Function Computation through a Bidirectional Relay

Jithin Ravi and Bikash Kumar Dey

Abstract

We consider a function computation problem in a three node wireless network. Nodes A and B observe two correlated sources X and Y respectively, and want to compute a function \( f(X, Y) \). To achieve this, nodes A and B send messages to a relay node C at rates \( R_A \) and \( R_B \) respectively. The relay C then broadcasts a message to A and B at rate \( R_C \). We allow block coding, and study the achievable region of rate triples under both zero-error and \( \epsilon \)-error.

As a preparation, we first consider a broadcast network from the relay to A and B. A and B have side information X and Y respectively. The relay node C observes both X and Y and broadcasts an encoded message to A and B. We want to obtain the optimal broadcast rate such that A and B can recover the function \( f(X, Y) \) from the received message and their individual side information X and Y respectively. For a special case of \( f(X, Y) = X \oplus Y \), we show equivalence between \( \epsilon \)-error and zero-error computations—this gives a rate characterization for zero-error XOR computation in the broadcast network. As a corollary, this also gives a rate characterization for zero-error in the relay network when the support set of \( p_{XY} \) is full. For the relay network, the zero-error rate region for arbitrary functions is characterized in terms of graph coloring of some suitably defined probabilistic graphs. We then give a single-letter inner bound to this rate region. Further, we extend the graph theoretic ideas to address the \( \epsilon \)-error problem and obtain a single-letter inner bound.

Index Terms

Distributed source coding, function computation, zero-error information theory.

I. INTRODUCTION

Distributed computation of distributed data over a network has been investigated in various flavours for a long time. Gathering all the data at the nodes where a function needs to be computed is wasteful in most situations. So intermediate nodes also help by doing some processing of the data to reduce the communication load on the links. Such computation frameworks are known as distributed function computation or in-network function computation \([1]-[6]\).

This paper was presented in part at the IEEE Information Theory Workshop (ITW), Jerusalem, Israel, April 2015 and is accepted in part to the IEEE GLOBECOM NetCod 2016, Washington, DC, USA, December 2016. This work was supported by the Department of Science and Technology under grant SB/S3/EECE/057/2013 and by Information Technology Research Academy under grant ITRA/15(64)/Mobile/USEAADWN/01.

J. Ravi and B. K. Dey are with the Department of Electrical Engineering at IIT Bombay, Mumbai, INDIA-400076. Email: \{rjithin,bikash\}@ee.iitb.ac.in.
We consider the problem of function computation in a wireless relay network (RN) with three nodes as shown in Fig. 1. Nodes A and B have two correlated random variables $X$ and $Y$ respectively. They have infinite i.i.d. realizations of these random variables. They can communicate directly to a relay node C over orthogonal error-free links. The relay node C can send a message to both A and B over a noise-less broadcast link. Nodes A and B want to compute a function $f(X, Y)$. We allow block coding of arbitrarily large block length $n$. We allow two phases of communication. In the first phase, both A and B send individual messages to C at rates $R_A$ and $R_B$ over the respective orthogonal links. In the second phase, the relay broadcasts a message to A and B at rate $R_C$.

The broadcasting relay in the model captures one aspect of wireless networks. We consider our function computation problem over this network under zero-error and $\epsilon$-error criteria. Under zero-error, both nodes want to compute the function with no error. Under $\epsilon$-error, the probability of error in computing the function should go to zero as block length tends to infinity. A special case of this problem have been studied in [7], [8]. Exchanging $X$ and $Y$, equivalently computing the function $X \oplus Y$, was considered in [7], and the rate region was characterized in the $\epsilon$-error setting. For this problem, some single-letter inner and outer bounds were given for the rate-distortion function in [8].

As a preparation to address the problem in Fig. 1, we first consider the broadcast function network with complementary side information (BFN-CSI) shown in Fig. 2. This problem arises as a special case of the function computation problem in the relay network, when A and B communicate $X$ and $Y$ to the relay node. In the relay network, rate $R_C$ attains its minimum when the relay has $X$ and $Y$. So the optimal broadcast rate for the problem in Fig. 2 is the minimum possible rate $R_C$ in the relay network. For the broadcast function network, the optimal $\epsilon$-error rate can be shown to be $\max\{H(Z|Y), H(Z|X)\}$ using the Slepian-Wolf result. We study this problem under zero-error criteria.

The problem of zero-error source coding with receiver side information was first studied for fixed length coding by Witsenhausen in [9] using a “confusability graph” $G_{X|Y}$. The minimum rate was characterized in terms of the
chromatic number of its AND product graphs $G_{X,Y}^{\land n}$. The same problem was later considered in [10] under variable length coding, and the minimum rate was shown to be the limit of the normalized chromatic entropy of $G_{X,Y}^{\land n}$. This asymptotic rate was later shown [11] to be the complementary graph entropy [12] of $G_{X,Y}$. However, a single-letter characterization for complementary graph entropy is still unknown.

In the absence of a single-letter characterization of zero-error source coding problems, many authors have studied their problems under a stricter decoding requirement, known as the “unrestricted input” setup [10], [13], [14]. In this setup, even for a vector which has some zero-probability components (and thus the vector itself having probability 0), the decoder is required to reproduce the desired symbols for the other components of the vector. In all these problems, the unrestricted input setup is represented by the OR product of a suitable confusability graph. Protocols for the unrestricted input setup are clearly also valid protocols for the original zero-error decoding problem. We note that for our function computation problem in the relay network, the unrestricted input setup is not represented by the OR products of the confusability graph.

For distributed coding of two sources and joint decoding, a single-letter characterization was given for the unrestricted input version in [13]. Most related recent work to our present work is [14], where a decoder having side information $Z$ wants to compute a function $f(X,Y,Z)$ using a message encoded by a relay, which in turn receives two messages encoded by two sources $X$ and $Y$. Single-letter inner and outer bounds were given for the unrestricted input setup.

The problem of broadcast with side information has been addressed in the name of index coding (see [15]-[19] and references therein). Here, a server has access to $K$ binary independent and uniformly distributed random variables and the receivers have access to different subsets of these messages. Each receiver wants to recover an arbitrary subset of the messages using its side information and the message broadcasted by the server. The goal is to minimize the broadcast rate of the message sent by the server. A computable characterization of the optimum broadcast rate for the general index coding problem is still unknown.

For the broadcast function network in Fig. 1 we first consider the special case of $f(X,Y) = X \oplus Y$. This problem is equivalent to recovering $Y$ at node A and $X$ at node B, known as the complementary delivery problem,
and is an instance of the index coding problem with two messages. This problem has been addressed under noisy
broadcast channel in [20]–[22] for $\epsilon$-error recovery of the messages. Lossy version of this problem was studied in
[23], [24]. For the lossless case, the optimal $\epsilon$-error rate can be shown to be $\max\{H(Y|X), H(X|Y)\}$ using the
Slepian-Wolf result. We show in this paper that this rate is also achievable with zero-error. For any index coding
problem with independent and uniformly distributed messages, the equivalence between zero-error and $\epsilon$-error rates
has been shown in [25]. Our result extends this to correlated sources with arbitrary distribution in the specific
case of complementary delivery. The technique followed in [25] does not directly extend to correlated sources. For
arbitrary function $Z = f(X, Y)$, the optimal $\epsilon$-error rate is $\max\{H(Z|X), H(Z|Y)\}$ (using Slepian-Wolf result),
thus it is a lower bound for the optimal zero-error rate. We provide a single-letter upper bound for the optimal
zero-error rate.

For the relay network, we study the function computation problem under zero-error. Suitable graphs are defined
to address the problem. We first consider computing XOR at both the end nodes. Building on our results on the
broadcast function network, we give a single-letter characterization of the rate region for computing XOR when
the support set of $p_{XY}$ is full. For arbitrary functions, we study the problem under unrestricted input setup and
provide a multiletter characterization of the rate region. Then we provide a single-letter inner bound for this region,
which is also an inner bound for the zero-error problem.

Next, we consider the function computation problem in the relay network under $\epsilon$-error. For this problem, we
use the graph theoretic ideas developed for zero-error, to get a single-letter inner bound for the rate region.

A. Contributions and organization of the paper

We list the contributions of this paper below.

- For the zero-error function computation problem shown in Fig. [2] in Theorem [1] we give a single-letter
  characterization for the optimum broadcast rate for computing $X \oplus Y$ at nodes A and B, and for general
  functions, we give upper and lower bounds for the optimum rate. Using these results, we give a single-letter
  characterization of the rate region for computing XOR in the relay network (Fig. [1] when the support set of
  $p_{XY}$ is full. We then argue that when $X$ and $Y$ are independent, exchanging $(X, Y)$ in the relay network has
  the same rate region under zero-error and $\epsilon$-error.

- We consider the zero-error function computation problem in the relay network (Fig. [1]) under the unrestricted
  input setup. This setup is a more constrained version of the zero-error problem. We give a multiletter charac-
  terization of the rate region under this setup as well as for the zero-error problem (Theorem [2]). The multiletter
  characterization is obtained using coloring of some suitably defined graphs. Our arguments based on coloring
are similar to [14]. We show that if $p_{XY}$ has full support, then the relay can also compute the function if A and B can compute it with zero-error (Theorem 4).

- For the unrestricted input setup, we propose two achievable schemes whose time sharing gives a single-letter inner bound for the corresponding rate region (Theorem 3).

- The function computation problem in Fig. 1 is then addressed under $\epsilon$-error. We extend the graph theoretic ideas used for zero-error computation to $\epsilon$-error computation. Similar to the two achievable schemes for zero-error computation, we give an inner bound for the rate region using two achievable schemes for $\epsilon$-error computation (Theorem 5). The cutset outer bound is given in Lemma 3.

- For two functions $f_1, f_2$ of $(X,Y)$, we give a graph theoretic sufficient condition under which the rate region for computing $f_1$ is a subset of the rate region for computing $f_2$. This condition holds for both zero-error and $\epsilon$-error computations (Theorem 6). Using this result, we give a class of functions for which the rate region is the same as the region for exchanging $(X,Y)$ (equivalently, computing $X \oplus Y$).

The organization of the paper is as follows. Problem formulations for zero-error and $\epsilon$-error are given in Section II-A and in Section II-B respectively. Some graph theoretic definitions are given in Section II-C and some well known results used in this paper are described in Section II-D. We provide our results for zero-error computation in Section III-A. The $\epsilon$-error results are given in Section III-B. The proof of the results for zero-error computation and $\epsilon$-error computation are given in Section IV and Section V respectively. We conclude our paper in Section VI.
II. PROBLEM FORMULATION AND PRELIMINARIES

A. Zero-error function computation

Nodes A and B observe X and Y respectively from finite alphabet sets \( \mathcal{X} \) and \( \mathcal{Y} \). \((X, Y)\) have a joint distribution \( p_{XY}(x, y) \), and their different realizations are i.i.d. In other words, \( n \) consecutive realizations \((X^n, Y^n)\) are distributed as \( Pr(x^n, y^n) = \prod_{i=1}^{n} p_{XY}(x_i, y_i) \) for all \( x^n = (x_1, x_2, \ldots, x_n) \) and \( y^n = (y_1, y_2, \ldots, y_n) \).

The support set of \((X, Y)\) is defined as

\[
S_{XY} = \{(x, y) : p_{XY}(x, y) > 0\}.
\]

On observing \( X^n \) and \( Y^n \) respectively, A and B send messages \( M_A \) and \( M_B \) using prefix free codes such that \( E|M_A| = nR_A \) and \( E|M_B| = nR_B \). Here \(|.|\) denotes the length of the respective message in bits. C then broadcasts a message \( M_C \) with \( E|M_C| = nR_C \) to A and B. Each of A and B then decode \( f(X_i, Y_i); i = 1, 2, \ldots, n \) from the information available to them. For the relay network, a \((2^{nR_A}, 2^{nR_B}, 2^{nR_C}, n)\) variable length scheme consists of three encoders

\[
\phi_A : \mathcal{X}^n \rightarrow \{0, 1\}^*, \quad \phi_B : \mathcal{Y}^n \rightarrow \{0, 1\}^*, \quad \phi_C : \phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n) \rightarrow \{0, 1\}^*,
\]

and two decoders

\[
\psi_A : \mathcal{X}^n \times \phi_C(\phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n)) \rightarrow \mathcal{Z}^n,
\]

\[
\psi_B : \mathcal{Y}^n \times \phi_C(\phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n)) \rightarrow \mathcal{Z}^n.
\]

Here \( \{0, 1\}^* \) denotes the set of all finite length binary sequences. Let us define \( \hat{Z}_A^n = \psi_A(X^n, \phi_C(\phi_A(X^n), \phi_B(Y^n))) \) and \( \hat{Z}_B^n = \psi_B(Y^n, \phi_C(\phi_A(X^n), \phi_B(Y^n))) \) to be the decoder outputs. The probability of error for a \( n \) length scheme is defined as

\[
P_e^{(n)} \triangleq Pr\{(\hat{Z}_A^n, \hat{Z}_B^n) \neq (Z^n, Z^n)\}.
\]

The rate triple \((R_A, R_B, R_C)\) of a code is defined as

\[
R_A = \frac{1}{n} \sum_{x^n} Pr(x^n) \mid \phi_A(x^n) \mid
\]

\[
R_B = \frac{1}{n} \sum_{y^n} Pr(y^n) \mid \phi_B(y^n) \mid
\]

\[
R_C = \frac{1}{n} \sum_{(x^n, y^n)} Pr(x^n, y^n) \mid \phi_C(\phi_A(x^n), \phi_B(y^n)) \mid.
\]

A rate triple \((R_A, R_B, R_C)\) is said to be achievable with zero-error if for any \( \epsilon > 0 \), there exists a scheme for
a large enough $n$ such that $\frac{1}{n}E|M_A| \leq R_A + \epsilon, \frac{1}{n}E|M_B| \leq R_B + \epsilon$ and $\frac{1}{n}E|M_C| \leq R_C + \epsilon$. The rate region $\mathcal{R}^0(f, X, Y)$ is the closure of the convex hull of all achievable rate triples. The above setup is known as restricted input setup in the literature.

We now define the function computation in the relay network under a stricter setting, known as the unrestricted input setup. A $(2^{nR_A}, 2^{nR_B}, 2^{nR_C}, n)$ code for unrestricted input setup consists of three encoders and two decoders which are defined as before. Let $(\psi_A(\cdot))_i$ and $(\psi_B(\cdot))_i$ denote the $i$-th components of $\psi_A(\cdot)$ and $\psi_B(\cdot)$ respectively. A scheme is called a unrestricted input scheme if for each $x^n \in X^n, y^n \in Y^n$, and $i = 1, 2, \cdots, n$,

\[
(\psi_A(x^n, \phi_C(\phi_A(x^n), \phi_B(y^n))))_i = f(x_i, y_i)
\]

and

\[
(\psi_B(y^n, \phi_C(\phi_A(x^n), \phi_B(y^n))))_i = f(x_i, y_i)
\]

if $(x_i, y_i) \in S_{XY}$. Note that this is a stricter condition than $P_e^{(n)} = 0$. A pair of vectors $(x^n, y^n)$ for which a component $(x_i, y_i)$ is outside the support set $S_{XY}$, does not contribute to $P_e^{(n)}$, and thus in the original zero-error problem setup, the decoders are also not required to correctly compute the other components. However, the unrestricted setup requires the decoders to compute the function correctly on all the components where $(x_i, y_i) \in S_{XY}$. Achievable rates and the rate region $\mathcal{R}^{(u)}(f, X, Y)$ under the unrestricted setup are defined similarly as before.

For the broadcast function network shown in Fig. 2, a variable length code for the function computation problem consists of one encoder

\[
\phi_C : X^n \times Y^n \rightarrow \{0, 1\}^n,
\]

and two decoders

\[
\psi_A : \phi_C(X^n \times Y^n) \times X^n \rightarrow Z^n, \tag{4}
\]
\[
\psi_B : \phi_C(X^n \times Y^n) \times Y^n \rightarrow Z^n. \tag{5}
\]

The rate of a code is defined as $\frac{1}{n} \sum_{(x^n, y^n)} \Pr(x^n, y^n) |\phi_C(x^n, y^n)|$, and the outputs of the decoders are given by $\hat{Z}_A^n = \psi_A(X^n, \phi_C(X^n, Y^n))$ and $\hat{Z}_B^n = \psi_B(Y^n, \phi_C(X^n, Y^n))$. A rate $R$ is said to be achievable with zero-error if there is a code of some length $n$ with rate $R$ and $P_e^{(n)} \triangleq \Pr\{|\hat{Z}_A^n, \hat{Z}_B^n) \neq (Z^n, Z^n)\} = 0$. Let $R_0^n$ denote the minimum achievable rate for $n$ length zero-error codes. Then the optimal zero-error rate $R_0^*$ is defined as $R_0^* = \lim_{n \rightarrow \infty} R_0^n$. 

B. $\epsilon$-error function computation

A fixed length $(2^{nR_A}, 2^{nR_B}, 2^{nR_C}, n)$ code for function computation in the relay network consists of three encoder maps

\[
\phi_A : \mathcal{X}^n \rightarrow \{1, 2, \ldots, 2^{nR_A}\},
\phi_B : \mathcal{Y}^n \rightarrow \{1, 2, \ldots, 2^{nR_B}\},
\phi_C : \phi_A(\mathcal{X}^n) \times \phi_B(\mathcal{Y}^n) \rightarrow \{1, 2, \ldots, 2^{nR_C}\}
\]

and two decoder maps as defined in (1), (2). A rate triple $(R_A, R_B, R_C)$ is said to be achievable with $\epsilon$-error if there exists a sequence of $(2^{nR_A}, 2^{nR_B}, 2^{nR_C}, n)$ codes such that probability of error $P_e(n) \rightarrow 0$ as $n \rightarrow \infty$. The achievable rate region $\mathcal{R}(f, x, y)$ is the closure of the convex hull of all achievable rate triples.

For the broadcast function network, a $(2^{nR}, n)$ code consists of one encoder map

\[
\phi_C : \mathcal{X}^n \times \mathcal{Y}^n \rightarrow \{1, 2, \ldots, 2^{nR}\}
\]

and the two decoder maps as defined in (4), (5). A rate $R$ is said to be achievable with $\epsilon$-error if there exists a sequence of $(2^{nR}, n)$ codes for which $P_e(n) \rightarrow 0$ as $n \rightarrow \infty$. The optimal broadcast rate $R^*_\epsilon$ in this case is the infimum of the set of all achievable rates.

C. Graph theoretic definitions

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $I \subseteq V(G)$ is called an independent set if no two vertices in $I$ are adjacent in $G$. Let $\Gamma(G)$ denote the set of all independent sets of $G$. A clique of a graph $G$ is a complete subgraph of $G$. A clique of the largest size is called a maximum clique. The number of vertices in a maximum clique is called clique number of $G$ and is denoted by $\omega(G)$. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number of colors required to color the graph $G$. A graph $G$ is said to be perfect if for any vertex induced subgraph $G'$ of $G$, $\omega(G') = \chi(G')$. Note that the vertex disjoint union of perfect graphs is also perfect.

The $n$-fold OR product of $G$, denoted by $G^{\lor n}$, is defined by $V(G^{\lor n}) = (V(G))^n$ and $E(G^{\lor n}) = \{(v^n, v'^n) : (v_i, v'_i) \in E(G) \text{ for some } i\}$. The $n$-fold AND product of $G$, denoted by $G^{\land n}$, is defined by $V(G^{\land n}) = (V(G))^n$ and $E(G^{\land n}) = \{(v^n, v'^n) : v^n \neq v'^n, \text{ and either } v_i = v'_i \text{ or } (v_i, v'_i) \in E(G) \text{ for all } i\}$.

For a graph $G$ and a random variable $X$ taking values in $V(G)$, $(G, X)$ represents a probabilistic graph. Chromatic entropy \[10\] of $(G, X)$ is defined as

\[
H_\chi(G, X) = \min \{H(c(X)) : c \text{ is a coloring of } G\}.
\]
Let \( W \) be distributed over the power set \( 2^X \). The graph entropy \([26], [27]\) of the probabilistic graph \((G, X)\) is defined as

\[
H_G(X) = \min_{X \in W, X \in \Gamma(G)} I(W; X),
\]

where \( \Gamma(G) \) is the set of all independent sets of \( G \). Here the minimum is taken over all conditional distributions \( p_{W|X} \) which are non-zero only for \( X \in W \). The following result was shown in [10].

\[
\lim_{n \to \infty} \frac{1}{n} H_X(G^\land n, X^n) = H_G(X).
\]

The complementary graph entropy of \((G, X)\) is defined as

\[
\bar{H}_G(X) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log_2 \{\chi(G^\land n, X^n)\},
\]

where \( T_{P_X, \epsilon}^n \) denotes the \( \epsilon \)-typical set of length \( n \) under the distribution \( P_X \). Unlike graph entropy, no single-letter characterization of the complementary graph entropy is known. It was shown in [11] that

\[
\lim_{n \to \infty} \frac{1}{n} H_X(G^\land n, X^n) = \bar{H}_G(X).
\]

The definition of graph entropy was extended to the conditional graph entropy in [3]. For a pair of random variables \((X, Y)\) and for a graph \( G \) defined on the support set of \( X \), the conditional graph entropy of \( X \) given \( Y \) is defined as

\[
H_G(X|Y) = \min_{W - X - Y \in \Gamma(G)} I(W; X|Y),
\]

where the minimization is over all conditional distribution \( p_{W|X,Y} = p_{W|X,Y} \) which is non-zero only for \( X \in W \).

We now define some graphs suitable for addressing our problem. For a function \( f(x, y) \) defined over \( X \times Y \), we define a graph called \( f \)-modified rook’s graph. A rook’s graph \( G_{XY} \) over \( X \times Y \) is defined by the vertex set \( X \times Y \) and edge set \( \{(x, y), (x', y') : x = x' \text{ or } y = y', \text{ but } (x, y) \neq (x', y')\} \).

**Definition 1** For a function \( f(x, y) \) the \( f \)-modified rook’s graph \( G_{X,Y}^f \) has its vertex set \( X \times Y \), and two vertices \((x_1, y_1) \) and \((x_2, y_2) \) are adjacent if and only if i) they are adjacent in the rook’s graph \( G_{X,Y} \), ii) \((x_1, y_1), (x_2, y_2) \in S_{XY} \), and iii) \( f(x_1, y_1) \neq f(x_2, y_2). \)

\( f \)-confusability graph \( G_{X|Y}^f \) of \( X,Y \) and \( f \) was used in [3], [14] to study some function computation problems. Its vertex set is \( X \), and two vertices \( x \) and \( x' \) are adjacent if and only if \( \exists \ y \in Y \) such that \( f(x, y) \neq f(x', y) \) and \((x, y), (x', y) \in S_{XY} \). \( G_{Y|X}^f \) is defined similarly.
Example 1 Let us consider $X, Y \in \{0, 1, 2, 3, 4\}$ with distribution

$$p(x, y) = \begin{cases} \frac{1}{10} & \text{if } y = x \text{ or } y = x + 1 \text{ mod } 5, \\ 0 & \text{otherwise} \end{cases},$$

and the equality function

$$f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \tag{10}$$

The $f$-modified rook’s graph for this function is shown in Fig. 3a. Both $G_{X|Y}^f$ and $G_{Y|X}^f$ are the pentagon graph which is shown in Fig. 3b.

Next we extend the definition of $G_{XY}^f$ to $n$ instances:

Definition 2 $G_{XY}^f(n)$ has its vertex set $X^n \times Y^n$, and two vertices $(x^n, y^n)$ and $(x'^n, y'^n)$ are adjacent if and only if

(i) $x^n = x'^n$ or $y^n = y'^n$,

(ii) $Pr(x^n, y^n).Pr(x'^n, y'^n) > 0$,

(iii) $\exists$ an $i \in [1, \ldots, n]$ such that $f(x_i, y_i) \neq f(x'_i, y'_i)$.

To address the unrestricted input setup, we define the following graph for $n$ instances.

Definition 3 $G_{XY}^{f,(n)}$ has its vertex set $X^n \times Y^n$, and two vertices $(x^n, y^n)$ and $(x'^n, y'^n)$ are adjacent if and only if

(i) $x^n = x'^n$ or $y^n = y'^n$,

(ii) $\exists$ an $i \in [1, \ldots, n]$ such that $f(x_i, y_i) \neq f(x'_i, y'_i)$ and $(x_i, y_i), (x'_i, y'_i) \in S_{XY}$.

It is easy to see that the graph $G_{XY}^f(n)$ is a subgraph of $G_{XY}^{f,(n)}$. Note that for $n = 1$, these two graphs are the same.
We define the chromatic entropy region of a $f$-modified rook’s graph $G_{XY}^f$. If $c_1$ and $c_2$ are two maps of $\mathcal{X}$ and $\mathcal{Y}$ into $\{0,1\}^*$ respectively, then $c_1 \times c_2$ denotes the map given by $(c_1 \times c_2)(x,y) = (c_1(x),c_2(y))$.

A triple $(c_A,c_B,c_C)$ of functions defined over $\mathcal{X},\mathcal{Y},\mathcal{X} \times \mathcal{Y}$ respectively is called a color cover for $G_{XY}^f$ if

i) $c_A \times c_B$ and $c_C$ are colorings of $G_{XY}^f$

ii) $c_A \times c_B$ is a refinement of $c_C$, i.e., $\exists$ a mapping $\theta : (c_A \times c_B)(\mathcal{X} \times \mathcal{Y}) \to c_C(\mathcal{X} \times \mathcal{Y})$ such that $\theta \circ (c_A \times c_B) = c_C$.

**Chromatic entropy region** $R_X(G_{XY}^f,X,Y)$ of $G_{XY}^f$ is defined as

$$R_X(G_{XY}^f,X,Y) \triangleq \bigcup_{(c_A,c_B,c_C)} \{(b_A,b_B,b_C) : b_A \geq H(c_A(X)), b_B \geq H(c_B(Y)), b_C \geq H(c_C(X,Y))\},$$

where the union is taken over all color covers of $G_{XY}^f$. From the chromatic entropy region, we define the following three dimensional regions

$$Z_{X,Y}^f \triangleq \bigcup_n \frac{1}{n} R_X(G_{XY}^f(n),X^n,Y^n),$$

$$Z_{X,Y}^{f(u)} \triangleq \bigcup_n \frac{1}{n} R_X(G_{XY}^{f(u)}(n),X^n,Y^n).$$

(D. Some known lemmas)

We use the notion of robust typicality [3] in the following. For $x^n \in A^n$, let us denote the number of occurrences of $x \in A$ in $x^n$ by $N(x|x^n)$. The set of sequences $x^n \in A^n$ satisfies

$$\left| \frac{1}{n} N(x|x^n) - p(x) \right| \leq \epsilon p(x)$$

for $\epsilon > 0$, is called $\epsilon$-robustly typical sequences and is denoted by $T^\epsilon_n(X)$. Next we present some known lemmas which are used in our achievable schemes.

**Lemma 1** (Covering Lemma, [3]). Let $(U,X,\hat{X}) \sim p(u,x,\hat{x})$ and $\epsilon' \leq \epsilon$. Let $(U^n,X^n) \sim p(u^n,x^n)$ be a pair of random sequences with $\lim_{n \to \infty} P\{(U^n,X^n) \in T^{\epsilon'}_n(U,X)\} = 1$, and let $\hat{X}^n(m), m \in A$, where $|A| \geq 2^{nR}$, be random sequences, conditionally independent of each other and of $X^n$ given $U^n$, each distributed according to $\prod_{i=1}^n p_{\hat{X}|U}(\hat{x}_i|u_i)$. Then, there exists $\delta(\epsilon)$ that tends to zero as $\epsilon \to 0$ such that

$$\lim_{n \to \infty} P\{(U^n,X^n,\hat{X}^n(m)) \notin T^\epsilon_n \text{ for all } m \in A\} = 0,$$

if $R > I(X;\hat{X}|U) + \delta(\epsilon)$.

**Lemma 2** (Markov Lemma, [3]). Suppose that $X \to Y \to Z$ form a Markov chain. Let $(x^n,y^n) \in T^\epsilon_n(X,Y)$, and $Z^n \sim p(z^n|y^n)$, where the conditional pmf $p(z^n|y^n)$ satisfies the following conditions:
1) \[ \lim_{n \to \infty} P\{(y^n, Z^n) \in T^n_{\epsilon}(Y, Z)\} = 1. \]

2) For every \( z^n \in T^n_{\epsilon}(Z|y^n) \) and \( n \) sufficiently large

\[ 2^{-n(H(Z|Y)+\delta(\epsilon'))} \leq p(z^n|y^n) \leq 2^{-n(H(Z|Y)-\delta(\epsilon'))} \]

for some \( \delta(\epsilon') \) that tends to zero as \( \epsilon' \to 0 \).

Then, for some sufficiently small \( \epsilon' < \epsilon \),

\[ \lim_{n \to \infty} P\{(x^n, y^n, Z^n) \in T^n_{\epsilon}(X, Y, Z)\} = 1. \]

III. RESULTS

A. Results for zero-error computation

We first give the results for the broadcast function computation problem shown in Fig. 2. We show that the optimal rate under zero-error and \( \epsilon \)-error are the same for computing XOR (equivalently for the complementary delivery problem) in Fig. 2. Then we give upper and lower bounds for the zero-error optimal rate for general functions. Proofs of all the theorems in this subsection are given in Section IV.

**Theorem 1 (BFN-CSI)** For the broadcast function computation problem with complementary side information (Fig. 2),

(a) [XOR function] The optimal zero-error broadcast rate for computing XOR is given by,

\[ R^*_0 = \max\{H(Y|X), H(X|Y)\}. \]

(b) [Arbitrary function] For any function \( f(X, Y) = Z \), the optimal zero-error rate \( R^*_0 \) satisfies

\[ \max\{H(Z|X), H(Z|Y)\} \leq R^*_0 \leq H_{G_{XY}}(X, Y). \]

Using Theorem 1 we get a single-letter characterization for computing XOR in the relay network (Fig. 1) when the support set \( S_{XY} \) is the full set.

**Corollary 1 (XOR in RN)** If \( S_{XY} = \mathcal{X} \times \mathcal{Y} \), then the zero-error rate region for computing XOR at nodes A and B (equivalently exchanging X and Y) in the relay network is given by

\[ \mathcal{R}^0(f, x, y) \triangleq \{(R_A, R_B, R_C) : R_A \geq H(X), R_B \geq H(Y), R_C \geq \max\{H(Y|X), H(X|Y)\}\}. \]

* In Section IV before proving Theorem 1 we argue that the scheme of binning which achieves the optimal \( \epsilon \)-error rate does not work with zero-error.
We note that the problem of exchanging $X$ and $Y$ through a relay has been addressed in [7] under $\epsilon$-error criteria. The rate region for this problem under the $\epsilon$-error criteria is given by

$$\{(R_A, R_B, R_C) : R_A \geq H(X|Y), R_B \geq H(Y|X), R_C \geq \max\{H(Y|X), H(X|Y)\}\}. \quad (14)$$

When the sources are independent, the rate regions are clearly the same under $\epsilon$-error and zero-error criteria. When the sources are dependent with full support, smaller rates are possible for $R_A$ and $R_B$ under $\epsilon$-error compared to zero-error. Even in this case, the minimum possible rate for $R_C$ is the same in both the cases.

**Theorem 2 (RN, multiletter characterization)**  
(a) The zero-error rate region is given by $R^0(f, X, Y) = Z^f_{X,Y}$.  
(b) The rate region under unrestricted input setup is given by $R^{(u)}(f, X, Y) = Z^{f,(u)}_{X,Y}$, where $Z^f_{X,Y}$ and $Z^{f,(u)}_{X,Y}$ are as defined in (11) and (12) respectively.

Since a scheme under the unrestricted input setup is also a zero-error scheme, $R^{(u)}(f, X, Y) \subseteq R^0(f, X, Y)$. The multi letter expressions for the rate regions given in Theorem 2 are difficult to compute. We give a single-letter inner bound for $R^{(u)}(f, X, Y)$ in Theorem 3. This is achieved by time sharing between two schemes as discussed in the proof. We first give some definitions.

**Definition 4** Let $U_1$ and $U_2$ be two random variables such that $X \in U_1 \in \Gamma(G^f_{X|Y})$ and $Y \in U_2 \in \Gamma(G^f_{Y|X})$. The random variable $(U_1, U_2)$ over $U_1 \times U_2$ has joint distribution with $(X, Y)$ as $p_{X,U_1,Y,U_2}(x, u_1, y, u_2) = p(x, y)p(u_1|x)p(u_2|y)$. We define a graph $G^f_{U_1,U_2}$ with vertex set $U_1 \times U_2$. Two vertices $(u_1, u_2)$ and $(u'_1, u'_2)$ in $G^f_{U_1,U_2}$ are connected if $\exists (x, y)$ and $(x', y')$ such that

1) $p_{X,U_1,Y,U_2}(x, u_1, y, u_2), p_{X,U_1,Y,U_2}(x', u'_1, y', u'_2) > 0$,  
2) $x = x'$, $u_1 = u'_1$ and $f(x, y) \neq f(x', y')$

or

$y = y'$, $u_2 = u'_2$ and $f(x, y) \neq f(x', y').$

Note that by Definition 4 two nodes $(u_1, u_2)$ and $(u'_1, u'_2)$ are connected in $G^f_{U_1,U_2}$ only if either $u_1 = u'_1$ or $u_2 = u'_2$, i.e., all connections are either row wise or column wise. $G^f_{U_1,U_2}$ can also be viewed as a $f$-modified rook’s graph on the vertex set $U_1 \times U_2$. Next we give an example to illustrate the above definitions. The function in Example 2 was used in [3] to explain the conditional graph entropy. Let us consider the same function for our function computation problem in the relay network.

**Example 2**  
Consider $X, Y \in \{1, 2, 3\}$

$$p(x, y) = \begin{cases} 
\frac{1}{6} & \text{if } x \neq y \\
0 & \text{otherwise}
\end{cases}$$
and

\[ f(x, y) = \begin{cases} 
1 & \text{if } x > y \\
0 & \text{if } x \leq y.
\end{cases} \]

Both the confusability graphs are the same graph which is shown in Fig. 4a. The \( f \)-modified rook’s graph for this function is shown in Fig. 4b.

In Example 2, the distribution of \((X, Y)\) is symmetric in \(X\) and \(Y\) and the function values are also symmetric. For this example, let us consider an instance of \(U_1\) and \(U_2\) as follows. Let \(U_1 = \{1, 2\}, \{2, 3\}\) and let us denote it by \(\{a, b\}\) where \(a = \{1, 2\}\) and \(b = \{2, 3\}\). Similarly, we choose \(U_2\) and we denote it by \(\{c, d\}\), where \(c = \{1, 2\}\) and \(d = \{2, 3\}\). The conditional distributions are given by \(p_{U_1|X}(a|2) = p_{U_1|X}(b|2) = p_{U_2|Y}(c|2) = p_{U_2|Y}(d|2) = \frac{1}{2}\).

Now let us consider the graph \(G^f_{U_1,U_2}\) for this function. The nodes \((a, c)\) and \((a, d)\) are connected in \(G^f_{U_1,U_2}\) because \(p_{XU_1YU_2}(2, a, 1, c), p_{XU_1YU_2}(2, a, 3, d) > 0\) and \(f(2, 1) \neq f(2, 3)\). By considering other pairs of nodes in \(G^f_{U_1,U_2}\), we can verify that the graph \(G^f_{U_1,U_2}\) is a “square” graph which is shown in Fig. 4c.

**Theorem 3 (RN, zero-error inner bound)** (a) Let

\[ \mathcal{R}_{I1} \triangleq \{(R_A, R_B, R_C) : R_A \geq I(X; U_1|Q), R_B \geq I(Y; U_2|Q), R_C \geq I(W; U_1, U_2|Q)\} \]

for some \(p(q) p(w|u_1, u_2, q)p(u_1|x, q)p(u_2|y, q)\) such that

(i) \(X \in U_1 \in \Gamma(G^f_{X|Y})\)

(ii) \(Y \in U_2 \in \Gamma(G^f_{Y|X})\)

(iii) \((U_1, U_2) \in W \in \Gamma(G^f_{U_1U_2}).\)

Let \(\mathcal{R}_{I2} = \{(R_A, R_B, R_C) : R_A \geq H_{G^f_{X|Y}}(X), R_B \geq H_{G^f_{Y|X}}(Y), R_C \geq \max\{H_{G^f_{X|Y}}(X), H_{G^f_{Y|X}}(Y)\}\} \).

Let \(\mathcal{R}_I\) be the convex closure of \(\mathcal{R}_{I1} \cup \mathcal{R}_{I2}\). Then \(\mathcal{R}_I \subseteq \mathcal{R}^{(a)}(f, x, y)\).

(b) Neither of \(\mathcal{R}_{I1}\) and \(\mathcal{R}_{I2}\) is a subset of the other in general.
The proof of Theorem 3 is given in Section IV-C. To prove part (b), it is shown that for the function computation problem in Example 1, $(\log 5, \log 5, \log 2) \in \mathcal{R}_{I_1} - \mathcal{R}_{I_2}$, and for the function computation problem in Example 2, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in \mathcal{R}_{I_2} - \mathcal{R}_{I_1}$.

Next we provide a sufficient condition on the joint distribution $p_{XY}$ under which the relay can also compute the function whenever nodes A and B compute it with zero-error.

**Theorem 4 (RN, relay’s knowledge)** If $p(x, y) > 0 \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, then for any zero-error scheme the relay can also compute the function with zero-error.

Theorem 4 does not hold if $S_{XY} \neq \mathcal{X} \times \mathcal{Y}$. We show an instance of encoding for the function given in Example 2 to demonstrate this. Let $\phi_A, \phi_B$ and $\phi_C$ be as follows.

$$\phi_A = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_B = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_C = \begin{cases} 1 & \text{if } \phi_A = \phi_B \\ 0 & \text{otherwise.} \end{cases}$$

Here nodes A and B recover the function with zero-error, but the relay can not reconstruct the function. When $\phi_A = \phi_B = 0 ((x, y)$ is either $(2, 3)$ or $(3, 2))$, the function value can be both 0 and 1. So $H(f|\phi_A, \phi_B) > 0$.

**B. Results for $\epsilon$-error computation**

In this section, we give our results for $\epsilon$-error function computation in the relay network (RN). In general, the rate region for computing a function in RN with $\epsilon$-error is strictly larger than the rate region for computing the function with zero-error. We give an explicit example below to show this fact. Proofs of all the theorems in this subsection are given in Section V.

**Example 3** Let us consider computing $X \oplus Y$ for a doubly symmetric binary source (DSBS($p$)) $(X, Y)$ where $p_{X,Y}(0,0) = p_{X,Y}(1,1) = (1-p)/2$ and $p_{X,Y}(0,1) = p_{X,Y}(1,0) = p/2$. From Corollary 1 we have the zero-error rate region as $\{(R_A, R_B, R_C) : R_A \geq 1, R_B \geq 1, R_C \geq H(p)\}$. As noted before, computing $X \oplus Y$ in the relay network is same as exchanging $X$ and $Y$. The $\epsilon$-error rate region for exchanging $X$ and $Y$ through the relay is given in (14). Computing this for DSBS($p$) $(X, Y)$ gives the rate region as $\{(R_A, R_B, R_C) : R_A, R_B, R_C \geq H(p)\}$. 
For arbitrary functions, we do not have a single-letter characterization for the \( \epsilon \)-error rate region. Next lemma gives a cutset outer bound for the \( \epsilon \)-error rate region.

**Lemma 3** (a) [Cutset outer bound] Any achievable rate triple \( (R_A, R_B, R_C) \in \mathcal{R}^\epsilon(f, X, Y) \) for RN satisfies the following.

\[
R_A \geq H_{G'_{\chi|Y}}(X|Y), \quad R_B \geq H_{G'_{\gamma|x}}(Y|X), \quad R_C \geq \max\{H(Z|X), H(Z|Y)\}. \tag{15}
\]

(b) Equality in (15) can be achieved individually for either \( R_A, R_B \) or \( R_C \).

**Remark 1** We suspect the cutset bound to be loose, though we do not have an example to show this. For all the example functions where we have a single-letter characterization of the rate region, the cutset outer bound in (15) is seen to be tight. Example 4 provides a class of functions for which the cutset outer bound is tight.

Next we propose two achievable schemes for the \( \epsilon \)-error computation problem. These two schemes are the extensions of the zero-error schemes given in Theorem 3.

**Theorem 5** (RN, \( \epsilon \)-error inner bound) (a) Let

\[
\mathcal{R}_{I_1}^{\epsilon} \triangleq \{(R_A, R_B, R_C) : R_A \geq I(X; U_1|U_2, Q), R_B \geq I(Y; U_2|U_1, Q), R_A + R_B \geq I(X, Y; U_1, U_2|Q), R_C \geq \max\{I(W; U_1|U_2, Y, Q), I(W; U_2|U_1, X, Q)\}\}
\]

for some \( p(q)p(w|u_1, u_2, q)p(u_1|x, q)p(u_2|y, q) \) such that

(i) \( X \in U_1 \in \Gamma(G^f_{I|Y}) \)

(ii) \( Y \in U_2 \in \Gamma(G^f_{Y|X}) \)

(iii) \( (U_1, U_2) \in W \in \Gamma(G^f_{U_1, U_2}) \).

\[
\mathcal{R}_{I_2}^{\epsilon} \triangleq \{(R_A, R_B, R_C) : R_A \geq H_{G'_{\chi|Y}}(X|Y), R_B \geq H_{G'_{\gamma|x}}(Y|X), R_C \geq \max\{H_{G'_{X|Y}}(X|Y), H_{G'_{Y|X}}(Y|X)\}\}.
\]

Let \( \mathcal{R}_I^{\epsilon} \) be the convex closure of \( \mathcal{R}_{I_1}^{\epsilon} \cup \mathcal{R}_{I_2}^{\epsilon} \). Then \( \mathcal{R}_I^{\epsilon} \subseteq \mathcal{R}(f, x, y) \).

(b) Neither of \( \mathcal{R}_{I_1}^{\epsilon} \) and \( \mathcal{R}_{I_2}^{\epsilon} \) is a subset of the other in general.

The proof of Theorem 5 is given in Section V-A. To prove part (b), we show that for computing AND for a DSBS(p) source, the rate triple \( (H(p), H(p), H(p)) \in \mathcal{R}_{I_2}^{\epsilon} - \mathcal{R}_{I_1}^{\epsilon} \), and \( (1, H(p), \frac{1}{2}H(p)) \in \mathcal{R}_{I_1}^{\epsilon} - \mathcal{R}_{I_2}^{\epsilon} \).

**Example 4** Let us consider the functions where one of the confusability graphs is empty. W.l.o.g., let us assume that \( G_{Y|X}^f \) is empty. Then on the support set \( S_{XY} \), the function \( f \) can be computed from \( X \) alone. This implies that
node $A$ can compute the function with zero-error from $X$, and $H_{G_{Y|X}^f}(Y|X) = 0$. Let us consider $H_{G_{Y|X}^f}(X|Y)$. In general, $H_{G_{X|Y}^f}(X|Y) \geq H(Z|Y)$. For a given $Z = z$, let us consider the set of all $x$, $A_z = \{x : f(x, y) = z, \text{ for some } y \text{ s.t. } (x, y) \in S_{XY}\}$. Since here for $X = x$, $f(x, y') = f(x, y'')$ for any $(x, y'), (x, y'') \in S_{XY}$, $A_z$ is an independent set of $G_{X|Y}^f$. Let $A$ denote the set of all $A_z$, and $W = A_Z$. Since $Z$ is a function of $X$, we have $W = g(X)$ for some function $g$. This $W$ in (9) gives that $I(W; X|Y) = H(Z|Y)$. So we get $H_{G_{X|Y}^f}(X|Y) = H(Z|Y)$. Then we get $R_{f,2}$ in Theorem 5 as \{(R_A, R_B, R_C) : R_A \geq H(Z|Y), R_B \geq 0, R_C \geq H(Z|Y)\}. It is easy to check that the cutset outer bound in (15) also gives the same rate region. This shows that for functions where one of the confusability graph is empty, the cutset outer bound is tight.

**Theorem 6** Let $f_1, f_2$ be two functions of $(X, Y)$.

(a) If $E(G_{XY}^{f_1}) \subseteq E(G_{XY}^{f_2})$, then (i) $R^0(f_1, x, y) \supseteq R^0(f_2, x, y)$, (ii) $R^c(f_1, x, y) \supseteq R^c(f_2, x, y)$.

(b) If $G_{XY}^{f_1}$ is isomorphic to $G_{XY}^{f_2}$, then (i) $R^0(f_1, x, y) = R^0(f_2, x, y)$, (ii) $R^c(f_1, x, y) = R^c(f_2, x, y)$.

For any arbitrary function $f$ of $(X, Y)$, if $G_{XY}^{f}$ is isomorphic to the the $f$-modified rook’s graph for exchanging $X$ and $Y$ (i.e. computing $X \oplus Y$), then the rate region $R^c(f, x, y)$ is given by (14). $R^c(f_1, x, y) = R^c(f_2, x, y)$ does not imply the isomorphism between $G_{XY}^{f_1}$ and $G_{XY}^{f_2}$. We show this through the following example.

**Example 5** For a DSBS$(p)$ $(X, Y)$, let functions $f_1, f_2$ of $(X, Y)$ be defined as $f_1 = X + Y$ and $f_2 = Y \cdot (X + Y)$. For these functions, $G_{XY}^{f_1}$ and $G_{XY}^{f_2}$ are shown in Fig. 5. The graph $G_{XY}^{f_1}$ is same as $G_{XY}^{X \oplus Y}$. Using Theorem 6 we get $R^c(f_1, x, y) = \{(R_A, R_B, R_C) : R_A, R_B, R_C \geq H(p)\}$. For function $f_2$, since graphs $G_{XY}^{f_2}$ and $G_{XY}^{f_2}$ are complete graphs, $H_{G_{Y|X}^{f_2}}(X|Y) = H(X|Y)$, and $H_{G_{Y|X}^{f_2}}(Y|X) = H(Y|X)$. Further, we have $H(Z_2|X) = H(p)$ and $H(Z_2|Y) = \frac{1}{2}H(p)$. This implies that $\max\{H_{G_{Y|X}^{f_2}}(X|Y), H_{G_{Y|X}^{f_2}}(Y|X)\} = \max\{H(Z|X), H(Z|Y)\} = H(p)$. Then the region given by $R_{f,2}$ in Theorem 5 is same as the region given by the cutset outer bound in (15). So we get $R^c(f_2, x, y) = \{(R_A, R_B, R_C) : R_A, R_B, R_C \geq H(p)\}$ which is same as $R^c(f_1, x, y)$. Here, even though $R^c(f_1, x, y) = R^c(f_2, x, y)$, $G_{XY}^{f_1}$ is not isomorphic to $G_{XY}^{f_2}$.

| $X$ | 0 | 1 |
|-----|---|---|
| 0   |   |   |
| 1   |   |   |

(a) $G_{XY}^{f_1}$

| $X$ | 0 | 1 |
|-----|---|---|
| 0   |   |   |
| 1   |   |   |

(b) $G_{XY}^{f_2}$

Fig. 5: Graphs $G_{XY}^{f_1}$ and $G_{XY}^{f_2}$ in Example 5

\(^\dagger\)Here $+$ is sum, not XOR. In particular, $f_1(1, 1) = 2$. 
In [2], Han and Kobayashi considered the function computation problem where two encoders encode $X^n$ and $Y^n$, and a decoder wants to compute $f(X, Y)$ from the encoded messages. They gave necessary and sufficient conditions under which the function computation rate region coincides with the Slepian-Wolf region. The conditions were based on a probability-free structure of the function $f(X, Y)$, assuming that $S_{XY} = \mathcal{X} \times \mathcal{Y}$. For our function computation problem, in general, if $G_{f|XY}^{n} \neq G_{X^n \oplus Y^n}^{n}$, then the equality $\mathcal{R}^{e}(f, x, y) = \mathcal{R}^{e}(X \oplus Y, x, y)$ also depends on $p_{XY}$ even when $S_{XY} = \mathcal{X} \times \mathcal{Y}$. In particular, for the function $f_2$ in Example 5, the equality $\mathcal{R}^{e}(f, x, y) = \mathcal{R}^{e}(X \oplus Y, x, y)$ depends on the distribution $p_{XY}$. This is illustrated in Example 6. Thus we observe that the characterization of $\mathcal{R}^{e}(f, x, y) = \mathcal{R}^{e}(X \oplus Y, x, y)$ in the relay network cannot have a probability-free structure.

**Example 6** Let us consider the function $f_2$ in Example 5. When $p_{XY}$ is DSBS(p), it is shown in Example 5 that $\mathcal{R}^{e}(f, x, y) = \mathcal{R}^{e}(X \oplus Y, x, y)$ . Let us consider the same function for the following distribution:

\[
\begin{align*}
p(0, 0) &= p(1, 0) = \frac{1}{6}, \\
p(0, 1) &= p(1, 1) = \frac{1}{3}.
\end{align*}
\]

We have $H(X|Y) = H(X) = 1$ and $H(Y|X) = H(Y) = H(\frac{1}{3})$. So we get $\mathcal{R}^{e}(X \oplus Y, x, y) = \{(R_A, R_B, R_C) : R_A \geq 1, R_B \geq H(\frac{1}{3}), R_C \geq 1\}$. For $Z = f_2(X, Y)$, $H(Z|Y) = \frac{2}{3}$, and $H(Z|X) = H(1/3) \approx 0.91$. Let us consider an instance of encoding where $A$ and $B$ communicate $X^n$ and $Y^n$ to the relay with rates $R_A = H(X)$ and $R_B = H(Y)$ respectively; and the relay computes $Z^n$ and use Slepian-Wolf binning to compress it at a rate $R_C = \max\{H(Z|X), H(Z|Y)\}$. Then the function computation at $A$ and $B$ follows from the Slepian-Wolf decoding. For this scheme, the rate triple $(1, H(1/3), H(1/3))$ is achievable. Clearly, $(1, H(1/3), H(1/3)) \notin \mathcal{R}^{e}(X \oplus Y, x, y)$ and we get $\mathcal{R}^{e}(f, x, y) \neq \mathcal{R}^{e}(X \oplus Y, x, y)$.

IV. ZERo ERROR COMPUTATION: PROOFS OF THEOREMS 1-4

A. Proof of Theorem 1

**Remark 2** To achieve rates $R$ close to $\max\{H(X|Y), H(Y|X)\}$, let us first consider the obvious scheme of random binning $X^n \oplus Y^n$ into $2^{Rn}$ bins. The decoders can run joint typicality decoding of $X^n \oplus Y^n$ similar to Slepian-Wolf scheme. However, there are two sources of errors. The decoding errors for non-typical sequences $(x^n, y^n)$ can be avoided by transmitting those $x^n \oplus y^n$ unencoded, with an additional vanishing rate. However, for the same $y^n$, there is a non-zero probability of two different $x^n \oplus y^n, x'^n \oplus y^n$, both of which are jointly typical with $y^n$, being in the same bin; leading to an error in decoding for at least one of them. It is not clear how to avoid this type of error with the help of an additional vanishing rate.
To prove part (a), we first consider the problem for single receiver case as shown in Fig. 6. Witsenhausen \cite{9} studied this problem under fixed length coding, and gave a single-letter characterization of the optimal rate. For variable length coding, optimal rate $R_0^*$ can be argued to be $R_0^* = H(Y|X)$ by using one codebook for each $x$. Here, we give a graph theoretic proof for this, and later extend this technique to prove part (a) of Theorem 1.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {Encoder};
  \node at (4,0) {Decoder};
  \node at (0,-1) {$X^n, Y^n$};
  \node at (4,-1) {$X^n$};
  \draw[->] (0,0) -- (4,0);
  \draw[->] (0,-1) -- (0,0);
  \draw[->] (4,0) -- (4,-1);
\end{tikzpicture}
\end{center}

Fig. 6: One receiver with side information

**Lemma 4** For the problem depicted in Fig. 6, $R_0^* = H(Y|X)$.

To prove Lemma 4, we first prove some claims. The graph $G$ that we use to prove Lemma 4 is defined as follows. Graph $G$ has its vertex set $S_{XY}$, and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $x_1 = x_2$ and $y_1 \neq y_2$. Similarly, the $n$-instance graph $G(n)$ for this problem has its vertex set $S_{X^nY^n}$, and two vertices $(x^n, y^n)$ and $(x'^n, y'^n)$ are adjacent if and only if $x^n = x'^n$ and $y^n \neq y'^n$.

It is easy to observe that $G$ is the disjoint union of complete row graphs $G_i$ for $i = 1, 2, \ldots, |X|$, where each $G_i$ has vertex set $\{(x_i, y) : (x_i, y) \in S_{XY}\}$.

**Claim 1** For any $n$, the decoder can recover $Y^n$ with zero-error if and only if $\phi$ is a coloring of $G(n)$.

**Proof**: The decoder can recover $Y^n$ with zero-error $\iff$ for any $(x^n, y^n), (x^n, y'^n) \in S_{X^nY^n}$ with $y^n \neq y'^n$, $\phi(x^n, y^n) \neq \phi(x^n, y'^n) \iff$ for any $((x^n, y^n), (x^n, y'^n)) \in E(G(n))$, $\phi(x^n, y^n) \neq \phi(x^n, y'^n) \iff \phi$ is a coloring of $G(n)$.

In the following claim, we identify the vertices of $G(n)$ with the vertices of $G^\wedge n$ by identifying $(x^n, y^n)$ with $((x_1, y_1), \ldots, (x_n, y_n))$.

**Claim 2** $G(n) = G^\wedge n$.

**Proof**: For both the graphs, $(x^n, y^n)$ is a vertex if and only if $p(x_i, y_i) > 0$ for all $i$. Thus both the graphs have the same vertex set.

Next we show that both the graphs have the same edge set. Suppose $(x^n, y^n), (x'^n, y'^n) \in S_{X^nY^n}$ are two distinct pairs. $((x^n, y^n), (x'^n, y'^n)) \in E(G(n)) \iff x^n = x'^n$ and $y^n \neq y'^n \iff x_i = x'_i$ for all $i$, and $y_j \neq y'_j$ for some $j \iff$ for each $i$, either $(x_i, y_i) = (x'_i, y'_i)$ or $((x_i, y_i), (x'_i, y'_i)) \in E(G) \iff (((x_1, y_1), \ldots, (x_n, y_n)), ((x'_1, y'_1), \ldots, (x'_n, y'_n))) \in E(G^\wedge n)$. This shows that $G(n) = G^\wedge n$. \hfill \blacksquare
Claim 3 $R_0^* = \bar{H}_G(X, Y)$.

Proof: Claim $\square$ and the definition of chromatic entropy imply that

$$\frac{1}{n} H_X(G(n), (X^n, Y^n)) \leq R_0^* \leq \frac{1}{n} H_X(G(n), (X^n, Y^n)) + \frac{1}{n}.$$ 

Using Claim $\square$ and taking limit, we get $R_0^* = \lim_{n \to \infty} \frac{1}{n} H_X(G^n, (X^n, Y^n))$. Using (8), this implies $R_0^* = \bar{H}_G(X, Y)$. $lacksquare$

Claim 4 $G$ is a perfect graph.

Proof: As mentioned before, $G$ is disjoint union of complete graphs. Since a complete graph is a perfect graph, it follows that $G$ is also a perfect graph. $lacksquare$

We now state a lemma from [28].

Lemma 5 Let the connected components of the graph $G$ be subgraphs $G_i$. Let $Pr(G_i) = \sum Pr(x), x \in V(G_i)$. Further, set

$$Pr_i(x) = Pr(x)[Pr(G_i)]^{-1}, \ x \in V(G_i).$$

Then $H_G(X) = \sum_i Pr(G_i) H_G(X_i)$.

We now prove Lemma $\square$

Proof of Lemma $\square$. For any perfect graph $A$, it is known that $\bar{H}_A(X) = H_A(X)$ [33], [34]. So Claims $\square$ and $\square$ imply that $R_0^* = H_G(X, Y)$. We now use Lemma 5 to compute $H_G(X, Y)$. Recall that each connected component of graph $G$ is a complete graph, and the connected component $G_i$, for each $i$, has vertex set $\{(x_i, y) : (x_i, y) \in S_{XY}\}$ and $Pr(G_i) = Pr(x_i)$. So we can set the probability of each vertex $(x_i, y) \in G_i$ as $Pr(x_i, y)/Pr(x_i)$. Since all the vertices in $G_i$ are connected, we get $H_{G_i}(x_i, Y) = H(Y|X = x_i)$. Then by using Lemma 5 we get $H_G(X, Y) = H(Y|X)$. This completes the proof of Lemma $\square$.

Now let us consider a special case of the the problem shown in Fig. 2 with $Z = X \oplus Y$. In this case, the $f$-modified rook’s graph $G_{XY}^f$ has its vertex set $S_{XY}$, and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if either $x_1 = x_2$ and $y_1 \neq y_2$, or $y_1 = y_2$ and $x_1 \neq x_2$. Now onwards, we denote $G_{XY}^f$ and the $n$-instance graph $G_{XY}^f(n)$ by $G$ and $G(n)$ respectively.

We now state a Theorem from [29] which is used to prove Theorem $\square$

Theorem 7 Let $G = (G_1, \ldots, G_k)$ be a family of graphs on the same vertex set. If $R_{\min}(G, P_X) := \lim_{n \to \infty} \frac{1}{n} H_X(\bigcup_i G_i^{X^n}, P_X^n)$, then $R_{\min}(G, P_X) = \max_i R_{\min}(G_i, P_X)$ where $R_{\min}(G_i, P_X) = \bar{H}_G(X)$.
We are now ready to prove Theorem 1.

Proof of part (a): For $i = 1, 2$, let $G_i$ be the modified rook’s graphs corresponding to decoding with side information at decoder $i$. So the modified rook’s graph for the problem with two decoders is given by $G = G_1 \cup G_2$. Two vertices $(x^n, y^n)$ and $(x'^n, y'^n)$ are connected in the corresponding $n$ instance graph $G(n)$ if and only if they are connected either in $G_1(n)$ or in $G_2(n)$. This implies that $G(n) = G_1(n) \cup G_2(n)$. This shows that both the decoders can decode with zero-error if and only if $\phi$ is a coloring of $G(n)$. This fact and the definition of chromatic entropy imply that $R_0^* = \lim_{n \to \infty} H(X(n), (X^n, Y^n))$. From Claim 2, it follows that $G(n) = G_1^\wedge n \cup G_2^\wedge n$.

Then by using Theorem 7, we get $R_0^* = \max \{H_{G_1}(X, Y), H_{G_2}(X, Y)\}$. As argued in the proof of Lemma 4, $H_{G_1}(X, Y) = H(Y|X)$ and $H_{G_2}(X, Y) = H(X|Y)$. Thus $R_0^* = \max \{H(Y|X), H(X|Y)\}$.

Proof of part (b): $\max \{H(Z|X), H(Z|Y)\}$ is a lower bound for $R_0^*$ from the cut-set bound. This implies that $\max \{H(Z|X), H(Z|Y)\}$ is also a lower bound for $R_0^*$. Next we show that $R_0^* \leq H_{G_{XY}}(X, Y)$. From the definition of $G_{XY}^f(n)$, it can be observed that the nodes A and B can compute functions Z if and only if $\phi$ is a coloring of $G_{XY}(n)$. This shows that $R_0^* = \lim_{n \to \infty} H_X \left(G_{XY}(n), (X^n, Y^n)\right)$. To get an upper bound for this limit, we consider the $n$-fold OR product graph $(G_{XY}^f)^\wedge n$. As noted before, $G_{XY}^f(n)$ is a subgraph of $(G_{XY}^f)^\wedge n$. Thus $R_0^* \leq \lim_{n \to \infty} H_X \left((G_{XY}^f)^\wedge n, (X^n, Y^n)\right) = H_{G_{XY}^f}(X, Y)$.

Proof of Corollary 7: First let us consider the converse for the rate region. For $R_A$, let us consider the cut between node A and a super node consisting of B and C. This situation arises when the relay node broadcasts the message sent by node A. Then the problem reduces to the problem of decoding with side information studied in [10], where the decoder with side information Y wants to recover X. They showed that if the support set is full, the optimal zero-error rate for this problem is $H(X)$. Similarly, $R_B \geq H(Y)$. Now let us consider the rate $R_C$. $R_C$ attains its minimum value when relay has X and Y. Theorem 1(a) shows that if relay has both X and Y, the minimum achievable broadcast rate is $\max \{H(Y|X), H(X|Y)\}$. This completes the converse. Now let us consider a scheme where nodes A and B communicate X and Y respectively to the relay. The relay can recover X and Y with zero-error if $R_A > H(X)$ and $R_B > H(Y)$. If the relay has X and Y, Theorem 4(a) shows that the rate $\max \{H(Y|X), H(X|Y)\}$ is achievable for $R_C$ for complementary delivery. This proves the achievability of the rate region.

B. Proof of Theorem 2

To prove Theorem 2, we first present some lemmas.

Lemma 6: For any $n \geq 1$, and given the encoding functions $\phi_A, \phi_B, \phi_C$, the nodes A and B can recover $f(X^n, Y^n)$ with zero-error if and only if $\phi_C \circ (\phi_A \times \phi_B)$ is a coloring of $G_{XY}^f(n)$.
Proof: Let \( E(G_{XY}^f(n)) \) denote the set of edges of \( G_{XY}^f(n) \). Note that
\[
E(G_{XY}^f(n)) = \{(x^n, y^n), (x^n, y^m) \in S_{X^nY^n}; f(x_i, y_i) \neq f(x'_i, y'_i) \text{ for some } i\}
\[
\cup \{(x^n, y^n), (x^m, y^n) \in S_{X^nY^n}; f(x_i, y_i) \neq f(x'_i, y'_i) \text{ for some } i\}.
\]
(16)
Observe that each edge is of the form \(((x^n, y^n), (x^n, y^m))\) or \(((x^n, y^n), (x^m, y^n))\). We note that

(i) A can recover \( f(X^n, Y^n) \) with zero-error \( \iff \) for any \((x^n, y^n), (x^n, y^m) \in S_{X^nY^n} \) with \( f(x_i, y_i) \neq f(x'_i, y'_i) \)
for some \( i, \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^n), \phi_B(y^m)) \).

(ii) B can recover \( f(X^n, Y^n) \) with zero-error \( \iff \) for any \((x^n, y^n), (x^m, y^n) \in S_{X^nY^n} \) with \( f(x_i, y_i) \neq f(x'_i, y'_i) \)
for some \( i, \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^m), \phi_B(y^n)) \).

From (i) and (ii) above, it follows that A and B can recover \( f(X^n, Y^n) \) with zero-error \( \iff \) for any \( ((x^n, y^n), (x^m, y^n)) \in E(G_{XY}^f(n)) \), \( \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^m), \phi_B(y^n)) \)
\( \iff \phi_C \circ (\phi_A \times \phi_B) \) is a coloring of \( G_{XY}^f(n) \).

Lemma 7 For any \( n \geq 1 \), and given the encoding functions \( \phi_A, \phi_B, \phi_C \), the nodes A and B can recover \( f(X^n, Y^n) \)
derived under the unrestricted input setup if and only if \( \phi_C \circ (\phi_A \times \phi_B) \) is a coloring of \( G_{XY}^{f(u)}(n) \).

Proof: Let \( E(G_{XY}^{f(u)}(n)) \) denote the set of edges of \( G_{XY}^{f(u)}(n) \). Observe that
\[
E(G_{XY}^{f(u)}(n)) = \{(x^n, y^n), (x^n, y^m) : \text{ for some } i((x_i, y_i), (x'_i, y'_i)) \in E(G_{XY}^{f(u)})\}
\[
\cup \{(x^n, y^n), (x^m, y^n) : \text{ for some } i((x_i, y_i), (x'_i, y'_i)) \in E(G_{XY}^{f(u)})\}.
\]
(17)
We note that

(i) A can recover \( f(X^n, Y^n) \) under the unrestricted input setup \( \iff \) for any \((x^n, y^n), (x^n, y^m) \) such that \( f(x_i, y_i) \neq f(x'_i, y'_i) \)
for some \( i \) where \( (x_i, y_i), (x'_i, y'_i) \in S_{XY}, \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^n), \phi_B(y^m)) \).

(ii) B can recover \( f(X^n, Y^n) \) under the unrestricted input setup \( \iff \) for any \((x^n, y^n), (x^m, y^n) \) such that \( f(x_i, y_i) \neq f(x'_i, y'_i) \)
for some \( i \) where \( (x_i, y_i), (x'_i, y'_i) \in S_{XY}, \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^m), \phi_B(y^n)) \).

From (i) and (ii) above, it follows that A and B can recover \( f(X^n, Y^n) \) with zero-error \( \iff \) for any \( ((x^n, y^n), (x^m, y^n)) \in E(G_{XY}^{f(u)}(n)) \), \( \phi_C(\phi_A(x^n), \phi_B(y^n)) \neq \phi_C(\phi_A(x^m), \phi_B(y^n)) \)
\( \iff \phi_C \circ (\phi_A \times \phi_B) \) is a coloring of \( G_{XY}^f(n) \).

Proof of part \( [\theta] \): Lemma \( \Box \) implies that for encoding functions \( \phi_A, \phi_B, \phi_C \) of any zero-error scheme, \( \phi_A, \phi_B, \phi_C \circ (\phi_A \times \phi_B) \) is a color cover for \( G_{XY}^f(n) \). Similarly, for any color cover \( (c_A, c_B, c_C) \) of \( G_{XY}^f(n) \), let \( \phi_A, \phi_B \) be any prefix-free encoding functions of \( c_A \) and \( c_B \) respectively. Since \( c_A \times c_B \) is a refinement of \( c_C \), there exists a mapping \( \theta_C \) such that \( c_C = \theta_C \circ (c_A \times c_B) \). Taking \( \phi_C \) as any prefix-free encoding of \( c_C \) yields a scheme with encoding functions \( (\phi_A, \phi_B, \phi_C) \). Thus the result follows from the definition of the region \( Z_{G_{XY}^f}(X, Y) \). \( \Box \)
Proof of part (b) follows along the similar lines as that of part (a) using Lemma [7].

C. Proof of Theorem [3]

We first give the proof of part (a) where we use the following lemmas.

Lemma 8 [3, Lemma 4] There exists a function $g$ such that $\forall (x, y) \in S_{XY}$, $u_2 \in \Gamma(G^f_{Y|X})$ s.t. $y \in u_2$, $g(x, u_2) = f(x, y)$.

Lemma 9 There exists functions $g_1$ and $g_2$ such that for all $(x, y, u_1, u_2, w) \in \mathcal{X} \times \mathcal{Y} \times \Gamma(G^f_{X|Y}) \times \Gamma(G^f_{Y|X}) \times \Gamma(G^f_{U_1U_2})$ satisfying $(u_1, u_2) \in w$ and $p(x, y)p(u_1|x)p(u_2|y) > 0$, $f(x, y) = g_1(x, u_1, w) = g_2(y, u_2, w)$.

Proof: For a given $X = x, U_1 = u_1$ and $W = w$, let us consider the set of possible $y$, $A_{x,u_1,w} = \{y': (x, y') \in S_{XY}, \text{ and } p(u_2'|y') > 0 \text{ for some } u_2' \text{ s.t. } (u_1, u_2') \in w\}$. Then we show that

Claim: $f(x, y') = f(x, y'') \\forall y', y'' \in A_{x,u_1,w}$.

Proof of the claim: Let us assume that for some $y', y'' \in A_{x,u_1,w}$, $f(x, y') \neq f(x, y'')$. By definition of $A_{x,u_1,w}$, $\exists u_2', u_2'' \in \Gamma(G^f_{Y|X})$, such that $y' \in u_2', y'' \in u_2''$, and $(u_1, u_2'), (u_1, u_2'') \in w$. But $(y', y'') \in E(G^f_{Y|X})$, and so $y'' \not\in u_2'$, and thus $u_2' \neq u_2''$. From the conditions in the lemma and the definition of $A_{x,u_1,w}$, we have $p(x, u_1, y', u_2'), p(x, u_1, y'', u_2'') > 0$. Then by Definition [4] $(u_1, u_2')$ and $(u_1, u_2'')$ are connected in $G^f_{U_1U_2}$. This implies that $w$ is not an independent set of $G^f_{U_1U_2}$, which is a contradiction. This proves the claim.

Now, $g_1$ (resp. $g_2$) is defined as the unique function value $f(x, y)$ for all $y \in A_{x,u_1,w}$ (resp. $x \in A_{y,u_1,w}$).

Proof of part (b): The inner bounds $R_{I1}$ and $R_{I2}$ are obtained by using two encoding schemes, scheme 1 and scheme 2 respectively. For both the schemes, the encoding operations at nodes A and B are the same. In the following, we assume $\epsilon > \epsilon' > \epsilon'' > 0$ and $|Q| = 1$. Let $\{U^1_{i1}(m_1)|m_1 \in \{1, \ldots, 2^{nR_1}\}\}$ be a set of independent sequences, each distributed according to $\prod_{i=1}^n p_{U_1}(u_{1i})$. Similarly, let $\{U^2_{i2}(m_2)|m_2 \in \{1, \ldots, 2^{nR_2}\}\}$, be a set of independent sequences, each distributed according to $\prod_{i=1}^n p_{U_2}(u_{2i})$.

Scheme 1:

Let $\{W^m(m_3)|m_3 \in \{1, \ldots, 2^{nR_3}\}\}$, be a set of independent sequences, each distributed according to $\prod_{i=1}^n p_W(w_i)$.

Encoding at node A:

For a given $x^n$, node A chooses an index $m_1$ (if any) such that $(x^n, U^n_1(m_1)) \in T^n_{\phi}(X, U_1)$. The encoding at node A is given by

$$ \phi_A(x^n) = \begin{cases} m_1 & (x^n, U^n_1(m_1)) \in T^n_{\phi}(X, U_1) \\ x^n & (x^n, U^n_1(m_1)) \not\in T^n_{\phi}(X, U_1) \forall m_1 \end{cases} $$
By the covering lemma, if \( R_A' > I(X; U_1) + \delta(\epsilon'') \) then

\[
\lim_{n \to \infty} \Pr(\exists m_1, (X^n, U_1^n(m_1)) \in T_{\epsilon'}(X, U_1)) = 1,
\]

where \( \delta(\epsilon'') \to 0 \) as \( \epsilon'' \to 0 \). Rate of the overall encoding is \( R_A < R_A' + \delta(\epsilon'') \) for large enough \( n \) such that \( \Pr(\forall m_1, (X^n, U_1^n(m_1)) \not\in T_{\epsilon'}(X, U_1)) < \delta(\epsilon'')/\log |\mathcal{X}|. \) Thus, any rate \( R_A > I(X; U_1) + 2\delta(\epsilon'') \) is sufficient.

Encoding at relay:

The relay receives either an index \( m_1 \) or a \( x^n \) sequence from node A. Similarly, from node B the relay receives \( m_2 \) or a \( y^n \) sequence. If \( m_1 \) and \( m_2 \) are received, and \((u_1^n(m_1), u_2^n(m_2), w^n(m_3)) \in T^n_e(U_1, U_2, W) \) for some \( m_3 \), then any such \( m_3 \) is broadcasted by the relay. In any other case, the relay broadcasts both the received sequences. So the encoding at the relay is given by

\[
\phi_C = \begin{cases} 
  m_3 & \text{for } (u_1^n(m_1), u_2^n(m_2), w^n(m_3)) \in T^n_e(U_1, U_2, W) \\
  (\phi_A(x^n), \phi_B(y^n)) & \text{otherwise.}
\end{cases}
\]

Let \( E_{n,\epsilon'} \) be the event \((U_1^n(m_1), U_2^n(m_2)) \in T^n_{U_1 U_2, \epsilon'} \) at the relay. Then from the Markov lemma, we have

\[
\lim_{n \to \infty} \Pr(E_{n,\epsilon'}) = 1. \]

By the covering lemma, if \( R'_C > I(W; U_1, U_2) + \delta(\epsilon) \) then

\[
\lim_{n \to \infty} \Pr(\exists m_3, (U_1^n(m_1), U_2^n(m_2), W^n(m_3)) \in T^n_e(U_1, U_2, W) \mid E_{n,\epsilon'}) = 1,
\]

where \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Rate of the overall encoding is \( R_C < R'_C + 2\delta(\epsilon) \) for large enough \( n \) such that

\[
\Pr(E_{n,\epsilon'} \cap \forall m_3, (U_1^n(m_1), U_2^n(m_2), W^n(m_3)) \not\in T^n_e(U_1, U_2, W)) < \delta(\epsilon)/\log(U_1 \cdot U_2),
\]

and \( \Pr(E_{n,\epsilon'}^c) < \delta(\epsilon)/\log(a \cdot b) \), where \( a = \max\{|\mathcal{X}|, |U_1|\} \) and \( b = \max\{|\mathcal{Y}|, |U_2|\} \). Thus, any rate \( R_C > I(W; U_1, U_2) + 3\delta(\epsilon) \) is sufficient.

Decoding at node A:

Node A receives either \( m_3, m_2 \) or \( y^n \). We show that node A computes \( f(x_i, y_i) \) with zero-error \( \forall i \) for which \((x_i, y_i) \in S_{XY} \). Let us consider a pair \((x^n, y^n)\) such that \((x_i, y_i) \in S_{XY} \) for some \( i \).

Case I: Node A receives \( m_3 \)

In this case, node A and node B had chosen \( m_1 \) and \( m_2 \) such that \((x^n, u_1^n(m_1)) \in T^n_e(X, U_1) \) and \((y^n, u_2^n(m_2)) \in T^n_e(Y, U_2) \) respectively, and at the relay \((u_1^n(m_1), u_2^n(m_2), w^n(m_3)) \in T^n_e(U_1, U_2, W) \).

As a robustly typical sequence can not have a zero-probability component (see (13)), the sequence \( u_1^n(m_1) \) chosen by node A satisfies \( p(u_1|x_i) > 0 \ \forall i \), since \((x^n, u_1^n(m_1)) \in T^n_e(X, U_1) \). Similarly, the sequence \( u_2^n(m_2) \) chosen by node B satisfies \( p(u_2|y_i) > 0 \ \forall i \), and the sequence \( w^n(m_3) \) chosen by the relay satisfies \( p(w_i|u_{1i}, u_{2i}) > 0 \ \forall i \).
Hence \( p(u_1|x)p(u_2|y)p(w_1|u_1,u_2) > 0 \). Thus by Lemma \[9\] if \( p(x_i,y_i) > 0 \) then node A can compute \( f(x_i,y_i) \) from \( x_i, u_{1i} \) and \( w_i \) with zero-error.

**Case 2:** Node A receives \( m_2 \)

In this case node B had \( m_2 \) such that \((y^n,u^n_2(m_2)) \in T^n_{e}(Y,U_2)\). This shows that \( p(u_2, m_2|y_i) > 0 \) for all \( i \).
Thus by Lemma \[8\] node A can recover \( f(x_i,y_i) \) with zero-error for all \( i \) such that \( p(x_i,y_i) > 0 \).

**Case 3:** Node A receives \( y^n \)

In this case node A can compute \( f(x_i,y_i) \) for all \( i \).

Node B follows the similar decoding procedure as that of node A. Now using time sharing random variable \( Q \) gives the achievability of every triple \((R_A,R_B,R_C)\) in \( R_{11} \) for some \( p(q)p(w|u_1,u_2,q)p(u_1|x,q)p(u_2|y,q) \).

**Scheme 2:**

Let \( U_1 \) and \( U_2 \) be the random variables which minimize \( R_A \) and \( R_B \) respectively in scheme 1. Since \( U_1 - X - Y - U_2 \) forms a Markov chain, rates \( R_A \) and \( R_B \) can be minimized independently. For this choice of \( U_1 \) and \( U_2 \), nodes A and B use the same encoding procedure of scheme 1. Then any rates \( R_A > H_{G^c_{x|y}}(X) + \delta(\epsilon'') \) and \( R_B > H_{G^c_{y|x}}(Y) + \delta(\epsilon'') \) are achievable.

**Encoding at relay:**

Let us consider the case where the relay receives \( m_1 \) and \( m_2 \) from node A and B respectively, such that \((u^n_1(m_1), u^n_2(m_2)) \in T^n_{U_1,U_2}\). Then the relay broadcasts the XOR of the binary representations of \( m_1 \) and \( m_2 \) (after padding zeros to the shorter sequence). In any other case, as in scheme 1, the relay broadcasts both the received sequences. So the encoding at the relay is given by

\[
\phi_C = \begin{cases} 
\phi_A(x^n), \phi_B(y^n) & \text{otherwise.} \\
1 \oplus m_2 & (u^n_1(m_1), u^n_2(m_2)) \in T^n_{U_1,U_2} 
\end{cases}
\]

By using the Markov lemma as before, rate of the overall encoding is \( R_C < \max\{R_A,R_B\} + 2\delta(\epsilon') \) for large enough \( n \) such that \( Pr((U^n_1(m_1), U^n_2(m_2)) \notin T^n_{U_1,U_2}) < \delta(\epsilon')/\log(|U_1||U_2|) \). Thus any rate \( R_C > \max\{R_A,R_B\} + 2\delta(\epsilon') \) is sufficient.

**Decoding at node A:**

Node A receives either \( m_1 \oplus m_2, m_2 \) or \( y^n \). Let us first consider the case where node A receives \( m_1 \oplus m_2 \). Since node A has \( m_1 \), it can decode \( m_2 \) from \( m_1 \oplus m_2 \) by XORing the received message with \( m_1 \). Zero-error function computation at node A from \( m_2 \) and \( x^n \) follows from Lemma \[8\]. Decoding for all other cases is the same as decoding in scheme 1.

Node B follows the similar decoding procedure as that of node A. This completes the proof of part (a).
Proof of part (b): To prove $R_{I2} \not\subset R_{I1}$, we show that for the function computation problem in Example 1, (log 5, log 5, log 2) $\in R_{I1} - R_{I2}$. In this example, graphs $G_{X|Y}$ and $G_{Y|X}$ are pentagon graphs. The complementary graph entropy of a pentagon graph with uniform distribution is shown to be $\frac{1}{2} \log 5$ [11]. Since the graph entropy is greater than or equal to the complementary graph entropy, we get the achievable region using scheme 2 as a subset of the region $\{(R_A, R_B, R_C) : R_A, R_B, R_C \geq \frac{1}{2} \log 5 \}$. Since log 2 $< \frac{1}{2} \log 5$, we get (log 5, log 5, log 2) $\notin R_{I2}$.

Let us consider scheme 1 for the choice of $U_1 = \{X\}$ and $U_2 = \{Y\}$. Then $R_A = R_B = \log 5$. For this choice of $(U_1, U_2)$, the graph $G'_{U_1U_2}$ is same as the graph $G'_{X|Y}$ which is shown in Fig. 3a. Let us choose

$$W = \begin{cases} 
\{(u_1, u_2)|u_1 = u_2\} & \text{if } U_1 = U_2 \\
\{(u_1, u_2)|u_1 \neq u_2\} & \text{if } U_1 \neq U_2.
\end{cases}$$

Then $W$ is a binary random variable with uniform distribution and satisfies all the conditions in Theorem 1. Here, since $W$ is a function of $(U_1, U_2)$, we get $I(W; U_1, U_2) = H(W) = \log 2$. This shows that the rate triple (log 5, log 5, log 2) $\in R_{I1}$.

To prove $R_{I2} \not\subset R_{I1}$, we show that for the function computation problem in Example 2, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in R_{I2} - R_{I1}$. To prove this, we consider the same of choices of $U_1$ and $U_2$ given in Section III-A and we use the following claim.

**Claim 5** The only conditional distribution $p_{U_1|X}$ achieving $R_A = \frac{2}{3}$ for the function computation problem in Example 2 is $p_{U_1|X}(a|2) = p_{U_1|X}(b|2) = \frac{1}{2}$.

**Proof:** To prove the above claim, we need to show that $I(X; U_1)$ is strictly convex in $p_{U_1|X}$. Let us take $p_{U_1|X}(a|2) = p$, for $0 < p < 1$. Then $I(X; U_1)$ is a function of $p$ which can be written as

$$I(X; U_1) = f(p) = -\frac{1}{3} (1 + p) \log \frac{1}{3} (1 + p) - \frac{1}{3} (2 - p) \log \frac{1}{3} (2 - p) + \frac{1}{3} p \log p + \frac{1}{3} (1 - p) \log (1 - p).$$

Next we show that $f''(p) > 0$, for $0 < p < 1$.

$$f'(p) = \frac{1}{3} \log \frac{1}{3} (1 + p) - \frac{1}{3} \log \frac{1}{3} (2 - p) + \frac{1}{3} \log p - \frac{1}{3} \log (1 + p).$$

$$f''(p) = \frac{1}{1 + p} + \frac{1}{2 - p} + \frac{11}{3} \frac{1}{1 - p}.$$  

Then we have $f''(p) > 0$, for $0 < p < 1$. This proves the claim. 

For Example 2, the confusability graphs $G'_{X|Y}$ and $G'_{Y|X}$ are the same and it is shown in Fig. 4a. For uniform distribution on its vertices, the graph entropy of the graph shown in Fig. 4a is computed as $\frac{2}{3}$ in Example 1 in [3]. So we have $H_{G'_{X|Y}}(X) = H_{G'_{Y|X}}(Y) = \frac{2}{3}$. Then we get $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in R_{I2}$. Claim 5 shows that we have to choose $p_{U_1|X}(a|2) = p_{U_1|X}(b|2) = p_{U_2|Y}(c|2) = p_{U_2|Y}(d|2) = \frac{1}{2}$ to achieve the rates $R_A = R_B = \frac{2}{3}$. For this
choice of \((U_1, U_2)\), let us compute the joint distribution of \((U_1, U_2)\). Note that \((U_1, U_2) = (a, c)\) has non zero joint probability with \((X, Y)\) when either \((X, Y) = (1, 2)\) or \((X, Y) = (2, 1)\). By marginalizing over \((X, Y)\), we get \(p_{U_1, U_2}(a, c) = \frac{1}{6}\). Similarly, we get \(p_{U_1, U_2}(b, d) = \frac{1}{6}\), \(p_{U_1, U_2}(a, d) = p_{U_1, U_2}(b, c) = \frac{1}{3}\).

As we have seen before the graph \(G^f_{U_1, U_2}\) is a “square” graph which is shown in Fig. 4c. The minimum \(R_C\) achievable by Scheme 1 in this case is \(H_{G^f_{U_1, U_2}}(U_1, U_2)\). For this graph \(G^f_{U_1, U_2}\), the only two maximal independent sets are \\{\((a, c), (b, d)\)\} and \\{\((a, d), (b, c)\)\}. Let \(W\) be a random variable distributed over \\{\((a, c), (b, d)\), \((a, d), (b, c)\)\}. Since each node of the graph \(G^f_{U_1, U_2}\) is contained in only one of the maximal independent set, we have \(w\) as a function of \((u_1, u_2)\). Then \(R_C = I(W; U_1, U_2) = H(W) = H(\frac{1}{3}) \approx 0.91\). This shows that for the above choice of \((U_1, U_2)\), the minimum \(R_C\) achievable using scheme 1 is \(H(\frac{1}{3})\). Since \(H(\frac{1}{3}) > \frac{2}{3}\), we get \((\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \not\in \mathcal{R}_{I1}\). This completes the proof of part (b).

\[\square\]

D. Proof of Theorem 4

Let us consider any \(n\). With abuse of notation, we denote the messages sent by the nodes A, B, and C by \(\phi_A, \phi_B, \phi_C\) respectively. In the following, we omit the arguments, and denote \(f(X^n, Y^n)\) by simply \(f\). Since the function is computed with zero-error at nodes A and B, we have \(H(f|\phi_C, X^n) = 0\) and \(H(f|\phi_C, Y^n) = 0\). We want to show that \(H(f|\phi_A, \phi_B) = 0\). We prove this by contradiction. Let us assume that \(H(f|\phi_A, \phi_B) > 0\). Then \(\exists (x^n, y^n)\) and \((x'^n, y'^n)\) such that

\[
Pr(X = x^n, Y = y^n, \phi_A = k_1, \phi_B = k_2) > 0 \tag{18}
\]

\[
Pr(X = x'^n, Y = y'^n, \phi_A = k_1, \phi_B = k_2) > 0 \tag{19}
\]

and \(f(x_i, y_i) \neq f(x'_i, y'_i)\) for some \(i\). \(\tag{20}\)

We consider two cases.

Case 1: \(x_i = x'_i = x\). Since we have \(Pr(X^n = x^n, \phi_A = k_1) > 0\) (using (18)), we get

\[
Pr(X^n = x^n, Y^n = y'^n, \phi_A = k_1) = Pr(X^n = x^n, \phi_A = k_1)Pr(Y^n = y'^n|X^n = x^n) > 0.
\]

So we get

\[
Pr(X^n = x^n, Y = y'^n, \phi_A = k_1, \phi_B = k_2) > 0 \tag{21}
\]

as \(\phi_B(y'^n) = k_2\).

Taking \(k_0 = \phi_C(k_1, k_2)\), (18), (21) imply that \(Pr(X^n = x^n, Y^n = y^n, \phi_C = k_0) > 0\) and \(Pr(X^n = x^n, Y^n = y'^n, \phi_C = k_0) > 0\).
This, together with (20) gives $H(f|\phi_C, X^n) > 0$. Thus A can not recover $f(X, Y)$ with zero-error - a contradiction.

Case 2: $x_i \neq x_i'$ and $y_i \neq y_i'$. Using (20), we get either $f(x_i, y_i') \neq f(x_i, y_i)$ or $f(x_i', y_i') \neq f(x_i', y_i')$. W.l.o.g., let us assume $f(x_i, y_i') \neq f(x_i, y_i)$. Then by combining (21) and (18), and using the fact that $f(x_i, y_i') \neq f(x_i, y_i)$, we get $H(f|X^n, \phi_C) \neq 0$. Thus A can not recover $f(X^n, Y^n)$ with zero-error - a contradiction. This completes the proof of the theorem.

V. $\epsilon$-error computation: Proofs of Theorems 5, 6

Proof of Lemma (3), part (a): Let us consider the cut between node A and the super-node consisting of B and C. Then it is the function computation problem with side information considered in [3] where the decoder with side information $Y$ wants to compute a function $f(X, Y)$. They showed that the optimal $\epsilon$-error rate for this problem is $H_{G_{X|Y}}(X|Y)$. This implies that $R_A \geq H_{G_{X|Y}}(X|Y)$. Similarly, $R_B \geq H_{G_{Y|X}}(Y|X)$. The lower bound for $R_C$ follows from the cut set bound by considering the cut $(\{C\}, \{A, B\})$ and assuming that the relay knows $(X, Y)$.

Proof of part (b): Let us consider a scheme where nodes A and B encode $X^n$ and $Y^n$ to messages $m_1$ and $m_2$ by the scheme given by Orlitsky and Roche in [3], and the relay broadcasts both these messages. From the result of [3], $R_A = H_{G_{X|Y}}(X|Y), R_B = H_{G_{Y|X}}(Y|X)$, and $R_C = H_{G_{X|Y}}(X|Y) + H_{G_{Y|X}}(Y|X)$ are achievable using this scheme. Now let us consider another scheme where $X$ and $Y$ are communicated to the relay from A and B. Then the relay first computes $f(X, Y)$ and then uses Slepian-Wolf binning to compress it at a rate $R_C = \max\{H(Z|X), H(Z|Y)\}$. Then nodes A and B can compute $f(X, Y)$ with negligible probability of error. The rates $R_A = H(X), R_B = H(Y)$, and $R_C = \max\{H(Z|X), H(Z|Y)\}$ are achievable for this scheme.

A. Proof of Theorem 5

Proof of part (a): The scheme used to prove the achievability of $R_{f1}'$ is similar to that of $R_{f1}$ in Theorem 4. Nodes A and B follow Berger-Tung coding scheme [32]. (We refer the reader to Theorem 12.1 in [31].) At node A, like in scheme 1 in Theorem 5, a codebook $\{U^n_i(m_1)|m_1 \in \{1, \ldots, 2^{nR_A}\}\}$ is used. The codebook is randomly binned into $2^{nR_A}$ bins. If a $u^n_i(m_1)$ is found which is jointly typical with $x^n$, then its bin index $b_1$ is sent. If no such $u^n_i$ is found in the codebook, then a randomly chosen bin index is sent. Node B encodes in a similar way.

The relay can correctly recover $(m_1, m_2)$ from $b_1$ and $b_2$ with high probability if $R_A > I(X; U_1|U_2)$, $R_B > I(Y; U_2|U_1)$ and $R_A + R_B > I(X, Y; U_1, U_2)$. Let the reconstructed messages be $(\hat{m}_1, \hat{m}_2)$. The relay follows Wyner-Ziv coding scheme where a codebook $\{W^n(m_3)|m_3 \in \{1, \ldots, 2^{nR_C}\}\}$ is randomly binned into $2^{nR_C}$ bins. If the relay finds a $w^n(m_3)$ which is jointly typical with $(u^n_1(\hat{m}_1), u^n_2(\hat{m}_2))$, then it broadcasts the bin index of
$w^n(m_3)$. Otherwise a randomly chosen bin index is broadcasted. Node A can decode $m_3$ correctly with high probability if

$$R_C \overset{(a)}{=} I(W; U_1, U_2) - I(W; X, U_1) + \epsilon = H(W|X, U_1) - H(W|U_1, U_2) + \epsilon$$

$$= H(W|X, U_1) - H(W|U_1, U_2, X) + \epsilon$$

$$\overset{(b)}{=} I(W; U_2|X, U_1) + \epsilon.$$

Here in (a), we have taken the size of the bin as $2^{n(I(W; X, U_1) + \epsilon)}$, and (b) follows from the Markov chain $W - U_1 U_2 - X$. Similarly node B can decode $m_3$ with high probability if $R_C \geq I(W; U_1|Y, U_2)$.

Let the reconstructed messages at nodes A and B be $\hat{m}_3^A$ and $\hat{m}_3^B$ respectively. Then $w^n(\hat{m}_3^A)$ will be jointly typical with $(x^n, u_1^n(m_1), y^n, u_2^n(m_2))$ with high probability. For such a $w^n(\hat{m}_3^A)$, for all $i$ such that $p(x_i, y_i) > 0$, we get $p(u_{1i}|x_i)p(u_{2i}|y_i)p(w_i|u_{1i}, u_{2i}) > 0$ using robust typicality. Thus by Lemma 9, node A can compute $f(x_i, y_i)$ from $x_i, u_{1i}$ and $w_i$. Node B computes the function in a similar way.

Now let us consider the encoding schemes used to obtain the rate region $R_{f_2}^e$. Node A encodes $X^n$ to an index $m_1$ using the scheme given by Orlitsky and Roche in [3]. Using the same scheme, node B encodes $Y^n$ to an index $m_2$ with rate $R_B$. Once the relay receives both the messages, it broadcasts the XOR of the binary representation of $m_1$ and $m_2$ (after appending zeros to the shorter sequence). Nodes A recovers message $m_2$ from $(m_1, m_1 \oplus m_2)$. Then node A follows the decoding operation given in [3] to compute the function. Similar decoding operation is performed at node B. By the result of [3], $R_A = H_{G_{X|Y}}^f(X|Y), R_B = H_{G_{Y|X}}^f(Y|X)$, and $R_C = \max\{H_{G_{X|Y}}^f(X|Y), H_{G_{Y|X}}^f(Y|X)\}$ are achievable using this scheme.

**Proof of part [b]:** Let us consider computing $X \cdot Y$ (AND function) for DSBS($p$) $(X, Y)$. Here both the confusability graphs $G_{X|Y}^f$ and $G_{Y|X}^f$ are complete. This implies $H_{G_{X|Y}}^f(X|Y) = H(X|Y)$ and $H_{G_{Y|X}}^f(Y|X) = H(Y|X)$. Since $H(X|Y) = H(Y|X) = H(p)$, we get $R_{f_2}^e = \{(R_A, R_B, R_C) : R_A, R_B, R_C \geq H(p)\}$. Now let us consider the achievable scheme of $R_{f_1}^e$ in Theorem 5 for this example. Since both the confusability graphs are complete, the only choice for $U_1$ and $U_2$ are $U_1 = \{X\}$ and $U_2 = \{Y\}$. For this choice of $U_1$ and $U_2$, the relay can recover $X$ and $Y$ by Berger-Tung coding scheme. Then the relay can compute the function $Z = f(X, Y)$. For a given $Z = z$, let us consider the set of all $(x, y), A_z = \{(x, y) : f(x, y) = z, \text{ and } (x, y) \in S_{XY}\}$. Let us choose $W = A_Z$. Then we get $R_C = \max\{H(Z|X), H(Z|Y)\} = \frac{1}{2}H(p)$, which is the minimum possible $R_C$ by Lemma 3. So we get $R_{f_1}^e$ as

$$\{(R_A, R_B, R_C) : R_A \geq H(p), R_B \geq H(p), R_A + R_B \geq 1 + H(p), R_C \geq \frac{1}{2}H(p)\}.$$
Then we have \((H(p), H(p), H(p)) \in \mathcal{R}_{f_2} - \mathcal{R}_{f_1} \) and \((1, H(p), \frac{1}{2} H(p)) \in \mathcal{R}_{f_1} - \mathcal{R}_{f_2} \).

### B. Proof of Theorem 6

We use the following lemma to prove Theorem 6. For \(f_1, f_2 \) of \((X, Y)\), let the random variables \(Z_1 \) and \(Z_2 \) denote \(f_1(X, Y) \) and \(f_2(X, Y) \) respectively.

**Lemma 10** If \(E(G_{XY}^{f_1}) \subseteq E(G_{XY}^{f_2})\), then \(H(Z_1|Z_2, X) = 0 \) and \(H(Z_1|Z_2, Y) = 0 \).

**Proof:** We prove that if \(E(G_{XY}^{f_1}) \subseteq E(G_{XY}^{f_2})\), then \(H(Z_1|Z_2, X) = 0 \). The other case follows similarly. For a given \(X = x \) and \(Z_2 = z_2 \), let us consider the set of all \(y\), \(A_{xz_2} = \{y' : f_2(x, y') = z_2 \) and \((x, y') \in S_{XY}\} \). Then by the definition of \(G_{XY}^{f_2} \), \(f_2(x, y') = f_2(x, y'') \) \(\forall y', y'' \in A_{xz_2} \). Further, since \(E(G_{XY}^{f_1}) \subseteq E(G_{XY}^{f_2})\), \(f_1(x, y') = f_1(x, y'') \). Let us denote this unique value by \(z_1 := f_1(x, y') \). Then we have \(Pr\{Z_1 = z_1|X = x, Z_2 = z_2\} = 1 \) and \(H(Z_1|Z_2, X) = 0 \).

**Proof of part a:** Lemma 10 shows that if \(E(G_{XY}^{f_1}) \subseteq E(G_{XY}^{f_2})\), then \(Z_1 \) is a function of \((Z_2, X)\) as well as a function of \((Z_2, Y)\). This implies that if node A can recover \(Z_2^n\) from \(M_C\) and \(X^n\) with some probability of error, then it can compute \(Z_1^n\) with at most the same probability of error. Similar arguments hold for computing \(Z_1^n\) at node B. This shows that \(\mathcal{R}(f_1, x, y) \supseteq \mathcal{R}(f_2, x, y)\) and \(\mathcal{R}(f_1, x, y) \supseteq \mathcal{R}(f_2, x, y)\).

Part (b) follows from part (a).

### VI. Conclusion

In this work, we studied the function computation problem in a bidirectional relay network (Fig. 1). Function computation problem has been addressed from an information theoretic point of view for unidirectional networks before, e.g. \([1]–[3], [14]\). To the best of our knowledge, this is the first work which addressed the function computation problem for a bidirectional network from an information theoretic point of view. We considered our function computation problem on this network for correlated sources under zero-error and \(\epsilon\)-error criteria and proposed single-letter inner and outer bounds for achievable rates. We studied the function computation problem in a broadcast network (Fig. 2), where we showed that the optimal broadcast rate is the same for computing XOR under zero-error and \(\epsilon\)-error criteria.

### References

[1] J. Körner, and K. Marton, “How to encode the modulo-two sum of binary sources,” *IEEE Transactions on Information Theory*, vol. 25, no. 2, pp. 219-221, Mar. 1979.

[2] T. Han and K. Kobayashi, “A dichotomy of functions \(F(X, Y)\) of correlated sources \((X, Y)\) from the viewpoint of the achievable rate region,” *IEEE Transactions on Information Theory*, vol. 33, no. 1, pp. 69-76, Jan. 1987.
[3] A. Orlitsky and J. R. Roche, “Coding for computing," IEEE Transactions on Information Theory, vol. 47, no. 3, pp. 903-917, Mar. 2001.
[4] H. Kowshik and P. R. Kumar, “Optimal function computation in directed and undirected graphs,” IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3407–3418, Jun. 2012.
[5] B. K. Rai and B. K. Dey, “On network coding for sum-networks,” IEEE Transactions on Information Theory, vol. 58, no. 1, pp. 50–63, Jan. 2012.
[6] V. Shah, B. K. Dey and D. Manjunath, “Network flows for function computation,” IEEE Journal on Selected Areas in Communications, vol. 31, no. 4, pp. 714–730, Apr. 2013.
[7] A. D. Wyner, J. K. Wolf and F. M. J. Willems, “Communicating via a processing broadcast satellite,” IEEE Transactions on Information Theory, vol. 48, no. 6, pp. 1243-1249, Jun. 2002.
[8] H. Su, and A. El Gamal, “Two-way source coding through a relay,” IEEE International Symposium on Information Theory, Jun. 2010.
[9] L. H. Witsenhausen, “The zero-error side information problem and chromatic numbers,” IEEE Transactions on Information Theory, vol. 22, no. 5, pp. 592–593, Jan. 1976.
[10] N. Alon and A. Orlitsky, “Source coding and graph entropies,” IEEE Transactions on Information Theory, vol. 42, no. 5, pp. 1329–1339, Sept. 1996.
[11] P. Koulgi, E. Tuncel, S. L. Regunathan, and K. Rose, “On zero-error source coding with decoder side information,” IEEE Transactions on Information Theory, vol. 49, no. 1, pp. 99-111, Jan. 2003.
[12] J. Körner and G. Longo, “Two-step encoding of finite memoryless sources”, IEEE Transactions on Information Theory, vol. 19, no. 6, pp. 778-782, Nov. 1973.
[13] P. Koulgi, E. Tuncel, S. L. Regunathan, and K. Rose, “On zero-error coding of correlated sources,” IEEE Transactions on Information Theory, vol. 49, no. 11, pp. 2856-2873, Nov. 2003.
[14] O. Shayevitz, “Distributed computing and the graph entropy region,” IEEE Transactions on Information Theory, vol. 60, no. 6, pp. 3435-3449, Jun. 2014.
[15] Z. Bar-Yossef, Y. Birk, T. S. Jayram, and T. Kol, “Index coding with side information,” IEEE Trans. Inf. Theory, vol. 57, no. 3, pp. 1479-1494, Mar. 2011.
[16] N. Alon, E. Lubetzky, U. Stav, A. Weinstein, and A. Hassidim, “Broadcasting with side information,” in 49th Ann. IEEE Symp. Found. Comput. Sci., Philadelphia, PA, Oct. 2008, pp. 823-832.
[17] M. Effros, S. El Rouayheb, and M. Langberg, “An Equivalence Between Network Coding and Index Coding” IEEE Trans. Inf. Theory, vol. 61, no. 5, pp. 2478-2487, May. 2015.
[18] F. Arbabjolfaei and Y. H. Kim, “Structural properties of index coding capacity using fractional graph theory,” in Proc. IEEE International Symposium on Information Theory, Hong Kong, Jun. 2015.
[19] H. Maleki, V. R. Cadambe, and S. A. Jafar, “Index coding an interference alignment perspective,” IEEE Trans. Inf. Theory, vol. 60, no. 9, pp. 54025432, Sep. 2014.
[20] E. Tuncel, “Slepian-Wolf coding over broadcast channels,” IEEE Transactions on Information Theory, vol. 52, no. 4, pp. 1469-1482, Apr. 2006.
[21] Y. Wu, “Broadcasting when receivers know some messages a priori,” in Proc. IEEE International Symposium on Information Theory, Nice, France, Jun. 2007.
[22] G. Kramer and S. Shamai, “Capacity for classes of broadcast channels with receiver side information,” in Proc. IEEE Information Theory Workshop, California, USA, Sep. 2007.
[23] A. Kimura, T. Uyematsu, and S. Kuzuoka, “Universal coding for correlated sources with complementary delivery,” IEICE Transactions Fundamentals, vol. E90-A, no. 9, pp. 1840–1847, Sep. 2007.
[24] R. Timo, A. Grant, and G. Kramer, “Lossy broadcasting with complementary side information,” *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 104-131, Jan. 2013.

[25] M. Langberg and M. Effros, “Network coding: Is zero error always possible?” in *Proc. 49th Ann. Allerton Conf. Comm. Control Comput.*, Monticello, IL, Sep. 2011, pp. 1478-1485.

[26] J. Körner, “Coding of an information source having ambiguous alphabet and the entropy of graphs”, in *Proc. 6th Prague Conf Inf. Theory*, 1973, pp. 411-425.

[27] G. Simonyi, “Graph entropy: A survey,” in *Proc. DIMACS*, vol. 20, 1995, pp. 399-441.

[28] J. Körner, “Fredman-Komlós bounds and information theory”, *SIAM J. Algebraic and Discrete Methods*, vol. 7, no. 4, pp. 560-570, Oct. 1986.

[29] E. Tuncel, J. Nayak, P. Koulgi, and K. Rose, “On Complementary Graph Entropy,” *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2537-2546, Jun. 2009.

[30] H. H. Permuter, Y. Steinberg, and T. Weissman, “Two-way source coding with a helper,” *IEEE Transactions on Information Theory*, vol. 56, no. 6, pp. 2905-2919, Jun. 2010.

[31] A. El Gamal and Y. H. Kim, *Network Information Theory*, Cambridge, U.K, Cambridge Univ. Press, 2011.

[32] T. Berger, “Multiterminal source coding,” in G. Longo, editor, *The Information Theory Approach to Communications*, pp. 171-231, Springer-Verlag, New York, 1977.

[33] I. Csiszár, J. Körner, L. Lovász, K. Marton, and G. Simonyi, “Entropy splitting for antiblocking corners and perfect graphs,” *Combinatorica*, vol. 10, no. 1, 1990.

[34] G. Simonyi, “Perfect graphs and graph entropy. An updated survey,” *J. Ramirez-Alfonsin, B. Reed (Eds.), Perfect Graphs*, pp. 293-328. John Wiley & Sons, 2001.