On the polynomial Hardy–Littlewood inequality

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Abstract. We investigate the behavior of the constants of the polynomial Hardy–Littlewood inequality.

Mathematics Subject Classification. 47H60, 47A63, 46G25.

Keywords. Hardy–Littlewood inequality, Bohnenblust–Hille inequality, Absolutely summing operators.

1. Introduction. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Also, $x^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality asserts that, given $m, n \geq 1$, if $P$ is a homogeneous polynomial of degree $m$ on $\ell_\infty^n$ given by

$$P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha,$$

then

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{pol}} \|P\|$$

for some positive constant $B_{\mathbb{K},m}^{\text{pol}}$ which does not depend on $n$ (the exponent $\frac{2m}{m+1}$ is optimal), where $\|P\| := \sup_{z \in \ell_\infty^n} |P(z)|$. Precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\text{pol}}$ are crucial for different applications. The following diagram shows the evolution of the estimates of $B_{\mathbb{K},m}^{\text{pol}}$ for complex scalars.

G. Araújo, D. Pellegrino and J. B. Seoane-Sepúlveda was supported by CNPq Grant 401735/2013-3 (PVE-Linha 2). G.A. Muñoz-Fernández was supported by MTM2012-34341. D. Nunez and D. Serrano were supported by CNPq Grant 461797/2014-3.
In the table above, \(C(\varepsilon) (1 + \varepsilon)^m\) means that given \(\varepsilon > 0\), there is a constant \(C(\varepsilon) > 0\) such that \(B_{C,m}^{\text{pol}} \leq C(\varepsilon)(1 + \varepsilon)^m\) for all \(m\).

For real scalars it is shown in [7, Theorem 2.2] that
\[
(1.1)^m \leq B_{R,m}^{\text{pol}} \leq C(\varepsilon)(2 + \varepsilon)^m,
\]
and this means that for real scalars the hypercontractivity of \(B_{R,m}^{\text{pol}}\) is optimal.

From now on, for any map \(f : \mathbb{R} \to \mathbb{R}\) we define
\[
f(\infty) := \lim_{p \to \infty} f(p).
\]

When replacing \(\ell^n_\infty\) by \(\ell^n_p\), the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality and the optimal exponents are \(\frac{2mp}{mp+p-2m}\) for \(2m \leq p \leq \infty\). More precisely, given \(m, n \geq 1\), if \(P\) is a homogeneous polynomial of degree \(m\) on \(\ell^n_p\) with \(2m \leq p \leq \infty\) given by
\[
P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha,
\]
then there is a constant \(C_{K,m,p}^{\text{pol}} \geq 1\) such that
\[
\left( \sum_{|\alpha|=m} |a_\alpha|^\frac{2mp}{mp+p-2m} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{K,m,p}^{\text{pol}} \|P\|,
\]
and \(C_{K,m,p}^{\text{pol}}\) does not depend on \(n\), where \(\|P\| := \sup_{z \in B_{\ell^n_p}} |P(z)|\).

This is a consequence of the multilinear Hardy–Littlewood inequality (see [2,10]). More precisely, given an integer \(m \geq 1\), the multilinear Hardy–Littlewood inequality (see [1,12,14]) asserts that for \(2m \leq p \leq \infty\) there exists a constant \(C_{K,m,p}^{\text{mult}} \geq 1\) such that, for all continuous \(m\)-linear forms \(T : \ell^n_p \times \cdots \times \ell^n_p \to K\) and all positive integers \(n\),
\[
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^\frac{2mp}{mp+p-2m} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{K,m,p}^{\text{mult}} \|T\|
\]
and the exponents \(\frac{2mp}{mp+p-2m}\) are optimal, when \(\|T\| := \sup_{z^{(1)}, \ldots, z^{(m)} \in B_{\ell^n_p}} |T(z^{(1)}, \ldots, z^{(m)})|\). When \(p = \infty\) we recover the classical multilinear Bohnenblust–Hille inequality (see [6]). More precisely, it asserts that there exists a constant \(B_{K,m}^{\text{mult}}\) such that for all continuous \(m\)-linear forms \(T : \ell^n_\infty \times \cdots \times \ell^n_\infty \to K\) and all positive integers \(n\),

| Authors                                | Year      | Estimate                                           |
|-----------------------------------------|-----------|----------------------------------------------------|
| Bohnenblust and Hille                   | 1931, [6] | \(B_{C,m}^{\text{pol}} \leq m^{\frac{m+1}{2m}} (\sqrt{2})^{m-1}\) |
| Defant, Frerick, Ortega-Cerdá,          | 2011, [9] | \(B_{C,m}^{\text{pol}} \leq \left(1 + \frac{1}{m-1}\right)^{m-1}\) |
| Ounaïes, and Seip                       |           | \(\sqrt{m} (\sqrt{2})^{m-1}\)                     |
| Bayart, Pellegrino, and Seoane-Sepúlveda| 2014, [5] | \(B_{C,m}^{\text{pol}} \leq C(\varepsilon)(1 + \varepsilon)^m\) |