Single-electron transport through the vortex core levels in clean superconductors

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We develop a microscopic theory of single-electron transport in N-S-N hybrid structures in the presence of applied magnetic field introducing vortex lines in a superconductor layer. We show that vortex cores in a thick and clean superconducting layer are similar to mesoscopic conducting channels where the bound core states play the role of transverse modes. The transport through not very thick layers is governed by another mechanism, namely by resonance tunneling via vortex core levels. We apply our method to calculation of the thermal conductance along the magnetic field.

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Electron transport through various hybrid structures is in the focus of current nanoscale physics research. Of a special interest are normal metal (N) - superconductor (S) - normal metal (N) trilayers where a superconducting gap $\Delta_0$ suppresses single-particle transport, making charge transfer transparency very sensitive to the external controlling parameters. The electron transmission through such N-S-N structure at low energies is associated with two-particle Andreev processes. If the thickness of the superconducting slab is much larger than the coherence length $\xi$ the electrons incident on the slab are reflected as holes, and the normal current converts into the supercurrent. Single-electron tunneling through an N-S-N structure decays exponentially with the slab thickness giving rise, in particular, to the exponential drop off of the electronic contribution to the thermal conductance.

A single-particle transport through NSN recovers by applying a strong magnetic field, which creates vortex lines where the gap in the spectrum is suppressed. In the present Letter we develop a regular theoretical description of the low-energy single-particle transfer through the vortex core states in clean type II superconductors. Since the single particle contribution to electric conductivity is short-circuited by supercurrent, we focus on the thermal conductivity which in this case is the experimentally accessible characteristic of the one-electron transport.

Taking the simplest view of a vortex core as a normal channel with a density of states as in the normal metal, we arrive at a single-electron Sharvin conductance of a normal wire with the radius of the order of $\xi$: $G_{sc} \propto (e^2/\pi\hbar)(k_F \xi)^2$ where $k_F = p_F/\hbar$ is the Fermi wave vector. A little bit more attentive second thought shows that in clean superconductors the only trajectories that do not hit vortex core “walls” contribute to a single-particle conductivity. Indeed, if a vortex core is a normal cylinder surrounded by a superconductor, an electron flying into the core boundary is Andreev reflected as a hole back along its incidental path. Thus the single-electron transport along such a trajectory is blocked, and the only contribution to the conductance of the “Andreev wire” comes from trajectories that traverse freely the normal region. Within this model, such trajectories must have incident angles $\theta \lesssim \xi/d$, i.e., are confined to a solid angle $\xi^2/d^2$. This would result in $G_{sc} \propto (e^2/\pi\hbar)^2(\xi/d)^2$.

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As we already mentioned, the single-electron transport determines the behavior of thermal conductivity. The Wiedemann–Franz law would result in \( \kappa \propto T G_{sc}/e^2 \) for the thermal conductance along the vortex line. In the vicinity of \( H_{c2} \), the thermal conductivity has been studied theoretically in a number of papers (see, for instance, \[8\]). In dirty superconductors, the electron contribution to the thermal conductance along the vortices for a small concentration of vortex lines \[8\] agrees conceptually with Eq. \[1\]: \( \kappa(B) \approx (B/H_{c2}) \kappa_N \), where \( \kappa_N \) is the electron thermal conductivity in the normal state. Unfortunately, in clean superconductors this simple estimate fails to describe the experimental data \[3, 10\]: the thermal conductance in the magnetic field direction appears to be two orders of magnitude smaller. It was noted first in \[10\] that this obvious conflict can be caused by a very small group velocity of the BdG modes as discussed above. Analysis of quantum transport through individual vortices is of particular importance for understanding the properties of mesoscopic superconductors. The exotic vortex states in these systems are nowadays the focus of a considerable attention \[11, 12, 13\].

Hereafter we concentrate on the low-energy single-particle transport through vortex cores in clean \( \ell \gg d \) type II superconductors in the low-field limit of separated vortices and develop a systematic approach for calculation of the thermal conductance along vortex cores. We study the transmission of an electron wave incident on the superconducting slab placed between two bulk normal metal electrodes assuming ideally transparent boundaries and neglecting the normal scattering. Considering two extremes of infinite and finite slab thicknesses we confirm the intuitive picture discussed above: For a not very thick slab, the transmission is determined by the semi-classical resonant tunneling through the energy gapped region where the energy coincides exactly with one of the levels in the vortex core. The transmission is proportional to the large Shervin conductance; however, it decays exponentially with the slab thickness except for the trajectories that go almost parallel to the vortex axis. The exponential decay is thus replaced with a \((\xi/d)^6\) power law, which gradually transforms the Andreev-like thermal conductance into a thickness-independent expression Eq. \[2\] as \( d \) increases.

**Wave functions.** Quasiparticles in the superconductor obey the Bogolubov-de Gennes (BdG) equations:

\[
\left[ \frac{1}{2m} \left( -i\hbar \nabla - \frac{e}{c} A \right)^2 - E_F \right] u + \Delta v = cu, \quad \text{(3)}
\]

\[
\left[ \frac{1}{2m} \left( -i\hbar \nabla + \frac{e}{c} A \right)^2 - E_F \right] v - \Delta^* u = -cv. \quad \text{(4)}
\]

The wave vector \( k_z \) along the vortex axis is a good quantum number, \( u = e^{ik_z x} u_k \), \( v = e^{i(k_z x + \Phi)} v_k \). We put \( \Delta = |\Delta| e^{i\phi} \), and \( u_{k_z} = e^{i\phi/2 + i\Phi} u \), \( v_{k_z} = e^{-i\phi/2 + i\Phi} v \). Equations \[3, 4\] have eigenvalues \( \epsilon_{\mu}(k_z) \) for the CdGM bound states. We establish first a simple identity for the localized states. Calculating the derivative with respect to \( k_z \) from the both sides of Eqs. \[3, 4\] and using the normalization of the CdGM wave functions we find

\[
\int \left[ u^*_{\mu k_z} \left( \hbar k_z - \frac{e}{c} A_z \right) u_{\nu k_z} - v^*_{\mu k_z} \left( \hbar k_z + \frac{e}{c} A_z \right) v_{\nu k_z} \right] \, d^2r = \frac{m}{\hbar} \frac{\partial \epsilon_{\mu}}{\partial k_z}. \quad \text{(5)}
\]

There is a considerable cancellation in Eq. \[5\]: the r.h.s. is by a factor of \((k_F \xi)^{-1}\) smaller than each term in the l.h.s. This is why Eq. \[2\] gives much smaller conductance than Eq. \[1\]. We will use Eq. \[3\] later to derive the thermal conductance for a thick slab.

However, the cancellation does not take place for a finite-thickness slab where the CdGM states are not truly localized. To consider this in more detail we use a semi-classical approach and look for \( \hat{U} = (U, V) \) in the form

\[
\hat{U} = H_1^{(1)}(k_r r) \hat{w}^{(+)} + H_1^{(2)}(k_r r) \hat{w}^{(-)}
\]

where \( H_1^{(1,2)} \) are the Hankel functions, \( l = \sqrt{\mu^2 + 1/4} \), assuming that \( \hat{w} = (w_1, w_2) \) are slow functions of \( r \) and \( z \). Following \[14\] we put \( x = \sqrt{x^2 + b^2} \), \( b = -\mu k_r \), and define the trajectories \( dx = \pm ds \pm \sin \theta, \quad dz = ds \cos \theta \) for \( \hat{w}^{(\pm)} \), respectively, where \( k_z = k_F \cos \theta \) and \( ds \) is the distance along the corresponding trajectory. For a point \((x, z)\) on the trajectory \( z = z_0 \pm x \cos \theta \) we obtain

\[
\pm \frac{i\hbar^2 k_r}{m} \frac{dv^{(\pm)}}{dx} - \Delta \left( \epsilon + \frac{\hbar^2 k_r b}{2m(x^2 + b^2)} \right) \hat{w}^{(\pm)} + i\epsilon \Delta |\Delta| \hat{w}^{(\pm)} = 0.
\]

(6)

For the functions \( u_{1,2}^{(+)}(x, z) \), the limit \( x \to \infty \) corresponds to a wave radiating from the vortex into the bulk while \( w_{1,2}^{(-)}(x, z) \) corresponds to a wave incident on the vortex. The condition of regularity at \( r = 0 \) requires \( \hat{w}^{(+)}(0, z) = \hat{w}^{(-)}(0, z) \) at the classical turning point, \( x = 0 \). Let us put \( \hat{w}^{(+)}(x, z) = \hat{w}(x, z) \), \( \hat{w}^{(-)}(x, z) = \hat{w}(-x, z) \). The functions \( w_{1,2}(x, z) \) satisfy Eq. \[6\] with the upper sign along the entire \( x \) axis.

We introduce new functions \( \eta \) and \( \zeta \) through \( w_1 = e^{\xi + i\eta}/2, \quad w_2 = e^{-\xi - i\eta}/2 \) and arrive at the equations

\[
\frac{d\eta}{dx} = \frac{2m}{\hbar^2 k_r} + \frac{b}{(x^2 + b^2)} - \frac{2m|\Delta|}{\hbar^2 k_r} \cos \eta, \quad \text{(7)}
\]

\[
\frac{d\zeta}{dx} = -m|\Delta|^{-1} \sin \eta. \quad \text{(8)}
\]

The requirement that \( \omega \) vanishes at \( x \to \pm \infty \) is \( \eta = \pm \pi/2 - (\epsilon/|\Delta|) + 2\pi k \). These values, however, are not stable for a general choice of the integration constants. A general solution for not very large \( \eta \) is

\[
\eta = \arctan \frac{x}{b} + \eta_0 e^{2K(x)}
\]

\[
+ \frac{2m}{\hbar^2 k_r} \int_0^x \left[ \epsilon - |\Delta(x')| \frac{b}{|x'|} \right] e^{2K(x)-2K(x')} \, dx' (9)
\]
where \( \eta_0 \) is a constant and

\[
K(x) = \frac{m \hbar^2 k_{r}^{-1}}{2} \int_{0}^{\infty} |\Delta(x')| \, dx'.
\]

For \( \eta_0 \ll 1 \) and \( \epsilon \ll \Delta_0 \), the function \( \eta \) is close to \( \frac{\pi}{2} \) for \( b \ll x \ll x_0 \) where \( x_0 \sim \xi \ln(1/|\gamma|) \) and

\[
\gamma = 2m \hbar^2 k_{r}^{-1} \int_{0}^{\infty} \left[ \epsilon - (b/x)|\Delta(x)| \right] e^{-2K(x)} \, dx.
\]

\( \gamma \) measures the distance from a CdGM level; \( \gamma = 0 \) exactly when \( \epsilon = \epsilon_{\mu}(k_{r}) \). The validity of Eq. 6 is restricted by the condition \( \gamma \ll 1 \) which generally holds if \( \epsilon \ll \Delta_0 \). The function \( \eta \) grows with \( |x| \) at distances \( x \gg x_0 \). To find its behavior in the region \( \eta \sim 1 \) we can neglect small terms with \( \epsilon \) and \( b/x \) in Eq. 7. The solution is

\[
\tan \left( \frac{\gamma}{2} - \frac{\pi}{4} \right) = C e^{2K(x)}.
\]

Matching with Eq. 9 at \( \xi \ll x \ll x_0 \) gives \( C \approx (\gamma + \eta_0)/2 \). For \( \gamma + \eta_0 > 0 \), the function \( \eta \to 3\pi/2 \) while \( w \) diverges exponentially as \( x \to \infty \). If \( \gamma + \eta_0 < 0 \), the function \( \eta \) approaches \( -\pi/2 \), and \( w \) diverges again. However, if \( \gamma + \eta_0 = 0 \), the value \( \eta = \pi/2 \) is stable (see Fig. 11) and the wave function decays for \( x \to \infty \).

The solution at \( |x| \ll x_0 \) for negative \( x \) is obtained from Eq. 9 by replacing \( K(x) \) with \( -K(x) \). The function \( \eta \) is close to \( -\pi/2 \) for \( b \ll |x| \ll x_0 \). Its behavior for \( |x| \gg x_0 \) is determined by Eq. 11 where \( C = 2/(\gamma - \eta_0) \). Equation 11 then shows that \( -3\pi/2 < \eta(-\infty) < -\pi/2 \) if \( \gamma - \eta_0 > 0 \). The function \( \eta \) grows and approaches \( \eta(-\infty) \approx -3\pi/2 \) while \( w \) diverges. Similarly, \( -\pi/2 < \eta(-\infty) \) if \( \gamma - \eta_0 < 0 \), and \( w \) also diverges. The value \( \eta = -\pi/2 \) is stable if only \( \gamma - \eta_0 = 0 \). The wave function thus decays at both ends if \( \gamma = \eta_0 = 0 \), which corresponds to a standard CdGM state.

**Reflection and transmission probabilities.**—For a superconducting slab with a thickness \( d \) we thus have a linear combination of two solutions \( \tilde{w} = A_r \tilde{w}^r + A_c \tilde{w}^c \) where \( A_r \) and \( A_c \) are constants. The first solution \( \tilde{w}^r \) has \( \eta_0 = -\gamma \), and decays at \( x \to \infty \). Thus \( \eta_r(+|x|) = \pi/2 \) for \( x \gg \xi \) while \( \eta_r(-|x|) \) satisfies

\[
\tan \left[ \frac{\gamma}{2} \eta_r(-|x|) - \frac{\pi}{4} \right] = e^{-2K(|x|)}
\]

for \( \gamma \neq 0 \). For \( \gamma \neq 0 \) the phase \( \eta_r = -3\pi/2 + 2\pi n \) at \( x = -|x| \to -\infty \). The amplitude factor for \( x = -|x| \) is found from Eqs. 6 and 11

\[
\zeta_r(-|x|) = K(|x|) + \frac{1}{2} \ln \left( \frac{\gamma^2 + e^{-4K(|x|)}}{1 + \gamma^2} \right).
\]

For \( x = |x| \) it is simply \( \zeta_r(+|x|) = -K(|x|) \). The other solution \( \tilde{w}^c \) grows at \( x \to +\infty \).

The particle transmission \( D_e \) and hole (Andreev) reflection \( R_h \) probabilities, \( 1 = R_h + D_e \), are determined such that \( D_e = |w_1(z = d)/w_1(z = 0)|^2 \) provided there are no transmitted holes, \( w_2 = 0 \) at \( z = d \). We denote the \( x \)-coordinates of the end points of the trajectory at \( z = 0 \) and \( z = d \) as \( x_- \) and \( x_+ \), respectively, such that \( d \tan \theta = x_+ - x_- \). For trajectories crossing the vortex axis, \( 0 < x_+ < d \tan \theta, x_- = -|x_-| \), we find

\[
D_e = (\gamma^2 + a^2)^{-1} \cosh^{-2} \left[ K(|x_+|) + K(|x_-|) \right]
\]

where

\[
a = \cosh \left[ K(|x_+|) - K(|x_-|) \right] / \cosh \left[ K(|x_+|) + K(|x_-|) \right].
\]

is the half-width of the level \( \gamma = 0 \) proportional to the escape rate of excitations from the level through the gapped region far from the vortex. For \( \gamma \to 0 \), the transmission coefficient becomes \( D_p = \cosh^{-2} \left[ K(|x_+|) - K(|x_-|) \right] \). It is \( D_p = 1 \) for resonant trajectories that go through the middle of the vortex, \( |x_-| = |x_+| = (d/2) \tan \theta \). For trajectories that do not cross the vortex axis, \( x_- < x_+ < 0 \) or \( x_+ < x_- > 0 \), we find \( D_p = \cosh^{-2} \left[ K(x_+) - K(x_-) \right] \) where \( x_- - x_+ = d \tan \theta \).

Equation 13 suggests that the largest contribution to the transmission through the energy gapped region comes from the resonant tunneling when the energy coincides exactly with one of the levels in the vortex core. The transmission probability is then unity if the trajectory crosses the vortex close to the half of its length. However, the width of the resonance \( a \) is exponentially small which leads to a small number of transmitted particles. This exponent, however, corresponds to only a half of the slab thickness, not to the entire slab thickness as it would be without vortices. The exponential decay of the number of transmitted particles disappears for trajectories which go almost parallel to the vortex axis because the projection of the trajectory on the plane perpendicular to the vortex axis shrinks as \( \theta \to 0 \). The transport is thus determined by trajectories almost parallel to the vortex axis; its exponential dependence on the slab thickness is replaced with a power-law behavior.
The heat current.— The energy current along \( z \) is

\[
I_E = \int d^2 r \sum_{\mu} \int \frac{dk_z}{2\pi m} \left[ \epsilon_\mu u^*_\mu k_z \left( \hbar k_z - \frac{e}{c} A_z \right) u_{\mu k_z} n(\epsilon_\mu) \right. \\
- \left. \epsilon_\mu v^*_{\mu k_z} \left( \hbar k_z + \frac{e}{c} A_z \right) v_{\mu k_z} [1 - n(-\epsilon_\mu)] \right].
\]

Particles \( u^* u \) with the distribution \( n(\epsilon) \) carry the energy \(+\epsilon\) while the holes \( v^* v \) with the distribution \( 1 - n(-\epsilon) \) carry the energy \(-\epsilon\). If the electrodes are in equilibrium, \( 1 - n(-\epsilon) = n(\epsilon) \) in each electrode.

Consider first an infinitely thick slab. Using Eq. (5) the energy current between the two electrodes becomes

\[
I_E = \sum_{\mu} \int \epsilon_\mu n(\epsilon_\mu) \frac{\partial \epsilon_\mu}{\partial k_z} \frac{dk_z}{2\pi\hbar}.
\]

Excitations with positive group velocity \( \hbar^{-1} \partial \epsilon_\mu / \partial k_z \) have the distribution \( n_1 = [e^{\epsilon/T_1} + 1]^{-1} \) as in the electrode 1. For those with negative group velocity the distribution is \( n_2 = [e^{\epsilon/T_2} + 1]^{-1} \) as in the electrode 2. Therefore

\[
I_E = \sum_{\mu} \int_{\mu > 0} \epsilon_\mu \left[ n_1(\epsilon_\mu) - n_2(\epsilon_\mu) \right] \frac{\partial \epsilon_\mu}{\partial k_z} \frac{dk_z}{2\pi\hbar}.
\]

By the order of magnitude, the heat current is \( I_E \sim (T/\hbar) (k_F \xi) (T/T_c) (T_1 - T_2) \) with the thermal conductance

\[
\kappa \sim (T/\hbar) (k_F \xi) (T/T_c)
\]

in compliance with Eq. (2).

For a finite slab thickness, \( I_E \) can be expressed through the transmission and reflection coefficients,

\[
I_E = \sum \int d\epsilon \langle v_+ \rangle \left[ \epsilon n_1(\epsilon) - \epsilon R_\hbar [1 - n_1(\epsilon)] - \epsilon D_\hbar n_2(\epsilon) \right]
\]

\[
= 2\nu_F \int_{\epsilon > 0} \frac{d\Omega}{4\pi} \langle v_+ \rangle \int dx_- db \int D_\hbar \epsilon (n_1 - n_2) \epsilon d\epsilon,
\]

where the sum is over all the trajectories; \( \nu_F \) is the single-spin density of states at the Fermi level. The first two terms in the upper line are due to incoming particles and Andreev reflected holes on one side of the slab. The third term is due to transmitted particles from the other side.

Using Eq. (13) we find that for the resonant trajectories \(-d\tan\theta < x_- < 0\) the integration over \( dx_- \) selects \( |x_-| \) close to \( |x_+| \). The angles along the vortex core axis are small that \( k_r = k_F \sin \theta \ll k_F \) and \( K(x) = (m/2\hbar^2 k_F) \Delta(0)x^2 \). The localization radius of the wave function in the \( x \) direction is \( \lambda = \hbar (k_F/m \pi n \Delta(0))^{1/2} \theta^{1/2} \ll \xi^{1/2} \). Let us put \( x_0 = (|x_-| - |x_+|)/2 \) while \( |x_-| + |x_+| = d\tan\theta \). For small \( \theta \) we have \( K(x_+) = K(x_-) = x_0 (md/2h^2 k_F) \Delta(0) \) and \( K(x_-) + K(x_+) \sim \theta (md^2/2h^2 k_F) \Delta(0) \sim d^2 \theta / \xi^2 \). Therefore, \( x_0 \sim \xi^2 / d \) and \( \theta \sim (\xi / d)^2 \). As a result,

\[
I_E \sim \nu_F \hbar^4 v_0^5 d^{-6} (\Delta'(0))^{-4} (T_1^2 - T_2^2).
\]