Resummation of double logarithms in electroweak high energy processes

V.S. Fadin, ¹ L.N. Lipatov, ² A.D. Martin, ³ and M. Melles ⁴

1) Budker Institute of Nuclear Physics and Novosibirsk State University, 630090 Novosibirsk, Russia
2) St. Petersburg Nuclear Physics Institute, 188350 and St. Petersburg State University, St. Petersburg, Russia
3) Department of Physics, University of Durham, Durham, DH1 3LE, UK
4) Paul Scherrer Institute (PSI), CH-5232 Villigen, Switzerland.

Abstract

At future linear $e^+e^-$ collider experiments in the TeV range, Sudakov double logarithms originating from massive boson exchange can lead to significant corrections to the cross sections of the observable processes. These effects are important for the high precision objectives of the Next Linear Collider. We use the infrared evolution equation, based on a gauge invariant dispersive method, to obtain double logarithmic asymptotics of scattering amplitudes and discuss how it can be applied, in the case of broken gauge symmetry, to the Standard Model of electroweak processes. We discuss the double logarithmic effects to both non-radiative processes and to processes accompanied by soft gauge boson emission. In all cases the Sudakov double logarithms are found to exponentiate. We also discuss double logarithmic effects of a non-Sudakov type which appear in Regge-like processes.

* fadin@inp.nsk.su
† lipatov@thd.pnpi.spb.ru
‡ A.D.Martin@durham.ac.uk
§ Michael.Melles@psi.ch
1 Introduction

The Next Linear Collider (NLC) will explore $e^+e^-$ processes in the TeV energy regime, and probe the Standard Model of elementary particles to great accuracy. Electroweak processes which may reveal New Physics, such as supersymmetry, are especially interesting. Therefore the accurate calculation of the scattering amplitudes of such high energy processes in the Standard Model is very important. The main corrections to the Born amplitudes at high energies, $\sqrt{s}$, are double logarithmic (DL) contributions of the form $(g^2 \log^2(s/m^2))^n$ which arise from soft vector boson exchanges. Here we derive the DL asymptotics of scattering amplitudes, for a sequence of gauge theories leading up to the Standard electroweak Model.

The DL correction for QED processes dates back many years to the paper by Sudakov \[1\]. In Section 2 we present the original Sudakov form factor for QED in a form in which it can be generalized to non-Abelian gauge theories, a subject to which we then turn. We start with the processes where all the scalar products $p_ip_j$ of the momenta of participating particles are of the same order. In Section 3 we compute the DL corrections to processes governed by massless non-Abelian gauge theories, and then, in Section 4, we turn to the interesting case of broken gauge symmetries, discussing, in particular, the DL effects for Standard Model electroweak processes.

In all these cases the DL terms can be resummed in exponential forms which are generalizations of

$$
\mathcal{M} = \mathcal{M}_{\text{Born}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} W_i(s, \mu^2) \right). \tag{1}
$$

Here $W_i$ is the probability of the emission of a soft and quasi-collinear gauge boson from an external particle $i$, and the summation is over all $n$ external charged particles participating in the process. The virtual particles are subject to an infrared cut-off $\mu$ on their transverse momenta.

We use an equation for the evolution of the scattering amplitudes as a function of the infrared cut-off $\mu$ of the transverse momenta of virtual particles. Our approach is based on a gauge invariant dispersive method which is a generalization of a powerful theorem on photon Bremsstrahlung due to Gribov \[2\]. His remarkable theorem, proved by dispersive methods, states that, due to gauge invariance, the region of applicability of well known formulas for accompanying photon Bremsstrahlung is considerably extended at high energies. In these formulas the amplitude for a process with the emission of a soft photon is expressed, in a factorized form, in terms of the amplitude for the non-radiative process, with the particles taken on-mass-shell. Since the amplitude is taken on-shell, gauge invariance is guaranteed. The evolution equation approach greatly simplifies the computation of DL effects for electroweak processes, which are mediated by broken symmetry with the photon having components in both the $SU(2)$ and $U(1)$ gauge groups.

The evolution equations for the amplitude as a function of the infrared cut-off
parameter $\mu$ are analogous to the Renormalization Group Equations (RGE). Basically we start from the domain of very large $\mu$, where the Born amplitude applies, and evolve down to small $\mu$, matching the solution at the mass thresholds bounding every new kinematic domain. It is therefore not surprising that the structure of the exponentiation of the DL effects has a clear physical interpretation.

Of course for observable processes it is necessary to consider the DL corrections to both the non-radiative and the radiative processes. For radiative processes we again have exponentiation of the Sudakov DL corrections in a form similar to (1). The dependence on the infrared cut-off parameter $\mu$ is canceled in the measurable semi-inclusive process, and is replaced by a dependence on the experimental acceptance cuts on the soft boson emissions. In Sections 3.1, 4.1 and 4.2 we address these issues.

As well as DL corrections of the Sudakov type, there are DL contributions specific to forward or backward scattering, when a final particle is quasi-collinear to an initial particle and their energies are almost equal. This is the domain of Regge kinematics and is the subject of Section 5. Here we have the DL corrections related to the exchange of soft pairs of fermions or of bosons in the crossed channel. The infrared evolution equations are now non-linear in the amplitude and in the $j$-plane representation are of Ricatti-type form. In simple cases the latter forms can be reduced to a Schrödinger-type equation with a harmonic potential. This approach can also be applied to production processes in the multi-Regge kinematics. In this case the evolution equations are solved in a sequence of domains starting from the region where the Born amplitude applies and evolving down in the cut-off $\mu$ to the region where soft particles are emitted. The vector boson and fermion reggeization can be also easily verified in the DL approximation. Moreover, the infrared evolution equation allows the construction of scattering amplitudes with quasi-elastic unitarity.

2 Sudakov form factor in QED

The high energy asymptotics of electromagnetic processes was calculated many years ago within the framework of QED. In particular the amplitude for $e^+e^-$ elastic scattering at a fixed angle ($s \sim |t| \sim |u| \gg m^2 \gg \lambda^2$, where $m$ is the electron and $\lambda$ a fictitious photon mass) in the DL approximation has the form

$$\mathcal{M} = \mathcal{M}_{\text{Born}} \Gamma^2 \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right),$$

1 An analogy is now to the non-linear RGE for the coupling constant, whereas for the Bremsstrahlung processes an analogy is the linear RGE for the moments of structure functions.

2 $\lambda$ plays the role of the infrared cut-off. In physical cross sections the divergence in $\lambda$ of the elastic amplitude is canceled with the analogous divergences in processes with soft photon emissions.
where $\mathcal{M}_{\text{Born}}$ is the Born amplitude for $e^+e^-$ scattering and $\Gamma$ is the Sudakov form factor. The DL approximation applies in the energy regime

$$\alpha \log^2 \frac{s}{m^2} \sim \alpha \log \frac{s}{m^2} \log \frac{m^2}{\lambda^2} \sim 1,$$

(3)

where the QED coupling $\alpha = e^2/4\pi \ll 1$. Thus each charged external particle effectively contributes $\sqrt{\Gamma}$ to the total amplitude. The Sudakov form factor appears in the elastic scattering of an electron off an external field [1]. It is of the form:

$$\Gamma \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right) = \exp \left( -\frac{\alpha^2}{2\pi} R \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right) \right),$$

(4)

To specify $R$ it is convenient to use the Sudakov parametrization of the momentum of the exchanged virtual photon:

$$k = v p_1 + u p_2 + k_\perp,$$

(5)

where $p_1$ and $p_2$ are the initial and final momenta of the scattered electron. $R(s)$ can then be written as the integral over $u$ and $v$:

$$R \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right) = \int_0^1 du \int_0^1 dv \left( \frac{1}{u + m^2 v/s} \right) \left( \frac{1}{v + m^2 u/s} \right) \theta(sv - \lambda^2),$$

(6)

where $s \sim |t| \sim 2p_1 p_2$. The first two factors in the integrand correspond to the propagators of the virtual fermions which occur in the one-loop triangle Sudakov diagram. The $\theta$ function appears as a result of the integration of the propagator of the photon over its transverse momentum $k_\perp$, noting that the main contribution comes from the region near the photon mass shell [1]:

$$svu = \lambda^2 + k_\perp^2.$$

(7)

To DL accuracy (6) gives for $\lambda \ll m$:

$$R \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right) = \frac{1}{2} \ln^2 \frac{s}{m^2} + \ln \frac{s}{m^2} \ln \frac{m^2}{\lambda^2},$$

(8)

where the result comes equally from two different kinematical regions, $v \gg u$ and $u \gg v$. Therefore one can write $R = 2r$.

We can obtain physical insight by presenting the two equal contributions separately. In the first region, with $v \gg u$, the virtual photon is emitted along $p_1$ and the parameter $v$ is given by the ratio of energies of the photon and the initial electron. Here instead of $u$, it is convenient to use Eq. (7) to replace it by the square of the transverse momentum component of the photon. Then integrating over $v$ and $k_\perp^2$ gives

$$r \left( \frac{s}{m^2}, \frac{m^2}{\lambda^2} \right) = \int_{\lambda/\sqrt{2}}^{1} \frac{dv}{v} \int_{\lambda^2}^{sv^2} \frac{dk_\perp^2}{k_\perp^2 + m^2 v^2} \sim \int_{\lambda^2}^{\infty} \frac{dk_\perp^2}{k_\perp^2} \int_{|k_\perp|/\sqrt{2}}^{\min(|k_\perp|/m, 1)} \frac{dv}{v}$$

(9)
in the DL approximation, which may be evaluated to give half of \( R \). The quantity \( r \) is proportional to the probability \( w_i \) of the emission of a soft and almost collinear photon from an external particle with energy \( \sqrt{s} \) and mass \( m_i \), i.e.

\[
w_i(s, \lambda^2) = \frac{\alpha}{\pi} r \left( \frac{s}{m_i^2}, \frac{m_i^2}{\lambda^2} \right),
\]

If several charged particles participate in a process, for example \( e^+e^- \rightarrow f\bar{f}f\bar{f} \), then analogous contributions appear for each external line, provided all external invariants are large and of the same order. This leads to the general result

\[
\mathcal{M} = \mathcal{M}_{\text{Born}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} w_i(s, \lambda^2) \right),
\]

where \( n \) is the number of external lines corresponding to charged particles. In summary the soft emissions described by the Sudakov form factor is a quasi-classical effect which does not depend on the hard dynamics of the process. In particular there are no quantum mechanical interference effects in the DL Sudakov corrections, for large scattering angles.

3 Generalization to non-Abelian gauge theories

Sudakov effects have been widely discussed for non-Abelian gauge theories, such as \( SU(N) \) and can be calculated in various ways (see, for instance, [3]). We consider here the scattering amplitude in the simplest kinematics when all its invariants \( s_{lj} = 2p_lp_j \) are large and of the same order \( s_{lj} \sim s \). A general method of finding the DL asymptotics (not only of the Sudakov type) is based on the infrared evolution equations describing the dependence of the amplitudes on the infrared cutoff \( \mu \) of the virtual particle transverse momenta [4]. This cutoff plays the same role as \( \lambda \) in QED, but, unlike \( \lambda \), it is not necessary that it vanishes and it may take an arbitrary value. It can be introduced in a gauge invariant way by working, for instance, in a finite phase space volume in the transverse direction with linear size \( l \sim 1/\mu \). Instead of calculating asymptotics of particular Feynman diagrams and summing these asymptotics for a process with \( n \) external lines it is convenient to extract the virtual particle with the smallest value of \( |k_\perp| \) in such a way, that the transverse momenta \( |k'_\perp| \) of the other virtual particles are much bigger

\[
k'_\perp^2 \gg k_\perp^2 \gg \mu^2.
\]

For the other particles \( k_\perp^2 \) plays the role of the initial infrared cut-off \( \mu^2 \).

In particular, the Sudakov DL corrections are related to the exchange of soft gauge bosons, see Fig. 1. For this case the integral over the momentum \( k \) of the soft (i.e.
Figure 1: Feynman diagrams contributing to the infrared evolution equation (13) for a process with \( n \) external legs. In a general covariant gauge the virtual gluon with the smallest value of \( k_\perp \) is attached to different external lines. The inner scattering amplitude is assumed to be on the mass shell.

\(|k^0| \ll \sqrt{s}\) virtual boson with the smallest \( k_\perp \) can be factored off, which leads to the following infrared evolution equation:

\[
\mathcal{M}(p_1, ..., p_n; \mu^2) = \mathcal{M}_{\text{Born}}(p_1, ..., p_n) - i \frac{g^2}{2(2\pi)^4} \sum_{j,l=1, j \neq l}^n \int_{k^2 \gg \mu^2} \frac{d^4k}{k^2 + i\epsilon} \frac{p_j p_l}{(kp_j)(kp_l)} \times T^a(j)T^a(l)\mathcal{M}(p_1, ..., p_n; k_\perp^2),
\]

where the amplitude \( \mathcal{M}(p_1, ..., p_n; k_\perp^2) \) on the right hand side is to be taken on the mass shell, but with the substituted infrared cutoff: \( \mu^2 \rightarrow k_\perp^2 \). The generator \( T^a(l)(a = 1, ..., N) \) acts on the color indices of the particle with momentum \( p_l \). The non-Abelian gauge coupling is \( g \). In Eq. (13), and below, \( k_\perp \) denotes the component of the gauge boson momentum \( k \) transverse to the particle emitting this boson. Note that in Sudakov DL corrections there are no interference effects, so that we can talk about the emission (and absorption) of a gauge boson by a definite (external) particle, namely by a particle with momentum almost collinear to \( k \). It can be expressed in invariant form as \( k_\perp^2 \equiv \min((kp_l)(kp_j)/(pp_j)) \) for all \( j \neq l \). The above factorization is related to a non-Abelian generalization of the Gribov theorem\(^3\) for the amplitude of the Bremsstrahlung of a photon with small transverse momentum \( k_\perp \) in high energy hadron scattering\(^3\).

The form in which we present Eq. (13) corresponds to a covariant gauge for the gluon with momentum \( k \). Formally this expression can be written in a gauge

\(^3\)The non-Abelian generalization of Gribov’s theorem is given in (21) below, together with a description of its essential content.
invariant way if we include in the sum the term with \( j = l \) (which does not give a DL contribution). Indeed, in this case we can substitute \( p_i p_j \) by \(-p_i^\mu p_j^\nu d_{\mu\nu}(k)\), where the polarization matrices of the boson \( d_{\mu\nu}(k) \) in the various gauges differ by the terms proportional to \( k^\mu \) or \( k^\nu \) giving a vanishing contribution due to the conservation of the total color charge \( \sum_a T^a = 0 \). Thus we have the possibility of choosing appropriate gauges for each kinematical region of quasi-collinearity of \( k \) and \( p_l \). We can, however, use (13) as well, noting that in this region for \( j \neq l \) we have 
\[
p_j p_l / k p_j \approx E_l / \omega,
\]
where \( E_l \) is the energy of the particle with momentum \( p_l \) and \( \omega \) the frequency of the emitted gauge boson, so that:

\[
\mathcal{M}(p_1, ..., p_n; \mu^2) = \mathcal{M}_{\text{Born}}(p_1, ..., p_n) - \frac{2g^2}{(4\pi)^2} \sum_{l=1}^n \int_{\mu^2}^s \int_{|k_\perp|/\sqrt{s}}^{\min(|k_\perp|/m_l, 1)} \frac{dv}{v} \times C_l \mathcal{M}(p_1, ..., p_n; k_\perp^2),
\]

where \( C_l \) is the eigenvalue of the Casimir operator \( T^a(l)T^a(l) \) (\( C_l = C_A \) for gauge bosons in the adjoint representation of the gauge group \( SU(N) \) and \( C_l = C_F \) for fermions in the fundamental representation).

The differential form of the infrared evolution equation follows immediately from (14):

\[
\frac{\partial \mathcal{M}(p_1, ..., p_n; \mu^2)}{\partial \log(\mu^2)} = K(\mu^2) \mathcal{M}(p_1, ..., p_n; \mu^2),
\]

where

\[
K(\mu^2) \equiv -\frac{1}{2} \sum_{l=1}^n \frac{\partial W_l(s, \mu^2)}{\partial \log(\mu^2)}
\]

with

\[
W_l(s, \mu^2) = \frac{g^2}{4\pi^2} C_l r \left( \frac{s}{m_l^2} \right) \left( \frac{m_l^2}{\mu^2} \right).
\]

As in the Abelian case, \( W_l \) is the probability to emit a soft and almost collinear gauge boson from the particle \( l \) with mass \( m_l \), subject to the infrared cut-off \( \mu \) on the transverse momentum. Note again that the cut-off \( \mu \) is not taken to zero. The function \( r \) is determined by (9) for arbitrary values of the ratio \( m_l/\mu \). To logarithmic accuracy, we obtain from (17):

\[
\frac{\partial W_l(s, \mu^2)}{\partial \log(\mu^2)} = \frac{g^2}{8\pi^2} C_l \log \frac{s}{\max(\mu^2, m_l^2)}.
\]

The infrared evolution equation (15) should be solved with an appropriate initial condition. In the case of large scattering angles, if we choose the cut-off to be the large scale \( s \) then clearly there are no Sudakov corrections. The initial condition is therefore

\[
\mathcal{M}(p_1, ..., p_n; \mu) = \mathcal{M}_{\text{Born}}(p_1, ..., p_n),
\]
and the solution of (15) is thus given by the product of the Born amplitude and the Sudakov form factors:

\[
\mathcal{M}(p_1, \ldots, p_n; \mu^2) = \mathcal{M}_{\text{Born}}(p_1, \ldots, p_n) \exp \left( -\frac{1}{2} \sum_{l=1}^{n} W_l(s, \mu^2) \right)
\]

(20)

Therefore we obtain an exactly analogous Sudakov exponentiation for the gauge group \(SU(N)\) to that for the Abelian case, see (11). Theories with semi-simple gauge groups can be considered in a similar way.

### 3.1 DL corrections to processes with soft emissions

Since ultimately we are interested in measurable cross sections we have to consider the DL corrections to amplitudes of processes with soft emissions, as well as those without. Only in inclusive cross sections will the dependence on the infrared cut-off \(\mu^2\) disappear, being replaced by parameters specifying the experimental acceptance. To put it in another way, cross sections of the emission processes receive large (DL) contributions from regions where the emitted bosons are soft and the emission angles are small. Therefore, to be consistent, we need to calculate cross sections of such processes as well. This is easy to do for QED processes, where the single gauge boson (the photon) is neutral and does not possess self-interactions. Therefore soft photons are emitted independently according to a Poisson distribution. In non-Abelian theories, gauge bosons are not neutral and interact with each other. Consequently, the soft emission does not follow a Poisson distribution [8].

We again consider the simplest situation, when the additional soft gauge boson is emitted in the process with all invariants \(s_{ij}\) large. Of course, for the emission of a boson almost collinear to the particle the direction of the particle with momentum \(p_i\), the invariant \(2kp_i\) is small in comparison with \(s\). In the case of non-Abelian gauge theories the corresponding amplitude for the emission of a soft gauge boson with small \(k^2_\perp \ll \mu^2\) has, according to the Gribov theorem, the following form:

\[
\mathcal{M}^a(p_1, \ldots, p_n; k; \mu^2) = \sum_{j=1}^{n} g \frac{\varepsilon^a_{pj}}{kp_j} T^a(j) \mathcal{M}(p_1, \ldots, p_n; \mu^2).
\]

(21)

The possible corrections to this factorized expression are of the order of \(k^2_\perp/\mu^2\). However, to DL accuracy, we can substitute \(\mu^2\) in the arguments of the scattering amplitudes by its boundary value \(k^2_\perp\). Notice that the amplitude on the r.h.s. of (21) is taken on-the-mass shell, which guarantees its gauge invariance. The result (21) is highly non-trivial in the Feynman diagram approach. It means, that the region of applicability of the classical formulas for the Bremsstrahlung amplitudes is significantly enlarged at high energies. V. Gribov proved this result by using dispersion relations in the variables \((k + p_j)^2\), and demonstrating that for small \(|k_\perp|\) the pole terms in these
invariants are much larger than the corresponding cut contributions due to the gauge invariance of the theory [3]. The region of applicability of (21) corresponds to the situation when the momentum of the emitted soft boson does not spoil the kinematics of the non-radiative process. This implies that the frequency $\omega$ of the boson emitted off the particle with momentum $p_i$ should be much smaller than the energy $E_i$, and that the emission angle $\vartheta_i \simeq |k_\perp|/\omega$ should be much smaller than the angle between $p_i$ and all other momenta $p_j$ (otherwise one needs to include interference effects). For the physical Coulomb gauge ($\varepsilon_0 = 0$, $\varepsilon \cdot k = 0$) in the kinematical region where the gauge boson is emitted at a small angle $\vartheta_i$ with respect to the particle with momentum $p_i$ in (21) only the term with $j = i$ contributes. The method based on the infrared evolution equation allows us to calculate, in the DL approximation, the amplitudes of the hard processes accompanied by the emission of any number of soft gauge bosons [8]. Let us consider the amplitude for the emission of one soft gauge boson in the region of quasi-collinearity of its momentum with momentum $p_i$ (i.e., the emission of a soft boson by a particle with momentum $p_i$). When the transverse momentum $|k_\perp|$ of this boson is much less than the infrared cut-off $\mu$, used for the virtual particles, the amplitude is given by the term with $j = i$ in the sum in (21). But we need to know the emission amplitude in the opposite case, $|k_\perp| \gg \mu$. It can be found from the evolution equation in this region using expression (21) at $\mu = |k_\perp|$ as the initial condition. The kernel of the evolution equation in this region differs from the corresponding kernel in the region $|k_\perp| \ll \mu$ (that is the kernel (16) of the evolution equation for the amplitude without emission) by a term connected with the emission of a virtual boson from the
real gauge boson with momentum \( k \):

\[
\Delta K(\mu^2) = -\frac{1}{2} \frac{\partial W_A(k_\perp^2, \mu^2)}{\partial \log(\mu^2)} = \frac{g^2}{(4\pi)^2} C_A \log \frac{k_\perp^2}{\mu^2},
\]

where \( W_A \) is given by (17) with \( C_l = C_A \) and \( m_l = \mu \), see Fig. 2. It is clear, that this new term in the kernel for evolution from \( k_\perp^2 \) to \( \mu^2 \) leads to an additional term \( W_A(k_\perp^2, \mu^2) \) in the Sudakov exponential. Thus, the amplitude for the emission of one gauge boson with small transverse momentum from the hard scattering process is of the form:

\[
\mathcal{M}^a(p_1, ..., p_n; k; \mu^2) = \sum_{j=1}^{n} g \frac{\varepsilon^* p_j}{kp_j} T^a(j) \mathcal{M}_{\text{Born}}(p_1, ..., p_n) 
\times \exp \left( -\frac{1}{2} \sum_{i=1}^{n} W_i(s, \mu^2) - \frac{1}{2} W_A(k_\perp^2, \mu^2) \right). \tag{23}
\]

We note again that here \( k_\perp^2 \) means the square of the component of the three-dimensional momentum transverse to the momentum of the emitting particle, say for example, \( p_l \). We can write \( k_\perp^2 \) in the invariant form \( (kp_l)(kp_j)/(p_l p_j) \) with \( j \neq l \), which does not depend on \( j \).

Again we see that we have the exponentiation of the Sudakov DL corrections. Note that in the Abelian QED case we have \( W_A = 0 \) and the exponent for the photon Bremsstrahlung amplitude remains the same as that for the process without photon emission. It is related to the Poisson distribution for soft photon production.

The exponentiation of virtual DL corrections holds for multiple emission processes as well. In QED it is trivial, since the soft photons are emitted independently. In non-Abelian gauge theories it is not so simple. The main complexity is connected with the nontrivial structure of the amplitudes for multiple emission of real soft gauge bosons, arising from their self-interaction. But in the DL approximation these amplitudes can be calculated. Due to the coherence effect, the branching cascade develops only in the region of sequentially shrinking angular cones [8]. In this region the Born amplitudes for multiple emission processes have a factorized form and the virtual corrections exponentiate [8]. It is proved by solving the infrared evolution equation in a series of regions where the infrared cut-off \( \mu \) is bounded between a sequence of decreasing transverse momenta of the emitted gluons, using in each region the solution of the previous region as the initial condition. The final result is

\[
\mathcal{M}(p_1, ..., p_n; k_1, ..., k_r; \mu^2) = \mathcal{M}_{\text{Born}}(p_1, ..., p_n; k_1, ..., k_r) 
\times \exp \left( -\frac{1}{2} \sum_{i=1}^{n} W_i(s, \mu^2) - \frac{1}{2} \sum_{i=1}^{r} W_A(k_{i\perp}^2, \mu^2) \right), \tag{24}
\]

where \( k_i \) are the momenta of the emitted gluons with strongly ordered energies and \( k_{i\perp} \) are their components transverse to the momenta of the emitting jets.
4 Sudakov effects in broken gauge theories

The same method, based on the infrared evolution equation, is also applicable to broken gauge theories. Let us consider for definiteness the Standard electroweak theory, where the physical gauge bosons are a massless photon (described by the field $A_\nu$) and massive $W^\pm$ and $Z$ bosons (described correspondingly by fields $W^\pm_\nu$ and $Z_\nu$). To DL accuracy, all masses can be set equal:

$$M_Z \sim M_W \sim M_{\text{Higgs}} \sim M$$

and the energy considered to be much larger, $\sqrt{s} \gg M$. In the unbroken phase the corresponding Abelian and Yang-Mills fields are denoted by $B_\nu$ and $W^a_\nu$, with $a = 1, 2, 3$. The physical fields are linear combinations of the fields in the unbroken theory with coefficients depending on the Weinberg angle $\theta_w$. The left and right handed fermions are correspondingly doublets ($T = 1/2$) and singlets ($T = 0$) of the $SU(2)$ weak isospin group and have hypercharge $Y$ related to the electric charge $Q$, measured in units of the proton charge, by the Gell-Mann-Nishijima formula $Q = T^3 + Y/2$.

In the evolution equation in the DL approximation the value of the infrared cutoff $\mu$ can be chosen in two different ranges: 1) $\sqrt{s} \gg \mu \gg M$ and 2) $\mu \ll M$. The second case is universal in the sense that it does not depend on details of the electroweak theory. It will be discussed below. In the first region we can neglect spontaneous symmetry breaking effects, in particular gauge boson masses, and consider the evolution equation in the unbroken phase with effectively massless particles $B$ and $W^a$. Of course one could calculate everything also in terms of the physical fields $A_\nu, Z_\nu$ and $W^\pm_\nu$. In the unbroken phase this is equivalent to the description in terms of the original fields $B^\nu$ and $W^a_\nu$ and leads to the same final result, but the intermediate steps will be more complicated because there are cancellations between non-exponentiating terms from diagrams with $Z$ and $\gamma$ exchanges. The separation of these contributions is not gauge invariant and if we would consider the diagrams without virtual photons we would violate $SU(2) \times U(1)$ symmetry. Taking into account only such diagrams leads to nonexponentiating DL effects in an axial gauge [3]. Fig. 3 illustrates this at the two loop level. The loss of gauge invariance is related to the fact that the photon field contains the component $W^3_\nu$ of the non-Abelian field $W^a_\nu$, and so omitting the virtual photons would violate the conservation of the weak isospin current (in the unbroken theory). In region 1) the infrared evolution equation, written in terms of the unbroken fields, is of a form analogous to (15) if we assume, for simplicity, that all the charged particles have masses $m_i \leq M$

$$\frac{\partial \mathcal{M}(p_1, ..., p_n; \mu^2)}{\partial \log(\mu^2)} = \frac{\log(s/\mu^2)}{(4\pi)^2} \sum_{i=1}^{n} \left( g^2 T_i (T_i + 1) + g'^2 \left( \frac{Y_i}{2} \right)^2 \right) \mathcal{M}(p_1, ..., p_n; \mu^2). \quad (25)$$

Here $T_i$ is the total weak isospin of particle $i$, $Y_i$ is its weak hypercharge, and $g$ and $g' = g \tan \theta_w$ are the couplings of the $SU(2)$ and $U(1)$ gauge groups, respectively. The
Figure 3: Two-loop ‘rainbow’ Feynman diagrams contributing to DL corrections in an axial gauge. The photon contribution has DL corrections in both the regions $\mu^2 \ll M^2$ and $M^2 \ll \mu^2 \ll s$. Taken together with the $W$- and $Z$-contributions, it yields the exponentiation of the Sudakov DL terms in the electroweak theory. In the region $M^2 \ll \mu^2$, the spontaneously broken gauge symmetry is restored and omitting the photon contributions would lead to a non-gauge invariant result.
sum in (25) is to be performed over all n external particles. As before, the initial condition is given by the requirement that for the infrared cut-off \( \mu^2 = s \) we obtain the Born amplitude. The solution of (25) is thus given by

\[
M(p_1, \ldots, p_n; \mu^2) = M_{\text{Born}}(p_1, \ldots, p_n) \times \exp \left[ -\frac{\log^2(s/\mu^2)}{2(4\pi)^2} \sum_{i=1}^{n} \left( g^2 T_i(T_i + 1) + g^2 \left( \frac{Y_i}{2} \right)^2 \right) \right]. \tag{26}
\]

The expression in the brackets in the exponential can be written in terms of the parameters of the broken theory as follows:

\[
g^2 T_i(T_i + 1) + g^2 \left( \frac{Y_i}{2} \right)^2 = e_i^2 + g^2 \left( T_i(T_i + 1) - (T_i^3)^2 \right) + \frac{g^2}{\cos^2 \theta_w} \left( T_i^3 - \sin^2 \theta_w Q_i \right)^2,
\]

where the three terms on the r.h.s. correspond to the contributions of the soft photon (interacting with the electric charge \( e_i = Q_i g \sin \theta_w \)), the \( W^\pm \) and the \( Z \) bosons, respectively. Although we may rewrite solution (26) in terms of the parameters of the broken theory in the form of a product of three exponents corresponding to the exchanges of photons, \( W^\pm \) and \( Z \) bosons, it would be wrong to identify the contributions of the diagrams without virtual photons with this expression for the particular case \( e_i^2 = 0 \). This becomes evident when we note that if we were to omit photon lines then the result would depend on the choice of gauge, and therefore be unphysical. Only for \( \theta_w = 0 \), where the photon coincides with the \( B \) gauge boson, would the identification of the \( e_i^2 \) term with the contribution of the diagrams with photons be correct.

Choosing the cutoff \( \mu \) in the second region, \( \mu \ll M \), the infrared evolution equation takes the following form:

\[
\frac{\partial M(p_1, \ldots, p_n; \mu^2)}{\partial \log(\mu^2)} = \sum_{i=1}^{n} \frac{e_i^2}{(4\pi)^2} \log \left( \frac{s}{\text{max}(m_i^2, \mu^2)} \right) \frac{1}{M^2} M(p_1, \ldots, p_n; \mu^2). \tag{27}
\]

Evidently, only the photon contribution remains in this region. Now the appropriate initial condition is given by (26) evaluated at the matching point \( \mu = M \). The solution is thus

\[
M(p_1, \ldots, p_n; \mu^2) = M_{\text{Born}}(p_1, \ldots, p_n) \times \exp \left[ -\frac{g^2}{2(4\pi)^2} \sum_{i=1}^{n} \left( T_i(T_i + 1) + \tan^2 \theta_w \left( \frac{Y_i}{2} \right)^2 \right) \log^2 \frac{s}{M^2} \right] \times \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} Q_i^2 \left( w_i(s, \mu^2) - w_i(s, M^2) \right) \right] = M_{\text{Born}}(p_1, \ldots, p_n) \times \exp \left[ -\frac{g^2}{2(4\pi)^2} \sum_{i=1}^{n} \left( T_i(T_i + 1) + \tan^2 \theta_w \left( \frac{Y_i}{2} \right)^2 \right) \log^2 \frac{s}{M^2} \right] \times \exp \left[ -\sum_{i=1}^{n} \left( \frac{e_i^2}{(4\pi)^2} \log \frac{s}{m_i} \log \frac{M^2}{m_i^2} + \log \frac{s}{m_i^2} \log \frac{m_i^2}{\mu^2} \right) \right], \tag{28}
\]
where the last equality holds for $\mu \ll m_i$ and $m_i^2 \ll M^2$ from the respective expansions of $w_i$ in (10). Let us stress that (24) and (28) are applicable for processes involving chiral fermions as well as gauge bosons, provided that all the invariants are large ($\mathcal{O}(s)$) compared to $M^2$. Note, that in the case, when quarks or gluons participate in the reaction, we should multiply these expressions by the Sudakov factors corresponding to the virtual gluons emitted by these colored particles. The infrared evolution equations (25) and (27) have a clear physical meaning analogous to the renormalization group equations and therefore it is natural to expect that the next-to-leading corrections to the kernels can be calculated.

For physical observables soft real photon emission must be taken into account in an inclusive way and effectively the parameter $\mu^2$ in (28) will be replaced by parameters depending on the experimental requirements.

4.1 DL effects for electroweak processes with soft emission

The calculation of amplitudes for processes with the emission of a gauge boson in the kinematical region which gives DL contributions to the cross sections (i.e. the region of soft quasi-collinear emission) is similar to the analogous calculation for unbroken gauge theories, with complications of the type that we discussed above. One is that we have to consider separately two regions of $|k_\perp|$ of the emitted boson: first $|k_\perp| \ll M$ and second $|k_\perp| \gg M$. At high energies the cross sections of the emission processes receive DL contributions from both of these regions. Of course, $W^\pm$ and $Z$ boson emissions contribute in the second region only. Therefore consideration of the first case is very simple. For values of the infrared cut-off $\mu^2 \gg k_\perp^2$, the amplitude for the emission of a soft photon by a particle with momenta $p_l$ (i.e. emission within the cone along $p_l$ not containing the momenta of the other particles) has, in the physical (Coulomb) gauge, a factorized form

$$
\mathcal{M}(p_1, ..., p_n; k; \mu^2) = e_\ell \frac{\varepsilon^* p_l}{k p_l} \mathcal{M}(p_1, ..., p_n; \mu^2),
$$

(29)

according to the Gribov theorem of (24). However, if $k_\perp^2 \ll M^2$, the kernel of the infrared evolution equation does not change when the cut-off $\mu^2$ changes from the domain $\mu^2 \gg k_\perp^2$ to $\mu^2 \ll k_\perp^2$ (since the $W^\pm$ and $Z$ bosons do not contribute in this first region). Therefore, (29) remains valid at arbitrary values of the cut-off $\mu$.

In the second region, $|k_\perp| \gg M$, the result is more involved. We need to start from $\mu^2 \gg k_\perp^2$, where we can use a generalization of Gribov’s theorem. Consider again the case when a gauge boson is emitted by a particle with momentum $p_l$. We have

$$
\mathcal{M}^a(p_1, ..., p_n; k; \mu^2) = G_0^a(1) \frac{\varepsilon^* p_l}{k p_l} \mathcal{M}(p_1, ..., p_n; \mu^2),
$$

(30)
where $\mathcal{M}(p_1, ..., p_n; \mu^2)$ is given by (24) and

\[
G_0^\pm = \frac{g}{\sqrt{2}} T^\pm \quad \text{for } W^\pm \text{ emission},
\]

\[
G_0^Z = \frac{g}{\cos \theta_w} \left( T^3 - Q \sin^2 \theta_w \right) \quad \text{for } Z \text{ emission},
\]

\[
G_0^\gamma = Q g \sin \theta_w \quad \text{for } \gamma \text{ emission},
\]

(31)

with $Q = (T^3 + Y/2)$. Then we have to solve the evolution equation in the region $M^2 \ll \mu^2 \ll k_\perp^2$ with the initial condition given by (30) at the matching point $\mu^2 = k_\perp^2$. In fact, it is more appropriate to work in terms of the fields $W_\nu^a$ and $B_\nu$. The kernel of the evolution equation remains unchanged for the emission of the $B$-particle due to its Abelian nature. On the other hand the emission of the $W_\nu^a$-particle leads to the same additional contribution as in the unbroken theory (see (22)). Therefore in the cut-off region $k_\perp^2 \gg \mu^2 \gg M^2$, we obtain

\[
\mathcal{M}^a(p_1, ..., p_n; k; \mu^2) = G_0^a(l) \frac{\varepsilon^* p_l}{k_{pl}} \mathcal{M}(p_1, ..., p_n; \mu^2),
\]

(32)

with the same amplitude $\mathcal{M}(p_1, ..., p_n; \mu^2)$ as in (26) and

\[
G_1^\pm = G_0^\pm \exp \left( -\frac{1}{2} W_A(k_\perp^2, \mu^2) \right),
\]

\[
G_1^Z = G_0^Z + g \cos \theta_w T^3 \left( \exp \left( -\frac{1}{2} W_A(k_\perp^2, \mu^2) \right) - 1 \right),
\]

\[
G_1^\gamma = G_0^\gamma + g \sin \theta_w T^3 \left( \exp \left( -\frac{1}{2} W_A(k_\perp^2, \mu^2) \right) - 1 \right).
\]

(33)

Finally we study the region of the infrared cut-off $\mu^2 \ll M^2$. In this region the kernel of the evolution equation is determined by the electromagnetic interaction only; therefore, the only contribution related to the emitted particles is that for $W^\pm$ emission. This contribution is given by

\[
\Delta K_W(\mu^2) = -\frac{1}{2} \frac{\partial w_W(k_\perp^2, \mu^2)}{\partial \log(\mu^2)} = \frac{e^2}{(4\pi)^2} \log \frac{k_\perp^2}{M^2}
\]

(34)

where $w_W(k_\perp^2, \mu^2)$ is defined by (11) and (9) with $M_W^2 = M^2$. Consequently, for values of the infrared cut-off $\mu^2 \ll M^2$ we obtain:

\[
\mathcal{M}^a(p_1, ..., p_n; k; \mu^2) = G_0^a(l) \frac{\varepsilon^* p_l}{k_{pl}} \mathcal{M}(p_1, ..., p_n; \mu^2),
\]

(35)

where $\mathcal{M}(p_1, ..., p_n; \mu^2)$ is given by (28) and

\[
G^\pm = G_0^\pm \exp \left( -\frac{1}{2} W_A(k_\perp^2, M^2) - \frac{1}{2} w_W(k_\perp^2, \mu^2) + \frac{1}{2} w_W(k_\perp^2, M^2) \right),
\]

14
\[ G^Z = G_0^Z + g \cos \theta_w T^3 \left( \exp \left( -\frac{1}{2} W_A(k_{\perp}^2, M^2) \right) - 1 \right), \]
\[ G^\gamma = G_0^\gamma + g \sin \theta_w T^3 \left( \exp \left( -\frac{1}{2} W_A(k_{\perp}^2, M^2) \right) - 1 \right). \] (36)

The important point here is the difference of the DL exponent for a nonradiative process and the process with photon emission, which leads to a violation of the Poisson distribution for photons in the DL approximation at high energies. It is a direct consequence of the fact that the photon has a non-Abelian component.

### 4.2 DL effects in semi-inclusive cross sections

Measurable cross sections have an inclusive nature (at least with respect to photons, since only cross sections with an infinite number of emitted soft photons are observable). Let us consider the DL corrections to such cross sections.

It is clear that for the emission of real gauge bosons the same cut-off \( \mu^2 \) must be used as for virtual ones. Therefore, to calculate an experimentally measured cross section we have to take the cut-off \( \mu^2 \) less than the lower bound \( \mu^2_{\exp} \) of those photons emitted in processes which are not included in the cross section.

The calculation of the cross section is simple if the experimental conditions are such that only processes with emission of photons with \( k_{\perp}^2 < M^2 \) are allowed. In this case the non-Abelian component of the photon is not essential, so that photon emissions obey a Poisson distribution. Therefore for the cross section with an arbitrary number of emitted photons with momenta lying inside regions \( \Omega_i \) of the momentum space around the emitting particles with momenta \( p_i \), we obtain:

\[ d\sigma(p_1, \ldots, p_n) = d\sigma_{\text{elastic}}(p_1, \ldots, p_n) \exp(w_{\exp}^\gamma), \]

where \( d\sigma_{\text{elastic}} \) is the cross section of the non-radiative process and \( w_{\exp}^\gamma \) is the probability of the emission of photons with \( k_{\perp}^2 > \mu^2 \) inside the allowed region

\[ w_{\exp}^\gamma = \sum_{i=1}^n \frac{e_i^2}{(2\pi)^3} \int_{\Omega_i} \frac{d^3k}{2\omega} \frac{2E_i}{\omega(kp_i)} \theta(k_{\perp}^2 - \mu^2) \]
\[ = \sum_{i=1}^n \frac{e_i^2}{4\pi^3} \int_{\Omega_i} \frac{d^2k_{\perp}}{k_{\perp}^2 + m_i^2\omega^2/E_i^2} \frac{d\omega}{\omega} \theta(k_{\perp}^2 - \mu^2). \] (37)

Since the upper bound on \( k_{\perp}^2 \) of the photons which are allowed to be radiated is less than \( M^2 \), we must use the cut-off \( \mu^2 < M^2 \) and, consequently, \( (28) \) for the matrix element of the non-radiative process. Therefore, we obtain

\[ d\sigma(p_1, \ldots, p_n) = d\sigma_{\text{Born}}(p_1, \ldots, p_n) \]
\[ \times \exp \left[ -\frac{g^2}{(4\pi)^2} \sum_{i=1}^n \left( \left[ T_i(T_i + 1) + \tan^2 \theta_w \frac{Y_i}{2} \right]^2 \right) \log^2 \frac{s}{M^2} \right] \]

15
\[ \times \exp \left[ - \sum_{i=1}^{n} Q_i^2 \left( w_i(s, \mu^2) - w_i(s, M^2) \right) + w_{\gamma}^{\text{exp}} \right], \]  

(38)

where \( w_i(s, \mu^2) \) is given by (10) and (9). Evidently, the dependence on \( \mu \) in \( \sum Q_i^2 w_i(s, \mu^2) \) and \( w_{\gamma}^{\text{exp}} \) cancels in the exponential.

If we include in the observed cross section the emission of gauge bosons with transverse momenta larger than \( M \), then the problem becomes much more complicated because of the non-Poisson distribution of soft emission in non-Abelian gauge theories \[8\]. Let us consider here the simplest example of the cross section completely inclusive of photons emission in the two-loop approximation. Then the cross section can be written as

\[ d\sigma(p_1, ..., p_n) = d\sigma_{\text{Born}}(p_1, ..., p_n) \left( 1 + \delta_v + \frac{\delta_v^2}{2} + \delta_r + \delta_{rv} + \frac{\delta_r^2}{2} \right), \]  

(39)

where \( \delta_v \) is the one-loop virtual correction, \( \delta_r \) comes from one-photon emission taken in the Born approximation and \( \delta_{rv} \) from the one-loop correction to one-photon emission. Due to exponentiation of the DL terms of the Sudakov-type in virtual corrections, the term \( \delta_v^2/2 \) in (39) gives the two-loop virtual correction. The term \( \delta_r^2/2 \) gives the correction from two-photon emission in the Born approximation, since in this approximation the two photons are emitted independently.

In the considered case of the cross section completely inclusive of photon emission the cut-off parameter \( \mu^2 \) can be taken as large as \( M^2 \) (but not greater, because the cross section does not include \( W^\pm \) and \( Z \) emission). Each of the corrections considered above depends on \( \mu^2 \), but their sum in (39) does not; therefore, we can take the most suitable value of the cut-off to calculate the cross section. It is easy to see that the most convenient choice is \( \mu^2 = M^2 \). In this case from (26) we have:

\[ \delta_v = - \frac{g^2}{(4\pi)^2} \sum_{i=1}^{n} \left( T_i(T_i + 1) + \tan^2 \theta_w \left( \frac{Y_i}{2} \right)^2 \right) \log^2 \frac{s}{M^2}. \]  

(40)

The correction due to one-photon emission taken in the Born approximation is \( \delta_r = w_{\gamma}^{\text{exp}} \) where \( w_{\gamma}^{\text{exp}} \) is given by (37) with \( \mu^2 = M^2 \), \( E_i \sim \sqrt{s} \) and the region \( \Omega_i \) defined by inequality \( \omega < E_i \). It gives

\[ \delta_r = \sum_{i=1}^{n} Q_i^2 w_i(s, M^2) = \sum_{i=1}^{n} Q_i^2 \frac{\epsilon^2}{(4\pi)^2} \log^2 \frac{s}{M^2}. \]  

(41)

The one-loop contribution to this correction \( \delta_{rv} \) can be found with help of (26), (31), (32) and (33):

\[ \delta_{rv} = \sum_{i=1}^{n} \frac{\epsilon^2}{4\pi^2} \int_{M^2}^{s} \frac{dk_{\perp}^2}{k_{\perp}^2} \int_{|k_{\perp}|}^{\sqrt{s}} d\omega \frac{d\omega}{\omega} \left[ Q_i^2 \delta_v - Q_i T_i^3 W_A(k_{\perp}, M^2) \right]. \]
\[ = \sum_{i=1}^{n} \frac{e^2}{(4\pi)^2} \left( \delta_i Q_i^2 \log^2 \frac{s}{M^2} - Q_i T_i^3 \frac{g^2}{3(4\pi)^2} \log^4 \frac{s}{M^2} \right). \]  

(42)

The important point here is that the virtual correction to the one-photon emission does not coincide with the corresponding correction to the non-radiative process (which means violation of the Poisson distribution) and depends on the momentum of the emitted photon. It is a consequence of the fact that the photon has a non-Abelian component.

One can check that using an arbitrary cut-off \( \mu^2 < M^2 \) gives the same result, but the calculation is more complicated.

## 5 DL effects for amplitudes with Regge kinematics

For two particle scattering in either the forward \( s \gg (-t) \) or the backward \( s \gg (-u) \) directions, corresponding to the Regge regime, there are situations, where the amplitudes have DL contributions different from those of the Sudakov type. These logarithms appear from Feynman diagrams in which the soft particles with the minimal transverse momenta are pairs of virtual fermions or bosons exchanged in the crossed \( t \) or \( u \) channels. For QED and QCD the infrared evolution equations for such Regge processes are known [4, 5, 7]. The new features presented by the Standard Model of weak interactions are that the gauge group is semi-simple, \( SU(2) \times U(1) \), and that there is a large difference in the particle mass scales. The general strategy of finding DL asymptotics in this case is to first solve the evolution equation in the region \( \mu \gg M_1 \), where \( M_1 \) is the largest particle mass in the theory; then to solve it in the region \( M_2 \ll \mu \ll M_1 \), where \( M_2 \) is the next largest particle mass, using the solution of the previous equation at \( \mu = M_1 \) as the initial condition, and to proceed with these steps until the intermediate particles are light quarks, electrons, neutrinos and photons.

For simplicity we consider only the evolution equation in the region where the cut-off parameter \( \mu \) is much larger than all particle masses. In this case it is natural to calculate the amplitudes in terms of the massless particles of the unbroken theory, that is the leptons and quarks, the Higgs, \( B \) and \( W^a \) bosons. By solving the generalized infrared evolution equation in the DL approximation we sum up contributions of ladder-type diagrams in the crossed channel, with all possible insertions of the gauge bosons, leading to the Sudakov double logarithms. Because the transverse momenta of the virtual particles are strongly ordered, the integral kernels in the ladder diagrams are given by the splitting kernels of the DGLAP evolution equation [10] which describes all possible transitions among fermions and bosons in the Standard Model. These splitting kernels can be simplified since the Regge kinematic domain corresponds to a strong ordering of the Bjorken variables \( x_i \) along the ladder.

For definiteness we study the simple case of the backward lepton (\( l \)) - antilepton (\( \bar{l} \)) scattering for \( s \simeq -t \gg -u \). In the Born approximation, the contribution of the
Feynman diagrams with the Higgs, photon, \( B \) and \( W^a \) boson exchanges depend on the helicities \( \zeta_i = \lambda_i/2 \) (with \( \lambda_i = \pm 1 \)) of the initial and final fermions, which have momenta

\[
p_i, \ p_i, \ p_\nu \simeq p_i, \ p_\bar{\nu} \simeq p_i. \tag{43}
\]

We consider the scattering of the leptons \( e \) and \( \nu_e \) belonging to the first generation with the smallest masses. Because the coupling of the Higgs particle to leptons is proportional to their mass, we can neglect the Higgs contribution. The Dirac spinors describing the \( l \) and \( \bar{7} \) states with the definite helicities \( \lambda \) are

\[
w_\lambda(p) = \sqrt{|p|} \left( \varphi_\lambda \lambda \varphi_\lambda \right), \quad v_\lambda(-p) = \sqrt{|p|} \left( \varphi_{-\lambda} - \lambda \varphi_{-\lambda} \right), \tag{44}
\]

where

\[
\frac{\sigma p}{|p|} \varphi_\lambda = \lambda \varphi_\lambda. \tag{45}
\]

With the use of these expressions we can calculate the matrix elements of the \( \gamma \)-matrix structures which appear in the Born approximation for backward lepton-antilepton scattering, \( \bar{l}l \rightarrow \bar{t}t' \),

\[
\frac{1}{s} \overline{\gamma}_\nu(-p_\nu) \gamma_\sigma u_{\lambda_1}(p_1) \overline{\gamma}_{\lambda_1'}(p_1') \gamma_\sigma v_{\lambda_1'}(-p_\nu) = a_s^{\lambda_1 \lambda_1' \lambda_\nu \lambda_\nu} = 2 \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_\nu \lambda_\nu} + \delta_{\lambda_1 \lambda_\nu} \delta_{\lambda_1' \lambda_\nu} \tag{46}
\]

\[
-\frac{1}{t} \overline{\gamma}_\nu(-p_\nu) \gamma_\sigma v_{\lambda_1'}(-p_\nu) \overline{\gamma}_{\lambda_1'}(p_1') \gamma_\sigma u_{\lambda_1}(p_1) = a_t^{\lambda_1 \lambda_1' \lambda_\nu \lambda_\nu} = 2 \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_\nu \lambda_\nu} + \delta_{\lambda_1 \lambda_\nu} \delta_{\lambda_1' \lambda_\nu}. \tag{47}
\]

It is important, to note that both the helicity structures of \( a_s \) and \( a_t \) are proportional to \( \delta_{\lambda_1 \lambda_\nu} \), which means, that the helicities of the two fermions in the crossed \( u \) channel are opposite in sign. Therefore the Born diagram with a virtual \( W^a \) boson gives a negligible contribution to the backward scattering. Because the \( B \) boson interacts with hypercharge \( Y \), its contribution to the Born amplitude is also zero for the right-handed neutrinos (if they were to exist).

Below we consider only the non-trivial case, when only \( e^\pm_1, \nu_- \) and \( \gamma_+ \) can participate in the reaction. Because the hypercharge \( Y = -1 \) for left-handed fermions and \( Y = -2 \) for the right-handed electron, in this case the Born amplitude is given by

\[
\mathcal{M}_{\text{Born}}^{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau} = \frac{g^2}{2} \left( \frac{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau}{a_s} + \frac{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau}{a_t} \right). \tag{48}
\]

In the DL approximation we also obtain a significant simplification of the helicity structure of the scattering amplitude:

\[
\mathcal{M}_{\text{DL}}^{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau} = \frac{g^2}{2} \left( a_s^{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau} f(s/\mu^2) + a_t^{\lambda_1 \lambda_\nu \lambda_\tau \lambda_\tau} f(-s/\mu^2) \right), \tag{49}
\]

\[
f(s/\mu^2) = f^+(s/\mu^2) + f^-(s/\mu^2), \tag{50}
\]
where the functions \( f^\pm(s/\mu^2) \) describe contributions with positive and negative signature:

\[
\begin{align*}
  f^\pm(-s/\mu^2) &= \pm f^\pm(s/\mu^2), \\
  f^+_{\text{Born}}(s/\mu^2) &= 1, \quad f^-_{\text{Born}}(s/\mu^2) = 0. 
\end{align*}
\]

The amplitudes \( f^\pm \) are assumed to satisfy dispersion relations of the form

\[
f^\pm(s/\mu^2) = -\frac{1}{\pi} \int_0^\infty \left( \frac{s}{s-s'} \pm \frac{s}{s+s'} \right) \text{Im} f^\pm(s'/\mu^2) \frac{ds'}{s'},
\]

where in the Born approximation \( \text{Im} f^\pm_{\text{Born}}(x) \) is different from zero only in the region where \( x \) is of the order of unity. In particular, from the known behavior of the Born contribution \( f_{\text{Born}}(s/\mu^2) \) at large \( s \) and its degeneracy in the signature we obtain

\[-\frac{2}{\pi} \int_0^\infty \text{Im} f^\pm_{\text{Born}}(s'/\mu^2) \frac{ds'}{s'} = 1.\]

The absence of degeneracy of the amplitude with respect to signature is a result of the presence of the non-planar diagrams which arise from the virtual soft \( B \)-bosons in the infrared evolution equations (cf. the QCD case [7]):

\[
\begin{align*}
  \frac{df^+(s/\mu^2)}{d\log(\mu^2)} &= -\frac{g'^2}{8\pi^3} \int_0^{\log \frac{s}{s'}} f^+(s_1/\mu^2) \left( \int_0^{\log \frac{s'}{s_2}} \text{Im} f^+(s_2/\mu^2) \frac{d\log s_2}{\mu^2} \right) d\log \frac{s_1}{\mu^2} + \\
  \frac{\log^+(-s/\mu^2)}{8\pi^2} \left( \frac{3g^2 + 9g'^2}{4} \right) f^+(s/\mu^2) + \frac{\log^+(s/\mu^2)}{8\pi^2} \left( \frac{3g^2 + g'^2}{4} \right) f^-(s/\mu^2), \\
  \frac{df^-(s/\mu^2)}{d\log(\mu^2)} &= -\frac{g'^2}{8\pi^3} \int_0^{\log \frac{s}{s'}} f^-(s_1/\mu^2) \left( \int_0^{\log \frac{s'}{s_2}} \text{Im} f^-(s_2/\mu^2) \frac{d\log s_2}{\mu^2} \right) d\log \frac{s_1}{\mu^2} + \\
  \frac{\log^+(s/\mu^2)}{8\pi^2} \left( \frac{3g^2 + 9g'^2}{4} \right) f^-(s/\mu^2) + \frac{\log^+(s/\mu^2)}{8\pi^2} \left( \frac{3g^2 + g'^2}{4} \right) f^+(s/\mu^2),
\end{align*}
\]

where we introduced the notation

\[
\log^\pm(-s/\mu^2) = \frac{\log(-s/\mu^2) \pm \log(s/\mu^2)}{2}.
\]

The linear terms in \( f^\pm \), on the right-hand side of the evolution equations, are the Sudakov contributions which arise from soft \( W^a \) and \( B \) virtual bosons coupling to external lines with different momenta \( p_l \simeq p_{l'} \) and \( p_{l'} \simeq p_l \). The non-linear terms appear due to soft pairs of leptons exchanged in the crossed \( u \) channel.

Note that, in the above equations, the term related to the s-channel Sudakov vertex with \( W^a \) exchange is proportional to the Casimir operator \( T(T+1) = 3/4 \) for the weak isospin group \( SU(2) \), and the total \( B \) boson contribution from the four Sudakov vertices
having singularities in the $s$ and $t$ channels is proportional to $(-1 - 2)^2/4 = 9/4$. The corresponding contribution of these vertices to the terms mixing the amplitudes with different signatures is proportional to $3/4$ for $W^a$ exchange and $(-1 + 2)^2/4 = 1/4$ for $B$ exchange. Non-linear terms take into account the contribution of the ladder diagrams for two leptons interacting through $B$ exchanges.

The above equations can be simplified if we use the Mellin representation for the amplitudes:

$$f^\pm(s_{\mu^2}) = \int_L \frac{dj}{4i} \left( \frac{s}{\mu^2} \right)^j \xi^\pm(j) f^\pm_j, \quad (55)$$

where $\xi^\pm(j)$ are the signature factors

$$\xi^\pm(j) = \frac{\exp(-i\pi j) \pm 1}{\sin(\pi j)},$$

and where the integration contour $L$ is situated along the imaginary axes to the right of all singularities of the $t$-channel partial waves $f^\pm_j$. In the $j$-representation, taking into account the degeneracy of the partial waves with different signatures in the Born approximation, the evolution equations can be written in the form of Ricatti equations [7]:

$$f^+_j = 1 - \frac{g'^2}{16\pi^2} \frac{1}{j^2} \left( f^+_j \right)^2 + \left( \frac{3g^2 + 9g'^2}{32\pi^2} \right) \frac{d}{dj}(f^+_j/j), \quad (56)$$

$$f^-_j = 1 - \frac{g'^2}{16\pi^2} \frac{1}{j^2} \left( f^-_j \right)^2 + \left( \frac{3g^2 + 9g'^2}{32\pi^2} \right) \frac{1}{j} \frac{df^-_j}{dj} - \left( \frac{3g^2 + g'^2}{32\pi^2} \right) \frac{f^+_j}{j^2}, \quad (57)$$

if we take into account only the terms with effective DL variables

$$\frac{1}{\omega^2} = \frac{g'^2}{16\pi^2} \frac{1}{j^2}, \quad a = -\frac{2g'^2}{3g^2 + 9g'^2}, \quad (58)$$

and neglect other terms. Furthermore, using the following definition for the $t$-channel partial waves

$$\varphi^\pm(\omega) = f^\pm_j, \quad (59)$$

the equations can be written as follows

$$\varphi^+(\omega) = 1 - \left( \frac{\varphi^+(\omega)}{\omega} \right)^2 - \frac{1}{a} \frac{d}{d\omega} \left( \frac{\varphi^+(\omega)}{\omega} \right), \quad (60)$$

$$\varphi^-(\omega) = 1 - \left( \frac{\varphi^-(\omega)}{\omega} \right)^2 - \frac{1}{a} \frac{1}{\omega} \frac{d\varphi^-(\omega)}{d\omega} + \frac{1}{a} \frac{4a \varphi^+(\omega)}{\omega^2}. \quad (61)$$
Both equations can be reduced to linear form after introducing new functions $\psi(\omega)$ and $\chi(\omega)$ according to the definitions
\[
\frac{\varphi^+(\omega)}{\omega} = \frac{d\psi(\omega)/d\omega}{a\psi(\omega)} - \frac{\omega}{2}, \quad \frac{\varphi^-(\omega)}{\omega} = \frac{d\chi(\omega)/d\omega}{a\chi(\omega)} - \frac{\omega}{2} - \frac{1}{2a\omega}. \quad (62)
\]
They have the form of Schrödinger equations
\[
\left(-\frac{d^2}{d\omega^2} + \frac{a^2}{4}\omega^2 + a^2 + \frac{a}{2}\right)\psi(\omega) = 0,
\]
\[
\left(-\frac{d^2}{d\omega^2} + \frac{a^2}{4}\omega^2 - \frac{1}{4\omega^2} + a + a^2 + (1 + 4a) \left(\frac{1}{\omega} \left(\frac{d\psi(\omega)/d\omega}{\psi(\omega)}\right) - \frac{a}{2}\right)\right)\chi(\omega) = 0 \quad (64)
\]
The first is an equation for the harmonic oscillator. Therefore taking into account the boundary condition
\[
\lim_{\omega \to +\infty} \psi(\omega) \sim \exp \left(\frac{a}{4} \omega^2\right) \omega^a, \quad (65)
\]
so as to match onto perturbation theory, we obtain the solution
\[
\psi(\omega) = D_a(\sqrt{-a\omega}), \quad (66)
\]
where $D_p(z)$ is the parabolic cylinder function:
\[
D_p(z) = \frac{\exp(-z^2/4)}{\Gamma(-p)} \int_0^\infty e^{-xz - x^2/2} x^{p-1} dx. \quad (67)
\]
Therefore for the scattering amplitude with positive signature we have, to DL accuracy,
\[
f^+(s/\mu^2) = \int_L \frac{dj}{2\pi i} \left(\frac{s}{\mu^2}\right)^j \frac{1}{a} \frac{d}{dj} \log \left(\exp(-a\omega^2/4)D_a(\sqrt{-a\omega})\right), \quad (68)
\]
with $\omega = 4\pi j/g'$. The equation for negative signatures can be easily solved for $a = -1/4$ (corresponding to complex values of the Weinberg angle):
\[
\chi(\omega) = \sqrt{\omega} \exp(-\omega^2/16) \Psi \left(\frac{1}{8}, 1, \frac{\omega^2}{8}\right), \quad (69)
\]
where $\Psi(\alpha, \gamma, z)$ is the confluent hypergeometric function:
\[
\Psi(\alpha, \gamma, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zx} x^{\alpha-1}(1 + x)^{\gamma-\alpha-1} dx. \quad (70)
\]
Therefore for $a = -1/4$, the amplitude with negative signature, in the DL approximation, is
\[
f^-(s/\mu^2) = -\frac{i\pi}{2} \int_L \frac{dj}{2\pi} \left(\frac{s}{\mu^2}\right)^j \frac{1}{a} \frac{d}{dj} \log \Psi \left(\frac{1}{8}, 1, \frac{\omega^2}{8}\right), \quad (71)
\]
with $\omega = 4\pi j/g'$. For general values of the parameter $a$ the solution can be obtained as a perturbation series in $\omega^{-1}$. This perturbative expansion is convergent for all values of the DL parameter $g^2 \log^2(s/\mu^2)$.

Because the function $D_p(z)$ does not have any singularities in the complex plane, the high energy asymptotics of the scattering amplitude with positive signature

$$f^+(s/\mu^2) \sim s^{j_i}, \quad j_i = \frac{g'}{4\pi} \frac{z_i}{\sqrt{-a}}$$

(72)

is determined by the position of its zeroes, $D_p(z_i) = 0$, in the left-half plane, $\text{Re} z_i < 0$. The function $\Psi(\alpha, 1, z)$ has a singularity $\sim \ln z$ as $z \to 0$, and therefore the scattering amplitude with negative signature behaves as

$$f^-(s/\mu^2) \sim \ln^{-1} s$$

(73)

at high energies. Such behavior of $f^-(s/\mu^2)$ is valid also for other values of the parameter $a$. Note, that similar DL asymptotics were obtained in QED for $e^+e^-$ backward scattering amplitudes. For energies smaller than the masses of $W^a$ and $Z$ bosons, only diagrams with virtual photons give rise to a DL contribution. The photons interact with electric charge $Q = T_3 + Y/2$, which is the same for left- and right-handed electrons. It is interesting, that at very large energies, the behavior of $e^+e^-$ backward scattering amplitudes is different from the QED prediction even in the case of the Abelian Standard Model with $g = 0$.

The DL asymptotics of lepton-antilepton forward scattering amplitudes, which is related to $t$-channel exchange of two fermions and an arbitrary number of $W^a$ and $B$ bosons, can be obtained in a similar way. In this case one should take into account the diagram with a virtual $W^a$ boson even in the Born approximation. The reggeization of the fermions and $W^a$ boson can also be verified in the DL approximation with the use of the infrared evolution equation. Moreover, one can construct amplitudes with quasi-elastic unitarity [11], which is important for the theory of the BFKL Pomeron [12]. Some other applications of the DL asymptotics in QCD are given in [13], [14].

6 Discussion

We have calculated the double logarithmic (DL) corrections to the amplitudes of a number of different high energy processes, in particular, of electroweak processes in the Standard Model. These Sudakov-type corrections, which are found to exponentiate, are crucial for the high precision studies planned at the Next Linear ($e^+e^-$) Collider. Our approach is based on the use of an evolution equation in the infrared cut-off parameter $\mu$, which in turn is based on a generalization of a gauge invariant dispersive method for photon Bremsstrahlung originally due to Gribov.
We have assumed that the energy of the process is much larger than all the masses in the theory. However New Physics could appear in the TeV region and it will modify our DL asymptotic forms. Because our approach is based on the infrared evolution equation, we need simply to change the initial conditions in this region to incorporate the new particles and/or new interactions of the existing particles. The infrared evolution equation has the form of a renormalization group equation in the two-dimensional impact parameter subspace.

The usual BFKL equation can be also considered as a simplified version of the renormalization group equation but in two-dimensional longitudinal subspace. In this case, in the leading logarithmic approximation $g^2 \log s \sim 1$, instead of their dependence from the infrared cut-off $\mu$, we have conformal invariance of the $t$-channel partial waves in impact parameter space. However, in the next-to-leading approximation, the effect of the running coupling leads to violation of conformal invariance \cite{15}. After the breaking of conformal invariance the equation takes the form of a quantized Callan-Symanzik equation \cite{16}. Therefore, even in the case of the BFKL equation, the renormalization group has its peculiarities. It is related to the fact, that the equation determines not only the anomalous dimensions of the operators of different twists, but also their relative contributions.

The expressions for the DL asymptotics that we have obtained in the Standard Model can be used in perturbation theory to verify the first order, and to predict the higher order, expressions for scattering amplitudes, as has been done for the DL asymptotics in QCD \cite{13}, \cite{17}. Finally, we reiterate that the accurate calculation of scattering amplitudes is important because the effects of New Physics can be rather small.

### Added Note

After this work was completed a paper by P. Ciafaloni and D. Comelli \cite{18} appeared, in which the electroweak DL corrections are considered in the particular case of massless $l\bar{l}$ production by a source which is a singlet with respect to the $SU(2) \times U(1)$ gauge group. The results of this paper (in particular, nonexponentiating DL corrections) are not in agreement with our corresponding results. The difference appears due to the fact that the method in \cite{18} consists of factorizing off the virtual gauge boson with the softest (i.e. with smallest frequency) momentum $k^\mu$ by computing external line insertions only, and in iterating this procedure by setting the virtual momentum $k = 0$ in the left-over diagram. We believe that this approach, which is in disagreement with the method based on the infrared evolution equation, is not valid.
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