entropy of the $0^+_1$ eigenstate with respect to a single label (as indicated) of the $U(5)$, $SU_-(3)$, and $SO(6)$ limits for the ECQF Hamiltonian with boson number $N_b = 15$. 
IV. QUASI DYNAMICAL SYMMETRIES

Quasi dynamical symmetries constitute another extension of the concept of dynamical symmetry. They can be given a mathematical definition in terms of embedded representations \[45\]. The (admittedly loose) physical interpretation of quasi dynamical symmetries is that observables can be consistent with a certain symmetry, which is in fact broken in the Hamiltonian. Typically, this situation occurs for a Hamiltonian that is transitional between two limits and which retains, for a certain range of its parameters, the characteristic patterns of one of those dynamical symmetries \[46–49\]. In more mathematical terms a coherent mixing of representations in a subset of eigenstates is at the basis of this ‘apparent’ symmetry.

The validity of a quasi dynamical symmetry must be probed by examining the similarity in the decomposition of certain eigenstates. A quantitative measure of quasi dynamical symmetry can be introduced by rewriting the expansion \[13\] in yet another way,

\[ |k⟩ ≡ |rL⟩ = \sum_{i=1}^{D_L} \alpha_{rL}^i |i⟩, \tag{19} \]

where \( \{k\} \equiv \{rL\} \), that is, \( r \) contains all labels except the angular momentum \( L \). The measure of quasi dynamical symmetry is defined as \( \Omega_r \equiv \sqrt{1 - \Theta_r} \) where \( \Theta_r \) is the average over pairs \( L \neq L' \) of the quantities

\[ \Theta_r^{LL'} ≡ \sum_{i=1}^{D_L} \alpha_{rL}^i \alpha_{rL'}^i. \tag{20} \]

A vanishing \( \Omega_r \) therefore indicates a perfect correlation between the expansion coefficients \( \alpha_{rL}^i \) with different angular momenta \( L \). In a typical application of quasi dynamical symmetry one wishes to probe the similarity of

FIG. 6: Illustration of the SO(5) partial dynamical symmetry in the IBM-1. The plots show the wave-function entropy for the eigenstates of the ECQF Hamiltonian \[\text{11}\] with angular momentum \( L = 0 \) and boson number \( N_b = 15 \). Top: average wave-function entropy of all \( 0^+ \) eigenstates with respect to the SO(5) label \( \nu \) (PDS-2). Bottom: wave-function entropy of the \( 0^+_1 \) eigenstate with respect to \( \nu \) (PDS-3).

FIG. 7: Where in the IBM-1 does a partial dynamical symmetry occur? The plot shows the lowest value of the wave-function entropy, as calculated with respect to five possible labels \( n_d, (\lambda, \mu), (\mu, \lambda), \sigma, \) or \( \nu \), for the \( 0^+_1 \) eigenstate of the ECQF Hamiltonian \[\text{11}\] for boson number \( N_b = 15 \) (PDS-3).
the structure of yrast states, which implies the identification of $r$ with the labels of the ground-state band, that is, for $n_d = L/2$ in $U(5)$, $(\lambda, \mu) = (2N, 0)$ in $SU(3)$, and $\sigma = N$ in $SO(6)$. Figure 8 shows the quantity $\Omega_r$ for the ECQF Hamiltonian (11) in the three different bases $U(5)$, $SU_-(3)$, and $SO(6)$, for the angular momenta $L = 0, 2, \ldots, 10$ and boson number $N_b = 15$.

It is obvious that connections exist between the concepts of partial and dynamical symmetry. For example, the band structure in the wave-function entropy of the ground state with respect to $\sigma$ in Fig. 5 also shows up in the SO(6) quasi dynamical symmetry of Fig. 8. A remarkable finding of this analysis is that the partial conservation of one symmetry may occur simultaneously with the coherent mixing of another, incompatible symmetry.

Again one can summarize these findings in a single Fig. 9 which displays the lowest value of the measure $\Omega_r$ in any of the four bases, $U(5)$, $SU_-(3)$, $SU_+(3)$, or $SO(6)$. Large areas of the parameter space are seen to be blue, that is, to display a quasi dynamical symmetry.

The contrast between the results shown for dynamical symmetries on the one hand, Fig. 2 and those for partial and quasi dynamical symmetries on the other, Figs. 7 and 9, is startling. Dynamical symmetries are restricted to small regions in the parameter space (the blue areas in Fig. 2) and therefore are expected to have only restricted applicability in nuclei. This is not the case for the extended concepts of partial and quasi dynamical symmetries, as illustrated in Figs. 7 and 9, where large bands of blue are found in the triangle.

V. CONCLUDING REMARKS

Dynamical symmetries are scarce while partial dynamical symmetries and quasi dynamical symmetries are ubiquitous. This has been the main theme of this contribution. It has been examined in the context of the interacting boson model for the schematic Hamiltonian of the extended consistent-$Q$ formalism and illustrated by a graphical representation of wave-function entropy in various bases. In no way do these results represent the complete symmetry analysis of the IBM. A general Hamiltonian of the interacting boson model with up to two-body interactions allows the occurrence of exact dynamical symmetries of various partialities, some of which are not or only approximately present in the schematic Hamiltonian of the extended consistent-$Q$ formalism. Also, given the composite nature of the bosons three-body interactions between them are to be expected, further enriching the symmetry features of the model. It is remarkable that more than forty years after the proposal by Arima...
and Iachello, the full symmetry content of the interacting boson model still remains to be uncovered.

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