Fractional Yamabe problem on locally flat conformal
infinities of Poincaré-Einstein manifolds

Martin MAYER\textsuperscript{a}, Cheikh Birahim NDIAYE\textsuperscript{b}

\textsuperscript{a} Scuola Superiore Meridionale,
Via Mezzocannone 4, Naples, ITALY.

\textsuperscript{b} Department of Mathematics of Howard University
Annex 3, Graduate School of Arts and Sciences, # 217
DC 20059 Washington, USA.

Abstract

We study the fractional Yamabe problem first considered by González-Qing\textsuperscript{36} on the conformal infinity 
\((M^n, [h])\) of a Poincaré-Einstein manifold \((X^{n+1}, g^+ )\) with either \(n = 2\) or \(n \geq 3\) and \((M^n, [h])\) locally flat - namely \((M, h)\) is locally conformally flat. However, as for the classical Yamabe problem, because of the involved quantization phenomena, the variational analysis of the fractional one exhibits a local situation and also a global one. The latter global situation includes the case of conformal infinities of Poincaré-Einstein manifolds of dimension either \(n = 2\) or of dimension \(n \geq 3\) and which are locally flat, and hence the minimizing technique of Aubin\textsuperscript{4}-Schoen\textsuperscript{48} in that case clearly requires an analogue of the positive mass theorem of Schoen-Yau\textsuperscript{49}, which is not known to hold. Using the algebraic topological argument of Bahri-Coron\textsuperscript{8}, we bypass the latter positive mass issue and show that any conformal infinity of a Poincaré-Einstein manifold of dimension either \(n = 2\) or of dimension \(n \geq 3\) and which is locally flat admits a Riemannian metric of constant fractional scalar curvature.

Key Words: Fractional scalar curvature, Poincaré-Einstein manifolds, Variational methods, Algebraic
topological argument, Bubbles, Barycenter spaces.

AMS subject classification: 53C21, 35C60, 58J60, 55N10.
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## 1 Introduction and statement of the results

In recent years there has been a lot of study about fractional order operators in the context of elliptic theory with nonlocal operators, nonlinear diffusion involving nonlocal operators, nonlocal aggregations and balance between nonlinear diffusions and nonlocal attractions. The elliptic theory of fractional order operators is well understood in many directions like semi-linear equations, free-boundary value problems and non-local minimal surfaces (see [13], [14], [15], [16], [17], [25]). Furthermore a connection to classical conformally covariant operators arising in conformal geometry and their associated conformally invariant geometric variational problems is established (see [18], [24], [26], [28], [33], [35], [36]). In this paper we are interested in the latter aspect of fractional order operators, precisely in the fractional Yamabe problem first considered by Gonzalez-Qing [36].

To discuss the fractional Yamabe problem, we first recall some definitions in the theory of asymptotically hyperbolic metrics. Given $X = X^{n+1}$ a smooth manifold with boundary $M = M^n$ and $n \geq 2$, we say that $\varrho$ is a defining function of the boundary $M$ in $X$, if

$$\varrho > 0 \text{ in } X, \quad \varrho = 0 \text{ on } M \text{ and } d\varrho \neq 0 \text{ on } M.$$  

A Riemannian metric $g^+$ on $X$ is said to be conformally compact, if for some defining function $\varrho$ the Riemannian metric

$$g := \varrho^2 g^+$$  

extends to $\overline{X} := X \cup M$ so that $(\overline{X}, g)$ is a compact Riemannian manifold with boundary $M$ and interior $X$. Clearly this induces a conformal class of Riemannian metrics

$$[h] = [g|_{TM}]$$  

on $M$, where $TM$ denotes the tangent bundle of $M$, when the defining functions $\varrho$ vary. The resulting conformal manifold $(M, [h])$ is called conformal infinity of $(X, g^+)$. Moreover a Riemannian metric $g^+$ in $X$ is said to be asymptotically hyperbolic, if it is conformally compact and its sectional curvature tends to $-1$ as one approaches the conformal infinity of $(X, g^+)$, which is equivalent to

$$|d\varrho|_{g^+} = 1$$
on $M$, see [14], and in such a case $(X, g^+)$ is called an asymptotically hyperbolic manifold. Furthermore a Riemannian metric $g^+$ on $X$ is said to be conformally compact Einstein or Poincaré-Einstein (PE), if it is asymptotically hyperbolic and satisfies the Einstein equation

$$\text{Ric}_{g^+} = -ng^+, \tag{1}$$

where $\text{Ric}_{g^+}$ denotes the Ricci tensor of $(X, g^+)$. On one hand for every asymptotically hyperbolic manifold $(X, g^+)$ and every choice of the representative $h$ of its conformal infinity $(M, [h])$ there exists a unique geodesic defining function $y$ of $M$ in $X$ such that in a tubular neighborhood of $M$ in $X$ the Riemannian metric $g^+$ takes the following normal form

$$g^+ = \frac{dy^2 + h_y}{y^2}, \tag{2}$$

where $h_y$ is a family of Riemannian metrics on $M$ satisfying $h_0 = h$. We say that the conformal infinity $(M, [\hat{h}])$ of an asymptotically hyperbolic manifold $(X, g^+)$ is locally flat, if $h$ is locally conformally flat, and clearly this is independent of the representative $h$ of $[\hat{h}]$. Moreover we say that $(M, [h])$ is umbilic, if $(M, h)$ is umbilic in $(X, g)$ where $g$ is given by (1) and $y$ is the unique geodesic defining function given by (2), and this is again independent of the representative $h$ of $[\hat{h}]$, as easily seen from the uniqueness of the normal form (2) or Lemma 2.3 in [36]. Similarly we say that $(M, [h])$ is minimal if $H_y = 0$ with $H_y$ denoting the mean curvature of $(M, h)$ in $(X, g)$ with respect to the inward direction, and this is again clearly independent of the representative of $h$ of $[\hat{h}]$, as easily seen from Lemma 2.3 in [36]. Finally we say that $(M, [h])$ is totally geodesic, if $(M, [h])$ is umbilic and minimal.

**Remark 1.1.** We remark that in the conformally compact Einstein case $h_y$ as in (2) has an asymptotic expansion which contains only even powers of $y$, at least up to order $n$, see [18]. In particular the conformal infinity $(M, [h])$ of any Poincaré-Einstein manifold $(X, g^+)$ is totally geodesic.

**Remark 1.2.** As every 2-dimensional Riemannian manifold is locally conformally flat, we will say locally flat conformal infinity of a Poincaré-Einstein manifold to mean just the conformal infinity of a Poincaré-Einstein manifold when $n = 2$, or which is furthermore locally flat, when $n > 2$.

On the other hand to any asymptotically hyperbolic manifold $(X, g^+)$ with conformal infinity $(M, [h])$ Graham-Zworsky [28] have attached a family of scattering operators $S(s)$, which is a meromorphic family of pseudo-differential operators on $M$ defined on $\mathbb{C}$, by considering Dirichlet-to-Neumann operators for the scattering problem for $(X, g^+)$ and a meromorphic continuation argument. Indeed it follows from [28] and [15] that for every $f \in C^\infty(M)$ and for every $s \in \mathbb{C}$ such that $\text{Re}(s) > \frac{n}{2}$ and $s(n - s)$ is not an $L^2$-eigenvalue of $-\Delta_{g^+}$ the following generalized eigenvalue problem

$$-\Delta_{g^+} u - s(n - s) u = 0 \quad \text{in} \quad X \tag{3}$$

has a solution of the form

$$u = F y^{n-s} + G y^s, \quad F, G \in C^\infty(\overline{X}), \quad F|_{y=0} = f,$$

where $y$ is given by (2) and for those values of $s$ the scattering operator $S(s)$ on $M$ is defined as

$$S(s) f = G|_{M}. \tag{4}$$

Furthermore, using a meromorphic continuation argument, Graham-Zworsky [28] extend $S(s)$ defined by (1) to a meromorphic family of pseudo-differential operators on $M$ defined on all $\mathbb{C}$ and still denoted by $S(s)$ with only a discrete set of poles including the trivial ones $s = \frac{n}{2}, \frac{n}{2} + 1, \ldots$, which are simple poles of finite rank, and possibly some others corresponding to the $L^2$-eigenvalues of $-\Delta_{g^+}$. Using the regular part of the scattering operators $S(s)$, to any $\gamma = s - \frac{n}{2} \in (0, 1)$ such that

$$\left(\frac{n}{2}\right)^2 - \gamma^2 < \lambda_1(-\Delta_{g^+})$$
with \( \lambda_1(-\Delta_{g^+}) \) denoting the first eigenvalue of \(-\Delta_{g^+} \), Chang-Gonzalez\cite{18} have associated the following fractional order pseudo-differential operators, referred to as fractional conformal Laplacians or fractional Paneitz operators

\[
P^\gamma[g^+, h] := -d_\gamma S \left( \frac{n}{2} + \gamma \right),
\]

where \( d_\gamma \) is a positive constant depending only on \( \gamma \) and chosen such that the principal symbol of \( P^\gamma[g^+, h] \) is exactly the same as the one of the fractional Laplacian \((-\Delta_h)\gamma\), when

\[
X = \mathbb{R}_{n+1}^+, \quad M = \mathbb{R}^n, \quad h = g_{\mathbb{R}^n}, \quad \text{and} \quad g^+ = g_{\mathbb{R}^{n+1}}.
\]

When there is no possible confusion with the metric \( g^+ \), we just use the simple notation

\[
P^\gamma_h := P^\gamma[g^+, h].
\]

Similarly to the other well studied conformally covariant differential operators Chang-Gonzalez\cite{18} associate to each \( P^\gamma_h \) the curvature quantity

\[
Q^\gamma_h := P^\gamma_h(1).
\]

The functions \( Q^\gamma_h \) are referred to as fractional scalar curvatures, fractional \( Q \)-curvatures or simply \( Q \)-curvatures. Of particular importance to conformal geometry is the covariance property

\[
P_{h_v}^\gamma(v) = v^{-\frac{4-2\gamma}{n}} P^\gamma_h(uv) \quad \text{for} \quad h_v = v^{-\frac{4-2\gamma}{n}} h \quad \text{and} \quad 0 < v \in C^\infty(M),
\]

verified by \( P^\gamma_h \), see \cite{13} or Subsection 3.2 in \cite{13}. As for the classical scalar curvature, for the \( Q \)-curvature Gonzalez-Qing\cite{36} have introduced the fractional \( \gamma \)-Yamabe problem which asks for conformal metrics of constant fractional scalar curvature \( Q^\gamma \). Moreover, for an asymptotically hyperbolic manifold \((X, g^+)\) with conformal infinity \((M, h)\) being minimal in case \( \gamma \in (\frac{1}{2}, 1) \), Chang-Gonzalez\cite{18} showed the equivalence between the Dirichlet-to-Neumann operators of the scattering problem \cite{35} and the ones of some uniformly degenerate elliptic boundary value problems defined on \((X, g)\), which coincide with the extension problem of Caffarelli-Silvestre\cite{14} when

\[
(X, g^+, h) = (\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}}, g_{\mathbb{R}^n}),
\]

and hence

\[
(-\Delta_{g_{\mathbb{R}^n}})^\gamma = P^\gamma[g_{\mathbb{R}^{n+1}}, g_{\mathbb{R}^n}].
\]

The latter established relation allows Gonzalez-Qing\cite{36} to derive a Hopf-type maximum principle by taking inspiration from the work of Cabre-Sire\cite{12}, which deals with the Euclidean half-space, see Theorem 3.5 and Corollary 3.6 in \cite{36}. Clearly the latter Hopf-type maximum principle of Gonzalez-Qing\cite{36} opens the door to variational arguments for existence for the \( \gamma \)-Yamabe problem, as explored by Gonzalez-Qing\cite{36}, Gonzalez-Wang\cite{37} and Kim-Musso-Wei\cite{39}.

In terms of geometric differential equations the fractional Yamabe problem is equivalent to finding a positive smooth solution to the semi-linear pseudo-differential equation with critical Sobolev nonlinearity

\[
P_h^\gamma u = c u^{\frac{4+2\gamma}{n-2}} \quad \text{on} \quad M
\]

for some constant \( c \). The non-local equation (7) has a variational structure and thanks to the regularity theory for uniformly degenerate elliptic boundary value problems (see \cite{12}, \cite{25}, \cite{36}), the above cited local interpretation of \( P^\gamma_h \) of Chang-Gonzalez\cite{18} and the Hopf-type maximum principle of Gonzalez-Qing\cite{36}, we have that positive smooth solutions to (7) can be found by looking at critical points of the following fractional Yamabe functional

\[
\mathcal{E}_h^\gamma(u) := \frac{\langle u, u \rangle_{P^\gamma_h}}{\left( \int_M u^{\frac{2n}{n-2\gamma}} dV_h \right)^{\frac{n-2-2\gamma}{n}}}, \quad u \in W_+^{\gamma, 2}(M, h) := \{ v \in W^{\gamma, 2}(M, h) : v \geq 0, \, v \not\equiv 0 \},
\]
where \( W^{\gamma,2}(M, h) \) denotes the usual fractional Sobolev space on \( M \) with respect to the Riemannian metric \( h \) and

\[
\langle u, u \rangle_{P^\gamma_h} = \langle P^\gamma_h u, u \rangle_{L^2_h(M)},
\]

with \( L^2_h(M) \) denoting the usual \( L^2 \)-space on \( M \) with respect to \( h \) and \( \langle \cdot, \cdot \rangle_{L^2_h(M)} \) denoting the scalar product on \( L^2_h(M) \). For more informations see [11], [31] and [50].

However, as for the classical Yamabe problem and for the same reasons, the variational analysis of \( C^\gamma_h \) has a local regime, namely the situation where the local geometry can be used to ensure a solution (even a minimizer), and a global one, where the local geometry cannot be used to find a solution and just a minimizer, can be used to apply the Aubin-Schoen’s minimizing technique. We refer to the introduction of [46] for a precise definition of local and global regimes.

Furthermore, still as for the classical Yamabe problem, there is a natural invariant called the \( \gamma \)-Yamabe invariant of \( (M, [h]) \), denoted by \( \gamma(M, [h]) \) and defined by the formula

\[
\gamma(M, [h]) := \inf_{u \in W^{\gamma,1}(M)} \frac{\langle u, u \rangle_{P^\gamma_h}}{(\int_M u^{2n-2\gamma} dV_h)^{\frac{n-\gamma}{n-2\gamma}}}
\]

From the work of Gonzalez-Qing [36] it is known that \( \gamma(M, [h]) \) satisfies the rigidity estimate

\[
\gamma(M, [h]) \leq \gamma(S^n, [g_{S^n}]),
\]

provided \( (M, [h]) \) is minimal in case \( \gamma \in (\frac{1}{2}, 1) \), where \( (S^n, [g_{S^n}]) \) is the conformal infinity of the Poincaré ball model of the hyperbolic space. Moreover, as mentioned in the abstract, the global situation clearly includes the case of a locally flat conformal infinity of a Poincaré-Einstein manifold, as observed by Kim-Musso-Wei [39], while the existence results of Gonzalez-Qing [36], Gonzalez-Wang [37] and some part of the existence results in Kim-Musso-Wei [39] deal with situations which clearly belong to the local regime. Moreover in the global situation, as for the classical Yamabe problem, to run the minimizing technique of Aubin [4]-Schoen [48] one needs an analogue of the positive mass theorem of Schoen-Yau [49], while the existence results of Gonzalez-Qing [36], Gonzalez-Wang [37] and some work by Kim-Musso-Wei [39] that the variational analysis of the fractional Yamabe problem for a locally flat conformal infinity

\[
\text{Theorem 1.3. Let } n \geq 2 \text{ be a positive integer, } (X^{n+1}, g^+) \text{ be a Poincaré-Einstein manifold with conformal infinity } (M^n, [h]), \gamma \in (0, 1), \text{ and } \left( \frac{\gamma}{2} \right)^2 - \gamma^2 < \lambda_1(-\Delta_{g^+}). \text{ Assuming that either } n = 2 \text{ or } n \geq 3 \text{ and } (M, [h]) \text{ is locally flat, then } (M, [h]) \text{ carries a Riemannian metric of constant } Q^\gamma\text{-curvature.}
\]

\[
\text{Remark 1.4. We point out that, as observed by Gonzalez-Qing [36], the } \frac{1}{2}\text{-Yamabe problem for Poincaré-Einstein manifolds is equivalent to the Riemann mapping problem of Cherrier [20]-Escobar [23], which is completely solved after the series of works of [1], [19], [23], [41], [42], [47].}
\]

As already mentioned, to prove Theorem 1.3 we use variational arguments by applying a suitable scheme of the Barycenter Technique of Bahri-Coron [8]. Indeed, exploiting that the conformal infinity is locally
flat, the ambient space is Poincaré-Einstein, the conformal covariance property \([14]\), the works of Caffarelli-Silvestre\([14]\) and Chang-Gonzalez \([18]\), the Hopf-type maximum principle of Gonzalez-Qing \([36]\) and the standard bubbles attached to the related optimal trace Sobolev inequality, we define some bubbles and show that they can be used to run a suitable scheme of the barycenter technique of Bahri-Coron \([8]\) for existence, which among others has been used in the works \([29, 30, 47, 46]\). We give below a brief discussion of the main ideas behind the argument and refer to the introduction of our paper \([42]\) for a more geometric description as well as to \([39]\) for a detailed and concise exposition.

The barycenter technique of Bahri-Coron \([8]\) is an argument by contradiction, thus we assume the problem has no solution. Then, denoting for \(1 \leq p \in \mathbb{N}\) the limiting energy of \(p\)-many \textit{non collapsing} bubbles, see \([41]\), by

\[
l_p = p \frac{2\pi}{n} \gamma(S^n),
\]

putting \(L_0 = \emptyset\) and considering for some \(\epsilon > 0\) the sublevels

\[
L_p := \{ u : \mathcal{E}_n^p[u] \leq l_p + \epsilon \},
\]

on one hand we construct recursively singular chains \(X_p\) in \(L_p\), which generate non zero classes in the relevant \(\mathbb{Z}_2\)-homologies of the topological pairs \((L_p, L_{p-1})\), precisely

\[
(11) \quad 0 \neq [X_p] \in H_{np+p-1}(L_p, L_{p-1}, \mathbb{Z}_2),
\]

as follows. The starting point is the existence of \(X_1\) and non triviality of

\[
[X_1] \in H_n(L_1, L_0, \mathbb{Z}_2) = H_n(L_1, \mathbb{Z}_2),
\]

which follow from \(H_n(M, \mathbb{Z}_2) \neq 0\), embedding \(M\) into \(L_1\) via bubbling

\[
M \hookrightarrow L_1 : a \mapsto v_{a, \lambda}
\]

and, that based on the quantization phenomenon, which \(\mathcal{E}_n^0\) enjoys, \(M\) survives via the deformation Lemma \([5, 3]\) and selection map \([58]\) topologically in \(L_1\), see Lemma \([5, 6]\). We then start \textit{piling up masses} \(v_{a, \lambda}\) over \(X_1\), thereby iteratively moving from the level \(l_p\) to the level \(l_{p+1}\). At each step one constructs a singular chain \(X_{p+1}\) with a non zero class \([X_{p+1}]\), which reads

\[
(1 - t)u + tv_{a, \lambda}, u \in X_p, t \in [0, 1],
\]

see Lemma \([5, 7]\). Indeed, denoting by \(\delta_a\) for \(a \in M\) the Dirac measure at \(a\) and recalling the space of formal barycenter of \(M\), defined as

\[
B_p(M) = \{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} : a_i \in M, \alpha_i \geq 0, \ i = 1, \ldots, p, \ \sum_{i=1}^{p} \alpha_i = 1 \}, \quad B_0(M) = \emptyset,
\]

the set \(B_{p+1}(M)\) as a cone over \(B_p(M)\) with top \(M\) survives as a non trivial cone in \((L_{p+1}, L_p)\), when embedding \(B_{p+1}(M)\) into \(L_{p+1}\) via \((p + 1)\)-convex combinations of the bubbles \(v_{a, \lambda}\). Again the latter survival is based on the quantization phenomenon, which \(\mathcal{E}_n^q\) enjoys, via the deformation Lemma \([5, 3]\) and the selection map \([58]\). Since for all \(q \geq 1\) we have the existence of

\[
(13) \quad 0 \neq w_q \in H_nq+q-1(B_q(M), B_{q-1}(M), \mathbb{Z}_2),
\]

see \([8]\), we then obtain \([X_{p+1}] \neq 0\) for some \(X_{p+1}\), which is the image of a representative of \(w_{p+1} \neq 0\), and \((11)\) is established. On the other hand, because of the strong interaction phenomenon, for some \(p_0\) large we are actually passing from the level \(l_{p_0} + \epsilon\) to the level \(l_{p_0+1} - \epsilon_0\) for some \(\epsilon_0 > 0\), that is

\[
X_{p_0+1} \text{ is a chain in } L_{p_0+1} = \{ u : \mathcal{E}_n^p \leq l_{p_0+1} - \epsilon_0 \} \subset L_{p_0+1}.
\]
Moreover, since in absence of solutions and due to the quantization phenomenon the Palais-Smale condition holds on the sets \( \{ u : t_p + \varepsilon < E^+_h < t_{p+1} - \varepsilon \} \) for all \( p \geq 1 \) and \( 0 < \varepsilon \ll 1 \), the pair \((L_{p_0+1}, L_{p_0})\) retracts by deformation onto the pair \((L_{p_0}, L_{p_0})\) and we conclude
\[
[X_{p_0+1}] \in H_{n(p_0+1)+p_0}(L_{p_0+1}, L_{p_0}, \mathbb{Z}_2) = 0.
\]
In particular \([X_{p_0+1}] = 0\) in contradiction to (11) for \( p = p_0 + 1 \) and so a solution must exist.

**Remark 1.5.** Clearly the Poincaré-Einstein structure reflects the flatness of the conformal infinity \((M, \lbrack h \rbrack)\) into the interior of \( X \), namely, as observed by Kim-Musso-Wei[39], the metric \( g \) on \( X \) takes locally the form \( g = \delta + O(|y|^n) \), i.e. \( g \) is flat to order \( n \). However, since we base the calculation of the fractional Yamabe energy of a bubble on comparison via maximum principle, the appearance of a logarithmic term in the construction of a suitable barrier solution, cf. (24), in case \( g = \delta + O(|y|^n) \) highly suggests the limitation of our argument to the latter order of flatness.

We would like to make some comments about the application of the barycenter technique of Bahri-Coron[8], in our situation and in the case of the classical Yamabe problem for locally conformally flat closed Riemannian manifolds, as studied by Bahri[6]. The latter situations are counterpart to each other, however the nonlocal aspect of our situation creates an additional difficulty, that is not in its counterpart for the classical Yamabe problem, which is of local nature. In fact, even if both problems are conformally invariant and after a conformal change we are locally in the corresponding model space of singularity - truly in the classical case and up to a critical, but handleable lower order term in the fractional scenario, cf. Remark 1.5 - we have that the lower order term, i.e. the scalar curvature, vanishes locally for the classical Yamabe problem, because of its local nature, while for the fractional Yamabe problem that does not necessarily imply that the \( Q^n \)-curvature vanishes locally, because of the nonlocal aspect of the problem. This is the source of the difficulty we mentioned before, which is similar to the one encountered by Bahri-Brezis[7] and in a different framework Brendle[10]. To overcome the latter issue, we use the works of Caffarelli-Silvestre[14], Chang-Gonzalez[18] and Gonzalez-Qing[36] to reduce ourselves to a local situation. Having done that, we then encounter the problem of not having an explicit knowledge of the standard bubble corresponding to the reduced local situation and clearly such an explicit knowledge plays an important role in the corresponding situation of the classical Yamabe problem. To deal with this lack of explicit knowledge of the standard bubbles corresponding to the reduced local situation on the 1-dimension augmented half space, we observe that its integral representation given in Caffarelli-Silvestre[14] can be interpreted as a suitable interaction of standard bubbles on the boundary of the latter augmented half space with different points and scales of concentration. This interpretation provides the required estimates of the latter argument, which one gets for free from an explicit knowledge, and this is made rigorous in this work, see Lemma 3.1 and Corollary 3.3. We point out that the role of interaction of bubbles in the existence mechanism of Yamabe type problems has been observed for the first time by Bahri-Coron[8]. The idea behind is, that interaction pushes the energy down from the expected critical value at infinity for multiple bubbles. This has been successfully used in the study of other Yamabe type problems (see [7], [29], [30], [42], [47], [46]) and in the study of Yamabe flow (see [9], [10]).

**Remark 1.6.** We would like to emphasize the analogy between our function \( \mathcal{M}_\gamma \) given by Definition 4.3 and the notion of mass appearing in the context of the classical Yamabe problem. Moreover we point out that our work in [45] answers the first part of the Conjecture of Kim-Musso-Wei[39] about the structure of the Green’s function, see Theorem 1.4 in [43] and gives rise to the definition of \( \mathcal{M}_\gamma \). We add that under the global assumption \( \mathcal{M}_\gamma > 0 \) our sharp estimate in Lemma 5.4 gives directly the existence of a fractional Yamabe minimizer.

The structure of the paper is as follows. In Section 2 we recall the standard bubbles of the variational problem, give some preliminaries and fix notations. In Section 3 we analyse the standard bubbles on \( \mathbb{R}^{n+1}_+ \), introduce the relevant Schoen’s and Projective bubbles in the curved scenario and prove some interaction estimates of the latter. In Section 4 we establish sharp \( L^\infty \)-estimates for the difference between these bubbles and derive a sharp selfaction estimate for the Projective one. In Section 5 we present the variational and algebraic topological argument to prove Theorem 1.3. It is divided into two subsections,
In Subsection 5.1 we present a variational principle which extends the classical variational principle by taking into account the non-compactness phenomena via our Projective bubbles. In Subsection 5.2, we present the barycenter technique or algebraic topological argument for existence. Finally in Section 6 we collect the proofs of some technical lemmas and estimates.

Acknowledgements

The authors worked on this project when they were visiting the department of Mathematics of the University of Ulm in Germany. Parts of this paper were written when the authors were visiting the Mathematical Institute of Oberwolfach in Germany as Research in Pairs and the Institut des Hautes Études Scientifiques in Paris. We are very grateful to all these institutions for their kind hospitality.

2 Preliminaries and notations

In this section we give some preliminaries and fix notations. We start with the standard bubbles. For \( a \in \mathbb{R}^n \) and \( \lambda > 0 \) we define the standard bubbles on \( \mathbb{R}^n \) as

\[
\delta_{a,\lambda}(x) = \left( \frac{\lambda}{1 + \lambda^2|x-a|^2} \right)^{\frac{n-2\gamma}{2}}, \quad x \in \mathbb{R}^n.
\]

They are solutions of the pseudo-differential equation

\[
(-\Delta_{\mathbb{R}^n})^\gamma \delta_{a,\lambda} = c_{n,\gamma} \delta_{a,\lambda}^{n+2\gamma} \quad \text{on} \quad \mathbb{R}^n,
\]

where \( c_{n,\gamma} \) is a positive constant depending only on \( n \) and \( \gamma \). From \( \delta_{a,\lambda} \) we then define \( \hat{\delta}_{a,\lambda} \) via

\[
\begin{aligned}
\text{div}(y^{1-2\gamma}\nabla \delta_{a,\lambda}) &= 0 \quad \text{in} \quad \mathbb{R}_{+}^{n+1} \\
\hat{\delta}_{a,\lambda} &= \delta_{a,\lambda} \quad \text{on} \quad \mathbb{R}^n
\end{aligned}
\]

and the quantities

\[
c_{n,1}^\gamma = \int_{\mathbb{R}^n} \left( \frac{1}{1 + |x|^2} \right)^n \, dx \quad \text{and} \quad c_{n,2}^\gamma = d_{\gamma}^* \int_{\mathbb{R}_{+}^{n+1}} y^{1-2\gamma} \left| \nabla \delta_{0,1}(y,x) \right|^2 \, dxdy,
\]

where

\[
d_{\gamma}^* = \frac{d_\gamma}{2\gamma},
\]

cf. \[5\], which relate via \(-d_{\gamma}^* \lim_{y \to 0} (\partial_y y^{1-2\gamma} \partial_y \hat{\delta}_{a,\lambda}) = c_{n,\gamma} \delta_{a,\lambda}^{n+2\gamma}\) and

\[
c_{n,1}^\gamma = \int_{\mathbb{R}^n} \delta_{a,\lambda}^{2\gamma} \, dx = \frac{1}{c_{n,\gamma}} \int_{\mathbb{R}^n} (-\Delta_{\mathbb{R}^n})^\gamma \delta_{a,\lambda} \delta_{a,\lambda}dx = -d_{\gamma}^* \lim_{y \to 0} \int_{\mathbb{R}_{+}^{n+1}} \partial_y (y^{1-2\gamma} \partial_y \hat{\delta}_{a,\lambda}) \delta_{a,\lambda} \, dxdy
\]

\[
= \frac{d_{\gamma}^*}{c_{n,\gamma}} \int_{\mathbb{R}_{+}^{n+1}} y^{1-2\gamma} \left| \nabla \delta_{a,\lambda} \right|^2 \, dxdy = \frac{c_{n,2}^\gamma}{c_{n,\gamma}}.
\]

The standard bubbles \( \delta_{a,\lambda} \) after stereographic projection also minimize

\[
\mathcal{Y}_\gamma(S^n) = \mathcal{Y}_\gamma(S^n, [gS^n])
\]

with \( \mathcal{Y}_\gamma(S^n, [gS^n]) \) as in \[10\], and therefore

\[
c_{n,2}^\gamma = c_{n,\gamma} c_{n,1}^\gamma \quad \text{and} \quad \mathcal{Y}_\gamma(S^n) = \frac{c_{n,2}^\gamma}{(c_{n,1}^\gamma)^{\frac{n-2\gamma}{n}}}.
\]
See also Section 3 and we refer to our previous work \[43\] for details. Furthermore we set

\[
\gamma_n,3 = \int_{\mathbb{R}^n} \left( \frac{1}{1+|x|^2} \right)^{\frac{n+2n}{2}} \, dx, \quad \gamma_n,4 = \gamma_n,3, \quad \gamma_n^{*} = c_n^{*} \gamma_n,3. \tag{19}
\]

We remark that by \[14\]

\[
\delta_{0,\lambda}(y,x) = \left[ K(\cdot, y) * \delta_{0,\lambda}(x) \right] = p_{n,\gamma} \int_{\mathbb{R}^n} \frac{y^{2\gamma}}{y^2 + |x - \xi|^2} \left( \frac{\lambda}{1 + \lambda^2 |\xi|^2} \right)^{\frac{n+2n}{2}} \, d\xi
\]

\[
= p_{n,\gamma} y^{-\frac{n-2}{2}} \int_{\mathbb{R}^n} \frac{y^{-1}}{1 + y^{-2}|x - \xi|^2} \left( \frac{\lambda}{1 + \lambda^2 |\xi|^2} \right)^{\frac{n+2n}{2}} \, d\xi
\]

\[
= p_{n,\gamma} y^{-\frac{n-2}{2}} \int_{\mathbb{R}^n} \delta_{x,y^{-1}}(\xi) \delta_{0,\lambda}(\xi) \, d\xi
\]

(20)

with \( K \) being the Poisson kernel at the origin \( 0 \in \mathbb{R}^{n+1} \) of the operator

\[
D = -\text{div}(y^{1-2\gamma}\nabla(\cdot))
\]

and given by

\[
K(y,x) = K\gamma(y,x) = p_{n,\gamma} \frac{y^{2\gamma}}{|y|^2 + |y|^2} \frac{y^{2\gamma}}{y^2 + |y|^2}, \quad p_{n,\gamma} = \frac{1}{c_n^{*}}
\]

see \[19\]. We also have

\[
\hat{\delta}_{a,\lambda}(\cdot) = \delta_{0,\lambda}(\cdot - a) \quad \forall a \in \mathbb{R}^n.
\]

Following \[36\], for an asymptotically hyperbolic manifold \((X,g^+)\) of dimension \(n+1 \geq 3\) and with conformal infinity \((M,[h])\) we introduce

\[
D_g U = -\text{div}_g(y^{1-2\gamma}\nabla_g U) + E_g U
\]

with \( g = y^2 g^+ \), \( y \) the unique geodesic defining function associated to \( h \) and

\[
E_g := y^{1-2\gamma} L_g y^{1-2\gamma} - \left( \frac{R_g}{c_n} + s(n-s) \right)y^{(1-2\gamma)-2},
\]

where

\[
L_g = -\Delta_g + \frac{R_g}{c_n}, \quad c_n = \frac{4n}{n-1}
\]

denotes the conformal Laplacian of \((X,g)\). This operator realizes via the Dirichlet to Neumann map

\[
u \rightarrow \left\{ \begin{array}{ll} D_g U = 0 & \text{in } X \\ U = u & \text{on } M \end{array} \right\} \rightarrow -d_{-}^{*} \lim_{y \rightarrow 0} y^{1-2\gamma} \partial_y U = f
\]

(24)

the conformal fractional Laplacian \( P_{\gamma}^h u = f \), provided

\[
0 < \gamma < \frac{1}{2} \quad \text{or, if } \frac{1}{2} < \gamma < 1, \quad \text{then additionally } H_g = 0.
\]

We refer to our work \[43\] for details, where the existence and asymptotic behavior of the Poisson kernel \( K_g := K_g\gamma \) of \( D_g \), the Green’s functions \( \Gamma_g := \Gamma_g\gamma \) of \( D_g \) under weighted normal boundary condition and \( G_h := G_h\gamma \) of the fractional conformal Laplacian \( P_{\gamma}^h \) have been studied.
The fundamental solutions $K_g$, $\Gamma_g$ and $G_h$ are defined via

\begin{align}
(25) \quad & 
\begin{cases}
D_gK_g(\cdot,\xi) = 0 \quad \text{in} \quad X \quad \text{and for all} \quad \xi \in M \\
\lim_{y \to 0} K_g(y, x, \xi) = \delta_x(\xi) \quad \text{and for all} \quad x, \xi \in M,
\end{cases} \\
(26) \quad & 
\begin{cases}
D_g\Gamma_g(\cdot,\xi) = 0 \quad \text{in} \quad X \quad \text{and for all} \quad \xi \in M \\
d^*\lim_{y \to 0} y^{1-2\gamma} \partial_y \Gamma_g(y, x, \xi) = \delta_x(\xi) \quad \text{and for all} \quad x, \xi \in M,
\end{cases} \\
(27) \quad & 
P^h_k G_h(x, \xi) = \delta_x(\xi), \quad x, \xi \in M,
\end{align}

where $d^*$ is as in (16), and are linked by

$$\Gamma_g(z, \xi) = \int_M K_g(z, x)G_h(x, \xi)d\nu_h(x), \quad z \in X, \xi \in M.$$ 

In particular

$$\lim_{y \to 0} \Gamma_g(y, \cdot) = G_h(\cdot).$$

**Notation:** We conclude this section with fixing some notations, used in this paper.

$\mathbb{N}$ denotes the set of non negative integers, $\mathbb{N}^*$ the set of positive integers and for $k \in \mathbb{N}^*$, $\mathbb{R}^k$ stands for the standard $k$-dimensional Euclidean space, $\mathbb{R}^k_+$ the open positive half-space of $\mathbb{R}^k$ and $\mathbb{R}^k_+$ its closure in $\mathbb{R}^k$. For simplicity we let $\mathbb{R}_+ = \mathbb{R}^1_+$ and $\mathbb{R}^1_- = \mathbb{R}^1_+$. For $r > 0$ we denote respectively by

$$B^k_r(0) \quad \text{and} \quad B^k_r(0) = B^k_r(0) \cap \mathbb{R}^k_+$$

the open and open upper half ball of $\mathbb{R}^k$ of center 0 and radius $r$, and set

$$B_r = B_r(0) = B^k_r(0) \quad \text{and} \quad B^+ = B^+_r(0) = B^{k+1}_r(0).$$

We recall that $X = X^{n+1}$ with $n \geq 2$ is a manifold of dimension $n+1$ with boundary $M = M^n$ and closure $\overline{X}$. For any Riemannian metric $\bar{h}$ defined on $M$, $a \in M$ and $r > 0$ we use the notation $B^k_r(a)$ to denote the geodesic ball with respect to $\bar{h}$ of radius $r$ and center $a$. We also denote by $d_{\bar{h}}(x, y)$ the geodesic distance with respect to $\bar{h}$ between two points $x, y \in M$. $d\nu_{\bar{h}}$ denotes the Riemannian measure associated to the metric $\bar{h}$ on $M$. For $a \in M$ we use the notation $\exp_{\bar{h}}^a$ to denote the exponential map with respect to $\bar{h}$ on $M$.

Similarly for any Riemannian metric $\bar{g}$ defined on $\overline{X}$, $a \in M$ and $r > 0$ we use the notation $B^k_r(a)$ to denote the geodesic half ball with respect to $\bar{g}$ of radius $r$ and center $a$. We also denote by $d_{\bar{g}}(x, y)$ the geodesic distance with respect to $\bar{g}$ between two points $x, y \in \overline{X}$. $d\nu_{\bar{g}}$ denotes the Riemannian measure associated to the metric $\bar{g}$ on $\overline{X}$. For $a \in M$ we use the notation $\exp_{\bar{g}}^{a+}$ to denote the exponential map with respect to $\bar{g}$ on $\overline{X}$.

For $p \in \mathbb{N}^*$ let $\sigma_p$ denote the permutation group of $p$ elements and let $M^p$ denote the Cartesian product of $p$ copies of $M$. We define

$$\text{(29)} \quad (M^2)^* := M^2 \setminus \text{Diag}(M^2),$$

where $\text{Diag}(M^2) = \{(a, a) : a \in M\}$ is the diagonal of $M$.

For $\bar{g}$, a Riemannian metric defined on $\overline{X}$, let $C^\infty(\overline{X}, \bar{g})$ denote the space of infinitely differentiable functions on $\overline{X}$ with respect to $\bar{g}$.

Large positive constants are usually denoted by $C$ and the value of $C$ is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are denoted by $c$ and their values may vary from formula to formula and also within the same line.

$O(1)$ stands for quantities, which are bounded. For $\epsilon > 0$ we denote by $o_\epsilon(1)$ any quantity, which tends to 0, as $\epsilon \to 0$. For $x \in \mathbb{R}$ we denote by $O(x)$ and $o(x)$ respectively $|x|O(1)$ and $|x|o(1)$. For a topological space $Z$ let $H_*(Z)$ denote the singular homology of $Z$ with $\mathbb{Z}_2$ coefficients and for a subspace $Y$ of $X$ let $H_*(X, Y)$ denote the relative homology. For a map $f : Z \to Y$ with $Z$ and $Y$ topological spaces we denote by $f_*$ the induced map in homology.
3 Bubbles and related interaction estimates

In this section we recall the definition of the standard bubbles on $\mathbb{R}^{n+1}_+$ and their interpretation as a suitable interaction of standard bubbles on $\mathbb{R}^n$. Furthermore we establish some new sharp estimates of independent interest for the interaction of the standard bubbles on $\mathbb{R}^n$ and use the latter to derive sharp estimates for the standard bubbles on $\mathbb{R}^{n+1}_+$. Moreover we recall the Schoen’s bubbles associated to the standard bubbles on $\mathbb{R}^{n+1}_+$ and use them to define other bubbles, called them Projective bubbles, which talk to the local formulation of the problem. Finally we derive sharp interaction estimates for the Projective bubbles.

3.1 The bubbles on $\mathbb{R}^{n+1}_+$ as an interaction of standard ones on $\mathbb{R}^n$

In this subsection we deal with the standard bubbles $\hat{\delta}_{a,\lambda}$ on $\mathbb{R}^{n+1}_+$. They are the natural extension of the fractional bubbles $\delta_{a,\lambda}$ on $\mathbb{R}^n$ with respect to $D$, cf. (21), and are given by the convolution of the Poisson kernel $K$ of $D$, cf. (20), and can be treated as a standard bubble interaction on $\mathbb{R}^n$. Indeed (20) is equivalent to

$$\hat{\delta}_{a,\lambda}(y, x) = p_{n, \gamma}^{-\frac{n-2\gamma}{2}} \int_{\mathbb{R}^n} \frac{\alpha_{\lambda}}{n-2\gamma} \delta_{a,\lambda}.$$  

This interpretation will be used to derive sharp estimates for $\hat{\delta}_{a,\lambda}$ relating it to the scale of the Green’s function $\lambda^{-\frac{2\gamma}{n-2\gamma}} \Gamma(a, \cdot)$, which is necessary for sharp energy estimates. Recalling that interaction always relates to suitable scales of the Green’s function, see Lemma 3.1 for instance, then clearly (30) provides a way to achieve that and we will follow this approach.

We then need the following Lemma 3.1 on interaction estimates for standard bubbles on $\mathbb{R}^n$, which are new in the fractional setting, sharp and of independent interest. While these estimates are completely analogous to the classical ones, e.g. in case of the conformal Laplacian and famously to those in [5], they do not only serve the purpose of understanding the interaction of different bubbles on $\mathbb{R}^n$, but by virtue of (30), as explained above, give also rise to sharp estimates on their extensions on $\mathbb{R}^{n+1}_+$. We anticipate that the proof of Lemma 3.1 is postponed to the appendix given by Section 6.

**Lemma 3.1.** For $c_{\alpha,3}^\gamma$ given by (19), $i \neq j$ and $\varepsilon_{i,j} = (\frac{a_i}{\lambda_j} + \frac{a_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2)^{\frac{2\gamma}{n-2\gamma}}$ there holds

(i) $\int_{\mathbb{R}^n} \delta_{a_i, \lambda_i}^{\frac{n-2\gamma}{2}} \delta_{a_j, \lambda_j} = c_{\alpha,3}^\gamma \varepsilon_{i,j} + O(\min(\frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i})^\gamma \varepsilon_{i,j}^{\frac{n-2\gamma}{2}})$

(ii) $\lambda_i \partial_{a_k} \int_{\mathbb{R}^n} \delta_{a_i, \lambda_i}^{\frac{n+2\gamma}{2}} \delta_{a_j, \lambda_j} = c_{\alpha,3}^\gamma \lambda_i \lambda_j \lambda_i \varepsilon_{i,j} + O(\min(\frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i})^\gamma \varepsilon_{i,j}^{\frac{n+2\gamma}{2}})$

(iii) $\nabla_{a_k} \int_{\mathbb{R}^n} \delta_{a_i, \lambda_i}^{\frac{n+2\gamma}{2}} \delta_{a_j, \lambda_j} = c_{\alpha,3}^\gamma \lambda_i \lambda_j (a_k - a_i) \varepsilon_{i,j} + O(\min(\frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i})^\gamma \lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+2\gamma}{2}})$

(iv) $\nabla_{a_k, a_l} \int_{\mathbb{R}^n} \delta_{a_i, \lambda_i}^{\frac{n+2\gamma}{2}} \delta_{a_j, \lambda_j} = c_{\alpha,3}^\gamma \lambda_i \lambda_j (id - \frac{(n+2\gamma)\lambda_i \lambda_j (a_i - a_j)(a_i - a_j)}{\lambda_i + \lambda_j \lambda_i |a_i - a_j|^2} \varepsilon_{i,j}^{\frac{n+2\gamma}{2}}$

$$+ O(\min(\frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i})^\gamma \lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+2\gamma}{2}})$$

for $k, l \in \{i, j\}$ and $\{k, l\} = \{i, j\}$.

Lemma 3.1 has the following impacts on the standard bubbles $\hat{\delta}_{a,\lambda}$ on $\mathbb{R}^{n+1}_+$.

**Corollary 3.2.** For $a \in \mathbb{R}^n$ and $r_a = |z - a| = |(y, x - a)|$ we have on $\mathbb{R}^{n+1}_+$

(i) $\hat{\delta}_{a,\lambda}(y, x) = O((\frac{1}{1 + r_a^2})^{\frac{n-2\gamma}{2}})$

(ii) $\partial_y \hat{\delta}_{a,\lambda}(y, x) = O(\lambda \gamma y^{2\gamma-1}(\frac{1}{1 + r_a^2})^{\frac{n}{2}})$
(iii) \( \nabla_x \delta_{a, \lambda}(y, x) = O(\sqrt{\lambda} (\frac{\lambda}{1+\lambda^2 r_a^2})^{n+1/2}) \)

(iv) \( \nabla_x^2 \delta_{a, \lambda}(y, x) = O(\lambda (\frac{\lambda}{1+\lambda^2 r_a^2})^{n+2/2}) \).

PROOF. Let \( a_i = x, \lambda_i = \frac{1}{y}, a_j = a \) and \( \lambda_j = \lambda \). Then (30) and Lemma 3.1 (i) show

\[
\delta_{a, \lambda}(y, x) \leq C y^{-\frac{n-2\gamma}{2}} y^{-\frac{n-2\gamma}{2}} = O(\frac{y^{-\frac{n-2\gamma}{2}}}{(1/y) + \lambda y + \lambda |x-a|^2})^{\frac{n-2\gamma}{2}},
\]

whence (i) follows. From (22), (30) and Lemma 3.1 (ii) we infer using \( y\partial_y = y^{-1} \partial_y^{-1} \) that

\[
y\partial_y \delta_{a, \lambda}(y, x) = p_{n, \gamma} y\partial_y \left( y^{-\frac{n-2\gamma}{2}} \int_{\mathbb{R}^n} \frac{\delta_{a, \lambda, n}}{(1/y + \lambda y + |x-a|^2)^{\frac{n-2\gamma}{2}}} \right)
\]

\[
y\partial_y \delta_{a, \lambda}(y, x) = y\partial_y \left( y^{-\frac{n-2\gamma}{2}} \right) \int_{\mathbb{R}^n} \frac{\delta_{a, \lambda, n}}{(1/y + \lambda y + |x-a|^2)^{\frac{n-2\gamma}{2}}} + O\left( \frac{\min(1/y, \lambda y)}{(1/y + \lambda y + |x-a|^2)^{\frac{n-2\gamma}{2}}} \right)
\]

\[
= -(n-2\gamma) \left( \frac{\lambda}{1 + \lambda^2 r_a^2} \right)^{-\frac{n-2\gamma}{2}} \left[ \frac{\lambda^2 y^2}{1 + \lambda^2 r_a^2} + O\left( \frac{\min(1, \lambda^2 y^2)}{1 + \lambda^2 r_a^2} \right) \right],
\]

whence (ii) follows. Likewise (iii),(iv) follow from (30) and Lemma 3.1 (iii),(iv) respectively.

Corollary 3.3. For \( a \in \mathbb{R}^n \) and \( r_a = |z-a| = |(y, x-a)| \) we have on \( B_{e}(a)^c \cap \mathbb{R}^{n+1} \)

(i) \( \delta_{a, \lambda}(y, x) = \frac{1}{\lambda^2 r_a^{n-2\gamma}} + o\left( \frac{1}{1+\lambda^2 r_a^{2\gamma}} \right) \)

(ii) \( y\partial_y \delta_{a, \lambda}(y, x) = y\partial_y \left( \frac{1}{\lambda^2 r_a^{n-2\gamma}} + o\left( \frac{1}{1+\lambda^2 r_a^{2\gamma}} \right) \right) \)

(iii) \( x\nabla_x \delta_{a, \lambda}(y, x) = x\nabla_x \left( \frac{1}{\lambda^2 r_a^{n-2\gamma}} + o\left( \frac{1}{1+\lambda^2 r_a^{2\gamma}} \right) \right) \)

PROOF. Let \( a_i = x, \lambda_i = \frac{1}{y}, a_j = 0 \) and \( \lambda_j = \lambda \). Then (22), (30) and Lemma 3.1 (i) give

\[
\delta_{a, \lambda}(y, x) = y^{-\frac{n-2\gamma}{2}} \left( \varepsilon_{i,j} + O\left( \frac{\lambda_i}{\lambda_j} \frac{\lambda_j}{\lambda_i} \varepsilon_{i,j} \right) \right)
\]

\[
= y^{-\frac{n-2\gamma}{2}} \left( \frac{1}{(1/y) + \lambda y + |x-a|^2} \right)^{\frac{n-2\gamma}{2}} + O\left( \frac{1}{(1/y) + \lambda y + |x-a|^2} \right)^{\frac{n-2\gamma}{2}}
\]

\[
= \frac{\lambda}{1 + \lambda^2 r_a^2} \left( \frac{1}{1/y + \lambda y + |x-a|^2} \right)^{\frac{n-2\gamma}{2}} + O\left( \frac{\min(1, \lambda^2 y^2)}{1 + \lambda^2 r_a^2} \right),
\]

whence (i) follows. Moreover we find from (30) and Lemma 3.1 (ii) as before

\[
y\partial_y \delta_{a, \lambda}(y, x) = -(n-2\gamma) \left( \frac{\lambda}{1 + \lambda^2 r_a^2} \right)^{-\frac{n-2\gamma}{2}} \left[ \frac{\lambda^2 y^2}{1 + \lambda^2 r_a^2} + O\left( \frac{\min(1, \lambda^2 y^2)}{1 + \lambda^2 r_a^2} \right) \right],
\]

whence

\[
y\partial_y \delta_{a, \lambda}(y, x) = y\partial_y \left( \frac{1}{\lambda^2 r_a^{n-2\gamma}} + \frac{1}{\lambda^2 r_a^{n-2\gamma}} \right) + \frac{1}{\lambda^2 r_a^{n-2\gamma}} O\left( \frac{1}{\lambda^2 r_a^{n-2\gamma+2}} + \frac{1}{\lambda^2 r_a^{n}} \right)
\]

and (ii) follows. (iii) follows analogously from (30) and Lemma 3.1 (iii).
3.2 Schoen’s bubbles and Projective bubbles

In this subsection we recall the Schoen’s bubbles associated to the standard bubbles on $\mathbb{R}^{n+1}$ for an asymptotically hyperbolic manifold $(X, g^+)$ of dimension $n + 1$ with $n \geq 2$ and minimal conformal infinity $(M, [h])$. Moreover we use them to define another type of bubbles, called them Projective bubbles, whose construction is motivated by the local interpretation of the problem under study. Furthermore we establish some interaction estimates for our Projective bubbles, which by the way coincide with the Schoen’s bubbles on the conformal infinity.

First of all, because of (2) and minimality of the conformal infinity, we can consider a geodesic defining function $y$ splitting the metric 

$$g = y^2 g^+, \quad g = dy^2 + h_y$$

near $M$ and $h = h_y|_M$

in such a way, that $H_g = 0$. Moreover, using the existence of conformal normal coordinates, cf. [32], there exists for every $a \in M$ a conformal factor

$$0 < u_a \in C^\infty(M)$$

satisfying $\frac{1}{C} \leq u_a \leq C$, $u_a(a) = 1$ and $\nabla u_a(a) = 0$,

inducing a conformal normal coordinate system close to $a$ on $M$, in particular in normal coordinates with respect to

$$h_a = u_a^{\frac{4}{n-2}} h$$

we have for some small $\epsilon > 0$, that $h_a = \delta + O(|x|^2)$, $ \det h_a \equiv 1$ on $B_{\epsilon}^h(a)$. As clarified in Subsection 3.2 of [43] the conformal factor $u_a$ then naturally extends onto $X$ via

$$u_a = \left(\frac{y_a}{y}\right)^{\frac{n-2}{2}},$$

where $y_a$ close to the boundary $M$ is the unique geodesic defining function, for which 

$$g_a = y_a^2 g^+, \quad g_a = dy_a^2 + h_{a,y}$$

near $M$ with $h_a = h_{a,y}|_M$

and there still holds $H_{g_a} = 0$. Consequently

$$g_a = \delta + O(y + |x|^2) \quad \text{and} \quad \det g_a = 1 + O(y^2) \quad \text{in} \quad B_{\epsilon}^{g_a}(a).$$

With respect to $h_a$-normal coordinates $x = x_a$ centered at $a$ we define the standard bubble

$$\delta_{a,\lambda} = \left(\frac{\lambda}{1 + \lambda^2|x|^2}\right)^{\frac{n-2}{2}}$$

on $B_{4\rho_0}^{h_a}(a)$

for some small $\rho_0 > 0$, identifying $M \ni a \sim 0 \in \mathbb{R}^n$, and the Schoen’s bubble associated to $\delta_{a,\lambda}$ by

$$\varphi_{a,\lambda,\varepsilon} = \eta_{a,\varepsilon}\delta_{a,\lambda} + (1 - \eta_{a,\varepsilon})\lambda^{\frac{2-n}{2}} \frac{G_a}{g_{n,\gamma}}$$

on $M$, $G_a = G_{h_a}(\cdot, a)$

with $G_{h_a}$ as [27], $\eta_{a,\varepsilon}$ a cut-off function defined in $g_a$-normal Fermi-coordinates by

$$\eta_{a,\varepsilon} = \eta\left(\frac{y^2 + |x|^2}{\varepsilon^2}\right), \quad \eta \equiv 1 \quad \text{on} \quad [0, 1], \quad \eta \equiv 0 \quad \text{on} \quad [2, \infty) \quad \text{and} \quad 0 < \varepsilon \ll 1$$

and $g_{n,\gamma} > 0$ such that

$$G_a/g_{n,\gamma} = (1 + o_\kappa(1))r_a^{2\gamma-n}$$

cf. [32] and see [13] for an expansion of $G_a$. 

13
In what follows we will always choose \( \varepsilon \sim \lambda^{-\frac{1}{4}} \) with \( 1 < k \in \mathbb{N} \), for instance \( \varepsilon = \lambda^{-\frac{1}{4}} \), and hence may write \( \varphi_{a,\lambda,e} = \varphi_{a,\lambda} \) by abusing the notation. We define our Projective bubbles on \( X \) by

\[
\varphi_{a,\lambda} = K_{g_{a}} \ast \varphi_{a,\lambda},
\]

 cf. \([24]\), i.e. \( \varphi_{a,\lambda} \) uniquely solves

\[
\begin{cases}
D_{g_{a}} \varphi_{a,\lambda} = 0 \quad \text{in} \quad X \\
\varphi_{a,\lambda} = \varphi_{a,\lambda} \quad \text{on} \quad M
\end{cases}
\]

(37)

and is the canonical extension of \( \varphi_{a,\lambda} \) to \( X \) with respect to \( D_{g_{a}} \) as in \([23]\). Using \( g_{a} \)-normal Fermi-coordinates \( z = z_{a} = (y, x_{a}) = (y, x) \), we consider as a second extension

\[
\varphi_{a,\lambda}^{a} = \eta_{a,e} \delta_{a,\lambda} + (1 - \eta_{a,e}) \lambda^{\frac{2n-a}{4}} \frac{\Gamma_{a}}{g_{n,\gamma}}, \quad \Gamma_{a} = \Gamma_{g_{a}}(\cdot,a),
\]

with \( \Gamma_{g_{a}} \) as in \([26]\), namely the Schoen’s bubble associated to the standard bubble \( \hat{\delta}_{a,\lambda} \) on \( \mathbb{R}^{n+1} \), cf. \([14]\) and \([15]\), solving

\[
\begin{cases}
D \hat{\delta}_{a,\lambda} = 0 \quad \text{in} \quad \mathbb{R}^{n+1} \\
\hat{\delta}_{a,\lambda} = \delta_{a,\lambda} \quad \text{on} \quad \mathbb{R}^{n} \\
-d_{\gamma}^{*} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} \hat{\delta}_{a,\lambda} = c_{n,\gamma} \eta_{a,e} \delta_{a,\lambda}^{\frac{n+2\gamma}{4}} \quad \text{on} \quad \mathbb{R}^{n},
\end{cases}
\]

(39)

where \( D \) is as in \([21]\). Readily \( \varphi_{a,\lambda}^{a} \) enjoys analogous, but weaker identities, namely

\[
\begin{cases}
D_{g_{a}} \varphi_{a,\lambda}^{a} = 0 \quad \text{in} \quad X \setminus B_{2\varepsilon}^{g_{a}}(a) \\
\varphi_{a,\lambda}^{a} = \varphi_{a,\lambda} \quad \text{on} \quad M \\
-d_{\gamma}^{*} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} \varphi_{a,\lambda}^{a} = c_{n,\gamma} \eta_{a,e} \delta_{a,\lambda}^{\frac{n+2\gamma}{4}} \quad \text{on} \quad M.
\end{cases}
\]

(40)

In fact, recalling \((38)\), the first property in \((40)\) is due to \([20]\) and \([35]\), while the second one is by definition. Furthermore, due to \( \partial_{y} \eta_{a,e} = \varepsilon^{-2}O(y), \gamma \in (0,1) \) and, since \( \Gamma_{a} \) extends smoothly to the boundary away from \( a \), cf. \([28]\), there holds

\[-d_{\gamma}^{*} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} \varphi_{a,\lambda}^{a} = -d_{\gamma}^{*} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} (\eta_{a,e} \delta_{a,\lambda}),\]

whence by \((i)\) of Corollary \(3.2\) and again by \( \partial_{y} \eta_{a,e} = \varepsilon^{-2}O(y) \) and \( \gamma \in (0,1) \)

\[-d_{\gamma}^{*} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} \varphi_{a,\lambda}^{a} = -d_{\gamma}^{*} \eta_{a,e} \lim_{y \to 0} y^{1-2\gamma} \partial_{y} \hat{\delta}_{a,\lambda} = c_{n,\gamma} \eta_{a,e} \delta_{a,\lambda}^{\frac{n+2\gamma}{4}} |_{y=0},\]

where the last equality is due to \( \hat{\delta}_{a,\lambda} \) being written \( g_{a} \)-normal Fermi-coordinates and the third property in \((39)\). Finally we set

\[
v_{a,\lambda} = u_{a} \varphi_{a,\lambda} \quad \text{on} \quad M \quad \text{and} \quad \overline{\omega} = K_{g_{a}} \ast u \quad \text{on} \quad X.
\]

### 3.3 Interaction estimates for \( v_{a,\lambda} \) and related identities

In this subsection we derive several interaction estimates for \( v_{a,\lambda} \) in \((41)\). Indeed we give estimates for the higher exponent interaction as well as for the linear and nonlinear interaction and relate the latter two. Recalling \((36)\), we start with the higher exponent interaction estimates.

**Lemma 3.4.** For \( i \neq j \) there holds

\[
(i) \quad \int_{M} v_{a_i,\lambda}^{\alpha} v_{a_j,\lambda_j}^{\beta} \ dV = O(\varepsilon^{\frac{\beta}{2}}) \quad \text{for all} \quad \alpha + \beta = \frac{2n}{n-2\gamma}, \quad \alpha > \frac{n}{n-2\gamma} > \beta > 0
\]

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(ii) \[ \int_M v_{a_i,\lambda_i}^{\frac{a}{n}} v_{a_j,\lambda_j}^{\frac{a}{n}} dV_h = O(\varepsilon_{i,j}^{\frac{2}{n}} \ln \varepsilon_{i,j}), \]

where

\[ \varepsilon_{i,j} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_{n,\gamma} G_{h}^{\frac{2}{n}} (a_i, a_j) \right)^{\frac{2-n}{2}}, \quad \gamma_{n,\gamma} = g_{n,\gamma}^{\frac{2}{n}}. \]

The type of estimates stated in the latter lemma are standard, so we delay the proof to the appendix and pass to establishing the linear and nonlinear interaction estimates mentioned above.

**Lemma 3.5.** For \( i \neq j \) and letting

(i) \[ \varepsilon_{i,j} = \langle v_{a_i,\lambda_i}, v_{a_j,\lambda_j} \rangle_{P_{h}^{\gamma}} \]

(ii) \[ \varepsilon_{i,j} = \int_M v_{a_i,\lambda_i}^{\frac{a}{n}} v_{a_j,\lambda_j}^{\frac{a}{n}} dV_h \]

(iii) \[ \varepsilon_{i,j} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_{n,\gamma} G_{h}^{\frac{2}{n}} (a_i, a_j) \right)^{\frac{2-n}{2}}, \quad \gamma_{n,\gamma} = g_{n,\gamma}^{\frac{2}{n}} \]

there holds

(i) \( c \leq \varepsilon_{i,j} \leq C \)

(ii) \( \varepsilon_{i,j} = c_{n,\gamma} \varepsilon_{i,j} (1 + o_{\max}(\frac{\lambda_i}{\lambda_j}, \frac{1}{\lambda_j}) (1)) \)

(iii) \( \varepsilon_{i,j} = c_{n,4}^{\gamma} \varepsilon_{i,j} (1 + o_{\varepsilon_{i,j}} (1)) \) and \( \varepsilon_{i,j} = c_{n,3}^{\gamma} \varepsilon_{i,j} (1 + o_{\varepsilon_{i,j}} (1)) \),

where \( c_{n,3}^{\gamma}, c_{n,4}^{\gamma} \) and \( c_{n,\gamma} \) are given by (14) and (19).

**Proof.** Writing abbreviatively \( \varphi_{k} = \varphi_{a_k,\lambda_k} \) we first remark

\[ \varepsilon_{i,j} = \varepsilon_{j,i} = o_{\max}(\frac{\lambda_i}{\lambda_j}, \frac{1}{\lambda_j}) (\varepsilon_{i,j}). \]

We will prove (42) in the appendix. Then, since \( \varepsilon_{i,j} \) and \( \varepsilon_{i,j} \) are symmetric in \( i \) and \( j \), we may assume \( \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \) for the rest of the proof. Consider

\[ \varepsilon_{i,j} = \int_M P_{h}^{\gamma} (u_{a_i,\varphi_i} (u_{a_j,\varphi_j}) dV_h = \int_M P_{h_{a_i}}^{\gamma} \frac{u_{a_j}}{u_{a_i}} \varphi_j dV_{h_{a_i}} = \langle \varphi_i, \frac{u_{a_j}}{u_{a_i}} \varphi_j \rangle_{P_{h_{a_i}}^{\gamma}}, \]

where for the second equality we have applied the covariance property (9) to the conformal metric

\[ h_{a_i} = u_{a_i}^{\frac{4}{n}} h, \]

cf. (31) and (32). Denoting by \( y = y_{a_i} \) the geodesic defining function attached to \( h_{a_i} \) and choosing close to \( a_i \) a corresponding \( g_{a_i} \)-normal Fermi-coordinate system \( z = z_{a_i} = (y_{a_i}, x_{a_i}) = (y, x), r = |z|, \)

we find from (24), (37) and the divergence theorem, that

\[ \frac{1}{d_r} \varepsilon_{i,j} = - \lim_{y \to 0} \int_M y^{1-2\gamma} \partial_{y} \varphi_i \frac{u_{a_j}}{u_{a_i}} \varphi_j dV_{h_{a_i}} \]

\[ + \int_X d_{g_{a_i}} (y^{1-2\gamma} \nabla_{g_{a_i}} (\frac{u_{a_j}}{u_{a_i}} \varphi_j - \varphi_i \frac{u_{a_j}}{u_{a_i}})) dV_{g_{a_i}} \]

\[ = I_1 + I_2. \]

We first show smallness of \( I_2 \), i.e. (46). Recalling (23), since \( D_{g_{a_i}} \frac{u_{a_j}}{u_{a_i}} \varphi_j = 0 \) due to (37), we have

\[ I_2 = - \int_X D_{g_{a_i}} \left( \varphi_i - \varphi_i \frac{u_{a_j}}{u_{a_i}} \right) \frac{u_{a_j}}{u_{a_i}} \varphi_j dV_{g_{a_i}} \]

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and there holds, as we will prove in the appendix,

\[
\begin{aligned}
\frac{\psi_{a_1}}{u_{a_1}} \xi_j \leq C \xi_j \quad \text{on} \quad B_{p_0}^{D_1}(a_1).
\end{aligned}
\]

Let us estimate

\[
D_{g_{a_i}}(\varphi_i - \varphi_i^{a_i}) = -D_{g_{a_i}} \varphi_i^{a_i}.
\]

Recalling (40), \(D_{g_{a_i}} \varphi_i^{a_i} = 0\) on \(B_{2r_i}^{g_{a_i}}(a_i)\), whereas on \(B_{2r_i}^{g_{a_i}}(a_i)\) we have

\[
D_{g_{a_i}} \varphi_i^{a_i} = D(\eta_{a_i, \varepsilon_i}(\delta_{a_i, \lambda_i} - \frac{2\gamma - n}{p^{2\gamma - n}})) + (D_{g_{a_i}} - D)(\eta_{a_i, \varepsilon_i} \delta_{a_i, \lambda_i} + (1 - \eta_{a_i, \varepsilon_i}) \frac{2\gamma - n}{p^{2\gamma - n}})
\]

\[
= J_1 + J_2 + J_3
\]

by writing \(\Gamma_{a_i} = g_{n, \lambda}(r^{2\gamma - n} + H_{a_i})\) in \(g_{a_i}\)-normal Fermi-coordinates, see Subsection 4.2 of [43]. Then

\[
J_1 = - \text{div}(y^{1-2\gamma} \nabla(\eta_{a_i, \varepsilon_i}(\delta_{a_i, \lambda_i} - \frac{2\gamma - n}{p^{2\gamma - n}})))
\]

\[
= - \partial y y^{1-2\gamma} \partial y \eta_{a_i, \varepsilon_i}(\delta_{a_i, \lambda_i} - \frac{2\gamma - n}{p^{2\gamma - n}}) + 2 y^{1-2\gamma} \nabla \eta_{a_i, \varepsilon_i} \nabla(\delta_{a_i, \lambda_i} - \frac{2\gamma - n}{p^{2\gamma - n}})
\]

\[
y^{1-2\gamma} \Delta \eta_{a_i, \varepsilon_i}(\delta_{a_i, \lambda_i} - \frac{2\gamma - n}{p^{2\gamma - n}})
\]

due to \(D \delta_{a_i, \lambda_i} = D r^{2\gamma - n} = 0\). Then (35) and Corollary 3.3 show

\[
J_1 = o\left(\frac{1}{\varepsilon_i^{2\lambda_1^{2\gamma - n}}}\right) = O(y^{1-2\gamma}(\varphi_i^{a_i}),
\]

whenever \(\varepsilon_i \sim \lambda_1^{-2\gamma} \) and \(k \gg 1\) is chosen sufficiently large. Letting

\[
p, q = 1, \ldots, n + 1 \quad \text{and} \quad k, l = 1, \ldots, n,
\]

we evaluate

\[
D - D_{g_{a_i}} = \frac{\partial y}{\sqrt{g_{a_i}}} (\sqrt{g_{a_i}} g_{a_i}^{p, q} y^{1-2\gamma} \partial y^{q}) - E_{g_{a_i}} - \partial y (\delta^{p, q} y^{1-2\gamma} \partial y^{q})
\]

\[
= -E_{g_{a_i}} + \partial y \sqrt{g_{a_i}} y^{1-2\gamma} \partial y^{q} + \partial k \sqrt{g_{a_i}} g_{a_i}^{k, l} y^{1-2\gamma} \partial y^{l} + \partial k ((g_{a_i}^{k, l} - \delta^{k, l}) y^{1-2\gamma} \partial y^{l})
\]

\[
= O(y^{1-2\gamma}) + O(y^{2-2\gamma}) \partial y + O(y^{1-2\gamma} r) \nabla_x + O(y^{1-2\gamma} r^2 + y^{2-2\gamma} \nabla_x^2),
\]

where we made use of formula (14) of [43], and

\[
g_{a_i} = \delta + O(y + |x|^2) \quad \text{and} \quad \det g_{a_i} = 1 + O(y^2) \quad \text{on} \quad B_{2r_i}^{g_{a_i}}(a_i).
\]

From (45) and Corollary 3.2 we then find

\[
J_2 = O(y^{1-2\gamma}(\varphi_i^{a_i})) + O(y^{1-2\gamma} \lambda_1 y((\varphi_i^{a_i})^{n+2-2\gamma}))
\]

for \(\varepsilon_i \sim \lambda_1^{-2\gamma}\) and \(k \gg 1\) chosen sufficiently large. Moreover from (45) we find

\[
D_{g_{a_i}} H_{a_i} = -D_{g_{a_i}} r^{2\gamma - n} = (D - D_{g_{a_i}}) r^{2\gamma - n} = O(y^{1-2\gamma} r^{2\gamma - n - 1}),
\]

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whence $H_{a_i} = O(r^{1+2\gamma-n})$, cf. Subsection 4.2 of [43]. Thus, using (45), we obtain

$$|J_3| \leq \frac{Cy^{1-2\gamma}B_{r_i(a_i)} \epsilon}{\lambda_i^{-2\gamma} r^{n-2\gamma}} = O(y^{1-2\gamma} \varphi_i^{a_i}) \quad \text{on} \quad B_{2r_i(a_i)}^+(a_i)$$

for any choice $\epsilon \simeq \lambda_i^{-\frac{1}{2}}$, $k \gg 1$. Collecting terms we conclude

$$D g_i(a_i) \varphi_i^{a_i} = O(y^{1-2\gamma} \varphi_i^{a_i}) + O(y^{1-2\gamma} \lambda_i y(\varphi_i^{a_i})^{\frac{a_i+2-2\gamma}{n-2\gamma}}) \quad \text{on} \quad B_{2r_i(a_i)}^+(a_i),$$

which in conjunction with (44) implies

$$I_2 = O(\int_{B_{2r_i(a_i)}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} + O(\int_{B_{2r_i(a_i)}^+(a_i)} y^{1-2\gamma} \lambda_i y(\varphi_i^{a_i})^{\frac{a_i+2-2\gamma}{n-2\gamma}} \varphi_j^{a_j}).$$

As we will prove in the appendix, this gives

$$I_2 = o\left(\varepsilon_{i,j}\right).$$

Recalling (43) we thus arrive at

$$\frac{1}{d_{\gamma}} \varphi_{i,j} = - \lim_{y \to 0} \int_{B_{r_i(a_i)}^+(a_i)} y^{1-2\gamma} \partial_y \varphi_i^{a_i} u_a \varphi_j dV_{h_{a_i}} + o_{\frac{1}{\lambda_i}}(\varepsilon_{i,j}),$$

whence by virtue of (44)

$$\frac{1}{d_{\gamma}} \varphi_{i,j} = \int_{B_{r_i(a_i)}^+(a_i)} \eta a_i \partial_{\gamma} \varphi_j dV_{h_{a_i}} + o_{\frac{1}{\lambda_i}}(\varepsilon_{i,j}),$$

where $\delta_i = \delta_{a_i, \lambda_i}$. Moreover, using Corollary 3.3 we have

$$\int_{B_{r_i(a_i)}^+(a_i)} \eta a_i \partial_{\gamma} \varphi_j dV_{h_{a_i}} = \int_{B_{r_i(a_i)}^+(a_i)} \eta a_i \partial_{\delta_i} \varphi_j dV_{h_{a_i}} + O_{\frac{1}{\lambda_i}}(\varepsilon_{i,j})$$

for any choice $\varepsilon_i \sim \lambda_i^{-\frac{1}{2}}$, $k$ large. We conclude

$$\epsilon_{i,j} = \epsilon_{a_i, \gamma} \epsilon_{i,j} + o_{\frac{1}{\lambda_i}}(\varepsilon_{i,j}).$$

We turn to analyse $\epsilon_{i,j}$. From (47) we have

$$\epsilon_{i,j} = \int_{B_{r_i(a_i)}^+(a_i)} \eta a_i \partial_{\delta_i} \varphi_j dV_{h_{a_i}} + o_{\frac{1}{\lambda_i}}(\varepsilon_{i,j})$$

Changing coordinates and rescaling we obtain

$$\epsilon_{i,j} = \int_{B_{r_i(a_i)}^+(a_i,0)} \frac{u_{a_i}(\exp_{h_{a_i}} \varphi_j \lambda_i)}{1 + y^2} \frac{1}{\lambda_i} \left(\frac{h_{a_i}(\varphi_j)}{\lambda_i} + \lambda_i \lambda_j G_{a_j} \frac{h_{a_j}(\varphi_j)}{\lambda_j}\right) dV_{h_{a_i}} + o_{\frac{1}{\lambda_i}}(\varepsilon_{i,j}).$$
In particular
\[ \epsilon_{i,j} \ll 1 \Leftrightarrow \epsilon_{i,j} \ll 1 \]
and, as we will prove in the appendix,
\[ \epsilon_{i,j} = \epsilon_{i,j}(1 + o_\epsilon_{i,j}(1)), \]
Now (49) and (50) show \( c \leq \epsilon_{i,j} / \epsilon_{i,j} \leq C \), whence by virtue of (48) for \( \frac{1}{\lambda_i} \ll 1 \) sufficiently small \( c \leq \epsilon_{i,j} / \epsilon_{i,j} \leq C \).

This shows (i), i.e. that \( \epsilon_{i,j}, \epsilon_{i,j} \) and \( \epsilon_{i,j} \) are pairwise comparable. In particular (ii) follows from (48).

In view of (50) we are left with verifying \( \epsilon_{i,j} = \epsilon_{i,j}(1 + o_\epsilon_{i,j}(1)) \), but this follows easily from (ii) and (50) combined with (19). The proof of the lemma is thereby complete.

4 Locally flat conformal infinities of Poincaré-Einstein manifolds

In this section we discuss Fermi-coordinates in case of a Poincaré-Einstein manifold \((X, g^+)\) with locally flat conformal infinity \((M, [h])\). Furthermore, in this particular case, we establish sharp \( L^\infty\)-estimates for \( D_g(\phi_{a,\lambda} - \tilde{\phi}_{a,\lambda}) \) and \( \phi_{a,\lambda} - \tilde{\phi}_{a,\lambda} \) as well as selfaction estimates for \( v_{a,\lambda} \).

4.1 Fermi-coordinates in the particular case

By our assumptions we have
(i) a geodesic defining function \( y \) splitting the metric
\[ g = y^2 g^+ \quad g = dy^2 + h_y \quad \text{near} \quad M \quad \text{and} \quad h = h_y|_M \]
and for every \( a \in M \) a conformal factor \( u_a > 0 \) as in (31), whose conformal metric \( h_a = u_a^{-\lambda} h \) close to \( a \) admits an Euclidean coordinate system, \( h_a = \delta \) on \( B^{h_a}_{a}(a) \). As clarified in Subsection 3.2 in [43] and recalling Remark 1.1, this gives rise to a geodesic defining function \( y_a \), for which
\[ g_a = y_a^2 g^+ \quad g_a = dy_a^2 + h_{a,y_a} \quad \text{near} \quad M \quad \text{with} \quad h_a = h_{a,y_a}|_M \quad \text{and} \quad \delta = h_{a,y_a}|_{B^{h_a}_{a}(a)}, \]
the boundary \((M, [h_a])\) is totally geodesic and the extension operator \( D_{g_a} \) is positive.

(ii) in \( g_a \)-normal Fermi-coordinates around \( a \) for some small \( \epsilon > 0 \)
\[ g_a = \delta + O(|y_a|^\alpha) \quad \text{on} \quad B^{g_a}_{a\epsilon}(a), \]
as observed by Kim-Musso-Wei in case \( n \geq 3 \), cf. Lemma 43 in [39], and for \( n = 2 \) due to Remark 1.1 and the existence of isothermal coordinates.

4.2 Comparing Schoen’s bubbles and the Projective bubbles

In this subsection we compare the Schoen’s bubbles \( \tilde{\phi}_{a,\lambda} \) and our Projective bubbles \( \phi_{a,\lambda} \). Indeed we establish sharp \( L^\infty\)-estimates for \( D_{g_a}(\phi_{a,\lambda} - \tilde{\phi}_{a,\lambda}) \) and, using the maximum principle for \( D_{g_a} \) under Dirichlet boundary conditions, sharp \( L^\infty\)-estimates for \( \phi_{a,\lambda} - \tilde{\phi}_{a,\lambda} \).

Lemma 4.1. Writing in \( g_a \)-normal Fermi-coordinates
\[ \frac{\Gamma_a}{g_{a,\lambda}} = r^{2\gamma-n} + H_a = r^{2\gamma-n} + M_a + O(\max(r, r^{2\gamma})), \]
where \( M_a \) depends on \( a \in M \) only, cf. Theorem 1.4 in [43], there holds
\((i)\) \(D_g a(\varphi_{a,\lambda} - \tilde{\varphi}_{a,\lambda}^{-\alpha}) = -D_g a(\varphi_{a,\lambda}^{-\alpha}) = D_g a(\eta_{a,\varepsilon} \frac{H_a}{\lambda^{\gamma/2}}) + O\left(\frac{\lambda^{2\gamma-n}}{y^{r\gamma+1}}\right)
\]
\[
= \frac{M_a}{\lambda^{\gamma/2}} \text{div}_g a(y^{1-2\gamma} \nabla_{g,a} \eta_{a,\varepsilon}) + O\left(\frac{\max(1,\varepsilon^{1-2\gamma})}{\lambda^{\gamma/2}}\right)
\]
\[
(ii) \quad \tilde{\varphi}_{a,\lambda}^{-\alpha} - \varphi_{a,\lambda}^{-\alpha} = O\left(\frac{1}{\lambda^{\gamma/2}}\right),
\]
provided \(\varepsilon \sim \lambda^{-\frac{1}{\gamma}}\) for some \(k = k(n,\gamma)\) sufficiently large.

**Proof.** Recalling (33) and (58), since \(D_g a \Gamma_a = 0\), we have
\[
D_g a(\varphi_{a,\lambda}^{-\alpha}) = D_g a(\eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}}))
\]
\[
= D(\eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}})) + (D_g a - D)(\eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}})) - D_g a(\eta_{a,\varepsilon} \frac{H_a}{\lambda^{\gamma/2}})
\]
\[
= I_1 + I_2 + I_3
\]
and start evaluating
\[
I_1 = - \text{div}(y^{1-2\gamma} \nabla(\eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}})))
\]
\[
= - \partial_y y^{1-2\gamma} \partial_y \eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}}) + 2y^{1-2\gamma} \nabla \eta_{a,\varepsilon} \nabla(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}}) + y^{1-2\gamma} \Delta \eta_{a,\varepsilon}(\delta_{a,\lambda} = \lambda^{\frac{2\gamma-n}{r-2\gamma}}),
\]
where we made use of \(D\delta_{a,\lambda} = D\gamma^{2\gamma-n} = 0\). Then (33) and Corollary 3.3 show
\[
I_1 = o\left(\frac{y^{1-2\gamma}}{\varepsilon^{2\gamma/\gamma}}\right) = O\left(\frac{\lambda^{2\gamma-n}}{\varepsilon^{\gamma/2\gamma}y^{r\gamma+1}}\right),
\]
whenever \(\varepsilon \sim \lambda^{-\frac{1}{\gamma}}\) and \(k \gg 1\) is chosen sufficiently large. Secondly, letting
\[
p, q = 1, \ldots, n + 1\quad \text{and} \quad i, j = 1, \ldots, n,
\]
we use of formula (14) of [49], (51) and the splitting of the metric to evaluate
\[
D - D_g a = \frac{\partial_i}{\sqrt{g_a}}(\sqrt{g_a} g^{a,b} y^{1-2\gamma} \partial_y) - E_g a - \partial_p (\delta^{p,q} y^{1-2\gamma} \partial_y)
\]
\[
= \partial_i \sqrt{g_a} y^{1-2\gamma} \partial_y + \partial_i \sqrt{g_a} g^{i,j} y^{1-2\gamma} \partial_y + \partial_i ((g^{i,j} - \delta^{i,j}) y^{1-2\gamma} \partial_y) - E_g a
\]
\[
= O(y^{n-1-2\gamma}) + O(y^{n-2\gamma}) \partial_y + O(y^{n+1-2\gamma}) \nabla_x + O(y^{n+1-2\gamma}) \nabla_x^2,
\]
cf (15) for the non flat case. From (52) and Corollary 3.2 we then find
\[
I_2 = O\left(\frac{\lambda^{2\gamma-n}}{y^{r\gamma+1}}\right).
\]
Moreover (52) and Theorem 1.4 in [43] imply \(D_g a H_a = (D - D_g a)^{2\gamma-n} = O\left(\frac{1}{y^{r\gamma+1}}\right)\), whence
\[
I_3 = - \frac{H_a}{\lambda^{\gamma/2}} \text{div}_g a(y^{1-2\gamma} \nabla_{g,a} \eta_{a,\varepsilon}) - 2y^{1-2\gamma} \nabla_{g,a} \eta_{a,\varepsilon} \nabla_{g,a} H_a \frac{H_a}{\lambda^{\gamma/2}} / g_a + O\left(\frac{\lambda^{2\gamma-n}}{y^{r\gamma+1}}\right)
\]
Due to the structure of \(H_a\), cf. Theorem 1.4 in [43], we have
\[
H_a = M_a + O(\max(r, \varepsilon^{2\gamma}))\quad \text{and} \quad \nabla_{g,a} H_a = O(\max(1, \varepsilon^{2\gamma-1})
\]

with $M_a$ denoting the constant in the expansion of the Green’s function $\Gamma_a$. Recalling (35) we then get

$$I_3 = -\frac{M_a}{\lambda^{n-2\gamma}} \text{div}_{g_a}(y^{1-2\gamma}\nabla_{g_a}\eta_{a,\varepsilon}) + O\left(\max(1,\varepsilon^{1-2\gamma})\right).$$

Collecting terms we conclude

$$D_{g_a}\tilde{\varphi}_{a,\lambda} = -D_{g_a}(\eta_{a,\varepsilon}) \frac{H_a}{\lambda^{n-2\gamma}} + O\left(\frac{\max(1,\varepsilon^{1-2\gamma})}{y^{\gamma r^{1-\gamma}}}\right)$$

(53)

This and, that $D_{g_a}\tilde{\varphi}_{a,\lambda} = 0$ by definition, show (i). And (ii) follows from the maximum principle for $D_{g_a}$ under Dirichlet boundary conditions. Indeed consider

$$\chi_{a,\rho} = \chi(\rho^{-2}y^2), \quad \chi = 1 \text{ on } [0, 1] \text{ and } \chi = 0 \text{ on } (2, +\infty),$$

i.e. a cut-off function for the boundary, and

$$\psi = \psi_{a,\lambda,\varepsilon,\rho} = \tilde{\varphi}_{a,\lambda} - \varphi_{a,\lambda} - \eta_{a,\varepsilon} \frac{H_a}{\lambda^{n-2\gamma}} \pm C\eta_{a,\rho} y^{2\gamma} \ln y$$

with constants $\rho, C > 0$. We then find from (37), (38), (53) and $D(y^{2\gamma} \ln y) = -\frac{2\gamma}{y}$, that

$$\begin{cases} 
D_{g_a}\psi = O\left(\lambda^{\frac{n-2\gamma}{2}}\right) \quad \text{in } X \\
D_{g_a}\psi \leq 0 \quad \text{in } [d_{g_a}(\cdot, M) \leq \rho] \\
\psi = O\left(\lambda^{\frac{n-2\gamma}{2}}\right) \quad \text{on } M
\end{cases}$$

for suitable $0 < \rho, C^{-1} \ll 1$ and according to Proposition 3.1 in [43] we may solve

$$\begin{cases} 
D_{g_a}u = c_1 \quad \text{in } X \\
u = c_2 \quad \text{on } M
\end{cases}$$

for constants $c_1, c_2 \in \mathbb{R}$ with $u \in C^\infty(X, g_a) + y^{2\gamma}C^\infty(X, g_a)$. □

### 4.3 Selfaction estimates for $v_{a,\lambda}$ and the emergence of mass

In this short subsection we derive two selfaction estimates for $v_{a,\lambda}$. In particular we identify the fractional analogue of the mass for the classical Yamabe problem in the locally conformally flat case.

**Lemma 4.2.** Writing in $g_a$-normal Fermi-coordinates $z = z_a = (y_a, x_a) = (y, x)$

$$\Gamma_a = \frac{\partial}{\partial_n} = x^{2\gamma-n} + H_a = r^{2\gamma-n} + M_a + O(\max(r, r^{2\gamma})), $$

where $M_a$ depends on $a \in M$ only, cf. Theorem 1.4 in [43], there holds

$$\begin{align*}
(i) \quad & \langle v_{a,\lambda}, v_{a,\lambda} \rangle_{r_h} = \langle \varphi_{a,\lambda}, \varphi_{a,\lambda} \rangle_{r_h} = c^2_{n,2} - c^*_n + \frac{M_a}{\lambda^{n-2\gamma}} + a_\lambda\left(\frac{1}{\lambda^{n-2\gamma}}\right), \\
(ii) \quad & \int_M v_{a,\lambda}^{2\gamma} dV_h = \int_M \varphi_{a,\lambda}^{2\gamma} dV_{h_a} = c^1_{n,1} + a_{\lambda}\left(\lambda^{2\gamma-n}\right)
\end{align*}$$

provided $\varepsilon \sim \lambda^{-\frac{2}{\gamma}}$ for $k \in \mathbb{N}$ sufficiently large.
PROOF. Recalling (37) and (10), we calculate the quadratic form
\[
\frac{1}{d^2} \langle \varphi_{a, \lambda}, \varphi_{a, \lambda} \rangle_{P^\gamma_{h_a}} = \frac{1}{d^2} \int_M P_{g_a} \varphi_{a, \lambda} \varphi_{a, \lambda} dV_{h_a}
\]
\[
= -\lim_{y \to 0} \int_M y^{1-2\gamma} \partial_y \varphi_{a, \lambda} \varphi_{a, \lambda} \, dV_{h_a} - \lim_{y \to 0} \int_M y^{1-2\gamma} \partial_y (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda} \, dV_{h_a}
\]
\[
= c_{n, \gamma} \frac{a^{n+2}}{d} \int_M \eta_a \epsilon^{\frac{n+2}{2}} \varphi_{a, \lambda} \, dV_{h_a} + \int_X \text{div}_{g_a} (y^{1-2\gamma} \nabla_{g_a} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda}) \, dV_{g_a}
\]
and from (53), (54) and \(h_a = g_a\{M = 0\) close to \(a\) we easily find
\[
\int_M \eta_a \epsilon^{\frac{n+2}{2}} \varphi_{a, \lambda} \, dV_{h_a} = \int_{\mathbb{R}^n} \delta_{0, \lambda} + O(\int_{\mathbb{R}^n \setminus B_{\lambda}(0)} \delta_{0, \lambda}^{\frac{a}{\lambda}}) = c_{n, \gamma} + O(\frac{1}{\epsilon^a}),
\]
whence by integration by parts and any choice \(\epsilon \sim \lambda^{-\frac{1}{\gamma}}\), \(k\) sufficiently large
\[
\frac{1}{d^2} \langle \varphi_{a, \lambda}, \varphi_{a, \lambda} \rangle_{P^\gamma_{h_a}} = \frac{c_{n, \gamma}}{d^2} + \int_X y^{1-2\gamma} \left(\nabla_{g_a} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}), \nabla_{g_a} (\tilde{\varphi}_{a, \lambda})\right)_{g_a} + E_{g_a} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda} \, dV_{g_a}
\]
\[
= \frac{c_{n, \gamma}}{d^2} + \int_X \text{div}_{g_a} (y^{1-2\gamma} \nabla_{g_a} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda}) \, dV_{g_a}
\]
\[
- \int_X D_{g_a} (\varphi_{a, \lambda}) - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda} \, dV_{g_a} + O(y^\frac{1}{\lambda} (\lambda^{2\gamma - n})
\]
\[
= \frac{c_{n, \gamma}}{d^2} + \int_X \text{div}_{g_a} (y^{1-2\gamma} \nabla_{g_a} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda}) \, dV_{g_a}
\]
\[
- \int_X D_{g_a} (\varphi_{a, \lambda}) - \tilde{\varphi}_{a, \lambda}) \tilde{\varphi}_{a, \lambda} \, dV_{g_a} + \int_X D_{g_a} \tilde{\varphi}_{a, \lambda} (\varphi_{a, \lambda} - \tilde{\varphi}_{a, \lambda}) \, dV_{g_a} + O(y^\frac{1}{\lambda} (\lambda^{2\gamma - n})
\]
\[
= I_1 + \ldots + I_5.
\]
Clearly \(I_2 = 0\) and from Lemma 4.4 (i) we obtain
\[
I_3 = -\int_X D_{g_a} (\eta_{a, \epsilon} \frac{H_a}{\lambda^{\frac{2 - 2\gamma}{2}}} \, \tilde{\varphi}_{a, \lambda} \, dV_{g_a} + O(y^\frac{1}{\lambda\lambda^{2\gamma - n}}),
\]
since, whenever we choose \(\epsilon \sim \lambda^{-\frac{1}{\gamma}}\), then
\[
\int_{B_{2\epsilon}^+} \frac{\tilde{\varphi}_{a, \lambda}^a}{g_{\gamma} y^{1-\gamma}} \leq C \frac{1}{\lambda^{\frac{2 - 2\gamma}{2}}} \int_{B_{2\epsilon}^+} \frac{1}{y^{\gamma} (y^2 + |x|) \frac{1}{\lambda^{\frac{1}{\gamma}}} (1 + y^2 + |x|^2)^{\frac{a - 2\gamma}{2}}} \leq C \frac{1}{\lambda^{\frac{2 - 2\gamma}{2}}} \int_0^{\lambda^{\frac{1}{\gamma}}} \frac{dy}{y} \int_{Q^n} (1 + |x|^2)^{\frac{a - 2\gamma}{2}} = \frac{e^{2\gamma}}{\lambda^{\frac{2 - 2\gamma}{2}},
\]
where \(Q^n = [-1/2, 1/2]^n\). Moreover from Lemma 4.4 we easily find
\[
D_{g_a} \tilde{\varphi}_{a, \lambda} = O\left(\frac{e^{-2\gamma}}{y^{\gamma} y^{1-\gamma}}\right) \quad \text{and} \quad \tilde{\varphi}_{a, \lambda} - \tilde{\varphi}_{a, \lambda} = O\left(\frac{1}{\lambda^{\frac{2 - 2\gamma}{2}}},
\]
whence \(I_4 = O(\frac{e^{n-2\gamma}}{\lambda^{\frac{2 - 2\gamma}}}).\) Collecting terms we obtain
\[
\frac{1}{d^2} \langle \varphi_{a, \lambda}, \varphi_{a, \lambda} \rangle_{P^\gamma_{h_a}} = \frac{c_{n, \gamma}}{d^2} - \int_X D_{g_a} (\eta_{a, \epsilon} \frac{H_a}{\lambda^{\frac{2 - 2\gamma}{2}}} \, \tilde{\varphi}_{a, \lambda} \, dV_{g_a} + O\left(\frac{\max(\epsilon, e^{2\gamma})}{\lambda^{\frac{2 - 2\gamma}}},
\]

From Lemma 4.4 (i) we then find
\[
\int_X D_{g_a} (\eta_{a, \epsilon} \frac{H_a}{\lambda^{\frac{2 - 2\gamma}}}) \tilde{\varphi}_{a, \lambda} \, dV_{g_a} = \frac{M_a}{\lambda^{\frac{2 - 2\gamma}{2}}} \int_X \text{div}_{g_a} (y^{1-2\gamma} \nabla_{g_a} \eta_{a, \epsilon} \, \tilde{\varphi}_{a, \lambda} \, dV_{g_a} + O\left(\frac{\max(\epsilon, e^{2\gamma})}{\lambda^{\frac{2 - 2\gamma}}},
\]

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where integrating by parts, using \((35)\), we have

\[
\int_X \text{div}_{g_a}(y^{1-2\gamma} \nabla_{g_a} \eta_{a,\varepsilon} \varphi_{a,\lambda} - a) \, dV_{g_a} = -\int_X \text{div}_{g_a}(y^{1-2\gamma} \eta_{a,\varepsilon} \nabla_{g_a} \varphi_{a,\lambda} - a) \, dV_{g_a} + \int_X \eta_{a,\varepsilon} \text{div}_{g_a}(y^{1-2\gamma} \nabla_{g_a} \varphi_{a,\lambda} - a) \, dV_{g_a}
\]

with the latter integral being of order \(O(\varepsilon^{n-2\gamma}/(\lambda^{n-2\gamma}))\), since

\[
div_{g_a}(y^{1-2\gamma} \nabla_{g_a} \varphi_{a,\lambda} - a) = -D_{g_a} \varphi_{a,\lambda} - E_{g_a} \varphi_{a,\lambda}
\]

\[
= -\frac{M_\lambda}{\lambda^{n-2\gamma}} \text{div}_{g_a}(y^{1-2\gamma} \eta_{a,\varepsilon} \nabla_{g_a} \varphi_{a,\lambda} - a) + O(\max(1, \varepsilon^{n-2\gamma}) = O(\varepsilon^{n-2\gamma}/(\lambda^{n-2\gamma}))
\]
doing to formula 14 in \([43], [51], [53]\). Collecting terms we derive

\[
\frac{1}{d_\gamma}(\varphi_{a,\lambda} \varphi_{a,\lambda})_{b_\alpha} = c_{b_\alpha}^{\gamma} - \frac{M_\lambda}{\lambda^{n-2\gamma}} \int_X \text{div}_{g_a}(y^{1-2\gamma} \eta_{a,\varepsilon} \nabla_{g_a} \varphi_{a,\lambda} - a) \, dV_{g_a} + o\left(\frac{1}{\lambda^{n-2\gamma}}\right),
\]

provided we choose \(\varepsilon \sim \lambda^{-\frac{k}{n}}\) and \(k \gg 1\). Recalling \(18\), from \((40)\) we then find

\[
\langle \varphi_{a,\lambda} \varphi_{a,\lambda} \rangle_{b_\alpha} = c_{b_\alpha}^{\gamma} - \frac{M_\lambda}{\lambda^{n-2\gamma}} \int_M \eta_{a,\varepsilon} \delta_{a,\lambda}^{\gamma} \, dV_{B_\alpha} + o\left(\frac{1}{\lambda^{n-2\gamma}}\right)
\]

and arguing as for \((55)\) we conclude

\[
\langle \varphi_{a,\lambda} \varphi_{a,\lambda} \rangle_{b_\alpha} = c_{b_\alpha}^{\gamma} - \frac{M_\lambda}{\lambda^{n-2\gamma}} + o\left(\frac{1}{\lambda^{n-2\gamma}}\right),
\]

cf. \((59)\). This shows (i) and to see (ii) we have, again arguing as for \((55)\), that

\[
\int_M \varphi_{a,\lambda}^{2n} \, dV_{B_\alpha} = \int_{B_\alpha(a)} \delta_{a,\lambda}^{2n} \, dV_{B_\alpha} + \int_{M \setminus B_\alpha(a)} O\left(\frac{1}{\lambda^{n-2\gamma}}\right) \, dV_{B_\alpha} = \int_{\mathbb{R}^n} \delta_{a,\lambda}^{2n} \, dV_a + \int_{\mathbb{R}^n \setminus B_\alpha(a)} \delta_{a,\lambda}^{2n} + O\left(\frac{1}{\lambda^n}\right) = c_{b_\alpha}^{\gamma} + o\left(\lambda^{2\gamma - n}\right)
\]

provided \(\varepsilon \sim \lambda^{-\frac{k}{n}}\) for \(k \gg 1\) sufficiently large. The assertion then follows immediately from conformal covariance properties \((6)\) and \((32)\). \(\blacksquare\)

**Definition 4.3.** In analogy to the classical Yamabe problem we define the fractional mass map \(M_\gamma\) as

\[
M_\gamma(a) = M_\gamma
\]

where \(M_a\) is given by Lemma \([4.1]\) and the fractional minimal mass as \(M_\gamma^{\text{min}} = \min_{a \in M} M_\gamma(a)\).

### 5 Variational and algebraic topological argument

In this section we present the proof of Theorem \(1.3\) and therefore assume that we are under the assumptions of Theorem \(1.3\). Furthermore, because of Remark \(1.4\) and the work of Gonzalez-Qing \([36]\), we assume also that \(\gamma \neq \frac{1}{2}\) and \(\gamma^*(M, [h]) > 0\), and hence \(G_{h'} > 0\) for the Green’s function. With the above agreement we carry out the variational and algebraic topological argument for existence. We point out that the algebraic topological argument of Bahri-Coron \([8]\) has been used \([29, 30, 47, 42, 16]\). Hence we will omit some standard proofs and encourage readers to find them in \([42]\).
5.1 Variational principle in a non-compact setting

In this subsection we extend the classical variational principle to this non-compact setting. Clearly our bubbles \( v_{n,\lambda} \) can be used to replace the standard bubbles in the analysis of diverging Palais-Smale sequences of \( E_h^n \) in [27] with \( E_h^n \) as in [8]. Relying on this fact, we derive a deformation lemma that takes into account the bubbling phenomena via our bubbles \( v_{n,\lambda} \), cf [11].

To do so, we first define for \( p \in \mathbb{N} \)

\[
W_p = W_{p,\varepsilon} := \{ u \in W^{1,2}(M, h) : E_h^n(u) \leq (p + 1) \frac{2^n}{\varepsilon^p} \mathcal{F}(S^n) \},
\]

where \( \mathcal{F}(S^n) \) is as in [17].

Next we introduce the *neighborhood of potential critical points at infinity of* \( E_h^n \), which depends on a universal constant \( \nu_0 > 1 \) determined by Proposition 5.5 below.

**Definition 5.1.** For \( p \in \mathbb{N}^* \) and \( 0 < \varepsilon \leq \varepsilon_0 \) (for some small \( \varepsilon_0 > 0 \)) we call

\[
V(p, \varepsilon) := \{ u \in W^{1,2}(M, h) : \exists a_1, \ldots, a_p, M, \alpha_1, \ldots, \alpha_p, \lambda_1, \ldots, \lambda_p \geq \frac{1}{\varepsilon}, \| u - \sum_{i=1}^p \alpha_i \varphi_{a_i, \lambda_i} \| \leq \varepsilon, \frac{\alpha_i}{\alpha_j} \leq \nu_0, \varepsilon_i, j \leq \varepsilon, i \neq j = 1, \ldots, p \},
\]

the \((p, \varepsilon)\)-neighborhood of potential critical points at infinity of \( E_h^n \), where \( \| \cdot \| \) denotes the norm associated to the scalar product \( \langle \cdot, \cdot \rangle_{p,\varepsilon} \) defined by [11].

Concerning the sets \( V(p, \varepsilon) \), we have

**Lemma 5.2.** For every \( p \in \mathbb{N}^* \) there exist \( 0 < \varepsilon_p \leq \varepsilon_0 \) and \( C_p > 1 \) such that for every \( 0 < \varepsilon \leq \varepsilon_p \)

\[
\forall u \in V(p, \varepsilon) \text{ the minimization problem } \min_{B_p^{\varepsilon}} \| u - \sum_{i=1}^p \alpha_i \varphi_{a_i, \lambda_i} \|
\]

has a solution \((\hat{\alpha}, A, \hat{\lambda}) \in B_p^{\varepsilon} \), which is unique up to permutations,

where \( B_p^{\varepsilon} \) for \( \gamma > 0 \) is defined by

\[
B_p^{\varepsilon} := \{ (\hat{\alpha} = (\alpha_1, \ldots, \alpha_p), A = (a_1, \ldots, a_p), \hat{\lambda} = (\lambda_1, \ldots, \lambda_p)) \in \mathbb{R}_+^p \times M^p \times (0, +\infty)^p
\]

such that \( \lambda_i \geq \frac{1}{\gamma} \) for \( i = 1, \ldots, p \), \( \frac{\alpha_i}{\alpha_j} \leq \nu_0 \) and \( \varepsilon_i, j \leq \gamma \) for \( i \neq j = 1, \ldots, p \).

Furthermore, for every \( p \in \mathbb{N}^* \) and \( 0 < \varepsilon \leq \varepsilon_p \), we consider the selection map

\[
(58) \quad s_p : V(p, \varepsilon) \longrightarrow (M)^p / \sigma_p : u \longrightarrow s_p(u) = |A|.
\]

Here \( \sigma_p \) denotes the permutation group acting on \( M^p \) and \( A = (a_1, \ldots, a_p) \in M^p \) is derived from a minimizer \( \sum_{i=1}^p \alpha_i \varphi_{a_i, \lambda_i} \) associated to \( u \in V(p, \varepsilon) \) by virtue of Lemma 5.2. Since Lemma 5.2 not only assures existence of a minimizer, but also its uniqueness up to permutations of indices, the class \([A] = A / \sigma_p\) is uniquely determined by \( u \) and \( s_p \) therefore well defined.

Now, having introduced the neighborhoods of potential critical points at infinity of \( E_h^n \), we are ready to state a deformation lemma, which follows from the same arguments as for its counterparts in classical application of the barycenter technique for existence of Bahri-Coron [8] and the fact that the \( v_{n,\lambda} \) can replace the standard bubbles in the analysis of diverging Palais-Smale sequences of \( E_h^n \). Indeed we have the following result, which was the aim of this subsection.

**Lemma 5.3.** Assuming that \( E_h^n \) has no critical points, then for every \( p \in \mathbb{N}^* \), up to taking \( \varepsilon_p \) given by \( 57 \) smaller, we have that for every \( 0 < \varepsilon \leq \varepsilon_p \) the topological pair \((W_p, W_{p-1})\) retracts by deformation onto \((W_{p-1} \cup A_p, W_{p-1})\) with \( V(p, \varepsilon) \subset A_p \subset V(p, \varepsilon) \), where \( 0 < \varepsilon < \frac{\varepsilon_1}{4} \) is a very small positive real number and depends on \( \varepsilon \).
5.2 Algebraic topological argument of Bahri-Coron

In this subsection we present the algebraic topological argument or barycenter technique of Bahri-Coron [8] for existence to prove Theorem 1.3. To that end we start establishing sharp $\mathcal{E}_h^\gamma$-energy estimates for $v_{a,\lambda}$. As in [10], we will follow the presentation of the barycenter technique in [42] and hence omit some standard proofs and direct the readers to [42] for details. Recalling [8], we have

**Lemma 5.4.** With $c_{n, \gamma}^*$ as in (19) there holds

$$\mathcal{E}_h^\gamma(v_{a, \lambda}) = \mathcal{V}_\gamma(S^n) \left( 1 - c_{n, \gamma}^* \frac{\mathcal{M}_\gamma(a)}{\lambda^{n-2\gamma}} \right) \left( 1 + o_\lambda(1) \right).$$

**Proof.** The proof is a direct application of Lemma 4.2. Next we define for $p \in \mathbb{N}^*$ and $\lambda > 0$

$$f_p(\lambda) : B_p(M) \rightarrow W^{2,2}_p(M) : \sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \mapsto f_p(\lambda)(\sigma) = \sum_{i=1}^p \alpha_i v_{a_i, \lambda}$$

with $B_p(M)$ as in (12). Using Lemmas 5.5 and 5.6 we will prove that at true infinity the quantity

$$\Delta^\lambda \mathcal{E}_h^\gamma(\sigma) := \lambda^{n-2\gamma} \left( \frac{\mathcal{E}_h^\gamma(f_p(\sigma))}{p^{\gamma} \mathcal{V}_\gamma(S^n)} - 1 \right), \quad \sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(M)$$

has at most a linear growth in $p$ with a negative slope and this uniformly in $\sigma \in B_p(M)$ and $\lambda$ large, namely at infinity

$$(59) \sup_{\lambda \gg 1, \sigma \in B_p(M)} \Delta^\lambda \mathcal{E}_h^\gamma(\sigma) \leq -cp + C.$$ 

About estimate (59) we precisely show the following proposition.

**Proposition 5.5.** There exists $\nu_0 > 1$ such that for every $p \in \mathbb{N}^*$, $p \geq 2$ and every $0 < \varepsilon \leq \varepsilon_0$ there exists $\lambda_p = \lambda_p(\varepsilon)$ such that for every $\lambda \geq \lambda_p$ and for every $\sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(M)$ we have,

(i) if $\sum_{i \neq j} \varepsilon_{i,j} > \varepsilon$ or there exist $i_0 \neq j_0$ such that $\frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu_0$, then

$$\mathcal{E}_h^\gamma(f_p(\lambda)(\sigma)) \leq \frac{2}{p^{\gamma}} \mathcal{V}_\gamma(S^n).$$

(ii) if $\sum_{i \neq j} \varepsilon_{i,j} \leq \varepsilon$ and for every $i \neq j$ we have $\frac{\alpha_i}{\alpha_j} \leq \nu_0$, then

$$\mathcal{E}_h^\gamma(f_p(\lambda)(\sigma)) \leq \frac{\mathcal{M}^{\min}_{\gamma}}{p^{\gamma}} \mathcal{V}_\gamma(S^n) \left( 1 - c_{n, \gamma}^* \frac{\mathcal{M}^{\min}_{\gamma}}{\lambda^{n-2\gamma}} - c_h^\gamma (p-1) \right),$$

where $c_h^\gamma := \frac{c_{n, \gamma}^*}{4G_\gamma}$, $G_\gamma$, $\mathcal{M}^{\min}_{\gamma}$ is as in Definition 4.3 and $(M^2)^*$ defined in (29).

**Proof.** The proof is the same as the one of Proposition 3.1 in [42] using Lemmas 3.3, 4.2, 5.4, 3.3. Now we start transporting the topology of the manifold $M$ into sublevels of the Euler-Lagrange functional $\mathcal{E}_h^\gamma$ by bubbling via $v_{a, \lambda}$. Recalling (13) and (56), we have

**Lemma 5.6.** Assuming that $\mathcal{E}_h^\gamma$ has no critical points and $0 < \varepsilon \leq \varepsilon_1$, then up to taking $\varepsilon_1$ smaller and $\lambda_1$ larger, we have that for every $\lambda \geq \lambda_1$

$$f_1(\lambda) : (B_1(M), B_0(M)) \rightarrow (W_1, W_0)$$

is well defined and satisfies

$$(f_1(\lambda))_*(w_1) \neq 0 \text{ in } H_n(W_1, W_0).$$
Proof. The proof follows from the same arguments as the ones used in the proof of Lemma 4.2 in [12] by using the selection map $s_1$ (see (68)), Lemma 5.3 and Lemma 5.4. 

Next we use the previous lemma and pile up masses by bubbling via $v_{a,\lambda}$ in a recursive way. Still recalling (14) we have

**Lemma 5.7.** Assuming that $\mathcal{E}_h^\gamma$ has no critical points and $0 < \varepsilon \leq \varepsilon_{p+1}$, then up to taking $\varepsilon_{p+1}$ smaller and $\lambda_p$ and $\lambda_{p+1}$ larger, we have that for every $\lambda \geq \max\{\lambda_p, \lambda_{p+1}\}$

$$f_{p+1}(\lambda) : (B_{p+1}(M), B_p(M)) \rightarrow (W_{p+1}, W_p)$$

and

$$f_p(\lambda) : (B_p(M), B_{p-1}(M)) \rightarrow (W_p, W_{p-1})$$

are well defined and

$$(f_p(\lambda)_*(w_p) \neq 0 \text{ in } H_{np+p-1}(W_p, W_{p-1})$$

implies

$$(f_{p+1}(\lambda)_*(w_{p+1}) \neq 0 \text{ in } H_{n(p+1)+p}(W_{p+1}, W_p).$$

Proof. The proof follows from the same arguments as the ones used in the proof of Lemma 4.3 in [12], by using the selection map $s_p$ (see (58)), Lemma 5.3 and Proposition 5.5.

Finally we use the strength of Proposition 5.5 - namely point (ii) - to give a criterion ensuring that the recursive process of piling up masses via Lemma 5.7 must stop at least after a very large number of steps.

**Lemma 5.8.** Setting

$$p_h^\gamma := [1 - c_{n,\gamma}^* \frac{M_{n,\gamma}}{c_h^\gamma}] + 2$$

and recalling (56), then $\forall 0 < \varepsilon \leq \varepsilon_0$ and $\forall \lambda \geq \lambda_{p_h}$ there holds $f_{p_h}(\lambda)(B_{p_h}(M)) \subset W_{p_{h-1}}$.

Proof. The proof is a direct application of Proposition 5.5.

Proof of Theorem 1.3 It follows by a contradiction argument from Lemma 5.6 - Lemma 5.8.

### 6 Appendix

In this section we provide the proofs of Lemmas 3.1, 3.4 and of some technical estimates.

Proof of Lemma 3.1 Let us write $\delta_i = \delta_{a_i,\lambda_i}$, $\delta_j = \delta_{a_j,\lambda_j}$ for simplicity.

(i) Due to

$$\int_{\mathbb{R}^n} \delta_{n+2\gamma} \delta_j = \frac{1}{c_{n,\gamma}} \int_{\mathbb{R}^n} (-\Delta)^\gamma \delta_i \delta_j = \frac{1}{c_{n,\gamma}} \int_{\mathbb{R}^n} \delta_i (-\Delta)^\gamma \delta_j = \int_{\mathbb{R}^n} \delta_j \delta_{n+2\gamma}$$

we may assume $\lambda_i \leq \lambda_j$. Then we have with $r = |x|$

$$\int_{\mathbb{R}^n} \delta_{n+2\gamma} \delta_j = \int_{\mathbb{R}^n} \left( \frac{1}{1 + r^2} \right)^{n+2\gamma} \left( \frac{\lambda_i}{\lambda_j} \lambda_j |x| + \lambda_i |x| + a_i - a_j |x| \right)^{n-2\gamma} dx.$$

Since $\varepsilon_{i,j} = \lambda_i |a_i - a_j|^2$ or $\varepsilon_{i,j} = \lambda_j |a_i - a_j|^2$, we may expand on

$$A = \left[ \frac{x}{\lambda_i} \leq \varepsilon |a_i - a_j| \right] \cup \left[ \frac{x}{\lambda_j} \leq \varepsilon \right]$$

with

$$\int_{\mathbb{R}^n} \delta_{n+2\gamma} \delta_j = \int_{A} \left( \frac{1}{1 + r^2} \right)^{n+2\gamma} \left( \frac{\lambda_i}{\lambda_j} \lambda_j |x| + \lambda_i |x| + a_i - a_j |x| \right)^{n-2\gamma} dx.$$
for $\epsilon > 0$ sufficiently small

\[
\left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \frac{x}{\lambda_i} + a_i - a_j^2\right)^{\frac{2\gamma - n}{2}}
\]

\[
= \left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \right)^{\frac{2\gamma - n}{2}} + (2\gamma - n) \frac{\langle a_i - a_j^2, \lambda_j x \rangle + O(\lambda_j \gamma \gamma \varepsilon, \varepsilon)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}}.
\]

Thus \(\int_{\mathbb{R}^n} \delta_i^{\gamma - n} \delta_j = \sum_{k=1}^{4} I_k\) with

(i) \(I_1 = \frac{1}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} \int_{\mathbb{R}^n} \frac{1}{1 + r^2} dx\)

(ii) \(I_2 = \frac{2\gamma - n}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} \int_{\mathbb{R}^n} \frac{\langle a_i - a_j^2, \lambda_j x \rangle}{1 + r^2} dx\)

(iii) \(I_3 = \frac{1}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} \int_{\mathbb{R}^n} O(\lambda_j \gamma \gamma \varepsilon, \varepsilon) dx\)

(iv) \(I_4 = \int_{\mathbb{R}^n} \frac{1}{1 + r^2} (\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma - n} dx\).

We get

\[
I_1 = \frac{b_1}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} + O((\frac{\lambda_j}{\lambda_i}) \gamma \varepsilon),
\]

with \(b_1 = \int_{\mathbb{R}^n} \frac{1}{1 + r^2} = c_{n, \gamma}\), \(I_2 = 0\) and \(I_4 = O((\frac{\lambda_j}{\lambda_i}) \gamma \varepsilon)\). Moreover

\[
I_4 \leq \frac{C}{(\frac{\lambda_j}{\lambda_i})^{\gamma - n}} = O((\frac{\lambda_j}{\lambda_i}) \gamma \varepsilon) \quad \text{in case} \quad \varepsilon^{\gamma - n} \sim \frac{\lambda_j}{\lambda_i}.
\]

Otherwise we may assume \(\frac{\lambda_i}{\lambda_j} \leq \frac{\lambda_i}{\lambda_j} \ll \lambda_i \lambda_j |a_i - a_j^2|\) and decompose

\(A^c \subseteq B_1 \cup B_2\),

where for a sufficiently large constant \(E > 0\)

\(B_1 = \left[ |a_i - a_j| \leq \frac{\gamma}{\lambda_i} \leq E |a_i - a_j| \right]\) and \(B_2 = \left[ E |a_i - a_j| \leq \frac{\gamma}{\lambda_i} \right].\)

We then may estimate

\[
I_1 = \int_{B_1} \left(\frac{1}{1 + r^2}\right)^{\gamma - n} \frac{1}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} dx
\]

\[
\leq \frac{C}{(\frac{\lambda_j}{\lambda_i})^{\gamma - n}} \int_{|x_i - x_j| \leq E |a_i - a_j|} \left(\frac{1}{1 + r^2}\right)^{\gamma - n} dx
\]

\[
\leq \frac{C}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} \int_{B_{\lambda_i} |a_i - a_j|} \left(\frac{1}{1 + r^2}\right)^{\gamma - n} dx
\]

\[
\leq \frac{C}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j^2| \lambda_j^{-n})^{\gamma}} = O((\frac{\lambda_j}{\lambda_i}) \gamma \varepsilon).
\]
and
\[ I_{4,2} = \int_{B_2} \left( \frac{1}{1 + r^2} \right)^{\frac{n-2s}{2}} \frac{1}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2)} \frac{1}{|x|^2} \, dx \]
\[ \leq \frac{C}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2)} \int_{|x| \geq \lambda_i |a_i - a_j|} \left( \frac{1}{1 + r^2} \right)^{\frac{n-2s}{2}} \, dx = O((\frac{\lambda_j}{\lambda_i})^{\gamma} \varepsilon_{i,j}^{\frac{n}{2-s}}), \]
so \( I_4 \leq \varepsilon_{i,j}^2 = O((\frac{\lambda_j}{\lambda_i})^{\gamma} \varepsilon_{i,j}^{\frac{n}{2-s}}) \). Collecting terms the claim follows.

(ii) It is sufficient to consider the case \( k = j \), since
\[ \lambda_j \partial_{\lambda_j} \int_{\mathbb{R}^n} \delta_i \delta_j \, dx = \lambda_j \partial_{\lambda_j} \int_{\mathbb{R}^n} \delta_j \, dx. \]

First we deal with the case \( \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \). We have
\[ -\lambda_j \int_{\mathbb{R}^n} \frac{2}{\lambda_j} \frac{1}{(1 + r^2)^{\frac{n-2s}{2}}} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \, dx, \]
whence
\[ -\lambda_j \int_{\mathbb{R}^n} \frac{2}{\lambda_j} \frac{1}{(1 + r^2)^{\frac{n-2s}{2}}} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \, dx. \]
Since \( \varepsilon_{i,j}^2 \sim \lambda_i \lambda_j |a_i - a_j|^2 \) or \( \varepsilon_{i,j}^2 \sim \frac{1}{\lambda_j} \), we may expand on
\[ A = \left[ \frac{x}{\lambda_i} \leq \epsilon |a_i - a_j| \right] \cup \left[ \frac{x}{\lambda_j} \leq \epsilon \right] \]
for \( \epsilon > 0 \) sufficiently small.
\[ \frac{1}{(\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2)} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \]
\[ = (\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2) \frac{1}{\lambda_i} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \]
\[ + (2\gamma - n) \frac{(a_i - a_j, \lambda_j x)}{\lambda_j} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2) \frac{1}{\lambda_i} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \]
\[ + \frac{4}{\lambda_j} \frac{(a_i - a_j, \lambda_j x)}{\lambda_j} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2) \frac{1}{\lambda_i} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \]
From this we derive as before with \( b_2 = \frac{n-2s}{2} c_{i,j} \),
\[ -\lambda_j \int_{\mathbb{R}^n} \delta_i \frac{2}{\lambda_j} \frac{1}{(1 + r^2)^{\frac{n-2s}{2}}} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \, dx = O((\frac{\lambda_j}{\lambda_i})^{\gamma} \varepsilon_{i,j}^{\frac{n}{2-s}}), \]
whence
\[ -\lambda_j \int_{\mathbb{R}^n} \delta_i \frac{2}{\lambda_j} \frac{1}{(1 + r^2)^{\frac{n-2s}{2}}} \lambda_j^2 |x|^2 (\frac{\lambda_j}{\lambda_i} + \lambda_i |a_i - a_j|^2) \frac{1}{|x|^2} \lambda_j^2 |x - a_j|^2 \, dx. \]
We turn to the case $\frac{1}{\lambda_i} \geq \frac{1}{\lambda_j}$. We then have $-\lambda_j \int_{\mathbb{R}^n} \delta_i^{n+2\gamma} \partial_\lambda_j \delta_j = -\lambda_j \int_{\mathbb{R}^n} \delta_i \partial_\lambda_j \delta_j^{n+2\gamma}$ and

$$-\lambda_j \int_{\mathbb{R}^n} \delta_i \partial_\lambda_j \delta_j^{n+2\gamma} = \frac{n + 2\gamma}{2} \int_{\mathbb{R}^n} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x - a_i|^2} \right)^{n+2\gamma} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x - a_j|^2} \right)^{n+2\gamma} \frac{\lambda_j^2 |x - a_j|^2 - 1}{\lambda_j^2 |x - a_j|^2 + 1} dx,$$

whence

$$-\lambda_j \int_{\mathbb{R}^n} \delta_i \partial_\lambda_j \delta_j^{n+2\gamma} = \frac{n + 2\gamma}{2} \int_{\mathbb{R}^n} \frac{r^2 - 1}{r^2 + 1} \left( \frac{1}{1 + r^2} \right)^{n+2\gamma} \left( \frac{\lambda_i}{\lambda_i^2 + \lambda_i \lambda_j |x - a_i|^2 + a_j - a_i|a_j|^2} \right)^{n+2\gamma} dx.$$

We may expand on $A = \left[ \frac{|x|}{\lambda_i} \leq |a_j - a_i| \right] \cup \left[ \frac{|x|}{\lambda_j} \leq |x_j| \right]$ for $\epsilon > 0$ sufficiently small

$$= \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x - a_i|^2 \left( \frac{2\gamma}{n} \right) + (2\gamma - n) \frac{(a_j - a_i, \lambda_i x) + O(\frac{\lambda_i}{\lambda_j} |x_i|^2)}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_j - a_i|^2 \right)^{\frac{n+2\gamma}{2}}}.$$

As before we find with $b_2 = \frac{n-2\gamma}{2} \int_{\mathbb{R}^n} \left( \frac{1}{1 + r^2} \right)^{n+2\gamma} = \frac{n+2\gamma}{2} \int_{\mathbb{R}^n} \frac{r^2 - 1}{r^2 + 1} \left( \frac{1}{1 + r^2} \right)^{n+2\gamma} dx$

$$-\lambda_j \int_{\mathbb{R}^n} \delta_i \partial_\lambda_j \delta_j^{n+2\gamma} = b_2 \frac{1}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |x - a_i|^2 \right)^{\frac{n+2\gamma}{2}}} + O((\frac{\lambda_i}{\lambda_j})^\gamma \varepsilon_{i,j}^\gamma).$$

Collecting the results in cases $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$ and $\frac{1}{\lambda_j} \leq \frac{1}{\lambda_i}$ the claim follows.

(iii) By translation invariance and symmetry we may assume $a_i = 0$, $a_j = a$ and $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$. Then

$$\nabla_a \int_{\mathbb{R}^n} \delta_i \partial_\lambda_j \delta_j = \nabla_a \int_{\mathbb{R}^n} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x + a|^2} \right)^{n+2\gamma} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x + a|^2} \right)^{n+2\gamma} \frac{\lambda_j^2 (x + a)}{1 + \lambda_j^2 |x + a|^2} dx$$

$$= (2\gamma - n) \int_{\mathbb{R}^n} \frac{1}{1 + r^2} \left( \frac{\lambda_i}{1 + \lambda_i^2 r^2} \right)^{n+2\gamma} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x + a|^2} \right)^{n+2\gamma} \frac{\lambda_j^2 (x + a)}{1 + \lambda_j^2 |x + a|^2} dx$$

$$= (2\gamma - n) \int_{\mathbb{R}^n} \left( \frac{1}{1 + r^2} \right)^{\frac{n+2\gamma}{2}} \frac{\lambda_i \lambda_j (\frac{x}{a_i} + a)}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}} dx.$$

Since $\varepsilon_{i,j}^\frac{2}{n+2\gamma} \sim \lambda_i \lambda_j |a|^2$ or $\varepsilon_{i,j}^\frac{2}{n+2\gamma} \sim \frac{\lambda_i}{\lambda_j}$ we may expand on

$$A = \left[ \frac{|x|}{\lambda_i} \leq \epsilon |a| \right] \cup \left[ \frac{|x|}{\lambda_j} \leq |x_j| \right]$$

for $\epsilon > 0$ sufficiently small

$$\frac{\lambda_i \lambda_j \left( \frac{x}{a_i} + a \right)}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}}$$

$$= \frac{\lambda_i \lambda_j a}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}} - (n + 2 - 2\gamma) \frac{\lambda_i \lambda_j a |x, a|}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}}$$

$$+ \frac{\lambda_j x}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}} + \frac{\sqrt{\lambda_i \lambda_j O(\frac{\lambda_i}{\lambda_j} |x|^2)}}{\left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right)^{\frac{n+2\gamma}{2}}}.$$
Using radial symmetry we obtain with \( b_3 = (2\gamma - n)c_{n,3}^\gamma \)

\[
\nabla_a \int_{\mathbb{R}^n} \frac{\lambda_i \lambda_j a}{(\lambda_i + \lambda_j |a|^2) \frac{n+2}{2}} + O(\frac{\lambda_i \lambda_j}{\lambda_i^\gamma} \sqrt{\lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+1}{2}}})
\]

\[
+ (2\gamma - n) \int_{\mathcal{A}^c} \frac{\lambda_i \lambda_j (\frac{x}{\lambda_i} + a)}{(\lambda_i + \lambda_j |a|^2) \frac{n+2}{2}} \frac{|a|^2}{\frac{n+2}{2}} \ dx.
\]

In case \( \varepsilon_{i,j}^{\frac{n+2}{2}} \sim \frac{2}{\lambda_i} \) the last summand above is of order

\[
O(\frac{\lambda_i \lambda_j}{\lambda_i^\gamma} \sqrt{\lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+1}{2}}}).
\]

Thus we may assume \( \lambda_i \lambda_j \sim \lambda_i \lambda_j |a|^2 \). Letting \( 0 < \theta < \frac{1}{4} \) and

\[
\mathcal{B}_1 = \left[ \frac{x}{\lambda_i} + a \leq \theta |a| \right] \subset \mathcal{B}_{\varepsilon\lambda_i|a|}(0) = \mathcal{A}^c
\]

we may expand on \( \mathcal{B}_1 \)

\[
(1 + \lambda_i^2 |a|^2) \frac{n+2}{2} - \lambda_i \lambda_j (\frac{x}{\lambda_i} + a) - \frac{1}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}} - \frac{O(\lambda_i^2 |a|^2) \frac{n+2}{2} \lambda_i \lambda_j (\frac{x}{\lambda_i} + a)}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}}
\]

and find using radial symmetry

\[
\left| \int_{\mathcal{B}_1} \frac{1}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}} \lambda_i \lambda_j (\frac{x}{\lambda_i} + a) \ dx \right| \leq \frac{C}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}} \int_{\mathcal{B}_1} \frac{\lambda_i \lambda_j |\frac{x}{\lambda_i} + a|^2 |a|}{\frac{n+2}{2}} \ dx
\]

\[
\leq \frac{C\lambda_i}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}} \int_{\mathcal{B}_1} \frac{1}{\frac{n+2}{2} |\frac{x}{\lambda_i} + a| < \theta |a|} \left| (1 + \lambda_j^2 |\frac{x}{\lambda_j} + a|^2) \frac{n+2}{2} \right| \ dx
\]

\[
\leq \frac{C}{\lambda_i} \sqrt{\lambda_i \lambda_j |a|^2} \frac{\gamma}{\lambda_j} \left( \frac{\lambda_j}{\lambda_i} \right) \int_{\mathcal{A}^c} \left( \frac{1}{1 + r^2} \right) \frac{n+2}{2} = O((\lambda_j / \lambda_i)^\gamma \sqrt{\lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+1}{2}}}).
\]

Moreover letting \( \mathcal{B}_2 = \mathcal{A}^c \setminus \mathcal{B}_1 \) we may estimate

\[
\left| \int_{\mathcal{B}_2} \frac{1}{(1 + r^2) \frac{n+2}{2}} \lambda_i \lambda_j (\frac{x}{\lambda_i} + a) \ dx \right| \leq \frac{C}{\lambda_i} \sqrt{\lambda_i \lambda_j} \left( \frac{\lambda_j}{\lambda_i} \right) \int_{\mathcal{A}^c} \left( \frac{1}{1 + r^2} \right) \frac{n+2}{2} = O((\lambda_j / \lambda_i)^\gamma \sqrt{\lambda_i \lambda_j \varepsilon_{i,j}^{\frac{n+1}{2}}}).
\]

Collecting terms the claim follows.
(iv) By translation invariance and symmetry we may assume $a_i = 0$, $a_j = a$ and $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$. Then

$$
\nabla_a \int_{\mathbb{R}^n} \delta_i^{\frac{n+2}{2}} \delta_j = (2\gamma - n) \int_{\mathbb{R}^n} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x|^2} \right)^{\frac{n+2}{2}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x+a|^2} \right)^{\frac{n+2}{2}} \left[ \frac{\lambda_i^2 id}{1 + \lambda_i^2 |x|^2} - (n + 2 - 2\gamma) \frac{\lambda_i^2 (x + a) \cdot (x + a)}{(1 + \lambda_i^2 |x + a|^2)} \right] dx

= (2\gamma - n) \lambda_i \lambda_j \int_{\mathbb{R}^n} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |x|^2 + a|^2 \right) id - (n + 2 - 2\gamma) \lambda_i \lambda_j (\frac{x}{\lambda_i} + a) \cdot (\frac{x}{\lambda_i} + a) \right) \lambda_i \lambda_j |x|^2 + a|^2 \right) dx.

Since $\varepsilon_{i,j}^{\frac{n+2}{2}} \sim \lambda_i \lambda_j |a|^2$ or $\varepsilon_{i,j}^{\frac{n+2}{2}} \sim \frac{1}{\lambda_j}$ we may expand on

$$
A = [\frac{1}{\lambda_j} \leq |a|] \cup [\frac{1}{\lambda_i} \leq |a|]
$$

for $\epsilon > 0$ sufficiently small

$$
\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |x|^2 + a|^2 \right) id - (n + 2 - 2\gamma) \lambda_i \lambda_j a.a + O(\frac{\lambda_i}{\lambda_j})^2 |x|^2 + a|^2 \right) dx,

up to some integrable odd terms and thus obtain

$$
\nabla_a \int_{\mathbb{R}^n} \delta_i^{\frac{n+2}{2}} \delta_j = b_4 \lambda_i \lambda_j \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a|^2 \right) id - (n + 2 - 2\gamma) \lambda_i \lambda_j a.a + O((\frac{\lambda_i}{\lambda_j})^2 |x|^2 + a|^2 \right) dx,

where $b_4 = (2\gamma - n) c_{n,3}^\gamma$ and

$$
\ldots = \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |x|^2 + a|^2 \right) id - (n + 2 - 2\gamma) \lambda_i \lambda_j (\frac{x}{\lambda_i} + a) \cdot (\frac{x}{\lambda_i} + a) \right) \lambda_i \lambda_j |x|^2 + a|^2 \right) dx.

In case $\varepsilon_{i,j}^{\frac{n+2}{2}} \sim \frac{1}{\lambda_j}$ the integral above is of order

$$
O((\frac{\lambda_j}{\lambda_i})^2 \varepsilon_{i,j}^{\frac{n+2}{2}}).
$$

Thus we may assume $\frac{\lambda_i}{\lambda_j} \leq \frac{\lambda_i}{\lambda_j} \leq \lambda_i \lambda_j |a|^2$, i.e. for $0 < \theta < \frac{1}{4}$

$$
B_1 = \left[ \frac{x}{\lambda_j} + a \leq \theta |a| \right] \subset B_{\lambda_i \lambda_j |a|}^\epsilon (0) = A^\epsilon.
$$

On $B_1$ we then expand

$$
\left( \frac{1}{1 + r^2} \right)^{\frac{n+2}{2}} = \left( \frac{1}{1 + \lambda_i^2 |x|^2} + a - a |a| \right) \frac{n+2}{2}

= \left( \frac{1}{1 + \lambda_i^2 |a|^2} \right)^{\frac{n+2}{2}} + (n + 2\gamma) \lambda_i^2 \frac{(\frac{x}{\lambda_i} + a) \cdot (\frac{x}{\lambda_i} + a)}{(1 + \lambda_i^2 |a|^2) \frac{n+2}{2}}.
$$

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Proof of Lemma 3.4

Thereby (i)-(iv) are proven and so is the lemma.

Proof of Lemma 3.3

(i) Let \( \alpha' = \frac{n-2\alpha}{2}, \beta' = \frac{n-2\beta}{2} \), so \( \alpha' + \beta' = n \). We distinguish the cases

\[ n \varepsilon_{i,j}^{\alpha'} \sim \frac{\gamma_{i,j}}{\lambda_i} \quad \varepsilon_{i,j}^{\beta'} \sim \lambda_i \gamma_n G_h^{\frac{2}{\lambda_i^2}}(a_i, a_j) \quad \text{We estimate for } c > 0 \text{ small} \]

\[ \int_M v_i^\alpha v_j^\beta dV_h \leq C \int_{B_r(0)} \left( \frac{\lambda_i}{1 + \lambda_i^2 r^2 \gamma_n^2} \right)^{\alpha'} \left( \frac{1}{1 + \lambda_i^2 r^2 \gamma_n^2 G_h^{\frac{2}{\lambda_i^2}}} \right) dV_i \]

\[ \leq C \int_{B_{\lambda_i}(0)} \left( \frac{1}{1 + r^2} \right)^{\alpha'} \left( \frac{1}{\lambda_i} + \lambda_i \gamma_n G_h^{\frac{2}{\lambda_i^2}}(a_i, a_j) \right)^{\beta'} dV_i \]

up to some \( O(\frac{\gamma_{i,j}}{\lambda_i}) \). Thus by \( \int_{B_{\lambda_i}(0)} \left( \frac{1}{1 + r^2} \right)^{\beta'} \leq C \frac{1}{\gamma_n^{2\beta'-n}} \) we get

\[ \int_M v_i^\alpha v_j^\beta dV_h \leq C \int_{B_{\lambda_i}(0)} \left( \frac{1}{1 + r^2} \right)^{\alpha'} \left( \frac{1}{\lambda_i} + \lambda_i \gamma_n \frac{1}{G_h^{\frac{2}{\lambda_i^2}}} \right)^{\beta'} dV_i \]

up to some \( O(\frac{\gamma_{i,j}}{\lambda_i}) \), whence the claim follows in cases

\[ \frac{\lambda_j}{\lambda_i} + \lambda_i \gamma_n \frac{1}{G_h^{\frac{2}{\lambda_i^2}}} \sim \frac{\lambda_i}{\lambda_j} \quad \text{or} \quad d_{g_{u_j}}(a_j, a_i) > 3c. \]
Else we may assume \( d_{g_{a_j}}(a_j, a_i) < 3c \) and
\[
\lambda_i + \lambda_j + \lambda_i \lambda_j G_{\alpha_i - \gamma} (a_i, a_j) \sim \lambda_i \lambda_j d_{g_{a_j}}^2(a_j, a_i).
\]
We then get with \( B = [\frac{1}{2} d_{g_{a_j}}(a_j, a_i) \leq \frac{c}{h}] \leq 2 d_{g_{a_j}}(a_j, a_i) \)
\[
\int_M v_i^\alpha v_j^\beta dV_h \leq C \int_B \left( \frac{1}{1 + r^2} \right) \alpha' \left( \frac{1}{\lambda_i + \lambda_j d_{g_{a_j}}(a_j, \exp_{a_j}(\frac{x}{\lambda_j}))} \right) \beta' + O(\varepsilon_{i,j}^\beta)
\]
\[
\leq C \left( \frac{1}{1 + |\lambda_i d_h(a_i, a_j)|^2} \right) \alpha' \int_{[\frac{c}{h}] \leq d_h(a_i, a_j)} \left( \frac{1}{\lambda_i + \lambda_j r^2} \right) \beta' + O(\varepsilon_{i,j}^\beta)
\]
\[
\leq C \left( \frac{1}{1 + |\lambda_i d_h(a_i, a_j)|^2} \right) \alpha' \int_{r \leq d_h(a_i, a_j)} \left( \frac{1}{1 + r^2} \right) \beta' + O(\varepsilon_{i,j}^\beta).
\]
Note, that in case \( \lambda_j d_{g_{a_j}}(a_j, a_i) \) remains bounded, we are done. Else
\[
\int_M v_i^\alpha v_j^\beta dV_h \leq C \left( \frac{\lambda_i}{\lambda_j} \right)^{\beta'-n} \left( \lambda_j d_h(a_j, a_i) \right)^{n-2\beta'} + O(\varepsilon_{i,j}^\beta)
\]
\[
\leq C \left( \frac{1}{1 + |\lambda_i d_h(a_i, a_j)|^2} \right) \alpha' \left( \frac{\lambda_i}{\lambda_j} \right)^{\beta'} + O(\varepsilon_{i,j}^\beta),
\]
whence due to \( \alpha' > \frac{2}{\lambda_j} \) the claim follows.

\( \beta \) \quad \varepsilon_{i,j}^\frac{2}{\lambda_j - n} \sim \frac{\lambda_i}{\lambda_j} \quad \) We estimate for \( c > 0 \) small
\[
\int_M v_i^\alpha v_j^\beta dV_h \leq C \int_{B_{\lambda_j}(0)} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_{\alpha_i, \gamma} G_{\alpha_i - \gamma} (\exp_{a_j}(\frac{x}{\lambda_j})) \right) \alpha' \left( \frac{1}{1 + r^2} \right) \beta'
\]
\[+ C \left( \frac{1}{\lambda_j} \right) \int_{B_{\lambda_j}(0)} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma \right) \alpha' + O(\varepsilon_{i,j}^\beta),
\]
which by \( \int_{B_{\lambda_j}(0)} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma \right) \alpha' \leq C \left( \frac{1}{\lambda_j} \right)^{\alpha'} \) gives
\[
\int_M v_i^\alpha v_j^\beta \leq C \int_{B_{\lambda_j}(0)} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma \exp_{a_j}(\frac{x}{\lambda_j}) \right) \alpha' \left( \frac{1}{1 + r^2} \right) \beta'
\]
up to some \( O(\varepsilon_{i,j}^\beta). \) Since by assumption \( d_h(a_i, a_j) \leq \frac{2}{\lambda_j}, \) there holds
\[
\gamma_{\alpha_i, \gamma} G_{\alpha_i}^\frac{2}{\lambda_j - n} (\exp_{a_j}(\frac{x}{\lambda_j})) \sim d_h^2(a_i, \exp_{a_j}(\frac{x}{\lambda_j})) \quad \text{ on } B_{\lambda_j}(0).
\]
Thus for \( \gamma > 3 \)
\[
\int_M v_i^\alpha v_j^\beta dV_h \leq C \int_{[\gamma \frac{2}{\lambda_j} \leq |x| \leq \lambda_j]} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_h^2(a_i, \exp_{a_j}(\frac{x}{\lambda_j})) \right) \alpha' \left( \frac{1}{1 + r^2} \right) \beta'
\]
\[+ C \left( \frac{\lambda_i}{\lambda_j} \right) \int_{|x| < \gamma \frac{2}{\lambda_j}} \left( \frac{1}{1 + r^2} \right) \beta' + o(\varepsilon_{i,j}^\beta)
\]
\[
\leq C \int_{[\gamma \frac{2}{\lambda_j} \leq |x| \leq \lambda_j]} \left( \frac{\lambda_i}{\lambda_j} + \frac{1}{1 + r^2} \right) \alpha' \left( \frac{1}{1 + r^2} \right) \beta' + C \left( \frac{\lambda_i}{\lambda_j} \right)^{\alpha'} \left( \frac{\lambda_i}{\lambda_j} \right)^{n-2\beta} + O(\varepsilon_{i,j}^\beta),
\]
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since for $|x| \geq \gamma \frac{1}{\lambda_i}$ we may due to $d_h(a_i, a_j) \leq \frac{2}{\lambda_i}$ assume $d_h(a_i, \exp_{a_i}(\frac{x}{\lambda_i})) \geq \frac{|x|}{\lambda_i}$. Therefore

$$\int_M v_i a_j v_i dV_h \leq C(\frac{\lambda_i}{\lambda_i})^{\alpha'} \int_{|x| \geq \gamma \frac{1}{\lambda_i}} r^{-2n} + O(\epsilon_{i,j}^\beta) = O(\epsilon_{i,j}^\beta).$$

(ii) By symmetry we may assume $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$ and thus

$$\epsilon_{i,j}^\alpha \sim \frac{\lambda_i}{\lambda_j} \frac{\lambda_i}{1 + \lambda_i^2} \sim \lambda_i \lambda_j \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(a_i, a_j)$$

In case $d_{h_{a_j}}(a_j, a_i) > 3\epsilon$ for some $\epsilon > 0$ we estimate

$$\int_M v_i^{\frac{n}{n^2}} v_j^{\frac{n}{n^2}} dV_h \leq C \int_{B_{\epsilon}(0)} \left( \frac{\lambda_i}{1 + \lambda_i^2} \right)^{\frac{n}{n^2}} \left( \frac{\lambda_i}{1 + \lambda_i^2} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(x)) \right)^{\frac{n}{n^2}}$$

$$+ C \int_{B_{\epsilon}(0)} \left( \frac{\lambda_i}{1 + \lambda_i^2} \right)^{\frac{n}{n^2}} \left( \frac{1}{\lambda_i} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(\frac{x}{\lambda_i})) \right)^{\frac{n}{n^2}} + C \ln \lambda_i \epsilon_{i,j}^\alpha$$

$$\leq C \int_{B_{\epsilon}(0)} \left( \frac{1}{1 + r^2} \right)^{\frac{n}{n^2}} \left( \frac{1}{\lambda_i} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(\frac{x}{\lambda_i})) \right)^{\frac{n}{n^2}} + C \ln \lambda_i \epsilon_{i,j}^\alpha$$

Thus we assume, that $d_{h_{a_j}}(a_j, a_i)$ is arbitrarily small. Then for $d_{h_{a_j}}(a_j, a_i) \ll c \ll 1$ we estimate passing to normal coordinates around $a_i$

$$\int_M v_i^{\frac{n}{n^2}} v_j^{\frac{n}{n^2}} dV_h \leq C \int_{B_c(0)} \left( \frac{1}{1 + r^2} \right)^{\frac{n}{n^2}} \left( \frac{1}{\lambda_i} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(\frac{x}{\lambda_i})) \right)^{\frac{n}{n^2}} = C \int_{B_{c}(0)} I$$

up to some terms of order $\frac{1}{\lambda_i} \sim O(\epsilon_{i,j}^\alpha)$. Decompose $B_{c}(0)$ into

$(\alpha)$ $A = ||\frac{x}{\lambda_i}|| < \epsilon \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}$

$(\beta)$ $B = [\epsilon \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}, \epsilon \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}]$

$(\gamma)$ $C = [\epsilon \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}, \epsilon \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}]$ for some fixed $0 < \epsilon \ll 1 \ll E \ll \infty$. We then find

$$\int_A I \leq C \epsilon_{i,j}^{\alpha} \int_A \left( \frac{1}{1 + r^2} \right)^{\frac{n}{n^2}} = O(\ln \epsilon_{i,j}^{\alpha})$$

and

$$\int_B I \leq \frac{C}{\lambda_i} \int_B \left( \frac{1}{\lambda_i} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(\frac{x}{\lambda_i})) \right)^{\frac{n}{n^2}}$$

$$\leq C \epsilon_{i,j}^{\alpha} \int_B \left( \frac{1}{\lambda_i} \gamma_{\eta,n} G_h^{\frac{n^2}{2}}(\exp_{a_i}(\frac{x}{\lambda_i})) \right)^{\frac{n}{n^2}}$$

$$= C \epsilon_{i,j}^{\alpha} \int_{|x| < \lambda_i E \sqrt{G_{a_j}^{\frac{n^2}{2}}}(a_i) + \frac{1}{\lambda_j}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{n}{n^2}} = O(\ln \epsilon_{i,j}^{\alpha})$$

and

$$\int_C I \leq C \epsilon_{i,j}^{\alpha} \int_C \left( \frac{1}{1 + r^2} \right)^{\frac{n}{n^2}} = O(\ln \epsilon_{i,j}^{\alpha})$$

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and
\[ \int_c I \leq C \int_c \left( \frac{1}{1 + r^2} \right)^{\frac{p}{2}} \left( \frac{1}{\lambda_j^2 + \frac{1}{2} |x|^2} \right)^{\frac{p}{2}} \leq C \left( \frac{\lambda_i}{\lambda_j} \right)^{\frac{p}{2}} \int_c \left( \frac{1}{1 + r^2} \right)^{\frac{p}{2}} r^{-n} \]
\[ \leq C \left( \frac{\lambda_i}{\lambda_j} \right)^{\frac{p}{2}} \int_0^\infty \frac{r^{-1-n}}{E \lambda_i \sqrt{G_{\lambda_j}^{-\frac{3}{2}}(a_i) + \frac{1}{\lambda_j}}}. \]

Collecting terms the assertion follows. □

We turn now to the proofs of the estimates (42), (44), (46) and (50).

**Proof of the estimate (42).** First we notice, that due to Corollary 4.6 in [43] and [34] we have
\[ \varphi_k = (1 + o_{\varepsilon_k}(1))(\frac{\lambda_k}{1 + \lambda_k^2 G_{\lambda_k}^{-\frac{3}{2}}})^{\frac{n-2}{2}} \]
for any choice \( \varepsilon_k \sim \lambda_k^{\frac{1}{4}} \), whence
\[ \varepsilon_{i,j} = \int_{B_{\varepsilon}(a_i)} (u_{a_i} \varphi_{a_i, \lambda_i}) \frac{\lambda_i}{1 + \lambda_i^2 G_{\lambda_i}^{-\frac{3}{2}}(a_i, a_j)} dV_h \]
\[ = (1 + o_{\varepsilon_k}(1)) \int_{B_{\varepsilon}(a_i)} \frac{1}{1 + r^2} \varphi_{a_i, \lambda_i} \frac{u_{a_i}}{u_{a_i} \exp_{a_i} \varphi_{a_i}} \frac{1}{\lambda_i \lambda_j G_{\lambda_j}^{-\frac{3}{2}}(a_i, a_j)} \]
up to some \( o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}} \). In case \( d_{a_i} (a_i, a_j) > \sqrt{\varepsilon} \) we thus get
\[ \varepsilon_{i,j} = (1 + o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}}(a_i, a_j)) \frac{1}{\lambda_i \lambda_j G_{\lambda_i}^{-\frac{3}{2}}(a_i, a_j)} \]
by conformal covariance of the Green’s function, i.e.
\[ G_{\lambda_i}(a_i) = G_{h_{a_i}}(a_i, a_j) = u_{a_i}^{-1}(a_i) u_{a_i}^{-1}(a_j) G_{h}(a_i, a_j) \quad \text{recalling} \quad h_{a_i} = \frac{1}{u_{a_i}} h. \]

Therefore (42) follows by symmetry in this case. Otherwise we may assume \( d_{a_i}(a_i, a_j) \leq \varepsilon_i \) and rewriting the Green’s function in (50) in \( h_{a_i} \)-normal coordinates via (61) we obtain
\[ \varepsilon_{i,j} = (1 + o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}}(a_i, a_j)) \frac{1}{\lambda_i \lambda_j G_{\lambda_i}^{-\frac{3}{2}}(a_i, a_j)} \]
up to some \( o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}} \). Using \( \exp_{a_i}^{-1} a_j = (1 + o_{\varepsilon_k}(1)) \exp_{a_j}^{-1} a_i \) and
\[ \int_{\mathbb{R}^n} \delta_i^{\frac{n+2}{2}} \delta_j = \frac{1}{c_{n, \gamma}} \int_{\mathbb{R}^n} (\Delta)^{\gamma} \delta_i \delta_j = \frac{1}{c_{n, \gamma}} \int_{\mathbb{R}^n} \delta_i (\Delta)^{\gamma} \delta_j = \int_{\mathbb{R}^n} \delta_i \delta_j^{\frac{n+2}{2}} \]
we thus find up to some \( o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}} \)
\[ \varepsilon_{i,j} = (1 + o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}}(a_i, a_j)) \frac{1}{\lambda_i \lambda_j G_{\lambda_i}^{-\frac{3}{2}}(a_i, a_j)} \]
\[ = (1 + o_{\varepsilon_k}(\lambda_i \lambda_j)^{\frac{n-2}{2}}(a_i, a_j)) \frac{1}{\lambda_i \lambda_j G_{\lambda_i}^{-\frac{3}{2}}(a_i, a_j)}. \]
This shows \( \epsilon_{i,j} = O(\varepsilon_{i,j}) \), cf. Lemma 3.1(i), and calculating back we find
\[
\epsilon_{i,j} = (1 + a_{\text{max}}(\frac{1}{\lambda_i}, \frac{1}{\lambda_j})(1))\epsilon_{i,j} + a_{\text{max}}(\frac{1}{\lambda_i}, \frac{1}{\lambda_j})(\lambda_i\lambda_j)^{\frac{2n-n}{2}} = (1 + a_{\text{max}}(\frac{1}{\lambda_i}, \frac{1}{\lambda_j})(1))\epsilon_{i,j}.
\]
Thereby (42) follows.

**Proof of** the estimate (44). Note, that due to Corollary 4.3 in [43] we have
\[
K_{g_{a_i}}(y, x, \xi) \leq C y^{2\gamma}(1 + \frac{1}{\gamma^2 + d_{g_{a_i}}^2(x, \xi)})^{\frac{n-2\gamma}{2}} + 1
\]
for \((y, x) \in B_{\rho_0}^{-1}(\alpha_i)\), whence
\[
\left|\frac{u_{a_i}}{u_{a_i}} \varphi_j\right|(y, x) \leq C \int_M \left( \frac{y^{-1}}{y^2 + d_{g_{a_i}}^2(x, \xi)} \right)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_{g_{a_i}}^2(x, \xi)} \right)^{\frac{n-2\gamma}{2}} dV_{g_{a_i}} + C y^{2\gamma} \frac{\lambda_j}{\lambda_j^2}
\]
and thus
\[
\left|\frac{u_{a_i}}{u_{a_i}} \varphi_j\right|(y, x) \leq C \int_M \left( \frac{y^{-1}}{y^2 + d_{g_{a_i}}^2(x, \xi)} \right)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_{g_{a_i}}^2(x, \xi)} \right)^{\frac{n-2\gamma}{2}} dV_{g_{a_i}} + C y^{2\gamma} \frac{\lambda_j}{\lambda_j^2}.
\]
As all the distances involved are comparable to \(d_g\), the first summand above vanishes for \(d_g(x, a_j) \geq C \rho_0\) and the claim follows. Else we may assume \(d_g(x, a_j) \leq C \rho_0\) and find passing to integration over \(\mathbb{R}^n\) by standard interaction estimates as given in Lemma 3.1
\[
\left|\frac{u_{a_i}}{u_{a_i}} \varphi_j\right|(y, x) \leq C \frac{1}{y^2 + d_{g_{a_i}}^2(x, \xi)} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_{g_{a_i}}^2(x, \xi)} \right)^{\frac{n-2\gamma}{2}} + C y^{2\gamma} \frac{\lambda_j}{\lambda_j^2}.
\]
The proof is thereby complete.

**Proof of** the estimate (46). We let \(B_r^+(a) = B_r^{-\alpha}(a)\) and start showing
\[
(62) \int_{B_{2r}^+(a_i)} y^{-2\gamma} \varphi_i^{-a_i} \varphi_j^{-a_j} = o(\frac{1}{\text{w}})(\varepsilon_{i,j}).
\]
In case \(d_g(a_i, a_j) > c \gg \varepsilon_i\) we have
\[
\int_{B_{2r}^+(a_i)} y^{-2\gamma} \varphi_i^{-a_i} \varphi_j^{-a_j} \leq C \frac{\lambda_j}{\lambda_j^2} \int_{B_{2r}^+(a_i)} y^{-2\gamma} \varphi_i^{-a_i} = O\left(\frac{\varepsilon_i^2}{(\lambda_i \lambda_j)\frac{2\gamma}{\alpha}}\right),
\]
so (62) holds true in this case for any choice \(\varepsilon_i \sim \lambda_i^{-\frac{2\gamma}{\alpha}}\). Thus we may assume \(d_g(a_i, a_j) \ll 1\) for the rest of the proof and moreover, that
\[
\varepsilon_{i,j}^{2\gamma}\sim \frac{\lambda_i}{\lambda_j} \quad \text{or} \quad \varepsilon_{i,j}^{2\gamma} \sim \lambda_i \lambda_j \gamma^{-\frac{2\gamma}{\alpha}}(a_i, a_j).
\]
Passing to \(g_{a_i}^{-}\)- normal Fermi-coordinates and rescaling we then have
\[
\int_{B_{2r}^+(a_i)} y^{-2\gamma} \varphi_i^{-a_i} \varphi_j^{-a_j} \leq C \frac{\lambda_j}{\lambda_j^2} \int_{B_{2r}^{-1}(a_i)} y^{-2\gamma} \left( \frac{1}{y^2 + \lambda_i \lambda_j \gamma^{-\frac{2\gamma}{\alpha}}(a_i, a_j)} \right)^{\frac{n-2\gamma}{2}}.
\]
In particular (62) holds in case \( \varepsilon_{i,j}^{\frac{2}{n} - n} \sim \frac{1}{\lambda_j} \), so we may assume

\[
\varepsilon_{i,j}^{\frac{2}{n} - n} \sim \lambda_i \lambda_j G_{\frac{2}{n}}^\gamma (a_i, a_j) \sim \lambda_i \lambda_j |a_i - a_j|^2.
\]

We subdivide the region of integration, i.e. \( B_{2r, \lambda_i} = B_{2r, \lambda_i}^+ (0) \) into

(i) \( B_1 = \{ |x| \leq \varepsilon |a_i - a_j| \} \cap B_{2r, \lambda_i}^+ \)

(ii) \( B_2 = \{ |a_i - a_j| < \frac{1}{|x|} \} \leq E|a_i - a_j| \} \cap B_{2r, \lambda_i}^+ \)

(iii) \( B_1 = \{ |x| > E|a_i - a_j| \} \cap B_{2r, \lambda_i}^+ \)

for \( z = (y, x) \) and suitable \( 0 < \epsilon, \epsilon^{-1} \ll 1 \) and obtain easily

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} \leq \frac{C}{\lambda_i^2} \int_{B_2} \left( \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}} \right) \frac{1}{|z|^{2\gamma - n}} \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}}
\]

up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \) for any choice \( \varepsilon_i \sim \lambda_i^{-\frac{1}{2}} \). Note, that on \( B_2 \) we have

\[
\epsilon |a_i - a_j| \leq |\frac{z}{\lambda_i}| \leq 2 \varepsilon_i,
\]

so \( |a_i - a_j| = O(\epsilon_i) \) for \( B_2 \neq \emptyset \). We then find up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \)

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} \leq \frac{C}{\lambda_i^2} \int_{B_2} \left( \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}} \right) \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}}
\]

up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \) for any choice \( \varepsilon_i \sim \lambda_i^{-\frac{1}{2}} \). Note, that on \( B_2 \) we have

\[
\epsilon |a_i - a_j| \leq |\frac{z}{\lambda_i}| \leq 2 \varepsilon_i,
\]

so \( |a_i - a_j| = O(\epsilon_i) \) for \( B_2 \neq \emptyset \). We then find up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \)

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} \leq \frac{C}{\lambda_i^2} \int_{B_2} \left( \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}} \right) \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}}
\]

up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \) for any choice \( \varepsilon_i \sim \lambda_i^{-\frac{1}{2}} \). Note, that on \( B_2 \) we have

\[
\epsilon |a_i - a_j| \leq |\frac{z}{\lambda_i}| \leq 2 \varepsilon_i,
\]

so \( |a_i - a_j| = O(\epsilon_i) \) for \( B_2 \neq \emptyset \). We then find up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \)

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} \leq \frac{C}{\lambda_i^2} \int_{B_2} \left( \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}} \right) \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}}
\]

up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \) for any choice \( \varepsilon_i \sim \lambda_i^{-\frac{1}{2}} \). Note, that on \( B_2 \) we have

\[
\epsilon |a_i - a_j| \leq |\frac{z}{\lambda_i}| \leq 2 \varepsilon_i,
\]

so \( |a_i - a_j| = O(\epsilon_i) \) for \( B_2 \neq \emptyset \). We then find up to some \( o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}) \)

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \varphi_i^{a_i} \varphi_j^{a_j} \leq \frac{C}{\lambda_i^2} \int_{B_2} \left( \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}} \right) \frac{y^{1-2\gamma}}{(1 + r^2)^{\frac{n-2\gamma}{2}} \left( \frac{\lambda_i + \lambda_j |x|}{\lambda_i} \right)^{2-\gamma}}
\]

whence (62) holds again and thus in any case. We are left with proving

\[
\int_{B_{2r, \lambda_i}^+(a_i)} y^{1-2\gamma} \lambda_i y(\varphi_i^{a_i}) \varphi_j^{a_j} = o^{\frac{1}{\lambda_i}}(\varepsilon_{i,j}).
\]

But this follows line by line as when showing (62). ■

**Proof of the estimate (50).** We know

\[
\epsilon_{i,j} = \int_{B_{2r, \lambda_i}^+(a_i)} \left( \frac{\exp_g h_{\frac{x}{\lambda_i}}} {\exp_g h_{\frac{x}{\lambda_i}}} \right)^{\frac{2}{n} - n} \frac{1} {\lambda_i + \lambda_j G_{\frac{2}{n}}} (\exp_g h_{\frac{x}{\lambda_i}})^{\frac{2}{n} - n} dV_{h_{\lambda_i}}.
\]

and have to show

\[
\epsilon_{i,j} = \epsilon_{i,j}^\gamma (1 + o_{i,j}(1)).
\]

Since \( \varepsilon_{i,j}^{\frac{2}{n} - n} \sim \lambda_i \lambda_j G_{\frac{2}{n}}^\gamma (a_i, a_j) \) or \( \varepsilon_{i,j}^{\frac{2}{n} - n} \sim \frac{1}{\lambda_i} \), we may expand on

\[
A = \left( \left( \frac{|x|}{\lambda_i} \right) \leq \epsilon |G_{\frac{2}{n}}^\gamma (a_i) \right) \cup \left( \frac{|x|}{\lambda_i} \leq \epsilon \frac{1}{\lambda_j} \right) \right) \cap B_{2r, \lambda_i}^+(0) \subseteq B_{2r, \lambda_i}(0)
\]

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for $\epsilon > 0$ small $\frac{u_{a_1}(\exp_{g_{a_1}} x)}{u_{a_1}} = u_{a_1}(a_1) + \nabla u_{a_1}(a_1) x + O(\frac{|x|^2}{\lambda_2})$ and

$$\left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(\exp_{g_{a_1}} x)\right)^{\frac{2}{n-2}}$$

$$= \left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i)\right)^{\frac{2}{n-2}} + 2\gamma - n \frac{\nabla G_{a_j}^{\frac{2}{n-2}}(a_i) x + O(\frac{|x|^2}{\lambda_2})}{2 \left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i)\right)^{\frac{n-2}{n}}}.$$ 

Thus $\epsilon_{i,j} = \sum_{k=1}^5 I_k + o_{\frac{1}{\lambda_2}}(\epsilon_{i,j})$ for

(i) $I_1 = \frac{u_{a_1}(a_1)}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} \int_{\mathcal{A}} \frac{1}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}}.$

(ii) $I_2 = \frac{1}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} \int_{\mathcal{A}} \frac{\nabla u_{a_1}(a_1) x}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}}.$

(iii) $I_3 = \frac{2 \gamma - n}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} \int_{\mathcal{A}} \frac{\nabla G_{a_j}^{\frac{2}{n-2}}(a_i) x}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}}.$

(iv) $I_4 = \frac{1}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} \int_{\mathcal{A}} \frac{O(\frac{|x|^2}{\lambda_i})}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}}.$

(v) $I_5 = \frac{u_{a_1}(a_1)}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i)^{\frac{n-2}{n}}} \int_{\mathcal{A}} \frac{1}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}}.$

We then find with $c_{a_1}^\gamma = \int_{\mathbb{R}^n} \frac{u_{a_1}(a_1)}{(\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} (1 + o_{\epsilon_{i,j}}(1)),$

where $I_2 = I_3 = 0$ by radial symmetry and $I_4 = o_{\epsilon_{i,j}}(\epsilon_{i,j}).$ Moreover

$I_5 = o_{\epsilon_{i,j}}(\epsilon_{i,j})$

in case $\frac{2 \gamma - n}{\lambda_i} \sim \frac{1}{\lambda_i}.$ Else we have $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_i} \leq \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i, a_j)$ and decompose

$\mathcal{A}^c = B_{\epsilon_1} \setminus \mathcal{A} \subseteq \mathcal{B}_1 \cup \mathcal{B}_2,$

where for a sufficiently large constant $E > 0$

$\mathcal{B}_1 = \{ x \sqrt{G_{a_j}^{\frac{2}{n-2}}(a_i)} \leq \frac{x}{\lambda_i} \leq E \sqrt{G_{a_j}^{\frac{2}{n-2}}(a_i)} \}$ and $\mathcal{B}_2 = \{ E \sqrt{G_{a_j}^{\frac{2}{n-2}}(a_i)} \leq \frac{x}{\lambda_i} \leq \epsilon | \lambda_i | \}$

We may assume $G_{a_j}(a_i)^{\frac{2}{n-2}} \sim d_{a_{a_j}}^2(a_j, a_i) \ll 1,$ since otherwise $I_5 = o_{\epsilon_{i,j}}(\epsilon_{i,j}),$ and estimate

$I_5 = \int \frac{u_{a_1}(\exp_{g_{a_1}} x)}{u_{a_1}} \frac{1}{(1+r^2)^{\frac{n-2}{2}}} dV_{h_{a_1}} \left( \frac{1}{\lambda_i + \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i)^{\frac{n-2}{n}}} \right)^{\frac{n-2}{n}}.$

$\leq c \int \frac{\frac{1}{\lambda_i}^{\frac{n-2}{2}}}{\frac{1}{(1+\lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i))^{\frac{n-2}{n}}} \left( \frac{\frac{1}{\lambda_i}}{1+ \lambda_i \lambda_j G_{a_j}^{\frac{2}{n-2}}(a_i)^{\frac{n-2}{n}}} \right)^{\frac{n-2}{n}}}.$

37
Changing coordinates via $d_{i,j} = \exp_{g_{a_i}} \exp_{g_{a_j}}$ we get

$$I_5^1 \leq \frac{C}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{-\gamma} (a_i))^{\frac{n+2\gamma}{2}}} \int_{\|x\|^2 \leq C G_{a_j}^{-\gamma} (a_i)} \left( \frac{1}{1+r^2} \right)^{\frac{n-2\gamma}{2}}$$

and thus $I_5^1 = o_{\epsilon_{i,j}} (\epsilon_{i,j})$. Moreover

$$I_{5,2} = \int_{B_2} \frac{u_{a_j} (\exp_{g_{a_i}} x)}{(1+r^2)^{\frac{n+2\gamma}{2}}} \left( \frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{-\gamma} (a_i)} \right)^{\frac{n-2\gamma}{2}}$$

$$\leq \frac{C}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{-\gamma} (a_i))^{\frac{n-2\gamma}{2}}} \int_{\|x\|^2 \geq C G_{a_j}^{-\gamma} (a_i)} \left( \frac{1}{1+r^2} \right)^{\frac{n-2\gamma}{2}} = o_{\epsilon_{i,j}} (\epsilon_{i,j}).$$

Therefore $I_5 = I_5^1 + I_5^2 = o_{\epsilon_{i,j}} (\epsilon_{i,j})$. Collecting terms we get

$$\epsilon_{i,j} = c_{n,3}^{\gamma} \frac{u_{a_i} (a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{-\gamma} (a_i))^{\frac{n-2\gamma}{2}}} + o_{\epsilon_{i,j}} (\epsilon_{i,j}).$$

By conformal covariance of the Green’s function, cf. [61], we conclude $\epsilon_{i,j} = c_{n,3}^{\gamma} \epsilon_{i,j} (1 + o_{\epsilon_{i,j}} (1))$. ■

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