Supersymmetry of Anti-de Sitter Black Holes

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Abstract

We examine supersymmetry of four-dimensional asymptotically Anti-de Sitter (AdS) dyonic black holes in the context of gauged $N = 2$ supergravity. Our calculations concentrate on black holes with unusual topology and their rotating generalizations, but we also reconsider the spherical rotating dyonic Kerr-Newman-AdS black hole, whose supersymmetry properties have previously been investigated by Kostelecký and Perry within another approach. We find that in the case of spherical, toroidal or cylindrical event horizon topology, the black holes must rotate in order to preserve some supersymmetry; the non-rotating supersymmetric configurations representing naked singularities. However, we show that this is no more true for black holes whose event horizons are Riemann surfaces of genus $g > 1$, where we find a nonrotating extremal solitonic black hole carrying magnetic charge and permitting one Killing spinor. For the nonrotating supersymmetric configurations of various topologies, all Killing spinors are explicitly constructed.

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Black holes in Anti-de Sitter space represent a subject of current interest, which, on the one hand, is based on Maldacena’s conjecture of AdS/Conformal field theory-correspondence [1], and, on the other hand, on the fact that AdS space admits black holes with unusual topology [2–7]. These so-called topological black holes have some intriguing properties [8]. Among them are the uncommon behaviour of the luminosity, another version of the information loss paradoxon due to the boundary conditions necessary in an asymptotically AdS space, and a mass spectrum that seems to be difficult to reconcile with string theory. In fact, the argument of Horowitz-Polchinski [9], which describes the transition from a highly excited string state to a black hole, and provides a microscopic interpretation of black hole entropy, does not seem to work for topological black holes, at least not in a naive manner [8]. The reason for this is that the mass levels of a string in AdS space are \( M \approx \ell^{-1} n \) for large quantum numbers \( n \) [10], \( \ell \) being related to the cosmological constant via \( \Lambda = -3/\ell^2 \). This yields a black hole entropy proportional to \( n \), whereas the string entropy goes with \( \sqrt{n} \) (c. f. [8] for details). So it would seem that a string had not enough degrees of freedom to account for black hole entropy. However, the correspondence principle of Horowitz-Polchinski, which has turned out to be very successful up to now, can not be rejected on the above, rather naive arguments, for example there are other mass spectrum regimes for a string in AdS (for \( \ell/l_s \gg 1 \), where \( l_s \) is the string length, the mass spectrum is like in flat space), or there could be a transition from a configuration of D-branes to a black hole. In short, it remains to see how exactly the argument of Horowitz-Polchinski works for topological black holes, and it seems quite challenging to try to give a microscopic description of the entropy of these objects within string theory, e. g. by using D-brane technology. A first step in this enterprise is to find supersymmetric configurations, as for BPS states we know that the degeneracy at weak string coupling constant \( g_s \) does not change if one increases \( g_s \). A natural candidate to address the issue of supersymmetry of topological AdS black holes is \( N = 2 \) gauged supergravity [11,12]. In this theory, the rigid SO(2) symmetry, rotating the 2 independent Majorana supersymmetries present in the ungauged theory, is made local. This requires a negative cosmological constant. Supersymmetry of AdS black holes with spherical event horizons have been studied before in the literature [1]. Romans [14] showed that the Reissner-Nordström-AdS black hole is supersymmetric in two cases. The first one appears for \( q_m = 0 \) and \( q_e^2 = m^2 \); \( q_m, q_e \) and \( m \) being the magnetic charge, electric charge and mass parameter, respectively. The second one emerges for \( m = 0 \) and \( q_m = \pm \ell/2 \) (we recall that \( \Lambda = -3/\ell^2 \) is the cosmological constant). However, all these supersymmetric configurations represent naked singularities. The situation is similar to the asymptotically flat Kerr-Newman black hole, which reaches the extreme limit \( m^2 = a^2 + q_e^2 \) (\( a \) denoting the rotation parameter) before the supersymmetry condition \( m^2 = q_e^2 \) is satisfied [15].

As far as the spherical Kerr-Newman-AdS solution is concerned, it was shown by Kostelecký

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1For BPS black holes in five-dimensional \( N = 2 \) gauged supergravity coupled to vector supermultiplets cf. [13].
and Perry [16] that only for nonvanishing rotation parameter $a$ it is possible to obtain supersymmetric extremal black holes. This means that in an AdS background, rotating black holes are the analogue of the asymptotically flat extreme Reissner-Nordström solution. This result was obtained by considering the Bogomol’nyi bound arising from the supersymmetry algebra $\text{osp}(4|2)$ of gauged $N = 2$ supergravity. It is given by $m = \sqrt{q_e^2 + q_m^2(1 \pm a/\ell)}$. In the present paper we will show that in this equation the magnetic charge is required to be zero, making thus the Kostelecký-Perry result more precise. Besides, we will see that there exists also a supersymmetric configuration with vanishing mass parameter, but nonvanishing magnetic charge. This represents a naked singularity, and it was not obtained in [16].

The main purpose of our paper, however, is to study supersymmetry of black holes with unusual topology, which has not been considered in the literature before. The rest of this article is organized as follows:

In section II we give a short introduction into the geometry of four-dimensional topological black holes and classify all known solutions.

In section III gauged $N = 2, d = 4$ supergravity is briefly reviewed.

In section IV we investigate supersymmetry of nonrotating black holes whose event horizons are Riemann surfaces of genus $g \geq 1$. The Killing spinors in the various cases are explicitly constructed.

In section V we generalize our results to rotating black holes of various topologies.

Finally, our results are summarized and discussed in VI.

II. ANTI-DE SITTER BLACK HOLES

In this section, we shall review four-dimensional asymptotically AdS black holes, which are solutions of the Einstein-Maxwell equations with negative cosmological constant. The cosmological constant is sufficient to avoid a few classic theorems forbidding nonspherical black holes [17, 18]. As a result, beyond the well-known Kerr-Newman-AdS black hole, there is a huge variety of black holes with unusual topology. We shall first show how topological black holes arise in AdS space in the simple nonrotating case, then we shall consider the rotating generalizations. Throughout the discussion, particular attention will be paid on the extreme black holes. Although not explicitly stated in the following, all the metrics we shall discuss are also solutions of $N = 2$ gauged supergravity, as will become clear in section III.

A. Nonrotating AdS Black Holes

We start from the class of metrics

$$d\sigma^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2d\sigma^2,$$  \hspace{1cm} (1)

where $d\sigma^2$ is the metric of a two-manifold $S$. The Einstein-Maxwell equations with negative cosmological constant $\Lambda = -3\ell^{-2}$ require $S$ to be a surface of constant curvature $\kappa$, and

$$V(r) = \kappa - \frac{2\eta}{r} + \frac{z^2}{r^2} + \frac{r^2}{\ell^2}, \quad z^2 = q_e^2 + q_m^2,$$  \hspace{1cm} (2)
where \( q_e \) and \( q_m \) denote the electric and magnetic charge parameters respectively. It is useful to define

\[
\eta_0(z) = \frac{\ell}{3\sqrt{6}} \left( \sqrt{\kappa^2 + 12z^2\ell^{-2}} + 2\kappa \right) \left( \sqrt{\kappa^2 + 12z^2\ell^{-2}} - \kappa \right)^{1/2}.
\]  

(3)

According to the sign of the curvature of \( S \) we obtain three cases:

1. \( \kappa = 1 \), \( \text{d} \sigma^2 = \text{d} \theta^2 + \sin^2 \theta \text{ d} \phi^2 \) corresponds to the charged spherical AdS black hole if \( \eta \geq \eta_0(z) \), to AdS space if \( \eta = z = 0 \) and otherwise to a naked singularity,

2. \( \kappa = 0 \), \( S \) flat, describes a flat charged black membrane in AdS when \( \eta \geq \eta_0(z) > 0 \). A genuine black hole should have a compact orientable event horizon, hence we should take some quotient of \( S \). As a result, \( \text{d} \sigma^2 = \text{d}x^2 + 2\text{Re} \tau \text{d}x \text{d}y + |\tau|^2 \text{d}y^2 \), \( x, y \in [0, 1] \) with 0 and 1 identified, and \( S \) is a torus with complex Teichmüller parameter \( \tau \), and the metric \( (\mathbb{I}) \) describes a charged toroidal black hole. For \( \eta < \eta_0(z) \), as well as for \( \eta = z = 0 \), the spacetime has a naked singularity,

3. \( \kappa = -1 \), \( \text{d} \sigma^2 = \text{d} \theta^2 + \sinh^2 \theta \text{ d} \phi^2 \), \( S \) is the hyperbolic plane \( H^2 \), and when \( \eta \geq \eta_0(z) \) we are again dealing with a charged black membrane in AdS. As is well known, \( H^2 \) is the universal covering space for all Riemann surfaces of genus \( g > 1 \). Quotienting \( S \) with a suitable discrete subgroup of its isometry group \( \text{SO}(2, 1) \), the metric \( (\mathbb{II}) \) will describe charged higher genus black holes. For \( \eta < \eta_0(z) \) the spacetime has a naked singularity.

The electromagnetic potential one-form is given by

\[
A = -\frac{q_e}{r} \text{d} t + q_m \cos \theta \text{d} \phi, \quad A = -\frac{q_e}{r} \text{d} t + q_m |\text{Im}\tau| \text{d} y, \quad A = -\frac{q_e}{r} \text{d} t + q_m \cosh \theta \text{d} \phi,
\]  

(4)

for the sphere, torus and higher genus case respectively.

The causal structure of these black holes has been studied in \([19]\), and we refer to that paper for the Penrose diagrams. In the three cases, for \( \eta > \eta_0(z) > 0 \) the black hole has an outer event horizon and an internal Cauchy horizon; the singularity is timelike, in analogy with the Reissner-Nordström black hole. For \( z = 0, \eta > 0 \), the black hole has a simple event horizon hiding a spacelike singularity, while for \( \eta = \eta_0(z) > 0 \) the lapse function has a double root, and the black hole is extreme. In addition, in the higher genus \( \kappa = -1 \) case, there is an uncharged nonrotating extreme black hole for \( (\eta = -\ell/3\sqrt{3}, z = 0) \), black holes with inner and outer horizons for \( (-\ell/3\sqrt{3} < \eta < 0, z = 0) \) and a locally AdS black hole with a single horizon for \( \eta = z = 0 \).

The computation of the mass of these black holes involves some subtleties, as a proper choice of the reference background has to be done \([1] \); in the spherical and toroidal cases the appropriate background is that obtained by putting \( \eta \) and the charges to zero, while for higher genus black holes one has to take the uncharged extreme black hole with mass parameter \( \eta_0 = -\ell/3\sqrt{3} \) as background, to have the Arnowitt-Deser-Misner (ADM) mass as a positive, concave function of the black hole’s temperature as defined by its surface gravity. Taking this into account, one obtains
\[ M = \eta, \quad M = \frac{\eta|\text{Im}\tau|}{4\pi}, \quad M = (\eta - \eta_0)(g - 1), \]  
for the mass of the spherical, toroidal, and genus \( g > 1 \) black holes respectively.

The total electric charge in the various cases is

\[ Q_e = q_e, \quad Q_e = q_e|\text{Im}\tau|, \quad Q_e = q_e(g - 1), \]

and the magnetic charge

\[ Q_m = q_m, \quad Q_m = q_m|\text{Im}\tau|, \quad Q_m = q_m(g - 1). \]

For \( \eta = z = 0 \), the metric (1) is locally AdS; for \( \kappa = 1 \) we obtain AdS space, for \( \kappa = 0 \) a quotient of AdS with a naked singularity, and finally, for \( \kappa = -1 \) we obtain a quotient of AdS space with a black hole interpretation [2], a four-dimensional analogue of the Banados-Teitelboim-Zanelli (BTZ) black hole [20].

The properties of these black holes have been extensively investigated in recent times. They can form by gravitational collapse [3], they emit Hawking radiation [8], and a consistent thermodynamics can be formulated for them: they respect the zeroth and first law, and obey the entropy-area law [7,19].

\textbf{B. Kerr-Newman-AdS Black Hole}

This is the usual charged rotating black hole in AdS. Its horizon is diffeomorphic to a sphere, and its metric, which is axisymmetric, reads in Boyer-Lindquist-type coordinates

\[ ds^2 = -\frac{\Delta_r}{\Xi^2\rho^2} \left[ dt - a \sin^2 \theta \, d\phi \right]^2 + \frac{\rho^2}{\Delta_r} \, dr^2 + \frac{\rho^2}{\Delta_\theta} \, d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\Xi^2\rho^2} \left[ adt - (r^2 + a^2) \, d\phi \right]^2, \]

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{\ell^2}, \]

\[ \Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{\ell^2} \right) - 2mr + z^2, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta, \]

\( a \) is the rotational parameter and \( z \) is defined by \( z^2 = q_e^2 + q_m^2 \), with \( q_e \) and \( q_m \) the electric and magnetic charge parameter respectively. This metric solves the Einstein-Maxwell field equations with an electromagnetic vector potential given by

\[ A = -\frac{q_e}{\Xi^2\rho^2} \left[ dt - a \sin^2 \theta \, d\phi \right] - \frac{q_m \cos \theta}{\Xi \rho^2} \left[ adt - (r^2 + a^2) \, d\phi \right] \]

\[ = -\frac{q_e r}{\rho \sqrt{\Delta_r}} \, e^0 - \frac{q_m \cos \theta}{\rho \sqrt{\Delta_\theta} \sin \theta} \, e^3, \]
where $e^a$ is the vierbein (see Appendix A). The associated field strength tensor is

$$F = -\frac{1}{\rho^4} [q_e(r^2 - a^2 \cos^2 \theta) + 2q_m r a \cos \theta] \ e^0 \wedge e^1$$

$$+ \frac{1}{\rho^4} [q_m(r^2 - a^2 \cos^2 \theta) - 2q_e r a \cos \theta] \ e^2 \wedge e^3. \quad (12)$$

Let us define the critical mass parameter $m_{extr}$,

$$m_{extr}(a, z) = \frac{\ell}{3 \sqrt{6}} \left( \sqrt{\left(1 + \frac{a^2}{\ell^2}\right)^2 + \frac{12}{\ell^2} (a^2 + z^2) + \frac{2a^2}{\ell^2} + 2} \right)$$

$$\times \left( \sqrt{\left(1 + \frac{a^2}{\ell^2}\right)^2 + \frac{12}{\ell^2} (a^2 + z^2) - \frac{a^2}{\ell^2} - 1} \right) \frac{1}{2}. \quad (13)$$

A study of the positive zeroes of the lapse function $\Delta_r$ shows that the metric (8) describes a naked singularity for $m < m_{extr}$ and a black hole with an outer event horizon and an inner Cauchy horizon for $m > m_{extr}$. Finally, for $m = m_{extr}$, the lapse function has a double root and (8) represents an extremal black hole.

Observing that $\partial_t$ and $\partial_\phi$ are Killing vectors, one can use Komar integrals to define mass and angular momentum of the Kerr-Newman-AdS black hole computed with respect to AdS space. For the results, as well as for the electric and magnetic charges, we refer to [16].

C. Rotating Generalization of the Charged $g > 1$ Topological Black Holes

A rotating generalization of the topological black holes with genus $g > 1$ has been obtained from the Kerr-AdS black hole by an analytical continuation [21]. Proceeding analogously from (8) (leaving in addition the charge $z$ unaffected by the analytical continuation), we easily obtain a charged generalization of the rotating topological black hole. The metric is given by

$$ds^2 = -\frac{\Delta_r}{\Xi^2 \rho^2} [dt + a \sinh \theta \ d\phi]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sinh^2 \theta}{\Xi^2 \rho^2} [adt - (r^2 + a^2) d\phi]^2, \quad (14)$$

where now

$$\rho^2 = r^2 + a^2 \cosh^2 \theta, \quad \Xi = 1 + \frac{a^2}{\ell^2}, \quad (15)$$

$$\Delta_r = (r^2 + a^2) \left( -1 + \frac{r^2}{\ell^2} \right) - 2\eta r + z^2, \quad \Delta_\theta = 1 + \frac{a^2}{\ell^2} \cosh^2 \theta. \quad (16)$$

Again, $a$ is the rotational parameter and $z$ is defined by $z^2 = q_e^2 + q_m^2$. The metric (14) is of Petrov type D, obtained as a special case of the most general known type D solution found
by Plebanski and Demianski [27]. (14) solves the Einstein-Maxwell field equations with an
electromagnetic vector potential given by

\[
A = -\frac{q_e r}{\rho^2} [dt + a \sinh^2 \theta d\phi] - \frac{q_m \cosh \theta}{\rho^2} [a dt - (r^2 + a^2) d\phi]
\]

\[
= -\frac{q_e r}{\rho \sqrt{\Delta_r}} e^0 - \frac{q_m \cosh \theta}{\rho \sqrt{\Delta_\theta \sinh \theta}} e^3 ,
\]  

(17)

and the associated field strength tensor is

\[
F = -\frac{1}{\rho^4} \left[ q_e (r^2 - a^2 \cosh^2 \theta) + 2q_m ra \cosh \theta \right] e^0 \wedge e^1
\]

\[
-\frac{1}{\rho^4} \left[ q_m (r^2 - a^2 \cosh^2 \theta) - 2q_e ra \cosh \theta \right] e^2 \wedge e^3.
\]  

(18)

(For the vierbein cf. Appendix A 2). The coordinates \( t \) and \( r \) range over \( \mathbb{R} \), while \( \theta \in \mathbb{R}^+ \) and \( \phi \in [0, 2\pi] \) (with endpoints identified) parametrize the sections of constant \( (t, r) \) in polar coordinates. Hence, our solution describes a charged rotating black membrane in AdS space.

The causal structure of these objects is very complicated and we are not interested in a complete analysis (see [21] for the study of the causal structure in the uncharged case).

Also the thermodynamic behaviour of these black membranes remains an open question, which will be discussed elsewhere.

**D. Charged Rotating Cylindrical Black Hole**

A rotating generalization of the toroidal black hole cannot be found by analytical continuation or by similar tricks, but it can be obtained from the general Petrov type D solution, by an appropriate choice of parameters [24]. Allowing in addition nonvanishing electromagnetic charges, we obtain a rotating generalization of the charged toroidal black hole. The metric is given by

\[
ds^2 = -\frac{\Delta_r}{\rho^2} [dt + aP^2 d\phi]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_P} dP^2 + \frac{\Delta_P}{\rho^2} [a dt - r^2 d\phi]^2 ,
\]

(19)

where

\[
\rho^2 = r^2 + a^2 P^2 ,
\]

\[
\Delta_r = a^2 + z^2 - 2\eta r + \frac{r^4}{\ell^2} , \quad \Delta_P = 1 + \frac{a^2}{\ell^2} P^4 .
\]

(20)

(21)

As usual, \( a \) is the rotational parameter and \( z \) is defined by \( z^2 = q_e^2 + q_m^2 \). This metric is a solution of the Einstein-Maxwell field equations with an electromagnetic vector potential given by
\[ A = -\frac{q_e r}{\rho^2} \left[ dt + a P^2 d\phi \right] - \frac{q_m P}{\rho^2} \left[ a dt - r^2 d\phi \right] \]
\[ = -\frac{q_e r}{\rho \sqrt{\Delta_r}} e^0 - \frac{q_m P}{\rho \sqrt{\Delta_P}} e^3, \]  
(22)

and a field strength tensor
\[ F = -\frac{1}{\rho^4} \left[ q_e (r^2 - a^2 P^2) + 2 q_m r a P \right] e^0 \wedge e^1 \]
\[ -\frac{1}{\rho^4} \left[ q_m (r^2 - a^2 P^2) - 2 q_e r a P \right] e^2 \wedge e^3, \]  
(23)

where the vierbein \( e^a \) is given in appendix A.3.

For a given rotational parameter \( a \) and charge parameter \( z \) we define the critical mass parameter
\[ \eta_{\text{extr}}(a, z) = \frac{2}{3^{3/4} \ell^{1/2}} \left( a^2 + z^2 \right)^{3/4}. \]  
(24)

If \( \eta < \eta_{\text{extr}} \), \( \Delta_r \) has no positive root and the metric (19) describes a naked singularity. For \( \eta = \eta_{\text{extr}} \), there is a double root in \( \Delta_r \), and we obtain an extremal black hole with a timelike singularity. Finally, for \( \eta > \eta_{\text{extr}} \) and \( a^2 + z^2 \neq 0 \), \( \Delta_r \) has two positive simple roots, and the metric (19) describes a black hole with an event horizon and an inner Cauchy horizon. The Penrose diagrams can be found in [21], substituting \( a^2 \) with \( a^2 + z^2 \).

Now, if an horizon is present, it is not compact. The ranges of the coordinates are \( t, P \in \mathbb{R} \), \( r \in \mathbb{R}^+ \) and \( \phi \in [0, 2\pi] \) with the extrema identified. Hence the black hole has a cylindrical event horizon. For \( a = 0 \), we can naturally compactify also the coordinate \( P \) (\( \partial P \) becomes a Killing vector in this case), whereupon (19) reduces to the static toroidal black hole spacetime (1) with \( \kappa = 0 \).

III. \( N = 2 \) GAUGED SUPERGRAVITY

The gauged version of \( N = 2 \) supergravity was found by Das and Freedman [11] and by Fradkin and Vasiliev [12]. In this theory, the rigid SO(2) symmetry rotating the two independent Majorana supersymmetries present in the ungauged theory, is made local by introduction of a minimal gauge coupling between the photons and the gravitini. Local supersymmetry then requires a negative cosmological constant and a gravitini mass term. The theory has four bosonic and four fermionic degrees of freedom; it describes a graviton \( e_a^m \), two real gravitini \( \psi_i^m \) (\( i = 1, 2 \)), and a Maxwell gauge field \( A_m \). As we said, the latter is minimally coupled to the gravitini, with coupling constant \( \ell^{-1} \). If we combine the two \( \psi_i^m \) to a single complex spinor \( \psi_m = \psi_m^1 + i \psi_m^2 \), the Lagrangian reads (cf. also [14])

\[ \text{Throughout this paper, the notation is as follows: } a, b, \ldots \text{ refer to } d = 4 \text{ tangent space indices, and } m, n, \ldots \text{ refer to } d = 4 \text{ world indices. The signature is } (-, +, +, +), \text{ and we use the real (Majorana) representation of the gamma matrices } \gamma_a \text{ (cf. appendix B). They satisfy } \{ \gamma_a, \gamma_b \} = 2\eta_{ab}. \text{ We antisymmetrize with unit weight, i.e. } \gamma_{ab} \equiv \gamma_a \gamma_b = \frac{1}{2}[\gamma_a, \gamma_b] \text{ etc. The parity matrix is } \gamma_5 = \gamma_{0123}. \]
\[ e^{-1}L = -\frac{1}{4}R + \frac{1}{2} \bar{\psi}_m \gamma^{mnp} D_n \psi_p + \frac{1}{4} F_{mn} F^{mn} + \frac{i}{8} (F^{mn} + \tilde{F}^{mn}) \bar{\psi}_p \gamma_{[m} \gamma^{pq} \gamma_{n]} \psi_q \]
\[-\frac{1}{2\ell} \bar{\psi}_m \gamma^{mn} \psi_n - \frac{3}{2\ell^2}. \tag{25}\]

We see that the cosmological constant is \( \Lambda = -3\ell^{-2} \). \( D_m \) denotes the gauge- and Lorentz covariant derivative defined by
\[ D_m = \nabla_m - i\ell^{-1} A_m, \tag{26} \]
where \( \nabla_m \) is the Lorentz-covariant derivative
\[ \nabla_m = \partial_m + \frac{1}{4} \omega_m^{ab} \gamma_{ab}. \tag{27} \]

The equation of motion for the spin connection \( \omega_{ab} \) reads
\[ \omega_{mab} = \Omega_{mab} - \Omega_{mba} - \Omega_{abm}, \tag{28} \]
where
\[ \Omega_{mn}^a \equiv \partial_a [\psi_m e_n^a] - \frac{1}{2} \Re(\bar{\psi}_m \gamma^a \psi_n). \tag{29} \]

\( \hat{F}_{mn} \) denotes the supercovariant field strenght given by
\[ \hat{F}_{mn} = F_{mn} - \Im(\bar{\psi}_m \psi_n). \tag{30} \]

The action is invariant under the local supertransformations
\[ \delta e_m^a = \Re(\epsilon^a \psi_m), \]
\[ \delta A_m = \Im(\bar{\epsilon} \psi_m), \]
\[ \delta \psi_m = \hat{\nabla}_m \epsilon. \tag{31} \]

In (31) \( \epsilon \) is an infinitesimal Dirac spinor, and \( \hat{\nabla}_m \) is the supercovariant derivative defined by
\[ \hat{\nabla}_m = D_m + \frac{1}{2\ell} \gamma_m + \frac{i}{4} \hat{F}_{ab} \gamma^{ab} \gamma_m. \tag{32} \]

The supersymmetry algebra of gauged \( N = 2 \) supergravity is \( \text{osp}(4|2) \). It has the ten bosonic generators \( M_{ab}, M_{a4} \) \((a = 0, 1, 2, 3)\) of the AdS subalgebra \( \text{so}(3, 2) \), two fermionic generators \( Q_i^\alpha \) \((i = 1, 2)\), plus one additional bosonic generator of \( \text{SO}(2) \) transformations, rotating the two supersymmetries into each other. The basic anticommutator is
\[ \{Q^i_\alpha, Q^j_\beta\} = \delta^{ij} \left((\gamma^a M_{a4} + i\gamma^{ab} M_{ab}) C\right)_{\alpha\beta} + i(C_{\alpha\beta} Q_e + i(C\gamma^5)_{\alpha\beta} Q_m) \epsilon^{ij}. \tag{33} \]

Here \( C \) denotes the charge conjugation matrix, \( Q_e \) and \( Q_m \) are central charges, and \( \epsilon^{ij} \) is the permutation symbol in two dimensions.

Let us now return to the lagrangian (25). As we are interested in the bosonic sector, we set \( \psi_m = 0 \). The field equations following from (25) are then the Einstein-Maxwell equations
with negative cosmological constant, thus the black hole spacetimes discussed in the previous section represent possible background solutions of gauged $N = 2$ supergravity. Invariance of these background solutions under the supertransformations (31) yields the equation for Killing spinors

$$\hat{\nabla}_m \epsilon = 0. \quad (34)$$

The integrability conditions for (34) read

$$\hat{R}_{mn} \epsilon = 0, \quad (35)$$

where

$$\hat{R}_{mn} = [\hat{\nabla}_m, \hat{\nabla}_n] \quad (36)$$

is the supercurvature. (35) is necessary, but not sufficient for the existence of Killing spinors. It assures that they exist locally, but globally there may exist no Killing spinor due to topological reasons. In the following sections, we shall solve (34) and (35) for the black hole spacetimes introduced in II.

**IV. SUPERSYMMETRY OF STATIC TOPOLOGICAL BLACK HOLES**

**A. Integrability Conditions**

Let us first consider the static black hole spacetimes (1), whose event horizons are Riemann surfaces of genus $g \geq 1$. Setting $a = 0$ in the spin connections given in the appendices A3 and A2, one finds the supercovariant derivatives

$$\hat{\nabla}_t = \partial_t - \frac{i q_e}{\ell} r + \frac{1}{2\ell} \sqrt{V(r)} \gamma_0 + \frac{i}{4} F_{ab} \gamma^{ab} \sqrt{V(r)} \gamma_0 + \frac{1}{2r} \left( \frac{\eta}{r} - \frac{z^2}{r^2} + \frac{r^2}{\ell^2} \right) \gamma_{01},$$

$$\hat{\nabla}_r = \partial_r + \frac{1}{2\ell} \sqrt{V(r)}^{-1} \gamma_1 + \frac{i}{4} F_{ab} \gamma^{ab} \sqrt{V(r)}^{-1} \gamma_1,$$

$$\hat{\nabla}_x = \partial_x - \frac{1}{2} \sqrt{V(r)} \gamma_{12} + \frac{1}{2\ell} r \gamma_2 + \frac{i}{4} r F_{ab} \gamma^{ab} \gamma_2,$$

$$\hat{\nabla}_y = \partial_y - \frac{1}{2} \sqrt{V(r)} \gamma_{13} - \frac{i}{\ell} q_m x + \frac{1}{2\ell} r \gamma_3 + \frac{i}{4} r F_{ab} \gamma^{ab} \gamma_3 \quad (37)$$

for toroidal topology (we consider here only the case $\tau = i$, $\tau$ denoting the Teichmüller parameter introduced in section II), and

$$\hat{\nabla}_t = \partial_t - \frac{i q_e}{\ell} r + \frac{1}{2\ell} \sqrt{V(r)} \gamma_0 + \frac{i}{4} F_{ab} \gamma^{ab} \sqrt{V(r)} \gamma_0 + \frac{1}{2r} \left( \frac{\eta}{r} - \frac{z^2}{r^2} + \frac{r^2}{\ell^2} \right) \gamma_{01},$$

$$\hat{\nabla}_r = \partial_r + \frac{1}{2\ell} \sqrt{V(r)}^{-1} \gamma_1 + \frac{i}{4} F_{ab} \gamma^{ab} \sqrt{V(r)}^{-1} \gamma_1,$$

$$\hat{\nabla}_\theta = \partial_\theta - \frac{1}{2} \sqrt{V(r)} \gamma_{12} + \frac{1}{2\ell} r \gamma_2 + \frac{i}{4} r F_{ab} \gamma^{ab} \gamma_2,$$

$$\hat{\nabla}_\phi = \partial_\phi - \frac{1}{2} \sqrt{V(r)} \gamma_{13} \sinh \theta - \frac{1}{2} \gamma_{23} \cosh \theta - \frac{i}{\ell} q_m \cosh \theta + \frac{1}{2\ell} r \gamma_3 \sinh \theta + \frac{i}{4} r F_{ab} \gamma^{ab} \gamma_3 \sinh \theta \quad (38)$$
for the higher genus \((g > 1)\) case. We recall that \(V(r)\) is given by

\[
V(r) = \delta_{g,1} - 1 - \frac{2\eta}{r} + \frac{r^2}{\ell^2} + \frac{z^2}{r^2}.
\] (39)

One verifies that the supercurvature \((36)\), like in the Reissner-Nordström-AdS case studied by Romans \([14]\), can be written as a product

\[
\hat{R}_{mn} = \mathcal{P} \mathcal{G}_{mn}(r) \mathcal{O},
\] (40)

where \(\mathcal{G}_{mn}(r)\) is \(\gamma_{mn}\) times some scalar-valued function of \(r\),

\[
\mathcal{P} \equiv \frac{r^2}{2z} i F_{ab} \gamma^{ab} \gamma_1
\] (41)

is an idempotent (and hence non-singular) operator, and \(\mathcal{O}\) is given by

\[
\mathcal{O} = \sqrt{V(r) + \frac{r}{\ell} \gamma_1 + \left(\frac{z}{r} - \frac{\eta}{z}\right) \mathcal{P}}.
\] (42)

As \(\mathcal{G}_{mn}(r)\) and \(\mathcal{P}\) are non-singular for \(z \neq 0\) (the case \(z = 0\) has to be considered separately), the integrability conditions for Killing spinors are equivalent to the vanishing of \(\det \mathcal{O}\). We obtain for the determinant

\[
\det \mathcal{O} = \left\{ \delta_{g,1} - 1 - \frac{2q_m}{\ell} - \frac{(\eta r - 2q_m \eta r \ell^{-1})}{z^2} \right\} \left\{ \delta_{g,1} - 1 + \frac{2q_m}{\ell} - \frac{(\eta^2 + 2q_m \eta r \ell^{-1})}{z^2} \right\}.
\] (43)

\(\det \mathcal{O}\) is a function of \(r\), and for supersymmetric configurations, this function must vanish identically. For genus \(g = 1\), this is fulfilled in two cases. The first one appears for

\[
q_m = 0 = \eta, \quad \Rightarrow \quad V(r) = \frac{r^2}{\ell^2} + \frac{q_e^2}{r^2},
\] (44)

and the second one for

\[
\eta = 0 \land \ell = \infty \quad (\Lambda = 0) \quad \Rightarrow \quad V(r) = \frac{z^2}{r^2}.
\] (45)

We observe that the lapse function is always positive in these cases, i.e. the corresponding spacetimes represent naked singularities. This result is very similar to the Reissner-Nordström-AdS black hole considered in \([14]\), where the supersymmetric configurations are also naked singularities.

For genus \(g > 1\), \((43)\) vanishes only for \(\eta = 0\) and \(q_m = \pm \ell/2\). This yields

\[
V(r) = \left(\frac{r}{\ell} - \frac{\ell}{2r}\right)^2 + \frac{q_e^2}{r^2}.
\] (46)

For vanishing electric charge, spacetime \([1]\), with \(V(r)\) given by \((46)\), describes an extremal black hole. Thus, unlike the case of spherical or toroidal event horizons, now we can hope to get a supersymmetric static extremal black hole. Clearly, this is not obvious, because it could be possible that the Killing spinors (which exist locally, as \((35)\) is satisfied), are
not compatible with the identifications which have to be carried out to get a compact event horizon. We shall see below, when we will construct explicitly the Killing spinors, that they depend only on the radial coordinate $r$, and consequently they do respect the identifications performed in the $(\theta, \phi)$-sector.

The extremal black hole found above is a solitonic object in the sense that the limit $\ell^{-1} \to 0$ (we recall that $\ell^{-1}$ is the coupling constant of the gauged theory, coupling the photon to the gravitini), does not exist.

For genus $g > 1$, there is still another case in which the integrability conditions (35) are fulfilled, namely for $\eta = 0 = q_m = q_e$. The spacetime is then a quotient space of AdS, and therefore locally admits Killing spinors. However, we will see that they do not exist globally, as the above mentioned identifications are not respected. Thus the corresponding black hole is not supersymmetric.

**B. Killing Spinors**

We now turn to the issue of solving the Killing spinor equation (34) explicitly for the diverse cases found above.

1. **Genus $g = 1$, $V(r) = \frac{r^2}{\ell^2} + \frac{q_e^2}{r^2}$**

The spacetime describes an electrically charged naked singularity with topology $\mathbb{R}^2 \times S^1 \times S^1$, and is asymptotically AdS. Using the integrability condition $\mathcal{O} \epsilon = 0$, (34) simplifies in this case to

\[
\nabla_r \epsilon = \left( \partial_r + \frac{1}{2\ell} \sqrt{V(r)^{-1}} \gamma_1 - \frac{iq_e}{2r^2} \sqrt{V(r)^{-1}} \gamma_0 \right) \epsilon = 0,
\]

\[
\nabla_t \epsilon = \partial_t \epsilon = 0,
\]

\[
\nabla_x \epsilon = \partial_x \epsilon = 0,
\]

\[
\nabla_y \epsilon = \partial_y \epsilon = 0.
\]

(47)

The Killing spinors thus depend only on $r$. One verifies that

\[
Q \equiv \frac{1}{2\sqrt{V(r)}} \mathcal{O} = \frac{1}{2} \left( 1 + \frac{r}{\ell \sqrt{V(r)}} \gamma_1 + \frac{q_e}{r \sqrt{V(r)}} \gamma_0 \right)
\]

(48)

is a projection operator. In the appendix of [14] one finds the solution of the spinorial differential equation

\[
\partial_r \epsilon(r) = (a(r) + b(r) \Gamma_1 + c(r) \Gamma_2) \epsilon(r),
\]

(49)

where

\[
(\Gamma_1)^2 = (\Gamma_2)^2 = 1, \quad \{\Gamma_1, \Gamma_2\} = 0,
\]

(50)

and $\epsilon(r)$ obeying the constraint.
\[ \mathcal{Q}\epsilon = 0, \]  

with a projector \( \mathcal{Q} \) given by

\[ \mathcal{Q} = \frac{1}{2}(1 + \xi(r)\Gamma_1 + \zeta(r)\Gamma_2), \]  

and

\[ \xi^2 + \zeta^2 = 1, \quad \zeta \neq 0. \]  

In our case, this solution reads

\[ \epsilon(r) = \left( \sqrt{V(r) + \frac{r}{\ell} + i\sqrt{V(r) - \frac{r}{\ell}\gamma_0}} \right) P(-\gamma_1)\epsilon_0, \]  

where \( \epsilon_0 \) is a constant spinor, and

\[ P(-\gamma_1) \equiv \frac{1}{2}(1 - \gamma_1) \]  

is another projection operator, which reduces the complex dimension of the space of Killing spinors from four to two. If the electric charge also vanishes, the spacetime is simply a quotient of AdS, representing the background (zero Hawking temperature) of uncharged toroidal black holes. Then the operators \( P \) (41) and \( O \) (42) are ill-defined, so we must consider this case separately. It is clear that locally as many Killing spinors as in AdS exist (four complex-dimensional solution space), but we find that the only ones respecting the identifications one carries out to compactify the \((x,y)\)-sector to a torus, are those resulting from (54) by setting \( q_e = 0 \), i.e.

\[ \epsilon(r) = \sqrt{r}P(-\gamma_1)\epsilon_0, \]  

so we have again a two complex-dimensional space of Killing spinors.

2. Genus \( g = 1 \), \( V(r) = \frac{z^2}{r^2} \)

This space is not asymptotically AdS and represents a dyonic naked singularity with topology \( \mathbb{R}^2 \times S^1 \times S^1 \). For completeness we give the Killing spinors also for this case. Making use of \( O\epsilon = 0 \), (34) reduces to

\[ \hat{\nabla}_r \epsilon = \left( \partial_r + \frac{1}{2r} \right) \epsilon = 0, \]

\[ \hat{\nabla}_t \epsilon = \partial_t \epsilon = 0, \]

\[ \hat{\nabla}_x \epsilon = \partial_x \epsilon = 0, \]

\[ \hat{\nabla}_y \epsilon = \partial_y \epsilon = 0. \]  

Taking into account the constraint \( O\epsilon = 0 \), one obtains the solution

\[ \epsilon(r) = r^{-1/2}P(-i\gamma_0 \frac{q_e}{z} + i\gamma_{123} \frac{q_m}{z})\epsilon_0. \]  

Again, a projection operator

\[ P(-i\gamma_0 \frac{q_e}{z} + i\gamma_{123} \frac{q_m}{z}) \equiv \frac{1}{2}(1 - i\gamma_0 \frac{q_e}{z} + i\gamma_{123} \frac{q_m}{z}) \]  

acts on \( \epsilon_0 \), reducing the dimension of the solution space from four to two.
3. Genus \( g > 1 \), \( V(r) = \left( \frac{r}{\ell} - \frac{\ell}{2r} \right)^2 + \frac{q_e^2}{r^2} \)

We now focus our attention on the case of spacetime topology \( \mathbb{R}^2 \times S_g \), \( S_g \) being a Riemann surface of genus \( g > 1 \). The mass parameter \( \eta \) is zero, and the magnetic charge \( q_m \) equals \( \pm \ell/2 \). We shall consider only \( q_m = +\ell/2 \), the other sign giving identical results. For nonvanishing electric charge, we have a dyonic naked singularity; for \( q_e = 0 \), however, we get an extremal magnetically charged black hole (a magnetic monopole hidden by an event horizon having the topology of a Riemann surface).

In this case the operator \( \mathcal{O} \) is not proportional to a projector, but rather is a linear combination of two projection operators \( P(-i\gamma_{23}) \equiv (1 - i\gamma_{23})/2 \) and

\[
\mathcal{Q} \equiv \frac{1}{2} + \frac{1}{2\sqrt{V(r)}} \left\{ \left( \frac{r}{\ell} - \frac{\ell}{2r} \right) \gamma_1 + i\gamma_0 \frac{q_e}{r} \right\}.
\]

We find

\[
\mathcal{O} = 2\sqrt{V(r)} Q + \frac{\ell}{r} \gamma_1 P(-i\gamma_{23})
\]

and

\[
-\frac{1}{2} \left[ \sqrt{V(r)} - \frac{r}{\ell} \gamma_1 - \frac{1}{r} (i\gamma_0 q_e - \frac{\ell}{2i}\gamma_{123}) \right] \mathcal{O} = P(-i\gamma_{23}),
\]

\[
\frac{\ell}{4\sqrt{V(r)r}} \gamma_1 \left[ \sqrt{V(r)} + \frac{r}{\ell} \gamma_1 - \frac{1}{r} (i\gamma_0 q_e - \frac{\ell}{2i}\gamma_{123}) \right] \mathcal{O} = \mathcal{Q},
\]

\[
[\mathcal{Q}, P(-i\gamma_{23})] = 0.
\]

The integrability condition \( \mathcal{O}\epsilon = 0 \) is thus equivalent to the two conditions

\[
P(-i\gamma_{23})\epsilon = 0, \quad \mathcal{Q}\epsilon = 0.
\]

The Killing spinor equations then simplify to

\[
\hat{\nabla}_r \epsilon = \left( \partial_r + \frac{1}{2r} + \frac{1}{\ell \sqrt{V(r)}} \gamma_1 \right) \epsilon = 0,
\]

\[
\hat{\nabla}_t \epsilon = \partial_t \epsilon = 0,
\]

\[
\hat{\nabla}_\theta \epsilon = \partial_\theta \epsilon = 0,
\]

\[
\hat{\nabla}_\phi \epsilon = \partial_\phi \epsilon = 0.
\]

The solution of the radial equation can again be constructed using the appendix of [14], yielding

\[\text{from now on, with } P(L), \text{ where } L \text{ is an operator, we always intend } (1 + L)/2.\]
\[
\epsilon(r) = \left( \sqrt{\frac{V(r)}{r}} + \frac{r}{\ell} - \frac{\ell}{2r} \sqrt{\frac{V(r)}{r}} - \frac{r}{\ell} + \frac{\ell}{2r} i \gamma_0 \right) P(-\gamma_1) P(i \gamma_{23}) \epsilon_0.
\] (67)

Now the constant spinor \( \epsilon_0 \) is subject to a double projection, so the solution space is one (complex)-dimensional. The Killing spinor does not depend on the coordinates \( \theta, \phi \), thus it respects the identifications we have done in the \((\theta, \phi)\)-sector to obtain a Riemann surface. Hence, for zero electric charge, we have obtained a supersymmetric extremal static black hole. This was not possible for spherical event horizons, i.e. for the Reissner-Nordström-AdS black hole, where all supersymmetric configurations were naked singularities [14]. So we see that admitting other spacetime topologies changes the supersymmetry properties.

According to (5) and (7), the mass and the magnetic charge of the supersymmetric higher genus black holes considered above, are given by
\[
M = \frac{\ell (g - 1)}{3\sqrt{3}} \quad \text{and} \quad Q_m = \frac{\ell (g - 1)}{2},
\]
i.e. we have
\[
M^2 = \frac{4}{27} Q_m^2
\] (68)
as Bogomol’nyi bound. This bound supports the view advocated in [7], namely that the mass of the higher genus black holes is not simply given by the parameter \( \eta \) appearing in (2), but rather by (5), i.e. the background which has to be subtracted in the mass calculation, is not simply the one with \( \eta = 0 \). Note that in [7], this conclusion emerges from thermodynamical considerations, and has nothing to do with supersymmetry. We found here that also supersymmetry as an independent argument supports this point of view.

4. Genus \( g > 1 \), \( V(r) = -1 + \frac{\ell^2}{r^2} \)

This is a quotient space of AdS describing an uncharged black hole. Without identifications in the \((\theta, \phi)\)-sector the spacetime is simply AdS viewed by a uniformly accelerated observer, the (noncompact) horizon being its acceleration horizon [7]. Only the compactification of the surfaces of constant \( r \) and \( t \) makes the spacetime to become a black hole, with the singularity at \( r = 0 \) being a causal one, i.e. the manifold cannot be continued beyond this singularity, otherwise one would have closed timelike curves [2]. It is clear that locally this spacetime admits as many Killing spinors as AdS, but we have to check if they respect the identifications. The Killing spinor equations read
\[
\hat{\nabla}_r \epsilon = \left( \partial_r + \frac{1}{2\ell \sqrt{V(r)}} \gamma_1 \right) \epsilon = 0, \\
\hat{\nabla}_t \epsilon = \left( \partial_t + \frac{r}{2\ell^2 \gamma_{01}} + \frac{1}{2\ell} \sqrt{V(r)} \gamma_0 \right) \epsilon = 0, \\
\hat{\nabla}_\theta \epsilon = \left( \partial_\theta + \frac{r}{2\ell^2 \gamma_2} - \frac{1}{2} \sqrt{V(r)} \gamma_{12} \right) \epsilon = 0, \\
\hat{\nabla}_\phi \epsilon = \left( \partial_\phi + \frac{1}{2} \gamma_3 \left( \frac{r}{\ell} + \sqrt{V(r)} \gamma_1 \right) \sinh \theta - \frac{1}{2} \gamma_{23} \cosh \theta \right) \epsilon = 0.
\] (69)

The solution is
\[ \epsilon(r, t, \theta, \phi) = \left( \sqrt{\frac{r}{\ell}} + 1 - \sqrt{\frac{r}{\ell}} - 1\gamma_1 \right) (\cosh \frac{t}{2\ell} - \gamma_{01} \sinh \frac{t}{2\ell}) (\cosh \frac{\theta}{2} - \gamma_2 \sinh \frac{\theta}{2}) (\cos \frac{\phi}{2} + \gamma_{23} \sin \frac{\phi}{2}) \epsilon_0. \] (70)

As an explicit \( \phi \)-dependence appears, the Killing spinors are not invariant under the transformations of the discrete group used in the identifications, and the black hole is not supersymmetric. Clearly this was to be expected, as a supersymmetric black hole necessarily must have zero temperature (note, however, that the converse is not true in general), whereas the hole considered above has nonvanishing Hawking temperature \( T = 1/(2\pi \ell) \).

Note that the minimal coupling of the photon and the gravitini in the action of gauged \( N = 2 \) supergravity gives rise to a Dirac quantization of the magnetic charge. In the spherical static case, this condition is automatically fulfilled for the supersymmetric solutions [14]. Finding the Dirac quantization condition in presence of unusual topologies, as is partially the case here, is a nontrivial task, which involves \( U(1) \)-bundles over Riemann surfaces of genus \( g \geq 1 \). We will discuss this problem in a forthcoming publication.

V. SUPERSYMMETRY OF ROTATING ADS BLACK HOLES

Now we turn to the rotating generalizations of the static black holes considered above. As the metrics are rather complicated, the investigation of supersymmetry properties of these spacetimes is a quite formidable task. However, as we shall see below, it is still possible to solve explicitly the integrability conditions, yielding some interesting results.

A. Cylindrical Event Horizons

Let us first consider the black hole spacetime (19). The supercovariant derivatives read

\[ \hat{\nabla}_t = \partial_t + \frac{1}{2} \omega_{12} \gamma_{12} + \frac{1}{2} \omega_{03} \gamma_{03} + \frac{\rho}{2\ell \sqrt{\Delta_r}} \gamma_1 + \frac{i}{4\sqrt{\Delta_r}} F_{ab} \gamma^{ab} \gamma_1, \]

\[ \hat{\nabla}_r = \partial_r + \frac{1}{2} \omega_{01} \gamma_{01} + \frac{1}{2} \omega_{23} \gamma_{23} + \frac{1}{2} \omega_{02} \gamma_{02} + \frac{1}{2} \omega_{13} \gamma_{13} + \frac{iq e r a P^2 - q_m P r^2}{\ell \rho^2}, \]

\[ \hat{\nabla}_\phi = \partial_\phi + \frac{1}{2} \omega_{03} \gamma_{03} + \frac{1}{2} \omega_{23} \gamma_{23} + \frac{1}{2} \omega_{02} \gamma_{02} + \frac{1}{2} \omega_{13} \gamma_{13} + \frac{i q e r a P^2 - q_m P r^2}{\ell \rho^2}, \]

\[ \hat{\nabla}_p = \partial_p + \frac{1}{2} \omega_{01} \gamma_{01} + \frac{1}{2} \omega_{23} \gamma_{23} + \frac{1}{2} \omega_{02} \gamma_{02} + \frac{1}{2} \omega_{13} \gamma_{13} + \frac{i q e r a P^2 - q_m P r^2}{\ell \rho^2}, \]

\[ + \left( \frac{1}{2\ell \rho} + \frac{i}{4\rho} F_{ab} \gamma^{ab} \right) \left( \sqrt{\Delta_r \gamma_0 - \sqrt{\Delta_p \gamma_3}} \right), \] (71)

with the electromagnetic field \( F_{ab} \) [23], and the spin connection \( \omega_{m}^{ab} \) given in appendix A3. Similarly to the nonrotating case, we find for the supercurvature

\[ \hat{R}_{mn} = \mathcal{P} \mathcal{G}_{mn}(r, P) \mathcal{O}, \] (72)

with the idempotent operator \( \mathcal{P} \) now given by
\[ P \equiv \frac{\rho^2}{2z} F_{ab} \gamma^{ab} \gamma_1, \]  

and

\[ O = \frac{1}{\rho^2} \left\{ -\sqrt{\Delta_P} r a \gamma_{03} - \sqrt{\Delta_P} r a \gamma_{0123} + \sqrt{\Delta} r (r + \sqrt{\Delta} a^2 P \gamma_{12}) \right\} + \frac{\rho}{\ell} \gamma_1 \]

\[ + \frac{1}{\rho^2} \left\{ -a P \left[ \frac{\eta}{z} (3r^2 - a^2 P^2) - 2r z \right] \gamma_{0123} + \left[ \frac{\eta^r}{z} (r^2 - 3a^2 P^2) + z (a^2 P^2 - r^2) \right] \right\} \]  

\( \mathcal{O} \). (73)

\( \mathcal{P} \) and \( \mathcal{G}_{mn}(r, P) \) are in general nonsingular, so the integrability condition is again \( \det \mathcal{O} = 0 \).

The determinant reads

\[ \det \mathcal{O} = \frac{1}{\ell^2 z^4} \left[ (\ell^2 \eta^4 - 4z^4 (q_m^2 + a^2)) + 8q_m^2 z^2 \eta r - 4q_m^2 \eta r^2 \right. \]

\[ + 8a \eta q_m q_e z^2 P + 4a^2 q_m^2 \eta^2 P^2 - 8a \eta^2 q_m q_e P r \]  

\( (75) \).

For \( z \neq 0 \) (which we presupposed, as otherwise \( \mathcal{P} \) is ill-defined), the requirement that \( \det \mathcal{O} \) be vanishing identically as a function of \( r \) and \( P \), yields the conditions

\[ q_m \eta = 0, \]

\[ \ell^2 \eta^4 = 4z^4 (q_m^2 + a^2). \]  

(76)

The case \( \eta = 0 \) is not of particular interest for us, as it does not represent a black hole spacetime. Therefore we will assume \( \eta > 0 \), from which follows \( q_m = 0 \) and

\[ \eta^2 = \frac{2a}{\ell} q_e^2 \]  

(77)

for supersymmetric configurations. We observe that for \( a = 0 \), (77) reduces correctly to (74), i.e. \( \eta = 0 \). We do not construct the Killing spinors explicitly for the rotating case. However, we have seen that for \( a = 0 \) there is a two-dimensional solution space of Killing spinors depending only on the radial coordinate \( r \) (cf. (54)), and we expect a similar behaviour also for the rotating black holes, especially we expect the Killing spinors not to depend on the angular coordinate \( \phi \), and hence to respect the identification \( \phi \sim \phi + 2\pi \), leading to cylindrical topology.

(77) can be compared with the extremality condition

\[ a^2 + q_e^2 = \frac{3 \eta^{4/3} \ell^{2/3}}{24^{1/3}}. \]  

(78)

Combining this with (77), we obtain the relation

\[ a^2 + q_e^2 = \frac{3}{22/3} q_e^{4/3} a^{2/3}. \]  

(79)

This is a homogenous equation, the solutions are therefore on straight lines \( q_e^2 = \beta^2 a^2 \), with \( \beta > 0 \). Inserting this into (79), one determines \( \beta = 2 \). Using the supersymmetry condition (77) finally yields that extremal supersymmetric rotating cylindrical black holes are parametrized by
\[ |q_e| = \frac{\ell^{1/3} \eta^{2/3}}{2^{1/6}} , \]
\[ a = \frac{|q_e|}{\sqrt{2}} = \frac{\ell^{1/3} \eta^{2/3}}{2^{2/3}} . \]  

We have obtained an interesting result: In order to get extremal supersymmetric black holes with cylindrical event horizon topology, we must allow the holes to carry angular momentum, for in the static case all the supersymmetric configurations are naked singularities. This behaviour is similar to that of the spherical Reissner-Nordström-AdS solution, for which Kostelecký and Perry showed by considering the Bogomol’nyi bound arising from the supersymmetry algebra, that solitonic black holes must be rotating \[16\].

**B. Generalization of the Higher Genus Case**

We turn now to the rotating generalizations \[14\] of the higher genus black hole space-times, i. e. to the rotating charged black membranes in AdS space. For the supercovariant derivatives, one gets

\[
\hat{\nabla}_t = \partial_t + \frac{1}{2} \omega_{t0} \gamma_{01} + \frac{1}{2} \omega_{t23} \gamma_{23} + i \frac{q_e r + q_m a \cosh \theta}{\ell \Xi \rho^2} \]
\[
+ \left( \frac{1}{2 \ell \Xi \rho} + \frac{i}{4 \Xi \rho} F_{ab} \gamma^{ab} \right) (\sqrt{\Delta_r} \gamma_{0} + \sqrt{\Delta_\theta} \rho \sinh \theta \gamma_3),
\]
\[
\hat{\nabla}_r = \partial_r + \frac{1}{2} \omega_{r0} \gamma_{03} + \frac{1}{2} \omega_{r12} \gamma_{12} + \frac{\rho}{2 \ell \sqrt{\Delta_\theta}} \gamma_1 + \frac{i \rho}{4 \sqrt{\Delta_\theta}} F_{ab} \gamma^{ab} \gamma_1,
\]
\[
\hat{\nabla}_\theta = \partial_\theta + \frac{1}{2} \omega_{\theta0} \gamma_{03} + \frac{1}{2} \omega_{\theta12} \gamma_{12} + \frac{\rho}{2 \ell \sqrt{\Delta_\theta}} \gamma_2 + \frac{i \rho}{4 \sqrt{\Delta_\theta}} F_{ab} \gamma^{ab} \gamma_2,
\]
\[
\hat{\nabla}_\phi = \partial_\phi + \frac{1}{2} \omega_{\phi0} \gamma_{01} + \frac{1}{2} \omega_{\phi12} \gamma_{12} + \frac{1}{2} \omega_{\phi}^{13} \gamma_{13} + \frac{1}{2} \omega_{\phi}^{23} \gamma_{23} + i \frac{q_e r a \sinh^2 \theta - q_m (r^2 + a^2) \cosh \theta}{\ell \Xi \rho^2} \]
\[
+ \frac{\sinh \theta}{2 \Xi \rho} \left( \frac{1}{\ell} + \frac{i}{2} F_{ab} \gamma^{ab} \right) (\sqrt{\Delta_r} a \sinh \theta \gamma_0 - \sqrt{\Delta_\theta} (r^2 + a^2) \gamma_3), \tag{81}
\]

with the electromagnetic field \( F_{ab} \) given by Eq. \[18\], and the spin connection \( \omega_{m}^{ab} \) given in appendix \[A2\]. Analogously to the previous cases, we find for the supercurvature

\[
\hat{R}_{mn} = P \mathcal{G}_{mn}(r, \theta) \mathcal{O}, \tag{82}
\]

with the idempotent operator \( P \) again given by Eq. \[73\], and

\[
\mathcal{O} = \frac{1}{\rho^2} \left\{ - \sqrt{\Delta_\theta} r a \sinh \theta \gamma_{03} - \sqrt{\Delta_r} r a \cosh \theta \gamma_{0123} + \sqrt{\Delta_r} r + \sqrt{\Delta_\theta} a^2 \sinh \theta \cosh \theta \gamma_{12} \right\} + \frac{\rho}{\ell} \gamma_1
\]
\[
+ \frac{1}{\rho^2} \left\{ -a \cosh \theta \left[ \frac{\eta}{z} (3r^2 - a^2 \cosh^2 \theta) - 2rz \right] \gamma_{0123}
\]
\[
+ \left[ \frac{\eta}{z} (r^2 - 3a^2 \cosh^2 \theta) + z (a^2 \cosh^2 \theta - r^2) \right] \mathcal{P} \right\}. \tag{83}
\]
Still $\mathcal{P}$ and $\mathcal{G}_{mn}(r, P)$ are in general nonsingular, so the integrability condition is again 
\[ \text{det } \mathcal{O} = 0. \]
The determinant reads 
\[ \text{det } \mathcal{O} = 1 \ell^4 z^4 \left[ ((a^2 + \ell^2)^2 - 4q_m^2 \ell^2)z^4 + 2\ell^2(\ell^2 - a^2)\eta^2 z^2 + \ell^4 \eta^4 + 4\ell^2 \eta^2_4 r(2z^2 - \eta r) 
+ 8a\ell^2 \eta q_m(z^2 - \eta r) \cosh \theta + 4a^2 \ell^2 \eta^2 \eta_4^2 \right]. \] (84)

We assume $z \neq 0$ in order that $\mathcal{P}$ being well-defined. Then the requirement that det $\mathcal{O}$ be vanishing identically as a function of $r$ and $\theta$, yields the conditions

\[ q_m \eta = 0, \]
\[ \eta^4 + 2 \left(1 - \frac{a^2}{\ell^2}\right) \eta^2 z^2 + \left(1 + \frac{a^2}{\ell^2}\right) - \frac{4q_m^2}{\ell^2} \right) z^4 = 0. \] (85)

The case $q_m = 0$ admits no solution, hence supersymmetry requires $\eta = 0$, from which it follows that 
\[ q_m^2 = \frac{\ell^2}{4} \left(1 + \frac{a^2}{\ell^2}\right)^2 \] (86)
for supersymmetric configurations. This yields 
\[ \Delta_r = \left[ \frac{r^2}{\ell} - \frac{\ell}{2} \left(1 - \frac{a^2}{\ell^2}\right) \right]^2 + q_m^2, \] (87)
which is a strictly positive function for $q_m^2 > 0$ and has a positive double root for vanishing electric charge and $a < \ell$. As long as there is an electric charge, these supersymmetric solutions represent a naked singularity. For $q_m = 0$ and $a < \ell$, however, we obtain a supersymmetric, magnetically charged, rotating, extremal black membrane, that has a solitonic interpretation. We observe that for $a = 0$, (86) reduces correctly to the result (46), and we have generalized this solution to a one-parameter family of extremal supersymmetric solutions.

In conclusion, we have shown that there exists also a rotating generalization of the extremal supersymmetric magnetic black hole found in section IV. Besides, we saw that supersymmetry requires $\eta = 0$, and that in order to get supersymmetric black objects, also the electric charge must vanish. It is interesting to compare this with the cylindrical topology considered in the previous subsection, where the magnetic charge was required to be zero.

**C. Revisitation of the Kerr-Newman-AdS Black Hole**

Now, we turn back to the Kerr-Newmann-AdS black hole (8), that has already been treated by Kostelecký and Perry [16], analyzing the Bogomol’nyi bound arising from the superalgebra. We shall reconsider the problem by solving the integrability condition, and show that the supersymmetry conditions are more restrictive than those found in [16]. The supercovariant derivatives read...
\[ \hat{\nabla}_t = \partial_t + \frac{1}{2} \omega^{01}_t \gamma_{01} + \frac{1}{2} \omega^{23}_t \gamma_{23} + \frac{i q_r + q_m a \cos \theta}{\ell \Xi \rho^2} + \left( \frac{1}{2 \Xi \rho} + \frac{i}{4 \Xi \rho} F_{ab} \gamma^{ab} \right) \left( \sqrt{\Delta_r} \gamma_0 + \sqrt{\Delta_{\theta a}} \sin \theta \gamma_3 \right), \]

\[ \hat{\nabla}_r = \partial_r + \frac{1}{2} \omega^{03}_r \gamma_{03} + \frac{1}{2} \omega^{12}_r \gamma_{12} + \frac{\rho}{2 \ell \sqrt{\Delta_r}} \gamma_1 + \frac{i \rho}{4 \sqrt{\Delta_r}} F_{ab} \gamma^{ab} \gamma_1, \]

\[ \hat{\nabla}_\theta = \partial_\theta + \frac{1}{2} \omega^{03}_\theta \gamma_{03} + \frac{1}{2} \omega^{12}_\theta \gamma_{12} + \frac{\rho}{2 \ell \sqrt{\Delta_\theta}} \gamma_2 + \frac{i \rho}{4 \sqrt{\Delta_\theta}} F_{ab} \gamma^{ab} \gamma_2, \]

\[ \hat{\nabla}_\phi = \partial_\phi + \frac{1}{2} \omega^{01}_\phi \gamma_{01} + \frac{1}{2} \omega^{02}_\phi \gamma_{02} + \frac{1}{2} \omega^{13}_\phi \gamma_{13} + \frac{1}{2} \omega^{23}_\phi \gamma_{23} - \frac{i q_r a \sin^2 \theta + q_m (r^2 + a^2) \cos \theta}{\ell \Xi \rho^2} \]

\[ - \frac{\sin \theta}{2 \Xi \rho} \left( \frac{1}{\ell} + \frac{i}{2} F_{ab} \gamma^{ab} \right) \left( \sqrt{\Delta_r} a \sin \theta \gamma_0 + \sqrt{\Delta_\theta} (r^2 + a^2) \gamma_3 \right), \tag{88} \]

with the electromagnetic field \( F_{ab} \) given by Eq. \( \text{(12)} \), and the spin connection \( \omega_m^{ab} \) given in Appendix A. Again, we find for the supercurvature

\[ \hat{R}_{mn} = \mathcal{P} g_{mn}(r, \theta) \mathcal{O}, \tag{89} \]

where, as usual, the idempotent operator \( \mathcal{P} \) is defined by \( \text{(73)} \), and

\[ \mathcal{O} = \frac{1}{\rho^2} \left\{ \sqrt{\Delta_r} a \sin \theta \gamma_{03} - \sqrt{\Delta_r} a \cos \theta \gamma_{0123} + \sqrt{\Delta_\theta} a^2 \sin \theta \cos \theta \gamma_{12} \right\} - \frac{\rho}{\ell} \gamma_1 \]

\[ + \frac{1}{\rho^3} \left\{ -a \cos \theta \left[ \frac{\eta}{z} (3 r^2 - a^2 \cos^2 \theta) - 2 r z \right] \gamma_{0123} \right. \]

\[ + \left[ \frac{\eta r}{z} (r^2 - 3 a^2 \cos^2 \theta) + z (a^2 \cos^2 \theta - r^2) \right] \} \mathcal{P}. \tag{90} \]

\( \mathcal{P} \) and \( g_{mn}(r, \theta) \) are in general nonsingular, so the integrability condition is again \( \det \mathcal{O} = 0 \). The determinant reads

\[ \det \mathcal{O} = \frac{1}{\ell^4 z^4} \left[ \left( \ell^2 - a^2 \right)^2 - 4 q_m^2 \ell^2 \right] z^4 - 2 \ell^2 (\ell^2 + a^2) m^2 z^2 + \ell^4 m^4 + 4 \ell^2 m q_m^2 r (2 z^2 - m r) \]

\[ + 8 a \ell^2 m q_m (z^2 - m r) \cos \theta + 4 a^2 \ell^2 m^2 q_m^2 \cos^2 \theta \]. \tag{91} \]

For \( z \neq 0 \) (which we still assume, as otherwise \( \mathcal{P} \) is ill-defined), the requirement that \( \det \mathcal{O} \) be vanishing identically as a function of \( r \) and \( \theta \), yields the conditions

\[ m q_m = 0, \]

\[ m^4 - 2 \left( 1 + a^2 \ell^2 \right) m^2 z^2 + \left( \left( 1 - a^2 \ell^2 \right)^2 - 4 q_m^2 \ell^2 \right) z^4 = 0. \tag{92} \]

To solve the integrability conditions, we have to put either \( q_m \) or \( m \) to zero. In the first case, \( q_m = 0 \), and the second condition of \( \text{(92)} \) yields

\[ m^2 = \left( 1 \pm \frac{a}{\ell} \right) q_m^2, \tag{93} \]

20
and we have electrically charged possibly supersymmetric configurations. In the limit case \( a = 0 \) we recover the usual condition \( m^2 = q^2 \).

Now one may wonder why the constraint (93) on the electric charge cannot be rotated into a similar constraint on the magnetic charge or a combination of the two by an electromagnetic duality transformation. This is a legitimate question, since dualities of this kind normally are also valid in supergravity theories [22] (see also [23] for a recent review), where a typical duality transformation is of the form

\[
\delta \hat{F}^{mn} = \frac{1}{2} ie^{-1} \lambda (\ast \hat{F})^{mn}.
\]

(94)

Here \( \lambda \) is a real parameter and \( \hat{F}^{mn} \) denotes the supercovariant field strength (30), involving also fermion fields in addition to \( F^{mn} \). (In general, also the fermions have to be transformed, cf. [23]). In fact, the Bogomol’nyi bound arising in ungauged supergravities usually involves electric and magnetic charges in a duality invariant way. Now it is clear that invariances of the kind (94) can hold only if the vector fields interact only through the field strength \( F^{mn} \) with the spinors of the theory, like it is the case e. g. in the trilinear coupling of the fourth term on the right-hand side of (25). However, if one introduces a minimal coupling of the vector fields to the fermions by the gauge potential \( A_m \) like in (29), electromagnetic duality invariance is broken, which means that gauged supergravity theories cannot have the usual duality symmetries present in the ungauged theories. Therefore one should not be surprised if the supersymmetry conditions found above break this invariance, and treat electric and magnetic charges in a different way. In our case the bosonic sector of (23) is duality invariant, hence performing a duality on the supersymmetric solution (93) we obtain again a solution of the supergravity equations; however the duality breaks the supersymmetry of the solution because the Killing spinor equation is not duality invariant, and the condition (34) does not hold anymore.

Inserting (93) into the extremality condition (13), we obtain the relation

\[
m^2 = \ell a \left( 1 + \frac{a}{\ell} \right)^4 ;
\]

(95)

the configurations that satisfy the latter equation are hence possibly supersymmetric extremal black holes, which carry electric charge and rotate. Expression (93) essentially coincides with the Bogomol’nyi bound found by Kostelecký and Perry [16], emerging from the susy algebra. However, we stress the fact that supersymmetry requires a vanishing magnetic charge in this case; the authors of [16] missed this condition in their paper. The reason for this is that the supersymmetry condition given in [16] is necessary, but not sufficient. If one derives the Bogomol’nyi bound from the superalgebra, one additionally has to satisfy the Witten equation [24] (see also [25,26]) on a three-dimensional spacelike hypersurface \( \Sigma \),

\[
^{(4)} \nabla_m \epsilon = 0,
\]

(96)

in order to assure the existence of Killing spinors. Here \( ^{(4)} \nabla_m \) is the projection into \( \Sigma \) of the four-dimensional supercovariant derivative \( \nabla_m \) (32), and \( \epsilon \) is a spinor field obeying the fall-off condition

\[
\nabla_m \epsilon = O \left( \frac{1}{r^2} \right).
\]

(97)
Now a priori it is not evident that (96) possesses a solution in the case under consideration (although a unique solution may exist in simpler cases, cf. [24,26] for a discussion), and we expect that the condition for (96) to have a solution will be just the vanishing of the magnetic charge.

Let us now return to the conditions (92). In the other case, \( m = 0 \), we obtain a supersymmetric solution with

\[ q_m^2 = \frac{\ell^2}{4} \left( 1 - \frac{a^2}{\ell^2} \right)^2, \]

which describes supersymmetric naked singularities, as can be seen from the function

\[ \Delta_r = \left[ \frac{r^2}{\ell} + \frac{\ell}{2} \left( 1 + \frac{a^2}{\ell^2} \right) \right]^2 + q_e^2. \]

This solution was not obtained in [16]; it is the spherical analogue of the rotating solitonic membrane described by (87). However, in the latter case we have an event horizon for \( q_e = 0 \), whereas for spherical topology the singularity is naked.

Summarized, we can state that in order to get extremal supersymmetric Kerr-Newman-AdS black holes, we must allow the holes to carry angular momentum, for in the static case all the supersymmetric configurations are naked singularities. Besides, the extremal supersymmetric holes carry electric charge only, making the result of Kostelecký and Perry more precise.

VI. SUMMARY AND DISCUSSION

In the present paper we considered four-dimensional asymptotically AdS dyonic black holes with various topologies in the context of gauged \( N = 2 \) supergravity. For toroidal or cylindrical topology, all static configurations preserving some amount of supersymmetry, are naked singularities, a behaviour common from the spherical Reissner-Nordström-AdS case studied previously by Romans. However, for black holes whose event horizons are Riemann surfaces of genus \( g > 1 \), we found an extremal supersymmetric black hole carrying purely magnetic charge, and admitting a one-(complex) dimensional solution space of Killing spinors. As we have seen, this solitonic object possesses also a rotating generalization, whose analogue in the Kerr-Newman-AdS case represents a naked singularity. However, for cylindrical or spherical topology, extremal supersymmetric black holes carrying only electric charge, can appear for nonvanishing angular momentum. Hence, in these cases, solitonic black holes must rotate. This is in agreement with the Kerr-Newman-AdS result of [16], emerging from considerations of the superalgebra. Yet, the authors of [16] did not obtain the condition of vanishing magnetic charge for these BPS states, so we have made more precise the Bogomol’nyi bound found in [16]. The rotating supersymmetric states with purely electric charge, appearing for cylindrical or spherical topology, have no analogue for the rotating generalizations of the higher genus solutions.

Summarized, we can state that admitting unusual black hole topologies, and allowing the holes also to carry angular momentum, can lead to a new variety of states preserving some
supersymmetry. It would be very interesting to understand the Bogomol’nyi bounds, found for unusual topology, in terms of the superalgebra. However, this requires a careful definition of the mass and angular momentum of these rotating black configurations. As such a definition is a rather delicate question [21], the mentioned enterprise becomes a nontrivial task, which we leave for future investigations.

All the metrics considered in this paper, are special cases of the most general known Petrov type D metric found by Plebanski and Demianski [27], and probably this metric can lead to other black configurations hitherto unknown. Therefore it would also be interesting to investigate, under which conditions this most general type D metric admits Killing spinors. Another future line of research would be to look for similar supersymmetric black hole solutions with unusual topology in the context of gauged $N = 4$ supergravity [28]. As was shown recently by Chamseddine [29] and Chamseddine and Volkov [30], gauged $N = 4$ SU(2) × SU(2) supergravity in four dimensions can be obtained by compactifying $N = 1$ supergravity in ten dimensions on the group manifold $S^3 \times S^3$, and hence can also be obtained by compactifying $N = 1$ supergravity in eleven dimensions, which is the low energy limit of M-theory, the most promising candidate for a theory unifying gravity with the other fundamental interactions. This connection would make possible to lift the supersymmetric black hole solutions of gauged $N = 4$ supergravity to ten or eleven dimensions, and thus to regard them as BPS solutions of string theory or M-theory. Viewed in this larger context, the black holes eventually would be accessible to a microscopic interpretation of their entropy, using the tools of string-/M-theory, like D-brane technology.

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APPENDIX A: VIERBEIN AND SPIN CONNECTION

In this section we give the choice of the vierbein and the spin connection used in this paper for the charged rotating AdS black objects. They solve the first Cartan equation $\text{d}e^a + \omega^a_{\ b} \wedge e^b = 0$. Setting $a = 0$ in these equations, one recovers the vierbein and the spin connection for the nonrotating black holes.

4In particular, for the rotating black membrane solution (14), the total mass and angular momentum are infinite. If one tries to define conserved quantities per unit brane-volume, as it is usually done for $p$-branes, they will depend on the coordinate $\theta$ on the membrane, and hence they are not constant.
1. Kerr-Newman-AdS Black Hole

One choice of the vierbein for the Kerr-Newman-AdS black hole is given by

\[ e^0 = \frac{\sqrt{\Delta_r}}{\Xi \rho} (dt - a \sin^2 \theta d\phi), \quad e^1 = \frac{\rho}{\sqrt{\Delta_r}} dr, \quad (A1) \]

\[ e^2 = \frac{\rho}{\sqrt{\Delta_\theta}} d\theta, \quad e^3 = \frac{\sqrt{\Delta_\theta} \sin \theta}{\Xi \rho} (adt - (r^2 + a^2) d\phi). \quad (A2) \]

This implies the spin connection

\[ \omega_t^{01} = \frac{1}{2\Xi \rho^4} \left[ \rho^2 \Delta_r' - 2r \Delta_r + 2a^2 r \sin^2 \theta \Delta_\theta \right], \quad (A3) \]

\[ \omega_\phi^{01} = -\frac{a \sin^2 \theta}{2\Xi \rho^4} \left[ \rho^2 \Delta_r' - 2r \Delta_r + 2r (r^2 + a^2) \Delta_\theta \right], \quad (A4) \]

\[ \omega_\phi^{02} = -\frac{a \sqrt{\Delta_r \Delta_\theta}}{\Xi \rho^2} \sin \theta \cos \theta, \quad (A5) \]

\[ \omega_r^{03} = \frac{a r \sin \theta}{\rho^2} \sqrt{\frac{\Delta_\theta}{\Delta_r}}, \quad \omega_\theta^{03} = -\frac{a \cos \theta}{\rho^2} \sqrt{\frac{\Delta_r}{\Delta_\theta}}, \quad (A6) \]

\[ \omega_r^{12} = -\frac{a^2}{\rho^2} \sqrt{\frac{\Delta_\theta}{\Delta_r}} \sin \theta \cos \theta, \quad \omega_\theta^{12} = -\frac{r}{\rho^2} \sqrt{\frac{\Delta_r}{\Delta_\theta}}, \quad (A7) \]

\[ \omega_\phi^{13} = \frac{r \sqrt{\Delta_r \Delta_\theta}}{\Xi \rho^2} \sin \theta, \quad (A8) \]

\[ \omega_t^{23} = -\frac{a}{2\Xi \rho^4} \left[ \rho^2 \Delta_\theta' \sin \theta + 2(r^2 + a^2) \Delta_\theta \cos \theta - 2a \Delta_r \cos \theta \right], \quad (A9) \]

\[ \omega_\phi^{23} = \frac{1}{2\Xi \rho^4} \left[ \rho^2 (r^2 + a^2) \Delta_\theta' \sin \theta + 2(r^2 + a^2)^2 \Delta_\theta \cos \theta - 2a^2 \Delta_r \sin^2 \theta \cos \theta \right]. \quad (A10) \]
2. Rotating Generalization of the $g > 1$ Charged Topological Black Hole

One choice of the vierbein for the rotating generalization (14) of the charged AdS black hole of genus $g > 1$ is given by

\begin{align}
e^0 &= \frac{\sqrt{\Delta_r}}{\Xi \rho} (dt + a \sinh^2 \theta d\phi), \\
e^1 &= \frac{\rho}{\sqrt{\Delta_r}} dr, \\
e^2 &= \frac{\rho}{\sqrt{\Delta_\theta}} d\theta, \\
e^3 &= \frac{\sqrt{\Delta_\theta} \sinh \theta}{\Xi \rho} (adt - (r^2 + a^2) d\phi),
\end{align}

leading to the spin connection

\begin{align}
\omega^0_1 &= \frac{1}{2 \Xi \rho^4} \left[ \rho^2 \Delta_r' - 2 r \Delta_r + 2 a^2 r \sinh^2 \theta \Delta_\theta \right], \\
\omega^0_3 &= \frac{a \sinh^2 \theta}{\Xi \rho^4} \left[ \rho^2 \Delta_r' - 2 r \Delta_r - 2 r (r^2 + a^2) \Delta_\theta \right], \\
\omega^0_2 &= \frac{a \sqrt{\Delta_r \Delta_\theta}}{\Xi \rho^2} \sinh \theta \cosh \theta, \\
\omega^{12}_r &= \frac{a^2}{\rho^2} \sqrt{\frac{\Delta_\theta}{\Delta_r}} \sinh \theta \cosh \theta, \\
\omega^{13}_\phi &= \frac{a}{\rho^2} \sqrt{\frac{\Delta_r}{\Delta_\theta}} \sinh \theta, \\
\omega^{23}_\phi &= \frac{a}{\rho^4} \left[ \rho^2 \Delta_\theta' \sinh \theta + 2 (r^2 + a^2) \Delta_\theta \cosh \theta + 2 \Delta_r \cosh \theta \right], \\
\omega^{23}_\phi &= \frac{1}{\rho^4} \left[ \rho^2 (r^2 + a^2) \Delta_\theta' \sinh \theta + 2 (r^2 + a^2)^2 \Delta_\theta \cosh \theta - 2 a^2 \Delta_r \sinh^2 \theta \cosh \theta \right].
\end{align}
3. Charged Rotating Cylindrical Black Hole

The vierbein used in this paper for the charged rotating cylindrical black hole \( [19] \) is

\[
e^0 = \frac{\sqrt{\Delta_r}}{\rho} (dt + aP^2 d\phi), \quad e^1 = \frac{\rho}{\sqrt{\Delta_r}} dr,
\]

\[
e^2 = \frac{\rho}{\sqrt{\Delta_P}} dP, \quad e^3 = \frac{\sqrt{\Delta_P}}{\rho} (adt - r^2 d\phi),
\]

which yields the spin connection

\[
\omega_{t'}^{01} = \frac{1}{2\rho^4} \left[ \rho^2 \Delta_r' - 2r \Delta_r + 2ra^2 \Delta_P \right],
\]

\[
\omega_{\phi}^{01} = \frac{1}{2\rho^4} \left[ \rho^2 aP^2 \Delta_r' - 2raP^2 \Delta_r - 2r^3 a \Delta_P \right],
\]

\[
\omega_{\phi}^{02} = \frac{aP \sqrt{\Delta_r \Delta_P}}{\rho^2},
\]

\[
\omega_r^{03} = \frac{ar}{\rho^2} \sqrt{\frac{\Delta_P}{\Delta_r}}, \quad \omega_P^{03} = \frac{aP}{\rho^2} \sqrt{\frac{\Delta_r}{\Delta_P}},
\]

\[
\omega_r^{12} = \frac{a^2 P}{\rho^2} \sqrt{\frac{\Delta_P}{\Delta_r}}, \quad \omega_P^{12} = -\frac{r}{\rho^2} \sqrt{\frac{\Delta_r}{\Delta_P}},
\]

\[
\omega_{\phi}^{13} = \frac{r \sqrt{\Delta_r \Delta_P}}{\rho^2},
\]

\[
\omega_t^{23} = -\frac{a}{2\rho^4} \left[ \rho^2 \Delta_P' - 2a^2 P \Delta_P + 2\Delta_r P \right],
\]

\[
\omega_{\phi}^{23} = \frac{1}{2\rho^4} \left[ \rho^2 r^2 \Delta_P' - 2a^2 r^2 P \Delta_P - 2a^2 P^2 \Delta_r \right].
\]
APPENDIX B: REAL REPRESENTATION OF DIRAC MATRICES

A real representation of Dirac matrices can be obtained by applying a unitary transformation $U$ to the standard representation (here denoted by $\gamma'_a$):

$$\gamma_a = U^{-1} \gamma'_a U,$$

where the transformation matrix $U$ is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{pmatrix}.$$

(The $\sigma_i$ denote the standard representation of the two-dimensional Pauli matrices, i. e. $\sigma_3 = \text{diag}(1, -1)$ etc.). Using this, one obtains

$$\gamma_0 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$\gamma_5 = \gamma_{0123} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. $$
REFERENCES

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231
[2] S. Åminneborg, I. Bengtsson, S. Holst and P. Peldán, Class. Quantum Grav. 13, 2707 (1996).
[3] R. B. Mann, Class. and Quantum Grav. 14, L109 (1997);
   R. B. Mann, gr-qc/9709033 (1997);
   R. B. Mann and W. L. Smith, Phys. Rev. D 56, 4942 (1997).
[4] J. P. S. Lemos, Class. Quantum Grav. 12, 1081 (1995);
   J. P. S. Lemos, Phys. Lett. B 353, 46 (1996);
   J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D 54, 3840 (1996).
[5] R. G. Cai and Y. Z. Zhang, Phys. Rev. D 54, 4891 (1996).
[6] C. G. Huang and C. B. Liang, Phys. Lett. A 201, 27 (1995);
   C. G. Huang, Acta Phys. Sin. 4, 617 (1996).
[7] L. Vanzo, Phys. Rev. D 56, 6475 (1997).
[8] D. Klemm and L. Vanzo, Phys. Rev. D 58, 104025 (1998).
[9] G. T. Horowitz and J. Polchinski, Phys. Rev. D 55, 6189 (1997).
[10] A. L. Larsen and N. Sánchez, Phys. Rev. D 52, 1051 (1995).
[11] A. Das and D. Z. Freedman, Nucl. Phys. B 120, 221 (1977).
[12] E. S. Fradkin and M. A. Vasiliev, Lebedev Institute preprint N 197 (1976), unpublished.
[13] K. Behrndt, A. H. Chamseddine, and W. A. Sabra, hep-th/9801857.
[14] L. J. Romans, Nucl. Phys. B 383, 395 (1992).
[15] K. P. Tod, Phys. Lett. B 121, 241 (1981).
[16] V. A. Kostelecký and M. J. Perry, Phys. Lett. B 371, 191 (1996).
[17] S. W. Hawking, Comm. Math. Phys. 25, 152 (1972).
[18] J. L. Friedman, K. Schleich and D. M. Witt, Phys. Rev. Lett. 71, 1486 (1993).
[19] D.R. Brill, J. Louko and P. Peldán, Phys. Rev. D 56, 3600 (1997).
[20] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[21] D. Klemm, V. Moretti and L. Vanzo, Phys. Rev. D 57, 6127 (1998).
[22] S. Ferrara, J. Scherk, and B. Zumino, Nucl. Phys. B 121, 393 (1977).
[23] Y. Tanii, hep-th/9802138.
[24] E. Witten, Comm. Math. Phys. 80, 381 (1981).
[25] G. W. Gibbons, C. M. Hull, and N. P. Warner, Nucl. Phys. B 218, 173 (1983).
[26] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry,
   Comm. Math. Phys. 88, 295 (1983).
[27] J. F. Plebanski and M. Demianski, Ann. of Phys. 98, 98 (1976).
[28] D. Z. Freedman and J. H. Schwarz, Nucl. Phys. B 137, 333 (1978).
[29] A. H. Chamseddine, Plenary talk given at the PASCOS-98 meeting at Northeastern University,
   Boston, March 21-29, 1998, hep-th/9806181.
[30] A. H. Chamseddine and M. S. Volkov, Phys. Rev. D 57, 6242 (1998).