WEIGHTED BLOWUPS AND MIRROR SYMMETRY FOR TORIC SURFACES

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Abstract. This paper explores homological mirror symmetry for weighted blowups of toric varieties. It will be shown that both the A-model and B-model categories have natural semiorthogonal decompositions. An explicit equivalence of the right orthogonal categories will be shown for the case of toric surfaces.

1. Introduction

The homological mirror symmetry (HMS) conjecture was proposed by Kontsevich [12] in 1994 as an attempt to gain a deeper mathematical understanding of mirror symmetry. Since this time, many papers have confirmed various versions of HMS. This paper explores the relation between the B-model of a toric stack and the A-model of its mirror Landau-Ginzburg model which is one version of HMS. Here the B-model of the toric stack $\mathcal{X}$ gives rise to the derived category $\mathcal{D}^b(\mathcal{X})$ of coherent sheaves on the stack (or equivariant sheaves on an atlas). The mirror is given by considering the complex torus with a superpotential $W$ and constructing the derived Fukaya category $\mathcal{D}^b(Fuk((\mathbb{C}^*)^n, W))$ as in [18]. For this version of HMS, the conjecture is that these two triangulated categories are equivalent.

This version of HMS has been confirmed for smooth Fano toric surfaces [21] as well as weighted projective planes and Hirzebruch surfaces [1]. In this paper we consider the case when $\mathcal{X}$ is obtained by taking the weighted projective blowup of a point on a toric variety $X$. For the standard blowup of a point, it is well known that $\mathcal{D}^b(\mathcal{X})$ has a natural semi-orthogonal decomposition in which one piece is independent of the original variety $X$ [4]. This result will be extended to the case of weighted blowups (Theorem 2) where now the independent piece of $\mathcal{D}^b(\text{Coh}(\mathcal{X}))$ also depends on the weights.

On the mirror side, there is a natural degeneration of the potential $W_X$ mirror to $\mathcal{X}$ into two potential functions, $W_X$ and $\tilde{W}$. The first of these, $W_X$ is the potential mirror to the original variety $X$ while $\tilde{W}$ depends only on the weights in the weighted blowup. It will be shown that the derived Fukaya category of $W_X$ admits a semi-orthogonal decomposition in which the two pieces are the derived Fukaya categories of $W_X$ and $\tilde{W}$ respectively (Theorem 6). Thus the strategy employed is to show the equivalence of these two categories with those given in the decomposition of $\mathcal{D}^b(\mathcal{X})$. By itself, this is not enough to prove the HMS conjecture for these Fano toric
stacks as the interaction between the two categories in the decomposition is neglected. However, it gives strong evidence for the truth of the conjecture.

We examine the two dimensional case in detail and show an explicit equivalence between the triangulated category associated to the weighted blow up and the derived Fukaya category associated to \( \tilde{W} \) (Theorem 8). This result along with the results established for Fano toric surfaces and weighted projective planes yields a large class of toric stacks for which our strategy proves successful.

In section 2 we will define the weighted projective blowup of a smooth variety. This procedure is analogous to the usual blowup, where in this case the (reduced) exceptional divisor is a weighted projective space. We then show that the derived category of the blowup admits a semi-orthogonal decomposition. The part of this decomposition corresponding to the exceptional divisor has an exceptional collection whose quiver algebra is described explicitly.

Section 3 is completely independent of the second section. Here we address issues related to the derived Fukaya category of a Landau-Ginzburg model. After a general introduction to the Fukaya category of a Landau-Ginzburg model, we define the notion of a partial Lefschetz fibration and its associated Fukaya category. The advantage of this definition is that the derived Fukaya categories of partial Lefschetz fibrations are invariant under perturbations of the potential. We then address the case of a superpotential \( W_\Delta \) on the complex torus which is a Laurent polynomial with Newton polytope \( \Delta \). In certain situations, there is a subdivision \( \Delta = \Delta_1 \cup \Delta_2 \) satisfying appropriate conditions and one can define two associated potentials \( W_{\Delta_i} \).

Using techniques from \([10]\), we show that the derived Fukaya category associated to \( W \) then admits a semiorthogonal decomposition into the derived Fukaya category associated to the Lefschetz fibration \( W_{\Delta_1} \) and the derived Fukaya category associated to the partial Lefschetz fibration \( W_{\Delta_2} \).

In section 4 we work through the example of a Fano stack which is a weighted projective blowup of a smooth toric Fano surface. We give an explicit isomorphism between the part of the derived Fukaya category associated to one piece of the decomposed polytope as detailed in section 3 and the part of the derived category of coherent sheaves of the toric stack associated to the weighted blowup as detailed in section 2.

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2. Weighted Projective Blowups

2.1. Definition of the weighted projective blowup. We first recall the notion of weighted projective space. Let \( \mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1} \) and define an action of \( \mathbb{C}^* \) on \( \mathbb{C}^{n+1} \setminus \{0\} \) via \( \lambda \cdot (z_0, z_1, \ldots, z_n) = (\lambda^{a_0} z_0, \lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n) \).

We define \( P(\mathbf{a}) \) to be the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by this action. In general
\( \mathbb{P}(a) \) is not smooth as a scheme, so we would like to regard it as the coarse space of a stack \( \mathbb{P}(a) \). There are several equivalent definitions of a stack, however in this paper we will confine ourselves to defining a particular atlas for the stacks under consideration.

Recall that a groupoid atlas for a stack consists of two schemes \( M, O \) and morphisms \( s, t, m, i, e \). One should think of the groupoid as a category with objects as points of \( O \) and morphisms as points of \( M \). The morphisms \( s \) and \( t \) yield “source” and “target” maps \( M \rightarrow O \), \( e : M \rightarrow O \) the identity map, \( m : M \times_O M \rightarrow M \) composition and \( i : O \rightarrow M \) the inverse. To obtain an Artin stack, all of the expected diagrams should commute, \( s \) and \( t \) should be flat morphisms and \( (s, t) : M \rightarrow O \times O \) ought to be separated and quasi-compact. By a coherent sheaf on the stack \( M \rightarrow O \) we will mean a coherent sheaf \( F \) on \( O \) together with a canonical isomorphism from \( s^*F \) to \( t^*F \).

A common example of a stack is an algebraic group \( G \) acting on a scheme \( X \). In this example we let \( R = X \times G \) and \( U = X \). The morphisms are \( s(x, g) = x \) and \( t(x, g) = g \cdot x \) and composition \( m((gx, h), (x, g)) = (x, hg) \).

This stack is often denoted \( [X/G] \) and called the quotient stack. In this case, coherent sheaves on \( [X/G] \) are simply \( G \)-equivariant coherent sheaves on \( X \).

Thus, to consider \( \mathbb{P}(a) \) as a stack, we define the stack \( \mathbb{P}(a) = \mathbb{C}^{n+1}/\{0\}/\mathbb{C}^* \) where the action is defined in the first paragraph. Alternatively, one can define the graded ring \( R = \mathbb{C}[x_0, \ldots, x_n] \) where \( \deg(x_i) = a_i \). Of course, this is simply the ring of functions on \( \mathbb{C}^{n+1} \) whose grading reflects the characters of the group action. In more generality, as discussed in [1], given a graded ring \( R = \bigoplus_{i \in \mathbb{Z}} R_i \) over a field \( k \) and the ideal \( I \) generated by all elements of positive degree in \( S \) one can give the following definition.

**Definition 1.** Define the quotient stack \( \text{Proj}(R) = [\text{Spec}(R) \setminus I)/G_m] \)

Now suppose we have a smooth \((n+1)\)-dimensional scheme \( X \) with a point \( p \) as the origin in an affine chart \( V = \text{Spec}\mathbb{C}[x_0, \ldots, x_n] \subset X \). The weighted blowup of a point is a procedure which takes \((X, p, a, V)\) and produces a stack \( X \). In the case of \( a = (1, 1, \ldots, 1) \), \( X \) will be the usual blowup of the scheme at the point \( p \). Although there is a more general procedure for weighted blowups of points which do not lie in an affine chart, we will only need this case for this paper. We have the following lemma whose proof occupies the rest of this subsection.

**Lemma 1.** Given \((X, p, a, V)\) as above, the weighted projective blowup \( X \) is a smooth stack which fits in a fiber square

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{i} & X \\
\downarrow & & \downarrow\pi \\
\{p\} & \hookrightarrow & X \\
\end{array}
\]

where \( \mathcal{E} \) is the exceptional divisor of the stack and \( \mathcal{E}_{\text{red}} \simeq \mathbb{P}(a) \). Furthermore, \( \pi : X \setminus \mathcal{E} \rightarrow X \setminus p \) is an equivalence of schemes.
Let \( \tilde{V} = (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \) and \( \tilde{W} = (X \setminus \mathcal{O}) \times \mathbb{C}^* \); the atlas \( \mathcal{X} \) will be a quotient stack with \( M = (\tilde{W} \cup \tilde{V}) \times \mathbb{C}^* \) and \( O = \tilde{W} \cup \tilde{V} \). The action of \( \mathbb{C}^* \) on \( \tilde{W} \) will be multiplication on the second factor. For \( \tilde{V} \) we define
\[
\lambda \cdot (x_0, \ldots, x_n, y) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n, \lambda^{-1} y)
\]
The intersection of \( \tilde{V} \) and \( \tilde{W} \) is \( (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}^* \) with group action of \( \mathbb{C}^* \) multiplication on the second factor. The equivariant inclusion maps are
\[
j_{\tilde{V}} : (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}^* \to \tilde{V}
\]
\[
j_{\tilde{W}} : (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}^* \to \tilde{W}
\]
The inclusion map \( j_{\tilde{W}} \) is just the inclusion of \( (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^* \) in \( \tilde{W} \), while \( j_{\tilde{V}}(s_0, \ldots, s_n, t) = (t^{a_0} s_0, \ldots, t^{a_n} s_n, t^{-1}) \). It is easily checked that \( j_{\tilde{V}} \) is equivariant with respect to the \( \mathbb{C}^* \) action.

**Definition 2.** Let \( U \) be the equivariant pushout:
\[
\begin{array}{ccc}
(C^{n+1} \setminus \{0\}) \times \mathbb{C}^* & \xrightarrow{j_{\tilde{V}}} & \tilde{V} \\
\downarrow j_{\tilde{W}} & & \downarrow \\
\tilde{W} & \subseteq & U
\end{array}
\]

and define \( \mathcal{X} = [U/\mathbb{C}^*] \).

As in the case of weighted projective spaces, one can view \( \tilde{V} \) as \( \mathbb{P} \text{Proj}(S) \) where \( S \) is the graded ring \( \mathbb{C}[z_0, \ldots, z_n, y] \) with \( \deg(z_i) = a_i \) and \( \deg(y) = -1 \). One sees that the irrelevant ideal is \( I = (z_0, \ldots, z_n) = S \cdot S_+ \) so that \( \text{Spec}(S) \setminus I = (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \) which is just \( \tilde{V} \). To define the map \( \pi \) we simply take the quotient map on \( \tilde{W} \) while on \( \tilde{V} \) we take the map induced by the homomorphism of graded rings \( \rho : \mathbb{C}[t_0, \ldots, t_n] \to \mathbb{C}[z_0, \ldots, z_n, y] \) which sends \( t_i \) to \( z_i \cdot y^{a_i} \).

Thus the exceptional divisor \( \mathcal{E} \) has ideal sheaf \( J = (z_0 y^{a_0}, \ldots, z_n y^{a_n}) \) and structure sheaf \( \mathcal{O}_\mathcal{E} = S/J \). If \( b = \max\{a_0, \ldots, a_n\} \) then the ideal \( (y^b) \) clearly satisfies \( I \cdot (y^b) \subseteq J \) where \( I \) is the irrelevant ideal of \( S \). So on \( \mathcal{E} \), the ideal sheaf \( (y^b) \) is supported on \( V(I) \cap \mathcal{E} \) and therefore torsion. Thus,
\[
\mathcal{E} = \mathbb{P} \text{Proj} \left( \frac{S}{J + (y^b)} \right)
\]
From this we see that \( y \in \text{Rad}(S/[J + (y^b)]) \) and
\[
\mathcal{E}_{\text{red}} = \mathbb{P} \text{Proj} \left( \frac{S}{(y)} \right) = \mathbb{P}(a)
\]
Which yields the expected diagram (1). We also have the map \( i : \mathbb{P}(a) = \mathcal{E}_{\text{red}} \to \mathcal{E} \to \tilde{V} \) which is induced by the homogeneous quotient map of graded rings \( \psi : S \to \frac{S}{(y)} \).
2.2. Weighted projective blowups of toric varieties. In this subsection we will see that the above definition has a simple expression when $X$ is a smooth toric variety. Useful references on the essentials of toric varieties are [9] and [10]; we will freely use results from [9] as well.

We recall some standard notation. Let $N \simeq \mathbb{Z}^{n+1}$ be a lattice, $M = \text{Hom}(N, \mathbb{Z})$, $N_\mathbb{R} := N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$. Let $\Delta$ be a fan with cones $\sigma \subset N_\mathbb{R}$. We will let $\Delta(k)$ be the set of $k$-dimensional cones in $\Delta$. A smooth toric variety $X_\Delta$ results from a fan $\Delta$ such that every $\sigma \in \Delta(n+1)$ can be represented as $\sigma = \mathbb{R}_{\geq 0} v_0 + \cdots + \mathbb{R}_{\geq 0} v_n$ where $\{v_0, \ldots, v_n\}$ generates $N$. For this subsection we will choose such a $\sigma \in \Delta(n+1)$. This yields an affine chart $U_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M] \simeq \text{Spec} \mathbb{C}[v_0^\vee, \ldots, v_n^\vee] \simeq \mathbb{A}^{n+1}$. Let $i : \mathbb{A}^{n+1} \hookrightarrow X_\Delta$ be the inclusion. If $p = i(0)$ then from the correspondence between orbit closures and cones, we have $X_\Delta \setminus p = X_{\Delta \setminus \sigma}$. Given $a = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ let $u = \sum_{i=0}^n a_i v_i$. For any proper subset $J \subset \{0, 1, \ldots, n\}$ let

$$
\tau_J = \mathbb{R}_{\geq 0} \cdot u + \sum_{i \in J} \mathbb{R}_{\geq 0} \cdot v_i
$$

We define a new fan in the expected manner

$$
\Delta(\sigma, a) = (\Delta \setminus \sigma) \cup (\cup_J \tau_J)
$$

Following the construction in subsection 1.1, we let $V = U_\sigma$ and $p = i(0)$. Let $\Sigma = \{\eta \in \Delta(\sigma, a)|\eta \subset \sigma\}$. Then, as $X_{\Delta(\sigma, a) \setminus \cup_J \tau_J} = X_{\Delta \setminus \sigma}$ we need only examine $X_\Sigma$ and the map induced by the identity $(N, \Sigma) \to (N, \sigma)$. In order for the fan $\Sigma$ to yield a stack as opposed to a singular scheme, one must regard $u$ as the element generating the one dimensional cone $\{\mathbb{R}_{\geq 0} \cdot u\} \subset \Sigma(1)$. For example, if $n = 1$ and $a = (2, 2)$ we consider $u$ to be the generating element of $\mathbb{R}_{\geq 0} \cdot u$ (as opposed to $v_1 + v_2$). We then define the map $\alpha : \mathbb{Z}^{\Sigma(1)} \to N$ as $\alpha(e_i) = v_i$ for $0 \leq i \leq n$ and $\alpha(e_{n+1}) = u$ where $\{e_0, \ldots, e_{n+1}\}$ is the standard basis. From this we obtain the exact sequence:

$$
0 \to M \xrightarrow{\alpha^\vee} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\beta} A_*(X_\Sigma) \to 0
$$

where, using the standard basis, we have identified $\mathbb{Z}^{\Sigma(1)}$ with it’s dual.

As in [9], for each basis element $e_i \in \mathbb{Z}^{\Sigma(1)}$, introduce the variable $z_i$ and consider the graded ring

$$
S = \mathbb{C}[z_0, \ldots, z_n, z_{n+1}]
$$

with the grading induced by $\deg(z_i) = \beta(e_i)$. A quick check shows that $A_*(X_\Sigma) \simeq \mathbb{Z}$ and that after a choice of generator, we have $\beta(e_i) = a_i$ for $0 \leq i \leq n$ and $\beta(e_{n+1}) = -1$. In the general case of a fan $F$, irrelevant ideal is the ideal generated by all monomials $z_{\rho(1)} \cdots z_{\rho(k)}$ such that $\{\rho(1), \ldots, \rho(k)\} = F(1) \setminus \sigma(1)$ with $\sigma \in F(n)$. Thus, in our situation the irrelevant ideal is $I = (z_0, \ldots, z_n) = S \cdot S_+$ so as a stack we have $X_\Sigma = [\text{Spec}(S) \setminus I, \mathbb{C}^*]$.

We recognize $\text{Spec}(S) \setminus I$ with the action of $\mathbb{C}^*$ given by the grading as $\tilde{V}$. Likewise we have $V = U_\sigma = \text{Spec}(\mathbb{C}[x_0, \ldots, x_n])$ where we have introduced the variable $x_i$ for $v_i^\vee$. The map $\tilde{V} \to V$ is induced by the identity $(N, \Sigma) \to (N, \sigma)$ and will be given by the map of homogeneous coordinate rings $\rho :$

\[\text{Spec}(\mathbb{C}[x_1]) \to \text{Spec}(\mathbb{C}[x_0, \ldots, x_n])\]
\( \mathbb{C}[x_0, \ldots, x_n] \to S \) where \( \rho(x_i) = z^{\alpha^i} = z_i^{a_i} \) as can be seen from examining the obvious diagram:

\[
\begin{array}{c}
0 \to M \xrightarrow{=} \mathbb{Z}^{\sigma(1)} \to 0 \\
\| \downarrow \alpha^v \\
0 \to M \xrightarrow{a^v} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\beta} A_n(X_{\Sigma}) \to 0
\end{array}
\]

For a more general procedure of obtaining stacks from fans, see [5].

### 2.3. Sheaves on weighted projective blowups.

In this section we will extend some of the results from [1] and [4] to obtain a semi-orthogonal decomposition of the derived category of coherent sheaves on \( X \) in terms of the derived categories of coherent sheaves on \( X \) and \( \mathbb{P}(a) \). There will be an explicit mirror decomposition on the symplectic side in section 3. We start with a theorem whose proof is given in either [1] or [6]. Recall that \( \tilde{V} = \text{Proj}(S) \) with \( S = \mathbb{C}[z_0, \ldots, z_n, y] \) and gradings and irrelevant ideal \( I \) given in section 2.1. Define \( gr(S) (gr_f(S)) \) to be the category of (finitely generated) graded modules over \( S \) and \( Tor(S) (Tor_f(S)) \) to be the full subcategory of modules \( M \) such that there exists a \( k \in \mathbb{Z}_+ \) with \( I^k M = 0 \). One sees that \( Tor(S) \) is a Serre subcategory of \( gr(S) \).

**Theorem 1.** The category of quasi-coherent (coherent) sheaves on \( \tilde{V} \) is equivalent to the categorical quotient of \( gr(S) (gr_f(S)) \) by \( Tor(S) \).

The same theorem applies if one replaces the ring \( S \) by \( R \) defined in section 1.1 and \( \tilde{V} \) by \( \mathbb{P}(a) \) (see [1]). We recall some homological properties of sheaves over \( \mathbb{P}(a) \). Given a graded module \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) we let \( M(k) \) be the graded module with \( M(k)_i = M_{k+i} \). Now, given \( a = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \) let \( l = \sum_{i=0}^n a_i \). We recall from [1] the following result on the cohomology of the sheaf \( \mathcal{O}(k) \) on \( \mathbb{P}(a) \) obtained from the graded \( R \)-module \( R(k) \).

**Proposition 1.** There are isomorphisms

\[
H^p(\mathbb{P}(a), \mathcal{O}(k)) \simeq \begin{cases} 
R_k & \text{for } p = 0, k \geq 0 \\
R^*_k & \text{for } p = n, k \leq -l \\
0 & \text{otherwise}
\end{cases}
\]

Our main concern in this section is with the derived category of coherent sheaves on \( X \). Given an Artin stack \( \mathcal{Y} \) we will denote the bounded derived category of coherent sheaves on \( \mathcal{Y} \) by \( D^b(\mathcal{Y}) \). We recall that this is the triangulated category obtained from the abelian category \( \text{Coh}(\mathcal{Y}) \) by localizing all quasi-isomorphisms. Given a proper (flat) map \( f \) between two stacks, we will write \( f_* \) (\( f^* \)) for the derived functors \( Rf_* \) and \( Lf^* \). Recall that if \( A^i \) is an object in \( D^b(\mathcal{Y}) \) then \( A[n]^i \) is the translated object \( A[n]^i = A^{i+n} \).

From diagram (1) we have a map \( i \colon \mathbb{P}(a) \to X \). Utilizing Theorem 1, we prove

**Proposition 2.** Given any \( j \) and \( k \), there are isomorphisms

\[
\text{Ext}^*(i_* \mathcal{O}(k), i_* \mathcal{O}(j)) \simeq \text{Ext}^*(\mathcal{O}(k), \mathcal{O}(j)) \oplus \text{Ext}^*(\mathcal{O}(k+1), \mathcal{O}(j))[-1]
\]
Proof. Since the image of \( i \) is contained in \( \tilde{V} \), it suffices to work locally with \( i : \mathbb{P}(a) \to \tilde{V} \) to compute the \( Ext \) groups of the pushforwards of \( O(p) \). Recall that the map \( i : \mathbb{P}(a) \to \tilde{V} \) is induced by the the homomorphism \( \psi : S \to R \) with kernel \( (y) \). Observe that, when regarded as a module over \( S \), the ideal \( (y) \) is isomorphic to \( S(1) \). Thus there is the exact sequence of \( S \)-modules

\[
0 \to S(1) \xrightarrow{m_y} S \to R \to 0
\]

where \( m_y \) is multiplication by \( y \). For arbitrary \( k \) we of course have

\[
0 \to S(k + 1) \xrightarrow{m_y} S(k) \to R(k) \to 0
\]

Now, as an \( S \)-module, \( R(k) \) is the module corresponding to \( i_*(O(k)) \) and \( i^* \circ i_*(O(k)) \) corresponds to the module \( R \otimes_S R(k) \). The first two modules in the sequence above are free and therefore acyclic for the functor \( R \otimes_S - \). Thus in \( gr(R) \) there is resolution of \( R \otimes_S R(k) \):

\[
\begin{array}{cccccccc}
0 & \to & R \otimes_S S(k+1) & \xrightarrow{id \otimes m_y} & R \otimes_S S(k) & \to & R \otimes_S R(k) & \to & 0 \\
& & \| & & \| & & \| & & \\
0 & \to & R(k+1) & \xrightarrow{0} & R(k) & \to & R \otimes_S R(k) & \to & 0 \\
\end{array}
\]

So one obtains \( R \otimes^L_S R(k) \simeq R(k) \oplus R(k+1)[1] \) in the bounded derived category \( \mathcal{D}^b(gr(R)) \). Thus we have the equivalence

\[
i^* \circ i_*(O(k)) \simeq O(k) \oplus O(k+1)[1]
\]

in \( \mathcal{D}^b(\mathbb{P}(a)) \). Using this and the fact that \( i_* \) is right adjoint to \( i^* \) yields the result. \( \square \)

We will need the following explicit corollary.

**Corollary 1.** Given any \( j \) and \( k \) such that \( 0 \leq k - j \leq l - 2 \) there is an natural isomorphism compatible with composition:

\[
Ext^*(i_*(O(j)), i_*(O(k))) \simeq R_{k-j} \oplus R_{k-j-1}[-1]
\]

Where \( R_{j-k} \) is placed in degree zero and \( R_{k-j-1}[-1] \) is placed in degree one. Furthermore, there is a natural \( \mathbb{C}^* \)-action which is also compatible with composition.

**Proof.** The isomorphism is immediate from the previous propositions. To see that it is compatible with composition we return to the proof of Proposition 2. As modules we have \( i_*(O(j)) \) is equivalent to \( R(j) \) which is quasi-isomorphic to the complex \( 0 \to S(j+1) \to S(j) \to 0 \) and likewise for \( i_*(O(k)) \). Following the proof of Proposition 2 we see that any monomial in \( S_{k-j} \) which does not lie in the ideal \( (y) \) gives rise to a degree zero chain map. Such monomials correspond naturally to the \( R_{k-j} \) summand in the stated isomorphism. Similarly, the degree one chain maps correspond to multiplication by monomials in \( S_{k-j-1} \) not in \( (y) \), or \( R_{k-j-1} \). It is clear that composing two such chain maps corresponds to multiplication of two such monomials (unless, of course, both are of degree 1, in which case the product is zero). The \( \mathbb{C}^* \)-action is induced by the \( \mathbb{C}^* \)-action on \( R \). \( \square \)
For basic definitions on triangulated categories, see [4]. We will denote the full triangulated category in \( \mathcal{D}^b(\mathcal{X}) \) generated by the object \( i_*(\mathcal{O}(k)) \) as \( \mathcal{T}_k \). Then we have the following proposition.

**Proposition 3.** The sequence \( \langle \pi^*(\mathcal{D}^b(\mathcal{X})), \mathcal{T}_0, \ldots, \mathcal{T}_{l-2} \rangle \) is an exceptional collection of triangulated subcategories in \( \mathcal{D}^b(\mathcal{X}) \).

**Proof.** One can use standard methods to show that these are all admissible subcategories of \( \mathcal{D}^b(\mathcal{X}) \) (see [4]). To see that \( \langle \mathcal{T}_0, \ldots, \mathcal{T}_{l-2} \rangle \) forms an exceptional collection we use Propositions 1 and 2. We note that it suffices to check that \( \text{Ext}^\bullet(i_*(\mathcal{O}(j)), i_*(\mathcal{O}(k))) = 0 \) for \( 0 \leq k < j \leq l - 2 \), for this will imply that the categories generated by these objects are semi-orthogonal.

By Proposition 2 and 1 we have

\[
\begin{align*}
\text{Ext}^\bullet(i_*(\mathcal{O}(j)), i_*(\mathcal{O}(k))) &= \text{Ext}^\bullet(\mathcal{O}(j), \mathcal{O}(k)) \oplus \text{Ext}^\bullet(\mathcal{O}(j+1), \mathcal{O}(k))[-1] \\
&= \text{Ext}^\bullet(\mathcal{O}, \mathcal{O}(k-j)) \oplus \text{Ext}^\bullet(\mathcal{O}, \mathcal{O}(k-j-1))[-1] \\
&= H^\bullet(\mathbb{P}(a), \mathcal{O}(k-j)) \oplus H^\bullet(\mathbb{P}(a), \mathcal{O}(k-j-1))
\end{align*}
\]

Since \( 0 \leq k < j \leq l - 2 \) we have \( 2 - l \leq k - j < 0 \) and \( 1 - l \leq k - j - 1 < -1 \). Applying Proposition 1 to each of these cases we see that the right hand side of the above equation is zero.

Thus we need only show that \( \langle \pi^*(\mathcal{D}^b(\mathcal{X})), \mathcal{T}_k \rangle \) is a semi-orthogonal sequence for \( 0 \leq k \leq l - 2 \). Again it suffices to check this on a sheaf \( \mathcal{F} \in \text{Coh}(\mathcal{X}) \) and \( \mathcal{O}(k) \), i.e. we need to check that

\[
\text{Ext}^\bullet(i_*(\mathcal{O}(k)), \pi^*(\mathcal{F})) = 0
\]

To do this we apply Serre Duality and the adjunction formula. Recall from [4] that the canonical sheaf of the stack \( \mathbb{P}(a) \) is \( \mathcal{O}(-1) \). Now, \( i : \mathbb{P}(a) \to \mathcal{X} \) is a closed embedding, and letting \( Y = i(\mathbb{P}(a)) \), we have that the sheaf \( \mathcal{O}(\mathcal{Y}) \) on \( \mathcal{Y} \) is equivalent to the module \( S(-1) \) (as \( (y) = S(1) \) is the ideal sheaf of the image). This implies \( i^*(\mathcal{O}([\mathcal{Y}])) = \mathcal{O}(-1) \). This fact and the adjunction formula yields

\[
\begin{align*}
i^*(\mathcal{O}(\mathcal{Y})) &= i^*(\mathcal{O}(\mathcal{Y})) \otimes \mathcal{O}(1) \\
&= i^*(\mathcal{O}(\mathcal{Y})) \otimes i^*(\mathcal{O}([\mathcal{Y}])) \otimes \mathcal{O}(1) \\
&= i^*(\mathcal{O}(\mathcal{Y})) \otimes \mathcal{O}(1) \\
&= \omega_{\mathbb{P}(a)} \otimes \mathcal{O}(1) \\
&= \mathcal{O}(1 - l)
\end{align*}
\]

Now we apply this with Serre Duality to see

\[
\begin{align*}
\text{Ext}^\bullet(i_*(\mathcal{O}(k)), \pi^*(\mathcal{F})) &= \text{Ext}^\bullet(\pi^*(\mathcal{F}), \omega_{\mathcal{X}} \otimes i_*(\mathcal{O}(k)))^\vee \\
&= \text{Ext}^\bullet(\pi^*(\mathcal{F}), i_*(\mathcal{O}(1 - l + k)))^\vee \\
&= \text{Ext}^\bullet(\mathcal{F}, (\pi \circ i)_*(\mathcal{O}(1 - l + k)))^\vee \\
&= 0
\end{align*}
\]

Indeed, to verify the last line of the equation observe that \( \pi \circ i = \text{inc} \circ \rho \) where \( \rho \) is projection to a point and \( \text{inc} \) is the inclusion of the point to \( p \in \mathcal{X} \). Thus \((\pi \circ i)_* \) factors through the derived global section functor
decomposition of $D$

To verify this claim let

By Proposition 3 we need only check that the collection is

Theorem 2. The sequence $\langle \pi^*(D^b(X)), \mathcal{T}_0, \ldots, \mathcal{T}_{l-2} \rangle$ gives a semi-orthogonal decomposition of $D^b(X)$.

Proof. By Proposition 3 we need only check that the collection is complete.

Claim 1: If $i_*(D^b(\mathbb{P}(a)))$ is contained in $\mathcal{T}$ then $\mathcal{T} = D^b(X)$.

To verify this claim let $Se$ be the Serre functor and suppose $A \in \perp \mathcal{T}$. Then for all $B \in D^b(X)$ we have $Hom(\pi^*(B), Se(A))^\vee = Hom(A, \pi^*(B)) = 0$ which implies $\pi_*(Se(A)) = 0$. Thus $Se(A)$ and therefore $A$ has support on $Y = i(\mathbb{P}(a))$. If $A \neq 0$ then the identity morphism implies $0 \neq Hom(i^*(A), i^*(A)) = Hom(A, i_*(i^*(A)))$ which implies $A \notin \perp \mathcal{T}$ contradicting our assumption. Thus $A = 0$ and $\perp \mathcal{T} = 0$. Since $\mathcal{T}$ is admissible, we have that $\mathcal{T} = D^b(X)$.

To complete the proof we need to establish that $i_*(D^b(\mathbb{P}(a)))$ is a subcategory of $\mathcal{T}$. For this recall from [1] that $\langle \mathcal{O}, \ldots, \mathcal{O}(l-1) \rangle$ is a full exceptional collection for $\mathbb{P}(a)$. So if $i_*(\mathcal{O}(l-1)) \in \mathcal{T}$ then $\langle i_*(\mathcal{O}), \ldots, i_*(\mathcal{O}(l-1)) \rangle = i_*(\langle \mathcal{O}, \ldots, \mathcal{O}(l-1) \rangle) = i_*(D^b(\mathbb{P}(a)))$ is contained in $\mathcal{T}$. Thus we need only show

Claim 2: $i_*(\mathcal{O}(l-1)) \in \mathcal{T}$.

Let us outline the proof of the claim. For each $0 \leq m \leq l$ we will introduce an object $K^*_m \in D^b(X)$ as well as a distinguished triangle $K^*_m \to K^*_{m-1} \to L^*_{m-1}$ where $L^*_{m-1}$ is an object of $\mathcal{T}$ for $1 \leq m < l$, $K^*_0 \in \pi^*(D^b(X))$, $K^*_l = 0$ and $L^*_l$ is a direct sum of objects in $\mathcal{T}$ and shift of $i_*(\mathcal{O}(l-1))$. An easy induction argument then shows that $L^*_l$ is in $\mathcal{T}$ and therefore $i_*(\mathcal{O}(l-1)) \in \mathcal{T}$. We will now fill in the details.

We require some notation to verify this claim. Let $0 \leq m \leq l$ and $c_m = (c_{m,0}, c_{m,1}, \ldots, c_{m,n})$ be the unique element in $\mathbb{Z}_{\geq 0}^{n+1}$ satisfying

\begin{align*}
(1) & \quad \Sigma_{j=0}^n c_{m,j} = m \\
(2) & \quad \Sigma_{j=0}^k c_{m,j} = \Sigma_{j=0}^k a_j \text{ if } \Sigma_{j=0}^k a_j \leq m \\
(3) & \quad c_{m,k} = 0 \text{ if } \Sigma_{j=0}^{k-1} a_j \geq m
\end{align*}

Alternatively, one can see that $c_m$ is the $m$-th term in the lexicographically ordered sequence in $\mathbb{Z}_{\geq 0}^{n+1}$ with each $j$-coordinate between 0 and $a_j$ and starting at $(0, \ldots, 0)$.
By Theorem 2, we can regard sheaves over $\tilde{V}$ as graded modules over $S = \mathbb{C}[z_0, \ldots, z_n, y]$. For $0 \leq m \leq l$, let

$$N_m = \bigoplus_{j=0}^{n} S(c_{m,j})$$

and define the degree zero element

$$s_m = \oplus_{j=0}^{n} y_j^{a_j-c_{m,j}} \in N_m$$

We now define the complex $K^\bullet_m$ to be the Koszul complex,

$$0 \to S^{\wedge s_m} N_m^{\wedge s_m} \wedge^2 N_m \to \cdots \wedge^m x^{n+1} N_m \to 0$$

One should note that $s_m$ does not generally come from a regular sequence for a given $m$. Following the above outline, we would like to define a chain map $f^\bullet_m : K^\bullet_m \to K^\bullet_{m-1}$. Observe that $c_{m-1} - c_m - 1 = (0, \ldots, 0, 1, 0, \ldots, 0)$ where $1$ is in the $i$-th coordinate for some $i$. We can define $\tilde{f}_m : N_m \to N_{m-1}$ by $\left(\oplus_{j \in J} i_d\right) \oplus m_y$ where $m_y$ is multiplication by $y$ in the $i$-th summand. One easily shows that $\tilde{f}_m(s_m) = s_{m-1}$ which implies that $\tilde{f}_m$ induces an injective chain map $f^\bullet_m : K^\bullet_m \to K^\bullet_{m-1}$ (by injective we will mean that $f^\bullet_m$ is injective for every $j$). By the proof of Proposition 2, we see that $\text{coker}(\tilde{f}_m) = R(c_{m-1}, i)$ and extrapolating this to the chain complex one can show

$$\text{coker}(f^\bullet_m) \cong \bigoplus_{i \in J, |J|=k} R(\Sigma_{j \in J} c_{m-1,j}) =: L^\bullet_{m-1}$$

where the direct sum is taken over all $J \subset \{0, \ldots, n\}$. As $\tilde{f}_m(s_m) = s_{m-1}$, the differential in the cokernel $L^\bullet_{m-1}$ must be zero. Thus, on the level of chain complexes, we have an exact sequence

$$0 \to K^\bullet_m \to K^\bullet_{m-1} \to L^\bullet_{m-1} \to 0$$

where $L^\bullet_{m-1}$ is a direct sum of the modules given above plus shifts. In the derived category, this gives us a distinguished triangle, $K^\bullet_m \to K^\bullet_{m-1} \to L^\bullet_{m-1}$. Now, for $1 \leq m < l$ and any subset $J \subset \{0, \ldots, n\}$, condition (1) implies that

$$0 \leq \Sigma_{j \in J} c_{m-1,j} \leq m - 1 < l - 1$$

Therefore, for $1 \leq m < l$, all of the summands in $L^\bullet_{m-1}$ are equal to $R(k)$ for $0 \leq k \leq l - 2$. But these modules are equivalent to the $i_*(\mathcal{O}(k))$ which appear in our exceptional collection. Thus $L^\bullet_{m-1} \in \mathcal{T}$ for all $1 \leq m < l$. For $m = l$ we have $c_{m-1} = (a_0, \ldots, a_n - 1)$ and we observe that for all subsets $J \neq \{0, \ldots, n\}$, the sum $\Sigma_{j \in J} c_{m-1,j} < l - 1$ implying that for each such $J$, the corresponding summand in $L^\bullet_{l-1}$ is contained in $\mathcal{T}$. On the other hand, for the summand corresponding to $J = \{0, \ldots, n\}$ we have $\Sigma_{j \in J} c_{m-1,j} = l - 1$.

Thus $L^\bullet_{l-1}$ is a direct sum of an object in $\mathcal{T}$ and a shift of $i_*(\mathcal{O}(l-1))$ as was desired.
Finally, we must examine the objects, $K_0^*$ and $K_1^*$. For $K_0^*$ observe that $c_0 = (0, \ldots, 0)$ so that $N_0 = S^{n+1}$. Also, the element $s_0 = (z_0 y^{a_0}, \ldots, z_n y^{a_n})$. Recall that the map $\pi : \bar{V} \to \mathbb{C}^{n+1}$ is induced by the ring homomorphism $\rho : \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[z_0, \ldots, z_n, y]$ which sends $x_i$ to $z_i y^{a_i}$. Thus $K_0^*$ is the pullback of the Koszul complex over $\mathbb{C}[x_0, \ldots, x_n]$ generated by the regular sequence $\langle x_0, \ldots, x_n \rangle$. One thus identifies $K_0^*$ as the pullback via $\pi$ of the skyscraper sheaf on $\mathbb{C}^{n+1} \subset X$ at zero. In particular $K_0^* \in \mathcal{T}$.

To see that $K_1^* = 0$ we observe that $c_1 = (a_0, \ldots, a_n)$ and $s_1 = (z_0, \ldots, z_n)$. This shows that $K_1^*$ is the Koszul complex associated to the regular sequence $\langle z_0, \ldots, z_n \rangle$. But this is just the regular sequence of the irrelevant ideal $I$ in $S$ so, modulo torsion, the complex $K_1^*$ is exact and thus equivalent to zero in the derived category.

The induction argument then goes as follows. For $m = 0$, we have seen that $K_1^* \in \mathcal{T}$. Assume $K_m^* \in \mathcal{T}$ for $0 \leq m < l - 1$, then this object fits into the distinguished triangle $K_{m+1}^* \to K_m^* \to L_m^*$ for all such $m$, we have seen that $L_m^* \in \mathcal{T}$ implying that $K_{m+1}^* \in \mathcal{T}$. By induction, we have $K_{l-1}^* \in \mathcal{T}$ and from the above observations $0 = K_l^* \in \mathcal{T}$. The distinguished triangle $K_{l-1}^* \to K_l^* \to L_{l-1}^*$ implies $L_{l-1}^* \in \mathcal{T}$. But as was observed, $L_{l-1}^* \simeq A \oplus i_* (O(l-1))[-(n+1)]$ where $A \in \mathcal{T}$. Thus the distinguished triangle, $A[n+1] \to L_{l-1}^*[n+1] \to i_* (O(l-1))$ shows that $i_* (O(l-1)) \in \mathcal{T}$ verifying the second claim and proving the theorem.

\qed

2.4. Koszul Duality for Weighted Projective Blowups. For the purpose of mirror symmetry, one often finds that the natural exceptional collection to use is the Koszul dual of one formed from line bundles (for examples see [1], [17]). The same is true in the case of the weighted projective blowup. We will need to calculate the $Ext$ groups and their compositions of the dual collection in order to see that the mirror derived directed Fukaya category forms an equivalent category. For more on Koszul duality, see [2].

Given any exceptional collection $E = \langle E_1, \ldots, E_r \rangle$ generating the derived category $\mathcal{T}$, one can regard the dual collection in the following way. Homological perturbation theory asserts that $\mathcal{T}$ is equivalent to the bounded derived category of graded right modules of the quiver algebra $Q_E$ associated to $E$ if $Q_E$ is formal as a dga (see, for example, [20], [11]). Recall

$$Q_E = \bigoplus_{1 \leq i \leq j \leq r} Ext^\bullet(E_i, E_j)$$

The exceptional collection in $D^b(gr Mod_{\mathbb{C}}(Q_E))$ corresponding to $E$ is simply the collection of projective objects $P_i = e_i \cdot Q_E$ where $e_i$ is the identity morphism in $Ext^\bullet(E_i, E_i)$. In this situation, the Koszul dual of this collection consists of the simple objects $S_i = e_i \cdot Q_E \cdot e_i$ (neglecting shifts). The collection $E' = \langle S_1, \ldots, S_l \rangle$ is full and exceptional for the same category and yields a dual quiver algebra $Q_{E'}$. It is this algebra which we will compute for
the exceptional collection $E = \langle i_*(\mathcal{O}(0)), \ldots, i_*(\mathcal{O}(l-2)) \rangle$ corresponding to the piece of $\mathcal{D}^b(\mathcal{X})$ from the weighted blowup $\mathbb{P}(a)$.

Corollary 1 gives the structure of $Q_E$ as a dga with zero differential. In order to apply duality in the straightforward way outlined above, it must be shown that $Q_E$ is formal. i.e. $Q_E$ carries an $A^\infty$-structure, unique up to quasi-isomorphism, which comes from the underlying dga via the Homological Perturbation Lemma. This structure is quasi-isomorphic to one in which higher products vanish if $HH^k(Q_E,Q_E)$ has no elements of degree $2 - k$ for $k > 2$ \cite{20}. In this case, $Q_E$ is intrinsically formal and we can pursue the above approach to Koszul duality.

In fact, we can restrict attention to an equivariant version of the Homological Perturbation Lemma. As was shown in Corollary 1, $Q_E$ has a natural $\mathbb{C}^*$-action, i.e. the morphisms in $Ext^*(i_*(\mathcal{O}(i)),i_*(\mathcal{O}(j))) = R_k = R_{k-j} \oplus R_{k-j+1}[-1]$ have weight in $\mathbb{Z}$ associated with the weights in $R = \mathbb{C}[x_0,\ldots,x_n]$ and composition respects these weights (we will use the term weight to distinguish from the degree given by the grading). It follows the $A^\infty$-structure must also respect this action and the intrinsic formality argument adapts using equivariant Hochschild cohomology. Furthermore, we take Hochschild cohomology over the semi-simple base $\oplus_{i=0}^{\infty} \mathbb{C} : 1_{\mathcal{O}(i)} = T$.

**Proposition 4.** The equivariant version of $HH^k(Q_E,Q_E)$ over $T$ is zero in degree $2 - k$ for $k > 2$.

**Proof.** Suppose $f : Q_E \otimes \cdots \otimes Q_E \rightarrow Q_E$ is a non-zero cocycle representing an element of degree $d$ in $HH^k(Q_E,Q_E)$. For $1 \leq i \leq k$, let $r_i \in Ext^*(i_*(\mathcal{O}(a_i)),i_*(\mathcal{O}(b_i)))$ be elements homogeneous in weight and degree such that $f(r_1 \otimes \cdots \otimes r_k) = r_{k+1} \neq 0$ and define $n_i = b_i - a_i$. Then, as we are working over $T$, we must have $\sum_{i=1}^k n_i = n_{k+1}$ and as $f$ is equivariant with respect to weight we have $\sum_{i=1}^k wt(r_i) = wt(r_{k+1})$. By the description of $Ext^*(i_*(\mathcal{O}(a_1)),i_*(\mathcal{O}(b_1)))$ in Corollary 1 we have that $n_i + \deg(r_i) = \deg(r_i)$ which implies $\sum_{i=1}^k \deg(r_i) = \deg(r_{k+1})$. Thus $d = \deg(f) = \deg(r_{k+1}) - \sum_{i=1}^k \deg(r_i) = 0$ confirming the claim. \hfill \Box

Now we will apply the above strategy and represent the dual quiver algebra to $Q_E$. To compute the $Ext$ groups between the simple modules of $Q_E$, we must find a projective resolution for each such object. In our case, the additional graded factor in the $Ext$ groups of $E$ poses a technical challenge in this computation. This is overcome by constructing a double complex of projective objects whose total complex resolves $S_i$ and whose spectral sequence converges at the third stage.

We start by resolving $S_i$ by modules over $Q_E$ which are modules over the degree zero piece of $Q_E$. By Corollary 1, we have that $Q_E = Q \oplus I$ where $Q$ consists of the morphisms graded at zero and $I$ is the ideal whose grading is 1. Given a graded module $M$ over $Q_E$, we let $\tilde{M} = M/MI$. Again by
Corollary 1, we have
\[ \tilde{P}_k = \bigoplus_{j=0}^{k} \operatorname{Ext}^\bullet(i_*(\mathcal{O}(j)), i_*(\mathcal{O}(k))) \approx \bigoplus_{j=0}^{k} R_j. \]

Recall that in \( R \), given an \( \alpha \in \mathbb{Z}_{\geq 0}^{n+1} \), one has \( \deg(x^\alpha) = \sum_{i=0}^{n+1} \alpha_i \cdot a_i \) where \( a = (a_0, \ldots, a_n) \) is the weight for \( \mathbb{P}(a) \). Thus \( \tilde{P}_k \) is a vector space generated by monomials \( \{x^\alpha \mid \deg(x^\alpha) \leq k\} \). One can also see that the \( Q_E \)-module morphisms \( \operatorname{Hom}(\tilde{P}_j, \tilde{P}_k) \) are generated by maps sending \( e_j \) to \( x^3 \) with \( \deg(x^3) = k - j \). We also note that \( S_k = e_k \cdot Q_E \cdot e_k \) is the quotient module of \( \tilde{P}_k \) by the submodule generated by all non-trivial monomials.

To resolve \( S_k \) by modules \( \tilde{P}_j \), we will exploit the interplay between these modules and sheaves on \( \mathbb{P}(a) \).

Now, let \([n] = \{0, \ldots, n\}\) and for any subset \( J \subset [n] \) let \( a_J = \sum_{i \in J} a_i \). We have the following exact Koszul resolution \( K^\bullet \) of sheaves on \( \mathbb{P}(a) \) generated by the regular sequence \( \langle x_0, \ldots, x_n \rangle \) in \( R \)

\[
0 \to \bigoplus_{|J| = n+1} \mathcal{O}(-a_J) \to \bigoplus_{|J| = n} \mathcal{O}(-a_J) \to \cdots \to \bigoplus_{|J| = 1} \mathcal{O}(-a_J) \to \mathcal{O} \to 0
\]

Here we will place \( \mathcal{O} \) in the zeroth position. One sees that this sequence remains exact after tensoring with any line bundle \( \mathcal{O}(i) \) so that the hypercohomology \( \mathbb{H}^*(K^\bullet \otimes \mathcal{O}(i)) \) is zero. Furthermore, given \( 1 \leq i \), \( 0 < r \) and any \( j \), Proposition 1 tells us that \( H^r(\mathbb{P}(a), K^j \otimes \mathcal{O}(i)) = 0 \). Thus, for \( i \geq 0 \), the first term in the spectral sequence

\[ E^1_{p,q} = H^q(\mathbb{P}(a), K^p \otimes \mathcal{O}(i))) \Rightarrow \mathbb{H}^*(K^\bullet \otimes \mathcal{O}(i)) = 0 \]

is concentrated on the \( q = 0 \) axis and yields an exact sequence. On the other hand, the cohomology of \( H^0(\mathbb{P}(a), K^\bullet) \) itself is clearly just \( \mathbb{C} \) concentrated in degree zero.

As a convention, we will take \( P_i = 0 \) and \( R_i = 0 \) if \( i < 0 \). Define

\[ M_{k,j} = \bigoplus_{J \subset [n], |J| = j} \tilde{P}_{k-a_J} \]

With these preliminaries in mind, we can state:

**Lemma 2.** The sequence \( H^0(\mathbb{P}(a), K^\bullet \otimes \bigoplus_{j=0}^{k} \mathcal{O}(j)) \) is naturally identified with a resolution of \( S_k \) by modules \( M_{k,j} \).

**Proof.** By the above definitions and Proposition 1 we have that

\[
H^0(\mathbb{P}(a), K^i \otimes \bigoplus_{j=0}^{k} \mathcal{O}(j)) = \bigoplus_{j=0}^{k} H^0(\mathbb{P}(a), K^i \otimes \mathcal{O}(j)) = \bigoplus_{j=0}^{k} \bigoplus_{|J| = -i} H^0(\mathbb{P}(a), \mathcal{O}(j - a_J)) = \bigoplus_{|J| = -i} \bigoplus_{j=0}^{k} \tilde{P}_{k-a_J} = M_{k,-i}.
\]
Furthermore, all maps in the Koszul sequence $K^\bullet$ are induced from multiplication by $\pm x_i$ and this descends to cohomology. As was pointed out, these are precisely the morphisms between the modules $\tilde{P}_i$, thus the maps in the sequence are $Q_E$-module morphisms. Examining the last map

$$
\oplus_{i=0}^n \tilde{P}_{k-a_i} \to \tilde{P}_k
$$

we see that this is simply the map that takes $e_i$ in each summand where $k \geq a_i$ to $x_i$ in $\tilde{P}_k$. So the cokernel of this map is just $S_k$. Rewriting the sequence in the language of $Q$-modules and observing that for all $0 \leq k \leq l-2$, $\tilde{M}_{k,n+1} = 0$ we have the sequence $C^\bullet_k$

$$
0 \to \tilde{M}_{k,n} \to \tilde{M}_{k,n-1} \to \cdots \to \tilde{M}_{1,1} \to \tilde{M}_{0,0} \to 0
$$

which is quasi-isomorphic to $S_k$ concentrated in degree zero. \qed

We now find truly projective resolutions for the modules $\tilde{M}_{k,j}$ and extend $C^\bullet_k$ to a double complex whose total complex resolves $S_k$. For this we start by resolving $\tilde{P}_k$ by projective modules. This is actually quite simple. By Corollary 1, we have

$$
P_k = \bigoplus_{j=0}^k (\text{Ext}^j(i_*(\mathcal{O}(j)), i_*(\mathcal{O}(k))) \oplus \text{Ext}^j(i_*(\mathcal{O}(j+1)), i_*(\mathcal{O}(k)))[-1])
$$

The summand of $P_k$ in degree 1 is generated by the identity element in $R_0[-1]$ which we will denote $e_k[-1]$. There is a degree 0 morphism $t_{k-1} : P_{k-1}[-1] \to P_k$ which sends $e_{k-1}$ to $e_k[-1]$. We see that the kernel of this morphism consists of all degree 2 elements of $P_{k-1}[-1]$ and the image consists of all degree 1 elements of $P_k$. Thus the cokernel is $\tilde{P}_k$ and one sees easily that the following is a projective resolution of $\tilde{P}_k$:

$$
0 \to P_0[-k] \xrightarrow{t_0} P_1[-k+1] \xrightarrow{t_1} \cdots \xrightarrow{t_{k-1}} P_k \to \tilde{P}_k \to 0
$$

Taking direct sums of these sequences yields a resolution

$$
0 \to M_{0,j}[-k] \to M_{1,j}[-k+1] \to \cdots \to M_{k,j} \to \tilde{M}_{k,j} \to 0
$$

Using these resolutions, we extend $C^\bullet_k$ to a third quadrant double complex $C^{\bullet\bullet}_k$ with

$$
C^{p,q}_k = \begin{cases} 
M_{k+p,q} & \text{for } q \leq 0 \\
0 & \text{otherwise}
\end{cases}
$$

Using the spectral sequence for $C^{\bullet\bullet}_k$ one sees that the total complex is a projective resolution for $S_k$. Define the $Q_E$ module

$$
(2) \quad N_{k,j} = \bigoplus_{p+q=-j} C^{p,q}_k = \bigoplus_{i=0}^j M_{k-i,j-i}[-i] = \bigoplus_{J \subseteq [n], |J| \leq j} \bigoplus_{J \subseteq [n], |J| \leq j} P_{k-j-1,1-j}[-|J|-j]
$$

Then we have proved

**Proposition 5.** $S_k$ has a projective resolution $D^\bullet_k$

$$
0 \to N_{k,n} \to N_{k,n-1} \to \cdots \to N_{k,0} \to S_k \to 0
$$
Now, one easily sees that $\text{Hom}(P_i, S_j) = \delta_{ij} \cdot \mathbb{C}$. Indeed if $i \neq j$ and $f \in \text{Hom}(P_i, S_j)$ then $0 = f(e_i \cdot e_j) = f(e_i) \cdot e_j = f(e_i)$. Using this and the above resolution, we can calculate the morphisms from $S_k$ to $S_i$. Let $V$ be a vector space over $\mathbb{C}$ with basis $\{e_0, \ldots, e_n\}$. On the exterior algebra $\bigwedge^* V$ we have two gradings generated by $\deg(e_i) = 1$ and $\deg_S(e_i) = a_i$. Thus we denote $\bigwedge^{r,s} V = \{w \in \bigwedge^* V | \deg(w) = r, \deg_S(w) = s\}$.

**Proposition 6.** There is a natural equivalence
\[
\text{Ext}^r(S_k, S_i) = \bigoplus_{s \leq k-i} \bigwedge^{r,s} V
\]
which is compatible with composition.

**Proof.** To calculate $\text{Ext}^r(S_k, S_i)$ we first examine the resolution $D_k^\bullet$. Given a $J \subset \{n\}$ one sees that $P_{k-j+|J|-a_J}[|J| - j]$ occurs as a summand in $D_k^\bullet$ for some $j$ if and only if $k \geq a_J$. Indeed, as $j \geq |J|$ in each such summand, if $k < a_J$ then $k - j + |J| - a_J < 0$ implying the associated module is zero. Conversely, if $k \geq a_J$ then for each $j$ satisfying $|J| \leq j \leq k - a_J + |J|$ one such summand occurs and these modules are $\{P_0[a_J-k], \ldots, P_{k-a_J}[0]\}$. For such a $J$, $P_i$ occurs in this list if and only if $i \leq k - a_J$. Thus, given any $J \subset \{n\}$, the projective module $P_i$ occurs once as a direct summand in $D_k^\bullet$ of the form $P_{k-j+|J|-a_J}[|J| - j]$ if and only if $a_J \leq k - i$. Such a summand occurs in $D_k^{\bullet-j}$ so that, as a graded module, the total grading on the summand is $P_i[|J| - j][j] = P_i[|J|]$. Now, as each map in $D_k^\bullet$ restricted to any summand either sends $e_p$ to $e_p \cdot x_q$ or to $e_p[-1]$, we have that the complex $\text{Hom}(D_k^\bullet, S_i)$ has a zero differential. This implies that $\text{Ext}^r(S_k, S_i) = \text{Hom}(D_k^\bullet, S_i)$ and recalling the above note that $\text{Hom}(P_j, S_i) = \delta_{ij} \mathbb{C}$ we have
\[
\text{Ext}^r(S_k, S_i) = \bigoplus_{J \subset \{n\}, a_J \leq k-i} \text{Hom}(P_i[|J|], S_i) = \bigoplus_{J \subset \{n\}, a_J \leq k-i} \mathbb{C}[-|J|]
\]

The identification is now clear. For any $J = \{j_1, \ldots, j_M\}$ with $a_J \leq k - i$ assign the projection map in $\text{Hom}(P_i[|J|], S_i) = \mathbb{C}[-|J|]$ to the element $v_J = e_{j_1} \wedge \cdots \wedge e_{j_M} \in \bigwedge V$. Then we see that $\deg(v_J) = |J|$ so that the grading of $v_J \in \mathbb{C}[-|J|]$ is $\deg(v_J)$ as claimed.

To see that these isomorphisms are compatible with composition, one observes that any element of $\text{Ext}^c(S_k, S_i)$ is a morphism from $D^\bullet_k$ to $D^\bullet_i$. Since the horizontal differential of $C^{\bullet\bullet}$ takes identity elements to degree 0 elements and the vertical differential takes identities to elements of degree 1, such maps naturally extend to maps of the double complexes giving each such map a bi-degree. Working out the degrees, one sees that an element corresponding to $v_J$ has bi-degree $(-\deg(v_J), \deg_S(v_J) + i - k)$. In particular, for $J = \emptyset$ we have the map which is simply a vertical shift plus projection of $D^\bullet_k$ to $D^\bullet_i$. One easily sees that this implies that $v_0$ acts as an identity element under compositions (after one has made the identification between the $\text{Ext}$ groups and the exterior algebra). Thus in examining relations, one needs only consider elements in $\text{Ext}^r(S_k, S_i)$ of bi-degree $(d, 0)$ which
correspond to elements for which \( a_j = k - i \). As all such elements are of degree zero, one can quotient the complex by the weight \(-1\) ideal in \( Q_E \). Viewing the double complex over \( Q \) gives a zero vertical differential, so examining the relations with respect to the complexes \( C^\bullet \) over \( Q \) is sufficient.

These relations have been worked out in [1], section 2.6, where it was found that for elements of bi-degree \((d, 0)\), there is a natural equivalence

\[
\text{Ext}_Q^\bullet(S_k, S_i) = \bigwedge^{r, k-i} V[-r]
\]

compatible with composition. □

Note: There are two alternative and less computational ways of approaching this problem. One could approach the above proof from the perspective of Koszul algebras. Indeed, letting \( k = \bigoplus_{i=0}^{l-2} \mathbb{C} \cdot e_i \) and viewing \( M \) as a module generated by monomials \( x_i: P_k \to P_j \) and \( e_i[-1] \) for all \( i, j \) and \( k \), \( Q_E \) can be viewed as a quadratic algebra \( T(M)/I \) where \( I \) is generated by elements in \( M \otimes_k M \). If one could show that this algebra is in fact Koszul, then the dual quadratic algebra is isomorphic to the opposite algebra of the Koszul dual. This algebra is easily seen to be the one exhibited in the previous proposition. For more on quadratic algebras over semi-simple rings, see [3].

Alternatively, one could view \( T \) as a subcategory of graded modules over the super symmetric algebra \( T = \text{Sym}(V_0 \oplus V_1) \) where the even part \( V_0 = V \) is the weighted vector space as above and and the odd \( V_1 = \mathbb{C} \) encodes the infinitesimal information of the exceptional divisor. One readily sees that \( i_* (\mathcal{O}(k)) \) could be identified with the module \( T_k \) by relating \( \text{Hom}(T_k, T_j) \) to \( \text{Ext}^\bullet(i_*(\mathcal{O}(k)), i_*(\mathcal{O}(j))) \) via Corollary 1. Then the Koszul dual collection is found by examining the dual collection to the dual algebra \( T^\vee = \text{Sym}(V[1]) \) which gives a Grassmanian in \( n + 1 \) variables tensored with a symmetric algebra in one variable. One can see that \( S_k \) can be associated to \( T^\vee_{-k} \) by rewriting Proposition 5 in terms of homogeneous elements of \( T^\vee \).

3. Degenerations of Lefschetz Pencils

3.1. The Directed Fukaya Category. This preliminary subsection will outline the definition of the directed Fukaya category corresponding to the Landau-Ginzburg model. There will be no proofs, but all of the stated results can be found in the literature, see for example [17], [1], [8]. We will impose some technical assumptions which greatly simplify the construction. These will always be satisfied in our applications. First, assume \((M, \omega)\) is an exact Kähler manifold. The condition of exactness is that there exists a 1-form \( \theta \) such that \( \omega = d\theta \). Now let \( W: M \to \mathbb{C} \) be a holomorphic function with isolated Morse singularities. We will assume that there is at most one critical point in any given fiber of \( W \) and will call \( W \) the potential. Given such a potential function and a regular point \( p \in M \), one can use the symplectic form to define a splitting of \( T_p M = T_p(W^{-1}(p)) \oplus U_p \). Here, \( U_p \) is the symplectic orthogonal to the tangent space of the fiber. One sees that
$U_p$ is mapped isomorphically onto the $T_{W(p)}C$ via the differential of $W$. So, given a tangent vector at $W(p)$, one can lift this to a vector field on $W^{-1}(p)$ in $M$ (at a critical point on a fiber, we take the vector field to equal zero).

Now suppose $\{q_0, \ldots, q_m\}$ are the critical values of $W$ and choose a regular value $q$. Let $\gamma_i : [0, 1] \to \mathbb{C}$ be a set of paths such that $\gamma_i(0) = q$, $\gamma_i(1) = q_i$, $\gamma_i((0, 1]) \cap \gamma_j((0, 1]) = \emptyset$ and the paths are oriented counter-clockwise around $q$. We will refer to such a collection $\{\gamma_i\}$ as a distinguished basis. Along each such path, one can lift the vector field $d/dt$ to a vector field on $W^{-1}(\gamma_i([0, 1]))$ which will define a parallel transport. We will assume for the moment that this vector field is integrable on $M$ and let $D_i$ be the set of points flowing into $\tilde{q}_i$, the critical point which is mapped to $q_i$. We let $L_i = D_i \cap W^{-1}(q)$ be its boundary. $D_i$ is often called a Lefschetz thimble and $L_i$ the vanishing cycle. One can show that the vanishing cycles $L_i$ are actually exact lagrangian sub-manifolds in the fiber $W^{-1}(q)$. Such a submanifold is one for which $\theta|_{L_i}$ is an exact one form on $L_i$. Assume that the vanishing cycles intersect transversely (indeed, one can always perturb them slightly to accomplish transversality). As the potential $W$ is not proper, the parallel transport vector field may not be integrable in general. To get around this one must occasionally perturb the parallel transport vector field. So long as such a perturbation results in an exact isotopy of the $L_i$, the theory will remain unaffected. For more on vanishing cycles, see [7], [15].

Now, one can form the Lagrangian Grassmannian bundle $\Lambda$ of the tangent bundle $T(W^{-1}(q))$ on the fiber $W^{-1}(q)$. Assuming certain obstructions vanish, we also have the fiberwise universal covering $\eta : \tilde{\Lambda} \to \Lambda$. For each vanishing cycle $L_i$, one has a natural lift

$$L_i \xrightarrow{\phi} \tilde{\Lambda}$$

where $\phi$ sends a point of $L_i$ to its tangent space. A grading on $L_i$ is a lift $\tilde{\phi} : L_i \to \tilde{\Lambda}$ satisfying $\eta \circ \tilde{\phi} = \phi$. As in [18], we define a graded lagrangian to be a lagrangian submanifold with such a lift $\tilde{\phi}$. For more details on gradings see Section 4.4.

The objects of the directed Fukaya category for $(W, \{\gamma_i\})$ are graded vanishing cycles $[L_i, \tilde{\phi}_i]$ where we have chosen one grading for each vanishing cycle. Given a point $p \in L_i \cap L_j$ one can choose a path $\delta_p : [0, 1] \to \tilde{\Lambda}_p$ defined by setting $\delta_p(0) = \tilde{\phi}_i(p)$ and $\delta_p(1) = \tilde{\phi}_i(p)$; as this path is in the simply connected universal cover of the lagrangian grassmanian, it is unique up to homotopy. Taking the Maslov index of $\eta \circ \delta_p$ gives an integer which we will call $\text{deg}(p)$. We can now define the Hom sets with their associated grading.
\( \text{Hom}^*(L_i, L_j) = \left\{ \begin{array}{ll}
\oplus_{p \in L_i \cap L_j} \mathbb{C} \langle p \rangle [-\deg(p)] & \text{for } i < j \\
\mathbb{C} \langle e_i \rangle & \text{for } i = j \\
0 & \text{otherwise}
\end{array} \right. \)

In order to proceed to products, one must confront the fact that the directed Fukaya category is actually an \( A^\infty \)-category. Thus, for every \( k \in \mathbb{Z}_{>0} \) one has a higher product \( m_k \) of degree \( 2-k \). To define these maps, one considers the moduli space of holomorphic maps from the disc with marked boundary points to the fiber \( W^{-1}(q) \) with certain boundary conditions. More specifically, Let \( \mathcal{M}_k(D) \) be the moduli space of the unit disc \( (D, \partial D) \) with \( k+1 \) distinct marked points \( \{z_0, \ldots, z_k\} \) oriented clockwise along the boundary. Now, assume \( i_0 < i_2 < \cdots < i_k \) and \( p_{i_j} \in L_{i_j} \cap L_{i_j+1} \) then we take \( \mathcal{M}(p_{i_0}, \ldots, p_{i_k}) \) to be the moduli space of all holomorphic maps \( u : D_k \to W^{-1}(q) \) such that \( D_k \in \mathcal{M}_k(D), u(z_j) = p_{i_j} \) and the arc along the boundary from \( z_j \) to \( z_{j+1} \) is sent to \( L_{i_{j+1}} \). In the above notation we take \( j \mod k + 1 \) and when \( k = 1 \), we need to quotient this space by the action of \( \mathbb{R} \) on such maps. In the situation described above, this moduli space has a natural compactification which is a manifold with corners. We will denote \( \mathcal{M}(p_{i_0}, \ldots, p_{i_k}) \) by \( \mathcal{M}_0(p_{i_0}, \ldots, p_{i_k}) \) if it has dimension zero, otherwise \( \mathcal{M}_0(p_{i_0}, \ldots, p_{i_k}) \) will be the empty set. This space has a natural orientation class \( \text{sgn} \) and defines our products. Namely, using the above notation one defines

\[
m_k : \text{Hom}^*(L_{i_{k-1}}, L_{i_k}) \otimes \cdots \otimes \text{Hom}^*(L_{i_0}, L_{i_1}) \to \text{Hom}^*(L_{i_0}, L_{i_1})[2-k]
\]

via

\[
m_k(p_{i_{k-1}} \otimes \cdots \otimes p_{i_0}) = \sum_{r \in L_{i_0} \cap L_{i_k}} \left( \sum_{u \in \mathcal{M}_0(p_{i_0}, \ldots, p_{i_{k-1}}, r)} \text{sgn}(u) \right) \text{sgn}(u)
\]

A Maslov index calculation shows that \( m_k \) has the indicated degree. As was noted in [1], one normally weights this sum by the exponential of the symplectic area of \( u(D) \) in \( W^{-1}(q) \). However, since the lagrangians are exact, this is not necessary.

The version of homological mirror symmetry examined in this paper proposes an equivalence between the derived category of coherent sheaves on a toric variety and the derived Fukaya category of the mirror Landau-Ginzburg model. To pass from the \( A^\infty \)-category described above to its derived category, one must take twisted complexes of formal sums and shifts of the above objects, their idempotent splittings and formal inverses of quasi-isomorphisms. This procedure was invented by Kontsevich [12] and is fully explained in Seidel’s recent book [18]. After forming such a construction, the objects \([L_i, \phi_i]\) form a complete exceptional collection in the derived category \( \mathcal{D}^b(Fuk(W, \{\gamma_i\}) \). A result of Seidel is that any other choice of paths \( \{\gamma'_i\} \) yields objects \([L_i, \phi'_i]\) that can be obtained from \([L_i, \phi_i]\) in \( \mathcal{D}^b(Fuk(W, \{\gamma_i\}) \) by a sequence of mutations [17]. As a consequence, the derived Fukaya category is independent of the choice of paths and one can simply write \( \mathcal{D}^b(Fuk(W)) \). We will need the following theorem for what follows which is
a standard fact whose proof can be found in Seidel’s book in the section on directed $A^\infty$-categories [19].

**Theorem 3.** (Seidel) $D^b(\text{Fuk}(W,\{\gamma_i\}))$ is invariant under exact perturbations of $\omega$. Furthermore, if $[L_i',\tilde{\phi}_i']$ are Hamiltonian isotopic to $[L_i,\tilde{\phi}_i]$, then $D^b(\text{Fuk}(W,\{\gamma_i\}))$ is equivalent to the derived Fukaya category generated by the graded lagrangians $[L_i',\tilde{\phi}_i']$.

This theorem is not only true on the derived level, but also up to $A^\infty$-quasi-isomorphism. By an exact perturbation we mean a perturbation of $\theta$ through one forms whose exterior derivative is non-degenerate.

In our case we will see that all $m_k$ vanish except $m_2$. This implies that the $A^\infty$-category is indeed a category. In such a situation, it is known that $D^b(\text{Fuk}(W,\{\gamma_i\}))$ is equivalent as a triangulated category to the bounded derived category of graded modules over the quiver algebra defined by the collection $[L_i,\tilde{\phi}_i]$. All of this will be explored in more detail in Section 3.

*Note:* In other accounts of this subject the distinguished basis of paths $\{\gamma_i\}$ is ordered clockwise and the moduli space $M(D)$ has curves with marked points oriented counterclockwise. Our formulation is equivalent and more advantageous for the examples worked through in Section 3; however, as will be seen, it will affect the definition of the Maslov index as well.

### 3.2. Partial Lefschetz Fibrations

Let $W$ be a potential function from a Kähler manifold $M$ to $\mathbb{C}$. In many cases of interest symplectic parallel transport is not well defined with respect to $W$ for all paths in $\mathbb{C}$. These cases arise when certain fibers of $W$ do not transversely intersect the divisor at infinity of a suitable compactification of $M$. This fact can make it difficult to define the directed Fukaya category for the pair $(M,W)$. However, when confronted with such a potential $W$, one can often find a connected open set $U \subset \mathbb{C}$ for which parallel transport can be defined. In the ideal case, all critical values of $W$ are contained in $U$ and $U$ is simply connected; this case yields the usual concept of a Landau-Ginzburg model explored in the last subsection. In the less ideal case, one can capture part of the Landau-Ginzburg model by examining non-simply connected domains $U$ containing only some of the critical values of $W$. The motivation for this procedure is that this part of the model will remain invariant under perturbation of $W$ yielding a semi-orthogonal piece of the directed Fukaya category corresponding to the perturbed model.

We now fix a Stein manifold $M$ with an exhaustive plurisubharmonic function $\rho$ and the associated (exact) symplectic structure $\omega$. We also take a potential function $W$ from $M$ to $\mathbb{C}$ associated to the Landau-Ginzburg model on $M$.

**Definition 3.** A partial Lefschetz fibration on $(M,\rho)$ consists of the data $(W,U,a)$ where $W$ is a holomorphic function on $M$, $U$ is a connected open
subset of \( \mathbb{C} \) with smooth boundary and \( a \in \mathbb{R} \) such that
(i) \( a \) is a regular value of \( \rho \)
(ii) \( W^{-1}(p) \) intersects \( \rho^{-1}(a) \) transversely for each \( p \in U \)
(iii) The critical points \( p_i \) such that \( W(p_i) \in U \) are Morse and no two such points lie on the same fiber
(iv) The set of critical values of \( W \) does not intersect \( \partial U \)
(v) \( \rho(p) < a \) for all critical points \( p \in M \) with \( W(p) \in U \)

Given a partial Lefschetz fibration, let \( M_{(U,a)} = W^{-1}(U) \cap \rho^{-1}((-\infty, a]) \).

The word "partial" refers to the part of \( M \) corresponding to \( M_{(U,a)} \) which is a Kähler manifold with boundary and codimension 2 corners. On this part of \( M \), the fibers of \( W \) are Kähler manifolds with contact type boundary and the setup looks similar to the one considered in [17]. A major flaw however is the possible failure of \( \rho^{-1}(a_i) \cap W^{-1}(U) \) being horizontal with respect to the symplectic orthogonal connection on \( M \). In other words, attempting parallel transport of a fiber \( W \) in \( M_{(U,a)} \) along a path in \( U \) may not be possible as one may be led outside the subspace \( M_{(U,a)} \). On can remedy this situation by adding the Liouville vector field to any given parallel transport [19]. While flowing downward along the Liouville vector field does not preserve the symplectic structure, it will be an exact isotopy for exact Lagrangian submanifolds. These are the manifolds of interest when defining the directed Fukaya category.

Let us start to pursue this line of reasoning by recalling some basic definitions and results. The symplectic form on \( M \) is \( \omega = -d(J^*d\rho) \) where \( J \) is the complex structure on \( M \). In the notation of the previous subsection \( \theta = -J^*d\rho \). The Liouville vector field \( X \) is defined by the equality
\[
\omega(X, Y) = \theta(Y)
\]
for all vector fields \( Y \). We have the classical result that \( L_X \theta = \theta \).

Alternatively, one can view \( X \) as the gradient vector field of \( \rho \) with respect to the Kähler metric. In particular, \( X \) is normal to the real hypersurface \( \rho^{-1}(a) \) for all regular values \( a \). Now suppose \( W \) is a potential function on \( M \). On each fiber \( F_p \), the restriction of the Kähler form \( \omega \) to \( F_p \) is generated by the restriction of the Kähler potential \( \rho \) to the fiber. Thus, for each fiber we can define a Liouville vector field which taken together on all of \( M \) we will call the fiberwise Liouville vector field \( X_f \). Although one no longer has \( L_{X_f} \theta = \theta \) for \( M \), this equality is certainly true fiberwise. One can see also that as \( X_f \) is the gradient vector field on \( F_p \) to \( \rho \) restricted to the fiber, it will be normal to \( \rho^{-1}(a) \) in \( M \) for those points at which \( F_p \) transversely intersects \( \rho^{-1}(a) \).

Now assume \( (W, U, a) \) is a partial Lefschetz fibration with critical values \( \{q_0, \ldots, q_m\} \subset U \) and corresponding critical points \( \{p_0, \ldots, p_m\} \). Choose a regular point \( q \in U \) of \( W \) and choose a distinguished basis of paths \( \{\delta_0, \ldots, \delta_m\} \) from \([0, 1]\) to \( U \) satisfying the conditions of the previous subsection. We will call such a choice of paths a distinguished basis for the partial
Lefschetz fibration \((W,U,a)\). For each such path we have a parallel transport vector field \(Y_i\) on \(W^{-1}(\delta_i([0,1]))\). As was discussed above, for \(p \in \rho^{-1}(a)\), one may have that \(Y_i\) at \(p\) enters the space \(M_{(U,a)}\), i.e. \(d\rho(Y_i) < 0\). However, there is a sufficiently large constant \(C_i\) such that \(d\rho(Y_i + C_i X_f) > 0\) for every point in the compact space \(\rho^{-1}(a) \cap M_{(U,a)}\) (this follows from the assumption of transversality in the definition of partial Lefschetz fibrations). It is necessary to add the fiberwise Liouville vector field \(C_i X_f\) to \(Y_i\) as opposed to the Liouville vector field as \(X\) is not generally defined on the tangent space of \(W^{-1}(\delta_i([0,1]))\), i.e. certain tangent vectors may lead one off of this subspace. Let \(C\) be the maximum of such \(C_i\) and \(Z_i = Y_i + C X_f\). We let 

\[D_i^{lu} = \text{the space of all points in } W^{-1}(\delta_i([0,1])) \text{ which flow to the critical point } p_i \text{ via } Z_i \text{ and } L_i^{lu} = \partial D_i^{lu} \subset W^{-1}(q).\]

Let \(D_i\) and \(L_i\) be defined as in the previous subsection. For the following proposition, we will assume that the vector fields \(Y_i\) are integrable.

**Proposition 7.** For any partial Lefschetz fibration \((W,U,a)\) on \(M\), one has \(D_i^{lu} \subset M_{(U,a)}\) for every \(i\). Furthermore, there is an exact isotopy from \(L_i^{lu}\) to \(L_i\) in \(W^{-1}(q)\) for each \(i\).

**Proof.** To see that \(D_i^{lu} \subset M_{(U,a)}\), observe that if \(f : [0,\infty) \to W^{-1}(\delta_i([0,1]))\) is a flow line for which \(\lim_{t \to \infty} f(t) = p_i\) and \(\rho(f(0)) \geq a\) then as \(p_i \in M_{(U,a)}\) there is some \(p \in \rho^{-1}(a) \cap M_{(U,a)}\) and some \(t_0\) for which \(f(t_0) = p\) and \(d/dt (\rho \circ f)_{t_0} \leq 0\). But then

\[0 \geq d/dt (\rho \circ f)_{t_0} = d\rho((df/dt)_{t_0}) = d\rho(Z_i(p)) > 0\]

Therefore, \(\rho(f(0)) < a\) and \(f(0) \in M_{(U,a)}\).

The second assertion follows from the fact that \(L_X \theta = \theta\) on the fibers of \(W\). Thus, flowing along \(X_f\) simply rescales \(\theta|_{L_i}\) so that exactness is never violated. \(\square\)

Now, given the collection \(\{L_i^{lu}, \ldots, L_m^{lu}\}\) in \(W^{-1}(q)\) we can define a directed \(\mathcal{A}\)-category following the procedure outlined in the last subsection where all products are defined using moduli spaces \(M_0(p_{i_0}, \ldots, p_{i_k})\) where the target is \(W^{-1}(q)\). As \(\rho\) is a subharmonic function on \(W^{-1}(q)\), the maximum principle shows that any holomorphic map from the unit disc into \(W^{-1}(q)\) such that \(\rho(\partial D) \subset M_{(U,a)}\) must have \(\rho(D) \subset M_{(U,a)}\). Thus the space \(M_0(p_{i_0}, \ldots, p_{i_k})\) only contains holomorphic maps whose image lies in the subspace \(M_{(U,a)}\). With this in mind we define

**Definition 4.** Let \((W,U,\rho)\) be a partial Lefschetz fibration and \(\{\delta_0, \ldots, \delta_m\}\) be paths as above. Define \(\mathcal{F}uk(W,U,a,\{\delta_i\})\) to be the \(\mathcal{A}\)-category generated by the lagrangians \(\{L_0^{lu}, \ldots, L_m^{lu}\}\) with higher products defined in 3.1

Note: Unlike the situation for ordinary Landau-Ginzburg models, the derived version of this category will depend on the choice of paths. This situation arises when one takes \(U\) to be a non-simply connected domain.
Then the monodromy prevents one from concluding that different choices correspond to mutations in the derived category.

From the arguments given above and Theorem 3 we have

**Corollary 2.** Let $W$ be a Lefschetz fibration on $M$. If $(W, U, a)$ is a partial Lefschetz fibration and $\{\delta_i\}$ a distinguished basis for $(W, U, a)$. Adding paths $\{\tau_j\}$ to obtain a distinguished basis for $W$, let $\text{Fuk}(W|_U)$ be the $A^\infty$-category contained in $\text{Fuk}(W)$ generated by the vanishing cycles associated to $\{\delta_i\}$. Then $\mathcal{D}^b(\text{Fuk}(W|_U))$ is equivalent to $\mathcal{D}^b(\text{Fuk}(W, U, a, \{\delta_i\}))$.

This corollary shows that $\mathcal{D}^b(\text{Fuk}(W, U, a, \{\delta_i\}))$ can be considered as a subcategory of $\mathcal{D}^b(\text{Fuk}(W))$ when $W$ is a Lefschetz fibration. Furthermore, if there is a distinguished basis of $W$ of the form $\{\tau_1, \ldots, \tau_r, \delta_0, \ldots, \delta_m\}$ we see that the triangulated category generated by the first $r$ vanishing cycles and $\mathcal{D}^b(\text{Fuk}(W, U, a, \{\delta_i\}))$ yields a semiorthogonal decomposition of $\mathcal{D}^b(\text{Fuk}(W))$.

### 3.3. Deformations of the potential.

The motivation behind the preceding definitions is not simply to take a piece of a single Landau-Ginzburg model $(M, W)$, but rather to take a piece that will remain somewhat stable under a perturbation of $W$. Of course, slight perturbations of the potential $W$ may introduce new critical points and new topology into the fibers. The simple example of $W_\varepsilon(z) = \varepsilon z^3 + z^2$ on $\mathbb{C}$ demonstrates such a situation. For $\varepsilon = 0$, we have one critical point and the regular fibers consist of 2 points. After slightly perturbing, a critical point enters in from infinity, and the generic fibers become 3 points. This basic example shows that the full derived Fukaya category of $M$ with a varying potential $W_\varepsilon$ is not generally invariant. In this paper, we will be considering such unruly types of perturbations, so it will be desirable to pick out a piece of the Fukaya category that does remain invariant. Of course, this piece will be precisely the one corresponding to a partial Lefschetz fibration. This motivates the following definition.

**Definition 5.** Suppose $(W, U, a)$ is a partial Lefschetz fibration on $(M, \rho)$. Letting $D_\varepsilon$ be the $\varepsilon$ disc in $\mathbb{C}$, a deformation of $(W, U, a)$ consists of a holomorphic map $W_- : D_\varepsilon \times M \to \mathbb{C}$ such that $W_0 = W$ and $(W_t, U, a)$ is a partial Lefschetz fibration for all $t \in D_\varepsilon$. Given such a deformation, we let $M_{(U, a, t)} = W_t^{-1}(U) \cap \rho^{-1}((-\infty, a])$.

Observe that a consequence of this definition is that no critical values or points will enter into $U$ or $M_{(U, a)}$ as we perturb $W$. Furthermore, the transversality condition insures that $M_{(U, a)}$ and its fibers are topologically unchanged. Moreover, we have the following proposition.

**Proposition 8.** Let $W_-$ be a deformation of a partial Lefschetz fibration $(W, U, a)$. If $q_0 \in U$ is a smoothly varying regular value of $W_t$, then $W_t^{-1}(q_0) \cap M_{(U, a, t)}$ contains a subspace $N_{(U, a, t)}$ that is an exact perturbation of $W^{-1}(q_0) \cap M_{(U, a)}$ up to a rescaling of $\omega$. 
Proof. The proof of this theorem follows the spirit of the proof of Proposition 7. We let \( F : D_\varepsilon \times M \rightarrow D_\varepsilon \times \mathbb{C} \) be defined as \( F = (\pi_1, W_{-}) \). Given a path \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) in \( D_\varepsilon \times \mathbb{C} \), we can form a parallel transport vector field in \( D_\varepsilon \times M \) along the fibers \( W_{\gamma_1(t)}^{-1}(\gamma_2(t)) \). Although this vector field will generally not be integrable, adding a fiberwise Liouville flow will force integrability in the subspaces \( M_{(U,a,t)} \). We have that parallel transport is an exact isotopy and the Liouville flow simply rescales the exact form \( \theta \), implying the result. The details of this argument follow the same line of reasoning as in Proposition 7. \( \square \)

Now suppose \( \{\delta_{i,t}\} \) is a smoothly varying set of distinguished bases for \( W_t \) and \( \{L_{i,t}^{iv}\} \) are their associated vanishing cycles in \( W_{t}^{-1}(q_t) \). Using the previous proposition, there is an embedding \( j_{0,t} \) from \( W_{t}^{-1}(q_0) \cap M_{(U,a,0)} \) into \( W_{t}^{-1}(q_t) \cap M_{(U,a,t)} \) which is an exact symplectic deformation. Thus we have the exact Lagrangians, \( \{j_{0,t}(L_{i,0}^{iv})\} \) contained in \( W_{t}^{-1}(q_t) \cap M_{(U,a,t)} \) as well.

**Proposition 9.** With the notation above, \( \{j_{0,t}(L_{i,0}^{iv})\} \) is exact isotopic in \( M_{(U,a,t)} \) to \( \{L_{i,t}^{iv}\} \) for every \( i \).

**Proof.** This result follows from considering the definitions of both \( j \) and \( L_{i,t}^{iv} \). Both such objects were constructed by adding a sufficiently large multiple of the fiberwise Liouville vector field and integrating the new parallel transport map. Indeed, given a smoothly varying critical value \( q_{i,t} \) one can consider the path \( \gamma(s) = (\gamma_1(s), \gamma_2(s)) \) such that \( \gamma_1(s) = 0 \) and \( \gamma_2(s) = \delta_{i,0}(2s) \) for \( 0 \leq s \leq 1/2 \). For \( 1/2 \leq s \leq 1 \) let \( \gamma_1(s) \) be the path from \( 0 \) to \( t \) and \( \gamma_2(s) = q_{i,1}(s) \). One may slightly perturb \( \gamma \) to make it a smooth map and add a sufficiently large multiple of the Liouville vector field to the parallel transport field. Observe that integrating the resulting field from the critical point \( p_{i,0} \) simply gives \( j_{0,t}(L_{i,0}^{iv}) \). On the other hand, one can deform \( \gamma \) through smooth paths \( \gamma_r \) to \( \gamma'(s) = (\gamma'_1(s), \gamma'_2(s)) \) such that for \( 0 \leq s \leq 1/2 \) one has \( \gamma'_1(s) = \gamma_1(1/2 + s) \) and \( \gamma'_2(s) = \delta_{i,1}(s)(0) \). For \( 1/2 \leq s \leq 1 \) let \( \gamma'_1(s) = t \) and \( \gamma'_2(s) = \delta_{i,t}(2s - 1) \). One sees that integrating the resulting vector field along this path from the point \( p_{i,0} \) yields \( L_{i,t}^{iv} \). For each path in the deformation from \( \gamma \) to \( \gamma' \) we can avoid critical values of \( F \) and obtain a smooth family of exact lagrangians in \( W_{t}^{-1}(q_t) \cap M_{(U,a,t)} \). This gives us the required isotopy. \( \square \)

Using this and Theorem 3, we have the following corollary which also holds on the level of quasi-isomorphic \( A^\infty \)-categories.

**Corollary 3.** Let \( W_- \) be a deformation of a partial Lefschetz fibration \( (W,U,a) \) and \( \{\delta_{i,t}\} \) be a smoothly varying family of distinguished bases for \( (W_t,U,a) \). Then \( \mathcal{D}(\text{Fuk}(W,U,a,\{\delta_i\})) \) is equivalent to \( \mathcal{D}(\text{Fuk}(W_t,U,a,\{\delta_{i,t}\})) \) for every \( t \in D_\varepsilon \).
Proof. Using Theorem 3 and Proposition 8, we have that the derived category generated by $L_{t,0}^{iv}$ in $M(U,a)$ is equivalent to that generated by $j_{(0,t)}(L_{t,0}^{iv})$ in the image of $N(U,a,t)$ in $M(U,a,t)$. By the previous proposition, these lagrangians are exact isotopic to $\{L_{t,0}^{iv}\}$ in $M(U,a,t)$ so that again by Theorem 3, their derived categories are equivalent in $M(U,a,t)$. To be precise, one should use the proposition which can be found in [18] under "Properties of the Fukaya Category" to see that all moduli spaces defining the higher products are equivalent (i.e. to see that the image of a holomorphic curve in $M(U,a,t)$ with boundary in $N(U,a,t)$ actually lies in $N(U,a,t)$). □

Utilizing this corollary and the results of previous subsections, we come to the main result of this section. To understand the statement we recall that the Fukaya category has an obvious exceptional collection corresponding to the ordered distinguished basis of paths. Given a connected open subset $U$ in $\mathbb{C}$ and a set of critical values of some potential $W$, we can choose a basis of paths from a regular point in $U$ to the critical points in $U$ such that each path is contained in $U$. We can then extend this set to a distinguished basis of $W$ so that all the paths of the basis which tend to points outside of $U$ occur before those in our set. Doing so yields a semi-orthogonal decomposition of the Fukaya category into two subcategories, the left piece being the subcategory generated by the vanishing cycles associated to points outside $U$ and the right piece the subcategory generated by vanishing cycles associated to the critical points in $U$.

**Theorem 4.** Let $W_-$ be a deformation of a partial Lefschetz fibration $(W,U,a)$ and $\{\delta_{i,t}\}$ be a smoothly varying family of distinguished basis for $(W_t,U,a)$. Suppose $W_{t_0}$ is a Lefschetz fibration for some $t_0$. Then there is a semi-orthogonal decomposition

$$\mathcal{D}^b(Fuk(W_{t_0})) = \langle \mathcal{T}, \mathcal{D}^b(Fuk(W,U,a,\{\delta_i\})) \rangle$$

where the triangulated category $\mathcal{T}$ is generated by the vanishing cycles associated to critical values of $W_{t_0}$ in $\mathbb{C} - U$.

Proof. We start by choosing a distinguished basis $\{\gamma_1, \ldots, \gamma_k, \delta_{0,t}, \ldots, \delta_{m,t}\}$ for $W_t$. Here, the paths $\gamma_j$ are from $q_t \in U$ to the critical values of $W_t$ in $\mathbb{C} - U$. We have that $\mathcal{D}^b(Fuk(W_t))$ decomposes into the category $\mathcal{T}$ generated by the vanishing cycles associated to the $\gamma_j$ and into the category generated by the vanishing cycles associated to $\delta_{i,t}$. By Corollary 2, the latter category is equivalent to $Fuk(W_t,U,a,\{\delta_i\})$, and by the Corollary 3, this category is equivalent to $Fuk(W,U,a,\{\delta_i\})$. □

3.4. The case $M = (\mathbb{C}^*)^n$. In this section we will be concerned with Laurent polynomials $W(z_1, \ldots, z_n) = \sum_{\alpha \in A} c_\alpha z^\alpha$ from $(\mathbb{C}^*)^n$ to $\mathbb{C}$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Also $A \subset \mathbb{Z}^n$ is the support of $W$, i.e. $c_\alpha \neq 0$ if and only if $\alpha \in A$. In certain situations, we will be able to decompose $A$ as the union of two subsets $A_1$ and $A_2$. Each subset comes with
it’s own potential \( W_i \) on \((\mathbb{C}^*)^n\) which can be taken to be \( W \) restricted to \( \mathcal{A}_i \), i.e., we set all coefficients with subscripts not in \( \mathcal{A}_i \) equal to zero. The idea is to apply Theorem 4 to this situation and obtain a semi-orthogonal decomposition of \( \mathcal{D}^b(\text{Fuk}(W)) \) into the categories \( \mathcal{D}^b(\text{Fuk}(W_1)) \) and \( \mathcal{D}^b(\text{Fuk}(W_2)) \). One of these categories, however, will require use of a partial Lefschetz fibration, as the associated symplectic vector field will not be integrable for all paths in \( \mathbb{C} \). In carrying out this plan, we will make heavy use of techniques from \([10]\).

We start by establishing some notation. Throughout this paper, we will take \( \rho(z_1, \ldots, z_n) = \sum_{i=1}^n 1/2(\log |z_i|^2)^2 \) to be the Kähler potential on \((\mathbb{C}^*)^n\). One can see that any \( a \neq 0 \) is a regular value for \( \rho \) and that \( \rho^{-1}((-\infty, a]) \) is compact for all such \( a \), i.e. \( \rho \) is an exhaustive plurisubharmonic function on \((\mathbb{C}^*)^n\). We also observe that \( \omega = \sum_{i=1}^n |z_i|^{-2} dz_i \wedge d \overline{z}_i \). Now, given any Laurent polynomial \( f(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \) we will call \( \mathcal{A} = \{ \alpha \in \mathbb{Z}^n | c_\alpha \neq 0 \} \) the support of \( W \) and \( \text{Newt}(W) \subseteq \mathbb{R}^n \) the Newton polytope of \( W \) where \( \text{Newt}(W) \) is the convex hull \( \text{Conv}(\mathcal{A}) \) of \( \mathcal{A} \).

Given any subset \( A \subseteq \mathbb{Z}^n \) we can consider the space \( \mathbb{C}^A \) consisting of all Laurent polynomials whose support is contained in \( A \). Recall from \([10]\) that there is a polynomial \( E_A : \mathbb{C}^A \to \mathbb{C} \) called the principal A-determinant whose zero locus consists generically of those polynomials \( f \in \mathbb{C}^A \) which have a solution to the system of equations

\[
\begin{align*}
f(z) &= 0 \\
z_1 \frac{\partial}{\partial z_1} f(z) &= 0 \\
& \vdots \\
z_n \frac{\partial}{\partial z_n} f(z) &= 0
\end{align*}
\]

Now, if \( 0 \in A \) and \( f = \sum_{\alpha \in A} c_\alpha z^\alpha \in \mathbb{C}^A \) then we let \( f_s = f - s \in \mathbb{C}^A \) where \( s \) is a constant. This gives a 1-dimensional subspace \( V_f \) parameterized by \( s \) of \( \mathbb{C}^A \) and restricting \( E_A \) to this subspace gives a one variable polynomial \( E_A(f_s) \). One can see that the set \( Z_f = \{ s \in \mathbb{C} | E_A(f_s) = 0 \} \) contains the set of critical values of \( f \). We will see that picking sufficiently generic coefficients for \( f \) and assuming some basic properties for \( A \), \( Z_f \) will equal the set of critical values of \( f \).

A crucial result in \([10]\) is the description of the Newton polytope of \( E_A \) as the secondary polytope \( \Sigma(A) \) of \( A \). For what follows we will require some notation and results on \( \Sigma(A) \) which are taken directly from \([10]\).

**Definition 6.** (i) A marked polytope \( (Q, A) \) is a subset \( A \subseteq \mathbb{Z}^n \) and a polytope \( Q \subseteq \mathbb{R}^n \) such that all vertices of \( Q \) lie in \( A \) and \( A \subseteq Q \).

(ii) A subdivision of a marked polytope \( (Q, A) \) consists of marked polytopes \( \{(Q_i, A_i)\} \) such that the dimension of \( Q_i \) is \( n \) for all \( i \), the intersection of any \( Q_i \) and \( Q_j \) gives a face of each, \( A_i \cap Q_i \cap Q_j = A_j \cap Q_i \cap Q_j \) and the union of all \( Q_i \) is \( Q \).

(iii) A subdivision \( \{(Q'_j, A'_j)\} \) of \( (Q, A) \) is a refinement of a subdivision \( \{(Q_i, A_i)\} \) of \( (Q, A) \) if the set of all \( Q'_j \subseteq Q_i \) forms a subdivision of \( (Q_i, A_i) \).
(iv) A subdivision \( \{ (Q_i, A_i) \} \) of \( (Q, A) \) will be called a triangulation if \( A_i \) is an affinely independent subset of \( \mathbb{Z}^n \) for all \( n \).

Observe that the definition of a triangulation implies that \( Q_i \) are simplices for all \( i \). In order to describe \( \Sigma(A) \), one needs the notion of coherent subdivisions. These subdivisions come from examining the upper boundary of a piecewise linear concave function. More precisely, consider an element \( \psi \in \mathbb{R}^A \) as a function from \( A \) to \( \mathbb{R} \). We can form the convex hull

\[
G_\psi = \text{Conv}\{(\alpha, y) : y \leq \psi(\alpha), \alpha \in A, y \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R}
\]

The upper boundary of \( G_\psi \) will be the graph of a piecewise linear function \( g_\psi : Q \to \mathbb{R} \) where \( Q = \text{Conv}(A) \). Form a subdivision \( S(\psi) = \{(Q_i, A_i)\} \) of \( (Q, A) \) by letting \( Q_i \) be the maximal domains of linearity for \( g_\psi \) and \( A_i = \{ \alpha \in A : \psi(\alpha) = g_\psi(\alpha) \} \). It is immediate that \( S(\psi) \) actually forms a subdivision and moreover, for generic \( \psi \), \( S(\psi) \) will give a triangulation.

**Definition 7.** A subdivision \( \{(Q_i, A_i)\} \) of \( (Q, A) \) is called coherent if there exists a \( \psi \) such that \( S(\psi) = \{(Q_i, A_i)\} \).

Now, given a triangulation \( T = \{(Q_i, A_i)\} \) of \( (Q, A) \) and any \( \psi \in \mathbb{R}^A \) we can construct a piecewise linear function \( g_{\psi,T} : Q \to \mathbb{R} \) which is defined by linearly interpolating \( \psi \) inside every individual \( Q_i \).

**Definition 8.** Let \( C(T) \subset \mathbb{R}^A \) consist of those \( \psi \in \mathbb{R}^A \) which satisfy

(i) The function \( g_{\psi,T} : Q \to \mathbb{R} \) is concave.

(ii) For any \( \alpha \in A \) which is not a vertex of any simplex from \( T \), we have \( g_{\psi,T}(\alpha) \geq \psi(\alpha) \).

Taking all coherent triangulations, the cones \( C(T) \) piece together to form a fan called the secondary fan. It is known that this fan is the normal fan of a polytope called the secondary polytope \( \Sigma(A) \) of \( A \). Given a coherent triangulation \( T \), one can define the element \( \varphi_T \) in \( (\mathbb{R}^A)^* \) via

\[
\varphi_T(v_\alpha) = \sum_{\alpha \in A_i} \text{Vol}(Q_i)
\]

where the sum is over all \( (Q_i, A_i) \) appearing in \( T \) with \( \alpha \in A_i \) and \( \{v_\alpha : \alpha \in A\} \) is the canonical basis of \( \mathbb{R}^A \). Define \( \Sigma(A) \subset (\mathbb{R}^A)^* \) to be the convex hull of \( \{\varphi_T : T \text{ a coherent triangulation of } (Q, A)\} \) and if \( S \) is a subdivision of \( (Q, A) \) let \( F(S) \) be the convex hull of all \( \varphi_T \) for which \( T \) is a refinement of \( S \). Dually, let \( C(S) \) be the set of \( \psi \in \mathbb{R}^A \) for which \( S \) is a refinement of \( S(\psi) \).

**Theorem 5.** ([10]) The secondary fan is the normal fan of \( \Sigma(A) \). Furthermore, the faces of \( \Sigma(A) \) are the polytopes \( F(S) \) for all coherent subdivisions of \( (Q, A) \). The normal cone of \( F(S) \) is \( C(S) \).

This theorem along with the fact that \( \text{Newt}(E_A) = \Sigma(A) \) and the product formula for \( E_A \) are all we will need on the secondary polytope in this paper.
Definition 9. Let $(Q, A)$ be a marked polytope in $\mathbb{R}^n$ such that $0 \in A$ and $0 \in \text{Int}(Q)$. We will say that a subdivision $\{(Q_0, A_0), (Q_1, A_1)\}$ is a bisection of $(Q, A)$ if $0 \in \text{Int}(Q_0)$ and $A_0 \cup A_1 = A$.

A bisection corresponds to breaking a given polytope into two pieces, with one piece containing the origin in its interior. Given such a bisection $S$, there is an element $\eta_S \in C(S)$ which is uniquely defined by the properties that $\eta_S \in \mathbb{Z}^A$, $\eta_S|_{A_0} = 0$ and $\int_Q \eta_S$ is maximal. Now, given a potential $W = \sum_{\alpha \in A} c_{\alpha} z^\alpha$ and a bisection $S$ of $(Q, A)$, we define a perturbation of $W$ via

$$W_t(z) = \sum_{\alpha \in A} c_{\alpha} t^{-\eta_S(\alpha)} z^\alpha$$

With such a perturbation of $W$, consider $E_A(W_t(z))$. From [10] we have that $E_A$ has Newton polytope $\Sigma(A)$, so

$$E_A(W_t(z)) = \sum_{\varphi \in \Sigma(A)} d_\varphi \prod_{\alpha \in A} c_{\alpha}^{\varphi(\alpha)} t^{-\varphi(\eta_S(\alpha))}$$

As $\eta_S \in C(S)$ Theorem 5 implies that $-\eta_S$ achieves its minimum on $F(S)$; say $-\sum_{\alpha \in A} \varphi(\alpha) \eta_S(\alpha) = k$ for $\varphi \in F(S)$, one can check that $k$ is positive. Then the above equation can be written

$$E_A(W_t(z)) = t^k \sum_{\varphi \in F(S)} \left( d_\varphi \prod_{\alpha \in A} c_{\alpha}^{\varphi(\alpha)} \right) + t^{k+1} q(t, c_\alpha)$$

Where $q(t, c_{\alpha})$ is the part of the principal $A$-determinant with support off of $F(S)$. Now let us recall the product formula from [10] for principal $A$-determinants. In general if $S = \{(Q_t, A_t)\}$ is a subdivision of a marked polytope $(Q, A)$ and $f$ is a Laurent polynomial with support on $A$, this theorem asserts that the part of $E_A(f)$ whose support is the face $F(S)$ is simply the product $\prod E_{A_t}(f|_{A_t})$. Here $f|_{A_t}$ is the part of $f$ whose support lies in $A_t$.

Using this, we can further reduce the above equation to:

$$E_A(W_t(z)) = t^k E_{A_0}(W|_{A_0}) E_{A_1}(W|_{A_1}) + t^{k+1} q(t, c_{\alpha})$$

Now let us set the constant $c_0$ equal the variable $s$, then we can regard $E_A(W_t(z))$ as a polynomial $g(s, t)$ in two variables. For a given value of $t$, assuming $W_t(z)$ is generic, the zeros of $g(s, t)$ will then be the critical values of $W_t(z)$. One can see this by investigating the secondary polytope which reveals that, for all $t \neq 0$, $g(s, t)$ has degree $Vol(Q)$ as a polynomial in $s$. To see this just choose a generic function $\psi$ on $A$ setting $\psi(0) = 0$ and all other elements negative. This will yield a triangulation $T$ with $\varphi_T(0) = Vol(Q)$ which is the degree of $s$ in $E_A(W_t(z))$ (again, assuming generic coordinates). In particular, the number of zeros up to multiplicity in the variable $s$ of $E_A(W_t(z))$ is $Vol(Q)$ for $t \neq 0$ which is in line with the number of critical points of $W_t(z)$ as given by Kouchnirenko’s Theorem [13]. Observe that for $t = 0$ we have that $g(s, t) = 0$, so $W_0$ is not generic. However, dividing
g(s, t) by \( t^k \) we have

\[
\frac{g(s, t)}{t^k} = E_{A_0}(W|A_0)E_{A_1}(W|A_1) + tq(t, c_0)
\]

Now, \( E_{A_1}(W|A_1) \) does not depend on \( s \) as \( 0 \notin A_1 \), so for a generic choice of coefficients this is a non-zero constant. On the other hand \( E_{A_0}(W|A_0) \) will be a polynomial in \( s \) with zeros at the critical values of \( W|A_0 \). So, in particular, as \( t \) tends to zero, \( g(s, t)/t^k \) converges to a non-zero polynomial in \( s \) with zeros at the critical values of \( W|A_0 \). Applying a convergence argument, one sees that all other zeros of \( g(s, t)/t^k \) must tend to infinity as \( t \to 0 \). For a given \( t \neq 0 \) let \( \{\nu_1(t), \ldots, \nu_r(t)\} \) be the zeros of \( g(s, t) \). The preceding argument shows that the critical values of \( W_t \) split into two groups as \( t \) tends to zero, the first group \( \{\nu_1(t), \ldots, \nu_m(t)\} \) corresponding to those critical points of \( W|A_0 \) and the second \( \{\nu_{m+1}(t), \ldots, \nu_r(t)\} \) tending out to infinity. One sees then that the number of critical values in the second group \( r - m \) is the number of critical values of \( W|A_1 \). Indeed, we would like to identify these critical values to those of \( W|A_1 \); this requires reparameterizing the deformation.

Given an affine map \( L : \mathbb{Z}^n \to \mathbb{Z} \), one can reparameterize \( W_t(z) \) to give the map

\[
W_{L,t}(z_1, \ldots, z_n) = t^{L(0)}W_t(t^{L(e_1)}-L(0)z_1, \ldots, t^{L(e_n)}-L(0)z_n)
\]

Using the homogeneity properties of \( E_A \), we have

\[
E_A(W_{L,t}) = t^{L_Q}E_A(W_t)
\]

Here \( L_Q = L(\int_Q xdx) + L(0) \cdot [(n + 1)Vol(Q) - 1] \). Now, one can define another element \( \tau_S \in C(S) \) by setting \( \tau_S = \eta_S - L \) where \( L \) is the affine map from \( \mathbb{Z}^n \) to \( \mathbb{Z} \) which is the restriction of \( \eta_S \) to \( A_1 \). One notes that \( \tau_S \) is zero when restricted to \( A_1 \) and \( \tau_S(0) < 0 \). The graphs of \( -\eta_S \) and \( -\tau_S \) for \( A = \{-1, 0, 1, 2\}, A_0 = \{-1, 0, 1\} \) and \( A_1 = \{1, 2\} \) are shown below.

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Graph of \( -\eta_S \)  

Graph of \( -\tau_S \)  

---

If we define a deformation of \( W \) using \( \tau_S \) via

\[
\tilde{W}_t(z) = \sum_{\alpha \in A} c_\alpha t^{-\tau_S(\alpha)}z^\alpha
\]

then one can easily show that \( \tilde{W}_t(z) = W_{L,t}(z) \). Therefore \( E_A(\tilde{W}_t(z)) = t^{L_Q}g(s, t) \). However, fixing a \( t \neq 0 \), the \( s \) zeros of \( g(s, t) \) will not represent critical values of \( \tilde{W}_t(z) \) as the constant term is no longer \( c_0 = s \). Having reparameterized, the new constant term is \( t^{L(0)}c_0 = t^{L(0)}s \), so the critical
values of $\hat{W}_t(z)$ for a fixed $t \neq 0$ are the points $\{t^{L(0)}_1(\nu_1(t)), \ldots, t^{L(0)}_r(\nu_r(t))\}$. At $t = 0$ we see that $\hat{W}_t(z) = W|_{A_1}$ which has $r - m$ critical values. As $t$ varies slightly, there will be $r - m$ critical values an $\varepsilon$-distance away from these values. Assuming that we chose coefficients of $W$ in a sufficiently generic way, we may assume that all of the critical values of $W|_{A_1}$ are distinct. Let us assume also that all of these critical values are non-zero. Putting all of this together, we see that as $t$ tends towards zero, the first $m$ critical values $\{t^{L(0)}_1(\nu_1(t)), \ldots, t^{L(0)}_m(\nu_m(t))\}$ of $\hat{W}_t(z)$ tend towards zero since it was shown above that the values $\nu_i(t)$ for $1 \leq i \leq m$ converge to finite values. On the other hand, there must be $r - m$ values converging to the critical points of $W|_{A_1}$ and as these values are distinct, this forces the remaining critical values $\{t^{L(0)}_{m+1}(\nu_{m+1}(t)), \ldots, t^{L(0)}_r(\nu_r(t))\}$ to converge to these finite values. We codify these results in the following proposition.

**Proposition 10.** Given a generic potential $W$ with support $A$ and Newton polytope $Q$ and an elementary subdivision $\{(Q_0, A_0), (Q_1, A_1)\}$, one can define two deformations $W_t$ and $W_l$ of $W$. There is a labeling $\{\nu_1(t), \ldots, \nu_r(t)\}$ of the critical values of $W_t$ for $t \neq 0$ such that $\{t^{L(0)}_1(\nu_1(t)), \ldots, t^{L(0)}_r(\nu_r(t))\}$ are the critical values of $\hat{W}_t$ where $L(0) > 0$. Furthermore, as $t$ tends towards zero, there is an $m$ such that

(i) $\{\nu_1(t), \ldots, \nu_m(t)\}$ approaches the critical values of $W|_{A_0}$.

(ii) $\{t^{L(0)}_{m+1}(\nu_{m+1}(t)), \ldots, t^{L(0)}_r(\nu_r(t))\}$ approaches the critical values of $W|_{A_1}$, each of which is non-zero.

We now turn to the main result of this subsection. Suppose $(W, U, a)$ is a partial Lefschetz fibration. If $W_t$ is a perturbation of the potential, one sees that for a sufficiently small $\varepsilon$ the openness of transversality ensures that for $t \in D_\varepsilon$, $(W_t, U, a)$ is a partial Lefschetz fibration if and only if no critical values of $W_t$ enter into $U$. This fact combined with the previous proposition gives us the following theorem.

**Theorem 6.** Suppose $W$ is a Laurent polynomial with support $A$ and Newton polytope $Q$ and $\{(Q_0, A_0), (Q_1, A_1)\}$ is a bisection of $(Q, A)$. Assume $W$ is generic in the following sense:

(i) $W$ is a Lefschetz fibration on $(\mathbb{C}^*)^n$

(ii) $W|_{A_0}$ is a Lefschetz fibration on $(\mathbb{C}^*)^n$

(iii) $(W|_{A_1}, \mathbb{C} - D_\varepsilon, a)$ is a partial Lefschetz fibration for sufficiently large $a$

(iv) $W|_{A_1}$ has nonzero critical values

Then, after choosing a distinguished basis of paths $\{\delta_i\}$ for $(W|_{A_1}, \mathbb{C} - D_\varepsilon, a)$, there is the following semi-orthogonal decomposition:

$$\mathcal{D}^b(\text{Fuk}(W)) = \left< \mathcal{D}^b(\text{Fuk}(W|_{A_0})), \mathcal{D}^b(\text{Fuk}(W|_{A_1}, \mathbb{C} - D_\varepsilon, a, \{\delta_i\})) \right>$$

**Proof.** By Proposition 10 and the above observation we have that $\hat{W}_t$ is a deformation of the partial Lefschetz fibration $(W|_{A_1}, \mathbb{C} - D_\varepsilon, a)$ for $|t| < \varepsilon$. But, by assumption (i), $W = \hat{W}_1$ is a Lefschetz fibration and from
Proposition 10 $\tilde{W}_t$ is a deformation of $W$ for $t \neq 0$. Thus
\[ D^b(Fuk(W)) \simeq D^b(Fuk(\tilde{W}_t)) = \langle T, D^b(Fuk(W|_{A_t}, C - D_\epsilon, a, \{\delta_i\}) \rangle \]
where the last equality follows from Theorem 4 and the first from Corollary 3. Now, the category $T$ is generated by those vanishing cycles coming from critical values of $\tilde{W}_\epsilon$ in $D_\epsilon$. Deforming to $W$ and then using the deformation $W_t$ to deform to $W_0$, Corollary 3 gives us that $T \simeq D^b(Fuk(W_0, D_R, a', \{\gamma_i\}))$ for some $R$. But as $W_0 = W|_{A_0}$ the latter category here is just $D^b(Fuk(W|_{A_0}))$.

4. Homological Mirror Symmetry for Weighted Projective Blowups of Toric Surfaces

4.1. The potential $W_a$ on $\mathbb{C}^2$. Recall that homological mirror symmetry for Fano toric varieties asserts an equivalence between $D^b(Coh(X_\Delta))$ and $D^b(Fuk(W_\Delta))$ where $W_\Delta$ is a Laurent polynomial with Newton polytope equal to the convex hull of $\Delta(1)$, the primitives of the one dimensional cones of the fan $\Delta$. Suppose $X$ is a weighted projective blowup of a smooth toric variety of a fan $\Delta$ at a $(\mathbb{C}^*)^n$ invariant point. In section 2 we saw that there is an $n + 1$ cone $\sigma$ in $\Delta$ generated by $\{v_0, \ldots, v_n\}$ such that $X$ is the toric stack of a fan $\tilde{\Delta}$ with $\tilde{\Delta}(1) = \Delta(1) \cup \{a_0v_0 + \cdots + a_nv_n\}$. Thus, if $W_\Delta$ is the potential associated to $\Delta$ then $W_\Delta = W_\Delta + Cz_0^{a_0} \cdots z_n^{a_n}$ (in fact, one could use other monomials in this expansion, but we will see that this is not necessary). The picture below shows the situation for the $a = (2,3)$ weighted projective blowup of $\mathbb{P}^2$.

Now, in section 2 we saw that given a weighted projective blowup $X'$ of $X$, the derived category can be expressed as a semi-orthogonal decomposition $\langle D^b(Coh(X')), T \rangle$ where $T$ has an exceptional collection Koszul dual to the pushforwards of line bundles. On the other hand, in section 3 we showed
that the derived Fukaya category of a potential $W$ on $(\mathbb{C}^*)^n$ also has a semi-orthogonal decomposition associated to a bisection. For the potential $W_{\tilde{\triangle}}$, one sees that in many situations there is a clear bisection of the convex hull of the primitives of $\triangle(1)$. Namely, take the convex hull of $\triangle$ and the convex hull $Q_a$ of $A_a = \{v_0, \ldots, v_n, a_0v_0 + \cdots + a_nv_n\}$. Note that in order for this to be a bisection one must have that $\text{Conv}(\triangle(1)) \cup Q_A = \text{Conv}(\tilde{\triangle}(1))$. This will be assumed for the rest of this paper. We illustrate this subdivision below for the potential associated to $Bl(2,3)(\mathbb{P}^2)$.

Now, there are several elementary examples where it is known that Homological mirror symmetry holds, e.g. weighted projective planes and smooth Fano toric surfaces ([1], [21]). Thus our strategy is to assume that homological mirror symmetry holds for $\triangle$ and show that the extra semi-orthogonal categories are equivalent as well. Although this does not fully prove mirror symmetry, as the interaction between the two semi-orthogonal categories is neglected, it does indicate that the mirror categories have equivalent pieces in natural decompositions. We will pursue this strategy for weighted projective blowups of smooth toric surfaces.

In general, the potential $W_a$ associated to $(A_a, Q_a)$ can be written

$$W_a(z_0, \ldots, z_n) = a_0z_0 + \cdots + a_nz_n - z_0^{a_0}\cdots z_n^{a_n}$$

It is the case that this potential is generic enough to fit into the definition of a partial Lefschetz fibration. It also has the advantage of having a good deal of symmetry which helps in its investigation. Observe that

$$\partial_iW_a(z_0, \ldots, z_n) = a_i - a_i z_0^{a_0} \cdots z_i^{a_i-1} \cdots z_n^{a_n}$$

so if $(c_0, \ldots, c_n)$ is a critical point of $W_a$ then $c_0^{a_0} \cdots c_i^{a_i-1} \cdots c_n^{a_n} = 1 = c_0^{a_0} \cdots c_j^{a_j-1} \cdots c_n^{a_n}$ for every $i$ and $j$. In particular, $c_i \neq 0$ for all $i$ and dividing the above equation by $c_0^{a_0} \cdots c_i^{a_i-1} \cdots c_j^{a_j-1} \cdots c_n^{a_n}$ we have $c_i = c_j$. Thus a critical point of $W_a$ can be written $(c, \ldots, c)$ where $c$ must satisfy the equation $c^{l-1} = 1$ where $l = \sum_{i=0}^n a_i$. Thus there are $l-1$ critical points $\{p_0, \ldots, p_{l-2}\}$ such that $p_i = (\zeta^i, \ldots, \zeta^i)$ where $\zeta$ is the $(l-1)$-th root of
union $e^{2\pi i/(l-1)}$. One can compute that the corresponding critical value $q_i$ of $p_i$ as $q_i = W_a(p_i) = (l - 1) \cdot \zeta^i$.

At first it appears that $W_a$ has smooth fibers outside the critical values $q_i$ and that parallel transport could be defined for paths. This is indeed the case if we consider $W_a$ as a fibration on the partial compactification $\mathbb{C}^{n+1}$ of $(\mathbb{C}^*)^{n+1}$. However, there are two objections to viewing $W_a$ as a Lefschetz fibration on $\mathbb{C}^{n+1}$. First, the standard Kähler structure on $\mathbb{C}^{n+1}$ is not an extension of that on the complex torus. This can easily be remedied by perturbing and using Theorem 4. The second and more serious objection is that the zero fiber of $W_a$ does not transversely intersect the normal crossing divisor $\bigcup z_i = 0$, so the topology of this fiber in $(\mathbb{C}^*)^{n+1}$ will abruptly change from those of neighboring fibers. Nevertheless, it will be helpful to first regard $W_a$ as a potential on $\mathbb{C}^{n+1}$ and obtain information on the vanishing cycles at the zero fiber. We then work backwards and perturb to $W_a$ to first regard

$$W_a(a) = \left( \begin{array}{c} z_0 \\ \vdots \\ z_n \end{array} \right) : (\mathbb{C}^*)^{n+1} \to \mathbb{C}^2$$

so that the discriminant variety is simply

$$D = \{(c, \ldots, c) \cup (\cup z_i = 0)\}$$

Thus the discriminant variety can be written as the union of two components $C_1$ and $C_2$. The diagonal component $C_1$ contains the critical points of $W_a$, and the other component $C_2$ is a union of divisors $\{z_i = 0\}$. Let us examine the critical values of $f$ restricted to the fiber $F_q$ associated to the second component of the discriminant variety. Since any element $p \in F_q \cap C_2$ must have at least one $z_i = 0$ we have that $z_0^{a_0} \cdots z_n^{a_n} = 0$ so that $q = W_a(p) = f(p)$. Thus $f(F_q \cap C_2) = q$; more generally, we have

$$f \big|_{F_q}^{-1}(q) = F_q \cap (\cup_{i=0}^{n} \{z_i = 0\})$$

If $C_2 = \emptyset$ (i.e. if $a_i = 1$ for all $i$) it will still be convenient to keep track of this image. Doing so will allow us to see which fibers have vanishing cycles intersecting the divisor $\cup_{i=0}^{n} \{z_i = 0\}$.

We now examine the the critical values of $f$ restricted to a fiber $F_q$ associated to the diagonal component $C_1$. These will be $f(c, \ldots, c) = l \cdot c$ where
c satisfies $W_a(c, \ldots c) = lc - d^l = q$. In other words, these critical values are the roots of the polynomial $h_q(x) = x^l - l^l x + l^l q$. Either by the theory of matching paths or by direct computation, one sees that $h_q(x)$ has multiple roots only if $q$ is among the critical values of $W_a$. Furthermore, $q$ is a root of $h_q(x)$ only for $q = 0$ which implies $f(C_1 \cap F_q) \cap f(C_2 \cap F_q)$ is empty for all $q \neq 0$. This implies that for a path $\gamma : [0, 1] \to \mathbb{C}^*$, the vanishing cycle in $F_{\gamma(t)}$ can be consistently isotoped so that it does not intersect the divisor $\cup_{i=0}^n \{z_i = 0\}$ and thus is contained in $(\mathbb{C}^*)^{n+1}$.

Now, observe that $\{0, l^{l/l-1}, l^{l/l-1} \zeta, \ldots, l^{l/l-1} \zeta^{l-2}\}$ are the roots of $h_0(x)$ and thus are the critical values of $f$ restricted to $F_0$. We would like to see the movement of the critical values of $f \mid F_q$ as $q$ moves from 0 to $l - 1$. These will be the zeros of $h_q(x)$ and the point $q$ itself. Let us focus on the real roots of $h_q(x)$ for a moment. We observe that, as a real function, $h_0(x)$ has two different graphs depending on whether $l$ is even or odd. These are drawn below:

\[
\begin{align*}
\text{h}_0(x) \text{ for } l \text{ even} & \quad \text{h}_0(x) \text{ for } l \text{ odd} \\
\end{align*}
\]

As $q$ tends from 0 to $l - 1$, we see that the real roots of $h_q(x)$ are simply the $x$-coordinates of the intersection of the horizontal line at $y = -l^l q$ with the above graphs. Thus, the two real roots 0 and $l^{l/l-1}$ contract together along the real axis without any other critical values associated to $C_1$ passing between them. Furthermore, the critical value associated to $C_2$ is simply the intersection of $y = -l^l q$ with the line $y = -l^l x$. Now, $h_0(x) - (-l^l x) = x^l$ is positive for all $x > 0$; so the real critical value associated to $C_2$ does not pass between the contracting roots either (although the 0 root "hits" this value for $q = 0$). To put this in the language of matching paths, we can say that to the path $\delta_0(t) = (l-1)t$ from the regular value 0 to the critical value $l - 1$, we associate the matching path $\eta_0(s) = t^{l/l-1} s$. These two paths occur on different lines, $\delta_0$ occurs in the range of $W_a$ while $\eta_0$ occurs on the image of $F_0$ via $f$ connecting the critical values. This is illustrated below:
Observe that $W_{\alpha}(\zeta z) = \zeta W_{\alpha}(z)$ and $f(\zeta z) = \zeta f(z)$. This symmetry implies that the matching path associated to $\hat{\delta}_i(t) = \zeta^i \delta_0(t)$ is $\eta_i(s) = \zeta^i \eta_0(s)$. Thus, in the picture above, one can rotate both the path in the image of $W_{\alpha}$ and its matching path in the image of $f|_{F_0}$ by the angle $\frac{2\pi}{l-1}$ to obtain all other pairs $(\hat{\delta}_i, \eta_i)$. Now, by the theory of matching paths, a matching path $\eta_i$ for $\hat{\delta}_i$ is isotopic to the image of the vanishing cycle $L_i$ under the map $f$, and the endpoints of the matching path are the critical values of a standard height function on the vanishing cycle $L_i \simeq S^n$. This machinery applies if the critical points of $f$ on $F_q$ corresponding to the endpoints are Morse. While this is the case for all $q \neq 0$, once $q = 0$ we acquire a more complicated singularity on $F_0$. We will ignore this fact as we are going to perturb to a neighboring fiber in the next section, however, the vanishing cycles we will describe in $F_0$ will necessarily have corners at their common intersection which is in $F_0 \cap f^{-1}(0)$. Thus we will identify the vanishing cycle $L_i$ associated to $\hat{\delta}_i$ with the component of the pre-image $f^{-1}(\eta_i)$ which contains the critical point $p_i$. After doing so, we see that all of the intersection points of vanishing cycles lie in the subspace $F_0 \cap f^{-1}(0)$.

At this point we restrict our attention to the case of $n = 1$. It will be convenient to adopt the convention that $\alpha_0 \leq \alpha_1$ which we will assume throughout. Observe that $F_0 \cap f^{-1}(0) = \{(0,0)\}$ so that all vanishing cycles $L_i$ in $F_0$ intersect in precisely one point, namely $\{(0,0)\}$. In what follows, we will represent $F_0$ as a $(l-1)$-fold branched covering of $\mathbb{P}^1 - \{0, \infty\}$ and describe the vanishing cycles as lifts of a single curve in $\mathbb{P}^1 - \{0, \infty\}$. To pursue this aim, we quotient $\mathbb{C}^2$ and $\mathbb{C}$ by the diagonal action of $G = \{\zeta^j : 0 \leq j \leq l - 2\}$. We have the diagram:

$$
\begin{array}{ccc}
\mathbb{C}^2 & \overset{W_{\alpha}}{\longrightarrow} & \mathbb{C} \\
\psi_i & \downarrow \zeta^{l-1} & \\
\mathbb{C}^2/G & \overset{\bar{W}_{\alpha}}{\hookrightarrow} & \mathbb{C}
\end{array}
$$

Here we take

$$
\psi(z_0, z_1) = (z_0^{\alpha_0-1} z_1^{\alpha_1}, z_0^{\alpha_0} z_1^{\alpha_1-1}) = (u_0, u_1)
$$
for \((z_0, z_1) \in (\mathbb{C}^*)^2\) and find that

\[
\tilde{W}_a(u_0, u_1) = u_0^{a_0} u_1^{a_1} \left( \frac{a_0}{u_0} + \frac{a_1}{u_1} - 1 \right)^{l-1}
\]

for \(u_0 \neq 0 \neq u_1\). Thus the closure of \(\tilde{W}_a^{-1}(0)\) is the variety \(V = \{(u_0, u_1) : a_0u_1 + a_1u_0 - u_1u_0 = 0\}\). One observes that \(V \cap \{u_0 = 0\} = \{(0, 0)\} = V \cap \{u_1 = 0\}\) which implies that \(\psi : F_0 \to V\) is a branched covering with branch point \((0, 0)\) whose ramification is \(l-1\). Furthermore, the defining equation for \(V\) is a quadratic which implies \(V \approx \mathbb{P}^1\) minus the two points intersecting the divisor at infinity. In other words, one can view the compactification of \(V\) in \(\mathbb{P}^2\) as the hypersurface \(Z = \text{Zero}(a_0u_1u_2 + a_1u_0u_2 - u_1u_0)\). Then we have \(V = Z - \{(0 : 1 : 0), [1 : 0 : 0]\}\).

Now, the fiber of \(\psi\) over any point \(p \in V - \{(0, 0)\}\) is a torsor over \(\mathbb{Z}/(l-1)\mathbb{Z} \approx G\) where the action is given by multiplication by \(\zeta\). We will describe the monodromy of a loop around the points \([0 : 1 : 0]\) and \([1 : 0 : 0]\) to give a complete description of the topology of \(F_0\). For this we give a coordinate map \(j : \mathbb{C} \to \mathbb{C}^2 \subset \mathbb{P}^2\) for \(Z\) where \(j(w) = (w, a_1w/w - a_0)\). Observe that in \(\mathbb{P}^2\), \(j(a_0) = [0 : 1 : 0]\). Now, in \((\mathbb{C}^*)^2\) there is a global multiple valued inverse of \(\psi\) which can be written

\[
\psi^{-1}(u_0, u_1) = \left( \frac{i-1}{i-1} \sqrt[1-1]{\frac{a_1^{a_1}}{u_0^{a_1-1}}}, \frac{i-1}{i-1} \sqrt[1-1]{\frac{a_0}{u_1^{a_0-1}}} \right)
\]

In the coordinate \(w\) this gives

\[
\psi^{-1}(w) = \left( \frac{i-1}{i-1} \sqrt[1-1]{\frac{a_1^{a_1}w}{w - a_0}}, \frac{i-1}{i-1} \sqrt[1-1]{\frac{w(w - a_0)^{a_0-1}}{a_1^{a_0-1}}} \right)
\]

From this one can see that the monodromy around \(a_0\) is multiplication by \(\zeta^{-a_1}\), or, from the torsor point of view, it is addition by \(-a_1\). To find the monodromy around the point \([1 : 0 : 0]\) we simply add the monodromies around \([0 : 1 : 0]\) and \([0 : 1 : 0]\) yielding \(a_0\) as \(1 - a_1 \equiv a_0 (\text{mod}(l-1))\). This gives a complete picture of the topology of \(F_0\).

Now we would like to investigate the vanishing cycles \(L_i \subset F_0\) (with corners). These are maps of \(S^3\) in \(F_0\), which are embeddings except at the point \((0, 0)\) which may have a corner. Each image contains the point \((0, 0)\) which is the only intersection point of any two; furthermore we have \(L_i = \zeta^i L_0\). Thus \(\psi(L_i) = \psi(L_j)\) for all \(i\) and \(j\), \((0, 0) \in \psi(L_0)\) and \(\psi(L_0)\) has no self intersection points in \(V\). Now, \(V\) is topologically a cylinder and the only closed curves with no intersection points in \(V\) up to isotopy are therefore the homotopically trivial curve or the curve wrapping around the cylinder. From standard theory on vanishing cycles, the middle dimensional homology of the fiber is generated by the vanishing cycles, so we can rule out the trivial curve. We summarize these results in the following proposition and picture.
Proposition 11. The zero fiber $F_0 \subset \mathbb{C}^2$ admits a $(l - 1)$-fold branched covering $\psi$ to $Z - \{(1 : 0 : 0), (0 : 1 : 0)\}$ with one branch point over $[0 : 0 : 1]$ of ramification $l - 1$. The monodromy around the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are $a_0$ and $-a_1$ respectively. All vanishing cycles $L_i$ have the same image $\psi(L_i)$ which is isotopic to a closed curve containing the point $[0 : 0 : 1]$, generating the homotopy of $Z - \{(1 : 0 : 0), (0 : 1 : 0)\}$ and containing no self intersections.

The base $V$ of $\psi$

One can see from the above picture that any holomorphic discs connecting intersection points of $L_i$ in a nearby fiber must map to a neighborhood of $[0 : 0 : 1]$. Thus we ought to examine the placement of the vanishing cycles in a neighborhood of $(0,0) \in F_0$. If we take an $\delta$-disc neighborhood $D$ of $(0,0)$ in $F_0$, then $\psi(z) = z^{l-1}$ for a local chart of $V$. Now, taking slits from $[0 : 0 : 1]$ to $[0 : 1 : 0]$ and from $[0 : 1 : 0]$ to $[1 : 0 : 0]$ gives $l - 1$ domains gluing together to form $W^\alpha_a(0)$. We fix the first slit to give the line segments $\{re^{2k+1} \pi i/2(l-1) : 0 \leq r \leq \delta\}$ in the neighborhood $D$. Then, after an isotopy, we have that the vanishing cycles in $D$ are approximately the lines $\ell_k = \{re^{2k+1} \pi i/2(l-1) : 0 \leq r \leq \delta\}$ for $0 \leq k \leq 2l - 3$ (in fact these would give the vanishing cycles with corners at 0). Multiplying this picture by a power of $\zeta$ if necessary, we can assume that $\ell_0 \subset L_0$. Then, following the curve $L_0$ around $V$ we see that it passes through the second slit from $[0 : 1 : 0]$ to $[0 : 0 : 1]$ so that it gains $-a_0$ monodromy. Additionally, the angle that $L_0$ returns with to $(0,0)$ acquires a $\zeta^{1/2}$ factor. Thus if $\ell_0$ is the outgoing part of $L_0$ in $D$ then $\zeta^{1/2 - a_0} \ell_0 = \ell_{1-2a_0}$ is the incoming part of $L_0$ where the subscript in the second line segment should be taken modulo $2(l-1)$. Utilizing the fact that $\zeta^i L_0 = L_i$ we have that

$$L_i \cap D = \ell_{2i} \cup \ell_{1-2a_0 + 2i}$$

We illustrate this for $a = (1, 4)$ and $a = (2, 3)$ below.
4.2. The potential $W_a$ on $(\mathbb{C}^*)^2$. We will now perturb the picture presented in the previous subsection to a fiber $W_a^{-1}(\varepsilon)$ in $(\mathbb{C}^*)^2$. The first observation we make is that the point $(0,0) \in F_0$ no longer exists when $W_a$ is considered as a potential on $(\mathbb{C}^*)^2$. Furthermore, this end of the fiber splits into two ends after perturbing slightly. More precisely, for a given $q \in \mathbb{C}$, we have that $F_q \cap \{ z_0 = 0 \} \cup \{ z_1 = 0 \} = \{ (0,q/a_1),(q/a_0,0) \}$. In the $(\mathbb{C}^*)^2$ picture these two points are not included in the fiber. Furthermore, if one parallel transports the fiber of $W_a$ around a small $\varepsilon$ circle about 0, the resulting monodromy map will be a full Dehn twist of the two ends. In the $\mathbb{C}^2$ picture, this monodromy map is isotopic to the identity; however, after removing the two points, the map becomes non-trivial. In what follows, we will give an explicit map which is symplectically isotopic to the parallel transport map for an arc about the origin in $\mathbb{C}$.

Fix $\varepsilon$ to be a sufficiently small real number and let $q_\theta = \varepsilon e^{i\theta}$. Given any fiber $W_a^{-1}(q_\theta)$, we can form a local chart $U_\theta = \{ w : |w| < 4\varepsilon, w \neq 0, w \neq q_\theta \}$ centered at one of the ends, say $(0,\varepsilon e^{i\theta}/a_1)$ such that the other end is $q_\theta$. Indeed, from the previous subsection we can identify $U_\theta = D - \{ 0, q_\theta \}$ where $D$ was a neighborhood of $(0,0)$ in $F_0$. This gives an identification of $W_a^{-1}(q_\theta)$ with $F_0 - \{ 0, q_\theta \}$. Utilizing such an identification, we will explicitly write a map $M_\theta : W_a^{-1}(q_0) \to W_a^{-1}(q_\theta)$ symplectically isotopic to the parallel transport map along the arc $\gamma(t) = \varepsilon e^{it\theta} = q_\theta$. To define $M_\theta$, let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a smooth non-increasing function which satisfies

$$h(x) = \begin{cases} 1 & \text{for } x < 2\varepsilon \\ 0 & \text{for } x > 3\varepsilon \end{cases}$$

Then we define

$$M_\theta(w) = \begin{cases} e^{h(|w|)\theta}w & \text{for } w \in U_\theta \\ w & \text{otherwise} \end{cases}$$

It is easy to see that $M_{2\pi} : W_a^{-1}(\varepsilon) \to W_a^{-1}(\varepsilon)$ gives a full Dehn twist about a circle encompassing the two ends and yields the isotopy class of the identity if these ends are included.
On the other hand, we have that multiplying a fiber $W^{-1}_a(q)$ by $\zeta^k$ will give a map to the fiber $W^{-1}_a(\zeta^k q)$ as well. We arrange the parameterization of $U_\theta$ in such a way that this map is symplectically equivalent to multiplication by $\zeta^k$. This, together with Proposition 11 yields the following proposition.

**Proposition 12.** For $\varepsilon$ sufficiently small and $q = \varepsilon e^{i\theta}$, the fiber $W^{-1}_a(q)$ is symplectically isotopic to the curve $F_0 - \{0, q\}$ where 0 and $q$ are regarded as elements of the disc neighborhood $D$. Letting $U_\theta = D - \{0, q\}$, the monodromy map along an arc from $q$ to $e^{i\varphi}q$ is symplectically isotopic $M_{\varphi}: W^{-1}_a(q) \to W^{-1}_a(e^{i\varphi}q)$ and the map from $W^{-1}_a(q)$ to $W^{-1}_a(\zeta^k q)$ induced by multiplication is given by the action of $\zeta^k$ on the fiber $F_0$ (i.e. the action on the fibers of the branched cover).

The motivation for describing this parallel transport map is to obtain a description for the vanishing cycles of $W_a$ on $(\mathbb{C}^\ast)^2$. Unlike the situation in $\mathbb{C}^2$, and indeed for all situations involving partial Lefschetz fibrations, we need to use more caution in choosing our distinguished basis of paths in the image of $W_a$. We saw in the last subsection that if one parallel transports the vanishing cycle to the origin, then the vanishing cycle will “fall off” the subspace $W^{-1}_a(0)$ (i.e. it will hit the point $(0, 0)$). Thus, any such path is not allowed. Therefore, instead of taking the paths $\tilde{\delta}_k$ we will take paths $\delta_k$ homotopic in $\mathbb{C} - \{0\}$ to

$$g_k(t) = \begin{cases} 
\varepsilon e^{4\pi kti/(l-1)} & \text{for } 0 \leq t \leq 1/2 \\
(2(1-t)\varepsilon + (2t-1)(l-1)) e^{2\pi i/(l-1)} & \text{for } 1/2 \leq t \leq 1
\end{cases}$$

An example of such a basis is illustrated below.

The curves $g_0(t)$ and $g_4(t)$ The distinguished basis $\{\delta_k\}$

By the previous proposition and the construction of the distinguished basis, we can describe the vanishing cycles $L_k$ associated to $\delta_k$ in terms of the vanishing cycle $L_0$.

**Corollary 4.** The vanishing cycle $L_k$ associated to $\delta_k$ is isotopic to $M_{2\pi k/(l-1)}(\zeta^k \cdot L_0)$
Proposition 13. Let \( s \) be the line segment in \( U_0 \) connecting 0 to \( q_0 \). Then, after a reparametrization of \( U_0 \) if necessary, the vanishing cycle \( L_0 \) obtained from the path \( \delta_0 \) intersects \( s \) in exactly one point.

Proof. To prove this proposition, we return to the matching paths argument. In the fiber \( W_{a}^{-1}(\epsilon) \) we have seen above that the matching path of \( \delta_0 \) is the line segment connecting the two positive real roots of \( h_\epsilon(x) \). Let us label the roots of \( h_\epsilon(x) \), \( \{r_0, r_1, \ldots, r_{l-1}\} \) where \( r_0 \) and \( r_1 \) are the two positive real roots with \( r_0 < r_1 \). We recall that these are the critical values of the map \( f \) restricted to the fiber \( W_{a}^{-1}(\epsilon) \). By arguments provided in the last subsection, we have that \( \epsilon < r_0 \) and \( f(W_{a}^{-1}(\epsilon)) = \mathbb{C} - \{\epsilon\} \). By the continuity of the roots of \( h_\epsilon(x) \) with respect to \( q \), we can assume that \( r_i \) is close to \( \langle l/(l-1) \rangle i^{-1} \) for \( 1 \leq i \leq l-1 \) and \( r_0 \) is close to \( \epsilon \). We start by examining the monodromy of \( f|_{W_{a}^{-1}(\epsilon)} \) about the critical values \( r_i \) and the missing point \( \epsilon \).

Recall that \( f(z_0, z_1) = a_0 z_0 + a_1 z_1 \) so if \( f(z_0, z_1) = s \) then \( z_0 = (s - a_1 z_1)/a_0 \). Thus for any \( q \) and \( s \in \mathbb{C} \)

\[
f|_{F_q}^{-1}(s) = \{((s - a_1 z_1)/a_0, z_1) \in F_q : C_{z_1} a_1 (s - a_1 z_1)^{a_0} = s - q\}
\]
In particular, for a regular value $s$ we have the fiber of $f$ restricted to $W_{\alpha}^{-1}(q)$ contains $l$ points so that $f|_{W_{\alpha}^{-1}(q)}$ is an $l$-fold branched cover. As was mentioned before, the critical values of $f$ restricted to a fiber in $(\mathbb{C}^*)^2$ are non-degenerate except for critical fibers. Thus $f$ restricted to $W_{\alpha}^{-1}(\varepsilon)$ is an $l$-fold branched cover of $\mathbb{C} - \{\varepsilon\}$ with ordinary double points over the critical values $r_i$. To find the monodromy around the point $\varepsilon$ we recall that there are two points $(0, \varepsilon/a_1)$, $(\varepsilon/a_0, 0)$ in the partial compactification $F_\varepsilon$ which have image $\varepsilon$ under $f$. From the above characterization of the fibers we see that

$$f|_{F_\varepsilon}^{-1}(\varepsilon) = \{((\varepsilon - a_1z_1)/a_0, z_1) \in F_q : C_{\varepsilon/a_1}^a((\varepsilon - a_1z_1)/a_0) = 0\}$$

Therefore the point $(0, \varepsilon/a_1)$ has ramification $a_0$ and the point $(\varepsilon/a_0, 0)$ has ramification $a_1$. So, as a permutation on the fibers, the monodromy around $\varepsilon$ can be written as a product of disjoint cycles $\tau_0$ and $\tau_1$ of order $a_0$ and $a_1$ respectively.

Now let us examine the monodromy around a curve $\gamma$ which encloses $r_0$ and $\varepsilon$ and no other $r_i$. Perturbing the fiber $W_{\alpha}^{-1}(\varepsilon)$ to the degenerate fiber $W_{\alpha}^{-1}(0)$, we see that as a permutation, the monodromy will be the same as that around the value 0 for the degenerate fiber. But $f|_{W_{\alpha}^{-1}(0)}^{-1}(0, 0)$ so the ramification of $f|_{W_{\alpha}^{-1}(0)}$ at $(0, 0)$ must be $l$. This implies that the monodromy of $\gamma$ is a cycle of order $l$. Now, $\gamma$ is also the composition of the monodromy around $\varepsilon$ and the monodromy around $r_0$ which is a transposition $\sigma$ (since $r_0$ is an ordinary double point). So if $s$ is a regular value with $\varepsilon < s < r_0$, and $f|_{W_{\alpha}^{-1}(s)}^{-1}(s) = \{p_1, \ldots, p_l\}$ labels the fiber in such a way that $\tau_0 = (p_1p_2\cdots p_{a_0})$ and $\tau_1 = (p_{a_0+1}\cdots p_{a_1})$ then we see that, as a permutation, the monodromy around $r_0$ must be $\sigma = (pjp_k)$ where $1 \leq j \leq a_0$ and $a_0 + 1 \leq k \leq l$.

As 0 and $q_0$ tend towards each other in $U_0 \subset W_{\alpha}^{-1}(\varepsilon)$ as $\varepsilon$ tends to zero, for sufficiently small $\varepsilon$, we have that $f(\vec{s})$ is contained in a small neighborhood of $\varepsilon$, say the interior $U$ of $\gamma$ (recall $\gamma$ is a curve going around $\varepsilon$ and $r_0$ in the image of $f|_{W_{\alpha}^{-1}(\varepsilon)}$). In particular, $f(\vec{s})$ stays sufficiently far away from $r_i$ for $i \neq 0$ as these values converge to the non-zero roots of $h_0(x)$. Now, in the domain $U_0$ we can find a curve $C$ connecting $(0, \varepsilon/a_1)$ to $(\varepsilon/a_0, 0)$. This is simply the curve which exits $(0, \varepsilon/a_1)$ along the fiber $p_j$, tends towards the double point over $r_0$ and loops back along the $p_k$ fiber towards $(\varepsilon/a_0, 0)$. It is clear from the description of the monodromy permutations above that any other curve connecting $(0, \varepsilon/a_1)$ to $(\varepsilon/a_0, 0)$ in $U_0$ without self intersection must be isotopic to $C$ relative boundary points. As $\vec{s}$ connects $(0, \varepsilon/a_1)$ to $(\varepsilon/a_0, 0)$ and is contained in $U_0$ we see that $\vec{s}$ is isotopic to $C$. Thus, reparameterizing $U_0$ if necessary we can assume that $\vec{s} = C$. Finally, recall that the vanishing cycle $L_0$ associated to $\delta_0$ has $f(L_0) = \vec{r_0}\vec{s}$ and must therefore contain the double point over $r_0$. Also, by the above description, $f(C) = \vec{s}\vec{r_0}$. This implies $L_0 \cap \vec{s}$ is precisely this double point yielding the result. \qed
The picture below shows the perturbed vanishing cycle $L_0$ in contrast to the degenerate case.

Combining this proposition and the corollary, we obtain a complete picture of the vanishing cycles $L_k$ in $W_a^{-1}(\varepsilon)$. Let us summarize this picture. The fiber $W_a^{-1}(\varepsilon)$ is symplectically equivalent $F_0 - \{0, \varepsilon\}$ which admits an $(l-1)$-fold cover to $\mathbb{P}^1 - \{0, 1, \infty\}$ except at the fiber containing $\varepsilon$ where there is a point missing. Outside the neighborhood $U_0$ of $0 \in F_0$, the description of the vanishing cycles $L_k$ stays the same as in the previous subsection, i.e. they are $l-1$ lifts of the loop around $\mathbb{P}^1 - \{1, \infty\}$. Inside $U_0$ we can describe $L_0$ as a curve which enters $U_0$ at the same points as it had entered $D$ but now passes once through the curve connecting 0 to $q$ (one may note that there are several such curves, but utilizing the smoothness of parallel transport of $W_a$ in $\mathbb{C}^2$, we are restricted to the picture above). Now all other vanishing cycles $L_k \subset W_a^{-1}(\varepsilon)$ are obtained in $U_0$ by multiplying $L_0$ by $\xi^k$ and then acting on $U_{2\pi k/(l-1)}$ by the monodromy map $M_{2\pi k/(l-1)}^{-1}$. The picture below shows the vanishing cycles $L_0$ and $L_3$ in $U_0$ for the cases $a = (1, 4)$ and $a = (2, 3)$.
In what follows, we will give an alternative representation of the vanishing cycles that yields the equivalence of $D^b(Fuk(W_a, \mathbb{C} - D^b_{\varepsilon, a}, \{\delta_k\}))$ with the category $\mathcal{T} \subset D^b(\text{Coh}(\mathcal{X}))$.

4.3. The derived category of the partial Lefschetz fibration $W_a$. From the previous subsection, we see that all intersection points of the vanishing cycles $L_k$ occur in $U_0$. It is also clear that any holomorphic disc occurring in the moduli spaces defining products in the Fukaya category are also contained in $U_0$; otherwise, upon projection to $\mathbb{P}^1 - U_0 - \{[1 : 0 : 0], [0 : 1 : 0]\}$ we would have that the image of $L_0$ is contractible. So in order to obtain the Fukaya category it suffices to consider the subspace $U_0$ and its intersections with the vanishing cycles $L_k$. Now, all of the data which gives the directed Fukaya category in our situation is invariant up to scaling and isotopy of the vanishing cycles. So instead of using $U_0$ we can take the space $D_e - \{0, 1\}$ by scaling by $\varepsilon^{-1}$ and fixing the intersection points of the vanishing cycles with the boundary. In the representation we will give we will take the logarithm $4(l - 1)/2\pi \log(z)$ of this setup and examine the images of the vanishing cycles in $\mathbb{C} - 4(l-1)i\mathbb{Z}$. Recall that at the end of subsection 4.1 we described the ordering of the vanishing cycles in $D$ in terms of the lines $\ell_k = \{re^{(2k+1)\pi i/2(l-1)} : 0 \leq r \leq \varepsilon\}$ and that $L_k \cap D = \ell_{2k} \cup \ell_{1-2a_0+2k}$. Thus the intersection of $L_k$ with the boundary of $D_e$ will be $e^{\zeta(2k+1)/4}$ and $e^{\zeta(4k-4a_0+3)/4}$. We will define a set of points in $\mathbb{C} - 4(l-1)i\mathbb{Z}$ as follows:

$$
P_{k-} &= (2k + 1 - 2(l-1))i \\
P_{k+} &= (2k + 1)i \\
Q_{k-} &= 1 + (4k + 1 - 2(l-1))i \\
Q_{k+} &= Q_{k-} + (4a_0 - 2)i
$$

Now let $s_{k\pm}$ be the class of line segments connecting $P_{k\pm} + 4(l-1)ai$ to $Q_{k\pm} + 4(l-1)ai$ for all integers $a$. Let $C_k = \{z : |z - P_{k-} + (l-1)i| = l - 1, \text{Im}(z) \leq 0\} + 4(l-1)ai$ be the set of half circles connecting $P_{k+} + 4(l-1)ai$ to $P_{k-} + 4(l-1)ai$ in the negative real half plane. Finally, take

$$\mathcal{L}_k = s_{k+} \cup s_{k-} \cup C_k$$

Then after re-examining Corollary 4, the comments of the last subsection and the definition of the monodromy map, we have the following proposition.

**Proposition 14.** The piecewise smooth curves $\exp(\frac{2\pi}{4(l-1)}\mathcal{L}_k)$ are isotopic to the vanishing cycles $L_k \cap U_0$.

By isotopic, we will mean that we have smoothed the corners on the imaginary line and that we have rotated the image to agree with the boundary conditions. We illustrate this set-up for our two examples.
The advantage of considering this representation of the vanishing cycles is that the morphisms \((L_i, L_k)\) have a natural decomposition which exhibits the isomorphism with the mirror category. We will now state this as a theorem modulo issues related to grading.

**Theorem 7.** Let \(V\) be a graded vector space generated by \(e_0\) and \(e_1\) with weights \(a_0\) and \(a_1\). Define \(\psi_{j,k} : (L_j, L_k) \to \oplus_{t \leq k-j} \wedge^t V\) as

\[
\psi_{j,k}(\exp(\frac{2\pi i}{4(l-1)} C_j \cap C_k)) = 1 \\
\psi_{j,k}(\exp(\frac{2\pi i}{4(l-1)} \tilde{s}_j^+ \cap \tilde{s}_k^-)) = e_0 \\
\psi_{j,k}(\exp(\frac{2\pi i}{4(l-1)} \tilde{s}_k^+ \cap \tilde{s}_j^-)) = e_1
\]

Then \(\psi_{j,k}\) is an isomorphism that commutes with composition in \(\text{Fuk}(W_a, \mathbb{C} - D_\epsilon, a, \{\delta_i\})\). Furthermore, all products \(m_i\) vanish for \(i \neq 2\).

The proof of this theorem will occupy the rest of this subsection. To see why this theorem is technically difficult, one can examine the figures above and find many embedded polygons with four or more edges that a priori could contribute to higher products. One point that will be proven is that all such polygons do not have the appropriate ordering of intersection points along the boundary to contribute to \(m_i\). Although this proof is lengthy, the techniques are completely elementary.

**Proof.** We will prove this theorem in several steps.

**Step 1:** The map \(\psi_{j,k}\) is a well defined isomorphism of vector spaces.
Recall that $\bigwedge^t V$ consists of the elements of the exterior algebra with weighted degree equal to $t$. So $\oplus_{t\leq k-j} \bigwedge^t V$ will always contain the identity element, it will contain the generator $e_1$ iff $a_0 \leq k-j$ and it will contain the generator $e_{j}$ iff $a_0 \leq k-j$. Note that $e_0 \wedge e_1$ will not be contained in this space as $k-j < l-2$. Now, it is clear that $C_j \cap C_k$ will always contain exactly one intersection point modulo $4(l-1)i\mathbb{Z}$. So to complete this step we must show that, modulo $4(l-1)i\mathbb{Z}$, $\tilde{s}_{j+} \cap \tilde{s}_{k-}$ contains 1 element iff $a_0 \leq k-j$ and is empty otherwise and $\tilde{s}_{k+} \cap \tilde{s}_{j-}$ contains 1 element iff $a_1 \leq k-j$ and is empty otherwise. We use brute force and calculate the equations for the lines containing the segments $\tilde{s}_{k\pm}$. Using cartesian coordinates we write the equations as follows

$$
\begin{align*}
\tilde{s}_{k-} & \quad y = 2kx + 2k + 1 - 2(l-1) + 4(l-1)b \\
\tilde{s}_{j+} & \quad y = (2j - 2l + 4a_0)x + (2k + 1) + 4(l-1)c
\end{align*}
$$

where $b$ and $c$ are integers. Then $(x, y) \in \tilde{s}_{j+} \cap \tilde{s}_{k-}$ iff $0 < x < 1$ and there exists an integer $d = b - c$ such that

$$
(k - j + l - 2a_0)x = (j - k) + (l - 1)(2d + 1)
$$

where we have used the above equations to solve for $x$.

**Case 1:** $\tilde{s}_{j+} \cap \tilde{s}_{k-}$

We have that $k - j > 0$ and $l - 2a_0 = a_1 + a_0 - 2a_0 = a_1 - a_0 \geq 0$ by our convention that $a_0 \leq a_1$. Thus $k - j + l - 2a_0 > 0$ and dividing equation (3) by this term, the inequality $0 < x < 1$ gives $0 < (j - k) + (l - 1)(2d + 1) < (k - j + l - 2a_0)$ which in turn gives

$$
k - j < (l - 1)(2d + 1) < 2(k - j) + l - 2a_0
$$

Now, we have $0 < k - j < l - 1$ and $l - 2a_0 = a_1 - a_0 < l - 1$ which implies

$$
0 < (l - 1)(2d + 1) < 3(l - 1)
$$

and therefore $d = 0$. So for $k > j$ there is an intersection point of $\tilde{s}_{j+} \cap \tilde{s}_{j-}$ iff the inequality $k - j < l - 1 < 2(k - j) + l - 2a_0$ is satisfied. The left hand side of this inequality is always true, while the right hand side gives $2a_0 - 1 < 2(k - j)$ which is the case iff $a_0 \leq k-j$. The fact that this intersection point is unique mod $4(l-1)i\mathbb{Z}$ follows from the fact that the integer $d$ was determined.

**Case 2:** $\tilde{s}_{k+} \cap \tilde{s}_{j-}$

For this case we have more options to consider. Our equation (3) is now:

$$
(j - k + l - 2a_0)x = (j - k) + (l - 1)(2d + 1)
$$

We start with the possibility that $j - k + l - 2a_0 > 0$. This will imply that $a_1 > a_1 - a_0 = l - 2a_0 > k - j$ so that we should expect no intersection points. Indeed, proceeding as in case 1 we obtain the inequality:

$$
j - k < (l - 1)(2d + 1) < 2(j - k) + l - 2a_0
$$
For what follows, we will give an orientation to each $L_i$. Step 2 follows from the uniqueness of $Q_i \in \mathcal{L}_{i^k}$ so that their concatenation gives the minimal curve in $\mathcal{L}_{i^k}$ along its orientation, the intersection points $C_{i^k} \cap C_j$ are decreasing (i.e. $n$ is decreasing). Thus if the first part is $C_i(+)C_j(-)$ then $k > i$. On the other hand, if the first part is $C_i(-)C_j(+) \text{ then } k < i$,
but such a situation is not allowed, as the ordering of the vanishing cycles in the word must be increasing. Thus any such combination occurring in \( Y_u \) must be of the form \( C_i(+)C_j(-)C_k(+) \). One can see by the same argument that there is no \( C_s(\pm) \) for which \( C_i(+)C_j(-)C_k(+)C_s(\pm) \) or \( C_s(\pm)C_i(-)C_j(+)C_k(-) \) occurs in \( Y_u \). But observe that there is a holomorphic triangle \( \tilde{u} \in \mathcal{M}_0(\mathcal{C}_i \cap \mathcal{C}_j, \mathcal{C}_j \cap \mathcal{C}_k, \mathcal{C}_k \cap \mathcal{C}_i) \) which has the word \( C_i(-)C_j(+)C_k(-) \). As no other intersection points for \( u \) consecutively occur before or after \( \mathcal{C}_i \cap \mathcal{C}_j \) and \( \mathcal{C}_j \cap \mathcal{C}_k \) in the negative real half plane we have that \( \text{im}(\tilde{u}) \subset \text{im}(u) \). But the boundary of \( \tilde{u} \) is also contained in the boundary of \( u \) which implies that \( u = \tilde{u} \). It is clear that such \( \tilde{u} \) defines the product \( m_2 \) for those intersection points sent to \( 1 \in \bigwedge^\bullet V \) and their composition is compatible.

Now let us assume that \( C_i(\pm)C_j(\mp) \) occurs in \( Y_u \) and \( u \) is not contained in the negative real half plane. Let us take the case \( C_i(+)C_j(-) \) as the alternative case has a similar argument. In this case we have that \( P_{i+}, P_{j+} \in \text{im}(u) \) so that \( \text{im}(u) \) intersects the positive real half plane \( H_+ \). Now, the component \( K \) of \( \text{im}(u) \cap H_+ \) containing \( P_{i+} \) is a convex set as the boundary consist of line segments with intersection angles \( < \pi \). Therefore, no \( P_{k-} \) or \( P_{i+} \) for \( t < i \) or \( t > j \) occurs in \( K \) as these points are not contained in the convex set lying above \( \tilde{s}_{i+} \) and below \( \tilde{s}_{j+} \). Furthermore, if \( P_{k+} \) occurs in the boundary of \( u \) with \( i < k < j \) then the symbol \( C_k(\pm) \) must occur somewhere in the word \( Y_u \). But as the indices of the letters in this word are increasing, this is not possible. Therefore, \( K \) contains \( P_{i+} \) and \( P_{j+} \). Taking the union \( \tilde{K} \) of \( K \) and the triangle \( P_{i+}(\mathcal{C}_i \cap \mathcal{C}_j)P_{j+} \) gives a holomorphic disc contained in the image of \( u \) whose boundary is also contained in the boundary of the image of \( u \). Thus, \( \tilde{K} = \text{im}(u) \) so that any \( u \) that is not a holomorphic triangle but contains two consecutive letters \( C_i(\pm)C_j(\mp) \) in \( Y_u \) contains no other \( C_k(\pm) \).

Now suppose that the first letter in \( Y_u \) is \( C_i(\pm) \) and the last letter is \( C_j(\pm) \) and that \( \text{im}(u) \) is not contained in the negative real half plane. Then we can see that the signs must be \( C_i(+) \) and \( C_j(-) \). From the above paragraph we know that \( C_i(+)\tilde{s}_{i+}(+) \) is the beginning of \( Y_u \) and \( \tilde{s}_{j-}(-)C_j(-) \) is the end of \( Y_u \). So \( P_{i+} \) and \( P_{j-} \) are elements of the boundary of \( u \). If we take \( K \) to be the convex set \( \text{im}(u) \cap H_+ \) containing \( P_{j-} \) then by the orientation of \( u \), \( K \) is contained in the set of elements above \( \tilde{s}_{j-} \). Now, \( K \) cannot contain \( 0 \) as this would imply that \( 0 \in \text{im}(u) \) so \( K \cap \mathbb{R}i \) must be a line segment from \( P_{j-} \) to \( P_{k-} \) with \( k > j \) and \( P_{k-} \) in the boundary of \( u \). But this implies that the letter \( C_k(\pm) \) occurs in \( Y_u \) and as \( C_j(-) \) is the last letter, it must occur before \( C_j(-) \). This violates the increasing order on the subscripts of letters in \( Y_u \). Therefore, there is no curve \( u \) for which the first and last letters are semi-circles.

Our last case to consider for this step is that \( C_i \) occurs in \( Y_u \) without any other consecutive letters being semi-circles (consecutive in the cyclic sense). This will imply that there are no intersection points lying in the negative real half plane. But then the interior of the region bounded by \( C_i \) and \( \mathbb{R}i \) is
contained in \( \text{im}(u) \) and, in particular, \( 0 \in \text{im}(u) \) which is not possible. So this case does not occur.

Summing up these results we have:

Given \( u \in \mathcal{M}_0(p_{i0}, \ldots, p_{ir}) \) there are three possibilities for \( Y_u \)

1. \( u \) is a holomorphic triangle contained in the negative real half plane
2. The word \( Y_u \) contains the expression \( C_i(\pm)C_j(\mp) \) and all other letters are of the type \( s_{k\pm}(\pm) \)
3. \( u \) contains no letters of the type \( C_i(\pm) \)

**Step 3** Holomorphic discs which intersect the positive real half plane.

Before pursuing this case in detail, let us note that as one moves along \( \vec{s}_{i+} \) in the positively oriented direction, \( j \) increases where \( \{p_j\} = \vec{s}_{i+} \cap \vec{s}_{j-} \).

Similarly, as one moves along \( \vec{s}_{i-} \) in the positively oriented direction, \( j \) increases where \( \{p_j\} = \vec{s}_{i-} \cap \vec{s}_{j+} \). Both of these facts can be seen from the equations defining \( s_{k\pm} \). Now, if \( i < j \) step 1 gives us \( \{p_j\} = \vec{s}_{i-} \cap \vec{s}_{j+} \) iff \( j - i = a_1 \). Now, if \( j - i = a_1 \) then there exists no \( 0 \leq k \leq l - 2 \) for which \( i - k \geq a_0 \) for then \( j - k \geq l \). Thus, if \( l - a_1 - 2 \geq i \) then the only intersection points \( p_j \) along \( \vec{s}_{i-} \) must satisfy \( j \geq i + a_1 \). A similar argument shows that if \( i \geq a_0 \) then the only intersection points \( p_j \) along \( \vec{s}_{i-} \) must satisfy \( j + a_0 \leq i \).

We summarize these and analogous results in the following table:

\[
\begin{array}{c|c}
\text{Condition} & \text{Intersection Points} \\
\hline
l - 2 - a_1 \geq i & j \geq i + a_1 \text{ for } p_j \in \vec{s}_{i-} \\
a_0 \leq i & j + a_0 \leq i \text{ for } p_j \in \vec{s}_{i-} \\
l - 2 - a_0 \geq i & j \geq i + a_0 \text{ for } p_j \in \vec{s}_{i+} \\
a_1 \leq i & j + a_1 \leq i \text{ for } p_j \in \vec{s}_{i+} \\
\end{array}
\]

We will use this table in what follows. Finally recall that \( \vec{s}_{i+} \cap \vec{s}_{j+} = \emptyset = \vec{s}_{i-} \cap \vec{s}_{j-} \) for all \( i \neq j \).

Our first case to consider for this step is that \( Y_u \) contains three consecutive letters of the type \( s_{i\pm}(\pm) \), say \( s_{i\pm}(\pm)s_{j\pm}(\pm)s_{k\pm}(\pm) \). Utilizing the clockwise orientation on \( u \), one sees that the possible combinations of two consecutive letter of this type are

\[
\begin{align*}
& s_{i-}(+)s_{j+}(-) & & s_{i+}(+)s_{j-}(+) \\
& s_{i-}(-)s_{j+}(+) & & s_{i+}(-)s_{j-}(-)
\end{align*}
\]

Let us examine the cases where either \( s_{i-}(+)s_{j+}(-) \) or \( s_{i+}(-)s_{j-}(-) \) occurs. Then, as we are moving from the intersection point \( \vec{s}_{i\pm} \cap \vec{s}_{j\pm} \) along \( \vec{s}_{i\pm} \) against the orientation, we see that \( k < i \). But as the indices in \( Y_u \) must be increasing, this is not possible, so these cases does not occur.

Now suppose \( s_{i+}(+)s_{j-}(+) \) occurs. If \( a_1 \leq i \) then by the above table, \( j < i \) which contradicts increasing ordering of indices, thus \( l - 2 - a_0 \geq i \) and \( j \geq i + a_0 \geq a_0 \). But again by the above table this implies that \( k + a_0 \leq j \) or \( k < j \) which contradicts the increasing ordering of indices in \( s_{i+}(+)s_{j+}(+)s_{k+}(-) \). Now suppose that \( s_{i-}(-)s_{j+}(+) \) occurs in \( Y_u \). If \( a_0 \leq i \) then the table shows that \( j < i \) contradicting increased ordering of
indices; so \( l - 2 - a_1 \geq i \) and \( j \geq i + a_1 \geq a_1 \). But then the table shows that \( k + a_1 \leq j \) or \( k < j \) again contradicting the increasing ordering of indices of \( Y_u \). As these are the only options, we see that there are never three consecutive letters of the type \( s_{i \pm}(\pm) \) in \( Y_u \). In particular, there is no acceptable holomorphic disc contained in the positive real half plane.

The above result, along with step 1, implies that the only acceptable holomorphic discs intersecting \( H_+ \) will contain either one point in \( H_- \) and one in \( H_+ \), one point in \( H_- \) and two in \( H_+ \) or one point in \( H_- \) and three in \( H_+ \). In each case \( Y_u \) must contain the word \( s_{i \pm}(\pm)C_i(\pm)C_j(\mp)s_{j \pm}(\mp) \) where the signs are all forced by the orientation. Let us eliminate the first and third options. For the first option we have \( Y_u = s_{i \pm}(\pm)C_i(\pm)C_j(\mp)s_{j \pm}(\mp) \). One quickly observes that the subscript sign for \( s_{i \pm} \) is the same as that for \( s_{j \pm} \). But then \( \bar{s}_{i \pm} \cap \bar{s}_{j \pm} = \emptyset \) implying that the boundary of \( u \) contains only one intersection point. This is not allowed, so no such \( u \) exists. The third option must have a word of the type

\[
Y_u = s_{i \mp}(\sigma_i)s_{i \pm}(\pm)C_i(\pm)C_j(\mp)s_{j \pm}(\mp)s_k(\mp)(\sigma_k)
\]

where \( \sigma_i \) and \( \sigma_k \) are the appropriate signs. But again we have that the signs of the subscripts for the beginning and end agree, so \( \bar{s}_{i \mp} \cap \bar{s}_{k \mp} = \emptyset \) and the curve \( u \) does not close.

At this point we can affirm that \( m_i = 0 \) for \( i \neq 2 \). Indeed, above we have shown that no \( u \) has image in \( H_+ \), so every \( u \) must have image intersecting \( H_- \). By step 1 we saw that any \( u \) with image intersecting \( H_- \) that is not a triangle contains precisely one point in \( H_- \). By the above paragraph, we have ruled out the case where \( u \) contributes to \( m_1 \) and the case where \( u \) contributes to \( m_2 \). As no three \( s_{i \pm}(\pm) \) occur consecutively in \( Y_u \), we have that the only remaining case consist of triangles with one point in \( H_- \) and two in \( H_+ \).

Let us start by assuming \( s_{i \pm}(+)C_i(+)C_j(-)s_{j \pm}(-) \) and \( s_{k \pm}(\pm) \) occurs in \( Y_u \). If \( k < i \) then \( s_{k \pm}(\pm) \) must occur before the word \( s_{i \pm}(+)C_i(+)C_j(-)s_{j \pm}(-) \) and if \( k > j \) it must occur after. Assume \( k < i \) so that

\[
Y_u = s_{k \pm}(-)s_{i \pm}(+)C_i(+)C_j(-)s_{j \pm}(-)
\]

The points \( p_0, p_1, p_2 \) occurring in the triangle satisfy \( \{p_0\} = \bar{s}_{k \pm} \cap \bar{s}_{i \pm}, \{p_1\} = C_i \cap C_j \) and \( \{p_2\} = \bar{s}_{j \pm} \cap \bar{s}_{k \pm} \). By the definition of \( \psi_{i \bullet} \) we have \( \psi_{k,i}(p_0) = e_1 \), \( \psi_{i,j}(p_1) = 1 \) and \( \psi_{k,j}(p_2) = e_1 \). Thus, if this triangle exists, it defines composition compatible with that in \( \wedge_{i \bullet}V \). One can see the existence of such a triangle by taking the convex hull of \( P_{i \pm}, P_{j \pm}, p_0 \) and \( p_2 \) and taking the union with the region contained in the triangle \( p_1 P_{i \pm}P_{j \pm} \).

Now suppose \( k > j \), then we have

\[
Y_u = s_{i \mp}(+)C_i(+)C_j(-)s_{j \pm}(-)s_{k \pm}(-)
\]

and \( \{p_0\} = C_i \cap C_j, \{p_1\} = \bar{s}_{j \pm} \cap \bar{s}_{k \pm} \) and \( \{p_2\} = \bar{s}_{i \pm} \cap \bar{s}_{k \pm} \). Again by the definition of \( \psi_{i \bullet} \) we have \( \psi_{i,j}(p_0) = 1 \), \( \psi_{j,k}(p_1) = e_0 \) and \( \psi_{i,k}(p_2) = e_0 \). So again we see that the part of \( m_2 \) that this triangle defines agrees via \( \psi \) with
multiplication in $\bigwedge^\bullet V$. The existence of such triangles can be shown in a similar way as above.

Following the procedure in the previous two paragraphs, one can take care of the case where $s_i(-)(-C_j)C_i(-)(+)s_j(-)(+)$ occurs in $Y_u$ to see that $\psi$ commutes with multiplication in $\text{Fuk}(W_a, C - D_\epsilon, a, \{\delta_i\})$. This completes the proof. □

4.4. Grading in $\text{Fuk}(W_a, C - D_\epsilon, a, \{\delta_i\})$. The theorem in the preceding section showed an equivalence between $\text{Fuk}(W_a, C - D_\epsilon, a, \{\delta_i\})$ and the category $T$ associated to our exceptional collection $\{S_k\}$ assuming we neglected grading. This subsection will finish the proof of this equivalence by explicitly grading the vanishing cycles $L_i$. The procedure for constructing these gradings is fairly straightforward; however, calculating the Maslov indices poses a technical challenge and is the most difficult part of the argument. There are two obstructions to giving vanishing cycles $\mathbb{Z}$-gradings. The first is the obstruction to lifting the lagrangian Grassmanian $\Lambda$ of the tangent bundle to a fiberwise universal covering $\tilde{\Lambda}$. Assuming this obstruction vanishes and we have such a lift, the second obstruction occurs as an obstruction of lifting the canonical section $\phi_i : L_i \to \Lambda$ to $\tilde{\phi}_i : L_i \to \tilde{\Lambda}$.

Now suppose we have a complex valued 1-form $\eta$ on the fiber $W^{-1}(\epsilon)$ that yields an isomorphism $\eta_p : T_p(W_a^{-1}(\epsilon)) \to \mathbb{C}$ for every $p \in W^{-1}(\epsilon)$. In most applications, $\eta$ is a holomorphic 1-form for which the above requirement is automatic. Observe that $\eta$ yields a fiberwise trivialization of $\Lambda$, $\eta_p : \Lambda_p \to S^1$ via $\eta_p((v)) = \arg(\eta_p(v))$ where $v \in T_p(W_a^{-1}(\epsilon))$ is a generator for the lagrangian line and $S^1$ is identified with $\mathbb{R}/\pi\mathbb{Z}$. Thus we can form the pullback:

$$
\begin{array}{c}
\tilde{\Lambda} \\
\pi \downarrow \\
\Lambda \xrightarrow{\text{arg } \eta} S^1
\end{array}
$$

So given any such $\eta$, the first obstruction will vanish (note however that different choices of $\eta$ may yield different bundles $\tilde{\Lambda}$). Let us assume for the moment that such a 1-form has been constructed. Let $P_\epsilon$ be the topological category of immersed paths in $W_a^{-1}(\epsilon)$. We will take an object of $P_\epsilon$ to be a point $p \in W_a^{-1}(\epsilon)$ along with a tangent vector $v \in T_p(W_a^{-1}(\epsilon))$ and morphisms to be immersed paths (we will systematically ignore the fact that this is only an $A^\infty$-category without units). Given a path $\alpha : [0, 1] \to W_a^{-1}(\epsilon)$ one has a canonical map $\phi_\alpha : [0, 1] \to \Lambda$ and if $\tilde{\phi}_\alpha : [0, 1] \to \tilde{\Lambda}$ is any lift then define

$$
\xi(\alpha) = v(\tilde{\phi}_\alpha(1)) - v(\tilde{\phi}_\alpha(0))
$$

where $v$ is the map defined in the above pullback diagram. It is not hard to show that $\xi$ is a functor from $P_\epsilon$ to $\mathbb{R}$ where $\mathbb{R}$ is the additive category with one object. One can also check that $\xi$ is constant on connected components of morphisms (i.e. invariant under isotopies of immersed paths with fixed tangent vectors at the endpoints). Furthermore, $\xi$ is also
constant on free isotopy classes of closed loops. Explicitly, we have that 
\[ \xi(\alpha) = \arg(\eta(\alpha'(1))) - \arg(\eta(\alpha'(0))) \]
where \( \arg(\eta(\alpha'(t))) \) is continuous on \([0,1]\). Using this language one can see that \( L_i \) has a \( \mathbb{Z} \)-grading if and only if \( \xi(L_i) = 0 \) (more generally, \( L_i \) has a \( \mathbb{Z}/n\mathbb{Z} \)-grading iff \( \xi(L_i) \equiv 0 \pmod{n} \)).

This language is also helpful in calculating the Maslov index of an intersection point of two vanishing cycles. Recall that a \( \mathbb{Z} \)-graded vanishing cycle is a vanishing cycle \( L_i \) along with a lift \( \tilde{\varphi}_i : L_i \to \tilde{\Lambda} \) of the canonical section \( \varphi_i \). Giving an orientation to each \( L_i \), we can take \( \tilde{\varphi}_i(p) = \arg(\eta(v_p)) \in \mathbb{R} \) where \( v_p \) is the positively oriented ray in \( T_pL_i \) and we have chosen a particular lift from \( \mathbb{R}/\pi\mathbb{Z} \) to \( \mathbb{R} \). If \( p \in L_i \cap L_j \) with \( j > i \) then one can define the Maslov index at \( p \) as \( \mu(p) = -\left\lfloor \frac{1}{\pi}(\tilde{\varphi}_j(p) - \tilde{\varphi}_i(p)) \right\rfloor \) where \( \lfloor a \rfloor \) is the least integer function. One should note that in other expositions this definition is actually \( 1 - \mu(p) \), but as we have taken a counter-clockwise orientation of our distinguished basis, we must take Maslov indices relative the conjugate holomorphic structure.

Now suppose we have oriented each \( L_i \) as in the previous subsection. Given \( p \in L_i \cap L_j \) with \( j > i \) let \( v_j \) and \( v_i \) be the tangent vectors at \( p \) to \( L_j \) and \( L_i \) respectively. Let \( f_i \) and \( f_j \) be oriented parameterizations of \( L_i \) and \( L_j \) respectively such that \( f_i(0) = p = f_j(0) \). Now, give \( W^{-1}(\varepsilon) \) the orientation induced by the complex conjugate structure (i.e. clockwise orientation). Define \( \text{or}(p) \) to be zero if \( v_j \wedge v_i \) is oriented and 1 otherwise. Given any sufficiently small \( \kappa \), we will define a path \( \alpha_\kappa \) which starts along \( L_j \) and ends along \( L_i \) and depends on \( \text{or}(p) \). Define \( \alpha_\kappa \) to be a smoothly embedded path contained in a sufficiently small neighborhood of \( p \) such that \( (\alpha_\kappa(0), \alpha'_\kappa(0)) = (f_j((-1)^{\text{or}(p)+1}\kappa), f'_j((-1)^{\text{or}(p)+1}\kappa)), (\alpha_\kappa(1), \alpha'_\kappa(1)) = (f_i((-1)^{\text{or}(p)}\kappa), f'_i((-1)^{\text{or}(p)}\kappa)) \) and \( \text{im}(\alpha) \) intersects the vanishing cycles in only these two points. We illustrate \( \alpha_\kappa \) below for the two cases.

One sees that from this construction that \( -\pi \leq \xi(\alpha_\kappa) + \text{or}(p)\pi \leq 0 \) and as \( \kappa \) approaches zero, \( \xi(\alpha_\kappa) + \text{or}(p)\pi \) approaches \( \arg(\eta(v_i)) - \arg(\eta(v_j)) \pmod{2\pi} \).
\[ \lim_{\kappa \to 0} \left[ \frac{1}{\pi} (\tilde{\phi}_j(p) - \tilde{\phi}_i(p)) + \xi(\alpha_\kappa) + \nu(p) \pi \right] = -\left[ \frac{1}{\pi} (\tilde{\phi}_j(p) - \tilde{\phi}_i(p)) \right] = \mu(p) \]

Now suppose \( p_i \in L_i \) and \( p_j \in L_j \) and take the oriented paths \( \beta_j \) from \( p_j \) to \( f_j((-1)^{\nu(p)+1} \kappa) \) and \( \beta_i \) from \( f_i((-1)^{\nu(p)} \kappa) \) to \( p_i \). Let \( \beta_{ji, \kappa} \) be the concatenation \( \beta_i \circ \alpha_\kappa \circ \beta_j \). Using the functorial properties of \( \xi \) we have

\[
\xi(\beta_{ji, \kappa}) + \nu(p) \pi = \xi(\beta_j) + \xi(\beta_i) + \xi(\alpha_{ji}) + \nu(p) \pi = \tilde{\phi}_j(f_j((-1)^{\nu(p)+1} \kappa)) - \tilde{\phi}_j(p_j) + \tilde{\phi}_i(p_i) - \tilde{\phi}_i(f_i((-1)^{\nu(p)} \kappa)) + \xi(\alpha_\kappa) + \nu(p) \pi = [\tilde{\phi}_j(f_j((-1)^{\nu(p)+1} \kappa)) - \tilde{\phi}_i(f_i((-1)^{\nu(p)} \kappa))] + \xi(\alpha_\kappa) + \nu(p) \pi] + \tilde{\phi}_i(p_i) - \tilde{\phi}_j(p_j)
\]

\[
= -\pi \nu(p) + \tilde{\phi}_i(p_i) - \tilde{\phi}_j(p_j)
\]

We see that the last equality follows from the invariance of \( \xi \) under isotopy. In other words, as \( \kappa \) tends to zero we saw that the second to last line approaches the last line, however, as these paths are isotopic, their value under \( \xi \) is constant, so we must have equality. This equation gives

\[
(5) \quad \mu(p) = -\frac{1}{\pi} [\xi(\beta_{ji, \kappa}) + \nu(p) \pi + \tilde{\phi}_j(p_j) - \tilde{\phi}_i(p_i)]
\]

Although this approach may seem cumbersome at first, its utility lies in the fact that we can calculate Maslov indices simply by knowing the path \( \beta_j \), the monodromy data associated to \( \eta \) and the values of \( \tilde{\phi}_i \) at a single point.

With these preliminaries in mind, we return to the fiber \( W_{a}^{-1}(\varepsilon) \) and prove our final theorem.

**Theorem 8.** The map \( \psi_{jk} \) induces an isomorphism of triangulated categories from \( D^b(Fuk(W_{a}, D^{*} - D_{s}, \{\delta_{i}\}) \) to \( T \).

**Proof.** The 1-form \( \eta \) is defined as the restriction to \( W_{a}^{-1}(\varepsilon) \) of a 1-form \( \tau \) on \( (C^*)^2 \) which satisfies

\[
\tau \wedge d\Omega = \frac{dz_1 \wedge dz_2}{z_1 z_2} := \Omega
\]

While \( \tau \) is not defined globally on \( (C^*)^2 \), it is well defined on the regular fibers \( W_{a}^{-1}(\varepsilon) \). One can see that \( \Omega \) yields a non-zero phase function on the Lefschetz thimbles \( D_i \) and the restriction of \( \Omega \) to \( L_i \) is precisely \( \eta(L_i) \) (assuming one has oriented \( L_i \) properly). As the thimbles are simply connected, one sees immediately that the vanishing cycles can be given \( Z \)-gradings via \( \eta \).

Now, as \( F_{\varepsilon} \) transversely intersects the divisor \( \{z_i = 0\} \), we have that near such an intersection point, \( \eta \) looks like \( \frac{d\hat{z}}{\hat{z}} \). Recall from the previous subsection that we have a neighborhood \( U_0 \subset W_{a}^{-1}(\varepsilon) \) where \( U_0 = \{w : |w| < \varepsilon, w \neq 0, w \neq 1\} \). As \( w \) tends to zero in \( U_0 \), \( z_i \) tends to zero, while as
$w$ tends to 1, $z_0$ tends to zero. This implies that on $U_0$,
\[
\eta = h(w) \frac{dw}{w(w-1)}
\]
where $h(w)$ is a nowhere vanishing holomorphic function. Of course, one can multiply $\eta$ by $1/h(w)$ and it will not effect the Maslov indices. Furthermore, perturbing $\eta$ near the boundary of $U_0$ we can assume that $\eta = \frac{dw}{w} = -dw^{-1}$ near the boundary. Although $\eta$ is no longer holomorphic, this perturbation will not effect Maslov indices. From the previous subsection, we have a description of the incoming and outgoing points of $L_i$ in $U_0$ as $R_{i\pm} = \exp(\frac{2\pi i}{2(l-1)}Q_{i\pm})$. We can assume that the tangent vector $v_{i\pm}$ at $R_{i\pm}$ is normal to the circular boundary of $U_0$, pointing inward for $R_{i+}$ and outward for $R_{i-}$. Thus, up to a positive real factor, $v_{i\pm} = -e^{i\frac{2\pi}{2(l-1)}(\text{Im}(Q_{i\pm})-2Q_{i\pm})}$ so that
\[
\arg(\eta(v_{i+})) = \pi - \frac{\pi}{2(l-1)}\text{Im}(Q_{i+})
\]
and
\[
\arg(\eta(v_{i-})) = -\frac{\pi}{2(l-1)}\text{Im}(Q_{i-})
\]
We will give $L_i$ a grading by letting
\[
\tilde{\phi}_i(R_{i+}) = \pi - \frac{\pi}{2(l-1)}\text{Im}(Q_{i+})
\]
We would like then to find the real number $\tilde{\phi}_i(R_{i-})$. Indeed, while we know that $\tilde{\phi}_i(R_{i-}) \equiv -\frac{\pi}{2(l-1)}\text{Im}(Q_{i-})(\text{mod} 2\pi)$, we must find the real number that is determined by our choice of $\tilde{\phi}_i(R_{i+})$. In order to do this we will find $\xi(S_r)$ for a certain path $S_r$ which we now describe. Define
\[
S_r(t) = 1 + re^{(2rt - r)}
\]
to be the oriented path of a semi-circle of radius $r$. We can pullback $\eta$ via the map $e^{\frac{2\pi iy}{2(l-1)}}$. For the half space $Re(z) \geq 1$ we obtain the form
\[
\tilde{\eta}_z = \frac{\pi}{2(l-1)}e^{-\frac{2\pi iy}{2(l-1)}}dz
\]
by extending $\eta$ to $-\bar{d}(w^{-1})$ outside of $U_0$ in the complex plane. Thus we can transfer all calculations to the covering space with parameter $z$ using the form $\tilde{\eta}$. One should note that around points $4(l-1)ai$, we have that $\tilde{\eta}_z = k(z)\tilde{\eta}_z$ with a non-zero holomorphic function $k(z)$. Now let $G_r(t) = 1 + i(2rt - r)$ be the straight line segment. One can see by perturbing the curve $S_r$ that $\xi(S_r) = \xi(G_r) + \pi$. Under the covering map, $G_r$ goes to a circular arc. Calculating, we have
\[
\tilde{\eta}(G_r(t)) = \tilde{\eta}_{G_r}(t)\partial_y = \frac{\pi \partial_y}{(l-1)}e^{-\frac{(1+i(2rt-r))\pi}{2(l-1)}}
\]
So that
\[
\arg(\tilde{\eta}(G_r'(t))) = \text{Im}(\log(\tilde{\eta}(G_r'(t)))) = \frac{\pi}{2} - \frac{\pi(2rt - r)}{2(l - 1)}
\]

Implying
\[
\xi(G_r) = -\frac{r\pi}{(l - 1)}
\]

Thus \(\xi(S_r) = \pi - r\pi/(l - 1)\). Also note that translating the curve \(S_r\) along the line \(\text{Re}(z) = 1\) will not effect \(\xi(S_r)\) as \(\tilde{\eta}\) only picks up a constant phase for such a translation. Using this calculation, we can find the values \(\tilde{\eta}(R_{i+})\).

To do this just assume that we have perturbed the corners of \(\mathcal{L}_i \subset \mathbb{C}\) so that the tangent vectors at \(Q_{i\pm}\) are normal to \(\text{Re}(z) = 1\). Then
\[
F_i = \mathcal{L}_i \cup (S_{\text{Im}((Q_{i+} - Q_{i-})/2)} + (Q_{i+} + Q_{i-})/2)
\]
gives an oriented embedded closed curve which loops around 0 precisely once. This curve is freely homotopic to a small curve around 0 and \(\tilde{\eta}(R_{i+})\) in a neighborhood of 0 we have that \(\xi(F_i) = 0\). Using the additivity of \(\xi\) we then have
\[
0 = \xi(\mathcal{L}_i) + \xi(S_{(Q_{i+} - Q_{i-})/2} + (Q_{i+} + Q_{i-})/2))
= \tilde{\phi}_i(R_{i+}) - \tilde{\phi}_i(R_{i+}) + \pi - \frac{\pi}{2l - 1}\text{Im}(Q_{i+}) + \frac{\pi}{2l - 1}\text{Im}(Q_{i-})
= \tilde{\phi}_i(R_{i+}) + \frac{\pi}{2l - 1}\text{Im}(Q_{i-})
\]
So that
\[
\tilde{\phi}_i(R_{i+}) = -\frac{\pi}{2l - 1}\text{Im}(Q_{i-})
\]

We can now compute the Maslov indices of the various intersection points. Let us start with the intersection points of the type \(p \in C_j \cap C_i\), i.e. those that are sent to the identity in \(\wedge V\). One can easily see either by computation or by examining the figures in the previous subsection that \(\text{or}(p) = 0\) for any such point. Now, choose \(p_j = Q_{j+}\) and \(p_i = Q_{i-}\) as in our setup for equation (5). Then the concatenated curve \(\beta_{jik}\) will be isotopic to an embedded curve \(\beta_1\) starting at \(Q_{j+}\), ending at \(Q_{i-}\) and going around 0. As in the derivation of \(\tilde{\phi}_i(R_{i+})\) we see that
\[
\xi(\beta_{jik}) = -\xi(S_{\text{Im}((Q_{j+} - Q_{i-})/2)}) = \frac{\pi}{2l - 1}\text{Im}(Q_{j+}) - \frac{\pi}{2l - 1}\text{Im}(Q_{i-}) - \pi
\]
implying by equation (5)
\[
\mu(p) = -\frac{1}{\pi}[\xi(\beta_1) + \tilde{\phi}_i(R_{j+}) - \tilde{\phi}_i(R_{i-})]
= -\frac{1}{\pi}[\frac{\pi}{2l - 1}\text{Im}(Q_{j+}) - \frac{\pi}{2l - 1}\text{Im}(Q_{i-}) - \pi + \tilde{\phi}_i(R_{j+}) - \tilde{\phi}_i(R_{i-})]
\]
as expected.

For all other intersection points we see that \(\text{or}(p) = 1\). Let \(p \in L_j \cap L_i\) be such a point that is sent to \(e_0\) under the isomorphism \(\psi_{ji}\). We can describe the curve \(\beta_{jik}\) as an immersed curve starting at \(Q_{j+}\), ending at \(Q_{i-}\), wrapping around 0 twice and containing an "extra" loop as drawn below.
Thus, by perturbing $\beta_{ji\kappa}$, it can be broken up into a curve $\beta_1$ from $Q_{j+}$ to $Q_{i-}$ around 0 as above, a loop $\beta_2$ around 0 and an extra contractible loop $\beta_3$ oriented clockwise. Then using the additivity of $\xi$ we have

$$\xi(\beta_{ji\kappa}) = \xi(\beta_1) + \xi(\beta_2) + \xi(\beta_3) = \xi(\beta_1) - 2\pi.$$ 

Thus, using the above computation we have

$$\mu(p) = -\frac{1}{4}[\xi(\beta_1) - 2\pi + \hat{\phi}_i(R_{j+}) - \hat{\phi}_i(R_{i-})] = -\frac{1}{4}[\xi(\beta_1) + \hat{\phi}_i(R_{j+}) - \hat{\phi}_i(R_{i-})] + 1 = 1$$

as expected.

Finally, assume $p \in L_j \cap L_i$ is sent to $e_1$ via $\psi_{ji}$. The curve $\beta_{ji\kappa}$ is then isotopic to a curve $\alpha_1$ that starts at $Q_{j+} - 4(l - 1)i$, ends at $Q_{i-}$ and a contractible loop $\alpha_2$ oriented clockwise as illustrated below.

Now $\alpha_1 \circ S_{Im(Q_{j+}-Q_{i-})/2-2(l-1)}$ forms a counter clockwise contractible loop so that

$$2\pi = \xi(\alpha_1) + \xi(S_{Im(Q_{j+}-Q_{i-})/2-2(l-1)})$$

$$= \xi(\alpha_1) + \pi - \frac{\pi}{2(l-1)} Im(Q_{j+}) + \frac{\pi}{2(l-1)} Im(Q_{i-}) + 2\pi$$

$$= 2\pi + \xi(\alpha_1) + \hat{\phi}_j(R_{j+}) - \hat{\phi}_i(R_{i-})$$
Therefore
\[
\xi(\alpha_1) = \tilde{\phi}_i(R_i -) - \tilde{\phi}_j(R_j +)
\]
and
\[
\mu(p) = -\frac{1}{\pi} \left[ \xi(\alpha_1) + \xi(\alpha_2) + \pi + \tilde{\phi}_i(R_j +) - \tilde{\phi}_i(R_i -) \right] = \frac{-1}{\pi} [-2\pi + \pi] = 1
\]
as expected. □

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