Dark Energy: Beyond General Relativity?

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Abstract. The late-time accelerated expansion of the universe is a major challenge for cosmology. It may well be that a solution to this problem will require a theory of gravitation beyond General Relativity. It is emphasized that precision cosmology will strongly constrain the possibilities by using observational data probing the background as well as the inhomogeneities.

INTRODUCTION

There is growing observational evidence for a late time accelerated expansion of our universe [1]. Such an expansion constitutes a radical departure from conventional cosmology but it is reassuring that a consistent picture of our universe seems to emerge from all the data [2]. This accelerated expansion might be due to a new component with sufficiently negative pressure, coined Dark Energy. The simplest candidate of this kind is a positive cosmological constant $\Lambda$ whose pressure satisfies $p_\Lambda = -\rho_\Lambda$. As well-known, its interpretation as the vacuum energy is problematic because of its exceeding smallness (see e.g. [3] for recent comprehensive reviews). One can also introduce phenomenologically a smooth component with constant equation of state and sufficiently negative pressure typically satisfying $w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} < -0.5$ [4]. Observations could force us to consider a smooth component with a varying equation of state. The most prominent candidate in this respect is a minimally coupled scalar field, Quintessence models [5]. A further constraint appears if observations force us to consider DE models of the phantom type, i.e. satisfying $w_{DE} < -1$ [6] as observations seem to suggest on small redshifts $z \lesssim 0.5$ [7]. The most striking consequence in that case is that quintessence models are ruled out.

However, the late-time acceleration might as well be caused by a change in the theory of gravitation, no longer described by General Relativity [8]. Interest for such candidates is increased by their ability to produce DE of the phantom type. We will review scalar-tensor DE models [9], [11] as a promising candidate belonging to this class of models. It is emphasized that observations probing the background and the inhomogeneities will establish whether such attempts provide a consistent description of our universe.

TWO BASIC DARK ENERGY MODELS INSIDE GENERAL RELATIVITY

Constant equation of state DE models

The simplest DE models are those in the framework of GR containing some unknown component with negative equation of state parameter $w_{DE} = \frac{p_{DE}}{\rho_{DE}}$. Our starting point are the well-known Friedmann equations for a homogeneous and isotropic universe,

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \sum_i \frac{8\pi G}{3} \rho_i ,$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i) .$$

(1)

(2)
In the equations above, the different components labelled $i$, are all isotropic perfect fluids while a dot stands for a derivative with respect to (cosmic) time $t$. We note from (2) that a necessary, but not yet sufficient, condition that a component $i$ induces an accelerated expansion is given by

$$\rho_i + 3p_i < 0.$$  \hspace{1cm} (3)

We concentrate on equations of state of the form $\rho_i = w_i p_i$, with constant $w_i$ and we specialize to a universe containing dustlike matter and some DE component. From (2), it is not hard to deduce that our universe is presently accelerating provided

$$w_{DE} < -\frac{1}{3} \left( 1 + \frac{\Omega_{m,0}}{\Omega_{DE,0}} \right),$$  \hspace{1cm} (4)

and in particular for a flat universe

$$w_{DE} < -\frac{1}{3} \Omega_{DE,0}^{-1}.$$  \hspace{1cm} (5)

For instance, for $\Omega_{m,0} = 0.3$, $\Omega_{DE,0} = 0.7$, $w_{DE} < -0.47$ is required. Therefore, present experimental evidence yielding $\Omega_{m,0} \sim 0.3$, combined with the location of the first acoustic peak in the CMB anisotropy first detected by balloon experiments and confirmed by the recent CMB data released by WMAP [12] suggesting a nearly flat universe, imply that our universe would be presently accelerating for a wide range of constant values, roughly $w_{DE} < -0.5$. Introducing the dimensionless quantity $x \equiv \frac{\dot{a}}{a_0}$, accelerated expansion starts at $x_a$ given by

$$x_a^{-3|w_{DE}|} = (-1 + 3|w_{DE}|) \frac{\Omega_{DE,0}}{\Omega_{m,0}},$$  \hspace{1cm} (6)

which corresponds to redshifts

$$z_a = (-1 + 3|w_{DE}|)^{\frac{1}{3|w_{DE}|}} \left( \frac{\Omega_{DE,0}}{\Omega_{m,0}} \right)^{\frac{1}{3|w_{DE}|}} - 1.$$  \hspace{1cm} (7)

For $-1 < w_{DE} \rightarrow -1/3$, $z_a$ is shifted towards smaller redshifts, for $(\Omega_{m,0}, \Omega_{DE,0}) = (0.3, 0.7)$, we have $z_a = 0.671$ for a constant $\Lambda$-term, and $z_a = 0.414$ when $w_{DE} = -0.6$. The fact that $z_a$ is so close to zero, is the cosmic coincidence problem. Finally, these models can yield a significant increase, depending on $w_{DE}, \Omega_{m,0}, \Omega_{DE,0}$ of the age of the universe for given Hubble parameter $H_0$ compared to an Einstein-de Sitter universe, i.e. a flat universe with $\Omega_{m,0} = 1$ [4].

**Quintessence**

The DE component could well be a time dependent minimally coupled scalar field $\phi(t)$ called Quintessence. This possibility is clearly inspired by the inflationary paradigm in which a scalar field is so successful in implementing the inflationary stage. Such a scalar field can be considered as a perfect fluid with

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \hspace{1cm} \text{and} \hspace{1cm} p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$  \hspace{1cm} (8)

and therefore the equation of state parameter $w_\phi$ is given by

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}.$$  \hspace{1cm} (9)

For $\rho_\phi \geq 0$, the equation of state must satisfy

$$w_\phi \geq -1,$$  \hspace{1cm} (10)

in other words $\phi$ cannot be of the Phantom type. It is possible to have scaling solutions with $\rho_\phi \propto a^m$, $m = -3(1+w_\phi) =$ constant. However this requires a very particular potential $V(\phi)$ for which,

$$V(\phi) = \frac{1 - w \ddot{\phi}^2}{1 + w}.$$  \hspace{1cm} (11)
Hence, the most natural thing for Quintessence is to have a time varying equation of state, however one that satisfies the condition \( w(z) \geq -1 \).

The two models reviewed in this section, though very different, are inside the framework of General Relativity (GR). A further dramatical departure is to modify the laws of gravity and to consider DE models outside the framework of GR. We will consider in details the example of scalar-tensor DE models.

**SCALAR-TENSOR DARK ENERGY MODELS**

**Full Reconstruction**

As we have seen, in quintessence models the equation of state parameter \( w_\phi \) must satisfy \( w_\phi \geq -1 \). The following inequality must be satisfied

\[
\frac{dH^2(z)}{dz} \geq 3\Omega_m H_0^2 (1+z)^2 .
\]  

(12)

Note that inequality (12) applies only to spatially flat universes, a full analysis should relax the flatness prior [13]. It is not clear from the existing data whether (12) is satisfied and actually the analysis of the most recent SN data supports a varying equation of state which is of the Phantom type, i.e. with \( w < -1 \), on very small redshifts \( 0 \leq z \lesssim 0.5 \). If confirmed, a striking consequence is that Quintessence models are ruled out. It is therefore important to consider a more general class of models, like scalar-tensor (ST) models, where the inequality (12) is no longer compulsory. Further ST theories are interesting to consider as they arise naturally from more fundamental theories like M-theory.

We now review the reconstruction program to scalar-tensor DE models [9]. We consider the following Lagrangian

\[
L = \frac{1}{2} \left( F(\Phi) R - g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right) - U(\Phi) + L_m(g_{\mu\nu}) ,
\]  

(13)

where \( L_m \) describes dustlike matter and \( F(\Phi) > 0 \). The Lagrangian as it is written in (13) can describe consistently models with a positive Brans-Dicke parameter \( \omega_{BD} = F/(dF/d\Phi)^2 > 0 \). If \( \omega_{BD} < 0 \), we must use the parametrization with \( Z = -1 \) where \( Z \) is the function in front of the kinetic term, the Lagrangian (13) corresponds to the \( Z = 1 \) parametrization. We will write all our equations using (13), i.e. in the JF using the \( Z = 1 \) parametrization. We do not introduce any direct coupling between \( \Phi \) and \( L_m \) so that, in particular, fundamental constants do not change with time. We can proceed in a way analogous to the Quintessence case, the main difference being that we now have to reconstruct two unknown functions instead of one. Again, specializing to a flat FRW universe, the corresponding (modified) Friedmann equations read

\[
3FH^2 = \rho_m + \frac{\dot{\Phi}^2}{2} + U - 3HF ,
\]  

(14)

\[
-2FH = \rho_m + \dot{\Phi}^2 + \dot{F} - HF .
\]  

(15)

As was the case in GR, the evolution equation for the scalar field \( \Phi \) is contained in the two Friedmann equations above. These background equations (14,15) can be combined to give the following master equation for \( F(z) \):

\[
F'' + \left[ (\ln H)' - \frac{4}{1+z} \right] F' + \left[ \frac{6}{(1+z)^2} - \frac{2(\ln H)'}{1+z} \right] F = \frac{2U}{(1+z)^2H^2} + 3 \left( 1 + z \right) \left( \frac{H_0}{H} \right)^2 F_0 \Omega_m ,
\]  

(16)

where a prime denotes a derivative with respect to \( z \).

In this theory, the effective value of Newton’s gravitational constant \( G_N \) is given by

\[
G_N = \frac{1}{8\pi F} .
\]  

(17)

It is natural to use its present value \( G_{N,0} \equiv \frac{1}{8\pi \rho_{cr}} \) in the definition of the critical density \( \rho_{cr} \). However \( G_N \) does not have the same physical meaning as in GR, it is no longer the coupling constant for the gravitational attraction between two
point masses. For a massless dilaton, the effective gravitational constant between two test masses is given by

\begin{equation}
G_{\text{eff}} = \frac{1}{8\pi F} \left( \frac{2F + 4(dF/d\Phi)^2}{2F + 3(dF/d\Phi)^2} \right). \quad (18)
\end{equation}

In our case, the dilaton is massive but (18) will still hold for physical scales \( R \) such that

\begin{equation}
R^{-2} \gg \max \left( \left| \frac{d^2U}{d\Phi^2} \right|, H^2, H^2 \left| \frac{d^2F}{d\Phi^2} \right| \right). \quad (19)
\end{equation}

The most recent solar system measurements [14] imply very stringent constraints on the Brans-Dicke parameter today

\begin{equation}
\omega_{BD,0} = F_0 \left( \frac{d\Phi}{dF} \right)_0^2 > 4 \times 10^4. \quad (20)
\end{equation}

As a consequence, \( G_{N,0} \) and \( G_{\text{eff},0} \) coincide with better than \( 1.25 \times 10^{-5} \) accuracy. On the other hand, the difference between \( G_N \) and \( G_{\text{eff}} \) could be larger at higher redshifts. Our interest for scalar-tensor theories of gravity is in the context of DE models. So our theory should satisfy the following requirements as any realistic DE model. First, the DE term should dominate today the energy density of the universe and satisfy

\begin{equation}
\Omega_{DE,0} \sim 0.7 \sim 2\Omega_{m,0}. \quad (21)
\end{equation}

If our model describes a universe whose expansion is presently accelerated, then it must satisfy

\begin{equation}
U_0 > (\rho_m + 2\Phi^2 + 3F + 3H\dot{F})_0. \quad (22)
\end{equation}

Finally it is important that DE remains essentially unclustered at scales up to \( R \sim 10^{-1}(1+z)^{-1} \) Mpc, though as we will see some limited clustering will arise on these scales. To achieve this, it is sufficient to assume that the inequality [19] is satisfied for all scales of interest.

Like in GR, we start with the determination of \( H(z) \) from \( D_L(z) \) using

\begin{equation}
\frac{1}{H(z)} = \left( \frac{D_L(z)}{1+z} \right)^{\gamma}. \quad (23)
\end{equation}

However, we need to recover the two functions \( F(z) \) and \( U(z) \), so the substitution of \( H(z) \) in (16) is no longer sufficient. For a complete reconstruction we must use a new equation based on independent observations. It is provided by \( \delta_m(z) \) data, measurement of the linear dustlike matter density fluctuations. We can expect in the near future accurate \( D_L(z) \) and \( \delta_m(z) \) data.

We consider the perturbation equations in the longitudinal gauge

\begin{equation}
ds^2 = -(1 + 2\phi)dt^2 + a^2(1 - 2\psi)dx^2, \quad (24)
\end{equation}

(see [9] for details). The idea is that, in the short wavelength limit, the leading terms are either those containing \( k^2 \), or those with \( \delta_m \). Then the following equation is obtained

\begin{equation}
\delta \Phi \sim (\phi - 2\psi) \frac{dF}{d\Phi} - \phi \frac{F dF / d\Phi}{F + 2(dF / d\Phi)^2}. \quad (25)
\end{equation}

Hence, unlike in GR, in ST gravity the dilaton remains partly clustered even for arbitrarily small scales, however this clustering is small because \( \omega_{BD} \) is large. In the same short wavelength limit, Poisson’s equation has the same form as in GR, with the important difference that Newton’s constant \( G_N \) is replaced by \( G_{\text{eff}} \), defined in (18) above. Hence, the equation for the evolution of dustlike matter linear density perturbations finally leads to

\begin{equation}
H^2 \delta_m'' + \left( \frac{H^2}{2} \right)' - \frac{H^2}{1+z} \right) \delta_m' \approx \frac{3}{2} \left( 1 + z \right) H_0^2 \frac{G_{\text{eff},0}}{G_{N,0}} \Omega_{m,0} \delta_m. \quad (26)
\end{equation}

So, we see here a second difference in the physical meaning of \( G_{\text{eff}} \) and \( G_N \): while it is \( G_N \) that appears in the equation for the evolution of the perturbations in GR, in ST models this role is played by \( G_{\text{eff}} \).
Let us now sketch briefly the reconstruction itself. Extracting $H(z)$ (through $D_L(z)$) and $\delta_m(z)$ from observations with sufficient accuracy, we first reconstruct $G_{\text{eff}}(z)/G_{N,0}$ analytically. Since, as follows from Eq. (20), the quantities $G_{\text{eff},0}$ and $G_{N,0}$ coincide with better than 0.00125% accuracy, Eq. (26) taken at $z = 0$ gives also the value of $\Omega_{m,0}$ with the same accuracy. Thus, in principle, no independent measurement of $\Omega_{m,0}$ is required, it follows from (26) taken at $z = 0$.

We get an equation $G_{\text{eff}}(z) = p(z)$, where $p(z)$ is a given function that can be determined solely using observational data, which can be transformed into a nonlinear second order differential equation for $F(z)$ using the background equation

$$
\Phi^2 = -F'' - \left( \ln H \right)' + \frac{2}{1 + z} F' + \frac{2(\ln H)'}{1 + z} F - 3(1 + z)^2 H^2 F_0 \Omega_{m,0}.
$$

(27)

Hence $F(z)$ can be determined by solving the equation $G_{\text{eff}}(z) = p(z)$ after we supply the initial conditions $F_0 = \frac{1}{8\pi G_{N,0}}$ and $F'_0$. Actually $F'_0$ must be very close to zero due to the solar system constraint (20). Once $F(z)$ is found, it can be substituted into equation (16) to yield the potential $U(z)$ in function of redshift. Then, using Eq. (27), $\Phi(z)$ is found by simple integration which, after inverting this relation, gives us $z = z(\Phi - \Phi_0)$. Finally, both unknown functions $F(\Phi)$ and $U(\Phi)$ are completely fixed as functions of $\Phi - \Phi_0$. Of course this reconstruction can only be implemented in the range probed by the data corresponding to $z \lesssim 2$.

Partial reconstruction

As one does not expect to have in the very near future data referring to perturbations that are as accurate as the distance-luminosity data, it is interesting to try to extract as much information as possible using only $D_L(z)$ data. If we make assumptions on either $F(z)$ or $U(z)$, or if we assume some functional relation between both, it is again possible to reconstruct the theory using only $D_L(z)$ data. This is what we mean by reconstruction of constrained models, or partial reconstruction: we reconstruct models where by assumption there is effectively only one unknown function. Several cases have been considered in [10] and powerful constraints can be obtained. In particular, it is interesting that these constraints can go beyond solar-system constraints just because they go back in time and probe the cosmological evolution. To see how powerful constraints can be obtained, we consider the interesting question whether it is possible to have a vanishing potential $U$ while the expansion of the universe still satisfies

$$
H^2(z) = H_0^2 [0.3 (1 + z)^3 + 0.7].
$$

(28)

Of course in the framework of GR an expression like (28) lends itself to the straightforward interpretation of a flat $\Lambda$ dominated universe with the cosmological parameters

$$
\Omega_{\Lambda,0} \approx 0.7, \quad \Omega_{m,0} \approx 0.3.
$$

(29)

The dynamical laws of GR are encoded in the definition of these cosmological parameters through the Friedmann equations. Now, we assume we have the same kinematics (expansion) but with modified dynamical laws as we deal with a modified gravity theory. When $U = 0$ equation (16) can be solved numerically and one can show that $F(z)$ will vanish at most at $z_{\text{max}} \approx 0.66$ if $F(z)$ evolves according to (16). Therefore, the predicted vanishing of $F(z)$ is well inside the range for which $H(z)$ is determined with sufficient accuracy. Hence, for $H(z)$ obeying (28) up to $z \approx 0.66$, these models with vanishing potential $U$ are excluded.

If observations force us to consider models that can have DE of the phantom type then quintessence models, i.e., models inside General Relativity with a minimally coupled scalar (quintessence) field, are ruled out and is led to consider models outside GR like scalar-tensor DE models that we have been discussing here. Many problems appearing in these models are representative of all DE models outside GR. Clearly discrimination between all the models will accurate data probing the background and the inhomogeneities: Supernovae data, Cosmic Microwave Background data, galaxy surveys, weak lensing data, etc. (see e.g. [15]).

Recently, investigation of $f(R)$ modified gravity theories led to the surprising result [16] that the cosmological history for a large class of such models is incompatible with observations because of the disappearance of the usual matter-dominated stage with $a \propto t^\frac{2}{3}$ which is replaced by a stage with $a \propto t^\frac{4}{3}$. So in this case, cosmological background constraints come from large $z$ behaviour and not just from SNIa data on small redshifts $z < 2!$
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