Life-time of minimal tubes and coefficients of univalent functions in a circular ring

Vladimir G. Tkachev

Abstract. We obtain various estimates of the life-time of two-dimensional minimal tubes in $\mathbb{R}^3$ by potential theory methods.

1. Introduction.

Let $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ be a point in Euclidean space $\mathbb{R}^{n+1}$ with the time axis $Ox_{n+1}$ and $M$ be a $p$-dimensional Riemannian manifold, $2 \leq p \leq n$.

Definition 1. We say that a surface $M = (M, u)$ given by $C^2$-immersion $u : M \to \mathbb{R}^{n+1}$ is a tube with the projection interval $\tau(M) \subset Ox_{n+1}$, if (i) for any $\tau \in \tau(M)$ the sections $\Sigma_\tau = f(M) \cap \Pi_\tau$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}^{n+1}_1 : x_{n+1} = \tau\}$ are not empty compact sets; (ii) for $\tau', \tau'' \in \tau(M)$ any part of $M$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

Definition 2. A surface $M$ is called minimal if the mean curvature of $M$ vanishes everywhere.

It is the well known fact (see [5], p.331) that the minimality condition of $M$ is equivalent to that all coordination functions of the immersion $u$ are harmonic. For this reason, the two-dimensional minimal tubes can be considered as direct analog of the closed relative string conception in the modern nuclear physics (cf. [2]). This approach was proposed by V.M.Miklyukov and the author in [7] for an arbitrary dimension $p$.

From this point of view many intrinsic geometric invariants of $M$ have the natural physical meaning. Namely, the length of the projection interval $|\tau(M)|$ can be interpreted as a life-time of the tube $M$.

To introduce the following important characteristic we denote by $v$ the unit normal to $\Sigma_\tau$ with respect to $M$ which is co-directed with the time-axis $Ox_{n+1}$. Then by virtue of the harmonicity of the coordinate functions $u_k(m) = x_k \circ u(m), \ldots$
1 \leq k \leq n + 1, the flow integrals

\[ J_k = \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma \]

are independent of \( \tau \in \tau(\mathcal{M}) \). Here \( d\Sigma \) is the 1-Hausdorff measure along \( \Sigma_\tau \).

**Definition 3.** We call \( Q(\mathcal{M}) = (J_1, J_2, \ldots, J_{n+1}) \in \mathbb{R}^{n+1} \) the **full flow-vector** of \( \mathcal{M} \).

We notice the positiveness of \( J_{n+1} \) as a consequence of the choice of \( \nu \) direction. Moreover, \( Q(\mathcal{M}) \) is an 1-homogeneous functional of \( \mathcal{M} \) under the homotheties group action in \( \mathbb{R}^{n+1} \). Let us denote by \( \alpha(\mathcal{M}) \) the angle between \( Q(\mathcal{M}) \) and the time-axis \( Ox_{n+1} \).

In this paper we are interested in the following question: What sufficient conditions yield the finiteness of the time-life of a two-dimensional minimal tube? As it shown in the series of papers [6]–[8], in the case \( p \geq 3 \) this quantity is always finite and the following estimation holds

\[ |\tau(\mathcal{M})| \leq g(\mathcal{M})c_p, \]

where \( c_p \) depends only on \( p \), and \( g(\mathcal{M}) \) is the smallest diameter of sections \( \Sigma_\tau \). The last relationship is sharp and the equality occurs if and only if \( \mathcal{M} \) is a minimal surface of revolution.

A special feature of the two-dimensional case is that there exist tubes with finite as well as infinite values of the life-time. Indeed, a family of slanted minimal surfaces with circular cross-sections \( \Sigma_\tau \) was discovered by B. Riemann [10]. Some other recent examples can also be found in [4].

In this paper we prove

**Theorem 1.** Let \( \mathcal{M}, \dim \mathcal{M} = 2 \) be a minimal two-connected tube with univalent Gaussian mapping. If the angle \( \alpha(\mathcal{M}) \) is different from zero, then the life-time \( |\tau(\mathcal{M})| \) of \( \mathcal{M} \) is finite and

\[ \tau(\mathcal{M}) \leq \frac{\pi \|Q\| \cos \alpha(\mathcal{M})}{\ln \tan(\frac{\pi}{4} + \frac{\alpha}{2})}. \]

Let us denote by \( a_0[f] \) the central coefficient of the Laurent decomposition of an holomorphic function \( f(z) \) in an annulus \( K_R = \{ z : 1/R < |z| < R \} \), i.e.

\[ a_0[f] \equiv \int_{C_1} \frac{g(\zeta) \, d\zeta}{\zeta}, \]

where \( C_1 \) is the unite circle \( \{ z \in \mathbb{C} : |z| = 1 \} \). The following auxiliary assertion is a key ingredient in the proof of Theorem 1.

**Theorem 2.** Let \( g(z) \) be a univalent holomorphic function defined in the annulus \( K_R \) omitting zero. Assume that

\[ a_0[g] = \lambda, \quad a_0[1/g] = -\lambda, \]

where

\[ 1 \leq k \leq n + 1, \]

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\[ a_0[g] = \lambda, \quad a_0[1/g] = -\lambda, \]
for some real positive \( \lambda \). Then

\[
(2) \quad \ln R \leq R_0(\lambda) = \frac{\pi^2}{\ln(\lambda + \sqrt{1 + \lambda^2})}
\]

Remark 1. We note that estimate (2) has well asymptotic behaviour for \( R \to \infty \) as shows Riemannian example mentioned above. But we cannot now present the sharp value of \( R_0(\lambda) \). Nevertheless, it seemed us very probably that the following conjecture is true.

Remark 2. The best upper bound in the left side of (2) is achieved for holomorphic function \( g_0(z) \) which provides a conformal map of the annulus \( K_R \) onto the plain \( \mathbb{C} \) with two slits: \((-1/\alpha; 0) \) and \((\alpha; +\infty)\), for the suitable choice of parameter \( \alpha \).

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2. Proof of Theorem 2

Let \( \Gamma = \{C_\rho : 1/R < \rho < R\} \) be a family of all concentric circles \( C_\rho = \{z : |z| = \rho\} \) in the annulus \( K_R \). It follows easily from the non-vanishing property of \( g(z) \) that the loop \( C_1 \) in the integrals (1) may be replaced by an arbitrary circle \( C_\rho \in \Gamma \). It follows from the mean value theorem and (1) that for every \( \rho \in (1/R; R) \) there exist \( t_1 \) and \( t_2 \) such that

\[
(3) \quad \text{Re } g(\rho e^{it_1}) = \lambda \quad \text{and} \quad \text{Re } \frac{1}{g(\rho e^{it_2})} = -\lambda.
\]

Let \( \gamma_\rho = g(C_\rho) \). Then by virtue of the univalence of \( g(z) \), the curve \( \gamma_\rho \) is the simple Jordan one. Let \( g(\rho e^{it}) = x(t) + iy(t) \) be the representation of \( \gamma_\rho \). Then we obtain from (3)

\[
x(t_1) = \lambda; \quad x^2(t_2) + y^2(t_2) + \frac{1}{\lambda} x(t_2) = 0.
\]

The last relations have the helpful geometric interpretation:

\( \star \) The curve \( \gamma_\rho \) has a non-empty intersection with the vertical rightline \( L_1 = \{z : \text{Re } z = \lambda\} \) and the circle \( L_2 = \{z : |z + 1/2\lambda| = 1/2\lambda\} \).

We shall make use the technique from the potential theory (the length-are method). Recall the exact definition. Let \( E \) be a family of locally rectifiable curves \( \gamma \) and \( \varphi(z) \geq 0 \) be a Baire function with the property

\[
\int_\gamma \varphi(z) |dz| \geq 1,
\]

for every \( \gamma \in E \). The infimum

\[
\text{mod } E = \inf \int \varphi^2(z) \, dx \, dy
\]

over all such \( \varphi(z) \) is called a conformal module of the family \( E \).
Then it is known (see [1]) that mod $E$ is the conformal invariant. As a consequence we obtain in our situation

$$\text{mod } \Gamma = \text{mod } \Gamma_1,$$

where $\Gamma_1 = \{ \gamma_\rho : 1/R < \rho < R \}$.

Let us denote by $D$ the two-dimensional domain

$$D = \left\{ z : \text{Re } z < \lambda; \frac{1}{2\lambda} < \frac{1}{2\lambda} \right\}. $$

Using the ($\star$)-property, we can find for every $\rho \in (1/R; R)$ the continuum $\gamma'_\rho \subset \gamma_\rho$ joining the boundary components of $D$. Then a family $\Gamma_2$ consisting of all continua $\gamma'_\rho$ is “shorter” than $\Gamma_1$ and it follows from Theorem 1.2, [1] that

$$\text{mod } \Gamma_1 \leq \text{mod } \Gamma_2.$$ 

On the other hand, $\Gamma_2$ is the subfamily of $\Gamma(D)$, where the last term means the family of all curves joining the boundary components of a domain $D$. The monotonicity property of infimum and Definition 4 lead to the following inequality

$$\text{mod } \Gamma_2 \leq \text{mod } \Gamma(D).$$

Now, combining the standard fact

$$\text{mod } \Gamma = \frac{\ln R}{\pi}$$

with relations (4), (5) and (6) we arrive at the following inequality

$$\frac{\ln R}{\pi} \leq \text{mod } \Gamma(D).$$

To compute the last module we note that the linear-fractional function

$$f(z) = \frac{1}{\lambda^*} \cdot \frac{z + \lambda^*}{1 - z \lambda^*}$$

maps $D$ onto an annulus $K_1 = \{ w : 1 < |w| < 1/\lambda^* \},$ where $\lambda^* = \sqrt{\lambda^2 + 1} - \lambda$. Thus, using the invariance property of conformal module we obtain

$$\frac{\ln R}{\pi} \leq \text{mod}(D) \equiv \frac{2\pi}{\ln (1/\lambda^* \pi)} = \frac{\pi}{\ln (\lambda + \sqrt{1 + \lambda^2})}.$$

and Theorem 2 is proved.

3. The Gaussian map of two-dimensional minimal tubes and their full-flow vector

In this section we express the full flow-vector of an arbitrary two-dimensional tube $\mathcal{M} \in \mathbb{R}^n$ via Chern-Weierstrass representation for minimal surfaces. Namely, if $\mathcal{M}$ is a two-connected surface then we can arrange that $\mathcal{M}$ is conformally equivalent
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to an annulus $K_R$ for the appropriate $R > 1$. Then there exist the corresponding parametrization of $\mathcal{M}$ (see [9]):

$$u(z) = \text{Re} \int_{z_0}^z F(\zeta) \, d\zeta : K_R \to \mathbb{R}^n,$$

where

$$F(z) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$$

and $\varphi_i(\zeta)$ are holomorphic functions satisfying the following conditions

$$(8) \quad \sum_{i=1}^n \varphi_i(\zeta)^2 = 0;$$

and

$$(9) \quad \text{Re} \int_{|z|=1} F(\zeta) \, d\zeta = 0.$$

**Lemma 1.** Under the above hypotheses we have

$$(10) \quad Q(\mathcal{M}) = \text{Im} \int_{|z|=1} F(\zeta) \, d\zeta.$$

**Proof.** It sufficient to show that

$$(11) \quad J_k = \int_{\Sigma^r} \langle \nabla u_k, \nu \rangle \, d\Sigma = \text{Im} \int_{|z|=1} \varphi_k(\zeta) \, d\zeta,$$

for every $k = 1, 2, \ldots, n + 1$.

To prove (11) we introduce the conjugate to $u_k(z)$ function $v_k(z)$ by

$$v_k^*(z) = \text{Im} \int_{z_0}^z \varphi_k(\zeta) \, d\zeta,$$

We notice that $v_k(z)$ in general is a multivalued function. On the other hand, the covariant derivative $\nabla v_k$ is well defined and using the properties of Hodge $*$- operator we have

$$\int_{\Sigma^r} \langle \nabla u_k, \nu \rangle \, d\Sigma = \int_{\Sigma^r} \langle *\nabla u_k, *\nu \rangle \, d\Sigma = \int_{\Sigma^r} \langle \nabla v_k, *\nu \rangle \, d\Sigma =$$

$$= \int_{\Sigma^r} d v_k = \text{Im} \int_{|z|=1} \varphi_k(\zeta) \, d\zeta,$$

and (11) is proved.

In our case $n = 2$, Chern-Weierstrass representation can be simplified in the following classic way. Namely, there exist a holomorphic function $f(z)$ and a meromorphic function $g(z)$ which are well defined in the annulus $K_R$ and such that

$$(12) \quad F(z) = ((1 - g^2)f; i(1 + g^2)f; 2g f).$$
Moreover, poles of $g(z)$ coincide with zeros of $f(z)$ and the order of a pole of $g(z)$ is precisely the order of the corresponding zero of $f(z)$. We emphasize that $g(z)$ is a composition of the stereographic projection and Gaussian map of $\mathcal{M}$.

**Lemma 2.** In our assumptions

\begin{equation}
2fg \equiv \frac{(Q(\mathcal{M}), e_3)}{2\pi z},
\end{equation}

and $g(z)$ omits the zero and infinity values.

**Proof.** We use the method proposed by M. Schiffman in [11]. We recall that the coordinate function $u_3(z)$ is harmonic in the annulus $K_R$ and by virtue of Definition 1,

\begin{equation}
\lim_{z \to 1/R} u_3(z) = \tau_1, \quad \lim_{z \to R} u_3(z) = \tau_2,
\end{equation}

where $\tau(\mathcal{M}) = (\tau_1; \tau_2)$ is the projection of the tube $\mathcal{M}$ onto $x_3$-axis.

We consider an auxiliary harmonic function

$$
h(z) = \tau_1 + \frac{\tau_2 - \tau_1}{2 \ln R} \ln |z|.
$$

It is easily seen that $h(z)$ satisfies (14). Thus $h_1(z) = u_3(z) - h(z)$ is harmonic in the annulus and

$$
\lim_{z \to \partial K_R} h_1(z) = 0.
$$

Then the maximum principle implies that $h_1(z) \equiv 0$ everywhere in $K_R$ and hence

\begin{equation}
\lim_{z \to \partial K_R} h_1(z) = 0.
\end{equation}

In particular, it follows from (15) that

$$
du_3(z) \equiv \frac{\tau_2 - \tau_1}{2 \ln R} \cdot \frac{z}{|z|^2}.
$$

doesn’t vanish in $K_R$. We have, as a consequence, the normal $n(z)$ to $\mathcal{M}$ isn’t parallel to $e_3$ at any point. Taking into account the above remark about the geometrical sense of $g(z)$ we obtain that $g(z) : K_R \to \mathbb{C} - \{0; \infty\}$.

By comparing of (15) and (12) we deduce that

\begin{equation}
2g(z)f(z) = \frac{\tau_2 - \tau_1}{2 \ln R} \cdot \frac{dz}{z}.
\end{equation}

In order to eliminate $\ln R$ from the latter equality we substitute (16) into (12), and after using (10) we obtain

\begin{equation}
\ln R = \frac{\pi(\tau_2 - \tau_1)}{J_3}.
\end{equation}

On substituting of the found relationship into (16) we arrive at the conclusion of the lemma. \qed
4. Proof of Theorem 1

Let us denote \( w = (J_1 + iJ_2)/J_3 \). Combining Lemma 2, (12) and (9) we obtain
\[
\int_{C_1} \frac{1 - g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_1 i,
\]
\[
\int_{C_1} \frac{1 + g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_2.
\]
Simplifying the last expressions and denoting \( w = |w| \cdot e^{i\theta} \), \( g_1(z) = -e^{-i\theta} g(z) \) give the following system
\[
\frac{1}{2\pi} \int_{C_1} g_1(\zeta) d\zeta = |w|,
\]
\[
\frac{1}{2\pi} \int_{C_1} \frac{d\zeta}{g_1(\zeta)} = -|w|.
\]
Applying Theorem 2 we arrive at the inequality
\[
\ln R \leq \frac{\pi^2}{|w| + \sqrt{1 + |w|^2}}
\]
where \( |w| = |J_1 + iJ_2|/|J_3| = \tan \alpha(\mathcal{M}) \). Using (17) we obtain the required estimate and the theorem is proved.

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