Exact analysis of scaling and dominant attractors beyond the exponential potential

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Abstract

By considering the potential parameter $\Gamma$ as a function of another potential parameter $\lambda$ (Zhou \textit{et al} 2008 \textit{Phys. Lett. B} \textbf{660} 7–12), we successfully extend the analysis of a two-dimensional autonomous dynamical system of a quintessence scalar field model to the analysis of a three-dimensional system, which enables us to study the critical points of a large number of potentials beyond the exponential potential exactly. We find that there are ten critical points in all, three points $P_3, P_5, P_6$ are general points which are possessed by all quintessence models regardless of the form of potentials and the rest of the points are closely connected to the concrete potentials. It is quite surprising that, apart from the exponential potential, there are a large number of potentials which can give a scaling solution when the function $f(\lambda) = \Gamma(\lambda) - 1$ equals zero for one or some values of $\lambda_*$ and if the parameter $\lambda_*$ also satisfies condition (16) or (17) at the same time. We give the differential equations to derive these potentials $V(\phi)$ from $f(\lambda)$. We also find that, if some conditions are satisfied, the de-Sitter-like dominant point $P_4$ and the scaling solution point $P_9$ (or $P_{10}$) can be stable simultaneously unlike $P_9$ and $P_{10}$. Although we survey scaling solutions beyond the exponential potential for ordinary quintessence models in standard general relativity, this method can be applied to other extensively scaling solution models studied in the literature (Copeland \textit{et al} 2006 \textit{Int. J. Mod. Phys. D} \textbf{15} 1753) including coupled quintessence, (coupled-)phantom scalar field, $k$-essence and even beyond the general relativity case $H^2 \propto \rho^n$. We also discuss the disadvantage of our approach.

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1. Introduction

Scalar fields have played an essential role in modern cosmology in the past half-century. This assumed scalar field has been used for various purposes in different cosmological research aspects [1], such as to drive inflation, to explain a time variable cosmological ‘constant’ and so on. Especially after the discovery of the accelerating expansion of universe, it has become another topic of hot discussion as the candidate for dark energy. There are so many scalar field dark energy models, such as the quintessence model [2–13], non-canonical scalar field model (including k-essence [14–17], phantom [18–22], B-I scalar field [23–29] and so on) and coupled scalar field model [30, 31]. There are also detailed studies on the multi-scalar field models which give an effective state equation $w_{\text{eff}}$ passing through the phantom divide line ($w = -1$) [32–34]. Some of these multi-scalar field models [35, 36] can always evolve into the regime of scalar field dominance $\lambda_{\text{eff}}^2 > 3\gamma$ even if each field has a potential which is too steep to drive the accelerating expansion. For all of these scalar field models we mentioned above, the important thing is to choose a different form of kinetic terms and different potentials from a fundamental physical motivation or directly from the observation. As expected, these different scalar field models will give different cosmological evolutions, different evolutions of state equation $w$, different values of sound speed $c_s^2$ and different cosmological perturbations. Hence, they can in principle be distinguished or excluded by the increasing observation data.

The phase-plane analysis of the cosmological autonomous system is an effective method to find the cosmological scaling and dominant attractor solutions. A phase-plane analysis of cosmologies containing a barotropic fluid and a scalar field with an exponential potential was presented [37]. Hao and Li studied the attractor solution of a phantom scalar field with the exponential potential [38, 39]. On the other hand, Amendola [31] considered the case of coupled quintessence. The case of a phantom scalar field interacting with dark matter was also investigated [40, 41]. Guo et al also investigated the properties of the critical points of a multi-field model with an exponential potential [42, 43] One may realize that the potentials investigated in all these papers are of an exponential form. Disregarding the important roles of the exponential potential in higher order or higher dimensional gravity theories and string or Kaluza–Klein-type models, the reason that why they choose the exponential potential may be that only the exponential potential can give a two-dimensional autonomous system. Since in this case the value of parameter $\Gamma$ equals 1 and that of another parameter $\lambda$ equals a constant (see equation (4) for the definition of parameters $\Gamma$ and $\lambda$), the system (see equations (5)–(7)) will reduce to the two-dimensional autonomous system. However, authors also considered the more complicated case when $\lambda$ is a dynamically changing quantity [44–46]. They applied the discussion of constant $\lambda$ to this case and obtained the so-called instantaneous critical points. For example, if $\Gamma$ is a constant (but does not equal one), say $\Gamma = (n + 1)/n$, then the corresponding potential is the inverse power-law potential $V(\phi) = V_0\phi^{-n}$ with $n > 0$. One of the critical points $(x_c, y_c) = (\lambda/\sqrt{6}, [1 - \lambda^2/6]^{1/2})$ will become the ‘instantaneous’ critical point $(x(N) = \lambda(N)/\sqrt{6}, y(N) = [1 - \lambda(N)^2/6]^{1/2})$. When $\Gamma > 0$, $\lambda(N)$ will decrease toward zero and then the ‘instantaneous’ critical points will eventually approach $x(N) \to 0$ and $y(N) \to 1$. This method is not exact here, and the critical point is obviously not a true critical point. Recently, a solution to a multiple attractor in a three-dimensional autonomous system of the quintessential models was studied in [47]. After writing the parameter $\Gamma$ as a function of $\lambda$, the author obtained a tracker solution which is different from that discovered before and found a solution to the multiple attractor. Here, we will extend the idea to an arbitrary function $\Gamma(\lambda)$. We will find out all the critical points of the dynamical autonomous system and then investigate the properties of the critical points and their cosmological implications in general. Regarding parameter $\Gamma$ as a function of $\lambda$ is quite an efficient approach since we
can investigate many quintessence models with different potentials. Giving a concrete form of function $\Gamma(\lambda)$ is equivalent to giving a concrete form of potential $V(\phi)$ since we can, in principle, figure out the potential via the relation between parameters $\Gamma$ and $\lambda$. What are the general properties of the critical points when we consider the three-dimensional autonomous system? Does a scaling solution also exist when we consider any function of $\Gamma(\lambda)$? Among all the critical points, which critical points are the critical points for all quintessence and which are only relative to the concrete potentials? In our paper, we will try to shed light on these issues. The paper is organized as follows. In section 2, we present the theoretical framework and give the differential relation between the function $\Gamma(\lambda)$ and potential $V(\phi)$. We find out all the critical points and investigate their properties in section 3. We try to give the cosmological implications of these critical points in section 4. We briefly display our conclusions in section 5.

2. Basic theoretical frame

We start with a spatially flat Friedman–Robertson–Walker universe containing a scalar field $\phi$ and a barotropic fluid (with state equation $p_b = w_b \rho_b$). To simplify, we give the Einstein equations directly:

\begin{align*}
H^2 &= \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + \rho_b \right] \quad (1) \\
H &= -\frac{\kappa^2}{2} \left[ \dot{\phi}^2 + (1 + w_b) \rho_b \right]. \quad (2)
\end{align*}

The motion equation of the scalar field $\phi$ is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (3)$$

Following [48], we define the following dimensionless variables:

$$x = \frac{\kappa \dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\kappa V^{\frac{1}{3}}}{\sqrt{3} H}, \quad \lambda = -\frac{V'}{V}, \quad \Gamma = \frac{VV''}{V'^2}, \quad (4)$$

where $V' = dV(\phi)/d\phi$ and $V'' = d^2V(\phi)/d\phi^2$. Using equation (4), equations (1)–(3) can be rewritten in the following dynamical form [37, 46, 48]:

\begin{align*}
\frac{dx}{dN} &= -3x + \sqrt{6} \lambda y^2 + \frac{3}{2} x [(1 - w_b)x^2 + (1 + w_b)(1 - y^2)] \quad (5) \\
\frac{dy}{dN} &= -\sqrt{6} \lambda xy + \frac{3}{2} y [(1 - w_b)x^2 + (1 + w_b)(1 - y^2)] \quad (6) \\
\frac{d\lambda}{dN} &= -\sqrt{6} \lambda^2 (\Gamma - 1)x, \quad (7)
\end{align*}

where $N = \ln(a)$. Here, we should emphasize that equations (5)–(7) are not a dynamical autonomous system since parameter $\Gamma$ is unknown. However, if we consider $\Gamma$ as a function of $\lambda$, namely

$$\Gamma(\lambda) = f(\lambda) + 1, \quad (8)$$

then equation (7) becomes

$$\frac{d\lambda}{dN} = -\sqrt{6} \lambda^2 f(\lambda)x. \quad (9)$$
Hereafter, equations (5), (6) and equation (9) are definitely a dynamical autonomous system. We will see that $\Gamma_1$ as a function of $\lambda$ can cover many quintessential potentials. The three-dimensional autonomous system reduces to two-dimensional autonomous systems when $f(\lambda) = 0$. In this case, the potential is of an exponential form which has been completely studied in many literature. When $f(\lambda)$ equals a nonvanishing constant $f_\lambda$, then the potential is proportional to $(c_1 \phi + c_2)^{-1/f_\lambda}$; this potential is the one which has been considered as ‘instantaneous’ critical points [48]. Generally speaking, we can analyze any explicit function. For some more complicated forms, $\Gamma_1(\lambda) = 1 + \frac{1}{n} - \frac{n \sigma^2}{\lambda^2}$ corresponds to $V(\phi) = V_0 \cosh(\sigma \phi)$, $\Gamma_1(\lambda) = 1 + \frac{1}{p} + \frac{\alpha}{\beta}$, which corresponds to $V(\phi) = V_0 e^{\frac{1}{\phi}}$. The form of $\Gamma_1(\lambda) = 1 + \frac{1}{\beta} + \frac{\alpha}{\phi}$, which corresponds to $V(\phi) = V_0 (\eta + e^{-\alpha \phi})^\beta$, was considered as an interesting cosmological model where the universe can evolve from a scaling attractor to a de-Sitter-like attractor by introducing a possible mechanism of spontaneous symmetry breaking [47].

In paper [47], the author gave an approach to obtaining the potential $V(\phi)$ as follows. Since the potential $V(\phi)$ is only a function of the field $\phi$, then the parameters $\lambda$ and $\Gamma$ can be written as a function of field: $\lambda = P(\phi), \Gamma = Q(\phi)$. If the inverse function of $P(\phi)$ exists, then we have

$$\Gamma = Q(P^{-1}(\lambda)) \equiv F(\lambda).$$

Using the definition of $\lambda$ and $\Gamma$, $V''$ can be written as $V'' = \frac{V^2}{V} F\left(-\frac{V}{V}\right) \equiv F(V, V')$. Let $h = V';$ then

$$\frac{dh}{dV} = \frac{1}{h} F(V, h) = \frac{h}{V} F\left(-\frac{h}{V}\right).$$

Now equation (11) is a one-order differential equation of $h$ and $V$. Figuring out $h(V)$, the potential can be solved from the equation $V'(\phi) = h(V(\phi))$.

Here we introduce another easier approach to obtaining the potential $V(\phi)$. We start with $\frac{d\lambda}{dV} = \frac{d\lambda}{d\phi} \frac{d\phi}{dV} = -\frac{d(V/V)}{d\phi} \frac{d\phi}{dV} = -\frac{1}{V} \frac{V''-V^2}{V'^2}$. Using the definition of $\lambda$ and $\Gamma$, and equation (8), we get a one-order differential equation of $\lambda$ and $V$:

$$\frac{d\lambda}{dV} = \frac{\lambda}{V} f(\lambda).$$

Integrating out $\lambda = \lambda(V)$, by using the definition of $\lambda$, we have the following differential equation of potential:

$$\frac{dV}{V\lambda(V)} = -d\phi.$$
3. Critical points and their properties

It is easily seen from equation (9) that \( \lambda = 0, x = 0 \) or \( f(\lambda) = 0 \) can make \( \frac{d\lambda}{dN} = 0 \). The critical points listed in table 1 can be found from equations (5), (6) and (9) after setting \( \frac{dx}{dN} = \frac{dy}{dN} = \frac{d\lambda}{dN} = 0 \). The properties of each critical point are determined by the eigenvalues of the Jacobi matrix of the three-dimensional autonomous system. For a general three-dimensional autonomous system,

\[
\begin{align*}
\frac{dx}{dN} &= f_1(x, y, \lambda) \\
\frac{dy}{dN} &= f_2(x, y, \lambda) \\
\frac{d\lambda}{dN} &= f_3(x, y, \lambda)
\end{align*}
\]

Functions \( f_1, f_2 \) and \( f_3 \) are only the functions of \( x, y \) and \( \lambda \), respectively, and not of variable \( N \) or other variables; we call this dynamical system an autonomous system. If \( f_1, f_2 \) and \( f_3 \) are only a linear combination of \( x, y \) and \( \lambda \) respectively, then equation (14) is a linear autonomous system. Its critical points \((x_c, y_c, \lambda_c)\) can be found from the set of functions \( f_1 = f_2 = f_3 = 0 \). Obviously, equations (5), (6) and (9) are not a linear autonomous system. However, the local behavior of the nonlinear autonomous system near a critical point can be deduced by linearizing the nonlinear system about this point and can be studied using the linear autonomous system analysis method. The properties of each critical point are determined by the eigenvalues of the Jacobi matrix \( A \), where

\[
A = \begin{bmatrix}
\frac{\partial f_1(x, y, \lambda)}{\partial x} & \frac{\partial f_1(x, y, \lambda)}{\partial y} & \frac{\partial f_1(x, y, \lambda)}{\partial \lambda} \\
\frac{\partial f_2(x, y, \lambda)}{\partial x} & \frac{\partial f_2(x, y, \lambda)}{\partial y} & \frac{\partial f_2(x, y, \lambda)}{\partial \lambda} \\
\frac{\partial f_3(x, y, \lambda)}{\partial x} & \frac{\partial f_3(x, y, \lambda)}{\partial y} & \frac{\partial f_3(x, y, \lambda)}{\partial \lambda}
\end{bmatrix}_{(x_c, y_c, \lambda_c)}.
\]

For a hyperbolic critical point\( ^6 \), if all the eigenvalues of \( A \) or the real part of these eigenvalues are negative, the critical point is stable. This is to say that as long as one of the eigenvalues or the real part of these eigenvalues is positive, the critical point must be unstable. However, if the critical point of the nonlinear autonomous system is a nonhyperbolic point\( ^7 \) and the rest of its eigenvalues have a negative real part, the properties of this point cannot be simply determined by the linearization method and need to resort to other more complicated methods [50]. From table 1, we can see that point \( P_4 \) is just this kind of point. In previous literature [30, 31, 37, 51], the authors generally neglected this nonhyperbolic point when they came across it. In fact, this point also has an important cosmological implication as other critical points and should not be ignored. We will explore the properties of this nonhyperbolic point \( P_4 \) in our paper using the center manifold theorem [50]. (The full analysis process is given in the appendix.) We list all the points and their properties in table 1. Note that we have neglected the cases with \( y < 0 \) since the system is symmetric under the reflection \( (\lambda, x, y) \rightarrow (\lambda, x, -y) \) and time reversal \( t \rightarrow -t \).

\( f(0) \) is the value of function \( f(\lambda) \) at \( \lambda = 0, \frac{df}{d\lambda} = \frac{df(\lambda)}{d\lambda} \). We limit the range of \( w_b(=\gamma - 1) \) to \( 0 \leq w_b < 1, w_b = 0 \) for matter and \( 1/3 \) for radiation. \( \lambda_a \) means an arbitrary value and \( \lambda_c \) is the value which makes \( f(\lambda_c) = 0 \). Thus, points \( P_{7-10} \) appear only if the function \( f(\lambda) \) can be zero for one or more values of \( \lambda_c \). Here, we simply consider the fact that only one value \( \lambda_c \) makes the function \( f(\lambda) \) zero.

\( ^6 \) Actually, the critical point in this paper is also called the ‘equilibrium point’ in mathematics or a ‘fixed point’ in some physical literature. A hyperbolic critical (equilibrium) point is the critical (equilibrium) point which has no eigenvalues with a zero real part.

\( ^7 \) That is, its eigenvalues have a zero value or zero real parts.
In order to investigate the expansive behavior of scale factor $a$, we also represent the decelerating factor:

\[
g = -\frac{\ddot{a}}{\dot{a}} = -\frac{\ddot{a}}{\dot{a}}/H^2 = \frac{\sum (1 + 3w_i) \rho_i}{2 \sum \rho_i} = \frac{1}{2} \sum (1 + 3w_i) \Omega_i = \frac{3}{2} \left[ 1 - w_b \right] x^2 - (1 + w_b) y^2 + \left( w_b + \frac{1}{3} \right) y^2.
\]

We list the other properties of these critical points in table 2.

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### Table 1. The properties of all critical points.

| $(\lambda_c, x_c, y_c)$ | Eigenvalues | Stability |
|--------------------------|-------------|-----------|
| $P_1$ $(0, 1, 0)$ | $3(1 - w_b), 3, 0$ | Unstable node |
| $P_2$ $(0, -1, 0)$ | $3(1 - w_b), 3, 0$ | Unstable node |
| $P_3$ $(0, 0, 0)$ | $-3(1 - w_b)/2, 3y^2/2, 0$ | Saddle point |
| $P_4$ $(0, 0, 1)$ | $-3, -3y, 0$ | Stable node for $f(0) > 0$ |
| $P_5$ $(\lambda_c, 0, 0)$ | $-3(1 - w_b)/2, 3y^2/2, 0$ | Saddle point |
| $P_6$ $(\lambda_c, 0, 0)$ | $-3(1 - w_b)/2, 3y^2/2, 0$ | Saddle point |
| $P_7$ $(\lambda_c, 1, 0)$ | $-\sqrt{6}\lambda_c^2 f_a, 3(1 - w_b), \frac{1}{2} (6 - \sqrt{6} \lambda_c)$ | Saddle point |
| $P_8$ $(\lambda_c, -1, 0)$ | $\sqrt{6} \lambda_c^2 f_a, 3(1 - w_b), \frac{1}{2} (6 + \sqrt{6} \lambda_c)$ | Saddle point |
| $P_9$ $(\lambda_c, \sqrt{\frac{6}{\lambda_c}}, \sqrt{1 - \frac{1}{\lambda_c^2}})$ | $\frac{1}{2} (\lambda^2 - 6), \lambda^2, 3y - \lambda_5^2 \lambda_c$, $df_a$ | Equation (16) |
| $P_{10}$ $(\lambda_c, \sqrt{\frac{6}{\lambda_c}}, \sqrt{\frac{6\lambda - \lambda_c}{\lambda_c}})$ | $-3 \lambda_c^2 \gamma d f_a, \frac{1}{2} (w_b - 1)$ | Equation (17) |

However, readers should keep in mind that to make $d\lambda/dN = 0$ in equation (9), we let $\lambda = 0, x = 0$ and $f(\lambda) = 0$ separately and then find out all the points listed in table 1. But we do not consider one special case, i.e. $\lambda^2 f(\lambda) = 0$ and $d\lambda/dN \neq 0$ when $\lambda = 0$. In this case, $P_1$ and $P_2$ are no longer critical points. For example, the product $\lambda^2 f(\lambda) = \frac{V_0}{\lambda} \neq 0$ even if $\lambda = 0$ for the potential $V(\phi) = V_0 [\cosh(\alpha \phi) - 1] + \Lambda$. So the necessary condition for the existence of equilibrium points with $x \neq 0$ is $\lambda^2 f(\lambda) = 0$.

$\lambda^2 < 6$ is the condition for critical point $P_9$ to exist and equation (16) is the condition for $P_9$ to be a stable node

\[
\lambda^2 < 3y \quad \text{and} \quad \lambda_c d f_a > 0.
\]

$\lambda^2 > 3y$ is the condition for critical point $P_{10}$ to exist and equation (17) is its stable condition $24y^2/(9y^2 - 2) > \lambda^2 > 3y$ and $\lambda_c d f_a > 0$ for $P_{10}$ to be a stable node

The density parameter of $\phi$ field and its equation of state are, respectively,

\[
\Omega_\phi = x^2 + y^2
\]

\[
w_\phi = \frac{x^2 - y^2}{x^2 + y^2}.
\]

In order to investigate the expansive behavior of scale factor $a$, we also represent the decelerating factor:

\[
g = -\frac{\ddot{a}}{\dot{a}} = -\frac{\ddot{a}}{\dot{a}}/H^2 = \frac{\sum (1 + 3w_i) \rho_i}{2 \sum \rho_i} = \frac{1}{2} \sum (1 + 3w_i) \Omega_i
\]

\[
= \frac{3}{2} \left[ 1 - w_b \right] x^2 - (1 + w_b) y^2 + \left( w_b + \frac{1}{3} \right) y^2.
\]
we can find from tables 1 and 2 the following facts. Though the critical points for all quintessence and the critical points which are only relative to understanding than from the view of the two-dimensional system. For example, we will point out the critical points for all quintessence and the critical points which are only relative to the concrete potentials. We can find from tables 1 and 2 the following facts. Though the stability of points \( P_{1,2} \) does not depend on the form of concrete potentials, points \( P_{1,2} \) only exist when \( \lambda^2 f(\lambda) = 0 \) at \( \lambda = 0 \). Points \( P_{3,5,6} \) always exist for all quintessence models and their stability are regardless of the form of concrete potentials. Point \( P_3 \) is also the critical point for all quintessence, but its stability depends on the form of concrete potentials. Points \( P_{7–10} \) can even be nonexistent if \( f(\lambda) \neq 0 \) for any \( \lambda \).

Of all the points, only points \( P_{3,5,6} \) are independent of function \( f(\lambda) \). In fact, they have the same properties and can be considered as one point. They are saddle points, which tell us that the barotropic-fluid-dominated solution \( (\lambda_c = 0, x_c = 0, y_c = 0) \), where \( \Omega_\phi = 0 \), is unstable. However, even though they are unstable, the phase-space trajectories may evolve in the vicinity of the barotropic-fluid-dominated solution for quite a long time and then leave this state to approach the possible future attractor. However, if \( y = 0 \), these points are found to be a stable attractor and can be used to alleviate the relic density problem in an inflation model [37].

Four of the critical points \( (P_{1,2}(\lambda_c = 0, x_c = \pm 1, y_c = 0)) \) and \( P_{7,8}(\lambda_c = \lambda_*, x_c = \pm 1, y_c = 0)) \) are all unstable nodes, which correspond to the solutions where the universe is dominated by the kinetic energy of the scalar field \( (\Omega_\phi = 1) \) with a stiff equation of state \( (w_\phi = 1) \).

In fact, we can conclude the above results with one brief sentence (see table 1): all the critical points with \( y_c \) being zero are not stable points. It tells us that, under the potential we

| \( (\lambda_c, x_c, y_c) \) | \( w_\phi \) | \( \Omega_\phi \) | Decelerating factor \( q \) |
|-----------------|--------|--------|-------------------|
| \( P_1 \) | \( (0, 1, 0) \) | 1 | 1 | 2 |
| \( P_2 \) | \( (0, -1, 0) \) | 1 | 1 | 2 |
| \( P_3 \) | \( (0, 0, 0) \) | Defined | 0 | \( (3w_\phi + 1)/2 \) |
| \( P_4 \) | \( (0, 0, 1) \) | -1 | 1 | -1 |
| \( P_5 \) | \( (\lambda_*, 0, 0) \) | Defined | 0 | \( (3w_\phi + 1)/2 \) |
| \( P_6 \) | \( (\lambda_*, 0, 0) \) | Defined | 0 | \( (3w_\phi + 1)/2 \) |
| \( P_7 \) | \( (\lambda_*, 1, 0) \) | 1 | 1 | 2 |
| \( P_8 \) | \( (\lambda_*, -1, 0) \) | 1 | 1 | 2 |
| \( P_9 \) | \( (\lambda_*, \frac{\sqrt{f}}{\Omega_\phi}, \sqrt{1 - \frac{1}{4}\lambda^2}) \) | \( \lambda^2/3 \) – 1 | 1 | \( \lambda^2/2 \) – 1 |
| \( P_{10} \) | \( (\lambda_*, \frac{\sqrt{f}1}{\Omega_\phi}, \sqrt{1 - \frac{1}{4}\lambda^2}) \) | \( w_\phi \) | 3\( y/\lambda^2 \) | \( (3w_\phi + 1)/2 \) |
considered here, the cosmological solution with the potential energy eventually evolving to zero will never be the final state of our universe. This is quite an interesting result since we know that the universe will never undergo a regime of accelerating expansion if there is no potential energy in quintessence models.

Therefore, there are only three critical points $P_{4,9,10}$ which correspond to possible late-time attractor solutions. We will study their properties and cosmological implications in more detail.

Points $P_{4,9}$ are both scalar-field-dominated solutions with $\Omega_\phi = 1$. We compare with points $P_4$, $P_9$ the well-known scalar-field-dominated solution which exists for $\lambda_\phi^2 < 6$. Table 1 has shown that this scalar-field-dominated solution is a late-time attractor in the presence of a barotropic fluid if we have $\lambda_\phi^2 < 3\gamma$ and $\lambda_\phi d_f > 0$. This solution will give an accelerating universe if $\lambda_\phi^2 < 2$ and $\lambda_\phi d_f > 0$. For example, $f(\lambda) = \frac{1}{\sigma} - \frac{\sigma^2}{\lambda^2}$ corresponds to $V(\phi) = \frac{V_0}{\cosh(\sigma\phi)^3}$. Obviously we have $\lambda_\phi = \pm[n\sigma]$ and $d_f = \frac{2n\sigma^2}{\lambda^2}$. The scalar-field-dominated solution with potential $V(\phi) = \frac{V_0}{\cosh(\sigma\phi)^3}$ is a late-time attractor if $n^2\sigma^2 < 3\gamma$ and $\frac{2n\sigma^2}{\lambda^2} > 0$. In addition, this solution admits an accelerating expansion of universe if $n^2\sigma^2 < 2$ and $\frac{2n\sigma^2}{\lambda^2} > 0$. Note that point $P_9$ means two stable critical points ({$\lambda_c = \pm[n\sigma]$, $x_c = \frac{\pm\sqrt{2}}{6}[n\sigma]$, $\gamma_c = \sqrt{1 \mp \frac{2}{6}n^2\sigma^2\sigma^2}$}) in this case.

$P_{10}$ is the scaling solution where neither the scalar field nor the barotropic fluid entirely dominates the universe. $P_{10}$ is a stable node for $24\gamma^2/(9\gamma - 2) > \lambda_\phi^2 > 3\gamma$ and $\lambda_\phi d_f > 0$ and a stable spiral for $\lambda_\phi^2 > 24\gamma^2/(9\gamma - 2)$ and $\lambda_\phi d_f > 0$. So $P_9$ and $P_{10}$ cannot be stable simultaneously. The scaling solution has drawn a lot of attention since it can alleviate the coincidence problem of dark energy. Many potentials have been proposed to give a scaling evolution regime [35, 36, 52–65]. Here we give a sufficient condition for a potential to possess a scaling solution, that is, as long as $f(\lambda)$ equals zero for one or more values of $\lambda(=\lambda_\phi)$ and these $\lambda_\phi$ also satisfy equation (17), then there must exist a scaling solution with $\Omega_\phi = 3\gamma/\lambda_\phi^2$. Obviously many potentials which satisfy this condition exist, such as potential $V(\phi) = \frac{V_0}{\cosh(\sigma\phi)^3}$ which corresponds to $f(\lambda) = \frac{1}{\sigma} - \frac{\sigma^2}{\lambda^2}$, potential $V(\phi) = \frac{V_0}{\cosh(\sigma\phi)^\lambda}$ which corresponds to $f(\lambda) = \frac{1}{\sigma} + \frac{\lambda}{\sigma^2}$ and so on. Our condition includes the potentials in [49] where the authors found that every positive and monotonous potential which was asymptotically exponential yielded a scaling solution. Our result also does not contradict the statement in [66, 67] where they assumed a scaling solution such as $P_{10}$ and found that the potential was unique and of the exponential form. This exponential potential is explicitly figured out from the assumption and the evolution of the universe with this potential being always the scaling solution (see equation (18) in [66]), while $P_{10}$ being a stable point means that all the evolution of the universe with a class of potentials which satisfy equation (17) will approach the scaling solution finally. It is just asymptotic behavior at late time. Unfortunately, for the scaling solution of $P_{10}$, the state equation of dark energy $w_\phi$ equals $w_m$, and therefore the accelerating expansion does not exist if $w_m$ is larger than zero. However, authors had obtained the exact quintessence potential $V(\phi) = \frac{1-w_m}{2} \rho_0 \left[ \sqrt{\frac{3\gamma}{1+w_m}} \sinh\left( \frac{3(w_m-w_\phi)}{2\sqrt{3(1+w_m)}} \frac{\phi-\phi_1}{m_\phi} \right) \right]^{-2(1+w_\phi)/(w_m-w_\phi)}$, which admitted a scaling solution with $w_\phi \neq w_m$ and $\Omega_\phi \neq 0$ [66]. With this potential, in principle, we can obtain a scaling solution with an accelerating expansion of the universe.

Finally, let us consider point $P_8$, which is a de-Sitter-like dominant attractor with $\Omega_\phi = 1$ and $w_\phi = -1$. The condition for $P_8$ to be a stable point is that the value of $f(\lambda)$ when $\lambda = 0$ must be larger than zero (i.e. $f(0) > 0$; see the appendix for details).

So, generally speaking, $P_4$ and $P_9$ (or $P_{10}$) also cannot be stable simultaneously. However, a possibility may exist for some potentials that their values at $\lambda = 0$ are larger than zero but
equal to zero for some others $\lambda_*$ ($\lambda_* \neq 0$); then this region of $\lambda$ in the phase space of the three dynamical autonomous system will lie in the basin of the attractor $P_{10}$. That means, in this case, there can exist two stable critical points simultaneously, but this is not to say that the universe can evolve continuously from one stable critical point to another one. Based on this fact, the author proposed a scenario of universe which could evolve from a scaling attractor to a de-Sitter-like attractor by introducing a field whose value changed by a certain amount in a short time [47]. In fact, we can also obtain these two asymptotical evolutions if the potential $V(\phi)$ can be approximated to two different potentials when $\phi$ evolves into different ranges: one admits the scaling solution and the other admits the de-Sitter-like solution [53, 58, 61]. For these potentials, the exit of the cosmological evolution from one attractor solution to another attractor is quite natural, but the explanation of why we have these special potentials is not quite natural.

5. Conclusion

In this paper, we extend the autonomous dynamical system analysis of the canonical scalar field from two dimensions to three dimensions by considering the potential parameter $\Gamma_1$ as a function of another potential parameter $\lambda$. There are ten critical points in all: three of these points $(P_3, 5, 6)$ are general points which are possessed by all quintessence models regardless of the form of potentials and the rest of the points, with their existence or/and stability, are closely connected to the concrete potentials. We surprisingly find that, apart from the exponential potential, there are a large number of potentials which can give the scaling solution when the function $f(\lambda) := \frac{\Gamma_1(\lambda)}{\Gamma_1(\lambda)} - 1$ equals zero for one or some values of $\lambda$ and the parameter $\lambda$ satisfies condition (16) or (17) at the same time. We give the explicit expression to derive these potentials $V(\phi)$ from $f(\lambda)$. We find that, if some conditions are satisfied, the de-Sitter-like dominant point $P_4$ and the scaling point $P_9$ (or $P_{10}$) can simultaneously be stable, but $P_9$ and $P_{10}$ cannot be stable at one time. As we have seen, the autonomous dynamical system analysis is a very powerful tool which helps us to extract useful cosmological information without solving the complicated background equations. Our method extends the analysis from a two-dimensional autonomous dynamical system to a three-dimensional one, which enables us to study a large number of potentials beyond the exponential potential. This method is quite effective and may be applied to a broad class of dark energy models studied in [46], including coupled quintessence, (coupled-)phantom scalar field, $k$-essence and even generalized background $H^2 \propto \rho^n$.

However, we should point out that our approach also has its drawbacks. First, as we have mentioned above, the approach cannot be applied for the potentials for which the function $\Gamma = \frac{V'}{V''}$ cannot be written as an explicit function of the variable $\lambda$. Second, the variable $\lambda$ is undefined if the potentials vanish at its minimum, so the approach cannot be applied for the potentials which vanish at its minimum. But, despite the second problem, it is actually not a fatal drawback. On one hand, the minimum of a potential is always associated with the late-time cosmological dynamics (future attractors). It is quite easy to discuss this special equilibrium point separately if we know that a given potential has minimum (it is usually not difficult to find out the minimum of a given function). On the other hand, we can still use our approach to analyze all the critical points $P_{1-10}$ since $\lambda$ is well defined around these critical points. We take the potential $V(\phi) = V_0[\cosh(\alpha \phi) - 1]$ for example; this potential has a minimum value 0 at $\phi = 0$ ($\lambda$ has no definition at $\phi = 0$). We can investigate the critical point corresponding to this minimum separately. The explicit function about this potential is $f(\lambda) = \frac{1}{2}(\frac{\alpha}{\lambda^2} - 1)$. The point corresponding to the potential’s minimum does not appear in...
We can still discuss the properties of the critical points \( P_{1-10} \) even if the variable \( \lambda \) sometimes has no definition.

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Appendix

In section 3, we pointed out that if the eigenvalues of the Jacobi matrix had one or more eigenvalues with zero real parts while the rest of the eigenvalues had negative real parts, then linearization failed to determine the stability properties of this critical point. From table 1 we realize that point \( P_4 \) is just such a point, so in this appendix we will show you how we get the stable condition of \( P_4 \) from the center manifold theorem. The point \( P_4 \) is \((\lambda_c = 0, x_c = 0, y_c = 1)\) and its three eigenvalues are \((0, -3, -3(1 + w_m))\). First, we transfer \( P_4 \) to \( P'_4 \) \((\lambda_c = 0, x_c = 0, y_c = y_c - 1 = 0)\) for convenience. In this case, equations (5)–(7) can be rewritten as

\[
\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 f(\lambda)x
\]

\[
\frac{dx}{dN} = -3x + \frac{1}{2}\sqrt{6}\lambda + \frac{1}{2}\sqrt{6}\lambda Y^2 + \sqrt{6}\lambda Y + \frac{3}{2}x^3(1 - w_m) - \frac{3}{2}(1 + w_m)xY^2 - 3(1 + w_m)xY
\]

\[
\frac{dY}{dN} = -3(1 + w_m)Y - \frac{1}{2}\sqrt{6}\lambda x(Y + 1)
+ \frac{3}{2}(1 - w_m)x^2Y - \frac{3}{2}Y^3 - \frac{3}{2}(3 + w_m)Y^2 + \frac{3}{2}(1 - w_m)x^2.
\]

Note that \((\lambda, x, Y)\) in equations (A.1)–(A.3) are very small variables around point \((\lambda_c = 0, x_c = 0, y_c = 0)\). So the function \( f(\lambda) \) in equation (A.1) should be taken as the Taylor series in \( \lambda \): \( f(\lambda) = f(0) + f'(0)\lambda + \frac{f''(0)}{2!}\lambda^2 + \cdots \), where \( f''(0) \) is the value of \( \frac{d^2f(\lambda)}{d\lambda^2} \) when \( \lambda = 0 \).

We can write down the Jacobi matrix \( \mathcal{A} \) of the dynamical system (equations (A.1)–(A.3)):

\[
\mathcal{A} = \begin{bmatrix}
0 & 0 & 0 \\
\frac{1}{2}\sqrt{6} & -3 & 0 \\
0 & 0 & -3(1 + w_m)
\end{bmatrix}.
\]

The eigenvalues of \( \mathcal{A} \) and the corresponding eigenvectors are

\[
\left\{0, \left[1, \frac{\sqrt{6}}{6}, 0\right]\right\}, \quad \{-3, [0, 1, 0]\}, \quad \{-3(1 + 3w_m), [0, 0, 1]\}.
\]

Let \( \mathcal{M} \) be a matrix whose columns are the eigenvectors of \( \mathcal{A} \); then we can write down \( \mathcal{M} \) and its inverse matrix \( T \):
\[ \mathcal{M} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{T} = \mathcal{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \] (A.6)

Using the similarity transformation \( \mathcal{T} \), we can transform \( \mathcal{A} \) into a block diagonal matrix, that is,

\[ \mathcal{T} \mathcal{A} \mathcal{T}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3(1 + w_m) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 \\ 0 \\ \mathcal{A}_2 \end{bmatrix}. \] (A.7)

where all eigenvalues of \( \mathcal{A}_1 \) have zero real parts and all eigenvalues of \( \mathcal{A}_2 \) have negative real parts. We make a change of variables:

\[ \begin{bmatrix} \lambda' \\ x' \\ Y' \end{bmatrix} = \mathcal{T} \begin{bmatrix} \lambda \\ x \\ Y \end{bmatrix} = \begin{bmatrix} \lambda \\ -\frac{\sqrt{6}}{6} \lambda + x \\ Y \end{bmatrix}. \] (A.8)

Then we can rewrite the dynamical system (equations (A.1)–(A.3)) in the form of new variables:

\[ \frac{d\lambda'}{dN} = \frac{d\lambda}{dN} = f_1(\lambda', x', Y') \] (A.9)

\[ \frac{dx'}{dN} = -\frac{\sqrt{6}}{6} \frac{d\lambda}{dN} + \frac{dx}{dN} = f_2(\lambda', x', Y') \] (A.10)

\[ \frac{dY'}{dN} = \frac{dY}{dN} = f_3(\lambda', x', Y'). \] (A.11)

The detailed forms of \( f_1(\lambda', x', Y') \), \( f_2(\lambda', x', Y') \), \( f_3(\lambda', x', Y') \) are easily obtained after we substitute transformations \( \lambda = \lambda' \), \( x = \frac{\sqrt{6}}{6} \lambda' + x' \) and \( Y = Y' \) into the right-hand sides of equations (A.1)–(A.3) respectively. According to the center manifold theorem, the stable condition of a dynamical system (equations (A.1)–(A.3)), i.e. the stability of \( P_4 \), will be finally determined by the following simple reduced system:

\[ \frac{d\lambda'}{dN} = \frac{d\lambda}{dN} = -\lambda^3 f(0) = -\lambda^3 f(0), \] (A.12)

where \( f(0) \) is the value of function \( f(\lambda) \) at \( \lambda = 0 \). This simple one-dimensional dynamical system (A.12) is stable if \( f(0) > 0 \).

So we conclude that \( P_4 \) is a stable de-Sitter-like dominant attractor when \( f(0) > 0 \), as shown in table 2.

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