Fine-grained uncertainty relations under relativistic motion

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Abstract – Among various uncertainty relations, the profound fine-grained uncertainty relation is used to distinguish the uncertainty inherent in obtaining any combination of outcomes for different measurements. In this letter, we explore this uncertainty relation in a relativistic regime. For an observer undergoing a uniform acceleration who is immersed in an Unruh thermal bath, we show that the uncertainty bound is dependent on the acceleration parameter and the choice of Unruh modes. We find that the measurements in mutually unbiased bases, sharing the same uncertainty bound in the inertial frame, could be distinguished from each other for a noninertial observer. In an alternative scenario, for the observer restricted in a single rigid cavity, we show that the uncertainty bound exhibits a periodic evolution with respect to the duration of acceleration. With properly chosen cavity parameters, the uncertainty bounds could be protected. Moreover, we find that the uncertainty bound can be degraded for specific quantum measurements to violate the bound exhibited in the nonrelativistic limit, which can be attributed to the entanglement generation between cavity modes during a particular epoch. Several implications of our results are discussed.

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Introduction. – The distinguishability of a quantum theory from its classical counterpart is formulated in the Heisenberg uncertainty principle [1], which bounds our prediction ability for a quantum system. In terms of entropic measures, it can be recast as \[ H(Q) + H(R) \geq \log_2 \frac{1}{\epsilon}, \] where \( H(Q) \) and \( H(R) \) are the Shannon entropy for the probability distribution of measurement outcomes. Since the complementarity \( \epsilon \) between observables \( Q \) and \( R \) does not depend on specific states to be measured, the r.h.s. of the inequality provides a fixed lower bound and a more general framework for quantifying uncertainty than standard deviations [3,4]. Moreover, once using quantum memory to store information about the measured system, the entropic uncertainty bound could even be violated [5], due to the entanglement between quantum memory and system. Such entropic uncertainty relation (EUR) plays an important role in many quantum information processes, e.g., quantum key distribution.

Nevertheless, entropic function is still a rather coarse way of measuring the uncertainty of a set of measurements (see ref. [6] for a recent review). For instance, with EUR, one cannot distinguish the uncertainty inherent in obtaining any combination of outcomes for different measurements. To overcome this defect, a new form of uncertainty relation, i.e., a fine-grained uncertainty relation (FGUR), has been recently proposed [7]. For a set of measurements labeled by \( t \), associating with every combination of possible outcomes \( x = (x^{(1)}, \ldots, x^{(n)}) \), there exists a set of inequalities,

\[
\left\{ \sum_{t=1}^{n} p(t)p(x^{(t)}|\rho) \leq \zeta_{x} \mid x \in \mathcal{B}^{\times n} \right\},
\]

where \( \mathcal{B}^{\times n} \) is the set involving all possible combinations of outcomes, \( p(t) \) is the probability of choosing a particular measurement, and \( p(x^{(t)}|\rho) \) is the probability that one obtains the outcome \( x^{(t)} \) after performing measurement \( t \) on the state \( \rho \). To measure the uncertainty, the maximum in function \( \zeta_{x} = \max_{\rho} \sum_{t=1}^{n} p(t)p(x^{(t)}|\rho) \) should be evaluated over all states allowed on a particular system. It can be proved that one cannot obtain outcomes with certainty for all measurements simultaneously when \( \zeta_{x} < 1 \).
Once inequality (1) is saturated, state $\rho$ is recognized as maximally certain state (MCS).

Since its introduction, many applications have been found for the FGUR. For instance, it was shown [8] that the FGUR could be used to discriminate among classical, quantum, and superquantum correlations involving two or three parties. Moreover, the uncertainty bound in (1) could be optimized once the measured system is assisted by a quantum memory [9]. Moreover, a profound link between the FGUR and the second law of thermodynamics has been found [10], which claims that a violation of the uncertainty relation implies a violation of the thermodynamical law. Other studies from various perspectives could be found in [11,12].

While most of the studies on uncertainty relations are nonrelativistic, a complete account of these relations requires one to understand them in the relativistic regime, which would link many different physical branches, e.g., quantum information, relativity, and may even shed new light on quantum gravity [13]. In the previous work, we have shown that, besides the choice on the observables, the entropic uncertainty bound should also depend on relativistic motion status of the observer who performs the measurement [14], or the global structure of curved spacetime background [15]. This new character of quantum-memory-assisted EUR is a direct result of entanglement generation in a relativistic system.

In this letter, we explore FGUR for a quantum system under relativistic motion, and find that the uncertainty bound does depend on the motion state of the system. We first consider that an observer undergoes a uniform acceleration relative to an inertial reference. Since two frames differ in their description of a given quantum state due to the so-called Unruh effect, the concept of measurement becomes observer-dependent, which implies a nontrivial relativistic modification to the FGUR. For a noninertial observer, we show that the measurements in general mutually unbiased bases (MUBs) could be distinguished from each other, while they correspond to the same uncertainty bound in the inertial frame. We extend the analysis to an alternative scenario, where, to prevent the Unruh decoherence, an observer is restricted in a single rigid cavity and undergoes a nonuniform acceleration. Remarkably, we show that the uncertainty could be drastically degraded by the nonuniform acceleration of cavity during a particular epoch, while the uncertainty bound itself exhibits a periodic evolution with respect to the duration of the acceleration. This phenomenon can be attributed to the entanglement generation between the field modes in single cavity that plays the role of quantum memory [9]. Except for the acceleration-duration time with integer periods, the measurements in different MUBs are also distinguishable by the corresponding uncertainty bounds, similar as in the scenario with uniform acceleration.

**FGUR for an accelerating observer.** We first explore the FGUR for an observer with uniform acceleration $a$, who performs projects measurements on the quantum state constructed from free field modes. For the noninertial observer traveling in, e.g., the right Rindler wedge (labeled as I), field modes in the left Rindler wedge (labeled as II) are unaccessible, as they are separated by the acceleration horizon. Therefore, the information loss associated with the horizon can result in a thermal bath perceived by the observer. From the view of quantum information [16], this celebrated Unruh effect could induce a nontrivial influence on the quantum entanglement between field modes.

We consider a free massive fermionic field with mass $m$, satisfying the equation $[i\gamma^\mu(\partial_\mu - \Gamma_\mu) + m}\psi = 0$, where $\gamma^\mu$ are Dirac matrices and $\Gamma_\mu$ are spin connection. Working in Rindler coordinates, the fermionic field can be expanded between Rindler wedges I and II. For particular construction to whole spacetime, the proper Unruh modes are symmetric between Rindler wedges I and II. For particular Rindler frequency $\Omega$, from (2) and (3), one can obtain

$$|0_{\Omega, R}\rangle = \cos r |0_{\Omega, I}\rangle - \sin r |0_{\Omega, I}\rangle^+, \quad |1_{\Omega, R}\rangle = \sin r |0_{\Omega, I}\rangle^+ + \cos r |0_{\Omega, I}\rangle^-,$$

where $|0\rangle$ and $|0\rangle^-$, similar for excited states.

The Unruh vacuum therefore becomes [18]

$$|0_{\Omega, U}\rangle = \cos^2 r (|0\rangle + |1\rangle + |0\rangle^+ + |0\rangle^- + |0\rangle^+^+ + |0\rangle^-^- + |0\rangle^-^+ + |0\rangle^+^-).$$

and the first excitation is

$$|1_{\Omega, U}\rangle = q_R (\cos r |1000\rangle - \sin r |1011\rangle) + q_L (\sin r |1101\rangle + \cos r |0001\rangle),$$

where we introduce the notations

$$|1111\rangle = \delta^+_c |1\rangle^+_c |1\rangle^+_c |1\rangle^+_c |0_{\Omega, I}\rangle^+ |0_{\Omega, I}\rangle^- |0_{\Omega, I}\rangle^- |0_{\Omega, I}\rangle^- |0_{\Omega, I}\rangle^+.$$

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It should be noted that different operator ordering in fermionic systems could lead to nonunique results in quantum information [19]. For instance, if we rearrange operator ordering in (7) as \( b_1^\dagger c_1 + b_2^\dagger c_1^\dagger \), then the Fock basis is changed to \(|1111\rangle = |1111\rangle\). In particular, we adopt this so-called physical ordering [20], as all region I operators appear to the left of all region II operators, which was proposed to that guarantee that the entanglement behavior of the above states yields physical results.

To explore how the relativistic motion of the observer could influence the FGUR, we consider a scenario in which the state to be measured is prepared in an inertial frame. The observer undergoing a uniform acceleration \( a \) can perform measurements on such a state in the Rindler wedge I, which now should be described in the corresponding Rindler frame. As information would be lost via the Unruh effect, we can expect that the uncertainty obtained by the accelerated observer would be motion-dependent.

We illustrate the above intuition by a complete measurements consisting of Pauli operators \( \{\sigma_i| i = x, y, z\} \). In particular, we select \( \sigma_x \) and \( \sigma_z \), behaving as the best measurement basis, where Pauli operators \( \sigma_x \) and \( \sigma_z \) with equal probability 1/2 are chosen [7]. Remarkably, along with \( \sigma_y \), three sets of their eigenvectors form the MUBs in Hilbert space with dimension \( d = 2 \), which plays a central role for theoretical and practical exploitations of complementarity properties [21].

For arbitrary pure states \(|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \) with \( \theta \in [0, \pi], \phi \in [0, 2\pi] \), the corresponding density matrix \( \rho = |\psi\rangle \langle \psi| \) should be rewritten according to the transformation (5) and (6). Since the field modes in the Rindler wedge II are inaccessible to the observer, after tracing over the modes in the wedge II, the reduced density matrix \( \rho_{\text{red}} \equiv \text{Tr}_I |\psi\rangle \langle \psi| \) becomes

\[
\rho_{\text{red}} = |00\rangle_1|00\rangle_2 \left( \cos^2 \frac{\theta}{2} c^2 + \sin^2 \frac{\theta}{2} q_L c^2 \right) + |11\rangle_1|11\rangle_2 \left( \cos^2 \frac{\theta}{2} s^2 + \sin^2 \frac{\theta}{2} q_R s^2 \right) + |01\rangle_1|01\rangle_2 \cos \frac{\theta}{2} c \left( \cos \frac{\theta}{2} q_L c + \sin \frac{\theta}{2} q_L s \right) + |10\rangle_1|10\rangle_2 \left[ \cos^2 \frac{\theta}{2} c^2 + \sin^2 \frac{\theta}{2} \left( q_R c^2 + q_L s^2 \right) \right] + \frac{e^{i\phi}}{2} \sin \theta q_L c (|10\rangle_1|11\rangle_2 - |00\rangle_1|01\rangle_2 |c^2|) + \frac{e^{-i\phi}}{2} \sin \theta q_R c (|00\rangle_1|10\rangle_2 + |01\rangle_1|11\rangle_2 |s^2|) + (\text{h.c.)}_{\text{non-diag}}
\]

(8)

with abbreviations \( c \equiv \cos \tau, s \equiv \sin \tau \). After performing the projective measurements \( \sigma_x \) and \( \sigma_z \) on the particle sector, we have the probabilities for the outcomes \( (0^x, 0^z) \),

\[
p(0^x|\sigma_z)_\rho \equiv \text{Tr}(|0\rangle^+_1|0\rangle_2 \rho_{\text{red}}) = c^2 \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right),
\]

\[
p(0^z|\sigma_x)_\rho \equiv \text{Tr}(|0\rangle^+_1|0\rangle_2 \rho_{\text{red}})
= \frac{1}{2} + \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \phi q_RC,
\]

(9)

where we chose the measurements in basis \(|+\rangle_1, |-\rangle_1\) and \(|0\rangle_1, |1\rangle_1\), the eigenstates of Pauli matrix \( \sigma_x \) and \( \sigma_z \) restricted in the Rindler wedge I. Therefore, the l.h.s. of (1) becomes

\[
U \equiv \frac{1}{2} \rho(0^x|\sigma_z)_\rho + p(0^z|\sigma_x)_\rho,
\]

\[
= \frac{1}{4} \left[ \sin \theta \cos \phi q_RC + \left( \cos \theta q_L^2 + q_R^2 + 1 \right) c^2 + 1 \right].
\]

(10)

For a particular Unruh mode with fixed acceleration, we estimate the uncertainty bound \( \zeta \equiv \max_{\theta} U \) and find that the maximum can always be achieved with \( \theta = \frac{\pi}{4} \) and \( \phi = 0 \) [22], i.e., the MCS saturating (1) is independent of the acceleration of the observer. This indicates that once we choose the bases enabling an optimal uncertainty in an inertial frame, the corresponding measurements should maintain their optimality for all noninertial observers. This could be useful in many real quantum processes, for instance, the BB84 states \(|+\rangle, |-\rangle \) and \(|0\rangle, |1\rangle \) in quantum cryptography [23]. Hereafter, we always adopt those MCS with \( \theta = \frac{\pi}{4} \) and \( \phi = 0 \), and explore the motion dependence of uncertainty bound \( \zeta \) in (1) for them.

Explicitly, the dependence of the fine-grained uncertainty bound with outcomes \( (0^x, 0^z) \) on the acceleration parameter and the choice of Unruh modes could be expressed as

\[
\zeta(x^*, o^*) = \frac{1}{4} \left[ c^2 (1 + q_L^2 \sqrt{2} + \frac{\sqrt{2}}{2} q_R c + 1) \right].
\]

(11)

The above calculation can extend to any other pairs of outcomes \( (0^x, 1^z), (1^x, 0^z) \) and \( (1^x, 1^z) \), which all give the same bound \( \zeta = \frac{1}{2} + \frac{\sqrt{2}}{2} \) in the inertial frame [7]. However, we find that the nontrivial Unruh effect could distinguish these four pairs of measurements into two categories. For instance, we have

\[
p(1^z|\sigma_x)_\rho \equiv \text{Tr}(|1\rangle^+_1|1\rangle_2 \rho_{\text{red}})
= \cos \frac{\theta}{2} s^2 + \sin \frac{\theta}{2} \left( q_R^2 + q_L^2 s^2 \right),
\]

\[
p(1^x|\sigma_z)_\rho \equiv \text{Tr}(|1\rangle^+_1|1\rangle_2 \rho_{\text{red}})
= \frac{1}{2} - \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \phi q_RC.
\]

(12)

which give

\[
\zeta(1^x, 1^z) = \frac{1}{4} \left[ \left( 1 + \frac{\sqrt{2}}{2} q_R^2 \right) q_R^2 + (1 + q_L^2) s^2 + \frac{\sqrt{2}}{2} q_R c + 1 \right]
\]

(13)

for the MCS. By straightforward calculations, it is easy to show that

\[
\zeta(0^x, 0^z) = \zeta(1^x, 0^z), \quad \zeta(1^x, 1^z) = \zeta(0^x, 1^z).
\]

(14)
In fig. 1, we depict the uncertainty bounds (11), (13) and (14) for three different choices of Unruh modes. For the case with $q = 0$, where the Minkowskian annihilation operator is taken to be one of the right or left moving Unruh modes, the noninertial observer would detect a single-mode state once the field is in a special superposition of Minkowski monochromatic modes from an inertial perspective [24]. Under this single-mode approximation (SMA), commonly assumed in the old literature on relativistic quantum information [16], we recover the standard result $\zeta = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ for vanishing acceleration. As $r$ grows, the value of $\zeta$ decreases, indicating an increment on the measurement uncertainty. This is not surprising as the thermality from inevitable Unruh decoherence introduces classical noise in quantum measurement. Therefore, the uncertainty bound given in (1) quantifies the total uncertainty involved in measurements. On the other hand, for general Unruh modes with $q \neq 0$, we observe a drastically increase of $\zeta$ with respect to growing acceleration for specific measurement outcomes $(1^x, 1^x)$ and $(0^x, 1^x)$, which means a degradation on the measurement uncertainty. Nevertheless, as $\zeta < \frac{1}{2} + \frac{1}{2\sqrt{2}}$ for infinite acceleration, no violation of standard uncertainty principle can be found in the relativistic scenario with global field modes.

As illustrated in fig. 1, we find that the distinguishability between the measurements in MUBs is a common feature for any choice of Unruh modes. To explain this, recall that, by definition, a set of orthonormal bases $\{B_k\}$ for a Hilbert space $H = \mathbb{C}^d$, where $B_k = \{|i_k\rangle\} = \{0, 1, \ldots, d-1\}$ is called unbiased if $|\langle i_k | j \rangle|^2 = \frac{1}{d}$, $\forall i, j \in \{0, 1, \ldots, d-1\}$ holds for all basis vectors $|i_k\rangle$ and $|j \rangle$ with $\forall k \neq l$. From an inertial perspective, the MUBs are intimately related to the complementarity principle [25], which indicates that the measurement of an observable reveals no information about the outcome of another one if their corresponding bases are mutually unbiased. However, for a noninertial observer, the bases $\{B_k\}$ should be transformed according to proper Bogoliubov transformations, which in general breaks the orthonormality. In other words, the MUBs in the inertial frame would become non-MUBs from a noninertial perspective. Therefore, for the observer undergoing a uniform acceleration, we expect that the Unruh effect could distinguish measurements in MUBs with respect to the inertial observer.

**FIGUR for a nonuniform-moving cavity.** We now discuss an alternative scenario in which an observer is localized in a rigid cavity, which is more flexible for implementing practical quantum information tasks. While the rigid boundaries of the cavity protect the inside observer from the Unruh effect, the relativistic motion of the cavity would still affect the entanglement between the free field modes inside [26–28], therefore leading to a motion-dependent uncertainty bound [14].

We consider a $(1+1)$-dimensional model, where a massless fermionic field is confined in a cavity with length $L = x_2 - x_1$, imposing Dirichlet conditions on the eigenfunctions $\psi_n(t, x)$ of the Hamiltonian. Once the cavity accelerating, it is convenient to use the Rindler coordinates $(\eta, \chi)$, defined in the wedge $x > |t|$ by $t = \chi \sinh \eta$ and $x = \chi \cosh \eta$, where $0 < \chi < \infty$ and $-\infty < \eta < +\infty$. The new orthonormal eigenfunctions $\tilde{\psi}_n(\eta, \chi)$ can be derived by solving the massless Dirac equation $i\gamma^\mu \partial_\mu \tilde{\psi} = 0$ in the Rindler coordinates.

A typical trajectory of nonuniform-moving cavity contains three segments as (I′) the cavity maintains its inertial status initially, then (II′) begins to accelerate at $t = 0$, following a Killing vector $\partial_\theta$, and finally (III′) the acceleration ends at Rindler time $\eta = \eta_1$. The duration of acceleration measured at the center of the cavity is $\tau_1 = \frac{1}{2} (x_1 + x_2) \eta_1$. The Dirac field can be expanded in quantized eigenfunctions as $\psi = \sum_{n \geq 0} c_n |\psi_n\rangle + \sum_{0 < n \leq 0} b_n |\psi_n\rangle$ in segment I′, and similarly be expressed by $\tilde{\psi}_n$ in segment II′ and by $\tilde{\psi}_n$ in segment III′. The anticommutators $[c_n, c_{m}^\dagger] = \delta_{mn}$, $[d_n, d_{m}^\dagger] = \delta_{mn}$ define the vacuum $c_n |0\rangle = d_n |0\rangle = 0$. Any two field modes in distinct region can be related by Bogoliubov transformations like $\tilde{\psi}_m = \sum_n a_{mn} \psi_n$ and $\tilde{\psi}_m = \sum_n a_{mn} \psi_n$, where the coefficients can be calculated perturbatively in the limit of small cavity acceleration [26]. More specifically, by introducing the dimensionless parameter $\tilde{h} = 2L/(x_1 + x_2)$, a product of cavity’s length and acceleration at the center of the cavity, the coefficients can be expanded in a Maclaurin series to $h^2$ order, $A = A^{(0)} + A^{(1)} + A^{(2)} + O(h^3)$, and similarly $\hat{A} = \hat{A}^{(0)} + \hat{A}^{(1)} + \hat{A}^{(2)} + O(h^3)$.

We start from a pure state $|\psi_0\rangle = \cos \frac{\theta}{2} |0_k\rangle + e^{i\phi} \sin \frac{\theta}{2} |1_k\rangle^+$ in segment I′. After a uniform acceleration, we can express this state in segment III′ via Bogoliubov transformations, which contains modes within all frequencies. Throughout the process, we assume that the observer can only be sensitive to particular modes within frequency $k$. Therefore, all other modes with frequency $k' \neq k$ should be traced out in the density matrix.
where the coefficients are \( f^k_\pm \equiv \sum_{p \geq 0} \beta_{kk}^{(1)p} \) and \( f^k_\pm \equiv \sum_{p < 0} \beta_{kk}^{(1)p} \). The probability of measurements \( (0^z,0^y) \) are

\[
p(0^z|\sigma_z) = \text{tr}[(\hat{0}_z)\langle \hat{0}_k | \hat{\rho}_{red}]
\]

\[
p(0^y|\sigma_x) = \text{tr}[(\hat{0}_x)\langle \hat{0}_k | \hat{\rho}_{red}]
\]

\[
(15)
\]

The uncertainty bound should be the maximum of the l.h.s. of (1), giving \( U \equiv \frac{1}{2}p(0^z|\sigma_z) + p(0^y|\sigma_x)\). Along a similar analysis as before, we know that the acceleration of the cavity would not change the MCS with parameters \( \theta = \frac{\pi}{4} \) and \( \phi = 0 \). Therefore, we obtain the uncertainty bound for the cavity system

\[
\tilde{\zeta}(0^z,0^y) = \frac{1}{4\sqrt{2}} \left[ 1 + 2\sqrt{2} - F_+ + \sqrt{2}F_- + \text{Re}(G_k + \Re^k) \right].
\]

(17)

The coefficients in the bound have been given in [26], which are

\[
F_+ = \sum_{p=-\infty}^{\infty} |E_1^{k-p} - 1|^2 |\beta_{kk}^{(1)p}|^2
\]

\[
= \frac{4\hbar^2}{\pi^3} [4(k + s)^2(Q_0(1) - Q_0(E_1)) + Q_4(1) - Q_4(E_1)]
\]

and [14]

\[
F_- \equiv f^k_+ - f^k_- = \left( \sum_{p \geq 0} - \sum_{p < 0} \right) |E_1^{k-p} - 1|^2 |\beta_{kk}^{(1)p}|^2
\]

\[
= \frac{16\hbar^2}{\pi^4} (2k + s) |Q_3(1) - Q_3(E_1)| + P(k, s, E_1)
\]

with \( s \in [0,1] \) characterizing the self-adjoint extension of the Hamiltonian. Here we use the notation \( Q_0(\beta) \equiv \text{Re}[\text{Li}_2(\beta) - \frac{1}{2} \text{Li}_2(\beta^2)] \), \( \text{Li} \) is the polylogarithm and \( E_1 \equiv \exp(-\ln(x_2/x_1)) = \exp(-\frac{i\beta\hbar y}{2L_x \tan(h/2)}) \). \( P \) is a polynomial summing for all terms with odd number

\[
\sum_{m=1, \text{odd}}^{k} \frac{4\hbar^2}{\pi^3} (1 - \text{Re}(E_1^{(m)})) \left[ 4(k + s) \left( \frac{k + s}{m} - 1 \right) + \frac{1}{m^2} \right]
\]

and

\[
\text{Re}(G_k + \Re^k) = 1 - h^2 \left\{ \left( \frac{1}{18} + \frac{x^2(k + s)^2}{120} \right) - \frac{2}{\pi^4} [4(k + s)^2 Q_0(E_1) + Q_4(E_1)] \right\}.
\]

In the previous section, we show that a uniformly accelerating observer can distinguish the measurements in MUBs which share the same uncertainty bounds in an inertial frame. Here we generalize this to the scenario with rigid cavity. To proceed, we calculate the probabilities of measurements \( (1^z,1^z) \)

\[
p(1^z|\sigma_z) \equiv \text{tr}[(\hat{1}_k)\langle \hat{1}_k | \hat{\rho}_{red}]
\]

\[
p(1^y|\sigma_x) \equiv \text{tr}[(\hat{1}_x)\langle \hat{1}_k | \hat{\rho}_{red}]
\]

\[
(16)
\]

which give the uncertainty bound \( \tilde{\zeta}(1^z,1^z) \) for MCS

\[
\tilde{\zeta}(1^z,1^z) = \frac{1}{4\sqrt{2}} \left[ 1 + 2\sqrt{2} - F_+ - \sqrt{2}F_- + \text{Re}(G_k + \Re^k) \right].
\]

(19)

Similarly as before, by straightforward calculations, we can further prove that

\[
\tilde{\zeta}(0^z,0^z) = \tilde{\zeta}(1^z,1^z) = \tilde{\zeta}(0^z,1^z).
\]

We depict the above uncertainty bounds for measurements performed within a cavity in fig. 2. Firstly, we find that the uncertainty bounds (17), (19) and (20) are now periodic in time \( \tau \), which measures the duration of the cavity acceleration, with the period \( T = 4L_x \tan(h/2)/\hbar \). By properly choosing the parameters to ensure that \( \tau_1 = nT \) with \( n \in \mathbb{N} \), the uncertainty bounds are protected [14], recovering the value \( \frac{1}{4\sqrt{2}} \) as in the inertial case. However, we observe an interesting violation of the uncertainty bound \( \frac{1}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \) for quantum measurements with specific outcomes, such as \( (0^z,0^z) \) and \( (1^z,0^z) \). This can be interpreted as a result of the entanglement generation between the field modes in the single rigid cavity that plays the role of quantum memory [9]. Therefore, by employing the relativistic effect on the localized quantum system, one may achieve lower uncertainty bound and higher precision of the outcomes prediction than in the nonrelativistic frame.

For arbitrary acceleration duration \( \tau_1 \neq nT \), measurements with outcomes \( (0^z,0^z) \) and \( (1^z,1^z) \) can be distinguished from each other by the corresponding uncertainty bounds (17) and (19). Therefore, as in the first scenario with the Unruh effect, we conclude that the relativistic motion of a rigid cavity can cause the distinguishability between the measurements in MUBs that share the same bound \( \frac{1}{4\sqrt{2}} + \frac{1}{2\sqrt{2}} \) for an inertial observer.

Finally, we would like to remark the experimental aspect of our results. While most of the experimental tests
for each pair of measurements $((0^s, 0^r)\text{ and } (1^s, 0^r), (1^s, 1^r)\text{ and } (0^s, 1^r))$, three curves from top to bottom correspond to parameters $s = 0, 0.3, 0.6$. The parameter $u = 1/(2\ln(x_2/x_1)) = h\tau_1/\left[4L\tau_1 \tanh h/2\right]$ characterizes the duration time of the cavity acceleration. To demonstrate the low acceleration approximation, the uncertainty is estimated by $h = 0.1$.

FIG. 2: (Colour online) The value of $\zeta$ depends on the duration time of the acceleration of the rigid cavity. We choose $k = 1$. For each pair of measurements $((0^s, 0^r)\text{ and } (1^s, 0^r), (1^s, 1^r)\text{ and } (0^s, 1^r))$, three curves from top to bottom correspond to parameters $s = 0, 0.3, 0.6$. The parameter $u = 1/(2\ln(x_2/x_1)) = h\tau_1/\left[4L\tau_1 \tanh h/2\right]$ characterizes the duration time of the cavity acceleration. To demonstrate the low acceleration approximation, the uncertainty is estimated by $h = 0.1$.

Discussions. – In this letter, we explored the nontrivial relativistic modification to the FGUR. We have shown that, for an observer undergoing a large acceleration, the associated Unruh effect could increase or reduce the fine-grained uncertainty bounds, depending on the choice of Unruh modes. Moreover, we have shown that the measurements in MUBs, sharing the same uncertainty bound in the inertial frame, could be distinguished from each other when the observer undergoes a nonvanishing acceleration. In an alternative scenario, we have investigated the FGUR for the measurements on fermionic field modes restricted in a single rigid cavity, where the uncertainty bound itself exhibits a periodic evolution with respect to the duration of the acceleration. For quantum measurements with specific outcomes, we find an interesting violation of the uncertainty bound in the inertial frame, attributed to the entanglement generation in a cavity that plays the role of quantum memory. Our results provide a novel way to investigate the relativistic effect in a quantum information context, which may be experimental tested by future quantum metrology.

Throughout the letter, we only discuss the influence of the Unruh effect on FGUR for a fermionic field, which we believe that manifests more abundant characteristics of the motion dependence of FGUR in a relativistic framework than in a bosonic field. For instance, while there is no substantial difference in the Unruh decoherence for a different choice of bosonic Unruh modes [18], it implies that the measurement uncertainty on bosonic states should always increase with growing acceleration of the observer. Therefore, no bosonic Unruh mode can degrade the measurement uncertainty like the fermionic Unruh mode with special $q_k \neq 0$ does in fig. 1. On the other hand, in a cavity scenario, the Unruh decoherence on localized bosonic modes exhibits a periodic dependence on the duration of acceleration, similarly as in the fermionic case [27]. Therefore, one could expect that for the bosonic field, comparing to fig. 2, no essential change but only numerical difference on the FGUR bound $\zeta$ would happen.

Our study raises several implications. Firstly, we can generalize the above analysis to fundamental MUBs in higher-dimensional Hilbert spaces [32,33], where a $n$-qubit can be truncated from free scalar field modes with infinite levels. On the other hand, the fascinating link between FGUR and the second thermodynamical law has been explored in [10], which proved that a deviation of the FGUR implies a violation of the second law of thermodynamics. In this spirit, by investigating the influence of the relativistic motion of an observer on a thermodynamical cycle, one could relate the relativistic effect to thermodynamics in an information-theoretic way. Finally, we can explore the FGUR in some dynamical spacetimes [15], e.g., a cosmological background, where the entanglement generated through the evolution of spacetime is expected to play a significant role in quantum measurements [34].

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