Renormalizable Expansion for Nonrenormalizable Theories: I. Scalar Higher Dimensional Theories

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Abstract

We demonstrate how one can construct renormalizable perturbative expansion in formally nonrenormalizable higher dimensional scalar theories. It is based on $1/N$-expansion and results in a logarithmically divergent perturbation theory in arbitrary high odd space-time dimension. The resulting effective coupling is dimensionless and is running in accordance with the usual RG equations. The corresponding beta function is calculated in the leading order and is nonpolynomial in effective coupling. It exhibits either UV asymptotically free or IR free behaviour depending on the dimension of space-time.

1 Introduction

Popular nowadays higher dimensional theories [1] suffer from the lack of renormalizable perturbative expansion. The usual coupling has a negative dimension, thus leading to power increasing divergencies which are out of control. Popular reasoning when dealing with such theories relies on higher energy (string) theory which is supposed to cure all the UV problems while the low energy one is treated as an effective theory basically at the tree level.

In our previous work [2] we considered the scalar theories in extra dimensions within the usual perturbative expansion and demonstrated that the leading divergences are governed by the one-loop diagrams even in the nonrenormalizable case, as it was shown
in [3]. Contrary to that work, we make here an attempt to construct renormalizable expansion in such formally nonrenormalizable theories. As in [2], we consider scalar higher dimensional theories as an example, but here we treat them in the framework of $1/N$-expansion (for review see [4]). The resulting perturbation theory is shown to be renormalizable, logarithmically divergent in any odd dimension $D$ and obtains an effective dimensionless expansion parameter.

We show below how one can construct such expansion and calculate the leading terms. One finds that resulting PT is nonpolynomial in effective coupling, but polynomial in $1/N$ and obeys the usual properties of renormalizable theory. It might be either UV asymptotically free or IR free depending on the space-time dimension $D$.

## 2 $1/N$-expansion

Let us start with the usual $N$ component scalar field theory in $D$ dimensions, where $D$ takes an arbitrary odd value, with $\phi^4$ self-interaction. The Lagrangian looks like

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \vec{\phi})^2 - \frac{1}{2} m^2 \vec{\phi}^2 - \frac{\lambda}{8N} (\vec{\phi}^2)^2,$$

where $N$ is the number of components of $\phi$. It is useful to rewrite it introducing a Lagrange multiplier $\sigma$ [5]

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \vec{\phi})^2 - \frac{1}{2} m^2 \vec{\phi}^2 - \frac{\sqrt{\lambda}}{2\sqrt{N}} \sigma (\vec{\phi}^2) + \frac{1}{2} \sigma^2.$$

Now one has two fields, one $N$ component and one singlet with triple interaction. Let us look at the propagator of the $\sigma$ field. At the tree level it is just "$i$", but then one has to take into account the corrections due to the loops of $\phi$ (see Fig.1).

![Figure 1](image1.png)

Figure 1: The chain of diagrams giving a contribution to the $\sigma$ field propagator in the zeroth order of $1/N$ expansion

If one follows the $N$ dependence of the corresponding graphs, one finds out that it cancels: they are all of the zeroth order in $1/N$. Thus, one can sum them up and get

$$\cdots = \frac{1}{1 - O_{- \nu}} = \frac{i}{1 + \lambda f(D)(-p^2)^{D/2 - 2}},$$

where

$$f(D) = \frac{\Gamma^2(D/2 - 1)\Gamma(2 - D/2)}{2^{D+1}\Gamma(D - 2)\pi^{D/2}}$$

and we put $m = 0$ for simplicity.
Notice that since $\lambda$ is positive, when $f(D) > 0$ the obtained propagator has no pole in the Euclidean region and has a cut for $p^2 > 0$. If $m \neq 0$, the cut starts at the threshold $p^2 = 4m^2$. On the contrary, when $f(D) < 0$ one has a pole in the Euclidean region which may cause trouble when integrating. Due to the factor $\Gamma(2 - D/2)$, $f(D)$ may have any sign depending on the value of $D$. In the case of negative $f(D)$ we take the integrals in a sense of a principle value. This is similar to what happens in logarithmic theories with the so-called renormalon chains [6], but contrary to $\log(p^2)$ the power of $p^2$ is positive in the whole Euclidean region. Notice also that $f(D)$ is finite for any odd $D$ despite naive power counting. This is due to the use of dimensional regularization: the one-loop diagrams in odd dimensions are finite since the gamma function has poles only at integer negative arguments and not at half-integer ones. This phenomenon can also be understood in other regularization techniques, but we do not discuss it here.

Thus, we have now the modified Feynman rules: the $\phi$ propagator is the usual one while the $\sigma$ propagator is given by eq. (3). One can now construct the diagrams using these propagators and the triple vertex having in mind that any closed cycle of $\phi$ gives an additional factor of $N$ and any vertex gives $1/\sqrt{N}$.

Let us first analyse the degree of divergence. Let us start with the $\phi$ propagator. If the diagram with two external $\phi$ lines contains $L$ loops, this means that it has $2L\phi$ vertices, $2L - 1$ $\phi$ lines and $L\sigma$ lines. Notice that each $\sigma$ line now behaves like $1/(p^2)^{D/2-2}$. Then, the degree of divergence is
$$\omega(G) = LD - (2L - 1)2 - L(D - 4) = 2!$$
for any $D$. Since this is a propagator, the divergence is proportional to $p^2$ and thus is reduced to the logarithmic one.

Let us now take the triple vertex. If it has $L$ loops, then one has $2L + 1$ vertices, $2L\phi$ lines and $L\sigma$ lines. Then, the degree of divergence is
$$\omega(G) = LD - (2L)2 - L(D - 4) = 0!$$
for any $D$. Thus, we again have only logarithmic divergence.

At last, consider the $\sigma$ propagator. In $L$ loops it has $2L$ vertices, $2L\phi$ lines and $L - 1\sigma$ lines. The degree of divergence is
$$\omega(G) = LD - (2L)2 - (L - 1)(D - 4) = D - 4.$$  

This means that in odd $D$ it has no global divergence (again we explore the properties of dimensional regularization) and the only possible divergencies are those of the subgraphs eliminated by renormalization of $\phi$ and the coupling. To see this, consider a genuine diagram for the $\sigma$-field propagator which is shown in Fig.2, where the blobs denote the 1PI vertex or propagator and vertex subgraphs.

![Figure 2: General type of the $\sigma$-field propagator](image-url)
After the $R'$ operation\footnote{The $R'$ operation means that we subtract from the diagram all divergent subgraphs} we do not have any poles in the integrand for the remaining one-loop integral. What is left is the finite part containing logarithms of momenta. This final integration has the following form:

$$\int \frac{\ln^n(k^2/\mu^2) \ln^m(k^2/p^2) \ln^k(k^2/(k-p)^2)}{k^2(k-p)^2} d^D k.$$ 

Due to the naive power counting of divergences in dimensional regularization we obtain the result proportional to $\Gamma(2-D/2)$ which is finite for any odd $D$. The logarithms can not change this property.

To demonstrate how this works explicitly, we consider a particular example of the two-loop diagram. The result of the $R'$-operation is shown in Fig.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{two_loop_diagram.png}
\caption{Demonstration of the global divergence cancellation in the two-loop diagram}
\end{figure}

After subtracting the divergence in a subgraph we have prior to the last integration

$$\int \frac{dk}{k^2(p-k)^2} \left[ \frac{1}{\Gamma(D/2 - 1 - \varepsilon)\Gamma(2 - \varepsilon)} \frac{1}{\Gamma(D/2 - 2)\Gamma(D/2 - 1)\Gamma(D/2 + 1 - 2\varepsilon)} \right] \frac{1}{(k^2)^\varepsilon},$$

where $D' = D - 2\varepsilon$. The pole terms in the integrand cancel and expanding it over $\varepsilon$ one gets $\log(k^2)$. The last integration gives

$$\Gamma(-1 + \varepsilon) \frac{\Gamma(D/2 - 1 - \varepsilon)\Gamma(2 - \varepsilon)\Gamma(2 + \varepsilon - D'/2)\Gamma(D'/2 - 1)\Gamma(D'/2 - 1 - \varepsilon)}{\Gamma(D/2 - 2)\Gamma(D/2 - 1)\Gamma(D/2 + 1 - 2\varepsilon)\Gamma(1 + \varepsilon)\Gamma(D' - 2 - \varepsilon)} \frac{p^2}{(p^2)^{2\varepsilon}} + \frac{1}{\varepsilon \Gamma(D/2 - 2)\Gamma(D/2 + 1)} \frac{\Gamma(2 - D'/2)\Gamma^2(D'/2 - 1)}{\Gamma(D' - 2)} \frac{p^2}{(p^2)^\varepsilon} = O(1).$$

Thus, after the $R'$ operation the diagram is finite and we do not need the $\sigma$ field renormalization.

This way one gets the perturbative expansion with only logarithmic divergences. We will show now that this is not expansion over $\lambda$ with a negative dimension equal to $D - 4$ but rather $1/N$ expansion with dimensionless coupling.

### 3 Properties of the $1/N$ expansion

Consider now the leading order calculations. We start with the $1/N$ terms for the propagator of $\phi$ and the triple vertex. One has the diagrams shown in Fig.4. Notice that besides the one-loop diagrams in the same order of $1/N$ expansion one has the two-loop diagram for the vertex.
Figure 4: The leading order diagrams giving a contribution to the $\phi$ field propagator and the triple vertex in $1/N$ expansion

Let us start with the diagram a). One has

$$I_a \sim \int \frac{d^{D'} k}{(2\pi)^D N} \frac{\lambda}{[(k-p)^2 - m^2][1 + \lambda f(D)(-k^2)^{D/2-2}]}$$

$$= \int \frac{d^{D'} k}{(2\pi)^D N} \frac{1}{[(k-p)^2 - m^2][1/\lambda + f(D)(-k^2)^{D/2-2}]}, \quad D' = D - 2\varepsilon.$$ 

This integral may have problems with evaluation if $f(D) < 0$. As we have already mentioned, we evaluate the integral in a sense of its principle value. Then the UV asymptotics is given by

$$I_a \Rightarrow \int \frac{d^{D'} k}{(2\pi)^D N f(D)(k-p)^2(-k^2)^{D/2-2}}.$$ 

One can see that the original coupling $\lambda$ plays the role of inverse mass and drops out from the UV expression. What is left is a dimensionless $1/N$ term.

Calculating the singular parts of the diagrams of Fig.4 in dimensional regularization with $D' = D - 2\varepsilon$ one finds

$$\text{Diag}.a \Rightarrow \frac{1}{\varepsilon N} A, \quad \text{Diag}.b \Rightarrow \frac{1}{\varepsilon N} B, \quad \text{Diag}.c \Rightarrow \frac{1}{\varepsilon N} C, \quad (7)$$

$$A = \frac{2\Gamma(D-2)}{\Gamma(D/2-2)\Gamma(D/2-1)\Gamma(D/2+1)\Gamma(2-D/2)}, \quad B = \frac{D}{4-D} A, \quad C = \frac{D(D-3)}{4-D} A.$$

The corresponding renormalization constants in the $\overline{MS}$ scheme are then

$$Z_2^{-1} = 1 - \frac{1}{\varepsilon N} A, \quad (8)$$

$$Z_1 = 1 - \frac{1}{\varepsilon N} \frac{B + C}{N}. \quad (9)$$

There is no any coupling in these formulas except for $1/N$. However, when renormalizing the coupling, i.e., replacing the bare coupling with the renormalized one times the corresponding $Z$ factors, we face the problem: one cannot take a bare (infinite) number of components and renormalize it. To overcome this difficulty, we introduce a new dimensionless coupling $h$ associated with the triple vertex (and not with the $\sigma$ propagator) as

$$\mathcal{L}_{\text{int}} = -\frac{\sqrt{h} \sqrt{\lambda}}{2\sqrt{N}} \sigma \phi^2.$$ 

5
Then in the leading order in $1/N$ the renormalization constants and the coupling take the form

\[
Z^{-1}_2 = 1 - \frac{hA}{\varepsilon N}, \\
Z_1 = 1 - \frac{hB}{\varepsilon N} - \frac{h^2 C}{\varepsilon N}, \\
h_B = (\mu^2)^\varepsilon h Z_1^2 Z_2^{-2} = h \left(1 - \frac{h}{\varepsilon} \frac{2(A + B)}{N} - \frac{h^2 C}{\varepsilon N}\right). 
\]

(10)

(11)

(12)

This is not, however, the final expression. To see this, we consider the next order of $1/N$ expansion. The corresponding diagrams for the $\phi$ propagator are shown in Fig.5. Again one can see that the $1/N^2$ terms contain not only the two-loop diagrams but also the three- and even four-loop ones.

![Diagrams](image)

Figure 5: The second order diagrams giving a contribution to the $\phi$ field propagator in $1/N$ expansion

All these diagrams are double logarithmically divergent, i.e., contain both single and double poles in dimensional regularization. We calculate the leading double pole after subtraction of the divergent subgraphs, i.e. perform the $R'$-operation. The answer is:

\[
\text{Diag.}a \Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{1}{2} A^2 h^2, \quad \text{Diag.}b \Rightarrow -\frac{1}{\varepsilon^2 N^2} AB h^2, \quad \text{Diag.}c \Rightarrow -\frac{1}{\varepsilon^2 N^2} A^2 h^2, \\
\text{Diag.}d \Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{1}{3} ACh^3, \quad \text{Diag.}e \Rightarrow \frac{1}{\varepsilon^2 N^2} \frac{2}{3} A^2 h^3, \quad \text{Diag.}f \Rightarrow \frac{1}{\varepsilon^2 N^2} \frac{2}{3} AB h^3, \\
\text{Diag.}g \Rightarrow \frac{1}{\varepsilon^2 N^2} ACh^4. 
\]

(13)

We performed the same calculation for the vertex diagrams, but there are too many of them to reproduce, so we present below only the final answer for the $Z$-factors.

Here we face a problem, namely, in subtracting the divergent subgraphs in the graphs e-g, we get the diagram which is absent in our expansion, since it is already included in our bold $\sigma$ line (see Fig.6).
There would be no problem unless this diagram is needed to match the so-called pole equations [7] which allow one to calculate the higher order poles in the Z factors from the single one. However, if we include this diagram in the $\sigma$ line, it will not change the latter, except for the additional $h$ factor coming from the vertex and not compensated by the propagator. Apparently, one can continue this insertion procedure and add any number of such loops not changing the order of $1/N$ expansion. The result is the sum of a geometrical progression

$$
\frac{1}{1/\lambda - \mathcal{O}} + \frac{1}{1/\lambda - \mathcal{O}} h \mathcal{O} \frac{1}{1/\lambda - \mathcal{O}} + \frac{1}{1/\lambda - \mathcal{O}} h \mathcal{O} \frac{1}{1/\lambda - \mathcal{O}} h \mathcal{O} \frac{1}{1/\lambda - \mathcal{O}} + \ldots
$$

$$= \frac{1}{1/\lambda - \mathcal{O}} \left( \frac{1}{1 - h \mathcal{O} / (1/\lambda - \mathcal{O})} \right) = \frac{1}{1/\lambda - (1 + h)\mathcal{O}}.
$$

Altogether this leads to the following effective Lagrangian for $1/N$ perturbation theory

$$L_{\text{eff}} = \frac{1}{2} (\partial_{\mu} \vec{\phi})^2 - \frac{\sqrt{h}}{2\sqrt{N}} \sigma (\vec{\phi}^2) + \frac{1}{2\lambda} \sigma^2 + \frac{1}{2} f(D) \sigma (\partial^2)^{D/2-2} \sigma (1 + h). \quad (14)$$

This means that in UV regime one should multiply every $\sigma$ line by $1/(1 + h)$.

Having all this in mind we come to the final expressions for the Z factors within the $1/N$ expansion:

$$Z_1 = 1 - \frac{1}{\varepsilon N} \left( \frac{Bh}{1 + h} + \frac{Ch^2}{(1 + h)^2} \right) + \frac{1}{\varepsilon^2 N^2} \left( \frac{3}{2} \frac{B^2 h^2}{(1 + h)^2} + \frac{ABh^2}{(1 + h)^2} \right) + \frac{11}{3} \frac{BCh^3}{(1 + h)^3} + \frac{4}{3} \frac{ACH^3}{(1 + h)^3} - \frac{2}{3} \frac{ABh^3}{(1 + h)^3} - \frac{2}{3} \frac{B^2 h^3}{(1 + h)^3} - \frac{2}{3} \frac{BCh^4}{(1 + h)^4} + \ldots + O\left( \frac{1}{\varepsilon N^2} \right), \quad (15)$$

$$Z_2^{-1} = 1 - \frac{1}{\varepsilon N} \frac{Ah}{1 + h} + \frac{1}{\varepsilon^2 N^2} \left( \frac{3}{2} \frac{A^2 h^2}{(1 + h)^2} + \frac{ABh^2}{(1 + h)^2} - \frac{2}{3} \frac{A^2 h^3}{(1 + h)^3} \right) + \frac{2}{3} \frac{ABh^3}{(1 + h)^3} + \frac{4}{3} \frac{ACH^3}{(1 + h)^3} - \frac{ACh^4}{(1 + h)^4} + \ldots + O\left( \frac{1}{\varepsilon N^2} \right). \quad (16)$$

In addition one has also the renormalization of $1/\lambda$ parameter which plays the role of a mass of the $\sigma$ field. In the leading order of $1/N$ expansion the relevant diagrams are shown in Fig.7.
Taking into account the previous discussion we obtain the following results

\[ \text{Diag.}a \Rightarrow \frac{h^2}{\varepsilon N(1+h)^2} F, \quad \text{Diag.}b \Rightarrow \frac{h^2}{\varepsilon N(1+h)^2} E, \quad \text{Diag.}c \Rightarrow \frac{h^3}{\varepsilon N(1+h)^3} G, \]

which gives the renormalization constant for \( \frac{1}{\lambda} \)

\[ Z_{1/\lambda} = 1 - \frac{1}{N\varepsilon} \left( \frac{(F+E)h^2}{(1+h)^2} + \frac{Gh^3}{(1+h)^3} \right). \]

4 Renormalization group in 1/N expansion

Having these expressions for the Z factors one can construct the coupling constant renormalization and the corresponding RG functions. One has as usual in the dimensional regularization

\[ h_B = (\mu^2)^\varepsilon h Z_1^2 Z_2^{-2} = (\mu^2)^\varepsilon \left( h + \sum_{n=1}^\infty \frac{a_n(h,N)}{\varepsilon^n} \right), \]

\[ Z_i = 1 + \sum_{n=1}^\infty \frac{c_n(h,N)}{\varepsilon^n}, \]

where the first coefficients \( a_n \) and \( c_n \) can be deduced from eqs.\((15,16)\).

This allows one to get the anomalous dimensions and the beta function defined as

\[ \gamma(h,N) = -\mu^2 \frac{d}{d\mu^2} \log Z = h \frac{d}{dh} c_1, \]

\[ \beta(h,N) = 2h(\gamma_1 + \gamma_2) = (h \frac{d}{dh} - 1) a_1. \]

With the help of eqs.\((15,16)\) one gets in the leading order of 1/N expansion\(^2\)

\[ \gamma_2(h,N) = -\frac{1}{N} \frac{Ah}{(1+h)^2}, \quad \gamma_1(h,N) = -\frac{1}{N} \left( \frac{Bh}{(1+h)^2} + \frac{2Ch^2}{(1+h)^3} \right), \]

\[ \beta(h,N) = -\frac{1}{N} \left( \frac{2(A+B)h^2}{(1+h)^2} + \frac{4Ch^3}{(1+h)^3} \right). \]

\(^2\)Note that the anomalous dimension of a field \( \gamma_2 \), is defined with respect to \( Z_2^{-1} \).
It is instructive to check the so-called pole equations \[7\] that express the coefficients of the higher order poles in $\varepsilon$ of the $Z$ factors via the coefficients of a simple pole. For $Z_2^{-1}$ one has, according to (16),

\[
c_1(h, N) = -\frac{1}{N} \frac{Ah}{1 + h}, \tag{25}
\]

\[
c_2(h, N) = \frac{1}{N^2} \left( \frac{3}{2} \frac{A^2 h^2}{(1 + h)^2} + \frac{AB h^2}{(1 + h)^2} - \frac{2}{3} \frac{A^2 h^3}{(1 + h)^3} - \frac{2}{3} \frac{AB h^3}{(1 + h)^3} + \frac{4}{3} \frac{ACH^3}{(1 + h)^3} - \frac{ACH^4}{(1 + h)^4} \right). \tag{26}
\]

At the same time the coefficient $c_2$ can be expressed through $c_1$ via the pole equations as

\[
h \frac{dc_2}{dh} = \gamma_2 c_1 + \beta \frac{dc_1}{dh}, \tag{27}
\]

which gives

\[
h \frac{dc_2}{dh} = \frac{1}{N^2} \frac{Ah}{1 + h} + \frac{1}{N^2} \left( \frac{2(A + B)h^2}{(1 + h)^2} + \frac{4Ch^3}{(1 + h)^3} \right) \frac{A}{(1 + h)^2}. \]

Integrating this equation one gets for $c_2$ the expression coinciding with (26) which was obtained by direct diagram evaluation. Notice that to get this coincidence the $h$-dependence in the denominator of eqs.(15,16) was absolutely crucial.

We have also checked the pole equations for the renormalized coupling. Combining eqs.(15) and (16) one gets for the renormalization constant $Z_h = Z_1^2 Z_2^{-2}$

\[
a_1(h, N) = -\frac{1}{N} \left( \frac{2(A + B)h^2}{1 + h} + \frac{4C h^3}{(1 + h)^2} \right), \tag{28}
\]

\[
a_2(h, N) = \frac{1}{N^2} \left( \frac{4(A + B)^2 h^3}{(1 + h)^3} + \frac{4(A + B)^2 h^4}{3(1 + h)^3} + \frac{28(A + B)C h^4}{3(1 + h)^3} \right.
\]

\[
-4 \frac{(A + B)C h^5}{(1 + h)^4} + \frac{6}{5} \frac{C^2 h^5}{(1 + h)^4} - \frac{16}{5} \frac{C^2 h^6}{(1 + h)^5} \right). \tag{29}
\]

At the same time from the pole equations one has

\[
(h \frac{d}{dh} - 1)a_n = \beta \frac{da_{n-1}}{dh}. \tag{30}
\]

We checked that the coefficient $a_2$ evaluated this way coincides with (29) which was obtained by direct diagram evaluation.

At last, in the leading order in $h$ when

\[
a_1(h, N) \simeq -\frac{2(A + B)h^2}{N} \quad \text{and} \quad \beta(h, N) \simeq -\frac{2(A + B)h^2}{N}
\]

eq.(30) gives a geometrical progression

\[a_n(h, N) = a_1(h, N)^n.\]
We have checked this relation up to third order in $1/N$ expansion and confirmed its validity.

Having expression for the $\beta$ function one may wonder how the coupling is running. The crucial point here is the sign of the $\beta$ function. One has

$$\frac{dh}{dt} = \beta(h, N) = -\frac{1}{N} \frac{4\Gamma(D-2)}{\Gamma(D/2-2)\Gamma(D/2-1)\Gamma(D/2+1)\Gamma(3-D/2)} \left( \frac{2h^2}{(1+h)^2} + \frac{D(D-3)h^3}{(1+h)^3} \right).$$

(31)

The beta function can also be rewritten as

$$\beta(h, N) = -\frac{1}{N} \frac{2^{D-1}\Gamma(D/2-1/2)(-)^{(D-1)/2}}{\Gamma(1/2)\pi\Gamma(D/2+1)} \left( \frac{2h^2}{(1+h)^2} + \frac{D(D-3)h^3}{(1+h)^3} \right),$$

(32)

that clearly indicates the theory is UV asymptotically free for $D = 2k + 1$, $k$ even and IR free for $k$-odd. Solution of the RG equation looks somewhat complicated, but for the small coupling in the leading order it simply equals the usual leading log approximation ($t = \log(Q^2/\mu^2)$

$$h(t, h) \simeq \frac{h}{1 - \beta_0 ht}, \quad \beta_0 = -\frac{1}{N} \frac{2^{D-1}\Gamma(D/2-1/2)(-)^{(D-1)/2}}{\Gamma(1/2)\pi\Gamma(D/2+1)}. \quad \text{(33)}$$

For example, for $D = 5, 7$ the beta function equals $\beta_0 = -256/15\pi^2N$ and $2^{12}/105\pi^2N$, respectively.

## 5 Conclusion

We conclude that even in a formally non-renormalizable theory it is possible to construct renormalizable $1/N$ expansion which obeys all the rules of a usual perturbation theory. The expansion parameter is dimensionless, the coupling is running logarithmically, all divergencies are absorbed into the renormalization of the wave function and the coupling. The propagator of the spurion field $\sigma$ behaves as $1/(p^2)^{D/2-2}$ which provides a better convergence of the diagrams. In the Euclidean space, depending on the value of $D$ it either contains no additional singularities, or has a simple renormalon type pole. The original dimensionful coupling plays a role of a mass and is renormalized multiplicatively. Expansion over this coupling is singular and creates the usual nonrenormalizable terms.

Properties of $1/N$ expansion do not depend on the space-time dimension if it is odd. For an even dimension our formulas have a singularity which originates from divergence of simple one-loop bubbles that are summed up in the denominator of the $\sigma$ field propagator. This requires a special treatment.

In fact, the $1/N$ expansion was considered in the same way in the 3 dimensional nonlinear $\sigma$-model which is also non-renormalizable in three dimensions [8]. It was shown that the resulting PT exhibits the properties of a renormalizable theory. Here we went further, we calculated the leading divergencies, checked the RG properties of the new expansion and revealed its nonpolynomial character.
There is one essential point that we omitted in our discussion. This is the analytical properties of the $\sigma$ field propagator and the unitarity of a resulting theory. As one can see from eq.(3) the propagator of the $\sigma$ field contains a cut starting from $p^2 = 0$ in massless case, thus leading to Källen-Lehmann representation with continuous spectral function. This can be interpreted as continuous spectrum of states. Then, the theory happens to be unitary in the full space including these states. We postpone discussion of this problem to a separate publication.

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References

[1] L.Randall and R.Sundrum, Phys.Rev.Lett., 83(1999) 4690; Phys.Rev.Lett., 83(1999) 3370; I.Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G.R.Dvali, Phys.Lett., B436 (1998) 257; N. Arkani-Hamed, S. Dimopoulos, and G.R.Dvali, Phys.Lett., B429 (1998) 263. see also Yu.A.Kybushin, Models with Extra Dimensions and Their Phenomenology, [hep-ph/0111027]; J.Hewett and M.Spiropulu, Particle Physics Probes of Extra Spacetime Dimensions, arXiv:hep-ph/0205106.

[2] D.I.Kazakov and G.S.Vartanov, J.Phys.A:Math.Gen., 39 (2006) 8051 (arXiv:hep-th/0509208).

[3] D.I.Kazakov, Theor.Math.Phys., 75 (1988) 440.

[4] M.Moshe, J.Zinn-Justin, Phys.Rept., 385 (2003) 69-228 (arXiv:hep-th/0306133).

[5] I.Ya.Aref’eva, Theor.Math.Phys., 29 (1976) 147; ibid 31 (1977) 3.

[6] M.Beneke, Phys.Rep., 317 (1999) 1; M.Beneke and V.Braun, ”Renormalons and power corrections” in B.Ioffe Festschrift, World Scientific, Singapore, 2001, v.3, p. 1719.

[7] G.t’Hooft, Nucl. Phys. B61 (1973) 455.

[8] I.Ya.Aref’eva, Ann. of Phys., 117 (1979) 393.