A generalization of circulant Hadamard and conference matrices

Ondřej Turek
Nuclear Physics Institute, Academy of Sciences of the Czech Republic
250 68 Řež, Czech Republic
Bogolyubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research
141980 Dubna, Russia
Laboratory of Unified Quantum Devices, Kochi University of Technology
Kochi 782-8502, Japan
email: o.turek@ujf.cas.cz

Dardo Goyeneche
Faculty of Physics, Warsaw University, Pasteura 5, 02-093, Warsaw, Poland
Faculty of Applied Physics and Mathematics, Technical University of Gdańsk
80-233 Gdańsk, Poland
Institute of Physics, Jagiellonian University, Kraków, Poland
email: dgoyeneche@cefop.udec.cl

March 21, 2016

Abstract
We study the existence and construction of circulant matrices $C$ of order $n \geq 2$ with diagonal entries $d \geq 0$, off-diagonal entries $\pm 1$ and mutually orthogonal columns. Matrices $C$ with $d$ on the diagonal generalize circulant conference ($d = 0$) and circulant Hadamard ($d = 1$) matrices. We demonstrate that matrices $C$ exist for every order $n$ and for $d$ chosen such that $n = 2(d + 1)$, and we find all solutions $C$ with this property. Furthermore, we prove that if $C$ is symmetric, or $n - 1$ is prime, or $d$ is not an odd integer, then the relation $n = 2(d + 1)$ holds. Finally, we conjecture that $n = 2(d + 1)$ holds for any matrix $C$, which generalizes the circulant Hadamard conjecture. We support the conjecture by computing all the existing solutions up to $n = 50$. 
1 Introduction

A circulant matrix is a square matrix in which each row is obtained as a cyclic shift of the precedent row by one position to the right. That is, a circulant matrix of order \( n \) takes the form

\[
C = \begin{pmatrix}
  c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\
  c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
  \vdots & c_{n-1} & c_0 & \ddots & \vdots \\
  c_2 & \cdots & \cdots & c_1 \\
  c_1 & c_2 & \cdots & c_{n-1} & c_0
\end{pmatrix}.
\]

A circulant matrix is fully specified by its first row, \((c_0, c_1, \ldots, c_{n-1})\), which we call the generator of \( C \).

Let us consider two special types of real circulant matrices, namely

- **circulant Hadamard matrices**, defined by conditions \( c_j \in \{1, -1\} \) for \( j = 0, 1, \ldots, n-1 \) and \( CC^T = nI \) (the superscript \( T \) denotes transposition);

- **circulant conference matrices**, defined by conditions \( c_j \in \{1, -1\} \) for \( j = 1, \ldots, n-1 \), \( c_0 = 0 \) and \( CC^T = (n-1)I \).

The circulant Hadamard conjecture says that circulant Hadamard matrices exist only for \( n = 1 \) and \( n = 4 \). The conjecture is open already for over half a century: according to Schmidt [19], “the conjecture was first mentioned in Ryser’s book [18] (1963), but goes back further to obscure sources”. Turyn [24] proved in 1965 that \( n \) can only take values \( 4u^2 \) for an odd \( u \) and derived further necessary conditions on \( n \). Schmidt [19, 20] showed that the circulant Hadamard conjecture is true for orders up to \( n = 10^{11} \) with three possible exceptions. On top of these results, it is known that a circulant Hadamard matrix cannot be symmetric for \( n > 4 \) (Johnsen [14], Brualdi and Newman [6], McKay and Wang [17], Craigen and Kharaghani [9]).

By contrast, the problem of existence of circulant conference matrices is fully solved. Stanton and Mullin [22] demonstrated that circulant conference matrices only exist of order \( n = 2 \); later Craigen [8] proposed a simpler proof of this fact.

The two kinds of matrices described above serve as a main motivation for our paper. We are concerned with their common generalization, in which we allow the diagonal entries of the matrix \( C \) to take an arbitrary value \( d \in \mathbb{R} \). For the sake of convenience, we assume \( d \geq 0 \) without loss of generality, and we exclude the trivial case \( n = 1 \). The aim of our work is thus to study matrices \( C \) defined by the following conditions:

\[
\begin{align*}
C \text{ is a circulant matrix of order } n \geq 2 \text{ with generator } \,(c_0, c_1, \ldots, c_{n-1})\,; \\
c_j \in \{1, -1\} \quad \forall \, j = 1, \ldots, n-1; \\
c_0 = d \geq 0; \\
CC^T = (d^2 + n - 1)I. \;
\end{align*}
\]

Matrices \( C \) for \( d = 1 \) and \( d = 0 \) correspond to circulant Hadamard matrices and circulant conference matrices, respectively. In this paper we find all solutions of problem (2) for any value \( d \geq 0 \) that is not an odd integer. The case of \( d \) being odd involves the circulant Hadamard conjecture and is thus much harder; for that case we conjecture that all matrices obeying conditions (2) satisfy the relation \( n = 2(d + 1) \). We verify the conjecture up to \( n = 50 \).
There exists another generalization of circulant Hadamard and conference matrices called circulant weighing matrices. A weighing matrix $W$ of order $n$ and weight $k$ is an $n \times n$ matrix having entries from the set $\{0, 1, -1\}$ such that $WW^T = kI$. Circulant weighing matrices and their classification were studied by several authors, see works of Eades and Hain [10], Arasu et al. [3, 2], Ang et al. [1].

To the best of our knowledge, the problem (2) has not been studied before. However, similar parametric matrix problems without the circulancy assumption were already considered. Seberry and Lam [21] examined symmetric matrices with orthogonal columns having a constant $m$ on the diagonal and $\pm 1$ off the diagonal, and Lam [16] later extended the study to the non-symmetric case. Recently, Hermitian unitary matrices with $\pm d$ on the diagonal and unit complex numbers off the diagonal were studied in mathematical physics in relation to scattering in quantum graph vertices (Turek and Cheon [23], Kurasov and Ogik [15]).

Our matrices are also closely related to Barker codes. A Barker code is a finite sequence of $n$ numbers $\{c_k\}$ with $c_k \in \{-1, 1\}$ and $0 \leq k \leq n - 1$ such that it satisfies the equation $|\sum_{k=0}^{m-1} c_k c_{k+m}| \leq 1$ for every $0 \leq m \leq n - 1$. It has been proven that only eight Barker codes exist for length $n \leq 13$ [5], if we assume $c_0 = c_1 = 1$ without loss of generality. Furthermore, the existence of a Barker code of length $n > 13$ would imply that a circulant Hadamard matrix of size $n$ exists (see [4, Chapter VI, §14]). This means that Barker codes of length $n > 13$ necessarily imply perfect auto-correlation for the sequence. We say that an auto-correlation is perfect if $|\sum_{k=0}^{m-1} c_k c_{k+m} \mod n| = 0$ for every $1 \leq m \leq n - 1$. Sequences with low auto-correlation have a fundamental importance in radar signals theory [7], data transmission and data compression [12]. It is thus interesting to search for new finite sequences having perfect auto-correlation, in a similar way as Huffman generalized Barker codes [13]. With this aim, in the present work we define sequences having the first element $c_0 \geq 0$ with absolute value different than one in general, i.e., the sequence $\{c_k\}$ does not have all its elements with constant amplitude. This perturbation in the amplitude of the signal allows us to find interesting novel results for sequences of any length $n$. From the point of view of correlations of finite sequences the main result of our paper can be stated as follows: We find the complete set of sequences $\{c_k\}$ of length $n$ with $c_0 = n/2 - 1$, $c_k \in \{-1, 1\}$ for $1 \leq k \leq n - 1$ and having perfect auto-correlation. These sequences exist for every $n \geq 2$. Furthermore, we conjecture that every finite sequence of length $n$, with $c_0 \geq 0$, $c_k \in \{-1, 1\}$ for $1 \leq k \leq n - 1$ and having perfect correlation satisfies $c_0 = n/2 - 1$. If this conjecture is true, then Barker codes of length $n > 13$ do not exist.

The paper is organized as follows. In Section 2 we review basic properties of matrices $C$ satisfying conditions (2). In particular, we prove that a matrix $C$ of order $n$ with diagonal entries $d$ exists only if $n \geq 2(d + 1)$. In Section 3 we derive further necessary conditions and bring in additional results obtained by a computer calculation. On the basis of our findings, we formulate a conjecture that extends the circulant Hadamard conjecture. In Section 4 we prove that a symmetric matrix $C$ with diagonal entries $d$ exists if and only if $n = 2(d + 1)$. In Section 5 we find all matrices $C$ that obey conditions (2) and have the property $n = 2(d + 1)$. Finally, we summarize the most important results in Section 6.

2 Preliminaries

Let $C$ be a circulant matrix of order $n$. Circulant matrices are diagonalized by the discrete Fourier transform. The vectors

$$v_k = \frac{1}{\sqrt{n}} (1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})^T,$$

where
where \( \omega = e^{2\pi i/n} \), are normalized eigenvectors of \( C \) for all \( k = 0,1,\ldots,n-1 \). If the matrix \( C \) has generator \((c_0,c_1,\ldots,c_{n-1})\), then the corresponding eigenvalues of \( C \) are
\[
\lambda_k = c_0 + c_1\omega^k + c_2\omega^{2k} + \cdots + c_{n-1}\omega^{(n-1)k}.
\] (3)

From now on we focus on circulant matrices \( C \) with generator \((c_0,c_1,\ldots,c_{n-1})\) satisfying conditions \(2\). For the sake of convenience, we will adopt the following convention.

**Convention 2.1.** The rows and columns of \( C \) are indexed from 0 to \( n-1 \), i.e., they will be refered to as 0th, 1st,\ldots, \((n-1)\)th.

In the rest of the section we prove two useful propositions. The first one relates the generator of \( C \) to the diagonal \( d \) and the order \( n \).

**Proposition 2.2.** If \( C \) satisfies conditions \(2\), then
\[
\left| \sum_{j=0}^{n-1} c_j \right| = \sqrt{d^2 + n-1}.
\]
(4)

Moreover, if \( n \) is even, then
\[
\left| \sum_{j=0}^{n-1} (-1)^j c_j \right| = \sqrt{d^2 + n-1}.
\]
(5)

**Proof.** The assumption \( CC^T = (d^2 + n-1)I \) implies that the eigenvalues \( \lambda_k \) of \( C \), given by equation (3), obey \( |\lambda_k| = \sqrt{d^2 + n-1} \) for all \( k = 0,1,\ldots,n-1 \). In the special case \( k = 0 \) we obtain equation (4). If \( n \) is even, then \( k = \frac{n}{2} \) leads to equation (5). \(\square\)

The following proposition gives a basic restriction on the diagonal \( d \).

**Proposition 2.3.** Let \( C \) satisfy conditions \(2\).

(i) If \( n \) is even, then \( d \leq \frac{n}{2} - 1 \) and \( d \equiv \frac{n}{2} - 1 \ (\text{mod} \ 2) \).

(ii) If \( n \) is odd, then \( d = \frac{n}{2} - 1 \) and the generator of \( C \) has the form \((d,-1,-1,\ldots,-1)\).

**Proof.** (i) If \( n \) is even, the orthogonality between the 0th and the \( \frac{n}{2} \)th row of \( C \) (recall that we use the numbering introduced in Convention 2.1) implies that
\[
2dc_{\frac{n}{2}} + 2 \sum_{j=1}^{\frac{n}{2}-1} c_j c_{\frac{n}{2}+j} = 0.
\]
We have \( c_j \in \{1,-1\} \) and \( d \geq 0 \); hence
\[
d = \left| \sum_{j=1}^{\frac{n}{2}-1} c_j c_{\frac{n}{2}+j} \right|.
\]
(6)

Since \( c_j c_{\frac{n}{2}+j} \in \{1,-1\} \) for all \( j = 1,\ldots,\frac{n}{2} - 1 \), the expression on the right hand side of equation (6) is an integer less than or equal to \( \frac{n}{2} - 1 \) and congruent to \( \frac{n}{2} - 1 \) modulo 2.
(ii) Let $n$ be odd. For any $k \in \{1, \ldots, n-1\}$, the scalar product of the 0th and $k$th row of $C$ must be zero, i.e.,

$$d(c_k + c_{n-k}) + \sum_{j=1, j \neq n-k}^{n-1} c_j c_{(j+k) \mod n} = 0. \quad (7)$$

The sum $\sum c_j c_{(j+k) \mod n}$ on the left hand side contains an odd number of summands of type $\pm 1$; therefore, it has a nonzero value. Equation $(7)$ thus cannot be satisfied unless $c_k + c_{n-k} \neq 0$ for all $k = 1, \ldots, n-1$. With regard to $c_j \in \{1, -1\}$, we conclude that $C$ must be symmetric.

Consider an arbitrary $k \in \{1, \ldots, n-1\}$ and denote $c_k = c_{n-k} = \gamma$. We write down the 0th and $k$th row of $C$ and rearrange the columns in the following way:

$$
\begin{array}{cccccccc}
  d & \gamma & +1 & \cdots & +1 & +1 & \cdots & +1 \\
\gamma & d & +1 & \cdots & +1 & -1 & \cdots & -1
\end{array}
\begin{array}{cccc}
\ell_1 \\
\ell_2 \\
\ell_3 \\
\ell_4
\end{array}
$$

We obviously have

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 = n - 2. \quad (8)$$

Since $C$ is circulant, every row of $C$ has the same sum of elements, i.e.,

$$d + \gamma + \ell_1 + \ell_2 - \ell_3 - \ell_4 = \gamma + d + \ell_1 - \ell_2 + \ell_3 - \ell_4. \quad (9)$$

Since $C$ is orthogonal, the scalar product of the 0th and the $k$th row must be 0; hence

$$2\gamma d + \ell_1 - \ell_2 - \ell_3 + \ell_4 = 0. \quad (10)$$

The system of equations $(8)$–$(10)$ implies

$$4\ell_2 = n - 2 + 2\gamma d. \quad (11)$$

Consequently,

$$n - 2 + 2\gamma d \equiv 0 \pmod{4}. \quad (11)$$

If $c_j = 1$ for all $j = 1, \ldots, n-1$, then the rows of $C$ would not be orthogonal. Therefore, there must exist a $k$ such that $c_k = c_{n-k} = -1$. Equation $(11)$ for $\gamma = -1$ leads to

$$n - 2 - 2d \equiv 0 \pmod{4}. \quad (12)$$

If there was also a $k'$ such that $c_{k'} = c_{n-k'} = +1$, then, with regard to equation $(11)$, we would have one more equation, namely,

$$n - 2 + 2d \equiv 0 \pmod{4}. \quad (12)$$

This equation together with equation $(12)$ implies $2n - 4 \equiv 0 \pmod{4}$, which is obviously in contradiction with the assumption that $n$ is odd. We conclude that $c_j = -1$ for all $j = 1, \ldots, n-1$, i.e., the generator of $C$ is $(d, -1, -1, \ldots, -1)$. The scalar product of any two rows of such matrix $C$ is equal to $-2d + n - 2$. This quantity must be 0 due to the orthogonality. Hence we obtain $d = \frac{n}{2} - 1$. \qed
3 Relations between the order $n$ and the diagonal $d$

As we have seen in Proposition 2.3, if a matrix $C$ satisfies conditions (2), then its order $n$ and the value $d$ on the diagonal are related. The aim of this section is to derive further restrictions that the pair $(n, d)$ has to obey. We start from a statement that follows straightforwardly from Proposition 2.3. The symbol $\mathbb{N}_0$ used in the text denotes the set of non-negative integers.

**Proposition 3.1.** If a matrix $C$ satisfies conditions (2), then $2d$ is an integer and $n \geq 2(d+1)$. Moreover, we have:

- If $d$ is an integer, then $n$ is even and the equivalence ($d$ is odd $\iff \frac{n}{2} - 1$ is odd).
- If $d$ is a half-integer, then $n$ is odd and $n = 2(d+1)$.

**Proof.** Proposition 2.3 implies that $2d$ is an integer and ($d$ is integer $\iff n$ is even). We distinguish two cases.

If $d$ is an integer, $n$ is even. Then Proposition 2.3 (i) gives the inequality $n \geq 2(d+1)$ and the equivalence ($d$ is odd $\iff \frac{n}{2} - 1$ is odd).

If $d$ is a half-integer, then $n$ is odd and we use Proposition 2.3 (ii) to infer that $n = 2(d+1)$.

**Remark 3.2.** A matrix $C$ obeying conditions (2) exists for every $d \geq 0$ such that $2d$ is an integer. To see the validity of the statement, it suffices to consider the generator $(d, -1, -1, \ldots, -1) \in \mathbb{R}^n$ for $n = 2(d+1)$.

Proposition 3.3 below gives a useful necessary condition of the existence of matrices $C$ for the case when $d$ is an integer.

**Proposition 3.3.** If a matrix $C$ satisfies assumptions (2) and $d \in \mathbb{N}_0$, then

$$n = k(2d + k) + 1$$

for a certain $k \in \mathbb{N}$.

**Proof.** From equation (4) we have

$$|d + c_1 + \cdots + c_{n-1}| = \sqrt{d^2 + n - 1}. \tag{14}$$

Since $c_j \in \{1, -1\}$ for all $j = 1, \ldots, n-1$, the left hand side of equation (14) is an integer. Therefore, there exists a $k \in \mathbb{Z}$ such that $|d + c_1 + \cdots + c_{n-1}| = d + k$. With regard to the right hand side of equation (14), $k$ is positive. To sum up, we have $d + k = \sqrt{d^2 + n - 1}$ for a certain $k \in \mathbb{N}$. Hence we obtain equation (13).

**Corollary 3.4.** If a matrix $C$ of order $n$ satisfies assumptions (2) and $n-1$ is a prime number, then $d = \frac{n}{2} - 1$.

**Proof.** If $n-1 = 2$, then $n = 3$ is odd and the statement follows from Proposition 2.3 (ii). From now on let $n-1$ be an odd prime. Since $n$ is even, Proposition 2.3 (i) implies $d \in \mathbb{N}_0$. We can thus use Proposition 3.3. According to equation (13), $d = \frac{1}{2} \left( \frac{n-1}{k} - k \right)$ for a certain $k$ that divides $n-1$. Since $n-1$ is a prime number, we have $k = 1$, which leads to $d = \frac{n}{2} - 1$.

We are going to demonstrate that matrices satisfying conditions (2) for even values $d$ on the diagonal exist only of orders $n = 2(d+1)$. We derive at first three auxiliary results on matrices $C$ of orders $n \equiv 2 \pmod{4}$.
Proposition 3.5. If $C$ satisfies conditions (2) and $n \equiv 2 \pmod{4}$, then $C$ is symmetric.

Proof. We prove the statement by contradiction. Assume that $n \equiv 2 \pmod{4}$ and $C$ is not symmetric. Then there exists a $j \in \{1, \ldots, n-1\}$ such that $c_j = 1$ and $c_{n-j} = -1$. We write down the 0th row and the $j$th row of $C$ and rearrange the columns as follows.

\[
\begin{array}{cccccccc}
  d & +1 & +1 & \cdots & +1 & -1 & \cdots & -1 \\
-1 & d & +1 & \cdots & +1 & -1 & \cdots & -1 \\
  \ell_1 & & & & & & & \\
  \ell_2 & & & & & & & \\
  \ell_3 & & & & & & & \\
  \ell_4 & & & & & & & \\
\end{array}
\]

We have

\[\ell_1 + \ell_2 + \ell_3 + \ell_4 = n - 2.\]  \hspace{1cm} (15)

Furthermore, all rows of $C$ have the same sum of elements, thus

\[d + 1 + \ell_1 + \ell_2 - \ell_3 - \ell_4 = -1 + d + \ell_1 - \ell_2 + \ell_3 - \ell_4.\]  \hspace{1cm} (16)

Finally, the scalar product of the two rows must be 0, i.e.,

\[\ell_1 - \ell_2 - \ell_3 + \ell_4 = 0.\]  \hspace{1cm} (17)

Solving the system of equations (15)–(17), we get in particular

\[\ell_2 = \frac{n}{4} - 1.\]

Consequently, $n$ is a multiple of 4. This contradicts the assumption $n \equiv 2 \pmod{4}$. \hfill \square

Proposition 3.6. If $C$ satisfies conditions (2) and $n \equiv 2 \pmod{4}$, then $d \equiv \frac{n}{2} - 1 \pmod{4}$.

Proof. According to Proposition 3.5, the matrix $C$ is symmetric. Hence $c_{n-j} = c_j$ for all $j \in \{1, \ldots, n-1\}$. Therefore, the 0th and the $\frac{n}{2}$th row of $C$ read

\[
\begin{array}{ccccccc}
d & c_1 & c_2 & \cdots & c_{\frac{n}{2}-1} & c_{\frac{n}{2}} & \cdots & c_1 \\
& c_{\frac{n}{2}} & c_{\frac{n}{2}-1} & c_{\frac{n}{2}-2} & \cdots & c_1 & d & c_1 & \cdots & c_{\frac{n}{2}-2} & c_{\frac{n}{2}-1} \\
\end{array}
\]

The two rows must be orthogonal, which implies

\[2dc_{\frac{n}{2}} + 2 \sum_{j=1}^{\frac{n}{2}-1} c_j c_{\frac{n}{2}-j} = 0.\]

Since $c_{\frac{n}{2}} \in \{1, -1\}$ and $d \geq 0$, we get

\[d = \left| \sum_{j=1}^{\frac{n}{2}-1} c_j c_{\frac{n}{2}-j} \right|.\]  \hspace{1cm} (18)

The assumption $n \equiv 2 \pmod{4}$ gives the identity

\[\sum_{j=1}^{\frac{n}{2}-1} c_j c_{\frac{n}{2}-j} = 2 \sum_{j=1}^{\frac{n-2}{2}} c_j c_{\frac{n}{2}-j};\]

therefore, we can rewrite equation (18) in the form

\[d = \left| \sum_{j=1}^{\frac{n-2}{2}} c_j c_{\frac{n}{2}-j} \right|.\]

Since the sum on the right hand side has the same parity as the number $\frac{n-2}{4}$, we have $\frac{d}{2} \equiv \frac{n-2}{4} \pmod{2}$. Hence we get trivially $d \equiv \frac{n}{2} - 1 \pmod{4}$. \hfill \square
Proposition 3.7. If $C$ satisfies conditions (2) and $n \equiv 2 \pmod{4}$, then
\[
\left( \sum_{j=1}^{\frac{n}{2}} c_{2j-1} \right)^2 = d^2 + n - 1.
\] (19)

Proof. Let us denote $a = \sum_{j=0}^{\frac{n}{2}-1} c_{2j}$, $b = \sum_{j=1}^{\frac{n}{2}} c_{2j-1}$. Equations (11) and (15) imply
\[
(a + b)^2 = d^2 + n - 1, \quad (a - b)^2 = d^2 + n - 1.
\]

Hence we get an alternative
\[
(a = 0 \land b^2 = d^2 + n - 1) \lor (b = 0 \land a^2 = d^2 + n - 1).
\]

Since $b$ consists of $\frac{n}{2}$ summands of type $\pm 1$ and $\frac{n}{2}$ is odd due to the assumption $n \equiv 2 \pmod{4}$, we have $b \neq 0$. Therefore, $a = 0$ and $b^2 = d^2 + n - 1$, as we set to prove.

Now we are ready to prove a result on matrices $C$ with the diagonal entries $d$ equal to an even integer. Theorem 3.8 below generalizes a theorem of Stanton and Mullin [22] which says that a circulant conference matrix exists only for $n = 2$. The idea of the proof is based on [22].

Theorem 3.8. If a matrix $C$ satisfies conditions (2) and $d$ is an even integer, then $n = 2(d + 1)$.

Proof. First of all we realize that if $d$ is even, then $n \equiv 2 \pmod{4}$ according to Proposition 3.1. Then we have that $C$ is symmetric due to Proposition 3.5. The 0th row of $C$ and the $\ell$th row for $\ell \in \{1, \ldots, \frac{n}{2} - 1\}$ thus take the form
\[
d \quad c_1 \quad \ldots \quad c_{\ell-1} \quad c_{\ell} \quad c_{\ell+1} \quad \ldots \quad c_{\frac{n}{2}} \quad c_{\frac{n}{2} - \ell + 1} \quad \ldots \quad c_{\frac{n}{2} - \ell} \quad \ldots \quad c_1.
\]

Their scalar product shall be zero, i.e.,
\[
2d c_\ell + \sum_{j=1}^{\ell-1} c_j c_{\ell-j} + 2 \sum_{j=1}^{\frac{n}{2} - \ell} c_j c_{j+\ell} + \sum_{j=\frac{n}{2} - \ell + 1}^{\frac{n}{2} - 1} c_j c_{n-\ell-j} = 0.
\] (20)

From now on let $\ell$ be odd. We have $\ell = 2h + 1$ for a certain $h$, and
\[
\sum_{j=1}^{\ell-1} c_j c_{\ell-j} = 2 \sum_{j=1}^{h} c_j c_{\ell-j};
\]
\[
\sum_{j=\frac{n}{2} - \ell + 1}^{\frac{n}{2} - 1} c_j c_{n-\ell-j} = 2 \sum_{j=\frac{n}{2} - h}^{\frac{n}{2} - 1} c_j c_{n-\ell-j}.
\]

With regard to these two identities, equation (20) implies
\[
d = \left| \sum_{j=1}^{\frac{n}{2} - \ell} c_j c_{\ell+j} + \sum_{j=1}^{h} c_j c_{\ell-j} + \sum_{j=\frac{n}{2} - h}^{\frac{n}{2} - 1} c_j c_{n-\ell-j} \right|.
\] (21)

Let us denote the sum appearing on the right hand side of equation (21) by $S$, i.e.,
\[
S := \sum_{j=1}^{\frac{n}{2} - \ell} c_j c_{\ell+j} + \sum_{j=1}^{h} c_j c_{\ell-j} + \sum_{j=\frac{n}{2} - h}^{\frac{n}{2} - 1} c_j c_{n-\ell-j}.
\] (22)
The sum $S$ consists of products $c_ic_j$ for $i,j \in \{1,\ldots,\frac{n}{2}\}$. It is easy to see that each product $c_ic_j$ for $i,j \in \{1,\ldots,\frac{n}{2}\}$ occurs at most once in $S$. Let us define a graph $G = (V,E)$ with the set of vertices $V = \{1,\ldots,\frac{n}{2}\}$ and the set of edges $E$ given by the condition

$$\{i,j\} \in E \iff \text{the product } c_ic_j \text{ is a summand of } S.$$ 

The graph $G$ has the following properties: (i) Vertex $\ell$ is incident with only one edge in $E$; (ii) vertex $\frac{n}{2}$ is incident with only one edge in $E$; (iii) each vertex in the set $\{1,\ldots,\frac{n}{2}-1\}\{\ell\}$ is incident with two edges in $E$. The properties are obvious from the following facts:

- If $j \in [\ell+1,\frac{n}{2}-%], then the factor $c_j$ occurs only in summands of $S_1$, namely, in the products $c_jc_{j+\ell}$ and $c_{j-\ell}c_j$.
- If $j \in [1,\ell-1]$, then the factor $c_j$ occurs once in $S_1$ in the product $c_jc_{j+\ell}$ and once in $S_2$ in the product $c_{j+\ell}c_j$.
- If $j \in [\frac{n}{2}-\ell+1,\frac{n}{2}-1]$, then the factor $c_j$ occurs once in $S_1$ in the product $c_{j-\ell}c_j$ and once in $S_3$ in the product $c_jc_{n-\ell-j}$.
- The factor $c_{\ell}$ occurs only in the sum $S_1$, namely, in the product $c_{\ell}c_{2\ell}$.
- The factor $c_{\frac{n}{2}}$ occurs only in the sum $S_1$, namely, in the product $c_{\frac{n}{2}}c_{2\ell}$.

Properties (i)–(iii) imply that the graph $G$ consists of connected components of two types:

- a simple path $P = (v_0,v_1,\ldots,v_L)$ with $v_0 = \ell$ and $v_L = \frac{n}{2}$;
- a certain number (possibly zero) of simple cycles $R_k = (v_0^{(k)},v_1^{(k)},\ldots,v_{L_k}^{(k)})$ with $v_0^{(k)} = v_{L_k}^{(k)}$, where $k \in K$. Elements of $K$ index the set of simple cycles in $G$. If the graph $G$ is connected, then $G$ consists of the simple path $P$ and the set $K$ is empty.

The lengths $L$ and $L_k$, as well as the number of simple cycles (the cardinality of $K$) are not important for our considerations.

Since the summands of $S$ represent the edges of $G$, we can rearrange them to follow the order of edges on the path $P$ and on the cycles $R_k$,

$$S = \sum_{i=0}^{L-1} c_{v_i}c_{v_{i+1}} + \sum_{k \in K} \sum_{i=0}^{L_k-1} c_{v_i^{(k)}}c_{v_{i+1}^{(k)}}. \quad (23)$$

The sum $S$ contains $\frac{n}{2}-1$ terms of type $\pm1$, cf. equation (22). Therefore, $S = \frac{n}{2} - 1 - 2s$, where $s$ is the total number of negative summands in $S$. Moreover, $S$ is an even integer, because $\frac{n}{2} - 1$ is even due to the assumption $n \equiv 2 \pmod{4}$. Equation (21) says that $d = |S|$, i.e.,

$$d = \left| \frac{n}{2} - 1 - 2s \right|. \quad (24)$$

The left hand side of equation (24) satisfies $d \equiv \frac{n}{2} - 1 \pmod{4}$ according to Proposition 3.6. The right hand side of equation (24) is an even integer; hence obviously $|\frac{n}{2} - 1 - 2s| \equiv \frac{n}{2} - 1 - 2s$ (mod 4). Combining these two facts, we obtain $\frac{n}{2} - 1 \equiv \frac{n}{2} - 1 - 2s$ (mod 4), i.e., $2s \equiv 0 \pmod{4}$. This means that $s$ is even, i.e., the sum $S$ must contain an even number of negative summands.
Equation (23) implies that the number of negative summands in $S$ is equal to the number of sign changes in the sequence $c_{v_0}, \ldots, c_{v_L}$ plus the number of sign changes in all the sequences $c_{v_0^{(k)}}, \ldots, c_{v_L^{(k)}}$ for $k \in K$. Since $v_0^{(k)} = v_L^{(k)}$ for each $k$ (recall that $R_k$ is a cycle), each sequence $c_{v_0^{(k)}}, \ldots, c_{v_L^{(k)}}$ contains an even number of sign changes. Therefore, there must be an even number of sign changes in the sequence $c_{v_1}, \ldots, c_{v_L}$ as well. This requirement is equivalent to $c_{v_0} = c_{v_L}$. We have $v_0 = \ell$ and $v_L = \frac{n}{2}$, whence we get the condition

$$c_\ell = c_{\frac{n}{2}}.$$  \hfill (25)

Equation (25) is valid for any odd number $\ell = 1, 3, \ldots, \frac{n}{2} - 2$. The symmetry of $C$ means $c_i = c_{n-i}$ for all $i = 1, \ldots, \frac{n}{2}$; therefore, equation (25) is satisfied also for $\ell = \frac{n}{2} + 2, \ldots, n - 3, n - 1$. Consequently,

$$\sum_{j=1}^{\frac{n}{2}} c_{2j-1} = \frac{n}{2} c_{\frac{n}{2}}.$$  \hfill (26)

At the same time we have, due to Proposition 3.7, \hfill (31)

$$\left(\sum_{j=1}^{\frac{n}{2}} c_{2j-1}\right)^2 = d^2 + n - 1.$$  \hfill (27)

Equations (26) and (27) imply $d^2 + n - 1 = \left(\frac{n}{2}\right)^2$; hence $d = \frac{n}{2} - 1$. \hfill \qed

Recall that whenever $2d \in \mathbb{N}_0$, there exists a matrix $C$ of order $n = 2(d+1)$ that satisfies conditions (2) for that value of $d$ (cf. Rem. 3.2), and the matrix can be chosen symmetric. As we will see in Section 3 matrices $C$ of order $n = 2(d+1)$ can be fully characterized. The situation is, however, very different for matrices $C$ of order $n > 2(d+1)$. Proposition 3.1 and Theorem 3.8 disprove their existence for any $d$ being a non-integer or an even integer, respectively, and Proposition 3.3 poses further restrictions on $n$. For example, up to $n = 50$, all pairs $(n, d)$ with $n > 2(d+1)$ are excluded except for \hfill (36)

$$(16, 1), \ (28, 3), \ (36, 1), \ (40, 5).$$

We carried out a computer calculation, which confirmed that there is no solution for any of the pairs $(n, d)$ in the above list. In other words, up to the order $n = 50$ there is no matrix $C$ obeying conditions (2) with $n \neq 2(d+1)$.

Our findings lead us to establishing the following conjecture.

**Conjecture 3.9.** A circulant matrix $C$ of order $n \geq 2$ having the generator $(d, c_1, \ldots, c_{n-1})$ with $d \geq 0$ and $c_j \in \{1, -1\}$ for all $j = 1, \ldots, n-1$ satisfies the condition $CC^T = (d^2 + n - 1)I$ only if $n = 2(d+1)$.

**Remark 3.10.** Let us summarize facts concerning the validity of Conjecture 3.9.

- We proved the conjecture in situations when $d$ is a half-integer (cf. Prop. 3.1), as well as when $d$ is even (cf. Thm. 3.8).
- If $2d$ is not an integer, the conjecture remains valid as well. Indeed, formula $n = 2(d+1)$ gives an $n \notin \mathbb{N}$ in this case, implying that a matrix $C$ with diagonal $d$ does not exist, which is consistent with the statement of Proposition 3.1.
- As a result of performed computer calculations, the conjecture is confirmed for matrices $C$ of orders up to $n = 50$. 


• Conjecture 3.9 generalizes the circulant Hadamard conjecture. Indeed, the relation \( n = 2(d + 1) \) applied on the special case \( d = 1 \) means that circulant Hadamard matrices of order \( n \geq 2 \) exist only for \( n = 4 \).

4 Symmetric solutions

In this section we prove that if a matrix \( C \) satisfying conditions (2) is symmetric, then the order \( n \geq 2 \) of \( C \) is related to the value \( d \) on its diagonal by equation \( n = 2(d+1) \). Our result generalizes the well-known theorem about the nonexistence of symmetric circulant Hadamard matrices of order \( n > 4 \).

Proposition 4.1. If a matrix \( C \) satisfies assumptions (2) for an odd \( d \) and \( C \) is symmetric, then \( d^2 - 1 \) is divisible by \( 2\sqrt{d^2 + n - 1} \).

Proof. Since \( d \) is an odd integer, \( n \) is even due to Proposition 3.1. Equation (1) then implies that \( \sqrt{d^2 + n - 1} \) is an even integer; let us denote this integer by \( \ell \). If \( C \) is symmetric, it has a generator \((d, c_1, \ldots, c_{\frac{d}{2}}, c_{\frac{d}{2}-1}, \ldots, c_1)\). Following an idea from [8] proof of Thm. 8, let us consider a circulant matrix \( M \) with the generator \((c_{\frac{d}{2}}, c_{\frac{d}{2}-1}, \ldots, c_1, d, c_1, \ldots, c_{\frac{d}{2}-1})\). The matrix \( M \) is obviously symmetric. Furthermore, since \( CC^T = (d^2 + n - 1)I \) and \( M = PC \) for a certain permutation matrix \( P \), we have \( MM^T = (d^2 + n - 1)I \). Therefore, \( M \) has eigenvalues \( \pm \ell \) for \( \ell = \sqrt{d^2 + n - 1} \). If we denote the multiplicity of the eigenvalue \( +\ell \) of \( M \) by \( m \), the sum of eigenvalues of \( M \) is \( 2 (m - \frac{1}{2}) \ell \). At the same time the sum of eigenvalues of \( M \) is equal to \( \text{Tr}(M) = nc_{\frac{d}{2}} \). Comparing these quantities, we obtain \( 2\ell \mid n \). Now we express \( n \) in terms of \( \ell \), i.e., \( n = \ell^2 + 1 - d^2 \). Since \( \ell \) is even, we have \( 2\ell \mid \ell^2 \). This allows us to transform the condition \( 2\ell \mid (\ell^2 + 1 - d^2) \) into \( 2\ell \mid (d^2 - 1) \).

Example 4.2. Proposition 4.1 implies that a symmetric matrix \( C \) satisfying (2) with \( d = 3 \) exists only for \( n = 8 \). Indeed, \( 2\sqrt{3^2 + 8 - 1} \mid (3^2 - 1) \) requires \( \sqrt{8 + n} = 4 \); hence \( n = 8 \).

Now we will take advantage of an idea of McKay and Wang [17], which they used for disproving the existence of symmetric Hadamard matrices of order \( n > 4 \). Namely, they found a strong inequality that relates the order \( n \) of a symmetric circulant Hadamard matrix with the prime factorization of \( n \). Following their approach, we prove a similar inequality valid for matrices \( C \) with a general \( d \in \mathbb{N} \) on the diagonal. The inequality, derived in Proposition 4.3 below, relates the prime factorization of \( n \) to the integer \( k \) appearing in formula (13). Recall that according to Proposition 3.3 matrices \( C \) satisfying conditions (2) can be only of orders \( n = k(2d + k) + 1 \), where \( d \) is the value on the diagonal and \( k \in \mathbb{N} \).

Proposition 4.3. Let a symmetric matrix \( C \) satisfy assumptions (2) with \( d \in \mathbb{N} \) and \( n = k(2d + k) + 1 \) for a certain \( k \in \mathbb{N} \). Let \( n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \) be the prime factorization of \( n \). Then

\[
k + 1 \leq 2^r.
\]

Proof. We will proceed in the same way as McKay and Wang did in [17] Proof of Thm. 3], with just minor modifications that are required with regard to the generality of \( d \).

The first step consists in proving the implication

\[
\gcd(j, n) = m \implies c_j = c_m.
\]

For each \( m \mid n \) we define the polynomial

\[
P_m(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} - \lambda_m,
\]
where \( \lambda_m \) are eigenvalues of \( C \), given by formula \( \text{(3.9)} \). Since \( C = CT \) and \( CCT = (d^2 + n - 1)I \), the eigenvalues of \( C \) satisfy \( \lambda_m = \pm \sqrt{d^2 + n - 1} \). The assumption \( n = k(2d + k) + 1 \) for a certain \( k \in \mathbb{N} \) then gives \( \lambda_m = \pm (d + k) \in \mathbb{Z} \) for all \( m \). Therefore, the polynomial \( P_m(x) \) has integer coefficients for each \( m \). Furthermore, \( P_m(\omega^m) = \lambda_m - \lambda_m = 0 \) for every \( m \), where \( \omega = e^{2\pi i/n} \).

Let \( \Phi_N(x) \) denote the \( N \)th cyclotomic polynomial. Then \( \Phi_N(e^{2\pi iK/N}) = 0 \) for every \( K \in \{0, 1, \ldots, N-1\} \) satisfying \( \gcd(K, N) = 1 \). If we set \( N = \frac{m}{d} \) and \( K = \frac{1}{d} \), we get \( \Phi_m(\omega^m) = 0 \). Since \( P_m(\omega^m) = 0 \) and \( \Phi_N(x) \) is irreducible by definition, necessarily \( \Phi_m(x) \mid P_m(x) \).

The fact \( \Phi_m(x) \mid P_m(x) \) implies that \( P_m(x) = 0 \) whenever \( \Phi_m(x) = 0 \). From now on let \( \gcd(j, n) = m \).

If we set \( N = \frac{m}{d} \) and \( K = \frac{j}{d} \), we have \( \gcd(K, N) = \frac{1}{d} \gcd(j, n) = 1 \). Therefore, \( \Phi_m(e^{2\pi ij/n}) = 0 \). Hence we infer \( P_m(e^{2\pi ij/n}) = 0 \). This means \( \lambda_j - \lambda_m = 0 \), i.e., \( \lambda_j = \lambda_m \).

Since the vector \( (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})^T \) is obtained by the discrete Fourier transform (DFT) of \( (c_0, c_1, \ldots, c_{n-1})^T \) and DFT is an invertible transformation, we have \( \lambda_j = \lambda_m \Rightarrow c_j = c_m \). We conclude: If \( \gcd(j, n) = m \), then \( c_j = c_m \).

Let us proceed to the second step. We keep following the approach of McKay and Wang, slightly modified to fit our problem. Equation \( \text{(3.9)} \) together with equation \( \text{(3.10)} \) allows us to express the eigenvalue \( \lambda_1 \) of \( C \) in the form

\[
\lambda_1 = c_0 + \sum_{j=1}^{n-1} c_j \omega^j = c_0 + \sum_{m|n} c_m \left( \sum_{\gcd(j, n) = m} \omega^j \right),
\]

where \( c_0 = d \). We have

\[
\sum_{1 \leq m \leq n-1} m \omega^j = \sum_{\gcd(j, n) = m} \omega^j \sum_{1 \leq q \leq \frac{n}{m}} \omega^{(\frac{n}{m})q},
\]

which is the sum of primitive \( \frac{n}{m} \)th roots of unity. According to a classical formula (cf. \[14\] eq. (16.6.4))], this sum is equal to \( \mu\left(\frac{n}{m}\right) \), where \( \mu \) is the Möbius function. Therefore,

\[
\lambda_1 = d + \sum_{1 \leq m \leq n-1} c_m \mu\left(\frac{n}{m}\right).
\]

Since \( \mu(1) = 1 \), we can rewrite the equation in the form

\[
\lambda_1 = d - 1 + \mu(1) + \sum_{m|n} c_m \mu\left(\frac{n}{m}\right). \tag{30}
\]

We have \( |\lambda_1| = d + k, d \in \mathbb{N} \) and \( |c_j| = 1 \) for all \( j \geq 1 \). Therefore, equation \( \text{(30)} \) implies

\[
d + k \leq d - 1 + \sum_{m|n} \left| \mu\left(\frac{n}{m}\right) \right| = d - 1 + \sum_{m|n} |\mu(m)|.
\]

Hence

\[
k + 1 \leq \sum_{m|n} |\mu(m)|. \tag{31}
\]

Let \( n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \) be the prime factorization of \( n \). By definition of \( \mu \), we have

\[
|\mu(\ell)| = \begin{cases} 
1, & \text{if } \ell \text{ is a square-free positive integer;} \\
0, & \text{if } \ell \text{ has a squared prime factor.}
\end{cases}
\]
Therefore, if \( n = q_1^{a_1}q_2^{a_2} \cdots q_r^{a_r} \), the sum on the right hand side of inequality (31) is equal to the number of subsets of \( \{q_1, \ldots, q_r\} \), i.e., to \( 2^r \). Hence we obtain inequality (28).

**Remark 4.4.** The original inequality of McKay and Wang, derived for \( d = 1 \) and \( n > 1 \), reads \( \sqrt{n} \leq 2^r \).

A technical Lemma 4.5 below contains a result that will be important two times in the sequel. At first, it will allow us to estimate \( n \) in the proof of Proposition 4.6. Secondly, it will be crucial for reducing the proof of Proposition 4.7 to an examination of a finite number of cases.

Let us recall that the existence of matrices \( C \) satisfying conditions \( 2 \) of orders \( n \neq 2(d + 1) \) is generally impossible whenever \( d \notin \mathbb{N}_0 \) or \( d \) is even, cf. Proposition 3.1 and Theorem 3.8. Therefore, we can impose an additional assumption that \( d \) is an odd integer wherever convenient.

**Lemma 4.5.** Let a symmetric matrix \( C \) satisfy assumptions \( 2 \) for an odd \( d \), and let \( n = k(2d + k) + 1 \) for a certain \( k > 1 \). Then there exist \( t, u, w, z \in \mathbb{N} \) such that \( w < t \) and

\[
\frac{k + 1}{2} = tu, \quad \frac{k - 1}{2} = wz \quad \text{and} \quad n = 4t(z(2tu - 1 - uw)).
\]

**Proof.** Since \( n = k(2d + k) + 1 \) is even, the number \( k \) is obviously odd. Hence we have \( \frac{k + 1}{2} \in \mathbb{N}, \frac{k - 1}{2} \in \mathbb{N} \) and \( \frac{d + k}{2} \in \mathbb{N} \). We set

\[
\frac{d + 1}{d + k} = \frac{s}{t} \quad \text{for} \quad s, t \in \mathbb{N}, \gcd(s, t) = 1.
\]

With regard to the assumption \( k > 1 \), we have \( s < t \). Equation (33) implies \( \frac{d + k}{2} = \frac{t}{s} \cdot \frac{d + 1}{2} \). Since \( \gcd(s, t) = 1 \), we have \( s | \frac{d + 1}{2} \), i.e., \( \frac{d + 1}{2} = zs \) for a certain \( z \in \mathbb{N} \). Then

\[
\frac{d + k}{2} = \frac{t}{s} \cdot \frac{d + 1}{2} = tz.
\]

According to Proposition 4.1, we have \( 2(d + k) | (d^2 - 1) \). Therefore, \( \frac{(d + 1)(d - 1)}{2(d + k)} = \frac{t}{s} \cdot \frac{d - 1}{2} \in \mathbb{N} \). We use again the assumption \( \gcd(s, t) = 1 \) to infer that \( \frac{d - 1}{2} = vt \) for a certain \( v \in \mathbb{N} \). Hence we get

\[
\frac{k + 1}{2} = \frac{d + k}{2} - \frac{d - 1}{2} = tz - vt = t(z - v);
\]

\[
\frac{k - 1}{2} = \frac{d + k}{2} - \frac{d + 1}{2} = tz - zs = z(t - s).
\]

If we set \( z - v =: u \) and \( t - s =: w \), we get \( \frac{k + 1}{2} = tu \) and \( \frac{k - 1}{2} = zw \). It remains to express \( n \) in terms of \( t, u, w, z \). For this purpose we rewrite

\[
n = k(2d + k) + 1 = 2(d + k)(k + 1) - 2(d + k) - k^2 + 1 = (d + k) \left( 2(k + 1) - 2 - \frac{(k + 1)(k - 1)}{d + k} \right)
\]

and take advantage of equations \( k + 1 = 2tu, k - 1 = 2uw \) and \( d + k = 2tz \) derived above. This gives

\[
n = 2tz \left( 4tu - 2 - \frac{2tu \cdot 2zw}{2tz} \right) = 4tz(2tu - 1 - uw).
\]

**Proposition 4.6.** Let \( d \) be odd and \( n = k(2d + k) + 1 \). If \( k \geq 2^7 \), then a symmetric matrix \( C \) satisfying assumptions \( 2 \) for those values of \( d \) and \( n \) does not exist.
Proof. Let \( n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \) be the prime factorization of \( n \). We will demonstrate that \( k + 1 > 2^r \) whenever \( k \geq 2^7 \). The statement then follows straightforwardly from Proposition 1.3.

The implication \( k \geq 2^7 \Rightarrow k + 1 > 2^r \) holds trivially for any \( n \) such that \( r \leq 7 \). Therefore, it remains to prove that \( k + 1 > 2^r \) for \( r \geq 8 \). According to Lemma 4.5, values \( n \) and \( k \) satisfy equations (32). In particular, we have

\[
tu = wz + 1 > z;
\]

hence we get

\[
n = 4tz(2tu - 1 - uw) < 4t \cdot tu \cdot 2tu = 8t^3u^2 \leq 8t^3u^3 = (2tu)^3 = (k + 1)^3.
\]  

(34)

Since \( d \) is odd, \( n \) is a multiple of 4 (cf. Prop. 3.1). Therefore, \( q_1 = 2 \) and \( \alpha_1 \geq 2 \). Then

\[
n \geq 2^2q_2q_3 \cdots q_r \geq 2p_r#,
\]

(35)

where \( p_r# = \prod_{j=1}^{r} p_j = 2 \cdot 3 \cdot 5 \cdots p_r \) is the \( r \)th primorial number (the product of the first \( r \) primes). We have

\[
p_r# > \frac{8^r}{2} \quad \text{for all } r \geq 8,
\]

(36)

which follows from the fact that \( p_8# = 9699690, \frac{8^8}{2} = 8388608 \) and \( p_j > 8 \) for all \( j > 8 \). When we combine inequalities (34), (35) and (36), we get

\[
k + 1 > 3^{\sqrt{2p_r#}} > 2^r \quad \text{for all } r \geq 8.
\]

\[\square\]

**Proposition 4.7.** Let \( d \) be odd and \( n = k(2d + k) + 1 \). If \( 1 < k \leq 2^7 \), then a symmetric matrix \( C \)
of order \( n \) satisfying assumptions (2) does not exist with possible exceptions for \( k = 7, n = 120 \) and \( k = 13, n = 924 \).

Proof. The proof relies on the fact that if \( C \) is symmetric, then for every \( k \in \mathbb{N} \) there are only finitely many possible orders \( n \) allowed in the formula \( n = k(2d + k) + 1 \), which follows from Lemma 4.5. This fact enables us to verify that for every \( k \leq 2^7 \) and for every \( n \) conforming system (32), except for \( k = 7, n = 120 \) and \( k = 13, n = 924 \), we have \( k + 1 > 2^r \), where \( q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \) is the prime factorization of \( n \). Once this is proved, the statement follows from Proposition 1.3.

The verification can be done step by step for each \( k = 3, 5, 7, \ldots, 2^7 - 1 \) (note that even values \( k \) do not obey equations (32)), using the following procedure.

1. Find all possible 4-tuples \((t, u, w, z) \in \mathbb{N}^4\) such that \( \frac{k + 1}{2} = tu \) and \( \frac{k - 1}{2} = wz \) with \( w < t \).

2. For each \((t, u, w, z)\), set \( n = 4tz(2tu - 1 - uw) \) and find the prime factorization \( n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \).

3. Check the inequality \( k + 1 > 2^r \) for all values \( r \) found in the previous step.

We will demonstrate the procedure for \( k = 3, 5, 7, \ldots \).

- Let \( k = 3 \), i.e., \( \frac{k + 1}{2} = 2 \). Step 1: The system \( tu = 2, wz = 1, w < t \) implies \( t = 2, u = 1, w = 1, z = 1 \). Step 2: \( n = 4 \cdot 2(2 \cdot 2 - 1 - 1) = 16 = 2^4 \); hence \( r = 1 \). Step 3: We have \( 2 \cdot 2 > 2^1 \).

- Let \( k = 5 \). Step 1: \( tu = 3, wz = 2, w < t \) implies \((t, u, w, z) \in \{(3, 1, 2, 1), (3, 1, 1, 2)\}\). Step 2: For \((3, 1, 2, 1)\) we get \( n = 12 \cdot 3 = 2^2 \cdot 3^2 \); hence \( r = 2 \). For \((3, 1, 1, 2)\) we get \( n = 24 \cdot 4 = 2^3 \cdot 3 \); hence \( r = 2 \). Step 3: In both cases we have \( 2 \cdot 3 > 2^2 \).
Let \( k = 7 \). Step 1: \( tu = 4, wz = 3, w < t \) implies \( (t, u, w, z) \in \{(4, 1, 3, 1), (4, 1, 1, 3), (2, 2, 1, 3)\} \).

Step 2: For \((4, 1, 3, 1)\) we get \( n = 16 \cdot 4 = 2^8\); hence \( r = 1 \). For \((4, 1, 1, 3)\) we get \( n = 48 \cdot 6 = 2^5 \cdot 3^2\); hence \( r = 2 \). For \((2, 2, 1, 3)\) we get \( n = 24 \cdot 5 = 2^3 \cdot 3 \cdot 5\); hence \( r = 3 \). Step 3: If \( r = 1 \) or \( r = 2 \), then \( 2 \cdot 4 > 2^r \). If \( r = 3 \), we have \( 2 \cdot 4 = 2^2 \), i.e., \( 2 \cdot 4 \not< 2^r \). Case \( r = 3 \) occurs for

\[
 n = 120, \quad d = \frac{1}{2} \left( \frac{n - 1}{k} - k \right) = \frac{1}{2} \left( \frac{119}{7} - 7 \right) = 5;
\]

therefore, there may exist a symmetric \( C \) for \((n, d) = (120, 5)\).

Repeating the procedure for the remaining odd values of \( k \) up to \( k = 127 \) is straightforward. The algorithm is very simple, and the calculation can be thus carried out on a computer, which will give results immediately. One finds that the inequality \( k + 1 > 2^r \) is satisfied for all \( k \geq 9 \) except for \( k = 13 \) and \((t, u, w, z) = (7, 1, 2, 3)\). In this case we have \( n = 924 = 2^2 \cdot 3 \cdot 7 \cdot 11 \), thus \( r = 4 \), and \( k + 1 = 14 \not< 2^r \). The corresponding value of \( d \) is \( d = \frac{1}{2} \left( \frac{924 - 1}{13} \right) = 29 \).

**Proposition 4.8.** There exists no symmetric matrix \( C \) satisfying conditions \( \square \) for \( n = 120, d = 5 \) or \( n = 924, d = 29 \).

**Proof.** Equation (30) together with \( |\lambda_j|=d+k \), obtained in the proof of Proposition 4.3 implies

\[
d + k = \left| d - 1 + \mu(1) + \sum_{\substack{m|n \\ 1 \leq m \leq n-1}} c_m \mu \left( \frac{n}{m} \right) \right|,
\]

where \( k \in \mathbb{N} \) is related to \( d \) and \( n \) by the formula \( n = k(2d + k) + 1 \). We have \( \mu(1) = 1, \mu(\ell) \in \{1, -1, 0\} \) for all \( \ell \in \mathbb{N} \) and \( \sum_{\substack{m|n \\ 1 \leq m \leq n}} |\mu \left( \frac{n}{m} \right)| = 2^r \), where \( n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} \) is the prime factorization of \( n \). Let \( s \) denote the number of proper divisors \( m \) of \( n \) such that \( c_m \mu \left( \frac{n}{m} \right) = -1 \). Then

\[
\mu(1) + \sum_{\substack{m|n \\ 1 \leq m \leq n-1}} c_m \mu \left( \frac{n}{m} \right) = \sum_{\substack{m|n \\ 1 \leq m \leq n}} |\mu \left( \frac{n}{m} \right)| - 2s = 2^r - 2s.
\]

This allows us to rewrite equation (37) in the form

\[
d + k = |d - 1 + 2^r - 2s|.
\]

With equation (38) in hand, we can proceed to disproving the existence of matrices \( C \) for \( n = 120, d = 5 \) and \( n = 924, d = 29 \).

Let \( n = 120, d = 5 \). Using equation \( n = k(2d + k) + 1 \) and the prime decomposition of \( n = 120 \), we get \( k = 7 \) and \( r = 3 \) (cf. the proof of Proposition 4.7). Equation (38) thus takes the form

\[
5 + 7 = |5 - 1 + 2^3 - 2s|.
\]

Hence we have \( s = 0 \) or \( s = 12 \). Let us start with the case \( s = 0 \) (as we will see below, the case \( s = 12 \) is not possible). By definition of \( s \), equation \( s = 0 \) means that \( c_m = \mu \left( \frac{m}{n} \right) \) for every \( m < n \) such that \( m \mid n \) and \( \mu \left( \frac{m}{n} \right) \neq 0 \). This allows us to find \( c_m \) explicitly for each proper divisor \( m \) of \( n \) that satisfies \( \mu \left( \frac{m}{n} \right) = \pm 1 \). Knowing \( c_m \) for an \( m \) being a divisor of \( n \), one can use equation (29) to find values \( c_j \) for all \( j \) such that \( \gcd(j, n) = m \). In this way we obtain Table II. The last column shows those \( j \) for which
\[ \gcd(j, n) = m < j. \] For the sake of brevity, we list only numbers \( j \leq \frac{n}{2}; \) values \( c_j \) for \( j > \frac{n}{2} \) can be found from the symmetry of \( C \) using equation \( c_j = c_{n-j} \) for all \( 1 \leq j \leq n - 1. \) Table 1 determines the matrix \( C \) up to eight parameters \( c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}, c_{30} \) that take values from \( \{1, -1\}. \) We performed a computer calculation for each possible 8-tuple \((c_1, c_2, c_3, c_5, c_6, c_{10}, c_{15}, c_{30}), \) finding that the rows of \( C \) are never mutually orthogonal. Therefore, a symmetric matrix \( C \) corresponding to \( s = 0 \) does not exist.

It remains to comment on the case \( s = 12. \) Table 1 above shows that there are only 7 proper divisors of 120 such that \( \mu\left(\frac{n}{m}\right) \neq 0, \) i.e., \( s \) cannot exceed 7. The case \( s = 12 \) can be thus excluded. To sum up, there exists no symmetric matrix \( C \) of order 120 satisfying conditions (2) for \( d = 5. \)

Let \( n = 924, d = 29. \) Then \( k = 13 \) and \( r = 4, \) and equation (38) takes the form

\[ 29 + 13 = |29 - 1 + 2^4 - 2^8| ; \]

hence \( s = 1 \) (the other solution, \( s = 43, \) is impossible, because 924 has only 23 proper divisors). Equation \( s = 1 \) means that there is a proper divisor \( m_0 \) of \( n \) such that

\[ \mu\left(\frac{n}{m_0}\right) \neq 0 \quad \land \quad c_{m_0} = -\mu\left(\frac{n}{m_0}\right), \]

and for all the other proper divisors of \( n \) we have

\[ \left(m_0 \neq m < n \quad \land \quad \mu\left(\frac{n}{m}\right) \neq 0 \right) \quad \Rightarrow \quad c_m = \mu\left(\frac{n}{m}\right). \]

Therefore, for each proper divisor of \( n \) such that \( \mu\left(\frac{n}{m}\right) \neq 0, \) we have \( c_m = b_m\mu\left(\frac{n}{m}\right), \) where the values \( b_m \) form a vector that is a permutation of \((-1, 1, 1, \ldots, 1)\). Properties of the Möbius function \( \mu \) imply that the size of the vector is \( 2^r - 1. \) Values \( c_j \) are shown in Table 2 They depend on parameters \( c_1, c_3, c_7, c_{11}, c_{21}, c_{33}, c_{77}, c_{231} \in \{1, -1\} \) and on the following permutation of \((-1, 1, 1, \ldots, 1)\),

\( (b_2, b_4, b_6, b_{12}, b_{14}, b_{22}, b_{28}, b_{42}, b_{44}, b_{66}, b_{84}, b_{132}, b_{154}, b_{308}, b_{462}) \).
Table 2: Values $c_j$ for $n = 924, d = 29$.

Entries of $C$ that are not listed in Table 2 can be obtained using equation (29). A computer calculation shows that for each choice of parameters $c_j$ and $b_j$, the rows of $C$ are not mutually orthogonal. Therefore, a symmetric matrix $C$ of order 924 satisfying conditions (2) for $d = 29$ does not exist. □

**Theorem 4.9.** If a symmetric matrix $C$ satisfies conditions (2) for a given $d \geq 0$, then $n = 2(d + 1)$.

**Proof.** If $d \notin \mathbb{N}_0$ or $d \in \mathbb{N}_0$ is even, then $n = 2(d + 1)$ according to results of Section 3, see Remark 3.10. If $d = 1$, the existence of symmetric Hadamard matrices of orders $n > 2(d + 1) = 4$ was disproved in papers [14, 6, 17, 9]. It remains to show the validity of the statement for odd numbers $d > 1$. According to Proposition 3.3, the order $n$ obeys $n = k(2d + k) + 1$ for a certain $k \in \mathbb{N}$. However,

- the case $k > 128$ is excluded by Proposition 4.6;
- the case $1 < k \leq 128$ is excluded by Proposition 4.7, except for $(k, n, d) = (7, 120, 5)$ and $(k, n, d) = (13, 924, 29);
- the existence of symmetric matrices obeying conditions (2) for $(n, d) = (120, 5)$ and $(n, d) = (924, 29)$ is disproved by Proposition 4.8.

To sum up, $k = 1$; hence $n = 2(d + 1)$. □

With regard to results of Section 3, we can also give a partial condition for matrices $C$ that are not symmetric:

**Proposition 4.10.** If a matrix $C$ satisfying conditions (2) is not symmetric, then $d$ is odd and $n \equiv 0 \pmod{4}$.

**Proof.** The existence of $C$ implies that $2d$ is an integer, see Proposition 3.1. If $d$ is a half-integer or an even integer, we have $n = 2(d + 1)$ according to Proposition 3.1 and Theorem 3.8; then $C$ is symmetric due to Theorem 5.3. Consequently, a matrix $C$ that is not symmetric exists only for odd $d$. Finally, $d$ is odd is equivalent to $n \equiv 0 \pmod{4}$, cf. Proposition 3.1. □
5 Matrices $C$ satisfying $n = 2(d + 1)$

According to Proposition 3.1, the smallest possible order of matrices $C$ obeying conditions (2) with a given value $d$ on the diagonal is $n = 2(d + 1)$. Other results of Sections 3 and 4 confirmed that this order is special and deserves a particular attention. Recall that the relation $n = 2(d + 1)$ is satisfied for matrices $C$ whenever $d$ is a half-integer or an even integer (cf. Rem. 3.10), as well as whenever $C$ is symmetric (Thm. 4.9). Conjecture 3.9 states that the relation $n = 2(d + 1)$ holds true generally for any matrix $C$ obeying conditions (2).

Considering the prominence of matrices $C$ satisfying $n = 2(d + 1)$, we devote this section to their full characterization. The problem is easy when $d$ is not an integer. Indeed, if $d$ is a half-integer, then the only possible generator of $C$ is $(n^2 - 1, -1, -1, \ldots, -1)$ (see Prop. 3.1 and Prop. 2.3), whereas there is no solution in case that $2d / \not\in \mathbb{N}_0$ (cf. Prop. 3.1). The examination of matrices $C$ for $d$ being an integer is more difficult, and we divide it into two steps. In the first step we address the special case when $c_j = 1 \lor c_{n-j} = 1$ for all $j = 1, \ldots, n-1$ (Proposition 5.1). In the second step we proceed to the characterization of matrices $C$ with the property ($\exists m \in \{1, \ldots, n-1\}$ $(c_m = c_{n-m} = -1)$ (Proposition 5.2).

**Proposition 5.1.** Let $C$ satisfy conditions (2) for $n = 2(d + 1)$. Let $c_j = 1 \lor c_{n-j} = 1$ for all $j = 1, \ldots, n-1$. Then

- either $n = 2$ and
  
  $$C = C_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

- or $n = 4$ and

  $$C = C_{4a} := \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \text{ or } C = C_{4b} := \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$  

**Proof.** Proposition 2.3 implies that $n$ is even. The assumption $c_j = 1 \lor c_{n-j} = 1$ applied on $j = \frac{n}{2}$ gives $c_\frac{n}{2} = c_{n-\frac{n}{2}} = 1$. Let $m \leq \frac{n}{2}$ denote the minimal number with the property

$$c_m = c_{n-m} = 1; \quad (39)$$

then

$$c_j = -c_{n-j} \text{ for all } j = 1, \ldots, m-1. \quad (40)$$

The $0$th and the $m$th row of $C$ take the form

$$\begin{pmatrix} \frac{n}{2} & -1 \\ 1 & c_1 & c_2 & \cdots & c_{m-1} & 1 & c_{m+1} & \cdots & c_{n-2} & c_{n-1} \\ c_{n-m+1} & c_{n-m+2} & \cdots & c_{n-1} & \frac{n}{2} & -1 & c_1 & \cdots & c_{n-m-2} & c_{n-m-1} \end{pmatrix}.$$  

Their scalar product must be zero; hence we obtain

$$\sum_{j=n-m+1}^{n-1} c_j c_{j+m-n} + \sum_{j=1}^{n-m-1} c_j c_{j+m} = -(n-2). \quad (41)$$
We see that each of the \( n - 2 \) summands on the left hand side of equation (41) must be equal to \(-1\). In particular,

\[
c_jc_{j+m} = -1 \quad \text{for all } j \in \{1, \ldots, n-m-1\}.
\]

(42)

Now let us show that \( m = \frac{n}{2} \). Indeed, if \( m < \frac{n}{2} \), then both \( j = m \) and \( j = n - 2m \) satisfy \( j \in \{1, \ldots, n-m-1\} \). If we use equation (42) with these two values of \( j \), we get \( c_mc_{2m} = -1 \) and \( c_{n-2m}c_{n-m} = -1 \). Taking equation (39) into account, we conclude that \( c_{2m} = c_{n-2m} = -1 \). This is, however, a contradiction with the assumptions of the proposition. Hence necessarily \( m = \frac{n}{2} \). Conditions (42) are thus equivalent to

\[
c_j = -c_{j+\frac{n}{2}} \quad \text{for all } j = 1, \ldots, \frac{n}{2}-1.
\]

(43)

It is easy to see that conditions (43) together with equations (40) and (39) imply that \( C \) has the block form

\[
C = \begin{pmatrix}
(\frac{n}{2} - 1)I + A & I - A \\
I - A & \left(\frac{n}{2} - 1\right)I + A
\end{pmatrix},
\]

(44)

where \( A \) is a Toeplitz matrix with the 0th row equal to \((0, c_1, \ldots, c_{\frac{n}{2}-1})\) and with the 0th column equal to \((0, -c_1, \ldots, -c_{\frac{n}{2}-1})^T\). Therefore, \( A \) is antisymmetric. Equation (44) together with the antisymmetry of \( A \) implies

\[
CC^T = \begin{pmatrix}
(d^2 + 1)I + 2AA^T & 2dI - 2AA^T \\
2dI - 2AA^T & (d^2 + 1)I + 2AA^T
\end{pmatrix},
\]

(45)

where \( d = \frac{n}{2} - 1 \). We see from equation (45) that the condition \( CC^T = (d^2 + n - 1)I \) is equivalent to

\[
AA^T = dI = \left(\frac{n}{2} - 1\right)I.
\]

(46)

Hence we can immediately find solutions for \( n = 2 \) and \( n = 4 \):

- If \( n = 2 \), then \( A \) is a 1 \( \times \) 1 matrix satisfying \( AA^T = 0 \), i.e., \( A = (0) \). When we substitute \( n = 2 \) and \( A = (0) \) into equation (44), we obtain the solution \( C_2 \).

- If \( n = 4 \), \( A \) has to be a 2 \( \times \) 2 antisymmetric matrix such that \( AA^T = I \). Hence

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

These two matrices, when substituted into (44), lead to the solutions \( C_{4a} \) and \( C_{4b} \).

In the rest of the proof we show that equation (46) is never satisfied by the above defined antisymmetric Toeplitz matrix \( A \) when \( n > 4 \). For the sake of convenience, we will keep using the symbol \( m = \frac{n}{2} \), assuming that \( m > 2 \).

First of all, \( m \) cannot be odd. Indeed, the orthogonality of the 0th and 1st row of \( A \) gives the condition

\[
\sum_{j=1}^{m-2} c_j c_{j+1} = 0.
\]

If \( m \) is odd, the left hand side is an odd number, and thus nonzero.

Let \( m > 2 \) be even. We use equations (43) and (40) to derive the relation

\[
c_j = -c_{j+m} = -(c_{n-(j+m)}) = c_{m-j} \quad \text{for all } j = 1, \ldots, m-1.
\]

(47)
Relation (47) implies that the 0th row of \( A \) takes the form

\[
0 \ c_1 \ c_2 \ c_3 \ \cdots \ c_{\frac{m}{2}-1} \ c_{\frac{m}{2}} \ c_{\frac{m}{2}-2} \ c_{\frac{m}{2}-3} \ \cdots \ c_2 \ c_1 .
\]

The 1st and 2nd row of \( A \) read

\[
-c_1 \ 0 \ c_1 \ c_2 \ \cdots \ c_{\frac{m}{2}-2} \ c_{\frac{m}{2}-1} \ c_{\frac{m}{2}} \ c_{\frac{m}{2}-2} \ c_{\frac{m}{2}-1} \ \cdots \ c_3 \ c_2 \\
-c_2 \ -c_1 \ 0 \ c_1 \ \cdots \ c_{\frac{m}{2}-3} \ c_{\frac{m}{2}-2} \ c_{\frac{m}{2}-1} \ c_{\frac{m}{2}} \ c_{\frac{m}{2}-1} \ \cdots \ c_4 \ c_3 .
\]

The scalar product of the 0th with the 1st row is equal to

\[
2 \left( \sum_{j=1}^{\frac{m}{2}-1} c_j c_{j+1}\right) .
\]

Similarly, the scalar product of the 0th with the 2nd row equals

\[
-c_1^2 + c_2^2 + 2 \left( \sum_{j=1}^{\frac{m}{2}-2} c_j c_{j+2}\right) = 2 \left( \sum_{j=1}^{\frac{m}{2}-2} c_j c_{j+2}\right) .
\]

Both scalar products should be zero. Hence we obtain the requirement

\[
\sum_{j=1}^{\frac{m}{2}-1} c_j c_{j+1} = 0 \quad \land \quad \sum_{j=1}^{\frac{m}{2}-2} c_j c_{j+2} = 0 .
\]

However, since the two sums have different parities, the two equations cannot be satisfied at the same time. \( \square \)

**Proposition 5.2.** Let \( C \) satisfy conditions (2) for \( n = 2(d + 1) \). Let there be an \( m \in \{1, \ldots, n-1\} \) such that \( c_m = c_{n-m} = -1 \). Then \( C \) is a block circulant matrix taking the form

\[
C = \begin{pmatrix}
B + \frac{n}{2} I & B & B & \cdots & B \\
B & B + \frac{n}{2} I & B & \cdots & B \\
B & B & B + \frac{n}{2} I & B & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
B & B & B & B + \frac{n}{2} I & \end{pmatrix},
\]

where the block \( B \) is either the \( 1 \times 1 \) matrix \((-1)\) or \( B \) is one of the matrices

\[
C_2 - I, \quad C_{4a} - 2I, \quad C_{4b} - 2I
\]

for \( C_2, C_{4a}, C_{4b} \) defined in Proposition 5.1.

**Proof.** Let \( m \) be the minimal number with the property \( c_m = c_{n-m} = -1 \). The 0th and the \( m \)th row of \( C \) take the form

\[
\begin{pmatrix}
\left( \frac{n}{2} - 1 \right) & c_1 & c_2 & \cdots & c_{m-1} & -1 & c_{m+1} & \cdots & c_{n-2} & c_{n-1} \\
-1 & c_{n-m+1} & c_{n-m+2} & \cdots & c_{n-1} & \left( \frac{n}{2} - 1 \right) & c_1 & \cdots & c_{n-m-2} & c_{n-m-1} \\
\end{pmatrix} .
\]

Their scalar product must be zero; hence

\[
\sum_{j=n-m+1}^{n-1} c_j c_{j+m-n} + \sum_{j=1}^{n-m-1} c_j c_{j+m} = n - 2 .
\]
Equation (50) is satisfied if and only if all the summands on the left hand side are equal to 1, i.e.,
\begin{equation}
    c_j = c_{(j+m) \mod n} \quad \text{for all } j = 1, \ldots, n-1, \ j \neq n-m .
\end{equation}

Let us show that \( m \) divides \( n \). Indeed, if \((n \mod m) = k \neq 0\), we get
\begin{align*}
    -1 &= c_m = c_{2m} = \cdots = c_{n-k}; \\
    -1 &= c_{n-m} = c_{n-2m} = \cdots = c_k,
\end{align*}
i.e., \( c_k = c_{n-k} = -1 \) for a certain \( k < m \). This would contradict the definition of \( m \). Therefore, \( m \) divides \( n \). Equation (51) thus gives the generator of \( C \) in the form
\begin{equation}
    \left( \frac{n}{2} - 1, c_1, \ldots, c_{m-1}, -1, c_1, \ldots, c_{m-1}, -1, c_1, \ldots, c_{m-1}, \ldots, -1, c_1, \ldots, c_{m-1} \right).
\end{equation}

This result implies that \( C \) has the block form (48). It remains to determine the matrix \( B \). With regard to the structure of the generator (52), \( \text{matrix } B \) is circulant and has the generator \((-1, c_1, \ldots, c_{m-1})\). If \( m = 1 \), we obtain immediately \( B = (-1) \). If \( m \geq 2 \), the minimality of \( m \) trivially implies that
\begin{equation}
    c_j = 1 \lor c_{m-j} = 1 \quad \text{for all } j = 1, \ldots, m-1 .
\end{equation}

Furthermore, a straightforward calculation for \( C \) given by equation (48) leads to
\[ CC^T = \begin{pmatrix} F & G & G & \cdots & G \\ G & F & G & \cdots & G \\ G & G & F & \cdots & G \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G & G & G & \cdots & F \end{pmatrix}, \]
where
\[ F = \frac{n}{m} BB^T + \frac{n}{2} (B + B^T) + \frac{n^2}{4} I, \quad G = \frac{n}{m} BB^T + \frac{n}{2} (B + B^T). \]

By assumption, matrix \( C \) satisfies \( CC^T = (d^2 + n - 1)I \) with \( d = \frac{n}{2} - 1 \), i.e., \( CC^T = \frac{n^2}{4} I \). Hence we obtain conditions \( F = \frac{n^2}{2} I \) and \( G = 0 \), which are both equal to
\begin{equation}
    \left( B + \frac{m}{2} I \right) \left( B + \frac{m}{2} I \right)^T = \frac{m^2}{4} I .
\end{equation}

To sum up, if \( m \geq 2 \), then \( B + \frac{m}{2} I \) is an \( m \times m \) circulant matrix with generator \((-1 + \frac{m}{2}, c_1, \ldots, c_{m-1})\) and with properties (52) and (54). In other words, the matrix \( B + \frac{m}{2} I \) satisfies all assumptions of Proposition 5.1. Therefore, \( B + \frac{m}{2} I \) is equal to \( C_2, C_{4a} \) or \( C_{4b} \). Hence we obtain three possible matrices \( B \), as given in equation (49).

Theorem 5.3 below summarizes results found in this section.

**Theorem 5.3.** A matrix \( C \) satisfies conditions (2) with \( n = 2(d + 1) \) if and only if the generator of \( C \) takes one of the forms below.
\begin{align*}
    &g_1 = \left( \frac{n}{2} - 1, -1, \ldots, -1 \right) \quad \text{(for any } n \geq 2); \\
    &g_2 = \left( \frac{n}{2} - 1, 1, -1, \ldots, -1, 1 \right) \quad \text{(for even } n); \\
    &g_{4a} = \left( \frac{n}{2} - 1, 1, -1, 1, \ldots, -1, 1, 1, -1 \right) \quad \text{(for } n \text{ being a multiple of } 4); \\
    &g_{4b} = \left( \frac{n}{2} - 1, -1, 1, 1, -1, 1, 1, \ldots, 1, 1, 1, -1 \right) \quad \text{(for } n \text{ being a multiple of } 4). \\
\end{align*}

In particular, a matrix \( C \) with \( n = 2(d + 1) \) exists for every \( n \geq 2 \).
Proof. One can easily check that all vectors in the list generate circulant matrices satisfying conditions (2). It suffices to verify that the generators \( g_1, g_2, g_{4a}, g_{4b} \) cover all possibilities found in Proposition 5.1 and Proposition 5.2. We have:

- the choice \( B = (-1) \) in Proposition 5.2 corresponds to vector \( g_1 \);
- matrix \( C_2 \) in Proposition 5.1 and the choice \( B = C_2 - I \) in Proposition 5.2 correspond to vector \( g_2 \);
- matrix \( C_{4a} \) in Proposition 5.1 and the choice \( B = C_{4a} - 2I \) in Proposition 5.2 correspond to vector \( g_{4a} \);
- matrix \( C_{4b} \) in Proposition 5.1 and the choice \( B = C_{4b} - 2I \) in Proposition 5.2 correspond to vector \( g_{4b} \).

\[ \square \]

**Remark 5.4.** A matrix \( C \) satisfying conditions (2) for \( n = 2(d + 1) \) may or may not be symmetric:

- If \( C \) is a circulant matrix with generator \( g_1 \) or \( g_2 \), then \( C^T = C \).
- Let \( C_a, C_b \) be circulant \( n \times n \) matrices with generators \( g_{4a} \) and \( g_{4b} \), respectively. Then \( C_a^T = C_b \neq C_a \).

**Remark 5.5.** Theorem 5.3 applied on \( d = 1 \) gives us four circulant Hadamard matrices of order 4. Multiplying each of them by \(-1\), we obtain four more matrices. These 8 matrices of order 4 together with the matrices (1), \((-1)\) of order 1 are the only known circulant Hadamard matrices. The Hadamard circulant conjecture says that no other solution exists.

If Conjecture 3.9 is true, then generators \( g_1, g_2, g_{4a}, g_{4b} \) from Theorem 5.3 determine all the matrices \( C \) satisfying conditions (2), giving thus a complete solution to the problem (2).

6 **Summary and conclusions**

We studied circulant matrices \( C \) of order \( n \geq 2 \) with diagonal entries \( d \geq 0 \), off-diagonal entries equal to \( \pm 1 \) and mutually orthogonal columns. These matrices generalize circulant Hadamard and circulant conference matrices, which correspond to \( d = 1 \) and \( d = 0 \), respectively. Matrices \( C \) can be constructed for every order \( n \) with the value \( d \) on the diagonal chosen such that \( n = 2(d + 1) \).

We demonstrated that a matrix \( C \) with diagonal \( d \) exists if and only if \( 2d \) is an integer. Furthermore, we proved that the order \( n \) is uniquely determined by the diagonal \( d \) via formula \( n = 2(d + 1) \) whenever \( d \) is an even integer or a half-integer; the case of \( d \) being an odd integer remains open. The formula \( n = 2(d + 1) \) for the special value \( d = 0 \) gives the well-known result obtained by Stanton and Mullin (1976), which says that circulant conference matrices exist only of order 2. In addition, we proved that the relation \( n = 2(d + 1) \) holds true whenever \( n - 1 \) is prime or \( C \) is symmetric. The latter result generalizes a well-known theorem that there is no symmetric circulant Hadamard matrix of order \( n > 4 \).

With regard to our findings, we conjectured that the relation \( n = 2(d + 1) \) holds for every matrix \( C \) defined above, including those \( C \) with odd diagonal values \( d \). The conjecture generalizes the circulant Hadamard conjecture, which corresponds to the special case \( d = 1 \). We further supported the conjecture by verifying it for all existing solutions \( C \) of order \( n \) up to \( n = 50 \).
Finally, we found all matrices $C$ of order $n \geq 2$ with the property $n = 2(d + 1)$. If the above stated conjecture is true, then those explicitly constructed matrices $C$ with $n = 2(d + 1)$ represent the complete set of solutions of the studied problem.

**Acknowledgements**

The authors thank J. Seberry and R. Craigen for useful comments on the topic. O. T. appreciates the hospitality at Jagellonian University in Krakow, where a part of this work was done. The research was supported by the Czech Science Foundation (GAČR) within the project 14-06818S, by the Polish National Science Center within the project DEC-2011/02/A/ST1/00119, by the European Union FP7 project PhoQuS@UW (Grant Agreement No. 316244), and by the John Templeton Foundation under the project No. 56033.

**References**

[1] M. H. Ang, K. T. Arasu, S. L. Ma, Y. Strassler, Study of proper circulant weighing matrices with weight 9, *Discrete Math.* 308 (2008) 2802–2809.

[2] K. T. Arasu, K. H. Leung, S. L. Ma, A. Nabavi, D. K. Ray-Chaudhuri, Determination of all possible orders of weight 16 circulant weighing matrices, *Finite Fields Th. App.* 12 (2006) 498–538.

[3] K. T. Arasu, J. Seberry, On circulant weighing matrices, *Australasian J. Combin.* 17 (1998) 21–37.

[4] T. Beth, D. Jungnickel, H. Lenz, *Design Theory* (2nd edition), Cambridge University Press, 1999.

[5] P. Borwein, M. Mossinghoff, Barker sequences and flat polynomials, in: J. McKee, C.s Smyth (Eds.), *Number Theory and Polynomials* (Bristol, U.K., 2006). London Math. Soc. Lecture Note Ser., vol. 352, Cambridge Univ. Press, 2008, pp. 71–88.

[6] R. A. Brualdi, A note on multipliers of difference sets, *J. Res. Natl. Bur. Stand.* 69B (1965) 87–89.

[7] C. Cook, M. Bernfeld, *Radar signals: An Introduction to Theory and Application*, Academic Press, New York, 1967.

[8] R. Craigen, Trace, symmetry and orthogonality, *Canad. Math. Bull.* 37 (1994) 461–467.

[9] R. Craigen, H. Kharaghani, On the nonexistence of Hermitian circulant complex Hadamard matrices, *Australas. J. Combin.* 7 (1993) 225–227.

[10] P. Eades, R.M. Hain, On circulant weighing matrices, *Ars Combin.* 2 (1976) 265–284.

[11] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers* (5th ed.), Oxford University Press, Oxford, 1980.

[12] D. Huffman, A method for the construction of minimum redundancy codes, *Proc. of the IRE*, vol. 40 (1952) pp. 1098–1101.

[13] D. Huffman, The generation of impulse-equivalent pulse trains, *IRE Trans. on Information Theory*, vol. 8 (1962) 10–16.
[14] E. C. Johnsen, The inverse multiplier for abelian group difference sets, *Canad. J. Math.* **16** (1964) 787–796.

[15] P. Kurasov, R. Ogik, On equi-transmitting matrices, *Research Reports in Mathematics*, no. 1 (2014), Stockholm University.

[16] C. W. H. Lam, Non-skew symmetric orthogonal matrices with constant diagonals, *Discrete Math.* **43** (1983) 65–78.

[17] J. H. McKay, S.S.-S. Wang, On a theorem of Brualdi and Newman, *Linear Algebra Appl.* **92** (1987) 39–43.

[18] H. J. Ryser, *Combinatorial mathematics*, Willey, New York, 1963.

[19] B. Schmidt, Cyclotomic integers and finite geometry, *J. Am. Math. Soc.* **12** (1999) 929–952.

[20] B. Schmidt, Towards Ryser’s conjecture, in: C. Casacuberta et al., eds., *Proc. of 3rd European Congress on Mathematics*, Progress in Mathematics, vol. 201, Birkhäuser 2001, pp. 533–541.

[21] J. Seberry, C. W. H. Lam, On orthogonal matrices with constant diagonal, *Linear Algebra Appl.* **46** (1982) 117–129.

[22] R. G. Stanton, R. C. Mullin, On the nonexistence of a class of circulant balanced weighing matrices, *SIAM J. Appl. Math.* **30** (1976) 98–102.

[23] O. Turek, T. Cheon, Hermitian unitary matrices with modular permutation symmetry, *Linear Algebra Appl.* **469** (2015) 569–593.

[24] R. Turyn, Character sums and difference sets, *Pacific J. Math.* **15** (1965) 319–346.