Electromagnetic field in an active slab: conditions for spontaneous oscillation

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Abstract. In this work we analyze the conditions for the propagation of plane waves due to spontaneous oscillation within an active slab of width $d$ and complex relative permittivity $\tilde{\varepsilon}_b$. The slab is immersed in transparent, semi-infinite media of relative permittivities $\varepsilon_a$ and $\varepsilon_c$, respectively, without external field sources –except, of course, the power source used for pumping the active medium in the slab. It is well known that, if there is enough gain in the active medium, it may sustain spontaneous oscillations of the electromagnetic field, which result in an impaired operation. This model gives a limiting value of the small signal gain to avoid this condition. It is applicable to disk lasers, amplifiers and other optoelectronic devices in which pumping is confined to a region with transverse dimensions much larger than its length.

1. Introduction

Spontaneous oscillation (SO) takes place in an active medium when the electromagnetic field fluctuations are enhanced by the gain mechanism. In general, the volume occupied by the active medium is limited. When the reflections in the front, back and lateral surfaces are taken into account, it is easy to see that for some radiation trajectories the gain-length product may exceed the oscillation threshold [1], giving rise to SO. Of course, when spontaneous (also called “parasitic”) oscillations occur the device will not work properly, if the threshold for the useful signal is higher. As the transverse dimensions of the active medium increase in comparison to its thickness, the feedback provided by the reflections from the periphery becomes increasingly relevant for the onset of SO. From the above is clear that the suppression of SO is a prime concern in the design of disk laser amplifiers and oscillators [2]. Therefore, numerous authors have presented models and/or experimental results where SO control is achieved by the elimination of the reflected waves, surrounding the active medium with a cladding or an absorbing layer [3–6]. However, if the gain is high enough, SO may be originated by reflections from the front and back surfaces only. In this work we study this limiting case, where the transverse dimensions of the active medium are very large in comparison to its thickness, and there are no reflected waves from the edges.

The onset of SO is studied by treating the electromagnetic field classically. Linear propagation in active media can be assumed for small amplitudes, where the gain may be considered as independent of the electric field magnitude. Also, the relative magnetic permeability of the
media is taken as unity. Under these assumptions the active media is described by a complex permittivity, with its imaginary part opposite in sign to that in usual dissipative materials [7]. The optical dispersion and gain lineshape may modeled by the frequency dependence of the real and imaginary parts of the complex permittivity. It must be remarked that, since in many cases of technological interest the gain is limited to a narrow band, in this work we will study the conditions for the onset of SO at a given frequency, considering the complex permittivity as a parameter.

2. Conditions for the existence of the spontaneous field

We will assume the existence of four plane waves (figure 1): two in the active medium and one in each of the transparent media. The wave vectors $k$, of the four plane waves have the same component $k_z$, parallel to the plane interfaces. This is due to the continuity conditions for the fields at the interfaces. Inside the active slab, the normal components of the wave vectors have the same magnitude but opposite directions. In the transparent media, the normal components of the wave vectors point away from the slab. Since the transparent media are semi-infinite and without field sources, we assume that there are no incident external waves impinging on the slab [8–11].

![Figure 1. Coordinate system and wave vectors of the transparent – active interfaces.](image)

In the transparent media ($a$ and $c$), we define $\alpha_a$ and $\alpha_c$ as the angles between the wave vector and the normal to the interface $x$ for each media, respectively. Also, we define $\eta_z$, a dimensionless parameter associated to the angles $\alpha_a$ and $\alpha_c$, given by:

$$\eta_z = \sqrt{\varepsilon l} \sin \alpha_l = n_l \sin \alpha_l \quad (l = a, c)$$

(1)

where $n_a$ and $n_c$ are the refractive indexes of the transparent media, respectively. It must be noted that, under the assumptions of this work, the value of the angles $\alpha_a$ and $\alpha_c$ (and therefore $\eta_z$) are limited by the ratio between the thickness and the largest tranverse dimension of the active medium. For example, for a circular slab in air of thickness $d$ and radius $R$, $\eta_z \leq \sin(\arctan(R/d))$. Therefore, $\eta_z$ may be large but remains bounded. If the refractive index
of any of the transparent media were greater than \( \eta_z \), this condition would not fulfilled at an interface, and the waves in the transparent medium would be evanescent (no outgoing wave propagating from the slab). This case will not be considered in this work.

The wave vector in each media will be normalized using the vacuum wavelength \( \lambda_v \), as:

\[
\vec{k}_a = \frac{2\pi}{\lambda_v} (\eta_{ax} \hat{x} + \eta_{az} \hat{z}) \quad \vec{k}_b^{(+)} = \frac{2\pi}{\lambda_v} (\eta_{bx} \hat{x} + \eta_{bz} \hat{z}) \\
\vec{k}_c = \frac{2\pi}{\lambda_v} (\eta_{cx} \hat{x} + \eta_{cz} \hat{z}) \quad \vec{k}_b^{(-)} = \frac{2\pi}{\lambda_v} (\eta_{bx} \hat{x} + \eta_{bz} \hat{z})
\]

where \( \eta_z \) and \( k_z \) are related by \( \eta_z = k_z \lambda_v/2\pi \). The remaining \( \eta \) parameters are defined through the following relationships:

\[
\eta_{lx}^2 + \eta_z^2 = \varepsilon_l \quad (l = a, c) \quad \eta_{lx}^2 + \eta_z^2 = \varepsilon_b
\]

It is important to bear in mind that the relative permittivities \( \varepsilon_a \) and \( \varepsilon_c \) are real, but \( \varepsilon_b \) is complex. In consequence, \( \eta_{lx} \) is the only complex \( \eta \) parameter (assuming outgoing waves). Thus, in the active medium, it is possible to define a complex refractive index \( \tilde{n}_b = \sqrt{\varepsilon_b} = n_{br} - i n_{bi} \), where \( n_{bi} \) is associated to the active medium gain value. In fact, in technological applications the small signal gain coefficient \( G \) is customarily used, and it is related to the imaginary part of the refractive index by \( n_{bi} = (1/2)(G \lambda_v/2\pi) \) [12].

The boundary conditions that must be satisfied by the fields at the interfaces impose constraints on the complex amplitudes of the plane waves. This leads to a homogeneous system of linear equations with non-trivial solutions only if the system determinant is zero [13]. For instance, if we consider that the electric field vector is perpendicular to the interface (perpendicular polarization) a dimensionless magnitude \( \rho_s \) can be defined:

\[
\rho_s = \frac{(\eta_{ax} + \eta_{ax})(\eta_{bx} + \eta_{cx})}{(\eta_{ax} - \eta_{ax})(\eta_{bx} - \eta_{cx})}
\]

as a function of the independent magnitude \( \eta_z \) and the media permittivities. Analogously, for parallel polarization, there is an equivalent dimensionless magnitude \( \rho_p \):

\[
\rho_p = \frac{(\eta_{ax} + \eta_{ax}^2)(\eta_{bx} + \eta_{cx}^2)}{(\eta_{ax} - \eta_{ax}^2)(\eta_{bx} - \eta_{cx}^2)}
\]

that also is a function of the same parameters. In consequence, the equations that define the existence of non-zero electromagnetic fields in the systems corresponding to perpendicular (s) and parallel (p) polarizations may be written as:

\[
\rho_{s,p} = e^{4\pi i \eta_{lx} d'}
\]

where \( d' \) is the slab thickness scaled to the vacuum wavelength: \( d' = d/\lambda_v \).

Equation (6) may be recast as:

\[
d' = \frac{\log |\rho|}{4\pi \text{Im}(\eta_{lx})}
\]

\[
\arg(\rho) - \frac{\text{Re}(\eta_{lx}) \log |\rho|}{\text{Im}(\eta_{lx})} = 2\pi m \quad m \in \mathbb{Z}
\]

where \( \rho = \rho_s \) or \( \rho_p \). Therefore, equation (8) gives the condition for non-trivial solutions that is independent of the width of the slab; moreover, for each value of \( \eta_z \) it depends only on the relative permittivities. For example, given the real part of the media permittivities, i.e. \( n_a \), \( n_{br} \) and \( n_c \), for each \( \eta_z \) there is a discrete set of \( n_{bi} \) values that verify equation (8), associated
to different values of \( m \). For each value of \( n_{bi} \) belonging to this discrete set, the value of \( d' \) is obtained from equation (7). It is worth mentioning that the values of \( m \) for which there are solutions of equation (8) belong to a bounded interval; the upper and lower bounds depend on the equation parameters and the interval length increases when \( d' \) decreases.

It is easy to see that equation (6) may be regarded as a generalization of the well-known threshold condition for oscillation (see for example [1] chap. 1), taking into account the polarization and the angular dependence of the phase and amplitude of the electromagnetic field. One of the most interesting consequences of equations (7) and (8) is the existence of a gain threshold for the existence of non-trivial solutions. This must be understood in the sense that the necessary condition for SO in not fulfilled for gain values corresponding to \( n_{bi} \) below those given by equations (7) and (8). It must be remarked that, for a given value of \( d' \), we can calculate for each \( \eta_z \) the gain value \( n_{bi} \) as a function of the real permittivities of the media, that is, \( n_a, n_{br}, n_c \). Figures 2 and 3 show these values for several combinations of permittivities and polarizations.

![Figure 2](image)

**Figure 2.** Values of \( n_{bi} \) from equations (7) and (8) as function of \( \eta_z \) for \( d' = 2 \) and the two polarizations. The values in the text box represent \( n_a, n_{br} \) and \( n_c \), respectively.

It must be stressed that the non-trivial solutions of equations (6) and (7), given by condition of equation (8), are a set of discrete values; for low values of \( d' \), the solutions are few and widely separated. On the contrary, for higher values of \( d' \) there is an increasing number of closely spaced solutions, and the set of non-trivial solution resembles a continuum. From the results presented in figure 3 it is evident that, for parallel polarization, there are certain critical values of \( \eta_z \) where the value of \( n_{bi} \) for SO is noticeably higher. This behaviour will be studied in detail in future works.

As an application example, we will consider a Nd:Yag slab as an active medium immersed in air, with a thickness of 2 cm and a maximum transverse dimension of 14 cm. The small signal gain coefficient at \( \lambda_v = 1.06 \) \( \mu m \) is taken as \( G = 8.4 \times 10^{-2} \) \( cm^{-1} \), and the real part of the refractive index of the active medium is \( n_{br} = 1.56 \). These parameters correspond to the active media of the laser described by McMahon et. al. in ref. [3]. These authors analyzed in detail the detrimental effects of SO originated in each slab by the reflections on the edges, even for \( G < 4 \times 10^{-2} \) \( cm^{-1} \). To supress the SO, the periphery of the active media was coated with absorbing glass, thus practically eliminating the reflections, as was confirmed by further measurements. McMahon et. al. reported that they used numerical methods based on intensity calculations only, to assure the absence of SO due to the reflections in the front and back surfaces of the slabs. Experimental data then confirmed the absence of SO even when \( G \) reached the
maximum value indicated above. According to our model, the scaled slab width is $d' = 1.89 \times 10^4$, the maximum value of $\eta_z$ given by geometrical considerations is 0.96 and the imaginary part of the refractive index of the active medium $n_{bi}$ has a value of $7.09 \times 10^{-7}$. From equations (6) and (7) the threshold values of $n_{bi}$ for the maximum values of $\eta_z$ are $3.14 \times 10^{-6}$ for perpendicular and $7.75 \times 10^{-6}$ for parallel polarization, respectively. These are well above the $n_{bi}$ corresponding to the maximum value of $G$. It is clear that our analytical calculation predicts satisfactorily the absence of SO for the experimental configuration described in ref. [3].

3. Conclusions
This work presents a study of the conditions for the onset of the spontaneous oscillations in an active slab immersed in transparent, semi-infinite media. The electromagnetic field is treated classically, and since gain saturation effects are neglected, linear propagation of electromagnetic fields is assumed. The active medium is described by a frequency-dependent complex permittivity with its the imaginary part opposite in sign to that in usual dissipative materials. Therefore, the optical dispersion and gain lineshape may modeled by the frequency dependence of the real and imaginary parts of the complex permittivity. It is assumed that the transverse dimensions of the active medium are very large and there are no reflected waves from the edges.

A system of equations describes the boundary conditions that must be satisfied by the field at the front and back interfaces. The values of thickness and gain that correspond to non-trivial solutions of this system determine the necessary conditions for the onset of SO at a given frequency. When the slab thickness is small in comparison to the wavelength, there are few solutions, widely separated; for thicker slabs the set of non-trivial solutions resembles a continuum. In practice, this implies that the gain value obtained for each thickness is in fact a threshold, since in most cases of technological interest saturation effects come into play when the gain exceeds the calculated value. The description of SO in the non-linear regime fall outside the scope of this work.

This model may be applied to estimate the maximum small signal gain without SO in a thin slab in the most favourable case, where reflections from the edges have been totally eliminated. This is a limiting condition for disk lasers, amplifiers and other optoelectronic devices in which pumping is confined to a region with transverse dimensions much larger than its length. The model predicted adequately the absence of SO in a high power device described in the literature.
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