Observability of a 1D Schrödinger Equation with Time-Varying Boundaries

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Abstract
We discuss the observability of a one-dimensional Schrödinger equation on certain time-dependent domain. In linear moving case, we give the exact boundary and pointwise internal observability for arbitrary time. For the general moving, we provide exact boundary observability when the curve satisfies some certain conditions. By duality theory, we establish the controllability of adjoint system.

Keywords Observability · Controllability · Schrödinger equation · Moving domain · Non-autonomous evolution equation

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1 Introduction

Let $\tau > 0$, and $\ell(t) : [0, \tau] \to \mathbb{R}_+$ a strictly positive $C^2$-function satisfying $\ell(0) = 1$ and $\frac{\ell'}{\tau} \in L_\infty$. We consider the following system as a initial boundary value problem in a time-dependent domain:

$$S_{\text{moving}} \quad \begin{cases}
i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 & x \in [0, \ell(t)] \\
u(0, t) = u(\ell(t), t) = 0 & t \geq 0 \\
u(x, 0) = u_0 & x \in [0, 1].
\end{cases}$$

For Neumann boundary observations, we obtain estimates like

$$c(\tau) \| u_0 \|_{H^1_0(0,1)}^2 \leq \int_0^\tau \left( |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 \right) dt \leq C(\tau) \| u_0 \|_{H^1_0(0,1)}^2.$$

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see Theorems 2.2, 2.3 and 2.4. We refer to the first estimate as observability estimate and to the second as admissibility estimate. The two first mentioned results rely on a transformation of \((\mathfrak{s}_{\text{moving}})\) into a non-autonomous equation on the fixed domain \([0, 1] \times [0, T]\); the change of variables \(y = \frac{x}{\ell(t)}\) (we remark that \(y(t) \in [0, 1]\) and \(y = 1\) if \(x = \ell(t)\), \(y = 0\) if \(x = 0\)) and new function \(w(y, t) := u(x, t)\) gives an equivalent differential equation for \(w\), namely

\[
\begin{aligned}
& i \frac{\partial w}{\partial t} = -\frac{1}{\ell(t)^2} \frac{\partial^2 w}{\partial y^2} + i \frac{\ell'(t)}{\ell(t)} y \frac{\partial w}{\partial y}, \\
& w(0, t) = w(1, t) = 0, \\
& w_y(0, t) = \ell(t) u_x(0, t), \\
& w_y(1, t) = \ell(t) u_x(\ell(t), t),
\end{aligned}
\]

which can be obtained by the chain rule.

To obtain Theorems 2.2 and 2.3 we apply the “multiplier technique”: this powerful method has been developed by Morawetz [15] and was later extended by Ho [8] and Lions [12]. We extend a version of Machtyngier [13] to time-dependent multipliers. The observability estimate relies then on the “compactness-uniqueness” property in Lemma 3.5. This lemma indicates the norms equivalences for two operators satisfying certain relations with another compact operator in some Hilbert spaces, which gives us the observability estimate. It can be used also to show the uniqueness of the solution to some inhomogeneous Fredholm equations. For more details, we could found the work of Tartar [20] and the proof of Lemma 3.5. The pitfall of this proof strategy is that it only proves existence of some positive constant, without explicit estimates. This is in contrast with Theorem 2.4 which is a specific result for the boundary curve \(\ell(t) = 1+xt\). In this linear moving wall case, we mimic a successful approach for a one-dimensional wave equation obtained by Haak and the second author in [6] and develop the solution of \((\mathfrak{s}_{\text{moving}})\) into a series of eigenfunctions. This allows to use results from Fourier analysis; the obtained admissibility estimates are sharper than those obtained in the previous results, and the observation estimate is provided with explicit constants. Moreover, we obtain in this case admissibility and exact observability of internal point observations:

\[
k(\tau)\|u_0\|_{L^2(0, 1)}^2 \leq \int_0^\tau |u(a, t)|^2 \, dt \leq K(\tau)\|u_0\|_{L^2(0, 1)}^2,
\]

see Theorem 2.7. It is remarkable that the lower estimate cannot be true when \(\varepsilon = 0\) on any rational point \(a\). On the contrary, when considering the extending domains, the lower estimate seem to be handled for any singular point. Closely related to this observation are works of Castro and Khapalov [3, 10, 11] where on a fixed domain \(\Omega\) a moving point observer is considered, with similar conclusions. We also mention results from Moyano [16, 17] where in a two-dimensional circle the radius \(\ell(t)\) is used as a control parameter.

An additional result on \(L^p\)-admissibility and observability of point observations are presented as well, see Theorem 2.9.

It is well known that exact observability for an (autonomous) wave equation implies observability for the associated Schrödinger equation, see, e.g., [23, Chapter 6.7]. An inspection of the proof gives several obstacles when one passes to non-autonomous problems, and we were not able to use this approach to directly infer our results from those for the wave equation in [6]. We mention that some results on the so-called Hautus-test have been obtained in [7].
2 Main Results

In this section we state explicitly our main results. We start by the result of the well-posedness of the system $S_{\text{moving}}$.

**Theorem 2.1** For all $u_0 \in L^2(0, 1)$ the system $S_{\text{moving}}$ is well-posed in the sense that there exists a unique solution, continuous in the time variable and solves $S_{\text{moving}}$.

Suppose now that we are given observation operators $C(t) : D \rightarrow Y$ where $Y$ is another Hilbert space. Define the output function $y(t) = C(t)w(t)$. The observability of the dual system of $(S)_{\text{fixed}}$ could be defined as:

$$(S)_{\text{fixed}}^* := \begin{cases} w'(t) = A(t)w(t) & x \in [0, 1] \\
y(t) = C(t)w(t) & t \geq 0 \\
w(0) = w_0. \end{cases}$$

By letting a new variable $z = x\ell(t)$, we transverse the system $(S)_{\text{fixed}}^*$ in fixed domain back to a moving domain system.

The operator $C(t)$ is called $(Y, Z)$-admissible if there exist $\gamma > 0$ such that:

$$\int_0^\tau \left\| C(t)w(t) \right\|^2_Y \, dt \leq \gamma \|w_0\|^2_Z.$$

We say that the system $(S)_{\text{fixed}}^*$ is exactly $(Y, Z)$-observable in time $\tau > 0$ if there exist $\delta > 0$ such that:

$$\int_0^\tau \left\| C(t)w(t) \right\|^2_Y \, dt \geq \delta \|w_0\|^2_Z.$$

If the spaces $Y, Z$ are fixed, we simply speak of admissibility and exact observability. Exact observation in time $\tau > 0$ means that the knowledge of $y_{[0, \tau]}$ allows to recover the initial value $w_0$. It is well known that exact observability is equivalent to exact controllability of the retrograde adjoint system:

$$z'(t) = -A(t)^*z(t) - C(t)^*w(t) \quad \text{with} \quad z(\tau) = 0.$$

Moreover, it is easy to see that admissibility or observability of $(S)_{\text{fixed}}$ is equivalent to those of $(S)_{\text{moving}}$.

2.1 Results on Neumann Observations

**Theorem 2.2** Let $\tau > 0$ and $\ell : [0, \tau] \rightarrow \mathbb{R}_+$ be a strictly positive, twice continuously differentiable function satisfying $\frac{\ell'}{\ell} \in L^\infty$ and $\ell(0) = 1$. Assuming that $u_0 \in H^1_0(0, 1)$, then there exists a constant $C(\tau)$ such that the following admissibility inequality hold:

$$\int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 \, dt \leq C(\tau) \|u_0\|^2_{H^1_0(0, 1)}.$$

An explicit estimate of the constant $C(\tau)$ is given in the proof, see Eq. 3.12.

Our main idea for the proof of the above result is to use the multiplier method and some energy estimates for $(S)_{\text{fixed}}$ (for more details, see Section 3).
Concerning observability, we will have the following result. Let \( \tau > 0 \) and \( \ell : [0, \tau] \to \mathbb{R}_+^* \) be a strictly positive, twice continuously differentiable function satisfying:

\[
\ell'(t) > 0, \quad \ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) < \frac{1}{\pi} \quad \forall t \in (0, \tau). \tag{2.1}
\]

Integrating between 0 and \( \tau \) of the second condition, we have \( 2\tau + \pi(1 - \ell(\tau)^2) \geq 0 \). From the condition Eq. 2.1, \( \ell(t) \) is an increasing function, and then \( \ell'(t) < \frac{1}{\pi} \). It follows that \( \ell'(t) \ell(t) < \frac{1}{\pi} \), and so, the condition \( \frac{\ell'}{\ell(t)} \in L^\infty \) guaranteeing admissibility is satisfied. In fact, this condition tell us that although \( \ell(t) \) is an increasing function, but its expanding rate will not exceed an exponential function.

**Theorem 2.3** Let \( \ell(t) \) and \( \tau \) satisfying Eq. 2.1. Assuming that \( u_0 \in H^1_0(0,1) \), the following observability inequality holds:

\[
c(\tau) \|u_0\|_{H^1_0(0,1)}^2 \leq \int_0^\tau \left( |u_x(0,t)|^2 + |u_x(\ell(t),t)|^2 \right) dt.
\]

Here \( c(\tau) \) is some positive constant depending on \( \tau \).

A direct application of Theorem 2.3 can be used for periodic moving boundary \( \ell(t) = 1 + \varepsilon \sin(\omega t) \) where \( \varepsilon \in (0,1) \) and \( \omega \in \left(0, \frac{1}{\pi \varepsilon(1+\varepsilon)}\right) \). For all \( \tau \in \left(0, \frac{\pi}{2\omega}\right) \), we have

\[
\ell'(t) = \varepsilon \omega \cos(\omega t) > 0 \quad \text{since} \quad \omega t \in \left(0, \frac{\pi}{2}\right) \quad \forall 0 \leq t \leq \tau
\]

\[
\ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) = \varepsilon \omega \cos(\omega t)(1 + \varepsilon \sin(\omega t)) < \varepsilon \omega (1 + \varepsilon) < \frac{1}{\pi}
\]

Hence, \( \ell(t) \) satisfies the condition Eq. 2.1, so the curve is admissible. The problem of particles moving inside one-dimensional square-well of oscillating width was proposed by Fermi and Ulam [14] in order to explain the mechanism of particles containing high energies. This model therefore plays an important role on theory of quantum chaos, and it seems difficult to give an exact solution formula. In [5], Glasser, Nengo, and Nieto investigated the behavior of wave functions and energy in a given instantaneous eigenstate by assumptions on the smoothness of the boundary. As far as we know, there are no results in the literature concerning observability and controllability with periodic boundary functions.

In the case that \( \ell(t) = 1 + \varepsilon t \), the condition Eq. 2.1 is ensured when \( \varepsilon \in \left(0, \frac{2}{\pi}\right) \) and \( 0 < t < \frac{1}{\varepsilon} \left( \frac{2}{\varepsilon \pi} - 1 \right) \). We have the following exact analytic solution for \( (S_{\text{moving}}) \), due to Doescher and Rice [4]

\[
u(x, t) = \sum_{n=1}^{+\infty} a_n \sqrt{\frac{2}{\ell(t)}} \sin \left( \frac{n\pi x}{\ell(t)} \right) e^{i \left( \frac{\varepsilon x^2}{\ell(t)} - \frac{n^2 \pi^2 t}{\ell(t)} \right)} \tag{2.2}
\]

where the coefficients \( (a_n) \) are defined by the sine-series development of the initial value \( u_0 \). A similar exact solution in the case of two-variable moving wall can be found in [24] where the author uses the fundamental transformation to change the moving boundary problem into a solvable one side fixed boundary problem.

Based on formula Eq. 2.2 we obtain a first result on Neumann observability at the boundary \( \{(x, t) : x \in [0, \ell(t)]\} \). Compared to Theorem 2.3 the admissibility constant is sharper.
In contrast with Theorem 2.3, where we can only prove existence of some positive constant $c(\tau)$, we obtain now an explicit estimate for the observability constant. The proof is presented in Section 3.

**Theorem 2.4** Let $\ell(t)$ satisfying Eq. 2.1 and assuming that $u_0 \in H^1_0(0, 1)$. For every $\tau > 0$ there exist explicit constants $c(\tau, \varepsilon), C(\tau, \varepsilon)$ such that:

$$
c(\tau, \varepsilon)\|u_0\|^2_{H^1_0(0, 1)} \leq \int_0^\tau \left| u_x(0, t) \right|^2 + \left| u_x(\ell(t), t) \right|^2 + \left| u_t(\ell(t), t) \right|^2 \, dt \leq C(\tau, \varepsilon)\|u_0\|^2_{H^1_0(0, 1)}. \quad (2.3)
$$

In particular, the Neumann observation at the boundary of the system ($S_{moving}$) is exact observable in any time $\tau > 0$. Moreover, we have sharp estimates for $c(\tau, \varepsilon)$ and $C(\tau, \varepsilon)$. In precisely, $C(\tau, \varepsilon)$ is proportional to $\frac{1}{\varepsilon \ell(\tau)} (1 + \varepsilon^2)$, whereas $c(\tau, \varepsilon)$ decays as $\exp\left(-\frac{2k\pi^2}{\varepsilon \tau}\right)$, where $k > \frac{3}{2}$.

**Remark 2.5** By the Dirichlet condition, $u(\ell(t), t) = 0$ for all $t$. Differentiating yields $\ell'(t)u_x(\ell(t), t) + u_t(\ell(t), t) = 0$, and so $u_x(\ell(t), t) = \frac{-1}{\varepsilon} u_t(\ell(t), t)$. As a result, observing $u_t(\ell(t), t)$ or $u_x(\ell(t), t)$ is, up to a constant, the same.

By converting back to the dual problem, the dual statement of the lower estimate in Theorems 2.4 and 2.3 should demonstrate the exact controllability of an adjoint Schrödinger system where Dirichlet control is applied to the boundaries.

**Theorem 2.6** Let $\ell(t)$ satisfying Eq. 2.1. For every $\tau > 0$, the following time-dependent Schrödinger equation is exactly controllable:

$$
\begin{aligned}
&ih_t = -h_{xx} + i\frac{\ell(t)}{\ell'(t)} h & (x, t) \in (0, \ell(t)) \times (0, \tau) \\
h(\bar{b}, t) = 0 & \{\bar{b}\} \cup \{b\} = \{0, \ell(t)\}, t \geq 0 \\
h(b, t) = -i\ell(t)^3 u_x(0, t) & t \geq 0 \\
h(x, \tau) = 0 & x \in [0, \ell(t)]
\end{aligned}
$$

(2.4)

**2.2 Point Observations**

We now focus on point observations $u \mapsto u(a, t)$ in the case of a linearly moving wall $\ell(t) = 1 + \varepsilon t$. Observe that in the “degenerate” case that is, $\varepsilon = 0$, the (then) autonomous Schrödinger equation has the well-known solution

$$
u(x, t) = \sum_{n=1}^{+\infty} \alpha_n e^{-i\pi^2 n^2 t} \sin(n\pi x).
$$

Clearly, there is no reasonable observability possible at rational points $x$ since infinitely many terms in the sum vanish, independently of the leading coefficient $\alpha_n$. This changes when $\varepsilon > 0$: from Eq. 2.2 we obtain

$$
u(a, t) = \sum_{n=1}^{+\infty} \alpha_n \left( \frac{2}{\ell(t)} \right) \frac{1}{t} \exp\left(i\frac{\varepsilon a^2}{4\ell(t)} - i\frac{n^2 \pi^2}{\ell(t)} \right) \sin\left(\frac{n\pi a}{\ell(t)}\right)
$$

and so

$$
\int_0^\tau \left| u(a, t) \right|^2 \, dt = \int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} \alpha_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 \, dt. \quad (2.5)
$$
Based on a remarkable result of Tenenbaum and Tucsnak \cite{22} we obtain the following result in Section 3.

Theorem 2.7 Assume \( \ell(t) = 1+e^t \). Then, for every \( \tau > 0 \), we have

\[
K(\tau)\|u_0\|_{L^2(0,1)}^2 \gtrsim \int_0^\tau |u(a, t)|^2 \, dt \gtrsim k(\tau)\|u_0\|_{L^2(0,1)}^2.
\] (2.6)

More precisely, \( k(\tau) \approx M e^{-T} \) where \( T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)} \) and \( M, c \) are some positive constants that appear in to proof.

Corollary 2.8 For sufficiently small, there exist two constants \( H_1(a, \varepsilon) \) and \( H_2(a, \varepsilon) \) such that:

\[
H_1(a, \varepsilon) \int_0^\tau |u(a, t)|^2 \, dt \leq \int_0^\tau \int_{a-\varepsilon}^{a+\varepsilon} |u(x, t)|^2 \, dx \, dt \leq H_2(a, \varepsilon) \int_0^\tau |u(a, t)|^2.
\]

Therefore, for all \( a \in (0, 1) \), the point observation \( C = \delta_a \) for the system \((S_{\text{moving}})\) is exactly observable in arbitrarily short interval time \((a - \varepsilon, a + \varepsilon)\).

2.3 \( L_p \)-Estimates of Point Observations

Finally, we have the following \( L_p \) admissibility and observability estimates.

Theorem 2.9 Let \( \ell(t) = 1+e^t \). We assume that \( u_0 \in H^{1}_0(0, 1) \). For \( 0 < p < 2 \) and \( a \in (0, 1) \), we have

\[
k_p(\tau)\|u_0\|_{L^2(0,1)}^{2/p} \|u_0\|_{H^1_0(0,1)}^{1-2/p} \leq \left( \int_0^\tau |u(a, t)|^2 \, dt \right)^{1/p} \leq k_p(\tau)\|u_0\|_{L^2(0,1)}^{2/p} \|u_0\|_{H^1_0(0,1)}^{1-2/p}
\]

where \( k_p(\tau) \) and \( k_p(\tau) \), are constants depending on \( \tau \) and \( p \).

The upper estimate is a direct consequence of Eq. 2.6. Indeed, by the continuity of the embeddings \( H^1_0 \hookrightarrow L^2 \hookrightarrow L^p \) and the boundedness of \( \ell(t) \), we obtain:

\[
\|u(a, t)\|_{L^p} \lesssim \|u(a, t)\|_{L^2} \lesssim \|u_0\|_{L^2} \lesssim \|u_0\|_{L^2(0,1)}^{2/p} \|u_0\|_{H^1_0(0,1)}^{1-2/p}.
\]

Hence, it serves only to show that the lower estimate is of the right order.

3 Proof of the Main Results

In this section we prove our main results. To do this, we first show several identities and estimates which will play an important role in the proofs of the main results.

3.1 The Multiplier Lemma

We follow E. Machtyngier \cite[Lemma 2.2]{13} by using multiplier method for \((S_{\text{fixed}})\): Let \( w \) be a solution to \((S_{\text{fixed}})\) and \( q \in C^2([0, 1] \times [0, \tau]) \) be a real valued function. Then, due to the differential \((S_{\text{fixed}})\),

\[
iw_t + \frac{1}{\ell(t)^2} w_{yy} - i \frac{\ell'(t)}{\ell(t)} yw_y = 0.
\] (3.1)
Hence,

$$\text{Re}\left(\int_0^\tau \int_0^1 (q \overline{w}_y + \frac{1}{2} \overline{w}_q)_y (i w_t + \frac{1}{\ell(t)^2} w_{yy} - i \frac{\ell'(t)}{\ell(t)} y w_y) \, dy \, dt\right) = 0. \quad (3.2)$$

We separate the left-hand side of Eq. 3.2 into three parts and simplify each of them.

**Lemma 3.1** The following identities hold.

\[
\begin{align*}
\text{Re}\left(\int_0^\tau \int_0^1 (q \overline{w}_y + \frac{1}{2} \overline{w}_q)_y i w_t \, dy \, dt\right) &= \text{Re}\left(\int_0^1 \left[\frac{1}{2} i q \overline{w}_y w\right]_{t=0}^T \, dy\right) - \frac{1}{2} \text{Re}\left(\int_0^1 \int_0^1 i w_q \overline{w}_y \, dy \, dt\right) \\
\text{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q \overline{w}_y + \frac{1}{2} \overline{w}_q)_y \, dy \, dt\right) &= \text{Re}\left(\int_0^1 \frac{1}{2\ell(t)^2} (q(1, t)|w_y(1, t)|^2 - q(0, t)|w_y(0, t)|^2) \, dt\right) \\
&\quad - \text{Re}\left(\int_0^1 \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y \, dy \, dt\right) - \text{Re}\left(\int_0^1 \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} \, dy \, dt\right) \\
-\text{Re}\left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} w_y (q \overline{w}_y + \frac{1}{2} \overline{w}_q)_y \, dy \, dt\right) &= -\text{Re}\left(\int_0^1 \int_0^1 \frac{i y \ell'(t)}{\ell(t)} |w_y|^2 \, dy \, dt\right) - \text{Re}\left(\int_0^1 \int_0^1 \frac{i y \ell'(t)}{\ell(t)} w_y \overline{w}_q \, dy \, dt\right). \quad (3.4)
\end{align*}
\]

**Proof** To prove Eq. 3.3, we use integration by parts. Using $\overline{w}(0, t) = \overline{w}(1, t) = 0$, we have:

\[
\text{Re}\left(\int_0^\tau \int_0^1 q_y \cdot \overline{w}_t \, dy \, dt\right) = \frac{1}{2} \text{Re}\left(\int_0^\tau \left[\overline{w}_y q\right]_{y=0}^{y=1} - \int_0^1 \overline{q} \cdot (\overline{w}_t w_t + \overline{w}_t w_y) \, dy \right) dt = -\frac{1}{2} \text{Re}\left(\int_0^\tau \int_0^1 \overline{w}_y w_t + \overline{w}_t w_y \, dy \, dt\right).
\]

Therefore, the left-hand side of Eq. 3.3 equals

\[
\text{Re}\left(\int_0^\tau \int_0^1 (q \overline{w}_y + \frac{1}{2} \overline{w}_q)_y i w_t \, dy \, dt\right) = \text{Re}\left(\int_0^\tau \int_0^1 \overline{w}_y w_t \, dy \right) dt + \frac{1}{2} \text{Re}\left(\int_0^\tau \int_0^1 \overline{w}_q w_t \, dy \, dt\right).
\]
\[ \frac{1}{2} \text{Re} \left( i \int_0^1 \int_0^\tau (w_t \cdot \overline{q w_y} - q \overline{w w_{ty}}) \, dy \, dt \right) \]

\[ = \frac{1}{2} \text{Re} \left( i \int_0^1 \left( \left[ q \overline{w w_y} \right]_t^\tau \right) \, dy \right) - \int_0^\tau w \left( q_i w_y + q \overline{w_y} \, dt \right) \, dy \]

\[ - \frac{1}{2} \text{Re} \left( \int_0^\tau \int_0^1 q_i \overline{w w_{ty}} \, dy \, dt \right) \]

\[ = \frac{1}{2} \text{Re} \left( i \int_0^1 \left( \left[ q \overline{w w_y} \right]_t^\tau \right) \, dy \right) - \frac{1}{2} \text{Re} \left( \int_0^1 \int_0^\tau i w q_i \overline{w_y} \, dy \, dt \right). \]

Here, we used the fact that

\[ -\text{Re} \left( \int_0^1 \int_0^\tau i q w \overline{w_{ty}} \, dt \, dy \right) = \text{Im} \left( \int_0^1 \int_0^\tau q w \overline{w_{ty}} \, dt \, dy \right) = \text{Re} \left( \int_0^1 \int_0^\tau i q \overline{w w_{ty}} \, dt \, dy \right). \]

To prove Eq. 3.4 we have

\[ \text{Re} \left( \int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} q \overline{w_y} \, dy \, dt \right) = \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{d_y} |w_y|^2 \cdot \frac{1}{2\ell(t)^2} q \, dy \, dt \right) \]

\[ = \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} \left( q(1, t)|w_y(1, t)|^2 - q(0, t)|w_y(0, t)|^2 \right) \, dt \right) \]

\[ - \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} q_y |w_y|^2 \, dy \, dt \right) \]

where we used \( \text{Re}(w_{yy} \overline{w_y}) = \text{Re}(\overline{w_{yy}} w_y) \). Again, integration by parts shows

\[ \text{Re} \left( \int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} \overline{w q_y} \, dy \, dt \right) = \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} \overline{w q_y} \, dy \, dt \right) \]

\[ = \text{Re} \left( \int_0^\tau \int_0^1 \left( \frac{1}{2\ell(t)^2} \overline{w q_y w_y} \right) \, dt \right) \]

\[ - \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y q_y} + \overline{w q_{yy}}) w_y \, dt \right) \]

\[ = -\text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y q_y} + \overline{w q_{yy}}) w_y \, dt \right). \]
Therefore
\[
\text{Re}\left( \int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q \overline{w_y} + \frac{1}{2} \overline{w q_y}) \, dy \, dt \right)
\]
\[
= \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (q(1, t)w_y^2(1, t) - q(0, t)w_y^2(0, t)) \, dt \right)
- \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y \, dy \, dt \right)
- \text{Re}\left( \int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} \, dy \, dt \right).
\]

Hence, the estimate in Eq. 3.4 is proved.

It is clear that
\[
-\text{Re}\left( \int_0^1 \frac{iy \ell'(t)}{\ell(t)} w_y (q \overline{w_y} + \frac{1}{2} \overline{w q_y}) \, dy \, dt \right)
\]
\[
= -\text{Re}\left( \int_0^\tau \int_0^1 \frac{iy \ell'(t)}{\ell(t)} q w_y \overline{w_y} \, dy \, dt \right) - \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} \frac{iy \ell'(t)}{\ell(t)} w_y \overline{w q_y} \, dy \, dt \right).
\]

Since \( w_y \overline{w_y} = |w_y|^2 \), then the estimate in Eq. 3.5 follows immediately.

Now summing up the three parts and using Eq. 3.2 yields the following result.

**Proposition 3.2** For any real valued function \( q \in C^2([0, 1] \times [0, \tau]) \) and a solution \( w \) to \((S_{\text{fixed}})\) we have
\[
0 = \text{Re}\left( \int_0^\tau \int_0^1 q w_y w \big|_{y=0}^{y=\tau} \, dy \right) - \frac{i}{2} \text{Re}\left( \int_0^\tau \int_0^1 i w q y \overline{w_y} \, dy \, dt \right)
+ \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (q(1, t)|w_y(1, t)|^2 - q(0, t)|w_y(0, t)|^2) \, dt \right)
- \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y \, dy \, dt \right) - \text{Re}\left( \int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} \, dy \, dt \right)
- \text{Re}\left( \int_0^\tau \int_0^1 \frac{iy \ell'(t)}{\ell(t)} q |w_y|^2 \, dy \, dt \right) - \text{Re}\left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} \frac{iy \ell'(t)}{\ell(t)} w_y \overline{w q_y} \, dy \, dt \right).
\]

### 3.2 Energy Estimates

For a solution \( w \) to \((S_{\text{fixed}})\) we define the first and second energy as
\[
E(t) = \frac{1}{2} \int_0^1 |w(y, t)|^2 \, dy \quad \text{and} \quad F(t) = \frac{1}{2} \int_0^1 |w_y(y, t)|^2 \, dy
\]
respectively.

**Lemma 3.3** We have \( \ell(\tau) E(\tau) = E(0) \).
Proof Taking the derivative with respect to $t$, we have
\[
\frac{dE(t)}{dt} = \frac{d}{dt} \frac{1}{2} \int_0^1 |w(y, t)|^2 \, dy = \frac{1}{2} \int_0^1 (w_t \bar{w} + w \bar{w}_t) \, dy
\]
\[
= (a) \frac{1}{2} \int_0^1 \left( \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} w_y \right) \bar{w} + w \left( \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} w_y \right) \, dy
\]
\[
= \frac{1}{2} \int_0^1 \left( \frac{i}{\ell(t)^2} (w_{yy} \bar{w} - w_{yy} w) + \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \bar{w}_y w) \right) \, dy,
\]
where, in (a), we used Eq. 3.1. Now integration by parts gives
\[
\int_0^1 \frac{i}{\ell(t)^2} (w_{yy} \bar{w} - w_{yy} w) \, dy = \int_0^1 \frac{i}{\ell(t)^2} \bar{w} d(w_y) - \int_0^1 \frac{i}{\ell(t)^2} w d(\bar{w}_y)
\]
\[
= \frac{i}{\ell(t)^2} \left( \left[ \bar{w} w_y \right]_{y=0}^{y=1} - \int_0^1 |w_y|^2 \right)
\]
\[
- \frac{i}{\ell(t)^2} \left( \left[ w \bar{w}_y \right]_{y=0}^{y=1} - \int_0^1 |\bar{w}_y|^2 \right)
\]
\[
= 0,
\]
whereas
\[
\int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \bar{w}_y w) \, dy = \int_0^1 \frac{\ell'(t)}{\ell(t)} \bar{w} d(w_y) - \int_0^1 \frac{\ell'(t)}{\ell(t)} w d(\bar{w}_y)
\]
\[
= \left[ \frac{\ell'(t)}{\ell(t)} \bar{w} w_y \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (w + y \bar{w}_y) w \, dy
\]
\[
+ \left[ \frac{\ell'(t)}{\ell(t)} \bar{w} w_y \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (w + y \bar{w}_y) \bar{w} \, dy
\]
\[
= -\frac{2 \ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 \, dy - \int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \bar{w}_y w) \, dy.
\]
Therefore,
\[
\int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \bar{w}_y w) \, dy = -\frac{\ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 \, dy,
\]
so that
\[
\frac{dE(t)}{dt} = -\frac{1}{2} \int_0^1 \frac{\ell'(t)}{\ell(t)} |w(y, t)|^2 \, dy = -\frac{\ell'(t)}{\ell(t)} E(t).
\]
Hence, \( E(\tau) = \frac{E(0) \ell(0)}{\ell(\tau)} \). Using \( \ell(0) = 1 \), this implies easily \( E(\tau) = \frac{E(0)}{\ell(\tau)} \).

Lemma 3.4 For all \( \tau > 0 \) and \( \tau \in \left( 0, \frac{\pi}{2\omega} \right) \), we have
\[
\frac{\pi^2}{\ell(\tau)} E(0) \leq F(\tau) \leq \ell(\tau) F(0).
\]
Proof We compute
\[
\frac{dF(t)}{dt} = \frac{d}{dt} \left( \frac{1}{2} \int_0^1 |w_y(y, t)|^2 \, dy \right) = \frac{1}{2} \int_0^1 (w_{yt} \overline{w_y} + w_y \overline{w_{yt}}) \, dy
\]
\[
= \left( a \right) \frac{1}{2} \int_0^1 \left( \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right) \overline{w_y} + w_y \left( \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right) \, dy
\]
\[
= \frac{i}{2\ell(t)^2} \int_0^1 (w_{yyy} \overline{w_y} - w_{yyy} w_y) \, dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 ((yw_y)_y \overline{w_y} + w_y (yw_y)_y) \, dy,
\]
where, in \(a\), we used Eq. 3.1. The first term on the right-hand side simplifies as
\[
\frac{i}{2\ell(t)^2} \int_0^1 (w_{yyy} \overline{w_y} - w_{yyy} w_y) \, dy
\]
\[
= \frac{i}{2\ell(t)^2} \int_0^1 \overline{w_y} \, d(w_{yy}) - \frac{i}{2\ell(t)^2} \int_0^1 w_y \, d(\overline{w_{yy}})
\]
\[
= \frac{i}{2\ell(t)^2} \left[ \overline{w_y} w_{yy} \right]_{y=0}^{y=1} - \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 \, dy - \frac{i}{2\ell(t)^2} \left[ \overline{w_{yy}} w_y \right]_{y=0}^{y=1} + \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 \, dy
\]
\[
= \left[ \frac{1}{2} \overline{w_y} (w_t - \frac{\ell'(t)}{\ell(t)} y w_y) \right]_{y=0}^{y=1} + \left[ \frac{1}{2} w_y (\overline{w_t} - \ell'(t) y \overline{w_y}) \right]_{y=0}^{y=1} = -\frac{\ell'(t)}{\ell(t)} |w_y(1, t)|^2,
\]
whereas the second term simplifies as follows
\[
\frac{\ell'(t)}{2\ell(t)} \int_0^1 ((yw_y)_y \overline{w_y} + w_y (yw_y)_y) \, dy = \frac{\ell'(t)}{2\ell(t)} \int_0^1 (w_y + y w_{yy}) \overline{w_y} + w_y (w_y + y w_{yy}) \, dy
\]
\[
= \frac{\ell'(t)}{2\ell(t)} \int_0^1 2|w_y|^2 + y(w_{yy} \overline{w_y} + w_y w_{yy}) \, dy
\]
\[
= \frac{\ell'(t)}{\ell(t)} \int_0^1 |w_y|^2 \, dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 y \, d(|w_y|^2)
\]
\[
= \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y|^2 \, dy + \frac{\ell'(t)}{2\ell(t)} |w_y(1, t)|^2.
\]
We add both parts to obtain
\[
\frac{dF(t)}{dt} = \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y(y, t)|^2 \, dt - \frac{1}{2} |w_y(1, t)|^2 \frac{\ell'(t)}{\ell(t)}
\]
\[
= \frac{\ell'(t)}{\ell(t)} \left( F(t) - \frac{1}{2} |w_y(1, t)|^2 \right).
\]
By variation of constants, we get an explicit solution
\[
F(t) = \ell(t) F(0) - \ell(t) \int_0^t \frac{\ell'(s)}{2\ell(s)^2} |w_y(1, s)|^2 \, ds.
\] (3.6)
One easily obtains an upper bound, namely \(F(t) \leq F(0) \ell(t)\). For the lower bound, we use the Poincaré (or Wirtinger) inequality on \([0, 1]\) to obtain
\[
F(t) = \frac{1}{2} \int_0^1 |w_y(y, t)|^2 \, dy \geq \frac{\pi^2}{2} \int_0^1 |w(y, t)|^2 \, dy = \frac{\pi^2}{\ell(t)} E(0).
\] (3.7)
\(\Box\)
3.3 Well-Posedness

Proof of Theorem 2.1 Let us start by proving that the Schrödinger (\(S_{\text{fixed}}\)) admits a unique solution: to this end, we reformulate it as an abstract non-autonomous Cauchy problem in the following way: let \(X = L^2(0, 1)\) and the family of operators \(\{A(t)\}\) be defined as

\[
A(t)w = \frac{i}{\ell'(t)} \xi yy + \frac{\xi'(t)}{\ell(t)} \xi y w_y
\]  \hspace{1cm} (3.8)

with natural domain \(D(A(t)) = H^2(0, 1) \cap H^1_0(0, 1) =: D\). Moreover, by assumption, the map \(t \mapsto A(t)u\) is continuously differentiable for all \(u \in D\). Let \(\omega > 0\). Then integration by parts gives

\[
\left((A(t) + \omega I)w, w\right) = \int_0^1 \left(\frac{i}{\ell(t)} w_{yy} \bar{w} + \frac{\xi'(t)}{\ell(t)} \xi w_y \bar{w} + \omega |w|^2\right) dy
\]

\[
= \frac{i}{\ell(t)} \int_0^1 |w_y|^2 dy + \frac{\xi'(t)}{\ell(t)} \int_0^1 \xi w_y \bar{w} dy + \omega \int_0^1 |w|^2 dy
\]

\[
= \frac{\xi'(t)}{\ell(t)} \int_0^1 |w_y|^2 dy + \omega \int_0^1 \left(|w|^2 + yw \bar{w}_y\right) dy + \omega \int_0^1 |w|^2 dy.
\]  \hspace{1cm} (3.9)

Taking real parts and observing that

\[
\text{Re} \left(\frac{\xi'(t)}{\ell(t)} \int_0^1 yw \bar{w}_y dy\right) = \text{Re} \left(\frac{\xi'(t)}{\ell(t)} \int_0^1 yw_y \bar{w} dy\right) = -\text{Re} \left(\frac{\xi'(t)}{2\ell(t)} \int_0^1 |w|^2 dy\right)
\]

we obtain

\[
\text{Re} \left(\left((A(t) + \omega I)w, w\right)\right) = \left(\omega - \frac{\xi'(t)}{2\ell(t)}\right) \int_0^1 |w|^2 dy. \hspace{1cm} (3.10)
\]

For \(\omega > \left\| \frac{\xi'}{\ell} \right\|_{L^\infty}, \) the left-hand side of Eq. 3.10 becomes positive, and the Lumer-Philips theorem (see, e.g., [19, Theorem 4.3, p. 14]) asserts that \(\omega + A(t)\) generates a contraction semigroup, i.e.,

\[
\forall t \geq 0, \quad \left\| e^{-\tau A(t)} \right\| \leq e^{\omega \tau}.
\]

This ensures in particular that the family \((A(t))_{t \in [0, \tau]}\) satisfies the Kato stability condition. The latter means that (see [19, p. 131])

\[
\left\| \prod_{j=1}^k e^{-s_j A(t_j)} \right\| \leq M \prod_{j=1}^k e^{\omega \sum_{j=1}^k s_j}
\]

for some \(M > 0\) and any finite sequence \(0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq \tau\) and \(s_j \geq 0, k \in \mathbb{N}^*\). We apply [19, Theorem V.4.8 p.145] to conclude that \((A(t))_{t \in [0, \tau]}\) generates a unique evolution family \(\{U(t, s)\}_{0 \leq s \leq t \leq \tau}\) on \(X\) satisfying \(w(t) = U(t, 0)w_0\) and \(w \in C([0, 1], L^2(0, 1))\). From this, we infer a unique solution to \((S_{\text{moving}})\) as well, by transforming the fixed domain back to the time-dependent domain. Then the existence of unique, strongly continuous and bounded evolution family \(\{U(t, s)\}_{0 \leq s \leq t \leq \tau}\) implies the well-posedness of \((S_{\text{moving}})\) (for more details we refer to [1, 2] and [18, 19]).
3.4 Admissibility of Neumann Observations at the Boundary

Proof of Theorem 2.2 We take the function \( q(y, t) = q(y)\ell(t) \) on \((0, 1)\) satisfying \( q(1) = 0 \) and \( q(0) = 1 \). By Proposition 3.2, we have

\[
\text{Re} \left( \int_0^\tau \frac{1}{2\ell(t)^2} q(0, t)|w_y(0, t)|^2 \, dt \right) = \text{Re} \left( \int_0^1 \left[ \frac{1}{2} i q\ell(t)\ol{w_y} w \right]_t^\tau \, dy \right) \\
- \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 q_y \, dy \, dt \right) \\
- \text{Re} \left( \int_0^\tau \int_0^1 \frac{w_y w}{2\ell(t)} q_{yy} \, dy \, dt \right) \\
- \text{Re} \left( \int_0^\tau \int_0^1 i y\ell'(t)q |w_y|^2 \, dy \, dt \right) \\
- \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2} iy\ell'(t) w_y \ol{w_q} \, dy \, dt \right) \\
- \frac{1}{2} \text{Re} \left( \int_0^\tau \int_0^1 i w\ell'(t)\ol{w_y} w \, dy \, dt \right)
\]

\[:= A + B + C + D + E + F.\]

Therefore,

\[
\int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 \, dt \leq |A| + |B| + |C| + |D| + |E| + |F|,
\]

where we estimate all five terms separately. Concerning \(|A|\), we separate the products in the real part by \( ab \leq \frac{1}{2}(a^2 + b^2) \), then use Lemmas 3.3 and 3.4 to obtain

\[
|A| = \left| \text{Re} \left( \int_0^1 \left[ \frac{1}{2} i q\ell(t)\ol{w_y} w \right]_t^\tau \, dy \right) \right|
\leq \frac{1}{4} \|q\|_{L_\infty(0, 1)} \left( \int_0^1 \ell(t)|w(y, \tau)|^2 + |w(y, 0)|^2 + \ell(t)|w_y(y, \tau)|^2 + |w_y(y, 0)|^2 \, dy \right)
\leq \frac{1}{4} \|q\|_{L_\infty(0, 1)} \left( \int_0^1 2|w(y, 0)|^2 + (1 + \ell(t)^2)|w_y(y, 0)|^2 \, dy \right)
\leq \frac{1}{4} \|q\|_{L_\infty(0, 1)} \left( \int_0^1 \left( 2\pi^2 + 1 + \ell(t)^2 \right) |w_y(y, 0)|^2 \, dy \right).
\]

Using the estimate \( \int_0^1 |w_y(y, \tau)|^2 \, dy \leq \ell(t)\int_0^1 |w_y(y, 0)|^2 \, dy \) of Lemma 3.4 one gets

\[
|B| = \left| \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 q_y \, dy \, dt \right) \right| \leq \|q_y\|_{L_\infty(0, 1)} \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 \, dy \, dt
\leq \|q_y\|_{L_\infty(0, 1)} \int_0^\tau \int_0^1 |w_y(y, 0)|^2 \, dy \, dt
= \|q_y\|_{L_\infty(0, 1)} \tau \int_0^1 |w_y(y, 0)|^2 \, dy.
\]
The term $|C|$ is decoupled by Cauchy-Schwarz and then estimated using Lemma 3.4 as follows:

$$|C| = \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 w_y w \frac{1}{2\ell(t)} q_{yy} \, dy \, dt \right) \right| \leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \int_0^1 \frac{1}{2\ell(t)} \left( \int_0^1 |w(y, t)|^2 \, dy \right)^{1/2} \left( \int_0^1 |w_y(y, t)|^2 \, dy \right)^{1/2} \, dt \right)$$

$$\leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \int_0^1 \frac{\pi}{2\ell(t)} \left( \int_0^1 |w_y(y, t)|^2 \, dy \right) \, dt \right)$$

$$\leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \frac{\pi}{2} \, dt \right) \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right)$$

$$= \|q_{yy}\|_{L^\infty(0,1)} \frac{\pi \tau}{2} \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right).$$

For the fourth term, we use Lemma 3.4 to obtain

$$|D| = \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 iy\ell'(t)w_y(y, t)^2 \, dy \, dt \right) \right| \leq \|q\|_{L^\infty(0,1)} \int_0^\tau \int_0^1 \ell'(t) |w_y(y, t)|^2 \, dy \, dt$$

$$\leq \|q\|_{L^\infty(0,1)} \left( \int_0^\tau \ell'(t) \ell(t) \, dt \right)$$

$$= \|q\|_{L^\infty(0,1)} \frac{\ell(t)^2 - 1}{2} \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right).$$

The estimate for $|E|$ is similar to the one for $|C|:

$$|E| = \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2} iy\ell'(t)w_y \bar{w} q_{yy} \, dy \, dt \right) \right| \leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \int_0^1 \frac{1}{2} \ell'(t) |w_y| \|\bar{w}\| \, dy \, dt \right)$$

$$\leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \frac{\pi \ell'(t)}{2} \left( \int_0^1 |w(y, t)|^2 \, dy \right)^{1/2} \left( \int_0^1 |w_y(y, t)|^2 \, dy \right)^{1/2} \, dt \right)$$

$$\leq \|q_{yy}\|_{L^\infty(0,1)} \left( \int_0^\tau \frac{\pi \ell'(t)}{2} \left( \int_0^1 |w_y(y, t)|^2 \, dy \right) \, dt \right)$$

$$\leq \|q_{yy}\|_{L^\infty(0,1)} \frac{\pi}{4} \left( \ell(t)^2 - 1 \right) \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right)$$

Finally, $|F|$ is treated like $|C|$ and $|E|:

$$|F| = \frac{1}{2} \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 iwq\ell'(t)\bar{w} \, dy \, dt \right) \right| \leq \frac{1}{2} \|q\|_{L^\infty(0,1)} \left( \int_0^\tau \int_0^1 \ell'(t) \|\bar{w}\| \|w\| \, dy \, dt \right)$$

$$\leq \|q\|_{L^\infty(0,1)} \frac{\pi}{4} (\ell(t)^2 - 1) \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right).$$
Summing up all five estimates, we obtain
\[ \int_0^\tau \frac{1}{2\ell(t)} \left| w_y(0, t) \right|^2 dt \leq C_1(\tau) \| w_0 \|^2_{H_0^1(0, 1)} \]  \hspace{1cm} (3.11)
where the constant \( C_1(\tau) \) is given by
\[
C_1(\tau) = \frac{(3 + \pi^2)\ell(\tau)^2 + \pi^2 - 1}{4} \| q \|_{L^\infty(0, 1)} + \left( \tau + \frac{\pi}{4} (\ell(\tau)^2 - 1) \right) \| q_y \|_{L^\infty(0, 1)}
+ \frac{\pi \tau}{2} \| q \|_{L^\infty(0, 1)}.
\]  \hspace{1cm} (3.12)

Replacing \( w_y(0, t) = \ell(t) u_x(0, t) \) in Eq. 3.11 yields the admissibility inequality
\[ \int_0^\tau \left| u_x(0, t) \right|^2 dt \leq \int_0^\tau \ell(t) \left| u_x(0, t) \right|^2 dt \leq 2C_1(\tau) \| u_0 \|^2_{H_0^1(0, 1)}. \]
The second admissibility estimate follows the same lines, using \( q(y, t) = q(y) \ell(t) \) on \((0, 1)\) with \( q(0) = 0 \) and \( q(1) = 1 \).

\textbf{3.4.1 Neumann Observability at the Boundary}

Before proving Theorem 2.3, let us recall the following preliminary Lemma.

\textbf{Lemma 3.5} Let \( E_1, E_2 \) and \( E_3 \) be the Hilbert spaces. We consider the continuous linear operators \( T : E_1 \rightarrow E_2, K : E_1 \rightarrow E_3 \) and \( L : E_1 \rightarrow E_1 \) such that \( K \) is compact, \( L \) is bounded below and:
\[ \| Lu \|_{E_1} \approx \| Tu \|_{E_2} + \| Ku \|_{E_3}. \]  \hspace{1cm} (3.13)

Then the kernel of \( T \) has finite dimension and \( \| Lu \|_{E_1} \approx \| Tu \|_{E_3} \)

\textbf{Proof} Note that Eq. 3.13 is equivalent to \( \| u \|_{E_1} \approx \| Tu \|_{E_2} + \| Ku \|_{E_3} \). Using [20, Lemma 1 p.1] we infer that the kernel of \( T \) has finite dimension. Applying now [20, Lemma 2 p.1] one gets that \( \| Lu \|_{E_1} \approx \| Tu \|_{E_3} \). \( \square \)

\textbf{Proof of Theorem 2.3} For all \( \tau \) satisfying \( 2\tau + \pi (1 - \ell(\tau)^2) > 0 \), we choose two positive constants \( \eta(\tau) \) and \( \delta(\tau) \) such that:
\[ \eta(\tau) + \delta(\tau) < \frac{4}{1 + \ell(\tau)^3} \left( \tau - \frac{\pi}{2} (\ell(\tau)^2 - 1) \right). \]  \hspace{1cm} (3.14)
We choose \( q(y) = (1 - y)\ell(t) \) where \( y \in (0, 1) \). Proposition 3.2 is then equivalent to:
\[
\int_0^\tau \frac{1}{2\ell(t)} \left| w_y(0, t) \right|^2 dt = \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 dy dt - \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2} i(1 - y)\ell(t) w_y \overline{w} dy dt \right)
+ \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2} i(1 - y)\ell(t) w_y \overline{w} dy dt \right) + \text{Re} \left( \int_0^\tau \int_0^1 \frac{1}{2} i(1 - y)\ell(t) w_y \overline{w} dy dt \right).
\]  \hspace{1cm} (3.15)
Taking the three last formula of the right-hand side to the left, then taking the absolute we get:

\[
\int_0^\tau \int_0^1 \frac{1}{\xi(t)} |w_y|^2 \, dy \, dt \leq \int_0^\tau \int_0^1 \frac{1}{2\xi(t)} |w_y(0, t)|^2 \, dt + \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} (1-y) \xi(t) |w_y|^2 \, dy \, dt \right) \right| \\
+ \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} y \xi(t) w_y \, dy \, dt \right) \right| \\
+ \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} y \xi(t) w_y, w \, dy \, dt \right) \right|.
\]

The sum of third and fourth terms in the right-hand side of above formula can be estimated as:

\[
\left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} (1-y) \xi(t) |w_y|^2 \, dy \, dt \right) \right| + \left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} y \xi(t) w_y, w \, dy \, dt \right) \right| \\
\leq \frac{1}{2} \int_0^\tau \int_0^1 \xi(t) |w_y| |w_y, w| \, dy \, dt + \frac{1}{2} \int_0^\tau \int_0^1 \xi(t) |w_y| |w_y, w| \, dy \, dt \\
\leq \int_0^\tau \xi(t) \left( \int_0^1 |w|^2 \, dy \right)^{1/2} \left( \int_0^1 |w_y|^2 \, dy \right)^{1/2} \, dt \\
\leq \int_0^\tau \pi \xi(t) \left( \int_0^1 |w|^2 \, dy \right) \, dt.
\]

Due to the energy estimate in Lemma 3.3 and 3.4, we have the upper bound for the second term:

\[
\left| \operatorname{Re} \left( \int_0^\tau \int_0^1 \frac{1}{\xi(t)} (1-y) \xi(t) |w_y|^2 \, dy \, dt \right) \right| \\
\leq \frac{1}{4} \int_0^1 \frac{1}{\eta(t)} \left( \frac{|w(y, 0)|^2}{\eta(t)} + \frac{|w(y, \tau)|^2}{\eta(t)} + \eta(t) |w_y(y, 0)|^2 + \eta(t) \xi(t)^2 |w_y(y, \tau)|^2 \right) \, dy \\
\leq \frac{1}{4\eta(t)} \left( \frac{1}{\xi(t)} + 1 \right) \int_0^1 |w(y, 0)|^2 \, dy + \frac{1 + \xi(t)^3}{4} \int_0^1 |w_y(y, 0)|^2 \, dy.
\]

As a result, we combine these estimates and use (3.6) to obtain

\[
\int_0^\tau \frac{1}{2\xi(t)} |w_y(0, t)|^2 \, dt + \int_0^\tau \frac{1}{4\eta(t)} \left( \frac{1}{\xi(t)} + 1 \right) \int_0^1 |w_y(y, 0)|^2 \, dy \\
\geq \int_0^\tau \int_0^1 \left( \frac{1}{\xi(t)} - \pi \xi(t) \right) |w_y(y, \tau)|^2 \, dy \, dt - \frac{1 + \xi(t)^3}{4} \eta(t) \int_0^1 |w_y(y, 0)|^2 \, dy \\
= \left( \int_0^\tau \left( 1 - \pi \xi(t) \xi(t) \right) dt - \frac{1 + \xi(t)^3}{4} \eta(t) \right) \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right) \\
- \int_0^\tau \left( 1 - \pi \xi(t) \xi(t) \right) \int_t^\tau \frac{1}{\xi(s)} |w_y(1, s)|^2 \, ds \, dt \\
= \left( \tau + \frac{\pi}{2} (1 - \xi(t)^2) - \frac{1 + \xi(t)^3}{4} \eta(t) \right) \left( \int_0^1 |w_y(y, 0)|^2 \, dy \right).
\]
\[-\int_0^\tau \left(1 - \pi \xi'(t)\xi(t)\right) \int_0^t \frac{\xi'(s)}{\xi(s)} \left|w_y(1, s)\right|^2 \, ds \, dt \geq \frac{(1 + \xi(t)^3)\delta(t)}{4} \left(\int_0^1 \left|w_y(y, 0)\right|^2 \, dy\right) - \int_0^\tau \left(1 - \pi \xi'(t)\xi(t)\right) \int_0^t \frac{\xi'(s)}{\xi(s)} \left|w_y(1, s)\right|^2 \, ds \, dt\]

where the last inequality come from Eq. 3.14. Therefore, there exist the constants \(A_\tau\) and \(B_\tau\) such that:

\[\int_0^1 \left|w_y(y, 0)\right|^2 \, dy \leq A_\tau \int_0^\tau \left(\left|w_y(0, t)\right|^2 + \left|w_y(1, t)\right|^2\right) \, dt + B_\tau \int_0^1 \left|w(y, 0)\right|^2 \, dy. \tag{3.16}\]

It is sufficient to prove that there exist a constant \(K > 0\) such that

\[\int_0^1 \left|w(y, 0)\right|^2 \, dy \leq K \left(\int_0^\tau \left|w_y(0, t)\right|^2 \, dt + \int_0^\tau \left|w_y(1, t)\right|^2 \, dt\right). \tag{3.17}\]

Let us denote the operator \(T\) from \(H_0^1(0, \tau)\) to \(L_2(0, \tau) \times L_2(0, \tau)\) and the operator \(K\) from \(H_0^1(0, 1)\) to \(L_2(0, 1)\) that maps:

\[(T w)(t) = \left(w_y(0, t), w_y(1, t)\right) \tag{3.18}\]

\[(K w)(y) = w(y, 0). \tag{3.19}\]

From admissibility and Eq. 3.16, we infer that

\[a_\tau \|T w\|^2_{L_2} + b_\tau \|K w\|^2_{L_2} \leq \|w_0\|^2_{H_0^1} \leq A_\tau \|T w\|^2_{L_2} + B_\tau \|K w\|^2_{L_2}. \tag{3.20}\]

It is easy to see that \(K\) is compact operator due to Rellich’s embedding lemma. In order to use the compactness-uniqueness Lemma 3.5 for \(L = K\), we need to check that \(T\) is injective. Observe that \(T w = 0\) means that \(w\) satisfies (\(S_{\text{fixed}}\)) with Dirichlet conditions and zero Neumann derivative. It is well known that \(w\) vanishes in this case, see for example [21, Theorem 3] or [9, Corollary 6.1]. As a consequence,

\[c_\tau \|T w\|^2_{L_2} \leq \|w_0\|^2_{H_0^1} \leq C_\tau \|T w\|^2_{L_2}\]

for some constants \(c_\tau, C_\tau > 0\). \(\square\)

### 3.5 Results for Linear Moving Walls

Recall the Doescher-Rice representation formula Eq. 2.2 that yields for \(t = 0\)

\[u(x, 0) = \sqrt{2} \sum_{n=1}^N a_n e^{\frac{i\pi x}{2}} \sin(n\pi x), \tag{3.21}\]

and denote by

\[u_n(x, t) := \sqrt{\frac{2}{\ell(t)}} \sin\left(\frac{n\pi x}{\ell(t)}\right).\]

For all fixed \(t > 0\), the functions \((u_n(\cdot, t))_{n \geq 1}\) form an orthonormal basis in \(L_2(0, \ell(t))\), since the change of variable \(y = \frac{x}{\ell(t)}\) reduces \(u_n(\cdot, t)\) to the standard trigonometric system on \(L_2([0, 1])\).
Lemma 3.6 For all finitely supported sequences \((a_n)\) we have the following relation between \((a_n)\) and the norms of the initial data \(u_0\).

\[
\|u(x, 0)\|_{L^2(0, 1)}^2 = \sum_{n=1}^{\infty} |a_n|^2, \quad \|u(x, 0)\|_{H^1_0(0, 1)}^2 \approx \sum_{n=1}^{\infty} |a_n|^2 n^2.
\]

Proof Observe that

\[
\|e^{-ix^2} u_N(x)\|_{L^2(0, 1)}^2 = \|u_N(x)\|_{L^2(0, 1)}^2 = 2 \int_0^1 \left| \sum_{n=1}^{N} a_n \sin(n\pi x) \right|^2 \, dx \\
= 2 \int_0^1 \left| \sum_{n=1}^{N} a_n \sin(n\pi x) \right|^2 \, dx = \sum_{n=1}^{\infty} |a_n|^2.
\]

Since \((a_n)\) is a finite sequence we may interchange differentiation and summation and obtain

\[
\frac{d}{dx} u(x) = \sqrt{2} \sum_{n=1}^{N} a_n e^{ix^2} (ix \frac{\varepsilon}{2} \sin(n\pi x) + n\pi \cos(n\pi x))
\]
so that, squaring real and imaginary parts, we find

\[
\|u(x)\|_{H^1_0(0, 1)}^2 = 2 \int_0^1 \left| \sum_{n=1}^{N} a_n n\pi \cos(n\pi x) \right|^2 \, dx + 2 \int_0^1 \left| \sum_{n=1}^{N} a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 \, dx \\
= \pi^2 \sum_{n=1}^{N} |a_n|^2 n^2 + 2 \int_0^1 \left| \sum_{n=1}^{N} a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 \, dx \\
\leq \pi^2 \sum_{n=1}^{N} |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \int_0^1 \left| \sum_{n=1}^{N} a_n \sin(n\pi x) \right|^2 \, dx \\
= \pi^2 \sum_{n=1}^{N} |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \sum_{n=1}^{N} |a_n|^2 \leq C(\varepsilon) \sum_{n=1}^{N} |a_n|^2 n^2.
\]

Lemma 3.7 Let \(\varepsilon \in (0, \frac{\pi}{2})\) and \(\tau = \frac{2}{\pi-2\varepsilon}\), then the functions \(b_n(t) = \frac{\sqrt{\pi}}{\sqrt{2\xi(t)}} e^{-i\pi^2 n^2 \frac{t}{\xi(t)}}\) for \(n \geq 1\) form an orthonormal system in \(L^2(0, \tau)\).

Proof Note that \(\frac{\tau}{\xi(t)} = \frac{\xi(t) - \xi(t')}{\xi(t')^2} = \frac{1}{\xi(t')^2}\). Therefore, the obvious change of variable \(x = \frac{t}{\xi(t)}\) reduces \(f_n\) to a standard trigonometric function on \([0, \frac{\tau}{\xi(t)}]\). Observe that \(\frac{\tau}{\xi(t)} = \frac{2}{\pi-2\varepsilon} \left(1 + \frac{2\varepsilon}{\pi-2\varepsilon}\right)^{-1} = \frac{2}{\pi}\). Now orthonormality easily follows.
3.5.1 Neumann Observation at the Boundary

Proof of Theorem 2.4 We start considering only the first term at $x = 0$. As in the proof of Lemma 3.6 we consider for a moment only initial data associated with finitely supported sequences $(a_n)$. Differentiating the representation formula Eq. 2.2 $u$ term by term yields

$$u_x(0, t) = \sum_{n=1}^{+\infty} a_n \left( \frac{2}{\ell(t)} \right)^{1/2} e^{-i\pi^2 t \frac{n^2}{\ell(t)}} \frac{n\pi}{\ell(t)},$$

and therefore

$$\|u_x(0, t)\|_{L^2(0, \tau)}^2 = \int_0^\tau \left( \sum_{n=1}^{+\infty} n a_n e^{-i\pi^2 t \frac{n^2}{\ell(t)}} \frac{n\pi}{\ell(t)} \right)^2 dt.$$

Using the monotonicity of $\ell(t)$ in $[0, \tau]$, we have $rac{2\pi^2}{\ell(t)} J \leq \|u_x(0, t)\|_{L^2(0, \tau)}^2 \leq 2\pi^2 J$ where

$$J = \int_0^\tau \left( \sum_{n=1}^{+\infty} n a_n e^{-i\pi^2 t \frac{n^2}{\ell(t)}} \frac{n\pi}{\ell(t)} \right)^2 \frac{dr}{\ell(t)^2}.$$

This allows to focus only on the integral $J$, where we abbreviate $b_n = n a_n e^{-i\pi^2 t \frac{n^2}{\ell(t)}}$ and make a change of variable $\xi = \frac{1}{\ell(t)} + \frac{1}{2} \left( \frac{1}{\ell(t)} + \frac{1}{2} \right)$. Letting $T = \frac{1}{\ell(\tau)} - \frac{1}{\ell(t)}$, the above double inequality rewrites as

$$\int_{-T/2}^{+T/2} \left( \sum_{n=1}^{+\infty} b_n e^{-i\pi^2 \frac{n^2}{T} \xi} \right)^2 d\xi \approx \|u_x(0, t)\|_{L^2(0, \tau)}^2.$$

The sequence $\lambda_n = \frac{\pi^2}{\ell(t)}$ satisfies the hypotheses of [22, Theorem 3.1 and Corollary 3.3] so that, for all $k > \frac{3}{2} \pi^2$ and $r = \epsilon/\pi^2$

$$\int_{-T/2}^{+T/2} \left( \sum_{n=1}^{+\infty} b_n e^{-i\pi^2 \frac{n^2}{T} \xi} \right)^2 d\xi \gg e^{-\frac{2k^2}{\pi^2}} \sum_{n=1}^{+\infty} |b_n|^2 = e^{-\frac{2k^2}{\pi^2}} \sum_{n=1}^{+\infty} |a_n|^2.$$

On the other hand side, if $T \in \{ m \frac{\epsilon}{\pi}, (m+1) \frac{\epsilon}{\pi} \}$, we have by periodicity and Parseval’s identity

$$\int_{-T/2}^{+T/2} \left( \sum_{n=1}^{+\infty} b_n e^{-i\pi^2 \frac{n^2}{T} \xi} \right)^2 d\xi \leq \int_{-(m+1) \frac{\epsilon}{\pi}}^{(m+1) \frac{\epsilon}{\pi}} \left( \sum_{n=1}^{+\infty} b_n e^{-i\pi^2 \frac{n^2}{T} \xi} \right)^2 d\xi = (m+1) \sum_{n=1}^{+\infty} |b_n|^2.$$

We conclude by Lemma 3.6 that

$$c(\epsilon) \|u_0\|^2_{H^1_0(0, 1)} \leq \|u_x(0, t)\|_{L^2(0, \tau)}^2 \leq C(\epsilon) \|u_0\|^2_{H^1_0(0, 1)}.$$

This inequality being true for all $u_0$ leading to finitely supported sequences $(a_n)$, it is true for any $u_0 \in H^1_0(0, 1)$ by density.

For second term at $x = \ell(t)$, we see for finitely supported sequences $(a_n)$ that

$$u_x(\ell(t), t) = \sum_{n=1}^{+\infty} (-1)^n a_n \left( \frac{2}{\ell(t)} \right)^{1/2} e^{-i\pi^2 t \frac{n^2}{\ell(t)}} \frac{n\pi}{\ell(t)} e^{t \frac{n^2}{2}}.$$
Taking the $L_2$-norm, one get the equivalent between $\|u_x(\ell(t), t)\|_{L_2}$ and $\|u_x(0, t)\|_{L_2}$

$$\|u_x(\ell(t), t)\|^2_{L_2(0, r)} = \int_0^T \left| \sum_{n=1}^{+\infty} (-1)^n a_n \left( \frac{2}{\ell(t)} \right)^{1/2} e^{-i\pi^2 n^2 t / \ell(t)} n \pi e^{i \xi \ell(t)} \right|^2 \, dt$$

$$= \int_0^r \frac{2 \pi^2}{\ell(t)} \left| \sum_{n=1}^{+\infty} (-1)^n n a_n \right|^2 e^{-i\pi^2 n^2 t / \ell(t)} \, dt \, d\ell(t).$$

Using Remark 2.5, we obtain

$$\|u_x(\ell(t), t)\|^2_{L_2(0, r)} = \frac{1}{\varepsilon^2} \|u_1(\ell(t), t)\|^2_{L_2(0, r)}. \quad (3.23)$$

Then, combining Eqs. 3.22 and 3.23 we get the desired result (the estimate Eq. 2.3).

### 3.5.2 Internal Point Observability

**Proof of Theorem 2.7** Since $\ell(t) \geq 1$ for all $t$,

$$\int_0^r \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 t / \ell(t)} \sin \left( \frac{n \pi a}{\ell(t)} \right) \right|^2 \, dt \geq \int_0^r \frac{2}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 t / \ell(t)} \sin \left( \frac{n \pi a}{\ell(t)} \right) \right|^2 \, dt.$$  

By definition, $\sin \left( \frac{n \pi a}{\ell(t)} \right) = \frac{1}{2i} \left( \exp \left( \frac{i n \pi a}{\ell(t)} \right) - \exp \left( -\frac{i n \pi a}{\ell(t)} \right) \right)$. Therefore,

$$\sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{r}} \sin \left( \frac{n \pi a}{\ell(t)} \right) = \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{r}} \left( e^{\frac{i n a}{\ell(t)}} - e^{-\frac{i n a}{\ell(t)}} \right)$$

$$= \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{r}} \left( e^{\frac{i n \pi a}{\ell(t)}} - e^{-\frac{i n \pi a}{\ell(t)}} \right).$$

For $n \in \mathbb{Z}$, we extend the series by $a_n = a_{-n}$, and $\lambda_n = \frac{\pi^2 n^2}{r} + \text{sign}(n)n \pi a$. The sequence $\lambda_n = \frac{\pi^2 n^2}{r}$ is regular and satisfies the hypotheses of [22, Theorem 3.1] with $r = \frac{\varepsilon}{\pi^2}$ and $C = a \pi$. We follow the lines of the proof of Theorem 2.4: changing the variable $\xi = \ell(t)$ gives with the notation $T = \frac{1}{\ell(t^0)} - \frac{1}{\ell(t)}$,

$$\int_0^T \frac{1}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{r}} \sin \left( \frac{n \pi a}{\ell(t)} \right) \right|^2 \, dt \geq \frac{1}{\varepsilon} \int_{-T/2}^{+T/2} \left| \sum_{n \in \mathbb{Z}} e^{-i\pi^2 n^2 \frac{1}{r}} a_n e^{i \lambda_n \xi} \right|^2 \, d\xi$$

we write $b_n = e^{-n^2 a^2} a_n$ and use [22, Corollary 3.3] with $k > \frac{3\pi^2}{2}$:

$$\frac{1}{\varepsilon} \int_{-T/2}^{+T/2} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\pi^2 \frac{1}{r}} e^{-i \lambda_n \xi} \right|^2 \, d\xi \gg e^{-2k} \sum_{n \in \mathbb{Z}} |a_n e^{-i\pi^2 \frac{1}{r}}|^2 \geq e^{-2k} \sum_{n=1}^{+\infty} |a_n|^2.$$  

For the upper estimate, we use similar method as in theorem Eq. 2.4. More precisely,

$$\|u(a, t)\|_{L_2(0, r)} \leq \int_0^T \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 t / \ell(t)} \right|^2 \, dt \lesssim (m + 1) \sum_{n=1}^{+\infty} |a_n|^2$$

where $m$ be the integer number such that $\frac{\pi^2}{T} \in [m, m + 1]$ with $T = \frac{1}{\ell(t^0)} - \frac{1}{\ell(t)}$. 

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3.5.3 \(L_p\)-Admissibility and Observability

**Proof of Theorem 2.9** The upper estimate yielding \(K_p(\tau)\) is obtained by interpolation of the two upper estimates in Theorem 2.4. We are left with the lower estimate. Since \(u \in H^1_0\), \((na_n) \in \ell_2\), and so \((a_n) \in \ell_1\) by the Cauchy-Schwarz inequality. Let \(p \in (0, 2)\) and let \(\theta = \frac{2}{\frac{2}{\theta} - 1} \in (0, 1)\) which is chosen to satisfy \(p\theta + 4(1 - \theta) = 2\). By Hölder’s inequality, we then have

\[
\int_0^\tau |u(a, t)|^2 dt = \int_0^\tau |u(a, t)|^{p\theta} |u(a, t)|^{4(1 - \theta)} dt \leq \left( \int_0^\tau |u(a, t)|^p dt \right)^{\theta} \left( \int_0^\tau |u(a, t)|^4 dt \right)^{1 - \theta}.
\] (3.24)

From trivial argument on boundedness of \(\sin\left(\frac{n\pi a}{\ell(t)}\right)\) and \(e^{i\pi a^2 - i\pi^2 x^2 t^{-\gamma}}\):

\[
|u(a, t)|^2 = \sum_{n=1}^{+\infty} a_n^2 e^{i\pi a^2 - i\pi^2 x^2 t^{-\gamma}} \sin\left(\frac{n\pi a}{\ell(t)}\right)^2 \leq \left( \sum_{n=1}^{+\infty} |a_n|^2 \right)^2.
\]

Combining this with the estimate Eq. 3.24, one get:

\[
\int_0^\tau |u(a, t)|^4 dt \leq \left( \sum_{n=1}^{+\infty} |a_n|^2 \right)^2 \left( \int_0^\tau |u(a, t)|^2 dt \right)^2.
\] (3.26)

From inequalities Eqs. 3.24 and 3.26 and Theorem (2.7) we deduce now

\[
\int_0^\tau |u(a, t)|^p dt \geq \left( \int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left( \int_0^\tau |u(a, t)|^4 dt \right)^{\theta - 1/\theta} \geq \left( \int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left( \sum_{n=1}^{+\infty} |a_n|^2 \right)^{2(\theta - 1/\theta)} \geq k \left( \sum_{n=1}^{+\infty} |a_n|^2 \right)^{2(\theta - 1/\theta)} \geq k \left\| u_0 \right\|_{L_2(0, 1)}^2 \left\| u_0 \right\|_{H^1_0}^{2(\theta - 1/\theta)}.
\]

Since \(\frac{\theta - 1}{\theta} = \frac{p - 2}{2}\), the result follows.

\[\square\]

4 Boundary Controllability of Dual Problem

This section is devoted to the proof of Theorem 2.6 on the exact controllability of the adjoint Schrödinger system Eq. 2.4.

**Proof of Theorem 2.6** Since we have already stated several theorems that can be interpreted as exact observation we will briefly sketch the duality theory that allows to rephrase these assertions in terms of exact control, then the solution \(z\) to adjoint problem

\[
z(t) = -A(t)^*z(t) - C(t)^*C(t)w(t) \quad z(0) = 0
\] (4.1)

satisfies \(<w_0, z(0)> \) = \(-\int_0^\tau \frac{d}{dt} (w(t), z(t)) dt = \int_0^\tau \|C(t)w(t)\|^2 dt\) by injection of the respective differential equations of \(w\) and \(z\). Hence exact observability implies that the Gramian \(Q : w_0 \mapsto z(0)\) satisfies \(\|Qw_0\| \geq \delta \|w_0\|\) to the effect that \(Q\)
has closed image. Moreover, if $Q^*w_0 = 0$, taking scalar product with $w_0$ reveals $w_0 = 0$, so $Q^*$ is injective and hence $Q$ has dense range. By the open mapping theorem, $Q$ is therefore an isomorphism on $X$. This means that the adjoint problem Eq. 4.1 can be steered to any state $z(0) \in X$ by an appropriate choice of the initial value $w_0$. Indeed, for $u, v \in D(A(t))$, we have

$$
(A(t)u, v)_X = \left( \frac{i}{\ell(t)} u_{yy} + \frac{\ell'(t)}{\ell(t)} yu_y, v \right)_X = \int_0^1 \frac{i}{\ell(t)^2} u_{yy} \overline{v} \, dy + \int_0^1 \frac{\ell'(t)}{\ell(t)} yu_y \overline{v} \, dy
$$

(int. by parts)

$$
= -\frac{i}{\ell(t)^2} \int_0^1 u_y \overline{v}_y \, dy - \int_0^1 \left( yu_y \overline{v} + y \overline{v} \right) \, dy
$$

(int. by parts)

$$
= \left( u, -\frac{i}{\ell(t)^2} u_{yy} + \frac{\ell'(t)}{\ell(t)} yu_y \right)_X = \left( u, \left( A(t) + \frac{\ell'(t)}{\ell(t)} \right) v \right)_X.
$$

It turns out that in our case $A(t)^* = -A(t) - \frac{\ell'(t)}{\ell(t)}$. So exact observation of the Schrödinger Eq. 1 can be reformulated as exact control for the Schrödinger equation with zero final time. We turn back to these ideas after stating our first theorem. In the case of linear moving $\ell(t) = 1 + \varepsilon t$, let $C(t) : D(A(t)) \to \mathbb{C}$ be given by $C(t)(\varphi)_b := \varphi_y(b)$ where $b \in \{0, 1\}$. The (lower) estimate in Theorems 2.4 and 2.3 then reformulates as exact observability of $C(t)$ for the non-autonomous Cauchy problem Eq. 3.8. Some care has to be taken since $C(t)$ is unbounded on $X$. Indeed, $C(t)^* : \mathbb{C} \to D(A(t))^\prime$ is given by $C(t)^* \alpha = -\alpha \frac{d}{dy} \delta_{y=b}$, then we obtain exact controllability of Eq. 4.1 in a distributional sense:

$$
z_t = \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} yz_y + \frac{\ell'(t)}{\ell(t)} z = w_y(b, t) \frac{d}{dy} \delta_{y=b} \quad \text{and} \quad z(y, \tau) = 0.
$$

Multiplying with a test function $\eta \in D((0, 1))$, and integrating on $[0, 1]$ we obtain by partial integration

$$
\int_0^1 z_t \eta(y) \, dy = \int_0^1 \left( \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} yz_y \right) \eta(y) \, dy - w_y(b, t) \eta'(b)
$$

$$
= \int_0^1 \left( \frac{i}{\ell(t)^2} z \eta''(y) - \frac{\ell'(t)}{\ell(t)} yz \eta'(y) \right) \, dy + \left( \frac{i}{\ell(t)^2} z(b, t) - w_y(b, t) \right) \eta'(b).
$$

This is possible for any test function $\eta$ only if the point evaluation vanishes. The dual statement of the lower estimate in Theorems 2.4 and 2.3 is thus exact controllability of a Schrödinger equation with Dirichlet control on the right boundary:

$$
\begin{cases}
    z_t = \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} yz_y + \frac{\ell'(t)}{\ell(t)} z & (y, t) \in (0, 1) \times (0, \tau) \\
    \left( \begin{array}{c}
    z(b, t) \\
    z(y, \tau)
    \end{array} \right) = \left( \begin{array}{c}
    0 \\
    0
    \end{array} \right) & t \geq 0 & \{b\} \cup \{y\} = \{0, 1\}
\end{cases}
$$

(4.2)

We reverse back to the moving boundary problem by taking $x = \ell(t)y$ and $h(x, t) = z(y, t)$. Then the problem can be written as

$$
\begin{cases}
    i\hbar_t + h_{xx} - i\frac{\ell'(t)}{\ell(t)} h = 0 & (x, t) \in (0, \ell(t)) \times (0, \tau) \\
    \hbar(\ell(t), t) = 0 & t \geq 0 \\
    h(0, t) = -i\ell(t)^3 u_x(0, t) & t \geq 0 \\
    h(x, \tau) = 0 & x \in (0, \ell(t)]
\end{cases}
$$

(4.3)
or
\[
\begin{align*}
  i\frac{\partial}{\partial t} + h_{xx} - i\frac{\partial}{\partial t} h &= 0 & (x, t) &\in (0, \ell(t)) \times (0, \tau) \\
  h(0, t) &= 0 & t &\geq 0 \\
  h(\ell(t), t) &= -i\ell(t)^3u_x(\ell(t), t) & t &\geq 0 \\
  h(x, \tau) &= 0 & x &\in [0, \ell(t)]. \\
\end{align*}
\]

In the general situation of \(\ell(t)\) satisfying condition Eq. 2.1, one takes \(C(t) : D(A(t)) \to \mathbb{C} \times \mathbb{C}\) be given by \(C(t)(\varphi) := (\varphi_y(0), \varphi_y(1))\). Therefore, the dual operator \(C(t)^* : \mathbb{C} \times \mathbb{C} \to D(A(t)^*)\) is given by \(C(t)^*(\alpha, \beta) = -\alpha \frac{d}{dx} \delta_{x=0} - \beta \frac{d}{dy} \delta_{y=1}\). Using similar arguments, we obtain exact controllability of a Schrödinger equation with Dirichlet control applied on both boundaries:
\[
\begin{align*}
  i\frac{\partial}{\partial t} + h_{xx} - i\frac{\partial}{\partial t} h &= 0 & (x, t) &\in (0, \ell(t)) \times (0, \tau) \\
  h(0, t) &= -i\ell(t)^3u_x(0, t) & t &\geq 0 \\
  h(\ell(t), t) &= -i\ell(t)^3u_x(\ell(t), t) & t &\geq 0 \\
  h(x, \tau) &= 0 & x &\in [0, \ell(t)]. \\
\end{align*}
\]

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