Logarithmic corrections to entropy for black holes with a hyperbolic horizon

Yun Soo Myung

Institute of Basic Science and School of Computer Aided Science, Inje University, Gimhae 621-749, Korea

E-mail: ysmyung@inje.ac.kr

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Abstract

We compute logarithmic corrections to entropy for black holes with a hyperbolic horizon. For this purpose, we introduce the topological black hole and Martinez–Troncoso–Zanelli (MTZ) black holes in four dimensions, while in five dimensions, the topological black holes and Gauss–Bonnet black hole with negative coupling are considered. As they stand, logarithmic corrections are problematic because small black holes with positive heat capacity have negative energies. However, introducing the background state of an extremal black hole, the logarithmic corrections are performed for a black hole with a hyperbolic horizon.

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1. Introduction

A number of authors have shown that for a large class of black holes, the Bekenstein–Hawking entropy receives logarithmic corrections due to thermodynamic fluctuations around thermal equilibrium [1, 2]. Up to now, a corrected-entropy formula takes the form

\[ S_c = S - \frac{1}{2} \ln (C T^2) + \cdots, \]

where \( C \) and \( T \) are the heat capacity and temperature of the given black hole, respectively, and \( S \) denotes the uncorrected Bekenstein–Hawking entropy. Here, \( C \) should be positive for equation (1) to be well defined. We note that for \( C > 0 \) (\( C < 0 \)), the system is thermodynamically stable (unstable). A black hole with negative specific heat is in unstable equilibrium with the heat reservoir of temperature \( T \) [3]. Its fate under small fluctuations will be either to decay to hot flat space or to grow without limit by emitting or absorbing thermal radiation in the heat reservoir [4]. Hence, it is meaningless to apply equation (1) to the black hole with \( C < 0 \). There exists a way to achieve a stable black hole in equilibrium with the heat reservoir. A black hole could be rendered thermodynamically stable by placing it in AdS...
spacetimes. An important point is to understand how a black hole with positive specific heat could emerge from thermal radiation through a phase transition. To this end, one introduces the Hawking–Page phase transition between the thermal AdS space and Schwarzschild–AdS black hole [5–7].

It is well known that the entropy-corrected formula of equation (1) is universal [2], implying that this prescription could apply to all black holes with positive specific heat. We note that the canonical ensemble was used to derive equation (1). In this case, the positive energy (mass) of the system is needed to define the canonical ensemble. However, small black holes with a hyperbolic horizon have negative energy, even though they have positive heat capacity. These may be counter examples, because they have negative masses in the range $M_e \leq M \leq 0$ with $M_e$ being the mass of the extremal black hole. Hence, a direct application of equation (1) to these black holes is not guaranteed for accounting their thermodynamic fluctuations.

In this work, we address this issue and resolve this by introducing the background state of an extremal black hole, similar to the charged black holes with a spherical horizon [8]. The difference is that for hyperbolic horizon, its background state energy is negative, whereas its background state energy is positive for a spherical horizon. In order to avoid negative mass, we use the subtraction scheme. The entropy-corrected formula is not changed because the subtraction scheme corresponds to a constant shifting from $M$ to $E = M - M_e$. We will use equation (1) to study thermal fluctuations of black holes with a hyperbolic horizon.

The organization of this work is as follows. Section 2 is devoted to performing the logarithmic corrections to the topological black hole (4DTBH) and Martinez–Troncoso–Zanelli (MTZ) black holes in four dimensions. We study the logarithmic corrections to the topological black hole (5DTBH) and Gauss–Bonnet (GB) black holes in five dimensions in section 3. We discuss the AdS/CFT correspondences for these black holes in section 4. Finally, we discuss our results in section 5.

2. 4DTBH and MTZ black holes

Topological black holes in asymptotically anti-de Sitter spacetimes were first found in three and four dimensions [9]. Their black hole horizons are Einstein spaces of spherical ($k = 1$), hyperbolic ($k = -1$) and flat ($k = 0$) curvature for higher dimensions more than three [10, 11]. The standard equilibrium and off-equilibrium thermodynamic analyses are possible to show that they are treated as the extended thermodynamic systems, even though their horizons are not spherical. The topological black holes in four-dimensional AdS spacetimes are given by

$$d\Sigma_{4}^2 = g_{\mu\nu} dx^\mu dx^\nu = -f_T(r) dr^2 + \frac{1}{f_T(r)} dr^2 + r^2 d\Theta^2,$$

where the metric function $f_T(r)$ is given by

$$f_T(r) = k - \frac{m}{r} + \frac{r^2}{l^2}.$$

$d\Sigma_{2}^2$ describes the 2D horizon geometry with a constant curvature

$$d\Sigma_{2}^2 = d\theta^2 + f_2^2(\theta) d\phi^2,$$

where $f_2(\theta)$ is given by

$$f_0(\theta) = \theta, \quad f_1(\theta) = \sin \theta, \quad f_{-1}(\theta) = \sinh \theta.$$

Here we define $k = 1, 0$ and $-1$ cases as the 4D Schwarzschild–AdS black hole (4DSAdS), 4D flat-AdS black hole and 4D hyperbolic-AdS black hole (=4DTBH) [12], respectively. In
the case of $k = 1$, $m = 0$, we have an AdS$_4$ spacetime with its curvature radius $l$. However, $m \neq 0$ generates the topological black holes. It is easy to check that the metric (2) satisfies Einstein’s equations with a negative cosmological constant:

$$R_{\mu\nu} = -\frac{3}{l^2} g_{\mu\nu},$$

(6)

when the horizon is an Einstein space:

$$R_{ij} = k h_{ij}.$$

(7)

In this work, we are interested in the negative curvature with $k = -1$ only. Then, the horizon space is a hyperbolic manifold of $\Sigma_{k=-1} = H^2 / \Gamma$, where $H^2$ is a 2D hyperbolic space and $\Gamma$ is a suitable discrete subgroup of the isometry group of $H^2$ [13].

First of all, the 4DTBH provides thermodynamic quantities of Hawking temperature $T_T$, mass $M_T$, entropy $S_T$, heat capacity $C_T$ and free energy $F_T = M_T - T_T S_T$

$$T_T = \frac{1}{4\pi \rho_+} \left( \frac{3\rho_+^2}{l^2} - 1 \right), \quad M_T = \frac{\sigma}{8\pi G_4} m = \frac{\sigma \rho_+}{8\pi G_4} \left( \frac{\rho_+^2}{l^2} - 1 \right),$$

(8)

$$S_T = \frac{\sigma \rho_+^2}{4G_4}, \quad C_T = 2 \left[ \frac{3\rho_+^2 - l^2}{3\rho_+^2 + l^2} \right] S_T, \quad F_T(\rho_+) = -\frac{\sigma \rho_+}{16\pi G_4} \left( \frac{\rho_+^2}{l^2} + 1 \right),$$

(9)

where $\sigma$ denotes the area of a unit 2D hyperbolic space $\Sigma_{k=-1}$ and $\rho_+$ is the outer horizon which satisfies $f_T = 0$ with $k = -1$. We note that the first law of thermodynamics $dM_T = T_T dS_T$ holds for the 4DTBH. Also, we observe that the entropy satisfies the area law.

Importantly, we observe that the temperature and heat capacity are positive for $\rho_+ > \rho_e = l/\sqrt{3}$, while the mass is positive only for $\rho_+ > l$. This means that any canonical ensemble is not defined for $\rho_+ < l$. Thus, it seems that the entropy-correction formula of equation (1) is useless for describing the thermal fluctuations of the small 4DTBH. This arises mainly because its horizon geometry is hyperbolic.

In order to resolve this problem, we have to choose an appropriate subtraction scheme. In the Reissner–Norström–AdS black hole, one has introduced the extremal black hole as the background state to define an appropriate free energy in the canonical ensemble [8]. Similarly, we wish to introduce the extremal black hole at $\rho_+ = \rho_ e$ as the background state even though the charge is absent in the 4DTBH [14]. In this case, we define the positive energy as

$$E_T = M_T - M^e_T$$

(10)

with respect to the negative background energy

$$M^e_T = M_T|_{\rho_+ = \rho_e} = -\frac{\sigma l}{12\sqrt{3}\pi G_4}.$$

(11)

Although one uses $E_T$ instead of $M_T$, all thermodynamics except $F_T$ remain unchanged. The free energy is an important quantity to discuss the global stability and phase transition. In this case, it is shifted by $F^e_T = E_T - T_M S_M$. Then, for $\rho_+ > \rho_e$, we may use the corrected-entropy formula to study thermodynamic fluctuations around the equilibrium 4DTBH. We prove it by showing that the root-mean-square energy fluctuations defined by the square root of

$$\langle (\Delta E_T)^2 \rangle = C_T T_T^2$$

(12)

remain unchanged under the change from $M_T$ to $E_T$. In deriving equation (1), one has used $S_e = S - \frac{1}{2} \ln S' + \cdots$ with $S' = \langle M^2 \rangle - \langle M \rangle^2 = C T^2$. It is easy to check $\langle E^2 \rangle - \langle E \rangle^2 = \langle M^2 \rangle - \langle M \rangle^2$ under the shifting $M \rightarrow E = M - M_e$ which is also confirmed by noting that $C T^2$ is an invariant quantity. Then, the mass (energy) becomes positive and thus one could
define the canonical ensemble to discuss its thermal fluctuations around the equilibrium configuration. Importantly, the entropy-corrected formula is not changed because the subtraction scheme corresponds to a constant shifting from $M$ to $E$.

For a large 4DTBH with $\rho_+ \gg l$, we have approximate forms

$$C_T \sim 2S_T, \quad T_T^2 \sim S_T,$$

(13)

which leads to an approximately correct-entropy formula

$$S'_T \sim S_T - \ln[S_T] + \cdots.$$  

(14)

This is the same form as for large 4DSAdS [2]. Furthermore, we note that the Smarr formula of $M = TS/2$ (Euler relation) is not satisfied for the choice of either $M = MT$ or $ET$.

On the other hand, the MTZ black hole dressed by a scalar could be obtained from the Einstein action minimally coupled to a scalar in AdS spacetimes [15]:

$$I_4[\hat{g}, \phi] = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\hat{R} - 2\Lambda_4}{16\pi G_4} - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right],$$

(15)

where the potential $V(\phi)$ is given by

$$V(\phi) = -\frac{3}{4\pi G_4 l^2} \sinh^2 \left[ \frac{\sqrt{4\pi G_4 / 3}}{\phi} \right].$$

(16)

Here $\Lambda_4 = -3/l^2$ with $l$ being the curvature radius of AdS$^4$ spacetimes. For the $\phi = 0$ case, we obtain the action for the 4DTBH. In order to understand the role of the scalar and its potential, it would be better to use the conformal frame by introducing a conformal factor $\frac{l}{\Psi_1} = \sqrt{3/4\pi G_4} \tanh \left[ \sqrt{4\pi G_4 / 3} \phi \right]$. Performing a conformal transformation as $\hat{g}_{\mu\nu} = \left( 1 - \frac{4\pi G}{\Psi_1^2} \right)^{-1} g_{\mu\nu}$, the action $I_4$ takes the form

$$I_4[\hat{g}, \Psi] = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\hat{R} - 2\Lambda_4}{16\pi G_4} - \frac{1}{2} \hat{\nabla} \Psi^2 - \frac{2\pi G_4}{3l^2} \Psi^4 \right],$$

(17)

where a conformally coupled scalar $\Psi$ appears with the conventional potential $\Psi^4$. In connection with higher derivative terms, this potential may play a role of a curvature-squared term $\hat{R}^2$. It provides a black hole dressed by a scalar with a hyperbolic horizon. On the other hand, for $\Psi = 0$, one has the action for the 4DTBH (equation (2)). Then, the solution of the MTZ black hole is given by

$$d\Sigma_M^2 = -f_M(r) \, dt^2 + \frac{dr^2}{f_M(r)} + r^2 \, d\Sigma_{k-1}^2,$$

(18)

where the metric function $f_M(r)$ is given by

$$f_M(r) = \frac{r^2}{l^2} - \left( 1 + \frac{G_4 \mu}{r} \right)^2$$

(19)

and a conformally coupled scalar has the configuration

$$\Psi(r) = \sqrt{\frac{3}{4\pi G_4} \frac{G_4 \mu}{r + G_4 \mu}}.$$  

(20)

Thermodynamic quantities of the MTZ are given by Hawking temperature $T_M$, mass $M_M$, entropy $S_M$, heat capacity $C_M$ and free energy $F_M = M_M - T_M S_M$ by [15–17]

$$T_M = \frac{1}{2\pi l} \left( \frac{2r_+}{l} - 1 \right) = \left( \frac{2G_3}{\pi l^3} \right) S_M, \quad M_M = \frac{\sigma}{4\pi l^2} = \frac{\sigma r_+}{4\pi G_4 l^2} \left( \frac{r_+}{l} - 1 \right),$$

$$S_M = \frac{\sigma l^2}{4G_4} \left( \frac{2r_+}{l} - 1 \right) = C_M, \quad F_M(r_+) = -\frac{\sigma}{8\pi G_4} \left( \frac{r_+^2}{l} - 2r_+ + l \right).$$

(21)
where $r_+$ is the outer horizon which satisfies $f_M = 0$. We note that the first law of thermodynamics $dM = T_M ds_M$ is satisfied for the MTZ. Also, we observe that the entropy $S_M$ was obtained by using Wald’s formula but it does not satisfy the area law. Importantly, the temperature and heat capacity are positive for $r_+ > r_e = l/2$, while the mass is positive for $r_+ > l$ only. This means that any canonical ensemble is not suitable for describing the case of $r_+ < l$ and thus the corrected-entropy formula of equation (1) may be useless for the small MTZ black hole.

In order to resolve this problem, we have to choose an appropriate background state. Similarly, we introduce the extremal black hole as the background state even though the charge is absent in the MTZ black hole, too. For the MTZ black hole, we define the positive energy as

$$E_M = M_M - M'_M$$

with the background state energy

$$M'_M = M_M|_{r_+ = \frac{l}{2}} = -\frac{\sigma l}{16\pi G_4}.$$  

(23)

Although one uses $E_M$ instead of $M_M$, all thermodynamics of the MTZ except $F_M = E_M - T_M S_M$ remain unchanged. Hence, we check that the Smarr formula

$$E_M = \frac{T_M S_M}{2}$$

(24)

is satisfied for $r_+ > r_e$ only when using the positive energy $E_M$. This implies that we could use the corrected-entropy formula to see thermodynamic fluctuations around the equilibrium MTZ, even though its entropy does not satisfy the area law. In this case, we have a corrected-entropy formula for $r_+ > r_e$:

$$S'_M = S_M - \frac{1}{2} \ln |S_M| + \cdots,$$

(25)

which is the same formula as for the non-rotating BTZ black hole [2] whose thermodynamic quantities are given by

$$T_B = \frac{r_+}{2\pi l^2} = \left[ \frac{G_3}{\pi^2 l^2} \right] S_B, \quad M_B = \frac{r_+^3}{8G_3 l^2} = E_B;$$

$$S_B = \frac{\pi r_+}{2G_3} = C_B, \quad F_B = -\frac{r_+^2}{8G_3 l^2}.$$  

(26)

Here we have the Smarr formula of $E_B = T_B S_B/2$ for the non-rotating BTZ black hole. Two are very similar in the sense that they satisfy the Smarr formula and have the same corrected-entropy formula. A difference is that these formulae are valid for the outer horizon $r_+ > r_e$ of the MTZ black hole and for any size of a non-rotating BTZ black hole. Furthermore, we derive thermodynamic quantities for a large MTZ black hole with $r_e \gg l$ as

$$T_M \simeq \frac{r_+}{\pi l^2}, \quad M_M \simeq \frac{\sigma r_+^2}{4\pi G_4}, \quad S_M = C_M \simeq \frac{\sigma r_+^2}{2G_4}, \quad F_M \simeq -\frac{\sigma r_+^2}{4\pi G_4},$$

(27)

which show nearly the same behavior as equation (26) shows. Particularly, choosing $\sigma = \pi/l$ leads to close relations of $T_M = 2T_B, M_M = 2M_B, F_M = 2F_B, S_M = S_B, C_M = C_B$. This suggests that the MTZ black hole dressed by a scalar is a cornerstone for the 4D black hole physics as the non-rotating BTZ black hole does play a key role in the 3D black hole physics.
3. 5DTBH and Gauss–Bonnet black holes

In five dimensions, the topological AdS black holes are given by
\[ ds_{5\text{DTBH}}^2 = -h(r)\, dt^2 + \frac{1}{h(r)}\, dr^2 + r^2\, d\Sigma_3^2, \]  
where \( d\Sigma_3^2 = d\chi^2 + h_1^2(\chi)\, (d\theta^2 + \sin^2\theta\, d\phi^2) \) describes the 3D horizon geometry with a constant curvature. Further, \( h(r) \) and \( h_1(\chi) \) are given by
\[ h(r) = k - \frac{m}{r^2} + \frac{r^2}{\ell^2}, \quad h_1(\chi) = \chi, \quad h_{-1}(\chi) = \sinh \chi. \]

Here we define \( k = 1, 0 \) and \( -1 \) cases as the 5D Schwarzschild–AdS black hole (5DSAdS) [19, 20], 5D flat-AdS black hole and 5D hyperbolic-AdS black hole (5DTBH) [21], respectively. We are interested in the \( k = -1 \) case only. In this case, the location of the event horizon is given by
\[ r^2 + \ell^2 = \frac{\ell^2}{2} (1 + \sqrt{1 + 4m/\ell^2}). \]

The relevant thermodynamic quantities of the Hawking temperature \( T_t \), mass \( M_t \), Bekenstein–Hawking entropy \( S_t \), heat capacity \( C_t \) and free energy \( F_t = M_t - T_t S_t \) are given by
\[ T_t = \frac{1}{2\pi r_+} \left( \frac{2r_+^2}{\ell^2} - 1 \right), \quad M_t \equiv \frac{3V_3 m}{16\pi G_5} = \frac{3V_3 r_+^2}{16\pi G_5} \left( \frac{r_+^2}{\ell^2} - 1 \right), \]
\[ S_t = \frac{V_3 r_+^3}{4G_5}, \quad C_t = 3 \left[ \frac{2r_+^2 - \ell^2}{2r_+^2 + \ell^2} \right] S_t, \quad F_t = -\frac{V_3 \ell^2}{16\pi G_5} \left( \frac{r_+^2}{\ell^2} + 1 \right), \]
where \( V_3 \) is the volume of a unit 3D hyperbolic space \( \Sigma_{k=-1} \) and \( G_5 \) is the five-dimensional Newton constant. Here we find that \( T_t, C_t > 0 \) for \( r_+ > r_c = l/\sqrt{2} \), while \( M_t > 0 \) for \( r_+ > l \). It seems that any canonical ensemble is not defined for \( r_+ < l \), and thus the entropy-correction formula of equation (1) is meaningless for the small 5DTBH.

In order to cure this problem, we have to find an appropriate subtraction scheme. Similarly, we introduce the extremal black hole as the background state. In this case, we define the positive energy as
\[ E_t = M_t - M_t^e \]
with the background state energy
\[ M_t^e = M_t|_{r_+ = l} = -\frac{3V_3 l^2}{64\pi G_5}. \]

Although one uses \( E_t \) instead of \( M_t \), all thermodynamics except \( F_t^e = E_t - T_t S_t \) remain unchanged. This implies that for \( r_+ > r_c \), we could use the corrected-entropy formula to study thermodynamic fluctuations around the equilibrium 5DTBH.

For a large 5DTBH with \( r_+ \gg l \), we have approximate forms
\[ C_t \sim 3S_t, \quad T_t^2 \sim S_t^3 \]
which leads to an approximately correct-entropy formula
\[ S_t^e \sim S_t - \frac{5}{6} \ln[S_t] + \cdots. \]
This is the same form as for large 5DSAdS [2, 22]. Furthermore, we note that the Smarr formula is not satisfied for the choice of either \( M = M_t \) or \( E_t \).
Hereafter, we consider the negative coupling of $Sg$ with the hyperbolic horizon. This may be found from a 5D gravitational action in the presence of a negative cosmological constant $\Lambda_5 = -6/l^2$ and the Gauss–Bonnet term as

$$I_5[g, c] = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R - 2\Lambda_5 + \frac{c}{2}L_{GB} \right],$$

(36)

where

$$L_{GB} = R^2 - 4R^\mu{}_{\nu\rho\sigma}R^\nu{}_{\rho\sigma} + R^\mu{}_{\nu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}.$$

(37)

Here, $c$ is a GB coupling constant having mass dimension $-2$. For $c = 0$, this action reduces to the action for the 5DTBH (equation (28)). The solution of the TGBAdS black hole is given by

$$d\bar{s}_5^2 = -f_\bar{g}(r) \, dr^2 + \frac{dr^2}{f_\bar{g}(r)} + r^2 \, d\Sigma_4^2.$$

(38)

The metric function is given by

$$f_\bar{g}(r) = k + \frac{r^2}{2c} \left[ 1 + \epsilon \sqrt{1 + \frac{4c}{3} \left( \frac{2\mu}{\rho} - \frac{3}{l^2} \right)} \right].$$

(39)

Hereafter, we consider the negative coupling of $c < 0$ with $k = -1$ and $\epsilon = -1$ in accordance with the hyperbolic horizon.

For the TGBAdS black hole, we have relevant thermodynamic quantities [23–25]:

$$T_g = \frac{r_*}{2\pi \left( r_*^2 - 2c \right)} \left( \frac{2r_*^2}{l^2} - 1 \right), \quad M_g = \frac{V_3 v}{8\pi G_5} = \frac{3V_3 r_*^2}{16\pi G_5} \left( \frac{r_*^2}{l^2} - 1 + \frac{c}{r_*^2} \right),$$

$$S_g = \frac{V_3 r_*^4}{4G_5} \left( 1 - \frac{6c}{r_*^2} \right), \quad C_g = \frac{3V_3 r_*^4}{4G_5} \left[ \frac{(2r_*^2 - l^2)(1 - 2c/r_*^2)^2}{(2r_*^2 + l^2) - (2c/r_*^2)(6r_*^2 - l^2)} \right],$$

$$F_g = M_g - T_g S_g = -\frac{V_3 r_*^4}{16\pi G_5 r_*^2} \left[ \left( \frac{r_*}{l^2} + 1 \right) - \frac{3c}{r_*^2} \left( 6r_*^2 - r_*^2 - 2c \right) \right],$$

(40)

where we point out that $S_g$ was derived using the first law of thermodynamics and thus it does not satisfy the area law. Here we find that $T_g, C_g > 0$ for $r_* > r_c = l/\sqrt{2}$. On the other hand, one has $M_g > 0$ for $r_* > r_0$ where

$$r_0 = \frac{l}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 4c/l^2}} > l$$

(41)

which is determined by the massless condition of $M_g = 0$. This means that any canonical ensemble is not defined for $r_* < r_0$ and thus the entropy-correction formula of equation (1) is useless for the small TGBAdS. In order to solve this problem, we have to find an appropriate background state. We introduce the extremal black hole as the background state. In this case, we define the positive energy as

$$E_g = M_g - M'_g,$$

(42)

with the background state energy

$$M'_g = M_g \big|_{r_* \to \infty} = -\frac{3V_3}{16\pi G_5} \left[ \frac{l^2}{4} - c \right].$$

(43)
At this stage, we find an important equality for energies:
\[ E_g = E_t = \frac{3Vr_g^2}{16\pi G_5} \left[ \frac{r_g^2}{l^2} - 1 + \frac{l^2}{4r_g^2} \right]. \]  
(44)

Although one uses \( E_g \) instead of \( M_g \), all thermodynamics except \( F_n g = E_g - T_g S_g \) remain unchanged. This implies that for \( r_e > r_e \), we could use the corrected-entropy formula to see thermodynamic fluctuations around the equilibrium TGBAdS. For a large TGBAdS with \( r_e \gg l, c \), we have approximate forms
\[ C_g \sim 3S_g, \quad T_g^2 \sim S_g^2, \]  
which leads to an approximately correct-entropy formula
\[ S'_g \sim S_g - \frac{5}{6} \ln[S_g] + \cdots. \]  
(46)

This is the same form as for large 5DAdS [2]. Furthermore, we note that the Smarr formula is not satisfied for the choice of either \( M = M_g \) or \( E_g \).

4. Boundary conformal field theories

The holographic principle means that the number of degrees of freedom associated with the bulk gravitational dynamics is determined by its boundary spacetime. We start with the bulk relation between the entropy and energy for the non-rotating BTZ black hole:
\[ S_B = \pi l \sqrt{\frac{2E_B}{G_3}}, \]  
(47)

which is derived using \( E_B = \frac{\pi l \sigma l^3}{24G_3} \) and the Smarr formula of \( E_B = \frac{T_B S_B}{2} \).

Also, this could be recovered from the CFT on the boundary at infinity using the Cardy formula:
\[ S_{\text{CFT}} = 2\pi \sqrt{\frac{cL_0}{6}} + 2\pi \sqrt{\frac{\bar{c}L_0}{6}} \]  
(48)

with \( c = \bar{c} = \frac{3M}{2G_3}, \quad L_0 = \bar{L}_0 = \frac{E_B}{2}. \)  
(49)

This means that the AdS\(_3\)/CFT\(_2\) correspondence was realized for counting the BTZ black hole entropy.

However, as far as we know, there is no definite realization of the AdS\(_4\)/CFT\(_3\) correspondence. Hence, we may try to understand this correspondence by analogy of the non-rotating BTZ black hole. First of all, we find the similar relation from \( E_M = \frac{\pi l \sigma l^3}{24G_4} \) and the Smarr formula of equation (24) as
\[ S_M = l \sqrt{\frac{\pi l \alpha E_M}{G_4}} = 2\pi l \sqrt{-4M_e^2}E_M, \]  
(50)

which is valid for the outer horizon \( r_e > r_e \). Assuming that the Cardy formula
\[ S_{\text{CFT}}^M = 2\pi l \sqrt{\frac{c_1L_0^M}{6}} + 2\pi l \sqrt{\frac{\bar{c}_1L_0^M}{6}} \]  
(51)

with \( c_3 = \bar{c}_3 = \frac{3\pi l}{2\pi G_4} = -24M_e^2, \quad L_0^M = L_0^M = \frac{E_M}{2} \)  
(52)
holds in three dimensions, we may recover the entropy $S_M$ from the boundary CFT$_3$ entropy $S_M^{\text{CFT}}$. We note that equations (51) and (52) are just our proposition. This is because we do not know what is the exact Cardy formula for the boundary CFT$_3$. One thing to show is that one may recover the bulk entropy $S_M$ from the presumed formula (51) with (52).

The AdS$_5$/CFT$_4$ correspondence also represents a concrete realization of the holographic principle. In this case, there is no Smarr formula to derive a direct relation like equations (47) and (50). Instead, one proposes the Cardy–Verlinde formula for strongly coupled CFT$_4$ with its AdS dual [26]. It is known that this formula holds for various kinds of asymptotically AdS spacetimes including the TAdS black holes [21]. The boundary spacetimes for CFT$_4$ are defined through the AdS$_5$/CFT$_4$ correspondence [27]:

$$ds^2_{\text{CFT}_4} = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2_{\text{5DTBH}} = -\frac{R^2}{l^2} dt^2 + R^2 d\Sigma_k^2.$$  (53)

From the above, the relation between the five-dimensional bulk and four-dimensional boundary quantities is given by $E_{\text{CFT}} = (l/R)E$ and $T_{\text{CFT}} = (l/R)T$ where the size of the boundary space $R$ satisfies $T_{\text{CFT}} > 1/R$. As is expected, we obtain the same entropy: $S_{\text{CFT}} = S$. We note that the boundary system at high temperature is described by the CFT radiation with the equation of state $p = E_{\text{CFT}}/3V$. Then, the Casimir energy defined by

$$E_c = 3(V_{\text{CFT}} + pV_3 - T_{\text{CFT}}S_{\text{CFT}})$$

[28] is necessary to obtain the Cardy–Verlinde formula

$$S_{\text{CFT}} = \frac{2\pi R}{3\sqrt{|k|}} \sqrt{E_c(2E_{\text{CFT}} - E_c)}.$$  (54)

The non-zero Casimir energy reflects that the Euler relation is not satisfied. We find the boundary thermal quantities for the 5DTBH as functions of $\hat{r} = r_+/l$ [12, 22, 29]:

$$E_{\text{CFT}}^t = \frac{3V_3\hat{r}^2}{R} \left[ \hat{r}^2 - 1 + \frac{1}{4\hat{r}^2} \right], \quad T_{\text{CFT}}^t = \frac{\hat{r}}{2\pi R} [2\hat{r}^2 - 1],$$

$$E_c^t = -\frac{3V_3\hat{r}^2(2\hat{r}^2 - 1)}{R},$$  (55)

with $\kappa = l^3/16\pi G$. On the other hand, for the TGBAdS case, we find the boundary thermal quantities

$$E_{\text{CFT}}^g = \frac{3V_3\hat{r}^2}{R} \left[ \hat{r}^2 - 1 + \frac{1}{4\hat{r}^2} \right], \quad T_{\text{CFT}}^g = \frac{\hat{r}}{2\pi R (\hat{r}^2 - 2\hat{c})} [2\hat{r}^2 - 1],$$

$$E_c^g = -\frac{3V_3\hat{r}^2(2\hat{r}^2 - 1)}{R} \left[ (1 - 8\hat{c})\hat{r}^2 - 2\hat{c} \right]/\hat{r}^2 - 2\hat{c}.$$  (56)

Concerning the AdS/CFT correspondence, we remind the reader that the boundary CFT$_4$ energy ($E_{\text{CFT}}$) should be positive in order for it to make sense. Also, the Casimir energy ($E_c$) is related to the central charge of the corresponding CFT$_4$. If it is negative, one may obtain non-unitary CFT$_4$. In this sense, for $\hat{r} > 1/\sqrt{2}$, the 5DTBH provides a non-unitary conformal field theory because of $E_{\text{CFT}}^t > 0$ and $E_c^t < 0$. Similarly, for TGBAdS with $\hat{r} < 0$ and $\hat{r} > 1/\sqrt{2}$, it provides a non-unitary conformal field theory with $E_{\text{CFT}}^g > 0$ and $E_c^g < 0$. For $\hat{r} < 1/\sqrt{2}$, the two cases are not well defined because of negative energy and temperature on the boundary. Hence, we do not have any Cardy–Verlinde formula for the 5DTBH and TGBAdS.

Finally, we introduce the Cardy–Verlinde formula (54) to show whether or not the subtraction scheme does improve the boundary CFT$_4$ of the topological black hole. However,
it does not resolve an original issue of the non-existence of Cardy–Verlinde formula for the topological black holes.

5. Discussion

We believe that for a large class of black holes, the Bekenstein–Hawking entropy receives logarithmic corrections due to thermodynamic fluctuations around thermal equilibrium. However, small black holes with a hyperbolic horizon have the negative energy, even though they have positive heat capacity. Hence, a direct application of equation (1) to these black holes is not suitable for investigating their thermodynamic fluctuations. We resolve this issue by introducing the mass of the extremal black hole as the background state energy for 4DTBH, MTZ, 5DTBH, TGBAdS black holes. In order to avoid a negative mass problem, we use the subtraction scheme. The entropy-corrected formula is not changed because the subtraction scheme corresponds to a constant shifting from $M$ to $E = M - M_e$. We have used equation (1) to study thermal fluctuations of black holes with a hyperbolic horizon.

Especially, we find that the MTZ black hole dressed by a scalar has the best thermodynamic property in AdS4 spacetimes as the BTZ black hole shows in AdS3 spacetimes. Hence, we propose the MTZ black hole with a hyperbolic horizon as the cornerstone to study the 4D black holes in AdS4 spacetimes, better than the Schwarzschild–AdS black hole. This is because the Schwarzschild–AdS black hole with a spherical horizon could not provide a completely thermodynamic picture as the MTZ black hole shows [20, 22].

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