Realisation of Linear Continuous-Time Fractional Singular Systems Using Digraph-Based Method. First approach.

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Abstract. In this paper, a new method for computation of a minimal realisation of a given proper transfer function of fractional continuous-time singular one-dimensional linear systems using one-dimensional digraphs theory $\mathcal{D}$ has been presented. For the proposed method, an algorithm was constructed. The proposed solution allows minimal digraphs construction for any one-dimensional system. The proposed method was discussed and illustrated with numerical examples.

Nomenclature

- $\mathbf{A}$ - matrices denoted by the bold font;
- $\mathbf{A}^{-1}$ – inverse matrix;
- $\det(\mathbf{A})$ – determinant of the matrix $\mathbf{A}$;
- $\mathcal{A}$ - the sets denoted by the double line;
- $a$ - lower/upper indices and polynomial coefficients will be written as a small font;
- $\mathcal{D}$ - one-dimensional digraph;
- $\mathfrak{M}(\mathcal{D})$ – digraph–mask;
- $\mathbb{R}^{n \times m}$ – the set $n \times m$ of real matrices;
- $\mathbb{Z}_+$ – the set of non-negative integers;
- $\mathbf{I}_n$ – the $n \times n$ identity matrix;
- $\mathcal{L}\{f(t)\}$ – $\mathcal{L}$-transform of the function $f(t)$;
- $T(s)$ - transfer function of one variable $s$;
- $p(s)$ – polynomial of the variable $s$;

1. Introduction
Singular continuous-time system were considered in many papers and books [1–8]. The properties and the use matrix theory of the singular system were established in [1,9,10]. There are many problems associated with the analysis and synthesis of the singular systems. One of the very important problems is realisation and minimal realisation problem. In many research studies, we can find a constant matrix form, which satisfies the system described by the transfer function [5,8,11–13]. In fact, there are many sets of matrices which fit into the system transfer function.

In the last two decades, integral and differential calculus of a fractional order has become a subject of great interest in different areas of physics, biology, economics and other sciences. Fractional calculus is a generalization of traditional integer order integration and differentiation.
actions onto a non-integer order. The idea of such a generalization was mentioned in 1695 by Leibniz and L'Hospital. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [14]. However, only just in the late 60’ of the 20th century, this idea drew attention of engineers. Fractional calculus was found to be a very useful tool for modelling the behaviour of many materials and systems. Mathematical fundamentals of fractional calculus are given in the monographs [8, 14–18]. Some other applications of fractional-order systems can be found in [19–24].

The main purpose of this paper is to present a method based on one-dimensional digraph theory for computation the set of minimal realisation of a given proper transfer function of the one-dimensional continuous-time fractional singular system. In the paper, a digraph structure corresponding to minimal realisations of the singular system is presented and discussed. Sufficient conditions for the existence of a minimal realisation of a given proper transfer function systems are established. In this paper, for the first time, a digraph-mask was introduced and used. It is very important in analysis of the singular systems, with the use of digraphs theory. As a result, we proposed the procedure for determining the set of realisations in the class $K_1$ of the digraph structure. The digraph classes in [25] have been defined and presented in details. This work is the next step in the research on the determination of the realisation problem $(A, E, B, C)$ of the continuous-time fractional singular systems by using a digraph theory, which was started in the publication [26], [27], [28], in which a minimal realisation of the continuous-time linear fractional system as a set of matrices $(A, B, C)$ was determined.

This work has been organised as follows: Section 1 presents some notations and basic definitions of a digraph theory; basic properties of the singular discrete-time linear system defined as the state-space representation. In the Section 2, we construct a procedure for determination minimal realisation of the singular continuous-time system. Finally, we demonstrate some numerical examples (Section 3), present concluding remarks, open problems and bibliography positions.

1.1. Digraphs

A directed graph (or just digraph) $D$ consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs [29]. We call $V(D)$ the vertex set and $A(D)$ the arc set of digraph $D$. We will often write $D = (V, A)$ which means that $V$ and $A$ are the vertex set and arc set of $D$, respectively. The order of $D$ is the number of vertices in $D$. The size of $D$ is the number of arcs in $D$. For an arc $(v_1, v_2)$ the first vertex $v_1$ is its tail and the second vertex $v_2$ is its head.

There are two well-known methods of representation of digraph: list and incidence matrix. In this paper we are using incidence matrix to represent all digraphs. Method of constructing digraphs by this method is presented for example in [29], [30] or [31].

1.2. Singular system

Let us consider the fractional singular continuous-time linear system

$$\begin{align*}
E \, D_a^\alpha x(t) &= A \, x(t) + B \, u(t), \\
y(t) &= C \, x(t),
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively and

$$E \in \mathbb{R}^{n \times n}, \ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}.$$  

(2)

The following Caputo definition of the fractional derivative will be used:

$$C_a^\alpha \, D_a^\alpha = \frac{d^\alpha}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha+1-n}} d\tau,$$

(3)
where \( \alpha \in \mathbb{R} \) is the order of a fractional derivative, \( f^{(n)}(\tau) = \frac{d^m f(\tau)}{d\tau^m} \) and \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \) is the gamma function.

**Theorem 1** The Laplace transform of the derivative-integral (3) has the form

\[
L \left[ \mathcal{D}_t^\alpha \right] = s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+).
\]

The proof of the Theorem 1 is given in [8].

After using the Laplace transform to (1), Theorem 1 and taking into account

\[
X(s) = L[x(t)] = \int_0^\infty x(t) e^{-st} dt,
\]

\[
L[\mathcal{D}_t^\alpha x(t)] = s^\alpha X(s) - s^{\alpha-1} x_0
\]

we obtain:

\[
X(s) = \left[ E s^\alpha - A \right]^{-1} \left[ s^{\alpha-1} x_0 + B U(s) \right],
\]

\[
Y(s) = C X(s), \quad U(s) = L[u(t)].
\]

After using (5) we can determine the transfer matrix of the system in the following form:

\[
T(s) = C \left[ E s^\alpha - A \right]^{-1} B \in \mathbb{R}^{p \times m}(s).
\]

It is assumed that \( \det E = 0 \) and the pencil of the system (1) is regular, that is

\[
\det \left[ E s^\alpha - A \right] \neq 0
\]

for some \( z \in \mathbb{C} \) (where \( \mathbb{C} \) is the field of the complex numbers).

**Definition 1** The system (1) is called singular system if and only if \( \det E = 0 \) (rank \( E = r < n \)).

The matrices (2) are called a realisation of a given transfer matrix \( T(s) \in \mathbb{R}^{p \times m}(s) \) if they satisfy the equality (6). The realisation is called minimal if the dimension of the state matrix \( A \) is minimal among all possible realisations of \( T(s) \).

**Task:** For the given transfer matrix (6), determine a minimal realisation of the system (1) using the one-dimensional \( D \) digraphs theory. The dimension of the system must be the minimal among possible.

2. Main results

The solution of the minimal positive realisation problem will be presented for one-dimensional single-input single-output (SISO) transfer function \( (m = p = 1) \). The proposed method will be based on the one-dimensional digraph theory, and it will determine solutions in class \( K_1 \). The classes of the digraph structure were considered in detail in [25].

Consider the irreducible transfer function:

\[
T(s) = \frac{n(s)}{d(s)} = \frac{b_q s^\alpha q + b_{q-1} s^{\alpha(q-1)} + b_{q-2} s^{\alpha(q-2)} + \ldots + b_1 s^\alpha + b_0}{a_r s^\alpha r + a_{r-1} s^{\alpha(r-1)} + a_{r-2} s^{\alpha(r-2)} + \ldots + a_1 s^\alpha + a_0},
\]

where
where \( b_i, \ i = 0, 1, \ldots, q \) and \( a_j, \ j = 0, 1, \ldots, r - 1 \) are given real coefficients and \( r < n \). The transfer matrix (8) can be considered as a pseudo-rational function of the variable \( \lambda = s^a \) in the form:

\[
T(\lambda) = \frac{n(\lambda)}{d(\lambda)} = \frac{b_q \lambda^q + b_{q-1} \lambda^{q-1} + b_{q-2} \lambda^{q-2} + \ldots + b_1 \lambda + b_0}{a_r \lambda^r + a_{r-1} \lambda^{r-1} + a_{r-2} \lambda^{r-2} + \ldots + a_1 \lambda + a_0},
\]

(9)

After multiplying the nominator and denominator of the transfer function (9) by \( \lambda^{-q} \) we obtain:

\[
T(\lambda) = \frac{b_q + b_{q-1} \lambda^{-1} + b_{q-2} \lambda^{-2} + \ldots + b_1 \lambda^{1-q} + b_0 \lambda^{-q}}{a_r \lambda^{r-q} + a_{r-1} \lambda^{r-q-1} + a_{r-2} \lambda^{r-q-2} + \ldots + a_1 \lambda^{1-q} + a_0 \lambda^{-q}}.
\]

(10)

In the first step, we must find matrix \( E \). The weight of the vertices in a digraph is associated with the digraph-mask. If the characteristic polynomial is in the form

\[
d(\lambda) = a_r \lambda^{r-q} + a_{r-1} \lambda^{r-q-1} + a_{r-2} \lambda^{r-q-2} + \ldots + a_1 \lambda^{1-q} + a_0 \lambda^{-q}.
\]

(11)

then we can determine digraph-mask by the use the following proposition.

**Proposition 1** A digraph-mask \( \mathcal{M}(D) \) corresponding to a characteristic polynomial (11) consists of minimum one vertex with the weight equal to 0.

**Proof** If digraph-mask \( \mathcal{M}(D) \) consists from all vertex with weight not equal to zero, this means that matrix \( E \) is diagonal with all non-zero entries. In this case the condition \( \det E \) is not satisfied and the system is not singular.

It should be noted, that we obtained a solution which is not minimal in case when more than one vertex in digraph-mask has a weight equal to zero. In this paper we will be assume, that digraph-mask consist from one vertex with weight equal to zero.

**Example 1** Let be given the characteristic polynomial \( d(\lambda) = \lambda^{-2} + \lambda^{-3} \). After using the Proposition 1, we obtain a set of a digraph-mask consisting of 2 vertices with the weight equal to 1. In Figure 1 vertices with weight equal to 1 are marked in blue and vertices with a weight equal to 0 are marked in red.

![Figure 1](image_url)

Figure 1: (a)–(d) All possible digraph-masks of the characteristic polynomial \( d(\lambda) = \lambda^{-2} + \lambda^{-3} \)

Using a digraph-mask we can write matrix \( E \). For example, from the digraph-mask presented in Figure 1(b), we have: \( E = \text{diag} \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right) \).

In the second step, we must find matrix \( A \) using decomposition of the characteristic polynomial \( d(\lambda) \) into a set of simple monomials:

\[
d(\lambda) = a_r \lambda^{r-q} + a_{r-1} \lambda^{r-q-1} + a_{r-2} \lambda^{r-q-2} + \ldots + a_1 \lambda^{1-q} + a_0 \lambda^{-q}.
\]

(12)

For each simple monomial using Proposition 2, we create digraphs representations.

**Proposition 2** The one-dimensional digraph \( D \) corresponding to monomial \( a_r \lambda^{r-q} \) where \( r = 0, 1, \ldots, q - 1 \) consists of one cycle, and it contains \( |r - q| \) vertices.
Introduce the definition of operation of the composition relative to vertices on digraphs that will be used in further considerations.

**Definition 2** Let \( G_1, G_2, \ldots, G_n \) be a digraphs with vertex sets \( V(G_n) = \{v_i(n) : i \in \mathbb{N}\} \). The composition relative to vertices \( \mathcal{D} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n \) is the digraph \( \mathcal{L} \) with the vertex set \( V(\mathcal{G}_1) \cup V(\mathcal{G}_2) \cup \cdots \cup V(\mathcal{G}_n) = \{v_j : j = 1, \ldots, \max\{i(n)\}, i \in \mathbb{N}\} \) and arc set \( A' = A(\mathcal{G}_1) \cup A(\mathcal{G}_2) \cup \cdots \cup A(\mathcal{G}_n) \), where \( A' \) denotes operation of the deleting multiple arcs.

Then, using Theorem 1 presented in paper [32,33] and Definition 2, we can create all digraph realisations of the characteristic polynomials. Each digraph corresponding to a characteristic polynomial must satisfy two conditions. The first condition (C1) relates to the existence in the common part of the digraph (vertex in blue), the second condition (C2) relates to non-existence of additional cycles in the digraph. To the vertices belonging to the common part of the digraph we assigned weight equal to 1 and for the other vertices – weight equal to 0.

**Example 2** (Continued Example 1) Characteristic polynomial \( d(\lambda) = \lambda^{-2} + \lambda^{-3} \) consisting of two monomials: \( M_1 = \lambda^{-2} \) and \( M_0 = \lambda^{-3} \). In Figure 2, possible realisations of the monomials are presented.

![Realisation of the monomial](image)

Then, we can determine all possible realisation (Figure 3) of the considering characteristic polynomial. Additionally, to the vertices belonging to the common part of the digraph we assigned weight equal to 1 (marked in blue) and 0 for other vertices (marked in red).

![All possible realisations](image)

Then, using Definition 2 and digraph-mask determined in the first step (Proposition 1), we create all possible digraph realisations as combinations of the monomial representations and a digraph-mask:

\[
d_i(\lambda) = \mathcal{M}_j(G) \circ (G_{M_r-1} \cup \cdots \cup G_{M_1} \cup G_{M_0}) = \mathcal{M}_j(G) \circ \left( \bigcup_{i=0}^{r-1} G_{M_i} \right)
\]

where: \( d_i(\lambda) \) means that we consider \( i \)-th realisation of the one-dimensional digraph \( \mathcal{D} \) corresponding to the characteristic polynomial \( d(\lambda) \); \( \mathcal{M}_j(G) \) is \( j \)-th possible digraph-mask; \( \circ \) is an operation of the superposition digraph-mask and one of the possible digraph realisations. It should be noted that superposition operation \( \circ \) on digraphs vertices corresponds to logical operation OR.

**Example 3** (Continued Example 2) Using composition relative to vertices and superposition operation to the digraph presented in Figure 3(a) and digraph-mask presented in Figure 1, we obtain the possible realisations presented in Figure 4. Two digraphs have all vertices filled with blue. The realisation which satisfies the characteristic polynomial is presented in Figure 4(b) and Figure 4(c).
Based on the above considerations we can obtain one of the possible digraph structures with assigned vertex weight, which satisfies conditions (C1) and (C2). By the use of the digraph structure (Figure 5), we can write one of the possible singular matrix and state matrix in the following form:

\[
A = \begin{bmatrix}
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
v_1 & w(v_1, v_1) & w(v_2, v_1) & \cdots & w(v_{q-2}, v_1) & w(v_{q-1}, v_1) \\
v_2 & w(v_1, v_2) & 0 & \cdots & 0 & 0 \\
v_3 & 0 & w(v_2, v_3) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_{q-2} & 0 & 0 & 0 & \cdots & \vdots \\
v_{q-1} & 0 & 0 & 0 & \cdots & w(v_{q-2}, v_{q-1}) \\
v_q & 0 & 0 & 0 & \cdots & 0 & w(v_{q-1}, v_q)
\end{bmatrix} \in \mathbb{R}^{q \times q} \tag{14}
\]

\[
E = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 1 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
v_{q-2} & 0 & 0 & 0 & 0 & 0 \\
v_{q-1} & 0 & 0 & 0 & 0 & 1 \\
v_q & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \in \mathbb{R}^{q \times q}. \tag{15}
\]

After finding the structure of the \(A\) and \(E\) matrices, we must find matrix \(B\) and \(C\) using the following polynomial:

\[
n(\lambda) = b_q + b_{q-1}\lambda^{-1} + b_{q-2}\lambda^{-2} + \ldots + b_1\lambda^{1-q} + b_0\lambda^{-q}. \tag{16}\]

For this purpose, we are expanding the digraph created in the first part of the algorithm. Because the degree of the polynomial \(n(\lambda)\) is higher than polynomial \(d(\lambda)\), then – to be able to realise it – we must add one vertex and an arc in such a way that there are no additional cycles (vertex and arc in green). In the considered digraph presented in Figure 5, vertex \(v_1\) is the common and from this vertex we determine a path of the maximum length equal to \(q\) (Figure 6).
Now we can add to the digraph the source vertex $s_1$ corresponding to matrix $B$ and output vertex $y_1$ corresponding to matrix $C$, and we combine them. This operation is presented in Figure 7.

By the using the digraph presented in Figure 7, we can write the set of the equations which we compare with the same power of the polynomial (16).

$$
\begin{align*}
\lambda^{-1} &\quad w(s_1, v_1) \cdot w(v_1, y_1) = b_q \\
\lambda^{-2} &\quad w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, y_1) = b_{q-1} \\
\vdots &\quad \vdots \\
\lambda^{2-q} &\quad w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, v_3) \cdots w(v_{q-1}, v_q) \cdot w(v_q, y_1) = b_2 \\
\lambda^{1-q} &\quad w(s_1, v_1) \cdot w(v_1, v_2) \cdots w(v_{q-2}, v_{q-1}) \cdot w(v_{q-1}, v_q) \cdot w(v_q, y_1) = b_1 \\
\lambda^{-q} &\quad w(s_1, v_1) \cdot w(v_1, v_2) \cdots w(v_{q-2}, v_{q-1}) \cdot w(v_{q-1}, v_q) \cdot w(v_q, y_1) \cdot w(v_{q+1}, y_1) = b_0
\end{align*}
$$
After solving the set of the equation (18) we can write \( B \) and \( C \) matrix in the following form:

\[
B = \begin{bmatrix}
  w(s_1, v_1) & 0 & 0 & \cdots & 0 \\
  w(v_1, y_1) & w(v_2, y_1) & w(v_3, y_1) & \cdots & w(v_{q-1}, y_1) \\
  0 & w(v_2, y_1) & w(v_3, y_1) & \cdots & w(v_{q-1}, y_1) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & w(v_{q-1}, y_1) \\
  0 & 0 & 0 & \cdots & w(v_q, y_1) \\
  0 & 0 & 0 & \cdots & w(v_{q+1}, y_1)
\end{bmatrix}
\]

where:

\[
w(v_1, y_1) = \frac{b_2}{w(s_1, v_1)}, \quad w(v_2, y_1) = \frac{b_{q-1}}{w(s_1, v_1)w(v_1, v_2)}, \quad w(v_3, y_1) = \frac{b_{q-2}}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3)},
\]

\[
w(v_{q-1}, y_1) = \frac{b_1}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3) \cdots w(v_{q-2}, v_{q-1})},
\]

\[
w(v_q, y_1) = \frac{b_0}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3) \cdots w(v_{q-2}, v_{q-1})w(v_{q-1}, v_q)w(v_q, v_{q+1})}
\]

\[
w(v_{q+1}, y_1) = \frac{b_0}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3) \cdots w(v_{q-2}, v_{q-1})w(v_{q-1}, v_q)w(v_q, v_{q+1})}
\]

The realisation of the (8) is given by (17) and (19).

3. Numerical Example

Example 4 Find a minimal realisation of the transfer function:

\[
T(z) = \frac{b_3s^{1.8} + b_2s^{1.2} + b_1s^{0.6} + b_0}{a_2s^{1.2} + a_1s^{0.6} + a_0}.
\]

Solution: Assuming that \( \lambda = s^{-0.6} \) and multiplying nominator and denominator of the transfer function by \( \lambda^{-3} \) we obtain:

\[
T(\lambda) = \frac{b_3 + b_2\lambda^{-1} + b_1\lambda^{-2} + b_0\lambda^{-3}}{a_2\lambda^{-1} + a_1\lambda^{-2} + a_0\lambda^{-3}}.
\]

Step 1: Using the characteristic polynomial \( d(\lambda) = a_2\lambda^{-1} + a_1\lambda^{-2} + a_0\lambda^{-3} \) and Proposition 1, we determine digraph-masks: \( \mathcal{M}_1(G) \) (Figure 8(a)), \( \mathcal{M}_2(G) \) (Figure 8(b)), \( \mathcal{M}_3(G) \) (Figure 8(c)).

![Figure 8: Digraph-masks of the characteristic polynomial](image)

Step 2: Then, we decompose the characteristic polynomial \( d(\lambda) \) into a set of simple monomials:

\[
d(\lambda) = a_2\lambda^{-1} + a_1\lambda^{-2} + a_0\lambda^{-3} = M_2 + M_1 + M_0
\]

For each monomial we determine all possible digraph-structure representations (see Figure 9).
Step 3: Now we can create all digraph realisations of the characteristic polynomial. Figure 10 presents all the possible realisations of the characteristic polynomial (22).

Then, using (13) we can determine all possible realisations of the characteristic polynomial $d(\lambda)$ as a combination of the digraph representation presented for example in Figure 10(a) and digraph-mask presented in Figure 8. After this we obtain other three possible realisations (Figure 11). It should be note that only one digraph-structure presented in Figure 11(c) satisfy characteristic polynomial $d(\lambda)$.

Step 4: Because the degree of the polynomial $n(\lambda)$ is higher than polynomial $d(\lambda)$, then – to be able to realise it – we must add one vertex and an arc in such a way that there are no additional cycles. Extended digraph is presented in Figure 12.

Now, we can write the following extended matrices

$$
E = \begin{bmatrix}
    v_1 & v_2 & v_3 & v_4 \\
    v_1 & 0 & 0 & 0 \\
    v_2 & 0 & 1 & 0 \\
    v_3 & 0 & 0 & 1 \\
    v_4 & 0 & 0 & 1
\end{bmatrix}, \quad
A = \begin{bmatrix}
    w(v_1, v_1) & w(v_2, v_1) & w(v_3, v_1) & 0 \\
    w(v_1, v_2) & w(v_2, v_2) & 0 & 0 \\
    w(v_1, v_3) & 0 & w(v_3, v_3) & 0 \\
    w(v_1, v_4) & 0 & 0 & w(v_3, v_4)
\end{bmatrix}.
$$

(23)
One of the possible numerical realisation is the following:

\[
E = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad A = \begin{bmatrix}
-a_0 & -a_1 & -a_0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

(24)

**Step 5:** We add source vertex \( s_1 \) and output vertex \( y_1 \) to the digraph presented in Figure 12, and we connect them (see Figure 13).

![Figure 13: One of the possible realisation of the transfer function (20).](image)

**Step 6:** Then, using the created digraph, we can write a set of equations in the form:

\[
\begin{align*}
\lambda^{-1} w(s_1, v_1) \cdot w(v_1, y_1) &= b_3 \\
\lambda^{-2} w(s_1, v_1) \cdot w(v_2, y_1) &= b_2 \\
\lambda^{-3} w(s_1, v_1) \cdot w(v_3, y_1) &= b_1 \\
\lambda^{-4} w(s_1, v_1) \cdot w(v_4, y_1) &= b_0 \\
\end{align*}
\]

(25)

After solving the set of the equation (25), we can write input and output matrix in the following form:

\[
B = \begin{bmatrix}
w(s_1, v_1) \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
b_3 & b_2 & b_1 & b_0 \\
w(s_1, v_1) & w(s_1, v_1) & w(s_1, v_1) & w(s_1, v_1) \\
\end{bmatrix}.
\]

(26)

The desired realisation of the (20) is given by (24) and (26) for \( w(s_1, v_1) \in \mathbb{R} \).

4. **Concluding Remarks**

A method for computation of a minimal realisation of a given proper transfer function of fractional singular one-dimensional continuous-time linear systems has been proposed. Sufficient conditions for the existence of a minimal realisation of a given proper transfer function have been established. A method based on one-dimensional digraph theory for computation of minimal realisations has been proposed. The effectiveness of the algorithm has been illustrated with some numerical examples. Extension of those considerations for fractional singular discrete-time linear systems and for fractional singular hybrid model is possible.

Formulation of the necessary and sufficient conditions for the existence of digraphs structures of the minimal positive realisation problem for one-dimensional singular discrete-time and continuous-time linear systems remains an open problem.
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