Generalized Localization for Spherical Partial Sums of Multiple Fourier Series

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Abstract
In this paper the generalized localization principle for the spherical partial sums of the multiple Fourier series in the \(L^2\) class is proved, that is, if \(f \in L^2(\mathbb{T}^N)\) and \(f = 0\) on an open set \(\Omega \subset \mathbb{T}^N\), then it is shown that the spherical partial sums of this function converge to zero almost-everywhere on \(\Omega\). It has been previously known that the generalized localization is not valid in \(L^p(\mathbb{T}^N)\) when \(1 \leq p < 2\). Thus the problem of generalized localization for the spherical partial sums is completely solved in \(L^p(\mathbb{T}^N)\), \(p \geq 1\): if \(p \geq 2\) then we have the generalized localization and if \(p < 2\), then the generalized localization fails.

Keywords Multiple Fourier series · Spherical partial sums · Convergence almost-everywhere · Generalized localization

Mathematics Subject Classification Primary 42B05 · Secondary 42B99

1 Introduction

Let \(\{f_n\}, n \in \mathbb{Z}^N\), be the Fourier coefficients of a function \(f \in L_2(\mathbb{T}^N), \ N \geq 2\), where \(\mathbb{T}^N\) is \(N\)-dimensional torus. Consider the spherical partial sums of the multiple Fourier series:

\[
S_\lambda f(x) = \sum_{|n|^2 < \lambda} f_n e^{inx}.
\]  

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The aim of this paper is to investigate convergence almost-everywhere (a.e.) of these partial sums. One of the first questions which arises in the study of a.e. convergence of the sums (1.1) is the question of the validity of the Luzin conjecture: is it true that the spherical sums (1.1) of the Fourier series of an arbitrary function \( f \in L^2(\mathbb{T}^N) \) converge a.e. on \( \mathbb{T}^N \)? In other words, does Carleson’s theorem extend to \( N \)-fold Fourier series when the latter is summed spherically? The answer to this question is unknown so far. What is known is only that Hunt’s theorem (convergence a.e. for \( L^p \) functions) does not extend to \( N \)-fold (\( N \geq 2 \)) series summed by circles (see [1] and references therein). Historically progress with solving the Luzin conjecture has been made by considering easier problems. One of such easier problems is to investigate convergence a.e. of the spherical sums (1.1) on \( \mathbb{T}^N \setminus \text{supp} f \).

Il’in [10] was the first to introduce the concept of generalized principle of localization for an arbitrary eigenfunction expansions. Following Il’in we say that the generalized localization principle for \( S_\lambda \) holds in \( L^p(\mathbb{T}^N) \), if for any function \( f \in L^p(\mathbb{T}^N) \) the equality

\[
\lim_{\lambda \to \infty} S_\lambda f(x) = 0
\]

holds a.e. on \( \mathbb{T}^N \setminus \text{supp} f \).

Recall that the validity of the classical Riemann localization principle means that the convergence at a point \( x_0 \) of the Fourier series of a function \( f \) depends only on the behavior of \( f \) in a small neighborhood of that point. More exactly, if \( f = 0 \) in an open set \( \Omega \subset \mathbb{T}^N \) then the Fourier series of \( f \) converges to 0 at every point of \( \Omega \). For generalized localization one requires that the convergence to 0 occurs a.e. on \( \Omega \).

According to the classical Riemann theorem, in the one-dimensional case, the localization principle is valid for any integrable function. The situation changes in the case of functions of two or more variables. In this case, there are examples of functions with high smoothness for which the classical principle of localization is not valid (see [1]). In this case, the principle of generalized localization, introduced by Il’in, may become a replacement for the classical principle of localization.

For the spherical partial integrals of multiple Fourier integrals (we denote by \( \sigma_\lambda f(x) \)) the generalized localization principle in \( L^p(\mathbb{R}^N) \) has been investigated by many authors (see [4–9,12]). In particular, in the remarkable paper of Carbery and Soria [6] the validity of the generalized localization for \( \sigma_\lambda \) has been proved in \( L^p(\mathbb{R}^N) \) when \( 2 \leq p < 2N/(N - 1) \). Note, that the method introduced by these authors can be easily applied to non-spherical partial integrals too [3].

If we turn back to the multiple Fourier series (1.1) and consider the classes \( L^p(\mathbb{T}^N) \) when \( 1 \leq p < 2 \), then as Bastys [5] has proved, following Fefferman in making use of the Kakeya’s problem, that the generalized localization for \( S_\lambda \) is not valid, i.e. there exists a function \( f \in L^p(\mathbb{T}^N) \), such that on some set of positive measure, contained in \( \mathbb{T}^N \setminus \text{supp} f \), we have

\[
\lim_{\lambda \to \infty} |S_\lambda f(x)| = +\infty.
\]
It may be worth mentioning that in [5] this result is also proved for the spherical partial integrals \( \sigma_\lambda f(x) \).

The main result of this paper is the following statement.

**Theorem 1.1** Let \( f \in L_2(\mathbb{T}^N) \) and \( f = 0 \) on an open set \( \Omega \subset \mathbb{T}^N \). Then the equality (1.2) holds a.e. on \( \Omega \).

Thus the problem of generalized localization for \( S_\lambda \) is completely solved in classes \( L_p(\mathbb{T}^N) \), \( p \geq 1 \): if \( p \geq 2 \) then we have the generalized localization and if \( p < 2 \), then the generalized localization fails.

In the study of a.e. convergence it is convenient to introduce the maximal operator

\[
S_\ast f(x) = \sup_{\lambda > 0} |S_\lambda f(x)|.
\]

The proof of Theorem 1.1 is based on the following estimate of this operator.

**Theorem 1.2** Let \( \Omega \) be an open subset of \( \mathbb{T}^N \). Then for any compact set \( K \subset \Omega \) there exists a constant \( C_K > 0 \) such that for any function \( f \in L_2(\mathbb{T}^N) \) with \( \text{supp } f \subset \mathbb{T}^N \setminus \Omega \) one has

\[
\|S_\ast f(x)\|_{L_2(K)} \leq C_K \|f\|_{L_2(\mathbb{T}^N)}.
\]

The formulated theorems are easily transferred to the case of non-spherical partial sums of multiple Fourier series (see [2,3]).

We should also note, that in the remarkable paper of Kenig and Tomas [11] the authors proved, by making use of transference techniques, the equivalence of convergence almost-everywhere of spherical partial sums of multiple Fourier series and integrals. It may be worth mentioning that Theorem 1.1 on the generalized localization does not follow immediately from the Carbery–Soria result [6], together with transferred theory [11], since applying this procedure one cannot control the support of the considering function. Nevertheless, to prove Theorem 1.1 we have used many of original ideas from Carbery and Soria [6].

### 2 Auxiliary Assertions

It is not hard to verify, that Theorem 1.2 is an easy corollary of the following theorem:

**Theorem 2.1** Let \( f \in L_2(\mathbb{T}^N) \) and \( f = 0 \) on the ball \( \{|x| < R\} \subset \mathbb{T}^N \). Then for any \( r < R \) there exists a constant \( C = C(R, r) \), such that

\[
\int_{|x| \leq r} |S_\ast f(x)|^2 dx \leq C \int_{\mathbb{T}^N} |f(x)|^2 dx. \tag{2.1}
\]
The proof of Theorem 2.1 is based on several auxiliary assertions, which are given in this section. So we assume that $f = 0$ on the fixed ball $\{|x| < R\} \subset \mathbb{T}^N$ and fix a number $r < R$.

Let $\chi_b(t)$ be the characteristic function of the segment $[0, b]$. We denote by $\varphi_1(t)$ a smooth function with $\chi_{(R-r)/3}(t) \leq \varphi_1(t) \leq \chi_{2(R-r)/3}(t)$ and put $\varphi_2(t) = 1 - \varphi_1(t)$.

Now we define a new function $\psi(x)$ as follows: $\psi(x) = \varphi_2(|x|)$, when $x \in \mathbb{T}^N$ and otherwise it is a $2\pi$-periodical on each variable $x_j$ function.

Let us denote $\theta(x, \lambda) = (2\pi)^{-N} \sum_{|n|^2 < \lambda} e^{inx}$.

Then by definition of the Fourier coefficients we may write

$$S_\lambda f(x) = \int_{\mathbb{T}^N} \theta(x - y, \lambda) f(y) dy.$$ 

If we define $\theta_\lambda(x) = \theta(x, \lambda) \psi(x)$, then we have

$$S_\lambda f(x) = \int_{\mathbb{T}^N} \theta_\lambda(x - y) f(y) dy, \text{ for all } x, \text{ with } |x| \leq r,$$

since $f$ is supported in $\{|x| \geq R\}$. Therefore, if we denote the last integral by $\theta_\lambda * f$, then to prove the estimate (2.1) it suffices to obtain the inequality

$$\int_{\mathbb{T}^N} \sup_{j > 0} |\theta_j * f|^2 dx \leq C \int_{\mathbb{T}^N} |f(x)|^2 dx,$$  \hspace{1cm} (2.2)

where sup is taken over all positive integers.

Now we need some estimates for the Fourier coefficients of the function $\theta_j(x)$, which we denote by $(\theta_j)_n, j \in \mathbb{N}$ (the set of positive integers), $n \in \mathbb{Z}^N$.

**Lemma 2.2** For an arbitrary $l \in \mathbb{N}$ there exists a constant $C_l$, depending on $l$, $r$ and $R$, such that for all $j \in \mathbb{N}$ and $n \in \mathbb{Z}^N$ one has

$$|\theta_j)_n| \leq \frac{C_l}{(1 + ||n| - \sqrt{j}|)^l}.$$ 

**Proof** Let $\{\psi_m\}$ be the Fourier coefficients of the function $\psi(x)$: $\psi_m = (2\pi)^{-N} \int_{\mathbb{T}^N} \psi(y) e^{-imy} dy$. Then

$$(\theta_j)_n = (2\pi)^{-2N} \int_{\mathbb{T}^N} \sum_{|m| < \sqrt{j}} e^{inx} \psi(x) e^{-inx} dx = (2\pi)^{-N} \sum_{|n - m| < \sqrt{j}} \psi_m.$$
If \(|n| > \sqrt{j}\) then we have

\[
(2\pi)^{-N} | \sum_{|n-m| < \sqrt{j}} \psi_m | \leq (2\pi)^{-N} \sum_{|m| > |n| - \sqrt{j}} |\psi_m|. 
\]

Similarly, if \(|n| \leq \sqrt{j}\) then making use of the equality (observe, \(\psi\) is an infinitely differentiable and \(2\pi\)-periodical function) \(\sum \psi_m = \psi(0) = 0\), we obtain

\[
(2\pi)^{-N} | \sum_{|n-m| < \sqrt{j}} \psi_m | = (2\pi)^{-N} - \sum_{|n-m| \geq \sqrt{j}} |\psi_m| \leq (2\pi)^{-N} \sum_{|m| \geq \sqrt{j} - |n|} |\psi_m|. 
\]

Now it is sufficient to note that for any integer \(q \geq 0\) there exists a constant \(c_q\), depending on \((R - r)\), such that

\[
|\psi_m| \leq \frac{c_q}{(1 + |m|)^q}, \tag{2.3}
\]

and to estimate the last sum by comparing it with the corresponding integral. \(\square\)

We will apply the estimate (2.3) further, so the corresponding constants will depend on \(r\) and \(R\). In addition, as we have done above, in order to estimate number series we compare them with the corresponding integrals.

Let \((\Theta_j)_n = (\theta_{j+1})_n - (\theta_j)_n\), that is,

\[
(\Theta_j)_n = (2\pi)^{-N} \sum_{|m|^2 = j} \psi_{m-n} = (2\pi)^{-N} \sum_{|n-m|^2 = j} \psi_m
\]

(if the Diophantine equation \(|m-n|^2 = j\) does not have a solution, then \((\Theta_j)_n = 0\).

These numbers have a better estimate than \((\theta_j)_n\) in the following sense. Suppose \(k \leq \sqrt{j} < k + 1\), i.e. \(k^2 \leq j < k^2 + 2k + 1\), or \(j = k^2 + p, 0 \leq p < 2k + 1\), then according to Lemma 2.2, \((\theta_j)_n\) has the same estimate. But, as we will see below, the numbers \((\Theta_j)_n\) vanish in some sense on the same interval. In particular, the following statement is true.

**Lemma 2.3** For any \(l \in \mathbb{N}\), there exists a constant \(C_l\) such that

\[
\sum_{k \leq \sqrt{j} < k + 1} |(\Theta_j)_n|^2 \leq \frac{C_l}{(1 + |n| - k)^l}. \tag{2.4}
\]

**Proof** Let \(|n| \leq k\); otherwise estimates are similar. By virtue of estimate (2.3) we have

\[
\sum_{k \leq \sqrt{j} < k + 1} |(\Theta_j)_n| \leq (2\pi)^{-N} \sum_{k \leq |n-m| < k + 1} |\psi_m| \\
\leq (2\pi)^{-N} \sum_{|m| > |n| - k} \frac{c_q}{(1 + |m|)^q} \leq \frac{C_l}{(1 + |n| - k)^l}.
\]

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Since \(|(\Theta_j)_n|^2 \leq C|(\Theta_j)_n|\), Lemma is proved. \(\square\)

**Corollary 2.4** Uniformly in \(n \in \mathbb{Z}^N\) one has

\[
\sum_{j=0}^{\infty} |(\Theta_j)_n|^2 = \sum_{k=0}^{\infty} \sum_{k \leq \sqrt{J} < k+1} |(\Theta_j)_n|^2 \leq C.
\]

Let us fix \(n \in \mathbb{Z}^N\) (say, from the first quadrant) and integer \(k \in \mathbb{N}\). Let \(k \leq \sqrt{J} < k + 1\), i.e. \(j = k^2 + p, p = 0, 1, \ldots, 2k\). To prove Theorem 2.1 we need the same estimate as (2.4) for the sum \(\sum_{p=0}^{2k} (p+1)^2 |(\Theta_{k^2+p})_n|^2\) (that is, the value under the sum (2.4) is multiplied by \((p+1)^2\)). To obtain this estimate we must show, that \(|(\Theta_{k^2+p})_n|^2\) decreases fast enough in \(p\). Unfortunately this is not always the case. Nevertheless if we change the order of summation in the last sum (since the summands are positive, this is always possible), say first take the sum over some sets \(Q_k^q, q = 0, 1, \ldots, 2k\). If \(q\) varies from 0 to 2, then we assign to the set \(Q_{k^2+p}\) the set \(\Theta_{k^2+p}\). Let \(\Theta_{k^2+p}\) be the tangential hyperplane to the ball \(B_j\) at the point \(y\). Let \(Q_k^q = \{x \in \mathbb{R}^N : |x - y| = (k+1)\}\) and divide it in to the following sets: \(P_k = K \cap C_j, j = 0, 1, \ldots, 2k - 1\).

Let us define the sets \(Q_k^q, q = 0, 1, \ldots, 2k - 1\), as follows. Let \(Q_k^q\) be the set of those integers \(p\), \(0 \leq p \leq 2k\), for which the Diophantine equation \(|m - n|^2 = k^2 + p\) has a solution in \(P_k^q\). If \(P_k^q\) does not contain any of solutions of equation \(|m - n|^2 = k^2 + p\), for any \(p\), then we assign to the set \(Q_k^q\) one of those parameters \(p\) that are not included in the previous sets \(Q_j^q, j = 0, 1, \ldots, q - 1\). If there are no such \(p's\) left, then we define \(Q_j^q, j = q, q + 1, \ldots, 2k - 1\) as the empty set.

In the proof of Lemma 2.8 we need to know how many at most parameters \(p\) does the set \(Q_k^q\) contain. The length of the projection of \(P_k^q\) on the axis of \(Ox_1\) is at most \(2\sqrt{q + T}\). Consequently, if, for a fixed \(p\), there is a solution of the Diophantine equation \(|m - n|^2 = k^2 + p\), provided \(m \in P_k^q\), then the first coordinates \(m_1\) of the numbers \(m\), take at most \(2\sqrt{q + T}\) \((|a|\) is the integer part of the number \(a)\) different values. When \(p\) varies from 0 to 2, each of these numbers \(m_1\) can repeat at most two times. Hence each set \(Q_k^q\) has at most \(4\sqrt{q + T}\) parameters \(p\) with the above property.

With this choice of \(Q_k^q\) we have the following statement.

**Lemma 2.5** Let \(q = 0, 1, \ldots, 2k - 1\) and \(S_p = \{m \in \mathbb{Z}^N : |m - n|^2 = k^2 + p\}\) \((p = 0, 1, \ldots, 2k)\). If \(|n| \geq k + 1\), then
\[
\min_{m \in \mathcal{S}_p, \; p \in Q^k_q} |m| \geq \sqrt{|n| - k - 1}^2 + q. \tag{2.5}
\]

If \( k < |n| < k + 1 \), then
\[
\min_{m \in \mathcal{S}_p, \; p \in Q^k_q} |m| \geq \sqrt{q}.
\]

If \( |n| \leq k \), then
\[
\min_{m \in \mathcal{S}_p, \; p \in Q^k_q} |m| \geq \frac{1}{2} \sqrt{(|n| - k)^2 + q}.
\]

**Proof** Note that it is sufficient to estimate the minimum distance from the origin to the set \( P^k_q \). If \( |n| \geq k + 1 \), then it is not hard to verify that the distance from the origin to the set \( B^k_q \) is equal to \( \sqrt{(|n| - k - 1)^2 + q} \). Obviously, this value is less or equal to the distance between the origin and \( P^k_q \). In case of \( k < |n| < k + 1 \), the arguments are similar.

If \( |n| \leq k \), then minimum distance from the origin to the set \( P^k_q \) is less than or equal to \( \sqrt{(|n| - \sqrt{k^2 - q})^2 + q} \). But we can estimate this number from below by \( \frac{1}{2} \sqrt{(|n| - k)^2 + q} \). \( \square \)

As we mentioned above for \( (\Theta_j)_n \) one has a more stronger result than Lemma 2.3.

**Lemma 2.6** For any \( l \in \mathbb{N} \), there exists a constant \( C_l \) such that
\[
\sum_{q=0}^{2k-1} (q+1)^2 \sum_{p \in Q^k_q} |(\Theta_{k^2+p})_n|^2 \leq \frac{C_l}{(1 + \sqrt{|n| - k})^l}. \tag{2.6}
\]

**Proof** From the definition of \( (\Theta_j)_n \), one has
\[
\sum_{q=0}^{2k-1} (q+1)^2 \sum_{p \in Q^k_q} |(\Theta_{k^2+p})_n| \leq (2\pi)^{-N} \sum_{q=0}^{2k-1} (q+1)^2 \sum_{p \in Q^k_q} \sum_{m-n^2=k^2+p} |\psi_m| \leq
\]

(and by virtue of estimates (2.3) and (2.5) (we assume that \( |n| \geq k + 1 \); otherwise arguments are similar) we finally have)
\[
\leq \sum_{q=0}^{2k-1} (q+1)^2 \sum_{|m| \geq \sqrt{|n| - k - 1}^2 + q} \frac{c_j}{(1 + |m|)^j} \leq \frac{C_l}{(1 + \sqrt{|n| - k})^l}. \]

Now (2.6) follows from the estimate \( |(\Theta_j)_n|^2 \leq C |(\Theta_j)_n| \). \( \square \)

The next statement is an easy consequence of this Lemma.
Corollary 2.7 Uniformly in $n \in \mathbb{Z}^N$, one has
\[
\sum_{k=0}^{2k-1} \sum_{q=0}^{(q+1)^2} \sum_{p \in Q^k_q} |(\Theta_{k^2+p})n|^2 \leq C. \tag{2.7}
\]

Now we turn back to the Fourier coefficients $(\theta_j)_n$. From Lemma 2.2 we have the following estimate.

Lemma 2.8 Uniformly in $n \in \mathbb{Z}^N$, one has
\[
\sum_{k=0}^{2k-1} \sum_{q=0}^{(q+1)^2} \sum_{p \in Q^k_q} |(\theta_{k^2+p})n|^2 \leq C. \tag{2.8}
\]

Proof As mentioned above, each $Q^k_q$ has at most $4[\sqrt{q+1}]$ parameters $p$. Therefore, by virtue of Lemma 2.2, one has
\[
\sum_{k=0}^{2k-1} \sum_{q=0}^{(q+1)^2} \sum_{p \in Q^k_q} |(\theta_{k^2+p})n|^2 \leq \sum_{k=0}^{2k-1} \sum_{q=0}^{(q+1)^2} \sum_{p \in Q^k_q} |(\Theta_{k^2+p})n|^2 \leq C.
\]
\]

3 Proofs of Theorems

First, we prove the estimate (2.2). Let $\Theta_j(x) = \theta_{j+1}(x) - \theta_j(x)$. Then $\theta_{j+1} * f + \theta_j * f = 2 \theta_j * f + \Theta_j * f$. Note the Fourier coefficients of the function $\Theta_j(x)$ are the numbers $(\Theta_j)_n$, introduced above.

If for a sequence of numbers $\{F_q\}, q \in \mathbb{N}$, we have $F_0 = 0$, then
\[
F_q^2 = \sum_{j=0}^{q-1} [F_{j+1} - F_j][F_{j+1} + F_j].
\]

Hence
\[
[\theta_q * f]^2 = \sum_{j=0}^{q-1} [\Theta_j * f]^2 + 2 \sum_{j=0}^{q-1} [\Theta_j * f] \theta_j * f,
\]
or
\[
\sup_{q \in \mathbb{N}} |\theta_q * f|^2 \leq \sum_{j=0}^{\infty} |\Theta_j * f|^2 + 2 \sum_{k=0}^{\infty} \sum_{q=0}^{2k-1} |\Theta_{k^2+p} * f |(q+1)|\Theta_{k^2+p} * f |(q+1)^{-1}.
\]
Integrating over $T^N$ and making use of the inequality $2ab \leq a^2 + b^2$ one obtains

$$\int_{T^N} \sup_{q \in \mathbb{N}} |\theta_q \ast f|^2 \leq \sum_{n} |f_n|^2 \sum_{j=0}^{\infty} |(\Theta_j)_{n}|^2$$

$$+ \sum_{n} |f_n|^2 \sum_{k=0}^{\infty} \sum_{q=0}^{2k-1} (q + 1)^2 \sum_{p \in Q_k^q} |(\Theta_{k^2+p})_{n}|^2$$

$$+ \sum_{n} |f_n|^2 \sum_{k=0}^{\infty} \sum_{q=0}^{2k-1} (q + 1)^{-2} \sum_{p \in Q_k^q} |(\theta_{k^2+p})_{n}|^2 \leq$$

(making use of Corollaries 2.4, 2.7, Lemma 2.8 and the fact that $f$ is an $L_2$-function)

$$\leq C \sum_{n} |f_n|^2 = C \int_{T^N} |f(x)|^2 dx.$$

Thus, the estimate (2.2) and, consequently, Theorem 1.2 is proved.

Theorem 1.1 can be proved by a standard technique based on Theorem 1.2 (see [13]).

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