On the Brezis-Nirenberg type critical problem for nonlinear Choquard equation

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Abstract

We establish some existence results for the Brezis-Nirenberg type problem of the nonlinear Choquard equation

\[-Δu = \left( \int_{Ω} \frac{|u|^{2^*_µ}}{|x-y|^µ} dy \right) |u|^{2^*_µ-2}u + λu \quad \text{in } Ω,\]

where Ω is a bounded domain of \(\mathbb{R}^N\) with Lipschitz boundary, λ is a real parameter, \(N ≥ 3\), \(2^*_µ = (2N - µ)/(N - 2)\) is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

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1 Introduction and main results

In the last decades many people studied the elliptic equation

\[
\begin{cases}
-Δu = |u|^{2^*-2}u + λu & \text{in } Ω, \\
u = 0 & \text{on } ∂Ω,
\end{cases}
\]

where Ω is a bounded domain of \(\mathbb{R}^N\), \(2^* = \frac{2N}{N-2}\) is the critical exponent for the embedding of \(H^1_0(Ω)\) to \(L^p(Ω)\), \(λ ∈ (0, λ_1)\), then problem (1.1) has a nontrivial solution; if \(N = 3\) then there exists a constant \(λ_* ∈ (0, λ_1)\) such that for any \(λ ∈ (λ_*, λ_1)\) problem (1.1) has a positive solution and if Ω is a ball, problem (1.1) has a positive solution if and only if \(λ ∈ (\frac{2N}{N-2}, λ_1)\). Capozzi, Fortunato and Palmieri [11] proved if \(N ≥ 4\) then the problem (1.1) has a nontrivial solution for all \(λ > 0\). In [13], Cerami, Solimini and Struwe proved if \(N ≥ 6\) and \(λ ∈ (0, λ_1)\), the existence of sign-changing solutions; if Ω is a ball, \(N ≥ 7\) and \(λ ∈ (0, λ_1)\), infinitely many radial solutions to problem (1.1). There is a great deal of work on elliptic equations with critical nonlinearity, see for example [10, 12, 16, 19, 20, 31, 33, 37] and the references therein.

In the present paper we are going to consider the existence and nonexistence of solutions for the following nonlocal equation:

\[
\begin{cases}
-Δu = \left( \int_{Ω} \frac{|u|^{2^*_µ}}{|x-y|^µ} dy \right) |u|^{2^*_µ-2}u + λu & \text{in } Ω, \\
u ∈ H^1_0(Ω),
\end{cases}
\]

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where $\Omega$ is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, $\lambda$ is a real parameter, $N \geq 3$, $0 < \mu < N$ and $2^*_\mu = (2N - \mu)/(N - 2)$. This nonlocal elliptic equation is closely related to the nonlinear Choquard equation

$$- \Delta u + V(x)u = \left(\frac{1}{|x|^{\mu}} + |u|^{p}\right)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3. \quad (1.3)$$

Different from the fractional Laplacian where the pseudo-differential operator causes the nonlocal phenomena, for the Choquard equation the nonlocal term appears in the nonlinearity and influences the equation greatly. For $p = 2$ and $\mu = 1$, it goes back to the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [29] and the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [21]. In some particular cases, this equation is also known as the Schrödinger-Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse of a quantum mechanical wave function [30].

The existence and qualitative properties of solutions of (1.3) have been widely studied in the last decades. In [21], Lieb proved the existence and uniqueness, up to translations, of the ground state. Later, in [23], Lions showed the existence of a sequence of radially symmetric solutions. In [15, 24, 25] the authors showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. Moreover, Moroz and Van Schaftingen in [26] considered the existence of ground states under the assumptions of Berestycki-Lions type. For periodic potential $V$ that changes sign and 0 lies in the gap of the spectrum of the Schrödinger operator $-\Delta + V$, the problem is strongly indefinite, and the existence of solution for $p = 2$ was considered in [17] by reduction arguments. In [3] Alves, Nóbrega and the second author studied the existence of multi-bump shaped solution for the nonlinear Choquard equation with deepening potential well. For a general case, Ackermann [1] proposed a new approach to prove the existence of infinitely many geometrically distinct weak solutions. For other related results, we refer the readers to [14, 17] for the existence of sign-changing solutions, [4, 5, 27, 32, 36, 39] for the existence of infinitely many geometrically distinct weak solutions. For other related results, we refer the readers to [14, 17] for the existence of sign-changing solutions, [4, 5, 27, 32, 36, 39] for the existence of infinitely many geometrically distinct weak solutions.

The starting point of the variational approach to the problem (1.3) is the following well-known Hardy-Littlewood-Sobolev inequality.

**Proposition 1.1.** (Hardy-Littlewood-Sobolev inequality). (See [22].) Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of $f, h$, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\mu}dxdy \leq C(t, N, \mu, r)|f|_t|h|_r. \quad (1.4)$$

If $t = r = 2N/(2N - \mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{N}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{\frac{\Gamma(N)}{\Gamma(N - \frac{\mu}{2})}\right\}^{-1 + \frac{\mu}{2}}.$$

In this case there is equality in (1.4) if and only if $f \equiv (\text{const.})h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Notice that, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x - y|^{\mu}}dxdy$$
is well defined if $|u|^q \in L'(\mathbb{R}^N)$ for some $t > 1$ satisfying
\[
\frac{2}{t} + \frac{\mu}{N} = 2.
\]
Thus, for $u \in H^1(\mathbb{R}^N)$, by Sobolev embedding Theorems, we know
\[
2 \leq tq \leq \frac{2N}{N-2},
\]
that is
\[
\frac{2N - \mu}{N} \leq q \leq \frac{2N - \mu}{N-2}.
\]
Thus, $\frac{2N-\mu}{N}$ is called the lower critical exponent and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

We need to point out that all the papers we mentioned above considered the nonlinear Choquard equation with superlinear subcritical nonlinearities. In a recent paper [28] by Moroz and Van Schaftingen, the authors considered the nonlinear Choquard equation (1.3) in $\mathbb{R}^N$ with lower critical exponent $\frac{2N-\mu}{N}$. There the authors investigated the existence and nonexistence of solutions to the equation with nonconstant potential by minimizing arguments. However, as far as we know there seems no result for the nonlinear Choquard equation with upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality. In [2], the authors studied the existence and concentrations of the solutions of a nonlocal Schrödinger with the critical exponential growth in $\mathbb{R}^2$, that problem is closely related to the Choquard equation. Recently many people also studied the Brezis-Nirenberg problem for elliptic equation driven by the fractional Laplacian, this type of problem are nonlocal in nature and we may refer the readers to [6, 34, 35] and the references therein for a recent progress. And so, it is quite natural to ask if the well-known results established by Brezis and Nirenberg in [9] for local elliptic equation still hold for the nonlocal Choquard equation. The main purpose of the present paper is to study the nonlinear Choquard equation with upper critical exponent $2^*_\mu = \frac{2N-\mu}{N-2}$ and give a confirm answer to the question of the existence and nonexistence of solutions.

From the Hardy-Littlewood-Sobolev inequality, for all $u \in D^{1,2}(\mathbb{R}^N)$ we know
\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dx \, dy \right)^{\frac{N-2}{2N}} \leq C(N, \mu)^{\frac{N-2}{2N}} |u|^2_{L^{2^*_\mu}(\mathbb{R}^N)},
\]
where $C(N, \mu)$ is defined as in the Proposition 1.1. We use $S_{H,L}$ to denote best constant defined by
\[
S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dx \, dy \right)^{\frac{N-2}{2N}}}, \tag{1.5}
\]

From commentaries above, we can easily draw the following conclusion.

\begin{lemma}
The constant $S_{H,L}$ defined in (1.5) is achieved if and only if
\[
u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}},
\]
where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. What’s more,
\[
S_{H,L} = \frac{S}{C(N, \mu)^{\frac{N-2}{2N}}},
\]
where $S$ is the best Sobolev constant.
\end{lemma}
Proof. By the Hardy-Littlewood-Sobolev inequality, we can see

\[ S_{H,L} \geq \frac{1}{C(N,\mu)^{\frac{N-2}{2}}} \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N} \frac{\nabla u^2}{|u|^2} dx = \frac{S}{C(N,\mu)^{\frac{N-2}{2}}}, \]

where \( S \) is the best Sobolev constant. Notice that the equality in the Hardy-Littlewood-Sobolev inequality holds if and only if \( u = C \left( \frac{b}{|x+b|^2} \right)^{\frac{N-2}{2}} \), where \( C > 0 \) is a fixed constant, \( a \in \mathbb{R}^N \) and \( b \in (0,\infty) \) are parameters. Meanwhile, it is well-known that the function \( u = C \left( \frac{b}{|x+b|^2} \right)^{\frac{N-2}{2}} \) is also a minimizer for \( S \), thus we get that \( S_{H,L} \) is achieved if and only if \( u = C \left( \frac{b}{|x+b|^2} \right)^{\frac{N-2}{2}} \) and

\[ S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2}}}. \]

In particular, let \( U(x) := \frac{N(N-2)}{(1+|x|)^{\frac{N-2}{2}}} \) be a minimizer for \( S \), then

\[ \tilde{U}(x) = S \left( \frac{(N-1)(N-2)}{N(N-2)-\mu} \right) C(N,\mu)^{\frac{2-N}{2}} U(x) \]

\[ = S \left( \frac{(N-1)(N-2)}{N(N-2)-\mu} \right) C(N,\mu)^{\frac{2-N}{2}} \left( \frac{N(N-2)}{2} \right) \frac{N-2}{4} \frac{1}{1+|x|^2} \]

is the unique minimizer for \( S_{H,L} \) and satisfies

\[-\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*}}{|x-y|^\mu} dy \right) |u|^{2^*-2} u \text{ in } \mathbb{R}^N. \]

Moreover,

\[ \int_{\mathbb{R}^N} |\nabla U|^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\tilde{U}(x)|^{2^*} |\tilde{U}(y)|^{2^*})}{|x-y|^\mu} dx dy = S_{H,L}^{\frac{2-N+\mu}{4}}. \]

We have some more words about the best constant \( S_{H,L} \).

**Lemma 1.3.** Let \( N \geq 3 \). For every open subset \( \Omega \) of \( \mathbb{R}^N \),

\[ S_{H,L}(\Omega) := \inf_{u \in D_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} dxdy \right)^{\frac{N-2}{N-2+\mu}}} = S_{H,L}, \]

\( S_{H,L}(\Omega) \) is never achieved except when \( \Omega = \mathbb{R}^N \).

**Proof.** It is clear that \( S_{H,L} \leq S_{H,L}(\Omega) \) by \( D_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^N) \). Let \( \{u_n\} \subset C_0^\infty(\mathbb{R}^N) \) be a minimizing sequence for \( S_{H,L} \). We make translations and dilations for \( \{u_n\} \) by choosing \( y_n \in \mathbb{R}^N \) and \( \tau_n > 0 \) such that

\[ u_n^{y_n,\tau_n}(x) := \tau_n^{\frac{N-2}{2}} u_n(\tau_n x + y_n) \in C_0^\infty(\Omega), \]

which satisfies

\[ \int_{\mathbb{R}^N} |\nabla u_n^{y_n,\tau_n}|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \]

and

\[ \int_{\Omega} \int_{\Omega} \frac{|u_n^{y_n,\tau_n}(x)\tau_n^{\frac{N-2}{2}}|^{2^*} |u_n^{y_n,\tau_n}(y)\tau_n^{\frac{N-2}{2}}|^{2^*}}{|x-y|^\mu} dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^\mu} dxdy. \]
Hence we obtain $S_{H,L}(\Omega) \leq S_{H,L}$. $S_{H,L}(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$ is due to the fact that $\hat{U}(x)$ is the only class of functions such that the equality holds in the Hardy-Littlewood-Sobolev inequality and attains the best constant. 

Next we will denote the sequence of eigenvalues of the operator $-\Delta$ on $\Omega$ with homogeneous Dirichlet boundary data by

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots$$

and

$$\lambda_j \to +\infty$$

as $j \to +\infty$. Moreover, $\{e_j\}_{j \in \mathbb{N}} \subset L^\infty(\Omega)$ will be the sequence of eigenfunctions corresponding to $\lambda_j$. We recall that this sequence is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1_0(\Omega)$. We denote

$$E_{j+1} := \{u \in H^1_0(\Omega) : \langle u, e_i \rangle_{H^1_0} = 0, \forall i = 1, 2, \ldots, j\},$$

while $\mathcal{Y}_j := \text{span}\{e_1, \ldots, e_j\}$ will denote the linear subspace generated by the first $j$ eigenfunctions of $-\Delta$ for any $j \in \mathbb{N}$. It is easily seen that $\mathcal{Y}_j$ is finite dimensional and $\mathcal{Y}_j \oplus E_{j+1} = H^1_0(\Omega)$.

In order to study the problem by variational methods, we introduce the energy functional associated to equation (1.2) by

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{22^{n}\mu} \int_\Omega \int_\Omega \frac{|u(x)|^{2^n}|u(y)|^{2^n}}{|x-y|^{\mu}} dx dy - \frac{\lambda}{2} \int_\Omega |u|^2 dx.$$

Then the Hardy-Littlewood-Sobolev inequality implies $J_\lambda$ belongs to $C^1(H^1_0(\Omega), \mathbb{R})$ with

$$\langle J'_\lambda(u), \phi \rangle = \int_\Omega \nabla u \nabla \phi dx - \int_\Omega \int_\Omega \frac{|u(x)|^{2^n}|u(y)|^{2^n-2u(y)\phi(y)}}{|x-y|^{\mu}} dx dy - \lambda \int_\Omega u \phi dx \quad (1.8)$$

for all $\phi \in C^\infty(\Omega)$. And so $u$ is a weak solution of (1.2) if and only if $u$ is a critical point of functional $J_\lambda$.

The main results of this paper are stated in the following two theorems.

**Theorem 1.4.** Assume $\Omega$ is a bounded domain of $\mathbb{R}^N$, with Lipschitz boundary and $0 < \mu < N$, the following result holds true:

(i) If $N \geq 4$, then problem (1.2) has a nontrivial solution for $\lambda > 0$, provided $\lambda$ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.

(ii) If $N = 3$, there exist $\lambda_*$ such that problem (1.2) has a nontrivial solution for $\lambda > \lambda_*$, provided $\lambda$ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.

**Theorem 1.5.** If $N \geq 3$, $\lambda < 0$ and $\Omega \neq \mathbb{R}^N$ is a smooth (possibly unbounded) domain in $\mathbb{R}^N$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^N$, then any solution $u \in H^1_0(\Omega)$ of problem (1.2) is trivial.

Throughout this paper we denote the norm $||u|| := (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$ on $H^1(\Omega)$ and write $|\cdot|_q$ for the $L^q(\Omega)$-norm for $q \in [1, \infty]$ and always assume $\Omega$ is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, $\lambda$ is a real parameter. We denote positive constants by $C, C_1, C_2, C_3 \cdots$.

**Definition 1.6.** Let $I$ be a $C^1$ functional defined on Banach space $X$, we say that $\{u_n\}$ is a Palais-Smale sequence of $I$ at $c$ ($\langle PS \rangle_c$ sequence, for short) if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to +\infty. \quad (1.9)$$

And we say that $I$ satisfies the Palais-Smale condition at the level $c$, if every Palais-Smale sequence at $c$ has a convergent subsequence.
An outline of the paper is as follows: In Section 2, we give some preliminary results and prove \((PS)\) condition. In Section 3, we prove the existence of solutions for \((1.2)\) when \(N \geq 4\) and \(0 < \lambda < \lambda_1\) by the Mountain pass theorem. In Section 4, we prove the existence of solutions for \((1.2)\) when \(N \geq 4\) and \(\lambda > \lambda_1\), provided \(\lambda\) is not an eigenvalue of \(-\Delta\) with homogeneous Dirichlet boundary data, by the Linking Theorem. In Section 5, we investigate the existence of solutions for \(\lambda > 0\) when \(N = 3\). In Section 6, we prove a Pohožaev identity for \((1.2)\) and use it to prove the nonexistence of solutions.

2 Preliminary results

To prove the \((PS)\) condition, we need a key lemma which is inspired by the Brézis-Lieb convergence lemma (see \([8]\)). The proof is analogous to that of Lemma 3.5 in \([1]\) or Lemma 2.4 in \([25]\), but we exhibit it here for completeness. First, we recall that pointwise convergence of a bounded sequence implies weak convergence (see \([38], \text{Proposition 5.4.7}\)).

**Lemma 2.1.** Let \(N \geq 3\), \(q \in (1, +\infty)\) and \(\{u_n\}\) is a bounded sequence in \(L^q(\mathbb{R}^N)\). If \(u_n \to u\) almost everywhere in \(\mathbb{R}^N\) as \(n \to \infty\), then \(u_n \rightharpoonup u\) weakly in \(L^q(\mathbb{R}^N)\).

**BLN Lemma 2.2.** Let \(N \geq 3\) and \(0 < \mu < N\). If \(\{u_n\}\) is a bounded sequence in \(L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)\) such that \(u_n \to u\) almost everywhere in \(\mathbb{R}^N\) as \(n \to \infty\), then the following hold,

\[
\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^{2^*_\mu})|u_n - u|^{2^*_\mu} dx \to \int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2^*_\mu})|u|^{2^*_\mu} dx
\]

as \(n \to \infty\).

**Proof.** First, similarly to the proof of the Brézis-Lieb Lemma \([8]\), we know that

\[
|u_n - u|^{2^*_\mu} \to |u|^{2^*_\mu}
\]  

in \(L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)\) as \(n \to \infty\). The Hardy-Littlewood-Sobolev inequality implies that

\[
|x|^{-\mu} * (|u_n - u|^{2^*_\mu} - |u_n|^{2^*_\mu}) \to |x|^{-\mu} * |u|^{2^*_\mu}
\]  

in \(L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)\) as \(n \to \infty\). On the other hand, we notice that

\[
\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^{2^*_\mu})|u_n - u|^{2^*_\mu} dx
\]

\[
= \int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}))(|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}) dx
\]

\[
+ 2 \int_{\mathbb{R}^N} (|x|^{-\mu} * (|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu}))|u_n - u|^{2^*_\mu} dx.
\]

By Lemma 2.2, we have that

\[
|u_n - u|^{2^*_\mu} \to 0
\]  

in \(L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)\) as \(n \to \infty\). From (2.2)-(2.5), we know that the result holds.

**EN Lemma 2.3.** Assume \(N \geq 3\) and \(0 < \mu < N\). Then

\[
\| \cdot \|_{NL} := \left( \int_{\Omega} \int_{\Omega} \frac{| \cdot |^{2^*_\mu} | \cdot |^{2^*_\mu}}{|x-y|^{\mu}} dxdy \right)^{\frac{1}{2^*_\mu}}
\]

defines a norm on \(L^{2^*_\mu}(\Omega_1)\).
Lemma 2.4. Let \( u \) be a weak solution of problem (1.2) such that \( u \in L^{2^*}(\Omega) \). Then, by the Minkowski inequality again, we have

\[
\left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{\frac{N\mu}{2}}} dy \right)^2 \leq \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{\frac{N\mu}{2}}} dy \right)^2 + \left( \int_{\Omega} \frac{|v(y)|^{2^*_\mu}}{|x-y|^{\frac{N\mu}{2}}} dy \right)^2 \cdot 22^*_\mu,
\]

for every \( u, v \in L^{2^*}(\Omega) \). Therefore, it is easy to verify that \( \cdot \| \cdot \|_{NL} \) is a norm on \( L^{2^*}(\Omega) \). \( \square \)

Lemma 2.4. Let \( N \geq 3, 0 < \mu < N \) and \( \lambda > 0 \). If \( \{u_n\} \) is a \((PS)_c\) sequence of \( J_\lambda \), then \( \{u_n\} \) is bounded. Let \( u_0 \in H^1_0(\Omega) \) be the weak limit of \( \{u_n\} \), then \( u_0 \) is a weak solution of problem (1.2).

Proof. It is easy to see \( c \geq 0 \) and there exists \( C_1 > 0 \) such that

\[
|J_\lambda(u_n)| \leq C_1, \quad |(J'_\lambda(u_n), \frac{u_n}{\|u_n\|})| \leq C_1.
\]

Let \( \beta \in (\frac{1}{2^*_\mu}, \frac{1}{2}) \). For \( n \) large enough, we have

\[
C_1(1 + \|u_n\|) \geq J_\lambda(u_n) - \beta(\beta')_\lambda(u_n) \geq \left(\frac{1}{2} - \beta(\|u_n\|^2 - \lambda|u_n|^{2^*_\mu}) + (\beta - \frac{1}{2^*_\mu}) \int \int_{\Omega} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^{N\mu}} dxdy \right. \geq \left. \left(\frac{1}{2} - \beta(\delta z_n^2 + (\lambda_1 - \lambda)|y_n|^{2^*_\mu}) + (\beta - \frac{1}{2^*_\mu}) \int \int_{\Omega} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^{N\mu}} dxdy \right),
\]

where \( u_n = z_n + y_n, z_n \in \mathbb{E}_{j+1}, y_n \in \mathbb{Y}_j \). It is then easy to verify that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \) using the fact that that \( \mathbb{Y}_j \) is finite dimensional and Lemma 2.3.

Since \( H^1_0(\Omega) \) is reflexive, up to a subsequence, still denoted by \( u_n \), there exists \( u_0 \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u_0 \) in \( H^1_0(\Omega) \) and \( u_n \rightarrow u_0 \) in \( L^{2^*}(\Omega) \) as \( n \rightarrow +\infty \). Then

\[
|u_n|^{2^*_\mu} \rightharpoonup |u_0|^{2^*_\mu} \quad \text{in} \quad L^{\frac{2N}{N+\mu}}(\Omega)
\]
as \( n \rightarrow +\infty \). By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N+\mu}}(\Omega) \) to \( L^\frac{2N}{N+\mu}(\Omega) \), we know that

\[
|x|^{-\mu} * |u_n|^{2^*_\mu} \rightharpoonup |x|^{-\mu} * |u_0|^{2^*_\mu} \quad \text{in} \quad L^{\frac{2N}{N+\mu}}(\Omega)
\]
as \( n \to +\infty \). Combining with the fact that
\[
|u_n|^{2^* - 2}u_n \to |u_0|^{2^* - 2}u_0 \quad \text{in} \quad L^{\frac{2N}{2^*-2}}(\Omega)
\]
as \( n \to +\infty \), we have
\[
(|x|^{-\mu} * |u_n|^{2^*}) |u_n|^{2^* - 2}u_n \to (|x|^{-\mu} * |u_0|^{2^*}) |u_0|^{2^* - 2}u_0 \quad \text{in} \quad L^{\frac{2N}{2^*-2}}(\Omega)
\]
as \( n \to +\infty \). Since, for any \( \varphi \in H^1_0(\Omega) \),
\[
0 \leftarrow \langle J'_\lambda(u_n), \varphi \rangle = \int_\Omega \nabla u_n \nabla \varphi dx - \lambda \int_\Omega u_n \varphi dx - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^* - 2}u_n(y) \varphi(y)}{|x-y|^{\mu}} dxdy.
\]
Passing to the limit as \( n \to +\infty \) we obtain
\[
\int_\Omega \nabla u_0 \nabla \varphi dx - \lambda \int_\Omega u_0 \varphi dx - \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^* - 2}u_0(y) \varphi(y)}{|x-y|^{\mu}} dxdy = 0
\]
for any \( \varphi \in H^1_0(\Omega) \), which means \( u_0 \) is a weak solution of problem [1,2].

Finally, taking \( \varphi = u_0 \in H^1_0(\Omega) \) as a test function in [1,2], we have
\[
\int_\Omega |\nabla u_0|^2 dx = \lambda \int_\Omega u_0^2 dx + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^* - 2}u_0(y)}{|x-y|^{\mu}} dxdy,
\]
and so
\[
J_\lambda(u_0) = \frac{N + 2 - \mu}{4N - 2\mu} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^*}}{|x-y|^{\mu}} dxdy \geq 0.
\]
\[
\square
\]

**Lemma 2.5.** Let \( N \geq 3, 0 < \mu < N \) and \( \lambda > 0 \). If \( \{u_n\} \) is a \((PS)_c\) sequence of \( J_\lambda \) with
\[
c < \frac{N + 2 - \mu}{4N - 2\mu} S^\frac{2^*-2}{2^*},
\]
then \( \{u_n\} \) has a convergent subsequence.

**Proof.** Let \( u_0 \) be the weak limit of \( \{u_n\} \) obtained in Lemma [24] and define \( v_n := u_n - u_0 \), then we know \( v_n \to 0 \) in \( H^1_0(\Omega) \) and \( v_n \to 0 \) a.e. in \( \Omega \). Moreover, by the Brézis-Lieb Lemma in [8] and Lemma [24] we know
\[
\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla v_n|^2 dx + \int_\Omega |\nabla u_0|^2 dx + o(1),
\]
and
\[
\int_\Omega |u_n|^2 dx = \int_\Omega |v_n|^2 dx + \int_\Omega |u_0|^2 dx + o_n(1)
\]
and
\[
\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy = \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^*}}{|x-y|^{\mu}} dxdy + o(1).
\]
Then, we have
\[
c \leftarrow J_\lambda(u_n) = \frac{1}{2} \int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2} \int_\Omega u_n^2 dx - \frac{1}{22^*} \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy
\]
\[
= \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{\lambda}{2} \int_\Omega v_n^2 dx + \frac{1}{2} \int_\Omega |\nabla u_0|^2 dx - \frac{\lambda}{2} \int_\Omega u_0^2 dx
\]
\[
- \frac{1}{22^*} \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy - \frac{1}{22^*} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2^*} |u_0(y)|^{2^*}}{|x-y|^{\mu}} dxdy + o_n(1)
\]
\[
= J_\lambda(u_0) + \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{\lambda}{2} \int_\Omega v_n^2 dx - \frac{1}{22^*} \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy + o_n(1)
\]
\[
\geq \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{22^*} \int_\Omega \int_\Omega \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^{\mu}} dxdy + o_n(1),
\]
(2.6) [C1]
since \( J'(u_0) \geq 0 \) and \( \int \nabla u_0^n \to 0 \), as \( n \to +\infty \). Similarly, since \( \langle J'_1(u_0), u_0 \rangle = 0 \), we have

\[
o_n(1) = \langle J'_n(u_0), u_0 \rangle \geq \int_{\Omega} |\nabla u_0|^2 dx - \lambda \int_{\Omega} \frac{u_0^n}{|x|^\mu} dx - \int_{\Omega} \int_{\Omega} \frac{|u_0^n(x)|^2 |u_0^n(y)|^2}{|x-y|^\mu} dxdy \geq \int_{\Omega} |\nabla u_0|^2 dx - \lambda \int_{\Omega} \frac{u_0^n}{|x|^\mu} dx - \int_{\Omega} \int_{\Omega} \frac{|u_0^n(x)|^2 |u_0^n(y)|^2}{|x-y|^\mu} dxdy + o_n(1)
\]

(2.7)

From (2.7), we know there exists a nonnegative constant \( b \) such that

\[
\int_{\Omega} |\nabla v_n|^2 dx \to b \quad \text{and} \quad \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^2 |v_n(y)|^2}{|x-y|^\mu} dxdy \to b,
\]
as \( n \to +\infty \). From (2.6) and (2.7), we obtain

\[
c \geq \frac{N-1}{N-2} \mu b.
\]

(2.8)

By the definition of the best constant \( S_{H,L} \) in (1.5), we have

\[
S_{H,L} \left( \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^2 |v_n(y)|^2}{|x-y|^\mu} dxdy \right)^{\frac{N-2}{N-\mu}} \leq \int_{\Omega} |\nabla v_n|^2 dx,
\]

which yields \( b \geq S_{H,L} b^{\frac{N-2}{N-\mu}} \). Thus we have either \( b = 0 \) or \( b \geq S_{H,L}^{\frac{N-2}{N-\mu}} \). If \( b = 0 \), the proof is complete. Otherwise \( b \geq S_{H,L}^{\frac{N-2}{N-\mu}} \), then we obtain from (2.6),

\[
\frac{N-1}{N-2} \mu S_{H,L}^{\frac{N-2}{N-\mu}} \leq \frac{N-1}{N-2} \mu b \leq c,
\]

which contradicts with the fact that \( c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{N-2}{N-\mu}} \). Thus \( b = 0 \), and

\[
\|u - u_0\| \to 0
\]
as \( n \to +\infty \). This ends the proof of Lemma 2.3.

\[\square\]

3 The case \( N \geq 4 \), \( 0 < \lambda < \lambda_1 \)

We devote this Section to prove Theorem 1.2 for the case \( N \geq 4 \) and \( 0 < \lambda < \lambda_1 \).

By Lemma 1.2 we know that \( U(x) = \frac{N(N-2)}{4(1+|x|^2)} \) is a minimizer for both \( S \) and \( S_{H,L} \). Without loss of generality, we may assume that \( 0 \in \Omega \) and \( B_\delta \subset \Omega \subset B_{2\delta} \). Let \( \psi \in C_0^\infty(\Omega) \) such that

\[
\begin{cases}
\psi(x) = 1 & \text{if } x \in B_\delta, \\
0 \leq \psi(x) \leq 1 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \\
0 \leq |D\psi(x)| \leq C = \text{const} & \forall x \in \mathbb{R}^N.
\end{cases}
\]

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We define, for $\varepsilon > 0$,
\[
U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U\left(\frac{x}{\varepsilon}\right), \\
u_\varepsilon(x) := \psi(x) U_\varepsilon(x).
\]
From Lemma 1.46 of [37] and Lemma 1.2 we know
\[
|\nabla U_\varepsilon|^2 = |U_\varepsilon|^2 = S^{\frac{N}{2}},
\]
and as $\varepsilon \to 0^+$,
\[
\int_\Omega |\nabla u_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) = C(N, \mu) \frac{N-2}{2N-2} S_H^{\frac{N}{2}} + O(\varepsilon^{N-2}),
\]
and
\[
\int_\Omega |u_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^N)
\]
and
\[
\int_\Omega |u_\varepsilon|^2 \, dx \geq \begin{cases} \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \end{cases}
\]
where $d$ is a positive constant.

Using the Hardy-Littlewood-Sobolev inequality, on one hand, we get
\[
\left(\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy \right)^{\frac{N-2}{2N-2}} \leq C(N, \mu) \frac{N-2}{2N-2} |u_\varepsilon|^2,
\]
\[
= C(N, \mu) \frac{N-2}{2N-2} \left(S^{\frac{N}{2}} + O(\varepsilon^N)\right)^{\frac{N-2}{2N-2}}
\]
\[
= C(N, \mu) \frac{N-2}{2N-2} \left(C(N, \mu) \frac{N-2}{2N-2} S_H^{\frac{N}{2}} + O(\varepsilon^N)\right)^{\frac{N-2}{2N-2}}
\]
\[
= C(N, \mu) \frac{N-2}{2N-2} S_H^{\frac{N}{2}} + O(\varepsilon^{N-2}).
\]
While on the other hand,
\[
\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy \geq \int_{B_\delta} \int_{B_\delta} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
\[
= \int_{B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
\[
= \int_{R^N} \int_{R^N} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
\[
- 2 \int_{R^N \setminus B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
\[
- \int_{R^N \setminus B_\delta} \int_{R^N \setminus B_\delta} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
\[
= C(N, \mu) \frac{N}{2} S_H^{\frac{N-\mu}{2N-2}} - 2D - E,
\]
where
\[
D = \int_{R^N \setminus B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy
\]
and
\[
E = \int_{R^N \setminus B_\delta} \int_{R^N \setminus B_\delta} \frac{|U_\varepsilon(x)|^2 |U_\varepsilon(y)|^2}{|x-y|^\mu} \, dx \, dy.
\]
By direct computation, we know

\begin{align}
D & = \int_{\mathbb{R}^N \setminus B_3} \int_{B_3} \frac{|U_\varepsilon(x)|^{2\mu} |U_\varepsilon(y)|^{2\mu}}{|x - y|^{\mu}} \, dx \, dy \\
& = \int_{\mathbb{R}^N \setminus B_3} \int_{B_3} \frac{\varepsilon^{2N}[N(N-2)]^{2N-\mu}}{2} \frac{1}{|x - y|^{\mu}} \, dx \, dy \\
& = \varepsilon^{2N-\mu}[N(N-2)]^{2N-\mu} \int_{\mathbb{R}^N \setminus B_3} \int_{B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dx \, dy \\
& \leq O(\varepsilon^{2N-\mu}) \left( \int_{\mathbb{R}^N \setminus B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dx \right)^{\frac{N}{2N-\mu}} \left( \int_{B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dy \right)^{\frac{N}{2N-\mu}} \\
& \leq O(\varepsilon^{2N-\mu}) \left( \int_{\mathbb{R}^N \setminus B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dx \right)^{\frac{N}{2N-\mu}} \left( \int_{B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dy \right)^{\frac{N}{2N-\mu}} \\
& = O(\varepsilon^{2N-\mu})
\end{align}

and

\begin{align}
E & = \int_{\mathbb{R}^N \setminus B_3} \int_{\mathbb{R}^N \setminus B_3} \frac{|U_\varepsilon(x)|^{2\mu} |U_\varepsilon(y)|^{2\mu}}{|x - y|^{\mu}} \, dx \, dy \\
& = \int_{\mathbb{R}^N \setminus B_3} \int_{\mathbb{R}^N \setminus B_3} \frac{\varepsilon^{2N}[N(N-2)]^{2N-\mu}}{2} \frac{1}{|x - y|^{\mu}} \, dx \, dy \\
& = \varepsilon^{2N-\mu}[N(N-2)]^{2N-\mu} \int_{\mathbb{R}^N \setminus B_3} \int_{\mathbb{R}^N \setminus B_3} \frac{1}{(x^2 + |y|^2)^{2N-\mu}} \, dx \, dy \\
& \leq \varepsilon^{2N-\mu}[N(N-2)]^{2N-\mu} \int_{\mathbb{R}^N \setminus B_3} \int_{\mathbb{R}^N \setminus B_3} \frac{1}{|x - y|^{2N-\mu}} \, dx \, dy \\
& = O(\varepsilon^{2N-\mu}).
\end{align}

It follows from (3.8) to (3.10) that

\begin{align}
\left( \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2\mu} |u_\varepsilon(y)|^{2\mu}}{|x - y|^{\mu}} \, dx \, dy \right)^{\frac{N-2}{2N-\mu}} 
& \geq (C(N, \mu) S_{H,L}^{2N-\mu})^{\frac{N-2}{2N-\mu}} - O(\varepsilon^{2N-\mu}) - O(\varepsilon^{2N-\mu}) \\
& = (C(N, \mu) S_{H,L}^{2N-\mu})^{\frac{N-2}{2N-\mu}}.
\end{align}

When \( N = 3 \), (3.2) and (3.3) also hold.

**Lemma 3.1.** If \( N \geq 4 \) and \( \lambda > 0 \), then, there exists \( v \in H^1_0(\Omega) \setminus \{0\} \) such that

\[ \frac{||\nabla v||^2_{L^2} - \lambda ||v||^2_{L^2}}{||v||^2_{L^2}} < S_{H,L}. \]
Proof. If $N = 4$, from (3.4), (3.2) and (3.9), we can obtain

$$\frac{\|\nabla u\|^2}{\|u\|^2} \leq \frac{C(4,\mu)\frac{1}{\epsilon^2}S_{H,L}^2 - \lambda \epsilon^2|\ln \epsilon| + O(\epsilon^2)}{(C(4,\mu)^2S_{H,L} - O(\epsilon^{4-\frac{5}{2}}))^{\frac{1}{2}}}$$

$$= S_{H,L} - \frac{\lambda \epsilon^2|\ln \epsilon|}{(C(4,\mu)^2S_{H,L} - O(\epsilon^{4-\frac{5}{2}}))^{\frac{1}{2}}} + O(\epsilon^2) \quad (3.10)$$

If $N \geq 5$, using (3.2), (3.4) and (3.9) again, we have

$$\frac{\|\nabla u\|^2}{\|u\|^2} \leq \frac{C(N,\mu)\frac{1}{\epsilon^2}\frac{N}{N-\mu}S_{H,L}^2 - \lambda \epsilon^2 + O(\epsilon^{N-2})}{(C(N,\mu)\frac{N}{N-\mu}S_{H,L} - O(\epsilon^{N-\frac{3}{2}}))^{\frac{N-2}{N-\mu}}}$$

$$\leq S_{H,L} - \frac{\lambda \epsilon^2}{(C(N,\mu)\frac{N}{N-\mu}S_{H,L} - O(\epsilon^{N-\frac{3}{2}}))^{\frac{N-2}{N-\mu}}} + O(\epsilon^{\frac{N}{2}}) \quad (3.11)$$

From the arguments above, we may take $v := u_\epsilon$ with $\epsilon$ small enough and then the conclusion follows immediately.

**Lemma 3.2.** If $N \geq 3$ and $\lambda \in (0,\lambda_1)$, then, the functional $J_\lambda$ satisfies the following properties:

(i) There exist $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.

(ii) There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.

**Proof.** (i) By $\lambda \in (0,\lambda_1)$, the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality, for all $u \in H_0^1(\Omega)\setminus \{0\}$ we have

$$J_\lambda(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2\lambda_1} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2\mu} C_0 |u|^{2(\frac{2N-\mu}{N-\mu})}$$

$$\geq \frac{1}{2}(1 - \frac{\lambda}{\lambda_1})\|u\|^2 - \frac{1}{2\mu} C_0 C_1 \|u\|^{2(\frac{2N-\mu}{N-\mu})}.$$

Since $2 < 2(\frac{2N-\mu}{N-\mu})$, we can choose some $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.

(ii) For some $u_1 \in H_0^1(\Omega)\setminus \{0\}$, we have

$$J_\lambda(tu_1) = \frac{t^2}{2} \int_{\Omega} |\nabla u_1|^2 dx - \frac{\lambda t^2}{2} \int_{\Omega} u_1^2 dx - \frac{t^2}{2\mu} \int_{\Omega} \int_{\Omega} \frac{|u_1(x)|^{2\mu} |u_1(y)|^{2\mu}}{|x-y|^\mu} dy dx < 0$$

for $t > 0$ large enough. Hence, we can take an $e := t_1 u_1$ for some $t_1 > 0$ and (ii) follows.

**Proposition 3.3.** By Lemma 3.2 and the mountain pass theorem without (PS) condition (cf. [27]), there exists a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n) \to c$ and $J_\lambda'(u_n) \to 0$ in $H_0^1(\Omega)^{-1}$ at the minimax level

$$c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > 0, \quad (3.12)$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$
Proof of Theorem 1.4 Case $N \geq 4$, $0 < \lambda < \lambda_1$. From Lemma 4.1 we know there exists $v \in H_0^1(\Omega) \setminus \{0\}$ such that
$$\frac{\|\nabla v\|_2^2 - \lambda|v|_2^2}{\|v\|_N^2} < S_{H,L}.$$ Therefore,
$$0 < \max_{t \geq 0} J_{\lambda}(tv) = \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\Omega} |\nabla v|^2 \, dx - \frac{\lambda^2}{2} \int_{\Omega} v^2 \, dx - \frac{t^{2\mu}}{22\mu} \int_{\Omega} \int_{\Omega} \frac{|v(x)|^{2\mu} |v(y)|^{2\mu}}{|x-y|^{\mu}} \, dx \, dy \right\} \leq \frac{N + 2 - \mu}{4N - 2\mu} \left( \frac{\|\nabla v\|_2^2 - \lambda|v|_2^2}{\|v\|_N^2} \right)^{\frac{2N-\mu}{N+\mu}}.$$

By the definition of $c^*$, we have $c^* < \frac{N + 2 - \mu}{4N - 2\mu} S_{H,L}^{\frac{2N-\mu}{N+\mu}}$. Let $\{u_n\}$ be the $(PS)$ sequence obtained in Proposition 3.3 Applying Lemma 2.5 we know $\{u_n\}$ contains a convergent subsequence. And so, we have $J_{\lambda}$ has a critical value $c^* \in (0, \frac{N + 2 - \mu}{4N - 2\mu} S_{H,L}^{\frac{2N-\mu}{N+\mu}})$ and the problem (1.2) has a nontrivial solution.

4 The case $N \geq 4$, $\lambda \geq \lambda_1$

We may suppose that $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, where $\lambda_j$ is the $j$-th eigenvalue of $-\Delta$ on $\Omega$ with boundary condition $u = 0$. $e_j$ is the $j$-th eigenfunctions corresponding to the eigenvalue $\lambda_j$.

Lemma 4.1. If $N \geq 3$ and $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then, the functional $J_{\lambda}$ satisfies the following properties:

(i) There exist $\alpha, \rho > 0$ such that for any $u \in \mathbb{Y}_{j+1}$ with $\|u\| = \rho$ it results that $J_{\lambda}(u) \geq \alpha$.

(ii) $J_{\lambda}(u) \leq 0$ for any $u \in \mathbb{Y}_j$.

(iii) Let $\mathbb{F}$ be a finite dimensional subspace of $H_0^1(\Omega)$. There exists $R > \rho$ such that for any $u \in \mathbb{F}$ with $\|u\| \geq R$ it results that $J_{\lambda}(u) \leq 0$.

Proof. (i) Since $\lambda \in [\lambda_j, \lambda_{j+1})$, by the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality, for all $u \in \mathbb{Y}_{j+1} \setminus \{0\}$ we have
$$J_{\lambda}(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{2\lambda_{j+1}} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{22\mu} C_0 |u|_{2\mu}^{2(2N-\mu)} \geq \frac{1}{2} (1 - \frac{\lambda}{\lambda_{j+1}}) \|u\|^2 - \frac{1}{22\mu} C_1 \|u\|^{2(2N-\mu)}.$$ Since $\lambda < 2(\frac{2N-\mu}{N-2})$, we can choose some $\alpha, \rho > 0$ such that $J_{\lambda}(u) \geq \alpha$ for any $u \in \mathbb{Y}_{j+1}$ with $\|u\| = \rho$.

(ii) Let $u \in \mathbb{Y}_j$, that is, $u = \sum_{i=1}^j l_i e_i$, where $l_i \in \mathbb{R}, i = 1, ..., j$. Since $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$, we have
$$\int_{\Omega} u^2 \, dx = \sum_{i=1}^j l_i^2 \quad \text{and} \quad \int_{\Omega} |\nabla u|^2 \, dx = \sum_{i=1}^j l_i^2 |\nabla e_i|^2_2.$$ Then, we get
$$J_{\lambda}(u) = \frac{1}{2} \sum_{i=1}^j l_i^2 (|\nabla e_i|^2_2 - \lambda) - \frac{1}{22\mu} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu}}{|x-y|^{\mu}} \, dx \, dy \leq \frac{1}{2} \sum_{i=1}^j l_i^2 (\lambda_i - \lambda) \leq 0.$$
thanks to $\lambda_i \leq \lambda_j \leq \lambda$.

(iii) For $u \in F \setminus \{0\}$, by the non-negativity of $\lambda$ gives

$$J_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \|u\|^2 - \frac{1}{2} \mu \|u\|^{22^*_\mu} = \frac{1}{2} \|u\|^2 - \frac{1}{2} \mu \|u\|^{22^*_\mu}$$

for some positive constant $C_1 > 0$, since all norms on finite dimensional space are equivalent. So, $J_\lambda(u) \to -\infty$ as $\|u\| \to +\infty$. Hence, there exists $R > \rho$ such that for any $u \in F$ with $\|u\| \geq R$ it results that $J_\lambda(u) \leq 0$ and (iii) follows.

From Lemma 3.1, if $N \geq 4$ and $\lambda > 0$, then for $\varepsilon$ small enough,

$$|\nabla u_\varepsilon|^2 - \lambda |u_\varepsilon|^2 \|u_\varepsilon\|^2_{NL} < S_{H,L}.\]$$

For any $j \in \mathbb{N}$, we define the linear space

$$G_{j,\varepsilon} := \text{span}\{e_1, \ldots, e_j, u_\varepsilon\}$$

and set

$$m_{j,\varepsilon} := \max_{u \in G_{j,\varepsilon}, \|u\|_{NL} = 1} \left( \int \Omega |\nabla u|^2 - \lambda \int \Omega |u|^2 \right),$$

where $\|\cdot\|_{NL}$ is defined in Lemma 2.3.

**Lemma 4.2.** If $N \geq 4$ and $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then,

(i) $m_{j,\varepsilon}$ is achieved at some $u_m \in G_{j,\varepsilon}$ and $u_m$ can be written as follows

$$u_m = v + tu_\varepsilon$$

with $v \in \mathbb{Y}_j$ and $t \geq 0$.

(ii) The following estimate holds true

$$m_{j,\varepsilon} \leq \begin{cases} (\lambda_j - \lambda |v|^2 \|v\|^2_{NL} + A_\varepsilon \left(1 + |v|^2 O(\varepsilon^{-3}) + O(\varepsilon^{-5})|v|^2\right) & \text{if } t = 0, \\ (\lambda_j - \lambda |v|^2 \|v\|^2_{NL} + A_\varepsilon ) & \text{if } t > 0, \end{cases}$$

as $\varepsilon \to 0$, where $v$ is given in (i), $u_\varepsilon$ is given in Section 3 and

$$A_\varepsilon = \frac{|\nabla u_\varepsilon|^2 - \lambda |u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}}.$$

**Proof.** (i) Since $G_{j,\varepsilon}$ is a finite dimensional space, then $m_{j,\varepsilon}$ is achieved at some $u_m \in G_{j,\varepsilon}$, that is,

$$m_{j,\varepsilon} = |\nabla u_m|^2 - \lambda |u_m|^2 \text{ and } \|u_m\|_{NL} = 1.$$

Obviously, $u_m \neq 0$. From the definition of $G_{j,\varepsilon}$ we have that

$$u_m = v + tu_\varepsilon$$

for some $v \in \mathbb{Y}_j$ and $t \in \mathbb{R}$. We can suppose that $t \geq 0$, otherwise, if $t < 0$ we can replace $u_m$ with $-u_m$. The result follows.
(ii) If $t = 0$, then $u_m = v \in \mathbb{V}_j$ and

$$m_{j,c} = |\nabla u_m|^2 - \lambda |u_m|^2 = |\nabla v|^2 - \lambda |v|^2 \leq (\lambda_j - \lambda) |v|^2.$$

We consider the case $t > 0$. Since $e_1, \ldots, e_j \in L^\infty(\Omega)$, we also have $v \in L^\infty(\Omega)$. By a direct computation, we have

$$\int_{B_{2\delta}} \int_{B_{2\delta}} \frac{|u_m(x)|^2 |u_m(y)|^{p-1}}{|x-y|^\mu} \, dx \, dy = \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{|U_m(x)|^2 |U_m(y)|^{p-1}}{|x-y|^\mu} \, dx \, dy$$

$$= \varepsilon^{2N-3N/2} \frac{1}{|N(N-2)|^{3N/2}} \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{1}{(1 + |x|^2)^{N/2} |x-y|^\mu (1 + |y|^2)^{N/2} - \int_{B_{2\delta}} \int_{B_{2\delta}} \frac{1}{(1 + |x|^2)^{N/2} |x-y|^\mu (1 + |y|^2)^{N/2}} \, dx \, dy.$$
provided $\varepsilon < 1$ and so
\[
\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^{\mu}}\,dx\,dy \geq O(\varepsilon^{\frac{N-\mu}{\mu}}).
\]
Then we can get
\[
\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^{\mu}}\,dx\,dy = O(\varepsilon^{\frac{N-\mu}{\mu}}).
\]
By convexity, we obtain
\[
1 = \int_\Omega \int_\Omega \frac{|u_m(x)|^{2^*_\mu}|u_m(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy
= \int_\Omega \int_\Omega \frac{|v(x) + tu_\varepsilon(x)|^{2^*_\mu}|v(y) + tu_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy
\geq \int_\Omega \int_\Omega \frac{|tu_\varepsilon(x)|^{2^*_\mu}|tu_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy + 2^*_\mu \|u\|_{2^*\mu,\Omega}^2 \int_\Omega \int_\Omega \frac{|tu_\varepsilon(x)|^{2^*_\mu-1}v(x)|tu_\varepsilon(y)|^{2^*_\mu-1}v(y)}{|x-y|^{\mu}}\,dx\,dy
+ 2^*_\mu \int_\Omega \int_\Omega \frac{|tu_\varepsilon(x)|^{2^*_\mu-1}v(x)|tu_\varepsilon(y)|^{2^*_\mu-1}v(y)}{|x-y|^{\mu}}\,dx\,dy
\geq \int_\Omega \int_\Omega \frac{|tu_\varepsilon(x)|^{2^*_\mu}|tu_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy + 2^*_\mu \|u\|_{2^*\mu,\Omega}^2 \int_\Omega \int_\Omega \frac{|tu_\varepsilon(x)|^{2^*_\mu-1}v(x)|tu_\varepsilon(y)|^{2^*_\mu-1}v(y)}{|x-y|^{\mu}}\,dx\,dy
\geq t^{2^*_\mu} \int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy + 2^*_\mu |u|_{2^*\mu,\Omega}^{2^*_\mu-1} \int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu-1}u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^{\mu}}\,dx\,dy
\geq t^{2^*_\mu} \int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy - C_{2^*_\mu} t^{2^*_\mu-1}|v|_2 O(\varepsilon^{\frac{N-2}{2}}), \tag{4.3}
\]
where we used the fact that $\mathcal{V}_j$ is a finite dimensional space and all norms on $\mathcal{V}_j$ are equivalent. This implies that $t < C_3$ for some constant $C_3 > 0$. Taking (4.3) into account, we have
\[
\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy \leq 1 + O(\varepsilon^{\frac{N-2}{2}})|v|_2.
\]
By (42), one can see that
\[
m_{j,\varepsilon} = \int \nabla(v + tu_\varepsilon)^2 \,dx - \lambda \int |v + tu_\varepsilon|^2 \,dx
\leq (\lambda_j - \lambda)|v|^2 + A_\varepsilon \left( \int \int \frac{|tu_\varepsilon(x)|^{2^*_\mu}|tu_\varepsilon(y)|^{2^*_\mu}}{|x-y|^{\mu}}\,dx\,dy \right)^{\frac{N-2}{2^*_\mu-\mu}} + C_4 |u_\varepsilon|_1 |v|_2
\leq (\lambda_j - \lambda)|v|^2 + A_\varepsilon \left( 1 + |v| \frac{\varepsilon^{\frac{N-2}{2}}}{\lambda_j} \right)^{\frac{N-2}{2^*_\mu-\mu}} + C_4 |u_\varepsilon|_1 |v|_2
\leq (\lambda_j - \lambda)|v|^2 + A_\varepsilon \left( 1 + |v|_2 O(\varepsilon^{\frac{N-2}{2}}) \right) + O(\varepsilon^{\frac{N-2}{2}})|v|_2,
\]
where we had used the estimate in Lemma 2.25 of [37] that $|u_\varepsilon|_1 = O(\varepsilon^{\frac{N-2}{2}})$.

**Lemma 4.3.** If $N \geq 4$ and $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then,
\[
\frac{\|\nabla u_\varepsilon\|^2_2}{\|u\|^2_{N,L}} < S_{H,L}
\]
for any $u \in \mathcal{G}_{j,\varepsilon}$.

**Proof.** We only need to check that
\[
m_{j,\varepsilon} = \max_{u \in \mathcal{G}_{j,\varepsilon}, \|u\|_{N,L} = 1} \left( \int \nabla u_\varepsilon^2 \,dx - \lambda \int |u|^2 \,dx \right) < S_{H,L}.
\]

\[\text{16}\]
If \( t = 0 \) in \([4,1]\), by the choice of \( \lambda \in (\lambda_j, \lambda_{j+1}) \), we get that
\[
m_{j,\varepsilon} \leq (\lambda_j - \lambda) |v|^2 < S_{H,L}.
\]

Now we suppose that \( t > 0 \) and discuss the cases \( N \geq 5 \) and \( N = 4 \) separately.

If \( N \geq 5 \), we have
\[
m_{j,\varepsilon} \leq (\lambda_j - \lambda) |v|^2 + \frac{\|\nabla u\|^2 - \lambda |u|}{\|u\|^2_{H,L}} \left(1 + |v|_2 O(\varepsilon^{N-2})\right) + O(\varepsilon^{N-2}) |v|_2
\]
\[
\leq (\lambda_j - \lambda) |v|^2 + \frac{C(\lambda,\mu) \varepsilon^{2N-2}}{C(\lambda,\mu) \varepsilon^{2N-2} - O(\varepsilon^{N-4})} \left(1 + |v|_2 O(\varepsilon^{N-2})\right) + O(\varepsilon^{N-2}) |v|_2
\]
\[
\leq \left(S_{H,L} - \frac{\lambda \varepsilon^2}{C(\lambda,\mu) \varepsilon^{2N-2} - O(\varepsilon^{N-4})}\right) \left(1 + |v|_2 O(\varepsilon^{N-2})\right) + (\lambda_j - \lambda) |v|^2 + O(\varepsilon^{N-2}) |v|_2
\]
for \( \varepsilon > 0 \) sufficiently small. Since \( \lambda \in (\lambda_j, \lambda_{j+1}) \), we know
\[
(\lambda_j - \lambda) |v|^2 + O(\varepsilon^{N-2}) |v|_2 \leq \frac{1}{4(\lambda_j - \lambda)} O(\varepsilon^{N-2}) = O(\varepsilon^{N-2}),
\]
therefore
\[
m_{j,\varepsilon} \leq S_{H,L} - \lambda \varepsilon^2 + O(\varepsilon^{N-2}) < S_{H,L}
\]
for \( \varepsilon > 0 \) sufficiently small.

If \( N = 4 \), by \([4,4]\), we have
\[
m_{j,\varepsilon} \leq (\lambda_j - \lambda) |v|^2 + \frac{\|\nabla u\|^2 - \lambda |u|}{\|u\|^2_{H,L}} \left(1 + |v|_2 O(\varepsilon)\right) + O(\varepsilon) |v|_2
\]
\[
\leq (\lambda_j - \lambda) |v|^2 + \frac{C(4,\mu) \varepsilon^4 S_{H,L}}{C(4,\mu) \varepsilon^4 S_{H,L} - O(\varepsilon^{4-2})} \left(1 + |v|_2 O(\varepsilon)\right) + O(\varepsilon) |v|_2
\]
\[
\leq \left(S_{H,L} - \frac{\lambda \varepsilon^2 |v|}{C(4,\mu) \varepsilon^4 S_{H,L} - O(\varepsilon^{4-2})}\right) \left(1 + |v|_2 O(\varepsilon)\right) + (\lambda_j - \lambda) |v|^2 + O(\varepsilon) |v|_2
\]
\[
\leq S_{H,L} - \lambda \varepsilon^2 |v| + O(\varepsilon^2)
\]
\[
< S_{H,L}
\]
for \( \varepsilon > 0 \) sufficiently small. The result follows.

**Proof of Theorem 1.4** \( N \geq 4, \lambda > \lambda_1 \). From the definition of \( G_{j,\varepsilon} \) we know
\[
u_m = \mathbf{\nabla} + tz,\]
where
\[
\mathbf{\tau} = v + t \sum_{i=1}^{j} \left( \int_{\Omega} u_{\epsilon} e_i dx \right) e_i \in \mathbb{Y}_j
\]
and
\[
z_{\epsilon} = u_{\epsilon} - t \sum_{i=1}^{j} \left( \int_{\Omega} u_{\epsilon} e_i dx \right) e_i,
\]
so that \(\mathbf{\tau}\) and \(z_{\epsilon}\) are orthogonal in \(L^2(\Omega)\). This implies that
\[
|u_m|_2^2 = |\mathbf{\tau}|_2^2 + t^2 |z_{\epsilon}|_2^2.
\]
Then,
\[
\mathbb{G}_{j,\epsilon} = \mathbb{Y}_j \oplus \mathbb{R} z_{\epsilon}.
\]
Applying Lemma 4.1, we know that \(J_\lambda\) satisfies the geometric structure of the Linking Theorem (see [31], Theorem 5.3), that is
\[
\inf_{u \in \mathbb{X}_{j+1}, \|u\| = \rho} J_\lambda(u) \geq \alpha > 0,
\]
\[
\sup_{u \in \mathbb{Y}_j} J_\lambda(u) < 0
\]
and
\[
\sup_{u \in \mathbb{G}_{j,\epsilon}, \|u\| \geq R} J_\lambda(u) \leq 0.
\]
where \(\alpha\) and \(R\) are as in Lemma 4.1. Define the Linking critical level of \(J_\lambda\), i.e.
\[
c^* = \inf_{\gamma \in \Gamma} \max_{u \in \mathbb{V}} J_\lambda(\gamma(u)) > 0,
\]
where
\[
\Gamma := \{ \gamma \in C(\mathbb{V}, \mathbb{H}_0^1(\Omega)) : \gamma = id \text{ on } \partial V \}
\]
and
\[
\mathbb{V} := (\mathbb{B}_R \cap \mathbb{Y}_j) \oplus \{ rz_{\epsilon} : r \in (0, R) \}.
\]
For any \(\gamma \in \Gamma\), we have
\[
c^* \leq \max_{u \in \mathbb{V}} J_\lambda(\gamma(u))
\]
and in particular, if we take \(\gamma = id\) on \(\overline{V}\), then
\[
c^* \leq \max_{u \in \mathbb{V}} J_\lambda(u) \leq \max_{u \in \mathbb{G}_{j,\epsilon}} J_\lambda(u).
\]
Note that for any \(u \in \mathbb{H}_0^1(\Omega) \backslash \{0\}\),
\[
\max_{t \geq 0} J_\lambda(tu) = \frac{N + 2 - \mu}{4N - 2\mu} \left( \frac{\|u\|_2^2}{\|u\|_N^2} \right)^{\frac{2N - \mu}{N + 2 - \mu}}.
\]
From \(\mathbb{G}_{j,\epsilon}\) is a linear space we have
\[
\max_{u \in \mathbb{G}_{j,\epsilon}} J_\lambda(u) = \max_{u \in \mathbb{G}_{j,\epsilon}, t \neq 0} \lambda(|t| \frac{u}{|t|}) = \max_{u \in \mathbb{G}_{j,\epsilon}, t > 0} J_\lambda(tu) \leq \max_{u \in \mathbb{G}_{j,\epsilon}, t \geq 0} J_\lambda(tu).
\]
Thus, by Lemma 4.3 we have
\[
c^* \leq \max_{u \in \mathbb{G}_{j,\epsilon}, t \geq 0} J_\lambda(tu)
\]
\[
= \max_{u \in \mathbb{G}_{j,\epsilon}} \frac{N + 2 - \mu}{4N - 2\mu} \left( \frac{\|u\|_2^2 - \lambda |u|_2^2}{\|u\|_N^2} \right)^{\frac{2N - \mu}{N + 2 - \mu}}
\]
\[
\leq \frac{N + 2 - \mu}{4N - 2\mu} S_{H,L}^{2N - \mu}.
\]
Therefore, the Linking Theorem and Lemma 2.5 yield that problem (1.2) admits a nontrivial solution \(u \in \mathbb{H}_0^1(\Omega)\) with critical value \(c^* \geq \alpha\). \(\square\)
5 The case $N = 3$

In this Section, we prove Theorem 1.4 for the case $N = 3$ by using the Mountain Pass Theorem and the Linking Theorem. We still denote $F$ be a finite dimensional subspace of $H^1_0(\Omega)$ and

$$
\mathcal{G}_{j,\varepsilon} := \text{span}\{e_1, \ldots, e_j, u_\varepsilon\}.
$$

for any $j \in \mathbb{N}$.

\textbf{Lemma 5.1.} Let $N = 3$ and $u_\varepsilon$ be as in Section 3. Then, there exists $\lambda_*$ such that for any $\lambda > \lambda_*$,

$$
\frac{|\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}} - \lambda \frac{|u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}} < S_{H,L}
$$

provided $\varepsilon > 0$ is sufficiently small.

\textit{Proof.} By the definition of $u_\varepsilon$, we can get

$$
\int_{\Omega} |u_\varepsilon|^2 \, dx \geq \int_{B_{\delta}} |U_\varepsilon|^2 \, dx \geq C_0 \varepsilon	ag{5.1}
$$

for $\varepsilon > 0$ sufficiently small. By (3.2), (3.9) and (5.1), we have

$$
\frac{|\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}} - \lambda \frac{|u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}} \leq \frac{C(3, \mu) \varepsilon^{-\frac{1}{2}} S_{H,L}^2 - \lambda C_0 \varepsilon + O(\varepsilon)}{(3, \mu) \varepsilon^{-\frac{1}{2}} S_{H,L}^2 - O(\varepsilon^{\frac{5}{4}})}
$$

$$
= S_{H,L} - \frac{(\lambda C_0 - O(1)) \varepsilon}{(3, \mu) \varepsilon^{-\frac{1}{2}} S_{H,L}^2 - O(\varepsilon^{\frac{5}{4}})}
$$

if $\lambda$ is large enough, say $\lambda > \lambda_*$, while $\varepsilon > 0$ is sufficiently small. \hfill \Box

We will show that $J_\lambda$ has the geometric structure of the Mountain Pass Theorem when $\lambda \in (0, \lambda_1)$ and the geometric structure of the Linking Theorem when $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$.

We set

$$
m_{j,\varepsilon} := \max_{u \in \mathcal{G}_{j,\varepsilon}, \|u\|_{NL} = 1} \left( \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right).
$$

Related to Lemma 4.2, we also have the corresponding result for $N = 3$, so, we have

\textbf{Lemma 5.2.} If $N = 3$ and $\lambda \in [\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$, then,

(i) $m_\varepsilon$ is achieved in $u_m \in \mathcal{G}_{j,\varepsilon}$ and $u_m$ can be written as follows

$$
u_m = v + tu_\varepsilon
$$

with $v \in \mathcal{V}_j$ and $t \geq 0$.

(ii) The following estimate holds true

$$
m_{j,\varepsilon} \leq \begin{cases} 
(\lambda_j - \lambda) |v|^2_2 & \text{if } t = 0, \\
(\lambda_j - \lambda) |v|^2_2 + A_\varepsilon \left( 1 + |v|_2 O(\varepsilon^{\frac{1}{2}}) \right) + O(\varepsilon^{\frac{5}{4}}) |v|^2_2 & \text{if } t > 0,
\end{cases}
$$

as $\varepsilon \to 0$, where $v$ is given in (i), $u_\varepsilon$ is given in Section 3 and

$$
A_\varepsilon = \frac{|\nabla u_\varepsilon|^2 - \lambda |u_\varepsilon|^2}{\|u_\varepsilon\|^2_{NL}}.
$$
Lemma 5.3. If $N = 3$, $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$ and $\lambda > \lambda_*$, then,

$$\frac{|\nabla u|^2 - \lambda |u|^2}{\|u\|^2_{N,L}} < S_{H,L}$$

for any $u \in \mathbb{G}_{j,*}$.

Proof. If $t = 0$ in (5.5), by the choice of $\lambda \in (\lambda_j, \lambda_{j+1})$, we get that

$$m_\varepsilon \leq (\lambda_j - \lambda)|v|^2 \leq 0 < S_{H,L}.$$ 

When $t > 0$, by (5.2), (3.9), (5.1) and Lemma 5.2 using similar estimate as in (4.1), we have

$$m_{j,\varepsilon} \leq (\lambda_j - \lambda)|v|^2 + \frac{|\nabla u_c|^2 - \lambda |u_c|^2}{\|u_c\|^2_{N,L}} \left(1 + |v|^2 O(\varepsilon^\frac{1}{2})\right) + O(\varepsilon^\frac{1}{2})|v|^2$$

$$\leq (\lambda_j - \lambda)|v|^2 + \frac{C(3, \mu) \frac{\varepsilon^\frac{1}{2}}{S_{H,L}} - \lambda C_0 \varepsilon + O(\varepsilon)}{\left(C(3, \mu) \frac{\varepsilon^\frac{1}{2}}{S_{H,L}} - O(\varepsilon^\frac{3}{2})\right)} \left(1 + |v|^2 O(\varepsilon^\frac{1}{2})\right) + (\lambda_j - \lambda)|v|^2 + O(\varepsilon^\frac{1}{2})|v|^2$$

$$\leq \left(S_{H,L} - \frac{(\lambda C_0 - O(1))\varepsilon}{C(3, \mu) \frac{\varepsilon^\frac{1}{2}}{S_{H,L}} - O(\varepsilon^\frac{3}{2})}\right) \left(1 + |v|^2 O(\varepsilon^\frac{1}{2})\right) + (\lambda_j - \lambda)|v|^2 + O(\varepsilon^\frac{1}{2})|v|^2$$

$$\leq \left(S_{H,L} - \frac{(\lambda C_0 - O(1))\varepsilon}{C(3, \mu) \frac{\varepsilon^\frac{1}{2}}{S_{H,L}} - O(\varepsilon^\frac{3}{2})}\right) + (\lambda_j - \lambda)|v|^2 + O(\varepsilon^\frac{1}{2})|v|^2$$

$$< S_{H,L}$$

for $\varepsilon > 0$ sufficiently small, since $\lambda > \lambda_*$ and $\lambda \in (\lambda_j, \lambda_{j+1})$. The result follows.

Proof of Theorem 1.4. Case $N = 3$. We consider the two cases: $\lambda_1 > \lambda_*$ and $\lambda_1 > \lambda_*$ separately.

Case 1. $\lambda_1 > \lambda_*$. 

For this case we will use the Mountain Pass Theorem if $\lambda \in (\lambda_*, \lambda_1)$ while the Linking Theorem if $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$.

If $\lambda \in (\lambda_*, \lambda_1)$, by Lemma 3.2 and the mountain pass theorem without $(PS)$ condition (cf. 3.71), there exists a $(PS)$ sequence $\{u_n\}$ such that $J_\lambda(u) \to c^*$ and $J_\lambda'(u_n) \to 0$ in $H^1_0(\Omega)^{-1}$ at the Mountain Pass level $c^*$. From Lemma 5.1 we have there exists $v \in H^1_0(\Omega) \setminus \{0\}$ such that

$$\frac{|\nabla v|^2 - \lambda |v|^2}{\|v\|^2_{N,L}} < S_{H,L}.$$ 

Thus,

$$0 < \max_{t \geq 0} J_\lambda(tv) = \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_\Omega |v|^2 dx - \frac{\lambda t^2}{2} \int_\Omega v^2 dx - \frac{t^{22\mu}}{22\mu} \int_\Omega \int_\Omega \frac{|v(x)|^{22\mu} |v(y)|^{22\mu}}{|x-y|^{12-2\mu}} dxdy \right\}$$

$$= \frac{5 - \mu}{12 - 2\mu} \left( \frac{|\nabla v|^2 - \lambda |v|^2}{\|v\|^2_{N,L}} \right)^{\frac{2+\mu}{2}}$$

$$< \frac{5 - \mu}{12 - 2\mu} S_{H,L}^{\frac{2+\mu}{\mu}}.$$ 

By the definition of $c$, we know $c < \frac{5 - \mu}{12 - 2\mu} S_{H,L}^{\frac{2+\mu}{\mu}}$. 

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From Lemma 2.5 we obtain \( \{u_n\} \) contains a convergent subsequence. So, we have \( J_\lambda \) has a critical value \( c^* \in (0, \frac{\mu}{12-2\mu} S_{H,L}^{-\mu}) \) and problem (1.2) has a nontrivial solution. If \( \lambda \in (\lambda_j, \lambda_{j+1}) \) for some \( j \in \mathbb{N} \), we define
\[
J_\varepsilon = u_\varepsilon - \sum_{i=1}^n \left( \int_{\Omega} u_\varepsilon e_i dx \right) e_i,
\]
then,
\[
G_{j,\varepsilon} = Y_j \oplus Ru_\varepsilon = Y_j \oplus Rz_\varepsilon.
\]

By Lemma 4.1 we get that \( J_\lambda \) has the geometric structure required by the Linking Theorem (see [31, Theorem 5.3]). Thus we can define the Linking critical level \( c_L \) of \( J_\lambda \) as in (4.4) and
\[
c_L \leq \max_{u \in V} J_\lambda(u) \leq \max_{u \in G_{j,\varepsilon}} J_\lambda(u).
\]
On the other hand, we note that for any \( u \in H^1_0(\Omega) \setminus \{0\} \)
\[
\max_{t \geq 0} J_\lambda(tu) = \frac{5 - \mu}{12 - 2\mu} \left( \frac{|\nabla u|^2_2 - \lambda |u|^2_2}{\|u\|_{N,L}^2} \right)^{\frac{\mu-\mu}{\mu}}.
\]
As the same arguments in Section 4, we have
\[
c_L \leq \max_{u \in G_{j,\varepsilon}, t \geq 0} J_\lambda(tu) \leq \max_{u \in G_{j,\varepsilon}} \frac{5 - \mu}{12 - 2\mu} \left( \frac{|\nabla u|^2_2 - \lambda |u|^2_2}{\|u\|_{N,L}^2} \right)^{\frac{\mu-\mu}{\mu}} \leq \frac{5 - \mu}{12 - 2\mu} S_{H,L}^{-\mu}.
\]

Therefore, the Linking Theorem and Lemma 2.5 yield that problem (1.2) admits a solution \( u \in H^1_0(\Omega) \) with critical value \( c_L \geq \alpha \). Since \( \alpha > 0 = J_\lambda(0) \), we deduce that \( u \) is not identically zero.

**Case 2** \( \lambda_1 < \lambda_* \)

In this case, we only consider \( \lambda \in (\lambda_j, \lambda_{j+1}) \) for some \( j \in \mathbb{N} \) and \( \lambda > \lambda_* \). We can argue as in the last part of Case 1. In this way we get that for any \( \lambda > \lambda_* \), different from an eigenvalue of \(-\Delta \), problem (1.2) admits a solution \( u \in H^1_0(\Omega) \) with critical value \( c_L \geq \alpha \) and \( u \) is not identically zero.

6 Nonexistence

In this Section, we discuss nonexistence of solutions for (1.2) by using Pohožaev identity. Firstly, we are going to show that the solutions for equation (1.2) possess some regularity which will be used to prove the Pohožaev identity.

**Lemma 6.1.** If \( N \geq 3, \lambda < 0 \) and \( u \in H^1(\Omega) \) solves (1.2), then \( u \in W^{2,p}_{loc}(\Omega) \) for any \( p \geq 1 \).

**Proof.** Denote by \( H = K = |u|^{2^*-1} = |u|^{\frac{N+2}{N-2}} \), then \( H,K \in L^{\frac{2N}{N-2}}(\Omega) \). Using Proposition 3.2 of [26], we know \( u \in L^p(\Omega) \) for every \( p \in [2, \frac{2N^2}{(N-2)(N-2)}] \). Moreover, there exists a constant \( C_p \) independent of \( u \) such that
\[
\left( \int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} \leq C_p \left( \int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}}.
\]
Proposition 6.2. If $N \geq 3$, $\lambda < 0$ and $u \in H^1(\Omega)$ solves (1.2), then the following equality holds

$$\frac{1}{2} \int_{\partial \Omega} (x \cdot \nu)|\nabla u|^2 ds + \frac{N-2}{2} \int_{\Omega} |\nabla u|^2 dx = \frac{2N-\mu}{22^*} \int_{\Omega} |u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu} \frac{dx dy}{|x-y|^{p\sigma}} + \frac{\lambda N}{2} \int_{\Omega} |u|^2 dx,$$

where $\nu$ denotes the unit outward normal to $\partial \Omega$.

**Proof.** Since $u$ is a solution of (1.2) and Lemma 6.1, then $u$ satisfies

$$-\Delta u = \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy \right) |u(x)|^{2^*_\mu-2}u + \lambda u,$$

then

$$-\int_{\Omega} (x \cdot \nabla u) \Delta u dx = \int_{\Omega} (x \cdot \nabla u) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy \right) |u(x)|^{2^*_\mu-1} dx + \lambda \int_{\Omega} (x \cdot \nabla u) u dx. \quad (6.2)$$

Calculating the first term on the right side, we know

$$\int_{\Omega} (x \cdot \nabla u(x)) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy \right) |u(x)|^{2^*_\mu-1} dx$$

$$= -\int_{\Omega} u(x) \nabla(x) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy |u(x)|^{2^*_\mu-1} \right) dx$$

$$= -\int_{\Omega} u(x) (N \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy |u(x)|^{2^*_\mu-1}$$

$$+ (2^*_\mu - 1) |u(x)|^{2^*_\mu-2} x \cdot \nabla u(x) \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy$$

$$+ |u(x)|^{2^*_\mu-1} \int (\mu) x \cdot (x-y) \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma+2}} dy |u(x)|^{2^*_\mu-1} dx$$

$$= -N \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dx dy$$

$$- (2^*_\mu - 1) \int_{\Omega} x \cdot \nabla u(x) \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy |u(x)|^{2^*_\mu-1} dx$$

$$+ \mu \int_{\Omega} \int_{\Omega} x \cdot (x-y) \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma+2}} |u(x)|^{2^*_\mu} dy dx. \quad (6.3)$$

This implies that

$$2^*_\mu \int_{\Omega} (x \cdot \nabla u(x)) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dy \right) |u(x)|^{2^*_\mu-1} dx$$

$$= -N \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^{p\sigma}} dx dy$$

$$+ \mu \int_{\Omega} \int_{\Omega} x \cdot (x-y) \frac{|u(y)|^{2^*_\mu}}{|x-y|^{p\sigma+2}} |u(x)|^{2^*_\mu} dy dx,$$
similarly,
\[ 2^*_\mu \int_{\Omega} (y \cdot \nabla u(y)) \left( \int_{\Omega} \frac{|u(x)|^{2^*_\mu}}{|x-y|^\mu} dx \right) |u(y)|^{2^*_\mu - 1} dy \]
\[ = -N \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{2^*_\mu} |u(x)|^{2^*_\mu}}{|x-y|^\mu} dy dx + \mu \int_{\Omega} \int_{\Omega} y \cdot (y-x) \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \]
and consequently, we get
\[ \int_{\Omega} (x \cdot \nabla u(x)) \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u(x)|^{2^*_\mu - 1} dx = \frac{\mu - 2N}{22^*_\mu} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy. \]  
\text{(6.4)}

Since
\[ \int_{\Omega} (x \cdot \nabla u) u dx = -\frac{N}{2} \int_{\Omega} u^2 dx \]  
\text{(6.5)}
and
\[ \int_{\partial \Omega} (x \cdot \nu) |\nabla u|^2 ds = (2 - N) \int_{\Omega} |\nabla u|^2 dx + 2 \int_{\Omega} (x \cdot \nabla u) \Delta u dx. \]  
\text{(6.6)}

From the equalities above, we know the result holds.

\textbf{Proof of Theorem 1.5.} We assume that \( u \) is a nontrivial solution of (1.2), then we have
\[ \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy + \lambda \int_{\Omega} u^2 dx. \]
From Proposition 6.2 we can obtain
\[ \int_{\partial \Omega} (x \cdot \nu) |\nabla u|^2 ds = 2\lambda \int_{\Omega} |u|^2 dx. \]
Since \( \Omega \) is strictly star-shaped with respect to the origin in \( \mathbb{R}^N \), then \( x \cdot \nu > 0 \). Thus, we obtain \( u \equiv 0 \) from \( \lambda < 0 \). Which is a contradiction.

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