Relativity Experiments in the Solar System

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Summary: Recent theoretical works on alternative metric theories of gravity give greater significance to solar-system tests of General Relativity. In particular, it is suggested that the post-Newtonian parameter $\gamma$ ought to be determined with great precision in order to discover possible preferred frame effects or effects related to a tensor-scalar theory of gravitation. In this work we focus on a future experiment, based on Doppler tracking of an interplanetary spacecraft near superior conjunction.

We simulate, under certain assumptions, the frequency of the signal received from a spacecraft equipped with the ultimate radio capabilities. The geometrical optics formalism adopted makes it particularly easy calculating the Doppler effect caused by a spherically symmetric refractive medium. This case includes both the post-Newtonian gravitational effect and a spherical solar corona model. Stochastic fluctuations in the plasma electron density, along with other sources of noise, were also considered in simulating the data.

Applying a least squares fit to the simulated Doppler data, we are able to determine $\gamma$ with a precision of $5 \times 10^{-4}$, a factor four better than the present value. The availability of a two-frequencies link to the spacecraft and back would allow an improvement by at least an order of magnitude. Our results seem to indicate that measuring the time derivative of the Shapiro ranging curve is more promising than the ranging technique itself, although additional work comparing the various methods is needed.

1. Introduction

The Parametrized Post-Newtonian (PPN) formalism is the standard framework for performing experiments and formulating gravitational theories in the weak field and slow motion limit, appropriate to the solar system. In fact, in this limit, the spacetime metric predicted by nearly every metric theory of gravity has the same
structure. In other words, the metric can be expanded about the Minkowskian metric through dimensionless gravitational potentials of varying degrees of smallness. Metric theories differ one from another only in the numerical values of the coefficients that appear in front of the metric potentials. The PPN formalism prescribes the values of these parameters for each particular theory under study. Usually, we consider only ten of these parameters, chosen in such a way that their values indicate general properties of the metric theories of gravity. Thus, the comparison of these theories with each other and with the experiments is straightforward in this framework [1].

After almost eighty years since General Relativity (GR) was born, we can conclude that Einstein’s theory has survived every test. This rare longevity, along with the absence of any adjustable parameters in the theory (namely two of the ten PPN parameters are equal to one, all others disappear), far from being satisfactory, seems to be the cause of a lasting search for a possible fatal discrepancy. Remarkably, as an upshot of these efforts most alternative theories have been put aside. Only those theories of gravity flexible enough to accommodate the resulting constraints have survived, the protective belt being provided by the free parameters and the coupling constants of the theory.

The Brans-Dicke (BD) theory is certainly the most famous among the alternative theories of gravity. It contains, besides the metric tensor \( g \), a scalar field \( \phi \) and an arbitrary coupling constant \( \omega \), related to the PN parameter \( \gamma \) by

\[
\gamma = \frac{1 + \omega}{2 + \omega}.
\]  

The present limit on \( \gamma \) (see eq. (3) below) gives the very tight bound on the coupling constant \( \omega > 500 \). In addition, many Scalar-Tensor (ST) theories, generalizing the original BD theory, have been proposed in which the coupling function depends on the value of the scalar field, \( \omega = \omega(\phi) \). These theories are considered as the most interesting alternatives to GR, due to recent developments in cosmology (e.g. inflationary models) and elementary-particle physics (e.g. string theory and Kaluza-Klein theories). As recently pointed out by various authors (see [2,3]), a large class of ST theories contain an attractor mechanism toward GR in a cosmological sense. If this is what actually happened, then today we can expect, from eq. (1), a small deviation of \( \gamma \) from the GR value \( \gamma = 1 \). The present level of this deviation is estimated to be in the interval

\[
\delta \gamma \approx 10^{-7} - 10^{-5}
\]  

the exact value depending on the particular ST theory under investigation [3].
2. A test based on Doppler technique

The previous arguments provide strong motivations for experiments able to push beyond the present empirical accuracy on \( \gamma \), which is \( \gamma = 1.000 \pm 0.002 \), (3)

where the quoted uncertainty is twice the standard deviation.

A number of advanced experiments and space missions have been proposed, in order to improve this level of accuracy by at least two orders of magnitude (see, for example, [5]). In addition to these, the interplanetary mission CASSINI, which will be launched in October 1997, offers a new possibility for a test based on a free-flying spacecraft. The instrumentation will include a coherent Doppler capability with simultaneous tracking at X (\( \sim 8.4\)GHz) and Ka (\( \sim 33\)GHz) radio bands. Since CASSINI is the first deep space mission to implement Ka-band communications and tracking, a significant improvement over the best published results using S-band (\( \sim 2.3\)GHz) and X-band is expected.

In Doppler experiments, a radio link is transmitted from the Earth to the spacecraft, coherently transponded and sent back to the Earth, where its frequency is measured with great accuracy, thanks to the availability of hydrogen masers at the stations used for interplanetary telecommunications. Comparing the transmitted and received frequencies, one can measure the Doppler shift

\[
y = \frac{\Delta \nu}{\nu_0} = \frac{1}{c} \frac{dl}{dt}, \tag{4}
\]

where \( l \) is the overall optical distance (including diffraction effects) traversed by the photon in both directions. This technique has various applications, from the estimate of the gravity field of planetary systems (see [6,7] for recent examples), to the study of interplanetary plasma [8,9], and the attempts to detect low frequency gravitational waves [10-12]. Interplanetary Doppler tracking near solar conjunction, as a tool for testing relativistic effects, has been discussed previously in [13].

Besides the ordinary Doppler effect, of order \( v/c \), two relativistic corrections of order \( (v/c)^2 \) contribute to the Doppler signal: the gravitational shift, which occurs when the gravitational potential at the source and the receiver is different, and the second order effect, well known in special relativity. Both effects have been tested using a space-borne hydrogen maser carried aboard a sub-orbital rocket, allowing a test of Einstein’s prediction to about \( 10^{-4} \) [14]. A different, third order relativistic effect is observable in interplanetary Doppler measurements and, indeed, is implicitly incorporated in the orbit determination numerical codes [15]. In experiments
near superior conjunction, with an impact parameter $b$, the ray suffers a deflection

$$\delta = \frac{4M_\odot}{b} = 8 \cdot 10^{-6} \frac{R_\odot}{b},$$

(5)

where $M_\odot = 1.5 \text{ km}$ is the gravitational radius of the sun. This deflection changes the velocity component along the line of sight and will produce an additional Doppler effect of order

$$v \delta = \frac{4M_\odot}{b} v \approx 8 \cdot 10^{-10} \frac{R_\odot}{b}.$$  

(6)

Here and in the following we have taken $G = c = 1$. Because of the different technique, this can be considered as an additional test of GR, related to the time-delay effect in radar propagation first pointed out by Shapiro [16]. Since in a generic metric theory the expressions (5) and (6) are multiplied by $(1+\gamma)/2$, the experiment amounts to a measurement of $\gamma$. The current stability of hydrogen masers ($10^{-15}$ or better) would allow a significant improvement in the accuracy of $\gamma$, assuming that the effect of the solar corona can be overcome using dual-frequency capability. Smaller effects, such as the gravito-magnetic force acting on photon propagation, can also be studied with a ‘differential’ Doppler technique [13].

Before the introduction of multi-link tracking signal, including links at Ka band, the technique of measuring the relativistic Doppler shift caused by solar gravity was not competitive with the ranging technique. Our initial assessment is that the Doppler technique will be more powerful than ranging for advanced experiments like CASSINI. The purpose of our work is to quantify that expectation. The plan of the paper is as follows. In sec.3 we derive the gravitational Doppler effect, using standard geometrical optics formalism. The same formalism gives, in sec.4.1, the steady-state contribution from the solar plasma. Sec.4.2 describes how to simulate a real experiment. The simulated data are then analyzed in sec.5, in order to predict the accuracy in the determination of the PN parameter $\gamma$, when single and dual-frequencies links are available. Some of the limits of our analysis are then presented and discussed. Finally, in sec.6, we consider the Doppler technique in connection with two well-known alternatives, namely ranging and Differential VLBI (DVLBI).

3. Gravitational Doppler shift

It is well known that null geodesics of a static spacetime are curves of extremal coordinate time interval between the two end-points [17]. In particular, null geodesics for the post-Newtonian metric

$$ds^2 = -(1 - 2U) dt^2 + (1 + 2\gamma U) \delta_{ij} dx^i dx^j.$$
generated by a massive central body \((U = U(r) = M_\odot/r)\) can be derived from the variational principle

\[
\delta \int_\Gamma \mu \, d\lambda = 0 ,
\]

where

\[
\mu = 1 + (1 + \gamma) \frac{M_\odot}{r}
\]

is the refractive index, \(d\lambda = \sqrt{\delta_{ij}dx^idx^j}\) is the geometrical arc-length (in flat space), and the spatial path \(\Gamma\) is to be varied with fixed end-points. The photon’s trajectory is thus obtained by integrating the ray-equation

\[
\frac{d}{d\lambda} (\mu \vec{p}) = \vec{\nabla} \mu ,
\]  

(7)

where \(\vec{p}\) is the unit vector tangent to the path. Since we are dealing with the first PN effect, we can integrate eq. (7) along the zero-th order trajectory, i.e. the straight line from the source to the receiver.

\[\text{Fig. 1: The deflection geometry.}\]

We choose the geometry as in fig.1; \(z = 0\) is the plane through the sun, the earth and the spacecraft at any given time, and the unperturbed ray is directed along the \(x\)-axis, so that \(\vec{r} = (\lambda, b, 0)\). We get, in the limit \(\ell_0, \ell_1 \gg b\),

\[
\Delta \vec{p} = \int_{-\infty}^{+\infty} \vec{\nabla} \mu \, d\lambda = -\frac{2(1 + \gamma)M_\odot}{b} \hat{y} .
\]  

(8)

Projecting eq. (8) along the \(y\)-axis we recognize the usual expression for the angular deflection of light, so that we will adopt the following helpful notation

\[
\Delta \vec{p} \equiv \Delta \theta_{gr}(b) \hat{y} , \quad \Delta \theta_{gr}(b) = -\frac{2(1 + \gamma)M_\odot}{b} .
\]  

(9)

The one-way Doppler shift is given by the invariant expression

\[
\frac{\nu_1}{\nu_0} = \frac{\vec{v}_1 \cdot \vec{p}(x_1)}{\vec{v}_0 \cdot \vec{p}(x_0) } 
\]

(10)

where \(x_i\) and \(v_i\) are, respectively, the position and velocity of the receiver (emitter). To lowest order the photon momentum is \(\vec{p} = (1, 1, 0, 0)\), which gives rise, when
substituted in eq. (10), to the ordinary special relativistic effect. This effect changes over the Keplerian time scale $\sqrt{\ell^3/M_\odot} \approx \ell/v$. The gravitational contribution to the Doppler signal comes instead from the first order correction $\delta \mathbf{p}$ to $\mathbf{p}$, and it changes over the time scale $b/v$. These corrections are needed at the end-points. Since we are dealing with a static metric, the time component $\delta p^0 = 0$. The space components are obtained from eq. (8) after imposing the aiming condition, i.e. the passage through $x_0$ and $x_1$:

$$\int_{-\ell_0}^{\ell_1} \delta \vec{p} \, d\lambda = 0. \quad (11)$$

The result is

$$\delta \vec{p}(x_0) = -\Delta \theta_{gr}(b) \frac{\ell_1}{\ell_0 + \ell_1} \hat{y}. \quad (12)$$

$$\delta \vec{p}(x_1) = +\Delta \theta_{gr}(b) \frac{\ell_0}{\ell_0 + \ell_1} \hat{y}. \quad (13)$$

Finally, substituting eqs. (12), (13) in eq. (10) we get the gravitational Doppler shift

$$y_{gr}(t) = -\left(\frac{v_y^0 \ell_0 + v_y^1 \ell_1}{\ell_0 + \ell_1}\right) \Delta \theta_{gr}(b). \quad (14)$$

The first factor in eq. (14) depends on the spacecraft and Earth’s orbits (according to fig.1, with $v_y$ we indicate the velocity orthogonal to the line of sight), while the second factor depends on time through the impact parameter $b$, which, near conjunction, changes almost linearly in time. Of course, the round trip, or two-way, Doppler signal is simply obtained multiplying eq. (14) by two.

Figure 2 shows an example of this signal, as a function of time, when the spacecraft is on a circular orbit around the sun, with Mercury’s orbital radius. We have assumed $\gamma = 1$ in this case, and chosen the origin of time at closest approach ($t_c = 0$). Moreover, the orbital plane is inclined by $\sim 1^0$ with respect to the ecliptic, so that the ray grazes the sun limb when $b$ is minimum (in other words, $b(t_c) \simeq R_\odot$). Note also that at closest approach the special relativistic effect, not shown here, has its minimum, since at this epoch the velocities are orthogonal to the photon’s path, and we have only a second order effect.

Fig. 2: The general relativistic Doppler shift due to the solar monopole. The inclination $I$ between the ecliptic plane and the spacecraft orbital plane is large enough to prevent the solar occultation, i.e. $I \simeq 1^0 \Rightarrow b(t_c) \simeq R_\odot$. The experiment is assumed to occur near the point where the satellite has the largest distance from the ecliptic.
4. Plasma effects

The solar corona plasma is the main source of noise in a radio experiment near conjunction. In general, we can express the Doppler signal as

\[ y = y_{gr} + y_{pl} + y_n, \]  

where \( y_{pl} \) is the dominant plasma noise, and \( y_n \) contains all non-dispersive sources of noise (clock, receiver, etc.), along with a minor dispersive contribution from the atmosphere.

The plasma contribution \( y_{pl} \) to the fractional frequency change (4) is related to the change in the optical path

\[ \Delta l = \frac{N_e e^2}{2\pi m_e \nu_0^2}, \]

where \( e \) is the electron’s charge, \( m_e \) its mass, and \( N_e \) the total columnar content along the beam, \( N_e = \int n_e \, d\lambda \). Therefore, in order to calibrate for the plasma term, we should know the electron density along the path. We start by decomposing the electron density \( n_e \) in a static, spherically symmetric part \( \langle n_e \rangle (r) \) plus a fluctuation \( \delta n_e \), i.e.

\[ n_e(\vec{r}, t) = \langle n_e \rangle (r) + \delta n_e(\vec{r}, t). \]  

As a matter of fact, the steady-state behavior is reasonably well known, and we can use one of the several plasma models found in the literature [18-21]. To be more explicit, we will refer to two particular models, namely

\[ \langle n_e \rangle (r) = \begin{cases} \left[ \frac{2.99}{\eta^{16}} + \frac{1.55}{\eta^6} \right] \times 10^8 + \frac{3.44 \times 10^5}{\eta^2} \text{ cm}^{-3} & \text{model (a)} \\ \left[ \frac{2.39 \times 10^8}{\eta^6} + \frac{1.67 \times 10^6}{\eta^2} \right] \text{ cm}^{-3} & \text{model (b)} \end{cases} \]

where \( \eta \equiv r/R_\odot \). The reason why we consider both models is two-fold: first, repeating the same analysis with two different electron distributions, we can test the model-dependency of the results presented here. The second reason is more subtle; in the following section we will describe a simulation and a fit of the real data, so that, if we want to obtain meaningful estimates for \( \delta \gamma \), we cannot use the same model in both operations. More on this later.

\footnote{We do not consider here a correction factor which depends on the heliographic latitude.}
4.1. Steady state coronal plasma

We will now determine the Doppler effect due to the solar corona plasma for a model of the form (17a) or (17b). More precisely, we consider a generic plasma model of the form

\[ \langle n_e \rangle (r) = \sum_k \frac{\alpha_k}{\eta^{\beta_k}}. \]

According to the electromagnetic law of phase propagation through a plasma medium, the refraction index \( \mu \) can be expressed in terms of the plasma frequency \( \nu_p \) and the carrier frequency \( \nu_0 \) as

\[ \mu = \sqrt{1 - \left( \frac{\nu_p}{\nu_0} \right)^2} \approx 1 - \frac{\langle n_e \rangle e^2}{2\pi m_e \nu_0^2}. \]

(18)

We can now insert eq. (18) in the eikonal equation (7), and integrate along the unperturbed trajectory \( \vec{r} = (\lambda, b, 0) \), exactly as we did in the previous section for the relativistic \( \mu \) (see fig.1). In the limit \( \ell_0, \ell_1 \gg b \) we easily find

\[ \Delta \vec{r} = \Delta \theta_{pl}(b) \hat{y}, \]

where now

\[ \Delta \theta_{pl}(b) = \frac{e^2}{2\pi m_e \nu_0^2} \sum_k \alpha_k \beta_k \left( \frac{R_\odot}{b} \right)^{\beta_k} B \left( \frac{1 + \beta_k}{2}, \frac{1}{2} \right), \]

(19)

and \( B(x, y) \) is the Euler’s Beta function. Comparing \( \Delta \theta_{pl} \) with \( \Delta \theta_{gr} \) we notice the opposite sign - gravity bends the ray outwards, plasma inwards - and the different dependence on \( b \), plasma effect being steeper. The plasma deflection as a function of the solar offset \( b \) is shown in fig.3. In this case, we adopted for the steady-state model the first of eqs. (17), and considered S, X, and K-bands radio frequencies. The absolute value of the GR bending is also shown. The same analysis for the second plasma model (eq. (17b)) does not yield any appreciable difference, so we shall omit it here.

Fig. 3: The deflection angle as a function of the impact parameter \( b \). The solid line is the plasma contribution, while the dashed line gives the absolute value of the general relativistic deflection, given by eq. (9) with \( \gamma = 1 \).

Imposing the boundary condition (11), as we did for the gravitational effect, we can now obtain \( \delta \vec{r}(x_i) \). Inserting these in eq. (10) we eventually get the steady-state...
part of the solar plasma Doppler shift. The details of this calculation have already been carried out in the previous section. The result is consequently analogous to the gravitational Doppler shift, i.e.

\[ y_{pl}(t) = - \left( \frac{v_{1y} \ell_0 + v_{0y} \ell_1}{\ell_0 + \ell_1} \right) \Delta \theta_{pl}(b) \]  
(steady-state plasma)  \hspace{1cm} (20)

Note from eqs. (14) and (20) that the first factor is the same as in \( y_{gr} \). Both effects are time dependent through the quantities \( \ell_0, \ell_1, \vec{v}_0, \vec{v}_1, \) and \( b \).

4.2. Towards a real experiment

In the previous paragraph we have considered a spherical and static corona model, and derived the resulting Doppler shift, according to the law of geometrical optics. Unfortunately, the true electron density (16) contains also the fluctuations \( \delta n_e \), which require particular attention. In fact, these fluctuations are carried along with the solar wind speed \( V \approx 400 \text{ km/sec} \), so that their spatial scale is \( V \tau \approx R_\odot/2 \), where \( \tau \approx 1000 \text{ sec} \) is the temporal scale. On the scale \( b \) typical of the gravitational Doppler effect, one expects \( \delta n_e \) to be of the same order of magnitude as its average \[ [22] \), i.e.

\[ \delta n_e(b, t) \approx \langle n_e \rangle(b). \]

With the differential Doppler technique, introduced in \[13\], the plasma contribution can be greatly reduced by at least three orders of magnitude. However, this technique requires a hydrogen maser aboard the spacecraft, which is beyond the scope of the present study.

In conclusion, in order to be realistic we should consider a stochastic signal of the same order of magnitude as the the steady-state plasma effect \( y_{pl} \), as given by eq. (20). Therefore, we have generated a random gaussian noise of amplitude \( y_{pl}(t) \), at each instant of time. For simplicity, we did not take into account the fact that this noise is not white; although this could be done quite easily, assuming for example a Kolmogorov’s spectrum, we claim that this assumption does not affect very much our results.

Finally, one must consider the non-plasma noise \( y_n \) which contributes to the overall signal (15). To be conservative, we have assumed here

\[ y_n \approx 5 \times 10^{-14} \]  \hspace{1cm} (21)

which takes into account the fact that the observations occur during day time, although the noise level (21) could be pessimistic in view, for example, of CASSINI specifications \[23\].
Summing up all these contributions, we can give a reasonable description of the real data. These have been simulated in S, X, and K-bands, assuming the same geometrical configuration as in fig. 2, and adopting the solar corona model given by eq. (17a). The simulated data are shown in fig. 4. A superficial look to the K-band plot could give an idea of the expected improvement over previous experiments, based on S and X-band tracking. Even a single K-band link could therefore provide a significant estimate of the PN parameter $\gamma$. The following section provides a tentative result for this expectation.

Fig. 4: The simulated data for the three bands. The axes and the flyby’s geometry are the same as in fig. 2, where only the gravitational Doppler shift was shown.

5. Regression analysis

As we have seen, the main difference between the gravitational and the plasma Doppler effects is that the latter is much more steeper, as a function of $b$ or $t$, than the former. This fact implies that we can try to isolate the gravitational contribution with a regression analysis. To be more explicit, given the $N$ data points $y_i$ and their variance $\sigma_i^2$, we want to minimize the $\chi^2$ function

$$
\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y_{\text{det}}(t_i; \gamma, \alpha_1, \alpha_2, \ldots)}{\sigma_i} \right)^2,
$$

where $y_{\text{det}}$ is the deterministic signal, sum of the gravitational term (14), with $\gamma$ as a parameter, plus the steady-state term (20), which contains the parameters $\alpha_1, \alpha_2, \ldots$. Thus

$$
y_{\text{det}}(t; \gamma, \alpha_1, \alpha_2, \ldots) = -\left( \frac{v_1 y_0 + v_y y_1}{\ell_0 + \ell_1} \right) (\Delta \theta_{gr} + \Delta \theta_{pl}). \tag{22}
$$

We have implicitly assumed that the exponents $\beta_k$ are not varied by the fitting process. As we anticipated, we have created the data $y_i$ using the three-component model (17a), but it would not be fair to use this information here, in the regression analysis. In other words, in order to take into account our ignorance about the ‘true’ spherical model that Nature has chosen, we will attempt a fit with the ‘wrong’ one, in this case the two-component model (17b), characterized by $\beta_1 = 6, \beta_2 = 2.3$. Despite its formal importance, this distinction is however irrelevant, as long as the models used are all reasonable approximations one of the other.
We stress the fact that the orbital factor in the relativistic Doppler signal (14), is multiplied by a quantity of order $O(M_\odot/b)$, so that its Newtonian value suffices. The analogous quantity in the time delay expression (see eq. (26) below) is added to the relativistic correction, so that they are both required at the same order. This constitutes a major advantage of the Doppler method compared to the ranging technique. Nonetheless, it is obvious that the orbital quantities appearing in $y_{det}$ are affected by statistical errors, due for example to non-gravitational forces. We will discuss these orbital stochastic effects, along with small variations resulting from dependence on the PN parameter $\beta$, in a subsequent paper.

Before we proceed, we should address the problem of the optimal time sequence to be used. Since the plasma, both the steady-state and the stochastic terms, become more and more important approaching the closest approach ($t = 0$ in fig.4), we can decide to cut-off that portion of the data, which introduces more noise than signal. Figure 5 represents the signal-to-noise ratio as a function of the cut-off impact parameter, indicated with $b_{cut}$. Of course, dealing with the $\chi^2$ function avoids this complication. In fact, the post-fit residuals give $\delta\gamma/\gamma$ as a function of the minimum impact parameter $b_{cut}$ considered (fig.6). Comparing figures 5 and 6 we can conclude that the accuracy with which we can determine $\gamma$ increases as we track closer to the sun, up to a certain point (in the K-band case around $b_{cut} \simeq 5R_\odot$), after which plasma effects begin to dominate and deteriorate the accuracy, so that we should not push the observations beyond this threshold.

Fig. 5: The Signal-To-Noise Ratio as a function of the cut-off impact parameter $b_{cut}$. For each value of $b_{cut}$ only the first portion of the data, from the beginning of the observations to the point where $b = b_{cut}$, are considered.

Fig. 6: The fractional error $\delta\gamma/\gamma$ as a function of the cut-off impact parameter $b_{cut}$, where $b_{cut}$ has the same role as in Fig. 5.

In conclusion, from our regression analysis we can infer that the availability of K-band tracking could allow us to determine the PN parameter $\gamma$ with a two-sigmas accuracy

$$\frac{\delta\gamma}{\gamma} \simeq 5 \times 10^{-4},$$  

(23)

a factor 4 better than the present value $2 \times 10^{-3}$, but still far from the optimal target quoted in sec.1 (see eq. (2)). However, this sensitivity would be enough to detect possible effects related to the existence of a preferred frame in the Universe. The solar system would have, with respect to this reference frame, a velocity $\sim 10^{-3}$,
which therefore represents a natural threshold for possible effects at variance with GR.

It seems very difficult to go much beyond the result (23) with a single link. Even in K-band, there are limitations imposed by plasma effects. However, since the plasma noise is dispersive, a drastic improvement can be achieved when simultaneous multifrequency links are available. For instance, if we track the spacecraft in X and K bands, then from the two Doppler observables $y^K(t)$ and $y^X(t)$ we can construct the quantity

$$\frac{y^K - \left(\frac{\nu_X}{\nu_K}\right)^2 y^X}{1 - \left(\frac{\nu_X}{\nu_K}\right)^2} \simeq y_{gr} + \frac{y^K_n - \left(\frac{\nu_X}{\nu_K}\right)^2 y^X_n}{1 - \left(\frac{\nu_X}{\nu_K}\right)^2}$$

(24)

which, in principle, does not contain the frequency-dependent plasma noise. A preliminary regression analysis shows that this observable offers the opportunity for a measurement of $\gamma$ with a precision at least an order of magnitude better than eq. (23).

Given our simple simulation of the real data, these estimates represent our best guess about the outcome of a gravitational test based on Doppler tracking of a spacecraft equipped with the ultimate radio capabilities. However, before we conclude, we would like to discuss the weak points in our analysis. To start with, we have simulated our data without imposing any strict obedience to an actual proposed experiment; although the opportunity offered by the CASSINI mission stimulated most of our work, we did not pay much attention to the CASSINI expected orbit, error budget, etc. [23]. When necessary, however, we always tried to assume realistic, if not pessimistic, numbers. A more crucial caveat is that we neglected some possible complications, like the presence of the solar magnetic field, the evolution of plasma properties over the duration of the experiment, non gravitational accelerations, and so on. In addition, our claim that we can get rid of plasma effects using multiple links is only partially true. For instance, one must consider that two radio signals at different frequencies traverse different paths, since the bending (19) is frequency-dependent. As a consequence, the plasma noise can not be completely removed from eq. (24). However, we can predict how relevant is this limitation. At closest approach, when the separation between the two frequencies is bigger, we have

$$\Delta b_0(b) \equiv b_0(X, b) - b_0(K, b) \approx \frac{0.1 R_\odot}{(b/R_\odot)^6},$$

(25)

where $b_0$ is the minimum distance of the ray from the sun (not to be confused with the impact parameter $b$). Comparing this separation with the typical length scale of
the plasma clouds, \( V\tau \approx R_\odot /2 \), we can conclude that this is not much of a problem. The result of this analysis is summarized in fig.7, where the actual difference in the closest approach distance of the K-band and X-band radio signals is shown as a function of the impact parameter.

\[ \Delta b_0 \]

Fig. 7: The difference \( \Delta b_0 \) between the closest approach distances in X and K bands, as a function of the impact parameter \( b \). The dotted line is the estimate (25).

6. Future developments

In the present work, we have considered a relativistic test based on Doppler technique. Of course, Doppler is not the only technique available when a free spacecraft is orbiting beyond the sun. Two other well established experiments may also be performed at conjunction, namely ranging experiments, aimed to observe the Shapiro time delay, and \( \Delta \text{VLBI} \) measurements of the deflection of light by the solar mass.

The Shapiro time delay [16] consists in the retardation of a light signal passing near a massive body (the sun in our case). The round trip light time (RTLT) from the Earth to the spacecraft and back is easily obtained from eq. (7) as

\[
\Delta t = \int \mu d\lambda \simeq (\Delta t)_{\text{Newt}} + 2(1 + \gamma)M_\odot \ln \left( \frac{4\ell_0\ell_1}{b^2} \right).
\]

Unfortunately, we do not have direct access to the Newtonian signal \( (\Delta t)_{\text{Newt}} \) (this is somewhat related to the fact that we are using a coordinate-dependent quantity), so that what people do is a differential measurement of the variations in RTLT as the spacecraft passes through superior conjunction. This requires a good knowledge of the orbital motion of the target relative to the Earth. For this reason, the best result, quoted in eq. (3), has been obtained with the Viking landers, anchored to the surface of Mars [4], while the more recent free-flying Voyager 2 test gave \( \delta \gamma \simeq 0.03 \) [24].

Improvements in the accuracy of Very-Long-Baseline-Interferometry (VLBI) made it possible to measure differential angular positions to the level of hundreds of microarcsec, making highly accurate measurements of the deflection of light possible [25]. This experiment is conceptually simple, and does not involve secondary parameters which limit the confidence of ranging experiments. A series of observations yielded a value \( \frac{1}{2}(1 + \gamma) = 1.000 \pm 0.001 \) [26], comparable to the Viking test of the Shapiro time delay.
In conclusion, every technique is based on different concepts, and presents its own problems and peculiarities. Therefore, deciding which one should be pursued during a given mission is a difficult task. The project’s management has to make a decision based on the particular instrumentation available, on the designated orbit, etc. It might also be advantageous to combine the three techniques in an optimal way, keeping in mind the technical difficulties involved in a multiple-task experiment. Here, without attempting such approach, we have just considered one particular technique, and tried to define its possibilities. The comparison with the others will be the object of future studies.

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