HODGE STRUCTURES AND WEIERSTRASS $\sigma$-FUNCTION

GRZEGORZ BANASZAK* AND JAN MILEWSKI**

Abstract. In this paper we introduce new definition of Hodge structures and show that $\mathbb{R}$-Hodge structures are determined by $\mathbb{R}$-linear operators that are annihilated by the Weierstrass $\sigma$-function.

1. Introduction

Classically a Hodge structure of a given weight can be defined in the four equivalent ways as follows (see e.g. [2, 5]):

Definition 1.1. A Hodge structure of a weight $n$ on a real vector space $V$ consists of a finite–dimensional $\mathbb{R}$–vector space $V$ together with any of the following equivalent data:

(i) A decomposition $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$, called the Hodge decomposition, such that $V^{p,q} = V^{q,p}$.

(ii) A decreasing filtration $F^r V_{\mathbb{C}}$ of $V_{\mathbb{C}}$, called the Hodge filtration, such that $F^r V_{\mathbb{C}} \oplus V^{n-r+1}_{\mathbb{C}} = V_{\mathbb{C}}$.

(iii) A homomorphism $h_n : S \to GL(V_{\mathbb{R}})$ of real algebraic groups, and also specifying that the weight of the Hodge structure is $n$.

(iv) A homomorphism $h_n : S \to GL(V_{\mathbb{R}})$ of real algebraic groups such that via the decomposition $\mathbb{G}_m / \mathbb{R} \to S \to GL(V_{\mathbb{R}})$ an element $t \in \mathbb{G}_m / \mathbb{R}$ acts as $t^{-n} \cdot Id$.

Throughout the paper we work with Hodge structures of various weights, hence by a Hodge structure we understand here a finite direct sum

$$(1) \quad \rho := \bigoplus_{j=1}^k h_{n_j}$$

of representations $h_{n_j}$ described in (iii) or (iv) of the Definition 1.1.

In this paper we consider Hodge structures on real vector space $V$ via representations of the Lie algebra of the real algebraic group $S$ (denoted also $\mathbb{C}^\times$) on $V$.

In section 2 we show that a Hodge structure can be treated as a pair of operators $E, T$ on $V$ satisfying certain conditions (see Theorem 2.1). In section 3 we show that a Hodge structure can be treated as a single operator $S := E + T$ on $V$ such that $\sigma(S) = 0$ for a Weierstrass $\sigma$-function which corresponds to decomposition of $V$ into eigenspaces of operators $E$ and $T$. Weirstrass $\sigma$-function does not have multiple zeros hence this corresponds to the fact that complexification of $S$ does not have generalized eigenvectors other than ordinary ones.

2010 Mathematics Subject Classification. 14D07.

Key words and phrases. Hodge structure, Weierstrass $\sigma$ function.

*Partially supported by the NCN (National center of Science for Poland) NN201 607440

**Partially supported by the NCN grant NN201 373236.
2. HODGE STRUCTURES AND LIE ALGEBRAS.

The following theorem gives another definition of the Hodge structure.

**Theorem 2.1.** Let $V$ be a finite dimensional vector space over $\mathbb{R}$. There is a one to one correspondence between the family of Hodge structures on $V$ and the family of pairs of endomorphisms $E, T \in \text{End}_\mathbb{R}(V)$ satisfying the following conditions:

1. $[E, T] = 0$, $\sin(\pi E) = 0$, $\sinh(\pi T) = 0$,
2. $\sin(\frac{\pi}{2}(E^2 + T^2)) = 0$

**Proof.** Consider a Hodge structure on $V$. By (1) (cf. Definition 1.1 (iii)) this gives a representation:

$$\rho : \mathbb{S} \to \text{GL}(V)$$

of real algebraic groups. The representation $\rho$ decomposes into irreducible representations $\rho_{p,q}$ with multiplicities $m_{p,q}$

$$\rho = \bigoplus_{q \leq p} m_{p,q} \rho_{p,q},$$

$$\rho_{p,q}(re^{i\phi}) := r^{p+q} \begin{bmatrix} \cos(p-q)\phi & -\sin(p-q)\phi \\ \sin(p-q)\phi & \cos(p-q)\phi \end{bmatrix} \text{ for } p \neq q, p, q \in \mathbb{Z}$$

$$\rho_{p,p}(re^{i\phi}) := r^{2p} \begin{bmatrix} 1 \end{bmatrix}.$$ 

Certainly, the complexification of the representation $\rho_{p,q}$ for $q < p$ decomposes into two one-dimensional $\mathbb{C}$-vector spaces:

$$\rho_{p,q} \otimes \mathbb{C} = \rho_{p,q}^\mathbb{C} \oplus \rho_{q,p}^\mathbb{C},$$

where

$$\rho_{m,n}^\mathbb{C}(z) = z^m \bar{z}^n.$$ 

Consider the real Lie algebra representation (the derivative of $\rho$):

$$\mathcal{L}(\rho) : \mathbb{C} \to \text{End}(V).$$

For $q \leq p$ the representation $\mathcal{L}(\rho_{p,q})$ is also two-dimensional

$$\mathcal{L}(\rho_{p,q})(1) = (p+q)I \text{ and } \mathcal{L}(\rho_{p,q})(i) = (p-q)J,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $p = q$

$$\mathcal{L}(\rho_{p,p})(1) = 2p \text{ and } \mathcal{L}(\rho_{p,p})(i) = 0.$$ 

If we put $E := \mathcal{L}(\rho)(1)$ and $T := \mathcal{L}(\rho)(i)$ then we get equations (2) and (3). The condition (3) is fulfilled because $p-q$ and $p+q$ have the same parity.

Now let us assume that conditions (2) and (3) hold. Observe that $\sinh(z)$ and $\sin(z)$ have single zeros in the complex plane. Moreover (2) and (3) imply that the complexifications $E \otimes 1$ and $T \otimes 1 \in \text{End}(\mathbb{C} \otimes \mathbb{R} \otimes V)$ have common eigenbasis. From this it follows that the endomorphisms $E, T \in \text{End}_\mathbb{R}(V)$ have common Jordan decomposition into eigenspaces of dimension 1 or 2. We define a representation

$$\rho : \mathbb{C}^\times \to \text{GL}(V),$$
\( \rho(e^{x+iy}) = \exp(xE + yT) \) for \( x, y \in \mathbb{R} \).

\( \rho \) is an algebraic representation, because the equality (3) holds. The representation \( \rho \) gives the Hodge structure on \( V \).

\[ \square \]

3. Hodge structures via single operator

Let \( \sigma(z) \) be the Weierstrass' sigma function for the lattice generated by \( \omega_1 = 1 - i \) and \( \omega_2 = 1 + i \):

\[
\sigma(z) := z \prod_{(k_1, k_2) \neq (0,0)} \left( 1 - \frac{z}{k_1 \omega_1 + k_2 \omega_2} \right) \exp \left[ \frac{z}{k_1 \omega_1 + k_2 \omega_2} + \frac{1}{2} \left( \frac{z}{k_1 \omega_1 + k_2 \omega_2} \right)^2 \right]
\]

**Theorem 3.1.** For operators \( E, T \in \text{End}_\mathbb{R}(V) \) considered above let \( S := E + T \). We get the following equality

(6) \( \sigma(S) = 0 \).

Conversely every \( S \in \text{End}_\mathbb{R}(V) \) satisfying condition (6) gives a unique pair \( (E, T) \) of operators in \( \text{End}_\mathbb{R}(V) \) such that \( S = E + T \) and the conditions (2) and (3) hold.

**Proof.** It is clear that \( S = E + T \) satisfies the equation (6). Conversely, assume that an operator \( S \in \text{End}_\mathbb{R}(V) \) satisfies (6). Since the \( \sigma \) function has zeros of order 1, we observe that the complexification of \( S \) is diagonalizable. We get the operators \( E \) and \( T \) considering equation

(7) \( S(v) = \lambda v \)

in the complexification of \( V \). The eigenvalues have integer real and imaginary parts with the same parity:

(8) \( \lambda = a + ib, \quad a, b \in \mathbb{Z}, \quad a - b \in 2\mathbb{Z}. \)

Moreover we define the operators \( E, T \) in such a way that their complexifications acting on the eigenvector \( v \) of \( S \) have form: \( E(v) = av \) i \( T(v) = bv \) where \( S(v) = (a + ib)v \). Operators \( E \) and \( T \) satisfy equations (2) and (3). The operators \( E \) and \( T \) are uniquely determined. Indeed, if \( S = E' + T' \) such that \( E' \) and \( T' \) satisfy (2) and (3) then it is clear that \([E', S] = 0\) and \([T', S] = 0\). \( \square \)

**Remark 3.2.** For certain Hodge structures the set of eigenvalues of the complexification of \( S \) has further obstructions beyond (8). In this case \( S \) satisfies the equation \( g(S) = 0 \), where \( g(z) \) is an analytic function that divides \( \sigma(z) \) in such a way that \( \sigma(z)/g(z) \) is also an analytic function on the whole complex plane.

**Remark 3.3.** In our work in progress we define certain deformations of Hodge structures that arise in a natural way in mathematical physics (see [1], [3], [4]).
References

[1] G. Banaszak, J. Milewski, Hodge structures in topological quantum mechanics, J. Phys. Conf. Ser. 213, 012017 (2010).

[2] B. Gordon, A survey of the Hodge conjecture for abelian varieties, Appendix B in “A survey of the Hodge conjecture”, by J. Lewis, 297–356 (1999) American Mathematical Society

[3] J. Milewski, Holomorphons and the standard almost complex structure on $S^6$, Commentationes Mathematicae, XLVI (2) (2006), 245–254.

[4] J. Milewski, Holomorphons on spheres, Commentationes Mathematicae, B XLVIII (2), 13-22, (2008).

[5] C. Peters, J. Steenbrink, Mixed Hodge structures Ergebnisse der Math. Springer 52, (2008).

Department of Mathematics and Computer Science, Adam Mickiewicz University, Poznań 61614, Poland
E-mail address: banaszak@amu.edu.pl

Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland
E-mail address: jsmilew@wp.pl