A HYPERDETERMINANT FOR $2 \times 2 \times 3$ ARRAYS

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Abstract. We use the representation theory of Lie algebras and computational linear algebra to determine the simplest nonconstant invariant polynomial in the entries of a general $2 \times 2 \times 3$ array. This polynomial is homogeneous of degree 6 and has 66 terms with coefficients ±1, ±2 in the 12 indeterminates $x_{ijk}$ where $i, j = 1, 2$ and $k = 1, 2, 3$. This invariant can be regarded as a natural generalization of Cayley’s hyperdeterminant for $2 \times 2 \times 2$ arrays.

1. Introduction

A fundamental object in multilinear algebra is Cayley’s hyperdeterminant [1], also called Kruskal’s polynomial [9], a homogeneous polynomial of degree 4 in the 8 entries of a $2 \times 2 \times 2$ array. This polynomial plays an important role in the calculation of the rank of such an array; see ten Berge [13], and the recent papers by de Silva and Lim [8], Stegeman and Comon [11], and Martin [10]. For a comprehensive survey of the topic of tensor decomposition, see Kolda and Bader [8].

Gelfand, Kapranov and Zelevinsky [5] pointed out that Cayley’s hyperdeterminant is the simplest (non-constant) polynomial in the entries of a $2 \times 2 \times 2$ array, regarded as an element of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, which is invariant under the action of the Lie group $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. Inspired by this perspective, we use the representation theory of Lie algebras and computational linear algebra to determine the simplest (non-constant) polynomial invariant of the entries of a general $2 \times 2 \times 3$ array. This polynomial is homogeneous of degree 6 and has 66 terms with coefficients ±1, ±2.

In Section 2 we recall some basic definitions, and in Section 3 we present the details of our calculations, which were done with the computer algebra system Maple. The necessary background in Lie algebras and representation theory is summarized in Section 4.

2. Preliminaries

We consider a $2 \times 2 \times 3$ array $X = (x_{ijk})$ with $i, j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$. We represent this array in terms of its three frontal slices:

$$
X = \begin{bmatrix}
  x_{111} & x_{112} & x_{113} \\
  x_{121} & x_{122} & x_{123} \\
  x_{211} & x_{212} & x_{213} \\
  x_{221} & x_{222} & x_{223}
\end{bmatrix}.
$$

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Geometrically, such an array can be represented by the following diagram:

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  111 112 113 123
 211 212 213 222 223
```

A monomial in the entries of the array $X$ has the form

$$M = x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}},$$

corresponding to a $2 \times 2 \times 3$ array $E = (e_{ijk})$ of non-negative integer exponents:

$$E = \begin{bmatrix}
e_{111} & e_{121} & e_{112} & e_{122} \\
e_{211} & e_{221} & e_{212} & e_{222}
\end{bmatrix}.$$

The degree of a monomial $M$ is the sum of its exponents:

$$e_{111} + e_{121} + e_{211} + e_{221} + e_{112} + e_{122} + e_{212} + e_{222} + e_{113} + e_{123} + e_{213} + e_{223}.$$

We define the weight of a monomial $M$ to be the ordered quadruple of integers,

$$[ w_1(M), w_2(M), w_{31}(M), w_{32}(M) ],$$

where the components are defined as follows:

$$w_1(M) = e_{111} + e_{121} - e_{211} - e_{221} + e_{112} + e_{122} - e_{212} - e_{222} + e_{113} + e_{123} - e_{213} - e_{223},$$

$$w_2(M) = e_{111} - e_{121} + e_{211} - e_{221} + e_{112} - e_{122} + e_{212} - e_{222} + e_{113} - e_{123} + e_{213} - e_{223},$$

$$w_{31}(M) = e_{111} + e_{121} + e_{211} + e_{221} - e_{112} - e_{122} - e_{212} - e_{222},$$

$$w_{32}(M) = e_{111} + e_{121} + e_{211} + e_{221} - e_{112} - e_{122} - e_{212} - e_{222}.$$

The motivation for this definition of the weight comes from the representation theory of Lie algebras (see Section 4). We note that:

- $w_1(M)$ is the difference between the upper and lower horizontal slices;
- $w_2(M)$ is the difference between the left and right vertical slices;
- $w_{31}(M)$ is the difference between the first and second frontal slices;
- $w_{32}(M)$ is the difference between the second and third frontal slices.

We write $P$ for the polynomial algebra generated by the entries of the array $X$ over the field of complex numbers:

$$P = \mathbb{C}[x_{111}, x_{121}, x_{211}, x_{221}, x_{112}, x_{122}, x_{212}, x_{222}, x_{113}, x_{123}, x_{213}, x_{223}].$$

We have the direct sum decompositions

$$P = \bigoplus_{n \geq 0} P_n, \quad P_n = \bigoplus_{a,b,c_1,c_2 \in \mathbb{Z}} P_n(a, b, c_1, c_2),$$

where $P_n$ is the subspace spanned by the monomials of degree $n$, and $P_n(a, b, c_1, c_2)$ is the subspace spanned by the monomials of weight $[a, b, c_1, c_2]$.

The representation theory of Lie algebras (see Section 4) shows that an invariant homogeneous polynomial of degree $n$ belongs to $P_n(0, 0, 0, 0)$. A basis of this subspace consists of the monomials for which parallel slices in the exponent array $E$, in each of the three directions, have the same entry sum. It is clear that such
monomials exist if and only if \( n \) is a multiple of \( \gcd(2, 2, 3) = 6 \). We also consider four other subspaces,

\[
P_n(0, 0, 0), \quad P_n(2, 0, 0), \quad P_n(0, 0, 2), \quad P_n(0, 0, -1, 2).
\]

We define four linear maps from \( P_n(0, 0, 0) \) to the other subspaces, again motivated by the representation theory of Lie algebras:

\[
U_1: P_n(0, 0, 0) \to P_n(2, 0, 0), \quad U_2: P_n(0, 0, 0) \to P_n(0, 0, 2),
\]

\[
U_{31}: P_n(0, 0, 0) \to P_n(0, 0, 2, -1), \quad U_{32}: P_n(0, 0, 0) \to P_n(0, 0, -1, 2).
\]

These maps are defined on basis monomials and extended linearly. For \( U_1 \), if \( e_{2jk} \geq 1 \) for some \( j, k \) then we multiply the monomial by \( e_{2jk} \), subtract 1 from the exponent of \( x_{2jk} \), and add 1 to the exponent of \( x_{1jk} \); the result of applying \( U_1 \) is the sum of these six terms (if \( e_{2jk} = 0 \) for some \( j, k \) then the corresponding term does not appear). For \( U_2 \), the definition is similar, but we consider the second index: if \( e_{1jk} \geq 1 \) for some \( i, k \) then we multiply the monomial by \( e_{1jk} \), subtract 1 from the exponent of \( x_{1jk} \), and add 1 to the exponent of \( x_{1ik} \). We have:

\[
U_1(x_{111} x_{112} x_{121} x_{122} x_{112} x_{122} x_{111} x_{123} x_{123} x_{123}) =
\]

\[
e_{211} x_{111} + e_{121} x_{121} x_{221} x_{111} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} +
\]

\[
e_{221} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} +
\]

\[
e_{212} x_{111} x_{121} x_{221} x_{112} x_{122} x_{122} x_{113} x_{123} x_{123} x_{123} +
\]

\[
e_{222} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} +
\]

\[
e_{213} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} x_{123} +
\]

\[
e_{223} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} x_{123} x_{123}.
\]

For \( U_{31} \), if \( e_{1jk} \geq 1 \) for some \( i, j \) then we multiply the monomial by \( e_{1jk} \), subtract 1 from the exponent of \( x_{ij2} \), and add 1 to the exponent of \( x_{ij1} \). For \( U_{32} \), if \( e_{1jk} \geq 1 \) for some \( i, j \) then we multiply the monomial by \( e_{1jk} \), subtract 1 from the exponent of \( x_{ij3} \), and add 1 to the exponent of \( x_{ij2} \). We have:

\[
U_{31}(x_{111} x_{121} x_{211} x_{212} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123}) =
\]

\[
e_{112} x_{111} + e_{121} x_{121} x_{221} x_{111} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} +
\]

\[
e_{122} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} x_{123} +
\]

\[
e_{121} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} x_{123} x_{123} +
\]

\[
e_{123} x_{111} x_{121} x_{221} x_{112} x_{122} x_{222} x_{113} x_{123} x_{123} x_{123} x_{123} x_{123} x_{123} x_{123}.
\]
With this notation, the 80 basis monomials of $P_{\text{lexicographical order}}$:

We also consider the four subspaces, with dimensions 63, 63, 60, 60 respectively.

We combine these four linear maps into a single map and consider

$$U: P_n(0, 0, 0) \rightarrow P_n(2, 0, 0, 0) \oplus P_n(0, 2, 0, 0) \oplus P_n(0, 0, 2, -1) \oplus P_n(0, 0, -1, 2),$$
defined on basis monomials by the equation

$$U(M) = \left( U_1(M), U_2(M), U_{31}(M), U_{32}(M) \right).$$

It follows from the representation theory of Lie algebras that the invariant polynomials of degree $n$ are the elements of the kernel of $U$.

3. Main Result

Ignoring the trivial invariant in degree 0 — the constant polynomial 1 — the lowest degree in which an invariant polynomial can exist is 6. We adopt the convention of flattening an array of exponents as follows:

$$\begin{bmatrix} e_{111} & e_{112} & e_{121} & e_{122} & e_{113} & e_{123} \\ e_{211} & e_{212} & e_{221} & e_{222} & e_{213} & e_{223} \end{bmatrix} \leftrightarrow \begin{bmatrix} e_{111} & e_{112} & e_{211} & e_{212} & e_{113} & e_{213} \\ e_{112} & e_{122} & e_{212} & e_{222} & e_{113} & e_{223} \\ e_{121} & e_{122} & e_{212} & e_{222} & e_{123} & e_{223} \\ e_{113} & e_{123} & e_{213} & e_{223} & e_{123} & e_{223} \end{bmatrix}$$

With this notation, the 80 basis monomials of $P_6(0, 0, 0, 0)$ are as follows, listed in lexicographical order:

We also consider the four subspaces, $P_6(2, 0, 0, 0), P_6(0, 2, 0, 0), P_6(0, 0, 2, -1), P_6(0, 0, -1, 2)$, with dimensions 63, 63, 60, 60 respectively.

In degree 6, we represent the linear map $U$ as the matrix $[U]$ with respect to the lexicographically ordered bases of the five subspaces. The matrix $[U]$ has 80
columns and 63 + 63 + 60 + 60 = 246 rows; it consists of a stack of four blocks, two of size 63 × 80 and two of size 60 × 80. We use the computer algebra system Maple to construct the matrix \([U]\) and compute its row canonical form; we find that the rank is 79 and hence the nullspace is 1-dimensional. This provides a computational verification that there exist invariant polynomials in degree 6, and that every such polynomial is a scalar multiple of one fundamental invariant. The \(1 \times 80\) coefficient vector of a nullspace basis can be represented by this \(4 \times 80\) matrix:

\[
0 \ 1 \ -1 \ -1 \ 0 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \\
1 \ -1 \ 0 \ 1 \ 1 \ 0 \ -2 \ -1 \ -2 \ 2 \ 1 \ -1 \ 1 \ 0 \ 1 \ -1 \ 0 \ 1 \\
1 \ -1 \ -1 \ -2 \ 2 \ 0 \ 2 \ 0 \ -1 \ 0 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \\
1 \ -1 \ -1 \ 0 \ 0 \ 1 \ -1 \ 1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 0 \ 1 \ -1 \ -1 \ 0 \ 1
\]

The 66 nonzero coefficients are \(±1\) and \(±2\). In each monomial, the exponents form a partition of 6, either 2211 or 21111 or 111111. We call this polynomial \(D\).

To understand the structure of the invariant polynomial \(D\), we consider the action of the group \(S_2 \times S_2 \times S_3\), where \(S_n\) is the group of permutations of \(n\) symbols, on the indices \((i, j, k)\) in the Cartesian product \(\{1, 2\} \times \{1, 2\} \times \{1, 2, 3\}\) corresponding to the indeterminates \(x_{ijk}\). Let \(\alpha\) (respectively \(\beta\)) be the transposition (12) in the first (respectively second) copy of \(S_2\), which we denote by \(S_{2,1}\) (respectively \(S_{2,2}\)). Let \(\sigma\) and \(\tau\) be the transposition (12) and the 3-cycle (123) in \(S_3\). We have

\[
S_{2,1} = \{1, \alpha\}, \quad S_{2,2} = \{1, \beta\}, \quad S_3 = \{1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2\}.
\]

Given a monomial \(M\) of degree 6 and weight \([0, 0, 0, 0]\), we consider the following element of \(P_6(0, 0, 0, 0)\), which we call the signed orbit of \(M\) under the action of \(S_{2,1} \times S_{2,2} \times S_3\) on the subscripts of its indeterminates:

\[
\text{orbit}(M) = M + \tau M + \tau^2 + \sigma M + \sigma \tau M + \sigma \tau^2 M \\
- \beta M - \beta \tau M - \beta \tau^2 - \beta \sigma M - \beta \sigma \tau M - \beta \sigma \tau^2 M \\
- \alpha M - \alpha \tau M - \alpha \tau^2 - \alpha \sigma M - \alpha \sigma \tau M - \alpha \sigma \tau^2 M \\
+ \alpha \beta M + \alpha \beta \tau M + \alpha \beta \tau^2 + \alpha \beta \sigma M + \alpha \beta \sigma \tau M + \alpha \beta \sigma \tau^2 M.
\]

The sign of each term is the product of the signs of the corresponding elements of \(S_{2,1}\) and \(S_{2,2}\) (we ignore the sign of the element of \(S_3\)).

In particular, we consider the following five monomials:

\[
M_1 = x_{111}^2 x_{122} x_{212}^2 x_{223}^2, \quad M_2 = x_{111}^2 x_{122} x_{222} x_{213} x_{223}, \\
M_3 = x_{111} x_{121} x_{112} x_{222} x_{213} x_{223}, \quad M_4 = x_{111} x_{211} x_{112} x_{222} x_{213} x_{223}, \\
M_5 = x_{111} x_{221} x_{112} x_{222} x_{213} x_{213}.
\]

Geometrically these monomials are represented by the following diagrams, where a solid (respectively open) vertex indicates that the corresponding indeterminate does (respectively does not) occur; a double solid vertex indicates the square:
The hyperdeterminant $\mathcal{D}$ of a $2 \times 2 \times 3$ array is a linear combination of the orbits generated by these five monomials. A straightforward calculation verifies the following expressions. We note that the orbits of $M_3$ and $M_4$ are interchanged by the transposition of the first two indices, which corresponds to the standard matrix transposition of the $2 \times 2$ frontal slices:

\[
\text{orbit}(M_1) = - x_{111}^2 x_{122} x_{212} x_{223}^2 + x_{111}^2 x_{122}^2 x_{123} x_{213} - x_{111} x_{221} x_{122}^2 x_{213}^2,
\]

\[
\text{orbit}(M_2) = - x_{111}^2 x_{122} x_{212} x_{223} x_{213}^2 - x_{111} x_{221} x_{122} x_{223} x_{123} - x_{111} x_{221} x_{122} x_{223} x_{123},
\]

\[
\text{orbit}(M_3) = x_{111} x_{121} x_{212} x_{222} x_{123} x_{213} - x_{111} x_{221} x_{122} x_{223} x_{123},
\]

\[
\text{orbit}(M_4) = - x_{111} x_{221} x_{122} x_{223} x_{123}^2 + x_{111} x_{221} x_{122} x_{223} x_{123},
\]

\[
\text{orbit}(M_5) = - x_{111} x_{221} x_{122} x_{223}^2 - x_{111} x_{221} x_{122} x_{223}^2
\]
The simplest (non-constant) invariant polynomial for $2 \times 2 \times 3$ arrays is a homogeneous polynomial $D$ of degree 6 with 66 terms and coefficients $\pm 1$ and $\pm 2$. It is equal to the following linear combination of the signed orbits of five monomials under the action of the group $S_2 \times S_2 \times S_3$:

$$D = \frac{1}{2} \text{orbit}(x_{111}^2 x_{122}^2 x_{213}^2) - \text{orbit}(x_{111} x_{122} x_{213}) + \frac{1}{2} \text{orbit}(x_{111} x_{121} x_{122} x_{213}) + \frac{1}{2} \text{orbit}(x_{111} x_{121} x_{122} x_{213}) - \frac{1}{2} \text{orbit}(x_{111} x_{211} x_{122} x_{213}).$$

The hyperdeterminant $D$ of the $2 \times 2 \times 3$ array

is the polynomial

$$D = a^2 f g i \ell + a^2 h^2 j k - a d f^2 k^2 - a d g^2 j^2 - b^2 c h k^2 - b^2 g^2 i \ell + b c e^2 \ell^2 + b c h^2 i^2 - c^2 e h j^2 - c^2 f^2 i \ell + d^2 c^2 j k + d^2 f g i^2 - a^2 f h k \ell - a^2 g h j \ell - a b e g i \ell^2 + a b f h k^2 + a b g^2 j \ell - a b h^2 i k - a c e \ell^2 + a c f^2 k \ell + a c g h j^2 - a c h^2 i j + b^2 e g k \ell + b^2 g h ik - b d e^2 k \ell + b d e f k^2 + b d g^2 i j - b d g h i^2 + c^2 e f j \ell + c^2 f h i j - c d e^2 j \ell + c d e g j^2 + c d f^2 i k - c d f h i^2 - d^2 e f i k - d^2 e g i j + a b c h k \ell - a b f g k \ell + a b g h i j + a d f k \ell + a d g h i j - b c e f k \ell - b c g h i j + c d e f i \ell - c d e f j k + c d e h i j - c d f g i j + a c e h j \ell - a c f g i j + a c f h i j - a c f h j k + a c d e g j \ell + a d e f k \ell + a d e f h i j - b c e g i j + b c f h i j + b d e g i j - b d e g j k + b d e h i j - b d f g i j - 2 a d e h j k - 2 a d f g i j + 2 a d f g j k - 2 b c e h i j + 2 b c e h j k + 2 b c f g i j.$$. 
4. Representation Theory of Lie Algebras

This section provides a brief summary of the necessary background material on Lie algebras and their representations. Details may be found in standard textbooks such as Jacobson [7], Humphreys [6], de Graaf [2], or Erdmann and Wildon [4].

We regard a $2 \times 2 \times 3$ array with complex entries as an element of the tensor product $T = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. The 14-dimensional semisimple Lie group

$$G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_3(\mathbb{C}),$$

acts on the 12-dimensional space $T$ by unimodular changes of basis along the three directions. ($SL_n(\mathbb{C})$ is the set of $n \times n$ complex matrices with determinant 1, with the usual definition of matrix multiplication.) Since we are concerned only with the action of this Lie group on finite-dimensional complex vector spaces, we can linearize the problem and consider instead the action of the Lie algebra

$$L = sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}) \oplus sl_3(\mathbb{C}).$$

($sl_n(\mathbb{C})$ is the vector space of $n \times n$ complex matrices with trace 0; two such matrices are composed using the commutator $[A, B] = AB - BA$.) For an elementary and attractive introduction to Lie theory, by which is meant the connection between Lie groups and Lie algebras, see Stillwell [12].

The most important elements of $sl_2(\mathbb{C})$ and $sl_3(\mathbb{C})$ are the diagonal matrices,

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and the superdiagonal matrices,

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the natural representation, the elements of $sl_2(\mathbb{C})$ and $sl_3(\mathbb{C})$ act by left matrix-vector multiplication on the vector spaces $\mathbb{C}^2$ and $\mathbb{C}^3$ with standard bases,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(The context will clarify this ambiguous notation.) From this we obtain the basis of $T$ consisting of the simple tensors

$$x_{ijk} = x_i \otimes x_j \otimes x_k \quad (i, j \in \{1, 2\}, k \in \{1, 2, 3\}).$$

Strictly speaking, we regard this element as a coordinate function on $T$, so we should use dual basis vectors, but this distinction will not matter for us.

We need to determine the action of the basis of $L$ on the basis of $T$. We have

$$H \cdot x_1 = x_1, \quad H \cdot x_2 = -x_2,$$
$$E \cdot x_1 = 0, \quad E \cdot x_2 = x_1,$$
$$H_1 \cdot x_1 = x_1, \quad H_1 \cdot x_2 = -x_2, \quad H_1 \cdot x_3 = 0,$$
$$H_2 \cdot x_1 = 0, \quad H_2 \cdot x_2 = x_2, \quad H_2 \cdot x_3 = -x_3,$$
$$E_1 \cdot x_1 = 0, \quad E_1 \cdot x_2 = x_1, \quad E_1 \cdot x_3 = 0,$$
$$E_2 \cdot x_1 = 0, \quad E_2 \cdot x_2 = 0, \quad E_2 \cdot x_3 = x_2.$$
The general element \((A, B, C)\) in \(L\) acts on simple tensors in \(T\) as follows:

\[
(A, B, C) \cdot (x_i \otimes x_j \otimes x_k) = (A \cdot x_i) \otimes x_j \otimes x_k + x_i \otimes (B \cdot x_j) \otimes x_k + x_i \otimes x_j \otimes (C \cdot x_k).
\]

This action extends to the monomial basis of the polynomial algebra

\[
P = \mathbb{C}[x_{111}, x_{121}, x_{211}, x_{112}, x_{122}, x_{212}, x_{221}, x_{123}, x_{213}, x_{223}],
\]

by induction on the degree using the derivation rule (which generalizes the product rule from elementary calculus),

\[
(A, B, C) \cdot (fg) = ((A, B, C) \cdot f)g + f((A, B, C) \cdot g);
\]

the action then extends linearly to \(P\). In particular, we obtain the following formulas (which generalize the extended power rule from elementary calculus):

\[
H \cdot x_i^2 = ex_i^1,
E \cdot x_i^1 = 0,
H_1 \cdot x_i^1 = ex_i^1,
H_2 \cdot x_i^1 = 0,
E_1 \cdot x_i^1 = 0,
E_2 \cdot x_i^1 = 0,
\]

\[
H \cdot x_i^2 = -ex_i^2,
E \cdot x_i^1 = ex_i^1e^{-1},
H_1 \cdot x_i^2 = -ex_i^2,
H_2 \cdot x_i^2 = ex_i^2,
E_1 \cdot x_i^2 = ex_i^1e^{-1},
E_2 \cdot x_i^2 = 0,
\]

Lie theory shows that the polynomials which are fixed by the action of the Lie group \(G\) coincide with the polynomials which are annihilated by the action of the Lie algebra \(L\). Furthermore, the representation theory of Lie algebras shows that a polynomial is annihilated by \(L\) if and only if it is annihilated by the elements

\[
(H, 0, 0), (E, 0, 0), (0, H, 0), (0, E, 0),
(0, 0, H_1), (0, 0, H_2), (0, 0, E_1), (0, 0, E_2).
\]

Combining the previous formulas to determine the action on a general monomial

\[
M = x_{111}^{e_1} x_{121}^{e_2} x_{211}^{e_3} x_{112}^{e_4} x_{122}^{e_5} x_{212}^{e_6} x_{221}^{e_7} x_{123}^{e_8} x_{213}^{e_9} x_{223}^{e_{10}},
\]

we obtain

\[
(H, 0, 0) \cdot M = w_1(M),
E \cdot M = w_2(M),
(0, 0, H_1) \cdot M = w_{31}(M),
(0, 0, H_2) \cdot M = w_{32}(M),
\]

where \(w_1(M), w_2(M), w_{31}(M), w_{32}(M)\) are defined in Section 2. Thus a homogeneous polynomial of degree \(n\) is annihilated by the diagonal matrices \(H, H_1, H_2\) in \(L\) if and only if it belongs to the subspace \(P_n(0, 0, 0)\). To conclude this summary of the representation theory, we observe that the action of the superdiagonal matrices \(E, E_1, E_2\) in \(L\) is given by the linear maps \(U_1, U_2, U_{31}, U_{32}\).

The dimension of the subspace \(P_n(a, b, c_1, c_2)\) is of combinatorial interest, since a basis of this subspace consists of the \(2 \times 2 \times 3\) arrays of non-negative integers with prescribed differences between the parallel slices in the three directions. Computational enumeration with Maple produced the dimensions in Table 2. Polynomial interpolation from the 17 data points in each column of Table 1 suggests the following conjecture for the dimensions of the weight spaces.
Table 1. Dimensions of weight spaces in degree $n$

| $n$ | $\dim P_n(0,0,0,0)$ | $\dim P_n(0,2,0,0)$ | $\dim P_n(0,0,-1,2)$ |
|-----|----------------------|----------------------|----------------------|
| 0   | 1                    | 0                    | 0                    |
| 6   | 80                   | 63                   | 60                   |
| 12  | 1323                 | 1206                 | 1180                 |
| 18  | 9832                 | 9354                 | 9240                 |
| 24  | 46733                | 45294                | 44940                |
| 30  | 167184               | 163629               | 162740               |
| 36  | 491384               | 483732               | 481800               |
| 42  | 1250576              | 1235700              | 1231920              |
| 48  | 2851065              | 2824308              | 2817480              |
| 54  | 5959216              | 5913963              | 5902380              |
| 60  | 11610467             | 11537658             | 11518980             |
| 66  | 21345336             | 21232926             | 21204040             |
| 72  | 37375429             | 37207794             | 37164660             |
| 78  | 62782448             | 62539737             | 62472220             |
| 84  | 101753199            | 101410632            | 101322320            |
| 90  | 159853600            | 159380712            | 159258720            |
| 96  | 244344689            | 243704520            | 243539280            |

Conjecture. We have the following dimension formulas:

\[
\dim P_n(0,0,0,0) = \frac{1}{58786560} (n + 6) \times (125n^6 + 4500n^5 + 68004n^4 + 552096n^3 + 2584224n^2 + 6811776n + 9797760),
\]

\[
\dim P_n(2,0,0,0) = \dim P_n(0,2,0,0) = \frac{1}{58786560} n (n + 6) (n + 12) \times (125n^4 + 3000n^3 + 28602n^2 + 127224n + 254664),
\]

\[
\dim P_n(0,0,2,-1) = \dim P_n(0,0,-1,2) = \frac{1}{11757312} n (n + 6) (n + 12) \times (5n^2 + 84n + 396) (5n^2 + 36n + 108).
\]

In each case the function is a polynomial of degree 7.

References

[1] A. Cayley: On the theory of linear transformations. Cambridge Mathematical Journal 4 (1845) 193–209. [www.archive.org/details/collectedmathema01cayluoft]

[2] W. A. de Graaf: Lie Algebras: Theory and Algorithms. North-Holland, Amsterdam, 2000.

[3] V. de Silva, L.-H. Lim: Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM Journal on Matrix Analysis and its Applications 30 (2008), no. 3, 1084–1127.

[4] K. Erdmann, M. J. Wildon: Introduction to Lie Algebras. Springer-Verlag, London, 2006.

[5] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky: Hyperdeterminants. Advances in Mathematics 96 (1992), no. 2, 226–263.

[6] J. E. Humphreys: Introduction to Lie Algebras and Representation Theory. Springer, New York, 1972.

[7] N. Jacobson: Lie Algebras. Interscience Publishers, New York, 1962.
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[8] T. G. Kolda, B. W. Bader: Tensor decompositions and applications. *SIAM Review* 51 (2009), no. 3, 455–500.

[9] J. B. Kruskal: Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics. *Linear Algebra and its Applications* 18 (1977), no. 2, 95–138.

[10] C. D. Martin: The rank of a $2 \times 2 \times 2$ tensor. *Linear and Multilinear Algebra* (to appear). <DOI:10.1080/03081087.2010.538923>

[11] A. Stegeman, P. Comon: Subtracting a best rank-1 approximation may increase tensor rank. *Linear Algebra and its Applications* 433 (2010) 1276–1300.

[12] J. Stillwell: *Naive Lie Theory*. Springer, New York, 2008.

[13] J. M. F. ten Berge: Kruskal’s polynomial for $2 \times 2 \times 2$ arrays and a generalization to $2 \times n \times n$ arrays. *Psychometrika* 56 (1991), no. 4, 631–636.

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