AUTOMORPHIC EQUIVALENCE OF LINEAR ALGEBRAS.

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Abstract

This research is motivated by universal algebraic geometry. We consider in universal algebraic geometry the some variety of universal algebras Θ and algebras \( H \in Θ \) from this variety. One of the central question of the theory is the following: When do two algebras have the same geometry? What does it mean that the two algebras have the same geometry? The notion of geometric equivalence of algebras gives a sort of answer to this question. Algebras \( H_1 \) and \( H_2 \) are called geometrically equivalent if and only if the \( H_1 \)-closed sets coincide with the \( H_2 \)-closed sets. The notion of automorphic equivalence is a generalization of the first notion. Algebras \( H_1 \) and \( H_2 \) are called automorphically equivalent if and only if the \( H_1 \)-closed sets coincide with the \( H_2 \)-closed sets after some "changing of coordinates".

We can detect the difference between geometric and automorphic equivalence of algebras of the variety Θ by researching of the automorphisms of the category \( Θ^0 \) of the finitely generated free algebras of the variety Θ. By \( \mathfrak{A} \) the automorphic equivalence of algebras provided by inner automorphism degenerated to the geometric equivalence. So the various differences between geometric and automorphic equivalence of algebras can be found in the variety Θ if the factor group \( \mathfrak{A}/\mathfrak{Y} \) is big. Hear \( \mathfrak{A} \) is the group of all automorphisms of the category \( Θ^0 \), \( \mathfrak{Y} \) is a normal subgroup of all inner automorphisms of the category \( Θ^0 \).

In [4] the variety of all Lie algebras and the variety of all associative algebras over the infinite field \( k \) were studied. If the field \( k \) has not nontrivial automorphisms then group \( \mathfrak{A}/\mathfrak{Y} \) in the first case is trivial and in the second case has order 2. We consider in this paper the variety of all linear algebras over the infinite field \( k \). We prove that group \( \mathfrak{A}/\mathfrak{Y} \) is isomorphic to the group \( (U(kS_2)/U(k\{e\}))\times Autk \), where \( S_2 \) is the
symmetric group of the set which has 2 elements, \( U(kS_2) \) is the group of all invertible elements of the group algebra \( kS_2 \), \( e \in S_2 \), \( U(k \{ e \}) \) is a group of all invertible elements of the subalgebra \( k \{ e \} \). \( \text{Aut} k \) is the group of all automorphisms of the field \( k \).

So even the field \( k \) has not nontrivial automorphisms the group \( A/Y \) is infinite. This kind of result is obtained for the first time.

The example of two linear algebras which are automorphically equivalent but not geometrically equivalent is presented in the last section of this paper. This kind of example is also obtained for the first time.

1 Introduction.

In the first two sections we consider some variety \( \Theta \) of one-sorted algebras of the signature \( \Omega \). Denote by \( X_0 = \{ x_1, x_2, \ldots, x_n, \ldots \} \) a countable set of symbols, and by \( \mathfrak{F}(X_0) \) the set of all finite subsets of \( X_0 \). We will consider the category \( \Theta^0 \), whose objects are all free algebras \( F(X) \) of the variety \( \Theta \) generated by finite subsets \( X \in \mathfrak{F}(X_0) \). Morphisms of the category \( \Theta^0 \) are homomorphisms of free algebras.

We denote some time \( F(X) = F(x_1, x_2, \ldots, x_n) \) if \( X = \{ x_1, x_2, \ldots, x_n \} \) and even \( F(X) = F(x) \) if \( X \) has only one element.

We assume that our variety \( \Theta \) possesses the IBN property: for free algebras \( F(X), F(Y) \in \Theta \) we have \( F(X) \cong F(Y) \) if and only if \( |X| = |Y| \). In this case we have [4, Theorem 2] this decomposition

\[
\mathfrak{A} = \mathfrak{P}\mathfrak{S},
\]

of the group \( \mathfrak{A} \) of all automorphisms of the category \( \Theta^0 \). Hear \( \mathfrak{P} \) is a group of all inner automorphisms of the category \( \Theta^0 \) and \( \mathfrak{S} \) is a group of all strongly stable automorphisms of the category \( \Theta^0 \).

**Definition 1.1** An automorphism \( \Upsilon \) of a category \( \mathcal{R} \) is **inner**, if it is isomorphic as a functor to the identity automorphism of the category \( \mathcal{R} \).

This means that for every \( A \in \text{Ob}\mathcal{R} \) there exists an isomorphism \( s_A^\Upsilon : A \to \Upsilon (A) \) such that for every \( \alpha \in \text{Mor}_\mathcal{R}(A, B) \) the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{s_A^\Upsilon} & \Upsilon (A) \\
\downarrow \alpha & & \downarrow \Upsilon (\alpha) \\
B & \xrightarrow{s_B^\Upsilon} & \Upsilon (B)
\end{array}
\]

commutes.

**Definition 1.2.** An automorphism \( \Phi \) of the category \( \Theta^0 \) is called **strongly stable** if it satisfies the conditions:

1. \( \Phi \) preserves all objects of \( \Theta^0 \),
A2) there exists a system of bijections \( \{ s_F^\Phi : F \to F \mid F \in \text{Ob}^0 \} \) such that \( \Phi \) acts on the morphisms \( \alpha : D \to F \) of \( \Theta^0 \) by this way:

\[
\Phi(\alpha) = s_F^\Phi \alpha (s_F^\Phi)^{-1},
\]

(1.2)

A3) \( s_F^\Phi \mid_X = id_X \), for every free algebra \( F = F(X) \).

The subgroup \( \mathcal{A} \) is a normal in \( \mathfrak{A} \). We will calculate the factor group \( \mathfrak{A}/\mathcal{A} \cong \mathfrak{S}/\mathcal{S} \cap \mathcal{A} \). This calculation is very important for universal algebraic geometry.

All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [1], [2] and [3]. In universal algebraic geometry we consider a "set of equations" \( T \subset F \times F \) in some finitely generated free algebra \( F \) of the arbitrary variety of universal algebras \( \Theta \) and we "resolve" these equations in \( \text{Hom}(F,H) \), where \( H \in \Theta \). The set \( \text{Hom}(F,H) \) serves as an "affine space over the algebra \( H \)". Denote by \( T'_{H} \) the set \( \{ \mu \in \text{Hom}(F,H) \mid T \subset \ker \mu \} \). This is the set of all solutions of the set of equations \( T \). For every set of "points" \( R \) of the affine space \( \text{Hom}(F,H) \) we consider a congruence of equations defined by this set: \( R'_{H} = \bigcap_{\mu \in R} \ker \mu \). For every set of equations \( T \) we consider its algebraic closure \( T''_{H} \) in respect to the algebra \( H \). A set \( T \subset F \times F \) is called \( H \)-closed if \( T = T''_{H} \). An \( H \)-closed set is always a congruence.

**Definition 1.3** Algebras \( H_1, H_2 \in \Theta \) are **geometrically equivalent** if and only if for every \( X \in \mathfrak{F}(X_0) \) and every \( T \subset F(X) \times F(X) \) fulfills \( T''_{H_1} = T''_{H_2} \).

Denote the family of all \( H \)-closed congruences in \( F \) by \( \text{Cl}_{H}(F) \). We can consider the category \( C_{\Theta}(H) \) of the coordinate algebras connected with the algebra \( H \in \Theta \). Objects of this category are quotient algebras \( F(X)/T \), where \( X \in \mathfrak{F}(X_0), T \in \text{Cl}_{H}(F(X)) \). Morphisms of this category are homomorphisms of algebras.

**Definition 1.4** Let \( \text{Id}(H,X) = \bigcap_{\varphi \in \text{Hom}(F(X),H)} \ker \varphi \) be the minimal \( H \)-closed congruence in \( F(X) \). Algebras \( H_1, H_2 \in \Theta \) are **automorphically equivalent** if and only if there exists a pair \( (\Phi, \Psi) \), where \( \Phi : \Theta^0 \to \Theta^0 \) is an automorphism, \( \Psi : C_{\Theta}(H_1) \to C_{\Theta}(H_2) \) is an isomorphism subject to conditions:

A. \( \Psi(F(X)/\text{Id}(H_1,X)) = F(Y)/\text{Id}(H_2,Y) \), where \( \Phi(F(X)) = F(Y) \),

B. \( \Psi(F(X)/T) = F(Y)/\bar{T} \), where \( T \in \text{Cl}_{H_1}(F(X)), \bar{T} \in \text{Cl}_{H_2}(F(Y)) \),

C. \( \Psi \) takes the natural epimorphism \( \tau : F(X)/\text{Id}(H_1,X) \to F(X)/T \) to the natural epimorphism \( \Psi(\tau) : F(Y)/\text{Id}(H_2,Y) \to F(Y)/\bar{T} \).

Note that if such a pair \( (\Phi, \Psi) \) exists, then \( \Psi \) is uniquely defined by \( \Phi \).

We can say, in certain sense, that automorphic equivalence of algebras is a coinciding of the structure of closed sets after some "changing of coordinates" provided by automorphism \( \Phi \).
Algebras $H_1$ and $H_2$ are geometrically equivalent if and only if an inner automorphism $\Phi : \Theta^0 \to \Theta^0$ provides the automorphic equivalence of algebras $H_1$ and $H_2$. So, only strongly stable automorphism $\Phi$ can provide us automorphic equivalence of algebras which not coincides with geometric equivalence of algebras. Therefore, in some sense, difference from the automorphic equivalence to the geometric equivalence is measured by the factor group $\mathcal{A}/\mathcal{G} \cong \mathcal{G}/\mathcal{G} \cap \mathcal{Z}$.

2 Verbal operations and strongly stable automorphisms.

For every word $w = w(x_1, \ldots, x_k) \in F(X)$, where $F(X) \in \text{Ob}\Theta^0$, $X = \{x_1, \ldots, x_k\}$ and for every algebra $H \in \Theta$ we can define a $k$-ary operation $w^*_H$ on $H$ by

$$w^*_H(h_1, \ldots, h_k) = w(h_1, \ldots, h_k) = \gamma_h(w(x_1, \ldots, x_k)),$$

where $\gamma_h$ is a homomorphism $F(X) \ni x_i \to \gamma_h(x_i) = h_i \in H$, $1 \leq i \leq k$. This operation we call the verbal operation induced on the algebra $H$ by the word $w(x_1, \ldots, x_k) \in F(X)$. A system of words $W = \{w_\omega | \omega \in \Omega\}$ such that $w_\omega \in F(X_\omega)$, $X_\omega = \{x_1, \ldots, x_{k_\omega}\}$, determines a system of $k_\omega$-ary operations $(w_\omega)_H^*$ on $H$. Denote the set $H$ with the system of these operation by $H_W^*$.

We have a correspondence between strongly stable automorphisms and systems of words which define the verbal operation and fulfill some conditions. This correspondence explained in [4] and [5]: We denote the signature of our variety $\Theta$ by $\Omega$, by $k_\omega$ we denote the arity of $\omega$ for every $\omega \in \Omega$. We suppose that we have the system of words $W = \{w_\omega | \omega \in \Omega\}$ satisfies the conditions:

Op1) $w_\omega(x_1, \ldots, x_{k_\omega}) \in F(X_\omega)$, where $X_\omega = \{x_1, \ldots, x_{k_\omega}\}$;

Op2) for every $F = F(X) \in \text{Ob}\Theta^0$ there exists an isomorphism $\sigma_F : F \to F^*_W$ such that $\sigma_F |_{X} = id_X$.

$F^*_W \in \Theta$ so isomorphisms $\sigma_F$ are defined uniquely by the system of words $W$.

The set $S = \{\sigma_F : F \to F | F \in \text{Ob}\Theta^0\}$ is a system of bijections which satisfies the conditions:

B1) for every homomorphism $\alpha : A \to B \in \text{Mor}\Theta^0$ the mappings $\sigma_B \alpha \sigma_A^{-1}$ and $\sigma_B^{-1} \alpha \sigma_A$ are homomorphisms;

B2) $\sigma_F |_{X} = id_X$ for every free algebra $F \in \text{Ob}\Theta^0$.

So we can define the strongly stable automorphism by this system of bijections. This automorphism preserves all objects of $\Theta^0$ and acts on morphism of $\Theta^0$ by formula \([1,2]\), where $s_F \Phi = \sigma_F$. 
Vice versa if we have a strongly stable automorphism $\Phi$ of the category $\Theta^0$ then its system of bijections $S = \{ s_F^\Phi : F \mapsto F \}$ defined uniquely. Really, if $F \in \text{Ob} \Theta^0$ and $f \in F$ then

$$s_F^\Phi (f) = s_F^\Phi (x) = \left( s_F^\Phi \left( s_F^\Phi \right)^{-1} \right) (x) = (\Phi (\alpha)) (x), \quad (2.1)$$

where $D = F(x)$ - 1-generated free linear algebra - and $\alpha : D \mapsto F$ homomorphism such that $\alpha(x) = f$. Obviously that this system of bijections $S = \{ s_F^\Phi : F \mapsto F \}$ fulfills conditions B1 and B2 with $\sigma_F = s_F^\Phi$.

If we have a system of bijections $S = \{ \sigma_F : F \mapsto F \}$ which fulfills conditions B1 and B2 than we can define the system of words $W = \{ w_\omega \mid \omega \in \Omega \}$ satisfies the conditions Op1 and Op2 by formula

$$w_\omega (x_1, \ldots, x_{k_\omega}) = \sigma_F \omega ((x_1, \ldots, x_{k_\omega})) \in F_\omega, \quad (2.2)$$

where $F_\omega = F(X_\omega)$.

By formulas (2.1) and (2.2) we can check that there are

1. one to one and onto correspondence between strongly stable automorphisms of the category $\Theta^0$ and systems of bijections satisfied the conditions B1 and B2
2. one to one and onto correspondence between systems of bijections satisfied the conditions B1 and B2 and systems of words satisfied the conditions Op1 and Op2.

So we can find a strongly stable automorphism $\Phi$ of the category $\Theta^0$ by finding a system of words which fulfills conditions Op1 and Op2.

3 Verbal operations in linear algebras.

From now on, we consider the variety $\Theta$ of all linear algebras over infinite field $k$. We consider linear algebras as one-sorted universal algebras, i.e., multiplication by scalar we consider as 1-ary operation for every $\lambda \in k$: $H \ni h \mapsto \lambda h \in H$ where $H \in \Theta$. Hence the signature $\Omega$ of algebras of our variety contains these operations: 0-ary operation 0; $|k|$ 1-ary operations of multiplications by scalars; 1-ary operation $- : h \mapsto -h$, where $h \in H$, $H \in \Theta$; 2-ary operation $\cdot$ and 2-ary operation $+$. We will finding the system of words $W = \{ w_\omega \mid \omega \in \Omega \}$ satisfies the conditions Op1 and Op2. We denote the words corresponding to these operations by $w_0$, $w_\lambda$ for all $\lambda \in k$, $w_-$, $w_+$. From now on, the word “ideal” means two sided ideal of linear algebra.
We denote the group of all automorphisms of the field $k$ by $\text{Aut} k$.

Our variety $\Theta$ possesses the IBN property, because $|X| = \dim F/F^2$ fulfills for all free algebras $F = F(x) \in \Theta$. So we have the decomposition (1) for group of all automorphisms of the category $\Theta^0$.

Now we need to prove one technical fact about 1-generated free linear algebra $F(x)$.

**Lemma 3.1** Let \{u_1, \ldots, u_r\} is the set of all monomials of degree $n$ in $F(x)$ (basis of $F_n$), \{v_1, \ldots, v_t\} is the set of all monomials of degree $m$ in $F(x)$ (basis of $F_m$), $\varphi$ is an arbitrary function from $\{1, \ldots, n\}$ to $\{1, \ldots, t\}$. Denote by $\varphi(u_l)$ the monomial which is a results of substitution into monomial $u_l$ ($1 \leq l \leq r$) instead $j$-th from left entry of $x$ the monomial $v_{\varphi(j)}$ ($1 \leq j \leq n$). All these monomials are distinct, i. e., $\varphi(u_l) = \varphi(u_l')$ if and only if $\varphi_1 = \varphi_2$ and $u_l = u_l'$, where $\varphi_1, \varphi_2 : \{1, \ldots, n\} \to \{1, \ldots, t\}$, $u_i, u_i' \in \{u_1, \ldots, u_r\}$.

**Proof.** We will prove this lemma by induction by $n$ - degree of monomials from \{u_1, \ldots, u_r\}. The claim of the lemma is trivial for $n = 1$. We assume that the claim of the lemma is proved for monomials which have degree $< n$. We suppose that $\varphi_1(u_{i_1}) = \varphi_2(u_{i_2})$, where $\deg u_{i_1} = \deg u_{i_2} = n > 1$, $\varphi_1, \varphi_2 : \{1, \ldots, n\} \to \{1, \ldots, t\}$. $u_i = u_i^{(1)} \cdot u_i^{(2)}$, where $i = 1, 2$. We denote $\deg u_i^{(1)} = c_i$, $1 \leq c_i < n$ for $i = 1, 2$. For $i = 1, 2$ we have $\varphi_i(u_i) = \varphi_i^{(1)}(u_i^{(1)}) \cdot \varphi_i^{(2)}(u_i^{(2)})$, where $\varphi_i^{(1)} : \{1, \ldots, c_i\} \to \{1, \ldots, t\}$, $\varphi_i^{(2)} : \{1, \ldots, n - c_i\} \to \{1, \ldots, t\}$, $\varphi_i^{(1)}(j) = \varphi_i(j)$ for $1 \leq j \leq c$, $\varphi_i^{(2)}(j) = \varphi_i(c_i + j)$ for $1 \leq j \leq n - c_i$. $\varphi_1(u_{i_1}) = \varphi_2(u_{i_2})$ if and only if $\varphi_1^{(1)}(u_i^{(1)}) = \varphi_2^{(1)}(u_i^{(1)})$ and $\varphi_1^{(2)}(u_i^{(2)}) = \varphi_2^{(2)}(u_i^{(2)})$. If $c_1 \neq c_2$ then $\deg \varphi_1^{(1)}(u_i^{(1)}) = c_1 m \neq \deg \varphi_2^{(1)}(u_i^{(1)}) = c_2 m$, hence $\varphi_1^{(1)}(u_i^{(1)}) \neq \varphi_2^{(1)}(u_i^{(1)})$ and $\varphi_1(u_{i_1}) \neq \varphi_2(u_{i_2})$. So $c_1 = c_2$ and, by our assumption, $\varphi_1^{(1)} = \varphi_2^{(1)}$, $u_i^{(1)} = u_i^{(2)}$, $\varphi_1^{(2)} = \varphi_2^{(2)}$, $u_i^{(2)} = u_i^{(2)}$. Therefore $\varphi_1 = \varphi_2$ and $u_i^{(1)} = u_i^{(2)}$.

**Corollary 1** Let $f(x), g(x) \in F(X)$. $f(g(x))$ is a result of substitution of $g(x)$ in $f(x)$ instead $x$. $f(g(x)) \in F_1$ if and only if $f(x), g(x) \in F_1$.

**Proof.** We write $f(x)$ and $g(x)$ as sum of its homogeneous components: $f(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$, $g(x) = g_1(x) + g_2(x) + \ldots + g_m(x)$, $f_i(x), g_i(x) \in F_i$. We assume that $n > 1$ or $m > 1$, $f_n(x) \neq 0$ and $g_m(x) \neq 0$. $f(g(x)) = f_1(g(x)) + f_2(g(x)) + \ldots + f_n(g(x))$. Addenda of the maximal possible degree of $x$, which can appear in $f(g(x))$, i. e., addenda of degree $nm$ can appear in $f_n(g(x))$. They coincide with addenda of $f_n(g_m(x))$. Denote $f_n(x) = \lambda_1 u_1 + \ldots + \lambda_r u_r$, $g_m(x) = \mu_1 v_1 + \ldots + \mu_r v_r$, where $\{u_1, \ldots, u_r\}$ is the set of all monomials of degree $n$ in $F(x)$, $\{v_1, \ldots, v_t\}$ is the set of all monomials of degree $m$ in $F(x)$, $\lambda_i, \mu_j \in k$. Not all $\{\lambda_1, \ldots, \lambda_r\}$ and not all $\{\mu_1, \ldots, \mu_r\}$ are equal to 0 by our assumption. $f_n(g_m(x)) = \lambda_1 u_1 (g_m(x)) + \ldots + \lambda_r u_r (g_m(x))$. If we open the brackets in $u_i (g_m(x)) = u_i (\mu_1 v_1 + \ldots + \mu_r v_r)$ ($1 \leq l \leq r$), we obtain addenda, which are results of substitution into monomial $u_i$ instead all entry of $x$ some monomial from $\mu_1 v_1, \ldots, \mu_r v_r$ in all possible options. We can say more
we take the addendum which is a result of substitution into monomial $\lambda \phi$ and $\lambda \phi$ is equal to 0, because $f$ another by Lemma $\text{3.1}$ discussed in and all entries of $\lambda$ and $\lambda$ the monomial $x$ which is a result of substitution into monomial $\sigma \nu$. There exists by Op2 an isomorphism $\lambda = 0$. So $\lambda \mu = 0$ for all $l \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, t\}$.

Theorem 3.1 The system of words

$$W = \{w_0, w_\lambda (\lambda \in k), w_-, w_+, w\}$$

satisfies the conditions Op1 and Op2 if and only if $w_0 = 0$, $w_\lambda = \phi (\lambda) x_1$, $w_- = -x_1$, $w_+ = x_1 + x_2$, $w = ax_1 x_2 + bx_2 x_1$, where $\phi$ is an automorphism of the field $k$, $a, b \in k$, $a \neq \pm b$.

Proof. Let $W$ (see $\text{3.1}$) satisfies the conditions Op1 and Op2.

$w_0$ is an element of the 0-generated free linear algebra. There is only one element in this algebra: 0. This is the only one opportunity for $w_0$.

$w_\lambda \in (F(x))_W$ for every $\lambda \in k$. Denote multiplications by scalars in $(F(x))_W$ by $*$, i. e., $\lambda * f = w_\lambda (f)$ for every $f \in (F(x))_W$ and every $\lambda \in k$. $(F(x))_W \in \Theta$, therefore, if $\lambda = 0$ then $0 * x = w_0 (x) = 0$. If $\lambda \neq 0$ then

$$1 * x = (\lambda^{-1} \lambda) * x = \lambda^{-1} (\lambda * x) = w_{\lambda^{-1}} (w_\lambda (x)) = x.$$ 

Hence $w_0 = \phi (\lambda) x$ by Corollary 1 from Lemma $\text{3.1}$, where $\phi (\lambda) \in k$. We can write $\phi (0) = 0$. Also we have that for all $\lambda_1, \lambda_2 \in k$ fulfills

$$(\lambda_1 \lambda_2) * x = \phi (\lambda_1 \lambda_2) x$$

and

$$(\lambda_1 \lambda_2) * x = \lambda_1 * (\lambda_2 * x) = \lambda_1 * (\phi (\lambda_2) x) = \phi (\lambda_1) (\phi (\lambda_2) x) = (\phi (\lambda_1) \phi (\lambda_2)) x.$$ 

So $\phi (\lambda_1) \phi (\lambda_2) = \phi (\lambda_1 \lambda_2)$. If $\mu \in k \setminus \{0\}$, then the 1-ary operation of multiplication by scalar $\mu$ is a verbal operation defined by some word $w_\mu^* (x) \in (F(x))_W$.

written be the operations defined by system of words $W$ - see $\text{4.2}$ Proposition 4.2]. Hence, $\mu f = w_\mu^* (f)$ holds for every $f \in (F(x))_W$. Also there is $w_{\mu^{-1}}^* (x) \in (F(x))_W$ such that $\mu^{-1} f = w_{\mu^{-1}}^* (f)$ for every $f \in (F(x))_W$. $x = \mu^{-1} (\mu x) = w_{\mu^{-1}}^* (w_\mu^* (x))$.

There exists by Op2 an isomorphism $\sigma_{F(x)} : (F(x))_W \to (F(x))_W$ such that $\sigma_{F(x)} (x) = x$. So $(F(x))_W$ is also 1-generated free linear algebra of $\Theta$ with
the free generator \( x \). Hence there exists a decomposition \( (F(x))^*_W = \bigoplus_{i=1}^\infty F_i^* \), where \( F_i^* \) are linear spaces of elements which are homogeneous according the degree of \( x \) but in respect of operations defined by system of words \( W \). Therefore \( w_+^*(x) = \lambda \cdot x \), where \( \lambda \in k \), by Corollary 1 from Lemma 3.3.1. So \( \mu x = \lambda \cdot x = \varphi(\lambda) x \) and \( \mu = \varphi(\lambda) \), hence \( \varphi: k \to k \) is a surjection.

\[
 w_+ \in F(x_1, x_2) = F. \text{ There exists } n \in \mathbb{N}, \text{ such that } \\
w_+ (x_1, x_2) = p_1 (x_1, x_2) + p_2 (x_1, x_2) + \ldots + p_n (x_1, x_2),
\]

where \( p_i (x_1, x_2) \in F_i, 1 \leq i \leq n \). We have for every \( \lambda \in k \) that

\[
 w_+ (\lambda \cdot x_1, \lambda \cdot x_2) = \lambda \cdot w_+ (x_1, x_2) = \varphi(\lambda) w_+ (x_1, x_2) = \\
\varphi(\lambda) p_1 (x_1, x_2) + \varphi(\lambda) p_2 (x_1, x_2) + \ldots + \varphi(\lambda) p_n (x_1, x_2)
\]

and

\[
w_+ (\lambda \cdot x_1, \lambda \cdot x_2) = p_1 (\lambda \cdot x_1, \lambda \cdot x_2) + p_2 (\lambda \cdot x_1, \lambda \cdot x_2) + \ldots + p_n (\lambda \cdot x_1, \lambda \cdot x_2) = \\
p_1 (\varphi(\lambda) x_1, \varphi(\lambda) x_2) + p_2 (\varphi(\lambda) x_1, \varphi(\lambda) x_2) + \ldots + p_n (\varphi(\lambda) x_1, \varphi(\lambda) x_2) = \\
\varphi(\lambda) p_1 (x_1, x_2) + (\varphi(\lambda))^2 p_2 (x_1, x_2) + \ldots + (\varphi(\lambda))^n p_n (x_1, x_2).
\]

We can take \( \lambda \in k \) such that \( \varphi(\lambda) \) is not a solution of any equation \( x^i = x \), where \( 2 \leq i \leq n \). So, \( p_i (x_1, x_2) = 0 \) for \( 2 \leq i \leq n \) by equality of the homogeneous components. Therefore \( w_+ = \alpha x_1 + \beta x_2 \), where \( \alpha, \beta \in k \). If we denote the operation defined by \( w_+ \) in \( (F(x_1, x_2))^*_W \) by \( \perp \), then \( x_1 \perp x_2 = x_2 \perp x_1 \) holds, so \( \alpha x_1 + \beta x_2 = \alpha x_2 + \beta x_1 \) and \( \alpha = \beta \). Also \( x_1 \perp 0 = x_1 \) holds and \( \alpha x_1 = x_1 \), so \( \alpha = \beta = 1 \). Now, by consideration of \( F(x) \), we can conclude that for all \( \lambda_1, \lambda_2 \in k \) fulfills

\[
\varphi(\lambda_1 + \lambda_2) x = (\lambda_1 + \lambda_2) \perp x = \lambda_1 \perp \lambda_2 \perp x = \\
\lambda_1 \perp x + \lambda_2 \perp x = \varphi(\lambda_1) x + \varphi(\lambda_2) x = (\varphi(\lambda_1) + \varphi(\lambda_2)) x,
\]

so \( \varphi(\lambda_1 + \lambda_2) = \varphi(\lambda_1) + \varphi(\lambda_2) \) and \( \varphi \) is an automorphism of the field \( k \).

It's clear now that \( w_- = -x \in F(x) \), because

\[
w_- (x) = -1 \perp x = \varphi(-1) x = (-1) x = -x.
\]

\( w \in F(x_1, x_2) \). We write \( w \) as sum of its homogeneous components according the degree of \( x_1 \):

\[
w_+ (x_1, x_2) = p_0 (x_1, x_2) + p_1 (x_1, x_2) + p_2 (x_1, x_2) + \ldots + p_n (x_1, x_2).
\]

We denote the operation defined by \( w \) in \( (F(x_1, x_2))^*_W \) by \( \times \). So we have for every \( \lambda \in k \) that

\[
(\lambda \cdot x_1) \times x_2 = \lambda \cdot (x_1 \times x_2) = \varphi(\lambda) w_+ (x_1, x_2) = \\
\varphi(\lambda) p_1 (x_1, x_2) + (\varphi(\lambda))^2 p_2 (x_1, x_2) + \ldots + (\varphi(\lambda))^n p_n (x_1, x_2).
\]
We can take, as above, we have for every λ

\text{op1 and op2 then}

\[ w (x_1, x_2) = \]

\[ \lambda = p_1 (x_1, x_2) + p_2 (x_1, x_2) + \ldots + p_n (x_1, x_2) . \]

\text{We can take, as above, } \lambda \in k \text{ such that by equality of the homogeneous components we obtain that } w (x_1, x_2) = \]

\[ \lambda = p_1 (x_1, x_2) + p_2 (x_1, x_2) + \ldots + p_n (x_1, x_2) . \]

\text{We can take, as above, } \lambda \in k \text{ such that by equality of the homogeneous components we obtain that } w (x_1, x_2) = \]

\[ \lambda = p_1 (x_1, x_2) + p_2 (x_1, x_2) + \ldots + p_n (x_1, x_2) . \]
(\lambda \mu) * x = \varphi (\lambda \mu) x = \varphi (\lambda) \varphi (\mu) x = \varphi (\lambda) (\mu * x) = \lambda * (\mu * x),
(\lambda + \mu) * x = \varphi (\lambda + \mu) x = (\varphi (\lambda) + \varphi (\mu)) x = \varphi (\lambda) x + \varphi (\mu) x = \lambda * x + \mu * x,
1 * x = \varphi (1) x = 1x = x,

x \times (y + z) = ax(y + z) + b(y + z) x = axy + axz + byx + bzx = x \times y + x \times z,
(y + z) \times x = a(y + z) x + bx(y + z) = ayx + axz + bxy + bxz = y \times x + z \times x,

\lambda * (x \times y) = \varphi (\lambda)(axy + byx) = a(\varphi (\lambda) x) y + by(\varphi (\lambda) x) = (\varphi (\lambda) x) \times y =
(\lambda * x) \times y = x \times (\lambda * y)

fulfills for every \( x, y, z \in H, \lambda, \mu \in k \).

Hence there exists a homomorphism \( \sigma_{F} : F \to F_{W}^{\ast} \) such that \( \sigma_{F} |_{X} = \text{id}_{X} \) for every \( F = F(X) \in \text{Ob} \Theta^{0} \). Our goal is to prove that these homomorphisms are isomorphisms. We will prove by induction by \( i \) that

\[ \sigma_{F}(F_{i}) = F_{i}. \tag{3.2} \]

for every \( i \in \mathbb{N} \). If \( X = \{ x_{1}, \ldots, x_{n} \} \) then every element of \( F_{1} \) has form \( \lambda_{1}x_{1} + \ldots + \lambda_{n}x_{n} \), where \( \lambda_{1}, \ldots, \lambda_{n} \in k \).

\( \sigma_{F}(\lambda_{1}x_{1} + \ldots + \lambda_{n}x_{n}) = \lambda_{1}\sigma_{F}(x_{1}) + \ldots + \lambda_{n}\sigma_{F}(x_{n}) = \varphi (\lambda_{1}) x_{1} + \ldots + \varphi (\lambda_{n}) x_{n} \),

so \( \sigma_{F}(F_{1}) \subset F_{1} \).

\[ \sigma_{F}(\varphi^{-1}(\lambda_{1}) x_{1} + \ldots + \varphi^{-1}(\lambda_{n}) x_{n}) = \lambda_{1}x_{1} + \ldots + \lambda_{n}x_{n}, \]

so \( \sigma_{F}(F_{1}) = F_{1} \).

Let \( \sigma_{F} \) proved for \( i \) such that \( 1 \leq i < r \). Every element of \( F_{r} \) is a linear combination of the monomials of the form \( uv, \) where \( u \in F_{i}, v \in F_{j}, i + j = r \).

\[ \sigma_{F}(uv) = \sigma_{F}(u) \times \sigma_{F}(v) = a\sigma_{F}(u)\sigma_{F}(v) + b\sigma_{F}(v)\sigma_{F}(u), \]

so \( \sigma_{F}(F_{r}) \subset F_{r} \), because, by our assumption, \( \sigma_{F}(u) \in F_{i}, \sigma_{F}(v) \in F_{j} \). Also, if \( u = \sigma_{F}(\bar{u}), v = \sigma_{F}(\bar{v}), \) where \( \bar{u} \in F_{r}, \bar{v} \in F_{r} \), then

\[ \sigma_{F}(\bar{u} \bar{v}) = \sigma_{F}(\bar{u}) \times \sigma_{F}(\bar{v}) = u \times v = auv + buv \]

fulfills. \( a \neq \pm b, \) so the matrix \( \begin{bmatrix} a & b \\ b & a \end{bmatrix} \) is regular, hence there exist \( \alpha, \beta \in k \) such that

\[ uv = \alpha \sigma_{F}(\bar{u} \bar{v}) + \beta \sigma_{F}(\bar{v} \bar{u}) = \sigma_{F}(\varphi^{-1}(\alpha) \bar{u} \bar{v} + \varphi^{-1}(\beta) \bar{v} \bar{u}). \]

Therefore \( \sigma_{F}(F_{r}) = F_{r} \). We can conclude that \( \sigma_{F} \) is an epimorphism.

Now we will prove that \( \ker \sigma_{F} = 0 \). Let \( f \in \ker \sigma_{F} \subset F(X) \). There exists \( m \in \mathbb{N} \) such that \( f \in \bigoplus_{i=1}^{m} F_{i}. \)

\[ \sigma_{F}\left( \bigoplus_{i=1}^{m} F_{i} \right) = \bigoplus_{i=1}^{m} F_{i} \] by \( \text{3.2} \). \( \sigma_{F} \) is a linear
mapping from the linear space $\bigoplus_{i=1}^{m} F_i$ with the original multiplication by scalars in $F$ to the $\left( \bigoplus_{i=1}^{m} F_i \right)^*_W$ - the linear space $\bigoplus_{i=1}^{m} F_i$ with the multiplication by scalars which we denote by $\ast$. From formulas $\sum_{i=1}^{k} (\lambda_i * e_i) = \sum_{i=1}^{k} \varphi (\lambda_i) e_i$ and $\sum_{i=1}^{k} \lambda_i e_i = \sum_{i=1}^{k} (\varphi^{-1} (\lambda_i) * e_i)$ we can conclude that if $E$ is a basis of the linear space $\bigoplus_{i=1}^{m} F_i$ then $E$ is a basis of the linear space $\left( \bigoplus_{i=1}^{m} F_i \right)^*_W$. So $\dim \left( \bigoplus_{i=1}^{m} F_i \right)^*_W < \infty$, therefore $\ker \left( \sigma_F \mid \bigoplus_{i=1}^{m} F_i \right) = 0$ and $f = 0$. 

4 Group $\mathcal{A}/\mathcal{Y}$.

From now on, $W$ is a system of words (3.1) which fulfills conditions $\text{Op1}$ and $\text{Op2}$.

The decomposition (1.1) is not split in general case, i.e. $\mathcal{S} \cap \mathcal{Y} \neq \{1\}$ in general case. The strongly stable automorphism $\Phi$ of the category $\Theta^0$ which corresponds to the system of words $W$ is inner, by [4, Lemma 3], if and only if for every $F \in \text{Ob} \Theta^0$ there exists an isomorphism $c_F : F \to F_W^*$ such that $c_F \alpha = \alpha c_D$ fulfills for every $(\alpha : D \to F) \in \text{Mor} \Theta^0$ (by [3] Remark 3.1) $\alpha$ is also a homomorphism from $D_W^*$ to $F_W^*$.

Hear we need to prove one technical lemma.

**Lemma 4.1** If $F = F (X) \in \text{Ob} \Theta^0$ and $c_F : F \to F_W^*$ is an isomorphism then there exists an isomorphism $c_i : F/F^i \to F_W^*/F^i$ such that $\chi_i c_F = c_i \chi_i$, where $\chi_i : F \to F/F^i$ and $\chi_i^* : F_W^* \to F_W^*/F^i$ are natural homomorphisms, $i \in \mathbb{N}$.

**Proof.** If $H \in \Theta$ and $I$ is an ideal of $H$. If $\lambda \in k$, $y \in I$, $h \in H$, then $\lambda \ast y = \varphi (\lambda) y \in I$, $y \ast h = ayh + bhy \in I$, analogously $h \ast y \in I$. Therefore $I$ is an ideal of $H_W^*$. Hence $F^i$ is an ideal of $F_W^*$.

If $\sigma_F : F \to F_W^*$ is an isomorphism such that $\sigma_F \mid X = \text{id} X$, then by (2.2) we have $c_F^{-1} (F^i) = c_F^{-1} \sigma_F^{-1} (F^i) = F^i$ because $c_F^{-1} \sigma_F : F \to F$ is an isomorphism. So $c_F (F^i) = F^i$. It finishes the proof. 

**Proposition 4.1** The strongly stable automorphism $\Phi$ which corresponds to the system of words $W$ is inner if and only if $\varphi = \text{id}_k$ and $b = 0$.

**Proof.** We suppose that strongly stable automorphism $\Phi$ which corresponds to the system of words $W$ is inner. We assume that $\varphi \neq \text{id}_k$, i.e., there exists $\lambda \in k$ such that $\varphi (\lambda) \neq \lambda$. We denote $F = F (x)$. We take $\alpha \in \text{End} F$, such that $\alpha (x) = \lambda x$. We suppose that $c_F : F \to F_W^*$ is an isomorphism. $c_2$ is defined as in the Lemma 4.1 and we by this Lemma we have:

$$
\chi_2^* c_F (x) = c_2 \chi_2 (x) = \mu \ast \chi_2^* (x) = \chi_2^* (\mu \ast x) = \chi_2^* (\varphi (\mu) x),
$$
where operations in algebra $F_W^*/F^2$ we denote by same symbols as operations in algebra $F_W^*$ and $\mu \in k \setminus \{0\}$. Therefore $c_F (x) \equiv \varphi (\mu) x \ (\text{mod} \ F^2)$, $\alpha (F^2) \subset F^2$ fulfills, so

$$\alpha c_F (x) = \alpha (\varphi (\mu) x + f_2) \equiv \alpha \varphi (\mu) x = \varphi (\mu) \alpha (x) = \varphi (\mu) \lambda x \ (\text{mod} \ F^2),$$

where $f_2 \in F^2$.

$$c_F \alpha (x) = c_F (\lambda x) = \lambda \ast c_F (x) = \varphi (\lambda) c_F (x) \equiv \varphi (\lambda) \varphi (\mu) x \ (\text{mod} \ F^2),$$

$\mu \neq 0$, so $\varphi (\mu) \neq 0$, $\varphi (\lambda) \neq \lambda$ hence $\alpha c_F \neq c_F \alpha$. This contradiction proves that $\varphi = \text{id}_k$.

Now we denote $F = F (x_1, x_2) \in \text{Ob} \Theta^0$. By our assumption there exists an isomorphism $c_F : F \to F_W^*$ such that $c_F \alpha = \alpha c_F$ fulfills for every $\alpha \in \text{End} F$. $c_2$ is defined as in the Lemma 4.1 $\alpha (F^2) \subset F^2$ so we can define the homomorphism $\tilde{\alpha} : F/F^2 \to F/F^2$ such that $\tilde{\alpha} x_2 = \tilde{\alpha} x_3 = \chi_2 \alpha$. From $c_F \alpha = \alpha c_F$ we can conclude $c_2 \tilde{\alpha} = \tilde{\alpha} c_2$ fulfills. By Lemma 4.1 $c_2$ is a regular linear mapping. We can take the endomorphisms $\alpha$ such that $\tilde{\alpha}$ will be an arbitrary linear mapping from $k^2$ to $k^2$. Therefore $c_2$ must be a regular linear mapping from $k^2$ to $k^2$ which commute with all linear mappings from $k^2$ to $k^2$. Hence $c_2$ must be a scalar mapping, i.e.,

$$\chi_2^* c_F (x_1) = c_2 \chi_2 (x_1) = \lambda \chi_2^* (x_1) = x_2^* (\lambda x_1),$$

where $\lambda \in k \setminus \{0\}$, $i = 1, 2$. Therefore $c_F (x_1) = \lambda x_1 + f_1$, where $f_i \in F^2$, $i = 1, 2$.

We can remark that now we consider the case when $\varphi = \text{id}_k$, hence we need not distinguish between multiplication by scalar in $F$ and $F_W$.

Now we take $\alpha \in \text{End} F$ such that $\alpha (x_1) = x_1 x_2$, $\alpha (x_2) = 0$. If $u$ is a monomial which contain only entries of $x_1$, then $\deg_{x_1} \alpha (u) + \deg_{x_2} \alpha (u) = 2 \deg_{x_1} u$. If a monomial $u$ contain at least one entry of $x_2$, then $\alpha (u) = 0$. Hence $\alpha (F^2) \subset F^3$. So we have

$$c_F \alpha (x_1) = c_F (x_1 x_2) = c_F (x_1) \times c_F (x_2) = c_F (x_1) c_F (x_2) + b c_F (x_2) c_F (x_1) = a (\lambda x_1 + f_1) (\lambda x_2 + f_2) + b (\lambda x_2 + f_2) (\lambda x_1 + f_1) \equiv a \lambda^2 x_1 x_2 + b \lambda^2 x_2 x_1 \ (\text{mod} \ F^3).$$

$$\alpha c_F (x_1) = \alpha (\lambda x_1 + f_1) \equiv \lambda x_1 x_2 \ (\text{mod} \ F^3).$$

Hence we conclude $b = 0$ from $c_F \alpha = \alpha c_F$.

If $b = 0$, i.e., $w = a x_1 x_2$, $a \neq 0$, then we take $c_F (f) = a^{-1} f$ for every $F \in \text{Ob} \Theta^0$ and every $f \in F$. It is obvious that $c_F$ is a regular linear mapping.

$$c_F (f_1) \times c_F (f_2) = a c_F (f_1) c_F (f_2) = a (a^{-1} f_1) (a^{-1} f_2) = a^{-1} f_1 f_2 = c_F (f_1 f_2).$$

for every $f_1, f_2 \in F$. So $c_F : F \to F_W^*$ is an isomorphism. It fulfills

$$c_F \alpha (d) = a^{-1} \alpha (d) = \alpha (a^{-1} d) = \alpha c_F (d)$$

for every $(\alpha : D \to F) \in \text{Mor} \Theta^0$ and every $d \in D$. ■
**Proposition 4.2** The group $\mathcal{S} \cong G \rtimes \text{Aut}_k$, where $G$ is the group of all regular $2 \times 2$ matrices over field $k$, which have a form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and every $\varphi \in \text{Aut}_k$ acts on the group $G$ by this way: $\varphi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(b) & \varphi(a) \end{pmatrix}$.

**Proof.** We will define the mapping $\tau : G \rtimes \text{Aut}_k \rightarrow \mathcal{S}$. If $g \varphi \in G \rtimes \text{Aut}_k$, where $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, then we define $\tau (g \varphi) = \Phi \in \mathcal{S}$, where $\Phi$ corresponds to the system of words $W$ with $w_\lambda = \varphi(\lambda) x_1$ for every $\lambda \in k$ and $w = ax_1 x_2 + bx_2 x_1$. By Section 2 and Theorem 3.1, $\Phi$ is bijection.

We consider $\tau (g_1 \varphi_1) = \Phi_1$ and $\tau (g_2 \varphi_2) = \Phi_2$, where $g_1 \varphi_1, g_2 \varphi_2 \in G \rtimes \text{Aut}_k$ and $g_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix}$, $g_2 = \begin{pmatrix} a_2 & b_2 \\ b_2 & a_2 \end{pmatrix}$. Both these strongly stable automorphisms preserves all objects of $\Theta^0$ and acts on morphisms of $\Theta^0$ by theirs systems of bijections $\left\{ s^F_{\Phi_i} : F \rightarrow F \mid F \in \text{Ob} \Theta^0 \right\}$, for $i = 1, 2$, according the formula $[1, 2]$. We have $\Phi_2 \Phi_1 (\alpha) = s^F_{\Phi_2} s^F_{\Phi_1} \alpha \left( s^F_{\Phi_1} \right)^{-1} \left( s^F_{\Phi_2} \right)^{-1}$ for every $(\alpha : D \rightarrow F) \in \text{Mor} \Theta^0$. So strongly stable automorphism $\Phi_2 \Phi_1 = \tau (g_2 \varphi_2) \tau (g_1 \varphi_1)$ preserves all objects of $\Theta^0$ and acts on morphisms of $\Theta^0$ by system of bijections

$$\left\{ s^F_{\Phi_2} s^F_{\Phi_1} : F \rightarrow F \mid F \in \text{Ob} \Theta^0 \right\}. $$

This system of bijections satisfies the conditions B1 and B2, so we can define the words $w^\Phi_\lambda$ for every $\lambda \in k$ and $w^\Phi \varphi_1$, which correspond to the automorphism $\Phi_2 \Phi_1$ by formula $[2, 2]$. The words $w^\Phi_\lambda (\lambda \in k)$ and $w^\Phi \varphi_1$ which correspond to the automorphism $\Phi_1$ have forms $w^\Phi_\lambda = \varphi_i (\lambda) x_1 (\lambda \in k)$ and $w^\Phi \varphi_1 = a_i x_1 x_2 + b_i x_2 x_1$ for $i = 1, 2$. So

$$w^\Phi_\lambda = s^F_{\Phi_2} s^F_{\Phi_1} (x_1 x_2) = s^F_{\Phi_2} \left( w^\Phi_\lambda \right) = s^F_{\Phi_2} ( \varphi_1 (\lambda) x_1 ) = \varphi_2 (\varphi_1 (\lambda)) x_1 = (\varphi_2 \varphi_1) (\lambda) x_1$$

for every $\lambda \in k$ and

$$w^\Phi \varphi_1 = s^F_{\Phi_2} s^F_{\Phi_1} (x_1 x_2) = s^F_{\Phi_2} (w^\Phi \varphi_1) = s^F_{\Phi_2} (a_1 x_1 x_2 + b_1 x_2 x_1) =$$

$$\varphi_2 (a_1) s^F_{\Phi_2} (x_1 x_2) + \varphi_2 (b_1) s^F_{\Phi_2} (x_2 x_1) =$$

$$\varphi_2 (a_1) (a_2 x_1 x_2 + b_2 x_2 x_1) + \varphi_2 (b_1) (a_2 x_2 x_1 + b_2 x_1 x_2) =$$

$$\varphi_2 (a_1) a_2 + \varphi_2 (b_1) b_2 x_1 x_2 + (\varphi_2 (a_1) + \varphi_2 (b_1) + \varphi_2 (a_2) + \varphi_2 (b_2)) x_2 x_1.$$ 

because $s^F_{\Phi_i} : F \rightarrow F_{W_i}$ is an isomorphism, $i = 1, 2$. Hence

$$\Phi_2 \Phi_1 = \tau (g_2 \varphi_2) \tau (g_1 \varphi_1) = \tau (g_2 \varphi_2 (g_1) \varphi_2 \varphi_1) = \tau (g_2 \varphi_2 \cdot g_1 \varphi_1).$$

■

**Corollary 1** Group $\mathcal{S} \cap \mathfrak{U}$ is isomorphic to the group $k^* I_2$ of the regular $2 \times 2$ scalar matrices over field $k$. 

13
Proof. By Propositions 4.1 and 4.2.

Corollary 2 \( \mathfrak{A}/\mathfrak{O} \cong (G/k^*I_2) \times \text{Aut}_k \).

Proof. By Proposition 4.2 and Corollary 1 we have that \( \varphi \)
and \( \text{Corollary 2} \).

By Proposition 5.1

5 Example of two linear algebras which are automorphically equivalent but not geometrically equivalent.

We take \( k = \mathbb{Q} \). \( \Theta \) will be the variety of all linear algebras over \( k \). \( H \) will be

the 2-generated linear algebra, which is free in the variety corresponding to the

identity \( (x_1x_1)x_2 = 0 \). We consider the strongly stable automorphism \( \Phi \) of the
category \( \Theta^0 \) corresponding to the system of words \( W \), where \( b \neq 0 \). Algebras
\( H \) and \( H^*_W \) are automorphically equivalent by [5] Theorem 5.1.

Proposition 5.1 Algebras \( H \) and \( H^*_W \) are not geometrically equivalent.

Proof. Let \( F = F(x_1, x_2) \). The ideal \( I = Id(H, \{x_1, x_2\}) \) of the all two-

variables identities which are fulfill in the algebra \( H \) will be the smallest \( H \)-
closed set in \( F \), because \( I = (0)^H \), where \( 0 \in F \). If algebras \( H \) and \( H^*_W \) are
gometrically equivalent then the structures of the \( H \)-closed sets and of the
\( H^*_W \)-closed sets in \( F \) coincide. Hence \( I \) must be the smallest \( H^*_W \)-closed set in \( F \).

By [5] Remark 5.1

\[ T \to \sigma_FT \]  

(5.1)

is a bijection from the structure of the \( H^*_W \)-closed sets in \( F \) to the structure of the
\( H \)-closed sets in \( F \). Hear \( \sigma_F : F \to F^*_W \) is an isomorphism from condition Op2.
It is clear that the bijection (5.1) preserves inclusions of sets. So it transforms
the smallest \( H^*_W \)-closed set to the smallest \( H \)-closed set, i. e. \( I = \sigma_F I \) must
fulfills.

It is obviously that \( I \subset F^3 \). By (3.2) \( \sigma_F I \subset F^3 \). We will compare the
linear subspaces \( I/F^4 \) and \( (\sigma_F I)/F^4 \). \( I = \langle \alpha((x_1x_1)x_2) | \alpha \in \text{End}F \rangle \). Let
\[ \alpha(x_i) \equiv \alpha_{1i}x_1 + \alpha_{2i}x_2 \pmod{F^2}, \text{ where } i = 1, 2, \alpha_{ji} \in k. \]

Then

\[ \alpha((x_1x_1)x_2) \equiv ((\alpha_{11}x_1 + \alpha_{21}x_2)(\alpha_{11}x_1 + \alpha_{21}x_2))(\alpha_{12}x_1 + \alpha_{22}x_2) \pmod{F^4}. \]

We achieve after the extending of brackets that \( I/F^4 \) is a subspace of the linear space spanned by the elements of \( F^3/F^4 \) which have form \( (x_1x_1)x_k + F^4 \), where \( i, j, k = 1, 2 \). But

\[
\sigma_F I \ni \sigma_F ((x_1x_1)x_2) = a\sigma_F (x_1x_1) \sigma_F (x_2) + b\sigma_F (x_2) \sigma_F (x_1x_1) = a(a + b)(x_1x_1)x_2 + b(a + b)x_2(x_1x_1).
\]

We have that \( a + b \neq 0, b \neq 0 \), so \( I/F^4 \neq (\sigma_F I)/F^4 \) and \( I \neq \sigma_F I \). This contradiction proves that algebras \( H \) and \( H_W^* \) are not geometrically equivalent.

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