Spatio-temporal correlation functions in scalar turbulence from functional renormalization group

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We provide the leading behavior at large wavenumbers of the two-point correlation function of a scalar field passively advected by a turbulent flow. We first consider the Kraichnan model, in which the turbulent carrier flow is modeled by a stochastic vector field with a Gaussian distribution, and then a scalar advected by a homogeneous and isotropic turbulent flow described by the Navier-Stokes equation, under the assumption that the scalar is passive, \textit{i.e.}, that it does not affect the carrier flow. We show that at large wavenumbers, the two-point correlation function of the scalar in the Kraichnan model decays as an exponential in the time delay, in both the inertial and dissipation ranges. We establish the expression, both from a perturbative and from a nonperturbative calculation, of the prefactor, which is found to be always proportional to $k^2$. For a real scalar, the decay is Gaussian in $t$ at small time delays, and it crosses over to an exponential only at large $t$. The assumption of delta-correlation in time of the stochastic velocity field in the Kraichnan model hence significantly alters the statistical temporal behavior of the scalar at small times.

I. INTRODUCTION

The advection of scalar fields by external flows enters in a wide range of phenomena, such as the fluctuations of temperature or salinity in the ocean, or the dispersion of pollutants in the atmosphere. When the carrier flow is turbulent, understanding the statistical properties and the mixing of the scalar become a difficult and fundamental issue. A simplifying assumption is to consider as passive the advected scalar field, which means that it has a negligible backreaction onto the carrier flow, such that the dynamics of the latter is unaffected. This assumption is realistic in particle-laden flows only at sufficiently small concentration. However, even in this simpler case, the complete characterization of the properties of the scalar is still challenging, and we focus on this case in this work.

For a high-Reynolds number carrier flow, one can distinguish several regimes for the scalar depending on the Schmidt number, which is the ratio of the fluid viscosity to the scalar diffusivity. We refer to Ref. \textsuperscript{1} for an overview. In this work, we focus on the inertial-convective range, which corresponds to Schmidt numbers of order one, and which spans scales between the energy injection scale of the scalar and the scale at which energy is dissipated, for typical scales of the velocity of the turbulent fluid in the inertial range. The spectrum of the scalar field in the inertial-convective range was established by Obukhov and Corrsin.\textsuperscript{12} Other pioneering studies are due to Yaglom, Batchelor, and Kraichnan.\textsuperscript{1,2} Barring possible corrections due to intermittency, the scalar spectrum in this range is determined via the energy cascade picture and shows a power-law decay with the same exponent $-5/3$ in three dimensions as the spectrum of the turbulent fluid carrying it.

The spectrum is related to the two-point correlation function of the scalar field at equal times. Characterizing its behavior at different times is also fundamental to understand the mixing properties of the scalar. In particular, determining the temporal dependence of the Eulerian correlations of the scalar is a difficult task and few is known so far on their properties. In this work, we use the framework of the functional renormalization group (FRG) (see Ref. \textsuperscript{7} for an introduction and Ref. \textsuperscript{8} for a recent review) to make progress on this issue. The FRG approach was recently employed to study Navier-Stokes (NS) turbulence, where it proved to be fruitful. By exploiting the symmetries of the field theoretical formulation of the stochastically forced Navier-Stokes equation, it yielded predictions for the general form of the spatio-temporal dependence of any $n$-point correlation function of the turbulent velocity field in the limit of large wavenumbers.\textsuperscript{9,10} These predictions were accurately confirmed by Direct Numerical Simulations.\textsuperscript{9,11,12} The FRG approach was also extended to study the direct cascade in 2D turbulence.\textsuperscript{13}

In this work, we follow a similar strategy for the passive scalar, in order to construct an approximation scheme which is not based on a small-coupling expansion, and thus offers a nonperturbative approach. Exploiting the symmetries of the field theory of the advected scalar, we provide the general expression of the temporal dependence of the two-point Eulerian correlations of the scalar field in the limit of large wavenumbers. These expressions are obtained both for a scalar advected by a turbulent NS flow, and for a scalar advected by a “synthetic” stochastic velocity field, as prescribed in the Kraichnan model.\textsuperscript{14} In the latter case, the velocity statistics is Gaussian and the associated covariance is white-in-time. Although these delta-correlations yield simpler explicit expressions, they have a significant impact on the temporal behavior of the scalar field at small time. Indeed, we show that for a scalar in a turbulent NS flow, the two-point correlation function decays at small $t$ as a Gaussian in the variable $tk$ where $t$ is the time delay and $k$ the wavenumber, and as an exponential at large $t$. The scalar field in the inertial-convective regime thus inherits in this case temporal correlations which are very similar to the ones of the turbulent fluid (reported \textit{e.g.} in Ref. \textsuperscript{13}). For the Kraichnan scalar, the two-point correlations always decay exponentially, the Gaussian small-time behavior is completely washed out by the instantaneous decor-
relation of the synthetic velocity field.

The paper is organized as follows. In Sec. III we introduce the Kraichnan model, the associated field theory, and we expound its symmetries and the related Ward identities. In Sec. IV we establish some preliminary results for this model that will be used to derive a closed flow equation for the correlation functions of the Kraichnan scalar in Sec. V. In Sec. VI we generalize the derivation and results to the case of scalar fields advected by turbulent NS flows. We summarize our findings in Sec. VII. In Appendix A, we discuss the Yaglom relation from a field theoretical viewpoint, and in Appendix B, we establish the expression of the spectrum in the near-dissipative range from the Dyson equation.

II. FIELD THEORY AND SYMMETRIES OF THE KRAICHNAN MODEL

The Kraichnan model was proposed as a simplified model of scalar turbulence where the advecting velocity field is taken as a random vector field with Gaussian statistics to replace the NS fluctuating turbulent velocity field. The key simplifying feature of this model is that the velocity covariance is white in time, \( \langle v^i(t,x) v^j(t',x') \rangle \equiv \delta(t - t') \). This model has been widely studied since many aspects can be investigated analytically. In particular, several techniques have been employed to compute intermittency corrections via suitable expansions. In the perturbative regime, intermittency corrections were computed also via various field theoretical approaches, including the functional renormalization group. In this section, we present the path integral formulation of the Kraichnan model, introduce the FRG formalism, and then discuss the symmetries of the Kraichnan field theory and the associated Ward identities. We shall focus on incompressible flows and take the velocity field to be divergenceless.

A. The Kraichnan model

The Kraichnan model describes the advection and diffusion of a scalar field \( \theta(t,\vec{x}) \) following the dynamics

\[
\partial_t \theta(t,\vec{x}) + v^i(t,\vec{x}) \partial_i \theta(t,\vec{x}) - \frac{\kappa}{2} \partial^2 \theta(t,\vec{x}) = f(t,\vec{x}),
\]

where \( \kappa \) is the molecular diffusivity and \( f(t,\vec{x}) \) is a stochastic forcing characterized by Gaussian statistics with covariance \( \langle f(t,\vec{x}) f(t',\vec{y}) \rangle = \delta(t-t') M \left( \frac{\vec{x} - \vec{y}}{L} \right) \). In this model, the velocity \( v^i(t,\vec{x}) \) is a stochastic vector field chosen with a Gaussian distribution. It has zero average and the following covariance:

\[
\langle v^i(t,\vec{x}) v^j(t',\vec{y}) \rangle = \delta(t-t') D_0 \int_{\bar{p}} \frac{\delta(\bar{p})}{\bar{p}^2 + m^2} P_{ij}(\bar{p}),
\]

with \( P_{ij}(\bar{p}) = \delta_{ij} - \frac{p_ip_j}{p^2} \) the transverse projector. The mass \( m \) in Eq. (2) prevents any infrared divergence in the velocity covariance. Note that we use for the integrals the shorthand notation

\[
\int_{\vec{x}} = \int dt \int d^dx, \quad \int_{\vec{q}} = \int \frac{d\omega}{2\pi} \int \frac{d^d\bar{q}}{(2\pi)^d}
\]

and for the Fourier transforms the convention

\[
g(\omega,\vec{p}) \equiv \int_{\vec{x}} e^{i\omega x} g(t,\vec{x}), \quad g(t,\vec{x}) \equiv \int_{\vec{x}} e^{-i\omega x} g(\omega,\vec{p}).
\]

In the limit \( \epsilon \to 0 \), the coinciding-point covariance \( \langle \epsilon \rangle \) diverges logarithmically. Formally, a finite limit can be obtained by defining \( D_0 = D_0^0 \epsilon \). We shall use either \( D_0 \) or \( D_0^0 \), keeping in mind that they are just fixed numbers for a given value of \( \epsilon \).

The stochastic differential equation (SDE) Eq. (1) is a multiplicative one and is interpreted in the Stratonovich sense. It can also be rewritten in the Ito sense, but we shall work here with the Stratonovich convention unless otherwise stated. The Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) formalism provides a well established procedure to map a SDE to an equivalent path integral formalism. The MSRJD approach allows one to compute an arbitrary correlation function by introducing the appropriate generating functionals. A key advantage of employing the MSRJD formalism is that it allows one to compute Eulerian correlation functions at different times in a simple way.

The generating functional for the Kraichnan model is obtained as

\[
Z = \int D\theta D\bar{\theta} D\varphi D\bar{\varphi} e^{-S_{K\theta} - \frac{1}{2} J_{ij} \psi \bar{J}^i \psi \bar{J}^j} \quad (4)
\]

with the following action:

\[
S_{K\theta} = \int_{\vec{x}} \left( \partial_t \theta(t,\vec{x}) + v^i(t,\vec{x}) \partial_i \theta(t,\vec{x}) - \frac{\kappa}{2} \partial^2 \theta(t,\vec{x}) \right)
\]

\[
- \frac{1}{2} \int_{\vec{x}} \bar{\theta}(t,\vec{x}) M \left( \frac{\vec{x} - \vec{y}}{L} \right) \bar{\theta}(t,\vec{y})
\]

\[
+ \int_{\vec{x}} c(t,\vec{x}) \left( \partial_i c(t,\vec{x}) + v^i(t,\vec{x}) \partial_i c(t,\vec{x}) - \frac{\kappa}{2} \partial^2 c(t,\vec{x}) \right)
\]

\[
+ \frac{1}{2} \int_{\vec{x}} \bar{v}(t,\vec{x}) \frac{(-\partial^2 + m^2)^2}{D_0} \bar{v}(t,\vec{x}). \quad (5)
\]

Several comments regarding this action are in order. The field \( \bar{\theta} \) is usually called the response field since it is related to response functions. The quadratic term in \( \bar{\theta} \) originates from integrating out the stochastic forcing \( f \) of the scalar: the function \( M \left( \frac{\vec{x} - \vec{y}}{L} \right) \) is essentially the forcing covariance and \( L \) represents the large length scale at which energy is injected to stir the scalar. The third line in (5) contains the Grassmannian odd fields \( c \) and \( \bar{c} \) which are introduced to express the functional determinant involved in the procedure via an action and are referred to as ghosts. This determinant is written in the Stratonovich convention. The source term \( J \) gathers the different sources, respectively \( J_\theta, J_\bar{\theta}, J, \eta, \bar{\eta} \), for the fields \( \theta, \bar{\theta}, v, c, \bar{c} \) gathered in the field multiplet \( \phi \).
B. The functional renormalization group formalism

The renormalization group has been instrumental for understanding the scaling properties of physical systems at criticality. Also away from equilibrium the RG constitutes a crucial conceptual and calculational framework to investigate several phenomena. As far as turbulence is concerned, the renormalization group approach has been employed in many studies. However, perturbative techniques have often been hampered by the lack of a small expansion parameter.

The functional renormalization group implements the Wilsonian renormalization program at a functional level by adding to the microscopic action a term \( \Delta S_k \) which introduces a wavenumber scale \( k \) and suppresses the integration over low wavenumber modes \( |\vec{p}| \lesssim k \). The fluctuations are hence progressively averaged thereby building up the effective theory at scale \( k \). The modified effective action in the presence of this scale-dependent term is called the effective average action (EEA) and its \( k \)-dependence is governed by an exact equation, which can be solved by implementing some approximation scheme. We refer to Ref. 7 for a pedagogical introduction and to Ref. 8 for a recent review.

To achieve the separation of fluctuation modes, the term \( \Delta S_k \) is chosen quadratic in the fields:

\[
\Delta S_k [\phi] = \int \frac{1}{2} R_k (\theta - \phi^2) \phi,
\]

where \( R_k (\theta - \phi^2) \) is a suitable cutoff kernel. It is required to be large for modes with wavenumber \( |\vec{p}| \lesssim k \) and to vanish for modes with \( |\vec{p}| \gtrsim k \) such that only these modes are integrated out:

\[
R_k (p^2) = \begin{cases} k^2 & |\vec{p}| \leq k, \\ 0 & |\vec{p}| \gtrsim k \end{cases}
\]

The modified functional integral takes the form

\[
Z_k = \exp \left[ W_k [J] \right] = \int \mathcal{D} \phi e^{-S[\phi] - \Delta S_k [\phi]} \int J e^{i \int x \phi},
\]

where \( W_k [J] \) is the (modified) generating functional of the connected correlation functions. The EAA is defined by the Legendre transform of \( W_k [J] \) and is denoted by \( \Gamma_k [\phi] \):

\[
\Gamma_k [\phi] + \Delta S_k [\phi] = \int J \phi - W_k [\phi],
\]

where \( \phi \) is the average field. The scale dependence of the EAA is governed by an exact RG flow equation that takes the following general form:

\[
\partial_t \Gamma_k [\phi] = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)} [\phi] + R_k \right)^{-1} \partial_t R_k \right],
\]

where \( s \equiv \log (k/\Lambda) \) with \( \Lambda \) a UV scale. At this scale, one can show that \( \Gamma_k \) identifies with the bare action, since no fluctuations is yet incorporated. When \( k \to 0 \), the regulator is removed and one obtains the actual properties of the model, when all fluctuations have been integrated over. Eq. (8) provides the exact interpolation between these two scales.

After this general introduction, let us discuss the application of the FRG formalism to the Kraichnan model. In particular, we follow the approach discussed in Ref. 9, where the scale \( k \) is identified with the forcing scale of the scalar, i.e., \( k = L^{-1} \), which is now running, and \( M \to M_k \) in Eq. (5). Indeed, since the forcing covariance is peaked at large length scales and vanishes at small length scales, it satisfies the requirements and can therefore be interpreted as a cutoff. Besides this "effective forcing" cutoff, we also introduce cutoff kernels for the other fields. The generalized cutoff action is defined by

\[
\Delta S_k [\phi] = \int \mathcal{D} \phi \left\{ \frac{k}{2} \theta R_k \left( \theta - \phi^2 \right) \theta - \frac{1}{2} \theta M_k \left( \theta - \phi^2 \right) \theta + \frac{1}{2D_0} \phi^2 R_k \left( \theta - \phi^2 \right) \phi \right\}.
\]

With this addition the total action finally reads

\[
S = \int \mathcal{D} \phi \left\{ \partial_t \phi + \phi \right\} + \int \mathcal{D} \phi \left\{ \partial_t \phi + \phi \right\} + \int \mathcal{D} \phi \left\{ \partial_t \phi + \phi \right\}
\]

where the last two lines contain the cutoff action which incorporates the scalar forcing term.

C. Vertex functions and notation

The EEA \( \Gamma_k \) introduced in section III is the generating functional of the one-particle-irreducible (1-PI) correlation functions, also called vertices, which are often introduced in field theoretical treatments. The set of \( n \)-point connected correlation functions, denoted generically \( W_k^{(n)} \), is equivalent to the set of \( n \)-point 1-PI correlation functions, denoted \( \Gamma_k^{(n)} \), in the sense that one can be constructed from the other. Both types of correlation functions are used in the following. Let us introduce some notation, in order to specify the fields entering these functions. By taking \( n \) functional derivatives of the EEA, one obtains the \( n \)-point corresponding vertex function

\[
\delta^n \Gamma_k \equiv \partial_{\phi_1} (t_1, \bar{x}_1) \cdots \partial_{\phi_2} (t_{n_1+1}, \bar{x}_{n_1+1}) \frac{\delta}{\delta \phi_1 (t_{n_1}, \bar{x}_{n_1} + 1)} \cdots \frac{\delta}{\delta \phi_2 (t_{n_1+1}, \bar{x}_{n_1+2})} \Gamma_k^{(n_1, n_2, \cdots)} (t_1, \bar{x}_1, \cdots, t_{n_1+1}, \bar{x}_{n_1+1}, \cdots),
\]
where }n = \sum n_i \text{ and } \phi_i \text{ denote any of the fields contained in the multiplet } \phi. \text{ The Fourier transform of a vertex is denoted by}
\Gamma_k^{(n \phi_1, n \phi_2, \ldots)}(\omega_1, \tilde{p}_1, \ldots) = \mathcal{F} \Gamma_k^{(n \phi_1, n \phi_2, \ldots)}(t_1, \tilde{x}_1, \ldots)(\omega_1, \tilde{p}_1, \ldots)

with }\mathcal{F} \text{ the Fourier transform defined as Eq. (13) for each variables } (\omega, \tilde{p}). \text{ A fully analogous notation will be employed for the connected correlation functions } W_k^{(n)}.

In order to take into account translation invariance in space and time, we also introduce the following generic }n\text{-point function:}
\Gamma_k^{(n \phi_1, n \phi_2, \ldots)}(\omega_1, \tilde{p}_1, \ldots) = (2\pi)^{d+1} \delta \left( \sum_{i=1}^{n} \omega_i \right) \delta^d \left( \sum_{i=1}^{n} \tilde{p}_i \right),

\tilde{\Gamma}_k^{(n \phi_1, n \phi_2, \ldots)}(\omega_1, \tilde{p}_1, \ldots, \omega_{n-1}, \tilde{p}_{n-1}),

where }\tilde{\Gamma}_k^{(n)} \text{ hence denotes a } 1\text{-PI vertex “stripped” of the wavenumber and frequency conserving delta functions. A fully analogous notation is introduced for the }n\text{-point correlation function, } W_k^{(n)}. \text{ In particular, a propagator stripped of the delta functions will be written as } \tilde{G}_{\phi_i \phi_0}(\omega, \tilde{p}).

In order not to overburden the notation in certain equations, we shall sometimes employ the DeWitt notation. In this case, the average field is denoted simply by } \phi_0 \text{ and the sources by } J^\mu. \text{ The summation convention is intended to mean } \phi_0 J^\mu \equiv \int d\tau \phi_0(t, \tilde{x}) J^\mu(t, \tilde{x}).

D. Symmetries and Ward identities

In this section we discuss the symmetries of the action (10) and the associated Ward identities for the EAA. It turns out that it is particularly useful to consider not only the strict symmetries of the action but also the field transformations which leave the action invariant up to terms linear in the fields since in this latter case one can actually write down Ward identities which are even more constraining. We first explicitly expound the details of the shift symmetry, and for the other symmetries, we only provide the field transformation, the action variation, and the ensuing Ward identity. We refer the interested reader to Refs. [13] and [30] where a similar analysis is detailed for the case of the NS field theory. With abuse of notation, we denote the average fields by employing the same symbols used for the fluctuating fields, i.e. } \phi \equiv (\theta, \tilde{\theta}, v, c, \tilde{c}).

Before considering continuous symmetries, let us note that the action (10) possesses the following }Z_2\text{-symmetry: } \theta \rightarrow -\theta \text{ and } \tilde{\theta} \rightarrow -\tilde{\theta}. \text{ An analogous symmetry is present in the ghost sector. Moreover, one can show that any term in the EAA must depend on at least one response field, the EAA cannot have terms depending on } \theta \text{ (or } c) \text{ only, see Ref. [41] for a detailed discussion on this point.}

Scalar field shifts. Let us consider the transformation } \theta(t, \tilde{x}) \rightarrow \theta(t, \tilde{x}) + \epsilon(t), \text{ which is a time gauged shift of the advected scalar field. The variation of the action reads}
\delta S = \int d\tau \left( \tilde{\theta}(t, \tilde{x}) \left( \partial_t + \frac{\kappa}{2} R_k(0) \right) \epsilon(t) \right).

By writing that this field transformation must leave the functional integral } Z \text{ unchanged, one obtains the following Ward identity
\int d\tau J_\theta(t, \tilde{x}) \epsilon(t) = \int d\tau \left( \tilde{\theta}(t, \tilde{x}) \left( \partial_t + \frac{\kappa}{2} R_k(0) \right) \epsilon(t) \right)

Expressing the sources as functional derivatives of } \Gamma_k \text{ using the definition (7) of the Legendre transform then yields
\int d\tau \left( \frac{\delta \Gamma_k}{\delta \theta(t, \tilde{x})} + \tilde{\theta}(t, \tilde{x}) \frac{\kappa}{2} R_k(0) \right) \epsilon(t) = \int d\tau \tilde{\theta}(t, \tilde{x}) \left( \partial_t + \frac{\kappa}{2} R_k(0) \right) \epsilon(t)

Since this relation holds for an arbitrary infinitesimal } \epsilon(t), \text{ this leads to the final Ward identity
\int d\tau \frac{\delta \Gamma_k}{\delta \theta(t, \tilde{x})} = - \int d\tau \partial_t \tilde{\theta}(t, \tilde{x}), \quad (11)

which is local in time. The Ward identity (11) has an immediate interpretation: the term } \int \partial_t \tilde{\theta} \text{ in the microscopic action } S \text{ is not renormalized and it appears identically in the associated EAA. By taking further functional derivatives of this identity with respect to the other fields, one obtains the following relations in Fourier space
\Gamma_k^{1,1,0,0,0}(\omega_1, \tilde{p}_1 = 0) = i\omega_1
\Gamma_k^{(n \phi_0 \geq 1, n \phi_1, n \phi_2, n \phi_3)}(\omega_0, \tilde{p}_0 = 0, \ldots) = 0. \quad (12)

Equation (12) entails that any vertex having at least one vanishing wavenumber carried by a } \theta \text{ field actually vanishes.}

Time gauge Galilei transformation. We now consider an extended Galilean transformation, which corresponds to the infinitesimal field transform
\begin{align*}
\nu^I(t, \tilde{x}) &\rightarrow \nu^I(t, \tilde{x}) - \epsilon^I(t) \partial_I \nu^I(t, \tilde{x}) + \tilde{\epsilon}^I(t) \\
\tilde{\theta}(t, \tilde{x}) &\rightarrow \tilde{\theta}(t, \tilde{x}) - \tilde{\epsilon}^I(t) \partial_I \tilde{\theta}(t, \tilde{x}) \\
\theta(t, \tilde{x}) &\rightarrow \theta(t, \tilde{x}) - \epsilon^I(t) \partial_I \theta(t, \tilde{x}) \quad , \quad (14)
\end{align*}

where
\begin{align*}
\nu^I(t, \tilde{x}) &\rightarrow \nu^I(t, \tilde{x}) - \epsilon^I(t) \partial_I \nu^I(t, \tilde{x}) + \tilde{\epsilon}^I(t) \\
\tilde{\theta}(t, \tilde{x}) &\rightarrow \tilde{\theta}(t, \tilde{x}) - \tilde{\epsilon}^I(t) \partial_I \tilde{\theta}(t, \tilde{x}) \\
\theta(t, \tilde{x}) &\rightarrow \theta(t, \tilde{x}) - \epsilon^I(t) \partial_I \theta(t, \tilde{x}) \quad , \quad (14)
\end{align*}
and similarly for the ghost fields. This transformation can be interpreted as a time-gauged extension of space translations. The standard Galilean transformation is recovered for $\xi(t) = \xi t$. One finds

$$\delta S = \int_{\mathcal{E}} \bar{v}'(t, \bar{x}) \left[ \frac{1}{D_0} m^{d+\varepsilon} + \frac{1}{D_0} R_{k,\nu\nu}(0) \right] \partial_\xi v'(t) .$$

The associated Ward identity constrains the vertices for which one of the velocity has a zero wavenumber. The explicit expression reads

$$\Gamma^{(n_\ell, n_{\bar{v}}, n_{\nu}, n_{\bar{\nu}})}_{\nu} \left( \cdots, \omega, \bar{n}_\ell = 0, \cdots \right) = - \sum_{i=1}^n \bar{p}_i \Gamma^{(n_\ell, n_{\bar{v}}, n_{\nu}, n_{\bar{\nu}})}_{\nu} \left( \cdots, \omega + \omega_i, \bar{n}_i, \cdots \right), \quad (15)$$

where $\gamma_1 \cdots \gamma_{n+1}$ are the spatial indices of the velocity fields, $\bar{n}_i$ is the wavevector of the $i$th field in the vertex, and $n = n_\ell + n_\nu + n_{\bar{v}} + n_{\bar{\nu}}$.

**Time-gauged rotations.** The time-gauged version of spatial rotations is also an extended symmetry of the action $S_{EAA}$. This symmetry was first identified for the two-dimensional NS equation, see Ref. [13]. In the present work, we will not exploit this symmetry and leave its study for future work.

**BRS (Becchi-Rouet-Stora) symmetry.** We consider the transformation $\theta \to \theta + \varepsilon c$, $\bar{c} \to \bar{c} - \bar{\theta}$ where $\varepsilon$ is now an anti-commuting Grassmann parameter. This transformation hence mixes the “bosonic” (scalar fields) and the “fermionic” (ghosts) sectors of the action. One observes that this transformation leaves the action invariant $\delta S = 0$. It follows that

$$\int_{\mathcal{E}} \left[ \frac{\delta \Gamma_k}{\delta \theta}(t, \bar{x}) c(t, \bar{x}) + \frac{\delta \Gamma_k}{\delta \bar{c}}(t, \bar{x}) c(t, \bar{x}) \right] = 0 .$$

By taking further functional derivatives one deduces relations between vertices involving ghosts. This is particularly useful to show that the kinetic term of the velocity field is not renormalized and to show that the terms in the EAA which depend on the velocity alone are at most quadratic in the fields.

Let us prove this last assertion, by explicitly using the flow equation of the EAA. Let us assume that at the UV scale $k = \Lambda$, the EAA is such that $\Gamma_\Lambda(0,0,v,0,0)$ is at most quadratic in the velocity fields, which is the case for the microscopic action $S$ in (10). Given this condition, we want to determine if the flow equation generates terms which are of higher order in $v$. The associated flow equation reads

$$\partial_\varepsilon \Gamma_k[0,0,v,0,0] = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_{k,\nu\nu}^{(2)} + R_k \right)^{-1} \dot{R}_{k,\nu\nu} \right] + \text{Tr} \left[ \left( \Gamma_{k,\theta\theta}^{(2)} + R_k \right)^{-1} \dot{R}_k \right] - \text{Tr} \left[ \left( \Gamma_{k,\nu\nu}^{(2)} + R_k \right)^{-1} \dot{R}_{k,\nu\nu} \right] .$$

where the last equality ensues from the BRS symmetry. Since by assumption $\Gamma_{k,\nu\nu}^{(2)}$ has no dependence on $v$, one finds that $\partial_\varepsilon \Gamma_k[0,0,v,0,0]$ is just a constant. Thus we can conclude that the quadratic-in-velocity nature of $\Gamma_k[0,0,v,0,0]$ is preserved by the flow, i.e. it holds true at any scale $k$.

It is instructive to sketch how the non-renormalization of the kinetic term of the velocity arises within the FRG approach. For this, one considers the flow equation for $\Gamma_{k,\nu\nu}^{(2)}$, which can be obtained by taking two functional derivatives of the exact flow Eq. (8) with respect to velocity fields. The vertices entering this flow equation are 3-point vertices with at least one velocity field (explicitly $\Gamma_{k,\nu\nu}^{(1,1,0,0)}$, $\Gamma_{k}^{(0,2,1,0,0)}$, $\Gamma_{k}^{(0,0,1,1,1)}$ and $\Gamma_{k}^{(0,0,3,0,0)}$), and 4-point vertices with at least two velocity fields (explicitly $\Gamma_{k}^{(1,1,2,0,0)}$, $\Gamma_{k}^{(0,2,2,0,0)}$, $\Gamma_{k}^{(0,0,2,1,1)}$ and $\Gamma_{k}^{(0,0,0,4,0,0)}$). The BRS Ward identities lead to the cancellation of all the diagrams with vertices including the scalar fields and the ghosts. One is left with only pure velocity vertices. However, since we have shown that $\Gamma_k$ is quadratic in $v$, it follows that $\Gamma_{k}^{(0,0,3,0,0)} = \Gamma_{k}^{(0,0,0,4,0,0)} = 0$, and this implies the non-renormalization of the velocity kinetic term.

**Further symmetries in the ghost sector.** The ghost sector shares the same symmetries as the scalar sector, i.e., shift and $Z_2$ symmetries. Moreover the ghost action is also invariant under $\bar{c} \rightarrow e^{i\varepsilon} \bar{c}$ and $c \rightarrow e^{-i\varepsilon} c$.

**Synthesis.** We have derived from the symmetries – in an extended sense – of the action, a set of Ward identities which will lead to the exact closure of the RG flow equation of any $n$-point correlation functions in the limit of large wavenumbers. The key feature that will be exploited is that the identities (12), (13), and (15) imply that any 1-PI $n$-point vertex carrying one vanishing wavevector is either zero or can be expressed in terms of lower-order $(n - 1)$-point vertices.

### III. ANOMALOUS DIMENSIONS

In this section, we establish certain fundamental properties which are needed in the derivation of the closed flow equation for $n$-point correlation functions.

#### A. Propagator and bare scaling dimensions

The exact flow equation (8) involves as a fundamental ingredient the regularized propagator $(\Gamma_{k,\nu\nu}^{(2)} + R_k)^{-1}$. To obtain it, one writes the matrix $(\Gamma_{k,\nu\nu}^{(2)} + R_k)$ of second functional derivatives of the EEA and regulator action. Since in Fourier space, it is diagonal in frequency and wavevector, one can then simply invert the matrix. Evaluated for vanishing values of the average field, $\varphi = 0$, the regularized propagator takes the fol-
(\bar{\Gamma}_k^{(2)} + R_k)^{-1} = \begin{pmatrix}
G_{\theta\theta} & \bar{G}_{\theta\theta} & 0 & 0 & 0 \\
0 & G_{\theta\theta} & 0 & 0 & 0 \\
0 & 0 & \bar{G}_{v\bar{v}} & 0 & 0 \\
0 & 0 & 0 & \bar{G}_{\epsilon\epsilon} & 0 \\
0 & 0 & 0 & 0 & \bar{G}_{\epsilon\epsilon}
\end{pmatrix}, \quad (16)

\begin{align}
G_{\theta\theta}(\omega, \bar{p}) &= \frac{1}{\Gamma_k^{(1,0,0,0)}(\omega, \bar{p}) + R_k(\bar{p})}, \\
G_{\theta\bar{v}}(\omega, \bar{p}) &= -\frac{\Gamma_k^{(1,0,0,0)}(\omega, \bar{p}) - M_k(\bar{p})}{\Gamma_k^{(1,0,0,0)}(\omega, \bar{p}) + R_k(\bar{p})}, \quad (17)
\end{align}

and similarly for the ghost sector. The field renormalizations can be obtained through the relations

\begin{align}
Z_{\theta\theta}^{1/2} \theta \equiv \frac{\theta}{\bar{p}^2 + m^2}, \\
Z_{\theta\bar{v}}^{1/2} \bar{v} \equiv \frac{\bar{v}}{\bar{p}^2 + m^2}, \\
Z_{\epsilon\epsilon}^{1/2} \epsilon \equiv \frac{\epsilon}{\bar{p}^2 + m^2}.
\end{align}

The field renormalizations can thus be obtained through the relations

\begin{align}
Z_{\theta\theta,k,0}^{1/2} Z_{\theta\bar{v},k,0}^{1/2} \theta \equiv \frac{\partial}{\partial (\bar{p})^{1/2}} \left[ \theta \left( \frac{\bar{p}^2 + m^2}{\bar{p}} \right) \right], \\
Z_{\theta\bar{v},k,0}^{1/2} Z_{\epsilon\epsilon,k,0}^{1/2} \bar{v} \equiv \frac{\partial}{\partial (\bar{p})^{1/2}} \left[ \bar{v} \left( \frac{\bar{p}^2 + m^2}{\bar{p}} \right) \right], \\
Z_{\epsilon\epsilon,k,0}^{1/2} \epsilon \equiv \frac{\epsilon}{\bar{p}^2 + m^2}.
\end{align}

The field renormalizations can thus be obtained through the relations

\begin{align}
Z_{\theta\theta}^{1/2} \theta \equiv \frac{\partial}{\partial (\bar{p})^{1/2}} \left[ \theta \left( \frac{\bar{p}^2 + m^2}{\bar{p}} \right) \right], \\
Z_{\theta\bar{v}}^{1/2} \bar{v} \equiv \frac{\partial}{\partial (\bar{p})^{1/2}} \left[ \bar{v} \left( \frac{\bar{p}^2 + m^2}{\bar{p}} \right) \right], \\
Z_{\epsilon\epsilon}^{1/2} \epsilon \equiv \frac{\epsilon}{\bar{p}^2 + m^2}.
\end{align}

The anomalous dimension of a field is then defined by $\eta_\theta \equiv -Z_{\theta\theta}^{-1} \partial \theta, Z_{\epsilon\epsilon}$, and the anomalous dimension associated to the molecular diffusivity by $\eta_\epsilon \equiv \eta_{\epsilon\epsilon}$. The Ward identities derived in Sec. [11D] provide crucial constraints on these running constants. We now spell out the consequences of these identities and of the non-renormalization theorems presented in Sec. [11D]. First, we showed that the velocity kinetic term is not renormalized. This immediately implies $Z_{\theta\bar{v}}^{1/2} = 1$ and $\eta_\epsilon = 0$. The Ward identity associated with the time-gauged shifts $\bar{p} \equiv \partial \bar{p} / \bar{p}$ and $\epsilon \equiv \eta_{\epsilon\epsilon}$ is $\eta_\epsilon = -\eta_{\epsilon\epsilon}$. It follows that the Ward identity associated with the time-gauged Galilean symmetry $\bar{p} \equiv \partial \bar{p} / \bar{p}$ imposes that this vertex is not renormalized, and hence $\lambda_k = 1$. Note that we redefine the cutoff action $\bar{p}^{1/2}$ such that the field renormalizations also appear in the cutoff kernels. In the present case, this simply amounts to the substitution $M_k \equiv \bar{p}^{1/2} \rightarrow Z_{\theta\theta}^{-1} \partial \theta$. To study the existence of a fixed point, one usually switches to a dimensionless formulation, by rescaling all quantities by suitable powers of the RG scale. In particular, dimensionless wavevectors $\bar{p}$ are defined by $\bar{p} \equiv \bar{p} / k$ and dimensionless frequencies by $\omega \equiv \omega / \nu_k^2$. At a fixed point, the RG flow of the dimensionless couplings vanishes. In this respect, the non-renormalization of the coupling $\lambda_k$ is a striking consequence. Indeed, the associated dimensionless coupling reads $\tilde{\lambda}_k \equiv \lambda_k \nu_k^{-1/2} \nu_k^2^{-1/2} = \lambda_k \nu_k^{-1/2}$. Hence, since $\partial_\epsilon \lambda_k = 0$, its flow equation is simply given by the linear contribution

$\partial_\lambda \tilde{\lambda}_k = \left( -\frac{1}{2} + \frac{\eta_k}{2} \right) \tilde{\lambda}_k. \quad (21)$

This implies that at a non-Gaussian fixed point, satisfying $\tilde{\lambda}_k \neq 0$, then $\eta_k = \epsilon$. Given the relations $\eta_\theta = -\eta_\epsilon, \eta_\epsilon = \epsilon$, and $\eta_\epsilon = 0$, one deduces that there is a single anomalous dimension left to be determined at a fixed point.

C. Energy budget

In this section, we outline an argument which allows one to fix the value of the remaining anomalous dimension $\eta_\theta$. We associate to each field in the EEA the corresponding field renormalization $Z_{\theta,k,0}^{1/2}$. Omitting the ghosts, the EEA then

\begin{equation}
\Gamma_k = \int k, \lambda \left[ \left( \partial_\lambda \right)^2 \bar{G}_{\theta\theta} \right] \left( \partial_\lambda \right)^2 \bar{G}_{\theta\theta} + \left( \partial_\lambda \right)^4 \bar{G}_{\theta\theta}
\end{equation}

reads

\begin{equation}
\Gamma_k = \int k, \lambda \left[ \left( \partial_\lambda \right)^2 \bar{G}_{\theta\theta} \right] \left( \partial_\lambda \right)^2 \bar{G}_{\theta\theta} + \left( \partial_\lambda \right)^4 \bar{G}_{\theta\theta}
\end{equation}
In principle, in order to determine $\eta_\theta$ within the FRG framework, one has to derive the flow equations for the two-point functions $\Gamma_k^{(2)}$, integrate them, and read off the value of $\eta_\theta$ at the fixed point.

In this work, we shall employ a shortcut first proposed in Ref. [23] for the case of NS equations. Since we are interested in the stationary turbulent state, this requires that the mean energy rate injected by the forcing is constant and matches the mean energy dissipated by the scalar field. The argument is thus based, in 3D, on imposing a finite value to $\langle f \theta \rangle$ in the ideal limit $L \to \infty$, i.e. $k \to 0$, or equivalently that $\langle \kappa \partial_i \theta \partial_i \theta \rangle$ remains finite in the limit $k \to 0$.

The averaged injected power for the scalar can be estimated as

$$\langle f(t,\vec{x}) \theta(t,\vec{x}) \rangle \approx \int_y \left\langle M_k (\vec{y} - \vec{y}) \hat{\theta}(t,\vec{y}) \theta(t,\vec{x}) \right\rangle$$

$$= \int_{\omega \tilde{q}} M_k (\tilde{q}) G_{\theta \theta} (\omega, \tilde{q}) \propto k^{d+2-\eta_\theta} \int_{\omega \tilde{q}} M_k (\tilde{q}) G_{\theta \theta} (\omega, \tilde{q}) \ .$$

For $\langle f \theta \rangle$ to have a finite limit when $k \to 0$ imposes $\eta_\theta = d + 2$. Let us remark that the estimate (22) is based on assuming standard scale invariance, in the sense that there is agreement between the RG scaling (in $k$) and the actual scaling properties of 1-PI vertices (in $\omega, \tilde{q}$). However, although such correspondence is true in general in critical phenomena, it can be violated in the case of turbulence, we refer to Ref. [23] for a discussion of this point in the context of the NS equations. Such violations may be at the source of deviations from the Kolmogorov scaling. In this work, we limit ourselves to assuming standard scale invariance to fix the anomalous dimension $\eta_\theta$. This in turn yields the standard Obukhov-Corrsin scaling.

Let us show that considering the energy rate dissipated by the scalar leads to the same result. In the Kraichnan model, one may define the analog of the Kolmogorov dissipation scale by $\eta_{\text{diss}} \sim (2\kappa/D^0)^{1/\varepsilon}$, see eg. Ref. [23]. The average dissipated power can be estimated as

$$\kappa \langle \partial_i \theta \partial_i \theta \rangle = \kappa \int_{\omega} \int_{\omega \tilde{q}} |\tilde{q}| < \eta_{\text{diss}}^{-1} q^2 G_{\theta \theta} (\omega, \tilde{q}) \ ,$$

where $\eta_{\text{diss}}^{-1}$ is a UV cutoff. The integral is dominated by the UV contribution, which we can estimate by assuming standard scale invariance. It follows that

$$\lim_{k \to 0} \kappa \langle \partial_i \theta \partial_i \theta \rangle \sim \lim_{k \to 0} \kappa \left( \frac{1}{\eta_{\text{diss}}} \right)^{d+2-\eta_\theta + \eta_k} \ .$$

By inserting $\eta_{\text{diss}} \sim (2\kappa/D^0)^{1/\varepsilon}$ and $\eta_k = \varepsilon$, one deduces that reaching a finite limit requires $\eta_\theta = d + 2 = 5$.

IV. CORRELATION FUNCTIONS IN THE KRAICHNAN MODEL

The (connected) correlation functions $W_k^{(n)}$ can be expressed in terms of the 1-PI correlation functions $\Gamma_k^{(1)}$ via a sum of tree diagrams. In the FRG framework, the $\Gamma_k^{(n)}$ are obtained by deriving the associated flow equations, solving them, and eventually taking the limit $k \to 0$. For instance, by taking two functional derivatives of the FRG flow equation (8), one obtains an exact flow equation for the 2-point functions $\Gamma_k^{(2)}$ (the inverse propagator). However, the RHS of this flow equation depends on the 3-point and 4-point vertices, which are governed by their own flow equation depending on higher-order vertices. It is then clear that an infinite hierarchy is generated and that a suitable approximation scheme must be implemented.

In the present section, we consider a closure scheme based on the Ward identities detailed in Sec. [11]. The crucial approximation underlying this scheme is the following one. All the vertices, which enter into the flow equation of a given vertex $\Gamma_k^{(n)} (\omega_k, \tilde{p}_i)$, are expanded in the loop wavevector $\tilde{q} \simeq 0$. The rationale of this approximation is that the loop wavevector is controlled by the cutoff derivative $\partial^i R_k (\tilde{q})$, which vanishes for $|\tilde{q}| \gtrsim k$. Hence, if one considers large external wavevectors $\tilde{p}_i$, one may expand all the vertices about zero $\tilde{q}$. This type of approximation is inspired by the Blaizot-Méndez-Galain-Wschebor (BMW) scheme [14-16]. It becomes exact in the limit where all $|\tilde{p}_i| \to \infty$. In the context of NS turbulence, this large wavenumber expansion was developed and analyzed in Refs. [9, 10, 13, and 39]. The crucial feature of this scheme for the NS equations, which renders it particularly powerful, is that all the vertices with one wavevector set to zero either vanish or are given exactly in terms of lower-order vertices through the Ward identities. This results in a closure of the flow equation of any $\Gamma_k^{(n)}$, which is thus expressed in terms of vertices with $m \leq n$ only, without any further approximation than the large wavenumber limit. We show in the following that this closure can also be achieved for the Kraichnan model, thanks to the Ward identities derived in Sec. [11]. In order to do so, we closely follow the derivation detailed in Ref. [14] to apply it to the Kraichnan model. Since the calculations which lead to the final form of the closed flow equation are formally identical, we limit ourselves to outlining the main steps of the derivation and refer to Ref. [14] for details. In this work, we derive the general closed flow equation for a generic $n$-point correlation function, but we only focus on the solution of the 2-point function of the scalar field.

A. Closed flow equations for $n$-point correlation functions

For this derivation, it is more convenient to consider the exact FRG equation for the generating functional of the connected correlation functions $W_k [J]$, which reads

$$\partial_j W_k [J] = -\frac{1}{2} \partial_i R_k [J] \left[ \frac{\delta^2 W_k [J]}{\delta J_i \delta J_j} + \frac{\delta W_k [J]}{\delta J_i} \frac{\delta W_k [J]}{\delta J_j} \right] ,$$

where we use deWitt notation, in particular the integrals are implicit. By taking functional derivatives of this equation, one obtains the exact flow equations for $n$-point correlation func-
tions as
\[
\frac{\partial}{\partial J^1 \cdots J^n} \frac{\delta^n W_k [J]}{\delta J^1 \delta J^1 \cdots J^n} = -\frac{1}{2} \frac{\partial}{\partial J^2} \frac{\delta^{n+2} W_k [J]}{\delta J^2 \delta J^1 \cdots J^n} + \sum_{\{a_k\} \neq \{a_k\}} \frac{\delta^{k+1} W_k [J]}{\delta J^k \delta J^1 \cdots J^n},
\]
where \(\{a_k\} \neq \{a_k\}\) indicates all possible bipartitions of the indices 1, \(\ldots\), \(n\).

Let us first consider the second term in Eq. (24). It is straightforward to check that in wavevector space the cutoff derivative term \(\partial R_{k,ij}\) is a function of the total wavevector carried by \(J^1, \ldots, J^k\), say \(\vec{p} = p_{a_1} + \cdots + p_{a_k}\). In the large wavenumber limit, the cutoff derivative \(\partial R_{k,ij}(\vec{p})\) is exponentially suppressed. Therefore, in this limit the second term of Eq. (24) is negligible and one can discard it.

Let us now turn to the first term in Eq. (24). We first rewrite it in terms of functional derivatives with respect to the average field \(\phi\), rather than functional derivatives with respect to the sources, since the former will generate 1-PI vertices to which one can apply the intended approximation scheme (\(q\)-expansion and Ward identities). This leads to
\[
\partial R_{k,ij} \frac{\delta^{n+2} W_k [J]}{\delta J^1 \delta J^1 \cdots J^n} = -\frac{1}{2} \partial R_{k,ab} \frac{\delta \phi_i \delta \phi_j}{\delta J^1 \delta J^1} + \frac{\delta}{\delta J^1} \frac{\delta^2 W_k [J]}{\delta \phi_i \delta \phi_j \delta J^1 \cdots J^n}.
\]

At vanishing sources and for corresponding vanishing average fields, the term \(\partial R_{k,ab} W_k^3\) in (25) is proportional to the flow of \(\frac{\delta W_k}{\delta J^1} = \phi_i\), which itself vanishes. Thus, one concludes that also the second term in (25) vanishes. The flow equation hence takes the form
\[
\partial J^1 \cdots J^n = -\frac{1}{2} \frac{\partial}{\partial J^2} \frac{\delta^2 W_k [J]}{\delta J^2 \delta J^1 \cdots J^n} + \frac{\delta}{\delta J^1} \frac{\delta^2 W_k [J]}{\delta \phi_i \delta \phi_j \delta J^1 \cdots J^n},
\]
where the RHS must be thought of as a functional of the average field which is eventually evaluated at \(\phi = 0\).

At this point, one can proceed and evaluate the functional derivatives \(\frac{\partial}{\partial \phi_i}\) and \(\frac{\partial}{\partial \phi_j}\) in (26). In order to implement the closure strategy, we observe that each of these functional derivatives carries a wavevector \(\vec{q}\) which is controlled by the cutoff derivative \(\partial R_{k,ab}(\vec{q})\). In turn this implies that \(\frac{\partial}{\partial \vec{q}}\) will generate a 1-PI vertex which has its wavevector carried by \(\phi_i\) equal to \(\vec{q}\). The large wavenumber expansion then leads to setting \(\vec{q} = 0\) in all such vertices. One can then apply the Ward identities studied in Sec. [10] They have an immediate striking consequence: the only functional derivatives \(\frac{\partial}{\partial \vec{q}}\) which lead to a non-zero contribution are those with respect to the velocity field only, since when \(\vec{q}\) is associated to any other fields, then the Ward identities yield that the corresponding vertex vanishes. Therefore, one can further simplify the flow equation, which takes the form
\[
\partial J^1 \cdots J^n = -\frac{1}{2} \left( \frac{\partial^2}{\partial \vec{q}^2} \frac{\delta^2 W_k [J]}{\delta \vec{q}^2} \right) W_k^3.
\]

As it is, Eq. (27) is not closed yet. More explicitly, this equation reads
\[
\partial J^1 \cdots J^n = -\frac{1}{2} \int_{\omega \vec{q}} \left( \frac{\partial^2}{\partial \vec{q}^2} \frac{\delta^2 W_k [J]}{\delta \vec{q}^2} \right) W_k^3.
\]

However, acting on \(W_k^3\) with \(\frac{\delta}{\delta \vec{q}}\) generates 1-PI vertices which are controlled by (15) after setting \(\vec{q} = 0\). One can show that the action of the velocity functional derivative yields
\[
\left( \frac{\delta}{\delta \vec{q}} \right) W_k^3(\omega, \vec{q}) \bigg|_{\vec{q}=0} = \sum_{\ell=1}^{n} -\frac{p_{\ell}^2}{\omega} W_k^3(\omega, \vec{p}_1, \cdots, \vec{p}_k, \cdots) = \frac{\partial}{\partial \vec{q}} (\omega) \bigg|_{\vec{q}=0}.
\]

and a further functional derivative can be expressed as
\[
\frac{\delta}{\delta \vec{q}} W_k^3(\omega, \vec{q}) \bigg|_{\vec{q}=0} = \partial \frac{\partial}{\partial \vec{q}} W_k^3(\omega, \vec{p}_1, \cdots) = \frac{\partial}{\partial \vec{q}} (\omega) W_k^3(\omega, \vec{p}_1, \cdots).
\]

Note that \(\vec{q}\) is set to zero only at the level of vertices (not in the \(W_k^3\)) and also the Ward identities are applied to these vertices. The proof of the resulting identities (28) and (29) at the level of the correlation functions \(W_k^3\) is quite lengthy, but it poses no difficulty as it is strictly similar to Ref. [10] to which we refer the reader for the detailed demonstration.

Using the identities (28) and (29) to express the RHS of Eq. (29) then leads to the final expression for the flow equation of \(W_k^3\) at large wavenumbers
\[
\partial J^1 \cdots J^n = -\frac{1}{2} \int_{\omega \vec{q}} \left( \frac{\partial^2}{\partial \vec{q}^2} \frac{\delta^2 W_k [J]}{\delta \vec{q}^2} \right) W_k^3(\omega, \vec{p}_k, \cdots) = \frac{1}{2} \int_{\omega \vec{q}} \left( \frac{\partial^2}{\partial \vec{q}^2} \frac{\delta^2 W_k [J]}{\delta \vec{q}^2} \right) W_k^3(\omega, \vec{q}) \bigg|_{\vec{q}=0}.
\]

which does not involved any higher-order correlation functions \(W_k^{(m>n)}\), showing that it is closed.

### B. Two-point correlation function in the large wavenumber limit

We now specify to the flow equation for \(\tilde{G}_{\theta \theta} (\omega, \vec{p}) = \tilde{W}_k^{(2,0,0,0)}(\omega, \vec{p})\). Starting from equation (30), a straightforward calculation leads to
\[
\partial J^1 \cdots J^n \tilde{G}_{\theta \theta} (\omega, \vec{p}) = \frac{1}{2} \int_{\omega \vec{q}} \left( \frac{\partial^2}{\partial \vec{q}^2} \frac{\delta^2 W_k [J]}{\delta \vec{q}^2} \right) W_k^3(\omega, \vec{q}) \times \frac{p_{\ell}^2}{\omega^2} \left( 2 \tilde{G}_{\theta \theta} (\omega, \vec{p}) - \tilde{G}_{\theta \theta} (\omega + \vec{p}) - \tilde{G}_{\theta \theta} (\omega - \vec{p}) \right).
\]
By inverse Fourier transforming to real time, one obtains
\[ \partial_s \tilde{G}_{\theta \theta}(t, \tilde{p}) = \frac{1}{2} \int_0^{\tilde{p}} \left( \tilde{G}_{\tilde{p}, \nu} \partial_s R_{k, \nu}(\tilde{p}) \right) (\omega, \tilde{q}) p^i p^j \times \left\{ 2 - 2 \cos(\omega) \right\} \tilde{G}_{\theta \theta}(t, \tilde{p}). \] (31)

Equation (31) is written in terms of the scalar field and the velocity propagators. At this point, one of the peculiarities of the Kraichnan model intervenes, namely that the velocity propagator is known exactly, Eq. \( \text{(18)} \), and it does not depend on the frequency. This allows one to perform explicitly the integration over the internal frequency, which gives
\[ \partial_s \tilde{G}_{\theta \theta}(t, \tilde{p}) = \frac{d-1}{2d} \int \frac{D_0}{(q^2 + m^2)^{\frac{d}{2}} + \frac{2}{2} + R_{k, \nu}(\tilde{q})^2} \partial_s R_{k, \nu}(\tilde{q}) \times p^2 |i| \tilde{G}_{\theta \theta}(t, \tilde{p}). \] (32)

We are interested in the fixed-point solution of this equation. It is thus convenient to switch to dimensionless variables. We thus introduce \( \tilde{p} = \tilde{p}/k, \tilde{t} = t k^2 \),
\[ \tilde{G}_{\theta \theta}(t, \tilde{p}) = k^{\eta_c - \eta_\theta} \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}), \]
and the dimensionless regulator following \( R_{k, \nu}(\tilde{q}) = k^{d+\varepsilon} R_{k, \nu}(\tilde{q}) \), and thus
\[ \partial_s R_{k, \nu}(\tilde{q}) = k^{d+\varepsilon} (d + \varepsilon) R_{k, \nu}(\tilde{q}) - \hat{q} \partial_q R_{k, \nu}(\tilde{q}) \).

We define
\[ \alpha_q \equiv \frac{d-1}{2d} \int \frac{D_0}{(q^2 + m^2)^{\frac{d}{2}} + \frac{2}{2} + R_{k, \nu}(\tilde{q})^2}, \]
which, at the fixed point, becomes a pure number that can be calculated given any explicit cutoff kernel \( R_{k, \nu} \), and eg. \( m = 0 \). The dimensionless flow equation then reads
\[ \left( \partial_s + (\eta_c - \eta_\theta) + (2 - \eta_c) i \partial_q - \hat{p} \partial_p \right) \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}) = \frac{D_0}{\alpha_q} \hat{\lambda}_k^2 p^2 |i| \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}) \] (33)

where \( \hat{\lambda}_k \) is the dimensionless coupling associated to the nonrenormalized bare one \( \hat{\lambda}_k = 1 \). By inserting \( \eta_c = \varepsilon \) and \( \eta_\theta = \delta \), and considering the fixed-point (i.e. \( \partial_s = 0 \) in Eq. \( \text{(33)} \)), one obtains
\[ \left( - (5 - \varepsilon) + (2 - \varepsilon) i \partial_q - \hat{p} \partial_p \right) \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}) = D_0 \hat{\lambda} \hat{\lambda}^2 p^2 |i| \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}) \] (34)
where \( \hat{\lambda} \) and \( \hat{\alpha} \) are the fixed-point values of \( \lambda_k \) and \( \alpha_q \).

At equal time \( \tilde{t} = 0 \), Eq. \( \text{(34)} \) takes a very simple form since the RHS and the logarithmic \( \tilde{t} \)-derivative vanish, and the solution is a power law
\[ \tilde{G}_{\theta \theta}(0, \tilde{p}) \sim \tilde{p}^{-(5 - \varepsilon)} \).

This power law corresponds to the scaling expected in the inertial range \( \tilde{G}_{\theta \theta}(0, x) - \tilde{G}_{\theta \theta}(0, 0) \sim x^{2 - \varepsilon} \).

The general solution of Eq. \( \text{(34)} \) is given by
\[ \tilde{G}_{\theta \theta}(\tilde{t}, \tilde{p}) = \tilde{p}^{-(5 - \varepsilon)} e^{-D_0 \hat{\lambda} \hat{\lambda}^2 p^2 |i|} f(\tilde{p}^{2 - \varepsilon}) \] (35)
where \( f(\tilde{p}^{2 - \varepsilon}) \) is a universal scaling function, which can be determined from FRG methods. To do so, it is not sufficient to solve the fixed-point equation but one has to evolve the flow equation from an initial condition corresponding to the bare action \( \text{(10)} \) down to \( k = 0 \) (in fact one can stop at the scale \( k \) where the fixed point is reached). The exponential present in the solution \( \text{(35)} \) does not have a smooth limit as \( \varepsilon \to 0 \).

This is not surprising since the velocity two-point correlation function \( \text{(2)} \) itself is not well defined as \( \varepsilon \to 0 \). In terms of the parameter \( D_0 \) the solution has a smooth behavior in \( \varepsilon \) (note that \( \hat{\alpha} \) is also a function of \( \varepsilon \)) which reads, switching back to the dimensionful physical quantities,
\[ \tilde{G}_{\theta \theta}(t, \tilde{p}) = p^{-(d + 2 - \varepsilon)} e^{-D_0 \hat{\lambda} \hat{\lambda}^2 p^2 |i|} f(\tilde{p}^{2 - \varepsilon}) \] (36)
Note that the exponential in \( \text{(36)} \) depends explicitly on \( k \). This is a sign of the breaking of standard scale invariance. When considering the fixed point for \( k \to 0 \), one typically identifies the RG scale \( k \) with the inverse of the energy injection scale, \( k = L^{-1} \) since the flow essentially stops when crossing this scale. However, for the Kraichnan model, it makes sense to identify \( k \) with the mass scale present in the velocity propagators since the coefficient \( \hat{\alpha} \) is fully determined by this. It yields
\[ \tilde{G}_{\theta \theta}(t, \tilde{p}) = p^{-(d + 2 - \varepsilon)} e^{-D_0 \hat{\lambda} \hat{\lambda}^2 p^2 |i|} f(\tilde{p}^{2 - \varepsilon}) \] (37)
which is one of the main results of this work. Let us emphasize that this expression is valid at large wavenumber \( p \) but for arbitrary time delays \( t \) in the stationary state. The temporal decorrelation of the scalar in the Kraichnan model is always exponential at any time scale. This is due to the complete decorrelation in time of the stochastic velocity field in this model. One can indeed remark that if this feature is relaxed by introducing some time correlation in the covariance Eq. \( \text{(2)} \), the expression \( \text{(37)} \) is no longer valid at small \( t \), but is replaced by a Gaussian decay as occurs for the real scalar, see Sec. \( \text{V} \).

C. Two-point correlation function from the Dyson equation

The simplifying assumptions on the velocity field in the Kraichnan model lead to the important consequence that the equal-time two-point correlation function \( \langle \theta(t, x) \theta(t, y) \rangle \) can be computed exactly, see Ref. \( \text{[47]} \) and references therein. Such an exact solution is obtained by deriving a closed form of the associated Hopf equation, which allows one to write an exact differential equation for the two-point correlation function, see, eg. Refs. \( \text{[47, 49]} \).

The connection between this exact differential equation for the two-point correlation function and the field theoretical calculation was worked out in Ref. \( \text{[52]} \), which showed how to retrieve the differential equation from the FRG equation in a...
steady state. In this section, we follow the approach used in Ref. [22] and extend it to compute the two-point Eulerian correlation function of the scalar at unequal times, thereby allowing for a comparison with the result obtained by FRG methods, Eq. (37).

The aim is to compute the self-energies, which can be defined from the propagators as

$$\bar{G}_{\theta\theta}(\sigma, \bar{p}) \equiv \frac{1}{-i\sigma + \frac{2}{\tau}p^2 - \Sigma_{\theta\theta}(\sigma, \bar{p})}$$

$$\bar{G}_{\theta\theta}(\sigma, \bar{p}) \equiv \frac{(M_k(\bar{p}) + \Sigma_{\theta\theta}(\sigma, \bar{p}))}{-i\sigma + \frac{2}{\tau}p^2 - \Sigma_{\theta\theta}(\sigma, \bar{p})^2}.$$

We consider the one-loop diagrams which enter in the determination of $\Sigma_{\theta\theta}$ and $\Sigma_{\theta\theta}$, and express them in terms of the exact propagators. Using the expression (2) for the velocity propagator, one finds

$$\Sigma_{\theta\theta}(\sigma, \bar{p}) = -\int_{\omega_{\theta\theta}} \bar{G}_{\theta\theta}(\omega + \sigma, \bar{q} + \bar{p}) p^\alpha \bar{G}_{\alpha,\nu}(\bar{q})(p^\beta + q^\beta)$$

$$= -p^2 \left[ D_0 \frac{(d-1)}{2d} \frac{\Gamma(\epsilon/2) m^{-\epsilon}}{(4\pi)^d / 2} \right].$$

This a striking feature of the Kraichnan model: The expression of $\Sigma_{\theta\theta}(\sigma, \bar{p})$ is frequency independent because the velocity propagator is itself frequency independent, allowing one to shift the frequency in the integral. The frequency integration simply yields $\int_{\omega_{\theta\theta}} \bar{G}_{\theta\theta}(\omega, \bar{p}) = \frac{1}{2}$, and the wavevector integral can then be carried out, leading to an extremely simple $\bar{p}$-dependence as well. The 1-PI vertex $\Gamma^{(1,1,0,0,0)}$ then takes the following simple form

$$\Gamma^{(1,1,0,0,0)}(\sigma, \bar{p}) = i\sigma + p^2 \left[ \frac{\kappa}{2} \frac{D_0}{2d} \frac{\Gamma(\epsilon/2) m^{-\epsilon}}{(4\pi)^d / 2} \right]$$

$$= i\sigma + p^2 \frac{\kappa_{\text{ren}}}{2},$$

where we introduced the renormalized molecular viscosity $\kappa_{\text{ren}}$.

The self-energy of the response field is given by

$$\Sigma_{\theta\theta}(\sigma, \bar{p})$$

$$= \int_{\omega_{\theta\theta}} \bar{G}_{\alpha,\nu}(\bar{q}) (p^\alpha + q^\alpha)(p^\beta + q^\beta) G_{\theta\theta}(\omega + \sigma, \bar{q} + \bar{p})$$

$$= \int_{\omega_{\theta\theta}} \frac{D_0}{(q^2 + m^2)^{d/2}} \left( p^2 - \frac{\bar{p} \cdot \bar{q}}{q^2} \right) \times$$

$$\times \frac{M_k(\bar{q} + \bar{p}) + \Sigma_{\theta\theta}(\sigma + \omega, \bar{q} + \bar{p})}{(\omega + \sigma)^2 + \left( \frac{\kappa_{\text{ren}}}{2} (\bar{q} + \bar{p})^2 \right)^2}.$$  

A shift in the integration frequency shows that this self-energy is also independent of $\sigma$. It is clear that the frequency independence of the self-energies is a peculiarity of the Kraichnan model.

At this point, one can express the two-point correlation function as

$$\bar{G}_{\theta\theta}(\sigma, \bar{p}) = \frac{1}{\sigma^2 + (\frac{k_{\text{ren}}}{2d} p^2)^2} \left( M_k(\bar{p}) + \int_{\bar{q}} (q^2 + m^2)^{-\epsilon/2} \right)$$

$$\times \left( p^2 - \frac{\bar{p} \cdot \bar{q}}{q^2} \right) \frac{M_k(\bar{q} + \bar{p}) + \Sigma_{\theta\theta}(\bar{q} + \bar{p})}{\kappa_{\text{ren}} (\bar{q} + \bar{p})^2}$$

$$= \frac{F(\bar{p})}{\sigma^2 + (\frac{k_{\text{ren}}}{2d} p^2)^2},$$

where $F(\bar{p})$ is simply a shorthand for the expression in the parentheses. It is straightforward to transform back to real time and obtain

$$\bar{G}_{\theta\theta}(t, \bar{p}) = e^{-\frac{k_{\text{ren}} p^2 |t|}{2d}} \frac{F(\bar{p})}{\kappa_{\text{ren}} p^2},$$

which provides another expression for the two-point correlation function of the scalar, that we compare with the FRG one in the following section.

D. Comparison of the two approaches and remarks

Let us comment on the two expressions we have derived for $\bar{G}_{\theta\theta}(t, \bar{p})$. First of all, the temporal dependence of (38) is of the same form as the one obtained by FRG methods, namely an exponential with an exponent proportional to $p^2 |t|$. It is also interesting to compare the coefficients appearing in front of this factor. In (38) the coefficient $\frac{k_{\text{ren}}}{2}$ has two contributions: one is just the microscopic molecular diffusivity and the other reads

$$\frac{\kappa_{\text{ren}}}{2} = D_0 \frac{(d-1)}{2d} \frac{\Gamma(\epsilon/2) m^{-\epsilon}}{(4\pi)^d / 2} \frac{\Gamma(d/2 + \epsilon/2)}{\Gamma(d/2 + \epsilon/2)}.$$

It is clear that the FRG result should be compared against this expression since $\kappa$ is negligible in the inertial range. The expression (39) exhibits three main features: it is proportional to $D_0$, it diverges as $1/\epsilon$ as $\epsilon \to 0$, and it is proportional to $m^{-\epsilon}$ (which signals an IR singularity as $m \to 0$). All these three aspects are shared by the coefficient appearing in the FRG formula (37).

Now, even neglecting the molecular viscosity, the two coefficients share many similarities but are not equal, which is to be expected, since the setting and assumptions are different. For instance the coefficient (39) is IR divergent if no IR mass/regularizer is added, which renders difficult a meaningful comparison beyond the qualitative features outlined above. However, note that if one chooses a mass cutoff kernel defined by $S_{\text{vel}} + \Delta S_k = \int \frac{d^d v}{(d-2)/2} (\partial^2 v + k^2 v)$ and eventually sets $k = m$, one finds that the exponent in (37) has the same expression as in (39). In principle, one can reconstruct the correlations of any microscopic theory by solving the FRG equation exactly, strictly taking the limit $k \to 0$, and keeping $k \neq m$. 
Let us now make two additional remarks. First, the same framework of the Dyson equations can be used to determine the behavior of the spectrum in the mildly non-linear regime, that is in the dissipative range, but close to the inertial range where the convection still plays a role. We find that the energy spectrum in this range exhibits an intermediate power law regime following $p^{-(d+2+\epsilon)}$, which is shown in Appendix B.

Finally, let us comment on the discretization scheme of the SDE (\ref{SDE}). This equation is written in the Stratonovich convention. In the Itô convention, it reads (see, e.g., Ref. \ref{Ref23})

$$\partial_t \theta + v^i \partial_i \theta - \left( \frac{\kappa}{2} \delta^{ij} + \frac{1}{2} D^{ij}(0) \right) \partial_j \partial_i \theta = f,$$  \hspace{1cm} (40)

where $D^{ij}(\vec{x} - \vec{y})$ denotes the spatial part of the velocity covariance, explicitly \(\langle v^i(t, \vec{x}) v^j(t', \vec{y}) \rangle = \delta(t - t') D^{ij}(\vec{x} - \vec{y})\). The new term in the SDE can be expressed as

$$\frac{1}{2} D^{ij}(0) = \frac{1}{2} D_0 \int \frac{P_{ij}(\vec{q})}{(q^2 + m^2)^{d/2}} \frac{\Gamma(\epsilon/2) m^{-\epsilon}}{4\pi^{d/2} \Gamma(d/2 + \epsilon/2)} \delta_{ij}.$$  

Hence, one observes that the coefficient in front of $\delta_{ij}$ is exactly the quantity appearing in \ref{Ref39}. It follows that Eq. (40) can be written as

$$\partial_t \theta(t, \vec{x}) + v^i(t, \vec{x}) \partial_i \theta(t, \vec{x}) - \frac{\kappa_{\text{ren}}}{2} \partial^2 \theta(t, \vec{x}) = f(t, \vec{x}).$$ \hspace{1cm} (41)

From the SDE \ref{SDE}, one can estimate the temporal dependence of $\tilde{G}_{\theta \theta}(t, \vec{p})$ via the following argument. Let us neglect the convection and the forcing term. In this approximation $\partial_t \langle \theta(t, \vec{x}) \theta(0, \vec{y}) \rangle = \frac{\kappa_{\text{ren}}}{2} \partial^2 \langle \theta(t, \vec{x}) \theta(0, \vec{y}) \rangle$, which leads precisely to the temporal behavior found via field theoretical methods, namely $\tilde{G}_{\theta \theta}(t, \vec{p}) \approx e^{-\frac{\kappa_{\text{ren}}}{2} p^2 |t|}$. Summarizing, the effect of computing $\Sigma_{\theta \theta}$ (or, equivalently, to renormalize the microscopic molecular diffusivity) simply amounts to introduce an “Itô correction” to the molecular diffusivity. In this sense, the Itô convention makes the mixing between the velocity and the scalar more transparent. A more refined approach has been developed in Ref. \ref{Ref54}, where a differential equation for the two-point correlation function at unequal time was found and investigated numerically confirming the exponential decay of the two-point correlation function.

V. CLOSED FLOW EQUATION FOR THE CORRELATION FUNCTIONS OF ADVECTED SCALARS

In the Kraichnan model, the temporal dependence of the two-point correlation function of the scalar can be obtained from various approaches, as shown in Sec. IV C. This is due to the key simplifying feature of this model, which is the white-in-time Gaussian statistics of the velocity. However, the resulting temporal dependence for the scalar is far from that of scalars in actual flows. However, the approximation scheme developed in Secs. IV A and IV B is not based on the white-in-time statistics of the velocity but on the symmetries of the action functional. As we shall show in Sec. IV A, the symmetries for scalars advected by a velocity field satisfying the NS equation are not much different from the symmetries of the Kraichnan model, such that the same scheme is also applicable to real scalars, contrary to the perturbative approach. The main objective of this section is thus to derive and solve the closed flow equation for the two-point correlation function of the scalar field advected by NS turbulence by exploiting the strategy already followed for the Kraichnan model.

A. Action and symmetries for scalars advected by a NS velocity

To obtain the action functional for the scalar field advected by a NS turbulent flow, we apply the MSRJD procedure to the system of equations composed by the Navier-Stokes equation for an incompressible fluid and by the advection-diffusion equation \ref{NS} for the scalar. In order to describe a stationary turbulent state, we consider the NS equation equipped with a Gaussian stochastic forcing. Following the conventions adopted in Ref. \ref{Ref39}, the action functional associated to the NS equation is given by

$$S_{\text{NS}} = \int_{\partial \Omega} \left[ \bar{v} \left( \partial_t + v^i \partial_i - \frac{\kappa}{2} \partial^2 \right) \bar{\nu} + \bar{v} \partial_i \bar{p} + \bar{p} \partial_i \bar{v} \right]$$

$$- \int_{\partial \Omega} \frac{1}{2} \bar{\nu} N_{kij} \bar{\nu}^i,$$ \hspace{1cm} (42)

where $\kappa$ denotes the kinematic viscosity (the notation $v$ is avoided to prevent confusion with the velocity $v$), $N_k$ is the covariance of the stochastic forcing of the fluid velocity in the NS equation, for which the integral scale $L$ of the fluid is identified with the inverse of the RG scale $k$, $p$ is the pressure, and $\bar{p}$ serves as a Lagrange multiplier which ensures the velocity to be divergenceless. The pressure sector is fully non-renormalized, as can be shown by considering gauged shifts in $p$ and $\bar{p}$, see Ref. \ref{Ref39} for the explicit Ward identities. The covariance $N_{kij}$ can be chosen diagonal in components $N_{kij} = \delta_{ij} N_k$ because of incompressibility, and $N_k$ is taken of the same form as the one of the scalar forcing $M_k$ defined in \ref{NS}.

The total action of the NS velocity plus scalar system is thus given by

$$S = \int_{\partial \Omega} \left[ \bar{\theta} \left( \partial_t + v^i \partial_i - \frac{\kappa}{2} \partial^2 \right) \bar{\theta} \right] - \int_{\partial \Omega} \frac{1}{2} \bar{\theta} M_k \bar{\theta} + S_{\text{NS}}.$$ \hspace{1cm} (43)

with the ghost sector implicit. The action \ref{NS} shares some similarity with the Kraichnan action \ref{K} with the main difference that now the velocity does not appear quadratically but in the more complicated form $S_{\text{NS}}$. However, the symmetries are essentially unaltered, as we now discuss.

In the FRG context, the symmetries and extended symmetries of the NS action \ref{NS} have been discussed in Ref. \ref{Ref39} to which we refer for a detailed presentation. Here we limit ourselves to discussing the most relevant points and the extension to the action \ref{NS}.
**Time-gauged Galilei transformation.** The Galilean transformation has the same expression as \[ \vec{\nabla} (t, \vec{x}) \rightarrow \vec{\nabla} (t, \vec{x}) - \vec{v} (t) \cdot \vec{\nabla} \phi (t, \vec{x}) \]
for \( \phi = v^i, p \) or \( \vec{\nabla} \). The associated Ward identity identifies
\[
0 = \int \vec{\nabla}^2 \vec{v} (t, \vec{x}) + \int \frac{\delta^2 \Gamma_k [\phi]}{\delta \phi (t, \vec{x})} \delta \phi (t, \vec{x}) + \frac{\delta \Gamma_k [\phi]}{\delta \vec{v} (t, \vec{x})} \delta \vec{v} (t, \vec{x})
\]
for \( \phi = v^i, p \) or \( \vec{\nabla} \). The associated Ward identity reads
\[
0 = \int \vec{\nabla}^2 \vec{v} (t, \vec{x}) + \int \frac{\delta^2 \Gamma_k [\phi]}{\delta \phi (t, \vec{x})} \delta \phi (t, \vec{x}) + \frac{\delta \Gamma_k [\phi]}{\delta \vec{v} (t, \vec{x})} \delta \vec{v} (t, \vec{x})
\]
for \( \phi = v^i, p \) or \( \vec{\nabla} \).

**Time-gauged shifts of the response fields.** An important extended symmetry of the NS action is related to the following time-gauged transformation \( \vec{\nabla} (t, \vec{x}) = \vec{\nabla} (t) + \vec{v} (t, \vec{x}) \) and \( \delta \vec{p} (t, \vec{x}) = \vec{v} (t, \vec{x}) \cdot \vec{\nabla} \phi (t, \vec{x}) \). It yields the Ward identity
\[
\int \frac{\delta \Gamma_k}{\delta \vec{v} (t, \vec{x})} = \int \frac{\delta \Gamma_k}{\delta \vec{v} (t, \vec{x})} \delta \vec{v} (t, \vec{x}) \]
which entails the non-renormalization of the Lagrangian time derivative term \( \vec{\nabla} (t, \vec{x}) + \vec{v} (t, \vec{x}) \) in the bare action, and the vanishing of all the vertices of \( n > 2 \) points with one zero wavevector carried by a response velocity.

**Shift symmetry of the scalar fields.** The time-gauged shift symmetry \( \vec{\nabla} (t, \vec{x}) \rightarrow \vec{\nabla} (t, \vec{x}) + \vec{v} (t, \vec{x}) \) is the same as in the Kraichnan model and the Ward identity \([14]\) is also valid for the NS scalar. Thus, any \( (n > 2) \)-point vertex with one zero wavevector carried by a scalar field vanishes.

For the response scalar field, the time-gauged shift now involves the coupled transformation \( \vec{\nabla} (t, \vec{x}) \rightarrow \vec{\nabla} (t, \vec{x}) + \vec{v} (t, \vec{x}) \) and \( \vec{\nabla} (t, \vec{x}) \rightarrow \vec{\nabla} (t, \vec{x}) + \vec{v} (t, \vec{x}) \), which leads to the modified Ward identity
\[
0 = - \int \left[ \frac{\delta \Gamma_k}{\delta \vec{\nabla} (t, \vec{x})} \delta \vec{\nabla} (t, \vec{x}) \right] + \int \frac{\delta \Gamma_k}{\delta \vec{\nabla} (t, \vec{x})} \delta \vec{\nabla} (t, \vec{x})
\]
This identity entails that the term \( \int \vec{\nabla} \vec{v} \cdot \delta \vec{\nabla} \phi \) is not renormalized. Furthermore, it yields that any vertex with a wavevector carried by a \( \vec{\nabla} \) set to zero is actually given by its bare expression, and thus vanishes when \( n > 2 \).

**Synthesis.** The key point is that the Ward identities for the scalar advected by the NS flow, although slightly modified compared to the ones of the Kraichnan scalar, imply the same consequence for 1-PI vertices, which is the cornerstone of the large wavenumber closure: if at least one wavevector of the vertex is set to zero, then it vanishes except when the wavevector is carried by a velocity field, in which case it is controlled by \([14]\).
C. Closed flow equation for the two-point correlation function

Since the general form of the flow equation and the structure of the Ward identities for the passive scalar in the NS flow are the same as in the Kraichnan model, one can closely follow the derivation of Secs. [45A] and [45B] to obtain a closed flow equation for any n-point correlation function of the scalar field in the limit of large wavenumber.

The inverse propagator \((\Gamma^{(2)}_k + R_k)\) is now a \(8 \times 8\) matrix, comprising the scalar field sector \((\theta, \tilde{\theta}, c, \tilde{c})\) and the velocity sector \((v_i, \tilde{v}_i, p, \tilde{p})\), but it has a simple structure since it is block-diagonal, and also diagonal in frequency and wavevector in Fourier space. It follows that the renormalized propagator barases a similar structure as Eq. (16), with the \(G_{\gamma\nu}\) now replaced by a \(4 \times 4\) matrix whose general form can be found in Ref. [39]. One can show that the pressure sector decouples (as the ghost sector) and the only part which plays a role is the velocity sector, whose propagator is given by

\[
\left(\Gamma^{(2)}_k + R_k\right)_{\nu \tilde{\nu}}^{-1} = \left(\begin{array}{cc} G_{\nu \alpha \beta} & \tilde{G}_{\nu \alpha \beta} \\
0 & 0 \end{array}\right),
\]

with

\[
\tilde{G}_{\nu \alpha \beta}(\omega, \tilde{q}) = \frac{P_{\alpha \beta}(\tilde{q})}{\Gamma^{(2)}_{\nu \alpha \beta}(-\omega, \tilde{q}) + R_{k, \nu \alpha \beta}(\tilde{q})},
\]

\[
G_{\nu \alpha \beta}(\omega, \tilde{q}) = -P_{\alpha \beta}(\tilde{q}) \Gamma^{(2)}_{\nu \alpha \beta}(\omega, \tilde{q}) - 2N_{k, \alpha \beta}(\tilde{q}),
\]

\[
\left|\Gamma^{(2)}_{\nu \alpha \beta} + R_{k, \nu \alpha \beta}(\tilde{q})\right|^2,
\]

where the field functional derivative are indicated as indices on the \(\Gamma^{(2)}\) to alleviate notation.

We now consider the generic flow equation \((24)\) for a n-point correlation function \(W^{(n)}_k\). In the large wavenumber limit, since we have shown that all vertices with one zero-wavevector vanish but the ones for which it is carried by a velocity field, the only remaining term in this flow equation is

\[
\partial_t \frac{\delta^n W_k[J]}{\delta J^1 \cdots \delta J^n} = -\frac{1}{2} H_{k,ij} \frac{\delta}{\delta v^i} \frac{\delta}{\delta \tilde{v}^j} \frac{\delta^n W_k[J]}{\delta J^1 \cdots \delta J^n},
\]

where we introduced the notation

\[
H_{k,ij}(\omega, \tilde{q}) \equiv \left\langle 2 \tilde{G}_{\nu \alpha \beta} \partial R_{k, \nu \alpha \beta} R(\tilde{G}_{\nu \alpha \beta}) + \tilde{G}_{\nu \gamma \alpha \beta} \partial N_{k, \gamma \alpha \beta} \tilde{G}_{\nu \gamma \beta} \right\rangle.
\]

Hence this equation can be closed exploiting the time-gauged Galilean Ward identity

\[
\frac{\delta}{\delta t} \frac{\delta}{\delta \tilde{v}^i(-\omega, -\tilde{q})} \frac{\delta}{\delta v^j(\omega, \tilde{q})} W^{(n)}_k(\omega_1, \tilde{p}_1, \cdots) \mid_{\tilde{q}=0} = \mathcal{D}^i(-\omega) \mathcal{D}^j(\omega) W^{(n)}_k(\omega_1, \tilde{p}_1, \cdots).
\]

One thus also obtains for the scalar in a turbulent NS flow a closed flow equation for any n-point correlation function of any fields since at this stage, the \(J^m\) are any of the sources.

In the following, we focus on the two-point correlation function of the scalar field. The explicit expression of its flow equation, with an inverse Fourier transform to revert to time-wavevector coordinates, is given at large \(p\) by

\[
\partial_t \tilde{G}_{\theta \theta}(t, \tilde{p}) = \frac{1}{2} \int \omega q H_{k,ij}(\omega, \tilde{q}) p^i p^j \left\{ \frac{2 - 2 \cos(\omega t)}{\omega^2} \right\} \tilde{G}_{\theta \theta}(t, \tilde{p}),
\]

where we introduced the notation \(\hat{H}_k = H_k / p^2\). One can further simplify Eq. (45) replacing \(H_{k,ij} = H_k \delta_{ij}\), one can focus for Eq. (45) being formally very similar to Eq. (16) obtained for the Kraichnan scalar. However, there are some crucial differences. First, in the case of the NS passive scalar, the velocity propagators are renormalized and are not known exactly. Second, the velocity propagators, which enter in (45) through \(H_k\), have a non-trivial frequency dependence, contrarily to the case of the Kraichnan scalar. Since their exact form is not known, one cannot obtain an explicit expression for the resulting frequency integral. However, these integrals can be simplified in both the short-time and long-time limits, which we study below.

1. Short-time limit of the two-point correlation function

Let us consider the limit \(t \to 0\) of Eq. (45). In this limit, the cosine can be expanded which yields

\[
\partial_t \tilde{G}_{\theta \theta}(t, \tilde{p}) = \frac{d - 1}{2d} \int \omega q H_k(\omega, \tilde{q}) p^2 \tilde{G}_{\theta \theta}(t, \tilde{p}),
\]

since the frequency integration converges thanks to the function \(H_k\). The dimensionless two-point function is defined by \(\tilde{G}_{\theta \theta}(t, \tilde{p}) \equiv k^{\kappa - \eta_\theta} \tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}})\), and it satisfies the flow equation

\[
\left(\partial_t + (\eta_\kappa - \eta_\theta) + (2 - \eta_\kappa) \hat{t} \partial_\hat{t} - \hat{\tilde{p}} \partial_\hat{\tilde{p}}\right) \tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}}) = \hat{\alpha}_k \hat{\lambda}^2 \hat{p}^2 \hat{\tilde{p}}^2 \tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}}),
\]

where we introduced the scale dependent parameter \(\hat{\alpha}_k \equiv \frac{d-1}{2d} \int \omega q \hat{H}_k(\omega, \tilde{q})\). At the fixed point, this parameter is just a number \(\hat{\alpha}_k \to \hat{\alpha}_k\) and the fixed-point equation reads, now specifying to \(d = 3\)

\[
\left(\frac{-11}{3} + \frac{2}{3} \hat{t} \partial_\hat{t} - \hat{\tilde{p}} \partial_\hat{\tilde{p}}\right) \tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}}) = \hat{\alpha}_k \hat{\lambda}^2 \hat{p}^2 \hat{\tilde{p}}^2 \tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}}).
\]

The solution of this equation is

\[
\tilde{G}_{\theta \theta}(\hat{t}, \hat{\tilde{p}}) = \hat{p}^{-(3/\kappa)} e^{-\frac{3}{2} \hat{p}^2 \hat{\tilde{p}}^2 \hat{\alpha}_k \hat{\lambda}^{2/3} \hat{t}},
\]

where \(\hat{f}(\hat{p}^{2/\kappa} \hat{t})\) is a universal scaling function, which could be computed by explicitly integrating the flow from the initial
condition. Reverting to dimensionful variables, and neglecting the subleading time dependence of the scaling function, one obtains

$$G_{\theta\theta}(t, \hat{p}) = C_\epsilon \epsilon_\theta \hat{p}^{-1/3} \rho \cdot \frac{12}{\lambda} e^{-\alpha_\ell (L/\ell_\alpha)^2 \rho^2 t^2},$$

(47)

where $\tau_0 \equiv (L^2/\epsilon_\theta)^{-1/3}$ denotes the eddy-turnover time at the energy injection scale and $C_\epsilon$ and $\alpha_\ell$ are non-universal constants (the factor $3/2$ was absorbed in the latter). In Eq. (47) $\epsilon_\theta$ and $\epsilon_\ell$ denote the mean rate of energy injection of the scalar and of the velocity, respectively.

2. Large-time limit of the two-point correlation function

We now consider the limit $t \to \infty$ of Eq. (45). In this limit, it is straightforward to check that the frequency integration is dominated by the term in the curly brackets of Eq. (45). Performing the frequency integration, one obtains

$$\partial_t \hat{G}_{\theta\theta}(t, \hat{p}) = \frac{d-1}{2d} \int_{\hat{q}} H_k(0, \hat{q}) \hat{p}^2 |t| \hat{G}_{\theta\theta}(t, \hat{p}).$$

In analogy with the short-time limit, we thus obtain the fixed point equation in $d = 3$

$$\left(-\frac{11}{3} + \frac{2}{3} \tilde{i} \partial_t - \hat{p} \partial_{\hat{p}}\right) \hat{G}_{\theta\theta}(\hat{t}, \hat{p}) = \hat{\alpha}_\ell \hat{p}^2 |\hat{t}| \hat{G}_{\theta\theta}(\hat{t}, \hat{p}).$$

(48)

where $\hat{\alpha}_\ell$ is the fixed point value of $\frac{1}{3} \int_{\hat{q}} \hat{H}_k(0, \hat{q})$. The solution to equation (48) is given by

$$\hat{G}_{\theta\theta}(\hat{t}, \hat{p}) = \hat{p}^{\frac{14}{11}} e^{-\frac{4}{11} \hat{\alpha}_\ell \hat{p}^2 |\hat{t}|} \hat{f}(\hat{p}^{2/3} \hat{t}),$$

which in dimensionful variables implies

$$G_{\theta\theta}(t, \hat{p}) = C_\ell \epsilon_\theta \hat{p}^{-1/3} \rho \cdot \frac{12}{\lambda} e^{-\alpha_\ell (L/\ell_\alpha)^2 \rho^2 t^2},$$

(49)

where $C_\ell$ and $\alpha_\ell$ are non-universal constants (with the numerical factor absorbed in the latter).

Equations (47) and (49) constitute the main results of this work and generalize the temporal dependence found for the Kraichnan model to scalar fields advected by a NS flow. As anticipated, when the carrier turbulent velocity field have some temporal correlations, which is always the case for any realistic flows, then the short-time behavior of the scalar is Gaussian in $t$. The exponential decay is only observed at sufficiently large time scales. This is the main difference with the scalar advected by the idealized Kraichnan stochastic velocity, where the short-time regime is eliminated by the time delta-correlation of the velocity covariance.

VI. SUMMARY AND PERSPECTIVES

In this work, we have studied the temporal dependence of the two-point correlation function of turbulent passive scalar fields in the inertial-convective range. We analyzed the symmetries, and extended symmetries, of the action functional of the system, and derived the corresponding Ward identities. These identities are the crucial ingredients which enabled us to obtain, within the FRG framework in the limit of large wavenumbers, a closed flow equation for generic Eulerian $n$-point correlation functions of the scalar field, advected both by the Gaussian stochastic velocity of the Kraichnan model, or by a turbulent NS velocity field.

We have focused on the solution at the fixed point of the two-point correlation function of the scalar. We have shown that its temporal decay exhibits two regimes: a Gaussian decay at small time delays, which crosses over to an exponential decay at large time delays, with a coefficient proportional to $k^2$ in both regimes. This time dependence explicitly breaks standard scale invariance. This demonstrates that the scalar field, in the inertial-convective range considered here, inherits the time correlations of the carrier fluid, since this exactly corresponds to the behavior of a NS velocity field.

The Kraichnan model replaces the carrier NS fluid velocity by a Gaussian stochastic field with delta-correlation in time. This leads to great simplifications, as was already recognized in many works. As a consequence, we obtained an explicit expression for the prefactor in the exponential, and we also derived the temporal behavior of the two-point function in this model using a perturbative approach, which is not possible for the NS flow. This approach yields results compatible with the FRG approach, and provides a useful connection with former results. The prize of this simplification is that it significantly alters the temporal behavior of the advected scalar, since the small-time regime is destroyed in this case.

The approximation scheme used in this work, based on a large wavenumber expansion, is rooted in the symmetries of the system. Since the symmetries of the advected scalar field are also present in other regimes, say the viscous-convective regime, it is conceivable that this approach can be extended outside the inertial-convective range. We leave this task for future work. Let us emphasize that all the results obtained in this work on the temporal behavior of turbulent passive scalar fields can be tested in direct numerical simulations, which is underway. These results may also initiate experimental investigations.

Another important direction which deserves further study is the combined use of numerical solutions to the FRG flow equations, via ansätze, and the symmetry arguments presented in this paper. This will allow one to address questions which include composite operators, and investigate their anomalous scalings. We hope to tackle these issues in the near future.

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**Appendix A: Yaglom relation from the path integral**

After the celebrated derivation by Kolmogorov of the exact formula for the third-order structure function from the Navier-Stokes equations, Yaglom established the analogous formula for scalar turbulence\(^{15}\). It was shown in Ref. 10 that the Kármán-Howarth relation on which is based Kolmogorov’s exact identity for the third-order structure function can be retrieved within a path integral formalism by considering a gauged shift of the response fields. In this appendix, we show that the Yaglom relation can also be inferred from the symmetries of the path integral.

Let us consider the following spacetime-dependent field transformation

\[ \tilde{\theta} (t, \vec{x}) \rightarrow \tilde{\theta} (t, \vec{x}) + \epsilon (t, \vec{x}), \quad \tilde{\nu} (t, \vec{x}) \rightarrow \tilde{\nu} (t, \vec{x}) + \epsilon (t, \vec{x}) \theta (t, \vec{x}) \]

in the action functional \(^{43}\). The variation of the action reads

\[
\delta S = \int \epsilon (t, \vec{x}) \left( \partial_t \theta (t, \vec{x}) + \partial_i (\nu^i (t, \vec{x}) \theta (t, \vec{x})) \right) - \frac{\kappa}{2} \partial^2 \theta (t, \vec{x}) - M_\theta \tilde{\theta} (t, \vec{x}) \right) \]

Hence, by performing this change of variables in the path integral, and writing that it must leave it unchanged, one obtains

\[
\left( \partial_t \theta (t, \vec{x}) + \partial_i (\nu^i (t, \vec{x}) \theta (t, \vec{x})) \right) - \frac{\kappa}{2} \partial^2 \theta (t, \vec{x}) - M_\theta \tilde{\theta} (t, \vec{x}) \right) = 0, \]

where the subscript indicates that the sources associated to \( \tilde{\theta} \) and \( \tilde{\nu} \) are taken to zero (while the other ones are kept generic). We now want to express Eq. (A1) in terms of functional derivatives of the generating functional \( W \left[ J \right] \). However, the second term in (A1) is nonlinear in the fields, it represents an insertion of a composite operator. Therefore, we introduce a further source conjugate to the composite operator \( \nu^i (t, \vec{x}) \theta (t, \vec{x}) \) by adding the term \( \int d_t \tilde{J}_i \langle L_i (t, \vec{x}) v^i (t, \vec{x}) \theta (t, \vec{x}) \rangle \) to the action. One can then rewrite equation (A1) as

\[
\left( \partial_t - \frac{\kappa}{2} \partial^2 \right) \frac{\delta W}{\delta J_\theta (t, \vec{x}, \vec{y})} + \partial_i \frac{\delta W}{\delta J_\nu (t, \vec{x}, \vec{y})} - \int d_t M_k (\vec{x} - \vec{y}) \frac{\delta W}{\delta J_\theta (t, \vec{y})} = 0, \]

where we have set \( J_\tilde{\theta} = J_\tilde{\nu} = 0 \) in \( W \left[ J \right] \).

We take a further differentiation with respect to \( J_\theta (t, \vec{y}) \) and obtain

\[
\left( \partial_t - \frac{\kappa}{2} \partial^2 \right) \frac{\delta W}{\delta J_\theta (t, \vec{x}, \vec{y})} + \partial_i \frac{\delta W}{\delta L_i (t, \vec{x}, \vec{y})} - \int d_t M_k (\vec{x} - \vec{y}) \frac{\delta W}{\delta J_\theta (t, \vec{y})} = 0, \]

that is, explicitly writing the corresponding connected correlation functions

\[
\left( \partial_t - \frac{\kappa}{2} \partial^2 \right) \left( \langle \theta (t, \vec{x}) \theta (t, \vec{y}) \rangle + \partial_i \langle \nu^i (t, \vec{x}) \theta (t, \vec{y}) \rangle \right) - \langle f (t, \vec{x}) \theta (t, \vec{y}) \rangle = 0. \]

This expression can be symmetrized with respect to \( x \leftrightarrow y \) as follows

\[
0 = \left( \partial_t - \frac{\kappa}{2} \partial^2 \right) \left( \langle \theta (t, \vec{x}) \theta (t, \vec{y}) \rangle - \langle f (t, \vec{x}) \theta (t, \vec{y}) \rangle \right) + \left( \partial_t - \frac{\kappa}{2} \partial^2 \right) \left( \langle \theta (t, \vec{x}) \theta (t, \vec{y}) \rangle - \langle f (t, \vec{y}) \theta (t, \vec{x}) \rangle \right) \]

\[
+ \partial_i \langle \nu^i (t, \vec{x}) \theta (t, \vec{y}) \rangle + \partial_i \langle \nu^i (t, \vec{y}) \theta (t, \vec{x}) \rangle \right) \quad (A2) \]

We now focus on the steady state and take the equal-time limit in Eq. (A2). Using incompressibility and translational invariance, one can rewrite the cubic terms exploiting the relation

\[
\frac{1}{2} \int \partial_t \langle \theta (t, \vec{x}) - \theta (t, \vec{y}) \rangle^2 \langle v_i (t, \vec{x}) - v_i (t, \vec{y}) \rangle = - \langle \theta (t, \vec{x}) \partial_i \langle \nu_i (t, \vec{y}) \theta (t, \vec{y}) \rangle - (\vec{x} \leftrightarrow \vec{y}) \rangle. \]

where \( \frac{\partial}{\partial (x-y)^i} = (\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i})/2 \). Thus one obtains

\[
0 = - \frac{\kappa}{2} \partial^2 \left( \langle \theta (t, \vec{x}) \theta (t, \vec{y}) \rangle - \langle f (t, \vec{x}) \theta (t, \vec{y}) \rangle \right) - \langle f (t, \vec{x}) \theta (t, \vec{y}) \rangle - \langle f (t, \vec{y}) \theta (t, \vec{x}) \rangle - \frac{1}{2} \partial \langle \theta (t, \vec{x}) - \theta (t, \vec{y}) \rangle^2 \langle v_i (t, \vec{x}) - v_i (t, \vec{y}) \rangle \right) \]

Following the reasoning as in Ref. 54, we impose \( \langle f (t, \vec{x}) \theta (t, \vec{y}) \rangle = \epsilon_0 \) in the limit of small \( |\vec{x} - \vec{y}| \). Moreover, by further neglecting the terms proportional to molecular diffusivity, and for \( |\vec{x} - \vec{y}| \to 0 \), we obtain

\[
- \frac{1}{2} \partial \langle \theta (t, \vec{x}) - \theta (t, \vec{y}) \rangle^2 \langle v_i (t, \vec{x}) - v_i (t, \vec{y}) \rangle = 2 \epsilon_0, \]

which is the Yaglom relation. We hence showed that this relation can in fact be inferred solely from a symmetry of the action functional (the invariance under gauged shifts of the response fields). This relation in turn implies that

\[
\langle \theta (t, \vec{x}) - \theta (t, \vec{y}) \rangle^2 \langle v_i (t, \vec{x}) - v_i (t, \vec{y}) \rangle = - 4 \epsilon_0 \frac{(x-y)^i}{d^{1/3}}. \]

Thus, if one assumes K41 scaling for the velocity, one deduces that the scalar field has the same scaling dimensions, \( \theta (t, \vec{x}) \sim \lambda^{1/3} \).

**Appendix B: Spectrum of the scalar in the dissipative range from Dyson equation**

The two-point correlation function \(^{38}\) is expressed in terms of the function \( \langle \tilde{p} \rangle \), which is a complicated function
given by a wavenumber integral. Since we did not assume the scalar to be in the inertial range, Eq. (38) also holds in the dissipative range. Let us show that one may estimate from it the behavior of $\tilde{G}(\tilde{p})$ in the dissipative regime.

If the convective term $\int \tilde{\theta} \cdot \tilde{\theta}$ is negligible then one is left with pure diffusion, implying that the two-point correlation function is given by the bare expression of the propagator $G(\tilde{p}, \tilde{\theta})$. When the convective term is not very efficient, let us assume that the behavior of $\tilde{G}(\tilde{p}, \tilde{\theta})$ is determined by only the first non-trivial correction due to the convection. The computation of $\tilde{\Sigma}$ is essentially unaffected, except that one assumes that the molecular diffusivity term dominates over $\tilde{\Sigma}$ in the dissipative range. As for $\tilde{\Sigma}$, the calculation can be simplified as follows

$$\tilde{\Sigma}(\tilde{p}) \approx D_0 \int_{\omega_d} \frac{D_0}{(\tilde{p}^2 + \tilde{m}^2)^2} \left( p^2 - \frac{\tilde{p}^2}{\tilde{q}^2} \right) \frac{M_k(\tilde{q})}{\omega^2 + \left( \frac{q}{\tilde{q}} \right)^2} .$$

We note that for large values of $\tilde{q}$, the forcing term $M_k(\tilde{q})$ suppresses the corresponding contribution in the integral. Hence, the range of $\tilde{q}$ which contributes is about the scale at which $M_k(\tilde{q})$ is peaked. For large $\tilde{p}$, i.e. for $|\tilde{p}|$ larger than this scale, one has $\tilde{q} - \tilde{p} \approx -\tilde{p}$ for values of $\tilde{q}$ relevant for the integral. For expanding small $\tilde{q}$ all the terms depending in $\tilde{q} - \tilde{p}$ one obtains

$$\tilde{\Sigma}(\tilde{p}) \propto \tilde{p}^{-d-\epsilon} .$$

It follows that for large wavenumbers in the dissipative range, $\tilde{\Sigma}(\tilde{p}) \propto \tilde{p}^{-d-\epsilon}$. The temporal dependence of $\tilde{G}(\tilde{p}, \tilde{\theta})$ is always of the form displayed in (38), both in the inertial and dissipative range. Thus, there exists a range where the convection term is perturbative but it nevertheless affects the spectrum. The spectrum in this range is proportional to

$$\tilde{G}(\tilde{p}, \tilde{\theta}) \approx \left( M_k(\tilde{p}) + \tilde{\Sigma}(\tilde{p}) \right) \int_{\omega_d} \frac{1}{\omega^2 + \left( \frac{\omega}{\tilde{p}} \right)^2} \approx \frac{\tilde{\Sigma}(\tilde{p})}{\kappa \eta p^2} \propto p^{-d-2-\epsilon} .$$
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