Molecular cloud collapse to stellar densities: models on moving geodesic vs. unstructured tetrahedron vs. nested meshes

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Abstract. The results of a study of a collapsing cold molecular cloud with three computational models based on different mesh methods are presented. Collapse of a gas with a molecular cloud core density ($\approx 10^{-18}$ g cm$^{-3}$) to stellar densities ($\approx 10^{-2}$ g cm$^{-3}$) is considered. To simulate the collapse, the equation of state for an ideal gas and three mesh computational models are used. These are: a model based on multilevel nested meshes, a model based on moving geodesic meshes, and a model based on nonuniform tetrahedral meshes. The restrictions on the resolution in solving the Poisson equation and on the reproduction of the gravitational potential gradient on geodesic and tetrahedral meshes do not allow reproducing well the onset of collapse (in the case of moving geodesic meshes) or the maximum density (in the case of tetrahedral meshes). The results of computational experiments show that the process of collapse is reproduced well on multilevel nested meshes with sufficient spatial resolution.

1. Introduction
The process of collapse and fragmentation of molecular clouds forms the basis of star formation. Extensive literature is devoted to the fundamental problem of formation of binaries or filaments (see [1, 2]) and numerical solution of similar problems by efficient methods such as SPH (see [3, 4, 5]). In the case of a collapsing rotating cloud, a major problem is finding the proper numerical solution at the center of the domain. It is well-known that, due to its artificial moment transfer, a numerical scheme may cause the formation of a torus, whereas the true solution is a disk. The problem becomes even more difficult due to the presence of a magnetic field and the effect of magnetorotational instability. This problem arises in the formation of protostars [6], binary stars [7, 8], and a single disk [9, 10, 11]. The magnetic field complicates the structure and mechanism of formation of jets in the case of misalignment of magnetic fields [12, 13]. A separate problem is modeling the chemical composition of a molecular cloud during its collapse and the formation of the first stars [14]. To verify such simulations, some particular analytical solutions have been constructed [15].

A major computational challenge in solving such problems is that they are solved with different scales. For instance, the simulation of the collapsing molecular cloud core with an initial density $\approx 10^{-18}$ g cm$^{-3}$ to a stellar density $\approx 10^{-2}$ g cm$^{-3}$ requires an efficient spatial resolution from $\approx 10^{17}$ cm to $10^{10}$ cm. This is equivalent to using an efficient mesh resolution of $10^{-7}$, which is impossible with a uniform mesh. To achieve such a resolution some specific approaches are
used, such as based on moving geodesic meshes, unstructured tetrahedron meshes, and nested meshes. Adaptive mesh refinement is not considered, since in this case nested meshes more adequately describe the computational domain.

A schematic description of the computational models used to simulate the collapse is given in Section 2. Section 3 is devoted to computational experiments. In Section 4, some controversial issues are discussed concerning advantages and disadvantages of the methods, as well as their applicability to the problem of collapse. Conclusions to the paper are given in Section 5.

2. Computational models

To simulate the hydrodynamics of a collapsing cold cloud, we need an efficient spatial resolution of the order of $1 : 10^{7}$. This can be achieved in different ways. We will consider three of the most promising and most commonly used approaches. We compare these approaches using a model problem and determine their advantages and disadvantages. Let us briefly describe the features of each approach.

**Moving Geodesic Meshes** This approach is based on a uniform triangulation of the sphere surface used to specify the computational domain [16]. A computational model for static geodesic meshes developed by us earlier [17] was complemented by a concept of co-collapse coordinates according to [18]. The mesh moves at a speed proportional to the speed of a collapsing pressureless gas. The Poisson equation is discretized by using the Gauss–Ostrogradsky formula applied to the Poisson equation integrated over all cells. This approach is rather complicated from the point of view of computations, since a SLAE matrix is formed at each time step due to changes in the mesh geometry. A schematic diagram of the meshes is shown in Fig. (1) from paper [17].

![Figure 1. Computations on Moving Geodesic Meshes.](image_url)

**Unstructured Tetrahedron Meshes** This approach is based on generating an unstructured tetrahedral mesh with arbitrary refinement at the center of the domain. It allows using both various software tools [19] and geodesic meshes [20]. In the case of a spherically symmetric collapse, the flow structure is known well. Therefore, we can construct a static tetrahedral mesh of any desired quality. To solve the equations of hydrodynamics, an approach similar to the above one is used. To solve the Poisson equation, a finite element formulation with piecewise-linear basic functions in spatial coordinates is used. A schematic diagram of the meshes is shown in Fig. (2) from paper [22].
Nested Meshes This approach is based on multilevel nested meshes [21] similar to the approaches used to construct numerical methods for polygonal meshes [22, 23]. An obvious advantage of this approach is that it can use the well-developed numerical methods for solving the hydrodynamic equations on regular meshes [24] and can be extended to more complex models [25]. Note that the Poisson equation is solved with a classical multigrid scheme in which the conjugate gradient method is used at each level by choosing an initial approximation for the potential from the evolution of the gravitational force. A schematic diagram of the meshes is shown in Fig. (3)

3. Computational experiments
Let us simulate a collapsing homogeneous cold ($T = 10 \text{ K}$) molecular cloud with mass $M = M_\odot$ and radius $R = 7 \times 10^{16} \text{ cm}$. Here we use the equation of state for an ideal gas with the following effective adiabatic index:

$$\gamma = \begin{cases} 
1, & \rho \leq 1.0 \times 10^{-13}, \\
7/5, & 1.0 \times 10^{-13} < \rho \leq 5.7 \times 10^{-8}, \\
1.15, & 5.7 \times 10^{-8} < \rho \leq 1.0 \times 10^{-3}, \\
5/3, & \rho > 1.0 \times 10^{-3}.
\end{cases}$$
where the density is measured in grams per cubic centimeter. The time is measured by the following relative free-fall time index for a pressureless gas cloud:

\[ t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho}}. \]

The simulated maximum density of the cold gas cloud collapse is shown in Fig. (4). This figure shows that with nested and moving meshes we have the required order of the density values, whereas with tetrahedral meshes the density is about one order of magnitude less. This is due to the insufficient accuracy of the numerical solution of the Poisson equation and the reconstruction of the gravity force when using a tetrahedral mesh. Nested meshes are capable of reproducing such solutions, since the numerical method for solving the Poisson equation and, hence, the gravitational force have a higher order of accuracy. A less significant but still important factor is the considerable slowdown in the reproduction of the collapse process when using tetrahedral meshes. This is also due to the lower accuracy in the numerical solution of the Poisson equation, which results in lower absolute values of the potential and, as a consequence, a later onset of the collapse. Moving meshes managed to reconstruct the potential well by restructuring the mesh. However, there was a side effect: an earlier onset of the collapse, which is also due to the restructuring of the mesh rather than to the physics of the process. Also, the movement of the meshes is very specific and related to the problem statement. A solution is considered to be most physically adequate if it is obtained with multilevel nested meshes, but with the required level of nesting. In the computational experiment, 15 levels of nesting have been used: one level of nesting per one order of density.

4. Discussion
In this section, we will consider advantages and disadvantages of each of the above approaches. Key factors are: quality of reconstruction of the gravitational potential and its gradient,
invariance of the solution under rotation, ability to reproduce discontinuous solutions and to be easily adapted to related problems, efficiency, and scalability.

(i) A major problem in the numerical solution of cold gas collapse is the accuracy of reproducing the gravitational potential. With nested meshes we used a 27-point stencil, which allowed obtaining a scheme of the fourth order of accuracy. Such accuracy cannot be achieved with polygonal meshes, and furthermore with tetrahedral meshes.

(ii) The low accuracy in determining the gravitational potential leads to a low accuracy in its gradient, which is explicitly present in the simulation of the hydrodynamic equations. In fact, with tetrahedral meshes we obtain a piecewise constant solution for the gradient, and with a polygonal mesh we obtain an approximation of the first order (and not higher). With nested meshes the gradient is approximated with the second order (at minimum) of accuracy in space.

(iii) An obvious advantage of geodesic and tetrahedral meshes is the Galilean invariance of the solution. Moreover, the use of geodesic meshes not only avoids the rotation problems, but also eliminates the singularities along the axis of symmetry, which often takes place in cylindrical and spherical coordinates. Only a special structure of the numerical method can solve the problem of non-invariance when using regular and nested meshes.

(iv) Note that despite all the advantages of geodesic meshes in terms of invariance, there arises a problem of formation of tight binary systems. When using tetrahedral and nested meshes this problem can be easily solved or does not exist at all. In fact, the problem of nested meshes is associated with the invariance of the numerical solution; it can be solved by choosing the proper numerical method [26].

(v) A fundamental problem of nested meshes is that of scalability, since in this approach there is a limitation on the potential number of supercomputer nodes that are equal to the number of levels of nested meshes. For geodesic meshes there is a simple approach to parallel
implementation based on 1D decomposition. Tetrahedral meshes can also be implemented rather efficiently.

It seems that nested meshes are the best option for simulating the star formation processes where collapse is the main component. Geodesic meshes have a potential in solving problems of supernova explosions based on core collapse. Tetrahedral meshes are optimal as a standard approach to solving the problem of gravitational hydrodynamics.

5. Conclusions
A study of three computational mesh models has been performed. These are: a model based on multilevel nested meshes, a model based on moving geodesic meshes, and a model based on nonuniform tetrahedral meshes. The restrictions on the resolution in solving the Poisson equation and on the reproduction of the gravitational potential gradient on geodesic and tetrahedral meshes do not allow reproducing well the onset of collapse (in the case of moving geodesic meshes) or the maximum density (in the case of tetrahedral meshes). The above computational experiments have shown that the process of collapse is reproduced well with multilevel nested meshes at a sufficient spatial resolution.

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