The entropy of near-extreme N=2 black holes

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Abstract

We give an explicit form of the classical entropy for four-dimensional static near-BPS-saturated black holes of generic N=2 superstring vacua. The expression is obtained by determining the leading corrections in the non-extremality parameter to the corresponding BPS-saturated black hole solutions. These classical results are quantitatively compared with the microscopic leading order corrections to the microscopic result of Maldacena Strominger and Witten for N=2 BPS-saturated black holes.

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1 Introduction

Over the past two years a dramatic progress has been made in understanding the microscopic origin of the black holes (for a review see for example [1] and references therein), in particular for the five-dimensional BPS-saturated solutions [2] of toroidally compactified string theory, i.e. vacua with \( N = 4 \) and \( N = 8 \) supersymmetry. Soon after the microscopics of near-BPS-saturated black hole solutions both static [3] and rotating ones [4] was addressed using the \( D \)-brane dynamics, however the arguments became more heuristic and less rigorous, in particular for four-dimensional BPS- and near-BPS-saturated solutions.

BPS-saturated solutions of \( N = 4, 8 \) vacua possess large symmetry, and corrections to the low-energy effective action can appear only through higher derivative terms like \( R^2 \) or \( F^4 \), however, there are no loop-corrections expected.

On the other hand BPS-saturated solution of \( N = 2 \) string vacua can receive corrections due to the quantum (loop) corrections already to the lowest derivative terms in the effective action. This rich structure makes it however much more difficult to find an explicit form or such corrections to the BPS-saturated black hole solutions. Nevertheless, solutions for general (stationary) BPS-saturated solutions which couple to arbitrary vector fields and trivial hyper-multiplets have been found [13]. These solutions are given in terms of an algebraic constraint for the symplectic section, which can be solved explicitly for many interesting cases. The existence-proof of such solutions heavily relies on the remaining supersymmetry of the solutions, i.e. they satisfy the Killing spinor equations. In a recent work of Maldacena, Strominger and Witten [5], and Vafa [6], the rigorous derivation of the microscopic entropy for a class of BPS-saturated black holes [7], [8] was given. It accounts for both the tree-level and loop corrections to the entropy of such black holes; they are microscopically described as a five-brane wrapping around \( C_4 \times S^1 \) of M-theory compactified on \( CY_3 \times S^1 \). Here \( CY_3 \) is the Calabi-Yau three-fold and \( C_4 \) is the four-cycle of \( CY_3 \). The additional electric charges correspond to the momentum along \( S^1 \) and the background value of the self-dual three-form of the five-brane. This work supersedes more heuristic approaches [3] which addressed the microscopic origin of these black holes within the (intersecting) \( D \)-brane approach.

In view of this recent progress a natural open question is to address the non-extreme black hole solutions of \( N = 2 \) vacua with the hope that once such classical solutions are understood, the next step should be to address the microscopics of such solutions, hopefully with a comparable rigor as the recent work on the BPS-saturated solutions. [1] While the explicit solutions for general non-extreme four-(and five-)dimensional static [11], [12] (13) and rotating [14] (15) black holes of \( N = 4, 8 \) string vacua have been constructed, the state of affairs for non-extreme black holes of \( N = 2 \) vacua is still a poorly explored territory.

\(^2\)In addition, the radiation rates in such black hole backgrounds could provide another dynamical test to probe the internal structure of such black holes, as it has already been done for black holes of \( N = 4, 8 \) vacua (for a review see [10] and references therein).
The work by Kastor and Win [16] constitutes the first attempt to address such non-extreme static solutions. However, their approach already reveals difficulties in finding an explicit solution of the equations of motion for a generic $N = 2$ vacuum. They took an Ansatz which coincides with that of the non-extreme black holes of toroidally compactified string vacua [11], [17] and it turned out to be a solution only for a very restricted class of $N = 2$ string vacua, i.e. Type II superstring [M-theory] compactified on a restricted Calabi-Yau three-fold [restricted Calabi-Yau three-fold $\times S^1$], which can effectively be reduced to that of a torus.

The aim of this paper is to further shed light on non-extreme static black holes for $N = 2$ string vacua. In particular, we concentrate on near-BPS-saturated static black hole solutions for generic four-dimensional $N = 2$ string vacua. The entropy of the near-BPS-saturated solution is terms of the entropy of the BPS-saturated solution and corrections in terms of the non-extremality parameter which parameterizes a deviation from the BPS-saturated limit. This classical result provides a starting point to address the microscopic origin of these corrections. Note, that the ultimate goal of the program is to gain the (macroscopic and microscopic) understanding of generic non-extreme solutions both static and rotating ones, both in four-and five dimensions. For a recent discussion for extreme solution or $N = 2$, $D = 5$ supergravity see e.g. [18]. Along these (ambitious) directions we therefore touch only the tip of the iceberg.

The paper is organized in the following way. In Section 2 we spell out the action and the constraints of a generic $N = 2$ supergravity theories in four-dimensions ($D = 4$) and discuss their BPS-saturated black hole solutions. In Section 3 we derive equations of motion for a generic $N = 2$ supergravity and employ them to obtain the leading corrections (in the non-extremality parameter) to the entropy for the near-BPS-saturated black holes for generic $N = 2$ vacua, i.e. for TypeII string theory compactified on a generic Calabi-Yau three-fold. In Section 4 we study microscopic origin of such corrections.

2 $N = 2$ Supergravity and BPS-Saturated Black Holes

We start with the action and the equations of motion of $N = 2$ supergravity in $D = 4$. Considering only gravity and vector multiplets the low energy action is given by

$$S \sim \int \sqrt{-g} \, d^4 x \left[ R - 2 g_{A\bar{B}} \partial^\mu z^A \partial_\mu \bar{z}^B - \frac{1}{4} F^I_{\mu\nu} (\ast G_I)^{\mu\nu} \right]$$

with the gauge field $G_{I \mu\nu}$ given by

$$G_{I \mu\nu} = \text{Re} N_{IJ} F^J_{\mu\nu} - \text{Im} N_{IJ}^* F^J_{\mu\nu}.$$  

and $I, J = 0, 1,..., n_v$, counts the number of vector multiplets. The complex scalar fields $z^A (A = 1..n_v)$ parameterize a special Kähler manifold with the metric $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$, where $K(z, \bar{z})$ is the Kähler potential. Both, the gauge field coupling as well as the Kähler
potential are expressed in terms of the holomorphic prepotential $F(X)$

$$e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I)$$

$$N_{IJ} = \bar{F}_{IJ} + 2i\frac{(\Im F_{IL} X^L) (\Im F_{JM} X^M)}{\Im F_{MN} X^M X^N},$$

with $F_I = \frac{\partial F(X)}{\partial X^I}$ and $F_{MN} = \frac{\partial^2 F(X)}{\partial X^M \partial X^N}$. The scalar fields $z^A$ are defined by

$$z^A = \frac{X^A}{X^0}.$$

The form of the static BPS-saturated solutions is of the form \[19\], \[20\]:

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dx^m dx^m, \quad e^{-2U} = e^{-K},$$

$$F^I_{mn} = \epsilon_{mnp} \partial_p H^I, \quad G_{mn} = \epsilon_{mnp} \partial_p H_I,$$

where $(H^I, H_I)$ are harmonic functions and the $X^I$ are determined by

$$i(X^I - \bar{X}^I) = H^I, \quad i(F_I - \bar{F}_I) = H_I.$$

We have assumed that $e^{-2U} = e^{-K} \to 1$ for $r \to \infty$, i.e. we have fixed the Kähler gauge. For the sake of concreteness we will consider the model defined by the prepotential

$$F(X) = \frac{d_{ABC} X^A X^B X^C}{X^0}.$$

If all harmonic functions are non-trivial, one cannot solve (6) explicitly, but as long as $H^0 = 0$ the complete solution to prepotential (7) is given by \[3\]

$$X^0 = \sqrt{\frac{d_{ABC} H^A H^B H^C}{H_0 + \frac{1}{12} D^{AB} H_A H_B}}, \quad X^A = \frac{1}{6} X^0 D^{AB} H_B - \frac{i}{2} H^A,$$

where $D^{AB} = (d_{ABC} H^C)^{-1}$. Inserting this into (4) one finds for the metric

$$e^{-2U} = \sqrt{\tilde{H}_0 d_{ABC} H^A H^B H^C}, \quad \tilde{H}_0 = H_0 + \frac{1}{12} D^{AB} H_A H_B.$$

The harmonic functions $H_{0,A}$, $H^A$ are parameterized as:

$$H_0 = 1 + \frac{q_0}{r}, \quad H_A = 1 + \frac{q_A}{r}, \quad H^A = 1 + \frac{p^A}{r}.$$

Microscopically, this solution corresponds to the case of an intersection of three $M5$-branes. ($p^A$ is the magnetic charge of a 5-brane wrapped around the $A$th 4-cycle of the Calabi-Yau three-fold $(CY_3)$.) Momentum modes traveling along the common intersection
produce the electric charge \( q_0 \). (In M-theory compactified on \( CY_3 \times S^1 \), it is the momentum along \( S^1 \).) The electric charge contributions \( (q_A) \) provide a shift in \( \tilde{H}_0 \) (see eq. \( (9) \)), which are due to the membrane excitations with their flux contribution along 2-cycle \( \times S^1 \) of the 5-branes \( \tilde{\mathfrak{E}} \), thus producing additional winding states on \( S^1 \).

The entropy of the above BPS-saturated solution is given by:

\[
S = 2\pi \sqrt{\tilde{q}_0 d_{ABC} p^A p^B p^C},
\]

where \( \tilde{q}_0 \) (determined in terms of \( q_0 \) and \( q_A \), \( p^A \)'s) is an "effective" electric charge obtained from \( H_0 \) in \( \tilde{\mathfrak{E}} \). The above entropy has been recently determined microscopically in terms of a 2-dimensional \((0,4)\) sigma model whose target space includes the 5-brane moduli space \( \tilde{\mathfrak{E}} \). (In \( \tilde{\mathfrak{E}} \) the microscopic origin of the loop corrections to the above tree level entropy \( \tilde{\mathfrak{E}} \) have also been discussed.)

### 3 Near-BPS-Saturated Black Holes

Our aim is to find the entropy of the near-BPS-saturated solutions. Its form should be determined in terms of the BPS-saturated entropy and corrections, parameterized by the non-extremality parameter \( \mu \), which quantified a deviation from the BPS-saturated limit.

While the BPS-saturated solutions are obtained by solving the Killing spinor equations (coupled first order differential equations), the non-extreme solutions solve general Euler-Lagrange equations of motion (coupled second-order differential equations), a much more formidable task. However, since the (bosonic) action and the equations of motions, including their symmetries, remain the same, we still assume that the symplectic structure remains intact.

#### 3.1 Equations of Motion

The gauge field equations and the Bianchi identities follow from \( \tilde{\mathfrak{E}} \) and can be written as

\[
dF^I = dG_I = 0.
\]

In order to solve these equations, it is sufficient to consider only the spatial components \( \{m,n\} \); \( F_{0m}^I \) can be obtained from \( G_{Imn} \) by using \( \tilde{\mathfrak{E}} \). We take the following Ansätze:

\[
G_{Imn} = \tilde{\chi}_I^J \epsilon_{mnp} \partial_p H_J, \quad F_{mn}^I = \chi_I^J \epsilon_{mnp} \partial_p H^J,
\]

where the matrices \( \chi \) and \( \tilde{\chi} \) are in general functions of the radius \( r \) and will determined later. The Ansätze \( \tilde{\mathfrak{E}} \) can be viewed as a (local) superposition of the extreme solution, however, note, that \( \chi \) and \( \tilde{\chi} \) are in general functions of \( r \).

For the discussion of the Einstein equations we use the relation \( \tilde{\mathfrak{E}} \) and write the gauge field part in terms of the magnetic fields. With this replacement the gauge field part in
the action (1) becomes

\[ \int \sqrt{-g} F_{\mu \nu}^I (\ast G_I)^{\mu \nu} = \int \sqrt{-g} \left( \text{Im} N_{IJ} F_{\mu \nu}^I F_J^{\mu \nu} + \text{Re} N_{IJ} F_{\mu \nu}^I (\ast F_J)^{\mu \nu} \right). \] (14)

The second part in this expression is topological, i.e. it is independent of the metric and does not contribute to the Einstein equations. The first part yields the energy momentum tensor and we use again (2) to replace the electric components by the magnetic ones, i.e.

\[ F_{\mu 0}^I = -\frac{1}{2} \text{Im} N_{IJ} \epsilon_{0mpn} \left( G_{J np} - \text{Re} N_{JK} F^K_{np} \right) \equiv -\frac{1}{2} \text{Im} N_{IJ} \epsilon_{0mpn} G_{J np}, \] (15)

with \( \text{Im} N_{IJ} = (\text{Im} N_{IJ})^{-1} \).

Thus the Einstein equations take the form

\[ R_{00} = -\frac{1}{8} (F^2 + G^2) g_{00}, \]
\[ R_{mn} = 2 g_{AB} (\partial_m z^A \partial_n z^B) + \frac{1}{2} \left( F^2_{mn} + G^2_{mn} - \frac{1}{4} (F^2 + G^2) g_{mn} \right), \] (16)

where \( F^2_{mn} = \text{Im} N_{IJ} (F^I)^{ml} (F^J)^{ln} \) and \( G^2_{mn} = \text{Im} N_{IJ} (G_I)^{ml} (G_J)^{ln} \). Insertion of the gauge fields (13) yields

\[ R_{00} = -\frac{1}{2} e^{4U} \left( \text{Im} N_{IJ} \chi^K_I \chi^K_J \partial_m H^K \partial_m H^L + \text{Im} N_{IJ} \tilde{\chi}^K_I \tilde{\chi}^K_J \partial_m \mathcal{H}_K \partial_m \mathcal{H}_L \right), \]
\[ R_\theta^\theta = R_\phi^\phi = -R_0^0, \]
\[ R_r^r = 2 g_{AB} (\partial_r z^A \partial_r z^B) + R_0^0, \] (17)

where

\[ \partial_m \mathcal{H}_I = \partial_m H_I - \text{Re} N_{IK} \partial_m H^K. \] (18)

(See (13) for the definition of \( G_{mn} \).)

In order to solve eqs. (17), we take the following metric Ansatz:

\[ ds^2 = -e^{2U} f \, dt^2 + e^{-2U} \left( \frac{dr^2}{f} + r^2 d\Omega \right). \] (19)

Consequently, we find the following identity for the \( R_0^0 + R_\theta^\theta \) combination:

\[ R_0^0 + R_\theta^\theta = \frac{1}{2} e^{2U} \left( f' + \frac{4}{r} f'' - \frac{2(1 - f)}{r^2} \right). \] (20)

It follows from (17) that this expression has to vanish and we obtain

\[ f = 1 - \frac{\mu}{r}. \] (21)

Here \( \mu \) is the non-extremality parameter which parameterizes a deviation from the BPS-saturated limit.
Inserting the explicit form (21) for \( f \) into the metric Ansatz (19) we find for the Ricci-tensor components

\[
R^0_0 = -R^\theta_\theta = -R^\phi_\phi = e^{2U} \left[ f \partial^2 U + \partial_m f \partial_m U \right], \tag{22}
\]

\[
R^r_r = e^{2U} \left[ -f (\partial^2 U - 2 \partial_m U \partial_m U) + \partial_m f \partial_m U \right].
\]

where \( \partial^2 U = U'' + \frac{2}{r} U' \) is the flat space Laplacian. The symmetries between the Ricci-tensor components are in agreement with eq. (17).

Finally, the scalar field equation is given by

\[
\frac{4}{\sqrt{g}} \partial^\mu \left( \sqrt{g} g^{\mu\nu} g_{AB} \partial^\nu \bar{z}^B \right) - 2 (\partial_A g_{BC}) \partial z^B \partial z^C \left[ (\partial_A \text{Im} \mathcal{N}_{IJ}) F_{mn}^I F_{Jmn}^J + (\partial_A \text{Im} \mathcal{N}^{IJ}) G_{I mn} G_{Jmn}^J \right] = 0,
\]

where \( \partial_A = \frac{\partial}{\partial z^A} \).

Since we assume that the symplectic structure remains intact, \( e^{-2U} \) in the metric is still given by the structure of the Kähler potential. However, now we allow in \( e^{-2U} \) the replacement of harmonic functions \((H^A, \bar{H}_I)\) by \textit{general functions} \((\hat{H}^A, \hat{H}_I)\). Thus, for the model with the prepotential (7) and the assumption that \( \hat{H}^0 = 0 \) the Ansatz is of the form:

\[
e^{-2U} = \sqrt{\hat{H}_0 d_{ABC} \hat{H}^A \hat{H}^B \hat{H}^C}, \quad z^A = \frac{1}{6} D^{AB} \hat{H}_B - i \hat{H}_0 \hat{H}^A e^{2U}.
\]

where \( \hat{H}_0 = \hat{H}_0 + \frac{1}{12} D^{AB} \hat{H}_B \hat{H}_A \). In terms of these functions the Einstein equations (17) reduce to one single equation

\[
-2 \partial_m f \partial_m e^{-2U} = \text{Im} \mathcal{N}^{IJ} \left( f \partial_m \hat{H}_I \partial_m \hat{H}_J - \bar{\chi}_I^K \bar{\chi}_J^L \partial_m \mathcal{H}_K \partial_m \mathcal{H}_L \right) - 8 \text{Re} \chi^I \partial^2 \hat{H}_I
\]

\[
+ \text{Im} \mathcal{N}_{IJ} \left( f \partial_m \bar{\hat{H}}^I \partial_m \bar{\hat{H}}^J - \chi^L \chi^K \partial_m \hat{H}^L \partial_m \hat{H}^K \right) + 8 \text{Re} F_{IJ} \partial^2 \hat{H}^I,
\]

where we have used the known expression for \( \partial^2 U \) (from the extreme solution). We can also simplify the scalar field equation

\[
-8 e^{-2U} \partial_m f g_{AB} \partial_m z^B =
\]

\[
(\partial_A \text{Im} \mathcal{N}^{IJ}) \left( f \partial_m \hat{H}_I \partial_m \hat{H}_J - \bar{\chi}_I^K \bar{\chi}_J^L \partial_m \mathcal{H}_K \partial_m \mathcal{H}_L \right) + O(\partial^2 \hat{H}_I) \tag{26}
\]

\[
+ (\partial_A \text{Im} \mathcal{N}_{IJ}) \left( f \partial_m \bar{\hat{H}}^I \partial_m \bar{\hat{H}}^J - \chi^L \chi^K \partial_m \hat{H}^L \partial_m \hat{H}^K \right) + O(\partial^2 \hat{H}^I).
\]

We omitted here the explicit expressions \( \sim \partial^2 \hat{H} \), because they do not have a simple form and we do not need them later. This structure of the Einstein and scalar equation suggests that one can still try to find a solution for \((\hat{H}^I, \hat{H}_I)\) in terms of harmonic function, at least in the neighborhood of the horizon.
### 3.2 The near-extreme solution

In order to obtain the complete solution one has to solve explicitly both the Einstein equations as well as the scalar field equation. However, in order to obtain the entropy for near-BPS-saturated black holes, it is sufficient to consider the behavior of the solution in the near-region of the outer-horizon. We will assume that in this region the general functions \( \bar{H}^I, \bar{H}_I \) of near-BPS-saturated solution can again be approximated by harmonic functions. Hence, we take

\[
\bar{H}^I = \frac{\tilde{p}^I}{r} + \bar{h}^A + \mathcal{O}(\mu),
\]
and a similar expression for \( \bar{H}_I \). In order to solve the Einstein equations (25) we consider the expansion

\[
\partial_m \bar{H}^I = \Omega^I_J \partial_m H^J, \quad \partial_m \bar{H}_I = \bar{\Omega}_I^J \partial_m H_J,
\]

where \( \Omega \) and \( \bar{\Omega} \) will be fixed later. Using the definition of Kähler potential (3) we find for \( \partial_m e^{-2U} \)

\[
-2 \partial_m e^{-2U} = U^I \partial_m \bar{H}_I + U_I \partial_m \bar{H}^I = -4 \text{Re}X^I \partial_m \bar{H}_I + 4 \left( \text{Re}F_I - \text{Re}X^J \text{Re}N_{JI} \right) \partial_m \bar{H}^I.
\]

It is straightforward to read off the coefficients \( (U^I, U_I) \) for the Ansatz given in (24).

Before we discuss the general solution, let us first recall the non-extreme solution for the toroidal compactification, which corresponds to the case where from the intersection numbers \( d_{ABC} \) \( (A, B, C = 1, 2, 3) \) only \( d_{123} = \frac{1}{6} \) is non-vanishing. If one furthermore assumes that all axions vanish, i.e. if \( H_A = 0 \), the matrices \( \Omega \) and \( \chi \) are known [16]

\[
\bar{H}^A = (\Omega_0)^A_B H^B = \tanh \beta(A) \delta^A_B H^B, \quad \bar{H}_0 = \tanh \beta(0) H_0, \quad \chi = 1,
\]

the charges are given by

\[
\tilde{p}^A = \mu \sinh^2 \beta(A), \quad p^A = \mu \sinh \beta(A) \cosh \beta(A),
\]

\[
\tilde{q}_0 = \mu \sinh^2 \beta(0), \quad q_0 = \mu \sinh \beta(0) \cosh \beta(0),
\]

and the electric gauge field are specified by \( F^0_0 = \text{Im}N^{00}(*G_0)_{00} = q_0/(\bar{H}_0^2 r^2) \) (see (2)).

In order to obtain the above “toroidal” solution, it was crucial that the matrix \( \text{Im}N_{AB} \) was diagonal. In general this is not the case, however, in the near-horizon region of a general near-BPS-saturated solution one can still diagonalize \( \text{Im}N_{IJ} \) by using the constant matrices \( \Omega \) and \( \chi \) and \( \text{Im}N^{IJ} \) by \( \bar{\Omega} \) and \( \bar{\chi} \). We take

\[
\Omega = \chi \Omega_0, \quad \bar{\Omega} = \bar{\chi} \bar{\Omega}_0
\]

with \( \chi \) and \( \bar{\chi} \) defined by

\[
(\chi^T \text{Im}N^{-1} \chi)^{IJ} = \text{diag}(\lambda^{(0)}, \lambda^{(1)}, \ldots), \quad \lambda^{(I)} = U^I/\bar{H}_I,
\]

\[
(\bar{\chi}^T \text{Im}N\bar{\chi})_{IJ} = \text{diag}(\rho^{(0)}, \rho^{(1)}, \ldots), \quad \rho^{(I)} = U_I/\bar{H}^I.
\]
where \((U_I, U^I)\) were introduced in (29). Note, that \(\lambda(I)\) and \(\rho(I)\) are not eigenvalues of \(\text{Im}N^{-1}\) and \(\text{Im}N\), but they are proportional to them. Thus, the harmonic functions are given by

\[
\bar{H}_I = \tilde{\chi}_I^J \left( 1 + \frac{\mu \sinh^2 \beta(J)}{r} \right), \quad H^I = \chi^I J \left( 1 + \frac{\mu \sinh \beta(J) \cosh \beta(J)}{r} \right).
\]

Finally, we should express the parameter in the metric \((\bar{q}_I, \bar{p}^I)\) by physical charges \(\tilde{q}_I = \tilde{\chi}_I^J q_J\) and \(\tilde{p}^I = \chi^I J p^J\), obtained from (33) and find

\[
\tilde{p}^I = \Omega^I J p^J = (\chi \Omega_0 \chi^{-1})^I J \tilde{p}^J, \quad \tilde{q}_I = \tilde{\Omega}_I^J q_J = (\tilde{\chi} \tilde{\Omega}_0 \tilde{\chi}^{-1})^I J \tilde{q}_J.
\]

where \(\Omega_0\) and \(\tilde{\Omega}_0\) are given by

\[
\Omega_0 = \text{diag}(\tanh \beta^{(1)}, \tanh \beta^{(2)}, ...), \quad \tilde{\Omega}_0 = \text{diag}(\tanh \beta^{(1)}, \tanh \beta^{(2)}, ...)
\]

On the horizon \(r = \mu\), the harmonic functions \((\bar{H}_I, \bar{H}^I)\) become

\[
\mu \bar{H}_I \bigg|_{\text{horizon}} = \tilde{\chi}_I^J \mu \cosh^2 \beta(J), \quad \mu \bar{H}^I \bigg|_{\text{horizon}} = \chi^I J \mu \cosh^2 \beta(J).
\]

Then the Hawking entropy for the model specified by (24) is of the form

\[
S = 2\pi \sqrt{\mu \bar{H}_0 \, d_{ABC} \, \mu \bar{H}^A \, \mu \bar{H}^B \, \mu \bar{H}^C} \bigg|_{\text{horizon}},
\]

and the Hawking temperature becomes

\[
T = \frac{\mu}{4\pi \sqrt{\mu \bar{H}_0 \, d_{ABC} \, \mu \bar{H}^A \, \mu \bar{H}^B \, \mu \bar{H}^C} \bigg|_{\text{horizon}}}. \tag{39}
\]

Replacing the charges \((\bar{q}_0, \bar{p}^A)\) by the physical charges \((\tilde{q}_0, \tilde{p}^A)\) from eq. (35) we obtain corrections to the extremal (BPS-saturated) entropy.

### 4 Microscopic Interpretation of the Near-Extreme Black Hole Entropy

A microscopic interpretation of the classical entropy in the BPS-saturated limit (11) was given in [5] and [6]. Let us shortly summarize the main points. The model at hand appears as an intersection of three M5-branes, which intersect over a common string and momentum modes travelling along this common string. The momentum modes are parameterized by the electric charge \(q_0\) and the 5-branes are related to the magnetic charges \(p^A\). In addition, the model described by (8) contains further electric charges \(q_A\), which correspond to the membranes, specified by the 5-brane world-volume excitations of the self-dual antisymmetric tensor. For the common string these modes appear as
winding states shifting the vacuum energy and momentum. So, the microscopic model can be described by a (1+1)-dimensional sigma model for the common string which has in the BPS-saturated limit a (4,0) supersymmetry. As a consequence of the world-sheet supersymmetry, all the momentum modes are chiral, e.g., only the left-moving excitations are present. In addition, the 4-cycles of the Calabi-Yau three-folds, around which the 5-branes are wrapped, are holomorphic. Both of these properties are lost in the near-BPS-saturated case, which we will discuss now.

In comparison to the black hole solutions of toroidally compactified string theory \(N = 4, 8\) string vacua), the near-extreme black holes of \(N = 2\) string vacua possess one obvious modification: the charges undergo a rotation, which is caused by the non-diagonal gauge coupling matrix \(N_{I,J}\) and is expressed by the mixing matrices \(\chi\) and \(\tilde{\chi}\) in (37). However, barring this rotation of charges, the physical microscopic interpretation of the solution resembles the toroidal case. First, the physical charges, which are defined as integral over the gauge fields (13) with the harmonic function defined in (34), split into two contributions

\[
\sim \mu \sinh \beta \cosh \beta = \mu (e^{2\beta} - e^{-2\beta}) = Q - \bar{Q}, \quad (40)
\]

where the first part coincides with the extreme charge and the second part is the near-extreme perturbation. In [3] the second part has also been interpreted as a small contribution with the opposite charge (e.g., anti-brane). This splitting takes place for all charges: (i) for the magnetic charges \(p^A\) it implies, that the 5-brane wraps also an anti-holomorphic part of the 4-cycle and in the BPS-saturated limit \((\mu \rightarrow 0, \beta^A \rightarrow \infty, \text{with } p^A \sim \mu e^{2\beta^A} \text{ finite})\) the anti-holomorphic parts vanish, (ii) for the momentum modes (associated with the electric charge \(q_0\)) this result simply implies that there are left- as well as right-moving momentum modes along \(S^1\), and finally (iii), the electric charges \((\sim q_A)\) of the membranes with non-trivial fluxes through a 2-cycles \(\times S^1\), yield winding as well as “anti-winding” states on \(S^1\).

Although this splitting is valid for all the three types of charges, in the subsequent discussion of the microscopic entropy we concentrate on the momentum part associated with \(q_0\), i.e. we will consider the microscopic in a dilute gas approximation. This means we will take large magnetic boosts so that

\[
\Omega_0 \simeq 1 \quad \text{or} \quad \tilde{p}^I \simeq \bar{p}^I \simeq p^I. \quad (41)
\]

For the sake of simplicity we consider the axion-free case with \(\bar{H}_A = 0\). Because in this case \(N_{0A} = 0\) also \(\chi^0_A = \bar{\chi}_0^A = 0\) and hence the 0-components factorize. In this case the

\[3\]There is a subtlety associated with the membrane modes, which can possess one right-moving mode coming from the two-form on the membrane, and which are paired with the three translational zero modes and four goldstinos in a (4,0) multiplet.
form of the classical entropy takes the form:

\[
S = S_L + S_R = 2\pi \left( \sqrt{(N_L)c_L/6} + \sqrt{(N_R)c_R/6} \right),
\]

\[
S_L = 2\pi \sqrt{x_0^0 \mu e^{2\beta(0)}} \sqrt{d_{ABC}p^Ap^Bp^C} = 2\pi \sqrt{N_Lc_L/6}
\]

\[
S_R = 2\pi \sqrt{x_0^0 \mu e^{-2\beta(0)}} \sqrt{d_{ABC}p^Ap^Bp^C} = 2\pi \sqrt{N_Rc_R/6}.
\]

(42)

i.e. both, the left and right-moving modes feel the identical central charge \(c_L = c_R\), since in the dilute gas approximation we consider the 5-branes wrapped around supersymmetric (holomorphic) 4-cycles (see [3]). The momentum modes \(N_{L,R}\) are respectively the left- and right-moving momenta along \(S^1\).

We would like to conclude with a few remarks.

- **Duality invariant structure of the thermodynamic quantities** The result for the entropy and the temperature of the near-BPS-saturated solution is cast in a duality invariant form, since by construction the Kähler potential is duality invariant.

- **General non-extreme solutions** The equations of motion may have an explicit solution for a (large) class of non-extreme solutions. In particular, there is a possibility of solving explicitly the scalar equation (26) simultaneously with the Einstein equations (25) and a large class of the solutions can be expressed in terms of the harmonic functions, only.

- **Rotating non-extreme solutions** Another interesting avenue to pursue are rotating solutions. As a first step, of most interest are the near-BPS-saturated rotating solutions. (In the BPS-saturated limit, only static four-dimensional solutions have regular horizons.) From the point of view of the microscopic interpretation one expects the angular momentum quanta to be identified with the \(U(1)_R \in SU(2)_R\) world-sheet charges of the right-moving modes. (The right-moving sector of the 2-dimensional sigma model possesses \(N = 4\) world-sheet supersymmetry with the \(SU(2)_R\) world-sheet current algebra.)

The ultimate goal would be to obtain non-extreme rotating solutions (both in four- and five-dimensions), and hopefully arrive at the same suggestive structure of the thermodynamic quantities, as in the case of black holes of \(N = 4, 8\) superstring vacua \([12]\); in the latter case these quantities split into two, suggesting a microscopic interpretation in terms of effective string degrees of freedom.

- **Calabi-Yau phase transitions** The structure of this near-extreme solution has a remarkable property, that the Bekenstein-Hawking entropy and the Hawking temperature depend on the gauge coupling matrix \(N_{IJ}\). As long as it is regular, a non-vanishing \(\mu\) can be treated as a small fluctuation around the extremal limit. However at points where \(N_{IJ}\) or \(N^{TJ}\) becomes singular, already small fluctuation could cause an infinite Hawking temperature making the black hole unstable. These
are the interesting points where Calabi-Yau phase transition can occur. The corresponding BPS-saturated solutions have been investigated in [21] and they remain finite as long as one keeps at least two charges. It would be very interesting to investigate these properties further.

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