Generalised group field theories and quantum gravity transition amplitudes

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We construct a generalised formalism for group field theories, in which the domain of the field is extended to include additional proper time variables, as well as their conjugate mass variables. This formalism allows for different types of quantum gravity transition amplitudes in perturbative expansion, and we show how both causal spin foam models and the usual a-causal ones can be derived from it, within a sum over triangulations of all topologies. We also highlight the relation of the so-derived causal transition amplitudes with simplicial gravity actions.

I. INTRODUCTION

Spin foam (SF) models \[1, 2\] are a promising recent approach to quantum gravity, in any dimension, as an algebraic and combinatorial sum-over-histories. They are defined by a sum over all possible spacetime geometries, encoded in a 2-complex \(\Gamma\) labeled by representations of the Lorentz group, weighted by a quantum amplitude function of these algebraic data and in general depending also on the combinatorics of the underlying 2-complex, with given boundary data. In turn, SF models are Feynman amplitudes of so-called group field theories (GFT), field theories defined on appropriate group manifolds, and the sum over spin foams is the sum over Feynman graphs of the corresponding field theory. The most studied model in 4-d is the Barrett-Crane (BC) model \[4\]:

\[
Z(\Gamma) = \sum \prod_f \Delta_J \prod_e A_e(J_f|e) \prod_v A_v(J_f|v)
\]

in which \(\Gamma\) is topologically dual to a simplicial complex, and where the sum is over class I representations \(J\) of \(SL(2, \mathbb{C})\) or \(Spin(4)\), depending on the signature, assigned to the faces of \(\Gamma\), with \(\Delta_J\) their Plancherel measure for \(SL(2, \mathbb{C})\) or their dimension for \(Spin(4)\), and the quantum amplitude is factorised into face \(f\), edge \(e\) and vertex \(v\) contributions. The vertex amplitude is \[3\]:

\[
A_v^{BC} = \prod_{e|v} \int_G dg_e \prod_{f|v} H^{J_f}(g_{e1} g_{e2})
\]

where the five edges of \(\Gamma\) incident to the vertex \(v\) are represented by a point and the 10 faces incident to the same vertex, each bounded by 2 of the edges, are represented by a line; the integral kernel \(H^{n}(g_{e1}, g_{e2})\) is a zonal spherical function \(D^{n}_{00}(g_{e1} g_{e2}^{-1})\) for the group \(G\). The various versions of this model, differing for the edge amplitude \(A_e\), can all be obtained through a GFT \[1, 2, 3\]. The relevant field is a complex scalar field defined over the product of 4 copies of \(G\), with a global gauge invariance property: \(\phi(g_1, g_2, g_3, g_4) = \int_G dg \phi(g_1 g, g_2 g, g_3 g, g_4) = \phi(g_i)\), and symmetric under even permutations of its arguments; the action for the theory \[6, 15\] is:

\[
S(\lambda) = \frac{1}{4} \sum_{i=1}^4 \int dg_i P_{h} \phi(g_i) P_{h} \phi(g_i) + \frac{\lambda}{4} \sum_{ij} \int d g_{ij} \{ P_{h}^2 \phi(g_{1j}) P_{h} \phi(g_{2j}) P_{h} \phi(g_{3j}) P_{h} \phi(g_{4j}) \}
\]

where \(P_{h} \phi(g_i) = \int_{SU(2)} dh \phi(g_i h_i)\); if the projections \(P_{h}\) are dropped from the kinetic term, one obtains the version \[10\]. \(\phi\) represents a 2nd quantized tetrahedron, its 4 arguments its 4 triangles, and the interaction term has indeed the combinatorial structure of a 4-simplex with 5 tetrahedra glued along triangles. The perturbative expansion produces a sum over Feynman graphs, 2-complexes dual to 4d triangulations, with amplitudes given by the BC model:

\[
Z(\lambda) = \sum_{\Gamma} \frac{\lambda^N}{\text{sym}(\Gamma)} Z(\Gamma).
\]

In \[3, 5\] a class of spin foam models, incorporating extra causality restrictions, and interpreted as the quantum gravity analogue of the Feynman propagator or causal 2-point function of QFT, has been constructed \[5\]. In this paper we present a generalised GFT formalism from which models of the type introduced in \[5\] as well as the usual ones can be derived, differing in their causal properties in the sense of \[5\], confirming their respective interpretation as causal and a-causal transition amplitudes for quantum gravity. Our construction and results apply to any spacetime dimension and any signature; however we present here only the 4d case because of more direct physical interest, and work in Riemannian signature for simplicity of notation, but the translation of notation and results to the Lorentzian signature is straightforward. Also, we do not discuss in detail the analytic continuation needed to define some of the integrals involved in our construction, of the very same type as that for Feynman propagators in QFT.
II. MOTIVATION AND BROADER PICTURE

Our main motivation for looking for new SF models for quantum gravity is the idea that one should be able to define more than one type of transition amplitudes for it, just as in QFT (or the relativistic particle). A related motivation is to incorporate causality in SFs, given that the difference between various transition amplitudes in QFT is in their different causal properties. The idea of doing this in quantum gravity is not new. In [8], it is shown that in a phase space path integral for gravity the choice of the range of integration over the lapse function characterizes the difference between the path integral definition of an analogue of the Hadamard function (infinite range \((-\infty, +\infty)\)), a projection onto solutions of the Hamiltonian constraint, and of the analogue of the Feynman propagator (half-infinite range \((0, +\infty)\)), a causal transition amplitude \(^1\). Both can be given a Lagrangian path integral realization \([10]\): given that opposite signs of the lapse correspond to opposite signs for the gravity action (opposite global orientations), the “Feynman propagator” for quantum gravity is obtained by a sum over 4d geometries with amplitude given by the exponential of the gravity action, and the “Hadamard propagator” is obtained summing over the same set of geometries with amplitude given by the sum of two exponentials of the gravity action with opposite signs:

\[
Z_F = \int \mathcal{D}g e^{iS(g)} \quad \text{vs} \quad Z_H = \int \mathcal{D}g \left( e^{iS(g)} + e^{-iS(g)} \right).
\]

As in QFT, the difference is a causal restriction: considering only positive proper times or symmetrizing over them. This is equivalent to summing only over one choice of global spacetime orientation, even if locally any space orientation is considered, or symmetrizing over it. The analogy with QFT is even more clear from the point of view of 3rd quantization of gravity \([11]\], where the gravitational Lagrangian path integral is indeed a Feynman 2-point function for the field in superspace.

We find a similar situation also in SF models, as it is clear from the asymptotic analysis of the BC vertex amplitude \([12]\), that gives indeed the cosine of the Regge action, and by the fact that all known SF models define real amplitudes and are symmetric under change of orientation, because of the use in each dual face of the kernels \(H^J(g_1, g_2)\) that are the real part of an orientation dependent complex amplitude \([8]\). A class of SF models that breaks this symmetric structure by substituting the kernels \(H^J\) with appropriate generalisation of them using a proper time parametrisation, and that reduce to the \(H^J\) is the infinite range of integration over it is chosen, while producing new causal amplitudes when the half-infinite range is selected, was constructed in \([8]\). The motivation was to realise in spin foam quantum gravity what is only formally realised in the continuum path integral quantization. The importance of these new models lies also in that they are given by the exponential of it times a complicated measure \([8]\), and thus in a direct link to other approaches to simplicial quantum gravity like quantum Regge calculus and causal dynamical triangulations \([13]\).

\(^1\) The first is a solution of the Hamiltonian constraint operator, while the second is a Green function for it; however, both are fully invariant with respect to 4-dimensional diffeomorphisms (the positive half-infinite range is diffeo-invariant and isomorphic to the negative one), so that their differences come from the difference between diffeomorphisms and canonical symmetries.

What is the need for a GFT derivation of the causal models and for a GFT implementation of the above ideas? First of all, one cannot really speak of a causal model, without a clear prescription for all its amplitudes, thus until some sort of derivation is performed. In fact, \([8]\) gave a proposal for the vertex amplitudes of causal models, but not a complete definition of them. Also, the GFT formalism allows to get rid of the dependence on a fixed 2-complex and to realise the sum over 2-complexes in a natural way. There is a more fundamental motivation, however: the idea that the group field theory represents the truly fundamental definition of a quantum gravity theory based on spin foam structures. Let us clarify what is at stake here, from this perspective, and thus what is the importance of our results. GFTs can be seen as providing a simplicial 3rd quantization of gravity \([14, 15]\), purely algebraic and combinatorial, in which both geometry and topology are dynamical. From this point of view, the idea of different types, causal and a-causal, of transition amplitudes with the same role as in ordinary QFT is indeed more than an analogy, and one expects to realise here rigorously what was formally proposed in continuum 3rd quantization theories and in the path integral formalism. There is more. Group field theories have all the ingredients that enter other approaches to quantum gravity: boundary states given by spin networks, as in loop quantum gravity, a simplicial description of spacetime and a sum over geometric data, as in Quantum Regge calculus, a sum over triangulations dual to 2-complexes, as in dynamical triangulations, a sum over topologies like in matrix models, of which GFTs are indeed higher-dimensional analogues, an ordering of fundamental events (vertices of Feynman diagrams), given by the orientation of the 2-complex, which has similarities to that defining causal sets. Therefore one can envisage GFTs as a general framework for non-perturbative quantum gravity. This is at present not much more than a dream, but clearly a direct connection to the Regge action, obtained through a GFT that has the exponential of it as quantum amplitude, and an explicit dependence on orientation data, would be an important step in establishing links with the other approaches. A GFT that produces causal spin foam amplitudes, which seem to have all these properties, is therefore crucial.
III. CAUSAL SF VERTEX AMPLITUDES

Let us review the construction of $G$, and highlight the ingredients we need in the generalised GFT formalism. Given a scalar particle with mass $m^2$, living in the homogeneous space $G = \text{Spin}(4)/\text{SU}(2)$, one defines a propagator by: $G_H(g_1, g_2, m^2) = \int ds K(g_1, g_2, s)e^{im^2s}$, where $K$ is the evolution kernel in proper time $s$. According to the range of integration chosen, one obtains either a propagator or a generator by:

$$G_H(g_1, g_2, m^2) \propto H^J(g_1, g_2) = \frac{4e^{i2(J+1)^2}}{8\sin \theta} + \frac{4ie^{i-2(J+1)^2}}{8\sin \theta},$$

i.e. the Hadamard propagator, with momentum square $C(J) = 2J(2J+2) = -m^2$ where $C(J)$ is the Casimir of $\text{Spin}(4)$ in the representation $J$, or the Feynman propagator $G_F(g_1, g_2, m^2) \propto \frac{1}{\sin \theta}$.

The idea is therefore of substituting $G_H(g_1, g_2, m^2)$ for $H^J(g_1, g_2)$ in the BC vertex amplitude to obtain a generalised model and from this (integrating $s$ over an half-infinite range only) the causal models. Therefore the resulting model will treat the masses $m^2$ for the “particles” associated to the faces of the dual 2-complex as variables, conjugate to the proper time variable $s$.

The new kernels $G(g_1, g_2, m^2)$ can be expanded in modes using harmonic analysis, giving:

$$G_H(g_1, g_2, m^2) = \int_{-\infty}^{+\infty} ds \sum_J \Delta_J D_0^{J}(g_1 g_2^{-1}) e^{-i(C(J)+m^2)s} \rightarrow \delta(C(J) + m^2) D_0^{J}(g_1 g_2^{-1})$$

so that in a model with variable mass, thus involving an integral over $m^2$, we get back the usual face contribution to the BC amplitude, and

$$G_F(g_1, g_2, m^2) = \int_{0}^{+\infty} \sum_J \Delta_J D_0^{J}(g_1 g_2^{-1}) e^{-i(C(J)+m^2)s} \rightarrow \frac{1}{C(J)+m^2 - i\epsilon} D_0^{J}(g_1 g_2^{-1}),$$

with $\frac{1}{C(J)+m^2 - i\epsilon}$ being indeed the Feynman propagator in momentum space. The vertex amplitude we seek to reproduce from a generalised GFT is thus of the type:

$$A_v = \prod_{e|v} \int dg_e \prod_{f|v} \frac{1}{C(J_f) + m_f^2 - i\epsilon} D_0^{J_f}(g_e g_{e^{-1}}) = \prod_{f|v} \frac{1}{C(J_f) + m_f^2 - i\epsilon}$$

within a model involving both a sum over representations $J$ associated to the faces of the 2-complex and an integral over the real line for the $m^2$ variables also associated to the faces of the 2-complex. The new causal vertex amplitudes are given then by a product of Feynman propagators in momentum space with variable mass, one for each face of the 2-complex, intertwined by the usual BC vertex amplitude. This is a full momentum space representation, while we expect a configuration space representation GFT to involve on equal footing group variables and proper time variables. On top of these, additional data encoding the orientation of the 2-complex will affect non-trivially the amplitudes $G$; these can enter the amplitudes in various ways, but it will be uniquely determined from the generalised GFT formalism.

IV. GENERALISED GFT FORMALISM

Consider a complex field: $\phi(g_1, s_1; g_2, s_2; g_3, s_3; g_4, s_4) : (\text{Spin}(4) \times \mathbb{R})^{\otimes 4} \rightarrow \mathbb{C}$, i.e. a simple extension of the usual field of GFTs to include a dependence on a proper time variable for each of its arguments. The field is required to be invariant under even permutation of its arguments, while odd permutations correspond to complex conjugation; this ensures that only orientable 2-complexes are obtained in the perturbative expansion of the theory; we label by a parameter $\alpha = \pm 1$ the field to represent both the field and its complex conjugate: $\phi^\alpha(g_1, s_1) = \phi(g_1, s_1)$, $\phi^{-\alpha}(g_1, s_1)$ (so that $\alpha$ labels the order of the permutation of the arguments of the field). The field is required to be invariant under the natural extension of the usual GFT gauge symmetry, i.e. we now require invariance under $G \times \mathbb{R}$, imposed through a projector $P_g$ defined as: $P_g \phi^\alpha(g_1, s_1; g_2, s_2; g_3, s_3; g_4, s_4) = \int dg \int ds \phi^\alpha(g_1 g, s_1 + s; g_2 g, s_2 + s; g_3 g, s_3 + s; g_4 g, s_4 + s)$. The usual projector $P_h$ is also imposed in the action. The field is expanded in modes (momentum space) as:

$$\phi^\alpha(g_1, s_1) = \sum_{J_1, \Lambda_1} \int_{\mathbb{R}} dm_1^2 \left( \phi_{J_1, \Lambda_1}^\alpha(m_1^2) \right)^\alpha$$

$$\prod_i D_{k_1, \ldots, k_4}^{J_{1, \ldots, 4}}(g_i) C_{k_1, \ldots, k_4}^{J_{1, \ldots, 4}} \prod_i e^{-im_i^2s_i} \delta \left( \sum_i m_i^2 \right)$$

where we have redefined the modes of the field contract with a $\text{Spin}(4)$ intertwiner $C_{k_1, \ldots, k_4}$ among the representations $J_1, \ldots, J_4$, with $\Lambda$ labelling a basis in the space of such intertwiners. Because of the gauge invariance we have imposed, the field is a combination of $\text{Spin}(4)$ and $\mathbb{R}$ intertwiners (delta functions constraining the sum of the 4 representations $m_i^2$ to be zero). The geometric interpretation of the field is again of a 2nd quantized tetrahedron, now parametrised by 4 extra proper time variables (treated as independent variables) one for each of its 4 triangles, with the requirement that the triangles close to form indeed a tetrahedron (gauge invariance under $G$) and that the proper time evolution of each of them is defined up to a global shift of the origin.
(gauge invariance under $\mathbb{R}$). The action is given by:

$$
S_{\text{gen}}(\{\lambda\}) = \sum_{\mu, \mu', \alpha, \beta} \frac{1}{4} \prod_{i=1}^{4} \left( \int dg_i \int ds_i \right) \left\{ \sum_{i,j} \frac{1}{5} \int_{g_{ij}} \int_{s_{ij}} \phi^{\mu \alpha}(g_{ij}, s_{ij}) \right\} + \sum_{i,j} \frac{1}{5} \int_{g_{ij}} \int_{s_{ij}} \phi^{\mu \alpha}(g_{ij}, s_{ij}) \right\} + \sum_{i,j} \frac{1}{5} \int_{g_{ij}} \int_{s_{ij}} \phi^{\mu \alpha}(g_{ij}, s_{ij}) \right\} \left( P_h \phi^{\mu \alpha}(g_{ij}, s_{ij}) \right) \right]\right\} \left( 1 \right) \right\} \left( 2 \right) \right\} \left( 3 \right) \right\} \left( 4 \right) \right\} \left( 5 \right) \right\}
$$

with $\lambda_{\alpha_i} = \lambda_{\alpha_i}^{(-\alpha_i)}$, in order to have an action that is real, where $\nabla_i$ is the Laplacian on the group manifold $G_i$, and we have assumed an ordering of the fields in the vertex term, corresponding to an ordering of the tetrahedra in the corresponding 4-simplex. We have also introduced an extra set of data $\mu = \pm 1$ and $\epsilon_{ij} = \pm 1$, being used in the same way as the $\alpha$ to indicate complex conjugation. We interpret the $\alpha$ labelling the field $\phi^\alpha$ as characterizing the orientation of the tetrahedron corresponding to it with respect to the 4-simplex corresponding to the potential term in the action, whose orientation is labelled by the variable $\mu_\alpha$, while the $\epsilon_{ij}$ characterizes the orientation of the triangle dual to the face bounded by the dual edges labeled $i$ and $j$.

Consistency of the above interpretation requires then these data to be restricted in the same way in which the orientation data for a 2-complex are restricted: $\epsilon_{ij} = \alpha_i^{\ast} \alpha_j$ in the vertex term and $\mu_\alpha, \alpha'^{\ast} = -\mu, \bar{\alpha}$ in the kinetic term. With this choice of orientation variables, the action is real as it should. This action, therefore, generates oriented 2-complexes as Feynman graphs, with orientation given by the variables $\mu$ and $\alpha$, that are summed over in the perturbative expansion, with amplitudes depending non-trivially on this orientation, as we are now going to show. From the kinetic term one deduces the propagator in momentum space:

$$
P(J_i, m^2_{ij}, k_i; \tilde{J}_i, \tilde{m}^2_{ij}, l_i) = \prod_{i} \frac{i\delta_{J_i, \tilde{J}_i} \delta(m^2_{ij} + \tilde{m}^2_{ij}) \delta_{k_i, l_i}}{C_{J_i} + m^2_{ij}} \times \prod_{i<j} \delta \left( \sum_{i,j} \right), \quad (4)$$

i.e. a product of Feynman propagators (with variable mass square, taking both positive and negative values, and appropriate analytic continuation, using Feynman prescription) one for each dual face intertwined in both their variables by the ‘eye diagram’and by a delta function relating the masses. Had we imposed the $P_h$ projectors also in the kinetic term the resulting propagator would have been the same, but with the inverse of the eye diagram instead of the eye diagram in the above formula, with an extra factor $\Delta_{J_i}$ for each face. The difference in the results between the two choices is therefore exactly the same as in the usual GFTs. The vertex amplitude for the various possible choices of interactions, labeled by the $\alpha_i$s and the $\mu$s, is easily read out of the action to be:

$$
\mathcal{V}(g_{ij}, s_{ij}, \mu, \alpha_i) = \prod_{i<j=1}^{5} \left( \prod_{i<j} \left( \theta(\alpha_i \alpha_j (s_{ij} + s_i - s_j - s_{ji}))) \right) \right) \left( P_h \phi^{\mu \alpha}(g_{ij}, s_{ij}) \right) \right\} \left( 1 \right) \right\} \left( 2 \right) \right\} \left( 3 \right) \right\} \left( 4 \right) \right\} \left( 5 \right) \right\}
$$

in configuration space. In momentum space it reads:

$$
\mathcal{V}(J_i, m^2_{ij}, k_i, \{\alpha_i\}) = \prod_{i<j=1}^{5} \left( \prod_{i<j} \left( \delta_{J_i, \tilde{J}_i} \delta(m^2_{ij} + \tilde{m}^2_{ij}) \delta_{k_i, l_i} \right) \right) \prod_{i<j=1}^{5} \left( \prod_{i<j} \left( \delta \left( \sum_{i,j} \right) \right) \right) \right\} \left( 1 \right) \right\} \left( 2 \right) \right\} \left( 3 \right) \right\} \left( 4 \right) \right\} \left( 5 \right) \right\}
$$

with the representations of $Spin(4)$ entering the formula being only the class I ones, because of the $P_h$ projectors in the action. The partition function of the theory is then expanded in Feynman diagrams (we restrict for simplicity to the special case of all $\lambda$’s being equal) as:

$$
Z_{\text{gen}}(\lambda) = \sum_{\Gamma_{\alpha, \mu}} \frac{\Lambda^N}{\text{sym}(\Gamma_{\alpha, \mu})} Z(\Gamma_{\alpha, \mu}). \quad (5)
$$

and the new type of spin foam models for given 2-complex $\Gamma_{\mu, \alpha}$, now characterized by its orientation encoded in the $\alpha$’s and the $\mu$’s, is given by:

$$
Z(\Gamma) = \prod_{f} \left( \int dm^2_{J_f} \sum_{J_f} \right) \Delta_{J_f} \prod_{e} \left( \prod_{f, e} \frac{1}{C_{J_f} + m^2_{J_f}} A_{e}(J_f) \right) \prod_{e} \left( \prod_{f, e} \left( \delta \left( \sum_{J_f} \right) \right) \right) \prod_{f, e} \left( \frac{1}{\mu \alpha_1 \alpha_2 C_{J_f} + m^2_{J_f}} \right) \right\} \left( 1 \right) \right\} \left( 2 \right) \right\} \left( 3 \right) \right\} \left( 4 \right) \right\} \left( 5 \right) \right\}
$$

where there are two independent variables for each face of $\Gamma$: a mass variable and a representation of $Spin(4)$. The $A_{e}(J_f)$ is either the eye diagram function or its inverse according to whether one imposes the $P_h$ projectors in the kinetic term of the action or not, and we have dropped two redundant deltas relating mass variables for each dual edge, arising from a redundant use of $P_h$ projectors in the action. We see that the above choice of action for this generalised field produces a spin foam model with the causal vertex amplitudes of $\mathbb{R}$ in momentum space, as desired. The amplitude $Z_{\text{gen}}$ is in general a complex number (recall the analytic continuation implicit in the above expression), while the usual spin foam models are real, in agreement with the interpretation of them as defining an analogue of the Feynman propagator and of the Hadamard function for (third quantized)
quantum gravity. As usual, there is some freedom, given the type of field one is considering and the symmetries one wants to impose, in the choice of the action for the theory: in particular, given that the interaction term chosen above to reproduce the causal spin foam models, one can choose different kinetic terms. A possible choice, that becomes almost a necessity when dealing with real fields, is to take as kinetic term the ‘square’ of the previous one:

\[
\prod_{i=1}^{4} \int dg_i \int ds_i \phi^{-\mu \alpha}(g_i, s_i) \left[ \prod_{i} \left( + \partial_{s_i}^2 + (\nabla_i)^2 \right) \right] \phi^{\mu \alpha}(g_i, s_i)
\]

(6)

(the 4th order nature of the differential operator not being of concern, given that we are not dealing with fields living in spacetime, but we are at a more abstract level). This produces in perturbation theory the same type of field one is considering and the symmetries of the tetrahedron dual to it, and depending on the orientation of the other tetrahedra in the same 4-simplex, which deserves to be explored. In any case, this shows the exact sense in which the usual spin foam models, thus the type of quantum gravity transition amplitudes they correspond to, can be obtained in the generalised group field theory formalism we are presenting. Notice that the same amplitudes, modulo additional delta divergences, could be obtained using the kinetic term \( - \) but using Hadamard propagators for each face in the edge instead of Feynman ones, i.e. going ‘on-shell’, while dropping at the same time the \( \theta \) functions in the vertex term; this would be a somewhat awkward construction and not easy to interpret in terms of usual field theoretic perturbation theory (perturbative expression of the anticommutator of field operators?), but it may nevertheless shed more light on the nature of the usual spin foam models from the point of view of this generalised formalism. Notice also that dropping the \( \theta \) functions in the vertex term turns each vertex amplitude into a real function and the poles of the complex functions appearing in it into zeros of delta functions, i.e. means going ‘on-shell’, which in turns has the effect of confining the orientation dependence of the amplitude in the additional constraints on the face representations. As we have discussed, one of the motivations for developing this generalised GFT formalism was to obtain models that could be more easily related to classical simplicial actions. We can indeed check that the vertex amplitude we have proposed above has an expression very close to the exponential of the Regge action in 1st order formalism \( \frac{\alpha}{4 \pi} \). Going to a mixed representation in terms of the variables \( g_i \) and \( m_i^2 \), i.e. performing the Fourier transform with respect to the proper time variables, this looks like:

\[
\prod_{\eta \mu \nu} \int_{-\infty}^{+\infty} ds_f \theta(\alpha_i, \alpha_j, s_f) K(\vartheta_{f_{i,j}}, \mu, \alpha_i, \alpha_j, s_f)e^{-im_i^2 s_f} = \prod_{\eta \mu \nu} \frac{1}{4\pi \sin \vartheta_{f_{i,j}}} \frac{1}{\sqrt{1-\mu \alpha_i \alpha_j m_i^2}} \theta_{f_{i,j}}(7)
\]

Interpreting the quantity \( \sqrt{1-\mu \alpha_i \alpha_j m_i^2} \) as the area of the triangle dual to the face \( f_{ij} \), which is consistent with the stationary point analysis of usual spin foam models (since on-shell, i.e. when \( C_{f_{i,j}} = - \mu \alpha_i \alpha_j m_i^2 \), the above quantity is equal to \( 2J_{f_{i,j}} + 1 \), we would have then a vertex amplitude given by the exponential of the Regge action in 1st order form \( \frac{\alpha}{4 \pi} \theta_{f_{i,j}} \), if the angle \( \vartheta_{f_{i,j}} \) was the dihedral angle between the tetrahedra \( i \) and \( j \). However, this angle measures the holonomy around the portion of the face dual to their shared triangle inside the 4-simplex \( v \), so it receives a contribution not only from the group elements labelling the parallel transport around dual edges inside the 4-simplex (that indeed correspond to the dihedral angle), but also from the boundary holonomies represented by the group elements being the arguments of the field in configuration space. Therefore we clearly get the exponential of the Regge action for each 4-simplex in the special case of flat boundary holonomies, but more work
is needed to show that this is also the result in the general case and after integrating out the boundary variables to obtain the quantum amplitude for the whole Feynman graph, i.e. for the whole simplicial complex. The above calculation however suggests that this is likely to happen.

V. CONCLUSIONS

We have constructed a generalised formalism for group field theories that extends the usual one by the use of a field depending on a set of extra proper time variables and their conjugate mass variables. The first motivation was the wish to be able to derive both a-causal and causal transition amplitudes for quantum gravity, of the type presented in \[1\]. We have shown that indeed one can derive this type of spin foam models with an appropriate choice of interaction term in the generalised group field action. We have also identified two different choices for the kinetic term that can be used in conjunction with this interaction term to define causal transition amplitudes (spin foam models), and producing slightly different edge amplitudes. The static-ultra-local limit of the generalised group field theory action, together with a removal of the causal restriction on the proper time variables in the interaction term, reproduces naturally the usual (a-causal) spin foam models, thus clarifying their role and interpretation within the generalised formalism. Finally, we have shown how the vertex amplitudes for the generalised group field theory action with causality restrictions are given by complex functions of the variables of the theory, with an expression very closely related to the exponential of the Regge action, and then understanding the meaning of the alternative expressions that the generalised formalism provides for the same quantities (a ‘parametrised Regge action’?); by studying parametrised formulations of gravity and the consequent extension of superspace \[22\]; by considering alternative descriptions and quantizations of the “quantum tetrahedron” \[23\]. Also, the form of the action in this generalised formalism may facilitate the analysis of the classical equations of motion as well as the Hamiltonian analysis of the theory. The coupling of particles in the 3d version of this generalised models, along the lines of \[24\], may produce Feynman diagrams for the matter fields that involve correctly feynan propagators for the field, instead of Hadamard ones as in \[24\]. Finally, it will be crucial to show how other approaches to quantum gravity can be obtained from this generalised GFT formalism.

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