Ultrasolitons: Multistability and subcritical power threshold from higher-order Kerr terms

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Abstract – We show that an optical system involving competing higher-order Kerr nonlinearities can support the existence of ultrasolitons, namely extremely localized modes that only appear above a certain threshold for the central intensity. Such new solitary waves can be produced for powers below the usual collapse threshold, but they can also coexist with ordinary, lower-intensity solitons. We derive analytical conditions for the occurrence of multistability and analyze the dynamics of the different kinds of fundamental eigenmodes that can be excited in these nonlinear systems. We also discuss the possible transitions between solitary waves belonging to different nonlinear regimes through the mechanism of soliton switching.

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Introduction. – A recent measurement of the instantaneous higher-order Kerr (HOKE) coefficients in gases\textsuperscript{[1]} has led to a revolutionary description of the filamentation of ultrashort laser pulses\textsuperscript{[2]}. The filament stabilization is attributed to the competition of the HOKE focusing and defocusing contributions to the refractive index alone, rather than to their interplay with the ionization-induced plasma defocusing, which had a key role in the traditional interpretation\textsuperscript{[3]}. This new paradigm has stimulated an increasing amount of work\textsuperscript{[4–11]}, aimed either at testing the controversial results reported in refs.\textsuperscript{[1,2]}, or at the theoretical exploration of its rich phenomenological implications\textsuperscript{[12–17]}.

In this letter, we will demonstrate that, within a well-defined parameters region, a system involving just local HOKE nonlinearities can support the existence of a new branch of localized solutions that will be called ultrasolitons. Such stationary states coexist with solitons of lower intensity, similar to those found for common optical media\textsuperscript{[12]}. Very remarkably, this implies the emergence of optical soliton multistability (OSM), \textit{i.e.} the existence of two or more stationary states with the same power and different propagation constants and profiles, like in the systems reported in refs.\textsuperscript{[18–21]} and in the recent work\textsuperscript{[22]}, that has appeared while we were preparing the revised version of the present paper. We will also derive an analytical condition on the HOKE coefficients for the emergence of OSM, and show that the ultrasolitons can be subcritical, \textit{i.e.} they may exhibit powers even below the ordinary collapse threshold\textsuperscript{[23]}. Finally, we will discuss the transitions among multistable states through efficient soliton switching processes.

Mathematical model. – Let us consider a wave system evolving along the $\eta$ direction in the space of transverse coordinates $\xi$ and $\chi$, and assume that the complex wavefunction $\Psi(\xi, \chi, \eta)$ satisfies the (dimensionless) nonlinear Schrödinger equation

$$\frac{\partial \Psi}{\partial \eta} + \frac{1}{2} \nabla_{\perp}^2 \Psi + F(|\Psi|^2)\Psi = 0,$$

(1)

where $\nabla_{\perp}^2 = \partial^2/\partial \xi^2 + \partial^2/\partial \chi^2$ and

$$F(|\Psi|^2) = \sum_q (-1)^{q+1} f_{2q} |\Psi|^{2q}.$$

(2)

This formalism applies to the paraxial propagation of the linearly polarized electric field $E(x, y, z)$ of a laser pulse of mean wave number in vacuum $k_0 = 2\pi/\lambda_0$, being $\lambda_0$ the central wavelength, in a nonlinear optical medium whose refractive index depends upon the intensity $I = c_0 E^2$.
as \( n = n_0 + \Delta n = n_0 + \sum_{q=1}^{4} n_{2q} f^q \). In fact, motivated by the results of ref. [1] for the optical response of common gases, we will assume that the refractive index involves 4 terms of increasing powers in the intensity \( I \) of the beam, with alternating sign coefficients \( n_2, n_4 > 0 \) and \( n_4, n_8 < 0 \), that contribute to focusing and defocusing respectively. We will choose the dimensionless quantities such that \( f_2 = f_4 = 1 \). The relations between the dimensional and dimensionless physically relevant quantities are then \( (\xi, \chi, \eta) = (n_0/n_4)^{1/2} (k_0 n_2)(x, y), \eta = (k_0 n_2^2/n_4)z, \Psi(\xi, \chi, \eta) = (\alpha \xi n_4/n_2)^{1/2} E(x, y, z), \Delta n = (n_2^2/n_4)F, n_6 = (n_3^2/n_2)f_6 \) and \( n_8 = (n_3^2/n_2^2) f_8 \). The only free parameters included in eq. (1) are then \( f_6 \) and \( f_8 \), that will be assumed to be positive.

For the sake of clarity, and motivated by refs. [1,2,12], in this letter we will neglect the effects of multiphoton absorption, ionization and temporal dispersion of the optical pulses. However, we emphasize that our results may also be applied to multilevel atomic media, where the dependence of \( \Delta n \) given by eq. (2) may be achieved by the coherent control of the atomic ensemble via quantum-engineering techniques [24].

Analytical condition for multistability. – Let us now derive an analytical condition for the possible emergence of OSM. Assuming radial symmetry, we search for soliton solutions of the form \( \Psi(\xi, \chi, \eta) = \Phi(r)e^{-i\mu \theta} \), where \( r = \sqrt{\xi^2 + \chi^2} \) and \( \mu \) is the propagation constant. As discussed in refs. [12,25], \( \mu \) can be identified with the chemical potential of an equivalent thermodynamical 2D system of \( N = \int \rho d\xi d\chi \equiv \int |\Phi|^2 d\xi d\chi \) particles for the optical system, \( N = n_0 k_0^2 n_2 \rho \), where \( \rho \) is the total power of the optical field). Equation (1) can then be derived by minimizing the Landau’s grand potential [26] \( \Omega = -\int \rho d\xi d\chi \), where the pressure field \( p \) is

\[
p = -\frac{1}{2} \nabla \Phi^2 + \mu |\Phi|^2 + \int_0^\infty F(U) dU.
\]

In particular, assuming the dependence given in eq. (2), the integral term in eq. (3) can be written as \( \sum_{q=1}^{4} (-1)^q + \frac{1}{2^q} \frac{\pi^2}{\pi^2} |\Phi|^{2(q+1)} \).

In the case of a high-power solution with a flat-top profile of radius \( R \), calling \( A = \Phi(0) \) the amplitude of the large and homogeneous central region, eq. (1) implies

\[
\mu = -F(A^2),
\]

where we choose the arbitrary phase of the solution such that \( A \) is positive real. On the other hand, as shown in refs. [12,25], any such flat-top solitons obey the Young-Laplace (YL) equation [26], \( p_c = 2\sigma/R \), where \( R \) is the radius of the droplet, \( p_c = p(0) \) is the central pressure and the effective surface tension can be computed as \( \sigma = -R^{-1} \int_0^\infty \rho r dp(r) dr \). In the large \( R \) limit, \( p_c = 0 \), the gradient term in eq. (3) can be neglected close to the origin, and using eq. (4) we get

\[
\int_0^{A_\infty^2} F(U) dU - A_\infty^2 F(A_\infty^2) = 0,
\]

being \( \mu_\infty \) and \( A_\infty \) the asymptotic values corresponding to the \( R \rightarrow \infty \) droplet.

In particular, for media whose nonlinear response is described by eq. (2), including terms up to \( f_8 \), we get the condition

\[
\frac{1}{2} - \frac{2}{3} U + f_6 \frac{3}{4} U^2 - f_8 \frac{4}{5} U^3 = 0,
\]

where \( U = A_\infty^2 \). This cubic equation, giving \( f_6 \) and \( f_8 \), can have either one or three real roots. The latter case will eventually correspond to OSM. After a long but straightforward algebra, we get the following necessary and sufficient condition on \( f_6 \) and \( f_8 \) for eq. (6) to have three real solutions:

\[
18225 f_6^8 - 5400 f_6^6 - 77760 f_6 f_8 + 20480 f_8 + 93312 f_8^2 < 0.
\]

In fig. 1, we plot the region of the \((f_6, f_8)\) plane that satisfies such a condition, assuming \( f_6, f_8 > 0 \). In particular, for \( f_6 > 0.38 \) or \( f_8 > 0.05 \), condition 7 is not fulfilled and only one solution can be found. Conversely, for pairs \((f_6, f_8)\) lying within the shaded region in fig. 1, three different branches of solitons can be found in principle, whose amplitude \( A_\infty \) for \( R \rightarrow \infty \) can be obtained from eq. (6). In the case of oxygen (air), that was examined in ref. [12], \( f_6 = 2.8 \) and \( f_8 = 3.9 \) \((f_6 = 11.2 \) and \( f_8 = 34.1)\). These values fall out of the shaded region in fig. 1, and in fact only one branch of solitons was found in ref. [12].

Hereafter, we will fix the values \( f_6 = 0.3 \) and \( f_8 = 0.02 \), that lie in the OSM domain, although we have verified that similar results can be obtained for different
choices satisfying eq. (7). The nonlinear refractive index
dependence on the intensity of the input beam acquires
then a double-hump structure (see inset in fig. 1). In
fact, the emergence of OSM can be heuristically related
to the appearance of the two maxima of the refractive
index, suggesting the possible existence of families of
fundamental solitons with limiting intensities close to
the values corresponding to the two local maxima of
F. However, the double-hump structure alone does not
guarantee the emergence of OSM, as we have checked by
finding values of \((f_a, f_b)\) not satisfying the multistability
condition 7, that nevertheless lead to a similar double-peak
structure for \(F\).

With the choice \(f_a = 0.3\), \(f_b = 0.02\), we calculate the
three roots of eq. (7), \(A_{\infty}^a = 1.087\), \(A_{\infty}^b = 1.601\), \(A_{\infty}^c = 3.212\),
together with their corresponding values for the
propagation constant, as given by eq. (4), \(\mu_{\infty}^a = -0.241\),
\(\mu_{\infty}^b = -0.182\), \(\mu_{\infty}^c = -6.72\). The superscripts \(a\), \(g\), \(u\) refer
to the names that will be given to the three branches
of solitons corresponding to these limiting values, namely
ordinary solitons, ghost solitons and ultrasolitons,
respectively.

**Localized stationary solutions. Ultrasolitons.** –
We have solved eq. (1) numerically, and found the localized
stationary states shown in fig. 2. We obtain two different
branches of solutions that cannot be connected with each
other, i.e., there are no bifurcations in the eigenstates’
structure. The red dotted line stands for the ordinary
soliton branch, i.e., the branch of solutions similar to those
reported in [12], whose lower (upper) limit corresponds
to the critical power \(P_c\) for self-focusing \((A_{\infty}^0\) plane
wave). The presence of HOKE nonlinearities gives rise
to a new family of solutions, represented by black lines
in fig. 2, whose upper limit corresponds to the \(A_{\infty}^c\)
plane wave. Their lower limit does not correspond to the
Kerr limit, which means that their existence cannot
be explained by a balance between diffraction and the
leading Kerr nonlinearity driven by \(f_2\), but rather as an
interplay between competing HOKE nonlinearities. To our
best knowledge, such solutions do not have counterparts
in any other nonlinear optical system ruled by local
intensity-dependent nonlinearities because they exist over a
certain intensity threshold and feature both amplitudes
and propagation constants higher (in absolute value) than
those of the ordinary ensemble. In other words, these
solitons belong to a completely different nonlinear regime
as compared with that of the ordinary branch. For all
the previous reasons, we have called them ultrasolitons.
On the other hand, according to the Vakhitov-Kolokolov
(VK) criterium [27] supported by systematic simulations
of propagation, we find that the eigenstates represented
by the solid (dashed) line are dynamically stable (unstable).

The eigenstates \(b\), \(c\) and \(d\), that are represented
in the lower-left inset of fig. 2, feature the same optical
power but different propagation constants and radial
profiles. Analogously, the flat-top eigenstates of \(\mu\)\)
d and \(f\) also correspond to the same power, thus
demonstrating OSM even in the high power regime. Both
cases demonstrate the emergence of OSM, generalizing to
our multibranch situation the definition given in [18] for
a single continuous branch.

On the other hand, the solution displayed in inset \(a\) of
fig. 2 is the first stable ultrasoliton, and corresponds to a
local minimum of the ultrasoliton power curve depicted in
fig. 2, being this a trace of the uniqueness of this solution.
Very remarkably, this soliton features a subcritical power
\((N = 3.83\) in dimensionless units\) below the ordinary
collapse threshold \((N = 5.85)\). To our best knowledge, this
is the first example of a stable subcritical soliton in an
optical system with an instantaneous nonlinear response.
Moreover, this minimum power state turns out to have
the lowest possible central intensity \((I = 9.98)\) among
the stable ultrasolitons, although we have found unstable
ultrasolitons (corresponding to the dashed curve in fig. 2)
starting from central intensities as small as \(I = 4.88\).

In fig. 2 we have only plotted two branches of solutions,
the ordinary solitons and ultrasolitons. In fact, we have not
been able to find numerically the third family of solitary
waves linked to the plane wave with amplitude \(A_{\infty}^c\) that
we have theoretically predicted above. We can understand
its non-existence by using three different arguments.
First, we recall that, according to refs. [12,25], for large
but finite flat-topped solutions the central pressure \(p_c\) does
not vanish, being compensated by the surface tension \(\sigma\).
like in an ordinary liquid. The actual values of $\sigma$ for the three branches can be computed as

$$\sigma = \frac{1}{\sqrt{2}} \int_0^{\Lambda_{\infty}} \left(-\mu_{\infty} - \sum_{q=1}^{4} (-1)^{q+1} \int_0^{\Lambda_{\infty}} \frac{A^{2q} - 1}{q+1} \right)^{1/2} dA. \tag{8}$$

We find: $\sigma^1 = 0.0948$, $\sigma^2 = 0.0484 + 0.120i$ and $\sigma^3 = 7.21$. The fact that for the ghost family the surface tension $\sigma^\infty$ is not real reflects the impossibility of equilibrating the inner pressure, which is a real magnitude, thus forbidding the existence of finite ghost solitons. In other words, the ghost solitons would not fulfill the YL equilibrium condition.

Second, we have applied to the plane wave solutions of our system a linear stability analysis similar to that of refs. [28,29]. After a straightforward study, we have found that the solutions with $A_{\infty}^b, A_{\infty}^c$ are linearly stable, i.e., they do not undergo modulational instability under small perturbations, while the $A_{\infty}^d$ lies within an instability window.

Third, we have studied numerically the propagation of three flat-top beams belonging to the high power regime of each of the three branches. The initial condition of our simulations is modeled by the following function:

$$\phi = \Lambda (\frac{1}{2}[1 + \tanh(r + \omega)]\left[1 - \tanh(r - \omega)\right]), \tag{9}$$

where $\Lambda = A_{\infty}^b, A_{\infty}^c, A_{\infty}^d$ is the amplitude of the beam envelope and $\omega = 50$ is the mean radius. We perturb the initial beam profiles with a 5% random noise in order to stimulate the onset of instability. The final states arising from the propagation of such initial conditions are displayed in fig. 3. In the left (right) picture, we show the surface amplitude plot of the flat-top beam having an initial amplitude $\Lambda = A_{\infty}^b (A_{\infty}^c)$, at a propagation distance $\eta = 2000$. We see that such beam is stable, as it quickly couples to a stable eigenstate of the low-intensity (high-intensity) branch of ordinary solitons (ultrasolitons) discussed above. In the middle picture, we show the outcome of the propagation of an initial beam having $\Lambda = A_{\infty}^d$, at a propagation distance $\eta = 100$. In this case, we observe how the spatial beam profile has been destabilized by the growth of the perturbations, yielding to multiple filamentation like in the Cubic-Quintic model [29]. The arising filaments correspond to perturbed quasi-stationary ultrasolitons. In this context, the onset of filamentation can be considered as an additional trace of the non-existence of the flat-top soliton with $\Lambda \approx A_{\infty}^b$, although it would not be conclusive if taken alone.

However, our present analysis is not sufficient to exclude that triple stability might be found for a different choice of $f_6$ and $f_8$ within the multistability region of fig. 1.

**Soliton switching.** – The existence of OSM, as shown in fig. 2, suggests the possibility of observing transitions between different multistable states. Such beam-reshaping mechanisms may lead to soliton switching processes with a great potential for all-optical communications [20]. In order to study the soliton switching in our system, we have simulated the free propagation of three perturbed solitary waves featuring the radial profiles b), c) and d) of fig. 2. The optical power of these beams is $N \approx 30$. The results of the numerical computations are summarized in fig. 4, plotting the peak amplitude evolution of the eigenstates b) (blue dashed line), c) (green dashed-dotted line) and d) (red solid line). Even though we have added a 5% random noise to their initial profiles, both fields b) and d) remain stable, in agreement with the prediction.

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Fig. 3: (Colour on-line) 3D pseudocolour plots of the amplitude of three flat-topped beams modeled by eq. (9) with amplitudes $\Lambda = A_{\infty}^b$ (left), $A_{\infty}^c$ (middle), $A_{\infty}^d$ (right), after a propagation distance of $\eta = 2000, 100$ and $2000$, respectively. They have been initially perturbed with a 5% random noise. We see that both beams featuring $A_{\infty}^c$ and $A_{\infty}^d$ remain stable while the one having $A_{\infty}^b$ undergoes filamentation. The spatial scales spanned are $(\xi, \chi) \in [-100, 100]$.

Fig. 4: (Colour on-line) Evolution of the peak amplitude $A = \Phi(0, 0)$ of the eigenstates b) (blue dashed line), c) (green dashed-dotted line) and d) (red solid line) displayed in fig. 2. We observe that both b) and d) fields are stable against small perturbations, while the eigenmode c) becomes unstable, decaying to the lower branch through a soliton switching mechanism. Alternatively, the unstable mode c) can excite a stable ultrasoliton (black dotted line) by adding an initial wavefront curvature to the optical field. The upper (lower) row of inner snapshots show several pseudocolour amplitude plots of the optical field during the soliton switching procedure, where an ultrasoliton (ordinary soliton) is excited from the unstable field c). The square window displayed in the snapshots has a width $\omega_{\xi, \chi} = 25$. 
of the VK criterion applied to our system. On the other hand, the unstable field c) rapidly decays to a nonlinear mode similar to b), even in the absence of external random noise. The conversion efficiency between both modes, measured as the ratio between the optical powers of the fields, is above 90%.

We have observed that the unstable fields always decay to the lower branch (down-switching), while the up-switching to the ultrasolitons realm does not occur spontaneously. However, we can force such a transition by including a focusing quadratic phase term \( e^{-i0.01r^2} \), similar to that introduced by a thin lens. In this case, the unstable beam d) can be promoted to the upper branch (black dotted line in fig. 4) with a maximum conversion efficiency of around 30%, while the energy excess is radiated as a low-intensity reservoir resembling that generated during the excitation of Townes-like waves in air [30]. The difference between the efficiencies of both processes can be understood by mode-coupling arguments. The modal energy transfer becomes more efficient whenever both profiles and phases of the corresponding waves of the system, that could have potential applications in ultrafast optical circuits intended for all-optical communications [20]. We hope that these results will contribute to stimulate the quest for a clarification of the mechanism of ultrashort pulses filamentation.

Conclusions. – We have shown that an optical system involving competing higher-order Kerr nonlinearities can support the existence of power multistability in the absence of any external potential, yielding a new class of solitary waves, called ultrsolitons, that may exhibit powers below the ordinary collapse threshold. We have also proposed a mechanism of soliton switching for inducing transitions between different multistable nonlinear waves of the system, that could have potential applications in ultrafast optical circuits intended for all-optical communications [20]. We hope that these results will contribute to stimulate the quest for a clarification of the mechanism of ultrashort pulses filamentation.

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