We construct a one-parametric family of the double-scaling limits in the hermitian matrix model $\Phi^6$ for 2D quantum gravity. The known limit of Bresin, Marinari and Parisi [1] belongs to this family. The family is represented by the Gurevich-Pitaevskii solution of the Korteweg-de Vries equation which describes the onset of nondissipative shock waves in media with small dispersion. Numerical simulation of the universal Gurevich-Pitaevskii solution is made.

In this paper we construct a one-parametric family of the double-scaling limits in the hermitian matrix model $\Phi^6$ for two-dimensional quantum gravity. This family is described by a common solution of the Korteweg-de Vries (KdV) equation

$$v_t + vv_x + v_{xxx} = 0$$

and the following ordinary differential equation (ODE)

$$v_{xxxx} + 5v_{xx}/3 + 5(v_x)^2/6 + 5(x - tv + v^3)/18 = 0.$$
The leading term of the asymptotics of this solution as $|x|\to\infty$ is the solution of the cubic equation

$$x - tf + f^3 = 0. \quad (3)$$

In the particular case $t = 0$ the solution of $(3)$ is reduced to the known solution of $[1]$. On the other hand, the common solution of $(1)$ and $(2)$ exactly coincides $[2]$ with the well known Gurevich-Pitaevskii (G-P) special solution of the KdV equation $[3], [4]$ which universally describes the onset of oscillating dissipationless shock waves in media with small dispersion.

2. An essential development in the theory of two-dimensional quantum gravity based on the study of $n\times n$ hermitian matrix models

$$Z_n = \int dH \exp(-\beta Tr U(H)), \quad (4)$$

where $H\sim n\times n$-hermitian matrix, $U(z) = \sum_{j}^{N} g_j z^j$, was made in the series of works $[5] - [10]$. The progress was achieved, considering the double-scaling limits in models $(4)$, ($h \to 0$): $\beta = h^{-4-2/m} B$, $n/\beta = (1 + \delta x h^{4}) A$. The following remarkable circumstance was then observed $[7], [8]$:

It turns out $[7], [8]$ that the calculation of the second derivative of the limit of the non-regular part of $\ln Z_n$ (its regular part is irrelevant) is reduced to the finding of the limit of solutions $R_n$ of nonlinear difference equations

$$n/\beta = Q(R_n, R_{n+1}, R_{n-1}, ..., R_{n+l}, R_{n-l}), \quad (5)$$

which the right hand sides $Q$ uniquely defined by the potentials $U(z)$. It was established that for any $U(z)$ there exist a natural $m > 1$ and constants $A$, $B$ and $\delta$, such that the asymptotics of solutions $R_n$ of $(5)$ is described by the formula ($\rho$ is a constant):

$$R_n = \rho(1 + h^{4/m} v(x) + ...) / 3,$$

where functions $v(x)$ satisfy of the first Painleve ODE (for $m = 2$) and its higher analogues (for $m > 2$). Before passing to a detailed discussion of a particular case of the potential

$$U(z) = z^2/2 + g_1 z^4 + g_2 z^6, \quad (6)$$
which we consider in this article, we would like to give the following useful general statement which should be taken into account when calculating double-scaling limits:

For any integer \( k \) the functions \( v(n + k) \) are expanded into Tailor series

\[
v(n + k) = v(x + hk) = v(x) + \sum_{l=1}^{\infty} (hk)^l v^{(l)}(x)/(l!),
\]

in which the constant \( c \) does not depend on the set of \( g_j \) of (4), the natural \( m \), and the constants \( A, B, \beta, \delta, \rho \).

Remark. To make the results of the present paper compatible with \([3]\) and \([4]\), we use notations different from \([1]\). This is the reason why we fix the following particular value of the constant \( c \): \( c = \sqrt{6} \).

3. The equation (5) for the potential (6) has the form \([11]\):

\[
n/\beta = R_n(1 + 4J_1(n) + 6J_2),
\]

\[
J_1(n) = R_{n+1} + R_n + R_{n-1}, J_2(n) = (J_1(n))^2 + R_{n+2}R_{n+1} - R_{n+1}R_{n-1} + R_{n-1}R_{n-2}.
\]

In the case of the general position, one has the case \( m = 2 \). Assuming that the constants \( \delta, \rho, A, g_1 \) and \( g_2 \) do not depend of \( h \), the substitution of

\[
n/\beta = A(1 + \delta h^4 x), R_n = \rho(1 + h^2 v(x) + ...)/3
\]

into (3) and equating the coefficients at different powers of \( h \) to the zero gives the following sequence of relations \([11]\):

\[
h^0 : \rho(60\rho^2 g_2 + 12\rho g_1 + 1) = A,
\]

\[
h^2 : \rho^2(240\rho g_2 + 24g_1) = -2A,
\]

\[
h^4 : -(g_1 + 10\rho g_2)\delta x = (15\rho g_2 + g_1)(2v_{xx} + v^2).
\]

It follows from the first two relations that \( \rho \) is a solution of the quadratic equation

\[
180g_2\rho^2 + 24g_1\rho + 1 = 0.
\]

Except for the degenerate case

\[
g_2 = -g_1/(15\rho) = 4g_1^2/5,
\]
(i.e. the case of the multiplicity of roots in (13)), the function $v(x)$ turns out to be a solution of the first Painleve equation.

In the degenerated case, one needs another limiting transition. Assuming that (14) is true, Bresin, Marinari and Parisi considered in [1] the following limiting transition:

$$R_n = -(1 + h^2 v(x))/12(g_1), \ n/\beta = -(1 - h^6 x)/36(g_1),$$

where $v$ is solution of ODE

$$v_{xxxx} + 5v_{xx}/3 + 5(v_x)^2/6 + 5(x + v^3)/18 = 0,$$

$$v(x) \sim -x^{1/3}, \ |x| \to \infty.$$ 

The numerical solution of the boundary value problem (16),(17) was obtained in [1] (it was based on a difference scheme in which the ODE (16) was replaced by the starting discrete equation (8).) That numerical calculation has shown the uniqueness of the solution of the boundary value problem (16), (17).

The work [1] had a wide response. The limiting transition considered in that paper was studied later from different points of view in a series of articles (see e.g. example [11],[12],). An unexpected connection of that transition with the an old problem of the onset of dissipationless shock waves was discovered in 1994 in [2]. It was observed that the solution of the boundary value problem (16) coincided with the G-P special solution of the equation (1) at $t = 0$.

However, in spite of the significance of [1], the given analysis of the limiting transition in the degenerated case was not satisfactory from the viewpoint of a general requirement for investigations of degenerate cases of that kind. According to the same requirement going back to H. Poincare, "... the investigation of degenerate systems should not be restricted by the study of the picture in the point of degeneration, but should include the description of the reorganizations which take place when the parameter passes through the degenerated value" [13].

It will be shown in the next section if we try to satisfy that requirement and take into account the additional statement (4), we are led to the replacement of the boundary value problem (16),(17) by the one-parametric family of boundary value
problems (2), (3). The particular case of the double-scaling limit in the degenerate case (14) exactly corresponds to $t = 0$.

4. Assume that the critical difference $g_1 - 4g_2^2/5$ has the order $O(h^p)$:

$$g_1 = a + bh^p + ..., g_2 = 4a^2/5 + rh^p + ....$$

(Here $p, a, b$ and $r$ are constants independent on $h$.) It follows from the square equation (13) follows that in this case

$$\rho = -1/(12a) + O(b^{1/2}h^{p/2}),$$

and the relation (12) implies that the parameter $\delta$ should also be small:

$$\delta = \gamma h^q + ...$$

($q, \gamma$ are constants independent on $h$). Assuming as before that (10), (11) are true, substituting the relations (9), (19) and (20) into (8), and then equating to zero the coefficients at $h^i$, we automatically (upon excluding of irrelevant trivial cases) come to the conclusion that

$$p = 4, q = 2, g_1 = a(1 - th^4/3 + ...), g_2 = 4a^2(1 - th^4 + ...) / 5.$$

The parameter $t$ is calibrated in accordance with notations of [4].

Taking into account the relations (9), (19) and (20), we obtain after some changes of notation a one-parametric family of double-scaling limits

$$n/\beta = -(1 - h^6x + ...)/(36a), R_n = -(1 + h^2v(t, x) + ...)/(12a),$$

where $v$ is a solution of the ODE (2). (The last statement is obtained if one substitutes (7), (21), (22) into the discrete string equation (8) and equates coefficient at $h^6$ to zero.)

5. The solution $v(x)$ of the ODE (16) of [1] satisfies the boundary conditions (17).

On the other hand, the ODE (2) is isomonodromic [2], and the complete set of its monodromic data is completely defined by the boundary condition (17). (It has been calculated at $t = 0$ by Moore in [12].) This implies (see [2]) that the solution $v(t, x)$ of the ODE (2) is at the same time the solution of the KdV equation (1), and that the
leading term of its asymptotics as \(|x| \to \infty\) is the solution of the cubic equation (3). According to [3], exactly this feature defines a special solution of the KdV equation which describes in a general way the onset of dissipationless shock waves in problems with small dispersion.

Actually, the corresponding solution (3) is the leading term of the asymptotics of \(v(t, x)\) as \(t \to -\infty\) for any \(x\) as well. If \(t \to \infty\), this asymptotics is true for any \(x\) except for a limited (but expanding with the growth of \(t\)) area filled with high frequency oscillations which correspond to the process of dissipationless shock wave generation.

For the description of the behavior of \(v(t, x)\) in this area it was suggested in [4] to use the self-similar solutions of the Whitham equations [14] which arise after averaging with respect to the period of a "knoidal" wave (one-phase periodic solution of (1)). The self-similar substitution suggested by Gurevich and Pitaevskii (correlating to the self-similarity of the solution of (3)) reduces the Whitham equations to the ODE, which was numerically investigated in [4]. Later a corresponding solution of the ODE was found in an explicit form by Potemin [15]. (This solution of Potemin, as has been pointed in [4], may be easily found, using the results of [10] and the fact that the ODE (2) is valid for \(v(t, x)\).) Potemin also has found the exact values of the so-called trailing \(s_-\) and leading \(s_+\) fronts, which define the boundary of the oscillation region \((s = x/t^{3/2})\):

\[
\begin{align*}
    s_- &= -\sqrt{2}, \\
    s_+ &= \sqrt{10}/27
\end{align*}
\]

numerically established in [4] before.

The next step to clarify the behaviour of \(v(t, x)\) was done in the recent paper [17] in which the uniform behavior of the GP solution in the neighborhood of the trailing edge was studied (the answer is obtained in the terms of the separatrix solution of the second Painleve solution). Besides, the speculations of [17] show that the "average" description of [4] and the real leading term of G-P solution asymptotics are probably the same within the oscillation region up to the shift of phase of \(\pi/2\). However, those results concern only to the behavior of \(v(t, x)\) with great \(t\). Taking into account the universal nature of the G-P special solution, the problem of numerical simulation of the G-P solution’s behavior with a finite \(|t|\) (solved in the last section of this article)
is of interest as well. To solve this problem, we use an elementary iterative scheme similar to one of [1] in which ODE (2) is replaced by a discrete equation equivalent to the discrete string equation (8). Namely, the following procedure is used (\([-L, L]\) is a numerically investigated interval, \(\epsilon = L/N\) is the step of the uniform net):

\[
v_{n+1}(k) = v_n(k) + P[v_n(k), k],
\]

\[
P[v(k), k] = R_k[1 + 4g_1J_1(k) + 6g_2J_2(k)] - (1 - \epsilon^7k\sqrt{6})/3,
\]

\[
J_1(k) = R_{k+1} + R_k + R_{k-1}, J_2(k) = (J_1(k))^2 + R_{k+2}R_{k+1} - R_{k+1}R_{k-1} + R_{k-1}R_{k-2},
\]

\[
g_1 = -(1 - t\epsilon^4/3)/12, g_2 = (1 - t\epsilon^4)/180, R_k = (1 + \epsilon^2v_k),
\]

\[
v_0(0) = 0, v_0(k) = -(\epsilon k\sqrt{6})^{1/3} - t(\epsilon k\sqrt{6})^{-1/3}/3 (k \neq 0)
\]

and in all approximations for \(|k| > N - 1\): \(v_n(k) = v_0(k)\). Results of the numerical analysis 1, 2 correspond to the results of [4] on the qualitative level. In particular, a good correspondence has been observed between theoretical results and the positions of the trailing \(s_-\) and the leading \(s_+\) fronts of the oscillation region.

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