LINEAR RAMSEY NUMBERS FOR BOUNDED-DEGREE HYPERGRAPHS

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ABSTRACT. We show that the Ramsey number of every bounded-degree uniform hypergraph is linear with respect to the number of vertices. This is a hypergraph extension of the famous theorem for ordinary graphs which Chvátal et al. [8] showed in 1983. Our result may demonstrate the potential of a new hypergraph regularity lemma by [18].

1. Introduction

A \( k \)-uniform hypergraph is a family of \( k \)-element subsets (called ‘edges’) of the underlying set, whose members are called ‘vertices.’ It is complete if and only if it contains all the \( k \)-element subsets. For a \( k \)-uniform hypergraph \( H \) and a positive integer \( b \), the Ramsey number of \( H \), denoted by \( R_b(H) \), is the least integer \( R \) such that for any \( b \)-coloring of the edges of the \( k \)-uniform complete hypergraph on \( R \) vertices, there exists a monochromatic copy of \( H \). The study of this number is a main theme of Ramsey Theory, which has been considered to be a central field of combinatorics or discrete mathematics. Ramsey theory started by the following theorem.

**Theorem 1.A** (Ramsey (1930) [26]). Let \( b, k \) and \( N \) be positive integers. For any \( k \)-uniform hypergraph \( H \) on \( N \) vertices, its Ramsey number \( R_b(H) \) exists.

As one of the earliest deep applications of the regularity lemma by Szemerédi, the following fundamental theorem in Ramsey theory was obtained. It was a conjecture of Burr and Erdős [1].

**Theorem 1.B** (Chvátal-Rödl-Szemerédi-Trotter (1983) [8]). Let \( b \geq 1 \) be a constant integer and \( N \geq 1 \) be a (large) integer. For any ordinary graph (i.e. a \( 2 \)-uniform hypergraph) on \( N \) vertices with maximum degree \( O(1) \), we have \( R_b(H) = O_b(N) \).

For a hypergraph, we say that a vertex is a neighbor of another different vertex if-and-only-if there exists an edge containing the two vertices. The degree of a vertex is the number of neighbors of the vertex. The maximum degree of a hypergraph is defined to be the largest degree over all vertices.

Very recently, two groups obtained the following independently by different methods, though both depend on the hypergraph regularity platform of Frankl-Rödl (2002) [12].

**Theorem 1.C** (Cooley et al. [10] and Nagle et al. [24]). Let \( N \) be a (large) integer. For any \( 3 \)-uniform hypergraph on \( N \) vertices with maximum degree \( O(1) \), we have \( R_2(H) = O(N) \).

Kostochka-Rödl (2006) [23] showed that \( R_2(H) \leq N^{1+\alpha(1)} \) for any \( O(1) \)-uniform hypergraph on \( N \) vertices with maximum degree \( O(1) \). In this paper, we will prove the following theorem.

**Theorem 1.1 (Main Theorem).** Let \( b \geq 1 \) be a constant integer and \( N \geq 1 \) be a (large) integer. For any \( O(1) \)-uniform hypergraph on \( N \) vertices with maximum degree \( O(1) \), we have \( R_b(H) = O_b(N) \).

I uploaded the first draft [19] of this result to the preprint server, arxiv.org (http://arxiv.org/), on 20 Dec. 2006. After writing almost all parts of it, I learned the existence of a preprint by Cooley et al. [11] uploaded to the preprint server on 13 Dec. 2006. They obtained the two-color case of the main theorem independently from us. However, our method is different from theirs. Their method relies on a regularity lemma with a counting lemma by Rödl-Schacht [27], which need long proofs. (27) is not self-contained. It uses results from [21], Th.6.5,Cor.6.11 and omits technical proofs ([27 Prop.28,29,30,32,33]) which are straightforward or similar to proofs in [12,25,28].) On the other hand, the version of the regularity lemma from [18] which we will use has a short proof. While our proof is simple, the main lemma(Lemma 2.2 or Corollary 2.3 counting lemma for blowups) is stronger than their corresponding main lemma(they called the embedding lemma), since our regularity setting is weaker in a sense.

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The main purpose of this paper is not only to prove the fundamental theorem in Ramsey theory but also to show the potential of the framework of \(\{18\}\). Although another proof of Theorem \(\{13\}\) was found later in \(\{14\}\), the techniques developed in \(\{8\}\) have been used for many applications. It may be why Theorem \(\{13\}\) is considered as a milestone in the survey \(\{22, 22\}\). §5.1) says that \(\{8\}\) was probably the first deep application of the regularity lemma. (On the other hand, Chvatal-Szemeredi \(\{9\}\) was published earlier and also deep, and some techniques of \(\{8\}\) appeared already in \(\{9\}\). The main theorem in \(\{9\}\) is extended in \(\{17\}\).) I believe that the technique of this paper will be used for other applications. Such an example can be seen already in \(\{20\}\).

The regularity lemma by \(\{18\}\) gives a new proof of the Szemerédi theorem on progressions which is shorter than previous proofs. Due to the simplicity of the proof, it is not hard to modify the proof of the regularity lemma for deeper applications if necessary. Although we need only the surface of the theorem for the purpose of this paper, we already have an application which needs a slight modification of our regularity lemma. See \(\{18\}\) for discussion on differences from earlier hypergraph regularity lemmas \(\{23, 25, 13, 30, 27\}\).

Cooley et al. \(\{10, 11\}\) and Nagle et al. \(\{24\}\) treated only the case of 2-coloring. Although their methods may be essentially extendable to the multicolor case, it should need more technical work and pages in their setting. On the other hand, from the beginning plan of our regularity lemma, we have considered the multicolor case because it is natural for both of regularity lemma and its applications.

2. Statements of Regularity Lemma and Main Lemma

In this paper, we denote by \(\mathbb{P}\) and \(\mathbb{E}\) the probability and expectation, respectively. We denote the conditional probability and expectation by \(\mathbb{P}[\cdots|\cdots]\) and \(\mathbb{E}[\cdots|\cdots]\).

**Setup 2.1.** Throughout this paper, we fix a positive integer \(r\) and an ‘index’ set \(\Gamma\) with \(|\Gamma| = r\). Also we fix a probability space \((\Omega_i, B_i, \mathbb{P})\) for each \(i \in \Gamma\). Assume that \(\Omega_i\) is finite (but its cardinality may not be constant) and \(B_i = 2^{\Omega_i}\) for the sake of simplicity. Write \(\Omega := (\Omega_i)_{i \in \Gamma}\). \(\square\)

In order to avoid using technical words like measurability or Fubini’s theorem frequently to readers who are interested only in applications to discrete mathematics, we assume \(\Omega_i\) as a (non-empty) finite set. However our argument should be extendable to a more general probability space. For applications, \(\Omega\) would contain a huge number of vertices, though we do not use the assumption in our proof.

For an integer \(a\), we write \([a] := \{1, 2, \cdots, a\}\), and \(\binom{[a]}{i} := \bigcup_{i \in [a]} \binom{[i]}{i} = \bigcup_{i \in [a]} \{I \subseteq [i]||I| = i\}\). When \(r\) sets \(X_i, i \in \Gamma\), with indices from \(\Gamma\) called **vertex sets**, we write \(X_J := \{e \in \bigcup_{i \in J} X_i \mid e \cap X_j = 1 \forall j \in J\}\) whenever \(J \subseteq \Gamma\).

**Definition 2.1.** [(Colored hyper)graphs] Suppose Setup 2.1. A \((k, b)\)-bound \((\{k\})\)-colored \((\Gamma\)-partite hyper)graph \(H\) is a triple \((\{X_i\}_{i \in \Gamma}, (C_I)_{I \in \binom{[k]}{\Gamma}}, (\gamma_I)_{I \in \binom{[\Gamma]}{\Gamma}})\) where \((1)\) each \(X_i\) is a set called a ‘vertex set,’ \((2)\) \(C_I\) is a set with at most \(b||I||\) elements, and \((3)\) \(\gamma_I\) is a function from \(X_I\) to \(C_I\). We write \(V(H) = \bigcup_{i \in \Gamma} V_i(H) = \bigcup_{i \in \Gamma} X_i\) and \(C_I(H) = C_I\). Each element of \(V(H)\) is called a vertex. Each element \(e \in V_i(H) = X_i, I \in \binom{[\Gamma]}{\Gamma}\), is called an **(index-I size-\(|I|\)) edge**. Each member in \(C_I(H)\) is a (face-color) of (index-I). Write \(H(e) = \gamma_I(e)\) for each \(I\).

Let \(I \in \binom{[\Gamma]}{\Gamma}\) and \(e \in V_i(H)\). For another index \(\emptyset \neq J \subset I\), we denote by \(e|_J\) the index-\(J\) edge \(e \in \bigcup_{I \in \binom{[\Gamma]}{\Gamma}} X_J \in V_i(H)\). We define the **frame-color** of \(e\) by \(H(\partial e) := (\emptyset \neq \emptyset \subset J \subseteq I\) and \(I\) and \(H(e) := (\emptyset \neq J \subseteq I\) and \(I\) and \(H(e|_J) := (\emptyset \neq J \subseteq I\) and \(I\). Write \(TC_I(H) := \{H(e|_J)| e \in X_I\}, TC_i(H) := \bigcup_{J \in \binom{[\Gamma]}{\Gamma}} TC_I(H),\) and \(TC_i(H) := \bigcup_{J \in \binom{[\Gamma]}{\Gamma}} TC_i(H)\).

A \((\text{\(b\)-bound})\) \((\text{simplicial})\)-complex is a \((\text{\(k\)-bound})\) \((\text{colored})\) \((\text{\(\Gamma\)-partite})\) hypergraph such that for each \(I \in \binom{[\Gamma]}{\Gamma}\), there exists at most one index-\(I\) color called ‘invisible’ and that if (the color of) an edge \(e\) is invisible then any edge \(e^* \supset e\) is invisible. An edge or its color is **visible** if it is not invisible.

For a \((\text{\(k\)-bound})\) \text{\(\text{\(G\)-graph})\} \text{on} \text{\(\Omega\)} and \(s \leq k\), let \(S_{r,s,h,G} = S_{r,s,h,G}\) be the set of \(s\)-bound simplicial-complexes \(S\) such that \((1)\) each of the \(r\) vertex sets contains exactly \(h\) vertices and that \((2)\) for any \(I \in \binom{[k]}{\Gamma}\), there exists a function from another index-\(I\) visible colors of \(S\) to the index-\(I\) colors of \(G\). (When a visible color \(\text{\(c\)}\) of \(S\) corresponds to another color \(\text{\(c\)}\) of \(G\), we simply write \(\text{\(c\)} = \text{\(c\)}\) without presenting the injection explicitly.) For \(S \in S_{s,h,G}\), we denote by \(\mathbb{V}_s(S)\) the set of index-\(I\) visible edges. Write \(\mathbb{V}_s(S) := \bigcup_{I \in \binom{[\Gamma]}{\Gamma}} V_I(S)\) and \(\mathbb{V}(S) := \bigcup_{s \in \mathbb{V}_s(S)} V_s(S)\).

For a complex \(S\) and \(U \subset V(S)\), we denote by \(S \setminus U\) the complex obtained from \(S\) by deleting the vertices in \(U\) and the edges containing a vertex in \(U\). When \(U\) consists of a single vertex \(u\), we write \(S \setminus \{u\} = S \setminus u\). Also write \(S \setminus V(N) = S \setminus N\) for another complex \(N\). Sometimes we write...
\[ S|_U = S \setminus (V(S) \setminus U) \text{ and call it the complex of } S \text{ induced by } U. \]

**Definition 2.2.** [Partitionwise maps] A partitionwise map \( \varphi \) is a map from \( r \) vertex sets \( W_i, i \in \mathfrak{r} \), with \(|W_i| < \infty \) to the \( r \) vertex sets (probability spaces) \( U_i, i \in \mathfrak{r} \), such that each \( w \in W_i \) is mapped into \( U_i \). We denote by \( \Phi([W_i],i \in \mathfrak{r}) \) or \( \Phi(\bigcup_{i \in \mathfrak{r}} W_i, \bigcup_{i \in \mathfrak{r}} U_i) \) the set of partitionwise maps from \((W_i),i \in \mathfrak{r}\) to \((U_i),i \in \mathfrak{r}\). If \( U_i = \Omega_i \) or \( U_i \) is obvious then we omit them. A partitionwise map is random if and only if each \( w \in W_i \) is mutually-independently mapped at random according to the probability space \( \Omega_i \).

For two partitionwise maps \( \varphi \in \Phi([W_i],i \in \mathfrak{r}) \) and \( \varphi' \in \Phi([W'_i],i \in \mathfrak{r}) \), we denote by \( \varphi \cup \varphi' \) the partitionwise map \( \varphi^* \in \Phi([W_i \cup W'_i],i \in \mathfrak{r}) \) such that \( \varphi'(w) = \varphi(w) \) and \( \varphi'(w') = \varphi'(w') \) for all \( w \in W_i, w' \in W'_i, i \in \mathfrak{r} \), where if \( W_i \cap W'_i \neq \emptyset \) for some \( i \) then we consider a copy of \( W'_i \) so that the two domains are disjoint.

Sometimes for a graph (a complex, usually) \( S \), we write \( \Phi(V(S)) = \Phi(S) \) when it is not confusing.

For two \( r \)-partite graphs \( S, G \) and for a partitionwise map \( \varphi \in \Phi(W, V(G)) \) with some \( W \supset V(S) \), we say that \( \varphi \) embeds \( S \) in \( G \), or write

\[ S \overset{\varphi}{\hookrightarrow} G \]

if and only if \( S(e) = G(\varphi(e)) \) for all \( e \in \mathcal{V}(S) \).

Suppose that \( \varphi \) is random and that any two events \( S(e) = G(\varphi(e)) \) and \( S(e') = G(\varphi(e')) \) are mutually independent unless \( e = e' \). (This happens if all edges of \( G \) are colored uniformly at random.) Then we observe that

\[
\mathbb{P}_{\varphi \in \Phi(S)}[S \overset{\varphi}{\hookrightarrow} G] = \prod_{I \in \binom{[k]}{r}} \prod_{e \in \mathcal{E}(I)} \mathbb{P}_{\varphi \in \Phi(S)}[G(\varphi(e)) = S(e) \forall e \in \mathcal{V}(S)]
\]

where \( e |_I \) and \( e |_J \) are the edges restricted in index \( J \). With this observation, we define the regularity of hypergraphs.

**Definition 2.3.** [Regularity] Let \( G \) be a \( k \)-bound graph on \( \Omega \). For \( \tilde{c} = (c_I), I \subseteq \mathfrak{r} \) \( \in \mathcal{C}_r(G), I \in \binom{[k]}{k} \), we define relative density

\[
d_G(\tilde{c}) := \mathbb{P}_{e \in \Omega}[G(e) = c_I|G(\delta e) = (c_J), J \subseteq I].
\]

For a positive integer \( h \) and a function \( \varepsilon : [k] \times \mathbb{N} \to (0,1] \), we say that \( G \) is \((\varepsilon,h)\)-regular if and only if there exists a function \( \delta : \mathcal{C}(G) \to [0,\infty) \) such that

\[
\begin{align*}
(\text{i}) & \quad \mathbb{P}_{\varphi \in \Phi(S)}[S \overset{\varphi}{\hookrightarrow} G] = \prod_{c \in \mathcal{V}(S)} \left( \mathbb{P}_{\varphi \in \Phi(S)}[G(\varphi(c)) = S(c) \pm \delta(S(c))] \right) \quad \forall S \in \mathcal{S}_{k,h,G}, \\
(\text{ii}) & \quad \mathbb{E}_{e \in \mathcal{E}(\Omega)}[|\delta(G(e))|] \leq \varepsilon \left( |I|, \max_{J \subseteq \mathcal{V}(I)} |C_J(G)| \right) \quad \forall I \in \binom{[k]}{k},
\end{align*}
\]

where \( a \pm b \) means (the interval of) numbers \( c \) with \( \max\{0, a - b\} \leq c \leq \min\{1, a + b\} \).

A subdivision of a \( k \)-bound graph \( G \) on \( \Omega \) is a \( k \)-bound graph \( G^* \) on the same \( \Omega \) such that

(i) for any size-\( k \) edge \( e \in \Omega_I \) with \( I \in \binom{[k]}{k} \), it holds that \( G^*(e) = G(e) \), and

(ii) for any two edges \( e,e' \in \Omega_I \) with \( I \in \binom{[k]}{k} \), if \( G^*(e) = G^*(e') \) then \( G(e) = G(e') \).

**Theorem 2.A (Hypergraph Regularity Lemma in [18]).** Let \( r \geq k, h, \tilde{b} = (b_i), i \in [k] \) be positive integers, and \( \varepsilon : [k] \times \mathbb{N} \to (0,1] \) a function. Then there exist integers \( \tilde{b}_1 \geq \cdots \geq \tilde{b}_{k - 1} \) such that if \( G \) is an \( \tilde{b} \)-colored \((k \text{-bound r-partite hyper)graph on } \Omega \) then there exists an \((\varepsilon,h)\)-regular \((\tilde{b}_1,\cdots,\tilde{b}_{k-1},b_k)\)-colored subdivision \( G^* \) of \( G \).

Two earliest versions of the hypergraph regularity lemmas were obtained by Rödl and his collaborators [25, 27] and by Gowers [13] independently, and another one was obtained by Tao [30]. Rödl-Schacht [27] obtained a variant of their earlier one so that it would be more appropriate for applications. (We discuss the differences between these regularity lemmas in [18].) (For earlier results about (weaker) hypergraph regularity lemmas, see [2, 3, 4, 5, 6, 7, 15, 16].)

Theorem 2.A lacks an important part of the main theorem in [18], the simple way to construct the subdivision. Although it is very important, we will not need it for our purpose of this paper.
Definition 2.4. [Blowup] For a positive integer $\Delta$, a $\Delta$-blowup of a complex $S$ is an $(r$-partite) $k$-bound complex $B$ on a finite set of vertices with maximum degree $\Delta(B) \leq \Delta$ such that $B$ is embeddable in $S$ (i.e. there exists a map $\phi$ which embeds $B$ in $S$) where the maximum degree of $B$ is defined by
\[
\Delta(B) := \max_{e \in V(B)} ||\{w \in V(B) \setminus \{v\}||\{v, w\} \in V_2(B)\}||.
\]

Note that $\max_{e \in V(B)} ||\{e \in V_k(B)\}|v \in e|| \leq \left(\frac{\Delta(B)}{k-1}\right)$. \hfill $\square$

Our main theorem will be obtained as a corollary of Theorem 2.5 and the following.

Lemma 2.2 (Main Lemma - Counting Lemma for Blowups). For any positive integers $k$ and $\Delta$, there exist $k$ functions $\eta_1 = \eta_1(\rho_i)$, $i \in [k]$, (independent from $\Omega$) such that the following holds for any real
\[
0 < \rho_1 \leq \rho_2 \leq \cdots \leq \rho_k < 1.
\]

Let $r \geq k$ and $h$ be positive integers. Let $G$ be an $(r$-partite) $k$-bound $((1/\rho_i)$-colored) hypergraph on (any probability space) $\Omega = (\Omega_i)_{i \in [r]}$. Let $S \in S_r,k,h,G$. Suppose, for any $(2\Delta)$-blowup $S'$ of $S$ with $|V(S')| \leq 2\Delta^{2k}$, the property that
\[
P_{\phi \in \Phi(V(S'))}[S' \xrightarrow{\phi} G] = \prod_{e \in V(S')} (1 \pm \eta_{|e|}(\rho_{|e|}))d_G(S'(e))
\]
and further suppose that
\[
d_G(S(e)) > \rho_{|e|} \forall e \in V(S).
\]

Let $B$ be a $\Delta$-blowup of $S$. Then for any vertex $u \in V(B)$,
\[
E_{\varphi \in \Phi(B \setminus u)} \left[\sum_{\phi \in \Phi_u} |B \xrightarrow{\phi} G|B \setminus u \xrightarrow{\varphi} G\right] \geq (1 - \eta_k^{1/4}) \prod_{e \in V(B) \cup u \in e} d_G(B(e)).
\]

Of course, in the above, the exact value $1/4$ of $\eta_k$ is not important here. Note that each $\eta_i$ is independent from $\rho_j$, $j < i$, and $|V(B)| < \infty$.

Corollary 2.3. In Lemma 2.2, if each $\Omega_i$ is a finite set and if $|V_i(B)| < \eta_1(\rho_1)|\Omega_i|$ for each $i \in r$ then the left hand side of (4) can be replaced by
\[
E_{\varphi \in \Phi(B \setminus u)} \left[\sum_{\phi \in \Phi_u} |B \xrightarrow{\phi \cup \varphi} G \text{ and } \phi(u) \notin \varphi(V(B \setminus u))|B \setminus u \xrightarrow{\varphi} G\right].
\]

In particular,
\[
P_{\varphi \in \Phi(B)} \left[B \xrightarrow{\varphi} G \text{ and } \varphi \text{ is an injection}\right] \geq (1 - \eta_k^{1/4}|V(B)|) \prod_{e \in V(B)} d_G(B(e)) > 0.
\]

3. Proof of Main Lemma

Our proof concept is to repeat $k - 1$ times of an argument which Cooley et al. [10] repeated twice for the 3-uniform case. Cooley et al. [11] avoided the iteration and employed the ‘half’ dense version of the regularity lemma with the counting lemma by Rödl-Schacht [27]. However the iteration will work smoothly in the platform of the regularity lemma by [18].

Definition 3.1. [Abbreviation] We write 'iff' for 'if and only if'.

- For a complex $B$ and its edge $e$, we write $d_G^{+(\delta)}(B(e)) := \min\{1, (1 + \eta_{|e|})d_G(B(e))\}$, $d_G^{-(\delta)}(e) := \max\{0, (1 - \eta_{|e|})d_G(B(e))\}$, and $d_G^{(\delta)}(B(e)) := (1 \pm \eta_{|e|})d_G(B(e))$.

- For a $k$-bound complex $B$ and an integer $i \leq k$, we denote by $B^{(i)}$ the complex obtained from $B$ by invariable all edges of size at least $i + 1$. That is, $V_j(B^{(i)}) = V_j(B)$ for all $j \leq i$ and $V_j(B^{(i)}) = \emptyset$ for all $j > i$.

- For a $k$-bound complex $B$ and an integer $i \leq k$, write $V^{(i)}(B) := \bigcup_{j \leq i} V_j(B)$.

- A complex $S'$ is a subcomplex of another complex $S$ iff there exists an injection which embeds $S'$ in $S$. \hfill $\square$

We will prove Lemma 2.2 by induction on $k$ and on $|V(B)|$. If $k = 1$ then it is trivial. We assume that $k \geq 2$ and the assertion holds for $k - 1$ or less, since $B$ has no edge of size $k$ in those cases. When $|V(B)| < k$ then it is clear from the induction hypothesis. Assume that $|V(B)| \geq k$. 

Definition 3.2. Let $B', B''$ be subcomplexes of $B$ such that $B' = B'' \vert_{V(B')}$ and $B'' = B'' \backslash V(B')$. Let $\varphi \in \Phi(B')$ (or $\varphi \in \Phi(W)$ for some $W \supset V(B')$ with $(V(B') \setminus V(B')) \cap W = \emptyset$). If $B' \xrightarrow{\varphi} G$ then we define the extension error of $\varphi$ from $B'$ to $B''$ by

$$
\beta(\varphi, B', B'') := \left| \mathbb{P}_{\varphi \in \Phi(B' \setminus V(B'))} [B'' \xrightarrow{\varphi} G] \left( \prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \right)^{-1} - 1 \right|^2.
$$

When $B'$ is empty (or when all visible edges contain no vertex in $B'$), we can naturally define $\beta(B'') = \left| \mathbb{P}_{\varphi \in \Phi(B' \setminus V(B'))} [B'' \xrightarrow{\varphi} G] \left( \prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \right)^{-1} - 1 \right|^2$. \qed

Claim 3.1 (Extension error is usually small). Let $\ell \leq k$ and $B', B''$ be $\ell$-bounded subcomplexes of $B$. Suppose that $B' = B'' \vert_{V(B')}$ and that $|V(B')| \leq |V(B'')| \leq \Delta^{2k}$. (Without loss of generality, $\Delta \geq 2$.) Then we see that

$$
\mathbb{E}_{\varphi \in \Phi(B')} \left[ \beta(\varphi, B', B'') \mid B' \xrightarrow{\varphi} G \right] \leq \beta_{\ell}
$$

where $\beta_{\ell} := 8.1 \cdot 2^{\Delta^{2k}} \eta_{\ell}$.

Proof. We consider the following complex $B^*$. We let $V(B^*) = V(B') \cup W \cup W'$ where $W, W'$ are two disjoint copies of $V(B'') \setminus V(B')$. Every edge $e$ with $|e \cap W| \cdot |e \cap W'| > 0$ is invisible in $B^*$. Any other edges have the same colors as the corresponding edges of $B''$. (That is, $B^*$ is obtained from $B''$ by ‘splitting’ $B'' \setminus V(B')$.) Since $|V(B^*)| \leq 2\Delta^{2k}$, the assumption (2) yields the property that

$$
\mathbb{E}_{\varphi \in \Phi(B^*)} \left[ B^* \xrightarrow{\varphi} G \mid B' \xrightarrow{\varphi} G \right] + \left( \prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \right)^2
$$

With $\frac{1}{2}$

$$
\mathbb{P}_{\varphi \in \Phi(B' \setminus V(B'))} [B^* \xrightarrow{\varphi} G] \left( \prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \right)^2
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) + \left( \prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \right)^2
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'') \setminus V(B')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$

By $\frac{1}{2}$

$$
\prod_{e \in V(B'')} d_G(B(e)) \prod_{e \in V(B')} d_G(B(e)) \prod_{e \in V(B'')} d_G(B(e))
$$
\( \prod_{e \in V(B')} d_G(B(e)) \)^2 \cdot 8.1 \cdot 2^{|V(B')|} \eta_{\ell} 
leq \eta_1 \leq \cdots \leq \eta_{\ell} \leq 1/k, 1/\Delta. \quad (6)

It completes the claim.

Fix \( u \in B \). For a set of positive integers \( A \), we denote by \( N_A \) the \((k\text{-bound})\) subcomplex of \( B \) induced by the set of vertices \( v \) whose distances from \( u \) belong to \( A \) in the ordinary (i.e., 2-uniform) graph \( B^{(2)} \). Dropping the symbol \( \{ \} \) we simply write \( N_{(a,b)} = N_{a,b} \). (Note that there is no visible (hyper)edge in \( B \) containing vertices from both of \( N_1 \) and \( N_3 \), since \( B \) is a complex.)

For \( \ell < k \) and \( i < k \), we say that \( \varphi \in \Phi(N_{2i-1}) \) is \( \ell\text{-bad} \) iff

(i) \( N_{2i-1}^{(\ell)} \twoheadrightarrow G \) but,

(ii) \( E_{\varphi \in \Phi(N_{2i-1})} [\beta(\varphi \cup \psi, N_{2i-1, 2i+1}^{(\ell)}) | N_{2i-1, 2i+1}^{(\ell)} \twoheadrightarrow G] > \beta_{k}^{2/3} \).

Since \( |V(N_{2i-1, 2i+1})| \leq \Delta^{2\ell} \), we can apply Claim 5.1 and obtain that

\[
P_{\varphi \in \Phi(N_{2i-1})} [\varphi \text{ is } \ell\text{-bad} | N_{2i-1}^{(\ell)} \twoheadrightarrow G] \leq P_{\varphi \in \Phi(N_{2i-1})} [\varphi \text{ is } (\ell + 1)\text{-bad}] \leq \beta_{\ell+1}^{1/3} P_{\varphi \in \Phi(N_{2i-1})} [N_{2i-1}^{(\ell+1)} \twoheadrightarrow G]. \quad (7)
\]

For a \( \varphi \in \Phi(N_{2i-1}) \), we define the \textbf{rank} of \( \varphi \), \( \text{rank}(\varphi) \in [0, k] \), as follows:

(i) \( \text{rank}(\varphi) := 0 \) if \( N_{2i-1}^{(\ell)} \twoheadrightarrow G \) does not hold,

(ii) \( \text{rank}(\varphi) := k \) if \( i = 1 \) and \( \beta(\varphi, N_1, N_{0,1}) \leq \beta_{k}^{2/3} \),

(iii) otherwise, \( \text{rank}(\varphi) \) is the largest \( \ell \in [k - 1] \) such that \( \varphi \) is not \( \ell'\text{-bad} \) for all \( \ell' \leq \ell \).

(Note that there is no \( \psi \) which is 1-bad, because \( \beta(\varphi \cup \psi, N_{2i-1, 2i+1}^{(1)}) = 0 \) for any \( \varphi \in \Phi(N_{2i+1}) \).

It follows for \( 1 \leq \ell \leq k - 2 \) that

\[
P_{\varphi \in \Phi(N_{2i-1})} [\text{rank}(\varphi) = \ell] \leq P_{\varphi \in \Phi(N_{2i-1})} [\varphi \text{ is } (\ell + 1)\text{-bad}] \leq \beta_{\ell+1}^{1/3} P_{\varphi \in \Phi(N_{2i-1})} [N_{2i-1}^{(\ell+1)} \twoheadrightarrow G]. \quad (8)
\]

A calculation similar to (7) with Claim 5.1 yields that

\[
P_{\varphi \in \Phi(N_1)} [\text{rank}(\varphi) \in [k - 1]] \leq \beta_{k}^{1/3} P_{\varphi \in \Phi(N_1)} [N_1 \twoheadrightarrow G]. \quad (9)
\]

For a \( \varphi \in \Phi(N_{2i+1}) \) with rank \( \ell \in [k - 1] \) and for \( \ell' \in [\ell] \), we say that \( \psi \in \Phi(N_{2i+1}) \) is \( \ell''\text{-}\psi\text{-bad} \) iff

(i) \( N_{2i+1}^{(\ell')} \twoheadrightarrow G \) (thus, \( N_{2i-1, 2i+1}^{(\ell')} \twoheadrightarrow G \)) but,

(ii) \( \beta(\varphi \cup \psi, N_{2i-1, 2i+1}^{(\ell')}) > \beta_{\ell'}^{1/3} \).

Furthermore for each \( i \), the \textbf{\( \psi \)-rank} of \( \psi \in \Phi(N_{2i+1}) \), denoted by \( \text{rank}^{\psi}(\psi) \), is defined as follows:

(i) the rank is 0 if \( N_{2i+1}^{(\ell)} \twoheadrightarrow G \) does not hold,

(ii) otherwise, it is the largest \( \ell'' \in [\ell] \) such that \( \psi \) is not \( \ell''\text{-}\psi\text{-bad} \) for all \( \ell'' \leq \ell' \).

(Note that there is no 1-\( \psi\text{-bad} \( \psi \).)

For \( \ell' \in [\ell - 1] \), we see that

\[
P_{\psi \in \Phi(N_{2i+1})} [\text{rank}^{\psi}(\psi) = \ell'] \leq P_{\psi \in \Phi(N_{2i+1})} [N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G] \cdot P_{\psi \in \Phi(N_{2i+1})} [\psi \text{ is } (\ell' + 1)\text{-}\psi\text{-bad} | N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G]
\]

\[
= P_{\psi \in \Phi(N_{2i+1})} [N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G] \cdot P_{\psi \in \Phi(N_{2i+1})} [\beta(\varphi \cup \psi, N_{2i-1, 2i+1}^{(\ell' + 1)}) > \beta_{\ell' + 1}^{1/3} | N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G]
\]

\[
\leq P_{\psi \in \Phi(N_{2i+1})} [N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G] \cdot E_{\varphi \in \Phi(N_{2i+1})} [\beta(\varphi \cup \psi, N_{2i-1, 2i+1}^{(\ell' + 1)}) | N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G] \cdot \beta_{\ell' + 1}^{1/3}
\]

\[
\leq \beta_{\ell' + 1}^{1/3} P_{\psi \in \Phi(N_{2i+1})} [N_{2i+1}^{(\ell' + 1)} \twoheadrightarrow G]. \quad (10)
\]
For \( \phi \in \Phi(B \setminus u) \) with \( B \setminus u \xrightarrow{\phi} G \), we define the label of \( \phi \), \( \text{label}(\phi) = (a_1, \ldots, a_{k-1}) \in [k]^{k-1} \) so that
- each \( a_i \) is the minimum number among the ranks of \( \phi|_{N_i}, \phi|_{N_3}, \ldots, \phi|_{N_{2i-1}} \) and among the \( (\phi|_{N_{2j-3}}) \)-rank of \( \phi|_{N_{2j-1}}, j = 2, \ldots, i \).

Since any label is a non-increasing sequence, if \( a_1 = \text{rank}(\phi|_{N_1}) \leq k - 1 \) then it satisfies that
(a) \( a_{k-1} = 1 \) or
(b) \( a_{i-1} = a_i = \ell, \exists i > 1, \exists \ell \in [2, k-1] \).

Thus we define the type of \( \phi \in \Phi(B \setminus u) \), denoted by \( \text{type}(\phi) \in [0, k] \) as follows.

(i) The type is 0 iff \( B \setminus u \xrightarrow{\phi} G \) does not hold.
(ii) The type is \( k \) if \( \text{rank}(\phi|_{N_1}) = k \).
(iii) Otherwise, if condition (a) holds then \( \text{type}(\phi) := 1 \), and if (a) does not hold but (b) holds then \( \text{type}(\phi) \) is the largest \( \ell \in [2, k-1] \) with property (b).

**Claim 3.2** (Case of type 1). For any fixed \( \varphi \in \Phi(N_1) \) with its rank in \([k - 1]\), we have

\[
\Pr_{\psi \in \Phi(N_{2, \infty})} [\text{type}(\varphi \cup \psi) = 1] \leq \eta_2^{1/3} \Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G].
\]

*Proof.* We divide it into two cases:
(i) \( \text{rank}(\phi|_{N_{2i-1}}) = 1 \) for \( i \geq 1 \) or
(ii) \( \text{rank}(\phi|_{N_{2i-3}}) \geq 2 \) but \( \text{rank}(\phi|_{N_{2i-1}}) = 1 \) for some \( i \geq 2 \).

Therefore it follows from (9) and (11) and from \(|V(N_{[2,2i-1]})| \leq \Delta^{2k}\) that

\[
\begin{aligned}
\Pr_{\phi \in \Phi(N_{2, \infty})} [\text{type}(\varphi \cup \psi) = 1] &= \frac{1}{\Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G]} \times \\
&\leq \sum_i \Pr_{\phi \in \Phi(N_{2, \infty})} [\text{type}(\phi) = 1] \Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G] \\
&\leq \sum_i \beta_i \Pr_{\phi \in \Phi(N_{2, \infty})} [\text{type}(\phi) = 1] \Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G] \\
&\leq \sum_i \beta_i \Pr_{\phi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G] \\
&\leq \sum_i \beta_i \Pr_{\phi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G]
\end{aligned}
\]

(by repeating the induction hypothesis (on \(|V(B \setminus \{u\})| > |V(N_{[2,2i-1]})| \)) \(|V(N_{[2,2i-1]})| \) times)

\[\leq \eta_2^{1/3} \frac{1}{\Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G]} \times \]

since \( \eta_2 \ll \rho_2 \leq \cdots \leq \rho_k, 1/k, 1/\Delta \).

**Claim 3.3** (Case of full types). Let \( \ell_0 \in [2, k-1] \). For any fixed \( \varphi \in \Phi(N_1) \) with its rank \( \ell_0 \), we have

\[
\Pr_{\psi \in \Phi(N_{2, \infty})} [\text{type}(\varphi \cup \psi) = \ell_0] \leq \eta_2^{-0.01} \Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G].
\]

*Proof.* We see that

\[
\begin{aligned}
\Pr_{\phi \in \Phi(N_{2, \infty})} [\text{type}(\varphi \cup \psi) = \ell_0] &= \frac{1}{\Pr_{\psi \in \Phi(B \setminus u)} [B \setminus u \xrightarrow{\psi} G|N_1 \xrightarrow{\psi} G]} \\
&\leq \left(1 + \beta_0^{1/6}\right) \prod_{e \in \psi(N_{[1,3]\cup N_{[1,3]}) \cup \{ \gamma \psi(N_{[1,3]) \cup \{ \gamma \} \}}} d_G(e) \cdot \frac{1}{\Pr_{\psi \in \Phi(N_{[3, \infty])} [\text{type}(\varphi) = \ell_0]} \\
&\leq \frac{1}{\Pr_{\psi \in \Phi(N_{[3, \infty])} [\text{type}(\varphi) = \ell_0]} \prod_{e \in \psi(N_{[1,3]\cup N_{[1,3]}) \cup \{ \gamma \psi(N_{[1,3]) \cup \{ \gamma \} \}}} d_G(e) \\
&\leq \left(1 + \beta_0^{1/6}\right) \prod_{e \in \psi(N_{[1,3]\cup N_{[1,3]}) \cup \{ \gamma \psi(N_{[1,3]) \cup \{ \gamma \} \}}} d_G(e) \cdot \frac{1}{\Pr_{\psi \in \Phi(N_{[3, \infty])} [\text{type}(\varphi) = \ell_0]} \\
&\leq \left(1 + \beta_0^{1/6}\right) \prod_{e \in \psi(N_{[1,3]\cup N_{[1,3]}) \cup \{ \gamma \psi(N_{[1,3]) \cup \{ \gamma \} \}}} d_G(e)
\end{aligned}
\]

(because of (6) and definition (ii) of \( \ell, \star, \varphi, \text{badness} \)).
\begin{align}
\leq & \frac{(1 + \beta_{\ell_0}^{1/6}) \prod_{e \in V(N_3)} (1 + \eta_{|e|})}{0.9 \prod_{|e| \geq \ell_0} \prod_{e \in V(N_{[1,3]}) \setminus V(N_3)} \rho_{|e|}} \\
\leq & \frac{1}{\eta_{\ell_0 + 1}^{0.01}}
\end{align}

since

\[ \eta_{\ell_0 + 1} \leq \cdots \leq \rho_{k}, 1/k, 1/\Delta. \]  

\[ \text{(11)} \]

Claim 3.4 (Case of degenerate types). Let \( \ell \in [2, k - 2] \). For any fixed \( \varphi \in \Phi(N_1) \) with its rank in \( \ell + 1, k - 1 \), we have

\[ \mathbb{P}_{\varphi \in \Phi(N_{[2, \infty]})} \{ \text{type}(\varphi \cup \varphi) = \ell \} \leq \frac{\eta_{\ell_0 + 1}^{\text{type}} \mathbb{P}_{\psi \in \Phi(B \setminus \psi)} [B \setminus \psi \rightarrow G \mid N_1 \rightarrow G]}. \]

Proof. Suppose that \( \text{label}(\varphi \cup \varphi) = (a_1, \cdots, a_{k-1}) \). Let \( i_0 \in [2, k - 2] \) be the smallest integer with \( a_{i_0} = \ell \). It follows from \( a_{i_0} = \ell \) and from the minimality of \( a_{i_0} \) that

\[ \min_{j \in [2, i_0 + 1]} \left\{ \min \{ \text{rank}(\varphi \cup \varphi)\mid N_{2j - 3}, \text{rank}(\varphi \cup \varphi)\mid N_{2j - 3} \} \right\} \geq \ell, \text{ and} \]

\[ \text{rank}(\varphi)_{N_{2i_0 - 1}} = \ell \text{ or } \text{rank}(\varphi)_{N_{2i_0 - 3}} > \text{rank}(\varphi)_{N_{2i_0 - 1}} = \ell. \]

When \( s = 1, \ell + 1 \), let \( B_s \) be the complex obtained from \( B \setminus u \) by invisualizing all the edges of size at least \( s \) containing a vertex of \( N_{2j}, j \in [i_0] \). It follows that

\[ \mathbb{P}_{\varphi \in \Phi(N_{[2, \infty]})} \{ \text{type}(\varphi \cup \varphi) = \ell \} \leq \frac{\mathbb{P}_{\psi \in \Phi(B \setminus \psi)} [B \setminus \psi \rightarrow G \mid N_1 \rightarrow G]}{\mathbb{P}_{\psi \in \Phi(B \setminus \psi)} [B \setminus \psi \rightarrow G \mid N_1 \rightarrow G]} \]

\[ \leq \sum_{i_0} \prod_{j \in [i_0]} (1 + \beta_{\ell}^{1/6}) \prod_{e \in V(\psi) (N_{[2j - 1, 2j + 1]} \mid V(N_{2ij - 1})} d_{\psi}(B(e)) \quad \text{(: I.H.)} \]

\[ \mathbb{P}_{\varphi \in \Phi(N_{[2, \infty]})} [B_1 \varphi \cup \varphi \rightarrow G \mid \text{and} \quad \psi \in \Phi(B \setminus \psi)] \leq \sum_{i_0} \prod_{j \in [i_0]} (1 + \beta_{\ell}^{1/6}) \prod_{e \in V(\psi) (N_{[2j - 1, 2j + 1]} \mid V(N_{2ij - 1})} d_{\psi}(B(e)). \]

\[ \mathbb{P}_{\varphi \in \Phi(N_{[2, \infty]})} [B_1 \varphi \cup \varphi \rightarrow G \mid \text{and} \quad \psi \in \Phi(B \setminus \psi)] \leq \sum_{i_0} \prod_{j \in [i_0]} (1 + \beta_{\ell}^{1/6}) \prod_{e \in V(\psi) (N_{[2j - 1, 2j + 1]} \mid V(N_{2ij - 1})} d_{\psi}(B(e)) \quad \text{(: I.H.)} \]

where we used, in the last two inequalities, the assumption that

\[ \eta_{\ell_0 + 1} \leq \rho_{\ell_0 + 2} \leq \cdots \leq \rho_{k}, 1/k, 1/\Delta. \]  

\[ \text{(14)} \]
Finally we obtain the inequalities that
\[ P_{\phi \in \Phi(B)}[B \stackrel{\phi}{\rightarrow} G] \]
\[ \geq P_{\varphi \in \Phi(N_1)}[\text{rank}(\varphi) = \ell_0]P_{\phi \in \Phi(B)}[B \stackrel{\phi}{\rightarrow} G] P_{\phi \in \Phi(B \setminus u)}[B \setminus u \stackrel{\phi}{\rightarrow} G] \]
and
\[ \geq (1 - \beta_k^{1/3}) \prod_{c \in \mathbb{V}(B); u \in e} d_G(B(e)) \cdot P_{\phi \in \Phi(B \setminus u)}[B \setminus u \stackrel{\phi}{\rightarrow} G] - \sum_{\ell_0 \in [k-1]} P_{\phi \in \Phi(B \setminus u)}[\text{type}(\phi) = \ell_0] \]
\[ \geq (1 - \eta_k^{1/4,1}) \prod_{c \in \mathbb{V}(B); u \in e} d_G(B(e)) \cdot (1 - \eta_k^{1/2,2}) P_{\phi \in \Phi(B \setminus u)}[B \setminus u \stackrel{\phi}{\rightarrow} G] \]
\[ \geq (1 - \eta_k^{1/4,2}) \prod_{c \in \mathbb{V}(B); u \in e} d_G(B(e)) \cdot P_{\phi \in \Phi(B \setminus u)}[B \setminus u \stackrel{\phi}{\rightarrow} G] \]
where we used the fact that
\[ \sum_{\ell_0 \in [k-1]} P_{\varphi \in \Phi(N_1)}[\text{rank}(\varphi) = \ell_0] \left[ P_{\phi \in \Phi(N_1)}[\text{type}(\varphi \cup \phi) = \ell_0] + \sum_{\ell \in [\ell_0-1]} P_{\phi}[\text{type}(\varphi \cup \phi) = \ell] \right] \text{[rank}(\varphi) = \ell_0] \]
\[ \leq \sum_{\ell_0 \in [k-1]} \eta^{1/3,1}_{\ell_0} P_{\varphi \in \Phi(N_1)}[N_1^{(\ell_0+1)} \not\rightarrow \mathbb{G}] \]
\[ \cdot \left( \eta^{0,1}_{\ell_0+1} + \sum_{\ell \in [\ell_0-1]} \eta^{1/3,1}_{\ell+1} \right) P_{\psi \in \Phi(B \setminus u)}[B \setminus u \not\rightarrow \mathbb{G}] |N_1 \not\rightarrow \mathbb{G}| \]
\[ \leq P_{\psi \in \Phi(B \setminus u)}[B \setminus u \not\rightarrow \mathbb{G}] \sum_{\ell_0 \in [k-1]} \eta^{1/3,1}_{\ell_0+1} \left( \eta^{0,1}_{\ell_0+1} + \sum_{\ell \in [\ell_0-1]} \eta^{1/3,1}_{\ell+1} \right) \frac{\prod_{c \in \mathbb{V}(\ell_0+1)(N_1)} d_G^{(+\delta)}(B(e))}{\prod_{c \in \mathbb{V}(N_1)} d_G^{-\delta}(B(e))} \]
\[ \leq P_{\psi}[B \setminus u \not\rightarrow \mathbb{G}] \sum_{\ell_0 \in [k-1]} \eta^{1/3,1}_{\ell_0+1} \left( \eta^{0,1}_{\ell_0+1} + \sum_{\ell \in [\ell_0-1]} \eta^{1/3,1}_{\ell+1} \right) \frac{\prod_{c \in \mathbb{V}(\ell_0+1)(N_1)(1 + \eta_{|c|})}{\prod_{c \in \mathbb{V}(N_1)} (1 - \eta_{|c|}) \prod_{c \in \mathbb{V}(N_1)} \rho_{|c|}} \]
\[ \leq P_{\psi \in \Phi(B \setminus u)}[B \setminus u \not\rightarrow \mathbb{G}] \eta^{1/3,1}_k , \]
\[ (15) \]
since
\[ \eta_{\ell_0+1} \ll \rho_{\ell_0+1} \leq \rho_{\ell_0+2} \leq \cdots \leq \rho_k, 1/k, 1/\Delta. \]
It completes the proof of the main lemma.
\[ \square \]

4. Proof of the Main Theorem

Let \( B \) be a \( k \)-uniform hypergraph on \( n \) vertices with maximum degree \( \Delta \), where each vertex is contained in at most \( \left( \begin{array}{c} \Delta \\ k-1 \end{array} \right) \) of size-\( k \) visible ‘white’ edges, and all non-white edges are invisible. It is clear that \( B \) can be seen as an \( r \)-partite hypergraph on \( V(B) = V_1(B) \cup \cdots \cup V_r(B) \) where \( r = \Delta + 1 \). (In other words, \( B \) is a \( \Delta \)-blowup of the \( k \)-uniform complete hypergraph on \( r \) vertices.) Let \( G \) be a \( k \)-uniform hypergraph on \( mN \) vertices, where each size-\( k \) edge has one of \( b_k \) visible colors. Our purpose is to find a monochromatic copy of \( B \) in \( G \). We set the following parameters
\[ r = \Delta + 1, k, b_k \ll m \ll 1/\alpha \ll 1/\eta(e) \ll 1/\varepsilon(\cdot, \cdot) \ll b_{k-1} \leq \cdots \leq b_1 \]
with an auxiliary function
\[ \rho(\cdot) = \rho(b^*_k) = \alpha/b^*_k \]
which will be used at \( \{(10, 17), (19), (20), (21)\}. \)

We set \( V(G) = \Omega_1 \cup \cdots \cup \Omega_m \) with \( |\Omega_i| = N \), and delete all ‘non-partitionwise’ edges. That is, any edge contains at most one vertex in a partite set \( \Omega_i \). And color in black all the edges of size at most
For this resulting $m$-partite $k$-bound $(1, \cdots, 1, b_k)$-colored graph, we apply the regularity lemma (Theorem 2.4) with $r = m, k = k, h = 2\Delta^{2k}, \bar{b} = (1, \cdots, 1, b_k),$ and

$$\varepsilon(\cdot, \cdot) = \varepsilon(i, b_i^*),$$

and obtain an $(\varepsilon(\cdot, \cdot), 2\Delta^k)$-regular subdivision $G^*$ which is $(\bar{b}_1, \cdots, \bar{b}_{k-1}, \bar{b}_k = b_k \geq 2)$-colored where $m, k, \Delta, b_k, 1/\varepsilon(\cdot, \cdot) \ll \bar{b}_1, \cdots, \bar{b}_{k-1}$.

Let

$$b_i^* := \max_{j : i \in \bar{b}_j} |C_{f,j}(G^*)| \leq \bar{b}_i \text{ and } \rho_i(b_i^*) = \frac{\alpha}{b_i^*} \geq \frac{\alpha}{\bar{b}_i}.$$  \hspace{2cm} (18)

A size-$i$ edge $e$ is called exceptional iff $d_{f,i}(G^*(e)) = \rho_i(b_i^*)$ or $d(G^*(e)) > \varepsilon(i, b_i^*)/\alpha = \rho_i(b_i^*)\eta_i(\rho_i(b_i^*))$ where $\delta(\cdot)$ is a function associated with $G^*$. For any index $I$, it easily follows that

$$P_{e \in \Omega I}(G^*(e) \text{ is exceptional}) \leq \rho_i(b_i^*)b_i^* + \frac{\varepsilon(i, b_i^*)}{\varepsilon(i, b_i^*)/\alpha} = \alpha + \alpha = 2\alpha.$$

Take $m$ vertices $v_i \in \Omega_i, i \in [m], \text{ randomly.}$ Then in the average, the number of exceptional edges of the hypergraph induced by the $m$ vertices is at most $\sum_{i \in [k]} (\binom{m}{i})2\alpha < 1$ since $m \ll 1/\alpha$. Thus there exist $m$ vertices $v_i \in \Omega_i, i \in [m], \text{ such that all the edges in the graph induced by them are not exceptional.}$ By Ramsey Theorem, Theorem 1.1 with

$$r = \Delta + 1 \ll m,$$

there exist $r$ vertices among the $m$ vertices such that in the induced hypergraph, all of the size-$k$ edges have the same color, say red. Consider $S \in S_{r, k, 1}, \text{ the } k\text{-bound } r\text{-partite complex on those } r = \Delta + 1 \text{ vertices such that the color of each edge of } S \text{ is given by the corresponding color in } G^*.$ (Note that all size$-k$ edges of $S$ are red.) Denote again by $B$ the complex obtained from the given $B$ (i) by recoloring each size-$k$ white edge of $B$ in red, and (ii) by coloring each edge of $B$ of size at most $k - 1$ in the color of corresponding edge in $S$ so that $B$ is a $\Delta$-blowup of $S$.

Finally we can apply Corollary 2.3 (with $r = r, k = k, h = 1, \Delta = \Delta, \bar{b} = \bar{b}^*, \rho_i = \rho_i = \alpha/b_i^*, S = S, B = B, G = G^*$) where

$$k, \Delta, b_1^*, \cdots, b_{k-1}^*, b_k^* = b_k, 1/\alpha \ll 1/\eta_i(\rho_i(b_i^*))$$

We get the desired injection $\varphi \in \Phi(B)$ which embeds $B$ in $G$, yielding a red copy of the original $k$-uniform hypergraph $B$.

The above argument can be applied for any $B$ as far as

$$|V_i(B)| < \eta_i(\rho_i(b_i^*))|V(G^*)| \text{ for each } i \in [r]$$

in which by (18) the right hand side is at least

$$\eta_i(\rho_i(\bar{b}_i))N = \frac{\eta_i(\rho_i(\bar{b}_i))}{m}|V(G)| = \Theta(|V(G)|).$$

It completes the proof of Theorem 1.1. \hspace{2cm} \Box

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