THE 2-ND HESSIAN TYPE EQUATION ON ALMOST HERMITIAN MANIFOLDS

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Abstract. In this paper, we derive the second order estimate to the 2-nd Hessian type equation on a compact almost Hermitian manifold.

1. Introduction

As a generalization of Laplace equation and complex Monge-Ampère equation on a complex manifold $M$, the following $k$-th complex Hessian equation (1 < $k$ < $n$) has been studied extensively,

$$
\begin{align*}
(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^k \wedge \omega^{n-k} &= e^F \omega^n \\
\omega + \sqrt{-1}\partial\bar{\partial}\varphi &\in \Gamma_k(M) \\
\sup_M \varphi &= 0,
\end{align*}
$$

where $\Gamma_k(M)$ is the space of $k$-th convex $(1,1)$-forms (cf. Section 2). When $(M, \omega)$ is a compact Kähler manifold, the second order a priori estimate was obtained by Hou [10] and Hou-Ma-Wu [11]. Lately, by using Hou-Ma-Wu’s result, Dinew-Kołodziej [3] solved the existence of (1.1). Székelyhidi [18] extended Dinew-Kołodziej’s result to a Hermitian manifold (see also [20] by Zhang).

The 2-nd complex Hessian type equation plays an important role in Strominger system from the string theory [17]. In [5], Fu-Yau reduced the Strominger system to an equation

$$
\sqrt{-1}\partial\bar{\partial}(e^\varphi - f e^{-\varphi}) \wedge \omega^{n-1} + n\alpha \sqrt{-1}\partial\bar{\partial}\varphi \wedge \sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega^{n-2} + \mu \frac{\omega^n}{n!} = 0,
$$

where $\alpha \in \mathbb{R}$ is a slope parameter and $f, \mu \in C^\infty(M)$ satisfy some admissible conditions. They found that (1.2) can be written as a general 2-nd Hessian equation,

$$
\begin{align*}
((e^\varphi + f e^{-\varphi})\omega + 2n\alpha \sqrt{-1}\partial\bar{\partial}\varphi)^2 \wedge \omega^{n-2} &= e^{F(z,\partial\varphi,\bar{\varphi})} \omega^n \\
(e^\varphi + f e^{-\varphi})\omega + 2n\alpha \sqrt{-1}\partial\bar{\partial}\varphi &\in \Gamma_2(M),
\end{align*}
$$

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where 
\[ e^{F(z, \partial \phi, \phi)} = e^{2\phi}(1 - 4\alpha e^{-\phi}|\partial \phi|^2_g) + 4\alpha f e^{-\phi}|\partial \phi|^2_g + 2f + e^{-2\phi} f^2 - 4\alpha \mu \frac{e^{-\phi}}{n-1} + 4\alpha e^{-\phi} \left( \Delta f - 2 \text{Re}(g^{ij} f_i \varphi^j) \right). \]

By (1.3), Fu-Yau [4, 5] solved (1.2) on a toric fibration over a K3 surface. Recently, Phong-Picard-Zhang [13] obtained a priori estimates of (1.3) with slope parameter \( \alpha > 0 \) on a compact Kähler manifold. In [15], they also solved the existence of (1.3) with slope parameter \( \alpha < 0 \).

In this paper, we generalize the 2-nd complex Hessian equation to an almost Hermitian manifold \((M, \omega, J)\) and consider equation,

\[
\begin{aligned}
\{ \chi(z, \varphi) + \sqrt{-1} \partial \overline{\partial} \varphi \}^2 \wedge \omega^{n-2} &= e^{F(z, \partial \varphi, \varphi)} \omega^n \\
\chi(z, \varphi) + \sqrt{-1} \partial \overline{\partial} \varphi &\in \Gamma_2(M),
\end{aligned}
\]

where \( \chi(z, \varphi) \) is a positive \((1,1)\)-form which may depend on the solution \( \varphi \).

We prove the following \( C^2 \)-estimate.

**Theorem 1.1.** Let \((M, \omega, J)\) be a compact almost Hermitian manifold. Suppose that \( \chi(z, \varphi) \geq \varepsilon_0 \omega \) for a positive constant \( \varepsilon_0 > 0 \) and \( \varphi \) is a smooth solution of (1.4). Then the following estimate holds,

\[
\sup_M |
\nabla^2 \varphi|_g \leq C,
\]

where \( \nabla \) is the Levi-Civita connection of \( g \) and \( C \) is a uniform constant depending only on \( \varepsilon_0, \| \varphi \|_{C^1}, \| F \|_{C^2}, \| \chi \|_{C^2} \) and \((M, \omega, J)\).

We note that Theorem 1.1 holds for any solution \( \varphi \) with \( (\chi(z, \varphi) + \sqrt{-1} \partial \overline{\partial} \varphi) \in \Gamma_2(M) \) and we do not need to assume that \( \varphi \) is \( \chi(z, \varphi) \)-convex. When \( M \) is Kählerian, an analogy of (1.3) was obtained for some special function \( F \) by Phong-Picard-Zhang [13, 15]. In another paper [14], they also got similar estimate (1.5) for \( \chi(z, \varphi) \)-convex solutions for general \( k \)-th complex Hessian equation on a Kähler manifold.

Compared to the work of Phong-Picard-Zhang [13, 15], our method is quite different. First, for general right hand side \( F(z, \partial \varphi, \varphi) \), there are more troublesome terms when one differentiates the equation (1.4). We overcome this new obstacle by investigating the structure of \( \log \sigma_2 \) (see Lemma 3.3). Second, since the almost complex structure \( J \) may be not integrable, there are more "bad" third order terms. In order to deal with these terms, we need to analyse the concavity of the operator \( \log \sigma_2 \). More precisely, we estimate the eigenvalues and eigenvectors of the matrix \((-G^{i\overline{j},J})\) (see Lemma 4.4).

The structure of \( \log \sigma_2 \) plays an important role in the proof, which involves some delicate calculations. We expect that the analogous argument can be extended to study \( \log \sigma_k \) (\( k > 2 \)).

More recently, Chu-Tosati-Weinkove [1] studied the Monge-Ampère equation on compact almost Hermitian manifolds and proved the existence and uniqueness of solutions for generalized Calabi-Yau equation. Since the manifold is just almost Hermitian, they gave an approach to estimate the Hessian.
of solution instead of its complex Hessian. Our motivation is from their work. In addition, the almost Hermitian manifold is a natural research object in non-Kähler geometry. The motivation of study is from differential geometry as well as mathematical physics. We refer the reader to interesting papers such as [6, 8, 2, 16, 7], etc.

At present, our computations just work for (1.4), not available for general $k$-th complex Hessian equation. On the other hand, the constant $C$ in (1.5) depends on the norm of $\partial \varphi$. We hope that there exists a $C^2$-estimate to (1.4) which may give an explicit dependence on $\partial \varphi$, so that it can be applied to study the existence of (1.4) as in [15].

The organization of paper is as follows. In Section 2, we introduce an auxiliary function $\hat{Q}$ in order to estimate the largest eigenvalue of Hessian matrix of solution of $\sigma_2$-equation. Then in Section 3, we estimate the lower bound of $L(\hat{Q})$ for the linear elliptic operator $L$ of $\sigma_2$-equation. The main estimate will be given in Section 4, where Theorem 1.1 will be proved at the end. In Section 5, we give the proof of Lemma 4.4.

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2. Preliminaries

2.1. $\Gamma_k(M)$-space. On an almost Hermitian manifold $(M, \omega, J)$ with real dimension $2n$, $\partial$ and $\overline{\partial}$ operators can be also defined for any $(p, q)$-form $\beta$ (cf. [7, 11]). In particular, for any $f \in C^2(M)$, $\sqrt{-1} \partial \overline{\partial} f = \frac{1}{2} (dJdf)^{(1,1)}$ is a real $(1, 1)$-form in $A^{1,1}(M)$, where $A^{1,1}(M)$ is the space of smooth real $(1,1)$ forms on $(M, \omega, J)$. Let $\{e_i\}_{i=1}^n$ be a local frame for $T^{(1,0)} C_M$ associated to Riemannian metric $g$ on $(M, \omega, J)$. Then (cf. [7, (2.5)])

$$f_\sigma = \sqrt{-1} \partial \overline{\partial} f(e_i, \overline{e}_j) = e_i \overline{e}_j (f) - [e_i, \overline{e}_j]^{(0,1)}(f).$$

As usually, we let $\sigma_k (1 \leq k \leq n)$ and $\Gamma_k$ be the $k$-th elementary symmetric function and the $k$-th Garding cone on $\mathbb{R}^n$, respectively. Namely, for any $\eta = (\eta_1, \eta_2, \cdots, \eta_n) \in \mathbb{R}^n$, we have

$$\sigma_k(\eta) = \sum_{1 < i_1 < \cdots < i_k < n} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_k},$$

$$\Gamma_k = \{ \eta \in \mathbb{R}^n \mid \sigma_j(\eta) > 0 \text{ for } j = 1, 2, \cdots, k \}.$$

Clearly $\sigma_k$ is a $k$-multiple functional. Then one can extend it to $A^{1,1}(M)$ by

$$\sigma_k(\alpha) = \binom{n}{k} \frac{\alpha^k}{\omega^n} \wedge \frac{\omega^{n-k}}{\omega^n}, \forall \alpha \in A^{1,1}(M).$$

Define a cone $\Gamma_k(M)$ on $A^{1,1}(M)$ by

$$\Gamma_k(M) = \{ \alpha \in A^{1,1}(M) \mid \sigma_j(\alpha) > 0 \text{ for } j = 1, 2, \cdots, k \}.$$
Thus we can introduce a $\sigma_k(\cdot)$ operator for any $\varphi \in C^\infty(M)$ with $\tilde{\omega} = (\chi + \sqrt{-1}\partial\bar{\partial}\varphi) \in \Gamma_k(M)$ by

$$\sigma_k(\chi + \sqrt{-1}\partial\bar{\partial}\varphi),$$

where $\chi$ is a real $(1,1)$-form, which may depend on $\varphi$.

In this paper, we are interested in $\sigma_2$ operator. We use the following notation

$$G^\tau = \frac{\partial \log \sigma_2(\tilde{\omega})}{\partial \tilde{g}^\tau_{ij}} \quad \text{and} \quad G^{\tau,kl} = \frac{\partial^2 \log \sigma_2(\tilde{\omega})}{\partial \tilde{g}^\tau_{ij} \partial \tilde{g}^\tau_{kl}},$$

where $\tilde{g}^\tau_{ij} = \chi^\tau_{ij} + \varphi^\tau_{ij}$. For any point $x_0 \in M$, let $\{e_i\}_{i=1}^n$ be a local unitary frame (with respect to $g$) such that $\tilde{g}^\tau_{ij}(x_0) = \delta_{ij}\tilde{g}_{ii}(x_0)$. We denote $\tilde{g}_{ii}(x_0)$ by $\eta_i$ and assume

$$\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n.$$

Then at $x_0$, we have

$$G^\tau = G^\tau_{ii} \delta_{ij} = \frac{\sigma_1(\eta|i)}{\sigma_2(\eta)} \delta_{ij},$$

where $\sigma_1(\eta|i) = \sum_{j \neq i} \eta_j$. Also we have

$$G^{\tau,kl} = \left\{ \begin{array}{ll}
G^{\tau,kl}, & \text{if } i = j, k = l; \\
G^{\tau,kl}, & \text{if } i = l, k = j, i \neq k; \\
0, & \text{otherwise.}
\end{array} \right.$$  

Moreover,

$$G^{\tau,kl} = (1 - \delta_{ik})(\sigma_2(\eta))^{-1} - (\sigma_2(\eta))^{-2} \sigma_1(\eta|i)\sigma_1(\eta|k),$$

$$G^{\tau,kl} = -(\sigma_2(\eta))^{-1}.$$

Without a confusion, we use $\sigma_1$, $\sigma_2$ and $\sigma_1(i)$ to denote $\sigma_1(\eta)$, $\sigma_2(\eta)$ and $\sigma_1(\eta|i)$, respectively. The following inequalities are very useful.

**Lemma 2.1.** At $x_0$, we have

$$\sum_i G^{\tau}_{ii} \geq \frac{2(n-1)}{n} \frac{1}{\sigma_2^{\frac{1}{n}}},$$

$$\eta_1 \sigma_1(1) \geq \frac{2}{n} \sigma_2,$$

$$G^{\tau}_{ii} \geq C \sum_k G^{\tau,kl}_{ij} \text{ for } i \geq 2.$$

**Proof.** (2.3) and (2.4) are direct consequences of Maclaurin’s inequality. For a proof of (2.5), see [12, Theorem 1].
We define a second order operator on \((M, \omega, J)\) by
\[
L(f) = G^{i\bar{j}} f_{i\bar{j}},
\]
where \(f \in C^2(M)\). It is clear that \(L\) is the linearized operator of (1.4). Since \(\chi + \sqrt{\Omega} \partial \bar{\partial} \varphi \in \Gamma_2(M)\), \(L\) is a second order elliptic operator. Here we use Einstein notation convention for convenience.

2.2. An auxiliary function. As mentioned in Section 1, we follow the argument in [1] to obtain estimate (1.5). For any smooth function \(\varphi\), we denote the eigenvalues of \(\nabla^2 \varphi\) by
\[
\lambda_1(\nabla^2 \varphi) \geq \lambda_2(\nabla^2 \varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2 \varphi).
\]
Since \((\chi + \sqrt{\Omega} \partial \bar{\partial} \varphi) \in \Gamma_2(M) \subset \Gamma_1(M)\),
\[
|\nabla^2 \varphi|_g \leq C \lambda_1(\nabla^2 \varphi) + C,
\]
for a uniform constant \(C\). Hence, it suffices to estimate \(\lambda_1(\nabla^2 \varphi)\). On the open set \(M_+ = \{ x \in M \mid \lambda_1(\nabla^2 \varphi) > 0 \}\), we consider the following quantity
\[
Q = \log \lambda_1(\nabla^2 \varphi) + h(|\partial \varphi|_g^2) + e^{-Ax}\varphi,
\]
where \(A\) is a constant to be determined. Without loss of generality, we may assume that \(M_+\) is nonempty. Otherwise, we get upper bound of \(\lambda_1(\nabla^2 \varphi)\) directly. Here
\[
h(s) = \frac{1}{2} \log(1 + \sup_M |\partial \varphi|_g^2 - s), \quad \forall \ s \geq 0.
\]
Then
\[
\frac{1}{2} \geq h' \geq \frac{1}{2 + 2 \sup_M |\partial \varphi|_g^2} \quad \text{and} \quad h'' \geq 2(h')^2.
\]
We assume that \(Q\) attains its maximum at \(x_0\) on \(M_+\). Near \(x_0\), there exists a local unitary frame \(\{ e_i \}_{i=1}^n\) (with respect to \(g\)) such that at \(x_0\), we have
\[
g_{i\bar{j}} = \delta_{ij}, \quad \tilde{g}_{i\bar{j}} = \delta_{ij} \tilde{g}_{i\bar{j}} \quad \text{and} \quad \tilde{g}_{i\Gamma} \geq \tilde{g}_{i\bar{\Gamma}} \geq \cdots \geq \tilde{g}_{i\bar{m}}.
\]
For convenience, we denote \(\tilde{g}_{i\alpha}(x_0)\) by \(\eta_i\). On the other hand, since \((M, \omega, J)\) is almost Hermitian, we can find a normal coordinate system \((U, \{ x^\alpha \}_{i=1}^{2n})\) around \(x_0\) such that it holds at \(x_0\).
\[
e_i = \frac{1}{\sqrt{2}} (\partial_{2i-1} - \sqrt{-1} \partial_{2i}) \quad \text{for} \quad i = 1, 2, \cdots, n
\]
and
\[
\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} = 0 \quad \text{for} \quad \alpha, \beta, \gamma = 1, 2, \cdots, 2n.
\]
Let \(V_1, V_2, \cdots, V_n\) be \(g\)-unit eigenvectors of \(\nabla^2 \varphi\) corresponding to eigenvalues \(\lambda_1(\nabla^2 \varphi), \lambda_2(\nabla^2 \varphi), \cdots, \lambda_{2n}(\nabla^2 \varphi)\) at \(x_0\). We assume that \(V_{\alpha} = V_{\alpha}^\beta \partial_{\beta}\) at \(x_0\) and extend vector \(V_{\alpha}\) to vector fields on \(U\) by taking the components \(V_{\alpha}^\beta\) to be constant.
When \( \lambda_1(\nabla^2\varphi)(x_0) = \lambda_2(\nabla^2\varphi)(x_0) \), \( \lambda_1(\nabla^2\varphi) \) is not smooth near \( x_0 \). To avoid this non-smooth case, we apply a perturbation argument as in [1,18,19]. We define an endomorphism \( \Phi \) of \( T\mathcal{M} \) on \( U \) by

\[
\Phi = \Phi^\beta_\alpha \frac{\partial}{\partial x^\alpha} \otimes dx^\beta,
\]

where \( B_{\gamma\beta} = \delta_{\gamma\beta} - V_1^\gamma V_1^\beta \). Let \( \lambda_1(\Phi) \geq \lambda_2(\Phi) \geq \cdots \geq \lambda_{2n}(\Phi) \) be the eigenvalues of \( \Phi \). Then \( V_1, V_2, \cdots, V_{2n} \) are still eigenvectors of \( \Phi \), corresponding to eigenvalues \( \lambda_1(\Phi), \lambda_2(\Phi), \cdots, \lambda_{2n}(\Phi) \) at \( x_0 \). Moreover, \( \lambda_1(\Phi)(x_0) > \lambda_2(\Phi)(x_0) \), which implies \( \lambda_1(\Phi) \) is smooth near \( x_0 \). On \( U \), we replace \( Q \) by the following smooth quantity

\[
\hat{Q} = \log \lambda_1(\Phi) + h(|\partial\varphi|_g^2) + e^{-A\varphi}.
\]

Since \( \lambda_1(\nabla^2\varphi)(x_0) = \lambda_1(\Phi)(x_0) \) and \( \lambda_1(\nabla^2\varphi) \geq \lambda_1(\Phi) \), \( x_0 \) is still the maximum point of \( \hat{Q} \). For convenience, we denote \( \lambda_\alpha(\Phi) \) by \( \lambda_\alpha \) for \( \alpha = 1, 2, \cdots, 2n \).

The following formulas give the first and second derivatives of \( \lambda_1 \) at \( x_0 \) (see e.g. [1, Lemma 5.2]).

**Lemma 2.2.**

\[
\lambda_1^{\alpha\beta} := \frac{\partial \lambda_1}{\partial \Phi_\alpha^\beta} = V_1^\alpha V_1^\beta,
\]

\[
\lambda_1^{\alpha\beta,\gamma\delta} := \frac{\partial^2 \lambda_1}{\partial \Phi_\alpha^\beta \partial \Phi_\delta^\gamma} = \sum_{\mu > 1} \frac{V_1^\alpha V_1^\beta V_1^\gamma V_1^\delta + V_1^\alpha V_1^\beta V_1^\gamma V_1^\delta}{\lambda_1 - \lambda_\mu},
\]

where \( \alpha, \beta, \gamma, \delta = 1, 2, \cdots, 2n \).

3. Lower bound of \( L(\hat{Q}) \)

In this section we compute \( L(\hat{Q}) \) by using equation (1.4). Since the right hand side \( F \) of (1.4) depends on \( \partial\varphi \), a trouble is that a bad term \(-C\lambda_1\) will appear when we differentiate (1.4) twice. We use the structure of the operator \( \log \sigma_2 \) to overcome it (see Lemma 3.3).

Locally, \( F(z, \partial\varphi, \varphi) \) can be regarded as a real-valued function on the set \( \Gamma = U \times \mathbb{C}^n \times \mathbb{R} \). We denote points in \( \Gamma \) typically by \( \gamma = (z, p, r) \) where \( z \in U \), \( p = (p_1, p_2, \cdots, p_n) \in \mathbb{C}^n \) and \( r \in \mathbb{R} \). For convenience, we use the following notations

\[
F_r = \frac{\partial F}{\partial r}, F_{p_i} = \frac{\partial F}{\partial p_i}, F_{p_i}^r = \frac{\partial F}{\partial p_i},
\]

\[
F_i = e_i(F(\cdot, p, r)), F_i^r = \widehat{e_i}(F(\cdot, p, r)), F_W = W(F(\cdot, p, r)),
\]

where \( W \) is a vector field. In the following, we always compute derivatives at the maximal point \( x_0 \) of \( \hat{Q} \). First we show
Lemma 3.1.

\[
L(|\partial \varphi|_g^2) \geq \frac{1}{2} \sum_k G^{\overline{\varphi}}(|e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2) - C \sum_i G^{\overline{\varphi}} \\
+ 2 \sum_{k,i} \text{Re} \left( \varphi_k (F_p e_k e_i(\varphi) + F_{\overline{k}} e_k e_i(\varphi)) \right).
\]  

(3.1)

Proof. By (3.4), we have

\[
\log \sigma_2(\tilde{\omega}) = \log \left(\frac{n}{2}\right) + F(z, \partial \varphi, \varphi),
\]

where \( \tilde{\omega} = \chi + \sqrt{-1} \partial \bar{\partial} \varphi \). For any vector field \( W \), differentiating (3.2) along \( W \) at \( x_0 \), we get

\[
G^{\overline{\varphi}} W(\tilde{g}) = W(F),
\]

which implies

\[
\sum_k G^{\overline{\varphi}} (W e_i \overline{e}_i(\varphi) - W[e_i, \overline{e}_i]^{(0,1)}(\varphi))
\]

(3.4)

\[
= -G^{\overline{\varphi}} W(\chi) + F_W + F_{\bar{z}} W(\varphi) + F_{\bar{z}} e_i(\varphi) + F_{\bar{z}} e_i(\varphi).
\]

By choosing \( W = \overline{\omega} \), it follows

\[
\sum_k G^{\overline{\varphi}} (\varphi_k e_k e_i(\varphi) + \varphi_k \overline{e}_k e_i(\varphi) - C \sum_i G^{\overline{\varphi}}.
\]

(3.5)

On the other hand,

\[
L(|\partial \varphi|_g^2) = \sum_k G^{\overline{\varphi}} \left( e_i \overline{e}_i(\varphi \varphi_k) - [e_i, \overline{e}_i]^{(0,1)}(\varphi \varphi_k) \right)
\]

(3.6)

\[
= \sum_k G^{\overline{\varphi}}(|e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2)
\]

\[
+ \sum_k G^{\overline{\varphi}} \left( \varphi_k e_i \overline{e}_i e_k(\varphi) - \varphi_k [e_i, \overline{e}_i]^{(0,1)} e_k(\varphi) \right)
\]

\[
+ \sum_k G^{\overline{\varphi}} \left( \varphi_k e_i \overline{e}_i e_k(\varphi) - \varphi_k [e_i, \overline{e}_i]^{(0,1)} e_k(\varphi) \right).
\]

Note

\[
\sum_k G^{\overline{\varphi}} \left( \varphi_k e_i \overline{e}_i e_k(\varphi) - \varphi_k [e_i, \overline{e}_i]^{(0,1)} e_k(\varphi) \right)
\]

\[
\geq \sum_k G^{\overline{\varphi}} \left( \varphi_k e_i \overline{e}_i e_k(\varphi) - \varphi_k [e_i, \overline{e}_i]^{(0,1)}(\varphi) \right) - C \sum_i G^{\overline{\varphi}}
\]

\[
- C \sum_k G^{\overline{\varphi}}(|e_i e_k(\varphi)| + |e_i \overline{e}_k(\varphi)|).
\]
By (3.5) and the Cauchy-Schwarz inequality, it follows
\[
\sum_k G^{\tilde{\alpha}} \left( \varphi_k e_i \tilde{\tau}_k (\varphi) - \varphi_k [e_i, \tilde{\tau}_i]^{(0,1)} (\varphi) \right)
\geq 2 \sum_{i,k} (\varphi_k F_{\mu_i} \tilde{\tau}_k (\varphi) + \varphi_k F_{\tilde{\tau}_k} \tilde{\tau}_i (\varphi)) - C \sum_i G^{\tilde{\alpha}}
- \frac{1}{4} \sum_k G^{\tilde{\alpha}} (|e_i e_k (\varphi)|^2 + |e_i \tilde{\tau}_k (\varphi)|^2).
\]
Similarly,
\[
\sum_k G^{\tilde{\alpha}} \left( \varphi_k e_i e_k (\varphi) - \varphi_k [e_i, \tilde{\tau}_i]^{(0,1)} e_k (\varphi) \right)
\geq 2 \sum_{i,k} (\varphi_k F_{\mu_i} e_k (\varphi) + \varphi_k F_{\tilde{\tau}_k} e_i (\varphi)) - C \sum_i G^{\tilde{\alpha}}
- \frac{1}{4} \sum_k G^{\tilde{\alpha}} (|e_i e_k (\varphi)|^2 + |e_i \tilde{\tau}_k (\varphi)|^2).
\]
Substituting the above two inequalities into (3.6), we get (3.1) immediately. \qed

Next, we compute \( L(\lambda_1) \).

**Lemma 3.2.**

\[
L(\lambda_1) \geq 2 \sum_{\alpha > 1} \frac{G^{\tilde{\alpha}} |e_1 (\varphi_{V_1})|}{(\lambda_1 - \lambda_\alpha)} - G^{\tilde{\alpha}} \tilde{\tau} V_1 (\tilde{g}_\sigma) V_1 (\tilde{\tau}_k)
- 2G^{\tilde{\alpha}} [V_1, e_i] V_1 e_i (\varphi) - 2G^{\tilde{\alpha}} [V_1, \tilde{\tau}_i] V_1 e_i (\varphi)
- C\lambda_1 \sum_i G^{\tilde{\alpha}} - C\lambda_1^2 + F_{\mu_i} V_1 e_i (\varphi) + F_{\tilde{\tau}_i} V_1 e_i (\varphi).
\]

**Proof.** The proof is similar to one of [1 Lemma 5.3]. In fact, by Lemma 2.2, we have
\[
L(\lambda_1)
= G^{\tilde{\alpha}} \lambda_1^{\alpha,\gamma} e_i (\Phi^\gamma) e_i (\Phi^\alpha) + G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} e_i \tilde{\tau}_i (\Phi^\alpha) - G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} [e_i, \tilde{\tau}_i]^{(0,1)} (\Phi^\alpha)
= G^{\tilde{\alpha}} \lambda_1^{\alpha,\gamma} e_i (\varphi_{\alpha}) e_i (\varphi_{\beta}) + G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} e_i \tilde{\tau}_i (\varphi_{\alpha}) + G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} \varphi_{\beta} e_i \tilde{\tau}_i (g^{\alpha,\gamma})
- G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} B_{\gamma,\beta} e_i \tilde{\tau}_i (g^{\alpha,\gamma}) - G^{\tilde{\alpha}} \lambda_1^{\alpha,\beta} [e_i, \tilde{\tau}_i]^{(0,1)} (\varphi_{\alpha,\beta})
\geq 2 \sum_{\alpha > 1} \frac{G^{\tilde{\alpha}} |e_1 (\varphi_{V_1})|}{\lambda_1 - \lambda_\alpha} - C\lambda_1 \sum_i G^{\tilde{\alpha}}
+ G^{\tilde{\alpha}} e_i \tilde{\tau}_i (\varphi_{V_1}) - G^{\tilde{\alpha}} [e_i, \tilde{\tau}_i]^{(0,1)} (\varphi_{V_1}).
\]
We need to deal with last two terms in (3.7). Note \(|e_i \bar{e}_i (\nabla V_1 V_1)(\varphi) - (\nabla V_1 V_1)e_i \bar{e}_i(\varphi)| \leq C \lambda_1\). Then by (3.4), we have
\[
\left| G^{\bar{\alpha}} e_i \bar{e}_i (\nabla V_1 V_1)(\varphi) \right| \leq C \lambda_1 \sum_i G^{\bar{\alpha}} + C \lambda_1.
\]
It follows
\[
G^{\bar{\alpha}} e_i \bar{e}_i (\varphi V_1 V_1) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)}(\varphi V_1 V_1) \\
= G^{\bar{\alpha}} e_i \bar{e}_i V_1 V_1(\varphi) - G^{\bar{\alpha}} e_i \bar{e}_i (\nabla V_1 V_1)(\varphi) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)} V_1 V_1(\varphi) \\
+ G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)} (\nabla V_1 V_1)(\varphi) \\
\geq G^{\bar{\alpha}} e_i \bar{e}_i V_1 V_1(\varphi) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)} V_1 V_1(\varphi) - C \lambda_1 \sum_i G^{\bar{\alpha}} - C \lambda_1.
\]
By using the Lie bracket for vector fields, we further get
\[
G^{\bar{\alpha}} e_i \bar{e}_i V_1 V_1(\varphi) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)} V_1 V_1(\varphi) \\
\geq G^{\bar{\alpha}} \left( V_1 e_i \bar{e}_i V_1(\varphi) + [e_i, V_1] \bar{e}_i V_1(\varphi) - [V_1, \bar{e}_i] e_i V_1(\varphi) - V_1 [e_i, \bar{e}_i]^{(0,1)}(\varphi) \right) \\
- C \lambda_1 \sum_i G^{\bar{\alpha}} \\
\geq G^{\bar{\alpha}} V_1 V_1 \left( e_i \bar{e}_i(\varphi) - [e_i, \bar{e}_i]^{(0,1)}(\varphi) \right) - 2G^{\bar{\alpha}}[V_1, e_i] V_1 \bar{e}_i(\varphi) \\
- 2G^{\bar{\alpha}}[V_1, \bar{e}_i] V_1 e_i(\varphi) - C \lambda_1 \sum_i G^{\bar{\alpha}}.
\]
Thus
\[
G^{\bar{\alpha}} e_i \bar{e}_i (\varphi V_1 V_1) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)}(\varphi V_1 V_1) \\
\geq G^{\bar{\alpha}} V_1 V_1(\tilde{g}_{\bar{\alpha}}) - 2G^{\bar{\alpha}}[V_1, e_i] V_1 \bar{e}_i(\varphi) - 2G^{\bar{\alpha}}[V_1, \bar{e}_i] V_1 e_i(\varphi) \\
- C \lambda_1 \sum_i G^{\bar{\alpha}} - C \lambda_1.
\]
(3.8)

On the other hand, differentiating (3.2) along \( V_1 \) twice at \( x_0 \), we have
\[
G^{\bar{\alpha}} V_1 V_1(\tilde{g}_{\bar{\alpha}}) + G^{\bar{\alpha}, k\bar{\alpha}} V_1(\tilde{g}_{\bar{\alpha}}) V_1(\tilde{g}_{k\bar{\alpha}}) = F_{p_1} V_1 V_1 e_i(\varphi) + F_{\bar{p}_1} V_1 V_1 \bar{e}_i(\varphi) + E,
\]
where \( E \) denotes a term satisfying \(|E| \leq C \lambda_1^2\) for a uniform constant \( C \). Thus by (3.8), we get
\[
G^{\bar{\alpha}} e_i \bar{e}_i (\varphi V_1 V_1) - G^{\bar{\alpha}}[e_i, \bar{e}_i]^{(0,1)}(\varphi V_1 V_1) \\
\geq - G^{\bar{\alpha}, k\bar{\alpha}} V_1(\tilde{g}_{\bar{\alpha}}) V_1(\tilde{g}_{k\bar{\alpha}}) - 2G^{\bar{\alpha}}[V_1, e_i] V_1 \bar{e}_i(\varphi) - 2G^{\bar{\alpha}}[V_1, \bar{e}_i] V_1 e_i(\varphi) \\
- C \lambda_1 \sum_i G^{\bar{\alpha}} - C \lambda_1^2 + F_{p_1} V_1 V_1 e_i(\varphi) + F_{\bar{p}_1} V_1 V_1 \bar{e}_i(\varphi).
\]
Substituting the above inequality into (3.7), we prove the lemma. \( \Box \)
By Lemma 3.1 and Lemma 3.2, we get
\[
L(\log \lambda_1(\Phi) + h(|\partial \phi|^2_g))
\]
\[
= \frac{L(\lambda_1)}{\lambda_1} + h' L(|\partial \phi|^2_g) - \frac{G^{\alpha i} e_i(\phi V_i V_i)}{\lambda_1^2} + h'' G^{\alpha i} e_i(\partial \phi)^2_g
\]
\[
\geq 2 \sum_{\alpha > 1} \frac{G^{\alpha i} e_i(\phi V_i V_i)}{\lambda_1 (\lambda_1 - \lambda_\alpha)} - \frac{G^{\alpha i} V_i (\tilde{\phi}_{\gamma} V_i (\tilde{\phi}_{\gamma}))}{\lambda_1}
\]
\[
+ \frac{h'}{2} \sum_k G^{\alpha i} \left( |e_i e_k(\phi)|^2 + |e_i \tilde{\phi}_k(\phi)|^2 \right) + h'' G^{\alpha i} e_i(\partial \phi)^2_g
\]
(3.9)
\[
- \frac{G^{\alpha i} e_i(\phi V_i V_i)}{\lambda_1^2} - C \sum_i G^{\alpha i}
\]
\[
- 2G^{\alpha i} [V_i, e_i] V_i \tilde{\phi}(\phi) + 2G^{\alpha i} [V_i, \tilde{\phi}] V_i e_i(\phi)
\]
\[
+ \left[ F_{\alpha}(\frac{V_i V_i e_i(\phi)}{\lambda_1}) + h' (\phi \tilde{\phi}_k e_i(\phi) + \phi_k \tilde{\phi}_k e_i(\phi)) \right]
\]
\[
+ \left[ F_{\alpha}(\frac{V_i V_i \tilde{\phi}_i(\phi)}{\lambda_1}) + h' (\phi \tilde{\phi}_k e_i(\phi) + \phi_k \tilde{\phi}_k e_i(\phi)) \right]
\]
\[- C\lambda_1.
\]
We need to deal with last four terms in (3.9) where three parts are about the 3th-derivative of \(\phi\) and one is an eigenvalue function. The term
\[
- 2G^{\alpha i} [V_i, e_i] V_i \tilde{\phi}(\phi) + 2G^{\alpha i} [V_i, \tilde{\phi}] V_i e_i(\phi)
\]
can be handled as
\[
2G^{\alpha i} [V_i, e_i] V_i \tilde{\phi}(\phi) + 2G^{\alpha i} [V_i, \tilde{\phi}] V_i e_i(\phi)
\]
(3.10)
\[
\leq \frac{\varepsilon G^{\alpha i} e_i(\phi V_i V_i)}{\lambda_1^2} + \frac{\varepsilon \sum_{\alpha > 1} G^{\alpha i} e_i(\phi V_i V_i)}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{C}{\varepsilon} \sum_i G^{\alpha i}. \tag{3.11}
\]
Here \(\varepsilon \in (0, \frac{1}{2})\) is a constant to be determined later. We refer the reader to a similar argument in [1] Lemma 5.4.

To control the term \((\frac{V_i V_i e_i(\phi)}{\lambda_1}) + h' (\phi \tilde{\phi}_k e_i(\phi) + \phi_k \tilde{\phi}_k e_i(\phi))\) in (3.9). We use the fact \(d\tilde{Q}(x_0) = 0\). In fact,
\[
\frac{e_i(\phi V_i V_i)}{\lambda_1} = Ae^{-A\phi} e_i(\phi) - h' e_i(|\partial \phi|^2_g)
\]
\[
= Ae^{-A\phi} e_i(\phi) - h' (\phi \tilde{\phi}_k e_i(\phi) + \phi_k e_i \tilde{\phi}_k(\phi)).
\]
Note
\[
|V_i V_i e_i(\phi) - e_i(\phi V_i V_i)| \leq C\lambda_1.
\]
Thus
\[
|F_p \left( \frac{V_1 V_1 e_i(\varphi)}{\lambda_1} + h'(\varphi \overline{\varphi} e_k e_i(\varphi)) \right) |
\leq |F_p| \cdot \left| \frac{V_1 V_1 e_i(\varphi)}{\lambda_1} \right| - \frac{e_i(\varphi V_1 V_1 e_i(\varphi))}{\lambda_1} + A e^{-A \varphi} e_i(\varphi)
\leq C A e^{-A \varphi}.
\]
(3.12)

Similarly, we have
\[
|F_p \left( \frac{V_1 V_1 \overline{e}_i(\varphi)}{\lambda_1} + h'(\varphi \overline{\varphi} e_k \overline{e}_i(\varphi)) \right) |
\leq C A e^{-A \varphi}.
\]
(3.13)

The following lemma gives a control to \( \lambda_1 \) for the solution \( \varphi \) in (1.4).

**Lemma 3.3.**
\[
C \lambda_1 \leq \frac{h'}{4} \sum_k G^{ii}(|e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2) + C \sum_i G^{ii}.
\]

**Proof.** At \( x_0 \), by (2.1), we have
\[
G^{ii} = \frac{\sigma_1(i)}{\sigma_2} \delta_{ij},
\]
where \( \eta_i = \widetilde{g}_{ii} \) and \( \sigma_1(i) = \sum_{k \neq i} \eta_k \). It is clear that
\[
\sigma_1(1) \sigma_1(\eta) = (\sigma_1(1))^2 + \eta_1 \sigma_1(1)
\]
\[
= \sum_{i \geq 2} \eta_i^2 + 2 \sum_{i > j \geq 2} \eta_i \eta_j + \sum_{i \geq 2} \eta_1 \eta_i
\]
\[
= \sum_{i \geq 2} \eta_i^2 + \sum_{i > j \geq 2} \eta_i \eta_j + \sigma_2
\]
\[
\geq \sigma_2,
\]
which implies
\[
\frac{1}{G^{ii}} \leq \frac{1}{G^{11}} = \frac{\sigma_2}{\sigma_1(1)} \leq \sigma_1 = \frac{\sigma_2}{n - 1} \sum_k G^{kk} \leq C \sum_k G^{kk}, \ i = 1, 2, \ldots, n.
\]
Combining this with the Cauchy-Schwarz inequality and (2.6), we have
\[
C \lambda_1 \leq \frac{h'}{4} G^{11} \lambda_1^2 + \frac{C}{h' G^{11}}
\]
\[
\leq \frac{h'}{4} \sum_k G^{ii}(|e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2) + C \sum_i G^{ii},
\]
as required. \( \square \)

Substituting the above relations into (3.9), we get the main estimate in this section.
Proposition 3.4. Let $\varphi$ be the solution of (1.4). Then at $x_0$, there exists a uniform constant $C$ such that for any $\varepsilon \in (0, \frac{1}{2}]$, it holds

$$0 \geq (2 - \varepsilon) \sum_{\alpha > 1} G_i^i |e_i(\varphi V_\alpha V_1)|^2 \frac{\lambda_1 (\lambda_1 - \lambda_\alpha)}{\lambda_1} - G^{j,k,l} V_1 (\tilde{g}_{jkl}) V_1 (\tilde{g}_{jkl}) \lambda_1 \frac{\lambda_1 (\lambda_1 - \lambda_\alpha)}{\lambda_1} - \sum_{\alpha > 1} G_i^i |e_i(\varphi V_\alpha V_1)|^2 \lambda_1 ^2$$

(3.14)

Proof. At $x_0$, we have

$$0 \geq L(\hat{Q}) = L(\log \lambda_1 + h(|\partial \varphi|^2_g)) - A e^{-A \varphi} L(\varphi) + A^2 e^{-A \varphi} G_i^i |e_i(\varphi)|^2.$$ 

Note

$$L(\varphi) = G_i^i (\tilde{g}_\pi - \chi_\pi) = 2 - G_i^i \chi_\pi \leq 2 - \varepsilon_0 \sum_i G_i^i.$$ 

Thus by (3.10) together with estimates (3.12), (3.13) and Lemma 3.3, one get (3.14) immediately.

By concavity of $\log \sigma_2$ and (2.2), we see that $-G^{k,l,k} > 0$ and $(-G_i^i,k)$ is a non-negative definite matrix. Hence, the "good" positive terms at the right hand of (3.14) is

$$I = (2 - \varepsilon) \sum_{\alpha > 1} G_i^i |e_i(\varphi V_\alpha V_1)|^2 \frac{\lambda_1 (\lambda_1 - \lambda_\alpha)}{\lambda_1} - G^{j,k,l} V_1 (\tilde{g}_{jkl}) V_1 (\tilde{g}_{jkl}) \lambda_1 \frac{\lambda_1 (\lambda_1 - \lambda_\alpha)}{\lambda_1} - \sum_{\alpha > 1} G_i^i |e_i(\varphi V_\alpha V_1)|^2 \lambda_1 ^2.$$ 

In next section, we will use this "good" positive terms to control the "bad" term in (3.14),

$$II = (1 + \varepsilon) \sum \frac{G_i^i |e_i(\varphi V_\alpha V_1)|^2}{\lambda_1 ^2}.$$ 

As an application of Proposition 3.4, we get the following partial estimate of real Hessian $\nabla^2 \varphi$.

Corollary 3.5. There exists a uniform constant $C_A$ depending on $A$ such that

$$\sum_{i=2}^n \sum_{k=1}^n (|e_i e_k(\varphi)|^2 + |e_i e_k(\varphi)|^2) \leq C_A, \quad \sum_{i=2}^n |\eta_i| \leq C_A$$

and

$$\lambda_1 \leq C_A \eta_1 + C.$$
where $\eta_i = \tilde{g}_{i\bar{i}} = \chi_i + \varphi_i$ for $i = 1, 2, \ldots, n$.

Proof. By (3.11), we have

$$
- \frac{3}{2} G_{\mu\nu} |e_i(\varphi V_1 V_1)|^2 = - \frac{3}{2} G_{\mu\nu} |A e^{-A\varphi_i} - h'| e_i(|\partial\varphi|^2_{\tilde{g}})|^2 
\geq - C_A \sum_i G_{\mu\nu} - 2(h')^2 G_{\mu\nu} |e_i(|\partial\varphi|^2_{\tilde{g}})|^2.
$$

Recall that the matrix $(-G_{\alpha\beta})$ is non-negative and $-G_{k\bar{l}, l\bar{l}} > 0$. Then

$$(2 - \varepsilon) \sum_{\alpha > 1} \frac{G_{\alpha\alpha} |e_i(\varphi V_1 V_1)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} - \frac{G^{k\bar{l}, \bar{l}} V_1 (\tilde{g}_{k\bar{l}})}{\lambda_1} \geq 0.$$

Thus by choosing $\varepsilon = \frac{1}{2}$ in (3.14), we obtain

$$
0 \geq \frac{h'}{4} \sum G_{\mu\nu} |e_i e_k(\varphi)|^2 + |e_i e_{\bar{k}}(\varphi)|^2 + h'' G_{\mu\nu} |e_i(|\partial\varphi|^2_{\tilde{g}})|^2
- 2(h')^2 G_{\mu\nu} |e_i(|\partial\varphi|^2_{\tilde{g}})|^2 - C_A \sum_k G^{k\bar{k}} - C_A.
$$

By (2.3) and (2.6), it follows

$$
0 \geq \sum_k G_{\mu\nu} |e_i e_k(\varphi)|^2 + |e_i e_{\bar{k}}(\varphi)|^2 - C_A \sum_k G^{k\bar{k}}.
$$

Combining this with (2.5), we obtain

$$
\sum_{i=2}^n \sum_{k=1}^n |e_i e_k(\varphi)|^2 + |e_i e_{\bar{k}}(\varphi)|^2 \leq C_A.
$$

In particular, for $i \geq 2$, it is clear that

$$
\eta_i = \chi_i + \varphi_i = \chi_i + e_i e_{\bar{i}}(\varphi) - [e_i, e_{\bar{i}}]^{(0,1)}(\varphi) \leq C_A.
$$

Hence (3.15) is true.

Next, we prove (3.16). By (2.1) and (2.4), we see

$$
G^{n\bar{n}} \geq \cdots \geq G^1 \geq \frac{1}{C_{\eta_i}}.
$$
Combining this with (3.17), we have
\[
\lambda_1^2 = (V_1 V_1(\varphi) - (\nabla V_1 V_1)(\varphi))^2
\leq \sum_{i,k} (|\varepsilon_i e_k(\varphi)|^2 + |\varepsilon_i \varepsilon_k(\varphi)|^2) + C
\leq C\eta_1 \sum_{i,k} G^{ii} (|\varepsilon_i e_k(\varphi)|^2 + |\varepsilon_i \varepsilon_k(\varphi)|^2) + C
\leq C_A\eta_1 \sum_k G^{kk} + C
= C_A\eta_1 \sum_k (\sigma_1(k) + C)
\leq C_A\eta_1^2 + C,
\]
where we used \(\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n\) in the last inequality. Thus (3.16) is true.

\[\square\]

Corollary 3.5 will be used in next section.

4. Estimate of \( II \)

We decompose \( II \) into three parts as follows,
\[
(1 + \varepsilon) \frac{G^{T1} |e_1(\varphi V_1 V_1)|^2}{\lambda_1^2} + 3\varepsilon \sum_{i \geq 2} \frac{G^{ii} |e_i(\varphi V_1 V_1)|^2}{\lambda_1^2} + (1 - 2\varepsilon) \sum_{i \geq 2} \frac{G^{ii} |e_i(\varphi V_1 V_1)|^2}{\lambda_1^2}
= II_1 + II_2 + II_3.
\]

In the following, we always use \( C_A \) to denote a uniform constant depending on \( A \). Without loss of generality, we may assume that \( \lambda_1 \geq \frac{C_A}{\varepsilon} \).

We first estimate \( II_1 \) and \( II_2 \).

**Lemma 4.1.**
\[
II_1 \leq C_A + 2(h')^2 G^{T1} |e_1(\varphi V_1 V_1)|^2
\]
and
\[
II_2 \leq 12\varepsilon A^2 e^{-2A\varphi} \sum_{i \geq 2} G^{ii} |e_i(\varphi)|^2 + 2(h')^2 \sum_{i \geq 2} G^{ii} |e_i(\varphi V_1 V_1)|^2.
\]

**Proof.** Using (3.11), we have
\[
II_1 = \frac{G^{T1} |e_1(\varphi V_1 V_1)|^2}{\lambda_1^2} = G^{T1} |A e^{-A\varphi} e_1(\varphi) - h' e_1(\varphi V_1 V_1)|^2.
\]
Since \( G^{T1} = \frac{\sigma_1(1)}{\sigma_2} \leq C \) by Corollary 3.5, we get
\[
II_1 \leq C_A + 2(h')^2 G^{T1} |e_1(\varphi V_1 V_1)|^2.
\]
Similarly, 

\[ H_2 = 3\varepsilon \sum_{i \geq 2} \frac{G^n_i |e_i(\varphi_{V_1} V_1)|^2}{\lambda^2_i} \]

\[ = 3\varepsilon G^n_1 |A e^{-A \varphi} e_i(\varphi) - h' e_i(\varphi)|^2 \]

\[ \leq 12\varepsilon A^2 e^{-2A \varphi} \sum_{i \geq 2} G^n_i |e_i(\varphi)|^2 + 4\varepsilon (h')^2 \sum_{i \geq 2} G^n_i |\partial \varphi|^2_g|^2 \]

\[ \leq 12\varepsilon A^2 e^{-2A \varphi} \sum_{i \geq 2} G^n_i |e_i(\varphi)|^2 + 2(h')^2 \sum_{i \geq 2} G^n_i |\partial \varphi|^2_g|^2. \]

Here we used \(0 < \varepsilon \leq \frac{1}{2}\) in the last inequality.

In order to estimate \(H_3\), we need several lemmas below. Let 

\[ \tilde{e} = \frac{1}{\sqrt{2}}(V_1 - \sqrt{-1}JV_1). \]

be \((1, 0)\)-type vector field in the coordinate system \((U, \{x^\alpha\}_{\alpha=1}^{2n})\). Since \(\tilde{e}\) is \(g\)-unit, we can write \(\tilde{e}\) at \(x_0\) as

\[ \tilde{e} = \sum_q \nu_q e_q \text{ and } \sum_{q=1}^n |\nu_q|^2 = 1, \]

for complex number \(\nu_1, \nu_2, \cdots, \nu_n\). There are also numbers \(\mu_\alpha (\alpha > 1)\) with \(\sum_{\alpha > 1} \mu_\alpha^2 = 1\) such that

\[ JV_1 = \sum_{\alpha > 1} \mu_\alpha V_\alpha. \]

Then we have

(4.1) \[ e_i(\varphi_{V_1} V_1) = \sqrt{2} \sum_q \nu_q V_1(\tilde{g}_q) - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha e_i(\varphi_{V_1} V_\alpha) + E, \]

where \(E\) denotes a term satisfying \(|E| \leq C\lambda_1\). A similar computation of (4.1) can be found in [1, (5.31)].

**Lemma 4.2.**

\[ |\nu_q| \leq \frac{C_A}{\lambda_1} \text{ for } q \geq 2. \]

**Proof.** By (3.15), we have

\[ \sum_{i=2}^n \sum_{k=1}^n (|e_i e_k(\varphi)|^2 + |e_i \varphi_k(\varphi)|^2) \leq C_A. \]

Combining this with (2.7), we obtain

(4.2) \[ \sum_{\alpha \geq 3} \sum_{\beta \geq 1} |\nabla^2_{\alpha \beta} \varphi| \leq C_A. \]
This means
\[ |\Phi_\alpha^\beta| \leq C_A \text{ for } 3 \leq \alpha \leq 2n, \ 1 \leq \beta \leq 2n. \]

Recalling that \( V_1 \) is the eigenvector of \( \Phi \) corresponding to \( \lambda_1 \), we have
\[ |V_1^\alpha| = \left| \frac{1}{\lambda_1} \sum_{\beta=1}^{2n} \Phi_\beta^\alpha V_1^\beta \right| \leq \frac{C}{\lambda_1} \text{ for } 3 \leq \alpha \leq 2n. \]

Thus for any \( q \geq 2 \), we get
\[ |\nu_q| = |V_1^{2q-1}| + |V_1^{2q}| \leq \frac{C_A}{\lambda_1}. \]

By Corollary 3.5 and Lemma 4.2, we get an upper bound of \( G_i^\alpha \) for \( \alpha \geq 2 \).

**Lemma 4.3.** For \( i \geq 2 \), at \( x_0 \), if \( \lambda_1 \geq \frac{C_A}{\varepsilon} \), we have
\[ (1 - \varepsilon)G_i^\alpha \leq \frac{1}{2\sigma^2 \lambda_1} \left( \lambda_1 + \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2 \right). \]

**Proof.** By the definition of \( \tilde{e} \), we see
\[ \tilde{g}(\tilde{e}, \tilde{e}) = \sum_q |\nu_1|^2 \eta_q = |\nu_1|^2 \eta_1 + \sum_{q=2}^n |\nu_1|^2 \eta_1. \]

By Corollary 3.5 and Lemma 4.2, it follows
\[ \tilde{g}(\tilde{e}, \tilde{e}) \geq \left( 1 - \frac{C_A}{\lambda_1^2} \right) \eta_1 - \frac{C_A}{\lambda_1^2}. \]

On the other hand,
\[ \tilde{g}(\tilde{e}, \tilde{e}) = g(\tilde{e}, \tilde{e}) + \tilde{e} \tilde{e}(\varphi) - [\tilde{e}, \tilde{e}]^{(0,1)}(\varphi) \]
\[ = 1 + \frac{1}{2} (V_1 V_1(\varphi) + (J V_1)(J V_1)(\varphi) + \sqrt{-1} [V_1, J V_1](\varphi)) - [\tilde{e}, \tilde{e}]^{(0,1)}(\varphi) \]
\[ = \frac{1}{2} \left( \lambda_1 + \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2 \right) + 1 + (\nabla V_1 V_1)(\varphi) + (\nabla J V_1, J V_1)(\varphi) \]
\[ + \sqrt{-1} [V_1, J V_1](\varphi) - [\tilde{e}, \tilde{e}]^{(0,1)}(\varphi) \]
\[ \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2 \right) + C. \]

Note \( \lambda_1 \geq \frac{C_A}{\varepsilon} \). Thus we deduce
\[ \left( 1 - \frac{C_A}{\lambda_1^2} \right) \eta_1 \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2 \right) + C. \]
As a consequence,

\[ \eta_1 \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) + C + \frac{C_A}{\lambda_1} \cdot \frac{\eta_1}{\lambda_1} \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) + C. \]

Hence, for \( i \geq 2 \), we obtain

\[ (1 - \varepsilon)G^i = (1 - \varepsilon) \frac{\sigma_1(i)}{\sigma_2} \leq \frac{\eta_1}{\sigma_2} - \frac{\varepsilon \eta_1}{\sigma_2} + C \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right), \]

where we used (3.16) and \( \lambda_1 \geq \frac{C_A}{\varepsilon} \) in the last inequality. \( \square \)

At \( x_0 \), we assume that the eigenvalues of matrix \((-G^i, \tilde{\mathcal{J}})\) are

\[ \kappa_1 \geq \kappa_2 \cdots \geq \kappa_n. \]

Let \( \xi_i = (\xi^1_i, \xi^2_i, \cdots, \xi^n_i) \) be the \( g \)-unit eigenvector corresponding to \( \kappa_i \) for \( i = 1, 2, \cdots, n \). Some estimates for eigenvalues \( \kappa_i \) and its eigenvectors \( \xi_i \) are given in the following lemma, which plays important role in the estimate of \( I_3 \).

**Lemma 4.4.**

(1) \( C_A^{-1} \lambda_1^{-2} \leq \kappa_n \leq C_A \lambda_1^{-2} \) and \( \kappa_i \geq C_A^{-1} \) for \( i \leq n - 1 \).

(2) \( \sum_{i=2}^{n} |\xi_{i,n}|^2 \leq C_A \lambda_1^{-2} \).

**Proof.** Since the proof of Lemma 4.4 is a little tedious, we give it in Appendix. \( \square \)

Now we begin to estimate \( I_3 \).

**Lemma 4.5.** For any positive number \( \gamma > 0 \), we have

\[
I_3 \leq C_A \varepsilon \sum_{i \geq 2} \sum_{q \geq 2} \frac{G^i |V_1(\tilde{g}_q)|}{\lambda^4_{1}} + \frac{C}{\varepsilon} \sum_{i} \frac{G^i}{\lambda^2_{1}} + \frac{2(1 - \varepsilon)(1 + \gamma)}{\lambda^2_{1}} \sum_{i \geq 2} \frac{G^i |V_1(\tilde{g}_{i})|^2}{\lambda^2_{1}} \\
+ (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \left( \sum_{i \geq 2} \sum_{\alpha > 1} \frac{G^i |e_i(\varphi V_\alpha V_1)|^2}{\lambda^2_{1}} \lambda_1 - \lambda_\alpha \right).
\]
Proof. By the relation (4.1) and the Cauchy-Schwarz inequality, we have

\[
II_3
\]

\[
= (1 - 2\varepsilon) \sum_{i \geq 2} G^\bar{\alpha} [\sqrt{2} \sum_q \nu_q V_1(\bar{g}_\bar{\alpha}) - \sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha e_i} (\varphi_{V_1 V_\alpha}) + E]^2 \lambda_1^2
\]

\[
= (1 - \varepsilon) \sum_{i \geq 2} G^\bar{\alpha} [\sqrt{2} \sum_{q \geq 2} \nu_q V_1(\bar{g}_\bar{\alpha}) + E]^2 \lambda_1^2
\]

\[
\leq (1 - \varepsilon) \sum_{i \geq 2} G^\bar{\alpha} [\sqrt{2} \sum_{q \geq 2} \nu_q V_1(\bar{g}_\bar{\alpha}) + E]^2 \lambda_1^2
\]

\[
+ C \epsilon \sum_{i \geq 2} G^\bar{\alpha} |V_1(\bar{g}_\bar{\alpha})|^2 \lambda_1^4 + C \epsilon \sum_i G^\bar{\alpha}.
\]

(4.3)

Here we used Lemma 4.2 in the last inequality. On the other hand, by the Cauchy-Schwarz inequality, we have

\[
(1 - \varepsilon) \sum_{i \geq 2} G^\bar{\alpha} [\sqrt{2} \sum_{q \geq 2} \nu_q V_1(\bar{g}_\bar{\alpha}) + E]^2 \lambda_1^2
\]

\[
\leq 2(1 - \varepsilon) (1 + \gamma) \sum_{i \geq 2} G^\bar{\alpha} |V_1(\bar{g}_\bar{\alpha})|^2 \lambda_1^2
\]

\[
+ (1 - \varepsilon) \left(1 + \frac{1}{\gamma}\right) \sum_{i \geq 2} G^\bar{\alpha} |\sum_{\alpha > 1} \mu_{\alpha e_i} (\varphi_{V_1 V_\alpha})|^2 \lambda_1^2,
\]

and

\[
\left| \sum_{\alpha > 1} \mu_{\alpha e_i} (\varphi_{V_\alpha V_1}) \right|^2 \leq \left( \sum_{\alpha > 1} (\lambda_1 - \lambda_\alpha) \mu_{\alpha}^2 \right) \left( \sum_{\alpha > 1} \frac{|e_i (\varphi_{V_\alpha V_1})|^2}{\lambda_1 - \lambda_\alpha} \right)
\]

\[
= \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) \left( \sum_{\alpha > 1} \frac{|e_i (\varphi_{V_\alpha V_1})|^2}{\lambda_1 - \lambda_\alpha} \right).
\]
Thus

\[
(1 - \varepsilon) \sum_{i \geq 2} G_{\tilde{\sigma}} |\sqrt{2\nu_1} V_1(\tilde{g}_\tilde{\sigma}) - \sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha \tilde{e}_i}(\varphi V_1 V_\alpha)|^2
\]

\[
\leq 2(1 - \varepsilon)(1 + \gamma) \sum_{i \geq 2} G_{\tilde{\sigma}} |V_1(\tilde{g}_\tilde{\sigma})|^2
\]

\[
+ (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) \left( \sum_{i \geq 2} \sum_{\alpha > 1} \frac{G_{\tilde{\sigma}} |\epsilon_i(\varphi V_\alpha V_1)|^2}{\lambda_1^2} \lambda_1 - \lambda_\alpha \right).
\]

Inserting the above inequality into (4.3), the lemma is proved. □

**Lemma 4.6.** At \( x_0 \), if \( \lambda_1 \geq \frac{C_A}{\varepsilon} \), then we have

\[
- \frac{G_{\tilde{\sigma},\tilde{k}\tilde{l}} V_1(\tilde{g}_{\tilde{k}\tilde{l}}) V_1(\tilde{g}_{\tilde{\sigma}})}{\lambda_1} \geq \frac{C_A}{\varepsilon} \sum_{i \geq 2} G_{\tilde{\sigma}} |V_1(\tilde{g}_{\tilde{\sigma}})|^2 \lambda_1^4.
\]

**Proof.** Recall that \( \xi_i = (\xi_1^i, \xi_2^i, \cdots, \xi_n^i) \) are the \( g \)-unit eigenvector corresponding to \( \kappa_i \) for \( i = 1, 2, \cdots, n \). Then there are complex numbers \( \tau_1, \tau_2, \cdots, \tau_n \) such that

\[
V_1(\tilde{g}_{\tilde{\sigma}}) = \sum_{q=1}^n \tau_q \xi_q^i \text{ for } i = 1, 2, \cdots, n.
\]

Since

\[
G_{\tilde{\sigma}} = \frac{\sigma_1(i)}{\sigma_2} \leq C \lambda_1 \text{ for } i = 1, 2, \cdots, n,
\]

we derive

\[
(4.4) \quad - \frac{C_A}{\varepsilon} \sum_{i \geq 2} G_{\tilde{\sigma}} |V_1(\tilde{g}_{\tilde{\sigma}})|^2 \lambda_1^4 \geq - \sum_{i \geq 2} \sum_{q=1}^n \frac{C_A}{\varepsilon \lambda_1^3} |\tau_q| |\xi_q^i|^2.
\]

Also we have

\[
(4.5) \quad - \frac{G_{\tilde{\sigma},\tilde{k}\tilde{l}} V_1(\tilde{g}_{\tilde{k}\tilde{l}}) V_1(\tilde{g}_{\tilde{\sigma}})}{\lambda_1} = \frac{1}{\lambda_1} \sum_{q=1}^n \kappa_q |\tau_q|^2.
\]
Thus Lemma 4.6 follows from the above inequalities and (4.6).

Moreover, by $\lambda_1 \geq \frac{C_4}{\varepsilon}$ and Lemma 4.3, we see

$$-\sum_{i \geq 2} \frac{C_A}{\varepsilon \lambda_1^3} |\tau_n|^2 |\xi_n|^2 + \frac{1}{\lambda_1} \kappa_n |\tau_n|^2 \geq \left( \frac{1}{C_A \lambda_1^3} - \frac{C_A}{\varepsilon \lambda_1^3} \right) |\tau_n|^2 \geq 0,$$

and

$$\sum_{q=1}^{n-1} \left( -\sum_{i \geq 2} \frac{C_A}{\varepsilon \lambda_1^3} |\tau_q|^2 |\xi_q|^2 + \frac{1}{\lambda_1} \kappa_q |\tau_q|^2 \right) \geq \sum_{q=1}^{n-1} \left( \frac{1}{C_A \lambda_1} - \frac{C_A}{\varepsilon \lambda_1} \right) |\tau_q|^2 \geq 0.$$

Thus Lemma 4.6 follows from the above inequalities and (4.6). \hfill \square

Lemma 4.6 gives an estimate for the term $\frac{C_A}{\varepsilon} \sum_{i \geq 2} G^n |V_i(\tilde{g}_k\tau_i)|^2$ in Lemma 4.5. We need to deal with other terms there. By the definition of $\lambda_\alpha$ and $\mu_\alpha$, it is clear that $\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 > 0$. From Lemma 4.3, we see $\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 > 0$. Recalling that the constant $\gamma > 0$ in Lemma 4.5 is arbitrary, now we choose

$$\gamma = \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}.$$

Thus by Lemma 4.3 and the definition of $\gamma$, we obtain

$$2 \sum_{i \geq 2} \frac{|V_i(\tilde{g}_k\tau)|^2}{\sigma_2 \lambda_1} \geq 2(1 - \varepsilon)(1 + \gamma) \sum_{i \geq 2} \frac{G^n |V_i(\tilde{g}_k\tau)|^2}{\lambda_1^2}$$

and

$$2 - 2\varepsilon \sum_{i \geq 2} \sum_{\alpha > 1} \frac{G^n |e_i(\varphi_{\lambda_\alpha} V_1)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} \lambda_1 (\lambda_1 - \lambda_\alpha) \lambda_1 (\lambda_1 - \lambda_\alpha)$$

$$= (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \geq 2} \sum_{\alpha > 1} \frac{G^n |e_i(\varphi_{\lambda_\alpha} V_1)|^2}{\lambda_1^2} \lambda_1 (\lambda_1 - \lambda_\alpha).$$
**Lemma 4.7.** At \( x_0 \), if \( \lambda_1 \geq \frac{C_A}{\varepsilon} \), we have

\[
II_3 \leq I + \frac{C}{\varepsilon} \sum_i G^{\eta i}.
\]

**Proof.** By the definition of \( G^{\eta j, k^i} \) (see (2.2)), it is clear that

\[
- \sum_{k \neq l} G^{\eta j, k^i} |V_1(\tilde{g}_{kl})|^2 = \sum_{k \neq l} |V_1(\tilde{g}_{kl})|^2 = \frac{2}{\sigma_2 \lambda_1} \sum_{i \geq 2} |V_1(\tilde{g}_{i})|^2 + \sum_{i \geq 2} \sum_{q \geq 2, q \neq i} \frac{|V_1(\tilde{g}_{i})|^2}{\sigma_2 \lambda_1}.
\]

On the other hand, by (3.16), we see

\[
C^{-1}_{A_1} \eta_1 \leq \lambda_1 \leq C_A \eta_1.
\]

Note \( \lambda_1 \geq \frac{C_A}{\varepsilon} \). Then by Lemma 3.5, we have

\[
G^{\eta i} = \frac{\sigma_1(i)}{\sigma_2} \leq \frac{\eta_1 + C}{\sigma_2} \leq \frac{\varepsilon \lambda^3_i}{C_A \sigma_2} \text{ for } i \geq 2.
\]

This implies

\[
\sum_{i \geq 2} \sum_{q \geq 2, q \neq i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1} \geq \frac{C_A}{\varepsilon} \sum_{i \geq 2} \sum_{q \geq 2, q \neq i} G^{\eta i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1^4}.
\]

Hence by (4.7), we deduce

\[
- \sum_{k \neq l} G^{\eta j, k^i} |V_1(\tilde{g}_{kl})|^2 \geq 2(1 - \varepsilon)(1 + \gamma) \sum_{i \geq 2} G^{\eta i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1^2} + \frac{C_A}{\varepsilon} \sum_{i \geq 2} \sum_{q \geq 2, q \neq i} G^{\eta i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1^4}.
\]

By (4.8) and (4.9), we see

\[
(2 - \varepsilon) \sum_{\alpha > 1} G^{\eta i} \frac{|e_i(\varphi V_\alpha V_1)|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} - \sum_{k \neq l} G^{\eta j, k^i} |V_1(\tilde{g}_{kl})|^2 \geq 2(1 - \varepsilon)(1 + \gamma) \sum_{i \geq 2} G^{\eta i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1^2} + \frac{C_A}{\varepsilon} \sum_{i \geq 2} \sum_{q \geq 2, q \neq i} G^{\eta i} \frac{|V_1(\tilde{g}_{i})|^2}{\lambda_1^4} + (1 - \varepsilon) \left(1 + \frac{1}{\gamma}\right) \left(\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu^2_\alpha\right) \sum_{i \geq 2} \sum_{\alpha > 1} \frac{G^{\eta i} |e_i(\varphi V_\alpha V_1)|^2}{\lambda_1^2}.
\]

Then Lemma 4.7 follows from Lemma 3.5, Lemma 4.6 and (4.10). \( \square \)

Combining Lemma 4.11 and Lemma 4.7 we finally obtain

**Proposition 4.8.** If \( \lambda_1 \geq \frac{C_A}{\varepsilon} \), we have

\[
II = II_1 + II_2 + II_3
\]

\[
\leq I + 12\varepsilon A^2 e^{-2A\varepsilon} G^{\eta i} |e_i(\varphi)|^2 + 2(h')^2 G^{\eta i} |e_i(\partial \varphi)|^2 + \frac{C}{\varepsilon} \sum_i G^{\eta i} + C_A.
\]
By Proposition 3.4 and Proposition 4.8, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we assume that \( \sup_M \phi = 0 \). Then by Proposition 3.4 and Proposition 4.8, we see that at \( x_0 \) there exists a uniform constant \( C_1 \) such that

\[
0 \geq \left( \varepsilon_0 A e^{-A \phi} - \frac{C_1}{\varepsilon} \right) \sum_i G_{\vec{\alpha}}^i + \frac{h'}{4} \sum_k G_{\vec{\alpha}}^i (|e_i e_k(\phi)|^2 + |e_i \overline{e}_k(\phi)|^2)
\]

\[
+ (A^2 e^{-A \phi} - 12 \varepsilon A^2 e^{-2A \phi}) G_{\vec{\alpha}}^i |e_i(\phi)|^2 - C_1 A e^{-A \phi}.
\]

Choose \( A = 12 C_1 + 1 \) and \( \varepsilon = \frac{e^{A \phi(x_0)}}{12} \in (0, \frac{1}{12}] \) so that

\[
A e^{-A \phi} - \frac{C_1}{\varepsilon} \geq 1 \quad \text{and} \quad A^2 e^{-A \phi} - 12 \varepsilon A^2 e^{-2A(\phi)} \geq 0.
\]

We get from (4.11),

\[
\sum_i G_{\vec{\alpha}}^i + \frac{h'}{4} \sum_k G_{\vec{\alpha}}^i (|e_i e_k(\phi)|^2 + |e_i \overline{e}_k(\phi)|^2) \leq C.
\]

As a consequence, \( \sum_i G_{\vec{\alpha}}^i \leq C \). Combining this with Maclaurin’s inequality, we obtain (for more details, cf. [11, Lemma 2.2]),

\[
G_{\vec{\alpha}}^i \geq C^{-1} \quad \text{for} \quad i = 1, 2, \cdots, n.
\]

Thus we get

\[
\lambda_1^2 \leq C \sum_k G_{\vec{\alpha}}^i (|e_i e_k(\phi)|^2 + |e_i \overline{e}_k(\phi)|^2) \leq C,
\]

as required. \( \square \)

5. Appendix

In this appendix, we give a proof of Lemma 4.4. Here we use the same notations in Section 4. We need the following algebraic Lemma for \( \sigma_2 \) polynomial function.

Lemma 5.1. At \( x_0 \), we have

\[
\det(-G_{\vec{\alpha}, \vec{j}}) = (n - 1) \sigma_2^{-n}
\]

Proof. For convenience, we define \( \vec{\sigma} = (\sigma_1(1), \cdots, \sigma_1(n)) \) and \( M_1 = \vec{\sigma}^T \overline{\vec{\sigma}} \), where \( \vec{\sigma}^T \) denotes the transpose of the vector \( \vec{\sigma} \). By the definition of \( G_{\vec{\alpha}, \vec{j}} \), it is clear that

\[
(-\sigma_2^2 G_{\vec{\alpha}, \vec{j}}) = M_1 - M_2,
\]

where

\[
M_2 = \begin{pmatrix}
0 & \sigma_2 & \sigma_2 & \cdots & \sigma_2 \\
\sigma_2 & 0 & \sigma_2 & \cdots & \sigma_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_2 & \sigma_2 & \sigma_2 & \cdots & 0
\end{pmatrix}.
\]
Since the rank of matrix $M_1$ is one, any two columns of $M_1$ are proportional. Combining this and properties of the determinant, we have

\[(5.2) \quad \det(M_1 - M_2) = \sum_{i=1}^{n} \det A_i + (-1)^n \det M_2,\]

where

\[
A_i = \begin{pmatrix}
0 & -\sigma_2 & -\sigma_2 & \cdots & \sigma_1(1)\sigma_1(i) & \cdots & -\sigma_2 \\
-\sigma_2 & 0 & -\sigma_2 & \cdots & \sigma_1(2)\sigma_1(i) & \cdots & -\sigma_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\sigma_2 & -\sigma_2 & -\sigma_2 & \cdots & \sigma_1(n)\sigma_1(i) & \cdots & 0
\end{pmatrix}.
\]

Applying some elementary row operations to $A_i$, we obtain

\[
\det A_i = \sigma_1(i)\sigma_2^{n-1} \left( \sum_{k=1}^{n} \sigma_1(k) - (n-1)\sigma_1(i) \right).
\]

Therefore,

\[(5.3) \quad \sum_{i=1}^{n} \det A_i = \sum_{i=1}^{n} \sigma_1(i)\sigma_2^{n-1} \left( \sum_{k=1}^{n} \sigma_1(k) - (n-1)\sigma_1(i) \right)
= \sigma_2^{n-1} \left( \left( \sum_{i=1}^{n} \sigma_1(i) \right)^2 - \sum_{i=1}^{n} (n-1)(\sigma_1(i))^2 \right)
= \sigma_2^{n-1} \left( (n-1)^2\sigma_2^2 - (n-1) \sum_i (\sigma_1 - \eta_i)^2 \right)
= (n-1)\sigma_2^{n-1} \left( \sigma_1^2 - \sum_i \eta_i^2 \right)
= 2(n-1)\sigma_2^n.
\]

On the other hand, it is clear that

\[(5.4) \quad \det M_2 = (-1)^{n-1}(n-1)\sigma_2^n.
\]

Then Lemma 5.1 follows from (5.1), (5.2), (5.3) and (5.4). □

**Proof of (1) in Lemma 4.4.** Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be the eigenvalues of $M_1$ and $M_2$, respectively. Then

\[
a_1 = ||\vec{\sigma}||^2, \quad a_2 = a_3 = \cdots = a_n = 0
\]

and

\[
b_1 = (n-1)\sigma_2, \quad b_2 = b_3 = \cdots = b_n = -\sigma_2.
\]

By Weyl’s inequality in matrix theory (cf. [19, Theorem 4.3.1]), we see

\[
\frac{a_1}{\sigma_2^2} - \frac{b_1}{\sigma_2^2} \leq \kappa_1 \leq \frac{a_1}{\sigma_2^2} - \frac{b_n}{\sigma_2^2}
\]
and

\[ \kappa_i \leq \frac{a_i}{\sigma_2^2} - \frac{b_n}{\sigma_2^2} \text{ for } i \geq 2. \]

It follows

\[ C_A^{-1} \lambda_1^2 \leq \kappa_1 \leq C_A \lambda_1^2 \text{ and } \kappa_i \leq C_A \text{ for } i \geq 2. \]

Thus by Lemma 5.1, we get

\[ \kappa_n = \frac{\det(-G_{ii,jj})}{\kappa_1 \kappa_2 \cdots \kappa_{n-1}} \geq \frac{1}{C_A \lambda_1^2}. \]

On the other hand, since \( \kappa_n \) is the smallest eigenvalue of matrix \((-G_{ii,jj})\), by (2.2) and Corollary 3.5, we have

\[ \kappa_n \leq -G_{11,11} = \left( \frac{\sigma_1(1)}{\sigma_2^2} \right)^2 \leq \frac{1}{C_A \lambda_1^2}. \]

Then by (5.7) and (5.5), we have

\[ \kappa_i \geq \frac{\det(-G_{ii,jj})}{\kappa_1 \kappa_2 \cdots \kappa_{n-1}} \geq C_A^{-1}, \quad \forall \ i \leq n - 1. \]

The first part (1) of Lemma 4.4 is proved. \( \square \)

**Proof of (2) in Lemma 4.4** For simplicity, we prove the case when \( n = 4 \). The general case can be proved by the same way.

Recall that the vector \( \xi_4 \) is the eigenvector of matrix \((-G_{ii,jj})\) corresponding to \( \kappa_4 \). We use the following elementary row operation of \((-G_{ii,jj})\) to compute the components \( \xi_4^i \) of \( \xi_4 \),

\[ \begin{pmatrix} \kappa_4 & -\frac{\sigma_1(1)^2}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(1)\sigma_1(2)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(1)\sigma_1(3)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(1)\sigma_1(4)}{\sigma_2^2} \\ \frac{\sigma_2-\sigma_1(2)\sigma_1(1)}{\sigma_2^2} & \kappa_4 & -\frac{\sigma_1(2)^2}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(2)\sigma_1(3)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(2)\sigma_1(4)}{\sigma_2^2} \\ \frac{\sigma_2-\sigma_1(3)\sigma_1(1)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(3)\sigma_1(2)}{\sigma_2^2} & \kappa_4 & -\frac{\sigma_1(3)^2}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(3)\sigma_1(4)}{\sigma_2^2} \\ \frac{\sigma_2-\sigma_1(4)\sigma_1(1)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(4)\sigma_1(2)}{\sigma_2^2} & \frac{\sigma_2-\sigma_1(3)\sigma_1(4)}{\sigma_2^2} & \kappa_4 & -\frac{\sigma_1(4)^2}{\sigma_2^2} \end{pmatrix}, \]

where \( I_4 \) denotes the identity matrix. There are four steps.
Step 1. For $i = 1, 2, 3$, multiplying the 4-th row by $-\frac{\sigma_1(i)}{\sigma_1(4)}$, and adding that to the $i$-th row, we obtain

$$
\begin{pmatrix}
\kappa_4 - \frac{(\sigma_1(1))}{\sigma_1(4)\sigma_2} & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \frac{\sigma_1(1)}{\sigma_1(4)\sigma_2}\sigma_2\kappa_4 \\
\sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \frac{\sigma_1(1)}{\sigma_1(4)\sigma_2}\sigma_2\kappa_4 \\
\sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \sigma_1(1) & \sigma_1(4) - \frac{\sigma_1(1)}{\sigma_1(4)\sigma_2}\sigma_2\kappa_4 \\
\frac{\sigma_2 - \sigma_1(4)\sigma_1(1)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(4)\sigma_1(2)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(3)\sigma_1(1)\sigma_2}{\sigma_2^2} & \kappa_4 - \frac{(\sigma_1(4))^2}{\sigma_2^2} \\
\end{pmatrix}
$$

Step 2. For $i = 1, 2, 3$, multiplying the $i$-th row by $\sigma_1(4)$, we obtain

$$
\begin{pmatrix}
\frac{\sigma_1(4)\sigma_2\kappa_4 - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{\sigma_2 - \sigma_1(4)\sigma_1(1)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(4)\sigma_1(2)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(3)\sigma_1(4)}{\sigma_2^2} & \kappa_4 - \frac{(\sigma_1(4))^2}{\sigma_2^2} \\
\end{pmatrix}
$$

Step 3. For $i = 2, 3$, multiplying the 1-st row by $-\frac{\sigma_1(4) - \sigma_1(i)}{\sigma_1(4) - \sigma_1(1)}$, and adding that to the $i$-th row, we obtain

$$
\begin{pmatrix}
\frac{\sigma_1(4)\sigma_2\kappa_4 - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_2} & \frac{\sigma_1(4) - \sigma_1(1)\sigma_2\kappa_4}{\sigma_2} \\
\frac{a_{21}}{\sigma_2} & \frac{a_{22}}{\sigma_2} & 0 & \frac{a_{24}}{\sigma_2} \\
\frac{a_{31}}{\sigma_2} & 0 & \frac{a_{33}}{\sigma_2} & \frac{a_{34}}{\sigma_2} \\
\frac{\sigma_2 - \sigma_1(4)\sigma_1(1)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(4)\sigma_1(2)}{\sigma_2^2} & \frac{\sigma_2 - \sigma_1(3)\sigma_1(4)}{\sigma_2^2} & \kappa_4 - \frac{(\sigma_1(4))^2}{\sigma_2^2} \\
\end{pmatrix}
$$

where

$$
a_{i1} = \frac{\sigma_1(4) - \sigma_1(i)}{\sigma_2} - \frac{\sigma_1(4) - \sigma_1(1)}{\sigma_1(4) - \sigma_1(1)} \cdot \frac{\sigma_1(4)\sigma_2\kappa_4 - \sigma_1(1)}{\sigma_2},$$
$$
a_{ii} = \frac{\sigma_1(4)}{\sigma_2} - \frac{\sigma_1(4)}{\sigma_2} \cdot \frac{\kappa_4 - \sigma_1(i)\kappa_4 - \frac{\sigma_1(4) - \sigma_1(i)}{\sigma_1(4) - \sigma_1(1)} \cdot \frac{\sigma_1(4) - \sigma_1(i)\sigma_2\kappa_4}{\sigma_2}}{\sigma_2},$$

for $i = 2, 3$. 

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Step 4. For \( i = 2, 3 \), multiplying the \( i \)-th row by \(-\frac{\sigma_2 - \sigma_1(i)\sigma_1(4)}{\sigma_2^2a_{ii}}\), and adding that to the 4-th row, we obtain

\[
\begin{pmatrix}
\sigma_1(4)\sigma_2\kappa_4 - \sigma_1(1) \\
\sigma_2 \\
a_{21} \\
a_{31} \\
a_{41}
\end{pmatrix} = 
\begin{pmatrix}
\sigma_1(4) - \sigma_1(1) \\
\sigma_2 \\
a_{22} \\
a_{33} \\
a_{44}
\end{pmatrix} - 
\begin{pmatrix}
\sigma_1(4) - \sigma_1(1) \\
\sigma_2 \\
a_{41} \\
a_{44}
\end{pmatrix},
\]

where

\[
a_{41} = \frac{\sigma_2 - \sigma_1(1)\sigma_1(4)}{\sigma_2^2} - \sum_{i=2}^3 \frac{\sigma_2 - \sigma_1(i)\sigma_1(4)}{\sigma_2^2a_{ii}}a_{i1},
\]

\[
a_{44} = \kappa_4 - \frac{(\sigma_1(4))^2}{\sigma_2^2} - \sum_{i=2}^3 \frac{\sigma_2 - \sigma_1(i)\sigma_1(4)}{\sigma_2^2a_{ii}}a_{i4}.
\]

By the part (1) of Lemma 44, a direct calculation shows

\[
|a_{i1}| \leq C_A, \quad a_{ii} = O(\lambda_1), \quad a_{i4} = O(\lambda_1) \quad \text{for} \quad i = 2, 3,
\]

\[
|a_{41}| \leq C\lambda_1, \quad a_{44} = O(\lambda_1^2),
\]

where \( O(\lambda_1^s) \) denotes a term satisfying \( C_A^{-1}\lambda_1^s \leq |O(\lambda_1^s)| \leq C_A\lambda_1^s \). Moreover, we see that

\[
d = (d_1, d_2, d_3, d_4) = \left(1, \frac{a_{24}a_{41} - a_{21}a_{44}}{a_{22}a_{44}}, \frac{a_{34}a_{41} - a_{31}a_{44}}{a_{33}a_{44}}, -a_{41}, -a_{44}\right)
\]

is an eigenvector of matrix \((-G^{i,j})\) corresponding to \( \kappa_4 \). By (5.9), we obtain

\[
|d_i| \leq \frac{C_A}{\lambda_1} \quad \text{for} \quad i = 2, 3, 4.
\]

Thus \( \xi_4 = \frac{\vec{d}}{||\vec{d}||} \) and each \( |\xi_4|^2 \leq C_A\lambda_1^{-2} \), \( i = 2, 3, 4 \). The proof of (2) in Lemma 44 is proved. \( \square \)

References

[1] J. Chu, V. Tosatti and B. Weinkove, The Monge-Ampère equation for non-integrable almost complex structures, preprint, [arXiv:1603.00706].
[2] P. De Bartolomeis and A. Tomassini, On solvable generalized Calabi-Yau manifolds, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 5, 1281–1296.
[3] S. Dinew and S. Kołodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, Amer. J. Math. 139 (2017), no. 2, 403–415.
[4] J.-X. Fu and S.-T. Yau, A Monge-Ampère-type equation motivated by string theory, Comm. Anal. Geom. 15 (2007), no. 1, 29–75.
[5] J.-X. Fu and S.-T. Yau, The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation, J. Differential Geom. 78 (2008), no. 3, 369–428.
[6] B. R. Greene, A. Shapere, C. Vafa and S.-T. Yau, Stringy cosmic strings and non-compact Calabi-Yau manifolds, Nuclear Phys. B 337 (1990), no. 1, 1–36.

[7] F. R. Harvey and H. B. Lawson, Potential theory on almost complex manifolds, Ann. Inst. Fourier (Grenoble) 65 (2015), no. 1, 171–210.

[8] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math. 54 (2003), no. 3, 281–308.

[9] R. A. Horn and C. R. Johnson, Matrix analysis, Second edition. Cambridge University Press, Cambridge, 2013. xviii+643 pp. ISBN: 978-0-521-54823-6.

[10] Z. Hou, Complex Hessian equation on Kähler manifold, Int. Math. Res. Not. IMRN 2009 (2009), no. 16, 3098–3111.

[11] Z. Hou, X.-N. Ma and D. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold, Math. Res. Lett. 17 (2010), no. 3, 547–561.

[12] M. Lin and N. S. Trudinger, On some inequalities for elementary symmetric functions, Bull. Aust. Math. Soc. 50 (1994), 317–326.

[13] D. H. Phong, S. Picard and X. Zhang, On estimates for the Fu-Yau generalization of a Strominger system, to appear in J. Reine Angew. Math.

[14] D. H. Phong, S. Picard and X. Zhang, A second order estimate for general complex Hessian equations, Anal. PDE 9 (2016), no. 7, 1693–1709.

[15] D. H. Phong, S. Picard and X. Zhang, The Fu-Yau equation with negative slope parameter, to appear in Invent. Math.

[16] J. Streets and G. Tian, Generalized Kähler geometry and the pluriclosed flow, Nuclear Phys. B, 858(2012), no. 2, 366-376.

[17] A. Strominger, Superstrings with torsion, Nuclear Phys. B 274 (1986), no. 2, 253-284.

[18] G. Székelyhidi, Fully non-linear elliptic equations on compact Hermitian manifolds, to appear in J. Differential Geom.

[19] G. Székelyhidi, V. Tosatti and B. Weinkove, Gauduchon metrics with prescribed volume form, preprint, arXiv:1503.04491.

[20] D. Zhang, Hessian equations on closed Hermitian manifolds, to appear in Pacific J. Math.

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