ON A CONJECTURE OF SUN ABOUT SUMS OF RESTRICTED SQUARES

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ABSTRACT. In this paper, we investigate sums of four squares of integers whose prime factorizations are restricted, making progress towards a conjecture of Sun that states that two of the integers may be restricted to the forms $2^a3^b$ and $2^c5^d$. We obtain an ineffective generalization of results of Gauss and Legendre on sums of three squares and an effective generalization of Lagrange’s four-square theorem.

1. Introduction And Statement Of Results

The study of representations of integers by sums of integral squares goes back to antiquity and has a storied history. To give one famous example, Legendre (in 1797) and Gauss (in 1796–1801) separately proved that every natural number not of the form $4q(8\ell + 7)$ can be represented as the sum of three squares of non-negative integers. Moreover, Gauss’ work culminated in a formula that relates the number of representations $r_3(m)$ of $m$ as a sum of three squares to a class number of an associated imaginary quadratic field. Letting $H(D)$ denote the Hurwitz class number, Gauss’ result may be stated as

$$r_3(m) = \begin{cases} 12H(-4m) & \text{if } n \equiv 1, 2 \pmod{4}, \\ 24H(-m) & \text{if } n \equiv 3 \pmod{8}, \\ r_3(\frac{m}{4}) & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 7 \pmod{8}. \end{cases}$$

Prior to Gauss and Legendre’s works on sums of three squares, Lagrange established in 1770 that every natural number can be represented as the sum of four squares of non-negative integers. Jacobi later found a formula in 1834 analogous to that of Gauss for the number of representation $r_4(m)$ of $m$ as a sum of four square, yielding

$$r_4(m) = 8 \sum_{d | m, 4 | d} d. \quad (1.1)$$

Lagrange’s four-square theorem and Jacobi’s formula (1.1) have been generalized in numerous directions through the years, with some results extending the types of sums being taken and others restricting the integers being squared into certain subsets. Along this vein, Sun recently stated a four square conjecture where some of the integers are restricted to be products of powers of 2, 3, and 5, as stated below.

Conjecture. Every $n = 2, 3, \cdots$ can be written as $x^2 + y^2 + (2^a3^b)^2 + (2^c5^d)^2$, where $x, y, a, b, c, d$ are non-negative integers.

This conjecture seems to be out of reach with current techniques, but, following results of Brüdern–Fouvry [2], some progress can be made in restricting the number of prime divisors of the last two squares. The goal of this article is to demonstrate how to use such techniques to generalize both Gauss/Legendre’s three-square theorem and Lagrange’s four-square theorem in the direction of Sun’s four square conjecture. The following result generalizes Gauss’ three square theorem ineffectively in the sense that every sufficiently large integer not of the form $4q(8\ell + 7)$ can be represented by sum of three squares where the last variable can be restricted to almost prime inputs. Here an almost prime of order $n$ is a product of at most $n$ primes.

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Theorem 1.1. Every sufficiently large integer \( m \) not of the form \( 4^q (8\ell + 7) \) can be represented in the form
\[
m = x^2 + y^2 + (2^a z)^2,
\]
where \( x, y, a \) and \( z \) are any integers with a non-negative and \( z \) has at most 118 prime factors. Moreover, the number of such representation exceeds \( cm^{1/2-\epsilon} (\log m)^{-1} \) for some positive constant \( c \).

The next result provides a generalization of Lagrange’s four square theorem ineffectively in the sense that every sufficiently large integer can be represented by sum of four squares with restricted inputs.

Corollary 1.2. Every sufficiently large integer \( m \) can be represented in the form
\[
m = x^2 + y^2 + 2^a + (2^b z)^2
\]
where \( x, y, a, b \) and \( z \) are any integers with \( a, b \) non-negative and \( z \) has at most 118 prime factors.

The proof of the above corollary follows from Theorem 1.1 by a simple argument. Namely, writing \( m = 4^k m' \) with \( 4 \nmid m' \), we can choose \( a \) such that \( m' - 4a \not\equiv 7 \pmod{8} \). Hence Corollary 1.2 follows from Theorem 1.1.

In connection to Sun’s four square conjecture, the following result holds directly from Theorem 1.1.

Corollary 1.3. Every sufficiently large integer \( m \) can be represented in any of the following form :
\[
\begin{align*}
m &= x^2 + y^2 + (2^a 3^b z_1)^2 + (2^c 5^d z_2)^2, \\
&\text{or} \\
m &= x^2 + y^2 + (2^a 3^b z_1)^2 + (2^c 5^d z_2)^2,
\end{align*}
\]
where \( x, y, a, b, c, d \) are non-negative integers and \( z_1, z_2 \) each either vanish or have at most 369 prime factors.

Combining Gauss’s result with Siegel’s lower bound for the class numbers \( \text{[13]} \) yields that for arbitrarily small \( \epsilon_1 > 0 \)
\[
r_3(m) \gg h(-m) \gg m^{1/2-\epsilon_1}, \tag{1.2}
\]
where \( h(-m) \) denotes the class number of the number field \( \mathbb{Q}(\sqrt{-m}) \). However, Siegel’s lower bound is ineffective, so the bound on \( m \) for which Theorem 1.1 is not effective. Under the assumption of the generalized Riemann hypothesis, Siegel’s result can be made effective and Ono and Soundarajan \( \text{[8]} \) worked out an explicit bound in order to obtain an conjectural proof of a conjecture of Ramanujan about sums of the form \( x^2 + y^2 + 10z^2 \). Following this method, an effective but conjectural version of Theorem 1.1 can be obtained.

By further relaxing the conditions on the last two integers being squared, we obtain an effective unconditional version of Theorem 1.1. Moreover, using a quantitative version \( \text{[1]} \) of results of Briënd–Fouvry \( \text{[3]} \) (see also \( \text{[14]} \) for th current state of the art), one can make this effective constant explicit, leading to the conclusion that indeed every integer may be written in a certain shape.

Theorem 1.4. Every natural number \( m \) can be represented in the form of
\[
m = x^2 + y^2 + (2^a 3^b z_1)^2 + (2^c 5^d z_2)^2,
\]
where \( x, y, a, b, c, d \) are non-negative integers and \( z_1, z_2 \) each either vanish or have at most 369 prime factors.

The paper is organized as follows. In \( \text{§2} \) we introduce the preliminaries needed for the rest of the paper. In \( \text{§3} \) we prove bounds on the coefficients of theta functions. We apply a linear sieve to prove Theorem 1.1 in \( \text{§4} \) In \( \text{§5} \) we obtain the bounds required to prove Theorem 1.4. Finally, Theorem 1.4 is proved in \( \text{§6} \).

2. Preliminaries

Let \( Q \) be any \( \ell \)-ary positive definite integer valued diagonal quadratic form with \( \ell \geq 3 \) and \( r_Q(m) \) denotes the number of solutions to \( Q(x) = m \) for \( x \in \mathbb{Z}^\ell \).
2.1. **Theta function.** Let $\mathbb{H}$ be the complex upper half-plane. For $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$, the theta function associated to the quadratic form $Q$ can be defined as
\[
\Theta_Q(\tau) := \sum_{m \geq 0} r_Q(m) q^m = \sum_{x \in \mathbb{Z}} q^{Q(x)}.
\]
The theta function $\Theta_Q$ is a modular form of weight $\ell/2$ on a particular congruence subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ with a certain Nebentypus $\chi$ (cf. [11, Proposition 2.1]). It naturally decomposes as
\[
\Theta_Q = E + f,
\]
where $E$ is an Eisenstein series of weight $\ell/2$ on $\Gamma$ with Nebentypus $\chi$ and $f$ is a cusp form of same weight $\ell/2$ on $\Gamma$ with Nebentypus $\chi$. It turns out that the $m$th coefficient $a_E(m)$ of $E$ grows faster than the $m$th coefficient $a_f(m)$ of $f$ for $\ell \geq 3$ and for those $m$ for which $a_E(m)$ is non-zero. Moreover, it can be shown that $a_E(m) > 0$ if and only if $m$ is represented locally i.e. represented modulo any natural number. Therefore, $E$ behaves as the main term of $\Theta_Q$ and $f$ as the error term, with $r_Q(m) > 0$ for sufficiently large $m$ that are locally represented.

2.2. **Eisenstein series part.** Siegel mainly used the ideas to give quantitative meaning to the Fourier coefficients of Eisenstein series associated to the quadratic form $Q$ both in terms of the underlying space of modular forms and also in terms of local densities associated to $Q$. The Eisenstein series
\[
E(\tau) := \sum_{m \geq 0} a_E(m) q^m
\]
can be expressed in two different ways:

Firstly, for $G(Q)$ denoting a set of representatives of the classes in the genus of $Q$ and $w_Q$ denoting the number of automorphs of $Q$, the Eisenstein series $E$ can be recovered as a weighted sum of theta series over $G(Q)$ by
\[
E = \frac{1}{\sum_{Q' \in G(Q)} w_Q' \prod_{Q'' \in G(Q)} \Theta_{Q''}} \sum_{Q' \in G(Q)} \frac{\Theta_{Q'}}{w_Q'}.
\]
This famous identity is known as Siegel – Weil average (the first identity is due to Siegel [12] and a generalization by Weil [15]).

We recall here local representation densities for an $\ell$-ary quadratic form $Q$ at $m$ which can be defined by the limit
\[
\beta_{Q,p}(m) := \lim_{U \to \{m\}} \frac{\text{vol}_{\mathbb{Z}_p}(Q^{-1}(U))}{\text{vol}_{\mathbb{Z}_p}(U)},
\]
where $U \subseteq \mathbb{Z}_p$ runs over open subsets of $\mathbb{Z}_p$ containing $m$ and for $p = \infty$ we have open subsets of $\mathbb{R}$. The Fourier coefficients $a_E(m)$ of $E$ can be expressed as an infinite local product
\[
a_E(m) = \prod_p \beta_{Q,p}(m),
\]
where the product runs over all the primes including $\infty$.

3. **Representation of sufficiently large integer by certain ternary quadratic forms**

In this section, we will set up the background and notations for Theorem [11]. The set to be sieved is the following
\[
A = \{z \in \mathbb{N} : x^2 + y^2 + (2^a z)^2 = m\},
\]
where $x, y, a$ are any non-negative integers. Setting $x_1 = x$, $x_2 = y$ and $x_3 = 2^a z$, we have
\[
A = \{x_3 \in \mathbb{N} : x_1^2 + x_2^2 + x_3^2 = m\}.
\]
In order to apply sieve theory, we need an asymptotic formula for the cardinality of the set
\[
A_d = \{x_3 \in A : x_3 \equiv 0 (\text{mod } d)\} = \{x_3 \in \mathbb{N} : x_1^2 + x_2^2 + d^2 x_3^2 = m\},
\]
where \( 2 \nmid d \). It needs some preparation to express the above cardinality in terms of main term and error term. We first consider the quadratic form

\[
Q_d(x) = x_1^2 + x_2^2 + d^2x_3^2,
\]

where \( x = (x_1, x_2, x_3) \) and \( 2 \nmid d \). For simplicity, we abbreviate \( Q_1 = Q \). Let, the theta function associated to \( Q_d \) be \( \Theta_{Q_d} \) with the Fourier expansion

\[
\Theta_{Q_d}(\tau) = \sum_{m \geq 0} r_{Q_d}(m)q^m
\]

for \( \tau \in \mathbb{H} \). Here the Fourier coefficient \( r_{Q_d}(m) \) denotes the number of representation of \( m \) by \( Q_d \). As mentioned earlier, \( \Theta_{Q_d} \) can be decomposed into two parts which are the Eisenstein series part and the cuspidal part respectively.

3.1. Eisenstein series contribution. We denote the Eisenstein series part associated to \( Q_d \) by \( E_d \). It follows from \(^{(2.3)}\) that the \( m \)-th Fourier coefficient of \( E_d \) can be expressed as

\[
a_{E_d}(m) = \prod_p \beta_{Q_d,p}(m),
\]

where the product runs over all the primes including \( \infty \). The definition \(^{(2.2)}\) of local representation density yields

\[
\beta_{Q_d,p}(m) = \lim_{U \to \{m\}} \frac{\text{vol}_\mathbb{Z}_p(Q^\tau(U))}{\text{vol}_\mathbb{Z}_p(U)},
\]

where \( U \subseteq \mathbb{Z}_p \) runs over open subsets of \( \mathbb{Z}_p \) containing \( m \) and for \( p = \infty \) we have open subsets of \( \mathbb{R} \). For \( p \neq \infty \), the local representation density may be realized by choosing \( U \) to be a ball of radius \( p^{-\tau} \) around \( m \), in which case we may write

\[
\beta_{Q_d,p}(m) = \lim_{r \to \infty} \frac{|R_{Q_d,p'}(m)|}{p^{2r}}
\]

with

\[
R_{Q_d,p'}(m) := \{ x \in (\mathbb{Z}/p^r\mathbb{Z})^3 : Q_d(x) \equiv m \mod p^r \}.
\]

The following lemma is crucial to determine the local density of \( Q_d \) at \( p = 2 \).

**Lemma 3.1.** For \( p \nmid d \), the local density of \( Q_d \) satisfies

\[
\beta_{Q_d,p} = \beta_{Q,p}.
\]

In particular, we have

\[
\beta_{Q_d,2} = \beta_{Q,2}.
\]

**Proof.** The proof of the lemma is straightforward. For \( p \nmid d \), there exist \( d^{-1} \in \mathbb{Z} \) such that \( dd^{-1} \equiv 1 \mod p \). It follows that there exist a bijection between the sets \( R_{Q,p'}(m) \) and \( R_{Q_d,p'}(m) \) under the map \((x_1, x_2, x_3) \mapsto (x_1, x_2, d^{-1}x_3)\). This implies \(|R_{Q,p'}(m)| = |R_{Q_d,p'}(m)|\). We now use \(^{(3.2)}\), which yields the lemma. The lemma follows in particular for \( p = 2 \) since by assumption we have considered \( d \) to be odd.

We fix some notations here before proceeding to the next lemma. Set for any \( d \) odd,

\[
\varepsilon_d := \begin{cases} 
1 & \text{for } d \equiv 1 \pmod{4}, \\
i & \text{for } d \equiv 3 \pmod{4}.
\end{cases}
\]

Let the symbols \((\cdot)_d\) and \([\cdot]\) denote the Legendre–Jacobi–Kronecker symbol and the greatest integer function respectively. We set \( d = p^dd' \) with \( p \nmid d' \) and \( m = p^mm' \) with \( p \nmid m' \). Let \( \delta_\ell \) denotes the standard characteristic function which counts 1 if \( \ell \) happens and vanishes otherwise. In the next lemma, we compute the local representation density \( \beta_{Q_d,p} \) at each odd prime \( p \).
Lemma 3.2. For any odd prime \( p \), we have

\[
\beta_{Q_d,p}(m) = \begin{cases} 
1 + (1 - p^{-1})(\left[ \frac{R}{p} \right] + \epsilon_p^2 \left[ \frac{R+1}{2} \right]) - \delta_{2|R} p^{-1} - \delta_{2|R} \epsilon_p^2 p^{-1} & \text{for } R < 2\alpha, \\
1 + (1 + \epsilon_p^2)(1 - p^{-1})\alpha + p^{-1} - p^{\alpha-1-\frac{R}{2}} \\
- \delta_{2|R} p^{\alpha-1-\frac{R}{2}} + \delta_{2|R} \epsilon_p^2 p^{\alpha-1-\frac{R}{2}} \left( \frac{m!}{p} \right) & \text{for } R \geq 2\alpha.
\end{cases}
\]

Proof. The orthogonality of roots of unity, namely

\[
\frac{1}{p^r} \sum_{n \pmod{p^r}} \epsilon^{2\pi i n m/p^r} = \begin{cases} 
1 & \text{if } p^r \mid m, \\
0 & \text{otherwise},
\end{cases}
\]

leads to

\[
|R_{Q_d,p'}(m)| = \sum_{x \in (\mathbb{Z}/p'\mathbb{Z})^3 \atop Q_d(x) = m \pmod{p'}} \frac{1}{p^r} \sum_{n \pmod{p^r}} e^{2\pi i n m/p^r} = \begin{cases} 
1 & \text{if } p^r \mid m, \\
0 & \text{otherwise},
\end{cases}
\]

where in the last step we used the definition of the quadratic Gauss sum, which is given by

\[
G_2(A, B, C) := \sum_{x \pmod{c}} e^{2\pi i (Ax^2 + Bx)}.
\]

We next split the sum over \( n \) by writing \( n = p^kn' \) with \( p \nmid n' \) and then make the change of variables \( k \mapsto r - k \). The fact that

\[
G_2(gA, gB, gC) = gG_2(A, B, C)
\]

yields

\[
|R_{Q_d,p'}(m)| = \frac{1}{p^r} \sum_{k=0}^{r} p^{3k} \sum_{n' \in (\mathbb{Z}/p'^{-k}\mathbb{Z})^3} e^{2\pi i n'm/p'^{-k}} G_2(n', 0, p'r^{-k})^2 G_2(n'd^2, 0, p'^{-k})
\]

For \( d = p^ad' \) with \( p \nmid d' \), it follows from (3.3) that

\[
G_2(n'd^2, 0, p^k) = (p^{2\alpha}, p^k) G_2 \left( \frac{n'd^2}{(p^{2\alpha}, p^k)}, 0, \frac{p^k}{(p^{2\alpha}, p^k)} \right)
\]

\[
= \begin{cases} 
p^k & \text{for } k \leq 2\alpha, \\
p^{2\alpha} G_2(n'd^2, 0, p^{k-2\alpha}) & \text{for } k > 2\alpha.
\end{cases}
\]
We can therefore rewrite
\[
\frac{1}{p^{2r}} |R_{Q_d,p^r}(m)| = \sum_{k=0}^{2\alpha} p^{-2k} \sum_{n' \in (\mathbb{Z}/p^k\mathbb{Z})^\times} e^{-\frac{2\pi i n'm}{p^k}} G_2(n', 0, p^k)^2 \\
+ \sum_{k=2\alpha+1}^{r} p^{-3k+2\alpha} \sum_{n' \in (\mathbb{Z}/p^k\mathbb{Z})^\times} e^{-\frac{2\pi i n'm}{p^k}} G_2(n', 0, p^k)^2 G_2(n'd^2, 0, p^{k-2\alpha}). \tag{3.6}
\]

Utilizing the fact that for \(C\) odd and for gcd(\(A, C\)) = 1,
\[
G_2(A, 0, C) = \varepsilon_C \sqrt{C} \left( \frac{A}{C} \right),
\]
we have, for \(p \neq 2\),
\[
G_2(n', 0, p^k) = \begin{cases} 
\frac{p^k}{\varepsilon_p(n'} & \text{if } k \equiv 0 \pmod{2}, \\
\varepsilon_p \left( \frac{n'}{p} \right) p^k & \text{if } k \equiv 1 \pmod{2}.
\end{cases} \tag{3.7}
\]
and
\[
G_2(n'd^2, 0, p^{k-2\alpha}) = G_2(n', 0, p^{k-2\alpha}) = \begin{cases} 
\frac{p^{k-\alpha}}{\varepsilon_p} & \text{if } k \equiv 0 \pmod{2}, \\
\varepsilon_p \left( \frac{n'}{p} \right) p^{k-\alpha} & \text{if } k \equiv 1 \pmod{2}.\end{cases} \tag{3.8}
\]

For a multiplicative character \(\chi\) and an additive character \(\psi\), both of modulus \(c\), we set
\[
\tau(\chi, \psi) := \sum_{x \pmod{c}} \chi(x)\psi(x).
\]

Let \(\chi = \chi_{a,b}\) denote a character of modulus \(b\) induced from a character of conductor \(a\). For simplicity, we abbreviate \(\chi_{a,a} = \chi_a\) (we will always have either the principal character \(\chi_{1,p^k}\) or the real Dirichlet character \(\chi_{p,p^k} = \left( \frac{\cdot}{p} \right)\) coming from the Legendre symbol) and take \(\psi(x) = \psi_{m,p^k}(x) := e^{\frac{2\pi i n'm}{p^k}}\). Inserting (3.7) and (3.8) into (3.6), we obtain
\[
\frac{1}{p^{2r}} |R_{Q_d,p^r}(m)| = \sum_{k=0}^{2\alpha} \left( \delta_{2k} + \delta_{2(k-2)} \right) p^{-k} \tau(\chi_{1,p^k}, \psi_{-m,p^k}) \\
+ \sum_{k=2\alpha+1}^{r} p^{-3k+\alpha} \tau(\chi_{1,p^k}, \psi_{-m,p^k}) + \varepsilon_p^2 \sum_{k=2\alpha+1}^{r} p^{-3k+\alpha} \tau(\chi_{p,p^k}, \psi_{-m,p^k}). \tag{3.9}
\]

We next evaluate \(\tau(\chi, \psi)\). Letting \(\chi^*\) denote the primitive character of modulus \(m^*\) associated to the character \(\chi\) of modulus \(m\) and abbreviating \(\tau(\chi^*) := \tau(\chi^*, \psi_{1,m^*})\), a corrected version of \([4, \text{Lemma 3.2}]\) yields
\[
\tau(\chi, \psi_{a,m}) = \tau(\chi^*) \sum_{d|\gcd(a, \frac{m}{m^*d})} d\chi^* \left( \frac{a}{m^*d} \right) \left( \frac{a}{d} \right) \mu \left( \frac{m}{m^*d} \right). 
\]

Hence we have (noting that \(\left( \frac{n}{p} \right) = 0\) if \(p | n\))
\[
\tau(\chi_{1,p^k}, \psi_{-m,p^k}) = \sum_{d|\gcd(m,p^k)} d\mu \left( \frac{p^k}{d} \right) = \begin{cases} 
1 & \text{if } k = 0, \\
-p^{k-1} & \text{if } \gcd(m, p^k) = p^{k-1}, \\
p^k - p^{k-1} & \text{if } \gcd(m, p^k) = p^k, \\
0 & \text{otherwise},
\end{cases} \tag{3.10}
\]

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\[
\tau (\chi_{p, p^k}, \psi_{-m, p^k}) = \tau (\chi_p) \sum_{d | \gcd(m, p^{k-1})} d \chi_p \left( \frac{p^{k-1}}{d} \right) \chi_p \left( -\frac{m}{d} \right) \mu \left( \frac{p^{k-1}}{d} \right)
\]
\[
= \begin{cases} 
p^{k-1} \tau (\chi_p) \chi_p \left( -\frac{m}{p^{k-1}} \right) & \text{if } \text{ord}_p(m) = k - 1 \\
0 & \text{otherwise.}
\end{cases}
\]
\[
= \begin{cases} 
\varepsilon_p^3 p^{k-\frac{1}{2}} \chi_p \left( \frac{m}{p^{k-1}} \right) & \text{if } \text{ord}_p(m) = k - 1 \\
0 & \text{otherwise,}
\end{cases}
\]
(3.11)
where in the last step we have used Gauss’s evaluation \( \tau (\chi_p) = \varepsilon_p \sqrt{p} \). Now for \( m = p^R m' \) with \( p \mid m' \), it follows from (3.10) and (3.11) that each term with \( k > R \) follows from (3.10) and (3.11) that each term with \( k > R + 1 \) in (3.9) vanishes. If \( R < 2\alpha \), we use (3.10) to obtain
\[
\lim_{r \to \infty} \frac{1}{p^{2r}} |R_{Q_d, p^r}(m)| = 1 + \sum_{k=1}^{R} (\delta_{2|k} + \delta_{2|k} \varepsilon_p^2) p^{-k}(p^k - p^{k-1}) - \delta_{2|R} p^{-1} - \delta_{2|R} \varepsilon_p^2 p^{-1}
\]
\[
= 1 + (1 - p^{-1}) \left[ \left( \frac{R}{2} \right) + \varepsilon_p^2 \left( \frac{R + 1}{2} \right) \right] - \delta_{2|R} p^{-1} - \delta_{2|R} \varepsilon_p^2 p^{-1}.
\]
Finally for \( R \geq 2\alpha \), we insert (3.10) and (3.11) into (3.9) to conclude
\[
\lim_{r \to \infty} \frac{1}{p^{2r}} |R_{Q_d, p^r}(m)| = 1 + (1 - p^{-1}) \alpha + \sum_{k=2\alpha + 1}^{R} p^{-\frac{2k}{R} + \alpha} (p^k - p^{k-1})
\]
\[
- \delta_{2|R} p^{\alpha - \frac{2k+3}{R}} + \delta_{2|R} \varepsilon_p^2 p^{\alpha - \frac{2k+1}{R}} \chi_p(m')
\]
\[
= 1 + (1 + \varepsilon_p^2)(1 - p^{-1}) \alpha + p^{-1} - p^{-\frac{2k+3}{R}} \chi_p(m')
\]
\[
- \delta_{2|R} p^{\alpha - \frac{2k+3}{R}} + \delta_{2|R} \varepsilon_p^2 p^{\alpha - \frac{2k+1}{R}} \chi_p(m')
\]
This completes the proof of the Lemma.

We next shift our attention in computing the local densities at \( p = \infty \). In the following lemma, we relate the local densities of \( Q_d \) and \( Q \) at \( p = \infty \).

**Lemma 3.3.** We have
\[
\beta_{Q_d, \infty} = \frac{1}{d} \beta_{Q, \infty}. \tag{3.12}
\]

**Proof.** We compute \( \beta_{Q_d, \infty} \) using the open sets \( U = U_\epsilon := (m - \epsilon, m + \epsilon) \). It follows from the definition (2.2) that
\[
\beta_{Q_d, \infty} = \lim_{\epsilon \to 0} \frac{\text{vol}_{3}(Q_{d}(U_\epsilon))}{\text{vol}_2(U_\epsilon)} = \lim_{\epsilon \to 0} \frac{\text{vol}(B_{Q_d, m+\epsilon}) - \text{vol}(B_{Q_d, m-\epsilon})}{2\epsilon}, \tag{3.13}
\]
where \( B_{Q_d, \ell} \) denotes the set \( \{ x \in \mathbb{R}^2 : Q_d(x) \leq \ell \} \) with \( x = (x_1, x_2, x_3) \). Now, it remains to compute the volumes to conclude the lemma. We have
\[
\text{vol}(B_{Q_d, \ell}) := \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \int_{-\sqrt{\ell}}^{\sqrt{\ell}} \frac{\sqrt{r-x_1^2-x_2^2}}{d} dx_3 dx_2 dx_1.
\]
The change of variable \( x_3 \to \frac{x_3}{d} \) yields
\[
\text{vol}(B_{Q_d, \ell}) := \frac{1}{d} \int_{-\sqrt{\ell}d}^{\sqrt{\ell}d} \int_{-\sqrt{\ell}d}^{\sqrt{\ell}d} \int_{-\sqrt{\ell}d}^{\sqrt{\ell}d} \frac{\sqrt{r-x_1^2-x_2^2}}{d} dx_3 dx_2 dx_1 = \frac{1}{d} \text{vol}(B_{Q, \ell}). \tag{3.14}
\]
Hence the lemma follows by plugging (3.14) with \( \ell = m + \epsilon \) and \( \ell = m - \epsilon \) simultaneously into (3.13). \( \square \)
3.2. Main term computation. We define the multiplicative function $\omega(m,d)$ for each square-free $d$ with $2 \nmid d$ by the following
\[ \omega(m,d) := \prod_{p|d} \omega(m,p) \] (3.15)
where
\[ \omega(m,p) = \frac{\beta_{Q_d,p}(m)}{\beta_{Q_d}(m)}. \]

We next compare the coefficients $a_{E_d}(m)$ and $a_E(m)$ by plugging (3.15) and (3.12) into (3.1) to obtain the main term of $|A_d|$. It follows from Lemma 3.1 that
\[ a_{E_d}(m) = \frac{a_{E_d}(m)}{a_E(m)} a_E(m) = \frac{\omega(m,d)}{d} r_3(m). \] (3.16)

We can now proceed to find explicit formulas for $\omega(m,p)$, which can be evaluated from Lemma 3.2.

**Lemma 3.4.** For $p \nmid m$, we have
\[ \omega(m,p) = \begin{cases} \frac{1-p^{-1}}{1+p} & \text{if } p \equiv 1 \pmod{4}, \left(\frac{m}{p}\right) = 1 \\ \frac{1+p^{-1}}{1-p^{-1}} & \text{if } p \equiv 3 \pmod{4}, \left(\frac{m}{p}\right) = 1 \\ 1 & \text{if } p \equiv 1 \pmod{4}, \left(\frac{m}{p}\right) = -1 \\ 1 & \text{if } p \equiv 3 \pmod{4}, \left(\frac{m}{p}\right) = -1. \end{cases} \]

**Lemma 3.5.** For $m = p^{2\theta}m'$ with $\theta \geq 1$ and $(m',p) = 1$, we have
\[ \omega(m,p) = \begin{cases} \frac{3-p^{-1}}{1+p} & \text{if } p \equiv 1 \pmod{4}, \left(\frac{m'}{p}\right) = 1 \\ \frac{1+p^{-1}-2p^{-\theta}}{1-p^{-1}+2p^{-\theta}} & \text{if } p \equiv 3 \pmod{4}, \left(\frac{m'}{p}\right) = 1 \\ \frac{3-p^{-1}-2p^{-\theta}}{1+p^{-1}-2p^{-\theta}} & \text{if } p \equiv 1 \pmod{4}, \left(\frac{m'}{p}\right) = -1 \\ \frac{1+p^{-1}-2p^{-\theta}}{1-p^{-1}+2p^{-\theta}} & \text{if } p \equiv 3 \pmod{4}, \left(\frac{m'}{p}\right) = -1. \end{cases} \]

**Lemma 3.6.** For $p^{2\theta-1} \mid m$ with $\theta \geq 1$, we have
\[ \omega(m,p) = \begin{cases} \frac{3-p^{-1}}{1+p^{-1}} \cdot \frac{1+p^{-1}+2p^{-\theta}}{1-p^{-1}-2p^{-\theta}} & \text{if } p \equiv 1 \pmod{4} \\ \frac{1+p^{-1}-2p^{-\theta}}{1-p^{-1}+2p^{-\theta}} & \text{if } p \equiv 1 \pmod{4}. \end{cases} \]

3.3. Cuspidal form contribution. We denote the cuspidal part associated to $Q_d$ by $f_d$. Let the $m$-th Fourier coefficient of $f_d$ be $R(m,d)$, which can be expressed as
\[ R(m,d) = r_{Q_d}(m) - a_{E_d}(m). \] (3.17)

The following lemma provides an upper bound of $R(m,d)$.

**Lemma 3.7.** For any $\epsilon > 0$
\[ R(m,d) \ll d^{15/14} m^{13/28+\epsilon} \]
uniformly for $4d^2 \leq m^{1/2}$.

**Proof.** It follows from [7, 102:10] that the quadratic form $Q_d(x) = x_1^2 + x_2^2 + d^2 x_3^2$ with $\mu^2(d) = 1$ and $2 \nmid d$, have only one spinor genus per genus. On the other hand, the last part of [2, Theorem 1] yields that for any $\epsilon > 0$,
\[ r(\text{spn } Q_d, m) - r_{Q_d}(m) \ll (4d^2)^{45/28} m^{13/28+\epsilon} \]
uniformly for $4d^2 \leq m^{1/2}$. Hence these two facts together imply
\[ r(\text{gen } Q_d, m) - r_{Q_d}(m) \ll d^{45/14} m^{13/28+\epsilon}. \quad (3.18) \]
We can therefore conclude our lemma by inserting (3.18) into (2.1). \qed

4. Application of linear sieve

Let $\mathcal{P}$ denote the set of all odd primes. In this section, we seek estimates for the sifting function $S(\mathcal{A}, \mathcal{P}, z_0)$, which represents the number of elements in $\mathcal{A}$ that have no prime factors $p < z_0$ in $\mathcal{P}$. More formally, letting
\[ P(z_0) = \prod_{p \leq z_0, p \in \mathcal{P}} p, \]
we want to estimate the following cardinality
\[ S(\mathcal{A}, \mathcal{P}, z_0) := |\{x_3 \in \mathcal{A} : (x_3, P(z_0)) = 1\}|. \]
The following proposition provides an asymptotic formula for the cardinality of the set $\mathcal{A}_d$.

**Proposition 4.1.** Let $r_3(m)$ be the number of representation of $m$ with sum of three squares. Then for $2 \nmid d$, we have
\[ |\mathcal{A}_d| = \frac{\omega(m, d)}{d} r_3(m) + R(m, d), \quad (4.1) \]
where $\omega(m, d)$ and $R(m, d)$ are defined in (3.15) and (3.17) respectively. For $0 < \theta < 1/118$, the error term $R(m, d)$ satisfies
\[ \sum_{d \leq m^\theta} \tilde{\mu}(d)^2 |R(m, d)| \ll m^{1/2-\epsilon_2} \]
where $\tilde{\mu}(d)$ is an arithmetic function defined as
\[ \tilde{\mu}(d) = \begin{cases} \mu(d) & \text{for } 2 \nmid d, \\ 0 & \text{otherwise}, \end{cases} \]
and $\epsilon_2 > 0$ is sufficiently small in terms of $\theta$.

**Proof.** The asymptotic formula (4.1) of $|\mathcal{A}_d|$ is an easy consequence of (3.16) and (3.17). It follows from Lemma 3.7 that
\[ \sum_{d \leq D} \tilde{\mu}(d)^2 |R(m, d)| \ll D^{59/14} m^{13/28+\epsilon}. \]
Now the conditions $0 < \theta < 1/118$ and $D = m^\theta$ will immediately imply
\[ \sum_{d \leq D} \tilde{\mu}(d)^2 |R(m, d)| \ll m^{1/2-\epsilon}. \]
This completes the proof the proposition. \qed

We define
\[ V(z_0) = \prod_{p | P(z_0)} \left( 1 - \frac{\omega(m, p)}{p} \right). \]
Let $F$ and $f$ denotes the classical functions of linear sieve which are the continuous solutions of the following system of differential-difference equations
\[
\begin{align*}
  sF(s) &= 2e^s, & \text{if } 0 < s \leq 3, \\
  sf(s) &= 0, & \text{if } 0 < s \leq 2, \\
  (sF(s))' &= f(s-1), & \text{if } s > 3, \\
  (sf(s))' &= F(s-1), & \text{if } s > 2,
\end{align*}
\]
where $\gamma$ is the Euler constant. The following proposition provides the upper and lower bound of the sifting function $S(A, \mathcal{P}, z_0)$.

**Proposition 4.2.** For $z_0 \geq 3$ and $D^2 \geq z_0$, we have

$$S(A, \mathcal{P}, z_0) \geq r_3(m) V(z_0) \left( f(s) + O \left( e^{\sqrt{L-s} (\log D)^{-1/3}} \right) \right) - \sum_{d \leq D \atop d | P(z_0)} \bar{\mu}(d)^2 |R(m, d)|$$

and

$$S(A, \mathcal{P}, z_0) \leq r_3(m) V(z_0) \left( F(s) + O \left( e^{\sqrt{L-s} (\log D)^{-1/3}} \right) \right) + \sum_{d \leq D \atop d | P(z_0)} \bar{\mu}(d)^2 |R(m, d)|$$

where $L$ is an absolute constant and $s = \frac{\log D}{\log z_0}$.

We prove the proposition by using Rosser’s weights (cf. [5], [6]). Fixing a positive integer $D$, we define two sequences $\{\lambda_d^\pm\}$ in the following way.

(i) $\lambda_1^\pm = 1$.

(ii) $\lambda_d^\pm = 0$ if $d$ is not square-free.

(iii) For $d = p_1 p_2 \cdots p_r$ with $p_1 > p_2 > \cdots > p_r$ and $2, 5 \nmid d$

$$\lambda_d^+ = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l} p_{2l+1}^3 < D \text{ whenever } 0 \leq l \leq \frac{r-1}{2} \\ 0 & \text{Otherwise} \end{cases}$$

and

$$\lambda_d^- = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l-1} p_{2l}^3 < D \text{ whenever } 0 \leq l \leq \frac{r}{2} \\ 0 & \text{Otherwise} \end{cases}.$$

It can be deduced from Lemma 3.4, Lemma 3.5 and Lemma 3.6 that $\omega(m, p)$ satisfies the following two inequalities, which are

$$0 \leq \omega(m, p) < p$$

for all primes $p$ and there exist an absolute constant $L$ independent of $m$ such that

$$\prod_{w \leq p < z_0} \left( 1 - \frac{\omega(m, p)}{p} \right)^{-1} < \left( \frac{\log z_0}{\log w} \right) \left( 1 + \frac{L}{\log w} \right)$$

for every $2 \leq w < z_0$.

Hence it follows from [6, Lemma 3] that the inequalities (4.2) and (4.3) yields the following Lemma which is crucial to obtain the upper and lower bound of the main term of the sifted function.

**Lemma 4.3.** We have

$$V(z_0) \geq \sum_{d | P(z_0)} \lambda_d^- \frac{\omega(m, d)}{d} \geq V(z_0) \left( f(s) + O \left( e^{\sqrt{L-s} (\log D)^{-1/3}} \right) \right)$$

whenever $z_0 \leq D^{1/2}$ and

$$V(z_0) \leq \sum_{d | P(z_0)} \lambda_d^+ \frac{\omega(m, d)}{d} \leq V(z_0) \left( F(s) + O \left( e^{\sqrt{L-s} (\log D)^{-1/3}} \right) \right)$$

whenever $z_0 \leq D$. Here $L$ is an absolute constant arising from (4.3) and $s = \frac{\log D}{\log z_0}$.  

4.1. Proof of the Proposition 4.2. The basic inclusion-exclusion principle yields
\[ S(A, \mathcal{P}, z_0) = \sum_{d \mid P(z_0)} \tilde{\mu}(d) |A_d| \]  
(4.4)

Inserting the cardinality of the set \( A_d \) from Proposition 4.1 into (4.4), we obtain
\[ S(A, \mathcal{P}, z_0) = \sum_{d \mid P(z_0)} \tilde{\mu}(d) \frac{\omega(m, d)}{d} r_3(m) + \sum_{d \mid P(z_0)} \tilde{\mu}(d) \mathcal{R}(m, d). \]

Thus, Lemma 4.3 yields for \( z_0 \leq D^{1/2} \), the main term of \( S(A, \mathcal{P}, z_0) \) can be bounded as
\[ r_3(m) V(z_0) \left( f(s) + O \left( e^{\sqrt{-s}} \log D \right)^{-1/3} \right) \leq \sum_{d \mid P(z_0)} \tilde{\mu}(d) \frac{\omega(m, d)}{d} r_3(m) \]
\[ \leq r_3(m) V(z_0) \left( F(s) + O \left( e^{\sqrt{-s}} \log D \right)^{-1/3} \right). \]
(4.5)

The error term of \( S(A, \mathcal{P}, z_0) \) can be estimated using Rosser’s weights as
\[ \sum_{d \mid P(z_0)} \tilde{\mu}(d) \mathcal{R}(m, d) \leq \sum_{d \mid P(z_0)} \lambda_d^+ \mathcal{R}(m, d). \]

Now observe that \( \lambda_d^+ = 0 \) for \( d > D \). Hence we have
\[ \sum_{d \mid P(z_0)} \tilde{\mu}(d) \mathcal{R}(m, d) \leq \sum_{d \leq D} \lambda_d^+ \mathcal{R}(m, d). \]

We can therefore provide the bound for the absolute value of the error term as
\[ \left| \sum_{d \mid P(z_0)} \tilde{\mu}(d) \mathcal{R}(m, d) \right| \leq \sum_{d \leq D} \left| \lambda_d^+ \right| |\mathcal{R}(m, d)| \leq \sum_{d \leq D} \tilde{\mu}(d)^2 |\mathcal{R}(m, d)|. \]
(4.6)

Thus (4.5) and (4.6) together concludes the proof of the proposition.

4.2. Proof of Theorem 1.1. Letting \( \theta < 1/118 \) and fixing \( D = m^\theta \), the Proposition 4.1 yields that the error term in Proposition 4.2 can be bounded as
\[ \sum_{d \leq D} \tilde{\mu}(d)^2 |\mathcal{R}(m, d)| \ll m^{1/2-\epsilon_2}. \]
(4.7)

On the other hand, for \( z_0 = m^\gamma \) with \( \gamma < 1/236 \), Lemma 4.3 implies that
\[ V(z_0) \gg \frac{1}{\log m} \]
(4.8)

where \( s = \frac{\log D}{\log z_0} > 2 \). Applying the bounds (4.2), (4.7) and (4.8) together in Proposition 4.2, we see that in order to prove Theorem 1.1 it remains to show that \( f(s) > 0 \), where \( f \) is a classical function of linear sieve. A numerical calculation shows that \( f(s) > 0 \) for \( s > 2 \). This completes the proof of Theorem 1.1.
5. Representation of every integer by certain quaternary quadratic forms

The set to be sieved is the following
\[ A := \{(z_1, z_2) \in \mathbb{N}^2 : x^2 + y^2 + (2^a 3^b z_1)^2 + (2^c 5^d z_2)^2 = m\}, \]
where \( x, y, a, b, c, d \) are any integers with \( a, b, c \) and \( d \) non-negative. Setting \( x_1 = x \), \( x_2 = y \), \( x_3 = 2^a 3^b z_1 \) and \( x_4 = 2^c 5^d z_2 \), we can write
\[ A = \{(x_3, x_4) \in \mathbb{N}^2 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = m\}. \]

For \( \bar{x} = (x_3, x_4) \) and \( d = (d_3, d_4) \), let \( \bar{x} \equiv 0 \pmod{d} \) denotes the simultaneous conditions \( x_3 \equiv 0 \pmod{d_3} \) and \( x_4 \equiv 0 \pmod{d_4} \). In order to apply sieve theory, we need an asymptotic formula for the cardinality of the following set
\[ A_d := \{\bar{x} \in A : \bar{x} \equiv 0 \pmod{d}\} = \{(x_3, x_4) \in \mathbb{N}^2 : x_1^2 + x_2^2 + d_3^2 x_3^2 + d_4^2 x_4^2 = m\}, \]
where \( 2, 3 \nmid d_3 \) and \( 2, 5 \nmid d_4 \). It needs some preparation to express the above cardinality in terms of main term and error term. We first consider the quadratic form
\[ Q_d(x) = x_1^2 + x_2^2 + d_3^2 x_3^2 + d_4^2 x_4^2 \]
where \( x = (x_1, x_2, x_3, x_4) \) and \( d = (d_3, d_4) \) satisfies \( 2, 3 \nmid d_3 \) and \( 2, 5 \nmid d_4 \). For simplicity, we abbreviate \( Q_{(1,1)} = Q \). Let, the theta function associated to \( Q_d \) be \( \Theta_{Q_d} \) with the Fourier expansion
\[ \Theta_{Q_d}(\tau) = \sum_{m \geq 0} r_{Q_d}(m) q^m \]
for \( \tau \in \mathbb{H} \). Here the Fourier coefficient \( r_{Q_d}(m) \) denotes the number of representation of \( m \) by \( Q_d \). As mentioned earlier, \( \Theta_{Q_d} \) can be decomposed into two parts which are the Eisenstein series part and the cuspidal part respectively.

5.1. Eisenstein series contribution. We denote the Eisenstein series part associated to \( Q_d \) by \( E_d \). It follows from (2.3) that the \( m \)-th Fourier coefficient of \( E_d \) can be expressed as
\[ a_{E_d}(m) = \prod_p \beta_{Q_d, p}(m) \]  
(5.1)
where the product runs over all the primes including \( \infty \). The definition (2.2) of local representation density yields
\[ \beta_{Q_d, p}(m) = \lim_{r \to \infty} \frac{|R_{Q_d, p^r}(m)|}{p^{3r}} \]
where\n\[ R_{Q_d, p^r}(m) := \{x \in (\mathbb{Z}/p^r\mathbb{Z})^4 : Q_d(x) \equiv m \pmod{p^r}\}. \]
The following lemma is crucial to determine the local density of \( Q_d \) at \( p = 2 \).

**Lemma 5.1.** For \( p \nmid d_3 d_4 \), the local density of \( Q_d \) satisfies
\[ \beta_{Q_d, p} = \beta_{Q, p}. \]
In particular, we have
\[ \beta_{Q_d, 2} = \beta_{Q, 2}. \]

**Proof.** The proof of the lemma follows similarly as of the proof of Lemma 3.1. \( \Box \)

We set \( d_j = p^{r_j} d_j' \) with \( p \nmid d_j' \) and \( m = p^{r_j} m' \) with \( p \nmid m' \). In the next lemma, we compute the local representation density \( \beta_{Q_d, p} \) at each odd prime \( p \).
Lemma 5.2. Let \( \alpha_3 \leq \alpha_4 \). Then for any odd prime \( p \), we have

\[
\beta_{Q, p}(m) = \begin{cases} 
1 + (1 - p^{-1}) \left( \left[ \frac{R}{2} \right] + \varepsilon_p^2 \left[ \frac{R+1}{2} \right] \right) - \delta_{2|R} p^{-1} - \delta_{2|R} \varepsilon_p^2 p^{-1} & \text{if } R < 2\alpha_3, \\
1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + p^{-1} - p^{\alpha_3 - 1 - \left[ \frac{R}{2} \right]} - \delta_{2|R} p^{\alpha_3 - 1 - \left[ \frac{R}{2} \right]} + \delta_{2|R} \varepsilon_p^2 p^{\alpha_3 - 1 - \left[ \frac{R}{2} \right]} \left( \frac{m'}{p} \right) & \text{if } 2\alpha_3 \leq R < 2\alpha_4, \\
1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + p^{-1} - p^{\alpha_3 + \alpha_4 - 1 - R} (1 + p^{-1}) & \text{if } R \geq 2\alpha_4.
\end{cases}
\]

Proof. We first evaluate the absolute value of \( R_{Q, p'}(m) \). It follows from (3.3) that

\[
|R_{Q, p'}(m)| = \sum_{x \in (\mathbb{Z}/p' \mathbb{Z})^4} \frac{1}{p'} \sum_{n \equiv m \pmod{p'}} e^{2\pi i m p' Q_4(x)} = \frac{1}{p'} \sum_{n \equiv m \pmod{p'}} e^{2\pi i m p' Q_4(x)} = \frac{1}{p'} \sum_{n \equiv m \pmod{p'}} G_2(n, 0, p')^2 \prod_{j=3}^4 G_2(nd_j^2, 0, p')
\]

We next split the sum over \( n \) by writing \( n = p^kn' \) with \( p \nmid n' \) and make the change of variables \( k \mapsto r - k \) to arrive at

\[
|R_{Q, p'}(m)| = \frac{1}{p'} \sum_{k=0}^r p^{4k} \sum_{n' \in (\mathbb{Z}/p' \mathbb{Z})^4} e^{-2\pi i m p' G_2(n', 0, p')^2} \prod_{j=3}^4 G_2(n'd_j^2, 0, p')
\]

where in the last step we applied (3.4). We can derive similarly as in (3.5) that

\[
G_2(n'd_j^2, 0, p') = \begin{cases} 
p^k & \text{for } k \leq 2\alpha_j, \\
p^{2\alpha_j} G_2(n'd_j^2, 0, p^{k-2\alpha_j}) & \text{for } k > 2\alpha_j.
\end{cases}
\]

Therefore for the assumption \( \alpha_3 \leq \alpha_4 \), we can write

\[
\frac{1}{p'^4} |R_{Q, p'}(m)| = \sum_{k=0}^{2\alpha_3} p^{-2k} \sum_{n' \in (\mathbb{Z}/p' \mathbb{Z})^4} e^{-2\pi i m p' G_2(n', 0, p')^2} + \sum_{k=2\alpha_3+1}^{2\alpha_4} p^{-3k+2\alpha_3} \sum_{n' \in (\mathbb{Z}/p' \mathbb{Z})^4} e^{-2\pi i m p' G_2(n', 0, p')^2} G_2(n'd_3^2, 0, p^{k-2\alpha_3}) + \sum_{k=2\alpha_4+1}^r p^{-4k+2\alpha_3} \sum_{n' \in (\mathbb{Z}/p' \mathbb{Z})^4} e^{-2\pi i m p' G_2(n', 0, p')^2} \prod_{j=3}^4 G_2(n'd_j^2, 0, p^{k-2\alpha_j}).
\]
Thus the evaluation of quadratic Gauss sums in (3.7) and (3.8) yields
\[
\frac{1}{p^{3r}}|R_{Q_d,p^r}(m)| = \sum_{k=0}^{2\alpha_3} (\delta_2[k + \delta_2(k^2)\ p^{-k}\tau(\chi_{1,p^k}, \psi_{-m,p^k})] + \sum_{k=2\alpha_3+1}^{2\alpha_4} p^{-\frac{3k}{2}+\alpha_3}\tau(\chi_{1,p^k}, \psi_{-m,p^k}) 
+ \varepsilon_p^3 \sum_{k=2\alpha_3+1}^{2\alpha_4} p^{-\frac{3k}{2}+\alpha_3}\tau(\chi_{p^k,p^k}, \psi_{-m,p^k}) + \sum_{k=2\alpha_4+1}^{R} p^{-2k+\alpha_3+\alpha_4}\tau(\chi_{1,p^k}, \psi_{-m,p^k}).
\]
Inserting the value of \(\tau(\chi_{1,p^k}, \psi_{-m,p^k})\) and \(\tau(\chi_{p^k,p^k}, \psi_{-m,p^k})\) from (3.10) and (3.11) in the above equation, it can be derived for \(R < 2\alpha_3\) that
\[
\lim_{r \to \infty} \frac{1}{p^{3r}}|R_{Q_d,p^r}(m)| = 1 + \sum_{k=1}^{R} (\delta_2[k + \delta_2(k^2)\ p^{-k}(p^k - p^{k-1})] - \delta_2[R] p^{-1} - \delta_2[R] \varepsilon_p^2 p^{-1} 
= 1 + (1 - p^{-1}) \left( \left\lfloor \frac{R}{2} \right\rfloor + \varepsilon_p^2 \left\lfloor \frac{R + 1}{2} \right\rfloor \right) - \delta_2[R] p^{-1} - \delta_2[R] \varepsilon_p^2 p^{-1}.
\]
For \(2\alpha_3 \leq R < 2\alpha_4\), we have
\[
\lim_{r \to \infty} \frac{1}{p^{3r}}|R_{Q_d,p^r}(m)| = 1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + \sum_{k=2\alpha_3+1}^{R} p^{-\frac{3k}{2}+\alpha_3}(p^k - p^{k-1}) - \delta_2[R] p^{\alpha_3 - \frac{R+1}{2}} + \delta_2[R] \varepsilon_p^2 p^{\alpha_3 - \frac{R}{2}} \chi_p(m') 
= 1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + p^{-1} - p^{\alpha_3 - \frac{R+1}{2}} \chi_p(m') 
- \delta_2[R] p^{\alpha_3 - \frac{R+1}{2}} + \delta_2[R] \varepsilon_p^2 p^{\alpha_3 - \frac{R}{2}} \chi_p(m').
\]
Finally we consider the case \(R \geq 2\alpha_4\) and obtain
\[
\lim_{r \to \infty} \frac{1}{p^{3r}}|R_{Q_d,p^r}(m)| = 1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + p^{-1} - p^{\alpha_3 - \alpha_4 - 1} \]
\[
+ \sum_{k=2\alpha_4+1}^{R} p^{-2k+\alpha_3+\alpha_4}(p^k - p^{k-1}) - p^{-2(R+1)+\alpha_3+\alpha_4+R} 
= 1 + (1 + \varepsilon_p^2)(1 - p^{-1})\alpha_3 + p^{-1} - p^{\alpha_3 + \alpha_4 - 1 - R} - p^{\alpha_3 + \alpha_4 - 2 - R}
\]
This concludes the proof of the lemma. \(\square\)

The next lemma relates the local densities of \(Q_d\) and \(Q\) at \(p = \infty\).

\textbf{Lemma 5.3.} We have
\[
\beta_{Q_d,\infty} = \frac{1}{d_3d_4} \beta_{Q,\infty}.
\]

\textit{Proof.} The proof of the lemma follows similarly as of the proof of Lemma 5.3.

\textbf{5.2. Main term computation.} We next define the multiplicative function \(\omega(m, d)\) for each square-free \(d_3\) and \(d_4\) with \(2, 3 \nmid d_3\) and \(2, 5 \nmid d_4\) by the following :
\[
\omega(m, d) := \prod_{p^i | d_3d_4} \omega_p(m, p) \quad (5.2)
\]
where
\[
\omega_p(m, p) = \frac{\beta_{Q_d,p}(m)}{\beta_{Q,p}(m)}.
\]
Here $\nu$ can take the values 1 or 2. Lemma 5.1 together with Lemma 5.3 implies that (5.1) can be rephrased as

$$a_{E_d}(m) = \frac{a_{E_d}(m)}{a_E(m)} a_E(m) = \frac{\omega(m, d)}{d_3d_4} r_4(m).$$

We can now proceed to find explicit formulas for $\omega_{\nu}(m, p)$. Invoking Lemma 5.2 inside the definition of $\omega_{\nu}(m, p)$ the following evaluations can be obtained.

**Lemma 5.4.** For $p \nmid m$, we have

$$\omega_{\nu}(m, p) = \begin{cases} 
\frac{1 + \varepsilon_p^2 \chi_p(m') p^{-1}}{1 - p^{-2}} & \text{for } \nu = 1 \\
\frac{1 - \varepsilon_p^2 p^{-1}}{1 - p^{-2}} & \text{for } \nu = 2.
\end{cases}$$

**Lemma 5.5.** For $p^R | m$ with $R \geq 1$, we have

$$\omega_{\nu}(m, p) = \begin{cases} 
\frac{1 - p^{-R}}{1 - p^{-R-1}} & \text{for } \nu = 1 \\
\frac{(1 - p^{-1}) (1 + \varepsilon_p^2) (1 + p^{-1}) (1 - p^{1-R})}{(1 + p^{-1}) (1 - p^{-R-1})} & \text{for } \nu = 2.
\end{cases}$$

We next define another function

$$\Omega(m, p) := 2\omega_1(m, p) - \frac{\omega_2(m, p)}{p}$$

and

$$\Omega(m, d) := \prod_{p \mid d} \Omega(m, p).$$

In the sieve theory calculation, the main term involves the following term:

$$W(z_0) := \prod_{p < z_0} \left(1 - \frac{\Omega(m, p)}{p}\right).$$

Therefore, we finally need the lower bound of $W(z_0)$ i.e, the upper bound of $\Omega(m, p)$.

**Lemma 5.6.** We have

$$\Omega(m, p) \leq \begin{cases} 
2 + \frac{p^{1.5}}{p^2 - 1} & \text{for } p \nmid m \\
2 & \text{for } p \mid m.
\end{cases}$$

In particular, for $p \geq 5$ we have $\Omega(m, p) \leq \frac{5}{2}$.

**Proof.** We first apply Lemma 5.4 and Lemma 5.5 to obtain trivial bounds of $\omega_{\nu}(m, p)$ and then combine and simplify the bounds to conclude the lemma.

### 5.3. Cusp form contribution

We denote the cuspidal part associated to $Q_d$ by $f_d$. Let the $m$-th Fourier coefficient of $f_d$ be $R(m, d)$, which can be expressed as

$$R(m, d) = r_{Q_d}(m) - a_{E_d}(m).$$

The following lemma provides an upper bound of $R(m, d)$.

**Lemma 5.7.** We have

$$|R(m, d)| \leq 7.07 \times 10^{23} (d_3d_4)^{6.01} m^{1.5}.$$

**Proof.** We write the quadratic form $Q_d(x) = \frac{1}{2} x^T A x$ where $x = (x_1, x_2, x_3, x_4)$ and $A$ is the diagonal matrix such that $A = [2, 2, 2d_2^2, 2d_4^2]$. For $N_d$ and $\Delta_d$ denoting the level and the discriminant respectively of the quadratic forms $Q_d$, we have

$$N_d = 4 \text{lcm}(d_3^2, d_4^2) \quad \text{and} \quad \Delta_d = 2^4 d_3^2 d_4^2.$$  

**Proof.** We write the quadratic form $Q_d(x) = \frac{1}{2} x^T A x$ where $x = (x_1, x_2, x_3, x_4)$ and $A$ is the diagonal matrix such that $A = [2, 2, 2d_2^2, 2d_4^2]$. For $N_d$ and $\Delta_d$ denoting the level and the discriminant respectively of the quadratic forms $Q_d$, we have

$$N_d = 4 \text{lcm}(d_3^2, d_4^2) \quad \text{and} \quad \Delta_d = 2^4 d_3^2 d_4^2.$$  

(5.4)
It follows from \cite[Lemma 4.2]{[1]} that
\[ |R(m, d)| \leq 1.797 \times 10^{21} m \frac{3}{4} N_d^{\frac{3}{4}} (27\pi \Delta_d + 16N_d^{\frac{1}{2}}). \]

Thus by inserting the value of $N_d$ and $\Delta_d$ from \cite[(5.4)]{[10]} into the above equation, we can bound the absolute value of $R(m, d)$ as
\[ |R(m, d)| \leq 1.448 \times 10^{22} m \frac{3}{4} (\text{lcm}(d_3^2, d_4^2))^{\frac{3}{2}} + 1024(\text{lcm}(d_3^2, d_4^2))^3. \]
\[ \leq 5.791 \times 10^{22}(27\pi + 64) \frac{3}{2} m \frac{3}{4} (d_3d_4)^6 + 4 \times 10^{-6} + \frac{1}{100}. \]
\[ \leq 7.07 \times 10^{23} m \frac{3}{4} (d_3d_4)^6.01. \]

This completes the proof of the lemma. \hfill \square

6. Application of vector sieve

Let $\mathcal{P}_7$ be the set of all primes starting from 7. For
\[ P(z_0) := \prod_{\substack{p \leq z_0 \\text{ and } \log p < \log w \\text{ for } \text{ all } w \leq z_0}} p, \]
we let $P_3(z_0) := 5P(z_0)$ and $P_4(z_0) := 3P(z_0)$ and seek estimates for the sifting function
\[ S(A, \mathcal{P}_7, z_0) := |\{(x_3, x_4) \in A : (x_j, P_j(z_0)) = 1 \text{ for } j = 3, 4\}|. \]

In the following proposition, we provide an asymptotic expansion for the cardinality of the set $A_d$, which follows from \cite[(5.2)]{[5]} and \cite[(5.3)]{[5]}.

**Proposition 6.1.** Let $r_4(m)$ be the number of representation of $m$ with sum of four squares. Then for $2, 3 \nmid d_3$ and $2, 5 \nmid d_4$, we have
\[ |A_d| = \frac{\omega(m, d)}{d_3d_4} r_4(m) + R(m, d). \]

The following bound on $\omega_1(m, p)$ is crucial to prove Theorem \cite{[14]}

**Lemma 6.2.** For $3 < w \leq z_0$, we have
\[ \prod_{w \leq p < z_0} \left( 1 - \frac{\omega_1(m, p)}{p} \right)^{-1} \leq \left( \frac{\log z_0}{\log w} \right) \left( 1 + \frac{6}{\log w} \right). \]

**Proof.** The evaluation of $\omega_1(m, p)$ in Lemma \cite[(5.4)]{[5]} and Lemma \cite[(5.5)]{[5]} implies that
\[ \omega_1(m, p) \leq \frac{p}{p - 1}. \] (6.1)

We can therefore bound
\[ \prod_{w \leq p < z_0} \left( 1 - \frac{\omega_1(m, p)}{p} \right)^{-1} \leq \prod_{w \leq p < z_0} \left( 1 - \frac{1}{p - 1} \right)^{-1} \leq \prod_{w \leq p < z_0} \frac{p}{p - 1} \prod_{w \leq p < z_0} \left( 1 + \frac{3}{p^2} \right)^{-1}, \] (6.2)
where in the last inequality we used the bound $\frac{1}{p(p - 2)} \leq \frac{3}{p^2}$ for $p \geq 3$. We next bound both the products of the above equation separately. It follows from \cite[(3.30)]{[10]} and \cite[(3.26)]{[10]} that
\[ \prod_{w \leq p < z_0} \frac{p}{p - 1} = \left( \prod_{p < z_0} \frac{p}{p - 1} \right) \left( \prod_{p < w} \frac{p - 1}{p} \right) \leq \left( \frac{\log z_0}{\log w} \right) \left( 1 + \frac{1}{\log^2 w} \right). \] (6.3)
Let \( \omega(n) \) denote the number of distinct prime divisors of \( n \). We have
\[
\prod_{w \leq p < z_0} \left(1 + \frac{3}{p^2}\right)^{-1} \leq \prod_{p \geq w} \left(1 + \frac{3}{p^2}\right)^{-1} = 1 + \sum_{p \mid n} \mu^2(n)3^{\omega(n)} \frac{n}{n^2}
\]
Applying the bound \( 3^{\omega(n)} \leq 1.614n^{1/2} \) (cf. [1, Lemma 2.5]) and bounding the sum against the Riemann integral, the above bound can be reduced as
\[
\prod_{w \leq p < z_0} \left(1 + \frac{3}{p^2}\right)^{-1} \leq 1 + 1.614 \sum_{n \geq w} \frac{1}{n^{3/2}} \leq 1 + \frac{3.228}{\sqrt{w}}
\] (6.4)
We next insert the bounds (6.3) and (6.4) of both the products into (6.2) to obtain
\[
\prod_{w \leq p < z_0} \left(1 - \frac{\omega_1(m,p)}{p}\right)^{-1} \leq \left(\frac{\log z_0}{\log w}\right) \left(1 + \frac{1}{\log^2 z_0}\right) \left(1 + \frac{1}{2 \log^2 w}\right) \left(1 + \frac{3.228}{\sqrt{w}}\right)
\] (6.5)
Finally, we utilize the bound
\[
\left(1 + \frac{1}{\log^2 z_0}\right) \left(1 + \frac{1}{2 \log^2 w}\right) \left(1 + \frac{3.228}{\sqrt{w}}\right) \leq \left(1 + \frac{6}{\log w}\right)
\]
in (6.5), to conclude our lemma.

For \( \beta, D > 0 \), we define two sequences \( \{\lambda_d^\pm\} \) in a following way.
(i) \( \lambda_d^+ = 1 \).
(ii) \( \lambda_d^+ = 0 \) if \( d \) is not square-free.
(iii) For \( d = p_1p_2\cdots p_r \) with \( p_1 > p_2 > \cdots > p_r \),
\[
\lambda_d^+ = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l} p_{2l+1}^\beta < D \text{ whenever } 0 \leq l \leq \frac{r-1}{2} \\ 0 & \text{Otherwise} \end{cases}
\]
and
\[
\lambda_d^- = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l-1} p_{2l}^\beta < D \text{ whenever } 0 \leq l \leq \frac{r}{2} \\ 0 & \text{Otherwise} \end{cases}.
\]
We define
\[
V_j(z_0) := \prod_{p \mid P_j(z_0)} \left(1 - \frac{\omega_1(m,p)}{p}\right)
\]
for \( j = 3, 4 \). As is standard, we consider \( D \) and \( \beta \) to be fixed throughout. For \( \beta > 1 \), we define
\[
a_\beta := e^{\beta \frac{\log (\beta - 1)}{\beta - 1}}, \quad r_\beta := \frac{\log \left(1 + \frac{6}{\log \beta}\right)}{\log \left(\frac{\beta}{\beta - 1}\right)}
\] (6.6)
and
\[
C_\beta(s) := e^{r_\beta - 1} \left(1 + \frac{6}{\log \beta}\right) a_\beta^{s-\beta} \frac{\beta^s}{\beta - 1} - 1 \quad \text{for } s \geq \beta.
\] (6.7)

**Lemma 6.3.** Let \( D > 0 \) and \( \beta \geq 5 \) be given and set \( s := \frac{\log D}{\log z_0} \). Then for \( s \geq \beta \) and \( z_0 \geq 7 \), we have
\[
\sum_{d \mid P_j(z_0)} \lambda_d^- \frac{\omega_1(m,d)}{d} > V_j(z_0)(1 - C_\beta(s)) \quad \text{and} \quad \sum_{d \mid P_j(z_0)} \lambda_d^+ \frac{\omega_1(m,d)}{d} < V_j(z_0)(1 + C_\beta(s))
\] (6.7)
Proof. For simplicity, we first denote
\[ V_j^-(z_0) := \sum_{d \mid P_j(z_0)} \lambda_d \frac{\omega_1(m, d)}{d} \quad \text{and} \quad V_j^+(z_0) := \sum_{d \mid P_j(z_0)} \lambda_d \frac{\omega_1(m, d)}{d}. \]

Letting
\[ y_m := \left( \frac{D}{p_1 p_2 \cdots p_m} \right)^{1/\beta}, \]
we define
\[ V_{j,n}(z_0) := \sum_{y_m < p_1 < \cdots < p_1 < z_0 \atop p_1 p_2 \cdots p_m \equiv ( \mod 2)} \frac{\omega_1(m, p_1 p_2 \cdots p_m)}{p_1 p_2 \cdots p_m} V_j(p_n). \tag{6.8} \]

It follows by inclusion-exclusion as in \[4\] (6.29) and (6.30)] that
\[ V_j^-(z_0) = V_j(z_0) - \sum_{n \text{ even}} V_{j,n}(z_0) \quad \text{and} \quad V_j^+(z_0) = V_j(z_0) + \sum_{n \text{ odd}} V_{j,n}(z_0). \tag{6.9} \]

For \( z_n := z_0^{\frac{\beta-1}{\beta}} \), we can bound \( V_{j,n}(z_0) \) as (cf. \[4\, \text{p. 157}])
\[ V_{j,n}(z_0) \leq \frac{V_j(z_0)}{n!} \left( \log \left( \frac{V_j(z_0)}{V_j^+(z_0)} \right) \right)^n, \tag{6.10} \]
where \( V_j(z_0) \) can be expressed as
\[ V_j(z_0) = \prod_{p \mid P_j(z_0)} \left( 1 - \frac{\omega_1(m, p)}{p} \right) = V_j(z_0) \prod_{z_n \leq p < z} \left( 1 - \frac{\omega_1(m, p)}{p} \right)^{-1}. \]

Applying Lemma \[6.2\] into the above equation, we obtain
\[ V_j(z_n) \leq V_j(z_0) \left( \frac{\log z_0}{\log z_n} \right) \left( 1 + \frac{6}{\log z_n} \right). \tag{6.11} \]

Note that \( z_n \geq 7 \). Thus by inserting (6.11) into (6.11), \( V_{j,n}(z_0) \) can be bounded as
\[ V_{j,n}(z_0) \leq \frac{V_j(z_0)}{n!} \left( \frac{\log z_0}{\log z_n} \right) \left( 1 + \frac{6}{\log z_n} \right) \left( \log \left( \frac{\log z_0}{\log z_n} \left( 1 + \frac{6}{\log z_n} \right) \right) \right)^n. \]

It follows from well-known Stirling’s bound (a more precise version by Robbins \[9\]) that
\[ n! \geq \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \geq e \left( \frac{n}{e} \right)^n. \]

Therefore by utilizing \( z_n = z_0^{\frac{\beta-1}{\beta}} \), we have
\[ V_{j,n}(z_0) \leq \frac{V_j(z_0)}{en^n} \left( e^{\frac{\beta}{\beta-1}} \right)^n \left( 1 + \frac{6}{\log z_n} \right) \left( \log \left( \frac{\left( \frac{\beta}{\beta-1} \right)^n \left( 1 + \frac{6}{\log z_n} \right) \right)}{n \log \left( \frac{\beta}{\beta-1} \right)} \right)^n \]
\[ = \frac{V_j(z_0)}{e} \left( e^{\frac{\beta}{\beta-1}} \log \left( \frac{\beta}{\beta-1} \right) \right)^n \left( 1 + \frac{6}{\log z_n} \right) \left( 1 + \frac{\log \left( 1 + \frac{6}{\log z_n} \right)}{n \log \left( \frac{\beta}{\beta-1} \right)} \right)^n \]
\[ = \frac{V_j(z_0)}{e} \cdot d_\beta^n \left( 1 + \frac{6}{\log z_n} \right) \left( 1 + \frac{r_\beta}{n} \right)^n \]
\[ \leq V_j(z_0) a_\beta^n e^{r_\beta-1} \left( 1 + \frac{6}{\log z_n} \right), \]
where in the penultimate step we used (6.6) and in the last step we have applied the bound $(1 + \frac{r_\beta}{n})^n \leq e^{r_\beta}$. We next take the sum over all $n \in \mathbb{N}$ on both the sides of the above equation. The definition (6.8) yields that $V_{j,n}(z_0) = 0$ for $n \leq s - \beta$. Therefore for $\beta \geq 5$, we have

$$
\sum_{n \geq 1} V_{j,n}(z_0) \leq V_j(z_0)e^{r_\beta - 1}\left(1 + \frac{6}{\log 7}\right)\sum_{n > s - \beta} a_\beta^n \leq V_j(z_0)C_\beta(s),
$$

where in the last step we have applied (6.7). Finally, we insert the above bound into (6.9) to conclude

$$V_j^-(z_0) > V_j(z_0)(1 - C_\beta(s)) \quad \text{and} \quad V_j^+(z_0) < V_j(z_0)(1 + C_\beta(s)).$$

This completes the proof of the lemma.

We next bound the sums of the type in Lemma 6.3 under the additional restriction that we only sum over those $d$ with $\delta \mid d$, for some $\delta \in \mathbb{N}$.

**Lemma 6.4.** Let $D > 0$ and $\beta \geq 5$ be given and set $s := \frac{\log(D)}{\log(5)}$. Then for $s \geq \beta$, $z_0 \geq 7$ and square-free $\delta \in \mathbb{N}$, we have

$$\sum_{d \mid P_j(z_0)} \lambda_d^\pm \frac{\omega_1(m, d)}{d} \geq \mu(\delta) \left(\prod_{p \mid \delta} \frac{\omega_1(m, p)}{p - \omega_1(m, p)}\right) V_j(z_0)(1 - C_\beta(s))$$

and

$$\sum_{d \mid P_j(z_0)} \lambda_d^\pm \frac{\omega_1(m, d)}{d} \leq \mu(\delta) \left(\prod_{p \mid \delta} \frac{\omega_1(m, p)}{p - \omega_1(m, p)}\right) V_j(z_0)(1 + C_\beta(s)).$$

**Proof.** We first define two characteristic functions

$$f_\delta(n) := \begin{cases} 1 & \text{if } \delta \mid n, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{f}_\delta(n) := \begin{cases} 1 & \text{if } \gcd(n, \delta) = 1, \\ 0 & \text{otherwise} \end{cases}.$$

Thus for a prime $p$, we have $f_p(n) = 1 - \bar{f}_p(n)$. For $\delta$ being square-free,

$$f_\delta(n) = \prod_{p \mid \delta} f_p(n) = \prod_{p \mid \delta} \left(1 - \bar{f}_p(n)\right) = \sum_{u \mid \delta} \mu(u) \bar{f}_u(n).$$

Therefore by denoting $\omega_u(d) := \bar{f}_u(d)\omega_1(m, d)$, we can write from the above equation that

$$\sum_{d \mid P_j(z_0)} \lambda_d^\pm \omega_1(m, d) = \sum_{d \mid P_j(z_0)} f_\delta(d)\lambda_d^\pm \frac{\omega_1(m, d)}{d} = \sum_{u \mid \delta} \mu(u) \sum_{d \mid P_j(z_0)} \bar{f}_u(d)\lambda_d^\pm \frac{\omega_1(m, d)}{d} = \sum_{u \mid \delta} \mu(u) \sum_{d \mid P_j(z_0)} \lambda_d^\pm \frac{\omega_u(d)}{d}. \quad (6.12)$$

Letting $V_{j,u}(z_0) := \prod_{p \mid P_j(z_0)} \left(1 - \frac{\omega_u(p)}{p}\right)$, it follows from Lemma 6.3 that

$$\sum_{d \mid P_j(z_0)} \lambda_d^\pm \frac{\omega_u(d)}{d} > V_{j,u}(z_0)(1 - C_\beta(s)) \quad \text{and} \quad \sum_{d \mid P_j(z_0)} \lambda_d^\pm \frac{\omega_u(d)}{d} < V_{j,u}(z_0)(1 + C_\beta(s)).$$
Thus (6.12) can be derived as
\[ \sum_{d|P_1(z_0) \delta|d} \lambda_d \frac{\omega_1(m, d)}{d} > \sum_{u|\delta} \mu(u)V_{j,u}(z_0)(1 - \xi_\beta(s)) \] (6.13)
and
\[ \sum_{d|P_1(z_0) \delta|d} \lambda_d \frac{\omega_1(m, d)}{d} < \sum_{u|\delta} \mu(u)V_{j,u}(z_0)(1 + \xi_\beta(s)) \] (6.14)

The sum on the right hand side of both (6.13) and (6.14) evaluates as
\[ \sum_{u|\delta} \mu(u) \prod_{p|P_1(z_0)} \left(1 - \frac{\bar{f}_u(p)\omega_1(m, p)}{p}\right) = V_j(z_0) \sum_{u|\delta} \prod_{p|u} \left(1 - \frac{\omega_1(m, p)}{p}\right) \]
\[ = V_j(z_0) \prod_{p|\delta} \left(1 - \frac{p}{p - \omega_1(m, p)}\right) = V_j(z_0) \prod_{p|\delta} \left(\frac{\omega_1(m, p)}{p - \omega_1(m, p)}\right). \] (6.15)

Therefore by inserting (6.15) into (6.13) and (6.14), we can conclude our lemma. 

We next define two functions
\[ \Sigma^-(D, z_0) := \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda^+_d \lambda^+_d \frac{\omega(m, d)}{d_3d_4} \] (6.16)
and
\[ \Sigma^+(D, z_0) := \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda^-_d \lambda^-_d \frac{\omega(m, d)}{d_3d_4}. \] (6.17)

In the following lemma we provide an upper and lower bound of \( S(A, \mathcal{P}_7, z_0) \).

**Lemma 6.5.** For \( D > 0, \beta \geq 6, \) we have
\[ \Sigma^-(D, z_0)r_4(m) - \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} |R(m, d)| \leq S(A, \mathcal{P}_7, z_0) \leq \Sigma^+(D, z_0)r_4(m) + \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} |R(m, d)| \]

**Proof.** We can write \( S(A, \mathcal{P}_7, z_0) \) as
\[ S(A, \mathcal{P}_7, z_0) = \sum_{(x_3, x_4) \in A} 1 = \sum_{(x_3, x_4) \in A} \left( \sum_{d_3|(x_3, P_3(z_0))} \mu(d_3) \right) \left( \sum_{d_4|(x_4, P_4(z_0))} \mu(d_4) \right) \]
\[ = \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \mu(d_3)\mu(d_4)|A_d| \]
Thus the inequality \( \lambda^-_d \leq \mu(d) \leq \lambda^+_d \) immediately yields
\[ \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda^-_d \lambda^+_d |A_d| \leq S(A, \mathcal{P}_7, z_0) \leq \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda^-_d \lambda^+_d |A_d|. \]

Invoking Proposition [6.1] in the above equation, we arrive at
\[ \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda^-_d \lambda^+_d \left( \frac{\omega(m, d)}{d_3d_4}r_4(m) + R(m, d) \right) \leq S(A, \mathcal{P}_7, z_0) \]
Therefore we can write \( \Sigma \)

\[
\leq \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda_{d_3}^{+} \lambda_{d_4}^{+} \left( \frac{\omega(m, d)}{d_3d_4} r_4(m) + R(m, d) \right).
\]

It follows from the definition of the Rosser weights that \(|\lambda_{d_3}^{\pm}| \leq 1\), \(\lambda_{d_3}^{\pm} = 0\) for \(d_3 \geq \frac{D}{3^6}\) and \(\lambda_{d_4}^{\pm} = 0\) for \(d_4 \geq \frac{D}{3^6}\) where \(\beta \geq 6\). Therefore by applying definitions of \(\Sigma^{-}(D, z_0)\) and \(\Sigma^{+}(D, z_0)\) from (6.16) and (6.17) respectively and and inserting the absolute value termwise for the sum on \(R(m, d)\), we can conclude our lemma.

6.1. **Bounds for the main term from sieving.** We define a multiplicative function

\[
g(\eta) := \prod_{p|\eta} \omega_2(m, p)
\]

and

\[
\Sigma_{MT}(D, z_0) := \sum_{d_3|P(z_0)} g(d_3) \sum_{d_4|P_4(z_0)} \mu(\ell) \prod_{j=3}^{4} V_j(z_0) \mu(\xi_j) \prod_{p|\xi_j} \omega_1(m, p) \frac{\omega_1(m, p)}{p - \omega_1(m, p)}.
\]

In the following lemma we bound \(\Sigma^{-}(D, z_0)\) from below to obtain a lower bound for \(S(\mathbb{A}, \mathbb{P}_7, z_0)\) from Lemma [5.4]

**Lemma 6.6.** For \(\beta \geq 5\) and \(C_\beta(s) < 1\), we have

\[
\Sigma^{-}(D, z_0) \geq (1 - C_\beta(s))^2 \Sigma_{MT}(D, z_0).
\]

**Proof.** It follows from the definition of \(\omega(m, d)\) that

\[
\omega(m, d) = \omega_1(m, d_3) \omega_1(m, d_4) g(d_34).
\]

Therefore we can write \(\Sigma^{-}(D, z_0)\) as

\[
\Sigma^{-}(D, z_0) = \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0)} \lambda_{d_3}^{+} \lambda_{d_4}^{+} \frac{\omega_1(m, d_3) \omega_1(m, d_4)}{d_3d_4} g(d_34)
\]

\[
= \sum_{d_3|P(z_0)} g(d_3) \sum_{d_4|P_4(z_0)} \lambda_{d_3}^{+} \lambda_{d_4}^{+} \frac{\omega_1(m, d_3) \omega_1(m, d_4)}{d_3d_4}
\]

\[
= \sum_{d_4|P_4(z_0)} g(d_34) S^{-}(d_34)
\]

where

\[
S^{-}(d_34) := \sum_{d_3|P_3(z_0)} \sum_{d_4|P_4(z_0) \gcd(d_3, d_4) = d_34} \lambda_{d_3}^{+} \lambda_{d_4}^{+} \frac{\omega_1(m, d_3) \omega_1(m, d_4)}{d_3d_4}.
\]

We next handle the sum \(S^{-}(d_34)\). Rewriting the condition \(\gcd(d_3, d_4) = d_34\), the sum can be written as

\[
S^{-}(d_34) = \left( \sum_{d_3|P_3(z_0) \left( \frac{d_4}{d_3} \frac{d_34}{d_3} \right) = 1} \lambda_{d_3}^{+} \frac{\omega_1(m, d_3)}{d_3} \right) \left( \sum_{d_4|P_4(z_0) \left( \frac{d_34}{d_3} \frac{d_3}{d_34} \right) = 1} \lambda_{d_4}^{+} \frac{\omega_1(m, d_4)}{d_4} \right)
\]

\[
= \left( \sum_{d_3|P_3(z_0) \left( \frac{d_4}{d_3} \frac{d_34}{d_3} \right)} \mu(\ell) \lambda_{d_3}^{+} \frac{\omega_1(m, d_3)}{d_3} \right) \left( \sum_{d_4|P_4(z_0) \left( \frac{d_34}{d_3} \frac{d_3}{d_34} \right)} \mu(\ell) \lambda_{d_4}^{+} \frac{\omega_1(m, d_4)}{d_4} \right).
\]
Now setting $\xi = \ell d_{34}$, the above equation reduces to

$$S^-(d_{34}) = \sum_{\ell | P(z_0)} \prod_{j=3}^{4} \left( \sum_{d_j | P_j(z_0)} \lambda_{d_j}^+ \omega_1(m, d_j) \right) \quad \text{(6.16)}$$

We next apply Lemma 6.4 inside the product of the above equation to bound $S^-(d_{34})$ from below as

$$S^-(d_{34}) \geq \sum_{\ell | P(z_0)} \prod_{j=3}^{4} \mu(\ell) \left( \mu(\xi) \prod_{p|\xi} \frac{\omega_1(m, p)}{p - \omega_1(m, p)} V_j(z_0)(1 - C_\beta(s)) \right). \quad \text{(6.20)}$$

Finally inserting (6.20) into (6.19), we can conclude our lemma.

The next lemma provides the upper bound of $\Sigma^+(D, z_0)$.

**Lemma 6.7.** For $\beta \geq 5$, we have

$$\Sigma^+(D, z_0) \leq (1 + C_\beta(s))^2 \Sigma_{MT}(D, z_0).$$

**Proof.** We can rephrase $\Sigma^+(D, z_0)$ from (6.18) that

$$\Sigma^+(D, z_0) = \sum_{d_{34} | P(z_0)} g(d_{34}) S^+(d_{34})$$

where

$$S^+(d_{34}) := \sum_{d_3 | P_3(z_0)} \sum_{d_4 | P_4(z_0)} \sum_{\gcd(d_3, d_4) = d_{34}} \lambda_{d_3}^+ \lambda_{d_4}^+ \frac{\omega_1(m, d_3) \omega_1(m, d_4)}{d_3 d_4}.$$

Now applying Lemma 6.4, the proof follows similarly as in the proof of Lemma 6.6.

**Lemma 6.8.** We have

$$\Sigma_{MT}(D, z_0) = \left(1 - \frac{\omega_1(m, 3)}{3}\right) \left(1 - \frac{\omega_1(m, 5)}{5}\right) \prod_{p | P(z_0)} \left(1 - \frac{\Omega(m, p)}{p}\right)$$

**Proof.** It follows from Lemma 6.6 and Lemma 6.7 that

$$(1 - C_\beta(s))^2 \Sigma_{MT}(D, z_0) \leq \Sigma^-(D, z_0) \leq \Sigma^+(D, z_0) \leq (1 + C_\beta(s))^2 \Sigma_{MT}(D, z_0)$$

Now as $D \to \infty$, we have $\lambda_{d_j}^- = \lambda_{d_j}^+ = \mu(d_j)$. Therefore the definitions (6.16) and (6.17) together imply

$$\lim_{D \to \infty} \Sigma^-(D, z_0) = \lim_{D \to \infty} \Sigma^+(D, z_0) = \left(1 - \frac{\omega_1(m, 3)}{3}\right) \left(1 - \frac{\omega_1(m, 5)}{5}\right) \prod_{p | P(z_0)} \left(1 - \frac{\Omega(m, p)}{p}\right).$$

On the other hand, for $\beta \geq 5$, $a_\beta < 1$ and $s = \frac{\log D}{\log z_0} \to \infty$ as $D \to \infty$, thus $C_\beta(s) \to 0$ as $D \to \infty$. This completes the proof of our lemma.

We next provide the lower bound for $\Sigma_{MT}(D, z_0)$.

**Lemma 6.9.** For $z_0 \geq 7$, we have

$$\Sigma_{MT}(D, z_0) \geq \frac{1.39}{(\log z_0)^3}$$
Proof. The bound of $\omega_1(m, p)$ in (6.1) along with Lemma 5.6 yields

$$\Sigma_{\text{MT}}(D, z_0) \geq \frac{3}{8} \prod_{p \mid P(z_0)} \left(1 - \frac{2.5}{p}\right).$$

It can be observed that for $p \geq 7$, we have $1 - \frac{2.5}{p} \geq \left(1 - \frac{1}{p}\right)^3$. Thus, we can write

$$\Sigma_{\text{MT}}(D, z_0) \geq \frac{3}{8} \prod_{p \mid P(z_0)} \left(1 - \frac{1}{p}\right)^3 \geq 19.77 \prod_{p < z_0} \left(1 - \frac{1}{p}\right)^3.$$  \hspace{1cm} (6.21)

For $\gamma$ denoting the Euler’s constant, the result [10, Equation 3.30, p. 70]

$$\prod_{p < z_0} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log z_0} \left(1 + \frac{1}{\log^2 z_0}\right)^{-1},$$

reduces the bound of $\Sigma_{\text{MT}}(D, z_0)$ in (6.21) as

$$\Sigma_{\text{MT}}(D, z_0) \geq 19.77 \frac{e^{-3\gamma}}{(\log z_0)^3} \left(1 + \frac{1}{\log^2 z_0}\right)^{-3} \geq 19.77 \frac{e^{-3\gamma}}{(\log z_0)^3} \left(1 - \frac{1}{\log^2 7}\right)^3 \geq \frac{1.39}{(\log z_0)^3},$$

where in the penultimate step we have used the fact that $z_0 \geq 7$. This completes the proof of the lemma. \hfill \Box

6.2. Bounds for the error term from sieving. We next bound the cuspidal contribution to obtain a bound for $S(\mathcal{A}, \mathcal{P}_7, z_0)$.

Lemma 6.10. For $\beta \geq 7$, we have

$$\sum_{d_3 \mid P_3(z_0)} \sum_{d_4 \mid P_4(z_0) \text{ and } d_3 < \frac{D}{d_4}} |R(m, d)| \leq 1.38 \times 10^{-34} m^{\frac{3}{2}} D^{14.02}$$

Proof. The proof of the lemma follows immediately from Lemma 5.7 by applying the trivial bound $d_3 d_4 < \frac{D^2}{15}$ for $\beta \geq 7$. \hfill \Box

6.3. The proof of Theorem 1.4. We next invoke Lemma 6.10 into Lemma 6.5 to obtain

$$\Sigma^{-}(D, z_0) r_4(m) - 1.38 \times 10^{-34} m^{\frac{3}{2}} D^{14.02} \leq S(\mathcal{A}, \mathcal{P}_7, z_0) \leq \Sigma^{+}(D, z_0) r_4(m) + 1.38 \times 10^{-34} m^{\frac{3}{2}} D^{14.02}.$$  

The following lemma provides the lower bound of $S(\mathcal{A}, \mathcal{P}_7, z_0)$ for $\beta = 7$ and $D \geq z_0^{21}$.

Lemma 6.11. For $\beta = 7$ and $D \geq z_0^{21}$, we have

$$S(\mathcal{A}, \mathcal{P}_7, z_0) \geq \frac{0.23 r_4(m)}{(\log z_0)^3} - 1.38 \times 10^{-34} m^{\frac{3}{2}} D^{14.02}$$

Proof. It follows from the definition (6.7) that for $\beta = 7$ and $D \geq z_0^{21}$,

$$\mathcal{E}_{\beta}(s) \leq \frac{3}{5}.$$  

Inserting the above bound and the bound from Lemma 6.9 into Lemma 6.6 we can bound $\Sigma^{-}(D, z_0)$ from below as

$$\Sigma^{-}(D, z_0) \geq 0.23 (\log z_0)^{-3}.$$  

This completes the proof of the lemma. \hfill \Box

We are now ready to prove Theorem 1.4.
Proof of Theorem 1.4. It follows from (1.1) that for 4 \mid m, one can bound trivially \( r_4(m) \) as
\[
r_4(m) \geq 8m.
\] (6.22)
Therefore Lemma 6.11 together with the bound (6.22) implies that for 4 \mid m, the number of solutions to the equation
\[
x^2 + y^2 + (2^a3^b z_1)^2 + (2^c5^d z_2)^2 = m
\]
with \( p \mid z_1, z_2 \) as long as \( p \geq z_0 \), can be written as
\[
S(A, \mathcal{P}_7, z_0) \geq \frac{1.84 m}{(\log z_0)^3} - 1.38 \times 10^{-34} m^{\frac{3}{2}} D^{14.02},
\]
where \( D \geq z_0^{21} \). We next choose \( D = z_0^{21} \) and \( z_0 = m^\frac{1}{10} \) to obtain
\[
S(A, \mathcal{P}_7, z_0) \geq \frac{1.84 \cdot (738)^3 m}{(\log m)^3} - 1.38 \times 10^{-34} m^{0.99895}.
\]
Applying the bound
\[
\log m \leq \frac{1}{r} m^r
\]
for \( r = 10^{-6} \), we obtain
\[
S(A, \mathcal{P}_7, z_0) \geq 7.39 \times 10^{-10} m^{0.99999} - 1.38 \times 10^{-34} m^{0.99895}.
\]
Clearly, \( S(A, \mathcal{P}_7, z_0) \) is positive as long as
\[
m^{0.00104} \geq 1.86 \times 10^{-25},
\]
which holds trivially for any natural number \( m \). Therefore, for every \( m \in \mathbb{N} \) with 4 \mid m we have a representation
\[
m = x^2 + y^2 + (2^a3^b z_1)^2 + (2^c5^d z_2)^2
\] (6.23)
where \( x, y, a, b, c, d \) are non-negative integers and \( z_1, z_2 \) has at most 369 prime factors.

Now, for 4 \mid m, we write \( m = 4^\ell m_0 \) such that \( \gcd(4, m_0) = 1 \). It follows from (6.23) that we can represent \( m_0 \) as
\[
m_0 = x'^2 + y'^2 + (2^a3^b' z_1')^2 + (2^c5^d' z_2')^2
\]
for some non-negative integers \( x', y', a', b', c', d' \) and \( z_1', z_2' \) with at most 369 prime factors. Therefore
\[
m = (2^\ell x')^2 + (2^\ell y')^2 + (2^{\ell + \ell}3^{\ell} z_1')^2 + (2^{\ell + \ell}5^{\ell} z_2')^2,
\]
which concludes that every \( m \) can be represented in the form of
\[
m = x^2 + y^2 + (2^a3^b z_1)^2 + (2^c5^d z_2)^2,
\]
for some non-negative integers \( x, y, a, b, c, d \) and \( z_1, z_2 \) with at most 369 prime factors. This completes the proof of our theorem.

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References
[1] S. Banerjee and B. Kane, Finiteness theorems for universal sums of squares of almost primes, to appear in Trans. Amer. Math. Soc.
[2] V. Blomer, Uniform bounds for Fourier coefficients of theta-series with arithmetic applications, Acta. Arith. 114 (2004), 1–21.
[3] J. Brüdern and E. Fouvry, Lagrange’s Four Squares Theorem with almost prime variables, J. reine angew. Math. 454 (1994), 59–96.
[4] H. Iwaniec and E. Kowalski, Analytic number theory, Colloq. Publ. 53, Amer. Math. Soc., 2004.
[5] H. Iwaniec, Rosser’s sieve, Acta Arith. 36 (1980), 171–202.
[6] H. Iwaniec, A new form of the error term in the linear sieve, Acta Arith. 37 (1980), 307–320.
[7] O. T. O’Meara, Introduction to Quadratic Forms, Springer, 1973.
[8] K. Ono, K. Soundararajan, *Ramanujan's ternary quadratic form*, Invent. Math. 130 (1997) 415–454.
[9] H. Robbins, *A remark on Stirling’s formula*, Amer. Math. Monthly 62 (1955), 26–29.
[10] J. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.
[11] G. Shimura, *On modular forms of half-integral weight*, Ann. Math. 97 (1973), 440–481.
[12] C. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. Math. 36 (1935), 527–606.
[13] C. Siegel, *Über die Klassenzahl algebraischer Zahlenkörper*, Acta. Arith. 1 (1935), 83–86.
[14] K. Tsang and L. Zhao, *On Lagrange’s four squares theorem with almost prime*, J. für die Reine Angew. Math. 726 (2017), 129–171.
[15] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. 113 (1965), 1–87.

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