Strong restriction on inflationary vacua from the local gauge invariance II: Infrared regularity and absence of secular growth in the Euclidean vacuum

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We investigate the initial state of the inflationary universe. In our recent publications, we have shown that requesting the gauge invariance in the local observable universe guarantees the infrared (IR) regularity of loop corrections in a general single clock inflation. Following this study, in this paper, we show that choosing the Euclidean vacuum ensures the gauge invariance in the local universe and hence the IR regularity of loop corrections. It has been suggested that loop corrections to inflationary perturbations may yield secular growth, which can lead to the breakdown of the perturbative analysis in an extremely long-term inflation. The absence of secular growth has been claimed by picking up only the IR contributions, which we think is incomplete because the non-IR modes that are comparable to or smaller than the Hubble scale can potentially contribute to the secular growth. We prove the absence of secular growth without neglecting these non-IR modes to a certain order in the perturbative expansion. We also discuss how the regularity of the $n$-point function for the genuinely gauge-invariant variable constrains the initial states of the inflationary universe. These results apply in a fully general single-field model of inflation.

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1. Introduction

1.1. Motivation and the current status of IR issues

Initial states of the observable universe. How did our universe begin? This is one of the biggest questions in cosmology. Observation of the cosmic microwave background tells us about the cosmological perturbation at the last scattering, which realistic scenarios of the early universe should explain. If inflation took place preceding big-bang nucleosynthesis, the quantum fluctuation of the inflaton can generate a seed of cosmological fluctuation that is consistent with the scale-invariant spectrum at large scales. Therefore, in the context of the inflationary scenario, which is currently the most successful scenario of the early universe, the period of inflation is the earliest part of our observable universe. In this series of papers, we pursue the question, “What can we claim about the initial quantum state of the observable universe if we require that the theoretical prediction should be stable against the infrared (IR) loop contribution?” The adiabatic vacuum is widely accepted as the most natural vacuum, at least for a free-field theory, since it mimics the vacuum of flat spacetime in the ultraviolet (UV) limit. However, in a number of publications [1–23], it has been suggested
that the adiabatic vacuum may not be stable against IR contributions in the presence of non-linear interactions.

**Non-locality of the action and IR divergence problem.** When we assume that the free field has a scale-invariant spectrum in the IR limit, a naive consideration can easily lead to IR divergence due to loop corrections. Here we illustrate how IR divergence can appear from loop corrections to the curvature perturbation in single-field models. Choosing the time slicing on which the inflaton field is homogeneous, we can express the action in terms of the unique dynamical degrees of freedom \( \zeta \), the curvature perturbation, and the Lagrange multipliers \( N \) and \( N_i \), the lapse function and the shift vector. The Hamiltonian and the momentum constraint equations relate the dynamical variable \( \zeta \) to the multipliers \( N \) and \( N_i \). As is explicitly shown in various papers, for instance in Refs. [24–26], these constraint equations are elliptic-type equations, and are schematically written as

\[
\partial^2 N = f[\zeta], \quad \partial^2 N_i = f_i[\zeta],
\]

where \( \partial^2 \) denotes the spatial Laplacian. By requesting regularity at spatial infinity, the boundary conditions of these elliptic-type equations are uniquely fixed. Substituting the expressions for \( N \) and \( N_i \) into the action, we obtain

\[
S = \int d^4x \mathcal{L}[\zeta, N, N_i] = \int d^4x \mathcal{L}[\zeta, \partial^{-2} f[\zeta], \partial^{-2} f_i[\zeta]],
\]

and hence the evolution of \( \zeta \) is described by the above non-local action. Here the inverse Laplacian \( \partial^{-2} \) is usually supposed to be defined as multiplying the inverse of the eigenvalue of the Laplacian operator by using harmonic decomposition. When we evaluate the loop corrections to the \( n \)-point functions, expanding them in terms of the interaction picture field \( \zeta_I \), we need to evaluate the expectation values such as

\[
\langle \zeta_I^2 \rangle, \quad \langle \zeta_I \partial^{-2} \zeta_I \rangle, \quad \ldots \quad (1.3)
\]

Inserting the scale-invariant spectrum into \( \langle \zeta_I^2 \rangle \) leads to the logarithmic divergence as \( \langle \zeta_I^2 \rangle \propto \int d^3k/k^3 \). The second expression of Eq. (1.3), which may arise as a consequence of the operation of \( \partial^{-2} \), is more singular as \( \langle \zeta_I \partial^{-2} \zeta_I \rangle \propto \int d^3k/k^5 \), which diverges quadratically. The presence of non-local interactions enhances the long-range correlations, and hence the singular behavior in the IR. When we introduce an IR cutoff, say at the Hubble scale at a particular time \( t_0 \), the variance \( \langle \zeta_I^2 \rangle \) shows the logarithmic secular growth as \( \langle \zeta_I^2 \rangle \propto \int_{a_0H_0}^{H} \frac{dk}{k} \sim \ln a/a_0 \) where \( a_0 \) and \( H_0 \), respectively, denote the scale factor and the Hubble scale at \( t = t_0 \). If the IR divergence exists, the loop corrections, which are suppressed by an extra power of the amplitude of the power spectrum \( (H/M_{pl})^2 \), may dominate if inflation continues sufficiently long, leading to the breakdown of perturbation.

**The dilatation symmetry as a necessary ingredient for IR regularity.** The regularization of the IR contributions has been discussed in a number of publications [25–38]. The important aspect in discussing the long wavelength mode of \( \zeta \) is the dilatation symmetry of the system. As is expected from the fact that the spatial metric is given in the form \( a^2e^{2\xi}dx^2 \), a constant shift of the dynamical variable \( \zeta \) can be absorbed by the overall rescaling of the spatial coordinates. Hence, the action for \( \zeta \) preserves the dilatation symmetry:

\[
x^i \rightarrow e^{-s}x^i, \quad \zeta(t, x) \rightarrow \zeta(t, e^{-s}x) - s,
\]

where \( s \) is a constant parameter. (There are a number of examples in the literature where this dilatation symmetry is addressed; see, for instance, Refs. [39,40] and references therein.) One may naively expect that we can absorb the IR divergent contribution of \( \zeta \) using this constant shift. As an
example, we set the parameter $s$ to the spatial average of the curvature perturbation within the Hubble patch at $t_0$, $\bar{\zeta}(t_0)$, where the size of the Hubble patch in comoving coordinates is given by $1/(a_0 H_0)$. Then, the logarithmically divergent two-point function $\langle \zeta_1^2 \rangle$ seems to be replaced with $\langle (\zeta_1 - \bar{\zeta}_1)^2 \rangle \propto \int_{a_0 H_0}^{a H} dk / k$, which is finite but still grows logarithmically in time. One may think that, if the system is described in such a way that the symmetry under the time-dependent dilatation transformation is manifest, setting $s(t)$ to the time-dependent spatial average in the Hubble patch might eliminate the logarithmic growth of $\bar{\zeta}(t)$. However, the reduced action (1.2) does not preserve the invariance under the dilatation transformation with the time-dependent parameter $s(t)$. For example, in recent work [40], the authors showed that, when we consider the whole universe with infinite spatial volume, the dilatation transformation should be time independent to keep the action invariant. In addition, the two-point function with $d^{-2}$ cannot be regularized by considering the dilatation symmetry alone. This quick consideration tells us that the presence of the dilatation symmetry of the system may play an important role in the regularization of the IR contributions but is not sufficient to guarantee IR regularity and the absence of secular growth.

**Residual gauge degrees of freedom in the local universe.** A missing piece in the above discussion is the need to pay careful attention to identifying the quantities that we can actually observe. Since our observable region is a limited portion of the whole universe, the observable fluctuations must be composed of local quantities. Furthermore, as the information that we can access is limited to our observable region, there is no reason to request regularity at spatial infinity in solving the elliptic constraint equations (1.1). Then, there arise degrees of freedom in choosing the boundary conditions of Eqs. (1.1). The degrees of freedom in solutions of $N$ and $N_i$ can be understood as the degrees of freedom in choosing coordinates. As we showed in Refs. [25,26], these residual coordinate transformations are expressed in terms of homogeneous solutions to the Laplace equation as

$$x^i \rightarrow x^i - s(t)x^j \sum_{m=1} x^{j_1 \cdots j_m}(t) x^{j_1} \cdots x^{j_m} + \cdots,$$  \hspace{1cm} (1.5)

where $s^{j_1 \cdots j_m}(t)$ are symmetric traceless tensors, which satisfy $\delta^{j_1} \cdots \delta^{j_m} s^{j_1 \cdots j_m}(t) = 0$. Here, we have abbreviated the non-linear terms in the coordinate transformation. Note that these coordinate transformations include the dilatation transformation with the time-dependent function $s(t)$. Since the transformations in Eq. (1.5) are nothing but coordinate transformations, the diffeomorphic invariant action $S = \int d^4x \mathcal{L}[\zeta, N, N_i]$ should preserve the symmetry under these transformations. Thus, when we consider only the local observable region, which is a portion of the whole universe, we find an infinite number of coordinate transformations, which keep the action invariant. We consider that the dilatation transformation in the whole universe is subtle in the sense that the transformation diverges at spatial infinity, even if the parameter $s$ is very small. By contrast, restricted to the local region, the magnitude of the coordinate transformations in Eq. (1.5) is kept perturbatively small. In this paper, we refer to the local observable (spacetime) region as $O$. The size of the observable region on each time slicing is supposed to be of order $1/a(t)H(t)$, at least in the far past since the past light cone asymptotes to that size. We should note that, once we insert the expressions for $N$ and $N_i$ into the action to obtain the action for the curvature perturbation $\zeta$, the symmetry under the residual coordinate transformation is lost, because specific boundary conditions are chosen for $N$ and $N_i$ in fixing coordinates. To emphasize the distinction between the coordinate transformations associated with the change in the boundary conditions and the usual gauge transformation, which keeps the action invariant, we denote the former by setting the gauge transformation in italics.
Removing the residual gauge degrees of freedom. One way to realize the invariance under the gauge transformation is to fix the gauge conditions completely. The residual gauge degrees of freedom introduced above can also be removed by employing additional gauge conditions, i.e., by fixing the boundary conditions of $N$ and $N_1$ at the boundary of the local region $\mathcal{O}$. Then, we naturally expect that the IR regularity may be explicitly shown by performing quantization in this local region, since the wavelengths that fit within this local region $\mathcal{O}$ will be bounded by the size of the region. Although quantization in the local region is an interesting approach, it is not so clear how to perform the quantization after removing the residual gauge degrees of freedom. One of the difficulties is that even the translation symmetry of the quantum state cannot be easily guaranteed in the local system, since it is broken by introducing the boundary condition at a finite distance (see also the discussion in Ref. [27]).

As an alternative way, in Ref. [28], we first set the initial state considering the whole universe, and then we performed the residual gauge transformation (1.5) to fix the coordinates so that the IR contributions are absorbed. Through the transformation with $s(t) = \bar{\zeta}(t)$, the curvature perturbation is transmitted as

$$\zeta(t, x) \rightarrow \zeta(t, e^{-\bar{\zeta}(t)} x) - \bar{\zeta}(t) = \zeta(t, x) - \bar{\zeta}(t) + \mathcal{O}(\zeta^2).$$

(1.6)

Here, $\zeta(x)$ is the original curvature perturbation defined in the whole universe and its spatial average over the whole universe is set to 0 as in conventional cosmological perturbation theory. By contrast, $\zeta(t, e^{-\bar{\zeta}(t)} x) - \bar{\zeta}(t)$ is the curvature perturbation relevant to the local universe, and its spatial average over the local region $\Sigma_t \cap \mathcal{O}$ is set to 0, where $\Sigma_t$ is a time constant surface. In Ref. [28] we considered the fluctuation of the inflaton, using the flat gauge, but the same discussion also follows for the curvature perturbation $\zeta$. In the recent publication by Senatore and Zaldarriaga [38], the same degrees of freedom in choosing coordinates are used in a slightly different way to absorb the IR divergent contributions. If the non-linear terms in the residual gauge transformation at the initial time (1.6) did not yield IR divergent contributions, the discussion in Ref. [28] would have proved the absence of IR divergence in general. What was shown there is that, once the field operator after the residual gauge transformation is guaranteed to be regular at the initial time, its succeeding evolution does not produce IR divergence. The heart of the proof is that $\zeta_I(x)$ is replaced with $\zeta_I(x) - \bar{\zeta}_I(t)$ in the expansion of the composite operators in terms of the interaction picture field, after the residual gauge transformation, and hence the IR contributions from $\zeta_I(x)$ are always canceled by those from $\bar{\zeta}_I(t)$. However, the non-linear part of the transformation at the initial time contains $\bar{\zeta}(t)$, whose IR contributions logarithmically diverge. The lesson is that it is not straightforward to reformulate the method of quantization so that the IR divergent contributions therein are all absorbed by the residual gauge transformation. (The absorption of the IR modes of the curvature perturbation was intended in other frameworks such as $\delta N$ formalism [31,32] and the semi-classical approach [33].)

Secular growth. The appearance of IR divergence due to the residual gauge transformation mentioned above might be avoided by sending the initial time to the past infinity. This is because the size of the local region $\Sigma_t \cap \mathcal{O}$ in comoving coordinates becomes infinitely large in this limit, making the discrepancy between the average in the local region and that in the global universe smaller and smaller. Then it might be effectively unnecessary to perform the residual gauge transformation at the initial time, although this statement is not very rigorous. We should note that, when we send the initial time to the past infinity, it is too naive to neglect the non-IR modes that are comparable to or shorter than the Hubble length scale, because all the modes were much shorter than the Hubble length scale in the distant past. This makes the issue regarding secular growth much more complicated. For
instance, once we include the contributions from non-IR modes, we cannot use the conservation of $\zeta_k$ in the limit $k/aH \ll 1$, where $k$ is the comoving wavenumber of the external leg, relying on the long wavelength approximation such as $\delta N$ formalism. Here, in a simple example, we show that vertex integrations can yield apparent secular growth through the non-linear contributions from the modes around the Hubble scale. Even if the vertex is confined in the region $O$, the integration region of each vertex is still infinite in the time direction as $\int d^3x a^3 (\cdots) \simeq \int d(\ln a)/H^4(\cdots)$, which may cause secular growth. Roughly speaking, the integrand $(\cdots)$ will be written in terms of the dimensionless time-dependent slow roll parameters and the wavenumber of the fields in this vertex $k_m/aH$, normalized by the Hubble scale. If we focus on the non-linear interaction composed of the modes with $k_m/aH$ of order unity, the integrand $(\cdots)$ is expressed only in terms of parameters that are supposed to change very slowly in time and then the contribution from the interaction vertex seems to yield logarithmic growth. This is another origin of secular growth, which should be distinguished from that inherited from the IR behavior of $\langle (\zeta_I)^2 \rangle$. Of course, the above argument is too naive, but it shows that the absence of secular growth from the vertex integration is rather subtle, requiring more careful treatment of the modes around the Hubble scale. Because of this subtlety, introducing the UV cutoff at a length scale longer than or equal to the Hubble length scale by hand makes the discussion incomplete. In fact, if it were allowable to simply neglect the short wavelength modes, the discussion in Ref. [28] with the initial time $t_i$ sent to $-\infty$ would have given a rough proof of the absence of IR divergence without any limitation on the quantum state by sending the initial time to the past infinity, which contradicts our current claim that the quantum state is restricted in order to avoid IR divergence. Recently, the absence of secular growth was claimed, relying on the conservation of the curvature perturbation in Refs. [37,38], but the aspects mentioned above were not discussed. In addition, even if the conservation of $\zeta_k$ in the limit $k/aH \ll 1$ is proved, logarithmic enhancement in the form $(k/aH)^2 \ln(k/a_i H_i)$ may arise, where $a_i$ and $H_i$ are the scale factor and the Hubble parameter at the initial time. The factor $\ln(k/a_i H_i)$ can become large enough to overcome the suppression by $(k/aH)$ when we send the initial time to the past infinity.

1.2. Summary of upcoming results

Short summary of the results. In this subsection, we summarize what we will show in this paper. Taking account of the current status of IR issues mentioned above, we will establish the following three statements in this paper:

1. There is an alternative equivalent Hamiltonian that describes the quantum dynamics of interest and whose interaction part is solely composed of IR irrelevant operators (which means the field operators associated with the operations that manifestly suppress the IR contribution such as $\partial_t/aH$ and $\partial_t/H$).
2. The Euclidean vacuum state, which is specified by the regularity when the time coordinates in the $n$-point functions are analytically continued to the imaginary in the complex plane, is physically the same both in the alternative description mentioned in item 1, and in the original description.
3. The $n$-point functions in the Euclidean vacuum state respect the spatial translation invariance and are regular in the IR. The secular growth is absent, even if we include the vertices with non-IR modes, as long as very high orders of loop corrections are not concerned.

Below we add some more detailed explanations of the above three items.
Gauge issue. In this paper, we choose to perform the quantization and fix the initial quantum state in the original system that describes the whole universe, leaving the residual gauge degrees of freedom in the local universe unfixed. Then, following Refs. [25–27,30,41], we introduce a field operator that preserves the invariance under any spatial coordinate transformations, including residual gauge transformations. We refer to such an operator as a genuine gauge-invariant curvature perturbation, $\delta R$. As long as such genuine gauge-invariant operators are concerned, we can perform the residual gauge transformation without affecting the results of computations. We will show that, using this residual gauge transformation, the boundary conditions of the non-local operator $\partial^{-2}$ in the action can be modified to be regular in the IR.

Requirement of the gauge invariance in the quantum state. To calculate the $n$-point functions that preserve the invariance under the residual gauge transformations, the initial state should also be specified in a genuinely gauge-invariant manner. However, when we perform the quantization considering the whole universe, preserving the residual gauge invariance becomes obscure, because these residual gauge degrees of freedom are not present as long as we deal with the whole universe. In our previous paper [27], we discovered a correspondence between the IR regularity and the invariance under the residual gauge transformations, which will provide an important clue to the guiding principle in choosing the genuinely gauge-invariant initial state. To discuss this point, aside from the original canonical variable $\zeta(x)$ and its conjugate momentum $\pi(x)$, whose evolution is governed by the action (1.2), we introduced another set of canonical variables corresponding to the description in the coordinates shifted by a constant dilatation transformation:

$$\bar{\zeta}(x) := \zeta(t, e^{-s}x), \quad \bar{\pi}(x),$$

where $s$ is a time-independent complex number and $\bar{\pi}(x)$ is the conjugate momentum of $\bar{\zeta}(x)$. In Ref. [27], we showed that requesting equivalence between the two quantum systems described by $\{\zeta, \pi\}$ and $\{\bar{\zeta}, \bar{\pi}\}$ guarantees the IR regularity of loop corrections. Here, the equivalence of two quantum systems means that the same iteration scheme (or formally the same initial condition of the interacting system) gives physically the same quantum state in both systems related to each other by the dilatation transformation. Namely, all the expectation values evaluated in both systems are equivalent if we take into account how they transform under dilatation transformation. Requesting this equivalence will be thought of as the invariance of the initial state under dilatation transformation. In Ref. [27], we employed the iteration scheme, in which the interaction is turned on at a finite past. Then, it turned out that the IR regularity/gauge invariance condition cannot be consistently imposed, unless we choose a fully scale-invariant spectrum, which does not provide a physically natural ultra-violet behavior. In the present paper, we will set the initial quantum state at the infinite past. We will show that the above transformation can be extended to allow the time dependence of the parameter $s$. As we described in the previous subsection, this extension plays a crucial role in discussing the absence of secular growth.

The Euclidean vacuum. The second and third items are related to each other, once we establish the correspondence between the gauge invariance and the IR regularity. We will show that the two quantum systems described by $\{\zeta, \pi\}$ and $\{\bar{\zeta}, \bar{\pi}\}$ are equivalent if we choose the Euclidean vacuum, which is defined by requesting the regularity of the $n$-point functions at the distant past with the time path rotated toward the complex plane. To be more specific, as the second item, we will show that the $n$-point functions for $\zeta(x)$ calculated by the canonical variables $\{\zeta, \pi\}$ with the boundary condition of the Euclidean vacuum agrees with the $n$-point functions for $\bar{\zeta}(t, e^{i(t)}x)$ calculated by the canonical
variables \( \{ \tilde{\zeta}, \tilde{\pi} \} \) under formally the same boundary condition, i.e.,

\[
\langle \xi(t, x_1) \xi(t, x_2) \cdots \xi(t, x_n) \rangle_{\{ \tilde{\zeta}, \tilde{\pi} \}} = \langle \xi(t, e^{i(t)} x_1) \xi(t, e^{i(t)} x_2) \cdots \xi(t, e^{i(t)} x_n) \rangle_{\{ \tilde{\zeta}, \tilde{\pi} \}}.
\]  

(1.8)

Combined with the previously mentioned technique for dealing with the gauge issue, we will show that, when we choose the Euclidean vacuum, the Hamiltonian density for \( \{ \tilde{\zeta}, \tilde{\pi} \} \) can be expressed only in terms of the IR irrelevant operators.

**The IR regularity and the absence of secular growth.** As for the third item, we evaluate the \( n \)-point function of the genuinely gauge-invariant operator. Performing the quantization in the canonical system of \( \{ \tilde{\zeta}, \tilde{\pi} \} \), we will show that the IR contributions do not diverge and that secular growth is suppressed. We carefully investigate the contributions from the modes that are comparable to or less than the Hubble scale, i.e., \( k \lessgtr aH \), without employing the asymptotic expansion with respect to \( k/aH \). As is stressed at the end of the preceding subsection, this point is one of the necessary ingredients to show the absence of secular growth. One may naively expect that the UV modes with \( k/aH \gtrsim 1 \) will not effectively contribute to the vertex integration because of the oscillatory behavior. A more careful consideration tells us that this naive expectation is not necessarily correct. In general, vertex integrations become a mixture of the positive and negative frequency mode functions, which yields the phase in the UV limit \( e^{i \eta (k_1 - k_2 + k_3 - \cdots)} \), where \( \eta \) represents the conformal time that runs from \( -\infty \) to 0. Then, the phase does not necessarily exhibit rapid oscillation even for the modes with \( k_m / aH \approx -k_m \eta \gtrsim 1 \), which can be a cause of secular growth. Intriguingly, choosing the Euclidean vacuum plays a crucial role not only in the IR limit but also in the UV limit. One can show that there is no mixing between the positive and the negative frequency modes if we choose the Euclidean vacuum. Therefore, secular growth is avoided in this case.

**The outline of the paper.** The outline of this paper is as follows. In Sect. 2, we will briefly review the method of constructing the genuinely gauge-invariant operator \( \delta R \), following Refs. [25,26]. Then, we will introduce the canonical variables \( \{ \tilde{\zeta}, \tilde{\pi} \} \) and derive the Hamiltonian for these variables. In Sect. 3, we will discuss items 1 and 2, mentioned above. In Sect. 3.1, we will describe the boundary conditions of the Euclidean vacuum and prove Eq. (1.8), which implies that the boundary conditions of the Euclidean vacuum select the same ground state both in \( \{ \zeta, \pi \} \) and \( \{ \tilde{\zeta}, \tilde{\pi} \} \). In Sect. 3.2 and Sect. 3.3, we will formulate the canonical quantization in terms of \( \{ \tilde{\zeta}, \tilde{\pi} \} \) and will show that the interacting vertices for these canonical variables consist only of the IR irrelevant operators. In particular, in Sect. 3.3, we will show that, using the residual gauge degrees of freedom, the non-local operator \( \delta^{-2} \) can be made IR regular. In Sect. 4, we will discuss item 3. In Sect. 4.1, we will show that the boundary condition of the Euclidean vacuum leads to the so-called \( i\epsilon \) prescription in a perturbative expansion. In Sect. 4.2, we will calculate the Wightman propagator, by which the \( n \)-point functions are expanded. Then, in Sect. 4.3, we explicitly evaluate \( n \)-point functions to investigate the IR regularity and secular growth. In Sect. 5, as concluding remarks, we discuss another possibility of the initial state that satisfies the IR regularity/gauge invariance conditions. We will also mention related papers to clarify what is new in this paper.

**The advantage of the in–in formalism.** In our previous publications [27–29], in calculating the \( n \)-point functions, we used the retarded Green function to solve the non-linear Heisenberg equation. This is because we thought that using the retarded Green function, whose Fourier mode is regular in the IR limit, makes the proof of the IR regularity transparent. However, the perturbative expansion using the retarded Green function is not suitable for the present purpose, because the positive and negative frequency modes are mixed in the vertex integrations once the retarded Green function is used. Therefore, the boundary conditions of the Euclidean vacuum do not guarantee the convergence.
of the time integrations for all the vertices. By contrast, when we calculate the $n$-point functions in the in–in formalism, all vertex integrals can be made manifestly convergent by adopting the boundary condition of the Euclidean vacuum (see Sect. 3.1). Since the $n$-point functions obtained from the solution written in terms of the retarded Green function agree with those obtained in the in–in formalism, the vertices that do not converge should vanish in the final result of the $n$-point functions. However, the cancellation is obscured in an explicit perturbative expansion. Therefore, in this paper, we calculate the $n$-point function totally based on the in–in formalism, without using the retarded Green function.

2. Constructing the gauge-invariant quantity

In this paper, as an explicit model of inflation, we consider a standard single-field inflation model whose action takes the form

$$S = \frac{M_{\text{pl}}^2}{2} \int \sqrt{-g} \left[ R - g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right] d^4 x,$$  \hspace{1cm} (2.1)

where $M_{\text{pl}}$ is the Planck mass and we set $\phi$ to a dimensionless scalar field, dividing it by $M_{\text{pl}}$. However, as long as we consider a scalar field with the second-order kinetic term, the extension proceeds in a straightforward way. In Sect. 2.1, we will construct the genuine gauge-invariant operator corresponding to the spatial curvature of a $\phi$-constant surface. In Sect. 2.2, we will introduce the canonical system $\{\tilde{\zeta}, \tilde{\pi}\}$ whose Hamiltonian density is composed only of the IR irrelevant operators.

2.1. Gauge-invariant operator and quantization

We fix the time slicing by adopting the uniform field gauge $\delta \phi = 0$. Under the metric decomposition in the Arnowitt, Deser, and Misner (ADM) formalism, which is given by

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N_i dt)(dx^j + N_j dt),$$  \hspace{1cm} (2.2)

we take the spatial metric $h_{ij}$ as

$$h_{ij} = e^{2(\rho + \zeta)} \left[ e^{\delta \gamma} \right]_{ij},$$  \hspace{1cm} (2.3)

where $a := e^\rho$ is the scale factor, $\zeta$ is the so-called curvature perturbation, and $\delta \gamma_{ij}$ is a traceless tensor:

$$\delta \gamma^i_j = 0.$$  \hspace{1cm} (2.4)

As spatial gauge conditions we impose transverse conditions on $\delta \gamma_{ij}$:

$$\partial_i \delta \gamma^i_j = 0.$$  \hspace{1cm} (2.5)

Since the time slicing is fixed by the gauge condition $\delta \phi = 0$, there are remaining residual gauge degrees of freedom only in choosing the spatial coordinates. In this paper, we neglect the vector and tensor perturbations. The tensor perturbation, which is massless, can also contribute to the IR divergence of loop corrections. We will address this issue in a future publication.

Following Refs. [25,26], we construct a genuine gauge-invariant operator, which preserves the gauge invariance in the local observable universe. For the construction, we note that the scalar curvature $^8R$, which transforms as a scalar quantity under spatial coordinate transformations, becomes genuinely gauge invariant, if we evaluate it in the geodesic normal coordinates on each time slice.
The geodesic normal coordinates are introduced by solving the spatial 3D geodesic equation:

\[
\frac{d^2 x^i_{gl}}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j_{gl}}{d\lambda} \frac{dx^k_{gl}}{d\lambda} = 0,
\]

(2.6)

where \( \Gamma^i_{jk} \) is the Christoffel symbol with respect to the 3D spatial metric on a constant time hypersurface and \( \lambda \) is the affine parameter. Here we put the index \( gl \) on the global coordinates, to reserve the simple notation \( x \) for the geodesic normal coordinates, which will be mainly used in this paper. We consider the 3D geodesics whose affine parameter ranges from \( \lambda = 0 \) to \( 1 \) with the initial “velocity” given by

\[
\frac{dx^i_{gl}(x, \lambda)}{d\lambda} \bigg|_{\lambda=0} = e^{-\zeta(\lambda=0)} x^i.
\]

(2.7)

A point \( x^i \) in the geodesic normal coordinates is identified with the end point of the geodesic, \( x^i_{gl}(x, \lambda = 1) \) in the original coordinates. Using the geodesic normal coordinates \( x^i \), we perturbatively expand \( x^i_{gl} \) as \( x^i_{gl} = x^i + \delta x^i(x) \). Then, we can construct a genuinely gauge-invariant variable as

\[
\delta R(t, x) := \delta R(t, x^i_{gl}(x)) = \delta R(t, x^i + \delta x^i(x)),
\]

(2.8)

where \( t \) denotes the cosmological time.

2.2. Dilatation symmetry in the global universe

The focus of this subsection is on the dilatation transformation, shifting to the rescaled spatial coordinates:

\[
\tilde{x}^i := e^{s(t)} x^i.
\]

(2.9)

Solving the Hamiltonian and momentum constraint equations, we can derive the action that is expressed only in terms of the curvature perturbation \( \zeta(x) \), which is schematically written as

\[
S = \int dt \, d^3x \, \mathcal{L}[\partial_t \zeta(x), \zeta(x)].
\]

(2.10)

Using the curvature perturbation \( \zeta \) and the conjugate momentum defined by \( \pi := \delta \mathcal{L}/\delta(\partial_t \zeta) \), the Hamiltonian density is given by the Legendre transform as

\[
\mathcal{H}[\zeta(x), \pi(x)] := \pi(x) \partial_t \zeta(x) - \mathcal{L}[\partial_t \zeta(x), \zeta(x)].
\]

(2.11)

What is important here is only the fact that the curvature perturbation \( \zeta \) appear in the action either with differentiation or in the form of the combination of the physical distance \( e^{\rho+\zeta} dx \) [27]. In the new coordinates (2.9), the physical distance is written as \( e^{\rho+\tilde{\zeta}(t, \tilde{x})-s(t)} d\tilde{x} \), with the definition of a new variable

\[
\tilde{\zeta}(t, \tilde{x}) := \zeta(t, x).
\]

(2.12)

Thus, if the field \( \zeta(x) \) is replaced with \( \tilde{\zeta}(t, \tilde{x}) - s(t) \) under the change of the coordinates from \( x \) to \( \tilde{x} \), the action basically remains invariant. To express \( \partial_t \zeta(x) \) in terms of the new variable \( \tilde{\zeta} \), we denote the partial differentiation with the spatial coordinates \( x \) fixed as \( (\partial_t \tilde{\zeta}(t, \tilde{x}))_x \). The subscript associated with the parentheses specifies the spatial coordinates that we fix in taking the partial differentiation.
Then, we have
\[ (\partial_t \tilde{\zeta}(t, \tilde{x}))_x = \partial_t \zeta(x). \]  
(2.13)

For brevity, when the fixed spatial coordinates are identical to the ones in the argument of the variable, we simply use \( \partial_t \). Then, we can establish an identity
\[ \int dt \, d^3x \, L[\partial_t \zeta(x), \zeta(x)] = \int dt \, d^3\tilde{x} \, L[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)]. \]  
(2.14)

Recalling the relation between \( x \) and \( \tilde{x} \) (2.9), this equality also means an equality at the level of the Lagrangian density, \( e^{-3s(t)} L[\partial_t \zeta(x), \zeta(x)] = L[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)]. \)

We introduce the canonical conjugate momentum corresponding to \( \tilde{\zeta}(t, \tilde{x}) \) in the standard way as
\[ \tilde{\pi}(t, x) := \frac{\partial L[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)]}{\partial (\partial_t \tilde{\zeta}(t, \tilde{x}))}. \]  
(2.15)

Noting the relation
\[ \partial_t \tilde{\zeta}(t, \tilde{x}) = (\partial_t \tilde{\zeta}(t, \tilde{x}))_x - \dot{s}(t) \tilde{x} \cdot \partial_x \tilde{\zeta}(t, \tilde{x}), \]  
(2.16)

we have
\[ \tilde{\pi}(t, \tilde{x}) = \frac{\partial L[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)]}{\partial (\partial_t \tilde{\zeta}(t, \tilde{x}))} = e^{-3s(t)} \frac{\partial L[\partial_t \zeta(x), \zeta(x)]}{\partial (\partial_t \tilde{\zeta}(t, \tilde{x}))} = e^{-3s(t)} \pi(x). \]  
(2.17)

As is expected, using the commutation relations for \( \zeta \) and \( \pi \) together with Eqs. (2.12) and (2.17), we can verify
\[ \left[ \tilde{\zeta}(t, \tilde{x}), \tilde{\pi}(t, \tilde{y}) \right] = e^{-3s(t)} i \delta^{(3)}(x - y) = i \delta^{(3)}(\tilde{x} - \tilde{y}), \]  
(2.18)

as well as
\[ \left[ \tilde{\zeta}(t, \tilde{x}), \tilde{\zeta}(t, \tilde{y}) \right] = \left[ \tilde{\pi}(t, \tilde{x}), \tilde{\pi}(t, \tilde{y}) \right] = 0. \]  
(2.19)

The Hamiltonian density for \( \tilde{\zeta}(\tilde{x}) \) and \( \tilde{\pi}(\tilde{x}) \) is obtained in the standard way as
\[ \tilde{\mathcal{H}} \left[ \tilde{\zeta}(t, \tilde{x}), \tilde{\pi}(t, \tilde{x}) \right] := \tilde{\pi}(t, \tilde{x}) \partial_t \tilde{\zeta}(t, \tilde{x}) - \mathcal{L}[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)] \]
\[ = \tilde{\pi}(t, \tilde{x}) (\partial_t \tilde{\zeta}(t, \tilde{x}))_x - \mathcal{L}[(\partial_t \tilde{\zeta}(t, \tilde{x}))_x, \tilde{\zeta}(t, \tilde{x}) - s(t)] \]
\[ - \dot{s}(t) \tilde{x} \cdot \partial_x \tilde{\zeta}(t, \tilde{x}) = \mathcal{H}[\tilde{\zeta}(t, \tilde{x}) - s(t), \tilde{\pi}(t, \tilde{x})] - \dot{s}(t) \tilde{x} \cdot \partial_x \tilde{\zeta}(t, \tilde{x}), \]  
(2.20)

where in the equality on the second line we have used Eq. (2.16). The last equality is exactly the same Legendre transformation as in the original system and therefore we can use the same functional form of the Hamiltonian density \( \mathcal{H} \).

Assuming that \( s(t) \) is as small as \( \tilde{\zeta}(x) \) and \( \tilde{\pi}(x) \), we decompose the Hamiltonian densities \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) into non-interacting parts, which include only the quadratic terms, and interacting parts as
\[ \mathcal{H}[\zeta(x), \pi(x)] = \mathcal{H}_0[\zeta(x), \pi(x)] + \mathcal{H}_I[\zeta(x), \pi(x)], \]  
(2.21)

and
\[ \tilde{\mathcal{H}} \left[ \tilde{\zeta}(x), \tilde{\pi}(x) \right] = \mathcal{H}_0 \left[ \tilde{\zeta}(x), \tilde{\pi}(x) \right] + \tilde{\mathcal{H}}_I \left[ \tilde{\zeta}(x), \tilde{\pi}(x) \right]. \]  
(2.22)

In the above we have used the coordinates \( x \) instead of \( \tilde{x} \) for the \( \{ \zeta, \pi \} \) system, but it will not cause any confusion after the relations between the \( \{ \zeta, \pi \} \) and \( \{ \tilde{\zeta}, \tilde{\pi} \} \) systems have been established. Here, we have replaced \( \mathcal{H}_0[\tilde{\zeta}(x) - s(t), \tilde{\pi}(x)] \) with \( \mathcal{H}_0[\tilde{\zeta}(x), \tilde{\pi}(x)] \), since \( \zeta(x) \) always appears with the
spatial derivative in $\mathcal{H}_0[\zeta(x), \pi(x)]$. Remarkably, the non-interacting part of the Hamiltonian density does not change at all under the dilatation transformation. Using Eq. (2.20), we find that the interaction Hamiltonian $\tilde{\mathcal{H}}_I[\tilde{\zeta}, \tilde{\pi}]$ is given by

$$\tilde{\mathcal{H}}_I[\tilde{\zeta}(x), \tilde{\pi}(x)] := \mathcal{H}_I\left[\tilde{\zeta}(x) - s(t), \tilde{\pi}(x) - \dot{s}(t)\pi(x)\right] - \dot{s}(t)\pi(x)\mathbf{x} \cdot \partial_x \tilde{\zeta}(x). \quad (2.23)$$

In this way, we can write down $\tilde{\mathcal{H}}_I$ only in terms of $\tilde{\zeta}(x) - s(t)$, $\tilde{\zeta}$ with differentiation, $\tilde{\pi}$ and $\dot{s}(t)$. In Ref. [27], we introduced the two sets of the canonical conjugate variables that are connected by the dilatation transformation with a constant parameter $s$. When we take the limit where $s(t)$ is constant, the Hamiltonian density $\tilde{\mathcal{H}}(x)$ takes the same functional form as $\mathcal{H}(x)$ except for the constant shift of $\tilde{\zeta}(x)$ by $-s$. This is because, without modifying the gauge condition, we can also perform the dilatation transformation with the constant parameter $s$ in the whole universe. Then the action that preserves the diffeomorphic invariance becomes invariant under the change from $\zeta(x)$ to $\zeta(t, e^{-\epsilon}x) - s$. Here we have extended the argument in Ref. [27] to allow $s$ to depend on time. As we mentioned in Sect. 1, this extension plays a crucial role in our discussion about secular growth. In the next section, we will show that all the interaction vertices in the canonical system $\{\zeta, \pi\}$ are composed only of the IR irrelevant operator.

3. Interaction Hamiltonian with IR irrelevant operators

In this section, we describe the first two of the three items we raised in Sect. 1. In the preceding section, we derived the Hamiltonian for the canonical variables $\tilde{\zeta}(x)$ and $\tilde{\pi}(x)$. Since $\{\zeta, \pi\}$ and $\{\tilde{\zeta}, \tilde{\pi}\}$ are connected by the canonical transformation, if we choose the same initial state in both of the canonical systems, the $n$-point functions for the same operator, for instance $g^R$, calculated in these canonical systems should agree with each other. However, even if we adopt operationally the same scheme to select the initial state in these two systems, it does not guarantee that the selected initial states are the same. In Sect. 3.1, after we describe the definition of the Euclidean vacuum, we will show that the condition of the Euclidean vacuum operationally selects the same quantum state irrespective of the choice of the canonical variables. This ensures the equivalence of these two canonical systems including the choice of the initial quantum state, which we mentioned in item 2. In Sect. 3.2, we will perform quantization using the canonical variables $\{\tilde{\zeta}, \tilde{\pi}\}$. As we will show in Sect. 3.3, by virtue of the equivalence between the two canonical systems, the interaction vertices for $\{\tilde{\zeta}, \tilde{\pi}\}$ can be expressed in terms of operator products composed only of the IR irrelevant operators.

3.1. Euclidean vacuum and its uniqueness

In the case with a massive scalar field in de Sitter spacetime, the boundary condition specified by rotating the time path in the complex plane can be understood as requesting the regularity of correlation functions on the Euclidean sphere that can be obtained by the analytic continuation from those on de Sitter spacetime. The vacuum state thus defined is called the Euclidean vacuum state. Because of the similarity, here we also refer to the state that is specified by a similar boundary condition as the Euclidean vacuum. To be more precise, we define the Euclidean vacuum as follows. In the in–in formalism, the insertion of interaction vertices is ordered along a closed time path. By rotating the time path toward the imaginary plane, the forward time evolution begins at $\eta(t_i) = -\infty(1 - i\epsilon)$ and ends at the final time $t_f$, and the backward time evolution begins at $t_f$ and ends at $\eta(t_i) = -\infty(1 + i\epsilon)$. Here we set $\epsilon$ to a small positive number. Since rotating the time path can be better understood by
using the conformal time $\eta$, we introduce the conformal time $\eta$ as

$$\eta(t) := \int_t^i \frac{dr'}{e^{\rho(t')}} = \int_{\rho(t)}^{\rho(t')} \frac{d\rho'}{e^{\rho(t')}\dot{\rho}(\rho')}.$$  

(3.1)

We define the Euclidean vacuum, requesting the regularity of the $n$-point functions with an arbitrary natural number $n$ in the limit of $\eta(t_e) \to -\infty(1 \pm \epsilon)$, i.e.,

$$F_n(x_1, \ldots, x_n) := \langle T_e \xi(x_1) \cdots \xi(x_n) \rangle \to \infty \quad \text{as} \quad \eta(t_e) \to -\infty(1 \pm \epsilon),$$  

(3.2)

where $a = 1, \ldots, n$ and $T_e$ denotes the time ordering along the closed time path. We first show that the $n$-point functions of $\zeta$ are uniquely fixed by requesting the condition (3.2). In this paper, for simplicity, we assume that $e^{\rho(t)} \dot{\rho}(\rho)$ is rapidly increasing in time so that

$$|\eta(t)| = O(e^{\rho(t)} \dot{\rho}(t)).$$  

(3.3)

Next, we show that the boundary condition of the Euclidean vacuum uniquely determines the $n$-point functions $F_n(x_1, \ldots, x_n)$. We schematically describe the Heisenberg equation for $\zeta(x)$ as

$$\mathcal{L} \zeta = \mathcal{S}_{NL}[\zeta],$$  

(3.4)

where $\mathcal{L}$ is the second-order differential operator:

$$\mathcal{L} := \partial_{\rho}^2 + (3 - \epsilon_1 + \epsilon_2) \partial_\rho - \frac{\partial^2}{e^{2\rho} \dot{\rho}^2}.$$  

(3.5)

For notational convenience, we introduce the horizon flow functions,

$$\epsilon_1 := -\frac{1}{\rho} \frac{d}{d\rho} \dot{\rho}, \quad \epsilon_n := \frac{1}{\epsilon_{n-1}} \frac{d}{d\rho} \epsilon_{n-1},$$  

(3.6)

with $n \geq 2$, but we do not assume that these functions are small to keep the background evolution unconstrained, except for requesting Eq. (3.3), which is valid, for instance, when $\epsilon_n$ are constant in time. Using the Heisenberg equation (3.4), we can obtain the evolution equation of the path-ordered $n$-point functions $F_n(x_1, \ldots, x_n)$ as

$$\mathcal{L}_{x_a} F_n(x_1, \ldots, x_n) = \mathcal{V}^{(a)}_{NL}[F_m]_{m>n},$$  

(3.7)

where $\mathcal{L}_{x_a}$ is the derivative operator $\mathcal{L}$ given in Eq. (3.5) with the coordinates $x$ replaced with $x_a$. Since the equation of motion for $\zeta(x)$ is non-linear, Eq. (3.7) includes the source term (the right-hand side) composed of $m$-point functions of $\zeta(x)$ with $m > n$. We can verify the uniqueness of the $n$-point functions for $\zeta(x)$ by showing that the solution of Eq. (3.7) is uniquely fixed by the boundary condition (3.2). To show this uniqueness, we formally solve the equation (3.7) as

$$F_n(x_1, \ldots, x_n) = f_n(x_1, \ldots, x_n) + \mathcal{L}_{x_a}^{-1} \mathcal{V}^{(a)}_{NL}[F_m]_{m>n},$$  

(3.8)

where $f_n(x_1, \ldots, x_n)$ is a homogeneous solution, while we assume that the specific solution $\mathcal{L}_{x_a}^{-1} \mathcal{V}^{(a)}_{NL}[F_m]_{m>n}$ satisfies the regularity condition in the limits $\eta(t_e) \to -\infty(1 \pm \epsilon)$. Now the question is whether the boundary condition (3.2) allows us to add any homogeneous solutions. In the Fourier space, $f_n$ can be expanded by $e^{-ik_\eta(t_a)}$ or $e^{ik_\eta(t_a)}$ in the limits $\eta(t_a) \to -\infty(1 \pm \epsilon)$ accepts $e^{-ik_\eta(t_a)}$ only, while the regularity at $\eta(t_a) \to -\infty(1 + i\epsilon)$ accepts $e^{ik_\eta(t_a)}$ only. The regularity condition in the two limits does not allow us to add any homogeneous solutions $f_n$, which implies that the $n$-point functions $F_n(x_1, \ldots, x_n)$ are uniquely fixed by the boundary condition of the Euclidean vacuum.
Next, we show that this uniqueness is ensured independent of whether we use the canonical variables \( \{\zeta, \pi\} \) or \( \{\tilde{\zeta}, \tilde{\pi}\} \). We employ the boundary condition of the Euclidean vacuum for the canonical variable \( \tilde{\zeta} \) as well, requesting
\[
\langle T_c \tilde{\zeta}(x_1) \cdots \tilde{\zeta}(x_n) \rangle_{\{\tilde{\zeta}, \tilde{\pi}\}} < \infty \quad \text{as} \quad \eta(t_0) \rightarrow -\infty (1 \pm i\epsilon). \tag{3.9}
\]
Then, we can show that the path-ordered \( n \)-point functions
\[
\tilde{F}_n(x_1, \ldots, x_n) := \langle T_c \tilde{\zeta}(t_1, e^{s(t_1)} x_1) \cdots \tilde{\zeta}(t_n, e^{s(t_n)} x_n) \rangle_{\{\tilde{\zeta}, \tilde{\pi}\}} \tag{3.10}
\]
agree with the \( n \)-point functions \( F_n(x_1, \ldots, x_n) = \langle T_c \zeta(x_1) \cdots \zeta(x_n) \rangle_{\{\zeta, \pi\}} \) fixed by the boundary condition (3.2), i.e.,
\[
\tilde{F}_n(x_1, \ldots, x_n) = F_n(x_1, \ldots, x_n). \tag{3.11}
\]
Here, putting the suffixes \( \{\zeta, \pi\} \) or \( \{\tilde{\zeta}, \tilde{\pi}\} \), we denote the canonical variables used in imposing the boundary condition explicitly. We again schematically describe the Heisenberg equation for \( \tilde{\zeta} \) as
\[
\mathcal{L}\tilde{\zeta} = \tilde{S}_{NL}[\tilde{\zeta}]. \tag{3.12}
\]
Since \( \zeta(x) \) and \( \tilde{\zeta}(x) \) are connected by the canonical transformation, the equation of motion obtained by operating \( \mathcal{L} \) on
\[
\zeta(x) = \tilde{\zeta}(t, e^{s(t)} x) = \tilde{\zeta}(x) + s(t) x \cdot \partial_x \tilde{\zeta}(x) + \cdots \tag{3.13}
\]
can be recast into Eq. (3.4) by using Eq. (3.12). A similar argument follows for the equations of motion for the correlation functions \( F_n \) and \( \tilde{F}_n \). Using the equation of motion for the \( n \)-point functions of \( \tilde{\zeta}(x) \), which can be derived from Eq. (3.12), we can confirm that an operation of \( \mathcal{L}_{x_a} \) on
\[
\tilde{F}_n(x_1, \ldots, x_n) = \langle T_c \tilde{\zeta}(x_1) \cdots \tilde{\zeta}(x_n) \rangle_{\{\tilde{\zeta}, \tilde{\pi}\}} + s(t_1) \langle T_c x_1 : \partial_{x_1} \tilde{\zeta}(t_1, x_1) \cdots \tilde{\zeta}(t_n, x_n) \rangle_{\{\tilde{\zeta}, \tilde{\pi}\}} + \cdots \tag{3.14}
\]
leads to
\[
\mathcal{L}_{x_a} \tilde{F}_n(x_1, \ldots, x_n) = \mathcal{V}_{NL}^{(a)}[\{\tilde{F}_m\}_{m>n}]. \tag{3.15}
\]
This equation takes the same form as the equation of motion (3.7). We also note that the boundary condition of the Euclidean vacuum (3.9) implies
\[
\tilde{F}_n(x_1, \ldots, x_a, \ldots, x_n) < \infty \quad \text{as} \quad \eta(t_a) \rightarrow \infty (1 \pm i\epsilon). \tag{3.16}
\]
The equivalence (3.11) is now transparent, because the equations of motion (3.7) and (3.15), and the boundary conditions (3.2) and (3.16) are the same, and the latter specify the solutions of the former uniquely. This equivalence is a distinctive property of the Euclidean vacuum. Here we took the boundary conditions for the \( n \)-point functions as the definition of the Euclidean vacuum state, assuming the existence of such a quantum state. In Sect. 4.1, we explain that such a Euclidean vacuum, if it exists, should be the one given by the ordinary \( i\epsilon \) prescription.

\[1\] The uniqueness of the Euclidean vacuum becomes intuitively clear when the Hamiltonian is time independent and the lowest energy eigenstate is non-degenerate, because the \( i\epsilon \) prescription selects the unique ground state of the system.
3.2. Rewriting the n-point functions

In this subsection, we rearrange the expression for the \( n \)-point functions of the genuinely gauge-invariant variable \( gR \) into a more suitable form to examine the regularity of the IR contributions. First, solving the 3D geodesic equations, we obtain the relation between the global coordinates \( x^i_{gl} \) and the geodesic normal coordinates \( x^i \) as

\[
x^i_{gl} = e^{-\zeta(t,e^{-\zeta}x)}x^i + \cdots, \tag{3.17}
\]

where the ellipsis means the terms that vanish when \( \zeta(x) \) is spatially homogeneous, i.e., the terms suppressed in the IR limit. Note that changing the spatial coordinates into geodesic normal coordinates also modifies the UV contributions. Tsamis and Woodard [45] showed that using the geodesic normal coordinates can introduce an additional origin of UV divergence, which may not be able to be renormalized by local counter-terms [46]. It should be clarified whether this issue is a serious problem or not, but we defer this to a future study. Instead, to keep the UV contributions under control, we replace \( \zeta(x) \) in Eq. (3.17) with the smeared curvature perturbation \( \bar{\zeta}(t) \), i.e.,

\[
x^i_{gl} = e^{-\bar{\zeta}(t)}x^i, \tag{3.18}
\]

with

\[
\bar{\zeta}(t) := \frac{\int d^3x W_L(x) \zeta(t, e^{-\bar{\zeta}}x)}{\int d^3x W_L(x)}, \tag{3.19}
\]

where \( W_L(x) \) is a window function that is non-vanishing only in the local region \( \Sigma_t \cap \mathcal{O} \). We approximate the averaging scale at each time \( t \) by the Hubble scale, i.e., \( L_t \simeq 1/\{e^{\rho(t)} \dot{\rho}(t) \} \). Although \( \bar{\zeta} \) appears on the right-hand side of Eq. (3.19), \( \bar{\zeta} \) is defined iteratively at each order of the perturbation. We calculate the \( n \)-point functions of \( R x g \zeta(t, x) \), instead of \( gR \), with

\[
g\zeta(t, x) := \zeta(t, e^{-\bar{\zeta}(t)}x). \tag{3.20}
\]

Here, \( R_x \) denotes the IR suppressing operator such as

\[
\partial_\rho, \quad \frac{\partial_x}{e^{\rho(t)} \dot{\rho}(t)}, \quad \left( 1 - \frac{\int d^3x W_L(x)}{\int d^3y W_L(y)} \right), \quad \cdots. \tag{3.21}
\]

where \( x \) is the spacetime coordinates of the field on which these operators act. Although \( R_x \zeta(t, x) \) is not genuinely gauge invariant, it is still invariant under the dilatation transformation, which is associated with the dominant IR contributions. In fact, since the smeared curvature perturbation \( \bar{\zeta}(t) \) transforms into \( \bar{\zeta}(t) - f \) under the dilatation transformation: \( x \to e^{-f}x \) with a constant \( f \), \( R_x \bar{\zeta}(x) \) is kept invariant under this transformation. By contrast, the constant part of \( \zeta(x) \) can be modified under the dilatation transformation as \( \zeta(x) \to \zeta(x) - f \). Since the genuine gauge-invariant variable \( gR(x) \) should not be affected by the dilatation transformation, which is a part of the residual gauge transformations, \( \zeta(x) \) appears only in the form of \( R_x \zeta(x) \) when we express \( gR(x) \) in terms of \( \zeta(x) \). As we can compute \( gR(x) \) from \( R_x \zeta(x) \), our goal is to prove that the expectation values of the products of \( R_x \zeta(x) \) are IR regular.
First, we calculate the \( n \)-point functions of \( \xi \) without the IR suppressing operator \( \mathcal{R}_\epsilon \):

\[
\langle 0 |^{\xi} \xi(t_f, x_1) \cdots \xi(t_f, x_n) | 0 \rangle.
\] (3.22)

Using the eigenstates of \( \xi \) \( |s\rangle \) that satisfy \( \xi \langle s | H s \rangle = s \langle s | H s \rangle \), we can construct a unit operator

\[
1 = \int ds |s \rangle H s \langle s |.
\] (3.23)

Inserting this into the expression for the \( n \)-point functions, we obtain

\[
\langle 0 |^{\xi} \xi(t_f, x_1) \cdots \xi(t_f, x_n) | 0 \rangle = \int ds \left( \langle 0 | \xi(t_f, e^{-\xi} x_1) \cdots \xi(t_f, e^{-\xi} x_n) | s \rangle \right)_H \langle s | 0 \rangle
\]

\[
= \int ds \left( \langle 0 | \tilde{\xi}(t_f, x_1) \cdots \tilde{\xi}(t_f, x_n) | s \rangle \right)_H \langle s | 0 \rangle.
\] (3.24)

In the first line we could simply replace \( \xi \) with \( s \), because \( \xi \) and \( \xi(t_f, x) \) commute with each other. Since the Heisenberg picture field \( \tilde{\xi}(t, x) \) is related to the interaction picture field \( \tilde{\xi}_I(t, x) \) as

\[
\tilde{\xi}(t, x) = \tilde{U}_I(t) \tilde{\xi}_I(t, x) \tilde{U}_I(t),
\] (3.25)

where the unitary operator \( \tilde{U}_I(t) \) is given by

\[
\tilde{U}_I(t) := \lim_{\eta(t) \to -\infty (1 - i\epsilon)} T \exp \left[ -i \int_{t_1}^t dt \int d^3x \tilde{H}_I \left[ \tilde{\xi}_I(x), \tilde{\pi}_I(x) \right] \right]
\]

\[
= \lim_{\eta(t) \to -\infty (1 - i\epsilon)} \sum_{n=0}^{\infty} (-i)^n \int_{t_1}^t dt_n \int_{t_1}^{t_n} dt_{n-1} \cdots \int_{t_1}^{t_2} dt_1 \times \int d^3x_1 \cdots \int d^3x_n \tilde{H}_I \left[ \tilde{\xi}_I(x_n), \tilde{\pi}_I(x_n) \right] \tilde{H}_I \left[ \tilde{\xi}_I(x_1), \tilde{\pi}_I(x_1) \right].
\] (3.26)

Thus, the \( n \)-point function can be rewritten as

\[
\langle 0 |^{\xi} \xi(t_f, x_1) \cdots \xi(t_f, x_n) | 0 \rangle = \int ds \left( \langle 0 | \tilde{\xi}_I(t_f, x_1) \cdots \tilde{\xi}_I(t_f, x_n) T \exp \left[ -i \int dt \int d^3x \tilde{H}_I \left[ \tilde{\xi}(x), \tilde{\pi}(x) \right] \right] \right)_H \langle s | 0 \rangle.
\] (3.27)

where \( \tilde{T} \) denotes the anti-time-ordered product. Note that the interaction Hamiltonian \( \tilde{H}_I \) does not contain a second or higher derivative of \( s(t) \). Using the eigenstates \( |s(t)\rangle \) and \( |\dot{s}(t)\rangle \) that satisfy

\[
\xi \xi I(t)|s(t)\rangle = s(t)|s(t)\rangle, \quad \xi \xi I(t)|\dot{s}(t)\rangle = \dot{s}(t)|\dot{s}(t)\rangle
\] (3.28)

where

\[
\xi \xi I(t) := \frac{\int d^3x W_L(x) \xi I(t, e^{-\xi I(t)} x)}{\int d^3x W_L(x)}
\] (3.29)

is the smeared interaction picture field, we construct unit operators as

\[
1 = \int ds(t) |s(t)\rangle \langle s(t)|, \quad 1 = \int d\dot{s}(t) |\dot{s}(t)\rangle \langle \dot{s}(t)|.
\] (3.30)

We next replace all \( s(t) \) and \( \dot{s}(t) \) with \( \xi \xi I(t) \) and \( \xi \xi I(t) \), respectively, by inserting the unit operators. To perform this replacement without ambiguity, we fix the operator ordering in \( \tilde{H}_I \) to Weyl ordering,
in which \( \tilde{\zeta}_I(x) - s(t) \) and \( \tilde{\pi}_I(x) \) are symmetrized. Instead of considering the explicit form of the interaction Hamiltonian, we use a schematic expression of \( \tilde{\mathcal{H}}_I \) that is expanded in a power series of \( \tilde{s}(t) \) as

\[
\tilde{\mathcal{H}}_I \left[ \tilde{\zeta}_I(x), \tilde{\pi}_I(x) \right] = \sum_{\alpha=0} \left[ \tilde{s}(t) \right]^\alpha \tilde{\mathcal{H}}_I(\alpha) \left[ \tilde{\zeta}_I(x) - s(t), \tilde{\pi}_I(x) \right],
\]

(3.31)

although \( \alpha \) is at most 1. Here, we stress that the perturbations \( \tilde{\zeta}_I(x) \) and \( s(t) \) appear in the Hamiltonian density \( \tilde{\mathcal{H}}_I \) only in the form of \( \tilde{\zeta}_I(x) - s(t) \) or its spatial differentiations. Inserting the unit operators, we obtain

\[
\tilde{\mathcal{H}}_I \left[ \tilde{\zeta}_I(x), \tilde{\pi}_I(x) \right] = \sum_{\alpha=0} \int ds(t) \int d\tilde{s}(t) \left[ \tilde{s}(t) \right]^\alpha \tilde{\mathcal{H}}_I(\alpha) \left[ \tilde{\zeta}_I(x) - s(t), \tilde{\pi}_I(x) \right]
\]

\[
\times \left| s(t) \right\langle s(t) \left| \tilde{s}(t) \right\rangle \left( \tilde{s}(t) \right). \tag{3.32}
\]

After we replace \( s(t) \) with \( \tilde{s} \tilde{\zeta}_I(t) \), \( \tilde{\xi}_I(t) \) is located next to the operator \( \left| s(t) \right\rangle \langle s(t) \right| \). Noting the fact that \( s(t) \left| s(t) \right\rangle \) can be expressed as

\[
s(t) \left| s(t) \right\rangle = \tilde{s} \tilde{\zeta}_I(t) \left| s(t) \right\rangle = \frac{\int d^3x \, W_{L_i}(x) \zeta_I(t, e^{-s(t)}x)}{\int d^3x \, W_{L_i}(x)} \left| s(t) \right\rangle,
\]

(3.33)

where, in the second equality, we have replaced \( \tilde{s} \tilde{\zeta}_I(t) \) in the argument of \( \zeta_I \) with \( s(t) \), we use \( \tilde{s} \tilde{\zeta}_I(t) \) expressed as

\[
\tilde{s} \tilde{\zeta}_I(t) = \frac{\int d^3x \, W_{L_i}(x) \zeta_I(t)}{\int d^3x \, W_{L_i}(x)},
\]

(3.34)

instead of the expression given in Eq. (3.29), when we replace \( s(t) \) with \( \tilde{s} \tilde{\zeta}_I(t) \). Using the formula

\[
\left( \zeta_I(t, x) - s(t) \right) \mathcal{A} \left| s(t) \right\rangle = \left( \zeta_I(t, x) - \tilde{s} \tilde{\zeta}_I(t) \right) \mathcal{A} \left| s(t) \right\rangle + \left[ \tilde{s} \tilde{\zeta}_I(t), \mathcal{A} \right] \left| s(t) \right\rangle,
\]

(3.35)

we replace \( (\zeta_I(t, x) - s(t)) \) with \( (\zeta_I(t, x) - \tilde{s} \tilde{\zeta}_I(t)) \) one by one. By induction, the operator \( \mathcal{A} \) is supposed to be composed of \( \tilde{\zeta}_I(x) - s(t) \) and \( \tilde{\pi}_I(x) \). Since \( \tilde{s} \tilde{\zeta}_I(t) \) commute with \( \tilde{\zeta}_I(x) - s(t) \), the non-vanishing commutation relation is only the following:

\[
\left[ \tilde{s} \tilde{\zeta}_I(t), \tilde{\pi}_I(t, x) \right] = \frac{1}{\int d^3x \, W_{L_i}(x)} \int d^3y \, W_{L_i}(y) \left[ \tilde{\zeta}_I(t, x), \tilde{\pi}_I(t, y) \right] = i \frac{W_{L_i}(x)}{\int d^3x \, W_{L_i}(x)}, \tag{3.36}
\]

where we have used

\[
\left[ \tilde{\zeta}_I(t, x), \tilde{\pi}_I(t, y) \right] = \tilde{U}_I(t) \left[ \zeta(t, x), \tilde{\pi}(t, y) \right] \tilde{U}_I^*(t) = i \delta^{(3)}(x - y). \tag{3.37}
\]

Since the commutator including \( \tilde{s} \tilde{\zeta}_I(t) \) yields only a local function, we can conclude that operators left after exchanging \( s(t) \) with \( \tilde{s} \tilde{\zeta}_I(t) \) are also composed of \( \tilde{\zeta}_I(x) - s(t) \) and \( \tilde{\pi}_I(x) \). Repeating this
procedure, we can replace all $s(t)$ with $\delta \xi_I(t)$ as
\[
\tilde{\mathcal{H}}_I \left[ \tilde{\xi}_I(x), \tilde{\pi}_I(x) \right] = \sum_{\alpha=0} \int \left. ds(t) \int \hat{d}s(t) \right| \hat{\xi}_I(x) - \delta \xi_I(t), \tilde{\pi}_I(x) \right| \left. \right| \hat{s}(t) \left. \right| \hat{\dot{s}}(t) \right| \left( \hat{s}(t) \right) \left( \hat{\dot{s}}(t) \right) \right| \left( \hat{s}(t) \right) \left( \hat{\dot{s}}(t) \right) \right| \left( 3.38 \right)
\]
where to denote the modification after the replacement of $s(t)$ with $\delta \xi_I(t)$, we put ' on the interaction Hamiltonian. Replacing $\dot{s}(t)$ with $\dot{\delta} \xi(t)$, we obtain
\[
\tilde{\mathcal{H}}_I \left[ \tilde{\xi}_I(x), \tilde{\pi}_I(x) \right] = \sum_{\alpha=0} \int \left. ds(t) \int \hat{d}s(t) \right| \hat{\xi}_I(x) - \delta \xi_I(t), \tilde{\pi}_I(x) \right| \left. \right| \hat{s}(t) \left. \right| \hat{\dot{s}}(t) \right| \left( \hat{s}(t) \right) \left( \hat{\dot{s}}(t) \right) \right| \left( 3.39 \right)
\]
We repeat this procedure for all integrating Hamiltonian densities that appear in the perturbative expansion of the $n$-point functions (3.27). After these replacements, the possible dependence of the $n$-point functions on $s(t)$ and $\dot{s}(t)$ remains only in $|s(t)\rangle \langle s(t)|$ and $|\dot{s}(t)\rangle \langle \dot{s}(t)|$. Since requesting the Euclidean vacuum uniquely determines the initial state independent of $s(t)$ and $\dot{s}(t)$, we can remove the identity operators $\int ds(t) |s(t)\rangle \langle s(t)|$ and $\int \hat{d}s(t) |\dot{s}(t)\rangle \langle \dot{s}(t)|$ as long as we choose the Euclidean vacuum. (From the same argument, we can remove the identity operator $\int ds |s\rangle H_{H} H |s\rangle$.) Then, the Hamiltonian density is recast into
\[
\tilde{\mathcal{H}}_I \left[ \tilde{\xi}_I(x), \tilde{\pi}_I(x) \right] \rightarrow \sum_{\alpha=0} \tilde{\mathcal{H}}_I^{(\alpha)} \left[ \tilde{\xi}_I(x) - \delta \xi_I(t), \tilde{\pi}_I(x) \right] \left( \delta \xi_I(t) \right) \left( 3.40 \right)
\]
Note that we can express $\delta \xi_I(t)$ as
\[
\delta \xi_I(t) = \int d^3x \partial_t \left( \frac{W_L(x)}{\int d^3x W_L(x)} \right) \tilde{\xi}_I(x) + \int \frac{d^3x W_L(x) \partial_t \tilde{\xi}_I(x)}{\int d^3x W_L(x)}
\]
\[
= \int d^3x \partial_t \left( \frac{W_L(x)}{\int d^3x W_L(x)} \right) \left[ \tilde{\xi}_I(x) - \delta \xi_I(t) \right] + \int \frac{d^3x W_L(x) \partial_t \tilde{\xi}_I(x)}{\int d^3x W_L(x)}, \left( 3.41 \right)
\]
where in the last equality we have inserted $0 = \delta \xi_I(t) \partial_t \left\{ \int d^3x W_L(x) / \int d^3x W_L(x) \right\}$ and the last term in the last line can be written in terms of $\tilde{\pi}_I(x)$. In this way, we can show that all $\xi_I$s in the interaction vertices are multiplied by an IR suppressing operator $R_x$. Note that, replacing the c-number parameter $s(t)$ with the operator $\delta \xi_I(t)$, we rewrote the Hamiltonian density as in Eq. (3.40). In this procedure, we used the fact that the initial state specified by the boundary condition of the Euclidean vacuum does not depend on the choice of the canonical variables. We should emphasize that if this equivalence of the initial state were not guaranteed, we could not express the interaction Hamiltonian only in terms of $\tilde{\xi}_I$'s with an IR suppressing operator.

3.3. Restricting the interaction vertices to the local region
In the above discussion, we found that the interaction picture fields that appear in the interaction vertices can be expressed only in terms of $\tilde{\pi}_I(x)$ and $\tilde{\xi}_I(x) - \delta \xi_I(t)$. Now, we can verify item 1 presented in Sect. 1, which claims that the interaction vertices are constructed only from the IR irrelevant operators. As we showed in the previous subsection, all the interaction picture fields are associated with an IR suppressing operator $R_x$, which increases the power law index with respect
to the wavenumber $k$ in the IR limit. To complete the proof of the argument given in item 1, we need to show that the inverse Laplacian $\partial^{-2}$, which appears in solving the constraint equations to obtain the lapse function and the shift vector, does not reduce the power law index with respect to $k$ in the IR limit. The potential danger can be understood as follows. When we choose the boundary condition specified by the regularity at spatial infinity following the standard procedure, the action of the operator $\partial^{-2}$ yields a multiplicative factor $1/k^2$. This IR singular behavior arises because the information from the outside of our observable region is used to determine the lapse function and the shift vector.

To remove this potential IR singular behavior originating from the inverse Laplacian, we need to discuss the causality. The causality is basically maintained even at the quantum level in the sense that the interaction vertices located outside our observable region $\mathcal{O}$ are decoupled in the in–in formalism. In the ordinary field theory with a local interaction, this can be shown by systematically replacing the Wightman function $G^+$ with the retarded Green function plus $G^-$ (see Appendix of Ref. [27]). However, when the gravitational perturbation is taken into account, it becomes less transparent whether the causality is maintained owing to the issue of the lapse function and the shift vector mentioned above.

Here, we should recall that what we really need to evaluate is the expectation values of genuinely gauge-invariant variables, which do not depend on the choice of the residual gauge degrees of freedom. As we explicitly showed in Appendix A, using the residual gauge degrees of freedom, we can modify the boundary conditions of the lapse function $N$ and the shift vector $N_i$ so that the terms associated with $\partial^{-2}$ are completely specified by the fields within the local region $\mathcal{O}$. Then, the operation of the non-local operator $\partial^{-2}$ no longer reduces the power law index with respect to $k$. In this way, using the degrees of freedom in the choice of boundary conditions, we can localize all the interaction vertices within the causally connected local region $\mathcal{O}$. Since $\mathcal{R}_x \zeta(x)$ is not invariant under the residual gauge transformations, their $n$-point functions are not invariant in general under the change of boundary conditions of $N$ and $N_i$. However, when we calculate $n$-point functions for the genuinely gauge-invariant operator $\mathcal{R}$ using those for $\mathcal{R}_x \zeta$, changing the boundary conditions should not affect the result.

### 4. The IR regularity and the absence of secular growth

In this section, we will calculate the $n$-point functions of $\mathcal{R}_x \zeta(x)$, properly taking into account not only the IR modes but also the modes with $k|\eta(t)| \gtrsim 1$. As stressed in Sect. 1, to prove the absence of secular growth, we need to evaluate the contribution of the latter modes carefully. In the preceding section, we showed that, using the canonical variables $\tilde{\zeta}(x)$ and $\tilde{\pi}(x)$, we can expand the $n$-point functions of $\mathcal{R}_x \zeta(x)$ for the Euclidean vacuum only in terms of the IR irrelevant operators. In this section, based on the perturbative expansion in the $\{\tilde{\zeta}, \tilde{\pi}\}$ system, we will discuss the IR regularity and the absence of secular growth in the $n$-point functions.

For our current discussion, the explicit form of the interaction Hamiltonian density $\tilde{\mathcal{H}}_I$ is not necessary. We use a formal expression

$$\tilde{\mathcal{H}}_I[\tilde{\zeta}_I(x), \tilde{\pi}_I(x)] = M^2_{\text{pl}} e^{-3\rho} \tilde{\rho}^2 \epsilon_1(t) \sum_{n=3}^\infty \lambda(t) \prod_{m=1}^n \mathcal{R}_x^{(m)} \tilde{\zeta}_I(x), \quad (4.1)$$

where $\lambda(t)$ is an $O(1)$ dimensionless time-dependent function that can be expressed only in terms of the horizon flow functions. To discriminate different IR suppressing operators, we associate a superscript $(m)$ with $\mathcal{R}_x$. 
4.1. Euclidean vacuum as obtained by the iε prescription

In the preceding section, we introduced the Euclidean vacuum as a vacuum state that satisfies the boundary condition (3.2)/(3.9). Here we show that this condition leads to the ordinary perturbative description of the iε prescription. We expand the curvature perturbation $\tilde{\zeta}_I(x)$ as

$$
\tilde{\zeta}_I(x) = \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot x} v_k(t) \tilde{a}_k + \text{h.c.},
$$

(4.2)

where $\tilde{a}_k$ is the annihilation operator, which satisfies

$$
[\tilde{a}_k, \tilde{a}_k^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad [\tilde{a}_k, \tilde{a}_k] = 0.
$$

(4.3)

The mode function $v_k(t)$ should satisfy

$$
\left[ \frac{d^2}{d\rho^2} + (3 - \varepsilon_1 + \varepsilon_2) \frac{d}{d\rho} + \left( \frac{k}{e^{\rho} \dot{\rho}} \right)^2 \right] v_k = 0.
$$

(4.4)

Since the boundary condition (3.2)/(3.9) should hold at the tree level, the asymptotic form of the positive frequency mode function $v_k(t)$ should be $\propto e^{-ik\eta(t)}$. Factoring out this time dependence at $\eta \to -\infty$, we express $v_k(t)$ as

$$
v_k(t) = \frac{A(t)}{k^{3/2}} f_k(t) e^{-ik\eta(t)},
$$

(4.5)

where we introduced

$$
A(t) := \frac{\dot{\rho}(t)}{\sqrt{\varepsilon_1(t)} M_{\text{pl}}},
$$

(4.6)

as an approximate amplitude of the fluctuation. The function $f_k(t)$ satisfies the regular second-order differential equation with the boundary condition

$$
f_k(t) \to \frac{k}{\sqrt{2} e^{\rho} \dot{\rho}} \quad \text{for} \quad -k\eta(t) \to \infty.
$$

(4.7)

Since both the differential equation and the boundary condition of $f_k(t)$ are analytic in $k$ for any $t$, the resulting function should be analytic as well. Namely, $f_k(t)$ does not have any singularity such as a pole on the complex $k$-plane. We suppose that a positive frequency function for a general vacuum except for the Euclidean vacuum is given by a linear combination of $v_k$ and $v_k^*$ with the Bogoliubov coefficients that have some nontrivial structure of singularities in the complex $k$-plane or diverge at infinity. The only exception to avoid the singularity is setting the Bogoliubov coefficients to constants, but then the UV behavior does not agree with that in the Minkowski vacuum.

On the other hand, in the limit $-k\eta(t_k) \ll 1$, the function $f_k(t)$ is proportional to $A(t_k)/A(t)$, where $t_k$ is the Hubble crossing time defined by $-k\eta(t_k) = 1$, because the curvature perturbation should be constant in this limit. Hence, the expansion for small $k$ is in general given by

$$
A(t) f_k(t) = A(t_k) \left[ 1 + O(k|\eta(t)|) \right].
$$

(4.8)

By using Eq. (4.5), the Wightman function is given by

$$
G^+(x, x') = \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot (x - x')} v_k(t) v_k^*(t') = A(t) A(t') \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^3} e^{ik \cdot (x - x')} f_k(t) f_k^*(t') e^{ik(\eta(t') - \eta(t))}.
$$

(4.9)

Using the in–in formalism, the $n$-point functions can be expanded by the Wightman function. At this point, the vertex integrals should start with $\eta = -\infty$ to be able to impose the boundary
We perform vertex integrations from the closer vertices to the future or past end of the closed time pass. The left figure represents the integration about the vertex that is closest to the past end and the right figure represents the integration about that which is next to the closest to the past end.

condition of the Euclidean vacuum \( (3.2)/(3.9) \). Although the integrands of the vertex integrals are infinitely oscillating in the limit \( \eta \to -\infty \), the time integration can be made convergent by adding a small imaginary part to the time coordinate, which is nothing but the ordinary \( i\epsilon \) prescription. To see the convergence of the time integration more explicitly, we first consider the integral for the vertex that is closest to the past infinity \( \eta \to -\infty \) (see Fig. 1). The interaction picture fields \( \tilde{\xi}_I(x) \) included in this vertex are contracted with \( \tilde{\xi}_I(x_m) \) contained in vertices labeled \( m = 1, 2, \ldots, n \), and give the Wightman function \( G^+(x_m, x) \). Then, the vertex integration with \( n \) interaction picture fields is given by

\[
V^{(1)}(t', \{x_m\}) := M_{pl}^2 \int_{t_i}^{t_f} dt \int d^3x \ e^{3\rho(t)} \epsilon_1(t) \hat{\rho}(t) \lambda(t) \prod_{m=1}^{n} \mathcal{R}_{x_m} \mathcal{R}_{x}^{(m)} G^+(x_m, x). \tag{4.10}
\]

The Euclidean vacuum condition \( (3.2)/(3.9) \) requires the convergence of this integral when we send \( \eta(t_i) \to -\infty \). Since the Wightman functions contain the exponential factor \( e^{i\eta(t)} \sum_k \), the integral can be made convergent by changing the integration contour, as shown in the left panel of Fig. 1, which is exactly what is known as the \( i\epsilon \) prescription.

The vertex integration next to the closest to the past infinity

\[
V^{(2)}(t'', \{x_m\}, \{x_m'\})
\]

\[
:= M_{pl}^2 \int_{t_i}^{t''} dt' \int d^3x' e^{3\rho(t')} \epsilon_1(t') \hat{\rho}(t') \lambda(t') \prod_{m'=1}^{n'} \mathcal{R}_{x_m} \mathcal{R}_{x}^{(m')} G^+(x_m', x') V^{(1)}(t', \{x_m\}) \tag{4.11}
\]

can be done in a similar manner, where \( n' \) is the number of propagators connecting between this second vertex and vertices other than the first one. If we perform the integration over the time coordinate of the first vertex \( t \) up to \( t' \), the exponential factor in \( G^+(x_m, x) \) can be replaced as

\[
e^{i\eta_m(\eta(t) - \eta(t_i))} \to e^{i\eta_m(\eta(t') - \eta(t_i))}. \tag{4.12}
\]

Therefore, all the Wightman functions connecting the vertices at \( t' \) or before \( t' \) with the vertices after \( t' \) give an exponential factor that is suppressed by adding \( +i\epsilon \) to \( \eta \). This is again consistent with the boundary condition of the Euclidean vacuum. The same argument can be made for the other vertices as well. In this subsection, we have performed the time integration, focusing on particular momenta of the Wightman propagators to illustrate the perturbative description of the boundary condition of the Euclidean vacuum. However, in our proof of the IR regularity, described below, we will perform the momentum integration of the propagator prior to the vertex integration.
4.2. The IR/UV suppressed Wightman function

Since all \( \hat{\xi}(x) \)’s in the interaction Hamiltonian are multiplied by the IR suppressing operators \( R_x \), the \( n \)-point function of \( R_x \hat{\xi}(x) \) can be expanded by the Wightman function \( R_x R_{x'} G^+(x, x') \) and its complex conjugate \( R_x R_{x'} G^-(x, x') \). In this subsection, we calculate the Wightman functions multiplied by the IR suppressing operator, \( R_x R_{x'} G^+(x, x') \) for \( t > t' \). After integration over the angular part of the momentum, the Wightman function \( R_x R_{x'} G^+(x, x') \) can be expressed as

\[
R_x R_{x'} G^+(x, x') = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} R_x R_{x'} A(t) f_k(t) \hat{A}(t') f_k^*(t') \left[ e^{ik\sigma_+(x, x')} - e^{ik\sigma_-(x, x')} \right],
\]

where we have introduced

\[
\sigma_\pm(x, x') := \eta(t') - \eta(t) \pm |x - x'|.
\]

We first show the regularity of the \( k \) integration in Eq. (4.13). Since the function \( f_k(t) \) is not singular, the regularity can be verified if the integration converges both in the IR and UV limits. The regularity in the IR limit is guaranteed by the presence of the IR suppressing operator. The IR suppressing operators \( R_x \) add at least one extra factor of \( k|\eta(t)| \) or eliminate the leading \( t \)-independent term in the IR limit, and yield

\[
R_x A(t) f_k(t) \left[ e^{ik\sigma_+(x, x')} - e^{ik\sigma_-(x, x')} \right] = A(t) e^{ik\eta(t')} \mathcal{O}(k|\eta(t)|)
\]

where we have introduced the spectral index \( n_s - 1 := d \ln(|A(t_k)|^2)/d \ln k \). Thus, the operation of \( R_x \) makes the \( k \) integration in Eq. (4.13) regular in the IR limit. Next, we consider the convergence in the UV limit. In Eq. (4.13), the integration contour of \( k \) should be appropriately modified at \( k \to \infty \) so that the integral becomes convergent. This modification of the integration contour can also be understood as part of the \( i\epsilon \) prescription, because adding a small imaginary part to all the time coordinates as \( \eta \to \eta \times (1 - i\epsilon) \) leads to the replacement \( \eta(t') - \eta(t) \to \eta(t') - \eta(t) + i\epsilon \), where we note \( \eta(t') - \eta(t) < 0 \), and hence to introducing an exponential suppression factor for large \( k \). This UV regulator makes the integral finite for the large \( k \) contribution except for the case \( \sigma_\pm(x, x') = 0 \), where \( x \) and \( x' \) are mutually light-like. Since the expression of the Wightman function obtained after the \( k \) integration is independent of the value of \( \epsilon \), the regulator makes the UV contributions convergent even after \( \epsilon \) is sent to zero. For \( \sigma_\pm(x, x') = 0 \), the integral becomes divergent in the limit \( \epsilon \to 0 \), but the divergence related to the behavior of the Wightman functions in this limit is to be interpreted as the ordinary UV divergences, whose contribution to the vertex integrals must be renormalized by introducing local counter-terms. Thus, the Wightman function \( R_x R_{x'} G^\pm(x, x') \) is now shown to be a regular function.

Since the amplitude of the Wightman function with the IR suppressing operator is bounded from above, we can show the regularity of the \( n \)-point functions, if the non-vanishing support of the integrands of the vertex integrals is effectively restricted to a finite spacetime region. Since the causality has been established with the aid of the residual gauge degrees of freedom, the question to address is whether vertices at the distant past are shut off or not. To address the presence of such a long-term correlation, we discuss the asymptotic behavior of the Wightman function \( R_x R_{x'} G^\pm(x, x') \), sending \( t' \) to a distant past. Recall that when \( \sigma_\pm(x, x') \neq 0 \), we can rotate the integration contour with
respect to \( k \) even toward the direction parallel to the imaginary axis. Rotating the direction of the path appropriately depending on the sign of \( \sigma_{\pm}(x, x') \), the integrand becomes an exponentially decaying function of \( k \). This rotation of the integration contour can be done without hitting any singularity in the complex \( k \)-plane, because the function \( f_k(t) \) is guaranteed to be analytic by construction. If we choose other vacua, this operation induces extra contributions from singularities. Since we send \( t' \) to the past infinity, assuming \(|\eta(t')| \gg |\eta(t)|, \sigma_{\pm}(x, x') = O(|\eta(t')|)\), except for the region where the two points are mutually light-like\(^2\). Then, the integration of \( k \) on the right-hand side of Eq. (4.13) is totally dominated by wavenumbers with \( k \lesssim 1/|\eta(t')| \ll 1/|\eta(t)| \). Using Eq. (4.14), which gives the asymptotic expansion in the limit \( |\eta(t)| \ll 1 \), we obtain

\[
\mathcal{R}_x \mathcal{R}_{x'} G^+(x, x') = A(t)O \left[ \int_0^\infty \frac{dk}{k} \left\{ \frac{k}{e^{\rho(t)} \hat{\rho}(t)} \right\} \left( \frac{n+1}{2} \right) \mathcal{R}_x \mathcal{A}(t') f_k(t') e^{ik\eta(t')} \right]
\]

(4.15)

where, in the second equality, we have performed the \( k \) integration, rotating the integration contour. We should emphasize that we did not employ the long wavelength approximation regarding the Hubble scale at \( t' \) to properly evaluate the modes \( k \) of \( O(1/|\eta(t')|) \) as well.

4.3. Secular growth

In this subsection, focusing on the long-term correlation, we discuss the convergence of the vertex integrals of the \( n \)-point functions for the Euclidean vacuum. We start with the integration of the \( n \)-point interaction vertex, which is the closest to \( \eta \), integrating of the \( n \)-point functions for the Euclidean vacuum. We start with the integration of the \( n \)-point \( \eta \)-function, which is the closest to \( \eta = -\infty(1 - i\epsilon) \). By inserting the expression of the Wightman function \( \mathcal{R}_x \mathcal{R}_{x'} G^+(x, x') \) with \( t \gg t' \), given in Eq. (4.15), into Eq. (4.10), the vertex integral \( V^{(1)}(t', \{x_m\}) \) can be estimated as

\[
V^{(1)}(t', \{x_m\}) = \mathcal{O} \left[ M_{\text{Pl}}^2 \int_{t_i}^{t_f} dt \int d^3x e^{3\rho(t)} \varepsilon_1(t) \hat{\rho}(t)^2 \lambda(t) (A(t))^{n} \prod_{m=1}^{n} A(t_m) \left( \frac{\eta(t_m)}{\eta(t)} \right)^{\frac{n+1}{2}} \right].
\]

(4.16)

As we have explained in Sect. 3.3, the interaction vertices are confined within the observable region, i.e., the non-vanishing support of the integrand is bounded by \( |x| \lesssim L_t \simeq |\eta(t)| \). Thus, we obtain

\[
V^{(1)}(t', \{x_m\}) = \mathcal{O} \left[ \int_{-\infty}^{\eta(t')} \frac{d\eta}{\eta} \lambda(\eta) (A(\eta))^{n-2} \prod_{m=1}^{n} A(t_m) \left( \frac{\eta(t_m)}{\eta} \right)^{\frac{n+1}{2}} \right].
\]

(4.17)

Since we have performed the momentum integral first, the exponential suppression for large \( |\eta| \) is no longer present. However, picking up the \( \eta \) dependence of the integrand of Eq. (4.17), we still find

\(^2\) Let us introduce a physical length scale \( \lambda_{\text{UV}} \) to remove the contributions from the vicinity of the light cone. On the time slice specified by \( \eta' \), we neglect the region within the distance \( \lambda_{\text{UV}} \) from the intersection of the light cone emanating from \( x \) with this time slice. Under this restriction, we have \( \sigma_{\pm}(x, x') > \lambda_{\text{UV}}/e^{\rho(t')} \simeq \lambda_{\text{UV}} |\eta(t')| \hat{\rho}(t') \) and hence \( \sigma_{\pm}(x, x') \) turns out to grow in proportion to \( |\eta(t')| \). This argument might be too heuristic, but we believe that the contribution from the region neglected here will not change our discussion about the IR regularity of the \( n \)-point functions. In order to clarify this point, it would be necessary to incorporate a discussion about UV renormalization, which is beyond the scope of this paper.
that the contribution from the distant past is suppressed if
\[ \lambda(\eta) \{ A(\eta) \}^{n_{1} - 2 \eta^{-1 - \frac{n_{1} + 1}{\epsilon}}} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow -\infty. \] (4.18)

Then, the time integral converges, and the amplitude of \( V^{(1)}(\eta', \{ x_{m} \}) \) is estimated by the value of the integrand at the upper end of the integration as
\[ V^{(1)}(t', \{ x_{m} \}) = \mathcal{O} \left[ \lambda(t') \{ A(t') \}^{n_{2} - 2} \prod_{m=1}^{n} A(t_{m}) \left( \frac{\eta(t_{m})}{\eta(t')} \right)^{\frac{n_{2} + 1}{2}} \right]. \] (4.19)

Therefore, when a Wightman propagator is connected to a vertex located in the future of \( x' \), i.e., when \( t_{m} > t' \), the \( t \) integration yields the suppression factor \( \{ \eta(t_{m})/\eta(t') \}^{\frac{n_{2} + 1}{2}} \). We denote the number of such propagators by \( \tilde{n} \).

Similarly, we can evaluate the amplitude of \( V^{(2)} \) as
\[ V^{(2)}(t'', \{ x_{m} \}, \{ x_{m'} \}) = \mathcal{O} \left[ \int_{-\infty}^{\eta(t'')} \frac{d\eta'}{\eta'} \lambda(\eta') \lambda' (\eta') \{ A(\eta') \}^{n_{1} + n' - d} |\eta'|^{-\frac{n_{2} + 1}{2}} \left( |\eta|^{n_{2} + 1} \right) V^{(1)}(t(\eta'), \{ x_{m} \}) \right]. \] (4.20)

Extracting the \( \eta' \)-dependent part in the above expression, we obtain
\[ \lambda(\eta) \{ A(\eta) \}^{N_{f} - 2N_{v} - 1} |\eta|^{-\frac{n_{2} + 1}{2}}. \] (4.21)

Note that all the Wightman propagators that are connected to the field \( \tilde{\xi}_{f} \) located in the future of \( x' \) yield the suppression factor \( |\eta(t'')|^{-\frac{n_{2} + 1}{2}} \).

Now the generalization becomes easy. For the \( N_{v} \)th vertex, the temporal integration becomes
\[ \int_{-\infty}^{\eta(t'')} \frac{d\eta_{v}}{\eta_{v}} \lambda(\eta_{v}) \{ A(\eta_{v}) \}^{N_{f} - 2N_{v} - 1} |\eta_{v}|^{-\frac{n_{2} + 1}{2}} \lambda |M|, \] (4.22)

where \( N_{f} \) denotes the number of \( \tilde{\xi}_{f}s \) contained in the vertices up to the \( N_{v} \)th, \( M \) denotes the number of the Wightman propagators connected to a vertex with \( \eta > \eta_{N_{v}} \), and \( \lambda \) denotes the product of the interaction coefficient up to the \( N_{v} \)th vertex. Thus, the convergence condition is given by
\[ \lambda(\eta) \{ A(\eta) \}^{N_{f} - 2N_{v} - 1} |\eta|^{-\frac{n_{2} + 1}{2}} |M| \rightarrow 0 \quad \text{as} \quad \eta \rightarrow -\infty \] (4.23)

with \( N := N_{f} - 2N_{v} \). Since all interaction vertices have at least one Wightman propagator connected with their future vertices, \( M \) should satisfy \( M \geq 1 \).

As a simple example, we consider the case where \( \varepsilon_{1} \) is constant. In this case, \( \lambda \) is expressed only in terms of \( \varepsilon_{1} \) and takes a constant value. By assuming \( M = 1 \) and using \( n_{s} - 1 = -2\varepsilon_{1} \), the convergence condition yields
\[ -\varepsilon_{1} N + (1 - \varepsilon_{1})^{2} > 0. \] (4.24)

In the slow roll limit \( \varepsilon_{1} \ll 1 \), the above condition is recast into
\[ N < \mathcal{O}(1/\varepsilon_{1}). \] (4.25)

The intuitive understanding of the above suppression mechanism is as follows. In the Euclidean vacuum case, only the contributions around the Hubble scale at each time are left unsuppressed (as shown in Fig. 2). When only the modes around the Hubble scale, i.e., \( k|\eta| \simeq k/e^{\rho} \dot{\rho} = \mathcal{O}(1) \), are
These figures show which modes can contribute to the loop integrals in the $n$-point function of $\zeta$ for the Euclidean vacuum. The horizontal axis represents the wavenumber $\ln k$ and the vertical axis represents the time $\ln(e^{\rho} \dot{\rho}) \simeq \ln(1/|\eta|)$, which becomes the number of e-folding in the limit $\epsilon_1 \ll 1$. The red region is suppressed because of the operation of the IR suppressing operator $R_x$ and the blue region is suppressed because of the exponential suppression of the $i\epsilon$ prescription. The dotted line with $\log(e^{\rho} \dot{\rho}) = \log k$ is the mode of the Hubble scale. The left figure (a) is for the case with $M \sim 1$ and the right figure (b) is for the case with $M \gg 1$.

relevant, the Wightman function $R_x R_x' G^+(x, x')$ is necessarily suppressed when $\eta(t)/\eta(t') \ll 1$. This is because, if $x$ and $x'$ are largely separated in time, any Fourier mode in the Wightman function cannot be of the order of the Hubble scale simultaneously at $t$ and $t'$. When we consider the contribution of vertices located far in the past, at least one Wightman function should satisfy $\eta(t)/\eta(t') \ll 1$, and therefore it is suppressed. However, when we consider a diagram for which a cluster of vertices in a distant past is connected to the vertices around the observation time by a single propagator, i.e., in the case with $M = 1$, the IR suppression comes only from this propagator. When the number of operators in the cluster of vertices in the past is sufficiently large, the suppression due to this propagator can be overwhelmed by the large amplitude of the fluctuation, which increases as the energy scale of inflation increases in the past direction. This corresponds to the case when the condition (4.23) is broken. However, we should also stress that the contributions from the distant past are suppressed and secular growth never appears in the slow roll inflation, unless the order of perturbative expansion $N$ takes an extremely large value such as $1/\epsilon_1 \simeq O(10^2)$. When the convergence condition (4.23) is satisfied, all the time integrations are dominated by the contributions near its upper end. The order of magnitude of the $n$-point functions of $R_x \zeta(t_f, x)$ is then given by

$$\langle 0 | R_x \zeta(t_f, x_1) R_x \zeta(t_f, x_2) \cdots R_x \zeta(t_f, x_n) | 0 \rangle \simeq \hat{\lambda}(t_f) [A(t_f)]^N. \quad (4.26)$$

5. Conclusion and discussion

5.1. Euclidean vacuum satisfies the strong constraint on the initial states

In this paper, we have shown that, when we choose the Euclidean vacuum as the initial state, the vertex integration in the $n$-point functions for the genuinely gauge-invariant curvature perturbation is regular unless a very high order in the perturbative expansion is concerned. Figure 3 shows the outline of the proof. We should emphasize that the regularity of the $n$-point functions in the limits $\eta \rightarrow -\infty (1 \pm i\epsilon)$ plays a crucial role in the proof: (i) Requesting this regularity guarantees the equivalence between two quantum systems, i.e., the original quantum system in which the Hamiltonian contains the IR relevant operators and the quantum system in which the Hamiltonian is totally composed of IR irrelevant operators. (ii) It guarantees the analyticity of the mode function $v_k(t)$ with
Choosing the Euclidean vacuum

Regularity of $n$-point functions for $\zeta$ and $\bar{\zeta}$ in the limits $\eta \to -\infty (1 \mp i c)$

Sects. 3.1, 4.1

(i) Equivalence between the 2 canonical systems
(ii) Analyticity of the mode function

Sect. 4.2

Convergence of $k$ integration in Wightman functions
(i) $\to$ IR suppression, (ii) $\to$ UV suppression

Sect. 4.3

Interaction vertices are localized within the past light cone

Sect. 5

Suppression of the long-term correlation

Regularity of vertex integration in the $n$-point functions

Preserving the gauge invariance in the local universe

Fig. 3. The outline of the proof that shows the regularity of the $n$-point functions of the genuinely gauge-invariant variable for the Euclidean vacuum. Since we have left a possibility that the $n$-point functions can become regular without requesting the boundary condition of the Euclidean vacuum, we have used a dotted arrow.

respect to the wavenumber $k$ for arbitrary $t$. By virtue of aspect (i), we can rewrite the $n$-point functions of $\zeta$ into those expressed in $\{\tilde{\zeta}, \tilde{\pi}\}$, in which all the field operators are manifestly associated with the IR suppressing operators, $R_x$. Aspect (ii) leads to exponential suppression in the UV so that the non-vanishing support of the $k$ integration is restricted to $-k\eta \lesssim O(1)$. It is intriguing that choosing the Euclidean vacuum plays the crucial role in discussing the suppressions, both in the IR and UV components. Since these suppressions cause the Wightman function (in the position space) to be multiplied by an IR suppressing operator regular everywhere except for the light cone limit, the missing piece to prove the regularity of the $n$-point functions is to show that the integration region of each vertex integral is effectively confined to a finite portion of the spacetime. Using the residual gauge degrees of freedom, we can confine the interaction vertices within the past light cone. Since the long-term correlation is shut off because of the suppression both in the IR and UV, the integration region of the vertex integrals is ensured to be effectively finite. Therefore, the $n$-point functions for the Euclidean vacuum are expressed by integrals whose integrand and integration region are both finite, and hence they are manifestly regular. Thus, we conclude that the Euclidean vacuum is a suitable initial state of the universe, and is free from IR pathology even in the presence of non-linear interactions.

In this section, we further address the converse question, “When we request that the $n$-point functions are finite and free from secular growth, is the Euclidean vacuum the unique possible initial quantum state?” To be precise, the condition we impose here is the regularity of $n$-point functions on the real time axis including the distant past. We naively expect that, in this case, the Euclidean vacuum is the unique possibility. Since any excitations are blue shifted at an earlier time, any small deviation from the Euclidean vacuum state at a finite time will lead to some singular behavior in the
However, we do not have any rigorous proof of this argument yet. There might be a fundamental obstacle when we try to make this statement precise. When we trace back the history of the universe, it should inevitably enter the regime in which the background energy density and hence the amplitude of the vacuum fluctuation are so high that the perturbative analysis would not make sense any more.

As an alternative setup of the problem, one may require the regularity of the $n$-point functions just for $\eta > \eta_i$ with a certain initial time, $\eta_i$. The relaxed requirement of the regularity allows us to take other states, if correlation functions for these states can be reinterpreted as correlation functions for the Euclidean vacuum. We introduce a new operator

$$A_{(m)} = \int d^3x W_{L,\xi}(x) R^{(m)}(x),$$

with an arbitrary choice of the IR suppression operator $R^{(m)}$ where $m$ is just a label for distinction. Then, we can define the 1-particle state by $|1\rangle := N A_{(m)}^\dagger |0\rangle$ with an appropriate normalization factor $N$. The $n$-point functions of $R^{(m)}(x)$ at the initial time $\eta = \eta_i$ for the 1-particle state $|1\rangle$ defined at the initial time can be expressed in terms of the $(n + 2)$-point functions for the products of $R^{(m)}(x)$ for the Euclidean vacuum, whose regularity is proved in this paper. When the initial distribution is regular, as we showed in Ref. [28], the distribution at late times will be kept regular as well. Similarly, we can construct excited states with plural particles. (Similar excited states are discussed in de Sitter spacetime in Ref. [44].) In this manner, one can construct various excited states that are IR regular and free from secular growth. However, here the allowed number of inserted operators might be bounded because our proof of regularity does not apply when the order of perturbation becomes very high. We leave the question of whether a more general construction of the IR regular excited states is possible or not for a future study.

5.2. Comparison to recent publications

In the recent papers [38,42], the absence of secular growth is also claimed. It would be profitable to make a comparison between these works and our present work. First, in these papers, item 1 raised in Sect. 1, i.e., the presence of the canonical system, which is equivalent to the original canonical system and whose interaction Hamiltonian is composed only of IR irrelevant operators, is postulated, while this is not automatically guaranteed from the symmetry of the classical system. Second, in these papers, the mode function in de Sitter spacetime, whose amplitude at large scales is given by a constant Hubble parameter, is used in proving the conservation of the curvature perturbation. This leads to a quantitative discrepancy in the evaluation of secular growth from ours. For instance, in Ref. [38], the locality of the solution $\tilde{\xi}_L^{(n)}(x, t)$ given in Eq. (22) of the paper is crucial in their proof. However, the locality is not necessarily valid, once we take into account the fact that, in the chaotic inflation, the amplitude of the fluctuation becomes larger and larger in the distant past as $\dot{\rho} \propto e^{-\int d\rho / \varepsilon_1}$. When we neglect this effect by setting $A^N \propto (\dot{\rho} / \varepsilon_1)^N$ in Eq. (4.23) to constant, the convergence condition is always satisfied (unless the interaction coefficient $\lambda$, composed of the horizon flow functions, varies rapidly). Therefore, our result does not contradict the conservation of the curvature perturbation they claimed. The third point is about the treatment of the UV contributions. In this paper, we have not directly discussed the UV renormalization. We simply assumed that the UV divergent contributions, which are shown to be localized to the region where the two arguments of the Wightman functions are mutually almost light-like, can be renormalized by introducing local...
counter-terms. As long as the renormalization does not break the dilatation symmetry of the classical action, our discussion can hold. Recently, an interesting investigation about UV renormalization has been pursued in Ref. [42]. It is claimed that a decaying composite operator in the free theory is also kept decaying after the renormalization of loops. Although non-trivial assumptions such as the locality must be removed or verified, if this statement is correct, the conservation of the curvature perturbation can also be shown in the presence of the loop corrections. We should, however, emphasize that the conservation of the curvature perturbation does not prohibit the appearance of the logarithmic amplification, as we mentioned in Sect. 1.

Finally, we also make a comment on the recent progress regarding the IR issues of a test field in the exact de Sitter spacetime, which can be interpreted as an approximation of the entropy mode. The regularity of the loop corrections for the Euclidean vacuum is shown for the massive scalar field by Hollands [47] and independently by Marolf and Morrison [48–50]. By contrast, for a massless scalar field, the IR regularity has not been shown and the absence of secular growth is unclear [51–54] (see also Ref. [55]). Although the adiabatic curvature perturbation is a sort of massless field in the sense that the Wightman function $G^+(x, x')$ possesses IR divergence and the long-term correlation, the operation of the IR suppressing operators $R_{\xi}$, which appear by virtue of the residual gauge symmetry and by choosing the Euclidean vacuum, cures the singular behavior. Hence, it would be intriguing to discuss a massless field with the exact shift symmetry in de Sitter spacetime, in comparison with the case of the adiabatic mode.

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Appendix A. Solving constraint equations

In this section, we discuss the boundary conditions of the constraint equations, which are of elliptic type. By expanding the metric perturbations as $\tilde{\zeta} = \tilde{\zeta}_I + \tilde{\zeta}_2 + \cdots$, $\tilde{N} = 1 + \tilde{N}_1 + \tilde{N}_2 + \cdots$, and $\tilde{N}_I = \tilde{N}_{i,1} + \tilde{N}_{i,2} + \cdots$, the Hamiltonian constraint and the momentum constraints yield

$$V \tilde{N}_n - 3 \dot{\bar{\rho}} \tilde{\zeta}_n + e^{-2\rho} \partial^2 \tilde{\zeta}_n + \dot{\bar{\rho}} e^{-2\rho} \partial^i \tilde{N}_{i,n} = H_n, \quad (A1)$$

$$4 \partial_i \left( \dot{\rho} \tilde{N}_n - \tilde{\zeta}_n \right) - e^{-2\rho} \partial^2 \tilde{N}_{i,n} + e^{-2\rho} \partial_i \partial^j \tilde{N}_{j,n} = M_{i,n}, \quad (A2)$$

where $H_n$ and $M_{i,n}$ vanish for $n = 1$ and, for $n \geq 2$, they are composed of $n$ interaction picture fields $\tilde{\zeta}_I$ in the combination $\tilde{\zeta}_I - s$ or with differentiation. Eliminating $\tilde{N}_n$ from these constraint equations, we obtain

$$\left( 1 - \frac{4\dot{\rho}^2}{V} \right) \partial_i \partial^j \tilde{N}_{j,n} - \partial^2 \tilde{N}_{i,n} + 2 \frac{\dot{\bar{\rho}}^2}{V} e^{2\rho} \partial_i \tilde{\zeta}_n - 4 \frac{\dot{\bar{\rho}}}{V} \partial_i \partial^2 \tilde{\zeta} = C_{i,n}, \quad (A3)$$

where we have defined

$$C_{i,n} := e^{2\rho} \left( M_{i,n} - \frac{4\dot{\rho}}{V} \partial_i H_n \right). \quad (A4)$$
Operating $\partial^i$ on Eq. (A3), we obtain
\[
\partial^2 \partial^i \tilde{N}_{i,n} = \frac{\phi^2}{2\rho^2} e^{2\rho} \partial^2 \zeta_n - \frac{1}{\rho} \partial^4 \zeta_n - \frac{V}{4\rho^2} \partial^i C_{i,n}. \tag{A5}
\]
We solve this equation as follows:
\[
\partial^i \tilde{N}_{i,n}(x) = \frac{\phi^2}{2\rho^2} e^{2\rho} \partial^2 \zeta_n(x) - \frac{1}{\rho} \partial^2 \zeta_n(x) - \frac{V}{4\rho^2} \left[ \partial^{-2} \partial^i C_{i,n}(x) - G^L_n(x) \right], \tag{A6}
\]
where $G^L_n(x)$ is an arbitrary solution of the Laplace equation, i.e., $\partial^2 G^L_n(x) = 0$. Inserting this solution into Eq. (A3), we obtain
\[
\partial^2 \tilde{N}_{i,n} = \partial_i \left[ \frac{\phi^2}{2\rho^2} e^{2\rho} \partial^2 \zeta_n(x) - \frac{1}{\rho} \partial^2 \zeta_n(x) - \frac{V}{4\rho^2} \left( \partial^{-2} \partial^i C_{i,n}(x) - G^L_n(x) \right) \right] + \partial_i \left( \partial^{-2} \partial^i C_{i,n} - G^L_n \right) - C_{i,n}. \tag{A7}
\]
Again, introducing an arbitrary solution of the Laplace equation $G_{i,n}(x)$, we solve Eq. (A7) as
\[
\tilde{N}_{i,n}(x) = \partial_i \partial^{-2} \left[ \frac{\phi^2}{2\rho^2} e^{2\rho} \partial^2 \zeta_n(x) - \frac{1}{\rho} \partial^2 \zeta_n(x) - \frac{V}{4\rho^2} \left( \partial^{-2} \partial^i C_{i,n}(x) - G^L_n(x) \right) \right] + \partial_i \partial^{-2} \left( \partial^{-2} \partial^i C_{i,n}(x) - G^L(x) \right) - \partial^{-2} C_{i,n}(x) + G_{i,n}(x). \tag{A8}
\]
Comparing the expression obtained by operating $\partial^i$ on Eq. (A8) with Eq. (A7), we obtain
\[
\partial^i G_{i,n} = \partial^i \partial^{-2} C_{i,n} - \left( \partial^{-2} \partial^i C_{i,n} - G^L \right). \tag{A9}
\]
Using this expression, we rewrite the longitudinal part of $G_{i,n}$ as
\[
G_{i,n} = \partial_i \partial^{-2} \left[ \partial^i \partial^{-2} C_{i,n} - \left( \partial^{-2} \partial^i C_{i,n} - G^L \right) \right] + G_{i,n} - \partial_i \partial^{-2} \partial^i G_{i,n}. \tag{A10}
\]
Inserting Eq. (A10) into Eq. (A8), we obtain
\[
\tilde{N}_{i,n}(x) = \partial_i \partial^{-2} \left[ \frac{\phi^2}{2\rho^2} e^{2\rho} \partial^2 \zeta_n(x) - \frac{1}{\rho} \partial^2 \zeta_n(x) - \frac{V}{4\rho^2} \left( \partial^{-2} \partial^i C_{i,n}(x) - G^L_n(x) \right) \right] - \left( \partial_i \partial^{-2} - \partial_i \partial^{-2} \partial^i \right) \left( \partial^{-2} C_{i,n}(x) - G_{i,n}(x) \right). \tag{A11}
\]
When we perform quantization in the whole universe, it is natural to request the regularity of the perturbation at spatial infinity. This requirement uniquely fixes $G^L_n$ and the transverse part of $G_{i,n}$. Then, the shift vector depends on the curvature perturbation $\zeta$ of the whole universe. To show the IR regularity, here we employ another boundary condition that requests that the integration region of the inverse Laplacian $\partial^{-2}$ is confined to around the local observable region $\mathcal{O}$. As is shown in Refs. [25,26], the degrees of freedom in changing the boundary condition can be understood as the gauge degrees of freedom in the local universe. Therefore, the operator $\tilde{R}$ is invariant under the change of the boundary condition.

Adjusting the solutions of the Laplace equations $G^L_n(x)$ and $G_{i,n}(x)$, we can change the boundary condition for $\partial^{-2}$ so that the integration region is limited. We fix the function $G^L_n(x)$, requesting
\[
\partial^{-2} W_{L,n}(x) \partial^i C_{i,n}(x) = \partial^{-2} \partial^i C_{i,n}(x) - G^L_n(x), \tag{A12}
\]
where we have inserted a window function that takes a non-vanishing value only within the vicinity of the observable region $\mathcal{O}$. If we evaluate the term in the first line of Eq. (A11) by using the Laplacian
inverse with two different boundary conditions, \( \partial_1^{-2} \) and \( \partial_2^{-2} \), the difference satisfies

\[
\partial^2 \left( \partial_1 \partial_1^{-2} \cdots - \partial_2 \partial_2^{-2} \cdots \right) = 0, \quad \partial^j \left( \partial_1 \partial_1^{-2} \cdots - \partial_2 \partial_2^{-2} \cdots \right) = 0, \quad (A13)
\]

where we have abbreviated the terms in the square brackets. Therefore, the change of the boundary condition for the Laplacian inverse can be absorbed by the transverse mode of \( G_{i,n}(x) \). Fixing the boundary condition of \( \partial^{-2} \) so that the integration region is restricted to the vicinity of the observable region, we obtain

\[
\tilde{N}_{i,n}(x) = \partial_i \partial^{-2} W_{L_i}(x) \left[ \frac{\partial^2}{2 \rho^2} \delta^2 \varphi_\rho(x) - \frac{1}{\rho} \partial^2 \varphi_{\rho}(x) - \frac{V}{4 \rho^2} \partial^{-2} W_{L_i}(x) \partial^j C_{j,n}(x) \right]
\]

\[
- \left( \delta^i - \partial_i \partial^{-2} \partial^j \right) \partial^{-2} W_{L_i}(x) C_{j,n}(x). \quad (A14)
\]

Inserting this solution into Eq. (A1), we can also obtain the lapse function whose support of the Laplacian inverse \( \partial^{-2} \) is also confined.

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