The dynamical picture of a quark-antiquark interaction in light mesons, which provides linearity of radial and orbital Regge trajectories (RT), is studied with the use of the relativistic string Hamiltonian with flattened confining potential and taking into account the self-energy and string corrections. Due to the flattening effect both slopes, $\beta_n$ of the radial and $\beta_l$ of the orbital RT, decrease by $\sim 30\%$ with the value of $\beta_n = 1.30(5)$ GeV$^2$ being larger than $\beta_l = 0.95(5)$ GeV. The self-energy correction provides the linearity of RT and remains important up to very high excitations; the string correction decreases the slope of the orbital RT, while the intercept $\beta_0 = 0.51(1)$ GeV$^2$ is equal to the squared centroid mass of $\rho(1S)$. If the universal gluon-exchanged potential without fitting parameters and screening function, as in heavy quarkonia, is taken, then the slope of the radial RT decreases, $\beta_n = 1.25(8)$ GeV$^2$, and its value coincides with the slope of the orbital RT, $\beta_l = 1.08(8)$ GeV$^2$ within theoretical errors, producing the universal RT.

PACS numbers:

I. INTRODUCTION

The spectroscopy of light mesons refers to the field where non-perturbative QCD dominates and the Regge trajectories (RT), both orbital and radial, appear to be the most explicit manifestation of non-perturbative effects. It is known that the leading RT in the $(M^2, J)$-plane has a linear behavior with the slope $\beta_J^{(\text{exp.})} = 2\pi\sigma = 1.13(1)$ GeV$^2$, which corresponds to the value of the string tension $\sigma = 0.18(2)$ GeV$^2$ in the string models [1, 2], and precisely this $\sigma$ has been used in the realistic potential model with linear confining potential (CP) [3, 4]. Also, systematization of radial excitations of light mesons has shown that their squared masses lie on linear, or approximately linear, trajectories in the $(M^2, n)$-plane ($n = n_r$ is the radial quantum number) with a larger slope, $\beta_n = (1.25 \pm 0.15)$ GeV$^2$ [5], although later in Refs. [6, 7] a smaller slope $\beta_n = (1.143 \pm 0.013)$ GeV$^2$ was extracted from the Crystal Barrel data [8]. It was also observed that the slopes of the $(M^2, J)$-trajectories for the masses with spin $S = 0$ and $S = 1$ differ only by $\sim 10\%$ [9] and assumed that within this accuracy a universal RT can exist in the $(l, n)$-plane,

$$M^2(n, l) = a(l + n) + c,$$  \hspace{1cm} (1)

with the slope $a = 1.10(1)$ GeV$^2$ and the intercept $c = 0.68$ GeV$^2$. From Eq. (1) it follows that the masses of resonances with equal quantum number $N = l + n$ have to be equal and this assumption agrees with the experimental values of the masses of the resonances with $N = 3, 4, 5$ (see Table I).

| $N$ | $M_2$ (n,l) (in MeV) |
|-----|---------------------|
| 2   | $\rho_3(1690)$      |
| 3   | $\rho_3(1990)$      |
|     | $\rho_3(2250)$      |
| 4   | $\rho_3(2250)$      |

However, in another analysis of the experimental data, where the PDG masses and widths were used, a larger $\beta_n = (1.35 \pm 0.04)$ GeV$^2$ was extracted [11] and later, after re-analysis of the experimental data, the same authors
have given a smaller $\beta_n = 1.28(5) \text{GeV}^2$ \cite{12} with the conclusion that the universality of the radial and orbital RTs is not fulfilled at the level of 2.4 standard deviations. These results, irrespective of the fact whether slopes of radial and orbital RT’s are equal or not, raise an important theoretical issue, namely, what dynamical effects are responsible for the values of the slopes observed in experiments, and recently new studies of RTs were presented \cite{13, 14}.

The problem why the slope of the radial RT is small, was partly resolved in Refs. \cite{15, 16}, where the centroid masses of the iso-vector light mesons (with $l = 0, 1, 2$) were calculated with the use of a simplified version of the relativistic string Hamiltonian (RSH) $H_0$ \cite{17, 18}, starting with the general form of the squared mass,

$$M_{\text{cog}}^2 = \beta_l l + \beta_n n + \beta_{\text{cog}}.$$  

These calculations have shown that in a linear CP the slope of the radial RT, $\beta_n \approx (1.9 - 2.0) \text{GeV}^2$ is too large and can decrease only due to the flattening (screening) of the CP, while the resulting slope $\beta_n$ appears to be very sensitive to the parameters of the flattened CP $V_f(r)$. Also, the choice of a GE potential $V_{ge}$ seems to be important and in Ref. \cite{10} the GE potential with strong exponential screening was taken, while in Ref. \cite{15} the Coulomb potential with small coupling (another kind of screening) was used. However, in these papers arguments why such a strong screening occurs, have not been presented. Moreover, there exists the opposite point of view that there is no screening of $V_{ge}$ at large distances \cite{19}. Note, that screening of the GE potential is not seen in high excitations of charmonium and bottomonium and therefore the role of the GE interaction in light and other mesons needs to be clarified. This topic is one of the goals of our paper. Also, here we mostly concentrate on high excitations of light mesons to present the physical picture in a clearer way.

The spectrum of light mesons was already studied with the use of a generalized form of the RSH, which describes the QCD string with spinless quarks at the ends ($m_q = 0$) \cite{17, 22, 21}, but neglects the GE and spin-dependent potentials, i.e., reducing the instantaneous $q \bar{q}$ potential to the linear CP $V_c(r)$. This Hamiltonian is complicated and even with the linear potential included the spectrum was calculated only in quasi-classical approximation \cite{21}, giving for all orbital RTs the value $\beta_l = 2.2 \pi\sigma$, which within 10\% accuracy coincides with the experimental number. However, the same quasi-classical calculations give a large slope of the radial RT, $\beta_n = 2.3 \text{GeV}^2$ (for any $l$ and $\sigma = 0.18 \text{GeV}^2$), if the linear CP is used. It shows that the large slope of the radial RT is determined by dynamical effects but not by the model used. In another approximation to the generalized RSH \cite{20}, the ground state masses with large $l$, $M^2(l, n = 0) = 2\pi\sigma\sqrt{l(l + 1)}$ were calculated, giving for $l \geq 3$ the slope of leading RT $\beta_l$ in agreement with the experimental number 2\pi\sigma.

From lattice QCD it is known that the linear behavior of the CP is observed in the region $r \lesssim 1.2 \text{fm}$ \cite{22, 23} and at larger distances the CP is flattened due to the creation of the light $q \bar{q}$ holes (loops) in the Wilson loop \cite{18}. Due to these loops, the surface of the Wilson loop decreases and the string tension becomes dependent on the separation $r$. For the ground states, like 1S and 1P, which have relatively small sizes, $\lesssim 1.2 \text{fm}$, the flattening effect is not important and the linear CP can be applied in this case. But for high excitations, having sizes $\gtrsim 1.5 \text{fm}$, the flattening effect is crucially important and provides the mass shifts of $\sim - (300 - 400) \text{MeV}$. Unfortunately, there exist no lattice data in this region and the form and parameters of the flattened CP are not derived on a fundamental level yet. Therefore, here we use a phenomenological flattened CP, suggested in Ref. \cite{15}. This potential possesses an important property - linear behavior with reduced but finite string tension $\sigma \sim 0.10 \text{GeV}^2$ at large distances and therefore a quark and antiquark remain confined in a meson. There exists a different form of flattening (screening) of the CP, where at asymptotic distances the CP goes to a constant \cite{24} and this screened CP was used in many papers devoted to heavy quarkonia \cite{25, 26}.

An important feature of the RSH is that it does not contain a fitting (negative) constant, which violates the linearity of the RTs. Instead, it is important to take into account the self-energy negative contribution to the mass, $\delta_m \sim - 300 \text{MeV}$, even for high excitations of light mesons \cite{27, 28}; this correction is proportional to $1/M(nl)$ and provides linearity of the RT.

II. THE MASS FORMULAS

The spectrum is calculated here with the use of the simplified version of the RSH, $H_0$, where the spin-dependent potentials are considered as a perturbation and an approximate expression is used for the string corrections, so that in a strict sense $H_0$ can be applied only to the resonances with the angular momentum $l \leq 3$ \cite{10}. Then the Hamiltonian with $m_q = 0$,

$$H_0 = \mu + \frac{P^2}{\mu} + V_0(r),$$

(3)
is expressed via the variable $\mu$, determined by the extremum condition, $\frac{\partial \mu}{\partial \mu} = 0$. It gives $\mu = \sqrt{p^2}$, i.e., $\mu$ is the kinetic energy of a quark, and $H_0$ reduces to the spinless Salpeter equation (SSE),

$$ (2\sqrt{p^2} + V_0(r))\varphi_{nl}(r) = M_0(nl)\varphi_{nl}(r). $$

The eigenvalue (e.v.) $M_0(nl)$ is an important part of the centroid mass $M_{cog}(nl)$ (see below). The solutions of this equation were calculated with different potentials, a purely linear confining potential $V_c(r)$ and a flattened CP $V_f(r)$,

$$ V_c(r) = \sigma r, \quad \sigma = 0.18 \text{ GeV}^2; \quad V_f(r) = \sigma_t(r)r, $$

and for the sum, $V_0(r) = V_c(r) + V_ge(r)$, where in the general case the GE term is

$$ V_{ge}(r) = -\frac{4\alpha_v(r)}{3r}. $$

The solutions with $V_{ge}$ can be found either in exact calculations or considering $V_{ge}(r)$ as a perturbation. In our further analysis we mostly take into account the GE potential as a perturbation, because its contribution is not large, $\sim -100$ MeV, i.e., its magnitude is smaller than the self-energy correction. Therefore the correction due to the GE potential, $\delta_{ge}$, has to be defined on the same grounds as other corrections, i.e., using the lowest approximation in $\alpha_v$.

Our calculations show that in high excitations of light mesons the exact and the approximate values of $\delta_{ge}$ coincide within $\sim 10$ MeV.

Then, the centroid mass $M_{cog}(nl)$ includes the e.v. $M_0(nl)$ and three negative corrections: the self-energy and string corrections, and $\delta_{ge}$,

$$ M_{cog}(nl) = M_0(nl) + \delta_{ge} + \delta_{str}(nl) + \delta_{se}, $$

where these corrections together give a large negative contribution, $\sim -(400-500)$ MeV, while the e.v.s of the ground states are the following: $M_0(1S) = 1.339$ GeV, $M_0(1P) = 1.792$ GeV, $M_0(1D) = 1.155$ GeV. It is worth to underline that the mass $M_{cog}(nl)$ does not contain a fitting negative constant $C_0$, usually introduced in potential models; this constant produces a non-linear term $C_0M_0(nl)$ in the squared mass. On the contrary, an important negative contribution comes from the self-energy correction, $\delta_{se}(nl)$

$$ \delta_{se}(nl) = -\frac{\eta f\sigma}{\mu(nl)}, $$

which through $1/\mu(nl)$ is proportional to $1/M$ (see below) and does not give a non-linear term in RT. In Eq. 8 the factor $\eta(f)$ depends on the quark flavour and in light mesons $\eta_f = (0.90 - 0.95)$, see Refs. 27, 28. In heavy quarkonia the self-energy correction is neglected, because e.g. in bottomonium $\eta_b \sim 1$ MeV is very small, while the kinetic energy m.e. $\mu_0(nl)$ is large, giving a small, $\sim 1$ MeV, self-energy correction. In a light meson $\delta_{se}$ is large, $\sim -(300-400)$ MeV, and in higher excitations its magnitude slightly decreases due to the growth of the kinetic energy, but still remains very important.

On the contrary, a negative string correction $\delta_{str}(nl)$ ($l = 1, 2$) 13, 10,

$$ \delta_{str}(nl) = -\frac{l(l+1)\sigma r^{-1}}{8\mu^2(nl)}, $$

increases for orbital excitations with growing $l$ (while it decreases for larger $n$): its values are $\sim -(40-150)$ MeV for $l = 1, 2, 3$. The expression of $\delta_{str}$, Eq. 9 does not change if the flattened CP is used, but then $\sigma$ has to be replaced by the averaged m.e. $\langle \sigma_t(r) \rangle_{nl}$.

High excitations with $l \neq 0$ have large sizes, giving small fine-structure splitting, and therefore the centroid mass $M_{cog}(nl)$ practically coincides with the masses of other members of the multiplet with $J = \mathbf{L} + \mathbf{S}$, with exception of the $nS$ and $1P$ states, where the spin-spin and fine-structure splitting are not small. In particular, in the $nS$ states the hyperfine correction,

$$ \delta_{hf}(nS) = \frac{8}{9}\alpha_{hf}\tau(nS), $$

with $\tau(nS) = \frac{|R_{nS}(0)|^2}{\mu^2(nS)}$, is not small even for the $4S$ resonance. Calculations show that the ratio $\tau(nS)$, Eq. 10, weakly depends on the parameters of the GE potential taken, e.g. for the ground 1S state $\tau(1S) = (0.84-0.94)$ GeV is obtained for different
types of GE potentials. This fact allows to extract \( M_{cog}(1S) \) from experiment with accuracy \( \sim (10-15) \) MeV. Notice that knowledge of \( M_{cog}(1S) \) is very important since it determines the intercept of the leading \( l \)-trajectory \( (n=0) \),
\[
M_{cog}^2(l) = \beta_l + \beta_{cog}, \quad (n=0),
\]

namely, the intercept \( \beta_{cog} = M_{cog}^2(1S) \) with \( M_{cog}(1S) = M(\rho(1S)) - \frac{1}{2} \delta_{hf}(1S) \). The hyperfine correction can be determined with \( \sim 10 \) MeV accuracy, if the universal hyperfine coupling \( \alpha_{hf} = 0.33(1) \) is taken (this coupling was used in heavy-light mesons and bottomonium [29]). Taking the theoretical value \( \tau(1S) = 0.90(5) \) GeV one obtains \( \delta_{hf}(1S) = 264(15) \) MeV and \( M_{cog}(1S, \exp.) = (775 - \frac{1}{2} 264(15)) = 709(4) \) MeV. It gives the “experimental” intercept,
\[
\beta_{cog}(\exp.) = 0.709(4)^2 \text{ GeV}^2 = 0.50(1) \text{ GeV}^2,
\]

which is smaller than the slope of the leading RT in the \((M^2, J)\)-plane, where \( \beta_0(\exp.) = M^2(\rho(1S, \exp.)) = 0.60 \) GeV\(^2\), determined by the mass of \( \rho(1^3S_1) \), is larger.

In an analysis of radial RT’s the difference between neighboring squared masses, \( b_n^2 = M_{cog}^2(n+1, l) - M_{cog}^2(n, l) \), which in general can depend on the radial quantum number \( n \), may be useful. If for all states with a given \( l \) the numbers \( b_n^2 = b \) are equal, then the radial RT reduces to the radial RT suggested in Ref. [8]:
\[
M(n, l)^2 = M_b^2 + b n, \quad (l \text{ fixed}),
\]

where \( M_b(n = 0, l) \) is the mass of the ground state.

### III. LINEAR CONFINING POTENTIAL

The structure of the light meson spectrum can be illustrated, taking a purely linear CP, where the mass formulas and the corrections are simplified. Notice that the linear CP plays a special role in string theory and also in the AdS approach [30]. For a linear CP
\[
M_0(nl) = 4\mu_0(nl); \quad \sigma \langle r \rangle_{nl} = 2\mu_0(nl) = 1/2M_0(nl),
\]

and then the correction
\[
\delta_{se}(nl) = -\frac{0.90\sigma}{M_0} \tag{15}
\]

is proportional to \( M_0^{-1}(nl) \) and therefore it does not change the slopes of the radial and orbital RT, but strongly decreases the intercept.

For further analysis it is very convenient to use the following representation of \( M_0^2(nl), \sigma = 0.180 \) GeV\(^2\) [31],
\[
M_0^2(nl) = \sigma \langle 8l + 4\pi n, r \rangle + 3\pi \xi(nl) = (1.44l + 2.26n + 1.70\xi(nl)) \text{ GeV}^2;
\]

where the numbers \( \xi(nl) \) are equal to 1.0 with an accuracy better 1% for all states, with the exception of \( \xi(1S) = 1.057 \) and \( \xi(1P) = 1.045 \). Notice that in the representation Eq. (16) the slopes \( \beta_l = 1.44 \) GeV\(^2\), \( \beta_n = 2.26 \) GeV\(^2\), and the intercept \( \beta_{cog} = 1.70 \) GeV\(^2\) are significantly larger than those in the experimental RTs.

As seen from Eq. (15), the self-energy correction affects only the intercept, while the GE correction appears to be proportional to the e.v. \( M_0(nl) \) due to the relation,
\[
\langle r^{-1} \rangle_{nl} = \frac{0.262M_0(l + 2)}{(l + 1)(l + n + 2)}, \quad (l \neq 0),
\]

found with a good accuracy for the states with \( l \neq 0 \). Therefore a negative GE correction, \( \delta_{ge} = -\frac{4}{3\sigma_{eff}}\langle r^{-1} \rangle \) is also proportional to \( M_0(nl) \) and provides a small decrease of the slopes of the radial and orbital RTs, and also of the intercept.

Moreover, owing to the relation Eq. (17), the string correction \( (n = 0) \) can be rewritten as
\[
\delta_{str} = -0.524l(l + 1)\frac{\sigma\langle r^{-1} \rangle}{M_0^2} = -0.524\frac{\sigma l}{M_0}
\]

i.e., it is negative and proportional to \( l \), thus decreasing the slope \( \beta_l \) and gives \( M_{cog}^2(l, n = 0) = M_0^2 - 1.05\sigma l + \delta_{str}^2 \). Then for \( \sigma = 0.180 \) GeV\(^2\) the slope of the orbital RT,
\[
\beta_l = (8 - 1.05)\sigma = 1.25 \text{ GeV}^2,
\]

\[
\]
is decreased, being still by 11% larger than $\beta_1(\text{exp.}) = 1.13(1) \text{ GeV}^2$; this relatively large slope $\beta_1$ is obtained because the GE correction was not taken into account yet. Notice that the slope of the radial RT $\beta_n = 4\pi\sigma$ is not affected by the self-energy and the string corrections.

Let us consider now the 1S state in a purely linear CP, where the kinetic energy $\mu(1S) = 335 \text{ MeV}$ is small and gives large values $\delta_{se}(1S) = -0.513 \text{ GeV}$ and the mass $M_{cog}(1S) = 0.826 \text{ GeV}$. Then the intercept $\beta_{cog} = M_{cog}^2(1S) = 0.682 \text{ GeV}^2$ appears to be larger than $\beta_0(\text{exp.}) = 0.51 \text{ GeV}^2$ in Eq. (12). This mass $M_{cog}(1S) = 0.826(3) \text{ GeV}$ is large because the GE potential is not taken into account and this fact shows that the value of the intercept depends on the GE dynamics.

The GE interaction manifests itself in two ways: first, through the correction $\delta_{ge}(1S)$ and also via the growth of the averaged kinetic energy $\mu(1S)$, if the GE potential is used. It has typical values $\mu_g(1S) \sim (0.38 - 0.41) \text{ GeV}$ and due to the larger $\mu_g$ the self-energy correction decreases. However, theoretical uncertainties $\sim (10 - 30) \text{ MeV}$ in the kinetic energies and in $\delta_{se}$ do not allow to extract a precise value of the vector coupling $\alpha_v$ from the mass of $\rho(1^3S_1)$ and here we estimate this coupling, taking a typical value $\mu_g = 0.39 \text{ GeV}$. It gives a smaller $\delta_{se}(1S) = -0.415(15) \text{ GeV}$ and using the m.e. $\langle r^{-1}(1S) \rangle = 0.3637 \text{ GeV}$ and $M_{cog}(1S) = 1.3933 \text{ GeV}$, one obtains the condition to extract the Coulomb coupling $\kappa(\text{eff.}) = -\frac{4}{3} \alpha(\text{eff.})$ from the experimental mass: $M_{cog}(1S)(\text{exp.}) = 0.709(5) \text{ GeV} = (1.339 - 0.415(15) - 0.364 \kappa) \text{ GeV}$. It gives a rather large $\kappa = 0.59(5)$, or $\alpha(\text{eff.}) = 0.44(4)$ (with a theoretical error of $\sim 10\%$), and $\delta_{ge}(1S) = -215(15) \text{ MeV}$. The extracted value of $\alpha(\text{eff.})$ appears to be significantly larger than $\alpha(\text{eff.}) = 0.30$, used in Ref. (15).

For the ground states $M(1l)$ with $l \neq 0$ the correction $\delta_{ge}$ can be rewritten, using the relation Eq. (13),

$$\delta_{ge}(l) = -\frac{0.262M_0}{l+1}, \quad (l \neq 0, n = 0),$$  \hspace{1cm} (20)

which weakly depends on $l$. With all corrections included and introducing the factor $Z(nl)$ as $M_0(nl) - \kappa(\langle r^{-1} \rangle)_{nl}$, i.e.,

$$Z_0(l) = Z(l, n = 0) = 1 - \frac{0.262\kappa(\text{eff.})}{l+1}; \quad Z(n, l) = 1 - \frac{0.262\kappa(\text{eff.})(l+2)}{(l+1)(l+n+2)},$$  \hspace{1cm} (21)

the centroid mass of the ground states can be presented by the analytical expression,

$$M_{cog}(1l) = M_0(1l)Z_0(l) - \frac{3.60\sigma}{M_0(1l)} - \sigma l \frac{0.524}{M_0}. \hspace{1cm} (22)$$

From Eq. (20) one can see that $\delta_{ge}$, proportional to $M_0$, slightly decreases the slopes $\beta_l$ and $\beta_n$ by the factor $Z_1$, while the negative string correction,

$$\delta_{str.}(n = 0, l) = -\sigma l \frac{0.524}{M_0}, \hspace{1cm} (23)$$

proportional to $l$ and $M_0^{-1}$, decreases only $\beta_l$. With all corrections taken into account the squared centroid mass $M_{cog}^2(1l)$ is described by the leading RT

$$M_{cog}^2(1l) = M_0^2Z_0^2(l) - 1.048\sigma l Z_0(l) - 7.2\sigma Z_0(l) + \delta_{se}^2$$  \hspace{1cm} (24)

with the slope of the orbital RT given by

$$\beta_l = \sigma \left(8Z_0^2(l) - 1.048Z_0(l)\right), \hspace{1cm} (25)$$

which weakly depends on $l$. Taking $\kappa(\text{eff.}) = 0.58$ one has $\beta_l(l = 1) = 5.862\sigma$, $\beta_l(l = 2) = 6.215\sigma$, $\beta_l(l = 3) = 6.396\sigma$, so that their averaged value $\bar{\beta}_l = 6.16\sigma$ coincides with the experimental slope $2\pi\sigma = 6.28\sigma$.

Thus, owing to the string and Coulomb corrections the slope of the leading RT is obtained in agreement with experiment [27] and the calculated centroid masses of the ground states (in MeV): $M_{cog}(1S) = 709(5)$, $M(\rho(1^3S_1)) = 775(15)$; $M_{cog}(1P) = 1282$, $M_{cog}(1D) = 1667$, $M_{cog}(1F) = 1978$ appear to be in agreement with the experimental masses of $\rho(1690)$ and $a_4(2040)$ (its $M(\text{exp.}) = 1995^{+8}_{-6} \text{ MeV}$) [10]), although in our calculations the fine-structure splittings have been neglected. This agreement with experimental masses shows that even for ground states with $l \neq 0$ fine-structure splittings are small and therefore our assumption, that these splittings are small in high excitations, is justified. Notice that the calculated string corrections: $-48 \text{ MeV}$, $-81 \text{ MeV}$, and $-142 \text{ MeV}$ respectively, grow for larger $l = 1, 2, 3$.

Thus the RSH $H_0$ with linear CP and the mass formula Eq. (7) provides the correct slope of the orbital leading RT, if the relatively large Coulomb constant $\kappa \cong 0.59(5)$ ($\alpha(\text{eff.}) \cong 0.44(4)$) is taken, while the intercepts $\beta_{cog} =$
0.51(1) GeV² of the \((l, n)\)-trajectory and \(\beta_0 = 0.60\) GeV² of the \((J, n)\) trajectory, are determined by the \(\rho\) masses, \(M_{\text{cog}}^2(1S)\) and \(M^2(\rho(1^{3}S_1))\).

As seen from Eqs. (17,24) and the definition of \(Z(nl)\), the slope \(\beta_l\) of the daughter RTs \((n \geq 1)\) depends on the radial quantum numbers \(l\) and \(n\), and the Coulomb constant \(\kappa\). In the radial excitations the value of \(Z(nl)\) appears to be larger than that for ground states, because for them the GE and the string corrections are smaller. Then the factor \(Z(nl)\) in Eq. (24) increases for growing \(n\) and \(l\) and its averaged value, denoted as \(\bar{Z}\), depends on the Coulomb constant \(\kappa\) and \(\bar{Z} \approx 1.0\). For a weak GE potential and \(\bar{Z} = 0.94(2), (\bar{Z}^2 = 0.88(5))\) for \(\kappa = 0.72\). Then collecting all corrections, the mass \(M_{\text{cog}}^2(n, l)\) can be described by the RT (\(\kappa = 0.72\)):

\[
M_{\text{cog}}^2(n, l) = 1.13(4)l + 2.0(1)n + 1.7(2) + \delta_{\text{sc}}^2.
\] (26)

Here the slope of the orbital daughter RTs, \(\beta_l = 1.13(4)\) GeV², is obtained in agreement with the experimental value, while the slope of the radial RTs, \(\beta_n = 2.0(1)\) GeV² is large, being only by \(\sim 10\%\) smaller than that for a purely linear CP.

Notice that a small value of the intercept, \(\beta_{\text{cog}} = 0.3(1)\) GeV², is obtained, if the term \(\delta_{\text{sc}}^2\) is neglected (if the \(\delta_{\text{sc}}^2\) correction is taken into account, then the calculated intercept coincides with \(\beta_{\text{cog}} = 0.51(1)\) GeV² within a theoretical error). Thus in the linear CP, with all three corrections taken into account, non-equal slopes of the orbital and the radial RTs are derived.

Notice, that the ground states with large \(l\) were already studied with the use of the generalized RSH \[17, 21\] and the calculated masses,

\[
M_{\text{cog}}^2(n = 0) = 2\pi\sigma\sqrt{1(l+1)}, \; (l \geq 2)
\] (27)

give the correct slope of the orbital RT: \(\beta_l = 2\pi\sigma\sqrt{1/(l+1)} \approx 2\pi\sigma\; (l \geq 3)\), while a small intercept was neglected.

Thus we conclude that for the linear-plus-GE potential with large \(\kappa \approx 0.72\) the slope of the radial RT slightly depends on \(l\) and remains large, \(\beta_n(l) \approx 2.0(1)\) GeV² and this result confirms our assumption that a smaller \(\beta_n\) can be reached only by taking the flattened CP at large distances, when in high excitations the characteristic values of the string tension, \(\langle \sigma \rangle_{nl} \sim (0.135 - 0.14)\) GeV² are small.

IV. FLATTENED POTENTIAL

The flattened CP \(V_l(r)\) was suggested in Ref. [15] to explain the “small” masses of the radial excitations in light mesons and the slope of the radial RT which was found to be dependent on the angular momentum \(l\), e.g. \(\beta_n = 1.38(5)\) GeV² for \(l = 1\) and \(\beta_n = 1.29(5)\) GeV² for \(l = 3\) exceed the slope of the orbital RT \(\beta_l \approx 1.0\) GeV² by \(\sim 30\%\), i.e., there a universal RT was not observed. In Ref. [15] the light meson spectrum was calculated using the Coulomb potential with a small value of the strong coupling constant, while later [16] the GE potential with exponential screening function was taken and the smaller slopes, \(\beta_n = 1.25\) GeV² for \(l = 1\) and \(\beta_l = 1.11(5)\) GeV² for \(l = 2\), were obtained. This result shows that although the GE potential gives a small contribution to the mass of a light meson, it affects the slopes. However, a strong screening of GE potential is neither proved yet in theoretical studies nor seen in heavy quarkonia. Therefore to study the spectrum of light mesons here we take the same universal GE potential as in heavy quarkonia and the phenomenological flattened CP \(V_l(r)\) as in Ref. [15].

\[
V_l(r) = \sigma_l(r)r, \; \text{with} \; \sigma_l(r) = \sigma(1 - \gamma f(r)),
\] (28)

The potential \(V_l(r)\) is defined by three parameters: first the characteristic distance \(R_0 \sim (1.2 - 1.4)\) fm, where the flattening effect starts and string breaking becomes possible. Its value \(\sim (1.2 - 1.4)\) fm, can be taken from the lattice calculations [22] and here will be varied in the range \((5.0 - 8.0)\) GeV⁻¹. The second parameter, \(\gamma\), determines the derivative of \(V_l(r)\) and also the asymptotic value of the string tension,

\[
\sigma_l(r \to \infty) = \sigma(1 - \gamma).
\] (29)

The variation of the parameter \(\gamma\) has confirmed the result of Ref. [16] that values of \(\gamma = (0.40 - 0.45)\) provide the best description of the spectrum. The third parameter - the constant \(B\) is present in the function \(f(r)\), Eq. (28),

\[
f(r) = \frac{\exp(\sqrt{\sigma}(r - R_0))}{B + \exp(\sqrt{\sigma}(r - R_0))},
\] (30)

characterizes the function \(f(r = R_0) = (B + 1)^{-1}\) at the point \(r = R_0\) and how fast the increasing function \(f(r)\) is approaching to its asymptotic value, \(f(\text{as.}) = 1.0\), at large distances, \(r \sim 3.0\) fm. In other aspects \(f(r)\) can be
rather arbitrary. In our analysis all parameter are varied in wide ranges: $\gamma = 0.35 - 0.50$, $B = 15 - 25$, $R_0 = (5 - 8)\ \text{GeV}^{-1} = (1.0 - 1.6)\ \text{fm}$, with $\sigma_{\text{as.}}(0) = (0.090 - 0.11)\ \text{GeV}^2$. The best agreement with experiment is reached for $\gamma = (0.40 - 0.45)$, $B \sim (20 - 25)$, $R_0 = 6.0\ \text{GeV}^{-1}$.

From the physical point of view it is also important that at large distances the potential, $V_l(r) \to \sigma_{\text{as.}}(r)$ becomes again linear with small $\sigma_{\text{as.}}$, thus still keeping a quark and anti-quark confined to a meson. Also, to keep $\langle \sigma_{\text{as.}}(1S) \rangle = 0.180\ \text{GeV}^2$ in the 1S state, one needs to take $\sigma = 0.182\ \text{GeV}^2$ in Eq. (28). The other (mostly used) parameter values are the following,

$$
\gamma = 0.40, \quad B = 20, \quad R_0 = 6.0\ \text{GeV}^{-1}, \quad \sigma = 0.182\ \text{GeV}^2.
$$

(31)

From our point of view this phenomenological potential represents the physical picture rather well, but it has a negative feature, namely the non-monotonic behavior near the point $r = R_0$. To obtain a smooth behavior of some matrix elements (m.e.s), a numerical regularization is needed.

With the flattened potential the spectrum significantly changes, as compared to the linear CP: first, the sizes $\langle r^2(n\ell) \rangle$ of highly excited states increase and can reach $\sim 2.5\ \text{fm}$ (see Table III), while the sizes of low states (1S, 2S, 1P) are not large, in particular, the r.m.s of $\rho(1S)$ meson, $\sqrt{\langle r^2(1S) \rangle} = 0.71\ \text{fm}$, is in good agreement with predictions in other approaches [32] and for them the linear CP can be used. Secondly, due to the flattening effect the e.v.s $M_{0f}(nl)$ have a different structure and, namely, the slopes of the linear trajectory built for the squared masses $M_{0f}^2(n\ell)$, appear to be by (30–40)% smaller than those in Eq. (10) with the linear CP,

$$
M_{0f}^2(n\ell)(\text{in GeV}^2) = 0.93(4)\ l + 1.23(16)\ n + 2.74(10).
$$

(32)

(in this trajectory the masses of the $nS$ and $1P$ states are not included). Here different slopes $\beta_1 \sim 0.93(4)\ \text{GeV}^2$ and $\beta_n \sim 1.23(16)\ \text{GeV}^2$ are obtained and they have large theoretical errors because of the irregular behavior of the flattened CP, so that the trajectory Eq. (32) can be considered only as approximately linear. It is of interest to notice that the calculated slopes $\beta_1$ and $\beta_n$ have values very close to those extracted from the PDG data in Refs. [11, 12].

Meanwhile, in Eq. (32) the large value of the intercept $\sim 2.74(10)\ \text{GeV}^2$, indicates that in the flattened CP other corrections, the self-energy and the Coulomb corrections, must give large contributions to $\beta_{\text{cog}}$. These corrections are defined by the same general formulas Eqs. (29, 31), if there the parameters $\mu(n\ell)$ and $\sigma$ are replaced by the m.e.s $\langle \mu_l \rangle$ and $\langle \sigma_1 \rangle$, respectively.

TABLE II: The r.m.s. (in fm) of high excitations of light mesons ($m_q = 0$) for the linear potential (LP) ($\sigma = 0.18\ \text{GeV}^2$), the flattened confining potential (FCP) with the parameters Eq. (31) and for the FCP + GE potential with the parameters from Eq. (32).

| state | r.m.s (LP) | r.m.s.(FCP) | r.m.s.(FCP+GE) |
|-------|------------|-------------|----------------|
| 1S    | 0.82       | 0.86        | 0.71           |
| 2S    | 1.47       | 1.53        | 1.30           |
| 3S    | 1.65       | 2.42        | 2.12           |
| 4S    | 1.78       | 2.67        | 2.61           |
| 5S    | 2.08       | 2.94        | 2.79           |
| 1P    | 1.06       | 1.13        | 1.00           |
| 2P    | 1.43       | 1.95        | 1.69           |
| 3P    | 1.72       | 2.64        | 2.53           |
| 4P    | 1.97       | 2.78        | 2.69           |
| 1D    | 1.24       | 1.41        | 1.28           |
| 2D    | 1.56       | 2.42        | 2.18           |
| 3D    | 1.83       | 2.70        | 2.67           |
| 4D    | 2.06       | 2.94        | 2.83           |
| 1F    | 1.41       | 1.77        | 1.59           |

Another specific feature of $V_l(r)$ refers to the m.e.s of high excitations ($l \geq 2$, $n \geq 1$), where the m.e.s $\langle \sigma_{\ell}(nl) \rangle = 0.149(3)\ \text{GeV}^2$ (with $\gamma = 0.40$) are equal within 6% accuracy (see Table III) (if $\gamma = 0.45$, then $\langle \sigma_{\ell}(nl) \rangle = 0.137(4)$) GeV$^2$ is smaller). Also the kinetic energy, as a function of $n$, grows slowly and appears to be by $\sim (100 - 200)\ \text{MeV}$ smaller than $\mu_0(n\ell)$ in the purely linear CP. Due to this feature the self-energy correction, proportional to $\mu_f^{-1}$,
TABLE III: The averaged \(\langle \sigma_1(nl) \rangle\) (in GeV\(^2\)) for the flattened CP with the parameters Eq. 31

\[
\begin{array}{cccc}
 n/l & 1 & 2 & 3 & 4 \\
 1 & 0.170 & 0.158 & 0.153 & 0.147 \\
 2 & 0.165 & 0.153 & 0.149 & 147 \\
 3 & 0.156 & 0.148 & 0.146 & 0.145 \\
\end{array}
\]

TABLE IV: The kinetic energies \(\mu_g(nl)\) (in MeV) for the potential \(V_{0f}(r) = V_I(r) + V_{ge}(r)\)

\[
\begin{array}{cccc}
 n/l & 0 & 1 & 2 & 3 \\
 0 & 400 & 491 & 539 & 564 \\
 1 & 500 & 480 & 525 & 580 \\
 2 & 520 & 482 & 536 & 594 \\
 3 & 550 & 500 & 560 & 620 \\
\end{array}
\]

remains large and important for high states. However, owing to the GE interaction the averaged kinetic energy, \(\mu_g(nl) \approx \mu_I(nl) + \delta_{ge}/4\) increases (see Table [V]).

With small GE corrections, \(\delta_{ge} \sim -(80 - 150)\) MeV, to the masses, the \(M_g^2(nl)\) trajectory has a larger slope \(\beta_l\) of the orbital RT and a smaller slope \(\beta_n\), and the intercept, as compared to that in Eq. (32),

\[
M_g^2(nl) \text{ (in GeV}^2\text{)} = 0.99(3)l + 1.20(13)n + 2.1(1)(l \neq 0).
\]

Although the flattening effect produces large mass shifts, \(\Delta_f(nl) \sim -(300 - 350)\) MeV, nevertheless, in Eq. (33) the masses \(M_g(nl)\), defined without the self-energy and the string corrections, are still by \(\sim (300 - 400)\) MeV larger than their experimental values (see Tables [IV][V]).

In the previous section we have presented results with the linear CP and in the Coulomb potential the effective constant \(\kappa_{(\text{eff.})} = 0.59\) was extracted from the \(\rho(1^3S_1)\) mass. Here we take the universal GE potential with the strong coupling \(\alpha_v(n_f = 3)\), defined in QCD by terms of fundamental parameters only \([33, 34]\); in the coordinate space the coupling,

\[
\alpha_v(r) = \frac{2}{\pi} \int_0^\infty dq q \sin(qr) \alpha_v(q^2),
\]

is expressed via the two-loop coupling in momentum space,

\[
\alpha_v(q^2) = \frac{12\pi}{27t} \left(1 - \frac{64 \ln t}{81t}\right),
\]

with \(t = \ln[(q^2 + M_B^2)/\Lambda^2]\) and the parameters, taken from Ref. [16], are

\[
\Lambda_V(n_f = 3) = 0.455 \text{ GeV}, \quad M_B = 1.15 \text{ GeV}, \quad \alpha(\text{as.}) = \alpha_v(r \to \infty) = 0.555; \quad \frac{4}{3} \alpha(\text{as.}) = 0.74.
\]

Here the value of the QCD vector constant \(\Lambda_V(n_f = 3) = 455\) MeV is a bit smaller than the constant \(\Lambda_V(n_f = 3) = 480 \pm 20\) MeV, derived in pQCD \([33]\) and discussed in detail in Ref. [34]. Our calculations show that the GE potential with this strong coupling, Eq. (30), and the Coulomb potential with the Coulomb constant \(\kappa = 0.72\), \(\alpha(\text{eff.}) = 0.54\) give equal values of the corrections \(\delta_{ge}(nl)\) (within 2\% accuracy) for all states, with the exception of the \(nS\) and \(1P\) states, where the asymptotic freedom (AF) behavior of the strong coupling remains important.

For these states the following effective Coulomb constants are defined: \(\alpha_{\text{eff.}}(1S) = 0.45\ (n = 0), \quad \alpha_{\text{eff.}}(1P) = 0.46, \quad \alpha_{\text{eff.}}(nS) = 0.48\ (n \geq 1)\), or \(\kappa(1S) = 0.60, \quad \kappa(1P) = 0.613, \quad \kappa(nS) = 0.64\). Notice that the use of the effective Coulomb constant simplifies the physical picture and provides \(\sim 15\) MeV accuracy in the values of \(\delta_{ge}(nl)\). These corrections are given in Tables [V] and [VI] together with the mass shifts due to the flattening effect, \(\Delta_f(nl) = M_{gf}(nl) - M_{gf}(nl)\). For the \(nS\) excitations all corrections, including the hyperfine correction, and the masses \(M(n^3S_1)\) are given in Table [V]. From the masses, given in Table [V] which agree with the experimental values, one can define the slope \(\beta_n\) of the
was discussed in Sec. III, and this difference characterizes the accuracy of our calculations with the flattened CP. The ratio \( \frac{Y}{g} \) written as Eq. (10), that this approximately universal RT has slopes which by 5% differ from those in the leading RT \((n = 0)\), which has large, has a large uncertainty and therefore the \( \rho(n^3S_1) \) trajectory can be considered only as approximately linear.

Notice that the masses of high orbital excitations \((l \neq 0)\) have also large shifts \( \Delta_l \sim -300 \) MeV and the Coulomb corrections \( \sim -100 \) MeV, but their masses, \( M_g(nl) = M_0(nl) + \Delta_l(nl) + \delta_{ge}(nl) \), are still exceed by \( \sim (300-400) \) MeV the experimental values (see Table VII). This means that in the flattened CP the self-energy and string corrections remain very important. The masses \( M_g^2(nl) \) from Table VII, with exception of the ground states, lie on an approximately linear trajectory Eq. (33).

V. THE UNIVERSAL REGGE TRAJECTORIES

As shown in the previous section, the solutions \( M_g(nl) \) of the SSE with the GE correction exceed the experimental masses by \( (300-400) \) MeV and here we take into account other two corrections, \( \delta_{se} \) and \( \delta_{str} \), which are defined by the general formulas Eqs. (37,38) and weakly depend on the quantum numbers \( l \) and \( n \). It is convenient to introduce the ratio \( \frac{Y(nl)}{\langle \sigma_f(r) \rangle_{nl}} / \mu(nl) \) \((l, n \geq 1)\), weakly dependent on \( n \) and \( l \). Then the self-energy correction can be written as \( \delta_{se} = -0.90Y(nl) \) and the string correction,

\[
\delta_{str}(nl) = -l(l+1) \frac{Y(nl)(r^{-1})_{nl}}{8\mu(nl)} \approx \left(-0.13(1)Y(nl)\right) \frac{l+2+n}{l+2+n},
\]

is proportional to \( l \) and also depends on the quantum numbers \( l, n \). The calculated masses \( M_{cog}(nl) \) of light mesons with all corrections included are given in Table VIII. These masses allow to built the trajectory in the \((M^2_{cog}, nl)\)-plane \((l \neq 0)\), where the slopes \( \beta_l \) and \( \beta_n \) of the orbital and radial RTs coincide within the theoretical uncertainties. Notice that this approximately universal RT has slopes which by \( \sim 5\% \) differ from those in the leading RT \((n = 0)\), which was discussed in Sec. III and this difference characterizes the accuracy of our calculations with the flattened CP,

\[
M^2_{cog}(n,l) (\text{in GeV}^2) = 1.03(9) l + 1.15(12) n + 0.70(20).
\]

| State  | \( M_0(nS) \) (MeV) | \( \Delta_l(nS) \) | \( \delta_{se}(nS) \) | \( \delta_{str}(nS) \) | \( M(n^3S_1) \) (MeV) | \( M(\text{exp.})[25] \) |
|--------|-----------------|----------------|----------------|----------------|-----------------|----------------|
| \( 1^3S_1 \) | 1339 | 0 | -405 | -225 | 66 | 775 | \( \rho(775), M = 775.5(3) \) |
| \( 2^3S_1 \) | 1998 | -55 | -325 | -190 | 40 | 1468 | \( \rho(1465), M = 1465(25) \) |
| \( 3^3S_1 \) | 2498 | -198 | -291 | -143 | 26 | 1892 | \( \rho(1900), M = 1880(30) \) |
| \( 4^3S_1 \) | 2915 | -346 | -244 | -135 | 20 | 2210 | \( \rho(2150), M = 2254(22) \) |

| state | \( M_0(nl) \) (MeV) | \( \Delta_l(nl) \) | \( \delta_{se}(nl) \) | \( M_g(nl) \) (MeV) |
|-------|-----------------|----------------|----------------|----------------|
| \( 1P \) | 1792 | 0 | -140 | 1652 |
| \( 2P \) | 2315 | -102 | -144 | 2069 |
| \( 3P \) | 2750 | -278 | -123 | 2349 |
| \( 4P \) | 3129 | -398 | -114 | 2617 |
| \( 1D \) | 2155 | 0 | -124 | 2031 |
| \( 2D \) | 2601 | -173 | -113 | 2315 |
| \( 3D \) | 2900 | -342 | -96 | 2552 |
| \( 4D \) | 3337 | -448 | -94 | 2795 |
| \( 1F \) | 2465 | 0 | -97 | 2368 |
| \( 2F \) | 2861 | -256 | -86 | 2519 |
| \( 3F \) | 3215 | -394 | -84 | 2737 |
In Table VII the masses, described by the RT Eq. (38), are compared with those from Table VII. Thus, we have shown that a universal RT can exist if the flattened CP and the GE potential with the strong frozen $\alpha_{as.} = 0.55$ are used, and also other corrections are taken into account. However, uncertainties in the parameters of the flattened CP and the self-energy term, give rise to uncertainties in the values of the slopes and the intercept of this RT.

We can conclude that if the GE interaction is strongly suppressed, then the parameters of the RTs occur to be the same as for the CP, either with or without flattening effect: $\beta_n \sim 1.23(16)$ GeV$^2$ is larger than the slope of the orbital RTs, $\beta_l(n) \sim 0.93(4)$ GeV$^2$ (n $\geq 1$). The conclusion whether the GE potential is screened or not, depends on the interpretation of the existing experimental data. If the slopes of the radial and orbital RTs, extracted from experiment [6, 7], are equal, then there is no screening of $V_{ge}$ (or the screening is very weak). If from the analysis of the experimental data the slope $\beta_n$ of the radial RTs, is found to be larger than $\beta_l$, then screening of the GE potential is very probable and this effect has to be explained in theory. Knowledge of the GE interaction at large distances may be also important for identification of high excitations in heavy quarkonia, where screening of $V_{ge}$ is not observed yet, while the flattening effect is seen in the decrease of the masses of the high $\chi_c J(nP)$ states, where the mass shifts $\sim -(100 - 200)$ MeV are expected [35]. Notice that our results do not confirm the assumption that the screening of $V_{ge}$ and the flattening of the CP can be described by the same function [26].

TABLE VIII: Comparison of the masses $M_{cog}$ (in MeV) from Table VII and those defined in the RT given in Eq. (38).

| State | $M_{cog}(nl)$ | $M(nl)$ | $M_{cog}(nl)$ Eq. (38) | exp. |
|-------|---------------|---------|-------------------------|------|
| $1P$  | 1282          | 1315    | $a_1(1260)$             |      |
| $2P$  | 1726          | 1697    | $a_1(1640)$             |      |
| $3P$  | 2028          | 2007    | absent                  |      |
| $4P$  | 2324          | 2276    | absent                  |      |
| $1D$  | 1662          | 1661    | $\rho_3(1690)$          |      |
| $2D$  | 1993          | 1977    | $\rho_3(1900)$          |      |
| $3D$  | 2244          | 2249    | absent                  |      |
| $1F$  | 1991          | 1977    | $a_4(2000)$             |      |
| $2F$  | 2244          | 2249    | absent                  |      |
| $3F$  | 2442          | 2467    | absent                  |      |

VI. CONCLUSIONS

The spectrum of light mesons was studied with the use of the RSH with the flattened confining potential and taking into account the GE, the self-energy, and the string corrections. We have confirmed that the flattening effect, existing due to light $q\bar{q}$-pairs creation, produces large mass shifts, which can reach $\sim -300$ MeV for the $3P$ and $3D$ excitations.
and the best set of the flattened potential parameters was defined. Our calculations show that agreement with the experimental values of the masses can be reached, if all corrections are taken into account, but only the self-energy correction provides the linearity of the RT.

A special accent was laid on the role of the GE potential by performing calculations with the universal GE potential without screening, like the one that is used in heavy quarkonia. Our analysis has shown that for a weak GE potential (with strong screening) the spectrum of light mesons is described by a RT with different slopes $\beta_l \sim 1.0 \text{ GeV}^2$ and $\beta_n = 1.25(5) \text{ GeV}^2$ of the orbital and the radial RTs. If the strong universal GE potential, as in heavy quarkonia, is taken, then the light meson masses with $l \neq 0$ can be described by a RT with equal slopes within the theoretical error: $\beta_l = 1.07(9) \text{ GeV}^2$ and $\beta_n = 1.15(8) \text{ GeV}^2$, which agrees with the existing experimental data. It means that the flattened CP plus the GE potential without screening (or with very weak screening) can give rise to the universal RT. However, the $n^3S_1$ resonances do not belong this RT, since their masses are strongly affected by the GE interaction and the spin-spin interaction, providing a large slope of the radial $\rho(n^3S_1)$ trajectory, $\beta_n \approx 1.43 \text{ GeV}^2$.

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