Quantum Image Representation Through Two-Dimensional Quantum States and Normalized Amplitude

Madhur Srivastava, and Prasanta K. Panigrahi

Abstract—In this paper, we have proposed a novel method of image representation on quantum computers. The proposed method uses the two-dimensional quantum states to locate each pixel in an image matrix with the normalized amplitude values of the image signal being coefficients of the quantum states. The two-dimensional quantum states are linear superposition of outer product of m-qubits row-location vector and n-qubits column-location vector of each pixel location in the image matrix. In classical systems, these quantum states are two-dimensional binary matrices having 1 at the targeted pixel location and 0 at the other pixel locations. The fact that linear superposition, tensor product and qubits form the fundamental basis of quantum computing, the proposed method presents the machine level representation of images on quantum computers.

Index Terms—Quantum Image Processing, Image Representation, Quantum Computing, Linear Superposition, Tensor Product.

I. INTRODUCTION

Images are one of the major sources of information transmitted and stored in the digital media. They are widely used in satellite communication, social media, medical diagnosis, and other applications where visual content is required. In classical computers, image processing is a well-developed field applying Fourier transforms, wavelet transforms and other various transforms on images for different applications. Likewise, the development of quantum Fourier transform [1-2] and quantum wavelet transform [3] have enabled the development of image processing on quantum computers. However, to apply these transforms, images on quantum computer need to be suitably represented and stored.

Quantum computers, first proposed by Richard Feynman in 1982 [4], has led to the advent of quantum computing. During the last two decades, extensive research has been conducted in quantum computing theory, resulting in the development of primary algorithms like Grover’s database search algorithm [5] and Shor’s integer factoring algorithm [6]. It has been proved theoretically that quantum computers perform operations much faster than classical computers. With the view of developing a quantum computer in the future, the fields of Quantum Information Processing, Quantum Information Theory, Quantum Cryptography and Quantum Communication are also explored, and intensive research is being conducted in these areas [7-8]. Quantum Image Processing (QIP) is a new addition to the quantum computation and information theory. Although QIP is currently at a nascent stage, its full scale development would enable various multimedia applications pertaining to images and videos on quantum computers.

Venegas-Andraca and Ball [9] proposed a method for storing binary shapes in images on a quantum computer through quantum entanglement. Although this paper provided a method for efficient and generic storage of geometrical shapes, it did not suggest any method to store the complete image. Moreover, the paper only concentrated on binary images, whereas real life images possess multiple intensity levels. Subsequently, Le et al. [10] provided a flexible representation of quantum images for multiple intensity levels and different colors. It used the information of colors and its corresponding position to represent a pixel value. However, this representation can be restructured. A different set of parameters instead of the normalized color information can be used, which conforms better to quantum systems. In addition, the single dimension n-qubits position identifier can be replaced with a two-dimensional multiple qubit position identifier. As the images are two-dimensional signals, a two-dimensional multiple qubit identifier would be more appropriate to represent image pixels.

In this paper, we propose a new representation of images on quantum computers based on the location of pixel values and the normalized amplitude of the signal values of a gray scale or a color image. Considering the fact that the grayscale images or the different color channels of images are two-dimensional vector space of X (row) and Y (column), the method uses the dual representation of row-location vector and column-location vector, separately, to identify a pixel in an image. Each location vector can be defined as the quantum state at nth row and nth column using multiple qubits. The direct product of the row-location vector and the column-location vector results in a two-dimensional quantum state of pixel in the Liouvillian space. Subsequently, the coefficients of different quantum states—consisting of both row and column information—are the normalized amplitude values of image signal. The normalization of amplitudes is derived from the constraint that the coefficients of the quantum states should be unit vector. The proposed representation is general and can be utilized to represent all types of color channels which represent color images for different applications. Unlike other quantum image representation techniques and all known classical image representation techniques, the proposed repre-
sentation provides a new approach of storing and transmitting information of locations in the image matrix where a particular amplitude value occurs, instead of storing and transmitting the content or amplitudes in the image matrix.

The following is the outline of this paper. The representation of pixel location/s using two-dimensional multiple qubits is shown in section II. Section III of this paper deals with the normalization of image signal amplitudes. In section IV, a method for generic representation of gray-scale and color images is given by merging the normalized amplitudes of the image signal with corresponding two-dimensional multiple qubits pixel location. In the end, section V discusses the advantages and limitations along with the future scope of this work in the proposed quantum image representation.

II. TWO-DIMENSIONAL QUANTUM STATE OF PIXEL LOCATION

One of the important features of this paper is the representation of a two-dimensional image by row and column vectors. Each pixel location in a classical image is specified by a row and a column number. Similarly, any pixel can also be defined by row and column vectors on a quantum computer. This can be achieved by generating $M$-length row-location vector and $N$-length column-location vector with $m$-qubits and $n$-qubits, respectively. For row vector, $m$-qubits can be generated with orthonormal binary qubits which are defined as,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ (1)

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ (2)

The above binary qubits represent the constituent vectors for row location. The notation $|$ is called ket. The recursive tensor products of these constituent vectors form $M$-length row-location vector using $m$-qubits. Mathematically, it is defined as,

$$|I_p\rangle = |i\rangle^\otimes m$$ (3)

where, $|I_p\rangle$ is the row-location vector or the state of $m$-qubit, $i \in \{0, 1\}$, $m = \log_2 M$ and $p$ is the row number of pixel.

Likewise, orthonomal constituent vectors for $N$-length column-location vector can be written as,

$$\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ (4)

$$\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ (5)

Here, notation $\langle |$ is called bra. Similar to the row-location vector, the column-location vector is also generated by taking the recursive tensor products of the constituent vectors using bra notation,

$$\langle J_q| = \langle j|^\otimes n$$ (6)

where, $\langle J_q|$ is the column-location vector or the state of $n$-qubit, $j \in \{0, 1\}$, $n = \log_2 N$ and $q$ is the column number of pixel.

To represent a pixel location in its complete two-dimensional format, the outer product of the row-location vector and the column-location vector is taken. Hence, the two-dimensional location of a pixel can be identified by the following,

$$L_{p,q} = |I_p\rangle \otimes \langle J_q|$$ (7)

where, $L_{p,q}$ is the pixel at $p$th row and $q$th column.

To illustrate with an example, the pixel location of any $4 \times 4$ image matrix in terms of two-dimensional quantum states is given in Table I – II. As can be seen, Table I expands equation 7 for each pixel location representing bra part as a $m$-qubit quantum state of the row-location vector, and ket part as a $n$-qubit quantum state of the column-location vector. Furthermore, through Table I, the orthonormality of each quantum state can also be derived. It should be noted that Table II is another way of representing Table I. Table I uses the binary representation of the quantum states, whereas Table II uses the decimal representation. The reason for showing Table II is that it occupies less space and would be used in section IV for a compact quantum image representation, instead of Table I.

| TABLE I |
|---|---|---|---|
| 1 | $|00\rangle \otimes |00\rangle$ | $|00\rangle \otimes |01\rangle$ | $|00\rangle \otimes |10\rangle$ | $|00\rangle \otimes |11\rangle$ |
| 2 | $|01\rangle \otimes |00\rangle$ | $|01\rangle \otimes |01\rangle$ | $|01\rangle \otimes |10\rangle$ | $|01\rangle \otimes |11\rangle$ |
| 3 | $|10\rangle \otimes |00\rangle$ | $|10\rangle \otimes |01\rangle$ | $|10\rangle \otimes |10\rangle$ | $|10\rangle \otimes |11\rangle$ |
| 4 | $|11\rangle \otimes |00\rangle$ | $|11\rangle \otimes |01\rangle$ | $|11\rangle \otimes |10\rangle$ | $|11\rangle \otimes |11\rangle$ |

| TABLE II |
|---|---|---|---|
| 1 | $|0\rangle \otimes |0\rangle$ | $|0\rangle \otimes |1\rangle$ | $|0\rangle \otimes |2\rangle$ | $|0\rangle \otimes |3\rangle$ |
| 2 | $|1\rangle \otimes |0\rangle$ | $|1\rangle \otimes |1\rangle$ | $|1\rangle \otimes |2\rangle$ | $|1\rangle \otimes |3\rangle$ |
| 3 | $|2\rangle \otimes |0\rangle$ | $|2\rangle \otimes |1\rangle$ | $|2\rangle \otimes |2\rangle$ | $|2\rangle \otimes |3\rangle$ |
| 4 | $|3\rangle \otimes |0\rangle$ | $|3\rangle \otimes |1\rangle$ | $|3\rangle \otimes |2\rangle$ | $|3\rangle \otimes |3\rangle$ |
III. NORMALIZATION OF IMAGE SIGNAL AMPLITUDES

Digital images are quantized analog images. The number of quantization levels depends on the total number of bits used to represent all possible values. In a gray scale image, quantized values represent intensity or luminance of the image. On the other hand, color images are expressed by different color spaces, like Red-Green-Blue (RGB), Hue-Saturation-Value (HSV) and Luminance-Chrominance (YUV), to name a few, depending on the application. In the case of a gray scale image, the amplitude of the image signal is the intensity value, but the same might not be true for the different color channels of various color spaces. For example, chrominance color spaces in YUV color space contains normalized color difference between the two color channels of RGB color space. Hence, considering various color spaces, the normalized intensity value proposed in [10] can be generalized with the normalized amplitude value of digitized image signal.

In quantum computation theory, the scalar amplitudes (α) of the quantum states are complex numbers (α = αR + iαI). Like the quantum states and the superposition of quantum states, these amplitudes are also constrained to be a unit vector, i.e.

\[ \sum_{p=1}^{M} \sum_{q=1}^{N} \alpha_{p,q} \times \alpha_{p,q}^* = 1 \]  

or in the matrix representation,

\[ \alpha \alpha^\dagger = 1 \]

However, in the case of any color channel of the image, amplitudes are always on the real line. Furthermore, in most color channels, they lie on the positive real axis. Thus, the following can be stated without the loss of generality,

\[ \alpha_{p,q} = \alpha_{p,q}^* \]

Now, let us consider the digitized image signal - whether gray scale or any channel of color image - a two-dimensional matrix of size M x N, having some value, called amplitude, at each pixel location. Let Amplitude_{p,q} be the amplitude of the digital image signal at p\textsuperscript{th} row and q\textsuperscript{th} column, where p ∈ [1, M] and q ∈ [1, N]. After incorporating the constraints of equations 8 and 9, the normalized amplitude matrix (α) can be defined as,

\[ \alpha_{p,q} = \frac{\text{Amplitude}_{p,q}}{\sqrt{\sum_{p=1}^{M} \sum_{q=1}^{N} \text{Amplitude}_{p,q}}} \]

or,

\[ \alpha_{p,q} = \frac{\sqrt{\text{Amplitude}_{p,q}}}{\sum_{p=1}^{M} \sum_{q=1}^{N} \text{Amplitude}_{p,q}} \]

Continuing with the example in section II, Table III shows the expansion of the normalized amplitudes (α) of image matrix (Amplitude) in equation 12 for the image size 4 x 4.

| Amplitude Matrix (Amplitude) | 1  | 2  | 3  | 4  |
|-----------------------------|----|----|----|----|
| 1 | 0.11 | 0.12 | 0.13 | 0.14 |
| 2 | 0.21 | 0.22 | 0.23 | 0.24 |
| 3 | 0.31 | 0.32 | 0.33 | 0.34 |
| 4 | 0.41 | 0.42 | 0.43 | 0.44 |

After achieving the normalized amplitudes of the image signal in equation 12, the next task is to represent these amplitudes also by multiple qubits quantum states. The reason for this approach is to reduce the normalized amplitude values to quantum states so that the amplitudes can easily be stored and transmitted in a quantum environment. Also, this reduction to quantum states allows storage and transmission directly at machine level.

For typographical convenience, the numerator and the denominator of the normalized amplitudes for the numerator and the denominator, respectively. Thus, the equation 12 can be rewritten as,

\[ \alpha_{p,q} = \frac{\alpha_{p,q}^\eta}{\alpha_D} \]

Both, α_{p,q}^\eta and α_D are real values, and can be integer or fractional values. In either case, the number of qubits required to represent these values will be \([\log_2 \alpha_{p,q}^\eta]\) and \([\log_2 \alpha_D]\) for the numerator and the denominator, respectively. Thus, the numerator and the denominator of the normalized amplitudes is represented by qubits in the following way:

\[ |\alpha_{p,q}^\eta\rangle = i^i \otimes \log_2 \alpha_{p,q}^\eta \]

\[ |\alpha_D\rangle = i^i \otimes \log_2 \alpha_D \]

As can be seen, the denominator in equation 13 is constant for all the normalized amplitudes of a particular color channel or image. Hence, it can be segregated and represented separately. The qubit representation of the numerator expressed in equation 14 is shown in Table IV for the image size 4 x 4. Unlike the qubit representation shown in Table I - II, the quantum states at different pixel locations can be same. This is due to the fact that the amplitudes of the image signal are completely independent of each other, and the pixel locations. On the other hand, the quantum states of the pixel locations are mutually exclusive of each other. Another fact which can be
observed from equation 14 and Table IV is that the quantum state representation of both the numerator and the denominator are individually unit vectors, conforming to the constraints of quantum computing theory.

IV. PROPOSED QUANTUM IMAGE REPRESENTATION METHOD

After calculating the normalized amplitude of the image signal, and deriving each location of two-dimensional images, the quantum images can be represented as the superposition of the products of the normalized amplitude (numerator) and its corresponding two-dimensional quantum state. The denominator is stored separately to avoid redundancy of information and extra storage space. The following equation defines the proposed quantum image representation mathematically,

$$|Y\rangle = \sum_{p=1}^{M} \sum_{q=1}^{N} \alpha_{p,q}^\eta \times L_{p,q}$$

(16)

where $Y$ is a two-dimensional gray scale quantum image, or one of the color channels of the quantum color image. After substituting $L_{p,q}$ from equation 7, the above equation is rewritten as,

$$|Y\rangle = \sum_{p=1}^{M} \sum_{q=1}^{N} \alpha_{p,q}^\eta \times (|I_p\rangle \otimes |J_q\rangle)$$

(17)

The above equation is further simplified by using equations 3 and 6 to replace $(|I_p\rangle \otimes |J_q\rangle)$, and equation 14 to replace $\alpha_{p,q}^\eta$, to be written as,

$$|Y\rangle = \sum_{p=1}^{M} \sum_{q=1}^{N} |i_p\rangle \otimes \log_2 a_n^\eta \times (|j_q\rangle \otimes m \otimes n)$$

(18)

or by substituting $m$ and $n$ with $\log_2 M$ and $\log_2 N$, we can write

$$|Y\rangle = \sum_{p=1}^{M} \sum_{q=1}^{N} |i_p\rangle \otimes \log_2 a_n^\eta \times (|j_q\rangle \otimes M \otimes N)$$

(19)

In Table V, each expression of equation 19 for $4 \times 4$ size image matrix is shown with respect to its pixel location. It can be seen that each pixel contains the normalized amplitude (numerator) of image signal and a two-dimensional quantum state corresponding to pixel’s location.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $|i_1\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_1\rangle \otimes |2\rangle \otimes (0) \otimes (3)$ | $|i_1\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_1\rangle \otimes (0) \otimes (1) \otimes (3)$ | $|i_3\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_3\rangle \otimes (0) \otimes (1) \otimes (3)$ | $|i_3\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_3\rangle \otimes (0) \otimes (1) \otimes (3)$ |
| 2 | $|i_2\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_2\rangle \otimes (0) \otimes (1) \otimes (3)$ | $|i_2\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_2\rangle \otimes (0) \otimes (1) \otimes (3)$ | $|i_4\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_4\rangle \otimes (0) \otimes (1) \otimes (3)$ | $|i_4\rangle \otimes \log_2 a_n^\eta \rangle \times (|j_4\rangle \otimes (0) \otimes (1) \otimes (3)$ |

V. DISCUSSION, CONCLUSION AND FUTURE WORK

The proposed representation focusses on two features of images, the amplitude of the image signal and its corresponding pixel location on the image space. The main idea behind this representation is to decorrelate both the features mentioned earlier. This is due to the fact that for a given or fixed image dimension, pixel positions would be identified using the same quantum states of the row-location vector and the column-location vector. Thus, only information which needs to be stored or transmitted in quantum systems would be all the pixel locations corresponding to the same amplitude value. Moreover, this representation is also able to decorrelate individual amplitude values with its neighborhood amplitudes. It may initially seem to be a disadvantage of this representation because the image signal amplitudes are highly correlated and this correlation produces high image compression. But it should be observed that instead of storing and transmitting amplitudes of the image signal, what actually stored and transmitted are all the pixel locations having the same amplitude values, for all the amplitudes present in the image. One reason for this approach is that the two different amplitudes cannot occur at the same two-dimensional pixel location. Hence, once a position has been allocated to an amplitude value, it would not be used again for other amplitude values resulting in the reduction of information to be stored or transmitted by introducing the conditional entropy. The second reason for this representation is that for a given image dimension, all the pixels would have fixed qubit location irrespective of the image content. Therefore, the things that would change during the storage and transmission of different images are the number of locations occupying a particular amplitude and its respective two-dimensional quantum state of column and row.

There are two major advantages of this representation: first, it is independent of bit depth of input image, and second, it is valid for any color channel of any color space which represents an image, in part or whole. As mentioned earlier in the paper, a quantum computer would store all the two-dimensional quantum states for a particular amplitude, and hence, increasing the bit depth would only increase the number of amplitudes for which locations have to be specified. The total number of locations would always remain constant for any fixed image dimension. In the case of other representations, more storage space is required to represent an image, because the number of bits needed to represent each pixel is directly proportional to the bit depth. In addition, the proposed representation only requires information for the amplitudes present in an image and not all possible amplitudes generated by bit depth.
The second major advantage of comprehensive representation for any color channel is due to the consideration of the amplitude of color channel representing the image signal as the coefficient of the joint quantum state of row and column vectors. Until now, all the quantum image representation techniques considered RGB color space for representation while excluding other colors spaces. There are many color channels, like chrominance in YUV, which do not store the luminance of an image, or the luminance of different colors which are combined to form an image. However, these color channels are quantized to a fixed number of values so that they cannot possess any value ranging in the set of real numbers. These fixed number of values act as amplitudes for the respective color channel in the proposed representation resulting in the generalization for all the color spaces.

In addition to the two major advantages, the proposed representation provides primitive level security to the image. To elaborate, the original image cannot be reconstructed successfully until the denominator ($\alpha D$) of the normalized amplitude is available. This is because the normalized numerator value can have many valid reconstructions. The denominator ($\alpha D$) acts as a key which results in the desired image. Hence, applications like watermarking, primary level cryptography, and password protection, to name a few, can be inherently used in this representation.

Nonetheless, there are a couple of limitations in the presented quantum image representation. First, the requirement for the number of qubits increases with the increase in image dimension resulting in more storage space per pixel compared to the bit depth per pixel in other representations. Second, the number of qubits required for identifying the quantum state of the row-location vector or the column-location vector depends on the ceiling of the logarithmic value of the length of each vector. This leads to extra fractional bits per pixel ranging between 0 and 1.

These limitations lead to future work. It is suggested that the limitations can be overcome by employing effective lossless compression technique on the qubits representing quantum states.

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