AXION ELECTRODYNAMICS AND DARK MATTER FINGERPRINTS IN THE TERRESTRIAL MAGNETIC AND ELECTRIC FIELDS

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Abstract

We consider mathematical aspects of the axion electrodynamics in application to the problem of evolution of geomagnetic and terrestrial electric fields, which are coupled by relic axions born in the early Universe and (hypothetically) forming now the cold dark matter. We find axionic analogs of the Debye potentials, well-known in the standard Faraday - Maxwell electrodynamics, and discuss exact solutions to the equations of the axion electrodynamics describing the state of axionically coupled electric and magnetic fields in a spherical resonator Earth-Ionosphere. We focus on the properties of the specific electric and magnetic oscillations, which appeared as a result of the axion-photon coupling in the dark matter environment. We indicate such electric and magnetic field configurations as longitudinal electro-magnetic clusters.

Keywords: dark matter, axion - photon coupling, geomagnetism.

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1 Introduction

1.1 On the physical aspects of the problem of the axion - photon coupling

The term dark matter axions, which came into general use during the last decade, reflects a synthesis of two trends based on ideas of modern astrophysics and cosmology on the one hand, and of high-energy particle physics on the other hand. Dark matter, which neither emits nor scatters electromagnetic radiation, is considered to be one of the main cosmic substrates accumulating about 23% of the Universe energy. Mass density distribution of the dark matter is now well-studied due to observations and theoretical modeling (see, e.g., [1]-[4] for details, review and references). The origin of the dark matter is not yet established, but there are several hypotheses about its nature. The most attractive is the hypothesis that the dark matter is formed by massive pseudo-Goldstone bosons (axions) belonging to the class of WIMPs (Weakly Interacting Massive Particles). The story of axions, which were predicted by Peccei and Quinn [5] and introduced into the high-energy physics as new light bosons by Weinberg [6] and Wilczek [7], is well-described: one can find reviews and references, e.g., in [8]-[15].

We are interested in studying of the axion-photon coupling; more precisely, we consider the response of the photon system to the action of the axions and possible experimental consequences of such action. First of all, it is worth reminding what type of axions we are going to consider in this paper. It is well-known that axions can be generated if there exist strong magnetic field, strong electric field, and they are not orthogonal to one another. Such axion sources can exist in the Solar plasma, in the magnetosphere of neutron stars, and in laboratories on the Earth. Other mechanisms of the axion production are predicted to occur in the early Universe; the axions born in the early Universe form now the unique cosmological substrate indicated as relic (or primordial, or background) axions; these are the cosmological dark matter axions, and their density distribution depends on time. Since the axions are massive particles, their gravitational attraction forms non-homogeneous configurations; for instance, the density distribution in the galactic halos is stationary but depends on the distance from the galactic center. In other words we live (hypothetically) in an axionic environment and have a right to pose a question: how serious the influence of the axions on the terrestrial electrodynamical systems is.

Wave propagation in the axion electrodynamics is studied in detail for two cases: first, when one deals with running plane waves, second, when one uses the approximation of geometrical optics, i.e., the problem of dispersion relations can be effectively reduced to the analysis of the Fresnel equation (see, e.g., [16] [17] [18]). It was shown that the axion-photon...
interactions induce the effect of polarization rotation, if the electromagnetic waves travel through the axion system (see, e.g., [19]-[22] for details). The inertial-gradient extension of the Einstein-Maxwell-axion model [23] predicts that the optical activity induced by relic cosmological axions should be a nonstationary process and should keep information about the rate of the Universe expansion. Nonminimal extension of the Einstein-Maxwell-axion model [24] allows us to describe the birefringent wave propagation. Minimal two-parameter gradient-type extension of the Einstein-Maxwell-axion model [25] also predicts the existence of birefringence, as well as, could explain abnormal growth of the number of axions in the early Universe.

In this paper we focus on the electrodynamic effects, which can occur in the terrestrial electric and magnetic fields under the influence of the relic dark matter axions. Relic axion distribution is considered to be homogeneous but time dependent. We work here within the Weinberg - Wilczek - Ni version of the axion electrodynamics, and focus on the axionically coupled electromagnetic oscillations in a spherical resonator. The obtained results are new, they can have numerous applications and are planned to be generalized in the framework of nonminimal and gradient-type extensions of the Einstein - Maxwell - axion model.

1.2 On the mathematical aspects of the problem of the axion - photon coupling

The axion-photon interaction can be described both in terms of high-energy physics and in terms of macroscopic axion electrodynamics. We analyze the problem in the framework of axion electrodynamics in the Weinberg - Wilczek - Ni (WWN) version [6, 7, 26, 27]. The WWN model is the particular case of the Carroll - Field - Jackiw (CFJ) model [28], based on the Chern - Simons ansatz [29]. These models are often discussed in context of the Equivalence Principle and Lorentz violation (see, e.g., [30] - [34] for review and references). Two models: CFJ and WWN, are known to be effectively coinciding, when the pseudovector \( p_i \) appearing in the first model is proportional to the four-gradient of the pseudoscalar field \( \phi \), on which the WWN model is based (e.g., \( p_i = \nabla_i \phi \)). Let us mention that in the cosmological context, when pseudoscalar field is the function of time only, \( \phi(t) \), and the pseudovector \( p_i \) is time-like and has only one component, \( p_i = \delta^0_i \psi(t) \), both electrodynamic extensions: WWN and CFJ, - give the same results. It gave us a supplementary motivation to use the WWN version of the axion electrodynamics in this paper.

Generally, the axion electrodynamics in the Weinberg - Wilczek - Ni version is more
sophisticated from the mathematical point of view than the Faraday - Maxwell theory. The main extension is that the axion electrodynamics includes additional equation for the pseudoscalar (axion) field $\phi$, which, in its turn, enters the equations for the electromagnetic field strength $F_{ik}$ (the Maxwell tensor), as well as, the constitutive equations. The total self-consistent system of master equations is nonlinear. The set of boundary and initial conditions is extended accordingly. We are interested to solve the master equations of the axion electrodynamics, which guide the evolution of the magnetic and electric field in the spherical resonator "Earth-Ionosphere". In the framework of the Faraday - Maxwell electrodynamics this problem is solved and discussed in detail, in particular, in terms of Hertz and Debye potentials [35]. These results are classical and they were used in numerous applications, in particular, in radio- and tele-communication, navigation, magnetic storm predictions, etc. When the axion-photon coupling is taken into account, these results require serious modifications, since in this case the electromagnetic wave propagate in chiral (quasi)medium. For instance, in classical terrestrial electrodynamics the so-called $E$-waves and $H$-waves are introduced [35], for which the radial components of the magnetic and electric fields, respectively, are equal to zero. From the mathematical point of view, such decomposition of the total electromagnetic field essentially facilitates the analysis of the problem. In the axion electrodynamics the introduction of pure $E$- wave or pure $H$- wave is not possible, since the axionic environment rotates the polarization of the electromagnetic waves.

Our goal is to find some analogs of the Debye potentials for the model, which takes into account the axion - photon coupling. Specific feature of our approach is that the pseudoscalar field is considered to be formed by relic cosmological axions. This means, first, that the axion field $\phi$ is specified, and thus the corresponding evolutionary equation for the pseudoscalar field is decoupled from the total set of equations of the axion electrodynamics. This approximation is well-motivated, since the density of dark matter axions is much higher than the number of axions produced by terrestrial electromagnetic field itself. The second specific detail, which also is well-motivated in the context of relic cosmological axions, is that the pseudoscalar field $\phi(t)$ evolves with time much more slowly than the typical terrestrial electric and magnetic fields oscillate. More precisely, in our model we consider the time derivative of the pseudoscalar field $\dot{\phi}$ as a constant, and introduce appropriate quantity $q$, which is directly connected with actual mass density of the axionic dark matter in the vicinity of the Earth.

In other words, we focus on a typical problem of mathematical physics, which came from the theory of evolution of the terrestrial electric and magnetic fields, the axion - photon coupling being the new key element of theory.
1.3 Organization of the paper

The paper is organized as follows. In Section 2 we formulate the model equations of axion electrodynamics for the case, when the pseudoscalar field is formed by relic cosmological axions born in the early Universe and is revealed now as axionic dark matter. In Subsection 2.3 we consider two illustrations: one exact and one approximate solutions to these equations, which show that relic axions can form specific electrodynamic configurations with parallel coupled electric and magnetic fields. In Section 3 we introduce the so-called truncated (φ-independent) model and represent the reduced model equations in spherical coordinates. Section 4 is devoted to the analysis of static solutions to the basic equations: we discuss the axion electrostatics in Subsection 4.1, the axion magnetostatics in Subsection 4.2. In Subsection 4.3 we reduce the obtained results to the dipole-type model of the terrestrial magnetic field, and consider some properties of this model in Subsection 4.4. In Section 5 we analyze time-dependent solutions to the basic equations of the truncated (φ-independent) model in the spherical resonator: in Subsection 5.1 we obtain eigenfunctions for the boundary-value problem, frequencies and amplitudes of the axionically coupled modes; in Subsection 5.2 we solve the problem for the case, when the azimuthal electric current induces forced oscillations. In Section 6 we focus on the description of axionically coupled electric and magnetic oscillations near the Earth surface. Section 7 is devoted to discussion of the obtained results.

2 Axion-photon coupling in the axionic dark matter environment

2.1 The model

Let us start with the action functional

\[ S = \frac{\hbar}{c} \int d^4x \sqrt{-g} \left\{ \frac{R + 2\Lambda}{\kappa} + \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} \phi F^*_{mn} F^{mn} - \Psi_0^2 \left[ \nabla_m \phi \nabla^m \phi - V(\phi^2) \right] \right\}. \]  

(1)

Here, \( g \) is the determinant of the metric tensor \( g_{ik} \), \( \nabla_m \) is the covariant derivative, \( R \) is the Ricci scalar, \( \kappa \equiv \frac{8\pi G}{c^4} \), \( G \) is the gravitational Newtonian coupling constant, \( \Lambda \) is the cosmological constant, \( \hbar \) is the Planck constant and \( c \) is the speed of light in vacuum. The Maxwell tensor \( F_{mn} \) is given by

\[ F_{mn} \equiv \nabla_m A_n - \nabla_n A_m, \quad \nabla_k F^{*ik} = 0, \]  

(2)

where \( A_m \) is a potential four-vector of the macroscopic electromagnetic field; \( F^{*mn} \equiv \frac{1}{2} \epsilon^{mnpq} F_{pq} \) is the tensor dual to \( F_{pq} \); \( \epsilon^{mnpq} \equiv \frac{1}{\sqrt{-g}} E^{mnpq} \) is the Levi-Civita tensor, \( E^{mnpq} \) is
the absolutely antisymmetric Levi-Civita symbol with $E^{0123}=1$.

The first term in (1) is the Hilbert-Einstein Lagrangian for the gravity field with the account of the cosmological constant. The second term $\frac{1}{2} F_{mn} F^{mn}$ relates to the standard linear electrodynamics in vacuum. The term $\frac{1}{2} \phi F^*_{mn} F^{mn}$ in (1) describes the pseudoscalar-photon interaction [26]. The symbol $\phi$ stands for an effective pseudoscalar field; this quantity is dimensionless providing the second and third terms in the integral to be of the same dimensionality. The macroscopic effective axion field, $\Phi$, is considered to be proportional to this quantity $\Phi=\Psi_0\phi$, and the constant $\Psi_0$ is the subject of a special discussion. The parameter $\Psi_0^2$ has the dimensionality of a force, and its estimation depends substantially on the model of the axion coupling, on the value of the axion-photon coupling constant $g_{(\Lambda\gamma\gamma)}$, on the individual axion mass $m_{(\Lambda)}$, on the number of axions in the unit volume $N_{(\Lambda)}$, etc. (see, e.g., [10, 36]). The function $V(\phi^2)$ describes the potential of the pseudoscalar field; in the simplest case we use the quadratic potential $V=\mu^2 \phi^2$, where $\mu=\frac{m_{(\Lambda)c}}{h}$ has the dimensionality of inverse length.

Below we assume that the gravitational background is given, the space-time being of the Friedmann-Lemaître-Robertson-Walker (FLRW) type

$$ds^2 = a^2(x^0) \left[(dx^0)^2 - dl^2\right], \quad (3)$$

with the scale factor $a$ and the Hubble function

$$H = \frac{1}{a^2} \frac{da}{dx^0} = \dot{a} = \frac{\dot{a}}{a}. \quad (4)$$

Here and below the dot denotes the derivative with respect to time $t$, which is connected with $x^0$ by the differential relation

$$a(x^0) \, dx^0 = cdt. \quad (5)$$

The metric in the three-space is assumed to be represented in the spherical coordinates

$$dl^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (6)$$

and we use the formula $\sqrt{-g} = a^4 \, r^2 \sin \theta$. This means that we do not consider back-reaction of the electromagnetic field on the gravitational field; moreover, we neglect the direct influence of the terrestrial gravity field on the electric and geomagnetic fields in comparison with the influence of the relic cosmological axions.

The effective macroscopic pseudoscalar field, appearing in the action functional (1), satisfies the equation

$$\nabla_k \nabla^k \phi + \phi V'(\phi^2) = -\frac{1}{4\Psi_0^2} F^*_{mn} F^{mn}, \quad (7)$$
however, we assume that the number of axions produced by the macroscopic electromagnetic field $F_{ik}$ is much less than the number of relic (primordial) axions created in the early Universe. This means that we neglect the electromagnetic source in the right-hand side of the equation (7) and consider the function $\phi(t)$ to satisfy the decoupled equation

$$\ddot{\phi} + 3H\dot{\phi} + \mu^2c^2\phi = 0. \quad (8)$$

The last and very important element of this macroscopic model comprises the relation between the effective pseudoscalar field $\phi$ and the dark matter state functions, the energy density $W(t)$ and pressure $P(t)$. These relations appear when we put equal the effective stress-energy tensor of the macroscopic pseudoscalar field on the one hand, and the phenomenological stress-energy tensor of the dark matter fluid, on the other hand (see, e.g., [37, 25] for details); they have the following form

$$\frac{1}{2}\Psi_0^2 \left[ \frac{q^2}{a^2} + \mu^2\phi^2 \right] = W, \quad (9)$$

$$\frac{1}{2}\Psi_0^2 \left[ \frac{q^2}{a^2} - \mu^2\phi^2 \right] = P. \quad (10)$$

The quantity $q$ is defined as

$$q \equiv \frac{d\phi}{dx^0} = \frac{a}{c}\dot{\phi}, \quad (11)$$

and can be readily found as

$$q = \pm \frac{a}{\Psi_0} \sqrt{W + P}. \quad (12)$$

For the cold dark matter $P \to 0$ and $W \to \rho c^2$, thus

$$q \to \pm \frac{ca}{\Psi_0} \sqrt{\rho_{(DM)}}, \quad (13)$$

where $\rho_{(DM)}$ is the mass density of the cold dark matter.

### 2.2 Electrodynamic equations

The equations of the axion electrodynamics have the following form

$$\nabla_k \left[ F^{ik} + \phi F^{*ik} \right] = -\frac{4\pi}{c} J^i, \quad (14)$$

where $J^i$ is the electric current four-vector. Taking into account the equations (2) one can rewrite these equations as follows:

$$\nabla_k F^{ik} = -\frac{4\pi}{c} \left[ J^i + J^i_{(A)} \right], \quad (15)$$

where

$$\frac{4\pi}{c} J^i_{(A)} \equiv F^{*ik} \nabla_k \phi. \quad (16)$$
When the electric current four-vector $J^i$ describes the standard conductivity, we can write

$$J^i = \sigma E^i = \sigma F^{ik} U_k = \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) \sigma U_k F_{mn},$$

(17)

where $U^k$ is the four-vector of the macroscopic velocity of the system as a whole. Similarly, we can write

$$\frac{4\pi}{c} J_i^{(A)} = \frac{1}{2} \epsilon^{ikmn} \nabla_k \phi F_{mn},$$

(18)

and link this term with axionic conductivity. Clearly, the four-pseudovector $\frac{4\pi}{c} \nabla \phi$ replaces the four-vector $\sigma U_k$, when the axionic conductivity is discussed.

### 2.3 Prologue: 3-dimensional representation of the model and spatially homogeneous solutions

In 1987 Wilczek (see the paper [27]) considered the application of axion electrodynamics to the problem of spherically symmetric static dyons. The dyon-type solution was obtained using two ingredients: a magnetic monopole plus axion field. In this case the static axion field produces static radial electric field using the radial magnetic field of the monopole. In fact, Wilczek presented the first (static) example of the longitudinal electro-magnetic cluster, since the static electric and magnetic fields are parallel in this spherically symmetric configuration. Now we illustrate the idea of longitudinal clusters by the example, in which the electric, magnetic and axion fields are spatially homogeneous, but depend on time.

Equations of the axion electrodynamics can be written in the three-vectorial form as follows:

$$\text{rot} \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B}, \quad \text{div} \vec{E} = 4\pi \rho - \vec{B} \cdot \vec{\nabla} \phi,$$

(19)

$$\text{div} \vec{B} = 0, \quad \text{rot} \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{E} + \frac{4\pi}{c} \vec{J} + \frac{1}{c} \vec{B} \frac{\partial}{\partial t} \phi - \vec{E} \times \vec{\nabla} \phi.$$

(20)

We assume here that the pseudoscalar (axion) field depends on time only, i.e., $\phi=\phi(t)$, and $\vec{\nabla} \phi=0$. Also, we assume that $\rho=0$ and $\vec{J}=0$, i.e., we study electrodynamic effects in the regions out of the electric charges. Then there exists the following exact constant solution: the magnetic field is constant, i.e., $\vec{B}=\vec{B}_0=\text{const}$, and the electric field is homogeneous, i.e., $\vec{E}=\vec{E}(t)$. Indeed, all the equations (19) and (20) are satisfied, if

$$0 = \frac{d}{dt} \vec{E} + \vec{B} \frac{d}{dt} \phi.$$

(21)

Clearly, the exact solution for the electric field reads

$$\vec{E}(t) = \vec{E}(0) - [\phi(t) - \phi(0)] \vec{B}_0.$$

(22)
This means that in the presence of constant magnetic field in addition to the initial arbitrary directed electric field \( \vec{E}(0) \) the electric field collinear to this magnetic field appears due to the influence of the evolving system of relic axions.

Now, let us consider an illustrative approximate model, in which the magnetic field is not constant, \( \vec{B}=\vec{B}(t) \), but it varies with time very slowly. Such situation is typical for the problem of infra-low-frequency variations of the terrestrial electric and magnetic fields. For instance, the variations of the geomagnetic field on the frequency \( \omega/2\pi=10^{-3} \) Hz have the effective wavelength of the order \( \lambda=300 \cdot 10^6 \) km, which is the distance to the Sun and back. Thus, the right-hand side of the first equation in (19) can be considered negligible, and the electric field, again, can be treated as homogeneous. Then, one obtains from (20)

\[
\vec{E}(t) = \vec{E}(0) - c \int_0^t d\tau q(\tau) \vec{B}(\tau). \tag{23}
\]

In comparison with the infra-low-frequency variations of the geomagnetic field the function \( q(t) \) behaves as a constant, and we can use the simplified formula

\[
\vec{B}(t) = \vec{B}_0 + \delta \vec{B} \cdot \cos \omega t \rightarrow \vec{E}(t) = \vec{E}(0) - [\phi(t) - \phi(0)] \vec{B}_0 - \frac{cq}{\omega} \delta \vec{B} \cdot \sin \omega t, \tag{24}
\]

where \( \delta \vec{B} \) denotes the amplitude of the geomagnetic field variation, the quantity \( \omega \) being its frequency. This is a good illustration of our idea to focus on the analysis of the electric field variations produced by the geomagnetic field variations in the relic axion background. Clearly, this electric field induced by the axions is parallel to the geomagnetic field variation, that is why we use the term “longitudinal electro-magnetic cluster” to indicate this specific mode in the combined electric and magnetic fields variations.

3 Electrodynamics in the relic axion background

3.1 Truncated (\( \varphi \)-independent) model

Let us now apply the model to the case of terrestrial electric and magnetic fields. In fact, the geomagnetic field is non-homogeneous, and we have to take into account the curvature of the magnetic field lines when consider global effects in the Earth’s magnetosphere. Below we discuss the non-stationary electric and magnetic fields, and start with the potentials, which do not depend on the azimuthal variable \( \varphi \), i.e., \( A_i=A_i(t,r,\theta) \) for \( i=0,r,\theta,\varphi \). This model is truncated, nevertheless, it illustrates all the important properties of the axion-photon coupling in the Earth’s electrodynamic system.

One can introduce the truncated (\( \varphi \)-independent) model using the following approach. Let us remind that the flat space-time metric admits the existence of the Killing vector...
\( \xi_{(\varphi)} = \delta_{\varphi} \) and the Lie derivative of the metric is equal to zero,

\[
\mathcal{L}_{\xi_{(\varphi)}} g_{mn} = \xi^l_{(\varphi)} \frac{\partial}{\partial x^l} g_{mn} + g_m \frac{\partial}{\partial x^m} \xi^l_{(\varphi)} + g_m l \frac{\partial}{\partial x^m} \xi^l_{(\varphi)} = 0 ,
\]

so in the standard spherical coordinates the metric does not depend on \( \varphi \). Let us assume that the potential four-vector \( A_k \) inherits this symmetry, i.e.,

\[
\mathcal{L}_{\xi_{(\varphi)}} A_k = \xi^l_{(\varphi)} \frac{\partial}{\partial x^l} A_k + A_l \frac{\partial}{\partial x^l} \xi^l_{(\varphi)} = 0 .
\]

This provides the condition \( \frac{\partial A_k}{\partial \varphi} = 0 \).

### 3.2 Master equations

The equations (14) can be written as follows. The first equation (for \( i=0 \))

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_{0r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_{\theta \varphi}) = 4 \pi \rho ,
\]

where \( \rho \equiv \frac{a^4}{c} J^0 \), does not contain the quantity \( q \). Three equations for \( i=r, \theta, \varphi \) give, respectively,

\[
\frac{\partial}{\partial x^0} F_{0r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_{r \varphi}) = \frac{q}{r^2 \sin \theta} F_{\theta \varphi} - \frac{4 \pi}{c} I^r ,
\]

\[
\frac{\partial}{\partial x^0} F_{0\theta} + \frac{1}{r} \frac{\partial}{\partial r} F_{\theta \varphi} = \frac{q}{\sin \theta} F_{r \varphi} + \frac{4 \pi}{c} I^\theta ,
\]

\[
\frac{\partial}{\partial x^0} F_{0\varphi} + \frac{1}{r} \frac{\partial}{\partial r} F_{\varphi \theta} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{F_{\varphi \theta}}{r^2 \sin \theta} \right) = \frac{q}{\sin \theta} F_{r \theta} - \frac{4 \pi}{c} r^2 I^\varphi \sin^2 \theta .
\]

Here the following notations have been used

\[
I^r = a^4 J^r , \quad I^\theta = a^4 J^\theta , \quad I^\varphi = a^4 J^\varphi ,
\]

in these terms the equations (27)-(30) do not contain the scale factor \( a(x^0) \) at all, so that the only quantity \( q \) reminds that these are the model equations of axion electrodynamics.

The components of the electric current four-vector are linked by the charge conservation law

\[
c \frac{\partial}{\partial x^0} \rho + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 I^r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta I^\theta) = 0 .
\]

The second subset of equations in (2) gives one equation of the form

\[
\frac{\partial}{\partial \theta} F_{0r} + \frac{1}{r} \frac{\partial}{\partial r} F_{0\theta} + \frac{1}{r x^0} F_{r \varphi} = 0 ,
\]

which does not include the symbol \( \varphi \). Other equations

\[
\frac{\partial}{\partial \theta} F_{r \varphi} + \frac{\partial}{\partial r} F_{\theta \varphi} = 0 , \quad \frac{\partial}{\partial \theta} F_{\varphi \theta} + \frac{\partial}{\partial x^0} F_{r \varphi} = 0 , \quad \frac{\partial}{\partial r} F_{\varphi \theta} + \frac{\partial}{\partial x^0} F_{\varphi \theta} = 0 ,
\]

contain only two terms for the truncated model under consideration.

Linearity of the electrodynamic equations allows us to consider separately static and time-dependent solutions of the model. Let us start with the static ones.
4 Static electric and magnetic fields

Since the axion field changes extremely slow, and we suppose \( q = \text{const} \), it is convenient to reconstruct the electric and magnetic fields in terms of two potential components, namely, \( A_0(r, \theta) \) and \( A_\varphi(r, \theta) \). Let us demonstrate the procedure and the obtained results for electric and magnetic fields out of domain of charges distribution.

4.1 Axion electrostatics

As usual, we distinguish the coordinate and physical (with subscripts in parentheses) components of the electric field, defined as follows

\[
E^r \equiv F^{r0} = -\frac{\partial A_0}{\partial r} \equiv E^\text{(rad)} ;
\]

\[
E^\theta \equiv F^{\theta 0} = -\frac{1}{r^2} \frac{\partial A_0}{\partial \theta} \equiv \frac{1}{r} E^\text{(merid)} ;
\]

\[
E^\varphi \equiv F^{\varphi 0} = -\frac{1}{r^2 \sin^2 \theta} \frac{\partial A_0}{\partial \varphi} \equiv \frac{1}{r \sin \theta} E^\text{(azim)} = 0 .
\]

Let us remind the motivation of these definitions. Since there exist two sets of quantities: the covariant and contravariant components of the electric field three-vector \( (E_r, E_\theta, E_\varphi) \) and \( E^r, E^\theta, E^\varphi \), respectively), one needs to introduce the so-called physical components, which are assumed to be measured in experiments. We follow the Synge’s concept (see, e.g., [38]) and introduce the tetrad four-vectors

\[
\lambda^i_{(0)} = \frac{1}{a} \delta^i_0 , \quad \lambda^i_{\text{(rad)}} = -\frac{1}{a} \delta^i_r , \quad \lambda^i_{\text{(merid)}} = -\frac{1}{ar} \delta^i_\theta , \quad \lambda^i_{\text{(azim)}} = -\frac{1}{ar \sin \theta} \delta^i_\varphi ,
\]

which are orthogonal one to another and satisfy the normalization conditions

\[
g_{ik} \lambda^i_{(0)} \lambda^k_{(0)} = 1 , \quad g_{ik} \lambda^i_{\text{(rad)}} \lambda^k_{\text{(rad)}} = -1 , \quad g_{ik} \lambda^i_{\text{(merid)}} \lambda^k_{\text{(merid)}} = -1 , \quad g_{ik} \lambda^i_{\text{(azim)}} \lambda^k_{\text{(azim)}} = -1 ,
\]

when the space-time metric has the form (3) with (6). The corresponding components of the electric field three-vector are defined as

\[
E_i_{\text{(rad)}} \equiv \frac{1}{a} E_i \lambda^i_{\text{(rad)}} , \quad E_i_{\text{(merid)}} \equiv \frac{1}{a} E_i \lambda^i_{\text{(merid)}} , \quad E_i_{\text{(azim)}} \equiv \frac{1}{a} E_i \lambda^i_{\text{(azim)}} ,
\]

providing the relationships (35)-(37). In terms of standard three-dimensional metric (6) these quantities can be formally expressed as

\[
E_i_{\text{(rad)}} \equiv \sqrt{|E^r E_r|} , \quad E_i_{\text{(merid)}} \equiv \sqrt{|E^\theta E_\theta|} , \quad E_i_{\text{(azim)}} \equiv \sqrt{|E^\varphi E_\varphi|} .
\]

The equation (27) with \( \rho = 0 \) yields the truncated Laplace equation

\[
\Delta_{(0)} A_0 = 0 .
\]
Here and below we use the following truncated Laplace operator for the sake of convenience:
\[
\Delta_{(m)} \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} \right].
\] (43)

The equation (42) does not contain the quantity \( q \), so, the corresponding solution is well-known in classical mathematical physics. E.g., for the domain \( r > R \) it has the form
\[
A_0(r, \theta) = \sum_{n=0}^{\infty} C_n \left( \frac{R}{r} \right)^{n+1} P_n(\cos \theta),
\] (44)
where \( R \) is the Earth radius. We use the standard definitions: \( P_n(x) \) for the Legendre polynomials [39, 35] and \( P_n^{(m)}(x) = (1-x^2)^{\frac{m}{2}} \left( \frac{d}{dx} \right)^m P_n(x) \) for the adjoint Legendre polynomials. As usual, the coefficients \( C_n \) can be found from boundary conditions (we do not consider this problem here). Clearly, the static electric field components
\[
E_{(\text{rad})} = \sum_{n=0}^{\infty} \frac{(n+1)}{R} C_n \left( \frac{R}{r} \right)^{n+2} P_n(\cos \theta),
\] (45)
\[
E_{(\text{merid})} = \sum_{n=0}^{\infty} \frac{C_n}{R} \left( \frac{R}{r} \right)^{n+2} P_n^{(1)}(\cos \theta), \quad E_{(\text{azim})} = 0,
\] (46)
do not feel the influence of axions.

### 4.2 Axion magnetostatics

Magnetic field components are defined as follows:
\[
B^r \equiv F^{s0} = -\frac{1}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \theta} = B_{(\text{rad})},
\] (47)
\[
B^\theta \equiv F^{r0} = \frac{1}{r^2 \sin \theta} F_{r\theta}, \quad B_{(\text{merid})} = -\frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial r},
\] (48)
\[
B^\varphi \equiv F^{\varphi0} = -\frac{1}{r^2 \sin \theta} F_{\varphi}, \quad B_{(\text{azim})} = -\frac{1}{r} \left[ \frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta} \right],
\] (49)

where the so-called physical components of the magnetic field
\[
B_{(\text{rad})} \equiv \sqrt{|B_r B^r|}, \quad B_{(\text{merid})} \equiv \sqrt{|B_\theta B^\theta|}, \quad B_{(\text{azim})} \equiv \sqrt{|B_\varphi B^\varphi|}
\] (50)
are obtained in analogy with the electric field components (see [38]–[41]). Since the equations (28) and (29) convert now into
\[
\frac{\partial}{\partial \theta} (\sin \theta F_{r\theta} - q A_\varphi) = 0, \quad \frac{\partial}{\partial r} (\sin \theta F_{r\theta} - q A_\varphi) = 0,
\] (51)
we obtain immediately that
\[
F_{r\theta} = \frac{q}{\sin \theta} A_\varphi.
\] (52)
Then the equation (30) reduces to
\[
\left[ r^2 \frac{\partial^2}{\partial r^2} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) + q^2 r^2 \right] A_\varphi = 0 ,
\]
(53)
and using the new potential \( U(r, \theta) \), defined by
\[ A_\varphi = r \sin \theta \ U, \quad U = \sqrt{|A_\varphi A^2|} , \]
(54)
we obtain the truncated (\( \varphi \)-independent) Helmholtz equation \[39\] \[35\]
\[ \Delta(1) U + q^2 U = 0 . \]
(55)
The solution of (55) has the form
\[ U = \sum_{n=1}^{\infty} P_n^{(1)}(\cos \theta) \ \mathcal{R}_n(r, q) , \]
(56)
where the radial function \( \mathcal{R}_n(r, q) \) depends essentially on the parameter \( q \). When \( q = 0 \), i.e., we neglect the axion influence at all, this function is \[39\] \[35\]
\[ \mathcal{R}_n(r, 0) = A_n \left( \frac{r}{R} \right)^n + B_n \left( \frac{r}{R} \right)^{-(n+1)} . \]
(57)
When \( q \neq 0 \), the radial function can be expressed in terms of Bessel functions
\[ \mathcal{R}_n(r, q) = \sqrt{\frac{R}{r}} \left\{ A_n \left[ \Gamma \left( n+\frac{1}{2} \right) \left( \frac{1}{2} qR \right)^{-(n+\frac{1}{2})} \right] J_{n+\frac{1}{2}}(qr) + \\
B_n \left[ (-1)^n \frac{\pi}{\Gamma \left( n+\frac{1}{2} \right)} \left( \frac{1}{2} qR \right)^{n+\frac{1}{2}} \right] J_{-(n+\frac{1}{2})}(qr) \right\} , \]
(58)
where \( J_{n+\frac{1}{2}}(qr) \) is the Bessel function of the first kind with the half-integer index \[39\] \[35\].
The constants of integration are chosen so that \( \lim_{q \to 0} \mathcal{R}_n(r, q) = \mathcal{R}_n(r, 0) \) (see (57)).

The corresponding physical components of the magnetic field are
\[ B_{(\text{rad})} = - \sum_{n=1}^{\infty} \frac{n(n+1)}{r} P_n(\cos \theta) \ \mathcal{R}_n(r, q) , \]
(59)
\[ B_{(\text{merid})} = - \sum_{n=1}^{\infty} P_n^{(1)}(\cos \theta) \frac{1}{r} \frac{d}{dr} [r \mathcal{R}_n(r, q)] , \]
(60)
\[ B_{(\text{azim})} = -q \sum_{n=1}^{\infty} P_n^{(1)}(\cos \theta) \mathcal{R}_n(r, q) . \]
(61)
The main new feature is the following: the azimuthal component \( B_{(\text{azim})} \), being equal to zero at \( q=0 \), becomes nonvanishing at \( q \neq 0 \).
4.3 Dipole-type magnetic field

In the zeroth-order approximation the terrestrial magnetic field can be described in terms of geomagnetic dipole. In the framework of this model we obtain basic formulas if put \( n = 1 \) into (56)-(61), what yields

\[
B_{(\text{rad})} = -\frac{2}{r} \cos \theta \ \Re_1(r, q), \tag{62}
\]

\[
B_{(\text{merid})} = -\frac{\sin \theta}{r} \frac{d}{dr} [r \Re_1(r, q)], \tag{63}
\]

\[
B_{(\text{azim})} = -q \sin \theta \ \Re_1(r, q). \tag{64}
\]

We consider two auxiliary angles. The first auxiliary angle \( \gamma \), defined as

\[
\tan \gamma \equiv \frac{B_{(\text{azim})}(R)}{B_{(\text{rad})}(R)} = \frac{qR}{2} \tan \theta, \tag{65}
\]

is linear in the parameter \( q \) and depends on the meridional angle \( \theta \). As for the second auxiliary angle \( \delta \), defined as

\[
\tan \delta \equiv \frac{B_{(\text{azim})}(R)}{B_{(\text{merid})}(R)} = qR \left[ 1 + R \frac{\Re_1(R, q)}{\Re_1(R, R)} \right]^{-1}, \tag{66}
\]

it does not depend on \( \theta \). For the domain \( r > R \) by substitutions \( A_n = 0 \) and \( B_1 = \frac{\mu}{R^2} \) we obtain

\[
\Re_1(r, q) = \frac{\mu}{r^2} \left( \cos qr + qr \sin qr \right), \tag{67}
\]

and thus

\[
B_{(\text{rad})}(r, q) = -\frac{2\mu}{r^3} \cos \theta \left( \cos qr + qr \sin qr \right), \tag{68}
\]

\[
B_{(\text{merid})}(r, q) = \frac{\mu \sin \theta}{r^3} \left[ \left( \cos qr + qr \sin qr \right) - q^2 r^2 \cos qr \right], \tag{69}
\]

\[
B_{(\text{azim})}(r, q) = -q \sin \theta \frac{\mu}{r^2} \left( \cos qr + qr \sin qr \right). \tag{70}
\]

In the approximation \( q \to 0 \) these formulas give

\[
B_{(\text{rad})}(r, 0) = -\frac{2\mu}{r^3} \cos \theta, \quad B_{(\text{merid})}(r, 0) = \frac{\mu \sin \theta}{r^3}, \quad B_{(\text{azim})}(r, 0) = 0, \tag{71}
\]

as it should be in classical theory of static dipole geomagnetic field. Let us emphasize that in the approximation linear in \( q \) the angle \( \delta \) is

\[
\tan \delta = -qR, \tag{72}
\]

thus, the axion-photon coupling can be considered as one of the explanations of the (visual) magnetic poles drift.
4.4 Axionically coupled magnetic field as a function of altitude

Classical dipole-type magnetic field decreases monotonically at $r \to \infty$ according to the law $1/r^3$ (see, e.g., (71)). In the axionic environment, i.e., when $q \neq 0$, the magnetic field components behave non-monotonically (see (70)). Indeed, there is infinite number of values of the radius $r=R^*_m$, for which the radial function $\Re_1(r,q)$ takes zero value; they can be found from the equation

$$\cos[qR^*_m] + qR^*_m \sin[qR^*_m] = 0,$$

or equivalently,

$$\cot z = -z, \quad z = qR^*_m.\quad (73)$$

At the same altitudes $r=R^*_m$ the radial and azimuthal components of the magnetic field change the signs. The position of the first null is defined by the inequalities $z < \pi$ and $R^*_1 < \pi/q$. The radial function has infinite number of extrema at $r = R^{**}_{(j)}$; these quantities satisfy the equation

$$\tan z = \frac{z}{2} - \frac{1}{z}, \quad z = qR^{**}_{(j)}.\quad (74)$$

The first extremum is placed at $z < 3\pi/2$, $R^{**}_{(1)} < 3\pi/2q$. In other words, the relic axions change essentially the configuration of the magnetic field.

5 Time dependent electric and magnetic fields in the relic axion background

Taking into account the linearity of the model of terrestrial axion electrodynamics in the relic axion background, we distinguish two cases in the non-stationary problem. First of all, we analyze natural oscillations and consider the electric current four-vector to be vanishing. Then we discuss the effects induced by the azimuthal electric current.

5.1 Natural oscillations of electric and magnetic fields in the spherical resonator Earth-Ionosphere

5.1.1 Key equations

When the electric current four-vector is equal to zero, the equations (27)-(30) and (33), (34) can be reduced to the coupled pair of equations for two new potentials $U$ and $V$, ...
which play in axion electrodynamics the same role as the Debye potentials \[35\] play in the electrodynamics of Faraday - Maxwell. Let us use the substitutions

\[
F_{0r} = -\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (V \sin \theta) = E_{(\text{rad})},
\]

\[
F_{\theta 0} = -\frac{\partial}{\partial r} (rV) = -rE_{(\text{merid})},
\]

\[
F_{\varphi 0} = -r \sin \theta \frac{\partial}{\partial x^0} U = -r \sin \theta E_{(\text{azim})},
\]

\[
F_{r \theta} = r \left( qU + \frac{\partial}{\partial x^0} V \right) = -rB_{(\text{azim})},
\]

\[
F_{r \varphi} = r \sin \theta \frac{\partial}{\partial r} (rU) = -r \sin \theta B_{(\text{merid})},
\]

\[
F_{\theta \varphi} = r \frac{\partial}{\partial \theta} (U \sin \theta) = -r^2 \sin \theta B_{(\text{rad})},
\]

\[
A_{\varphi} = rU \sin \theta.
\]

Then the equations (27), (28), (29) and (34) turn into identities, the equation (33) gives

\[
\Delta (1) V - \frac{\partial^2}{\partial x^0 \partial x^0} V = q \frac{\partial}{\partial x^0} U,
\]

and the equations (30) takes the form

\[
\Delta (1) U - \frac{\partial^2}{\partial x^0 \partial x^0} U = -q^2 U - q \frac{\partial}{\partial x^0} V.
\]

We indicate the pair of equations (83) and (84) as key equations. When \( q \neq 0 \), using the decoupling procedure for the equations (83) and (84), we obtain the key equation of the fourth order for the function \( V \):

\[
\left\{ \left[ \Delta (1) - \frac{\partial^2}{\partial x^0 \partial x^0} \right]^2 + q^2 \Delta (1) \right\} V = 0.
\]

When \( V \) is found, the function \( U \) can be extracted from (83) by integration over time.

5.1.2 Boundary value problem, eigenvalues and eigenfunctions

Let us consider electromagnetic oscillations in the spherical resonator bounded by the Earth surface \( (r=R) \) and the bottom edge of the Earth Ionosphere \( (r=R_*) \) in the framework of the truncated model.

Let the quantities \( F_{nj}(r, \theta) \) be the eigenfunctions of the operator \( \Delta (1) \) satisfying the boundary conditions

\[
F_{nj}(R, \theta) = 0, \quad F_{nj}(R_*, \theta) = 0.
\]
They have the well-known multiplicative form

\[ F_{nj}(r, \theta) = P_n^{(1)}(\cos \theta) \mathcal{H}_{nj}(r), \tag{87} \]

where the radial functions \( \mathcal{H}_{nj}(r) \) are

\[ \mathcal{H}_{nj}(r) = \frac{1}{\sqrt{r}} \left[ J_{n+\frac{1}{2}}(\nu_j^{(n)} R) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R) - J_{n+\frac{1}{2}}(\nu_j^{(n)} R) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R) \right]. \tag{88} \]

Boundary conditions \(\text{(86)}\) are satisfied, when the parameters \( \nu_j^{(n)} \) are found from the equation

\[ J_{n+\frac{1}{2}}(\nu_j^{(n)} R) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R_s) = J_{n+\frac{1}{2}}(\nu_j^{(n)} R_s) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R), \tag{89} \]

where the index \( j = 0, 1, 2, \ldots \) counts the positive zeros of the equations \(\text{(89)}\). Of course, the boundary conditions \(\text{(86)}\) are chosen as the simplest example, which can illustrate our main statements; as for applications, we intend to use more sophisticated boundary conditions in special papers.

We search for the potential functions \( U \) and \( V \) in the form

\[ U(\tilde{t}, r, \theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} u_{nj}(\tilde{t}) P_n^{(1)}(\cos \theta) \mathcal{H}_{nj}(r), \tag{90} \]

\[ V(\tilde{t}, r, \theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} v_{nj}(\tilde{t}) P_n^{(1)}(\cos \theta) \mathcal{H}_{nj}(r), \tag{91} \]

where we put for the sake of simplicity that \( x^0 \equiv c\tilde{t} \). The key equations \(\text{(84)}\) and \(\text{(83)}\) give the following coupled pair of equations for the time dependent amplitudes:

\[ \ddot{u}_{nj} + c^2 \left( \nu_j^{(n)} \right)^2 u_{nj} - q c \dot{v}_{nj} = 0, \tag{92} \]

\[ \ddot{v}_{nj} + c^2 \left( \nu_j^{(n)} \right)^2 v_{nj} + q c \dot{u}_{nj} = 0. \tag{93} \]

The axionic parameter enters these equation non-equivalently.

5.1.3 Solutions for the mode amplitudes

In order to simplify the analysis we suppose here that the axionic parameter \( q \) is positive, \( q > 0 \). The system of coupled equations \(\text{(92)}\) and \(\text{(93)}\) has standardly the solutions of three types. When \( q \) is so small that \( q < \nu_j^{(n)} \) for all \( j \) and \( n \), we obtain harmonic oscillations for both functions

\[ v_{nj}(\tilde{t}) = \left[ C_{1nj}^{(1)} \cos \omega_{1nj} \tilde{t} + C_{2nj}^{(1)} \sin \omega_{1nj} \tilde{t} \right] + \left[ C_{1nj}^{(2)} \cos \omega_{2nj} \tilde{t} + C_{2nj}^{(2)} \sin \omega_{2nj} \tilde{t} \right], \tag{94} \]
\[ u_{nj}(\tilde{t}) = \frac{1}{\sqrt{1 - \frac{q}{\nu_j^{(n)}}}} \left[ C_{1n,1}^{(1)} \cos \omega_{1n,1} \tilde{t} - C_{1n,2}^{(1)} \sin \omega_{1n,1} \tilde{t} \right] + \]
\[ + \frac{1}{\sqrt{1 + \frac{q}{\nu_j^{(n)}}}} \left[ C_{1n,1}^{(2)} \sin \omega_{2n,1} \tilde{t} - C_{1n,2}^{(2)} \cos \omega_{2n,1} \tilde{t} \right]. \]  
\[(95)\]

The constants \( C_{1n,1}^{(1)}, C_{1n,2}^{(1)}, C_{1n,1}^{(2)} \) and \( C_{1n,2}^{(2)} \) can be found from initial data. The hybrid frequencies
\[ \omega_{1n,j} = c \nu_j^{(n)} \sqrt{1 - \frac{q}{\nu_j^{(n)}}}, \quad \omega_{2n,j} = c \nu_j^{(n)} \sqrt{1 + \frac{q}{\nu_j^{(n)}}}, \]  
\[(96)\]
coincide when \( q = 0 \), i.e., when there is no axion-photon coupling.

Formally speaking, for some mode numbers, say, \( n < n^* \) and \( j < j^* \), the quantity \( q \) can exceed \( \nu_j^{(n*)} \). The corresponding modes are non-harmonic and one should replace the first pair \( \cos \omega_{1n,j} \tilde{t} \) and \( \sin \omega_{1n,j} \tilde{t} \) by hyperbolic functions \( \cosh \Gamma_{1n,j} \tilde{t} \) and \( \sinh \Gamma_{1n,j} \tilde{t} \), where
\[ \Gamma_{1n,j} = \sqrt{\nu_j^{(n*)} - 1}. \]

When \( q = \nu_j^{(n*)} \), the corresponding mode amplitudes
\[ v_{\ast}(\tilde{t}) = \frac{1}{2} \left[ v_{\ast}(0) - \frac{\dot{u}_{\ast}(0)}{cq} \right] + \frac{1}{2} \left[ v_{\ast}(0) + \frac{\dot{u}_{\ast}(0)}{cq} \right] \cos \sqrt{2cq} \tilde{t} + \frac{\dot{v}_{\ast}(0)}{\sqrt{2cq}} \sin \sqrt{2cq} \tilde{t}, \]  
\[ u_{\ast}(\tilde{t}) = \left[ u_{\ast}(0) + \frac{\dot{v}_{\ast}(0)}{2cq} \right] - \frac{cq}{2} \left[ v_{\ast}(0) - \frac{\dot{u}_{\ast}(0)}{cq} \right] + \]
\[ + \frac{1}{2\sqrt{2}} \left[ v_{\ast}(0) + \frac{\dot{u}_{\ast}(0)}{cq} \right] \sin \sqrt{2cq} \tilde{t} - \frac{\dot{v}_{\ast}(0)}{2cq} \cos \sqrt{2cq} \tilde{t}, \]  
\[(97)\]
display the possibility of oscillations on the specific (axionic) frequency \( \omega_A = \sqrt{2cq} \), as well as, the linear growth with time of the \( U \)-potential.

5.2 Forced oscillations produced by an azimuthal electric current

Let us consider the models in which the only azimuthal component of the electric current four-vector \( J^\varphi \) is non-vanishing. This case is very illustrative just for the truncated (\( \varphi \)-independent) model. Indeed, the charge conservation law (32) is now satisfied identically. The electrodynamic equations (83), (84) with the definitions (76)-(82) transform into
\[ \Delta_{(1)} V - \frac{\partial^2}{\partial x_{02}^2} V = q \frac{\partial}{\partial x^0} U, \]  
\[(99)\]
\[ \Delta_{(1)} U - \frac{\partial^2}{\partial x_{02}^2} U = -q^2 U - q \frac{\partial}{\partial x^0} V + \frac{4\pi}{c} I, \]  
\[(100)\]
where

\[ I \equiv \sqrt{\left| I^\varphi I^\varphi \right|} = I^r \sin \theta. \]  

(101)

Decoupled equation for \( V \) is modified accordingly as

\[
\left\{ \Delta(1) - \frac{\partial^2}{\partial x^2} \right\} V = \frac{4\pi q}{c} \frac{\partial}{\partial x} I.
\]

(102)

Repeating the decomposition of the potentials \( V \) and \( U \) fulfilled in Subsection 5.1., we obtain the differential equations

\[
\ddot{u}_{nj} + c^2 \left( \nu_j^{(n)} \right)^2 u_{nj} - q \dot{c} v_{nj} = I_{nj}(\tilde{t}),
\]

(103)

\[
\ddot{v}_{nj} + c^2 \left( \nu_j^{(n)} \right)^2 v_{nj} + q c \dot{u}_{nj} = 0,
\]

(104)

where the functions \( I_{nj}(\tilde{t}) \) denote the coefficients in the decomposition

\[
\frac{4\pi}{c} I(\tilde{t}, r, \theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} I_{nj}(\tilde{t}) P_n^1(\cos \theta) \mathcal{H}_{nj}(r).
\]

(105)

When \( q=0 \), the key equations are not coupled, so the only \( U \) potential feels the influence of the stimulating current of this type. The case \( q \neq 0 \) is much more sophisticated.

The particular solutions \( v^*_{nj}(\tilde{t}) \), which satisfy (103) and (104) and null initial data \( v^*_{nj}(0) = 0 = \dot{v}^*_{nj}(0) \), have the following form

\[
v^*_{nj}(\tilde{t}) = \frac{1}{\sqrt{2}} \int_0^\tilde{t} d\tau I_{nj}(\tau) \left[ \cos \omega_{2nj}(\tilde{t}-\tau) - \cos \omega_{1nj}(\tilde{t}-\tau) \right].
\]

(106)

Clearly, this solution vanishes at \( q \to 0 \), since in this limit \( \omega_{2nj} \to \omega_{1nj} \).

The particular solutions \( u^*_{nj}(\tilde{t}) \) with null initial data \( u^*_{nj}(0) = 0 = \dot{u}^*_{nj}(0) \) is

\[
 u^*_{nj}(\tilde{t}) = \frac{1}{2} \int_0^\tilde{t} d\tau I_{nj}(\tau) \left[ \frac{\sin \omega_{1nj}(\tilde{t}-\tau)}{\omega_{1nj}} + \frac{\sin \omega_{2nj}(\tilde{t}-\tau)}{\omega_{2nj}} \right].
\]

(107)

Thus, when \( q=0 \), the azimuthal current produces stimulated azimuthal electric field, meridional and radial magnetic field. When \( q \neq 0 \), additional radial and meridional electric field, as well as the azimuthal magnetic field also appears as a result of the axion-photon coupling. When the azimuthal component of the electric current is periodic, and the frequency is, say, \( \Omega \), i.e.,

\[ I_{nj}(\tilde{t}) = I_{nj} \sin \Omega \tilde{t}, \]

(108)

we obtain stimulated oscillations on the heterodyne frequencies \( \omega_{1nj} \pm \Omega \) and \( \omega_{2nj} \pm \Omega \).

For instance, for the amplitude \( u^*_{nj}(\tilde{t}) \) one obtains:

\[
u^*_{nj}(\tilde{t}) = \frac{1}{4\omega_{1nj}^2 \omega_{2nj}^2} I_{nj} \left\{ \right. \frac{1}{2} \left( \omega_{1nj}^2 + \omega_{2nj}^2 \right) \sin \Omega \tilde{t} + \omega_{2nj}^2 \left[ \sin \omega_{1nj}(\tilde{t}) - \sin \omega_{1nj}(\Omega) \tilde{t} \right] + \omega_{1nj}^2 \left[ \sin \omega_{2nj}(\tilde{t}) - \sin \omega_{2nj}(\Omega) \tilde{t} \right] \left. \right\}. \]

(109)

For the mode amplitude \( v^*_{nj}(\tilde{t}) \) the result is similar.
6 An application: axionically coupled electric and magnetic fields near the Earth’s surface

6.1 Longitudinal cluster formed by meridional electric and magnetic fields

Let us consider the electric and magnetic fields at the Earth surface \( r = R \), using the definitions (76)-(81) and the solutions (90)-(96) to the key equations. The radial and azimuthal components of the electric field in this model vanish at the Earth surface

\[
E_{(\text{rad})}(\tilde{t}, R, \theta) = 0, \quad E_{(\text{azim})}(\tilde{t}, R, \theta) = 0. \tag{110}
\]

The radial and azimuthal components of the magnetic field also take zero values

\[
B_{(\text{rad})}(\tilde{t}, R, \theta) = 0, \quad B_{(\text{azim})}(\tilde{t}, R, \theta) = 0. \tag{111}
\]

Only meridional components of electric field

\[
E_{(\text{merid})}(\tilde{t}, R, \theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} v_{nj}(\tilde{t}) P_n^{(1)}(\cos \theta) \mathcal{G}_{nj}(R), \tag{112}
\]

and magnetic field

\[
B_{(\text{merid})}(\tilde{t}, R, \theta) = -\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} u_{nj}(\tilde{t}) P_n^{(1)}(\cos \theta) \mathcal{G}_{nj}(R), \tag{113}
\]

are non-vanishing and thus attract attention in this model. In the formulas (112) and (113) we used the auxiliary function

\[
\mathcal{G}_{nj}(R) = \frac{\nu_x^{(n)}}{\sqrt{R}} \left[ J_{n+\frac{1}{2}}(\nu_j^{(n)} R) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R) - J_{n+\frac{1}{2}}(\nu_j^{(n)} R) J_{-(n+\frac{1}{2})}(\nu_j^{(n)} R) \right]. \tag{114}
\]

Clearly, the meridional electric and magnetic fields are coupled by the axion field, and we deal with longitudinal cluster of electromagnetic modes. Let us analyze the time dependent variations of these fields by two examples.

6.2 Perturbations of meridional electric field produce variations of the meridional magnetic field

Let us consider, first, the situation when initial data are the following

\[
v_{nj}(0) \neq 0, \quad \dot{v}_{nj}(0) \neq 0, \quad u_{nj}(0) = 0, \quad \dot{u}_{nj}(0) = 0. \tag{115}
\]
This means that initially only the meridional electric field is perturbed. The corresponding solution for the $v_{nj}$ amplitude is (see (114))

$$v_{nj}(\tilde{t}) = \frac{1}{2} v_{nj}(0) \left[ \cos \omega_{1nj} \tilde{t} + \cos \omega_{2nj} \tilde{t} \right] +$$

$$+ \frac{1}{2} \dot{v}_{nj}(0) \frac{1}{\omega_{1nj} \omega_{2nj}} \gamma_{(n)j}^2 \left[ \omega_{1nj} \sin \omega_{1nj} \tilde{t} + \omega_{2nj} \sin \omega_{2nj} \tilde{t} \right].$$

(116)

The solution for the $u_{nj}$ amplitude gives now non-vanishing result, though these functions started from zero initial values (see (115)):

$$u_{nj}(\tilde{t}) = \frac{1}{2} \sqrt{\frac{\omega_{1nj} \omega_{2nj}}{2\omega_{1nj} \omega_{2nj}}} \gamma_{(n)j}^{-1} \left[ v_{nj}(0) \left[ \omega_{1nj} \sin \omega_{2nj} \tilde{t} - \omega_{2nj} \sin \omega_{1nj} \tilde{t} \right] + \right.$$

$$+ \dot{v}_{nj}(0) \gamma_{(n)j}^2 \left[ \cos \omega_{1nj} \tilde{t} - \cos \omega_{2nj} \tilde{t} \right].$$

(117)

Here we introduced the following convenient constants:

$$\gamma_{(n)j} \equiv \left[ 1 - q^2 \left( \nu_j^{(n)} \right)^{-2} \right]^{-\frac{1}{4}}.$$

(118)

For the resonant case, when $q = \nu_j^{n*}$ for some mode numbers $n*$ and $j*$, one can illustrate the $U$-mode amplitude generation by the formula

$$u_*(\tilde{t}) = \frac{v_*(0)}{2\sqrt{2}} \left[ \sin \sqrt{2}cq\tilde{t} - \sqrt{2}cq\tilde{t} \right] + \frac{\dot{v}_*(0)}{cq} \sin^2 \frac{cq\tilde{t}}{\sqrt{2}}.$$ 

(119)

Clearly, there is the term linear in time, and for small value of $cq\tilde{t}$ the expression (119) has the leading order term $\propto cq\tilde{t}^2$.

If $q = 0$, so that $\omega_{1nj}=\omega_{2nj} \equiv \omega_{nj}$, one obtains that $u_{nj}(\tilde{t}) = 0$, thus the magnetic field remains unperturbed. In the approximation linear with respect to $q$ one obtains

$$v_{nj}(\tilde{t}) = v_{nj}(0) \cos \omega_{0nj} \tilde{t} + \frac{\dot{v}_{nj}(0)}{\omega_{0nj}} \sin \omega_{0nj} \tilde{t},$$

(120)

$$u_{nj}(\tilde{t}) = \frac{qc}{2\omega_{0nj}} \left\{ v_{nj}(0) \left[ \omega_{0nj} \tilde{t} \cos \omega_{0nj} \tilde{t} - \sin \omega_{0nj} \tilde{t} \right] + \dot{v}_{nj}(0) \tilde{t} \sin \omega_{0nj} \tilde{t} \right\}.$$ 

(121)

In particular, we obtain very convenient formula for estimations

$$u_{nj}(\tilde{t}) = \frac{1}{2} qct \, v_{nj}(\tilde{t}),$$

(122)

when the initial data have specific form $v_{nj}(0) = 0$. 


6.3 Perturbations of meridional magnetic field produce variations of the meridional electric field

Let us consider now the situation when initial data are the following

\[ u_{nj}(0) \neq 0, \quad \dot{u}_{nj}(0) \neq 0, \quad v_{nj}(0) = 0, \quad \dot{v}_{nj}(0) = 0. \quad (123) \]

This means that initially only the meridional magnetic field is perturbed. Now the \( v_{nj} \) amplitudes become non-vanishing at \( \tilde{t} > 0 \) due to the axion-photon coupling:

\[ v_{nj}(\tilde{t}) = \frac{\gamma(n)j}{2\sqrt{\omega_{1nj}\omega_{2nj}}} \left\{ \dot{u}_{nj}(0) \left[ \cos \omega_{2nj} \tilde{t} - \cos \omega_{1nj} \tilde{t} \right] + \right. \]
\[ \left. + u_{nj}(0) \gamma(n) j^2 \left[ \omega_{2nj} \sin \omega_{1nj} \tilde{t} - \omega_{1nj} \sin \omega_{2nj} \tilde{t} \right] \right\}. \quad (124) \]

The \( u_{nj} \) amplitudes evolve as follows:

\[ u_{nj}(\tilde{t}) = \frac{1}{2\omega_{1nj}\omega_{2nj}} \left\{ u_{nj}(0) \gamma(n) j^2 \left[ \omega_{2nj}^2 \cos \omega_{1nj} \tilde{t} + \omega_{1nj}^2 \cos \omega_{2nj} \tilde{t} \right] + \right. \]
\[ \left. + \dot{u}_{nj}(0) \left[ \omega_{2nj} \sin \omega_{1nj} \tilde{t} + \omega_{1nj} \sin \omega_{2nj} \tilde{t} \right] \right\}. \quad (125) \]

Again, when \( q = \nu_j^{*+} \), one can illustrate the \( v \)-mode amplitude generation by the formulas

\[ v_\ast(\tilde{t}) = -\frac{\dot{u}_\ast(0)}{cq} \sin \frac{cq \tilde{t}}{\sqrt{2}}, \quad u_\ast(t) = u_\ast(0) + \frac{\dot{u}_\ast(0)}{2\sqrt{2}cq} \left[ \sin \sqrt{2}cq \tilde{t} + \sqrt{2}cq \tilde{t} \right]. \quad (126) \]

The term linear in time appears now in \( u_\ast(t) \).

Clearly, if \( q = 0 \), so that \( \omega_{1nj} = \omega_{2nj} \), one obtains that \( v_{nj}(\tilde{t}) = 0 \), thus the electric field remains unperturbed. Analogously, in the linear approximation we obtain

\[ u_{nj}(\tilde{t}) = u_{nj}(0) \cos \omega_{0nj} \tilde{t} + \frac{\dot{u}_{nj}(0)}{\omega_{0nj}} \sin \omega_{0nj} \tilde{t}, \quad (127) \]
\[ v_{nj}(\tilde{t}) = -\frac{qc}{2\omega_{0nj}} \left\{ u_{nj}(0) \left[ \omega_{0nj} \tilde{t} \cos \omega_{0nj} \tilde{t} - \sin \omega_{0nj} \tilde{t} \right] + \dot{u}_{nj}(0) t \sin \omega_{0nj} \tilde{t} \right\}, \quad (128) \]

with

\[ v_{nj}(\tilde{t}) = -\frac{1}{2}qct \ u_{nj}(\tilde{t}), \quad (129) \]

when the initial data are chosen as \( u_{nj}(0) = 0 \).

7 Discussion

1. We studied one model of coupling of terrestrial magnetic and electric fields with a relic axion background. Exact solutions obtained in the framework of the axion electrodynamics illustrate three important conclusions.
1.1. Relic dark matter axions produce in the terrestrial electrodynamic system oscillations of a new type, which belong to the class of Longitudinal Electro-Magnetic Clusters (LEMCs). LEMC has the following specific feature: the electric and magnetic fields are parallel to one another and are coupled by axionic field (see, e.g., the formulas (110)-(114)); in the absence of axions such oscillations decouple. Electric and magnetic fields of this type are correlated: the oscillations are characterized by identical frequencies but different phases (see, e.g., the formulas (116)-(117) and (124)-(125)). Generally, there are two sets of hybrid frequencies of LEMCs (see the formulas (96)), and these frequencies coincide when the axionic dark matter influence is negligible.

1.2. Relic dark matter axions deform the static terrestrial magnetic field. First, in case when the original geomagnetic field has the radial and meridional components only, the axion-photon coupling produces a supplementary azimuthal component (see, e.g., (61), (70)); this effect contributes to the phenomenon of the Earth’s magnetic pole drift. Second, due to the axion-photon coupling the dependence of the magnetic field on the altitude becomes non-monotonic: the surfaces appear, on which the radial and/or meridional components of the geomagnetic field change the signs, as well as reach maxima and minima (see Subsection 4.4).

1.3. The truncated (ϕ-independent) model can be effectively described in terms of two electromagnetic potentials $U$ and $V$, which can be considered as axionic generalizations of the well-known Debye potentials.

2. Estimations of the effect produced by relic axion background in the terrestrial electrodynamic system depend on how small the dimensionless parameter $\xi \equiv qR$ is. It can be estimated as follows. Let us remind that in the laboratory frame of reference $|q| = \frac{\sqrt{\rho_{(DM)}}}{\Psi_0}$, and use the natural units, in which $\hbar=1$ and $c=1$. According to [40] the mass density of the dark matter in the Solar system is estimated to be $\rho_{(DM)} \simeq 0.033 \, M_{(\text{Sun})} \text{pc}^{-3}$ or in the natural units $\rho_{(DM)} \simeq 1.25 \, \text{GeV} \cdot \text{cm}^{-3}$. According to [10] [36] the parameter $\Psi_0$ is reciprocal to the axion-photon-photon coupling constant $\rho_{A\gamma\gamma}$, i.e., $\frac{1}{\Psi_0} = \rho_{A\gamma\gamma}$ and $\rho_{A\gamma\gamma}$ itself can be (optimistically) estimated as $\rho_{A\gamma\gamma} \simeq 10^{-9} \text{GeV}^{-1}$. Thus, we obtain $\xi \simeq 10^{-7}$. This means that the effective frequency of LEMCs is about $\nu_A = cq = \frac{c\xi}{R} \simeq 0.5 \cdot 10^{-5} \text{Hz}$. This frequency belongs to the range of infra-low frequencies; variations of electric and magnetic fields of this type are studying in the experimental group of Vladimir University during more than 40 years [41] [42] [43].

3. We consider this paper as theoretical grounds for the program of testing of the LEMCs in the Earth’s atmosphere. Detailed technical description of the experiments and first experimental results are planned to be discussed in a special paper.
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