ON ELLIPTIC OPERATOR PENCILS
WITH GENERAL BOUNDARY CONDITIONS

Preprint No. 37

Moscow 1999
Abstract

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In this paper operator pencils $A(x, D, \lambda)$ are investigated which depend polynomially on the parameter $\lambda$ and act on a manifold with boundary. The operator $A$ is assumed to satisfy the condition of $N$-ellipticity with parameter which is an ellipticity condition formulated with the use of the Newton polygon. We consider general boundary operators $B_1(x, D), \ldots, B_m(x, D)$ and define $N$-ellipticity for the boundary value problem $(A, B_1, \ldots, B_m)$ analogously to the Shapiro–Lopatinskii condition. It is shown that the boundary value problem is $N$-elliptic if and only if an a priori estimate holds, where the norms in the estimate are again defined in terms of the Newton polygon. These results are closely connected with singular perturbation theory and lead to uniform estimates for problems of Vishik–Lyusternik type containing a small parameter.

\footnote{Supported in part by the Deutsche Forschungsgemeinschaft and by Russian Foundation of Fundamental Research, Grant 97-01-00541}
1. Introduction

In this paper we consider an operator pencil depending polynomially on the complex parameter $\lambda$ and being of the form

$$A(x, D, \lambda) = A_{2m}(x, D) + \lambda A_{2m-1}(x, D) + \cdots + \lambda^{2m-2\mu} A_{2\mu}(x, D),$$  \hspace{1cm} (1.1)

where $m$ and $\mu$ are integer numbers with $m > \mu > 0$ and $A_{2\mu}, \ldots, A_{2m}$ are partial differential operators of the form

$$A_j(x, D) = \sum_{|\alpha| \leq j} a_{\alpha j}(x) D^\alpha \quad (j = 2\mu, 2\mu + 1, \ldots, 2m).$$  \hspace{1cm} (1.2)

We assume that the pencil (1.1) acts on a compact manifold $M$ with boundary $\partial M$; the coefficients $a_{\alpha j}$ in (1.2) are complex-valued. The manifold, its boundary and the coefficients are assumed to be infinitely smooth. In (1.2) and in the following, we write $x = (x_1, \ldots, x_n)$ and use the standard multi-index notation

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = -i \frac{\partial}{\partial x_j}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

The Dirichlet boundary value problem corresponding to the pencil (1.1) was studied in detail in the paper [5]. The aim of the present paper is to obtain an a priori estimate for general boundary value problems connected with (1.1). So we assume that we have boundary operators, for simplicity independent of the complex parameter $\lambda$, of the form

$$B_j(x, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x) D^\beta \quad (j = 1, \ldots, m),$$  \hspace{1cm} (1.3)

where the numbering is chosen such that for the orders of the operators $B_j$ we have $m_1 \leq m_2 \leq \cdots \leq m_m$. Additionally, we assume that

$$m_\mu < m_{\mu+1}.$$  \hspace{1cm} (1.4)

The coefficients of $B_j$ are supposed to be defined in $\overline{M}$ and to be infinitely smooth.

We will show that an a priori estimate holds if the boundary value problem $(A, B_1, \ldots, B_m)$ satisfies the condition of $N$-ellipticity with parameter which will be explained in Section 2. Moreover, we will prove that $N$-ellipticity is also necessary for the validity of the a priori estimate (see Section 6).
The principal symbol $A^{(0)}(x, \xi, \lambda)$ of (1.1) is defined as
\[
A^{(0)}(x, \xi, \lambda) := A^{(0)}_{2m}(x, \xi) + \lambda A^{(0)}_{2m-1}(x, \xi) + \ldots + \lambda^{2m-2\mu} A^{(0)}_{2\mu}(x, \xi),
\]
where
\[
A^{(0)}_{j}(x, \xi) := \sum_{|\alpha| = j} a_{\alpha j}(x)\xi^\alpha \quad (j = 2\mu, \ldots, 2m)
\]
stands for the principal symbol of $A_j$. In (1.4) we have set $\xi^\alpha = \xi^\alpha_1 \cdot \ldots \cdot \xi^\alpha_n$ for $\xi = (\xi_1, \ldots, \xi_n)$. The principal symbols (1.5) and (1.6) are invariant under change of coordinates and thus globally defined on the cotangent bundle $T^* M \setminus \{0\}$. The principal symbols $B^{(0)}_j$ of the boundary operators $B_j$ are defined analogously.

In (1.4) the Newton polygon approach was used to formulate and prove an a priori estimate for the Dirichlet boundary value problem. This method (which was also applied to Douglis–Nirenberg systems in (4)) turns out to be suitable for general boundary conditions, too. The concept of the Newton polygon makes it possible to define the general notion of $N$-ellipticity with parameter which is a generalization of the classical definition of ellipticity with parameter given by Agmon (1) and Agranovich–Vishik (3). For the connection to $N$-parabolic problems and Douglas–Nirenberg systems, the reader is referred to (3), Section 1.

Replacing in (1.1) $\lambda$ by $\varepsilon^{-1}$, we obtain a problem of singular perturbation theory as it was studied, for instance, by Vishik and Lyusternik (10). The a priori estimate stated below in Section 4 corresponds to a uniform (with respect to $\varepsilon$) estimate in the Vishik–Lyusternik theory (see also (6), (9)). We will show this close connection in the Appendix.

2. The Shapiro–Lopatinskii condition

As the manifold $M$ is compact we may fix a finite number of coordinate systems. Locally in each of these coordinate systems the operator pencil $A(x, D, \lambda)$ is of the form (1.1) and acts in $\mathbb{R}^n$. We can suppose without loss of generality that the coefficients of $A(x, D, \lambda)$ are (in local coordinates) of the form
\[
a_{\alpha j}(x) = a_{\alpha j} + a'_{\alpha j}(x), \quad a'_{\alpha j} \in \mathcal{D}(\mathbb{R}^n). \quad (2.1)
\]

**Definition 2.1** Let $x^0 \in \overline{M}$ be fixed. The interior symbol $A(x^0, \xi, \lambda)$ is called $N$-elliptic with parameter in $[0, \infty)$ at $x^0$ (cf. (3)) if the estimate
\[
|A^{(0)}(x^0, \xi, \lambda)| \geq C|\xi|^{2\mu} (\lambda + |\xi|)^{2m-2\mu} \quad (\xi \in \mathbb{R}^n, \lambda \in [0, \infty))
\]
(2.2)
holds with a constant $C$ which does not depend on $\xi$ or $\lambda$. If this is true for every $x^0 \in \overline{M}$, the symbol $A(x, \xi, \lambda)$ and the operator $A(x, D, \lambda)$ are called $N$-elliptic with parameter in $[0, \infty)$.

By continuity and compactness, for an $N$-elliptic operator the constant $C$ in (2.2) can be chosen independently of $x^0$.

Now we shall define the analogue of the Shapiro–Lopatinskii condition for our problem. For this, we fix a point $x^0 \in \partial M$ and a coordinate system in the neighbourhood of $x^0$ such that in this system locally the boundary $\partial M$ is given by the equation $x_n = 0$. We use in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ the coordinates $x = (x', x_n)$ and the dual coordinates $\xi = (\xi', \xi_n)$. If $A$ is $N$-elliptic with parameter, it follows from [5], Lemma 3.2, that for every $x^0 \in M$ we have

$$A^{(0)}(x^0, \xi, \lambda) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in [0, \infty)) .$$

In the case $n > 2$ this implies that $A^{(0)}$, considered as a polynomial in $\xi_n$, has exactly $m$ roots with positive imaginary part for every $\xi' \neq 0$. In the case $n = 2$ this is an additional condition which we assume to hold in the following. Similar considerations hold for $A^{(0)}_{2\mu}$.

Let $A$ be $N$-elliptic with parameter in $[0, \infty)$, fix $x^0 \in \partial M$ and write $A$ in local coordinates corresponding to $x^0$ as considered above. Then we define the polynomial in $\tau \in \mathbb{C}$

$$Q(x^0, \tau) = \tau^{-2\mu} A^{(0)}(x^0, 0, \tau, 1) .$$

**Definition 2.2** The operator $A(x, D, \lambda)$ degenerates regularly at the boundary $\partial M$ if for every $x^0 \in \partial M$ the polynomial (2.4) has exactly $m - \mu$ roots in the upper half-plane of the complex plane.

**Remark 2.3** a) It is easily seen that if for a fixed $x^0 \in \partial M$ and a fixed coordinate system polynomial (2.4) has $m - \mu$ roots in the upper half-plane, then this polynomial has this property for arbitrary $x^0 \in \partial M$ and for an arbitrary coordinate system.

b) The condition of regular degeneration has its direct counterpart in the theory of singular perturbations (see, e.g., [14], Section 6).

c) Some examples where the condition of regular degeneration (Definition 2.2) holds automatically can be found in [5], Remark 3.4.

If $A$ is $N$-elliptic with parameter in $[0, \infty)$, then for any fixed $x^0 \in \overline{M}$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, we see from (2.3) that we can factorize the principal
symbol \( A^{(0)}(x^0, \xi, \lambda) \) in the form
\[
A^{(0)}(x^0, \xi, \lambda) = A_+^{(0)}(x^0, \xi, \lambda) A_-^{(0)}(x^0, \xi, \lambda).
\]

Here
\[
A_+^{(0)}(x^0, \xi', \tau, \lambda) := \prod_{j=1}^{m} (\tau - \tau_j(x^0, \xi', \lambda)), \tag{2.5}
\]
where \( \tau_1, \ldots, \tau_m \) are the zeros of \( A^{(0)} \) with positive imaginary part.

Now let \( x^0 \in \partial M \) and denote by \( B_j^{(0)}(x^0, \xi, \lambda) \) the remainder of \( B_j^{(0)}(x^0, \xi) \) after division by \( A_+^{(0)}(x^0, \xi, \lambda) \), where all polynomials are considered as polynomials in \( \xi_n \). We write \( B_j^{(0)} \) in the form
\[
B_j^{(0)}(x^0, \xi', \xi_n, \lambda) = \sum_{k=1}^{m} b_{jk}(x^0, \xi', \lambda) \xi_n^{k-1}. \tag{2.6}
\]
and define the Lopatinskii determinant by
\[
\text{Lop}(x^0, \xi', \lambda) := \det \left( b_{jk}(x^0, \xi', \lambda) \right)_{j,k=1,\ldots,m}. \tag{2.7}
\]
Then the condition
\[
\text{Lop}(x^0, \xi', \lambda) \neq 0 \tag{2.8}
\]
means that the polynomials \( B_j^{(0)}(x^0, \xi', \cdot) \) are linearly independent modulo \( A_+^{(0)}(x^0, \xi', \cdot, \lambda) \). It is well-known that condition (2.8) is satisfied if and only if the ordinary differential equation on the half-line
\[
A(\xi', D_t, \lambda) w(t) = 0 \quad (t > 0), \tag{2.9}
\]
\[
B_k(\xi', D_t) w(t)|_{t=0} = h_k \quad (k = 1, \ldots, m), \tag{2.10}
\]
\[
w(t) \to 0 \quad (t \to +\infty),
\]
is uniquely solvable for every \( (h_1, \ldots, h_m) \in \mathbb{C}^m \). Here \( D_t \) stands for \( -i \frac{\partial}{\partial t} \).

**Definition 2.4** Let \( A \) satisfy the regular degeneration condition. Then the boundary problem \( (A, B_1, \ldots, B_m) \) is called \( N \)-elliptic with parameter \( \lambda \in [0, \infty) \) if the following conditions hold:

a) The interior symbol \( A(x, \xi, \lambda) \) is \( N \)-elliptic with parameter in \( [0, \infty) \) in the sense of Definition 2.1.
b) For every fixed \( x^0 \in \partial M \), every \( \xi' \neq 0 \) and every \( \lambda \in [0, \infty) \) the polynomials \((B_j^{(0)}(x^0, \xi', \cdot))_{j=1,\ldots,m}\) are linearly independent modulo \( A_+(x^0, \xi', \cdot, \lambda) \), i.e. (2.8) holds.

c) For every fixed \( x^0 \in \partial M \), the boundary problem 
\[
(A_{2\mu}^{(0)}(x^0, D), B_1(x^0, D), \ldots, B_\mu(x^0, D))
\]
fulfills the Shapiro–Lopatinskii condition, i.e. \((B_j^{(0)}(x^0, \xi))_{j=1,\ldots,\mu}\) are linearly independent modulo \((A_{2\mu}^{(0)})_+(x^0, \xi)\). Here \((A_{2\mu}^{(0)})_+\) is defined in analogy to (2.5) with \( A \) replaced by \( A_{2\mu} \).

d) Let \( Q_+(x^0, \tau) := \prod_{j=\mu+1}^{m}(\tau - \tau_j^{(0)}(x^0)) \) where \( \tau_\mu+1, \ldots, \tau_m \) denote the zeros of \( Q(x^0, \tau) \) with positive imaginary part. Then the polynomials \((B_j^{(0)}(x^0, 0, \tau))_{j=\mu+1,\ldots,m}\) are linearly independent modulo \( Q_+(x^0, \tau) \) for every \( x^0 \in \partial M \).

**Remark 2.5**  
a) Note that the degree of \( B_j^{(0)}(x^0, 0, \cdot) \) is \( m_j \) which may be greater than \( 2m - 2\mu \).

b) Note that condition b) in Definition 2.4 differs from the Agmon–Agranovich–Vishik condition of ellipticity with parameter. If the symbols \( A(x, \xi, \lambda) \) and \( B_j(x, \xi) \) are homogeneous with respect to \((\xi, \lambda)\), the Agmon–Agranovich–Vishik condition means that
\[
\text{Lop}(x^0, \xi', \lambda) \neq 0 \quad \text{for} \quad |\xi'|^2 + \lambda^2 = 1, \quad \lambda \geq 0.
\]
In particular, in this case the inequality (2.8) holds for \( \lambda = 1 \) and \( \xi' = 0 \). In the case of \( N\)-ellipticity, however, the Lopatinskii determinant is in general not defined for \( \xi' = 0 \) and may tend to zero as \( \xi' \to 0 \).

c) Taking in 2.4 b) \( \lambda = 0 \) and \( |\xi'| = 1 \), we obtain the standard Shapiro–Lopatinskii condition for the boundary value problem \((A_{2m}, B_1, \ldots, B_m)\).

d) In the Vishik–Lyusternik theory, the analogue of condition 2.4 d) leads to the existence of solutions of boundary layer type (see [10], Section 6).

3. The basic ODE estimate

In a first step we consider the model problem in the half space. Let \((A, B_1, \ldots, B_m)\) be of the form (1.1), (1.3) and acting in \( \mathbb{R}^n_+ \). We suppose that \( A \) is homogeneous in \((\xi, \lambda)\), i.e. has the form
\[
A(\xi, \lambda) = A_{2m}(\xi) + \lambda A_{2m-1}(\xi) + \ldots + \lambda^{2m-2\mu} A_{2\mu}(\xi),
\]

(3.1)
where $A_j(\xi)$ is a homogeneous polynomial in $\xi$ of degree $j$. Similarly we assume that $B_j$ is given by

$$B_j(\xi) = \sum_{|\beta| = m_j} b_{j\beta} \xi^\beta \quad (j = 1, \ldots, m). \quad (3.2)$$

For fixed $\lambda \geq 0$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ we investigate the boundary problem

$$A(\xi', D_t, \lambda) w_j(t) = 0 \quad (t > 0), \quad (3.3)$$
$$B_k(\xi', D_t) w_j(t)|_{t=0} = \delta_{jk} \quad (k = 1, \ldots, m), \quad (3.4)$$
$$w_j(t) \to 0 \quad (t \to +\infty).$$

In [5], the following lemma on the roots of the polynomial $A(\xi', \cdot, \lambda)$ is proved.

**Lemma 3.1** Let the polynomial $A(\xi, \lambda)$ in (3.1) be $N$-elliptic with parameter in $[0, \infty)$ and assume that $A$ degenerates regularly. Then, with a suitable numbering of the roots $\tau_j(\xi', \lambda)$ of $A(\xi', \tau, \lambda)$ with positive imaginary part, we have:

(i) Let $S(\xi') = \{\tau_0(\xi'), \ldots, \tau_\mu(\xi')\}$ be the set of all zeros of $A_{2\mu}(\xi', \tau)$ with positive imaginary part. Then for all $r > 0$ there exists a $\lambda_0 > 0$ such that the distance between the sets $\{\tau_1(\xi', \lambda), \ldots, \tau_\mu(\xi', \lambda)\}$ and $S(\xi')$ is less than $r$ for all $\xi'$ with $|\xi'| = 1$ and all $\lambda \geq \lambda_0$.

(ii) Let $\tau_{\mu+1}^1, \ldots, \tau_m^1$ be the roots of the polynomial $Q(\tau)$ (cf. (2.4)) with positive imaginary part. Then

$$\tau_j(\xi', \lambda) = \lambda \tau_j^1 + \tilde{\tau}_j(\xi', \lambda) \quad (j = \mu + 1, \ldots, m), \quad (3.5)$$

and there exist constants $K_j$ and $\lambda_1$, independent of $\xi'$ and $\lambda$, such that for $\lambda \geq \lambda_1$ the inequality

$$|\tilde{\tau}_j^1(\xi', \lambda)| \leq K_j |\xi'|^{\frac{1}{k_1}} \lambda^{1 - \frac{1}{k_1}} \quad (|\xi'| \leq \lambda) \quad (3.6)$$

holds, where $k_1$ is the maximal multiplicity of the roots of $Q(\tau)$.

**Theorem 3.2** Assume that the operator $(A, B_1, \ldots, B_m)$ is of the form (3.1) – (3.2). Assume that condition (1.4) holds and that $A$ degenerates regularly at the boundary (cf. Definition 2.2) and $(A, B_1, \ldots, B_m)$ is $N$-elliptic with parameter in $\mathbb{R}_+^n$ in the sense of Definition 2.4. Then for
every \( \xi' \in \mathbb{R}^{n-1} \setminus \{0\} \) and \( \lambda \in [0, \infty) \) the ordinary differential equation (3.3)–(3.4) has a unique solution \( w_j(t, \xi', \lambda) \), and the estimate

\[
\|D^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \leq C
\]

holds with a constant \( C \) not depending on \( \xi' \) and \( \lambda \).

**Proof.** The existence and the uniqueness of the solution follows immediately from conditions a) and b) in Definition 2.4. From the homogeneity of the symbols and from the uniqueness of the solution we see that

\[
w_j(t, \xi', \lambda) = r^{-m_j} w_j\left( rt, \frac{\xi'}{r}, \frac{\lambda}{r} \right)
\]

holds for every \( r > 0 \). If we set \( r = |\xi'| \) and \( \omega' = \frac{\xi'}{|\xi'|} \), we obtain

\[
\|D^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{l-m_j-\frac{1}{2}} \|D^l w_j\left( \cdot, \omega', \frac{\lambda}{|\xi'|} \right)\|_{L_2(\mathbb{R}_+)}.
\]

The theorem will be proved if we show that for \(|\omega'| = 1\) we have

\[
\|D^l w_j(\cdot, \omega', \Lambda)\|_{L_2(\mathbb{R}_+)} \leq C
\]

for \( \Lambda \geq 1 \) and that the left-hand side is bounded by a constant for \( \Lambda \leq 1 \).

The boundedness for \( \Lambda \leq 1 \) follows easily from conditions a) and b) of Definition 2.4. We have to consider the case of large \( \Lambda \).

To find an estimate in this case, we represent the solution in a form suggested in a paper of Frank [6]. This representation is different from the formula used for the Dirichlet boundary value problem in [3] and allows us to separate the two parts of the zeros of the polynomial \( A_+(\omega', \tau, \Lambda) \) in a more adequate form.

Due to Lemma 3.1, the roots of this polynomial consist of two groups, the first group, denoted by \( \{\tau_1(\omega', \Lambda), \ldots, \tau_m(\omega', \Lambda)\} \), being bounded for
\( \Lambda \to \infty \), the other group, denoted by \( \{ \tau_{\mu+1}(\omega', \Lambda), \ldots, \tau_m(\omega', \Lambda) \} \), being of order \( \Lambda \) for \( \Lambda \to \infty \).

We define
\[
A_1(\omega', \tau, \Lambda) := \prod_{j=1}^{\mu} (\tau - \tau_j(\omega', \Lambda)). \tag{3.11}
\]

Let \( \gamma^{(1)} \) be a contour in the upper half of the complex plane enclosing the zeros \( \tau_1, \ldots, \tau_{\mu} \). From Lemma 3.1 we see that \( \gamma^{(1)} \) can be chosen independently of \( \omega' \) and \( \Lambda \) for all \( |\omega'| = 1 \) and \( \Lambda \geq \Lambda_0 \).

From the same lemma we see that \( A_1(\omega', \tau, \Lambda) \to (A_{2\mu})_+(\omega', \tau) \) as \( \Lambda \to \infty \). Therefore we obtain from condition c) in Definition 2.4 that for all \( |\omega'| = 1 \) and \( \Lambda \geq \Lambda_0 \) and \( j = 1, \ldots, \mu \),
\[
\frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{B_k(\omega', \tau)N_j(\omega', \tau, \Lambda)}{A_1(\omega', \tau, \Lambda)} d\tau = \delta_{kj} \quad (k,j = 1, \ldots, \mu). \tag{3.12}
\]

(For the construction of \( N_j \) cf., e.g., [2], p. 634.)

Analogously, we define
\[
A_2(\omega', \tau, \Lambda) := \prod_{j=\mu+1}^{m} (\tau - \tau_j(\omega', \Lambda)). \tag{3.13}
\]

Let \( \tilde{\gamma}^{(2)}(\omega', \Lambda) \) be a contour in the upper half of the complex plane enclosing the zeros \( \tau_{\mu+1}(\omega', \Lambda), \ldots, \tau_m(\omega', \Lambda) \). From Lemma 3.1 we know that this contour is of order \( \Lambda \) for \( \Lambda \to \infty \). Therefore we may fix a contour \( \gamma^{(2)} \), independent of \( \omega' \) and \( \Lambda \) such that \( \gamma^{(2)} \) encloses all values \( \tau_j/\Lambda \) with \( j = \mu+1, \ldots, m \). We also remark that due to the regular degeneration we may choose \( \gamma^{(2)} \) with a positive distance to the real axis (cf. also (3.5)).

From condition d) in 2.4 we know that \( \{ B_j(0, \tau) \}_{j=\mu+1, \ldots, m} \) is linearly independent modulo \( Q_+(\tau) \). From Lemma 3.1 b) we know that
\[
A_2\left(\frac{\omega'}{\Lambda}, \tau, 1\right) \to Q_+(\tau) \quad (\Lambda \to \infty).
\]

Due to continuity, the polynomials \( \{ B_j(\frac{\omega'}{\Lambda}, \tau, 1) \}_{j=\mu+1, \ldots, m} \) are for sufficiently large \( \Lambda \) linearly independent modulo \( Q_+(\tau) \). Therefore there exist polynomials (in \( \tau \)) \( N_j(\omega', \tau, \Lambda) \) for \( j = \mu + 1, \ldots, m \), depending continuously on \( \omega' \) and \( \Lambda \), such that
\[
\frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{B_k(\frac{\omega'}{\Lambda}, \tau)N_j(\omega', \tau, \Lambda)}{A_2(\frac{\omega'}{\Lambda}, \tau, 1)} d\tau = \delta_{kj} \quad (k,j = \mu + 1, \ldots, m). \tag{3.14}
\]
Now we need a lemma which will be proved below.

**Lemma 3.3** The solution \( w_j(t, \omega', \Lambda) \) of the problem (3.3)–(3.4) can be represented in the form

\[
\begin{align*}
  w_j(t, \omega', \Lambda) &= \frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{M_j^{(1)}(\omega', \tau, \Lambda)}{A_1(\omega', \tau, \Lambda)} e^{it\tau} d\tau \\
  &\quad + \frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{M_j^{(2)}(\omega', \tau, \Lambda)}{A_2(\frac{\omega'}{\Lambda}, \tau, 1)} e^{it\Lambda\tau} d\tau 
\end{align*}
\]

(3.15)

where for \(|\tau| = O(1)\) and \(|\omega'| = 1\) we have

\[
M_j^{(1)}(\omega', \tau, \Lambda) \leq \begin{cases} 
  C, & j \leq \mu, \\
  \Lambda^{m_j - m_j}, & j > \mu,
\end{cases}
\]

and

\[
M_j^{(2)}(\omega', \tau, \Lambda) \leq \begin{cases} 
  \Lambda^{-\mu+1}, & j \leq \mu, \\
  \Lambda^{-m_j}, & j > \mu,
\end{cases}
\]

As a direct corollary of the lemma we obtain

\[
\| (D_1 w_j)(\cdot, \omega', \Lambda) \|_{L^2(\mathbb{R}^+)} \leq \begin{cases} 
  O(1) + O(\Lambda^{-m_j - \frac{1}{2}}), & j \leq \mu, \\
  O(\Lambda^{-m_j}) + O(\Lambda^{-m_j - \frac{1}{2}}), & j > \mu.
\end{cases}
\]

The estimate (3.10) trivially follows from these relations.

**Proof of Lemma 3.3.** Let \( w(t, \omega', \Lambda) \) be a solution of the problem (3.3)–(3.4) with \( \delta_{jk} \) replaced by \( \phi = (\phi_1, \ldots, \phi_m) \in \mathbb{C}^m \). We seek the solution in the form

\[
\begin{align*}
  w(t, \omega', \Lambda) &= \sum_{k=1}^{\mu} \psi_k(\omega', \Lambda) \frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{N_k(\omega', \tau, \Lambda)}{A_1(\omega', \tau, \Lambda)} e^{it\tau} d\tau \\
  &\quad + \sum_{k=\mu+1}^{m} \psi_k(\omega', \Lambda) \frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{N_k(\omega', \tau, \Lambda)}{A_2(\frac{\omega'}{\Lambda}, \tau, 1)} e^{it\Lambda\tau} d\tau
\end{align*}
\]

(3.16)

where the functions \( \psi_k \) still have to be found.

Applying the boundary operator \( B_1(\xi', D_1) \) to both sides of (3.14) and taking \( t = 0 \) we obtain the following system for the unknown functions \( \psi_k(\omega', \Lambda) \):
\[ \psi_l(\omega', \Lambda) + \Lambda^m \sum_{k=\mu+1}^m \psi_k(\omega', \Lambda) h_{lk}(\omega', \Lambda) = \phi_l \]

\[ (l = 1, \ldots, \mu), \quad (3.17) \]

\[ \sum_{k=1}^\mu \psi_k(\omega', \Lambda) h_{lk}(\omega', \Lambda) + \Lambda^m \psi_l(\omega', \Lambda) = \phi_l \]

\[ (l = \mu + 1, \ldots, m). \quad (3.18) \]

Here we have set

\[ h_{lk}(\omega', \Lambda) = \frac{1}{2\pi i} \int_{\gamma(2)} \frac{B_l(\omega', \tau) N_k(\omega', \Lambda, \tau)}{A_2(\omega', \Lambda, \tau, 1)} d\tau \]

\[ (l = 1, \ldots, \mu; \; k = \mu + 1, \ldots, m), \quad (3.19) \]

\[ h_{lk}(\omega', \Lambda) = \frac{1}{2\pi i} \int_{\gamma(1)} \frac{B_l(\omega', \tau) N_k(\omega', \Lambda, \tau)}{A_1(\omega', \Lambda, \tau)} d\tau \]

\[ (l = \mu + 1, \ldots, m; \; k = 1, \ldots, \mu). \quad (3.20) \]

We remark that we have used \( B_l(\omega', \Lambda) = \Lambda^m B_l(\omega', \Lambda). \)

Now we write \( \psi = (\psi', \psi'') \), where \( \psi' \) consists of the first \( \mu \) components of the vector \( \psi \), and \( \psi'' \) consists of the other \( m - \mu \) components. In the same way we write \( \phi = (\phi', \phi'') \). In these notations the system (3.17)–(3.18) can be rewritten in the form

\[ \psi' + \Delta_1 H_{12} \psi'' = \phi', \]

\[ H_{21} \psi' + \Delta_2 \psi'' = \phi'', \]

where we use the notation

\[ \Delta_1 := \begin{pmatrix} \Lambda^{m_1} & \ddots & \Lambda^{m_\mu} \\ \vdots & \ddots & \vdots \\ \Lambda^{m_\mu} & \ddots & \Lambda^{m_m} \end{pmatrix}, \quad \Delta_2 := \begin{pmatrix} \Lambda^{m_{\mu+1}} & \ddots & \Lambda^{m_m} \\ \vdots & \ddots & \vdots \\ \Lambda^{m_m} & \ddots & \Lambda^{m_m} \end{pmatrix} \]

and

\[ H_{12} := \left( h_{lk} \right)_{k=\mu+1, \ldots, m}^{l=1, \ldots, \mu}, \quad H_{21} := \left( h_{lk} \right)_{l=\mu+1, \ldots, m}^{k=1, \ldots, \mu}. \]

If we multiply the second equation by the matrix \( \Delta_1 H_{12} \Delta_2^{-1} \) from the left and subtract it from the first equation we obtain

\[ (I - \Delta_1 H_{12} \Delta_2^{-1} H_{12}) \psi' = \phi' - \Delta_1 H_{12} \Delta_2^{-1} \phi''. \]
In a similar way we obtain

\[(I - \Delta_2^{-1} H_{21} \Delta_1 H_{12})\psi'' = -\Delta_2^{-1} H_{21} \phi' + \Delta_2^{-1} \phi''\,.
\]
The matrices in brackets in the left-hand sides of above relations differ from the identity by matrices whose elements can be estimated by a constant times \(\Lambda^{m_\mu - m_{\mu+1}}\). According to (1.4), their norms tend to zero as \(\Lambda \to \infty\). From this it follows that the matrices in brackets for large \(\Lambda\) have inverses which we denote by \(G_1\) and \(G_2\), respectively. Then we obtain

\[
\psi' = G_1 \phi' - G_1 \Delta_1 H_{12} \Delta_2^{-1} \phi'', \quad \psi'' = -G_2 \Delta_2^{-1} H_{21} \phi' + G_2 \Delta_2^{-1} \phi''.
\]

If we take \(\phi = e_j\) \((1 \leq j \leq \mu)\), where \(e_j\) stands for the \(j\)-th unit vector, and denote by \(e'_j\) the first \(\mu\) components of \(e_j\), we obtain

\[
\psi'_{(j)} = G_1 e'_j, \quad \psi''_{(j)} = -G_2 \Delta_2^{-1} H_{21} e'_j.
\]

In the same way if \(j > \mu\) and \(e''_{(j)}\) denotes the components \(\mu + 1, \ldots, m\) of \(e_j\), we obtain

\[
\psi'_{(j)} = -G_1 \Delta_1 H_{12} \Lambda^{-m_j} e''_{(j)}, \quad \psi''_{(j)} = G_2 \Lambda^{-m_j} e''_{(j)}.
\]

The statement of the lemma directly follows from these relations. \(\square\)

4. A priori estimates

Theorem 3.2 is the key result for proving a priori estimates. The norms used in these estimates are based on the Newton polygon \(N_{r,s}\) (cf. Fig. 1) defined for \(r > s \geq 0\) as the convex hull of the set

\[
\{(0, 0), (0, r-s), (s, r-s), (r, 0)\}
\]

The weight function \(\Xi_{r,s}(\xi, \lambda)\) is defined by

\[
\Xi_{r,s}(\xi, \lambda) := \sum_{(i, k) \in N_{r,s}} |\xi|^i |\lambda|^k,
\]

where the summation on the right-hand side is extended over all integer points of \(N_{r,s}\). For a discussion of general Newton polygons we refer the reader to [4], [5], [7].

It is easily seen that we have the equivalence

\[
\Xi_{r,s}(\xi, \lambda) \approx (1 + |\xi|)^r (\lambda + |\xi|)^{-s}.
\]
The sign \( \approx \) means that the quotient of the left-hand and the right-hand side is bounded from below and from above by positive constants independent of \( \xi \) and \( \lambda \). Taking the right-hand side of (4.2) as a definition, we may define \( \Xi_{r,s} \) for every \( r, s \in \mathbb{R} \). The Sobolev space \( H^{(r,s)}(\mathbb{R}^n) := H^{\Xi_{r,s}}(\mathbb{R}^n) \) is defined as

\[
\{ u \in \mathcal{S}'(\mathbb{R}^n) : \Xi_{r,s}(\xi, \lambda) Fu(\xi) \in L^2(\mathbb{R}^n) \}
\]

with the norm

\[
\| u \|_{(r,s), \mathbb{R}^n} := \| u \|_{\Xi_{r,s}, \mathbb{R}^n} := \| F^{-1} \Xi_{r,s}(\xi, \lambda) Fu(\xi) \|_{L^2(\mathbb{R}^n)}
\] (4.3)

Here \( Fu \) stands for the Fourier transform of \( u \) and \( \mathcal{S}'(\mathbb{R}^n) \) denotes the space of all tempered distributions. The space \( H^{\Xi_{r,s-1/2}}(\mathbb{R}^{n-1}) \) is defined analogously with the weight function \( \Xi_{r,s}(\xi', \lambda) := \Xi_{r,s}(\xi', 0, \lambda) \). These spaces can be defined on the half-space \( \mathbb{R}^n_+ \) in accordance with the general theory of Sobolev spaces with weight functions as it can be found, e.g., in [11]. On the manifold \( M \) and the boundary \( \partial M \), the spaces \( H^{\Xi_{r,s}}(M) \) and \( H^{\Xi_{r,s}}(\partial M) \), respectively, are defined in the usual way, using a partition of unity.

In [5], Section 2, Sobolev spaces connected with Newton polygons were investigated in detail. In particular, for \( r \in \mathbb{N} \) it was shown that \( (\frac{\partial}{\partial \nu})^j : u \mapsto (\frac{\partial}{\partial \nu})^j u|_{\partial M} \) acts continuously from \( H^{\Xi_{r,s}}(M) \) to \( H^{\Xi_{r-s-j-1/2}}(\partial M) \) for \( j = 0, \ldots, r-1 \). Here \( \Xi_{r,s}^{(-j-1/2)} \) denotes the weight function corresponding to the Newton polygon which is constructed from \( N_{r,s} \) by a shift of length \( j + 1/2 \) to the left parallel to the abscissa.
While in the basic Sobolev space was $H^\Xi_{m,\mu}$, we here consider more general a priori estimates. In the following, we fix integer numbers

$$r \geq m_m + 1 \quad \text{and} \quad m_\mu + 1 \leq s \leq m_\mu + 1$$  \hspace{1cm} (4.4)

and consider the Newton polygon $N_{r,s}$, its weight function $\Xi := \Xi_{r,s}$ and the corresponding Sobolev space. We remark that for the Dirichlet problem the values $r = m$ and $s = \mu$ used in $\Xi$ are included as an example.

Analogously to (4.1), we define the function $\Phi = \Phi_{r,s}$ by

$$\Phi(\xi, \lambda) := \sum_{i,k} |\xi|^i \lambda^k,$$  \hspace{1cm} (4.5)

where the sum runs over all integer points $(i, k)$ belonging to the side of $N_{r,s}$ which is not parallel to one of the coordinate lines. This means that we have

$$\Phi(\xi, \lambda) \approx |\xi|^s (\lambda + |\xi|)^{-s}.$$  \hspace{1cm} (4.6)

By $\Phi(-l)$ we again denote the corresponding function for the shifted Newton polygon. From Theorem 3.2 we obtain the following estimate for the fundamental solution $w_j$ defined in (3.3)–(3.4):

**Lemma 4.1** For the solution $w_j(t, \xi', \lambda)$ considered in Theorem 3.2 we have the estimate

$$\|D^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \leq C \frac{\Phi(-m_j - 1/2)(\xi', \lambda)}{\Phi(-l)(\xi', \lambda)}.$$  \hspace{1cm} (4.7)

**Proof.** To see this, we only have to remark that the right-hand side of (4.7) is equivalent to

$$\begin{cases}
|\xi'|^{l - m_j - \frac{1}{2}}, & j \leq \mu, \ l \leq s, \\
|\xi'|^{s - m_j - \frac{1}{2}(\lambda + |\xi'|)^{-s}}, & j \leq \mu, \ l > s, \\
|\xi'|^{l - s}(\lambda + |\xi'|)^{-s - m_j - \frac{1}{2}}, & j > \mu, \ l \leq s, \\
(\lambda + |\xi'|)^{l - m_j - \frac{1}{2}}, & j > \mu, \ l > s.
\end{cases}$$

The first and fourth lines above coincide with the corresponding lines in the right-hand side of (3.7). The ratio of the second line in (3.7) and the second line above is equal to

$$\left(\frac{|\xi'|}{\lambda + |\xi'|} \right)^{m_\mu + 1 - s + 1/2}.$$
Respectively, the ratio of the third line in (3.7) and the third line above is equal to
\[ \left( \frac{\lvert \xi' \rvert}{\lambda + \lvert \xi \rvert} \right)^{s-m-1/2}. \]
Now our statement follows from (4.4).

\[ \square \]

**Theorem 4.2** Let \( A(x, D, \lambda) \) be an operator pencil of the form (1.1), acting on the manifold \( M \) with boundary \( \partial M \). Let \( B_j(x, D) \), \( j = 1, \ldots, m \), be boundary operators of the form (1.3). Assume that \( A \) degenerates regularly at the boundary and that \( (A, B_1, \ldots, B_m) \) is \( N \)-elliptic with parameter in the sense of Definition 2.4. Set \( \Xi = \Xi_{r,s} \) with \( r \) and \( s \) satisfying (4.4). For simplicity, assume that \( r \) and \( s \) are integers. Then for \( \lambda \geq \lambda_0 \) there exists a constant \( C = C(\lambda_0) \), independent of \( u \) and \( \lambda \), such that
\[ \| u \|_{\Xi, M} \leq C \left( \| A(x, D, \lambda) u \|_{(r-2m, s-2\mu), M} \right. \]
\[ \left. + \sum_{j=1}^{m} \| B_j(x, D) u \|_{\Xi(-m_j-1/2), \partial M} + \lambda^{r-s} \| u \|_{L^2(M)} \right). \] (4.8)

**Proof.** The proof of this theorem is similar to the proof of Theorem 5.6 in [5], therefore we only indicate the main steps.

By the localization method (“freezing the coefficients”), it is possible to reduce the proof to the proof of the corresponding results for model problems in \( \mathbb{R}^n \) and \( \mathbb{R}^n_{+} \). The case of the whole space \( \mathbb{R}^n \) is quite elementary and needs only slight changes in comparison with [5], Proposition 5.2. The key result is the a priori estimate in the half space \( \mathbb{R}^n_{+} \).

So we assume that \( u \in H^\Xi(\mathbb{R}^n_{+}) \) is a solution of
\[ A(D, \lambda) u(x) = f \quad \text{in} \; \mathbb{R}^n_{+}, \]
\[ B_j(D) u|_{x_n=0} = g_j \quad (j = 1, \ldots, m) \quad \text{on} \; \mathbb{R}^{n-1}. \] (4.9) (4.10)

Let \( \psi \in C^\infty(\mathbb{R}^n) \) be a cut-off function, i.e. \( \psi(\xi) = 1 \) for \( \lvert \xi \rvert \leq 1 \) and \( \psi(\xi) = 0 \) for \( \lvert \xi \rvert \geq 2 \). As in [5] we write \( u \) in the form
\[ u = u_1 + u_2 + v = R \psi(D) E u + R(1 - \psi(D)) A^{-1}(D, \lambda) Ef + v. \] (4.11)

Here we fixed an extension operator \( E \) from \( \mathbb{R}^n_{+} \) to \( \mathbb{R}^n \), being continuous from \( L^2(\mathbb{R}^n_{+}) \) to \( L^2(\mathbb{R}^n) \) and from \( H^\Xi(\mathbb{R}^n_{+}) \) to \( H^\Xi(\mathbb{R}^n) \); we define the distribution \( Ef \) as \( A(D, \lambda) Ef \). By \( R \) we denote the operator of restriction of functions on \( \mathbb{R}^n \) onto \( \mathbb{R}^n_{+} \). In [11], the pseudodifferential operator (ps.d.o.) \( \psi(D) \) in \( \mathbb{R}^n \) is defined as
\[ \psi(D) := F^{-1} \psi(\xi) F. \]
It is easily seen that we have

\[ \|u_1\|_{\Xi, R_n^+} + \|u_2\|_{\Xi, R_n^+} \leq C \left( \|f\|_{(r-2m,s-2\mu), R_n^+} + \lambda^{r-s} \|u\|_{L_2(\mathbb{R}_n^+)} \right). \quad (4.12) \]

We still have to estimate \( v \) defined in (4.11). By definition, \( v \) is a solution of

\[ A(D, \lambda) v = 0 \quad \text{in} \quad \mathbb{R}_n^+, \]

\[ B_j(D)v|_{x_n=0} = h_j \quad (j = 1, \ldots, m) \quad \text{on} \quad \mathbb{R}^{n-1}, \quad (4.14) \]

where we set \( h_j(x') := B_j(D)u(x', 0) - B_j(D)u_1(x', 0) - B_j(D)u_2(x', 0). \)

Now we use the fact that

\[ \|v\|_{\Xi, R_n^+} \approx \|v\|_{\Phi, R_n^+} + \lambda^{r-s}\|v\|_{L_2(\mathbb{R}_n^+)} \]

As

\[ \lambda^{r-s}\|v\|_{L_2(\mathbb{R}_n^+)} \leq \lambda^{r-s}\|u_1\|_{\Xi, R_n^+} + \|u_2\|_{\Xi, R_n^+}, \]

it is sufficient to estimate \( \|v\|_{\Phi, R_n^+} \).

It can be seen, using the binomial formula, that

\[ (\Phi(\xi, \lambda))^2 \approx \sum_{l=0}^r (\xi^l_n \Xi(-l)(\xi', \lambda))^2, \]

and therefore the semi-norm \( \|v\|_{\Phi, R_n^+} \) is equivalent to

\[ \left[ \sum_{l=0}^r \int_0^{\infty} \|D_n^l v|_{\cdot, x_n}\|_{\Phi, -l, \mathbb{R}^{n-1}}^2 \, dx_n \right]^{1/2}. \quad (4.15) \]

Taking the partial Fourier transform \( F' \) with respect to \( x' \in \mathbb{R}^{n-1} \), we obtain from (4.13)–(4.14) that for \( \xi' \neq 0 \) the function \( w := F'v \) is a solution of

\[ A(\xi', D_n, \lambda)w(x_n) = 0, \]

\[ B_j(\xi', D_n)w(x_n)|_{x_n=0} = (F'h_j)(\xi'). \quad (4.17) \]

Due to Theorem 3.2, this solution is unique and given by

\[ w = w(x_n, \xi', \lambda) = \sum_{j=1}^m w_j(x_n, \xi', \lambda)(F'h_j)(\xi'). \quad (4.18) \]
with \( w_j(x_n, \xi', \lambda) \) being the solution of (3.3)–(3.4). Now we can apply Lemma 4.1 to obtain
\[
(\Phi^{(-l)}(\xi', \lambda))^2 \int_0^\infty |D^l_n w(x_n, \xi', \lambda)|^2 \, dx_n
\leq C \sum |\Xi^{(-m_j - \frac{1}{2})}(\xi', \lambda)(F^j h_j)(\xi')|^2.
\]
Integrating this inequality with respect to \( \xi' \) and using the norm (4.15) we get the desired estimate for \( \|v\|_{\Phi, \mathbb{R}^n_+} \), which finishes the proof of the theorem. \( \square \)

5. The parametrix construction

In this section, we will construct a right (rough) parametrix for the operator \( (A, B) = (A, B_1, \ldots, B_m) \), generalizing the result of [5]. We restrict ourselves to the construction of local parametrices in \( \mathbb{R}^n \) and \( \mathbb{R}^n_+ \); after this the definition of the parametrix on the manifold is standard (cf. also [5] for the Dirichlet problem).

For the remainder of this section, we fix integer numbers \( r \) and \( s \) satisfying (4.4) and set \( \Xi = \Xi_{r,s} \). The following result is a slight generalization of [5], Proposition 6.1, which can be proved literally in the same way.

**Lemma 5.1** Let \( A(x, D, \lambda) \) in (2.1) be \( N \)-elliptic in \( \mathbb{R}^n \) with coefficients of the form (2.1). Then there exists a bounded operator
\[
P_0 : H^{(r-2m,s-2\mu)}(\mathbb{R}^n) \to H^{(r,s)}(\mathbb{R}^n)
\]
such that
\[
AP_0 = I + T
\]
where \( I \) denotes the identity operator in \( H^{(r-2m,s-2\mu)}(\mathbb{R}^n) \) and
\[
T : H^{\Theta}(\mathbb{R}^n) \to H^{(r-2m,s-2\mu)}(\mathbb{R}^n)
\]
is bounded. Here we have set
\[
\Theta(\xi, \lambda) := \Xi_{r-2m+1, s-2\mu+1}(\xi, \lambda) = (1 + |\xi|) \Xi_{r-2m, s-2\mu}(\xi, \lambda). \quad (5.1)
\]

Throughout this section, by a bounded operator we understand a continuous operator with norm bounded by a constant independent of \( \lambda \).
To define a parametrix, we use a cut-off function $\psi_{A,B}$ of the form (2.1) and that $(\Xi)$ acts in the half space $\mathbb{R}^n_+$ with coefficients of the form (2.1) and that $(A,B)$ is $N$-elliptic in the sense of Definition 2.4.

Due to Lemma 3.3, for large $\lambda$, $w_{j}$ is equivalent norm $C$ for some function $w_{j}$ we define the ps.d.o. $\psi_{A,B}$ with

$$\psi_{A,B}(\xi') = \begin{cases} 0, & |\xi'| \leq 1, \\ 1, & |\xi'| \geq 2. \end{cases}$$

For $j = 1, \ldots, m$ we define the ps.d.o. $P_{j}$ in $\mathbb{R}^{n-1}$ (with $x_n$ as parameter) by

$$(P_{j}g)(x',x_n) := \psi'(D') w_{j}(x',x_n,D',\lambda) g,$$

where $w_{j}(x',x_n,\xi',\lambda)$ is the unique solution of (3.3)–(3.4) with

$$A(\xi',D_t,\lambda) = A^{(0)}(x',0,\xi',D_t,\lambda),$$

$$B_{k}(\xi',D_t) = B^{(0)}_{k}(x',0,\xi',D_t).$$

Due to Lemma 3.3, for large $\lambda$ the symbol of $w_{j}(x',x_n,D',\lambda)$ can be written in the form

$$w_{j}(x',x_n,\xi',\lambda) = \frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{M_{1}^{(1)}(x',\xi',\tau,\lambda)}{A^{(1)}(x',\xi',\tau,\lambda)} e^{ix_n \tau} d\tau + \frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{M_{1}^{(2)}(x',\xi',\tau,\lambda)}{A^{(2)}(x',\xi'/\lambda,\tau,1)} e^{ix_n \lambda \tau} d\tau. \quad (5.4)$$

**Lemma 5.2** The operator $P_{j}$ defined in (5.2) is continuous from $H^{m_{j}}(\mathbb{R}^{n-1})$ to $H^{m_{j}}(\mathbb{R}^{n}_{+})$.

**Proof.** Let $g \in H^{m_{j}}(\mathbb{R}^{n-1})$ and set $u := P_{j}g$. Using the equivalent norm

$$\left[ \sum_{l=0}^{r} \int_{0}^{\infty} \|D^{l}_{n} u(\cdot,x_n)\|_{\Xi^{(-l)}(\mathbb{R}^{n-1})}^{2} dx_n \right]^{1/2} \quad (5.5)$$

in $H^{m_{j}}(\mathbb{R}^{n}_{+})$, we see that we have to show that

$$\left\| \Xi^{(-l)}(D',\lambda)D^{l}_{n} P_{j} \Xi^{(-m_{j}+1/2)}(D',\lambda) \right\|_{L^{2}(\mathbb{R}^{n-1}) \rightarrow L^{2}(\mathbb{R}^{n-1})} \leq C(x_n)$$

for some function $C = C(x_n)$ whose $L^{2}(\mathbb{R}_{+})$-norm is bounded by a constant independent of $\lambda$. For this it is sufficient to show that for $|\xi'| \geq 1$ we have

$$\left( \int_{0}^{\infty} |D^{l}_{n} w_{j}(x',x_n,\xi',\lambda)|^{2} dx_n \right)^{1/2} \leq C \frac{\Xi^{(-m_{j}+1/2)}(\xi',\lambda)}{\Xi^{(-l)}(\xi',\lambda)}.$$
As we have for $|\xi'| \geq 1$ the equivalence

$$\Xi_{r,s}(\xi', \lambda) \approx \Phi_{r,s}(\xi', \lambda)$$

for all $r, s \in \mathbb{R}$ (with $\Phi_{r,s}$ defined by the right-hand side of (1.4)), the case $\alpha' = 0$ is already covered by Lemma 4.1. Here we take into account that, due to condition (2.1), the constant $C$ in Lemma 4.1 applied to the symbols (5.3) may be chosen independently of $x' \in \mathbb{R}^{n-1}$.

The case $\alpha' > 0$ follows after differentiation of (5.4) with respect to $x'$ along the same lines as in the proof of Lemma 3.3.

□

**Lemma 5.3** The operator

$$C_j := A(x, D, \lambda) P_j : H^{\Xi^{(\alpha', l) - 1/2}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+)$$

is bounded. Here $\Theta(\xi, \lambda)$ is defined in (5.1).

**Proof.** The symbol of the ps.d.o. $C_j$ in $\mathbb{R}^{n-1}$ with parameter $x_n$ is given by

$$\sum_{|\alpha'| = 1}^{2m} \frac{1}{(\alpha')!} \partial_{\xi'}^\alpha A(x, \xi', D_n, \lambda) D_x^\alpha D_x^\alpha P_j(x', x_n, \xi', \lambda)$$

with

$$P_j(x, \xi', \lambda) = \psi(\xi') w_j(x, \xi', \lambda).$$

Consider the family $\mathcal{F} = \{A(x, \xi, \lambda) : x \in \mathbb{R}^n_+\}$ of polynomials in $(\xi, \lambda) \in \mathbb{R}^{n+1}$. As the degree of the polynomial $A(x, \cdot)$ is equal to $2m$ for all $x \in \mathbb{R}^n_+$, the family $\mathcal{F}$ is a subset of the finite-dimensional vector space of all polynomials in $(\xi, \lambda)$ of degree not greater than $2m$. Therefore, there exists a finite set $x^{(1)}, \ldots, x^{(K)} \in \mathbb{R}^n_+$ such that every $A \in \mathcal{F}$ may be represented in the form

$$A(x, \xi, \lambda) = \sum_{k=1}^{K} c_k(x) A(x^{(k)}, \xi, \lambda)$$

with smooth coefficients $c_k(x)$.

Taking into account that the operators of multiplication by $c_k(x)$ are bounded in $H^{\Theta}(\mathbb{R}^n_+)$, we reduce our problem to the proof of the boundedness of operators of the form

$$C_{\alpha', l} := a_{\alpha', l}(D', \lambda) D_n^l D_x^\alpha P_j(x, D', \lambda) : H^{\Xi^{(\alpha', l) - 1/2}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+),$$
where
\[
|a_{\alpha',l}(\xi', \lambda)| \leq C \Xi^{\frac{-1}{2m} - \frac{1}{2\mu}}(\xi', \lambda).
\] (5.6)

Literally repeating the proof of Lemma 5.2 we establish the boundedness of the operator
\[
D_{x', p}(x, D, \lambda) : H^\Xi(R^{n+1}) \rightarrow H^\Xi(R^n).
\]
According to (5.6) the operator
\[
a_{\alpha',l}(D', \lambda) : H^\Xi(R^n) \rightarrow H^\Theta(R^n)
\]
is bounded. As $C_{\alpha', l}$ is the product of the above operators this operator is also bounded. □

**Theorem 5.4** Consider in the half space $R^n_+$ the boundary value problem \((A, B) = (A_1, B_1, \ldots, B_m)\) of the form (1.1), (1.3) with coefficients of the form (2.1). Assume that $A$ degenerates regularly at the boundary and that \((A, B)\) is $N$-elliptic with parameter in $[0, \infty)$ in the sense of Definition 2.4. Then there exists a bounded operator $P : H^{r-2m, s-2\mu}(R^n_+) \times \prod_{j=1}^m H^\Xi(-m_j, \frac{1}{2})(R^{n-1}) \rightarrow H^\Xi(R^n_+)$ such that
\[
(A, B)P = I + T
\]
where $I$ stands for the identity operator in the space
\[
H^{r-2m, s-2\mu}(R^n_+) \times \prod_{j=1}^m H^\Xi(-m_j, \frac{1}{2})(R^{n-1})
\]
and $T$ is a continuous operator from the space (5.7) to the space
\[
H^\Theta(R^n_+) \times \prod_{j=1}^m H^\Xi(-m_j, \frac{1}{2})(R^{n-1})
\]
with $\Theta(\xi, \lambda)$ being defined in (5.1).

**Proof.** We define
\[
P(f, g_1, \ldots, g_m) := P_0 f + \sum_{j=1}^m P_j (g_j - B_j P_0 f)
\]
with $P_0$ from Lemma 5.1 and $P_j$ given by (5.2). The continuity of $P$ follows from Lemma 5.1 and Lemma 5.2. In order to see that the operator $T$ is continuous with respect to the spaces given in the theorem, we denote the components of $T$ by $T_0, T_1, \ldots, T_m$. The operator $T_0$ is given by

$$T_0(f, g_1, \ldots, g_m) = A P_0 f - f + \sum_{j=1}^{m} A P_j (g_j - B_j P_0 f).$$

We see from Lemma 5.1 and Lemma 5.3 that $T_0$ maps the space (5.7) continuously into $H^0(\mathbb{R}^n_+)$. Turning to the other components $T_1, \ldots, T_m$, we remark that for $j, k = 1, \ldots, m$ the operator $B_k P_j$ equals $\delta_{kj} I$ up to operators of lower order. More precisely, the operator

$$B_k(x, D) P_j - \delta_{kj} I$$

is a ps.d.o. in $\mathbb{R}^{n-1}$ which is continuous from

$$H^{\xi(-m_j, -1/2)}(\mathbb{R}^{n-1})$$

to

$$H^{\xi(-m_k, 1/2)}(\mathbb{R}^{n-1}).$$

This is due to the fact that $w_j(x', x_n, \xi', \lambda)$ satisfies (3.3)–(3.4); the estimates for the lower order terms of the ps.d.o. $B_k P_j$ can be found in the same way as it was done for $A P_j$ in the proof of Lemma 5.3. From the continuity of $B_k P_j - \delta_{kj} I$ the continuity of $T_k$ in the spaces given in the theorem immediately follows. $\Box$

6. Proof of the necessity

The aim of this section is to prove the following theorem.

**Theorem 6.1** Let $A$ degenerate regularly at the boundary $\partial M$ and assume that inequality (1.4) holds. Let $r$ and $s$ be integers satisfying (4.4) and assume, in addition, that

$$r \geq m \quad \text{and} \quad \mu \leq s \leq r - m + \mu. \quad (6.1)$$

If the a priori estimate (4.8) holds, then $(A, B_1, \ldots, B_m)$ is $N$-elliptic with parameter in the sense of Definition 2.4.

Note that if the orders $m_j$ of the boundary operators $B_j$ are all different (e.g., if the boundary operators are normal), then (6.1) is satisfied.
and (6.1) is a consequence of (4.4). The proof of this theorem is divided into several steps.

Necessity of condition 2.4 a). First of all note that applying estimate (4.8) to functions whose support does not intersect with the boundary, we obtain the estimate in \( \mathbb{R}^n \)

\[
\|u\|_{r,s} \leq C \left( \|A(x,D,\lambda)u\|_{r-2m,s-2\mu} + \lambda^{r-s}\|u\|_{L^2} \right) \quad (6.2)
\]

**Proposition 6.2** Suppose that (6.2) takes place and \( x^0 \) is an arbitrary point of \( \mathbb{R}^n \). Denote \( A(D,\lambda) = A^0(x^0,D,\lambda) \). Then the estimate

\[
\left\| D^{{s}(|D| + \lambda)^{r-s}}u \right\|_{L^2} \leq C \left\| D^{s-2\mu}(|D| + \lambda)^{r-s-2m+2\mu}A(D,\lambda)u \right\|_{L^2} \quad (6.3)
\]

holds, where the constant \( C \) does not depend on \( x^0 \) or \( \lambda \).

The necessity of a) easily follows from (6.3). Indeed, applying the Fourier transform, we can rewrite (6.3) in the form

\[
\int_{\mathbb{R}^n} \left[ \xi \right]^{2s}(|\xi| + \lambda)^2(r-s) \\left( -C^2 \xi^{2(s-2\mu)}(|\xi| + \lambda)^2(r-s-2m+2\mu)|A(\xi,\lambda)|^2 \right) \left| (Fu)(\xi) \right|^2 d\xi \leq 0 .
\]

Since \( u \in \mathcal{D} \) is arbitrary, the expression in the square brackets is nonpositive. From this part a) follows.

To prove (6.3) we replace in (6.2) \( \lambda \) by \( \rho \lambda \) with \( \rho > 0 \) and \( u(x) \) by

\[
u_\rho(x) = \rho^{-r+n/2}u(\rho(x - x^0)) \quad (6.4)
\]

and tend \( \rho \) to \( +\infty \). To carry out the calculations we need the following

**Lemma 6.3** Denote

\[
(S_{\rho,x^0}u)(x) = u(\rho(x - x^0)) . \quad (6.5)
\]

Then for an arbitrary ps.d.o. \( a(x,D) \) we have

\[
\left[ a(x,D)S_{\rho,x^0}u \right](x) = \left[ S_{\rho,x^0}a(x^0 + \rho^{-1}x,\rho D)u \right](x) . \quad (6.6)
\]
Proof. Direct calculation shows that
\[(FS_{\rho,x^0}u)(\xi) = \rho^{-n}\exp(-ix^0\xi)(Fu)\left(\frac{\xi}{\rho}\right).\]
If we substitute the last expression in the left-hand side of (6.6) and change \(\xi\) to \(\rho\xi\) we obtain the right-hand side of (6.6).

\[\square\]

Proof of Proposition 6.2. Applying the a priori estimate (6.2) to the
function \(u_{\rho}\) (cf. (6.4)), we obtain, according to the lemma,
\[
((1 + |D|) s (\rho\lambda + |D|)^{r-s} u_{\rho})(x)
= \rho^{n/2}S_{\rho,x^0}((\rho^{-1} + |D|)^s (\lambda + |D|)^{r-s} u)(x).
\]
The \(L_2(\mathbb{R}^n)\) norm of this expression tends to the left-hand side of (6.3),
as \(\rho\) tends to +\(\infty\).

Now we turn to the right-hand side of (6.2). We have
\[
(1 + |D|)^{s-2\mu} (\rho\lambda + |D|)^{r-s-2m+2\mu} A(x, D, \rho\lambda) u_{\rho}(x)
= \rho^{n/2}S_{\rho,x^0}((\rho^{-1} + |D|)(\lambda + |D|)^{r-s-2m+2\mu} h_{\rho})(x)
\]
where
\[
h_{\rho}(x) = \rho^{-2m} A(x^0 + \rho^{-1} x, \rho D, \rho\lambda) u(x) \rightarrow A(D, \lambda) u
\]
as \(\rho \to +\infty\).

It is easy to check that the limit of the second term of the right-hand
side of (6.2) is equal to zero.

\[\square\]

To prove the necessity of 2.4 b), c) and d) we consider (4.8) for
functions with supports belonging to a small neighbourhood of a point
\(x^0 \in \partial M\). In this case the norms in (4.8) can be taken in \(\mathbb{R}^n\) and \(\mathbb{R}^{n-1}\),
respectively. Now we use the fact that we have the norm equivalence
(5.5) and the equivalence
\[
\Xi^{-t}(\xi, \lambda) \approx \begin{cases} 
(1 + |\xi|)^{s-l} (\lambda + |\xi|)^{-s}, & l \leq s, \\
(\lambda + |\xi|)^{-t}, & l > s.
\end{cases}
\]
According to [5], Section 2, the norm
\[
\|(iD_n + \sqrt{1 + |D'|^2})^q (iD_n + \sqrt{\lambda^2 + |D'|^2})^{r-q} u\|_{L_2(\mathbb{R}^n)}
\]
is defined for any \( p, q \in \mathbb{R} \) and is equivalent to \( \| u \|_{(p, q, \mathbb{R}^n)} \) (cf. (6.3)). Substituting these expressions into the a priori estimate (6.8) for the half space, we obtain in explicit form

\[
\sum_{l=0}^{s} \left\| (1 + |D'|^2)^{(s-l)/2}(\lambda^2 + |D'|^2)^{(r-s)/2}D^l_n u \right\|_{L^2(\mathbb{R}^n_+)} + \sum_{l=s+1}^{r} \left\| (\lambda^2 + |D'|^2)^{(r-l)/2}D^l_n u \right\|_{L^2(\mathbb{R}^n_+)} \\
\leq C \left( \| \sigma(D, \lambda)A(x, D, \lambda)u \|_{L^2(\mathbb{R}^n_+)} + \lambda^{r-s} \| u \|_{L^2(\mathbb{R}^n_+)} + \sum_{j=1}^{\mu} \left\| (1 + |D'|^2)^{(s-m_j-1/2)/2}(\lambda^2 + |D'|^2)^{(r-s)/2}B_j(x', D) u \right\|_{L^2(\mathbb{R}^{n-1})} + \sum_{j=\mu+1}^{m} \left\| (\lambda^2 + |D'|^2)^{(r-m_j-1/2)/2}B_j(x', D) u \right\|_{L^2(\mathbb{R}^{n-1})} \right),
\]

where we used the abbreviation

\[ \sigma(D, \lambda) := (iD_n + \sqrt{1 + |D'|^2})^{s-2\mu}(iD_n + \sqrt{\lambda^2 + |D'|^2})^{r-s-2m+2\mu}. \]

**Proposition 6.4** Suppose estimate (6.7) holds. Let \( x^0 \) be an arbitrary point in \( \mathbb{R}^{n-1} \) and set \( A(D, \lambda) = A^{(0)}(x^0, D, \lambda) \), \( B_j(D, \lambda) = B_j^{(0)}(x^0, D, \lambda), j = 1, \ldots, m. \) Then the following estimate holds

\[
\sum_{l=0}^{s} \left\| |D'|^{s-l}(\lambda^2 + |D'|^2)^{(r-s)/2}D^l_n u \right\|_{L^2(\mathbb{R}^n_+)} + \sum_{l=s+1}^{r} \left\| (\lambda^2 + |D'|^2)^{(r-l)/2}D^l_n u \right\|_{L^2(\mathbb{R}^n_+)} \\
\leq C \left( \| \tilde{\sigma}(D, \lambda)A(D, \lambda)u \|_{L^2(\mathbb{R}^n_+)} + \sum_{j=1}^{\mu} \left\| |D'|^{s-m_j-1/2}(\lambda^2 + |D'|^2)^{(r-s)/2}B_j(D) u \right\|_{L^2(\mathbb{R}^{n-1})} + \sum_{j=\mu+1}^{m} \left\| (\lambda^2 + |D'|^2)^{(r-m_j-1/2)/2}B_j(D) u \right\|_{L^2(\mathbb{R}^{n-1})} \right),
\]

(6.8)
where we have set
\[ \tilde{\sigma}(D, \lambda) := (iD_n + |D'|)^s - 2 \mu (iD_n + \sqrt{\lambda^2 + |D'|^2})^{r-s-2m+2 \mu}. \]

Proof. We apply (6.7) with \( \lambda \) replaced by \( \rho \lambda \) to the function \( u_\rho \) defined in (6.4), noting that \( S_{\rho, x_0} u \) is again defined in \( \mathbb{R}^n_+ \) because of \( x^0 \in \mathbb{R}^{n-1} \) and \( \rho > 0 \). From Lemma 6.3 and the fact that for any function \( v \in L^2(\mathbb{R}^n_+) \) we have \( \frac{\rho}{\rho^2} \| S_{\rho, x_0} v \|_{L^2(\mathbb{R}^n_+)} = \| v \|_{L^2(\mathbb{R}^n_+)} \), we see that the \( l \)-th term in the first sum in (6.7) is equal to
\[
\left( \frac{1}{\rho^2} + |D'|^2 \right)^{(s-l)/2} \left( \lambda^2 + |D'|^2 \right)^{(r-s)/2} D_n^l u \quad \|_{L^2(\mathbb{R}^n_+)}
\]
which tends to the corresponding term in (6.8) for \( \rho \to \infty \). The remaining expressions in (6.7) can be treated analogously; the term \( (\rho \lambda)^{r-s} \| u_\rho \|_{L^2(\mathbb{R}^n_+)} \) tends to zero for \( \rho \to \infty \).

For the terms involving the boundary operators we remark that \( \gamma_0 S_{\rho, x_0} = S_{\rho, x_0} \gamma_0 \) where \( \gamma_0 : u \mapsto u(\cdot, 0) \) stands for the trace operator. Therefore we may apply Lemma 6.3 to the function \( B_j(x', D) u_\rho \) defined in \( \mathbb{R}^{n-1} \).

If we apply (6.8) to a function of the form \( u(x) = \phi(x') V(x_n), \quad \phi(x') \in D(\mathbb{R}^{n-1}) \)
we obtain an estimate on the half-line (cf. [8], Chapter 3, Proposition 2 in Subsection 2.3):
\[
\sum_{l=0}^{s} |\xi'|^{s-l} (\lambda^2 + |\xi'|^2)^{(r-s)/2} \| D_n^l V \|_{L^2(\mathbb{R}^n_+)} + \sum_{l=s+1}^{r} (\lambda^2 + |\xi'|^2)^{(r-l)/2} \| D_n^l V \|_{L^2(\mathbb{R}^n_+)} \leq C \left( \| \tilde{\sigma}(\xi', D_n, \lambda) A(\xi', D_n, \lambda) V \|_{L^2(\mathbb{R}^n_+)} + \sum_{j=1}^{m} |\xi'|^{s-m_j-1/2} (\lambda^2 + |\xi'|^2)^{(r-s)/2} |B_j(\xi', D_n) V(0)| + \sum_{j=m+1}^{m} (\lambda^2 + |\xi'|^2)^{(r-m_j-1/2)/2} |B_j(\xi', D_n) V(0)| \right). \quad (6.9)
\]


**Necessity of condition b.** Suppose \( V(x_n) \in L^2(R_+) \) is a solution of the homogeneous equation

\[
A(\xi', D_n, \lambda)V(x_n) = 0, \quad x_n > 0.
\]

Then this function satisfies the equation

\[
A_+(\xi', D_n, \lambda)V(x_n) = 0, \quad x_n > 0.
\]

(6.10)

Now from (6.9) we deduce the estimate

\[
c(\xi', \lambda) \sum_{l=0}^r ||D^l_n V||_{L^2(R_+)} \leq m \sum_{j=1}^m |B'_j(\xi', \lambda, D_n)V(0)|
\]

(6.11)

Here \( B'_j \) are remainders of \( B_j \) after the division by \( A_+ \) and \( c(\xi, \lambda) > 0 \) for \( \xi' \neq 0 \) and \( \lambda \geq 0 \). From a standard trace result for Sobolev spaces on \( R_+ \) we know that

\[
\sum_{j=1}^m |D^{j-1}_n V(0)| \leq C \sum_{j=1}^{r+1} ||D^{j-1}_n V||_{L^2(R_+)}.
\]

(6.12)

From this and (6.11) we obtain, using \( r \geq m \) (see (6.3)),

\[
\tilde{c}(\xi', \lambda) \sum_{j=1}^m |D^{j-1}_n V(0)| \leq \sum_{j=1}^m \left| \sum_{k=1}^m b_{jk}(\xi', \lambda)D^{k-1}_n V(0) \right|,
\]

(6.13)

where

\[
B'_j(\xi', \lambda, z) = \sum_{k=1}^m b_{jk}(\xi', \lambda)z^{k-1}.
\]

(6.14)

The constant \( \tilde{c}(\xi', \lambda) \) in (6.13) is positive for \( \xi' \neq 0 \) and \( \lambda \geq 0 \). Note that the Cauchy problem

\[
D^{k-1}_n V(0) = \zeta_k, \quad k = 1, \ldots, m
\]

for ODE (6.10) has a unique solution for arbitrary \( \zeta = (\zeta_1, \ldots, \zeta_m) \in C^m \). This means that for an arbitrary complex vector \( \zeta \) we have the estimate

\[
\tilde{c}(\xi', \lambda)||\zeta|| \leq |B(\xi', \lambda)\zeta|
\]

where \( B(\xi', \lambda) := (b_{jk}(\xi', \lambda))_{j,k=1,\ldots,m} \). The last inequality means that the matrix \( B(\xi', \lambda) \) is nonsingular as \( |\xi'| \neq 0, \lambda \geq 0 \), i.e. the necessity of b) is proved.
Proposition 6.5 Suppose the estimate (6.7) holds. Let $x^0$ be an arbitrary point of $\mathbb{R}^{n-1}$. Then the inequality

$$
\sum_{l=0}^{s} \left| D' \right|^{s-l} \left| D_l' V \right|_{L_2(\mathbb{R}^n_+)}
\leq
C \left( \left\| (iD_n + |D'|)^{s-2\mu} A_{2\mu}(D)_n u \right\|_{L_2(\mathbb{R}^n_+)} 
+ \sum_{j=1}^{\mu} \left| D' \right|^{s-m_j-1/2} B_j(D)_n \left\| u \right\|_{L_2(\mathbb{R}^{n-1})} \right) \tag{6.15}
$$

holds, where $A_{2\mu}(D) = A_{2\mu}^{(0)}(x^0,D)$, $B_j(D) = B_j^{(0)}(x^0,D)$, $j = 1, \ldots, \mu$.

Proof. This can be seen in exactly the same way as Proposition 6.4, now applying the a priori estimate (6.7) with $\rho^t \lambda$ instead of $\lambda$ to the function

$$
u_{\rho}(x) := \rho^{t(r-s)-s+n/2} u(\rho(x - x^0))
$$

where $t > 1$ is fixed and $\rho > 0$ tends to infinity. $\square$

Necessity of condition 2.4 c). From Proposition 6.3 the estimate on the half-line can be obtained

$$
\sum_{l=0}^{s} \left| \xi' \right|^{s-l} \left| D_l' V \right|_{L_2(\mathbb{R}^n_+)}
\leq
C \left( \left\| (iD_n + |\xi'|)^{s-2\mu} A_{2\mu}(\xi',D_n) V \right\|_{L_2(\mathbb{R}^n_+)} 
+ \sum_{j=1}^{\mu} \left| \xi' \right|^{s-m_j-1/2} B_j(\xi',D_n) V(0) \right) .
$$

As above we see that for solutions $V(x_n) \in L_2(\mathbb{R}^n_+)$ of

$$
A_{2\mu}(\xi',D_n)V(x_n) = 0, \quad x_n > 0
$$

we obtain the inequality

$$
\sum_{l=1}^{s} \left| D_n^{l-1} V(0) \right| \leq C \sum_{j=1}^{\mu} \left| B_j'(\xi',\lambda, D_n) V(0) \right| \tag{6.16}
$$

with a constant $C$ independent of $\xi'$, $|\xi'| = 1$, and $\lambda$, where now $B_j'$ denotes the remainder of $B_j$ after division by $(A_{2\mu})_+$. Replacing in (6.16)
the germ of $V$ in 0 by an arbitrary vector $\zeta \in \mathbb{C}^n$ and using $s \geq \mu$ (see (6.1)), we obtain the necessity of c).

**Proposition 6.6** Suppose the estimate (6.4) holds and $x^0$ is an arbitrary point of $\mathbb{R}^{n-1}$. Then the estimate

$$
\sum_{l=s}^{r} \|D_n^l u\|_{L^2(\mathbb{R}^n_+)} \leq C \left( \|(D_n - i)^{r-s-2m+2m} D_n^s Q(x^0, D_n) u\|_{L^2(\mathbb{R}^n_+)} + \sum_{j=\mu+1}^{m} \|B_j^{(0)}(x^0, 0, D_n) u\|_{L^2(\mathbb{R}^{n-1})} \right)
$$

(6.17)

holds.

**Proof.** We apply (6.7) with $\lambda$ replaced by $\rho$ to the function

$$u(\rho \varepsilon (x' - x^0), \rho x_n) = \left[ S^{(x')}_{\rho', x, 0} S^{(x_n)}_{\rho, 0} u \right](x),$$

where $S^{(x')}_{\cdot}$ indicates that the operator $S_{\cdot}$ acts on the first $n-1$ variables (and analogously that $S^{(x_n)}_{\cdot}$ acts on the last variable), and apply Lemma 6.3 twice. For the $l$-th term in the first sum of (6.7) we obtain the expression

$$\rho^{(1-s)(l-s)} \left( \rho^{-2s} + |D'|^2 (s-1)^2 (1 + \rho^2 (\varepsilon-1)|D'|^2)^{(r-s)/2} D_n^l u \right)_{L^2(\mathbb{R}^n_+)}.$$

For $l \leq s - 1$ this expression tends to zero for $\rho \to \infty$, for $l = s$ its limit equals $\|D_n^s u\|_{L^2(\mathbb{R}^n_+)}$.

The remaining terms can be treated analogously; to finish the proof we use

$$\rho^{-2m} A \left( x^0 + \frac{x'}{\rho'}, \frac{x_n}{\rho}, \rho^\varepsilon D', \rho D_n, \rho \right) \to D_n^{2\mu} Q(x^0, D_n)$$

and

$$\rho^{-m+1} B_j \left( x^0 + \frac{x'}{\rho'}, \rho^\varepsilon D', \rho D_n \right) \to B_j^{(0)}(x^0, 0, D_n)$$

as $\rho \to \infty$. □
Necessity of condition 2.4 d). From Proposition 5.6 we obtain the estimate on the half-line

$$\sum_{l=s}^{r} \|D_{n}^{l}V\|_{L_{2}(\mathbb{R}^+)} \leq C\left(\left\|(D_{n}-i)^{r-s-2m+2\mu}D_{n}^{s}Q(x^{0},D_{n})V\right\|_{L_{2}(\mathbb{R}^+)} + \sum_{j=\mu+1}^{m} |B_{j}^{(0)}(x^{0},0,D_{n})V(0)|\right).$$

(6.18)

Since $m_{j} \geq s$ for $j \geq \mu + 1$, each $B_{j}^{(0)}(x^{0},0,\tau)$ contains the factor $\tau^{s}$, and it is easily seen that condition d) follows from the analogous condition for

$$\tilde{B}_{j}^{(0)}(x^{0},0,\tau) := \tau^{-s}B_{j}^{(0)}(x^{0},0,\tau).$$

Now we apply (6.18) to a solution $V \in L_{2}(\mathbb{R}^+)$ of

$$Q(x^{0},D_{n})V(x_{n}) = 0, \quad x_{n} > 0$$

and substitute $W(x_{n}) := D_{n}^{s}V(x_{n})$. We obtain

$$\sum_{l=1}^{r-s+1} \|D_{n}^{l-1}W\|_{L_{2}(\mathbb{R}^+)} \leq C \sum_{j=\mu+1}^{m} |B_{j}^{(0)}(x^{0},0,D_{n})W(0)|,$$

(6.19)

where now $B_{j}^{(0)}$ stands for the remainder of $\tilde{B}_{j}^{(0)}$ after division by $Q_{+}$. Using $r-s \geq m-\mu$ (cf. (5.14)) and the trace result (5.13), we obtain the linear independence of $B_{j}^{(0)}$ modulo $Q_{+}$ from (6.19) and therefore condition d).

Appendix. Singularly perturbed problems

One of the most important features of the Newton polygon approach is to provide an easy formulation and proof of a priori estimates in the theory of singularly perturbed problems. All results of the previous sections can be rewritten for boundary value problems with small parameter as treated by Vishik–Lyusternik [11], Nazarov [9], Frank [6] and others. (Cf. also [5], Appendix, for the Dirichlet problem.) As an example, we formulate an a priori estimate for such problems.

Consider for $\varepsilon > 0$ the operator

$$A_{\varepsilon}(x,D) := \varepsilon^{2m-2\mu}A_{2m}(x,D) + \varepsilon^{2m-2\mu-1}A_{2m-1}(x,D) + \ldots + A_{2\mu}(x,D)$$
with $A_j$ of the form (1.2). Let $A_\varepsilon$ act on a smooth compact manifold $M$ with boundary $\partial M$ and assume that we have boundary conditions $B_1(x, D), \ldots, B_m(x, D)$ of the form (1.3) satisfying (1.4).

We fix integer numbers $r$ and $s$ satisfying (4.4) and consider the weight function

$$\Xi_\varepsilon(\xi) := \Xi_{\varepsilon, (r, s)}(\xi) := (1 + |\xi|)^s (1 + \varepsilon |\xi|)^{r-s}.$$  

The norms corresponding to this weight function will be denoted by $\| \cdot \|_{\Xi_\varepsilon, M} = \| \cdot \|_{\varepsilon, (r, s), M}$.

**Definition A.1**  

a) The operator $A_\varepsilon(x, D)$ is called $N$-elliptic if

$$|A_\varepsilon^{(0)}(x, \xi)| \geq C|\xi|^{2\mu} (1 + \varepsilon |\xi|)^{2m-2\mu} \quad (\xi \in \mathbb{R}^n, \varepsilon > 0, x \in \overline{M})$$

holds where $C$ does not depend on $x, \xi$ or $\varepsilon$.

b) The operator $A_\varepsilon$ is said to degenerate regularly at the boundary if the polynomial

$$Q(x^0, \tau) := \tau^{-2\mu} A^{(0)}_\varepsilon(x^0, 0, \tau)$$

has exactly $m - \mu$ roots in the upper half plane.

**Definition A.2** The boundary problem $(A_\varepsilon, B_1, \ldots, B_m)$ is called $N$-elliptic if the following conditions hold:

a) The operator $A_\varepsilon(x, D)$ is $N$-elliptic in the sense of Definition A.1.

b) For every fixed $x^0 \in \partial M$ the boundary problem

$$\left( A^{(0)}_\varepsilon(x^0, \xi', D_n), B^{(0)}_1(x^0, \xi', D_n), \ldots, B^{(0)}_m(x^0, \xi', D_n) \right)$$

for each $\varepsilon > 0$ and $\xi' \neq 0$ is uniquely solvable on the half-line $x_n \geq 0$ in the space of functions tending to zero as $x_n \to \infty$. Moreover we suppose that the problem

$$\left( A^{(0)}_{2\mu}(x^0, \xi', D_n), B^{(0)}_1(x^0, \xi', D_n), \ldots, B^{(0)}_m(x^0, \xi', D_n) \right)$$

(corresponding to $\varepsilon = \infty$) has the same property.

c) For every $x^0 \in \partial M$ the boundary problem

$$(A_{2\mu}(x^0, D), B_1(x^0, D), \ldots, B_m(x^0, D))$$

fulfills the Shapiro–Lopatinskii condition.

d) For every $x^0 \in \partial M$ the polynomials $(B^{(0)}_j(x^0, 0, \tau))_{j=\mu+1, \ldots, m}$ are linearly independent modulo $Q_+(x^0, \tau)$ with $Q_+$ defined in Definition 2.4.
If the conditions of Definition A.1 and A.2 hold, we can apply Theorem 4.2 to the operator

\[ A(x, D, \lambda) := \lambda^{2m-2\mu} A_{1/\lambda}(x, D). \]

The connection between \( \Xi_{\varepsilon}(\xi) \) and \( \Xi(\xi, \varepsilon^{-1}) \) (defined in (4.2)) is given by

\[ \Xi_{\varepsilon}(\xi) = \varepsilon^{r-s} \Xi(\xi, \varepsilon^{-1}) \]

and

\[ \Xi_{\varepsilon}((-m_j - 1/2)(\xi)) = \begin{cases} \varepsilon^{r-s} \Xi((-m_j - 1/2)(\xi), \varepsilon^{-1}) & \text{if } j \leq \mu, \\ \varepsilon^{r-m_j - 1/2} \Xi((-m_j - 1/2)(\xi), \varepsilon^{-1}) & \text{if } j > \mu. \end{cases} \]

Using these relations, we obtain from Theorem 4.2 the following result which can be found (without the notation of the Newton polygon) in [3]:

**Theorem A.3** Assume that \( A_{\varepsilon} \) degenerates regularly and that \( (A_{\varepsilon}, B_1, \ldots, B_m) \) is \( N \)-elliptic in the sense of Definition A.2. Then the following a priori estimate holds with a constant \( C \) independent of \( \varepsilon > 0 \):

\[
\|u\|_{\Xi_{\varepsilon}, M} \leq C \left( \|A_{\varepsilon} u\|_{\varepsilon, (r-2m, s-2\mu), M} + \sum_{j=1}^{\mu} \|B_j u\|_{\Xi_{\varepsilon}((-m_j - 1/2), \partial M)} \right.
\]

\[
+ \sum_{j=\mu+1}^{m} \varepsilon^{m_j + 1/2 - s} \|B_j u\|_{\Xi_{\varepsilon}((-m_j - 1/2), \partial M)} + \|u\|_{L^2(M)} \bigg). 
\]

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