Dirac operator index and topology of lattice gauge fields

David H. Adams
Dept. of Pure Mathematics, University of Adelaide
Adelaide, S.A. 5005, Australia.
Email: dadams@maths.adelaide.edu.au

November 11, 2018

The fermionic topological charge of lattice gauge fields, given in terms of a spectral flow of the Hermitian Wilson–Dirac operator, or equivalently, as the index of Neuberger’s lattice Dirac operator, is shown to have analogous properties to Lüscher’s geometrical lattice topological charge. The main new result is that it reduces to the continuum topological charge in the classical continuum limit. (This is sketched here; the full proof will be given in a sequel to this paper.) A potential application of the ideas behind fermionic lattice topological charge to deriving a combinatorial construction of the signature invariant of a 4-manifold is also discussed.

PACS. 11.15.Ha, 11.30.Rd, 02.40.-k.

I. Background

In the continuum, a smooth SU(2) gauge field $A(x) = A_\mu^a(x)T^a dx^\mu$ on the Euclidean hyper-torus $T^4$ is from a mathematical point of view a connection 1-form on a principal SU(2) bundle $P$ over $T^4$. The bundle is characterised (up to topological equivalence) by an integer $Q_P$ and this number can be recovered from the gauge field:

$$Q_P = Q(A) = \frac{1}{8\pi^2} \int_{T^4} \text{tr}(F \wedge F) = \frac{1}{32\pi^2} \int d^4 x \epsilon_{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}(x)F_{\rho\sigma}(x))$$

(1)

where $F = dA + \frac{i}{2}[A, A]$ is the curvature of $A$. Thus the topological structure of $P$ is encoded in the gauge field $A$. It is also encoded in the space of zero-modes of the Dirac operator $\mathcal{D}_A = \gamma^\mu(\partial_\mu + A_\mu)$: the Atiyah–Singer index theorem \[\text{gives}\]

$$\text{index}(\mathcal{D}_A) \equiv \text{Tr}(\gamma_5 |_{\ker\mathcal{D}_A}) = Q(A).$$

(2)

The space of all SU(2) gauge fields on $T^4$ is a disjoint union of components (topological sectors) labelled by $Q \in \mathbb{Z}$. 

1
Now put a lattice (i.e. hyper-cubic cell decomposition) on $T^4$ with lattice spacing $a$. In [2] M. Lüscher showed that a lattice gauge field $U_\mu(x)$, defined on the links $[x,x+ae_\mu]$ ($e_\mu=$unit vector on the positive $\mu$-direction) and taking values in SU(2), also has encoded in it the structure of a principal SU(2) bundle $P$ over $T^4$: transition functions for $P$ on the overlaps of a collection of regions covering $T^4$ were explicitly constructed from $U$. An integer topological charge for $U$ is then given by $Q_{geo}(U) = Q_P$. This is often referred to as the geometrical lattice topological charge (to distinguish it from the fermionic topological charge discussed below). The construction of $P$ from $U$ is ambiguous for certain “exceptional” lattice gauge fields; roughly speaking, these are the fields for which it is not possible to canonically write $U(p) = \exp(\tau(p))$ for each plaquette $p$, where $U(p)$ is the product of the link variables around $p$. (Example: in the analogous case of U(1) gauge fields on the 2-torus [3], $U(p) \in \text{U}(1)$ can be canonically written as $e^{i\tau(p)}$, $i\tau(p) \in (\pi, \pi)$ except for the exceptional fields which have $U(p) = -1$ for some $p$.) The exceptional fields form a lower-dimensional manifold in the space of lattice gauge fields. They can be excluded by restricting to the lattice gauge fields whose plaquette variables are sufficiently close to the identity 1 in SU(2); a sufficient condition is $||1 - U(p)|| < 0.015$ (3) where we are using the matrix norm defined by $||M||^2 = \frac{1}{d} \sum_I |M_I|^2$ for $d \times d$ matrix $M$ with the sum running over the column vectors $M_I$ of $M$ (the normalisation factor $\frac{1}{d}$ is so that unitary matrices have norm 1). For the non-exceptional fields the bundle $P$ is uniquely determined so $Q_{geo}(U) = Q_P$ is well-defined. Furthermore, the topological structure of $P$ is unchanged under continuous deformation of $U$ provided that no exceptional fields are encountered [2]. Thus after excising the exceptional fields the space of lattice gauge fields acquires a non-trivial topological structure, decomposing into disjoint topological sectors which are again labelled by $Q \in \mathbb{Z}$. In the continuum $Q$ can take any value in $\mathbb{Z}$, but in the lattice setting $Q$ can only take values in a finite subset of $\mathbb{Z}$ depending on how fine the lattice is [3, 4]. When $U$ is the lattice transcript of a smooth continuum field $A$ (and $U$ is non-exceptional) the bundle $P_U$ specified by $U$ need not coincide with the bundle $P_A$ specified by $A$ in general. However, Lüscher showed that $P_U$ and $P_A$ do coincide when the lattice is sufficiently fine: he showed that $Q_{geo}(U) \rightarrow Q(A)$ in the classical continuum limit $a \rightarrow 0$ [2].

In light of the Index Theorem (2) in the continuum, it is natural to ask if there is a

* For $U \in \text{SU}(2)$ we have $\text{tr}(1 - U) \leq 2 ||1 - U||$.  

2
lattice Dirac operator $D$ such that

$$\text{index}(D_U) = Q^{geo}(U)$$  \hspace{1cm} (4)

Such a lattice version of the Index Theorem would certainly be mathematically interesting. It would also be of physical relevance: 't Hooft’s solution of the U(1) problem \cite{2} uses in a crucial way the connection between the topological structure of $A$ and zero-modes of $\mathcal{D}_A$ implied by the Index Theorem (\cite{2}). Also, an alternative “fermionic” description of $Q^{geo}(U)$ as the index of a lattice Dirac operator could be practically useful for numerical work, seeing as the expression for $Q^{geo}(U)$ is quite complicated and time-consuming to implement numerically. Traditional lattice Dirac operators are a bit problematic in this context. To avoid doubler zero-modes, operators such as the Wilson–Dirac operator \cite{6} include a chiral symmetry-breaking term; this results in the nullspace $\ker D$ not being invariant under $\gamma_5$ so the zero-modes do not have definite chirality and index $D$ is not defined. Nevertheless, a lattice version of the index can still be defined from the eigenvectors of the Wilson–Dirac operator $D_W^U$ with low-lying real eigenvalues \cite{7}. Essentially the same lattice index is obtained \cite{8} as minus the spectral flow of the Hermitian Wilson–Dirac operator

$$H_U(m) = \gamma_5(D_W^U - \frac{m}{a})$$  \hspace{1cm} (5)

coming from the crossings of the origin by eigenvalues $\lambda_k(m)$ at low-lying values of $m$.\footnote{Note that if $\lambda_k(m_0) = 0$ then the corresponding eigenvector $\psi_k(m_0)$ is an eigenvector for $D_W^U$ with eigenvalue $\frac{m_0}{a}$.) This “fermionic” definition of the topological charge of a lattice gauge field, which we denote by $Q^f(U)$, can be made more precise by defining $Q^f(U)$ to be minus the spectral flow of $H_U(m)$ as $m$ varies from 0 to 1; this is justified by the fact \cite{8} that for “sufficiently smooth” lattice gauge fields the real eigenvalues of $D_W^U$ are positive and are localised around $\frac{m}{a}$; $m = 0, 2, 4, 6, 8$ (an analytic explanation of this was recently sketched in \cite{9}). Note that $Q^f(U)$ has properties analogous to those of $Q^{geo}(U)$ discussed above: It is defined for all $U$ except those for which $H_U(1)$ has a zero-mode; these are exceptional in the sense that they form a lower-dimensional manifold in the space of lattice gauge fields. It is clear from the definition in terms of spectral flow that $Q^f(U)$ is constant under continuous variations of $U$ provided that no exceptional fields are encountered. Thus $Q^f(U)$ determines a topological structure in the space of

\footnote{The notation in \cite{8} is different; instead of $m$ they have a different parameter $K$.}

\footnote{In the continuum, taking the $\gamma$-matrices to be hermitian so $\mathcal{D}_A$ is anti-hermitian, crossings of the origin by eigenvalues of the hermitian operator $H_A(m) = \gamma_5(\mathcal{D}_A - m)$ only happen at $m = 0$ and the spectral flow from these is $-\text{index}(\mathcal{D}_A)$.}
lattice gauge fields in the same way as $Q^{geo}(U)$. The exceptional fields for $Q^f(U)$ can be excluded by imposing a condition on the plaquette variables [11, 12]:

$$||1 - U(p)|| < 0.03$$

(6)

Note the similarity with (3). (Neither (3) nor (6) are optimal.)

$Q^f(U)$ has subsequently appeared in other guises, as the overlap topological charge in [12] and more recently as the index of a new lattice Dirac operator: Neuberger’s Overlap-Dirac operator [13], given by

$$D_U = \frac{1}{a} \left( 1 + \gamma_5 \frac{H_U}{\sqrt{H_U^2}} \right)$$

(7)

where $H_U \equiv H_U(1)$. This operator satisfies [14] the Ginsparg–Wilson relation [15] $D\gamma_5 + \gamma_5 D = aD\gamma_5 D$, which implies that ker $D$ is invariant under $\gamma_5$ (since $D\psi = 0 \Rightarrow D(\gamma_5\psi) = (aD\gamma_5 D - \gamma_5 D)\psi = 0$) so the zero-modes of $D$ have definite chirality and index $D = \text{Tr}(\gamma_5 \text{ker } D)$ is well-defined, as was first noted in [16]. A formula for the index gives [16, 17, 18]

$$\text{index}(D_U) = -\frac{a}{2} \text{Tr}(\gamma_5 D_U) = -\frac{1}{2} \text{Tr} \left( \frac{H_U}{\sqrt{H_U^2}} \right) = Q^f(U)$$

(8)

where the last equality follows from the facts that Tr$\left( \frac{H_U}{\sqrt{H_U^2}} \right)$ is the spectral asymmetry of $H_U(m)$ at $m = 1$ and $H_U(m)$ has symmetric spectrum for $m < 0$ [12, 19].

The fermionic lattice topological charge, in its various guises, has been an ongoing subject of study since the original works [7, 8, 20] in the mid 1980’s; for a selection of recent works see, e.g., [21, 19, 22, 23, 24]. The emphasis has tended to be on numerical investigations though, and there are a number of interesting questions which have been partially answered by numerically studies but which currently lack a complete analytical resolution. Perhaps the most fundamental of these is the question of whether $Q^f$ and $Q^{geo}$ are equal (at least under suitable conditions on the lattice gauge field, such as (3),(6)). This is equivalent to the question of whether the Lattice Index Theorem (4) holds with $D_U$ being the Overlap-Dirac operator (7). Another basic question is whether $Q^f(U)$ reduces to the continuum topological charge $Q(A)$ ( = index$(\partial A)$) in the classical continuum limit. (This is a necessary condition for equality between $Q^f$ and $Q^{geo}$ since, as discussed above, $Q^{geo}(U) \rightarrow Q(A)$ in this limit [22].) This has sometimes been taken for granted in the

§ This question has received attention in numerical studies, e.g. in [8] where $Q^f(U)$ was compared to a simpler version of $Q^{geo}(U)$ due to P. Woit [25].
literature; e.g. in [8] it is claimed (without proof) that the eigenvectors of $D^{Wilson}$ with low-lying real eigenvalues will “reduce to” the zero-modes of $\mathcal{A}$ in the classical continuum limit. To give a precise mathematical formulation and proof of this statement is not so easy though. From a mathematical point of view, showing $Q^f(U) \to Q(A)$ in the classical continuum limit is a long-standing open problem, and the main purpose of this article is to announce and sketch a proof of this:

**Theorem.** For SU(N) gauge fields on the Euclidean 4-torus, $Q^f(U) \to Q(A)$ in the classical continuum limit. Equivalently, $\text{index}(D_U) \to \text{index}(\mathcal{A})$ in this limit, where $D_U$ is the Overlap-Dirac operator (7) above.

This is the fermionic analogue of Lüscher’s result $Q^{geo}(U) \to Q(A)$. The deeper question of whether $Q^f$ and $Q^{geo}$ are equal (at least when $U(p)$ satisfies a bound of the form (8),(9)) is a challenging mathematical problem which we will not attempt here.

**II. Classical continuum limit of the fermionic lattice topological charge**

The proof of the theorem above, which we sketch in the following, grew out of suggestions by Martin Lüscher [26]. The full details will be given in a forthcoming paper [27]. (An alternative argument for a restricted class of topologically non-trivial fields in a slightly different setting was previously given in [28, v4].) The general idea is to start with the last equality in (8),

$$Q^f(U) = -\frac{1}{2} \text{Tr} \left( \frac{H_U}{\sqrt{H_U^2}} \right),$$

(9)
carry out a certain power series expansion of the inverse square root, and then evaluate the trace in an explicit basis for the spinor fields. But we do this in a slightly indirect way: First, we use the locality result of [10] for the Overlap-Dirac operator to derive a relation $Q^f(U) = Q^f(U)^{(2p+1)} + O(e^{-c/a})$ (with $c > 0$) where $Q^f(U)^{(2p+1)}$ is the fermionic topological charge in a setting in which an infinite volume limit $p \to \infty$ can be taken. Then $\lim_{p \to \infty} Q^f(U) = \lim_{a \to 0} \lim_{p \to \infty} Q^f(U)^{(2p+1)}$ and the latter quantity is easier to evaluate because the sum resulting from the trace in (9) becomes a tractable integral in the $p \to \infty$ limit.

**II-1. Preliminaries**

The 4-torus $T^4$ is taken to be $[-\frac{1}{2}L, \frac{1}{2}L]/\sim \times \cdots \times [-\frac{1}{2}L, \frac{1}{2}L]/\sim$ (where $\sim$ means identify endpoints). Then a continuum $SU(2)$ gauge field $A_\mu(x)$ on $T^4$ can be viewed as
a gauge field on $\mathbb{R}^4$ satisfying

$$A_\mu(x + Le_\nu) = \Omega(x, \nu)A_\mu(x)\Omega(x, \nu)^{-1} + \Omega(x, \nu)\partial_\mu\Omega(x, \nu)^{-1}$$  \hspace{1cm} (10)

where $\Omega(x, \nu), \nu = 1, 2, 3, 4$, are the SU(2)-valued monodromy fields which specify the principal SU(2) bundle over $T^4$. Now put a hyper-cubic lattice on $T^4$ with $2N$ sites along each edge, so the lattice spacing is $a = L/2N$. This extends to a hyper-cubic lattice on $\mathbb{R}^4$ with sites $a\mathbb{Z}^4$. The lattice transcript of $A$, 

$$U_\mu(x) = T \exp\left(\int_0^1 aA_\mu(x + tae_\mu) \, dt\right)$$  \hspace{1cm} (11)

(T=t-ordering and for simplicity the coupling constant has been set to unity) satisfies

$$U_\mu(x + Le_\nu) = \Omega(x, \nu)U_\mu(x)\Omega(x + ae_\mu, \nu)^{-1}.$$  \hspace{1cm} (12)

The finite-dimensional complex vector-space $\mathcal{C}$ of lattice spinor fields on $T^4$ consists of the spinor fields $\psi(x), x \in a\mathbb{Z}^4$, satisfying

$$\psi(x + Le_\nu) = \Omega(x, \nu)\psi(x).$$  \hspace{1cm} (13)

The covariant finite difference operators $\frac{1}{a}\nabla^\pm_\mu$, given by

$$\nabla^+_\mu \psi(x) = U_\mu(x)\psi(x + ae_\mu) - \psi(x)$$  \hspace{1cm} (14)

$$\nabla^-_\mu \psi(x) = \psi(x) - U_\mu(x - ae_\mu)^{-1}\psi(x - ae_\mu),$$  \hspace{1cm} (15)

preserve (13) and are therefore well-defined on $\mathcal{C}$, as is the Wilson–Dirac operator, given by

$$D_U^{\text{Wilson}} = \frac{1}{a}\nabla_U + \frac{1}{2}a\left(\frac{1}{a}\Delta_U\right)$$  \hspace{1cm} (16)

where $\frac{1}{a}\nabla_U = \frac{1}{a}\sum_\mu \gamma^\mu \nabla_\mu$ is the naive lattice Dirac operator (the $\gamma^\mu$'s are taken to be hermitian so $\nabla_U$ is anti-hermitian), $\frac{1}{a}\Delta = \frac{1}{a^2}\sum_\mu \nabla^-_\mu - \nabla^+_\mu = \frac{1}{a^2}\sum_\mu (\nabla^+_\mu)^* \nabla^+_\mu = \frac{1}{a^2}\sum_\mu (\nabla^-_\mu)^* \nabla^-_\mu$ is the lattice Laplace operator (hermitian, positive). Likewise $H_U = \gamma_5(D_U^{\text{Wilson}} - \frac{1}{a})$ is well-defined on $\mathcal{C}$, and so is the Overlap-Dirac operator

$\text{II}$ These also satisfy a cocycle condition which ensures that $A_\mu(x + Le_\nu + Le_\rho)$ is unambiguous. It is always possible to make a gauge transformation so that $\Omega(x, \nu) = 1$ for $\nu = 1, 2, 3$ and $\Omega(x, 4)$ is periodic in $x_1, x_2, x_3$. Then for fixed $x_4$ $\Omega(x, 4)$ determines a map $T^3 \to \text{SU}(2)$. The degree of this map (which is independent of $x_4$ since $\Omega(x, 4)$ depends smoothly on $x_4$) is precisely the integer $Q_P$ specifying the SU(2) bundle $P$ over $T^4$, and is therefore also the topological charge $Q(A) = Q_P$ of $A$.

$\text{**}$ For simplicity we have taken the Wilson parameter to be $r = 1$. 

6
\[ D_U = \frac{1}{a}(1 + \gamma_5 H_U H_U^{-1}) \] provided that \( H_U \) has no zero-modes. \( H_U \) is guaranteed not to have zero-modes when \( a \) is sufficiently small. Indeed, \( 1 - U(p, \mu, \nu) = -a^2 F_{\mu\nu}(x) + O(a^3) \) vanishes uniformly for \( a \to 0 \), implying that (9) is satisfied for sufficiently small \( a \), which in turn implies that [10, 11]

\[ H_U^2 \geq b > 0 \quad (17) \]

(This holds, e.g., with \( b = 0.1 \) but we will not need the explicit value in the following.) We henceforth assume \( a \) to be small enough that (9) – and thereby (17) – holds.

### II-2. Passing to a setting with an infinite volume limit

A general linear operator \( \mathcal{O} \) on lattice spinor fields corresponds to a kernel function \( \mathcal{O}(x, y) \) via \( \mathcal{O}\psi(x) = a^4 \sum_y \mathcal{O}(x, y)\psi(x) \). By (8) we have

\[ Q^f(U) = a^4 \sum_{x \in \Gamma} q_U(x) \quad (18) \]

where

\[ q_U(x) = -\frac{a}{2} \text{tr}(\gamma_5 D_U(x, x)) \quad (19) \]

and the summation is over

\[ \Gamma = \{ x = an \mid n_\mu = -N, -N + 1, \ldots, N - 1 \} \quad (20) \]

Now, for arbitrary whole number \( p \), set

\[ \Omega^{(p)}(x, \nu) = \Omega(x + pLe_\nu, \nu)\Omega(x + (p - 1)L e_\nu, \nu) \cdots \Omega(x + L e_\nu, \nu)\Omega(x, \nu) \quad (21) \]

and define \( C_p \) to be the space of lattice spinor fields \( \psi(x) \), \( x \in a\mathbb{Z}^4 \), satisfying

\[ \psi(x + (p + 1)L e_\nu) = \Omega^{(p)}(x, \nu)\psi(x) \quad (22) \]

Note that (13) implies (22), so \( C \) is contained in \( C_p \) for all \( p \). Note also that the covariant finite difference operators (14)–(15) preserve (22) and are therefore well-defined operators on \( C_p \); it follows that the Overlap-Dirac operator is well-defined as an operator on \( C_p \). We denote the Overlap-Dirac operator on \( C_p \) by \( D_U^{(p)} \) in the following, with \( D_U \) denoting the operator on \( C \). The fact that \( D_U \) is the restriction of \( D_U^{(p)} \) to \( C \subseteq C_p \) for all \( p \) implies

\[ D_U \psi(x) = D_U^{(2p+1)} \psi(x) = a^4 \sum_{y \in \Gamma_{2p+1}} D_U^{(2p+1)}(x, y)\psi(y) \quad \text{for } \psi \in C \quad (23) \]
\[
\Gamma_{2p+1} = \{ x = an \mid n_\mu = -(2p+1)N, -(2p+1)N+1, \ldots, (2p+1)N-1 \}
= \{ x + Lm \mid x \in \Gamma, m_\mu = -p, -p+1, \ldots, p \}. \tag{24}
\]
From this it is straightforward to derive [27], using (13), that for \( p \geq 1 \) the norm of
\[
R_U^{(2p+1)}(x, y) := D_U(x, y) - D_U^{(2p+1)}(x, y) \tag{25}
\]
has a bound
\[
\|R_U^{(2p+1)}(x, y)\| \leq \sum_{|m_\mu| \in \{1, 2, \ldots, p\}} \|D_U^{(2p+1)}(x, y + Lm)\|. \tag{26}
\]
The locality result in [10] now gives
\[
\||aD_U^{(2p+1)}(x, x + Lm)|| \leq \frac{c_1}{a^4} \exp\left(-\frac{c_2}{a} \sum_{\mu} |m_\mu|\right) \tag{27}
\]
where \( c_1 \) and \( c_2 > 0 \) are constants independent of \( a, p \) and \( U \). It follows that
\[
\||aR_U^{(2p+1)}(x, x)|| \leq 2^4 \sum_{m_\mu \in \{1, 2, \ldots, p\}} \frac{c_1}{a^4} \exp\left(-\frac{c_2}{a} \sum_{\mu} m_\mu\right) \leq 2^4 \frac{c_1}{a^4} \prod_{\mu} \int_{1/2}^{p+1/2} dt_\mu \exp\left(-\frac{c_2}{a} t_\mu\right) \leq c_1 \left(\frac{2}{2c_2 L}\right)^4 \exp\left(-\frac{c_2 L}{2a}\right). \tag{28}
\]
Set
\[
q_U^{(2p+1)}(x) = -\frac{a}{2} \text{tr}(\gamma_5 D_U^{(2p+1)}(x, x)) \tag{29}
\]
then substituting (25) in (19) and taking account of (28) we see that the norm of
\[
r_U^{(2p+1)}(x) := q_U(x) - q_U^{(2p+1)}(x) \tag{30}
\]
has a bound of the form
\[
|r_U^{(2p+1)}(x)| \leq O(e^{-c/a}) \quad c > 0 \tag{31}
\]
where \( O(e^{-c/a}) \) is independent of \( U \) and \( p \). It follows that
\[
\lim_{a \to 0} q_U(x) = \lim_{p \to \infty} \lim_{a \to 0} q_U^{(2p+1)}(x) \quad (x \in \Gamma) \tag{32}
\]
\[
\lim_{a \to 0} Q^f(U) = \lim_{p \to \infty} \lim_{a \to 0} a^4 \sum_{x \in \Gamma} q_U^{(2p+1)}(x) \tag{33}
\]
The infinite volume limit \( p \to \infty \) in these expressions will facilitate their evaluation, as we will see in the following subsection.
II-3. Evaluation of the classical continuum limit

We now exploit the locality of the Overlap-Dirac operator in the gauge field \([10]\) to replace \(A_\mu(x)\), or rather, its lattice transcript \(U_\mu(x)\), in (32)–(33) by the lattice transcript \(\tilde{U}_\mu(x)\) of a gauge field \(\tilde{A}_\mu(x)\) which coincides with \(A_\mu(x)\) in a neighbourhood of \([-\frac{1}{2}L, \frac{1}{2}L]^4\) but which vanishes outside a bounded region in \(\mathbb{R}^4\). Specifically, choose a smooth function \(\lambda(x)\) on \(\mathbb{R}^4\) which equals 1 in a neighbourhood of \([-\frac{1}{2}L, \frac{1}{2}L]^4\) and vanishes outside a region contained in \([-\frac{3}{2}L, \frac{3}{2}L]^4\), and set \(\tilde{A}_\mu(x) = \lambda(x)A_\mu(x)\). For each \(p \geq 1\) we take \(\tilde{U}_\mu\) to be the lattice transcript of \(\tilde{A}_\mu\) in \([-2p+1, 2p+1)^4\) and extend \(\tilde{U}_\mu\) to the rest of the lattice on \(\mathbb{R}^4\) by requiring that \(\tilde{U}_\mu(x)\) be periodic in all directions with period \((2p+1)L\).

Then the Overlap-Dirac operator with lattice gauge field \(\tilde{U}\) is a well-defined operator \(D_{\tilde{U}}^{(2p+1)}\) on the space \(\tilde{C}_{2p+1}\) of lattice spinor fields satisfying the periodicity condition \(\psi(x+(2p+1)L\epsilon_\nu) = \psi(x)\), \(\nu = 1, 2, 3, 4\). A version of the locality result of [10] leads to

\[
||aD_{\tilde{U}}^{(2p+1)}(x, x) - aD_{\tilde{U}}^{(2p+1)}(x, x)|| \leq \frac{1}{a^4} O(e^{-\tilde{c}/a}) \quad x \in \Gamma \quad p \geq 1
\]

where \(\tilde{c} > 0\) is a constant and \(O(e^{-\tilde{c}/a})\) is independent of \(x\), \(p\), \(U\), \(\tilde{U}\). It follows that (32)–(33) are unchanged if \(q_{U}^{(2p+1)}(x)\) is replaced by

\[
q_{\tilde{U}}^{(2p+1)}(x) = -\frac{1}{a} \text{tr}(\gamma_5 D_{\tilde{U}}^{(2p+1)}(x, x)) .
\]

There are several reasons for making the replacement \(A \to \tilde{A}\), \(U \to \tilde{U}\). (i) The rigorous justification in [27] of the steps that follow uses the fact that \(\tilde{A}_\mu(x)\) vanishes outside a bounded region. (This is not the case in general for \(A_\mu(x)\) which can diverge in the \(|x| \to \infty\) limit.) (ii) It leads to \(\mathcal{C}_p\) being replaced by a space \(\tilde{\mathcal{C}}_p\) of periodic spinor fields, thereby allowing the trace in (36) below to be evaluated in a “plane wave” basis.

Substituting the expression for the Overlap-Dirac operator in (35) we find (cf. (3))

\[
q_{\tilde{U}}^{(2p+1)}(x) = \frac{-1}{2} \text{tr} \left( \frac{H_{\tilde{U}}}{\sqrt{H_{\tilde{U}}^2}} (x, x) \right) \\
= \frac{-1}{2} \text{tr} \left( \gamma_5 \left\{ X_{\tilde{U}} \frac{1}{\sqrt{X_{\tilde{U}}^* X_{\tilde{U}}}} \right\} (x, x) \right) \\
= \frac{-1}{2a^4} \text{Tr} \left( \gamma_5 X_{\tilde{U}} \frac{1}{\sqrt{X_{\tilde{U}}^* X_{\tilde{U}}}} \tilde{\delta}_x \right)
\]

where

\[
X_{\tilde{U}} = a(D^{Wilson}_{\tilde{U}} - \frac{1}{a}) = \nabla_{\tilde{U}} + \frac{1}{2}(\Delta_{\tilde{U}} - 2)
\]
and the operator \( \hat{\delta}_x \) is defined by \( (\hat{\delta}_x \psi)(y) = \psi(x)\delta_{xy} \). The strategy now is to carry out a power series expansion of the inverse square root in (36). A calculation gives

\[
X_{\tilde{U}}^* X_{\tilde{U}} = L_{\tilde{U}} + V_{\tilde{U}}
\]

where

\[
L = -\nabla^2 + \frac{1}{2}(\Delta - 2)^2
\]

and

\[
V = -\frac{1}{4}[\gamma^\mu, \gamma^\nu][\nabla_\mu, \nabla_\nu] - \frac{1}{2}[\nabla, \Delta].
\]

\( V \) is a linear combination of commutators of the \( \nabla^\pm \)’s. As pointed out in [10], the norms of these commutators are bounded by \( \max_p ||1 - \tilde{U}(p)|| \). In the present case, this together with the expansion

\[
\tilde{U}(p_{x,\mu,\nu}) = 1 + a^2 \bar{F}_{\mu\nu}(x) + O(a^3)
\]

show that

\[
||V_{\tilde{U}}|| \sim O(a^2)
\]

As in (17) we have a bound \( X_{\tilde{U}}^* X_{\tilde{U}} \geq b > 0 \) when \( a \) is sufficiently small; furthermore, due to (12) we can assume that \( ||V|| \leq b/2 \). Then from (38) we get a lower bound \( L_{\tilde{U}} \geq b/2 > 0 \) on the positive hermitian operator \( L_{\tilde{U}} \). It follows that \( L \) is invertible and \( ||L^{-1}V|| < 1 \) when \( a \) is sufficiently small. The inverse square root in (36) can then be expanded as

\[
\frac{1}{\sqrt{X^*X}} = \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{X^*X + \sigma^2}
\]

\[
= \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \left( \frac{1}{L + \sigma^2} \right) \left( \frac{1}{1 + (L + \sigma^2)^{-1}V} \right)
\]

\[
= \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{L + \sigma^2} \sum_{k=0}^{\infty} (-1)^k ((L + \sigma^2)^{-1}V)^k.
\]

(43)

Note that the \( \gamma \)-matrices in (43) are all contained in \( V \). Since the trace of \( \gamma_5 \) times a product of less than 4 \( \gamma \)-matrices vanishes, the terms with \( k = 0 \) and \( k = 1 \) in (43) give vanishing contributions to (36). On the other hand, by (12) the terms with \( k \geq 3 \) are \( O(V^3) \sim O(a^6) \), hence the contribution of these in (36) is \( \sim O(a^2) \). (This is rigorously established in [14] using the presence of \( \hat{\delta}_x \) in (36) together with the fact that \( \hat{A}_\mu(x) \) has compact support.) Thus the only relevant contribution to (36) from the expansion (43) comes from the \( k = 2 \) term:

\[
\int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{(L + \sigma^2)^2} V \frac{1}{L + \sigma^2} V
\]

(44)
By making a power series expansion of $1/(L + \sigma^2)$ and using $[L, V] \sim O(a)$ we find in \[27\] that, modulo an $O(a)$ term, the contribution of (44) in (36) is the same as that of

$$
V^2 \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{(L + \sigma^2)^3} = V^2 L^{-5/2} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{(1 + \sigma^2)^3} = \frac{3}{8} V^2 L^{-5/2}
$$

(45)

We hereby see that (36) reduces to

$$
q_{(2p+1)}(x) = \frac{-3}{16a^4} \text{Tr}(\gamma_5 X_U V^2 L^{-5/2} \tilde{\delta}_x) + O(a)
$$

(46)

Now, from (11) with $A \rightarrow \tilde{A}$ we see that the $a$-dependence of $U$ enters through a product $a\tilde{A}$. It follows that

$$
X_U = X_1 + O(a), \quad L_U = L_1 + O(a)
$$

(47)

Since $V \sim O(a^2)$ we can replace $X_U \rightarrow X_1$ and $L_U \rightarrow L_1$ in (10) at the expense of another $O(a)$ term. With the expressions (37) and (39) for $X$ and $L$ we find

$$
q_{(2p+1)}(x) = \frac{-3}{16a^4} \text{Tr}(\gamma_5 \tilde{\delta}_x V^2_U (\nabla_1 + \frac{1}{2} (\Delta_1 - 2)) (-\nabla_1^2 + \frac{1}{2} (\Delta_1 - 2))^{-5/2}) + O(a)
$$

(48)

The rigorous derivation of this in \[27\] again uses the fact that $\tilde{A}_\mu(x)$ has compact support. The trace in (48) can now be evaluated using the “plane wave” orthonormal basis $\{\phi_k\}$ for the lattice scalar fields with periodicity $\phi(x + (2p + 1) Le^\nu) = \phi(x)$ (i.e. the fields in the scalar version of $\tilde{C}_{2p+1}$), given by

$$
\phi_k(x) = \frac{1}{\sqrt{N}} e^{ik \cdot x}, \quad N = ((2p + 1)2N)^4
$$

$$
k_\mu \in \frac{2\pi}{a(2p + 1)2N} \{ - (2p + 1)N, -(2p + 1)N + 1, \ldots, (2p + 1)N - 1 \}
$$

(49)

(50)

Note that the volume per $k$ is

$$
\Delta^4 k = \left( \frac{2\pi}{a(2p + 1)2N} \right)^4 = \frac{(2\pi)^4}{a^4N^4}
$$

(51)

Using the plane wave basis, the trace in (48) can now be evaluated as in \[28, v4\], leading to

$$
q_{(2p+1)}(x) = q_A(x) \sum_k a^4 \Delta^4 k \tilde{I}(ak) + O(a)
$$

(52)
where the summation region for $k$ is (50), $\Delta^4k$ is given by (51),

$$q_A(x) = \frac{1}{32} \epsilon_{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}(x)F_{\rho\sigma}(x))$$  \hspace{1cm} (53)

and

$$\hat{I}(k) = \frac{-3}{8\pi^2} \prod_\nu \cos K_\nu \left(-1 + \sum_\mu (1 - \cos k_\mu) - \sum_\mu \frac{\sin^2 k_\mu}{\cos k_\mu}\right) \left[ \sum_\mu \sin^2 k_\mu + (-1 + \sum_\mu (1 - \cos k_\mu))^2 \right]^{5/2}$$  \hspace{1cm} (54)

and we have exploited the fact that $\tilde{A}_\mu(x) = A_\mu(x)$ for $x \in \Gamma$. Changing summation variable from $k$ to $\tilde{k} = ak$ in (52) and taking the $p \to \infty$ limit gives

$$\lim_{p \to \infty} q_U^{(2p+1)}(x) = q_A(x) \int_{-\pi}^{\pi} d^4\tilde{k} \hat{I}(\tilde{k}) + O(a)$$  \hspace{1cm} (55)

The integral over $\tilde{k}$ in this expression was evaluated in [28], and independently in [29], and was found to be 1. Recalling (32)–(33) we finally get

$$\lim_{a \to 0} q_U(x) = \lim_{a \to 0} \lim_{p \to \infty} q_U^{(2p+1)}(x) = \lim_{a \to 0} \lim_{p \to \infty} q_U^{(2p+1)}(x) = q_A(x) \quad (x \in \Gamma)$$  \hspace{1cm} (56)

and

$$\lim_{a \to 0} Q^f(U) = \lim_{a \to 0} \lim_{p \to \infty} a^4 \sum_{x \in \Gamma} q_U^{(2p+1)}(x) = \lim_{a \to 0} \lim_{p \to \infty} a^4 \sum_{x \in \Gamma} (q_A(x) + O(a)) = \int_{T^4} d^4x q_A(x) = Q(A)$$  \hspace{1cm} (57)

This completes the sketch of the proof of the theorem. The latter part of the derivation has similarities with, and was inspired by, the calculation of the classical continuum limit of the axial anomaly for Wilson–Dirac fermions done many years ago by W. Kerler [30] and E. Seiler and I. O. Stamatescu [31].

The Overlap-Dirac operator determines a lattice Dirac fermion action $\bar{\psi}D_U\psi$ [13], which has an exact symmetry under a new kind of lattice chiral transformations [17]. The corresponding axial anomaly (=the infinitesimal chiral jacobian) was found in [17] to be (in the flavour singlet case)

$$A_U(x) = 2i q_U(x)$$

Its classical continuum limit was calculated in a perturbative setting in [32], and subsequently in a non-perturbative setting in [28, v1], [18], [29]. The perturbative setting was further studied in [33]. However, these calculations are all problematic in the case where
the continuum gauge field \( A_\mu(x) \) is topologically non-trivial (e.g. \( A_\mu(x) \) does not have a well-defined Fourier expansion in this case.) This case is covered by the arguments above though: we have found that

\[
q_U(x) = q_A(x) + O(a)
\]

in complete generality. As we have seen, the locality of the Overlap-Dirac operator (which was not used in the previous references) is a key ingredient in the derivation of this result.

The arguments and results above, which are for SU(N) gauge fields on the 4-torus, can be easily extended to U(1) fields on the 2-torus, thereby providing analytic verification of results from numerical studies in [12] where the fermionic topological charge and axial anomaly were seen to reduce to the correct continuum quantities for topologically non-trivial gauge fields in this setting.

**III. Index of lattice Dirac operators and combinatorial approach to topological invariants**

In this section a potential mathematical application of the ideas behind fermionic lattice topological charge to the construction of certain topological invariants of manifolds is discussed. Consider Kähler–Dirac spinor fields [34, 35] with the “spacetime” being an arbitrary smooth oriented riemannian 4-dimensional manifold \( M \). In the continuum the fields are the differential forms on \( M \), the space of which we denote by \( \Omega(M) = \bigoplus_{p=0}^{4} \Omega^p(M) \). (Locally, a p-form \( \omega \in \Omega^p(M) \) is of the form \( \omega_{\mu_1...\mu_p}(x)dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \).)

The Dirac operator on \( \Omega(M) \) is

\[
D = d + d^*
\]

where \( d_p : \Omega^p(M) \to \Omega^{p+1}(M) \) is the exterior derivative. The standard way to define the chirality operator \( \Gamma_5 \) (analogue of \( \gamma_5 \)) is \( \Gamma_5 = (-1)^p \) on \( \Omega^p(M) \). This has the chirality properties \( (\Gamma_5)^2 = 1 \) and \( \Gamma_5 D = -D \Gamma_5 \), and it is not difficult to show that the corresponding index of \( D \) is the Euler characteristic \( \chi(M) \) of \( M \) (see, e.g., [36, Ch.4]). A discretisation of Kähler–Dirac theory can be obtained via a triangulation \( K \) of \( M \) [35]. In the discrete theory the fields are the cochains of \( K \) (=the functions on the oriented simplexes of \( K \)), the space of which we denote by \( C(K) = \bigoplus_{p=0}^{4} C^p(K) \). The analogue of \( d \) is the coboundary operator \( d^K : C^p(K) \to C^{p+1}(K) \), the Dirac operator is \( D^K = d^K + (d^K)^* \) and \( \Gamma_5 = (-1)^p \) on \( C^p(K) \). Using the de Rham theorem it can be shown that index \( D^K = \text{index } D = \chi(M) \) so index \( D^K \) is a combinatorial construction of the topological invariant \( \chi(M) \). This is nothing new though; in fact we have just recovered Euler’s original combinatorial construction of \( \chi(M) \). Now, there is another way to define
Note that in the continuum, namely by $\Gamma_5 = (-1)^{s*}$ where $*: \Omega^p(M) \to \Omega^{4-p}(M)$ is the Hodge star operator and $s = (-1)^{1+p(p-1)/2}$ on $\Omega^p(M)$; this also satisfies the chirality properties $(\Gamma_5)^2 = 1$ and $\Gamma_5 D = -D \Gamma_5$ and the corresponding index is again a topological invariant of $M$, namely the Hirzebruch signature $\sigma(M)$ (see, e.g., [36, Ch.4]). Unlike $\chi(M)$ there is no known combinatorial construction of $\sigma(M)$ — this is an interesting open problem in mathematics. The problem is reflected in the fact that there is no natural discretisation of $*$, and thereby of $\Gamma_5$, in this setting. However, a natural discretisation is obtained after introducing $\hat{K}$: the cell decomposition of $M$ dual to $K$. We can then consider the doubled discretisation $\Omega(M) \to C(K) \oplus C(\hat{K})$ with

$$D \to D^K\hat{K} = \begin{pmatrix} D^K & 0 \\ 0 & D\hat{K} \end{pmatrix}, \quad \Gamma_5 \to \Gamma^K\hat{K}_5 = \begin{pmatrix} 0 & (-1)^{s*}\hat{K} \\ (-1)^{s*}K & 0 \end{pmatrix}$$

(59)

where $*^K_p : C^p(K) \to C^{4-p}(\hat{K})$ and $*^q_{\hat{K}} : C^q(\hat{K}) \to C^{4-q}(K)$ are the duality operators (see, e.g., [37], where topological quantities were exactly reproduced in a similar discrete setting). The index of $D^K\hat{K}$ can be seen to vanish though; this is essentially due to the fact that $\Gamma^K\hat{K}_5$ is skew-diagonal in (59). This is reminiscent of the vanishing of the index of the usual naive lattice Dirac operator. The necessity of introducing $C(\hat{K})$ is reminiscent of the appearance of fermion doubling in the usual lattice theory. Therefore, in light of the preceding sections, it may be possible to obtain a combinatorial construction of $\sigma(M)$ from the spectral flow of a “Hermitian Wilson–Dirac operator” $H^{K\hat{K}}_{(m)} = \Gamma^K\hat{K}(D^K\hat{K}_{\text{Wilson}} - mf(a))$. The main problem is to find a suitable “Wilson term” $W^{K\hat{K}}$ on $C(K) \oplus C(\hat{K})$ for the “Wilson–Dirac operator” $D^K\hat{K}_{\text{Wilson}} = iD^K\hat{K} + aW^{K\hat{K}}$ (where $a$ is now the mesh size of $K$), and a suitable function $f(a)$ in $H^{K\hat{K}}_{(m)}$. (In the usual lattice setting $f(a) = 1/a$.) One idea is to embed $C(K) \oplus C(\hat{K})$ into $C(BK)$ where $BK$ = the barycentric subdivision of $K$, and then try to construct $W^{K\hat{K}}$ from the discrete Laplace operator $\Delta^{BK}$ on $C(BK)$.

Finally, we remark that after “twisting” by a flat gauge field $A$, the signature $\sigma_A(M)$ of a 4-manifold $M$ with boundary $N$ is closely related to the Atiyah–Patodi–Singer rho invariant $\rho(N, A)$ of a 3-manifold $N$ together with a representation $\alpha : \pi_1(N) \to \mathcal{O}(n)$. A combinatorial construction of the signature invariant $\sigma_A(M)$ would in all likelihood lead to a combinatorial construction of the rho invariant as well.

Acknowledgements. I thank Ting-Wai Chiu and everyone else involved in the running and organisation of Chiral’99 for a very enjoyable and stimulating conference, and the

†† Note that in the continuum, $\sigma(M) = \text{index} D$ equals minus the spectral flow of the hermitian operator $H_{(m)} = \Gamma_5(iD - m)$ as $m$ varies from any negative value to any positive value.
National Center for Theoretical Sciences in Hsinchu for hospitality and financial support in the weeks leading up to Chiral’99. Also, I thank Martin Lüscher for discussions on the classical continuum limit, from which I benefitted greatly, and for hospitality during a visit to DESY. Thanks also go to Varghese Mathai for discussions on the signature invariant. The support of an ARC postdoctoral fellowship is gratefully acknowledged.

References

[1] M. Atiyah and I. Singer, Ann. Math. 87 (1968) 484
[2] M. Lüscher, Comm. Math. Phys. 85 (1982) 39
[3] A. Phillips, Ann. Phys. 161 (1985) 399
[4] A. Phillips and D. Stone, Comm. Math. Phys. 103 (1986) 599
[5] G. ’t Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D 14 (1976) 3432; R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172; C. Callan, R. Dashen and D. Gross, Phys. Lett. B 63 (1976) 334; Phys. Rev. D 17 (1978) 2717
[6] K. Wilson, Phys. Rev. D 14 (1974) 2445
[7] J. Smit and J. Vink, Nucl. Phys. B 286 (1987) 485
[8] S. Itoh, Y. Iwasaki and T. Yoshié, Phys. Rev. D 36 (1987) 527
[9] D. Adams, hep-lat/9907005
[10] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B 552, 363 (1999)
[11] H. Neuberger, hep-lat/9911004
[12] H. Neuberger and R. Narayanan, Phys. Lett. B 302 (1993) 62; Nucl. Phys. B 412 (1994) 574; 443 (1995) 305
[13] H. Neuberger, Phys. Lett. B 417 (1998) 141
[14] H. Neuberger, Phys. Lett. B 427 (1998) 353
[15] P. Ginsparg and K. G. Wilson, Phys. Rev. D 25 (1982) 2649
[16] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B 427 (1998) 125
[17] M. Lüscher, Phys. Lett. B 428 (1998) 342
[18] K. Fujikawa, Nucl. Phys. B 546 (1999) 480
[19] R. Edwards, U. Heller and R. Narayanan, Nucl. Phys. B 522 (1998) 285
[20] F. Karsch, E. Seiler and I. Stamatescu, Nucl. Phys. B 271 (1986) 349
[21] R. Narayanan and P. Vranas, Nucl. Phys. B 506 (1997) 373
[22] C. Gattringer, I. Hip and C. Lang, Nucl. Phys. B 63 (1998) 498
[23] P. Hernández, Nucl. Phys. B 536 (1998) 345
[24] T.-W. Chiu, Phys. Rev. D 58 (1998) 074511; 60 (1999) 114510
[25] P. Woit, Phys. Rev. Lett 51 (1983) 638
[26] M. Lüscher, private communication
[27] D. Adams, in preparation
[28] D. Adams, hep-lat/9812003
[29] H. Suzuki, Prog. Theor. Phys. 102 (1999) 141
[30] W. Kerler, Phys. Rev. D 23 (1981) 2384
[31] E. Seiler and I. Stamatescu, Phys. Rev. D 25 (1982) 2177; 26 (1982) 534 (E)
[32] Y. Kikukawa and A. Yamada, Phys. Lett. B 448 (1999) 265
[33] T.-W. Chiu and T.-H. Hsieh, hep-lat/9901011
[34] E. Kähler, Rendiconti di Matematica 21 (1962) 425; W. Graf, Ann. Inst. Henri Poincaré A 29 (1978) 85
[35] P. Becher and H. Joos, Z. Phys. C 15 (1982) 343; J. Rabin, Nucl. Phys. B 201 (1982) 315
[36] C. Nash, Differential topology and quantum field theory. (Academic Press, London, 1976)
[37] D. Adams, Phys. Rev. Lett. 78 (1997) 4155; hep-th/9612003
[38] M. Atiyah, V. Patodi and I. Singer, Math. Proc. Camb. Phil. Soc. 78 (1975) 405

16