Exceptional points in a topological waveguide-cavity coupled system

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Abstract

Exceptional points (EPs) as branch singularities describe peculiar degeneracies of non-Hermitian systems, widely studied in topological and non-topological optical architectures with introducing gain or loss technically. This work focuses on the EPs in a topological waveguide (TW)-cavity coupled structure, where there is no need to introduce practical gain or loss. The topological cavity contains two degenerate counter-propagation topological whispering gallery modes, whose coupling with the TW leads to the effective gain and loss, responsible for the EP. Such a photonic architecture is designed practically by crystal-symmetry-protected topological photonic insulators based on air rods in conventional dielectric materials. The relevant EP reveals the breaking of the parity-time symmetry, reflected by the change of the transmission-dip number in the optical transmission spectra of the system. Achieving EPs in topological photonic systems possibly opens a new avenue toward robust optical devices with exceptional-point-based unique properties and functionalities.

1. Introduction

Hermiticity is required by plenty of physical models, if they are assumed to be energy conservative and time-reversal symmetric. However, non-Hermitian physics, as a counterpart of Hermitian physics [1, 2], has attracted a lot of interest in recent years, from quantum physics [3, 4] to optics or photonics [5–7]. Non-Hermitian phenomena have been revealed to be able to dramatically alter the properties of a system. One of the best known examples is the so-called exceptional points (EPs), which are branch point singularities in the parameter space of the system. More than one eigenvalues at the EPs and associated eigenvectors coalesce simultaneously and therefore, the system becomes degenerate. If the system holds the parity-time (PT) symmetry, the corresponding Hamiltonian can support purely real eigenvalue spectra [3]. Moreover, the PT-symmetric Hamiltonian can reach a spontaneously-broken regime through the phase transition, where eigenvalues become complex. Since the PT-symmetric systems require gain and loss channels, the photonic structures [8] are ideal platforms for exploring the non-Hermitian physics, for example, integrated photonic waveguides [9], coupled micro-resonators [10], optomechanical architectures [11], and so on [12, 13]. With these architectures, researchers have achieved coherent laser absorber [14], unidirectional invisibility [15], negative refraction [16], and anisotropic transmission resonances [17] at the EPs. The singularity of EPs can not only be used for mode discrimination in multimode laser cavity [18], but also provide a way for enhancing the sensitivity of optical structures with respect to external perturbations. This is due to that the frequency splitting of optical structures in the EP case has a sharp change as external perturbations increase a little. Accordingly, the optical structures in the EP case can work as a sensor with high sensitivity [19–21]. Such an enhance of sensitivity can be up to hundreds of times of those optical structures near conventional degeneracies, so-called diabolic points [21].
Topological photonics triggered by topological electronics [22] can simulate a number of electrical topological phenomena, such as quantum (spin) Hall effect [23], high-order topological insulators [24, 25], Weyl semimetals [26], photonic topological valley Hall effect [27], and so on [28, 29]. The remarkable application of topological photonic insulators (ToPIs) rests in the robustness of optical properties against perturbations. The topological optical interfaces are often used to design ideal waveguides for topological edge states (TESs) [30, 31] and the topological corners can work as the amazing optical cavities [24, 25]. Other types of robust optical devices have also been attempted, for example, topological lasers [32, 33] and perfect reflectors [34]. Accordingly, the ToPIs have attracted extensive attention [30, 35] and provide an extraordinary platform for exploring and understanding topological protection, as well as for the EPs in these topological systems [6, 36]. In recent years, non-Hermitian topological systems and phenomena have been widely studied [37–39] with the burgeoning topological field. Many types of photonic architectures have been proposed in theory or fabricated in experiment [32, 33, 36, 39–41], including one-dimensional non-Hermitian photonic lattice [6], evanescently coupled waveguides [36, 42], one-dimensional tight-binding model [43], and two dimensional periodic photonic crystals [44]. These works commonly contain gain and loss technically introduced.

The topology of the system can supply a protection for the EP against perturbations. However, these previous works commonly require additionally-introduced gain or loss, which make manufacturing process complicated. Accordingly, it may be beneficial for the EP to design an optical system without gain or loss. This inspires us to focus on the nontrivial EP in the topological photonic system, as a whole, without gain or loss. Here, we consider the topological waveguide (TW)–cavity coupled structure. The topological cavity contains two degenerate counter-propagation topological whispering-gallery modes (TWGMs). This structure is Hermitian as a whole, but the internal subsystem, i.e. the whispering-gallery cavity, can be regarded as non-Hermitian due to its energy exchange with the waveguide and the coupling between the two modes. In fact, trivial EPs widely exist in the systems without gain or loss, such as critical angle of the total internal reflection at the interface between two dielectric materials, cut-off frequency of a closed waveguide, and band edge of a photonic crystal [45]. The emergent EP in the non-Hermitian subsystem reveals the breaking of the $PT$ symmetry, reflected by the change of the transmission-dip number in the transmission spectra of the system in this work. Since the optical architecture is topological, the relevant EP should be robust against the system imperfections, such as waveguide bending, disorder, and rod missing.

### 2. Model and formulas

The architecture considered is composed of one TW and one topological whispering-gallery cavity (TWGC), see figure 1(a). They both are achieved by the topological interfaces of two topology-different photonic insulators. We define that the rightward-moving (leftward-moving) TES in the TW and the clockwise (counter clockwise) rotating TES in the TWGC carry the up (down) spin or pseudospin. Photons as carriers of information transporting in the bus waveguide can be effectively adjusted by the quantum emitters (coupled with the bus waveguide), for example, optical cavities, two-level atoms, and Jaynes–Cummings models [46–48]. The TWGC in figure 1(a) plays such a role. Owing to spin conservation, the rightward-moving (leftward-moving) TES in the bus TW only couples with the clockwise (counter clockwise) TWGM in the TWGC with the strength $V$. This coupled architecture is described by the following Hamiltonian [31, 49],

$$H = \sum_{\sigma = \pm} \left\{ \int dx \left[ \hat{\psi}_\sigma^\dagger(x) \hat{\omega}_\sigma (\hat{\psi}_\sigma(x) + \omega_\sigma \hat{c}_\sigma^\dagger \hat{c}_\sigma) \right] + \int dx V \delta(x) \left[ \hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma(x) + \hat{\psi}_\sigma^\dagger(x) \hat{c}_\sigma \right] \right\},$$

where $x$ is the coordinate along the bus waveguide and the TWGC is placed at the original point. For convenience, the Planck constant is set to be $\hbar = 1$ henceforth. $\sigma = +(-)$ denotes the spin-up (down) TESs. $\hat{c}_\sigma$ ($\hat{c}_\sigma^\dagger$) is the annihilation (creation) operator of the TWGM with spin $\sigma$ and eigenfrequency $\omega_\sigma$. Note that the clockwise and counter clockwise TWGMs are degenerate, between which the backscattering strength is measured by the parameter $g$. The value of $g$ can be controlled by designing the geometry of the TWGC. $\hat{\psi}_\sigma$ ($\hat{\psi}_\sigma^\dagger$) is the annihilation (creation) field operator for the TES with spin $\sigma$ in the bus TW, whose dispersion $\hat{\omega}_\sigma (\hat{\psi}_\sigma^\dagger (x))$ can be linearized as $\omega_\sigma (k) = \omega_\xi + \nu_\sigma (\sigma k - k)$, corresponding to $\omega_\xi (\sigma k_e) = \omega_\xi$ when $k = \sigma k_e$, $\nu_\sigma$ and $k$ are the group velocity and the wave vector of the TES, respectively. Here, the coupling between the bus TW and TWGC is taken as $\delta$-type with the strength $V$, expressed as $V \delta(x)$ [50, 51]. It is valid when the cavity width near the waveguide is much narrower than the wavelength of the guided light [49].
To derive the TES (photon) transmission, the following single-particle wave function for $H$ is adopted,

$$|\Psi(t)\rangle = \sum_{\sigma=\pm} \int dx \mathcal{W}_\sigma(x, t) \hat{\psi}_\sigma^\dagger(x)|0\rangle + \sum_{\sigma=\pm} \mathcal{C}_\sigma(t) \hat{c}_\sigma^\dagger|0\rangle$$

(2)

where $|0\rangle$ represents the vacuum state with zero photon in the TWGC or bus TW. $\mathcal{W}_\sigma(x, t)$ and $\mathcal{C}_\sigma(t)$ are the wave function of the pseudospin-$\sigma$ TES in the bus TW and the excitation amplitude of the pseudospin-$\sigma$ TWGM. Substituting equations (1) and (2) into the Schrödinger equation,

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle,$$

(3)

leads to the coupled equation set for $\mathcal{W}_\sigma(x, t)$ and $\mathcal{C}_\sigma(t)$ as follows:

$$i\frac{\partial}{\partial t} \mathcal{W}_+(x, t) = \tilde{\omega}_+ (-i\partial_x) \mathcal{W}_+(x, t) + V\delta(x)\mathcal{C}_+(t),$$

$$i\frac{\partial}{\partial t} \mathcal{W}_-(x, t) = \tilde{\omega}_- (-i\partial_x) \mathcal{W}_-(x, t) + V\delta(x)\mathcal{C}_-(t),$$

$$i\frac{\partial}{\partial t} \mathcal{C}_+(t) = \omega_+ \mathcal{C}_+(t) + g\mathcal{C}_-(t) + VV\mathcal{W}_+(0, t),$$

$$i\frac{\partial}{\partial t} \mathcal{C}_-(t) = \omega_- \mathcal{C}_-(t) + g\mathcal{C}_+(t) + VV\mathcal{W}_-(0, t).$$

(4)

In the steady state case, $\mathcal{W}_\sigma(x, t) = \mathcal{W}_\sigma(x)e^{-i\omega t}$ and $\mathcal{C}_\sigma(t) = \mathcal{C}_\sigma e^{-i\omega t}$ with the oscillation frequency $\omega$ and therefore, one can directly obtain the time-independent equation set for the time-independent functions of $\mathcal{W}_\sigma(x)$ and $\mathcal{C}_\sigma$ by making the transform,

$$i\frac{\partial}{\partial t} \mathcal{W}_\sigma(x, t) = \omega \mathcal{W}_\sigma(x)e^{-i\omega t},$$

$$i\frac{\partial}{\partial t} \mathcal{C}_\sigma(t) = \omega \mathcal{C}_\sigma e^{-i\omega t},$$

(5)

for the equation set (4).

Since the TWGC is placed at the original point, $\mathcal{W}_\sigma(x)$ can be constructed as the following form through the system reflection and transmission coefficients, $r$ and $t$, i.e.

$$\mathcal{W}_+(x) = e^{i\theta(x)}[r(-x) + t\theta(x)], \quad \mathcal{W}_-(x) = r e^{-i\theta(-x)},$$

(6)
where $\theta(x)$ is the unit step function. Using $\omega_\pm(-i\partial_\chi) = \omega_\chi - v_\chi k_\chi - i\sigma v_\chi \partial_x$, the transmission coefficient $t$ can be found as,

$$t = \frac{\left[ \omega - \left( \omega_\chi - i\frac{\Gamma}{2} \right) \right] \left[ \omega - \left( \omega_\chi + i\frac{\Gamma}{2} \right) \right] - g^2}{\left( \omega - \omega_\chi + i\frac{\Gamma}{2} \right)^2 - g^2}, \quad (7)$$

where $\Gamma \equiv V^2/\nu_\chi$ describes the effective coupling between the bus $TW$ and $TWGC$, see figure 1(a). The corresponding transmissivity reads

$$T = |t|^2. \quad (8)$$

The transmission spectra are plotted as functions of $g$ and $\Gamma$ in figures 1(b), (c), (e) and (f). One or two minimum transmission points (MTPs) can be observed in the frequency domain, determined by the ratio $\eta \equiv \Gamma/2g$. For $\eta < 1$ there are two MTPs at which the transmissivities are exactly equal to zero, while for $\eta > 1$ there is only one at which the transmissivity is greater than zero, see figures 1(c) and (f). Two regions are separated by the transition point of $\eta = 1$, as demonstrated by the grey dash-dotted curves in figures 1(b) and (e). Such a transition point, intuitively, is just the EP in the present topological architecture. It can be confirmed by mapping the numerator of equation (7) to the following two-level symmetric Hamiltonian,

$$H_{PT} = \begin{pmatrix} \omega_\chi + i\frac{\Gamma}{2} & g \\ g & \omega_\chi - i\frac{\Gamma}{2} \end{pmatrix}, \quad (9)$$

where $\Gamma$ measures the gain and loss rates of the two levels. The origins of the gain and loss can be argued as follows. Referred to figure 1(a), the clockwise (i.e. pseudospin-up) TWGM in the $TWGC$ gains excitation or energy from the incident rightward-moving pseudospin-up TES through their coupling, then transfers energy to the counter clockwise (i.e. pseudospin-down) TWGM due to the backscattering $g$, and finally the energy of the counter clockwise (i.e. pseudospin-down) TWGM is lost to the leftward-moving (i.e. pseudospin-down) TES. The rates for gain and loss are exactly equal to each other, namely, $\Gamma$, which is attributed to the spin conservation in the coupling between the bus $TW$ and $TWGC$ and the time-reversal symmetry of the whole system. The two modes in equation (9) are just the clockwise and counter clockwise TWGMs. A simple derivation for this effective Hamiltonian is provided in appendix A.

Though the whole coupled $TW$-$TWGC$ structure is Hermitian (see equation (1)), the TWGC as a subsystem could be non-Hermitian since it couples with the bus TW. The zero-transmission points in equation (7) are exactly determined by the eigenfrequencies of $H_{PT}$, reading as

$$\omega_\pm = \omega_\chi \pm \sqrt{g^2 - \frac{\Gamma^2}{4}}. \quad (10)$$

The evaluation of the imaginary parts of $\omega_\pm$ are given in figures 1(d) and (g). When $\eta$ increases from 0 to 1, the two eigenfrequencies coalesce and the EP is achieved at $\eta = 1$. In the region with $\eta > 1$, the $PT$ symmetry is spontaneously broken [45, 52], resulting in complex eigenfrequencies, see figures 1(d) and (g). The coupling with the TESs in the bus $TW$ is responsible for the emergence of the $PT$ symmetric Hamiltonian in equation (9), which confirms that the transition point at $\eta = 1$ is the EP in the present system. The EP is reflected by the coalescence of the two MTPs as $\eta$ increases from 0 to that greater than 1, see figures 1(b), (c), (e) and (f). Furthermore, the topology of the system would provide protection for the EP against perturbations. The achievement of the EP on a Hermitian topological platform could bring about convenience for optical applications. Next section will show a practical example for achieving the topological EP, based on ToPIs.

### 3. Achieve topological EPs

The ToPIs used here are the crystal-symmetry-protected systems [53, 54], based on the hexagonal air-rod lattice in the silicon plate with relative dielectric constant $\varepsilon_1 = 11.7$, whose unit cells are denoted in figures 2(a) and (b). The silicon-photonic crystals are widely fabricated in experiments with advanced micro/nano-processing technology [55]. If the central air rod is smaller than those around it, the lattice behaves as a $TrPI$, otherwise the lattice behaves as a topological one [31, 35], reflected by the band inversion of $p$ and $d$ orbitals at $\Gamma$ point, see figures 2(a) and (b). In simulations, the finite element method (FEM) based on the code of COMSOL Multiphysics is used to calculate these photonic bands, including the transport properties of the waveguide modes [31]. The $C_{6v}$ symmetry of the system leads to the degeneracy of the two $p$ or two $d$ bands at $\Gamma$ point, whose field distributions are plotted in figure 2(c). The $p_x$ and $p_y$
\((d_{xy} \text{ and } d_{x^2−y^2})\) orbitals are the bases of the two-dimensional irreducible representation \(E_1\) (\(E_2\)) of \(C_{6v}\). With them the pseudospin up and down states could be constructed as \([31, 35]\)

\[
p\pm = (p\pm \pm ip\pm) / \sqrt{2}, \quad d\pm = (d_{xy} \pm id_{x^2−y^2}) / \sqrt{2}. \tag{11}
\]

The corresponding time reversal operator is expressed as \(\mathcal{T} = −\sigma_y \mathcal{K}\) with complex conjugate operator \( \mathcal{K} \) and Pauli matrix \( \sigma_y \) operating on \( p_{\pm} \) and \( d_{\pm} \) \([31, 35]\). On the bases of \( p_{\pm} \) and \( d_{\pm} \), \( \mathcal{T}^2 = −1 \) is responsible for the nontrivial topology in figure 2(b). The topology of system can also be confirmed by analyzing the Chern numbers of the pseudospin up and down states, being \( C = \pm 1 \), while \( C = 0 \) for the trivial one, respectively \([35]\).

In order to match the band gaps the radii of the air rods for the TrPI and ToPI are set to be 0.32a for the black, 0.42a for the light blue, 0.45a for the red, and 0.35a for the green, where \( a \) is the distance between the adjacent rods, see the insets in figures 2(a) and (b). The band gap is \((0.2281, 0.2468)\) for the TrPI and is \((0.2265, 0.2453)\) for the ToPI (\( c \) is the speed of light in vacuum), matching well. The bands of the TESs for the zigzag and armchair interfaces between the TrPI and ToPI are shown in figures 2(d) and (e), where the rightward-moving (leftward-moving) TESs are assumed to carry the up (down) pseudospin. Owing to the breaking of the crystal-symmetry on the interfaces, there exists a gap at the cross point of the pseudospin-up and -down dispersions, which indicates that the crystal-symmetry could not provide perfect protection for the TES against perturbations. The value of this gap can be adjusted by controlling the geometry of the interface \([35]\). On the other hand, only when the frequencies of the perturbation-induced states fall into the bulk band gap, the perturbations would exert a strong influence on the transport of the TESs \([31]\). Consequently, these two aspects guarantee that the TESs constructed by the crystal-symmetry-protected topological systems are immune to most of common perturbations, such as waveguide bending, rod missing, and local disorder \([34]\). In fact, this characteristic will be used to adjust the backscattering strength, \( g \), between the clockwise and counter clockwise TWGMs. Figures 2(d) and (e) show that this gap value is much smaller for the armchair interface \((\sim 2.1 \times 10^{-3}c/a)\), than that for the zigzag one \((\sim 4.5 \times 10^{-3}c/a)\) in the present crystal-symmetry-protected topological systems.

The zigzag interface is designed as the bus TW, while the closed armchair one is used to achieve the TWGC, referred to figure 3(a). The air-rod number along one side of the TWGC is denoted as \( N \) and that between the TW and TWGC is \( L \). Owing to the pseudospin conservation, the incident pseudospin-up (reflected pseudospin-down) wave only couples with the clockwise (counter clockwise) TWGM, see figures 1(a) and 3(a). The coupling strength \( \Gamma \) between the bus TW and TWGC and the backscattering strength \( g \) between the two TWGMs both decrease with the increasing \( L \). The former is intuitive, since \( L \) measures the distance between the bus TW and TWGC. The later is attributed to that the bus TW shows

\[\nu = \frac{2\pi}{\sqrt{N^2+L^2}}\]
Figure 3. (a) Schematic of the TW-TWGC architecture. The radii of air rods are identical to those given in figure 2. (b)–(d) Transmission spectra of the TW-TWGC architecture for different $N$ and $L$. Hollow scatter dots are calculated by the FEM within COSMOL code and the solid lines present the theoretical fitting using equation (7) with the fitted parameters listed in table 1.

Table 1. Fitted parameters for the transmission spectra in figures 3(b)–(d) and corresponding $\eta = \Gamma/2g$. The $PT$ transition is at $\eta = 1$.

| $L$ | $N$  | $\omega_c/(c/a)$ | $\Gamma/(10^{-4}c/a)$ | $g/(10^{-4}c/a)$ | $\eta$ |
|-----|------|-------------------|------------------------|------------------|--------|
| 4   | 14   | 0.235789          | 0.418410               | 0.342990         | 0.61   |
|     | 23   | 0.236341          | 0.364198               | 0.262572         | 0.69   |
|     | 35   | 0.236613          | 0.311221               | 0.209549         | 0.74   |
| 3   | 14   | 0.235648          | 1.818314               | 1.021660         | 0.89   |
|     | 23   | 0.236242          | 1.505573               | 0.767392         | 0.98   |
|     | 35   | 0.236547          | 1.186634               | 0.557715         | 1.06   |
| 1   | 14   | 0.235364          | 5.864978               | 1.960157         | 1.50   |
|     | 23   | 0.236101          | 3.582460               | 1.122943         | 1.60   |
|     | 35   | 0.236475          | 2.539101               | 0.723024         | 1.76   |

weaker and weaker perturbation on the armchair interface in the TWGC as $L$ increases. On the other hand, as $N$ increases they both decrease also. The former originates from the extension of the TWGMs and the later is attributed to the overlap decrease of the TWGMs across the TWGC. These variations are not linear and therefore, $\eta$ is different for different $L$ and $N$, which can be confirmed by fitting the transmission spectra in figures 3(b)–(d) using equation (7). In addition, the eigenfrequency $\omega_c$ exhibits a blueshift for increasing $L$ and $N$, which is due to the decrease of the bus TW influence and that of the TWGM overlapping across the TWGC. In figures 3(b)–(d), three different $L$ and three different $N$ are adopted, for which the hollow scatters are calculated from the FEM within COMSOL code and the solid curves are their fitted ones by equation (7) with the parameters listed in table 1.

For $L = 4$ the spectra present two MTPs where the transmission is zero. Since $g$ decreases when $N$ increases, the two MTPs in the spectra become closer but do not coalesce, see figure 3(b), which tells that the $PT$ symmetry is satisfied, reflected by $\eta < 1$, see the rows with $L = 4$ in table 1. Obviously, the theoretical fitting is in good agreement with the numerical data from the FEM. As $L$ decreases to $L = 3$, both $\Gamma$ and $g$ increase, leading to the changes of the spectral line shapes, as well as $\eta$, see figure 3(c) and the rows with $L = 3$ in table 1. For the spectrum with $L = 3$ and $N = 23$ and that with $L = 3$ and $N = 35$, $\eta = 0.98$ and 1.06, respectively, both of which approach the EP point where $\eta = 1$, confirmed by the zero-transmission at the MTP in the spectra, see the red and green curves in figure 3(c).

Decreasing $L$ to $L = 1$, both $\Gamma$ and $g$ increase further, resulting in that the three spectra for $N = 14$, 23, and 35 all hold only one MTP, see figure 3(d). The non-zero transmission at the MTP tells that the $PT$ symmetry is broken, corresponding to $\eta > 1$. This is confirmed by the fitted parameters in the rows with $L = 1$ in table 1. The weak mismatch between the FEM data and theoretical fitting in figure 3(d) should be attributed to that the $\delta$-type interaction is not well enough to cover the coupling between the bus TW and TWGC when $L$ is very small. Instead, an extended one can be used, for example, Gaussian function [31, 49]. Considering complex coupling functions would bring about difficulties for analyzing the physics, the $\delta$-type coupling is always used here, whose availability is confirmed by the acceptable fitting results in figures 3(b)–(d). If fixing $N = 23$, one can definitely observe the $PT$ phase transition, reflected by the
number of the MTPs varies from two to one as \( L \) decreases from \( L = 4 \) to \( L = 1 \). Accordingly, the EP can be achieved in the present topological photonic system. Since the sensitivity-enhanced sensor is a typical application of EPs [56], it can also be realized by this TW-cavity coupled system. The degeneracy of the two TWGMs, that can be lifted by a small perturbation, plays a basic role of the sensor. When an external perturbation leads to a small change of the backscattering between the clockwise and counter clockwise TWGMs at the EP, the two TWGMs present a large frequency splitting, i.e. \( \Delta \omega \equiv \omega_+ - \omega_- \). The change of the backscattering is defined as \( \Delta g \equiv g_p - g \) where \( g \) is the original backscattering strength and \( g_p \) represents that after exerting perturbation. Let us take the cases of \( N = 23 \) with \( L = 4 (\eta < 1) \), \( 3 (\eta \sim 1) \), and \( 1 (\eta > 1) \) as examples. For them, the sensitivities \( \Delta \omega / \Delta g \) are 4, 32, and 0, respectively, where \( \Delta g = 1 \times 10^{-7} c / a \) is used. It is obvious that the situation near the EP (i.e. \( \eta \sim 1 \)) has a high sensitivity.

The role of topology is to ensure that the system is robust against local perturbations, such as rod missing, disorders and waveguide bending. Accordingly, such an EP should be immune to the local perturbations too, as long as the eigenfrequencies of their induced optical modes is not within the topological band gap [31, 34]. To show this, we take the case with \( N = 23 \) and \( L = 3 \) as an example. The architectures of the rod missing, waveguide bending, and local disorder are shown in figures 4(b)–(d), respectively, and the case without perturbation (i.e. the perfect case) is provided in figure 4(a) for comparison. The transmission spectra of these four cases are summarized in figure 4(e) where the black solid curve is identical to the red circle dot curve in figure 3(c) with \( \eta = 0.98 \). Obviously, the transmission spectra maintain the line shape near the transmission valley when the perturbations are introduced, comparing the red dashed, green dotted, and blue dash-dotted curves with the black solid one. The line shape did not change a lot on the whole with these local disorders, potentially decreasing the complexity of structure fabrication. Accordingly, the EP in this system is topologically protected and robust against the local perturbations. Note that for the crystalline symmetry protected ToPIs (see figure 2) the local perturbations can strongly influence the transmission spectra when the defect modes induced by local perturbations are in the band gap [31]. Otherwise, their influences are weak, even if the local perturbations are very strong. This is due to that the local perturbations do not change the system topology. For the trivial waveguides (constructed by two TrPIs), the local perturbations almost always break the transmission spectra, see one example in reference [34]. Accordingly, though the same EP proposed in this work can be intuitively expected in a straight precisely-designed trivial waveguide, it is not robust against the local perturbations.

4. Conclusion

To summarize, a nontrivial EP is achieved in a topological photonic system without introduced gain or loss technically, which reduces the complexity of fabrication process. The topological system is based on the hexagonal optical architecture, comprised of a topological photonic waveguide and a TWGC. The zigzag topological interface is taken as the topological bus waveguide and the closed armchair one is regarded as the TWGC. The effective coupling strength between the waveguide and cavity, \( I' \), plays the roles of gain and
loss rates in the subsystem of cavity, which guarantees the realization of EPs. The backscattering strength between the two TWGMs (clockwise and counter clockwise), \( g \), together determine the TES transmission. The parameter \( \eta = \Gamma/2g = 1 \) gives the EP. When \( \eta < 1 \), the parity-time symmetry of the system is satisfied, otherwise is broken. \( \Gamma \) and \( g \) can be adjusted by controlling the distance between the waveguide and cavity or the cavity size. The achievement of EPs in topological photonic systems paves a way for robust optical devices with exceptional-point-based unique properties and functionalities, for example, sensitivity-enhanced sensors.

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**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

**Appendix A. Derivation of \( H_{PT} \) in equation (9)**

For convenience, we transform the Hamiltonian in equation (1) into the momentum space, i.e.

\[
H = H_c + \int \frac{dk}{2\pi} \sum_{\sigma = \pm} \left[ \omega_{k\sigma} \hat{\psi}^\dagger_k \hat{\psi}_{k\sigma} + V \left( \hat{c}^\dagger_k \hat{\psi}_{k\sigma} + \hat{\psi}^\dagger_k \hat{c}_{k\sigma} \right) \right], \tag{A1}
\]

where \( H_c = \sum_{k} \omega_{k\sigma} \hat{c}^\dagger_k \hat{c}_k + g \left( \hat{c}^\dagger_k \hat{c}_{-k} + \hat{c}^\dagger_{-k} \hat{c}_k \right) \) and \( \hat{\psi}_{k\sigma} = \int dx \hat{\psi}_\sigma(x) e^{-ikx} \). Using equation (A1), one can find the motion equations for \( \hat{\psi}_{k\sigma} \) and \( \hat{c}_k \) as follows,

\[
\frac{d}{dt} \hat{\psi}_{k\sigma} = -i\omega_{k\sigma} \hat{\psi}_{k\sigma} + iV \hat{c}_k, \tag{A2}
\]

\[
\frac{d}{dt} \hat{c}_k = -i[\hat{c}_k, H_c] + iV \int \frac{dk}{2\pi} \hat{\psi}_{k\sigma}. \tag{A3}
\]

If the incident field takes \( \sigma = + \), using equation (A2) it is intuitive to express \( \hat{\psi}_{k+} \) at time \( t \) by its initial field \( \hat{\psi}_{k+}(0) \) at time \( t_0 \), i.e.

\[
\hat{\psi}_{k+}(t) = \hat{\psi}_{k+}(0) e^{-i\omega_{k+}(t-t_0)} - iV \int_{t_0}^{t} dt' \hat{c}_{k+}(t') e^{-i\omega_{k+}(t-t')}, \tag{A4}
\]

while \( \hat{\psi}_{k-} \) at time \( t \) can be expressed by its final field \( \hat{\psi}_{k-}(t_1) \) at time \( t_1 \),

\[
\hat{\psi}_{k-}(t) = \hat{\psi}_{k-}(t_1) e^{-i\omega_{k-}(t-t_1)} + iV \int_{t_1}^{t} dt' \hat{c}_{k-}(t') e^{-i\omega_{k-}(t-t')}. \tag{A5}
\]

Substituting equations (A4) and (A5) into equation (A3), one can obtain

\[
\frac{d}{dt} \hat{c}_{+} = -i[\hat{c}_{+}, H_c] - \frac{\Gamma}{2} \hat{c}_{+} - iV \int \frac{dk}{2\pi} \hat{\psi}_{k+}(0) e^{-i\omega_{k+}(t-t_0)}, \tag{A6}
\]

\[
\frac{d}{dt} \hat{c}_{-} = -i[\hat{c}_{-}, H_c] + \frac{\Gamma}{2} \hat{c}_{-} - iV \int \frac{dk}{2\pi} \hat{\psi}_{k-}(t_1) e^{-i\omega_{k-}(t-t_1)}. \tag{A7}
\]

after some algebraic derivations. Here, the following relations

\[
\int dt' e^{-i\omega_{k\sigma}(t-t')} = \frac{2\pi}{\omega_k} \delta(t-t'), \tag{A8}
\]

\[
\int_{t_0}^{t} dt' f(t') \delta(t-t') = \int_{t_0}^{t} dt' f(t') \delta(t-t') = \frac{1}{2} f(t).
\]

are used. If we cast the \( \Gamma \) terms in equations (A6) and (A7) into the Hamiltonian \( H_c \) and neglect the waveguide fields, one can immediately get the effective Hamiltonian \( H_{PT} \) in equation (9).
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