Sensitivity analysis for variational inequalities

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ABSTRACT

We investigate existence and regularity of the solution of a parametric variational inequality for small changes in the perturbation parameters. More precisely, assuming (pseudo) monotonicity of the operator, we extend a result of Kyparissis on the differentiability of the solution.

Moreover, following an idea of Ralph and Dempe, we show that the solution is locally piecewise continuously differentiable, the directional derivative of the solution exists and it can be evaluated by solving a linear variational inequality.

Keywords: Parametric Variational Inequalities, Solution Differentiability, Sensitivity Analysis.

1. INTRODUCTION

Several authors ([2, 4, 7, 13]) have recently developed sensitivity results for the following parametric variational inequality problem:

Find \( x^* \in R(\varepsilon) \) such that:

\[
\langle F(x^*, \varepsilon); z - x^* \rangle \geq 0, \quad \forall z \in R(\varepsilon)
\]
where \( R(\varepsilon) := \{ x \in \mathbb{R}^n \mid g(x, \varepsilon) \geq 0, \ h(x, \varepsilon) = 0 \} \), \( F : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n \), \( g : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^m \), \( h : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^p \) and, finally, \( \varepsilon \in \mathbb{R}^r \) is the vector of perturbation parameters.

Classical sensitivity analysis has been earlier developed, under various regularity assumptions ([10]), for a parametric nonlinear programming problem of the form:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x, \varepsilon) \\
g(x, \varepsilon) & \geq 0 \\
h(x, \varepsilon) & = 0
\end{align*}
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R} \) and \( g, h, \varepsilon \) are as in \( VI(\varepsilon) \).

Since \( VI(\varepsilon) \) and \( NLP(\varepsilon) \) are traditionally linked together, it has been quite common to translate classical results of the latter problem to the former one. For instance, we shall recall that a function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^r \) can be seen as the gradient of some function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), i.e. \( F(x) = \nabla f(x) \), provided the Jacobian matrix \( \nabla F(x) \) is symmetric for all \( x \in \text{dom}(f) \). If that is the case (the variational inequality is then said to be integrable), we have that the Karush-Kuhn-Tucker type conditions for the variational inequality coincide with the ordinary Karush-Kuhn-Tucker conditions for the optimization problem. Referring to \( NLP(\varepsilon) \), the following assumptions are now usual:

- The functions \( f, g \) and \( h \) are \( C^2 \) in both variables \((x, \varepsilon)\), at least in a neighborhood of \((x^*, \varepsilon^*)\), local solution of the unperturbed problem \( NLP(\varepsilon^*) \).
- Let \( x^* \) be a local solution of \( NLP(\varepsilon^*) \) and some constraint qualification condition hold at \( x^* \). Then, the Karush-Kuhn-Tucker optimality conditions (KKT) hold at \((x^*, \varepsilon^*)\), i.e. there exist multipliers \( \lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p \) such that:

\[
\nabla_x L(x^*, \lambda, \mu, \varepsilon^*) = 0
\]

\[
\lambda_i \geq 0, \quad \lambda_i g_i(x^*, \varepsilon^*) = 0, \quad i = 1, \ldots, m
\]

\[
g_i(x^*, \varepsilon^*) \geq 0, \quad i = 1, \ldots, m; \quad h_j(x^*, \varepsilon^*) = 0, \quad j = 1, \ldots, p,
\]

where \( L(x, \lambda, \mu, \varepsilon) = f(x, \varepsilon) - \lambda^T g(x, \varepsilon) + \mu^T h(x, \varepsilon) \) is the Lagrange function.

- The linear independence condition (LI) holds at \( x^* \) when the vectors \( \nabla_x g_i(x^*, \varepsilon^*), \quad i \in I(x^*, \varepsilon^*), \nabla_x h_j(x^*, \varepsilon^*), \quad j = 1, \ldots, p, \) are
linearly independent, \( I(x^*, \varepsilon^*) = \{ i \mid 1 \leq i \leq m, \ g(x^*, \varepsilon^*) = 0 \} \) being the set of active constraints at \((x^*, \varepsilon^*)\).

- The Mangasarian-Fromovitz constraint qualification condition (MFCQ) holds at \( x^* \) when:
  i) \( \nabla_x h_j(x^*, \varepsilon^*), j = 1, \ldots, p, \) are linearly independent;
  ii) \( \exists z \in \mathbb{R}^n \) such that:
    \[
    \nabla_x g_i(x^*, \varepsilon^*)z > 0, \quad i \in I(x^*, \varepsilon^*)
    \]
    \[
    \nabla_x h_j(x^*, \varepsilon^*)z = 0, \quad j = 1, \ldots, p.
    \]

- The strict Mangasarian-Fromovitz constraint qualification condition (SMFCQ) holds at \( x^* \) when:
  i) For each index \( i \) such that \( \lambda_i > 0 \) and \( j = 1, \ldots, p, \)
  \( \nabla_x g_i(x^*, \varepsilon^*) \) and \( \nabla_x h_j(x^*, \varepsilon^*) \), are linearly independent;
  ii) \( \exists z \in \mathbb{R}^n \) such that:
    \[
    \nabla_x g_i(x^*, \varepsilon^*)z > 0, \quad \lambda_i = 0, \quad i \in I(x^*, \varepsilon^*)
    \]
    \[
    \nabla_x g_i(x^*, \varepsilon^*)z = 0, \quad \lambda_i > 0
    \]
    \[
    \nabla_x h_j(x^*, \varepsilon^*)z = 0, \quad j = 1, \ldots, p.
    \]

Let us define the index subset \( I^*(\lambda, \mu) := \{ i \mid 1 \leq i \leq m, \ \lambda_i > 0 \} \), for each pair of Lagrange multipliers \((\lambda, \mu)\).

- The constant rank condition (CR) holds at \( x^* \) when there exists a neighborhood \( W \) of \((x^*, \varepsilon^*)\) such that, for any subset \( K \) of \( I(x^*, \varepsilon^*) \) and \( J \) of \( \{1, \ldots, p\} \), the family of vectors \( \{\nabla_x g_i(x, \varepsilon), i \in K\} \cup \{\nabla_x h_j(x, \varepsilon), j \in J\} \) has constant rank (possibly depending on \( K \) and \( J \)) for all vectors \((x, \varepsilon) \in W\).

- The strong second order conditions (SSOC) hold at \( x^* \), with respect to \((\text{w.r.t.}) \ (\lambda, \mu)\), when \( z^T \nabla^2_{xx} L(x^*, \lambda, \mu, \varepsilon^*) z > 0 \), for all \( z \neq 0 \) such that:
  \[
  \nabla_x g_i(x^*, \varepsilon^*)z = 0, \quad \lambda_i > 0
  \]
  \[
  \nabla_x h_j(x^*, \varepsilon^*)z = 0, \quad j = 1, \ldots, p.
  \]

- The general strong second order conditions (GSSOC) hold at \( x^* \) when SSOC hold at \( x^* \) for all \((\lambda, \mu)\) which satisfy KKT.

It is commonly known (see e.g. [10]) that SMFCQ is a necessary and sufficient condition to ensure uniqueness of multipliers \((\lambda, \mu)\) in KKT. Moreover, it is easy to prove the following implication:
Finally, if we let \( M(x, \varepsilon) \) denote the set of multipliers \((\lambda, \mu)\) which satisfy KKT at \((x, \varepsilon)\), MFCQ ensures that \( M(x, \varepsilon) \) is a nonempty polyhedral uniformly bounded set for each \((x, \varepsilon)\) in a neighborhood of \((x^*, \varepsilon^*)\) (see, e.g., [4]).

Definition 1. A function \( x : \mathbb{R}^r \to \mathbb{R}^n \) is said to be B-differentiable at \( \varepsilon_0 \) in the direction \( d \in \mathbb{R}^r \) when the directional derivative

\[
\mathcal{D}x(\varepsilon_0, d) = \lim_{t \to 0} \frac{x(\varepsilon_0 + td) - x(\varepsilon_0)}{t}
\]

is uniform for any direction \( d \) of unit length, i.e.

\[
x(\varepsilon_0 + d) = x(\varepsilon_0) + \mathcal{D}x(\varepsilon_0, d) + o(d).
\]

Definition 2. A function \( x() \) is a \( PC^1 \) function in a neighborhood of \( \varepsilon_0 \) when it is continuous and there exists a finite family of \( C^1 \) functions, say \( x^1() \), ..., \( x^k() \), defined in a neighborhood of \( \varepsilon_0 \), such that \( x() \in \{x^1(), \ldots, x^k()\} \) for all \( \varepsilon \) in such a neighborhood.

Clearly, if \( x() \) is a \( PC^1 \) function, then it is B-differentiable, but the converse does not necessarily hold true.

With reference to NLP(\( \varepsilon \)) problems, some authors ([10, 4]) proved the following basic sensitivity results:

Theorem 1. (Dempe [1], Kyparisis [4], Kojima [6], Shapiro [12]) Let \( f, g \) and \( h \) be \( C^2 \) functions in their variables, MFCQ, GSSOC, CR hold at \( x^* \) for the unperturbed problem NLP(\( \varepsilon^* \)). Then, for any \( \varepsilon \) in a neighborhood of \( \varepsilon^* \), there exists (locally) a continuous function \( x() \) which is the unique local solution of NLP(\( \varepsilon \)). Moreover, \( x(\varepsilon) \) is directionally differentiable at \( \varepsilon^* \) w.r.t. any direction \( d \) and the directional derivative \( \mathcal{D}x(\varepsilon^*, d) \) is the unique solution of the following convex quadratic programming problem

\[
\min_{x} \frac{1}{2} z^T \nabla^2_{xx}L(x^*, \lambda, \mu, \varepsilon^*) z + z^T \nabla_{x} L(x^*, \lambda, \mu, \varepsilon^*) d
\]

subject to:

\[
\begin{align*}
\nabla_{g_i}(x^*, \varepsilon^*) z + \nabla_{g_i}(x^*, \varepsilon^*) d & = 0, \quad i \in \Gamma(x^*, \mu) \\
\nabla_{g_i}(x^*, \varepsilon^*) z + \nabla_{g_i}(x^*, \varepsilon^*) d & \geq 0, \quad i \notin \Gamma(x^*, \varepsilon^*) \backslash \Gamma(x^*, \mu) \\
\nabla_{h_j}(x^*, \varepsilon^*) z + \nabla_{h_j}(x^*, \varepsilon^*) d & = 0, \quad j = 1, \ldots, p, 
\end{align*}
\]

for some vector \((\lambda, \mu) \in M(x^*, \varepsilon^*)\).
This result, however, does not allow to evaluate explicitly \( \nabla x(e^*, d) \), since we do not know which multiplier \((\lambda, \mu) \in M(x, e)\) has to be used in \(QP(\lambda, \mu)\) (clearly if we should assume \(LI\) or \(SMFCQ\) instead of \(MFCQ\), we would avoid such a problem). Kyparisis has presented a result which partially improves the problem, in the sense that it is possible to restrict the choice of \((\lambda, \mu)\) to the set of extreme points of the polytope \(M(x^*, e^*)\). However, the following results, due to Ralph and Dempe [10], solve definitively the problem:

**THEOREM 2.** Let us assume the same hypothesis of Theorem 1. Then the conclusions of Theorem 1 still hold true and, moreover:

1. \(x^*\) is locally \(PC^2\) and hence locally Lipschitz and \(B\)-differentiable;
2. \(\nabla x(e^*, d)\) uniquely solves \(QP(\lambda, \mu)\) for all \((\lambda, \mu) \in S(e^*, d), \) where \(S(e^*, d) := \arg\max_{\lambda, \mu} \{-\lambda \nabla_x g(x^*, e^*) d + \mu \nabla_x h(x^*, e^*) d \mid (\lambda, \mu) \in M(x^*, e^*)\} \).

2. **SENSITIVITY FOR VARIATIONAL INEQUALITIES**

For \(VI(e)\) we shall introduce a set of \(KKT\) type conditions, by means of the following Lagrangian function ((8)):

\[
L_D(x, \lambda, \mu, e) = F(x, e) - \lambda \nabla_x g(x, e) + \mu \nabla_x h(x, e).
\]

- Let \(x^*\) be a solution of \(VI(e^*)\); we say that the generalized Karush-Kuhn-Tucker (GKKT) conditions hold at \(x^*\) if and only if:
  
  \[
  L_D(x^*, \lambda, \mu, e^*) = 0
  \]
  \[
  \lambda_i \geq 0, \quad \lambda_i g(x^*, e^*) = 0, \quad i = 1, \ldots, m
  \]
  \[
  g(x^*, e^*) \geq 0, \quad i = 1, \ldots, m, \quad h_j(x^*, e^*) = 0, \quad j = 1, \ldots, p.
  \]

Conditions \(LI, MFCQ, SMFCQ, CR\) are applied to \(VI(e^*)\) without any change. Some second order conditions for \(VI(e)\) are introduced.

- The strong second order conditions (SSOC) hold at \(x^*\) with \((\lambda, \mu)\) if and only if:
  
  \[
  \epsilon^T \nabla_x L_D(x^*, \lambda, \mu, e^*) \epsilon > 0
  \]
  for all \(\epsilon \neq 0\) such that:
  
  \[
  \nabla_x g_i(x^*, e^*) \epsilon = 0, \quad i \in I^*(\lambda, \mu)
  \]
\[ \nabla_j h_j(x^*, \varepsilon^*)z = 0, \quad j = \ldots, p. \]

- The modified strong second order conditions (MSSOC) hold at \( x^* \) with \( (\lambda, \mu) \) if and only if \( z^T \nabla_x F(x^*, \varepsilon^*)z > 0 \) for all vectors \( z \) as in SSOC;
- The general modified strong second order conditions (GMSSOC) hold at \( x^* \) if and only if MSSOC hold at \( x^* \) with \( (\lambda, \mu) \), for all \( (\lambda, \mu) \in M(x^*, \varepsilon^*) \).

We are now ready to introduce the partial counterpart of Theorem 1 to variational inequalities.

**Theorem 3.** (Kyparisis [4]) Let GKKT, MFCQ, CR and GMSSOC hold at \( x^* \). Moreover, assume for any fixed \( \varepsilon, g_i, i = 1, \ldots, m, \) are concave functions and \( h_j, j = 1, \ldots, p, \) are affine functions in \( x \) and \( F \) is a \( C^1 \) function in both arguments. Then, for any \( \varepsilon \) in a suitable neighborhood of \( \varepsilon^* \), there exists (locally) a unique solution \( x(\varepsilon) \) of \( V(\varepsilon) \) and this solution is a continuous function of \( \varepsilon \). Moreover, \( x(\varepsilon) \) is directionally differentiable at \( \varepsilon^* \) w.r.t. any direction \( d \neq 0 \) and its directional derivative \( \nabla_x x(\varepsilon^*, d) \) uniquely solves, for some \( (\lambda, \mu) \) in the set of extreme points of \( M(x^*, \varepsilon^*) \), the following linear variational inequality:

Find \( z^* \in LR(\lambda, \mu) \) so that \( \forall z \in LR(\lambda, \mu), \)

\[ \langle \nabla L_D(x^*, \lambda, \mu, \varepsilon^*)z + \nabla L_D(x^*, \lambda, \mu, \varepsilon^*)d; z - z^* \rangle \geq 0 \quad LVI(\lambda, \mu) \]

where \( LR(\lambda, \mu) \) is the region defined in Theorem 1.

**Remark 1.** There are some slight differences between Theorem 1 and Theorem 3. GMSSOC is a stronger requirement than GSSOC and the assumptions of convexity of \( g_i \) and affinity of \( h_j \) are made to have that GMSSOC implies GSSOC (however the converse implication does not hold true).

First we shall improve the previous result in a very special case.

**Proposition 1.** Let the assumptions of Theorem 3 hold and let \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) be convex and such that \( F = \nabla f \). Then the conclusions of Theorem 3 still hold true and, moreover:

1. \( x(\varepsilon) \) is (locally) \( PC^1 \) and then locally Lipschitz and \( B \)-differentiable;
2. \( \nabla x(\varepsilon, d) \) uniquely solves \( LVI(\lambda, \mu) \) for all \( (\lambda, \mu) \in S(\varepsilon, d) \), where

\[
S(\varepsilon, d) := \arg\max_{\lambda, \mu} \{-\lambda \nabla_x g(x^*, \varepsilon^*)d + \mu \nabla_x h(x^*, \varepsilon^*)d \mid (\lambda, \mu) \in M(x^*, \varepsilon^*)\}. 
\]
PROOF. Obviously the results in Theorem 3 hold true. Moreover, $VI(\varepsilon)$ is integrable and, by the convexity assumption on $f$, it is equivalent to the following optimization problem:

$$\min_{x \in f(\mathbf{e})} f(x, \varepsilon).$$

(1)

MFCQ, CR and the regularity of $g$ and $h$ still hold true for the $\text{NLP}(\varepsilon)$ problem (1). Since $F$ is continuously differentiable, $f \in \mathcal{C}^2$. Moreover, $\text{GMSSOC}$ implies $\text{GSSOC}$, as:

$$\nabla_x L_D(x, \lambda, \mu, \varepsilon) = \nabla_x f(x, \varepsilon) - \lambda \nabla^2_{xx} g(x, \varepsilon) + \mu \nabla^2_{xx} h(x, \varepsilon)$$

$$= \nabla^2_{xx} f(x, \varepsilon) - \lambda \nabla^2_{xx} g(x, \varepsilon) + \mu \nabla^2_{xx} h(x, \varepsilon)$$

$$= \nabla^2_{xx} L(x, \lambda, \mu, \varepsilon).$$

Therefore we can apply Theorem 2 to (1) to conclude that $x(\varepsilon)$ is $PC^1$ and uniquely solves $QP(\lambda, \mu)$.

To complete our proof, we shall obtain $\text{LVI}(\lambda, \mu)$ from $QP(\lambda, \mu)$. First, we have the classical first order necessary (and sufficient, thanks to the convexity of $f$) condition:

$$\langle \nabla^2_{xx} L(x^*, \lambda, \mu, \varepsilon) z^* + \nabla^2_{xx} L(x^*, \lambda, \mu, \varepsilon) d; z - z^* \rangle \geq 0$$

$$\forall z \in LR(\lambda, \mu).$$

(2)

As already observed, $\nabla^2_{xx} L = \nabla_x L_D$ and, in a similar way, we deduce

$$\nabla^2_{xx} L(x, \lambda, \mu, \varepsilon) = \nabla^2_{xx} f(x, \varepsilon) - \lambda \nabla^2_{xx} g(x, \varepsilon) + \mu \nabla^2_{xx} h(x, \varepsilon)$$

$$= \nabla^2_{xx} f(x, \varepsilon) - \lambda \nabla^2_{xx} g(x, \varepsilon) + \mu \nabla^2_{xx} h(x, \varepsilon)$$

$$= \nabla_x L_D(x, \lambda, \mu, \varepsilon)$$

which completes the proof. □

A similar result can be proved in a more general setting. First, we note that $\text{LVI}(\lambda, \mu)$ can be seen as the first order condition of the following quadratic programming problem:

$$\min_{z \in LR(\lambda, \mu)} \frac{1}{2} z^T \nabla_x L_D(x, \lambda, \mu, \varepsilon) z + z^T \nabla_x L_D(x, \lambda, \mu, \varepsilon) d.$$

Define $\Phi(x, \varepsilon; y) = \frac{1}{2} \langle F(x, \varepsilon); x - y \rangle + \frac{1}{2} \langle F(y, \varepsilon); x - y \rangle$, for $y \in R(\varepsilon)$ fixed.
Definition 3. A function \( F : \mathbb{R}^n \to \mathbb{R}^n \) is said to be monotone if and only if:
\[
(F(x) - F(y); x - y) \geq 0 \quad \forall x, y \in \text{dom} F.
\]

Theorem 4. Let \( F \) be continuous and monotone in \( x^* \). Then \( x^* \) solves \( VI(\varepsilon^*) \), if and only if \( x^* \in \text{argmin}_{x \in R(\varepsilon^*)} \Phi(x, \varepsilon^*; x^*) \).

Proof. If \( x^* \) solves \( VI(\varepsilon^*) \), thanks to the monotonicity of \( F \), it solves also the following Minty variational inequality:
\[
(F(y, \varepsilon); y - x^*) \geq 0 \quad \forall y \in R(\varepsilon) \quad \text{MI}(\varepsilon)
\]
Then
\[
0 = \Phi(x^*, \varepsilon^*; x^*) \leq \frac{1}{2} \langle F(x^*, \varepsilon^*); x - x^* \rangle + \frac{1}{2} \langle F(x^*, \varepsilon^*); x - x^* \rangle
\]
for all \( x \in R(\varepsilon^*) \), i.e. \( x^* \) is a minimum point of \( \Phi(\cdot, \varepsilon^*, x^*) \).

Let now \( x^* \) be a minimum point of \( \Phi(\cdot, \varepsilon^*, x^*) \). For all \( x \in R(\varepsilon^*) \) we have
\[
\langle F(x^*, \varepsilon^*); x - x^* \rangle + \langle F(x, \varepsilon^*); x - x^* \rangle \geq 0.
\]
Adding up this inequality with the monotonicity condition, we get, for all \( x \in R(\varepsilon^*) \):
\[
\langle F(x^*, \varepsilon^*) + F(x, \varepsilon^*) + F(x, \varepsilon^* - F(x^*, \varepsilon^*); x - x^* \rangle \geq 0
\]
which implies that \( x^* \) solves the Minty inequality. By continuity, we conclude that \( x^* \) solves also the variational inequality \( VI(\varepsilon^*) \). \( \square \)

Now we prove the following result:

Proposition 2. If \( F(x, \varepsilon) \) satisfies GMSSOC, \( g(x, \varepsilon) \) is concave and \( h(x, \varepsilon) \) is affine, then \( \Phi(x, \varepsilon, x^*) \) satisfies GSSOC.

Proof. The vectors \( z \), used for both conditions, depend only on the constraints, thus the set of such vectors is the same in both cases. The Lagrange function related to the minimization of \( \Phi(\cdot, \varepsilon^*, x^*) \) is:
\[
L(x, \lambda, \mu, \varepsilon) = \Phi(x, \varepsilon, x^*) - \lambda g(x, \varepsilon) + \mu h(x, \varepsilon).
\]
Thus we have
\[
\nabla_x L(x, \lambda, \mu, \varepsilon) = \nabla_x \Phi(x, \varepsilon; x^*) - \lambda \nabla_x g(x, \varepsilon) + \mu \nabla_x h(x, \varepsilon)
\]
\[

\]
where

$$V_\epsilon \Phi(x, \epsilon; x^*) = \frac{1}{2} [\langle \nabla_x F(x, \epsilon), x - x^* \rangle + F(x, \epsilon) + F(x^*, \epsilon)].$$

Finally we get

$$V_\epsilon^2 L(x, \lambda, \mu, \epsilon) = \frac{1}{2} [2 \nabla_x F(x, \epsilon) + \langle \nabla_{xx} F(x, \epsilon), x - x^* \rangle]$$

$$- \lambda \nabla_{xx} g(x, \epsilon) + \mu \nabla_{xx}^2 h(x, \epsilon)$$

which must be evaluated at \((x^*, \epsilon^*)\) to conclude:

$$V_\epsilon^2 L(x^*, \lambda, \mu, \epsilon^*) = \nabla_x F(x^*, \epsilon^*) - \lambda \nabla_{xx}^2 g(x^*, \epsilon^*) + \mu \nabla_{xx}^2 h(x^*, \epsilon^*)$$

$$= \nabla_x L_0(x^*, \lambda, \mu, \epsilon^*).$$

By the concavity of \(g\) and affinity of \(h\) we obtain the thesis. \(\square\)

We shall now prove the following results, which may be considered as an extension to variational inequalities of Theorem 2.

**Theorem 5.** Under the same assumptions of Theorem 3 and assuming \(F\) is monotone, we have, besides the thesis of Theorem 3, that:

1. \(x_0\) is (locally) PC and thus locally Lipshitz and B-differen­
tiable:
2. \(\Delta x(\epsilon^*, d)\) uniquely solves LVI(\(\lambda, \mu\)) for all \((\lambda, \mu) \in S(\epsilon^*, d)\).

**Proof.** We begin to prove the last part of the theorem.

By the previous results, we conclude \(V_\epsilon(\epsilon)\) is equivalent to \(NL_\epsilon(\epsilon)\) for the function \(\Phi\) defined above. Moreover, we have proved that the hypotheses of our theorem ensure we can apply Theorem 2 to \(\Phi\) and thus we get \(\Delta x(\epsilon^*, d)\) is the unique solution of the problem:

$$\min_{y \in LR(\lambda, \mu)} \frac{1}{2} y^\top \gamma_{xx}^2 L(x^*, \lambda, \mu, \epsilon^*) y + y \nabla_{xx}^2 L(x^*, \lambda, \mu, \epsilon^*) d \quad QP(\lambda, \mu)$$

for all \((\lambda, \mu) \in S(\epsilon^*, d)\). By calculating the necessary (and sufficient) first order conditions for the said problem, we get the inequality:

$$\langle \nabla_{xx}^2 L(x^*, \lambda, \mu, \epsilon^*) y^* + \nabla_{xx}^2 L(x^*, \lambda, \mu, \epsilon^*) d, y - y^* \rangle \geq 0 \quad \forall y \in LR(\lambda, \mu)$$

for which we get \(\nabla_{xx}^2 L(x^*, \lambda, \mu, \epsilon^*) = \nabla_x L_0(x^*, \lambda, \mu, \epsilon^*),\) as already seen. In a similar way we have
\[ \nabla^2 \phi(x; \varepsilon, x^*) = \nabla^2 \phi(x; \varepsilon, x^*) = \frac{1}{2} \left[ (\nabla^2 \phi(x; \varepsilon, x^*) - \nabla^2 \phi(x; \varepsilon, x^*) + \nabla^2 \phi(x; \varepsilon, x^*) \right]. \]

Evaluating the expression at \((x^*, \varepsilon^*)\) we obtain

\[ \nabla^2 \phi(x^*, \lambda, \mu, \varepsilon^*) = \nabla^2 \phi(x^*, \lambda, \mu, \varepsilon^*) - \nabla^2 \phi(x^*, \lambda, \mu, \varepsilon^*) + \nabla^2 \phi(x^*, \lambda, \mu, \varepsilon^*). \]

Thus it is also proved the first part of the theorem, which is obtained by the thesis of Theorem 2. □

**Remark 2.** The monotonicity assumption in Theorem 5 is needed to recover the results of Theorem 4. In the proof of this last theorem we have basically applied a result, known as Minty Lemma, in order to ensure the equivalence between the two inequalities \(VI(\varepsilon)\) and \(MI(\varepsilon)\). This Lemma holds true also with the weaker assumption of pseudo-monotonicity. Therefore, also Theorem 5 can be stated with this weaker assumption on \(\phi\), without changing anything in the proof.

We recall that a function \(\phi: K \subseteq \mathbb{R}^n \rightarrow \mathbb{R}\) is said pseudo-monotone at \(x^*\) if, for all \(y \in K\), we have

\[ \langle F(x^*); y - x^* \rangle \geq 0 \Rightarrow \langle F(y), y - x^* \rangle \geq 0. \]  

**Remark 3.** In proposition 1 we have assumed the convexity of \(f\), which implies \(F = \nabla f\) is a monotone map, by a classical characterization of convexity (see e.g. [9]). It is not difficult to build up examples where Theorem 5 fails in absence of monotonicity of the operator.

The following example shows how Theorem 3 does not give a constructive way to evaluate directional derivatives of \(x^*(\varepsilon)\), whereas Theorem 5 works.

**Example 1.** Let \(F(x, \varepsilon) = x + 4 + \varepsilon\) and \(R(\varepsilon)\) be defined by the following inequalities

\[ g_1(x, \varepsilon) = -x^2 + 9 \]
\[ g_2(x, \varepsilon) = -3x + \varepsilon. \]

The unperturbed problem \((\varepsilon = 0)\) is trivially solved by \(x^* = 3\).
We shall consider the GKKT system to conclude that:

\[ \lambda_1 \geq 0 \quad \text{and} \quad \lambda_1 (9 - x^*) = 0 \implies \lambda_1 \geq 0 \]

\[ \lambda_2 \geq 0 \quad \text{and} \quad \lambda_2 (-3x^*) = 0 \implies \lambda_2 = 0. \]

Since only \( i = 1 \in I(x^*, \varepsilon^*) \), \( R(\varepsilon^*) \) satisfies MFCQ at \( x^* \).

Moreover, the linearity of \( F \) implies GMSSOC; the concavity of \( g_i \), \( i = 1, 2 \) and \( CR \) are trivially verified. Theorem 3 ensures \( x^*(\varepsilon) \) locally exists and, for any \( d \in \mathbb{R}^n \), the directional derivative \( D_x^*(\varepsilon^*; d) \) uniquely solves \( LVI(\lambda) \) for some \( \lambda \). However, this theorem allows to choose \( \lambda' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) to construct the linear variational inequality \( LVI(\lambda') \). If this is the choice, the feasible region would be:

\[
LR(\lambda') = \begin{cases} 
\nabla_x g_1(-3, 0)^T z = 0 \\
\nabla_x g_2(-3, 0)^T z + \nabla_e g_2(-3, 0)^T d \geq 0
\end{cases}
\]

and it is easy to conclude that for some \( d \in \mathbb{R}^n \), e.g. \( d = 1 \), \( LR(\lambda') \) is empty.

On the contrary, according to Theorem 5, one has to choose \( \lambda \) in \( S(-3, 0) = \{ [0 0] \} \) and thus it is possible to construct a non empty feasible region for \( LVI(\lambda) \).

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