Abstract

We present a new calculation of the CP violation parameter $\epsilon'/\epsilon$. The results reported in this paper have been obtained by using the $\Delta S = 1$ effective Hamiltonian computed at the next-to-leading order, including QCD and QED penguins. The matrix elements of the relevant operators have been taken from lattice QCD, at a scale $\mu = 2$ GeV. At this relatively large scale, the perturbative matching between the relevant operators and the corresponding coefficients is quite reliable.

The effect of the next-to-leading corrections is to lower the prediction obtained at the leading order, thus favouring the experimental result of E731. We analyze different contributions to the final result and compare the leading and next-to-leading cases.
In this paper we present a theoretical prediction of $\epsilon'/\epsilon$ obtained from the effective weak hamiltonian $H_\text{eff}^{\Delta S=1}$, up to next-to-leading QCD and QED corrections. The Wilson coefficients of the operators of $H_\text{eff}^{\Delta S=1}$ have been computed using the $(10 \times 10)$ anomalous dimension matrix which governs the mixing of the relevant current-current and penguin operators, renormalized in the $\overline{\text{MS}}$ t’Hooft-Veltman dimensional regularization scheme (HV). The anomalous dimension matrix includes orders $(\alpha_s t)^{n}$, $\alpha_s (\alpha_s t)^{n}$, $\alpha_s t (\alpha_s t)^{n}$ and $\alpha_e (\alpha_s t)^{n}$ (where $t = \ln M_W^2/\mu^2$). The coefficients have been obtained by integrating numerically the renormalization group equations from a scale $\mu \sim M_W \sim m_t$ down to a scale $\mu = 2$ GeV $> m_c$. The coefficients at the initial scale $\mu \sim M_W$ are those computed by Inami and Lim in ref. [1] and by Flynn and Randall [2], as corrected in ref. [3]. The initial effective hamiltonian includes QCD, $Z_0$ and electro-magnetic penguins and box diagrams. We have neglected the running of the coefficients between $m_t$ and $M_W$, but this corresponds to a negligible error in the final result.

The matrix elements of the operators, expressed in terms of the so-called $B$-factors, have been taken, when possible, from lattice calculations. It turns out that the most important $B$-factors have indeed been determined on the lattice, with either Wilson and staggered fermions [4]-[9]. One notable exception is the $B$-factor of the operator $O^{-}$, believed to be responsible for the $\Delta I = 1/2$ enhancement. For this reason we take the non-leptonic $\Delta I = 1/2$ amplitude from experiments. For those matrix elements not yet determined by lattice calculations, we allowed a variation of the $B$-factor in the range 1-6. The next-to-leading relation between the operators in the lattice renormalization scheme and $\overline{\text{MS}}$ naive dimensional regularization (NDR) is known [10]-[15] (the corrections are of the order of 30% in current lattice calculations) and is usually included in the lattice results. We have only computed at one loop the factors which relate the operators renormalized in NDR to the corresponding ones in HV. These corrections give a negligible shift to the values of the $B$-parameters. Given the uncertainties in the evaluation of the $B$-parameters, we have decided to use the same values as in ref. [16], see Table 1. We have chosen the renormalization scale $\mu = 2$ GeV, corresponding to the typical inverse lattice spacing at which actual calculations of weak matrix elements are performed in lattice QCD ($a^{-1} = 2 - 3.5$ GeV).

There are great advantages in using weak matrix elements of operators from lattice QCD. From the theoretical point of view, the matching of the
coefficients with the matrix elements of the operators, renormalized at the scale \( \mu \), is exact at the next-to-leading order in \( \alpha_s \). This is not the case with other methods. For example, in the \( 1/N \) expansion, one has to match the coefficients, computed by renormalizing the operators on quark states, with the matrix elements calculated in the meson theory. However, it is not clear to us how this matching can be implemented at the next-to-leading order.

A second important point is that the scale \( \mu \) can be taken as large as \( 2 - 3 \) GeV, where the perturbative evaluation of the coefficients is expected to be accurate and the final result quite stable for variations of \( \mu \) in the above range, see below. This is to be contrasted with other approaches where values of \( \mu \sim 0.6 - 0.8 \) GeV are chosen. At such low scales the results are not stable against a variation of \( \mu \). Moreover, at low values of \( \mu \), next-to-leading order (NLO) corrections to the coefficients of penguin operators are large, so that one can question the convergence of the perturbative expansion.

The calculation of \( \epsilon'/\epsilon \) has been combined with a next-to-leading order calculation of \( \epsilon \) and of the \( B^0-\bar{B}^0 \) mixing amplitude, following the approach of ref. [16]. For these quantities the next-to-leading perturbative corrections to the Wilson coefficients have been known for quite a while and we do not have much to add to previous analyses. We notice that, with the inclusion of the coefficients and of the anomalous dimensions computed at second order, the analysis of ref. [16] is here consistently done at the next-to-leading accuracy. The strategy is the same as in ref. [16]: from the comparison of the theoretical value of \( \epsilon \) with the experimental number, given the uncertainties on the matrix element of the \( \Delta S = 2 \) operator and on the CKM parameters, we find a range of allowed values of the CP violating phase \( \delta \). Correspondingly we compute \( \epsilon'/\epsilon \), which will also be affected by a theoretical error, see Figs. (1)-(4) below.

The paper is organized as follows. We first give the effective Hamiltonians responsible for \( \Delta S = 2 \) and \( \Delta S = 1 \) transitions, expressed in terms of the relevant operators and their Wilson coefficients. From these Hamiltonians we derive the expressions for \( \epsilon \) and \( \epsilon'/\epsilon \), written as combinations of the Wilson coefficients times the B-parameters of the different operators, i.e. their matrix elements. We do not discuss the determination of the coefficients of the operators relevant for \( \epsilon \), since this point was explained in great detail in ref. [17]. We instead focus on the evaluation of the coefficients of the \( \Delta S = 1 \)
Hamiltonian beyond the leading order including the effects of the electroweak penguins. A comparison of the predictions in the leading and next-to-leading cases is presented. We also study the uncertainties in the final evaluation of $\epsilon'/\epsilon$ coming from the choice of the scale and $\Lambda_{QCD}$ and the dependence on the top quark mass. Finally the theoretical predictions will be confronted with the experimental results coming from NA31 [17] and E731 [18].

We have also considered the $B^0 - \bar{B}^0$ transition parameter $x_d$, following the analysis done in ref. [16], but with a different range of $\Lambda_{nf} = 340 \pm 120$ MeV [19, 20]. It remains true that a large value of $f_B$, for $m_t \geq 140-150$ GeV, favours a positive value for $\cos \delta$. Since the comparison of $x_d$ with the theoretical prediction has no other effect on $\epsilon'/\epsilon$ we will not discuss it any more.

1) $\epsilon$: The effective Hamiltonian governing the $\Delta S=2$ amplitude is given by:

$$H_{\Delta S=2}^{\text{eff}} = \frac{G_F^2}{16\pi^2} M_W^2 (\bar{d}\gamma_\mu s)^2 \left\{ \lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t) \right\}$$

(1)

where $G_F$ is the Fermi coupling constant and $\gamma_\mu^L = \gamma^\mu(1 - \gamma_5)$; $\lambda_q$’s are related to the CKM matrix elements by $\lambda_q = V_{qi}^* V_{qf}$, where ‘$i$’ and ‘$f$’ are the labels of the initial and final states respectively (in the present case $i = s$ and $f = d$). $x_q = m_q^2/M_W^2$ and the functions $F(x_i)$ and $F(x_i, x_j)$ are the so-called Inami-Lim functions [1], obtained from the calculation of the basic box-diagram and including QCD corrections [21]. $F(x_i)$ is known at the next-to-leading order, which has been included in our calculation. From eq. (1) we can derive the CP violation parameter $\epsilon$:

$$|\epsilon|_{\xi = 0} = C_{\epsilon} B_K A^2 \lambda^6 \hat{\sigma} \left\{ F(x_c, x_t) + F(x_t) [A^2 \lambda^4 (1 - \rho)] - F(x_c) \right\}$$

(2)

where

$$C_{\epsilon} = \frac{G_F^2 f_K^2 m_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K}$$

(3)

$\Delta M_K$ is the mass difference between the two neutral kaon mass eigenstates. In eq. (2) $\hat{\sigma} = \sqrt{\rho^2 + \eta^2} \sin \delta$ and $A, \rho, \eta$ and $\delta$ are the parameters of the CKM
matrix in the Wolfenstein parametrization \cite{22}. $B_K$ is the renormalization
group invariant B-factor, defined as:

$$B_K = B_K(\mu) [\alpha_{QCD}(\mu)]^{-6/25} \quad (4)$$

$B_K$ takes into account all the possible deviations from the vacuum insertion
approximation in the evaluation of the $\langle K^0|(d\gamma^\mu_s)^2|K^0 \rangle$ matrix element
($B_K = 1$ corresponding to an exact vacuum insertion approximation).

2) $\epsilon'/\epsilon$: Most of the discussion in this paper is devoted to the Wilson
coefficients of the operators appearing in the effective $\Delta S = 1$ Hamiltonian,
which we have computed at the next-to-leading order, including QCD and
QED corrections.

The $\Delta S = 1$ effective hamiltonian is given by:

$$H_{\text{eff}}^{\Delta S=1} = \sum_i C_i(\mu) Q_i(\mu) \quad (5)$$

The complete basis of operators when QCD and QED corrections are taken
into account is given by:

$$Q_1 = (\bar{s}_\alpha d_\alpha)(V-A)(\bar{u}_\beta u_\beta)(V-A)$$
$$Q_2 = (\bar{s}_\alpha d_\beta)(V-A)(\bar{u}_\beta u_\alpha)(V-A)$$
$$Q_{3,5} = (\bar{s}_\alpha d_\alpha)(V-A) \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)(V\pm A)$$
$$Q_{4,6} = (\bar{s}_\alpha d_\beta)(V-A) \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)(V\mp A)$$
$$Q_{7,9} = \frac{3}{2} (\bar{s}_\alpha d_\alpha)(V-A) \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\beta)(V\pm A)$$
$$Q_{8,10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)(V-A) \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\alpha)(V\mp A)$$
$$Q_1' = (\bar{s}_\alpha d_\alpha)(V-A)(\bar{c}_\beta c_\beta)(V-A)$$
$$Q_2' = (\bar{s}_\alpha d_\beta)(V-A)(\bar{c}_\beta c_\alpha)(V-A) \quad (6)$$

where the subscript $(V \pm A)$ indicates the chiral structure and $\alpha$ and $\beta$ are
colour indices.

The operators $Q_i(\mu)$ are renormalized at the scale $\mu$ in $\overline{MS}$, using the
HV regularization scheme. The corresponding coefficients, $C_i(\mu)$ are scheme
dependent. The dependence on the regularization scheme appears at one loop, when we express the original current-current product in terms of the Wilson OPE:

\[
< F | H_{\text{eff}}^{[\Delta S = 1]} | I > = \frac{g_{W}^{2}}{8} \int d^{4}x D_{W}(x^{2}, M_{W}^{2}) < F | T \left( J_{\mu}(x), J_{\mu}^{\dagger}(0) \right) | I > \\
\rightarrow \sum_{i} C_{i}(M_{W}) < F | Q_{i}(M_{W}) | I >
\]  

(7)

The Wilson coefficients \( \vec{C}(M_{W}) = (C_{1}(M_{W}), C_{2}(M_{W})... \) are found by matching, at \( O(\alpha_{e}) \) and \( O(\alpha_{s}) \) in \( HV \), the current-current and penguin diagrams computed with the \( W \) and top propagators to those computed with the local four-fermion operators in the effective theory.

\( \vec{C}(\mu) \) are expressed in terms of \( \vec{C}(M_{W}) \) through the renormalization evolution matrix \( \hat{W}[\mu, M_{W}] \) :

\[
\vec{C}(\mu) = \hat{W}[\mu, M_{W}] \vec{C}(M_{W})
\]  

(8)

where:

\[
\hat{W}[\mu, M_{W}] = \hat{M}[\mu] \hat{U}[\mu, M_{W}] \hat{M}'[M_{W}]
\]  

(9)

with:

\[
\hat{M}[\mu] = \left( \hat{1} + \frac{\alpha_{e}}{4\pi} \hat{K} \right) \left( \hat{1} + \frac{\alpha_{s}(\mu)}{4\pi} \hat{j} \right) \left( \hat{1} + \frac{\alpha_{e}}{\alpha_{s}(\mu)} \hat{P} \right)
\]  

(10)

and

\[
\hat{M}'[M_{W}] = \left( \hat{1} - \frac{\alpha_{e}}{\alpha_{s}(M_{W})} \hat{P} \right) \left( \hat{1} - \frac{\alpha_{s}(M_{W})}{4\pi} \hat{j} \right) \left( \hat{1} - \frac{\alpha_{e}}{4\pi} \hat{K} \right)
\]  

(11)

At the next-to-leading accuracy \( \hat{W}[\mu, M_{W}] \) is regularization scheme dependent.

Eqs. (9), (10) and (11) require a detailed explanation. At the leading order, the QCD anomalous dimension matrix, including QCD penguins, has been computed in refs. [23, 24]. The electro-weak anomalous dimension matrix at the same order can be found in refs. [3], [25] and [26]. We have computed the anomalous dimension matrix at the next-to-leading order, by

\footnote{We have properly taken into account the beauty threshold in the evolution matrix.}
calculating all the current-current and penguin operators at two loops up to order $\alpha_s^2 t$ and $\alpha_e \alpha_s t$. This corresponds to all the diagrams with four external quark legs, where one of the operators in the list given in eqs. (6) is inserted and two gluons or one gluon and one photon are exchanged. At $O(\alpha_s^2 t)$, the two loop anomalous dimension matrix was computed in refs. [27, 28] for current-current diagrams and in ref. [29] for penguin diagrams. The explicit expression of the anomalous dimension matrix alone would take more space than that allowed for a letter and will be presented in a separate publication [30]. Here we simply explain the meaning of all the terms appearing in eq. (9)-(11). To obtain the expression in eq. (9)-(11) we have neglected the running of the coefficients between the top quark mass and the $W$ mass. We have also expanded the formula at first order in $\alpha_e$ and neglected the running of the electro-magnetic coupling. These approximations are immaterial for the final numerical result.

The matrix $\hat{U}[\mu, M_W]$ in eq. (9) is given by

$$\hat{U}[\mu, M_W] = \left[ \frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right]^{\gamma_s(0)T/2\beta_0}$$

(12)

The matrices $\hat{P}$, $\hat{J}$ and $\hat{K}$ are solutions of the equations:

$$\hat{P} + \left[ \hat{P}, \frac{\hat{\gamma}_s(0)T}{2\beta_0} \right] = \frac{\hat{\gamma}_e(0)T}{2\beta_0}$$

(13)

$$\hat{J} - \left[ \hat{J}, \frac{\hat{\gamma}_s(0)T}{2\beta_0} \right] = \frac{\beta_1}{2\beta_0^2} \hat{\gamma}_s(0)T - \frac{\hat{\gamma}_s(1)T}{2\beta_0}$$

(14)

$$[\hat{K}, \hat{\gamma}_s(0)T] = \hat{\gamma}_e(1)T + \hat{\gamma}_e(0)T \hat{J} + \hat{\gamma}_s(1)T \hat{P} + \left[ \hat{\gamma}_s(0)T, \hat{J} \hat{P} \right] - 2\beta_1 \hat{P} - \frac{\beta_1}{2\beta_0} \hat{P} \hat{\gamma}_s(0)T$$

(15)

The anomalous dimension matrix, which includes gluon and photon corrections has been separated in several pieces which appear in the above equations:

$$\hat{\gamma} = \frac{\alpha_s}{4\pi} \hat{\gamma}_s(0) + \frac{\alpha_e}{4\pi} \hat{\gamma}_e(0) + \left( \frac{\alpha_s}{4\pi} \right)^2 \hat{\gamma}_s(1) + \frac{\alpha_s}{4\pi} \frac{\alpha_e}{4\pi} \hat{\gamma}_e(1)$$

(16)

where each of the $\hat{\gamma}_{s,e}^{(0,1)}$ is a $10 \times 10$ matrix. In eqs. (12-15), $\beta_0$ and $\beta_1$ are the first two coefficients of the $\beta$-function of $\alpha_s$. 

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From $\mathcal{H}_{\text{eff}}^{\Delta S=1}$ we can derive the expression for $\epsilon'$:

$$\epsilon' = e^{i\pi/4} \frac{\omega}{\sqrt{2} \text{Re} A_0} \left[ \omega^{-1} (\text{Im} A_2)' - (1 - \Omega_{IB}) \text{Im} A_0 \right]$$  \hfill (17)

where $(\text{Im} A_2)'$ and $\text{Im} A_0$ are given by:

$$\text{Im} A_0 = -\frac{G_F}{\sqrt{2}} \text{Im}(V_{ts}^* V_{td}) \left\{ - \left( C_6 B_6 + \frac{1}{3} C_5 B_5 \right) Z + \left( C_4 B_4 + \frac{1}{3} C_3 B_3 \right) X + 
C_7 B_7^{1/2} \left( \frac{2Y}{3} + \frac{Z}{6} - \frac{X}{2} \right) + C_8 B_8^{1/2} \left( 2Y + \frac{Z}{2} + \frac{X}{6} \right) - 
C_9 B_9^{1/2} \frac{X}{3} + \left( \frac{C_1 B_1^c}{3} + C_2 B_2^c \right) X \right\}$$  \hfill (18)

and

$$(\text{Im} A_2)' = -G_F \text{Im}(V_{ts}^* V_{td}) \left\{ C_7 B_7^{3/2} \left( \frac{Y}{3} - \frac{X}{2} \right) + 
C_8 B_8^{3/2} \left( Y - \frac{X}{6} \right) + C_9 B_9^{3/2} \frac{2X}{3} \right\}$$  \hfill (19)

$\omega = \text{Re} A_2/\text{Re} A_0 = 0.045$ and we have introduced $(\text{Im} A_2)'$ defined as:

$$\text{Im} A_2 = (\text{Im} A_2)' + \Omega_{IB} (\omega \text{Im} A_0)$$  \hfill (20)

$\Omega_{IB} = +0.25 \pm 0.10$ represents the isospin breaking contribution, see for example ref. [31].

The numerical evaluation of $\epsilon'/\epsilon$ requires the knowledge of the Wilson coefficients of the operators and of the corresponding matrix elements. The Wilson coefficients have been evaluated, using eq. (8), combined with the evolution matrix of eq. (11), and the initial conditions computed in refs. [2, 3] (and given for HV in ref. [29]). The matrix elements of the operators have been written in terms of the three quantities (see eqs. (18) and (19)):

$$X = f_\pi \left( M_K^2 - M_\pi^2 \right)$$  \hfill (21)

$$Y = f_\pi \left( \frac{M_K^2}{m_s(\mu) + m_d(\mu)} \right)^2 \sim 12 X \left( \frac{0.15 \text{GeV}}{m_s(\mu)} \right)^2$$  \hfill (22)

$$Z = 4 \left( \frac{f_K}{f_\pi} - 1 \right) Y$$  \hfill (23)
Table 1: Values of the $B$-parameters. Entries with a (*)& are educated guesses; the others are taken from lattice QCD calculations.

| $B_K$, $B_9^{(3/2)}$ | $B_{1-2}^c$ | $B_{3,4}$ | $B_{5,6}$ | $B_{7-8-9}^{(1/2)}$ | $B_{7-8}^{(3/2)}$ |
|-----------------------|-------------|-----------|-----------|---------------------|---------------------|
| 0.8 ± 0.2             | 0 − 0.15(*) | 1 − 6(*) | 1.0 ± 0.2 | 1(*)               | 1.0 ± 0.2           |

and a set $\{B_i\}$ of B-parameters (in our normalization $f_\pi = 132$ MeV). The numerical value of the B-parameters have been taken from lattice calculations and multiplied by suitable renormalization factors to take into account the difference between HV and the lattice regularization scheme. For those B-factors which have not been computed yet on the lattice we have used an educated guess, which will be discussed below. We observe that in eqs. (13) and (19) only nine coefficients (B-parameters) appear since we have used the relation $Q_{10} = -Q_3 + Q_4 + Q_9$.

We will call “central” results obtained by using the central values of the B-parameters reported in Table 1, $\mu = 2$ GeV and $\Lambda_{QCD} = 340$ MeV (4 flavours). We will study the uncertainty of the theoretical prediction by varying the B-parameters, $m_s$ and the experimental quantities, such as $\Lambda_{QCD}$, the CKM mixing parameters $A$, $\rho$, etc., in the range indicated by the errors on the quantities reported in Tables 1 and 2. In the study of the differences between $\epsilon'/\epsilon$, computed at the leading order (LO) or next-to-leading order (NLO), we will keep fixed the coefficients at the scale $M_W$. This means that, also at the LO, the coefficients include the $O(\alpha_s)$ and $O(\alpha_e)$ corrections which arise from the matching between the effective hamiltonian and the original current-current product. We also keep fixed $\Lambda_{QCD}$ in passing from the LO calculation to the NLO one.

We first consider the relative contribution of different operators to $\epsilon'/\epsilon$. Following the standard notation we write:

$$\epsilon'/\epsilon \sim R \times C_6 B_6 \left( 1 - \sum_i \Omega_i \right)$$  \hspace{1cm} (24)
Figure 1: Band of allowed values for $\epsilon'/\epsilon$ at $m_t = 100$ GeV (next-to-leading order). The dashed lines represent the experimental results of NA31, $(2.3 \pm 0.7) \cdot 10^{-3}$ and E731, $(0.6 \pm 0.7) \cdot 10^{-3}$.  

where the coefficient $R$ includes $\sin \delta$. In eq.(24) $\Omega_2 = \Omega_4, \Omega_8^{3/2}$, etc. indicate the relative corrections due to the matrix elements of the operators, different from $Q_6$, appearing in eqs. (18)-(19). For example, the central value of $R$, averaged over the values of $\cos \delta$ allowed by the analysis of $\epsilon$, is $|R| \sim 1.0 \cdot 10^{-2}$ for $m_t = 140$ GeV.

In the following discussion we fix $\mu = 2$ GeV and $m_t = 140$ GeV and we allow a variation of the B-parameters, computed on the lattice, around their central values. For those matrix elements still to be computed we proceed as follows. We fix $B_{7-9}^{1/2}$ equal to one, since their contribution is of the order of few percent. If we take $B_{3,4} = 1$ and $B_{1,2}^c = 0$, at the LO the largest contributions to $\epsilon'/\epsilon$ come from $Q_6$, $Q_8$ and $Q_9$, $\Omega_8^{3/2} \sim 28\%$ and $\Omega_9^{3/2} \sim -18\%$. $C_9$ is much larger than $C_8$ and compensates for the matrix element of $Q_9$ which is much smaller than the matrix element of $Q_8$. With $B_{3,4} = 1$, $\Omega_3 \sim -0.01$ and $\Omega_4 \sim 0.04$. The operators $Q_{3,4}$ have a chiral structure similar to $Q_2$ and, at scales larger than $m_c$, give rise to the same “eye” diagrams, like $Q_2$ does. In order to explain the experimental $\Delta I = 1/2$ enhancement, $B_2$ must be of the order 5-6. For this reason in ref. [16] and
Figure 2: Same as in Fig.() at $m_t = 140$ GeV.

in the present calculation we have allowed a variation of $B_{3,4}$ between 1 and 6. The central values in this case are $\Omega_3 \sim -0.04$ and $\Omega_4 \sim 0.28$. Thus, if $Q_4$ has a large B-factor, its contribution is as important as the contribution of $Q_8$ or $Q_9$. $B_{1,2}^c$ are zero in the vacuum saturation approximation. Their coefficients however are so large that even a small B-factor can give a sizeable contribution to $\epsilon'/\epsilon$. By varying $B_{1,2}^c$ between 0-0.15 we find that $\Omega_2^c \sim -0.18$ ($\Omega_1^c \sim 0.02$), comparable to other large terms.

In the following we will focus on the main contributions to $\epsilon'/\epsilon$, which are due to few operators, $Q_{4,6,7,8,9}$ and $Q_5^c$. All the other terms are of the order of a few %, with alternating signs and we will not discuss them any more.

We now come to the effects of the next-to-leading corrections. If we only include next-to-leading corrections due to two gluon exchanges (corresponding to $\hat{\gamma}_s^{(1)}$), all the coefficients are slightly changed, without modifying the pattern of the different contributions observed at the leading order. For example $\Omega_4$ goes from 0.28 to 0.34, $\Omega_5$ from 0.06 to 0.08 and $\Omega_8^{3/2}$ from 0.28 to 0.26. $\Omega_9^{3/2}$ increases from $-0.18$ to $-0.22$. The gluon-photon corrections on $\Omega_i$ are always very tiny and, given the uncertainties on the B-factors, practically invisible. The only exceptions are $\Omega_{7,8}^{3/2}$: $\Omega_7^{3/2}$ is very small at the leading
order, $-0.01$, becomes larger including two-gluon exchanges, $-0.05$ and becomes $-0.09$ with the complete next-to-leading corrections; $\Omega_{8}^{3/2}$ is 0.28 at the LO and increases to 0.51 with the complete NLO corrections. We notice that the sum of the contributions of $Q_{7,8}$ and $Q_{9}$ almost cancel at the leading, as well as at the next to leading order. $\Omega_{7}^{3/2} + \Omega_{8}^{3/2} + \Omega_{9}^{3/2} \sim 0.10$ at the leading order and 0.20 at the next-to-leading one. Similarly $\Omega_{4} + \Omega_{2}^{c} \sim +10\%$ both at the LO and at NLO. We thus find that at $m_{t} = 140$ GeV and for the “central” values of the B-parameters, the result is essentially the same as the original Gilman-Wise prediction, since the most important electro-penguin corrections almost cancel and their sum is of the order of a few per cent.\footnote{The cancellation between different contributions may not occur for values of the B-parameters different from their central ones. This will correspond to a band of uncertainty in the theoretical prediction for $\epsilon'/\epsilon$, which is reported in the Figs. (1)-(4).} Thus $\epsilon'/\epsilon$ is essentially determined by $C_{6}$ and the corresponding B-parameter, $B_{6}$. $C_{6}$ varies from $-6.2 \times 10^{-2}$ to $-5.1 \times 10^{-2}$, at fixed $\Lambda_{QCD} = 340$ MeV, so that the average value of $\epsilon'/\epsilon$ decreases from $6.5 \times 10^{-4}$ (LO) to $4.5 \times 10^{-4}$ (NLO). The decrease of $C_{6}$ is mainly due to the variation of the value of $\alpha_{s}$ between LO and NLO, if we insist in using the same value of $\Lambda_{QCD}$. The relative uncertainty on $C_{6}$, by using different values of $\Lambda_{QCD}$, remains roughly
the same between the LO, $\delta C_6/C_6 \sim 18\%$ and NLO, $\delta C_6/C_6 \sim 17\%$.

As observed already in ref. [2], and confirmed by all other analyses [3, 16], for increasing values of the top mass, the contribution of the electropenguin operators tends to cancel the contribution coming from $Q_6$. We have already observed that at $m_t = 140$ GeV the main corrections come from $\Omega_{7,8,9}^{3/2}, \Omega_4$ and $\Omega_5^6$. $\Omega_4$ and $\Omega_5^6$ vary very little with $m_t$ and their values remain essentially the same in going from LO to NLO. We know very little about their matrix elements from non-perturbative calculations. We estimate that the uncertainty on the final result, coming from the poor knowledge of the matrix elements of $Q_4$ and $Q_2$ is of the order of 30%. On the other hand $\Omega_{7,8,9}^{3/2}$ vary with $m_t$ and change considerably from LO to NLO. For example, at $m_t = 140$ GeV $\Omega_8^{3/2}$ is $\sim 0.28$ at LO and $\sim 0.50$ at NLO, at $m_t = 200$ GeV it becomes 0.68 at LO and 0.90 at NLO. We also observe large variations for $\Omega_7^{3/2}$ and $\Omega_9^{3/2}$. The sum $\Omega_7^{3/2} + \Omega_8^{3/2} + \Omega_9^{3/2}$ is $\sim 20\%$ at $m_t = 140$ GeV and $\sim 65\%$ at $m_t = 200$ GeV. The same can be said also at low values of the top mass, for example $m_t = 100$ GeV. In summary, in the range of $m_t$

\footnote{The increase of the relative contribution of $\Omega_{7-9}^{3/2}$ is due to the decrease of $C_6$ combined with an increase of $C_{7-9}$ between the leading and next-to-leading cases.}
considered in this work, even though single contributions from the electropenguin operators may change by $\sim 40\%$ because of the next-to-leading corrections, they give globally more or less the same relative contribution at NLO as at the leading order. Since for fixed $\Lambda_{QCD}$, $C_6$ decreases, the net effect is the the central value of the theoretical prediction for $\epsilon'/\epsilon$ is smaller at the NLO. In Figs. (1)-(3), we report our results at NLO for $\epsilon'/\epsilon$, at three different values of $m_t$, 100, 140 and 200 GeV. The predictions are given for different values of $\cos \delta$, which are compatible with the analysis of $\epsilon$ [16]. They have been obtained by comparing the theoretical expressions for $\epsilon$, eqs. (2), to the corresponding experimental value. The dashed lines indicate the results by the NA31 and E731 experiments at the 1-$\sigma$ level. The numbers and errors reported in the figures are the average and the theoretical variance (for $\cos \delta$ positive or negative) computed by varying the B-parameters and the other quantities in the ranges reported in Table 1 and 2. Since the value of $\epsilon'/\epsilon$ is lowered at the next-to-leading order, our theoretical prediction, at $m_t = 140$ GeV, is now centered on the experimental result of E731 ($\cos \delta \geq 0$)\footnote{We recall that the most recent lattice results [32]-[34] and QCD sum rules calculations [35]-[36] suggest a large value of $f_B$ which corresponds to $\cos \delta \geq 0$ when $m_t \geq 140$-150 GeV [14].}. For a comparison between the LO and the NLO, we report in Fig. (4) the LO result, at $m_t = 100$ GeV. The LO result in this case sits in the middle between

| parameter     | value                      |
|---------------|----------------------------|
| $\Lambda_{QCD}$ | 340 $\pm$ 120 GeV         |
| $m_s$(2 GeV)   | (170 $\pm$ 30) MeV        |
| $m_c$(2 GeV)   | 1.5 GeV                   |
| $m_b$(2 GeV)   | 4.5 GeV                   |
| $\sqrt{\rho^2 + \eta^2} = V_{ub}/(\lambda V_{cb})$ | 0.50 $\pm$ 0.14          |
| $\epsilon_{exp}$ | 2.28 $\cdot$ 10$^{-3}$   |
| $\text{Re}A_0$ | 2.7 $\cdot$ 10$^{-7}$ GeV |

Table 2: Values of experimental parameters used in this work.
Figure 5: \( C_6 \) and \( C_8 \) as a function of \( \mu \) for \( \Lambda_{QCD} = 220 \) (dotted), 340 (solid) and 460 (dashed) MeV.

the measurements of NA31 and E731\(^5\). We also observe that the band of error is slightly reduced by the inclusion of the next-to-leading corrections. Indeed the relative error on \( \epsilon'/\epsilon \) at LO and NLO is basically the same. Since the central value of \( \epsilon'/\epsilon \) is decreased at NLO, the band in Fig. (4) appears narrower than in Fig. (4).

Before concluding this paper we want to discuss the stability of the Wilson coefficients, i.e. how much their values are affected by the presence of NLO corrections and their sensitivity with respect to a variation of \( \mu \) and \( \Lambda_{QCD} \). We observe that for \( \mu \leq 1 \) GeV the values of the coefficients start to vary wildly if one changes \( \Lambda_{QCD} \) or \( \mu \). As an example, in Fig. (6), we report the variation of \( C_6(\mu) \) and \( C_8(\mu) \) as a function of \( \mu \), at three different values of \( \Lambda_{QCD} \) and \( m_t = 140 \) GeV. The coefficients change by a factor of 2 or more, for \( \mu \leq 1 \) GeV, if we vary \( \Lambda_{QCD} \) from 220 to 460 MeV. Moreover, at small values of \( \mu \), the difference between the leading and next-to-leading results is very large, and we cannot trust the perturbative expansion. To illustrate this point, we report in Fig. (6) \( C_8(\mu) \), for 0.8 GeV \( \leq \mu \leq 4.0 \) GeV at the leading and next-to-leading order. On the basis of the above discussion we

\(^5\)The corresponding figure in ref. \[16\] was obtained with different values of \( \Lambda_{QCD} \).
believe that a realistic prediction for $\epsilon'/\epsilon$ can only be obtained by matching the Wilson coefficients and the operators matrix elements at scales larger than $m_c$, as one can do by taking the matrix elements from lattice QCD.

The conclusions of this work are the following. It is reassuring that by taking a renormalization scale $\mu$ of the order of 2 GeV, the Wilson coefficients of the effective Hamiltonian do not vary by more than $\sim 35\%$ and that the final result on $\epsilon'/\epsilon$ is quite stable in going from the leading to the next-to-leading order. At NLO one gives a precise meaning to the value of $\Lambda_{QCD}$ to be used. It is now consistent to take $\Lambda_{QCD}$ from the measurements done in deep inelastic scattering or at LEP. The NLO calculation of the anomalous dimension matrix makes also consistent the evolution of the Wilson coefficients with their matching at $\mu \sim m_t, M_W$. Indeed the matching procedure is a next-to-leading order effect. Finally, in the operator product expansion, only the NLO calculation fixes unambiguously the scale at which the operators must be renormalized. In the lattice case the scale is dictated, up to higher order effects, by the inverse lattice spacing at which the numerical simulations are performed. The major source of theoretical uncertainty remains now the evaluation of the matrix elements, which are determined with large errors or are still to be determined, like it is the case for $O^-$. 

Figure 6: $C_8$ as a function of $\mu$ for $\Lambda_{QCD} = 340$ MeV at the LO and NLO.
From the phenomenological point of view, at fixed $\Lambda_{QCD}$ and $m_t$, the next-to-leading corrections lower the theoretical predictions, thus favouring the experimental result by E731.

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