Superconducting qubits coupled to torsional resonators

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Abstract

We propose a scheme of strong and tunable coupling between a superconducting phase qubit and a nanomechanical torsional resonator. In our scheme the torsional resonator directly modulates the largest energy scale (the Josephson coupling energy) of the phase qubit, and the coupling strength is very large. We analyze the quantum correlation effects in the torsional resonator as a result of the strong coupling to the phase qubit.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Probing the quantum mechanical properties of macroscopic objects is believed to be the key to understanding the border between classical and quantum physics. Nanomechanical resonators, which have a high frequency of gigahertz and low dissipation, provide a tangible system to study such macroscopic quantum phenomena [1–3]. Coupling the resonator to the superconducting qubits has attracted great theoretical interest as it provides a way of controlling and detecting the quantum behavior of the resonator [4–12] and a prototypical experiment has recently been demonstrated [13, 14].

Besides the fundamental aspect of the system, a nanomechanical resonator prepared in a squeezed state can improve its noise properties, upon which the limit of force detection sensitivity is based, beyond the standard quantum limit [15]. An architecture for a scalable quantum computation has also been suggested based on the integration of the nanomechanical resonators with the superconducting phase qubits [16, 17].

In this paper, we propose a scheme of strong and tunable coupling between a superconducting phase qubit and a nanomechanical torsional resonator. In our scheme, the direct modulation of the largest energy scale of the phase qubit enables a large coupling strength. This distinguishes our scheme from other previously proposed schemes. For example, in [18], the flexural vibrational modes were coupled to a charge qubit by modulating the Josephson energy, which in their case is one of the smallest energy scales of the qubit system. We analyze the quantum correlation effects in the torsional resonator and also provide the noise analysis, which shows that our scheme is feasible experimentally at the level of present technology.

The rest of the paper is organized as follows. In section 2 we summarize the basic operational mechanism of the phase qubit and the characteristics of the torsional resonator. In section 3 we analyze the coupling mechanism between the phase qubit and the torsional resonator. The reduced coupling constant is expressed in terms of the control parameters of the phase qubit and the torsional resonator. In section 4 we discuss the possible quantum correlation effects, especially, the squeezing of the torsional vibration mode, in the strong coupling limit. In section 5 we provide a detailed noise analysis in a possible experimental realization of the scheme. Finally the paper is concluded in section 6.

2. Qubit and resonator

A superconducting phase qubit consists of a double Josephson junction (figure 1) of small size, and is described by the Hamiltonian of the form

\[ H_{\text{qubit}} = E_C n^2 - 2E_J \cos(\pi f) \cos(\phi - \pi f), \]  

(1)
where the number $n$ of Cooper pairs that has tunnelled through the double junction and the phase difference $\varphi$ across the junction are quantum mechanical conjugate variables, i.e. $[\varphi, n] = i$. Here $E_C = (2\varepsilon)^2/2C \sim 10$ meV is the charging energy of the double junction with total capacitance $C$. $E_J \sim 50$ meV is the Josephson coupling energy of each junction\(^5\), $f$ is the external flux (in units of the flux quantum $\Phi_0 = h/2e$) threading the loop. In equation (1), the effective Josephson coupling $E_C^{\text{eff}} = 2E_J \cos(\pi f)$ of a phase qubit is controlled by the external flux $f$. A phase qubit typically operates in the range where $k_B T \ll E_C \ll E_J$, and uses as its computation basis the two lowest-energy states $|0\rangle$ and $|1\rangle$ confined in the potential well around $\varphi = \pi f$; see figure 1(b). Within the subspace spanned by the computational basis, the qubit Hamiltonian (1) can be written as

$$H_{\text{qubit}} \approx -\frac{1}{2} \Omega \sigma_z$$

where $\sigma_x$, $\sigma_y$, $\sigma_z$ are the Pauli matrices. The level splitting $\Omega$ can be estimated by $\Omega \approx \sqrt{2E_C E_J^{\text{eff}}} \approx 40$ $\mu$eV $\sim 2\pi \times 10$ GHz\(^6\).

The torsional vibration mode of the substrate is described by an harmonic oscillator Hamiltonian

$$H_{\text{osc}} = \frac{P^2}{2I} + \frac{1}{2} I \omega_0^2 \theta^2$$

where $P_\theta$ is the (angular) momentum conjugate to $\theta$, $I \sim 10^{-28}$–$10^{-32}$ kg m$^2$ is the rotational moment of inertia of the torsional resonator, and $\omega_0/2\pi \sim 8$–800 MHz is the vibration frequency. The fluctuations of the angle $\theta$ can be characterized by the parameter $\theta_0 \equiv \sqrt{\hbar/I \omega_0}$, which is the fluctuation in the ground state. In typical experimental situations $\theta_0 \sim 10^{-6}$–$10^{-7}$ rad depending on the values of $I$ and $\omega_0$.

### 3. Spin–resonator coupling

With the qubit put on the torsional resonator as in figure 1, the effective flux $f$ in the qubit Hamiltonian (1) is modulated as

$$f = f_0 \sin \theta$$

where $\theta$ is measured relative to the direction of the external magnetic field, and hence the qubit is coupled to the torsional vibration mode. We point out a key advantage of this qubit–resonator coupling scheme: as mentioned above, the phase qubit operates in the regime, where $E_J$ is the largest energy scale. The torsional vibration directly modulates this largest energy scale. This means that the coupling between the qubit and the torsional vibrational mode can be large, as demonstrated below.

If we apply the external magnetic field parallel to the phase qubit plane, then the flux modulation is given by $f = f_0(\theta - \theta_e)$. Here $f_0$ is the maximum magnetic flux (i.e. the value when the field is perpendicular to the qubit plane) and $\theta_e$ is the angle at equilibrium measured from the direction of external field. We assume that $\theta_e = 0$ (non-zero $\theta_e$ slightly decreases the coupling strength by a factor $\sin \theta_e$). Within the two-level approximation, the total Hamiltonian is given by

$$H = -\frac{1}{2} \omega \sigma_z + \frac{1}{2} g \sqrt{\Omega_0} \sigma_x + \omega_\theta \sigma_z$$

where $g$ is the dimensionless reduced coupling constant between the phase qubit and the torsional resonator. Note that the oscillator frequency has been slightly renormalized from $\omega_0$ to

$$\omega \equiv \omega_0 \sqrt{1 + 2(\pi f_0)^2 E_J/\omega_0^2}$$

due to the coupling to the phase qubit. The renormalization of the frequency $\omega_0 \rightarrow \omega$ also renormalizes the quantum fluctuation angle $\theta_0$ to

$$\theta_1 \equiv \sqrt{\hbar/1 \omega}.$$  

The coupling constant $g$ in this case is given by

$$g = \pi f_0 \sqrt{2E_J/\omega_0^2}.$$  

The effective Hamiltonian (4) is the well-known cavity-QED (quantum electrodynamics) Hamiltonian for the atom–light interaction in an optical cavity. For optical cavities, the two-level system (or ‘spin’) is at resonance with the oscillator.
and the coupling energy $g \sqrt{\Omega \omega}$ is $10^{-6}$ times smaller, at best, than $\omega$. In such cases, it is customary to make a so-called rotating-wave approximation (RWA), which leads to the Jaynes–Cummings model [20],

$$H \approx -\frac{i}{2} \Omega \sigma_z + \frac{1}{2} g \sqrt{\Omega \omega} (a \sigma_+ + a^\dagger \sigma_-) + \omega a^\dagger a$$

where $\sigma_{\pm} = (\sigma_x \pm i \sigma_y)/2$. The ground state of the Jaynes–Cummings model (8) does not exhibit any quantum correlation effects of particular interest. As the coupling energy $g \sqrt{\Omega \omega}$ increases ($g \gtrsim 10^{-3}$), however, the RWA breaks down, and the ground states start to exhibit strong quantum correlation effects such as squeezing of the oscillation mode (figure 2), as discussed below.

### 4. Strong coupling limits

Before we discuss the ‘strong’ coupling limit of the qubit–resonator composite system, we need to distinguish the limit from the conventional strong coupling limit. The effective qubit–resonator Hamiltonian (4) is an example of a more general class of spin-boson models, which are commonly achieved in optical cavities. Conventionally, for optical cavities, the strong coupling limit means the coupling constant $g$ is large enough for modes to be highly correlated with each other, and this coupling energy $E_J$ and the ratio $b/a$ of the lateral sizes of the phase qubit.

![Figure 2](image-url)

**Figure 2.** (a) Squeezing of the torsional resonator as a function of the reduced coupling strength $g$ for $\Omega = 10 \omega$ and $\Omega = 10^4 \omega$. (b) Squeezing of the torsional resonator as a function of the Josephson energy $E_J$ and the ratio $b/a$ of the lateral sizes of the phase qubit.
superconducting circuits [29], the theoretical limit is known to be \( g \sim 1 \) as well.) The strong coupling in our scheme is possible because the vibrational mode directly modulates the largest energy scale (\( E_1 \)) of the phase qubit. A similar idea has been explored in [8, 30–33] but using the flexural vibrations of nanomechanical beams. Note that even for moderate values of \( Q \)-factor, \( Q \sim 10^3 \) [26], the condition for the conventional strong coupling limit (\( g \sqrt{\Delta \omega} \gg \gamma \)) is easily satisfied in our case, \( g \sqrt{\Delta \omega} / \gamma \sim 10^3 \).

5. Noise analysis

It is important to calculate and compare the angle detection limit, for example that of an optical interferometer, and the fundamental limits due to thermal and quantum fluctuations. The deflection angle of a torsional resonator can be detected by measuring the displacement at the end of its wing with a fiber-optic interferometer. A low-power (~1 \( \mu \)W) laser light from a fiber is focused by a lens system to a micron spot on the backside of the resonator, reflected to couple back into the fiber, giving a displacement-sensitive signal.

For the following analysis, we consider a silicon resonator, coupled to a superconducting qubit, consisting of a \( 2 \times 2 \mu m^2 \) rectangular paddle suspended by narrow beams on both edges, which are 2 \( \mu m \) long with a (100) nm\(^2\) cross section. To minimize the heating effect due to the probing light, it may be necessary to deposit a high-purity silver thin film on the backside, which will play the role of an excellent reflector and a heat sink at a low temperature of 20 mK. From our estimation, a 50 nm-thick silver coating may reduce the temperature difference from a cryogenic bath down to 5 mK, while increasing a torsional spring constant by 7%. Recently, a Fabry–Perot (FP) interferometer with a miniaturized semi-focal cavity, developed for micrometer-sized cantilevers, was reported to have a remarkably small noise floor of 1 fm Hz\(^{-1/2}\) at 1 MHz with a decreasing tail at higher frequencies [34]. The effective noise bandwidth of the torsional oscillator of interest to us is estimated to be \( \Delta \omega = o/Q \sim 2\pi \times 2.4 \text{ kHz} \) if \( Q \approx 5000 \), a typical value for a micrometer-sized oscillator [27]. The angle detection limit for our oscillator is

\[
\Delta \theta_{FP} = 2 \sqrt{\frac{S_{FP} \Delta \omega}{2\pi d^2}} \sim 5 \times 10^{-8} \text{ rad},
\]

where \( d \approx 2 \mu m \) is the lateral size of the qubit and \( \sqrt{S_{FP}} \approx 1 \text{ fm Hz}^{-1/2} \) is the noise floor of the FP interferometer. The detection limit is comparable or even smaller than the quantum fluctuation angle \( \theta_q \sim 10^{-7} \text{ rad} \) (for \( I \sim 10^{-28} \text{ kg m}^2 \)) or larger, and enough to measure the quantum fluctuations.

The thermal fluctuation of angle originates from the thermal energy stored in the mechanical vibration energy of the torsional resonator, and can be estimated as

\[
\theta_T = \sqrt{k_B T / I \omega^2}.
\]

At an experimentally accessible low temperature of 20 mK and with \( I \sim 10^{-28} \text{ kg m}^2 \), for instance, the torsional resonator is predicted to vibrate up to \( \theta_T \approx 6.2 \times 10^{-7} \text{ rad} \). Therefore, the thermal fluctuation would be the main limitation to detecting the quantum vibration. The ratio of the thermal to quantum fluctuation is only \( \sim 7 \) at 20 mK, and can be improved even further by lowering the cryogenic temperature or by using optical or microwave cooling techniques [35]. The quantum temperature \( T_Q = \hbar \omega / k_B \) is a border where the torsional resonator enters the quantum regime, and yields 0.37 mK. So, the thermal occupation factor \( N = T / T_Q \) is \( \sim 60 \) when the torsional resonator is at a temperature of \( T = 20 \text{ mK} \), which can also be found from an experimentally observed fluctuation angle \( \theta_T \) by \( T = I \omega^2 \theta_T^2 / k_B \).

6. Conclusion

We have proposed a scheme of strong and tunable coupling between a superconducting phase qubit and a nanomechanical torsional resonator. The torsional resonator directly modulates the largest energy scale (the Josephson coupling energy) of the phase qubit, and the coupling strength is achievable. We have analyzed the quantum correlation effects in the torsional resonator as a result of the strong coupling to the phase qubit. We have also provided the noise analysis, which shows that our scheme is feasible experimentally at the level of present technology.

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References

[1] Huang X M H, Zorman C A, Mehregany M and Roukes M L 2003 Nature 421 496
[2] Knobel R G and Cleland A N 2003 Nature 424 291
[3] LaHaye M D, Buo O, Camarota B and Schwab K C 2004 Science 304 74
[4] Armour A D, Blencowe M P and Schwab K C 2002 Phys. Rev. Lett. 88 148301
[5] Irish E K and Schwab K 2003 Phys. Rev. B 68 155311
[6] Martin I, Shnirman A, Tian L and Zoller P 2004 Phys. Rev. B 69 125339
[7] Tian L 2005 Phys. Rev. B 72 195411
[8] Buks E and Blencowe M P 2006 Phys. Rev. B 74 174504
[9] Wei L F, Liu Y x, Sun C P and Nori F 2006 Phys. Rev. Lett. 97 237201
[10] Jacobs K, Lougovski P and Blencowe M 2007 Phys. Rev. Lett. 98 147201
[11] Hauss J, Fedorov A, Hutter C, Shnirman A and Schön G 2008 Phys. Rev. Lett. 100 037003
[12] Utami D W and Clerk A A 2008 Phys. Rev. A 78 042323
[13] LaHaye M D, Suh J, Eichermann P M, Schwab K C and Roukes M L 2009 Nature 459 960
[14] O’Connell A D, Hoheinz M, Ansmann M, Bialczak R C, Lenander M, Lucero E, Neeley M, Sank D, Wang H, Weides M, Wenner J, Martinis J M and Cleland A N 2010 Nature 464 697
[15] Rabl P, Shnirman A and Zoller P 2004 Phys. Rev. B 70 205304
[16] Cleland A N and Geller M R 2004 Phys. Rev. Lett. 93 070501
[17] Geller M R and Cleland A N 2005 Phys. Rev. A 71 032311
[18] Zhou X and Mizzel A 2006 Phys. Rev. Lett. 97 267201
[19] Blanch G 1972 Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables ed M Abramowitz and I A Stegun (New York: Wiley)
[20] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89
[21] Irish E K 2007 Phys. Rev. Lett. 99 173601
[22] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[23] Ashhab S and Nori F 2010 Phys. Rev. A 81 042311
[24] Hwang M-J and Choi M-S 2010 arXiv:1006.1989
[25] Gallop J, Josephs-Franks P W, Davies J, Hao L and Macfarlane J 2002 Physica C 368 109
[26] Evoy S, Carr D W, Sekaric L, Olkhovets A, Parpia J M and Craighead H G 1999 J. Appl. Phys. 86 6072
[27] Cleland A N and Roukes M L 1998 Nature 392 160
[28] Poole C P Jr (ed) 2000 Handbook of Superconductivity (San Diego, CA: Academic)
[29] Schuster D I, Houck A A, Schreier J A, Wallraff A, Gambetta J M, Blais A, Frunzio L, Majer J, Johnson B, Devoret M H, Girvin S M and Schoelkopf R J 2007 Nature 445 515
[30] Xue F, Liu Y-x, Sun C P and Nori F 2007 Phys. Rev. B 76 064305
[31] Etaki S, Poot M, Mahboob I, Onomitsu K, Yamaguchi H and van der Zant H S J 2008 Nat. Phys. 4 785
[32] Buks E, Segev E, Zaitsev S, Abdo B and Blencowe M P 2008 Europhys. Lett. 81 10001
[33] Pugnetti S, Blanter Y M, Dolcini F and Fazio R 2009 Phys. Rev. B 79 174516
[34] Hoogenboom B W, Frederix P L T M, Yang J L, Martin S, Pellmont Y, Steinacher M, Züch S, Langenbach E, Heimbeck H-J, Engel A and Hug H J 2005 Appl. Phys. Lett. 86 074101
[35] Rocheleau T, Ndukum T, Macklin C, Hertzberg J B, Clerk A A and Schwab K C 2010 Nature 463 72