EFFECTIVE DISCRETENESS RADIUS OF STABILISERS FOR STATIONARY ACTIONS

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Dedicated to Gopal Prasad on the occasion of his 75th birthday.

Abstract. We prove an effective variant of the Kazhdan–Margulis theorem generalized to stationary actions of semisimple groups over local fields: the probability that the stabilizer of a random point admits a non-trivial intersection with a small r-neighborhood of the identity is at most $\beta r^\delta$ for some explicit constants $\beta, \delta > 0$ depending only the group. This is a consequence of a key convolution inequality. We deduce that vanishing at infinity of injectivity radius implies finiteness of volume. Further applications are the compactness of the space of discrete stationary random subgroups and a novel proof of the fact that all lattices in semisimple groups are weakly cocompact.

1. Introduction

Let $G$ be a semisimple real Lie group without compact factors. Recall

Theorem 1.1 (Kazhdan–Margulis [KM68]). There exists an identity neighborhood $V \subset G$ such that if $\Gamma$ is a discrete subgroup of $G$ then $\Gamma g \cap V = \{e\}$ for some $g \in G$.

This theorem establishes the existence of a single point $g\Gamma$ in the probability space $G/\Gamma$ whose stabilizer $\text{Stab}_G(g\Gamma) = \Gamma g$ intersects trivially a given identity neighborhood. In fact, by [Gel18], the same conclusion holds true for most points in any probability space admitting a measure preserving action of $G$. Our main results make this statement quantitative, and apply more generally to stationary measures rather than just invariant one.

Let $K$ be a maximal compact subgroup of the Lie group $G$. Let $\mu$ be a bi-$K$-invariant probability measure on the group $G$ whose support $\text{supp}(\mu)$ is generating. Recall that a probability $G$-space $(Z, \nu)$ is called $\mu$-stationary if $\mu \ast \nu = \nu$. In particular any $G$-invariant probability measure is such.

Fix a left-invariant Riemannian metric on the Lie group $G$ and denote by $B_r$ the $r$-ball at the identity element of $G$ with respect to this metric.

Theorem 1.2. There are constants $\beta, \rho, \delta > 0$ such that every $\mu$-stationary probability $G$-space $(Z, \nu)$ with $\nu$-almost everywhere discrete stabilizers $G_z$ satisfies

\begin{equation}
\nu(\{z \in Z : G_z \cap B_r \neq \{\text{id}_G\}\}) \leq \beta r^\delta \quad \forall r < \rho.
\end{equation}

The constants $\beta$ and $\rho$ depend on the choice of the metric on the Lie group $G$ but are independent of the fixed probability measure $\mu$ and of the particular $\mu$-stationary $G$-space $(Z, \nu)$. The constant $\delta$ depends only on the Lie group $G$ and admits the explicit lower bound

\begin{equation}
\delta \geq (3ht(g) \dim_G)^{-(\text{rank}(K)+1)}
\end{equation}
where \( \text{rank}(K) \) is the rank of the maximal compact subgroup \( K \) and \( \text{ht}(g) \) is the largest height \( 1 \) of any positive root in the relative root system associated to \( G \).

The non-effective analogue of Theorem 1.2 is called \textit{weak uniform discreteness}. It was previously established in [Gel18] for invariant probability measures by an abstract compactness argument in the space of invariant random subgroups. Clearly one recovers Theorem 1.1 for any lattice \( \Gamma \) in the Lie group \( G \) as a special case of [Gel18, Theorem 1.3] (or of Theorem 1.2) by applying it to the probability space \( G/\Gamma \) with respect to any sufficiently small radius \( r > 0 \).

We remark that assuming the measure \( \nu \) is invariant and non-atomic, if the Lie group \( G \) is simple, or more generally if the \( G \)-action is irreducible (i.e. every non-central normal subgroup acts ergodically), then \( \nu \)-almost every stabilizer subgroup will be discrete [ABB+17].

While the conclusion of Theorem 1.2 is indeed independent of the choice of the probability measure \( \mu \), our proof relies on the properties of a specific and carefully chosen such probability measure, see Theorem 1.3 below.

**Remark 1.3.** The constant \( \rho \) appearing in the statement of Theorem 1.2 is strictly speaking redundant, in the sense that Equation (1.2) holds for all values of \( r > 0 \) provided that the constant \( \beta \) is replaced by \( \max\{\beta, \rho^{-1}\} \). We prefer to keep our sharper statement as is, since the constant \( \rho \) plays an essential role in the proof.

**The positive characteristic case.** Let \( k \) be a non-Archimedean local field and \( G \) a connected simply-connected semisimple \( k \)-algebraic linear group without \( k \)-anisotropic factors. Take \( G = G(k) \) so that \( G \) is a \( k \)-analytic group.

If \( \text{char}(k) = 0 \) then \( G \) has no small discrete subgroups [Ser09, Part II, Chapter V.9, Theorem 5], namely there is a compact open subgroup \( U \subset G \) such that every discrete subgroup \( \Gamma \) satisfies \( \Gamma \cap U = \{1\} \). This fact is already much stronger than any potential analogue of Theorems 1.1 or 1.2.

Assume therefore that \( \text{char}(k) \) is positive and is a good prime\(^2\) for \( G \). Our main result naturally generalizes to the \( k \)-analytic group \( G \), answering in the positive [Gel18, Question 4.3] and extending Theorem 1.1 to this setting.

We now state our main result in the positive characteristic case. Let \( \mathcal{O} \) be the ring of integers of the non-Archimedean local field \( k \) and \( \mathfrak{m} \) be the maximal ideal of \( \mathcal{O} \). Let \( G(\mathfrak{m}^i) \) denote the congruence subgroup\(^1\) of the compact group \( G(\mathcal{O}) \) modulo the ideal \( \mathfrak{m}^i \) for every \( i \in \mathbb{N} \). The subgroups \( G(\mathfrak{m}^i) \) form a basis of identity neighborhoods for the topology of \( G \).

Let \( \mu \) be a bi-\( G(\mathcal{O}) \)-invariant probability measure on the \( k \)-analytic group \( G \) such that the support of \( \mu \) generates the group \( G \).

**Theorem 1.4.** There are constants \( \beta, \rho, \delta > 0 \) such that every \( \mu \)-stationary probability \( G \)-space \((Z, \nu)\) with \( \nu \)-almost everywhere discrete stabilizers \( G_z \) satisfies

\[
\nu(\{z \in Z : G_z \cap G(\mathfrak{m}^i) \neq \{1\}\}) \leq \beta |\mathcal{O}/\mathfrak{m}|^{-\delta i}
\]

for all \( i \in \mathbb{N} \) with \( |\mathcal{O}/\mathfrak{m}|^{-i} < \rho \).

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1. The family \( A_n \) has largest height 1. The other classical families \( B_n, C_n, BC_n \) and \( D_n \) all have largest height 2. The exceptional semisimple Lie algebras have largest height 6 at most.

2. Good primes are discussed e.g. in [SS70, §4]. The only primes which fail to be good for some semisimple group \( G \) are 2, 3 and 5.

3. The exact definition of each congruence subgroup \( G(\mathfrak{m}^i) \) depends (up to finite index) on the particular matrix realization of the \( k \)-algebraic group \( G \).
The constants $\beta, \rho$ and $\delta$ are independent of the particular probability measure $\mu$ as well as of the $\mu$-stationary probability $G$-space $(Z, \nu)$. The constant $\delta$ admits the explicit lower bound

$$\delta \geq \frac{1}{ht(g)^2 \dim_k G}.$$  

Alternatively, one may consider the norm $\| \cdot \|_m$ on the group $G(m)$ given by

$$\|g\|_m = \inf \{ |O/m|^i : i \in \mathbb{N} \text{ and } g \in G(m^i) \}$$

for all elements $g \in G(m)$ and reformulate Equation (1.3) in terms of $r$-balls with respect to $\| \cdot \|_m$ in a manner analogous to Equation (1.1).

Just as in the Archimedean case, if $(Z, \nu)$ is a non-atomic irreducible probability measure preserving $G$-space then $\nu$-almost every stabilizer subgroup is discrete [GL18, Theorem 1.9].

We remark that while an analog of the Kazhdan–Margulis theorem over positive characteristic local fields was proved for products of rank-one groups in [Rag89, Lub90, Lub91] and for simply connected Chevalley groups of any rank in [Gol09], the general case has not been available in the literature until now.

**The key inequality.** A fundamental part of this work is the inequality stated in Theorem 1.5 below. The above results Theorems 1.2 and 1.4 are both derived in a rather straightforward manner from this inequality, which is of independent interest and admits various other applications.

To fix notations, let $k$ be a local field and $G$ a connected simply-connected semisimple $k$-algebraic linear group. Denote $G = G(k)$ so that $G$ is $k$-analytic. Let $K$ be a maximal compact subgroup of $G$ (if $k$ is non-Archimedean assume moreover the subgroup $K$ is good).

Let $\mu_s$ be the following bi-$K$-invariant probability measure on the $k$-analytic group $G$

$$\mu_s = \eta_K * \delta_s * \eta_K$$

where $\eta_K$ is the normalized Haar probability measure of the compact group $K$ and $\delta_s$ is an atomic probability measure supported on some sufficiently expanding regular semisimple element $s \in G$, see §8 for details. Note that $\mu_s$ is compactly supported.

To state the key inequality we consider the *discreteness radius* function $I_G(\Gamma)$ defined for every discrete subgroup $\Gamma$ of the $k$-analytic group $G$ by

$$I_G(\Gamma) = \sup\{0 \leq r \leq \rho : \Gamma \cap B_r = \{id_G\} \}$$

for some constant $\rho > 0$ (determined in §7). Here $B_r = \exp(\{X \in g : \|X\| \leq r\})$ with respect to some norm $\| \cdot \|$ on the Lie algebra $g = \text{Lie}(G)$ in the Archimedean case and $B_r = \{g \in G(m) : \|g\|_m \leq r\}$ in the non-Archimedean case.

**Theorem 1.5 (The key inequality).** There are constants $0 < c < 1$ and $b, \delta > 0$ such that every discrete subgroup $\Gamma$ of the group $G$ satisfies

$$\int G I_G^{-\delta}(\Gamma^g) \, d\mu_s(g) \leq c I_G^{-\delta}(\Gamma) + b.$$

The constant $\delta$ depends only on the analytic group $G$ in question. Moreover $\delta$ and $\rho$ coincide with the eponymous constants appearing in Theorems 1.2 and 1.4.
The space of discrete stationary random subgroups is compact. We consider the space Sub\((G)\) of all closed subgroups of the group \(G\) equipped with the Chabauty topology where \(G\) is a \(k\)-analytic group as above. This is a compact space on which \(G\) acts continuously by conjugation. Let Sub\(_d\)(\(G\)) be the subspace of Sub\((G)\) consisting of discrete subgroups. It is \(G\)-invariant but no longer compact.

A discrete random subgroup of \(G\) is a Borel probability measure \(\nu\) on the Chabauty space Sub\((G)\) satisfying \(\nu(\text{Sub}_d(G)) = 1\).

Any probability \(G\)-space \((Z, \nu)\) with \(\nu\)-almost surely discrete stabilizers determines a discrete random subgroup of \(G\) by pushing forward the measure \(\nu\) via the stabilizer map \(Z \to \text{Sub}(G), z \mapsto G_z\). Vice versa, every discrete random subgroup of \(G\) can be realized in this way with respect to some probability \(G\)-space.

This alternative viewpoint allows Theorems 1.2, 1.4 and 1.5 to be reformulated as results about discrete stationary random subgroups. In particular, the weak uniform discreteness of the discrete stationary random subgroups can be stated as a compactness result.

**Corollary 1.6.** Let \(\mu\) be a bi-\(K\)-invariant probability measure on the group \(G\). Then the set of \(\mu\)-stationary probability measures on the space \(\text{Sub}_d(G)\) is compact in the weak-\(*\) topology.

We remark that in the Archimedean case it is possible to go in the opposite direction and deduce weak uniform discreteness from the weak-\(*\) compactness of the set of discrete invariant random subgroups, see \[Gel18\].

**Evanescence and finiteness of volume.** Let \(M\) be a Riemannian manifold of non-positive sectional curvature. The following notion is introduced in \[BGS13\]: the manifold \(M\) has \(\text{InjRad} \to 0\) (hereafter evanescent) if its injectivity radius \(\text{Inj-Rad}_M\) vanishes at infinity. In other words \(M\) is evanescent if and only if its \(\varepsilon\)-thick part \(M_{\geq \varepsilon} = \{x \in M : \text{Ind-Rad}_M(x) \geq \varepsilon\}\) is compact for every \(\varepsilon > 0\). This is equivalent to the finiteness of the \(\varepsilon\)-essential volume\(^6\) for every \(\varepsilon > 0\).

Gromov proved that an evanescent analytic manifold of bounded non-positive sectional curvature is diffeomorphic to the interior of a compact manifold with boundary \[BGS13\, Theorem 2\].

It is clear that finite volume implies evanescence. The converse of this fact for locally symmetric spaces is a consequence of our key inequality (Theorem 1.5).

**Theorem 1.7.** A locally symmetric manifold is evanescent if and only if it has finite volume.

**Weak cocompactness of lattices.** A lattice \(\Gamma\) in a locally compact group \(H\) is weakly cocompact if the quasiregular unitary representation of \(H\) in the Hilbert space \(L^2_\mathbb{H}(H/\Gamma)\) does not admit almost invariant vectors, or in other words the \(H\)-space \(H/\Gamma\) has spectral gap.

Cocompact lattices are weakly cocompact \[Mar91, III.1.8\]. It was asked in \[Mar91, III.1.12\] whether all lattices are weakly cocompact. This is true for Lie

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\(^4\)If the local field \(k\) is Archimedean then the subspace Sub\(_d\)(\(G\)) is Chabauty open.

\(^5\)See \[ABB*17\, Theorem 2.6\] for a proof of the analogue fact for invariant random subgroups. The same proof applies to any locally compact group \(G\) and any measure \(\nu\) on Sub\((G)\) as long as \(\nu\)-almost every subgroup is unimodular.

\(^6\)We use \(\varepsilon\)-essential volume in the sense of \[BGS13\, §12.B\].
groups \cite{Bek98, BC10}, simple algebraic groups over non-Archimedean local fields \cite[Theorem 1]{BL11} and groups with Kazhdan’s property (T).

Some locally compact groups however do admit non-weakly cocompact lattices. A first explicit example in the group of tree automorphisms was constructed in \cite[Theorem 2]{BL11}. A lattice in an amenable second countable locally compact group is cocompact if and only if it is weakly cocompact \cite[Proposition 1.8]{BCGM19}. Thus, the non-uniform lattices constructed in \cite[Theorem 1.11]{BCGM19} provide further non-weakly cocompact examples.

We obtain the following result relying on the key inequality \eqref{key_inequality} and on the methods developed towards its proof.

**Theorem 1.8.** Let $k$ be a local field. Let $G$ be a connected simply-connected semisimple $k$-algebraic group without $k$-anisotropic factors. Then every lattice in the locally compact group $G(k)$ is weakly cocompact.

The conclusion of Theorem 1.8 has been previously known in most (but not all) cases. Its novelty lies in obtaining a unified and a direct proof.

Interestingly Theorem 1.8 is deduced in a rather straightforward manner from a stronger and a more general statement, namely Theorem 2 below. Roughly speaking, this last theorem says that the norm of a convolution operator restricted to the "thin part" of any probability measure preserving action of the group $G(k)$ is bounded away from one uniformly over all such actions. This result is novel even in the presence of Kazhdan’s property (T).

2. Averaging operators

Let $G$ be a locally compact second countable group and $\mu$ a probability measure on $G$. Let $Z$ be a locally compact topological space admitting a continuous action of the group $G$.

We study the averaging operator $A_\mu$ acting on continuous functions on the space $Z$. This operator is used to bound the $\nu$-measure of super-level sets of continuous functions satisfying a certain key inequality, where $\nu$ is any $\mu$-stationary measure on the space $Z$.

The current discussion uses ideas from \cite{EMM98, Mar04}. See also the works \cite{EM01, EM02, Ath06, EK09, MM10, BGHM10, BQ12, KLKM14, EMM15, HLMK16, MO20} for related ideas and further applications.

**Bounds on super-level sets.** Consider the averaging operator $A_\mu$ corresponding to the probability measure $\mu$. That is, given any continuous function $F : Z \to [0, \infty]$ we define the function $A_\mu F : Z \to [0, \infty]$ by

\begin{equation}
A_\mu F(z) = \int G F(gz) \, d\mu(g) \quad \forall z \in Z.
\end{equation}

Let $\nu$ be a $\mu$-stationary Borel probability measure on $Z$.

**Lemma 2.1.** Assume that the space $Z$ is $\sigma$-compact. Let $F : Z \to (0, \infty)$ be a continuous function. If there are constants $0 < c < 1$ and $b > 0$ such that

\begin{equation}
A_\mu F(z) \leq cF(z) + b
\end{equation}

then

\begin{equation}
\nu(\{ z \in Z \mid F(z) \geq M \}) \leq \frac{b}{(1-c)M} \quad \forall M > 0.
\end{equation}
Proof. Consider the sequence of probability measures \( \mu_m = \frac{1}{m} \sum_{i=1}^{m} \mu^{*i} \) on the group \( G \) for all \( m \in \mathbb{N} \). Note that the probability measure \( \nu \) is \( \mu_m \)-stationary for all \( m \in \mathbb{N} \), namely \( \mu_m * \nu = \nu \).

Since the space \( Z \) is assumed to be \( \sigma \)-compact it is possible to write \( Z = \bigcup_{n \in \mathbb{N}} Z_n \) where each \( Z_n \) is a compact subspace and \( Z_n \subset Z_{n+1} \) for all \( n \in \mathbb{N} \). Fix an index \( n \in \mathbb{N} \) and denote \( \nu_n = \nu|_{Z_n} \). Decompose the probability measure \( \nu \) as

\begin{equation}
\nu = \mu_m * \nu = \mu_m * \nu_n + \mu_m * (\nu - \nu_n).
\end{equation}

Up to passing twice to a subsequence, we may assume that \( \mu_m * \nu_n \xrightarrow{m \to \infty} \nu_n^\infty \) and \( \mu_m * (\nu - \nu_n) \xrightarrow{m \to \infty} \eta_n^\infty \) in the weak-* topology for some positive \( \mu \)-stationary measures \( \nu_n^\infty \) and \( \eta_n^\infty \) on the space \( Z \). Generally speaking, the Portmanteau theorem says that \( \nu_n^\infty (Z) \leq \nu_n (Z) \) and \( \eta_n^\infty (Z) \leq (\nu - \nu_n) (Z) = 1 - \nu_n (Z) \). Note however that passing to the limit in Equation (2.4) gives \( \nu = \nu_n^\infty + \eta_n^\infty \). Therefore it must be the case that \( \nu_n^\infty (Z) = \nu_n (Z) \) and \( \eta_n^\infty (Z) = 1 - \nu_n (Z) \). In particular

\begin{equation}
\inf_n \nu_n^\infty (Z) = 1 - \sup_n \nu_n^\infty (Z) = 1 - \sup_n \nu_n (Z) = 0.
\end{equation}

Clearly \( \sup Z_n, F < \infty \) as the subspace \( Z_n \) is compact and the function \( F \) is continuous. Iterating Equation (2.2) gives

\begin{equation}
A^i \mu_n F(z) \leq \frac{b}{1 - c} + c^i F(z) \leq \frac{b}{1 - c} + \sup_{Z_n} F \cdot c^i
\end{equation}

for all points \( z \in Z_n \). Therefore

\begin{equation}
\int_Z F d\nu_n^\infty = \lim_{m \to \infty} \int_Z F d(\mu_m * \nu_n) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \int_Z A^i \mu_n F d\nu_n \leq \frac{b}{1 - c} + \sup_{Z_n} F \cdot \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} c^i = \frac{b}{1 - c}.
\end{equation}

Markov’s inequality applied with respect to the measure \( \nu_n^\infty \) for each \( n \in \mathbb{N} \) and combined with Equation (2.5) implies that

\begin{equation}
\nu \{ z \in Z : F(z) \geq M \} = \inf_n (\nu_n^\infty + \eta_n^\infty) \{ z \in Z : F(z) \geq M \} \leq \frac{b}{(1 - c) M} + \inf_n \eta_n^\infty (Z) = \frac{b}{(1 - c) M}
\end{equation}

for all \( M > 0 \) as required.

Remark. It is possible to prove an analog of Lemma 2.1 in the measurable setting and without assuming the continuity of the function \( F \). This alternative relies on the von Neumann ergodic theorem and requires the averaging operator \( A_\mu \) to admit no non-trivial invariant vectors.

From expansion to contraction. Let \( f : Z \to (0, \infty) \) be a non-negative continuous function. Our goal is to show that, under suitable assumptions, if \( f \) is pointwise expanded by the averaging operator \( A_\mu \) on some subset of large measure, then the function \( F = f^{-\delta} \) is pointwise contracted by \( A_\mu \) for some sufficiently small exponent \( \delta > 0 \). We begin with an elementary calculation.

Calculation 2.2. Let \( a_1 > 1 > a_2 > 0 \) and \( 0 < p < 1 \). The real function \( \varphi \) given by

\begin{equation}
\varphi(\delta) = p a_1^{-\delta} + (1 - p) a_2^{-\delta}
\end{equation}

...
has a global minimum point at

\[
\delta_0 = \delta_0(a_1, a_2; p) = -\frac{\ln \left( -\frac{1-p}{p} \ln a_2 \right)}{\ln \left( \frac{a_1}{a_2} \right)}.
\]

Moreover if

\[
(1-p) \ln \frac{1}{a_2} < p \ln a_1
\]

then \(\delta_0 > 0\) and \(\varphi(\delta_0) < 1\).

**Proof.** Note that \(\varphi(0) = 1\) and \(\lim_{\delta \to \pm \infty} \varphi(\delta) = +\infty\). The derivative \(\varphi'\) of the function \(\varphi\) is given by

\[
\varphi'(\delta) = -p \ln a_1 \cdot a_1^{-\delta} - (1-p) \ln a_2 \cdot a_2^{-\delta}.
\]

A real number \(\delta_0\) is a critical point of the function \(\varphi\) if and only if \(\varphi'(\delta_0) = 0\). This condition is equivalent to

\[
\frac{a_1}{a_2}^{-\delta_0} = -\frac{1-p}{p} \ln a_2 \ln a_1 > 0.
\]

Therefore \(\varphi\) has a unique critical point \(\delta_0 = \delta(a_1, a_2; p)\) whose exact value is given by Equation (2.10). As \(a_1/a_2 > 1\) this critical point satisfies \(\delta_0 > 0\) if and only if

\[-\frac{1-p}{p} \ln a_2 \ln a_1 < 1 \iff -(1-p) \ln a_2 < p \ln a_1.
\]

Moreover, in this case \(\varphi(\delta_0) < \varphi(0) = 1\). \(\square\)

**Proposition 2.3.** Let \(a_1 > 1 > a_2 > 0\) and \(0 < p < 1\) be such that Equation (2.11) holds. Let \(\rho_0 > 0\) be such that every point \(z \in Z\) satisfies

1. if \(f(z) < \rho_0\) then the subset

\[
L_z = \{ g \in G : f(gz) \geq a_1 f(z) \}
\]

satisfies \(\mu(L_z) \geq p\) and

2. \(f(gz) \geq a_2 f(z)\) holds for \(\mu\)-almost every element \(g \in G\).

Then there are constants \(0 < c < 1\) and \(b > 0\) such that

\[
A_{\mu} f^{-\delta} \leq cf^{-\delta} + b
\]

where \(\delta = \delta_0(a_1, a_2; p)\) is as given by Equation (2.10).

**Proof.** Let \(\delta = \delta_0(a_1, a_2; p)\) be as in Equation (2.10). Take the constant \(c\) to be

\[
c = pa_1^{-\delta} + (1-p)a_2^{-\delta}.
\]

According to Calculation 2.2 this constant satisfies \(0 < c < 1\).

Consider the partition of the topological space \(Z\) into a pair of sublevel and superlevel sets

\[
Z = Z_{<\rho_0} \cup Z_{\geq\rho_0}
\]

where

\[
Z_{<\rho_0} = \{ z \in Z : f(z) < \rho_0 \} \quad \text{and} \quad Z_{\geq\rho_0} = \{ z \in Z : f(z) \geq \rho_0 \}.
\]

The subsets \(Z_{<\rho_0}\) and \(Z_{\geq\rho_0}\) are respectively open and closed in the space \(Z\).
On the one hand, take the constant \( b = (a_2 p_0)^{-\delta} \) and observe that every point \( z \in Z_{\geq p_0} \) and \( \mu \)-almost every element \( g \in G \) satisfy
\[
 f(gz) \geq a_2 f(z) \quad \Rightarrow \quad f^{-\delta}(gz) \leq (a_2 f(z))^{-\delta} \leq (a_2 p_0)^{-\delta} = b.
\]

On the other hand, observe that every point \( z \in Z_{< p_0} \) satisfies
\[
 A_\mu f^{-\delta}(z) = \int_G f^{-\delta}(gz) \, d\mu(g) = \int_{L_z} f^{-\delta}(gz) \, d\mu(g) + \int_{G \setminus L_z} f^{-\delta}(gz) \, d\mu(g) \leq \rho a_2^{-\delta} f^{-\delta}(z) = \rho f^{-\delta}(z).
\]

We conclude that the two inequalities \( A_\mu f^{-\delta} \leq b \) and \( A_\mu f^{-\delta} \leq cf^{-\delta} \) hold true on the two subsets \( Z_{\geq p_0} \) and \( Z_{< p_0} \) respectively. Since \( f^{-\delta} \geq 0 \) it follows that the inequality given in Equation (2.14) holds at every point \( z \in Z \).

**The limiting behavior of \( \delta \).** We estimate the asymptotic behavior of the function \( \delta_0(a_1, a_2; p) \) within a certain regime of the parameters \( a_1, a_2 \) and \( p \) that will be used below, see e.g. Proposition 6.12 and Probability 7.3. This is needed in order to compute the explicit value of the constant \( \delta \) appearing in our main results.

**Calculation 2.4.** Let \( h, \alpha, \zeta, a_0 > 0 \) as well as \( a_1 > 1 \) be some fixed constants. For all \( \lambda > 1 \) denote
\[
 (2.15) \quad a_{2, \lambda} = a_0 \lambda^{-h} \quad \text{and} \quad p_\lambda = 1 - \zeta \lambda^{-\alpha}.
\]

Then for all sufficiently large \( \lambda > 1 \) we have that Equation (2.11) holds and the critical point
\[
 (2.16) \quad \delta_\lambda = \delta_0(a_1, a_{2, \lambda}; p_\lambda)
\]
determined by Equation (2.11) satisfies \( \delta_\lambda > 0 \). Moreover
\[
 (2.17) \quad \lim_{\lambda \to +\infty} \delta_\lambda = \frac{\alpha}{h}.
\]

**Proof.** Recall that Equation (2.11) requires
\[
 (2.18) \quad (1 - p_\lambda) \ln \frac{1}{a_{2, \lambda}} < p_\lambda \ln a_1.
\]

Consider the behavior of Equation (2.18) in the limit as \( \lambda \to +\infty \). Note that
\[
 \lim_{\lambda \to +\infty} p_\lambda \ln a_1 = \ln a_1.
\]

On the other hand
\[
 \lim_{\lambda \to +\infty} (1 - p_\lambda) \ln(1/a_{2, \lambda}) = \lim_{\lambda \to +\infty} \zeta \lambda^{-\alpha} \ln(a_0 \lambda^h) = 0.
\]

Therefore the inequality given in Equation (2.18) is satisfied and \( a_{2, \lambda} < 1 \) for all \( \lambda \) sufficiently large. As shown in Calculation 2.2 it follows that
\[
 (2.19) \quad \delta_\lambda = -\ln \left( \frac{a_1}{a_{2, \lambda}} \right) \left( \frac{1 - p_\lambda \ln a_{2, \lambda}}{p_\lambda \ln a_1} \right) > 0
\]
for all \( \lambda \) sufficiently large. Lastly, the following computation
\[
 (2.20) \quad \lim_{\lambda \to +\infty} \delta_\lambda = \lim_{\lambda \to +\infty} \frac{-\ln \left( \frac{1 - p_\lambda \ln a_{2, \lambda}}{p_\lambda \ln a_1} \right)}{-\ln \left( \frac{a_1 a_{2, \lambda}}{a_{2, \lambda}} \right)} = \lim_{\lambda \to +\infty} \frac{-\ln \left( \frac{\zeta \lambda^{-\alpha} \ln(a_0 \lambda^h)}{\ln a_1} \right)}{-\ln \left( a_1 a_{2, \lambda} \right)} = \frac{\alpha}{h}.
\]
establishes Equation (2.17). \[\square\]

3. The Order of an Analytic Function on a Manifold

Let \( k \) be a local field. Consider a \( k \)-analytic manifold \( M \) of dimension \( d \in \mathbb{N} \). Let \( \mathcal{A}(M, k^m) \) denote the \( k \)-vector space of all \( k \)-analytic maps \( f : M \to k^m \) for each dimension \( m \in \mathbb{N} \).

The order \( \text{ord}_p f \) of any given \( k \)-analytic function \( f \in \mathcal{A}(M, k) \) at the point \( p \in M \) is defined as follows. Fix an arbitrary local chart \( \varphi : U \to V \) where \( p \in U \subset M \) and \( 0 \in V \subset k^d \) are open subsets such that \( \varphi(p) = 0 \). The \( k \)-analytic function \( f \) can be written in local coordinates as

\[
(f \circ \varphi^{-1})(x) = \sum_{\beta} c_{\beta} x^\beta \quad \forall x \in V
\]

where the \( \beta \)'s are multi-indices and the coefficients \( c_{\beta} \) belong to the local field \( k \). Then

\[
\text{ord}_p f = \inf_{\beta} \{ |\beta| : c_{\beta} \neq 0 \}.
\]

If the function \( f \) is identically zero on a neighborhood of the point \( p \) then all the coefficients \( c_{\beta} \) are zero and we set \( \text{ord}_p f = \infty \). The fact that transition maps are \( k \)-analytic isomorphisms implies that the definition of \( \text{ord}_p f \) is independent of the local chart.

The order of any \( k \)-analytic function \( f = (f_1, \ldots, f_m) \in \mathcal{A}(M, k^m) \) is

\[
\text{ord}_p(f) = \inf_{i \in \{1, \ldots, m\}} \text{ord}_p(f_i).
\]

Let \( \mathcal{F} \subset \mathcal{A}(M, k^m) \) be a family of \( k \)-analytic functions. The order of the family \( \mathcal{F} \) at the point \( p \in M \) is

\[
\text{ord}_p \mathcal{F} = \sup_{f \in \mathcal{F}} \text{ord}_p f.
\]

Moreover denote

\[
\text{ord}_M \mathcal{F} = \sup_{p \in M} \text{ord}_p \mathcal{F}.
\]

Remark. The order of the product \( fg \) of any pair of \( k \)-analytic functions \( f, g \in \mathcal{A}(M, k) \) satisfies

\[
\text{ord}_p(fg) = \text{ord}_p(f) + \text{ord}_p(g) \quad \forall p \in M.
\]

Lemma 3.1. Let \( N \) be an embedded \( k \)-analytic submanifold of \( M \). Then every \( k \)-analytic function \( f \in \mathcal{A}(M, k^m) \) satisfies \( \text{ord}_p f \leq \text{ord}_p f_{|N} \) for all points \( p \in N \).

Proof. Denote \( e = \dim_k N \). Consider an arbitrary point \( p \in N \). By assumption there is a relative neighborhood \( p \in U_1 \subset N \) and a local chart \( \varphi_1 : U_1 \xrightarrow{\sim} V_1 \subset k^e \) with \( \varphi_1(p) = 0 \) such that the power series expressing the function \( f \circ \varphi_1^{-1} \) at the point 0 admits a non-zero monomial of degree \( \text{ord}_p f_{|N} \). The \( k \)-vector space \( k^d \) admits a direct sum decomposition \( k^d = k^e \oplus k^{d-e} \) such that, up to replacing \( U_1 \) and \( V_1 \) by suitable smaller neighborhoods, there is an open neighborhood \( p \in U \subset M \) and a local chart \( \varphi : U \to k^d \) with \( \varphi(p) = 0 \), \( \varphi(U) = V_1 \times V_2 \) for some subset \( V_2 \subset k^{d-e} \) so that \( U_1 \subset U \) and \( \varphi|_{U_1} = \varphi_1 \) [Ser09, Part II, Chapter III, §10.1]. It is clear that

\[\footnote{We refer the reader to [Bou07] and [Ser09, Part II] for basic information on \( k \)-analytic functions and manifolds.}\]
in this situation \( \text{ord}_p f \leq \text{ord}_p f_j \). As the point \( p \in N \) was arbitrary the conclusion follows.

\[ \square \]

**Strong order.** In working with matrix coefficients of unipotent groups we will find it convenient to use the following sharper notion.

**Definition 3.2.** The \( k \)-analytic function \( f \in \mathcal{A}(M,k^m) \) has strong order at most \( d \in \mathbb{N} \) at the point \( p \in M \) if there exists an embedded \( k \)-analytic submanifold \( N \) admitting a local chart \( \varphi : U \rightarrow V \) where \( p \in U \subset N \) and \( V \subset k^{\dim N} \) are open subsets and such that \( f \circ \varphi^{-1} : V \rightarrow k^m \) is a non-zero polynomial mapping of degree at most \( d \).

Lemma 3.4 implies that strong order gives an upper bound on order. Namely the following is true.

**Lemma 3.3.** If \( f \in \mathcal{A}(M,k^m) \) has strong order at most \( d \in \mathbb{N} \) at the point \( p \in M \) then \( \text{ord}_p f \leq d \).

**Order of matrix coefficients of compact Lie groups.** Let \( V \) be a real finite dimensional inner product space. Let \( K \) be a compact connected subgroup of the orthogonal group \( O(V) \). In particular \( K \) is a real Lie group. Let \( T \) be a maximal compact torus of the group \( K \) with real Lie algebra \( \mathfrak{t} = \text{Lie}(T) \).

**Lemma 3.4.** Let \( \Omega \subset \mathfrak{t}^* \) be a finite set of analytically integral forms. Let \( c_{\omega} \in \mathbb{C} \) be an arbitrary coefficient for each \( \omega \in \Omega \). Consider the real analytic function \( \zeta \in \mathcal{A}(T,\mathbb{C}) \) given by

\[
\zeta(\exp(X)) = \sum_{\omega \in \Omega} c_{\omega} e^{\omega(X)} \quad \forall X \in \mathfrak{t}.
\]

If the function \( \zeta \) is not identically zero on the torus \( T \) then \( \text{ord}_T \zeta < |\Omega| \).

The function \( \zeta \) is defined on the real Lie group \( T \) and is regarded as a real analytic mapping into the two-dimensional real vector space \( \mathbb{C} \cong \mathbb{R}^2 \).

**Proof of Lemma 3.4** Fix an arbitrary element \( t_0 \in T \) with \( t_0 = \exp(X_0) \) for some \( X_0 \in \mathfrak{t} \). We will compute the order of the function \( \zeta \) at the point \( t_0 \) and show that \( \text{ord}_{t_0} \zeta < |\Omega| \).

Choose neighborhoods \( t_0 \in U \subset T \) and \( 0 \in V \subset \mathfrak{t} \) such that the mapping \( \varphi : U \xrightarrow{\cong} V \) given by \( \varphi : \exp(X_0 + X) \mapsto X \) for all \( X \in V \) is a local chart. For a given element \( t \in U \) with \( \varphi(t) = X \in V \) write

\[
\zeta(t) = \zeta(\exp(X_0 + X)) = \sum_{\omega \in \Omega} c_{\omega} e^{\omega(X_0)} \sum_{l=0}^{\infty} \frac{\omega(X)^l}{l!} = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\omega \in \Omega} \tilde{c}_{\omega} \omega(X)^l
\]

where \( \tilde{c}_{\omega} = c_{\omega} e^{\omega(X_0)} \). The part \( \zeta_0 \) of the real analytic function \( \zeta \) consisting of all monomials with multi-index \( \beta \) satisfying \( |\beta| < |\Omega| \) equals

\[
\zeta_0(t) = \sum_{l=0}^{|\Omega| - 1} \frac{1}{l!} \sum_{\omega \in \Omega} \tilde{c}_{\omega} \omega(X)^l
\]

for each given element \( t = \exp(X_0 + X) \in U \) as above.

The subspace \( \text{ker}(\omega_1 - \omega_2) \) of the real Lie algebra \( \mathfrak{t} \) is proper for each pair of distinct forms \( \omega_1, \omega_2 \in \Omega \). Therefore there is an element \( Y \in V \) such that
\( \omega_1(Y) \neq \omega_2(Y) \) for all such pairs of distinct indices. The Vandermonde determinant formula gives

\[
\det \left( \frac{\omega(Y)^l}{l!} \right)_{l \in \{0, \ldots, |\Omega| - 1\}} = \prod_{l=0}^{(|\Omega|-1) \over 1!} \prod_{\substack{\omega_1, \omega_2 \in \Omega \\omega_1 \neq \omega_2}} (\omega_1(Y) - \omega_2(Y)) \neq 0.
\]

The non-vanishing of the above determinant implies \( \zeta_0(Y) \neq 0 \) so that \( \text{ord}_u \zeta < |\Omega| \) as required. As the point \( t_0 \in T \) in the above argument was arbitrary we conclude that \( \text{ord}_T f < |\Omega| \).

Consider the Hermitian space \( V_C = V \otimes \mathbb{C} \) which is the complexification of the inner product space \( V \). The maximal compact torus \( T \) preserves the Hermitian form on \( V_C \). Let \( \Omega = \Omega(T, V) \) be the weights of the torus \( T \) in its representation on the Hermitian space \( V_C \). Every weight \( \omega \in \Omega \) is an analytic integral form belonging to \( i t^* \). Let

\[
V_C = \bigoplus_{\omega \in \Omega} W_\omega
\]

be the associated weight space decomposition satisfying

(3.6) \( \exp(X)w = e^{\omega(X)}w \quad \forall \omega \in \Omega, w \in W_\omega, X \in t. \)

**Lemma 3.5.** Consider the matrix coefficient \( \zeta_{u,v} \in \mathcal{A}(K, \mathbb{R}) \) given by

\[
\zeta_{u,v} : K \to V, \quad \zeta_{u,v}(g) = (gu,v) \quad \forall g \in K.
\]

for some pair of vectors \( u, v \in V \). If \( \zeta_{u,v} \) is not identically zero on the group \( K \) then \( \text{ord}_K \zeta_{u,v} < |\Omega(T,V)| \leq \dim \mathbb{R} V \).

**Proof.** Every element of the compact connected Lie group \( K \) belongs to some maximal torus and all such tori are conjugate [Kna13, Corollary 4.35, Theorem 4.36]. Up to replacing the vectors \( u \) and \( v \) by \( gu \) and \( gv \) for some element \( g \in K \), we may assume without loss of generality that the restriction of the matrix coefficient \( \zeta_{u,v} \) to the maximal torus \( T \) is not identically zero.

Denote \( \Omega = \Omega(T, V) \) and consider the weight space decompositions \( u = \bigoplus_{\omega \in \Omega} u_\omega \) and \( v = \bigoplus_{\omega \in \Omega} v_\omega \) where \( u_\omega, v_\omega \in W_\omega \). The matrix coefficient \( \zeta_{u,v} \) can be expressed at the point \( g = \exp X \in T \) with \( X \in t \) as

(3.7) \( \zeta_{u,v}(g) = \left\langle \exp(X)u, \sum_{\omega \in \Omega} u_\omega, \sum_{\omega \in \Omega} v_\omega \right\rangle = \sum_{\omega \in \Omega} \left\langle u_\omega, w_\omega \right\rangle e^{\omega(X)}. \)

It follows from Lemmas [3.4 and 3.5] applied at the identity element that

\[
\text{ord}_{eK} \zeta_{u,v} \leq \text{ord}_{eT} \zeta_{u,v}|_T < |\Omega|.
\]

Observe that \( \zeta_{u,v}(gh) = \zeta_{hu,v}(g) \) for all pairs of elements \( g, h \in K \). By repeating the above argument with respect to each matrix coefficient of the form \( \zeta_{hu,v} \) we show that \( \text{ord}_K \zeta_{u,v} < |\Omega| \) as required.

Fix some \( l \in \mathbb{N} \). Consider the \( l \)-exterior power inner product space \( \bigwedge^l V \). The compact connected group \( K \) can naturally be regarded as a subgroup of \( \text{O}(\bigwedge^l V) \). The weights \( \Omega(T, \bigwedge^l V_C) \) of the torus \( T \) in its representation on the Hermitian space \( \bigwedge^l V_C \) satisfy

\[
\Omega(T, \bigwedge^l V_C) \subset \Omega^l = \{ \omega_1 + \cdots + \omega_l : \omega_1, \ldots, \omega_l \in \Omega \}. \]
The following follows immediately from the previous Lemma 3.5

**Lemma 3.6.** Consider the matrix coefficient ζ_{u,v}^t ∈ A(K, R) given by

\[
ζ_{u,v}^t(g) = \left\langle \bigwedge^t gu, v \right\rangle \quad \forall g ∈ K.
\]

for some pair of vectors u, v ∈ \(\Lambda^t V\). If \(ζ_{u,v}^t\) is not identically zero on the group K then \(\text{ord}_K ζ_{u,v}^t < |Ω(T, \Lambda^t V_C)| ≤ |Ω|^t\).

**Remark.** The above proofs of Lemmas 3.5 and 3.6 apply more generally without assuming the compact group K is connected at each connected component where the matrix coefficient in question is non-zero.

**Strong order of unipotent groups in positive characteristic.** Assume that k is a non-Archimedean local field with \(p = \text{char}(k) > 0\). Let \(V\) be a finite dimensional \(k\)-vector space of dimension \(d = \dim_k V\).

Consider an arbitrary element \(s ∈ GL(V)\). Write \(n = s - \text{Id} ∈ \text{End}(V)\). Note that \(sp = \text{Id}\) if and only if \(n^p = 0\). If this is the case then the Cayley–Hamilton theorem implies that \(n^d = 0\). In particular

\[
(3.8) \quad \log(s) = \log(\text{Id} + n) = \sum_{i=1}^{d-1} (-1)^{i+1} \frac{n^i}{i}.
\]

The matrix \(x = \log(s)\) is a linear combination of positive powers of the matrix \(n\) and as such \(x^d = 0\). Therefore

\[
(3.9) \quad s = \exp(x) = \exp(\log s) = \sum_{i=0}^{d-1} \frac{x^i}{i!}.
\]

**Lemma 3.7.** Let \(T\) be a diagonal \(k\)-split torus subgroup of \(GL(V)\). Let \(θ : k^r → GL(V)\) be a \(k\)-rational representation for some \(r ∈ \mathbb{N}\) so that \(U = θ(k^r) ≤ GL(V)\) is a connected \(k\)-unipotent subgroup. Assume that \(T\) normalizes \(U\) and that there is a \(k\)-character \(α ∈ X(T)\) so that

\[
(3.10) \quad tθ(\alpha^{-1}) = θ(α(t)z) \quad ∀t ∈ T, z ∈ k^r.
\]

Then the inclusion mapping \(U → End(V)\) has strong order at most \(2drI\) at all points, where \(I ∈ \mathbb{N}\) is the maximal number so that \(Iα = β - β'\) for any pair of distinct \(k\)-characters \(β, β' ∈ X(T)\) corresponding to diagonal entries of \(T\).

**Proof.** Assume to begin with that \(r = 1\). In other words \(θ\) is a \(k\)-rational representation of the additive group. The explicit form of such representations is known [DG80 II §2.2.6]. Namely there is some \(N ∈ \mathbb{N} \cup \{0\}\) and a family of pairwise commuting matrices \(s_0, \ldots, s_N ∈ GL(V)\) satisfying \(s_i^p = \text{Id}\) for all \(i ∈ \{0, \ldots, N\}\) such that

\[
(3.11) \quad θ(z) = \prod_{i=0}^{N} \exp(z^{p^i} \log s_i) \quad ∀z ∈ k.
\]

Denote \(x_i = \log s_i\). The expression \(tθ(z)t^{-1}\) for each element \(t ∈ T\) and \(z ∈ k\) can be computed in two different ways. On the one hand Equation (3.10) gives

\[
(3.12) \quad tθ(z)t^{-1} = θ(α(t)z) = \prod_{i=0}^{N} \exp((α(t)z)^{p^i} x_i) = \prod_{i=0}^{N} \exp(z^{p^i} α(t)^{p^i} x_i) \quad ∀z ∈ k.
\]
On the other hand

\[(3.13) \quad t \theta(z) t^{-1} = \prod_{i=0}^{N} t \exp(z_{i} x_{i}) t^{-1} = \prod_{i=0}^{N} \exp(z_{i} t x_{i} t^{-1}) \quad \forall z \in k.\]

We apply logarithm to both Equations (3.12) and (3.13). Since the matrices \(x_{i}\) pairwise commute we obtain the equality

\[(3.14) \quad \sum_{i=0}^{N} p^{i} (\alpha(t)^{p^{i}} x_{i}) = \sum_{i=0}^{N} p^{i} (t x_{i} t^{-1}) \quad \forall z \in k, t \in T.\]

Note that Equation (3.14) holds true if and only if \(\alpha(t)^{p^{i}} x_{i} = t x_{i} t^{-1}\) for all \(i \in \{0, \ldots, N\}\). This only be the case for each given index \(i\) provided there are two distinct \(k\)-characters \(\beta, \beta' \in X(T)\) corresponding to diagonal entries of the torus \(T\) such that \(p^{i} \alpha = \beta - \beta'\). Therefore the number \(N\) must be such that \(p^{N} \leq I\) where \(I\) is as in the statement.

The discussion preceding the Lemma shows that each function \(\exp(z_{i} \log s_{i})\) is a polynomial map in the variable \(z\) into the \(k\)-vector space \(\text{End}(V)\) of degree at most \(p^{i} \dim_{k} V\) for every \(i \in \{0, \ldots, N\}\). Therefore \(\theta\) is a polynomial map in the variable \(z\) into the \(k\)-vector space \(\text{End}(V)\) of degree at most \(N \sum_{i=0}^{N} p^{i} \dim_{k} V \leq p^{N+1} - 1\).

Lastly if \(r > 1\) then \(\theta\) is obtained as a product in the endomorphism ring \(\text{End}(V)\) of \(r\)-many \(k\)-rational representations as above. The result follows. \(\square\)

### 4. Sublevel set estimates of analytic functions on manifolds

Let \(k\) be either the field \(\mathbb{R}\) or a non-Archimedean local field and \(|\cdot|\) be an absolute value on \(k\). Fix a dimension \(m \in \mathbb{N}\). Endow the \(k\)-vector space \(k^{m}\) with the supremum norm \(\|\cdot\|_{\infty}\) given by

\[(4.1) \quad \|(x_{1}, \ldots, x_{m})\|_{\infty} = \sup_{i} |x_{i}| \quad \forall x_{1}, \ldots, x_{m} \in k.\]

Let \(M\) be a fixed compact \(k\)-analytic manifold. Let \(\eta\) be a probability measure in the canonical measure class of \(M\). For example, if \(M\) is a compact \(k\)-analytic group then \(\eta\) can be taken to be normalized Haar measure.

Recall from \(\S 3\) that \(\mathcal{A}(M, k^{m})\) denotes the \(k\)-vector space of all \(k\)-analytic maps \(f : M \to k^{m}\). Every non-empty subset \(X \subset M\) defines a seminorm \(\|\cdot\|_{X}\) on \(\mathcal{A}(M, k^{m})\) by

\[(4.2) \quad \|f\|_{X} = \sup_{x \in X} \|f(x)\|_{\infty} \quad \forall f \in \mathcal{A}(M, k^{m}).\]

Endow \(\mathcal{A}(M, k^{m})\) with the topology coming from the norm \(\|\cdot\|_{M}\).

The goal of the current \(\S 4\) is to prove Theorem \(\|\|\). This establishes an upper bound on the \(\eta\)-measure of sublevel sets of \(k\)-analytic maps on the compact \(k\)-analytic manifold \(M\).

---

8Our discussion and results easily extend to the complex case via a restriction of scalars. This has the effect of doubling the dimension of the domain in Theorem \(\|\|\) and its applications below.
Theorem 4.1. Let $\mathcal{F} \subset A(M, k^m)$ be a compact family of $k$-analytic maps. If $\dim_k \text{span}_k \mathcal{F} < \infty$ then there are constants $\kappa_{\mathcal{F}}, \varepsilon_{\mathcal{F}} > 0$ such that every map $f \in \mathcal{F}$ satisfies

$$\eta(\{x \in M : \|f(x)\|_\infty < \varepsilon\}) \leq \kappa_{\mathcal{F}} \varepsilon^{\frac{1}{\dim_k \text{span}_k \mathcal{F}}} \forall 0 < \varepsilon < \varepsilon_{\mathcal{F}}.$$ 

Note that the dimension $m$ of the target space does not appear in Equation (4.3).

We deduce Theorem 4.1 from known results on $(C, \alpha)$-good functions that are discussed below. In fact Theorem 4.1 can be seen as an analogue of Theorem 4.3 in the setting of $k$-analytic manifolds.

Sublevel set estimates. Let $d \in \mathbb{N}$ be fixed. A ball in the $d$-dimensional $k$-vector space $k^d$ with center point $y \in k^d$ and radius $r > 0$ is the subset $B = B(y, r) = \{x \in k^d : \|x - y\|_\infty \leq r\}.$

Let $V \subset k^d$ be a fixed open subset. Given a continuous function $f : V \to k$ and a ball $B \subset V$ denote

$$\|f\|_B = \sup_{x \in B} |f(x)|.$$ 

In addition, for every $\varepsilon > 0$ denote

$$B^{f, \varepsilon} = \{x \in B : |f(x)| < \varepsilon\}.$$ 

Let $\lambda$ be a Haar measure on the $k$-vector space $k^d$ regarded as an additive locally compact group. The following terminology has been introduced in [KM98].

Definition 4.2. A continuous function $f : V \to k$ is $(C, \alpha)$-good if every open ball $B \subset V$ satisfies

$$\lambda(B^{f, \varepsilon}) \leq C \left(\frac{\varepsilon}{\|f\|_B}\right)^\alpha \lambda(B)$$ 

for every $\varepsilon > 0.$

Polynomials are $(C, \alpha)$-good with parameters $C$ and $\alpha$ depending only on degree. In the real case this goes back to the Remez inequality [Rem36, BG73]. See also [DM93 Lemma 4.1], [KM98 Proposition 3.2] and [ABRDS15 Proposition 2.8].

Analytic functions are also known to be $(C, \alpha)$-good under suitable assumptions:

Theorem 4.3. Let $f : V \to k$ be a $k$-analytic function. Assume that there exist constants $A_1, A_2 > 0$ and some $l \in \mathbb{N}$ such that

$$\|\partial_\beta f\|_V \leq A_1 \quad \forall \text{multi-index } \beta \text{ with } |\beta| \leq l$$ 

and

$$|\partial_i^l f(x)| \geq A_2 \quad \forall x \in V, \forall i = 1, \ldots, d.$$ 

Then there is a constant $C > 0$ such that $f$ is $(C, \frac{1}{2l})$-good. The constant $C$ depends only on the values of $A_1, A_2, l$ and on the dimension $d.$

Proof. The real case is precisely [KM98 Lemma 3.3]. Note that our “balls” are “cubes” in the terminology of [KM98]. The non-Archimedean case is contained in [KT05 Theorem 3.2].

The following argument is well-known. In particular, it is used implicitly in the proof of [KM98 Proposition 3.4, p. 349] and explicitly in [KT05 Proposition 4.2]. We provide a detailed proof for the reader’s convenience.
Proposition 4.4. Let \( f : V \to k \) be a \( k \)-analytic function. Assume that there is a constant \( A'_1 > 0 \) and some \( l \in \mathbb{N} \) such that
\[
\| \partial_\beta f \|_V \leq A'_1 \quad \forall \text{ multi-index } \beta \text{ with } |\beta| \leq l.
\]
Assume moreover that \( 0 \in V \) and that there is a multi-index \( \beta_0 \) with \( |\beta_0| = l \) as well as a constant \( A'_2 > 0 \) satisfying
\[
|\partial_{\beta_0} f(0)| \geq A'_2.
\]
Then there constants \( A_1, A_2, K \geq 1 \) as well as a linear map \( g \in \text{GL}(k^d) \) with \( 1 \leq |\det g| \leq K \) such that
\[
\| \partial_\beta (f \circ g) \|_V \leq A_1 \quad \text{ and } \quad |\partial^i_{\beta} (f \circ g)(0)| \geq A_2 \quad \forall i \in \{1, \ldots, d\}.
\]
The constants \( A_1, A_2 \) and \( K \) depend only on the two constants \( A'_1 \) and \( A'_2 \) and on the local field \( k \).

Proof. Let \( \beta \) be an arbitrary multi-index with \( |\beta| = l \). Consider the value of the partial derivative \( \partial_\beta (f \circ g)(0) \) where the element \( g \) varies over the ring of endomorphisms \( \text{End}(k^d) \). Identify the space \( \text{End}(k^d) \) with the \( k \)-affine space of matrices \( M_d(k) \). A repeated application of the chain rule shows that the value of the partial derivative \( \partial_\beta (f \circ g)(0) \) is given by a homogeneous polynomial \( P_{\beta,f} \) in the matrix entries of \( g \). The coefficients of the polynomial \( P_{\beta,f} \) are the partial derivatives \( \partial_{\beta'} f(0) \) where \( \beta' \) varies over all possible multi-indices with \( |\beta'| = l \). In particular, the polynomial \( P_{\beta,f} \) is non-trivial as one of its coefficients is given by \( \partial_{\beta_0} f(0) \).

The above discussion applied with respect to the multi-indices corresponding to the partial differentiation operators \( \partial_\beta \) implies that there exists a non-empty Zariski open subset of \( \text{End}(k^d) \) consisting of endomorphisms \( g \) satisfying
\[
\partial^i_{\beta} (f \circ g)(0) \neq 0 \quad \forall i \in \{1, \ldots, d\}.
\]

Therefore we may find a linear map \( g \) belonging to the Zariski open subset \( \text{GL}(k^d) \) and satisfying Equation \( (4.7) \). As the polynomials \( P_{\beta,f} \) are all homogeneous we may assume, up to replacing the matrix \( g \) by a scalar multiple, that \( 1 \leq |\det g| = K \) for some constant \( K \geq 1 \) depending only on the local field \( k \).

The coefficients of the polynomials in the family \( P_{\beta,f} \) satisfy \( |\partial_{\beta'} f(0)| \leq A'_1 \) and \( |\partial_{\beta_0} f(0)| \geq A'_2 \) for some multi-index \( \beta_0 \). The family of all polynomials satisfying such coefficient bounds is compact. This implies the existence of the constants \( A_1 \) and \( A_2 \) depending only on the pair of constants \( A'_1 \) and \( A'_2 \) and the local field \( k \). \( \square \)

We are now ready to complete the proof of the main result for \( \mathbb{A}^1 \).

Proof of Theorem 4.1. Fix an arbitrary point \( x \in M \). Consider a compatible chart given by an open subset \( U_x \subset M \) containing the point \( x \) and a map \( \varphi_x : U_x \to V_x \) with \( \varphi_x(x) = 0 \) where \( V_x \subset k^d \) is a ball at the point \( 0 \). Let \( \psi_x : V_x \to U_x \) denote the inverse mapping of \( \varphi_x \).

For every multi-index \( \beta \) consider the two seminorms \( \| \cdot \|_{x,\beta} \) and \( \| \cdot \|_{\infty,x,\beta} \) on the \( k \)-vector space \( \mathcal{A}(M,k^m) \) given by
\[
\| f \|_{x,\beta} = \| \partial_\beta (f \circ \psi_x) \|_{V_x} \quad \text{ and } \quad \| f \|_{\infty,x,\beta} = \| \partial_\beta (f \circ \psi_x)(0) \|_\infty \quad \forall f \in \mathcal{A}(M,k^m).
\]

\footnote{In the Archimedean case \( K = 1 \). Namely we may assume that \( |\det g| = 1 \).}
Our assumptions that the family $\mathcal{F}$ is compact and that $\text{dim span}_k(\mathcal{F}) < \infty$ imply that there are constants $A'_{1,x}, A'_{2,x} > 0$ such that

$$\|f\|_{x,\beta} \leq A'_{1,x} \quad \forall f \in \mathcal{F} \quad \forall \text{multi-index } \beta \text{ with } 0 \leq |\beta| \leq \text{ord}_x \mathcal{F} + 1$$

and

$$\|f\|_{x,\beta_0}^0 \geq A'_{2,x} \quad \forall f \in \mathcal{F} \quad \text{and for some multi-index } \beta_0 = \beta_0(f) \text{ with } 0 \leq |\beta_0| \leq \text{ord}_x \mathcal{F}.$$

According to Proposition 4.12 there are constants $A_{1,x}, A_{2,x} > 0$ such that for every function $f \in \mathcal{F}$ there exists a linear isomorphism $g = g(x,f) \in \text{GL}(k^d)$ with $\|f \circ g\|_{x,\beta} \leq A_1$ and

$$\|f \circ g\|_{x,\beta_l}^0 \geq A_{2,x}^l$$

for some $l = l(f) \in \{0,\ldots, \text{ord}_x \mathcal{F}\}$ and $\forall i \in \{1,\ldots, d\}$.

We may assume that the collection of elements $g(x,f)$ for $f \in \mathcal{F}$ is relatively compact.

Finally, there is a constant $A_{2,x} > 0$ such that, up to replacing $U_x$ by a smaller neighborhood of the point $x \in M$ and $V_x$ by a corresponding ball of smaller radius, the function $f \circ \psi_x \circ g(x,f)$ is well-defined on the ball $V_x$ and satisfies

$$\inf_{y \in V_x} \|\partial_i^\ell (f \circ \psi_x \circ g(x,f))(y)\|_\infty \geq A_{2,x} \quad \forall i \in \{1,\ldots, d\}$$

for all functions $f \in \mathcal{F}$ and for some $l = l(f) \in \{0,\ldots, \text{ord}_x \mathcal{F}\}$. In the real case this follows from the error estimate in the multi-dimensional variant of Taylor's series. In the non-Archimedean case this follows from the non-Archimedean triangle inequality.

The map $\psi_x$ pushes forward the measure class of the Haar measure $\lambda$ on the ball $V_x$ to the measure class of $\eta$ on $V$. Let $0 < D_x < \infty$ be the supremum of the $L_\infty$-norm of the Radon–Nikodym derivative $\frac{d(\psi_x \circ g) \lambda}{dm}$ as $g$ ranges over all possible elements $g = g(f, x)$.

We know from Theorem 4.3 that each map $f \circ \varphi_x \circ g : V_x \to U_x$ has at least one coordinate which is $(C_x, \frac{1}{d!})$-good on $V_x$ where $C_x > 0$ is some constant depending only on the constants $A_{1,x}$ and $A_{2,x}$. This means that

$$\eta(V_x) \leq D_x \lambda(V_x) \leq C_x D_x \left(\frac{\varepsilon}{\|f \circ \psi_x \circ g\|_{V_x}}\right) \overset{\text{def}}{=} \lambda(V_x) \quad \forall f \in \mathcal{F}, \forall \varepsilon > 0. \quad (4.8)$$

Recall that the $k$-analytic variety $M$ is assumed to be compact. Therefore there is some $N \in \mathbb{N}$ and some points $x_1, \ldots, x_N \in M$ such that $M \subset \bigcup_{i=1}^N U_{x_i}$. Denote $D = \max_{i=1}^N D_{x_i}$ and $C = \max_{i=1}^N C_{x_i}$.

We know that $\sup_{f \in \mathcal{F}} \|f\| < \infty$. Moreover, for all $i \in \{1,\ldots, N\}$, we have $\inf_{f \in \mathcal{F}} \|f\|_{U_{x_i}} > 0$, as $\text{ord}_x \mathcal{F} \leq l$. Therefore there is some constant $B > 0$ such that

$$\inf_{i \in \{1,\ldots, N\}} \inf_{f \in \mathcal{F}} \frac{\|f \circ \psi_{x_i}\|_{B_{x_i}}}{\|f\|_{M}} = \inf_{i \in \{1,\ldots, N\}} \inf_{f \in \mathcal{F}} \frac{\|f\|_{U_{x_i}}}{\|f\|_{M}} \geq B. \quad (4.9)$$

It remains to deduce the global Equation 4.13 from the local information of Equation 4.8. For every function $p \in \mathcal{F}$ and every $\varepsilon > 0$ we have that

$$\eta(M_p) \leq \sum_{i=1}^N \eta(U_{x_i}^{p, \varepsilon}) \leq CD \sum_{i=1}^N \left(\frac{\varepsilon}{\|f\|_{U_{x_i}}}\right) \overset{\text{def}}{=} \lambda(B_{x_i}). \quad (4.10)$$
The proof follows for an appropriate choice of the constant \( \kappa_F > 0 \) by putting together Equations (4.8), (4.9) and (4.10).

\[ \square \]

5. Grassmannians of normed vector spaces

Let \( k \) be the field \( \mathbb{R} \) or a non-Archimedean local field with absolute value \( | \cdot | \).
Let \( V \) be a \( k \)-vector space of dimension \( n = \dim_k V \) equipped with the norm \( \| \cdot \| \).
Fix a non-trivial direct sum decomposition

\[ V = U \oplus U' \]

and a pair of projections

\[ P : V \rightarrow U \quad \text{and} \quad P' : V \rightarrow U'. \]

Fix some \( l \in \mathbb{N} \). Let \( K \) be a fixed compact \( k \)-analytic subgroup of \( \text{GL}(V) \) with the Haar probability measure \( \eta_K \). For each vector \( x \in \bigwedge^l V \) consider the \( k \)-analytic function \( Q_x \) on the compact group \( K \) defined by

\[ Q_x \in \mathcal{A}(K, \bigwedge^l U), \quad Q_x(g) = \left( \bigwedge^l P g \right) x \quad \forall g \in K. \]

\[ \text{Theorem 5.1.} \quad \text{Let } \mathcal{N} \text{ be a closed subset of the Grassmannian } \mathfrak{g}(l, V). \text{ Consider the family } \mathcal{F}(\mathcal{N}) \subset \mathcal{A}(K, \bigwedge^l U) \text{ of } k \text{-analytic functions given by} \]

\[ \mathcal{F}(\mathcal{N}) = \{ Q_{w_1 \wedge \cdots \wedge w_l} : \| w_i \| \leq 1 \text{ and span}_k \{ w_1, \ldots, w_l \} \in \mathcal{N} \}. \]

Then there are constants \( \kappa_{\mathcal{N}}, \varepsilon_{\mathcal{N}} > 0 \) such that

\[ \eta_K(\{ g \in K : \inf_{0 \neq w \in gW} \frac{\| Pw \|}{\| w \|} \leq \varepsilon \}) < \kappa_{\mathcal{N}} \exp\left( \frac{-\min_{0 \neq w \in gW} \frac{1}{\| Pw \|}}{\varepsilon_{\mathcal{N}}} \right) \]

for every subspace \( W \in \mathcal{N} \) and every \( 0 < \varepsilon < \varepsilon_{\mathcal{N}} \).

In proving Theorem 5.1 we may and will assume without loss of generality that \( \text{ord}_K \mathcal{F}(\mathcal{N}) < \infty \). In other words, for every subspace \( W \in \mathcal{N} \) and every open subset \( U \subset K \) there is an element \( g \in U \) with \( gW \cap U' = \{ 0 \} \). In particular we may assume that that \( l \leq \dim_k U \).

**Bijections and contraction.** In \( \S 6 \) below we will be relying on Theorem 5.1 in combination with the following elementary observation.

**Lemma 5.2.** Let \( L : V \rightarrow V \) be a linear bijection that preserves the two subspaces \( U \) and \( U' \). Then

\[ \inf_{0 \neq w \in gW} \frac{\| Lw \|}{\| w \|} \geq \inf_{0 \neq w \in U} \frac{\| Lw \|}{\| w \|} \quad \text{inf}_{0 \neq w \in gW} \frac{\| Pw \|}{\| w \|} \]

for all elements \( g \in K \).

**Proof.** Consider a group element \( g \in K \). Let \( w \in gW \) be any non-zero vector. Write \( w = Pw + P'w \). Therefore \( Lw = LPw + LP'w \) with \( LPw \in U \) and \( LP'w \in U' \).
Using the fact that \( Pw \in U \) we obtain

\[ \| Lw \| \geq \| LPw \| \geq \| Pw \| \inf_{0 \neq w \in U} \frac{\| Lw \|}{\| w \|} \geq \| w \| \inf_{0 \neq w \in U} \frac{\| Lw \|}{\| w \|} \inf_{0 \neq w \in gW} \frac{\| Pw \|}{\| w \|}. \]

The conclusion follows by taking the infimum over all such non-zero vectors \( w \). \( \square \)

The remainder of \( \S 6 \) is dedicated to setting up the necessary mechanism towards deducing Theorem 5.1 as a consequence of Theorem 4.1.
Norms of projections. The statement of Theorem 5.1 is independent of the choice of the norm \( \| \cdot \| \) on the finite dimensional \( k \)-vector space \( V \) up to modifying the value of the constant \( c_{pq} \). It will be convenient to assume for the remainder of the current \$\text{II} \$ that the norm \( \| \cdot \| \) is given as follows:

Fix a basis \( \beta = \{ e_1, \ldots, e_n \} \) for the \( k \)-vector space \( V \) such that the subspace \( U \) is spanned by \( \{ e_1, \ldots, e_{\dim_k U} \} \). We will assume that \( \| \cdot \| \) is the Euclidean norm making the basis \( \beta \) orthonormal in the real case or the maximum norm with respect to the basis \( \beta \) in the non-Archimedean case.

Recall that \( l \in \mathbb{N} \) is a fixed integer satisfying \( l \leq \dim_k U \). The exterior power \( \Lambda^l V \) is a \( k \)-vector space satisfying \( \dim_k \Lambda^l V = \binom{n}{l} \). There is a \( k \)-linear subspace \( \Lambda^l U \leq \Lambda^l V \) and a projection \( \Lambda^l P : \Lambda^l V \to \Lambda^l U \). Endow the vector space \( \Lambda^l V \) with the supremum norm \( \| \cdot \|_\beta \) with respect to the fixed basis

\[
\tilde{\beta} = \{ e_{i_1} \land e_{i_2} \land \cdots \land e_{i_l} : 1 \leq i_1 < i_2 < \cdots < i_l \leq n \}.
\]

Consider the continuous function \( q \) on the Grassmannian \( \mathfrak{G}(l, V) \) given by

\[
q(W) = \sup \{ \| \Lambda^l P(w_1 \land \cdots \land w_l)\|_\beta : w_1, \ldots, w_l \in W \text{ and } \| w_i \| \leq 1 \}.
\]

for any \( l \)-dimensional \( k \)-linear subspace \( W \leq V \). Note that \( q(W) = 0 \) if and only if \( \text{rank}_k P|_W < l \). This happens if and only if \( \dim_k (W \cap U') > 0 \).

**Proposition 5.3.** Every subspace \( W \leq V \) the projection \( P : W \to U \) satisfies

\[
\inf_{0 \neq w \in W} \frac{\| Pw \|}{\| w \|} \geq q(W).
\]

Before proceeding, we will need an upper bound on the absolute value of the determinant of a matrix in terms of, say, the norms of its column vectors.

**Lemma 5.4.** Let \( A \) be an \( l \)-by-\( l \) matrix over the local field \( k \) for some \( l \in \mathbb{N} \). Let \( v_i \in k^l \) denote the \( i \)-th column of the matrix \( A \) for every \( i \in \{1, \ldots, l\} \). Then

\[
| \det A | \leq \prod_{i=1}^{l} \| v_i \|
\]

where \( \| \cdot \| \) is the Euclidean norm if \( k = \mathbb{R} \) and for the supremum norm if \( k \) is non-Archimedean.

**Proof of Lemma 5.4.** The real case is the Hadamard determinant inequality. In the non-Archimedean case Equation \( 5.6 \) follows immediately from the Leibniz determinant formula and the non-Archimedean triangle inequality.

**Proof of Proposition 5.3.** Fix a \( k \)-linear subspace \( W \in \mathfrak{G}(l, V) \). Consider any unit vector \( w \in W \) and any ordered basis \( \omega = \{ w_1, \ldots, w_l \} \) for the subspace \( W \) consisting of unit vectors. To establish Equation \( 5.5 \) it suffices to verify that

\[
\| \Lambda^l P(w_1 \land \cdots \land w_l)\|_\beta \leq \| Pw \|.
\]

We claim that there is a linear transformation \( B \in \text{SL}(W) \) satisfying \( Bw_1 = xw \) for some \( x \in k \) with \( |x| \leq 1 \) and \( \| Bw_i \| = \| w_i \| = 1 \) for all \( i \in \{2, \ldots, l\} \). The claim follows in the real case with \( x = 1 \) as the \( \text{SO}(W) \)-action is transitive on Euclidean unit vectors in \( W \). To prove the claim in the non-Archimedean case let \( O_k \) be the ring of units and \( \pi \in O_k \) a uniformizing element. Let \( i \in \mathbb{N} \cup \{0\} \) be
a minimal integer such that $\pi^i w$ belongs to the $O_k$-module spanned by $w_1, \ldots, w_l$. Write $\pi^i w = \sum_{j=1}^l x_j w_j$ for some coefficients $x_j \in O_k$. The minimality of $i$ implies that $|x_j| = 1$ for some $j$. Assume without loss of generality $|x_1| = 1$ and write

$$(x_1^{-1}) \pi^i w - w_1 = \sum_{j=2}^l x_j^{-1} x_j w_j.$$  

By considering the suitable product of elementary matrices, we obtain the matrix $B \in SL(W)$ taking the vector $w_1$ to $xw$ with $x = x_1^{-1} \pi^i$ and preserving all other vectors $w_j$. This completes the claim.

Now let $B \in SL(W)$ be any transformation as provided by the claim. Up to replacing the ordered basis $\omega$ with the ordered basis $B \omega$ and observing that $w_1 \wedge \cdots \wedge w_l = Bw_1 \wedge \cdots \wedge Bw_l$ we may assume without loss of generality that $w_1 = xw$ with some $x \in K, |x| \leq 1$ and maintain the assumption that $\|w_i\| = 1$ for all $i \in \{2, \ldots, l\}$.

The coordinates of the $l$-vector $\bigwedge^l P(w_1 \wedge \cdots \wedge w_l) \in \bigwedge^l V$ in the basis $\beta$ are the determinants of the corresponding $l$-minors of the matrix representing the linear transformation $P|_W : W \to U$ with respect to the basis $\omega$ for the subspace $W$ and the basis $e_1, \ldots, e_{\dim U}$ for the subspace $U$. Let therefore $e_{i_1} \wedge \cdots \wedge e_{i_l} \in \beta$ with $1 \leq i_1 < \cdots < i_l \leq n$ be a particular basis element satisfying

$$\| \bigwedge^l P(w_1 \wedge \cdots \wedge w_l) \|_\beta = | \det A_{i_1, \ldots, i_l} |$$

where $A_{i_1, \ldots, i_l}$ is the corresponding $l$-minor of the above-mentioned matrix. The determinant bound established in Lemma 5.4 gives

$$| \det A_{i_1, \ldots, i_l} | \leq \prod_{i=1}^l \| Pw_i \| \leq \| Pw_1 \| = x \| Pw \| \leq \| Pw \|.$$  

We have used the fact that $P$ is contracting so that $\| Pw_i \| \leq 1$ for all $i \in \{1, \ldots, l\}$ as well as that $|x| \leq 1$.

**Proof of the main result.** Let $\mathcal{R}$ be a closed subset of the Grassmannian $\mathfrak{G}(l, V)$. Assume that every subspace $W \in \mathcal{R}$ satisfies $gW \cap U' = \{0\}$ for some element $g \in K$.

Consider the family $\mathcal{F}(\mathcal{R})$ of $k$-analytic functions as introduced in Equation (5.2). Denote $\|Q\|_K = \sup_{g \in K} \| Q(g) \|_\beta$ for every function $Q \in \mathcal{F}(\mathcal{R})$.

**Proposition 5.5.** There is a constant $E > 0$ such that every subspace $W \in \mathcal{R}$ satisfies

$$E \leq \sup_{\|w_i\| \leq 1} \| Q_{w_1 \wedge \cdots \wedge w_l} \|_K \leq 1.$$  

**Proof.** The existence of the upper bound follows from Lemma 5.4. To establish the existence of the lower bound, assume towards contradiction that there is a sequence of subspaces $W_i \in \mathcal{R}$ such that

$$\lim_{i} \sup_{\|w_j\| \leq 1} \| Q_{w_1 \wedge \cdots \wedge w_l} \|_K = 0.$$  

Up to passing to a subsequence, we may assume that $W_i \to W_0 \in \mathcal{R} \subset \mathfrak{G}(l, V)$. Consider an arbitrary ordered basis $\omega_0 = \{w_1, \ldots, w_l\}$ consisting of unit vectors for the limiting vector subspace $W_0$. There is a sequence $\omega_i$ of ordered bases for $W_i$
consisting of unit vectors such that \( \omega_i \rightarrow \omega_0 \). It follows that \( \| Q_{w_1, \ldots, w_l} \|_K = 0 \). Since the basis \( \omega_0 \) was arbitrary it follows that \( q(gW_0) = 0 \) for all elements \( g \in K \) in the sense of Equation (5.4). This is a contradiction to the above assumption. \( \square \)

**Proposition 5.6.** For every constant \( E > 0 \) the subfamily

\[
\mathcal{F}_E(\mathfrak{N}) = \{ Q \in \mathcal{F}(\mathfrak{N}) : \| Q \|_K \geq E \}
\]

is compact in the topology of uniform convergence on the compact group \( K \).

**Proof.** Consider a sequence of functions \( Q_i = Q_{w_i^1, \ldots, w_i^j} \in \mathcal{F}_E(\mathfrak{N}) \) where \( w_i^1, \ldots, w_i^j \) is an ordered basis for some subspace \( W_i \in \mathfrak{N} \subset \mathfrak{gl}(l, V) \). Assume that the sequence \( Q_i \) uniformly converges to some continuous function \( \Psi : K \rightarrow \bigwedge^l U \). Up to passing to a subsequence, we may assume that \( W_i \rightarrow W_0 \in \mathfrak{N} \), that \( w_i^j \rightarrow w_0^j \) for each \( j \in \{1, \ldots, l\} \) and that \( w_0^1, \ldots, w_0^l \) is an ordered \( l \)-tuple of unit vectors contained in \( W_0 \). Denote \( Q_0 = Q_{w_0^1, \ldots, w_0^l} \). There are elements \( g_i \in K \) with \( \| Q_i(g_i) \|_\beta \geq E \). Up to passing to a further subsequence, we may assume that \( g_i \rightarrow g_0 \in K \). Therefore \( \| Q_0(g_0) \|_\beta \geq E \) and the ordered \( l \)-tuple \( w_0^1, \ldots, w_0^l \) is linearly independent, hence a basis of \( W_0 \). In particular \( Q_0 \in \mathcal{F}_E(\mathfrak{N}) \). Finally it is clear that the uniform limit \( \Psi \) of the sequence \( Q_i \) coincides with \( Q_0 \). \( \square \)

We are ready to complete the proof of the main result of [5].

**Proof of Theorem 5.1.** Let \( \mathcal{F}(\mathfrak{N}) \) be the family of polynomial mappings associated to the subset \( \mathfrak{N} \) as introduced in Equation (5.2). Fix a sufficiently small constant \( E > 0 \) as provided by Proposition 5.5. Namely every subspace \( W \in \mathfrak{N} \) admits some ordered basis \( w_1, \ldots, w_l \) of unit vectors such that \( Q_{w_1, \ldots, w_l} \in \mathcal{F}_E(\mathfrak{N}) \). The family \( \mathcal{F}_E(\mathfrak{N}) \) of \( k \)-analytic maps from the compact group \( K \) to the \( k \)-vector space \( \bigwedge^l U \) is compact in the topology of uniform convergence according to Proposition 5.6.

The Haar measure \( \eta_K \) belongs to the canonical measure class of \( K \) regarded as a \( k \)-analytic manifold. Taking into account Proposition 5.5, the desired Equation (5.3) follows from Theorem 4.1 providing estimates on the measures of sublevel sets of \( k \)-analytic functions with respect to the Haar measure \( \eta_K \) on the compact \( k \)-analytic group \( K \). \( \square \)

6. Nilpotent subalgebras of semisimple Lie algebras

Let \( k \) be either the field \( \mathbb{R} \) or a non-Archimedean local field with absolute value \( | \cdot | \). Let \( G \) be a connected simply-connected semisimple \( k \)-algebraic linear group without \( k \)-anisotropic factors. Denote \( G = G(k) \) so that \( G \) is a \( k \)-analytic group.

In the positive characteristic case we make the additional assumption that \( \text{char}(k) \) is a good prime for the semisimple group \( G \) in the sense of [SS70, Definition 4.1]. This means that \( \text{char}(k) \) does not divide the coefficient of any simple root in the highest root. We point out that the only primes which fail to be good for some semisimple group are 2, 3 and 5.

**Maximal split torus.** Let \( T \) be a maximal \( k \)-split torus of \( G \). Let \( \Phi = \Phi(T, G) \) be the relative \( k \)-root system of the group \( G \) with respect to the torus \( T \).

Choose an ordering on the \( k \)-root system \( \Phi \) and let \( \Phi^+ \) and \( \Phi^- = -\Phi^+ \) denote the subsets of positive and negative roots respectively. Let \( \Phi_0^+ \subset \Phi^+ \) denote the subset of the simple positive roots.
Let $\mathbb{B}$ be the minimal parabolic $k$-subgroup corresponding to $\Phi^+$. We have $\mathbb{B} = Z(T)U$ where $U$ is the unipotent radical of $\mathbb{B}$. Every minimal parabolic $k$-subgroup is conjugate to $\mathbb{B}$ via an element of $G$ \cite{Bor12 §I.21.12}. The subgroup $U$ is defined over $k$ \cite{Bor12 §V.21.11}.

Fix notations for the corresponding groups of rational points, namely

$$B = \mathbb{B}(k), \quad T = T(k) \quad \text{and} \quad U = U(k).$$

The relative Weyl group. Let $W = W(T, G)$ be the relative Weyl group with respect to the maximal $k$-split torus $T$ in the group $G$. It is a finite Coxeter group containing an involution $s_\alpha$ for each root $\alpha \in \Phi$ \cite{Bor12 §21}. In fact the set involutions $\Sigma = \{s_\alpha\}_{\alpha \in \Phi^+_0}$ corresponding to the simple roots $\Phi^+_0$ generates the Weyl group $W$.

Let $L_\Sigma : W \to \mathbb{N} \cup \{0\}$ be the length function on the relative Weyl group $W$ with respect to the generating set $\Sigma$. Let $w_0 \in W$ be the longest element of $W$, i.e. the unique element satisfying $L_\Sigma(w_0) = \max\{L_\Sigma(w) \mid w \in W\}$ \cite{Hum90 §1.8}. It is the unique element satisfying $w_0\Phi^+ = \Phi^-$ and such that $Bw_0$ is the opposite parabolic $k$-subgroup to $B$. The longest element $w_0$ is independent of the choice of ordering.

The compact subgroup $K$. Let $K$ be a maximal compact subgroup of the $k$-analytic group $G$. In the real case assume that $K$ is compatible with the Cartan involution. In the positive characteristic case assume that $K$ is a good maximal compact subgroup, namely it admits a representative of every element of the spherical Weyl group. In either case, let $\eta_K$ denote the Haar measure on the compact group $K$ normalized so that $\eta_K(K) = 1$.

The group $G$ admits an Iwasawa decomposition $G = KB$ \cite{BT65}. In particular the compact subgroup $K$ is transitive in its action on the homogeneous space $G/B$.

Unipotent subgroups — positive characteristic case. We state an important theorem concerning unipotent subgroups of simply-connected semisimple groups in positive characteristic.

**Theorem 6.1** (Gille). Assume that the characteristic of $k$ is a good prime for $G$. Then every unipotent subgroup $V$ of $G$ is contained in the group of $k$-points of some minimal parabolic $k$-subgroup.

**Proof.** This follows from the assumption that $G$ is simply connected and relying on the work \cite{Gil02}. See also \cite{Gol09 Theorem 2.7}. □

Intersections with minimal parabolic $k$-subgroups. We show that any torus $S$ in $G$ as well as the unipotent subgroup $U$ admit a zero-dimensional intersection with a generic minimal parabolic $k$-subgroup.

The proofs are based to a large extent on the ideas of \cite[Lemma 3]{KM68}. In particular, we rely on the Zariski density of $k$-points — namely if $H$ is any connected reductive linear algebraic $k$-group then $H(k)$ is Zariski dense, see \cite[18.3]{Bor12} or \cite[§2.1 Theorem 2.2]{PR92} in the real case. Therefore the $k$-dimension of the group of $k$-points $H(k)$ is equal to the dimension of $H$.

**Proposition 6.2.** The subset

$$\{g \in G : \dim(U^g \cap B) = 0\}$$

is non-empty and Zariski open in $G$. 



Proof. By the Zariski density of $k$-points it will suffice to show that

$$\Psi = \{ g \in G : \dim(U^g \cap B) = 0 \}$$

is a non-empty Zariski open subset of $G$. The longest element $w_0$ of the relative Weyl group satisfies $B^{w_0} \cap B = T$. Therefore $\dim(U^{w_0} \cap B) = 0$ and in particular $\dim(U^{w_0 b} \cap B) = 0$ for every pair of elements $u \in U$ and $b \in B$. In other words, the big Bruhat cell $U w_0 B$ is contained in $\Psi$. This Bruhat cell is Zariski open and non-empty.

Proposition 6.3. Let $S = S(k)$ where $S \leq G$ is any $k$-torus. Then the subset

$$\{ g \in G : \dim(S^g \cap B) = 0 \}$$

is non-empty and Zariski open in $G$.

We do not assume that the $k$-torus $S$ is split.

Proof of Proposition 6.3. By the preceding discussion about the Zariski density of $k$-points it will suffice to show that

$$\Omega = \{ g \in G : \dim(S^g \cap B) > 0 \}$$

is a proper Zariski closed subset of $G$. Up to conjugation by an element of $G$ we may assume that $S \leq T$ [Bor12 §IV.11.3].

Consider any element $g \in \Omega$. There is a Zariski closed subgroup $S_1 \leq S$ depending on $g$ such that $\dim S_1 > 0$ and $S_1^T \leq B$. Since all maximal tori in the solvable group $B$ are conjugate there is an element $h \in B$ with $S_1^h \leq T$ [Bor12 §III.10.16]. As $S_1 \leq T$ this condition implies that $gh \in Z_G(S_1)$ [Bor12 §III.8.10 Corollary 1]. We conclude that

$$\Omega \subset \bigcup_{S_1 \leq S \atop \dim S_1 > 0} Z_G(S_1) B.$$

Our assumption that $G$ has no $k$-anisotropic factors implies that each $Z_G(S_1) B$ is a proper standard parabolic $k$-subgroup [Bor12 §V.21.11]. There are only finitely many such subgroups. In particular $\Omega$ is proper and Zariski closed.

Lie algebras. Let $g = Lie(G)$ denote the Lie algebra of the Lie group $G$. Let $g_\alpha$ be the relative $k$-root space corresponding to each relative $k$-root $\alpha \in \Phi$, namely

$$Ad(s)X = \alpha(s)X \quad \forall s \in T, \quad \forall X \in g_\alpha.$$

Consider the two Lie subalgebras

$$u^- = \bigoplus_{\alpha \in \Phi^-} g_\alpha \quad \text{and} \quad u^+ = \bigoplus_{\alpha \in \Phi^+} g_\alpha.$$

The Lie algebra $g$ admits a direct sum decomposition

$$g = b \oplus u^- \quad \text{and} \quad B = u^+ \oplus g_0$$

where $b = Lie(B)$ and $g_0 = Lie(Z_G(T))$ [Bor12 §V.21.7]. Fix some $Ad(K)$-invariant norm $\| \cdot \|$ on the Lie algebra $g$. In the real case assume that this norm is coming from an $Ad(K)$-invariant inner product on $g$.

\footnote{In the Archimedean case, the Lie algebra $g$ taken in the sense of Lie theory coincides with the real points of the Lie algebra of $G$ taken in the sense of algebraic groups [PR92 Lemma §3.3.1].}
Nilpotent Lie subalgebras. Let $\mathcal{N}(g)$ denote the subset of the Grassmannian $\mathcal{O}(g)$ consisting of all the nilpotent Lie subalgebras of $g$.

The subspace of $\mathcal{O}(g)$ consisting of Lie subalgebras is closed. This follows from the fact that the Lie bracket operation $\{\cdot,\cdot\} : \mathcal{O}(g) \times \mathcal{O}(g) \to \mathcal{O}(g)$ is lower semicontinuous. By induction on the nilpotency degree we deduce moreover that $\mathcal{N}(g)$ is a closed subset of $\mathcal{O}(g)$.

There is an alternative argument in the real case. Namely, recall that $X \in g$ is called ad-nilpotent if $\text{ad}(X)\dim g = 0$. Engel’s theorem says that a Lie subalgebra $h \leq g$ is nilpotent if and only if every $X \in g$ is ad-nilpotent. The adjoint map $\text{ad} : g \to \text{End}(g)$ is continuous. Therefore being ad-nilpotent is a closed condition in $g$, and it follows that $\mathcal{N}(g)$ is closed in $\mathcal{O}(g)$.

We remark that $\mathcal{N}(g)$ is Zariski closed in $\mathcal{O}(g)$ but we will not need this.

Algebraic Lie subalgebras. A Lie subalgebra $h \leq g$ is algebraic if $h = \text{Lie}(H(k))$ for some algebraic subgroup $H$ of $G$. For more on this notion see [Bor12, §I.7].

We rely on Propositions 6.2 and 6.3 to deduce the following statement concerning almost every conjugate of an algebraic nilpotent Lie subalgebra.

Proposition 6.4. If $n \in \mathcal{N}(g)$ is an algebraic nilpotent Lie subalgebra then the subset $\{g \in G : \text{Ad}(g)n \cap b = \{0\}\}$ is non-empty and Zariski open in $G$.

Proof. Let $n$ be an algebraic nilpotent Lie subalgebra of $g$. Let $N$ be a Zariski closed nilpotent subgroup of $G$ with $\text{Lie}(N) = n$. The nilpotent subgroup $N$ decomposes as a direct product $N = N_s \times N_u$ where $N_s$ and $N_u$ are the semi-simple and unipotent parts of $N$, respectively [Bor12 10.6]. Both $N_s$ and $N_u$ are defined over $k$. The subgroup $N_s$ is a $k$-torus.

The intersection $N \cap B^g$ is nilpotent and Zariski closed for every element $g \in G$. By the same reasoning as above, these intersections admit direct product decompositions

$$N \cap B^g = N_s(g) \times N_u(g)$$

for some Zariski closed subgroups $N_s(g) \leq N_s$ and $N_u(g) \leq N_u$ depending on the element $g \in G$. By Proposition 6.2 and Proposition 6.3 $\dim N_s(g) = \dim N_u(g) = 0$ for every element $g$ belonging to a non-empty Zariski open subset of $G$. The result follows.

Nilpotent Lie subalgebras — real case. Assume that the local field $k$ is $\mathbb{R}$ in the following two Propositions 6.5 and 6.6.

Proposition 6.5. Every nilpotent Lie subalgebra $n \in \mathcal{N}(g)$ is contained in some algebraic nilpotent Lie subalgebra.

Proof. Let $n$ be a nilpotent Lie subalgebra of $g$. Lie’s third theorem provides us with a simply connected Lie group $N_0$ with Lie algebra $\text{Lie}(N_0) \cong n$. There is a Lie group homomorphism $f : N_0 \to G$ such that the differential $df : \text{Lie}(N_0) \to n$ is an isomorphism. Let $N$ denote the Zariski closure of the nilpotent subgroup $f(N_0)$ inside $G$. The subgroup $N$ is nilpotent and Zariski closed. Its Lie algebra is algebraic by definition and contains $n$. □
Let \( \text{rank}(K) \) denote the \textit{compact rank} of the maximal compact group \( K \), in other words \( \text{rank}(K) \) is the dimension of any maximal compact torus of \( K \). Let \( P \) denote a fixed projection from the Lie algebra \( g \) to its Lie subalgebra \( u^- \).

**Proposition 6.6.** Let \( n \in \mathfrak{n} \) be any nilpotent Lie subalgebra with basis \( \beta_n \). Then the real analytic function \( q_n \in A(G, \bigwedge^{\dim_{\mathbb{R}} n} u^-) \) given by

\[
 q_n(g) = \bigwedge_{X \in \beta_n} (P \circ \text{Ad})(g)X \quad \forall g \in G
\]

satisfies \( \text{ord}_K q_n < (6 \text{ht}(g) \dim_{\mathbb{R}} n + 1)^{\text{rank}(K)} \).

**Proof.** The real analytic function \( q_n \) is non-zero on a Zariski open subset of \( G \) as can be seen by combining Proposition 6.4 and Proposition 6.5. Note that for any pair of elements \( g \in G \) and \( b \in B \) the function \( q_n \) satisfies \( q_n(g) = 0 \) if and only if \( q_n(bg) = 0 \). The Iwasawa decomposition \( G = KB \) shows that the function \( q_n \) is not identically zero on each connected component of the compact subgroup \( K \).

We wish to bound the order \( \text{ord}_K q_n \) relying on Lemma 3.6. With this goal in mind, let \( T_K \) be a maximal compact torus of \( K \) and \( \Lambda_{\text{analytic}} \) be the lattice of analytically integral forms on \( \text{Lie}(T_K) \). Note that \( \Lambda_{\text{analytic}} \cong \mathbb{Z}^{\text{rank}(K)} \).

Let \( S = S(\mathbb{R}) \) be a maximal \( \mathbb{R} \)-torus containing the compact torus \( T_K \) and \( \Delta \) be the absolute root system associated to \( S \) in the semisimple group \( G \). The restrictions to \( T_K \) of the simple positive roots (with respect to some notion of positivity on \( \Delta \)) are analytic and are a \( \mathbb{Z} \)-basis for some lattice \( \Lambda \leq \Lambda_{\text{analytic}} \), see Proposition A.1 of the appendix. It follows that the weights of the \( T_K \)-action on the complexification of the vector space \( \bigwedge^{\dim_{\mathbb{R}} n} g \) are contained in a ball of radius \( \text{ht}(\Delta) \dim_{\mathbb{R}} n \) in the lattice \( \Lambda \).

A comparison of the possibilities for absolute and restricted root systems in the real case [OV12, Reference Chapter, Table 9] shows\(^{11}\) that \( \text{ht}(\Delta) \leq 2 \text{ht}(\Phi) = 2 \text{ht}(g) \). The size of the ball of radius \( 3 \text{ht}(g) \dim_{\mathbb{R}} n \) in the lattice \( \Lambda \cong \mathbb{Z}^{\text{rank}(K)} \) is at most \( (6 \text{ht}(g) \dim_{\mathbb{R}} n + 1)^{\text{rank}(K)} \). This concludes the proof. \( \square \)

**Nilpotent Lie subalgebras — positive characteristic case.** Assume that the local field \( k \) has positive characteristic. Let \( P \) be a projection from the Lie algebra \( g \) to its Lie subalgebra \( u^- \). We establish the following the following order estimate.

**Proposition 6.7.** Let \( \beta^+ \) be a \( k \)-basis for \( u^+ \). Then the \( k \)-analytic function

\[
 q_u \in A(K, \bigwedge^{\dim_{\mathbb{R}} u^-} u^-)
\]

given by

\[
 q_u(g) = \bigwedge_{X \in \beta^+} (P \circ \text{Ad})(g)X \quad \forall g \in K
\]

satisfies

\[
 \text{ord}_K q_u \leq 4 \text{ht}(g) \dim_{\mathbb{R}} n \dim_{\mathbb{R}} g.
\]

\(^{11}\)If an absolute root system \( \Psi \) is classical then each real restricted root system \( \Phi \) obtained from it is classical as well. More generally \( \text{ht}(\Psi) \leq 2 \text{ht}(\Phi) \) unless \( \Psi \) and \( \Phi \) have types \( E_6 \) and \( F_4 \) respectively.
Some preparation is needed before giving the proof of Proposition 6.7. To begin with, recall that the relative $k$-root system $\Phi$ may not be reduced in general. For each relative $k$-root $\alpha \in \Phi$ denote

$$\bar{\alpha} = \begin{cases} 2\alpha & 2\alpha \in \Phi, \\ \alpha & 2\alpha \notin \Phi \end{cases}$$

and consider the subset $(\alpha) \subset \Phi$ given by

$$\text{(6.5)} \quad (\alpha) = \{\alpha \in \Phi : \{\alpha, \bar{\alpha}\} = \{\alpha, 2\alpha\} \cup \{\alpha\} \notin \Phi. \}$$

Generally speaking, the group $G$ admits a metabelian unipotent $k$-subgroup $U_{(\alpha)}$ whose center is the connected unipotent $k$-subgroup $U_{(\bar{\alpha})}$. The groups $U_{(\alpha)}$ and $U_{(\bar{\alpha})}$ are normalized by the maximal $k$-split torus $T$. \text{[Bor12 Proposition 21.9].}

The following result uses the notion of strong order introduced in Definition 3.2.

**Proposition 6.8.** Let $\alpha \in \Phi$ be a relative $k$-root. Then the adjoint representation of the group $U_{(\bar{\alpha})}$ regarded as a $k$-analytic map

$$\text{Ad} \in \mathcal{A}(U_{(\bar{\alpha})}, \text{End}(g))$$

has strong order at most $4ht(g) \dim_k U_{(\bar{\alpha})} \dim_k g$ at each point of $U_{(\bar{\alpha})}$.

**Proof.** The connected unipotent $k$-subgroup $U_{(\bar{\alpha})}$ is abelian and admits a $k$-isomorphism $\theta_{\alpha} : k^n \to U_{(\bar{\alpha})}$ where $n = \dim_k U_{(\bar{\alpha})}$ \text{[Bor12 Lemma 21.17].} Since the adjoint representation $\text{Ad} : G \to \text{GL}(g)$ is a central isogeny onto its image, the $k$-rational representation $\text{Ad} \circ \theta_{\alpha} : k^n \to \text{GL}(g)$ must be a $k$-isomorphism onto its image \text{[Bor12 Proposition 22.4].} It follows that the map $\text{Ad} \circ \theta_{\alpha} \in \mathcal{A}(k^n, \text{End}(g))$ is an immersion of $k$-analytic manifolds \text{[Mar91 §1.2.5.3].}

We wish to conclude relying on Lemma 3.7. Indeed, the subgroup $\text{Ad}(T)$ is a $k$-split torus of $\text{GL}(g)$ \text{[Bor12 Corollary 8.4].} It satisfies

$$\text{Ad}(t)\text{Ad}(\theta_{\alpha}(z))\text{Ad}(t^{-1}) = \text{Ad}(\theta_{\alpha}(z)t^{-1}) = \text{Ad}(\bar{\alpha}(t)z)) \quad \forall t \in T, z \in k^n.$$
For any pair of elements $w' \in W$ and $s \in \Sigma$ the condition $L_{\Sigma}(w's) > L_{\Sigma}(w')$ implies that $Bw'BSB \subset Bw'sB$ \cite[Proposition 21.22]{Bor12}. Observe that

$$U(\alpha) \setminus \{e\} \subset Bs\alpha B$$

for all simple positive roots $\alpha \in \Phi^+_0$, see \cite[Theorem 21.15]{Bor12} and its proof. Applying these facts inductively shows that

$$Bwu_1 \cdots u_{i-1}Bu_i \subset Bwu_1 \cdots u_iB$$

for all $i \in \{1, \ldots, L_{\Sigma}(w^{-1}w_0)\}$ provided that the unipotent elements $u_i \in U(\alpha_i)$ are all non-trivial. The final induction step gives

$$BwBu_1 \cdots u_{L_{\Sigma}(w^{-1}w_0)} \subset Bwu_1 \cdots u_{L_{\Sigma}(w^{-1}w_0)}B = Bw(w^{-1}w_0)B = Bw_0B$$

as required. \hfill \Box

We are ready to complete the following argument.

Proof of Proposition \ref{Weyl}. Fix an arbitrary element $g \in K$. There is a unique relative Weyl group element $w(g) \in W$ satisfying $g \in Bw(g)B$ \cite[Theorem 21.15]{Bor12}. Denote

$$w' = w'(g) = w(g)^{-1}w_0$$

where $w_0 \in W$ is the longest element of $W$. Further denote $L = L_{\Sigma}(w'(g))$. Express the Weyl group element $w'$ as a reduced word $w' = s_1 \cdots s_L$ where each $s_i$ belongs to the generating set $\Sigma$ with $s_i = s_{\alpha_i}$ for some root $\alpha_i \in \Phi_0^+$.

For each relative $k$-root $\alpha \in \Phi$ recall that $U(\alpha)$ is a connected $k$-unipotent group $k$-isomorphic to a $k$-vector space. Let $Id_{\alpha}, Fr_{\alpha} : U(\alpha) \to U(\alpha)$ be the identity and the Frobenius maps, respectively. For each index $i \in \{1, \ldots, L\}$ define a map $f_{g,i} : U(\bar{\alpha}_i) \to U(\bar{\alpha}_i)$ by

$$f_{g,i} = \begin{cases} 
Id_{\alpha_i}, & s_j \neq s_i \text{ for all } j < i, \\
Fr_{\alpha_i}, & \text{otherwise.}
\end{cases}$$

Consider the subset of simple positive roots

$$\Phi_g = \{\alpha \in \Phi_0^+: s_i = s_{\alpha_i} \text{ for some } 1 \leq i \leq L\}$$

and the $k$-analytic product group $H_g = \prod_{\alpha \in \Phi_g} U(\bar{\alpha})$. We are interested in the $k$-analytic map $F_g : H_g \to GL(g)$ given by

$$F_g : (u_{\alpha})_{\alpha \in \Phi_g} \mapsto g_{f_{g,1}(u_{\alpha_1})} \cdots f_{g,L}(u_{\alpha_L}).$$

The differential of the map $Ad \circ F_g$ at the identity element $e \in H_g$ can be computed relying on the chain-rule formula \cite[§3.2]{Bor12}. Namely

$$d(Ad \circ F_g)|_e (X_{\alpha})_{\alpha \in \Phi_g} = Ad(g) \sum_{\alpha \in \Phi_g} \text{ad}(X_{\alpha}).$$

Note that $\text{ad}(X_{\alpha}) \subset \text{ad}(g_{-\alpha})$ for all roots $\alpha \in \Phi_g$. Therefore the differentials $\text{ad}(X_{\alpha})$ are linearly independent provided that the $X_{\alpha}$’s are non-zero. It follows that the map $Ad \circ F_g$ is a $k$-analytic immersion at the identity element of $H_g$.

Let $H_0 \leq H$ be a sufficiently small open subgroup such that $(Ad \circ F_g)|_{H_0}$ is a $k$-analytic embedding into the compact subgroup $Ad(K)$. We know from Lemma \ref{embed} that every element of $F_g(H_0)$ obtained as a product of non-trivial elements in

\footnote{Strictly speaking, the objects $f_{g,i}, \Phi_g, H_g$ and $F_g$ all depend on the reduced word $w'(g)$ rather than on the element $g$.}
each coordinate of $H_0$ belongs to the big Bruhat cell $B w_0 B$. In particular $q_u| F_g| H_0)$ is not identically zero according to Proposition 6.2 and its proof.

The $k$-analytic maps $\text{Ad} \circ f_g : U_i(\tilde{\alpha}_i) \to \text{End}(\mathfrak{g})$ have strong order at most $4 \text{ht}(\mathfrak{g}) \dim_k U(\tilde{\alpha}_i) \dim_k \mathfrak{g}$ at each point of their domain and for all $i \in \{1, \ldots, L\}$, see Proposition 6.8. As the map $(\text{Ad} \circ F)| H_0$ is a $k$-embedding it has strong order at most

$$\sum_{i=1}^{L} 4 \text{ht}(\mathfrak{g}) \dim_k U(\tilde{\alpha}_i) \dim_k \mathfrak{g} \leq 4 \text{ht}(\mathfrak{g}) \dim_k^2 U \dim_k \mathfrak{g}$$

at each point of $H_0$. The above computation relies on the fact that $L \leq L_{\Sigma}(w_0) = |\Phi^+| \leq \dim_k U$.

To conclude we estimate the order of the map $q_u$ at the element $g \in K$. Denote $\dim_k u^+ = |\beta^+| = l$ and enumerate $\beta^+ = \{X_1, \ldots, X_l\}$. Note that

$$(q_u \circ F_g)(h) = \det \left( \begin{array}{ccc} (\text{Ad} \circ F_g)(h)X_1 & \cdots & (\text{Ad} \circ F_g)(h)X_l \\ \vdots & \ddots & \vdots \\ (\text{Ad} \circ F_g)(h)X_l & \cdots & (\text{Ad} \circ F_g)(h)X_1 \end{array} \right) \quad \forall h \in H_0.$$

In light of Lemma 6.8 we therefore have that

$$\text{ord}_g q_u \leq \text{ord}_g q_u| H_0 \leq 4 \text{ht}(\mathfrak{g}) \dim_k^2 U \dim_k \mathfrak{g}.$$

Since the point $g \in K$ was arbitrary this concludes the proof. $\square$

**Conjugation by semisimple elements.** Let $s \in T$ be any element. The linear operator $\text{Ad}(s)$ as well as its inverse $\text{Ad}(s^{-1})$ preserve the direct sum decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{b}$. It is easy to compute the norms of the restrictions of these operators to, say, $\mathfrak{u}^-$. 

**Proposition 6.10.** Let $s \in T$ be such that $|\alpha(s)| < 1$ for all $\alpha \in \Phi^+$. Then

$$(6.8) \quad \|\text{Ad}(s)\| = \|\text{Ad}(s^{-1})\| = \max_{\alpha \in \Phi^-} |\alpha(X)|$$

and

$$(6.9) \quad \|\text{Ad}(s^{-1})|_{\mathfrak{u}^-}\| = \frac{1}{\min_{\alpha \in \Phi^-} |\alpha(X)|}.$$

**Proof.** The operator $\text{Ad}(s)$ is diagonalizable, and the corresponding weights for its action on the Lie subalgebra $\mathfrak{g}_\alpha$ are given by the relative $k$-root $\alpha \in \Phi$, see Equation (6.2). Therefore

$$\|\text{Ad}(s)\| = \max_{\alpha \in \Phi} |\alpha(s)| = \max_{\alpha \in \Phi^-} |\alpha(s)|.$$

This implies that Equation (6.8) holds. Since $\Phi = -\Phi$, the same reasoning shows that $\|\text{Ad}(s^{-1})\| = \|\text{Ad}(s)\|$. The linear isomorphism $\text{Ad}(s)$ preserves $\mathfrak{u}^-$. Therefore

$$\|\text{Ad}(s^{-1})|_{\mathfrak{u}^-}\| = \max_{\alpha \in \Phi^-} \frac{1}{|\alpha(s)|} = \frac{1}{\min_{\alpha \in \Phi^-} |\alpha(s)|}.$$

The validity of Equation (6.9) follows. $\square$

Recall that $\Phi_0^+$ is the subset of the simple positive roots. Every positive root $\alpha \in \Phi^+$ can be written as an integral linear combination $\alpha = \sum_{\alpha_0 \in \Phi_0^+} n_{\alpha_0} \alpha_0$ of...
simple roots in a unique way. The height $ht(\alpha)$ is the sum of the coefficients $n_{\alpha_0}$ in the above expression. Denote
\begin{equation}
ht(g) = \sup_{\alpha \in \Phi^+} ht(\alpha).
\end{equation}

**Proposition 6.11.** There is a semisimple element $s_0 \in T$ and some $\lambda_0 > 1$ such that
\begin{equation}
\|Ad(s_0^n)\| = \lambda_0^{hn(g)} \quad \text{and} \quad \|Ad(s_0^{-n})|_{u^-}\| = \lambda_0^{-n}
\end{equation}
for all $n \in \mathbb{N}$.

**Proof.** For each simple relative root $\alpha \in \Phi^+$ there is a one-parameter subgroup $\chi_\alpha$ of the torus $T$ so that
$$\langle \alpha, \chi_\beta \rangle = N\delta_{\alpha\beta}$$
for some fixed $N \in \mathbb{N}$ and all simple relative $k$-roots $\beta \in \Phi_0^+$ [Bor12, Proposition 8.6]. Fix any element $x_0 \in k$ with $|x_0| < 1$. Consider the semisimple element $s_0 \in T$ given by
\begin{equation}
s_0 = \prod_{\alpha \in \Phi_0^+} \chi_\alpha(x_0).
\end{equation}
Note that $\alpha(s_0^n) = |x_0|^n N^{ht(\alpha)} < 1$ for every positive root $\alpha \in \Phi^+$ and all $n \in \mathbb{N}$. The result follows relying on Proposition 6.10 and taking $\lambda_0 = |x_0|^{-N}$. □

**Norms of restrictions to nilpotent subalgebras.** Consider the Lie algebra $\mathfrak{g}$ with its direct sum decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{b}$. The current setup fits into the framework of our discussion in §5 where
- the Lie algebra $\mathfrak{g}$ plays the role of the vector space $V$,
- the subgroup $Ad(K)$ plays the role of the compact subgroup of $GL(V)$,
- the two Lie subalgebras $\mathfrak{u}^-$ and $\mathfrak{b}$ respectively play the roles of the two direct summands $U$ and $U'$,
- any fixed nilpotent subalgebra $\mathfrak{n}$ plays the role of the fixed subspace $W$, and
- the linear operator $Ad(s)$ for any fixed semisimple element $s \in T$ plays the role of the linear isomorphism $L$ preserving both direct summands $U$ and $U'$ as in Lemma 5.2.

In light of this correspondence, Theorem 5.1 can be reformulated to obtain the following result. Consider the subset $\mathfrak{M} \subset \mathfrak{O}(\mathfrak{g})$ consisting of all
- the non-zero nilpotent Lie subalgebras $\mathfrak{n} \in \mathfrak{M}(\mathfrak{g})$ if $k = \mathbb{R}$, or
- the Lie subalgebras $Ad(g)\mathfrak{u}^+$ for some $g \in K$ if $k$ is non-Archimedean.

The collection $\mathfrak{M}$ of Lie subalgebras is closed in the Grassmannian $\mathfrak{O}(\mathfrak{g})$. This follows from the preceding discussion in the real case and from the compactness of the group $K$ in the non-Archimedean case. Let $\mathcal{F}(\mathfrak{M})$ be the family on $k$-analytic functions from the group $K$ associated to the collection $\mathfrak{M}$ as considered in Equations 5.1 and 5.2. Namely
\begin{equation}
\mathcal{F}(\mathfrak{M}) = \{ g \mapsto \bigwedge_{i=1}^t Pgw_i : \|w_i\| \leq 1 \text{ and span}_k\{w_1, \ldots, w_t\} \in \mathfrak{M}\}
\end{equation}
where $P : \mathfrak{g} \rightarrow \mathfrak{u}^-$ is some fixed projection.
Proposition 6.12. There is a constant $\zeta > 0$ with the following property — for all sufficiently large $\lambda > 1$ there exists a semisimple element $s_\lambda \in T$ such that

\[
(6.13) \quad \eta_K \left( \{ g \in K : \inf_{0 \neq X \in \text{Ad}(g)n} \frac{\|\text{Ad}(s_\lambda)X\|}{\|X\|} \geq 2 \} \right) \geq 1 - \zeta \lambda^{-\frac{1}{\text{dim}_K \text{ord}_K F(N)}} \quad \text{and}
\]

\[
(6.14) \quad \|\text{Ad}(s_\lambda)\| \leq \lambda^{\text{ht}(g)}
\]

for all Lie subalgebras $n \in \mathfrak{N}$.

Proof. Consider the semisimple element $s_0 \in T$ and the constant $\lambda_0 > 1$ given in Proposition 6.11. We have

\[
(6.15) \quad \|\text{Ad}(s^n_0)\| = \lambda_0^{n\text{ht}(g)} \quad \text{and} \quad \|\text{Ad}(s^{-n}_0)\|_{u^{-n}} = \lambda_0^{-n}
\]

for all $n \in \mathbb{N}$.

Given any sufficiently large fixed $\lambda > 1$ there is an integer $n_0 \in \mathbb{N}$ such that $\lambda_0^{n_0} \leq \lambda \leq \lambda_0^{n_0+1}$. Take $s_\lambda = s^{n_0}_0$. Observe that Equation (6.14) holds with respect to the element $s_\lambda$ as $\|\text{Ad}(s_\lambda)\| = \lambda_0^{n_0\text{ht}(g)} \leq \lambda^{\text{ht}(g)}$.

We are now in a position to apply our results from the previous §5. According to Theorem 5.1 combined with Lemma 5.2 there exists a constant $\kappa_\mathfrak{N} > 0$ such that every Lie subalgebra $n \in \mathfrak{N}$ and every constant $\varepsilon > 0$ satisfy

\[
(6.16) \quad \eta_K \left( \{ g \in K : \inf_{0 \neq X \in \text{Ad}(g)n} \frac{\|\text{Ad}(s_\lambda)X\|}{\|X\|} \geq \varepsilon \inf_{0 \neq X \in u^-} \frac{\|\text{Ad}(s^{-n}_0)\|}{\|X\|} \} \right) \geq 1 - \kappa_\mathfrak{N} \varepsilon \lambda^{-\frac{1}{\text{dim}_K \text{ord}_K F(N)}}.
\]

Note that Equation (6.15) implies

\[
(6.17) \quad \varepsilon \inf_{0 \neq X \in u^-} \frac{\|\text{Ad}(s_\lambda)X\|}{\|X\|} \geq \varepsilon \lambda_0^n \geq \varepsilon \frac{\lambda}{\lambda_0}.
\]

Take $\varepsilon = \frac{2\lambda}{\lambda_0}$. We conclude from the two Equations (6.16) and (6.17) that the desired Equation (6.13) holds true with the constant $\zeta = \kappa_\mathfrak{N}(2\lambda_0)^{-\frac{1}{\text{dim}_K \text{ord}_K F(N)}}$. □

The parameters $\lambda(G), \delta(G)$ and $s(G)$. We now fix some parameters for the remainder of this work. Let the constant $\zeta$ be as provided by Proposition 6.12. According to Calculation 2.4 there is a sufficiently large value

\[
(6.18) \quad \lambda = \lambda(G) > 0
\]

such that the parameters

\[
(6.19) \quad a_1 = 2, \quad a_2(G) = \lambda(G)^{-\text{ht}(g)} \quad \text{and} \quad p(G) = p_\lambda = 1 - \zeta \lambda^{-\frac{1}{\text{dim}_K \text{ord}_K F(N)}}
\]

satisfy Equation (2.11) and such that the corresponding constant

\[
(6.20) \quad \delta = \delta(G) = \delta_0(a_1, a_2, \lambda; p_\lambda)
\]
is bounded from below as follows\footnote{The constant 2 appearing in the first line of Equation (6.21) can be replaced by any real number in the large $[1, \infty)$. The integral value of 2 is only a matter of convenience.}

$$\delta^{-1} \leq 2ht(g) \dim_k K \ord_K \mathcal{F}(\mathfrak{N}) \leq \begin{cases} 2ht(g) \dim_K K (6ht(g) \dim_U + 1)^{\text{rank}(K)} & k \text{ is } \mathbb{R}, \\ 8ht(g)^2 \dim_k K \dim_k G \dim_k^3 U & k \text{ is non-Arch.} \end{cases} \leq \begin{cases} (3ht(g) \dim_G)^{(\text{rank}(K)+1)} & k \text{ is } \mathbb{R}, \\ \text{ht}(g)^2 \dim_k^2 G & k \text{ is non-Arch.} \end{cases}$$

(6.21)

The upper bounds on $\ord_K \mathcal{F}(\mathfrak{N})$ follow from Propositions 6.6 and 6.7 respectively. Lastly, for the remainder of this work fix the semisimple element

$$s = s(G) = s_\lambda \in T$$

such that the two Equations (6.13) and (6.14) of Proposition 6.12 are satisfied.

7. Subgroups generated by small elements

We continue using the notations introduced in §6 so that in particular $G$ is a semisimple $k$-analytic group over the local field $k$. The goal of this section is to study discrete subgroups of $G$ generated by small elements. We introduce a function $I_G$ defined on the Chabauty space of discrete subgroups $\text{Sub}_d(G)$ which, roughly speaking, assigns to every discrete subgroup $\Gamma$ of $G$ the norm of its smallest non-trivial element.

Let $g = \text{Lie}(G)$ be the Lie algebra of the $k$-analytic group $G$. We consider $g$ with some $\text{Ad}(K)$-invariant norm $\| \cdot \|$ as in §6. It will be useful to introduce the following notation

$$\mathfrak{B}_g(r) = \{ X \in g : \|X\| \leq r \}$$

for every radius $r > 0$.

The real case. Zassenhaus lemma. Assume that the local field $k$ is $\mathbb{R}$ so that the group $G$ is a semisimple real Lie group without compact factors.

The exponential map $\exp$ sets up a local diffeomorphism from $0 \in g$ to $\text{id}_G \in G$. We will use $\log$ to denote the inverse of the exponential map $\exp$ where defined.

We rely on the classical lemma due to Zassenhaus [Zas37]. It was proved also in [KM68, Lemma 2]. Roughly speaking, it says that a discrete subgroup generated by small elements of $G$ is nilpotent. The precise formulation is as follows.

**Lemma 7.1 (Zassenhaus).** There is a radius $R = R(G) > 0$ and an identity neighborhood $V = \exp \mathfrak{B}_g(R)$ such that for every discrete subgroup $\Gamma$ of $G$ the group generated by $\Gamma \cap V$ is contained in a connected nilpotent Lie subgroup.

For every discrete subgroup $\Gamma$ of $G$ let $N(\Gamma)$ denote the minimal connected nilpotent Lie subgroup of $G$ containing $\langle \Gamma \cap V \rangle$. Denote $n(\Gamma) = \text{Lie}(N(\Gamma))$. 
The non-Archimedean case. Springer isomorphism. Assume that \( k \) is a non-Archimedean local field of positive characteristic and that \( \text{char}(k) \) is a good prime for the semisimple group \( G \).

Let \( \mathcal{O} \) be the ring of integers of the non-Archimedean local field \( k \). Let \( m \) be the maximal ideal of \( \mathcal{O} \). Let \( G(m^i) \) denote the congruence subgroup of the compact group \( G(\mathcal{O}) \) with respect to the ideal \( m^i \) for every \( i \in \mathbb{N} \).

We define a norm \( \| \cdot \|_m \) on the group \( G(m^i) \) by letting
\[
\|g\|_m = \inf \{|\mathcal{O}/m|^{-i} : i \in \mathbb{N} \text{ and } g \in G(m^i)\}
\]
for all elements \( g \in G(m^i) \).

It is well-known that every discrete subgroup of \( G(m^i) \) is finite and unipotent. Indeed the subgroup \( G(m^i) \) is a pro-\( p \)-group \([PR92, Lemma \S3.3.8]\). Therefore every discrete subgroup \( \Gamma \) of \( G(m^i) \) is a finite \( p \)-group and is in particular nilpotent. We obtain
\[
(g - \text{id}_G)_p^l = g_p^l - \text{id}_G^p = 0
\]
for some \( l \in \mathbb{N} \) and all \( g \in \Gamma \). Therefore \( g - \text{id}_G \) is nilpotent and \( g \) is unipotent.

Let \( U \) be the subset of \( G \) consisting of all unipotent elements. Then \( U \) is an irreducible closed subvariety of \( G \) defined over \( k \) and \( \dim U = \dim G - \text{rank} G \) \([Spr69, Proposition 1.1]\). Likewise, the subset \( \mathfrak{B} \subset \text{Lie}(G) \) of all nilpotent elements is an irreducible closed subvariety of \( \text{Lie}(G) \) defined over \( k \) \([Spr69, Proposition 2.1]\).

Theorem 7.2 (Springer isomorphism). Assume that the characteristic of \( k \) is a good prime for \( G \). Then there exists a \( G \)-equivariant \( k \)-isomorphism \( \Sigma : U \to \mathfrak{B} \). Moreover \( \Sigma(U) = u^+ \).

Here \( U \) is the group of \( k \)-rational points of the unipotent radical of a minimal parabolic \( k \)-subgroup. The Lie subalgebra \( u^+ \) is discussed in \([\mathfrak{3}]\).

Proof of Theorem 7.2. Springer constructed in \([Spr69, Theorem 3.1]\) a \( G \)-equivariant \( k \)-morphism \( \Sigma : U \to \mathfrak{B} \) such that \( \Sigma(U(k)) \to \mathfrak{B}(k) \) is a homeomorphism. This statement was strengthened in \([BR85, Corollary 9.3.4]\) where \( \Sigma \) is shown to be a \( k \)-isomorphism. The fact that \( \Sigma(U) = u^+ \) follows from the proof. \( \square \)

Note that \( \Sigma(U(k)) = \mathfrak{B}(k) \). The restriction of \( \Sigma \) to the compact Hausdorff space \( U(m) \) is a \( k \)-morphism which is a homeomorphism onto its image. Therefore there is a constant \( \sigma > 1 \) such that
\[
\sigma^{-1}\|u\|_m \leq \|\Sigma(u)\| \leq \sigma\|u\|_m
\]
for every unipotent element \( u \in U(m) \).

Abusing notation, we will from now on use \( \log \) and \( \exp \) to denote the Springer isomorphism \( \Sigma \) and its inverse where defined, even though \( \Sigma \) is not a logarithmic map in the strict sense.

The function \( I_G \). In the Archimedean case let \( R > 0 \) be the radius given by the Zassenhaus lemma (i.e. Lemma \([\mathfrak{7.1}]\)). In the positive characteristic case let \( R = \sigma^{-1} |\mathcal{O}/m|^{-1} \) where the constant \( \sigma \) is as given in Equation \((7.2)\). Denote
\[
V = \exp B_g(R).
\]
Namely \( V \) is a Zassenhaus neighborhood in the Archimedean case and is an identity neighborhood contained in the subgroup \( G(m) \) in the non-Archimedean case.
Let $0 < \rho < R$ be a sufficiently small radius such that the identity neighborhood
\begin{equation}
V_0 = \exp B_g(\rho)
\end{equation}
satisfies
\begin{equation}
V_0 \subseteq V \cap V^s \cap V^{s^{-1}}
\end{equation}
where $s = s(G) \in T$ is the particular semisimple element fixed in the final paragraph of §6, see Equation (6.22).

We introduce a real-valued discreteness radius function $I_G$ on the Chabauty space $\text{Sub}_d(G)$ of discrete subgroups of the group $G$. Given a discrete subgroup $\Gamma$ of $G$ define
\begin{equation}
I_G(\Gamma) = \begin{cases} 
\min_{\gamma \in (\Gamma \cap V_0) \setminus \{e\}} \| \log \gamma \| & \Gamma \cap V_0 \neq \{e\} \\
\rho & \Gamma \cap V_0 = \{e\}
\end{cases}
= \sup \{0 < r < \rho : \Gamma \cap \exp(B_g(r)) = \{e\}\}
\end{equation}
The function $I_G$ takes values in the interval $(0, \rho]$. In the non-Archimedean case $\log$ denotes the Springer isomorphism $\Sigma$ and Equation (7.6) depends on the fact that $\Gamma \cap V_0$ is unipotent combined with Theorems 6.1 and 7.2.

The logarithm map (and the Springer isomorphism) are $G$-equivariant, namely
\begin{equation}
\log(x^g) = \text{Ad}(g) \log(x)
\end{equation}
for every element $x \in V \cap V^{g^{-1}}$. As the norm $\| \cdot \|$ on the Lie algebra $\mathfrak{g}$ is $\text{Ad}(K)$-invariant, the function $I_G$ is $K$-invariant with respect to the compact subgroup $K$, namely
\begin{equation}
I_G(\Gamma^k) = I_G(\Gamma) \quad \forall k \in K.
\end{equation}

**Contraction on the average.** Let $\Gamma \in \text{Sub}_d(G)$ be a discrete subgroup of $G$. We show that conjugation by the semisimple element $s$ expands the function $I_G$ by a definite amount, when it is evaluated at most conjugates of $\Gamma$ by an element of the compact subgroup $K$.

Let $\eta_K$ denote the Haar measure of the compact group $K$ normalized to be a probability measure so that $\eta_K(K) = 1$. The precise statement is as follows.

**Proposition 7.3.** If the discrete subgroup $\Gamma$ satisfies $I_G(\Gamma) \leq \frac{\rho}{2}$ then
\begin{equation}
\eta_K(\{g \in K : I_G(\Gamma^g) \geq 2I_G(\Gamma^g)\}) \geq p(G)
\end{equation}
where the parameter $p(G) > 0$ is as given in Equation (6.19).

We postpone the proof of Proposition 7.3 until the end of the current §7.

**Proposition 7.4.** For every discrete group $\Gamma$ we have
\begin{equation}
I_G(\Gamma^s) \geq \frac{I_G(\Gamma)}{\|\text{Ad}(s^{-1})\|}.
\end{equation}

**Proof.** As $\|\text{Ad}(s^{-1})\| > 1$ and $0 < I_G(\Gamma) \leq \rho$ the result clearly holds if $I_G(\Gamma^s) = \rho$. Assume therefore that $I_G(\Gamma^s) < \rho$. There is a non-trivial element $\gamma \in \Gamma$ with $\gamma^s \in V_0$ and $I_G(\Gamma^s) = \| \log \gamma^s \|$. Since $V_0^{s^{-1}} \subseteq V$ we have that $\gamma \in V$. The equivariance of the logarithm map (or the Springer isomorphism) formulated in Equation (7.7) implies that
\begin{equation}
I_G(\Gamma^s) = \| \log \gamma^s \| = \frac{\| \log \gamma \|}{\|\text{Ad}(s^{-1})\|} \geq \frac{I_G(\Gamma)}{\|\text{Ad}(s^{-1})\|}
\end{equation}
Proposition 7.5. Let $\mathfrak{h} \leq \mathfrak{g}$ be any Lie subalgebra with $\log(\Gamma \cap V) \subset \mathfrak{h}$. If the discrete group $\Gamma$ and the element $s \in T$ satisfy

\[
\inf_{0 \neq X \in \mathfrak{h}} \frac{\|\text{Ad}(s)X\|}{\|X\|} \geq 2 \quad \text{and} \quad I_G(\Gamma) \leq \frac{\rho}{2}
\]

then

\[
I_G(\Gamma^s) \geq 2I_G(\Gamma).
\]

Proof. Assume that two conditions given in Equation (7.12) are satisfied. This implies $I_G(\Gamma) < \rho$ so that $\Gamma \cap V_0 \neq \{e\}$. Observe that

\[
\|\log \gamma\| = \|\text{Ad}(s) \log \gamma\| \geq 2\|\log \gamma\|
\]

for every element $\gamma \in \Gamma \cap V \cap V_0^{-1}$. In particular, as $V_0 \subset V \cap V_0^{-1}$ it follows that Equation (7.14) holds for every $\gamma \in \Gamma \cap V_0$. On the other hand, note that

\[
\Gamma^s \cap V_0 = (\Gamma \cap V \cap V_0^{-1})^s \cap V_0.
\]

The containment in the non-trivial direction of Equation (7.15) holds true since $V_0 \subset V \cap V_0^{-1}$. The required Equation (7.13) follows by combining Equations (7.12), (7.14) and (7.15) with the definition of the function $I_G$. □

We are ready to complete the proof of the main result of the current §7.

Proof of Proposition 7.3. Assume that the subgroup $\Gamma$ satisfies $I_G(\Gamma) \leq \frac{\rho}{2}$. In particular $I_G(\Gamma^g) \leq \frac{\rho}{2}$ for all elements $g \in K$, see Equation (7.8).

In the Archimedean case take $N = N(\Gamma)$ relying on the Zassenhaus Lemma and let $n = n(\Gamma) = \text{Lie}(N)$ be the corresponding Lie algebra. In the non-Archimedean case let $U$ be the unipotent radical of some parabolic subgroup containing $\Gamma \cap V$ relying on Theorem 6.1 and let $u = u^+$ be the corresponding Lie subalgebra. In both cases $\log(\Gamma \cap V) \subset n$, see Theorem 7.2 for the non-Archimedean case.

Let $p(G) > 0$ be the parameter fixed in the last paragraph of §6. We deduce from Proposition 6.12 that

\[
\eta_K \left( \{g \in K : \inf_{0 \neq X \in \text{Ad}(g)n} \frac{\|\text{Ad}(s)X\|}{\|X\|} \geq 2 \} \right) \geq p(G).
\]

The desired Equation (7.16) follows by combining the above Equation (7.16) with Proposition 7.5. □

8. THE KEY INEQUALITY AND ITS APPLICATIONS

We maintain the notations introduced in the previous §6 and §7 so that $G$ is a semisimple analytic group, $K$ is a compact subgroup with Haar probability measure $\eta_K$ and $s$ is the particular semisimple element fixed in the last paragraph of §6.

We will work with the Borel probability $\mu_s$ on the group $G$ given by

\[
\mu_s = \eta_K * \delta_s * \eta_K.
\]

The main goal of the current §8 is to prove the key inequality (Theorem 1.5) and derive some of its applications. For the reader’s convenience we restate the inequality using the convolution operator $A_{\mu_s}$ introduced in Equation (2.1).
**Theorem (The key inequality).** Consider the discreteness radius function $I_G: \text{Sub}_d(G) \to (0, \rho]$, given by

\begin{equation}
I_G(\Gamma) = \sup \{0 < r < \rho : \Gamma \cap \exp(B_\rho(r)) = \{\text{id}_G\}\}
\end{equation}

for any discrete subgroup $\Gamma$ of $G$. There are constants $0 < c < 1$ and $b > 0$ such that the function $I_G$ satisfies

\begin{equation}
A_\mu I_G^{-\delta} \leq c I_G^{-\delta} + b
\end{equation}

where the parameter $\delta = \delta(G)$ is as in Equation (6.20).\[\tag{8.2}\]

**Proof.** Let the constants $a_1 = 2, 0 < a_2(G) < 1$ and $0 < p(G) < 1$ be as in Equation (6.19). Moreover denote $\rho_0 = \frac{c}{2}$. Observe that every discrete subgroup $\Gamma \in \text{Sub}_d(G)$ satisfies

1. if $I_G(\Gamma) \leq \rho_0$ then $\mu_s(\{g \in G : I_G(\Gamma^g) \geq a_2 I_G(\Gamma)\}) \geq p(G)$ and
2. $I_G(\Gamma^{\rho_0}) \geq a_2(G) I_G(\Gamma)$ holds every element $g \in KsK = \text{supp}(\mu_s)$.

Statements (1) and (2) follow from Propositions 7.3 and 7.4 respectively, combined with Equation (7.8) to take into account conjugation by elements of the compact subgroup $K$. Putting all this together and applying Proposition 8.8 we immediately deduce the desired Equation (8.3). □

**Effective weak uniform discreteness.** The two main Theorems 1.2 and 1.4 appearing in the introduction follow immediately from Theorem 8.1 which we are now in a position to state and prove.

Let $\mu$ be any bi-$K$-invariant probability measure on the group $G$ whose support $\text{supp}(G)$ generates the group.

**Theorem 8.1.** There are constants $\beta, \delta > 0$ such that every $\mu$-stationary probability $G$-space $(Z, \nu)$ with $\nu$-almost everywhere discrete stabilizers satisfies

\begin{equation}
\nu(\{z \in Z : I_G(z) < \varepsilon\}) \leq \beta \varepsilon^{\delta}
\end{equation}

for all $\varepsilon > 0$. The constant $\delta$ satisfies a lower bound as in Equation (6.24).\[\tag{8.4}\]

Prior to proving Theorem 8.1 we observe that any $\mu$-stationary $G$-space $(Z, \nu)$ is also $\mu'$-stationary for any other probability measure $\mu'$ as above. In other words, the statements of Theorems 1.2, 1.3, and 8.1 are independent of the choice of $\mu$.

Indeed, the Poisson boundary of the pair $(G, \mu)$ can be identified with the homogenous space $(G/B, \eta_B)$ where $B$ is a minimal $k$-parabolic subgroup and $\nu_B$ is the unique $K$-invariant probability measure\[\tag{8.1}\] on $G/B$. From the universal property of the Poisson boundary, a measure $\nu$ on a $G$-space $Z$ is $\mu$-stationary if and only if $\nu$ is the $\eta_B$-barycentre of some measurable map from $G/B$ to the space $\text{Prob}(Z)$ of probability measures on $Z$. This condition depends only on $\eta_B$ and not on the specific choice of the measure $\mu$.

**Proof of Theorem 8.1.** Let $(Z, \nu)$ be any $\mu$-stationary probability $G$-space with $\nu$-almost surely discrete stabilizers. By the preceding discussion we may assume that $\nu$ is $\mu_s$-stationary where $\mu_s = \eta_K * \delta_s * \eta_K$ as in Equation (8.1).

Consider the probability measure $\nu_s$ on the Chabauty space of discrete subgroups $\text{Sub}_d(G)$ obtained as the push-forward\[\tag{8.1}\] of the measure $\nu$ via the stabilizer map

---

\[14\] The uniqueness of $\eta_B$ follows from the Iwasawa decomposition, i.e. the transitivity of the $K$-action on the homogenous space $G/B$.

\[15\] Varadarajan’s compact model theorem [Zim13, 2.1.19] implies that the stabilizer map is $\nu$-measurable.
As the stabilizer map is $G$-equivariant, the measure $\nu_*$ is $\mu$-stationary (as well as $\mu_*$-stationary).

To conclude the proof we apply Lemma 4.4 with respect to the function $I_G^\beta$ on the Chabauty space of discrete subgroups $\text{Sub}_d(G)$. Observe that the discreteness radius function $I_G$ is continuous on $\text{Sub}_d(G)$ with respect to the Chabauty topology [DH08]. The key equality coincides with Equation (2.2). We obtain the estimate

$$\nu_*(\{\Gamma \in \text{Sub}_d(G) : I_G^\beta(\Gamma) \geq M\}) \leq \beta M^{-1}$$

for all $M > 0$ and with $\beta = \frac{b}{1-c}$. Taking $\varepsilon = \frac{1}{M}$ and rearranging gives

$$\nu_*(\{\Gamma \in \text{Sub}_d(G) : I_G(\Gamma) \geq \varepsilon\}) \leq \beta \varepsilon^b$$

for all $\varepsilon > 0$. The desired conclusion follows since $\nu_*$ is the push-forward of $\nu$. \qed

In the remainder of the current §8 we complete the proofs of other results stated in the introduction.

**The real Lie group case.** Let $G$ be a real semisimple Lie group without compact factors. For the purpose of proving the real case of our main result (Theorem 1.2) we may replace the Lie group $G$ by the group $G^0/(G^0 \cap Z(G))$. In other words, we may assume without loss of generality that the Lie group $G$ is connected and center-free. There exists a connected $\mathbb{R}$-linear semisimple algebraic group $\mathbb{G}$ such that $G = \mathbb{G}(\mathbb{R})^0$. The algebraic group $\mathbb{G}$ is in fact defined over $\mathbb{Q}$ [Zim13, 3.1.6]. The semisimple algebraic group $\mathbb{G}$ has no $\mathbb{R}$-anisotropic factors since the Lie group $G$ has no compact factors.

The norm on $B_0(R)$ given by $g \mapsto \| \log g \|$ is bi-Lipschitz equivalent to any fixed left-invariant Riemannian (or Finsler) metric on the Lie group $G$. However note that the value of the function $I_G$ for a discrete subgroup $\Gamma$ can only be used to determine whether $\Gamma \cap B_\varepsilon \neq \{e\}$ provided that $\varepsilon < \rho$ where $\rho$ is as in §7.

We conclude that Theorem 1.2 of the introduction follows immediately from the real case of Theorem 8.1.

**The positive characteristic case.** Theorem 8.1 of the introduction is essentially stated in terms of the norm $\| \cdot \|_m$ on the $k$-analytic group $G$ defined in Equation (7.1). The Springer isomorphism $\Sigma$ is bi-Lipschitz with constant $\sigma > 1$ according to Equation (7.2). Therefore Theorem 8.1 follows from the non-Archimedean case of Theorem 8.1. Note that the value of the function $I_G$ for a discrete subgroup $\Gamma$ can only be used to determine whether $\Gamma \cap G(m^i) \neq \{e\}$ provided that $|\mathcal{O}/m|^i < \rho$.

**Discrete $\mu$-stationary random subgroups.** Let $\mu$ be a probability measure on the group $G$ such that $\text{supp}(\mu)$ generates $G$. Let $\text{DRS}_\mu(G)$ denote the space of all discrete $\mu$-stationary random subgroups of the group $G$, i.e. all $\mu$-stationary probability measures $\nu$ on $\text{Sub}(G)$ satisfying $\nu(\text{Sub}_d(G)) = 1$. We show that the space $\text{DRS}_\mu(G)$ is compact in the weak-* topology of probability measures.

**Proof of Corollary 8.7** Any weak-* limit of $\mu$-stationary random subgroups (i.e. probability measures on $\text{Sub}(G)$) is also a $\mu$-stationary random subgroup. Assume towards contradiction that $\nu_n$ is a sequence of discrete $\mu$-stationary random subgroups satisfying $\nu_n \to \nu_0$ where $\nu_0(\text{Sub}_d(G)) < 1$. In other words

$$\liminf_{r \to 0} \nu_0(\{H \in \text{Sub}(G) : H \cap B_r \neq \{\text{id}_G\}\}) > 0.$$
Since the condition of intersecting $B$ non-trivially is Chabauty open, it follows from the Portmanteau theorem that Equation (8.7) contradicts Theorem 8.1. □

**Evanescent for locally symmetric manifolds.** Recall that a Riemannian manifold of non-positive sectional curvature is called *evanescent* if its injectivity radius function vanishes at infinity. We now prove that evanescence implies finite volume for locally symmetric spaces.

**Proof of Theorem 1.7.** Let $M = K \backslash G/\Gamma$ be a locally symmetric space for some semisimple Lie group $G$ with maximal compact subgroup $K$ and a discrete torsion-free subgroup $\Gamma$ of $G$. The homogenous space $G/\Gamma$ admits a natural quotient map

$$\pi : G/\Gamma \to M, \quad \pi : g\Gamma \to Kg\Gamma \in M.$$ 

Furthermore the natural $G$-invariant measure $\sigma$ on $G/\Gamma$ pushes forward to give a Riemannian volume $\pi^* \sigma$ on the manifold $M$. Note that the locally symmetric space $M$ is evanescent if and only if the continuous function $g\Gamma \rightarrow L_G(g\Gamma g^{-1})^{-\delta}$ is proper on the homogenous space $G/\Gamma$. The desired conclusion follows from the key inequality (Theorem 1.5) combined with the argument of [Mar04, Corollary 1.5]. □

9. **Weakly cocompact lattices**

We briefly discuss the notion weakly cocompact lattices in general. We then apply our main result Theorem 1.4 towards the study of this notion.

**Weakly cocompact lattices.** Let $H$ be a locally compact group and $\Gamma$ be a lattice in the group $H$. The lattice $\Gamma$ is *weakly co-compact* if the unitary quasiregular representation of the group $H$ on the Hilbert space $L_2^0(H/\Gamma)$ admits no almost invariant vectors.

**Lemma 9.1.** Let $\mu$ be a compactly supported probability measure on the group $H$. Let $D \subset H/\Gamma$ be a compact subset. Let $P : L^2(H/\Gamma) \to L^2((H/\Gamma) \backslash D)$ be the orthogonal projection operator. If

$$(9.1) \quad \|PA_\mu P_{|L^2(H/\Gamma \backslash D)}\| < 1$$

then the lattice $\Gamma$ is weakly cocompact.

**Proof.** Denote $\hat{D} = \text{supp}(\mu)D$. Let $\hat{P}$ be the orthogonal projection operator

$$\hat{P} : L^2(H/\Gamma) \to L^2((H/\Gamma) \backslash \hat{D}).$$

It is clear that $PA_\mu \hat{P} = A_\mu \hat{P}$ and that $\|\hat{P}f\| \leq \|Pf\|$ for every vector $f \in L^2(H/\Gamma)$.

Assume towards contradiction that the lattice $\Gamma$ is not weakly cocompact in the group $H$. According to [Mar91, Lemma 1.9, §III.1] there exists a sequence $v_n \in L^2(H/F)$ of functions with $\|v_n\| = 1$, $\|A_\mu v_n\| \to 1$ and $d_n = \|\hat{P}v_n - v_n\| \to 0$. Since $A_\mu$ are $P$ are both contracting operators we obtain that

$$\|A_\mu v_n\| \leq \|A_\mu \hat{P}v_n\| + \|A_\mu (\hat{P}v_n - v_n)\| \leq \|PA_\mu P_{|L^2(H/\Gamma \backslash D)}\| + d_n \leq \|PA_\mu P_{|L^2(H/\Gamma \backslash D)}\| + d_n$$

for all $n \in \mathbb{N}$. This is a contradiction to the assumption stated in Equation (9.1). □
The thin part. Let us return to our main case of interest. Namely, let $k$ be either the field $\mathbb{R}$ or a non-Archimedean local field. Let $G$ be a connected simply-connected linear $k$-algebraic semisimple group with $k$-anisotropic factors. Denote $G = G(k)$ so that $G$ is a $k$-analytic group.

Let $\mu$ be any bi-$K$-invariant probability measure on the group $G$ whose support supp($\mu$) generates the group. Let $\nu$ be any $\mu$-stationary probability measure on the Chabauty space Sub$_d(G)$ of discrete subgroups.

The following elementary integrability result is an immediate consequence of our main result Theorem 8.1. Its proof is general and can be applied to any function satisfying the conclusion of Lemma 2.1.

**Proposition 9.2.** The function $I_G^{-\frac{\delta}{2}}$ is $\nu$-integrable.

**Proof.** Let $D : [0, 1] \to \mathbb{R}_{\geq 0}$ be the inverse cumulative distribution function corresponding to $I_G^{-\frac{\delta}{2}}$. It is defined as

$$D(t) = \inf \{ x \in \mathbb{R}_{\geq 0} : \nu(\{ \Gamma \in \text{Sub}_d(G) : I_G^{-\frac{\delta}{2}}(\Gamma) \leq x \}) \geq t \}. $$

In other words, the function $D$ satisfies

$$\nu(\{ \Gamma \in \text{Sub}_d(G) : I_G^{-\frac{\delta}{2}}(\Gamma) \leq D(t) \}) = t \quad \forall t \in [0, 1].$$

Theorem 8.1 implies

$$\nu(\{ \Gamma \in \text{Sub}_d(G) : I_G^{-\frac{\delta}{2}}(\Gamma) \leq M^{-\frac{\delta}{2}} \}) \geq 1 - \beta M$$

for some constant $\beta > 0$ and all sufficiently small $M > 0$. It follows that

$$D(1 - \beta t) \leq t^{-\frac{\delta}{2}}$$

for all sufficiently small values of $t > 0$. It follows from Fubini’s theorem applied to the “area under the graph” of the function $I_G^{-\frac{\delta}{2}}$ regarded as a subset of Sub$_d(G) \times \mathbb{R}_{\geq 0}$ that

$$\int_Z I_G^{-\frac{\delta}{2}}(\Gamma) \, d\nu(\Gamma) = \int_{[0, 1]} D(t) \, d\lambda(t)$$

where $\lambda$ is the Lebesgue measure. As the function $D$ is monotone non-decreasing and the integral $\int_0^1 x^{-\frac{\delta}{2}} \, dx$ converges, we conclude from Equation (9.6) that the function $I_G^{-\frac{\delta}{2}}$ is indeed $\nu$-integrable. \qed

A careful examination of (§6 and §7) shows that all of our arguments and proofs apply if the semisimple element $s = s(G)$ is replaced by its inverse $s(G)^{-1}$. Consider therefore the Borel probability measure

$$\hat{\mu} = \eta_K \ast \frac{1}{2}(\delta_{s(G)} + \delta_{s(G)^{-1}}) \ast \eta_K = \frac{1}{2}(\mu_s + \mu_s^\ast)$$

on the $k$-analytic group $G$. Clearly $\hat{\mu} = \mu_s^\ast$ so that the averaging operator $A_{\hat{\mu}}$ acting on the Hilbert space $L^2(Z, \nu)$ is self-adjoint.

To ease our notations let $Z = \text{Sub}_d(G)$ so that $\nu$ is a $\hat{\mu}$-stationary measure on the space $Z$. Recall that $I_G^{-\frac{\delta}{2}} \in L^1(Z, \nu)$ according to Proposition 9.2. This implies that

$$F_G = I_G^{-\frac{\delta}{2}} \in L^2(Z, \nu).$$
Therefore the lattice $\Gamma$ is weakly cocompact by Lemma 9.1.

\[ \square \]

It follows from Theorem 9.3 that $\rho$ has spectral gap. Every lattice $\Gamma$ depends only on the $k$-analytic group $G$ and is independent of the measure $\nu$.

**Corollary 9.4.** Every lattice $\Gamma$ in the group $G$ is weakly cocompact, i.e., $L^2(G/\Gamma)$ has spectral gap.

**Proof.** Let $\Gamma$ be a lattice in the $k$-analytic group $G$. Let $D = (G/\Gamma)_{>0}$ be the $\rho$-thick part of the homogenous $G$-space $G/\Gamma$. In particular $D$ is a compact subset. It follows from Theorem 9.3 that

\[ \|PA_\mu P|_{L^2((G/\Gamma)_{<\rho})}\| \leq \hat{c} < 1. \]

Therefore the lattice $\Gamma$ is weakly cocompact by Lemma 9.1. $\square$
Appendix A. Maximally compact Cartan subalgebras

Let $g_0$ be a real semisimple Lie algebra with Cartan involution $\theta$. Let $g_0 = l_0 \oplus p_0$ be the corresponding Cartan decomposition. Let $t_0$ be a maximal abelian Lie subalgebra of $l_0$. Then $h_0 = Z_{g_0}(t_0)$ is a maximally compact $\theta$-stable Cartan subalgebra of $g_0$. It satisfies $h_0 = t_0 \oplus a_0$ where $a_0 \leq p_0$ is some abelian Lie subalgebra (not necessarily maximal).

We will use $g, h, t, l$ and $p$ to denote the suitable complexifications. In particular $g$ is a complex semisimple Lie algebra with Cartan subalgebra $h$. Moreover $g = l \oplus p$. Let $\Delta = \Delta(g, h)$ be the absolute root system. The roots $\Delta$ are real valued on the real form $h_R = it_0 \oplus a_0$ of $h$. We may write $g = h \oplus \bigoplus_{\alpha \in \Delta} g_0$ where each root space satisfies $\dim_C g_0 = 1$.

The Lie algebra $l_0$ is compact. Therefore $l_0$ is reductive [Kna13, Corollary 4.25]. Denote $t_0 = Z(l_0)$ so that $l_0 = c_0 \oplus [t_0, l_0]$. Write $t_0 = c \oplus t_0'$ and let denote $c, t'$ be the complexifications of $c_0, t_0$, respectively. Then $t'$ is Cartan subalgebra of the complex semisimple Lie algebra $[t, t]$. Let $\Psi = \Psi([t, t], t')$ be the corresponding root system. We will regard each root $\beta \in \Psi$ to be defined on $t$ by extending it by zero on the center $c$. Note that the roots $\Psi$ take real values on $it_0$.

Let $\Lambda_{\text{root}}$ and $\Lambda_{\text{algebraic}}$ respectively be the root lattice (i.e. the $\mathbb{Z}$-span of $\Psi$) and the weight lattice (i.e. the set of algebraically integral forms) in $(t_0')^\ast$. We have $\Lambda_{\text{root}} \subseteq \Lambda_{\text{algebraic}}$ and $[\Lambda_{\text{algebraic}} : \Lambda_{\text{root}}]$ equals the determinant of the Cartan matrix of $\Psi$.

Let $G$ be a real Lie group with maximal compact subgroup $K$ containing a compact torus $T$ with Lie algebras $g_0, l_0$ and $t_0$, respectively. Consider the lattice $\Lambda_{\text{analytic}}$ of analytic roots in $t_0'$. It satisfies $\Lambda_{\text{root}} \subset \Lambda_{\text{analytic}} \cap (t_0')^\ast \subset \Lambda_{\text{algebraic}}$. By definition $\dim_C t_0 = \text{rank}(K)$ so that $\Lambda_{\text{analytic}} \cong \mathbb{Z}^\text{rank}(K)$.

The adjoint action of the compact torus $T$ on the complexification $g$ admits a simultaneous eigenvalue decomposition given by restricting the absolute roots $\Delta$ to $t_0$. Let $r : h^\ast \to t^\ast$ be the restriction map. For every root $\alpha \in \Delta$ we have

$$\text{Ad}(\exp(X))Y = r(\alpha)(X)Y \quad \forall X \in t_0, Y \in g_0.$$  

We conclude that $r(\alpha) \in \Lambda_{\text{analytic}}$ for every root $\alpha \in \Delta$.

Generally speaking, a root $\alpha \in \Delta$ is called real (imaginary resp.) if $\alpha(h_0)$ belongs to $\mathbb{R}$ ($i\mathbb{R}$ resp.). Otherwise $\alpha$ is called complex. In other words $\alpha$ is real if and only if $\alpha$ vanishes on $t_0$. Likewise $\alpha$ is imaginary if and only if $\alpha$ vanishes on $a_0$.

Since the Cartan subalgebra $h_0$ is maximally compact $\Delta$ admits no real roots [Kna13, Proposition 6.70]. Equivalently $r(\alpha) \neq 0$ for all $\alpha \in \Delta$. The Cartan involution $\theta$ determines an involution of $\Delta$ fixing pointwise the imaginary roots and permuting the complex roots in 2-cycles. Every imaginary root $\alpha \in \Delta$ can be classified as being either compact if $g_0 \leq l$ or non-compact if $g_0 \leq p$ (one of these possibilities must occur since $g$ is $\theta$-stable). Moreover $\Psi \subset r(\Delta)$ for every root of $\Psi$ is the restriction of some compact imaginary root.

Fix a lexicographic notion of positivity $\Delta^+$ taking $t_0$ before $a_0$. Let $\Phi^+$ be the induced notion of positivity. It follows that $r(\alpha) > 0$ for every $\alpha \in \Delta^+$ (however $r(\alpha)$ need not be a root of $\Phi$ in general, just a positive analytic form).

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16 It is not possible in general to assume that $K$ is adjoint, even if $G$ is. E.g. the real semisimple Lie group $SO(3, 4)$ is adjoint but its maximal compact subgroup $SO(3) \times SO(4)$ is not. Therefore we are forced to allow the set of analytical forms to be strictly larger than the root lattice.
Let \( \Pi^+ \subset \Delta^+ \) be the set of simple positive roots. Write

\[
\Pi^+ = \Pi^\text{complex} \cup \Pi^\text{compact} \cup \Pi^\text{non-compact}
\]

where \( \Pi^\text{complex}, \Pi^\text{compact} \) and \( \Pi^\text{non-compact} \) consist of the complex, imaginary compact and imaginary non-compact simple positive roots, respectively. Note that \( \Pi^\text{complex} \neq \emptyset \) if and only if \( \theta \) induces an non-trivial automorphism of \( \Pi^+ \). By the Borel–de Siebenthal theorem we may assume without loss of generality that the notion of positivity has been chosen so that \( |\Pi^\text{non-compact}| \leq 1 \), i.e. there is at most one imaginary non-compact root. See [BDS49] and [Kna13] Theorem 6.96.

**Proposition A.1.** The restriction of the simple positive absolute roots \( \Pi^+ \) to the maximal compact abelian Lie subalgebra \( t_0 \) spans a lattice \( \Lambda \) with \( \Lambda_{\text{root}} \leq \Lambda \leq \Lambda_{\text{analytic}} \) and \( [\Lambda_{\text{analytic}} : \Lambda] < \infty \).

Note that the second conclusion does not automatically follow from the first, for \( [\Lambda_{\text{analytic}} : \Lambda_{\text{root}}] < \infty \) only provided \( \dim \mathfrak{c}_0 = 0 \).

**Proof of Proposition A.1.** The Lie subalgebra \( \mathfrak{c}_0 = Z(t_0) \) satisfies \( \dim \mathfrak{c}_0 \leq 1 \) [Kna13] Appendix C.3. If \( \mathfrak{c}_0 \) is non-zero let \( \gamma \in \Lambda_{\text{analytic}} \) denote the generator of the infinite cyclic group of analytic integral forms on \( \mathfrak{c}_0 \).

We now consider the restrictions \( r(\alpha) = \alpha|_{t_0} \) of the various types of simple positive roots \( \alpha \in \Pi^+ \):

- Let \( \alpha \in \Pi^\text{compact} \) be a compact imaginary root. Then \( g_\alpha \leq 1 \) which means that \( r(\alpha) \in \Phi^+ \). We claim that \( r(\alpha) \) is simple. Indeed if \( r(\alpha) = \beta_1 + \beta_2 \) for some pair of positive roots \( \beta_1, \beta_2 \in \Phi^+ \) then \( \beta_i = r(\alpha_i) \) for some compact imaginary \( \alpha_i \in \Delta^+ \) so that \( \alpha = \alpha_1 + \alpha_2 \), a contradiction.

- Let \( \alpha \in \Pi^\text{complex} \) be a complex root. Then \( \beta = r(\frac{\alpha + \theta \alpha}{2}) \in \Phi^+ \). We claim that \( \beta \) is simple. Indeed if \( \beta = \beta_1 + \beta_2 \) for some pair of positive roots \( \beta_1, \beta_2 \in \Phi^+ \) then \( \beta_i = r(\alpha_i) \) for some compact imaginary \( \alpha_i \in \Delta^+ \). Then \( 2\alpha_1 + 2\alpha_2 = \alpha + \theta \alpha \), a contradiction as both \( \alpha \) and \( \theta \alpha \) are simple and not imaginary.

- Let \( \alpha \in \Pi^\text{non-compact} \) be a non-compact imaginary root.
  - Assume \( \dim \mathfrak{c}_0 = 0 \). Let \( \alpha' \in \Delta^+ \) be the smallest complex root such that \( \alpha < \alpha' \) if \( \Pi^\text{complex} \neq \emptyset \) and \( 2\alpha < \alpha' \) otherwise. Write \( \alpha' = \sum_{\lambda \in \Pi^+} n_\lambda \lambda \) so that \( n_\lambda \geq 0 \). In addition \( n_\alpha \geq 1 \) or \( n_\alpha \geq 2 \) in the first and second cases, respectively. Then \( \beta = r(\frac{\alpha' + \theta \alpha}{2}) \in \Phi^+ \) is simple [Kna13] Appendix C.3. Rearranging gives

\[
r(\alpha) = \sum_{\alpha \in \Pi^\text{complex} \cup \Pi^\text{non-compact}} \frac{1}{n_\alpha} \beta + Zr(\alpha).
\]

All the roots of \( \Phi^+ \) that appear on the right hand side are simple.

- Assume \( \dim \mathfrak{c}_0 = 1 \). In particular \( \Pi^\text{complex} = \emptyset \) so that the other simple roots of \( \Pi^+ \) are all compact imaginary [Kna13] Appendix C.3 and restrict to simple roots of \( \Phi^+ \). As \( \dim t_0 = \dim t_0 - 1 = |\Pi^+| - 1 = |\Pi^\text{compact}| \) the roots \( r(\Pi^\text{compact}) \) span \( (t_0)^* \). It follows that the restriction \( r(\alpha) \) is non-trivial on \( \mathfrak{c}_0 \).

Let \( \Lambda \leq \Lambda_{\text{analytic}} \) be the lattice spanned by \( r(\Phi^+) \). If \( \Pi^\text{non-compact} = \emptyset \) then \( r(\Phi^+) \) coincides with set of simple positive roots for \( \Phi^+ \) and so \( \Lambda = \Lambda_{\text{root}} \). In the general
case the above discussion shows that $r(\Phi^+)$ is a basis of $\Lambda$, $\Lambda_{\text{root}} \leq \Lambda \leq \Lambda_{\text{analytic}}$ and $[\Lambda_{\text{analytic}} : \Lambda] < \infty$, as required.

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