1. Introduction

Let \( f(x, y) \in \mathbb{C}\{x, y\} \) irreducible. The germ of irreducible analytic curve (branch) of equation \( f(x, y) = 0 \) is denoted by \( C \equiv f(x, y) = 0 \). Observe that the curves \( f(x, y) = 0 \) and \( u(x, y)f(x, y) = 0 \) are the same, for any unit \( u(x, y) \in \mathbb{C}\{x, y\} \). The multiplicity of \( C \), denoted by \( m(C) \), is by definition the order of the power series \( f(x, y) \). Suppose that \( C \) has multiplicity \( n > 1 \). We will say that \( C \) is singular if \( n > 1 \). Otherwise \( C \) is a smooth curve. The initial form of \( f(x, y) \) is the sum of all terms of \( f(x, y) \) of degree \( n \). Since \( f \) is irreducible its initial form is a power of a linear form. After a linear change of coordinates, if necessary, we can suppose that the initial form of \( f(x, y) \) is \( y^n \). Suppose that \( C \) has multiplicity \( n > 1 \). We denote by \( N^* \) the set of positive integers. By Newton’s theorem ([Hef, Theorem 3.8]) there is \( \alpha(x^{1/n}) \in \mathbb{C}\{x\}^* = \bigcup_{m \in \mathbb{N}} \mathbb{C}\{x^{1/m}\} \) with \( \alpha(0) = 0 \) such that \( f(x, \alpha(x^{1/n})) = 0 \) and we say that \( \alpha(x^{1/n}) \in \mathbb{C}\{x\}^* \) is a Newton-Puiseux root of \( C \). Let us denote by \( \text{Zer}(f) \) the set of Newton-Puiseux roots of \( C \). Let \( \alpha(x^{1/n}) \in \text{Zer}(f) \). After Puiseux theorem ([Hef, Corollary 3.12]) we have that \( \text{Zer}(f) = \{ \alpha_j := \alpha(\omega^j x^{1/n}) \}_{j=1}^n \), where \( \omega \) is a \( n \)-th-primitive root of the unity. Hence

\[
(1) \quad f(x, y) = u(x, y) \prod_{j=1}^n \left( y - (\alpha(\omega^j x^{1/n})) \right),
\]

where \( u \in \mathbb{C}\{x, y\} \) is a unit. After a change of coordinates, if necessary, we can write \( \alpha(x) = \sum_{i \geq s_1} a_i x^{i/n} \), where \( s_1 > n \) and \( s_1 \not\equiv 0 \mod n \).
If we put \( x = t^n \), where \( t \) is a new variable, the Newton-Puiseux root \( \alpha(x^{1/n}) \) can be written as
\[
\begin{align*}
x(t) &= t^n \\
y(t) &= \sum_{i \geq s_1} a_i t^i,
\end{align*}
\]
what we will call \textit{Puiseux parametrisation} of \( C \).

There are \( g \in \mathbb{N} \) and a sequence \( (\beta_0 = n < \beta_1 = s_1 < \beta_2 < \cdots < \beta_g) \) of nonnegative integers such that
\[
(2) \quad \{ \text{ord}(\alpha_i - \alpha_j) : \alpha_i, \alpha_j \in \text{Zer}(f), i \neq j \} = \left\{ \frac{\beta_l}{\beta_0} : 1 \leq l \leq g \right\} \subseteq \mathbb{Q} \setminus \mathbb{Z}.
\]

The sequence \( (\beta_0, \ldots, \beta_g) \subseteq \mathbb{N} \) is called sequence of \textit{characteristic exponents} of \( C \). The number \( g \) is a topological invariant called \textit{genus} of the branch \( C \).

Consider the set
\[
S(C) := \{ i_0(f,h) : h \in \mathbb{C}\{x,y\}, h \not\equiv 0 \mod f \},
\]
where \( i_0(f,h) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/(f,h) \) is the \textit{intersection number} (or \textit{intersection multiplicity}) of \( f(x,y) = 0 \) and \( h(x,y) = 0 \) at the origin. It is well-known that \( S(C) \) is a semigroup called \textit{semigroup of values} of the branch \( C \). The complementary of \( S(C) \) in \( \mathbb{N} \) is finite. The \textit{conductor} of \( S(C) \) is by definition the greatest natural number \( c \in \mathbb{N} \) such that for every natural number \( N \in \mathbb{N} \), with \( N \geq c \), is an element of \( S(C) \).

The semigroup \( S(C) \) admits a minimal system of generators \( (s_0, s_1, \ldots, s_g) \), where \( s_i-1 < s_i, g \) is the genus of \( C \), \( s_0 = n = i_0(f,x) \) and \( s_1 = m =: i_0(f,y) \). It is a well-known property of \( S(C) \) ([Hef, page 88, inequality (6.5)]) that \( e_k := \gcd(s_0, \ldots, s_k) = \gcd(\beta_0, \ldots, \beta_k) \) for \( 0 \leq k \leq g \) and \( e_k s_{k+1} < e_k s_k \) for \( 1 \leq k \leq g - 1 \).

If \( n > 2 \) we have \( c \geq s_1 + 1 \). Let \( q \) be the number of natural numbers between \( s_1 \) and \( c \) which are not in \( S(C) \). We can verify (see [Z2, page 21]) that \( q = \frac{c^2 - s_1 + 1}{s_0} \), for \( s_0 = n > 2 \), where \( [z] \) denotes the integral part of \( z \in \mathbb{R} \).

Let \( f, h \in \mathbb{C}\{x,y\} \) be irreducible power series. After Halphen-Zeuthen formula we get
\[
(3) \quad i_0(f,h) = \sum_{i,j} \text{ord}(\gamma_j - \alpha_i),
\]
where \( \text{Zer} f = \{ \alpha_i \} \) and \( \text{Zer} h = \{ \gamma_j \} \).

Two branches \( C \) and \( D \) have the \textit{same topological type} (or they are \textit{equisingular}) if they are topologically equivalent as embedded surfaces in \( \mathbb{C}^2 \). It is well-known ([Z2, Chapter II]) that two branches are equisingular if and only if they have the same semigroup of values or equivalently they have the same characteristic exponents. Denote by \( \mathcal{E}(C) \) the set of branches which are equisingular to \( C \). In the set \( \mathcal{E}(C) \) we define the next equivalence relation: two branches \( D_1 \) and \( D_2 \) in \( \mathcal{E}(C) \) are \textit{analytically equivalent}, and we will denote it by \( D_1 \equiv D_2 \) if there exists an analytic isomorphism \( T : U_1 \rightarrow U_2 \) such that \( U_i \) are neighbourhoods of the origin, \( D_i \) is defined in \( U_i \), \( 1 \leq i \leq 2 \) and \( T(D_1 \cap U_1) = D_2 \cap U_2 \). The \textit{moduli space} of the equisingularity class \( \mathcal{E}(C) \) is the quotient space \( \mathcal{E}(C)/\equiv \). Let \( \nu_1 < \nu_2 < \cdots < \nu_q \) be the integers of the set \( \{ s_1 + 1, \ldots, c - 1 \} \) which are not in \( S(C) \). Zariski proved [Z2, Proposition 1.2, Chapter III] that there exists a branch \( \overline{C} \) analytically equivalent to \( C \), parametrized as follows:
\[
(4) \quad \begin{align*}
\bar{x} &= t^n \\
\bar{y} &= \sum_{i=1}^q a_i t^{\nu_i}.
\end{align*}
\]
Put \( \Omega := \{ \omega = g(x,y)dx + h(x,y)dy : g, h \in \mathbb{C}\{x,y\} \} \). If \( (x(t), y(t)) \) is a Puiseux parametrisation of \( C \) we put
\[
\nu(\omega) := \text{ord} \left( (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t)) \right) + 1.
\]
Let \( \Lambda := \{ \nu(\omega) : \omega \in \Omega \} \). If \( \Lambda \setminus S(C) \neq \emptyset \) then the number \( \lambda := \min (\Lambda \setminus S(C)) - \nu_0 \) is an analytical invariant of \( C \) called \textit{Zariski invariant}. 

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After [Z2, Lemma 2.6, Chapter IV] we can rewrite the parametrization (4) in the next form:

\[
\begin{cases}
\bar{x} = t^n \\
\bar{y} = t^{s_1} + at^\lambda + \text{a finite sum of terms } a_it^{n_i},
\end{cases}
\]

where \(a \neq 0, \nu_i > \lambda > s_1\).

Let \(f(x, y) = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{C}\{x, y\}\). The support of \(f\) is \(\text{supp}(f) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}\). The Newton polygon of \(f\), denoted by \(N(f)\), is by definition the convex hull of \(\text{supp}(f) + \mathbb{R}_{\geq 0}\). Observe that \(N(f) = N(uf)\) for any unit \(u \in \mathbb{C}\{x, y\}\). Nevertheless the Newton polygon depends on coordinates. The inclination of any compact face \(L\) of \(N(f)\) is by definition the quotient of the length of the projection of \(L\) over the horizontal axis by the length of its projection over the vertical axis. The Newton polygon of \(f\) gives information on the Newton-Puiseux roots of \(f(x, y) = 0\). More precisely, if \(L\) is a compact face of \(N(f)\) of inclination \(i\) and the length of its projection over the vertical axis is \(\ell_2\) then \(f\) has \(\ell_2\) Newton-Puiseux roots of order \(i\) (see [Ch, Lemme 8.4.2]).

We say that \(f(x, y) \in \mathbb{C}\{x, y\}\) is non degenerate in the sense of Kouchnirenko, with respect to the coordinates \((x, y)\), if for any compact edge \(L\) of \(N(f)\) the polynomial \(f_L(x, y) := \sum_{(i,j) \in L \cap \text{supp}(f)} a_{ij}x^iy^j\) does not have critical points outside the axes \(x = 0\) and \(y = 0\), or equivalently, the polynomial \(F_L(z) := \frac{f_L(1+z)}{z^{j_0}}\) has no multiple roots, where \(j_0 := \min\{j \in \mathbb{N} : (i, j) \in L\}\). Since \(N(f) = N(uf)\), for any unit \(u \in \mathbb{C}\{x, y\}\), the notion of non degeneracy is extended to curves. The topological type of non degenerate plane curves are completely determined by their Newton polygons (see [0, Proposition 4.7] and [GB-L-P, Theorem 3.2]).

Let \(\ell(x, y) = 0\) be a smooth curve and \(f(x, y) = 0\) defining an isolated singularity at \(0 \in \mathbb{C}^2\). Assume that \(\ell(x, y)\) does not divide \(f(x, y)\) and consider the morphism

\[(\ell, f) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \quad (x, y) \rightarrow (u, v) := (\ell(x, y), f(x, y)).\]

There are two curves associated with \((\ell, f)\): the polar curve \(\frac{\partial \ell}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \ell}{\partial y} \frac{\partial f}{\partial x} = 0\) and its direct image \(D(u, v) = 0\) which is called the discriminant curve of the morphism \((\ell, f)\).

The topological type of the polar curve depends on the analytical type of \(\ell(x, y) = 0\) and \(f(x, y) = 0\). In [Hef-Her-Hil] the authors completely determine the topological type of the generic polar curve when the multiplicity of \(f(x, y) = 0\) is less than five.

The Newton polygon of \(D(u, v)\) in the coordinates \((u, v)\) is called jacobian Newton polygon of the morphism \((\ell, f)\). This notion was introduced by Teissier in [T], who proved that the inclinations of this jacobian polygon are topological invariants of \((\ell, f)\) called polar invariants. After Merle [M], when \(f\) is irreducible with semigroup of values \(S(f) = \{s_0, s_1, \ldots, s_g\}\) then the jacobian Newton polygon of \((\ell, f)\) has \(g\) compact edges \(\{E_i\}_{i=1}^g\). The length of the projection of \(E_i\) on the vertical axis is \(\left(\frac{e_i-1}{e_i} - 1\right) \cdot \frac{e_i-1}{e_0}\). The length of the projection of \(E_i\) on the horizontal axis is \(\left(\frac{e_i-1}{e_i} - 1\right) \cdot s_i\). Hence the inclinations (quotient between the length of the horizontal projection and the length of the vertical projection) of the compact edges of the jacobian polygon are \(s_1 < \frac{s_1}{e_0}s_2 < \frac{s_1}{e_0}s_3 < \cdots < \frac{s_1}{e_0}s_g\).
In [GB-Gw-L] the authors study the pairs \((\ell, f)\) for which the discriminant curve is non degenerate in the Kouchnirenko sense. In particular, when \(f\) is irreducible, after [GB-Gw-L, Corollary 4.4] the discriminant curve \(D(u,v) = 0\) is non degenerate if and only if the multiplicity of \(f(x,y) = 0\) equals two or equals four and genus equals two. Otherwise the discriminant curve is degenerate. Our aim in this paper will be to describe the topological type of the discriminant curve \(D(u,v) = 0\) of the morphism \((\ell, f)\), where \(f\) is irreducible and belonging to some special families, as for example, branches of multiplicity less than five, branches \(C\) such that the difference between its Milnor number \(\mu(C)\) and Tjurina number \(\tau(C)\) is less than 3, with \(\mu(C) = i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \dim \mathbb{C}\{x,y\}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\) and \(\tau(C) = \dim \mathbb{C}\{x,y\}/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\); where \(\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\) (resp. \(\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\right)\)) denotes the ideal of \(\mathbb{C}\{x,y\}\) generated by \(\frac{\partial f}{\partial x}\) and \(\frac{\partial f}{\partial y}\) (resp. by \(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\) and \(f\)).

For these families of plane branches we determine the topological type of their discriminant curves, in the spirit of [Hef-Her-HI1]. We prove that the topological type of the discriminant curve \(D(u,v) = 0\) is determined, at most, by the semigroup of values \(S(f)\), the Zariski invariant and two other analytical invariants of the curve \(f(x,y) = 0\). In all cases we explicitly determine such analytical invariants. Hence, in order to describe the topological type of the discriminant curve of a branch, it is necessary the same number of analytical invariants of the initial branch, as it happens for its generic polar curves (see [Hef-Her-HI1]). Finally in Section 5 we summarize the different topological types of the discriminant curve in some tables.

2. Equation of the discriminant curve

An analytic change of coordinates does not affect the discriminant curve of the morphism defined in (5) (see for example [Ca, Section 3]). Hence in what follows we assume that \(\ell(x,y) = x\). Then \(\frac{\partial f}{\partial y} = 0\) is the polar curve of the morphism \((x,f)\).

In this paper we will determine the topological type of the discriminant curve of the morphism (5) for \(\ell(x,y) = x\) and \(f(x,y) \in \mathbb{C}\{x,y\}\) irreducible belonging to some special families. The corresponding study relative to the polar curves was done in [Hef-Her-HI2] and [HI], where the authors characterize the equisingularity classes of irreducible plane curve germs whose general members have non degenerate general polar curves. In addition, they give explicit Zariski open sets of curves in these equisingularity classes whose general polars are non degenerate and describe their topology.

Suppose that the Newton-Puiseux factorizations of \(f(x,y)\) and \(\frac{\partial f}{\partial y}(x,y)\) are of the form

\[
(6) \quad f(x,y) = u_1(x,y) \prod_{i=1}^{n} [y - \alpha_i(x)],
\]

\[
(7) \quad \frac{\partial f}{\partial y}(x,y) = u_2(x,y) \prod_{j=1}^{n-1} [y - \gamma_j(x)],
\]

where \(u_1(x,y), u_2(x,y) \in \mathbb{C}\{x,y\}\) are units, \(n = \text{ord} f\), \(\text{Zer}(f) = \{\alpha_i(x)\}_i\) and \(\text{Zer}\left(\frac{\partial f}{\partial y}\right) = \{\gamma_j(x)\}_j\). If \(f\) is irreducible of order \(n\) then \(n\) is the smallest natural number such that \(\{\alpha_i(x)\}_i \subset \mathbb{C}\{x^{1/n}\}\). Moreover if we fix \(\alpha_i(x^{1/n})\) then \(\alpha_j(x^{1/n}) = \alpha_i(\omega x^{1/n})\) for any \(1 \leq j \leq n\), where \(\omega\) is a \(n\)-th root of the unity.
Following [GB-Gwo, Lemma 5.4] the discriminant curve of the morphism \((x, f)\) can be written as

\begin{equation}
D(u, v) = \prod_{j=1}^{n-1} (v - f(u, \gamma_j(u))).
\end{equation}

3. Discriminants of branches of small multiplicities

In this section we determine the topological type of the discriminant of the morphism given in (5), where \(C \equiv f(x, y) = 0\) has small multiplicity. For this we will make use the results of [Z2] and the analytic classification of plane branches of multiplicity less than or equal to four, given in [Hef-Her].

3.1. Discriminants of branches of multiplicity 2. Let \(C \equiv f(x, y) = 0\) be a branch of multiplicity 2. The minimal system of generators of the semigroup of \(C\) is \((2, s_1)\), where \(s_1\) is an odd natural number. By [Z2, Chapitre V] the moduli space of branches of multiplicity two have a unique point which parametrization is given by \((t^2, t^{s_1})\), that is the branch \(y^2 - x^{s_1} = 0\). Then \(f_\varphi(x, y) = 2y\) whose Newton-Puiseux root is \(y = 0\). Hence, after (8) we have \(D(u, v) = v - f(u, 0) = v + u^{s_1}\), that is the discriminant curve is smooth. The Newton polygon of \(D(u, v)\) has only one compact edge. The univariable polynomial associated with this edge is \(z + 1\), so \(D(u, v)\) is non degenerate.

3.2. Discriminants of branches of multiplicity 3. Let \(C \equiv f(x, y) = 0\) be a branch of multiplicity 3. The minimal system of generators of the semigroup of \(C\) is \((3, s_1)\), where \(s_1 \in \mathbb{N}\) such that \(s_1 \not\equiv 0 \text{ mod } 3\). By [Z2, Chapitre V] the moduli space of branches of multiplicity three is completely determined by the semigroup of the branch and its Zariski invariant \(\lambda\). The corresponding normal forms are:

\[ x = t^3, \quad y = t^{s_1} + t^\lambda, \]

where \(\lambda = 0\) or if \(\lambda \neq 0\) we have

\begin{equation}
\lambda = \begin{cases} 
3e + 3k + 4 & \text{when } s_1 = 3e + 2, \\
3e + 3k + 2 & \text{when } s_1 = 3e + 1,
\end{cases}
\end{equation}

where \(0 \leq k \leq e - 2\).

Observe that if \(\lambda \neq 0\) then \(\frac{s_1 + \lambda}{2}\) is a natural number greater than 2.

Proposition 3.1. Let \(C \equiv f(x, y) = 0\) be a branch of semigroup \((3, s_1)\) and Zariski invariant equals \(\lambda\). The discriminant curve \(D(u, v) = 0\) is degenerate and its topological type is determined by \((3, s_1, \lambda)\) in the next way:

1. If \(\lambda = 0\) then the discriminant is the double smooth branch \((v + u^{s_1})^2 = 0\).
2. If \(\lambda \neq 0\) then
   a. when \(\gcd(2, s_1 + \lambda) = 2\), the discriminant curve is the union of two smooth branches \(D_1(u, v) = 0\), \(1 \leq i \leq 2\), with intersection number \(\eta(D_1, D_2) = \frac{s_1 + \lambda}{2}\).
   b. When \(\gcd(2, s_1 + \lambda) = 1\), the discriminant is a branch of semigroup \((2, s_1 + \lambda)\).

Proof. Suppose \(\lambda = 0\). The implicit equation of the normal form is \(f(x, y) = y^3 - x^{s_1} = 0\). Then \(f_\varphi(x, y) = 3y^2\) whose Newton-Puiseux root is \(y = 0\), with multiplicity two. Hence, after (8) we have \(D(u, v) = (v - f(u, 0))^2 = (v + u^{s_1})^2\), that is the discriminant is a double smooth branch.

Suppose now \(\lambda \neq 0\). The normal forms are \(x = t^3, y = t^{s_1} + t^\lambda\), with \(\lambda\) as in (9). After (1) the implicit equation is \(f(x, y) = y^3 - 3x^{\frac{3s_1 + 3\lambda}{6}} y - (x^{s_1} + x^\lambda)\), then \(f_\varphi(x, y) = 3y^2 - 3x^{\frac{3s_1 + 3\lambda}{6}}\), whose Newton-Puiseux roots are \(\gamma_i(x) = \pm x^{\frac{s_1 + \lambda}{6}}\) for \(1 \leq i \leq 2\). Using (8) we have \(D(u, v) = \left(v + u^{s_1} - 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda\right) \left(v + u^{s_1} + 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda\right)\).

Hence \(\text{Zer}(D) = \{\eta_1 := -u^{s_1} - 2u^{\frac{s_1 + \lambda}{2}} - u^\lambda, \eta_2 := -u^{s_1} + 2u^{\frac{s_1 + \lambda}{2}} - u^\lambda\}\) and \(\text{ord}(\eta_1 - \eta_2) = \frac{s_1 + \lambda}{2}\). We distinguish two cases: if \(\gcd(2, s_1 + \lambda) = 2\) then by (2) we conclude that \(D(u, v) = 0\) has two smooth branches of equations \(D_1(u, v) := v + u^{s_1} - 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda\) and \(D_2(u, v) := v + u^{s_1} + 2u^{\frac{s_1 + \lambda}{2}} + u^\lambda\) such that, after
Halphen-Zeuthen formula, the intersection multiplicity is \(i_0(D_1, D_2) = \frac{a_2}{s_1 + \lambda}\). If \(\gcd(2, s_1 + \lambda) = 1\) then the discriminant curve \(D(u, v) = 0\) is a singular branch of semigroup \((2, s_1 + \lambda)\).

The Newton polygon of \(D(u, v)\) is elementary (it has only one compact edge) and the univariate polynomial associated with its compact edge is \((z + 1)^2\). Hence the discriminant \(D(u, v) = 0\) is degenerate. \(\square\)

**Corollary 3.2.** If \(C\) is a branch of multiplicity 2 or 3 and non-zero Zariski invariant then the discriminant curve \(D(u, v) = 0\) has not multiple irreducible branches.

Corollary 3.2 does not hold for branches of multiplicity 4 as the proof of Proposition 3.5 shows.

### 3.3. Discriminants of branches of multiplicity 4

Let \(C \equiv f(x, y) = 0\) be a branch of multiplicity 4. The branch \(C\) may have genus one or two.

**Proposition 3.3.** Let \(C \equiv f(x, y) = 0\) be a branch of semigroup \((4, s_1, s_2)\). Then the discriminant curve \(D(u, v) = 0\) is non-degenerate. Moreover \(D(u, v) = D_1(u, v)D_2(u, v)\), where \(D_1\) is a smooth branch, \(D_2\) is a singular branch of semigroup \((2, s_2)\) and the intersection multiplicity between both branches is \(i_0(D_1, D_2) = 2s_1\).

**Proof.** For genus two, and after the second part of [GB-Gw-L, Corollary 4.4] we get that \(D(u, v) = 0\) is non-degenerate and we can determine its topological type from its Newton polygon (see [O, Proposition 4.7] and [GB-L-P, Theorem 3.2]), which is the jacobian Newton polygon of \((x, f)\) (see Figure 1).

![Figure 1. Jacobian Newton polygon of a branch with semigroup \((4, s_1, s_2)\).](image)

Since \(s_2\) is an odd natural number then \(\gcd(s_1 + s_2, 2) = 1\). Moreover, after the properties of \(S(f)\), we get \(e_0s_1 = 4s_1 < 2s_2 = e_1s_2\), hence \(s_2 > 2s_1\) and \(D(u, v) = D_1(u, v)D_2(u, v)\), where \(D_1(u, v) = 0\) is a smooth curve admitting as parametrization \((t, t^{s_1} + \cdots)\) and \(D_2(u, v) = 0\) is a singular curve of genus 1 and semigroup of values equals \((2, s_2)\). Finally, after Halphen-Zeuthen formula, the intersection multiplicity between both branches is \(i_0(D_1, D_2) = 2s_1\). \(\square\)

Suppose now that the branch \(C\) has genus 1 and semigroup of values equals \((4, s_1)\). By [Hef-Her] the moduli space of branches of multiplicity four and genus 1 has five families of normal forms:

\[\text{NF}\ 4.1: \ x = t^4, \ y_i = \sigma_i(t), \quad 1 \leq i \leq 5,\]

where \(\lambda_i\) is the Zariski invariant of the \(i\)th-normal form family. More precisely we have

1. \((\lambda_1, \sigma_1(t)) = (0, t^{s_1})\),
2. If \(2 \leq i \leq 4\) then \(\lambda_i = 2s_1 - 4j\) for \(2 \leq j \leq \left[\frac{4}{s_1}\right]\) and

\[\sigma_2(t) = t^{s_1} + t^{\lambda_2} + a_k t^{3s_1 - 4\left[\frac{4}{s_1}\right] + j + 1 - k} + \cdots + a_j - \left[\frac{4}{s_1}\right] - 2 \cdot 3^{s_1 - 4\left[\frac{4}{s_1}\right] + 3 - k},\]

with \(a_k \neq 0\), and \(1 \leq k \leq \left[\frac{4}{s_1}\right] - j\).
\[ \sigma_3(t) = t^{s_1} + t^{\lambda_3} + \frac{3s_1 - 4j}{2s_1}t^{3s_1-8j} + a_{[\frac{s_1}{4}]-j+2}t^{3s_1-4(2j-1)} + \ldots + a_{[\frac{s_1}{4}]-j+1}t^{3s_1-(j+1)}. \]

\[ \sigma_4(t) = t^{s_1} + t^{\lambda_4} + a_{[\frac{s_1}{4}]-j+1}t^{3s_1-8j} + a_{[\frac{s_1}{4}]-j+2}t^{3s_1-4(2j-1)} + \ldots + a_{[\frac{s_1}{4}]-1}t^{3s_1-4(j+2)}, \]

where \( a_{[\frac{s_1}{4}]-j+1} \neq \frac{3s_1-4j}{2s_1} \).

(3) If \( i = 5 \) then \( \lambda_5 = 3s_1 - 4j \) for \( 2 \leq j \leq \left[ \frac{s_1}{2} \right] \) and

\[ \sigma_5(t) = t^{s_1} + t^{3s_1-4j} + a_{k}t^{2s_1-4(j-[\frac{s_1}{4}]-k)} + a_{k+s}t^{2s_1-4(j-[\frac{s_1}{4}]-k-s)} + \ldots \]

After the Newton-Puiseux Theorem, the \( i \)th-normal form admits the equation

\[ f_i(x,y) = \prod_{\omega^i=1} (y - \sigma_i(\omega x^{1/4})), \]

where \( \omega \) is a 4th primitive root of the unity.

Hence, we obtain the implicit equation for each normal form family:

**NF 4.1:** \( f_1(x,y) = y^4 - x^{s_1} \).

**NF 4.2:** For \( 2 \leq i \leq 4 \) we get

\[ f_i(x,y) = y^4 + P_i(x)y^2 + Q_i(x)y + x^{s_1}u(x), \]

where \( u(x) \in \mathbb{C}\{x\} \) is a unit (that is \( u(0) \neq 0 \)), \( \lambda_i = 2s_1 - 4j \) for \( 2 \leq j \leq \left[ \frac{s_1}{4} \right], Q_i(x) = -4x^{s_1-j} + \ldots \) and

\[
P_i(x) = \begin{cases} 
-4a_{k}s_1^i \left( [\frac{s_1}{4}]+j+1-k \right) + \ldots & \text{for } i = 2 \\
bxs_1^i - 2j + \ldots & \text{for } i = 3 \text{ where } b = \frac{-8(s_1-j)}{s_1} \\
-2 - 4a_{[\frac{s_1}{4}]-j+1}s_1^i - 2j + \ldots & \text{for } i = 4 \text{ and } -2 - 4a_{[\frac{s_1}{4}]-j+1} \neq 0 \\
0 & \text{for } i = 4, -2 - 4a_{[\frac{s_1}{4}]-j+1} = 0 \text{ and } a_{[\frac{s_1}{4}]-j+2} = \ldots = a_{[\frac{s_1}{4}]-1} = 0 \\
-4a_{[\frac{s_1}{4}]-j+k_0}s_1^i - (2j-(k_0-1)) + \ldots & \text{for } i = 4, -2 - 4a_{[\frac{s_1}{4}]-j+1} = 0 \text{ and } a_l \neq 0 \text{ for some } l = [\frac{s_1}{4}]-j+2, \ldots, [\frac{s_1}{4}]-1 \\
\end{cases} \]

where \( 2 \leq k_0 \leq j - 1 \) such that

\[ k := \min \left\{ l : a_l \neq 0, \left[ \frac{s_1}{4} \right] - j + 2 \leq l < \left[ \frac{s_1}{4} \right] \right\} = \left[ \frac{s_1}{4} \right] - j + k_0. \]

Observe that

\[ \text{ord} P_i(x) \geq \frac{2}{3}(s_1 - j). \]

**NF 4.5:** \( f_5(x,y) = y^4 + P(x)y^2 + Q(x)y + R(x), \) where

\[ P(x) = -4x^{s_1-j} - 2a_{k}^2x^{s_1-2(j-[\frac{s_1}{4}]-k)} - 4a_{k}a_{k+s}x^{s_1-2(j-s_1-[\frac{s_1}{4}]-k)+s} - 2a_{k+s}^2x^{s_1-2(j-[\frac{s_1}{4}]-k-s)} + \ldots \]

\[ (14) \]
\[ Q(x) = -4akx^{s_1-j+[\frac{2}{3}]+k} - 4ak_{k+s}x^{s_1-j-[\frac{2}{3}]+k+s} \]

(15)

\[ R(x) = -x^{s_1} + 2x^{2(s_1-j)} - x^{3s_1-4j} + akx^{2s_1-4(j-[\frac{2}{3}]-k)} + \ldots \]

(16)

for \(2 \leq j \leq [\frac{4}{3}]\).

**Proposition 3.4.** Let \(C \equiv f(x, y) = 0\) be a branch belonging to the family NF 4,i, for \(1 \leq i \leq 4\). Then the discriminant curve \(D(u, v) = 0\) is degenerate and its topological type is determined by the semigroup \(S(f) = \langle 4, s_1 \rangle\) and the Zariski invariant \(\lambda_i\) of \(C\). Moreover

1. If \(\lambda_i = 0\) then the discriminant curve is the triple smooth branch \((v + u^{s_1})^3 = 0\).
2. If \(\lambda_i \neq 0\) then \(\lambda_i = 2s_1 - 4j\) for \(2 \leq j \leq [\frac{4}{3}]\) and
   
   a) when \(\gcd(3, 2s_1 + \lambda_i) = 1\) the discriminant curve \(D(u, v) = 0\) is a branch of semigroup \(\langle 3, 2s_1 + \lambda_i \rangle\).
   
   b) when \(\gcd(3, 2s_1 + \lambda_i) = 3\) the discriminant curve is the union of three smooth branches \(D_i(u, v) = 0\), \(1 \leq i \leq 3\), with intersection number \(i_0(D_l, D_r) = \frac{2s_1 + \lambda_i}{3}\) for \(l \neq r\).

**Proof.** Suppose \(\lambda_i = 0\). By (11) the implicit equation of the normal form is \(f_1(x, y) = y^4 - x^{s_1}\). Then \((f_1)_y(x, y) = 4y^3\) whose Newton-Puiseux root is \(y = 0\), with multiplicity three. Hence, after (8) we have \(D(u, v) = (v - f_1(u, 0))^3 = (v + u^{s_1})^3\), that is the discriminant curve is a triple smooth branch.

Suppose now \(\lambda_i \neq 0\). Then \(\lambda_i = 2s_1 - 4j\) for \(2 \leq j \leq [\frac{4}{3}]\) and the normal form of \(C\) is \(x = t^4\), \(y = \sigma_i(t)\) for \(2 \leq i \leq 4\).

From the implicit equations \(f_i(x, y), 2 \leq i \leq 4\), given in (12) and from the inequality (13) we get that the Newton polygon of \((f_i)_y(x, y)\) has only one compact edge whose vertices are \((0, 3)\) and \((s_1 - j, 0)\). All the parametrisations of \((f_i)_y(x, y) = 0\), \(2 \leq i \leq 4\), have the same order and we can write them by \(\gamma_r(u) = \varepsilon u^{4j - \frac{1}{3}} + \ldots\), where \(\varepsilon\) is a 3th-root of the unity. From (8) and for a fix \(i \in \{2, 3, 4\}\), we have

\[ D(u, v) = \prod_{(f_i)_y = 0} (v - f_i(u, \gamma_r(u))), \]

and considering the development of \(f_i(u, \gamma_r(u))\) we obtain:

\[ D(u, v) = \prod_{\varepsilon^i} (v + u^{s_1} + 3\varepsilon u^{4j - \frac{1}{3}} + \ldots) = \prod_{\varepsilon^i} (v + u^{s_1} + 3\varepsilon u^{2s_1 + \lambda_i} + \ldots), \]

so \(\text{Zer}(D) = \{\eta_l(\varepsilon) = -u^{s_1} - 3\varepsilon u^{2s_1 + \lambda_i} + \ldots\}_{j=1}^{i=1}\) and \(\text{ord}(\eta_l - \eta_r) = \frac{2s_1 + \lambda_i}{3}\) for \(1 \leq l \neq r \leq 3\). The topological type of \(D(u, v) = 0\) is determined by the semigroup of the branch \(C \equiv f_i(x, y) = 0\) and its Zariski analytical invariant \(\lambda_i = 2s_1 - 4j\) for \(2 \leq j \leq [\frac{4}{3}]\). We distinguish two cases: if \(3\) and \(2s_1 + \lambda_i\) are coprime then the discriminant \(D(u, v) = 0\) is a branch of semigroup \(\langle 3, 2s_1 + \lambda_i \rangle\). Otherwise, by (2), we conclude that the discriminant curve is the union of three different smooth branches \(D_i(u, v) = 0\) with intersection multiplicity \(i_0(D_l, D_r) = \frac{2s_1 + \lambda_i}{3}\) for \(l \neq r\).

In all cases the Newton polygon of \(D(u, v)\) is elementary with vertices \((0, 3)\) and \((3s_1, 0)\). The polynomial associated with its compact edge is \((z + 1)^3\), so the discriminant curve \(D(u, v) = 0\) is degenerate.

**Proposition 3.5.** Let \(C \equiv f(x, y) = 0\) be a branch belonging to the family NF 4.5. Then the discriminant curve \(D(u, v) = 0\) is degenerate and its topological type is determined by the semigroup \(S(f) = \langle 4, s_1 \rangle\), the Zariski invariant \(\lambda_5\) and at most two other analytical invariants of \(C\).
Proof. The implicit equation of $C$ has the form $f_5(x, y) = y^4 + P(x)y^2 + Q(x)y + R(x)$, where $P(x), Q(x), R(x)$ are as in (14), (15) and (16). Hence $(f_5)_y(x, y) = 4y^3 + 2P(x)y + Q(x)$.

We distinguish different cases:

**Case A.** If $a_i = 0$ in $\sigma_5(t)$ for all $i$ then we have that $(f_5)_y(x, y) = 4y^3 - 8x^j - jy = 4y(y^2 - 2x^j)$ whose Newton-Puiseux roots are $\left\{0, \pm \sqrt{2}x^j \right\}$. Therefore $f_5(u, 0) = R(u) = -u^3 + 2u^{2(s_1-j)} - u^{3s_1-4j}$ and $f_5(u, \pm \sqrt{2}u^{\frac{3s_1-4j}{2}}) = -u^{s_1} - 2u^{2(s_1-j)} - u^{3s_1-4j}$. From (8) we conclude that the discriminant curve is the union of three smooth curves $D_1(u, v) = 0$, where $D_1(u, v) = v + u^{s_1} - 2u^{2(s_1-j)} + u^{3s_1-4j}$, $D_2(u, v) = D_3(u, v) = v + u^{s_1} + 2u^{2(s_1-j)} + u^{3s_1-4j}$, and $i_0(D_1, D_2) = 2(s_1 - j)$ for $l \in \{2, 3\}$.

**Case B.** If $a_k \neq 0$ and $a_{k+l} = 0$ in $\sigma_5(t)$ for $l > 0$. Then

$$
(f_5)_y(x, y) = 4y^3 + 2(-4x^j - 2a_k^2x^{s_1-2j-\left[\frac{j}{3}\right]k} + \ldots)y - 4a_k \left( x^{s_1-j+\left[\frac{j}{3}\right]k} + x^{2(s_1-j)-\left[\frac{j}{3}\right]k} + \ldots \right).
$$

So the Newton polygon of $(f_5)_y(x, y)$ depends on the position of the point $M = (s_1 - j, 1)$ with respect to the line passing by $E = (0, 3)$ and $F = (s_1 - j + [\frac{3k}{2}], k), 0)$. We get three situations:

**B.1.** If $\frac{2}{s_1-j} < \frac{1}{[\frac{k}{2}]+k}$ then $\mathcal{N}((f_5)_y)$ has only one compact edge of vertices $E$ and $F$. Hence the order of the Newton-Puiseux roots $\left\{\gamma_i\right\}_{i=1}^3$ of $(f_5)_y(x, y) = 0$ equals $\frac{s_1-j+\left[\frac{j}{3}\right]k}{3}$. After (8) we obtain

$$
D(u, v) = \prod_{u_i^2 = 1} \left( v - 3a^4u_iu \right)^4 \left( \frac{s_1-j+\left[\frac{j}{3}\right]k}{3} + u^{s_1} + \ldots \right),
$$

for some nonzero complex number $a$ and where $u_i$ is a cubic root of the unity. If $\gcd(3(s_1-j + [\frac{3k}{2}]+k)) = 1$ then the discriminant curve is irreducible with semigroup $(3, 4(s_1-j + [\frac{3k}{2}]+k))$. On the other case, we get $\gcd(3(s_1-j + [\frac{3k}{2}]+k)) = 3$ and the discriminant curve is the union of three smooth curves $D_1(x, y) = 0$ such that $i_0(D_1, D_2) = \frac{4(s_1-j+\left[\frac{j}{3}\right]+k)}{3}$.

**B.2.** If $\frac{2}{s_1-j} > \frac{1}{[\frac{k}{2}]+k}$ then $\mathcal{N}((f_5)_y)$ has two compact edges of vertices $M, E$ and $F$. By (8) we get

$$
D(u, v) = (v + u^{s_1} - 2u^{2(s_1-j)} + \ldots)(v + u^{s_1} + 2u^{2(s_1-j)} + \ldots) + 4\sqrt{2}a_ku^{\frac{3}{2}(s_1-j)+\left[\frac{j}{3}\right]+k} + \ldots.
$$

If $s_1 - j$ is odd then the discriminant curve is the union of a smooth branch $D_1(u, v) = 0$ and a singular branch $D_2(x, y) = 0$ with semigroup $(2, 3(s_1-j) + 2([\frac{j}{3}]+k))$. Moreover $i_0(D_1, D_2) = 4(s_1-j)$. Otherwise, if $s_1 - j$ is even then the discriminant curve is the union of three smooth branches $D_1(x, y) = 0$ such that $i_0(D_1, D_l) = 2(s_1-j)$ for $2 \leq l \leq 3$ and $i_0(D_2, D_3) = \frac{3}{2}(s_1-j) + [\frac{3k}{2}] + k$.

**B.3.** If $\frac{2}{s_1-j} = \frac{1}{[\frac{k}{2}]+k}$ then $\mathcal{N}((f_5)_y)$ has only one compact edge of vertices $E$ and $F$ and $M$ is an interior point of this edge. The polynomial in one variable, associated with the compact edge of $\mathcal{N}((f_5)_y)$, is $p(z) = z^2 - 2z - a_k$. After the $z$-discriminant of $p(z)$ we have that the roots of $p(z)$ are simple if and only if $a_k \neq \pm \left(\frac{4\sqrt{2}}{9}\right)$. We will study both cases:

**B.3.1** Suppose that $a_k \neq \pm \left(\frac{4\sqrt{2}}{9}\right)$. Denote by $z_i$ the three different roots of the polynomial $p(z)$. For any $\gamma_i(x) \in \text{Zer}(f_5)\gamma_i$ we have:

$$
f(u, \gamma_i(u)) = -u^{s_1} + q(z_i)u^{2(s_1-j)} + \ldots, \text{ where } q(z) = z^4 - 4z^2 - 4a_kz + 2.
$$
Since \( z^3 - 2z = a_k \) then \( q(z) = -3z^4 + 4z^2 + 2 \) and if \( z_r \neq z_i \) then \( q(z_r) \neq q(z_i) \). Hence \( D(u,v) = \prod_i (z + u_i^*) + q(z_i)u^{2(s_1-j) + \cdots} \) and the discriminant curve \( D(u,v) = 0 \) is the union of three smooth branches \( D_l(x,y) = 0 \) such that \( i_0(D_l, D_r) = 2(s_1-j) \).

\section*{B.3.2} Suppose that \( a_k = \frac{4\sqrt{3}}{n} \). The polynomial \( p(z) \) has \( z_1 = 2\sqrt[3]{\frac{2}{3}} \) as a simple root and \( z_2 = -\sqrt[3]{\frac{2}{3}} \) as a double root. If \( \gamma_i \in \text{Zer}(f_3)_y \) corresponds to \( z_i \)

\begin{align*}
    f_3(u, \gamma_1(u)) &= u^{s_1} + q(z_1)u^{2(s_1-j) + \cdots}, \\
    f_3(u, \gamma_2(u)) &= u^{s_1} + q(z_2)u^{2(s_1-j) + r_2u^{2(s_1-j)} + \cdots}, \\
    f_3(u, \gamma_3(u)) &= u^{s_1} + q(z_2)u^{2(s_1-j) + r_3u^{2(s_1-j)} + \cdots},
\end{align*}

where \( r_2, r_3 \in \mathbb{C} \) are different. Observe that \( q(z_1) \neq q(z_2) \). Hence, if \( s_1 - 2j \) is odd then the discriminant curve is the union of a smooth branch \( D_1(u,v) = 0 \) and a singular branch \( D_2(u,v) = 0 \) with semigroup \( (2, 5s_1 - 6j) \), where \( i_0(D_1, D_2) = 4(s_1 - j) \). Otherwise, if \( s_1 - 2j \) is even then the discriminant curve is the union of three smooth branches \( D_l(u,v) = 0 \) such that \( i_0(D_l, D_l) = 2(s_1 - j) \), for \( 2 \leq l \leq 3 \) and \( i_0(D_2, D_3) = \frac{5s_1 - 7j}{2} \).

\section*{B.3.3} Suppose that \( a_k = -\frac{4\sqrt{3}}{n} \). The polynomial \( p(z) \) has \( 2\sqrt[3]{\frac{2}{3}} \) as a double root and \( -\sqrt[3]{\frac{2}{3}} \) as a simple root. After a similar procedure we conclude, in this case, that the topological type of the discriminant curve, is as in B.3.2.

\subsection*{Case C.} If \( a_k \neq 0 \neq a_{k+\frac{1}{3}} \) then we have \( (f_3)_y(x,y) = 4y^3 + 2P(x)y + Q(x) \). The Newton polygon of \((f_3)_y(x,y)\) depends on the position of the point \( M = (s_1 - j, 1) \) with respect to the line passing by \( E = (0,3) \) and \( F = (s_1 - j + \frac{1}{2} + k, 0) \). We have the following situations:

\begin{itemize}
    \item \textbf{C.1.} If \( \frac{2}{s_1-j} < \frac{1}{|\frac{1}{2}| + k} \) then the topological type of the discriminant curve \( D(u,v) = 0 \) is as in the case B.1.
    \item \textbf{C.2.} If \( \frac{2}{s_1-j} > \frac{1}{|\frac{1}{2}| + k} \) then the topological type of the discriminant curve is as in the case B.2.
    \item \textbf{C.3.} If \( \frac{2}{s_1-j} = \frac{1}{|\frac{1}{2}| + k} \) then the Newton polygon of \((f_3)_y(x,y)\) has only one compact edge containing the points \( E, F, M \), as in the case B.3. The polynomial associated with this compact edge is \( p(z) = z^3 - 2z - a_k \) whose roots are simple if and only if \( a_k \neq \pm \left( \frac{4\sqrt{3}}{n} \right) \). Let us study the different cases:
        \begin{itemize}
            \item \textbf{C.3.1.} If \( a_k \neq \pm \left( \frac{4\sqrt{3}}{n} \right) \) then the topological type of the discriminant curve \( D(u,v) = 0 \) is as in B.3.1.
            \item \textbf{C.3.2.} Suppose that \( a_k = \pm \left( \frac{4\sqrt{3}}{n} \right) \). The polynomial \( p(z) \) has \( z_1 = 2\sqrt[3]{\frac{2}{3}} \) as a simple root and \( z_2 = z_3 = -\sqrt[3]{\frac{2}{3}} \) as a double root. If \( \gamma_i \in \text{Zer}(f_3)_y \) corresponds to \( z_i \) then we can write

\begin{equation}
    \gamma_i = z_iu^{\frac{1}{2}+|k|} + \cdots
\end{equation}

Hence

\begin{align*}
    \delta_1 &:= f(u, \gamma_1(u)) = -u^{s_1} + q(z_1)u^{2(s_1-j)} + \cdots, \\
    \delta_2 &:= f(u, \gamma_2(u)) = -u^{s_1} + q(z_2)u^{2(s_1-j)} + \cdots, \\
    \delta_3 &:= f(u, \gamma_3(u)) = -u^{s_1} + q(z_2)u^{2(s_1-j)} + \cdots,
\end{align*}

where \( \delta_2, \delta_3 \in \text{Zer}D(u,v) \) are different. Observe that \( q(z_1) \neq q(z_2) \). So \( \text{ord}(\delta_1 - \delta_l) = 2(s_1-j) \) for \( 2 \leq l \leq 3 \). Let us determine \( \text{ord}(\delta_2 - \delta_3) \). For that we need to precise new terms in \( \delta_2 \) and \( \delta_3 \). We apply the Newton procedure for \((f_3)_y\): let \( y_1 \) a new variable. Substituting \((x,y) := (x, x^{\frac{2}{3}} + k(z_2 + y_1)) \) in \((f_3)_y(x,y) \) we get

\begin{align*}
    (f_3)_y(x, x^{\frac{2}{3}} + k(z_2 + y_1)) &= x^{3\left(\frac{2}{3}\right) + k}g(x, y_1),
\end{align*}
with
\[ g(x, y_1) = 4 \left[ \left( \frac{2}{3} \right) z_2 + 2y_1 + 3z_2y_1^2 + y_1^3 \right] - (4a_k + 4a_{k+s}x^s + 4a_{k,s}x^{s_1-2j} + \ldots) \]
- \[ [8 + 4a_k^2x^{s_1-2j} + 8a_kak_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \ldots]y_1 \]
- \[ [8 + 4a_k^2x^{s_1-2j} + 8a_kak_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \ldots]z_2 \]
= \[ 4 \left[ 3z_2y_1^2 + y_1^3 \right] - (4a_kx^{s_1-2j} + \ldots) \]
- \[ [4a_k^2x^{s_1-2j} + 8a_kak_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \ldots]y_1 \]
- \[ [4a_k^2x^{s_1-2j} + 8a_kak_{k+s}x^{s_1-2j+s} + 4a_{k+s}^2x^{s_1-2j+2s} + \ldots]z_2, \]
(18)
where the last equality follows from \( p(z_2) = 0 \).

Hence, in the next step of the Newton procedure it is enough to consider the polynomial

\[ H(x, y_1) = 12z_2y_1^2 - 4a_k^2x^{s_1-2j}y_1 + (-4z_2a_k^2 - 4a_k)x^{s_1-2j} - 4a_{k+s}x^s. \]

The topological type of the discriminant curve will depend on the relation between \( s \) and \( s_1 - 2j \):

**C.3.2.1** Suppose that \( s_1 - 2j > s \). We get:

\[ \gamma_2(u) = -\frac{\sqrt{6}}{3} u^{\frac{s_1-j}{2}} + \frac{\sqrt{-a_{k+s}}}{\sqrt{6}} u^{\frac{s_1-j}{2}} + \frac{\sqrt{z}}{6} + \ldots, \]
\[ \gamma_3(u) = -\frac{\sqrt{6}}{3} u^{\frac{s_1-j}{2}} - \frac{\sqrt{-a_{k+s}}}{\sqrt{6}} u^{\frac{s_1-j}{2}} + \frac{\sqrt{z}}{6} + \ldots. \]

Hence

\[ f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u_1^{2(s_1-j)} + p(z_2)u_1^{2(s_1-j)+\frac{s}{2}} + l(a_{k+s})u_1^{3s_1-4j} + \frac{8}{3}ca_{k+s}u_1^{2(s_1-j)+\frac{s}{2}} + \ldots, \]
and

\[ f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u_1^{2(s_1-j)} - p(z_2)u_1^{2(s_1-j)+\frac{s}{2}} + l(a_{k+s})u_1^{3s_1-4j} - \frac{8}{3}ca_{k+s}u_1^{2(s_1-j)+\frac{s}{2}} + \ldots, \]
where \( c := \frac{-a_{k+s}}{\sqrt{6}} \), \( l(z) = -4z_2z - 1 \). As a consequence, when \( s \) is odd then the discriminant curve \( D(u, v) = 0 \) is the union of a smooth branch \( D_1(u, v) = 0 \) and a singular branch with semigroup \( (2, 4(s_1 - j) + 3s) \), with \( i_0(D_1, D_2) = 4(s_1 - j) \). Otherwise, if \( s \) is even then the discriminant curve \( D(u, v) = 0 \) is the union of three smooth branches \( D_1(u, v) = 0 \) such that \( i_0(D_1, D_1) = 2(s_1 - j) \) for \( 2 \leq l \leq 3 \) and \( i_0(D_2, D_3) = 2(s_1 - j) + \frac{3}{2} s \).

**C.3.2.2** Suppose that \( s_1 - 2j < s \). After (19) and for the next step of the Newton procedure we only need the polynomial

\[ H(Z) = 12z_2Z^2 - 4(z_2a_k^2 + a_k), \]
whose roots are \( \pm \frac{\sqrt{3}}{2} \). Let \( d = \frac{\sqrt{3}}{2} \). We get:

\[ \gamma_2(u) = z_2u^{\frac{s_1}{2}+k} + du^{\frac{s_1}{2}+k+s_1-2j} + \ldots, \]
\[ \gamma_3(u) = z_2u^{\frac{s_1}{2}+k} - du^{\frac{s_1}{2}+k+s_1-2j} + \ldots. \]

Hence
\[ f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} + p(z_2)u_2^{2(s_1-j)+s} + l(a_k)u_3^{s_1-4j} + \frac{80}{81} z_2 \cdot 2^{s_1-10j}u^3 + \ldots, \]
and
\[ f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} - p(z_2)u_2^{2(s_1-j)+s} + l(a_k)u_3^{s_1-4j} - \frac{80}{81} z_2 \cdot 2^{s_1-10j}u^3 + \ldots \]

Since 4 and \( s_1 \) are coprime then \( 7s_1 - 10 \) is odd and the discriminant curve \( D(u, v) = 0 \) is the union of a smooth branch \( D_1(u, v) = 0 \) and a singular branch with semigroup \( \langle 2, 7s_1 - 10j \rangle \), where \( i_0(D_1, D_2) = 4(s_1 - j) \).

**C.3.2.3** Suppose that \( s_1 - 2j = s \). After (19), in order to obtain the next term in the power series \( \gamma_i \) it is enough to consider the polynomial
\[ \bar{H}(Z) = 12z_2Z^2 - 4(z_2a_k^2 + a_{k+s} + a_k). \]
The topological type of the discriminant will depend on the value of \( a_{k+s} \):

**C.3.2.3.1** For \( a_{k+s} \neq -4\sqrt{\frac{6}{81}} \), we get
\[ \bar{H}(Z) = 12z_2Z^2 - 4 (a_{k+s} + 4\sqrt{\frac{6}{81}}), \]
whose roots are \( \pm b := \pm \left( \frac{-d_{k+s}}{\sqrt{6}} \right) \), where \( d_{k+s} = a_{k+s} + 4\sqrt{\frac{6}{81}} \). Hence
\[ \gamma_2(u) = z_2 u_1^{\frac{3}{2} + k} + bu_1^{\frac{3}{2} + k + s} + \ldots, \]
\[ \gamma_3(u) = z_2 u_1^{\frac{3}{2} + k} - bu_1^{\frac{3}{2} + k + s} + \ldots \]
We conclude that
\[ f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} + l(a_{k+s})u_3^{s_1-4j} + \frac{80}{81} z_2u_2^{s_1-j+\frac{3}{2}s} + \ldots, \]
and
\[ f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} + l(a_{k+s})u_3^{s_1-4j} + \frac{80}{81} z_2u_2^{s_1-j+\frac{3}{2}s} + \ldots \]
So, if \( s \) is odd then \( D(u, v) = 0 \) is the union of a smooth branch \( D_1(u, v) = 0 \) and a singular branch of semigroup \( \langle 2, 7s_1 - 10j \rangle \), with \( i_0(D_1, D_2) = 4(s_1 - j) \). On the other case, if \( s \) is even then \( D(u, v) = 0 \) is the union of three smooth branches \( D_1(u, v) = 0 \) such that \( i_0(D_1, D_1) = 2(s_1 - j) \), and \( i_0(D_2, D_3) = \frac{7s_1 - 10j}{2} \).

**C.3.2.3.2** For \( a_{k+s} = -4\sqrt{\frac{6}{81}} \), after (18), in order to obtain the next term in the power series \( \gamma_i \) it is enough to consider the polynomial
\[ \bar{H}(Z) = 12z_2Z^2 - 4a_k^2Z - 4a_{k+s}(1 + 2a_kz_2), \]
whose roots \( t_1, t_2 \) are simple. Hence
\[ \gamma_2(u) = z_2 u_1^{\frac{3}{2} + k} + t_1 u_1^{\frac{3}{2} + k + s} + \ldots, \]
and
\[ \gamma_3(u) = z_2 u_1^{\frac{3}{2} + k} - t_2 u_1^{\frac{3}{2} + k + s} + \ldots \]
We conclude that
\[ f_5(u, \gamma_2(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} - u_2^{2(s_1-j)+s} + h(t_1)u_2^{2(s_1-j)+s^2} + \ldots, \]
and
\[ f_5(u, \gamma_3(u)) = -u_1^s + q(z_2)u_2^{2(s_1-j)} - u_2^{2(s_1-j)+s} + h(t_2)u_2^{2(s_1-j)+s^2} + \ldots, \]
where \( h(z) = -4a_{k+s}z + a_k \). The discriminant curve \( D(u, v) = 0 \) is the union of three smooth branches \( D_i(u, v) = 0 \) such that \( i_0(D_1, D_1) = 2(s_1 - j) \) and \( i_0(D_2, D_3) = 4s_1 - 6j \).

From the above computations we achieve:

**Theorem 3.6.** Let \( f(x, y) = 0 \) be a plane branch of multiplicity \( n \). We have
(1) If \( n = 2 \) then the discriminant curve \( D(u, v) = 0 \) is non degenerate.

(2) If \( n = 3 \) then the discriminant curve \( D(u, v) = 0 \) is degenerate and its topological type depends on the semigroup \( S(f) \) and its Zariski invariant (when it is not zero).

(3) If \( n = 4 \) then the discriminant curve \( D(u, v) = 0 \) is degenerate and its topological type depends on the semigroup \( S(f) \), the Zariski invariant (when it is not zero) and at most two other analytical invariants.

Remark 3.7. Since there is an explicit normal form for branches of multiplicity 2, 3 or 4, the description of the topological type of the discriminant curve of such a branch was possible. The normal form for branches of multiplicity greater than 4 is not completely determined.

4. Discriminant of branches \( C \) with \( \mu(C) - \tau(C) \leq 2 \)

Let \( C : f(x, y) = 0 \) be a plane branch. Put \( r(C) := \mu(C) - \tau(C) \), where \( \mu(C) \) and \( \tau(C) \) are the Milnor number and the Tjurina number of \( C \), respectively. Observe that \( r(C) \) is a nonnegative integer number. In zero characteristic, from [Z1, Theorem 4] we get \( r(C) = 0 \) if and only if \( C \) is analytically equivalent to the curve \( y^{s_0} - x^{s_1} = 0 \), for two coprime integers \( s_0 \) and \( s_1 \) greater than one. Later, in [B-Hef], the authors describe all plane branches defined over an algebraically closed field of characteristic zero, modulo analytic equivalence, having the property that the difference between their Milnor and Tjurina numbers is one or two. In particular the authors determined the normal forms of the branches of this family, which show us that the Zariski invariant of these branches are determined by the two first generators of their semigroup. By [B-Hef, Corollary 5] we know that if \( r(C) \neq 0 \) then \( r(C) \geq 2^{r-1} \). Hence, if \( r(C) = 1 \) then \( g = 1 \) and if \( r(C) = 2 \) then \( g \leq 2 \). In this section we will describe the topological type of the discriminant curve \( D(u, v) = 0 \) of branches \( C \) with \( \mu(C) - \tau(C) \leq 2 \).

Theorem 4.1. Let \( C : f(x, y) = 0 \) be a plane branch with \( r(C) = \mu(C) - \tau(C) \leq 2 \). Then the discriminant curve \( D(u, v) = 0 \) is degenerate and its topological type is given by the semigroup \( S(f) \).

Proof. Suppose first that \( r(C) = 1 \). By [B-Hef, Corollary 8] the branch \( C \) is analytically equivalent to the curve defined by the equation \( f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-2}y^{s_0-2} \), where \( 2 \leq s_0 < s_1 \) are coprime integers. Hence \( f_y(x, y) = s_0y^{s_0-1} + (s_0 - 2)x^{s_1-2}y^{s_0-3} = y^{s_0-3}(s_0y^2 + (s_0 - 2)x^{s_1-2}) \), which Newton-Puiseux roots are \( \alpha_1 = 0 \) (with multiplicity \( s_0 - 3 \)), \( \alpha_2 = \sqrt{\frac{2 - s_0}{s_0}x^{s_1-2}} \) and \( \alpha_3 = -\sqrt{\frac{2 - s_0}{s_0}x^{s_1-2}} \). From (8) we obtain that the Newton-Puiseux roots of the discriminant curve are

(1) \( \delta_1 = u^{s_1} \) with multiplicity \( s_0 - 3 \),

(2) \( \delta_2 = -u^{s_1} \left( \left( -\frac{2 - s_0}{s_0} \right)^{s_0} + \left( -\frac{2 - s_0}{s_0} \right)^{s_0 - 2} \right) u^\left( \frac{(s_1 - 2)s_0}{2} \right) \),

(3) \( \delta_3 = -u^{s_1} \left( \left( -\frac{2 - s_0}{s_0} \right)^{s_0} + \left( -\frac{2 - s_0}{s_0} \right)^{s_0 - 2} \right) u^\left( \frac{(s_1 - 2)s_0}{2} \right) \).

Hence, if \( s_1 \) and \( s_0 \) are odd then the discriminant curve is given by \( D(u, v) = D_1(u, v)^{s_0-3}D_2(u, v) \), where \( D_1(u, v) = (v - u^{s_1}) \) and \( D_2(u, v) \) is a branch of semigroup \( (2, s_1 - 2) \) and the intersection multiplicity \( i_0(D_1, D_2) = \min \{s_1, (s_1 - 2)n\} \). Otherwise \( D(u, v) \) is the product of \( D_1(u, v)^{s_0-3} \) and two smooth branches \( D_2(u, v) \) and \( D_3(u, v) \), where \( i_0(D_1, D_k) = \min \{s_1, (s_1 - 2)s_0 \} \) for \( 2 \leq k \leq 3 \) and \( i_0(D_2, D_3) = \frac{(s_1 - 2)s_0}{2} \).

Suppose now that \( r(C) = 2 \). In this case we get \( g \leq 2 \). If \( g = 2 \) then by [B-Hef, Corollary 13] the multiplicity of \( C \) is 4 and we study this case in Proposition 3.3.

For \( g = 1 \), after [B-Hef, Theorem 17, Corollary 18], we have two normal forms

(A) \( f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-3}y^{s_0-2} \), with \( 2 < s_0 < s_1 \).

(B) \( f(x, y) = y^{s_0} - x^{s_1} + x^{s_1-2}y^{s_0-3} + \left( \sum_{k=2}^{\alpha_{s_1-1}} a_kx^{s_1-k} \right) y^{s_0-2} \), with \( 4 \leq s_0 < s_1, \frac{2s_0}{s_0 - 3} < s_1 \) and \( a_k \in \mathbb{C} \).

In the case (A) we get \( f_y(x, y) = y^{s_0-3}(s_0y^2 + (s_0 - 2)x^{s_1-3}) \), which Newton-Puiseux roots are \( \alpha_1 = 0 \) (with multiplicity \( s_0 - 3 \)), \( \alpha_2 = \sqrt{\frac{2 - s_0}{s_0}x^{s_1-3}} \) and \( \alpha_3 = -\sqrt{\frac{2 - s_0}{s_0}x^{s_1-3}} \). From (8) we obtain that the Newton-Puiseux roots of the discriminant curve are
(1) \( \delta_1 = u^{s_1} \) with multiplicity \( s_0 - 3 \).
(2) \( \delta_2 = -u^{s_1} \left( \left( \frac{2 - s_0}{s_0} \right)^{s_0} + \left( \frac{2 - s_0}{s_0} \right)^{s_0-2} \right) u^{(s_1-3)s_0} \).
(3) \( \delta_3 = -u^{s_1} \left( \left( \frac{2 - s_0}{s_0} \right)^{s_0} + \left( \frac{2 - s_0}{s_0} \right)^{s_0-2} \right) u^{(s_1-3)s_0} \).

Hence, if \( s_0 \) is odd and \( s_1 \) is even then the discriminant curve is given by \( D(u, v) = D_1(u,v)^{s_0-3}D_2(u,v) \), where \( D_1(u,v) = (v-u^{s_1}) \) and \( D_2(u,v) \) is a branch of semigroup \( (2, (s_1 - 3)s_0) \) and the intersection multiplicity \( i_0(D_1, D_2) = min \{s_1, (s_1 - 3)s_0 \} \). Otherwise \( D_2(u,v) \) is the product of \( D_1(u,v)^{s_0-3} \) and two smooth branches \( D_2(u,v) \) and \( D_3(u,v) \), where \( i_0(D_1, D_k) = min \{s_1, \frac{(s_1-3)s_0}{2} \} \) for \( 2 \leq k \leq 3 \) and \( i_0(D_2, D_3) = \frac{(s_1-3)s_0}{2} \).

In the case (B) we have \( f_y(x,y) = y^{s_0-4} \left( s_0^3 + (s_0 - 3)x^{s_1-2} + (s_0 - 2) \left( \sum_{k \geq 2} a_k x^{s_1-k} \right) y \right) \), hence its Newton polygon coincides with the Newton polygon determined by \((0,3), (s_1 - 2, 0) \) and \((s_1 - 2 - \frac{s_1}{s_0}, 1) \). But, after the inequality \( \frac{s_0}{s_0-3} < s_1 \), we get that this Newton polygon has only two points which are its vertices: \((0,3)\) and \((s_1 - 2, 0)\). The Newton-Puiseux roots of \( f_y(x,y) = 0 \) are \( \alpha_i = 0 \) (with multiplicity \( s_0 - 4 \)) and \( \alpha_i = \xi_i^{\frac{s_0-3}{s_0}} u^{-\frac{2}{s_0}} \cdots \), where \( \xi_i \) is a cubic root of the unity, \( 1 \leq i \leq 3 \). Then, after (8) the Newton-Puiseux roots of the discriminant curve are \( \delta_i = u^{s_1} \) with multiplicity \( s_0 - 4 \) and

\[
\delta_i = -u^{s_1} + \left[ \xi_i^{\frac{s_0-3}{s_0}} \right]^\frac{s_0}{s_0-2} + \left[ \xi_i^{\frac{s_0-3}{s_0}} u^{\frac{(s_1-2)(s_0-2)}{s_0}} \right] + \cdots ,
\]

for \( 1 \leq i \leq 3 \).

If \( s_1 - 2 \) (respectively \( s_0 \)) and \( 3 \) are coprime then the discriminant curve is given by \( D(u,v) = D_1(u,v)^{s_0-4}D_2(u,v) \), where \( D_1(u,v) = (v-u^{s_1}) \) and \( D_2(u,v) \) is a branch of semigroup \( (3, (s_1 - 2)s_0) \) and, after Halphen-Zeuthen formula, the intersection multiplicity \( i_0(D_1, D_2) = 3s_1 \). Otherwise \( D(u,v) \) is the product of \( D_1(u,v)^{s_0-4} \) and three smooth branches \( D_k(u,v) \), \( 2 \leq k \leq 4 \), where \( i_0(D_1, D_k) = s_1 \) for \( 2 \leq k \leq 4 \) and \( i_0(D_l, D_r) = \frac{(s_1-2)s_0}{3} \) for \( 2 \leq l \neq r \leq 4 \).

**Remark 4.2.** Observe that in Theorem 4.1 the Case (B) with \( s_0 = 4 \) coincides with the case \( \sigma_3 \) in (10) for \( j = 2 \). Hence this case was studied in Proposition 3.4.

5. **Tables**

The following tables collect the topological type of the discriminants for branches studied in this paper:

**Table 1.** Discriminants of branches of semigroup \( (2, s_1) \)

| Multiplicity 2 | Normal form | Discriminant \( D(u,v) \) |
|---------------|-------------|-----------------------|
| \( \lambda = 0 \) | \( v + u^{s_1} \) | \( u^{s_0} \) |

**Table 2.** Discriminants of branches of semigroup \( (3, s_1) \)

| Multiplicity 3 | Normal form | Discriminant \( D(u,v) \) |
|----------------|-------------|-----------------------|
| \( \lambda = 0 \) | \( (v + u^{s_1})^3 \) | \( D_1, D_2, m(D_i) = 1, i_0(D_1, D_2) = \frac{s_0}{s_0-2} \) if \( \gcd(2, s_1 + \lambda) = 2 \) \( D_1, S(D_i) = (2, s_1 + \lambda) \) if \( \gcd(2, s_1 + \lambda) = 1 \) |
| \( \lambda \neq 0 \) | \( D_1, D_2, m(D_i) = 1, i_0(D_1, D_2) = \frac{s_0}{2} \) if \( \gcd(2, s_1 + \lambda) = 2 \) \( D_1, S(D_i) = (2, s_1 + \lambda) \) if \( \gcd(2, s_1 + \lambda) = 1 \) |
**Table 3.** Discriminants of branches of semigroup \(\langle 4, 6, s_2 \rangle\)

| Normal form | Discriminant \(D(u, v)\) |
|-------------|-------------------------|
| \(\lambda = s_2 - 6\) | \(D_1 D_2\), \(m(D_1) = 1, S(D_2) = \{2, s_2\}, i_0(D_1, D_2) = 2s_1\) |

**Table 4.** Discriminants of branches of semigroup \(\langle 4, s_1 \rangle\)

| Multiplicity 4 and \(g = 1\) |
|-------------------------------|
| \(\lambda = 0\) | \((v + u^s)^2\) |
| \(\lambda = 2s_1 - 4j \neq 0\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = \frac{2s_1 + j}{3}\) if \(\gcd(3, 2s_1 + \lambda) = 3\), \(D_1\) |
| \(\lambda = 3s_1 - 4j \neq 0\) | See Table 5 |

Depending on the analytical invariants

| Multiplicity 4, \(g = 1\), \(\lambda = 3s_1 - 4j \neq 0\) |
|-----------------------------------------------|
| \(2 \leq j \leq \lceil \frac{9}{4} \rceil\) |

**Case B of Proposition 3.5**

| Discriminant \(D(u, v)\) |
|--------------------------|
| \(\frac{2}{s_1 - j} < \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = \frac{2(s_1 - j)}{3} + k\) if \(\gcd(3, s_1 - j + \frac{9}{4} + k) = 3\), \(D_1\) |
| \(\frac{2}{s_1 - j} > \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = 2(s_1 - j), i_0(D_2, D_3) = \frac{2(s_1 - j)}{3} + k\) if \(\gcd(2, s_1 - j) = 2\), \(D_1\) |
| \(\frac{2}{s_1 - j} \leq \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_2) = 1, S(D_2) = \{2, 3(s_1 - j) + 2(\frac{9}{4} + k)\}, i_0(D_1, D_j) = 4(s_1 - j)\) if \(\gcd(2, s_1 - j) = 1\), \(D_1\) |

**Case C of Proposition 3.5**

| Discriminant \(D(u, v)\) |
|--------------------------|
| \(\frac{2}{s_1 - j} < \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = \frac{2(s_1 - j)}{3} + k\) if \(\gcd(3, s_1 - j + \frac{9}{4} + k) = 3\), \(D_1\) |
| \(\frac{2}{s_1 - j} > \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = 2(s_1 - j), i_0(D_2, D_3) = \frac{2(s_1 - j)}{3} + k\) if \(\gcd(2, s_1 - j) = 2\), \(D_1\) |
| \(\frac{2}{s_1 - j} \leq \frac{1}{3}\) | \(D_1 D_2 D_3\), \(m(D_1) = 1, i_0(D_1, D_j) = 2(s_1 - j)\) if \(a_k \neq \pm \frac{3s_7}{4}\) | see Table 8 |

if \(a_k = \pm \frac{3s_7}{4}\)

**Table 5.**
Discriminant $D(u,v)$

$s_1 - 2j > s$

$D_1D_2; m(D_1) = 1, \ S(D_2) = (2, 4(s_1 - j) + 3s_2), \ i_0(D_1, D_2) = 4(s_1 - j)$

if $s$ is odd

$D_1D_2D_3; m(D_1) = 1, \ i_0(D_1, D_2) = 2(s_1 - j), \ i_0(D_2, D_3) = 2(s_1 - j) + \frac{4}{3}s_2$

if $s$ is even

$s_1 - 2j < s$

$D_1D_2; m(D_1) = 1, \ S(D_2) = (2, 4(s_1 - j)), \ i_0(D_1, D_2) = 4(s_1 - j)$

$s_1 - 2j = s$ and $a_{k+e} \neq \pm \frac{1}{2}D

D_1D_2D_3; m(D_1) = 1, \ i_0(D_1, D_2) = 2(s_1 - j), \ i_0(D_2, D_3) = 4(s_1 - j)$

if $\gcd(2, s) = 2$

$D_1D_2; m(D_1) = 1, \ S(D_2) = (2, 7s_1 - 10j), \ i_0(D_1, D_2) = 4(s_1 - j)$

if $\gcd(2, s) = 1$

$s_1 - 2j = s$ and $a_{k+e} = \pm \frac{1}{2}s_1$

$D_1D_2D_3, \ m(D_1) = 1, \ i_0(D_1, D_2) = 2(s_1 - j), \ i_0(D_2, D_3) = 4s_1 - 6j.$

Table 8.

$r(C) = \mu(C) - \tau(C) \leq 2$

| Normal form | Discriminant $D(u,v)$ |
|-------------|------------------------|
| $r(C) = 1$  | $D_1^{0,-3}D_2; \ D_1 = (v - u^{r-1}), \ S(D_2) = (3, s_1 - 2), \ i_0(D_1, D_2) = \min \{s_1, (s_1 - 2)s_0\}$ |
| $S(f) = (s_0, s_1)$ | if $s_1, s_0$ coprime |
| $r(C) = 2$  | $D_1^{0,-3}D_2D_3; \ D_1 = (v - u^{r-1}), \ m(D_1) = 1, \ i_0(D_1, D_2) = \min \{s_1, 1, \frac{1}{2}(s_1 - 2)s_0\}; \ i_0(D_2, D_3) = \frac{1}{2}(s_1 - 2)s_0$ |
| $S(f) = (4, s_1, s_2)$ | otherwise |
| $r(C) = 2$  | $D_1D_2; \ m(D_1) = 1, \ S(D_2) = (2, s_1), \ i_0(D_1, D_2) = 2s_1$ |
| $S(f) = (s_0, s_1)$ | see Table 10 and Table 11 |

Table 9. Discriminants of branches with $r(C) = \mu(C) - \tau(C) \leq 2$

$\sigma_0, \ odd; \ s_1 \ even$

$D_1^{0,-3}D_2; \ D_1 = (v - u^{r-1}), \ S(D_2) = (2, s_1 - 3)s_0, \ i_0(D_1, D_2) = \min \{s_1, (s_1 - 3)s_0\}$

otherwise

$D_1^{0,-3}D_2D_3; \ m(D_1) = 1, \ i_0(D_1, D_2) = \min \{s_1, 1, \frac{1}{2}(s_1 - 3)s_0\}; \ i_0(D_2, D_3) = \frac{1}{2}(s_1 - 3)s_0$.

Table 10.

$s_1 - 2j = 0$ (resp. $s_0$) and 3 coprime

$D_1^{0,-1}D_2; \ D_1 = (v - u^{r-1}), \ S(D_2) = (3, (s_1 - 2)s_0), \ i_0(D_1, D_2) = 3s_1$

otherwise

$D_1^{0,-1}D_2D_3D_1; \ m(D_1) = 1, \ i_0(D_1, D_2) = s_1, \ i_0(D_1, D_3) = \frac{1}{2} \frac{1}{6} 27s_0^2 2 \leq i \neq r \leq 4$

Table 11.

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