GENERALIZED α-VARIATION AND LEBESGUE EQUIVALENCE TO DIFFERENTIABLE FUNCTIONS

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Abstract. We find an equivalent condition for a real function \( f : [a, b] \to \mathbb{R} \) to be Lebesgue equivalent to an \( n \)-times differentiable function (\( n \geq 2 \)); a simple solution in the case \( n = 2 \) appeared in an earlier paper. For that purpose, we introduce the notions of \( C BV G_{1/n} \) and \( SBV G_{1/n} \) functions, which play analogous rôles for the \( n \)-th order differentiability as the classical notion of a \( V B G_* \) function for the first order differentiability, and the classes \( C BV_{1/n} \) and \( SBV_{1/n} \) (introduced by Preiss and Laczkovich) for \( C^n \) smoothness. As a consequence of our approach, we obtain that Lebesgue equivalence to \( n \)-times differentiable function is the same as Lebesgue equivalence to a function \( f \) which is \( (n-1) \)-times differentiable with \( f^{(n-1)}(\cdot) \) being pointwise Lipschitz. We also characterize the situation when a given function is Lebesgue equivalent to an \( n \)-times differentiable function \( g \) such that \( g' \) is nonzero a.e.

As a corollary, we establish a generalization of Zahorski’s Lemma for higher order differentiability.

1. Introduction

Let \( f : [a, b] \to \mathbb{R} \). We say that \( f \) is Lebesgue equivalent to \( g : [a, b] \to \mathbb{R} \) provided there exists a homeomorphism \( h \) of \([a, b]\) onto itself such that \( g = f \circ h \). This terminology is taken from [3]. Zahorski [15] and Choquet [4] (see also Tolstov [14]) proved a result characterizing paths \( (f : [a, b] \to \mathbb{R}^n) \) that allow a differentiable parametrization (resp. a dif. parametrization with almost everywhere nonzero derivative) as those paths that have the \( V B G_* \) property (resp. which are also not constant on any interval). Fleissner and Foran [9] reproved this later (for real functions only and not considering the case of a.e. nonzero derivatives) using a different result of Tolstov. The definition of \( V B G_* \) is classical; see e.g. [13]. The mentioned results were generalized by L. Zajiček and the author [6] to paths with values in Banach spaces (and also metric spaces using the metric derivative instead of the
usual one). Laczkovich, Preiss [11], and Lebedev [12] studied (among other things) the case of \(C^n\)-parametrizations of real-valued functions \((n \geq 2)\). Lebedev proved that a continuous function \(f : [a, b] \rightarrow \mathbb{R}\) is Lebesgue equivalent to a \(C^n\) function provided

\[
\lambda(f(K_f)) = 0 \text{ and } \sum_{\alpha \in A} (\omega^f_\alpha)^{1/n} < \infty,
\]

where \(K_f\) is the set of point of varying monotonicity of \(f\) (see the definition below) and \(\omega^f_\alpha\) is the oscillation of \(f\) on \(I_\alpha\), where \((I_\alpha)_{\alpha \in A}\) are all the intervals contiguous to \(K_f\) in \([a, b]\). Laczkovich and Preiss showed that the same is true for a continuous \(f\) provided

\[
(1.1) \quad V_{1/n}(f, K_f) < \infty,
\]
or

\[
(1.2) \quad SV_{1/n}(f, K_f) = 0.
\]

(See Definition 2.3 in Section 2). They define \(CBV_{1/n}\) (resp. \(SBV_{1/n}\)) as the class of continuous function which satisfy (1.1) (resp. (1.2)). Moreover, in [11] and [12] also the case of \(C^{n,\alpha}\) \((0 < \alpha \leq 1)\) parametrizations is settled (where \(C^{n,\alpha}\) is the class of functions such that \(f^{(n)}\) is \(\alpha\)-Hölder).

Differentiability via a homeomorphic change of variable was studied by other authors (see e.g. [1], [2]). For a nice survey of differentiability of real-valued functions via homeomorphisms, see [10]. L. Zajíček and the author [7, 8] characterized the situation when a Banach space-valued path (for Banach spaces with a \(C^1\) norm) admits a \(C^2\)-parametrization or a parametrization with finite convexity. In the corresponding situations, also the case of the first derivative being almost everywhere nonzero is treated in \(7, 8\).

In [5], we characterized the situation when a function \(f : [a, b] \rightarrow \mathbb{R}\) is Lebesgue equivalent to a twice differentiable function. We introduced the notion of \(VBG_{1/2}\) functions for that purpose. We also established that for a real function \(f\) defined on a closed interval, being Lebesgue equivalent to a twice differentiable function is equivalent to being Lebesgue equivalent to a differentiable function whose derivative is pointwise Lipschitz.

In the present article, we characterize the situation when \(f\) is Lebesgue equivalent to an \(n\)-times differentiable function for \(n \geq 3\) (our approach in the present article gives a certain condition also in case \(n = 2\) which can be seen to be equivalent to the one proved in [5], but the present general proof is much more complicated than the arguments of [5] in that interesting special case). We introduce two new classes of functions: \(CBV_{1/n}G_{1/2}\) and \(SBV_{1/n}G_{1/2}\), which are analogous to the classes \(CBV_{1/n}\) and \(SBV_{1/n}\) introduced by Preiss and Laczkovich in [11] in order to characterize the situation when a given
function is Lebesgue equivalent to a $C^n$ function. In the main Theorem 4.1, we prove that $f$ is Lebesgue equivalent to an $n$-times differentiable function if and only if $f$ is $CBV G_{1/n}$ (resp. $f$ is $SBV G_{1/n}$). As a corollary, we obtain that the classes $CBV G_{1/n}$ and $SBV G_{1/n}$ coincide (which seems to be difficult to establish directly). Our approach also yields that $f$ is Lebesgue equivalent to an $n$-times differentiable function if and only if $f$ is $CBV G_{(n-1)/n}$ (resp. $f$ is $SBV G_{(n-1)/n}$). As a corollary, we obtain that the classes $CBV G_{1/n}$ and $SBV G_{1/n}$ coincide (which seems to be difficult to establish directly). Our approach also yields that $f$ is Lebesgue equivalent to an $n$-times differentiable function if and only if $f$ is $CBV G_{1/n}$, but which is not Lebesgue equivalent to any $C^n$ function. In Theorem 4.4, we characterize Lebesgue equivalence to an $n$-times differentiable function whose first derivative is a.e. nonzero.

The classical Zahorski’s Lemma (see e.g. [15] or [10, p. 27]) claims that if $M \subset [a, b]$ has Lebesgue measure 0, then there exists a (boundedly) differentiable homeomorphism $h$ from $[a, b]$ onto itself such that $h^{-1}(M) \subset \{x \in [a, b] : h'(x) = 0\}$. If $M$ is closed, then $h$ can be taken $C^1$. In Theorem 5.1, we show a higher order analogue of this fact; i.e., a closed set $M \subset [a, b]$ is the image by an $n$-times differentiable homeomorphism such that $h^{(i)}(x) = 0$ for all $x \in h^{-1}(M)$ and $i = 1, \ldots, n$, if and only if there exists a decomposition of the set $M$ such that certain variational conditions closely related to the definition of the class $CBV G_{1/n}$ (respectively, $SBV G_{1/n}$) are satisfied. See Theorem 5.1 for details.

The current paper is structured as follows. Section 2 contains basic facts and definitions. Section 3 contains facts about the generalized variation $GV_{1/n}$ (and related notions) and classes $CBV G_{1/n}$, resp. $SBV G_{1/n}$; there is also a definition of an auxiliary class $SBV G_{1/n}$. Section 4 contains the main Theorems 4.1 and 4.4. In section 5 we prove Theorem 5.1, which is an analogue of the Zahorski’s Lemma for higher order differentiability.

In the proofs, we need many auxiliary results. Let us point out that the main ingredients for our results are the estimate of Lemma 3.8, and the method of construction of a suitable variation in Lemmata 3.10, 3.11.

2. Preliminaries

By $C$ (resp. $C_x$, ...) we will denote an absolute constant (resp. constant depending on $x$, ...) that can change between lines. By letter $n$ we will always denote a positive integer. By $\lambda$ we will denote the Lebesgue measure
on \( \mathbb{R} \). For \( x, r \in \mathbb{R} \) with \( r > 0 \) we will denote by \( B(x, r) := \{ y \in \mathbb{R} : |x-y| < r \} \) the open ball with center \( x \) and radius \( r \). Let \( K \subset [a, b] \) be closed, and such that \( \{a, b\} \subset K \). We say that the interval \((c, d) \subset [a, b]\) is contiguous to \( K \) (in \([a, b]\)) provided \( c, d \in K \) and \((c, d) \cap K = \emptyset \) (i.e. it is a maximal open component of \([a, b] \setminus K \) in \([a, b]\)). Let \( f : [a, b] \to \mathbb{R} \). By \( K_f \) we will denote the set of points of varying monotonicity of \( f \), i.e. the set of points \( x \in [a, b] \) such that there is no open neighbourhood \( U \) of \( x \) such that \( f|_U \) is either constant or strictly monotone (see e.g. [11]). Obviously, \( K_f \) is closed and \( \{a, b\} \subset K_f \). We will also frequently use the simple fact that if \( h \) is a homeomorphism of \([a, b]\) onto itself, \( g = f \circ h \), then \( K_g = h^{-1}(K_f) \).

Let \( f : [a, b] \to \mathbb{R} \). We say that \( f \) is pointwise Lipschitz at \( x \in [a, b] \) provided

\[
\limsup_{t \to 0, t \neq 0} \frac{|f(x+t) - f(x)|}{|t|} < \infty.
\]

We say that \( f \) is pointwise Lipschitz provided it is pointwise Lipschitz at each point \( x \in [a, b] \). We will define the derivative \( f'(x) \) of \( f \) at \( x \in [a, b] \) as usual; at the endpoints we consider the corresponding unilateral derivative. The \( n^{\text{th}} \) derivative \( f^{(n)}(x) \) of \( f \) at \( x \) is defined by induction as \( f^{(0)}(x) := f(x) \), and \( f^{(n)}(x) := (f^{(n-1)})'(x) \) for \( n \geq 1 \). We say that \( f \) is \( C^n \) for \( n \geq 1 \) provided \( f^{(n)} \) exists and is continuous in \([a, b]\). We will often use the following easy fact: if \( f \) is \( C^1 \), and \( x \in K_f \cap (a, b) \), then \( f'(x) = 0 \).

The following version of Sard’s theorem is proved in e.g. [6, Lemma 2.2].

**Lemma 2.1.** If \( f : \mathbb{R} \to \mathbb{R} \), then \( \lambda(f(\{x \in \mathbb{R} : f'(x) = 0\})) = 0 \).

The following simple lemma is proved in [5, Lemma 9].

**Lemma 2.2.** Let \( h_m : [a, b] \to [c_m, d_m] \) \((m \in \mathcal{M} \subset \mathbb{N})\) be continuous increasing functions such that \( \sum_{m \in \mathcal{M}} h_m(x) \in \mathbb{R} \) for all \( x \in [a, b] \). Let \( K \subset [a, b] \) be closed and such that \( \lambda(h_m(K)) = 0 \) for all \( m \in \mathcal{M} \). Then \( h : [a, b] \to [c, d] \), defined as \( h(x) := \sum_{m \in \mathcal{M}} h_m(x) \), is a continuous and increasing function (for some \( c, d \in \mathbb{R} \)) such that \( \lambda(h(K)) = 0 \).

The following definition is taken from [11].

**Definition 2.3.** Let \( f : [a, b] \to \mathbb{R} \) be continuous. For \( \alpha \in (0, 1] \), \( \delta > 0 \), and \( K \subset [a, b] \), we will define \( V_{\alpha}^{\delta}(f, K) \) as a supremum of the sums

\[
(2.1) \quad \sum_{i=1}^{N} |f(d_i) - f(c_i)|^\alpha,
\]

where \(([c_i, d_i])_{i=1}^{N}\) is any sequence of non-overlapping intervals in \([a, b]\) such that \( c_i, d_i \in K \), \( d_i - c_i \leq \delta \) for all \( i = 1, \ldots, N \). We define \( V(f, [a, b]) := \)
Lemma 2.7. Let 
\[ V^\alpha_{b-a}(f, [a, b]), \quad V^\alpha_0(f, K) := V^\alpha_{b-a}(f, K), \]
and
\[ SV^\alpha_0(f, K) := \lim_{\delta \to 0^+} V^\delta_0(f, K). \]

See the paper [11] for basic properties of these fractional variations. Note that [11, Lemma 3.13] implies that if \( K \subset [a, b] \) is closed, and \( V^\delta_0(f, K) < \infty \), then the function \( g(x) = V^\delta_0(f, K \cap [a, x]) \) is continuous on \( K \).

We have the following simple lemma.

**Lemma 2.4.** Suppose that \( H \subset \mathbb{R} \) is bounded, \( f : H \to \mathbb{R} \) is uniformly continuous and such that \( V^\alpha_0(f, H) < \infty \) for some \( \alpha \in (0, 1) \) and \( \delta > 0 \). Then \( \lambda(\overline{f(H)}) = 0 \).

**Proof.** Let \( \varepsilon > 0 \) and put \( \eta := (\frac{\varepsilon}{\lambda(\overline{f(H)})})^{\frac{1}{1-\alpha}}. \) Choose \( 0 < \zeta < \delta \) such that for all \( x, y \in H \) with \( |x - y| < \zeta \) we have \( |f(x) - f(y)| < \eta \). Let \( ([a_i, b_i])_{i=1}^N \) be non-overlapping intervals with \( a_i, b_i \in H \) and \( b_i - a_i < \zeta \). Then

\[
\sum_{i=1}^N |f(b_i) - f(a_i)| \leq \eta^{(1-\alpha)} \cdot V^\delta_0(f, H) < \varepsilon.
\]

Thus we have \( SV^1_0(f, H) = 0 \); [11, Theorem 2.9] implies \( \lambda(\overline{f(H)}) = 0 \). \( \square \)

We shall need the following lemma. For a proof, see e.g. [6, Lemma 2.7].

**Lemma 2.5.** Let \( \{a, b\} \subset B \subset [a, b] \) be closed, and \( f : [a, b] \to \mathbb{R} \) be continuous. If \( \lambda(f(B)) = 0 \), then \( V(f, [a, b]) = \sum_{i \in \mathcal{I}} V(f, [c_i, d_i]) \), where \( I_i = (c_i, d_i) \), \( (i \in \mathcal{I} \subset \mathbb{N}) \) are all intervals contiguous to \( B \) in \([a, b]\).

**Lemma 2.6.** Let \( f : [a, b] \to \mathbb{R} \) be continuous, \( \{a, b\} \subset K \subset [a, b] \) be closed, such that \( V^\alpha_0(f, K) < \infty \) for some \( \alpha \in (0, 1) \), \( \delta > 0 \), and \( V(f, [c, d]) = |f(d) - f(c)| \) whenever \( (c, d) \) is an interval contiguous to \( K \) in \([a, b]\). Then \( V(f, [a, b]) < \infty \).

**Proof.** Let \( (u_p, v_p) (p \in \mathcal{P} \subset \mathbb{N}) \) be all the intervals contiguous to \( K \) in \([a, b]\).

By Lemma 2.4, we have that \( \lambda(f(K)) = 0 \), and thus by Lemma 2.5 and the assumptions, we obtain

\[
V(f, [a, b]) = \sum_{p \in \mathcal{P}} V(f, [u_p, v_p]) = \sum_{p \in \mathcal{P}} |f(v_p) - f(u_p)| \leq 2NM + V^\delta_0(f, K) < \infty,
\]

where \( N \) is the number of \( p \in \mathcal{P} \) such that either \( |f(v_p) - f(u_p)| > 1 \) or \( v_p - u_p \geq \delta \) (which is finite since \( V^\alpha_0(f, K) < \infty \) and \( b - a < \infty \)), and \( M := \max_{x \in [a, b]} |f(x)| \). \( \square \)

**Lemma 2.7.** Let \( f : [a, b] \to \mathbb{R} \) be continuous, \( \{a, b\} \subset K \subset [a, b] \) closed, \( V^\delta_0(f, K) < \infty \) for some \( \alpha \in (0, 1) \), \( \delta > 0 \), and \( V(f, [u, v]) = |f(v) - f(u)| \) whenever \( (u, v) \) is an interval contiguous to \( K \). Let \( g(x) := V^\delta_0(f, K \cap [a, x]) \). Then \( \lambda(g(K)) = 0 \).
Remark 2.8. Lemmata 2.2 and 2.7 together with the methods from [11] show that \( \lambda(f(K)) = 0 \), and by Lemma 2.6 it follows that \( V(f, [a, b]) < \infty \). Let \( \tilde{g}(x) = g(x) \) for \( x \in K \). Now, [11, Lemma 3.13] shows that \( g \) is continuous on \( K \). Extend \( \tilde{g} \) to \([a, b]\) in such a way that \( \tilde{g} \) is affine and continuous on every \([u, v]\), whenever \((u, v)\) is an interval contiguous to \( K \). Since \( g(K) = \tilde{g}(K) \), it is enough to prove that \( \lambda(\tilde{g}(K)) = 0 \). Let \( \varepsilon > 0 \) and \((c_i, d_i)_{i=1}^N\) be non-overlapping intervals with \( c_i, d_i \in K \), \( d_i - c_i \leq \delta \) for each \( i = 1, \ldots, N \), and such that \( V_\alpha^c(f, K) \leq \sum_{i=1}^N |f(d_i) - f(c_i)|^\alpha + \varepsilon / 2 \). For each \( i = 1, \ldots, N \), by Lemma 2.5 find intervals \(([c^i_j, d^i_j])_{j=1}^{J_i}\) contiguous to \( K \) in \([c_i, d_i]\) such that

\[
|f(d_i) - f(c_i)| \leq V(f, [c_i, d_i]) \leq \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_j)| + \left( \frac{\varepsilon}{2N} \right)^{1/\alpha}.
\]

Thus

\[
\tilde{g}(b) - \tilde{g}(a) = V_\alpha^c(f, K) \leq \sum_{i=1}^N \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_j)|^\alpha + \varepsilon \leq \sum_{p \in P} |f(v_p) - f(u_p)|^\alpha + \varepsilon,
\]

where \((u_p, v_p) (p \in \mathcal{P} \subset \mathbb{N})\) are all the intervals contiguous to \( K \) in \([a, b]\). Now, send \( \varepsilon \to 0^+ \) to conclude that \( \tilde{g}(d) - \tilde{g}(c) \leq \sum_{p \in \mathcal{P}} \tilde{g}(v_p) - \tilde{g}(u_p) \). Since \( \tilde{g}(K) \cap \tilde{g}(\bigcup_{p \in \mathcal{P}} (u_p, v_p)) \) is countable, we obtain that

\[
\lambda(\tilde{g}(K)) = (\tilde{g}(b) - \tilde{g}(a)) - \sum_{p \in \mathcal{P}} (\tilde{g}(v_p) - \tilde{g}(u_p)) = 0.
\]

\( \Box \)

Remark 2.8. Lemmata 2.2 and 2.7 together with the methods from [11] can be used to give a proof of the following: Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous, \( n \geq 2 \). Then the following are equivalent:

(i) \( f \) is Lebesgue equivalent to a \( C^n \) function \( g \) such that \( g'(x) \neq 0 \) for almost all \( x \in [a, b] \);

(ii) \( V_{1/n}(f, K_f) < +\infty \) and \( f \) is not constant on any interval;

(iii) \( SV_{1/n}(f, K_f) = 0 \) and \( f \) is not constant on any interval.

See Definition 2.3 for the definitions of the fractional variations. In (i), we can actually replace “\( C^n \)” with “\( C^s \)”, where \( s > 1 \) (we did not define this class), if we replace \( 1/n \) by \( 1/s \) in the variations in (ii) and (iii). See [11, Definition 3.1] for the definition of this class.

Let \( K \subset \mathbb{R} \) be closed. As usual, by \( K' \), we will denote the set of all accumulation points of \( K \). It is easy to see that \( K' \) is closed, and \( K \setminus K' \) is countable.

We have the following easy consequence of the classical Rolle’s theorem.
Lemma 2.9. Let \( f : [a, b] \to \mathbb{R} \) be \((n - 1)\)-times differentiable for some \( n \geq 2 \), and suppose that there are \( n \) distinct points \((x_i)_{i=1}^n \) in \([a, b]\) such that \( f'(x_i) = 0 \) for \( i = 1, \ldots, n \). Then there exists \( y \in [\min_{1 \leq i \leq n} x_i, \max_{1 \leq i \leq n} x_i] \) such that \( f^{(n-1)}(y) = 0 \).

The following lemma shows that the derivatives are zero at all accumulation points of a given set.

Lemma 2.10. Let \( f : [a, b] \to \mathbb{R} \) be \((n - 1)\)-times differentiable for some \( n \geq 2 \), let \( K \subset \{ x \in [a, b] : f'(x) = 0 \} \) be a closed set. Then \( f^{(i)}(x) = 0 \) whenever \( x \in K' \cap (a, b) \) and \( i \in \{ 1, \ldots, n - 1 \} \). If \( x \in K' \cap (a, b) \), and \( f^{(n)}(x) \) exists, then \( f^{(n)}(x) = 0 \).

Proof. By assumptions, we have that \( f'(x) = 0 \) for all \( x \in K \). Let \( i \in \{ 2, \ldots, n-1 \} \). Without any loss of generality, assume that \( x \) is a right-hand-side accumulation point of \( K \). Fix \( k \in \mathbb{N} \). There are \( x < z_1 < \cdots < z_i < x + 1/k \) with \( z_j \in K \). By Lemma 2.9 there exists \( w_k \in [z_i, z_i] \) with \( f^{(i-1)}(w_k) = 0 \), and thus (by induction) we have \( f^{(i)}(x) = \lim_{k \to \infty} \frac{f^{(i-1)}(w_k)}{w_k - x} = 0 \) for \( i = 2, \ldots, n-1 \).

If \( f^{(n)}(x) \) exists, then the argument above implies that it is equal to 0. \( \square \)

Next lemma will allow us to construct suitable extensions of functions.

Lemma 2.11. Let \( \alpha, \beta, A, B \in \mathbb{R} \), with \( \alpha < \beta \), \( A < B \), \( n \in \mathbb{N} \), \( n \geq 2 \). Then there exists an \( n \)-times differentiable homeomorphism \( H : [\alpha, \beta] \to [A, B] \) such that

(i) \( H^{(i)}(\alpha) = H^{(i)}(\beta) = 0 \) for \( i = 1, \ldots, n \), \( H'(x) > 0 \) for all \( x \in (\alpha, \beta) \),
(ii) \( H(\alpha) = A, H(\beta) = B \),
(iii) \( |H^{(n-1)}(t)| \leq C \cdot \frac{|B-A|}{(\beta-\alpha)^n} \cdot \min(t-\alpha, \beta-t) \) for \( t \in (\alpha, \beta) \), where \( C > 0 \) is an absolute constant.

Proof. As in [11, p. 420], denote

\[
w(x) = c \int_0^x \exp(-t^2 - (1 - t)^2) \, dt, \quad (x \in [0, 1]),
\]

where the positive constant \( c \) is chosen such that \( w(1) = 1 \). Then \( w \in C^\infty([0, 1]) \), \( w(0) = 0 \), \( w \) is strictly increasing on \([0, 1]\), \( w'(x) \neq 0 \) for all \( x \in (0, 1) \), and \( w^{(i)}(0) = w^{(i)}(1) = 0 \) for \( i = 1, 2, \ldots \). For \( x \in [\alpha, \beta] \) define

\[
H(x) = A + (B - A) \cdot w\left(\frac{x - \alpha}{\beta - \alpha}\right).
\]

It is easy to see that the conditions (i) and (ii) hold. Condition (iii) follows from the fact that the function \( w \) is \( C^\infty \) and thus \( H^{(n-1)}(\cdot) \) is \( C \cdot \frac{|B-A|}{(\beta-\alpha)^n} \) Lipschitz on \([\alpha, \beta] \). \( \square \)
3. Generalized fractional variation

We need the following generalized fractional variation.

**Definition 3.1.** Let $f : [a, b] \to \mathbb{R}$. Let $\emptyset \neq A \subset K$ be closed sets. We define the generalized $(1/n)$-variation $GV_{1/n}(f, A, K)$ (resp. $\overline{GV}_{1/n}(f, A, K)$ for $\delta > 0$) as the supremum of sums

$$
(3.1) \quad \sum_{i=1}^{N} \left( V_{\frac{1}{n-1}}(f, K \cap [x_i, y_i]) \right)^{\frac{n-1}{n}},
$$

where the supremum is taken over all collections of non-overlapping intervals $([x_i, y_i])_{i=1}^{N}$ in $[a, b]$ with $x_i, y_i \in A$ for all $i = 1, \ldots, N$ (resp. over all collections $([x_i, y_i])_{i=1}^{N}$ of non-overlapping intervals in $[a, b]$ such that $y_i - x_i \leq \delta$, $x_i, y_i \in K$, and $\{x_i, y_i\} \cap A \neq \emptyset$ for all $i = 1, \ldots, N$).

We put $GV_{1/n}(f, \emptyset, K) = \overline{GV}_{1/n}(f, \emptyset, K) = 0$. Similarly, we also define auxiliary variation $rGV_{1/n}^\delta(f, A, K)$ (resp. $lGV_{1/n}^\delta(f, A, K)$) as a supremum of the sums in (3.1) for $\overline{GV}_{1/n}^\delta$, but requiring that $x_i \in A$ (resp. $y_i \in A$) for all $i = 1, \ldots, N$, whenever $([x_i, y_i])_{i=1}^{N}$ is a sequence of admissible intervals for $\overline{GV}_{1/n}$ in (3.1).

In all cases, when there is no admissible sequence $([x_i, y_i])_{i=1}^{N}$, we define the corresponding variation to be equal to 0.

If $A \subset K \subset [a, b]$ are closed sets, $rGV_{1/n}^\delta(f, A, K) < \infty$ (respectively, $lGV_{1/n}^\delta(f, A, K) < \infty$), and $x \in A$, then $rGV_{1/n}^\delta$ (resp. $lGV_{1/n}^\delta$) is “additive at $x$”, i.e.

$$
(3.2) \quad rGV_{1/n}^\delta(f, A, K)
= rGV_{n}^\delta(f, A \cap [a, x], K \cap [a, x]) + rGV_{n}^\delta(f, A \cap [x, b], K \cap [x, b]);
$$

(and similarly for $lGV_{n}^\delta$). This is easily seen from the definition. Also, we have that

$$
(3.3) \quad \max \left(lGV_{1/n}^\delta(f, A, K), rGV_{1/n}^\delta(f, A, K) \right)
\leq \min \left( \overline{GV}_{1/n}^\delta(f, A, K), GV_{1/n}(f, A \cup \{a, b\}, K \cup \{a, b\}) \right).
$$

Further, if $0 < \delta < \gamma$, then $\overline{GV}_{1/n}^\delta(f, A, K) \leq \overline{GV}_{1/n}^\gamma(f, A, K)$.

We will need some properties of the “unilateral” variations.

**Lemma 3.2.** Let $\emptyset \neq A \subset K \subset [a, b]$ be closed sets, $\{a, b\} \subset K$, $f : [a, b] \to \mathbb{R}$ continuous and such that $\overline{GV}_{1/n}^\delta(f, A, K) < \infty$ for some $\delta > 0$ and $n \geq 2$. Then $v(x) := rGV_{1/n}^\delta(f, A \cap [a, x], K \cap [a, x])$ and $\bar{v}(x) := \overline{GV}_{1/n}^\delta(f, A \cap [a, x], K \cap [a, x])$ for some $x \in [a, b]$. Then

$$
\forall \emptyset \neq A \subset K : v(x) = \bar{v}(x) = \frac{1}{n} \max \left(v(x), \bar{v}(x) \right) = \frac{1}{n} \min \left(v(x), \bar{v}(x) \right) = \frac{1}{n} \overline{GV}_{1/n}^\delta(f, A \cap [a, x], K \cap [a, x]).
$$
$lGV^f_{1/n}(f, A \cap [x, b], K \cap [x, b])$ are continuous functions on $K$ such that $v$ is increasing, $\bar{v}$ is decreasing, and

\begin{equation}
\max(v(b) - v(a), \bar{v}(a) - \bar{v}(b)) \leq \overline{GV}^f_{1/n}(f, A, K).
\end{equation}

**Proof.** Clearly, $v$ is increasing and $\bar{v}$ is decreasing on $K$. Also, we easily see that (3.4) holds. We will only establish the continuity of $v$ on $K$, as the case of $\bar{v}$ is similar. To show that $v$ is continuous on $K$, let $x \in K$ be such that $x$ is not a left-hand-side accumulation point of $A$. If $[x - \delta, x) \cap A = \emptyset$, then $v$ is constant on $[x - \delta, x]$. If $[x - \delta, x) \cap A \neq \emptyset$, then

\begin{equation}
v(t) = v(r) + \left(V_{1/n}^{\frac{1}{n}}(f, K \cap [r, t])\right)^{\frac{n-1}{n}},
\end{equation}

for all $t \in [r, x]$, where $r := \max([a, x) \cap A)$ provided $[a, x) \cap A \neq \emptyset$ (the case when $[a, x) \cap A = \emptyset$ is trivial). Obviously, we have that $v(t) - v(r) \geq \left(V_{1/n}^{\frac{1}{n}}(f, K \cap [r, t])\right)^{\frac{n-1}{n}}$. The other inequality in (3.5) follows easily from the definition of the fractional variation together with the fact that $[x - \delta, x) \cap A = \emptyset$. The unilateral continuity of $v$ at $x$ in this case follows from [11, Lemma 3.13]. We have shown that $v$ is continuous from the left at all $x \in K$ which are not left-hand-side accumulation points of $A$.

If $x \in K$ is not right-hand-side accumulation point of $A$, then either $(x - \delta, x] \cap A = \emptyset$, in which case $v$ is constant on $[x, x + \eta]$ for some $\eta > 0$, or if $(x - \delta, x] \cap A \neq \emptyset$, then (3.5) holds and continuity of $v$ at $x$ from the right follows again from [11, Lemma 3.13].

Now suppose that $x \in A$ is a left-hand-side accumulation point of $A$. Fix $\varepsilon > 0$. Choose the sequence $([x_i, y_i])_{i=1}^N$ as in (3.1) for $rGV^f_{1/n}(f, A \cap [a, x], K \cap [a, x])$ and $([c_j^i, d_j^i])_{j=1}^{J_i}$ from (2.1) for $V_{1/n}^{\frac{1}{n}}(f, [x_i, y_i] \cap K)$ so that

\begin{equation}
\sum_{i=1}^N \left(\sum_{j=1}^{J_i} |f(c_j^i) - f(d_{j-1}^i)|^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} > v(x) - \varepsilon.
\end{equation}

We can assume that $([x_i, y_i])$ and $([c_j^i, d_j^i])$ are ordered in the natural sense (i.e. $y_i < x_{i'}$ whenever $i < i'$; similarly for $c_j^i, d_j^i$). We can suppose that $d_{J_i}^N < x$ because if $d_{J_i}^N = x$, then by continuity of $f$ and by the fact that $x$ is a left-hand-side accumulation point of $A$, we can make $d_{J_i}^N$ slightly smaller (and adjust $y_N$) without violating (3.6). Then we have $v(z) > v(x) - \varepsilon$ for every $d_{J_i}^N < z < x$ and hence $v$ is continuous from the left at $x$.

Now suppose that $x \in A$ is a a right-hand-side point of accumulation of $A$, and fix $\varepsilon > 0$. Choose $([x_i, y_i])_{i=1}^N$ and $([c_j^i, d_j^i])_{j=1}^{J_i}$ from (3.1) for $rGV^f_{1/n}(f, A \cap [x, b]) = v(b) - v(x)$ (the equality holds because $x \in A$) so
that
\[
(3.7) \quad \sum_{i=1}^{N} \left( \sum_{j=1}^{J_i} |f(c^i_j) - f(d^i_{j-1})|^{1/n} \right)^{n-1} > v(b) - v(x) - \varepsilon.
\]

As before, we can assume that \([x_i, y_i]\) and \([c^i_j, d^i_j]\) are ordered in the natural sense (see the remark after (3.6)) and that \(c^1_1 > x\) (with \(x_1 > x\)). Let \(y \in (x, x_1) \cap K\). Take any \([\tilde{x}_i, \tilde{y}_i]\) as in (3.1) for \(rGV_n^\delta(f, A \cap [a, y]) = v(y)\). We have to prove that
\[
(3.8) \quad \sum_{i=1}^{\tilde{N}} \left( V_{1/n}^\delta(f, [\tilde{x}_i, \tilde{y}_i] \cap K) \right)^{n-1} \leq v(x) + \varepsilon.
\]

For \(i = 1, \ldots, \tilde{N}\) take \((\tilde{c}^i_j, \tilde{d}^i_j)_{j=1}^{J_i}\) as in (2.1) for \(V_{1/n}^\delta(f, [\tilde{x}_i, \tilde{y}_i] \cap K)\). Since \(a \leq \tilde{x}_1 < \cdots < \tilde{x}_N < x_1 < \cdots < x_N \leq b\), we have
\[
(3.9) \quad \sum_{i=1}^{\tilde{N}} \left( \sum_{j=1}^{J_i} |f(\tilde{c}^i_j) - f(\tilde{d}^i_j)|^{1/n} \right)^{n-1} + \sum_{i=1}^{N} \left( \sum_{j=1}^{J_i} |f(c^i_j) - f(d^i_j)|^{1/n} \right)^{n-1} \leq v(b).
\]

The left hand side of (3.9) is by (3.7) greater than
\[
\sum_{i=1}^{\tilde{N}} \left( \sum_{j=1}^{J_i} |f(\tilde{c}^i_j) - f(\tilde{d}^i_j)|^{1/n} \right)^{n-1} + v(b) - v(x) - \varepsilon,
\]
and this easily implies (3.8). This concludes the proof of continuity of \(v\) on \(K\) since it is clearly an increasing function.

Now we are ready to define our classes of functions. The following class plays a similar rôle for \(n\)-times differentiables as the class \(CBV_1/n\) from [11] in case of continuous derivatives.

**Definition 3.3.** We say that a continuous \(f : [a, b] \to \mathbb{R}\) is \(CBVG_1/n\) for \(n \geq 2\) provided \(V_{1/n}^\delta(f, K_f) < \infty\), and there exist closed \(A_m \subset K_f \) (\(m \in \mathcal{M} \subset \mathbb{N}\)) such that \(K_f = \bigcup_{m \in \mathcal{M}} A_m\), and \(GV_1/n(f, A_m, K_f) < \infty\) for all \(m \in \mathcal{M}\).

It is easy to see that if \(f\) is a \(CBVG_1/n\) function for some \(n \geq 2\), then \(f\) has bounded variation. If \(n = 2\), then it is not difficult to prove that the class \(CBVG_{1/2}\) coincides with the class \(VBG_{1/2}\) from [5].

The analogue of the class \(SBV_1/n\) from [11] in the case of continuous derivatives is given by the following definition for the case of \(n\)-times differentiable functions.
Definition 3.4. We say that a continuous $f : [a, b] \to \mathbb{R}$ is $SBVG_{1/n}$ for $n \geq 2$ provided $V_{n, 1}(f, K_f) < \infty$ and there exist closed sets $A_m \subset K_f$, numbers $\delta_m > 0$ ($m \in \mathbb{N}$) such that $K_f = \bigcup_{m \in \mathbb{N}} A_m$ and we have

(i) $\lim_{m \to \infty} \delta_m = 0$,
(ii) $A_m \subset A_{m+1}$ for each $m \in \mathbb{N}$,
(iii) $\lim_{m \to \infty} GV_{1/n}^m(f, A_m, K_f) = 0$.

We will need the following auxiliary class:

Definition 3.5. We say that a continuous $f : [a, b] \to \mathbb{R}$ is $CBVG_{1/n}$ for $n \geq 2$ provided $V_{n, 1}(f, K_f) < \infty$ and there exist closed sets $A_m^k \subset K_f$, numbers $\delta_m^k > 0$ ($k, m \in \mathbb{N}$) such that $K_f = \bigcup_{k, m} A_m^k$ and for each $k \in \mathbb{N}$ we have

(i) $\lim_{m \to \infty} \delta_m^k = 0$,
(ii) $A_m^k \subset A_{m+1}^k$ for each $m \in \mathbb{N}$,
(iii) $\lim_{m \to \infty} GV_{1/n}^m(f, A_m^k, K_f) = 0$.

Since every continuous function on a compact interval is uniformly continuous, it is easy to see that if $f$ is $SBVG_{1/n}$ (resp. $SBVG_{1/n}$ or $CBVG_{1/n}$) and $g$ is Lebesgue equivalent to $f$, then $g$ is $SBVG_{1/n}$ (resp. $SBVG_{1/n}$ or $CBVG_{1/n}$).

We need the following observation:

Lemma 3.6. Let $f : [a, b] \to \mathbb{R}$. Then $f$ is $SBVG_{1/n}$ if and only if $f$ is $SBVG_{1/n}$.

Proof. Suppose that $f$ is $SBVG_{1/n}$. Let $A_m$ and $\delta_m$ ($m \in \mathbb{N}$) be as in Definition 3.4 for $f$. We define $A_m^1 := A_m$ and $A_m^k := \emptyset$ for $k > 1$. Similarly, $\delta_m^1 := \delta_m$ and $\delta_m^k := 1/m$ for $k > 1$. Then it is easy to see that $(A_m^k)$ and $(\delta_m^k)$ satisfy Definition 3.5 for $f$.

Now, suppose that $f$ is $SBVG_{1/n}$, and let $A_m^k$ and $\delta_m^k$ be as in Definition 3.5. By relabeling, we can assume that $A_m^k \neq \emptyset$ for all $m, k \in \mathbb{N}$ (since the case when there exists $k_0 \in \mathbb{N}$ such that $\bigcup_m A_m^k = \emptyset$ for all $k \geq k_0$ is simple to handle using (3.10)). By a standard diagonalization argument (using the property (ii) from Definition 3.5), we can also assume that $GV_{1/n}^{k_m}(f, A_m^k, K_f) \downarrow 0$ when $m \to \infty$ for all $k \in \mathbb{N}$. Define $A_1 := A_1^1$, $\delta_1 := \delta_1^1$, and $N_1 := 1$. By induction, we will construct closed sets $A_p \subset K_f$, $\delta_p > 0$, and $N_p \in \mathbb{N}$. Suppose that $A_1, \ldots, A_{p-1}$ (together with $\delta_i$ and $N_i$ for $i < p$) were constructed. Note that for closed $B_1, \ldots, B_l \subset K_f$, and $\xi > 0$, we have

$$GV_{1/n}^\xi \left( f, \bigcup_{j=1}^l B_j, K_f \right) \leq \sum_{j=1}^l GV_{1/n}^\xi(f, B_j, K_f).$$

(3.10)
For $i = 1, \ldots, p$, find $l_i > N_{p-1}$ such that $\overline{GV}_{1/n}^{\delta_i}(f, A_i^l, K_f) \leq p^{-2}$, put $A_p := \bigcup_{i=1}^p A_i^l$, $\delta := \min(p^{-1}, \min_{i=1,\ldots,p} \delta_i)$, and using (3.10) conclude that

\[(3.11) \quad \overline{GV}_{1/n}^{\delta}(f, A_p, K_f) \leq p^{-1}.
\]

Finally, put $N_p := 1 + \max_{i=1,\ldots,p} l_i$, and proceed with induction. Since $K_f = \bigcup_{k,m} A^k_{m}$, we obtain $K_f = \bigcup_p A_p$. By construction, it is easy to see that $A_p \subset A_{p+1}$ and $\delta_p \to 0$. Finally, (3.11) shows that

$$
\lim_{p \to \infty} \overline{GV}_{1/n}^{\delta_p}(f, A_p, K_f) = 0.
$$

We will also need the following property.

**Lemma 3.7.** Let $f : [a, b] \to \mathbb{R}$ be a function which is either $CBVG_{1/n}$, $SBVG_{1/n}$, or $SBVG_{1/n}$. Then $\lambda(f(K_f)) = 0$.

**Proof.** If $f$ is in one of the three classes, then $V_{1/n}(f, K_f) < \infty$, and thus Lemma 2.4 implies the conclusion. \hfill \Box

Next lemma contains our basic estimate.

**Lemma 3.8.** Let $n \geq 2$, and $f : [a, b] \to \mathbb{R}$ be $(n-1)$-times differentiable with $f^{(n-1)}(\cdot)$ being pointwise Lipschitz, let $K \subset \{x \in [a, b] : f'(x) = 0\}$ be closed with $a, b \in K$, and such that $|f(c) - f(d)| = V(f, [c, d])$ for all intervals $(c, d)$ contiguous to $K$ in $[a, b]$, let

$$
A \subset \{x \in K' : |f^{(n-1)}(y)| \leq k|y - x| \text{ for all } y \in B(x, 1/m)\}
$$

be a closed set, where $k, m \in \mathbb{N}$, $a \leq x < x' \leq b$ are such that $x, x' \in K$, $\{x, x'\} \cap A \neq \emptyset$ and $0 < x' - x < 1/m$. Then

\[(3.12) \quad \left(V_{1/n}(f, K \cap [x, x'])\right)^{\frac{n-1}{n}} \leq C_{kn}(x' - x)^{\frac{n}{2}} \left(\sum_{p \in \mathcal{P}} (v_{p} - u_{p})\right)^{\frac{n-1}{n}},
\]

where $(u_{p}, v_{p})$ ($p \in \mathcal{P} \subset \mathbb{N}$) are all the intervals contiguous to $K$ in $[x, x']$, and $C_{kn} = k^{\frac{1}{n}}(2n)^{\frac{n-1}{n}}$.

**Proof.** Without any loss of generality, assume that $x \in A$ (if $x' \in A$, then work with $f(-\cdot)$ instead). By Lemma 2.10 we have

\[(3.13) \quad f^{(i)}(x) = 0 \quad \text{for all } i = 1, \ldots, n - 1.
\]

Let

\[(3.14) \quad ([c_j, d_j])_{j=1}^J \text{ be non-overlapping intervals with } c_j, d_j \in K \cap [x, x'] \text{ for all } j \in \{1, \ldots, J\}.
\]
Assume first that $\#(K \cap [x, x']) < 2n + 1$. Then
\[
|f(c_j) - f(d_j)| \leq \int_{c_j}^{d_j} |f'(s)| \, ds \leq \int_{c_j}^{d_j} |f^{(n-1)}(\xi_{n-1})|(x' - x)^{n-2} \, ds
\]
\[\leq k(x' - x)^n,
\]
where $\xi_i = \xi_i(s)$ is chosen inductively (using (3.13)) such that $\xi_1 = s$, and
\[
|f^{(i-1)}(\xi_{i-1})| = |f^{(i-1)}(\xi_{i-1}) - f^{(i-1)}(x)| = |f^{(i)}(\xi_i)||\xi_{i-1} - x|
\]
for $i = 2, \ldots, n - 1$. We obtain $|f(c_j) - f(d_j)|^{n-1} \leq k^{1/n-1}(x' - x)^{n-1}$, and it follows that
\[
\sum_{j=1}^{J} |f(c_j) - f(d_j)|^{n-1} \leq k^{1/n-1} \sum_{j=1}^{J} (x' - x)^{n-1}
\]
\[\leq k^{1/n-1}(x' - x)^{1/n-1}(2n)\left(\sum_{p \in \mathcal{P}} (v_p - u_p)\right),
\]
where we used that $x' - x = \sum_{p \in \mathcal{P}} (v_p - u_p)$ and $J \leq 2n$ since $\#(K \cap [x, x']) < 2n + 1$. Thus
\[
\left(\sum_{j=1}^{J} |f(c_j) - f(d_j)|^{n-1}\right)^{\frac{n}{n-1}} \leq k^{\frac{1}{n}}(2n)^{\frac{n}{n-1}}(x' - x)^{\frac{1}{n}}\left(\sum_{p \in \mathcal{P}} (v_p - u_p)\right)^{\frac{n-1}{n}},
\]
and (3.12) holds in this case.

Now assume that $\#(K \cap [x, x']) \geq 2n + 1$. Let $([c_j, d_j])_{j=1}^{J}$ be as in (3.14). Since $\lambda(f(K)) = 0$ by Lemma 2.1, Lemma 2.5 implies that
\[
|f(d_j) - f(c_j)| \leq V(f, [c_j, d_j]) = \sum_{p \in \mathcal{P}_j} V(f, [\gamma_p^j, \delta_p^j]) = \sum_{p \in \mathcal{P}_j} |f(\delta_p^j) - f(\gamma_p^j)|,
\]
where we used the assumptions in the last equality, and where $(\gamma_p^j, \delta_p^j)$ $(p \in \mathcal{P}_j \subset \mathbb{N})$ are all the intervals contiguous to $K \cap [c_j, d_j]$ in $[c_j, d_j]$. By adding (3.15) for $j = 1, \ldots, J$, and using the subadditivity of $g(t) = t^{1/n-1}$ for $t \geq 0$, we obtain
\[
\sum_{j=1}^{J} |f(c_j) - f(d_j)|^{n-1} \leq \sum_{p \in \mathcal{P}} |f(v_p) - f(u_p)|^{n-1};
\]
thus the conclusion of the lemma will follow from (3.16) once we establish
\[
\left(\sum_{p \in \mathcal{P}} |f(v_p) - f(u_p)|^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} \leq C_{kn}(x' - x)^n\left(\sum_{p \in \mathcal{P}} (v_p - u_p)\right)^{\frac{n}{n-1}}.
\]
In the rest of the proof, we will prove (3.17). We need the following definition. If \((\alpha, \beta)\) is an interval contiguous to \(K \cap [x, x']\) in \([x, x']\), then put \(r_1(\beta) := \beta\), and

\[
  r_i(\beta) := \inf\{t \in [r_{i-1}(\beta), x'] : \#(K \cap [\beta, t]) \geq i \text{ or } t = x'\}
\]

for \(i = 2, \ldots, n - 2\). Similarly, define \(l_1(\alpha) := \alpha\), and

\[
  l_i(\alpha) := \sup\{t \in [x, l_{i-1}(\alpha)] : \#(K \cap [t, \alpha]) \geq i \text{ or } t = x\}
\]

for \(i = 2, \ldots, n - 2\). We have the following easy observation:

\((*)\) for each interval \((\alpha, \beta)\) contiguous to \(K \cap [x, x']\) in \([x, x']\) and each \(i \in \{1, \ldots, n - 2\}\) there exists \(w_i \in [l_i(\alpha), r_i(\beta)]\) such that \(f^{(i)}(w_i) = 0\).

We will now prove \((*)\). If \(i = 1\), then take \(w_1 = \alpha\) and \((*)\) follows. Suppose that \(1 < i < n - 1\). If \([l_i(\alpha), r_i(\beta)] \cap K' \neq \emptyset\), then \((*)\) follows by Lemma 2.10, otherwise \(\#([l_i(\alpha), r_i(\beta)] \cap K) \geq i + 1\) by the choice of \(l_i(\alpha), r_i(\beta),\) and \((*)\) follows from Lemma 2.9.

Fix \(p \in \mathcal{P}\), and suppose that \(s \in [u_p, v_p]\). Then \(|f'(s)| = |f'(s) - f'(u_p)| = |f''(\xi_1)| \cdot |s - u_p|\), and by induction for \(k = 2, \ldots, n - 2\), choose \(\xi_k \in [l_k(u_p), r_k(v_p)]\) so that

\[
  (3.18) \quad |f^{(k)}(\xi_{k-1})| = |f^{(k)}(\xi_{k-1}) - f^{(k)}(w_k)| = |f^{(k+1)}(\xi_k)| \cdot |\xi_{k-1} - w_k|,
\]

where \(w_k\) is chosen by applying \((*)\) to \((u_p, v_p)\). Using (3.18), we obtain

\[
  |f'(s)| \leq |f^{(n-1)}(\xi_{n-2})| \cdot \prod_{l=1}^{n-2} (r_l(v_p) - l_l(u_p)).
\]

Thus we obtain

\[
  |f(u_p) - f(v_p)| \leq \int_{u_p}^{v_p} |f'(s)| ds \leq \int_{u_p}^{v_p} |f^{(n-1)}(\xi_{n-2})| \prod_{l=1}^{n-2} (r_l(v_p) - l_l(u_p)) ds \leq k(v_p - x)(r_{n-2}(v_p) - l_{n-2}(u_p))^{n-2}(v_p - u_p).
\]

From this we get

\[
  |f(u_p) - f(v_p)| \leq (k(v_p - x))^{1 \over n-1} (r_{n-2}(v_p) - l_{n-2}(u_p))
\]

for each \(p \in \mathcal{P}\), and thus

\[
  \left(\sum_{p \in \mathcal{P}} |f(u_p) - f(v_p)| \right)^{1 \over n-1} \leq k^{1 \over n} (x' - x)^{1 \over n} \left(\sum_{p \in \mathcal{P}} (r_{n-2}(v_p) - l_{n-2}(u_p)) \right)^{1 \over n},
\]

but since \(\sum_{p \in \mathcal{P}} (r_{n-2}(v_p) - l_{n-2}(u_p)) \leq 2n \cdot \sum_{p \in \mathcal{P}} (v_p - u_p)\), where \((u_p, v_p)\)

\((p \in \mathcal{P} \subset \mathbb{N})\) are all the intervals contiguous to \(K\) in \([x, x']\), we obtain

\[
  \left(\sum_{p \in \mathcal{P}} |f(v_p) - f(u_p)| \right)^{1 \over n-1} \leq k^{1 \over n} (2n)^{1 \over n} (x' - x)^{1 \over n} \left(\sum_{p \in \mathcal{P}} (v_p - u_p) \right)^{1 \over n-1}.
\]
Thus, (3.17) and also (3.12) follow. \qed

The following lemma contains a sufficient condition guaranteeing that a function belongs to the classes \( \overline{SBV}_{1/n} \) and \( CBV_{1/n} \).

**Lemma 3.9.** Suppose that \( f : [a, b] \to \mathbb{R} \) is \((n-1)\)-times differentiable (for \( n \geq 2 \)) so that \( f^{(n-1)}(\cdot) \) is pointwise-Lipschitz. Then \( f \) is both \( \overline{SBV}_{1/n} \) and \( CBV_{1/n} \).

**Proof.** First, note that \( V_{\frac{1}{n}}(f, K_f) < \infty \) by [11, Remark 3.6] since \( f \) is \( C^{n-1} \) by the assumption. Since \( K_f \subset \{ x \in [a, b] : f'(x) = 0 \} \), we have that \( f^{(n-1)}(x) = 0 \) for \( x \in K'_f \) by Lemma 2.10. Denote
\[
B^k_m := \{ x \in K'_f : |f^{(n-1)}(y)| \leq k|y - x| \text{ for all } y \in B(x, 1/m) \}.
\]

It is easy to see that each \( B^k_m \) is closed, \( \bigcup_{k,m} B^k_m = K'_f \), and \( B^k_m \subset B^{k+1}_m \).

Let \( \delta^k_m := \frac{1}{2m} \).

First, we will show that \( \lim_{m \to \infty} \overline{GV}_{1/n}^{\delta^k_m}(f, B^k_m, K_f) = 0 \). Fix \( k, m \in \mathbb{N} \) with \( B^k_m \neq \emptyset \). Let the collection \( ([x_i, y_i])_{i=1}^N \) as in (3.1) for \( A = B^k_m \), \( K = K_f \) and \( \delta = \delta^k_m \) in the definition of \( \overline{GV}_{1/n}^{\delta^k_m}(f, B^k_m, K_f) \). Let
\[
\mathcal{P}_i := \{ p \in \mathcal{P} : (u_p, v_p) \subset [x_i, y_i] \},
\]
where \( (u_p, v_p) \ (p \in \mathcal{P} \subset \mathbb{N}) \) are all the intervals contiguous to \( K_f \) in \([a, b] \). Applying Lemma 3.8 to \( [x, x'] = [x_i, y_i] \) for a fixed \( i = 1, \ldots, N \), summing over \( i \in \{1, \ldots, N\} \), and using Hölder’s inequality (with exponents \( p = n \) and \( p' = \frac{n}{n-1} \)), we obtain
\[
\begin{align*}
\sum_{i=1}^N \left( V_{\frac{1}{n}}(f, K_f \cap [x_i, y_i]) \right)^{\frac{n-1}{n}} &\leq C_{kn} \sum_{i=1}^N (y_i - x_i)^{\frac{1}{p}} \left( \sum_{p \in \mathcal{P}_i} (v_p - u_p) \right)^{\frac{n-1}{n}} \\
&\leq C_{kn} (b-a)^{\frac{1}{p}} \left( \sum_{i=1}^N \sum_{p \in \mathcal{P}_i} (v_p - u_p) \right)^{\frac{n-1}{n}} \\
&\leq C_{kn} (b-a)^{\frac{1}{p}} \left( \sum_{v_p - u_p \leq \delta^k_m} (v_p - u_p) \right)^{\frac{n-1}{n}},
\end{align*}
\]
and since we have \( \lim_{m \to \infty} \delta^k_m = 0 \) for a fixed \( k \in \mathbb{N} \), by (3.20) we obtain that
\[
\lim_{m \to \infty} \overline{GV}_{1/n}^{\delta^k_m}(f, B^k_m, K_f) = 0,
\]
for each \( k \in \mathbb{N} \).
Index $K_f \setminus K'_f$ as $\{x_j\}_{j \in J}$, where $J \subset \mathbb{N}$ and each $j \in J$ is even. Now, define $A_m^{2k+1} := B_m^{2k+1}$, and take $\tilde{\delta}_m^{2k+1} := \delta_m^{2k+1}$. For $k \in J$ define $A^k := \{x_k\}$ for all $m \in \mathbb{N}$. For each $k \in J$ find $\gamma_k > 0$ such that $B(x_k, \gamma_k) \setminus \{x_k\} \cap K_f = \emptyset$ and put $\delta^k_m := \min(\gamma_k, 1/m)$. For $k \in \mathbb{N} \setminus J$ which are even, put $A^k := \emptyset$ for all $m \in \mathbb{N}$ and $\tilde{\delta}^k := 1/m$. Using (3.21), it is easy to see that $f$ satisfies Definition 3.5 with $A^k_m$ and $\tilde{\delta}^k_m$.

To show that $f$ is $CBVG_{1/n}$, write each $B^k_m$ as $B^k_m = \bigcup_i B^k_{ml}$, where each $B^k_{ml}$ is closed, and $\text{diam}(B^k_{ml}) < 1/m$. Fix $k, m, l \in \mathbb{N}$ such that $B^k_{ml} \neq \emptyset$. Let $([x_i, y_i])_{i=1}^N$, $x_i, y_i \in B^k_{ml}$, be as in (3.1) for $f$, $A = B^k_{ml}$, $K = K_f$ and the definition of $GV_{1/n}(f, B^k_{ml}, K_f)$. If $i \in \{1, \ldots, N\}$, then Lemma 3.8 applied to $[x, x'] = [x_i, y_i]$ shows that

$$
(3.22) \quad \left( \frac{V_{n-1}(f, [x_i, y_i] \cap K_f)}{n} \right)^{n-1} \leq C_{k,n} (y_i - x_i)^{n/2} \left( \sum_{i \in P_i} (v_p - u_p) \right)^{n/2-n}
\leq C_{k,n} (y_i - x_i),
$$

where $P_i$ is defined as in (3.19). By summing over $i \in \{1, \ldots, N\}$ in (3.22), we obtain

$$
(3.23) \quad \sum_{i=1}^N \left( \frac{V_{n-1}(f, [x_i, y_i] \cap K_f)}{n} \right)^{n-1} \leq C_{k,n} \sum_{i=1}^N (y_i - x_i) \leq C_{k,n} (b - a) < \infty.
$$

We proved that $GV_{1/n}(f, B^k_{ml}, K_f) < \infty$ (for each $k, m, l \in \mathbb{N}$). If we reorder the sequence $(B^k_{ml})_{k,m,l}$ together with the sequence $\{\{x\}\}_{x \in K_f \setminus K'_f}$ into a single sequence which we call $A_m$ (while omitting the empty sets), where $m \in M \subset \mathbb{N}$, by (3.23) we see that $f$ is $CBVG_{1/n}$. \hfill \Box

The following lemma will allow us to construct certain variations, which play a key rôle in establishing differentiability.

**Lemma 3.10.** Let $f : [a, b] \to \mathbb{R}$ be continuous, $\emptyset \neq A \subset K \subset [a, b]$ be closed sets, $\{a, b\} \subset K$, and $\delta > 0$. Suppose that $G\!V^\delta_{1/n}(f, A, K) < \infty$. Then there exists a continuous increasing function $v$ on $[a, b]$ with $v(a) = 0$, $v(b) \leq G\!V^\delta_{1/n}(f, A)$, and such that for $x \in A$ and $y, z \in K$ with $x \leq y < z < x + \delta$ we have

$$
(3.24) \quad |f(y) - f(z)| \leq n^{n-1}(v(z) - v(y))^{n-1}(v(z) - v(x)).
$$

If $f$ has bounded variation, and $V(f, [\alpha, \beta]) = |f(\beta) - f(\alpha)|$ whenever $(\alpha, \beta)$ is an interval contiguous to $K$, then $\lambda(v(K)) = 0$.

**Proof.** Define $v(x) := rG\!V^\delta_n(f, A \cap [a, x], K \cap [a, x])$ for $x \in K$. The continuity of $v$ on $K$ follows from Lemma 3.2. Now, extend $v$ to a continuous
function on $[a, b]$ such that $v$ is continuous and affine on each $[\alpha, \beta]$, whenever $(\alpha, \beta)$ is an interval contiguous to $K$ in $[a, b]$.

To prove (3.24), let $x \leq y < z < x + \delta$ where $x \in A$, and $y, z \in K$. By continuity, there is no loss of generality in assuming that $x < y$. Fix $\varepsilon_0 > 0$. Choose $([x_i, y_i])_{i=1}^N$, and $((c^i_j, d^i_j))_{j=1}^J$ such that

$$v(y) = v(x) + \sum_{i=1}^N \left( \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_{j-1})| \right)^{\frac{n-1}{n}} + \varepsilon,$$

where $0 \leq \varepsilon < \varepsilon_0$. This can be done because $x \in A$ (see (3.2)). We can also assume that $([x_i, y_i])$ and $((c^i_j, d^i_j))$ are ordered in the natural sense (see the remark after (3.6)). Then

$$v(z) \geq v(x) + \sum_{i=1}^{N-1} \left( \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_{j-1})| \right)^{\frac{n-1}{n}} + \left( \sum_{j=1}^{J_N} |f(d^N_j) - f(c^N_{j-1})| \right)^{\frac{n-1}{n}} + |f(y) - f(z)|^{\frac{1}{n-1}}.$$

To simplify the notation, put $b := \sum_{j=1}^{J_N} |f(d^N_j) - f(c^N_{j-1})|^{\frac{1}{n-1}}$, and $a := b + |f(y) - f(z)|^{\frac{1}{n-1}}$. Because of the algebraic identity

$$u^{\frac{n-1}{n}} - w^{\frac{n-1}{n}} = (u - w) \cdot \frac{\sum_{i=0}^{n-2} u^i w^{n-2-i} - \sum_{i=0}^{n-1} u^i w^{n-1-i}}{\sum_{i=0}^{n-1} u^i w^{n-1-i}},$$

which is easily seen to be valid for all $u, w \geq 0$ with $u + w > 0$, we obtain

$$v(z) - v(y) \geq a^{\frac{n-1}{n}} - b^{\frac{n-1}{n}} - \varepsilon = (a - b) \cdot \frac{\sum_{i=0}^{n-2} a^i b^{n-2-i} - \sum_{i=0}^{n-1} a^i b^{n-1-i}}{\sum_{i=0}^{n-1} a^i b^{n-1-i}} - \varepsilon.$$

Because $a \geq b$, we obtain $\sum_{i=0}^{n-1} a^i b^{n-1-i} \leq na^{n-1}$, and this together with the inequality $v(z) - v(x) \geq a^{\frac{n-1}{n}}$ implies

$$v(z) - v(y) \geq |f(y) - f(z)|^{\frac{1}{n-1}} \cdot a^{\frac{n-2}{n-1}} \cdot \frac{a^{\frac{n-1}{n}}}{na^{n-1}} - \varepsilon \geq \frac{|f(y) - f(z)|^{\frac{1}{n-1}}}{n(v(z) - v(x))^{\frac{n-1}{n}}} - \varepsilon.$$

To finish the proof of (3.24), send $\varepsilon_0 \to 0$.

Now, suppose that $f$ has bounded variation and $V(f, [\alpha, \beta]) = |f(\beta) - f(\alpha)|$ whenever $(\alpha, \beta)$ is an interval contiguous to $K$. We will show that
\[ v(b) - v(a) \leq \sum_{p \in P} (v(d_p) - v(c_p)). \]

To prove (3.25), fix \( \varepsilon_0 > 0 \), and let \((x_i, y_i)\) be non-overlapping intervals as in Definition 3.1 for \( rG_{1/n}^V(f, A, K) \) such that

\[ v(b) - v(a) = \sum_{i=1}^{N} \left( V_{\frac{1}{n}}(f, K \cap [x_i, y_i]) \right)^{\frac{n-1}{n}} + \varepsilon, \]

for some \( 0 \leq \varepsilon < \varepsilon_0/3 \). Now for each \( i = 1, \ldots, N \), find non-overlapping intervals \( (c^i_j, d^i_j) \) such that \( c^i_j, d^i_j \in K \cap [x_i, y_i] \) and

\[ V_{\frac{1}{n}}(f, K \cap [x_i, y_i]) \leq \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_j)|^{\frac{1}{n-1}} + \left( \frac{\varepsilon_0}{3N} \right)^{\frac{n}{n-1}}. \]

For \( i \in \{1, \ldots, N\} \), we can assume that \( d^i_j \leq c^i_{j+1} \) for \( j = 1, \ldots, J_i - 1 \). For a moment, fix \( i \in \{1, \ldots, N\} \). By splitting and regrouping the intervals \( (c^i_j, d^i_j) \), we can assume that there is a sequence of finite families of intervals \( A^i_k \), \( k = 1, \ldots, K_i \), and points \( a^i_k \in A \cap [x_i, y_i] \) such that if \((\alpha, \beta) \in A^i_k \), then \( \alpha, \beta \in K \cap [x_i, y_i] \), if \((\sigma, \tau) \in A^i_k \) where \( k < l \), then \( \alpha < \beta \leq a^i_l \leq \sigma < \tau \),

\[ \sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_j)|^{\frac{1}{n-1}} \leq \sum_{k=1}^{K_i} \sum_{(\alpha, \beta) \in A^i_k} |f(\beta) - f(\alpha)|^{\frac{1}{n-1}}, \]

and

\( (c^i_j, d^i_j) \subset \bigcup \{(\alpha, \beta) : (\alpha, \beta) \in A^i_k, k = 1, \ldots, K_i\}. \)

By Lemma 2.5 applied to \( f \) on \([a, b] = [\alpha, \beta] \) for \( (\alpha, \beta) \in A^i_k \), and \( B = (A \cup \{\alpha, \beta\}) \cap [\alpha, \beta] \) (note that \( \lambda(f(A)) = 0 \) since \( \lambda(f(K)) = 0 \), and thus \( \lambda(f(B)) = 0 \)), let \((\alpha^\alpha_l, \beta^\alpha_l) \) \((l \in \{1, \ldots, L^i_k\}) \) be a finite collection of intervals contiguous to \((A \cup \{\alpha, \beta\}) \cap [\alpha, \beta] \) in \([\alpha, \beta] \) such that

\[ |f(\beta) - f(\alpha)| \leq V(f, [\alpha, \beta]) \leq \sum_{l=1}^{L^i_k} V(f, [\alpha^\alpha_l, \beta^\alpha_l]) + \left( \frac{\varepsilon_0}{3NK_i |A^i_k|} \right)^n. \]

On the set \( \{(\alpha^\alpha_l, \beta^\alpha_l) : l = 1, \ldots, L^i_k, k = 1, \ldots, K_i, (\alpha, \beta) \in A^i_k\} \) define an equivalence relation \(~\) in the following way: \( I \sim J \) whenever \((\max I, \min J) \cap A = \emptyset \) and \((\max J, \min I) \cap A = \emptyset \) (note that one of the
conditions always holds). By $B^i_q (q = 1, \ldots, Q_i)$ denote the equivalence classes of $\sim$. We have

$$
\sum_{k=1}^{K_i} \sum_{(\alpha,\beta) \in A^i_k} |f(\beta) - f(\alpha)|^{\frac{1}{n-1}} 
\leq \sum_{k=1}^{K_i} \sum_{(\alpha,\beta) \in A^i_k} L_{ik}(V(f, [\alpha^i, \beta^i]))^{\frac{1}{n-1}} + \left(\frac{\varepsilon_0}{3N}\right)^{\frac{n}{n-1}}
$$

(3.29)

$$
\leq \sum_{q=1}^{Q_i} \sum_{(\eta,\theta) \in B^i_q} (V(f, [\eta, \theta]))^{\frac{1}{n-1}} + \left(\frac{\varepsilon_0}{3N}\right)^{\frac{n}{n-1}}.
$$

(3.30)

By Lemma 2.5 and the assumptions, we obtain

$$
\sum_{(\eta,\theta) \in B^i_q} (V(f, [\eta, \theta]))^{\frac{1}{n-1}} \leq \sum_{\xi \in \Xi^i_q} |f(\omega_\xi) - f(\Omega_\xi)|^{\frac{1}{n-1}} \leq V_{1-n}(f, [\tau^i_q, T^i_q]),
$$

where $(\omega_\xi, \Omega_\xi)$ (for $\xi \in \Xi^i_q \subset \mathbb{N}$) are all the intervals contiguous to $K \cap [\tau^i_q, T^i_q]$ for $\tau^i_q = \inf_{x \in I \in B^i_q} x$, and $T^i_q = \sup_{x \in I \in B^i_q} x$. By putting the inequalities (3.26), (3.27), (3.28), (3.29) and (3.30), we obtain

$$
v(b) - v(a) \leq \sum_{i=1}^{N} \left(\sum_{j=1}^{J_i} |f(d^i_j) - f(c^i_j)|^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} + \frac{2\varepsilon_0}{3}

\leq \sum_{i=1}^{N} \left(\sum_{k=1}^{K_i} \sum_{(\alpha,\beta) \in A^i_k} |f(\beta) - f(\alpha)|^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} + \frac{2\varepsilon_0}{3}

\leq \sum_{i=1}^{N} \sum_{q=1}^{Q_i} \left(\sum_{(\eta,\theta) \in B^i_q} (V(f, [\eta, \theta]))^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} + \varepsilon_0

\leq \sum_{i=1}^{N} \sum_{q=1}^{Q_i} \left(V_{1-n}(f, [\tau^i_q, T^i_q] \cap K)\right)^{\frac{n-1}{n}} + \varepsilon_0.
$$

(3.31)

Since $(\tau^i_q, T^i_q) \cap A = \emptyset$, for $x \in [\tau^i_q, T^i_q] \cap K$ we have

$$
v(x) = v(z) + \left(V_{1-n}(f, K \cap [z, x])\right)^{\frac{n-1}{n}},
$$

where $z = \max(A \cap [a, \tau^i_q])$, Lemma 2.7 shows that $\lambda(v([\tau^i_q, T^i_q] \cap K)) = 0$. Now, Lemma 2.5 applied to $\zeta(x)$ (where $\zeta(x) = \left(V_{1-n}(f, K \cap [z, x])\right)^{\frac{n-1}{n}}$ for $x \in [\tau^i_q, T^i_q] \cap K$, and $\zeta$ is continuous and affine on intervals contiguous
to \( [\tau_q^i, T_q^i] \cap K \) implies that

\[
\left( \frac{V}{n-1} (f, [\tau_q^i, T_q^i] \cap K) \right)^n = v(T_q^i) - v(\tau_q^i) \leq \sum_{p \in P_q^i} (v(d_p) - v(c_p)),
\]

where \( P_q^i = \{ p \in P : (c_p, d_p) \subset [\tau_q^i, T_q^i] \} \). Combining this inequality with (3.31), we get \( v(b) - v(a) \leq \sum_{p \in P} (v(d_p) - v(c_p)) + \varepsilon_0 \), and by sending \( \varepsilon_0 \to 0 \) it follows that (3.25) holds. Since \( v(K) \cap v(\bigcup_{p \in P} (c_p, d_p)) \) is countable, we have \( \lambda(v(K)) = v(b) - v(a) - \lambda(\bigcup_{p \in P} (c_p, d_p)) = 0 \). \( \square \)

By a symmetric argument (this time defining \( \tilde{v}(x) := lGV_n^\delta(f, A \cap [x, b]) \)), we obtain the following:

**Lemma 3.11.** Let \( f : [a, b] \to \mathbb{R} \) be continuous, \( \emptyset \neq A \subset K \subset [a, b] \) be closed sets, \( \{a, b\} \subset K, \) and \( \delta > 0 \). Suppose that \( G_{1/n}^\delta(f, A, K) < \infty \). Then there exists a continuous decreasing function \( \tilde{v} \) on \( [a, b] \) with \( \tilde{v}(b) = 0 \), \( \tilde{v}(a) \leq G_{1/n}^\delta(f, A, K) \), and such that for \( x \in A \) and \( y, z \in K \) with \( x - \delta < z < y \leq x \) we have

\[
|f(y) - f(z)| \leq n^{n-1}(\tilde{v}(z) - \tilde{v}(y))^{n-1}(\tilde{v}(z) - \tilde{v}(x)).
\]

If \( \{a, b\} \subset K, \) \( f \) has bounded variation, and \( V(f, [\alpha, \beta]) = |f(\beta) - f(\alpha)| \) whenever \( (\alpha, \beta) \) is an interval contiguous to \( K \), then \( \lambda(\tilde{v}(K)) = 0 \).

We have the following proposition.

**Proposition 3.12.** Let \( f : [a, b] \to \mathbb{R} \) be a \( SBV G_{1/n} \) function. Then \( f \) is Lebesgue equivalent to an \( n \)-times differentiable function \( \varphi \) such that \( \varphi^{(i)}(x) = 0 \) whenever \( i \in \{1, \ldots, n\} \) and \( x \in K_\varphi \). Also, \( \varphi'(x) \neq 0 \) whenever \( x \in [a, b] \setminus K_\varphi \).

If \( f \) is not constant on any interval, then \( \lambda(K_\varphi) = 0 \).

**Proof.** For a moment, assume that the function \( f \) is not constant in any interval. Lemma 2.6 shows that \( V(f, [a, b]) < \infty \). Then put \( g := f \circ v_f^{-1} : [0, \ell] \to \mathbb{R} \) (where \( \ell := v_f(b) \)). Since \( \lambda(f(K_f)) = 0 \) by Lemma 2.4, and \( v_f(K_f) = K_g \), by Lemma 2.5 we have that

\[
\ell = V(f, [a, b]) = \sum_{p \in P} V(g, [u_p, v_p]) = \sum_{p \in P} (v_p - u_p),
\]

where \( (u_p, v_p) \) \((p \in P \subset \mathbb{N})\) are all the intervals contiguous to \( K_g \) in \([0, \ell] \), and thus \( \lambda(K_g) = \ell - \lambda(\bigcup_{p \in P} (u_p, v_p)) = 0 \). Putting \( G(t) = g(\frac{t}{v_f(b)} \cdot (t - a)) \), \( t \in [a, b] \), we can assume that \( f \) satisfies \( \lambda(K_f) = 0 \) (since \( f \) is clearly Lebesgue equivalent to \( G \)) provided \( f \) is not constant in any interval.
Let \((A_m)_m\) and \((\delta_m)_m \subset \mathbb{R}_+ \setminus \{0\}\) be the sequences from Definition 3.4 for \(f\). Find a monotone sequence \((m_j)_{j \in \mathbb{N}} \subset \mathbb{N}\) such that \(\lim_{j \to \infty} m_j = \infty\), and

\[
\sum_j j \cdot \overline{GV}_{1/j}(f, A_{m_j}, K_f) < \infty. \tag{3.34}
\]

Relabel \((A_{m_j})_j\) as \((A_m)_m\), and \((\delta_{m_j})_j\) as \((\delta_m)_m\). Then by (3.34) we have \(\sum_m m \cdot \overline{GV}_{1/m}(f, A_m, K_f) < \infty\). For \(x \in [a, b]\) define

\[
v(x) := x + \sum_m m(v_m(x) - \tilde{v}_m(x)),
\]

where \(v_m(x)\) (resp. \(\tilde{v}_m(x)\)) are the functions \(v\) (resp. \(\tilde{v}\)) obtained by applying Lemma 3.10 (resp. Lemma 3.11) to \(f, K = K_f, A = A_m,\) and \(\delta = \delta_m\). Note that \(v : K_f \to [c, d]\) is a continuous strictly increasing function, which is onto \([c, d]\), where \(c = v(a), d = v(b)\).

In case that \(f\) is not constant on any intervals, by Lemmata 3.10 and 3.11, we have that \(\lambda(v_m(K_f)) = 0, \lambda(\tilde{v}_m(K_f)) = 0\) for each \(m \in \mathbb{N}\). Also, \(\lambda(K_f) = 0,\) and thus Lemma 2.2 shows that

\[
\lambda(v(K_f)) = 0. \tag{3.35}
\]

For \(x \in K_f,\) we will show that for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(x \leq y < z < x + \delta\) or \(x - \delta < z < y \leq x\), and \(y, z \in K_f,\) then

\[
|f(y) - f(z)| \leq \varepsilon|v(z) - v(y)|^{n-1}|v(z) - v(x)|. \tag{3.36}
\]

To prove (3.36), fix \(x \in K_f,\) and \(\varepsilon > 0\). Find \(m_0 \in \mathbb{N}\) such that \(x \in A_{m_0}\) (and thus \(x \in A_m\) for all \(m \geq m_0\)), and pick \(m > m_0\) such that \(\frac{n^{n-1}}{m^n} < \varepsilon\).

Define \(\delta := \delta_m\). Let \(y, z \in K_f\) be such that \(x < y < z < x + \delta\). Then (3.24) implies that \(|f(y) - f(z)| \leq n^{n-1}(v_m(z) - v_m(y))^{n-1}(v_m(z) - v_m(x))\). But since \(m(v_m(\tau) - v_m(\sigma)) \leq v(\tau) - v(\sigma)\) for all \(a \leq \sigma < \tau \leq b\), we obtain

\[
m^a|f(y) - f(z)| \leq n^{n-1}(v(z) - v(y))^{n-1}(v(z) - v(x)).
\]

By the choice of \(m\) we have \(|f(y) - f(z)| \leq \varepsilon(v(z) - v(y))^{n-1}(v(z) - v(x))\).

By continuity, the above argument shows that (3.36) holds also for \(y, z \in K_f\) such that \(x = y < z < x + \delta\). Finally, by using (3.33) (instead of (3.24)) in the previous argument, we obtain (3.36) for \(x - \delta < z < y \leq x\) with \(y, z \in K_f\).

We will define \(F : [c, d] \to \mathbb{R}\) as

\[
F(x) := \begin{cases} 
  f \circ v^{-1}(x) & \text{for } x \in v(K_f), \\
  H_{\alpha, \beta}(x) & \text{for } x \in (\alpha, \beta),
\end{cases} \tag{3.37}
\]

whenever \((\alpha, \beta)\) is an interval contiguous to \(v(K_f)\) in \([c, d]\), and \(H = H_{\alpha, \beta}\) is is chosen by applying Lemma 2.11 to \(\alpha, \beta, A = f \circ v^{-1}(\alpha), B = f \circ v^{-1}(\beta)\). It
follows that $F$ is Lebesgue equivalent to $f$, and $F$ is $n$-times differentiable at all $x \in [c, d] \setminus v(K_f)$ (by Lemma 2.11). To prove that $F$ is $n$-times differentiable, it remains to show that $F^{(i)}(x) = 0$ for all $x \in v(K_f)$, $i = 1, \ldots, n$.

Now, (3.36) implies that for each $x \in v(K_f)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| F(y) - F(z) \right| \leq \varepsilon |y - z|^{-1} |x - z|,$$

whenever $x \leq y < z < x + \delta$ or $x - \delta < z < y \leq x$, and $y, z \in v(K_f)$. Fix $x \in v(K_f)$. First, we will show that for each $x \in v(K_f)$ we have that

$$|F^{(n-1)}(t)| \leq \varepsilon |t - x| \text{ holds for all } t \in [c, d] \setminus v(K_f) \text{ with } |x - t| < \delta.$$

To prove (3.39), if $x$ is not a right-hand-side accumulation point of the set $v(K_f)$, then (3.39) follows from (3.37) and the fact that $H^{(n)}_{\alpha, \beta}(x) = 0$ (where $x = \alpha$, and $(\alpha, \beta)$ is the corresponding interval contiguous to $v(K_f)$). If $x$ is a right-hand-side accumulation point of $v(K_f)$, let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that (3.38) holds and with $x + \delta \in v(K_f)$. Let $(\alpha, \beta)$ be an interval contiguous to $v(K_f)$ in $[c, d]$ with $(\alpha, \beta) \subset [x, x + \delta]$, and let $t \in (\alpha, \beta)$ (the case $t < x$ being treated symmetrically). Let $l_1(\alpha) := \xi(x - \alpha)$ for $x \in [\alpha, \alpha + \beta/2]$, and $l_2(x) := \xi(\beta - x)$ for $x \in [\alpha + \beta/2, \beta]$, where $\xi = C \cdot |f(\beta - x) - f(x)|/(\beta - \alpha)^n$, and $C = C_{\alpha, \beta}$ comes from condition (iii) of Lemma 2.11 applied on $[\alpha, \beta]$. By (3.38), we have that

$$l_i \left( \frac{\alpha + \beta}{2} \right) = C \cdot \frac{|f \circ v^{-1}(\beta) - f \circ v^{-1}(\alpha)|}{(\beta - \alpha)^n} \cdot \frac{\beta - \alpha}{2} = C \cdot \frac{|f \circ v^{-1}(\beta) - f \circ v^{-1}(\alpha)|}{(\beta - \alpha)^n} \leq C \cdot \varepsilon \left( \frac{\alpha + \beta}{2} - \alpha \right),$$

for $i = 1, 2$, and this inequality together with the equalities $l_1(\alpha) = l_2(\beta) = 0$, and condition (iii) from Lemma 2.11 easily implies that $|F^{(n-1)}(t)| = |H^{(n-1)}_{\alpha, \beta}(t)| \leq \min(l_1(t), l_2(t)) \leq C \varepsilon |t - x|$ for all $t \in (\alpha, \beta)$. To see this, we use the fact that if two continuous affine functions $a_1, a_2 : [a, b] \to \mathbb{R}$ satisfy $a_1(a) \leq a_2(a)$, and $a_1(b) \leq a_2(b)$, then $a_1(t) \leq a_2(t)$ for all $t \in [a, b]$.

We apply this fact to $a_1(t) = l_1(t)$ for $t \in \left[ \alpha, \alpha + \frac{\beta}{2} \right]$ (resp. $a_1(t) = l_2(t)$ for $t \in \left[ \alpha + \frac{\beta}{2}, \beta \right]$), and $a_2(x) = C \varepsilon |t - x|$. Similarly for $(\alpha, \beta) \subset [x - \delta, x]$.

Let $\varepsilon > 0$, and let $\delta > 0$ be as in (3.39). It follows easily by induction (using (3.39)) that

$$|F^{(i)}(t)| \leq C \varepsilon |t - x| \quad \text{for all } t \in (x, x + \delta) \setminus v(K_f) \text{ and } i = 1, \ldots, n - 1.$$
To show (3.40), let $t' := \max(v(K_f) \cap [x, t])$. Then (using the Mean Value Theorem and the fact that on $[t', t]$ it holds that $F = H_{\alpha, \beta}$ for some $\alpha, \beta$) we obtain $|F^{(i)}(t)| = |F^{(i)}(t) - F^{(i)}(t')| \leq |F^{(i+1)}(\xi_{i+1})| \cdot |t - t'| \leq \cdots \leq |F^{(n)}(\xi_{n-1})| \cdot |t - t'|^{n-1}$, where $\xi_j \in [t', t]$, and (3.40) easily follows.

Using (3.38), (3.40), and the fact that $F^{(i)}(\alpha) = H^{(i)}_{\alpha, \beta}(\alpha) = 0$ for $i = 1, \ldots, n$, by a simple induction argument we obtain that

\begin{equation}
F^{(i)}(x) = 0 \quad \text{for all } x \in v(K_f), \quad i = 1, \ldots, n-1.
\end{equation}

To prove this, note that the case $i = 1$ follows directly from (3.38). For $i = 2$, $F'(t) - F'(x) = 0$ provided $t \in v(K_f)$ and given $\varepsilon > 0$ then for $t \in (x - \delta, x + \delta) \setminus v(K_f)$ (where $\delta$ is chosen so that (3.38) and (3.40) hold) we have $|F'(t) - F'(x)| = |F'(t)| \leq C\varepsilon |t - x|$, and thus $F''(x) = 0$. Similarly for higher $i$'s.

To finish the proof of differentiability of $F$, we will show that $F^{(n)}(x) = 0$ for each $x \in v(K_f)$. But since $F^{(n-1)}(w) = 0$ for all $w \in v(K_f)$, (3.39) together with (3.41) easily imply this assertion.

If $f$ is not constant on any interval, then (3.35) implies that $\lambda(K_f) = \lambda(v(K_f)) = 0$.

By (3.37) and by the property (i) of Lemma 2.11, it follows that $F'(x) \neq 0$ for all $x \in [c, d] \setminus K_f$. It follows that there exists a linear homeomorphism $\eta : [a, b] \rightarrow [c, d]$, which is onto. Now, it clearly suffices to put $\phi := F \circ \eta$. \hfill $\Box$

**Proposition 3.13.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a CBV $G_{1/n}$ function. Then $f$ is Lebesgue equivalent to an $(n - 1)$-times differentiable function $\phi$ such that $\phi^{(n-1)}(\cdot)$ is pointwise Lipschitz, $\phi^{(i)}(x) = 0$ for all $x \in K_\phi$, $i = 1, \ldots, n - 1$, and $\phi'(x) \neq 0$ whenever $x \in [a, b] \setminus K_\phi$.

If $f$ is not constant on any interval, then $\lambda(K_\phi) = 0$.

**Proof.** It is similar to the proof of Proposition 3.12, so we will only sketch it. If $f$ is not constant on any interval, similarly as in the proof of Proposition 3.12 we can assume that $\lambda(K_f) = 0$.

Let $(A_m)_{m \in \mathcal{M}}$ be the sets from Definition 3.3 for $f$. If $\mathcal{M}$ is finite, then use $A_m = K_f$ and $\mathcal{M} = \mathbb{N}$ instead of $A_m$ and $\mathcal{M}$ (it is easy to see that in this case $G_{1/n}(f, K_f, K_f) < \infty$). Find a sequence of $a_m > 0$ such that

$$\sum_{m \in \mathcal{M}} a_m \cdot G_{1/n}(f, A_m \cup \{a, b\}, K_f) < \infty.$$  

For $x \in [a, b] \cap K_f$ define $v(x)$ as $x + \sum_{m \in \mathcal{M}} a_m \cdot (v_m(x) - \bar{v}_m(x))$, where $v_m$ (resp. $\bar{v}_m$) are the functions obtained from Lemma 3.10 (resp. Lemma 3.11) applied to $f$ and $\delta = b - a$ (since it is easy to see that $\overline{G_{1/n}}(f, A_m, K_f) \leq 2 G_{1/n}(f, A_m \cup \{a, b\}, K_f)$; see also (3.3)). As in the proof of Proposition 3.12, we obtain that $v$ is a continuous strictly increasing function. We
have that for each $x \in K_f$ there exists $m \in M$ such that

$$|f(y) - f(z)| \leq C_m |v(z) - v(y)|^{n-1} |v(z) - v(x)|$$

for all $y, z \in K_f$ with $x \leq y < z$ or $z < y \leq x$. This follows from (3.24), and (3.33) in a similar way as (3.36) in the proof of Proposition 3.12.

Define a new function $F : [a, b] \to \mathbb{R}$ as

$$F(x) := \begin{cases} f \circ v^{-1}(x) & \text{for } x \in v(K_f), \\ H_{\alpha, \beta}(x) & \text{for } x \in (\alpha, \beta), \end{cases}$$

whenever $(\alpha, \beta)$ is an interval contiguous to $v(K_f)$ in $[c, d]$, and $H = H_{\alpha, \beta}$ is is chosen by applying Lemma 2.11 to $\alpha, \beta, A = f \circ v^{-1}(\alpha), B = f \circ v^{-1}(\beta)$. It follows that $F$ is $n$-times differentiable at all $x \in [c, d] \setminus v(K_f)$. It remains to show that $F^{(i)}(x) = 0$ for all $x \in v(K_f), i = 1, \ldots, n-1$, and that $F^{(n-1)}$ is pointwise Lipschitz at points $x \in v(K_f)$.

From (3.42), we have that for each $x \in v(K_f)$ there exists $C_x > 0$ such that

$$|F(y) - F(z)| \leq C_x |z - y|^{n-1} |z - x|$$

for $y, z \in v(K_f)$ with $z < y \leq x$ or $x \leq y < z$. From this we obtain that

$$|F^{(n-1)}(t)| \leq C'_x |t - x|$$

for all $t \in [c, d] \setminus v(K_f)$. By induction, we obtain that $F^{(i)}(x) = 0$ for all $i = 1, \ldots, n-1$ (since (3.44) together with (3.45) imply that $F'(x) = 0$, and then (3.45) easily implies that $F^{(i)}(x) = 0$ for $i = 2, \ldots, n-1$). Now (3.45) (together with the fact that $F^{(n-1)}(x) = 0$ for each $x \in v(K_f)$) easily implies that $F^{(n-1)}$ is pointwise Lipschitz at all points of $v(K_f)$.

If $f$ is not constant on any interval, then as in Proposition 3.12, we establish that $\lambda(K_F) = \lambda(v(K_f)) = 0$.

By (3.43) and by the property (i) of Lemma 2.11, it follows that $F'(x) \neq 0$ for all $x \in [c, d] \setminus K_F$. It follows that there exists a linear homeomorphism $\eta : [a, b] \to [c, d]$, which is onto. Now, it clearly suffices to put $\phi := F \circ \eta$. $\square$

4. MAIN RESULT

The case $n = 2$ is handled in [5]. The following main theorem gives a slightly different characterization in that case (see Introduction).

**Theorem 4.1.** Let $f : [a, b] \to \mathbb{R}$ be continuous, $n \geq 2$, and $n \in \mathbb{N}$. Then the following are equivalent:

(i) $f$ is Lebesgue equivalent to a function $g$ which is $n$-times differentiable.
(ii) $f$ is Lebesgue equivalent to a function $g$ which is $n$-times differentiable and such that $g^{(i)}(x) = 0$ whenever $i \in \{1, \ldots, n\}$ and $x \in K_g$, and $g'(x) \neq 0$ whenever $x \in [a, b] \setminus K_g$.

(iii) $f$ is Lebesgue equivalent to a function $g$ which is $(n - 1)$-times differentiable and such that $g^{(n-1)}(\cdot)$ is pointwise-Lipschitz.

(iv) $f$ is $\mathrm{SBV}_G 1/n$.

(v) $f$ is $\mathrm{CBV}_G 1/n$.

(vi) $f$ is $\mathrm{SBV}_G 1/n$.

Proof. The implications (ii) $\implies$ (i), and (i) $\implies$ (iii) are trivial. The implications (iii) $\implies$ (iv), and (iii) $\implies$ (v) follow from Lemma 3.9. The implication (vi) $\implies$ (ii) follows from Proposition 3.12, and the implication (v) $\implies$ (iii) from Proposition 3.13. Finally, the implication (i) $\implies$ (vi) follows from Lemmata 3.9 and 3.6; and the implication (iv) $\implies$ (vi) from Lemma 3.6.

We have the following corollary:

**Corollary 4.2.** Let $f : [a, b] \to \mathbb{R}$, $n \geq 2$, $n \in \mathbb{N}$. Then $f$ is $\mathrm{CBV}_G 1/n$ if and only if $f$ is $\mathrm{SBV}_G 1/n$.

The following example shows that for each $n \geq 2$ there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f$ is $\mathrm{CBV}_G 1/n$ (and thus $f$ is Lebesgue equivalent to an $n$-times differentiable function by Theorem 4.1), but $V_{1/n}(f, K_f) = \infty$ (and thus $f$ is not Lebesgue equivalent to any $C^n$ function by the results of [11]). It is a simplified version of [11, Example 8.3].

**Example 4.3.** Let $n \geq 2$ be an integer. Let $a_m \subset (0, 1)$ be such that $a_m \downarrow 0$. Define $f(a_{2m}) = m^{-n}$ and $f(2m-1) = 0$ for all $m = 1, 2, \ldots$, $f(0) = f(1) = 0$, and extend $f$ onto $[0, 1]$ such that it is continuous and affine on each interval contiguous to $K = \{0, 1\} \cup \{a_m : m \in \mathbb{N}\}$. Then $K_f = K$, $f$ is $\mathrm{CBV}_G 1/n$, but $V_{1/n}(f, K_f) = \infty$.

Proof. Obviously, the function $f$ is continuous, has bounded variation, and $V_{n+1}(f, K_f) < \infty$. Also, it is easy to see that $f$ is $\mathrm{CBV}_G 1/n$ (using $A_1 = \{0, 1\}$ and $A_m = \{a_{2m-2}, a_{2m-3}\}$ for $m = 2, 3, \ldots$). On the other hand,

$$V_{1/n}(f, K_f) \geq \sum_{m \in \mathbb{N}} |f(a_{2m}) - f(a_{2m-1})|^+ = \sum_{m} \frac{1}{m} = +\infty.$$  

The following theorem characterizes the situation when we require the first derivative to be nonzero almost everywhere.

**Theorem 4.4.** Let $f : [a, b] \to \mathbb{R}$ be continuous, $n \geq 2$, and $n \in \mathbb{N}$. Then the following are equivalent:
(i) $f$ is Lebesgue equivalent to a function $g$ which is $n$-times differentiable and such that $g'(x) \neq 0$ for a.e. $x \in [a, b]$.

(ii) $f$ is Lebesgue equivalent to a function $g$ which is $n$-times differentiable, such that $g^{(i)}(x) = 0$ whenever $i \in \{1, \ldots, n\}$, $x \in K_g$, $g'(x) \neq 0$ whenever $x \in [a, b] \setminus K_g$, and such that $\lambda(K_g) = 0$.

(iii) $f$ is Lebesgue equivalent to a function $g$ which is $(n-1)$-times differentiable and such that $g^{(n-1)}(\cdot)$ is pointwise-Lipschitz, $g'(x) \neq 0$ for a.e. $x \in [a, b]$.

(iv) $f$ is $\text{SBV}_{1/n}$ and $f$ is not constant on any interval.

(v) $f$ is $\text{CBV}_{1/n}$ and $f$ is not constant on any interval.

(vi) $f$ is $\text{SBV}_{1/n}$ and $f$ is not constant on any interval.

Proof. The proof is similar to the proof of Theorem 4.1 while we also use the fact that $g'(x) \neq 0$ for a.e. $x \in [a, b]$ implies that $g$ is not constant on any interval, the fact that being nonconstant on any interval is invariant with respect to the Lebesgue equivalence, and the corresponding assertions in Propositions 3.12 and 3.13. \qed

5. Generalized Zahorski Lemma

Our methods yield the following theorem, which can be viewed as a generalization of Zahorski’s Lemma; see e.g. [15] or [10, p. 27].

Theorem 5.1. Let $K \subset [a, b]$ be a closed set, $n \geq 2$, $n \in \mathbb{N}$. Then the following are equivalent:

(i) There exists an $n$-times differentiable homeomorphism $h$ of $[a, b]$ onto itself such that

$$K = h(\{x \in [a, b] : h^{(i)}(x) = 0 \ \forall i = 1, \ldots, n\}).$$

(ii) $V_{1/n}(\text{id}, K) < \infty$ and there exist closed sets $A_m \subset [a, b]$ ($m \in \mathcal{M} \subset \mathbb{N}$) such that $K = \bigcup_m A_m$ and $GV_{1/n}(\text{id}, A_m, K) < \infty$ for all $m \in \mathcal{M}$.

(iii) $V_{1/n}(\text{id}, K) < \infty$ and there exist closed sets $A_m \subset [a, b]$ and numbers $\delta_m > 0$ ($m \in \mathbb{N}$) such that $K = \bigcup_m A_m$, $A_m \subset A_{m+1}$ for all $m \in \mathbb{N}$, and $\lim_{m \to \infty} GV_{1/n}(\text{id}, A_m, K) = 0$.

Proof. Since the proof is similar to the considerations above, we will only sketch it. Without any loss of generality, we can assume that $\{a, b\} \subset K$.

The proof of the implication (i) $\implies$ (ii) is similar to the proof that every $(n-1)$-times differentiable function $f$ such that $f^{(n-1)}$ is pointwise Lipschitz, is $\text{CBV}_{1/n}$ (see Lemma 3.9).

To prove that (ii) $\implies$ (iii), note that an argument similar to the proof of Proposition 3.13 shows that there exists a homeomorphism $\varphi$ of $[a, b]$
onto itself which is \((n - 1)\)-times differentiable with \(\varphi^{(n-1)}\) being pointwise Lipschitz, and such that

\[
K = \varphi\left\{ x \in [a, b] : \varphi^{(i)}(x) = 0 \text{ for all } i = 1, \ldots, n - 1 \right\}.
\]

By the argument of the proof of Lemma 3.9, we find a decomposition \(\tilde{A}_m^k\) with some \(\delta_m^k > 0\) for \(\varphi^{-1}(K)\) such that \(\tilde{A}_m^k \subset \tilde{A}_{m+1}^k\) for all \(m, k \in \mathbb{N}\),

\[
\lim_{m \to \infty} \text{GV}_{\delta_m^k} (id, \tilde{A}_m^k, \varphi^{-1}(K)) = 0
\]

for each \(k\), and then using the diagonal argument from Lemma 3.6, we find \(\tilde{A}_m^1\) and \(\delta_m^1\) such that \(\varphi^{-1}(K) = \bigcup_m \tilde{A}_m^1\), and

\[
\lim_{m \to \infty} \text{GV}_{\delta_m^1} (id, \tilde{A}_m^1, \varphi^{-1}(K)) = 0.
\]

Now, we put \(A_m := \varphi(\tilde{A}_m^1)\), and find suitable \(\delta_m > 0\) using the uniform continuity of \(\varphi^{-1}\). This shows that (iii) holds.

Finally, to show that (iii) \(\implies\) (i), one can use a similar construction as in the proof of Proposition 3.12.

\[\square\]

Remark 5.2. It is also not very difficult to see that in the previous theorem, we can replace (i) with

(i') There exists an \(n\)-times differentiable homeomorphism \(h\) of \([a, b]\) onto itself such that \(h^{-1}\) is absolutely continuous, \(h^{-1}(K) = 0\), and

\[
K = h\left\{ x \in [a, b] : h^{(i)}(x) = 0 \text{ for all } i = 1, \ldots, n \right\}.
\]

The proof uses ideas from the proof of Theorem 4.4. Let us only remark that it is well known that \(h^{-1}\) can be taken absolutely continuous in the classical Zahorski’s Lemma; see e.g. [2].

Acknowledgment

The author would like to thank Professor Luděk Zajíček for several remarks that led to considerable improvements of the presentation.

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