Abstract

We present recently obtained results in the theory of pseudoanalytic functions and its applications to elliptic second-order equations. The operator \((\text{div} p \, \text{grad} + q)\) with \(p\) and \(q\) being real valued functions is factorized with the aid of Vekua type operators of a special form and as a consequence the elliptic equation

\[
(\text{div} p \, \text{grad} + q) u = 0, \tag{1}
\]

reduces to a homogeneous Vekua equation describing generalized analytic (or pseudoanalytic) functions. As a tool for solving the Vekua equation we use the theory of Taylor and Laurent series in formal powers for pseudoanalytic functions developed by L. Bers. The series possess many important properties of the usual analytic power series. Their applications until recently were limited mainly because of the impossibility of their explicit construction in a general situation.
We obtain an algorithm which in a really broad range of practical applications allows us to construct the formal powers and hence the pseudoanalytic Taylor series in explicit form precisely for the Vekua equation related to equation (1). In other words, in a bounded domain this gives us a complete (in $C$-norm) system of exact solutions of (1).

1 Introduction

The foundations of pseudoanalytic function theory have been created by a considerable number of mathematicians among which Lipman Bers and Ilya Vekua played the most prominent role. In the works of I. Vekua and many other researchers pseudoanalytic functions are called generalized analytic. Nevertheless in the present work we use the term “pseudoanalytic” in order to emphasize the fact that we mainly use the part of the theory developed by L. Bers and his collaborators.

In the recent works of the author [29], [30] and [32] a close connection between the second-order elliptic equation

$$\text{div}(p \text{grad } + q)u = 0$$

and a Vekua equation of a special form was presented. This connection is a direct generalization of a relation which exists between harmonic and analytic functions. The special form of the arising Vekua equation (which we call the main Vekua equation) allows us to apply well developed methods of pseudoanalytic function theory ([4], [8], [17], [41], [43], [44] and others) and of $p$-analytic function theory [39] to the analysis of the corresponding second-order equations. In this work we restrict ourselves to the development of the theory of series in formal powers for the main Vekua equation and as a consequence of the corresponding theory for equation (2).

Formal powers were defined by L. Bers (see [8]) and represent a generalization of the usual powers $(z - z_0)^n$ which play a crucial role in the one-dimensional complex analysis. As their name reveals formal powers in general are not powers. They behave as $(z - z_0)^n$ only locally, near the center, and in fact can be complex functions of a quite arbitrary nature. Nevertheless they are solutions of a corresponding Vekua equation and under quite general conditions represent a complete system of its solutions in the same sense as any analytic function under quite general and well known conditions can be represented as its normally convergent Taylor series.
The main result of the present work is a procedure for explicit construction of formal powers corresponding to the main Vekua equation in a very general situation. From the relation of the main Vekua equation to equation (2) we obtain that under quite general conditions we are able to construct explicitly a complete system of solutions of (2). More precisely, let us consider, e.g., the conductivity equation

\[ \text{div} (p \text{grad} u) = 0. \]

Our result then gives us the possibility to construct explicitly a complete system of solutions of this equation if \( p \) has the form

\[ p = \Phi(\varphi)\Psi(\psi) \quad (3) \]

where \((\varphi, \psi)\) is any orthogonal coordinate system, \(\Phi\) and \(\Psi\) are arbitrary positive differentiable functions.

In the case of the stationary Schrödinger equation

\[ (-\Delta + q)u = 0 \quad (4) \]

in order to construct a complete system of solutions explicitly we need a particular solution of this equation of the form (3). Note that before this result has been obtained the knowledge of one particular solution of a second-order equation in two dimensions like (4) had not given much information about the general solution. Now one particular solution of (4) is a generator of a complete system of solutions of (4) which in a sense and for many purposes represents the general solution of the equation.

The present work is an introduction to this new method and includes explanation of the theory behind it and some first examples of application. The paper is organized as follows. In section 2 we present the already known results from pseudoanalytic function theory concerning mainly the theory of series in formal powers. In section 3 we give our recent results on the relationship between equation (2) and the main Vekua equation. In section 4 we explain how a complete system of solutions of equation (2) is obtained from a complete system of pseudoanalytic formal powers. Section 5 is dedicated to a classical but unfortunately not widely known relation between orthogonal coordinate systems in the plane and complex analytic functions. In section 6 we first explain and then prove the main new result of this work which opens the way for explicit construction of formal powers for the main Vekua equation.
In section 7 we explain how this construction gives us complete systems of solutions of second-order elliptic equations. We present some examples which were calculated using the Matlab tool for symbolic calculation. The choice of the examples was inspired by the desire to show another interesting relation between our work and the theory of so called $\nu$-regular functions (solutions of Vekua equations with constant coefficients) started by R. J. Duffin in [18] and [19]. We show that the complete systems of solutions of Yukawa and Helmholtz equations obtained in [11] represent an example of a complete system of solutions for a second-order elliptic equation constructed on the base of pseudoanalytic formal powers.

Note that some of the results presented in this work are sufficiently simple that may serve as a nice and useful addition to a standard complex analysis course, for example, the construction of conjugate metaharmonic functions (see subsection 3.3). The author has already done this didactical experiment successfully. At the same time it is clear that very close to these simple things there are many open and not so simple problems. First of all, the theory of pseudoanalytic (or generalized analytic) functions is very rich and meanwhile some of the well known results from this theory like, e.g., the Cauchy integral theorem and the Morera theorem have been already transferred to the corresponding second-order elliptic equation (see [29]), many others still wait for their application. For example, it is of great interest and importance to construct not only positive formal powers (as we do in the present work) but also the negative formal powers for the main Vekua equation and hence the Cauchy kernel and integral representations for solutions of second-order equations. Moreover, in the present work we consider equation (2) with real valued coefficients. Nevertheless as was shown in [32] the same scheme works in the case of complex valued coefficients in (2), but the corresponding main Vekua equation is bicomplex. The theory of bicomplex Vekua equations (which are closely related to the Dirac equation with electromagnetic potential) we started to develop in [12], but it is clear that it is much more complicated than the theory of complex Vekua equations and represents a separate and important challenge.

The development in three or more dimensions of a theory comparable with the theory of pseudoanalytic functions in the plane was an object of study of many researchers (see, e.g., [36]). In general, the results obtained in this direction are much less complete than their two-dimensional counterparts. In [32] a quaternionic generalization of the main Vekua equation was obtained, an equation which clearly preserves the basic important properties which
made the present work possible. It is interesting to develop the L. Bers theory for this main quaternionic Vekua equation which seems to be possible due to its quite special form.

Finally, behind the results presented here there is an essential and in a sense universal idea of factorization. The main Vekua equation (as we show in section 3) is a result of a factorization of a corresponding second-order operator. This kind of factorization is studied not only in the case of elliptic operators ([5], [7], [27], [28], [29], [32]) but as well in the case of operators of other types ([6] and [13]). Thus, it seems quite possible that similar results can be obtained, e.g., not only for a stationary Schrödinger equation (as in the present work) but also for a time-dependent Schrödinger equation.

2 Some definitions and results from pseudo-analytic function theory

This section is based on notions and results presented in [8] and [9]. Let Ω be a domain in $\mathbb{R}^2$. Throughout the whole paper we suppose that Ω is a simply connected domain.

2.1 Generating pair, derivative and antiderivative

**Definition 1** A pair of complex functions $F$ and $G$ possessing in $\Omega$ partial derivatives with respect to the real variables $x$ and $y$ is said to be a generating pair if it satisfies the inequality

$$\text{Im}(FG) > 0 \quad \text{in } \Omega. \quad (5)$$

Denote $\partial_\tau = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ and $\partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$. The following expressions are known as characteristic coefficients of the pair $(F,G)$

$$a_{(F,G)} = -\frac{FG_\tau - F_\tau G}{FG - F \bar{G}}, \quad b_{(F,G)} = \frac{FG_\tau - F_\tau G}{FG - F \bar{G}},$$

$$A_{(F,G)} = -\frac{FG_z - F_z \bar{G}}{FG - F \bar{G}}, \quad B_{(F,G)} = \frac{FG_z - F_z \bar{G}}{FG - F \bar{G}},$$

where the subindex $\tau$ or $z$ means the application of $\partial_\tau$ or $\partial_z$ respectively.
Every complex function \( W \) defined in a subdomain of \( \Omega \) admits the unique representation \( W = \phi F + \psi G \) where the functions \( \phi \) and \( \psi \) are real valued. Thus, the pair \((F, G)\) generalizes the pair \((1, i)\) which corresponds to usual complex analytic function theory.

Sometimes it is convenient to associate with the function \( W \) the function \( \omega = \phi + i\psi \). The correspondence between \( W \) and \( \omega \) is one-to-one.

The \((F, G)\)-derivative \( \dot{W} = \frac{d(F,G)W}{dz} \) of a function \( W \) exists and has the form
\[
\dot{W} = \phi_z F + \psi_z G = W_z - A_{(F,G)} W - B_{(F,G)} \overline{W}
\]  

if and only if
\[
\phi_z F + \psi_z G = 0.
\]

This last equation can be rewritten also in the following form
\[
W_z = a_{(F,G)} W + b_{(F,G)} \overline{W}
\]

which we call a Vekua equation. Solutions of this equation are called \((F, G)\)-pseudoanalytic functions. If \( W \) is \((F, G)\)-pseudoanalytic, the associated function \( \omega \) is called \((F, G)\)-pseudoanalytic of second kind.

**Remark 2** The functions \( F \) and \( G \) are \((F, G)\)-pseudoanalytic, and \( \dot{F} \equiv \dot{G} \equiv 0 \).

**Definition 3** Let \((F, G)\) and \((F_1, G_1)\) be two generating pairs in \( \Omega \). \((F_1, G_1)\) is called successor of \((F, G)\) and \((F, G)\) is called predecessor of \((F_1, G_1)\) if
\[
a_{(F_1,G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)}.
\]

The importance of this definition becomes obvious from the following statement.

**Theorem 4** Let \( W \) be an \((F, G)\)-pseudoanalytic function and let \((F_1, G_1)\) be a successor of \((F, G)\). Then \( W \) is an \((F_1, G_1)\)-pseudoanalytic function.

This theorem shows us that to the difference of analytic functions whose derivatives are again analytic, the \((F, G)\)-derivatives of pseudoanalytic functions are in general solutions of another Vekua equation.
Definition 5 Let \((F,G)\) be a generating pair. Its adjoint generating pair 
\((F,G)^* = (F^*,G^*)\) is defined by the formulas

\[
F^* = -\frac{2F}{FG - FG}, \quad G^* = \frac{2G}{FG - FG}.
\]

The \((F,G)\)-integral is defined as follows

\[
\int_{\Gamma} Wd_{(F,G)}z = F(z_1) \text{ Re} \int_{\Gamma} G^* Wdz + G(z_1) \text{ Re} \int_{\Gamma} F^* Wdz
\]

where \(\Gamma\) is a rectifiable curve leading from \(z_0\) to \(z_1\).

If \(W = \phi F + \psi G\) is an \((F,G)\)-pseudoanalytic function where \(\phi\) and \(\psi\) are 
real valued functions then

\[
\int_{z_0}^{z} Wd_{(F,G)}z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z), \quad (8)
\]

and as \(F = G = 0\), this integral is path-independent and represents the \((F,G)\)-
antiderivative of \(W\).

A continuous function \(w\) defined in a domain \(\Omega\) is called \((F,G)\)-integrable 
if for every closed curve \(\Gamma\) situated in a simply connected subdomain of \(\Omega\),

\[
\int_{\Gamma} wd_{(F,G)}z = 0
\]

or what is the same

\[
\text{Re} \int_{\Gamma} G^* wdz + i \text{ Re} \int_{\Gamma} F^* wdz = 0. \quad (9)
\]

Theorem 6 An \((F,G)\)-derivative \(\dot{W}\) of an \((F,G)\)-pseudoanalytic function \(W\) 
is \((F,G)\)-integrable.

Theorem 7 Let \((F,G)\) be a predecessor of \((F_1,G_1)\). A continuous function 
is \((F_1,G_1)\)-pseudoanalytic if and only if it is \((F,G)\)-integrable.

Remark 8 It is easy to see that in the case \(F \equiv 1, G \equiv i\), equality \((9)\) turns 
into the Cauchy integral theorem for analytic functions: \(\int_{\Gamma} wdz = 0\).
2.2 Generating sequences and Taylor series in formal powers

In order to introduce the notion of pseudoanalytic derivatives of arbitrary order the following definition is necessary.

**Definition 9** A sequence of generating pairs \{\((F_m, G_m)\)\}, \(m = 0, \pm 1, \pm 2, \ldots\), is called a generating sequence if \((F_{m+1}, G_{m+1})\) is a successor of \((F_m, G_m)\). If \((F_0, G_0) = (F, G)\), we say that \((F, G)\) is embedded in \{\((F_m, G_m)\)\}.

**Theorem 10** Let \((F, G)\) be a generating pair in \(\Omega\). Let \(\Omega_1\) be a bounded domain, \(\overline{\Omega_1} \subset \Omega\). Then \((F, G)\) can be embedded in a generating sequence in \(\Omega_1\).

**Definition 11** A generating sequence \{\((F_m, G_m)\)\} is said to have period \(\mu > 0\) if \((F_{m+\mu}, G_{m+\mu})\) is equivalent to \((F_m, G_m)\) that is their characteristic coefficients coincide.

Let \(W\) be an \((F, G)\)-pseudoanalytic function. Using a generating sequence in which \((F, G)\) is embedded we can define the higher derivatives of \(W\) by the recursion formula

\[
W^{[0]} = W; \quad W^{[m+1]} = \frac{d[(F_m, G_m)]}{dz} W^{[m]}, \quad m = 1, 2, \ldots.
\]

A generating sequence defines an infinite sequence of Vekua equations. If for a given (original) Vekua equation we know not only a corresponding generating pair but the whole generating sequence, that is a couple of exact and independent solutions for each of the Vekua equations from the infinite sequence of equations corresponding to the original one, we are able to construct an infinite system of solutions of the original Vekua equation as is shown in the next definition. Moreover, as we show in the next subsection under quite general conditions this infinite system of solutions is complete.

**Definition 12** The formal power \(Z^{(0)}_m(a, z_0; z)\) with center at \(z_0 \in \Omega\), coefficient \(a\) and exponent 0 is defined as the linear combination of the generators \(F_m, G_m\) with real constant coefficients \(\lambda, \mu\) chosen so that \(\lambda F_m(z_0) + \mu G_m(z_0) = a\). The formal powers with exponents \(n = 1, 2, \ldots\) are defined by the recursion formula

\[
Z^{(n+1)}_m(a, z_0; z) = (n + 1) \int_{z_0}^{z} Z^{(n)}_{m+1}(a, z_0; \zeta)d[(F_m, G_m)]\zeta.
\]

(10)
This definition implies the following properties.

1. $Z^{(n)}_m(a, z_0; z)$ is an $(F_m, G_m)$-pseudoanalytic function of $z$.

2. If $a'$ and $a''$ are real constants, then
   \[ Z^{(n)}_m(a' + ia'', z_0; z) = a'Z^{(n)}_m(1, z_0; z) + a''Z^{(n)}_m(i, z_0; z). \]

3. The formal powers satisfy the differential relations
   \[ \frac{d}{dz}Z^{(n)}_m(a, z_0; z) = nZ^{(n-1)}_m(a, z_0; z). \]

4. The asymptotic formulas
   \[ Z^{(n)}_m(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0 \tag{11} \]
   hold.

Assume now that
\[ W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a, z_0; z) \tag{12} \]
where the absence of the subindex $m$ means that all the formal powers correspond to the same generating pair $(F, G)$, and the series converges uniformly in some neighborhood of $z_0$. It can be shown that the uniform limit of pseudoanalytic functions is pseudoanalytic, and that a uniformly convergent series of $(F, G)$-pseudoanalytic functions can be $(F, G)$-differentiated term by term. Hence the function $W$ in (12) is $(F, G)$-pseudoanalytic and its $r$th derivative admits the expansion
\[ W^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1) \cdots (n-r+1) Z^{(n-r)}_r(a_n, z_0; z). \]

From this the Taylor formulas for the coefficients are obtained
\[ a_n = \frac{W^{[n]}(z_0)}{n!}. \tag{13} \]
Definition 13 Let $W(z)$ be a given $(F,G)$-pseudoanalytic function defined for small values of $|z - z_0|$. The series
\[ \sum_{n=0}^{\infty} Z^{(n)}(a, z_0; z) \] (14)
with the coefficients given by (13) is called the Taylor series of $W$ at $z_0$, formed with formal powers.

The Taylor series always represents the function asymptotically:
\[ W(z) - \sum_{n=0}^{N} Z^{(n)}(a, z_0; z) = O\left(|z - z_0|^{N+1}\right), \quad z \to z_0, \] (15)
for all $N$. This implies (since a pseudoanalytic function can not have a zero of arbitrarily high order without vanishing identically) that the sequence of derivatives $\{W^{[n]}(z_0)\}$ determines the function $W$ uniquely.

If the series (14) converges uniformly in a neighborhood of $z_0$, it converges to the function $W$.

2.3 Convergence theorems

The statements given in this subsection were obtained by L. Bers [8], [10] and S. Agmon and L. Bers [1].

Theorem 14 [8] The formal Taylor expansion (14) of a pseudoanalytic function in formal powers defined by a periodic generating sequence converges in some neighborhood of the center.

This theorem means only a local completeness of the system of formal powers. The following definition due to L. Bers describes the case when corresponding formal powers represent a globally complete system of solutions of a Vekua equation much as in the case of usual powers of the variable $z$ and the Cauchy-Riemann equation.

Definition 15 [8] A generating pair $(F, G)$ is called complete if these functions are defined and satisfy the Hölder condition for all finite values of $z$, the limits $F(\infty)$, $G(\infty)$ exist, $\text{Im}(F(\infty)G(\infty)) > 0$, and the functions $F(1/z)$, $G(1/z)$ also satisfy the Hölder condition. A complete generating pair is called normalized if $F(\infty) = 1$, $G(\infty) = i$. 
A generating pair equivalent to a complete one is complete, and every complete generating pair is equivalent to a uniquely determined normalized pair. The adjoint of a complete (normalized) generating pair is complete (normalized).

From now on we assume that \((F, G)\) is a complete normalized generating pair. Then much more can be said on the series of corresponding formal powers. We limit ourselves to the following completeness results (the expansion theorem and Runge's approximation theorem for pseudoanalytic functions).

Following [8] we shall say that a sequence of functions \(W_n\) converges normally in a domain \(\Omega\) if it converges uniformly on every bounded closed subdomain of \(\Omega\).

**Theorem 16** Let \(W\) be an \((F, G)\)-pseudoanalytic function defined for \(|z - z_0| < R\). Then it admits a unique expansion of the form \(W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z)\) which converges normally for \(|z - z_0| < \theta R\), where \(\theta\) is a positive constant depending on the generating sequence.

The first version of this theorem was proved in [1]. We follow here [10].

**Remark 17** Necessary and sufficient conditions for the relation \(\theta = 1\) are, unfortunately, not known. However, in [10] the following sufficient conditions for the case when the generators \((F, G)\) possess partial derivatives are given. One such condition reads:

\[
|F_z(z)| + |G_z(z)| \leq \frac{\text{Const}}{1 + |z|^{1+\varepsilon}}
\]

for some \(\varepsilon > 0\). Another condition is

\[
\int \int_{|z| < \infty} \left( |F_z|^{2-\varepsilon} + |F_z|^{2+\varepsilon} + |G_z|^{2-\varepsilon} + |G_z|^{2+\varepsilon} \right) dx dy < \infty
\]

for some \(0 < \varepsilon < 1\).

**Theorem 18** [10] A pseudoanalytic function defined in a simply connected domain can be expanded into a normally convergent series of formal polynomials (linear combinations of formal powers with positive exponents).

**Remark 19** This theorem admits a direct generalization onto the case of a multiply connected domain (see [10]).
In posterior works \cite{24}, \cite{37}, \cite{20} and others deep results on interpolation and on the degree of approximation by pseudopolynomials were obtained. For example,

**Theorem 20** \cite{37} Let $W$ be a pseudoanalytic function in a domain $\Omega$ bounded by a Jordan curve and satisfy the Hölder condition on $\partial \Omega$ with the exponent $\alpha$ ($0 < \alpha \leq 1$). Then for any $\varepsilon > 0$ and any natural $n$ there exists a pseudopolynomial of order $n$ satisfying the inequality

$$|W(z) - P_n(z)| \leq \frac{\text{Const}}{n^\alpha - \varepsilon}$$

for any $z \in \overline{\Omega}$

where the constant does not depend on $n$, but only on $\varepsilon$.

### 3 Solutions of second order elliptic equations as real components of complex pseudoanalytic functions

#### 3.1 Factorization of the stationary Schrödinger operator

It is well known that if $f_0$ is a nonvanishing particular solution of the one-dimensional stationary Schrödinger equation

$$\left( -\frac{d^2}{dx^2} + \nu(x) \right) f(x) = 0$$

then the Schrödinger operator can be factorized as follows

$$\frac{d^2}{dx^2} - \nu(x) = \left( \frac{d}{dx} + \frac{f_0'}{f_0} \right) \left( \frac{d}{dx} - \frac{f_0'}{f_0} \right).$$

We start with a generalization of this result onto a two-dimensional situation. Consider the equation

$$(-\Delta + \nu) f = 0$$

in some domain $\Omega \subset \mathbb{R}^2$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\nu$ and $f$ are real valued functions. We assume that $f$ is a twice continuously differentiable function. By $C$ we denote the complex conjugation operator.
Theorem 21 \[30\] Let \( f \) be a positive in \( \Omega \) particular solution of (16). Then for any real valued function \( \varphi \in C^2(\Omega) \) the following equalities hold

\[
\frac{1}{4}(\Delta - \nu) \varphi = \left( \partial_z + \frac{f_z}{f} C \right) \left( \partial_z - \frac{f_z}{f} C \right) \varphi = \left( \partial_z + \frac{f_z}{f} C \right) \left( \partial_z - \frac{f_z}{f} C \right) \varphi.
\]

(17)

Proof. Consider

\[
\left( \partial_z + \frac{f_z}{f} C \right) \left( \partial_z - \frac{f_z}{f} C \right) \varphi = \frac{1}{4} \Delta \varphi - \frac{\left| \partial_z f \right|^2}{f^2} \varphi - \partial_z \left( \frac{\partial_z f}{f} \right) \varphi
\]

\[
= \frac{1}{4} (\Delta \varphi - \frac{\Delta f}{f} \varphi) = \frac{1}{4} (\Delta - \nu) \varphi.
\]

(18)

Thus, we have the first equality in (17). Now application of \( C \) to both sides of (18) gives us the second equality in (17).

The operator \( \partial_z - \frac{f_z}{f} I \), where \( I \) is the identity operator, can be represented in the form

\[
\partial_z - \frac{f_z}{f} I = f \partial_z f^{-1} I.
\]

Let us introduce the notation \( P = f \partial_z f^{-1} I \). Due to Theorem 21 if \( f \) is a positive solution of (16), the operator \( P \) transforms real valued solutions of (16) into solutions of the Vekua equation

\[
\left( \partial_z + \frac{f_z}{f} C \right) w = 0.
\]

(19)

Note that the operator \( \partial_z \) applied to a real valued function \( \varphi \) can be regarded as a kind of gradient, and if we know that \( \partial_z \varphi = \Phi \) in a whole complex plane or in a convex domain, where \( \Phi = \Phi_1 + i\Phi_2 \) is a given complex valued function such that its real part \( \Phi_1 \) and imaginary part \( \Phi_2 \) satisfy the equation

\[
\partial_y \Phi_1 + \partial_x \Phi_2 = 0,
\]

(20)

then we can reconstruct \( \varphi \) up to an arbitrary real constant \( c \) in the following way

\[
\varphi(x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta - \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c
\]

(21)

where \((x_0, y_0)\) is an arbitrary fixed point in the domain of interest.
By $A$ we denote the integral operator in (21):

$$A[\Phi](x,y) = 2 \left( \int_{x_0}^{x} \Phi_1(\eta,y) \, d\eta - \int_{y_0}^{y} \Phi_2(x_0,\xi) \, d\xi \right) + c.$$  

Note that formula (21) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve $\Gamma$ leading from $(x_0,y_0)$ to $(x,y)$

$$\varphi(x,y) = 2 \left( \int_{\Gamma} \Phi_1 \, dx - \Phi_2 \, dy \right) + c.$$  

Thus if $\Phi$ satisfies (20), there exists a family of real valued functions $\varphi$ such that $\partial_z \varphi = \Phi$, given by the formula $\varphi = A[\Phi]$.

In a similar way we define the operator $\overline{A}$ corresponding to $\partial_z$:

$$\overline{A}[\Phi](x,y) = 2 \left( \int_{x_0}^{x} \Phi_1(\eta,y) \, d\eta + \int_{y_0}^{y} \Phi_2(x_0,\xi) \, d\xi \right) + c.$$  

Consider the operator $S = f Af^{-1}I$ applicable to any complex valued function $w$ such that $\Phi = f^{-1}w$ satisfies condition (20). Then it is clear that for such $w$ we have that $PSw = w$.

**Proposition 22** [31] Let $f$ be a positive particular solution of (16) and $w$ be a solution of (19). Then the real valued function $g = Sw$ is a solution of (16).

**Proof.** First of all let us check that the function $\Phi = w/f$ satisfies (20). Let $u = \text{Re} \, w$ and $v = \text{Im} \, w$. Consider

$$\partial_y \Phi_1 + \partial_x \Phi_2 = \frac{1}{f} \left( (\partial_y u + \partial_x v) - \left( \frac{\partial_y f}{f} u + \frac{\partial_x f}{f} v \right) \right). \quad (22)$$

Note that equation (19) is equivalent to the system

$$\partial_x u - \partial_y v = -\frac{\partial_x f}{f} u + \frac{\partial_y f}{f} v, \quad \partial_y u + \partial_x v = \frac{\partial_y f}{f} u + \frac{\partial_x f}{f} v$$

from which we obtain that expression (22) is zero. Thus the function $\Phi$ satisfies (20) and hence the real valued function $\varphi = A[w/f]$ is well defined and satisfies the equation $\partial_z \varphi = w/f$.  

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Consider the expression
\[
\partial_y \partial_z (Sw) = \partial_y \left( (\partial_z f) A\left[\frac{w}{f}\right] + w \right) \\
= \left( \frac{1}{4} \Delta f \right) A\left[\frac{w}{f}\right] + (\partial_z f) \partial_z A\left[\frac{w}{f}\right] - \frac{\partial_z f}{f} w. \tag{23}
\]
For the expression \( \partial_z A\left[\frac{w}{f}\right] \) we have
\[
\partial_z A\left[\frac{w}{f}\right] = \partial_z A\left[\frac{w}{f}\right] + i \partial_y A\left[\frac{w}{f}\right] \\
= \frac{w}{f} - 2i \frac{v}{f} = \frac{\overline{w}}{f} \tag{24}
\]
where the following observation was used
\[
\partial_y A\left[\frac{u + iv}{f}\right](x, y) = 2 \left( \int_{x_0}^{x} \partial_y \left( \frac{u(\eta, y)}{f(\eta, y)} \right) d\eta - \frac{v(x_0, y)}{f(x_0, y)} \right) = \\
= -2 \left( \int_{x_0}^{x} \partial_\eta \left( \frac{v(\eta, y)}{f(\eta, y)} \right) d\eta - \frac{v(x_0, y)}{f(x_0, y)} \right) = -\frac{2v(x, y)}{f(x, y)}. 
\]
Thus substitution of (24) into (23) gives us the equality
\[
\Delta(Sw) = \nu f A\left[\frac{w}{f}\right] = \nu Sw.
\]

**Proposition 23** \[31\] Let \( g \) be a real valued solution of (16). Then
\[
SPg = g + cf
\]
where \( c \) is an arbitrary real constant.

**Proof.** Consider
\[
SPg = f A \partial_z \left[\frac{g}{f}\right] = f\left(\frac{g}{f} + c\right) = g + cf.
\]

Theorem 21 together with Proposition 22 show us that equation (16) is equivalent to the Vekua equation (19) in the following sense. Every solution of one of these equations can be transformed into a solution of the other equation and vice versa.
3.2 Factorization of the operator \( \text{div} p \text{ grad} + q \).

The following statement is known in a form of a substitution (see, e.g., \[38\]). Here we formulate it as an operational relation.

**Proposition 24** Let \( p \) and \( q \) be complex valued functions, \( p \in C^2(\Omega) \) and \( p \neq 0 \) in \( \Omega \). Then

\[
\text{div} p \text{ grad} + q = p^{1/2}(\Delta - r)p^{1/2} \quad \text{in } \Omega,
\]

where

\[
r = \frac{\Delta p^{1/2}}{p^{1/2}} - \frac{q}{p}.
\]

**Proof.** The easily verified relation

\[
\text{div} p \text{ grad} = p^{1/2}(\Delta - \frac{\Delta p^{1/2}}{p^{1/2}})p^{1/2}
\]

is well known (see, e.g., \[42\]). Adding to both sides of (26) the term \( q \) (and representing it on the right-hand side as \( p^{1/2} \left( \frac{q}{p} \right) p^{1/2} \)) gives us (25).

**Theorem 25** [32] Let \( p \) and \( q \) be real valued functions, \( p \in C^2(\Omega) \) and \( p \neq 0 \) in \( \Omega \), \( u_0 \) be a positive particular solution of the equation

\[
(\text{div} p \text{ grad} + q)u = 0 \quad \text{in } \Omega.
\]

Then for any real valued continuously twice differentiable function \( \varphi \) the following equality holds

\[
\frac{1}{4}(\text{div} p \text{ grad} + q)\varphi = p^{1/2} \left( \partial_z + \frac{fz}{f} C \right) \left( \partial_z - \frac{fz}{f} C \right)p^{1/2}\varphi,
\]

where

\[
f = p^{1/2}u_0.
\]

**Proof.** This is based on (17). From (25) we have that if \( u_0 \) is a solution of (27) then the function (29) is a solution of the equation

\[
(\Delta - r)f = 0.
\]

Then combining (25) and (17) we obtain (28). ■
Remark 26 According to (26), \[ \Delta - r = f^{-1} \text{div} f^2 \text{grad} f^{-1} \] where \( f \) is a solution of (30). Then from (25) we have
\[
\text{div} \ p \text{grad} + q = p^{1/2} f^{-1} \text{div} f^2 \text{grad} f^{-1} p^{1/2}.
\] (31)

Taking into account (29) we obtain
\[
\text{div} \ p \text{grad} + q = p u_0^{-1} \text{div} p u_0^2 \text{grad} u_0^{-1} \quad \text{in} \ \Omega.
\]

Remark 27 Let \( q \equiv 0 \). Then \( u_0 \) can be chosen as \( u_0 \equiv 1 \). Hence (28) gives us the equality
\[
\frac{1}{4} \text{div}(p \text{grad} \varphi) = p^{1/2} \left( \partial_z + \frac{\partial \varphi}{p^{1/2}} \right) \left( \partial_{\bar{\varphi}} - \frac{\partial \varphi}{p^{1/2}} \right) (p^{1/2} \varphi).
\]

In what follows we suppose that in \( \Omega \) there exists a positive particular solution of (27) which we denote by \( u_0 \).

Let \( f \) be a real function of \( x \) and \( y \). Consider the Vekua equation
\[
W \bar{z} = f \bar{W} \quad \text{in} \ \Omega.
\] (32)

This equation plays a crucial role in all that follows hence we will call it the main Vekua equation.

Denote \( W_1 = \text{Re} W \) and \( W_2 = \text{Im} W \).

Remark 28 Equation (32) can be written as follows
\[
f \partial_z(f^{-1}W_1) + if^{-1} \partial_{\bar{z}}(fW_2) = 0.
\] (33)

Theorem 29 Let \( W = W_1 + iW_2 \) be a solution of (32). Then \( U = f^{-1}W_1 \) is a solution of the conductivity equation
\[
\text{div}(f^2 \nabla U) = 0 \quad \text{in} \ \Omega,
\] (34)

and \( V = fW_2 \) is a solution of the associated conductivity equation
\[
\text{div}(f^{-2} \nabla V) = 0 \quad \text{in} \ \Omega,
\] (35)

the function \( W_1 \) is a solution of the stationary Schrödinger equation
\[
- \Delta W_1 + r_1 W_1 = 0 \quad \text{in} \ \Omega
\] (36)

with \( r_1 = \Delta f / f \), and \( W_2 \) is a solution of the associated stationary Schrödinger equation
\[
- \Delta W_2 + r_2 W_2 = 0 \quad \text{in} \ \Omega
\] (37)

where \( r_2 = 2(\nabla f)^2 / f^2 - r_1 \) and \((\nabla f)^2 = f_x^2 + f_y^2\).
Proof. To prove the first part of the theorem we use the form of equation (32) given in Remark 28. Multiplying (33) by \( f \) and applying \( \partial_z \) gives

\[
\partial_z \left( f^2 \partial_z \left( f^{-1} W_1 \right) \right) + \frac{i}{4} \Delta \left( f W_2 \right) = 0
\]

from where we have that \( \text{Re} \left( \partial_z \left( f^2 \partial_z \left( f^{-1} W_1 \right) \right) \right) = 0 \) which is equivalent to (34) where \( U = f^{-1} W_1 \).

Multiplying (33) by \( f^{-1} \) and applying \( \partial_z \) gives

\[
\frac{1}{4} \Delta \left( f^{-1} W_1 \right) + i \partial_z \left( f^{-2} \partial_z (f W_2) \right) = 0
\]

from where we have that \( \text{Re} \left( \partial_z \left( f^{-2} \partial_z (f W_2) \right) \right) = 0 \) which is equivalent to (35) where \( V = f W_2 \).

From (26) we have

\[
(\Delta - r_1) W_1 = f^{-1} \text{div} \left( f^2 \nabla \left( f^{-1} W_1 \right) \right).
\]

Hence from the just proven equation (34) we obtain that \( W_1 \) is a solution of (36).

In order to obtain equation (37) for \( W_2 \) it should be noticed that

\[
f \text{div} \left( f^{-2} \nabla (f W_2) \right) = (\Delta - r_2) W_2.
\]

Remark 30 Observe that the pair of functions

\[
F = f \quad \text{and} \quad G = \frac{i}{f}
\]

is a generating pair for (32). This allows us to rewrite (32) in the form of an equation for pseudoanalytic functions of second kind (equation (7))

\[
\varphi \overline{f} + \psi \frac{i}{f} = 0, \tag{39}
\]

where \( \varphi \) and \( \psi \) are real valued functions. If \( \varphi \) and \( \psi \) satisfy (39) then \( W = \varphi f + \psi \frac{i}{f} \) is a solution of (32) and vice versa.

Denote \( w = \varphi + \psi i \). Then from (39) we have

\[
(w + \overline{w}) \pi f + (w - \overline{w}) \frac{1}{f} = 0,
\]
which is equivalent to the equation

\[ u_\varphi = \frac{1 - f^2}{1 + f^2} w_\varphi \]  

(40)

The relation between (40) and (34), (35) was observed in [3] and resulted to be essential for solving the Calderón problem in the plane.

**Theorem 31** [32] Let \( W = W_1 + iW_2 \) be a solution of (32). Assume that \( f = p^{1/2}u_0 \), where \( u_0 \) is a positive solution of (27) in \( \Omega \). Then \( u = p^{-1/2}W_1 \) is a solution of (27) in \( \Omega \), and \( v = p^{1/2}W_2 \) is a solution of the equation

\[ (\text{div} \frac{1}{p} \text{grad} + q_1)v = 0 \quad \text{in} \ \Omega, \]  

(41)

where

\[ q_1 = -\frac{1}{p} \left( \frac{q}{p} + 2 \left\langle \nabla p, \nabla u_0 \right\rangle + 2 \left( \frac{\nabla u_0}{u_0} \right)^2 \right). \]  

(42)

**Proof.** According to theorem 29, the function \( f^{-1}W_1 \) is a solution of (34). From (31) we have that

\[ p^{-1/2} (\text{div} p \text{grad} + q_1) (p^{-1/2}W_1) = f^{-1} \text{div}(f^2 \nabla (f^{-1}W_1)) \]

from which we obtain that \( u = p^{-1/2}W_1 \) is a solution of (27).

In order to obtain the second assertion of the theorem, let us show that

\[ p^{1/2}(\text{div} \frac{1}{p} \text{grad} + q_1)(p^{1/2} \varphi) = f \text{div}(f^{-2}\nabla(f\varphi)) \]

for any real valued \( \varphi \in C^2(\Omega) \). According to (26),

\[ f \text{div}(f^{-2}\nabla(f\varphi)) = \left( \Delta - \frac{\Delta f^{-1}}{f^{-1}} \right) \varphi = (\Delta - r_2) \varphi. \]

Straightforward calculation gives us the following equality

\[ \frac{\Delta f^{-1}}{f^{-1}} = \frac{3}{4} \left( \frac{\nabla p}{p} \right)^2 - \frac{1}{2} \frac{\Delta p}{p} + \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle - \frac{\Delta u_0}{u_0} + 2 \left( \frac{\nabla u_0}{u_0} \right)^2. \]

From the condition that \( u_0 \) is a solution of (27) we obtain the equality

\[ -\frac{\Delta u_0}{u_0} = \frac{q}{p} + \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle. \]
Thus,
\[
\frac{\Delta f^{-1}}{f^{-1}} = \frac{3}{4} \left( \frac{\nabla p}{p} \right)^2 - \frac{1}{2} \frac{\Delta p}{p} + 2 \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle + \frac{q}{p} + 2 \left( \frac{\nabla u_0}{u_0} \right)^2.
\]

Notice that
\[
\frac{\Delta p^{-1/2}}{p^{-1/2}} = \frac{3}{4} \left( \frac{\nabla p}{p} \right)^2 - \frac{1}{2} \frac{\Delta p}{p}.
\]

Then
\[
\frac{\Delta f^{-1}}{f^{-1}} = \frac{\Delta p^{-1/2}}{p^{-1/2}} + 2 \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle + \frac{q}{p} + 2 \left( \frac{\nabla u_0}{u_0} \right)^2.
\]

Now taking \( q_1 \) in the form (42) we obtain the result from (25).

### 3.3 Conjugate metaharmonic functions

Theorems 29 and 31 show us that as much as real and imaginary parts of a complex analytic function are harmonic functions, the real and imaginary parts of a solution of the main Vekua equation (32) are solutions of associated stationary Schrödinger equations being also related to conductivity equations as well as to more general elliptic equations (27) and (41). The following natural question arises then. We know that given an arbitrary real valued harmonic function in a simply connected domain, a conjugate harmonic function can be constructed explicitly such that the obtained couple of harmonic functions represent the real and imaginary parts of a complex analytic function. What is the corresponding more general fact for solutions of associated stationary Schrödinger equations (which we slightly generalizing the definition of I. N. Vekua call metaharmonic functions) and of other aforementioned elliptic equations. The precise result for the Schrödinger equations is given in the following theorem.

**Theorem 32** [30] Let \( W_1 \) be a real valued solution of (36) in a simply connected domain \( \Omega \). Then the real valued function \( W_2 \), solution of (37) such that \( W = W_1 + iW_2 \) is a solution of (32), is constructed according to the formula
\[
W_2 = f^{-1} \mathcal{A}(i f^2 \partial_\tau(f^{-1} W_1)).
\]

Given a solution \( W_2 \) of (37), the corresponding solution \( W_1 \) of (36) such that \( W = W_1 + iW_2 \) is a solution of (32), is constructed as follows
\[
W_1 = -f \mathcal{A}(i f^{-2} \partial_\tau(f W_2)).
\]
Proof. Consider equation (32). Let $W = \phi f + i\psi / f$ be its solution. Then the equation

$$\psi_{\tau} - if^2 \phi_{\tau} = 0$$

(45)

is valid. Note that if $W_1 = \text{Re}W$ then $\phi = W_1 / f$. Given $\phi$, $\psi$ is easily found from (45):

$$\psi = \overline{A}(if^2 \phi_{\tau}).$$

It can be verified that the expression $\overline{A}(if^2 \phi_{\tau})$ makes sense, that is $\partial_x(f^2 \phi_x) + \partial_y(f^2 \phi_y) = 0$.

By theorem 29 the function $W_2 = f^{-1}\psi$ is a solution of (37). Thus we obtain (43). Let us notice that as the operator $\overline{A}$ reconstructs the real function up to an arbitrary real constant, the function $W_2$ in the formula (43) is uniquely determined up to an additive term $cf^{-1}$ where $c$ is an arbitrary real constant.

Equation (44) is proved in a similar way. ■

Remark 33 When in (36) $r_1 \equiv 0$ and $f \equiv 1$, equalities (43) and (44) turn into the well known formulas in complex analysis for constructing conjugate harmonic functions.

Corollary 34 Let $U$ be a solution of (34). Then a solution $V$ of (35) such that

$$W = fU + if^{-1}V$$

is a solution of (32), is constructed according to the formula

$$V = \overline{A}(if^2 U_{\tau}).$$

(46)

Conversely, given a solution $V$ of (37), the corresponding solution $U$ of (34) can be constructed as follows:

$$U = -\overline{A}(if^{-2}V_{\tau}).$$

Proof. Consists in substitution of $W_1 = fU$ and of $W_2 = f^{-1}V$ into (43) and (44). ■

Corollary 35 Let $f = p^{1/2}u_0$, where $u_0$ is a positive solution of (27) in a simply connected domain $\Omega$ and $u$ be a solution of (27). Then a solution
v of (41) with \( q_1 \) defined by (42) such that \( W = p^{1/2}u + ip^{-1/2}v \) is a solution of (32), is constructed according to the formula

\[
v = u_0^{-1}A(ipu_0^2 \partial_z(u_0^{-1}u)).
\]

Let \( v \) be a solution of (41), then the corresponding solution \( u \) of (27) such that \( W = p^{1/2}u + ip^{-1/2}v \) is a solution of (32), is constructed according to the formula

\[
u = -u_0A(ip^{-1}u_0^{-2} \partial_z(u_0v)).
\]

**Proof.** Consists in substitution of \( f = p^{1/2}u_0, W_1 = p^{1/2}u \) and \( W_2 = p^{-1/2}v \) into (43) and (44).

3.4 The main Vekua equation

The results of this section show us that the theory of the elliptic equation

\[( \text{div } p \text{ grad } + q )u = 0 \]

is closely related to the equation (32):

\[
W_z = \frac{f}{\overline{f}} W.
\]  

(47)

As was pointed out in remark 30 the pair of functions: \( F = f \) and \( G = \frac{i}{f} \) is a generating pair for this equation. Then the corresponding characteristic coefficients \( A_{(F,G)} \) and \( B_{(F,G)} \) have the form

\[A_{(F,G)} = 0, \quad B_{(F,G)} = \frac{f \overline{z}}{f},\]

and the \((F,G)\)-derivative according to (6) is defined as follows

\[
\dot{W} = W_z - \frac{f \overline{z}}{f} W = \left( \partial_z - \frac{f \overline{z}}{f} C \right) W.
\]

Comparing \( B_{(F,G)} \) with the coefficient in (19) and due to Theorem 4 we obtain the following statement.

**Proposition 36** Let \( W \) be a solution of (47). Then its \((F,G)\)-derivative, the function \( w = \dot{W} \) is a solution of (19).
This result can be verified also by a direct substitution. According to (8) and taking into account that
\[ F^* = -i \frac{f}{f} \quad \text{and} \quad G^* = \frac{1}{f}, \]
the \((F,G)\)-antiderivative has the form
\[
\int_{z_0}^z w(\zeta) d_{(F,G)}\zeta = f(z) \text{Re} \int_{z_0}^z \frac{w(\zeta)}{f(\zeta)} d\zeta - i \frac{f(z)}{f(\zeta)} \text{Re} \int_{z_0}^z i f(\zeta) w(\zeta) d\zeta
\]
\[
= f(z) \text{Re} \int_{z_0}^z \frac{w(\zeta)}{f(\zeta)} d\zeta + i \frac{f(z)}{f(\zeta)} \text{Im} \int_{z_0}^z f(\zeta) w(\zeta) d\zeta, \quad (48)
\]
and we obtain the following statement.

**Proposition 37** Let \( w \) be a solution of (19). Then the function
\[
W(z) = \int_{z_0}^z w(\zeta) d_{(F,G)}\zeta
\]
is a solution of (47).

### 3.5 \( p \)-analytic functions

**Definition 38** A function \( \Phi = u + iv \) of a complex variable \( z = x + iy \) is said to be \( p \)-analytic in some domain \( \Omega \) iff
\[
\frac{u_x}{p} = v_y, \quad \frac{u_y}{p} = -\frac{v_x}{p} \quad \text{in} \ \Omega \quad (49)
\]
where \( p \) is a given positive function of \( x \) and \( y \) which is supposed to be continuously differentiable.

The theory of \( p \)-analytic functions was presented in [39]. \( p \)-analytic functions in a certain sense represent a subclass of generalized analytic (or pseudoanalytic) functions studied by L. Bers [8, 9] and I. N. Vekua [43] and it should be noticed that this subclass preserves some important properties of usual analytic functions which are not preserved by a too ample class of generalized analytic functions (corresponding details can be found in [39]). \( p \)-analytic functions found applications in elasticity theory (see, e.g., [2, 22]) and in axisymmetric problems of hydrodynamics (see, e.g., [24, 45]).

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As was pointed out in remark 30 the function \( W = \phi f + i\psi / f \) is a solution of the main Vekua equation (47) if and only if \( \phi \) and \( \psi \) satisfy the equation (39) which is equivalent to the system

\[
\frac{\phi_x}{f^2} = \psi_y, \quad \frac{\phi_y}{f^2} = -\frac{\psi_x}{f^2}.
\]

In other words \( W \) is a solution of the main Vekua equation iff its corresponding pseudoanalytic function of second kind is \( f^2 \)-analytic. Thus, we obtain the following connection between the stationary Schrödinger equation and the system defining \( p \)-analytic functions.

**Theorem 39** Let \( f \) be a positive solution of the equation

\[
-\Delta g + \nu g = 0 \quad (50)
\]

where \( \nu \) is a real valued function and let \( W_1 \) be another real valued solution of this equation. Then the function \( \Phi = W_1/f + i f W_2 \), where \( W_2 \) is defined by (43) is an \( f^2 \)-analytic function, and vice versa, let \( \Phi \) be an \( f^2 \)-analytic function then the function \( W_1 = f \operatorname{Re} \Phi \) is a solution of (50).

The following relation between solutions of the conductivity equation and \( p \)-analytic functions is valid also.

**Theorem 40** Let \( f \) be a positive continuously differentiable function in a domain \( \Omega \) and let \( U \) be a real valued solution of the equation

\[
\operatorname{div}(f^2 \nabla U) = 0 \quad \text{in } \Omega. \quad (51)
\]

Then the function \( \Phi = U + iV \) is \( f^2 \)-analytic in \( \Omega \), where \( V \) is defined by (46), and vice versa, let \( \Phi \) be \( f^2 \)-analytic in \( \Omega \) then \( U = \operatorname{Re} \Phi \) is a solution of (51).

Thus, solutions of the stationary Schrödinger equation and of the conductivity equation can be converted into \( p \)-analytic functions and vice versa. In some cases this relation leads to a simplification of a part of \( p \)-analytic function theory.

**Example 41** A considerable part of bibliography dedicated to \( p \)-analytic functions consists of studying the case \( p = x^k \), where \( k \in \mathbb{R} \) (see, e.g., [16], [31]).
Let us see what is the form of the corresponding Schrödinger equation. For this we should calculate the potential \( \nu \) in (50) when \( f = x^{k/2} \). It is easy to see that

\[
\nu = \frac{k^2 - 2k}{4x^2}.
\]

The Schrödinger equation with this potential is well studied. Separation of variables leads us to the equation

\[
X''(x) + \left( \beta^2 - \frac{4\alpha^2 - 1}{4x^2} \right) X(x) = 0,
\]

where \( \beta^2 \) is the separation constant and \( \alpha = (k - 1)/2 \). The function

\[
X(x) = \sqrt{x}Z_\alpha(\beta x)
\]

is a solution of (53) (see [23, 8.491]) where \( Z_\alpha \) denotes any cylindric function of order \( \alpha \) (Bessel functions of first or second kind). Thus the study of \( x^k \)-analytic functions reduces to the Schrödinger equation (16) with \( \nu \) defined by (52) which in its turn after having separated variables reduces to a kind of Bessel equation (53).

**Example 42** In the work [26] boundary value problems for \( p \)-analytic functions with \( p = x/(x^2 + y^2) \) were studied. Considering

\[
f = \sqrt{p} = \sqrt{\frac{x}{x^2 + y^2}}
\]

we see that this function is a solution of the Schrödinger equation (16) with \( \nu \) having the form

\[
\nu = -\frac{1}{4x^2},
\]

that is we obtain again the potential of the form (52) where \( k = 1 \) and as was shown in the previous Remark the study of corresponding \( p \)-analytic functions in a sense reduces to the Bessel equation (53).

## 4 Complete systems of solutions for second-order equations

In what follows let us suppose that the real valued function \( f \) is defined in a somewhat bigger domain \( \Omega_e \) with a sufficiently smooth boundary. Then we
change the function \( f \) for \( z \in \Omega \setminus \Omega \) and continue it over the whole plane in such a way that \( f \equiv 1 \) for large \( |z| \) (see [10]). In this way the generating pair \((F,G) = (f, i/f)\) becomes complete and normalized.

Then the following statements are direct corollaries of the relations established in section 3 between pseudoanalytic functions (solutions of (32)) and solutions of second-order elliptic equations, and of the convergence theorems from subsection 2.3.

**Definition 43** Let \( u(z) \) be a given solution of the equation (27) defined for small values of \( |z - z_0| \), and let \( W(z) \) be a solution of (32) constructed according corollary 35, such that \( \Re W = p^{1/2}u \). The series

\[
\sum_{n=0}^{\infty} \Re Z^{(n)}(a_n, z_0; z)
\]

with the coefficients given by (13) is called the Taylor series of \( u \) at \( z_0 \), formed with formal powers.

**Theorem 44** Let \( u(z) \) be a solution of (27) defined for \( |z - z_0| < R \). Then it admits a unique expansion of the form

\[
u(z) = p^{-1/2}(z) \sum_{n=0}^{\infty} \Re Z^{(n)}(a_n, z_0; z)\]

which converges normally for \( |z - z_0| < R \).

**Proof.** This is a direct consequence of theorem 16 and remark 17. Both necessary conditions in remark 17 are fulfilled for the generating pair (38). ■

**Theorem 45** An arbitrary solution of (27) defined in a simply connected domain where there exists a positive particular solution \( u_0 \) can be expanded into a normally convergent series of formal polynomials multiplied by \( p^{-1/2} \).

**Proof.** This is a direct corollary of theorem 18. ■

More precisely the last theorem has the following meaning. Due to Property 2 of formal powers we have that \( Z^{(n)}(a, z_0; z) \) for any Taylor coefficient \( a \) can be expressed through \( Z^{(n)}(1, z_0; z) \) and \( Z^{(n)}(i, z_0; z) \). Then due to theorem 18 any solution \( W \) of (32) can be expanded into a normally convergent
series of linear combinations of $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$. Consequently, any solution of (27) can be expanded into a normally convergent series of linear combinations of real parts of $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$ multiplied by $p^{-1/2}$.

Obviously, for solutions of (27) the results on the interpolation and on the degree of approximation like, e.g., theorem 20 are also valid.

Let us stress that theorem 45 gives us the following result. The functions
\[ \left\{ p^{-1/2}(z) \Re Z^{(n)}(1, z_0; z), \quad p^{-1/2}(z) \Re Z^{(n)}(i, z_0; z) \right\}_{n=0}^{\infty} \] (54)
represent a complete system of solutions of (27) in the sense that any solution of (27) can be represented by a normally convergent series formed by functions (54) in any simply connected domain $\Omega$ where a positive solution of (27) exists. Moreover, as we show in section 6, in many practically interesting situations these functions can be constructed explicitly.

5 A remark on orthogonal coordinate systems in a plane

Orthogonal coordinate systems in a plane are obtained (see [35]) from Cartesian coordinates $x$, $y$ by means of the relation
\[ u + iv = \Phi(x + iy) \]
where $\Phi$ is an arbitrary analytic function. Quite often a transition to more general coordinates is useful
\[ \xi = \xi(u), \quad \eta = \eta(v). \]
\[ \xi \text{ and } \eta \text{ preserve the property of orthogonality. Some examples taken from [35] illustrate the point.} \]

Example 46 Polar coordinates
\[ u + iv = \ln(x + iy), \]
\[ u = \ln \sqrt{x^2 + y^2}, \quad v = \arctan \frac{y}{x}. \] (55)

Usually the following new coordinates are introduced
\[ r = e^u = \sqrt{x^2 + y^2}, \quad \varphi = v = \arctan \frac{y}{x}. \]
Example 47 *Parabolic coordinates*

\[ \frac{u + iv}{\sqrt{2}} = \sqrt{x + iy}, \]

\[ u = \sqrt{r + x}, \quad v = \sqrt{r - x}. \]

More frequently the parabolic coordinates are introduced as follows

\[ \xi = u^2, \quad \eta = v^2. \]

Example 48 *Elliptic coordinates*

\[ u + iv = \arcsin \frac{x + iy}{\alpha}, \]

\[ \sin u = \frac{s_1 - s_2}{2\alpha}, \quad \cosh v = \frac{s_1 + s_2}{2\alpha} \]

where \( s_1 = \sqrt{(x + \alpha)^2 + y^2} \), \( s_2 = \sqrt{(x - \alpha)^2 + y^2} \). The substitution

\[ \xi = \sin u, \quad \eta = \cosh v \]

is frequently used.

Example 49 *Bipolar coordinates*

\[ u + iv = \ln \frac{\alpha + x + iy}{\alpha - x - iy}, \]

\[ \tanh u = \frac{2\alpha x}{\alpha^2 + x^2 + y^2}, \quad \tan v = \frac{2\alpha y}{\alpha^2 - x^2 - y^2}. \]

The following substitution is frequently used

\[ \xi = e^{-u}, \quad \eta = \pi - v. \]

6 Explicit construction of a generating sequence

We suppose that the function \( f \) in the main Vekua equation (32) has the following form

\[ f = U(u)V(v) \] (56)
where \( u \) and \( v \) represent an orthogonal coordinate system and according to
the explained in the previous section we assume that \( \Phi = u + iv \) is an analytic
function of the variable \( z = x + iy \). \( U \) and \( V \) are arbitrary differentiable
nonvanishing real valued functions.

The present section is structured in the following way. First we explain
how one can naturally arrive at the form of generating sequences in the main
result of the section, theorem 5.40 and then we give a rigorous proof of this
result.

The first step in the construction of a generating sequence for the main
Vekua equation (32) is the construction of a generating pair for the equation
(19) which as was shown in subsection 3.4 is a successor of the main Vekua
equation. For this one of the possibilities consists in constructing another
pair of solutions of (32). Then their \((F,G)\)-derivatives will give us solutions
of (19). Let us consider equation (32) in its equivalent form (39):

\[
\varphi_z + \frac{i}{f^2} \psi_z = 0 \tag{57}
\]

and look for a solution in the form

\[
\varphi = \varphi(u), \quad \psi = \psi(v).
\]

Then we have the following equation

\[
\varphi'(u)u_z + \frac{i}{f^2} \psi'(v)v_z = 0.
\]

Taking into account that \( \Phi = u + iv \) is an analytic function, that is \( u_z + iv_z = 0 \)
we observe that (57) is fulfilled if \( \varphi'(u) = \psi'(v)/f^2 \). Now using (56) we obtain
that \( U^2(u)V^2(v) = \psi'(v)/\varphi'(u) \) and hence

\[
\varphi(u) = \int \frac{du}{U^2(u)} \quad \text{and} \quad \psi(v) = \int V^2(v)dv.
\]

The corresponding solution \( W_1 \) of (32) has the form

\[
W_1 = \int \frac{du}{U^2(u)} U(u)V(v) + \int V^2(v)dv \frac{i}{U(u)V(v)}.
\]

Its \((F,G)\)-derivative is obtained according to (6) as follows

\[
W_1 = \frac{V}{U} u_z + \frac{V}{U} v_z = \frac{V}{U} \Phi_z.
\]
By analogy, we can look for a solution of (57) in the form

\[ \varphi = \varphi(v), \quad \psi = \psi(u). \]

Then we have the following equation

\[ \varphi'(v) v + i f^2 \psi'(u) u = 0. \]

This equation is fulfilled if \( \varphi'(v) = -1/V^2(v) \) and \( \psi'(u) = U^2(u) \). Consequently,

\[ \varphi(v) = -\int \frac{dv}{V^2(v)} \quad \text{and} \quad \psi(u) = \int U^2(u) du. \]

The corresponding solution \( W_2 \) of (32) has the form

\[ W_2 = -\int \frac{dv}{V^2(v)} U(u)V(v) + \int U^2(u)du \frac{i}{U(u)V(v)} \]

with its corresponding \((F,G)\)-derivative

\[ \dot{W}_2 = -\frac{U}{V} v_z + \frac{U}{V} u_z = i \frac{U}{V} \Phi_z. \]

Denote \( F_1 = \dot{W}_1 \) and \( G_1 = \dot{W}_2 \). It is easy to see that the pair \( F_1, G_1 \) fulfils (51) and by construction satisfies equation (19). Thus, \( (F_1, G_1) \) is a successor of \((F,G)\) defined by (38).

The following step is to construct the generating pair \((F_2, G_2)\). For this we should find another pair of solutions of (19) and apply the \((F_1, G_1)\)-derivative to them. Consider equation (19) written in the form

\[ \varphi \tau F_1 + \psi \tau G_1 = 0 \]

which in our case can be represented as follows

\[ \varphi \frac{V}{U} + \psi \frac{U}{V} = 0. \]  

(58)

Again, let us look for a solution in the form

\[ \varphi = \varphi(u), \quad \psi = \psi(v). \]
Then equation (58) is satisfied if $\varphi'(u) = \left(\frac{U(u)}{V(v)}\right)^2 \psi'(v)$ from where we obtain

$$\varphi(u) = \int U^2(u)du \quad \text{and} \quad \psi(v) = \int V^2(v)dv.$$ 

Then the corresponding solution of (19) has the form

$$w_1 = \int U^2(u)du\Phi_z \frac{V(v)}{U(u)} + \int V^2(v)dv\Phi_z \frac{iU(u)}{V(v)}.$$ 

Its $(F_1, G_1)$-derivative is obtained as follows

$$\dot{w}_1 = \Phi_z UVu_z + i\Phi_z UVv_z = UV (\Phi_z)^2.$$ 

Analogously, looking for a solution of (58) in the form

$$\varphi = \varphi(v), \quad \psi = \psi(u)$$

we obtain that

$$\varphi(v) = -\int \frac{dv}{V^2(v)} \quad \text{and} \quad \psi(u) = \int \frac{du}{U^2(u)}.$$ 

The corresponding solution of (19) has the form

$$w_2 = -\int \frac{dv}{V^2(v)}\Phi_z \frac{V(v)}{U(u)} + \int \frac{du}{U^2(u)}\Phi_z \frac{iU(u)}{V(v)}.$$ 

Its $(F_1, G_1)$-derivative is obtained as follows

$$\dot{w}_2 = -\Phi_zv_z + \Phi_z iu_z \frac{i}{UV} = \frac{i}{UV} (\Phi_z)^2.$$ 

We have that the pair $F_2 = \dot{w}_1$ and $G_2 = \dot{w}_2$ is a successor of $(F_1, G_1)$. Observe that

$$(F_1, G_1) = \left(\frac{\Phi_z}{U^2}F, \quad U^2\Phi_z G\right)$$

and

$$(F_2, G_2) = \left((\Phi_z)^2 F, \quad (\Phi_z)^2 G\right).$$

From these formulae it is easy to guess a general form of the corresponding generating sequence which we give in the following statement.
Theorem 50  Let \( F = U(u)V(v) \) and \( G = \frac{i}{U(u)V(v)} \) where \( U \) and \( V \) are arbitrary differentiable nonvanishing real valued functions, \( \Phi = u + iv \) is an analytic function of the variable \( z = x + iy \) in \( \Omega \) such that \( \Phi_z \) is bounded and has no zeros in \( \Omega \). Then the generating pair \((F, G)\) is embedded in the generating sequence \((F_m, G_m)\), \( m = 0, \pm 1, \pm 2, \ldots \) in \( \Omega \) defined as follows

\[
F_m = (\Phi_z)^m F \quad \text{and} \quad G_m = (\Phi_z)^m G \quad \text{for even } m \]

and

\[
F_m = \left(\Phi_z\right)^m U^2 F \quad \text{and} \quad G_m = \left(\Phi_z\right)^m U^2 G \quad \text{for odd } m.
\]

Proof. First of all let us show that \((F_m, G_m)\) is a generating pair for \( m = \pm 1, \pm 2, \ldots \). Indeed we have

\[
\text{Im}(F_m G_m) = \text{Im}(\left|\Phi_z\right|^{2m} F G) > 0.
\]

Consider \( a_{(F_m, G_m)} \). For both \( m \) being even or odd we obtain

\[
a_{(F_m, G_m)} = \left|\Phi_z\right|^{2m} a_{(F, G)} \equiv 0.
\]

We should verify the equality

\[
b_{(F_m, G_m)} = -B_{(F_m-1, G_{m-1})} \quad (59)
\]

Consider first the case of an odd \( m \). A direct calculation gives us

\[
b_{(F_m, G_m)} = \left(\frac{\Phi_z}{\Phi_z}\right)^m \left(b_{(F, G)} - 2u_z U'\right)
\]

and

\[
B_{(F_m-1, G_{m-1})} = \left(\frac{\Phi_z}{\Phi_z}\right)^{m-1} B_{(F, G)}.
\]

Thus equality (59) is true iff \( \frac{\Phi_z}{\Phi_z} \left(b_{(F, G)} - 2u_z U'\right) \) is equal to \( -B_{(F, G)} \). It is easy to see that

\[
B_{(F, G)} = u_z \left(\frac{U'}{U} - i\frac{V'}{V}\right).
\]

Consider

\[
\frac{\Phi_z}{\Phi_z} \left(b_{(F, G)} - 2u_z U'\right) = \frac{u_z + iv_z}{u_z - iv_z} \left(\frac{U'}{U} u_z + \frac{V'}{V} v_z - 2u_z U'\right)
\]

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Thus, equality (59) is proved in the case of \( m \) being odd.

Now let \( m \) be even. Then

\[
\Phi \frac{b(F, G)}{\Phi} = \left( \frac{\Phi}{\Phi} \right)^m b(F, G)
\]

and

\[
B_{(F_m, G_m)} = \left( \frac{\Phi}{\Phi} \right)^m \left( B_{(F, G)} - 2 \frac{U'}{U} u_z \right).
\]

Equality (59) is valid iff the expression \( \Phi \frac{b(F, G)}{\Phi} \) is equal to \( -B_{(F, G)} + 2 \frac{U'}{U} u_z \).

We have

\[
\frac{\Phi}{\Phi} b(F, G) = \frac{u_z + iv_z}{u_z - iv_z} \left( \frac{U'}{U} u_z + \frac{V'}{V} v_z \right) = \frac{u_z}{u_z} \left( \frac{U'}{U} u_z + \frac{V'}{V} iu_z \right) = u_z \left( \frac{U'}{U} + \frac{V'}{V} \right)
\]

and from the other side

\[
-B_{(F, G)} + 2 \frac{U'}{U} u_z = -u_z \left( \frac{U'}{U} - i \frac{V'}{V} \right) + 2 \frac{U'}{U} u_z = u_z \left( \frac{U'}{U} + i \frac{V'}{V} \right).
\]

Thus, equality (59) is proved in all cases and the sequence \((F_m, G_m), m = 0, \pm 1, \pm 2, \ldots\) satisfies the conditions of Definition 9. Therefore it is a generating sequence. 

Remark 51 This result obviously generalizes the explicit construction of a generating sequence in the case when \( u = x \) and \( v = y \) presented in a number of works by L. Bers (see, e.g., [8] and [17]).

The last theorem opens the way for explicit construction of formal powers corresponding to the main Vekua equation (32) in the case when \( f \) has the form (56) and hence for explicit construction of complete systems of solutions for corresponding second-order elliptic equations. We give more details on this as well as some examples in the next section.
7 Explicit construction of complete systems of solutions of second-order elliptic equations

In section 4 we explained how complete systems of solutions of second-order elliptic equations can be constructed from systems of formal powers for a corresponding main Vekua equation (32). In the preceding section we established a result which allows us to make this construction possible in the case when \( f \) in (32) has the form (56). Then formal powers are constructed simply by definition (12). The meaning of this result for the stationary Schrödinger equation and for the conductivity equation we discuss separately in the next two subsections.

7.1 Explicit construction of complete systems of solutions for a stationary Schrödinger equation

Consider the equation

\[-\Delta g + \nu g = 0 \quad \text{in } \Omega.\]

(60)

where \( \nu \) is a real valued function. In order to start the procedure of construction of a complete system of solutions of this equation we need a positive particular solution \( f \) having the form (56).

**Example 52** Let \( \nu \) in (60) depend on one Cartesian variable: \( \nu = \nu(x) \). Suppose we are given a particular solution \( f = f(x) \) of the ordinary differential equation

\[-\frac{d^2 f}{dx^2} + \nu f = 0.\]

(61)

It is sufficient for the application of our result from the preceding section for constructing the corresponding generating sequence and hence the system of formal powers for the main Vekua equation which in this particular case has the form

\[W_x = \frac{f_x}{2f}W.\]

**Example 53** A number of works (see, e.g., [14] and [15]) are dedicated to construction (in our terms) of a particular solution for the Schrödinger equation with a radially symmetric potential. This solution \( f \) is precisely the only
necessary ingredient in order to obtain a complete system of formal powers and hence of solutions of the Schrödinger equation. Here an important restriction is that the analytic function \( \Phi_z \) corresponding to the polar coordinate system has the form \( 1/z \) and hence our procedure works in any domain not including the origin where \( f \) is positive.

**Example 54** Consider the Yukawa equation

\[
(-\Delta + c^2)u = 0
\]  

with \( c \) being a real constant. Take the following particular solution of (62): \( f = e^{cy} \). Let us construct the first few corresponding formal powers with center at the origin. We have

\[
Z^{(0)}(1, 0; z) = e^{cy}, \quad Z^{(0)}(i, 0; z) = ie^{-cy},
\]

\[
Z^{(1)}(1, 0; z) = xe^{cy} + \frac{i \sinh(cy)}{c}, \quad Z^{(1)}(i, 0; z) = -\frac{\sinh(cy)}{c} + ixe^{-cy},
\]

\[
Z^{(2)}(1, 0; z) = \left( x^2 - \frac{y}{c} \right) e^{cy} + \frac{\sinh(cy)}{c^2} + \frac{2ix \sinh(cy)}{c},
\]

\[
Z^{(2)}(i, 0; z) = -\frac{2x \sinh(cy)}{c} + i \left( \left( x^2 + \frac{y}{c} \right) e^{-cy} - \frac{\sinh(cy)}{c^2} \right), \ldots
\]

It is a simple exercise to verify that indeed the asymptotic formulas (11) hold. Now taking real parts of the formal powers we obtain a complete system of solutions of the Yukawa equation:

\[
u_0(x, y) = e^{cy}, \quad u_1(x, y) = xe^{cy}, \quad u_2(x, y) = -\frac{\sinh(cy)}{c},
\]

\[
u_3(x, y) = \left( x^2 - \frac{y}{c} \right) e^{cy} + \frac{\sinh(cy)}{c^2}, \quad u_4(x, y) = -\frac{2x \sinh(cy)}{c}, \ldots
\]

Formal powers of higher order can be constructed explicitly using a computer system of symbolic calculation. For this particular example (together with Maria Rosalía Tenorio) Matlab 6.5 allowed us to obtain analytic expressions for the formal powers up to the order ten, that gave us the first twenty one functions \( u_0, \ldots, u_{20} \). We used them for a numerical solution of the Dirichlet problem for the Yukawa equation with very satisfactory results. For example, in the case when \( \Omega \) is a unit disk with centre at the origin, \( c = 1 \) and \( u \) on the boundary is equal to \( e^{x} \) (this test exact solution gave us the worst precision
because of its obvious “disparateness” from functions $u_0, u_1 \ldots$) the maximal error $\max_{z \in \Omega} |u(z) - \tilde{u}(z)|$ where $u$ is the exact solution and $\tilde{u} = \sum_{n=0}^{20} a_n u_n$, the real constants $a_n$ being found by the collocation method, was of order $10^{-7}$. A very fast convergence of the method was observed.

Although the numerical method based on the usage of explicitly or numerically constructed pseudoanalytic formal powers still needs a much more detailed analysis these first results show us that it is quite possible that in due time and with a further development of symbolic calculation systems it can rank high among other numerical approaches. The use of complete systems of exact solutions based on pseudoanalytic formal powers has an important advantage of universality. The system does not depend on the choice of a domain whenever the function $f$ keeps to have no zeros.

In [11] the following system of solutions of the Yukawa equation (62) was obtained from completely different reasonings,

$$
u_0 = \cosh(cy), \quad u_1 = x \cosh(cy), \quad u_2 = x^2 \cosh(cy) - \frac{y}{c} \sinh(cy),$$

$$u_3 = x^3 \cosh(cy) - \frac{3xy}{c} \sinh(cy), \ldots$$

The same system is easily obtained (our Matlab program gives symbolic representations of these functions up to the subindex 21) if instead of the particular solution $f = e^{cy}$ considered in the above example we choose $f = \cosh(cy)$.

In the case of the Helmholtz equation

$$(\Delta + c^2)u = 0$$

L. R. Bragg and J. W. Dettman constructed the following system

$$u_0 = \cos(cy), \quad u_1 = x \cos(cy), \quad u_2 = x^2 \cos(cy) - \frac{y}{c} \sin(cy),$$

$$u_3 = x^3 \cos(cy) - \frac{3xy}{c} \sin(cy), \ldots$$

This system is also easily obtained as explained above choosing $f = \cos(cy)$.

Thus, the construction of the systems of solutions for the Yukawa and for the Helmholtz equation as well as the proof of their completeness in [11] are corollaries of a more general theory presented here.
In fact, having a positive particular solution \( f \) of the form (56) for equation (60) allows us to construct explicitly a complete system of solutions of (60) in any simply connected bounded domain where the corresponding to a chosen coordinate system analytic function \( \Phi_z \) is bounded and different from zero.

### 7.2 Explicit construction of complete systems of solutions for a conductivity equation

Consider the equation

\[
\text{div}(f^2 \nabla U) = 0.
\]

Usually in practical applications the function \( f \) is positive and equals 1 outside a certain disk. We suppose additionally that \( f \) is continuously differentiable on the whole plane. Under these conditions \( F = f \) and \( G = i/f \) is a complete normalized generating pair and all the convergence and Runge type results are valid. Using results of section 6 one can construct explicitly a complete system of solutions of the conductivity equation when the function \( f \) has the form (56) in any domain where the analytic function \( \Phi_z \) from theorem 50 is bounded and has no zeros.

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