FINITE SPEED OF PROPAGATION FOR STOCHASTIC POROUS MEDIA EQUATIONS.

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Abstract. We prove finite speed of propagation for stochastic porous media equations perturbed by linear multiplicative space-time rough signals. Explicit and optimal estimates for the speed of propagation are given. The result applies to any continuous driving signal, thus including fractional Brownian motion for all Hurst parameters. The explicit estimates are then used to prove that the corresponding random attractor has infinite fractal dimension.

0. Introduction

In this paper we prove finite speed of propagation for solutions to stochastic porous media equations (SPME) driven by linear multiplicative space-time rough signals, i.e. to equations of the form

\[ dX_t = \Delta (|X_t|^m \text{sgn}(X_t)) \, dt + \sum_{k=1}^{N} f_k X_t \circ d\kappa_t^{(k)}, \quad \text{on } \mathcal{O}_T, \]

\[ X(0) = X_0, \quad \text{on } \mathcal{O}, \]

with homogeneous Dirichlet boundary conditions on a bounded, smooth domain \( \mathcal{O} \subseteq \mathbb{R}^d \), \( m \in (1, \infty) \), rough driving signals \( \kappa_t^{(k)} \in C([0, T]; \mathbb{R}) \) and diffusion coefficients \( f_k \in C^\infty(\overline{\mathcal{O}}) \). We assume the number of signals \( N \) to be finite and high regularity for \( f_k \) for simplicity only. In fact, the proofs only require \( \sum_{k=1}^{N} f_k(\xi) \kappa_t^{(k)} \in C([0, T]; C^2(\overline{\mathcal{O}})) \). The stochastic Stratonovich integral \( \circ \) occurring in (0.0) is informal but justified by a transformation technique and stability results analyzed in detail in [20, 22].

Recently, a hole-filling property for SPME driven by multiplicative space-time Brownian noise has been shown in [8], which may be seen as an important step towards proving finite speed of propagation. However, no explicit control on the rate of growth of the support of the solution could be established, which made it impossible to deduce finite speed of propagation. In the present paper, we prove explicit (and locally optimal) estimates on the speed of hole-filling and thus deduce finite speed of propagation for SPME. Moreover, we will completely remove the non-degeneracy assumption on the noise as it was conjectured to be possible in [8], which allows to analyze the dependence of the speed of propagation on the strength of the
noise. In particular, we prove convergence to the deterministic, optimal estimates when the noise-intensity converges to zero (cf. Remark 2.6 below). In [8] restrictions on the dimension \(d\) and on the order of the nonlinearity \(m\) had to be supposed for technical reasons and it was conjectured that these could be completely removed. In the present paper we prove that this indeed is the case.

Our methods are purely local and thus apply without change to the homogeneous Cauchy-Dirichlet problem to (0.0) on not necessarily bounded domains \(O \subseteq \mathbb{R}^d\), as soon as the problem of unique existence of corresponding solutions is solved. Since up to now this problem remains open, we restrict to bounded domains for simplicity (cf., however, Remark 2.10 below).

The stochastic case is contained in our setup by choosing \(z^{(k)}\) to be given as paths of some continuous stochastic process. Therefore, our results yield purely pathwise results for the stochastic case. However, due to the explicit form of our estimates, moment estimates also immediately follow.

In the deterministic case it is well-known that the attractor corresponding to

\[
dX_t = \Delta (|X_t|^m \text{sgn}(X_t)) \, dt + \lambda X_t dt,
\]

with Dirichlet boundary conditions has infinite fractal dimension \(\text{iff} \lambda > 0\) (cf. [19]). Generally speaking, it highly depends on the drift of an SPDE as well as on the type of random perturbation, whether the noise has a regularizing effect on the long-time dynamics of the unperturbed system.

In [21] it has been shown that sufficiently non-degenerate additive Wiener noise stabilizes the dynamics of (0.1) in the sense that the random attractor consists of a single random point and thus is zero dimensional. Moreover, it is well-known that multiplicative Itô noise may stabilize the long-time dynamics due to the Itô correction term. For example, this has been realized in [12] in case of the Cahee-Infante equation perturbed by spatially homogeneous, linear multiplicative Itô noise. The more intriguing case of space-time, linear multiplicative Itô noise has been analyzed in [6] for fast diffusion equations (cf. also the references therein), where a regularizing effect due to the Itô correction term has been observed in [6, Theorem 3.5].

This correction term is absent in the case of linear multiplicative Stratonovich noise. In this spirit, it has been shown in [12] that spatially homogeneous, linear multiplicative Stratonovich noise does not have any regularizing effect on the long-time behavior of the Cahee-Infante equation. On the other hand, each linear PDE with non-negative, self-adjoint drift having negative trace (possibly \(-\infty\)) may be stabilized by linear multiplicative space-time Stratonovich noise (cf. [13]). For these reasons, it is an intriguing question, whether including linear multiplicative space-time Stratonovich noise in (0.1) stabilizes the long-time behavior, or whether the random attractor associated to

\[
dX_t = \Delta (|X_t|^m \text{sgn}(X_t)) \, dt + \lambda X_t dt + \sum_{k=1}^{N} \int_{\mathbb{R}^d} f_k X_t \circ dz^{(k)}_t,
\]

remains infinite dimensional. Based on the explicit bounds on the rate of propagation obtained in this paper, we prove lower bounds for the Kolmogorov \(\varepsilon\)-entropy of the random attractor corresponding to (0.0) and thus conclude that the random attractor remains infinite dimensional.

The SPME (0.0) with driving signals \(z^{(k)}\) given as paths of independent Brownian motions \(\beta^{(k)}\) has been intensively studied in the recent history (cf. e.g. [2,11,10,17,23,25,28,29] and references therein). The construction of a random dynamical system (RDS) associated to (0.0) and the proof of existence of a corresponding random attractor has been given in [20,22]. In case of porous media equations (PME) perturbed by additive noise, the existence of a random attractor has been shown
in \[1\] and has subsequently been generalized to more general additive perturbations \[24\] and spatially rougher noise \[21\].

The sublinear, fast diffusion case \( (m \in [0, 1]) \) exhibits completely different propagation properties. In particular, finite speed of propagation does not hold for fast diffusion equations, but the positivity set of non-trivial solutions will cover the hole domain of definition after an arbitrarily small timespan (cf. \[33\] and references therein). On the other hand, solutions to the fast diffusion equation become extinct in finite time (cf. \[34\] for the deterministic case, \[5, 30\] for the stochastic case).

In the following let \( \mathcal{O} \subseteq \mathbb{R}^d \) be a bounded domain with smooth boundary \( \Sigma := \partial \mathcal{O} \). For \( T > 0 \) we define the space-time domain \( \mathcal{O}_T := [0, T] \times \mathcal{O} \), the lateral boundary \( \Sigma_T := [0, T] \times \partial \mathcal{O} \) and the parabolic boundary \( \mathcal{P}_T := \Sigma_T \cup \{(T) \times \mathcal{O}\} \). Let \( \vartheta \) be the surface measure on \( \Sigma \). Then, for two non-empty subsets \( A, B \) we define \( \text{dist}(A, B) := \inf \{d(a, b) \mid a \in A, \ b \in B\} \).

Moreover, we define \( H \) to be the dual of the first order Sobolev space with zero boundary \( H^0_0(\mathcal{O}) \). For two non-empty subsets \( A, B \) of a metric space \( (E, d) \) we define \( \text{dist}(A, B) := \inf \{d(a, b) \mid a \in A, \ b \in B\} \). If \( X \) is a Banach space, then \( L^p_{\text{loc}}((0, T]; X) \) denotes the space of all \( X \)-valued functions \( f \) such that \( f \in L^p([\tau, T]; X) \) for all \( \tau \in (0, T] \). As usual in probability theory we often denote the time-dependency of functions by a subscript \( X_t \) rather than by \( X(t) \) in order to keep the equations at a bearable length.

Let us start by recalling the finite speed of propagation properties for deterministic PME

\[ \partial_t u = \Delta \Phi(u), \]

where for simplicity of notation we have set \( \Phi(u) := |u|^m \text{sgn}(u) \). Finite speed of propagation for deterministic PME has been known for a long time and was first proved in \[27\]. For a more detailed study on interfaces for the one dimensional case we refer to \[22\].

Our main reference for the deterministic PME and main source of inspiration for the stochastic case will be \[33\] where a beautiful account on the propagation and expansion properties for deterministic PME is given.

**Definition 0.1** (Notions of solutions for \(0.3\))

i. A function \( u \in L^1_{\text{loc}}(\mathcal{O}_T) \) with \( \Phi(u) \in L^1_{\text{loc}}(\mathcal{O}_T) \) is said to be a local, very weak subsolution to \(0.3\) if

\[ \int_{\mathcal{O}_T} u \partial_t \eta \ d\xi dr \geq - \int_{\Sigma_0} \Phi(u) \Delta \eta \ d\xi dr, \]

for all non-negative \( \eta \in C^1_c(\mathcal{O}_T) \).

ii. If, in addition, \( u \in L^1(\mathcal{O}_T) \) with \( \Phi(u) \in L^1(\mathcal{O}_T) \) and there are functions \( u_0 \in L^1(\mathcal{O}) \) and \( \Phi(g) \in L^1(\Sigma_0) \) such that

\[ \int_{\mathcal{O}_T} u \partial_t \eta \ d\xi dr + \int_{\mathcal{O}} u_0 \eta_0 \ d\xi \geq - \int_{\Sigma_0} \Phi(u) \Delta \eta \ d\xi dr + \int_{\Sigma_0} \Phi(g) \partial_\nu \eta \ d\nu dr, \]

for all non-negative \( \eta \in C^1(\Sigma_0) \) with \( \eta|_{\Sigma_T} = 0 \), then \( u \) is said to be a very weak subsolution to the (inhomogeneous) Dirichlet problem to \(0.3\) with initial condition \( u_0 \) and boundary value \( g \).

iii. If \( \Phi(u) \in L^2((0, T]; H^1_0(\mathcal{O})) \) then \( u \) is said to be a (local) weak subsolution to the homogeneous Dirichlet problem to \(0.3\).

iv. If \( \Phi(u) \in L^2_{\text{loc}}((0, T]; H^1_0(\mathcal{O})) \) then \( u \) is said to be a generalized (local) weak subsolution to the homogeneous Dirichlet problem to \(0.3\).
Analogous definitions are used for (local) very weak supersolutions. (Local) very weak solutions to \( u \) are functions that are supersolutions and subsolutions simultaneously.

We note that each essentially bounded, generalized weak solution \( u \) is a generalized weak solution to \( \text{[1.3]} \) on each smooth subdomain \( K \subseteq \Omega \) with initial data \( u_0|_K \) and boundary data \( \Phi(g) = \Phi(u) \) in the sense of traces.

The proof of finite speed of propagation is a direct consequence of the so-called hole-filling problem

**Lemma 0.2** (Deterministic hole-filling, [33], Lemma 14.5). Let \( \xi_0 \in \mathbb{R}^d \), \( T, R > 0 \) and \( u \in C((0, T) \times B_R(\xi_0)) \) be an essentially bounded, non-negative, very weak subsolution to \( \text{[1.3]} \) with vanishing initial value \( u_0 \) on \( B_R(\xi_0) \) and boundary value \( g \) satisfying \( H := \|g\|_{L^\infty((0,T)\times\partial B_R(\xi_0))} < \infty \). Define \( C_{\text{det}} = \frac{m-1}{2m(m-1)+4m} \) and

\[
T_{\text{det}} := R^2 \frac{C_{\text{det}}}{H^{m-1}}.
\]

Then \( u(t) \) vanishes in \( B_{R_{\text{det}}(t)}(\xi_0) \) for all \( t \in [0, T_{\text{det}} \wedge T] \), where

\[
R_{\text{det}}(t) = R - \sqrt{t \left( \frac{H^{m-1}}{C_{\text{det}}} \right)^\frac{1}{m-1}}.
\]

For boundary value \( g \) given as \( g \equiv H \) for some \( H > 0 \), the bound on the rate of hole-filling from Lemma 0.2 is optimal (cf. [33, p. 339]).

From the hole-filling Lemma one may deduce

**Theorem 0.3** (Deterministic finite speed of propagation, [33], Theorem 14.6). Let \( u \in C((0, T) \times \Omega) \) be an essentially bounded, non-negative, very weak subsolution to the homogeneous Dirichlet problem to \( \text{[1.3]} \) and set \( H = \|u\|_{L^\infty(\Omega_T)} \). Then

i. For every \( s \in [0, T] \) and every \( h > 0 \) there is a time-span \( T_h > 0 \) such that

\[
\supp(u_{s+t}) \subseteq B_h(\supp(u_s)), \quad \forall t \in [0, T_h \wedge (T - s)].
\]

More precisely, \( T_h \) is given by

\[
T_h := h^2 \frac{C_{\text{det}}}{H^{m-1}}.
\]

ii. For every \( s \in [0, T] \)

\[
\supp(u_{s+t}) \subseteq B \frac{1}{\sqrt{T(\frac{m-1}{C_{\text{det}}})}} \left( \supp(u_s) \right), \quad \forall t \in [0, T - s].
\]

**Proof.** For each non-negative \( u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) there is a unique non-negative, essentially bounded, weak solution \( u_C \in C([0,T]; L^1(\mathbb{R}^d)) \) to the Cauchy problem for \( \text{[1.3]} \) (cf. [33, Theorem 9.3]). This in turn is a weak supersolution to the Cauchy-Dirichlet problem on \( \Omega \). Since \( t \mapsto \|u_C(t)\|_\infty \) is non-increasing, we have \( \|u_C\|_{L^\infty(\Omega_T)} = \|u\|_{L^\infty(\Omega_T)} \). Without loss of generality one may thus assume that \( u \) is a solution to the Cauchy problem, which simplifies the argument since no difficulties at the boundary appear. Noticing that \( u \) in particular is an essentially bounded, very weak solution on each \( B_R(\xi_0) \), the claim becomes a direct consequence of Lemma 0.2. \( \square \)

1. **Real-valued linear multiplicative noise**

We start the analysis of the stochastically perturbed case by the much simpler situation of spatially homogeneous noise, i.e. we consider the homogeneous Dirichlet problem to

\[
(1.4) \quad dX_t = \Delta \Phi(X_t) dt + \sum_{k=1}^{N} f_k X_t \circ dW_t^{(k)}, \quad \text{on } \Omega_T,
\]
where \( f_k \in \mathbb{R} \) are \( \mathbb{R} \)-valued constants and \( \mathcal{O} \subseteq \mathbb{R}^d \) is as before. We will prove below that (1.4) reduces to the deterministic PME (0.3) by rescaling and a random transformation in time. Since the bounds on the rate of propagation are known to be optimal in the deterministic case, we deduce optimal bounds for the case of spatially homogeneous perturbations.

Let \( \mu_t = -\sum_{k=1}^N f_k z^{(k)}_t \) and \( Y_t := e^{\mu t} X_t \). Then (informally)

\[
\partial_t Y_t = e^{-(m-1)\mu} \Delta \Phi(Y_t), \quad \text{on } \mathcal{O}_T.
\]

Solutions to (1.4) are then defined by the reverse transformation, i.e. a function \( X \) is a solution to (1.3) with initial value \( X_0 \in L^1(\mathcal{O}) \) and boundary value \( g \) iff \( Y_t := e^{\mu t} X_t \) is a solution to (1.7) with initial value \( Y_0 := e^{\mu t} X_0 \) and boundary value \( e^{\mu t} g \).

In [20] it has been shown that this transformation can be made rigorous if the signals \( z^{(k)} \) are given as paths of continuous semimartingales or are of bounded variation. In addition, in case of continuous driving signals, solutions to (1.4) were obtained in [20] as limits of approximating solutions driven by smoothed signals \( z^{(\delta)} \in C^\infty([0,T];\mathbb{R}^N) \) with \( z^{(\delta)} \to z \) in \( C([0,T];\mathbb{R}^N) \).

We set \( F(t) := \int_0^t e^{-(m-1)\mu r} dr \in C^1(\mathbb{R}_+;\mathbb{R}_+) \). Since \( F \) is strictly increasing we may define \( G(t) := F^{-1}(t) \) to be the inverse of \( F \) and \( u_t := Y_{G(t)} \). An informal computation suggests

\[
\partial_t u_t = \Delta \Phi(u_t), \quad \text{on } \mathcal{O}_T.
\]

A rigorous justification of this temporal transformation can easily be given by considering an artificial viscosity approximation, i.e. \( \partial_t u^{(\varepsilon)} = \Delta \Phi(u^{(\varepsilon)}) + \varepsilon \Delta u^{(\varepsilon)} \).

Local uniform continuity of \( u^{(\varepsilon)} \) (cf. [18,33]) allows to pass to the limit pointwisely and thus implies the claim.

Vice versa, solutions \( X \) to (1.4) can be expressed by solutions to (1.6) via:

\[
X_t := e^{-\mu t} u_{F(t)}.
\]

Lemma 0.2 implies

**Proposition 1.1** (Hole-filling for spatially homogeneous noise). Let \( \xi_0 \in \mathbb{R}^d \), \( T, R > 0 \) and \( X \in C([0,T] \times B_R(\xi_0)) \) be an essentially bounded, non-negative, very weak subsolution to (1.3) with vanishing initial value \( X_0 \) on \( B_R(\xi_0) \) and boundary value \( g \) satisfying \( H := \|e^{\mu t} g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))} < \infty \). Define

\[
T_{\text{stoch}} := F^{-1} \left( \frac{R^2 C_{\text{det}}}{H^{m-1}} \right),
\]

where \( F(t) := \int_0^t e^{-(m-1)\mu r} dr \).

Then \( X_t \) vanishes in \( B_{R_{\text{stoch}}(t)}(\xi_0) \) for all \( t \in [0,T_{\text{stoch}} \wedge T] \), where

\[
R_{\text{stoch}}(t) = R - \sqrt{F(t) \left( \frac{H^{m-1}}{C_{\text{det}}} \right)}.
\]

As pointed out above, the rates and constants given in Proposition 1.1 are optimal. Analogously, bounds on the rate of expansion of the support of solutions to (1.4) may be derived from (1.7) and Theorem 0.3.

**Remark 1.2.** For \( T \approx 0 \) we have \( \mu_t \approx \mu_0 \) on \([0,T]\) and thus

\[
\sqrt{F(t)} = \left( \int_0^t e^{-(m-1)\mu r} dr \right)^{\frac{1}{2}} \approx e^{-\frac{(m-1)\mu t}{2}} \sqrt{t}, \quad \text{on } [0,T]
\]
and \( H = \|e^\mu g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))} \approx e^{\mu_0} \|g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))} \). Thus
\[
R_{\text{stoch}}(t) \approx R - \sqrt{t} \left( \frac{\|g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))}^{m-1}}{C_{\text{det}}} \right)^{\frac{1}{2}}, \quad \forall t \in [0, T_{\text{stoch}} \wedge T].
\]
Consequently, we recover the deterministic rate of expansion for small times \( T \approx 0 \).

2. Linear multiplicative space-time noise

We now turn to the case of SPME perturbed by spatially inhomogeneous noise \((0,0)\). Since spatially homogeneous noise is contained as a special case, the precise bounds derived in the last section will serve as optimal bounds for the inhomogeneous case. Let
\[
\mu_t(\xi) := -\sum_{k=1}^{N} f_k(\xi) z_t^{(k)}.
\]
As in the case of spatially homogeneous noise, solutions to \((0,0)\) are defined via the transformation \( Y_t := e^{\mu_t} X_t \) which (informally) leads to the transformed equation (first studied in [5,7])
\[
\begin{align*}
\partial_t Y_t &= e^{\mu_t} \Delta \Phi(e^{-\mu_t} Y_t), \text{ on } \mathcal{O}_T \\
Y(0) &= Y_0 = e^{\mu_0} X_0, \text{ on } \mathcal{O},
\end{align*}
\]
with homogeneous Dirichlet boundary conditions. Note that \( \mu_t \) now depends on the spatial variable \( \xi \in \mathcal{O} \). As before, this transformation can be made rigorous if the driving signals \( z^{(k)} \) are given as paths of continuous semimartingales or are of bounded variation. In case of continuous driving signals this notion of solution is justified in a limiting sense via approximation of the driving signal (cf. [20]).

Similar results and methods as presented in this section may be applied to the more general equation
\[
\begin{align*}
\partial_t Y_t &= \rho_1 \Delta \Phi(\rho_2 Y_t), \text{ on } \mathcal{O}_T \\
Y(0) &= Y_0, \text{ on } \mathcal{O},
\end{align*}
\]
with \( \rho_1, \rho_2 \in C^{0,2}(\mathcal{O}_T) \) and zero Dirichlet boundary conditions. For simplicity and in order to derive locally optimal estimates we restrict to equations of the form (2.8) and postpone the treatment of the more general case to the Appendix A.4.

The unique existence of weak solutions to (2.8) for bounded initial data has been given in [20] in the following sense

**Definition 2.1** (weak & very weak solutions for (2.8)).

i. A function \( Y \in L^1_{\text{loc}}(\mathcal{O}_T) \) with \( \Phi(e^{-\mu} Y) \in L^1_{\text{loc}}(\mathcal{O}_T) \) is called a local, very weak subsolution to (2.8) if
\[
\int_{\mathcal{O}_T} Y \partial_\tau \eta \, d\xi \, dr \geq - \int_{\mathcal{O}_T} \Phi(e^{-\mu} Y) \Delta(\epsilon^{\mu} \eta) \, d\xi \, dr,
\]
for all non-negative \( \eta \in C^{1,2}_{\text{loc}}(\mathcal{O}_T) \).

ii. If, in addition, \( Y \in L^1(\Omega_T) \) with \( \Phi(e^{-\mu} Y) \in L^1(\Omega_T) \) and there are functions \( Y_0 \in L^1(\Omega) \) and \( \Phi(g) \in L^1(\Sigma_T) \) such that
\[
\begin{align*}
\int_{\mathcal{O}_T} Y \partial_\tau \eta \, d\xi \, dr &+ \int_{\mathcal{O}} Y_0 \eta_0 \, d\xi \geq - \int_{\mathcal{O}_T} \Phi(e^{-\mu} Y) \Delta(\epsilon^{\mu} \eta) \, d\xi \, dr \\
&+ \int_{\Sigma_T} \Phi(e^{-\mu} g) \partial_\tau(\epsilon^{\mu} \eta) \, d\nu \, dr,
\end{align*}
\]
for all non-negative \( \eta \in C^{1,2}_{\text{loc}}(\Omega_T) \) with \( \eta_{\partial \tau} = 0 \) then \( Y \) is said to be a very weak subsolution to the (inhomogeneous) Dirichlet problem to (2.8) with initial condition \( Y_0 \) and boundary value \( g \).
A.2. 

In particular, essentially bounded, very weak solutions to (0.0) are unique.

The proof of finite speed of propagation will rely on local comparison to supersolutions. We now present the required comparison result for essentially bounded, very weak solutions to the inhomogeneous Dirichlet problem.

Theorem 2.4 (Comparison for very weak solutions). Let $X^{(1)}, X^{(2)}$ be essentially bounded sub/supersolutions to (0.0) with initial conditions $X_0^{(1)} \leq X_0^{(2)}$ and boundary data $g^{(1)} \leq g^{(2)}$, a.e. in $\Omega$ respectively. Then,

$$X^{(1)} \leq X^{(2)}, \quad \text{a.e. in } \Omega.$$ 

In particular, essentially bounded, very weak solutions to (0.0) are unique.

The proof of a more general version of Theorem 2.4 may be found in the Appendix A.2.

2.1. Finite speed of propagation. We are going to prove bounds on the speed of propagation for (0.0), based on estimates for the rate of hole-filling as in the deterministic case. As we have seen in Section 1, the optimal bounds on the rate of collapse of balls have to depend on the driving signal. Since the perturbation now is spatially dependent, we expect worse estimates than in Proposition 1.4.

On the other hand, since $\xi \mapsto \mu_t(\xi)$ is continuous and thus $\mu_t(\xi) \approx \mu_t(\xi_0)$ on small balls $B_R(\xi_0)$, locally in space the rate of expansion should be given as in
Proposition 1.1 with $\mu_r \equiv \mu_\nu(\xi_0)$. This line of thought leads to optimal bounds on the rate of collapse of asymptotically small balls, proven in Theorem 2.5 below.

Moreover, due to the continuity of $t \mapsto \mu_\nu(t)$ we have $\mu_\nu(t) \approx \mu_\nu(0)$ on small time intervals $[0,T]$. Therefore, we expect to recover the optimal bounds from the deterministic case at least for asymptotically small times $T$, which indeed is proven in Theorem 2.7 below. In case of spatially homogeneous perturbations this has been observed in Remark 1.2.

**Theorem 2.5 (Hole-filling theorem for small balls).** Let $\xi_0 \in \mathbb{R}^d$, $T > 0$ and $X \in C((0,T] \times B_R(\xi_0))$ be an essentially bounded, non-negative, very weak subsolution to (0.0) with vanishing initial value $X_0$ on $B_R(\xi_0)$ and boundary value $g$ satisfying $H := \|e^{\nu}g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))} < \infty$. Define $F(t) := \int_0^t e^{-(m-1)\nu_\nu(\xi)} \, dt$ and

$$T_{\text{stoch}} := F^{-1}\left(\frac{\tilde{C}_{\text{det}}}{H^{m-1}C_R}\right),$$

where $R \mapsto C_R$ is a continuous, non-increasing function with $\lim_{R \downarrow 0} C_R = 1$.

Then $X_t$ vanishes in $B_{R_{\text{stoch}}(t)}(\xi_0)$ for all $t \in [0, T_{\text{stoch}} \wedge T]$, where

$$R_{\text{stoch}}(t) = R - \left.\sqrt{F(t)}\right| \left.\left(\frac{H^{m-1}}{C_{\text{det}}}\right)^{\frac{1}{2}} C_R^{\frac{1}{2}}.\right|$$

Note that for $R \approx 0$ we recover the optimal rate from Proposition 1.1 with $\mu_r \equiv \mu_\nu(\xi_0)$.

**Proof.** Since $X$ is a very weak subsolution to (0.0) with initial value $X_0 \equiv 0$ and boundary value $g$, $Y := e^{\nu}X$ is a very weak subsolution to (2.8) with initial value $Y_0 \equiv 0$ and boundary value $e^{\nu}g$.

For $\xi_1 \in B_R(\xi_0)$, $\tilde{T} \in (0,T]$, $r \in (0, \text{dist}(\xi_1, \partial B_R(\xi_0))]$ we construct an explicit supersolution to (2.8) in $(0, \tilde{T}] \times B_r(\xi_1)$. Let

$$W(t, \xi, r) := \tilde{C}|\xi - \xi_1|^\frac{m}{2} \left(F(\tilde{T}) - F(t)\right) \frac{1}{m-1}, \quad t \in [0, \tilde{T}], \xi \in B_r(\xi_1),$$

where $\tilde{C}$ will be chosen below (only depending on $R$) and $F$ is as in Section 1 for noise frozen at $\xi_0$, i.e.

$$F(t) := \int_0^t e^{-(m-1)\nu_\nu(\xi)} \, dt.$$

We compute:

$$\partial_t W(t, \xi, \xi_1) = \frac{1}{m-1} \tilde{C}|\xi - \xi_1|^\frac{m}{2} \left(F(\tilde{T}) - F(t)\right) \frac{1}{m} e^{-(m-1)\nu_\nu(\xi)},$$

for all $(t, \xi) \in [0, \tilde{T}] \times B_r(\xi_1)$ and

$$\Delta \left(e^{-\nu_\nu(\xi)}W(t, \xi, \xi_1)\right)^m \leq \frac{\tilde{C}^m}{(m-1)C_{\text{det}}} \left(F(\tilde{T}) - F(t)\right)^{\frac{m}{m-1}} |\xi - \xi_1|^\frac{m}{2} \left(e^{-m\nu_\nu(\xi)} + 2(m-1)\frac{\nu_\nu(\xi)}{d(m-1)+2}|\nu e^{m\nu_\nu(\xi)}| + \frac{2(m-1)}{d(m-1)+2}(m(m-1)C_{\text{det}} |\Delta e^{m\nu_\nu(\xi)}|)\right),$$

and

$$\Delta \left(e^{-\nu_\nu(\xi)}W(t, \xi, \xi_1)\right)^m \leq \frac{\tilde{C}^m}{(m-1)C_{\text{det}}} \left(F(\tilde{T}) - F(t)\right)^{\frac{m}{m-1}} |\xi - \xi_1|^\frac{m}{2} e^{-m\nu_\nu(\xi)} \left(1 + \frac{2(m-1)}{d(m-1)+2}(m(m-1)C_{\text{det}} |\Delta \mu_\nu(\xi)|)\right),$$

and

$$\Delta \left(e^{-\nu_\nu(\xi)}W(t, \xi, \xi_1)\right)^m \leq \frac{\tilde{C}^m}{(m-1)C_{\text{det}}} \left(F(\tilde{T}) - F(t)\right)^{\frac{m}{m-1}} |\xi - \xi_1|^\frac{m}{2} e^{-m\nu_\nu(\xi)} \left(1 + \frac{2(m-1)}{d(m-1)+2}(m(m-1)C_{\text{det}} |\Delta \mu_\nu(\xi)|)\right).$$
\[
\left(1 + C(d, m)R(1 + R)\|\mu\|_{C^{0,2}[0, \tilde{T}] \times B_R(\xi_1))}\right),
\]
for all \((t, \xi) \in [0, \tilde{T}) \times B_r(\xi_1)\). We conclude that
\[
\partial_t W(t, \xi, \xi_1) \geq e^{\mu(t)} \Delta \left(e^{-\mu(t)} W(t, \xi, \xi_1)\right)^m
\]
on \([0, \tilde{T}) \times B_r(\xi_1)\) if
\[
e^{-A^0(\xi_0) - \mu(t)} \|e^{A^0(\xi_0) + B_R(\xi_0)} \geq \frac{C^{m-1}}{C_{det}} \left(1 + C(d, m)R(1 + R)\|\mu\|_{C^{0,2}[0, \tilde{T}] \times B_R(\xi_0))}\right),
\]
which is satisfied if we choose \(C^{m-1} = C_{det}C_R\) with
\[
C_R := \frac{1 + C(d, m)R(1 + R)\|\mu\|_{C^{0,2}[0, \tilde{T}] \times B_R(\xi_0))}.\]
We note that \(R \to C_R\) is continuous, non-increasing in \(R\) and \(\lim_{R \to 0} C_R = 1\). In contrast, \(C_R\) does not necessarily converge to 1 for \(T \to 0\). Thus, the bounds become optimal locally in space but not locally in time.

In order to derive the upper bound \(Y(t, \xi) \leq W(t, \xi, \xi_1)\) on \([0, \tilde{T}) \times B_r(\xi_1)\) we need \(g(t, \xi)e^{\mu(t)} \leq W(t, \xi, \xi_1)\) for a.a. \((t, \xi) \in [0, \tilde{T}) \times \partial B_r(\xi_1)\). For this it is sufficient to have
\[
W(t, \xi, \xi_1) = C_{\xi - \xi_1} \left(F(\tilde{T}) - F(t)\right) \geq H,
\]
for a.a. \((t, \xi) \in [0, \tilde{T}) \times \partial B_r(\xi_1)\). This is satisfied if we choose \(\tilde{T} = \tilde{T}_r\) by
\[
T_r := \sqrt{\frac{F^{-1} \left(\frac{C^{m-1}}{H^{m-1}}\right)}{\det C_{\xi}}} = \sqrt{\frac{F^{-1} \left(\frac{C^{m-1}}{H^{m-1}}\right)C_R}{C_{det}}}.\]
By Theorem 2.4 and by continuity of \(Y, W\) we conclude
\[
0 \leq Y(t, \xi_1) \leq W(t, \xi_1, \xi_1) = 0, \quad \forall t \in [0, \tilde{T}].
\]
Let \(R_1 \in (0, R)\), \(\xi_1 \in B_{R_1}(\xi_0)\) and \(r = \text{dist}(\xi_1, \partial B_R(\xi_0)) \geq R - R_1 > 0\). Resolving (2.3) for \(r\) yields
\[
R(T) := R_1 = R - \frac{\sqrt{F(T)}}{C_{\xi}} \left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}} C_R^{\frac{1}{2}}.
\]
Hence,
\[
Y(t, \xi) = 0, \quad \forall \xi \in B_{R(T)}(\xi_0), \ t \in [0, T_{\text{stoch}} \wedge T],
\]
where \(T_{\text{stoch}} = T_R = F^{-1} \left(R^2 \frac{C_{det}}{H^{m-1}}\right)C_R\).

**Remark 2.6.** Due to the explicit form of the estimates and the constant \(C_R\) in Theorem 2.4, the dependence of the bounds on the strength of the noise is obvious. In particular, when the noise intensity \(\sum_{k=1}^N \|f_k\|_{C^{2}[B_R(\xi_0)]}\) decreases to 0, then the bounds from Theorem 2.4 approach the corresponding deterministic, optimal ones.

We will now derive a second bound on the rate of collapse of balls for (0.4). In contrast to Theorem 2.4, the construction of a suitable supersolution will be based on a temporal discretization, i.e. on freezing the noise at time \(t = 0\).

**Theorem 2.7** (Hole-filling theorem for small times). Let \(\xi_0 \in \mathbb{R}^d\), \(T, R > 0\) and \(X \in C([0, T) \times B_R(\xi_0))\) be an essentially bounded, non-negative, very weak subsolution to (0.4) with vanishing initial value \(X_0\) on \(B_R(\xi_0)\) and boundary value \(g\) satisfying \(H := \|g\|_{L^\infty([0, T] \times \partial B_R(\xi_0))} < \infty\). Define \(T_{\text{stoch}}\) by
\[
T_{\text{stoch}} := \sup \left\{ \tilde{T} \in [0, T] \mid \tilde{T}C_{\xi} \leq R^2 \frac{C_{det}}{H^{m-1}} \right\}.
\]
where \( t \mapsto C_t \) is a continuous, non-decreasing function with \( \lim_{t \to 0} C_t = 1 \).

Then \( X_t \) vanishes in \( B_{R\text{stoch},(t)}(\xi_0) \) for all \( t \in [0, T_{\text{stoch}}] \), where

\[
R_{\text{stoch}}(t) = R - \sqrt{T \left( \frac{H^{m-1}}{C_{\det}} \right)} \sqrt{C_t}.
\]

Note that for \( t \approx 0 \) we recover the optimal rate from the deterministic case.

**Proof.** The proof proceeds similarly to Theorem 2.5. Hence, let \( Y := e^{\mu} X \) be a very weak subsolution to (2.8) with initial value \( Y_0 \equiv 0 \) and boundary value \( e^g \).

For \( \xi_1 \in B_R(\xi_0), \tilde{T} \in (0, T], \tilde{r} \in (0, \text{dist}(\xi_1, \partial B_R(\xi_0))] \) we again construct an explicit supersolution to (2.8) in \([0, \tilde{T}) \times B_r(\xi_1)\):

\[
W(t, \xi, \xi_1) := \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} (\tilde{T} - t)^{-\frac{m}{m-1}}, \quad t \in [0, \tilde{T}), \xi \in B_r(\xi_1),
\]

where \( \tilde{C} \) will be chosen below, depending on \( \tilde{T}, R \) only. We compute:

\[
\partial_t W(t, \xi, \xi_1) = \frac{1}{m-1} \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} (\tilde{T} - t)^{-\frac{m}{m-1}},
\]

for all \( (t, \xi) \in [0, \tilde{T}) \times B_r(\xi_1) \) and

\[
\Delta \left( e^{-\mu_1(\xi)} W(t, \xi, \xi_1) \right)^m \leq \frac{\tilde{C}^m}{(m-1)C_{\det}} (\tilde{T} - t)^{-\frac{m}{m-1}} |\xi - \xi_1|^{\frac{2}{m-1}} \left( e^{m(\mu_0(\xi) - \mu_1(\xi))} + \frac{2(m-1)}{2 + d(m-1)} |\nabla e^{m(\mu_0(\xi) - \mu_1(\xi))}| R + \frac{(m-1)C_{\det} R^2 |\Delta e^{m(\mu_0(\xi) - \mu_1(\xi))}|}{1 + C(d, m) R(1 + R)} \right)
\]

\[
\leq \frac{\tilde{C}^m}{(m-1)C_{\det}} (\tilde{T} - t)^{-\frac{m}{m-1}} |\xi - \xi_1|^{\frac{2}{m-1}} e^{m(\mu_0(\xi) - \mu_1(\xi))} \left( 1 + C(d, m) R(1 + R) \right) \| \mu_0 - \mu \|_{C^{0, 2}([0, \tilde{T}] \times B_R(\xi_0))}
\]

for all \( (t, \xi) \in [0, \tilde{T}) \times B_r(\xi_1) \). We conclude that

\[
\partial_t W(t, \xi, \xi_1) \geq e^{\mu_1(\xi)} \Delta \left( e^{-\mu_1(\xi)} W(t, \xi, \xi_1) \right)^m
\]

on \([0, \tilde{T}) \times B_r(\xi_1)\) if

\[
e^{(m-1)(\mu_1(\xi) - \mu_0(\xi))} \geq \frac{e^{(m-1)\mu_1(\xi) - \mu_0(\xi)}}{C_{\tilde{T}}} \left( 1 + C(d, m) R(1 + R) \right) \| \mu_0 - \mu \|_{C^{0, 2}([0, \tilde{T}] \times B_R(\xi_0))},
\]

for all \( (t, \xi) \in [0, \tilde{T}) \times B_r(\xi_1) \), which is satisfied for the choice

\[
\tilde{C}^{m-1} = \frac{C_{\det} e^{(m-1)\mu_0(\xi)}}{C_{\tilde{T}}} e^{2(m-1)\| \mu_0 - \mu \|_{C^{0,2}([0, \tilde{T}] \times \partial B_R(\xi_0))}}
\]

with

\[
C_{\tilde{T}} := \frac{1 + C(d, m) R(1 + R) \| \mu_0 - \mu \|_{C^{0, 2}([0, \tilde{T}] \times B_R(\xi_0))}}{e^{-2(m-1)\| \mu_0 - \mu \|_{C^{0,2}([0, \tilde{T}] \times B_R(\xi_0))}}}.\]

In order to derive the upper bound \( Y(t, \xi) \leq W(t, \xi, \xi_1) \) on \([0, \tilde{T}) \times B_r(\xi_1)\) we need \( g(t, \xi) e^{\mu_1(\xi)} \leq W(t, \xi, \xi_1) \) for a.e. \((t, \xi) \in [0, \tilde{T}) \times \partial B_r(\xi_1)\). For this to be true it is sufficient to have

\[
W(t, \xi, \xi_1) = \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} (\tilde{T} - t)^{-\frac{m}{m-1}} \geq H e^{\mu_1(\xi)},
\]
for a.a. \((t, \xi) \in [0, \hat{T}] \times \partial B_r(\xi_0)\). This is satisfied if
\[
C_{m-1}^{-1} \frac{\int_{\partial B_r(\xi_0) \times (0, t]} \mu \cdot \nabla \rho \, d\tau}{H} \leq r^2 \frac{1}{H^{m-1}},
\]
for which in turn it is sufficient to have
\[
(2.10) \quad \hat{T} C_{\hat{T}} \leq r^2 \frac{C_{\text{det}}}{H^{m-1}}.
\]
Since the left hand side is continuous in \(\hat{T}\) we may choose \(\hat{T}\) as
\[
\hat{T} := \sup \left\{ \hat{T} \in [0, T] \mid \hat{T} C_{\hat{T}} \leq r^2 \frac{C_{\text{det}}}{H^{m-1}} \right\}.
\]
Note that \(\hat{T} := \sup \\{ \hat{T} \in [0, T] \mid \hat{T} C_{\hat{T}} \leq r^2 \frac{C_{\text{det}}}{H^{m-1}} \}\) is a continuous, non-decreasing and
\[
\hat{T} \to C_{\hat{T}} > 0 \text{ is continuous}, \quad \text{for } \hat{T} \to 0
\]
i.e. we recover the optimal constant from the deterministic case for asymptotically small time, while locally in space the estimates will not be optimal.

Let now \(R_1 \in (0, R), \xi_1 \in B_{R_1}(\xi_0)\) and \(r = \text{dist}(\xi_1, \partial B_{R_1}(\xi_0)) \geq R - R_1 > 0\). By Theorem 2.4 and by continuity of \(Y, W\) we conclude
\[
0 \leq Y(t, \xi_1) \leq W(t, \xi_1) = 0, \quad \forall t \in [0, \hat{T}(r)].
\]
Resolving \((2.10)\) for \(R_1\) yields
\[
R(T) := R_1 = R - \sqrt{\hat{T} \left( \frac{H^{m-1}}{C_{\text{det}}} \right)^{\frac{1}{2}}} \sqrt{C_T}.
\]
Hence,
\[
Y(t, \xi) = 0, \quad \forall \xi \in B_{R(T)}(\xi_0), \; t \in [0, T_{\text{stoch}}],
\]
where
\[
T_{\text{stoch}} := \hat{T}(R)
\]
\[
:= \sup \left\{ \hat{T} \in [0, T] \mid \hat{T} C_{\hat{T}} \leq R^2 \frac{C_{\text{det}}}{H^{m-1}} \right\}.
\]

We are now ready to derive bounds on the speed of propagation for \((0.0)\). We give two formulations of this property

**Theorem 2.8** (Finite speed of propagation). Let \(X \in C([0, T] \times \Omega)\) be an essentially bounded, non-negative, very weak subsolution to the homogeneous Dirichlet problem to \((0.0)\) and set \(H := \|e^{\mu X}\|_{L^\infty(\Omega)}\). Then, for each \(s \in [0, T]\) and every \(h > 0\) there is a time-span \(T_h > 0\) such that
\[
(2.11) \quad \text{supp}(X_{s+t}) \subseteq B_h(\text{supp}(X_s)), \quad \forall t \in [0, T_h \wedge (T - s)].
\]
More precisely, \(T_h\) is given by
\[
T_h := F_h^{-1} \left( h^2 \frac{C_{\text{det}}}{H^{m-1}} C_h \right),
\]
where \(F_h(t) := \int_0^t e^{-(m-1) \int_{\text{supp}(X_s)} \mu \cdot \nabla \rho \, d\tau} \, dr\) and \(h \mapsto C_h\) is a continuous, non-increasing function satisfying \(\lim_{h \to 0} C_h = 1\). In particular,
\[
\left| T_h - F_h^{-1} \left( h^2 \frac{C_{\text{det}}}{H^{m-1}} \right) \right| \to 0, \quad \text{for } h \to 0,
\]
with \(F_0(t) = \int_0^t e^{-(m-1) \int_{\text{supp}(X_s)} \mu \cdot \nabla \rho \, d\tau} \, dr\).
Proof. Without loss of generality we assume \( s = 0 \). In order to avoid difficulties at the boundary we first replace \( X \) by a solution to \( (1.0) \) on some large ball \( B_R(0) \supseteq \mathcal{O} \), where we choose \( R > 0 \) large enough, such that the boundary \( \partial B_R(0) \) becomes “invisible” for the solution on \( [0, T] \). Let \( h > 0, R > 0 \) such that

\[
\bar{\mathcal{O}} := B_R(0) \supseteq B_{2h}(\text{supp}(X_0)) \cup \mathcal{O}.
\]

By [20, Theorem 1.3, Theorem 1.4, Theorem 1.12] there is a unique, essentially bounded, non-negative, weak solution \( X \in C((0, T] \times \bar{\mathcal{O}}) \) to \( (1.0) \) on \( \bar{\mathcal{O}} \) with zero Dirichlet boundary conditions and initial condition \( X_0 := X_0 |_{\mathcal{O}} \in L^\infty(\bar{\mathcal{O}}) \). Since \( \bar{X} \) is a supersolution to the homogeneous Dirichlet problem to \( (1.0) \) on \( \mathcal{O} \), by Theorem 2.4 we have

\[
X \leq \bar{X}, \quad \text{on } \mathcal{O}_T.
\]

Thus, it is sufficient to prove the claim for \( \bar{X} \). Hence, without loss of generality we may assume \( X \) to be an essentially bounded, weak solution to \( (1.0) \) and

\[
\text{dist}(\text{supp}(X_0), \partial \mathcal{O}) > 2h.
\]

Let \( \xi_0 \in \partial B_h(\text{supp}(X_0)) \subseteq \mathcal{O} \). Then, \( X_0 = 0 \) on \( B_h(\xi_0) \subseteq \mathcal{O} \) and Theorem 2.5 (with \( R = h \)) implies that \( X_t \) vanishes on \( B_{R_{\text{stock}}(t)}(\xi_0) \) for all \( t \in [0, T_{\text{stock}} \wedge T] \), where \( R_{\text{stock}}(t) \) and \( T_{\text{stock}} \) given in Theorem 2.5 depend on \( \xi_0 \) via the constant \( C_h \) and the function \( F \). We note that \( C_h \) may be uniformly estimated by

\[
C_h \geq \tilde{C}_h := \frac{e^{-h\|\mu\|_{C^{0,1}([0, T] \times (\partial \mathcal{O} \setminus \text{supp}(X_0)))}}}{1 + C(m)h(1 + h)\|\mu\|_{C^{0,2}([0, T] \times (\partial \mathcal{O} \setminus \text{supp}(X_0)))}}
\]

and \( F \) by

\[
\tilde{F}_h(t) = \int_0^t e^{-(m-1)\inf_{\xi_0 \in \partial B_h(\text{supp}(X_0))} \mu_r(\xi_0) dr} \geq F(t).
\]

Therefore, \( T_{\text{stock}} \) is uniformly bounded from below by

\[
\tilde{T}_h := \tilde{F}_h^{-1}\left( h^2 \frac{C_{\text{det}}}{H_{m-1}^{\tilde{C}_h}} \right),
\]

and \( R_{\text{stock}}(t) \) by

\[
\tilde{R}_h(t) = h - \sqrt{\tilde{F}_h(t)} \left( \frac{H_{m-1}^{\tilde{C}_h}}{C_{\text{det}}} \right)^{\frac{1}{2}} \tilde{C}_h^{-\frac{1}{2}}.
\]

Hence, \( X_t \) vanishes on \( B_{\tilde{R}_h(t)}(\partial B_h(\text{supp}(X_0))) \) for all \( t \in [0, \tilde{T}_h \wedge T] \).

In particular, this implies that \( X \) is a weak solution to the homogeneous Dirichlet problem to \( (1.0) \) on \( [0, \tilde{T}_h \wedge T] \times \mathcal{O} \cap B_h(\text{supp}(X_0)) \). Since \( X_0 \equiv 0 \) on \( \mathcal{O} \cap B_h(\text{supp}(X_0)) \) this implies \( X_t \equiv 0 \) on \( \mathcal{O} \cap B_h(\text{supp}(X_0)) \) for all \( t \in [0, T_{\text{stock}} \wedge T] \).

**Theorem 2.9** (Finite speed of propagation). Let \( X \in C((0, T] \times \mathcal{O}) \) be an essentially bounded, non-negative, very weak subsolution to the homogeneous Dirichlet problem to \( (1.0) \) and set \( H := \|X\|_{L^\infty(\mathcal{O}_T)} \). Then, for every \( s \in [0, T] \)

\[
\text{supp}(X_{s+t}) \subseteq B_{\sqrt{t}(\frac{H_{m-1}^{\tilde{C}_h}}{C_{\text{det}}} \sqrt{\tilde{C}_h} \text{supp}(X_s))}, \quad \forall t \in [0, T-s],
\]

where \( t \mapsto C_t \) is a continuous, non-decreasing function with \( C_t \to 1 \) for \( t \to 0 \).

**Proof.** We argue as for Theorem 2.8 but apply Theorem 2.7 instead of Theorem 2.5. Let \( D := \text{diam}(\mathcal{O}) \). We then estimate \( C_T \) uniformly by

\[
C_T \leq \tilde{C}_T := \frac{1 + C(d, m)D(1 + D)\|\mu_0 - \mu\|_{C^{0,2}([0, T] \times (\partial \mathcal{O} \setminus \text{supp}(X_0)))}}{e^{-2(m-1)\|\mu_0 - \mu\|_{C^{0,2}([0, T] \times (\partial \mathcal{O} \setminus \text{supp}(X_0)))}}},
\]

as in [20, Theorem 1.3, Theorem 1.4, Theorem 1.12].
Hence, for  
\[
\bar{T}(h) := \sup \left\{ \bar{T} \in [0, T] \mid \bar{T} C_{\bar{T}} \leq h^2 \frac{C_{\det H}}{H^{m-1}} \right\},
\]
we have \( \bar{T}(h) \leq T_{\text{stoch}} \) for all \( \xi_0 \in \partial (B_h(\text{supp}(X_0))) \) as in the proof of Theorem 2.8 and for  
\[
\bar{R}(t) := h - \sqrt{t} \left( \frac{H^{m-1}}{C_{\det}} \right)^{\frac{1}{2}} \sqrt{C_t},
\]
we have \( \bar{R}(t) \leq R_{\text{stoch}}(t) \) for all \( t \in [0, T] \). In particular, for all \( \xi_0 \in \partial (B_h(\text{supp}(X_0))) \) we deduce  
\[
X_t(\xi_0) = 0, \quad \forall t \in [0, \bar{T}(h)].
\]
Arguing as for Theorem 2.8 this implies \( \text{supp}(X_t) \subseteq B_h(\text{supp}(X_0)) \) for all \( t \leq \bar{T}(h) \).

\[\square\]

**Remark 2.10** (Unbounded domains \( \mathcal{O} \subseteq \mathbb{R}^d \)). In case of unbounded domains \( \mathcal{O} \subseteq \mathbb{R}^d \) no pathwise uniqueness and existence theory (in the sense of existence of a stochastic flow) has been established for (0.0) so far. We note, however, that the simpler problem of constructing probabilistic solutions to (0.0) with \( z^{(k)} \) being given as paths of Brownian motions has been solved in [28] for \( d \geq 3 \).

If the support of the initial condition \( X_0 \in L^\infty(\mathcal{O}) \) is compact and bounded away from \( \partial \mathcal{O} \) then the existence of corresponding essentially bounded, weak solutions \( X \) to the homogeneous Cauchy-Dirichlet problem on short time intervals \([0, T]\) follows from the finite speed of propagation properties proved in this paper. The time of existence \( T \) allowed by this approach is limited due to the support \( \text{supp}(X_t) \) reaching the boundary \( \partial \mathcal{O} \). In particular, for the Cauchy problem no restriction on the time of existence has to be made.

For initial conditions \( X_0 \) with compact support, also uniqueness of essentially bounded, very weak solutions may be deduced from the methods of this paper at least on short time intervals \([0, T]\). Again, for the Cauchy problem no restriction on the time interval has to be supposed.

The case of initial conditions with unbounded support, however, remains open.

### 3. Infinite dimensional random attractor

In this section we use the result of finite speed of propagation for SPME of the form (0.0) to prove that the random attractor associated to

\[
\begin{align*}
\frac{dX_t}{dt} &= \Delta \left( |X_t|^{m-1} \text{sgn}(X_t) \right) dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \quad \text{on } \mathcal{O}, \\
X(0) &= X_0, \quad \text{on } \mathcal{O},
\end{align*}
\]

with homogeneous Dirichlet boundary conditions and \( \lambda > 0 \) has infinite fractal dimension. First, we will prove the existence of an RDS corresponding to (3.12) in Proposition 3.1, then we will obtain the existence of an associated random attractor (Proposition 3.2) and provide lower bounds on its Kolmogorov \( \varepsilon \)-entropy (Theorem 3.3).

In the following we assume the driving signals \( z^{(k)} \) to be given as paths of a stochastic process with strictly stationary increments. More precisely, let \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \)
be a filtered probability space, \((z_t)_{t \in \mathbb{R}}\) be an \(\mathbb{R}^N\)-valued adapted stochastic process and \(((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{R}})\) be a metric dynamical system. For notions and results from the theory of RDS and random attractors we refer to [20, Section 1.2.1], [11][14][15][31]. We suppose:

1. (Strictly stationary increments) For all \(t, s \in \mathbb{R}, \omega \in \Omega\):
   \[ z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s \omega), \]
   where we assume \(z_0 = 0\) for notational convenience only.

2. (Regularity) \(z_t\) has continuous paths.

3. (Sublinear growth) \(z_t(\omega) = o(|t|)\) for \(t \to -\infty\), for all \(\omega \in \Omega\).

3.1. Generation of an RDS and existence of a random attractor. If we set \(f_{N+1} := \lambda\) and \(z_t^{(N+1)} := t\), then Proposition 2.5 implies the unique existence of a generalized weak solution \(X(\cdot, s; \omega)x\) with \(X(s, s; \omega)x = x\) for each \(s \in \mathbb{R}, x \in L^1(\Omega)\) and driving signals \(t \mapsto z_t^{(k)}(\omega)\). Recall that \(X(\cdot, s; \omega)x\) is defined to be a solution to (3.12) (resp. (0.0)) if

\[ Y(t, s; \omega)(e^{\mu_t(\omega) - \lambda s}x) := e^{\mu_t(\omega) - \lambda s}X(t, s; \omega)x, \quad t \in [s, \infty), \]

is a solution to \((2.8)\) with initial condition \(Y(s, s; \omega)(e^{\mu_s(\omega) - \lambda s}x) = e^{\mu_s(\omega) - \lambda s}x\). We set

\[ \varphi(t - s, \theta_s \omega)x := X(t, s; \omega)x, \quad t \geq s, \omega \in \Omega, x \in L^1(\Omega) \]

and note that in [20, Theorem 3.1] strict stationarity of \(z^{(k)}\) was only needed to prove the stochastic flow property for the solutions \(X(t, s; \omega)x\). Since the additional term \(\lambda X_t dt\) in (3.12) does not depend on time, the same proof as in [20, Theorem 3.1] still yields

**Proposition 3.1.** The map \(\varphi\) is a continuous RDS on \(X = L^1(\Omega)\) and thus a quasi-weakly-continuous RDS on each \(L^p(\Omega), p \in [1, \infty)\). In addition, \(\varphi\) is a quasi-weakly*-continuous RDS on \(L^{\infty}(\Omega)\). \(\varphi\) satisfies comparison, i.e. for \(x_1, x_2 \in X\) with \(x_1 \leq x_2\) a.e. in \(\Omega\)

\[ \varphi(t, \omega)x_1 \leq \varphi(t, \omega)x_2, \quad \text{a.e. in } \Omega. \]

Moreover, \(\varphi\) satisfies \(\varphi(t, \omega)0 = 0 = 0\) and

i. \(x \mapsto \varphi(t, \omega)x\) is Lipschitz continuous on \(X\), locally uniformly in \(t\).

ii. \(t \mapsto \varphi(t, \omega)x\) is continuous in \(x\).

In the following let \(\mathcal{D}\) be the universe of all random closed sets in \(X\).

As pointed out above we may rewrite (3.12) in the form of (2.8). From [20, Theorem 1.12, Theorem 3.1] we deduce that there is a piecewisely smooth function \(U(\omega) : (0, T] \to \mathbb{R}_+\) such that

\[ \|\varphi(t, \omega)x\|_{L^\infty(\Omega)} \leq U(t, \omega), \quad \forall (t, \omega) \in (0, T] \times \Omega. \]

Note that \(U\) does not depend on the initial condition \(x \in L^1(\Omega)\). This implies \(\mathcal{D}\)-bounded absorption for \(\varphi\) at time \(t = 0\) with absorbing set being bounded with respect to the \(\|\cdot\|_{L^\infty(\Omega)}\)-norm. Moreover, for each \(D \in \mathcal{D}\), \(\varphi(t, \omega)D\) is locally equicontinuous in \((0, T] \times \Omega\), i.e. \(\varphi(t, \omega)D = \{\varphi(t, \omega)x| x \in D\}\) is a set of equicontinuous functions on each compact set \(K \subseteq (0, T] \times \Omega\). This yields \(\mathcal{D}\)-asymptotic compactness for \(\varphi\) as in [20, Lemma 3.2]. We conclude:

**Proposition 3.2 (Existence of a random attractor).** The RDS \(\varphi\) has a \(\mathcal{D}\)-random attractor \(\mathcal{A}\) (as an RDS on \(L^1(\Omega)\)). \(\mathcal{A}\) is compact in each \(L^p(\Omega)\) and attracts all sets in \(\mathcal{D}\) in \(L^p\)-norm, \(p \in [1, \infty)\).

Moreover, \(\mathcal{A}(\omega)\) is a bounded set in \(L^{\infty}(\Omega)\) and the functions in \(\mathcal{A}(\omega)\) are equicontinuous on every compact set \(K \subseteq \Omega\).
3.2. Lower bounds on the Kolmogorov $\varepsilon$-entropy. We will now prove that the random attractor constructed in Proposition 3.2 has infinite fractal dimension in $L^1(\mathcal{O})$.

A precompact set $\mathcal{A} \subseteq X$ can be covered by a finite number of balls of radius $\varepsilon$ for each $\varepsilon > 0$. Let $N_\varepsilon(\mathcal{A})$ be the minimal number of such balls. Then, the Kolmogorov $\varepsilon$-entropy of $\mathcal{A}$ is defined by

$$H_\varepsilon(\mathcal{A}) := \log_2(N_\varepsilon(\mathcal{A})).$$

The fractal dimension of $\mathcal{A}$ is defined by

$$d_f(\mathcal{A}) = \limsup_{\varepsilon \to 0} \frac{H_\varepsilon(\mathcal{A})}{\log_2(\frac{1}{\varepsilon})}.$$ 

We obtain

Theorem 3.3 (Lower bounds on the Kolmogorov $\varepsilon$-entropy). Let $\mathcal{A}$ be the random attractor for $\varphi$ constructed in Proposition 3.2. Then, the Kolmogorov $\varepsilon$-entropy of $\mathcal{A}$ is bounded below by

$$H_\varepsilon(\mathcal{A}(\omega)) \geq C(\omega)\delta \frac{d(m-1)}{2(m+1)}, \quad \forall \omega \in \Omega,$$

where $C(\omega) > 0$ is a constant which may depend on $m, d$. In particular, the fractal dimension $d_f(\mathcal{A}(\omega))$ is infinite for all $\omega \in \Omega$.

Proof. The proof is inspired by [19] Theorem 4.1 and [21] Theorem 3.3. In order to prove the lower bound on the Kolmogorov $\varepsilon$-entropy we consider the unstable manifold of the equilibrium point 0 defined by

$$\mathcal{M}^+(0, \omega) := \{ u_0 \in X \mid \exists \text{ function } u : (-\infty, 0] \to X, \text{ such that } \varphi(t; \theta_{-t})u(-t) = u_0 \text{ for all } t \geq 0 \text{ and } \|u(t)\|_X \to 0 \text{ for } t \to -\infty \}.$$ 

Since $\mathcal{A}(\omega)$ attracts all deterministic sets we have

$$\mathcal{M}^+(0, \omega) \subseteq \mathcal{A}(\omega), \quad \forall \omega \in \Omega.$$ 

Therefore, it is sufficient to derive a lower bound on the Kolmogorov $\varepsilon$-entropy for the unstable manifold of 0.

In order to construct an element $u_0 \in \mathcal{M}^+(0, \omega)$ we need to find a function $u : (-\infty, 0] \to X$ converging to 0 for $t \to -\infty$ such that

$$u_0 = \varphi(t; \theta_{-t})u(-t) = X(0, -t; \omega)u(-t) = Y(0, -t; \omega) \left( e^{-\mu(\omega) + \lambda t}u(-t) \right), \quad \forall t \geq 0,$$

where we used (5.13). By defining $u(-t) := e^{-\mu(\omega) - \lambda t}v(-t)$, due to (S3) it is enough to find a bounded function $v : (-\infty, 0] \to X$ such that

$$u_0 = Y(0, -t; \omega)v(-t), \quad \forall t \geq 0.$$ 

We note that (2.3) in case of (3.12) reads

$$\partial_t Y(t, s; \omega)x = e^{\mu(\omega) - \lambda t} \Delta \Phi(e^{-\mu(\omega) + \lambda t}Y(t, s; \omega)x),$$

$$Y(s, s; \omega)x = x,$$

for a.e. $t \geq s$. For $x \in L^\infty(\mathcal{O})$ let $Y(t, s; \omega)x \in C((0, \infty) \times \mathcal{O})$ denote the corresponding essentially bounded, weak solution to (3.15) given by Proposition A.2.

In order to find a function $v$ satisfying (3.14), we use a time scaling to transform (5.15) from the infinite time interval $(-\infty, 0]$ into a PDE on a finite time interval. Let $\delta > 0$ small enough such that $(m - 1)\lambda - \delta > 0$ and set $\eta := \frac{(m - 1)\lambda - \delta}{m + 1}$. Then (5.15) may be rewritten as

$$\partial_t Y(t, s; \omega)x = e^{\delta t} e^{\mu(\omega) - \mu t} \Delta \Phi(e^{-\mu(\omega) + \mu t}Y(t, s; \omega)x).$$
We define $T = \frac{1}{\delta}$ and
\[ F(t) := \frac{e^{\delta t}}{\delta}: (-\infty, 0] \to (0, T], \]
\[ G(t) = F^{-1}(t) = \frac{\log(\delta t)}{\delta}: (0, T] \to (-\infty, 0]. \]

We note $G \in C^1(0, T]$ with $G'(t) > 0$, $G(T) = 0$ and $G(t) \to -\infty$ for $t \to 0$. Let
\[ U(t, s; \omega)x := Y(G(t), G(s); \omega)x \text{ for } t \geq s, t, s \in (0, T]. \]
Then $U(t, s; \omega)x$ is a weak solution to
\begin{equation}
(3.16) \quad \partial_t U(t, s; \omega)x = e^{\mu G(t) + \eta G(t)} \Delta \Phi(e^{-\mu G(t) + \eta G(t)} U(t, s; \omega)x), \quad \text{on } [s, \infty) \times \mathcal{O}.
\end{equation}

The rigorous proof of this transformation proceeds by considering a non-degenerate approximation $\Phi^{(\delta)}(r) := \Phi(r) + \delta r$ and smoothed coefficients $\mu^{(\delta)}$. In this case the transformation is a direct consequence of the classical chain-rule. One may then use local equicontinuity and uniform boundedness of the approximating solutions $Y^{(\delta)}$ to pass to the limit.

Thus, we can solve (3.16) on each interval $[\tau, T]$ with $\tau > 0$. In order to construct the required function $v : (-\infty, 0] \to X$ we aim to solve (3.16) on the whole interval $[0, T]$. Let $\rho_1(t) := e^{\mu G(t) + \eta G(t)}$, $\rho_2(t) := e^{-\mu G(t) + \eta G(t)}$. Due to condition (S3), for each $\varepsilon > 0$ there is a $t_0(\varepsilon) < 0$ small enough, such that
\[ \|\mu G(t)\|_{C^\infty(\mathcal{O})} \leq \varepsilon \left( \sum_{k=1}^N \|f_k\|_{C^\infty(\mathcal{O})} \right) |G(t)|, \quad \forall t \leq t_0(\varepsilon), \ n \in \mathbb{N}. \]

Choosing $\varepsilon > 0$ small enough we thus obtain
\[ \|\rho_1(t)\|_{C^\infty(\mathcal{O})} \leq e^{\|\mu G(t)\|_{C^\infty(\mathcal{O})} + \eta G(t)} P(\|\mu G(t)\|_{C^\infty(\mathcal{O})}) \to 0, \quad \text{for } t \to 0, \]
for some polynomial $P$. Similarly,
\[ \frac{|\partial_{\xi_{i_1} \cdots \xi_{i_n}} \rho_1(t)|^2}{\rho_1(t)^2} \leq \rho_1(t)^2 P(\|\mu G(t)\|_{C^\infty(\mathcal{O})}) \leq e^{\|\mu G(t)\|_{C^\infty(\mathcal{O})} + \eta G(t)} P(\|\mu G(t)\|_{C^\infty(\mathcal{O})}) \to 0, \quad \text{for } t \to 0, \]
for all $i_1, \ldots, i_n \in \{1, \ldots, d\}$. The same reasoning applies for $\rho_2$. In particular, $\rho_1, \rho_2 \in C^{0,n}([0, T] \times \mathcal{O})$ for all $n \in \mathbb{N}$. Hence, (3.10) is of the form (3.17) and Proposition 3.2 implies the existence of a very weak solution
\[ U(\cdot, 0; \omega)x \in L^\infty([0, T] \times \mathcal{O}) \cap C((0, T) \times \mathcal{O}) \]
with homogeneous Dirichlet boundary conditions for each initial condition $x \in L^\infty(\mathcal{O})$.

Reversing the time transformation we define
\[ v(t) := U(F(t), 0; \omega)x, \quad t \in (-\infty, 0]. \]

Uniqueness of essentially bounded, very weak solutions to (3.10) (Theorem 3.3) implies
\[ U(t, t; \omega)x = U(t, r; \omega)U(r, s; \omega)x, \quad \forall 0 \leq s \leq r \leq t \leq T. \]

Hence,
\[ v(0) = U(F(0), 0; \omega)x = U(F(0), F(s); \omega)U(F(s), 0; \omega)x = U(F(0), F(s); \omega)v(s) = Y(0, s; \omega)v(s), \]
for all $s < 0$. Consequently, $v(0) \in M^+([0, \omega])$ for each $x \in L^\infty(\mathcal{O})$.

In order to use this construction of elements $v(0) \in M^+([0, \omega])$ to derive a lower bound on the Kolmogorov $\varepsilon$-entropy of $M^+([0, \omega])$ we consider solutions to (3.16).
so that the final values \( v(0) = U(F(0), 0; \omega)x \) are sufficiently far apart (w.r.t. the \( L^1 \)-norm): For \( \varepsilon > 0 \) small enough we can find a finite set \( R_\varepsilon = \{ \xi_i \} \subseteq \Omega \) such that

\[
B(\varepsilon, \xi_i) \cap B(\varepsilon, \xi_j) = \emptyset, \quad \text{for} \ i \neq j,
\]

\[
|R_\varepsilon| \geq C\varepsilon^{-d},
\]

\[
\bar{B}(\varepsilon, \xi_i) \subseteq \Omega, \quad \forall i.
\]

Let \( x_0^i := M_{B(\varepsilon, \xi_i)} \) and \( M = (m\varepsilon)^{\frac{1}{d+1}} \), where \( m > 0 \) will be specified below. By Proposition \text{A.2}.

\[
H^i := ||U^i(\cdot, 0; \omega)x||_{L^\infty(\Omega_T)} \leq C||x_0^i||_{L^\infty(\Omega)} \leq C(m\varepsilon)^{\frac{1}{d+1}}.
\]

Thus, the bound on the rate of expansion of the support of \( U^i \) given in Theorem \text{A.6} becomes

\[
\text{supp}(U_t^i) \subseteq B_{C\varepsilon m \sqrt{\varepsilon T} \parallel \varepsilon}} \subseteq B_{C\varepsilon m \sqrt{\varepsilon T} + \bar{\varepsilon}}(\xi_i), \quad \forall t \in [0, T],
\]

where \( t \mapsto C_t \) is a continuous function. Thus, choosing \( m \) small enough yields

\[
\text{supp}(U_t^i) \subseteq B_{2}(\xi_i), \quad \forall t \in [0, T].
\]

Hence, \( U^i, U^j \) have disjoint support on \([0, T]\). Therefore, also

\[
U^m(t, \xi) = \sum_{i=1}^{\lfloor |R_\varepsilon| \rfloor} m_i U^i(t, \xi),
\]

for each \( m \in \{0, 1\}^{\lfloor |R_\varepsilon| \rfloor} \) is a very weak solution to \( (3.16) \) with homogeneous Dirichlet boundary conditions. For \( m^1 \neq m^2 \) let \( i \) such that \( m_1^i \neq m_2^i \). By Proposition \text{A.4} we observe

\[
||U^{m^1}(T) - U^{m^2}(T)||_{L^1(\Omega)} \geq ||U^i(T)||_{L^1(\Omega)} \geq e^{-CT}||U^i(0)||_{L^1(\Omega)} \geq C e^{-CT} \varepsilon^{-\frac{1}{d+1}-1}.
\]

Hence,

\[
\mathbb{H}_\delta(A(\omega)) \geq \mathbb{H}_\delta(M^+(0, \omega)) \geq \log_2 2^{\lfloor |R_\varepsilon| \rfloor} \geq C(\omega)\delta^{-\frac{d(d+1)}{2(d+1)(d-1)}},
\]

and

\[
d_f(A(\omega)) \geq d_f(M^+(0, \omega)) = \lim_{\delta \rightarrow 0} \frac{\mathbb{H}_\delta(M^+(0, \omega))}{\log_2(\frac{1}{\delta})} = \infty.
\]

\[\Box\]

**Appendix A. Finite Speed of Propagation for More General Perturbations**

In Section \text{2.1} we proved finite speed of propagation for \( (1.6) \) via the transformed equation \( (2.8) \). The precise structure of the spatially dependent perturbing factors \( e^\rho, e^{-\rho} \) has been used to provide explicit and locally optimal bounds on the rate of hole-filling. By disregarding the optimality of the estimates, more general perturbations may be allowed. Such an extension of the results of Section \text{2.1} is required in Section \text{3} in order to prove lower bounds for the Kolmogorov \( \varepsilon \)-entropy of the random attractor. In this section we provide some details on the proof of finite speed of propagation for more general perturbing factors. We consider the homogeneous Dirichlet problem for

\[
\partial_t Y_t = \rho_1 \Delta \Phi(p_2 Y_t), \quad \text{on} \ \Omega_T
\]

\[
Y(0) = Y_0, \quad \text{on} \ \Omega,
\]

\[\text{(A.17)}\]
where $\rho_1, \rho_2 \in C^{0,2}(\bar{O}_T)$ are non-negative. (Local, generalized, very) weak solutions to (A.17) are defined analogously to Definition 2.1. In particular, a function $Y \in L^1(\bar{O}_T)$ with $\Phi(\rho_2 Y) \in L^1(\bar{O}_T)$ satisfying

$$
\int_{\partial T} Y \delta_t \eta \, d\xi dr + \int_{\bar{O}} Y \eta_0 \, d\xi \geq -\int_{\partial T} \Phi(\rho_2 Y) \Delta(\rho_1 \eta) \, d\xi dr
$$

(A.18)

for all non-negative $\eta \in C^{1,2}(\bar{O}_T)$ with $\eta \rho_T = 0$ and for some functions $Y_0 \in L^1(\bar{O})$, $\Phi(g) \in L^1(\Sigma_T)$ is said to be a very weak subsolution to the (inhomogeneous) Cauchy-Dirichlet problem to (A.17).

A.1. Existence of very weak solutions to (A.17). Let $Y_0 \in L^\infty(\bar{O})$. We will only require the existence of solutions to (A.17) with homogeneous Dirichlet boundary conditions (i.e. $g \equiv 0$) and for $\rho_1, \rho_2$ satisfying one of the following conditions

(A1) $\rho_2$ is strictly positive on $[0, T] \times \bar{O}$,
(A2) $\rho_2$ is strictly positive on $(0, T] \times \bar{O}$ and $\|\rho_2(t)\|_{C^2(\bar{O})} \to 0$ for $t \to 0$.

The construction of solutions for (A.17) relies on a smooth, non-degenerate approximation of $\Phi$. I.e. for $\delta > 0$ let

$$
\Phi^{(\delta)}(r) := \Phi(r) + \delta r,
$$

$\rho_1^{(\delta)}, \rho_2^{(\delta)} \in C^\infty(\bar{O}_T)$ be approximations of $\rho_1, \rho_2$ in $C^{0,2}(\bar{O}_T)$ and let $Y_0^{(\delta)} \in C^\infty(\bar{O})$ be smooth approximations of $Y_0$ in $L^\infty(\bar{O})$. We consider the approximating problems

$$
\partial_t Y_t^{(\delta)} = \rho_1^{(\delta)} \Delta \left( \Phi(\rho_2^{(\delta)}) \Phi^{(\delta)}(Y_t^{(\delta)}) \right), \quad \text{on } \bar{O}_T
$$

(A.19)

$$
Y_0^{(\delta)} = Y_0, \quad \text{on } \bar{O},
$$

with homogeneous Dirichlet boundary conditions. Since (A.19) is a non-degenerate, quasilinear PDE with smooth coefficients, standard results imply the unique existence of a classical solution $Y^{(\delta)}$ (cf. e.g. [20]).

The main ingredient of the construction of solutions to (A.17) is the following a-priori $L^\infty$ bound

**Lemma A.1.** Let $M := \|Y_0\|_{L^\infty(\bar{O})} < \infty$ and assume (A1) or (A2). Then, there are constants $C, \delta_0 = \delta_0(M) > 0$ such that

$$
\sup_{\delta \in [0, \delta_0]} \|Y^{(\delta)}\|_{C^\infty([0, T] \times \bar{O})} \leq C\|Y_0\|_{L^\infty(\bar{O})} < \infty.
$$

**Proof.** Case (A1): The proof relies on a combination of explicit supersolutions to (13) with an interval splitting technique as it has been used in [9, 20]. In the following let $\varphi \in C^2(\bar{O})$ be the solution to

$$
\Delta \varphi = -1, \quad \text{on } \bar{O}
$$

$$
\varphi = 1, \quad \text{on } \partial O.
$$

By the maximum principle we have $\varphi \geq 1$.

Since $\{\rho_2^{(\delta)}\}_{\delta \in [0, 1]}$ is a compact set in $C^{0,2}(\bar{O}_T)$ and may be chosen such that

$$
\inf_{\delta \in [0, 1], \ (t, \xi) \in [0, T] \times \bar{O}} \rho_2^{(\delta)}(t, \xi) > 0,
$$

we have

$$
\bar{\eta}_t^{(\delta)} := \Phi \left( \frac{\rho_2^{(\delta)}}{\rho_2^{(\delta)}(t_\delta)} \right) \in C^{0,2}(\bar{O}_T)
$$
with \( \eta^{(\delta)}(t) \to 1 \) in \( C^2(\mathcal{O}) \) for \( t \to t_1 \) uniformly in \( \delta \in [0, 1] \) and \( t_1 \in [0, T] \). Hence,

\[
\Delta(\varphi \eta^{(\delta)}_i) = -\eta^{(\delta)}_i + 2 \nabla \varphi \cdot \nabla \eta^{(\delta)}_i + \varphi \Delta \eta^{(\delta)}_i \leq -\frac{1}{2}, \quad \forall \xi \in \mathcal{O}, \ \delta \in [0, 1]
\]

and all \( |t - t_1| \) small enough. We can thus choose a finite partition \( 0 = t_0 < t_1 < t_2 < \ldots < t_N = T \) of \([0, T]\) such that

\[
\sup_{\delta \in [0, 1]} \Delta \left( \varphi \Phi \left( \frac{\rho_1^{(\delta)}}{\rho_2^{(\delta)}(t_i)} \right) \right) \leq -\frac{1}{2}, \quad \text{on } [t_i, t_{i+1}] \times \mathcal{O},
\]

for all \( i = 0, \ldots, N - 1 \).

We will prove the bound iteratively over \( i = 0, \ldots, N - 1 \). Suppose the bound has been shown on \([0, t_i]\) for some \( i \geq 0 \) and let \( ||Y_{t_i}||_{L^\infty(\mathcal{O})} \leq C_i M \). Choosing

\[
K^{(i)}(t, \xi) := \varphi(\xi) \frac{\rho_2^{(\delta)}}{\rho_2^{(\delta)}(t_i, \xi)},
\]

we have \( K^{(i)}(t_1, \xi) \geq ||Y_{t_i}||_{L^\infty(\mathcal{O})}, \ \partial_t K^{(i)} = 0 \) and

\[
\begin{align*}
\rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} \rho_1^{(\delta)}(K^{(i)}) \right) \right) &= \rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} K^{(i)} \right) \right) + \delta \rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} K^{(i)} \right) \right) \\
&\leq \|\rho_2^{(\delta)}\|_{C^{\alpha,2}(\mathcal{O}_R)} C_i M \rho_1^{(\delta)} \left( -\frac{1}{2} \|\rho_2^{(\delta)}\|^1_{C^{\alpha,1}(\mathcal{O}_R)} (C_i M)^m - \delta \Delta \left( \Phi \left( \rho_2^{(\delta)} \varphi \right) \right) \right) \\
&\leq 0,
\end{align*}
\]

by the choice of the partition \( \{\tau_i\}_{i=0, \ldots, N} \), for all \( \delta \leq \delta_0(M) \) small enough.

Consequently, \( K^{(i)} \) is a supersolution to (A.19) on \([\tau_i, \tau_{i+1}] \times \mathcal{O}\) and the upper bound follows since \( K^{(i)}(t, \xi) \leq C_{i+1} M \), with \( C_{i+1} \) depending on the data only. The derivation of the lower bound proceeds analogously.

Case (A2): We only need to prove the claim on some small interval \([0, \tau_1]\) with \( \tau_1 > 0 \), since case (A1) may be applied on \([\tau_1, T]\) subsequently. Choose \( \tau_1 \in (0, T] \) such that

\[
\sup_{\delta \in [0, 1]} \Delta \left( \varphi \Phi \left( \frac{\rho_2^{(\delta)}}{\rho_2^{(\delta)}(t_1)} \right) \right) \leq -\frac{1}{2}, \quad \text{on } [0, \tau_1] \times \mathcal{O}.
\]

This is possible since \( ||\rho_2(t)||_{C^{1,1}(\mathcal{O})} \to 0 \) for \( t \to 0 \) by assumption. Let \( K^{(0)}(t, \xi) := \varphi(\xi) \frac{\rho_2}{\rho_2(\tau_1)} M \). Then \( \partial_t K = 0 \) and

\[
\begin{align*}
\rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} \rho_1^{(\delta)}(K^{(i)}) \right) \right) &= \rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} K^{(i)} \right) \right) + \delta \rho_1^{(\delta)} \Delta \left( \Phi \left( \rho_2^{(\delta)} K^{(i)} \right) \right) \\
&= M \rho_1^{(\delta)} \left( M^{-\delta} - \delta \Delta \left( \Phi \left( \rho_2^{(\delta)} \varphi \right) \right) \right) \leq 0,
\end{align*}
\]

on \([0, \tau_1] \times \mathcal{O}\) for \( \delta \leq \delta_0(M) \) small enough. Hence, \( K^{(0)} \) is a supersolution to (A.19) on \([0, \tau_1] \times \mathcal{O}\) and

\[
Y^{(\delta)} \leq K^{(0)} \leq CM, \quad \text{on } [0, \tau_1] \times \mathcal{O}.
\]

The lower bound may be derived analogously. \( \square \)

**Proposition A.2 (Existence of very weak solutions to (A.19)).** Let \( Y_0 \in L^\infty(\mathcal{O}) \) and assume (A1) or (A2). Then, there exists a very weak solution \( Y \in C((0, T] \times \mathcal{O}) \) to (A.17) with Dirichlet boundary conditions satisfying

\[
Y_{L^\infty(\mathcal{O}_R)} \leq C ||Y_0||_{L^\infty(\mathcal{O})},
\]

for some constant \( C > 0 \).
Proof. Based on the uniform \( L^\infty \) estimate for the approximating solutions \( Y^{(i)} \) derived in Lemma A.1, we obtain local equicontinuity of \( Y^{(i)} \) in \( \mathcal{O} \) by [18] (cf. also [20] Theorem 1.12). I.e. \( Y^{(i)} \in C(K) \) for each compact set \( K \subseteq (0, T] \times \mathcal{O} \) with modulus of continuity independent of \( \delta > 0 \).

By a diagonal argument it follows that there exists a \( Y \in C((0, T] \times \mathcal{O}) \) with \( \| Y \|_{L^\infty(\mathcal{O}_T)} \leq C\| Y_0 \|_{L^\infty(\mathcal{O})} \) such that \( Y^k \to Y \) (passing to a subsequence if necessary) locally uniformly on \( \mathcal{O} \). By dominated convergence, this implies that \( Y \) is a very weak solution to \( (A.17) \). \( \square \)

A.2. Comparison and uniqueness for \((A.17)\). We now prove a comparison result for \((A.17)\). In particular, this implies Theorem 2.4 since sub/supersolutions to \((0.0)\) are defined in terms of solutions to \((2.8)\) and thus it is enough to prove the comparison result for \((2.8)\) and \((A.17)\). We will assume either of

\( (A1') \) \( \rho_1 \) is strictly positive on \([0, T] \times \mathcal{O} \),
\( (A2') \) \( \rho_1 \) is strictly positive on \([0, T] \times \mathcal{O} \) and

\[ \left\| \frac{\nabla \rho_1(t)}{\rho_1(t)} \right\|_{C^0(\mathcal{O})} + \left\| \frac{\Delta \rho_1(t)}{\rho_1(t)} \right\|_{C^0(\mathcal{O})} \to 0, \text{ for } t \to 0. \]

**Theorem A.3** (Comparison for very weak solutions). Let \( Y^{(1)}, Y^{(2)} \) be essentially bounded sub/supersolutions to \((A.17)\) with initial conditions \( Y^{(1)}_0 \leq Y^{(2)}_0 \) and boundary data \( g^{(1)} \leq g^{(2)} \) a.e. in \( \mathcal{O} \) respectively. Assume either \((A1')\) or \((A2')\). Then,

\( Y^{(1)} \leq Y^{(2)}, \text{ a.e. in } \mathcal{O} \).

In particular, essentially bounded, very weak solutions are unique.

**Proof.** The proof proceeds similar to [20] Theorem 1.3. Let \( Y^{(1)}, Y^{(2)} \) be as in the statement, \( Y := Y^{(1)} - Y^{(2)} \) and \( g := g^{(1)} - g^{(2)} \). Then

\[ \int_{\mathcal{O}_T} Y \partial_\nu \eta \ d\xi dr \]
\[ \geq - \int_{\mathcal{O}} (Y^{(1)}_0 - Y^{(2)}_0) \eta_0 d\xi - \int_{\mathcal{O}_T} \left( \Phi(\rho_2 Y^{(1)}) - \Phi(\rho_2 Y^{(2)}) \right) \Delta(\rho_1 \eta) \ d\xi dr \]
\[ + \int_{\Sigma_T} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu(\rho_1 \eta) d\nu dr \]
\[ \geq - \int_{\mathcal{O}} Y^{(1)}_0 \eta_0 d\xi - \int_{\mathcal{O}_T} aY \Delta(\rho_1 \eta) \ d\xi dr + \int_{\Sigma_T} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu(\rho_1 \eta) d\nu dr, \]

for all non-negative \( \eta \in C^{1,2}(\bar{\mathcal{O}}_T) \) with \( \eta = 0 \) on \( \mathcal{P}_T \), where

\[ a_T := \begin{cases} \frac{\Phi(\rho_2(t) Y^{(1)}_t) - \Phi(\rho_2(t) Y^{(2)}_t)}{Y^{(1)}_t - Y^{(2)}_t} & , \text{for } Y^{(1)}_t \neq Y^{(2)}_t \\ 0 & , \text{otherwise.} \end{cases} \]

**Case \((A1')\):** Let \( \rho_1^{(c)} \in C^\infty(\mathcal{O}_T) \) be a smooth approximation of \( \rho_1 \) in \( C^{0,2}(\mathcal{O}_T) \), such that \( \| \rho_1^{(c)} - \rho_1 \|_{C^{0,2}(\mathcal{O}_T)} \leq \varepsilon^2 \). By equicontinuity of \( t \mapsto \rho_1^{(c)}(t) \) in \( C^2(\mathcal{O}) \) we can choose a partition \( 0 = \tau_0 < \ldots < \tau_N = T \) such that

\[ \text{(A.20)} \]
\[ C_1 \| \rho_1^{(c)}(\tau_i) \|_{C^0(\mathcal{O})} \left( \left\| \nabla \left( \frac{\rho_1^{(c)}}{\rho_1(\tau_i)} \right) \right\|_{C^0(\tau_i, \tau_i+1) \times \mathcal{O})}^2 + \left\| \Delta \left( \frac{\rho_1^{(c)}}{\rho_1(\tau_i)} \right) \right\|_{C^0(\tau_i, \tau_i+1) \times \mathcal{O})}^2 \right) \]
\[ \leq C \frac{\varepsilon}{4}, \forall i = 0, \ldots, N-1, \varepsilon > 0, \]

where \( c, C_1 > 0 \) are constants that will be specified below (depending on \( \| a \|_{L^\infty(\mathcal{O}_T)} \) only). Let \( \gamma := \max_{i=0, \ldots, N-1} |\tau_{i+1} - \tau_i| \).
We prove $Y \leq 0$ a.e. via induction over $i = 0, \ldots, N - 1$. Thus, assume $Y \leq 0$ on $[0, \tau_i] \times \Omega$ almost everywhere. We can modify $\tau_i$ so that (A.20) is preserved and $Y(\tau_i) \leq 0$ a.e. in $\Omega$. Define $\Omega_i := [\tau_i, \tau_{i+1}] \times \Omega$, $\Sigma_i = [\tau_i, \tau_{i+1}] \times \partial\Omega$, $\mathcal{P}_i = \Sigma_i \cup (\{Y\} \times \Omega)$. Then

$$
\int_{\Omega_i} Y(\partial_r \eta + a \Delta(\rho_1 \eta))\, d\xi d\tau \geq -\int_{\Omega_i} Y \eta, \eta_r d\xi + \int_{\Sigma_i} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_r (\rho_1 \eta) d\nu d\tau,
$$

for all non-negative $\eta \in C^{1,2}([\tau_i, \tau_{i+1}] \times \bar{\Omega})$ with $\eta = 0$ on $\mathcal{P}_i$. Since $\eta \geq 0$ on $\Omega_i$, we have $\partial_r (\rho_1 \eta) \leq 0$ on $\Sigma_i$ and thus

$$
-\int_{\Omega_i} Y \eta, \eta_r d\xi + \int_{\Sigma_i} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_r (\rho_1 \eta) d\nu d\tau \geq 0.
$$

We conclude,

$$
\int_{\Omega_i} Y(\partial_r \eta + a \Delta(\rho_1 \eta))\, d\xi d\tau \geq 0,
$$

for all non-negative $\eta \in C^{1,2}([\tau_i, \tau_{i+1}] \times \bar{\Omega})$ with $\eta = 0$ on $\mathcal{P}_i$.

For $Y^{(1)}_t \neq Y^{(2)}_t$ we have $a_t = \rho_2(t) \Phi(\zeta_t)$ with $\zeta_t \in [\rho_2(t)Y^{(1)}_t, \rho_2(t)Y^{(2)}_t]$ and thus $\|a\|_{L^\infty(\Omega_t)} < \infty$ by essential boundedness of $Y^{(1)}$. We consider a non-degenerate, smooth approximation of $a$. Set $\bar{a}_\varepsilon := a \vee \varepsilon$ and let $a_{\varepsilon, \delta}$ be a smooth approximation of $\bar{a}_\varepsilon$ such that $a_{\varepsilon, \delta} \geq \varepsilon$ and $\int_{\Omega_t} |Y|^2 (a_{\varepsilon, \delta})^2\, d\xi d\tau < \delta$. Then choose $a_{\varepsilon} = a_{\varepsilon, \varepsilon^2}$.

Let $\eta = \frac{\bar{a}_\varepsilon^2}{\rho_1^{(1)}(\tau_i)} \in C^{0,2}(\Omega_i)$ with $\varphi$ being the classical solution to

$$
\begin{align*}
\partial_t \varphi + a_{\varepsilon} \rho_1^{(c)}(\tau_i) \Delta \left( \frac{\rho_1^{(c)}}{\rho_1^{(1)}(\tau_i)} \varphi \right) - \theta &= 0, &\text{on } \Omega_i \\
\varphi &= 0, &\text{on } [\tau_i, \tau_{i+1}] \times \partial \Omega \\
\varphi(\tau_{i+1}) &= 0, &\text{on } \Omega,
\end{align*}
$$

(A.21)

where $\theta$ is an arbitrary, non-positive, smooth testfunction and for simplicity of notation we suppress the $\varepsilon$-dependency of $\varphi$. Time inversion transforms (A.21) into a uniformly parabolic linear equation with smooth coefficients. Thus, unique existence of a non-negative classical solution follows from standard results (cf. e.g. [26]).

Consequently,

$$
\begin{align*}
0 &\leq \int_{\Omega_i} Y(\partial_r \eta + a \Delta(\rho_1 \eta))\, d\xi d\tau \\
&= \int_{\Omega_i} Y(\partial_r \eta + a_{\varepsilon} \Delta(\rho_1^{(c)} \eta))\, d\xi d\tau + \int_{\Omega_i} Y(a - a_{\varepsilon}) \Delta(\rho_1^{(c)} \eta)\, d\xi d\tau \\
&\quad + \int_{\Omega_i} Ya \Delta((\rho_1 - \rho_1^{(c)}) \eta)\, d\xi d\tau \\
&= \int_{\Omega_i} \frac{1}{\rho_1^{(c)}(\tau_i)} Y \theta\, d\xi d\tau + \int_{\Omega_i} Y(a - a_{\varepsilon}) \Delta \left( \frac{\rho_1^{(c)}}{\rho_1^{(1)}(\tau_i)} \varphi \right)\, d\xi d\tau \\
&\quad + \int_{\Omega_i} Ya \Delta \left( \frac{\rho_1 - \rho_1^{(c)}}{\rho_1^{(c)}(\tau_i)} \varphi \right)\, d\xi d\tau.
\end{align*}
$$

(A.22)
We need to prove that the last two terms vanish for $\varepsilon \to 0$. We note

$$
\int_{\Omega_t} Y(a - a_\varepsilon) \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \varphi \right) \, d\xi \, dr
$$

(A.23)

$$
\leq C \left( \int_{\Omega_t} a_\varepsilon \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right)^2 \, d\xi \, dr \right)^{\frac{1}{2}} \sqrt{\varepsilon}
$$

and

$$
\int_{\Omega_t} Y a \Delta \left( \frac{\rho_1^{(e)} - \rho_1^{(e)}(\tau_i)}{\rho_1^{(e)}(\tau_i)} \varphi \right) \, d\xi \, dr \leq C \left\| \frac{\rho_1^{(e)} - \rho_1^{(e)}(\tau_i)}{\rho_1^{(e)}(\tau_i)} \right\|_{H^2(\Omega_t)} \| \varphi \|_{H^2(\Omega_t)}
$$

(A.24)

\[ \leq C \varepsilon^2 \| \varphi \|_{H^2(\Omega_t)}. \]

Therefore, we first derive a bound for $\| \varphi \|_{H^2(\Omega_t)}$ with explicit control on the possible explosion for $\varepsilon \to 0$. Let $\zeta \in C^\infty(\mathbb{R})$ with $\zeta(\tau_i) = 0$, $\zeta \leq 1$ on $[0, T]$ and $\zeta \geq c > 0$, for some $c \leq \frac{1}{\tilde{c}}$. Multiplying (A.21) by $\zeta \Delta \varphi$ and integrating yields

$$
\int_{\Omega_t} (\partial_t \zeta) \varphi \, d\xi \, dr
$$

(A.25)

\[ = \int_{\Omega_t} \left( -a_\varepsilon \rho_1^{(e)}(\tau_i) \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \varphi \right) \zeta \Delta \varphi + \theta \zeta \Delta \varphi \right) \, d\xi \, dr.
\]

We compute

$$
- \int_{\Omega_t} a_\varepsilon \rho_1^{(e)}(\tau_i) \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \varphi \, d\xi \, dr
$$

$$
= - \int_{\Omega_t} \zeta a_\varepsilon \rho_1^{(e)}(\tau_i) \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) |\Delta \varphi|^2 \, d\xi \, dr
$$

$$
+ \int_{\Omega_t} \zeta a_\varepsilon \rho_1^{(e)}(\tau_i) \left( 2 \nabla \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \nabla \varphi + \varphi \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \right) \Delta \varphi \, d\xi \, dr
$$

\[ \leq - \frac{1}{4} \int_{\Omega_t} \zeta a_\varepsilon \rho_1^{(e)}(\tau_i) |\Delta \varphi|^2 \, d\xi \, dr
\]

$$
+ C_1 \| \rho_1^{(e)}(\tau_i) \|_{C^0(\Omega_t)} \left\| \nabla \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \right\|_{C^1(\Omega_t)}^2 \int_{\Omega_t} |\nabla \varphi|^2 \, d\xi \, dr
$$

$$
+ C_1 \| \rho_1^{(e)}(\tau_i) \|_{C^0(\Omega_t)} \left\| \Delta \left( \frac{\rho_1^{(e)}}{\rho_1^{(e)}(\tau_i)} \right) \right\|_{C^0(\Omega_t)}^2 \int_{\Omega_t} |\varphi|^2 \, d\xi \, dr
$$

\[ \leq - \frac{1}{4} \int_{\Omega_t} \zeta a_\varepsilon \rho_1^{(e)}(\tau_i) |\Delta \varphi|^2 \, d\xi \, dr + \frac{c}{4} \int_{\Omega_t} |\nabla \varphi|^2 \, d\xi \, dr,
\]

where we use (A.20). Using this in (A.25) together with the arbitrariness of $\zeta$ with the above properties, Fatou’s Lemma and strict positivity of $\rho_1^{(e)}(\tau_i)$ we deduce

$$
\frac{c}{2} \int_{\Omega_t} |\nabla \varphi|^2 \, d\xi \, dr + \frac{1}{4} \int_{\Omega_t} a_\varepsilon |\varphi|^2 \, d\xi \, dr \leq C \int_{\Omega_t} |\nabla \varphi|^2 \, d\xi \, dr
$$

\[ + \frac{c}{4} \int_{\Omega_t} |\nabla \varphi|^2 \, d\xi \, dr,
\]
and \( \|\varphi\|_{H^2(\Omega)} \leq \frac{C}{\varepsilon} \int_{\Omega} |\nabla \theta|^2 \varepsilon \) due to \( a_\varepsilon \geq \varepsilon \). For (A.23) this implies
\[
\int_{\Omega_1} Y(a - a_\varepsilon) \Delta \left( \frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_1)} \varphi \right) \, d\xi \, dr \leq C \|\theta\|^2_{H^2(\Omega_1)} \sqrt{\varepsilon},
\]
for (A.24)
\[
\int_{\Omega_1} Y a \Delta \left( \frac{\rho_1 - \rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_1)} \varphi \right) \, d\xi \, dr \leq C \varepsilon \|\theta\|^2_{H^2(\Omega_1)}.
\]
Taking \( \varepsilon \to 0 \) in in (A.22) thus yields
\[
0 \leq \int_{\Omega_1} \frac{1}{\rho_1^{(\varepsilon)}(\tau_1)} Y \theta \, d\xi \, dr,
\]
for any non-positive, smooth testfunction \( \theta \). Thus \( Y^{(1)} \leq Y^{(2)} \) in \( \Omega_t = [\tau_t, \tau_{t+1}] \times \Omega \) almost everywhere. Induction finishes the proof.

Case (A2'): It is sufficient to prove comparison for a short time-interval \([0, \tau_1]\) for some \( \tau_1 > 0 \), since case (A1') may be applied on \([\tau_1, T]\) subsequently. Let \( 0 = \tau_0 < \tau_1 \). As for case (A1') we note
\[
\int_{\Omega_0} Y(\partial_t \eta + a_\varepsilon \Delta(\rho_1^{(\varepsilon)})) \, d\xi \, dr \geq 0,
\]
for all non-negative \( \eta \in C^{1,2}([0, \tau_1] \times \bar{\Omega}) \) with \( \eta = 0 \) on \( \partial \Omega \).

We follow the same idea of prove as in the case of (A1'). Hence, let \( a^{(\varepsilon)}, \rho_1^{(\varepsilon)} \) be smooth approximations as before and \( \varphi \) be the classical solution to
\[
\partial_t \varphi + a_\varepsilon \Delta(\rho_1^{(\varepsilon)} \varphi) - \theta = 0, \quad \text{on } \Omega_0
\]
(\( A.26 \))
\[
\varphi = 0, \quad \text{on } [0, \tau_1] \times \partial \Omega
\]
\[
\varphi(\tau_1) = 0, \quad \text{on } \Omega,
\]
where \( \theta \) is an arbitrary, non-negative, smooth testfunction. As for (A.22) this yields
\[
0 \leq \int_{\Omega_0} Y \theta \, d\xi \, dr + \int_{\Omega_0} Y(a - a_\varepsilon) \Delta(\rho_1^{(\varepsilon)} \varphi) \, d\xi \, dr + \int_{\Omega_0} Y a \Delta((\rho_1 - \rho_1^{(\varepsilon)}) \varphi) \, d\xi \, dr.
\]
We thus aim to show that the last two terms vanish for \( \varepsilon \to 0 \). Due to the degeneracy of \( \rho_1(t) \) for \( t \to 0 \) care has be taken in establishing the required a-priori bound on \( \varphi \). Multiplying (A.26) by \( \zeta \Delta \varphi \) as before and noting
\[
\Delta(\rho_1^{(\varepsilon)} \varphi) = \rho_1^{(\varepsilon)} \Delta \varphi + 2 \nabla \rho_1^{(\varepsilon)} \cdot \nabla \varphi + \varphi \Delta \rho_1^{(\varepsilon)}
\]
we obtain
\[
\int_{\Omega_0} (\partial_t \varphi) \zeta \Delta \varphi \, d\xi \, dr
\]
\[
= \int_{\Omega_0} \left( -a_\varepsilon \Delta(\rho_1^{(\varepsilon)} \varphi) \zeta \Delta \varphi + \theta \zeta \Delta \varphi \right) \, d\xi \, dr
\]
\[
\leq \frac{1}{2} \int_{\Omega_0} a_\varepsilon \rho_1^{(\varepsilon)} |\Delta \varphi|^2 \, d\xi \, dr
\]
\[
+ \left( C_1 \left\| \nabla \rho_1^{(\varepsilon)} \right\|_{C^1(\Omega_0)} + C_1 \left\| \Delta \rho_1^{(\varepsilon)} \right\|_{C^1(\Omega_0)} + \frac{c}{4} \right) \int_{\Omega_0} |\nabla \varphi|^2 \, d\xi \, dr
\]
\[
+ C \int_{\Omega_0} |\nabla \theta|^2 \, d\xi \, dr,
\]
Proof. It is easy to see that $C > \set$. Then, there is a constant and all non-negative
\begin{equation}
\tag{A.27}
K
\end{equation}
ϕ a test-function for all $0 \leq L$ in Section 3 in order to ensure that the constructed solutions with disjoint support
have a sufficiently large distance with respect to the $L^1$-norm.

A.3. Lower bound on $L^1$-decay for (A.17). In this section we provide a lower bound for the decay of the $L^1$ norm of solutions to (A.17). This estimate is required in Section 3 in order to ensure that the constructed solutions with disjoint support have a sufficiently large distance with respect to the $L^1$-norm.

Proposition A.4. Let $Y \in C((0, T) \times \set)$ be an essentially bounded, non-negative, very weak supersolution to the homogeneous Cauchy-Dirichlet problem for (A.17) with uniformly compact support, i.e. $K := \bigcup_{t \in [0, T]} \text{supp}(Y_t) \subseteq \set$ is a precompact set. Then, there is a constant $C > 0$ such that
\begin{equation}
\|Y_t\|_{L^1(\set)} \geq e^{-C(1)} \|Y_{t}^{m-1}\|_{L^1(\set)}, \quad \forall t \in [0, T].
\end{equation}

Proof. It is easy to see that $Y$ is a very weak supersolution to the homogeneous Cauchy-Dirichlet problem for (A.18) iff
\begin{equation}
\tag{A.27}
\int_{\set} Y_s \varphi \, d\xi - \int_{\set} Y_s \varphi \, d\xi \leq - \int_{s}^{t} \int_{\set} \Phi(\rho_2 Y) \Delta(\rho_1 \varphi) \, d\xi \, dr, \quad \forall 0 \leq s < t \leq T,
\end{equation}
and all non-negative $\varphi \in C^2(\set)$ with $\varphi_{|\partial \set} = 0$. Let $M := \|Y\|_{L^\infty(\set)}$. We choose a test-function $\varphi \in C^2(\set)$ with $\varphi \equiv 1$ on $K$. Then
\begin{equation}
\|Y_t\|_{L^1(\set)} = \int_{\set} Y_t \varphi \, d\xi
\end{equation}
\begin{align*}
&\leq \int_{\set} Y_s \varphi \, d\xi - \int_{s}^{t} \int_{\set} \Phi(\rho_2 Y) \Delta(\rho_1 \varphi) \, d\xi \, dr \\
&\leq \|Y_s\|_{L^1(\set)} - M^{m-1} \int_{s}^{t} \int_{\set} \Phi(\rho_2) \left( \frac{Y}{M} \right)^{m-1} |\Delta \rho_1| \, d\xi \, dr \\
&\geq \|Y_s\|_{L^1(\set)} - M^{m-1} \left( \|\rho_1\|_{C^0(\set)} + \|\rho_2\|_{C^0(\set)} \right) \int_{s}^{t} \|Y_r\|_{L^1(\set)} \, dr,
\end{align*}
for all $0 \leq s < t \leq T$. Gronwall’s Lemma finishes the proof.

A.4. Finite speed of propagation for (A.17). The proof of finite speed of propagation for (A.17) is very similar to Theorem 2.8 and is based on a bound for the speed of hole-filling as given in Theorem 2.7. The arguments remain the same with minor changes in the calculation. For the readers convenience we state the corresponding results in detail and give some short remarks on the proofs.

Theorem A.5. Let $\xi_0 \in \mathbb{R}^d, T, R > 0$ and $Y \in C((0, T] \times B_R(\xi_0))$ be an essentially bounded, non-negative, very weak subsolution to (A.17) with vanishing initial value $Y_0$ on $B_R(\xi_0)$ and boundary value $g$ satisfying $H := \|g\|_{L^\infty([0, T] \times \partial B_R(\xi_0))} < \infty$. Define $T_{\text{stoch}}$ by
\begin{equation}
T_{\text{stoch}} := \sup \left\{ \tilde{T} \in [0, T] \mid \tilde{T} C_{\tilde{T}} \leq R^2 H^{-(m-1)} \right\},
\end{equation}
where $C_1$ is a constant depending only on $\|a\|_{L^\infty(\set)}$ and $c > 0$ is a constant as in case (A1'). We now choose $\tau_1 > 0$ such that
\begin{equation}
C_1 \left( \|\nabla \rho_1^{c} \|_{C^0(\set)} + \|\Delta \rho_1^{c} \|_{C^0(\set)} \right) \leq \frac{c}{4}.
\end{equation}
By the choice of $\tau_1$ and Fatou’s Lemma we get
\begin{equation}
\frac{c}{2} \int_{\set} |\nabla \varphi|^2 \, d\xi dr + \frac{1}{2} \int_{\set} a_\rho \rho_1 \Delta \varphi \, d\xi dr \leq C \int_{\set} |\nabla \varphi|^2 \, d\xi dr
\end{equation}
and we conclude the proof as in case (A1').
where \( t \mapsto C_t \) is a continuous, non-decreasing function.

Then \( Y_t \) vanishes in \( B_{R\text{stoch}(t)}(\xi_0) \) for all \( t \in [0, T_{\text{stoch}}] \), where

\[
R_{\text{stoch}}(t) = R - \sqrt{t} \sqrt{C_t H}.
\]

**Proof.** As in Theorem 2.7, the proof is based on the construction of an appropriate supersolution to (A.17). For \( r \in (0, R] \), \( \xi_1 \in \mathbb{R}^d \), \( \tilde{T} > 0 \) let

\[
W(t, \xi, \xi_1) := \tilde{C} |\xi - \xi_1|^{\frac{2}{m-1}} \left( \tilde{T} - t \right)^{\frac{1}{m-1}}, \quad t \in [0, \tilde{T}), \xi \in B_r(\xi_1).
\]

Direct computations yield

\[
\partial_t W(t, \xi, \xi_1) \geq \rho_1 \Delta (\rho_2 W(t, \xi, \xi_1))^m
\]
on \( [0, \tilde{T}) \times B_r(\xi_1) \) if

\[
1 \geq C(d, m) \tilde{C}^{-1} (1 + R)^2 \rho_2 \|\mathcal{C} \|_{C^0([0, \tilde{T}] \times B_R(\xi_0))} \|\mathcal{R} \|_{C^{\infty}([0, \tilde{T}] \times B_R(\xi_0))},
\]

for all \( (t, \xi) \in [0, \tilde{T}) \times B_r(\xi_1) \) and some generic constant \( C(d, m) \). This is satisfied for the choice

\[
\tilde{C}^{-1} = \tilde{C}^{-1} := \left( C(d, m) (1 + R)^2 \rho_2 \|\mathcal{C} \|_{C^0([0, \tilde{T}] \times B_R(\xi_0))} \|\mathcal{R} \|_{C^{\infty}([0, \tilde{T}] \times B_R(\xi_0))} \right)^{-1}.
\]

Moreover,

\[
W(t, \xi, \xi_1) = \tilde{C} |\xi - \xi_1|^{\frac{2}{m-1}} \left( \tilde{T} - t \right)^{\frac{1}{m-1}} \geq H,
\]

for a.a. \( (t, \xi) \in [0, \tilde{T}) \times \partial B_r(\xi_1) \) is satisfied if \( \tilde{T} \tilde{C}^{-1} \leq r^2 H^{-(m-1)} \).

We conclude the proof as for Theorem 2.7. \( \square \)

As in Theorem 2.9, we may now use Theorem A.5 to deduce

**Theorem A.6.** Let \( Y \in C((0, T] \times \mathcal{O}) \) be an essentially bounded, non-negative, very weak subsolution to the homogeneous Dirichlet problem to (1.1) and set \( H := \|Y\|_{L^\infty(\mathcal{O}, T)} \). Then, for every \( s \in [0, T] \)

\[
\text{supp}(Y_{s+t}) \subseteq B_{\sqrt{t} \sqrt{C_t H}^{1-m}}(\text{supp}(Y_s)), \quad \forall t \in [0, T - s],
\]

where \( t \mapsto C_t \) is a continuous, non-decreasing function.

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