$L^2$-Theory of Linear Degenerate SPDEs and $L^p$ Estimates for the Uniform Norm of Weak Solutions

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Abstract

In this paper, we are concerned with possibly degenerate stochastic partial differential equations (SPDEs). An $L^2$-theory is introduced, from which we derive the Hörmander theorem with an analytical approach. With the method of De Giorgi iteration, we obtain the maximum principle which states the $L^p$ ($p \geq 2$) estimates for the time-space uniform norm of weak solutions.

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, on which a $d_1$-dimensional Wiener process $W = (W_t)_{t \geq 0}$ is well defined. We consider SPDE of the form

$$
\begin{aligned}
 du(t,x) &= \left[ \frac{1}{2}(L_k^2 + M_k^2)u + b^j D_j u + cu + f + L'_k g^k + M'_k h^k \right] (t,x) \, dt \\
 &\quad + \left[ M_k u + \beta^k u + h^k \right] (t,x) \, dW^k_t, \quad (t,x) \in Q := [0,T] \times \mathbb{R}^d;
\end{aligned}
$$

(1.1)

Here and throughout this paper, the summation over repeated indices is enforced unless stated otherwise, $T \in (0,\infty)$, $D = (D_1, \ldots, D_d)$ is the gradient operator, and $L_k = \sigma^{jk} D_j$, $M_k = \theta^{jk} D_j$, $L'_k = D_j (\sigma^{jk})$, $M'_k = D_j (\theta^{jk})$, for $k = 1, \ldots, d_1$. SPDE (1.1) is said to be degenerate when it fails to satisfy the super-parabolicity (SP): There exists $\lambda \in (0,\infty)$ such that

$$
\sigma^{ik} \sigma^{jk} (t,x) \xi^i \xi^j \geq \lambda |\xi|^2 \quad \text{a.s.,} \quad \forall (t,x,\xi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.
$$

We first investigate the solvability of linear, possibly degenerate SPDEs in $L^2$-spaces. An $L^2$-theory on linear degenerate SPDEs was initiated by Krylov and Rozovskii [16], and it was developed recently by [2] [7] [13] [18]. Along this line, to get a solution of SPDE (1.1) in space $L^2(\Omega; C([0,T]; H^m))$ not only requires that $f + L'_k g^k + M'_k h^k$ is $H^m$-valued but also assumes
that \( h^k \) is \( H^{m+1} \)-valued, while in this work, \( f, g \) and \( h \) are allowed to be just \( H^m \)-valued, with \( H^m := (1 - \Delta)^{-\frac{m}{2}} L^2(\mathbb{R}^d) \) for each \( m \in \mathbb{R} \). Moreover, we get the estimate \( L_k u \in L^2(\Omega \times [0, T]; H^m) \), and at the price of assuming a Hörmander-type condition, we further have \( u \in L^2(\Omega \times [0, T]; H^{m+\eta}) \) for some \( \eta \in (0, 1] \). For the proof, we apply the a priori estimates for solutions of the approximating super-parabolic SPDEs in line with the applications of pseudo-differential operator theory. As a byproduct, a Hörmander theorem for SPDE (1.1) is derived from the established \( L^2 \)-theory and an estimate on the Lie bracket (Lemma 3.4).

Most importantly, we prove the maximum principle for the weak solution of SPDE (1.1). More precisely, we obtain the \( L^p \) (\( p \geq 2 \)) estimates for the time-space uniform norm of weak solutions, i.e., under suitable integrability assumptions on \( u_0, f, g \) and \( h \), we have

\[
\text{Theorem 1.1. Let the Hörmander-type condition (H) hold. For the weak solution } u \text{ of SPDE (1.1), we have for any } p \geq 2 \\
E\|u^\pm\|_{L^p(Q)}^p \leq C \Xi(u_0^\pm, f^\pm, g, h),
\]

where \( \Xi(u_0^\pm, f^\pm, g, h) \) is expressed in terms of certain norms of \( (u_0, f^\pm, g, h) \), and the constant \( C \) depends on \( d, p, T \) and the quantities related to the structure coefficients of SPDE (1.1).

The novelty of our result is that it does not require the super-parabolic condition (SP), which, to the best of our knowledge, is always assumed in the existing literature on such kind of maximum principles for SPDEs.

For the super-parabolic SPDEs, Krylov [11, 12] established the \( L^p \)-theory (\( p \geq 2 \)), from which one can derive from the classical Sobolev embedding theorem the \( L^p \) estimates of time-space uniform norm for the strong solutions that require more smoothness assumptions on the coefficients. For the weak solutions of super-parabolic SPDEs in bounded domains, the maximum principle was obtained by Denis, Matoussi and Stoica [5] and further by [3, 4]. Their method relied on Moser’s iteration. Such method was also used by Denis, Matoussi and Zhang [6] to derive the maximum principle for weak solutions of super-parabolic SPDEs with obstacle. In comparison, we adopt a stochastic version of De Giorgi iteration scheme in this paper. We would also note that our method is inspired by the other two different versions of De Giorgi iteration used by Hsu, Wang and Wang [10] to investigate the regularity of strong solutions for super-parabolic SPDEs and by Qiu and Tang [20] to study the maximum principles of weak solutions for a class of backward SPDEs.

The remainder of this paper is organized as follows. In Section 2, we set some notations and state our main result. The \( L^2 \)-theory and the Hörmander theorem are proved in Section 3. In section 4, we prove the maximum principle.

## 2 Preliminaries and the main results

Let \( L^2(\mathbb{R}^d) \) (\( L^2 \) for short) be the usual Lebesgue integrable space with usual scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). For \( n \in (-\infty, \infty) \), we denote by \( H^n \) the space of Bessel potentials, that is \( H^n := (1 - \Delta)^{-\frac{n}{2}} L^2 \) with the norm

\[
\|\phi\|_n := \|(1 - \Delta)^{\frac{n}{2}} \phi\|, \quad \phi \in H^n.
\]

For each \( l \in \mathbb{N}^+ \) and domain \( \Pi \subset \mathbb{R}^l \), denote by \( C^\infty_c(\Pi) \) the space of infinitely differentiable functions with compact supports in \( \Pi \). For convenience, we shall use \( \langle \cdot, \cdot \rangle \) to denote the duality
between \((H^n)^k\) and \((H^{-n})^k\) \((k \in \mathbb{N}^+, n \in \mathbb{R})\) as well as that between the Schwartz function space \(\mathcal{S}\) and \(C_c^\infty(\mathbb{R}^d)\). Moreover, We always omit the superscript associated to the dimension when there is no confusion.

For Banach space \((\mathcal{B}, \|\cdot\|_{\mathcal{B}})\) and \(p \in [1, \infty]\), \(S^p(\mathcal{B})\) is the set of all the \(\mathcal{B}\)-valued, \((\mathcal{F}_t)\)-adapted and continuous processes \((X_t)_{t \in [0,T]}\) such that
\[
\|X\|_{S^p(\mathcal{B})} := \left\| \sup_{t \in [0,T]} \|X_t\|_{\mathcal{B}} \right\|_{L^p(\Omega)} < \infty.
\]
Denote by \(L^p(\mathcal{B})\) the totality of all the \(\mathcal{B}\)-valued, \((\mathcal{F}_t)\)-adapted processes \((X_t)_{t \in [0,T]}\) such that
\[
\|X\|_{L^p(\mathcal{B})} := \left\| \|X_t\|_{\mathcal{B}} \right\|_{L^p(\Omega \times [0,T])} < \infty.
\]
Obviously, both \((S^p(\mathcal{B}), \|\cdot\|_{S^p(\mathcal{B})})\) and \((L^p(\mathcal{B}), \|\cdot\|_{L^p(\mathcal{B})})\) are Banach spaces.

By \(C_b^\infty\), we denote the set of infinitely differentiable functions with bounded derivatives of any order. Denote by \(\mathcal{L}^\infty(C_b^\infty)\) the set of functions \(h\) on \(\Omega \times [0,T] \times \mathbb{R}^d\) such that \(h(t,x)\) is infinitely differentiable with respect to \(x\) and all the derivatives of any order belong to \(\mathcal{L}^\infty(\mathbb{R}^d)\).

Throughout this paper, we denote \(I^n = (1 - \Delta)^\frac{n}{2}\) for \(n \in \mathbb{R}\). Then \(I^n\) belongs to \(\Psi_n\) that is the class of pseudo-differential operators of order \(n\). By the pseudo-differential operator theory (see [9] for instance), the \(m\)-th order differential operator belongs to \(\Psi_m\) for \(m \in \mathbb{N}^+\), the multiplication by elements of \(C_b^\infty\) lies in \(\Psi_0\), and for the reader’s convenience, two basic results are collected below.

**Lemma 2.1.** (i). If \(J_1 \in \Psi_{n_1}\) and \(J_2 \in \Psi_{n_2}\) with \(n_1, n_2 \in \mathbb{R}\), then \(J_1 J_2 \in \Psi_{n_1 + n_2}\) and the Lie bracket \([J_1, J_2] := J_1 J_2 - J_2 J_1 \in \Psi_{n_1 + n_2 - 1}\).

(ii). For \(m \in (0, \infty)\), let \(\zeta\) belong to \(C_b^m\) which is defined as usual. Then for any \(n \in (-m, m)\) there exists constant \(C\) such that
\[
\|\zeta \phi\|_n \leq C \|\zeta\|_{C^m} \|\phi\|_n, \quad \forall \phi \in H^n.
\]

We introduce the definition for solution of SPDE (1.1).

**Definition 2.1.** A process \(u\) is called a solution to SPDE (1.1) if \(u \in \mathcal{S}^2(H^m)\) for some \(m \in \mathbb{R}\) and SPDE (1.1) holds in the distributional sense, i.e., for any \(\zeta \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^d)\) there holds almost surely
\[
\langle \zeta(t), u(t) \rangle - \int_0^t \langle \partial_x \zeta(s), u(s) \rangle ds - \int_0^t \langle \zeta(s), (M_k u + \beta^k u + h_k)(s) \rangle dW^k_s = \langle \zeta(0), u_0 \rangle + \int_0^t \langle \zeta, \left( \frac{1}{2} (L_k^2 + M_k^2) u + b^i D_j u + cu + f + L'_k g_k + M_k^i h_k \right)(s) \rangle ds, \quad \forall t \in [0,T].
\]
In particular, if \(u \in \mathcal{S}^2(L^2)\), it is said to be a weak solution.

Set
\[
\mathbb{V}_0 = \{ L_1, \ldots, L_{d_1} \} \quad \text{and} \quad \mathbb{V}_{n+1} = \mathbb{V}_n \cup \{ [L_k, V] : V \in \mathbb{V}_n, k = 1, \ldots, d_1 \}.
\]
Denote by \(L_n\) the set of linear combinations of elements of \(\mathbb{V}_n\) with coefficients of \(\mathcal{L}^\infty(C_b^\infty)\). We introduce the following Hörmander-type condition.
Theorem 2.2. Let assumption \( k \) and in the following, we set \( L \) with \( C \) have further \( u \) is a continuous function on \( (\Omega, \mathcal{F}_0; L^2) \), for \( k = 1, \ldots, d_1 \), and moreover, for some \( p > d + 2\eta \)
\[(f, g, h) \in L^\infty(\Omega; L^\frac{\rho(4+2\eta)}{(4+2\eta)n}(Q)) \times L^\infty(\Omega; L^\frac{\rho}{n}(Q)) \times \left(L^\infty(\Omega; L^\frac{2\rho(4+2\eta)}{(4+2\eta)n}(Q)) \cap L^\infty(\Omega; L^\frac{\rho}{n}(Q))\right),\]
where and in the following, we set \( \eta = 2^{-n_0} \). Throughout this paper, we denote
\[
\Lambda_\rho^{p, \infty} = \|u_{\rho, \infty}\|_{L^\infty(\Omega \times \mathbb{R}^d)} + \sup_{\omega \in \Omega} \|f^{\infty}(\omega, \cdot, \cdot)\|_{L^\frac{\rho(4+2\eta)}{(4+2\eta)n}(Q)} + \sup_{\omega \in \Omega} \|(g, h)(\omega, \cdot, \cdot)\|_{L^\frac{\rho}{n}(Q)}
+ \sup_{\omega \in \Omega} \|h(\omega, \cdot, \cdot)\|_{L^\frac{2\rho(4+2\eta)}{(4+2\eta)n}(Q)} ,
\Lambda_\rho^{p} = \|u_{\rho, \infty}\|_{L^p(\Omega; L^2)} + \|(f^{\infty}, g, h)\|_{L^p(\Omega; L^2(Q))},\quad p \in [2, \infty).
\]

We now state our main results.

Theorem 2.2. Let assumption (A1) hold. Given \( f \in L^2(H^m) \), \( g, h \in L^2((H^m)^{d_1}) \) and \( \mathbf{w} \in L^2(\Omega, \mathcal{F}_0; H^m) \) with some \( m \in \mathbb{R} \), the following three assertions hold:
(i) SPDE (1.1) admits a unique solution \( u \in S^2(H^m) \) with \( L_k u \in L^2(H^m) \), \( k = 1, \ldots, d_1 \), and
\[
E \sup_{t \in [0,T]} \|u(t)\|^2_m + \sum_{k=1}^{d_1} E \int_0^T \|L_k u(t)\|^2_m \, dt 
\leq C \left\{ E\|u_0\|^2_m + E \int_0^T (\|f(s)\|^2_m + \|g(s)\|^2_m + \|h(s)\|^2_m) \, ds \right\},
\]
with the constant \( C \) depending on \( T, m, \theta, \sigma, b, c \) and \( \beta \). In particular, if condition (H) holds, we have further
\[
E \int_0^T \|u(t)\|^2_{m+\eta} \, dt \leq C \left\{ E\|u_0\|^2_m + E \int_0^T (\|f(s)\|^2_m + \|g(s)\|^2_m + \|h(s)\|^2_m) \, ds \right\},
\]
with \( C \) depending on \( T, m, n_0, \theta, \sigma, b, c \) and \( \beta \).
(ii) Assume further (H) and \( f \in \cap_{n \in \mathbb{N}} L^2(H^n) \), \( g, h \in \cap_{n \in \mathbb{N}} L^2((H^n)^{d_1}) \). For any \( \epsilon \in (0, T) \), one has \( u \in \cap_{n \in \mathbb{N}} L^2(\Omega; C([\epsilon, T]; H^n)) \), and for each \( n \in \mathbb{R} \),
\[
E \sup_{t \in [\epsilon, T]} \|u(t)\|^2_n + E \int_\epsilon^T \|u(t)\|^2_{n+\eta} \, dt
\leq C \left\{ E\|G\|^2_m + E \int_\epsilon^T (\|f(s)\|^2_{m+(j-1)\eta} + \|g(s)\|^2_{m+(j-1)\eta} + \|h(s)\|^2_{m+(j-1)\eta}) \, ds \right\}, \quad (2.1)
\]
with the constant \( C \) depending on \( n, T, m, n_0, \theta, \gamma, b \) and \( c \). In particular, the random field \( u(t, x) \) is almost surely infinitely differentiable with respect to \( x \) on \( (0, T] \times \mathbb{R}^d \) and each derivative is a continuous function on \( (0, T] \times \mathbb{R}^d \).
Proposition 3.1. Let assumption (A2) and condition (H) hold. For the weak solution $u$ of SPDE (1.1), we have for any $p \geq 2$

$$E\|u^+\|^p_{L^\infty(Q)} \leq C \left( \Lambda_{p,\infty}^+ + \Lambda_{p,0}^+ \right)^p,$$

with the constant $C$ depending on $d, p, n_0, T$ and the quantities related to the coefficients $\sigma, \theta, b, c$ and $\beta$.

**Remark 2.1.** Assertion (i) is a summary of Theorem 3.3 and Corollary 3.5 in which an $L^2$-theory is presented for the linear, possibly degenerate SPDEs. Assertion (ii) is from Theorem 3.6, which states a Hörmander-type theorem. The most important result of this paper is the maximum principle of assertion (iii), which corresponds to Theorem 4.1 below and states the $L^p$ ($p \geq 2$) estimates for the time-space uniform norm of weak solutions for possibly degenerate SPDE (1.1) in the whole space.

3. **$L^2$-theory and Hörmander theorem for SPDEs**

3.1. **$L^2$-theory of SPDEs**

We consider the following SPDE

$$du(t, x) = \left[ \delta \Delta u + \frac{1}{2}(L_k^2 + M_k^2)u + b^i D_j u + cu + f + L_k' g^k + M_k' h^k \right] (t, x) dt$$

$$+ [M_k u + \beta^i u + h^k] (t, x) dW^k_t, \quad (t, x) \in Q;$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d,$$

with $\delta \in [0, \infty)$.

We first give an a priori estimate for the solution of SPDE (3.1).

**Proposition 3.1.** Let assumption (A1) hold. Assume $u_0 \in L^2(\Omega, \mathcal{F}_0; H^m)$ and $f, g^k, h^k \in \mathcal{L}^2(H^m)$ with $m \in \mathbb{R}$, for $k = 1, \ldots, d_1$. If $u \in \mathcal{S}^2(H^{m+1}) \cap \mathcal{L}^2(H^{m+2})$ is a solution of SPDE (3.1), one has

$$E \sup_{t \in [0, T]} \|u(t)\|^2_m + E \int_0^T \left( \delta \|Du(t)\|^2_m + \sum_{k=1}^{d_1} \|L_k u(t)\|^2_m \right) dt$$

$$\leq C \left\{ E \|u_0\|^2_m + E \int_0^T (\|f(s)\|^2_m + \|g(s)\|^2_m + \|h(s)\|^2_m) ds \right\},$$

with $C$ being independent of $\delta$.

**Proof.** We have decompositions $L_k = L_k' + c_k$ and $M_k = M_k' + \alpha_k$ with $c_k = -(D_i \theta^{ik})$, and $\alpha_k = -(D_i \theta^{ik})$, for $k = 1, \ldots, d_1$. Applying Itô formula for the square norm (see e.g. [17, Theorem 3.1]), one has almost surely for $t \in [0, T]$,

$$\|I^m u(t)\|^2 + \int_0^t 2\delta \|I^m Du(s)\|^2 ds - \int_0^t 2 \langle I^m u(s), I^m (Du \theta + \beta + h)(s) dW_s \rangle$$

$$= \|I^m u_0\|^2 + \int_0^t \left\langle I^m u, I^m \left( (L_k^2 + M_k^2)u + 2M_k' h^k + 2L_k' g^k \right) \right\rangle (s) ds$$
First, basic calculations yield

\[
\langle I^m u, I^m (L_k^2 u + 2L_k' g_k + 2cu + 2f) \rangle \\
= \langle I^m u, I^m (L_k' + c_k) L_k u \rangle + 2\langle I^m u, I^m L_k' g_k \rangle + 2\langle I^m u, I^m (cu + f) \rangle \\
= -\|I^m L_k u\|^2 + \langle I^m, [L_k, I^m L_k u] \rangle + \langle I^m u, I^m [L_k, L_k u] + I^m c_k L_k u \rangle \\
- 2\langle I^m L_k u, I^m g_k \rangle + 2\langle [I^m, L_k] u, I^m g_k \rangle + 2\langle I^m u, [I^m, L_k'] g_k \rangle + 2\langle I^m u, I^m (cu + f) \rangle \\
\leq -(1 - \varepsilon)\|I^m L_k u\|^2 + C_\varepsilon \left( \|I^m u\|^2 + \|I^m g_k\|^2 + \|I^m f\|^2 \right), \quad \varepsilon > 0,
\]

and

\[
\langle I^m u, I^m (M_k^2 u + 2M_k' h_k) \rangle \\
= -\|I^m M_k u\|^2 + \langle [I^m, M_k] u, I^m M_k u \rangle + \langle I^m u, [I^m, M_k'] M_k u + I^m \alpha_k M_k u \rangle \\
- 2\langle I^m M_k u, I^m h_k \rangle + 2\langle [I^m, M_k] u, I^m h_k \rangle + 2\langle I^m u, [I^m, M_k'] h_k \rangle \\
\leq -\|I^m M_k u\|^2 - 2\langle I^m M_k u, I^m h_k \rangle + \langle [I^m, M_k] u, M_k I^m u \rangle + \langle I^m u, [I^m, M_k] M_k u + \alpha_k M_k I^m u \rangle \\
+ C \left( \|I^m u\|^2 + \|I^m h_k\|^2 \right) \\
\leq -\|I^m M_k u\|^2 - 2\langle I^m M_k u, I^m h_k \rangle + C \left( \|I^m u\|^2 + \|I^m h_k\|^2 \right),
\]

where we have used the relation

\[
\langle I^m u, \alpha_k M_k I^m u \rangle = -\frac{1}{2} \langle I^m u, D_i (\alpha_k \theta^{ik}) I^m u \rangle.
\]

Notice that for \( i = 1, \ldots, d, \ k = 1, \ldots, d_1, \)

\[
\|I^m (h_k + \beta u + M_k u)\|^2 = \|I^m h_k\|^2 + 2\langle I^m h_k, I^m M_k u \rangle + \|I^m M_k u\|^2 \\
+ 2\langle I^m (h_k + M_k u), I^m (\beta u) \rangle + \|I^m (\beta u)\|^2,
\]

\[
\langle I^m u, I^m (b^j D_j u) \rangle = -\frac{1}{2} \langle I^m u, D_i b^j D_i u + 2[b^j D_i, I^m] u \rangle,
\]

\[
\langle I^m M_k u, I^m (\beta u) \rangle \leq \langle M_k I^m u, \beta I^m u \rangle + C \|I^m u\|^2 \\
= -\frac{1}{2} \langle I^m u, D_i (\beta \theta^{ik}) I^m u \rangle + C \|I^m u\|^2.
\]

Putting (3.3), (3.4) and (3.5) together, and taking expectations on both sides of (3.3), one gets by Gronwall inequality

\[
\sup_{t \in [0, T]} E \|u(t)\|_m^2 + E \int_0^T \left( \delta \|D u(t)\|_m^2 + \sum_{k=1}^{d_1} \|L_k u(t)\|_m^2 \right) dt \\
\leq C \left\{ E \|u_0\|_m^2 + E \int_0^T \left( \|f(s)\|_m^2 + \|g(s)\|_m^2 + \|h(s)\|_m^2 \right) ds \right\}. 
\]
On the other hand, one has for each \( t \in [0, T) \),
\[
E \sup_{\tau \in [0, t]} \left| \int_{0}^{\tau} 2 \langle I^m u(s), I^m (h + \beta u + Du \theta)(s) dW_s \rangle \right|
\leq C \left( E \sum_{k=1}^{d_1} \int_{0}^{t} \left( \langle I^m u(s), I^m (h^k + \beta u^k)(s) \rangle \right)^2 + \langle I^m u(s), (M_k I^m + [I^m, M_k]) u(s) \rangle \right) ds \right)^{1/2}
\leq C \left( E \int_{0}^{t} \left( \|I^m u(s)\|^2 \|I^m h(s)\|^2 + \|I^m u(s)\|^4 \right) ds \right)^{1/2}
\leq \varepsilon E \sup_{s \in [0, t]} \|I^m u(s)\|^2 + C_\varepsilon E \int_{0}^{t} \left( \|I^m h(s)\|^2 + \|I^m u(s)\|^2 \right) ds, \quad \varepsilon \in (0, 1).
\]
Together with (3.3), (3.4), (3.5) and (3.8), the above estimate implies (3.2).
\[\Box\]

**Remark 3.1.** The estimate (3.2) plays important role in our \( L^2 \)-theory for SPDEs, for which some unusual techniques are applied in the calculations of (3.4), (3.5) and (3.7). Especially, we treat the term \( 2 \langle I^m M_k u, I^m h^k \rangle \) as a unity and it allows us to weaken the assumptions on \( h \) in the \( L^2 \)-theory.

An immediate consequence of Proposition 3.7 is the following

**Corollary 3.2.** Let assumption (A1) hold. Given \( u_0 \in L^2(\Omega, \mathcal{F}_0; H^m) \) and \( f, g^k, h^k \in L^2(H^m) \) with \( m \in \mathbb{R} \), for \( k = 1, \ldots, d_1 \), the solution of SPDE (3.1) is unique.

**Theorem 3.3.** Let assumption (A1) hold. Assume \( u_0 \in L^2(\Omega, \mathcal{F}_0; H^m) \) and \( f, g^k, h^k \in L^2(H^m) \) with \( m \in \mathbb{R} \), for \( k = 1, \ldots, d_1 \). SPDE (3.1) with \( \delta = 0 \) (equivalently, SPDE (1.1)) admits a unique solution \( u \in S^2(H^m) \) with \( L_k u \in L^2(H^m) \), \( k = 1, \ldots, d_1 \), and
\[
E \sup_{t \in [0, T]} \|u(t)\|^2_m + \sum_{k=1}^{d_1} E \int_{0}^{T} \|L_k u(t)\|^2_m dt
\leq C \left( E \|u_0\|^2_m + E \int_{0}^{T} \left( \|f(s)\|^2_m + \|g(s)\|^2_m + \|h(s)\|^2_m \right) ds \right), \tag{3.9}
\]
with \( C \) depending on \( T, m, \sigma, \theta, b, c \) and \( \beta \).

**Proof.** Choose \( \{\delta_l\}_{l \in \mathbb{N}^+} \subset (0, 1), \{u_0^n\}_{n \in \mathbb{N}^+} \subset L^2(\Omega, \mathcal{F}_0; H^{m+5}) \) and \( \{f_n, g_n^k, h_n^k\}_{n \in \mathbb{N}^+} \subset L^2(H^{m+5}) \), for \( k = 1, \ldots, d_1 \), such that \( \delta_l \) converges down to 0 and
\[
\lim_{n \to \infty} \|u_0^n - u_0\|_{L^2(\Omega; H^m)} + \| (f_n - f, g_n - g, h_n - h) \|_{L^2(H^m)} = 0.
\]
By \( L^p \)-theory for SPDEs (see [11] for instance), SPDE (3.1) admits a unique solution \( u_{l,n} \in S^2(H^{m+5}) \cap L^2(H^{m+6}) \) associated with \( (\delta_l, f_n, g_n, h_n, u_0^n) \).

Fixing \( n \), one deduces from Proposition 3.1 that \( \{u_{l,n}, L_k u_{l,n}\}_{l \in \mathbb{N}^+} \) is bounded in \( S^2(H^{m+4}) \times L^2(H^{m+4}), k = 1, \ldots, d_1 \). Notice that \( \delta_l \Delta u_{l,n} \) tends to zero in \( L^2(H^{m+2}) \) as \( l \) goes to infinity. Therefore, letting \( l \) tend to infinity, we derive from Proposition 3.1 and Corollary 3.2 the unique solution \( u_n \) for SPDE (3.1) associated with \( (f_n, g_n, h_n, u_0^n) \) and \( \delta = 0 \) such that \( (u_n, L_k u_n) \in S^2(H^{m+2}) \times L^2(H^{m+2}), \) for \( k = 1, \ldots, d_1 \).

Furthermore, letting \( n \) go to infinity, again by Proposition 3.1 and Corollary 3.2 one obtains the unique solution \( u \) and associated estimates. This completes the proof. \[\Box\]
Remark 3.2. Consider the particular case $m = 0$ in Theorem 3.3. In view of the approximations in the above proof, through similar calculations as in the proof of Proposition 3.1, we can get the following estimate

$$
\|u(t)\|_2^2 \leq \|u_0\|_2^2 - (1 - \varepsilon) \int_0^t \sum_{k=1}^{d_1} \|L_k u(s)\|^2 \, ds + C_{\varepsilon} \int_0^t \|u(s)\|^2 \, ds
$$

where we note that $u$ can get more regularity properties of solutions of SPDE similar to the above estimate, we have

$$
\text{Itô formula for the square norm of the positive part of solution (see [21, Corollary 3.11]), in a first recall an estimate on the Lie bracket.}
$$

Assume further $c \geq 0$. Put $u_\lambda = (u - \lambda)^+ := \max\{u - \lambda, 0\}$ for $\lambda \in [0, \infty)$. If we start from the Itô formula for the square norm of the positive part of solution (see [21, Corollary 3.11]), in a similar way to the above estimate, we have

$$
\|u_\lambda(t)\|_2^2 \leq \|u_\lambda(0)\|_2^2 - (1 - \varepsilon) \int_0^t \sum_{k=1}^{d_1} \|L_k u_\lambda(s)\|^2 \, ds + C_{\varepsilon} \int_0^t \|u_\lambda(s)\|^2 + \langle \|u_\lambda\|, \lambda 1_{\{u_\lambda > 0\}}\rangle(s) \, ds
$$

where we note that $u \leq u_\lambda + \lambda 1_{\{u_\lambda > 0\}}$.

Note that we do not assume the Hörmander-type condition $\mathcal{H}$ in Theorem 3.3. In fact, we can get more regularity properties of solutions of SPDE (1.1) under condition $\mathcal{H}$, for which we first recall an estimate on the Lie bracket.

Lemma 3.4. ([19, Lemma 4.1]). For $\{J, L\} \subset \cup_{l \geq 0} \mathcal{V}_l$, $m \in \mathbb{R}$ and $\varepsilon \in [0, 1]$, there exists a positive constant $C$ such that almost surely for any $\phi \in H^m$ with $J \phi \in H^{m-1+\varepsilon}$ and $L \phi \in H^m$, it holds that

$$
\|[J, L]\phi\|_{m-1+\frac{\varepsilon}{2}} \leq C (\|[J\phi]\|_{m-1+\varepsilon} + \|L\phi\|_m + \|\phi\|_m).
$$

The above lemma basically generalizes [19, Lemma 4.2] from the deterministic case for $m = 0$ to the stochastic case for any $m \in \mathbb{R}$. Starting from estimate (3.9) of Theorem 3.3 and applying Lemma 3.4 iteratively to elements of $\mathcal{V}_0, \ldots, \mathcal{V}_n$, we have

Corollary 3.5. Assume the same hypothesis of Theorem 3.3. Let condition $\mathcal{H}$ hold. For the unique solution $u$ of SPDE (1.1), one has further $u \in L^2(H^{m+n})$ with

$$
E \int_0^T \|u(s)\|^2_{m+n} \, ds \leq C \left( \|G\|_m^2 + E \int_0^T (\|f(s)\|_m^2 + \|g(s)\|_m^2 + \|h(s)\|_m^2) \, ds \right),
$$

where the constant $C$ depends on $T, m, n_0, \sigma, \theta, b, c$ and $\beta$.

The estimate on solution of SPDE (1.1) for the case $m = 0$ in Corollary 3.5 plays an important role in Section 4 for the maximum principle of weak solutions. Therefore, for the reader’s convenience, we provide the following proof of Lemma 3.4 from which Corollary 3.5 follows immediately.
Hence, \( \phi \in \mathcal{F} \) through standard density arguments, one verifies that the above estimate also holds for any \( H^\circ \) ormander theorem for SPDE (1.1) surely for each \( n \) and \( \varepsilon \). Inspired by the filtering theory of partially observable diffusion processes, Krylov [14, 13] has just obtained the H\" ormander theorem for SPDEs, which states the spacial smoothness of solutions. Assume first

\[ \text{Proof of Lemma 3.4.} \]

Next to the above established L\( ^2 \)-theory, we intend to derive the following H\" ormander theorem for SPDE (1.1) under the condition (\( \mathcal{H} \)) with an analytical approach.

**Theorem 3.6.** Let assumptions (\( \mathcal{H} \)) and (A1) hold. If \( f \in \cap_{n \in \mathbb{R}} L^2(H^n), g, h \in \cap_{n \in \mathbb{R}} L^2((H^n)_\mathbb{R}), \) and \( u_0 \in L^2(\Omega; H^n) \) for some \( m \in \mathbb{R}, \) then for the unique solution \( u \) of SPDE (1.1) in Theorem 3.6, one has for any \( \varepsilon \in (0, T), \)

\[ u \in \cap_{n \in \mathbb{R}} L^2(\Omega; C([\varepsilon, T]; H^n)), \]

and for any \( n \in \mathbb{R}, \)

\[ E \sup_{t \in [\varepsilon, T]} \|u(t)\|_n^2 + E \int_{\varepsilon}^{T} \|u(t)\|_{n+\eta}^2 dt \leq C \left\{ \|G\|^2_m + E \int_{0}^{T} \left( \|f(s)\|_{m+(j-1)\eta}^2 + \|g(s)\|_{m+(j-1)\eta}^2 + \|h(s)\|_{m+(j-1)\eta}^2 \right) ds \right\}, \quad (3.12) \]

with the constant \( C \) depending on \( n, T, m, n_0, \sigma, \theta, \gamma, b \) and \( c. \) In particular, the random field \( u(t, x) \) is almost surely infinitely differentiable with respect to \( x \) on \( (0, T] \times \mathbb{R}^d \) and each derivative is a continuous function on \( (0, T] \times \mathbb{R}^d. \)
Proof. By Theorem 3.3, SPDE (1.1) admits a unique solution 
\( u \in S^2(H^m) \) and the random field 
\( \bar{u}(t, x) := tu(t, x) \) is the unique solution of SPDE

\[
\begin{aligned}
\frac{d\bar{u}}{dt}(t, x) &= \left[ \frac{1}{2}(L_k^2 + M_k^2)\bar{u} + b^iD_j \bar{u} + c\bar{u} + u + t \left( f + L_k^i g^k + M_k^i h^k \right) \right] (t, x) dt \\
&\quad + \left[ tM_k \bar{u} + \beta \bar{u} + t h^k \right] (t, x) dW_t^k, \quad (t, x) \in Q;
\end{aligned}
\]

(3.13)

with

\[
E \sup_{t \in [0, T]} \|\bar{u}(t)\|_m^2 + \sum_{k=1}^{d_1} E \int_0^T \|L_k \bar{u}(t)\|_m^2 dt \\
\leq C(T^2 + 1) E \int_0^T (\|f(s)\|_m^2 + \|g(s)\|_m^2 + \|h(s)\|_m^2 + \|u(s)\|_m^2) ds.
\]

Starting from the above estimate, we apply Lemma 3.4 iteratively to elements of \( \mathbb{V}_0, \ldots, \mathbb{V}_{n_0} \). Under condition (H), there arrives the estimate

\[
\int_0^T \|D\bar{u}\|_{m-1+\eta}^2 ds \leq C(T^2 + 1) E \int_0^T (\|f(s)\|_m^2 + \|g(s)\|_m^2 + \|h(s)\|_m^2 + \|u(s)\|_m^2) ds. \tag{3.14}
\]

Fix any \( \varepsilon \in (0, T \land 1) \) and define \( \varepsilon_l = \sum_{i=1}^{l} \frac{\varepsilon}{2^i} \) for \( l \in \mathbb{N}^+ \). By interpolation and Theorem 3.3, we have

\[
E \sup_{t \in [\varepsilon_1, T]} \|u(t)\|_{m+2-n_0}^2 + E \int_{\varepsilon_1}^T \|u(t)\|_{m+2-n_0}^2 dt \\
\leq \frac{C(2T^2 + 1)}{\varepsilon} E \int_0^T (\|f(s)\|_m^2 + \|g(s)\|_m^2 + \|h(s)\|_m^2 + \|u(s)\|_m^2) ds.
\]

Since \( f \in \cap_{n \in \mathbb{R}} L^2(H^n) \) and \( g, h \in \cap_{n \in \mathbb{R}} L^2((H^n)^{d_1}) \), by iteration we obtain for any \( j \in \mathbb{N}^+ \),

\[
E \sup_{t \in [\varepsilon_j, T]} \|u(t)\|_{m+(j-1)\eta}^2 + E \int_{\varepsilon_j}^T \|u(t)\|_{m+j\eta}^2 dt \\
\leq \frac{C(2j(T^2 + 1))}{\varepsilon} E \int_{\varepsilon_{j-1}}^T (\|f(s)\|_{m+(j-1)\eta}^2 + \|g(s)\|_{m+(j-1)\eta}^2 + \|h(s)\|_{m+(j-1)\eta}^2 + \|u(s)\|_{m+(j-1)\eta}) ds,
\]

which together with estimate (3.9), implies by iteration that

\[
E \sup_{t \in [\varepsilon_j, T]} \|u(t)\|_{m+(j-1)\eta}^2 + E \int_{\varepsilon_j}^T \|u(t)\|_{m+j\eta}^2 dt \\
\leq C(j) \left\{ E\|G\|_{m}^2 + E \int_0^T (\|f(s)\|_{m+(j-1)\eta}^2 + \|g(s)\|_{m+(j-1)\eta}^2 + \|h(s)\|_{m+(j-1)\eta}^2) ds \right\}.
\]

Hence, for any \( \varepsilon \in (0, T) \), one has \( u \in \cap_{n \in \mathbb{R}} L^2(\Omega; C(\varepsilon, T]; H^n) \) and the estimate (3.12) holds. In particular, by Sobolev embedding theorem, \( u(t, x) \) is almost surely infinitely differentiable with respect to \( x \) and each derivative is a continuous function on \( (0, T) \times \mathbb{R}^d \). □
Remark 3.3. By Theorem 3.6, we have the global spacial smoothness of the solution in time interval \((0, T]\). A similar result exists in Krylov’s recent work \([14, 13]\), which states the local spacial smoothness of solution under a Hörmander-type condition of local type. However, the method therein relies on the generalized Itô-Wentzell formula and associated results on deterministic PDEs, while herein, we use an analytical approach on the basis of our \(L^2\)-theory and an estimate on the Lie bracket (Lemma 3.4). In fact, our method has the potential to derive the associated local results, but we would not seek such a generality in the present paper. In addition, we would mention that, to the best of our knowledge, the hypoellipticity for SPDEs was first considered by Chaleyat-Maurel and Michel \([1]\), where the coefficients depend on \((t, \omega)\) only through a substituted Wiener process.

4 \(L^p\) estimates for the uniform norm of solutions

To the end, let assumptions \((A1), (A2)\) and \((H)\) hold. By Theorem 3.3, SPDE (1.1) has a unique weak solution. In this section, we shall prove the \(L^p\)-estimates for the time-space uniform norm of the weak solution.

Theorem 4.1. For the weak solution \(u\) of SPDE (1.1), we have for any \(p \geq 2\)

\[ E\|u^+\|_{L^\infty(Q)}^p \leq C \left( \Lambda^+_{p,\infty} + \Lambda^+_{p,\eta} \right)^p, \]

with the constant \(C\) depending on \(d, p, n_0, T\) and the quantities related to the coefficients \(\sigma, \theta, b, c\) and \(\beta\).

An immediate consequence is the following comparison principle.

Corollary 4.2. Suppose that random field \(u\) is the weak solution of SPDE (1.1). Let \(\tilde{u}\) be the solution of SPDE (1.1) with the initial value \(\tilde{u}_0\) and external force \(f\) being replaced by \(\tilde{u}_0\) and \(\tilde{f}\) respectively. Suppose further that \(f \leq \tilde{f},\ \mathbb{P} \otimes dt \otimes dx\)-a.e. and \(\tilde{u}_0 \leq \tilde{u}_0,\ \mathbb{P} \otimes dx\)-a.e.

Then, there holds \(u \leq \tilde{u},\ \mathbb{P} \otimes dt \otimes dx\)-a.e.

Before proving Theorem 4.1, we give the following embedding lemma that will be used frequently in what follows.

Lemma 4.3. For \(\psi \in L^2(0, T; H^0) \cap C([0, T]; L^2)\), one has \(\psi \in L^{\frac{2(d+2n)}{d}}(Q)\) and

\[ \|\psi\|_{L^{\frac{2(d+2n)}{d}}(Q)} \leq \|\psi\|_{L^2(0, T; H^0)}^\alpha \|\psi\|_{C([0, T]; L^2)}^{\frac{2q}{2q-2d\eta}} \] \( (4.1) \)

with the positive constant \(C\) depending on \(d\) and \(\eta\).

Proof. By the fractional Gagliard-Nirenberg inequality (see \([8, Corollary 2.3]\) for instance), we have

\[ \|\psi(s, \cdot)\|_{L^q}^q \leq C \|\psi(s, \cdot)\|_{H^\eta}^\alpha \|\psi(s, \cdot)\|_{H^{(1-\alpha)}}^q, \ \text{a.e.} \ s \in [0, T], \]

where \(\alpha = d/(d + 2\eta)\) and \(q = 2(d + 2\eta)/d\). Integrating on \([0, T]\), we obtain

\[ \int_Q |\psi(s, x)|^q dx ds \leq C \|\psi\|_{H^\eta}^\alpha \max_{s \in [0, T]} \|\psi(s, \cdot)\|_{H^{(1-\alpha)}}^q. \]

Therefore, \(\psi \in L^{\frac{2(d+2n)}{d}}(Q)\) and there holds (4.1). \(\square\)
For $\lambda > 0$ and $z \in \mathbb{N}_0$, set
\[
  u_z = (u - \lambda(1 - 2^{-z}))^+ \quad \text{and} \quad U_z = \sup_{t \in [0,T]} \|u_z(t)\|^2 + \int_0^T \left( \|u_z(t)\|^2 + \sum_{k=1}^{d_1} \|L_k u_z(t)\|^2 \right) dt.
\]

Obviously, for each $z \in \mathbb{N}^+$, one has $|D_t u_{z-1}| \geq |D_t u_z|$ for $i = 1, \ldots, d$, 
\[
  u_{z-1} \geq u_z, \ u_1^1 = u_z + \lambda(1 - 2^{-z}) 1_{\{u_z > 0\}} \quad \text{and} \quad 1_{\{u_z > 0\}} \leq \left( \frac{2^z u_{z-1}}{\lambda} \right)^q, \ \forall q > 0. \quad (4.2)
\]

As an immediate consequence of Lemma 4.3, there follows

Corollary 4.4.
\[
  \|u_z\|^2 \leq C U_z, \quad \text{a.s.}
\]

with the constant $C$ depending on $d$ and $\eta$.

In view of Remark 3.3, the weak solution $u$ of SPDE (1.1) satisfies
\[
  \|u_z(t)\|^2 - \int_0^t \left( \langle u_z(s), (-D_t \theta^{\beta^k} u_z + 2\beta^k u_z + 2h^k)(s) dW_s \right) \leq \|u_z(0)\|^2 - (1 - \varepsilon) \int_0^t \sum_{k=1}^{d_1} \|L_k u_z(s)\|^2 ds + C \varepsilon \int_0^t \left( \|u_z(s)\|^2 + \text{sup}_{t \in [0,t]} \lambda(1 - 2^{-z}) 1_{\{u_z > 0\}}(s) \right) ds \\
  + \int_0^t \left( \|h(s) 1_{\{u_z > 0\}}\|^2 + 2 \left( \langle u_z(s), (L'_k g + f^+)(s) \rangle \right) \right) ds, \ \text{a.s.}, \forall \varepsilon \in (0,1). \quad (4.3)
\]

Taking $\varepsilon = 1/2$, we have by Gronwall inequality
\[
  \sup_{s \in [0,t]} \|u_z(s)\|^2 + \int_0^t \sum_{k=1}^{d_1} \|L_k u_z(s)\|^2 ds \\
  \leq C \left\{ \lambda(1 - 2^{-z}) \int_0^t \langle u_z(s), 1_{\{u_z > 0\}} \rangle ds + \sup_{t \in [0,t]} \int_0^t \langle u_z(s), (-D_t \theta^{\beta^k} u_z + 2\beta^k u_z + 2h^k)(s) dW_s \right) \\
  + \int_0^t \left( \|h(s) 1_{\{u_z > 0\}}\|^2 + 2 \left( \langle u_z(s), (L'_k g + f^+)(s) \rangle \right) \right) ds + \|u_z(0)\|^2 \right\}, \ \text{a.s.}
\]

Under condition ($H$), starting from the above estimate and applying Lemma 3.4 iteratively to elements of $\mathcal{V}_0, \ldots, \mathcal{V}_{n_0}$, we get
\[
  \sup_{s \in [0,t]} \|u_z(s)\|^2 + \int_0^t \left( \|u_z(s)\|^2 + \sum_{k=1}^{d_1} \|L_k u_z(s)\|^2 \right) ds \\
  \leq C \left\{ \lambda(1 - 2^{-z}) \int_0^t \langle u_z(s), 1_{\{u_z > 0\}} \rangle ds + \sup_{t \in [0,t]} \int_0^t \langle u_z(s), (-D_t \theta^{\beta^k} u_z + 2\beta^k u_z + 2h^k)(s) dW_s \right) \\
  + \int_0^t \left( \|h(s) 1_{\{u_z > 0\}}\|^2 + 2 \left( \langle u_z(s), (L'_k g + f^+)(s) \rangle \right) \right) ds + \|u_z(0)\|^2 \right\}, \ \text{a.s.} \quad (4.4)
\]

Set
\[
  M_z(t) = \int_0^t \langle u_z(s), (-D_t \theta^{\beta^k} u_z + 2\beta^k u_z + 2h^k)(s) dW_s \rangle, \quad t \in [0, T].
\]

The proof of Theorem 4.4 is started from the iteration inequality of the following lemma.
Lemma 4.5. Assume $\lambda \geq 2\Lambda_{\bar{p},\infty}^+ > 1$. For the solution of SPDE (1.1), there exists a positive constant $N$ such that for any $z \in \mathbb{N}^+$,

$$U_z \leq \frac{N^z}{\lambda^{2a_0}} (U_{z-1})^{1+\alpha_0} + N \sup_{t \in [0,T]} M_z(t), \quad a.s. \quad (4.5)$$

where

$$0 < \alpha_0 := \frac{(\bar{p} - 2\eta)(d + 2\eta)}{2pd} - \frac{1}{2}$$

Proof. We estimate each item involved in relation (4.4). Since $\bar{p} > d + 2\eta$, basic calculations yield that $2 < 2 + 4\alpha_0 < \frac{2(d+2\eta)}{d}$. Then, it holds that

$$\lambda(1 - 2^{-z}) \int_0^t \langle |u_z|, 1_{\{u_z > 0\}} \rangle (s) \, ds$$

$$\leq \lambda(1 - 2^{-z}) \int_0^t \left( \left| u_{z-1} \right|, \left( \frac{2^z u_{z-1}}{\lambda} \right)^{1+2\alpha_0} \right) (s) \, ds$$

$$= \left( 1 - 2^{-z} \right) \frac{2^{(1+2\alpha_0)z}}{\lambda^{2a_0}} \| u_{z-1} \|^2_{L^{2+2\alpha_0}(Q)}$$

$$\leq \left( 1 - 2^{-z} \right) \frac{2^{(1+2\alpha_0)z}}{\lambda^{2a_0}} \| u_{z-1} \|_{L^2(\bar{p}+d+2\eta)^0}(Q)^{2+2\alpha_0}(1-\varepsilon)$$

$$\leq \frac{C(1 - 2^{-z}) 2^{(1+2\alpha_0)z}}{\lambda^{2a_0}} (U_{k-1})^{1+\alpha_0}, \quad a.s.$$

where by Lyapunov’s inequality, $\varepsilon \in (0, 1)$ is chosen to satisfy

$$\frac{1}{2 + 2\alpha_0} = \frac{d\varepsilon}{2(d + 2\eta)} + \frac{1 - \varepsilon}{2}.$$

Furthermore, we have

$$\int_0^T \langle u_z, f^+ \rangle (s) \, ds$$

$$\leq \| u_z \|_{L^\frac{2(d+2\eta)}{d}(Q)} \| f^+ \|_{L^{\frac{\bar{p}(d+2\eta)}{(\bar{p}+d+2\eta)^0}}(Q)} \left( \int_Q \left| 1_{\{u_z > 0\}} \right| \, dxds \right)^{\frac{1}{2} - \frac{2}{d}}$$

$$\leq \| u_z \|_{L^\frac{2(d+2\eta)}{d}(Q)} \| f^+ \|_{L^{\frac{\bar{p}(d+2\eta)}{(\bar{p}+d+2\eta)^0}}(Q)} \left( \int_Q \left| \frac{2^z u_{z-1}}{\lambda} \right|^{2(d+2\eta)} \, dxds \right)^{\frac{1}{2} - \frac{2}{d}}$$

$$\leq \left( \frac{2^z}{\lambda} \right)^{1+2\alpha_0} \| f^+ \|_{L^{\frac{\bar{p}(d+2\eta)}{(\bar{p}+d+2\eta)^0}}(Q)} \| u_{z-1} \|_{L^\frac{2+2\alpha_0}{2(d+2\eta)}(Q)}$$

$$\leq C \left( \frac{2^z}{\lambda} \right)^{1+2\alpha_0} \| f^+ \|_{L^{\frac{\bar{p}(d+2\eta)}{(\bar{p}+d+2\eta)^0}}(Q)} (U_{z-1})^{1+\alpha_0}, \quad a.s.$$

and

$$\int_0^T \langle u_z, L_k^g^k \rangle (s) \, ds$$

$$= \int_0^T \langle L_k u_z, g^k \rangle (s) \, ds$$

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\[
\leq \|L_k u_\varepsilon\|_{L^2(Q)} \left( \int_Q g^2 1_{\{u_\varepsilon > 0\}} \, dx \right)^{\frac{1}{2}} \\
\leq \|L_k u_\varepsilon\|_{L^2(Q)} \|g\|_{L^\frac{5}{2}(Q)} \left( \int_Q 1_{\{u_\varepsilon > 0\}} \, dx \right)^{\frac{1}{2} - \frac{2}{5}} \\
\leq \left( \frac{2^z}{\lambda} \right)^{1+2\alpha_0} \|g\|_{L^\frac{5}{2}(Q)} \|L_k u_{\varepsilon - 1}\|_{L^2(Q)} \|u_{\varepsilon - 1}\|_{L^2(Q)}^{1+2\alpha_0} \\
\leq C \left( \frac{2^z}{\lambda} \right)^{1+2\alpha_0} \|g\|_{L^\frac{5}{2}(Q)} (U_{\varepsilon - 1})^{1+\alpha_0}, \text{ a.s.}
\]

Let \( q = \frac{p(d+2\alpha_0)}{n(p+d+2\eta)} \) and \( \bar{q} = \frac{\eta}{q-1} \). There follows \( \frac{2(d+2\alpha_0)}{d} = 2 + 2\alpha_0 \) and thus

\[
\int_0^T \|h(s) 1_{\{u_\varepsilon > 0\}}\|^2 \, ds \\
\leq \|h\|_{L^{2\bar{q}}(Q)}^2 \left( \int_Q 1_{\{u_\varepsilon > 0\}} \, dx \right)^{\frac{1}{2}} \\
\leq \|h\|_{L^{2\bar{q}}(Q)}^2 \left( \int_Q \left( \frac{2^z u_{\varepsilon - 1}}{\lambda} \right)^{\frac{2(d+2\alpha_0)}{d}} \, dx \right)^{\frac{1}{2}} \\
= \left( \frac{2^z}{\lambda} \right)^{2+2\alpha_0} \|h\|_{L^{2\bar{q}}(Q)}^2 \|u_{\varepsilon - 1}\|_{L^2(Q)}^{2+2\alpha_0} \\
\leq \frac{C2^{2+2\alpha_0}z}{\lambda^{2+2\alpha_0}} \|h\|_{L^{2\bar{q}}(Q)}^2 (U_{\varepsilon - 1})^{1+\alpha_0}, \text{ a.s.}
\]

Since \( \lambda \geq 2\Lambda^+_{p,\infty} \), it follows that \( u_\varepsilon(0) \equiv 0 \) for any \( z \in \mathbb{N}^+ \). Choosing \( N \) to be big enough, we have by relation (4.4),

\[
U_z \leq \frac{N^z}{\lambda^{2\alpha_0}} (U_{\varepsilon - 1})^{1+\alpha_0} + N \sup_{t \in [0,T]} M_z(t), \text{ a.s.}
\]

\( \square \)

Next, let us deal with the martingale part \( M_z(\cdot) \) in the iteration inequality (4.5). We shall prove that \( M_z(\cdot) \) is comparable with \( (U_{\varepsilon - 1})^{1+\alpha_0} \).

**Lemma 4.6.** Let \( \lambda \geq \Lambda^+_{p,\infty} \). There exists \( N \in (1, \infty) \) such that for any \( \kappa, \zeta \in (0, \infty) \),

\[
\mathbb{P} \left( \sup_{t \in [0,T]} M_z(t) \geq \kappa \zeta, (U_{\varepsilon - 1})^{1+\alpha_0} \leq \zeta \right) \leq \exp \left\{ -\frac{\kappa^2 \lambda^{4\alpha_0}}{2N^z} \right\}, \quad \forall z \in \mathbb{N}^+.
\]

**Proof.** First, we have

\[
\langle M_z \rangle_T = \sum_{k=1}^{d_1} \int_0^T \left| \left\langle u_z, (-D_i \theta^{ik} u_z + 2\beta^k u_z + 2h^k)(s) \right\rangle \right|^2 \, ds \\
\leq C \int_0^T (\|u_z\|^4 + \|u_z\|^2 \|h_1 \{u_\varepsilon > 0\}\|^2) \, ds, \text{ a.s.}
\]

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with the constant \( C \) being independent of \( z \). On the other hand, we have
\[
\int_0^T \|u_z\|^2 \|h_1(u_z > 0)\|^2 ds \\
\leq \sup_{s \in [0,T]} \|u_z(s)\|^2 \int_0^T \|h_1(u_z > 0)\|^2 ds \\
\leq \sup_{s \in [0,T]} \|u_z(s)\|^2 \|h\|^2 \left( \int_Q \left| \frac{2z u_{z-1}}{\lambda} \right|^\frac{2(\lambda+2\alpha)}{\lambda} dx ds \right)^{1-\frac{2\alpha}{\lambda}} \\
\leq \|h\|^2 \sup_{s \in [0,T]} \|u_z(s)\|^2 \left( \int_Q \left| \frac{2z u_{z-1}}{\lambda} \right|^\frac{2(\lambda+2\alpha)}{\lambda} dx ds \right)^{1-\frac{2\alpha}{\lambda}} \\
= \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^{2+4\alpha} \sup_{s \in [0,T]} \|u_z(s)\|^2 \|u_z-1\|^{2+4\alpha} \\
\leq C \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^{2+4\alpha} (U_{z-1})^{2+2\alpha}, \text{ a.s.}
\]
and
\[
\int_0^T \|u_z(s)\|^4 ds \\
\leq \sup_{s \in [0,T]} \|u_z(s)\|^2 \int_0^T \|u_z\|^2 ds \\
\leq \sup_{s \in [0,T]} \|u_z(s)\|^2 \int_Q |u_z|^2 \left| \frac{2z u_{z-1}}{\lambda} \right|^\frac{4\alpha}{2+4\alpha} dx ds \\
\leq \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^4 \sup_{s \in [0,T]} \|u_z(s)\|^2 \|u_z-1\|^{2+4\alpha} \\
\leq C \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^4 (U_{z-1})^{2+2\alpha}, \text{ a.s.}
\]
Therefore, there exists \( N \in (1, \infty) \) such that for any \( z \in \mathbb{N}^+ \),
\[
\langle M_z \rangle_T \leq C \left( \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^4 + \left( \frac{2\sqrt[4]{\lambda}}{\lambda} \right)^{2+4\alpha} \right) (U_{z-1})^{2+2\alpha} \leq \frac{N^z}{\lambda^{4\alpha}} (U_{z-1})^{2+2\alpha}, \text{ a.s. (4.6)}
\]
with the constant \( C \) being independent of \( z \).

In view of relation (4.6), \( (U_{z-1})^{1+\alpha} \leq \zeta \) implies that \( \langle M_z \rangle_T \leq \gamma := \frac{N^z \zeta^2}{\lambda^{4\alpha}} \). Note that there exists a Brownian motion \( B \) such that \( M_t = B_{(M)_t} \). Hence,
\[
\mathbb{P} \left( \left\{ \sup_{t \in [0,T]} M_z(t) \geq \kappa \zeta, \ (U_{z-1})^{1+\alpha} \leq \zeta \right\} \right) \leq \mathbb{P} \left( \left\{ \sup_{t \in [0,T]} M_z(t) \geq \kappa \zeta, \langle M \rangle_T \leq \gamma \right\} \right) \\
\leq \mathbb{P} \left( \left\{ \sup_{t \in [0,\gamma]} B_t \geq \kappa \zeta \right\} \right) \\
\text{(by the reflection principle)} \leq 2 \mathbb{P} \left( \{ B_\gamma \geq \kappa \zeta \} \right) \\
\leq \exp \left\{ -\frac{\kappa^2 \zeta^2}{2\gamma} \right\} = \exp \left\{ -\frac{\kappa^2 \lambda^{4\alpha}}{2N^z} \right\},
\]
which completes the proof. \( \square \)
Combining the iteration inequality (4.5) and the estimate on martingale part $M_z(\cdot)$, we shall estimate the tail probability of $\|u^+\|_{L^\infty(Q)}$.

**Proposition 4.7.** There exist $\theta_0 \in (0, 1)$ and $\lambda_0 \in (1, \infty)$ such that for any $\lambda \geq \lambda_0$,

$$
\mathbb{P} \left( \left\{ \|u^+\|_{L^\infty(Q)} > \lambda, U_0 \leq \lambda^{2\theta_0} \right\} \right) \leq 2 \exp \left\{ -\lambda^{2\alpha_0} \right\}. \tag{4.7}
$$

**Proof.** For $z \in \mathbb{N}_0$, set

$$
A_z = \left\{ U_z \leq \frac{\lambda^{2\theta_0}}{\nu^z} \right\},
$$

with the parameter $\nu > 1$ waiting to be determined later. Observe that

$$
\left\{ \|u^+\|_{L^\infty(Q)} > \lambda, U_0 \leq \lambda^{2\theta_0} \right\} \subset \cup_{z \in \mathbb{N}_0} (A_z)^c \cap A_0 \subset \cup_{z \in \mathbb{N}^+} (A_z)^c \cap A_{z-1}
$$

which implies that

$$
\mathbb{P} \left( \left\{ \|u^+\|_{L^\infty(Q)} > \lambda, U_0 \leq \lambda^{2\theta_0} \right\} \right) \leq \sum_{z \in \mathbb{N}^+} \mathbb{P} (A_z^c \cap A_{z-1}). \tag{4.8}
$$

In view of Lemma 4.5, the event in $(A_z)^c \cap A_{z-1}$ implies that

$$
\sup_{t \in [0, T]} M_z(t) \geq \frac{\lambda^{2\theta_0}}{N^{\nu z}} - \frac{N^{z-1}}{\lambda^{2\theta_0 - 2\theta_0(1+\alpha_0)} \nu^z (z-1)(1+\alpha_0)} = \frac{\lambda^{2\theta_0 + (1+\alpha_0)}}{\nu (z-1)(1+\alpha_0)} \left[ \frac{\nu^\alpha z-1-\alpha_0}{N \lambda^{2\theta_0} (1+\alpha_0)} - \frac{N^{z-1}}{\lambda^{2\alpha_0}} \right].
$$

Put

$$
\zeta_z = \frac{\lambda^{2\theta_0 + (1+\alpha_0)}}{\nu (z-1)(1+\alpha_0)} \quad \text{and} \quad \kappa_z = \frac{\nu^\alpha z-1-\alpha_0}{N \lambda^{2\theta_0} (1+\alpha_0)} - \frac{N^{z-1}}{\lambda^{2\alpha_0}},
$$

and take

$$
\theta_0 = \frac{1}{4} \quad \text{and} \quad \nu = (2N+1)^{-\alpha_0}.
$$

There exists $\lambda_0 \in (1, \infty)$ such that for any $\lambda \geq \lambda_0$, one has

$$
\kappa_z \geq \frac{(2N+1)^z}{\lambda^{\alpha_0}}, \quad \forall z \in \mathbb{N}^+.
$$

By Lemma 4.6, it follows that for any $z \in \mathbb{N}^+$,

$$
\mathbb{P} (A_z^c \cap A_{z-1}) \leq \mathbb{P} \left( \left\{ \sup_{t \in [0, T]} M_z(t) \geq \kappa_z \zeta_z, (U_{z-1})^{1+\alpha_0} \leq \zeta_z \right\} \right)
$$

$$
\leq \exp \left\{ -\kappa_z^2 \lambda^{4\alpha_0} \right\} \leq \exp \left\{ -\frac{(2N+1)^2 \lambda^{2\alpha_0}}{2N^z} \right\}
$$

$$
\leq \exp \left\{ -2^z \lambda^{2\alpha_0} \right\} \leq \exp \left\{ -\frac{z \lambda^{2\alpha_0}}{2} \right\},
$$

which together with relation (4.8) implies estimate (4.7).

Finally, equipped with the above estimate on the tail probability, we are now at a position to prove the $L^p$-estimates for the time-space uniform norm of weak solutions.
Proof of Theorem 4.1. Taking \( z = 0 \) in relation (4.9) and applying Hölder inequality, we have for \( 0 \leq \tau \leq T \),

\[
\sup_{t \in [0, \tau]} \|u^+(t)\|^2 + \int_0^\tau \left( \|u^+(s)\|^2 + \sum_{k=1}^{d_1} \|L_k u^+(s)\|^2 \right) ds \leq C \left\{ \sup_{t \in [0, \tau]} \int_0^t \left\langle u^+(s), (-D_i \theta^i u^+ + 2 \beta^k u^+ + 2 h^k)(s) dW^k_s \right\rangle + \int_0^\tau \left( \|h(s)1_{\{u>0\}}\|^2 + 2 \left\langle u^+(s), (L_k^t g^k + f^+)(s) \right\rangle \right) ds + \|u_0^+\|^2 \right\}, \quad \text{a.s., (4.9)}
\]

which implies that

\[
\sup_{t \in [0, \tau]} \|u^+(t)\|^2 + \int_0^\tau \left( \|u^+(s)\|^2 + \sum_{k=1}^{d_1} \|L_k u^+(s)\|^2 \right) ds \leq C \left\{ \sup_{t \in [0, \tau]} \tilde{M}_t + \|(f^+, g, h)1_{\{u>0\}}\|^2_{L^2([0, \tau] \times \mathbb{R}^d)} + \|u_0^+\|^2 \right\}, \quad \text{a.s., (4.9)}
\]

with

\[
\tilde{M}_t := \int_0^t \left\langle u^+(s), (-D_i \theta^i u^+ + 2 \beta^k u^+ + 2 h^k)(s) dW^k_s \right\rangle, \quad t \in [0, T].
\]

Observe that for any \( t \in [0, T] \) and \( q \geq 1 \),

\[
\langle \tilde{M} \rangle_t^\frac{q}{2} = \left( \sum_{k=1}^{d_1} \int_0^t \left\langle u^+, (-D_i \theta^i u + 2 \beta^k u + 2 h^k)(s) \right\rangle^2 ds \right)^{\frac{q}{2}} \leq C \left( \int_0^t (\|u^+\|^4 + \|u^+\|^2 \|h1_{\{u>0\}}\|^2) (s) ds \right)^{\frac{q}{2}} \leq \varepsilon \sup_{s \in [0, t]} \|u^+(s)\|^{2q} + C_\varepsilon \left\{ \int_0^t \|u^+_z(s)\|^{2q} ds + \left( \int_0^t \|h1_{\{u>0\}}\|^2 ds \right)^q \right\}.
\]

By relation (4.9) and Doob’s inequality, we have for \( q \geq 1 \),

\[
E \sup_{t \in [0, \tau]} \|u^+(t)\|^{2q} + E \left[ \int_0^\tau \left( \|u^+(s)\|^2 + \sum_{k=1}^{d_1} \|L_k u^+(s)\|^2 \right) ds \right]^q \leq \frac{1}{2} E \sup_{s \in [0, \tau]} \|u^+(s)\|^{2q} + CE \left[ \int_0^\tau \|u^+(s)\|^{2q} ds + \left( \|(f^+, g, h)1_{\{u>0\}}\|^2_{L^2([0, \tau] \times \mathbb{R}^d)} + \|u_0^+\|^2 \right)^q \right],
\]

which implies by Gronwall inequality

\[
E (U_0)^q \leq CE \left[ \|(f^+, g, h)1_{\{u>0\}}\|^2_{L^2(Q)} + \|u_0^+\|^2 \right].
\]
Taking \( q = \frac{p}{2\theta_0} \) in the above inequality, we have by Proposition 4.7,

\[
E\|u^+\|_{L^\infty(Q)}^p = p \int_0^\infty P\left(\{\|u^+\|_{L^\infty(Q)} > \lambda\} \right) \lambda^{p-1} d\lambda
\leq \lambda_0^p + \int_{\lambda_0}^\infty P\left(\{U_0 > \lambda^{2\theta_0}\} \right) \lambda^{p-1} d\lambda + \int_{\lambda_0}^\infty P\left(\{\|u^+\|_{L^\infty(Q)} > \lambda, U_0 \leq \lambda^{2\theta_0}\} \right) \lambda^{p-1} d\lambda
\leq \lambda_0^p + \frac{1}{2\theta_0} E|U_0|_\frac{p}{p\theta_0} + \int_{\lambda_0}^\infty \exp\{\lambda^{2\alpha_0}\} \lambda^{p-1} d\lambda
< \infty.
\]

Hence, in view of Lemmas 4.5 and 4.7 we have by scaling

\[
E\|u^+\|_{L^\infty(Q)}^p \leq C \left( \Lambda_{p,\infty}^+ + \Lambda_{\frac{p}{\theta_0}}^+ \right)^p,
\]

with the constant \( C \) depending on \( d, p, n_0, T \) and the quantities related to the coefficients \( \sigma, \theta, b, c \) and \( \beta \). The estimate on \( u^- \) follows in a similar way. We complete the proof.

**Remark 4.1.** The most important result of this paper is the maximum principle of Theorem 4.1 which states the \( L^p \) \( (p \geq 2) \) estimates for the time-space uniform norm of weak solutions for possibly degenerate SPDE (1.1) in the whole space. It seems to be new, even for the super-parabolic case (that is \( n_0 = 0 \) in (H)), as the existing results on such kind of estimates for weak solutions of super-parabolic SPDEs are restricted in bounded domains (see [3, 4, 5]). In fact, our method of De Giorgi iteration in this section is applicable to the local maximum principle for weak solutions of SPDEs in either bounded or unbounded domains, by using the techniques of cut-off functions (see [20] for instance). On the other hand, in Theorem 4.1 as well as in assertion (i) of Theorem 2.2, we assume (A1) which requires the spacial smoothness of coefficients \( \sigma, \theta, b, c \) and \( \beta \); in fact, such assumption is made for the sake of simplicity and it can be relaxed in a standard way due to the properties of multipliers in (ii) of Lemma 2.1. We would postpone such generalizations in domains with relaxed assumption (A1) to a future work.

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