Some results and conjectures about recurrence relations for certain sequences of binomial sums.

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Abstract

In a previous paper [1] I have obtained some results about recurrences of certain binomial sums. The aim was to find an explanation of the remarkable identities

\[ \sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{n}{n+5k+2} \right) = F_n \quad \text{and} \quad \sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{n}{n+5k} \right) = F_{n+1} \]

for Fibonacci numbers \( F_n \) by putting these identities into a more general context. These identities have first been obtained by I. Schur [4] in his proof of the famous Rogers-Ramanujan identities. In the meantime computer experiments have led to several conjectures about the concrete form of these recurrences, some of which are proved in this paper.

1. Some general results

We recall the following known fact (cf. [1]):

Theorem 1

Let \( m \geq 2, i \geq 1 \) be integers, \( n \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \).

The sequences

\[ a(n,i,\ell,m,z) = \sum_{k \in \mathbb{Z}} \left( \frac{n}{n+ik+\ell} \right) z^k \in \mathbb{Q}[z,z^{-1}] \]

satisfy linear recurrences of order \( i \) with constant coefficients. More precisely there exist uniquely determined polynomials

\[ p_i(n,m,x,s) = x^n + \sum_{j=1}^{\left\lfloor \frac{n}{m} \right\rfloor} c_i(n,m,j)x^{n-mj}s^j \]

and
\[ p_k(n,m,x,s) = \sum_{j=1}^{\left\lfloor \frac{kn}{m} \right\rfloor} c_j(n,m,f)x^{kn-mj}x^j, \]  

with integer coefficients such that

\[ \sum_{k=1}^{m-1} z^{k-1} p_k(i,m,E,1)a(n,i,\ell,m,z) = \left( \frac{1}{z} + (-1)^{m(i-1)}z^{m-1} \right)a(n,i,\ell,m,z), \]

where \( E \) denotes the shift operator defined by \( E^k f(n) = f(n+k) \).

**Sketch of the proof**

We observe as in [3] that the recurrence relations for the binomial coefficients imply that

\[ t(m,n,k) = \binom{n}{m+k} \text{ satisfies the recurrence relation} \]

\[ t(m,n,k) = t(m,n-1,k-m+1) + t(m,n-1,k+1). \]

If we introduce the shift operator \( K \) defined by \( K^j f(k) = f(k-j) \), then (5) can be written in the form

\[ t(m,n,k) = Et(m,n-1,k) = (K^{m-1} + K^{-1})t(m,n-1,k) = (K^{m-1} + K^{-1})^n t(m,0,k). \]

This implies

\[ p_j(i,m,E,1)t(m,n,k) = p_j(i,m,K^{m-1} + K^{-1},1)t(m,n,k). \]

It turns out that in order to prove (4) it suffices to show

\[ \sum_{j=1}^{m-1} x^{(j-1)n}\ p_j(i,m,x^{m-1}+1,1) = \frac{1}{x} + (-1)^{m(i-1)}x^{(m-1)i}. \]

For then we get from (7)

\[ \sum_{j=1}^{m-1} p_j(i,m,E,1)t(m,n,k-(j-1)i) = t(m,n,k+i) + (-1)^{m(i-1)}t(m,n,k-(m-1)i) \]

for all \( n \in \mathbb{N} \).
This means that

\[
\sum_{j=1}^{m-1} p_j(i,m,E,1) \left( \begin{array}{c} n \\ m \\ n+k+i-j \end{array} \right) = (-1)^{m(j-1)} \left( \begin{array}{c} n \\ m \\ n+k-(m-1)i-j \end{array} \right).
\]

If we replace \( k \) by \( ik \), multiply each side by \( z^k \) and sum over all \( k \in \mathbb{Z} \) we get

\[
\sum_{j=1}^{m-1} z^{j-1} p_j(i,m,E,1)a(n,i,\ell,m,z) = \left( \frac{1}{z} + (-1)^{m(j-1)} z^{m-1} \right) a(n,i,\ell,m,z),
\]

Identity (8) is equivalent with

\[
\sum_{k=1}^{m-1} p_k(n,m,x+1,x) = 1 + (-1)^{m(n-1)} x^n. \quad (10)
\]

This follows from

\[
\sum_{k=1}^{m-1} x^{kn} p_k(n,m,x^{m-1} + \frac{1}{x},1) = \sum_{k=1}^{m-1} \sum_j c_k(n,m,j)x^{kn} \left( x^{m-1} + \frac{1}{x} \right)^{kn-mj} = \sum_{k=1}^{m-1} p_k(n,m,x^{m}+1,x^{m}) = 1 + (-1)^{m(n-1)} x^n.
\]

For \( n = 0 \) we set \( p_k(0,m,x,s) = (-1)^{k-1} \binom{m}{k} \). Then (10) remains true for \( n = 0 \).

It is easy to verify that the polynomials \( p_k(n,m,x,s) \) of the form (3) which satisfy (10) are uniquely determined.

Let now \( b(n,m,j) = (-1)^{m(n-1)-1} c_k(n,m,j) \) for 

\[
\frac{(k-1)n}{m} < j \leq \frac{kn}{m}.
\]

Then (10) may be reformulated as

\[
(-1)^{m(n-1)} x^n + (-1)^{m(n-1)} \sum_{k=1}^{m-1} \sum_{j=1}^{\lfloor \frac{kn}{m} \rfloor} b(n,m,j)(x+1)^{kn-mj} x^j = (1+x)^n - 1. \quad (11)
\]
If we consider the homomorphism which sends $1 + x$ to a primitive root of unity $\zeta_n$ of order $n$, then it has been shown in [1] that (11) implies

$$\sum_{j=1}^{n} b(n, m, j)x^j = \prod_{k=1}^{n} \left(x - \zeta_n^{-mk} (\zeta_n^k - 1)\right). \quad (12)$$

Let $s_j$ be the $j$'s elementary symmetric function of $\zeta_n^{-mk} (\zeta_n^k - 1)$, $1 \leq j < n$, and $\pi_j$ the $j$'s power sum of those numbers. Let $k(\text{mod} \ n)$ be the remainder modulo $n$ with $0 \leq k(\text{mod} \ n) < n$.

Then $\pi_j = \sum_{i=0}^{j} (-1)^{j-i} \left(\begin{array}{c} j \\ i \end{array}\right) \sum_{k=1}^{n} \zeta_n^{-mkj} = (-1)^{j-m(\text{mod} \ n)} \binom{j}{m(\text{mod} \ n)}$.

Newton's formula gives

$$\sum_{i=0}^{j-1} (-1)^{i} \pi_{j-i} s_i = (-1)^{j-1} j s_j.$$

Since $s_j = (-1)^{j} b_{n,m,n-j}$ we get

**Theorem 2**

Let $d(n, m, j) = (-1)^{j-m(\text{mod} \ n)} \binom{j}{m(\text{mod} \ n)}$ and let $b(n, m, n - j)$ be defined by

$$b(n, m, n - j) = -\frac{1}{j} \sum_{i=0}^{j-1} d(n, m, j-i) b(n, m, n - i) \text{ with } b(n, m, n) = 1.$$

Then for $n \geq 1$ we have

$$p_1(n, m, x, s) = x^n + \sum_{j} c_1(n, m, j)x^{n-mj}s^j = x^n - (-1)^{m(n-1)} \sum_{j=1}^{n} b(n, m, j)x^{n-mj}s^j = -(-1)^{m(n-1)} \sum_{j=1}^{n} b(n, m, j)x^{n-mj}s^j. \quad (13)$$

and

$$p_k(n, m, x, s) = \sum_{j} c_k(n, m, j)x^{kn-mj}s^j = -(-1)^{m(n-1)} \sum_{j=1}^{kn} b(n, m, j)x^{kn-mj}s^j. \quad (14)$$
2. Concrete evaluations
Now we want to obtain more concrete information about the polynomials $p_i(n,m,x,s)$.
First I derive some known results (cf. [1]) from the present point of view.

2.1. The case $m = 2$
For $m = 2$ formula (10) tells us that there are polynomials $p_i(n,2,x,s)$ of the form
\[ \sum_j c_j x^{n-2j} s^j \]
which satisfy
\[ p_i(n,2,x+1,x) = x^n + 1. \tag{15} \]

It is easy to see that the sequence $(p_i(n,2,x,1))_{n \geq 1}$ begins with
\[ x, x^2 - 2, x^3 - 3x, x^4 - 4x^2 + 2, x^5 - 5x^3 + 5x, x^6 - 6x^4 + 9x^2 - 2, \ldots. \]

Everyone familiar with the Lucas polynomials $L_n(x,s) = \sum_{j=0}^{n \over 2} \binom{n-j}{j} \frac{n}{n-j} x^{n-2j}$ will immediately guess that
\[ p_i(n,2,x,1) = \sum_{j=0}^{n \over 2} (-1)^j \binom{n-j}{j} \frac{n}{n-j} x^{n-2j} \]
and thus
\[ p_i(n,2,x,s) = L_n(x,-s) \]
holds.
It is well known that the Lucas polynomials satisfy the recurrence
\[ L_n(x,s) = xL_{n-1}(x,s) + sL_{n-2}(x,s) \]
with initial values $L_0(x,s) = 2$ and $L_1(x,s) = x$.

Therefore it remains only to show that $p_i(n,2,x,s) = L_n(x,-s)$ satisfies (15), i.e.
\[ L_n(x+1,-x) = x^n + 1. \tag{17} \]

This is of course a well-known result. It may be proved in the following way:
The polynomials $L_n(x+1,-x)$ satisfy the recurrence
\[ L_{n+2}(x+1,-x) - (x+1)L_{n+1}(x+1,-x) + xL_n(x+1,-x) = \left( E^2 - (x+1)E + x \right) L_n(x+1,-x) = 0. \]
If $p(E)f(n) = 0$ we call $p(z)$ the characteristic polynomial of the sequence $f(n)$. Thus the characteristic polynomial of the sequence $L_n(x+1,-x)$ is $z^2 - (x+1)z + x = (z-1)(z-x)$.
This implies that the constant sequence 1 and the sequence $(x^n)$ satisfy the same recurrence as $(L_n(x+1,-x))$.
Therefore we get
\[ \left( E^2 - (x+1)E + x \right) \left( L_n(x+1,-x) - 1 - x^n \right) = 0 \]
with initial values $L_0(x+1,-x) - 1 - x^0 = 2 - 1 - 1 = 0$ and
\[ L_1(x+1,-x) - 1 - x = (x+1) - 1 - x = 0. \]
This implies that \( L_n(x+1,-x) - 1 - x^n = 0 \) for all \( n \in \mathbb{N} \) as asserted.

Another way to derive this result is by considering the generating function
\[
\sum_{n \geq 0} p_i(n,2,x+1,x)z^n = \sum_{n \geq 0} (1 + x^n)z^n = \frac{1}{1-z} + \frac{1}{1-xz} = \frac{2-(x+1)z}{1-(x+1)z + xz^2}.
\]
Since the polynomials are uniquely determined we get the generating function of the Lucas polynomials
\[
\sum_{n \geq 0} p_i(n,2,x,s)z^n = \frac{2-xz}{1-xz + sz^2}.
\]

We have thus shown that
\[
L_i(E,-1)a(n,i,\ell,2,z)\left(\frac{z + 1}{z}\right)a(n,i,\ell,2,z)
\]
holds.

E.g. for \( i = 5 \) we get \( L_5(E,-1) = E^5 - 5E^3 + 5E \). Therefore
\[
(E^5 - 5E^3 + 5E + 2)a(n,5,\ell,2,-1) = 0.
\]
Now we have
\[
(E^5 - 5E^3 + 5E + 2) = (E + 2)(E^2 - E - 1)^2.
\]
The Fibonacci numbers satisfy the same recurrence. Considering the initial values we get
\[
\sum (-1)^k \left( \left\lfloor \frac{n}{n+5k+2} \right\rfloor \right) = F_{n+1}.
\]

In [3] we have also obtained the generating function
\[
\sum_{n \geq 0} a(n,i,0,2,z)x^n = \frac{\frac{x^{i-1}}{z} + F_i(1,-x^2) + xF_{i-1}(1,-x^2)}{L_i(1,-x^2) - x^i \left( z + \frac{1}{z} \right)}, \tag{19}
\]
where \( F_n(x,s) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} x^{n-2k-1} \) are the Fibonacci polynomials.

As shown in [1] and [3] for \( z = \pm 1 \) there are always simpler recurrences. The formulas are different depending whether \( i \) is odd or even. For example
\[
\sum_{n \geq 0} a(n,2m+1,0,2,-1)x^n \frac{F_{2m+1}(1,-x^2) + xF_{2m}(1,-x^2)}{L_{2m+1}(1,-x^2) + 2x^{2m+1}} = \frac{F_m(1,-x^2)}{F_{m+1}(1,-x^2) -xF_m(1,-x^2)}.
\]
Remark
It should be noted that \(a(n,2m+1,0,2,-1)\) has a simple combinatorial interpretation (cf. [2]). It is the number of the set \(A(n,m)\) of all lattice paths in \(\mathbb{R}^2\) of length \(n\) which are contained in the strip \(-m-1 < y < m\), start at the origin and consist of \(\left\lfloor \frac{n}{2} \right\rfloor\) northeast steps \((1,1)\) and \(\left\lfloor \frac{n+1}{2} \right\rfloor\) southeast steps \((1,-1)\). Define a peak as a vertex preceded by a northeast step and followed by a southeast step, and a valley as a vertex preceded by a southeast step and followed by a northeast step. The height of a vertex is its \(y\)–coordinate. The peaks with height at least 1 and the valleys with height at most \(-2\) are called extremal points. Define the weight of a path \(v\) by \(w(v,t) = t^{d(v)}\) where \(d(v)\) is the number of extremal points of the path \(v\). The weight of a set of paths is the sum of the weights of all paths of the set.

It has been shown in [2] that if we set \(\binom{n}{k} = 0\) for \(n < 0\) then the weight of the set \(A(n,m)\) is

\[
w(A(n,m),t) = \sum_{j \in \mathbb{Z}} (-1)^j \sum_{\ell \in \mathbb{Z}} \binom{n+(2m-3)j}{\frac{\ell-j}{2}} \binom{n+1-(2m-3)j}{\frac{\ell+j}{2}} t^\ell.
\]

For \(t = 1\) this reduces by Vandermonde’s formula to

\[
\sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\frac{n+2m+1}{2}j}.
\]

For example, \(w(A(n,2),t) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-2k}{k} t^k = F_{n+1}(1,t)\).

As shown in [2] with other methods for \(m \geq 2\) the polynomial \(w(A(n,m),t)\) satisfies the recurrence relation

\[
\left(F_m(E^2+1-t,-E^2)-(1+E)F_{m-1}(E^2+1-t,-E^2)+EF_{m-2}(E^2+1-t,-E^2)\right)w(A(n,m),t) = 0
\]

of order \(2m-2\) which for \(t = 1\) reduces to the recurrence relation

\[
(F_{m+1}(E,-1)-F_m(E,-1))w(A(n,m),1) = (F_{m+1}(E,-1)-F_m(E,-1))a(n,2m+1,0,2,-1) = 0
\]

of order \(m\).

It should also be noted that \((a(n,5,0,2,-1))_{n \geq 0}\), \((a(n,7,0,2,-1))_{n \geq 0}\) and \((a(n,9,0,2,-1))_{n \geq 0}\) are the sequences A000045, A028495 and A061551 of The On-Line Encyclopedia of Integer Sequences.
2.2. The case \( m = 3 \).

In [1] I have obtained explicit formulas for the recurrence relations of the sums

\[
a(n,i,\ell,3,z) = \sum_{k \in \mathbb{Z}} \left( \frac{n}{n+ik+\ell} \right)^k z^k.
\]

From Theorem 1 we know that there are polynomials \( v_i(x) \) and \( w_i(x) \) such that

\[
\left( v_i(E) - zw_i(E) - \left( \frac{1}{z} + (-1)^i z^2 \right) \right) a(n,i,\ell,z) = 0.
\]

Computer experiments led to the table

| \( v_i(x) \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|---------------|----|----|----|----|----|----|----|
|               | \( x \) | \( x^2 \) | \( x^3 - 3 \) | \( x^4 - 4x \) | \( x^5 - 5x^2 \) | \( x^6 - 6x^3 + 3 \) | \( x^7 - 7x^4 + 7x \) |

| \( w_i(x) \) | 0  | 2x | 3  | 2x^2 | 5x | 3 + 2x^3 | 7x^2 |

An inspection of these polynomials led to the conjecture that they satisfy the recurrences

\[
v_n(x) = xv_{n-1}(x) - v_{n-3}(x)
\]  

with initial values

\[
v_0(x) = 3, v_1(x) = x, v_2(x) = x^2
\]

and

\[
w_n(x) = xw_{n-2}(x) + w_{n-1}(x)
\]

with initial values

\[
w_0(x) = 3, w_1(x) = 0, w_2(x) = 2x.
\]

It is not difficult to determine these polynomials explicitly: For \( n > 0 \) we have

\[
v_n(x) = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n-2j}{j} \frac{n}{n-2j} x^{n-3j}
\]

and

\[
w_n(x) = \sum_{j \in \mathbb{Z}} \binom{n-f}{2n-3j} \frac{n}{n-j} x^{2n-3j}.
\]

For the recurrences are easily verified and the initial values coincide for \( n = 1,2,3 \).
Thus we have experimental evidence for

**Theorem 3**

*For* \( i \geq 1 \) *the sequence*

\[
a(n, i, \ell, 3, z) := \sum_{k \in \mathbb{Z}} \left( \frac{n}{n + ik + \ell} \right)^{3k} z^k
\]

*satisfies the recurrence*

\[
(v_j(E) - zw_j(E))a(n, i, \ell, z) - \left( \frac{1}{z} + (-1)^{i-1}z^2 \right)a(n, i, \ell, z) = 0
\]

*for all* \( n \in \mathbb{N} \).

**Remark**

In [1] unfortunately some typos corrupted the formulation of this theorem.

With the notations introduced above we have only to verify that the polynomials \( p_j(n, 3, x, s) \) are given by \( v_n(x, s) \) and \( w_n(x, s) \), i.e.

\[
p_j(n, 3, x, s) = v_n(x, s) = \sum_{j \geq 0} (-1)^j \binom{n-2j}{j} \frac{n}{n-2j} x^{n-3j} s^j
\]

and

\[
-p_3(n, 3, x, s) = w_n(x, s) = \sum_{j \geq 2n} \binom{n-j}{2n-3j} \frac{n}{n-j} x^{2n-3j} s^j.
\]

Note that \( v_n(x) = v_n(x, 1) \) and \( w_n(x) = w_n(x, 1) \).

With other words we have to show that

\[
v_n(1+x,x) - w_n(1+x,x) - 1 + (-x)^n = 0.
\]

These polynomials satisfy the recurrences

\[v_n(x, s) = xv_{n-1}(x, s) - sv_{n-3}(x) \quad \text{and} \quad w_n(x, s) = xs w_{n-2}(x) + s^3 w_{n-3}(x) \quad \text{for all} \ n \in \mathbb{N} \]

if we set \( v_n(x, s) = w_n(x, s) = 3 \).

The sequences \( v_n(1+x,x) \) and the constant sequence 1 satisfy the recurrence

\[(E^3 - (1+x)E^2 + x)v_n(1+x,x) = 0 \quad \text{and} \quad (E^3 - (1+x)E^2 + x)1 = 0 \quad \text{and the sequences}
\]
\( w_n(1+x,x) \) and \((-x)^n\) satisfy \((E^3-(1+x)xE-x^3)w_n(1+x,x) = 0\) and
\((E^3-(1+x)xE-x^3)(-x)^n = 0\).

Therefore \( f_n(x) = v_n(1+x,x) - w_n(1+x,x) - 1 + (-x)^n \) satisfies the recurrence
\((E^3-(1+x)E^2+x)(E^3-(1+x)xE-x^3)f_n(x) = 0\) of order 6. Since the first 6 initial values are 0 we get \( f_n(x) = 0 \) for all \( n \in \mathbb{N} \).

It is again instructive to consider the generating functions

\[
\sum_{n \geq 0} p_1(n,3,x,s)z^n = \frac{3-2xz}{1-xz+sz^2} \quad \text{and} \quad \sum_{n \geq 0} p_2(n,3,x,s)z^n = \frac{3-xsz^2}{1-xsz^2-s^2z^3}.
\]

This gives

\[
\sum_{n \geq 0} p_1(n,3,x+1,x)z^n = \frac{3-2(x+1)z}{1-(x+1)z+xz^2} = \frac{1}{1-z} + \frac{2-xz}{1-xz-xz^2}
\]

and

\[
\sum_{n \geq 0} p_2(n,3,x+1,x)z^n = \frac{3-(x+1)xz^2}{1-(x+1)xz^2-x^2z^3} = \frac{-1}{1+xz} - \frac{2-xz}{1-xz-xz^2}.
\]

These again imply

\[
\sum_{n \geq 0} (p_1(n,3,x+1,x) + p_2(n,3,x+1,x))z^n = \frac{1}{1-z} - \frac{1}{1+xz} = \sum_{n \geq 0} (1+(-1)^{n-1}x^n)z^n.
\]

In addition to Theorem 3 we compute the generating function

\[
\sum_{n \geq 0} a(n,i,0,3,z)x^n = \frac{b_i(x) + c_i(x)z + (-1)^{i-1}x^{i-1}z^2}{v_i(1,x^3) - \frac{z}{x^i}w_i(1,x^3) - x^i\left(\frac{1}{z} + (-1)^{i-1}z^2\right)} \quad \text{for } i = 1, 2, 3.
\]

Computer experiments suggest that \( b_i(x) \) satisfies the same recurrence as \( v_i(1,x^3) \), i.e.
\( b_i(x) = b_{i-1}(x) - x^3b_{i-3}(x) \).

The initial values are \( b_1(x) = 1, b_2(x) = 1+x, b_3(x) = 1+x+x^2 \).

The generating function of these polynomials is

\[
\sum_{i \geq 0} b_{i+1}(x)t^i = \frac{1+xt+x^2t^2}{1-t+x^3t^3}.
\]

Let \( \frac{1}{1-t+x^3t^3} = \sum_{n \geq 0} r_n(x)t^n \). Then it is easily verified that \( r_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-2k}{k}x^{3k} \). The first values of the sequence \( (r_n(x))_{n \geq 0} \) are \( 1,1,1-x^3,1-2x^3,1-3x^3,1-4x^3+x^6, \cdots \).

This gives \( b_{i+1}(x) = r_i(x) + xr_{i-1}(x) + x^2r_{i-2}(x) \).
The polynomials \( c_i(x) \) satisfy the same recurrence as \( w_i(1, x^3) \), i.e.
\[
c_i(x) = x c_{i-2}(x) + x^3 c_{i-3}(x).
\]
The initial values are \( c_1(x) = 1, c_2(x) = 1 - x, c_3(x) = x + 2x^2 \).

The generating function of these polynomials is
\[
\sum_{i=0}^{n} c_{i+1}(x)t^i = \frac{1 + (1-x)t + 2x^2t^2}{1 - xt^2 - x^3t^3}.
\]

Let
\[
\frac{1}{1 - xt^2 - x^3t^3} = \sum_{n=0}^{\infty} s_n(x)t^n.
\]
Then \( s_n(x) = \sum_{j=0}^{n} \binom{n+1}{j} x^{n-3j+\varepsilon(n)} \),

where \( \varepsilon(n) \equiv n(\text{mod } 3) \) with \( \varepsilon(n) \in \{-1, 0, 1\} \). This can easily be verified by considering each of the cases \( s_{3n}(x), s_{3n-1}(x), s_{3n+1}(x) \) separately. The first values of the sequence \( (s_n(x))_{n \geq 0} \) are
\[
1, 0, x, x^3, x^2, 2x^4, x^3 + x^6, 3x^5, x^4 + 3x^7, \ldots
\]

Another representation is
\[
s_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} [n + k \equiv 0(\text{mod } 3)].
\]

Finally we get \( c_{i+1}(x) = s_i(x) + (1-x)s_{i-1}(x) + 2x^2s_{i-2}(x) \).

In order to prove (31) we note first that
\[
\left( v_i(1, x^3) - \frac{z}{x} w_i(1, x^3) - x^i \left( \frac{1}{z} + (-1)^{i-1} z^2 \right) \sum_{n=0}^{\infty} a(n, i, 0, 3, z)x^n \right)
\]
is a polynomial \( d_i(x, z) \) in \( x \) of degree \( < i \). For a polynomial \( p(x) = \sum_{k=0}^{i-1} p_kx^k \) let
\[
P_i(p(x)) = \sum_{k=0}^{i-1} p_kx^k.
\]
Therefore
\[
d_i(x, z) = P_i \left( v_i(1, x^3) - \frac{z}{x} w_i(1, x^3) \sum_{n=0}^{\infty} a(n, i, 0, 3, z)x^n \right).
\]

Since no factor contains negative powers of \( z \) we see that the sum of the first \( i \) terms of
\[
P_i \left( v_i(1, x^3) \sum_{n=0}^{i-1} a(n, i, 0, 3, 0)x^n \right) = d_i(x, 0).
\]
Let \( a_i(x, z) = \sum_{n=0}^{i-1} a(n, i, 0, 3, z)x^n \).
Then

\[ a_i(x, z) = \sum_{n=0}^{i-1} \left( \frac{n}{3} \right) x^n + z \sum_{n=\left\lceil \frac{i-2}{2} \right\rceil}^{i-1} \left( \frac{n}{3} \right) x^n + z^2 x^{i-1}. \] (32)

This implies

\[ d_i(x, 0) - d_{i-1}(x, 0) + x^i d_{i-3}(x, 0) = P \left( v_{i-1}(1, x^3) (a_i(x, 0) - a_{i-1}(x, 0)) + x^i v_{i-3}(1, x^3) (a_i(x, 0) - a_{i-3}(x, 0)) \right) = 0. \]

Since the initial values coincide we get \( d_i(x, 0) = b_i(x) \).

Next we determine the coefficient of \( z^2 \) of \( d_i(x, z) \). From (32) we get

\[ \left[ z^2 \right] \left( v_i(1, x^3) \sum_{n=0}^{i-1} a(n, i, 0, 3, 0) x^n \right) = x^{i-1}. \]

On the other hand we have

\[ -\frac{z}{x^2} \sum_{n=\left\lceil \frac{i-2}{2} \right\rceil}^{2i-1} \left( \frac{n}{3} \right) x^n = -z^2 \sum_{n=\left\lceil \frac{i-2}{2} \right\rceil}^{2i-1} \left( \frac{n+2i}{3} \right) x^n = -z^2 (2x^i + \cdots) (x^{i-1} + \cdots) = -2x^{2i-1} - \cdots. \]

This implies \( \left[ z^2 \right] d_2(x, z) = -x^{2i-1} \).

For odd indices we get

\[ -\frac{z}{x^{2i+1}} \sum_{n=\left\lceil \frac{i+1}{2} \right\rceil}^{2i} \left( \frac{n}{3} \right) x^n = -z^2 x^{i+2} (\cdots) (x^{i-1} + \cdots) = -x^{2i+1} - \cdots. \]

This implies \( \left[ z^2 \right] d_{2i+1}(x, z) = x^{2i-1} \).

As above we verify that

\[ p_i(x) = P \left( \frac{1}{x} w_i(1, x^3) \sum_{n=0}^{i-1} \left( \frac{n}{3} \right) x^n \right) \]

satisfies the recurrence \( p_i(x) - xp_{i-2}(x) - x^3 p_{i-3}(x) = 0 \), i.e the same recurrence as

\[ \frac{1}{x^i} w_i(1, x^3). \]

In the same way we get that \( q_i(x) = P \left( v_i(1, x^3) \sum_{n=\left\lceil \frac{i-2}{2} \right\rceil}^{i-1} \left( \frac{n}{3} \right) x^n \right) \) satisfies the same recurrence.

Writing \( d_i(x, z) = b_i(x) + c_i(x)z + (-1)^{i-1} x^{i-1} z^2 \) we see therefore that \( c_i(x) - xc_{i-2}(x) - x^3 c_{i-3}(x) = 0 \), from which equation (31) follows.

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2.3. Some general observations.

As a special case of Theorem 1 we see that

\[ a(n, 1, 0, m, z) = \sum_{k \in \mathbb{Z}} \left( \frac{n}{n + k} \right)^m = \frac{1 - z^m}{1 - z} \left( \frac{1 + z^m}{z} \right)^n. \]

From

\[ p_k(n, m, x + 1, x) = x^n + \sum_{j=1}^{n \mod m} c_i(n, m, j)(x + 1)^{n-mj} x^j \]

and the fact that

\[ p_k(n, m, x + 1, x) = x^{(k-1)n/m-j} \cdot \ldots \text{ for } k \geq 2 \]

we see that (10) implies that all coefficients

\[ \left[ x^j \right] p_i(n, m, x + 1, x), 1 \leq j \leq \frac{n}{m}, \]

must vanish, i.e.

\[ P_{m \mid n} \left( p_i(n, m, x + 1, x) \right) = 1. \]

By these conditions \( p_i(n, m, x, s) \) is uniquely determined.

This observation implies that

\[ p_i(n, m, x, s) = \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \left(-1\right)^j \binom{n-(m-1)j}{j} \frac{n}{n-(m-1)j} x^{n-mj} x^j. \quad (33) \]

For

\[ p_i(n, m, x + 1, x) = \sum_{j=0}^{n \mod m} \left(-1\right)^j \binom{n-(m-1)j}{j} \frac{n}{n-(m-1)j} x^{n-mj} x^j \]

and therefore the coefficient of \( x^r \) is given by

\[ \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \left(-1\right)^j \binom{n-(m-1)j}{j} \frac{n}{n-(m-1)j} \left( \binom{n-mj}{r} \right) (n-(m-1)j-1) \ldots (n-(m-1)j-r+1). \]

for all \( r \) such that \( r \leq \left\lfloor \frac{n}{m} \right\rfloor \).

Let \( q(n) = (n-1)_{r-1} = (n-1)(n-2) \ldots (n-r+1) \). This is a polynomial in \( n \) of degree \( r-1 \).

With this notation the coefficient of \( x^r \) is \( \frac{n}{r!} (1 - E^{m-1})^r q(n) \). Since

\[ (1 - E^{m-1}) n^k = n^k - (n + m-1)^k = k(m-1)n^{k-1} + \ldots \]

the operator \( 1 - E^{m-1} \) decreases the degree of a polynomial by 1. Therefore \( \frac{n}{r!} (1 - E^{m-1})^r q(n) = 0 \) and the assertion is proved.
It is easily verified that (33) satisfies the recurrence relation

$$\left(E^m - xE^{m-1} - s\right) p_1(n,m,x,s) = 0.$$  

Now we observe that

$$p_{m-1}(n,m,x,s) = \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \binom{n-j}{(m-1)n-mj} \frac{n}{n-j} s^j x^{(m-1)n-mj}.$$  

We know that

$$d(n,m,j) = (-1)^{j-mj (\text{mod} m)} \binom{j}{mj (\text{mod} m)} n.$$  

This implies that $d(n,m,0) = n$ and $d(n,m,j) = 0$ for $j \leq \frac{n}{m}$. For in this case we have $mj (\text{mod} m) = mj$ and therefore $\binom{j}{mj} = 0$.

Thus also

$$b(n,m,n-j) = -\frac{1}{m} \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} d(n,m,j-i) b(n,m,n-i) = 0.$$  

Let now $\frac{n}{m} < j \leq \frac{2n}{m}$.

From

$$-jb(n,m,n-j) = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} d(n,m,j-i) b(n,m,n-i)$$

$$= d(n,m,j) b(n,m,n) + \sum_{i=1}^{\lfloor \frac{m}{n} \rfloor} d(n,m,j-i) b(n,m,n-i) + \sum_{i=1}^{\lfloor \frac{m}{n} \rfloor} d(n,m,j-i) b(n,m,n-i)$$

we deduce that $-jb(n,m,n-j) = d(n,m,j) = (-1)^{j-mj (\text{mod} m)} \binom{j}{mj (\text{mod} m)} n$,

because the first sum vanishes since $b(n,m,n-i) = 0$ and the second sum vanishes because $d(n,m,j-i) = 0$.

Therefore for $\frac{(m-2)n}{m} < j \leq \frac{(m-1)n}{m}$ we get

$$b(n,m,j) = -(-1)^{j-mj (\text{mod} m)} \binom{n-j}{m(n-j) (\text{mod} n)} \frac{n}{n-j} = -(-1)^{a+j+(m-1)n-mj} \binom{n-j}{(m-1)n-mj} \frac{n}{n-j}.$$  

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Therefore we get

\[ p_{m-1}(n,m,x,s) = (-1)^m \sum_{j=1}^{m-n} \left( \frac{n-j}{(m-1)n-mj} \right) \binom{m-1}{m-n-mj} (-1)^{m-j} x_j. \]  

(34)

In order to give a simple description of these results we introduce polynomials

\[ v_n(m,x,s) = \sum_{j \in \mathbb{N}} (-1)^j \left( \frac{n-(m-1)j}{n-(m-1)j} \right) n \frac{x^{n-mj} s^j}{n-(m-1)j}. \]  

(35)

and

\[ w_n(m,x,s) = \sum_{j \in \mathbb{N}} \left( \frac{n-j}{(m-1)n-mj} \right) n x^{(m-1)n-mj} s^j, \]  

(36)

where as always in this paper \( \binom{n}{k} = 0 \) for \( n < 0 \).

These polynomials satisfy the recurrence relations

\[ v_{n+m}(m,x,s) = xv_{n+m-1}(m,x,s) - sv_n(m,x,s) \]  

(37)

with initial values \( v_0(m,x,s) = m \) and \( v_i(m,x,s) = x^i \) for \( 1 \leq i \leq m-1 \) and

\[ w_{n+m}(m,x,s) = xsw_{n+m-1}(m,x,s) + s^{m-1}w_n(m,x,s) \]  

(38)

with initial values \( w_0(m,x,s) = m \), \( w_i(m,x,s) = 0 \) for \( 1 \leq i \leq m-2 \), and

\( w_{m-1}(m,x,s) = (m-1)s^{m-2}x. \)

Let \( c_{m,k}(x,s,z) \) be the characteristic polynomial of (the recurrence of) the sequence \( (p_k(n,m,x,s))_{n \geq 0} \), so that \( c_{m,k}(x,s,E)p_k(n,m,x,s) = 0. \)

The characteristic polynomial of \( v_n(m,x+1,x) \) is

\[ c_{m,1}(x+1,x,z) = z^{m-1}(z-1) - x(z^{m-1} - 1) = (z-1)(z^{m-1} - xz^{m-2} - xz^{m-3} - \cdots - xz - x) \]  

(39)

and that of \( w_n(m,x+1,(-1)^{m-1}x) \) is
\[ c_{m,m-1}(x+1,x,z) = z^m - (x+1)x^{m-2}z - (-1)^{m-1}x^{m-1} = (x - (-1)^m z) \left( z^{m-1} + x z^{m-2} + \cdots + x^{m-2} z - x^{m-2} \right). \]

It is easily verified that
\[ c_{m,m-1}(x+1,x,z) = \frac{z^m}{x} c_{m-1}(x+1,x,(-1)^m \frac{x}{z}). \] (40)

This is in accord with the fact that the right hand side of (10) has the characteristic polynomial \( c(z) = (z - 1)(z - (-1)^m x) \) satisfies \( c \left( (-1)^m \frac{x}{z} \right) = \frac{(-1)^m x}{z^2} c(z). \)

These results can be formulated as

**Theorem 4**

*For each \( m \geq 2 \) we have*
\[ p_1(n,m,x,s) = v_n(m,x,s) \] (41)

*and*
\[ p_{m-1}(n,m,x,s) = (-1)^m w_n(m,x,(-1)^{m-1}s). \] (42)

2.4. The case \( m = 4 \).

Now we want to consider the case \( m = 4 \) in more detail.

We already know that
\[ p_1(n,4,x,s) = \sum_{j=0}^{\lfloor \frac{n-1}{3} \rfloor} (-1)^j \binom{n-3j}{j} \frac{n}{n-3j} x^{n-4j} s^j. \]

and
\[ p_3(n,4,x,s) = \sum_{j=\lfloor \frac{2n}{3} \rfloor}^{\lfloor \frac{n-1}{3} \rfloor} (-1)^j \binom{n-j}{3n-4j} \frac{n}{n-j} x^{3n-4j} s^j. \]

The polynomials \( p_2(n,4,x,s) \) turn out to satisfy a recursion of order 6 instead of order 4. It is given by
\[ p_2(n,4,x,s) = s p_2(n-2,4,x,s) + sx^2 p_2(n-3,4,x,s) + s^2 p_2(n-4,4,x,s) - s p_2(n-6,4,x,s) \] (43)
with initial values
\[ p_2(0,4,x,s) = -6, \ p_2(1,4,x,s) = 0, \ p_2(2,4,x,s) = -2s, \ p_2(3,4,x,s) = -3sx^2, \]
\[ p_2(4,4,x,s) = -6s^2, \ p_2(5,4,x,s) = -5s^2x^2. \]

To prove this we observe that we know already the generating functions
\[ \sum_{n=0}^{\infty} p_1(n,4,x,s)z^n = \frac{4 - 3xz}{1 - xz + sz^4} \]
\[ \text{and} \ \sum_{n=0}^{\infty} p_1(n,4,x,s)z^n = \frac{4 - xs^2z^3}{1 - xs^2z^3 + s^3z^4}. \]

This gives
\[ \sum_{n=0}^{\infty} p_1(n,4,x+1,x)z^n = \frac{4 - 3(x+1)z}{1 - (x+1)z + xz^4} = \frac{1}{1 - xz} + \frac{3 - 2xz - x^2z^2}{1 - xz - xz^2 - xz^3} \]
\[ \text{and} \ \sum_{n=0}^{\infty} p_3(n,4,x+1,x)z^n = \frac{4 - (x+1)x^2z^3}{1 - (x+1)x^2z^3 + x^3z^4} = \frac{1}{1 - xz} + \frac{3 + 2xz + x^2z^2}{1 + xz + x^2z^2 - x^2z^4}. \]

Furthermore we know that \( p_3(n,4,x+1,x) = 1 + x^n - p_1(n,4,x+1,x) - p_2(n,4,x+1,x) \).

This gives the generating function
\[ \sum_{n=0}^{\infty} p_2(n,4,x+1,x)z^n = -\frac{3 - 2xz - x^2z^2}{1 - xz - xz^2 - xz^3} - \frac{3 + 2xz + x^2z^2}{1 + xz + x^2z^2 - x^2z^3} \]
\[ = \frac{-6 + 4xz^2 + 3x(x+1)^2z^3 + 2x^2z^4}{1 - xz^2 - x(x+1)^2z^3 - x^2z^4 + x^3z^6} \]

and thus also
\[ \sum_{n=0}^{\infty} p_2(n,4,x,s)z^n = \frac{-6 + 4sz^2 + 3sx^2z^3 + 2s^2z^4}{1 - sz^2 - sx^2z^3 - s^2z^4 + s^3z^6}. \]

We did not find a simple expression for the polynomials \( p_2(n,4,x,s) \) themselves.

2.5. The case \( m = 5 \).

For \( m = 5 \) we have found the following characteristic polynomials for \( p_k(n,5,x+1,x) \):
\[ c_{5,1}(x+1,x,z) = z^5 - (1 + x)z^4 + x = (z - 1)(z^4 - xz^3 - xz^2 - xz - x), \]
\[ c_{5,2}(x+1,x,z) = z^{10} - x(1 + x)z^7 - x(1 + x)^3z^6 - 2x^2z^5 - x^2(1 + x)^2z^4 + x^3(1 + x)z^2 + x^4 \]
\[ = (z^4 - xz^3 - xz^2 - xz - x)(z^6 + xz^5 + x(1 + x)z^4 + x^2(1 + x)z^3 - x^2(1 + x)z^2 + x^3z - x^3), \]
\[ c_{5,3}(x+1,x,z) = z^{10} + x(1 + x)z^8 - x^2(1 + x)^2z^6 + 2x^3z^5 - x^3(1 + x)^3z^4 + x^4(1 + x)z^3 + x^6 \]
\[ = (z^4 - xz^3 + x^2z^2 - x^3z - x^3)(z^6 + xz^5 + x(1 + x)z^4 + x^2(1 + x)z^3 - x^2(1 + x)z^2 + x^3z - x^3), \]
$c_{z,d}(x+1,x,z) = z^5 - (1+x)x^3z - x^4 = (z+x)(z^4 - xz^3 + x^2z^2 - x^3z - x^4)$.

Once found these identities can be verified as above by proving (10).

3. Some open problems

Extensive computer experiments suggest the following facts:

$\deg c_{m,k}(x+1,x,z) = \binom{m}{k}$; for $k < \frac{m}{2}$ the polynomial $c_{m,k}(x+1,x,z)$ factors in the form $c_{m,k}(x+1,x,z) = v_{m,k-1}v_{m,k}$ where $v_{m,0} = z - 1$; for $k > \frac{m}{2}$ the factorization is $c_{m,k}(x+1,x,z) = v_{m,k}v_{m,k+1}$; for $k = \frac{m}{2}$ we get $c_{m,k}(x+1,x,z) = v_{m,k-1}v_{m,k+1}$; $\deg v_{m,k} = \binom{m-1}{k}$;

for $m$ odd we have $v_{m,m} = v_{m,m} = z + (-1)^{m-1}x$. Furthermore it seems that as for $k = 0$ and $k = 1$ also in general $c_{m,m-k}(x+1,x,z)$ is the monic polynomial proportional to $c_{m,k}(x+1,x,(-1)^{m-k}\frac{x}{z})$. So it would suffice to compute $v_{m,k}$ for $k < \frac{m}{2}$.

But till now I could not prove these general results.

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