Classical geometry and gauge duals for fractional branes on ALE orbifolds

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Abstract

We investigate the classical geometry corresponding to a collection of fractional D3 branes in the orbifold limit $\mathbb{C}^2/\Gamma$ of an ALE space. We discuss its interpretation in terms of the world-volume gauge theory on the branes, which is in general a non conformal $\mathcal{N} = 2$ Yang-Mills theory with matter. The twisted fields reproduce the perturbative behaviour of the gauge theory. We regulate the IR singularities for both twisted and untwisted fields by means of an enhançon mechanism qualitatively consistent with the gauge theory expectations. The five-form flux decreases logarithmically towards the IR with a coefficient dictated by the gauge theory $\beta$-functions.

1 Introduction

The study of fractional branes, that is of branes wrapped on vanishing cycles at orbifold or conifold singularities, has attracted in the last period much interest [1]-[11]. The resulting world-volume gauge theories, in fact, possess fewer supersymmetries than the theories on unwrapped (or “bulk”) branes, and in general are not conformal. At the same time, it is often possible to derive explicit supergravity solutions describing the geometry created by collections of such wrapped branes. This is therefore an arena where one can try to extend the Maldacena gauge/gravity duality [12] to non-conformal cases, an important goal in present day string theory [13]-[19]. Although the supergravity solutions representing non-conformal situations typically exhibit IR singularities, it is hoped that such singularities are resolved or explained by some stringy phenomenon. Interesting physics has been elucidated by considering D3-branes on conifold singularities [1]-[5], corresponding to $\mathcal{N} = 1$ gauge theories, for instance the cascade of Seiberg dualities of [3]. We will focus on the case of fractional D3-branes solutions [1],[3],[4],[5] on orbifold backgrounds of the type $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$, where $\Gamma$ is a Kleinian subgroup of SU(2). These fractional branes can be interpreted as D5-branes wrapped on holomorphic two-cycles of an ALE orbifold.
manifold \([20]\), in the limit in which the volumes of such cycles vanish and the ALE space degenerates to \(\mathbb{C}^2/\Gamma\). The world-volume theories are \(\mathcal{N} = 2\) super-Yang-Mills theories in 4 dimensions, with, in general, matter content (hyper-multiplets). In the \(\mathcal{N} = 2\) context, the resolution of the IR singularities has been often attributed to the so-called enhançon mechanism \([21]\), and the fractional branes on \(\mathbb{C}^2/\mathbb{Z}_2\) orbifolds seem to follow this pattern \([6, 7, 11]\).

To search for a generalization of the Maldacena conjecture in the context of fractional branes on orbifold singularities, a possible approach is to start with a situation involving only “bulk” (or “regular”) branes. On such branes lives a conformal field theory \([22, 23]\), dual to \(\text{AdS}_5 \times S^5/\Gamma\) supergravity \([24]\). One can then break conformal invariance by properly decomposing the bulk branes in their fractional branes constituents, and placing some of the latter at a large distance \(\rho_0\) in the transverse \(z\)-plane \((z = x^4 + i x^5)\) fixed by the orbifold action \([4, 11, 14]\). This allows to discuss the gravity dual of the non-conformal theory living on the remaining fractional branes (of which \(\rho_0\) represents the UV cut-off), as a deformation of the \(\text{AdS}_5 \times S^5/\Gamma\) geometry \([1, 11]\). To describe a direct gauge/gravity duality for a generic configuration of (many) fractional branes, one should, in the above perspective, send the UV cut-off \(\rho_0\) to infinity so that the gauge theory is valid at all scales. It is argued in \([10, 11]\) that it is impossible to do so while retaining the supergravity approximation, which requires large numbers of branes, as it is generally to be expected in presence of UV free gauge factors.

Even with this limitation, several papers have investigated the case \(\mathbb{C}^2/\mathbb{Z}_2\), determining \([6, 7]\) the classical solutions\(^1\) and pointing out many intriguing features in their interpretation \([6, 10, 11]\). Here we will work out the supergravity solutions describing fractional branes in the case of a generic orbifold \(\mathbb{C}^2/\Gamma\) and discuss their field theory interpretation. One of the main features of these solutions (which already emerged in the \(\mathbb{Z}_2\) case) is the existence of non-trivial “twisted” complex scalar fields \(\gamma_i\), which exhibits a logarithmic behaviour that matches the one of the (complexified) running coupling constants of the world-volume gauge theories. This perfect correspondence at the perturbative level is basically due to the open/closed string duality of the one-loop cylinder diagrams that, in the field theory limit of the open string channel represent, because of the \(\mathcal{N} = 2\) supersymmetry, the only perturbative corrections to the gauge theory.

In working out the solution, however, one encounters IR singularities in both the twisted and untwisted equations of motion which need the introduction of appropriate regulators \(\Lambda_i\). On the gauge side these IR cut-offs can naturally be interpreted as the dynamically generated scales of the various factors of the gauge theory, while on the gravity side they represent enhançon radii at which fractional probes of type \(i\) become tensionless\(^2\). From the gauge theory point of view, drastic modifications

\(^1\)The classical solutions for fractional branes on \(\mathbb{C}^2/\mathbb{Z}_N\) case were investigated, with a different approach from ours, in \([8]\). The classical solutions for the compact orbifold \(T^4/\mathbb{Z}_2\) have been investigated in \([8, 23]\).

\(^2\)This generalizes the case of \(\mathbb{C}^2/\mathbb{Z}_2\), where there are only two types of fractional branes and a
with respect to the logarithmic behaviour of the effective couplings occur at the scales $\Lambda_i$. These modifications are due to the instanton corrections and should correspond, on the string theory side, to fractional D-instanton corrections \[1\]. For large number of fractional branes the instanton corrections are well implemented by assuming that there is no running of each gauge coupling below the corresponding scale $\Lambda_i$ \[11\]. This way of regulating the IR singularities suggests that the fractional branes that represent the source for the supergravity solution, initially thought of as being placed at the origin, dispose in fact themselves on (or near to) the enhançon radii used as cut-offs.

In the classical supergravity solution the twisted fields back-react, providing a source in the bulk for the untwisted fields (metric and RR 5-form). An intriguing role seems to be played by the RR five-form flux, which measures the total untwisted D3-charge within a certain scale. This flux varies logarithmically with the radial distance in the $z$-plane \[7, 10\]. Such a behaviour is quite general for fractional brane solutions in different contexts \[3\] and its interpretation at the level of the field theory is an important issue. The flux is expected to be related to the degrees of freedom of the gauge theory; the fact that it turns out to decrease with the scale towards the IR is consistent with this expectation. The untwisted charge of the branes of each type is proportional to $\gamma_i(z)$. At the enhançon scale $\Lambda_i$ it changes sign, so that below $\Lambda_i$ the system is no longer BPS. Here a difference with respect to the $\mathbb{C}^2/\mathbb{Z}_2$ case seems to occur. Indeed in the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold one can revert to a BPS situation by a shift of $\gamma$ corresponding to a shift of the $B$-flux under which the string theory background is periodic. However, the five-form flux is not invariant unless one decreases at the same time the number of bulk branes by an appropriate amount. In particular starting with a configuration of $N+M$ branes of one type and $N$ of the other, so that the world-volume theory is a $SU(N+M) \times SU(N)$ gauge theory with matter, one needs to add $M$ bulk branes modifying the gauge group to $SU(N) \times SU(N-M)$. This property has been interpreted \[7\] as suggesting an $\mathcal{N} = 2$ analogue of the duality cascades that take place \[3\] in the $\mathcal{N} = 1$ theories associated to fractional branes at the conifold. In \[10\] the reduction of the gauge group has been instead interpreted as due to a Higgs phenomenon. In both interpretations, the behaviour of the untwisted fields is taken to suggest a possible “extension” of the validity of the description beyond the perturbative regime of the original gauge theory, namely below the dynamically generated scale. The authors of \[11\] take a more conservative point of view. Relying on the comparison with the appropriate Seiberg-Witten curve which they construct explicitly, they argue that it is not necessary to try to extend the supergravity description below the enhançon.

By considering fractional branes on $\mathbb{C}^2/\Gamma$ orbifolds, with gauge groups consisting in general of more than two factors, we find that it is no longer true that all non-BPS situations, where the untwisted charges of the various types do not have all the same sign, can be amended by shifts of the twisted fields corresponding to periodicities single enhançon radius.
of the $B$-fluxes. A duality cascade seems therefore unlikely, at least if it should generalize directly the proposal of [7], and we tend to the "conservative" point of view that limits the validity of the classical supergravity solution to the perturbative regime of the field theory. Also remaining within this regime, the role of the RR flux could indeed be that of counting the degrees of freedom. To this purpose, we note that the logarithmic running of the flux corresponds to a differential equation with a strong analogy (totally at a formal level, though, at least for now) with the equation satisfied by the holographic $c$-function proposed in [20].

The paper is organized as follows. In section 2, we review some needed material about the $\mathbb{C}^2/\Gamma$-type orbifold of type IIB, the corresponding fractional D3-branes and their world-volume gauge theory. In section 3, we write the corresponding supergravity equation and solve them using an ansatz similar to [6, 9]. In section 4, we discuss the relation of the supergravity solution to the gauge theory. In appendix A, we give an analytic study of the function governing the untwisted fields in the supergravity solution, while in appendix B we review some material about closed strings on $\mathbb{C}^2/\Gamma$ orbifolds and the boundary states for the corresponding branes.

## 2 Fractional D3-branes on $\mathbb{C}^2/\Gamma$

Let us consider the bulk theory of type IIB strings on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$, where $\Gamma$ is a discrete subgroup of SU(2) acting on the coordinates $z^1 \equiv x^6 + ix^7$ and $z^2 \equiv x^8 + ix^9$ by

$$g \in \Gamma : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto \mathcal{Q}(g) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix},$$

with $\mathcal{Q}(g) \in$ SU(2) being the representative of $g$ in the defining two-dimensional representation.

This space admits obviously an exact description as a $\mathcal{N} = (4, 4)$ SCFT, see, e.g., [27]. The massless spectrum, in particular, will be briefly described in Appendix A. In the untwisted NS-NS and R-R sectors the $\Gamma$-invariant subset of the usual type IIB fields are kept. These fields may depend on the whole 10-dimensional space. The twisted fields, instead, only have a six-dimensional dynamics, as they cannot carry closed string momentum in the orbifolded directions. In the NS-NS twisted sectors one has 4 six-dimensional scalars. From the R-R twisted sector, we get a 0-form and a 2-form (always in the six-dimensional sense).

This exact background, preserving half of the bulk supersymmetries, is the singular limit of an (almost) geometrical background $\mathbb{R}^{1,5} \times \mathcal{M}_\Gamma$, where $\mathcal{M}_\Gamma$ is the ALE space obtained resolving the $\mathbb{C}^2/\Gamma$ orbifold [28]. Let us recall that the resolved space acquires a non-trivial homology lattice $H_2(\mathcal{M}_\Gamma, \mathbb{Z})$, entirely built by exceptional divisors. These latter are holomorphic cycles $e_i$, which topologically are spheres. They are in one–to–one correspondence with the simple roots $\alpha_i$ of a simply-laced Lie algebra $\mathcal{G}_\Gamma$ which is associated to $\Gamma$ by the McKay correspondence [29]. This correspondence can be stated as follows. Denote by $\mathcal{D}_\Gamma$ the irreducible
representations of $\Gamma$. Then the Clebsh-Gordon coefficients $\hat{A}_{IJ}$ in the decomposition

$$Q \otimes D_I = \oplus_J \hat{A}_{IJ} D_J$$

(2)

represent the adjacency matrix of the *extended* Dynkin diagram of $G_{\Gamma}$. The intersection matrix of the exceptional cycles is the negative of the Cartan matrix of $G_{\Gamma}$:

$$e_i \cdot e_j = -C_{ij}.$$ 

(3)

The trivial representation $D_0$ of $\Gamma$, on the other hand, is associated to the extra site in the Dynkin diagram, corresponding to the lowest root $\alpha_0 = -\sum_i d_i \alpha_i$, and thus to the (non-independent) homology cycle $e_0 = -\sum_i d_i e_i$. We will always use the convention that the index $I$ runs on the irreducible representations of $\Gamma$, while $i$ runs on the holomorphic cycles $e_i$; so, $I = (0, i)$. The Dynkin labels $d_i$ give the dimensions of the irreducible representations $D_i$; of course, $d_0 = 1$.

Let us notice that the orbifold CFT of $\mathbb{C}^2/\Gamma$ is obtained in the limit in which the volume of the exceptional cycles vanishes, but the integrals of the $B$-field on these cycles remain finite \[30\], with the values

$$b_i = \frac{1}{2\pi} \int_{e_i} B = \frac{d_i}{\Gamma}.$$ 

(4)

We are using here and in the following the convention that $2\pi \alpha' = 1$, otherwise in Eq. (4) the string tension $1/(2\pi \alpha')$ should further appear in front of the integral. The limits in which also the $B$-fluxes vanish correspond to more exotic phases in which tensionless strings appear (and one has an effective six-dimensional theory which includes so-called “little strings”) \[31\]. At the string theory level, the $B$-fluxes $b_i$ are periodic variables of period 1; indeed, the contribution to the partition function of a world-sheet instanton on the vanishing cycle $e_i$ is $\exp(2\pi i b_i)$.

Our purpose is to discuss the effect of placing in this background D3-branes transverse to $\mathbb{C}^2/\Gamma$. In \[22\] (see also \[23\]) the $D_p$-branes associated to Chan-Paton factors transforming in the regular representation of $\Gamma$ were considered. Such $D_p$-branes are called “bulk” branes, as they can move in the orbifold space, and are just the counterparts of the usual $D_p$-branes of flat space.

The regular representation $R$ is not irreducible: it decomposes as $R = \oplus_I d_I D_I$, where $d_I$ denotes the dimension of $D_I$. It is natural to consider “fractional” branes \[20\] corresponding to Chan-Paton factors transforming in irreducible representations. The McKay correspondence associates the irreducible representations $D_i$ ($i \neq 0$) to the homology cycles\footnote{And the trivial representation $D_0$ to cycle $e_0 = -\sum_i d_i e_i$, corresponding to the lowest root of $G_{\Gamma}$, i.e., to the extra dot in the extended Dynkin diagram.} $e_i$. This points to a geometrical explanation of such branes \[22\]: a $D(p+2)$-brane of the theory defined on the ALE space $M_\Gamma$, which wraps the homology cycle $e_i$ gives rise in the orbifold limit to the fractional
Dp-brane\(^{\text{\footnotesize 4}}\) of type \(D_i\). Notice that the fractional D3 branes, stretching along the \(x^0, \ldots, x^3\) directions, are confined to the plane \((x^4, x^5)\) fixed by the orbifold action (it is only the regular representation which contains all the images of a brane, not the single irreducible representations).

The massless spectrum of the open strings stretched between the fractional branes determines the field content of the gauge field theory living on their world-volumes \(^{22}\). The spectrum is easily obtained by explicitly computing the \(\Gamma\)-projected 1-loop trace

\[
Z_{IJ}(q) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \text{Tr}_{IJ}(\hat{g} q^{L_0 - \frac{c}{24}}),
\]

(5)

where \(\hat{g}\) acts on the string fields \(X^\mu\) and \(\psi^\mu\), but also on the Chan-Paton labels at the two open string endpoints, transforming respectively in the representation \(D_I\) and \(D_J\). The trace in Eq. (5) is of course composed of GSO-projected NS and R contributions. The final result is that, with \(m^I\) D3-branes of each type \(I\), the massless fields are those of an \(\mathcal{N} = 2\) gauge theory. The gauge group is \(\otimes_I U(m^I)\), each \(U(m^I)\) gauge multiplet coming from the strings attached to the branes of type \(I\), and there are \(\hat{A}_{JK}\) hyper-multiplets in the bi-fundamental representation \((m_I, m_J)\) of two gauge groups \(U(m_I)\) and \(U(m_J)\). These latter arise from strings stretched between branes of type \(I\) and \(J\) whenever the link \(IJ\) is in the extended Dynkin of \(G_\Gamma\) (namely, \(\hat{A}_{JK}\) is the adjacency matrix of the diagram). Let us notice that the 1-loop coefficients of the \(\beta\)-functions of these gauge theories (which are also, because of the \(\mathcal{N} = 2\) supersymmetry, the only perturbative contributions), are simply expressed in terms of the extended Cartan matrix as

\[
2m_I - \hat{A}_{II} m^I = \hat{C}_{IJ} m^J,
\]

(6)

the positive contributions being of course from the non-abelian gauge multiplet fields and the negative ones from the charged hyper-multiplets.

A bulk brane corresponds, by decomposing the regular representation, to a collection of fractional branes with \(m^I = d^I\), and, in accordance with Eq. (2), it is conformal, as the vector \(d^I\) of the dimensions is the null eigenvector of the extended Cartan matrix. Being conformal, the field theory associated to \(N\) bulk branes is dual, for large \(N\), to type IIB supergravity on \(\text{AdS}_5 \times S^5 / \Gamma\) \(^{24}\). On the other hand, the moduli space of the field theory, beside the Higgs branch corresponding to the possible motion of the bulk branes inside \(\mathbb{C}^2 / \Gamma\), has a Coulomb branch in which the expectation values of the complex scalars sitting in the gauge multiplets represent the positions of the fractional branes into which the bulk branes can decompose when hitting the fixed plane. To study generic, non conformal, configurations of

\(^{4}\)The fractional D\(_p\)-brane associated to the trivial representation \(D_0\) is obtained by wrapping a D\((p+2)\)-brane on the cycle \(e_0 = - \sum_i d_i e_i\), but with an additional background flux of the world-volume gauge field \(F\) turned on: \(\int_{e_0} F = 2\pi\). This ensures that such brane gets an untwisted D\(_p\)-brane charge of the same sign of those of the branes associated to non-trivial representations.
fractional branes, one can start with a stack of bulk branes and then place some of the fractional branes arising from their decomposition at a large distance $\rho_0$ in the fixed plane $[7, 10, 11]$. For scales below $\rho_0$, the gauge theory is effectively the one of the remaining fractional branes. This gives the possibility of discussing the gravity dual of this latter, non-conformal, theory as a deformation of the AdS$_5 \times S^5/\Gamma$ geometry induced by the non-trivial v.e.v’s.

3 Classical supergravity solution

As we discussed in the introduction, it is already clear from the $C^2/Z_2$ case that the supergravity solution created by a generic collection of fractional branes does not lead straight-forwardly to a gauge/gravity correspondence, because sending the UV cutoff $\rho_0$ of the gauge theory to infinity while keeping finite the dynamically generated scale $\Lambda$, in order to decouple it, is inconsistent with the validity of the supergravity approximation which requires a large number of branes. It is nevertheless possible to derive the classical supergravity solution that describes the classical geometry created by a generic configuration of fractional D3 branes also in the generic orbifold of $C^2/\Gamma$, and we will do so in this Section. As we shall see, such a solution clearly encodes the perturbative behaviour of the world-volume gauge theory in the range in which perturbation theory makes sense, and presents interesting effects, like the non-constant flux of the RR 5-form, which is expected to be related in some way to the dependence of the degrees of freedom on the scale.

The supergravity action we consider has to be a consistent truncation of the effective action for type IIB strings on $C^2/\Gamma$ involving only the relevant fields for fractional D3-branes. These are the metric, the RR 4-form $C_4$, and the twisted fields, as suggested by the linear couplings obtained through the boundary state analysis in Appendix B.2. The dilaton and axion fields we take instead to be constant. Our starting point is thus the bulk action

$$S_b = \frac{1}{2\kappa^2} \left\{ \int d^{10} x \sqrt{-\det G} \, R - \frac{1}{2} \int \left[ G_3 \wedge \ast G_3 - \frac{i}{2} C_4 \wedge G_3 \wedge \bar{G}_3 + \frac{1}{2} \tilde{F}_5 \wedge \ast \tilde{F}_5 \right] \right\}, \quad (7)$$

where we have introduced a complex field notation, defining $\gamma_2 \equiv C_2 - i B_2$, $G_3 \equiv d\gamma_2 = F_3 - i H_3$ where $H_3 = dB_2$, $F_3 = dC_2$ and $F_5 = dC_4$ are respectively the field strengths corresponding to the NS-NS 2-form potential, and to the 2-form and the 4-form potentials of the R-R sector. As usual, the self-duality constraint on the gauge-invariant field-strength $\tilde{F}_5$ defined as $\tilde{F}_5 = dC_4 + C_2 \wedge H_3$ has to be implemented on shell. Finally, $\kappa = 8 \pi^{7/2} g_s \alpha'^2 = 2\pi^{3/2} g_s$, where $g_s$ is the string coupling constant. Notice that in order for the choice of constant dilaton and axion to be consistent, the three-form $G_3$ must satisfy the condition

$$G_3 \wedge \ast G_3 = 0. \quad (8)$$

$^{5}$The gauge transformations of the RR forms being given compactly by $\delta C_{p+1} = d\lambda_p + H_3 \wedge \lambda_{p-2}$. 
We search for the solution corresponding to wrapping a D5-brane on a collection \( m^J e_J \) of vanishing two-cycles\(^6\). In the orbifold limit, this gives indeed a fractional D3-brane to which open strings attach with Chan-Paton factors in the representation \( \oplus_J m^J \mathcal{D}_J \). Consider the harmonic decomposition of the complex 2-form field,

\[
\gamma_2 = 2\pi \gamma_j \omega^j \quad \text{with} \quad \gamma_j \equiv c_i - ib_i ,
\]

where the normalizable, anti-self-dual (1,1)-forms \( \omega^i \) defined on the ALE space are dual to the exceptional cycles \( e_i \), in the sense that

\[
\int_{e_i} \omega^j = \delta^j_i \quad \text{and} \quad \int_{\mathcal{M}_4} \omega^i \wedge \omega^j = -(C^{-1})^{ij} .
\]

Substituting the decomposition Eq. (9) into Eq. (7) one gets the following expression for the bulk action

\[
S_b = \frac{1}{2\kappa^2} \left\{ \int d^{10}x \sqrt{-\det G} R - \frac{1}{2} \int \left[ \frac{1}{4\pi^2} d\gamma_i \wedge \ast d\gamma_j \wedge \omega^i \wedge \ast \omega^j 
- \frac{i}{8\pi^2} C_4 \wedge d\gamma_i \wedge d\gamma_j \wedge \omega^i \wedge \omega^j 
+ \frac{1}{2} \tilde{F}_5 \wedge \ast \tilde{F}_5 \right] \right\} .
\]

(11)

The scalar fields \( b_i = \int_{e_i} B/(2\pi) \) and \( c_i = \int_{e_i} C_2/(2\pi) \) couple to the wrapped D5-branes, and correspond thus to appropriate combinations of the twisted fields in the orbifold limit (see Appendix B.2 for the explicit relations). In fact, let us start with the boundary action for a D5-brane with constant axion and dilaton and wrap it on the collection of exceptional 2-cycles:

\[
S_{\text{wv}} = - \frac{T_5}{\kappa} \left\{ \int d3 \int_{m^J e_J} \sqrt{-\det(G + B)} + \int d3 \int_{m^J e_J} (C(6) + C(4) \wedge B) \right\} ,
\]

(12)

where the tension of a \( p \)-brane is given by the usual expression \( T_p = \sqrt{\pi}(2\pi \sqrt{\alpha'})^{3-p} = \sqrt{\pi}(2\pi)^{(3-p)/2} \). The RR 6-form \( C_6 \) to which the D5 couples directly is the dual of \( C_2 \), the precise relation being modified with respect to the naïve Hodge duality so as to be compatible with the equation of motion of \( C_2 \):

\[
\ast dC_2 + C_4 \wedge H_3 = -dC_6 .
\]

(13)

The harmonic decomposition

\[
C_6 = 2\pi A_{4,j} \wedge \omega^j
\]

provides us with four-forms \( A_{4,j} \) that are dual in the six flat dimensions to the scalars \( c_i \), namely we have \( dA_{4,j} = \ast dc_i - C_4 \wedge db_i \). Carrying out the integrations

\(^6\)For the D5 wrapped on the cycle \( e_0 = - \sum_i d_i e_i \) the effect of a background flux \( \int_{e_0} F = 2\pi \) has to be taken into account.
over the exceptional cycles, in the limit in which they have vanishing volume, we obtain the following expression for the world-volume action:

$$S_{wv} = -\frac{T_3}{\kappa} \left\{ \int_{D3} \sqrt{-\text{det} G} m^J b_J + \int_{D3} C(4) m^J b_J + \int_{D3} m^J A_{4,J} \right\} ,$$  

(15)

explicitly exhibiting the coupling to the twisted fields.

From the bulk and boundary actions Eq. (7), Eq. (15), one can straightforwardly determine the equations of motion for the various fields emitted by a fractional D3-brane. Moreover, we will utilize the standard black 3-brane ansatz for the untwisted fields:

$$ds^2 = H^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{1/2} \delta_{ij} dx^i dx^j ,$$  

(16)

$$\tilde{F}_5 = -d \left( H^{-1} dx^0 \wedge ... \wedge dx^3 \right) - * d \left( H^{-1} dx^0 \wedge ... \wedge dx^3 \right) .$$  

(17)

The condition Eq. (8) requires, upon the harmonic decomposition Eq. (9), that $d\gamma_i \wedge * d\gamma_j \wedge \omega^i \wedge \omega^j = 0$. Introducing the complex coordinate $z \equiv x^4 + i x^5$, the above equation implies that

$$\partial_z \gamma_i \partial_{\bar{z}} \gamma_j (C^{-1})_{ij} = 0 .$$  

(18)

This constraint is satisfied either choosing $\gamma_i$'s to be analytic or anti-analytic functions. However consistency requirements impose that $\gamma_i$ be an analytic function of $z$ for fractional branes having a positive R-R untwisted charge, while it must be anti-analytic in the case of fractional anti-branes, with a negative R-R untwisted charge. In what follows we will explicitly refer always to the first case.

The equations for the twisted fields $\gamma_i$ is:

$$\partial_a \partial_a \gamma_i + 2i T_3 \kappa \frac{\hat{C}_{\bar{I}J} m^J}{4 \pi^2} \delta(x^4) \delta(x^5) = 0 ,$$  

(19)

with $a = 4, 5$. Therefore, the twisted fields are just harmonic functions of $z$; taking into account the explicit values of $T_3$ and $\kappa$, and including also the non-independent field $\gamma_0$, we have

$$\gamma_I(z) = -i \frac{g_s}{2 \pi} \hat{C}_{\bar{I}J} m^J \ln \frac{z}{\Lambda_I} .$$  

(20)

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7We introduce, for notational simplicity, the non-independent twisted field $b_0$ and $A_{4,0}$ arising by wrapping on the cycle $e_0 = - \sum_i e_i$ (with a non-trivial flux of $F$):

$$b_0 \equiv \frac{1}{2 \pi} \int_{e_0} (B + F) = 1 - \sum_j d_j b_j ; \quad A_{4,0} \equiv \frac{1}{2 \pi} \int_{e_0} C_0 = - \sum_j d_j A_{4,j} .$$

8The minus sign in the ansatz for $\tilde{F}_5$ is consistent to the sign of the coupling to $C_4$ that we chose in Eq. (13), whereas the plus sign describe the case of fractional anti-branes.

9Our convention is to define 'positive' the charge of a brane if it gives a positive R-R untwisted contribution to the $F_5$ flux.

10In writing Eq. (19) we use the fact that the entries $\hat{C}_{\bar{I}0}$ of the extended Cartan matrix are expressed in terms of the non-extended matrix by $\hat{C}_{\bar{I}0} = - \sum_j C_{\bar{I}j} d_j$. 

9
where \( \Lambda_I \) are IR regulators. Remembering Eq. (6), we see that the twisted fields run logarithmically with the (complexified) scale exactly as the coupling constants of the various gauge groups of the world-volume theory \([1, 36]\):

\[
\tau_I(z) \equiv \frac{4\pi i}{g_I^2} \ln \frac{z}{\Lambda_I} = -\frac{1}{g_s} \gamma_I(z)
\]

having identified the (complexified) energy scale as \( z/(2\pi\alpha') = z \). The IR regulators correspond in this perspective to the dynamically generated scales, where the non-perturbative effects in the gauge theory suddenly become important. On the other hand, the scale \( \rho_0 \) at which the twisted fields attain the values Eq. (4) appropriate for the orbifold CFT correspond to the UV cutoff at which the “bare” gauge theory, with coupling constants

\[
\bar{g}_I^2 = 4\pi g_s |\Gamma|/d_I .
\]

is defined. Indeed, it is in the orbifold CFT that the world-volume theory is determined to be the \( \mathcal{N} = 2 \) SYM theory we are discussing. From Eq. (21) we see that the UV cutoff is related to the dynamically generated scales by the relation

\[
\Lambda_I = \rho_0 \exp \left( -\frac{2\pi}{g_s \bar{C}_{IJ} m^J} \frac{d_I}{|\Gamma|} \right) = \rho_0 \exp \left( -\frac{8\pi^2}{\bar{g}_I^2 \bar{C}_{IJ} m^J} \right) .
\]

Let us notice that the agreement at the perturbative level between the twisted fields and the running gauge coupling constant holds for any configuration \( \{m^I\} \) of fractional branes, with gauge factors \( \text{U}(m^I) \) UV free, conformal or IR free according to the values of the coefficients \( \bar{C}_{IJ} m^J \).

Let us discuss now the behaviour of the untwisted fields. We restrict our attention to configuration of branes such that a set of \( |\Gamma| - 1 \) independent twisted fields (let’s say, the \( \gamma_i \)) run to asymptotic freedom in the UV (or vanish), while the remaining field \( \gamma_0 = -i - \sum_i d_i \gamma_i \) has obviously the opposite running. That is, we require the positivity of the \( \beta \)-function coefficients

\[
\bar{C}_{ij} m^j = C_{ij} (m^j - d^i m^0) = C_{ij} \tilde{m}^j ,
\]

where we introduced the notation \( \tilde{m}^j = m^j - d^i m^0 \) which will be convenient in the sequel. This case generalizes directly the case of the \( \mathbb{C}^2/\mathbb{Z}_2 \) orbifold\(^{[11]}\) considered in \([1,11,14]\).

The equation of motion for the RR 5-form involves a boundary term, obtained from the world-volume action Eq. (15). If one decomposes the twisted fields \( b_I \) into back-ground value plus fluctuation, tadpole regularization implies that only the back-ground value contributes to the equation of motion, so the inhomogeneous

\(^{[11]}\)In this case, with \( N + M \) branes of type 1 and \( N \) branes of type 0, the gauge group is \( \text{SU}(N + M) \times \text{SU}(N) \). The single independent twisted field \( \gamma_1(z) = c_1 - i b_1 \) corresponds to the running coupling \( \tau(z) \) of the UV free \( \text{SU}(N + M) \) theory, while the \( \text{SU}(N) \) factor is IR free, as its coupling described by \( \gamma_0 = -i - \gamma_1 \).
equation of motions depend on how one fixes the back-ground value. At the same 
time, the contributions from the twisted fields to the five–form equation of motion 
suffer from IR divergences, which need to be regularized by means of IR cutoffs. A 
natural procedure would be to identify the back-ground value of $b_I$ with the value 
$b_I = d I / \Gamma$ it assumes in the orbifold background; this value is attained by the actual 
solution Eq. (20) at the UV cutoff radius $\rho_0$. Indeed the stringy derivation [25] of 
the boundary action Eq. (15) is fully justified in the orbifold background only. Our 
approach is to assume that the boundary action nevertheless describes correctly the 
relevant degrees of freedom down to the radii $\Lambda_I$ that correspond to the dynamically 
generated scales, covering the entire range in which the perturbative approach to 
the world-volume gauge theory makes sense.

Thus, assuming that some sort of enhançon mechanism takes place, we imagine 
that the fractional branes of type $i$ dispose themselves on (or near) a circle of 
radius $\Lambda_i$, where their effective tension and charge, proportional to $b_i(\Lambda_i)$, vanish, 
see Eq. (15) while, for the branes of type 0, generalizing the $\mathbb{Z}_2$ set-up, we assume 
them to be located in the point in which $\gamma_0(z) = -i$. The tension and the D3-charge 
of the branes of type $i$ are however not lost; they are stored in the twisted fields. We are assuming that the branes of types $i$ correspond to UV free gauge theories, 
so the corresponding dynamically generated scales $\Lambda_i$ are small.

With such an interpretation, the equation of motion for the 5-form field strength 
takes the form

$$
d^* \tilde{F}_5 = i (2 \pi^2) d \gamma_i \wedge d \gamma_j (C^{-1})^{ij} \wedge \Omega^4 + 2 T_3 \kappa m_0 \Omega^2 \wedge \Omega^4 + 2 T_3 \kappa m_0 \Omega^2 \wedge \Omega^4,
$$

(25)

where the four-form $\Omega_4$ has delta-function support on $x^6 = \ldots = x^9 = 0$ and $\Omega_2$ on $x^4 = x^5 = 0$. Notice in the r.h.s. we included the explicit charge localized on the 
fractional D3 branes of type 0 only.

The (non constant) untwisted charge is measured by the flux $\Phi_5(\rho)$ of the RR 
5-form $\tilde{F}_5$ through a surface which intersects the $z$-plane in a circle of radius $\rho$. This can be obtained integrating directly Eq. (25) on a region bounded by such a 
surface. Substituting the twisted fields Eq. (20) we remain with integrals in the $z$-plane which diverge for small values of the radius. It is natural to regulate them 
by cutting the integration at the enhançon radii $\Lambda_I$ appropriate for each term in 
the r.h.s of Eq. (25):

$$
\Phi_5(\rho) = 4 \pi^2 g_s \left( m^0 + \frac{g_s}{2 \pi} \sum_{i,j} m^M \tilde{C}_{Mi}(C^{-1})^{ij} \tilde{C}_{jM} m^0 \int_{\max(\Lambda_i,\Lambda_j)}^{\rho} \frac{d \rho'}{\rho'} \right)
$$

$$
= 4 \pi^2 g_s \left( m^0 + \frac{g_s}{2 \pi} \sum_i \theta(\rho - \Lambda_i) m^M \tilde{C}_{Mi} \tilde{m}^i \ln \frac{\rho}{\Lambda_i} \right).
$$

(26)

The flux is thus proportional to the untwisted charge $Q = m^I b_I$ encoded in the 
boundary action Eq. (15). This can be seen, for instance, using Eq. (23) to write
the flux in terms of the UV cut-off, obtaining, for scales above the biggest enhançon radius, so that we can forget the theta-functions in Eq. (26),

$$\Phi_5(\rho) = 4\pi^2 g_s \left( \frac{m^I d_I}{|\Gamma|} + \frac{g_s}{2\pi} \tilde{C}_{IJ} m^I m^J \ln \frac{\rho}{\rho_0} \right) = 4\pi^2 g_s Q(\rho) . \quad (27)$$

In this form, the first contribution is attributed to the untwisted D3 charge localized on the fractional branes, due to the background value Eq. (4). This is the picture which most directly compares with the yield of the couplings to the boundary states described in Appendix B. The non-constant (and in particular, logarithmic) RR flux is a seemingly general feature of fractional brane solutions in various contexts [3]. Though it is generally believed to be related to the decrease of degrees of freedom towards the IR, its precise interpretation seems to depend on the specific context. We will discuss some properties of the flux Eq. (26) of our model in the next Section.

Through the ansatz Eqs. (16,17) the R-R untwisted field equation Eq. (25) gives rise, consistently with the yield of the Einstein equation that we omitted for the sake of shortness, to the following equation for the function $H$:

$$\partial_i \partial_i H + 4\pi^2 \partial_z \gamma_i \partial_{\bar{z}} \gamma_j (C^{-1})^{ij} \delta(x^i) \cdots \delta(x^9) + 2T_3 \kappa m^0 \delta(x^4) \cdots \delta(x^9) = 0 , \quad (28)$$

with $i = 4, \ldots 9$. The solution of this equation involves IR divergent integrations on the $z$ plane, which, in perfect analogy to the computation of the $\Phi_5$ flux, we regularize by means of the enhançon radii. In the end, for scales above the highest enhançon, the function $H$ can be expressed in terms of the UV cutoff $\rho_0$ as

$$H = 1 + \frac{g_s m^I d_I}{4\pi^2} \frac{1}{r^4} + \frac{g_s^2}{4\pi^2} \frac{m^I \tilde{C}_{IJ} m^J}{r^4} \left[ \ln \frac{r^4}{\rho_0^2 \sigma^2} - 1 + \frac{\rho^2}{\sigma^2} \right] , \quad (29)$$

where $\rho^2 = (x^4)^2 + (x^5)^2$, $\sigma^2 = \sum_{i=6}^9 (x^i)^2$ and $r^2 = \rho^2 + \sigma^2$. The first contribution would arise by a source term for $C_4$ in Eq. (25), localized on the fractional branes and due to the orbifold background values Eq. (4) of the twisted fields.

The curvature scalar for the metric of Eq. (16) takes the following simple form

$$R = -\frac{1}{2} H^{-3/2} \partial_i \partial_i H \quad (30)$$

and hence vanishes outside the fixed plane where the different sources for the function $H$ live (see Eq. (28)). The metric exhibits singularities whenever $H$ vanishes and an horizon on the fixed plane. In Appendix A we will briefly discuss further the form of $H$.

4 Relation to the gauge theory

We already put forward in Eq. (21) the main relation between the classical supergravity solution and the quantum behaviour of the world-volume gauge theory,
namely the identification between the twisted scalars and the one-loop running coupling constants. This relation can be understood, and attributed substantially to the usual open-closed string duality of the “cylinder” diagrams, by using a probe-brane approach. Let us thus consider the set-up described in Figure 1, in which a single test fractional brane of type $I$ moves slowly in the supergravity background generated by a configuration of $m_J$ branes in each representation $\mathcal{R}_J$ of a given orbifold group $\Gamma$.

Let us expand the world-volume action Eq. (15) for our test brane up to quadratic order in the ‘velocities’ $\frac{\partial x^a}{\partial \xi^\alpha}$, where $a = 4, 5$ are the transverse directions accessible to our fractional probe and $\xi^\alpha$ are the world-volume coordinates. The dependence from the function $H$ cancels at this order, and the part of order zero in the velocities is a constant, so that no potential opposes the motion of our probe in the $x^4, x^5$ directions, i.e., in the $z$-plane. In the dual gauge theory picture, this BPS property corresponds to the opening up of the Coulomb branch of the moduli space. As usual, the kinetic terms of the probe action,

$$S_{p,I} \sim \frac{T_3}{2\kappa} \int d^4\xi \left( \frac{\partial x^a}{\partial \xi^\alpha} \right)^2 b_I,$$

should describe the effective metric on this moduli space.

On the gauge theory side, the moduli space for the $U(m^I + 1)$ gauge factor associated to the branes of type $I$ is parametrized by the expectation values of the adjoint complex scalar in the vector multiplet, $\langle \Phi \rangle = \text{diag}(a_1, \ldots, a_{m_I+1})$, which generically break the gauge group to $U(1)^{m_I+1}$, and by the masses $M_k \ (k = 1, \ldots, N_h)$.
with \( \hat{N}_h = \hat{A}_{I,m^I} \) of the fundamental hyper-multiplets. As depicted in Figure [1], our probe brane configuration corresponds to the corner of the moduli space where one of the \( a \)'s, say \( a_{m_I+1} \), assumes a large value \( z \), with \(|z| \gg \Lambda_I\). Moreover, the (complexified) masses \( M_k \) of the hyper-multiplet fields, which are the lowest excitations of the open strings stretching from the probe to other types of fractional branes sitting at the origin, also are all equal to \( z \).

Substituting in Eq. (31) the classical solution Eq. (20) for the twisted field \( b_I \) and identifying \( z \equiv x^4 + i x^5 \) with the v.e.v \( a_{m_I+1} \) of the Higgs field, as required by the open string description of the gauge theory living on the branes, we obtain the kinetic term

\[
S_{p,I} = \int d^4\xi \frac{\hat{C}_{IJ}m^J}{8\pi^2} \ln \frac{\rho}{\Lambda_I} |a_{a_{m_I+1}}|^2 .
\] (32)

We see that the effective tension of the probe brane moving in the \( z \) plane, namely the effective metric on the moduli space corner parametrized by \( a_{m_I+1} \), coincides with the running coupling constant \( 1/g_I^2(\rho) \) of Eq. (21) for the \( U(m^I) \) gauge theory of the "background" branes of type \( I \). If, instead of considering only the terms in Eq. (15), we expand the full world-volume action for the probe brane, keeping track also of the world-volume gauge field \( F \), we find consistently that the resulting gauge coupling and theta-angle for the world-volume U(1) field are summarized in the \( \tau_I(z) \) of Eq. (21).

This is also the correct result from the field theory side. Indeed, the perturbative part of the pre-potential of the \( \mathcal{N} = 2 \) low-energy effective theory, which contains one-loop effects, is given by (see, e.g., [33])

\[
\mathcal{F}_{\text{pert}} = -\frac{1}{8\pi I} \left( \sum_{p=1}^{m_I+1} (a_p - a_q)^2 \ln \frac{(a_p - a_q)^2}{\Lambda_I^2} \right) - \sum_{p=1}^{m_I+1} \sum_{k=1}^{N_h} (a_p + M_k)^2 \ln \frac{(a_p + M_k)^2}{\Lambda_I^2} .
\] (33)

The effect of the non-trivial vev \( a_{m_I+1} \) corresponding to the position of the probe brane, is to Higgs \( U(m^I + 1) \) into \( U(m^I) \times U(1) \). The effective coupling of the \( U(1) \) (which also appears in the effective action as moduli space metric) is given by

\[
\frac{\partial^2 \mathcal{F}}{\partial a_{a_{m_I+1}}^2} \sim \frac{i}{2\pi} (2m^I - N_h) \ln \frac{z}{\Lambda_I} ,
\] (34)

namely it coincides indeed with \( \tau_I(z) \).

The origin of the identification Eq. (21) between the twisted scalar \( \gamma_I \) and the effective coupling \( \tau_I \) is quite simply understood by considering the one-loop diagrams that contribute to the perturbative pre-potential Eq. (33), as summarized in Figure [1]. Each such diagram arise in the field-theory limit from the one-loop

\footnote{There are no perturbative corrections from higher loops, but only instanton corrections.}
amplitude of open strings attached on one side to the probe brane and on the other side to a background brane of the same type (then \(W\)-bosons multiplets run in the loop) or of other types (then hyper-multiplets run in the loop). Upon open-closed duality, these cylinder diagrams are seen as tree-level closed string exchange diagrams which, in the supergravity limit, describe the interaction of the probe brane with the twisted fields emitted by the background.

To go beyond the perturbative level (which task we will not attempt here), on the field-theory side one may construct the Seiberg-Witten curve, as it is done in \[11\] in the \(\mathbb{C}^2/\mathbb{Z}_2\) case. In the moduli space corner corresponding to our probe brane configuration, the effective theory will receive corrections proportional to powers of the one-instanton contribution to the partition function

\[
\exp \left( -\frac{8\pi^2}{g_s^2} + i\theta_I \right) = \exp (2\pi i \tau_I) = \left( \frac{\Lambda_I}{z} \right)^{\hat{C}_{IJ}(m).} \tag{35}
\]

On the string theory side, such effects are likely due \[1\] to fractional D-instantons, namely to D1 (Euclidean) branes wrapped on the vanishing cycles \(e_i\), with action

\[
\exp \left( -\frac{2\pi i}{g_s} \gamma_I \right) = \exp (2\pi i \tau_I) . \tag{36}
\]

It would be very interesting to pursue this argument further and determine the coefficients of the instanton corrections.

The instanton corrections in the field theory description appear to be consistent with the choice of using the dynamically generated scales \(\Lambda_i\) as IR regulators for the contributions of the twisted fields to the untwisted supergravity equations of motion. Indeed, especially for large numbers of fractional branes, as pointed out in \[7, 11\], the instantonic contributions Eq. (35) become quite suddenly important near the enhançon radius \(|z| = \Lambda_i\) and the form of the Seiberg-Witten curve suggests that for smaller radii the coupling \(\tau_i\) stops running. Then Eq. (21) would be valid for \(|z| > \Lambda_i\) only, while for \(|z| < \Lambda_i\) \(\gamma_i\) would be constant so that \(\partial\gamma_i = 0\). This agrees with the cut off at \(\rho \equiv |z| = \Lambda_i\) of the twisted contributions (always proportional to derivatives of the \(\gamma\)'s) to the RR five-form flux, Eq. (26), and to \(H\), Eq. (29).

What field theory interpretation can be given of the behaviour of the untwisted fields? Namely, what information can be deduced from the full supergravity solution, and from its embedding in a consistent string theory? As mentioned in the introduction, interesting proposals and discussions, all focused on the \(\mathbb{C}^2/\mathbb{Z}_2\) case, have appeared in the literature \[4, 10, 11\]. In the \(\mathbb{C}^2/\mathbb{Z}_2\) case, at the enhançon radius \(\Lambda\) the \(b\) field, and so the untwisted charge of each brane of type 1, changes sign: below the enhançon, the system is no longer BPS. A BPS situation can be recovered by shifting \(b \rightarrow b + 1\) which, as discussed after Eq. (4), corresponds to a symmetry of the string background. Nevertheless, to preserve the overall untwisted charge \(Q\), \(M\) bulk branes must be subtracted, modifying the gauge group to \(\text{SU}(N) \times \text{SU}(N - M)\).
In [7] it is suggested that this picture implies a duality akin to the one discovered in [3] in the context of the $\mathcal{N} = 1$ theories associated to fractional branes at the conifold. Below the enhançon scale, the theory is dual to a theory with $\tau' = \tau + i/\alpha$, reduced gauge groups and a dynamically generated scale $\Lambda'$ rescaled by a factor of $\exp(-\pi/\alpha M)$ with respect to $\Lambda$. Below $\Lambda'$ the duality is to a theory with further reduced gauge groups (corresponding again to the removal of $M$ bulk branes), and so forth, until one cannot subtract any more $M$ bulk branes.

The change of the gauge group between scales related by a factor of $\exp(-\pi/\alpha M)$ is instead interpreted in [10] as due to a Higgs phenomenon implying that $M$ bulk branes are actually distributed within such two scales.

In [11] a more conservative point of view is supported; based on the comparison with the appropriate Seiberg-Witten curve which they construct explicitly, these authors argue that it is not necessary to try to extend the supergravity description below the “first” enhançon $\Lambda_1$, where (D-)instanton effects become all important both in field theory and in string theory.

What changes for a generic orbifold $\mathbb{C}^2/\Gamma$? Having in general more than two gauge factors, it turns out that when a $b_i$ field becomes negative (namely, at a radius below its enhançon $\lambda_i$), making the system non BPS, it is not always possible to recover a BPS system by shifting other $b_j$-fields by integers. As an example, consider the $\mathbb{C}^2/Z_3$ orbifold with three gauge groups, two of them (say 1 and 2) being UV free while the last one (say 0) is IR free. In such a situation, one has (see figure)

$$b_0 + b_1 + b_2 = 1 , \quad \Lambda_2 < \Lambda_1 < \Lambda_0 .$$

(37)
In the region where $\rho \lesssim \Lambda_1$, $b_1$ is small, negative while $0 < b_0, b_2 < 1$ and the system is not BPS. The shift $b_1 \rightarrow b_1 + 1$ has to be compensated with a negative shift of $b_0$ (or $b_2$). Such a shift would however render this latter field negative, so that one goes from a non-BPS situation to another non-BPS situation. It appears therefore that the possibility of relating the theory below an enhançon to another theory via a duality based on the periodicity of the $B$-fluxes [7] does not apply to the generic situation.

Let us notice that the 5-form flux $\Phi_5(\rho)$ is always decreasing towards the IR with the scale $\rho$, for any configuration of fractional branes. In fact, the coefficient of the logarithm in Eq. (27) satisfies

$$\hat{C}_{ij} m^I m^J \geq 0 \quad (38)$$

for any set of $m^I$'s, since the extended Cartan matrices of the ADE series are positive semi–definite. Let us moreover notice that the flux Eq. (27) or, equivalently, the charge $Q(\rho)$ satisfies the differential equation

$$\frac{dQ}{d \ln \rho} = \frac{g_s}{2\pi} \hat{C}_{ij} m^I m^J \propto \beta_i \beta_j G^{ij}, \quad (39)$$

where we introduced the logarithmic derivatives

$$\beta_i \equiv \frac{d\tau_i}{d \ln \rho} = -\frac{1}{g_s} \frac{d\gamma_i}{d \ln \rho} = \frac{i}{2\pi} \hat{C}_{ij} m^J \quad (40)$$

of the “twisted” supergravity scalars, which represent the beta-functions for the gauge theory couplings $\tau_i$ dual to these supergravity fields [14] and we denoted by $G^{ij}$ the metric appearing in their kinetic term. Indeed, carrying out the integration over the ALE space of the bulk action Eq. (11), one finds a 6-dimensional kinetic term proportional to

$$\int d^6 x \sqrt{-g_6} \partial_\mu \gamma_i \partial^\mu \bar{\gamma}_j (C^{-1})^{ij}, \quad (41)$$

identifying the kinetic matrix as $G^{ij} \propto (C^{-1})^{ij}$. The equation Eq. (39) satisfied by the untwisted charge $Q(\rho)$ is reminiscent of the equation for the holographic $c$-function of [26], which with the same notations (in a different context!) reads

$$\frac{d \ln c}{d \ln \rho} = 2\beta_i \beta_j G^{ij}. \quad (42)$$

Of course, this equation arises in the context of 5D supergravity, and the logarithmic derivative in the l.h.s. is justified only upon the identification between the scale factor of the 5D metric and $\ln \rho$, which holds at criticality only [26]. The similarity with equation Eq. (39) is thus mainly formal; nevertheless, it would be interesting to understand whether the relation of the charge $Q(\rho)$, i.e., of the 5-form flux, with the logarithm of a (holographic?) $c$-function could be more substantial.

14Recall the definition Eq. (21) of the couplings $\tau_i$. Then $\text{Im} \beta_i = 4\pi(-1/g_i^3)(d g_i/d \ln \rho)$ is related to the usual beta-function $d g_i/d \ln \rho = -b_i^{(i)}/g_i^3$ by $\text{Im} \beta_i = b_i^{(i)}/2\pi$. Using Eq. (3), this agrees with Eq. (44).
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A Analytic study of the function \(H\)

We shall give here a brief analytical study of the function \(H\) characterizing the ansatz Eq.s (16,17), whose expression was given in Eq. (29). We shall draw, in part qualitatively, its level curves. With reference to the notations of Section 3, let us define the two positive real variables

\[ x = \rho^2 > 0, \quad y = \sigma^2 > 0. \]

(43)

We rewrite then for commodity the function \(H\) as follows:

\[ H(x, y) = 1 + \frac{A}{(x + y)^2} + \frac{B}{(x + y)^2} \left( \ln \left( \frac{(x + y)^2}{\rho_0^2 y} \right) - 1 + \frac{x}{y} \right), \]

(44)

where \(A\) and \(B\) are two strictly positive constants, which can be deduced by comparison with Eq. (29). The partial derivatives with respect to \(x\) and \(y\) are given by

\[ \partial_x H = -\frac{2(H - 1)}{x + y} + B \frac{3y + x}{y(x + y)^2} ; \quad \partial_y H = -\frac{2(H - 1)}{x + y} + B \frac{2y^2 - (x + y)^2}{y^2(x + y)^3}. \]

(45)

These expressions imply that there is neither a local extremum, nor a multiple point for \(H\) in the open quarter plane. Such a point, if it exists, has to be located on the axes \(x = 0\) or \(y = 0\). Indeed, such a point is characterized by a vanishing gradient, which in turn requires that

\[ (x + y)(x + 2y) = 0, \]

(46)

a condition impossible to satisfy for \(x, y > 0\).

\(H\) on the axis \(x = 0\) From the previous expressions, we deduce that the function \(H(0, y)\) increases monotonically from \(-\infty\) in \(y = 0\) to a maximum \(\chi > 1\) reached in \(y_\chi\). Let us also denote by \(y_1\) the point on the \(x\) axis such that \(H(0, y_1) = 1\). Explicitly, one finds

\[ y_1 = \rho_0^2 \exp \left( 1 - \frac{A}{B} \right), \quad y_\chi = \sqrt{e} y_1, \quad \chi = 1 + \frac{B}{2\rho_0^4} \exp \left( \frac{2A}{B} - 3 \right). \]

(47)

For \(y > y_\chi\), \(H(0, y)\) decreases monotonically and one has \(\lim_{y \to +\infty} H(0, y) = 1\).
**H at constant** $x > 0$ In that case one has
\[
\lim_{y \to 0} H(x, y) = +\infty \quad \text{and} \quad \lim_{y \to +\infty} H(x, y) = 1 .
\]

On the one hand, this tells us that $H$ has no definite limit in $(0, 0)$. On the other hand, one would like to know if $H$ is single-valued on this axis. To this purpose, let us introduce the auxiliary function
\[
Q(x, y) = -\frac{(x + y)^3}{2} \partial_y H = A + B \left[ \ln \left( \frac{(x + y)^2}{\rho_0^2 y} \right) + \frac{x^2 + 4xy - 3y^2}{2y^2} \right]
\]
so that the question translates in the knowledge of the sign of $Q$ at $x$ fixed. The $y$-derivative of $Q$ reads
\[
\partial_y Q = \frac{B}{(x + y)y^3} (2y^3 - (x + y)^3) .
\]

From this expression, it follows that $Q$ decreases from $+\infty$ in $y = 0$ to a minimum in $y = \frac{x}{2^{1/3} - 1}$. Above this point, $Q$ increases so that $\lim_{y \to +\infty} Q(x, y) = +\infty$. Direct inspection shows that for
\[
X = y_1 \frac{2^{1/3} - 1}{2^{2/3}} \exp \left( -(2^{1/6} - 2^{-1/6})^2 \right) , \quad Y = \frac{X}{2^{1/3} - 1} \]

one has
\[
Q(X, Y) = 0 \quad \text{and} \quad Q(x, \frac{x}{2^{1/3} - 1}) \geq 0 \quad \text{for} \quad x \geq X .
\]

Hence, for $x \geq X$, $H(x, y)$ monotonically decreases while for $x < X$, it reaches a local minimum and then a local maximum and so is not single valued. One has
\[
\mu = H(X, Y) = 1 + (\chi - 1)2^{4/3}(2^{1/3} - 1)\exp \left( (2^{1/6} - 2^{-1/6})^2 - 1 \right) , \quad \chi > \mu > 1 .
\]

**Level curves** From the previous observations, we can reach a qualitative understanding of the level curves $H(x, y) = h$ which separate in three families as shown on Figure [4]. To the first family belong the curves with $\chi > h > -\infty$ stretching from the origin point $(0, 0)$ and the axis $x = 0$ for $-\infty < y < y_\chi$. To the second belong those with $h \geq \chi$, starting at $(0, 0)$ and going in the region where $x \gg X$. The last family contains the level curves starting from the axis $x = 0$ at a position $y > y_\chi$ (with $\chi > h > 1$) and going in the region where $x \gg X$. Near the origin point $(0, 0)$, any level curve $H(x, y) = h$ may be represented by
\[
x = yF_h(y) , \quad F_h(y) \approx |\ln y| .
\]

In the region where $x \gg X$, the level curves are asymptotically hyperboles of equation
\[
xy = \frac{B}{h - 1} .
\]
In the region where $x \gg X$ and $y \gg y_\chi$, the level curves are asymptotically straight lines of equation
\[ x + y = y_h, \quad H(0, y_h) = h. \] (56)
In this last region, we have an approximate six-dimensional spherical symmetry.

\section*{B Closed strings on $\mathbb{C}^2/\Gamma$}

\subsection*{B.1 Bulk massless spectrum in the orbifold theory}

The closed string theory on the orbifold space $\mathbb{C}^2/\Gamma$ admits untwisted and twisted sectors. Let us consider (the conjugacy class of) an element $g \in \Gamma$ of order $N$. We have then $N$ twisted sectors $T_k$, with $k \in [0, N - 1]$; $k = 0$ labels the untwisted sector. We define $\nu = k/N \in [0, 1]$. The conformal theory is as usual along the directions $0, 1, \ldots, 5$ while it depends on the twist in the orbifolded directions. In a given sector, the modings of the world-sheet complex fields $X^l$ and $\psi^l$ ($l = 1, 2$) are given in Table \[. The intercepts may be evaluated directly by computing the Virasoro algebra from the basic commutation relations between oscillators.

Let us recall that he $SO(4) \sim SU(2)_- \otimes SU(2)_+$ symmetry of the covering space
\[ X^l \ Z + \nu \ Z - \nu \ \psi^l \ Z + \nu \ Z - \nu \ \psi^l \ Z + \frac{1}{2} + \nu \ Z + \frac{1}{2} - \nu \]
\[ X^l \ Z - \nu \ Z + \nu \ \bar{\psi}^l \ Z - \nu \ Z + \nu \ \bar{\psi}^l \ Z + \frac{1}{2} - \nu \ Z + \frac{1}{2} + \nu \]
\[ \tilde{X}^l \ Z - \nu \ Z + \nu \ \bar{\psi}^l \ Z - \nu \ Z + \nu \ \bar{\psi}^l \ Z + \frac{1}{2} - \nu \ Z + \frac{1}{2} + \nu \]
\[ \tilde{X}^l \ Z + \nu \ Z - \nu \ \bar{\psi}^l \ Z + \nu \ Z - \nu \ \bar{\psi}^l \ Z + \frac{1}{2} + \nu \ Z + \frac{1}{2} - \nu \]

Table 1: Modings of the closed superstring fields in the sector twisted by \( \nu \).

The untwisted sector  The untwisted sector contains the states of the IIB theory that are invariant under the orbifold. Such states have a 10-dimensional dynamics. In particular in the NS-NS sector, the massless states

\[ \psi^\mu \bar{\psi}^{\nu} |0, 0\rangle \]

with \( \mu, \nu = 0, \ldots, 5 \), are invariant. They correspond, from the 6-dimensional point of view, to a dilaton, a metric and a Kalb-Ramond field \( B_{\mu\nu} \), and they obviously are singlets of the “internal” residual symmetry group \( SU(2)_+ \). Also invariant are the following 16 6-dimensional scalars

\[ \psi^a \bar{\psi}^b |0, 0\rangle \]

with \( a, b = 6, 7, 8, 9 \). These states are in the \( 4 \otimes 4 \) of the \( SO(4) \) symmetry group of the covering space, that is to say in the \([ (2, 1) + (1, 2)] \otimes [(2, 1) + (1, 2)]\) of \( SU(2)_- \otimes SU(2)_+ \). In terms of the residual \( SU(2)_+ \), these states thus organize as follows

\[ 1^5 \oplus 2^4 \oplus 3. \]

The total number of invariant massless states in the NS sector is thus \( 1 + 9 + 6 + 16 = 32 \).

The massless R-R sector of the IIB theory contains a zero-form potential \( C^{(0)} \), an antisymmetric two form \( C^{(2)} \) and an antisymmetric self-dual four form \( C^{(4)} \). From the 6-dimensional point of view, \( C^{(0)} \) is invariant and thus gives a 6-d zero-form in the 1 of \( SU(2)_+ \). \( C^{(2)} \) gives a 6-d two form in the 1 of \( SU(2)_+ \) and 6 6-d zero-forms organizing in the adjoint of the broken \( SO(4) \), hence in the \( 1^3 \oplus 3 \) of the residual
SU(2)$_{+}$. Finally, $C_{(4)}$ gives a 6-d four-form and a 6-d zero-form, both in the 1 and related by self-duality, and three 6-d 2-forms organizing in the 3. This gives a total of $(1) + (6 + 6) + (1 + 3 \times 6) = 32$ states.

**The NS-NS twisted sectors** There are no bosonic zero modes in the orbifolded directions, so the twisted fields have a six dimensional dynamics. The intercept being $a_{NS} = |1/2 - \nu|$, three cases have to be considered, depending on the value of $\nu$.

In orbifolds of even order, we may consider sectors twisted by an element of order 2, i.e., we may have $\nu = 1/2$. The intercept vanishes and the world-sheet fermions have zero modes in the orbifolded directions. The massless fundamental state is then a bi-spinor of SO(4). On each side, the GSO projection selects a spinor of positive chirality and we end with a space-time scalar in the 1 of SU(2)$_{+}$ and three space-time scalars in the 3 of SU(2)$_{+}$.

When we have $0 < \nu < 1/2$, the intercept is positive and there are no fermionic zero modes. Hence the non degenerated fundamental state is tachyonic and odd under the GSO projection. There exist massless states obtained by the action of some complex fermion modes on the fundamental one that we shall write in a vector notation. Considering the action of SU(2)$_{+}$ in (1), one obtains the two following SU(2)$_{+}$ doublets $(\bar{z}_1 - i z_2)$ and $(-i \bar{z}_2 z_1)$. Extending this property from world-sheet bosons to world-sheet fermions, we obtain the following four doublets

$$
\begin{align*}
\Psi^< &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\psi}^1 \\ -i \psi^2 \end{pmatrix}, \\
\Psi^> &= \frac{1}{\sqrt{2}} \begin{pmatrix} -i \bar{\psi}^2 \\ \psi^1 \end{pmatrix}, \\
\Psi^\leq &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\gamma}^1 \\ -i \gamma^2 \end{pmatrix}, \\
\Psi^\geq &= \frac{1}{\sqrt{2}} \begin{pmatrix} -i \bar{\gamma}^2 \\ \gamma^1 \end{pmatrix},
\end{align*}
$$

(61)

where in each doublet, the two complex fermions have the same modings. The four massless states may be thus written in the matrix form

$$
\Psi^\leq \begin{pmatrix} 1 \\ 2+\nu \end{pmatrix} \begin{pmatrix} 0, 0; \nu \end{pmatrix}
$$

(62)

on which SU(2)$_{+}$ has an adjoint action. Thus, as in the case $\nu = 1/2$, these states are space-time scalars that organize in the 1 $\oplus$ 3 of SU(2)$_{+}$.

The case in which $1/2 < \nu < 1$ is similar to the previous one. The four massless states are the matrix elements of

$$
\begin{align*}
\Psi^\leq \begin{pmatrix} 1 \\ 2-\nu \end{pmatrix} \begin{pmatrix} 0, 0; \nu \end{pmatrix}.
\end{align*}
$$

(63)

These states are space-time scalars that organize in the 1 $\oplus$ 3 of SU(2)$_{+}$.

**The R-R twisted sectors** There are neither bosonic nor fermionic zero modes in the orbifolded directions and the intercept is zero. The corresponding fields have a six dimensional dynamics and the massless fundamental is a bi-spinor of SO(1, 5). The GSO projection selects the same chirality in the left and right sectors and we obtain, as potential forms, a scalar and an antisymmetric self-dual two-form for a total of $1 + 3 = 4$ states. These states are scalars of SU(2)$_{+}$.
B.2 Fractional D3-branes as sources

The boundary state describing the fractional D3-brane \[34, 35\] associated to the representation \(D_I\) is schematically given as follows:

\[
|I\rangle = N_3 \frac{d_I}{\sqrt{|\Gamma|}} |a = 0\rangle + N_3^T \sum_{a \neq 0} \sqrt{\frac{|\Gamma|}{|I|}} \rho^a_1 2 \sin \pi \nu(a) |a\rangle.
\] (64)

Here \(N_3\) is the usual normalization of the D3 boundary state, \(N_3 = \sqrt{\pi/2}\), while the normalization in front of the twisted components is \[36, 35\] \(N_3^T = \sqrt{\pi (2\pi \sqrt{\alpha'})^{-2}} = \frac{N_3}{(2\pi \alpha')^{1/2}}\). The Ishibashi states \(|a\rangle\) are in correspondence to the sectors of the closed string theory twisted by an element of \(\Gamma\) in the \(a^\text{th}\) conjugacy class and \(\nu(a)\) defines the eigenvalues of the 2-dimensional representative of such an element. For the full expressions of the boundary states we refer to \[35\].

The fractional D3 branes act as sources of closed string fields. It is possible to deduce the linear couplings of a D-brane with the massless closed string states by simply projecting the latter onto its boundary state \[37, 38\]. The untwisted NS-NS component of the boundary state couples to the graviton \(h_{\mu \nu}\), while the R-R untwisted one couples to a 4-form potential \(C_{4}\), and one finds:

\[
\text{NS} \langle I|h\rangle = -\frac{T_3 d_I V_4}{\sqrt{|\Gamma|}} \sum_{a=0}^3 h^a, \quad \text{R} \langle I|C_4\rangle = \frac{\mu_3 d_I V_4}{\sqrt{|\Gamma|}} C_{0123}.
\] (65)

Here \(V_4\) is the world-volume of the D3-brane, \(T_3\) and \(\mu_3\) are the usual tension (in units of \(\kappa\)) and charge of a D3-brane, related to the boundary state normalization by \(\mu_3 = \sqrt{2} T_3 = 2\sqrt{2} N_3\). As we said before, in a non-trivially twisted NS-NS sector there are 4 massless scalar fields, organized in 1 + 3 with respect to the residual SU(2) geometrical symmetry. The boundary state couples only to the singlet part, which we denote as \(\tilde{b}_a\), as may be seen from the following discussion. The boundary conditions for world-sheet fermions may be written as follows

\[
\Psi^>_n = \eta \Psi^>_n, \quad \Psi^<_m = \eta \Psi^<_m
\] (66)

where \(n, m\) have the correct modings. Notice that one has

\[
\psi^>_n = \eta \psi^>_n, \quad \psi^>_n = \eta \psi^>_n, \quad \psi^<_n = \eta \psi^<_n.
\] (67)

The oscillator part of the boundary state thus reads

\[
\exp \left( \eta \sum_{n<0} \Psi^>_n \tilde{\Psi}^>_n + \eta \sum_{m<0} \Psi^<_m \tilde{\Psi}^<_m \right).
\] (68)

\[\text{Here we have restored the factor } 2\pi \alpha' \text{ that we put equal 1 in the paper}\]
The emission of massless states by a D3-brane, when $0 < \nu < \frac{1}{2}$ for instance, is obtained by saturating the boundary state with the corresponding bra and is thus reads

$$\langle 0, 0; \nu | \text{Tr}(\sigma^k \bar{\Psi}^{\frac{1}{2}+\nu} \Psi^{\frac{1}{2}+\nu}) e^{i\eta \sum_{n<0} \bar{\Psi}^m \Psi^m + i\eta \sum_{m<0} \bar{\Psi}_m \Psi_m} | 0, 0; \nu \rangle = \text{Tr}(\sigma^k \sigma^0) = \delta_{k,0}. \quad (69)$$

Hence only the state in the 1 of $SU(2)_+$ is emitted.

In the RR twisted sector, the boundary state couples to a 4-form field $A^a_4$; explicitly, one has

$$\text{NS} \langle I | \bar{b}_a \rangle = -\frac{\mu_3 V_4}{4\pi^2 \alpha'} \sum_{a \neq 0} \sqrt{\frac{n_a}{|\Gamma|}} \rho^a_I 2 \sin \pi \nu(a) \bar{b}_a, \quad (70)$$

$$\text{R} \langle I | \bar{A}^a_4 \rangle = \frac{\mu_3 V_4}{4\pi^2 \alpha'} \sum_{a \neq 0} \sqrt{\frac{n_a}{|\Gamma|}} \rho^a_I 2 \sin \pi \nu(a) A^a_{0123}. \quad (71)$$

If we limit ourselves to the case $I \neq 0$, corresponding to non-trivial representation of $\Gamma$, and use the following relation which connects the geometric twisted fields appearing in Section 3 (that we indicate with $\Phi^i$) with the ones appearing in the present discussion ($\Phi^a$)

$$\Phi^i = -\sum_{a \neq 0} \sqrt{\frac{n_a}{|\Gamma|}} \rho^a_I 2 \sin \pi \nu(a) \Phi^a \quad (72)$$

both for $\Phi = b$ and $\Phi = A_4$, one can easily verify that the couplings in Eq. (53)-Eq. (71) are perfectly consistent with those that one can read from the world-volume action Eq. (15) after transforming the latter in terms of canonically normalized fields. To this effect, one must take into account the fact that the tension and untwisted charge which one reads from the above boundary state are defined with respect to fields which are normalized correctly on the covering space of the orbifold (namely, when in the action we integrate over the covering space). These correspond to $1/\sqrt{|\Gamma|}$ times the fields correctly normalized on the orbifold, which we used in Section 3. This explains the further factor of $1/\sqrt{|\Gamma|}$ present in the untwisted couplings in Eq. (15) with respect to Eq. (53).

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