Tychonoff spaces and a ring theoretic order on $C(X)$

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Abstract. The reduced ring order (rr-order) is a natural partial order on a reduced ring $R$ given by $r \leq_{rr} s$ if $r^2 = rs$. It can be studied algebraically or topologically in rings of the form $C(X)$. The focus here is on those reduced rings in which each pair of elements has an infimum in the rr-order, and what this implies for $X$. A space $X$ is called rr-good if $C(X)$ has this property. Surprisingly both locally connected and basically disconnected spaces share this property. The rr-good property is studied under various topological conditions including its behaviour under Cartesian products. The product of two rr-good spaces can fail to be rr-good (e.g., $\beta R \times \beta R$), however, the product of a $P$-space and an rr-good weakly Lindelöf space is always rr-good. $P$-spaces, $F$-spaces and $U$-spaces play a role, as do Glicksberg’s theorem and work by Comfort, Hindman and Negrepontis.

Introduction. In a reduced ring, a ring with no non-zero nilpotent elements, such as $C(X)$, there is a partial order that generalizes the natural partial order on a boolean ring. The order relation is defined as $r \leq_{rr} s$ if $r^2 = rs$. The study of this order, here called the rr-order for the reduced ring order, goes back at least to 1958 in [S]. Since then it has been studied at various times (see [Ch] and [B], for example), but most recently in [BR1] and [BR2]. In these papers some of the most interesting examples and results are about rings of the form $C(X)$.

It is rare for a pair of elements in a reduced ring $R$ to have a supremum in the rr-order and the most natural generalization of the boolean ring case is where the ring has, for every pair of elements $r, s \in R$, an infimum in the rr-order, noted $r \land_{rr} s$, i.e., when $R$ is a lower semi-lattice in the order. Such rings are called rr-good. A space $X$ is called rr-good if the ring $C(X)$ is rr-good. The theme of this paper is the study of spaces that are rr-good and those that are not.

In the sequel, all topological spaces will be assumed to be Tychonoff spaces.

Not all spaces are rr-good but those that are form a surprisingly diverse family that includes locally connected spaces and those that are basically disconnected. To find a topological characterization of rr-good spaces would seem an unrealistic task but much can be said about them. There are connected spaces that are not

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rr-good (see Theorem 2.9 below) and even connected, compact metric spaces that are not rr-good ([BR1] Theorem 3.5(2)).

The paper is divided into five sections. The first gives some basic results and examples. The second deals with when a product of two spaces is rr-good. If \( X \times Y \) is rr-good it is easy to see that \( X \) and \( Y \) are also rr-good. The converse, false in general, turns out to be a rich subject.

If the real line \( \mathbb{R} \) is partitioned into two complementary dense subspaces, neither can be rr-good. The third section shows that \( \mathbb{R}^2 \) is quite different. Complementary dense subspaces of the plane are found, one of which is rr-good.

Section 4 examines basically disconnected and \( U \)-spaces with an emphasis on separation properties and their consequences for \( C(X) \).

Section 5 looks at a sufficient condition, called the B-property (for boundary property), for a space to be rr-good. Basically disconnected spaces that are not discrete do not have the B-property. Here it is shown how to find connected rr-good spaces without the B-property.

1. The definition of rr-good spaces: basic properties and examples.

To recall: a ring \( C(X) \) is partially ordered by the relation \( f \preceq_{rr} g \) if \( f^2 = fg \). When \( h \preceq_{rr} f \) and \( h \preceq_{rr} g \) this is abbreviated to \( h \preceq_{rr} f, g \). The following facts are obvious but are basic tools underlying many of the results used below.

**Lemma 1.1.** In a ring \( C(X) \),
(1) if \( f \preceq_{rr} g \) then \( f \) and \( g \) coincide on \( cl(coz f) \);

(2) if \( h \preceq_{rr} f, g \) then \( coz h \subseteq z(f - g) \cap coz f \).

It is clear that a free union of rr-good spaces is rr-good since a product of rr-good rings is rr-good. The following proposition quotes results that show how rr-good spaces can be found from a given rr-good space.

**Proposition 1.2.** (1) ([BR1] Proposition 3.10) A cozero set in an rr-good space is rr-good.

(2) ([BR1] Proposition 3.9) The ring \( C(X) \) is rr-good if and only if \( C^*(X) \) is rr-good, i.e., a space \( X \) is rr-good if and only if \( \beta X \) is rr-good.

The two main classes of examples of rr-good spaces are summarized here.

**Examples 1.3.** (1) [BR1] Theorem 3.5(1) If \( X \) is a locally connected space then \( X \) is rr-good.

(2) [NR] before Ex. 3.2] If \( X \) is basically disconnected then \( X \) is rr-good.

The second case will be expanded upon in Section 4.

It is not true that a quotient space of an rr-good space has to be rr-good.

**Example 1.4.** The space \( \beta\mathbb{N} \) is rr-good but its quotient space \( \mathbb{N} \cup \{\infty\} \) (the one-point compactification) is not.

**Proof.** The space \( \mathbb{N} \cup \{\infty\} \) is not rr-good by [BR1] Proposition 3.6, or see Lemma 3.1 below. \( \square \)

Sometimes quotients behave well.

**Proposition 1.5.** (1) If \( X \) is a locally connected space, all its quotient spaces are rr-good. (2) In particular, if \( X \) is a locally connected pseudocompact space then all its continuous images are rr-good.
Proof. (1) All the quotients spaces of a locally connected space are locally connected. (2) This is by [W] page 223. □

The long line is an example of part (2) of the proposition.

Note also that every space, rr-good or not, can be embedded in a direct product of copies of a closed interval, a compact, locally connected (rr-good) space.

This section closes with a pair of illustrative examples.

The space \( \Lambda = \beta \mathbb{R} \setminus (\beta \mathbb{N} \setminus \mathbb{N}) \) of [GJ, 6P] is pseudocompact and rr-good because \( \beta \Lambda = \beta \mathbb{R} \) is rr-good, but it is known not to be locally connected [W] pp 221,222. This space will appear again in Example 2.11 and at the end of Section 2.

On the other hand the pseudocompact Tychonoff plank \( T \) is not rr-good. If it were, \( \beta T \) would also be rr-good. However, \( \beta T \) has a clopen subset which is homeomorphic to the one-point compactification of \( \mathbb{N} \), a space which is not rr-good, showing \( \beta T \) is not rr-good by Proposition 1.2(1).

2. Product spaces and the rr-order.

In this section the question of rr-good product spaces will be examined. It will be easy to see that if a product is rr-good, so are its factors. The converse, false in general, will take up much of the section.

Proposition 2.1. Suppose \( Y \) is a retract of an rr-good space \( X \). Then, \( Y \) is rr-good.

Proof. Let \( \phi \) and \( \psi \) be continuous functions \( X \xrightarrow{\phi} Y \xrightarrow{\psi} X \) with \( \psi \circ \phi = 1_X \) and \( f, g \in C(X) \). It is easy to see that if \( h = f \psi \land_{rr} g \psi \) then \( h \circ \phi = f \land_{rr} g \). □

Corollary 2.2. If \( X \) and \( Y \) are spaces such that \( X \times Y \) is rr-good, then \( X \) and \( Y \) are rr-good.

As already mentioned, the converse is false but there are some cases where there are positive results.

Proposition 2.3. (1) If \( \{X_\alpha\}_{\alpha \in \Lambda} \) are locally connected spaces all but finitely many of which are connected then \( \prod_{\alpha \in \Lambda} X_\alpha \) is rr-good.

(2) If \( \{X_1, \ldots, X_n\} \) is a finite set of P-spaces then \( \prod_{i=1}^n X_i \) is rr-good.

Proof. (1) These products are locally connected and, hence, rr-good. (2) A finite product of P-space is a P-spaces and, hence, rr-good. □

As examples, all euclidean spaces are rr-good. Other types of rr-good products will be found at the end of this section.

The following will show that if a space \( X \) has enough clopen sets and is rr-good, then it is basically disconnected. This will play a role later in this section and again in Section 4.

Proposition 2.4. Let \( X \) be a space which has a clopen \( \pi \)-base. If \( X \) is rr-good then \( X \) is basically disconnected.

Proof. For \( f \in C(X) \) it will be shown that \( \text{cl}(\text{coz} f) \) is clopen. Since \( X \) is rr-good, \( h = 1 \land_{rr} (1 - f) \) exists. Because \( h \leq_{rr} 1 \), \( h \) is an idempotent and \( \text{coz} h = D \) is clopen. Moreover, \( h = h^2 = h(1 - f) \) implies that \( hf = 0 \). When \( E \subseteq x(f) \) is clopen, let the idempotent \( e \) have cozero set \( E \). It follows that \( e \leq_{rr} 1, (1 - f) \) and, from this, \( e \leq_{rr} h \), giving \( E \subseteq D \). Hence, \( D \) is the unique largest clopen set in \( z(f) \).
If \( \text{cl} \left( \text{coz} f \right) \neq X \setminus D \) then, because of the clopen \( \pi \)-base, there would be a non-empty clopen set in \((X \setminus D) \setminus \text{cl} \left( \text{coz} f \right)\). This would contradict the fact that \( D \) is the maximal clopen set in \( z(f) \).

A first step in finding examples is to recall two results of Negrepontis.

**Proposition 2.5.** (1) [\( \mathbb{N} \)] Theorem 7.3 For any \( P \)-space \( X \) there exists an extremally disconnected space \( Y \) for which \( X \times Y \) is not an \( F \)-space. (2) [\( \mathbb{N} \)] Theorem 6.3 The product of a \( P \)-space and a compact basically disconnected space is basically disconnected.

**Corollary 2.6.** (1) If \( X \) is a \( P \)-space and \( Y \) is extremally disconnected such that \( X \times Y \) is not an \( F \)-space, then \( X \times Y \) is not \( \text{rr} \)-good. (2) If \( X \) is a \( P \)-space and \( Y \) is compact and basically disconnected, then \( X \times Y \) is \( \text{rr} \)-good.

**Proof.** (1) If \( X \times Y \) were \( \text{rr} \)-good, Proposition 2.4 would say that it is basically disconnected and, hence, an \( F \)-space. (2) Proposition 2.5 (2) gives the result. \( \square \)

The case where neither space is a \( P \)-space can also be dealt with as follows.

**Theorem 2.7.** Let \( X \) and \( Y \) be spaces such that each has a clopen \( \pi \)-base and are not \( P \)-spaces. The space \( X \times Y \) is not \( \text{rr} \)-good.

**Proof.** Every non-empty open set in \( X \times Y \) contains a non-empty clopen. If \( X \times Y \) were \( \text{rr} \)-good it would be basically disconnected by Proposition 2.1, hence an \( F \)-space, so one of \( X \) and \( Y \) would be a \( P \)-space by [\( \text{Cu} \], Theorem p. 51] or by [\( \text{GJ} \], 14Q.1] \( \square \)

Theorem 2.7 yields families of examples.

**Examples 2.8.** If \( X \) and \( Y \) are basically disconnected but not \( P \)-spaces, then \( X \) and \( Y \) are \( \text{rr} \)-good but \( X \times Y \) is not \( \text{rr} \)-good. As an illustration, \( \beta \mathbb{N} \times \beta \mathbb{N} \) is not \( \text{rr} \)-good.

Another example of a product of \( \text{rr} \)-good spaces that is not \( \text{rr} \)-good is found in the next result. It is of a quite different sort than in Examples 2.8, indeed, the factors are connected. The functions needed in the proof are best presented by a description of their graphs.

**Theorem 2.9.** The space \( \beta \mathbb{R} \times \beta \mathbb{R} \) is not \( \text{rr} \)-good.

**Proof.** Consider a band of width 2 centred on the diagonal \( D = \{(x,x) \mid x \in \mathbb{R}\} \) in \( \mathbb{R} \times \mathbb{R} \), bounded by two lines parallel to \( D \), \( L_1 \) above and \( L_2 \) below. Functions \( f, g \in C(\mathbb{R} \times \mathbb{R}) \) will be defined.

1. In the region above and including line \( L_1 \), \( f(x,y) = 3 \) and \( g(x,y) = 2 \).
2. In the region below and including \( L_2 \), \( f(x,y) = g(x,y) = 0 \).
3. Let \( L_3 \) be the line parallel to \( D \), midway between \( D \) and \( L_1 \). On any line \( M \) perpendicular to \( D \), let \( f \) go linearly from 3 to 0 as \( (x,y) \) goes from \( L_1 \) to \( L_3 \). Similarly, \( g \) will go linearly from 2 to 0 on \( M \).
4. Everywhere below \( L_3 \) both \( f \) and \( g \) will be 0 except where indicated below.
5. For each \( n \in \mathbb{N} \) consider a disk \( \Delta_n \) of radius 1/4 around \((n,n)\). The functions \( f \) and \( g \) will coincide on \( \Delta_n \) and their graphs there will be a regular cone of height 1 and centre \((n,n)\).
There are several claims to be proved.

**Claim 1:** Both \( f \) and \( g \) extend to \( \beta R \times \beta R \).

As is customary in \( R \times R \), the first factor is the horizontal axis and the second the vertical one.

It must be shown that the oscillation condition of [W] Theorem, page 200 is satisfied so that \( f \) and \( g \) can be extended to \( \beta R \times \beta R \).

It is readily seen that the functions \( f \) and \( g \) are uniformly continuous because of the repeated patterns along the diagonal. This means that for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( |f(x, y) - f(u, v)| < \varepsilon \) if \( \| (x, y), (u, v) \| < \delta \). Now fix \( (x_0, y_0) \), set \( \zeta = (1/\sqrt{\delta}) \delta \) and consider the vertical lines through \( x_0 + n\zeta \) and horizontal lines through \( y_0 + n\zeta \), with \( m, n \in \mathbb{Z} \). Set \( U_1 = \bigcup_{m \in \mathbb{Z}} (x_0 + m\zeta, x_0 + (m + 1)\zeta) \), a union of intervals along the horizontal axis; and \( V_1 = \bigcup_{n \in \mathbb{Z}} (y_0 + n\zeta, y_0 + (n + 1)\zeta) \), a union of intervals along the vertical axis.

Similarly, construct \( U_2 \) and \( V_2 \) using \( \{ x_0 + (1/2 + m)\zeta \} \) and \( \{ y_0 + (1/2 + n)\zeta \} \), \( m, n \in \mathbb{Z} \). Then the open sets \( U_1 \times V_1 \) and \( U_2 \times V_2 \) cover \( R \times R \) and make a grid satisfying the oscillation conditions. Since this can be done for all \( \varepsilon > 0 \), \( f \) and \( g \) can be extended to elements of \( C(\beta R \times \beta R) \), say \( F \) and \( G \), respectively.

**Claim 2:** If \( H = F \cap \beta R \) and \( G \) then for \( n, m \in \mathbb{N} \), \( n \neq m \), \( H, n, m \) = 0. In the case where \( n > m \), this holds because both \( f \) and \( g \) vanish at \( (n, m) \). In the case where \( n < m \), this holds because \( f \) and \( g \) do not agree at \( (n, m) \) (Lemma [2.12]).

**Claim 3:** For \( n \in \mathbb{N} \) let \( h_n \in C(R \times R) \) be the function whose graph is the cone defined for \( (n, n) \). The function \( h_n \) extends to \( \beta R \times \beta R \) because of the same grid as used for \( f \) and \( g \). Let \( H_n \) denote its extension to \( \beta R \times \beta R \). Observe that \( H_n \leq_{rr} F, G \) since \( h_n \leq_{rr} f, g \) on the dense subset \( R \times R \).

**Claim 4:** It follows from Claim 3 that \( H(n, n) = 1 \) for all \( n \in \mathbb{N} \) since \( H_n(n, n) = 1 \) and \( H_n \leq_{rr} H \). This means that \( H \), restricted to \( \mathbb{N} \times \mathbb{N} \), is the Kronecker delta function, which is shown in [W] p. 196] not to extend to \( \beta \mathbb{N} \times \beta \mathbb{N} \). This is a contradiction.

**Corollary 2.10.** For any \( m, n \geq 1 \), \( \beta(\mathbb{R}^m) \times \beta(\mathbb{R}^n) \) is not rr-good.

**Proof.** The space \( R \) is a retract of \( \mathbb{R}^m \) and, hence, \( \beta R \) is a retract of \( \beta(\mathbb{R}^m) \) and, similarly, \( \beta R \) is retract of \( \beta(\mathbb{R}^n) \). From this, \( \beta R \times \beta R \) is a retract of \( \beta(\mathbb{R}^m) \times \beta(\mathbb{R}^n) \). The result follows from Theorem [2.9] and Proposition [2.11].

It would be interesting to know if \( R \times \beta R \) is rr-good or not. The methods used above do not apply to this space.

For spaces \( X \) and \( Y \) it is possible for \( \beta(X \times Y) \) to be homeomorphic to \( \beta X \times \beta Y \), where the homeomorphism does not fix \( X \times Y \) (see [W] 8.18) for such an example. In the case of a homeomorphism, all the spaces \( X \times Y, \beta(X \times Y) \) and \( \beta X \times \beta Y \) are simultaneously rr-good or none of them is. The latter possibility is illustrated in the next example.

**Example 2.11.** There is a connected non-compact rr-good pseudocompact space whose product with itself is pseudocompact but not rr-good.

**Proof.** The example is the space \( \Lambda = \beta R \setminus (\beta \mathbb{N} \setminus \mathbb{N}) \) mentioned at the end of Section 1. It is rr-good but \( \beta \Lambda \times \beta \Lambda = \beta R \times \beta R \) is not rr-good. On the other hand, \( \Lambda \times \Lambda \) is pseudocompact by [W] Proposition, p. 203]. Hence, Glicksberg’s theorem applies showing that \( \beta(\Lambda \times \Lambda) = \beta \Lambda \times \beta \Lambda \). This is not rr-good and therefore \( \Lambda \times \Lambda \) is not rr-good either.
This section ends with some \textit{rr}-good products. Unlike previous examples, the ones to be presented here need not be locally connected or basically disconnected. A simple lemma, whose proof follows by direct point-wise calculations, will be useful.

\textbf{Lemma 2.12.} Let $Y$ be an \textit{rr}-good space and $X$ any space. Suppose $f, g \in C(X \times Y)$. For each $x \in X$, let $f_x, g_x \in C(Y)$ be given by $f_x(y) = f(x, y)$ and $g_x(y) = g(x, y)$. Set $h_x = f_x \land g_x$ and let $h$ be defined by $h(x, y) = h_x(y)$, for all $x, y$. If $h$ is continuous on $X \times Y$ then $h = f \land g$.

Recall that a space $X$ is \textit{weakly Lindelöf} if for any open cover $\{U_a\}_{a \in A}$, there is a countable subfamily $\{U_{a_n}\}_{n \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} U_{a_n}$ dense in $X$. In the following, a result of Comfort, Hindman and Negrepontis (\textbf{CHN}) will be crucial.

\textbf{Theorem 2.13.} Let $X$ be an arbitrary $P$-space and $Y$ an \textit{rr}-good weakly Lindelöf space. The space $X \times Y$ is \textit{rr}-good.

\textbf{Proof.} Fix $f, g \in C(X \times Y)$. By \textbf{CHN} Lemma 3.2, each point in $X \times Y$ lies in an open set of the form $U \times V$, $U$ open in $X$, $V$ open in $Y$, such that for $x, x' \in U$ and all $y \in V$, $f_x(y) = f_{x'}(y)$ (the notation is as in the lemma). There are such open sets for $g$ as well and, by taking intersections, it may be assumed that these open sets work for both functions. Moreover, since $X$ is a $P$-space, it may also be assumed that $U$ is clopen. An open set $U \times V$ where $U$ is clopen and the \textbf{CHN} properties hold for both $f$ and $g$ will be here called a \textit{tile}.

Fix $p \in X$. For each $y \in Y$ there is a tile $U \times V$ with $(p, y) \in U \times V$. Hence, the set of tiles $\{U \times V \}_{(p, y) \in Y}$, with $p \in U_{\beta}$, is such that $\bigcup_{\beta \in B} V_\beta = Y$. By the weakly Lindelöf property, there is a countable subset $\{V_\alpha\}_{\alpha \in \mathbb{N}}$ whose union, $V_p$, is dense in $Y$. Put $A_p = \bigcap_{n \in \mathbb{N}} U_{\alpha_n}$. Since $X$ is a $P$-space, $A_p$ is clopen. For all $x \in A_p$, $f_x = f_p$ are equal on the dense open set $V_p$. Hence, $f_x = f_p$ on $Y$. Similarly, $g_x = g_p$ on $Y$. Put $h_p = f_p \land g_p$ and notice that $h_p = f_x \land g_x$, for all $x \in A_p$.

Now consider $p, p' \in X$. If $A_p \cap A_{p'} \neq \emptyset$, then $h_p = h_{p'}$. Indeed, for $x \in A_p \cap A_{p'}$, $f_x = f_p = f_p'$ and $g_x = g_p = g_{p'}$. Define $h(x, y) = h_p(y)$ whenever $x \in A_p$. This is well-defined and continuous on all the elements of the open cover of $\{A_p \times Y\}_{p \in X}$. Hence, $h = f \land g$ by Lemma 2.12. \hfill \Box

There are many examples of \textit{rr}-good Lindelöf spaces $Y$ which can be used in Proposition 2.13 for example $\mathbb{R}$. For any non-discrete $P$-space $X$, $X \times \mathbb{R}$ is \textit{rr}-good but neither locally connected nor basically disconnected. An example where the \textit{rr}-good space $Y$ is weakly Lindelöf but not Lindelöf is $\Lambda$, described at the end of Section 1. Another is found in \textbf{LR} Example 2, p. 237.

3. A partition of $\mathbb{R}^2$ into two dense subspaces, one \textit{rr}-good: this is impossible in $\mathbb{R}$.

The first thing to note is that if the real line $\mathbb{R} = A \cup B$, with $A \cap B = \emptyset$ and $A$ and $B$ both dense, then neither $A$ nor $B$ is \textit{rr}-good. This is a consequence of the following.

\textbf{Lemma 3.1.} \textbf{BR1} Proposition 3.6] Suppose, in a space $X$, there is a sequence $\{D_n\}_{n \in \mathbb{N}}$ of pairwise disjoint clopen sets such that $U = \bigcup_{n \in \mathbb{N}} D_n$ is not closed and there is $x \in \text{Fr} U$ (the boundary or frontier) such that every neighbourhood of $x$ meets all but finitely many of the $D_n$. Then, $X$ is not \textit{rr}-good.

To use the lemma in the case of $A$ and $B$ in $\mathbb{R}$, it suffices to take a convergent increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ in, say, $A$ and intersperse it with a sequence from $B$. 

Subsets of $\mathbb{R}^2$ will now be constructed to show a quite different situation in the plane.

**Definition 3.2.** A line $y = mx + b$ in $\mathbb{R}^2$ is called **matched** if $m, b \in \mathbb{Q}$ and $m \neq 0$. The graph of such a line is denoted $L_{m,b}$.

**Lemma 3.3.** Consider a matched line $L_{m,b}$ in $\mathbb{R}^2$ given by $y = mx + b$, where $m \neq 0$ and $m, b \in \mathbb{Q}$. Then if $(p, q) \in L_{m,b}$, both $p, q \in \mathbb{Q}$ or both are irrational.

**Proof.** If $x \in \mathbb{Q}$ then $y = mx + b \in \mathbb{Q}$. If $x \notin \mathbb{Q}$ then $y = mx + b \in \mathbb{Q}$ would imply $mx \in \mathbb{Q}$, but $m \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, which is impossible. □

It is also useful to note that if $(a, b)$ and $(c, d)$ are such that $a, b, c, d \in \mathbb{Q}$, $a \neq c$, then the line joining these points is a matched line.

**Theorem 3.4.** Consider the following two subsets of $\mathbb{R}^2$:

$$B = \bigcup_{m,b \in \mathbb{Q}, m \neq 0} L_{m,b} \text{ and } A = \mathbb{R}^2 \setminus B.$$  

Then,

(1) $B$ is dense in $\mathbb{R}^2$, locally connected and, hence, $rr$-good.

(2) $A$ is dense in $\mathbb{R}^2$ and has a basis of clopen sets. It is not $rr$-good.

**Proof.** (1) Since any open set in $\mathbb{R}^2$ contains points where both coordinates are rational, $B$ is dense in $\mathbb{R}^2$. Notice that $B$ also contains points $(a, b)$ where both $a$ and $b$ are irrational, but not all such points.

Consider a point $(a, b)$ in $B$ and an open disk $C$ with centre $(a, b)$. Suppose that $U$ and $V$ are open sets of $\mathbb{R}^2$ such that $U \cup V \supseteq C \cap B$, $U \cap V \cap C \cap B = \emptyset$, $U \cap C \cap B \neq \emptyset$ and $V \cap C \cap B \neq \emptyset$. In other words assume that there is a partition of $C \cap B$. Choose points $(p, q) \in U \cap C \cap B$ and $(u, v) \in V \cap C \cap B$, $p, q, u, v \in \mathbb{Q}$, $u \neq p$. The line segment joining these two points will lie in $C \cap B$ but this line segment is connected in $\mathbb{R}^2$, which is impossible. Hence, $B$ is locally connected and, hence, $rr$-good by [BR1] Theorem 3.5(1)]. (It can be seen that $B$ is even arcwise connected.)

(2) The set $A$ contains all points $(a, b)$ where one coordinate is rational and the other irrational, as well as some points where both coordinates are irrational. This shows that $A$ is dense in $\mathbb{R}^2$. Moreover, for any $(a, b) \in A$ and any open disk $C$ with centre $(a, b)$ there is a quadrilateral inside $C$ containing $(a, b)$ bounded by matched lines. The interior of such a quadrilateral, intersected with $A$, is a clopen set in $A$.

Since $A$ has a basis of clopen sets and has convergent sequences, it is not $rr$-good by Lemma 3.1. □

There are similar constructions in $\mathbb{R}^n$, $n > 2$.

### 4. Some separation properties and $C(X)$.

Two sorts of reduced rings will make an appearance in this section. The definitions are recalled here and, in the case of $C(X)$, the corresponding topological notions will follow.

**Definition 4.1.** (1) A ring $R$ is called *weakly Baer* or $wB$ if, for each $r \in R$, $\text{ann } r$ is generated by an idempotent $e = e^2$. (2) A ring $R$ is called *almost weakly Baer* or $awB$ if, for each $r \in R$, $\text{ann } r$ is generated by a set of idempotents.
In the literature the names “pp-ring” and “almost pp-ring” are also used for wB and awB rings, respectively.

The first thing to note is the following.

**Lemma 4.2.** [BRI] Theorem 2.6| An awB ring is rr-good if and only if it is wB.

Not all awB rings are wB.

**Example 4.3.** [NR] Example 3.2| The ring $C(\beta N \setminus N)$ is awB but not wB.

Even though awB rings need not be rr-good, a topological description of them nicely parallels that for wB rings of the form $C(X)$, and is given here.

The equivalence of the first two statements in the following is mentioned in [NR] but is also proved here.

**Proposition 4.4.** The following three statements about a space $X$ are equivalent. (1) $X$ is basically disconnected; (2) $C(X)$ is a wB ring; and (3) if $U$ is a cozero set and $V$ an open set with $U \cap V = \emptyset$ then $U$ and $V$ can be separated by a clopen set.

**Proof.** (1) $\Rightarrow$ (2): Consider $f \in C(X)$ and let $D = X \setminus (\text{cl}(\text{coz } f))$, a clopen set, and $e = e^2 \in C(X)$ such that $\text{coz } e = D$. For any $g \in \text{ann } f$, $ge = g$ and $fe = 0$. Hence, $\text{ann } f = eC(X)$. (2) $\Rightarrow$ (3): Let $U = \text{coz } f$ and $V$ be open with $U \cap V = \emptyset$. Since $\text{ann } f = eC(X)$ for some $e = e^2$, the clopen set $D = \text{coz } e$ is such that $\text{coz } f = U \subseteq X \setminus D$. For every $g \in C(X)$ with $\text{coz } g \subseteq V$, $fg = 0$ implying that $\text{coz } g \subseteq D$. Thus, the clopen set $D$ separates $U$ and $V$. (3) $\Rightarrow$ (1): If $U = \text{coz } f$, put $V = \text{int}(X \setminus U)$. There is a clopen set $D$ with $\text{coz } f \subseteq D$ and $V \subseteq X \setminus D$. It follows that $\text{cl } U = D$. □

The equivalence of (1) and (2) in the next result was obtained in [AE] Theorem 2.4], but the proof here is more direct. $U$-spaces were introduced in [GH]; they are spaces $X$ such that, for each $f \in C(X)$, there is a unit $u \in C(X)$ with $f = |f|u$.

**Proposition 4.5.** The following statements for a space $X$ are equivalent. (1) $X$ is a $U$-space; (2) $C(X)$ is an awB ring; and (3) if $U$ and $V$ are cozero sets with $U \cap V = \emptyset$ then $U$ and $V$ can be separated by a clopen set.

**Proof.** (1) $\Rightarrow$ (2): Let $0 \neq f, g \in C(X)$ with $fg = 0$. Replace $f$ by $k = -|f|$ and $g$ by $l = |g|$; the cozero sets do not change. There is a unit $u$ such that $k + l = |k + l|u = (-k + l)u$. Hence, for $x \in \text{coz } k$, $u(x) = -1$ and for $x \in \text{coz } l$, $u(x) = 1$. Since $u$ is a unit, there is a clopen set $D$ such that for $x \in D$, $u(x) > 0$ and for $x \notin D$, $u(x) < 0$. From this, $\text{coz } k = \text{coz } f \subseteq X \setminus D$ and $\text{coz } l = \text{coz } g \subseteq D$. Put $e = e^2$ with $\text{coz } e = D$. Then, $fe = 0$ and $g = eg$, showing that ann $f$ is generated by idempotents.

(2) $\Rightarrow$ (3): Let $U = \text{coz } f$ and $V = \text{coz } g$ be such that $U \cap V = \emptyset$. The product $fg = 0$. Since $C(X)$ is awB there are $e_i = e_i^2$ and $l_i \in C(X)$, $i = 1, \ldots, k$, with each $e_i$ such that $fe_i = 0$ and $g = \sum_{i=1}^{k} e_i l_i$. Since $D = \bigcup_{i=1}^{k} \text{coz } e_i$ is clopen, there is $e = e^2$ with $\text{coz } e = D$. From this, $fe = 0$ and $g = ge$. The clopen set $\text{coz } e$ separates $U$ and $V$.

(3) $\Rightarrow$ (1): It must be shown that for any $f \in C(X)$ there is a unit $u$ with $f = |f|u$. If $f$ does not change sign in $\text{coz } f$, the unit can be $\pm 1$. Otherwise, let
5. A sufficient but not a necessary condition for $rr$-good.

We begin by recalling the definition of the B-property from [BRI] Definition 3.3. It is a sufficient condition for a space to be $rr$-good ([BRI] Corollary 3.4]). It is implied by local connectedness. However, it is known not to be a necessary condition; a topic expanded upon here.

**Definition 5.1.** In a space $X$ let $\{U_\alpha\}_{\alpha \in A}$ be any family of non-empty cozero sets in $X$ with the following property: for $\alpha \neq \beta$ in $A$, $(\operatorname{Fr} U_\alpha) \cap U_\beta = \emptyset$. The space $X$ is said to satisfy the B-property (for boundary property) if the following holds for each such family of cozero sets. Let $z \in \operatorname{Fr} (\bigcup_{\alpha \in A} U_\alpha)$. For every neighbourhood $N$ of $z$ there is $\beta \in A$ such that $N \cap \operatorname{Fr} U_\beta \neq \emptyset$.

The motivation for this definition is as follows: Suppose in $C(X)$ that, for $f, g \in C(X)$, there are non-zero $rr$-lower bounds $\{h_\alpha\}_{\alpha \in A}$ for $f$ and $g$. Then, by Lemma 1.1, the set $\{\operatorname{coz} h_\alpha\}_{\alpha \in A}$ satisfies the demands of Definition 5.1.

The purpose here is to find connected $rr$-good spaces without the B-property. Before doing that, the next proposition shows that, at the other extreme, it is easy to find basically disconnected spaces without the B-property.

**Proposition 5.2.** If $X$ is a space that has the B-property then each union of clopen sets is clopen. If, in addition, $X$ has a clopen $\pi$-base, it is discrete.

**Proof.** Any set $\{U_\alpha\}_{\alpha \in A}$ of clopen sets satisfies the conditions of Definition 5.1. Set $U = \bigcup_{\alpha \in A} U_\alpha$. If $x \in \operatorname{Fr} U$, any neighbourhood of $x$ would meet $\operatorname{Fr} U_\alpha$, for some $\alpha$. However, $\operatorname{Fr} U_\alpha = \emptyset$ and, hence, $\operatorname{Fr} U = \emptyset$. For the second part, any open $V$ in $X$ has a union of clopen sets dense in it. From the first part, $V$ is clopen. □

Any basically disconnected space $X$ which is not discrete is $rr$-good and does not have the B-property.

The next proposition is the key tool for the construction of connected examples.

**Proposition 5.3.** Let $X$ be a space which is not compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an infinite family of pairwise disjoint non-empty cozero sets of $X$. For $\alpha \in A$, let $f_\alpha$ be such that $f_\alpha \in C(X)$ such that $\operatorname{coz} f_\alpha = U_\alpha$, for all $x \in X$, $0 \leq f_\alpha(x) \leq 1$ and for some $k_\alpha \in U_\alpha$, $f_\alpha(k_\alpha) = 1$. Assume that these data also satisfy the following properties:

(i) for $\alpha \neq \beta$, $(\operatorname{Fr} X U_\beta) \cap U_\alpha = \emptyset$,
(ii) $K_\alpha = \operatorname{cl} X U_\alpha$ is compact for all $\alpha \in A$,
(iii) the function $f$ defined by $f(x) = f_\alpha(x)$ for $x \in U_\alpha$ and $f(x) = 0$ if $x \notin U = \bigcup_{\alpha \in A} U_\alpha$ is continuous on $X$,
(iv) $K = f^{-1}(\{1\})$ is not compact in $X$.

Then, $\beta X$ does not have the B-property.

**Proof.** The condition (ii) says that $K_\alpha$ is compact and so it and $X^* = \beta X \setminus X$ are completely separated. There is $u_\alpha \in C(\beta X)$ such that $u_\alpha|_{K_\alpha}$ is constantly 1 and $u_\alpha|_{X^*}$ is constantly 0. Now $f_\alpha$ can be extended to $\beta f_\alpha \in C(\beta X)$. The product
\[ u_{\alpha} \cdot \beta f_{\alpha} \text{ coincides with } f_{\alpha} \text{ on } U_{\alpha} \text{ and is 0 elsewhere. This shows that } U_{\alpha} \text{ is a cozero set in } \beta X. \text{ Moreover, } Fr_{X} U_{\alpha} = Fr_{\beta X} U_{\alpha} \text{ because } K_{\alpha} \text{ is compact and, hence, also closed in } \beta X. \]

It follows that \( \{ U_{\alpha} \} \) is a family of cozero sets in \( \beta X \) which satisfies the condition to test for the B-property.

Since \( K \) is closed and not compact in \( X \), \( P = (cl_{\beta X} K) \cap X^* \neq \emptyset \). Now, extend \( f \) to \( \beta f \).

It follows that \((coz \beta f) \cap X = U \) but also \( P \subseteq coz \beta f \), since for \( p \in P, \beta f(p) = 1 \). Notice that any \( p \in P \) is in \( Fr_{\beta X} U \) since any neighbourhood \( N \) of \( p \) with \( N \subseteq coz \beta f \) will meet \( X \) and thus \( N \cap X \subseteq U \). However, any such \( N \) will not meet any \( Fr_{X} U_{\alpha} = Fr_{\beta X} U_{\alpha} \), contradicting the B-property. \( \Box \)

**Corollary 5.4.** Suppose that \( X \) satisfies the conditions of Proposition 5.3 and that \( X \) is \( rr \)-good. Then, \( \beta X \) is \( rr \)-good and does not have the B-property. Moreover, \( \beta X \) is not locally connected.

**Proof.** Since \( X \) is \( rr \)-good, so is \( \beta X \). Then Proposition 5.3 says that \( \beta X \) does not have the B-property. If \( \beta X \) were locally connected, it would have the B-property. \( \Box \)

**Example 5.5.** The connected space \( \beta \mathbb{R} \) is \( rr \)-good and does not have the B-property.

**Proof.** The space \( \mathbb{R} \) is \( rr \)-good. The cozero sets needed in the proposition can be taken to be the intervals \( \{(n, n + 1)\}_{n \in \mathbb{Z}} \) and, hence, Corollary 5.4 applies. \( \Box \)

Similarly, for any euclidean space \( \mathbb{R}^n \), \( \beta \mathbb{R}^n \) is a connected \( rr \)-good space which does not have the B-property.

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