Black rings in $U(1)^3$ supergravity and their dual 2d CFT

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Abstract
We study the near-horizon geometry of black ring solutions in five-dimensional $U(1)^3$ supergravity with three electric dipole charges and one angular momentum. We consider the extremal vanishing horizon (EVH) limit of these solutions and show that the near-horizon geometries develop $\text{AdS}_3$ throats locally. At the near-EVH near horizon limit, the $\text{AdS}_3$ factor turns into a BTZ black hole. By analysing the first law of thermodynamics for black rings we show that at the EVH limit, they reduce to the first law of thermodynamics for BTZ black holes. Using the $\text{AdS}_3/\text{CFT}_2$ duality, we propose a dual CFT to describe the near-horizon low energy dynamics of near-EVH black rings. We also discuss the connection between our CFT proposal and the Kerr/CFT correspondence in the cases where these two overlap.

Keywords: black rings, near-horizon geometries, $\text{AdS}/\text{CFT}$

1. Introduction
One of the main challenges that a quantum theory of gravity should address is to determine what degrees of freedom account for a black hole’s entropy. There has been a lot of progress in the understanding of black hole thermodynamics in some special cases within string theory, involving extremal and near-extremal black holes, following the eminent work of Strominger and Vafa [1]. This analysis has been generalized to many other extremal and near extremal black hole solutions in different theories of gravity in different dimensions.

If the near-horizon geometry of a black hole solution contains an $\text{AdS}_3$ factor, there exists a simpler microscopic model to describe black hole entropy. Since quantum gravity in asymptotic $\text{AdS}_3$ space has a dual conformal field theory (2d-CFT), the entropy of these extremal black holes can be counted, using the universal properties of the dual 2d-CFT [2].
More recently, there have been several attempts to find a holographic description for more generic extremal black holes, which have AdS2 throats in their near horizons. These have involved either the AdS2/CFT1 correspondence [3] or the Kerr/CFT correspondence [4]. For black holes with a compact horizon these proposals suggest the existence of a non-dynamical dual description. Thus, finite energy excitations are not allowed and vacuum degeneracy of the dual CFT accounts for the macroscopic black hole entropy.

Here, we continue our investigations of the extremal vanishing horizon (EVH) black holes by studying five-dimensional rotating black ring solutions within the U(1)3 supergravity. These solutions have been found recently [5]. They carry a single angular momentum, mass and three electric dipole charges. We show that the near-horizon geometry of these solutions at the EVH limit contains an AdS3 throat. The EVH limit can be achieved in two different ways, stationary and static limit. In the stationary limit, where the angular momentum vanishes, the AdS3 part of the near horizon turns to a BTZ black hole at near-EVH limit. Using AdS/CFT correspondence, we propose a 2d CFT that governs near-EVH black ring dynamics and accounts the entropy.

In the static EVH limit black rings where the angular momentum vanishes and we have static solutions, we find that near-horizon geometry is well-defined in five dimensions. This is in contrast with the stationary EVH black hole solutions where the near-horizon geometry is well-defined only after uplifting the solutions to higher dimensions [6]. However, we argue that the excitations do not lead to the BTZ in the near-horizon geometry.

We also study the first law of thermodynamics for these black rings. We show that, despite the existence of the conical singularity, one can define thermodynamic quantities properly and that they obey the first law. This reduces to the first law of thermodynamics for the BTZ black hole when we take the near-EVH limit.

2. Dipole charge black rings in five-dimensional U(1)3 supergravity

We consider a particular class of black ring solution within the five-dimensional U(1)3 supergravity. We obtain the theory by dimensional reduction of eleven-dimensional supergravity on a six-dimensional space T2 × T2 × T2, using the following ansatz for metric and the form field

\[ ds^2_{11} = ds^2_{5} + X_1^2 (dz_1^2 + dz_2^2) + X_2^2 (dz_3^2 + dz_4^2) + X_3^2 (dz_5^2 + dz_6^2), \]

\[ A = A_1 dz_1 \wedge dz_2 + A_2 dz_3 \wedge dz_4 + A_3 dz_5 \wedge dz_6, \]

assuming that the scalar fields \( X_i \) obey constraint \( X_1 X_2 X_3 = 1 \), which ensures that the internal 6-dimensional space has a fixed volume.4 The Bosonic part of the reduced action is given by

\[ S = \frac{1}{16\pi G_5} \int d^5 \sqrt{-g} \left( R - \frac{1}{2} \left( \partial \phi \right)^2 - \frac{1}{4} \sum_{i=1}^{3} X_i^{-2} F_i^2 + \frac{1}{4} \epsilon_{\mu \nu \rho \lambda} F_1^{\mu \nu} F_2^{\rho \lambda} A_3 \right), \]

where we defined \( F_i = dA_i, \hat{\phi} = (\phi_1, \phi_2) \) and

\[ X_1 = e^\frac{\phi_1}{\gamma_1}, \quad X_2 = e^\frac{\phi_2}{\gamma_2}, \quad X_3 = e^\frac{\phi_3}{\gamma_3}. \]

Alternatively, the five-dimensional theory can be obtained from \( T^4 \times S^1 \) compactification of IIB supergravity through the following ansatz:

4 By adding a warped factor and allowing a non-compact internal space the same KK-reduction ansatz also works for the gauged supergravity [7].
\[ ds_{10}^2 = X_2^2 dx_2^2 + X_3^2 (dz_5 + A_3)^2 + X_i X_7^2 (dz_i^2 + dz_3^2 + dz_4^2), \]  
\[ e^{2\phi} = \frac{X_2}{X_3}, \quad F_{(3)} = X_7^{-2} * F_1 + F_2 \wedge (dz_5 + A_3), \]  
where \( *F_1 \) is the dual form with respect to the five-dimensional metric, \( F_{(3)} \) is the RR 3-form field strength and \( \phi \) is the dilaton.

2.1. Neutral single-spin black ring solution

Let us start with the simplest black ring solution in five dimensions, i.e. neutral single-spin black ring solution. The solution has been constructed in [8]. We adapt the notation [9] which is more suitable for our study:

\[
\begin{aligned}
\rho_{nl} &= - F(y) \left( dt - \frac{RC(y)}{\sqrt{K_5(y)}} d\psi \right)^2 \\
&+ \frac{R^2 F(x)}{(x - y)^2} \left[ - \frac{G(y)}{K_5(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{K_5(x)} d\phi^2 \right],
\end{aligned}
\]  
where
\[
F(\xi) = 1 + \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 + \nu \xi),
\]
\[
C = \sqrt{\frac{\lambda (\lambda - \nu)}{(1 - \lambda)^2}}, \quad K = \frac{(1 - \nu)^2}{1 - \lambda}.
\]  
The dimensionless parameters \( \nu \) and \( \lambda \) lie in the range
\[
0 < \nu \leq \lambda < 1,
\]  
and \( x \in [-1, 1], \quad y \in (-\infty, -1] \) and \( \phi, \psi \in [0, 2\pi] \). Asymptotic infinity, which is a flat five-dimensional Minkowski space-time is located at \( x = y = -1 \) and black ring horizon at \( y = -\nu^{-1} \). The solution has a conical singularity at \( x = 1 \). The deficit/defect is given by
\[
\Delta = 2\pi \left( 1 - \frac{(1 + \nu)^2}{(1 - \nu)^2}(1 + \lambda) \right).
\]  
To avoid the conical singularity (\( \Delta = 0 \)) we must take it that
\[
\lambda = \lambda_\nu = \frac{2\nu}{1 + \nu^2}.
\]  
In our analysis we relax this condition. The ADM mass, angular momentum, angular velocity at the horizon, entropy and Hawking temperature for these solutions are given by
\[
\begin{aligned}
M_0 &= \frac{3\pi R^2}{4G_5} \frac{\lambda}{1 - \nu}, \quad J_\nu = \frac{\pi R^3}{2G_5} \sqrt{\frac{\lambda (\lambda - \nu)(1 + \lambda)}{(1 - \nu)^2}}, \\
\Omega_\psi &= \frac{1}{R} \sqrt{\frac{\lambda - \nu}{\lambda (1 + \lambda)}},
\end{aligned}
\]
\[
S = \frac{2\pi^2 R^3}{G_5} \frac{\nu^2 \sqrt{\lambda (1 - \lambda^2)}}{(1 - \nu)^2(1 + \nu)}, \quad T_H = \frac{1 + \nu}{4\pi R} \sqrt{\frac{1 - \lambda}{\lambda \nu (1 + \lambda)}}.
\]  
It is straightforward to check that for general values of the parameters we have the Smarr relation:

\[5\]  
To compare this solution to the general one discussed in [10] we take \( \sigma \to 0, \mu \to \nu, \xi \to \lambda, k \to \frac{1 - \lambda}{\lambda^{-\nu} - \nu} \).
When the conical singularity is absent $\lambda = \lambda_c$, the first law of thermodynamics
d\mathcal{M}_0 = TdS + \Omega_v dJ_v, \tag{2.14}
is obeyed. This is no longer valid for an unbalanced black ring with $\lambda \neq \lambda_c$. This is due to the
energy corresponding to the conical singularity in space-time, which needs to be considered in the
first law properly.

2.1.1. First law of thermodynamics in the presence of conical singularity. Although it seems
that the existence of a conical singularity is a pathology for defining thermodynamics of the
black ring, the solutions have well-defined Euclidean action, and therefore we expect they
have well-defined thermodynamic properties. In [11] it is argued that for asymptotic flat
solutions with conical singularity, if we work with the appropriate set of thermodynamic
variables, the area relation of Bekenstein–Hawking entropy still holds, but the mass which
enters in the first law of thermodynamics is different from the ADM mass and the difference
is the energy associated with the conical singularity, as seen by an asymptotic, static observer.
The first law of thermodynamics should be modified accordingly:

\[ d\mathcal{M} = TdS + \Omega_v dJ_v + \mathcal{P}dA. \tag{2.15} \]
The first two terms on the right-hand side are the standard ones, $J_I$ indicates charge/angular
momentum of the black hole and $\Omega_I$ is its momentum conjugate. The last term accounting for
the effect of the conical singularity, which is exerting a pressure $\mathcal{P}$ with world-volume spanning a space-time area $\beta A$, computed in the Euclidean section where $\beta$ is the periodicity of the Euclidean time.

The energy associated with the conical singularity which contributes to the mass defined by
(2.15) can be evaluated and is given by $E_{\text{int}} \equiv M_0 - \mathcal{M} = -\mathcal{P}A$.

For solution (2.7) we obtain

\[ \mathcal{A} = \frac{\pi R^2 \sqrt{1 - \lambda^2}}{4G_s(1 + \nu)}, \quad \mathcal{P} = \frac{(1 + \nu) \sqrt{1 - \lambda}}{(1 - \nu) \sqrt{1 + \lambda}} - 1. \tag{2.16} \]

Modified Smarr formula is given by

\[ M = \frac{3}{2} (TS + \Omega_v J_v) + PA. \tag{2.17} \]

2.1.2. Myers–Perry black hole limit. As a particular limit of metric (2.7) we can obtain the
Myers–Perry black hole with rotation in a single plane if $R \to 0$ and $\lambda, \nu \to 1$, while
maintaining the fixed parameters $a, m$. This is achieved by taking the limit

\[ \lambda = 1 - \epsilon^2, \quad \nu = 1 - \frac{me^2}{m - a^2}, \quad R = \frac{me}{\sqrt{2(m - a^2)}}, \quad \epsilon \to 0. \tag{2.18} \]

Note that in this limit, condition (2.11) is satisfied and the solution does not have conical
singularity while ADM mass, Hawking temperature, entropy and angular momentum remain
finite:

\[ M_0 = \frac{3}{8G_s} \pi m, \quad T = \sqrt{\frac{m - a^2}{2\pi m}}, \quad J = \frac{1}{4G_s} \pi am, \quad S = \frac{1}{2G_s} \pi^2 m \sqrt{m - a^2}. \tag{2.19} \]
It is more convenient to use the following coordinate transformation
\[
x = -1 + \frac{(m - a^2)\cos^2 \theta}{r^2 - (m - a^2)\cos^2 \theta} \epsilon^2, \quad y = -1 - \frac{(m - a^2)\sin^2 \theta}{r^2 - (m - a^2)\cos^2 \theta} \epsilon^2,
\]
leading to
\[
ds^2 = -dt^2 + \frac{m}{\Sigma} (dt - a^2 \sin^2 \theta d\psi)^2 + \frac{\Sigma}{\Gamma} dr^2 + \Sigma d\theta^2 + (a^2 + r^2) \sin^2 \theta d\psi^2 + r^2 \cos^2 \theta d\phi^2,
\]
where
\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Gamma = r^2 - (m - a^2).
\]

The coordinate transformation (2.20) looks like a small expansion near the asymptotic infinity which is located at \(x = y = -1\). In this limit, the horizon which is located at \(y = -\nu^{-1} \approx -1 - \epsilon^2\), may also seem close to the asymptotic infinity. But we must note that in this limit \(xx\) and \(yy\) components of the metric are also of the order \(\epsilon^2\) and one can show that the horizon is infinitely far from the asymptotic infinity. Moreover, for a generic value of \(r\), points in (2.20) are at finite distance from the horizon and arbitrarily far from the asymptotic infinity. It is also useful to compare (2.20) with taking the asymptotic limit of black ring solutions which is obtained by
\[
x = -1 + \epsilon^n \tilde{x}, \quad y = -1 - \epsilon^n \tilde{y}, \quad n > 2,
\]
and yields
\[
ds^2 = -dt^2 + m \epsilon^{2-n} \left( \frac{d\tilde{x}^2}{\tilde{x}} + \frac{d\tilde{y}^2}{\tilde{y}} + \tilde{x}d\phi^2 + \tilde{y}d\psi^2 \right).
\]
This is a five-dimensional flat space-time metric, written in the ring coordinate system.

2.1.3. The EVH limit. For the black rings (2.7) with finite mass, the extremal limit where Hawking temperature vanishes corresponds to \(\lambda \to 1\). However, from (2.12), in this limit entropy vanishes while the ADM mass and angular momentum remain finite with \(M^3 = \frac{27}{32G} \pi J^2\).

The extremal vanishing horizon (EVH) limit for many different of black hole/ring solutions in different dimensions is studied in detail (see, for example, [12]–[16]). Note that this limit does not necessarily give a small black hole or a naked singularly. One can obtain this limit properly and the resulting solution have a regular horizon everywhere, except at some isolated points, with finite mass. In [16], we have shown that, under very general assumptions, the near-horizon geometry of an EVH black hole develops an AdS\(_3\) factor. This AdS\(_3\) is replaced by a BTZ black hole when we take the near-horizon limit of a near-EVH black hole.

Taking \(\lambda = 1 - \epsilon^2\) and expanding thermodynamic quantities for small \(\epsilon\) we find that \(T \sim S \sim \epsilon\). Then, we are interested in studying the near-horizon geometry of the black ring solution in this limit. By taking the near-horizon limit, we want to focus on the region of space-time close to the horizon. Since we introduce a small parameter \(\epsilon\), while we take near-horizon geometry one can imagine three possibilities: where distance to the horizon is small but much larger than \(\epsilon\), is of the same order, or is much smaller than \(\epsilon\). As we have shown in [16], the first two possibilities correspond to the near-horizon EVH (NHEVH) geometry and near-EVH near-horizon limit, respectively. The last case would correspond too far from the EVH case and we will exclude it.
Finding the near horizon as defined above is subtle in the original ring coordinate system. It is, then, useful to consider the following coordinate transformation:

\[
\begin{align*}
x &= -1 + \frac{(1 - \nu)(1 - \lambda)\cos^2 \theta}{r^2 + \nu(1 - \lambda)\cos^2 \theta}, \\
y &= -1 - \frac{(1 - \nu)(1 - \lambda)\sin^2 \theta}{r^2 + \nu(1 - \lambda)\cos^2 \theta},
\end{align*}
\]

where \( \theta \in [0, \pi/2] \). In this coordinate system the horizon is located at \( r = 0 \) and asymptotic infinity at \( r = \infty \). Note that this coordinate system covers regions only outside of the horizon. Now to take near-horizon geometry we scale \( d \to r \) with \( \epsilon \ll \delta \ll 1 \) and expand the metric for small parameters \( \delta \) and \( \epsilon \). This ensures we are zooming into the near-horizon region of a distance of about \( \delta \). By expanding the metric we get

\[
d s^2 = -4r^2 \cos^2 \theta \delta^2 d t^2 + \frac{2R^2 \cos^2 \theta}{1 - \nu} d r^2 + \frac{2R^2 \cos^2 \theta}{1 - \nu} \delta^2 d \phi^2 \\
+ \frac{2R^2 \cos^2 \theta}{1 - \nu} d \theta^2 + \frac{R^2 \sin^2 \theta}{2(1 - \nu)\cos^2 \theta} \left( d \psi - \frac{\sqrt{2}(1 - \nu)}{R} d t \right)^2.
\]

The first line in the above metric has an AdS_3 factor, anticipated by theorems 1 and 2 in [16]. It is more convenient, however, to use the following coordinate transformation

\[
d \phi = \frac{d \tilde{\phi}}{\delta}, \quad d \psi = 2d \tilde{\psi} + \frac{\sqrt{2}(1 - \nu)}{2R} d t, \quad d t = \frac{R}{\sqrt{2 - 2\nu}} \frac{d x}{\delta}.
\]

The metric (2.26) then turns into

\[
d s^2 = \frac{2R^2}{1 - \nu} \cos^2 \theta \left[ -r^2 d t^2 + \frac{d r^2}{r^2} + r^2 d \phi^2 \right] + \frac{2R^2}{1 - \nu} \left( \cos^2 \theta d \theta^2 + \frac{\sin^2 \theta}{\cos^2 \theta} d \tilde{\psi}^2 \right).
\]

It is straightforward to check that this is a solution to the vacuum Einstein equations. It is regular everywhere, except isolated point \( \theta = \frac{\pi}{2} \). This is where the horizon touches the singularity in the black ring solution when we take the EVH limit. The AdS_3 angular direction \( \phi \) is pinching with range \( \phi \in [0, 2\pi \delta] \). It is also clear that the AdS_3 in the near horizon does not turn to a BTZ at near-EVH limit. This can be understood by looking at the more general black ring solution with two angular momenta and its EVH limit [15]. The only possible excitations are in the second angular momentum direction.

2.2. Adding dipole charges

The black ring solutions which we are studying here have mass, three dipole charges and one angular momentum. These solutions were constructed in [5]. A more general class of black ring solutions with two independent angular momenta and electric dipole charges was constructed in [17]:

\[
d s^2 = -\frac{y}{xU^2} \left[ d t + (y^{-1} - \eta_x^2) a Q_2 d \phi_1 \right]^2 + \frac{x}{y(x - y)^2 U^2} \left( -a^2 G(y) d \phi_1^2 - \frac{\mathcal{H}(y)}{G(y)} dy^2 \right) \\
+ \frac{U^2}{(x - y)^2} \left( \frac{\mathcal{H}(x)}{G(x)} dx^2 + a^2 G(x) d \phi_2^2 \right).
\]

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where \( x \in [\eta_1^2, \eta_2^2] \), \( y \in [-\eta_4^2, \eta_3^2] \) and we defined

\[
\mathcal{H}(\xi) = \xi h_1(\xi) h_2(\xi) h_3(\xi), \quad G(\xi) = -\mu^2 (\xi + \eta_1^2)(\xi + \eta_2^2)(\xi - \eta_4^2)(\xi - \eta_3^2),
\]

\[
a = \frac{2\eta_1 \sqrt{(\eta_1^2 + q_1)(\eta_1^2 + q_2)(\eta_1^2 + q_3)}}{\mu^2(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)(\eta_2^4 - \eta_3^4)}, \quad Q_i^2 = -G(-q_i), \quad Q_4^2 = \mu^2 \eta_1^2 \eta_2^2 \eta_3^2 \eta_4^2,
\]

\[
h_i(\xi) \equiv \xi + q_i, \quad U_i \equiv \frac{h_i(y)}{h_i(x)}, \quad U \equiv U_1 U_2 U_3, \quad i = 1, 2, 3.
\]

\[\eta_1 > \eta_2 > 0, \quad \eta_4 > \eta_3 > 0. \tag{2.30}\]

Scalar and gauge fields are

\[
X_i = \frac{U_i}{U^2}, \quad A_i = \frac{a Q_i}{h_i(x)} \frac{d \phi_2}{2} . \tag{2.31}\]

It is straightforward to check that the black ring solution (2.29) for a generic value of parameters has a conical singularity. However, as it has been shown in [5] this can be avoided by imposing the following constraint on the black ring parameter space:

\[
\frac{(\eta_1^2 + \eta_2^2)(\eta_2^2 + \eta_3^2) \sqrt{\mathcal{H}(\eta_1^2)}}{(\eta_1^2 + \eta_3^2)(\eta_2^2 + \eta_3^2) \sqrt{\mathcal{H}(\eta_2^2)}} = 1 . \tag{2.32}\]

The neutral black ring solution is obtained in the \( q_i \to \eta_1^2 \) limit. To get metric (2.7) one should perform following coordinate transformations:

\[
x \to \frac{c_1 + \eta_1^2 x}{c_2 - x}, \quad y \to \frac{c_1 + \eta_1^2 y}{c_2 - y}, \quad c_1 = \frac{\eta_1^2 \eta_2^2 + \eta_1^2 \eta_3^2 + 2 \eta_2^2 \eta_4^2}{\eta_4^2 - \eta_3^2} , \quad c_2 = \frac{2 \eta_1^2 + \eta_3^2 + \eta_4^2}{\eta_4^2 - \eta_3^2} . \tag{2.33}\]

The neural black ring parameters are given by

\[
\lambda = \frac{\eta_1^2 (\eta_4^2 - \eta_3^2)}{\eta_1^2 \eta_2^2 + \eta_1^2 \eta_3^2 + 2 \eta_2^2 \eta_4^2}, \quad \nu = \frac{(\eta_1^2 - \eta_3^2)(\eta_2^2 - \eta_4^2)}{\eta_1^2 \eta_2^2 + \eta_1^2 \eta_3^2 + \sum_{j > 1} \eta_1^2 \eta_j^2},
\]

\[
R^2 = \frac{4 (\eta_1^2 + \eta_3^2)(\eta_2^2 + \eta_4^2)}{\mu^2(\eta_4^2 - \eta_3^2)(\eta_1^2 \eta_2^2 + \eta_1^2 \eta_3^2 + \sum_{j > 1} \eta_1^2 \eta_j^2)} . \tag{2.34}\]

The solution (2.29) is over-parameterized. This can be also seen in (2.34). Among five parameters \( \eta_i \) and \( \mu \) only three combinations in (2.34) are independent which, along with three parameters \( q_i \), give the six independent parameters of the original dipole black ring. The absence of a conical singularity (2.32) eliminates one extra parameter and leaves five real parameters corresponding to three dipole charges, mass and angular momentum.

2.2.1. Charges and thermodynamics. The angular momentum and three dipole electric charges of the family of black ring solutions can be evaluated using Komar and Gaussian integrals respectively:
\[ J \equiv J_{\phi_1} = \frac{\pi a Q_4}{4 G_5 b^2 \eta_3}, 
J_{\phi_2} = 0 \]
\[ D_i = \frac{\pi a Q_4}{4 G_5} \left( \frac{1}{q_i + \eta_3} - \frac{1}{q_i + \eta_3^2} \right), \quad (2.35) \]

where
\[ b^2 = \frac{\mu^2 (\eta_1^2 + \eta_3^2) (\eta_2^2 + \eta_3^2) (\eta_1^2 - \eta_3^2)}{4 \eta_1^2 (\eta_1^2 + q_1) (\eta_2^2 + q_2) (\eta_3^2 + q_3)}. \quad (2.36) \]

The horizon is located at \( y = -\eta_3^2 \) and its topology is \( S^1 \times S^2 \) where \( S^1 \) is along the \( \phi_1 \) direction and \( S^2 \) corresponds to \((x, \phi_2)\) directions. The horizon structure determines the thermodynamic properties of the black ring. The Hawking temperature can be computed through the surface gravity at the horizon, leading to
\[ T_H = \frac{\mu \eta_3 (\eta_1^2 - \eta_2^2) (\eta_3^2 + \eta_3^2)}{4 \pi \eta_1 \eta_3 \sqrt{(\eta_1^2 - \eta_2^2)(\eta_2^2 - \eta_3^2)(\eta_3^2 - \eta_3^2)}}. \quad (2.37) \]

and the Bekenstein–Hawking entropy, which is proportional to the area of the black ring horizon, is given by
\[ S = \frac{\pi^2 a^2 Q_4 (\eta_1^2 - \eta_3^2)}{G_5 \eta_2 \eta_3^2 (\eta_2^2 + \eta_3^2)} \sqrt{(\eta_1^2 - \eta_2^2)(\eta_2^2 - \eta_3^2)(\eta_3^2 - \eta_3^2)). \quad (2.38) \]

The Killing horizon is generated by the Killing vector field \( \ell = \partial_t + \Omega \partial_{\phi_1} \) where \( \Omega \) stands for the angular velocity on the horizon:
\[ \Omega \equiv \Omega_{\phi_1} = \frac{\eta_2^2 \eta_3^2}{a Q_4 (\eta_2^2 + \eta_3^2)}. \quad (2.39) \]

The asymptotic region is located at \( x = y = \eta_3^2 \), where to get the Minkowski metric we make the coordinate transformation
\[ r \sin \theta = \frac{\sqrt{\eta_3^2 - y}}{b(x - y)}, \quad r \cos \theta = \frac{\sqrt{x - \eta_3^2}}{b(x - y)}, \quad (2.40) \]
then taking limit \( r \to \infty \) we get
\[ ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2). \quad (2.41) \]

The ADM mass can be read off from the asymptotic fall-offs of the metric
\[ M_0 = \frac{\pi}{8 b^2 G_5} \left( \frac{3}{\eta_3^2} - \frac{1}{\eta_3^2 + q_1} - \frac{1}{\eta_3^2 + q_2} - \frac{1}{\eta_3^2 + q_3} \right). \quad (2.42) \]

The electrostatic potentials \( \Phi_i \) associated with the electric dipole charges are given by
\[ \Phi_{D_i} = a Q \left( \frac{1}{q_i - \eta_2^2} - \frac{1}{q_i + \eta_3^2} \right). \quad (2.43) \]

One can check that the ADM mass defined by (2.42) obeys the Smarr formula:
\[ M_0 = \frac{3}{2} TS + \frac{3}{2} \Omega J_{\phi_1} + \frac{1}{2} \sum_{i=1}^{3} \Phi_{D_i} D_i. \quad (2.44) \]

It is noted in [5] that when the condition (2.32) is satisfied, and therefore solutions do not have the conical singularity, thermodynamic quantities defined above satisfy the first law of
thermodynamics:
\[ dM_0 = T_H dS + \Omega dJ + \sum_{i=1}^{3} \Phi_{\alpha_i} d\alpha_i. \] (2.45)

However, this is not valid for generic values of charges where the solutions (2.29) have conical singularity.

For the black ring solution (2.29) the first law of thermodynamics reads \(^6\)
\[ dM = T_H dS + \Omega dJ + \sum_{i=1}^{3} \Phi_{\alpha_i} d\alpha_i + P dA, \] (2.46)

where \(P\) and \(A\) are given by
\[ P = -\frac{\Delta}{2\pi}, \quad A = \frac{\text{Area}}{4G_5} \times T_H, \] (2.47)

where \(\Delta\) is the deficit/excess angle. For solution (2.29), \(P\) is given by
\[ P = \frac{(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)}{(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)\sqrt{\mathcal{H}(\eta_2^2)}} - 1. \] (2.48)

The world volume of the conical defect is a 3d surface and its area is given by
\[ \text{Area} = \int \sqrt{\det h} \, d\tau d\phi_1 dy, \] (2.49)

where \(h\) is the induced metric on the 3d surface spanned by the conical singularity for the solution (2.29). It reads
\[ A = \frac{\pi \eta_3 \eta_4 \sqrt{\mathcal{H}(\eta_2^2)\mathcal{H}(\eta_3^2)}}{G_5 \mu^2 (\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)^2}. \] (2.50)

From (2.46) we find that
\[ E_{\text{int}} = \frac{\pi \sqrt{\mathcal{H}(\eta_2^2)} \left[ (\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)\sqrt{\mathcal{H}(\eta_2^2)} - (\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_3^2)\sqrt{\mathcal{H}(\eta_3^2)} \right]}{G_5 \mu^2 (\eta_1^2 + \eta_2^2)^2 (\eta_1^2 + \eta_3^2) (\eta_1^2 + \eta_3^2)^2 (\eta_1^2 + \eta_2^2)} \] (2.51)

From (2.32) it is clear that in the absence of a conical singularity the last term in the first law of thermodynamics (2.46) and the correction to black ring mass (2.51) vanish and we get back to the familiar form of the first law. \(^7\) One can check these satisfy modified Smarr mass formula:
\[ M = \frac{3}{2} TS + \frac{3}{2} \Omega_4 J_{\psi_4} + \frac{1}{2} \sum_{i=1}^{3} \Phi_{\alpha_i} D_i + P A. \] (2.52)

Using the fact that \(E_{\text{int}} = -P A\), this reduces to (2.44).

2.2.2. Near-horizon geometry and Kerr/CFT correspondence. The extremal black ring solutions correspond to \(\eta_1 = \eta_2\) limit where Hawking temperature vanishes. To take the near-horizon geometry, we consider the following rescalings:

\(^6\) The ambiguity in defining \(A\) and \(P\) is resolved by noting that \(P\) corresponds to the pressure exerted by the strut. This can be obtained directly by integrating the corresponding component of energy-momentum tensor.

\(^7\) In \([5]\) authors noticed that the black ring solutions obey the first law (2.45) only in the absence of a conical singularity.
\[ y = - \eta_2^2 + x, \quad t = \frac{1}{\epsilon} k^{-}, \quad \phi_1 = \phi_1 - \omega t, \]
\[ k = \frac{\eta_2 \eta_4 \sqrt{(q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2)}}{\mu \eta_3^2 (\eta_2^2 + \eta_3^2)}, \quad \omega = \frac{\mu (\eta_2^2 \eta_3^2 + \eta_2^4 - \eta_3^2 \eta_4^2)}{2 \eta_4 \sqrt{(q_1 + \eta_2^2)(q_2 + \eta_2^2)(q_3 + \eta_2^2)}}, \]
and take limit \( \epsilon \to 0 \). We get
\[ ds^2 = A(x) \left( -Y^2 dr^2 + \frac{dY^2}{Y^2} + B(x) (d\phi_1 + k_{\phi_1} Y d\epsilon)^2 \right) + C(x) (dx^2 + E(x) d\phi_2^2). \]

The three-dimensional part of the near-horizon metric is a U(1) bundle over AdS_2. Functions \( A \) and \( B \) and constant \( k_{\phi_1} \) are given by
\[ k_{\phi_1} = \frac{\eta_3^2 (\eta_4^2 - \eta_2^2) \sqrt{(q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2)}}{2 \eta_2 \eta_4 \eta_3^2 \sqrt{\mathcal{H}(\eta_3^2)}}, \]
\[ A(x) = \frac{((q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2))^2}{\mu^2 (\eta_2^2 + \eta_4^2)(\eta_3^2 + \eta_4^2)(x + \eta_2^2)^2}, \]
\[ B(x) = \frac{4 \eta_2 \eta_4 \eta_3^2 (\eta_3^2 + \eta_4^2)^2 \mathcal{H}(\eta_3^2)(x + \eta_2^2)^2}{\eta_2^4 (q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2)(\eta_2^2 + \eta_4^2)(\eta_3^2 + \eta_4^2)(\eta_4^2 - \eta_3^2)^2 x^2}, \]
\[ C(x) = \frac{((q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2))^2}{\mu^2 (\eta_4^2 - x)(x - \eta_2^2)(x + \eta_2^2)^2}, \]
\[ E(x) = \frac{4 \eta_2^2 (x^2)(x - \eta_2^2)^2 (x + \eta_2^2)^4 \mathcal{H}(\eta_2^2)}{(\eta_2^2 + \eta_4^2)^4 (\eta_3^2 - \eta_4^2)^2 \mathcal{H}(x)}. \]

According to the Kerr/CFT dictionary, this fixes the dual CFT Frolov–Thorne temperature:
\[ T_{\phi_1} = \frac{1}{2 \pi k_{\phi_1}}, \]

The central charge of corresponding chiral Virasoro algebra is obtained from the asymptotic symmetry group analysis:
\[ c_{\phi_1} = \frac{12 \pi \eta_3 \eta_4 (q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2) \sqrt{\mathcal{H}(\eta_3^2)}}{G \mu^3 (\eta_2^2 + \eta_4^2)^4 (\eta_3^2 + \eta_4^2)^2}, \]
and entropy of the extremal black ring can be reproduced upon using Cardy’s formula
\[ S = \frac{\pi^2}{3} c_{\phi_1} T_{\phi_1} = \frac{4 \pi^2 \eta_2 \eta_4 \mathcal{H}(\eta_2^2) \sqrt{(q_1 - \eta_2^2)(q_2 - \eta_2^2)(q_3 - \eta_2^2)}}{G \mu^3 \eta_3 (\eta_2^2 - \eta_3^2)(\eta_2^2 + \eta_4^2)(\eta_3^2 + \eta_4^2)^4}, \]
which is equal to the Bekenstein–Hawking entropy (2.38) evaluated at the extremal point.

### 3. Extremal vanishing horizon (EVH) limits of dipole black rings

The extremal limit where Hawking temperature (2.37) vanishes corresponds to \( \eta_1 = \eta_2 \) or \( \eta_3 = 0 \). In the space of extremal black ring solutions we are interested in studying those with
vanishing horizon area. We are physically defining the subset of EVH configurations as a limit of near-extremal black rings in which the area of horizon $A_h \sim T_H \sim \epsilon \to 0$, keeping the ration $A_h/T_H$ finite. Inspection of (2.37) and (2.38) reveals $\eta_1 \sim \eta_2 \sim \epsilon$ or $\eta_3 \sim \epsilon$ as the EVH regions.

3.1. Stationary case: $\eta_3 \sim 0$

Taking the vanishing limit of $\eta_3$ while keeping other black ring parameters finite, we get the entropy and Hawking temperature vanishing at the same rate. Dipole charges are all vanishing in this limit while angular momentum remains finite. This branch is rather similar to the EVH limit of neutral single-spin black ring studied in section (2.1). The shrinking cycle of the horizon is $\phi_2$ which is inside the $S^2$ part of the horizon. To take the near-horizon limit we consider coordinate transformation (2.25) where $\lambda$, $\nu$ and $R$ are defined by (2.34) and then scale $r \to \delta r$ with $\epsilon \ll \delta \ll 1$. Expanding the metric for small parameters gives

$$ds^2 = \frac{4q_1q_2q_3}{\mu^2\eta_1^2\eta_2^2\eta_4^2} \cos^2 \theta \left[ -\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2 d\tilde{\phi}_2^2 \right] + \frac{4q_1q_2q_3}{\mu^2\eta_1^2\eta_2^2\eta_4^2} \cos^2 \theta d\tilde{\phi}_1^2 + \frac{\sin^2 \theta}{\cos^2 \theta} \frac{d\tilde{\phi}_2^2}{\rho^2},$$

(3.1)

where

$$\rho = \frac{2\eta_1^2 (\eta_2^2 + \eta_2^2)}{2\eta_1^2 \eta_2^2 + \eta_2^2 \eta_4^2 + \eta_2^2 \eta_4^2} r, \quad \tilde{\phi}_2 = \delta \phi_2, \quad \tau = \frac{\mu \eta_1 \eta_2 \eta_4}{\sqrt{4q_1q_2q_3}} \delta t,$$

$$\tilde{\phi}_1 = \phi_1 - \frac{\mu \eta_1 \eta_2 \eta_4}{\sqrt{4q_1q_2q_3}} \delta t.$$  

(3.2)

The first three terms in metric (3.1) form a pinching AdS$_3$ space. Similarly to what we discussed in section (2.1), this cannot be excited to BTZ.

3.2. Static case: $\eta_1 \sim \eta_2 \sim 0$

Assuming $\eta_{1,2}$ are small and the same order of $\epsilon$ and keeping the other black ring parameters finite, we obtain

$$T_H = \frac{\mu \eta_3 \eta_4 (\alpha_1^2 - \alpha_2^2)}{\pi \alpha_1 \sqrt{q_1q_2q_3}} \epsilon + O(\epsilon^3), \quad S = \frac{4\pi^2 \mathcal{H}(\eta_3^2) \sqrt{q_1q_2q_3}}{\mu^3 \eta_3 \eta_4 (\eta_4^2 - \eta_3^2) G_5} \frac{\alpha_1}{\epsilon} + O(\epsilon^3).$$  

(3.3)

The mass and electric dipole charges of black rings remain finite while the angular momentum vanishes at this limit $J \sim \epsilon^2$, in contrast to the stationary case. This is the static limit of solution (2.29). It is also worth noting that the cycle which is shrinking causes vanishing of the horizon area in the $S^1$ part of the horizon while the size of $S^2$ part remains finite.

3.2.1. Near-horizon geometry analysis. We study the near-horizon geometries corresponding to the EVH black rings together with their near-extremal versions. Assuming parameters $\eta_1 \sim \eta_2 \sim \epsilon$ are small and studying deep interior geometry of the EVH black rings by expanding in small $\delta$ for $y = \delta^2 Y^2$ where $\epsilon \ll \delta$, leading terms in the metric expansion are
\[ ds^2 = \frac{4(q_1 q_2 q_3)^2 h_3^2 h^3(x)}{\mu^2 \eta_1^2 \eta_2^2 \eta_3^2 \eta_4^2 (x - \eta_3^2)^2} \left[ -\delta^2 Y^2 dt^2 + \frac{dY^2}{Y^2} + \frac{\delta^2 Y^2 d\phi_4^2}{C^2} \right] + \frac{(q_1 q_2 q_3)^2 h_3^2 h(x)^2}{\mu^2 x^3 (x - \eta_3^2)^2} \left( dx^2 + \frac{4h(x) x^4 (x - \eta_3^2)^2 (\eta_4^2 - x)^2}{\eta_3^2 (\eta_4^2 - \eta_3^2)^2 h(x)} d\phi_4^2 \right) \]

where
\[ C^2 = \frac{\eta_1^2 q_1 q_2 q_3 (\eta_4^2 - \eta_3^2)^2}{\eta_3^2 h(\eta_3^2)}, \quad K^2 = \frac{4\eta_4^2 h(\eta_3^2)}{\mu^2 \eta_3^2 (\eta_4^2 - \eta_3^2)^2}. \] 

The extremality condition determines the scaling \( \delta^2 Y \delta t^2 \) together with \( \frac{dy^2}{dy} \) giving rise to an AdS_2 throat responsible for the SO(2, 1) isometry enhancement of the near-horizon geometry of black rings. However, the new feature here is the vanishing size of the one-cycle along the \( \phi_4 \) direction as \( \delta^2 Y^2 \). This is responsible for the vanishing of the entropy and transforms the standard AdS_2 throat into a local pinching AdS_3 throat. Thus, the near-horizon geometry is obtained by considering the limit
\[ \eta_{1,2} = \alpha_{1,2} \epsilon, \quad Y = C \rho, \quad dt = \frac{Kd\tau}{\delta}, \quad \phi_4 = \frac{\psi}{\delta}, \] 

The resulting metric is
\[ ds^2 = \frac{4(q_1 q_2 q_3)^2 h_3^2 h^3(x)}{\mu^2 \eta_1^2 \eta_2^2 \eta_3^2 \eta_4^2 (x - \eta_3^2)^2} \left[ -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + \rho^2 d\psi^2 \right] + \frac{(q_1 q_2 q_3)^2 h_3^2 h(x)^2}{\mu^2 x^3 (x - \eta_3^2)^2} \left( dx^2 + \frac{4h(x) x^4 (x - \eta_3^2)^2 (\eta_4^2 - x)^2}{\eta_3^2 (\eta_4^2 - \eta_3^2)^2 h(x)} d\phi_4^2 \right). \] 

The near-horizon geometry is a warped product of a deformed two-dimensional sphere and AdS_3 space. Due to 2\pi\delta periodicity in \( \psi \), the near-horizon geometry describes a locally AdS_3 with the unit radius. Besides the pinching, which does not introduce a curvature singularity, the geometry (3.7) is smooth everywhere.\(^8\)

It is straightforward to check that the above geometry is a solution to the U(1)^3 supergravity (2.3). In [16] we classified all solutions to the five-dimensional gauged supergravity with SO(2,2) isometry. To compare metric (3.7) with the generic solution in [16], we apply following coordinate transformations:
\[ x = \frac{\eta_1^2 \eta_3^2}{\eta_3^2 + (\eta_4^2 - \eta_3^2) \cos^2 \theta}, \quad q_i = \frac{\eta_i^2 \eta_4^2}{s_i (\eta_4^2 - \eta_3^2) - \eta_3^2}. \] 

In [6] the EVH limit of charged rotating black hole solutions to U(1)^3 gauged supergravity is studied. It has been shown that the limit contains two branches corresponding to stationary and static cases. For the static one, where the angular momenta vanish at the EVH limit, the near-horizon geometry of the five-dimensional solution is not well-defined. To get a regular geometry one needs to uplift the solution to higher dimensions. This is a rather generic property of all EVH static black hole solutions [6]. For static black hole solutions with a spherical horizon, one can argue that the vanishing horizon limit turns the solution to a singular geometry. However, if the solution can be uplifted to a higher dimension there is a

\(^8\) Although \( x = 0 \) is a singular point but note that \( x \in [\eta_3^2, \eta_4^2] \).
possibility to obtain the well-defined near-horizon geometry in higher dimensions. In this case, the shrinking cycle comes from higher dimensions. This gives room to find a regular geometry at the near horizon.

The black ring solutions (2.29) have \( S^2 \times S^1 \) horizon topology and as we can see from (3.4), the shrinking cycle is the \( S^1 \) part of the horizon which lies in five dimensions and we get the well-defined near-horizon geometry in five dimensions.

3.2.2 Near-EVH near-horizon geometry. Near-EVH black rings are excitations of the EVH vacua, therefore we expect them to be encoded in the near-horizon geometry as pinching BTZ black holes. Indeed, as we shall see in more detail in the next section these excitations are described by mass and angular momentum of the pinching BTZ. These expectations can be verified when we take \( \delta \sim \epsilon \) in taking the near-horizon limit:

\[
y = (\rho^2 - \rho_1^2 - \rho_2^2)C^2e^{2}, \quad \rho_{1,2} = \frac{\alpha_{1,2}}{C}, \quad dt = \frac{Kd\tau}{\epsilon}, \quad \phi_1 = \frac{\psi}{\epsilon},
\]

the metric expansion is

\[
dx^2 = \frac{4(q_1q_2q_3)^2}{\mu^2\eta_3^2\eta_4^2\eta_5^2\eta_7^2H(x)^3} \left[ -\frac{(\rho^2 - \rho_1^2)(\rho^2 - \rho_2^2)d\tau^2}{\rho^2} + \frac{\rho^2d\rho^2}{(\rho^2 - \rho_1^2)(\rho^2 - \rho_2^2)}
+ \rho^2(d\psi - \frac{\rho_1\rho_2d\tau}{\rho^2})^2 + \frac{(q_1q_2q_3)^2}{\mu^2\eta_3^4\eta_4^4\eta_5^4\eta_7^4H(x)^3} \right.
\]

\[
\times \left( \frac{4x^4(x - \eta_3^2)(\eta_4^2 - x)^2H(\eta_3^2)}{\eta_3^8(\eta_4^2 - \eta_5^2)^2H(x)} d\phi_2 \right)^2.
\]

The result is the same as we found in (3.7), but the \( \text{AdS}_3 \) part of the geometry is replaced by a (pinching) BTZ. Excitations appear only on the \( \text{AdS}_3 \) and the \( 2d \) part of the near-horizon geometry is intact.

Temperature and entropy. In order to use thermodynamics laws satisfied by BTZ black holes, we need to compactify our five-dimensional theory to three dimensions. This is achieved by considering the ansatz

\[
dx^2 = \frac{4(q_1q_2q_3)^2}{\mu^2\eta_3^2\eta_4^2\eta_5^2\eta_7^2H(x)^3} dx_5^2 + \frac{(q_1q_2q_3)^2}{\mu^2\eta_3^4\eta_4^4\eta_5^4\eta_7^4H(x)^3} 
\]

\[
\times \left( dx_1^2 + \frac{4x^4(x - \eta_3^2)(\eta_4^2 - x)^2H(\eta_3^2)}{\eta_3^8(\eta_4^2 - \eta_5^2)^2H(x)} d\phi_2 \right),
\]

and plugging it into the action of \( U(1)^3 \) supergravity, focusing on its Einstein–Hilbert term

\[
\frac{1}{16\pi G_3} \int d^5x \sqrt{-g_5} R_5 + \cdots = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g_3} R_3 + \cdots,
\]

where \( R_5 \) and \( R_3 \) are Ricci scalars for the five and three-dimensional metrics \( ds^2 \) and \( ds_3^2 \) respectively. We read the 3d Newton coupling constant \( G_3 \):
Thus, the temperature and entropy of the pinching BTZ black holes (3.10) are given by

\[
T_{\text{BTZ}} = \frac{\rho_1^2 - \rho_2^2}{2\pi \rho_1} = \frac{K}{\epsilon} T_H + \mathcal{O}(\epsilon), \quad S_{\text{BTZ}} = \frac{\pi \epsilon \rho_1}{2G_3} = S + \mathcal{O}(\epsilon^2),
\]

where \(T_H\) and \(S\) are original black ring temperature and entropy, given by (2.37) and (2.38). As expected, the BTZ entropy matches the original black ring entropy whereas its temperature agrees with the scaling of time in (3.9).

Angular momentum and horizon angular velocity of the emergent BTZ black hole are

\[
J_{\text{BTZ}} = \frac{\epsilon \rho_1 \rho_2}{4G_3} = \frac{J}{\epsilon} + \mathcal{O}(\epsilon), \quad \Omega_{\text{BTZ}} = \frac{\rho_2}{\rho_1} = K\Omega + \mathcal{O}(\epsilon^2),
\]

where \(J\) and \(\Omega\) are defined by (2.35) and (2.39), evaluated at the EVH point.

This confirms the expectations when interpreting the near-EVH temperature and entropy (3.3) as BTZ thermodynamic quantities.

BTZ mass is given by

\[
M_{\text{BTZ}} = \frac{\epsilon}{8G_3} (\rho_1^2 + \rho_2^2).
\]

Comparing it with the original black ring mass reveals an interesting relation between the first law in three and five-dimensional theories. We will discuss this in the next section.

4. First law analysis, 5d versus 3d

In this section we analyse the first law of thermodynamics for the class of EVH black rings we studied in the previous sections. We want to see how angular momentum and the dipole charges appears in the first law in the EVH limit and study whether our general first law analysis \[14\] applies to the near-EVH black ring case.

To do that, let us expand black ring thermodynamic quantities, such as mass, dipole charges and conjugate momenta as well as \(\Pi\) and \(\Phi\) appearing in the first law of the black ring thermodynamics (2.46), for the small perturbation above the EVH point denoted by \(\epsilon\). The entropy and Hawking temperature have the following expansions:

\[
S = \sum_{n=1}^{\infty} S^{(n)} \epsilon^n, \quad T_H = \sum_{n=1}^{\infty} T^{(n)} \epsilon^n,
\]

while the other quantities have the following generic form:

\[
Z = \sum_{n=0}^{\infty} Z^{(2n)} \epsilon^{2n},
\]

where coefficients of expansions are functions of \(\mu, q, \eta_{\lambda},\) and \(\alpha_{1,2}\) (or \(\rho_{1,2}\) equivalently).

Plugging these expansions into the first law (2.46), it reduces to an infinite series of equations corresponding to each order in \(\epsilon\) expansion. Noting that in the expansion of angular momentum \(J\) we get \(J^{(0)} = 0\). In \[14\] it is argued that for a generic EVH black hole, under certain general assumptions such as analyticity of thermodynamic quantities, we expect that

Note that angular direction \(\psi \in [0, 2\pi]\) is pinching now and therefore mass, angular momentum and the entropy of BTZ appear with a factor \(\epsilon\) in front.
the angular momentum along the pinching direction vanishes in the EVH limit. This is indeed what we have for the $\phi_1$ direction of the black rings solutions and the angular momentum along it, $J$, vanishes in the EVH limit. Therefore, at the zeroth order of $\epsilon$ we get

$$dM_\epsilon = \Phi_{\epsilon}^{(0)} dD_{\epsilon}^{(0)} + P^{(0)} dA^{(0)} , \quad \Phi_{\epsilon}^{(0)} = \frac{\partial M}{\partial D_{\epsilon}^{(0)}} \Bigg|_{\text{ext}} , \quad P^{(0)} = \frac{\partial M}{\partial A^{(0)}} \Bigg|_{\text{ext}}$$

(4.3)

This is the BPS condition for the extremal black ring.

The next order equations, which correspond to adding excitations to the EVH system, turn out to be

$$dM^{(2)} = T_H^{(1)} dS^{(1)} + \Omega^{(0)} dJ^{(2)} + D_{\epsilon}^{(0)} d\Phi_{\epsilon}^{(2)} + D_{\epsilon}^{(2)} d\Phi_{\epsilon}^{(0)} + P^{(0)} dA^{(2)} + P^{(2)} dA^{(0)},$$

(4.4)

where from (3.13) and (3.14), we get

$$\frac{1}{\epsilon K} T_{BTZ} dS_{BTZ} = T_H^{(1)} dS^{(1)} , \quad \frac{1}{\epsilon K} \Omega_{BTZ} dJ_{BTZ} = \Omega^{(0)} dJ^{(2)}.$$ (4.5)

It is a straightforward exercise to check that the mass of the near-EVH black ring and the emerged BTZ black hole satisfy the following equation:

$$dM^{(2)} = \frac{dM_{BTZ}}{\epsilon K} + \Omega^{(0)} dJ^{(2)} + D_{\epsilon}^{(0)} d\Phi_{\epsilon}^{(2)} + D_{\epsilon}^{(2)} d\Phi_{\epsilon}^{(0)} + P^{(0)} dA^{(2)} + P^{(2)} dA^{(0)},$$

(4.6)

To get this equation one should note that the variation of the three-dimensional Newton constant $G_3$ should be set to zero. These results, along with (4.5), imply that whenever the EVH black ring obeys the first law of thermodynamics, the emerging BTZ also satisfies the first law of thermodynamics for BTZ black holes, i.e.

$$d\tilde{M} = T_H dS + \Omega dJ + \sum_{i=1}^{3} \Phi_D dD_i + P dA \implies$$

$$dM_{BTZ} = T_{BTZ} dS_{BTZ} + \Omega_{BTZ} dS_{BTZ},$$

(4.7)

where we dropped all vanishing sub-leading terms in the $\epsilon \to 0$ limit. Although we have explicitly shown that in the near-horizon region the EVH and the near-EVH black rings develop AdS3 and BTZ factors, from the above analysis of the first law solely, one should expect to find an AdS3 throat in the near-horizon of EVH black rings, and that this AdS3 factor turns to a BTZ black hole for near-EVH cases.

### 5. The dual EVH/CFT formulation

In this section we study a possible connection between the 2d CFT appearing in the EVH/CFT correspondence and the 2d chiral CFT proposed in the Kerr/CFT correspondence, studied in section 2. Our proposal is that the 2d chiral CFT of Kerr/CFT is nothing but the discrete light-cone quantization (DLCQ) of a standard 2d CFT [13]. This has been studied for the case of static and rotating charged AdS5 EVH black hole solutions in [6].

The central charge of the 2d CFT describing the gravitational black rings can be obtained from the standard AdS3/CFT2 taking into account the pinching periodicity:

$$c = \frac{3 \epsilon}{2 G_3} = \frac{12 \pi q_1 q_2 q_3 \sqrt{\frac{\eta_3}{\eta_5^2} \sqrt{\frac{\eta_3}{\eta_5^2}}}}{\mu^3 \eta_3 \eta_5^2 G_5} \epsilon.$$ (5.1)

To keep the central charge finite and to have a finite gap in this 2d CFT, $G_3$ should scale the same as the horizon area and Hawking temperature:
This implies the black ring entropy \((3.3)\) remains finite in this limit. From \((3.12)\) and \((3.13)\) we find that the same holds for the excitations \(M_{BTZ}\) and \(J_{BTZ}\).

The quantum numbers of the associated 2d CFT \((L_0, \tilde{L}_0)\) turn out to be

\[
L_0 - \frac{c}{24} = \frac{\pi \eta_4 (\alpha_1 + \alpha_2)^2 \mathcal{H}^2 (\eta_3^2)}{2 \mathcal{G}_5^{1/3} \eta_3^{1/3} (\eta_4^2 - \eta_3^2)} \epsilon, \quad \tilde{L}_0 - \frac{c}{24} = \frac{\pi \eta_4 (\alpha_1 - \alpha_2)^2 \mathcal{H}^2 (\eta_3^2)}{2 \mathcal{G}_5^{1/3} \eta_3^{1/3} (\eta_4^2 - \eta_3^2)} \epsilon.
\]

Using Cardy’s formula we get

\[
S = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - \frac{c}{24} \right) + 2\pi \sqrt{\frac{c}{6}} \left( \tilde{L}_0 - \frac{c}{24} \right) = \frac{4\pi^2 \sqrt{q_1 q_2 q_3} \mathcal{H}(\eta_3^2)}{\mu^3 \eta_3^2 \eta_4^2 (\eta_4^2 - \eta_3^2) \mathcal{G}_5} \epsilon.
\]

We observe that the near-EVH black ring entropy \((3.3)\) is reproduced.

To compare this result with Kerr/CFT we note that the Kerr/CFT applies for extremal finite-sized black rings, while EVH/CFT works for near-EVH black rings which can be non-extremal. Therefore, we must compare them in a region of parameters where both apply. This can be done by considering the extremal excitations in the EVH/CFT side and restricting them to the vanishing entropy limit in the Kerr/CFT side. The latter step is subtle and may involve singular limits.

On the bulk side, we should note that taking the near-horizon limit of near-EVH black rings does not compute with taking the near-EVH limit of the near-horizon geometry of extremal finite horizon black rings. These two limits lead to different geometries. On the CFT side, to reproduce the appearance of a vanishing cycle to account for the vanishing entropy, the Kerr/CFT central charge tends to zero.

Despite these facts, we observe that the Kerr/CFT central charge \((2.57)\) associated with the vanishing U(1) isometry cycle \(\tilde{\phi}_1\) remains finite in the EVH limit and the result matches the AdS$_3$ Brown–Henneaux central charge \((5.1)\) computed in the EVH/CFT correspondence.

To this end, we note that at the EVH limit the leading term in the Kerr/CFT central charge is given by

\[
c_{\tilde{\phi}_1} = \frac{12\pi q_1 q_2 q_3 \mathcal{H}(\eta_3^2)}{\mu^3 \eta_3^2 \eta_4^2 \mathcal{G}_5} + \mathcal{O}(\epsilon^2),
\]

From coordinate scaling \((3.9)\), the central charge scales as

\[
c_{\phi_1} = \epsilon^{1/2} c_{\tilde{\phi}_1},
\]

which matches with the central charge \((5.1)\). Within the Kerr/CFT proposal one may then expect that in the near horizon of the extremal rings we have a chiral CFT description associated with the EVH/CFT via the DLCQ description.

6. Conclusion

In this work we analysed the near-horizon geometry of a recently discovered dipole electric charged black ring [5]. We have shown that the black ring parameter space contains a region in which an AdS$_3$ factor emerges in the near-horizon geometry. This happens when the Hawking temperature and the black ring’s horizon area are small and the same order. We have shown that there are two possibilities to obtain this limit corresponding to static or stationary limits. In the stationary limit, all dipole charges vanish and the near-horizon limit is similar to the neutral single rotating black ring. In the static limit where angular momentum
vanishes, the AdS$_3$ part of the near horizon turns to BTZ when we turn on infinitesimal non-extremalities while keeping the ratio of temperature to area of the horizon fixed. For some black ring solutions (see, for example, [18]) where uplifting to string/M-theory is known, one can find degeneracy of corresponding micro states. Here we used AdS$_3$/CFT$_2$ to find statistical entropy of black ring solutions and studied its relation to Kerr/CFT. The black ring solution at the generic point in parameter space has conical singularity. We have studied the first law of thermodynamics for this black ring solution and have shown how to modify it to account for the effect of a conical singularity. We have also studied the first law when temperature and horizon area are both small. By expanding both sides we have shown that it reduces to the first law for BTZ, therefore the appearance of BTZ in the near-horizon geometry is predicted indirectly in the first law analysis.

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Appendix A. Five-dimensional Myers–Perry black hole

The simplest example of a black hole with one angular momentum in five dimensions is the Myers–Perry black hole solution. The metric in Boyer–Lindquist type coordinates is given by

\[
ds^2 = -dt^2 + \Sigma \left( \frac{r^2}{\Delta} \, dr^2 + d\theta^2 \right) + r^2 \sin^2 \theta \, d\phi^2 + (r^2 + a^2) \cos^2 \theta \, d\psi^2 + \frac{m}{\Sigma} \left( dt - a \cos^2 \theta \, d\psi \right)^2,
\]

(A.1)

where

\[
\Sigma = r^2 + a^2 \sin^2 \theta, \quad \Delta = r^2 + a^2 - m,
\]

and $m$ is a parameter related to the physical mass of the black hole, while the parameter $a$ is associated with its angular momentum. This metric depends only on two coordinates, $0 < r < \infty$ and $0 \leq \theta \leq \pi/2$, and it is independent of time, $-\infty < t < \infty$, and the azimuthal angles, $0 < \phi, \psi < 2\pi$. The inner and outer horizons are located at non-negative zeroth of function $\Delta$

\[
r_- = 0, \quad r_+ = \sqrt{m - a^2}.
\]

(A.3)

Notice that the horizon exists if and only if $|a| \leq m$ otherwise, the metric describes a naked singularity. Hawking temperature is given by

\[
T_H = \frac{\sqrt{m - a^2}}{2\pi m},
\]

(A.4)

ADM mass, angular momentum and horizon angular velocity are given by

\[
M_0 = \frac{3}{8} \frac{\pi m}{G_5}, \quad J_\psi = \frac{\pi m a}{4 G_5}, \quad \Omega_\psi = \frac{a}{m}.
\]

(A.5)
Area of the horizon is given by the following integral
\[ A = \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{2\pi} d\theta \sqrt{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2} = 2\pi^2 m \sqrt{m - a^2}. \]  
(A.6)

Therefore the Bekenstein–Hawking entropy is
\[ S_{BH} = \frac{A}{4G_5} = \frac{\pi^2 m \sqrt{m - a^2}}{2G_5}. \]  
(A.7)

### A.1. EVH point and static BTZ

The EVH limit is defined by \( T \to 0 \) and \( S_{BH} \to 0 \) at the same time. In addition, one may also wish to keep the ratio \( \frac{S}{T} \) finite. For the Myers–Perry black hole solution this is given by \( r_+ \to 0 \) while keeping \( m \) finite. We introduce
\[ a \simeq \sqrt{m} - \alpha \epsilon^2. \]  
(A.8)

The near-horizon limit is obtained by taking the limit \( \epsilon \to 0 \) and performing the following change of coordinate:
\[ r = \epsilon \sqrt{m} \rho, \quad \phi = \frac{\tilde{\phi}}{\epsilon}, \quad \psi = \frac{\tilde{\psi}}{\sqrt{m}}, \quad t = \frac{\sqrt{m} \tau}{\epsilon}. \]  
(A.9)

The result is
\[ ds^2 = m \sin^2 \theta \left[ -(\rho^2 - \rho_+^2) dr^2 + \frac{d\rho^2}{\rho^2 - \rho_+^2} + \rho^2 d\tilde{\phi}^2 + \frac{\cos^2 \theta}{\sin^4 \theta} d\tilde{\psi}^2 + d\theta^2 \right]. \]  
(A.10)

where
\[ \rho_+^2 = \frac{2\alpha}{\sqrt{m}}. \]  
(A.11)

To get real \( \rho_+ \) we must impose \( \alpha > 0 \). For the emerged static BTZ, we get
\[ M_{BTZ} = \frac{\epsilon}{4G_3 \sqrt{m}}, \quad T_{BTZ} = \frac{\sqrt{2\alpha}}{2\pi m^{1/4}}, \quad S_{BTZ} = \frac{\pi \epsilon \sqrt{2\alpha}}{2G_3 m^{1/4}}. \]  
(A.12)

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