A bias in VLBI measurements of the core shift effect in AGN jets

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ABSTRACT

The Blandford and Königl model of AGN jets predicts that the position of the apparent opaque jet base — the core — changes with frequency. This effect is observed with radio interferometry and is widely used to infer parameters and structure of the innermost jet regions. The position of the radio core is typically estimated by fitting a Gaussian template to the interferometric visibilities. This results in a model approximation error, i.e. a bias that can be detected and evaluated through simulations of observations with a realistic jet model. To assess the bias, we construct an artificial sample of sources based on the AGN jet model evaluated on a grid of the parameters derived from a real VLBI flux-density-limited sample and create simulated VLBI data sets at 2.3, 8.1 and 15.4 GHz. We found that the core position shifts from the true jet apex are generally overestimated. The bias is typically comparable to the core shift random error and can reach a factor of two for jets with large apparent opening angles. This observational bias depends mostly on the ratio between the true core shift and the image resolution. This implies that the magnetic field, the core radial distance and the jet speed inferred from the core shift measurements are overestimated. We present a method to account for the bias.

Key words: techniques: interferometric – galaxies: active – galaxies: nuclei – galaxies: jets

1 INTRODUCTION

The Very Long Baseline Interferometry (VLBI) technique allows exploring Active Galactic Nuclei (AGN) jets with the highest angular resolution. Synthesised images of most of the radio-loud AGN observed with VLBI are dominated by an unresolved or barely-resolved feature called the core (or VLBI core, e.g. Zensus 1997; Kovalev et al. 2005). The core has a flat or inverted radio spectrum, characteristic of optically-thick synchrotron emission (Hovatta et al. 2014). Its frequency-dependent position was discovered by Marcaide & Shapiro (1984) and can be explained within the relativistic jet model (Blandford & Königl 1979; Königl 1981). The core appears in that model as a surface in a continuous flow where the optical depth of jets synchrotron radiation at a given frequency $\tau_\nu \approx 1$ — “photosphere”. However, there are other explanations of the core nature (e.g. Marscher 2008).

The measurements of the core shift effect are of a key importance in studying AGN jets. They were used to estimate the jet magnetic field and particle density (Lobanov 1998; Hirotani 2005; Kovalev et al. 2008; O’Sullivan & Gabuzda 2009; Sokolovsky et al. 2011; Alagba et al. 2012; Pushkarev et al. 2012; Fromm et al. 2013; Kutkin et al. 2014; Zdziarski et al. 2015; Nokhrina 2017; Finke 2019; Plavin et al. 2019a), the composition of the jet plasma (Hirotani 2005), the jet magnetization (Nokhrina et al. 2015), the jet power and its production efficiency (e.g. Pjanka et al. 2017; Nokhrina 2020), the location of core relative to the central black hole and the geometry of the outflow (e.g. Hada et al. 2011; Algaba et al. 2017). They were also used to constrain the spins of the supermassive black holes (Agarwal et al. 2017) and the nature of the central engine (Zamaninasab et al. 2014). If a source demonstrates flaring activity, it is possible to derive the jets innermost regions kinematics based on the measured core shift and multi-frequency time delays (Kudryavtseva et al. 2011; Kutkin et al. 2014, 2018, 2019; Plavin et al. 2019a). Differences in core positions should be taken into account for image registration when constructing the VLBI spectral index (e.g. Marr et al. 2001; Kovalev et al. 2008; Hovatta et al. 2014) and Faraday rotation maps (e.g. Hovatta et al. 2012). The core shift effect is also important for the radio (VLBI) and optical (Gaia) reference frame alignment (Porcas 2009; Kovalev et al. 2017; Plavin et al. 2019b), as it can partially explain the observed offsets.

However, conventionally the position of the core and, thus, the core shift is estimated by fitting a set of Gaussian templates to the interferometric visibility. This could result in a biased estimates of the core shift if the emission profile is non Gaussian. Plavin et al. (2019a) assumed a linear dependence between the measured core shift and the synthesised beam for a multi-epoch multi-frequency (2.3 and 8.4 GHz) sample of 40 sources and found that the measured core shift is positively correlated with resolution, i.e. the beam size. Their results could imply that core shifts are typically overestimated. However, Sokolovsky et al. (2011) did not find larger core shifts between 1.4 and 15.4 GHz in the direction of the elongated beam in their sample of 20 sources. In this paper we study the possible bias directly, using the jet model (Blandford & Königl 1979; Königl 1981) that is used to explain the core shift effect.
Throughout the paper, we adopt the ΛCDM cosmology with \( \Omega_m = 0.287 \), \( \Omega_{\Lambda} = 0.7185 \) and \( H_0 = 69.3 \text{ km s}^{-1} \text{ Mpc}^{-1} \) (Hinshaw et al. 2013). The spectral index is defined via the dependence of the flux density on the spectral index \( S \propto \nu^{-\alpha} \).

## 2 Jet Model and Core Shift Measurements

The jet model by Blandford & Königl (1979); Königl (1981) assumes a conical geometry of the outflow and the particle density amplitude and the magnetic field dependence on the distance \( r \) from the jet apex: \( K = K_1(r/r_1)^{-n} \) and \( B = B_1(r/r_1)^m \), where the reference position is \( r_1 = 1 \text{ pc} \) and \( B_1 \) and \( K_1 \) are the values at the \( r_1 \). In the general case of arbitrary exponents \( m, n \) and optically thin spectral index \( \alpha \) analytical expressions for the distribution of optical depth and brightness temperature can be found in, e.g. Pashchenko & Plavin (2019).

Lobanov (1998) has shown that the relativistic jet model can be used to estimate the jets’ physical parameters, e.g. magnetic field and particle density, using the measured core shift. However, this requires some further assumptions because core shift measurements alone do not constrain the parameters of the jet model. More on degeneracies between model parameters can be found in Pashchenko & Plavin (2019). The most widely used is the assumption of an equipartition between emitting particles and magnetic field energy densities (Lobanov 1998; O’Sullivan & Gabuzda 2009; Pushkarev et al. 2012). Zdziarski et al. (2015) used the optically thick approximation to avoid the equipartition assumption (for a similar approach, see also Nokhrina 2017). They obtained magnetic field estimates which significantly differ from the corresponding equipartition values. Both approaches need measurements of the core shifts between two frequencies \( \Delta r \). In the equipartition case, the magnetic field scales with the core shift as \( B_1^{\text{eq}} \propto r_1^{0.75}k_5 \) (Lobanov 1998), where \( k_5 = \frac{n+m(1.5+\alpha)-1}{1.5+\alpha} \). However, for a more general case, the dependence on the measured core shift is much stronger: \( B_1^{\text{noeq}} \propto r_1^{-5} \) (Zdziarski et al. 2015).

There are several methods to measure the core shift effect (for a brief review, see Kutkin et al. 2014). In general, one has to know the relative position of the core at several frequencies. The absolute position information is lost during self-calibration of the VLBI data (Pearson & Readhead 1984). Thus in the absence of a phase calibrator, the first step in deriving the core shift is matching images at different frequencies using a reference point that is stable with frequency. The optically thin structure of a jet is the most obvious reference for core shift measurements. Depending on the morphology of the extended jet structure, two methods are conventionally used. In the first one, the core position is measured with respect to an individual jet feature using a brightness distribution model (Lobanov 1998; Kovalen et al. 2008; Sokolovsky et al. 2011; Kutkin et al. 2014). The second measure core position with respect to a large jet section rich in structure using image cross-correlation (Walker et al. 2000; Croke & Gabuzda 2008; O’Sullivan & Gabuzda 2009; Algaba et al. 2012; Hovatta et al. 2012; Pushkarev et al. 2012; Fromm et al. 2013; Plavin et al. 2019a).

Regardless of the method used to align images, the final step in deriving the core shift is the measurement of the core position itself. Conventionally, the source structure is modelled with a set of Gaussian templates (e.g. Lister et al. 2009b). More recently, Plavin et al. (2019a) employed subtraction of the extended structure from the original data set and considered the resulting data as a contribution from the VLBI core only. In both approaches, the core brightness distribution is fitted with a simple model, i.e. usually a circular Gaussian. This could result in a bias of the measured core position if the true brightness distribution is significantly non-Gaussian. Here we define the bias as the difference between the observed and true values. However, before exploring the bias of the measurement, it is worth finding out what is supposed to be measured and what is actually measured.

## 3 Bias Due to \( \tau = 1 \) Region Being Offset from the Intensity Peak

To find the analytical expression for the position of the maximum intensity at some frequency, one has to differentiate the equation of the brightness temperature radial distribution. The brightness temperature distribution in a general case of arbitrary \( n, m \) and \( \alpha \) is 
(Pashchenko & Plavin 2019):

\[
T(r_{\text{obs}}, d) = \frac{c^2 C_1(\alpha) D_0 0.5^5 0.5^5 0.5 m_{\text{obs}} 0.5 m_{\text{obs}}}{8\pi k_B C_2(\alpha)(1 + z) 0.5^5 1^1 0.5^5 0.5 m_{\text{obs}}(\sin \theta) 0.5 m_{\text{obs}}(1 - e^{-\tau(r_{\text{obs}}, d)})},
\]

(1)

where the optical depth is

\[
\tau(r_{\text{obs}}, d) = C_2(\alpha)(1 + z)^{-1(\alpha+2.5)} D_1 1.5^5 a^5_{\text{obs}} K_1 B_1^{1.5^5 a^5_{\text{obs}} -2.5^5 a^5_{\text{obs}}} \times 2r_1 n^m(1.5^5 a^5_{\text{obs}})(\sin \theta) n^m(1.5^5 a^5_{\text{obs}} -1)^{-1} \sqrt{\varphi_\text{app}^2 - \left(\frac{d}{r_{\text{obs}}}ight)^2 r_{\text{obs}}^{-n^m(1.5^5 a^5_{\text{obs}} -1)}}.
\]

(2)

Here the position in the sky is described by \( r_{\text{obs}} \) and \( d \) — distances along and perpendicular to the jet direction, \( D \) — Doppler factor, \( \varphi_\text{app} \) is the observed jet half-opening angle, \( z \) — redshift, \( C_1(\alpha) \) and \( C_2(\alpha) \) — functions of optically thin spectral index \( \alpha \), \( c \) — speed of light and \( k_B \) — Boltzmann constant.

To this end, the following relation for the optical depth at the maximum intensity is obtained:

\[
\tau_{\text{max}} = \log \left(1 + \frac{n + m(1.5^5 + \alpha) - 1}{0.5^5}\right) \tau_{\text{max}}.
\]

(3)

For \( n = 2, m = 1, \alpha = 0.5 \) we obtain \( \tau_{\text{max}} \approx 2.92 \). This stays close to \( \tau = 3 \) for other physically meaningful power-law exponents \( n, m \) and spectral index \( \alpha \). Thus to obtain the position of the maximum \( r_{\text{max}} \) for some frequency, one has to solve the equation of the optical depth radial distribution for \( \tau = 2.92 \).

Some works (e.g. Blandford & Königl 1979) associate the core with this maximum intensity region, and derive parameter estimates based on that. However, (e.g. Lobanov 1998) the VLBI core is often interpreted as the region of unity optical depth, \( \tau = 1 \). These two approaches are not equivalent, and combining equations from papers adopting different conventions can become confusing and lead to subtle biases.

Both definitions of the core consider the on-axis distributions. An example of the on-axis intensity profile of the Blandford-Königl jet is given in Fig. 1. The ratio of the distances from the jet apex to the \( \tau = 1 \) region \( r_{\tau=1} \) and to the intensity maximum \( r_{\text{max}} \) can be calculated by substituting \( \tau_{\text{max}} \) into the equation for optical depth distribution along the jet axis (Pashchenko & Plavin 2019):

\[
\frac{r_{\text{max}}}{r_{\tau=1}} = \frac{r_{\tau=1}}{r_{\text{max}}}.\]

(4)

For a wide range of parameters, this ratio is \( r_{\text{max}}/r_{\tau=1} \approx 0.7 \), and \( r_{\text{max}} \) differs from \( r_{\tau=1} \) by \( \approx 40\% \). This implies that even if the
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4 BIAS DUE TO VLBI CORE GAUSSIAN FITTING

Blandford & Königl (1979) model has a significantly asymmetric radial profile (Fig. 1). To assess the bias due to its approximation with a Gaussian template, simulations with a realistic brightness distribution model are necessary. Here we describe a study of the systematics in core shift measurements using brightness distribution of the Blandford-Königl jet model, constructing artificial visibility data sets and processing them with conventional methods.

In this section, we use the results of MOJAVE observations to construct and study the properties of our artificial sample. We note that through the years MOJAVE group has constructed and published results for slightly different samples, including the MOJAVE-I sample (Lister et al. 2009a; Pushkarev et al. 2012; Clausen-Brown et al. 2013), its extension in (Pushkarev et al. 2017), as well as the MOJAVE 1.5 Jy Quarter-Century sample (1.5JyQC, Lister et al. 2019). All of them are VLBI flux-density-limited samples of Doppler-boosted AGN jets. Their properties are comparable and do not differ significantly for our goals.

4.1 Creation of the artificial sample

The brightness distribution of the Blandford & Königl (1979) model depends on several parameters. Only some combinations of these parameters are identifiable on the basis of the VLBI observations (Pashchenko & Plavin 2019). Using a uniform or random grid in the parameter space will result in sample parameter values, which are not typical for the sources observed in flux-density-limited samples (e.g. 1.5JyQC). Thus we construct a grid of model parameters conditioning them on the observed data in the following way. First, we draw a source with the parameters \((L, z, \theta, \Gamma)\) and the flux density \(> 1.5 \text{ Jy}\) from the intrinsic pure luminosity evolving luminosity function based on the 1.5JyQC sample. Here \(L\) — luminosity, \(z\) — redshift, \(\theta\) — viewing angle and \(\Gamma\) — bulk motion Lorentz factor. We draw \(z\) from constant co-moving density between \(z_{\text{min}} = 0.1\) and \(z_{\text{max}} = 3.4\). Thus we obtained the power-law low-luminosity cut-off of the unbeamed luminosity function from Lister et al. (2019). Then we draw \(L\) from the obtained power-law distributed luminosity function and \(\Gamma\) from the power-law with limits \(\Gamma_{\text{min}} = 1.25\) and \(\Gamma_{\text{max}} = 50\) and exponent \(-1.4\), obtained from the Monte-Carlo modelling of the 1.5JyQC sample in Lister et al. (2019). The viewing angle \(\theta\) was drawn from the isotropical (i.e. uniform in \(\cos\theta\)) distribution. From the unbeamed luminosity and Doppler factor \(\delta(\theta, \Gamma)\) we calculated the observed luminosity \(L\delta^{\alpha*}\), where we assume beaming exponent \(b = 2\) for a continuous flow and spectral index \(\alpha = 0\). If the observed flux corresponding to the parameters \((L, z, \theta, \Gamma)\) was larger than the threshold value 1.5 Jy, the source was accepted to the simulated sample. The distributions of redshifts and Lorentz factors of the simulated sources are presented in Fig. 2.

As expected, the redshift distribution is close to that of the 1.5JyQC sample, while the distribution of \(\Gamma\) corresponds to its best fit Monte-Carlo simulation (see fig. 11 in Lister et al. 2019). To obtain the value of the half-opening angle \(\phi_{1/2}\), we used relation \(\phi_{1/2} \approx \rho/\Gamma\) derived from statistical modelling of the opening angles distribution by Clausen-Brown et al. (2013), where \(\rho\) is drawn from the normal distribution \(N(\mu = 0.13, \sigma = 0.02)\). The resulted distribution of apparent opening angles \(\phi_{\text{app}}\) in the simulated sample is consistent with those obtained by Pushkarev et al. (2017): compare our median \(\phi_{\text{app}} = 20.3^\circ\) with their \(\phi_{\text{app}} = 21.5^\circ\).

We assumed a conical jet geometry with a radial velocity field and exponents of the magnetic field and particle density amplitude \(m = 1\) and \(n = 2\) (section 2). The first corresponds to a conical jet geometry with a constant bulk motion Lorentz factor \(\Gamma\) and isothermal particle distribution. The second implies a dominant transverse magnetic field component. Together they result in a constant ratio of the emitting particles to the magnetic field energy densities. This choice is justified because most of the core shift measurements were used in deriving the physical properties of the jet under the equipartition assumption. Note that the departures from the conical shape are seen close to the jet base up to the distance of \(10^6\) gravitational radii (Kovalev et al. 2020). This affects mostly nearby sources with the given limited VLBI resolution. We employed synchrotron radiation transfer coefficients for a tangled magnetic field model (Longair 1994) and assumed the optically thin spectral index \(\alpha = 0.5\), which corresponds to \(\zeta = 2.0\) in the particle energy spectra \(N(\gamma) \propto \gamma^{-\zeta}\). To limit the visible size of the jet, we included a steepening of the spectral index with the value 0.0001 pc\(^{-1}\). With this settings, the model spectrum is flat (Blandford & Königl 1979). We also assumed that the low- and high-frequency cut-offs in the spectrum fall outside of the chosen frequency range due to an external jet scale and break in the energy distribution of emitting particles in the magnetic field (Königl 1981).

To obtain the sample of the core shifts, we generated Stokes I images for frequencies 2.3, 8.1 and 15.4 GHz (S, X and U radio bands, respectively) numerically solving the radiative transfer equation using an adaptive Dormand-Prince 5 type of Runge-Kutta algorithm (Dormand & Prince 1980). As we assume radial velocity field, the Doppler factor \(\delta\) changes from the near to the far side of the jet. To obtain the synthetic VLBI data sets, we substituted the observed visibilities of a single template data set with their predicted model values and added a Gaussian noise estimated from the scatter of the
observed visibilities. We used real VLBI observations of the source 1458+718 from Lister et al. (2009b) and Sokolovskiy et al. (2011) as a template for \((u,v)\)-coverage and noise estimation. To estimate the thermal noise from the observed visibilities, we employed the successive differences approach (Briggs 1995). We choose a source with a high declination to reduce possible effects due to an elongated VLBA beam. We used major axes of naturally weighted CLEAN beams for calculations: 0.81, 1.61 and 6.51 mas for 15.4, 8.1 and 2.3 GHz. As will follow in subsection 4.3, for the straight jet model it is the core shift in units of the beam size that determines the value of the core shift bias. For elongated beams (i.e. sources with a low declination), we expect qualitatively the same behaviour with the effective resolution determined by the beam size along the core shift direction (subsection 4.4).

Following the common practice, we modelled the resulting data sets in the \((u,v)\)-plane at each frequency using simple Gaussian templates in Difmap (Shepherd 1997). We automated the modelling procedure routine using several stopping and filtering criteria to choose the “best” number of components, thus mimicking the modelling process by a human. At each step, the procedure analyses the residual map and suggests new components estimating their parameters by the method of moments (Wasserman 2010). We also employed the approach of Plavin et al. (2019a) of obtaining VLBI core parameters. It consists of estimating the extended emission by a list of CLEAN components outside of the core region and subtracting them from the data set in the \((u,v)\)-plane. The obtained residuals are then modelled with a single Gaussian component. We estimated the true core shift value using the model brightness distribution and determined the observed shift of the VLBI core in the corresponding Difmap model fitted to the synthetic VLBI data. Note that our brightness distributions are perfectly aligned, i.e. the jet cone apex is always positioned in the phase center of the VLBI images in the model.

Finally our simulated sample includes 1000 sources. We filtered 22 sources with core component flux densities larger than 10 Jy and less than 10 mJy. We also excluded 3 outliers with the ratio of the observed to true core shifts between 8.1 and 15.4 GHz less than 0.2 and one source with the observed core shift between 8.1 and 15.4 GHz larger than 1 mas.

### 4.2 Observed and true core shifts in the simulated sample

The median of the observed core shifts between 15.4 and 8.1 GHz for the MOJAVE-I sample is 0.114 mas (see tab. 1 in Pushkarev et al. 2012) for \(z > 0.1\). This agrees with the value of our simulated sample of 0.114 ± 0.003 mas for a circular Gaussian core model (hereafter, CG). Hereafter the error corresponds to the 95\% confidence interval. The confidence interval for the median was obtained by applying the Binomial distribution (Conover 1999). This agreement is expected because we run our simulations assuming population properties inferred from the MOJAVE 1.5Jy QC sample. The median core shift for the elliptical Gaussian core model (hereafter, EG) is 0.169 ± 0.005 mas.

Kovalev et al. (2008) found a median core shift between 2.3 and 8.6 GHz as 0.44 mas for their sample of 29 sources observed in geodetic sessions with a rather high dispersion (see their fig. 3). In our simulated sample, the median observed core shift between 2.3 and 8.1 GHz is 0.537 ± 0.013 mas. This is close to the value 0.61 mas extrapolated from the typical core shift between 15.4 and 8.1 GHz, assuming that the core shift \(\Delta r\) between frequencies \(v_1\) and \(v_2\) scales with \((v_1 - v_2)/(v_1 v_2)\) (Lobanov 1998). Plavin et al. (2019b) estimated the typical core shift between 2.3 and 8.6 GHz as 0.53 mas for a sample of 40 sources observed at 10 epochs or more, mainly during geodetic VLBI sessions. However, a different method of core parameters extraction was employed (subsection 4.1) there. In our simulated sample, this method provides the median core shift 0.91 mas. Note that both studies required specific criteria on the core shift measurement to be reliable, e.g. the same source structure at both bands (Kovalev et al. 2008), prominent extended structure (Plavin et al. 2019b), etc. These criteria are potentially capable of lowering observed typical core shifts relative to core shifts estimated for our simulated sample. Kovalev et al. (2008) estimated the theoretical mean core shift from 0.13 to 0.30 mas for a flux-density-limited sample between 2.3 and 8.6 GHz for the Blandford & Königl (1979) model. The specific value depends on the distribution of the Lorentz factors and assumes a mean synchrotron luminosity \(10^{44}\) erg/s and a mean redshift \(z = 1\). We note that this is significantly lower than in our simulated sample based on different underlying assumptions about the source population. This also could be the reason for a lower typical core shifts.

In Fig. 3 the observed and true core shifts obtained from the
simulated sample are compared. The positive bias of the core shift estimates is evident — the measured values are systematically larger than the true ones. The median of the true core shifts between 15.4 and 8.1 GHz is 0.09 mas, while for 2.3 and 8.1 GHz it is 0.43 mas.

Pushkarev et al. (2012) indicate a value of 0.05 mas as the random error of the core shift measurements between 15.4 and 8.1 GHz obtained using three approaches. This estimate includes a contribution from thermal noise, calibration error and mask size error. Plavin et al. (2019b) estimated the typical uncertainty of 8.6 - 2.3 GHz core shifts as 0.2 mas. The median bias of the estimated core shift between 8.1 and 15.4 GHz is 0.024 and 0.078 mas for circular and elliptical core models. For 8.6 - 2.3 GHz the corresponding median biases are 0.09 and 0.37 mas. These values are comparable to the random errors. However, even more important is the possible bias dependence on the source properties.

4.3 Bias dependency on the resolution

The core shift is determined by the difference in core positions at two frequencies. The dependence of the ratio of the observed $r_{\text{obs}}$ to the true $r_{\text{true}}$ projected core distance on the true core distance $r_{\text{true}}$ expressed in units of the restoring beam FWHM for our simulated sample is shown in Fig. 4 for 15.4, 8.1 and 2.3 GHz. Apparently, the core position is overestimated for the elliptical Gaussian core template. However, using circular Gaussians for core fitting may lead to an underestimation of $r_{\text{true}}$. In our simulations, this occurs at high jet viewing angles when the core is modelled with two close components. The position of the first one (more compact with a flatter spectra, thus, treated as the core) lays upstream of the maximum of the model brightness.

We found that the relative angular resolution is the main factor which determines the fractional core position bias. This dependence in Fig. 4 can be well described by:

$$\frac{r_{\text{obs}}}{r_{\text{true}}} = C_0 \left( \frac{r_{\text{true}}}{\text{FWHM}} \right)^{-k}$$

with $C_0 = 0.507 \pm 0.008$ and 1.099 ± 0.013 for CG and EG. Exponent $k = 0.41 \pm 0.02$ and $0.25 \pm 0.01$ for CG and EG. Note that the measurements at all frequencies are shown in Fig. 4. Assuming that the true position of the core for a given source is $r_{\text{true}}(\nu) = C_1 \nu^{-1/k}$ and FWHM$(\nu) = C_2 \nu^{-1}$, the observed core position of this source at some frequency $\nu$ yields:

$$r_{\text{obs}}(\nu) = C_0 C_1 \left( \frac{C_1}{C_2} \right)^{-k} \nu^{-1/k_{\text{eff}}}$$

and the ratio of the observed and true core shifts between two frequencies $\nu_1$ and $\nu_2$:

$$\frac{\Delta r_{\text{obs}}}{\Delta r_{\text{true}}} = C_0 \left( \frac{C_1}{C_2} \right)^{-k} \frac{\nu_1^{-1/k_{\text{eff}}} - \nu_2^{-1/k_{\text{eff}}}}{\nu_1^{-1/k} - \nu_2^{-1/k}}$$

where the effective coefficient of the frequency dependence is:

$$k_{\text{eff}} = k_{\nu} + k(k_{\nu} - 1)$$

equals to the true $k_{\nu}$ only for $k_{\nu} = 1$. In this case, the ratio of the observed to the true core shift is frequency independent. As $\Delta r_{\text{true}} \propto C_1$ and FWHM $\propto C_2$, this case also can be written via the true core shift in units of the mean restoring beam FWHM:

$$\frac{\Delta r_{\text{obs}}}{\Delta r_{\text{true}}} = C_{\Delta r} \left( \langle \text{FWHM} \rangle \right)^{-k}$$

where

$$C_{\Delta r} = \left( \frac{\nu_1^{1/k} + \nu_2^{1/k}}{2(\nu_1 - \nu_2)} \right)^{-k} C_0$$

Here $\langle \text{FWHM} \rangle$ — mean size of the beam for two frequencies and $C_0$ — constant from Equation 5. For 15.4 and 8.1 GHz we obtain $C_{\Delta r} = 0.41$ and for 8.1 and 2.3 GHz $C_{\Delta r} = 0.53$. This relation should hold at any pair of frequencies provided that the true core shift and the resolution scales linearly with the wavelength. The corresponding dependence is shown in Fig. 5 for the core shift between 15.4 and 8.1 GHz with its fit by Equation 9, which gives $C_{\Delta r} = 0.42 \pm 0.02$ and $k = 0.43 \pm 0.02$ for CG. This agrees with our estimates of $C_{\Delta r}$ from Equation 10 and fits of $C_0$ from Equation 5. For EG, we obtained $C_{\Delta r} = 0.87 \pm 0.03$ and $k = 0.30 \pm 0.02$. For both CG and EG core templates, the overestimation of the core shift is the most pronounced for sources with the largest apparent opening angles $\phi_{\text{app}}$ (Fig. 6). In our artificial sample, these are the sources with the smallest jet viewing angles, i.e., blazars. Radio galaxies are observed at larger viewing angles, have smaller $\phi_{\text{app}}$ (Pushkarev et al. 2017) and, consequently, practically unbiased core shifts.
For the core shift between 8.1 and 2.3 GHz, the corresponding values are $k = 0.34 \pm 0.02$ and $C_{\Delta r} = 0.57 \pm 0.02$ for CG. This is slightly different than estimated from Equation 5 and Equation 10. It could be due to a different VLBI array configuration at 2.3 GHz relative to other bands used as a template in our simulations.

4.4 Comparison with existing observational estimates of the bias

A recent extensive study of the core shift effect variability (Plavin et al. 2019a) analysed possible systematic biases as well. However, there are several differences in the analysis and the assumptions with our work. First, they focused on 2.3 and 8.6 GHz data obtained from different configurations of VLBI arrays. Moreover, they employed a different approach to the estimation of the core parameters.

Plavin et al. (2019a) estimated the proportionality coefficient between core shift bias and mean beam FWHM to be 0.14. Assuming that the effect at each frequency is proportional to its beam size, this implies that the bias of the core position is $0.10 \cdot \text{FWHM}$. Applying the same approach to the core parameter estimation as in Plavin et al. (2019a) to our simulated sample, we obtain the median bias of 0.11 · FWHM. Note that there are other possible sources of uncertainties (see Plavin et al. 2019a), which are out of the scope of this work. However, if the close agreement does not occur by chance, it suggests that the remaining measurement uncertainties only increase the statistical variance and do not make the results more biased. It also implies that significant beam elongation present in Plavin et al. (2019a) does not significantly change the effect.

The assumption adopted in Plavin et al. (2019a) leads to different dependence of the fractional core shift bias on the core shift between 2.3 and 8.1 GHz:

$$\frac{\Delta r_{\text{obs}}}{\Delta r_{\text{true}}} = 1 + 0.10 \left(\frac{\Delta r_{\text{true}}}{\text{FWHM}}\right)^{-1}. \quad (11)$$

This is more consistent with our results obtained for the elliptical Gaussian core model (Fig. 5, left), however, with a steeper dependence on the true core shift $-1$ vs. $-0.25$.

Finally, we also compared the different procedures of the core parameter estimation mentioned in section 2. It occurs that if a circular Gaussian is used to model the core, then the subtraction-based approach of Plavin et al. (2019a) almost always overestimates core shifts compared to the traditional approach of a modelling structure with a set of Gaussians. For 2.3–8.1 GHz, the median ratio and 95% interquantile ranges are 1.64 and (0.99, 3.04), while for 8.1–15.4 GHz, they are 1.59 and (0.26, 3.46). However, using an elliptical Gaussian as a core model leads to comparable results with the median ratio of 1.07 and 95% interquantile range (0.67, 1.99). We found that the core flux density is also overestimated — the median ratio of about 1.25 for all bands with interval (0.8, 2.2). At the same time, the most biased observable for subtraction-based core modelling is the core size. It overestimates true core sizes up to an order of magnitude, while the traditional approach estimates the true core size within a factor of 2. Here we calculated the true core size as half-width of the model jet at the position of the observed core. For all observables, the exact value of the overestimation depends on the core flux density fraction, with core dominant jets providing more consistent results. We interpret this as a result of modelling the extended structure by a set of CLEAN components prior to fitting a single Gaussian to the core region. CLEAN components, which are $\delta$-functions, always contribute to a correlated flux density at the longest baselines which could significantly influence the subsequent estimation of the core parameters.

5 METHOD TO COMPENSATE FOR THE BIAS

If only the measurements of the core shift between two frequencies $\nu_1$ and $\nu_2$ are available, one can assume $k = 1$ and obtain the relation for compensating the bias from Equation 9 and Equation 10:

$$\frac{\Delta r_{\text{true}}}{\text{FWHM}} = \left(\frac{\nu_1 + \nu_2}{2(\nu_1 - \nu_2)}\right)^{k/(1-k)} C_0^{1/(k-1)} \left(\frac{\Delta r_{\text{obs}}}{\text{FWHM}}\right)^{1/(1-k)}. \quad (12)$$
Here constants $C_0$ and $k$ are estimated in subsection 4.3 for both circular and elliptical core models. Note that Equation 12 can be employed for any pair of frequencies provided both true core shift and the resolution scale linearly with the wavelength. In Fig. 7, we plot the measured and true core shifts between 15.4 and 8.1 GHz together with those corrected for the bias using Equation 12. Here the residuals between the corrected and true values correspond to $\sigma_{\text{MAD}} = 0.02$ and 0.01 mas for circular and elliptical Gaussians (Fig. 8), where $\sigma_{\text{MAD}}$ is the robust estimate of standard deviation (Leys et al. 2013). The biases of both compensations are 0.010 and 0.005 mas for CG and EG, respectively.

If observations at more than two frequencies are analysed, then Equation 6 can be fitted to the measured core shift frequency dependence. Here $C_0$ and $k$ are estimated in this work (see subsection 4.3) and $C_2$ can be estimated from fitting the inverse frequency dependence to beam sizes. After $k_{\text{eff}}$ and $C_1$ are found, the can then be used to estimate true core shifts using the original relation $r_{\text{true}}(v) = C_1 v^{-1/k_{\text{eff}}}$. Finally, we warn readers that the estimates of $k$ and $C_0$ (thus $C_{\Delta r}$) could depend on the VLBI array configuration, as shown in subsection 4.3. This could introduce uncertainty in the core shift correction procedure described here. Thus the most rigorous way to compensate for the core shift bias for a particular VLBI array configuration would be to estimate coefficients $k$ and $C_{\Delta r}$ performing a series of simulations using a jet model (e.g. Blandford & Königl 1979) and a specific $(u, v)$-coverage.

6 IMPLICATIONS

6.1 Equipartition magnetic field estimates

The measured shifts are widely used in estimating the reference values (at 1 pc from the apex) of the magnetic field $B_1$ and particle density $K_1$ assuming equipartition (Lobanov 1998; Hirotani 2005; O’Sullivan & Gabuzda 2009; Pushkarev et al. 2012):

$$B_1^{\text{eq}} \propto (\Delta r_{\text{obs}})^{0.75k_1}.$$  
(13)

Here we should account for both biases: one is due to defining the core shift as the $r = 1$ region but measuring using the position of the Gaussian core (section 3), and the other is due to the measured position of the Gaussian core not representing the maximum intensity (section 4). They affect $\Delta r$ in different directions and compensate each other for $(\Delta r_{\text{true}})_{\text{FWHM}} = 0.1$ for CG and 0.4 for EG according to Fig. 5 and fits with Equation 9. Note that the magnetic field value $B_{\text{core}}$ at the core itself turns out less biased. The distance of the core from the jet apex $r_{\text{core}} \propto \Delta r_{\text{obs}}$ for $k_1 = 1$ (Lobanov 1998). Thus the magnetic field in the core weakly depends on the measured core shift as $B_{\text{core}}^{\text{eq}} = B_1^{\text{eq}} r_{\text{core}}^{-1} \propto (\Delta r_{\text{obs}})^{-1/4}$.

O’Sullivan & Gabuzda (2009) have estimated the 1$\sigma$ uncertainty of the magnetic field strength changing from 15% at 15.4 GHz to 100% and larger at 5 GHz, assuming equipartition and only considering the errors in the core shift measurements. More robust estimates should include others sources of errors as those of Doppler factors, jet opening and viewing angles, and spectral indices. Thus, the biases considered here should be treated as a lower limit of the accuracy of the magnetic field estimation. However, as averaging across multiple observations does not decrease a bias, it has certain astrophysical implications discussed below.
Figure 7. Raw observed (blue) and corrected (orange) core shifts vs true core shifts between 15.4 and 8.1 GHz for circular (left) and elliptical Gaussian (right) according to Equation 12. The dashed line represents the equality between the observed and true values.

Figure 8. Residuals between the true and measured but bias-corrected core shifts between 15.4 and 8.1 GHz for the circular (blue) and elliptical Gaussian core models (orange).

Nalewajko et al. (2014) found that magnetic fields inferred from Equation 13 are smaller than the $B_1$ estimates from the modelling of the spectral energy distribution (SED) by a factor of 2. This result is based on a sample of blazars from Zamaninasab et al. (2014) with a core shift estimated by Pushkarev et al. (2012). These authors propose that jets could have a non-uniform distribution of the magnetic field across the jet. An alternative explanation is that magnetic fields inferred from the core shift measurements are overestimated. Nalewajko et al. (2014) associate this with radio cores not being photospheres due to the synchrotron self-absorption (SSA) process, but due to a break in a lower energy cutoff of the energy distribution of the emitting particles. In our simulations, the sources with the smallest viewing angles are the most biased in terms of the core shift. This could reconcile the lower measured $B_1$ with the corresponding SED estimates within the SSA model of radio cores.

Zamaninasab et al. (2014) found observational evidence for the magnetically arrested disc (MAD) (Bisnovatyi-Kogan & Ruzmaikin 1974; Narayan et al. 2003) scenario in which magnetic fluxes of the jet $\Phi_j \propto B_p R_j^2$ should be comparable to magnetic fluxes $\Phi_{MAD}$ threading a black hole. Here $B_p$ is the poloidal magnetic field in the jet at some distance where the jet radius equals to $R_j$. As $\Phi_j$ is estimated using magnetic field amplitudes (Equation 13), it is overestimated for blazars. Another assumption lowering $\Phi_j$ was that $\Gamma_\theta \sim 1$. Jorstad et al. (2005); Pushkarev et al. (2009); Clausen-Brown et al. (2013) estimated $\Gamma_\theta \sim 0.1$ in the optically thin part of the jets. Accounting for this and for the biased core shift estimates results in the decrease of $\Phi_j$ on the order of the magnitude for blazars (see also Finke 2019).

6.2 Non-equipartition magnetic field estimates

A magnetic field can be also estimated from the core shift without equipartition assumptions using equation (8) from Zdziarski et al. (2015):

$$B_{\text{noeq}}(r) = \frac{3.35 \times 10^{-11} D_L \delta (\Delta r_{\text{obs}}) \tan^2 \phi}{r (v_1^{-1} - v_2^{-1})^3 \sin \theta} F_v \left(\frac{1}{s^2}ight)$$

(14)

where $r$ is distance in pc, $F_v$ — flux density of an inner jet integrated over both its optically thick and thin parts in Jy, $D_L$ — luminosity distance in Mpc, $v_1 < v_2$ — frequencies of observations in GHz, and the numerical coefficient is calculated for the exponent of the emitting particles energy distribution $s = 2$. Strong dependence on $\Delta r_{\text{obs}}$ implies that the found systematic error of the core shift could bias the magnetic field estimates up to an order of magnitude.

Zdziarski et al. (2015) estimated magnetic fields from core shift measurements using both equipartition (Equation 13) and a more general approach (Equation 14). They found $\langle B_{\text{noeq}} / B_{\text{eq}} \rangle = 1.6$ for blazars and $\langle B_{\text{noeq}} / B_{\text{eq}} \rangle = 0.09$ for radio galaxies. This ratio is modified in the presence of the core shift bias as $\langle B_{\text{noeq}} / B_{\text{eq}} \rangle = (\Delta r_{\text{obs}} / \Delta r_{\text{true}}) \times (B_{\text{noeq}} / B_{\text{eq}})$. Here $B_{\text{unbiased}}$ — magnetic field estimate which would be obtained if core shift measurements were unbiased. From our sample, the mean of $\Delta r_{\text{obs}} / \Delta r_{\text{true}}$
for sources with $\theta > 20^\circ$ (i.e. radio galaxies), corrected for bias estimated in section 3, is 0.73. Then, for blazars to have the same ratio of the unbiased estimates ($\tilde{R}_{\text{bias}}/\tilde{R}_{\text{true}}$), the mean ratio ($\Delta r_{\text{obs}}/\Delta r_{\text{true}}$) should be 1.44. In our simulated sample, this is achieved for sources with $\theta < 1.4^\circ$. Zdziarski et al. (2015) attribute the effect of different ($B_{\text{noeq}}/B_{\text{eq}}$) for blazars and radio galaxies to different viewing angles $\theta$ and the corresponding difference in relativistic beaming. However, it can also be explained by the systematics of the core shift measurements.

Pjanka et al. (2017) found that jet power $P_0$ values computed from the core shift estimated according to Zdziarski et al. (2015) are an order of magnitude lower than those estimated through the extended radio lobe luminosity. As the jet power $P_0 \propto B^2$ and $B \propto \Delta r^3$, this disagreement could be explained if the measured core shifts are overestimated by a factor of 1.26, consistent with our findings.

6.3 AGN jets: flares, speeds and geometry

Plavin et al. (2019a) found that core shifts vary significantly during flares. Their results imply that flares are due to an increase in the emitting particle density and a decrease of the magnetic field. As follows from the flare model presented in Plavin et al. (2019a), see their fig. 13, when a flare reaches the core at some frequency, it shifts the core downstream. This increases the core separation from the jet base and, according to subsection 4.2, decreases its fractional bias. Thus the quiescent state has the most fractionally biased core shift estimate. Contrary to expectations, it could be that the most accurate core shift measurements in terms of a fractional bias are possible during the flaring state. Not accounting for this effect could potentially underestimate departures from the equipartition in cores during flares, as found in Plavin et al. (2019a).

Algaba et al. (2017) found evidence for non-conical jet shapes in radio-loud AGN jets in the core region using core sizes $R$ measured at least four frequencies. The distances of radio cores from the central engines were estimated using measured core shifts from Pushkarev et al. (2012) and assuming $r_{\text{core}} \propto \Delta r_{\text{obs}}/\nu$ (i.e. $k_1 = 1$). For 56 sources with fit statistic $R^2 < 0.85$, the obtained outflow geometry is characterised by the coefficient $\epsilon$ of the radial core size dependence $R \propto r^\epsilon$, which is consistent with a quasi-parabolic outflow with mean ($\eta = 0.87$. From Equation 9, the true core shift is related to the observed one as $\Delta r_{\text{true}} \propto (\Delta r_{\text{obs}})^{1/(1-k)}$. Then, in the presence of a bias, the underlying dependence $R \propto r^\epsilon \propto \Delta r_{\text{true}}$ transforms to $R \propto (\Delta r_{\text{obs}})^{\epsilon/(1-k)}$, changing the effective exponent to $\epsilon_{\text{eff}} = \epsilon/(1-k)$. In subsection 4.3, we estimated $k \approx 0.42$ in the case of fitting the core with a circular Gaussian template. Thus the exponent ($\epsilon_{\text{eff}}$) was 0.87 measured by Algaba et al. (2017) corresponds to the intrinsic ($\epsilon$) = 0.50, implying parabolic jet shape. However, Algaba et al. (2017) also employed the elliptical core model in their estimation of ($\epsilon$). According to subsection 4.3, modelling the core with an elliptical Gaussian results in a lower $k$ and, consequently, higher ($\epsilon$). Thus our estimate of the intrinsic ($\epsilon$) should be considered as a lower bound.

The overestimation of $\epsilon$ due to the core shift bias could be interpreted as follows. In the parabolic accelerating outflow, the magnetic field and the particle density gradients are lower than those in a conical constant speed jet. This leads to a steeper than $\nu^{-1}$ frequency dependence of the core shift for distances closer to the jet base (i.e. higher frequencies), as demonstrated by Porth et al. (2011). Since resolution scales linearly with wavelength, and the bias is determined by the ratio of the true core shift to the resolution (subsection 4.3), core shift estimates at higher frequencies will have a larger fractional bias, effectively increasing the measured $\epsilon$. However, this assumes that the model brightness distribution is close to that of Blandford & Königl (1979) for a conical jet used in our simulations. In fact, Abellán et al. (2018) found more elongated cores at 43 GHz relative to 15 GHz for a fraction of sources in their complete SS polar cap sample (Eckart et al. 1986). This also could be a manifestation of a radio core at 43 GHz residing in the parabolic accelerating part of the jet.

If a source demonstrates flaring activity, it is possible to derive the kinematics of the jet innermost regions based on the measured core shift and multi-frequency time delays (Kudryavtseva et al. 2011; Kutkin et al. 2014, 2018, 2019). Assuming that the peak of a flare at a given frequency is observed when a moving jet feature passes through the region of the core at that frequency, it is possible to estimate the apparent projected speed of the component by comparing the light curve time delay, $\Delta t$, with the core shift for a pair of frequencies, $\Delta r$: $\nu_{\text{app}} = \Delta r / \Delta t$. Kutkin et al. (2014) obtained an estimate of the apparent angular velocity $\nu_{\text{app}} = 0.7$ mas yr$^{-1}$, which is 2–8 times larger than the speed estimated using VLBI kinematic data (Lister et al. 2013; Jorstad et al. 2010). These papers propose several explanations for this discrepancy, e.g. truly fast components or VLBI not measuring true plasma speed. Systematically overestimated core shifts could at least partially explain this controversy. This especially holds for blazars, inclined at small angles to the line of sight and demonstrating the largest fractional bias.

6.4 Astrometric implications

Precise positions of AGNs are measured with VLBI using either single-band phase delays or multi-band group delays of the arriving waves. The effects of the source structure and its frequency dependence on these positions were extensively studied: see, e.g. Porcas (2009), Plank et al. (2016), Xu et al. (2017), Anderson & Xu (2018). Single-band measurements are sensitive to the brightest part position itself; thus, they are affected by the full magnitude of the core position bias addressed in this paper.

Commonly used multi-band astrometry is sensitive to the frequency dependence of the core offset (Porcas 2009): the more it departs from $r \sim \nu^{-1}$, the larger the position measurement errors are. Such departures often arise due to intrinsic physical properties of AGNs: see Lobanov (1998) for theoretical modelling and Kudryavtseva et al. (2011); Algaba et al. (2012); Kutkin et al. (2014); Karmanavis et al. (2016) for specific observational results. Plavin et al. (2019a) have shown that this frequency dependence is also affected by strong flares which commonly happen in AGN jets. Our analysis suggests that systematic errors of astrometric positions in all these cases are effectively amplified. Indeed, as illustrated in Fig. 9, any departure of the exponent in $r \sim \nu^{-k}$ from $k_1 = 1$ gets magnified as an effect of the core shift measurement bias. Quantitative estimates of this influence on astrometric positions depend on specific methods used and are out of the scope of this paper.

In principle, the effects of the extended source structure on astrometric measurements are independent from the effects of the frequency-dependent core position (Porcas 2009). However, the extended structure is often described with a simple model like a set of Gaussian components (Charlot 1990). The determination of these components parameters can also be degraded by biases of the same nature as those studied here.
7 SUMMARY

In this paper, we examined and found significant biases in measurements of the frequency dependent core shift effect in AGN jets and suggested a method to compensate for them. To estimate the model approximation bias, we performed simulations using the artificially generated multi-frequency VLBI data sets based on the brightness distributions derived from the Blandford & Königl (1979) jet model and the parameters of the complete MOJAVE sample. We employed two approaches to modelling the core with a Gaussian template: fitting source structure with a number of Gaussians and fitting the core with a single Gaussian after subtracting the extended emission. Our results show that the estimated core positions and core shifts are positively biased. For example, for the 15.4–8.1 GHz frequency interval,Le, the typical bias is ≈ 0.02–0.08 mas, i.e. comparable to the core shift random error. In our simulated sample, the overestimation factor reaches the highest value of two for the sources with large apparent jet opening angles. The observed-to-true core shift ratio depends mostly on the true core shift in units of the beam size, which can be employed to compensate for the bias. The exponent $k_r$ of the core shift frequency dependence $r(\nu) \sim \nu^{-k_r}$ is biased for the $k_r \neq 1$ case. In fact, any departure from the canonical value $k_r = 1$ is underestimated. We also show that mixing two different definitions of the VLBI core — the position of the $\tau = 1$ surface and the position of the maximal intensity — could result in biased estimates of the magnetic field. Our results imply that the estimates of the jet magnetic fields made using measured values of the core shift could be positively biased up to an order of a magnitude depending on the estimation method. The apparent speed estimated from the core shift and the variability time lags as well as the distances of the core from jet origin obtained from the core shift might also be overestimated to some degree. Estimates of the jet geometry obtained from the core shift and size measurements are biased. They become consistent with a parabolic jet shape after correcting for the bias.

DATA AVAILABILITY

The datasets were derived from sources in the public domain: http://www.physics.purdue.edu/astro/MOJAVE/allsources.html, http://astrogeo.org/vlbi_images/.

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This research made use of Astropy, a community-developed core Python package for Astronomy (Astropy Collaboration et al. 2013), NumPy (Van Der Walt et al. 2011), Scipy (Jones et al. 2001), scikit-learn (Pedregosa et al. 2011) and PyMC3 (Salvatier et al. 2016). Matplotlib Python package (Hunter 2007) was used for generating all plots in this paper.

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Figure 9. Blue line shows the dependence of the observed vs the true $k_r$. The line of equality is shown by the orange color.
