On generalized category $O$ for a quiver variety

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Abstract. In this paper, we give a method for relating the generalized category $O$ defined by the author and collaborators to explicit finitely presented algebras, and apply this to quiver varieties. This allows us to describe combinatorially not just the structure of these category $O$'s but also how certain interesting families of derived equivalences, the shuffling and twisting functors, act on them.

In the case of Nakajima quiver varieties, the algebras that appear are weighted KLR algebras and their steadied quotients, defined by the author in earlier work. In particular, these give a geometric construction of canonical bases for simple representations, tensor products and Fock spaces. If the $C^*$-action used to define the category $O$ is a “tensor product action” in the sense of Nakajima, then we arrive at the unique categorifications of tensor products; in particular, we obtain a geometric description of the braid group actions used by the author in defining categorifications of Reshetikhin-Turaev invariants. Similarly, in affine type $A$, an arbitrary action results in the diagrammatic algebra equivalent to blocks of category $O$ for cyclotomic Cherednik algebras.

This approach also allows us to show that these categories are Koszul and understand their Koszul duals; in particular, we can show that categorifications of minuscule tensor products in types ADE are Koszul.

In the affine case, this shows that our category $O$'s are Koszul and their Koszul duals are given by category $O$’s with rank-level dual dimension data, and that this duality switches shuffling and twisting functors.

1. Introduction

In this paper, our aim is to introduce a new technique for relating geometric and algebraic categories. Since the categories on both sides of this correspondence are probably not familiar to many readers, we will provide a teaser for the results before covering any of the details. Amongst the results we will cover are:

- A new geometric construction of categories (using quiver varieties) equivalent to cyclotomic Khovanov-Lauda-Rouquier (KLR) algebras and their weighted generalizations. In particular, this gives geometric constructions of the canonical bases of simple representations and their tensor products in types ADE, and on higher level Fock spaces in affine type $A$ as the simple objects in a category of sheaves.

Alternatively, the reader can think of these as Kazhdan-Lusztig type character formulae for the decomposition multiplicities of category $O$ in these cases.

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The rôle of Kazhdan-Lusztig polynomials is played by the coefficients of the canonical basis.

- A geometric description of the braid group action used in [Webb] to construction categorifications of Reshetikhin-Turaev invariants.
- A new method for proving Koszulity of these algebras, providing the first proof that categorifications of minuscule tensor products in types D and E are Koszul, and giving a new proof that blocks of category $O$ for Cherednik algebras of $\mathbb{Z}/(\mathbb{Z}/\ell) \wr \mathfrak{S}_n$ are Koszul.

In recent years, a great deal of attention in representation theory has been directed toward generalizations of the BGG category $O$; perhaps the most famous of these is that for Cherednik algebras [GCOR03], but Braden, Licata, Proudfoot and the author have also considered these for “hypertoric enveloping algebras” [BLPW12]. The author and the same group of collaborators have given a definition subsuming all these examples into a uniform definition of a category $O$ attached to a conical symplectic singularity, a Hamiltonian $\mathbb{C}^*$-action commuting with the conical structure and a choice of period (which the reader should think of as a central character) [BPW, BLPW]. Many natural properties of this category follow from general principles, but at the moment, the techniques for understanding these categories are not well-developed.

It’s a consistent theme that these categories have simple realizations as representation categories of finite dimensional algebras with combinatorial presentations.

- For BGG category $O$, this follows from the combinatorial descriptions of the categories of Soergel bimodules by Elias, Khovanov and Williamson [EK, EW].
- For hypertoric category $O$, these are provided by certain combinatorial algebras defined by Braden, Licata, Proudfoot and the author [BLPW10].
- For a Cherednik algebra of type $G(r, 1, \ell)$, these are given in [Webc] by certain variants of Hecke and KLR algebras.

Obviously, one of the most interesting symplectic singularities is a Nakajima quiver variety for a quiver $\Gamma$, especially for a finite or affine Dynkin diagram. We have already proven that this category carries a categorical action of the Lie algebra attached to $\Gamma$ [Webal], and we know that such a structure has a lot of power over the category it acts on.

For any Hamiltonian $\mathbb{C}^*$-action commuting with the conical structure on a quiver variety, we have a diagrammatically defined algebra which is a candidate for category $O$: a reduced steadied quotient of a weighted Khovanov-Lauda-Rouquier algebra [Webd]. Of course, this is not such a familiar object, but it is purely combinatorial in nature, and certain special cases have had a bit more of a chance to filter into public consciousness.

To simplify things for the introduction, assume that $\Gamma$ is finite or affine type ADE. In this case, we separate appropriate $\mathbb{C}^*$-actions into 2 cases:

- Case 1: the action is a tensor product action as used by Nakajima [Nak01]; this is induced by a cocharacter $\mathbb{C}^* \rightarrow G_w$ in Nakajima’s notation.
- Case 2: $\Gamma$ is a cycle, and the sum of the weights of the edges of the cycle is non-zero.
In the first case, the Grothendieck group of category $O$ has a natural map to a tensor product of simple representations; in the second, it has a natural map to a higher-level Fock space.

**Theorem A.** For a generic integral period, category $O$ is equivalent to the heart of a $t$-structure on:

Case 1: the category of dg-modules over a tensor product algebra $T^\Lambda$ from \cite[§??]{Webb}.
Case 2: the category of dg-modules over a Fock space algebra $T^\theta$, as studied in \cite[§refR-sec:weighted-algebras]{Webc}.

In both cases, the abelian category is equivalent to the category of linear projective complexes over these algebras, as in \cite[MOS09]{MOS09}. In the latter case, we also have an explicit description of the abelian category as a block of category $O$ for a cyclotomic Cherednik algebra.

These results follow from a more general yoga for understanding category $O$ of symplectic reductions of vector spaces by studying the category of D-modules before reduction. We can use the same principles to recover the hypertoric results of \cite[BLPW12]{BLPW}. At the moment, we are not aware of any other especially interesting symplectic reductions of vector spaces on which to apply these techniques, but such examples may present themselves in the future. We also expect that the same ideas can apply to other categories of representations, the most obvious possibility being the Harish-Chandra bimodules considered in \cite[BPW]{BPW}. It is also worth noting that this approach only covers certain special parameters and changes will be needed to understand the structure of these categories at general parameters.

One interesting application of this approach is that it gives a new geometric proof of the Koszulity of these category $O$'s, conjectured in \cite[BLPW]{BLPW}.

**Theorem B.** If the Hamiltonian $C^*$-action has isolated fixed points on the symplectic quotient, and $E$ and $G$ satisfy certain geometric conditions (which we denote (✠) or (†)), then the category $O$ is Koszul.

In particular, if $\Lambda$ is a list of minuscule weights, then the tensor product algebra $T^\Lambda$ is Koszul.

This also gives a new proof of known Koszulity results for affine quiver varieties and hypertoric varieties.

Furthermore, there are two classes of auto-functors defined on categories $O$:

- **twisting functors**, which arise from tensoring with the sections of quantized line bundles, are defined in \cite[§6]{BPW}.
- **shuffling functors**, which arise from changing the $C^*$-action, are defined in \cite[§7]{BLPW}.

These can be identified with diagrammatically defined functors:

**Theorem C.** The twisting functors in Cases 1 and 2 are intertwined with the functors of Chuang and Rouquier's braid group action. The shuffling functors are intertwined with the diagrammatic functors given by reordering tensor factors defined in \cite[§??]{Webb} (in Case 1) or change-of-charge functors as defined in \cite[§??]{Webc} (in Case 2).

In Case 2, the Koszul duality between rank-level dual category $O$'s interchanges these two affine braid group actions.
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In particular, this theorem gives an indication of how to give a geometric construction of the knot invariants categorifying Reshetikhin-Turaev invariants from [Webb], using quiver varieties. The above theorem gives a geometric description as shuffling functors of the braid group action used in that paper; describing the cups and caps will require techniques we will not delve into here.

2. Reduction and quotients

2.1. Symplectic quotients. Let $E$ be a complex vector space, and let $G$ be a connected reductive algebraic group with a fixed faithful linear action on $E$. Let $H = \text{Aut}_G(E)$ and let $Z = H \cap G = Z(G)$. We assume throughout that the induced moment map $\mu: T^*E \to g^*$ is flat and that there are no non-constant $G$-invariant polynomial functions on $E$. In this case, the symplectic reduction $\mathcal{M} = T^*E//_{\alpha}G$ is rationally smooth for $\alpha$ a generic character of $G$ and is a symplectic rational resolution of $\mathfrak{N} = T^*E//_0G$. Examples of such varieties are Nakajima quiver varieties and hypertoric varieties (for more general discussions of these varieties, see [Nak98] and [Pro08], respectively). Let $S$ be the copy of $C^*$ acting with weight $-1$ on the cotangent fibers and trivially on the base.

Also fix a Hamiltonian $C^*$-action on $\mathcal{M}$ which factors through $H/Z$; to avoid confusion (and match notation from [BPW, BLPW]), we denote this copy of $C^*$ by $T$. If we are given two commuting $C^*$-actions, we take their product to be the pointwise product of their values. We let $T'$ be the action of $C^*$ on $\mathcal{M}$ given by the product of $T$ with $S$.

Let $\mathcal{D}_E$ be the ring of differential operators on $E$; we wish to think of this as a non-commutative version of the cotangent bundle $T^*E$. The action of $G$ extends to one on $\mathcal{D}_E$, which is inner; we have a non-commutative moment map $\mu_\lambda: U(g) \to \mathcal{D}_E$ given by $\mu_\lambda(X) = X_E - \lambda(X)$ where $X_E$ is the vector field on $E$ attached to the action of $X \in g$, thought of as a differential operator.

After fixing $\lambda$, there is a non-commutative algebra given by the quantum Hamiltonian reduction $A_\lambda = \left(\mathcal{D}_E/\mathcal{D}_E \cdot \mu_\lambda(g)\right)^G$; this algebra can also be constructed in a universal way as a quantization of the cone $\mathfrak{N}$. We let $\xi$ be the vector field induced by the action of $T$; you should think of this as a quantization of the moment map for the action of $T$ on $\mathfrak{N}$. Since the actions of $T$ and $G$ commute, the element $\xi$ induces an element of $A_\lambda$, which we also denote by $\xi$.

Definition 2.1 ([BLPW]). The category $O_{\xi}$ is the category of finitely generated $A_\lambda$-modules on which $\xi$ acts locally finitely with finite dimensional generalized eigenspaces, such that the real parts of the eigenvalues of $\xi$ are bounded above by some real number $b$.

This is what most representation theorists would probably think of as the basic object considered in this paper; however, its dependence on $\lambda$ is quite complicated, and beyond our ability to analyze in any generality in this work. Instead, we’ll replace it with a category which is easier to analyze, and gives the same answer for many $\lambda$. 

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Attached to $\lambda$, we also have a sheaf of algebras on $\mathcal{M}$ called $\mathcal{D}_\lambda$; you should think of $\mathcal{M}$ as “morally” a cotangent bundle and $\mathcal{D}_\lambda$ as an analogue of the sheaf of twisted microlocal differential operators.

More explicitly, if we let $\mathcal{D}_E$ denote the sheaf of microlocal differential operators on $E$ with $h^{-1}$ adjoined, then

$$\mathcal{D}_\lambda := \psi^* (\text{End}_{\mathcal{D}_E}(\mathcal{D}_E/\mathcal{D}_E \cdot \mu_\lambda(\mathfrak{g}))|_{\mu_\lambda^{-1}(0)^{ss}})$$

where $\psi$ is the quotient map $\mu_\lambda^{-1}(0)^{ss} \to \mathcal{M}$. This is a sheaf of free $\mathbb{C}((h))$-modules, and it is equipped with a free $\mathbb{C}[[h]]$-lattice $\mathcal{D}_\lambda(0)$. The category of good $\mathbb{S}$-equivariant modules over $\mathcal{D}_\lambda$ is related to the category of $A_\lambda$-modules by an adjoint pair of functors

$$\Gamma_\mathbb{S} : \mathcal{D}_\lambda \text{-mod} \to A_\lambda \text{-mod} \quad \text{Loc} : A_\lambda \text{-mod} \to \mathcal{D}_\lambda \text{-mod}$$

discussed in [BPW, §4.2].

**Definition 2.2 ([BLPW]).** The category $\mathcal{O}_g$ is the category of coherent $\mathcal{D}_\lambda$-modules which are supported on the subvariety

$$\mathcal{M}^+ = \{ p \in \mathcal{M} \mid \lim_{T \to 0} t \cdot p \text{ exists}\}$$

and which are $\mathbb{T}$-regular, that is, they possess a $\mathcal{D}_\lambda(0)$-lattice which is preserved by the induced action of the Lie algebra $\mathfrak{t}$. All these issues are discussed in great detail in [BPW, §2].

For those readers (probably most) who are not familiar with these sheaves and the general machinery of deformation quantization in algebraic geometry (see [BK04, KS]), a $\mathcal{D}_\lambda(0)$-lattice should be thought of as an equivalent of a good filtration on a $\mathcal{D}$-module. All these issues are discussed in great detail in [BPW, §2].

The category $\mathcal{O}_g$ is a “local” version of $\mathcal{O}_a$; our primary results in this paper will be stated in terms of this category, since it is more conducive to analysis from a topological perspective. The relation between these categories is close, but subtle. In particular, we have induced functors

$$\text{Loc} : \mathcal{O}_a \to \mathcal{O}_g \quad \Gamma_\mathbb{S} : \mathcal{O}_g \to \mathcal{O}_a$$

which are equivalences for many values of $\lambda$ (though far from always); in this case, we say localization holds at $\lambda$. Though its details are best left to other papers, we note for the benefit of those who worry that $\mathcal{O}_g$ is too far afield from representation theory that these functors show that:

- $\mathcal{O}_g$ for a given parameter $\lambda$ is always equivalent to $\mathcal{O}_a$ for some possibly different parameter $\lambda + na$ with $n > 0$ [BPW, 5.11].
- generically on $H^2(\mathcal{M}; \mathbb{C})$, the derived functors $\mathbb{L}\text{Loc}$ and $\mathbb{R}\Gamma_\mathbb{S}$ induce an equivalence between $\mathcal{D}(\mathcal{O}_a)$ and $\mathcal{D}(\mathcal{O}_g)$ [BPW, 4.13], see also [MNa].

### 2.2. Quotients of dg-categories

At the base of this work is the interplay between $G$-equivariant $\mathcal{D}_E$-modules on $E$ and $\mathcal{D}_\lambda$-modules.

The relevant category to consider is that of $(G, \lambda)$ strongly equivariant $\mathcal{D}_E$-modules.

**Definition 2.3.** We call a $\mathcal{D}_E$-module $M$ strongly $(G, \lambda)$-equivariant if the action of $\mathfrak{g}$ via left multiplication by $\mu_\lambda$ integrates to a $G$-action.
As usual, we wish to consider these as the heart of a $t$-structure on the dg-category of complexes of $\mathcal{D}_E$-modules on the Artin stack $E/G$ with strongly $(G, \lambda)$-equivariant cohomology bounded-below in degree; that is to say, we wish to work in the dg-enriched equivariant bounded-below derived category of $E$. This is perhaps most conveniently understood as the dg-category of $D$-modules on the simplicial manifold given by the Borel space of this action:

$$\cdots \to G \times G \times E \to G \times E \Rightarrow E$$

Recall that Drinfeld (and others) have introduced the quotient of a dg-category by a dg-subcategory; see [Dri04] for a more detailed discussion of this construction. In intuitive terms, the quotient of a dg-category by a dg-subcategory is the universal dg-category which receives a functor killing all objects in that subcategory.

Kashiwara and Rouquier [KR08, 2.8] have shown that for a free action $D\lambda$-modules on a quotient can be interpreted as strongly equivariant modules on the source. It is more useful for us to use a slightly stronger version of this result.

Consider the reduction functor (called the Kirwan functor in [BPW, §5.4])

$$\tau(-) = \psi^* \Hom_{\mathcal{D}_E}(\mathcal{D}_E/\mathcal{D}_E \cdot \mu_\lambda(g), -): \mathcal{D}_E\text{-mod} \to \mathcal{D}_\lambda\text{-mod}.$$ 

In [BPW, 5.19], we define a left adjoint $\tau_!$ to this reduction functor. The construction of this functor is in terms of $Z$-algebras, which are beyond the scope of this paper. Both left and right adjoints to $\tau$ are constructed in [MNB]. Its existence shows that:

**Proposition 2.4.** The functor $\tau$ realizes the dg-category of $S$-equivariant $\mathcal{D}_\lambda$-modules on $\mathcal{M}$ as the quotient of the dg-category of strongly $(G, \lambda)$-equivariant $\mathcal{D}_E$-modules by the subcategory of complexes whose cohomology has singular support in the unstable locus.

**Proof.** Here, we apply [Dri04, 1.4].

- The functor $\tau$ is essentially surjective after passing to the homotopy category, since $\tau_! : \mathcal{D}_E\text{-mod} \to \mathcal{D}_\lambda\text{-mod}$ is isomorphic to the identity functor by [BPW, 5.19].
- All complexes whose cohomology is supported on the unstable locus are sent to contractible complexes.
- The cone of the natural map $\tau_! \tau(M) \to M$ (which represents the functor given by the cone of the map $\text{Ext}^\bullet(M, -) \to \text{Ext}^\bullet(\tau(M), \tau(-))$) has cohomology supported on the unstable locus, since the induced map $\tau_! \tau(M) \to \tau(M)$ is an isomorphism.

Thus, the result follows immediately. $\square$

While psychologically satisfying, this proposition doesn’t directly give us a great deal of information about the category of $S$-equivariant $\mathcal{D}$-modules since the quotient is a category in which it is difficult to calculate. On the other hand, it serves as a hand-hold on the way to more concrete results.

In particular, the adjoint $\tau_!$ provides a slightly more concrete approach to calculation in $\mathcal{D}_\lambda\text{-mod}$, since $\text{Ext}^\bullet_{\mathcal{D}_\lambda}(M, N) \cong \text{Ext}^\bullet_{\mathcal{D}_E}(\tau_! M, \tau_! N)$; of course, the principle of conservation of trouble suggests that computing $\tau_! M$ is not particularly easy, but this at least allows one to work in a better understood category than $\mathcal{D}_\lambda\text{-mod}$.
Remark 2.5. We should also mention the work of Zheng and Li, which was an important inspiration for us. In this case, they work with the localization of the category of constructible sheaves on \( E/G \) by those with certain bad singular supports; this category can be compared with \( D\)-mod using the Riemann-Hilbert correspondence. Using \( D\)-modules instead of constructible sheaves has the very significant advantage that the quotient category is a concrete category of modules over a sheaf of algebras, rather than an abstraction.

2.3. Riemann-Hilbert and sheaves on the relative precore. We let the precore of a cocharacter \( \mathbb{T} \to H/Z \) be the union of the points in \( \mu^{-1}(0) \) which have a limit as \( t \to 0 \) for some rational lifting \( \mathbb{T}' \to H \times Z \to G \times S \) of our fixed cocharacter \( \mathbb{T}' \to H/Z \times S \).

Proposition 2.6. The precore is a Lagrangian subvariety with finitely many components. It is the vanishing locus on \( \mu^{-1}(0) \) of the functions which are \( G \)-invariant and positive weight for some (equivalently all) lifts of \( \mathbb{T}' \).

We make an observation which is helpful for matching with [BLPW]: the precore is unchanged when we replace \( \mathbb{T} \) by any power of its action. After all, a point has a limit under \( \mathbb{T}' \) if and only if it lies in the sum of the non-negative weight spaces for some lift, and replacing that lift with its power won’t change which weight spaces have non-negative weight.

Proof. For each lift \( \gamma \), the space \( V \) of points with limits under the action of \( \gamma(t) \) as \( t \to 0 \) is Lagrangian. Furthermore, the set of points in \( \mu^{-1}(0) \) which have a limit under a conjugate of \( \gamma \) is the \( G \)-saturation \( W = G \cdot (V \cap \mu^{-1}(0)) \) of \( V \cap \mu^{-1}(0) \). If \( x \) has a limit (and is generic), then the tangent space to \( W \) is \( g \cdot x + (T_x V \cap (g \cdot x)^\perp) \), so \( W \) is Lagrangian.

Thus, we need only prove that only finitely many different such components appear when we consider all lifts. This is the same as to say that only finitely many different spaces \( V \) come up as the spaces with limits under the cocharacters landing in a particular torus of \( H \times Z \) \( G \). But this is obvious, since \( V \) must be a sum of weight spaces for that torus, and there are only finitely many such spaces for any torus module.

Obviously if a function \( f \) is in the ideal generated by \( G \)-invariant functions of positive weight under all lifts, it is 0 on the pre-core.

Now assume \( x \) is a point in \( \mu^{-1}(0) \) where all such functions vanish. Let us abuse notation and let \( \mathbb{T} \times G \) denote the fiber product of the maps \( H \times Z \to H/Z \leftarrow \mathbb{T} \). If the \( \mathbb{T} \times G \)-orbit through \( x \) doesn’t have 0 in it’s closure, there is a \( \mathbb{T} \times G \)-invariant polynomial vanishing on 0 which doesn’t vanish on the orbit. Since \( S \) commutes with \( \mathbb{T} \times G \), the space of \( \mathbb{T} \times G \)-invariant functions is \( S \)-invariant, and \( S \) acts with positive weight, since there are no non-constant polynomial \( G \)-invariant functions on \( E \). Thus, \( \mathbb{T}' \) acts with positive weight on these functions and they must all vanish on \( x \).

Thus, this orbit does have 0 in its closure, and on this closure \( (\mathbb{T} \times G) \cdot x \), there are no non-zero functions with positive weight for \( \mathbb{T}' \). Since this is an affine variety, every point thus has a limit under \( \mathbb{T}' \) as \( t \to 0 \). □
Corollary 2.7. Every component of the precore is the closure of the conormal bundle to the space of points in $E$ which have a limit under at least one element of a fixed conjugacy class of rational lifts of $T$.

We call a $D_E$-module $M$ $T$-regular if the vector field $\xi$ acts locally finitely.

Definition 2.8. We let $pO_g$ be category of $T$-regular $(G, \lambda)$-equivariant $D_E$-modules with singular support on the precore.

Proposition 2.9. A $T$-regular $D_E$-module lies in $pO_g$ if and only if it is smooth along a Whitney stratification where each stratum closure is exactly the set of points with a limit as $t \to 0$ for at least one element of a fixed conjugacy class of rational lifts of $T'$.

This is the natural category whose reductions lie in $O_g$ for any different choice of GIT stability condition.

Proposition 2.10. The functor $r$ sends modules in $pO_g$ to sheaves in $O_g$. The functor $r_!$ sends sheaves in $O_g$ to modules in $pO_g$.

Proof. If a module $M$ is in $pO_g$ then obviously $r(M)$ is supported on $M^+$. Furthermore, $M$ has a good filtration which is invariant under the action of $\xi$ (since $M$ must be generated by a $\xi$-stable finite dimensional subspace). This induces a lattice on $r(M)$ which is also $\xi$-invariant. Thus $r(M)$ is in $O_g$.

By the same argument, it’s clear that a sheaf $M$ in $O_g$ has associated graded killed by positive weight $G$-invariant functions, so $K = r(M)$ has the same property, and thus has associated graded supported on the precore. Furthermore, since the bimodule $Y_\lambda = D_E/D_E \cdot \mu_\lambda(g)$ is $\xi$-locally finite for the adjoint action, tensor product with it preserves $\xi$-local finiteness. By the description in [BPW, 5.18], the $D_E$-module $r(M)$ can be realized by tensoring $M$ with a quantization of a line bundle, taking sections, and then tensoring with $Y_\lambda'$ for an appropriate $\lambda'$. Since the sections after tensoring with a line bundle form a $\xi$-locally finite module, so is $r(M)$. \qed

Let $D_{\mathcal{O}_g}$ and $D_{p\mathcal{O}_g}$ be the dg-subcategories of the bounded dg-category of all $D_{\mathcal{O}}$-modules and equivariant $D_E$-modules, respectively, generated by $\mathcal{O}_g$ and $p\mathcal{O}_g$. The same proof as Proposition 2.4 shows that:

Corollary 2.11. The category $D_{\mathcal{O}_g}$ is the dg-quotient of $D_{p\mathcal{O}_g}$ by the dg-subcategory of complexes whose cohomology is supported on the unstable components of the precore.

Let $\mathcal{L}$ the sum of one copy of each isomorphism class of simple objects in $p\mathcal{O}_g$, and let $\mathcal{Q} := \text{Ext}^*_{p\mathcal{O}_g}(\mathcal{L}, \mathcal{L})^{op}$; we’ll assume throughout that

(*) this dg-algebra is formal (i.e. quasi-isomorphic to its cohomology).

We have a quasi-equivalence $D_{p\mathcal{O}_g} \cong \mathcal{Q}$-dg-mod given by $\text{Ext}^*_{D_{\mathcal{O}_g}}(\mathcal{L}, -)$. Under this quasi-equivalence, the simples of $p\mathcal{O}_g$ are sent to the indecomposable summands of $\mathcal{Q}$ and modules supported on the unstable locus are exactly those that have a $\mathcal{K}$-projective resolution only using summands of $\mathcal{Q}$ corresponding to unstable simples.

Let $I$ be the 2-sided ideal in $\mathcal{Q}$ generated by every map factoring through a summand supported on the unstable locus and let $\mathcal{R} \cong \mathcal{Q}/I$. 

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**Proposition 2.12.** We have an quasi-equivalence \( D_{Q_\mathfrak{g}} \cong R\text{-dg-mod} \) intertwining \( \mathfrak{g} \). In particular, \( \text{Ext}_{D_{Q_\mathfrak{g}}} (\tau(\mathcal{L}), \tau(\mathcal{L})) \cong R \) as formal dg-algebras.

*Proof.* The functor \( R \otimes_{Q_{\mathfrak{g}}} - \) : \( \Omega\text{-dg-mod} \to \mathfrak{g}\text{-dg-mod} \) is a dg-quotient functor. By Corollary 2.11 and the uniqueness of dg-quotients up to quasi-equivalence ([Dri04, 6.1]), the result follows. \( \square \)

On any smooth variety over the complex numbers, there is an equivalence of categories between appropriate dg-categories of regular holonomic D-modules and constructible sheaves, typically called the **Riemann-Hilbert correspondence**.

Assume that every object in \( pO_{\mathfrak{g}} \) is a regular \( D_E \)-module; this will, for example, be the case when the supports of such modules are a finite collection of \( G \)-orbits.

**Proposition 2.13.** If \( \lambda \) is integral, the category \( D_{pO_\mathfrak{g}} \) is quasi-equivalent to the dg-category of sheaves of \( \mathbb{C} \)-vector spaces on \( E/G \) with constructible cohomology and singular support on the precore.

Thus, we can use the geometry of constructible sheaves in order to understand the modules on the precore, which in turn can lead to an understanding of category \( O_{\mathfrak{g}} \).

### 2.4. Koszulity and mixed Hodge modules

Given that we have assumed that the algebra \( \mathfrak{g} \) is formal, there is a natural graded lift of \( \mathfrak{g}\text{-dg-mod} \): the derived category of graded modules over \( H^*(\mathfrak{g}) \). One might naturally ask if this lift can be interpreted in a geometric way. At least in certain cases, this is indeed the case, using the formalism of mixed Hodge modules. The category of mixed Hodge modules on a quotient Artin stack such as \( E/G \) is considered by Achar in [Ach]. We will only use basic properties of this category, for example, that it possesses proper pushforwards and basic compatibility with Hodge filtrations. Assume that:

- \( \mathfrak{g} \)-modules Ext\( (\mathfrak{g}, \mathfrak{g}) \) can be endowed with a pure Hodge structure such that the induced Hodge structure on \( \text{Ext}_{\text{MHM}} (\mathfrak{g}, \mathfrak{g}) \) in the category \( \text{MHM} \) of mixed Hodge modules is pure.

This may seem like a rather abstruse assumption, but we’ll see below that it is quite geometrically natural in many situations.

We’ll call an object \( L \) in \( pO_{\mathfrak{g}} \) **gradeable** if it can be endowed with a mixed Hodge structure compatible with those on the simples. Note that this is equivalent to the \( Q \)-modules \( \text{Ext}(-, L) \) or \( \text{Ext}(L, -) \) being formal. In particular, projective objects are always gradeable.

We’ll call an object \( M \) of \( O_{\mathfrak{g}} \) **gradeable** if \( v_1 M \) is gradeable as defined above. Since \( v_1 \) preserves projectives, the projectives of \( O_{\mathfrak{g}} \) are always gradeable. Thus, assuming \( (\mathfrak{g}) \), we can give the projectives \( v_1 P_\alpha \) for the projectives of \( O_{\mathfrak{g}} \) unique mixed Hodge structures with the head pure of degree 0, and thus a natural grading on the endomorphism ring

\[
\mathcal{R}^i := \text{End}(\bigoplus_\alpha P_\alpha) \cong \text{End}(\bigoplus_\alpha v_1 P_\alpha).
\]

Similarly, we can define \( \Omega^i \) as the graded endomorphism ring of the sum of all projectives in \( pO_{\mathfrak{g}} \).
Proposition 2.14. If the assumption \(✠\) holds, the ring \(R'\) is the quadratic dual (in the sense of [MOS09]) of the ring \(R\), and similarly \(Q'\) the quadratic dual of \(Q\). That is, the induced graded lift of the category \(O_g\) is naturally equivalent to the category of linear complexes of projectives over \(R'\).

**Proof.** The category \(O_g\) is the heart of the standard \(t\)-structure on \(D_{O_g}\). Thus, we need only study the image of the standard \(t\)-structure under the functor \(\text{Ext}^\bullet(L,−)\). The subcategory \(D_{≥0}\) is sent to the category of complexes of projectives where the \(i\)th term is generated in degrees \(≥−i\); similarly \(D_{≤0}\) is sent to the category of such complexes generated in degree \(≤−i\). Thus, the intersection of these subcategories is precisely the linear complexes of projectives. The ring \(R'\) is the endomorphism algebra of a projective generator in this subcategory, and thus is the quadratic dual. □

Thus, from [MOS09, Thm. 30], we conclude:

**Theorem 2.15.** If the assumption \(✠\) holds, then the inclusion functor \(\text{D}^b(O_g) \to D_{O_g}\) is a quasi-equivalence if and only if the category \(O_g\) is Koszul. Similarly, the functor \(\text{D}^b(pO_g) \to D_{pO_g}\) is a quasi-equivalence if and only if the category \(pO_g\) is Koszul.

As shown in [BLPW], if the fixed points of \(T\) are isolated, the inclusion \(\text{D}^b(O_g) \to D_{O_g}\) is always an equivalence. Thus:

**Corollary 2.16.** If the fixed points of \(T\) are isolated and \(✠\) holds, the category \(O_g\) will be Koszul.

Of course, this is a very serious geometric assumption, but it holds in many cases, as we will see.

2.5. **Constructing objects in \(pO_g\).** There’s a general method for constructing objects in \(O_g\) in a way which is conducive to calculation; let \(γ\) be a rational lift \(T \to H \times_G G\) of our fixed cocharacter \(T \to H/Z\), that is, a lift of \(T\) after possibly replacing it by a multiple. Let \(g_γ\) be the subalgebra with non-negative weights under \(γ\) in the adjoint representation (this is a parabolic subalgebra) and let \(G_γ\) be corresponding parabolic subgroup. Let \(E_γ \subseteq E\) be the sum of the non-negative weight spaces of \(γ\) acting on \(E\), and \(E_γ^+\) be the sum of positive weight spaces.

Let \(X_γ\) be the fiber product \(G \times_{G_γ} E_γ\). This is a vector bundle over \(G/G_γ\) via the obvious projection map, and has a proper map \(p_γ : X_γ \to E\) induced by sending \((g,e)\) to \(g \cdot e\).

The set of rational lifts is infinite, but if we require our lift to be diagonal in some fixed basis, the space of lifts will be an affine space. There will be finitely many affine hyperplanes in this space given by the vanishing sets of weights in \(E\) and \(g_γ\). The spaces \(E_γ\) and \(G_γ\) are constant on the faces of this arrangement. Thus, only finitely many different \(G_γ\) and \(E_γ\) will occur.

**Remark 2.17.** Note that attached to this data, we have a generalized Springer theory in the sense defined by Sauter [Sau], for the quadruple \((G,\{G_γ\},E,\{E_γ\})\) where \(γ\) ranges over conjugacy classes of generic rational lifts.

Every interesting example we know of a generalized Springer theory is of this form or is obtained from it by the “Fourier transform” operation sending \((G,P_γ,V,F_γ)\) to \((G,P_γ,V^*,F_γ^+)\):
• the classical Springer theory of a group is obtained when $E = g$ and $T$ acts trivially. This is usually presented in the Fourier dual form, so $F_i = b^i$ instead of $F_i = b$.

• the quiver Springer theory is obtained by taking $E$ to be the space of representations of a quiver in a fixed vector space, and $G$ the group acting on such representations by change of basis, and $T$ acting trivially. Note that this definition recovers the generalization of Lusztig’s sheaves defined by Bozec in [Bozb].

Thus at the moment we believe this is the best scheme for viewing these examples.

**Theorem 2.18.** The pushforward $L_\gamma = (p_\gamma)_* \mathcal{S}_{X_\gamma}$ is a sum of shifts of simple modules in $pO_g$.

If there is a cocharacter $\omega : \mathbb{C}^* \to G$ such that $E_\omega < E_\gamma < E_\omega$ and $\langle \alpha, \omega \rangle > 0$, then $L_\gamma$ is supported on the unstable locus for $\alpha$ in $T^*E$.

When such a cocharacter $\omega$ exists, we call $\gamma$ **unsteady**.

**Proof.** This pushforward is a sum of shifts of simple modules by the decomposition theorem, and these are all regular since $\mathcal{S}_{X_\gamma}$ is regular. Thus, we need only show that the singular support of $(p_\gamma)_* \mathcal{S}_{X_\gamma}$ lies in the precore.

As in [Lus91, §13], we can compute the singular support of $(p_\gamma)_* \mathcal{S}_{X_\gamma}$ using the geometry of this situation. Let $Y = \{(x, \xi) \in p_\gamma^* T^*E \mid \langle T^*x, \xi \rangle = 0\}$.

By [Ber 9a & b], we have $SS((p_\gamma)_* \mathcal{S}_{X_\gamma}) \subset p_\gamma(Y)$; thus, we need only show that for any point in $y \in Y$, there is a rational lift of $T'$ which attracts $p_\gamma(y)$ to 0. Thus, let $x$ be a point in $E$ in the image of $p_\gamma$, $x$ is attracted to a limit by $g_\gamma g^{-1}$ for some $g \in G$. If $(x, \xi) \in Y$, then in particular, $\xi$ must kill $g \cdot E_\gamma = E_{g_\gamma g^{-1}}$. Thus, $\xi$ must be a sum of vectors of negative or zero weight for $g_\gamma g^{-1}$ (since the non-negative weight subspace of the dual space is the annihilator of the non-negative weight space of the primal). In particular, $p_\gamma(y)$ has a limit under this lift, and we are done.

Note that the same argument shows that any point in $p_\gamma(Y)$ has a limit under $\omega$; the Hilbert-Mumford criterion shows that this point is unstable in the sense of GIT. □
We should note that the pushforwards \((p_\gamma)_*\) have a natural lift to mixed Hodge modules, since the structure sheaf on a quasi-projective variety carries a canonical Hodge structure, and mixed Hodge modules have a natural pushforward.

**Proposition 2.19.** We have an quasi-isomorphism of dg-algebras

\[
Q := \text{Ext}^\bullet(L, L) \cong H^\text{BM}_G(X \times_E X)
\]

where the latter is endowed with convolution product and trivial differential; the induced Hodge structure on \(\text{Ext}^\bullet(L, L)\) is pure.

**Proof.** In general we have that \(\text{Ext}^\bullet(L, L)\) is quasi-isomorphic to the dg-algebra of Borel-Moore chains on \(X \times_E X\) endowed with the convolution multiplication by Ginzburg and Chriss \([CG97]\). We need only see that the latter is formal; the product \(X_\gamma \times_E X_{\gamma'}\) has a \(G\)-equivariant map \(X_\gamma \times_E X_{\gamma'} \to G/G_\gamma \times G/G_{\gamma'}\) with fibers given by vector spaces \(g_1 \cdot E_\gamma \cap g_2 \cdot E_{\gamma'}\). The variety \(G/G_\gamma \times G/G_{\gamma'}\) has finitely many \(G\)-orbits which are all affine bundles over partial flag varieties, so \(X \times_E X\) is a finite union of affine bundles over partial flag varieties. Since each of these pieces has a pure Hodge structure on its Borel-Moore homology, the Borel-Moore homology of \(X \times_E X\) is pure as well and the higher \(A_\infty\)-operations must vanish on any minimal model. Thus, we are done. \(\Box\)

In \([Sau]\), the algebra \(Q\) is called the **Steinberg algebra**. For every rational lift \(\gamma\), we have a diagonal embedding of \(X_\gamma \hookrightarrow X \times_E X\). Thus, we can view the fundamental class \(\Delta_\gamma^*[X_\gamma]\) as an idempotent element of \(Q\). Under the isomorphism to the Ext-algebra, this corresponds to the projection \(L \to L_\gamma\).

Now assume that:

\((\dagger)\) we have chosen a set \(B\) of rational lifts such that each simple module in \(pO_g\) is a summand of a shift of \(L\), and every simple with unstable support is a summand of \(L_\gamma\) for \(\gamma \in B\) unsteady.

Note that, by Proposition 2.19, we have that

\((\dagger)\) implies condition \((\dagger)\).

**Corollary 2.20.** The condition \((\dagger)\) implies condition \((\dagger)\).

Let \(I \subset Q\) be the ideal generated by the classes \(\Delta_\gamma[X_\gamma]\) for \(\gamma\) unsteady, and let \(R = Q/I\). Note that if \((\dagger)\) holds, then the algebras \(Q\) and \(O\) are Morita equivalent, since they are Ext-algebras of semi-simple objects in which the same simples appear. This further induces a Morita equivalence between \(R\) and \(Q\). Combining Corollary 2.11 and Proposition 2.19 we see that:

**Theorem 2.21.** Subject to hypothesis \((\dagger)\), category \(D_{O_g}\) is quasi-equivalent to \(R\)-dg-mod, via the functor \(M \mapsto \text{Ext}^\bullet(L, M)\).

In the remainder of the paper, we will consider particular examples, with the aim of confirming hypothesis \((\dagger)\) in these cases, and identifying the ring \(R\), thus giving an algebraic description of \(D_{O_g}\).
If $G$ is a torus, with $G \times T$ acting on $\mathbb{C}^n$ diagonally, then we can describe much of the geometry of the situation using an associated hyperplane arrangement. Since the structure of the associated category $O$ is set forth in great detail in [BLPW12, BLPW10], we’ll just give a sketch of how to apply our techniques in this case as a warm-up to approaching quiver varieties. Let $D$ be the full group of invertible diagonal matrices; we let $g_{\mathbb{R}}, d_{\mathbb{R}}$ be the corresponding Lie algebras, and $g_{\mathbb{R}}, d_{\mathbb{R}}$ be the Lie algebras of the maximal compact subgroups of these tori. Choosing a GIT parameter $\eta \in g_{\mathbb{R}}^{\ast}$ and the derivative $\xi : \mathbb{R} \to d_{\mathbb{R}}/g_{\mathbb{R}}$ of the $T$-action, we obtain a polarized hyperplane arrangement in the sense of [BLPW10], by intersecting the coordinate hyperplanes of $d_{\mathbb{R}}^{\ast}$ with the affine space of functionals which restrict to $\eta$ on $g_{\mathbb{R}}$. For simplicity, we assume that this arrangement is unimodular, i.e. all subsets of normal vectors that span over $\mathbb{R}$ span the same lattice over $\mathbb{Z}$.

For each rational lift $\gamma : T \to D$, the weights of $\gamma$ form a vector $a = (a_1, \ldots, a_n)$. The space $E_\gamma$ is the sum of coordinate lines where $a_i \geq 0$; we will encode this by replacing $a$ with a corresponding sign vector $\sigma$ (where for our purposes, 0 becomes a plus sign).

A coordinate subspace will appear in this way if and only if there’s an element of $\xi + g_{\mathbb{R}}$ which lies in the corresponding chamber, that is, if it is feasible.

For each sign vector, we let $g_{\sigma} = \{(b_1, \ldots, b_n) \in g_{\mathbb{R}}| b_i \geq 0\}$. That subspace will be unstable if there’s an element $\omega$ of $g_{\sigma}$ with $\langle \eta, \omega \rangle > 0$; that is, if $g_{\sigma}$ has no minimum for $\eta$ and thus is unbounded.

The arrangement on $\xi + g_{\mathbb{R}}$ is the Gale dual of the one usually used to describe the torus action; as discussed in [BLPW10], the fundamental theorem of linear programming shows that feasibility and boundedness switch under Gale duality.

Thus, in the usual indexing, we find that:

**Proposition 3.1.** The relative precore is a union of the conormal bundles to certain coordinate subspaces. A subspace lies in the relative precore if and only if its chamber is bounded, and contained in the unstable locus if and only if its chamber is infeasible.

Since $G$ is abelian, we have that $G_\gamma = G$ for every rational lift, and the pushforward $L_\gamma$ is just the trivial local system on $E_\gamma$. As before, let $L = \bigoplus L_\gamma$, where the set $B$ consists of one lift corresponding to each bounded sign vector.

**Proposition 3.2.** If $\lambda$ is integral, then hypothesis (†) holds for this set $B$. Every simple in $pO_{\mathbb{R}}$ is of the form $L_\gamma$; if $L_\gamma$ has unstable support, then $\gamma$ is unsteady.

**Proof.** Certainly, every simple object $M$ in $pO_{\mathbb{R}}$ is the intermediate extension of a local system on a coordinate subspace minus its intersections with smaller coordinate subspaces. By applying Fourier transform, we can assume that this coordinate subspace is all of $\mathbb{C}^n$.

The monodromy of moving around a smaller coordinate subspace is given by integrating the vector field $x_i \frac{\partial}{\partial x_i}$ where $x_i$ is the coordinate forgotten. If $x_i \frac{\partial}{\partial x_i} f = af$, then the monodromy acts by multiplication by $e^{2\pi i a}$, as we can see from the case where $f = x_i$. We wish to prove that this monodromy is trivial, which is equivalent to $x_i \frac{\partial}{\partial x_i}$ acting with integral eigenvalues on $M$. 

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The set of $i$ for which $x_i \partial_{x_i}$ acts with integral eigenvalues forms a subarrangement; we’ll call these coordinates integral. In order to prove that all coordinates are integral, it suffices to prove this for a basis of $\mathfrak{d}/\mathfrak{g}$ by unimodularity.

On the other hand, if $x_i$ is not an integral coordinate, then the vanishing cycles along $x_i = 0$ of $\mathcal{M}$ are non-trivial. Thus, we can Fourier transform, switching $x_i$ and $\partial_{x_i}$, and again obtain the intermediate extension of a local system on the complement of coordinate subspaces. Thus, between the corresponding chamber and any unbounded one, there must be at least one integral coordinate hyperplane. This will only be the case if $x_i \partial_{x_i}$ for $i$ integral span $\mathfrak{d}/\mathfrak{g}$. As noted before, unimodularity proves this is only possible if all coordinates are integral.

Thus, the local system induced on any component of the pre-core has trivial monodromy around any isotropic coordinate subspace; thus, we have that every simple is a summand of $L_{\gamma}$ for some $\gamma$. Since $L_{\gamma}$ is just the pushforward of the functions on a linear subspace, it is already simple, so every simple is of this form. Note that if $\lambda$ is not integral, we can have non-trivial local systems.

Thus, every simple in $pO_{\eta}$ is on the form $L_{\gamma}$. If this simple has unstable support, then every point in the subspace $N^*E_{\gamma}$ must have a limit as $t \to 0$ under some cocharacter $\omega$. The cocharacter $\omega$ thus unsteadies the rational lift $\gamma$. □

By Theorem [2.21], we can give a description description of $D_{O_{\eta}}$.

**Corollary 3.3.** The Steinberg algebra $Q = H^{BMG}(X \times_E X)$ is the ring $A_{pol}^i(-\xi, -)$ defined in [BLPW12, §8.6]. The quotient algebra $R = Q/I$ is isomorphic to $A^i(-\xi, -\eta)$.

**Proof.** First, consider the constant sheaves of all coordinate subspaces in the equivariant derived category for $D$. We can identify this with the algebra denoted $Q_n$ in [BLPW12]; for two sign vectors that differ by single entry, the element $\alpha \to \beta$ is given by the pushforward or pullback maps on the constant sheaves on coordinate spaces. In this situation, one will be codimension 1 inside the other, and the composition of pullback and pushforward in either order is the equivariant Euler class of the normal bundle. This is simply the usual polynomial generators of the cohomology of the classifying space of the full diagonal matrices, which we identify with the elements $\theta_i$ in the notation of [BLPW12].

Restricting to objects that lie in $pO_{\eta}$, we only consider the vertices of the $n$-cube corresponding to the bounded sign vectors for the corresponding hyperplane arrangement; furthermore restricting the group acting from $D$ to $G$ has the effect of imposing linear relations on the $\theta_i$’s, exactly those in the kernel of $\mathfrak{d}^* \to \mathfrak{g}^*$. Thus, we obtain the ring $A_{pol}^i(-\xi, -)$.

The classes $\Delta_*[X_\gamma]$ for $\gamma$ unsteady are exactly the idempotents that generate the kernel of the map $A_{pol}^i(-\xi, -) \to A^i(-\xi, -\eta)$. The result follows. □

**Remark 3.4.** Braden, Licata, Proudfoot and the author have proven this result in a different way, by analyzing the category of projectives in $O_\eta$ over the section algebra of $D$ twisted so that localization holds. We show that it is an algebra $A(\eta, \xi)$, which we had previously shown was Koszul dual to $A^i(-\xi, -\eta)$ in [BLPW10]. The result above gives a more direct geometric proof of this fact.
A bit more care in the non-integral case would also yield the general case of [BLPW12, 4.7], but since this is reproving an old result, we leave the details to the reader.

4. Quiver varieties: general structure

4.1. Background. Our primary application is the study of quiver varieties. Quiver varieties are perhaps the most interesting examples of symplectic singularities in the wild. Furthermore, since the work of Ringel and Lusztig in the late 80’s and early 90’s, it has been quite clear that they have a powerful tie to Lie theory, in particular to its categorification.

Definition 4.1. For each orientation \( \Omega \) of \( \Gamma \) (thought of as a subset of the edges of the oriented double), a representation of \((\Gamma, \Omega)\) with shadows is

- a pair of finite dimensional \( \mathbb{C} \)-vector spaces \( V \) and \( W \), graded by the vertices of \( \Gamma \), and
- a map \( x_e : V_{\omega(e)} \to V_{\alpha(e)} \) for each oriented edge (as usual, \( \alpha \) and \( \omega \) denote the head and tail of an oriented edge), and
- a map \( q : V \to W \) that preserves grading.

We let \( w \) and \( v \) denote \( \Gamma \)-tuples of integers.

For now, we fix an orientation \( \Omega \), though we will sometimes wish to consider the collection of all orientations. With this choice, we have the universal \((w,v)\)-dimensional representation

\[
E_{v,w} = \bigoplus_{i \to j} \text{Hom}(C^v_i, C^v_j) \oplus \bigoplus_i \text{Hom}(C^v_i, C^w_i).
\]

In moduli terms, this is the moduli space of actions of the quiver (in the sense above) on the vector spaces \( V = \bigoplus_i C^v_i, W = \bigoplus_i C^w_i \), with their chosen bases considered as additional structure. Let \( e_{ij} \) be the number of arrows with \( \alpha(e) = i \) and \( \omega(e) = j \).

This can be thought of in terms of usual quiver representations by adding a new vertex \( \infty \) with \( w_i \) edges from \( i \) to \( \infty \), forming the Crawley-Boevey quiver. See [Webd, §3.1] for a longer discussion.

If we wish to consider the moduli space of representations where \( V \) has fixed graded dimension (rather than of actions on a fixed vector space), we should quotient by the group of isomorphisms of quiver representations; that is, by the product \( G_v = \prod_i \text{GL}(C^v_i) \) acting by pre- and post-composition. The result is the moduli stack of \( v \)-dimensional representations shadowed by \( C^w \), which we can define as the stack quotient

\[
X^w_v = E_{v,w} / G_v.
\]

As before, this is a smooth Artin stack, which we can understand using the simplicial Borel space construction.

By convention, if \( w_i = \alpha_i^\vee(\lambda) \) and \( \mu = \lambda - \sum v_i \alpha_i \), then \( X^\lambda_\mu = X^\mu_v, E^\lambda_\mu = E^\mu_v \) (if the difference is not in the positive cone of the root lattice, then these spaces are by definition empty), and \( X^1 = \bigcup_i X^1_\alpha \). Let

\[
\mathfrak{M}_\mu^\lambda = T^*E^\lambda_\mu / \text{det}G_\mu = \mu^{-1}(0)^s / G_\mu \quad \mathfrak{M}_\mu^\lambda = T^*E^\lambda_\mu / 0 G_\mu
\]
be the Nakajima quiver variety attached to \( \lambda \) and \( \mu \). See [Nak94, Nak98] for a more detailed discussion of the geometry of these varieties. We are interested in categories of modules over quantizations of these varieties, and specifically, the categories \( \mathcal{O}_g \). In [Weba], we proved that for integral parameters, the categories \( \mathcal{O}_g \) carry a categorical \( G \)-action where \( G \) is the Kac-Moody algebra attached to the graph \( \Gamma \).

The Grothendieck group \( K(\mathcal{O}_g) \) for any conical symplectic variety carries a 2-sided cell filtration induced by the decomposition of \( \mathcal{H} \) into symplectic strata; following the notation of [BLPW, §7], we let \( \mathcal{O}_S^g \) be the subcategory in \( \mathcal{O}_g \) of objects supported on \( \pi^{-1}(S) \), and \( \mathcal{O}_{\partial S}^g \) the subcategory of objects supported on \( \pi^{-1}(\partial S) \). We call a stratum \( S \) special if \( \mathcal{O}_S^g \neq \mathcal{O}_{\partial S}^g \).

In the case of quiver varieties, all the varieties \( \mathcal{H}_\mu^\lambda \) are embedded simultaneously into the affine variety \( \mathcal{H}_\infty^\lambda \) of semi-simple representations of the preprojective algebra up to stabilization. Thus, we can think about these filtrations simultaneously on all the category \( \mathcal{O} \)'s of \( \mathcal{H}_\mu^\lambda \) for all \( \mu \). The most important example of strata are the subvarieties of the form \( \mathcal{H}_{\mu'}^\lambda \) for various \( \mu' \), but other subvarieties can occur if \( \Gamma \) is not of finite type.

**Lemma 4.2.** The 2-sided cell filtration on \( K(\mathcal{O}_g) \) is invariant under the induced action of \( G \).

**Proof.** Since the categorical action is given by convolution with Harish-Chandra sheaves by the construction of [Weba], this follows immediately from the same argument as [BLPW, 7.10]. \( \square \)

Thus, in order to understand this filtration in particular cases, we need only understand where highest weight simples lie. For any representation \( V \) of \( G \), we let the isotypic filtration be the filtration indexed by the poset of dominant weights where \( V_\mu \) is the sum of the isotypic components for \( \mu' \geq \mu \) in the usual partial order on dominant weights.

Obviously, every highest weight vector of weight \( \mu' \) in \( K(\mathcal{O}_g) \) is supported on \( \mathcal{H}_{\mu'}^\lambda \) and thus lies in \( K(\mathcal{O}_g^{\mu'}) \) for \( \mu \leq \mu' \); that is, the isotypic filtration is “smaller” than the 2-sided cell filtration. On the other hand, as discussed in [BLPW, §6], we can also compare this filtration with the BBD filtration on \( H_{BM}(\mathcal{H}_\mu^\lambda) \), which is of necessity “bigger.”

### 4.2. Weighted KLR algebras

As before, let \( H = \text{Aut}_{G}(E_v) \).

**Proposition 4.3.** The group \( H \) is the product \( \prod_{i,j \in \Gamma_w} GL(C^{e_{ij}}) \) over ordered pairs of vertices in the Crawley-Boevey graph \( \Gamma_w \), where \( e_{ij} \) is the number of arrows directed from \( i \) to \( j \). This acts by replacing the maps along edges with linear combinations of the maps along parallel edges; in particular, the contribution of the pair \( (i, \infty) \) is \( GL(C^{v_i}) \).

**Proof.** The group \( H \) is a product of general linear groups of the multiplicity spaces of \( G \) acting on \( E \). The spaces \( C^{e_{ij}} \) are precisely the multiplicity space of \( \text{Hom}(C^{v_i}, C^{v_j}) \) in \( E_{v_i,v_j} \). \( \square \)

If \( \Gamma \) is a tree, then \( H/Z \) is just \( PG_w = \prod GL(C^{v_i})/C^* \) where the \( C^* \) represents the diagonal embedding of scalar matrices. If \( \Gamma \) is a cycle, then \( H/Z \) is a quotient of
without loss of generality assume that these loadings are over finitely many to get the finite set of components of the precore. In fact, we could arrange on the set of lifts. Though there are infinitely many loadings, we only take one loading from each of equivalence classes. As discussed before, the sets \( \mathcal{E}_\sigma \) and \( \mathcal{G}_\sigma \) are constant on the chambers of an affine hyperplane arrangement on the set of lifts. Though there are infinitely many loadings, we only need finitely many to get the finite set of components of the precore. In fact, we could without loss of generality assume that these loadings are over \( \mathbb{Z} \). Let \( p: X_i \to E_v \) be the map forgetting the flags, and let

\[
Z = \bigsqcup_{i,j \in B(v)} X_i \times_{E_v} X_j \quad \text{and} \quad L_v := \bigoplus_{i \in B(v)} (p_i)_* \mathcal{E}_{X_i}[\dim X_i]
\]

In \[\text{[Webd]}, \S3.1\], we also defined a diagrammatic algebra, the reduced weighted KLR algebra, which depends on a choice of 1-cocycle in \( \mathbb{R} \) on \( \Gamma \); of course, for any
On generalized category $O$ for a quiver variety

rational cocharacter $\delta: \mathbb{T} \to H$, we can think of it as a 1-cocycle, and consider its reduced wKLR algebra $\bar{W}_\delta$. Applying Proposition 2.19 we arrive at the conclusion that:

**Theorem 4.6** ([Wead, 4.3]). $\operatorname{Ext}^*(L_\nu, L_\nu) \cong \bar{W}_\delta^\nu$.

In [Wead, §2.6], we considered the quotient of $\bar{W}_\delta^\nu$ which corresponds to killing the sheaves $L_\xi$ attached to unsteady cocharacters. For a GIT parameter $\omega$, we denote this **steadied quotient** by $\bar{W}_\delta^\nu(\omega)$. By Proposition 2.12 we have the immediate corollary:

**Corollary 4.7.** If the hypothesis [†] holds, then the category $D_{\mathcal{O}_k}$ is quasi-equivalent to $\bar{W}_\delta^\nu(\omega) - \text{dg-mod}$.

Note that Theorem A will follow from this corollary once we know that hypothesis [†] holds in the relevant cases.

4.3. **The case of $\mathbb{T}$ trivial.** Consider the special case where $\mathbb{T}$ acts trivially. If we orient $\Gamma$ so that there are no oriented cycles, the precore is the Lagrangian subvariety $\Lambda$ considered by Lusztig [Lus91]. Note that if $\Gamma$ has loops, we obtain the generalization of $\Gamma$ defined by Bozec [Boza]. In this case, the simple $D_k$-modules that appear as summands of $L_\nu$ are the images of under the Riemann-Hilbert correspondence of the sheaves considered by Lusztig [Lus91] in his categorification of the upper half of the universal enveloping algebra. The corresponding algebra is the original KLR algebra of [KL09] Rou.

One fact we will need to use in this paper is that:

**Proposition 4.8.** When $\mathbb{T}$ acts trivially and $\lambda$ is a character of $G$, every simple in $p\mathcal{O}_k$ is a summand of $L_\gamma$ for some lift $\gamma$.

The category $p\mathcal{O}_k$ is the category of strongly $G$-equivariant $D$-modules on $E$ whose characteristic cycle is contained in the subvariety Lusztig calls $\Lambda$; thus, this proposition shows that every such $D$-module arises from Lusztig’s construction. This is closely allied with the hypothesis [†] discussed earlier, but with no assumptions about stability.

This will be deduced from the following result of Baranovsky and Ginzburg [BG]:

**Theorem 4.9.** Let $\mathcal{M}$ be a conic symplectic resolution with quantization $\mathcal{D}$ and $L$ the preimage of the cone point in $\mathcal{M}$. Then the Grothendieck group of sheaves of modules over $\mathcal{D}$ with support in $L$ injects into $H_{\text{top}}(L_\pi; \mathbb{Z})$ under the characteristic cycle map.

This theorem has yet to be published; we’ll note that actually it can be gotten around in the finite and affine cases through the use of more ad hoc arguments.

**Proof of 4.8.** By Theorem 4.9 we can show that the only simples supported on the core of $\mathcal{M}_\lambda$ by simply proving that Lusztig’s construction supplies simples whose characteristic classes span $H_{\text{top}}(L_\pi; \mathbb{Z})$. Each component of $L \subset \mathcal{M}_\lambda$ is the quotient of the preimage of one of the components of Lusztig’s $\Lambda$ for the map $\pi: E_\lambda \to E$.

This follows since there is an order on the components $\Lambda$ such that for each component, one can construct one of Lusztig’s sheaves which has multiplicity 1 along that
component and trivial multiplicity along higher ones, by [KS97, 6.2.2(2)]. Thus, the
classes of the sheaves $\pi^*L_0$ surject to $H^{BM}_L(L; \mathbb{Z})$. This shows that if $pO_g$ for $E^1$ contains
any objects which are not produced from Lusztig’s construction, they are killed by
performing $\pi^*$ and then reduction.

Now, let $M$ be an arbitrary object in $pO_g$ for $E$. There is some $\lambda$ such that the
pullback of $M$ by $\pi: E^1 \to E$ is not killed by the reduction functor $r$. After all, we can
simply choose $\lambda$ large enough that any destabilizing subrepresentation at any point
in the singular support of $M$ has an injective map via the new edges. Since $\pi^*M$
is not killed by reduction, it’s a summand of $\pi^*L_j$, the pullback of one of Lusztig’s
sheaves. Thus $M$ is a summand of $L_j$ and we’re done. □

### 4.4. Twisting functors

The category $O$’s attached to different GIT parameters are related by functors, which we call twisting functors, introduced in [BPW §6]. In that paper, we focused more on the induced functors on the algebraic categories $O$, for different parameters, but these functors have a natural geometric interpretation.

For each quiver variety, we can identify the sets of GIT parameters with $\mathfrak{g}$, the dual
Cartan of $\mathfrak{g}$. Nakajima’s usual stability condition is identified with the dominant
Weyl chamber; choose $\eta \in \mathfrak{g}$, a strictly dominant integral weight.

**Proposition 4.10 ([B.1]BLPWquant).** For any fixed parameter $\xi$ and any finite subset $V \subset W$, there is an integer $n \gg 0$ such that localization holds for the GIT parameter $\xi + n\eta$ for every $w \in V$ on the GIT quotient for $w \cdot \eta$.

We can identify the category $O$ for different GIT parameters with category $O$ for
different quantization parameters using localization functors; throughout this
section, we’ll implicitly identify $O$ for different parameters that differ by integral
amounts, using tensor product with quantized line bundles (the geometric twisting functors of [BPW, §6]). We let $O_{\xi}^{\eta}$ be the geometric category $O$ of the GIT quotient $\mu^{-1}(0)/G_{\mu}$, and $L\operatorname{Loc}_{\xi}^{\eta}$ and $\operatorname{R} \Gamma_{\xi}^{\eta}$ be the localization and $S$-invariant sections functors on this GIT quotient at the period $\xi$.

In this notation, identifying $O_{\xi}^{\eta}$ with $O_8$ for the parameter $\xi + n\eta$ intertwines the twisting functors from [BPW, §6] with the functors given by

$$\mathcal{T}_{\xi}^{\eta} := L\operatorname{Loc}_{\xi}^{\eta} \circ \operatorname{R} \Gamma_{\xi}^{\eta}: O_{\xi}^{\eta} \to O_{\xi}^{\eta}$$

by [BPW 6.29]. Note that these functors depend on $\xi$; in particular, they generate a
very large group of autoequivalences of the derived category $D^b(O_8)$. On the other
hand, one cannot get so many different functors. In particular, the functors $\mathcal{T}_{\xi+\eta}^{\eta}$ and $\mathcal{T}_{\xi+\eta}^{\eta}$ stabilize for $n \gg 0$. We can also describe these functors in terms of reduction
functors and their adjoints.

**Proposition 4.11.** For $n \gg 0$, we have isomorphisms of functors $\mathcal{T}_{\xi+\eta}^{\eta} \equiv \mathcal{T}_{\xi}^{\eta} \circ \mathcal{T}_{\xi}^{\eta}$ and $\mathcal{T}_{\xi+\eta}^{\eta} \equiv \mathcal{T}_{\xi}^{\eta} \circ \mathcal{T}_{\xi}^{\eta}$.

**Proof.** By taking adjoints, it suffices to show the second isomorphism. Since local-
ization holds at $\xi + n\eta$ for the stability condition $\eta$, we have that $\mathcal{T}_{\xi}^{\eta}(D_{\xi+\eta})$ is just
the module $\mathcal{D}_E/\mathcal{D}_E \cdot \mu_{\xi+n\eta}(g)$. Of course, $\tau^\nu_\eta(\mathcal{D}_E/\mathcal{D}_E \cdot \mu_{\xi+n\eta}(g))$ is $\mathcal{D}_{\xi+n\eta}$ (this time on $\mu^{-1}(0)/_{\eta}G_V$), with the functor inducing the obvious isomorphism of endomorphism rings. On the other hand, by the localization we already observed, $\mathcal{D}_{\xi+n\eta}$ generates the category of $\mathcal{D}_{\xi+n\eta}$-modules, so $\tau^\nu_{\xi+n\eta}$ is the unique functor that sends $\mathcal{D}_{\xi+n\eta}$ (on $\mu^{-1}(0)/_{\eta}G_\mu$) to $\mathcal{D}_{\xi+n\eta}$ (on $\mu^{-1}(0)/_{\eta}G_\mu$) and we are done. □

Let $\mathcal{T}_w$ be the functor given by $\tau^\nu_{\xi+n\eta}$ for $n \gg 0$, for all $v \in W$.

**Proposition 4.12.** The functors $\mathcal{T}_w$ define a strong action of the Artin braid group of $g$ on the categories $\mathcal{O}_{g_\nu}^{\xi+n\eta}$ for $v \in W$.

**Proof.** This follows immediately from [BPW] 6.32; the braid relations for the Artin braid group are relations in the Weyl-Deligne groupoid of the Coxeter arrangement. The important point here is that the path $v\eta \to vs_i\eta \to vs_is_i\eta \to \cdots$ for any reduced expression $s_{i_m} \cdots s_i$ is minimal length (that is, it crosses the minimal number of hyperplanes to connect those two points); thus, the functor $\mathcal{T}_{s_{i_m}} \cdots \mathcal{T}_{s_i} = \mathcal{T}_w$ is independent of reduced expression. This establishes all of the Artin braid relations. Since the isomorphism $\mathcal{T}_w \mathcal{T}_{w'} = \mathcal{T}_{ww'}$ for $\ell(w) + \ell(w') = \ell(ww')$ is associative (it induces a unique isomorphism $\mathcal{T}_w \mathcal{T}_{w'} \mathcal{T}_{w''} = \mathcal{T}_{ww'w''}$ when $\ell(w) + \ell(w') + \ell(w'') = \ell(ww'w'')$), this action is strong. □

By work of Maffei [Maf02] Th. 26, we have a $H$-equivariant isomorphism of symplectic varieties $\phi: \mu^{-1}(0)/_{\eta}G_v \to \mu^{-1}(0)/_{\eta}G_{ww'}$. We let $(\mathcal{O}_{g_v}^{\xi})^\nu$ denote geometric category $\mathcal{O}$ for $\mu^{-1}(0)/_{\eta}G_v$ and an integral choice of $\xi$; if we omit $\eta$, it is assumed to be dominant, so the underlying variety is $\mathfrak{m}_v^\nu$.

**Proposition 4.13.** The isomorphism of varieties $\phi$ induces an equivalence of categories. $(\mathcal{O}_{g_v}^{\eta})^\xi \cong (\mathcal{O}_{g_{ww'}}^{\nu})$.

**Proof.** We have Kirwan maps

$$K_v: (g_v^{G_v})^* \to H^2(\mu^{-1}(0)/_{\eta}G_v) \quad K_{ww'}: (g_{ww'}^{G_{ww'}})^* \to H^2(\mathfrak{m}_{ww'}^{G_{ww'}}).$$

If $\alpha^\nu_i(\lambda - \nu)$ and $\alpha^\nu_i(\lambda - w \cdot \nu)$ are both positive, then $G_v$ and $G_{ww'}$ are products of equal numbers of general linear groups and we have a canonical isomorphism $(g^{G_v}_v)^* \cong (g^{G_{ww'}}_{ww'})^*$. In the degenerate cases where one of the vertices of the quiver gives 0 in the dimension vector, we add in a trivially acting $\mathbb{C}^*$ to fix this isomorphism (which is killed by the Kirwan map). Under Maffei’s isomorphism, we have that $\phi^*K_{ww'}(\xi) \cong K_v(w^{-1} \cdot \xi)$. This shows us how to compare quantizations on the two varieties by comparing their periods.

By [BPW] 3.14, the quantization $\mathcal{D}_{\xi-\rho/2}$ of $\mu^{-1}(0)/_{\eta}G_v$ has period $K_v(\xi)$ and the quantization $\mathcal{D}_{ww\xi-\rho/2}$ of $\mu^{-1}(0)/_{\eta}G_{ww'}$ has period $K_{ww'}(w\xi)$. Here $\rho_\mu$ is the character of $\otimes_i \det(V_i)$; this indexing is chosen so that $\mathcal{D}^{opp}_{\xi-\rho/2} \cong \mathcal{D}_{-\xi-\rho/2}$. Thus isomorphism $\phi$ identifies these two quantizations and thus category $\mathcal{O}$ over them by $H$-equivariance. □

Thus, in place of fixing a weight space and considering all GIT conditions, we can instead fix the dominant stability condition and vary the weight space.
Definition 4.14. We let the functors

\[ \mathcal{T}_w : (O_\mathfrak{g})^\eta_v \to (O_\mathfrak{g})^{w\cdot \eta}_v \cong (O_\mathfrak{g})^\eta_w \]

be the transport of the \( \mathcal{T}_w \) via this isomorphism. These again define a strong action of the Artin braid group.

When \( \eta \) is chosen to be dominant, there is another such braid action on any category with a categorical \( \Theta \)-action, that given by Rickard complexes \( \Theta_i \) as defined by Chuang and Rouquier [CR08]. These are compared in recent work of Bezrukavnikov and Losev [BL], building on work of Cautis, Dodd, and Kamnitzer [CDK].

Proposition 4.15 ([BL 3.5]). On \( \oplus \mu D^b(\mathcal{D}_{\mathfrak{g}}^\mu \mathfrak{g}-\text{mod}) \), we have an isomorphism of functors \( \Theta_i \cong \mathcal{T}_s_i \).

5. Quiver varieties: special cases

5.1. Tensor product actions. A tensor product action is one induced by a cocharacter of \( PC_\mathfrak{g} \). This is the same as assigning weakly increasing weights \( \delta_1, \ldots, \delta_\ell \) to the different new edges in the Crawley-Boevey quiver. These actions played an important role in Nakajima’s definition of the “tensor product quiver variety” [Nak01].

In this case, the algebra \( \bar{\mathcal{W}}_{\nu}^\lambda \) is isomorphic to one which appeared earlier in the work of the author [Webb, §?]; this is the algebra \( \mathcal{T}_{\nu}^\lambda \), where \( \lambda \) is the ordered list of the fundamental weights attached to the new edges. The steadied quotient of this algebra corresponding to positive powers of the determinant characters is the tensor product algebra \( \mathcal{T}_{\nu}^\lambda \) also defined in [Webb]. This is an algebra who representation category categorifies the tensor product of representations of highest weights \( \lambda_i \) in an appropriate sense.

Theorem 5.1. Assume \( \Gamma \) has no oriented cycles, has property L, and \( \mathbb{T} \) is given by a tensor product action. Then hypothesis \( (\dagger) \) holds in this case.

Lemma 5.2. The summands of \( L_i^\lambda \) where \( i \) ranges over all loadings are precisely those which appear as summands of \( L_i^{-\infty} \) where \( -\infty \) lies in the support of \( i \).

Proof. This is just a restatement of [Li 8.2.1(4)]; in Li’s notation \( N_{\mathcal{V}D^*} \) is the set of simples in \( O_\mathfrak{g} \) which are summands of \( L_i^{-\infty} \) where \( -\infty \) lies in the support of \( i \), and \( M_{\mathcal{V}D^*} \) is the set of simples with unstable microsupport, i.e., those killed by \( r \). \( \square \)

Lemma 5.3. For any tensor product action, any simple in \( pO_\mathfrak{g} \) with non-zero reduction is a summand of \( L \).

Proof. First, we restrict to the case where the eigenvalues of \( \mathbb{T} \) are distinct. In this case, we can reduce to the fundamental case, by studying the \( \mathbb{T} \)-action on \( \mathcal{M}^\lambda_{\mathcal{V}} \). The fixed points of this action is symplectomorphic to the product \( \mathcal{M}^\lambda_1 \times \cdots \times \mathcal{M}^\lambda_{\ell} \) of quiver varieties attached to the weights for the different eigenvalues. Furthermore, as in [BLPW, Rmk. 5.4], we can induce simple modules up from these fixed point loci to give all the simples of \( O_\mathfrak{g} \).
Now, by Lemma 4.8, the number of such modules on the fixed point locus is \( \prod \dim V_{\lambda_i} \). On the other hand, by work of Li [Li, 8.2.1(4)], this is exactly the number of summands of \( L \) which survive under \( r \). Thus, all simples in \( O_g \) must be of this form.

The general case follows from this one since for the relative core for any tensor product action is contained inside the relative core for one with distinct weights. Thus, any simple in the category \( O'_g \) for any tensor product action is a summand of \( L' \) for a possibly different action. But using Li’s work again, we see that the number of simples in \( O'_g \) that lie in the \( O_g \) is no more than the number that are summands of \( L \); thus they must all be summands of \( L \).

\[ \square \]

**Proof of 5.1.** This is sufficiently similar to the proof of Proposition 4.8 that we only give a sketch. By adding a sufficiently large weight \( \lambda' \) to \( \lambda \), and pulling back by the map \( E^{\lambda + \lambda'} \to E^\lambda \), we can assure that this pullback is not killed by reduction, and thus must be a summand of \( L_\gamma \) for some \( \gamma \). Thus, in turn, shows that our original sheaf was also of this type.

Having established the hypothesis (†), we can give a description of the category \( D_{O_g} \) as \( R \text{-}dg\text{-mod} \), where \( R \) is as defined in Section 2.5. Corollary 4.7 and [Webd, 3.6] will now establish Case 1 of Theorem A.

**Corollary 5.4.** The category \( D_{O_g} \) for a tensor product action is quasi-equivalent to \( T^{\Lambda}_g \)-dg-mod.

Furthermore (✠) holds in this case as well by Corollary 2.20. Thus, if \( T \) has isolated fixed points, then the algebra \( T^{\Lambda}_g \) must be Koszul.

**Theorem 5.5.** If all the weights \( \Lambda \) are minuscule, then the algebra \( T^{\Lambda}_g \) is Koszul.

**Proof.** As noted in Nakajima [Nak01], the fixed points of \( T \) acting on the quiver variety \( \mathcal{M} \) are the product of the quiver varieties attached to the weights \( \lambda_i \). If all \( \lambda_i \) are minuscule, then these quiver varieties are all finite sets of points, so \( T \) has isolated fixed points. Thus Theorem 2.15 implies the Koszulity of these algebras.

At least in finite type, we can also easily understand the cell filtration.

**Proposition 5.6.** If \( \Gamma \) is an ADE Dynkin diagram, then the 2-sided cell, isotypic and BBD filtrations all coincide.

**Proof.** Since the 2-sided cell filtration is sandwiched between the isotypic and BBD filtrations, if these two coincide, the 2-sided cell filtration must match them. By [Nak98, Rmk. 3.28], the isotypic and BBD filtrations coincide, so this is indeed the case.

We can generalize Corollary 5.4 a bit to include interactions between different category \( O \)'s for the different tensor product actions.

The set of cocharacters \( \delta : T \to T_w \) carries an \( S_n \) action which preserves the weight spaces of \( T \) and the set of weights which occur, while permuting the order of the weight spaces. There are functors relating the category \( O \)'s for different \( T \) actions in the most obvious way possible: we have a the obvious inclusion \( i^\delta : D_{O_{\overline{g}}} \to \)
For two different $T$-actions, we can always take the shuffling functors $\mathcal{S}_{\sigma',\sigma} := i_{\sigma'}^g \circ i^g : D_{O^g} \to D_{O^g}$.

Similarly, we have versions of these functors $I^g_i, I^g_e : \mathcal{D}_E \text{-mod} \to D_{pO}$ defined before reduction.

**Lemma 5.7.** The functor $x$ intertwines $I^g_i, I^g_e$ with $i^g_i, i^g_e$.

**Proof.** This follows instantly from the fact that both $x$, and $i$ intertwine $I^g$ with $i^g$.

**Theorem 5.8.** The shuffling functors between $D_{O^g}$ for all permutations $\sigma$ and a fixed tensor product action $\mathcal{S}$ give a weak action of the action groupoid for the $\ell$-strand braid group on total orders of an $\ell$ element set. This is intertwined by the equivalences with the action of this group on dg-modules over $T_{\mathcal{A}}$ defined in [Webb ??].

**Proof.** The functor $I_{\mathcal{A}}^g \circ \mathcal{L}$ is intertwined with the functor $W_{\mathcal{A}} \text{-dg-mod} \to W_{\mathcal{A}} \text{-dg-mod}$ given by tensor product with the bimodule $B_{\mathcal{A}}^g$ by [Webb 4.10]; for a tensor product action, this is exactly the bimodule $\mathcal{B}_g$. Thus, for a simple $M$ in $O_g$, there is a simple $M'$ in $pO_g$ which reduces to it, and

$$G(i^g_\sigma \circ i^g(M)) \cong T_{\mathcal{A}}^L \otimes W^g \mathcal{B}_g^L \otimes W^g G(M).$$

Thus, we need only check that $T_{\mathcal{A}}^L \otimes W^g \mathcal{B}_g^L \cong \mathcal{B}_g$. Obviously, the degree 0 part of this is correct; we just need to see that the higher $\text{Tor}$’s vanish. The algebra $T_{\mathcal{A}}$ is standardly stratified just as $T_{\mathcal{A}}$ is, and we can apply the triangular induction used in [Webb ??] to see that the higher $\text{Tor}$’s above vanish, and that $\mathcal{B}_g$ has a standard filtration. Thus, we are done. This checks the statement on a generating set of the action groupoid, and checks all necessary relations, since these are known for the functors induced by the bimodules above by [Webb ??].

5.2. **Affine type A.** Of course, the case of a non-tensor product action is more complicated; there are of necessity more simple modules. The first interesting such case is when $g$ is $\tilde{sl}_\ell$. In this case, we’ll identify the nodes of $\Gamma$ with the residues in $\mathbb{Z}/\ell\mathbb{Z}$ with arrows from $i$ to $i+1$ for every $i$. We’ll want to include the case where $\ell = 1$, that is, the Jordan quiver.

A careful reader might note that in this case, there will be $G$-invariant non-constant polynomial functions on $E$ (the coefficients of the characteristic polynomial of the composition of all edges of the cycle). Note that this condition was only necessary to prove the finiteness of the number of components of the precore, so as long as we assure that condition holds, we will have no issues.

In this case, we can think of the space $V \cong \bigoplus_i V_i$ as a single $\mathbb{Z}/\ell\mathbb{Z}$-graded space. For each $i$, we have a map $x_{i,i+1} : V_i \to V_{i+1}$. We can view the sum $x = \sum x_{i,i+1}$ of the maps along the edges. This is homogeneous of degree 1; we use this convention throughout the section.

As we indicated before, in this case $H/Z$ is a quotient of $PG \times \mathbb{C}^*$ by a finite central subgroup. Thus, a rational cocharacter into $H/Z$ is essentially the same as choosing rational numbers $\delta_i$ given by the weights of the new edges (though these are only
uniquely specified up to simultaneous translation), as well as the rational number $\kappa$ giving the weight of the projection to the last factor. We assume for the sake of simplicity that the weights $\vartheta_i$ are all distinct modulo $\kappa$. This is sufficient for the associated action to have isolated fixed points on $M^\lambda_\nu$ for all $\nu$ and for every component of the pre-core to come from a lift with no positive dimensional fixed subspaces. In fact, we could strengthen this a bit to a necessary and sufficient condition for isolated fixed points for all $\mu$: that $\vartheta_i - \vartheta_j - \kappa(r_i - r_j)$ is not divisible by $e\kappa$.

We wish to analyze the structure of the category $O$ for such an action. In order to do this, we should consider the structure of the varieties $E_{w,v}$ in this case. Obviously, if we forget the map $q: V \to W$, that is forget the new vertex, then the orbits of $G_v$ are classified by multisegments, giving the Jordan type of $x$. However, we must analyze the situation more carefully to take the map $q$ into account.

**Definition 5.9.** We call a lift $\gamma$ of $\vartheta$ and its associated grading **tight** if the map $x$ is fixed by $\gamma$; in the associated graded, this means that $x$ is homogeneous of weight $\kappa$. Obviously, each representation has a tight lift under which it has a limit. We call such a lift and grading **minimal** if the dimension of $V_{\vartheta \leq \beta}$ is as large as possible amongst small deformations of the grading which remain tight.

If we only consider usual lifts, many representations will have no minimal lift, but if we consider gradings where $-\infty$ is also allowed as a grade, then one will exist. From now on, we will only consider gradings valued in $[-\infty, \infty)$.

**Proposition 5.10.** For any representation, there is a multipartition $\xi = (\xi_1, \ldots, \xi_\ell)$ such that in the minimal grading, $\dim V_{\vartheta \leq \beta} + p\varphi$ is the number of boxes of content $p$ in $\xi_i$, and the only other degree that appears is $-\infty$.

**Proof.** Consider a minimal grading on $V$. We can divide the pieces of such a grading according to their residue modulo $\kappa$. Consider the structure of the pieces with a fixed residue modulo $\kappa$. Each of these is a sum of Jordan blocks for $x$, and we wish to analyze which of these can occur.

We cannot have a summand of this submodule such that $q$ sends $V_{\vartheta \leq \beta}$ to $W_{< \vartheta}$ or this would contradict minimality; that is, we must have that each Jordan block for $x$ in this subspace intersects $V_{\vartheta}$, since otherwise we can decrease the grading on such a block by $\varepsilon$.

Thus, we have a multisegment $m_i$ formed by the Jordan blocks of $x$ intersecting $V_{\vartheta_i}$. These are, in fact, defined by the pair of integers $(a_i, b_i)$ such that the block ranges from $V_{\vartheta_i + a_i}$ to $V_{\vartheta_i + b_i}$. We claim that we can order blocks in such a way that the $a_i$'s are strictly decreasing, and the $b_i$'s strictly increasing.

The obstruction to this would be if two blocks interlace in the sense that one has highest degree $a_i$ and lowest $b_i$ with

$$a_1 \geq a_2 \geq \vartheta_i \geq b_1 \geq b_2.$$ 

However, in this case, we have a graded map from the first block to the second. We can use this map to change the splitting so that $q$ sends the first block to $W_{< \vartheta_i}$. Thus, this grading is not minimal.
This shows that these blocks must be nested. There is a unique partition $\xi_i$ such that the pairs $(a, b)$ are the arm and leg lengths for the boxes in the column through the nadir of the partition i.e. the boxes $(j, j)$. Put a different way, the partition $\xi_i$ is produced by bending each block into an L-shape with the kink at $\vartheta_i$ and stacking these to make a diagram in Russian notation, as shown in Figure 1. In the case shown in Figure 1, the pairs of $(a, b)$ are $(3, -5), (1, -3), (0, 0)$ (as suggested by the black bars).

In the abacus model for partitions, this is placing beads in all negative positions except those of the form $b_i - \frac{1}{2}$, and only in the positive positions $a_i + \frac{1}{2}$. □

The structure of the set $Y_{\xi, m}$ of representations with a fixed multipartition $\xi$ and multi-segment $m$ arising from their minimal tight lift is easy to describe: it is an affine bundle over a homogeneous space for the group $G_v$. 

Proposition 5.11. Every simple in $pO_g$ is smooth along the sets $Y_{\xi, m}$ and is the intermediate extension of $\mathcal{Z}_{Y_{\xi, m}}$ for some choice of $\xi$ and $m$.

Proof. First, we should note that for any lift, whether a representation $(x, q)$ has a limit under a conjugate of this lift depends only on the structure of its minimal lift. A lift is essentially determined by assigning a collection of weights to each node. If we associate the usual diagram to $\xi$, and a series of rows of the corresponding length and residue to $m$, then we have a limit if and only if we can fill diagram with the weights, matching residues, so that in each of the rows for $m$, and each of the “L”s in the diagram for $\xi$ (as in Figure 1), for each pair of consecutive entries $w$ and $w'$ (reading right to left), we have $w' + \kappa \geq w$. Thus, every component of the precore is the conormal bundle to $Y_{\xi, m}$ for some $(\xi, m)$, and every simple in $O_g$ must be a local system on one of these varieties.

If we let $Z_{\xi, m}$ be the subspace of $Y_{\xi, m}$ where the map $q$ is homogeneous of degree 0, then we have a map $Y_{\xi, m} \rightarrow Z_{\xi, m}$ replacing $q$ by its degree 0 part. This map is an affine space bundle whose fiber is the spaces of appropriate maps of strictly negative degree. The space $Z_{\xi, m}$ has a transitive action of $G_{\mu}$, with stabilizer given by the automorphisms of the representation for the multi-segment $m$; in particular, this stabilizer is connected, so $Z_{\xi, m}$ and thus $Y_{\xi, m}$ are equivariantly simply connected. □
Weighted KLR algebras have natural quotients, which we call steadied quotients. The special case which is important for us is that denoted $T^\theta$ in [Wecb]. This is the quotient of $\bar{W}^{\theta}$ by all idempotents associated to loadings where the points of the loading can be divided into two sections, with a gap of distance $> k$ and the left group to the left of all red strands. The definition is somewhat complex, so we refer the reader to that given in [Wecb, §??].

**Proposition 5.12.** Every simple in $pO_g$ is a summand of $L_\nu$. The sheaves with unstable characteristic varieties are exactly the summands of such push forwards where $i$ has support at $-\infty$.

*Proof.* In order to check that every simple is a summand of $L_\nu$, we can simply use the pigeonhole principle. The simples (and thus indecomposable projectives) of $\bar{W}^{\theta}$ are in bijection with multipartitions and multi-segments with the right number of boxes: each simple module over $\bar{W}^{\theta}$ is the quotient of an induction of a unique simple module over $W^{\theta}_{\nu'}$ for $\nu'$ just supported on $\Gamma$ with a unique simple module over $T^\theta_{\nu''}$ with $\nu = \nu' + \nu''$. The former is Morita equivalent to a quiver Schur algebra by [Webd, 3.9], whose representations are in bijection with multisegments by [SW, 2.12]; the latter are in bijection with multipartitions by [Wecb, ??].

Thus, this is also the number of distinct simple summands of $L_\nu$. Similarly, by Proposition 5.11, simples in $pO_g$ are in bijection with the same objects. Since the summands of $L_\nu$ are all of the simples in $pO_g$ by the pigeonhole principle there can be no others.

Now, consider the sheaf $L_i$ with $i$ an unsteady loading; that is, the loading $i$ has some group of dots which are “far left” of all others, including the red lines. A point in the microsupport of $L_i$ is necessarily unstable since (thought of as a representation of the pre-projective algebra), it must have a submodule not containing $V_\infty$, the vector space on the new vertex $\infty$. Note that the number of these is the number of pairs as above with the multi-segment non-trivial (since the number of simples that survive is the number of multipartitions).

On the other hand, the number of unstable components of the precore is also the number of pairs with non-trivial multi-segments, so there can be no more than this many simples with unstable micro-support. Thus, all must be summands of $L_i$ for $i$ unsteady.

**Corollary 5.13.** For $\Gamma$ a cyclic quiver and a generic $T$-action, hypothesis $\nabla$ holds.

In particular, we have quasi-equivalences

$$\bar{W}^{\theta} \text{-dg-mod} \cong D_{pO_g} \quad T^\theta \text{-dg-mod} \cong D_{O_g}.$$  

*Proof.* This combines Proposition 5.11 and 5.12.

We can also understand the cell filtration in this case; unfortunately, this is more challenging than the finite type case, since the variety $\mathfrak{Y}_\mu^\lambda$ has strata which are not of the form $\mathfrak{Y}_{\mu'}^\lambda$ for $\mu' \geq \mu$ now. For example, $\mathfrak{Y}_0^{\lambda\mu} \cong C^{2n}/(\mathbb{Z}/\mathbb{Z} \wr S_n)$ and counting shows that there are not enough weight spaces to account for all the strata.

In particular, for a weight $\mu$ and integer $n$, there is a stratum closure we denote $\mathfrak{Y}_{\mu,n}^\lambda$ in $\mathfrak{Y}_{\mu}^\lambda$ given by representations of the preprojective algebra given by a sum of
• a simple representation with dimension vector $v_i$ with $v_\infty = 1$ and $\lambda = \mu + \sum_{i \in I} v_i \alpha_i$.
• a semi-simple representation with dimension vector given by $v'_\infty = 0$ and $v'_i \leq n$ for all $i \in I$, and
• a trivial representation.

To simplify notation, we denote the subcategory of $O_\ell$ supported on $\mathcal{H}_\mu^\lambda$ by $\mathcal{O}_\ell^{\mu,n}$. Note that for $n$ sufficiently large that $\mu - n \delta \leq \mu'$, this subset is independent of $n$, so we can use $n = \infty$ to indicate this stable range, and thus speak of $\mathcal{H}_\mu^\lambda$ and $\mathcal{O}_\ell^{\mu,\infty}$.

In this case, we can also refine the isotypic filtration to account for the presence of a $\hat{\mathfrak{g}}_\ell$-action on the vector space $K(O)$; there are both isotypic filtrations for $\hat{\mathfrak{g}}_\ell$ and $\hat{\mathfrak{g}}_L$. We let $J_{\mu,\gamma}$ be the intersection of the spaces generated under $\mathfrak{g}_L$ by highest weight vectors of weight $\geq \mu$ and under $\hat{\mathfrak{g}}_\ell$ by highest weight vectors of weight $\geq \gamma$. This filtration also has a geometric interpretation: it is the coarsening of the BBD filtration to only include pieces corresponding to the strata $\mathcal{H}_\mu^\lambda$.

**Theorem 5.14.** Assume $\mu$ is dominant. The special strata of $\mathcal{H}_\mu^\lambda$ are exactly those of the form $\mathcal{H}_{\nu,\mu}$ and the 2-sided cell filtration on $K(O_\ell)$ matches the refined isotypic filtration, with $K(O_\ell^{\mu,\gamma}) \equiv J_{\nu,\gamma}$ and more generally, $J_{\nu,\gamma} \equiv K(O_\ell^{\nu,\gamma}) \cap K(O_\ell^{\nu,\gamma})$.

Note that this shows that these quantizations of $\mathcal{H}_\mu^\lambda$ are *interleaved* in the sense of [BLPW, §6].

**Lemma 5.15.** Assume $\mu$ is dominant. The simples in $pO_\ell$ that have a closed free orbit in their support are precisely those corresponding to projectives in $\hat{T}_\mu^{\delta}$ which are not summands of a left or right induction.

**Proof.** Obviously, any summand of a left or right induction has a submodule at each point of the singular support, so each of these orbits is not closed if the submodule doesn’t split, or not free if it does.

Thus, every simple with a closed free orbit in its support is not an induction. On the other hand, the number of simples with a closed free orbit in their support is at least the number of components of $\mathcal{H}_\mu^\lambda$ which contain a closed free orbit, which is at least the dimension of the space of $U(\hat{\mathfrak{g}}_\ell)$-highest weight vectors of weight $\mu$ for the Nakajima action on $H_{BM}^{BM}(\mathcal{H}_\mu^\lambda)$, which can be identified with a level $\ell$ Fock space.

Now, consider the projectives of $\mathcal{H}_\mu^\lambda$ which are not inductions. These give to projectives over $\hat{T}_\mu^{\delta}$ whose simple quotients are a basis of highest weight vectors of weight $\mu$ for $U(\hat{\mathfrak{g}}_\ell)$ acting on $K^0(T_\delta^{\delta} \mod)$. Since this is also a level $\ell$-Fock space, this must coincide with the number of such highest weight vectors for $H^{BM}(\mathcal{H}_\mu^\lambda)$. By the pigeonhole principle, we must have that every projection which is not an induction corresponds to a simple with a closed free orbit in its support.

**Proof of Theorem 5.14** First, note that we must have $J_{\nu,\gamma} \subset K(O_\ell^{\nu,\gamma}) \cap K(O_\ell^{\nu,\gamma})$ since $K(O_\ell^{\nu,\gamma})$ is closed under the action of $\hat{\mathfrak{g}}_\ell$, and $K(O_\ell^{\nu,\gamma})$ under the action of $\hat{\mathfrak{g}}_\ell$. Thus, we need only establish the coincidence of dimensions to check equality.
What we need to establish is this: the number of simples of weight \( \nu - n \delta \) with support in \( \mathcal{R}_{v,n}^A \) and no smaller stratum is equal to the number of highest weight vectors of \( \hat{\mathfrak{g}}_\ell \) of weight \( \nu \) times the number of integer \( \ell \)-multipartitions of \( n \) (since each \( \hat{\mathfrak{g}}_\ell \)-highest weight vector generates a vacuum representation of the Heisenberg consisting of \( \hat{\mathfrak{sl}}_\ell \)-highest weight vectors). Furthermore, comparison with the BBD filtration shows that there can be no more than this number. For \( n = 0 \), this follows immediately from Lemma 5.15.

Applying Fourier transform if necessary, we may assume that \( \kappa > 0 \). Consider the D-module \( Y_i \) where \( i \) is the loading putting a dot labeled \( i \) at \( k + ei, \ldots, nk + ei \) for \( i \in I \) for \( k \gg \kappa \gg \epsilon \); that is, the loading with \( n \) clusters with each having dimension vector \( \delta \).

The space \( C_i \) given by the first \( i \) clusters is invariant in the usual sense and thus gives an invariant flag on each point in \( X_i \) with \( \dim C_i/C_{i-1} = \delta \). On \( C_i/C_{i-1} \), the map going around the cycle must give a scalar, which is the same at all points of the cycle. Thus, we have a natural map \( X_i \to C^n \) sending a representation with flag to the \( n \)-tuple of scalars associated to \( C_i/C_{i-1} \). If we let \( C^n_\xi = \{ (x_1, \ldots, x_n) \in C^n \mid x_i \neq x_j \neq 0 \text{ for all } i \neq j \} \) then we have an open inclusion \( C^n_\xi \to X_i \) sending \( (x_1, \ldots, x_n) \) to the representation where the spaces are all \( C^n \) equipped with the standard flag and the map along one edge is \( \text{diag}(x_1, \ldots, x_n) \) and along all the others is the identity. We have a Cartesian diagram

\[
\begin{array}{ccc}
C^n_\xi & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
C^n_\xi/S_n & \longrightarrow & E_{n\delta}/G_{n\delta}
\end{array}
\]

since two points in \( C^n_\xi \) will give isomorphic representations if and only if they differ by a permutation. Thus, to each representation of \( S_n \) (and thus to a partition \( \xi \) of \( n \)), we have an induced local system on \( C^n_\xi/S_n \), and thus an intermediate extension D-module \( Z_\xi \) on \( E_{n\delta}/G_{n\delta} \). The Cartesian diagram above shows that these are all summands of \( Y_i \) and lie in \( p\mathcal{O}_g \).

For every simple \( L \) in \( \mathcal{R}_{v,0}^A \) which is not an induction, we can consider the induction \( L \circ Z_\xi \); this lies in \( \mathcal{O}_g \) and has at least one simple summand whose support lies in \( \mathcal{R}_{v,0}^A \) and no smaller locus. Furthermore, these are all distinct, as comparing the induced local system on any piece of the support lying over the generic locus of \( \mathcal{R}_{v,0}^A \) shows. Thus, we have recovered the desired number of distinct simples with this property, and we are done. \( \square \)

One important consideration is how this filtration can be realized algebraically. Let \( \mathcal{F}_{\mu,\gamma} \) be the intersection of the subcategories generated by objects \( T^\delta \cdot\text{-dg-mod} \) of weight \( \geq \mu \) under induction with projective modules over the weighted KLR algebra of \( \Gamma \), and by objects of weight \( \geq \gamma \) under the action of projectives for Hecke loadings (i.e. of the usual KLR algebra). It’s manifest from the match of categorical \( g \)-actions on \( T^\delta \cdot\text{-dg-mod} \) and \( \mathcal{O}_g \) that:
Proposition 5.16. The quasi-equivalence $T^g$-dg-mod $\cong O_{\hat{g}}$ induces quasi-equivalences $\mathcal{J}_{\mu,\gamma} \cong O_{\hat{g}}^{\mu,\gamma} \cap O_{\hat{g}}^{\gamma,\mu}$. \hfill \Box

In [Webb], we introduced change-of-charge functors which are weighted analogues of the R-matrix functors from [Webb, §7]; they have a very similar geometric definition. These can be assembled into a strong action of the affine braid group $\hat{B}_\ell$.

By a re retard of the argument in Proposition 5.8, we see that:

Corollary 5.17. The quasi-equivalence $T^g$-dg-mod $\cong D_{O_{\hat{g}}}$ intertwines the change-of-charge functor $\mathcal{B}^{g,\gamma}_L \otimes -$ with the shuffling functor $\mathcal{J}^{g,\gamma}_L$. \hfill \Box

5.3. Koszul duality. The ring $T^g$ is Koszul and its Koszul dual is another algebra of the same type. In order to state this precisely, let us briefly describe the combinatorics underlying this duality. Fix an integer $w$ and a $\ell \times e$ matrix of integers $U = \{u_{ij}\}$, and let $s_i = \sum_{j=1}^\ell u_{ij}$ and $t_j = \sum_{i=1}^e u_{ij}$. Associated to each row of $U$, we have a charged $e$-core partition; let $v_i$ be the unique integer such that $v_i - w$ is the total number of boxes of residue $i$ in all these partitions. We wish to consider the affine quiver variety for the highest weight $\lambda := \sum_i w_i$, with the dimension vector $v_i$; that is, with weight $\mu := \lambda - \sum v_i a_i$.

In order to describe a category $O_{\hat{g}}$, we also need to consider a $T$-action on this quiver variety. This is equivalent to a weighting of the Crawley-Boevey graph for $\lambda$, where we enumerate the new edges so that $e_i$ connects to the node corresponding to the residue of $s_i$ (mod $e$). We let $\delta_U$ be the weighting where we give each edge of the oriented cycle weight $\ell$, and the new edge $e$, the weight $s_i \ell + ie$.

Proposition 5.18. Every generic cocharacter $T \to H/Z$ has a category $O$ which coincides with that of $\pm \delta_U$ for some $U$.

Proof. Scale the cocharacter until the weight of the cycle is $\ell e$, and choose a lift where all old edges have weight $\ell$. For each $i$, let $r_i$ be the unique integer in $\{1, \ldots, e\}$ indexing the node that $e_i$ attaches to. Now, for each edge $e_i$ with weight $\delta$, we write $\delta_i = s_i \ell + \delta'_i$ with $0 \leq \delta'_i < \ell e$ and $s_i \equiv r_i$ (mod $e$). We can reindex the edges so that $\delta'_1 < \cdots < \delta'_e$; by genericity, these are all distinct.

Having made this reindexing, the action corresponding to $s_i$ is the one we desire. We can see that this is equivalent to our original action, since it has the same sign of the weight on any function on the quiver variety: any such function is given by the trace of a loop in the Crawley-Boevey quiver. The weight of such a trace is of the form $(\delta_i - r_i \ell) - (\delta_j - r_j \ell) + gel$ for $g \in \mathbb{Z}$; since

$$(\delta'_i + (s_i - r_i) \ell) - (\delta'_j + (s_j - r_j) \ell) + gel.$$

On the other hand for our standard action, the same loop will have weight

$$(ie + (s_i - r_i) \ell) - (je + (s_j - r_j) \ell) + gel.$$

If $s_i - r_i + ge > s_j - r_j$, then this weight is positive in both cases (since $\delta'_i - \delta'_j < \ell e$ and similarly with $(i - j)e < \ell e$) and similarly with the opposite inequality. If $s_i - r_i + ge = s_j - r_j$, then $\delta'_i < \delta'_j$ if and only if $i < j$ (by definition), so we are done. \hfill \Box

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Let \( U' \) be the transpose of \( U \), and \( \mu' = \lambda(U'), \lambda' = \mu(U') \) the weights attached to \( U' \) and \( w \) by the recipe above. We have switched the roles of \( \mu \) and \( \lambda \) here, since changing \( \mu \) and holding \( \lambda \) constant will change \( \mu' \) and vice versa. More generally, we have an order reversing bijection from the interval \([\lambda, \mu]\) in the poset of weights to \([\mu', \lambda']\), which we denote by \( v \mapsto v' \).

**Theorem 5.19 ([Webe Th. C]).** The graded abelian categories \( T^{u \circ \delta_u}_{\mu} \)-mod and \( T^{u' \circ \delta_{u'}}_{\lambda'} \)-mod are Koszul dual.

Note that Theorem 2.15 gives a new proof that these algebras are Koszul. Using Theorem 5.12, we can give a reformulation of this result. We let \( O^{\lambda; \delta} \) be the category \( O \) for Nakajima’s stability condition on the quiver variety \( M^1 \) for the \( T \) action coming from \( \delta \).

**Corollary 5.20.** We have an equivalence of abelian categories \( O^{\lambda; \delta_u} \cong T^{u \circ \delta_u}_{\mu} \)-mod. In particular, categories \( O^{\lambda; \delta_u} \) and \( O^{\mu'; -\delta_{u'}} \) have graded lifts which are Koszul dual.

By Proposition 5.18 this describes the Koszul dual of every integral geometric category \( O \) for an affine type A quiver variety. Of course, this functor is quite inexplicit; it would be very interesting to define a more concrete functor between these categories.

We would like to understand how various geometric and algebraic constructions match under this Koszul duality. In particular, the categories \( T^{\mu} \)-mod are interesting not just on their own, but because they carry an interesting action of change-of-charge functors. These correspond to changing \( \delta_u \), while keeping \( \lambda^l \) constant; according to the prescription given above, this would be accomplished by acting on \( \mu \) by the affine Weyl group of \( \tilde{\mathfrak{k}}_e \), but as we already noted in Section 4.4, acting on this weight and keeping the stability condition constant has the same effect as changing stability condition while keeping the dimension vector constant. Thus, we let \( O^w \) denote the category \( O \) for the stability condition \( w^{-1} \cdot \xi \), and let \( v^w \) denote the corresponding quotient functor.

Thus, the change-of-charge functors relate category \( O \)'s which are actually quotients of a single common category \( pO \) for the \( G \)-action on \( E_{\mu}^l \) and the \( C^* \)-action corresponding to \( \delta_u \). Furthermore, it is relatively easy to understand the structure of these quotient functors. First note that:

**Lemma 5.21.** If \( L \) is a simple object in \( pO \) with \( v^w(L) \neq 0 \) and \( P \) is a projective cover of \( v^w(L) \), then \( v^w(P) \) is a projective cover of \( L \).

In particular, if \( L \) is a simple whose singular support contains a closed free \( G_{\mu'} \)-orbit, then \( v^w(L) \) is non-zero for all stability conditions, and there is a projective \( Q \) in \( pO \) such that \( v^w(Q) \) is the projective cover of \( v^w(L) \) in the category \( O \) for any \( v \in \hat{B}_e \).

**Proof.** Since \( v \) is exact, \( v^w(P) \) is projective, and obviously has a natural surjective map \( v^w(P) \to L \). Since \( \text{End}(v^w(P)) \cong \text{End}(P) \) is local, this projective is indecomposable and thus a projective cover of \( L \).

Now, turn to the second part; by the first part, we have \( Q \cong v^w(P) \), so \( P \cong v^w(Q) \).
Thus, one common tie between these different categories $O$ is the collection of simples whose singular supports contain a closed free orbit. We wish to understand how these simples in $p\mathcal{O}_g$ match up with simple modules over $T^{\hat{s}^\vee s'}$. As a first step, we consider which projectives of $T^{\hat{s}^\vee s'}$ they match with.

More generally, we should understand how the cell filtration behaves under Koszul duality. We expect that the quiver varieties for dual data $\mathcal{W}^{\Lambda}_{\mu}$ and $\mathcal{W}^{\Lambda'}_{\mu'}$ will have an order reversing bijection between their special strata.

**Proposition 5.22.** The map $\mathcal{W}^{\Lambda}_{\nu n} \leftrightarrow \mathcal{W}^{\Lambda'}_{\nu' n + \delta}$ is an order reversing bijection between special strata. Koszul duality induces a bijection between simple modules which sends modules with support contained in $\mathcal{W}^{\nu}_{\mu n}$ but not any smaller stratum to those supported in $\mathcal{W}^{\nu' + \delta}_{\mu' n}$ but not any smaller stratum.

**Proof.** The statement about strata follows immediately from Theorem 5.14 and the fact that $\nu \mapsto \nu'$ is an order reversing bijection and $(\nu - n\delta)^! = \nu' + n\delta$.

The statement about simple modules follows since a simple is in $\mathcal{W}^{\nu}_{\mu n}$ and no smaller stratum if it is in a component of the $\hat{\mathfrak{sl}}_\ell$-crystal structure with highest weight $\nu - n\delta$, and that this is in the $\hat{\mathfrak{sl}}_\ell \times \hat{\mathfrak{sl}}_\ell$ crystal orbit of $\tau(Z_\xi)$ for $\xi$ a partition of $n$. Under rank-level duality, the $\hat{\mathfrak{sl}}_\ell \times \hat{\mathfrak{sl}}_\ell$ crystal operators switch roles, and the set $\{|\tau(Z_\xi)|_\xi = n\}$ is preserved; so the conditions are symmetric, and the result follows from the interaction between $\hat{\mathfrak{gl}}_\ell$ and $\hat{\mathfrak{gl}}_\ell$-weights. □

The simples of $\mathcal{O}_g$ has a finer decomposition into sets called left and right cells. We say that $L$ and $M$ are in the same **left cell** if their sections $\Gamma_S(L)$ and $\Gamma_S(M)$ have the same annihilator (where we assume we have translated to a period where localization holds) and in the same **right cell** if $L$ is a composition factor in $B \otimes M$ for some Harish-Chandra bimodule and vice versa.

**Conjecture 5.23.** Two simples are in the same left cell if they have the same weight and the same string parametrization for $\hat{\mathfrak{sl}}_\ell$; similarly, two simples are in the same right cell if and only if they have the same weight, and the same string parametrization for $\hat{\mathfrak{sl}}_\ell$.

One important special case of this correspondence is where $\mu = \nu$; in this case, the simples whose support lie in no smaller stratum are exactly those whose support contains a free closed orbit. Proposition 5.22 shows these match under Koszul duality with simples supported over the point strata; these are the same as simples in the category generated by the highest weight object of weight $\mu'$. That is, they match with the projectives corresponding to Hecke loadings. Thus, if we let $P_0$ be the sum of these projectives as before, and abuse notation to let it denote the corresponding projective in $\mathcal{O}_g$, we have that:

**Corollary 5.24.** The projective covers of simples in $\mathcal{O}_g$ with support that contains a free and closed orbit correspond to the modules over $T^{\hat{s}^\vee s'}$ for Hecke loadings. In particular, $\tau_!^v(P_0)$ is independent of $v$, and has $\text{End}(\tau_!^v(P_0)) \cong T^{\hat{s}^\vee s'}$. 31
From this, we can deduce that:

**Theorem 5.25.** The equivalence $O_g \cong T_{\lambda^i}^{s^j} \mod$ intertwines twisting functors $\mathcal{T}_i$ with change-of-charge functors $\mathcal{B}^{s, i, j}$.

**Proof.** The first fact that we need is that both the twisting functor $\mathcal{T}_i$ and change-of-charge functors send the standard exceptional collection to a mutation. In the first case, this is proven algebraically for the Rickard complexes in [Webc ??]; in the second, this follows from [Webc ??]. In both cases, the change of order is that induced by the generator of the braid group $\hat{B}_\ell$.

Thus, if we consider the composition $\mathcal{B}^{s, i, j} \circ \mathcal{T}_i^{-1}$, this functor is exact and sends projectives to projectives. On the other hand, the twisting functors send $P^0$ to $P^0$ and induces the identity functor on morphisms by Corollary 5.24. The same is manifestly true for change-of-charge functors. By the faithfulness of the cover $\text{Hom}(P^0, -)$, the functor $\mathcal{B}^{s, i, j} \circ \mathcal{T}_i$ must thus be isomorphic to the identity and the result follows. □

**Proof of Theorem C.** We already know that the Koszul dual equivalence $D_{O_g} \cong T_{\mu^j} \text{-dg-mod}$ intertwines shuffling functors with change-of-charge functors by Corollary 5.17. Thus, Theorem 5.25 shows the desired Koszul duality of twisting and shuffling, completing the proof of Theorem C. □

### 5.4. Symplectic duality.

This theorem is part of a more general picture, laid out by Braden, Licata, Proudfoot and the author [BLPW, §9], called symplectic duality. The underlying idea is that there is a duality operation on symplectic cones which switches certain geometric data.

We regard Corollary 5.20 as evidence that affine quiver varieties come in dual pairs $\mathbb{N}_\mu^\lambda$ and $\mathbb{N}_{\mu, i}^\lambda$ indexed by rank-level dual weight spaces. The reader could rightly protest that $\lambda^i$ depends on the weighting $\overline{\varepsilon}$. However, different choices of Uglov weighting $\overline{\varepsilon}$ result in $\lambda^i$ which are conjugate under the action of the Weyl group, and the cone $\mathbb{N}_{\lambda, i}^\mu$ only depends on the Weyl orbit of $\lambda^i$ by work of Maffei [Maf02]. In fact, for purposes of understanding duality, it is better to fix the weights $\lambda^i$ and $\mu$ to be dominant and think of the varieties $\mathbb{N}_{w^i}^{\lambda}$ as $w$ ranges over the Weyl group $W$ as the GIT quotients of $E_{\mu}^\lambda$ at the GIT stability conditions $w^{-1} \cdot \det$. If we also consider $-w^{-1} \cdot \det$, this gives us a (redundant) list containing a representative of every GIT chamber.

The different weightings of the Crawley-Boevey quiver form a similar chamber structure when broken up according to their Uglovation; the walls that separate them are of the form

$$\overline{\varepsilon}_i - \overline{\varepsilon}_j - \kappa(r_i - r_j + me) = 0 \quad \text{for all} \quad m \in \mathbb{Z} \text{ and } i, j \in \mathbb{Z}/\ell \mathbb{Z}.$$

These walls are unchanged (just reindexed) if we replace $r_i$ by $s_i$ for any charge with $r_i \equiv s_i \pmod{e}$. 

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We can identify the Lie algebra $\mathfrak{t}_\ell$ of the torus $T_\ell \cong (\mathbb{C}^\times)^\ell$ of $H$ with the span $\mathfrak{s}\mathfrak{l}_\ell$ of the fundamental weights $\omega_i$ in the dual Cartan of $\hat{\mathfrak{s}\mathfrak{l}}_\ell$ via the map

$$u_s(\vartheta) = \kappa \vartheta_0 + \sum_{i=1}^{\ell} (\vartheta_i - \kappa s_i)(\omega_i - \omega_{i+1})$$

these walls are defined by $(u(\vartheta), \alpha) = 0$ for all positive roots $\alpha$ of $\hat{\mathfrak{s}\mathfrak{l}}_\ell$. Note that if $w \cdot s = s'$ for $w \in \hat{W}_\ell$, then $w \cdot u_s = u_s \cdot w = u_{s'}$. Note that

$$u_s(\vartheta_U) = e(\omega_1 - \omega_2) + 2e(\omega_2 - \omega_3) + \cdots + e(\ell - 1)(\omega_{\ell-1} - \omega_0) + e\ell \omega_0 = e \sum \omega_i.$$ 

Thus, under rank-level duality, the possible spaces of choices for GIT stability conditions and weightings switch; furthermore, these bijections preserve the appropriate chamber structures, sending the chamber containing our chosen weighting to the dominant Weyl chamber, by equation (1). The reader might object that not all simple roots genuinely contribute GIT walls; this is compensated for by the fact that the corresponding Uglov weightings have the same relative core $\mathfrak{m}^+$ and associated category $\mathcal{O}$.

**Proposition 5.26.** The hyperplane $(-, \alpha) = 0$ is a GIT wall for the reduction that presents $\mathfrak{m}_\lambda^\mu$ if and only if one of the multipartitions indexing a fixed point in $\mathfrak{m}_\mu^\lambda$ has a removable ribbon of residue $\alpha$.

**Proof.** Using the Weyl group action, we can rephrase the characterization of the GIT walls. Since the GIT walls for $\mu$ are sent to GIT walls for $w \cdot \mu$ by the action of $w \in \hat{W}_\ell$, we need only prove this for $\alpha = \alpha_i$ a simple root.

Our claimed characterization is equivalent to the statement that for any $\mu$, the locus $(-, \alpha_i)$ is a GIT wall if and only if one of the partitions indexing a fixed point has a removable box of residue $i$. The “if” direction is clear; the $T$-fixed point corresponding to any multipartition with such a removable box provides an example of a strictly semi-stable representation on this wall.

Now consider the “only if.” The points of $\mathfrak{m}_\mu^\lambda$ that become semi-stable on the wall are a closed $S \times T$-invariant subset which is by assumption non-empty. Thus, it must contain at least one point of the core, which we can assume is $T$ invariant. The corresponding multi-partition has a removable box of the right residue.

**Proposition 5.27.** The walls in the Hamiltonian torus $\mathfrak{t}_\ell$ are given by those which correspond to GIT walls of $\mathfrak{m}_\lambda^\mu$ under duality.

**Proof.** Consider the GIT wall attached to a root $\alpha_{k'} + \alpha_{k'+1} + \cdots + \alpha_{k-1} + \alpha_k + m \delta$. In terms of abaci, the appearance of a removable ribbon of the right residue says that there must be some abacus of the right residue such that a bead of residue $k$ can be moved down a runner $k - k' + 1 + m$ slots into an empty spot. The rank-level dual condition is that a bead of some fixed residue $r$ in the $k$th runner can to moved to a slot of residue $r$ in the $k'$th runner in the row $\ell m$ slots down if $k > k'$, or $\ell (m + 1)$ slots down if $k \leq k'$. This bead and slot it moves into are at the end of the leg and arm of some box in the $k$th partition, and the line in the tangent space corresponding to this box
in the formula of [ST12, 5.10] has trivial $T$ action. Thus the $T$-fixed locus is positive dimensional.

The same formula shows that if there is no such bead, the $T$ action on the tangent space at each fixed point has no invariants. Thus, all $T$-fixed points remain isolated, and there is no wall. □

Since the kernel of the $t_\ell$ action on $\mathcal{M}^\lambda_\mu$ is the intersection of all the walls with positive dimensional fixed locus, and the kernel of Kirwan map is the intersection of all the GIT walls, this further implies that:

**Corollary 5.28.** The map $u_s$ induces an injective map $t^\lambda_\mu \to H^2(\mathcal{M}^\lambda_\mu)$ from the quotient $t^\lambda_\mu$ of $t_\ell$ that acts faithfully on $\mathcal{M}^\lambda_\mu$. The image of this map coincides with that of the Kirwan map.

This allows us to restate Corollary 5.20 and Theorem C in a more symmetric way. Let $O_\xi$ be the category $O_g$ attached to $\xi \in t_w$ and the GIT stability condition $\eta$ (which can switch places to be the same data for the rank-level dual).

**Theorem 5.29.** There is a Koszul duality equivalence $D^b(O_\xi^\eta) \cong D^b(O_\eta^\xi)$, which intertwines the shuffling functor $\Psi^{\xi,\xi'} : D^b(O_\xi^\eta) \to D^b(O_\eta^\xi)$ with the twisting functor $\Phi^{\xi,\xi'} : D^b(O_\eta^\xi) \to D^b(O_\xi^\eta)$.

For ease of reference, we collect together the pieces of data we require for a symplectic duality. Here we employ the notation of [BLPW].

- The set of simples in $O^{\lambda;\mu}_\mu$ are indexed by $\ell$-multipartitions whose total content is fixed by $\mu$; these can be interpreted as $\ell$-strand abaci. As proven in [Webc, ??], the Koszul duality bijection is given by cutting the abacus into $\ell \times e$ rectangles and flipping, as in the picture below:

  ![Diagram of abaci]

- The poset $\mathcal{J}^{sp}_{\mathcal{M}^\lambda_\mu}$ of special strata is in bijection with weights $\nu$ and integers $n$ such that $\lambda \geq \nu \geq \nu - n\delta \geq \mu$, and the desired bijection to $\mathcal{J}^{sp}_{\mathcal{M}^\mu_\lambda}$ sends $(\nu, n) \mapsto (\nu + n\delta, n)$.

- We have described maps $u_\ell : t^\nu_\mu \to H^2(\mathcal{M}^\mu_\nu, \mathbb{C})$ and $-u_s : t^\lambda_\mu \to H^2(\mathcal{M}^\mu_\lambda, \mathbb{C})$. These are isomorphisms if Kirwan surjectivity holds for affine type A quiver varieties, which we will assume from now on.

- The Namikawa Weyl group $W$ is the stabilizer of $\mu$ under the action of $\tilde{W}_\ell$, and the Weyl group $W_\ell$ is the stabilizer of $s$ under $\tilde{W}_\ell$.

**Theorem 5.30.** These bijections together with Koszul duality of Corollary 5.20 define a symplectic duality in the sense of [BLPW, §10] between $\mathcal{M}^\lambda_\mu$ and $\mathcal{M}^\mu_\lambda$.

**Proof.** We must check that:
The bijections between simples and special leaves are compatible with the map sending a simple to the associated variety of its annihilator: This follows from Proposition 5.22.

The isomorphism $H^2(\mathcal{M}_{\mu}; \mathbb{C}) \cong t^!_{\mathcal{O}}$ is compatible with the action of the stabilizer of $\mu$ in $\tilde{W}_e$ on both sides: It is clear that the isomorphism $t_e \cong \mathfrak{S}_e$ intertwines the action of $\tilde{W}_e$ on $t_e$ by swapping coordinates and translating and the natural action on $\mathfrak{S}_e$, and the quotients inherit this action.

The ample cone in $H^2(\mathcal{M}_{\mu}; \mathbb{C})$ is sent to the chamber of the chamber of weightings with the same category $\mathcal{O}$ as $\mathfrak{S}_U$, and similarly for the ample cone of $H^2(\mathcal{M}_{\mu}; \mathbb{C})$ and $-\mathfrak{S}_U$: For $u_t$, this follows immediately from (1). The same argument works for the dual, since we have taken the negative map $-u_s$.

The Koszul duality switches twisting and shuffling functors: This follows from Theorem 5.25.

**References**

[Ach] Pramod Achar, *Equivariant mixed Hodge modules*, lecture notes.
[Ber] Joseph Bernstein, *Algebraic theory of D-modules*, preprint.
[BG] Vladimir Baranovsky and Victor Ginzburg, personal communication.
[BK04] Roman Bezrukavnikov and Dmitry Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J. 4 (2004), no. 3, 559–592, 782. MR MR2119140 (2006j:53130)
[BL] R. Bezrukavnikov and I. Losev, *Etingof conjecture for quantized quiver varieties*, arXiv:1309.1716.
[BLPW] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$*, arXiv:1407.6964.
[BLPW10] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Gale duality and Koszul duality*, Adv. Math. 225 (2010), no. 4, 2002–2049.
[BLPW12] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Hypertoric category $\mathcal{O}$*, Adv. Math. 231 (2012), no. 3-4, 1487–1545.
[Boza] Tristan Bozec, *Quivers with loops and Lagrangian subvarieties*, arXiv:1311.5396.
[Bozb] Tristan Bozec, *Quivers with loops and perverse sheaves*, arXiv:1401.5302.
[BPW] Tom Braden, Nicholas J. Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions I: local and global structure*, arXiv:1208.3863.
[CDK] Sabin Cautis, Christopher Dodd, and Joel Kamnitzer, in preparation.
[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR 98i:22021
[CR08] Joseph Chuang and Raphaël Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2) 167 (2008), no. 1, 245–298.
[Dri04] Vladimir Drinfeld, *DG quotients of DG categories*, J. Algebra 272 (2004), no. 2, 643–691. MR 2028075 (2006e:18018)
[EK] Ben Elias and Mikhail Khovanov, *Diagrammatics for Soergel categories*, arXiv:0902.4700.
[EW] Ben Elias and Geordie Williamson, *Soergel Calculus*, arXiv:1309.0865.
[GGOR03] Victor Ginzburg, Nicolas Guay, Eric Opdam, and Raphaël Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. 154 (2003), no. 3, 617–651.
[KL09] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory 13 (2009), 309–347.
[KR08] Masaki Kashihara and Raphaël Rouquier, *Microlocalization of rational Cherednik algebras*, Duke Math. J. 144 (2008), no. 3, 525–573. MR 2444305
[KS] Masaki Kashihara and Pierre Schapira, *Deformation quantization modules*, arXiv:1003.3304.
On generalized category $\mathcal{O}$ for a quiver variety

[KS97] Masaki Kashiwara and Yoshihisa Saito, Geometric construction of crystal bases, Duke Math. J. 89 (1997), no. 1, 9–36. MR MR1458969 (99e:17025)

[Li] Yiqiang Li, Tensor product varieties, perverse sheaves and stability conditions. \texttt{arXiv:1109.4578}

[Lus91] George Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), no. 2, 365–421. MR MR1088333 (91m:17018)

[Maf02] Andrea Maffei, A remark on quiver varieties and Weyl groups, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1 (2002), no. 3, 649–686. MR 1990675 (2004h:14051)

[MNa] Kevin McGerty and Thomas Nevins, Derived equivalence for quantum symplectic resolutions, \texttt{arXiv:1108.6267}

[MNb] , Morse decomposition for D-module categories on stacks, \texttt{arXiv:1402.7365}

[MOS09] Volodymyr Mazorchuk, Serge Ovsienko, and Catharina Stroppel, Quadratic duals, Koszul dual functors, and applications, Trans. Amer. Math. Soc. 361 (2009), no. 3, 1129–1172.

[Nak94] Hiraku Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, Duke Math. J. 76 (1994), no. 2, 365–416. MR MR1302318 (95i:53051)

[Nak98] , Quiver varieties and Kac-Moody algebras, Duke Math. J. 91 (1998), no. 3, 515–560. MR MR1604167 (99b:17033)

[Nak01] , Quiver varieties and tensor products, Invent. Math. 146 (2001), no. 2, 399–449. MR MR1865400 (2003e:17023)

[Pro08] Nicholas Proudfoot, A survey of hypertoric geometry and topology, Toric Topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 323–338.

[Rou] Raphael Rouquier, 2-Kac-Moody algebras, \texttt{arXiv:0812.5023}

[Sau] Julia Sauter, A survey on Springer theory, \texttt{arXiv:1307.0973}

[ST12] Steven V Sam and Peter Tingley, Combinatorial realizations of crystals via torus actions on quiver varieties, Journal of Algebraic Combinatorics (2012), 1–30.

[SW] Catharina Stroppel and Ben Webster, Quiver Schur algebras and q-Fock space, \texttt{arXiv:1111.1115}

[Weba] Ben Webster, A categorical action on quantized quiver varieties, \texttt{arXiv:1208.5957}

[Webb] , Knot invariants and higher representation theory, \texttt{arXiv:1309.3796}

[Webc] , Rouquier's conjecture and diagrammatic algebra, \texttt{arXiv:1306.0074}

[Webd] , Weighted Khovanov-Lauda-Rouquier algebras, \texttt{arXiv:1209.2463}

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