Heisenberg groups and their automorphisms over algebras with central involution

Robert W. Johnson
(Email: rwjcontact@aol.com)

Abstract

Heisenberg groups over algebras with central involution and their automorphism groups are constructed. The complex quaternion group algebra over a prime field is used as an example. Its subspaces provide finite models for each of the real and complex quadratic spaces with dimension 4 or less. A model for the representations of these Heisenberg groups and automorphism groups is constructed. A pseudo-differential operator enables a parallel treatment of spaces defined over finite and real fields.

Keywords: Heisenberg group, Schrödinger-Weil representation, involutive algebra.
I. INTRODUCTION

Our goal in this paper is to construct particular finite groups along with their representations that can be used as models for physical systems in dimensions greater than 1. The interrelationship between quantum mechanics and group theory is an old and vast subject. Recent developments in constructing Weil representations for $\text{SL}_n(2, A)$ for quite general involutive algebras $A$ and new insights into the relationship between the Jacobi group, that is formed by the semidirect product of $\text{SL}_n(2, A)$ with appropriate Heisenberg group, and the wavefunction and Wigner distribution of associated quantum systems motivate this paper. To simplify this treatment we work with algebras defined over finite fields. The finite Heisenberg groups that are formed over these algebras admit an elementary and constructive treatment.

Our starting point is an algebra $A$ with central involution $\ast$ that is used to define a trace and norm. We use elements from this algebra as entries in a $4\times4$ matrix presentation of a Heisenberg group $H_*(A)$. This presentation extends the treatment of Berndt and Schmidt (p 2) of Heisenberg groups over a commutative ring.

Groups of automorphisms of $H_*(A)$ are then introduced. We recover the automorphism group $\text{SL}_n(2, A)$ described by Pantoja and Soto-Andrade. For orientation, $\text{SL}_n(2, A)$ reduces to the symplectic group $\text{Sp}(2n, F)$ when $A$ is the ring of matrices $M_n(F)$ over a field $F$ and involution corresponds to matrix transpose. We then consider the semidirect product $\text{SL}_n(2, A) \ltimes H_*(A)$ that is the Jacobi group.

The Schrödinger representation for $H_*(A)$ is readily constructed and is unique for fixed nonzero central character. The Weil representation of $\text{SL}_n(2, A)$ is associated with automorphisms of the Schrödinger representation. Gutierrez, Pantoja and Soto-Andrade show how Weil representations for $\text{SL}_n(2, A)$ can be constructed when $\text{SL}_n(2, A)$ admits a Bruhat decomposition. Using this information one may then construct the Schrödinger-Weil representation for the semidirect product $\text{SL}_n(2, A) \ltimes H_*(A)$ for a wide range of algebras.

We may identify a function on the group algebra of $H_*(A)$ that transforms according to the Schrödinger representation with the wavefunction of a quantum system whose configuration space is $A$. The dynamical evolution of the wavefunction under $\text{SL}_n(2, A)$ automorphisms is described by the Schrödinger-Weil representation of $\text{SL}_n(2, A) \ltimes H_*(A)$ (see, de Gosson ch 15). From the wavefunction one may then construct the Wigner distribution for the quantum system.

Additional motivation for this paper includes the wish, addressed by Low, to treat relativistic...
quantum mechanics, with focus the Poincare group, and non-relativistic quantum mechanics, with focus the Heisenberg group, in a more uniform way. The work that we describe below provides a framework, different from Low’s, that combines these two cases in a natural way and that points towards further generalization.

Section II reviews some properties of the involutive algebras that are used as matrix entries in the construction of the Heisenberg groups $H_*(A)$ in Sec. IIIA. In Sec. IIIB we construct the group $SL_*(2, A)$ that preserves $H_*(A)$ under similarity transformation. When we restrict from $H_*(A)$ to $H_*(S)$ for $S \subset A$ we must also restrict $SL_*(2, A)$ to the subgroup that preserves $H_*(S)$ under automorphisms. In this paper we consider in detail the case $SL(2, F_N)$ where $F_N$ is the identity component of $A$. It is applicable to each of the Heisenberg groups that we consider. In Sec. IIIC we construct the diagonal subgroup $D(A^0)$ whose entries are from the set of norm 1 elements of $A$. Elements from $D(A^0)$ generate norm preserving transformations of the translational subgroups of $H_*(A)$. The group relations are summarized in Sec. IIID. In Sec. IIIE we introduce the complex quaternion group $C4 \otimes Q$ and its algebra $C4 \otimes Q(F_N)$ over the finite field $F_N$. Vector subspaces or subalgebras of $C4 \otimes Q(F_N)$ provide finite models for each of the real or complex quadratic spaces with dimension 4 or less.

In Sec. IV we construct representations for $H_*(A)$ and its automorphism groups for $A = C4 \otimes Q(F_N)$. In Sec. IVA, while defining notation, we treat the representations of important cyclic subgroups. Representations of the Heisenberg group $H_*(A)$ are described in Sec. IVB. In Sec. IVC we consider a subspace of the group algebra of $SL(2, F_N) \ltimes H_*(A)$ and show by direct calculation that it is closed under the action of the group. Details of this calculation are contained in Appendix A. Vectors in this subspace transform according to the Schrödinger-Weil representation. In Sec. IVD we describe how the Schrödinger-Weil representation can be extended to obtain representations of $(D(A^0) \otimes SL(2, F_N)) \ltimes H_*(A)$.

We define a pseudo-differential operator in Sec. VA that allows one to express group actions in a form that parallels the familiar setting of differential operators acting on functions defined over real spaces. In this way, while avoiding complications having a purely analytical origin by working over finite fields and rings, we may still gain insight into the structure of the physical case. These pseudo-differential operators are also needed when Wigner distributions for quantum systems based on these groups and representations are constructed in future work. In Sec. VB we consider eigenfunctions of particular group generators in the Schrödinger-Weil representation.
II. ALGEBRA WITH INVOLUTION

We consider Heisenberg groups whose group elements may be expressed as matrices with entries from an algebra $A$ with involution. An involution is a map $* : A \to A$ subject to the following conditions

$$x^{**} = x$$

$$(xy)^* = y^*x^*$$

$$(x + y)^* = x^* + y^*$$

for $x, y \in A$ ([9] p 13 and [10] p 139). Elements that are invariant under the involution are termed Hermitian and can be constructed in 2 standard ways

$$tr(x) = x + x^*$$

$$n(x) = xx^*$$

where $tr(x)$ is the trace mapping and $n(x)$ is the norm map. For central involutions the center of the algebra is Hermitian. For central involutions the norm map can be composed

$$n(xy) = (xy)(xy)^*$$

$$= (xy)(y^*x^*)$$

$$= n(x)n(y)$$

When $n(x)$ is invertible then $x$ is also invertible

$$x^{-1} = x^*n(x)^{-1}$$

When the center of the algebra is multidimensional, we denote the identity component of the trace as $Tr(x)$ and the identity component of the norm as $N(x)$.

We restrict to associative algebras in this paper since these provide examples that are already of physical interest and allow a simpler treatment.

III. GROUP CONSTRUCTION

A. Heisenberg group
We consider the group that is generated by the matrices

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
p & 1 & 0 & 0 \\
0 & 0 & 1 & \varepsilon p^* \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad t_y^q =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

by matrix multiplication, where \(p, q\) are elements in an algebra \(A\) with involution and \(\varepsilon = \pm 1\). We call this the Heisenberg group \(H_{\varepsilon}(A)\). We have the relations

\[
t_x^p t_x^{p'} = t_x^{p+p'} \\
t_y^q t_y^{q'} = t_y^{q+q'}
\]

We have the commutator

\[
t_x^{p} t_y^{q} t_{-z}^{p-q} = t_x^{pq^* - \varepsilon(pq^*)^*}
\]

for

\[
t_x^{pq^* - \varepsilon(pq^*)^*} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & pq^* - \varepsilon(pq^*)^* \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The subgroup with elements \(\{t_z^{t-\varepsilon t^*}\} \text{ for } t \in A\) forms the center of \(H_{\varepsilon}(A)\). When \(\varepsilon = -1\) the argument of \(t_z\) is hermitian with respect to *. This is the case of most interest in this paper and in the following we restrict to \(\varepsilon = -1\). We donote this group \(H_{-1}(A)\). The group multiplication can be written

\[
\left( t_x^{p} t_y^{q} t_z^{t(t)} \right) \left( t_x^{p'} t_y^{q'} t_z^{t'(t')} \right) = t_x^{p+p'} t_y^{q+q'} t_z^{t(t'+t'-qp'^*)}
\]

We may relax the requirement that \(p, q \in A\). The Heisenberg group \(H_{-1}(S)\) is still well-defined if we require, less stringently, that \(p, q\) reside in a subspace \(S \subset A\) that is closed under translations. The argument of \(t_z\) then resides in the subspace whose terms can be written \(\text{tr}(xy)\) for \(x, y \in S\).

Let us review the familiar construction of the Heisenberg group \(H(V)\) (see, e.g., Neuhauser\[11\] and Folland \[12\] p 19) over the vector space \(V\) whose elements \(v\) consist of pairs of vectors \((p, q)\)
defined over a ground field $F$ and equipped with a symplectic form $\langle p, q' \rangle - \langle q, p' \rangle$, where $\langle p, q' \rangle$ denotes a nondegenerate symmetric bilinear form on the vector space that contains $p$ and $q'$. This Heisenberg group contains elements $(p, q, t)$ for $t \in F$ equipped with the multiplication

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(\langle p, q' \rangle - \langle q, p' \rangle))$$

Now consider the Heisenberg group $H_*(A)$ of this paper. Identify the group element

$$t^p t^q t^{\text{tr}(t-\frac{1}{2}pq^*\lambda)} \in H_*(A)$$

with a triple $(p, q, \text{tr}(t))$ and obtain the multiplication

$$(p, q, \text{tr}(t))(p', q', \text{tr}(t')) = (p + p', q + q', \text{tr}(t + t' + \frac{1}{2}(pq^* - qp^*)))$$

Our construction reduces to the standard construction when $\text{tr}(pq^*) = \langle p, q \rangle$.

B. Semidirect product with $\text{SL}_*(2, A)$

We now construct the group $\text{SL}_*(2, A)$ described by Pantoja and Soto-Andrade [5] as a group of automorphisms of $H_*(A)$. Pantoja and Soto-Andrade [13] also consider the generalized case $\text{SL}^\varepsilon_*(2, A)$ for $\varepsilon = \pm 1$. One can show that $\text{SL}^\varepsilon_*(2, A)$ is an automorphism group of the generalized Heisenberg group $H^\varepsilon_*(A)$.

For $a, b, c, d \in A$ consider

$$g(a, b, c, d) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with inverse

$$g(a, b, c, d)^{-1} = g(x, y, z, w)$$

for some $x, y, z, w \in A$. We then have

$$1 = ax + bz = cy + dw$$

$$0 = ay + bw = cx + dz$$

(3)
Now consider the similarity transformation

\[ g(a, b, c, d) \begin{pmatrix} 1 & 0 & 0 & q^* \\ p & 1 & q \tr(t) + pq^* \\ 0 & 0 & 1 & -p^* \\ 0 & 0 & 0 & 1 \end{pmatrix} g(a, b, c, d)^{-1} \]

\[ = \begin{pmatrix} 1 & 0 & 0 & aq^* - bp^* \\ px + qz & 1 & py + qw \tr(t) + pq^* \\ 0 & 0 & 1 & cq^* - dp^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Closure of \( H_*(A) \) requires

\[-(px + qz)^* = cq^* - dp^* \]

\[(py + qw)^* = aq^* - bp^* \]

for each \( p, q \). This implies

\[ x^* = d; \quad -z^* = c; \quad y^* = -b; \quad w^* = a \]

Using the equalities in Eq. 3 we conclude

\[ 1 = ad^* - bc^* \]

\[ 0 = cd^* - dc^* = -ab^* + ba^* \]

and interchanging \( g(a, b, c, d) \) with \( g(a, b, c, d)^{-1} \) that

\[ 1 = a^*d - c^*b \]

\[ 0 = -c^*a + a^*c = -d^*b + b^*d. \]

One may show that the set of such \( g(a, b, c, d) \) compose a group under matrix multiplication. Denote this group \( \text{SL}_*(2, A) \). For \( t_{x,t}^p y_{t_2}^q t_{t_2}(t) \in H_*(A) \) we have

\[ g(a, b, c, d) \left( t_{x,t}^p y_{t_2}^q t_{t_2}(t) \right) g(a, b, c, d)^{-1} = t_{x}^{pd^* - qc^*} t_{y}^{qa^* - pb^*} t_{t_2}^{\tr(t) + pq^* - (pd^* - qc^*)(qa^* - pb^*)} \tag{4} \]

The semidirect product \( \text{SL}_*(2, A) \ltimes H_*(A) \) is the Jacobi group associated with \( A \). Berndt and Schmidt ([4] p 3) describe the Jacobi group when \( A \) is a commutative ring with identity.
When the Heisenberg group $H^*_s(S)$ for $S \subset A$ is constructed only over a vector space and not a ring we must restrict $\text{SL}_s(2, A)$ to the subgroup whose conjugate action preserves this Heisenberg group. In this paper we focus on Weil representations for $\text{SL}(2, F_N)$ where $F_N$ is the prime field $F_N$ for odd prime $N$. $\text{SL}(2, F_N)$ is an automorphism group for each of the Heisenberg groups $H^*_s(S)$ that we consider. The automorphisms that $\text{SL}(2, F_N)$ induce in $H^*_s(A)$ are summarized in Sect. 3.4 below.

C. Semidirect product with diagonal subgroup

Let $A^x_0$ denote the set of norm 1 elements of $A$

$$A^x_0 = \{ x \in A \mid xx^* = 1 \}.$$ 

When $A$ is a Clifford algebra, $xx^*$ is the spinor norm and $A^x_0$ composes the spin group \cite{14}. For $r, s, v, w \in A^x_0$ consider the similarity transformation

$$\begin{pmatrix} r & s & v & w \\ p & 1 & q \, \text{tr}(t) + pq^* & 0 \\ 0 & 0 & 1 & -p^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & q^* \\ p & 1 & q \, \text{tr}(t) + pq^* & 0 \\ 0 & 0 & 1 & -p^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & 0 & v^{-1} & 0 \\ 0 & 0 & 0 & w^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & rq^*w^{-1} \\ spr^{-1} & 1 & sqv^{-1} & s \, \text{tr}(t) + pq^* \, w^{-1} \\ 0 & 0 & 1 & -vp^*w^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

We require that the central subgroup of $H^*_s(A)$ also commute with this transformation. Since for central involutions $\text{tr}(t)$ is in the center of $A$, we obtain the condition $s = w$. Closure of $H^*_s(A)$ under this similarity transformation also requires $(spr^{-1})^* = vp^*w^{-1}$ and rearranging $p^* = r^{-1}vp^*w^{-1}s$ for all $p$. A similar relation holds for $q$. Using $s = w$ we obtain

$$p = p (rv^*)$$

For invertible $p$, this implies $r = v$. 

8
We conclude that $H_*(A)$ is closed under similarity transformations with

$$t^{r,s}_A \overset{\text{def}}{=} \begin{pmatrix} r \\ s \\ r \\ s \end{pmatrix}; r, s \in A_0^x$$

Let $D(A_0^x)$ denote the group that is composed of such matrices with their multiplication and form the semidirect product $D(A_0^x) \ltimes H_*(A)$. The automorphisms that $D(A_0^x)$ induces in $H_*(A)$ are summarized in Sect. 3.4 below.

When considering Heisenberg groups $H_*(S)$ for $S$ a vector subspace of an algebra, we restrict $t^{r,s}_A$ to the subgroup of $D(A_0^x)$ that preserves $H_*(S)$ under conjugation. Consider, for example, when $S$ corresponds to the subspace of $A$ with elements $x + x^*$. The Heisenberg group over this subspace has general element

$$\left( t^{p+p^*}_{x} t^{q+q^*}_{y} t^{t^*}_{z} \right).$$

Conjugating with $t^{r,s}_A \in D(A_0^x)$ leads to

$$t^{r,s}_A \left( t^{p+p^*}_{x} t^{q+q^*}_{y} t^{t^*}_{z} \right) t^{r,s*}_A = t^{s(p+p^*)}_{x} t^{q(q+q^*)r^*}_{y} t^{t^*}_{z}$$

$H_*(S)$ is stable under this transformation if we restrict to $s = r$.

D. Summary of group relations

The Heisenberg group $H_*(A)$ is composed of elements

$$\left\{ t^{p}_{x} t^{q}_{y} t^{t^*}_{z} \right\} = \begin{pmatrix} 1 & 0 & 0 & q^* \\ p & 1 & q & \text{tr}(t) + pq^* \\ 0 & 0 & 1 & -p^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid p, q, t \in A$$

with group multiplication

$$\left( t^{p}_{x} t^{q}_{y} t^{t^*}_{z} \right) \left( t^{p'}_{x} t^{q'}_{y} t^{t'^*}_{z} \right) = t^{p+p'}_{x} t^{q+q'}_{y} t^{t+t'^*+qp^*}_{z}$$

and commutation relation

$$t^{p}_{x} t^{q}_{y} = t^{q}_{y} t^{p}_{x} t^{qp^*}_{z}$$
The \( SL(2,F_N) \) subgroup is composed of the set of matrices

\[
\{g = \begin{pmatrix}
a & 0 & b & 0 \\
0 & 1 & 0 & 0 \\
c & 0 & d & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \mid a, b, c, d \in F_N; \det g = 1\}.
\]

It induces the following automorphisms of \( H(A) \)

\[
t^a_A \left( t^p_x t^q_y t^r_z(t) \right) t^{-a}_A = t^p_x t^q_y t^r_z(t)
\]

\[
t^a_d \left( t^p_x t^q_y t^r_z(t) \right) t^{-a}_d = t^p_x t^q_y t^r_z(t) + cqq^*
\]

\[
t^b_u \left( t^p_x t^q_y t^r_z(t) \right) t^{-b}_u = t^p_x t^q_y t^r_z(t) + bpp^*
\]

\[
j \left( t^p_x t^q_y t^r_z(t) \right) j^{-1} = t^p_x t^q_y t^r_z(t) + pqq^*
\]

\[
t^{a+\sqrt{d}b} t^p_x t^q_y t^r_z(t) t^{a-\sqrt{d}b} = t^p_x t^q_y t^r_z(t) + (pa - qb)(p\delta + qa)(t^r_z(t) + pq) - (pa - qb)(p\delta + qa)^*
\]

(5)

The \( SL(2,F_N) \) operators in these expressions are defined in section IV.A below.

The diagonal subgroup \( D(A^5_0) \) is composed of matrices

\[
\{t^r_s A = \begin{pmatrix} r \\ s \\ r \\ s \end{pmatrix} \mid r, s \in A^5_0\}
\]

and generates the following transformation of \( H_s(A) \)

\[
t^r_s A \left( t^p_x t^q_y t^r_z(t) \right) t^{-1,s-1}_A = t^s r^{-1} t^s p^{-1} t^r z(t)
\]

The action of \( D(A^5_0) \) preserves the norm and acts on both sides of the arguments of the translation subgroups of the Heisenberg group. \( SL(2,F_N) \) commutes with \( D(A^5_0) \).

E. Example: \( C4 \otimes Q \) group and algebra

Since our motivation derives primarily from physics and since the complex quaternion algebra suffices to treat cases of immediate interest let us focus on this algebra. The complex quaternion algebra is the algebra of 2x2 complex matrices. It is also known as the Pauli algebra. It is the Clifford algebra \( C_{3,0} \) in the notation of Benn and Tucker [14]. It is considered in a different context in [15].
The complex quaternion group $C4 \otimes Q$ has generators

$$\{e_1, e_2, e_3\}$$

and relations

$$e_1^2 = e_2^2 = e_3^2 = 1; (e_{ij})^4 = e_{123}^4 = 1 \text{ for } i \neq j$$

where we use the notation $e_i e_j = e_{ij}$. So that we may construct group representations in an elementary way, we consider the complex quaternion algebra over an odd prime field $F_N$. We restrict to the subspace whose representation maps $(e_{ij})^2 = -1$ for $i \neq j$. We then also have $(e_{123})^2 = -1$. $C4 \otimes Q$ is isomorphic with $C4 \otimes D4$ where $D4$ is the dihedral group with order 4. The center of the group is $\{1, e_{123}\}$.

Let us denote the group algebra for $C4 \otimes Q$ over the prime field $F_N$ as $C4 \otimes Q(F_N)$. A general element in this algebra can be written

$$x = x_0 1 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_{12} + x_5 e_{23} + x_6 e_{31} + x_7 e_{123}$$

(6)

We may express the basis elements of $C4 \otimes Q$ as 2x2 matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$e_{12} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_{23} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_{31} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_{123} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

The general element $x$ in the group algebra of $C4 \otimes Q$ can then be written in matrix form as

$$x = \begin{pmatrix} x_0 + x_3 + ix_4 + ix_7 & x_1 - ix_2 + ix_5 + x_6 \\ x_1 + ix_2 + ix_5 - x_6 & x_0 - x_3 - ix_4 + ix_7 \end{pmatrix}$$

We consider the following maps

$$x^\eta = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 + x_4 e_{12} + x_5 e_{23} + x_6 e_{31} - x_7 e_{123}$$

$$x^\xi = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 - x_4 e_{12} - x_5 e_{23} - x_6 e_{31} - x_7 e_{123}$$

$$x^{\xi \eta} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 - x_4 e_{12} - x_5 e_{23} - x_6 e_{31} + x_7 e_{123}$$

$= \text{def } x^*$$

The map $\eta$ is the main automorphism and is obtained by changing the signs of each of the group generators $\{e_1, e_2, e_3\}$. The reversal mapping $\xi$ is the anti-automorphism obtained by taking each
group element to its inverse. It corresponds to the matrix conjugate transpose (or adjoint) mapping when \( x \) is given in matrix form. We compose \( \eta \) and \( \xi \) to obtain the central involution \( \xi \eta \) that plays the role of the central involution \( * \).

We have the trace mapping

\[
\text{tr}(xy) = (xy) + (xy)^* = 2((x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7)1 + (x_0y_7 + x_7y_0 + x_1y_5 + x_5y_1 + x_2y_6 + x_6y_2 + x_3y_4 + x_4y_3)e_{123})
\]

and observe

\[
\text{tr}(xy) = \text{tr}(yx)
\]

We have the norm mapping

\[
n(x) = xx^* = (x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_7^2)1 + 2(x_0x_7 - x_1x_5 - x_2x_6 - x_3x_4)e_{123}
\]

with product rule

\[
n(xy) = xyy^*x^* = n(x)n(y),
\]

since \( yy^* \) is in the center of the algebra. The norm \( n(x) \) of an element \( x \in C4 \otimes Q(F_N) \) corresponds to the determinant of its matrix form.

The identity component of the trace can be expressed

\[
\text{Tr}(x) = \frac{1}{2}(x + x^{\xi \eta} + x^{\eta} + x^{\xi}) = 2x_0
\]

The identity component of the norm can be expressed

\[
N(x) = \frac{1}{2}(n(x) + n(x)^{\xi}) = (x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2 - x_7^2)
\]

The \( C4 \otimes Q(F_N) \) algebra with norm \( n(x) = xx^* \) can be viewed as a 4d complex space with Euclidean norm. Here \( e_{123} \) plays the role of the unit imaginary number. Subspaces of this algebra provide finite models for each of the real or complex quadratic spaces with dimension 4 or less.
Let us consider the case of a real vector space with signature (1,3) in more detail. The subspace $S \subset C^4 \otimes Q(F_N)$ that contains elements
\[ x + x^\xi = 2(x_01 + x_1e_1 + x_2e_2 + x_3e_3) \]
provides an example of such a vector space. We have the norm
\[ n(x + x^\xi) = 4 \left(x_0^2 - x_1^2 - x_2^2 - x_3^2\right). \]
and bilinear form
\[ \text{tr}((x + x^\xi)(y + y^\xi)^{\xi\eta}) = 8(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3). \]
We form the Heisenberg group over this subspace having elements
\[ \{ t^p_x t^q_y t^{r\xi}_z, \ \ t^\eta | p, q \in C^4 \otimes Q(F_N), \text{tr}(t) \in F_N \}. \]
This Heisenberg group is closed under the norm preserving automorphisms
\[ t^r_{A \xi} t^p_x t^q_y t^{r\xi}_z = t^r_{x} t^p_x t^q_y t^r_{z} t^{p+q\xi \xi, r\xi}_z \]
The subgroup $\{ t^r_{A \xi} | r \in C^4 \otimes Q(F_N); r r^{\xi\eta} = 1 \} \subset D(A_0^5)$ is isomorphic to $\text{SL}(2, C(F_N))$ that is our finite counterpart to $\text{SL}(2, C)$, the two fold cover of the proper orthochronous Lorentz group. The group formed by the semidirect product of $\text{SL}(2, C(F_N))$ with the translation subgroup $T_x = \{ t^p_x | p \in C^4 \otimes Q(F_N) \}$ is the Poincare group. In this way we place the Poincare group within the larger group structure $(\text{SL}(2, F_N) \otimes D(A_0^5)) \rtimes \text{H}_4(S)$.

Tables 1 and 2 show basis vectors in $C^4 \otimes Q$ that span subspaces with representative signatures. In these tables $C^2 = \{ 1, e_1 \}$ denotes the cyclic group with order 2, $C^4 = \{ 1, e_{123} \}$ in Table 1 and $C^4 = \{ 1, e_{12} \}$ in Table 2 denotes the cyclic group with order 4, $Q = \{ 1, e_{12}, e_{23}, e_{31} \}$ denotes the quaternion group, and $D^4 = \{ 1, e_1, e_2, e_{12} \}$ is the dihedral group of order 4.
Table 1. Complex vector spaces. $e_{123}$ plays role of unit imaginary number.

| basis elements | subspace       | signature |
|----------------|----------------|-----------|
| 1, $e_{123}$  | $C4(F_N)$      | (1,0)     |
| 1, $e_{12}$, $e_3$, $e_{123}$ | $C2 \otimes C4(F_N)$ | (2,0) |
| $e_1, e_2, e_3, e_{12}, e_{23}, e_{31}$ | $x-x^3 \subset C4 \otimes Q(F_N)$ | (3,0) |
| 1, $e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}$ | $C4 \otimes Q(F_N)$ | (4,0) |

Table 2. Vector spaces over $F_N$.

| basis elements | subspace       | signature |
|----------------|----------------|-----------|
| 1, $e_{12}$   | $C4(F_N)$      | (2,0)     |
| 1, $e_1$      | $C2(F_N)$      | (1,1)     |
| $e_{12}, e_{23}, e_{31}$ | $x-x^3 \subset Q(F_N)$ | (3,0) |
| 1, $e_1, e_2$ | $x+x^3 \subset D4(F_N)$ | (1,2) |
| 1, $e_{12}, e_{23}, e_{31}$ | $Q(F_N)$ | (4,0) |
| 1, $e_1, e_2, e_{12}$ | $D4(F_N)$ | (2,2) |
| 1, $e_1, e_2, e_3$ | $x+x^3 \subset C4 \otimes Q(F_N)$ | (1,3) |

IV. GROUP REPRESENTATIONS

In this section we construct a model for the representations of $(D(A_0^x) \otimes SL(2, F)) \rtimes H_+(S)$ when $S \subset C4 \otimes Q(F_N)$, $F$ is the field $F_N$ for odd prime $N$ and $A_0^x$ is the group of unit norm elements in the algebra containing $S$. In order that our construction be self-contained and to define notation let us start with the representations of important cyclic subgroups.

A. Representations of the cyclic subgroups

Let us describe a model for representations of the abelian translation subgroup

$$T_x = \{ t_x^h \mid h \in S \}$$

with multiplication

$$t_x^{h+h'} = t_x^h t_x^{h'}$$
We form eigenstates

$$\hat{x}_\eta = \frac{1}{V} \sum_{h \in S} \exp\left(\frac{-2\pi i}{N} \text{Tr} \eta h\right)t_x^h$$

(7)

where $V$ is the number of elements in the translation subgroup (For $S = C4 \otimes Q(F_N)$, $V = N^8$.) and $\eta \in S$. Note that $\text{Tr}(\eta h)$ projects to the identity component of the center of $S$ (in contrast to $tr(\eta h)$ that projects to the center of $S$ and that may be multidimensional). We have the group action and relationships

$$t_x^h \hat{x}_\eta = \exp\left(\frac{2\pi i}{N} \text{Tr} \eta h\right) \hat{x}_\eta$$

$$\sum_{\eta \in S} \hat{x}_\eta = 1$$

$$\hat{x}_\eta \hat{x}_{\eta'} = \delta(\eta - \eta') \hat{x}_\eta$$

In the same way we form eigenstates $\hat{y}_\eta$ of $t_y^h$ and eigenstates $\hat{z}_\omega$ of $t_z^h$ for $h, \eta, \omega \in S$.

Eigenstates for the additive subgroups of $\text{SL}(2, F_N)$ can be constructed in the same way. Writing elements of $\text{SL}(2, F_N)$ as $2 \times 2$ matrices, we introduce notation in the following. For the subgroup

$$\left\{ t_u^b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F_N \right\}$$

we form eigenstates

$$\hat{u}_\sigma = \frac{1}{N} \sum_{b \in F_N} \exp\left(\frac{-2\pi i}{N} \sigma b\right)t_u^b$$

(8)

Denote the order four element

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For the subgroup

$$\left\{ t_c^c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = j t_u^{-c} j^{-1} \mid c \in F_N \right\}$$

we form eigenstates

$$\hat{d}_\sigma = \frac{1}{N} \sum_{c \in F_N} \exp\left(\frac{-2\pi i}{N} \sigma c\right)t_d^c$$

(9)
Eigenstates for the multiplicative subgroups of SL(2, \( F_N \)) are constructed in a similar way. For the multiplicative group

\[ \{ \tau_a = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \mid a = z^{(N+1)t} \in F_N^{\times} \} \]

we form eigenstates

\[ \hat{s}_\chi = \frac{1}{N-1} \sum_{z^{(N+1)} \chi \in F_N^{\times}} \exp\left( -\frac{2\pi i}{N-1} \chi t \right) t^{-\chi t}_s \]

for \( z^{(N+1)} \chi \in F_N^{\times} \). Here, \( z \) denotes a generator of the multiplicative group \( F_N^{\times} \) in the quadratic extension \( F_N^{2} \). It has order \( N^2 - 1 \). The number \( z^{(N+1)} \) is a generator of the multiplicative group \( F_N^{\times} \) of \( F_N \) with order \( N - 1 \). We will often denote the action of \( t^a_s \) on \( \hat{s}_\chi \) as \( \chi(a) \) for \( \chi(a) = \exp(\frac{2\pi i}{N-1} \chi t) \). We denote the representation having \( \chi = 0 \) as \( \hat{s}_+ \) and the representation with \( \chi = \frac{N-1}{2} \) as \( \hat{s}_- \), where

\[ \hat{s}_- = \frac{1}{N-1} \sum_{a \in F_N^{\times}} \left( \frac{a}{N} \right) t^a_s \]

and \( \left( \frac{a}{N} \right) \) is the Legendre symbol.

The circle subgroup has elements

\[ \{ \tau^a_r = \begin{pmatrix} a & b \delta \\ b & a \end{pmatrix} \} \]

where \( a + \sqrt{\delta} b = z^{(N-1)m} \) and \( z^{(N-1)} \) is a generator of the multiplicative group of unit norm elements \( U \) in the quadratic extension \( F_N^{2} \)(see, e.g., [16] p 306). \( U \) has order \( N + 1 \). We form eigenstates

\[ \hat{r}_\mu = \frac{1}{N+1} \sum_{a+\sqrt{\delta} b = z^{(N-1)m} \in U} \exp\left( -\frac{2\pi i}{N+1} \mu m \right) t^{(N-1)m}_r \]

for each \( z^{(N-1)} \mu \in U \).

Additional information about \( SL(2, F_N) \subset GL(2, F_N) \) and its subgroups and representations are contained in Terras ([16], Ch. 21) and Piatetski-Shapiro [17].

### B. Representations of the Heisenberg group

Let us develop models for the representations of \( H_*(S) \). The representations with trivial central character can be understood very simply as arising from the group action on ideals in its...
group algebra with form \( \hat{x}_\sigma \hat{y}_\eta \hat{z}_0 \) for \( \sigma, \eta \in S \). \( \hat{z}_0 \) corresponds to the trivial representation of the central subgroup. The 1d subspaces \( \hat{x}_\sigma \hat{y}_\eta \hat{z}_0 \) are each stable under \( H^* (S) \).

Let us construct a model for the representations with nontrivial central character. Consider the maximal abelian subgroup \( \{ t_s t_y t_z | s, t \in S \} \) and form the 1 dimensional invariant subspace \( \hat{y}_\eta \hat{z}_{\text{tr}(\omega)} \) for \( \text{tr}(\omega) \neq 0 \).

For fixed \( \text{tr}(\omega) \) any choice for \( \hat{y}_\eta \) leads to an equivalent representation; for simplicity, let us choose \( \eta = 0 \). Act on \( \hat{y}_0 \hat{z}_{\text{tr}(\omega)} \) with a general element \( f \) in the \( H^* (S) \) group algebra

\[
 f = \sum_{h, k, l \in S} f(h, k, \text{tr}(l)) t_x^h t_y^k t_z^{\text{tr}(l)}.
\]

where \( f(h, k, \text{tr}(l)) \) is a complex number for each \( h, k, l \in S \). We obtain

\[
 f \hat{y}_0 \hat{z}_{\text{tr}(\omega)} = \left( \sum_{h, k, l \in S} f(h, k, \text{tr}(l)) t_x^h t_y^k t_z^{\text{tr}(l)} \right) \hat{y}_0 \hat{z}_{\text{tr}(\omega)}
 = \sum_{h \in S} f(h) t_x^h \hat{y}_0 \hat{z}_{\text{tr}(\omega)}
\]

after absorbing \( t_y^k t_z^{\text{tr}(l)} \) into \( \hat{y}_0 \hat{z}_{-\varepsilon \omega} \) and summing over \( k, l \). Here \( f(h) \) a complex number for each \( h \). Let us denote \( f \hat{y}_0 \hat{z}_{\text{tr}(\omega)} = f_{\text{tr}(\omega)} \). The action of the Heisenberg group on \( f_{\text{tr}(\omega)} \) is the Schrödinger representation with central character \( \text{tr}(\omega) \). The Schrödinger representation is unique and irreducible for each value of \( \text{tr}(\omega) \neq 0 \). It has dimensionality \( V \) the number of elements in \( S \). See, Terras ([16], ch 18) for additional description of the Heisenberg group and its representations.

Similarity transformation with

\[
 M = \begin{pmatrix}
 x & y & \frac{yw}{x} \\
 y & yw & w \\
 \frac{yw}{x} & w & w
\end{pmatrix}
\]

for \( x, y, w \in \text{center of A and nonzero} \)

leads to

\[
 M \left( t_x^p t_y^q t_z^{\text{tr}(t)} \right) M^{-1} = t_x^p t_y^q t_z^{\text{tr}(t)}
\]

This transformation can be used to translate between Schrödinger representations having different central character. In the following, we treat the special case when the central character \( \text{tr}(\omega) = \omega_0 \) is in the ground field \( F_N \). The action of \( H^*(S) \) on \( f_{\omega_0} \) is summarized below in Eqs. [15].
C. A representation of $\text{SL}(2, F_N) \rtimes H_\ast(S)$

We now describe a model for the Schrödinger-Weil representation of the semidirect product group $\text{SL}(2, F_N) \rtimes H_\ast(S)$ for $S$ a subspace of $C4 \otimes Q(F_N)$. This construction is similar to the case when $A$ is 1 dimensional [3]. The Weil representation for the group $\text{SL}(2, A)$ that contains $\text{SL}(2, F_N)$ is described by Gutierrez et. al. [1].

We consider a subspace of the regular representation of $\text{SL}(2, F_N) \rtimes H_\ast(S)$ having vectors with the form

$$f = \sum_{k \in S} f(k) t_x^k I$$

for

$$I = \hat{y}_0 \hat{z}_0 \hat{s}_\chi \hat{u}_0 \left\{ \sum_{p \in S} t_x^p \left[ 1 + \tau \sum_{l \in S; N(l) = 0} \exp\left(\frac{2\pi i}{N} 2\omega_0 \text{Tr} l p^*\right) \right] \right\} (1 + \alpha j) \hat{d}_0$$

(13)

In this expression $\tau, \alpha$ are numbers to be determined, $\hat{s}_\chi$ is either $\hat{s}_+ \text{or} \hat{s}_-$ and we restrict to central characters $\omega_0 \in F_N^\times$. We show that $f$ resides in a stable subspace under the action of $\text{SL}(2, F_N) \rtimes H_\ast(S)$ for appropriate choice of the numbers $\tau, \alpha$ and choice of $\hat{s}_\chi$. The action of $\text{SL}(2, F_N) \rtimes H_\ast(S)$ on $f$ then determines the sought after representation.

It is evident by inspection that $f$ is stable under $H_\ast(S)$. One can also show without difficulty that $f$ is stable under the action of $t_u^b \in \text{SL}(2, F_N), b \in F_N$. This is done by expanding out the terms in Eq. (12) to the left of $\hat{u}_0$, passing $t_u^b$ through these terms using the group relations, and then absorbing $t_u^b$ into $\hat{u}_0$. Since $\text{SL}(2, F_N)$ can be generated using the 2 operators $t_u^1$ and $j$ it is left to show that $f$ is stable under the action of $j$. We verify this by a direct calculation that is summarized in Appendix A.

1. Group actions in the Schrödinger-Weil representation

We have the following group actions for the Heisenberg group generators

$$t_x^h f = \sum_{k \in S} f(k - h) t_x^k I$$

(14)

$$t_y^b f = \sum_{k \in S} \exp\left(\frac{-2\pi i}{N} \omega_0 2\text{Tr} kh^*\right) f(k) t_x^b I$$

(15)

$$t_z^l f = \exp\left(\frac{2\pi i}{N} \omega_0 \text{Tr} l\right) f$$
For the SL(2, N) group operators we have
\begin{align}
  t^a_b f &= \sum_{k \in S} \exp \left( \frac{2\pi i}{N} \omega_0 b \text{Tr} n(k) \right) f(k) t^k_x I \\
  t^a_s f &= \left( \frac{a}{N} \right)^\text{dim} \sum_{k \in S} f(ak) t^k_x I \\
  j f &= \kappa \sum_{h \in S} \left( \sum_{k \in S} \exp \left( \frac{2\pi i}{N} \omega_0 2 \text{Tr} h^* k \right) f(k) \right) t^h_x I
\end{align}

where
\begin{equation}
  \kappa = \frac{1}{V} G(1, N)^\text{dim} \left( \frac{2 \omega_0}{N} \right)^\text{dim} \left( -\frac{1}{N} \right)^q
\end{equation}

In this expression, \text{dim} is the dimensionality of \( S \) and \( q \) is the number of negative terms in the quadratic form. The Gauss sum \( G(1, N) \) ([18] p 82) is given by
\begin{align}
  G(1, N) &= \sum_{a \in F^*_N} \left( \frac{a}{N} \right) \exp \left( \frac{2\pi i}{N} a \right) \\
          &= \sum_{a \in F^*_N} \exp \left( \frac{2\pi i}{N} a^2 \right) \\
          &= \{ \sqrt{N} \text{ for } N=1 \text{ mod } 4 \} \{ i\sqrt{N} \text{ for } N=3 \text{ mod } 4 \}
\end{align}

To calculate \( t^c_d f \) we use
\begin{equation}
  t^c_d f = jt^{-c}_u j^{-1} f
\end{equation}

and substitute into this expression the action of the operators \( j, t^{-c}_u \) and \( j^{-1} = jt^{-1}_s \). We obtain
\begin{equation}
  t^c_d f = \frac{1}{V} \sum_{l \in S} \left( \sum_{h, k \in S} \exp \left( \frac{2\pi i}{N} \omega_0 \text{Tr} \left( -cn(h) + 2h^*(k-l) \right) \right) f(k) \right) t^l_x I
\end{equation}

We may obtain \( t^{a+\sqrt{b}}_r \) by using either of the decompositions
\begin{align}
  t^{a+\sqrt{b}}_r &= t^{b/\sqrt{a}}_d \left( t^{a+\sqrt{b}}_u \right)^{b/\sqrt{a}} \\
                 &= t^{b/\sqrt{a}}_d \left( t^{a/\sqrt{b}}_u \right)^{b/\sqrt{a}}
\end{align}

We find for \( b \neq 0 \)
\begin{align}
  t^{a+\sqrt{b}}_r f \\
  &= \frac{1}{V} \sum_{h \in A} \sum_{k \in A} \exp \left( \frac{2\pi i}{N} \omega_0 \text{Tr} \left( (n(k) + n(h))(a - 1/b - b/4\omega_0 n(p)) \right) \exp \left( \frac{2\pi i}{N} \text{Tr} (k - h) - \frac{b}{4\omega_0} n(p) \right) f(k) t^h_x I
\end{align}
The actions for $t^c_d$ and $t^a_+ \sqrt{\bar{a}b}$ can be reexpressed by completing the norm and summing over $h$ in Eq. 19 and over $p$ in Eq. 20. The given form for $t^c_d$ is convenient for use below.

D. Representations including the diagonal subgroup

To obtain representations of the complete group $(\text{SL}(2,F_N) \otimes \text{D}(A_0^x)) \ltimes \text{H}(S)$ consider the action of $t_A^{r,s}$ on the function $f = \sum_{k \in S} f(k)t_x^k I$

$$t_A^{r,s} f = \sum_{k \in S} f(s^{-1}kr)t_x^k I_A^{r,s}$$

$t_A^{r,s}$ generates a norm-preserving transformation of the argument of $t_x^k$ (after redefining $k$ this becomes a change in the argument of $f(k)$) and commutes through the remainder of the expression. The function $f$ will be stable under this action if we place a left ideal of $\text{D}(A_0^x)$ on the right of $f$ to absorb $t_A^{r,s}$. Most simply, we can place the ideal

$$\rho_0 = \sum_{r, s \in A_0^x} t_A^{r,s}$$

on the right hand side of $f$. $t_A^{r,s}$ has trivial action on $\rho_0$. The function $f(k)$ then behaves like a scalar function of a vector variable.

Now I show schematically how finite dimensional representations of $\text{D}(A_0^x)$ arise in this construction. Form the affine-like group $\text{D}(A_0^x) \ltimes T_x$ (the Poincare group is of this type) with elements

$$\{t_x^{h,t_A^{r,s}}\}$$

For $\tau \in A(F_N)$ let $O_\tau = \{r\tau s^* \mid r, s \in A_0^x\}$. $O_\tau$ is the orbit of $\tau$ for this action. The subspace in the $\text{D}(A_0^x) \ltimes T_x$ group algebra that contains vectors

$$f_A = \sum_{\sigma \in O_\tau} f_A(\sigma) \hat{x}_\sigma \rho_0,$$

for $f_A(\sigma)$ a complex number for each $\sigma$, is stable under the group action:

$$t_x^{h,t_A^{r,s}} f_A = \sum_{\sigma \in O_\tau} \exp\left(\frac{2\pi i}{N} \text{Tr}h \sigma\right) f_A(r^{-1} \sigma s) \hat{x}_\sigma \rho_0.$$

Placing $f_A$ on the right side of Eq. 12 we find a representation for $(\text{D}(A_0^x) \otimes \text{SL}(2,F)) \ltimes \text{H}_s(A)$ given by the action of the group on the vector.

$$f = \sum_{k \in A, \sigma \in O_\tau} f(k, \sigma)t_x^k I \hat{x}_\sigma \rho_0$$
Now, express
\[ \hat{x}_\sigma = \frac{1}{\sqrt{N}} \sum_{h \in A} \exp\left(-\frac{2\pi i}{N} \text{Tr}(h)\right) t_x^h \]
and expand out the exponential in this expression. Observe that the \( m \)th order term in the expansion of the exponential corresponds to a vector in the space of homogeneous polynomials of degree \( m \); also, note that the space containing this vector is stable under \( t_A^{r,s} \). These spaces of homogeneous polynomials correspond to the representation spaces of finite dimensional representations of \( D(A_0^r) \).

When \( D(A_0^r) \ltimes T_x \) corresponds to the Poincare group the construction of the finite dimensional representations is well known (See, e.g., Sternberg [19] p 143.). Berndt describes the unitary (20) p 143) representations when \( D(A_0^r) \) is the Lorentz group. Examples of these representations can be realized in the above construction by placing an ideal in the group algebra of \( D(A_0^r) \) (rather than in the group algebra of \( D(A_0^r) \ltimes T_x \)) on the right hand side of Eq. 12.

V. PSEUDO-DERIVATIVE OPERATOR

In this section we construct a pseudo-derivative operator that is applicable to functions in the group algebra of a finite group. These operators enable a comparison of our results to those obtained using differential operators on functions defined over real spaces.

A. Construction of pseudo-derivative operator

Act on an element \( t_x^h \) in the additive cyclic group \( T_x(A) = \{ t_x^h \mid h \in A \} \) with 1 = \( \sum_{\sigma \in A} \hat{x}_\sigma \)

\[ t_x^h = t_x^h \sum_{\sigma \in A} \hat{x}_\sigma \]

\[ = \sum_{\sigma \in A} \exp\left(\frac{2\pi i}{N} \text{Tr}(h)\right) \hat{x}_\sigma \]

Since \( \hat{x}_\sigma^m = \hat{x}_\sigma \) we can write

\[ t_x^h = \sum_{\sigma \in A} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi i}{N} \text{Tr}(h) \hat{x}_\sigma\right)^m \]

and since \( \hat{x}_\sigma \hat{x}_{\sigma'} = \delta(\sigma - \sigma') \hat{x}_\sigma \) we switch the order of the summations and obtain

\[ t_x^h = \exp\left(\frac{2\pi i}{N} \text{Tr}(h \sum_{\sigma \in A} \sigma \hat{x}_\sigma)\right) \]

The exponential map Eq. 21 from an element in the group algebra to an element in the group is analogous to the exponential map between Lie algebra and Lie group elements. It motivates that
we consider the operator

\[ X = \sum_{\sigma \in A} \sigma \hat{x}_\sigma \]

that is analogous to an element in the Lie algebra of the translation group. \( X \) resides in an algebra that is formed by taking the direct product of \( A(F_N) \) with \( \sigma \in A(F_N) \), with the group algebra of \( T_x(A) \) over the complex numbers, where \( \hat{x}_\sigma \in \text{group algebra of } T_x(A) \). Properties of this direct product algebra are developed only as needed in the following; one may initially view the expression for \( X \), and similar expressions below, as a formal expression.

Consider the action of \( t^h_x \) on a function \( f = \sum_{k \in A} f(k) t^k_x \) in the group algebra of \( T_x(A) \) over the complex numbers

\[ t^h_x f = \exp\left(\frac{2\pi i}{N} \text{Tr}(h \sum_{\sigma \in A} \sigma \hat{x}_\sigma)\right) \sum_{k \in A} f(k) t^k_x \]

Expand out the exponential and let \( t^k_x \) act on \( \hat{x}_\sigma \) to obtain

\[ t^h_x f = \sum_{\sigma \in A} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{2\pi i}{N} \text{Tr} h \sigma \right)^m \sum_{k \in A} \exp\left(\frac{2\pi i}{N} \text{Tr} \sigma k\right) f(k) \hat{x}_\sigma \]

Now expanding \( \hat{x}_\sigma \) using Eq. 7 we obtain

\[ t^h_x f = \sum_{l \in A} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{2\pi i}{N} \text{Tr} h \sigma \right)^m \sum_{k \in A} \exp\left(\frac{2\pi i}{N} \text{Tr} \sigma (k - l)\right) f(k) \]

Since we also have \( t^h_x f = \sum_{l \in A} f(l - h) t^l_x \) we conclude

\[ f(l - h) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{2\pi i}{N} \right)^m \left( \frac{1}{V} \sum_{\sigma, k \in A} (\text{Tr} h \sigma)^m \exp\left(\frac{2\pi i}{N} \text{Tr} \sigma (k - l)\right) f(k) \right) \]

(22)

This is our analog to the Taylor series expansion of a function about a position \( l \) by a displacement \( h \). We use the first order term in this expression to define an operator that is analogous to the directional derivative or gradient operator

\[ \text{grad } f(l) = \text{def} \frac{1}{V} \sum_{\sigma, k \in A} \sigma \exp\left(\frac{2\pi i}{N} \text{Tr} \sigma (k - l)\right) f(k) \]

(23)

This is our counterpart to Eq. 2.31 in Folland ([12] p 93) that describes the Kohn-Nirenberg correspondence in the theory of pseudo-differential operators. \( \text{grad } f(l) \) resides in the algebra
formed by the direct product of $A(F_N)$, where $\sigma \in A(F_N)$, with the complex numbers. The $i^{th}$ component of grad $f(l)$ provides our definition of the partial derivative operation

$$\frac{\partial}{\partial l_i} f(l) \overset{\text{def}}{=} \text{grad } f(l) \mid_i = \frac{1}{V} \sum_{\sigma,k \in A} \sigma_i \exp\left(\frac{2\pi i}{N} \text{Tr}(\sigma(k - l)) \right) f(k)$$

Using these definitions, we have

$$\text{grad } f(l) = \sum_{e_i} e_i \frac{\partial}{\partial l_i} f(l)$$

One can develop the algebra of grad in a manner similar to the treatment of Doran and Lasenby ([21], p 168) in the context of geometric algebra. We limit the development in the following to our immediate applications.

Iterating the expression for $\text{Tr}(h \text{ grad}) f(l)$ we find

$$(\text{Tr}(h \text{ grad}))^m f(l) = \frac{1}{V} \sum_{\sigma,k \in A} (\text{Tr}h \sigma)^m \exp\left(\frac{2\pi i}{N} \text{Tr}(\sigma(k - l)) \right) f(k)$$

Exponentiating $\text{Tr}(h \text{ grad})$ we obtain

$$f(l - h) = \exp\left(\frac{2\pi i}{N} \text{Tr}(h \text{ grad}) \right) f(l) \quad (24)$$

We derive rules for differentiation by applying Eq. 24 to a chosen function and equating terms according to their powers of $h$ on the two sides of the equation. E.g., applying $\exp\left(\frac{2\pi i}{N} \text{Tr}(h \text{ grad}) \right)$ to the function $f(l) = f_1(l)f_2(l)$

$$\exp\left(\frac{2\pi i}{N} \text{Tr}(h \text{ grad}) \right)f(l) = f_1(l - h)f_2(l - h)$$
$$= \left[\exp\left(\frac{2\pi i}{N} \text{Tr}(h \text{ grad}) \right) f_1(l)\right]\left[\exp\left(\frac{2\pi i}{N} \text{Tr}(h \text{ grad}) \right) f_2(l)\right]$$

and equating the first order terms in $h$ we obtain the Leibnitz rule.

Comparing Eq. 24 with Eq. 14 we have the exponential form for the operator $t_x^h$

$$t_x^hf = \sum_{k \in A} \exp\left(\frac{2\pi i}{N} \text{Tr}h \text{ grad} \right) f(k)t_x^kI \quad (25)$$

We calculate the conjugate reversal of grad and iterate with grad to obtain

$$\text{grad } \text{grad}^* f(l) = \frac{1}{V} \sum_{\sigma,k \in A} \sigma \sigma^* \exp\left(\frac{2\pi i}{N} \text{Tr}(\sigma(k - l)) \right) f(k)$$

(26)

(27)
Just as for $\text{grad}$ in Eq. 24 we can form the exponential of the trace of this operator. Comparing the result with expression Eq. 19 we find
\[
t^{\epsilon} f = \sum_{l \in A} \exp(-2\pi i / N \text{Tr} \text{grad} \text{grad}^*) f(l) t_x^l I
\] (28)

### B. Eigenstates of group actions

We use these derivative operations to reexpress eigenstates of group actions in the Schrödinger-Weil representation in a differential form.

Projecting with an eigenstate of the translation operator
\[
\hat{x}_\sigma f = \frac{1}{V} \sum_{h \in S} \exp(-2\pi i / N \text{Tr} h \sigma) t_x^h f
\]
\[
= \frac{1}{V} \sum_{h,k \in S} \exp(-2\pi i / N \text{Tr}(h(\sigma - \text{grad}))) f(k) t_x^k I
\]
\[
= \sum_{k \in S} \delta(\text{grad} - \sigma) f(k) t_x^k I
\]
Therefore eigenstates $\hat{x}_\sigma f$ satisfy the operator identity
\[
0 = (\text{grad} - \sigma) f(k).
\]

Projecting with $\hat{d}_\eta = \frac{1}{N} \sum_{h \in F_N} \exp(2\pi i h \eta) t_d^h$ and using Eq. 28 for the derivative form for $t_d^h$ we find that eigenstates $\hat{d}_\eta f$ satisfy
\[
0 = (\text{Tr}(\text{grad} \text{grad}^*)) + 4\omega_0 \eta f(k)
\] (29)
This is the Klein-Gordon equation for the configuration space $A(F_N)$. We can verify this expression by writing $f$ in terms of its Fourier components
\[
f = \sum_{\sigma \in S} \bar{f}(\sigma) \hat{x}_{\sigma} I
\]
for $\bar{f}(\sigma) = \sum_{k \in S} \exp(2\pi i / N \text{Tr} k \sigma) f(k)$. Since
\[
\hat{d}_\eta \hat{x}_{\sigma} I = \delta(\eta + 1 / 4\omega_0 \text{Tr}(\sigma \sigma^*)) \hat{x}_{\sigma} I
\]
we have
\[
\hat{d}_\eta f = \sum_{\sigma \in S; \text{Tr}(\sigma \sigma^*) = -4\omega_0 \eta} \bar{f}(\sigma) \hat{x}_{\sigma} I
\]
\[
= \frac{1}{V} \sum_{k \in S} (\sum_{\sigma \in S; \text{Tr}(\sigma \sigma^*) = -4\omega_0 \eta} \bar{f}(\sigma) \exp(-2\pi i / N \text{Tr} k \sigma)) t_x^k I
\]
Applying $\text{grad } \text{grad}^*$ to the eigenfunction \( \sum_{\sigma \in S; \gamma} \tilde{f}(\sigma) \exp \left( \frac{-2\pi i}{N} \text{Tr} k \sigma \right) \) returns Eq. 29.

The operator \( t_h^a \in \text{SL}(2, F_N) \) that is conjugate to \( t_d^{-h} \)

\[
t_h^a = j t_d^{-h} j^{-1}
\]

eigenstates \( \hat{u}_\tau f \) in the Schrödinger-Weil representation

\[
\hat{u}_\tau f = \sum_{k \in S; N(k) = \frac{\tau}{2\omega_0}} f(k) t_k^\tau I
\]

We observe that \( \hat{u}_\tau \) projects out a function that is defined on the spherical shell \( N(k) = \frac{\tau}{2\omega_0} \).

We postpone a derivation of the pseudo-differential expressions for the multiplicative operators \( t^a \) and \( t^{a+\sqrt{b}} \) since this entails some additional development.

VI. DISCUSSION

We have described the construction of Heisenberg groups \( H_*(A) \) over an involutive ring \( A \) and shown how this extends the usual construction [4, 22]. One subgroup of automorphisms of this Heisenberg group recovers the group \( \text{SL}_*(2, A) \) described by J. Pantoja and J. Soto-Andrade [5].

Another subgroup of automorphisms of \( H_*(A) \) are generated by elements from a diagonal matrix group \( D(A_0^x) \) whose entries are from the set of norm 1 elements of \( A \). These automorphisms preserve the norm of \( A \).

We specialize to the case \( (D(A_0^x) \otimes \text{SL}(2, F_N)) \ltimes \text{H}_*(S) \) for \( S \) a subspace of the complex quaternion group algebra \( C4 \otimes Q(F_N) \) over the prime field \( F_N \). Vector subspaces or subalgebras of \( C4 \otimes Q(F_N) \) provide finite models for each of the real or complex quadratic spaces with dimension 4 or less. We construct a model that provides the Schrödinger-Weil representation of \( \text{SL}(2, F_N) \ltimes \text{H}_*(S) \) and indicate how it can be extended to the complete group \( (D(A_0^x) \otimes \text{SL}(2, F_N)) \ltimes \text{H}_*(S) \). Functions that transform according to the Schrödinger representation of \( \text{H}_*(S) \) can be associated with wave-functions of quantum systems having \( S \) as configuration space. The evolution of the wavefunction under quadratic Hamiltonians is described by the Weil representation of its automorphism group [6].

We define a pseudo-differential operator that can be applied to functions in the group algebra of a finite translation group. We obtain differential expressions for particular group actions in the Schrödinger-Weil representation of \( (D(A_0^x) \otimes \text{SL}(2, F_N)) \ltimes \text{H}_*(S) \). The pseudo-differential operator enables a parallel treatment of spaces defined over finite and real fields.
Appendix A: Action of $j$:

We determine the action of $j$ by direct calculation. The full calculation, though elementary, is long. We do the first portion below and then indicate key intermediate results towards the final result. The complete calculation for the 1 dimensional case is contained in the appendix of Johnson [3].

We use the identity

$$ t^k_x y^z_0 = \hat{y}_\sigma - 2\omega_0 k^* z^\omega_0 t^k_x \tag{A1} $$

that is derived by expanding out $\hat{y}_\sigma$ according to the definition Eq. 7 and passing $t^k_x$ through the expression using the commutator Eq. 2.

Consider the action of $j$ on $f$ given by Eq. 12

$$ jf = \sum_{k \in S} f(k) \frac{1}{V} \sum_{h \in S} t^h_x t^l_y t^m_z (l(r(t)))^{-1} t^h_x t^l_y t^m_z (l(r(t+p_q))) \hat{y}_\sigma \hat{z}^\chi \hat{s}^\sigma \hat{u}_0^{-} $$

where, to simplify notation, the underline in the preceding expression denotes all of the terms following $\hat{u}_0$. Expand out $\hat{y}_\sigma$ using $\hat{y}_\sigma = \frac{1}{V} \sum_{h \in S} t^h_x t^l_y t^m_z (l(r(t)))^{-1} t^h_x t^l_y t^m_z (l(r(t+p_q)))$. Noting that $j\hat{s}^\chi = \hat{s}^\chi j$ for the cases $\chi = 0, \frac{N-1}{2}$ that are of interest here, we obtain

$$ jf = \sum_{k \in S} f(k) \frac{1}{V} \sum_{h \in S} t^h_x t^l_y t^m_z (l(r(t)))^{-1} t^h_x t^l_y t^m_z (l(r(t+p_q))) \hat{y}_\sigma \hat{z}^\chi \hat{s}^\sigma \hat{u}_0^{-} $$

Now insert $1 = \sum_{\sigma \in S} \hat{y}_\sigma$

$$ jf = \sum_{k \in S} f(k) \frac{1}{V} \sum_{h \in S} t^h_x t^l_y t^m_z (l(r(t)))^{-1} \sum_{\sigma \in S} \hat{y}_\sigma \hat{z}^\chi \hat{s}^\sigma \hat{u}_0^{-} $$

Let $t^l_y$ act on $\hat{y}_\sigma$ and $t^m_z (l(r(t)))$ act on $\hat{z}^\omega_0$ to obtain the exponential term

$$ \exp(-\frac{2\pi i}{N} \text{Tr}(\sigma - 2\omega_0 h^*)) $$

Insert $1 = t^\sigma x t^\sigma x^{-1}$ to the left of $\hat{y}_\sigma$. We obtain

$$ jf = \sum_{k \in S} f(k) \frac{1}{V} \sum_{h, \sigma \in S} \exp(-\frac{2\pi i}{N} \text{Tr}(\sigma - 2\omega_0 h^*)) t^h_x t^l_y t^m_z \hat{y}_\sigma \hat{z}^\chi \hat{s}^\sigma \hat{u}_0^{-} $$
Now factor \( t_x \) through \( \hat{y}_\sigma \) and use the identity Eq. A1 to obtain

\[
jf = \sum_{k \in S} f(k) \frac{1}{V} \sum_{h, \sigma \in S} \exp\left(-\frac{2\pi i}{N} \text{Tr}k(\sigma - 2\omega_0 h^*)\right) t_x \frac{h_2 \omega_0 - \sigma^*}{2\omega_0} \hat{y}_0 \hat{z}_\omega t_x \frac{\sigma^*}{2\omega_0} \hat{s}_x j \hat{u}_0.
\]

Substitute \( h' = \frac{h_2 \omega_0 - \sigma^*}{2\omega_0} \) and sum over \( \sigma \). We obtain

\[
jf = \sum_{h' \in S} \left( \sum_{k \in S} f(k) \exp\left(\frac{2\pi i}{N} 2\omega_0 \text{Tr}h'^*\right)\right) t_x h'.
\]

\[
\cdot \hat{y}_0 \hat{z}_\omega \hat{s}_x (\tilde{x}_0 j) \hat{u}_0 \left\{ \sum_{p \in S} t_p^h \left( 1 + \tau \sum_{l \in S; N(l) = 0} \exp\left(\frac{2\pi i}{N} 2\omega_0 \text{Tr}p^*\right)\right) \right\} (1 + \alpha_j) \hat{d}_0
\]

(A2)

It remains to show that the second line of Eq. A2 is equal, up to a prefactor, to Eq. 13 for chosen values of \( \tau, \alpha \) and \( \hat{s}_x \). In outline, we pass \((\tilde{x}_0 j)\) to the right through the second line of expression Eq. A2 and compare with Eq. 12. We find three types of terms: terms that equate to the \( \alpha j \) portion of Eq. 12 up to a prefactor, terms that sum to zero, and terms that equate to the non-\( \alpha j \) portion of Eq. 12 up to a prefactor. The first two types provide conditions on \( \chi, \tau \) and \( \alpha \). We then find that the condition arising from the non-\( \alpha j \) portion is satisfied automatically.

The parameters \( \tau \) and \( \alpha \) can be expressed in terms of 2 sums

\[
c_0 = \sum_{l \in S; N(l) = 0} 1
\]

\[
c_1 = \sum_{l \in S; N(l) = 0} \exp\left(\frac{2\pi i}{N} 2\omega_0 \text{Tr}h^*\right)
\]

for fixed \( h \) such that \( h \neq 0, N(h) = 0 \). \( \tau \) satisfies the quadratic equation

\[
0 = \tau^2(c_0 - c_1) - \tau c_1 - 1
\]

and \( \alpha \) can be expressed

\[
\alpha = \frac{1}{N \tau (c_0 - c_1)} G(1, N)^\text{dim} \left(\frac{-2\omega_0}{N}\right)^\text{dim} \left(\frac{-1}{N}\right)^q
\]

where \( \text{dim} \) is the dimensionality of \( S \), \( G(1, N) \) is the Gauss sum, and \( q \) is the number of negative terms in the quadratic form. The representation \( \hat{s}_x \) is the trivial representation \( \hat{s}_+ \) when \( \text{dim} \) is even and the representation \( \hat{s}_- \) when \( \text{dim} \) is odd.

For all cases we can express the action of \( j \) as

\[
jf = \kappa \sum_{h \in S} \left( \sum_{k \in S} f(k) \exp\left(\frac{2\pi i}{N} 2\omega_0 \text{Tr}h^*\right)\right) t_x^h I
\]

(A3)
for
\[ \kappa = \frac{1}{V} G(1, N)^{\dim} \left( \frac{2\omega_0}{N} \right)^{\dim} \left( \frac{-1}{N} \right)^q \]

The trace of the linear transformation Eq. [A3] is the character of \( j \). It has value \( \left( \frac{-2}{N} \right)^{\dim} \) (cf., [11, 23]).

[1] L. Gutierrez, J. Pantoja, and J. Soto-Andrade: Contemporary Mathematics 544 (2011), 109-122.
[2] A. Ibar, V.I. Man’ko, G. Marmo, A. Simoni and F. Ventriglia: J. Phys. A 42 (2009), 155302.
[3] R. W. Johnson: arXiv:quant-ph/0604153 (2006).
[4] R. Berndt and R. Schmidt: Elements in the Representation Theory of the Jacobi Group, Birkhäuser Verlag, Basel 1998.
[5] J. Pantoja and J. Soto-Andrade: J. Algebra 262 (2003), 401-412.
[6] M. A. de Gosson: Symplectic Methods in Harmonic Analysis and in Mathematical Physics, Birkhauser, Berlin 2011.
[7] A. E. Krasowska and S. T. Ali: J. Phys. A 36 (2003), 2801.
[8] S. G. Low: J. Phys.: Conf. Ser. 343 (2012), 012069.
[9] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol: The Book of Involutions, Amer. Math. Soc. Colloq. Publ. V44, Amer. Math. Soc., Providence 1998.
[10] K. McCrimmon: A Taste of Jordan Algebras, Springer-Verlag, New York 2004.
[11] M. Neuhauser: J. Lie Theory 12 (2002), 15-30.
[12] G. B. Folland: Harmonic Analysis in Phase Space, Princeton University Press, Princeton 1989.
[13] J. Pantoja and J. Soto-Andrade: Commun. Algebra 37 (2009), 4170-4191.
[14] I. M. Benn and R. W. Tucker: An Introduction to Spinors and Geometry with Applications in Physics, Adam Hilger, Bristol 1987.
[15] R. W. Johnson: Found. Phys. 26 (1996), 197-222.
[16] A. Terras: Fourier Analysis on Finite Groups and Applications, Cambridge Univ. Press, Cambridge 1999.
[17] I. Piatetski-Shapiro: Contemp. Math. 16, Amer. Math. Soc., Providence, R.I., (1983), 1-71.
[18] S. Lang: Algebraic Number Theory, Springer-Verlag, New York 1994.
[19] S. Sternberg: Group Theory and Physics, Cambridge Univ. Press, Cambridge 1994.
[20] R. Berndt: Representations of Linear Groups, Friedr. Vieweg & Sohn Verlag, Wiesbaden 2007.
[21] C. Doran and A. Lasenby: Geometric Algebra for Physicists, Cambridge University Press, Cambridge 2003.
[22] S. G. Low: J. Math. Phys. 55 (2014), 0222105.
[23] T. Thomas: *J. London Math. Soc.* **77** (2008), 221-239.