Investigation of a compact model for squeeze-film air damping in the free molecular regime

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Abstract. Due to the huge computational cost, the Monte Carlo (MC) model is difficult to use to predict squeeze-film air damping in the free molecular regime. Sumali (2007) proposed an empirical compact formula relating MC model and the analytical energy transfer model (ETM) by fitting two curves. By using the efficient numerical model, this paper discusses the application of the compact formula for squeeze-film air damping of different microplate sizes in the free molecular regime. The results show that, for plates of various sizes, the quality factors predicted by the compact model match well with those predicted by the numerical model. The work of this paper has validated that the compact model for squeeze-film air damping proposed by Sumali (2007) can be used for plates of various sizes.

1. Introduction
The quality factor of squeeze-film air damping is a key parameter for Micro-Electro-Mechanical system (MEMS) devices [1]. The most effective measure to achieve high quality factors is to encapsulate the microplates in rarefied air, usually in the free molecular regime [2]. The prediction of squeeze-film air damping in the free molecular regime has received much attention in recent years [3, 4]. In the free molecular regime, the collision frequency of inter-molecule is much smaller than the collision frequency between gas molecules and plates, thus the collision of inter-molecule can be ignored in this regime and the gas can hardly be considered as a continuum. In this case, using of the Reynolds equation model based on continuum theory for squeeze-film air damping is questionable [5-7]. To predict squeeze-film air damping in the free molecular regime, many models has been proposed based on the free molecular method. With several assumptions for simplicity, Bao et al. [2] proposed an analytical energy transfer model (ETM) for squeeze-film air damping in the free molecular regime. This model considers the effect of the nearby substrate on air damping, and gives the direct relationship between quality factor and design parameters, such as air pressure, gas distance, and plate size. Due to the simple closed form, ETM has been widely used. However, ETM suffers the large quantitative discrepancy with experiment data pointed by Hutcherson and Ye [8]. By releasing some assumptions of ETM, Hutcherson and Ye [8] proposed a Monte Carlo (MC) numerical model, which shows higher accuracy than ETM. However, the huge computational cost makes MC model hard to use. By fitting two curves of Hutcherson and Ye [8], Sumali [9] gave an empirical compact formula relating MC model and ETM. However, noted by Sumali [9], this compact model has not been validated for plates of various sizes. Based on ETM and MC model, Lu and Li [10] proposed an efficient numerical model for squeeze-film damping (called “MC-S” for short as it bases on MC model, “S” means “simple”). Compared with ETM and MC model, MC-S model shows higher accuracy than ETM and less computationally time than MC model. By using the numerical model
proposed by Lu and Li [10], this paper firstly investigates the ability of Sumali’s compact model to apply for plates of various sizes. It should be noted that all plates discussed in this paper are rigid rectangular.

The outline of this paper is as follows. Section 2 gives brief introductions of ETM, MC model, MC-S model and the compact formula. In section 3, the ability of the compact model for plates of various sizes is investigated by using MC-S model. Section 4 gives a conclusion of the whole work.

2. Previous models

2.1. Energy transfer model (ETM)

Figure 1 is a schematic drawing of a microplate vibrating parallel to the substrate, where \( a, b \) and \( H \) are the plate length, width and thickness, \( d \) the gap distance between the vibrating plate and the substrate. The displacement of the plate is \( z = A \sin \omega t \), where \( A \) and \( \omega \) are the amplitude and vibration frequency of the plate. Bao et al. [2] have made several assumptions for simplicity.

![Figure 1. Squeeze-film air damping of a vibrating microplate.](image)

(1) Collisions of gas molecules and plate are fully elastic and specular, which means after each collision with the plate, the horizontal movement direction of gas molecules remains unchanged.

(2) Gas molecular velocity keeps constant in the gap.

(3) When the gas molecule is moving through the gap, the plate velocity keeps constant.

(4) As the displacement of the plate is much smaller than the gap distance, the gap distance is assumed to be constant.

Besides, Bao et al. proposed the gas molecular average traveling distance under the plate, \( \frac{l}{T} = \frac{2ab}{\pi} \). With these assumptions and average traveling distance, the 3-D squeeze-film model is simplified into a 0-D model. The analytical expression of quality factor of ETM is written as

\[
Q_{\text{bao}} = (2\pi)^{2} \rho_{p} H \omega (\frac{d}{L}) \left[ \frac{RT}{M_{m}p} \right] \left( \frac{1}{M_{m}} \right)
\]

where \( \rho_{p} \) is the plate density, \( L = 2(a+b) \) the peripheral length of the plate, \( R \) the universal gas constant, \( T \) the temperature, \( M_{m} \) the molar mass of the gas and \( p \) the gas pressure.

2.2. MC model and MC-S model

By releasing the assumptions (2)(3)(4) in ETM, Hutcherson and Ye proposed a free molecular model using Monte Carlo (MC) method. Firstly, the vibration period of the plate is divided into many time steps. Secondly, at each time step, a lot of sample molecules enter the gap between the plate and substrate, which the initial position of the molecule is assigned randomly on the basis of a uniform distribution and the initial velocity is assigned as the average velocity. By tracking the sample molecular motion, the energy change of molecule can be obtained in the gap. Then, the energy change of all the sample molecules is averaged to obtain the energy change of gas in this time step. Lastly, by summing up all energy dissipation of gas in all time steps, the quality factor of MC model can be obtained. Compared with experiment data, MC model shows higher accuracy than ETM. However, due to the complexity and huge computational cost, MC model is hard to use.

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Based on ETM and MC model, Lu and Li [10] proposed an efficient numerical model (MC-S) which just released the assumption (2), the assumption is proved to be a key factor that influences the accuracy of ETM. The process of MC-S model is similar to that of MC model. The vibration period of the plate is divided into $M$ time steps with a length of $\Delta t$. Using the average traveling distance, the time of molecule going through the gap is $\frac{\hat{t}}{v_{yo}}$, $\hat{t} = \frac{2ab}{\pi \cdot v_{yo}} = \frac{\sqrt{6}}{3 \cdot \sqrt{v}}$, $\overline{v} = \frac{\sqrt{8RT}}{(\pi M)}$ gas molecular average velocity. At $i$th time step, when gas molecule collides with the plate, the molecule gains a speed increment $2z_i$, $z_i = Ao\cos(oi\Delta t)$ is the plate velocity. Setting $K_i$ as the time of collisions of a molecule and the plate, the time cost should be less than or equal to the traveling time $\frac{\hat{t}}{v_{yo}}$.

$$\frac{2d}{v_{z0}} + \frac{2d}{v_{z0} + 2z_i} + \frac{2d}{v_{z0} + 2 \cdot 2z_i} \ldots \frac{2d}{v_{z0} + K_i \cdot 2z_i} \leq \frac{\hat{t}}{v_{yo}}$$

Figure 2. The calculation flow chart of $K_i$

The calculation flow chart of $K_i$ is shown in Figure 2. By calculating all values of $K_i$ at all time steps, the quality factor of MC-S model can be found [10].

$$Q_{MC-S} = \pi ab \rho_p A^2 \omega^2 / \Delta E \text{ and } \Delta E = \sum_{i=1}^{M} \left[ \frac{1}{4} n \bar{v} = Ld \cdot \frac{1}{2} m (4K_i z_i v_{yo} + 4(K_i z_i)^2) \Delta t \right]$$

2.3. Sumali’s compact model

By fitting the two curves of ETM and MC model, Sumali [9] proposed an empirical compact formula relating MC model and ETM

$$Q_{MC} \approx 1 / 2.233 \cdot Q_{Baso}$$

However, noted by Sumali [9], the accuracy of the compact model has not been verified for plates of various sizes. To our best knowledge, no paper has discussed the feasibility of the compact model for plates of various sizes.

3. Investigation of the compact model

To investigate the compact model in equation (4) for squeeze-film air damping in the free molecular regime, MC-S model is used for comparison. The accuracy of MC-S model has already been validated by Lu and Li. In this section, Zook’s experiment [11] is used and the parameters are shown in Table 1.

| Parameters         | Value  |
|--------------------|--------|
| Plate density $\rho_p$ (kg/m$^3$) | 2330   |
| Plate length $a$ (μm)    | 200    |
| Plate width $b$ (μm)     | 40     |
| Plate thickness $H$ (μm) | 1.8    |
| Gap distance $d$ (μm)    | 1.1    |
| Frequency $f$ (kHz)      | 550    |
| Amplitude $A$ (μm)       | <0.1   |
| Temperature $T$ (K)      | 293    |
Table 2. Comparison of quality factors in the free molecular regime [10].

| $Kn$ | EXP | ETM  | MC   | Compact | MC-S  | Error(%) |
|------|-----|------|------|--------|-------|----------|
|      |     |      |      |        |       |          |
| 1320 | 10000 | 30717 | 16724 | 13756  | 15902 | 67       |
| 561  | 5000  | 13055 | 7107  | 5846   | 6758  | 42       |
| 189  | 2400  | 4254  | 2369  | 1905   | 2277  | -1       |
| 83   | 1000  | 1931  | 1053  | 865    | 1000  | 5        |
| 41   | 600   | 954   | 526   | 427    | 494   | -12      |
| 19   | 330   | 418   | 236   | 187    | 229   | -28      |
| 10   | 240   | 232   | 132   | 104    | 120   | -45      |

Table 2 gives the comparison of quality factors as $Kn$ varies, $Kn$ is the Knudsen number, defined by the ratio of the molecular mean free path to the characteristic length. ETM overestimates the quality factors in the whole range of $Kn$. The results of the compact model, MC model and MC-S model are similar and match well with experiment data. The reason is that, MC model has released most assumptions of ETM; the compact model is obtained by fitting the results of ETM and MC model; MC-S model has released the assumption which is the key factor that influences the accuracy.

To investigate the compact model for plate of various sizes, the parameters in Table 1 keep constant except plate width $b$ varies from 40 to 1000μm, and the quality factors of various plate sizes at $Kn=100$ and $Kn=500$ are shown in Figure 3. Figure 3 shows that, as the plate size increases, the gas quality factor decreases. The results of compact model match well with those of MC-S model in the whole range of $b$. Table 3 gives the error of the compact model at $Kn=500$, and the error is less than 15%. It is proved that the compact model has relatively high accuracy for predicting the squeeze-film air damping of plate of various sizes in the free molecular regime.

Figure 3. Quality factor of plate of various sizes at (a) $Kn=100$, (b) $Kn=500$. 
Table 3. Quality factor of plate various sizes at $Kn=500$.

| $b(\mu m)$ | MC-S | Compact | Error(%) | $b(\mu m)$ | MC-S | Compact | Error(%) |
|------------|------|---------|----------|------------|------|---------|----------|
| 40         | 6023 | 5347    | -11      | 500        | 2011 | 1833    | -9       |
| 60         | 5565 | 4935    | -11      | 600        | 1761 | 1604    | -9       |
| 80         | 5372 | 4583    | -15      | 700        | 1563 | 1426    | -9       |
| 100        | 4693 | 4277    | -9       | 800        | 1401 | 1283    | -8       |
| 200        | 3603 | 3208    | -11      | 900        | 1271 | 1167    | -8       |
| 300        | 2822 | 2566    | -9       | 1000       | 1163 | 1069    | -8       |
| 400        | 2361 | 2139    | -9       |            |      |         |          |

4. Conclusions
To predict squeeze-film air damping of microplates in the free molecular regime, many models have been proposed based on the free molecular method. Based on ETM proposed by Bao et al. [2], Hutcherson and Ye [8] proposed a MC model which shows higher accuracy than ETM. However, the huge computational cost makes MC model hard to be used. By fitting the results of ETM and MC model, Sumali [9] proposed an empirical compact formula relating MC model and ETM. However, the compact model has not been validated for plates of different sizes. By using MC-S model proposed by Lu and Li [10], this paper has investigated the ability of the compact model for various sizes. The results show that the compact model for squeeze-film air damping in the free molecular regime proposed by Sumali can be used for plates of various sizes.

Future work may focus on the application of the compact model for flexible microplates.

Acknowledgment
This work was supported by the National Natural Science Foundation of China (Grant No 51375091).

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