The Beauty of the Commons: Optimal Load Sharing by Base Station Hopping in Wireless Sensor Networks
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Abstract—In wireless sensor networks (WSNs), the base station (BS) is a critical sensor node whose failure causes severe data losses. Deploying multiple fixed BSs improves the robustness, yet requires all BSs to be installed with large batteries and large energy-harvesting devices due to the high energy consumption of BSs. In this paper, we propose a scheme to coordinate the multiple deployed BSs such that the energy supplies required by individual BSs can be substantially reduced. In this scheme, only one BS is selected to be active at a time and the other BSs act as regular sensor nodes. We first present the basic architecture of our system, including how we keep the network running with only one active BS and how we manage the handover of the role of the active BS. Then, we propose an algorithm for adaptively selecting the active BS under the spatial and temporal variations of energy resources. This algorithm is simple to implement but is also asymptotically optimal under mild conditions. Finally, by running simulations and real experiments on an outdoor testbed, we verify that the proposed scheme is energy-efficient, has low communication overhead and is robust to network changes.

Index Terms—Wireless sensor networks, multiple base station, load sharing, renewable energy supply

I. INTRODUCTION

Wireless sensor networks (WSNs) are composed of autonomous sensor nodes that monitor physical conditions. Regular sensor nodes in WSNs perform sensing and transmit the captured data to a base station (BS) by using short-range communication, e.g., 802.15.4/Zigbee, in a multi-hop manner. The BS is the key sensor node that collects data across the WSN and then forwards it to a remote server by using long-range communication, e.g., GSM/GPRS. It serves as a communication bridge between the sensing field and the remote server. Therefore, a failure of the BS can cause severe data losses. As a remedy, researchers have previously proposed to deploy multiple fixed BSs [3], [16]. The energy consumption of short-range communication is also reduced because the average hop distance from regular sensor nodes to the BSs is decreased. In these works, BSs are implicitly assumed to have infinite energy supplies. In practice, this will require all BSs to be equipped with large batteries and large energy harvesting devices, because of the high energy consumption of long-range communication of BSs.

In this paper, we propose a novel scheme for coordinating the energy resources available to all the deployed BSs such that the sizes of energy sources for individual BSs can be substantially reduced. The scheme is shown in Figure 1. We keep one active BS at a time and we adaptively and iteratively select the BS that is activated. The active BS collects data and maintains long-range communication with the remote server. Meanwhile, passive BSs behave as regular sensor nodes. They turn off their long-range communication devices, only sampling and forwarding data by using short-range communication. In this scheme, the high energy consumption of using long-range communication for the active BS is shared among all BSs. The batteries of all BSs form a pool, virtually resulting in a larger global power source. To build a sustainable WSN, the requirement is that the total energy harvested by all BSs sustains the consumption of the active BS. Consequently, the size of the individual power sources can be reduced, which, while depending on the network characteristics, may result in...
a difference of orders of magnitude, as we will see in the paper.

When designing an algorithm for adaptively re-selecting the active BS, the first thought is to let all BSs be sequentially active with equal time. This algorithm evenly distributes the energy consumption of being the active BS onto all BSs. However, it is not the optimal solution when (i) the energy resources from batteries and energy harvesting devices are spatially and temporally varying and (ii) the energy consumption for short-range communication and sensing are not negligible. We propose a simple but powerful algorithm, which we call “Highest Energy First” (HEF), and which adaptively selects the BS with the highest available energy to be active. The appealing feature of HEF is that it requires little information as input and yet fits perfectly for the WSN paradigm. The active BS only needs to gather the battery levels of passive BSs. This algorithm is proved to be asymptotically optimal under mild conditions.

When implementing HEF on a real WSN, we face new challenges in the coordination of multiple BSs. In particular, we need to solve the following problems: (i) how to start the WSN into the state with only one active BS, (ii) how to gather the information required by HEF as input and (iii) how to manage the handover of the active BS to its successor. The solutions we provide to these practical issues are distributed and robust.

To evaluate our proposed scheme, we first run several simulations on the simulator Omnet++/Castalia [13] and next run real experiments on an outdoor testbed. Simulation results show that HEF is energy-efficient, is robust to the changes of the network and has low communication overhead. The real experiments lasted for 15 days, and their results show that by using HEF to coordinate 3 BSs, the lifetime of the WSN is prolonged by a factor of 3 to 4. The enhancement will be more pronounced if HEF is used on a larger number of cooperative BSs.

The main contributions of this paper are as follows:

1) We propose a novel scheme that deploys multiple BSs, keeps only one BS active at a time and adaptively re-selects the active BS. By using this scheme, the temporally and spatially varying energy resources available to all BSs are efficiently utilized, and therefore the energy supplies of individual BSs can be reduced substantially.

2) We propose an adaptive algorithm HEF for re-selecting the active BS. This algorithm requires little information exchange in the WSN and is easy to implement. We show that under certain mild conditions, this algorithm is asymptotically optimal.

3) We discuss the implementation issues of HEF on real WSNs. In particular, we discuss how to start the network, how to gather the needed information and how to hand over the active BS. The solutions we provide are distributed and robust.

4) To evaluate the proposed scheme, we run simulations on the simulator Omnet++/Castalia and real experiments on an outdoor testbed. To the best of our knowledge, it is the first installation of a real testbed with multiple cooperative BSs. The obtained results show that our proposed scheme is energy-efficient, is robust to network changes, and has low in communication overhead.

The outline of this paper is as follows. First, we show related work in Section II. Then, we formulate the adaptive BS selection problem in Section III. Next, we propose the HEF algorithm and prove its asymptotic optimality in Section IV. In Section V we describe the implementation issues on real WSNs. We show results from simulations in Section VI and from experiments in Section VII. Finally, we conclude in Section VIII.

II. RELATED WORK

This paper relates closely with the works on deploying multiple fixed BSs, the works on physically moving the BS and the works on energy management of energy harvesting WSNs.

Deploying multiple fixed BSs: Researchers have previously proposed to deploy multiple fixed and always-active BSs for enhancing the robustness of WSNs and for reducing the energy consumption of short-range communication. Vinceze et al. [16] optimize the locations of the multiple BSs in order to minimize the average distance from regular sensor nodes to BSs. Andrej et al. [3] show that the problem of finding the optimal locations of BSs to maximize the sensing data rate under energy constraints is NP-hard. They propose a greedy heuristic to solve it. These works all implicitly assume that BSs have infinite energy supplies, which requires the installation of large batteries and large energy harvesting devices on all BSs.

Physically moving the BS: This paper is also inspired by previous works on physically moving the BSs. Their goals are usually to mitigate the energy hole problem caused by the high energy consumption of sensor nodes around the only BS. Optimizing the continuous travel path of the BS to maximize the lifetime of the WSN is usually hard. Bi et al. [2] propose a simple strategy that intuitively moves the BS towards the nodes with high residual energy and away from the nodes with low residual energy. Shi et al. [14] reduce the infinite search space of the continuous travel path of the BS into a finite subset of discrete sites. They show that the simplification still guarantees the achieved network lifetime to be within $1 - \epsilon$ of the maximum network lifetime, where $\epsilon$ can be set arbitrarily small. However, adding mobility to BSs is often infeasible, for example, in remote environmental monitoring applications [12].

Energy management of energy harvesting devices: The works in this area implicitly assume that the locations of the BSs are fixed and design energy spending policies of energy harvesting WSNs. They either assume that: (i) the exact profiles of energy recharge rates are deterministically known at the very beginning [5, 12], (ii) the probability distributions of the energy recharge rates are known in advance [8], or (iii) the probability distribution of the energy recharge rates are unknown but are assumed to be stationary in some sense, for example, i.i.d. [7]. Our work falls into this third category, and we make a weaker assumption that the energy recharge rates have constant conditional expectations at all time.

In our work, we set up multiple BSs for enhancing the robustness. The active BS is adaptively re-selected, so that the
energy resources available to all BSs are fully utilized. This allows the size of individual batteries and energy harvesting devices to be substantially reduced.

III. ADAPTIVE BS SELECTION PROBLEM FORMULATION

In this section, we consider the problem of optimally re-selecting the active BS, so that the energy resources on all BSs are efficiently utilized. Consider that \( M \) BSs are deployed in the sensing field. Time is discretized into slots \( n \in \mathbb{N}^+ \), and we denote the length of a time slot by \( \tau \).

**Decision vector**: As we have mentioned before, the active BS is adaptively re-selected in different time slots. Let \( v_m^{(n)} \) indicate whether BS \( m \) is active during a given time slot \( n \), i.e., \( v_m^{(n)} = 1 \) BS \( m \) is active during time slot \( n \), where \( \mathbb{I}(A) \) denotes the indicator function: \( \mathbb{I}(A) = 1 \) if argument \( A \) is true and \( \mathbb{I}(A) = 0 \) otherwise. Collect all \( v_m^{(n)} \), \( 1 \leq m \leq M \), in an \( M \times 1 \) column vector \( v^{(n)} = [v_1^{(n)} \ v_2^{(n)} \ \cdots \ v_M^{(n)}]^\top \) with \( \top \) denoting transposition. Call \( v^{(n)} \) the decision vector during time slot \( n \). Because only one active BS is possible during one time slot, \( v^{(n)} \) has \( M-1 \) zero entries and one entry equal to 1. Denote the sequence of decision vectors up to time slot \( n \) by \( \mathcal{V}^{(n)} = \{v^{(1)} \ v^{(2)} \ \cdots \ v^{(n)}\} \). In particular, \( \mathcal{V}^{(\infty)} \) denotes the sequence of decision vectors over an infinite time horizon.

**Cost matrix**: The energy consumption of BSs might come from three parts: sensing, short-range communication, and long-range communication. We assume that the sensing costs are negligible. Let the MAC protocol and routing protocol of the WSN be predefined. Therefore, when a specific BS is selected to be active, both the energy consumption from short-range communication and from long-range communication of each BS is deterministic. Denote by \( C_{ml} \) the energy consumption rate of BS \( m \) \( (1 \leq m \leq M) \) when BS \( l \) \( (1 \leq l \leq M) \) is active. We group all energy consumption rates in an \( M \times M \) matrix \( C \), which we call the cost matrix. If we neglect the energy consumption from short-range communication, the passive BSs do not consume any energy, and therefore the cost matrix becomes diagonal. In practice, the ratio between the energy consumption from long-range communication and that from short-range communication might be 5 ~ 20, based on different settings of the network.

**Available energy**: We denote the remaining amount of energy of BS \( m \) at the end of time slot \( n \) by \( e_m^{(n)} \) and we call it available energy. We gather the available energy of all BSs in a vector \( e(n) = [e_1^{(n)} e_2^{(n)} \cdots e_M^{(n)}]^\top \). In practice, available energy is lower-bounded by zero and upper-bounded by the storage capacity. In the analysis of this paper, however, we assume that it is not upper-bounded for simplicity. Without loss of generality, we assume that all BSs have the same available energy \( e_{\text{init}} \) initially, with \( e^{(0)} = e_{\text{init}} u_M \), where \( u_M = [1 \cdot \cdots \cdot 1]^\top \) is the \( M \times 1 \) all-ones vector.

**Energy recharge rates**: During each time slot \( n \in \mathbb{N}^+ \), each BS \( m \) \( (1 \leq m \leq M) \) receives a certain amount of incoming energy. Denote the average rate of incoming energy during this time slot by \( s_m^{(n)} \) and call it the energy recharge rate. We group all the energy recharge rates during time slot \( n \) into a vector \( s^{(n)} = [s_1^{(n)} s_2^{(n)} \cdots s_M^{(n)}]^\top \). We denote the sequence of energy recharge rates up to time slot \( n \) by \( S^{(n)} = \{s^{(1)} \ s^{(2)} \ \cdots \ s^{(n)}\} \). In particular, \( S^{(\infty)} \) denotes the sequence of energy recharge rates over an infinite time horizon. We make the following realistic assumptions on \( S^{(\infty)} \):

- **Constant conditional expectation**:  
  \[
  \mathbb{E}_{n-1} s^{(n)} = \bar{s}, \forall n \in \mathbb{N}^+, 
  \]
  where \( \mathbb{E}_{n-1} \) represents the expectation conditioned on the sequence \( S^{(n-1)} \) and \( \bar{s} \) is a constant vector. This assumption is weaker than assuming \( S^{(\infty)} \) is an i.i.d process or a Poisson process.
- **Boundedness**: There is a constant \( 0 \leq S < +\infty \), such that  
  \[
  \|s^{(n)}\|_\infty \leq S, \forall n \in \mathbb{N}^+. 
  \]

**Relations among the aforementioned parameters**: During time slot \( n \), the amounts of energy recharged for all BSs given by \( \tau s^{(n)} \) and the amounts of energy consumed are given by \( \tau C \cdot v^{(n)} \). Therefore, the available energy evolves according to  
\[
 e^{(n)} = e^{(n-1)} - \tau C \cdot v^{(n)} + \tau s^{(n)}. 
\]

**Adaptive BS selection problem**: The energy recharge rates \( S^{(n)} \) from energy harvesting devices are not controllable by us. The goal of adaptively re-selecting the active BS is to re-distribute the energy consumption, so that the fastest net energy outflow among all BSs is minimized. We define the *energy decrease rates* as \( \theta^{(n)} = (e^{(0)} - e^{(n)})/n\tau \). By summing up the iterative equation over \( n \) iterations, we see that \( \theta^{(n)} = C \cdot \sum_{t=1}^{n} v^{(t)}/n - \sum_{t=1}^{n} s^{(t)}/n \). To make the dependency of \( \theta^{(n)} \) explicit on \( \mathcal{V}^{(n)} \), we also write \( \theta^{(n)} \) as \( \theta^{(n)}(\mathcal{V}^{(n)}) \) in the following. Therefore, the objective is to minimize the maximum energy decrease rate among all BSs  
\[
 f(\theta) = \max_{1 \leq m \leq M} \theta_m, 
\]
where \( \theta \in \mathbb{R}^M \) is the vector of energy decrease rates. In summary, the *adaptive BS selection problem* is formally defined as  
\[
 \begin{align*}
 \min_{\mathcal{V}^{(n)}} & \quad f\left(\theta^{(n)}(\mathcal{V}^{(n)})\right) \\
 \text{s.t.} & \quad \theta^{(n)}(\mathcal{V}^{(n)}) = \frac{1}{n} \sum_{t=1}^{n} v^{(t)} - \frac{1}{n} \sum_{t=1}^{n} s^{(t)}, \quad (5) \\
 & \quad u_M^\top v^{(t)} = 1, \ \forall t = 1, 2, \cdots, n, \\
 & \quad v^{(t)} \in \{0, 1\}^M, \ \forall t = 1, 2, \cdots, n.
\end{align*}
\]

If we fix energy recharge rates to be their means \( \bar{s} \), \( \forall n \in \mathbb{N}^+ \), and let the sensing period to be infinitely long, problem (5) becomes  
\[
 \begin{align*}
 \min_{\bar{v}} & \quad f\left(\theta(\bar{v})\right) \\
 \text{s.t.} & \quad \theta(\bar{v}) = R \cdot \bar{v}, \quad (6) \\
 & \quad u_M^\top \bar{v} = 1, \\
 & \quad \bar{v} \geq 0,
\end{align*}
\]
The intuition behind is that if at any time, the available energy of passive BSs increases and that of the active BS decreases compared to the average, HEF would drive the available energy of all BSs to be the same when $n \to \infty$. The condition reads
\[
(\Delta_M \cdot R)_{ij} < 0, \forall 1 \leq i \neq j \leq M, \tag{10}
\]
where $R$ is defined in (7), $\Delta_M = I_M - u_M \cdot u_M^T / M$, $I_M$ is the identity matrix of order $M$, and $A_{ij}$ denotes the $(i, j)$-th entry of matrix $A$. Condition (10) can be interpreted as follows: (i) The energy consumption rates of active BSs are much larger than those of passive BSs, i.e., $C_{ii} \gg C_{ij}, \forall i \neq j$. (ii) The variance of the average energy recharge rates $\bar{s}$ is small. We will show the validation of this condition in Section VI-C.

**Theorem 1:** If
- the sequence of energy recharge rates $S^{(n)}$ has constant conditional expectations (1) and is bounded (2),
- condition (10) is met,
by running HEF when $\tau \to 0$ and $n\tau \to \infty$, the variance of the available energy of all BSs is bounded in probability, i.e.,
\[
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( |\zeta^{(n)} - \delta| < \delta \right) = 1, \forall \delta > 0, \tag{11}
\]
where
\[
\zeta^{(n)} = \sum_{m=1}^{M} (\Delta_M \cdot e^{(n)})_{m}^{2} \tag{12}
\]
denotes the variance of the available energy of all BSs $e^{(n)}$.

The details of the proof are given in the technical report [7]. Theorem 1 shows that under certain conditions, the available energy of all BSs $e^{(n)}$ tends to be uniform when $n \to \infty$. However, this does not ensure that HEF is asymptotically optimal. We need one more condition, which is $R$ is invertible, $R^{-1} \cdot u_M > 0$ and $(R^T)^{-1} \cdot u_M > 0$, or $R$ is invertible, $R^{-1} \cdot u_M < 0$ and $(R^T)^{-1} \cdot u_M < 0$. \tag{13}

**Theorem 2:** If
- the sequence of energy recharge rates $S^{(n)}$ has constant conditional expectations (1) and is bounded (2),
- conditions (10) and (13) are met,
by running HEF when $\tau \to 0$ and $n\tau \to \infty$, HEF is asymptotically optimal in probability, i.e.,
\[
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( |f(\theta^{(n)}) - f(\theta(\bar{v}^*))| < \delta \right) = 1, \forall \delta > 0. \tag{14}
\]

We give the proof of this theorem in the technical report [7]. We will show the validation of the conditions in Section VI-C.

## V. Implementations

In the previous section, we discussed the problem of adaptive BS selection, and we proposed the HEF algorithm to solve it. In this section, we show how HEF is implemented on a real WSN. In particular, we tackle the following problems: (i) how the network starts into the state with only one active BS, (ii) how the active BS gathers the information needed to run HEF, and (iii) how the active BS hands over the role to the selected successor. Before discussing these issues, we first briefly review the overall architecture of the system.
A. General Architecture

In our architecture, time is partitioned into slots whose lengths $\tau$ are two hours each. At the beginning of each time slot, one BS is selected to be active. It begins broadcasting beacons and notifying the whole network. Upon receiving these beacons, passive BSs and regular sensor nodes update their routing tables and forward these beacons. Every sensor node takes sensing samples at a constant rate. The sensed data are then forwarded to the active BS by using short-range communication in a multi-hop fashion. The active BS collects all the data packets and forwards them to the remote server. In the next time slot, the active BS remains the same or it hands over the role to its successor, depending on the output result of HEF. Then, the new active BS starts broadcasting the beacons and the whole process is repeated.

In the following, we will show how the network manages synchronization, MAC protocols, routing protocols and the usage of GSM/GPRS. The interested reader can refer to our previously published work for more details and justifications of the choices [10].

1) Synchronization: All sensor nodes are synchronized on Universal Coordinated Time (UTC), retrieved by the active BSs when they connect to our server. The current time $T_c$ is inserted into beacons through MAC time-stamping [6]. To estimate $T_c$, we use the crystal of sensor nodes to compute the elapsed time since the last update of UTC. This mechanism, although simplistic, allows for a synchronization in the order of one millisecond, which is sufficient in our application.

2) MAC protocols: In the MAC layer, we adopt the commonly used S-MAC [17] and T-MAC [4]. S-MAC is a collision avoidance MAC protocol with fixed duty-cycles for all sensor nodes. With T-MAC, sensor nodes dynamically adjust their duty cycles based on the communication loads.

3) Routing protocols: We use the gradient routing where sensor nodes send the data packets to their neighbors who have the shortest hop-distances to the active BS. We also make a few modifications on the classic gradient routing protocol, so that control messages for updating the active BS is specially handled, as will be discussed later.

4) GSM/GPRS usage: As the GSM/GPRS chip is an energy-hungry device (two orders of magnitude more than the short-range radio transceiver), its connection to the server is duty-cycled. There is an obvious trade-off between real-time information and energy savings. The typical connection interval that we use is 3 to 5 minutes.

B. Starting The Network

In our architecture, starting the network is a bit more complex than that in a traditional WSN. Multiple BSs have to make a consensus on who should be the only active BS. In this subsection, we give a de-centralized solution to this problem.

Once a BS is booted, it is passive and listens for beacons from other sensor nodes. If, after some timeout, it still has not heard any beacon, it becomes active. Then, it connects to the server to retrieve UTC, and begins broadcasting beacons. Other sensor nodes receive the beacons and know their hop distances to the active BSs. Because sensor nodes use the gradient routing protocol, they join the nearest active BS. Notice that there might be several active BSs co-existing at this stage. The whole network is virtually split into several clusters, where each cluster has one active BS.

Then, we discuss how the network merges these clusters in a de-centralized way. For simplicity of showing the merging process, we assume that the network only has two clusters $B_i$ and $B_j$ with the active BSs $b_i$ and $b_j$, respectively. There are obviously some nodes at the boundaries, belonging to one cluster and having neighbors belonging to the other one. These nodes can detect the presence of the two active BSs due to the beacon messages, as those belonging to $B_i$ will eventually hear about $b_j$ from their neighbors belonging to $B_j$. When these nodes detect the presence of the two active BSs, it is their duty to fix the problem. To keep things simple, we arbitrarily decide that the active BS with the smaller identifier should be kept active. Assuming $i < j$, the boundary nodes belonging to $B_j$ would thus send a BS_DOWN message to $b_j$, asking it to become passive in favor of $b_i$. Upon reception of this request, the BS $b_j$ stops sending beacons and becomes passive. As a result, routes to $b_j$ in the cluster $B_j$ gradually disappear, while at the same time routes to $b_i$ propagate from $B_i$ to $B_j$. When the process is over, the cluster $B_j$ has been merged with $B_i$, resulting in only one cluster. This merging process is also applicable when multiple clusters are present.

Figure 2 provides an example of how the whole starting process operates. At the first step, BS 1 is started. As it cannot hear from any other sensor node, it becomes active, gathering its own data and sending them to the server. Then, a regular sensor node 2 is started. It detects the active BS 1 and joins it to form a two-node network. At step three, BS 3 is started. It is too far away to hear from BS 1 and regular sensor node 2, so it becomes active. At step four, another BS 4 is added. It hears both from the active BS 3 and from the regular sensor node 2, and it decides to join BS 3 rather than the small network $\{1, 2\}$ because of the shorter routing paths. Hence, there are two clusters: $B_1 = \{1, 2\}$ and $B_3 = \{3, 4\}$. The boundary nodes are regular sensor node 2 and BS 4, and respectively advertise about active BSs 1 and 3. When regular sensor node 2 hears about BS 3, it does nothing as regular sensor node 1, its active BS, has a lower identifier than BS 3. BS 4, however, sends a BS_DOWN message to BS 3. Once BS 3 becomes a passive BS and stops sending beacons, the route from BS 4
to BS 3 breaks, so that at some point, BS 4 joins $B_1$, as well as BS 3 later on, resulting in the final state of Figure 2.

C. Gathering the information for HEF

In Section III we describe the algorithm for adaptively selecting the active BS. Here, we show how our system gathers the information needed as input by HEF.

To correctly handle the handover process, the active BS needs to learn the existence of other passive BSs and their battery levels. For this purpose, we use a specific type of message, called BS_ADVERT. The BS_ADVERT messages are periodically generated by passive BSs, and then routed to the active BS like any other data message using gradient routing. The BS_ADVERT messages are specifically handled. All sensor nodes include their own IDs in the packet when forwarding the BS_ADVERT messages. When the active BS receives the BS_ADVERT messages, it knows exactly the paths that these messages have traveled through. By reverting these paths contained in the BS_ADVERT messages, the active BS stores a handover table, which is used when sending notifications to the next active BSs. This mechanism is well-known in ad hoc networks (e.g., dynamic source routing [11]) and is sometimes called piggybacking. The active BS also maintains a list of battery levels for all BSs. When the active BS receives a BS_ADVERT message, it updates the corresponding elements in the list or table if this message contains newer timestamps.

D. Executing the decisions made by HEF

The active BS knows the locations of all passive BSs and their battery levels. It is able to run the HEF algorithm to decide the next active BS. If the next active BS is not the same one, the current active BS will send out a BS_UP message for notifying its successor. This BS_UP message contains the routing information from the handover table. Once a regular sensor node receives a BS_UP message, it forwards the message if it is on the route, and drops the message otherwise. When a BS receives the BS_UP message, it checks whether it is the destination of the BS_UP message. If it is, this BS sends back a BS_UP_ACK message to the currently active BS and becomes active by advertising its status through the beacon messages. The previously active BS, upon reception of a BS_UP_ACK, becomes passive and stops its beacons. In the case where no BS_UP_ACK is received (e.g., node unreachable), the active BS tries again with the next best candidate. This process continues until a suitable candidate takes over the active role.

The whole process of executing the handover decision is illustrated in Figure 3. Initially, BS 1 is active. It selects BS 4 as its successor. During step 1, a BS_UP message is routed from BS 1 to BS 4 to inform BS 4 the decision made by BS 1. At step 2, BS 4 receives the BS_UP message and becomes active. At the same time, it sends back a BS_UP_ACK message to BS 1. Finally, BS 1 becomes passive and BS 4 is the only active BS.

Handover failures can happen for various reasons, but with our architecture, the network automatically recovers from them. In case of a failure during the handover process, the active BS either reboots or stops working; both cases lead to the disappearance of active BS beacons. Should this happen, all routes in the network would disappear, and one or multiple BSs would eventually decide to become active, just like during the starting process. If multiples of them become active, the merging process would apply, eventually leading to only one active BS.

VI. SIMULATIONS

In this section, we will show how we evaluate the proposed scheme by running several simulations on the simulator Castalia/OMNeT++ [13].

A. General Settings

The general settings of the simulations are chosen to closely approximate our hardware specifications, as listed in Table I. As shown in Figure 4, we simulate a 5 × 5 grid network composed of 5 BSs and 20 regular sensor nodes, with 4 BSs in the corners and 1 BS in the center. In the physical layer of all sensor nodes, we simulate the XE1205 radio transceiver, with the transmitting power fixed to 0 dbm. We adopt the ideal unit disk model for the wireless channel and choose the

2 The results of the simulations are reproducible: The datasets, codes on the simulator Castalia/Omnet++ and Matlab codes for generating figures and tables can be found in our reproducible repository at http://rr.epfl.ch/.
parameters so that the transmitting range is fixed to 40 m. Hence, only direct neighbours can communicate with each other in our WSN, as shown in Figure 4. In the MAC layer, the T-MAC protocol is used. All sensors generate data packets at a rate of 1 packet/sec. The BS_ADVERTISE message (Section V) is transmitted at a rate of 1 packet/5 min. The active BS connects to the remote server with GSM/GPRS every 3 min. Because the transmitted data volume during each connection is small, the major part of the energy consumption comes from starting, maintaining and closing the communication. We assume that for each GSM/GPRS connection, the active time and the average power consumption is 40 sec and 296 mW (we choose these values based on the measurements with a digital oscilloscope). The active BS decides whether to transfer its role every 2 hours, which amounts to τ = 2 hours for each time slot. We simulate a length of 20 days, which results in a total number of 240 time slots. Each BS is assumed to have a set of AA NiMH rechargeable batteries with an initial energy of 800 mAh × 5 V = 14400 J and a solar panel. The energy recharged from solar panels is synthesized by using the solar radiation data captured from project Sensorscope [10]. We let the energy conversion rates of these solar panels be drawn from [100 mW, 150 mW] uniformly at random under the peak solar radiation 1 kW/m². The settings discussed above are default unless other settings are explicitly mentioned.

### B. Performance of Different Algorithms

In the following, we show the performances of four different algorithms for choosing the active BS, i.e., FIXED, equal ratios (ER), OPT and HEF. FIXED denotes the scheme with the active BS fixed to be BS 1. ER denotes the algorithm where all BSs take turns to be active and have perfectly identical active times. OPT is the off-line optimal algorithm that achieves the optimal objective value $f(\theta^{(240)})$ in (4) with the energy recharge rates $S^{(240)}$ known at the very beginning. It is not applicable in real WSNs and is only used for comparisons. Finally, HEF is the “highest energy first” algorithm described in Section IV. In the following, we will compare the energy efficiency, robustness to sudden network changes and communication overheads of different algorithms, as summarized in Table II.

#### Energy Efficiency: First, we show the available energy $e^{(n)}(1 \leq n \leq 240)$ when running different algorithms in Figure 5. We see that HEF leads $e^{(n)}$ to be uniform despite different energy harvested for different BSs. ER evenly distributes the energy consumption on all BSs, but it cannot fully utilize the energy harvested on all BSs. HEF equalizes the available energy of all BSs despite different energy harvested on different BSs. It can substantially prolong the lifetime of the WSN compared to FIXED and ER.

![Fig. 5: The available energy $e^{(n)} (1 \leq n \leq 240)$ when running different algorithms for selecting active BSs. FIXED depletes the battery of the only active BS quickly, thus leading to an early death of the WSN. ER evenly distributes the energy consumption on all BSs, but it cannot fully utilize the energy harvested on all BSs. HEF equalizes the available energy of all BSs despite different energy harvested on different BSs. It can substantially prolong the lifetime of the WSN compared to FIXED and ER.](image)

| Simulation settings | Values                  |
|---------------------|-------------------------|
| Sensing field       | 120 m × 120 m           |
| Sensor node deployment | 5 × 5 in a grid         |
| BS deployment       | 5 BSs, with 4 in the corners and 1 in the center |
| Radio layer model   | XE1205 chip, unit disk model, the transmitting range is 40 m |
| Radio energy consumptions in TX/RX/Sleep mode | 79.45 $\times$ 10⁻¹, 1.4 mW |
| Sensing data rate   | 1 packet/sec            |
| Control message rate | 1 packet/5 min         |
| GSM/GPRS connection rate | once/ 3 min           |
| Average power consumption of GSM/GPRS per connection | 296 mW × 40 sec |
| Active BS handover interval | every 2 hours |
| Initial available energy | 14400 J               |
| Solar panel         | generate [100 mW, 150 mW] power in peak solar radiation 1 kW/m² |

**TABLE I: Simulation settings**
TABLE II: Comparisons of Different Algorithms

| Algorithms | Energy efficiency | Robustness | Communication overhead | Implementation complexity |
|------------|-------------------|------------|------------------------|--------------------------|
| FIXED      | low               | none       | none                   | low                      |
| ER         | medium            | high       | low                    | medium                   |
| HEF        | high              | high       | low                    | medium                   |
| OPT        | high              | none       | none                   | not applicable           |

TABLE III: Energy decrease rates $\theta_m^{(240)}$, $1 \leq m \leq 5$ incurred by different algorithms.

| Algorithms | FIXED | ER | HEF | OPT |
|------------|-------|----|-----|-----|
| $\theta_1^{(240)}/\text{mW}$ | 41.3 | 4.6 | 2.3 | 2.4 |
| $\theta_2^{(240)}/\text{mW}$ | -7.0 | 1.0 | 3.0 | 2.4 |
| $\theta_3^{(240)}/\text{mW}$ | -9.9 | 0.1 | 2.2 | 2.4 |
| $\theta_4^{(240)}/\text{mW}$ | -6.3 | 2.5 | 2.4 | 2.4 |
| $\theta_5^{(240)}/\text{mW}$ | -3.9 | 5.1 | 2.5 | 2.4 |
| $f(\theta^{(240)})/\text{mW}$ | 41.3 | 5.1 | 3.0 | 2.4 |

necessarily optimal. HEF always tends to optimize the usage of the available energy of the remaining BSs. In Figure 6 we show an example where BS 1 fails at the end of time slot 60 and is repaired at the end of time slot 180. We define the ratios of active time for all BSs $\bar{v}^{(n)} = \sum_{t=1}^{n} v^{(t)}/n$ as a metric of performance. In the first 60 time slots (5 days), the ratios of active time $\bar{v}^{(n)}$ incurred by HEF gradually converge to those of OPT. Then, at the end of time slot $n = 60$, BS 1 fails and the entries of $\bar{v}^{(n)}$ of OPT suddenly change. In the following time slots, HEF draws $\bar{v}^{(n)}$ to that of OPT with the remaining 4 BSs. At the end of time slot $n = 180$ (15 days), BS 1 is repaired and added back into the network. The ratios of active time $\bar{v}^{(n)}$ of OPT experiences a sudden change again. Then, HEF gradually adapts to this change. In summary, we see that HEF is robust against a failure of a BS. It can also adapt to other kinds of network changes, e.g., weak communication links. Due to the page limit, we do not show them here.

Communication Overhead: Figure 7 shows the overall number of packets transmitted per hour by using short-range communication when using different algorithms. FIXED only transmits data packets and does not need to exchange any other control messages. It serves as a baseline in the comparisons. HEF has additional packet exchanges of BS_ADVERT, BS_UP and BS_UP_ACK messages. Because these messages are sent at low rates, e.g., 1 packet per 5 minutes for BS_ADVERT and 1 packet every 2 hours for BS_UP and BS_UP_ACK, the communication overhead of HEF is almost negligible. The communication overhead of ER is the same as HEF because they have the same amount of control messages.

C. Validations of Optimality Conditions

Asymptotic Optimality for HEF: Conditions (10) and (13) are required to ensure the asymptotic optimality of HEF. In Figure 8 we test the validity of these conditions and the performance of HEF in different scenarios.

In Figure 8a, we keep the sequence of energy recharge rates $S^{(240)}$ the same as in the default settings and randomly generate instances of the cost matrix as follows: (i) Let the non-diagonal elements of the cost matrix $C$ (energy consumption rates of passive BSs) be generated from a Gaussian distribution. The mean and standard deviation of the Gaussian distribution are estimated by using the maximum likelihood estimation from the data in the default settings. (ii) The diagonal elements of $C$ (energy consumption rates of active BSs) are calculated by assuming different GSM/GPRS connection intervals. For each data point, we generate 50 instances of the cost matrix and then calculate the ratio of instances where the conditions (10) and (13) are met, which we call the condition fulfilled ratio (CFR). On the left axis, we show the CFR and on the right axis, we show the objective value $f(\theta^{(240)})$ as a metric of performance.
adaptively choose one active BS. The second network (3 of as discussed in Section V. The first network frequency bands and thus do not interfere with each other.

when running different algorithms. The CFR is 100% with the default settings. When the GSM/GPRS connection interval becomes large (the energy consumption rates of active BSs become small, hence the diagonal elements of $C$ become small), the CFR decreases, but the objective value of HEF is still close to that of OPT. In Figure 8, we keep the same cost matrix $C$ as in the default settings and randomly generate the vector $\tilde{s}$ by assuming that its elements are drawn from a Gaussian distribution. We estimate the mean by using the maximum likelihood estimation from the data in the default settings and we vary the standard deviation. For each data point, we generate 50 instances of the vector $\tilde{s}$. The CFR is 100% with the default settings. When the standard deviation of $\tilde{s}$ becomes large, CFR decreases, but the objective value incurred by HEF is still close to that of OPT.

VII. REAL EXPERIMENTS
We run a 15-day experiment on an outdoor testbed on our campus based on the project Sensorscope [10]. As shown in Figure 9, we deploy 2 different networks at the same 9 locations, resulting in a total number of 18 sensor nodes. These two networks use separately 868 MHz and 870 MHz frequency bands and thus do not interfere with each other. The general architecture of these two networks are the same as discussed in Section VII. The first network $N_1$ is composed of 3 BSs ($A_1$, $A_2$ and $A_3$) and 6 regular sensor nodes ($A_4$, $A_5$, $A_6$, $A_7$, $A_8$ and $A_9$). This network runs HEF to adaptively choose one active BS. The second network $N_2$ is also composed of 3 BSs ($B_1$, $B_2$ and $B_3$) and 6 regular sensor nodes ($B_4$, $B_5$, $B_6$, $B_7$, $B_8$ and $B_9$). It runs FIXED, which keeps BS $B_2$ active and BSs $B_1$, $B_3$ passive. The data packet is generated as follows. Each sensor node generates a 2-byte counter every 30 sec. The value of the counter changes according to a triangular waveform. Then, each counter is attached with a 4-byte timestamp and a 2-byte indicator for indicating message types. We duplicate them into four copies and then encapsulate them into data packets. Each data packet has a 3-byte header containing the node IDs and the hop distances to the active BS. The average data generating rate of each sensor node is $35 \text{ byte}/30 \text{ sec}$. All these data packets are
routed to the active BS who connects to the remote server by using GSM/GPRS every 5 min. On average, the active BS transmits $9 \times 35 \text{byte} \times 5 \text{min}/30 \text{sec} = 3150 \text{byte}$ data every 5 min. In practice, the data volume transmitted by the GSM/GPRS module has fluctuations, as shown in Figure 10. We see that in $N_1$, BSs $A_1$, $A_2$ and $A_3$ share the load of transmitting data with GSM/GPRS. Meanwhile, in $N_2$, the always-active BS $B_2$ has to take the load by itself.

In the experiment, we use the battery level as the indicator for the available energy. Every 5 min, each BS sends its battery level to the active BS in a BS_ADVERT message (Section V-C). The active BS then forwards this message to the remote server, hence we are able to observe the variations of the available energy in the WSN. Notice that this message is transmitted with a low rate and it will not add much communication burden to the network.

BSs and regular sensor nodes are equipped with solar panels with areas of 100 cm$^2$ and 50 cm$^2$, respectively. They are all equipped with 4 AA NiMH rechargeable batteries (each battery has a capacity of 800 mAh). In Figure 11, we show the battery levels of the six BSs. We see that in network $N_1$, the 3 BSs with ID $A_1$, $A_2$ and $A_3$ almost always keep the same battery levels, although their solar panels harvest different amounts of energy. During this period of 15 days, their batteries do not deplete. Meanwhile, in network $N_2$, the passive BSs $B_1$ and $B_3$ always have high battery levels because of their low energy consumption rates. The always-active BS $B_2$ consumes its battery quickly and on the 4th, 8th and 12th days, the batteries of $B_2$ drain out and we have to change them. From the experiment, we conclude that by deploying multiple BSs and adaptively choosing the active BS, the harvested energy is fully used and the network lifetime is prolonged.

VIII. CONCLUSION

In this paper, we have presented and evaluated a novel scheme for organizing WSNs, in which multiple BSs are deployed but only one BS is adaptively selected to be active. By using the proposed scheme, we efficiently utilize the temporally and spatially varying energy resources available to all BSs. Therefore, the large batteries and energy harvesting devices of individual BSs can be substantially reduced.

To adaptively choose the active BS, we have proposed a simple yet powerful algorithm HEF. We have proved its asymptotic optimality under mild conditions.

Through simulations on the simulator Omnet++/Castalia and real experiments on an outdoor testbed, we have shown that the proposed scheme is energy-efficient, is adaptable to network changes and is low in communication overhead.

In future work, we will discuss the scenarios where multiple BSs are active simultaneously in large WSNs. The coordination of multiple active BSs needs to be differently handled during the startup phase of the network as well as for handover of the active BSs.
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APPENDIX A
PROOF OF LEMMA 1

A. Sketch of The Proof

We present the proof in two steps:

• **Step 1:** We prove that $\forall \epsilon > 0$,
\[
\lim_{n \to \infty} P \left( f(\theta(\bar{v}^*)) - f(\theta((n)(V^*(n)))) < \epsilon \right) = 1. \tag{14}
\]

• **Step 2:** We prove the converse, i.e., $\forall \epsilon > 0$,
\[
\lim_{n \to \infty} P \left( f(\theta((n)(V^*(n)))) - f(\theta(\bar{v}^*)) < \epsilon \right) = 1. \tag{15}
\]

By combining the results from these two steps, we conclude the proof.

**B. Step 1: The Proof of (14)**

Denote the average of decision vectors in the set $V^*(n)$ by $\bar{v}^*(n) = \sum_{v \in V^*(n)} v/n$. We let $\theta((v^*(n))) = C\bar{v}^*(n) - \bar{s}$ and use the term $f(\theta((v^*(n))))$ as a bridge between $f(\theta(\bar{v}^*))$ and $f(\theta((n)(V^*(n))))$. We first observe that
\[
\theta((n)(V^*(n))) = \theta(\bar{v}^*(n)) - \frac{1}{n} \sum_{t=1}^{n} s(t) + \bar{s}. \tag{16}
\]

Because $\{s(n) - \bar{s}\}_{n \in \mathbb{N}^+}$ is a martingale difference process, we apply the weak law of large numbers for zero-mean martingale sequences \[9, p. 8\] and have that
\[
\lim_{n \to \infty} P \left( \left\| \frac{1}{n} \sum_{t=1}^{n} s(t) - \bar{s} \right\|_\infty < \epsilon \right) = 1, \forall \epsilon > 0. \tag{17}
\]

Because of (16) and (17), we see that
\[
\lim_{n \to \infty} P \left( \left\| \theta((n)(V^*(n))) - \theta(\bar{v}^*(n)) \right\|_\infty < \epsilon \right) = 1, \forall \epsilon > 0. \tag{18}
\]

Because the function $f(\cdot)$ is the max function, we have
\[
\left\| \theta((n)(V^*(n))) - \theta(\bar{v}^*(n)) \right\|_\infty \geq f(\theta((n)(V^*(n)))) - f(\theta(\bar{v}^*)) \geq f(\theta(\bar{v}^*)) - f(\theta((n)(V^*(n)))).
\]

Therefore, (18) yields that $\forall \epsilon > 0$,
\[
\lim_{n \to \infty} P \left( f(\theta(\bar{v}^*)) - f(\theta((n)(V^*(n)))) < \epsilon \right) = 1. \tag{19}
\]

Moreover, because for problem (15), $\bar{v}^*$ is an optimal solution and $\bar{v}^*(n)$ is a feasible solution, we have that
\[
f(\theta(\bar{v}^*)) \geq f(\theta(\bar{v}^*)).
\]

By combining (19) and (20), we obtain (14).

**C. Step 2: The Proof of (15)**

Because $u_1^T \cdot v^* = 1$, the elements of $v^*$ can be considered as probabilities. By independently choosing the decision vectors according to these probabilities in each time slot, we obtain a sequence of decision vectors $V'(n)$. Denote the average of decision vectors in the set $V'(n)$ by $\bar{v}'(n) = \sum_{v \in V'(n)} v/n$. Because of the weak law of large numbers, we have
\[
\lim_{n \to \infty} P \left( \left\| \bar{v}'(n) - v^* \right\|_\infty < \epsilon \right) = 1, \forall \epsilon > 0. \tag{21}
\]

We will use the term $f(\theta((n)(V'(n))))$ as a bridge between $f(\theta(\bar{v}^*))$ and $f(\theta(v^*))$.

By using the definition of $\theta(v^*)$ and $\theta((n)(V'(n)))$, then sequentially using (17) and (21), we have
\[
\lim_{n \to \infty} P \left( \left\| \theta((n)(V'(n))) - \theta(\bar{v}^*) \right\|_\infty < \epsilon \right) = 1. \tag{22}
\]

Because $f(\cdot)$ is the max function, we have that
\[
\left\| \theta((n)(V'(n))) - \theta(\bar{v}^*) \right\|_\infty \geq f(\theta((n)(V'(n)))) - f(\theta(v^*)) \geq f(\theta(\bar{v}^*)) - f(\theta((n)(V'(n)))).
\]

Therefore, (22) and (23) yield that
\[
\lim_{n \to \infty} P \left( f(\theta((n)(V'(n)))) - f(\theta(\bar{v}^*)) < \epsilon \right) = 1. \tag{24}
\]

Moreover, because for problem (15), $V^*(n)$ is an optimal solution and $V'(n)$ is a feasible solution, we have that
\[
f(\theta((n)(V'(n)))) \geq f(\theta((n)(V^*(n)))).
\]

By combining (24) and (25), we obtain (15).

APPENDIX B
SUPER-MARTINGALE CONVERGENCE THEOREM

In the following, we will use the super-martingale theorem \[15, p. 355\]. We hence repeat it here for convenience. If a deterministic sequence $\{W(n)\}_{n \in \mathbb{N}}$ is lower bounded and monotonically decreasing, we know that it converges. The super-martingale convergence theorem extends this result to the stochastic case when $\{W(n)\}_{n \in \mathbb{N}}$ is no longer deterministic. It states that, if the conditional expectation of $W(n)$ on the information up to time slot $n - 1$ is smaller than or equal to $W(n-1)$, then $W(n)$ is convergent almost surely. We repeat the theorem as follows.

**Theorem 3** \[15, p.355\]: If both the non-negative sequences $\{W(n)\}_{n \in \mathbb{N}}$ and $\{\gamma(n)\}_{n \in \mathbb{N}}$ are adapted to a sequence of increasing $\sigma$-fields $H(n)$ and if
\[
E(W(n) \mid H(n-1)) \leq W(n-1) - \gamma(n),
\]
then $W(n)$ converges with probability 1 and $\sum_{n=1}^{\infty} \gamma(n) < \infty$ with probability 1.
Appendix C
Proof of Theorem 1

A. Sketch of The Proof

We present the proof in the following two steps.

Step 1: We prove that \( \hat{\zeta}^{(n)} \) decreases in expectation under the given conditions. For the ease of discussion, we introduce a few notations. The maximum component of vector \( \Delta_M \cdot e^{(t)} \) during time slot \( t \) is \( \max_{1 \leq j \leq M} (\Delta_M \cdot e^{(t)})_j \). Denote its minimum over the first \( n \) time slots by

\[
\xi(n) = \min_{1 \leq t \leq n} \left[ \max_{1 \leq j \leq M} (\Delta_M \cdot e^{(t)})_j \right].
\]

(26)

Define three constants whose signs follow from (10)

\[
\begin{aligned}
d_1 &= \max_{m \neq j} (\Delta_M \cdot R)_{mj} < 0, \\
d_2 &= \max_{m \neq j} |(\Delta_M \cdot R)_{mj}| \geq 0, \\
d_3 &= -(d_2 + 4S)^2/(2d_1) - d_1/M > 0.
\end{aligned}
\]

(27)

(28)

(29)

Define two sequences

\[
\begin{aligned}
\kappa(n) &= \zeta^{(n)} \cdot I \left\{ \xi(n) > \tau d_3 \right\}, \\
\mu(n) &= 2\tau^2d_3^2 \cdot \mathbb{I} \left\{ \xi(n) > \tau d_3 \right\}.
\end{aligned}
\]

(30)

(31)

We will prove that the sequence \( \{\kappa(n)\} \) is a non-negative super-martingale with

\[
E_n-1\kappa(n) \leq \kappa(n-1) - \mu(n-1).
\]

(32)

Step 2: We apply the super-martingale convergence theorem (Theorem 3 in Appendix B) and see that the sequence \( \{\kappa(n)\} \) visits the range \([0, \tau d_3]\) at least once with probability 1. More precisely, there exists a finite time \( N_0 \geq 0 \) such that \( \xi(N_0) \leq \tau d_3 \).

Similarly, if at any time \( N_k^+ \geq N_0 \),

\[
\|\Delta_M \cdot e^{(N_k^+)}\|_\infty > (M-1)\tau d_3,
\]

we right shift the time axis by \( N_k^+ \) (i.e., replace \( n \) by \( n - N_k^+ \)) and apply the same super-martingale trick. We see that, with probability 1, there exists a finite time \( N_{k+1}^+ > N_k^+ \) such that

\[
\|\Delta_M \cdot e^{(N_{k+1}^+)}\|_\infty \leq (M-1)\tau d_3.
\]

(33)

In other words, the sequence \( \{\|\Delta_M \cdot e^{(n)}\|_\infty\} \) can only fulfill \( \|\Delta_M \cdot e^{(n)}\|_\infty \geq (M-1)\tau d_3 \) for finite consecutive time slots, as shown in Figure 12. We will use this result and Chebychev’s inequality to bound the probability that \( \zeta(n) \geq \delta \) for any arbitrary constant \( \delta > 0 \). By taking \( \tau \to 0 \) and \( n\tau \to \infty \), we will see that this probability goes to zero, as stated in (11).

B. Step 1: Sequence \( \{\zeta^{(n)}\} \) Decreases In Expectation

Let \( n > 1 \) and let us distinguish the two cases: i) \( \zeta^{(n-1)} \leq \tau d_3 \) and ii) \( \zeta^{(n-1)} > \tau d_3 \).

i) If \( \zeta^{(n-1)} \leq \tau d_3 \), because of (26), we have that \( \zeta^{(n)} \leq \zeta^{(n-1)} \leq \tau d_3 \), for all \( n' \geq n - 1 \). It implies that \( \kappa^{(n')} = 0 \) and \( \mu^{(n')} = 0 \) for all \( n' \geq n - 1 \), where \( \kappa^{(n')} \) and \( \mu^{(n')} \) are defined in (30) and (31). Hence, (32) holds trivially in this case.

ii) Now we look at the case where \( \zeta^{(n-1)} > \tau d_3 \). Let us denote

\[
\begin{aligned}
am_l &= (\Delta_M \cdot e^{(n-1)})_l, \forall 1 \leq l \leq M \\
b_l &= (\Delta_M \cdot R)_l, \forall 1 \leq l \leq M \\
i^* &= \arg\max_{1 \leq l \leq M} \{a_l\}, \\
j^* &= \arg\min_{1 \leq l \leq M} \{a_l\}.
\end{aligned}
\]

(33)

(34)

(35)

(36)

By definition (9), HEF selects BS \( i^* \) at time slot \( n \). Because

\[
\begin{aligned}
u_M^\top \cdot \Delta_M &= u_M^\top \cdot \left( I_M - \frac{1}{M} u_M^\top u_M \right) = u_M^\top - u_M = 0,
\end{aligned}
\]

it follows that

\[
\sum_{l=1}^{M} a_l = u_M^\top \cdot \Delta_M \cdot e^{(n-1)} = 0
\]

(37)

and similarly

\[
\sum_{l=1}^{M} b_l = 0.
\]

(38)

Because of (26), \( a_{i^*} \geq \zeta^{(n-1)} > \tau d_3 > 0 \). Hence, \( a_{j^*} < 0 \) because of (37). Similarly, because of (10), \( b_l < 0 \) for all \( l \neq i^* \), hence (38) implies that \( b_{i^*} > 0 \).

Plugging \( C = R + \bar{s} \cdot u_M^\top \) into (4), and multiplying it by \( \Delta_M = I_M - u_M \cdot u_M^\top / M \) yields

\[
\begin{aligned}
\Delta_M \cdot e^{(n)} &= \Delta_M \cdot e^{(n-1)} - \tau \Delta_M \cdot (R + \bar{s} \cdot u_M^\top) \cdot v^{(n)} \\
&\quad + \tau \Delta_M \cdot \bar{s}^{(n)} \\
&\quad + \tau \Delta_M \cdot (s^{(n)} - \bar{s}),
\end{aligned}
\]

(39)

where the second equality follows from \( u_M^\top \cdot v^{(n)} = 1 \). By taking the conditional expectation \( E_n-1 \) on both sides of (39), we therefore have that

\[
E_n-1 \Delta_M \cdot e^{(n)} = \Delta_M \cdot e^{(n-1)} - \tau \Delta_M \cdot R \cdot v^{(n)}.
\]

(40)

Using (33) and (36), we can now recast (40) component-wisely to get

\[
E_n-1 \left( \Delta_M \cdot e^{(n)} \right)_l = a_l - \tau b_l, \forall 1 \leq l \leq M.
\]

(41)
From (39), we have that
\[
\left\| \Delta_M \cdot e^{(n)} - \Delta_M \cdot e^{(n-1)} \right\|_\infty \\
\leq \tau \left\| \Delta_M \cdot R \cdot u^{(n)} \right\|_\infty + \tau \left\| \Delta_M \cdot (s^{(n)} - \bar{s}) \right\|_\infty \\
\leq \tau d_2 + \tau \left\| \Delta_M \right\|_\infty \left\| s^{(n)} - \bar{s} \right\|_\infty \\
\leq \tau d_2 + 2\tau \left( \left\| s^{(n)} \right\|_\infty + \left\| \bar{s} \right\|_\infty \right) \\
\leq \tau d_2 + 4\tau S, \\
\tag{42}
\] where the third inequality follows from (23) and \(\left\| \Delta_M \right\|_\infty \leq 2\) and the fourth inequality is because the energy recharge rates are bounded by the constant \(S\). Taking (42) component-wisely, we see that
\[
\left( \Delta_M \cdot e^{(n)} \right)_l - a_l \leq \tau(d_2 + 4S), \forall 1 \leq l \leq M. \\
\tag{43}
\]
By taking the conditional expectation \(E_{n-1}\) on both sides of the identity
\[
\left( \Delta_M \cdot e^{(n)} \right)_l^2 = \left( a_l + \left( \Delta_M \cdot e^{(n)} \right)_l - a_l \right)^2 \\
= a_l^2 + 2a_l \left( \Delta_M \cdot e^{(n)} \right)_l - a_l \left( \Delta_M \cdot e^{(n)} \right)_l - a_l \\
+ \left( \Delta_M \cdot e^{(n)} \right)_l - a_l \right)^2
\]
and plugging in (41) and (43), we have that
\[
E_{n-1} \left( \Delta_M \cdot e^{(n)} \right)_l^2 \leq a_l^2 - 2\tau a_l b_l + \tau^2(d_2 + 4S)^2. \\
\tag{44}
\]
By summing (44) up for all \(1 \leq l \leq M\), we have that
\[
\sum_{l=1}^{M} E_{n-1} \left( \Delta_M \cdot e^{(n)} \right)_l^2 \leq \sum_{l=1}^{M} a_l^2 - 2\tau a_l b_l + \tau^2(d_2 + 4S)^2. \\
\tag{45}
\]
Meanwhile,
\[
\sum_{l=1}^{M} a_l b_l = a_{i^*} b_{i^*} + \sum_{l=1, l \neq i^*}^{M} (a_l - a_{i^*}) b_l + \sum_{l=1, l \neq i^*}^{M} a_{i^*} b_l \\
= \sum_{l=1, l \neq i^*}^{M} (a_l - a_{i^*}) b_l \\
\geq (a_{j^*} - a_{i^*}) b_{j^*} \\
\geq -a_{i^*} b_{j^*} \\
\geq -\tau d_1 d_3, \\
\tag{46}
\]
where the equality on the second line follows from (38), the inequality on the third line is because \(a_l - a_{i^*} \leq 0\) and \(b_l < 0\) for any \(l \neq i^*, l \neq j^*\), the inequality on the fourth line is because \(a_{j^*}, b_{j^*} \leq 0\) and the inequality on the last line follows from \(a_{i^*} \geq \tau d_3\) and \(-b_{j^*} \geq -d_1\) because of (27).

Consequently, by using (12), (45), (46) and (29) successively, we obtain
\[
E_{n-1} \xi^{(n)} = \sum_{l=1}^{M} E_{n-1} \left( \Delta_M \cdot e^{(n)} \right)_l^2 \\
\leq \sum_{l=1}^{M} a_l^2 - \sum_{l=1}^{M} 2\tau a_l b_l + M\tau^2(d_2 + 4S)^2 \\
\leq \sum_{l=1}^{M} a_l^2 + \sum_{l=1}^{M} 2\tau^2 d_1 d_3 + M\tau^2(d_2 + 4S)^2 \\
= \xi^{(n-1)} - 2\tau^2 d_1^2. \\
\tag{47}
\]
With (30) and (47) can be re-casted as (32).
Combining cases i) and ii), we see that the sequence \(\{\xi^{(n)}\}_{n\in\mathbb{N}}\) is a super-martingale, as (32) always verified.

C. Step 2: Sequence \(\{\xi^{(n)}\}_n\) Is Bounded With Probability 1
The super-martingale convergence theorem (Theorem 3 in Appendix B) implies that, with probability 1, the sequence \(\{\mu^{(n)}\}_{n\in\mathbb{N}}\) is summable, i.e.,
\[
\sum_{n=1}^{\infty} \mu^{(n)} = \sum_{n=1}^{\infty} 2\tau^2 d_1^2 \cdot \mathbb{1} \{\xi^{(n)} > \tau d_3\} < \infty. \\
\tag{48}
\]
Because \(2\tau^2 d_1^2\) is a constant, if \(\xi^{(n)} > \tau d_3, \forall n\in\mathbb{N}\), we have that \(\sum_{n=1}^{\infty} \mu^{(n)} = \infty\), which contradicts (48). Consequently, we know that, with probability 1, there exists a finite time \(N_0 \geq 0\), such that \(\xi^{(N_0)} \leq \tau d_3\) and thus
\[
\max_{1 \leq j \leq M} \left\{ \left( \Delta_M \cdot e^{(N_0)} \right)_j \right\} \leq \tau d_3. \\
\tag{49}
\]
Therefore, since \(\sum_{j=1}^{M} \left( \Delta_M \cdot e^{(N_0)} \right)_j = 0\) by (37),
\[
\min_{1 \leq j \leq M} \left\{ \left( \Delta_M \cdot e^{(N_0)} \right)_j \right\} \geq -(M-1)\tau d_3. \\
\tag{50}
\]
Hence, (49) and (50) yield that
\[
\left\| \Delta_M \cdot e^{(N_0)} \right\|_\infty \leq (M-1)\tau d_3. \\
\]
Now, if at any time \(N_0^+ > N_0\), we have
\[
\left\| \Delta_M \cdot e^{(N_0^+)} \right\|_\infty > (M-1)\tau d_3, \\
\]
we right shift the time axis by \(N_0^+\) (i.e., replace \(n\) by \(n - N_0^+\)) and apply the same super-martingale technique above.
We know that, with probability 1, there exists a finite time \(N_1 > N_0^+\) such that
\[
\left\| \Delta_M \cdot e^{(N_1)} \right\|_\infty \leq (M-1)\tau d_3. \\
\]
Similarly, if at any time slot \(N_0^+\), \(\left\| \Delta_M \cdot e^{(N_0^+)} \right\|_\infty > (M-1)\tau d_3\), with probability 1, there exists a finite time slot \(N_{k+1} > N_0^+\), such that \(\left\| \Delta_M \cdot e^{(N_{k+1})} \right\|_\infty \leq (M-1)\tau d_3\), as shown in Figure 12.
We now prove (11) for any given \(\delta > 0\). Suppose first that for any \(n \in \mathbb{N}\),
\[
\left\| \Delta_M \cdot e^{(n)} \right\|_\infty \leq (M-1)\tau d_3. \\
\]
Then (12) yields that
$$\zeta^{(n)} \leq M \left\| \Delta_M \cdot e^{(n)} \right\|_{\infty}^2 \leq M(M-1)^2 \tau^2 d_3^2.$$ 
Taking \( \tau \to 0 \), we trivially prove (11).

From now on, we consider the case \( \|\Delta_M \cdot e^{(n)}\|_{\infty} \geq (M-1)\tau d_3 \), and will use the Chebyshev’s inequality to bound \( \zeta^{(n)} \) in this case.

Denote the last time slot before \( n \) that fulfills \( \|\Delta_M \cdot e^{(n)}\|_{\infty} \leq (M-1)\tau d_3 \), i.e.,
$$N_{k_0}^+ = \max \{ t < n, t \in \mathbb{N} : \|\Delta_M \cdot e^{(t)}\|_{\infty} \leq (M-1)\tau d_3 \},$$
which implies that
$$\|\Delta_M \cdot e^{(N_{k_0}^+-1)}\|_{\infty} \leq (M-1)\tau d_3,$$ and
$$\|\Delta_M \cdot e^{(t)}\|_{\infty} > (M-1)\tau d_3, \forall N_{k_0}^+ \leq t < n.$$
As we just saw above, with probability 1, there exists a time \( N_{k_0}^+ < N_{k_0+1} < +\infty \) such that
$$\|\Delta_M \cdot e^{(N_{k_0}^+)\}}_{\infty} \leq (M-1)\tau d_3.$$ Remember that \( N_{k_0}^+ \leq n \leq N_{k_0+1} \). Hence, with probability 1, \( n - N_{k_0}^+ \) is finite.

By iterating (47) from time slot \( N_{k_0}^+ \) to \( n \), we have that
$$\mathbb{E}_{N_{k_0}^+} \zeta^{(n)} - \zeta^{(N_{k_0}^+ - 1)} \leq -2(n - N_{k_0}^+)\tau^2 d_1^2 < 0.$$ Denoting by \( A = \zeta^{(n)} - \zeta^{(N_{k_0}^+ - 1)} \), we thus have that
$$\mathbb{E}_{N_{k_0}^+} A < 0.$$ Iterating (43) in a similar fashion and next applying (52), we have that
$$\|\Delta_M \cdot e^{(n)}\|_{\infty} \leq \|\Delta_M \cdot e^{(N_{k_0}^+)\}}_{\infty} + (n - N_{k_0}^+)\tau d_2 + 4S \leq (M-1)\tau d_3 + (n - N_{k_0}^+)\tau (d_2 + 4S) < \tau(n - N_{k_0}^+ + 1)(d_2 + 4S + (M-1)d_3),$$
where the third inequality is because of \( n \geq N_{k_0}^+ \) and \( n - N_{k_0}^+ + 1 > n - N_{k_0}^+ \). From (12) and (54), we deduce that
$$\zeta^{(n)} \leq M \left\| \Delta_M \cdot e^{(n)} \right\|_{\infty}^2 \leq M\tau^2(n - N_{k_0}^+ + 1)^2(d_2 + 4S + (M-1)d_3)^2.$$ (55)
Denote the probability and the variance conditioned on the information before time slots \( N_{k_0}^+ \) by \( \mathbb{P}_{N_{k_0}^+} \) and \( \text{Var}_{N_{k_0}^+} \), respectively. We have
$$\text{Var}_{N_{k_0}^+} A = \text{Var}_{N_{k_0}^+} \zeta^{(n)} \leq M^2\tau^2(n - N_{k_0}^+ + 1)^2(d_2 + 4S + (M-1)d_3)^4,$$ (56)
where the inequality on the second line is due to (55).

For any constant \( \delta > 0 \), we apply the Chebyshev’s inequality and obtain that
$$\mathbb{P}_{N_{k_0}^+} (A \geq \delta / 2) = \mathbb{P}_{N_{k_0}^+} \left( A - \mathbb{E}_{N_{k_0}^+} A \geq \delta / 2 - \mathbb{E}_{N_{k_0}^+} A \right) \leq \frac{(\delta/2 - \mathbb{E}_{N_{k_0}^+} A)^2}{(n - N_{k_0}^+ + 1)^4(d_2 + 4S + (M-1)d_3)^4},$$ (57)
where the inequality on the fourth line is because of (53) and (56). Remember that given any \( \tau > 0 \), when \( n \to \infty \), \( n - N_{k_0}^+ \) is finite with probability 1. Now, when we take \( \tau \to 0 \), we obtain from (57) that
$$\lim_{\tau \to 0, n \to \infty} \mathbb{P}_{N_{k_0}^+} \left( \zeta^{(n)} - \zeta^{(N_{k_0}^+ - 1)} \geq \delta / 2 \right) = 0.$$ (58)
Now, because of (52),
$$\lim_{\tau \to 0, n \to \infty} \zeta^{(N_{k_0}^+ - 1)} = \lim_{\tau \to 0, n \to \infty} \frac{1}{M} \sum_{t=1}^{M} \left( \Delta_M \cdot e^{(N_{k_0}^+ - 1)} \right)^2 = 0.$$ (59)
We hence have \( \zeta^{(N_{k_0}^+ - 1)} < \delta / 2 \) if \( \tau \) is taken small enough. Then (58) becomes
$$\lim_{\tau \to 0, n \to \infty} \mathbb{P}_{N_{k_0}^+} \left( \zeta^{(n)} \geq \delta \right) = 0.$$ (59)
Using the total probability theorem over all realizations of time \( N_{k_0}^+ - 1 \), we obtain (11) from (59) and conclude the proof.

**Appendix D**

**Proof of Theorem 2**

**A. Sketch of The Proof**

We present the proof in the following two steps.

**Step 1**: We first prove that HEF leads the average of decision vectors
$$\bar{v}^{(n)} = \frac{1}{n} \sum_{t=1}^{n} v^{(t)}$$
to converge to a constant vector
$$\bar{v}_{\text{hef}} = \frac{R^{-1} \cdot u_M}{u_M \cdot R^{-1} \cdot u_M}$$
in probability.

**Step 2**: We then prove Lemma 2 and see that \( \bar{v}_{\text{hef}} \) achieves the optimal objective value of (6), which concludes the proof of this theorem.

**B. Step 1: The Average Decision Vector Converges In Probability**

We first show that
$$\lim_{\tau \to 0, n \to \infty} \mathbb{P} \left( \|\Delta_M \cdot e^{(n)}\|_{\infty} < \delta \right) = 1, \forall \delta > 0.$$ (62)
To prove (62), let us first iterate (39) for time slots \( 0 \leq t \leq n \). Noticing that \( \Delta_M \cdot e^{(0)} = 0 \), we have that
$$\Delta_M \cdot e^{(n)} = -\tau \Delta_M \cdot R \cdot \sum_{t=1}^{n} v^{(t)} + \tau \sum_{t=1}^{n} \Delta_M \cdot (s^{(t)} - \bar{s}).$$ (63)
Multiplying by a factor of $1/(n\tau)$, it amounts to
\[
\Delta_M \cdot R \cdot \bar{v}(n) = \frac{1}{n} \sum_{t=1}^{n} \Delta_M \cdot (s^{(t)} - \bar{s}) - \frac{1}{n\tau} \Delta_M \cdot e^{(n)}. \tag{64}
\]
All the assumptions of Theorem 1 are included in the assumptions of Theorem 2 and therefore (11) holds. Now, since
\[
\left\| \Delta_M \cdot e^{(n)} \right\|_\infty^2 = \left( \max_{1 \leq i \leq M} \left( \Delta_M \cdot e^{(n)} \right) \right)^2 
\leq \sum_{i=1}^{M} \left( \Delta_M \cdot e^{(n)} \right)^2, \tag{65}
\]
(11) yields that $\forall \delta > 0,
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( \left\| \Delta_M \cdot e^{(n)} \right\|_\infty < \delta^{1/2} \right) = 1. \tag{66}
\]
From (64) and (65), we obtain (62).

Now, we will use (62) to prove that the average decision vector $\bar{v}(n)$ converges to the constant vector $\bar{v}_{\text{hef}}$ in probability, i.e., $\forall \delta > 0,
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( \left\| \bar{v}(n) - \bar{v}_{\text{hef}} \right\|_\infty < \delta \right) = 1. \tag{67}
\]
Indeed, with the triangle inequality, we see that
\[
\left\| \bar{v}(n) - \bar{v}_{\text{hef}} \right\|_\infty \leq \left\| \bar{v}(n) - k' R^{-1} \cdot u_M \right\|_\infty + \left\| k' R^{-1} \cdot u_M - \bar{v}_{\text{hef}} \right\|_\infty, \tag{68}
\]
where we pick $k' = u_M^\top \cdot R \cdot \bar{v}(n) / M$.

Now, since $u_M$ is the $M \times 1$ all-ones vector and the condition (13), we have that
\[
\left\| R^{-1} \cdot u_M \right\|_\infty \leq \left| u_M^\top \cdot R^{-1} \cdot u_M \right|. \tag{69}
\]
Therefore, we have that
\[
\left\| k' R^{-1} \cdot u_M - \bar{v}_{\text{hef}} \right\|_\infty = \left\| k' R^{-1} \cdot u_M - R^{-1} \cdot u_M \right\|_\infty + \left\| R^{-1} \cdot u_M - \bar{v}_{\text{hef}} \right\|_\infty 
\leq \left| u_M^\top \cdot R^{-1} \cdot u_M - 1 \right| + \left\| R^{-1} \cdot u_M - \bar{v}(n) \right\|_\infty 
\leq M \left\| k' R^{-1} \cdot u_M - \bar{v}(n) \right\|_\infty, \tag{70}
\]
where the inequality on the fourth line is because of (68) and the equality on the fifth line is because of $u_M^\top \cdot \bar{v}(n) = 1$. Plugging (69) in the right-hand side of (67), we get
\[
\left\| \bar{v}(n) - \bar{v}_{\text{hef}} \right\|_\infty \leq (1 + M) \left\| \bar{v}(n) - k' R^{-1} \cdot u_M \right\|_\infty. \tag{71}
\]
Because $\Delta_M = I_M - u_M \cdot u_M^\top / M,
R^{-1} \cdot \Delta_M \cdot R \cdot \bar{v}(n) = \bar{v}(n) - R^{-1} \cdot u_M \cdot u_M^\top \cdot R \cdot \bar{v}(n) / M
= \bar{v}(n) - k' R^{-1} \cdot u_M$.

Hence (71) becomes
\[
\left\| \bar{v}(n) - \bar{v}_{\text{hef}} \right\|_\infty \leq (1 + M) \left\| R^{-1} \cdot \Delta_M \cdot R \cdot \bar{v}(n) \right\|_\infty \leq (1 + M) \left\| R^{-1} \right\|_\infty \cdot \left\| \Delta_M \cdot R \cdot \bar{v}(n) \right\|_\infty. \tag{72}
\]
Given $\delta > 0$, let us pick $\delta' = \frac{(1 + M) R^{-1} \cdot \Delta_M \cdot R \cdot \bar{v}(n)}{\delta}$.
Let $\tau \to 0 \quad n\tau \to \infty$, then (62) implies that
\[
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( \left\| \Delta_M \cdot R \cdot \bar{v}(n) \right\|_\infty < \delta' \right) = 1, \forall \delta' > 0,
\]
and with (71), it further implies that
\[
\lim_{\tau \to 0, n\tau \to \infty} \mathbb{P} \left( \left\| \bar{v}(n) - \bar{v}_{\text{hef}} \right\|_\infty < \delta \right) = 1, \forall \delta > 0,
\]
which establishes (66).

C. Step 2: Asymptotic Optimality of HEF
In the following lemma, we show that $\bar{v}_{\text{hef}}$ is an optimal solution of (6).

Lemma 2: Under the conditions (13), $\bar{v}_{\text{hef}}$ is the optimal solution of problem (6) achieving the optimal value
\[
f(\theta(\bar{v}_{\text{hef}})) = \frac{1}{u_M^\top \cdot R^{-1} \cdot u_M}. \tag{73}
\]

Proof: The general idea of the proof is to use the duality properties of linear programming problems. Because $R^{-1} \cdot u_M > 0$ or $R^{-1} \cdot u_M < 0$ (13), $\bar{v}_{\text{hef}}$ is a feasible solution of problem (6). The corresponding objective value is $f(\theta(\bar{v}_{\text{hef}}))$ (73).

The dual problem of (6) is written as:
\[
\max_{w, \lambda} w \quad \text{s.t.} \quad u_M^\top \cdot \lambda = 1, \quad R^\top \cdot \lambda \geq w u_M, \quad w \geq 0, \quad \lambda \geq 0. \tag{74}
\]
Because $(R^\top)^{-1} \cdot u_M > 0$ or $(R^\top)^{-1} \cdot u_M < 0$ (13), we see that
\[
\lambda = \frac{(R^\top)^{-1} \cdot u_M}{u_M^\top \cdot (R^\top)^{-1} \cdot u_M}, \quad w = \frac{1}{u_M^\top \cdot (R^\top)^{-1} \cdot u_M}.
\]

is a feasible solution of the dual problem (74). Consequently, the objective value $1 / (u_M^\top \cdot (R^\top)^{-1} \cdot u_M)$ reached by this feasible solution of the dual problem provides a lower bound of the objective value for the original problem (6). By noticing that
\[
\frac{1}{u_M^\top \cdot (R^\top)^{-1} \cdot u_M} = \left( \frac{1}{u_M^\top \cdot (R^\top)^{-1} \cdot u_M} \right)^\top = u_M^\top \cdot (R^\top)^{-1} \cdot u_M,
\]
we see that $\bar{v}_{\text{hef}}$ achieves the lower bound and hence is the optimal solution of the original problem (6).}

Finally, because of (66) and because of Lemma 2, we see that HEF is asymptotically optimal in probability.