DECOMPOSABLE PARTIAL ACTIONS

FERNANDO ABADIE, EUSEBIO GARDELLA, AND SHIRLY GEFFEN

Abstract. We define the decomposition property for partial actions of discrete groups on $C^*$-algebras. Decomposable partial systems appear naturally in practice, and many commonly occurring partial actions can be decomposed into partial actions with the decomposition property. For instance, any partial action of a finite group is an iterated extension of decomposable systems.

Partial actions with the decomposition property are always globalizable and amenable, regardless of the acting group, and their globalization can be explicitly described in terms of certain global sub-systems. A direct computation of their crossed products is also carried out. We show that partial actions with the decomposition property behave in many ways like global actions of finite groups (even when the acting group is infinite), which makes their study particularly accessible. For example, there exists a canonical faithful conditional expectation onto the fixed point algebra, which is moreover a corner in the crossed product in a natural way. (Both of these facts are in general false for partial actions of finite groups.) As an application, we show that freeness of a topological partial action with the decomposition property is equivalent to its fixed point algebra being Morita equivalent to its crossed product. We also show by example that this fails for general partial actions of finite groups.

1. Introduction

The computation of the $K$-theory of $C^*$-algebras is a challenging task, whose origins date back to Cuntz’s seminal work [7]. Obtaining descriptions of the $K$-groups is useful when trying to distinguish $C^*$-algebras, and in many cases of interest, algebras with isomorphic $K$-theory are themselves isomorphic (see [10] for a survey on this topic). There are a number of tools that help us compute the $K$-groups of a given $C^*$-algebra, and the Pimsner-Voiculescu six-term exact sequence for crossed products by $\mathbb{Z}$-actions [24] is arguably one of the most powerful ones. A classical result of Landstad provides an abstract characterization of $\mathbb{Z}$-crossed products as those algebras carrying a circle action with suitably large spectral subspaces, thus granting access to the Pimsner-Voiculescu exact sequence without knowing beforehand that the algebra in question is a $\mathbb{Z}$-crossed product. However, many naturally occurring circle actions tend to have rather “small” spectral subspaces (this is typically the case for gauge actions), and Landstad’s result therefore does not apply. This problem was tackled by Exel in [12], where he showed that under very mild assumptions on the circle action, such algebras are isomorphic to the crossed product of what he called a partial automorphism, that is, an isomorphism between two ideals. This, in combination with a version of Pimsner-Voiculescu for partial automorphisms also obtained in [12], allows one to compute the $K$-groups of a large class of $C^*$-algebras, including all Cuntz-Krieger algebras.

The notion of a partial action of a general discrete group $G$ on a $C^*$-algebra $A$ was introduced by McClanahan in [21]: it is a collection $(A_g)_{g \in G}$ of ideals of $A$ and isomorphisms $\alpha_g: A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \text{id}_A$ and $\alpha_{gh}$ extends $\alpha_g \circ \alpha_h$ wherever defined.
the decomposition is well-defined. By taking all domains to be $A$ we recover the usual definition of a group action (which we call a global action, to avoid confusion). The gener-ality of these objects is illustrated by the fact that large classes of $C^*$-algebras can be described as partial crossed products of commutative $C^*$-algebras, including AF-algebras [13], Bunce-Deddens algebras [11], Exel-Laca algebras [19], or the Jiang-Su algebra [8], thus considerably enlarging the toolkit available to study them.

The study of partial actions unveiled a number of unexpected connections, particularly as partial actions were realized to be closely related to inverse semigroup actions. Indeed, Exel showed in [14] that to every discrete group $G$ one can canonically associate a semigroup $E(G)$, sometimes referred to as the Exel semigroup of $G$, in such a way that partial actions of $G$ on a topological space $X$ are in natural one-to-one correspondence with inverse semigroup morphisms $E(G) \to \text{Homeo}_{\text{par}}(X)$ from $E(G)$ to the inverse semigroup of partial homeomorphisms of $X$. Moreover, the inverse semigroup $C^*$-algebra $C^*(E(G))$ can be canonically identified with the partial group algebra $C^*_{\text{par}}(G)$ of $G$. For further connections to inverse semigroups and related objects, we refer the reader to Exel-Steinberg’s Proceedings of the ICM [17]. Connections to groupoids have been explored in [2, 3].

Despite the numerous advances within the theory of partial dynamical systems, a number of aspects remain unexplored, and many others exhibit behaviors that differ dramatically from the case of global actions. This is in spite of the tight connections with global systems discovered in [1] in the context of the globalization problem. For example, in the case of finite groups, and even for actions that admit a globalization, virtually all averaging arguments (as well as their consequences) that are standard for global actions, completely break down in the partial setting. Among others, the fixed point algebra may be very small; there is in general no conditional expectation onto it; and it is in general not a corner in the crossed product. The lack of approximate identities that are compatible with the partial action is also a source of difficulties in this setting.

The present work originates in the attempts by the authors to obtain a better and more systematic understanding of the internal structure of partial actions. To illustrate this, let us consider first the smallest nontrivial finite group $\mathbb{Z}$ more systematic understanding of the internal structure of partial actions. To illustrate with the partial action is also a source of difficulties in this setting. A corner in the crossed product. The lack of approximate identities that are compatible with the fixed point algebra may be very small; there is in general no conditional expectation onto it; and it is in general not a corner in the crossed product. The lack of approximate identities that are compatible with the partial action is also a source of difficulties in this setting.

The present work originates in the attempts by the authors to obtain a better and more systematic understanding of the internal structure of partial actions. To illustrate this, let us consider first the smallest nontrivial finite group $\mathbb{Z}$, and let $\alpha$ be a partial action of $\mathbb{Z}$ on a compact Hausdorff space $X$. This amounts to a choice of an open subset $U \subseteq X$ and an order two homeomorphism $\sigma$ of it. The restriction of this partial system to $U$ is a global system, while the “remainder” $Y = X \setminus U$ is acted upon trivially. In other words, there is an equivariant topological extension $U \xhookrightarrow{} X \xleftarrow{} Y$. The equivariant structure of $U$ and $Y$ is completely understood, and the complexity of the partial action $\alpha$ is determined by the way how $U$ and $Y$ are glued together. Regardless of how complicated this gluing is, many aspects of $C(X) \rtimes_{\alpha} \mathbb{Z}$ can be deduced from the fact that it is an extension of $C(Y)$ by $C_0(U) \rtimes_{\sigma} \mathbb{Z}$, the latter being a global crossed product.

For larger groups, one has to iterate the process of decomposing the action into simpler sub-systems. For example, an action of $\mathbb{Z}_2$ on a compact Hausdorff space $X$ is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1: U_2 \rightarrow U_1$ satisfying $\sigma_1^2 = \text{id}$ wherever the composition is well-defined. The restriction of this partial system to $U = U_1 \cap U_2$ is thus global, and we get an extension

$$0 \rightarrow C_0(U) \rightarrow C(X) \rightarrow C(Y) \rightarrow 0,$$

where $Y = X \setminus U$. In this case, however, the induced partial action on $Y$ is not trivial. Instead, we may decompose it further: with $V_1 = U_1 \setminus U_2$ and $V_2 = U_2 \setminus U_1$, the homeomorphisms induced by $\sigma_1$ and $\sigma_2$ “exchange” $V_1$ and $V_2$ (and act internally in a global manner), while the complement $Z = Y \setminus (V_1 \cup V_2)$ carries the trivial partial action. Thus, we get an equivariant extension

$$0 \rightarrow C_0(V_1 \cup V_2) \rightarrow C(Y) \rightarrow C(Z) \rightarrow 0,$$
where now the action on \( V_1 \cup V_2 \) is a combination of a translation and an order-3 homeomorphism, and the action on \( Z \) is trivial. We have therefore reduced the understanding of \( Z_3 \cap X \) to the understanding of the systems \( Z_3 \cap U \) and \( Z_3 \cap V_1 \cup V_2 \), together with the understanding of the above equivariant extensions.

For general finite groups, the decomposition process is rather subtle: one obtains a larger number of extensions, and the structure of the intermediate \( G \)-equivariant ideals becomes very complicated. Problems of this nature already appeared in the work of the third-named author \cite{19}. In particular, for such a decomposition to be useful in practice, one would like to be able to compute the crossed products of the intermediate systems.

Our attempts to shed light over the problem described above led us to isolate and study a property that the intermediate systems satisfy, which we call the decomposition property. For a discrete group \( G \) and \( n \in \mathbb{N} \), we set

\[
T_n(G) = \{ \tau \subseteq G : 1 \in \tau, |\tau| = n \},
\]

endowed with the partial action of \( G \) by left multiplication. For a partial action \( \alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G}) \) on a \( C^* \)-algebra \( A \), we set \( A_\tau = \bigcap_{g \in \tau} A_g \) for \( \tau \in T_n(G) \).

**Definition A.** Let \( n \in \mathbb{N} \) and let \( \alpha \) be a partial action of \( G \) on \( A \). We say that \( \alpha \) has the \( n \)-decomposition property if

(a) \( A = \sum_{\tau \in T_n(G)} A_\tau \), and

(b) \( A_\tau \cap A_g = \{0\} \) for all \( \tau \in T_n(G) \) and all \( g \notin \tau \).

We say that \( \alpha \) is decomposable if \( \alpha \) has the \( n \)-decomposition property for some \( n \in \mathbb{N} \).

Decomposable partial actions form a rich class of systems whose essential structure is roughly a combination of a global action of a finite group (even if the original group is infinite), a translation, and a trivial action. In this sense, actions with the decomposition property are much more accessible than general partial actions, the upshot being that many naturally occurring partial actions can be approximated by systems with the decomposition property. This is particularly the case for arbitrary partial actions of finite groups. This fact is extensively exploited in \cite{3}.

Let \( \alpha \) be a partial action with the \( n \)-decomposition property. Then \( \alpha \) decomposes as a direct sum of restrictions to sub-systems of the form \( A_{G,\tau} = \bigoplus_{g \in \tau} A_g \). Thus, it suffices to understand these in order to understand \( \alpha \). For a given \( \tau \), we set \( H_\tau = \{ h \in G : h \tau = \tau \} \), which is a finite subgroup of \( G \) that acts globally on \( A_\tau \). Moreover, there are \( x_0, x_1, \ldots, x_m \in G \) such that \( \tau = \bigcup_{j=0}^m H_\tau x_j \). This data determines most of the relevant features of the action on \( A_{G,\tau} \) (and hence on \( A \)), which we study in detail in this work.

The main result of Section 3 \cite{3} asserts that actions with the decomposition property are always amenable and globalizable, and provides an explicit description of the globalization in terms of induced systems (see [Definition 3.3]).

**Theorem B.** Let \( \alpha \) be a partial action of \( G \) on a \( C^* \)-algebra \( A \) with the \( n \)-decomposition property. Then each \( G \circ A_{G,\tau} \) is amenable and globalizable, and its globalization is \( G \circ \text{Ind}^G_{H_\tau} (A_\tau) \). It follows that \( \alpha \) is amenable and globalizable, and globalization is a direct sum of actions of the form \( G \circ \text{Ind}^G_{H_\tau} (A_\tau) \).
Combining the above theorem with results of the first-named author from [1], it follows that the partial crossed product \( A \rtimes_\alpha G \) can be computed, up to Morita equivalence, using the crossed products of the global sub-systems \( H_\tau \rtimes A_\tau \). However, for some applications it is necessary to have a computation of \( A \rtimes_\alpha G \) up to isomorphism and not just up to Morita equivalence, and we do this in [Theorem 5.1].

**Theorem C.** Let \( \alpha \) be a partial action of \( G \) on a \( C^* \)-algebra \( A \) with the \( n \)-decomposition property. Then \( A_{G,\tau} \rtimes_\alpha G \cong M_{m_n+1}(A_\tau \rtimes H_\tau) \). In particular, \( A \rtimes_\alpha G \) is a direct sum of algebras of the form \( M_{m_\tau+1}(A_\tau \rtimes H_\tau) \), for \( \tau \in T_n(G) \).

Using the above result, we provide an explicit computation of the partial group algebra \( C^*_p(G) \) of a finite group \( G \) from [12]; see [Theorem 5.2]. Indeed, \( C^*_p(G) \) can be realized as the crossed product of the partial action of \( G \) on \( \bigcup_{n=1}^{\infty} T_n(G) \). Moreover, \( G \rtimes T_n(G) \) has the \( n \)-decomposition property, so its crossed product can be computed using Theorem C, see [Theorem 5.2]. An equivalent description of \( C^*_p(G) \) was obtained in [9].

Perhaps some of the most surprising features of actions with the decomposition property refer to their \( G \)-invariant elements and fixed point subalgebras. For global actions, the theory works best in the setting of finite groups: for instance, there is a faithful conditional expectation \( E: A \to A^G \), and there is an injective homomorphism \( c: A^G \to A \rtimes G \) whose image is a corner in \( A \rtimes G \). Both of these maps fail to exist for infinite groups. For partial actions, even of finite groups, additional complications arise and these maps rarely exist; see [Example 4.3]. The following [Theorem 4.4] (Theorem 4.8) is thus unexpected:

**Theorem D.** Let \( \alpha \) be a decomposable partial action of \( G \) on a \( C^* \)-algebra \( A \). Then there are a faithful conditional expectation \( E: A \to A^G \) and an injective homomorphism \( c: A^G \to A \rtimes G \) whose image is a corner in \( A \rtimes G \).

Finally, in [Theorem 5.5] we give a characterization of freeness for topological partial actions with the decomposition property, in terms of the corner map \( c \):

**Theorem E.** Let \( \sigma \) be a decomposable partial action of \( G \) on a locally compact Hausdorff space \( X \). Then \( \sigma \) is free if and only if the image of \( c: C_0(X)^G \to C_0(X) \rtimes_\sigma G \) is full.

In Theorem E, we could have used \( C_0(X/G) \) in place of \( C_0(X)^G \), since the two agree for a decomposable action. For general partial actions, even of finite groups, the above characterization fails (even if one considers \( C_0(X/G) \)); see [Example 5.3].

2. The decomposition property

In this section, we define the decomposition property of a partial action and study some of its basic features. We fix a discrete group \( G \) and \( n \in \mathbb{N} \).

**Definition 2.1.** Given \( n \in \mathbb{N} \), we define the space of \( n \)-tuples of \( G \) to be

\[
T_n(G) = \{ \tau \subseteq G : 1 \leq \tau \text{ and } |\tau| = n \}.
\]

(Note that \( T_n(G) = \emptyset \) whenever \( G \) is finite and \( n > |G| \).) For \( g \in G \), we set \( T_n(G)_g = \{ \tau \in T_n(G) : g \in \tau \} \). There is a canonical partial action \( Lt \) of \( G \) on \( T_n(G) \) with \( Lt_g : T_n(G)_{g^{-1}} \to T_n(G)_g \) given by \( Lt_g(\tau) = g\tau \) for all \( \tau \in T_n(G)_{g^{-1}} \).

We regard \( T_n(G) \) as a topological space, equipped with the discrete topology.

**Notation 2.2.** Let \( \alpha = (A_g)_{g \in G}, (\alpha_g)_{g \in G} \) be a partial action of a discrete group \( G \) on a \( C^* \)-algebra \( A \), and let \( n \in \mathbb{N} \). For \( \tau \in T_n(G) \), we write \( A_\tau \) for the ideal \( A_\tau = \bigcap_{g \in \tau} A_g \). Observe that for \( g \in G \) and \( \tau \in T_n(G)_{g^{-1}} \), we have \( \alpha_g(A_\tau) = A_{g\tau} \). For \( \tau \in T_n(G) \), we write \( G \cdot \tau \subseteq T_n(G) \) for the orbit of \( \tau \) (with respect to the partial action \( Lt \) from Definition 2.1), and we set \( A_{G,\tau} = \sum_{g \in \tau^{-1}} A_{g\tau} \).

The following is the main technical definition of this work.
Lemma 2.7. Let actions; this is in particular true for partial actions of finite groups by Theorem 6.1. In this sense, decomposable partial actions are much more accessible than general combinations of the trivial action, an action by translation, and a global action of a finite group. So condition (a) is also satisfied.

We say that $\alpha$ has the decomposition property if it has the $n$-decomposition property for some $n \in \mathbb{N}$. A partial action on a locally compact space $X$ is said to have the $(n)$-decomposition property if its induced partial action on $C_0(X)$ has it.

The following easy observation will be needed later.

Remark 2.4. In the context of Definition 2.3 condition (b) implies that $A_\tau \cap A_\sigma = \{0\}$ whenever $\tau, \sigma \in \mathcal{T}_n(G)$ are distinct. In particular, we think of $A$ as a direct sum of the orthogonal ideals appearing in condition (a). In fact if $n > 1$, then $A$ has the $n$-decomposition property if and only if $A$ is isomorphic to the $C^*$-algebraic direct sum $\bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$, in such a way that $A_g$ corresponds to $\bigoplus_{\tau \in \mathcal{T}_n(G)} A_{\tau g}$. Indeed, assume that the latter holds, and let $\tau \in \mathcal{T}_n(G)$ and $g \in G \setminus \tau$. Pick any $g' \in \tau \setminus \{1\}$ and set $\sigma = (\tau \cup \{g\}) \setminus \{g'\}$. Then $0 = A_\sigma A_\tau = A_g A_\tau$, so condition (b) holds.

We now turn to examples. The extreme cases are easy to describe:

Example 2.5. Let $\alpha$ be a partial action of a discrete group $G$ on a $C^*$-algebra $A$.

1. $\alpha$ has the 1-decomposition property if and only if $A_g = \{0\}$ for all $g \in G \setminus \{1\}$. This is the trivial partial action of $G$ on $A$.

2. If $G$ is finite, then $\alpha$ has the $|G|$-decomposition property if and only if $\alpha$ is global.

Next, we show that the canonical partial action of $G$ on $\mathcal{T}_n(G)$ has the $n$-decomposition property. This is the prototypical partial action with the $n$-decomposition property.

Proposition 2.6. Let $G$ be a discrete group and let $n \in \mathbb{N}$. Then the partial action $\text{Lt}$ of $G$ on $\mathcal{T}_n(G)$ described in Definition 2.1 has the $n$-decomposition property.

Proof. For $\tau \in \mathcal{T}_n(G)$, it is easy to check that $C_0(\mathcal{T}_n(G))_{\tau} = C(\{\tau\})$, so condition (b) in Definition 2.3 is satisfied. Since $\mathcal{T}_n(G)$ is discrete, we have $C_0(\mathcal{T}_n(G)) \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} C(\{\tau\})$, so condition (a) is also satisfied.

Decomposable partial actions form a rich class whose structure is, in a rough sense, a combination of the trivial action, an action by translation, and a global action of a finite group. In this sense, decomposable partial actions are much more accessible than general partial actions. The upshot of this approach is the fact that many naturally occurring partial actions can be written either as iterated extensions or limits of decomposable partial actions; this is in particular true for partial actions of finite groups by Theorem 6.1.

The following lemma will be fundamental in most of our analysis.

Lemma 2.7. Let $\tau \in \mathcal{T}_n(G)$ and set $H_\tau = \{h \in G: h \tau = \tau\}$. Then

1. $H_\tau$ is finite a subgroup of $G$, and $|H_\tau|$ divides $n$;
2. With $m_\tau = \frac{n}{|H_\tau|} - 1$, there exist $x_1^\tau, \ldots, x_{m_\tau}^\tau \in G$ distinct such that $\tau = H_\tau \cup H_\tau x_1^\tau \cup \ldots \cup H_\tau x_{m_\tau}^\tau$;
3. If $y_1^\tau, \ldots, y_{m_\tau}^\tau \in G$ satisfy $\tau = H_\tau \cup H_\tau y_1^\tau \cup \ldots \cup H_\tau y_{m_\tau}^\tau$, then there exist a permutation $\sigma \in S_{m_\tau}$ and $h_1, \ldots, h_{m_\tau} \in H_\tau$ such that $y_j = h_j x_{\sigma(j)}$ for all $j$.

Proof. It is clear that $H_\tau$ is a subgroup of $G$. We will prove items (1) and (2) simultaneously. The condition $h \tau = \tau$ for all $h \in H_\tau$ implies that $\tau$ is $H_\tau$-invariant, and that $H_\tau \subseteq \tau$, since $1 \in \tau$. Hence $H_\tau$ is finite. $H_\tau$ acts globally on the finite set $\tau$. Thus $\tau$ is a disjoint union of $H_\tau$ orbits, $\tau = H_\tau x_0^\tau \cup H_\tau x_1^\tau \cup \ldots \cup H_\tau x_{m_\tau}^\tau$, where $m_\tau + 1$ is the cardinality of the orbit space, and $x_0^\tau, x_1^\tau, \ldots, x_{m_\tau}^\tau \in \tau$ are the representatives. One of
these disjoint orbits must be $H_\tau$, so we assume without loss of generality that $x_0^\tau = 1$.

Since the action of $H_\tau$ on $\tau$ is free, all orbits have the same cardinality as $H_\tau$, from which it follows that $n = |\tau| = (m_\tau + 1)|H_\tau|$.

Finally, let $y_1^\tau, \ldots, y_r^\tau \in G$ be as in (3) in the statement. Then these elements determine a decomposition of $\tau$ as a disjoint union of $H_\tau$-orbits, so up to a permutation they must agree modulo $H_\tau$ with $x_1^\tau, \ldots, x_m^\tau$, as desired.

**Notation 2.8.** Let $G$ be a discrete group and let $n \in \mathbb{N}$. For $\tau \in T_n(G)$, we set $H_\tau = \{ h \in G : h\tau = \tau \}$. Using Lemma 2.7 we set $m_\tau = \frac{n}{|H_\tau|} - 1$ and fix elements $x_0^\tau = 1, x_1^\tau, \ldots, x_m^\tau \in G$ satisfying

$$
\tau = H_\tau \sqcup H_\tau x_1^\tau \sqcup \ldots \sqcup H_\tau x_m^\tau.
$$

We write $A_{g,\tau}$ for the ideal $\sum_{g \in \tau} A_{g,\tau}$ of $A$. Whenever $\tau$ is understood from the context, we will omit it from the notation for $H_\tau$, $m_\tau$ and $x_j^\tau$, for $j = 1, \ldots, m_\tau$. For example, the above identity will be written $\tau = H \sqcup H x_1 \sqcup \ldots \sqcup H x_m$.

Let $O_n(G)$ be the orbit space for the partial system described in Definition 2.1. We denote by $\kappa : T_n(G) \to O_n(G)$ the canonical quotient map, and fix, for the rest of this work, a global section $s : O_n(G) \to T_n(G)$ for it. For $z \in O_n(G)$, we write $\tau_z$ for $s(z)$; we write $H_z$ for $H_{\tau_z}$; we write $m_z$ for $m_{\tau_z}$. (Note that $m_z$ is really independent of the choice of the section, unlike $H_z$ or $\tau_z$.)

**Proposition 2.9.** Let $n \in \mathbb{N}$, let $G$ be a discrete group, and let $\tau \in T_n(G)$. Set $X = \{ 1, x_1, \ldots, x_m \}$ to be the set of representatives of $H_\tau$-classes. For $g \in G$, set $X_g = \{ x \in X : g(x) = x \} \setminus \tau$. Then

(1) For $g \in G$ and $x \in X_{g^{-1}}$ there is a unique $\sigma_g(x) \in X_g$ such that $g \sigma_g(x) = x^{-1} H x$.

(2) $(X_g)_{g \in G}, (\sigma_g)_{g \in G}$ is a partial action of $G$ on $X$.

**Proof.** Since $x_j^{-1} = \bigcup_{k=0}^m x_j^{-1} H x_k$, part (1) follows. As for (2), note first that $X_1 = X$ and $\sigma_1 = \text{id}_X$. Suppose that $x \in X_{g^{-1}}$ satisfies $\sigma_{g_1}(x) \in X_{g_1^{-1}}$, that is

$$
g_2^{-1} \in x^{-1} H \sigma_{g_2}(x) \quad \text{and} \quad g_1^{-1} \in \sigma_{g_2}(x)^{-1} H \sigma_{g_1}(\sigma_{g_2}(x)).
$$

Then $(g_1 g_2)^{-1} \in x^{-1} H \sigma_{g_1}(\sigma_{g_2}(x)) \subseteq x^{-1} \tau$. This shows that $x \in X (g_1 g_2)^{-1}$, and $\sigma_{g_1 g_2}(x) = \sigma_{g_1}(\sigma_{g_2}(x))$ by uniqueness of the left member of the equality.

**Proposition 2.10.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, let $A = (A_g)_{g \in G}, (A_{g,\tau})_{g \in G}$ be a partial action of $G$ on $A$ with the $n$-decomposition property, and let $\tau \in T_n(G)$. Adopt the conventions from Notation 2.8. Then:

(1) The restriction of $\alpha|_H^\tau$ to $A_\tau$ is a global action;

(2) The ideal $A_{G,\tau}$ is $G$-invariant, and for $g \in G$ one has

$$
(A_{G,\tau})_g = \begin{cases} 
\{0\} & \text{if } g \notin \tau^{-1} \cdot \tau \\
\sum_{0 \leq j \leq m : g \in x_j^{-1} \tau} A_{x_j^{-1} \tau} & \text{if } g \in \tau^{-1} \cdot \tau.
\end{cases}
$$

(3) For $\sigma \in T_n(G)$, we have $A_{G,\sigma} \cap A_{G,\tau} = \{0\}$ if $\sigma \notin G \cdot \tau$.

(4) There is a natural $G$-equivariant isomorphism

$$
\varphi : \bigoplus_{z \in O_n(G)} A_{G,\tau_z} \to A
$$

given by $\varphi(a) = \sum_{z \in O_n(G)} a_z$ for all $a = (a_z)_{z \in O_n(G)}$.

**Proof.** (1). Since $H \tau = \tau$, it follows for every $h \in H$ that $A_h \cap A_\tau = A_\tau$ and moreover $\alpha_h(A_\tau) = A_\tau$. Hence $\alpha|_H$ induces a global action on $A_\tau$. 

Fix $g \in G$. We have $A_{g^{-1}} \cap A_{\tau} = \{0\}$ if $g^{-1} \notin \tau$; and $A_{g^{-1}} \cap A_{\tau} = A_{\tau}$ if $g^{-1} \in \tau$, in which case $\alpha_g(A_{\tau}) = A_{g\tau}$. Hence $A_{g\tau}$ is invariant and there is a well-defined restricted partial action of $G$ on it. For $g \in G$, we have

$$(A_{G\tau})_g = A_{G\tau} \cap A_g = \sum_{j=0}^{m} A_{x_j g^{-1} \tau} \cap A_g = \sum_{0 \leq j \leq m : g \in x_j g^{-1} \tau} A_{x_j g^{-1} \tau},$$

where at the last equality we use the decomposition property. In particular, if $g \notin \tau^{-1} \cdot \tau$, this domain is trivial.

We prove the contrapositive, so we suppose that $A_{G\sigma} \cap A_{G\tau} \neq \{0\}$ and will show that $\sigma \in G \cdot \tau$. Fix $g \in \tau^{-1}$ and $h \in \sigma^{-1}$ with $A_{g\tau} \cap A_{h\sigma} \neq \{0\}$. By Remark 2.3, we have $gh = h\sigma$. In particular, $h \in g\tau$ and thus $(h^{-1}g)\tau = \sigma$, as desired.

Note that $\varphi$ is a homomorphism by part (3) above. Moreover, it is clearly injective and equivariant, and it is surjective by condition (a) of Definition 2.3.

Remark 2.11. Using part (4) of the proposition above, a number of facts about decomposable partial actions can be reduced to the $G$-invariant direct summands $A_{G\tau}$. In practice, for many purposes it suffices to work with a single tuple $\tau \in T_n(G)$ and the induced partial action on $A_{G\tau}$.

The following lemma will be used repeatedly (see Notation 2.8).

Lemma 2.12. Let $G$ be a discrete group, let $A$ be a $C^\ast$-algebra, let $n \in \mathbb{N}$, and let $\alpha$ be a partial action of $G$ on $A$ with the $n$-decomposition property. Fix $\tau \in T_n(G)$.

1. There are canonical quotient maps $\pi_j : A_{G\tau} \to A_{x_j^{-1} \tau}$, for $j = 0, \ldots, m$, which can be explicitly described as follows: if $(e_\lambda)_{\lambda \in \Lambda}$ is any approximate identity for $A_{x_j^{-1} \tau}$, then $\pi_j(a) = \lim_{\lambda \to a} \tau_j$ for all $a \in A_{G\tau}$.

2. The map $\pi_j$ is independent of the choice of the representative in $Hx_j$.

Proof. (1). Follows from Remark 2.3 which implies that $A_{G\tau} \cong \bigoplus_{j=0}^{m} A_{x_j^{-1} \tau}$.

(2). Let $y_0, \ldots, y_m \in G$ and $h_1, \ldots, h_m \in H$ be elements satisfying $y_j = h_j x_j$ for all $j = 0, \ldots, m$. The claim follows since $A_{y_j^{-1} \tau} = A_{x_j^{-1} \tau}$. □

In the context of the above lemma, and whenever the tuple $\tau$ is not clear from the context, we will write $\pi_j^\tau$ instead of $\pi_j$.

Our next goal is to show that the domains of partial actions with the decomposition property admit particularly well-behaved kind of approximate identities, which we call an “equivariant system of approximate identities”.

Definition 2.13. Let $G$ be a discrete group, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on a $C^\ast$-algebra $A$. A system of equivariant approximate identities for $\alpha$ is a choice, for every $g \in G$, of an approximate identity $(e_\lambda^g)_{\lambda \in \Lambda}$ of $A_g$, satisfying $\alpha_g(e_h^g e_{\lambda^{-1}}^g) = e_{gh}^g e_h^g$ for all $g, h \in G$ and all $\lambda \in \Lambda$.

The above definition is inspired by the case of unital partial actions, where the units of the respective domains form a system of equivariant approximate identities. Explicitly, if $1_g$ denotes the unit of $A_g$, then one has $\alpha_g(1_h 1_{g^{-1}}) = 1_{gh} 1_g$ for all $g, h \in G$. On the other hand, global actions of finite groups always admit such systems: it suffices to consider a $G$-invariant approximate identity of $A$.

Systems of equivariant approximate identities fail to exist in general, but we show in the following proposition that decomposable systems always possess them.

Proposition 2.14. Let $G$ be a discrete group, let $A$ be a $C^\ast$-algebra, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a decomposable partial action of $G$ on $A$. Then there exists a system of equivariant approximate identities for $\alpha$. 
Proof. Let \( n \in \mathbb{N} \) be such that \( \alpha \) has the \( n \)-decomposition property. By Remark 2.11 it suffices to show that \( A_{G,T} \) possesses a system of equivariant approximate identities for every \( \tau \in \mathcal{T}_n(G) \). Fix \( \tau \in \mathcal{T}_n(G) \) and fix an approximate identity \( (e^\lambda)_{\lambda \in \Lambda} \) for \( A_{\tau} \). We use Notation 2.8. For \( g \in G \) and \( \lambda \in \Lambda \), define

\[
e^\lambda_g = \frac{1}{|H|} \sum_{h=0}^{m} \left( \sum_{h \in H} \alpha_{x_j^{-1}h}^\lambda(e^\lambda) \right)
\]

Notice that \( e^\lambda_g \in (A_{G,T})_g \) is a positive contraction. We claim that \( (e^\lambda_g)_{\lambda \in \Lambda} \) is an approximate identity for \( (A_{G,T})_g \). Since \( (A_{G,T})_g = \{0\} \) whenever \( g \not\in \tau^{-1} \cdot \tau \), we may assume that \( g \in \tau^{-1} \cdot \tau \). By part (2) of Proposition 2.10, it is enough to show that \( (e^\lambda_g) \) is an approximate identity for \( A_{x_j^{-1} \tau} \) for any \( j = 0, \ldots, m \) such that \( g \in x_j^{-1} \tau \). Fix such \( j \) and let \( a \in A_{x_j^{-1} \tau} \). Using at the first step that \( \sum_{h \in H} \alpha_{x_j^{-1}h}^\lambda(e^\lambda) \) belongs to the ideal \( A_{x_j^{-1} \tau} \) which is orthogonal to \( A_{x_j^{-1} \tau} \) when \( k \neq j \), and at the second that \( (\alpha_{x_j^{-1}h}^\lambda(e^\lambda))_{\lambda \in \Lambda} \) is an approximate identity of \( A_{x_j^{-1} \tau} \), we conclude that

\[
\lim_{\lambda \in \Lambda} e^\lambda_g a = \lim_{\lambda \in \Lambda} \frac{1}{|H|} \sum_{h \in H} \alpha_{x_j^{-1}h}^\lambda(e^\lambda)a = a.
\]

Let \( g_1, g_2 \in G \) and let \( \lambda \in \Lambda \). We will show that \( \alpha_g(e^\lambda_{g_1} e^\lambda_{g_2}) = e^\lambda_{g_1 g_2} e^\lambda_{g_1} \). To ease notation, we drop the superscript \( \lambda \) everywhere, as well as the denominator \( |H| \). For \( \ell = 0, \ldots, m \), let \( \pi_\ell : A_{G,T} \to A_{x_k^{-1} \tau} \) be the map from Lemma 2.12. Then

\[
\pi_\ell(e_{g_1} e_{g_2} e_{g_1}) = \sum_{j=0}^{m} \sum_{h,t \in H} \pi_\ell \left( \left( \sum_{h \in H} \alpha_{x_j^{-1}h}^\lambda(e) \right) \cdot \left( \sum_{t \in H} \alpha_{x_j^{-1}t}^\lambda(e) \right) \right)
\]

On the other hand,

\[
\pi_\ell(\alpha_g(e_{g_2} e_{g_1}^{-1})) = \sum_{k=0}^{m} \sum_{s,r \in H} \pi_\ell \left( \left( \sum_{s \in H} \alpha_{x_k^{-1}s}^\lambda(e) \right) \cdot \left( \sum_{r \in H} \alpha_{x_k^{-1}r}^\lambda(e) \right) \right)
\]

where at the second and last steps we use that if \( g \in x_k^{-1} \tau \), then there is a unique \( j \in \{0, \ldots, m\} \) such that \( g \in x_j^{-1} H x_j \). The result follows since \( \sum_{j=0}^{m} \pi_j = \text{id}_{A_{G,T}} \).
3. Enveloping actions

In this section, we show that decomposable partial actions are always globalizable, and we give an explicit construction of their globalizations. We point out that the results in this section apply without major modifications to algebraic partial actions.

We begin by recalling the notion of an enveloping action (Definition 3.4 in [1]).

Definition 3.1. Let $G$ be a discrete group and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on a $C^*$-algebra $A$. A triple $(B, \beta, \iota)$ consisting of a $C^*$-algebra $B$, a (global) action $\beta : G \to \text{Aut}(B)$ and an embedding $\iota : A \to B$ as an ideal, is said to be an enveloping action of $\alpha$ if the following conditions are satisfied:

1. $A_g = A \cap \beta_g(A)$ for all $g \in G$;
2. $\alpha_g(a) = \beta_g(a)$ for all $a \in A_{g^{-1}}$ and all $g \in G$;
3. $B = \overline{\text{sp}}(\beta_g(a) : a \in A, \ g \in G)$.

We say that $\alpha$ is globalizable if there exists an enveloping action.

By Theorem 3.8 in [1], enveloping actions are unique up to an equivariant isomorphism, extending the identity on $A$; for this reason, we will always refer to the enveloping action of a given globalizable partial action.

Not every partial action is globalizable, and even when it is, identifying its enveloping action may turn out to be challenging. Using Ferraro’s recent abstract characterization of globalizability [13], we show next that decomposable partial actions are always globalizable. Obtaining an explicit description of its enveloping action is rather involved, and this is done in Theorem 3.6.

Proposition 3.2. Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha$ be a decomposable partial action of $G$ on $A$. Then $\alpha$ is globalizable.

Proof. By the equivalence between (a) and (c) in Theorem 4.5 of [18], it suffices to show that given $(g, a, b) \in G \times A \times A$, there exists $u_{g,a,b} \in A_g$ such that $c u_{g,a,b} \cdot c u_{g^{-1}, a^{-1}, c^{-1}} = a b$ for all $c \in A_g$. Let $n \in \mathbb{N}$ such that $\alpha$ has the n-decomposition property. By condition (a) in Definition 2.3, we may take $a \in A_r$ for $r \in T_n(G)$.

For $g \notin \tau^{-1}$, we set $u_{g,a,b} = 0$. In this case, by condition (b) in Definition 2.3, we have $A_r \cap A_{g^{-1}} = \{0\}$, and hence the identity $c u_{g,a,b} = c u_{g^{-1}, a^{-1}} = a b$ holds for all $c \in A_g$, since $\alpha_g^{-1}(c) a = 0$. For $g \in \tau^{-1}$, set $u_{g,a,b} = \alpha_g(a) b$, which is well-defined as $a \in A_{g^{-1}}$. Then $c u_{g,a,b} = c \alpha_g(a) b = \alpha_g(\alpha_g^{-1}(c) a) b$ for all $c \in A_g$, as desired. This finishes the proof.

Our next goal is to obtain an explicit description of the globalization of a decomposable partial action, which was shown to exist in the proposition above. We need to recall the notion of induced (global) actions; see Chapter 3 in [27].

Definition 3.3. Given a discrete group $G$, a subgroup $H$, a $C^*$-algebra $C$ and an action $\gamma : H \to \text{Aut}(C)$, the induced dynamical system $(\text{Ind}_G^H(C), \text{Ind}_G^H(\gamma))$ is

$$\text{Ind}_G^H(C) = \left\{ \xi \in C_b(G, C) : \xi(hg) = \gamma_h(\xi(g)) \text{ for all } g \in G \text{ and } h \in H, \right. \left. \text{ and the map } Hg \mapsto \|\xi(g)\| \text{ is in } C_b(G/H) \right\},$$

with $\text{Ind}_H^G(\gamma)_g(\xi)(k) = \xi(kg)$ for all $g, k \in G$ and all $\xi \in \text{Ind}_G(C)$.

Remark 3.4. For discrete groups, Green’s Imprimitivity Theorem (see, for example, Corollary 4.17 in [27]) can be deduced from the theory of enveloping actions of partial actions from [1] [6]. Indeed, in the context of Definition 3.3, the action $\gamma$ defines a partial action $\gamma$ of $G$ on $C$, where $\gamma_g = \gamma_0$ if $g \in H$, and $\gamma_g = 0$ otherwise. Then $C \rtimes \gamma = C \rtimes \gamma H = C \rtimes \gamma G$, and similarly for the reduced crossed products. Moreover, one readily checks that $(\text{Ind}_G^H(C), \text{Ind}_G^H(\gamma))$ is an enveloping action for $\gamma$, with inclusion map $\iota : C \to \text{Ind}_G^H(C)$ given by $\iota(c)(g) = \gamma_g(c)$ if $g \in H$, and 0 if $g \notin H$, for all $c \in C$. It follows from [1] and [6].
that the crossed products \( C \rtimes_\gamma H = C \rtimes_\gamma G \) and \( \text{Ind}^G_H(C) \rtimes_{\text{Ind}^G_H(\gamma)} G \) are Morita equivalent, as well as the reduced crossed products \( C \rtimes_{\gamma,r} H = C \rtimes_{\gamma,r} G \) and \( \text{Ind}^G_H(C) \rtimes_{\text{Ind}^G_H(\gamma),r} G \).

**Remark 3.5.** We recall \([6]\) that a partial action is said to be amenable if the canonical surjection \( A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G \) is injective. Adopt the notation and assumptions of [Remark 3.3] By Corollary 1.3 of \([9]\) (or Corollary 5.4 of \([6]\)), the actions \( \gamma, \tilde{\gamma} \) and \( \text{Ind}^G_H(\gamma) \) are simultaneously amenable or non-amenable. If \( H \) is amenable, then they are all amenable.

We are now ready for the main result of this section.

**Theorem 3.6.** Let \( G \) be a discrete group, let \( A \) be a C*-algebra, let \( n \in \mathbb{N} \), and let \( \alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G}) \) be a partial action of \( G \) on \( A \) with the \( n \)-decomposition property. For every \( \tau \in T_n(G) \), the restriction of \( \alpha \) to \( A_{G-\tau} \) is globalizable and amenable, and its enveloping action is \( (\text{Ind}^G_H(A_\tau), \text{Ind}^G_H(\alpha), \iota_{\tau}) \), where the inclusion \( \iota_{\tau} : A_{G-\tau} \to \text{Ind}^G_H(A_\tau) \) is given by

\[
\iota_{\tau}(a)(g) = \begin{cases} 
\alpha_g(\tau_k(a)) & \text{if } g \in H \cdot x_k \text{ for some } k = 0, \ldots, m \\
0 & \text{else.}
\end{cases}
\]

for all \( a \in A_{G-\tau} \) and all \( g \in G \). (In particular, \( \iota_{\tau}(a) \) is supported on \( \tau \).) It follows that \( \alpha \) is globalizable and amenable, and its globalization is \( \bigoplus_{z \in \mathcal{O}_n(G)} (\text{Ind}^G_H(A_{\tau_z}), \text{Ind}^G_H(\alpha), \iota_{\tau_z}) \).

**Proof.** As explained in [Remark 2.11] it suffices to prove the first part of the statement. Let \( \tau \in T_n(G) \). To lighten the notation, we abbreviate \( H_\tau \to H, m_\tau \to m, x_j \to x_j, \) and \( \iota_{\tau} \to \iota \). It is clear that \( \iota \) is an embedding. We check the remaining conditions in [Definition 3.1] in a series of claims.

**Claim:** the range of \( \iota \) is contained in \( \text{Ind}^G_H(A_\tau) \). It is clear that \( \iota_{\tau}(a)(g) \) belongs to \( A_\tau \) for all \( a \in A_{G-\tau} \) and all \( g \in G \). Let \( g \in G \), let \( h \in H \), and let \( a \in A_{G-\tau} \). Then \( \iota(a)(hg) = 0 = \iota(a)(g) \) unless \( g \in H x_k \) for some \( k = 0, \ldots, m \), in which case

\[
\iota(a)(hg) = \alpha_{hg}(\tau_k(a)) = \alpha_k(\iota(a)(g)),
\]

as desired. Moreover, the induced map \( Hg \to \|\iota(a)(g)\| \) belongs to \( C_0(G/H) \) because it is supported on the finite set \( \{Hx_0, \ldots, Hx_m\} \). This proves the claim.

**Claim:** the image of \( \iota \) is an ideal in \( \text{Ind}^G_H(A_\tau) \). Let \( \xi \in \text{Ind}^G_H(A_\tau) \), let \( k = 0, \ldots, m \), and let \( a \in A_{x_k^{-1}} \). It suffices to check that \( \iota(a)\xi \) belongs to the image of \( \iota \). For \( g \in G \), we have \( \iota(a)\xi(g)(h) = 0 \) unless \( g \in H x_k \), in which case we have

\[
(\iota(a)\xi(g))(h) = \alpha_g(\alpha_{x_k^{-1}}(\xi(g))) = \alpha_g(\alpha_{x_k^{-1}}(\xi(x_k)))(a)(\tau_k(a))(g),
\]

where at the last step we use that \( \alpha_{x_k^{-1}}(\xi(x_k)) \) belongs to \( A_{x_k^{-1}} \). We conclude that, \( \iota(a)\xi = \iota(\alpha_{x_k^{-1}}(\xi(x_k))) \), thus proving the claim. From now on, we set \( \beta = \text{Ind}^G_H(\alpha) \).

**Claim:** for \( g \in G \), we have

\[
\iota(A_{G-\tau} \cap A_g) = \beta_g(\iota(A_{G-\tau})) \cap \iota(A_{G-\tau}).
\]

We begin with the inclusion \( \subseteq \). For this, it is enough to show that \( \iota(a) \) belongs to \( \beta_g(\iota(A_{G-\tau})) \) for \( k = 0, \ldots, m \) and \( a \in A_{x_k^{-1}} \cap A_g \). This is trivial if \( g \notin x_k^{-1} \tau \), since the decomposition property implies that \( A_{x_k^{-1}} \cap A_g = \{0\} \). Assume that \( g \in x_k^{-1} \tau \) and let \( j = 0, \ldots, m \) satisfy \( g \in x_k^{-1} H x_j \). For \( t \in G \) we have

\[
\beta_g(\iota(\alpha^{-1}_g(a)))(t) = \iota(\alpha^{-1}_g(a))(tg) = \mathbb{1}_{t \in H x_j} \alpha_{tg}(\alpha^{-1}_g(a)) = \mathbb{1}_{t \in H x_k} \alpha_a(a) = \iota(a)(t).
\]

It follows that \( \iota(a) = \beta_g(\iota(\alpha^{-1}_g(a))) \). To prove the converse inclusion, it suffices to show

\[
\iota(A_{G-\tau} \cap A_g) \supseteq \beta_g(\iota(A_{x_k^{-1}} \cap \tau)) \cap \iota(A_{x_k^{-1}} \cap \tau)
\]

for all \( g \in G \) and \( \tau \in T_n(G) \).
for all $g \in G$ and all $j, k = 0, \ldots, m$. Let $a \in A_{x_{j-1}^G}$ and $b \in A_{x_{k-1}^G}$ satisfy $\beta_g(\iota(a)) = \iota(b)$. We will be done once we show that $b \in A_g$. For $t \in G$ we have
\[
\beta_g(\iota(a))(t) = \iota(a)(tg) = \begin{cases} 
\alpha_g(a) & \text{if } tg \in H x_j \\
0 & \text{else.}
\end{cases}
\]
On the other hand $\iota(b)(t) = \alpha_g(b)$ if $t \in H x_k$, and $0$ otherwise. If $\beta_g(\iota(a)) = \iota(b) = 0$, then by injectivity of $\iota$ we get $b = 0 \in A_g$, as required. Otherwise, there exists some $t \in H x_k$ such that $tg \in H x_j$. Then, $g \in x_k^{-1} H x_j \subseteq x_k^{-1} \tau$, and by definition $A_{x_k^{-1} \tau} \subseteq A_g$, so $b \in A_g$, as required. The claim is proved.

**Claim:** $\iota(\alpha_g(a)) = \beta_g(\iota(a))$ for all $a \in (A_{G, \tau})_g^{-1}$ and all $g \in G$. As $a = \sum_{j=0}^m \pi_j(a)$, we may assume that $a \in A_{x_{j-1}^G} \cap A_{g^{-1}}$ for some $j = 0, \ldots, m$, and that $g^{-1} \in x_{j}^{-1} \tau$ (otherwise, this intersection is trivial). Let $k \in \{0, \ldots, m\}$ be the unique element such that $g^{-1} \in x_k^{-1} H x_k$. Note that $a = \pi_j(a)$. For $t \in G$, we have
\[
\beta_g(\iota(a))(t) = \iota(\alpha_g(a))(t)
\]
unless $t$ belongs to $H x_k$, in which case we have
\[
\beta_g(\iota(a))(t) = \iota(a)(tg) = \alpha_g(a) = \iota(\alpha_g(a))(t).
\]
This proves the claim.

**Claim:** $\sum_{g \in G} \beta_g(\iota(A_{G, \tau})) = \operatorname{Ind}^G_H(A_\tau)$. Let $(g_\lambda)_{\lambda \in \Lambda}$ be a subset of $G$ satisfying $\bigcup_{\lambda \in \Lambda} H g_\lambda = G$. An element $\xi \in \operatorname{Ind}^G_H(A_\tau)$ is completely determined by its values on $g_\lambda$, for $\lambda \in \Lambda$. Moreover, the set of those $\xi \in \operatorname{Ind}^G_H(A_\tau)$ that take nonzero values on a finite subset of $(g_\lambda : \lambda \in \Lambda)$ is dense in $\operatorname{Ind}^G_H(A_\tau)$.

Let $\xi \in \operatorname{Ind}^G_H(A_\tau)$ satisfy $\xi(g_\lambda) = 0$ for all but finitely many $\lambda$. By the comments above, it suffices to show that $\sum_{\lambda \in \Lambda} \beta_{g_\lambda}^{-1}(\iota(\xi(g_\lambda))) = \xi$. To see this, let $\mu \in \Lambda$ and $h \in H$. Then
\[
\left( \sum_{\lambda \in \Lambda} \beta_{g_\lambda}^{-1}(\iota(\xi(g_\lambda))))(hg_\mu) = \sum_{\lambda \in \Lambda} \iota(\xi(g_\lambda))(hg_\mu g_\lambda^{-1}) = \xi(hg_\mu),
\]
where at the second step we use the fact that $hg_\mu g_\lambda^{-1} \in H$ if and only if $\mu = \lambda$. This proves the claim, and finishes the identification of the enveloping action. Amenability of the restriction of $\alpha$ to $A_{G, \tau}$ (and hence for $\alpha$) follows from Remark 3.5 since the global action $\alpha : H \to \operatorname{Aut}(A_\tau)$ is amenable because $H$ is finite.

In the next result, we write $\sim_M$ for Morita equivalence.

**Corollary 3.7.** Fix $\tau \in \mathcal{T}_n(G)$. With the assumptions of Theorem 3.6 we have
\[
A_{G, \tau} \rtimes_M A G = A_{G, \tau} \rtimes_M A, G \sim_M \operatorname{Ind}^G_{H_\tau}(A_\tau) \rtimes_{\operatorname{Ind}^G_{H_\tau}} (\alpha) G \sim_M A_\tau \rtimes_{\alpha|_{H_\tau}} H_\tau.
\]
Moreover, $A \rtimes_M A = A \rtimes_{r_{A, G}} A \sim_M \bigoplus_{\xi \in \mathcal{O}_n(G)} A_{r_\xi} \rtimes_{\alpha|_{H_\xi}} H_\xi$. In particular, the partial crossed products of $\alpha$ can be computed, up to Morita equivalence, using the crossed products of the global systems $H_\tau \rtimes A_\tau$.

4. **Fixed point algebras**

In this section, we study algebras of $G$-invariant elements. For global actions, the theory is well-known to work best for finite (or even compact) groups: there are a faithful conditional expectation $E : A \to A^G$, and an injective homomorphism $c : A^G \to A \times G$ whose image is a corner in $A \times G$. With $u_g \in M(A \rtimes G)$, for $g \in G$, denoting the canonical unitaries, these maps are given by
\[
E(a) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(a) \quad \text{and} \quad c(b) = \frac{1}{|G|} \sum_{g \in G} u_g b.$
for all $a \in A$ and all $b \in A^G$. Both of these maps fail to exist for infinite groups: in this case, $A^G$ is typically too small (and is often trivial). For partial actions, even of finite groups, additional complications arise; see Example 4.3 and Example 4.7.

In this section, we show that decomposable partial actions, even for infinite groups, behave similarly to global actions of finite groups from the point of view of fixed point algebras. For example, there are analogs of the maps $E: A \rightarrow A^G$ and $c: A^G \rightarrow A \times G$ mentioned above; see Theorem 4.3 and Theorem 4.6. We also give an explicit description of $A^G$ in terms of the fixed point algebras of the global systems $H \rightarrow A_r$; see Theorem 4.6.

**Definition 4.1.** Let $G$ be a discrete group, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on a $C^*$-algebra $A$. We say that an element $a \in A$ is $G$-invariant, if for every $g \in G$ and every $b_g \in A_g$ we have $\alpha_g^{-1}(b_g) = a \alpha_g^{-1}(b_g)$. Finally, the subalgebra $A^G$ of $A$ consisting of all $G$-invariant elements is called the fixed point algebra of $\alpha$.

In the context of the above definition, we could have instead defined an element $a \in A$ to be $G$-invariant if $\alpha_g^{-1}(b_g)a = a \alpha_g^{-1}(b_g)$ for all $g \in G$ and all $b_g \in A_g$. The existence of approximate identities in $C^*$-algebras implies that the two notions would be equivalent, as we show in the next remark.

**Remark 4.2.** Let $G$ be a discrete group, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on a $C^*$-algebra $A$. Let $a \in A^G$, let $g \in G$ and let $b_g \in A_g$; we claim that $\alpha_g^{-1}(b_g a) = \alpha_g^{-1}(b_g)a$. To see this, let $(e_j)_{j \in I}$ be an approximate identity of $A_g$. Then

$$\alpha_g^{-1}(b_g a) = \lim_{j \in I} \alpha_g^{-1}(b_g a) = \lim_{j \in I} \alpha_g^{-1}(b_g a) = \lim_{j \in I} \alpha_g^{-1}(b_g a) \alpha_g^{-1}(e_j) = \alpha_g^{-1}(b_g) \alpha_g^{-1}(e_j) = \alpha_g^{-1}(b_g)a,$$

as required. In particular, it follows that $A^G$ is closed under the adjoint operation, and is thus a $C^*$-subalgebra of $A$.

Perhaps surprisingly, conditional expectations as described at the beginning of this section do not always exist in the case of partial actions of finite groups:

**Example 4.3.** Set $X = (0, 2]$ and $U = (0, 1) \cup (1, 2) \subseteq X$, and let $\sigma: U \rightarrow U$ be the order-2 homeomorphism given by $\sigma(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x > 1. \end{cases}$ for $x \in U$. Then there is no conditional expectation $E: C_0(X) \rightarrow C_0(X)^{\mathbb{Z}_2}$.

**Proof.** We claim that $C_0(X)^{\mathbb{Z}_2} = \{ f \in C_0((0, 2]): f(x) = f(x + 1) \text{ for all } 0 < x < 1 \}.$ Since the inclusion $\supseteq$ is clear, we prove the reverse containment. Let $f \in C_0(X)^{\mathbb{Z}_2},$ and let $g \in C_0(U)$ be strictly positive. For $x \in (0, 1) \subseteq U$ we have

$$f(x + 1)g(x + 1) = \alpha_1(fg)(x) = (f(\alpha_1(g))(x) = f(x)g(x + 1).$$

It follows that $f(x) = f(x + 1)$ for all $x \in (0, 1)$, as desired.

In particular, any $f \in C_0(X)^{\mathbb{Z}_2}$ satisfies $f(1) = 0$.

Suppose that there exists a conditional expectation $C_0(X) \rightarrow C_0(X)^{\mathbb{Z}_2}$. Denote by $\iota \in C_0(X)$ the canonical inclusion $X \rightarrow \mathbb{C}$. We claim that $E(\iota)(1) \neq 0$, which will be a contradiction.

For every $n \in \mathbb{N}$, denote by $f_n \in C_0(X)^{\mathbb{Z}_2}$ any positive function with $f_n \leq \iota$ that satisfies $f_n(x) = x$ for $x \in (0, 2^{1/n}]$. Since conditional expectations are order-preserving, we must have $f_n = E(f_n) \leq E(\iota)$ for all $n \in \mathbb{N}$. In particular,

$$E(\iota)(1) = \lim_{n \to \infty} E(\iota)\left(\frac{n - 1}{n}\right) \geq \lim_{n \to \infty} f_n\left(\frac{n - 1}{n}\right) = \lim_{n \to \infty} \frac{n - 1}{n} = 1,$$

as desired. It follows that no such conditional expectation exists. \[\square\]
In contrast with the previous example, we show in Theorem 4.4 that decomposable actions, even of infinite groups, always admit canonical conditional expectations onto their fixed point algebras. We retain Notation 2.8.

**Theorem 4.4.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha$ be a partial action of $G$ on $A$ with the $n$-decomposition property. Fix $\tau \in \mathcal{T}_n(G)$.

1. For $h \in H, j, k = 0, \ldots, m$, and $a \in A_{G,\tau}^G$, we have $\pi_k(a) = \alpha_{x_k^{-1}hx_j}(\pi_j(a))$.

2. For $a \in A_\tau$, the following element belongs to $A_{G,\tau}^G$:

$$\varphi_\tau(a) = \frac{1}{|H|} \sum_{j=0}^{m} \sum_{h \in H} \alpha_{hx_j}^{-1}(a).$$

3. The map $E_\tau : A_{G,\tau} \to A_{G,\tau}^G$ given by

$$E_\tau(a) = \frac{1}{m+1} \sum_{j=0}^{m} \varphi_\tau(\alpha_{x_j}(\pi_j(a))),$$

for all $a \in A_{G,\tau}$, is a faithful conditional expectation.

4. $E_\tau$ is independent of the elements $x_1, \ldots, x_m$.

5. For $g \in \tau^{-1}$, we have $E_{g\tau} = E_\tau$.

It follows that the map $E = \bigoplus_{z \in \mathcal{O}_n(G)} E_{\tau_z} : A \to A^G$ is a canonical faithful conditional expectation, that is, $E$ is independent of the section $z \mapsto \tau_z$.

**Proof.** (1). Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $A_{x_k^{-1}h}^{-1}$. Then $(\alpha_{x_k^{-1}h}^{-1} e_\lambda)_{\lambda \in \Lambda}$ is an approximate identity for $A_{x_k^{-1}h}^{-1}$, since $\alpha_{x_k^{-1}h}^{-1} e_\lambda = e_\lambda$. Using this in combination with part (1) of Lemma 2.12 at the first and third step, and using invariance of $a$ at the second step, we get

$$\alpha_{x_k^{-1}h}^{-1}(\pi_k(a)) = \lim_{\lambda} \alpha_{x_k^{-1}h}^{-1}(ae_\lambda) = \lim_{\lambda} \alpha_{x_k^{-1}h}^{-1}(e_\lambda) = \pi_j(a).$$

(2). Let $\tau \in \mathcal{T}_n(G)$, let $a \in A_\tau$, let $g \in G$, and let $b_g \in (A_{\tau}G)_g$. We claim that

$$\alpha_{g^{-1}}(\varphi_\tau(a)b_g) = \varphi_\tau(a)\alpha_{g^{-1}}(b_g).$$

Note first that the identity is trivially satisfied if $g \notin \tau^{-1}$, since in this case $b_g$ must be zero. Since $b_g = \sum_{j=0}^{m} \pi_j(b_g)$, we may assume that $b_g \in A_{x_j^{-1}h} \cap A_g$ for some $j = 0, \ldots, m$ and that $g \in x_j^{-1}\tau$ (otherwise, this intersection is zero). Let $k \in \{0, \ldots, m\}$ and $h \in H$ be the unique elements satisfying $g = x_j^{-1}hx_k$. Then

$$\alpha_{g^{-1}}(\varphi_\tau(a)b_g) = \frac{1}{|H|} \sum_{h \in H} \sum_{\ell=0}^{m} \alpha_{g^{-1}}(\alpha_{\ell x_k}^{-1}(a)b_g).$$

The product $\alpha_{\ell x_k}^{-1}(a)b_g$ belongs to $A_{x_k^{-1}h} \cap A_g$, so it is zero unless $\ell = j$. Similarly,

$$\varphi_\tau(a)\alpha_{g^{-1}}(b_g) = \frac{1}{|H|} \sum_{h \in H} \sum_{\ell=0}^{m} \alpha_{\ell x_k}^{-1}(a)\alpha_{g^{-1}}(b_g)$$

and the product $\alpha_{\ell x_k}^{-1}(a)\alpha_{g^{-1}}(b_g)$ is zero unless $\ell = k$. Thus, it suffices to show that

$$\sum_{\ell \in H} \alpha_{\ell x_k^{-1}h^{-1}\ell x_j}^{-1}(a_{\ell x_k}^{-1}(a)b_g) = \sum_{\ell \in H} \alpha_{\ell x_k}^{-1}(a)\alpha_{\ell x_k^{-1}h^{-1}x_j}(b_g).$$

Fix $t \in H$. Then $\alpha_{t x_k}^{-1}(a)$ belongs to the domain of $\alpha_{x_k^{-1}h^{-1}x_j}$, and thus

$$\alpha_{x_k^{-1}h^{-1}x_j}(\alpha_{t x_k}^{-1}(a)b_g) = \alpha_{x_k^{-1}h^{-1}x_j}(\alpha_{t x_k}^{-1}(a))\alpha_{x_k^{-1}h^{-1}x_j}(b_g) = \alpha_{t x_k}^{-1}(a)\alpha_{x_k^{-1}h^{-1}x_j}(b_g),$$

which implies the identity above. Hence $\varphi_\tau(a)\alpha_{g^{-1}}(b_g)$ belongs to $A_{G,\tau}^G$. 

(3). Note that $E_\tau$ is well-defined, since $\alpha_{x_j}(\pi_j(a))$ belongs to $A_\tau$ for all $j = 0, \ldots, m$ and all $a \in A_{G \tau}$. Moreover, for $a \in A_{G \tau}$, we have

$$E_\tau(a) = \frac{1}{|H|(m + 1)} \sum_{j,k=0}^m \alpha_{x_j^{-1}hx_k}(\pi_k(a)).$$

We claim that $E_\tau$ is faithful. Observe that, for $a \geq 0$, we have $E_\tau(a) = 0$ if and only if $\pi_k(a) = 0$ for all $k = 0, \ldots, m$. The claim follows since $a = \sum_{k=0}^m \pi_k(a)$.

Fix $a \in A_{G \tau}^\sigma$. Then $\alpha_{x_j^{-1}hx_k}(\pi_k(a)) = \pi_j(a)$ for all $j, k = 0, \ldots, m$ and all $h \in H$, by part (1) of this theorem. Thus,

$$E_\tau(a) = \frac{1}{|H|(m + 1)} \sum_{j,k=0}^m \alpha_{x_j^{-1}hx_k}(\pi_k(a)) = \frac{1}{|H|(m + 1)} \sum_{j,k=0}^m \pi_j(a) = a,$$

as desired. It follows that $E_\tau$ is an idempotent map. Since it is easily seen to be contractive, it follows that $E_\tau$ is a conditional expectation.

(4). We write $E_\nu^\tau$ for the conditional expectation constructed from the elements $x_0 = 1, x_1, \ldots, x_m$ as in part (2). Let $y_0 = 1, y_1, \ldots, y_m \in G$ be elements satisfying $\tau = \bigsqcup_{y_0}^y y_j$, and write $E_\nu^\tau$ for the corresponding conditional expectation. We want to show that $E_\nu^\tau = E_\tau$. By part (3) of [Lemma 2.7], we may assume without loss of generality that there exist $h_1, \ldots, h_m \in H$ satisfying $y_j = h_j x_j$ for all $j = 1, \ldots, m$. Let $a \in A_{G \tau}$. Then

$$E_\nu^\tau(a) = \frac{1}{|H|(m + 1)} \sum_{j,k=0}^m \alpha_{x_j^{-1}}(h_1 x_j)(\alpha_{h_k x_k}(\pi_k(a)))$$

$$= \frac{1}{|H|(m + 1)} \sum_{j,k=0}^m \alpha_{x_j^{-1}}(h_1^{-1}h_k x_k)(\alpha_{x_k}(\pi_k(a)))) = E_\tau(a).$$

(5). Let $g \in \tau^{-1}$, and let $\ell = 0, \ldots, m$ be the unique element satisfying $g \in x_\ell^{-1}H$. Then $g\tau = x_\ell^{-1}\tau$. In particular, it suffices to assume that $g = x_\ell^{-1}$. Set $\sigma = x_\ell^{-1}$. By part (4) of this theorem, we may compute $E_\sigma$ using any decomposition $\sigma = H_\sigma \bigsqcup_H H_\sigma y_1 \bigsqcup_H \ldots \bigsqcup_H H_\sigma y_m$ as in [Lemma 2.7]. In this context, we must have $H_\sigma = x_\ell^{-1}H x_\ell$, and we take $y_j = x_\ell^{-1}x_j$ for $j = 0, \ldots, m$. Note that $\pi_j^\sigma = \pi_j^\tau$ for all $j = 0, \ldots, m$. Given $a \in A_{G \tau}$, we have

$$E_\sigma(a) = \frac{1}{|H\sigma|(m + 1)} \sum_{j,k=0}^m \alpha_{y_j^{-1}y_k}(\pi_j^\tau(a))$$

$$= \frac{1}{|H\tau|(m + 1)} \sum_{j,k=0}^m \alpha_{x_j^{-1}x_j x_\ell^{-1}x_k}(\pi_k^\tau(a)) = E_\tau(a).$$

The last statement of the theorem follows from part (4) of [Proposition 2.10].

Of particular importance is the existence of approximate identities consisting of $G$-invariant elements. In the setting of global actions, a straightforward averaging argument shows that finite group actions admit invariant approximate identities. On the other hand, invariant approximate identities fail to exist in general for partial actions of finite groups; see [Example 4.3]. For decomposable actions, even of infinite groups, we establish the existence of $G$-invariant approximate identities in the following proposition.

Proposition 4.5. Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha = (A_{\lambda})_{\lambda \in G}, (\alpha_\lambda)_{\lambda \in G}$ be a decomposable partial action of $G$ on $A$. Then there exists a $G$-invariant approximate identity for $A$.

Proof. Let $E: A \rightarrow A^G$ be the canonical conditional expectation constructed in [Theorem 4.4]. If $\alpha_\lambda$ is an approximate identity for $A$, then $E(\alpha_\lambda)$ is an approximate identity for $A^G$. We claim that it is also an approximate identity for $A$. Let $n \in \mathbb{N}$ be such that $\alpha$
has the \( n \)-decomposition property. Using part (4) of Proposition 2.10, it suffices to show that \((E_\tau (\alpha))(\lambda)_{\lambda \in \Lambda}\) is an approximate identity for \(A_{G, \tau}\) whenever \((\alpha)_{\lambda \in \Lambda}\) is an approximate identity for \(A_{G, \tau}\), for every \( \tau \in T_n(G) \).

Fix \( \tau \in T_n(G) \), fix \( b \in A_{G, \tau} \) and fix an approximate identity \((\alpha)_{\lambda \in \Lambda}\) of \(A_{G, \tau}\). For \( j, k = 0, \ldots, m \) and \( h \in H \), the net \((\alpha_{x_j^{-1}hx_k}(\pi_k(\alpha)))_{\lambda \in \Lambda}\) is an approximate identity for \(A_{x_j^{-1}hx_k}\). Using this and part (1) of Lemma 2.12 at the second step, and using the identity \( b = \sum_{j=0}^m \pi_j(b) \) at the third step, we get

\[
\lim_{\lambda \in \Lambda} E_\tau(a) b = \frac{1}{|H|(m+1)} \sum_{j, k=0}^m \sum_{\lambda \in \Lambda} \alpha_{x_j^{-1}hx_k}(\pi_k(\alpha)) b = \frac{1}{|H|(m+1)} \sum_{j=0}^m b = b.
\]

Analogously, one shows that \( \lim_{\lambda \in \Lambda} b E_\tau(\alpha) = b \). It follows that \((E_\tau(\alpha))(\lambda)_{\lambda \in \Lambda}\) is an approximate identity for \(A_{G, \tau}\), and the proof is finished. \( \square \)

Fixed point algebras of decomposable partial actions can be explicitly described using certain global subsystems of finite subgroups of \( G \), as we show below.

**Theorem 4.6.** Let \( G \) be a discrete group, let \( A \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \), and let \( (\alpha_x)_{x \in \mathbb{Z}^n} \) be a partial action of \( G \) on \( A \) with the \( n \)-decomposition property. For every \( \tau \in T_n(G) \), the fixed point algebra of \( A_{G, \tau} \) can be canonically identified with \( A_{G, \tau}^H \), via the restriction of \( \pi_0^\tau \) to \( A_{G, \tau}^H \). Under this identification, the canonical inclusion \( \tau: A_{G, \tau}^H \hookrightarrow A_{G, \tau} \) is given by \( \tau(a) = \sum_{j=0}^m \alpha_{x_j}^{-1}(a) \) for all \( a \in A_{G, \tau}^H \). In particular, \( A_G^\tau \) can be canonically identified with \( \bigoplus_{z \in \Omega_n(G)} A_{z}^H \).

**Proof.** By Remark 2.11 it suffices to prove the first statement. Fix \( \tau \in T_n(G) \); we make the abbreviations \( m = m_\tau \), \( H = H_\tau \), \( \pi_j = \pi_j^\tau \), and \( x_j = x_j^\tau \), for \( j = 0, \ldots, m \).

Let \( \pi_0: A_{G, \tau} \rightarrow A_{G, \tau} \) be the canonical quotient map described in Lemma 2.12. By part (1) of Theorem 4.4, \( \pi_0 \) restricts to a homomorphism \( A_{G, \tau}^H \rightarrow A_H^\tau \). We claim that this is an isomorphism. To show surjectivity, let \( a \in A_H^\tau \). By part (1) of Theorem 4.4, the element \( \varphi_\tau(a) = \sum_{j=0}^m \sum_{h \in H} \alpha_{x_j}^{-1}(a) \) is \( G \)-invariant, and we have

\[
\pi_0(\varphi_\tau(a)) = \frac{1}{|H|} \sum_{h \in H} \alpha_h^{-1}(a) = a,
\]

as required. To show injectivity, let \( a \in A_{G, \tau}^H \) satisfy \( \pi_0(a) = 0 \). By part (1) of Theorem 4.4, it follows that \( \pi_j(a) = 0 \) for all \( j = 1, \ldots, m \), and therefore \( a = \sum_{j=0}^m \pi_j(a) = 0 \). The remaining claims in the statement are immediate. \( \square \)

We remark that under the identification \( A_{G, \tau}^H \cong A_H^{\tau} \), the canonical conditional expectations \( E_{\tau} \) described in Theorem 4.4 become the natural conditional expectations \( A_{G, \tau} \rightarrow A_{G, \tau}^{H} \) given by \( a \mapsto \frac{1}{m_\tau+1} \sum_{j=0}^m \left( \frac{1}{|H|} \sum_{h \in H} \alpha_{h x_j}(\pi_j(a)) \right) \).

We turn to the existence of a corner embedding \( c: A_G^\tau \rightarrow A \rtimes_\alpha G \), which is well-known to exist for global actions of finite groups. In the setting of partial actions, such a map does not exist in general, even for finite groups.

**Example 4.7.** Let \( \alpha \) be the partial action of \( \mathbb{Z}_3 = \{0, 1, 2\} \) on \( X = [0, 1] \) given by letting \( U = (0, 1] \) be the domain of both \( \alpha_1 \) and \( \alpha_2 \), with \( \alpha_1 = \alpha_2 = id_U \). Then \( C(X)^G = C(X) \) and \( C(X) \rtimes_\alpha \mathbb{Z}_3 \) contains no nontrivial projections. In particular, there is no corner embedding \( c: C(X)^{\mathbb{Z}_3} \rightarrow C(X) \rtimes_\alpha \mathbb{Z}_3 \).
Proof. For $f \in C(X)$, the identity $\alpha_1(fg) = f\alpha_1(g)$ is automatically satisfied for all $g \in C_0(U) = C(X, 2)$, since $\alpha_1 = id_U$. Similarly, $\alpha_2(fh) = f\alpha_2(h)$ for all $h \in C(X)_1$, and hence $f \in C(X)G$. It follows that $C(X)^G = C(X)$. There is a short exact sequence

$$0 \longrightarrow C_0(U) \rtimes_{\alpha|_U} \mathbb{Z}_3 \longrightarrow C(X) \rtimes_{\alpha} \mathbb{Z}_3 \longrightarrow \mathbb{C} \longrightarrow 0,$$

where $C_0(U) \rtimes_{\alpha|_U} \mathbb{Z}_3$ is a global crossed product, and $\mathbb{C}$ is identified with the crossed product of $C(\{0\})$ by the trivial partial action of $\mathbb{Z}_3$. Since $\alpha_1 = \alpha_2 = id_U$, there is an isomorphism $C_0(U) \rtimes_{\alpha|_U} \mathbb{Z}_3 \cong C_0(U) \otimes C^*(\mathbb{Z}_3)$. In particular, this algebra has no projections other than zero.

Let $p$ be a projection in $C(X) \rtimes_{\alpha} \mathbb{Z}_3$. Then $\pi(p)$ is a projection in $\mathbb{C}$, so it is either 0 or 1. If $\pi(p) = 0$, then $p$ belongs to the kernel of $\pi$, which is $C_0(U) \rtimes_{\alpha|_U} \mathbb{Z}_3$, and hence is the zero projection. On the other hand, if $\pi(p) = 1$, then $1 - p$ is a projection in the kernel of $\pi$, and hence it must be zero. Thus $p = 1$.

To prove the last statement, if a map $c$ as in the statement exists then it must be an isomorphism, and thus $C(X) \cong C(X) \rtimes_{\alpha} \mathbb{Z}_3$. However, from the above short exact sequence it is clear that the spectrum of the abelian algebra $C(X) \rtimes_{\alpha} \mathbb{Z}_3$ contains a point whose complement has three connected components. In particular, this space is not homeomorphic to $X = [0, 1]$, and the claim follows. \(\square\)

For decomposable partial systems, even for infinite groups, there does in fact exist a canonical corner map $c : A^G \to A \rtimes_{\alpha} G$.

**Theorem 4.8.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha$ be a partial action of $G$ on $A$ with the $n$-decomposition property. Fix $\tau \in T_n(G)$.

1. There is a corner-embedding $c_{\tau} : A^G_{G, \tau} \to A_{G, \tau} \rtimes_{\alpha} G$ given on $a \in A^G_{G, \tau}$ by

$$c_{\tau}(a) = \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_j(a)\delta_{x_j^{-1}hx_k}. $$

2. $c_{\tau}$ is independent of the elements $x_1, \ldots, x_m$.

3. For $g \in \tau^{-1}$, we have $c_{g\tau} = c_{\tau}$.

Thus, the map $c = \bigoplus_{z \in G_{\tau}(G)} c_{z} : A^G \to A \rtimes_{\alpha} G$ is a canonical corner embedding.

**Proof.** (1). Note that $c_{z}$ is well-defined, since $\pi_j(a) \in (A_{G, \tau})_{x_j^{-1}hx_k}$ for all $h \in H$ and all $k = 0, \ldots, m$, so that the product $\pi_j(a)\delta_{x_j^{-1}hx_k}$ is defined in the partial crossed product $A_{G, \tau} \rtimes_{\alpha} G$. Moreover, $c_{z}$ is clearly injective, since $c_{z}(a) = 0$ if and only if $\pi_j(a) = 0$ for all $j = 0, \ldots, m$, which is equivalent to $a = 0$.

**Claim:** $c_{z}$ is a homomorphism. Let $a, b \in A^G_{G, \tau}$. Use orthogonality of $\alpha_{x_j^{-1}h_{-1}x_j}(\pi_j(a))$ and $\pi_i(a)$ for $k \neq i$ at the third step, and part (1) of Theorem 4.4 at the fifth step, to get

$$c_{z}(a)c_{z}(b) = \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_j(a)\delta_{x_j^{-1}hx_k}\pi_i(a)\delta_{x_i^{-1}tx_k},$$

$$= \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \alpha_{x_j^{-1}hx_k}(\alpha_{x_i^{-1}h_{-1}x_j}(\pi_j(a))\pi_i(b))\delta_{x_j^{-1}hx_kx_i^{-1}tx_k},$$

$$= \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \alpha_{x_j^{-1}hx_k}(\alpha_{x_j^{-1}hx_k}(\pi_j(a))\pi_k(b))\delta_{x_j^{-1}hx_kx_j^{-1}tx_k},$$

$$= \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_j(a)\alpha_{x_j^{-1}hx_k}(\pi_k(b))\delta_{x_j^{-1}hx_kx_j^{-1}tx_k},$$

$$= \frac{1}{|H|(m + 1)^2} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_j(a)\alpha_{x_j^{-1}hx_k}(\pi_k(b))\delta_{x_j^{-1}hx_kx_j^{-1}tx_k}.$$
it follows that the left-hand side is contained in the right-hand side. We prove the converse

\[
\alpha_{x^{-1}_j h x_t} (\pi_x(a)) \delta_{x^{-1}_j h x_t} = c_r(ab).
\]

The rest of the proof consists in proving that \( c_r(A_G^G) \) is a corner in \( A_G \rtimes \alpha G \). To
this end, we define a multiplier \( c_r(1) \) of \( A_G \rtimes \alpha G \) by setting

\[
c_r(1)(a \delta_g) = \frac{1}{|H|(m+1)} \sum_{j,k=0}^{m} \alpha_{x^{-1}_j h x_t} (\pi_x(a)) \delta_{x^{-1}_j h x_t}.
\]

for all \( a \in (A_G \rtimes \alpha G) \) and all \( g \in G \). For \( j = 0, \ldots, m \), denote by \( 1_j \) the unit of the multiplier
algebra of \( A_{x^{-1}_j} \). Then the map \( c_r(1) \) can be identified, in a way compatible with the
operations in \( A_G \rtimes \alpha G \), with the formal linear combination

\[
c_r(1) = \frac{1}{|H|(m+1)} \sum_{j,k=0}^{m} \alpha_{x^{-1}_j h x_t} \delta_{x^{-1}_j h x_t}.
\]

(When \( A_G \rtimes \alpha G \), and hence \( A_x \), is unital, \( c_r(1) \) is really the image of \( 1 \in A_G^G \) under \( c_r \), hence
the notation.) It is easy to check that \( c_r(1) \) is a multiplier of \( A_G \rtimes \alpha G \).

**Claim:** \( c_r(1) \) is a projection. Let \( a \in (A_G \rtimes \alpha G) \). Since \( a = \sum_{j=0}^{m} \pi_x(a) \) and \( c_r \) is
linear, it is enough to assume that \( a \in A_{x^{-1}_k} \cap \mathcal{A}_g \) for some \( k = 0, \ldots, m \) and show
\( c_r(1)(c_r(1)(a \delta_g)) = c_r(1)(a \delta_g) \). We may also assume that \( g \in x^{-1}_k \tau \), otherwise the identity
is satisfied. Let \( \ell \in \{0, \ldots, m\} \) and \( t \in H \) be the unique elements with \( g = x^{-1}_k t x_\ell \). Then

\[
c_r(1)(a \delta_g) = \frac{1}{|H|(m+1)} \sum_{j,k=0}^{m} \alpha_{x^{-1}_j h x_t} (\pi_x(a)) \delta_{x^{-1}_j h x_t}.
\]

so \( c_r(1) = c_r(1)^2 \). To check that \( c_r(1) \) is self-adjoint, we use its presentation as a formal
linear combination to get

\[
c_r(1)^* = \frac{1}{|H|(m+1)} \sum_{j,k=0}^{m} \alpha_{x^{-1}_j h x_t} (\pi_x(a)) \delta_{x^{-1}_j h x_t} = \frac{1}{|H|(m+1)} \sum_{j,k=0}^{m} \alpha_{x^{-1}_j h x_t} \delta_{x^{-1}_j h x_t} = c_r(1).
\]

**Claim:** \( c_r(A_G^G) = c_r(1)(A_G \rtimes \alpha G) c_r(1) \). Since \( c_r(1)(a \delta_g) = c_r(a) \) for all \( a \in A_G^G \),
it follows that the left-hand side is contained in the right-hand side. We prove the converse
inclusion. For this, it is enough to show that an element of the form \( c_r(1)(a \delta_g) c_r(1) \), for
\( a \in A_{x^{-1}_k} \cap \mathcal{A}_g \), belongs to the image of \( c_r \), for each \( k = 0, \ldots, m \). This is immediate if
\( g \notin x^{-1}_k \tau \), so assume that \( g \) has the form \( g = x^{-1}_k t x_\ell \) for unique \( \ell = 0, \ldots, m \) and \( t \in H \).
Note that \( 1_\ell a = 0 \) unless \( i = k \), in which case \( 1_\ell a = a \). Similarly, \( \alpha_{g^{-1}}(a) 1_\ell = 0 \) unless
\( p = \ell \), in which case \( \alpha_{g^{-1}}(a)1_{p} = \alpha_{g^{-1}}(a) \). We have
\[
c_{r}(1)(a\delta_{g})c_{r}(1) = \frac{1}{|H|^{2}(m+1)^{2}} \sum_{i,j,p,q=0}^{m} \sum_{h,s \in H} (1_{j}\delta_{e_{j}^{-1}h_{x_{j}}}((a\delta_{g})(1_{p}\delta_{e_{p}^{-1}sx_{q}}) \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{i,j,p,q=0}^{m} \sum_{h,s \in H} \alpha_{x_{j}^{-1}h_{x_{j}}(1_{a})\delta_{e_{j}^{-1}h_{x_{j}}}}((a\delta_{g})(1_{p}\delta_{e_{p}^{-1}sx_{q}}) \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,p,q=0}^{m} \sum_{h,s \in H} \alpha_{x_{j}^{-1}h_{x_{j}}(a)\delta_{e_{j}^{-1}htx_{j}}}}((a\delta_{g})(1_{p}\delta_{e_{p}^{-1}sx_{q}}) \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,p,q=0}^{m} \sum_{h,s \in H} \alpha_{x_{j}^{-1}htx_{j}}((a\delta_{g})(1_{p}\delta_{e_{p}^{-1}sx_{q}}) \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,p,q=0}^{m} \sum_{h,s \in H} \alpha_{x_{j}^{-1}htx_{j}}((a\delta_{g})(1_{p}\delta_{e_{p}^{-1}sx_{q}}) \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,q=0}^{m} \sum_{h,s \in H} \alpha_{x_{j}^{-1}h_{x_{j}}(a)\delta_{e_{j}^{-1}htsx_{q}}}} \]

Set \( b = \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,q=0}^{m} \sum_{h \in H} \sum_{x \in H} \sigma_{x_{j}^{-1}h_{x_{j}}(a)\delta_{e_{j}^{-1}htsx_{q}}} \). With \( E_{\tau} \) denoting the canonical conditional expectation constructed in [Theorem 4.3], we have \( b = E_{\tau}(a) \), and hence \( b \) belongs to \( A_{G,\tau}^{c} \). Finally, it is clear that \( c_{r}(b) = c_{r}(1)(a\delta_{g})c_{r}(1) \), as desired.

(2). We write \( c_{r}^{c} \) for the corner embedding constructed from the elements \( x_{0}, x_{1}, \ldots, x_{m} \) as in part (1). Let \( y_{0} = 1, y_{1}, \ldots, y_{m} \in G \) be elements satisfying \( \tau = \bigsqcup_{j=0}^{m} H y_{j} \), and write \( c_{r}^{c} \) for the corresponding corner embedding. We want to show that \( c_{r}^{c} = c_{r}^{c} \). By part (3) of [Lemma 2.7], we may assume without loss of generality that there exist \( h_{1}, \ldots, h_{m} \in H \) satisfying \( y_{j} = h_{j}x_{j} \) for all \( j = 1, \ldots, m \). Let \( a \in A_{G,\tau}^{c} \). Then \( \pi_{j}^{y}(a) = \pi_{j}^{y}(a_{h_{j}^{-1}}(a)) = \pi_{j}^{y}(a) \) for all \( j = 0, \ldots, m \). Using this at the second step, we get
\[
c_{r}^{c}(a) = \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}hty_{k}} \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}hty_{k}} \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}hty_{k}} \]
\[= \frac{1}{|H|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{h \in H} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}hty_{k}} \]

(3). Let \( g \in \tau^{-1} \), and let \( \ell = 0, \ldots, m \) be the unique element satisfying \( g \in x_{\ell}^{-1}H \). Then \( g\tau = x_{\ell}^{-1}\tau \). In particular, it suffices to assume that \( g \equiv x_{\ell}^{-1} \). Set \( \sigma = x_{\ell}^{-1} \). By part (2) of this theorem, we may compute \( c_{\sigma} \) using any decomposition \( \sigma = H_{\alpha} \sqcup H_{\beta} y_{1} \sqcup \ldots \sqcup H_{\beta} y_{m} \) as in [Lemma 2.7]. In this context, we must have \( H_{\sigma} = x_{\ell}^{-1}H_{\ell}x_{\ell} \), and we take \( y_{j} = x_{\ell}^{-1}x_{j} \) for \( j = 0, \ldots, m \). Note that \( \pi_{j}^{\tau} = \pi_{j}^{y} \) for all \( j = 0, \ldots, m \). Given \( a \in A_{G,\tau}^{c} \), we have
\[
c_{\sigma}(a) = \frac{1}{|H_{\tau}|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{t \in H_{\sigma}} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}ty_{k}} \]
\[= \frac{1}{|H_{\tau}|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{t \in H_{\sigma}} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}tx_{j}(x_{\ell}^{-1}h_{x_{j}}x_{\ell}^{-1}x_{h})} \]
\[= \frac{1}{|H_{\tau}|^{2}(m+1)^{2}} \sum_{j,k=0}^{m} \sum_{t \in H_{\sigma}} \pi_{j}^{y}(a)\delta_{e_{j}^{-1}tx_{j}(x_{\ell}^{-1}h_{x_{j}}x_{\ell}^{-1}x_{h})} = c_{r}(a) \]

The last statement of the theorem follows from part (3) of [Proposition 2.10] \qed
5. Computation of the crossed product

The crossed product of a decomposable action was computed, up to Morita equivalence, in [Corollary 3.7]. For some purposes, such as [Theorem 5.5], it is necessary to have a computation of $A \rtimes_{\alpha} G$ up to isomorphism and not just up to Morita equivalence. Obtaining such a description is the goal of this section; see [Theorem 5.1]. Using this calculation, we provide an alternative computation of the partial group algebra $C^*_\text{par}(G)$ of a finite group $G$ from [9]; see [Theorem 5.2]. Finally, in [Theorem 5.5], we combine the results from Section 4 and [Theorem 5.1] to give a characterization of freeness for decomposable topological partial actions, in terms of the corner map $c$ defined in [Theorem 4.8]. This characterization fails in general, even for free partial actions of finite groups; see [Example 5.3].

**Theorem 5.1.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on $A$ with the $n$-decomposition property. For every $\tau \in \mathcal{T}_n(G)$, there is a natural isomorphism

$$\psi_\tau : A_{G^\tau} \rtimes_{\alpha} G \rightarrow M_{m+1}(A_{\tau} \rtimes_{\alpha|_{\mathcal{H}_\tau}} H_{\tau})$$

which satisfies $\psi_\tau(a_{\delta}^{-1} h_{x_{\tau}}) = \alpha_{x_{\tau}}(a) v h \otimes e_{j,k}$ for all $a \in A_{\delta}^{-1}$, for all $j, k = 0, \ldots, m$, and all $h \in H$, where $v : \mathcal{H} \rightarrow M(A_{\tau} \rtimes_{\alpha|_{\mathcal{H}_\tau}} H_{\tau})$ denotes the canonical unitary representation. It follows that $A \rtimes_{\alpha} G$ can be naturally identified with $\bigoplus_{x \in \mathcal{O}_n(G)} M_{m+1}(A_{\tau} \rtimes_{\alpha|_{\mathcal{H}_\tau}} H_{\tau}).$

**Proof.** By [Remark 2.11], it suffices to prove the first statement. Fix $\tau \in \mathcal{T}_n(G)$; we use Notation 2.8 and [Lemma 2.12] to define maps $\varphi : A_{G^\tau} \rightarrow M_{m+1}(A_{\tau} \rtimes_{\alpha|_{\mathcal{H}_\tau}} H_{\tau})$ and $u : G \rightarrow M(M_{m+1}(A_{\tau} \rtimes_{\alpha|_{\mathcal{H}_\tau}} H_{\tau})) \otimes M_{m+1}$ by

$$\varphi(a) = \sum_{j=0}^{m} \alpha_{x_{j}}(\pi_{j}(a)) \otimes e_{j,j} \quad \text{and} \quad u_g = \sum_{j,k=0}^{m} \|_{g \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g x_{j}^{-1}} \otimes e_{j,k}$$

for $a \in A_{G^\tau}$ and $g \in G$. We will see that $(\varphi, u)$ is a covariant pair for $(A_{G^\tau}, \alpha)$.

**Claim:** $u$ is a partial representation of $G$. To check this, let $g \in G$. We have

$$u_{g}^* = \left( \sum_{j,k=0}^{m} \|_{g \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g x_{j}^{-1}} \otimes e_{j,k} \right)^* = \sum_{j,k=0}^{m} \|_{g^{-1} \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g^{-1} x_{j}^{-1}} \otimes e_{k,j} = u_{g^{-1}}.$$ 

Moreover, for $g_1, g_2 \in G$, the product $u_{g_1} u_{g_2} u_{g_2^{-1}}$ is equal to:

$$\left( \sum_{j,k=0}^{m} \|_{g_1 \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g_1 x_{j}^{-1}} \otimes e_{j,k} \right) \cdot \left( \sum_{j,k=0}^{m} \|_{g_2 \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g_2 x_{j}^{-1}} \otimes e_{j,k} \right) \cdot \left( \sum_{j,k=0}^{m} \|_{g_1^{-1} \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g_1^{-1} x_{j}^{-1}} \otimes e_{j,k} \right)$$

$$= \sum_{j,k,r=0}^{m} \|_{g_1 \in x_{j}^{-1} H_{x_{j}}} \|_{g_2 \in x_{j}^{-1} H_{x_{j}}} \|_{g_1^{-1} \in x_{k}^{-1} H_{x_{k}}} v_{x_{j} g_1 x_{j}^{-1}} v_{x_{j} g_2 x_{j}^{-1}} v_{x_{j} g_1^{-1} x_{j}^{-1}} \otimes e_{j,k}$$

$$= \sum_{j,k=0}^{m} \|_{g_1 \in x_{j}^{-1} H_{x_{j}}} \|_{g_2 \in x_{j}^{-1} H_{x_{j}}} v_{x_{j} g_1 x_{j}^{-1}} \otimes e_{j,k},$$
where at the last step, we use that $x^{-1}_k H \cap x^{-1}_t H = \emptyset$ for $k \neq t$, which implies that $\mathbb{1}_{(g_2 \in x^{-1}_k H x_1)} \mathbb{1}_{(g_2' \in x^{-1}_t H x_1)} = 0$ for $k \neq t$ and $r = 0, \ldots, m$. On the other hand,

$$u_{g_1 g_2}u_{g_2}^{-1} = \sum_{j,k,r=0}^m \mathbb{1}_{(g_1 g_2 \in x^{-1}_j H x_1)} \mathbb{1}_{(g_2' \in x^{-1}_k H x_1)} v_{g_1 g_2 x_1^{-1} x_2^{-1}} v_{g_2 x_1^{-1} x_2^{-1}} \otimes e_{j,k}$$

$$= \sum_{j,k,r=0}^m \mathbb{1}_{(g_1 g_2 \in x^{-1}_j H x_1)} \mathbb{1}_{(g_2' \in x^{-1}_k H x_1)} v_{g_1 g_2 x_1^{-1}} \otimes e_{j,k}.$$

**Claim:** the pair $(\varphi, u)$ is a covariant representation for $(A_{G,C}, \alpha)$. To prove the claim, fix $g \in G$. For $a \in (A_{G,C}, g^{-1})$, we must show that $u_g \varphi(a) u_g^{-1} = \varphi(\alpha_g(a))$. By linearity and the decomposition property, we may assume that there are $k \in \{0, \ldots, m\}$ and $g^{-1} \in x^{-1}_k$ such that $a \in A_{x^{-1}_k}$. There exist unique $j \in \{0, \ldots, m\}$ and $h \in H$ such that $g = x_j^{-1} h x_k$. Since $\alpha_g(a)$ belongs to $A_{x_j^{-1}}$, we get

$$\varphi(a) = \alpha_{x_j}(a) \otimes e_{k,k} \text{ and } \varphi(\alpha_g(a)) = \alpha_{x_j}(\alpha_g(a)) \otimes e_{j,j}.$$ 

Using at the first step the uniqueness of $j$, we conclude that

$$u_g \varphi(a) u_g^{-1} = (v_{h} \otimes e_{j,k}) (\alpha_{x_j}(a) \otimes e_{k,k}) (v_{h}^* \otimes e_{j,k}) = \alpha_{h x_k}(a) \otimes e_{j,j} = \alpha_{x_j g}(a) \otimes e_{j,j} = \varphi(\alpha_g(a)),$$

and the claim is proved.

By the universal property of the partial crossed product, there is a homomorphism

$$\psi_\tau: A_{G,T} \rtimes_{\alpha} G \to M_{m+1}(A_{\tau} \rtimes_{\alpha|_H} H).$$

satisfying $(\psi_\tau)(a \delta_g) = \varphi(a) u_g$ whenever $g \in G$ and $a \in (A_{G,T}, g)$. It is clear that $\psi_\tau$ satisfies the formula given in the statement.

**Claim:** the map $\psi_\tau$ is an isomorphism. We construct an inverse map to $\psi_\tau$. Since $H$ is a finite group, the $C^*$-algebra $M_{m+1}(A_{\tau} \rtimes_{\alpha|_H} H)$ is linearly spanned by elements of the form $a v_{h} \otimes e_{j,k}$, for $a \in A_{\tau}$, $h \in H$, and $j, k \in \{0, \ldots, m\}$. Define a linear map

$$\phi_\tau: M_{m+1}(A_{\tau} \rtimes_{\alpha|_H} H) \to A_{G,T} \rtimes_{\alpha} G,$$

by setting $\phi_\tau(a v_{h} \otimes e_{j,k}) = \alpha_{x_j^{-1}}(a) \delta_{x_j^{-1} h x_k}$, for all $a \in A_{\tau}$, for all $j, k = 0, \ldots, m$, and all $h \in H$. It is easily seen that $\phi_\tau \circ \psi_\tau = \text{id}_{A_{G,T} \rtimes_{\alpha} G}$ and $\psi_\tau \circ \phi_\tau = \text{id}_{M_{m+1}(A_{\tau} \rtimes_{\alpha|_H} H)}$, by checking these identities on (linear) generators. We conclude that $\psi_\tau$ is bijective, and hence an isomorphism.

As a first application of **Theorem 5.1**, give an alternative proof for the explicit description of the partial group algebra of a finite group from [9].

**Theorem 5.2.** Let $G$ be a finite group. Then there is a canonical identification

$$C^*_{\text{par}}(G) \cong \bigoplus_{n=1}^{[G]} \bigoplus_{z \in O_n(G)} M_{m+1}(C^*(H_2)).$$

**Proof.** Set $T(G) = \bigcup_{n=1}^{[G]} T_n(G)$, endowed with its canonical partial action of $G$. Recall from Section 6 in [14] that $C^*_{\text{par}}(G)$ can be canonically identified with $C(T(G)) \rtimes G$. On the other hand, we have $C(T(G)) \rtimes G \cong \bigoplus_{n=1}^{[G]} C(T_n(G)) \rtimes G$, and for $n = 1, \ldots, [G]$, the canonical action of $G$ on $C(T_n(G))$ has the $n$-decomposition property by **Proposition 2.6**. It follows from **Theorem 5.1** that

$$C^*_{\text{par}}(G) \cong \bigoplus_{n=1}^{[G]} \bigoplus_{z \in O_n(G)} M_{m+1}(C(\{z\}) \rtimes H_2) \cong \bigoplus_{n=1}^{[G]} \bigoplus_{z \in O_n(G)} M_{m+1}(C^*(H_2)).$$
We close this section with a second application, this time to topological partial actions; see Theorem 5.5. It is well-known that for a finite group action $G \curvearrowright X$ on a locally compact Hausdorff space $X$, freeness is equivalent to the canonical corner map $c: C_0(X)^G \to C_0(X \rtimes G)$ having full range\footnote{This means that $c(C_0(X)^G)$ is a full corner in $C_0(X \rtimes G)$.}. (In particular, $C_0(X)^G$ is Morita equivalent to $C_0(X \rtimes G)$.) This result fails in general for partial actions of finite groups, even if one uses $C_0(X/G)$ instead of $C_0(X)^G$.

Recall that a partial action $\sigma$ of a group $G$ on a topological space $X$ is said to be free if whenever $g \in G$ and $x \in X$ satisfy $\sigma_g(x) = x$, then $g = 1$.

**Example 5.3.** Set $X = \{0, 2\}$ and $U = (0, 1) \cup (1, 2) \subseteq X$, and let $\sigma: U \to U$ be the order-2 homeomorphism given by $\sigma(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x > 1 \end{cases}$. for $x \in U$. Denote by $\alpha$ the partial action of $G = \mathbb{Z}_2$ on $X$ determined by $\sigma$ (also used in Example 4.3). Then $\alpha$ is free, and $C_0(X) \rtimes \alpha G$ is not Morita equivalent to either $C_0(X)^G$ or $C(X/G)$.

**Proof.** That $\alpha$ is free is clear, since $\alpha_1(x) = x$ implies that $x$ belongs to $U$ and $\sigma(x) = x$, which is not possible. Note that $C_0(X)^G \cong C_0((0, 1))$ (see Example 4.3). Moreover, $X/G$ can be naturally identified as a set with $[1, 2]$. The topology on $[1, 2]$ induced by this identification is the usual one when restricted to $(1, 2)$, while a neighborhood base at $\{1\}$ is given by sets of the form $[1, 1 + \delta) \cup (2 - \delta, 2)$ for $\delta > 0$. This is not a Hausdorff topology since 1 and 2 cannot be separated, but $C(X/G)$ can be identified with $C(S^1)$, since any continuous function on $[1, 2]$ must take the same value at 1 and 2.

We show that the $K$-groups of $C_0(X) \rtimes \alpha G$ are not isomorphic to those of either $C_0(X)^G$ or $C(X/G)$, from which it will follow that it is not Morita equivalent to either of them. Note that $K_1(C_0(X)^G) \cong K_1(C(X/G)) \cong \mathbb{Z}$. There is a short exact sequence

$$0 \to C_0(U) \rtimes G \to C_0(X) \rtimes \alpha G \to C([0, 1]) \rtimes \text{trivial } G \to 0,$$

where the partial action of $G$ on $\{0, 1\}$ is the trivial one (with trivial domains). Its crossed product is thus $\mathbb{C} \oplus \mathbb{C}$. Moreover, since $U$ is equivariantly isomorphic to $(0, 2) \times G$ with $\sigma$ corresponding to the product of the identity on $(0, 2)$ and the left translation action on $G$, it follows that $C_0(U) \rtimes G \cong C((0, 2)) \otimes M_2$. The induced six-term exact sequence in $K$-theory takes the following form:

$$0 \to K_0(C_0(X) \rtimes \alpha G) \to \mathbb{Z} \oplus \mathbb{Z} \to K_1(C_0(X) \rtimes \alpha G) \to \mathbb{Z}.$$

If $K_1(C_0(X) \rtimes \alpha G) \cong \mathbb{Z}$, then the bottom-right map is an isomorphism, thus the vertical-right map is zero, and thus $K_0(C_0(X) \rtimes \alpha G) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is not isomorphic to the $K_0$-group of either $C_0(X)^G$ or $C(X/G)$. The proof is finished.

Having identified the crossed product $A_{G, \tau} \rtimes \alpha G$ with $M_{m+1}(A_{\tau} \rtimes \alpha H)$ in the previous theorem, and having identified $A_{G, \tau}^G$ with $A_{\tau}^G$ in Theorem 4.6 in the next lemma we give a description of the corner map $c_\tau: A_{G, \tau}^G \to A_{G, \tau} \rtimes \alpha G$ from Theorem 4.8.

**Lemma 5.4.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha = ((A_g)_{g \in G}, (\alpha_g)_{g \in G})$ be a partial action of $G$ on $A$ with the $n$-decomposition property. For $\tau \in T_n(G)$, let $\psi: A_{G, \tau} \rtimes \alpha G \to M_{m+1}(A_{\tau} \rtimes \alpha H)$ and $\phi: A_{G, \tau}^G \to A_{\tau}^G$ be the isomorphisms from Theorem 5.1 and Theorem 4.6 respectively. Let

$$c_\tau: A_{G, \tau}^G \to A_{G, \tau} \rtimes \alpha G \text{ and } c_H: A_{\tau}^H \to A_{\tau} \rtimes \alpha H$$
be the canonical corner embeddings described in Theorem 4.8. (Note that $H \triangleleft A$, has the $|H|$-decomposition property, since it is a global action.) Denote by $e \in M_{m+1}$ the rank-one projection given by $e = \frac{1}{m+1} \sum_{j,k=0}^m e_{j,k}$. Then the composition

$$\psi_\tau \circ c_\tau \circ \phi_\tau^{-1} : A^H \to M_{m+1}(A_\tau \rtimes_H H)$$

is given by $(\psi_\tau \circ c_\tau \circ \phi_\tau^{-1})(a) = c_H(a) \otimes e$ for all $a \in A^H$.

**Proof.** Let $a \in A^H$. Then $\phi_\tau^{-1}(a) = \sum_{j=0}^m a_{x_j}^{-1}(a)$. Then

$$c_\tau(\phi_\tau^{-1}(a)) = \frac{1}{|H| (m+1)} \sum_{j,k=0}^m \sum_{h \in H} a_{x_j}^{-1}(a) \delta_{x_j^{-1} h x_k},$$

and applying $\psi_\tau$ to the above expression gives

$$\psi_\tau \circ c_\tau \circ \phi_\tau^{-1}(a) = \frac{1}{|H| (m+1)} \sum_{j,k=0}^m (a \otimes e_{j,j})(\psi_h \otimes e_{j,k})$$

$$= \frac{1}{|H| (m+1)} \sum_{j,k=0}^m a_{\psi_h} \otimes e_{j,k}$$

$$= \left( \frac{1}{|H|} \sum_{h \in H} a_{\psi_h} \right) \otimes \left( \frac{1}{m+1} \sum_{j,k=0}^m e_{j,k} \right) = c_H(a) \otimes e. \quad \square$$

In the next theorem we obtain the desired characterization of freeness in terms of the map $c$. We also show that for partial actions with the decomposition property, $C_0(X)^G$ is canonically isomorphic to $C_0(X/G)$, while we have seen that this fails in general for partial actions even of finite groups; see Example 5.3.

**Theorem 5.5.** Let $G$ be a discrete group, let $X$ be a locally compact Hausdorff space, and let $\sigma = ((X_g)_{g \in G}, (\sigma_g)_{g \in G})$ be a partial action with the decomposition property. Then:

1. There is a natural isomorphism $C_0(X)^G \cong C_0(X/G)$.
2. $\sigma$ is free if and only if the corner embedding $c : C_0(X)^G \to C_0(X) \rtimes_{\sigma} G$ from Theorem 4.8 has full range.

**Proof.** Let $n \in \mathbb{N}$ be such that $\alpha$ has the $n$-decomposition property. For $\tau \in T_n(G)$, we set $X_\tau = \bigcap_{g \in G} X_g$ and $X_{G,\tau} = \bigcup_{g \in G} X_{g^{-1} \tau}$. By the $n$-decomposition property, the sets $X_\tau$ are pairwise disjoint, and there is an equivariant disjoint-union decomposition $X = \bigsqcup_{\tau \in T_n(G)} X_{G,\tau}$. We adopt the conventions from Notation 2.8.

1. It suffices to prove the statement for the restriction of $\sigma$ to $X_{G,\tau}$. We define a function $\varphi : X_\tau / H \to X_{G,\tau} / G$ given by $\varphi(\text{orb}_H(x)) = \text{orb}_G(x)$ for all $x \in X_\tau$. Note that $\varphi$ is well-defined. We claim that $\varphi$ is a homeomorphism.

Let $x, y \in X_\tau$ satisfy $\text{orb}_G(x) = \text{orb}_G(y)$, and let $g \in G$ satisfy $g \cdot x = y$. Then $g \cdot \tau = \tau$, and thus $g \in H$. We conclude that $\text{orb}_H(x) = \text{orb}_H(y)$ as desired. To check surjectivity, let $x \in X_{G,\tau}$. Choose $j = 0, \ldots, m$ such that $x \in X_{x_j^{-1} \tau}$. Then $x_j \cdot x$ belongs to $X_\tau$, and

$$\text{orb}_G(x) = \text{orb}_G(x_j \cdot x) = \varphi(\text{orb}_H(x_j \cdot x)),$$

as desired. Finally, we show that $\varphi$ is open. Consider the commutative diagram

$$\begin{array}{ccc}
X_\tau & \xrightarrow{\pi_H} & X_{G,\tau} \\
\downarrow{\pi_G} & & \downarrow{\pi_G} \\
X_\tau / H & \xrightarrow{\varphi} & X_{G,\tau} / G,
\end{array}$$

where $\pi_H$ and $\pi_G$ are the canonical quotient maps. Both these maps are open and finite-to-one: this is clear for $\pi_H$, while for $\pi_G$ it follows from the fact that the orbits of $G \cap X_{G,\tau}$ are finite. Since the inclusion $X_\tau \hookrightarrow X_{G,\tau}$ is open, we deduce that $\varphi$ is open, as desired.
Using [Theorem 4.8] at the first step; using that $H \rhd X_\tau$ is a global action of a finite group at the second step; and using the above claim at the third step, we obtain the following natural isomorphisms:

$$C_0(X_G)_\tau^G \cong C_0(X_\tau)^H \cong C_0(X_\tau)_H \cong C_0(X_{G_\tau}).$$

(2). We begin by proving that $\sigma$ is free if and only if $H_\tau \rhd \alpha^G_{\tau^1} X_\tau$ is free for every $\tau \in T_n(G)$. By the preceding comments, it suffices to show, for a fixed $\tau \in T_n(G)$, that the partial action of $G$ on $X_{G_\tau}$ is free if and only if $H_\tau \rhd X_\tau$ is free. Note that the “only if” implication is immediate. Conversely, assume that $H_\tau \rhd X_\tau$ is free, and let $g \in G$ and $x \in X_{G_\tau} \cap X_\tau = \bigsqcup_{0 \leq j \leq m} g \in x_j^{-1} \tau X_{x_j^{-1} \tau}$ satisfy $\sigma_{g^{-1}}(x) = x$. There exists a unique $j \in \{0, \ldots, m\}$ such that $x \in X_{x_j}^{-1} \tau$ and $g \in x_j^{-1} \tau$. As $x_j^{-1} \tau = \bigsqcup_{k=0}^m x_j^{-1} H x_k$, we let $k \in \{0, \ldots, m\}$ and $h \in H$ be the unique elements such that $g = x_j^{-1} h x_k$. Then $x = \sigma_{g^{-1}}(x) \in \sigma_{g^{-1}}(X_{x_j}^{-1} \tau) \subseteq X_{x_k}^{-1} \tau$. It follows that $j = k$. Set $y = \sigma_{x_j}(x) \in X_\tau$. Then

$$\sigma_h(y) = \sigma_{hx_j}(x) = \sigma_{x_j}(\sigma_{x_j^{-1} h x_j}(x)) = \sigma_{x_j}(x) = y.$$ 

Since the action $H_\tau \rhd X_\tau$ is free, it follows that $h = 1$ and hence $g = x_j^{-1} h x_j = 1$.

We turn to the statement of the theorem. We have just shown that $\sigma$ is free if and only if $H_\tau \rhd X_\tau$ is free for every $\tau \in T_n(G)$. Since these are global actions of finite groups on locally compact Hausdorff spaces, this is in turn equivalent to the canonical corner map $C_0(X_\tau)_H \to C_0(X_\tau) \rtimes H_\tau$ having full range for every $\tau \in T_n(G)$ (see Proposition 7.1.12 and Theorem 7.2.6 in [23], noting that saturation means precisely that the corner embedding from the fixed point algebra into the crossed product has full range). By [Lemma 5.3] this is itself equivalent to the canonical corner map $c_\tau: C_0(X_G)_\tau^G \to C_0(X_{G_\tau}) \rtimes_{\alpha} G$ from [Theorem 4.8] having full range, for all $\tau \in T_n(G)$. In turn, this is equivalent to the canonical corner map $c: C_0(X)^G \to C_0(X) \rtimes_{\alpha} G$ having full range, as desired. \hfill \Box

6. Decompositions of partial actions of finite groups

In this final section, we show that any partial action of a finite group can be written as an iterated extension of decomposable partial actions; see [Theorem 6.1]. This allows us to reduce the computation of such partial crossed products to an extension problem for global crossed products. As an application, we prove that crossed products of partial actions of finite groups preserve the property of having finite stable rank; see [Corollary 6.2].

**Theorem 6.1.** Let $G$ be a finite group, let $A$ be a $C^*$-algebra, and let $\alpha$ be a partial action of $G$ on $A$. Then there are canonical equivariant extensions

$$0 \longrightarrow (D(k), \delta(k)) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$, with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$ and satisfying the following properties:

(a) $\delta^{(k)}$ has the $k$-decomposition property;

(b) $A_\alpha^{(k)} = \{0\}$ for all $\sigma \in T_{k+1}(G)$;

(c) $\alpha^{(1)}$ has the 1-decomposition property.

Thus $\alpha$ can be written canonically as an iterated extension of decomposable partial actions.

**Proof.** Set $n = |G|$ and take $(A^{(n)}, \alpha^{(n)}) = (A, \alpha)$. Define $D^{(n)} = \bigcap_{g \in G} A_g^{(n)}$, which is a $G$-invariant ideal in $A^{(n)}$. The restriction $\delta^{(n)}$ of $\alpha^{(n)}$ to $D^{(n)}$ is a global action. By item (2) in [Example 2.5], we deduce that $\delta^{(n)}$ has the $n$-decomposition property. Denote by $A^{(n-1)}$ the quotient of $A^{(n)}$ by $D^{(n)}$, and let $\pi^{(n)}: A^{(n)} \to A^{(n-1)}$ be the canonical equivariant quotient map. Then

$$\bigcap_{g \in G} A_g^{(n-1)} = \pi^{(n)} \left( \bigcap_{g \in G} A_g^{(n)} \right) = \pi^{(n)}(D^{(n)}) = 0.$$
In other words, $A_{\sigma}^{(n-1)} = \{0\}$ for all $\sigma \in T_n(G) = \{G\}$. Assuming that $(A^{(k)})_{\alpha^{(k)}}$ has been constructed and satisfies condition (b) above, we will construct $(D^{(k)}, \delta^{(k)})$ and $(A^{(k-1)}, \alpha^{(k-1)})$. We set $D^{(k)} = \sum_{\tau \in T_k(G)} A^{(k)}_\tau$, which is a $G$-invariant ideal in $A^{(k)}$. Let $\delta^{(k)}$ be the induced action on it; we claim that $\delta^{(k)}$ has the $k$-decomposition property.

For $g \in G$, we have $D^{(k)}_g = \sum_{\tau \in T_k(G)} A^{(k)}_\tau$, and hence $D^{(k)}_\tau = A^{(k)}_\tau$ for all $\tau \in T_k(G)$. It follows that $D^{(k)}$ satisfies condition (a) in [Definition 2.3] in order to verify condition (b), let $\tau \in T_k(G)$ and let $g \notin \tau$. Set $\sigma = \tau \cup \{g\}$, which is a tuple in $T_{k+1}(G)$. Using condition (b) of the inductive step at the first step, we get

$\{0\} = A^{(k)}_0 = A^{(k)}_\tau \cap A^{(k)}_g = D^{(k)}_\tau \cap A^{(k)}_g$.

Since $D^{(k)}_\tau \subseteq A^{(k)}_\tau$, it follows that $D^{(k)} \cap A^{(k)}_g = \{0\}$ as desired. Thus $\delta^{(k)}$ has the $k$-decomposition property. Let $A^{(k-1)}$ denote the quotient of $A^{(k)}$ by $D^{(k)}$, and let $\pi^{(k)} : A^{(k)} \to A^{(k-1)}$ be the canonical equivariant quotient map. Then

$\sum_{\tau \in T_k(G)} A^{(k-1)}_\tau = \pi^{(k)} \left( \sum_{\tau \in T_k(G)} A^{(k)}_\tau \right) = \pi^{(k)}(D^{(k)}) = 0$.

In other words, $A^{(k-1)} = \{0\}$ for all $\tau \in T_k(G)$. We have thus established conditions (a) and (b) in the statement. Condition (c) follows from taking $k = 1$ in condition (b), since in this case we have $A^{(1)}_g = \{0\}$ for all $g \in G \setminus \{1\}$, which is equivalent to $\alpha^{(1)}$ having the 1-decomposition property by item (1) in [Example 2.3]. This concludes the proof. □

It follows that crossed products of partial actions of finite groups can be written as iterated extensions of crossed products of actions with the decomposition property, which in turn can be explicitly computed using [Theorem 5.1]. This fact has several applications to the structure of partial crossed products. We summarize a relatively direct consequence in the following corollary, while more involved applications are presented in [5].

**Corollary 6.2.** Let $G$ be a finite group, and let $P$ be a property for $C^*$-algebras which passes to ideals, quotients, extensions, is stable under tensoring with matrix algebras, and is preserved by formation of crossed products by global actions of $G$. Then $P$ is preserved by formation of crossed products by partial actions of $G$. This applies, in particular, to the properties of being nuclear, having finite stable rank, or being of type I.

**Proof.** Let $(D^{(k)}, \delta^{(k)})$ and $(A^{(k)}, \alpha^{(k)})$ be partial dynamical systems as in the conclusion of [Theorem 6.1]. For $k = 2, \ldots, |G|$, we apply crossed products to get the extension

$(6.1) \quad 0 \longrightarrow D^{(k)} \rtimes_{\delta^{(k)}} G \longrightarrow A^{(k)} \rtimes_{\alpha^{(k)}} G \longrightarrow A^{(k-1)} \rtimes_{\alpha^{(k-1)}} G \longrightarrow 0$.

Note that $A^{(k)}$ satisfies $P$ for all $k = 1, \ldots, |G|$, being a quotient of $A = A^{(|G|)}$, and that the same is true for $D^{(k)}$ for $k = 2, \ldots, |G|$, being an ideal in $A^{(k)}$.

We will show by induction that $A^{(k)} \rtimes_{\alpha^{(k)}} G$ satisfies $P$. For $k = 1$, we have $A^{(1)} \rtimes_{\alpha^{(1)}} G = A^{(1)}$, which satisfies $P$. Assume now that we have proved the claim for $k - 1$, and let us prove it for $k$. Since $P$ passes to extensions, the exact sequence in (6.1) implies that it suffices to show that $D^{(k)} \rtimes_{\delta^{(k)}} G$ satisfies $P$. Since $\delta^{(k)}$ has the $k$-decomposition property, it follows from [Theorem 5.1] that $D^{(k)} \rtimes_{\delta^{(k)}} G$ is isomorphic to a finite direct sum of algebras of the form $M_m(D^{(k)} \rtimes_{\delta^{(k)}} H_\tau)$, for $\tau \in T_k(G)$, where $D^{(k)}_\tau$ is an ideal in $D^{(k)}$, $H_\tau$ is a subgroup of $G$, and $\delta^{(k)}_\tau$ is the global action obtained as the restriction of $\delta^{(k)}$ to $H_\tau$ and to $D^{(k)}_\tau$. This proves the claim, since $P$ passes to direct sums, is stable under tensoring with matrix algebras, and is preserved by formation of global crossed products, by assumption.

Since $A \rtimes_{\alpha} G$ equals $A^{(|G|)} \rtimes_{\alpha^{(|G|)}} G$, the claim shows the first assertion in the statement.

Nuclearity, finiteness of the stable rank, and being of type I are known to satisfy the properties in the statement (see [20] Sections 4–6) for permanence properties of stable
is easily seen to be equivalent short exact sequence\(\pi\)

Thus \(\pi\)\(A/I\) (see item (1) in Example 2.5), so we have

partial actions of finite groups do not preserve exact sequences.

several helpful comments, and in particular for pointing out that

fixed point algebras of partial actions of finite groups preserves short exact sequences, as we briefly note in the following example:

Example 6.3. Set \(X = [-1, 1]\) and \(U = (-1, 1)\), and let \(\alpha\) be the partial action of \(\mathbb{Z}_2\)
on \(A = C(X)\) induced by the homeomorphism \(\sigma\) of \(U\) given by \(\sigma(x) = -x\) for all \(x \in U\).

Then \(I = C_0(U)\) is an \(\alpha\)-invariant ideal in \(A\), and the restriction of \(\alpha\) to it is the global action \(\alpha^*\).

The induced partial action of \(\mathbb{Z}_2\) on \(A/I \cong \mathbb{C}^2\) is the trivial partial action (see item (1) in Example 2.5), so we have \((A/I)^G = A/I\). Moreover, the quotient map \(\pi: A \to A/I\) is given by \(\pi(f) = (f(-1), f(1))\) for \(f \in A\), and the fixed point algebra \(A^G\) is easily seen to be

\[A^G = \{f \in C([-1, 1]): f(x) = f(-x) \text{ for all } x \in [-1, 1]\}\]

Thus \(\pi(A^G) = [(\lambda, \lambda) \in \mathbb{C}^2: \lambda \in \mathbb{C}]\) does not equal \((A/I)^G = A/I\). It follows that the equivariant short exact sequence

\[0 \longrightarrow (C_0(U), \sigma^*) \longrightarrow (C(X), \alpha) \longrightarrow (\mathbb{C}^2, \text{trivial}) \longrightarrow 0\]

does not induce a short exact sequence of fixed point algebras.

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