ON THE MODULI SPACES OF PARABOLIC SYMPLECTIC/ORTHOGONAL BUNDLES ON CURVES

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Abstract. We prove that the moduli spaces of parabolic symplectic/orthogonal bundles on a smooth curve are globally F regular type. As a consequence, all higher cohomology of theta line bundle vanish. During the proof, we develop a method to estimate codimension, and consider the infinite grassmannians for parabolic G bundles.

1. Introduction

Let X be a variety over an algebraically closed field of positive characteristic and \( F_X : X \rightarrow X \) be the absolute Frobenius map. In [14] Mehta and Ramanathan introduced the notion “F split”: X is said to be F split if the natural morphism \( F_X^* : \mathcal{O}_X \rightarrow F_X^* \mathcal{O}_X \) splits as an \( \mathcal{O}_X \) module morphism. Later in [21] Smith studied a special kind of F split varieties: globally F regular varieties (see Section 6 for details). F split varieties and globally F regular varieties have many nice properties, for example, the vanishing of higher cohomologies of ample line bundles (nef line bundles in the case of globally F regular varieties).

Although almost all varieties are not F split, some important kind of varieties are, such as flag varieties, toric varieties. In [13] Mehta and Ramadas proved that the moduli space of semistable parabolic rank two vector bundles with fixed determinant on a generic nonsingular projective curve, are F split. They conjectured that the generic condition can be moved. On the other hand, as mentioned in [25], this conjecture should be extended into the following: the moduli spaces of semistable parabolic bundles with fixed determinant on any nonsingular projective curve are globally F regular.

In [25] Sun and Zhou studied the characteristic zero analogy of this extended conjecture. A variety over a field of characteristic zero is said to be of globally F regular type if its modulo p reduction are globally F regular for all p \( \gg 0 \). They proved that the moduli spaces of semistable parabolic vector bundles on a smooth projective curve over an algebraically closed field of characteristic zero are of globally F regular type. As an application, they can give a finite dimensional proof of the so called Verlinde formula in \( GL_n \) and \( SL_n \) case (24).

Globally F regular type varieties have similar vanishing properties as globally F regular varieties, namely all higher cohomologies of nef line bundles are vanishing. Unlike the positive characteristic case, in characteristic zero, all Fano varieties with rational singularities are globally F regular type varieties (21). So globally F regular type varieties can be regarded as a generalization of Fano varieties in characteristic zero, with the vanishing properties retained and hence it is interesting to find examples of globally F regular type varieties.

On the other hand, properties of moduli spaces is a central topic in the study of moduli problems. We already know that for connected simply connected algebraic group G, the moduli space of semistable G bundles on a smooth curve is a Fano variety (11). However, if one consider the moduli space of semistable G bundles with parabolic structure on a smooth curve, then one may not get a Fano variety. As mentioned before, in the case of \( G = SL_n \), Sun and Zhou proved that the moduli spaces of semistable parabolic vector bundles with fixed determinant are globally F regular type varieties (24). So it encourage us to consider globally F regularity as a reasonable property of moduli spaces of G bundles with parabolic structure on curves.

In this paper, we consider parabolic symplectic and orthogonal bundles over smooth curves. Our main theorem is the following:
Theorem 1.1 (Main theorem, see Theorem [6,5].) The moduli spaces of semistable parabolic symplectic/orthogonal bundles over any smooth projective curve are globally F regular type varieties. As a consequence, any higher cohomologies of nef line bundles on these moduli spaces vanish.

We now describe how this paper is organized:

In Section 2 we recall some basics about parabolic vector bundles, parabolic symplectic/orthogonal bundles and the equivalence between parabolic bundles and orbifold bundles.

In Section 3 we construct the moduli space of semistable parabolic symplectic/orthogonal bundles explicitly, using Geometric Invariant Theory.

In Section 4 we develop a technique to estimate the codimension of unsemistable locus in a given family, not only for parabolic symplectic/orthogonal bundles, but also $G$ bundles and parabolic vector bundles.

In Section 5 to evaluate the canonical line bundle on the moduli spaces we constructed in Section 3, we introduce the infinite Grassmannians for parabolic vector bundles, here $G$ is a connected simply connected simple algebraic group; we also define the theta line bundles for any family of symplectic/orthogonal bundle then we can show that under certain choice of rank and weights, the moduli spaces we constructed in Section 3 are Fano varieties.

In Section 6 we recall definition and properties of globally F regular type varieties, with the help of key Proposition 6.9, we can prove our main theorem.

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2. Basics of parabolic principal bundle over curve

2.1. Parabolic vector bundles and parabolic symplectic/orthogonal bundles. Let $C$ be a smooth projective curve of genus $g \geq 0$. We fix a reduced effective divisor $D$ of $C$, and an integer $K > 0$.

$E$ is a vector bundle of rank $r$ and degree $d$ over $C$, by a parabolic structure on $E$, we mean the following:

(1) At each $x \in D$, we have a choice of flag of $E_x$:

$$0 = F_{l_x}(E_x) \subseteq F_{l_x-1}(E_x) \subseteq \cdots \subseteq F_0(E_x) = E_x$$

Let $n_i(x) = \dim F_{l_x}(E_x)/F_{l_x-1}(E_x)$ and $\bar{n}(x) = (n_1(x), n_2(x), \cdots, n_{l_x}(x))$. Notice that all these filtrations together are equivalent to a filtration:

$$E(-D) = E \subseteq F_1(E) \subseteq \cdots \subseteq F_0(E) = E$$

(2) At each $x \in D$, we fix a choice of sequence of integers, which are called weights:

$$0 \leq a_1(x) < a_2(x) \cdots < a_{l_x}(x) < K$$

Put $\bar{a}(x) = (a_1(x), a_2(x), \cdots, a_{l_x}(x))$.

We say that $(E, D, K, \{\bar{n}(x)\}_{x \in D}, \{\bar{a}(x)\}_{x \in D})$, or simply $E$, is a parabolic vector bundle, and $\sigma = (\{\bar{n}(x)\}_{x \in D}, \{\bar{a}(x)\}_{x \in D})$ is the parabolic type of $E$.

For any subbundle $F$ of the vector bundle $E$, it is clearly that there is an induced parabolic structure on $F$, with induced flags structures and same weights; similarly there is an induced parabolic structure on $E/F$.

Let $E_1$ and $E_2$ be two parabolic vector bundle with same weights, the space of parabolic homomorphisms $\text{Hom}_{par}(E_1, E_2)$ given by $O_C$-homomorphisms between $E_1$ and $E_2$ preserving filtrations at each $x \in D$. We can also define parabolic sheaf of parabolic homomorphisms $\text{Hom}_{par}(E_1, E_2)$ in a similar way, which inherits a parabolic structure naturally. In fact, in [26], it is shown that the category of parabolic bundles is contained in an abelian category with enough injectives. So we have the derived functors of parabolic homomorphism. We use $\text{Ext}^1_{par}(E_1, E_2)$ to denote the space of parabolic extensions.
Definition 2.1. The parabolic degree of $E$ is defined by

$$\text{pardeg}E = \deg E + \frac{1}{K} \sum_{x \in D} \sum_{i=1}^{l_x} a_i(x)n_i(x)$$

and $E$ is said to be stable (resp. semistable) if for all nontrivial subbundle $F \subset E$, concerning the induced parabolic structure, we have:

$$\frac{\text{pardeg}F}{\text{rank}F} < \frac{\text{pardeg}E}{\text{rank}E} \quad \text{(resp.} \leq)$$

Now let us talk about family of parabolic vector bundles. Let $S$ be a scheme of finite type, a family of parabolic vector bundle with type $\sigma$ over $C$ parametrized by $S$ is a vector bundle $\mathcal{E}$ over $S \times C$, together with filtrations of vector bundles on $\mathcal{E}_x$ of type $\vec{n}(x)$ and weights $\vec{d}(x)$ for each $x \in D$. As before, such filtrations are equivalent to the following:

$$\mathcal{E}(-(S \times D)) = F_1(\mathcal{E}) \subseteq F_{l-1}(\mathcal{E}) \subseteq \cdots \subseteq F_0(\mathcal{E}) = \mathcal{E}$$

where $S \times D$ is considered as an effective divisor of $S \times C$. Following [26], we say $\mathcal{E}$ is a flat family if all $F_i(\mathcal{E})$ are flat families.

Definition 2.2. $E$ is a vector bundle of rank $r$ degree $d$ over $C$. By a symplectic/orthogonal parabolic structure on $E$, we mean the following:

1. A non-degenerated anti-symmetric/symmetric two form

   $$\omega : E \otimes E \rightarrow \mathcal{O}_C(-D)$$

2. At each $x \in D$, a choice of flag:

   $$0 = F_{2l_x+1}(E_x) \subseteq F_{2l_x}(E_x) \subseteq \cdots \subseteq F_{l_x+1}(E_x) \subseteq F_{l_x}(E_x) \subseteq \cdots \subseteq F_0(E_x) = E_x$$

   where $F_i(E_x)$ are isotropic subspaces of $E_x$ respect to the form $\omega$ and $F_{2l_x+1-i}(E_x) = F_i(E_x)$ for $l_x + 1 \leq i \leq 2l_x + 1$.

3. At each $x \in D$, we fix a choice of weights:

   $$0 \leq a_1(x) < a_2(x) \cdots < a_{l_x}(x) < a_{l_x+1}(x) < \cdots < a_{2l_x+1}(x) \leq K$$

   satisfying $a_i(x) + a_{2l_x+1-i}(x) = K$, $1 \leq i \leq l_x + 1$.

As before, we put $n_i(x) = \dim(F_{i-1}(E_x)/F_i(E_x))$, and

$$\vec{n}(x) = (n_1(x), n_2(x), \cdots, n_{2l_x+1}(x))$$

$$\vec{d}(x) = (a_1(x), a_2(x), \cdots, a_{2l_x+1}(x))$$

We say that $(E, \omega, D, K, \{\vec{n}(x)\}_{x \in D}, \{\vec{d}(x)\}_{x \in D})$, or simply $E$, is a parabolic symplectic/orthogonal bundle and $\sigma = \{\{\vec{n}(x)\}_{x \in D}, \{\vec{d}(x)\}_{x \in D}\}$ is the parabolic type of $E$.

Convention: when talked about parabolic symplectic/orthogonal bundles, we always assume that $\deg D$ is even, and we fix a line bundle $L$ over $C$ and an isomorphism $L^\otimes 2 \cong \mathcal{O}_C(D)$.

Remark 2.3.

1. The original definition of parabolic principal bundles is just a principal bundle together with additional structures [18]. Later in [1] Balaji, Biswas and Nagaraj establish a different definition, which share some nice properties as in the case of parabolic vector bundles, for example, a parabolic symplectic/orthogonal bundle admits an Einstein–Hermitian connection if and only if it is polystable ([1]).

2. Although in our definition, $E$ is not a principal symplectic/orthogonal bundle, but $E \otimes L$ is, we call $E$ twisted orthogonal/symplectic bundle.

The parabolic degree of $E$ is given by

$$\text{pardeg}E = \deg E + \frac{1}{K} \sum_{x \in D} \sum_{i=1}^{2l_x+1} a_i(x)n_i(x)$$
By relations between $\widehat{\mu}(x)$ and $\widehat{a}(x)$ we see that \( \text{pardeg} E = \deg E + \frac{r}{2} \deg D \), noticing that \( \omega : E \otimes E \to \mathcal{O}_X(-D) \) is non-degenerated, so \( E \simeq E' \). Thus \( \deg E + \frac{r}{2} \deg D = 0 \) and then \( \text{pardeg} E = 0 \).

For any subbundle \( F \) of \( E \), we can define the parabolic degree of \( F \) by

\[
\text{pardeg} F = \deg F + \frac{1}{K} \sum_{x \in D} \sum_{i=1}^{2l_x+1} a_i(x) n_i^F(x)
\]

where \( n_i^F(x) = \dim \left( F_{i-1}(E_x) \cap F_x / F_i(E_x) \cap F_x \right) \).

**Definition 2.4.** A parabolic orthogonal/symplectic bundle \( E \) is said to be stable(resp. semistable) if for all nontrivial subbundle \( F \subset E \) (by isotropic we mean \( \omega(F \otimes F) = 0 \)), we have

\[
\text{pardeg} F < 0 \quad \text{(resp. \leq)}
\]

**Lemma 2.5.** A parabolic symplectic/orthogonal bundle is semistable iff for any subbundle \( F \), not necessarily isotropic, we have \( \text{pardeg} F \leq 0 \), i.e. semistable as a parabolic vector bundle.

**Proof.** If \( E \) is semistable as a parabolic vector bundle, then it is semistable as parabolic symplectic/orthogonal bundle.

Conversely, if \( E \) is a semistable parabolic symplectic/orthogonal bundle and a subbundle \( F \) is given. We want to show that \( \text{pardeg}(F) \leq 0 \).

If \( F \cap F^\perp = 0 \), then \( F = F \oplus F^\perp \). Hence \( 2 \text{pardeg}(F) = \text{pardeg}(F) + \text{pardeg}(F^\perp) = \deg(E) = 0 \) and we are done.

If \( F \cap F^\perp \neq 0 \), then we have the exact sequence of parabolic bundles:

\[
0 \to F \cap F^\perp \to F \oplus F^\perp \to F + F^\perp \to 0
\]

This shows that

\[
\text{pardeg}(F \cap F^\perp) + \text{pardeg}(F + F^\perp) = \text{pardeg}(F) + \text{pardeg}(F^\perp) = 2 \text{pardeg}(F)
\]

It is easy to see \( \text{pardeg}(F \cap F^\perp) \geq \text{pardeg}(F + F^\perp) \) and hence we have \( 2 \text{pardeg}(F) \leq 2 \text{pardeg}(F \cap F^\perp) \leq 0 \), where \( \text{pardeg}(F \cap F^\perp) \leq 0 \) since \( F \cap F^\perp \) is isotropic.

**2.2. Equivalence between parabolic bundles and orbifold bundles.** There is an interesting and useful correspondence between parabolic bundles and orbifold bundles, which is developed in \cite{L}, \cite{H} and \cite{E} for general case. We will recall the correspondence briefly as follows:

Given \( C, D, K \) as before, By Kawamata covering, there is a smooth projective curve \( Y \) and a morphism \( p: Y \to C \) such that \( p \) is only ramified over \( D \) with \( p^*D = K \sum_{x \in D} p^{-1}(x) \), moreover, if we put \( \Gamma = \text{Gal} \left( \text{Rat}(Y) / \text{Rat}(C) \right) \) to be the Galois group, then \( p \) is exactly the quotient map of \( Y \) by \( \Gamma \).

**Definition 2.6.** An orbifold bundle over \( Y \) is a vector bundle \( W \) over \( Y \) such that the action of \( \Gamma \) lifts to \( W \).

And an orbifold symplectic/orthogonal bundle is an orbifold bundle such that the correspondence 2-form \( \omega \) is a morphism of orbifold bundles.

Given an orbifold bundle \( W \), for any \( y = p^{-1}(x) \in p^*D \), the stabilizer \( \Gamma_y \), which is a cyclic group of order \( K \), acts on the fiber \( W_y \) by some representation(after choosing suitable basis):

\[
\xi_K \mapsto \text{diag}(\xi_K^{a_1(x)}, \ldots, \xi_K^{a_l(x)}, \xi_K^{a_1(x)}, \ldots, \xi_K^{a_l(x)})
\]

where \( 0 \leq a_1(x) < a_2(x) \cdots a_l(x) < K \) are integers, \( \xi_K \) is the \( K \)-th root of unity and the multiplicity of \( \xi_K^{a_i(x)} \) is given by \( n_i(x) \). Similar in the definition of parabolic bundle, we use \( \sigma = \left( \{ \overline{\mu}(x) \} \right)_{x \in D}, \{ \overline{a}(x) \} \right)_{x \in D} \) to denote the type of orbifold bundle \( W \).

**Proposition 2.7 (\cite{L}, \cite{H}).** There is an equivalence between the category of orbifold bundles over \( Y \) with type \( \sigma \) and the category of parabolic vector bundles over \( C \) with type \( \sigma \).
Roughly speaking, given an orbifold bundle \( W \), then \( (p_*W)^\Gamma \) is a parabolic vector bundle over \( C \), with parabolic structures given by the action of stabilizers. Conversely, \( E \) is a parabolic vector bundle, we put \( W_1 = p^*E \), after some elementary transformations of \( W_1 \), we would have an orbifold bundle of type \( \sigma \). Moreover, we have

\[
#\Gamma \cdot \text{pardeg} E = \text{deg} W
\]

and \( E \) is (semi)stable as parabolic bundle if and only if \( W \) is (semi)stable as orbifold bundle.

Now we will talk about orbifold symplectic/orthogonal bundles over \( Y \): an orbifold symplectic/orthogonal bundles is symplectic/orthogonal bundle \( W \) over \( Y \) such that the action of \( \Gamma \) lifts to \( W \) compatible with the symplectic/orthogonal structure. For any \( y = p^{-1}(x) \in p * (D) \), the action of stabilizer is given by:

\[
\xi_K \mapsto \text{diag}\{\xi_K^{a_1(x)}, \ldots, \xi_K^{a_{\mu}(x)}, \xi_K^{a_{\mu}(x)}, \ldots, \xi_K^{a_1(x)}\}
\]

As before, we use \( \sigma \) to denote the type of this orbifold symplectic bundle. Similarly, we have:

**Proposition 2.8.** There is an equivalence between the category of orbifold symplectic/orthogonal bundles over \( Y \) with type \( \sigma \) and the category of parabolic symplectic/orthogonal bundles over \( C \) with type \( \sigma \). Moreover, this equivalence induces an equivalence between orbifold isotropic subbundles and isotropic subbundles.

**Proof.** See [3].

\[\square\]

3. **Moduli space of semistable parabolic symplectic/orthogonal bundles**

In this section, we construct the moduli space of semistable parabolic symplectic/orthogonal bundles with fixed parabolic type \( \sigma \) over \( C \). Although the moduli space is already constructed in [3] for general algebraic groups, but for our purpose, we will construct the moduli spaces explicitly using GIT constructions.

3.1. **Construction of the moduli space.** In this section we will use \( E \) to denote a parabolic symplectic/orthogonal bundle of rank \( r \), degree \( d \) and parabolic type \( \sigma \).

We will fix an ample line bundle \( O(1) \) on \( C \) with degree \( c \), then the Hilbert polynomial of \( E \) is \( P_E(m) = cm + \chi(E) \).

Firstly we notice that by Lemma 2.3 of [8], the class of semistable parabolic orthogonal/symplectic bundles with fixed rank, degree and parabolic type are bounded. So we may choose an integer \( N_0 \) large enough so that \( E(N) \) is globally generated for all semistable parabolic bundle \( E \) and all integers \( N \geq N_0 \); which means, we have a quotient

\[
q : V \otimes O_X(-N) \to E
\]

where \( V \) is the vector space \( \mathbb{C}^{P(N)} \) and \( P \) is the Hilbert polynomial of \( E \).

Let \( Q \) be the Quot scheme of quotients of \( V \otimes O_X(-N) \) with Hilbert polynomial \( P \).

The orthogonal/symplectic structure on \( E \) will induce a morphism:

\[
(V \otimes O_C) \otimes (V \otimes O_C) \to E(N) \otimes E(N) \to O_C(2N - D)
\]

which is equivalent to a bilinear map on \( V \):

\[
\phi : V \otimes V \to H^0(C, O_C(2N - D))
\]

here \( O_C(2N - D) = O_C(2N) \otimes O_C(-D) \) and we use \( H \) to denote the space \( H^0(C, O_C(2N - D)) \).

Now we can regard every semistable \( E \) as a point in the space \( Q \times \mathbb{P}Hom(V \otimes V, H) \). However, not every element in \( \mathbb{P}Hom(V \otimes V, H) \) would give a nondegenerated form on \( E \). To fix this, we will use the following lemma:

**Lemma 3.1** (Lemma 3.1 of [8]). Let \( X \) be a smooth projective variety and \( Y \) be a scheme. Consider a morphism of sheaves \( f : \mathcal{E} \to \mathcal{F} \) over \( X \times Y \), moreover, we assume \( \mathcal{F} \) is flat over \( Y \). Then there is a unique closed subscheme \( Z \) of \( Y \) satisfying the following universal property: for any scheme \( S \) and a Cartesian diagram:
\[
\begin{array}{ccc}
X \times S & \xrightarrow{h} & X \times Y \\
\downarrow ps & & \downarrow p \\
S & \xrightarrow{h} & Y
\end{array}
\]

then \( \tilde{f}^*(f) = 0 \) if and only if \( h \) factors through \( Z \).

Now we let \( Z \subset Q \times \mathbb{P}^n \) be the closed subscheme such that every closed point \( (q, \phi) : V \otimes \mathcal{O}_X(-N) \to E, \phi : V \otimes V \to H) \) of \( Z \) represents a twisted symplectic/orthogonal bundle \( E \).

So over \( Z \times C \), we have a universal quotient \( q : V \otimes p_C^* \mathcal{O}_C(-N) \to E \to 0 \) on \( X \times Z \) and a nondegenerated anti-symmetric/symmetric two form \( \omega : E \otimes E \to p_C^* \mathcal{O}_C(-D) \) where \( p_C : Z \times C \to C \) is the projection. For any \( x \in D \), let \( E_x \) be the restriction of \( E \) on \( Z \times \{x\} \cong Z \) and we put \( Flag_{n(x)}(E_x) \to Z \) be the relative isotropic flag scheme of type \( n(x) \).

Let \( R := \times_Z Flag_{n(x)}(E_x) \to Z \), then a closed point of \( R \) is represented by

\[
((q, \phi), (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D})
\]

where \((q, \phi)\) is a point of \( Z \), and \( q_i(x) \) is the composition \( q_i(x) : V \otimes \mathcal{O}_X(-N) \to E \to E_x \to Q_i(x) \). We denote by \( Q_i(x) \) the quotients \( E_x/F_i(E_x) \), and let \( r_i(x) = \dim Q_i(x) \).

For \( m \) large enough, let \( G = Grass_{P(m)}(V \otimes W_m) \times \mathbb{P}^n \) be the restriction of \( E \) on \( Z \times \{x\} \cong Z \) and \( Flag \), where, \( W_m = H^0(V \otimes \mathcal{O}(m - N)) \), and \( Flag \) is defined as:

\[
Flag = \prod_{x \in D} (Grass_{r_1(x)}(V) \times \cdots \times Grass_{r_{2l_x}(x)}(V))
\]

Now, consider the \( SL(V) \)-equivariant embedding

\[
R \hookrightarrow G = Grass_{P(m)}(V \otimes W_m) \times \mathbb{P}^n \times \mathcal{O}(V \otimes V, H) \times Flag
\]

Which maps the point \((q, \phi), (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D}) \) of \( R \) to the point

\[
(g, \phi, (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D})
\]

of \( G \), where \( g : V \otimes W_m \to H^0(E(m - N)) \) and \( q_i(x) : V \to Q_i(x) \).

We give the polarisation on \( G \) by:

\[
n_1 \times 1 \times \prod_{x \in D} \prod_{i=1}^{2l_x} d_i(x)
\]

Where \( n_1 = \frac{\ell \chi_{(m-N)}}{\ell \chi_{(m-N)}} \), \( d_i(x) = a_{i+1}(x) - a_i(x) \) and \( l \) is the number satisfying

\[
\sum_{x \in D} \sum_{i=1}^{2l_x} d_i(x) r_i(x) + rl = K \chi
\]

We will analyse the action of \( SL(V) \) on \( R \) using a method in [S]. Let \( R^s \) (resp. \( R^{ss} \)) to denote the sublocus of \( R \) where the corresponding parabolic symplectic/orthogonal bundles are stable (resp. semistable) and the map \( H^0(q) : V \to H^0(C, E(m)) \) is an isomorphism. We are going to show \( R^s \) (respectively, \( R^{ss} \)) is the stable (respectively, semistable) locus of the action in the sense of GIT. Firstly let us recall a definition in [S]:

**Definition 3.2.** A weighted filtration \((E_\bullet, m_\bullet)\) of a parabolic symplectic/orthogonal bundle \( E \) consists of

1. a filtration of subsheaves

\[
0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E
\]

We denote \( rk(E_i) \) by \( s_i \);

2. a sequence of positive numbers \( m_1, m_2, \ldots, m_t \), called the weights of this filtration.
Let $\Gamma = \sum_{i=1}^{t} m_i \Gamma^{x_i} \in \mathbb{C}^r$, where

$$\Gamma^k = (k-r,k-r,\ldots,k-r,k,\ldots,k)$$

Now, given a weighted filtration $(E_\bullet,m_\bullet)$ of a parabolic symplectic/orthogonal bundle $E$, let $\Gamma_j$ be the $j$-th component of $\Gamma$, and we define

$$\mu(\omega,E_\bullet,m_\bullet) := \min \{ \Gamma_{s_1} + \Gamma_{s_2} : |E_{s_1} \otimes E_{s_2} \neq 0 \}$$

We have the following result (see [5], Lemma 5.6):

**Lemma 3.3.** If $\omega$ is nondegenerate, then $\mu(\omega,E_\bullet,m_\bullet) \leq 0$.

**Proof.** We can take the index $i$ and $j$ such that $\mu(\omega,E_\bullet,m_\bullet) = \Gamma_{s_i} + \Gamma_{s_j}$ and $\omega|_{E_i \otimes E_j} \neq 0$. Then there exist a point $x \in C$ away from $D$ such that the restriction $\omega_x = \omega|_{E_{s_i} \otimes E_{s_j}} \neq 0$.

Let $W = E_x$. Then the nondegenerate form $\omega$ over $E$ induces a nondegenerate form over the vector space $W$. We still write this form as $\omega : W \otimes W \to \mathbb{C}$. By the nondegenerate condition, using Hilbert-Mumford criterion, one can see that $\mathcal{F} \in \mathbb{P} \text{Hom}(W \otimes W, \mathbb{C})$ is GIT semistable with the natural $SL(W)$ action. It implies that $\mu(\omega,W_\bullet,m_\bullet) \leq 0$ for all weighted filtrations of $W$.

It’s easy to see $\mu(\phi,E_\bullet,m_\bullet) \leq \mu(\omega,W_\bullet,m_\bullet)$, hence $\mu(\omega,E_\bullet,m_\bullet) \leq 0$. \hfill \Box

In the following we use Hilbert-Mumford criterion to determine the (semi)stable locus for the action of $SL(V)$ of $\mathcal{R}$.

**Proposition 3.4.** A point $((q,\phi),(q_1(x),q_2(x),\ldots,q_{2n}(x))_{x \in D})$ of $\mathcal{R}$ is GIT stable (resp. GIT semistable) for the action of $SL(V)$, with respect to the polarisation defined in definition 2.1, if and only if for all weighted filtration $(E_\bullet,m_\bullet)$, we have

$$kP(N)(\sum_{i=1}^{t} \text{pardeg}(E_i)) + \mu(\omega,E_\bullet,m_\bullet) < 0 \quad \text{(resp.} \leq \text{)}$$

**Proof.** By the Hilbert-Mumford criterion, a point $((q,\phi),(q_1(x),q_2(x),\ldots,q_{2n}(x))_{x \in D})$ is GIT semistable if and only if any one parameter subgroup $\lambda : \mathbb{G}_m \to SL(V)$, the corresponding Hilbert-Mumford weight is greater or equal than zero. But a one parameter subgroup of $SL(V)$ is equivalent to a weighted filtration of $V$ and hence gives a weight filtration $(E_\bullet,m_\bullet)$ for the corresponding bundle $E$. In terms of weight filtration for $E$, we see that the Hilbert-Mumford weight is given by

$$s(E) := n_1(\sum_{i=1}^{t} m_i(\chi(E_i(N))P(m) - P(N)\chi(E_i(m)))) + \mu(\omega,E_\bullet,m_\bullet)$$

$$+ \sum_{x \in D} \sum_{j=1}^{2l_x} d_j(x) \left( \sum_{i=1}^{t} m_i(\chi(E_i(N))r_j(x) - P(N)r_j^{E_i}(x)) \right)$$

where $r_j^{E_i}(x) := \text{dim}(\text{Im}(E_i \to E \to Q_j(x)))$. Hence the point is GIT semistable if and only if $s(E) \leq 0$.

However, one can show that (see Proposition 2.9 of [23])

$$s(E) = kP(N)(\sum_{i=1}^{t} m_i \text{pardeg}(E_i)) + \mu(\omega,E_\bullet,m_\bullet)$$

In fact, the coefficients of $m_i$ in $s(E)$ is

$$n_1(\chi(E_i(N))P(m) - P(N)\chi(E_i(m))) + \sum_{x \in D} \sum_{j=1}^{2l_x} d_j(x) \left( \chi(E_i(N))r_j(x) - P(N)r_j^{E_i}(x) \right)$$
= (r l + rKcN)(deg(Ei) − \frac{r(Ei)}{r}deg(E)) + \sum_{x \in D} \sum_{j=1}^{2l_x} d_j(x)(\chi(Ei(N))r_j(x) − P(N)r_j E_i(x))

= KP(N)(deg(Ei) − \frac{r(Ei)}{r}deg(E)) + P(N)(\frac{r(Ei)}{r} \sum_{x \in D} \sum_{j=1}^{2l_x} d_j(x)r_j(x) − \sum_{x \in D} \sum_{j=1}^{2l_x} d_j(x)r_j E_i(x))

= KP(N)(deg(Ei) − \frac{r(Ei)}{r}deg(E)) + \frac{r(Ei)}{r}P(N)(r \sum_{x \in D} a_{2l_x + 1} − \sum_{x \in D} \sum_{j=1}^{2l_x + 1} a_j(x)n_j(x))

= KP(N)(pardeg(Ei))

\square

**Proposition 3.5.** A parabolic symplectic/orthogonal bundle \( E \) is stable (resp. semistable) if and only if the correspondence point \( ((q, \phi), (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D}) \) of \( \mathcal{R} \) is GIT stable (resp. semistable) for the action of \( SL(V) \).

**Proof.** Let \( E \) be a stable (resp. semistable) bundle. For any weighted filtration \( (E_i, m_i) \), we have \( \text{pardeg}(E_i) < 0 \) (resp. \( \leq \)) by Lemma 2.3. Furthermore, by Lemma 3.3, \( \mu(\omega, E_i, m_i) \leq 0 \), hence

\[ KP(N)(\sum_{i=1}^{l}(\text{pardeg}(E_i))) + \mu(\omega, E_i, m_i) < 0 \text{ (resp. } \leq \)\]

By Proposition 3.3, this tells that the corresponding point \( ((q, \phi), (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D}) \) of \( \mathcal{R} \) is GIT stable (resp. semistable).

Conversely, let \( E \) be a parabolic orthogonal/symplectic bundle such that the corresponding point \( ((q, \phi), (q_1(x), q_2(x), \ldots, q_{2l_x}(x))_{x \in D}) \) is GIT stable (resp. GIT semistable). We want to show that \( E \) is a stable (resp. semistable). That is, for any isotropic subbundle \( F \) of \( E \), we have \( \text{pardeg}(F) < 0 \) (resp. \( \leq \)).

Since \( E \) is stable (resp. semistable), the inequality in Proposition 3.3 must hold for all weighted filtrations \( (E_i, m_i) \). In particular, if we take the weighted filtration as: \( 0 \subset F \subset F^i \subset E \), and weights \( m_1 = m_2 = 1 \), then the inequality becomes

\[ KP(N)((\text{pardeg}(F) + \text{pardeg}(F^i^i))) + \mu(\omega, E_i, m_i) < 0 \text{ (resp. } \leq \)\]

However, in this case we have \( \mu(\omega, E_i, m_i) = 0 \) and \( \text{pardeg}(F) = \text{pardeg}(F^i) \), hence we have \( \text{pardeg}(F) < 0 \) (resp. \( \leq \)).

Therefore, let \( \mathcal{R}^{ss} \subset \mathcal{R} \) be the open set of \( \mathcal{R} \) which consists of semistable parabolic orthogonal (symplectic, resp.) sheaves. In the rest of this section, we will show that \( \mathcal{R}^{ss} \) is smooth. Therefore let \( M_{G,P} = \mathcal{R}^{ss} / SL(V) \) be the GIT quotient, then we have

**Theorem 3.6.** \( M_{G,P} \) is the coarse moduli space of semistable parabolic orthogonal/symplectic sheaves of rank \( r \) and degree \( d \) with fixed parabolic type \( \sigma \). Moreover, \( M_{G,P} \) is a normal Cohen-Macaulay projective variety, with only rational singularities.

**Proof.** Since we have show that \( \mathcal{R}^{ss} \) is smooth in the next subsection, especially \( \mathcal{R}^{ss} \) is normal with only rational singularities, so is its GIT quotient \( M_{G,P} \). Finally the fact that \( \mathcal{R}^{ss} \) is regular implies that \( M_{G,P} \) is Cohen-Macaulay (see [16]).

\square

### 3.2. Smoothness of \( \mathcal{R}^{ss} \)

The smoothness of \( \mathcal{R}^{ss} \) has essentially proved in [13]. We will reformulate the proof here.

Let \( Q_F \) be the open subscheme of \( Q \) consisting of quotients \( [q : V \otimes O_X(−N) → E] \in Q \) such that \( H^1(E(N)) = 0 \). Let \( Z_F \) be the inverse image of \( Q_F \) under the projection \( Z → Q \) and \( R_F \) be the inverse image of \( Z_F \) under the projection \( R → Z \). If we can show that \( Z_F \) is smooth, then \( \mathcal{R}_F \) is smooth because is a flag bundle over \( Z_F \). Thus \( \mathcal{R}^{ss} \) is smooth as it is an
open subscheme of $Z_F$. So the smoothness of $R^{ss}$ reduce to the smoothness of $Z_F$. We will prove the smoothness of $Z_F$ in the rest part of this subsection. First of all, let us recall the definition of Atiyah bundle of a principal $G$-bundle (5).

**Definition 3.7.** Let $p : E \to X$ be a principal $G$-bundle, the Atiyah bundle $At(E)$ of $E$ is defined as $At(E)(U) := H^0(p^{-1}U, T_{p^{-1}U})$ for any open subset $U \subseteq X$.

**Proposition 3.8.** Let $p : E \to X$ be a principal $G$-bundle and $At(E)$ is the Atiyah bundle, then

1. We have an exact sequence (the Atiyah sequence): $0 \to ad(E) \to At(E) \to T_X \to 0$.
2. There is a natural isomorphism $\mu : p^*At(E) \cong T_E$.

**Proof.** See [5] section 1.

**Remark 3.9.** For the Grassmann variety $G_{n,r}$, let $p : \mathcal{A} \to G_{n,r}$ be the universal $GL(r)$ bundle and $0 \to K \to V \otimes \mathcal{O}_{G_{n,r}} \to \mathcal{A} \to 0$ be the universal exact sequence, then we have an isomorphism $At(A) \cong V \otimes \mathcal{A}$ and the Atiyah sequence becomes $0 \to \mathcal{A}^* \otimes \mathcal{A} \to V \otimes \mathcal{A} \to \mathcal{K}^* \otimes \mathcal{A} \to 0$.

Let $G \to GL(r)$ be the orthogonal/symmetric subgroup of $GL(r)$. In [18], the author has construct the moduli space of principal $G$-bundles. The author also shows that $Q_F$ and $Z_F$ can be open subschemes of some Hilbert scheme: Consider the Grassmann variety $G_{n,r}$, where $n = \text{dim} V$. Denote the universal family over $G_{n,r}$ by $\mathcal{A} \to G_{n,r}$. Let $Y = GL(r) \sslash G$ and $\mathcal{A}(Y) = \mathcal{A} \sslash G$ be the fibre bundle with fibre $Y$ over $G_{n,r}$ associated to $\mathcal{A} \to G_{n,r}$.

Then we have the following proposition:

**Proposition 3.10.** $Q_F$ is an open subscheme of $\text{Hom}(C, \mathcal{A}(Y))$ and $Z_F$ is an open subscheme of $\text{Hom}(C, \mathcal{A}(Y))$.

**Proof.** See [18] section 4.13.

**Proposition 3.11.** The semistable locus $R^{ss}$ is smooth.

**Proof.** As mentioned before, we just need to show that $Z_F$ is smooth. By proposition 3.10, $Z_F \subset \text{Hom}(C, \mathcal{A}(Y))$. So let $f : C \to \mathcal{A}(Y)$ be a point of $Z_F$. We need to show that $Z_F$ is smooth at $f$, that is $\text{Hom}(C, \mathcal{A}(Y))$ is smooth at $f$. However by associating $f$ to the graph $\Gamma_f$, we may consider $\text{Hom}(C, \mathcal{A}(Y))$ as an open subscheme of $\text{Hilb}(C \times \mathcal{A}(Y))$. Hence we should prove the that $\text{Hilb}(C \times \mathcal{A}(Y))$ is smooth at $\Gamma_f$. By obstruction theory, this is equivalent to show that $H^1(\Gamma_f, N_{\Gamma_f}) = 0$ where $N_{\Gamma_f}$ is the normal bundle of $\Gamma_f$ in $C \times \mathcal{A}(Y)$. However, since $\Gamma_f \simeq C$ by the projection $C \times \mathcal{A}(Y) \to C$, we have $N_{\Gamma_f} \simeq f^*T_{\mathcal{A}(Y)}$ where $T_{\mathcal{A}(Y)}$ is the tangent bundle of $\mathcal{A}(Y)$. We will show that $H^1(C, f^*T_{\mathcal{A}(Y)}) = 0$.

Let $p : \mathcal{A}(Y) \to G_{n,r}$ and $q : \mathcal{A} \to G_{n,r}$ be the canonical maps. Denote by $\theta : \mathcal{A} \to \mathcal{A}(Y) = \mathcal{A} \sslash G$ the natural quotient. It is easy to see $q = p \circ \theta$. Then we get the following exact sequence by taking the differential of the projection $\theta : \mathcal{A} \to \mathcal{A}(Y)$

$$0 \to \mathcal{M} \to \theta_*T_{\mathcal{A}} \to T_{\mathcal{A}(Y)} \to 0$$

On the other hand, by proposition 3.8, we have

$$T_{\mathcal{A}} \cong q^*At(\mathcal{A}) = (p \circ \theta)^*At(\mathcal{A}) = \theta^*p^*At(\mathcal{A})$$

So we have a surjective map

$$\theta_*\theta^*p^*At(\mathcal{A}) \to T_{\mathcal{A}(Y)}$$

Compose the above map with the canonical map $p^*At(\mathcal{A}) \to \theta_*\theta^*p^*At(\mathcal{A})$, we get a surjective morphism

$$p^*At(\mathcal{A}) \to T_{\mathcal{A}(Y)}$$

Assume the kernel is $\mathcal{N}$, then we have an exact sequence

$$0 \to \mathcal{N} \to p^*At(\mathcal{A}) \to T_{\mathcal{A}(Y)} \to 0$$
Taking the pullback functor $f^*$, we get an exact sequence over $C$
\[ 0 \rightarrow f^*N \rightarrow f^*p^*At(A) \rightarrow f^*T_{At(Y)} \rightarrow 0 \]
We get an exact sequence of cohomologies from the above sequence
\[ H^1(C, f^*N) \rightarrow H^1(C, f^*p^*At(A)) \rightarrow H^1(C, f^*T_{At(Y)}) \rightarrow 0 \]
\[ \text{(3.1)} \]
However, by Remark 3.9 we have $At(A) \cong V \otimes A$. So
\[ f^*p^*At(A) \cong (p \circ f)^*(V \otimes A) \]
Notice that $p \circ f \in Z_F$, so it correspondence to a quotient bundle
\[ 0 \rightarrow F_f \rightarrow V \otimes O_C(-N) \rightarrow E_f \rightarrow 0 \]
Then we have $(p \circ f)^*(V \otimes A) \cong V \otimes E_f(N)$, so we have that
\[ H^1(C, f^*p^*At(A)) \cong H^1(C, V \otimes E_f(N)) = 0 \]
\[ \text{(3.2)} \]
Combine with (3.1) and (3.2) we finally get $H^1(C, f^*T_{At(Y)}) = 0$. \qed

4. Codimention estimate
In this section, we fix $S$ to be a scheme of finite type. Let $E$ be a flat family of vector bundle, principal $G$ bundle, parabolic vector bundle or parabolic symplectic/orthogonal bundle over $C$ parametrized by $S$, under certain conditions, we want to estimate the codimension of the unstable (unsemistable) locus, i.e. the locally closed subscheme $S^{us} \subset S$ ($S^{uss} \subset S$) parametrizing all $E_t$ which is not stable (semistable). Our main method is taken from [10].

4.1. The case of vector bundle and principal $G$ bundle. In fact, the case of vector bundle and principal $G$ bundle have been already done in [10] and [11]. For later use, we reformulate the results and give a short proof if necessary.

All the stories begin with the following proposition:

**Proposition 4.1.** $E$ is a flat family of vector bundles over $S \times C$. Let $\phi : Q \rightarrow S$ be the relative Quot-scheme parametrizing all flat quotients of $E$ with certain fixed rank and degree. For any $s \in S$ and $q \in \phi^{-1}(s)$, corresponding to exact sequence:
\[ 0 \rightarrow F \rightarrow E_s \rightarrow G \rightarrow 0 \]
we have the following exact sequence:
\[ 0 \rightarrow \text{Hom}(F, G) \rightarrow T_qQ \rightarrow T_sS \rightarrow \text{Ext}^1(F, G). \]
\[ \text{(4.1)} \]
**Proof.** See [9] Proposition 2.2.7. \qed

Let $E$ be a vector bundle over $C$, the classical Harder-Narasimhan filtration and Jordan-Holder filtration show that if $E$ is not stable (resp. semistable), then there is a maximal stable subbundle $F_0 \subset E$ with the property $\deg \text{Hom}(F_0, E/F_0) \leq 0$ (resp. $< 0$). $F_0$ is taken to be the first term of the Jordan-Holder filtration of the maximal destabilizing subbundle of $E$ (so different choice of $F_0$ have same slope). Moreover, if we say $F_0$ is of type $\mu = (r', d')$, i.e. $F$ is of rank $r'$ and degree $d'$. Then for a flat family of vector bundle $E$ over $S \times C$, the locus $S^\mu \subset S$ parametrizing $E_t$ having a subbundle described above with type $\mu$, is locally closed and non-empty for finitely many $\mu$.

Similarly properties hold for principal $G$ bundles. Let $E$ be a principal $G$ bundle, then there is a unique standard parabolic subgroup $P$ and a unique reduction $E_P$, and if we denote $E_s$ to be the vector bundle associated to $E_P$ by the natural representation of $P$ on the vector space $s := g/p$, where $g$ and $p$ are Lie algebras of $G$ and $P$, then $\deg E_s < 0$. More over, we have similar concept of $S^\mu$. For details and proof, please refer to [11].
Proposition 4.4. Let $E$ be a flat family of vector bundles or principal $G$ bundles over $S \times C$. Assume that for each closed point $t \in S$, the Kodaira-Spencer maps

$$T_tS \to \text{Ext}^1(\mathcal{E}_t, \mathcal{E}_t) \quad \text{or} \quad T_tS \to H^1(C, \mathcal{E}_t(Ad))$$

are surjective. Then:

1. In the vector bundle case, for any $s \in S^\mu$, the normal space $N_sS^\mu$ is isomorphic to $\text{Ext}^1(F_0, \mathcal{E}_s/F_0)$, where $F_0$ is a maximal stable bundle described above.
2. In the principal $G$ bundles case, for any $s \in S^\mu$, the normal space $N_sS^\mu$ is isomorphic to $H^1(C, \mathcal{E}_s)$ where $\mathcal{E}_{s, C}$ is described above.

Proof. For the vector bundle case, we first consider the Quot-scheme $\phi : Q \to S$ parametrizing all subbundles of type $\mu$, then analyse the exact sequence $\mathcal{E}_s$. Firstly the image of $\phi$ covers $S^\mu$, we see that the map $T_qQ \to T_sS$ factors as $T_qQ \to T_sS^\mu \to T_sS$. Secondly, by the proof of exactness of $\mathcal{E}_s$, we see that the map $T_sS \to \text{Ext}^1(F_0, \mathcal{E}_s/F_0)$ indeed factors as $T_sS \to \text{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \to \text{Ext}^1(F_0, \mathcal{E}_s/F_0)$.

The first map is Kodaira-Spencer map which is surjective by assumption; the second map is induced by the exact sequence:

$$0 \to F_0 \to \mathcal{E}_s \to \mathcal{E}_s/F_0 \to 0$$

which is surjective naturally. Thus we see that $\text{Ext}^1(F_0, \mathcal{E}_s/F_0)$ is isomorphic to the cokernel of $T_qQ \to T_sS$, i.e. the normal space $N_sS^\mu$.

The principal bundle case is similar, except we need a variety to parametrize all reductions to $P$. But this is already done in [ES], it is an open subscheme $U$ of $\text{Hilb}(\mathcal{E}/P)/S$, parametrizing all sections of $\mathcal{E}/P \to S$. Now we apply Proposition 4.4 to this $U$, with similar method above, we have our proposition.

Corollary 4.3. With same notation and assumptions as above, if we assume $S$ is smooth, we have:

1. In the vector bundle case, the rank of $\mathcal{E}$ is assumed to be $r$, then we have

$$\text{codim}(S^\mu) \geq (r - 1)(g - 1)$$

2. In the principal bundle case, we have

$$\text{codim}(S^\mu) > (r - 1)(g - 1)$$

Proof. Since $S^\mu$ is non-empty for only finitely many $\mu$, by proposition above, we only need to calculate $\text{dimExt}^1(F_0, \mathcal{E}_s/F_0)$ and $\text{dim} H^1(C, \mathcal{E}_s)$. Using Riemann-Roch, we have

$$\text{dimExt}^1(F_0, \mathcal{E}_s/F_0) = \text{dimHom}(F_0, \mathcal{E}_s/F_0) - \text{degHom}(F_0, \mathcal{E}_s/F_0) + r'(r - r')(g - 1)$$

$$\text{dim} H^1(C, \mathcal{E}_s) = \text{dim} H^0(C, \mathcal{E}_s) - \text{deg} \mathcal{E}_s + \text{rank} \mathcal{E}_s(g - 1)$$

where $r'$ is the rank of $F$. Thus our corollary holds by analyse of degrees of $\text{Hom}(F_0, \mathcal{E}_s/F_0)$ and $\mathcal{E}_s$ before.

4.2. The case of parabolic vector bundle. We fix $\mathcal{E}$ to be a flat family of parabolic vector bundles of type $\sigma$ over $S \times C$. To apply our method to parabolic vector bundle case, we need to construct an $S$-scheme parametrizing all flat quotients of $\mathcal{E}$, with fixed parabolic type $\sigma'$.

We begin with a functor

$$F : (\text{Sch}/S)^{op} \to (\text{Set})$$

as follows: for any $f : T \to S$, $F(f : T \to S)$ is the set of isomorphism classes of all quotients $f^*_s \mathcal{E} \to \mathcal{G} \to 0$, such that the induced parabolic structure on $\mathcal{G}$ makes $\mathcal{G}$ a flat family of parabolic vector bundle of rank $r'$ and degree $d'$ with fixed type $\sigma'$.

Proposition 4.4. $F$ is represented by a finite type scheme $\phi_P : Q_P \to S$. 

that parabolic bundle which is not stable (resp. semistable), there is a maximal stable subbundle. The Holder filtration for parabolic bundles. So similar as in the previous subsection, for a parabolic bundle as above, we have
\[ \text{Hom}(\tilde{\mathcal{E}}^1, \mathcal{G}) \rightarrow 0 \]
where \( \mathcal{G} \) is a flat family of parabolic vector bundles with type \( \sigma' \). Notice that this is a quotient since taking \( \Gamma \)-invariant sections of \( \mathbb{C} \)-modules is an exact functor.

**Remark 4.5.** In [7], a similar scheme is constructed in a different way.

**Corollary 4.6.** For any \( s \in S \) and \( q \in \phi_p^{-1}(s) \), corresponding to exact sequence:
\[ 0 \rightarrow F \rightarrow \mathcal{E}_s \rightarrow G \rightarrow 0 \]
Then we have an exact sequence:
\[ 0 \rightarrow \text{Hom}_{par}(F, G) \rightarrow T_q Q_P \rightarrow T_s S \rightarrow \text{Ext}^1_{par}(F, G). \]

**Proof.** Let \( 0 \rightarrow \tilde{F} \rightarrow W_s \rightarrow \tilde{G} \rightarrow 0 \) be the corresponding exact sequence of orbifold bundles over \( Y \). When we regard \( q \) as a point of \( Q \), apply the exact sequence \[ 4.1 \]
we have an exact sequence:
\[ 0 \rightarrow \text{Hom}(\tilde{F}, \tilde{G}) \rightarrow T_q Q \rightarrow T_s S \rightarrow \text{Ext}^1(\tilde{F}, \tilde{G}) \]
However, this sequence is in fact a \( \Gamma \)-exact sequence, Thus we have:
\[ 0 \rightarrow \text{Hom}(\tilde{F}, \tilde{G})^\Gamma \rightarrow (T_q Q)^\Gamma \rightarrow T_s S \rightarrow \text{Ext}^1(\tilde{F}, \tilde{G})^\Gamma \]
which is exact since taking \( \Gamma \)-invariant sections of \( \mathbb{C} \)-modules is an exact functor. Now, it is known that \( \text{Hom}(\tilde{F}, \tilde{G})^\Gamma = \text{Hom}_{par}(F, G) \) and \( (T_q Q)^\Gamma = T_q Q_P \). Finally, spectral sequence argument tells \( \text{Ext}^1(\tilde{F}, \tilde{G})^\Gamma = \text{Ext}^1_{par}(F, G) \), we are done.

Before going further, we mention that there are Harder-Narasimhan filtration and Jordan-Hölder filtration for parabolic bundles. So similar as in the previous subsection, for a parabolic bundle which is not stable (resp. semistable), there is a maximal stable subbundle \( F_0 \) such that \( \text{pardeg} \text{Hom}_{par}(F_0, E/F_0) \leq 0 \) (resp. < 0). Moreover, for a family of parabolic vector bundle as above, \( S^\mu \) defined as before, is locally closed and non-empty for finitely many \( \mu \).

**Proposition 4.7.** Assume that for any \( t \in S \), the Kodaira-Spencer map
\[ T_t S \rightarrow \text{Ext}^1_{par}(\mathcal{E}_t, \mathcal{E}_t) \]
is surjective. Let \( S^\mu \subset S \) be the locally closed described before. Then for any \( s \in S^\mu \), we have \( N_s S^\mu \cong \text{Ext}^1_{par}(F_0, \mathcal{E}_s/F_0) \).

**Proof.** Similar as Proposition [12]

**Corollary 4.8.** With same assumption as above, assuming that \( S \) is smooth and rank \( \mathcal{E} = r \) we have
\[
\text{codim}(S^{us}) \geq \text{deg}D/K + (r - 1)(g - 1) \\
\text{codim}(S^{uss}) > \text{deg}D/K + (r - 1)(g - 1).
\]
Proof. As before, it suffice to estimate \( \dim \text{Ext}^1_{\text{par}}(F_0, \mathcal{E}_s/F_0) \). By [26], we have \( \text{Ext}^1_{\text{par}}(F_0, \mathcal{E}_s/F_0) = H^1(C, \mathcal{H}om_{\text{par}}(F_0, \mathcal{E}_s/F_0)) \), so

\[
\dim \text{Ext}^1_{\text{par}}(F_0, \mathcal{E}_s/F_0) = \dim \mathcal{H}om_{\text{par}}(F_0, \mathcal{E}_s/F_0) - \deg \mathcal{H}om_{\text{par}}(F_0, \mathcal{E}_s/F_0) + r'(r - r')(g - 1)
\]

Since \( \text{par} \deg \mathcal{H}om_{\text{par}}(F_0, \mathcal{E}_s/F_0) \leq 0 \). We see that \(-\deg \mathcal{H}om_{\text{par}}(F_0, \mathcal{E}_s/F_0) \geq \deg D/K \). This would give our results. \( \square \)

**Remark 4.9.** Similar results have been given in [22] by a different way.

4.3. **The case of parabolic symplectic/orthogonal bundle.** The case of parabolic symplectic/orthogonal bundles is similar to those in former two sections, but we need define some notions first.

Let \( E \) be a parabolic symplectic bundle over \( C \), and \( W \) be the corresponding orbifold symplectic bundle over \( Y \). By the constructions before, we have \( W(Ad) \) and \( W_s \) for \( s = g/p \). \( W \) is an orbifold symplectic bundle, so \( W(Ad) \) and \( W_s \) are both orbifold vector bundles over \( Y \). We use \( E(Ad) \) and \( E_s \) to denote corresponding parabolic vector bundles over \( C \).

For any family of parabolic symplectic bundle \( E \) over \( C \) parametrized by a scheme \( S \), let \( W \) be the corresponding orbifold symplectic bundle on \( S \times Y \). For any \( t \in S \), we have the Kodaira-Spencer map

\[
T_tS \longrightarrow H^1(Y, W_t(Ad))
\]

for \( W \). This map is obviously \( \Gamma \)-invariant, so we have

\[
T_tS \longrightarrow H^1(Y, W_t(Ad))^{\Gamma} = H^1(C, \mathcal{E}_t(Ad))
\]

**Definition 4.10.** The Kodaira-Spencer map for \( \mathcal{E} \) at \( t \in S \) is given by

\[
T_tS \longrightarrow H^1(C, \mathcal{E}_t(Ad)).
\]

**Proposition 4.11.** Let \( S \) and \( \mathcal{E} \) be as before. Then there is a scheme \( \psi_{PS} : Q_{PS} \to S \) parametrizing all isotropic subbundles of \( \mathcal{E} \), flat over \( S \) with same fixed type \( \tau' \).

Moreover, for any \( s \in S \) and \( q \in \psi_{PS}^{-1}(s) \), corresponding to an isotropic subbundle \( F \subset \mathcal{E}_s \), which corresponds to a reduction to a parabolic subgroup \( P \) of \( W_s \), we have an exact sequence:

\[
0 \longrightarrow H^0(C, \mathcal{E}_{s,s}) \longrightarrow T_q Q_{PS} \longrightarrow T_s S \longrightarrow H^1(C, \mathcal{E}_{s,s})
\]

**Proof.** Similar to Corollary 4.6. \( \square \)

With similar method, we can show that:

**Corollary 4.12.** With notations as before, assume that the Kodaira-Spencer map is surjective for any \( s \in S \), then we have

\[
\text{codim}(S^{\text{ss}}) \geq \deg D/K + \text{rank}(\mathcal{E}_{s,s})(g - 1)
\]

\[
\text{codim}(S^{\text{usc}}) > \deg D/K + \text{rank}(\mathcal{E}_{s,s})(g - 1).
\]

5. **Infinite Grassmannians and the theta line bundle**

5.1. **Infinite Grassmannians.** In this subsection, we use \( G \) to denote a connected simply connected simple affine algebraic group, and the parabolic \( G \) bundle over \( C \) we considered in this subsection is given by a principal \( G \) bundle \( E \) together with choices of one parameter subgroups in \( E(G)_x \) for every \( x \in D \); a quasi-parabolic \( G \) bundle is just a choice of choices of parabolic subgroups of \( E(G)_x \), i.e. (quasi-)parabolic \( G \) bundles in the sense of [33].

We fix a point \( p \in X \), away from \( D \), let \( C^* = C - p \), following [12], we define

\[
\mathcal{G} = G(\hat{C}_p)
\]

\[
\mathcal{P} = G(\hat{P}_p)
\]

\[
\Lambda = G(\mathbb{C}[C^*])
\]

where \( \hat{O}_p \) is the completion of local ring \( O_p \) of \( p \in C \); \( \hat{C}_p \) is the field of quotient of \( \hat{O}_p \); \( \mathbb{C}[C^*] \) is the coordinate ring of \( C^* \). Similarly in [12], we have
Proposition 5.1. If we use \( \mathcal{X} \) to denote the set of isomorphism classes of quasi-parabolic \( G \) bundle with parabolic structure \( P_x \) at each \( x \in D \), we have a bijection of sets:
\[
\alpha : \Lambda \setminus (\mathcal{G}/\mathcal{P} \times \prod_{x \in D} G/P_x) \longrightarrow \mathcal{X}
\]

Proof. By proposition 1.5 of [12], there is a bijection between \( \Lambda \setminus \mathcal{G}/\mathcal{P} \) and the set of isomorphism classes of \( G \) bundles. Notice that a quasi-parabolic \( G \) bundle is nothing but a parabolic \( G \) bundle plus a point in \( \prod_{x \in D} G/P_x \), we have our bijection. □

Recall that in [12] the generalized flag variety \( X := \mathcal{G}/\mathcal{P} \) has a structure of ind-variety, more precisely,
\[
X = \lim_{\rightarrow} X_{\sigma}
\]
where \( X_{\sigma} \) are the generalised Schubert varieties they defined there. Moreover, there is an algebraic \( G \) bundle \( \mathcal{U} \to C \times X \) such that \( \mathcal{U}|_{C^* \times X} \) is trivial. So, for any \( x \in D \), we have a trivial \( G \) bundle \( \mathcal{U}_x \) over \( X \), then we define
\[
X_P = X \times \prod_{x \in D} G/P_x
\]
to be the relative flag variety over \( X \) defined by \( \{U_x\}_{x \in D} \). Let \( \pi : X_P \to X \) be the natural projection.

Proposition 5.2. There is a quasi-parabolic \( G \) bundle \( \mathcal{U}_P \) over \( C \times X_P \) such that for any \( x \in X_P \), the quasi-parabolic bundle \( (\mathcal{U}_P)_x := \mathcal{U}_P|_{C \times x} \) is exactly the parabolic \( G \) bundle corresponds to \( x \) through the bijection in Proposition 5.1. Moreover the bundle \( \mathcal{U}|_{C^* \times X_P} \) carries a trivialization \( \varepsilon : \tau \to \mathcal{U}|_{C^* \times X_P} \), where \( \tau \) is a trivial quasi-parabolic \( G \) bundle over \( C^* \times X_P \).

For any scheme \( T \) and any family of parabolic \( G \) bundle \( \mathcal{F} \) over \( C \times T \), if \( \mathcal{F}|_{C^* \times T} \) and \( \mathcal{F}|_{\text{Spec} \mathcal{O}_p \times T} \) are both trivial. Then if we choose a trivialization \( \varepsilon : \tau' \to \mathcal{F}|_{C^* \times T} \), we would have a Schubert variety \( X_{\sigma} \), and a morphism \( f : T \to X_{\sigma} \times \prod_{x \in D} G/P_x \) such that \( \varepsilon \) is exactly the trivialization pulled back from \( \varepsilon \) by \( f \).

Proof. This is just a parabolic analogy of proposition 2.8 in [12]. The quasi-parabolic \( G \) bundle \( \mathcal{U}_P \) is given by \( \pi^* \mathcal{U} \) with quasi-parabolic structure determined by universal property of flag variety.

To see the existence of the morphism \( f \), we firstly observe that by proposition 2.8 in [12], we have a morphism \( f' : T \to X \) Now since \( \mathcal{F}|_{C^* \times T} \) is trivial, we would have a point in \( \prod_{x \in D} G/P_x \) determined by this trivial parabolic \( G \) bundle. □

Corollary 5.3. There is an open subset \( X_P^{ss} \subset X_P \) and a morphism \( \phi : X_P^{ss} \to M_{G,P} \) to the moduli space of semistable \( G \) bundles.

By proposition 5.1, for any point \( m \in M_{G,P} \), the fibre \( \phi^{-1}(m) \) is a union of certain \( \Lambda \)-orbits. Next, we analyse the closure of these orbits.

Lemma 5.4. Let \( E \) be a semistable parabolic \( G \) bundle on \( C \) and we consider \( gr(E) \) defined in Proposition 3.1 of [3]. Then there exists a family of parabolic \( G \) bundle \( \mathcal{E} \) on \( C \times \mathbb{A}^1 \) such that:
- (a) \( \mathcal{E}|_{C \times (\mathbb{A}^1 \setminus \{0\})} \cong p_0^*(E) \), \( \mathcal{E}|_{C \times \{0\}} \cong gr(E) \) and
- (b) \( \mathcal{E}|_{C^* \times \mathbb{A}^1} \) and \( \mathcal{E}|_{\text{Spec} \mathcal{O}_p \times \mathbb{A}^1} \) are both trivial.

Where \( p_C \) is the projection from \( C \times \mathbb{A}^1 \) to \( C \).

Proof. This is the parabolic analogy of Proposition 3.7 of [12]. Proof is similar and we omit the it here. □

Now we have the following:

Proposition 5.5. The morphism \( \phi^* : \text{Pic}(M_{G,P}) \to \text{Pic}(X_P^{ss}) \) is injective.
There is an exact sequence:

\[ 0 \to \prod_{x \in D} T_{x_z}(G/P_x) \xrightarrow{f} D_E \xrightarrow{g} H^1(C, E(Ad)) \to 0 \]

where \( T_{x_z}(G/P_x) \) is the tangent space of \( G/P_x \) at \( t_x \).
Recall that $H^1(C, E(Ad))$ is the infinitesimal deformation space of $E$ as a twisted $G$ bundle, so the morphism $g$ is given by forgetting parabolic structures. Since every twisted $G$ bundle can be equipped with any parabolic structure, $g$ is a surjection.

To determine the kernel of $g$, we need to figure out how many parabolic structures we can impose on $E$ so that the restriction to $C$ are the parabolic structures $\{t_x \in G/P_x\}$. The question is local, so it is equivalent to find a parabolic subgroup $\tilde{P}_x \subset G(\mathbb{C}[\epsilon]/(\epsilon^2))$ such that $\tilde{P}_x|_x = t_x \in G/P_x$. The space of such groups is exactly $\prod_{x \in D} T_{t_x}(G/P_x)$. □

Corollary 5.8. For any family of stable parabolic $G$ bundle $F$ over $S \times C$, let $\pi_S : S \times C \rightarrow S$ be the projection and $\varphi_S : S \rightarrow M_{G,P}$ be the induced map, then

$$\varphi_S^*(\omega_{M,G,P}^{-1}) = \det(R\pi_S^*F(Ad))^{-1} \otimes \bigotimes_{x \in D} \left\{ \det(Q_x(F_{S\times x}))^{m_i(x)} \right\}$$

where $m_i(x) = n_i(x) + n_{i+1}(x)$ for $1 \leq x \leq l_x - 1$; $m_{l_x} = n_{l_x} + n_{l_x+1} + 1$ for $G = Sp(2n)$; $m_{l_x} = n_{l_x} + n_{l_x+1} - 1$ for $G = SO(2n)$ and $m_{l_x} = n_{l_x} + n_{l_x+1} + 1$ for $G = SO(2n+1)$.

The main results in this section is to under certain choices of weights, the moduli space of parabolic symplectic/orthogonal bundles is Fano varieties. A normal projective variety $X$ is call Fano if $\omega_X^{-1}$ is an ample line bundle. Our method is to compare the pull back of anti-canonical line bundle over $M_{G,P}$ to $X^ss_P$ with theta line bundle over $X^ss_P$. It is known that the Picard group of moduli space of symplectic/orthogonal bundles has rank one, so there exists positive integer $\chi_G$ such that $\det(R\pi_S^*F(Ad)) \cong (\det R\pi_S^*F)^{\otimes \chi_G}$. For $G = Sp(2n)$, $\chi_G = n + 1$, for $G = SO(2n)$, $\chi_G = 2n - 2$ and for $G = SO(2n+1)$, $\chi_G = 2n - 1$.

We first deal with symplectic case, since symplectic groups are simply connected. Combine Proposition 5.5 and Theorem 5.6 together, we have:

Proposition 5.9. Let $G = Sp(2n)$, $K = 2\chi_G$ and $a^*(x)$ satisfying $a_{i+1}(x) - a_i(x) = m_i(x)$ for $1 \leq i \leq l_x$, the moduli space of parabolic symplectic bundles are Fano.

Proof. We show that under the condition in the proposition, $\Theta_{D_i}$ is equal to $\omega_{M,G,P}^{-2\chi_G}$. The problem here is that we do not know whether $M_{G,P}$ is Gorenstein or not, i.e. whether $\omega_{M,G,P}$ is a line bundle. But we do know that $M_{G,P}$ is Cohen-Macaulay and normal. Let $M^o \subset M_{G,P}$ be the open subset where $\omega_{M,G,P}$ is a line bundle and points in $M^o$ representing stable bundles, then we have $\text{codim}(M_{G,P} \setminus M^o) \geq 2$. Apply Proposition 5.9 to $M^o$ we see that $\omega_{M,G,P}^{-2\chi_G}$ and $\Theta_{D_i}$ are coincide over $M^o$. Now we use Lemma 2.7 of [11], and we see that $\omega_{M,G,P}$ is a line bundle, moreover, $M_{G,P}$ is a Fano variety. □

The special orthogonal group case is different, since $SO(n)$ is not simply connected, and its universal cover is $Spin(n)$. For any one parameter subgroup of $SO(n)$, we choose a lift to be a one parameter subgroup of $Spin(n)$. Then if we consider the moduli space of parabolic $Spin(n)$ bundles with parabolic structure given by the lifts, by Lemma 1.4 of [3], we would have a natural map: $t : M_{Spin(n),P} \rightarrow M_{SO(n),P}$ which identifies $M_{SO(n),P}$ as a quotient by a finite group of $M_{Spin(n),P}$. By discussion in the section 6 of [2], we have:

Proposition 5.10. The map between Picard groups: $t^* : Pic(M_{SO(n),P}) \rightarrow Pic(M_{Spin(n),P})$ is injective on the subgroup of infinite order elements.

Similar as before, we have:

Proposition 5.11. Let $G = SO(n)$, $K = 2\chi_G$ and $a^*(x)$ satisfying $a_{i+1}(x) - a_i(x) = m_i(x)$ for $1 \leq i \leq l_x$, the moduli space of parabolic special orthogonal bundles are Fano.

6. Globally $F$ regular type varieties and Main theorem

Let $k$ be a perfect field of $\text{char}(k) = p > 0$ and $X$ be a normal variety over $k$. Consider

$$F : X \rightarrow X$$
to be the absolute Frobenius map and \( F^e : X \to X \) to be the \( e \)-th iteration of \( F \).

For any Weil divisor \( D \in \text{Div}(X) \), we have a reflexive sheaf
\[
\mathcal{O}_X(D) = j_\ast \mathcal{O}_{X^{sm}}(D)
\]
where \( j : X^{sm} \hookrightarrow X \) is the inclusion of smooth locus, and \( \mathcal{O}_X(D) \) is an invertible sheaf if and only if \( D \) is a Cartier divisor.

**Definition 6.1.** Let \( X \) and \( D \) be as above, \( X \) is called **stably Frobenius D-split** if the natural homomorphism
\[
\mathcal{O}_X \to F^e_\ast \mathcal{O}_X(D)
\]
is split as an \( \mathcal{O}_X \) homomorphism for some \( e > 0 \). And \( X \) is called **globally F-regular** if \( X \) is stably Frobenius \( D \)-split for any effective divisor \( D \).

We state the following lemma about globally F-regular varieties, for proof and more details, please refer to [25], [21].

**Lemma 6.2** (Corollary 6.4 of [19]). Let \( f : X \to Y \) be a morphism of normal varieties over \( k \). Assume that the natural map
\[
f^\# : \mathcal{O}_Y \to f_\ast \mathcal{O}_X
\]
splits as an \( \mathcal{O}_Y \) homomorphism, then if \( X \) is globally F-regular, so is \( Y \).

Now we let \( K \) be a field of characteristic zero.

For any scheme \( X \) over \( K \), there is a finitely generated \( \mathbb{Z} \)-algebra \( R \subset K \) such that \( X \) is ”defined” over \( R \). That is, there is a flat \( R \)-scheme
\[
X_R \to S = \text{Spec} R
\]
such that \( X_K := X_R \times_S \text{Spec} K \cong X \). \( X_R \to S \) is called an integral model of \( X/K \). For any closed point \( s \in S \), \( X_s := X_R \times_S \text{Spec}(k(s)) \) is called the ”modulo \( p \) reduction” of \( X \), where \( p = \text{char}(k(s)) > 0 \).

**Definition 6.3.** A variety \( X \) over \( K \) is called of **globally F-regular type** if its ”modulo \( p \) reduction” of \( X \) are globally F-regular for a dense set of \( p \) for some integral model \( X_R \to S \).

Globally F-regular type varieties have many nice properties, which we will state some of them as the following theorem. Again, for proof and more details, please refer to [25] and [21].

**Theorem 6.4.** Let \( X \) be a projective variety over \( K \), if \( X \) is of globally F-regular type, then:

1. \( X \) is normal, Cohen-Macaulay with rational singularities. If \( X \) is \( \mathbb{Q} \)-Gorenstein, then \( X \) has log terminal singularities.
2. For any nef line bundle \( L \) over \( X \), we have \( H^i(X, L) = 0 \), for any \( i > 0 \). In particular, \( H^i(X, \mathcal{O}_X) = 0 \) for any \( i > 0 \).

Our main theorem of this paper is:

**Theorem 6.5.** The moduli space of parabolic symplectic/orthogonal bundles \( M_P \) over a smooth projective curve \( C \) over \( \mathbb{C} \) is of globally F-regular type.

**Corollary 6.6.** Let \( \Theta_{D_i} \) be the theta line bundle over \( M_{G,P} \) define before, then
\[
H^i(M_P, \Theta_{D_i}) = 0
\]
for any \( i > 0 \).

Our beginning example of globally F-regular type variety is Fano variety.

**Proposition 6.7** (Proposition 6.3 in [21]). A Fano variety over \( K \) with at most rational singularities is of globally F-regular type.

With our beginning example, the next step is to ask whether Lemma 6.2 holds in characteristic zero. To answer such question, in [25], they introduced the following:
Definition 6.8. A morphism $f : X \to Y$ of varieties over $K$ is called $p$-compatible if there is an integral model
\[ f_R : X_R \to Y_R \]
such that, if for any $s \in S = \text{Spec} R$, we put $X_s = X_R \times_S \text{Spec}(s)$, $Y_s = Y_R \times_S \text{Spec}(s)$ and consider
\[
\begin{array}{c}
X_s \\
\downarrow j_s \\
Y_s
\end{array}
\xymatrix{
& X_R \\
Y_R 
\ar[ur]^{f_R} & \\
& Y_s
\ar[ul]_{f_s}}
\]
then we have that $i_s^* f_* \mathcal{O}_{X_R} = f_{ss} j_* \mathcal{O}_{X_R}$ holds for a dense set of $s$.

It can be shown that if $f : X \to Y$ is a flat proper morphisms such that $R^i f_* \mathcal{O}_X = 0$ for all $i \geq 1$, then $f$ is $p$-compatible.

To prove our main theorem, we need to introduce a key proposition from [25].

Let $(\mathcal{R}', L')$ and $(\mathcal{R}, L)$ be two polarized projective varieties over $K$, with linear actions by a reductive group scheme $G$ over $K$ respectively. We use $(\mathcal{R}')^{ss}(L') \subseteq \mathcal{R}'$ and $\mathcal{R}^{ss}(L) \subseteq \mathcal{R}$ to denote the GIT semistable locus, then there are projective GIT quotients:
\[ \psi : \mathcal{R}^{ss}(L) \to Y := \mathcal{R}^{ss}(L)/G, \varphi : (\mathcal{R}')^{ss}(L') \to Z := (\mathcal{R}')^{ss}(L')/G \]

**Proposition 6.9** (Proposition 2.10 of [25]). Let $\mathcal{R}, \mathcal{R}'$ as above. Considering the following diagram, assume

1. there is a $G$-invariant $p$-compatible morphism $\tilde{f} : \mathcal{R}' \to \mathcal{R}$ such that $\tilde{f}_* \mathcal{O}_{\mathcal{R}'} = \mathcal{O}_\mathcal{R}$;
2. there is a $G$-invariant open subset $W \subset (\mathcal{R}')^{ss}(L')$ such that
\[ \text{Codim}(\mathcal{R}' \setminus W) \geq 2, \hat{X} = \varphi^{-1}(\hat{X}) \]
where $\hat{X} = W \cap \tilde{f}^{-1}(\mathcal{R}^{ss}(L))$. And we put $X = \varphi(\hat{X})$.

Then if $Z$ is of globally F-regular type, so is $Y$.

Finally, we will prove our main theorem:

**Proof of Theorem**

We choose an effective divisor $D'$ of $C$ such that $D' \cap D = \emptyset$, $\deg D'$ being even and
\[ \frac{\deg(D) + \deg(D')}{2\chi_G} + (r - 1)(g - 1) \geq 2 \]
and for each $x \in D'$, we put $\overline{u}(x) = (1, \cdots, 1)$. Let $Z'$ be the scheme parametrizing symplectic/orthogonal bundles $(E, \omega)$ where $\omega : E \otimes E \to \mathcal{O}_C(-D - D')$ as we constructed in section 3. We see that $Z' \cong Z$. Then we let
\[ \mathcal{R}' = \times_{x \in D \cup D'} \text{Flag}_{\overline{u}(x)}(\mathcal{F}_x) = \mathcal{R} \times_{\mathcal{R}} \left( \times_{x \in D'} \text{Flag}_{\overline{u}(x)}(\mathcal{F}_x) \right) \xrightarrow{\tilde{f}} \mathcal{R}. \]

So $\tilde{f} : \mathcal{R}' \to \mathcal{R}$ is a flag bundle and hence $p$-compatible with $\tilde{f}_* \mathcal{O}_{\mathcal{R}'} = \mathcal{O}_\mathcal{R}$. We choose polarization for $\mathcal{R}'$ and $\mathcal{R}$ as the ones given in Section 3, say $L'$ and $L$. Clearly there are $SL(V)$ action on $\mathcal{R}'$ and $\mathcal{R}$ and $\tilde{f}$ is $SL(V)$-invariant.

Now we put $K = 2\chi_G$ and give weights for $\mathcal{R}'$ by $\overline{u}(x)$ satisfying $a_{i+1}(x) - a_i(x) = m_i(x)$ and $a_i(x) + a_{2l+2-i}(x) = K$ for $1 \leq i \leq l_x$ and any $x \in D \cup D'$. So by Proposition 5.9 and
we see that $Z := (\mathcal{R}')^s/(L')/SL(V)$ is a Fano variety. We use $\varphi : (\mathcal{R}')^s \to Z$ to denote the quotient map.

Moreover, if one let $W = (\mathcal{R})^s$, $\hat{X} = W \cap \hat{f}^{-1}(\mathcal{R})^s$ and $\hat{X} = \varphi^{-1}(\hat{X})$. By Corollary 4.12 and our assumption, we would have: $\text{Codim}(\mathcal{R}' \setminus W) \geq 2$. Now Proposition 6.9 shows that the moduli space of parabolic symplectic/orthogonal bundles $Y := \mathcal{R}(L')^s/SL(V)$ is of globally $F$-regular type.

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