Joint distributions of partial and global maxima of a Brownian bridge

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Abstract

We analyze the joint distributions and temporal correlations between the partial maximum $m$ and the global maximum $M$ achieved by a Brownian bridge on the subinterval $[0, t]$ and on the entire interval $[0, t]$, respectively. We determine three probability distribution functions: the joint distribution $P(m, M)$ of both maxima; the distribution $P(m)$ of the partial maximum; and the distribution $P(G)$ of the gap between the maxima, $G = M - m$. We present exact results for the moments of these distributions and quantify the temporal correlations between $m$ and $M$ by calculating the Pearson correlation coefficient.

Keywords: extreme value statistics, Brownian motion, joint distributions of extreme values

(Some figures may appear in colour only in the online journal)

1. Introduction

The Brownian bridge (BB) is a one-dimensional Brownian motion (BM) $B_s$, $0 \leq s \leq t$, which is conditioned to return to the starting point [1]. Without loss of generality one can postulate that the BB starts and returns to the origin (see figure 1): $B_0 = B_t = 0$. BBs admit numerous interpretations. For instance, a BB can be regarded as a stationary 1 + 1-dimensional Edwards–Wilkinson interface [2] in a box with periodic boundary conditions (see, e.g., [3]). BBs naturally arise in the analysis of convex hulls of planar BMs [4] and of dephasing due to electron–electron interactions in quasi-1D wires [5], they have been used to model a random potential in studies of diffusion in presence of a strong periodic disorder [6] and are
also relevant for diffusion in disordered non-periodic potentials as they are related to the
statistics of transients. BBs appear in mathematical statistics, e.g., in Kolmogorov–Smirnov
tests of the difference between the empirical distributions calculated from a sample and the
true distributions governing the sample process \([7–10]\) (see also \([11]\) for the applications in
mathematical finance). BBs are often used in computer science, e.g., in the analysis of the
maximal size reached by a dynamic data structure over a long period of time \([12]\). In ecology,
BBs have been used for an analysis of animal home ranges and migration routes, as well as
for estimating the influence of resource selection on movement \([13]\).

Extremal value statistics of the BBs, e.g., statistics of a maximum, a minimum, or a range
on the entire time interval \([0, t]\) were studied beginning with classical papers \([7–10]\), and were
subsequently generalized for Bessel process (the radius of a \(d\)-dimensional BM) with a bridge
constraint \([14–16]\), and also for some conditioned extremal values of BBs \([17, 18]\). The
statistics of longest excursions and various non-self-averaging characteristics of BBs have
been studied e.g. in \([19, 20]\). Using a real-space renormalization group technique, a wealth of
results on the extreme value statistics of BBs, reflected BBs, Brownian meanders and
excursions, as well as more general processes like Bessel Bridges, have been presented
in \([21]\).

In this paper we investigate the joint statistics and temporal correlations between the
partial maximum \(m = \max_{0 \leq s \leq t} B_s\) and the global maximum \(M = \max_{0 \leq s \leq t} B_s\) achieved by
the BB on the subinterval \([0, t]\) and on the entire interval \([0, t]\), see figure 1. We recently
studied similar problems for the unconstrained BM \([22]\), and below we compare the outcomes
for the BB and the standard BM starting at the origin. Due to the ubiquitousness of the BBs,
our results admit numerous reformulations. For instance, the analogy with the Edwards–
Wilkinson interface asserts that our problem is tantamount to studying the correlations of the
maximal height of the interface on the entire interval and the maximal height in a window
near one of the fixed boundaries.

In the next section we determine three probability distribution functions (pdfs): the joint
dpdf of both maxima, \(P(m, M)\); the pdf of the partial maximum, \(P(m)\); and the pdf of the gap
between the maxima, \(\Pi(G)\) with \(G = M - m\). Using these distributions, we derive exact
expressions for the moments \(\mathbb{E}(m^k)\) and \(\mathbb{E}((M - m)^k)\) with arbitrary \(k \geq 0\). We also cal-
culate the Pearson correlation coefficient \(\rho(m, M)\) which permits us to quantify the linear
correlations between \(m\) and \(M\).
To determine \( P_{mM} \) we use two auxiliary pdfs which describe the BM starting at the origin. One of these quantities is \( \Pi_t(m, x) \), the pdf that the BM is at \( x \) at time \( t \) and it has achieved the maximum \( m \) during the time interval \([0, t]\). This pdf is given by (see, e.g., [23, 24])

\[
\Pi_t(m, x) = \frac{2m - x}{2\sqrt{\pi Dt}t^{3/2}} \exp\left(-\frac{(2m - x)^2}{4Dt}\right).
\]

Another quantity is \( S_t(m, x) \), the pdf that the BM does not reach a fixed level \( m_0 \) within the time interval \([0, t]\), and appears at position \( x \) at time moment \( t \). This survival probability is given by (see, e.g., [25])

\[
S_t(m, x) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) - \exp\left(-\frac{(2m - x)^2}{4Dt}\right).
\]

2. Results

We compute \( P_{mM} \) using the same procedure [22] as for the BM. We denote by \( x \) the position of the BB at time \( s = t_1 \), and we integrate over \(-\infty < x < m\) to determine \( P_{mM} \). There are two contributions corresponding trajectories with \( m = M \) and with \( m < M \), the latter occurs e.g. for the trajectory shown in figure 1. Thus we represent \( P_{mM} \) as

\[
P_{mM} = A \delta(M - m) \int_{-\infty}^m dx \Pi_t(m, x) S_{t-\delta}(m, x)
+ \int_{-\infty}^m dx \Pi_t(m, x) \Pi_{t-\delta}(M, x).
\]

The normalization factor \( A \) is chosen to ensure that \( \int_0^\infty dm \int_{m}^\infty dMP_{mM} = 1 \). Using (1) and (2) we get \( A = 2/\sqrt{\pi Dt} \). Performing the integrals in (3), we arrive at

\[
P_{mM} = \frac{m}{Dt} \exp\left(-\frac{m^2}{Dt}\right) \left[1 - \text{erf}\left(\frac{(1 - 2z)m}{2\sqrt{(1 - z)Dt}}\right)\right] \delta(M - m)
+ \frac{(1 - z)(2M - m) + zm}{\sqrt{\pi z(1 - z)(Dt)^{3/2}}} \exp\left(-\frac{m^2}{4Dt} - \frac{(2M - m)^2}{4Dt(1 - z)}\right)
- \frac{2(M - m)^2 - Dt}{(Dt)^2} \exp\left(-\frac{(M - m)^2}{Dt}\right) \text{erfc}\left(\frac{z(2M - m) + (1 - z)m}{2\sqrt{(1 - z)Dt}}\right).
\]

Equation (4) is the chief result of this paper, it allows us to deduce most of other results. In figure 2 we plot \( P_{mM} \) (without the delta-peak) for several values of \( z \). The pdf \( P_{mM} \) is bimodal for \( z < 1/2 \) (due to the delta-peak for \( m = M \)) and unimodal for \( z \geq 1/2 \). In what follows we analyze the characteristic features of the pdf in (4) in more detail.

2.1. Distribution and moments of the partial maximum

We now compute \( P(m) \), the distribution of the partial maximum \( m \), viz. the maximum of the BB defined on the entire interval \([0, t]\) which is achieved on a subinterval \([0, t_1]\). This quantity can be calculated from (4) by integrating over \( M \geq m \):
In the limit \( z \to 1 \), i.e., when \( t \to t_{1} \) and \( m \to M \), equation (5) reduces to the classic result for the global maximum of the BB (see [7–10]):

\[
P(M) = \frac{2M}{Dt} \exp\left(-\frac{M^2}{Dt}\right)
\]

The distributions (5) and (6) are depicted in figure 3(a).
Using (5) we compute the moments

\[
\frac{\mathbb{E} \{ m^k \}}{(Dt)^{k/2}} = \frac{1}{2} \Gamma \left( \frac{k+1}{2} \right) + \frac{(1-z)^{2k}}{\sqrt{\pi}} \Gamma \left( \frac{k+3}{2} \right) F_1 \left( 1, -\frac{k}{2}, -\frac{3}{2}, (1-z)^2 \right),
\]

where \( F_1 \) is the hypergeometric function. We consider the moments with non-negative integer \( k \), although they are well-defined for all \( k > -1 \). For even integer \( k \geq 0 \) one can express the moments through the Gegenbauer polynomials, viz.

\[
\frac{\mathbb{E} \{ m^k \}}{(Dt)^{k/2}} = \frac{2^k}{\sqrt{\pi}} \Gamma \left( \frac{k+1}{2} \right) z^{k/2} (1-z)^{k/2+1} + \frac{1}{2} \Gamma \left( \frac{k+2}{2} \right) (1 + C_{k+1}^{-1/2}(1-2z)),
\]

where \( C_{\alpha}^{\beta}(x) \) are Gegenbauer polynomials. In particular, for \( k = 2, 4, 6 \)

\[
\begin{align*}
\frac{\mathbb{E} \{ m^2 \}}{(Dt)} &= 2z - z^2, \\
\frac{\mathbb{E} \{ m^4 \}}{(Dt)^2} &= 12z^2 - 16z^3 + 6z^4, \\
\frac{\mathbb{E} \{ m^6 \}}{(Dt)^3} &= 120z^3 - 270z^4 + 216z^5 - 60z^6.
\end{align*}
\]

The moments of the odd order have a more complicated structure and contain the inverse trigonometric function \( \arccos(1-2z) \). The first three odd moments read

\[
\frac{\mathbb{E} \{ m^3 \}}{(Dt)^{1/2}} = \frac{1}{2} \Gamma \left( \frac{3}{2} \right) + \frac{(1-z)^3}{\sqrt{\pi}} \Gamma \left( \frac{5}{2} \right) F_1 \left( 1, -\frac{3}{2}, -\frac{1}{2}, (1-z)^2 \right).
\]
Using explicit expressions for $\mathbb{E}\{m\}$ and $\mathbb{E}\{m^2\}$ we determine the variance of the partial maximum:

\[
\text{Var}(m) = Dt \ V(z) = 2z - z^2 + \frac{z^2 - z - \sqrt{z(1-z)} \arccos(1 - 2z)}{\pi} - \frac{1}{2} \arccos^2(1 - 2z).
\]

Interestingly enough, as shown in figure 3(b), the variance of $m$ appears to be a non-monotonic function of $z$: upon a gradual increase of $z$, $\text{Var}(m)/Dt$ first grows, crosses at $z \approx 0.454$ the dashed line which defines the corresponding value of the variance of the global maximum $M$, $\text{Var}(M)/Dt = (1 - \pi/4)$, attains a maximal value at $z \approx 0.695$ and then decreases reaching finally the level $\text{Var}(M)/Dt$ at $z = 1$. This is a rather intriguing behavior which shows that in some region the variance of the partial maximum of a BB can be bigger than the variance of the global maximum. Note that the skewness $\gamma(m, z)$ of the pdf $P(m)$ in equation (5), defined as

\[
\gamma(m, z) = \frac{\kappa_3}{\text{Var}^{3/2}(m)},
\]

where $\kappa_3$ is the third cumulant of the pdf $P(m)$, also exhibit a non-monotonic behavior as a function of $z$, see figure 4(a).

In the limit $z \to 0$, in the leading in $z$ order, we recover from (7) the standard expression for the moments of the maximum of an unconstrained BM on the interval $[0, t]$, i.e.
In the opposite limit $z \to 1$ we have
\[
\frac{\mathbb{E} \{m^k\}}{(Dt)^{k/2}} \to \Gamma\left(\frac{k+1}{2}\right),
\]
which is a standard expression for the moments of the global maximum $M$ of a BB. The moments $\mathbb{E} \{m^k\}$ as functions of $z$ are plotted in figure 4(b).

2.2. Distribution and moments of the gap between the partial and global maxima

From (4) we derive the distribution $\Pi(G)$ of the gap between $M$ and $m$. Rescaling the gap and the gap distribution
\[
\Pi(G) = \frac{1}{\sqrt{Dt}} \mathcal{P}(g), \quad g = \frac{G}{\sqrt{Dt}}
\]
we get
\[
\mathcal{P}(g) = z \delta(g) + 4 \sqrt{\frac{z(1-z)}{\pi}} (1 - g^2) \exp\left(-\frac{g^2}{1-z}\right)
+ 2g [2gz + 1 - 3z] e^{-g^2} \text{erfc}\left(\sqrt{\frac{z}{1-z}} g\right)
\]
(16)
This distribution is depicted in figure 5. For $z \geq 1/2$, the distribution $\mathcal{P}(g)$ is unimodal with maximum at $g = 0$. For $z < 1/2$, the distribution is bimodal—in addition to the maximum at $g = 0$ (due to the delta-peak) there is a second maximum which moves away from the origin as $z \to 0$. 

Figure 5. The rescaled gap distribution $\mathcal{P}(g)$, equation (16), versus rescaled gap $g = G/\sqrt{Dt}$ (the delta-peak at $g = 0$ is not shown). The dashed curves (top to bottom) correspond to $z = 1/2, 3/4, z = 7/8$. The solid curves (top to bottom) correspond to $z = 1/4, 1/8, 1/16$. 

\[\mathbb{E} \{m^k\} = \frac{2^k}{k!} \left(\frac{k+1}{2}\right) \Gamma\left(\frac{k}{2} + 1\right),\]
The moments of the gap are found from (16) to give

\[
\begin{align*}
\frac{\mathbb{E}\{G^k\}}{(Dt)^{k/2}} &= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{k + 1}{2}\right) (1 + z - (1 - z)k) \sqrt{z} (1 - z)^{(k + 2)/2} \\
&\quad + \frac{\Gamma(k + 4)}{2^k} z (1 - z)^{(k + 4)/4} p_{k/2 + 1}^{-k/2 - 2}(\sqrt{z}) \\
&\quad + \frac{\Gamma(k + 2)}{2^k} (1 - 3z)(1 - z)^{(k + 2)/4} p_{k/2}^{-k/2 - 1}(\sqrt{z}),
\end{align*}
\]

where \( p_k^\ell(\cdot) \) are the associated Legendre functions of the first kind. For even \( k \) the latter are polynomials, so that the moments of the even order are polynomials of \( \sqrt{z} \). For instance

\[
\begin{align*}
\frac{\mathbb{E}\{G^2\}}{(Dt)} &= (1 - \sqrt{z})^2, \\
\frac{\mathbb{E}\{G^4\}}{(Dt)^2} &= 2(1 - \sqrt{z})^3, \\
\frac{\mathbb{E}\{G^6\}}{(Dt)^3} &= \frac{3}{2} (1 - \sqrt{z})^4 (4 + \sqrt{z}).
\end{align*}
\]

The moments of odd order contain an additional inverse trigonometric function \( \text{arccos}(\sqrt{z}) \):

\[
\begin{align*}
\frac{\mathbb{E}\{G\}}{(Dt)^{1/2}} &= \frac{1}{\sqrt{\pi}} (\text{arccos}(\sqrt{z}) - \sqrt{z}(1 - z)), \\
\frac{\mathbb{E}\{G^3\}}{(Dt)^{3/2}} &= \frac{3}{2\sqrt{\pi}} (1 + 2z) \text{arccos}(\sqrt{z}) - 3\sqrt{z}(1 - z)), \\
\frac{\mathbb{E}\{G^5\}}{(Dt)^{5/2}} &= \frac{5}{4\sqrt{\pi}} (3(1 + 4z) \text{arccos}(\sqrt{z}) - (13 + 2z)\sqrt{z}(1 - z)).
\end{align*}
\]

Using these explicit results one can compute cumulants. For instance, the variance reads

\[
\frac{\text{Var}(G)}{Dt} = (1 - \sqrt{z})^2 - \frac{1}{\pi}(\text{arccos}(\sqrt{z}) - \sqrt{z}(1 - z))^2.
\]
In figure 6(a) we plot $\mathbb{E}\{G^k\}$ versus $z$ for several integer values of $k$, while figure 6(b) presents the variance of the gap versus $z$.

### 2.3. Two-time correlations between the partial and global maxima

Let us determine the Pearson correlation coefficient of partial $m$ and global $M$ maxima. By definition

$$
\rho(m, M) = \frac{\mathbb{E}\{mM\} - \mathbb{E}\{m\}\mathbb{E}\{M\}}{\sqrt{\text{Var}(m)\text{Var}(M)}}.
$$

(21)

We have already computed all terms in equation (21) apart from the cross-moment of two maxima $\mathbb{E}\{mM\}$. This cross-moment can be determined from (4) to give

$$
\mathbb{E}\{mM\} = \frac{Dt}{2} (2\sqrt{z} + z - z^2) - \frac{1}{2} \arccos(1 - 2z)
$$

(22)

leading to

$$
\rho(m, M) = \frac{2\sqrt{z} + z - z^2 - \sqrt{z(1 - z)} - \frac{1}{2} \arccos(1 - 2z)}{\sqrt{(4 - \pi)^{V(z)}}}
$$

(23)

with $V(z)$ defined in equation (11).

In figure 7 we plot the Pearson’s coefficient as a function of $z$. The Pearson coefficient approaches unity, $\rho(m, M) \rightarrow 1$, when $z \rightarrow 1$, i.e. $t_i \rightarrow t$. Indeed, $m$ and $M$ are almost completely correlated in this region. The more precise asymptotic behavior is

$$
\rho(m, M) = 1 - \frac{4\sqrt{1 - z}}{4 - \pi} + O((1 - z)^{3/2}).
$$

(24)

Conversely, $\rho(m, M) \rightarrow 0$ when $z \rightarrow 0$ implying that $m$ and $M$ become uncorrelated. More precisely, one gets

$$
\rho(m, M) = \sqrt{\frac{\pi}{2(\pi - 2)(4 - \pi)}} \sqrt{z} + O(z),
$$

(25)

implying that correlations vanish slowly, $\rho(m, M) \sim \sqrt{h/t}$. 
3. Conclusions

We have determined the joint statistics and temporal correlations between a partial and global extremes of one-dimensional BBs. We have calculated the joint probability distribution function of two maxima, the pdf of the partial maximum and the pdf of the gap \( G = M - m \). We also derived exact expressions for the moments \( \mathbb{E}\{m^k\} \) and \( \mathbb{E}\{(M - m)^k\} \) with arbitrary \( k \geq 0 \) and computed the Pearson correlation coefficient \( \rho(m, M) \) quantifying the correlations between \( m \) and \( M \). Our results for the one-dimensional BBs can be generalized to the general Bessel process—the radius of \( d \)-dimensional BM, with the bridge constraint. The calculations are very similar, one should use explicit expressions for \( P_{mx}(t) \) obtained in [21].

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References

[1] Mörters P and Peres Y 2010 Brownian Motion (Cambridge: Cambridge University Press)
[2] Edwards S F and Wilkinson D R 1982 Proc. R. Soc. A 381 17
[3] Majumdar S N and Comtet A 2004 Phys. Rev. Lett. 92 225501
[4] Randon-Furling J, Majumdar S N and Comtet A 2009 Phys. Rev. Lett. 103 140602
[5] Comtet A, Desbois J and Texier C 2005 J. Phys. A: Math. Theor. 38 R341
[6] Dean D S, Gupta S, Oshanin G, Rosso A and Schehr G 2014 J. Phys. A: Math. Theor. 47 372001
[7] Kolmogorov A N 1933 Giorn. Inst. Ital. Attuari 4 83
[8] Smirnov N V 1939 Bull. Math. Univ. Moscow 2 2
[9] Feller W 1948 Ann. Math. Stat. 19 177
[10] Doob J L 1949 Ann. Math. Stat. 20 393
[11] Chicheportiche R and Bouchaud J-P 2014 Some applications of first-passage ideas to finance
First-Passage Phenomena and Their Applications ed R Metzler et al (Singapore: World Scientific Publishers)
[12] Mari J-F and Schott R 2001 Probabilistic and Statistical Methods in Computer Science (New York: Springer Science and Business Media)
[13] Horne J, Garton E, Krone S and Lewis J 2007 Ecology 88 2354
[14] Gikhman I I 1957 Theory Probab. Appl. 2 369
[15] Kieler J 1959 Ann. Math. Stat. 30 420
[16] Pitman J and Yor M 1999 Electron. J. Probab. 4 1
[17] Majumdar S N, Randon-Furling J, Kearney M J and Yor M 2008 J. Phys. A: Math. Theor. 41 365005
[18] Perret A, Comtet A, Majumdar S N and Schehr G 2015 J. Stat. Phys. 161 1112
[19] Frachebourg L, Ispolatov I and Krapivsky P L 1995 Phys. Rev. E 52 R5727
[20] Derrida B 1997 Physica D 107 186
[21] Schehr G and Le Doussal P 2010 J. Stat. Mech. P01009
[22] Bénichou O, Krapivsky P L, Mejía-Monasterio C and Oshanin G 2016 Temporal correlations of the running maximum of a Brownian trajectory arXiv:1602.06770
[23] Lévy P 1948 Processus Stochastiques et Mouvement Brownien (Paris: Gauthier-Villars)
[24] Itô K and McKean H P 1965 Diffusion Processes and Their Sample Paths (New York: Springer)
[25] Redner S 2001 A Guide to First-Passage Processes (New York: Cambridge University Press)