4-Dimensional Euler-Totient Matrix Operator and Some Double Sequence Spaces

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Abstract
Our main purpose in this study is to investigate the matrix domains of the 4-dimensional Euler-totient matrix operator on the classical double sequence spaces $M_u, C_p, C_{bp}$ and $C_r$. Besides these, we examine their topological and algebraic properties and give inclusion relations about the new spaces. Also, the $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals of these spaces are determined and finally, some matrix classes are characterized.

Keywords: Euler function, Möbius function, 4-dimensional Euler-totient matrix operator, matrix domain, double sequence space, $\alpha-$, $\beta(\vartheta)$- and $\gamma-$duals, matrix transformations.

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1. Preliminaries, Background and Notations

The function $f$ described by $f : \mathbb{N} \times \mathbb{N} \to \varphi, (t,u) \mapsto f(t,u) = x_{tu}$ is entitled as double sequence, where $\varphi$ denotes any nonempty set and $\mathbb{N} = \{1, 2, ...\}$. $\Omega$ stands for the set of all complex valued double sequences. It is well known that this set is a vector space with coordinatewise addition and scalar multiplication. Any linear subspace of $\Omega$ is called as double sequence space. The set of all bounded complex valued double sequences is symbolized with $M_u$, that is,

$$M_u = \left\{ x = (x_{tu}) \in \Omega : \|x\|_\infty = \sup_{t,u \in \mathbb{N}} |x_{tu}| < \infty \right\}. $$

It should be noted that $M_u$ is a Banach space with the norm $\|x\|_\infty$. We say that the double sequence $x = (x_{tu})$ is convergent in the Pringsheim’s sense provided that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{tu} - L| < \varepsilon$ whenever $t,u > n_\varepsilon$. In that case, $L \in \mathbb{C}$ is called the Pringsheim limit of $x$ and stated by $p - \lim_{t,u \to \infty} x_{tu} = L$, where $\mathbb{C}$ denotes the complex field. $C_p$ represents the space of all such $x$ which are called shortly as $p$-convergent. Of particular interest is unlike single sequences, $p$-convergent double sequences need not be bounded. For example, if we consider the sequence $x = (x_{tu})$ identified by

$$x_{tu} = \begin{pmatrix}
1 & 2 & 3 & \cdots & u & \cdots \\
2 & 0 & 0 & \cdots & 0 & \cdots \\
3 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
t & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots
\end{pmatrix},$$

it can easily seen that $p - \lim x_{tu} = 0$ but $\|x\|_\infty = \infty$. As a conclusion $x \in C_p \setminus M_u$. The bounded sequences which are also $p$-convergent are indicated by $C_{bp}$, that is, $C_{bp} = C_p \cap M_u$. A double sequence $x = (x_{tu}) \in C_p$ is called as regularly convergent if the limits $x_t := \lim_{u \to \infty} x_{tu} (t \in \mathbb{N})$ and $x_u := \lim_{t \to \infty} x_{tu} (u \in \mathbb{N})$ exist, and the limits $\lim_t \lim_u x_{tu}$

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and \( \lim_{u} \lim_{t} x_{tu} \) exist and are equivalent to the \( p \)-\( \lim \) of \( x \). The space of all regularly convergent double sequences is denoted by \( C \). Obviously, the regular convergence of a double sequence \( x \) implies the convergence in Pringsheim's sense as well as the boundedness of the terms of \( x \), but the converse implication fails. A sequence \( x = (x_{tu}) \) is called \textit{double null sequence} if it converges to zero. Additionally, all double null sequences in the spaces \( C_{bp} \) and \( C \) are denoted by \( C_{0} \) and \( C_{0} \), respectively. Móricz [25] showed that the spaces \( C_{bp}, C_{0}, C \) and \( C_{0} \) are Banach spaces endowed with the norm \( \| \cdot \|_{\infty} \).

Let us take any \( x \in \Omega \) and describe the sequence \( K = (k_{rs}) \) defined by

\[
k_{rs} := \sum_{t=1}^{r} \sum_{u=1}^{s} x_{tu}, \quad (r, s \in \mathbb{N}).
\]

In that case, the pair \((x_{rs}, (k_{rs}))\) is entitled as \textit{double series}. Here, the sequence \( K = (k_{rs}) \) is the sequence of \textit{partial sums} of the double series.

Consider the double sequence space \( \Psi \) converging with respect to some linear convergence rule \( \vartheta - \lim : \Psi \to \mathbb{C} \). The sum of a double series \( \sum_{t,u} x_{tu} \) relating to this rule is defined by \( \vartheta - \sum_{t,u} x_{tu} = \vartheta - \lim_{r,s \to \infty} s_{rs} \). Here and thereafter, when needed we will use the summation \( \sum_{r,s} \) instead of \( \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \), assume that \( \vartheta \in \{ p, bp, r \} \) and \( p' \) denotes the conjugate of \( p \), that is, \( p' = p/(p - 1) \) for \( 1 < p < \infty \). With the notation of Zeltser [42], we describe the double sequences \( e^{rs} = (e^{rs}_{tu}) \) and \( e_{tu}^{rs} = 1 \) if \( (r, s) = (t, u) \) and \( e_{tu}^{rs} = 0 \) otherwise, and \( e = \sum_{r,s} e^{rs} \) (coordinatewise convergence) for every \( r, s, t, u \in \mathbb{N} \).

The \( \alpha - \text{dual} \) of \( \Psi \), \( \beta(\vartheta) - \text{dual} \) of \( \Psi \), and the \( \gamma - \text{dual} \) of \( \Psi \) with respect to the \( \vartheta - \text{convergence} \) and the \( \gamma - \text{dual} \) of \( \Psi \) of a double sequence space \( \Psi \) are described as

\[
\begin{align*}
\Psi^{\alpha} & := \left\{ c = (c_{tu}) \in \Omega : \sum_{t,u} |c_{tu}x_{tu}| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi \right\}, \\
\Psi^{\beta(\vartheta)} & := \left\{ c = (c_{tu}) \in \Omega : \vartheta - \sum_{t,u} c_{tu}x_{tu} \quad \text{exists for all} \quad (x_{tu}) \in \Psi \right\}, \\
\Psi^{\gamma} & := \left\{ c = (c_{tu}) \in \Omega : \sup_{r,s \in \mathbb{N}} \left| \sum_{t,u=1}^{r,s} c_{tu}x_{tu} \right| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi \right\},
\end{align*}
\]

respectively. It is well known that \( \Psi^{\alpha} \subset \Psi^{\gamma} \) and if \( \Psi \subset \Lambda \), then \( \Lambda^{\alpha} \subset \Psi^{\alpha} \) for the double sequence spaces \( \Psi \) and \( \Lambda \).

Let us remember the definition of triangle matrix. If \( b_{rstu} = 0 \) for \( t > r \) or \( u > s \) or both for every \( r, s, t, u \in \mathbb{N} \), it is said that \( B = (b_{rstu}) \) is a \textit{triangular matrix} and also if \( b_{rstu} \neq 0 \) for every \( r, s \in \mathbb{N} \), then the 4-dimensional matrix \( B \) is called \textit{triangle}. It should be noted by [11] that, every triangle has a unique inverse which is also a triangle.

Now, we shall deal with matrix mapping. Let us consider double sequence spaces \( \Psi \) and \( \Lambda \) and the 4-dimensional complex infinite matrix \( B = (b_{rstu}) \). In that case, we say that \( B \) defines a \textit{matrix mapping} from \( \Psi \) into \( \Lambda \) and it is written as \( B : \Psi \to \Lambda \), if for every sequence \( x = (x_{tu}) \in \Psi \), the \( B \)-transform \( Bx = \{(Bx)_{rs}\}_{r,s \in \mathbb{N}} \) of \( x \) exists and is in \( \Lambda \); where

\[
(Bx)_{rs} = \vartheta - \sum_{t,u} b_{rstu}x_{tu}, \quad (1.1)
\]

for each \( r, s \in \mathbb{N} \). \( (\Psi : \Lambda) \) stands for the class of all 4-dimensional complex infinite matrices from a double sequence space \( \Psi \) into a double sequence space \( \Lambda \). In that case, \( B \in (\Psi : \Lambda) \) if and only if \( B_{rs} \in \Psi^{\beta(\vartheta)} \), where \( B_{rs} = (b_{rstu})_{t,u \in \mathbb{N}} \) for all \( r, s \in \mathbb{N} \).

The \( \vartheta \)-summability domain \( \Psi^{(\vartheta)}_{B} \) of a 4-dimensional complex infinite matrix \( B \) in a double sequence space \( \Psi \) consists of whose \( B \)-transforms are in \( \Psi \); that is,

\[
\Psi^{(\vartheta)}_{B} := \left\{ x = (x_{tu}) \in \Omega : Bx := \left( \vartheta - \sum_{t,u} b_{rstu}x_{tu} \right)_{r,s \in \mathbb{N}} \quad \text{exists and is in} \quad \Psi \right\}.
\]

In the past, many authors were interested in double sequence spaces. Now, let us give some information about these studies. Zeltser [41] has fundamentally examined both the topological structure and the theory of summability of double sequences in her doctoral dissertation. Recently, Altay and Başar [3] defined the double sequence spaces \( BS, BS(t), CS_{p}, CS_{bp}, CS_{r}, \) and \( BV \) of double series whose sequences of partial sums are in the spaces \( M_{u}, M_{u}(t), C_{p}, C_{bp}, C_{r}, \) and \( L_{u} \), respectively, and also examined some properties of those spaces. Later, in [5], Başar and Sever
have defined the set $L_p$ of all absolutely $p$-summable double sequences which is a Banach space with the norm $\| \cdot \|_{L_p}$ defined in the following way:

$$\| \cdot \|_{L_p} = \left( \sum_{t,u} |x_{tu}|^p \right)^{\frac{1}{p}}.$$

It is also significant that the double sequence space $L_p$ which was defined by Zeltser [42] is the special case of the space $L_p$ for $p = 1$. For more details about the double sequences and related topics, the reader may refer to [1, 3–5, 13, 25–28, 31, 35–40, 43] and references therein.

In the rest of the study, $\varphi$ and $\mu$ represent Euler function and the Möbius function, respectively. For every $r \in \mathbb{N}$ with $r > 1$, $\varphi(r)$ is the number of positive integers less than $r$ which are coprime with $r$ and $\varphi(1) = 1$. If $a_1 b_1 a_2 b_2 a_3 b_3 ... a_m b_m$ is the prime factorization of a natural number $r > 1$, then

$$\varphi(r) = r(1 - \frac{1}{a_1})(1 - \frac{1}{a_2})(1 - \frac{1}{a_3})...(1 - \frac{1}{a_m}).$$

Also, the equality

$$r = \sum_{t | r} \varphi(t)$$

holds for every $r \in \mathbb{N}$ and $\varphi(r_1 r_2) = \varphi(r_1) \varphi(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime. Given any $r \in \mathbb{N}$ with $r > 1$, $\mu$ is defined as

$$\mu(r) := \begin{cases} (-1)^m & \text{if } r = a_1 a_2 ... a_m, \text{ where } a_1 a_2 ... a_m \text{ are non-equivalent prime numbers} \\ 0 & \text{if } a^2 | r \text{ for some prime number } a, \end{cases}$$

and $\mu(1) = 1$. If $a_1 b_1 a_2 b_2 a_3 b_3 ... a_m b_m$ is the prime factorization of a natural number $r > 1$, in this fact,

$$\sum_{t | r} t \mu(t) = (1 - a_1)(1 - a_2)(1 - a_3)...(1 - a_m).$$

If $r \neq 1$, then the equality

$$\sum_{t | r} \mu(t) = 0$$

holds and $\mu(r_1 r_2) = \mu(r_1) \mu(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime.

By using the regular 2-dimensional Euler-totient matrix $\Phi$, the Euler-totient sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ which consist of all sequences whose $\Phi$-transforms are in the spaces $\ell_p$ of absolutely $p$-summable and $\ell_\infty$ of bounded single sequences are introduced and examined by İlkhan and Kara [18].

The target of the existing study is to acquaint the matrix domains of the 4-dimensional Euler-totient matrix on some classical double sequence spaces.

### 2. Domain of Euler-Totient Matrix in Some Spaces of Double Sequences

In this section, we introduce the double sequence spaces $\Phi^* (\mathcal{M}_u)$, $\Phi^* (\mathcal{C}_p)$, $\Phi^* (\mathcal{C}_p)$ and $\Phi^* (\mathcal{C}_r)$ by using the 4-dimensional Euler-totient matrix $\Phi^*$ and give some properties and results on these spaces.

In [14], we have defined the 4-dimensional matrix $\Phi^* = (\Phi_{rstu})$ which is called Euler-totient matrix operator as follows:

$$\Phi_{rstu}^* := \begin{cases} \frac{\varphi(t) \varphi(u)}{rs} & \text{if } t | r, u | s, \\ 0 & \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. Thus, it is clear that $\Phi^*$ is a triangle and the $\Phi^*$-transform of a double sequence $x = (x_{rs})$ is given by

$$y_{rs} := (\Phi^* x)_{rs} = \frac{1}{r s} \sum_{t | r, u | s} \varphi(t) \varphi(u) x_{tu},$$

(2.2)
for every \( r, s \in \mathbb{N} \). Throughout the article, we suppose that the terms of the double sequences \( x = (x_{rs}) \) and \( y = (y_{rs}) \) are connected with the relation (2.2).

The inverse \((\Phi^*)^{-1} = (\phi_{rstu}^*)^{-1}\) of the triangle matrix \( \Phi^* \) is calculated as

\[
\phi_{rstu}^* := \begin{cases} 
\frac{\mu(\frac{r}{s}) \mu(\frac{u}{t})}{\varphi(r) \varphi(s)}, & \text{if } t \mid r, \ u \mid s, \\
0, & \text{otherwise},
\end{cases}
\]

for every \( r,s,t,u \in \mathbb{N} \). We introduce the sequence spaces \( \Phi^*(\mathcal{M}_u) \), \( \Phi^*(\mathcal{C}_p) \), \( \Phi^*(\mathcal{C}_{bp}) \) and \( \Phi^*(\mathcal{C}_r) \) as the sets of all double sequences such that \( \Phi^* \)-transforms of them are in the spaces \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp} \) and \( \mathcal{C}_r \), that is,

\[
\Phi^*(\mathcal{M}_u) = \left\{ x = (x_{rs}) \in \Omega : \sup_{r,s \in \mathbb{N}} \frac{1}{rs} \sum_{t \mid [r,u]} \varphi(t) \varphi(u) x_{tu} < \infty \right\},
\]

\[
\Phi^*(\mathcal{C}_p) = \left\{ x = (x_{rs}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{r,s \to \infty} \frac{1}{rs} \sum_{t \mid [r,u]} \varphi(t) \varphi(u) x_{tu} - L = 0 \right\},
\]

\[
\Phi^*(\mathcal{C}_{bp}) = \left\{ x = (x_{rs}) \in \Omega : \frac{1}{rs} \sum_{t \mid [r,u]} \varphi(t) \varphi(u) x_{tu} \in \mathcal{C}_{bp} \right\},
\]

\[
\Phi^*(\mathcal{C}_r) = \left\{ x = (x_{rs}) \in \Omega : \frac{1}{rs} \sum_{t \mid [r,u]} \varphi(t) \varphi(u) x_{tu} \in \mathcal{C}_r \right\}.
\]

It is immediately seen that \( \Phi^*(\mathcal{M}_u) \), \( \Phi^*(\mathcal{C}_p) \), \( \Phi^*(\mathcal{C}_{bp}) \) and \( \Phi^*(\mathcal{C}_r) \) are the domains of the 4-dimensional Euler-totient matrix \( \Phi^* \) in the spaces \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp} \) and \( \mathcal{C}_r \), respectively.

If \( \Psi \) is any normed double sequence space, then we call the matrix domain \( \Phi^*(\Psi) \) as the double Euler-totient sequence space.

**Definition 2.1** (See [16],[31]). A 4-dimensional matrix \( B \) is said to be RH-regular if it maps every bounded \( p \)-convergent sequence into a \( p \)-convergent sequence with the same \( p \)-limit.

**Lemma 2.1** (See [16],[31]). A 4-dimensional triangle matrix \( B = (b_{rstu}) \) is RH-regular iff

\[
RH_1 : p - \lim_{r,s \to \infty} b_{rstu} = 0 \quad \text{for each} \quad t,u \in \mathbb{N},
\]

\[
RH_2 : p - \lim_{r,s \to \infty} \sum_{t,u} b_{rstu} = 1,
\]

\[
RH_3 : p - \lim_{r,s \to \infty} \sum_{t} |b_{rstu}| = 0 \quad \text{for each} \quad u \in \mathbb{N},
\]

\[
RH_4 : p - \lim_{r,s \to \infty} \sum_{u} |b_{rstu}| = 0 \quad \text{for each} \quad t \in \mathbb{N},
\]

\[
RH_5 : \text{There exists finite positive integers} \quad M_1 \quad \text{and} \quad M_2 \quad \text{such that}
\]

\[
\sum_{t,u > M_1} |b_{rstu}| < M_2.
\]

It should be noted that the 4-dimensional Euler-totient matrix \( \Phi^* \) described by (2.1) is RH-regular [14].

Now, we may continue with the following two theorems which are the essential in the study.

**Theorem 2.1.** The sets \( \Phi^*(\mathcal{M}_u) \), \( \Phi^*(\mathcal{C}_{bp}) \) and \( \Phi^*(\mathcal{C}_r) \) are the linear spaces which are linearly norm isomorphic to the spaces \( \mathcal{M}_u, \mathcal{C}_{bp} \) and \( \mathcal{C}_r \), respectively, and are the Banach spaces with the norm

\[
\|x\|_{\Phi^*(\mathcal{M}_u)} = \|\Phi^* x\|_{\infty} = \sup_{r,s \in \mathbb{N}} \frac{1}{rs} \sum_{t \mid [r,u]} \varphi(t) \varphi(u) x_{tu}.
\]
Proof. To avoid the repetition of the similar statements, we give the proof only for the space \( \Phi^*(M_u) \). Since the initial assertion is routine verification and is easy to prove, we ignore its proof in here. To confirm the fact that \( \Phi^*(M_u) \) is linearly norm isomorphic to the space \( M_u \), we need to be sure the existence of a linear and norm preserving bijection between the spaces \( \Phi^*(M_u) \) and \( M_u \). For this purpose, let us take the transformation \( B \) defined from \( \Phi^*(M_u) \) into \( M_u \) by \( y = Bx \), where \( y = (y_{rs}) \) is the \( \Phi^* \)-transform of the sequence \( x = (x_{tu}) \). The linearity of \( B \) is clear. Consider the equality \( Bx = \theta \) which yields us that \( x_{tu} = 0 \) for every \( t,u \in \mathbb{N} \). So, \( x = \theta \). Therefore, \( B \) is injective. Let us consider \( y \in M_u \) and describe the double sequence \( x = (x_{rs}) \) by

\[
x_{rs} = \sum_{t|r,u|} \frac{\mu(t) \mu(s)}{\varphi(t) \varphi(s)} |tuy_{tu}|
\]

for every \( r,s \in \mathbb{N} \). By taking supremum over \( r,s \in \mathbb{N} \) on the following equality

\[
|\langle \Phi^*x \rangle_{rs}| = \left| \sum_{t|r,u|} \varphi(t) \varphi(u) x_{tu} \right| = |y_{rs}|
\]

it can be derived that \( B \) is surjective and norm preserving.

Now, we may prove that \( \Phi^*(M_u) \) is a Banach space with the norm \( \| \cdot \|_{\Phi^*(M_u)} \) described by (2.3). Since \( M_u \) is a Banach space from [25], we obtain the desired result from Section (b) of Corollary 6.3.41 in [6].

Theorem 2.2. The set \( \Phi^*(C_p) \) is linearly isomorphic to the space \( C_p \) and is a complete semi-normed space with the semi-norm

\[
\|x\|_{\Phi^*(C_p)} = \lim_{i \to \infty} \left( \sup_{r,s \geq i} |\langle \Phi^*x \rangle_{rs}| \right).
\]

Proof. Since the proof of the theorem is similar to the proof of Theorem 2.1, we ignore it.

Now, let us give our results about inclusion relations.

Theorem 2.3. The inclusion \( M_u \subset \Phi^*(M_u) \) holds.

Proof. Let us take a sequence \( x = (x_{tu}) \in M_u \). In that case, there exists a positive real number \( M_3 \) such that \( \sup_{t,u \in \mathbb{N}} |x_{tu}| \leq M_3 \). Therefore, one can immediately see that

\[
\|x\|_{\Phi^*(M_u)} = \sup_{r,s \in \mathbb{N}} \left| \sum_{t|r,u|} \varphi(t) \varphi(u) x_{tu} \right|
\]

\[
\leq \sup_{r,s \in \mathbb{N}} \left| \sum_{t|r,u|} \varphi(t) \varphi(u) |x_{tu}| \right|
\]

\[
\leq M_3 \sup_{r,s \in \mathbb{N}} \left| \sum_{t|r,u|} \varphi(t) \varphi(u) \right| = M_3.
\]

Thus, the inclusion is valid.

Theorem 2.4. The inclusion \( C_{bp} \subset \Phi^*(C_p) \) holds.

Proof. Let us take the sequence \( x = (x_{tu}) \in C_{bp} \) with \( p \lim_{t,u \to \infty} x_{tu} = L \). Since 4-dimensional Euler-totient matrix is RH-regular, \( p \lim_{t,u \to \infty} y_{tu} = L \), where \( y_{tu} = (\Phi^*x)_{tu} \). Hence, we see that \( C_{bp} \subset \Phi^*(C_p) \).
3. Dual Spaces

In the current section, we tend to compute the $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals of the new double Euler-totient sequence spaces.

**Theorem 3.1.** The $\alpha$-dual of the space $\Phi^*(\mathcal{M}_u)$ is $\mathcal{L}_u$.

**Proof.** Suppose that $c = (c_{rs}) \in \{ \Phi^*(\mathcal{M}_u) \}^\alpha$ but $c \notin \mathcal{L}_u$. Then, $\sum_{r,s} |c_{rs} x_{rs}| < \infty$ for all $x = (x_{rs}) \in \Phi^*(\mathcal{M}_u)$. If we consider $c \in \Phi^*(\mathcal{M}_u)$, in that case $cc = c \notin \mathcal{L}_u$, that is $c \notin \{ \Phi^*(\mathcal{M}_u) \}^\alpha$ and it is seen that this is a contradiction. Thus, $c$ must be in $\mathcal{L}_u$.

Conversely, let us take sequences $c = (c_{rs}) \in \mathcal{L}_u$ and $x = (x_{rs}) \in \Phi^*(\mathcal{M}_u)$. In that case, there exists a double sequence $y = (y_{rs}) \in \mathcal{M}_u$ such that $y = \Phi^* x$ and $\sup_{r,s} |y_{rs}| < M_4$, where $M_4 \in \mathbb{R}^+$. Then, we have from the following inequality

$$
\sum_{r,s} |c_{rs} x_{rs}| = \sum_{r,s} |c_{rs}| \left| \sum_{t,u} \frac{\mu(t)}{\varphi(t)} \frac{\mu(u)}{\varphi(u)} t uy_{tu} \right|
\leq M_4 \sum_{r,s} |c_{rs}| \left| \sum_{t,u} \frac{\mu(t)}{\varphi(t)} \frac{\mu(u)}{\varphi(u)} t uy_{tu} \right|
= M_4 \sum_{r,s} |c_{rs}| < \infty,
$$

that $c \in (\Phi^*(\mathcal{M}_u))^\alpha$ and this completes the proof. \qed

Now, we give some lemmas which characterize the classes of 4-dimensional matrix mappings(see [16], [42] and [43]). With the help of these lemmas, we will calculate the $\beta(\vartheta)$, $\beta(bp)$, $\beta(p)$ and $\gamma$-duals of our new double sequence spaces.

**Lemma 3.1.** Suppose that $B = (b_{rstu})$ is a 4-dimensional infinite matrix. Then, $B \in (\mathcal{C}_{bp} : \mathcal{C}_\vartheta)$ iff following conditions hold:

$$\sup_{r,s \in \mathbb{N}} \sum_{t,u} |b_{rstu}| < \infty, \quad (3.1)$$

$$\exists b_{tu} \in \mathcal{C} \ni \vartheta - \lim_{r,s \to \infty} b_{rstu} = b_{tu} \text{ for all } t, u \in \mathbb{N}, \quad (3.2)$$

$$\exists L \in \mathcal{C} \ni \vartheta - \lim_{r,s \to \infty} \sum_{t,u} b_{rstu} = L \text{ exists,} \quad (3.3)$$

$$\exists \psi_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s \to \infty} \sum_{t} |b_{rstu} - b_{tu}| = 0, \quad (3.4)$$

$$\exists \psi_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s \to \infty} \sum_{t} |b_{rstu} - b_{tu}| = 0. \quad (3.5)$$

In the case of (3.5), $b = (b_{tu}) \in \mathcal{L}_u$ and

$$\vartheta - \lim_{r,s \to \infty} [B]_{rs} \vartheta = \sum_{t,u} b_{tu} x_{tu} + \left( L - \sum_{t,u} b_{tu} \right) bp - \lim_{r,s \to \infty} x_{rs}$$
satisfies for $x \in \mathcal{C}_{bp}$.

**Lemma 3.2.** Suppose that $B = (b_{rstu})$ is a 4-dimensional infinite matrix. Then, $B \in (\mathcal{C}_p : \mathcal{C}_\vartheta)$ iff (3.1)-(3.3) hold and the following conditions hold, too:

$$\forall t \in \mathbb{N}, \quad \exists u_0 \in \mathbb{N} \ni b_{rstu} = 0 \text{ for every } u > u_0 \text{ and } r, s \in \mathbb{N}, \quad (3.6)$$

$$\forall u \in \mathbb{N}, \quad \exists t_0 \in \mathbb{N} \ni b_{rstu} = 0 \text{ for every } t > t_0 \text{ and } r, s \in \mathbb{N}. \quad (3.7)$$

In the case of (3.7), $\exists_0, u_0 \in \mathbb{N}$ such that $b = (b_{tu}) \in \mathcal{L}_u$ and $(b_{tu})_{u \in \mathbb{N}}, (b_{tu})_{u \in \mathbb{N}} \in \zeta$, where $\zeta$ represents the space of every finitely sequences which are non-equivalent zero and

$$\vartheta - \lim_{r,s \to \infty} [B]_{rs} \vartheta = \sum_{t,u} b_{tu} x_{tu} + \sum_{t} \left( L - \sum_{t,u} b_{tu} \right) p - \lim_{r,s \to \infty} x_{rs}$$
satisfies for \( x \in \mathcal{C}_p \).

**Lemma 3.3.** Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. Then, \( B \in (\mathcal{C}_r : \mathcal{C}_\vartheta) \) iff (3.1)-(3.3) hold and the following conditions hold, too:

\[
\exists u_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s \to \infty} \sum_t b_{rstu_0} = \rho_{u_0}, \tag{3.8}
\]

\[
\exists t_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s \to \infty} \sum_u b_{rst_0u} = \varrho_{t_0}. \tag{3.9}
\]

In the case of (3.9), \( b = (b_{tu}) \in \mathcal{L}_u \) and \( \rho_{u_0}, \varrho_{t_0} \in \ell_1 \)

\[
\vartheta - \lim_{r,s \to \infty} [Bx]_{rs} = \sum_{t,u} b_{tu}x_{tu} + \sum_{t} \left( \varrho_{t} - \sum_u b_{tu} \right) x_t + \sum_u \left( \rho_{u} - \sum_t b_{tu} \right) x_u + \left( L + \sum_{t,u} b_{tu} - \sum_{u} \varrho_{t} - \sum_{t} \rho_{u} \right) r - \lim_{r,s \to \infty} x_{rs}
\]

satisfies for \( x \in \mathcal{C}_r \).

**Lemma 3.4.** \([36]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. Then, \( B \in (\mathcal{C}_{bp} : \mathcal{M}_u) \) iff the condition (3.1) holds.

**Lemma 3.5.** \([12]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. Then, \( B \in (\mathcal{M}_u : \mathcal{C}_{bp}) \) iff the conditions (3.1), (3.2) hold and the following conditions hold, too:

\[
\exists b_{tu} \in \mathcal{C} \ni bp - \lim_{r,s \to \infty} \sum_{u=0}^{s} b_{rstu} \text{ exists for each } t \in \mathbb{N}, \tag{3.10}
\]

\[
bp - \lim_{r,s \to \infty} \sum_{t=0}^{r} b_{rstu} \text{ exists for each } u \in \mathbb{N}, \tag{3.11}
\]

\[
\sum_{t,u} |b_{rstu}| \text{ converges}. \tag{3.12}
\]

**Lemma 3.6.** \([38]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. Then, \( B \in (\mathcal{M}_u : \mathcal{M}_u) \) iff the condition (3.1) holds.

**Lemma 3.7.** \([39]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. Then, \( B \in (\mathcal{M}_u : \mathcal{C}_p) \) iff the conditions (3.2), (3.6) and (3.7) hold.

**Lemma 3.8.** \([40]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. In that case:

(i) If \( 0 < p \leq 1 \), then \( B \in (\mathcal{L}_p : \mathcal{M}_u) \) iff

\[
\sup_{r,s,t,u \in \mathbb{N}} |b_{rstu}| < \infty, \tag{3.14}
\]

(ii) If \( 1 < p < \infty \), then \( B \in (\mathcal{L}_p : \mathcal{M}_u) \) iff

\[
\sup_{r,s \in \mathbb{N}} \sum_{t,u} |b_{rstu}|^p < \infty. \tag{3.15}
\]

**Lemma 3.9.** \([40]\) Suppose that \( B = (b_{rstu}) \) is a 4-dimensional infinite matrix. In that case:

(i) If \( 0 < p \leq 1 \), then \( B \in (\mathcal{L}_p : \mathcal{C}_{bp}) \) iff the conditions (3.2) and (3.14) hold with \( \vartheta = bp \),

(ii) If \( 1 < p < \infty \), then \( B \in (\mathcal{L}_p : \mathcal{C}_{bp}) \) iff the conditions (3.2) and (3.15) hold.
Theorem 3.2. Consider the set \( w_1 \) defined by

\[
w_1 = \left\{ c = (c_{rs}) \in \Omega : \sup_{r,s} \left| \sum_{t,u} |c(r,s,t,u,m,n)| \right| < \infty \right\},
\]

where

\[
\sigma(r,s,t,u,m,n) = \sum_{m=t,t|m=n=u|n} \sum_{u,u|n} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} \mu(t) \mu(u) \phi(m) \phi(n) t u c_{mn}.
\]

Then, \( (\Phi^*(C_{bp}))^\gamma = w_1 = (\Phi^*(M_u))^\gamma \).

Proof. Suppose that \( c = (c_{rs}) \in \Omega \) and \( x = (x_{rs}) \in \Phi^*(C_{bp}) \). Then, we can conclude from (2.2) that \( y = (y_{rs}) \in C_{bp} \).

Now, let us define the 4-dimensional matrix \( O = (o_{rstu}) \) by

\[
o_{rstu} := \begin{cases} \sigma(r,s,t,u,m,n), & t \mid m, \quad u \mid n, \\ 0, & \text{otherwise}, \end{cases}
\]

for every \( r,s,t,u \in \mathbb{N} \). Therefore, we obtain by using the relation (2.4) that

\[
z_{rs} = \sum_{t,u=1}^{r,s} c_{tu} x_{tu} = \sum_{t,u=1}^{r,s} c_{tu} \left[ \sum_{m=t,t|m=n=u|n} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} m n y_{mn} \right] \]

\[
= \sum_{t,u=1}^{r,s} \left[ \sum_{m=t,t|m=n=u|n} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} t u c_{mn} \right] y_{tu} = (Oy)_{rs}
\]

for every \( r,s \in \mathbb{N} \). Then, by considering the equality (3.16), we deduce that \( cx = (c_{rs} x_{rs}) \in BS \) whenever \( x \in \Phi^*(C_{bp}) \)

iff \( z = (z_{rs}) \in M_u \) whenever \( y \in C_{bp} \). This leads us to the fact that \( c = (c_{rs}) \in (\Phi^*(C_{bp}))^\gamma \) iff \( O \in (C_{bp} : M_u) \). Hence, we achieve that \( (\Phi^*(C_{bp}))^\gamma = w_1 \). The other part of the theorem can be proven by using similar technique. So, we omit it. \( \square \)
Theorem 3.3. Consider the sets \( w_2 - w_{13} \) defined by

\[
\begin{align*}
\Phi & (i) \quad w_2 = \left\{ c = (c_{rs}) \in \Omega : \exists b_{tu} \in C \ni \exists \vartheta - \lim_{r,s \to \infty} \sigma(r, s, t, u, m, n) = b_{tu} \right\}, \\
\Phi & (i) \quad w_3 = \left\{ c = (c_{rs}) \in \Omega : \exists \vartheta \ni \lim_{r,s \to \infty} \sum_{t,u} \sigma(r, s, t, u, m, n) = L \Rightarrow \text{exists} \right\}, \\
\Phi & (i) \quad w_4 = \left\{ c = (c_{rs}) \in \Omega : \exists u_0 \in \mathbb{N} \ni \lim_{r,s \to \infty} \sum_{t,u} |\sigma(r, s, t, u, m, n) - b_{tu_0}| = 0 \right\}, \\
\Phi & (i) \quad w_5 = \left\{ c = (c_{rs}) \in \Omega : \exists \vartheta \ni \lim_{r,s \to \infty} \sum_{u} |\sigma(r, s, t, u, m, n) - b_{tu_0}| = 0 \right\}, \\
\Phi & (i) \quad w_6 = \left\{ c = (c_{rs}) \in \Omega : \forall t \ni \exists u_0 \ni \exists \vartheta \ni \sigma(r, s, t, u, m, n) = 0, \forall u > u_0, \forall r, s \in \mathbb{N} \right\}, \\
\Phi & (i) \quad w_7 = \left\{ c = (c_{rs}) \in \Omega : \forall u \ni \exists \vartheta \ni \exists \sigma(r, s, t, u, m, n) = 0, \forall t > t_0, \forall r, s \in \mathbb{N} \right\}, \\
\Phi & (i) \quad w_8 = \left\{ c = (c_{rs}) \in \Omega : \exists u_0 \ni \exists \vartheta \ni \lim_{r,s \to \infty} \sum_{t,u} \sigma(r, s, t, u, m, n) = b_{u_0} \right\}, \\
\Phi & (i) \quad w_9 = \left\{ c = (c_{rs}) \in \Omega : \exists \vartheta \ni \lim_{r,s \to \infty} \sum_{u} \sigma(r, s, t, u, m, n) = b_{u_0} \right\}, \\
\Phi & (i) \quad w_{10} = \left\{ c = (c_{rs}) \in \Omega : \exists b_{tu} \in C \ni \exists bp \ni \lim_{r,s,t \to \infty} \sum_{t,u} \sigma(r, s, t, u, m, n) = b_{tu} \right\}, \\
\Phi & (i) \quad w_{11} = \left\{ c = (c_{rs}) \in \Omega : \forall t \ni \exists bp \ni \lim_{r,s \to \infty} \sum_{u=1}^{s} \sigma(r, s, t, u, m, n) \ni \text{exists} \right\}, \\
\Phi & (i) \quad w_{12} = \left\{ c = (c_{rs}) \in \Omega : \forall u \ni \exists bp \ni \lim_{r,s,t \to \infty} \sum_{t=1}^{r} \sigma(r, s, t, u, m, n) \ni \text{exists} \right\}, \\
\Phi & (i) \quad w_{13} = \left\{ c = (c_{rs}) \in \Omega : \sum_{t,u} |\sigma(r, s, t, u, m, n)| \ni \text{converges} \right\}.
\end{align*}
\]

In that case, following statements are satisfied:

(i) \((\Phi^*(C_{bp}))^{\beta(\vartheta)} = \bigcap_{k=1}^{5} w_k,\)

(ii) \((\Phi^*(C_p))^{\beta(\vartheta)} = \bigcap_{k=1}^{3} w_k \cap w_6 \cap w_7,\)

(iii) \((\Phi^*(C_*))^{\beta(\vartheta)} = \bigcap_{k=1}^{3} w_k \cap w_8 \cap w_9,\)

(iv) \((\Phi^*(M_{*}))^{\beta(bp)} = w_1 \cap w_2 \bigcap_{k=1}^{13} w_k,\)

(V) \((\Phi^*(M_*))^{\beta(p)} = w_2 \cap w_6 \cap w_7.\)

Proof.

(i) Suppose that \( c = (c_{rs}) \in \Omega \) and \( x = (x_{rs}) \in \Phi^*(C_{bp}). \) In that case, there exists a double sequence \( y = (y_{rs}) \in C_{bp} \) with \( \Phi^* x = y. \) Since (3.16) holds, we deduce that \( \exists c \in CS_{\vartheta} \) whenever \( x \in \Phi^*(C_{bp}) \) if \( z \in C_{\vartheta} \) whenever \( y \in C_{bp}. \) This leads us to the fact that \( c = (c_{rs}) \in (\Phi^*(C_{bp}))^{\beta(\vartheta)} \) if \( O \ni (C_{bp} : C_{\vartheta}). \) Therefore, the conditions of Lemma 3.1 are satisfied with \( O = (o_{rstu}) \) defined as in Theorem 3.2. Hence, we achieve that the \( \beta(\vartheta) \)-dual of the space \( \Phi^*(C_{bp}) \) is \( \bigcap_{k=1}^{5} w_k. \)

The other parts of the Theorem can be done analogously by using the Lemmas 3.2, 3.3, 3.5 and 3.7, respectively. So, we pass the details.
4. Charactarization of Some Classes of 4-Dimensional Matrices

In the current section, we deal with some 4-dimensional matrix mapping classes related to the double sequence spaces $\Phi^*(M_u), \Phi^*(C_p), \Phi^*(C_{bp})$ and $\Phi^*(C_r)$ by using dual summability methods for double sequences which have been presented and examined by Başar [4] and Yeşilkayagil and Başar [37] and which have been applied by Tuğ [36].

**Theorem 4.1.** Assume that the elements of 4-dimensional infinite matrices $B = (b_{rstu})$ and $H = (h_{rstu})$ are connected with the relation

$$h_{rstu} = \sum_{m=t,t|m=n,u,u}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} t u b_{rsmn}.$$  \hspace{1cm} (4.1)

Then, $B \in (\Phi^*(\Psi) : \Lambda)$ iff $B_{rs} \in [\Phi^*(\Psi)]^{\beta(\psi)}$ for every $r, s \in \mathbb{N}$ and $H \in (\Psi : \Lambda)$, where $\Psi$ and $\Lambda \in \{M_u, C_p, C_{bp}, C_r\}$.

**Proof.** Assume that $B \in (\Phi^*(\Psi) : \Lambda)$. In that case, $Bx$ exists and is in $\Lambda$ for every $x \in \Phi^*(\Psi)$ and it also implies that $B_{rs} \in [\Phi^*(\Psi)]^{\beta(\psi)}$ for every $r, s \in \mathbb{N}$. Thus, we have the following equality derived from partial sums of the series $\sum_{t,u} b_{rstu} x_{tu}$ with relation (4.2)

$$\sum_{t,u=1}^{i,j} b_{rstu} x_{tu} = \sum_{t,u=1}^{i,j} b_{rstu} \left[ \sum_{m=t,t|m=n,u,u} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} m n y_{mn} \right]$$

$$= \sum_{t,u=1}^{i,j} \left[ \sum_{m=t,t|m=n,u,u} \sum_{n=u,u|n} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} t u b_{rsmn} \right] y_{tu}$$

for every $i, j \in \mathbb{N}$. In that case, if we take $\beta$-limit on equality above as $i, j \to \infty$, we have $Bx = H y$. Therefore, we obtain that $H y \in \Lambda$ whenever $y \in \Psi$, that is $H \in (\Psi : \Lambda)$.

Conversely, suppose that $B_{rs} \in [\Phi^*(\Psi)]^{\beta(\psi)}$ for every $r, s \in \mathbb{N}, H \in (\Psi : \Lambda)$ and $x \in \Phi^*(\Psi)$ such that $y = \Phi^*x$. In that case, $Bx$ exists and therefore, the $(k,l)$th rectangular partial sums of the series $\sum_{t,u} b_{rstu} x_{tu}$ obtained as

$$(Bx)_{r,s}^{k,l} = \sum_{t,u=1}^{k,l} b_{rstu} x_{tu}$$

$$= \sum_{t,u=1}^{k,l} b_{rstu} \left[ \sum_{m=t,t|m=n,u,u} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} m n y_{mn} \right]$$

$$= \sum_{t,u=1}^{k,l} \left[ \sum_{m=t,t|m=n,u,u} \sum_{n=u,u|n} \frac{\mu(m)}{\varphi(m)} \frac{\mu(n)}{\varphi(n)} t u b_{rsmn} \right] y_{tu}$$

(4.2)

for every $r, s, k, l \in \mathbb{N}$. By taking $\beta$-limit on (4.2) while $k, l \to \infty$, it can be easily obtain from the following equality

$$\sum_{t,u} b_{rstu} x_{tu} = \sum_{t,u} h_{rstu} y_{tu}$$

for every $r, s \in \mathbb{N}$ that $Bx = H y$ which leads us to the fact that $B \in (\Phi^*(\Psi) : \Lambda)$. \hfill \Box

**Corollary 4.1.** Suppose that $B = (b_{rstu})$ is a 4-dimensional matrix. In that case the following statements are satisfied:

(i) $B \in (\Phi^*(C_p) : C_0)$ iff the conditions (3.1)-(3.3), (3.6) and (3.7) are satisfied with $h_{rstu}$ in place of $b_{rstu}$,

(ii) $B \in (\Phi^*(C_{bp}) : C_0)$ iff the conditions (3.1)-(3.5) are satisfied with $h_{rstu}$ in place of $b_{rstu}$,

(iii) $B \in (\Phi^*(C_r) : M_u)$ iff the condition (3.1) is satisfied with $h_{rstu}$ in place of $b_{rstu}$,

(iv) $B \in (\Phi^*(C_r) : C_0)$ iff the conditions (3.1)-(3.3), (3.8) and (3.9) are satisfied with $h_{rstu}$ in place of $b_{rstu}$,

(v) $B \in (\Phi^*(M_u) : C_{bp})$ iff the conditions (3.1), (3.2), (3.10)-(3.13) are satisfied with $h_{rstu}$ in place of $b_{rstu}$,
(vi) $B \in (\Phi^*(\mathcal{M}_u) : \mathcal{C}_p)$ iff the conditions (3.2), (3.6) and (3.7) are satisfied with $b_{rstu}$ in place of $b_{rstuv}$.

**Lemma 4.1.** [10] Let $\Psi$ and $\Lambda$ be two double sequence spaces, $B = (b_{rstu})$ be any 4-dimensional matrix and $F = (f_{rstu})$ also be a 4-dimensional triangle matrix such that $f_{rstu} = 0$ if $t > r$ and $u > s$ for every $r, s, t, u \in \mathbb{N}$. In that case, $B \in (\Psi : \Lambda_F)$ iff $FB \in (\Psi : \Lambda)$.

Now, let us define the 4-dimensional matrix $G = (g_{rstu})$ by

$$g_{rstu} = \sum_{m|r,n|s} \phi^*_{rstmn} b_{mntu}$$

for every $r, s, t, u \in \mathbb{N}$ and give following corollary.

**Corollary 4.2.** Suppose that $B = (b_{rstu})$ is a 4-dimensional matrix. In that case the following statements are satisfied:

(i) $B \in (\mathcal{C}_p : \Phi^*(\mathcal{C}_q))$ iff the conditions (3.1)-(3.3), (3.6) and (3.7) are satisfied with $g_{rstu}$ in place of $b_{rstu}$.
(ii) $B \in (\mathcal{C}_{bp} : \Phi^*(\mathcal{C}_q))$ iff the conditions (3.1)-(3.5) are satisfied with $g_{rstu}$ in place of $b_{rstu}$.
(iii) $B \in (\mathcal{C}_r : \Phi^*(\mathcal{C}_q))$ iff the conditions (3.1)-(3.3), (3.8) and (3.9) are satisfied with $g_{rstu}$ in place of $b_{rstu}$.
(iv) $B \in (\mathcal{L}_p : \Phi^*(\mathcal{C}_{bp}))$ iff the conditions (3.2) and (3.14) are satisfied for $0 < p \leq 1$ and $\vartheta = bp$ with $g_{rstu}$ in place of $b_{rstu}$.
(v) $B \in (\mathcal{L}_p : \Phi^*(\mathcal{C}_{bp}))$ iff the conditions (3.2) and (3.15) are satisfied for $1 < p < \infty$ and $\vartheta = bp$ with $g_{rstu}$ in place of $b_{rstu}$.
(vi) $B \in (\mathcal{L}_p : \Phi^*(\mathcal{C}_u))$ iff the condition (3.14) is satisfied for $0 < p \leq 1$ with $g_{rstu}$ in place of $b_{rstu}$.
(vii) $B \in (\mathcal{L}_p : \Phi^*(\mathcal{C}_u))$ iff the condition (3.15) is satisfied for $1 < p < \infty$ with $g_{rstu}$ in place of $b_{rstu}$.
(viii) $B \in (\mathcal{M}_u : \Phi^*(\mathcal{C}_{bp}))$ iff the conditions (3.1)-(3.2), (3.10)-(3.13) are satisfied with $g_{rstu}$ in place of $b_{rstu}$.
(ix) $B \in (\mathcal{M}_u : \Phi^*(\mathcal{C}_p))$ iff the conditions (3.2), (3.6) and (3.7) are satisfied with $g_{rstu}$ in place of $b_{rstu}$.
(x) $B \in (\mathcal{C}_{bp} : \Phi^*(\mathcal{C}_u))$ iff the condition (3.1) is satisfied with $g_{rstu}$ in place of $b_{rstu}$.

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