Thick morphisms of supermanifolds and oscillatory integral operators

Th. Th. Voronov

In [3] and [4] we introduced non-linear pullbacks of functions with respect to microformal or thick morphisms, which generalize ordinary smooth maps of (super)manifolds. By definition, thick morphisms are formal canonical relations of a special kind between the cotangent bundles, and are specified by generating functions that depend on the position coordinates on the source manifold and the momentum coordinates on the target manifold. Such a generating function is regarded as a power series expansion near the zero section. The thick morphisms make up a formal category; this means that the composition law for generating functions is given by a formal power series. The pullback with respect to a thick morphism is also a formal power series whose terms are non-linear differential operators. (A parallel construction based on anticotangent bundles yields a non-linear pullback for odd functions; even functions are meant above.) Our first application is to $L^\infty$-morphisms between homotopy Poisson structures. Another application is the construction of an ‘adjoint operator’ for non-linear maps of vector bundles. (We remark that thick morphisms in their even version are close to the symplectic micromorphisms of Cattaneo–Dherin–Weinstein [1], defined as germs of canonical relations between germs of symplectic manifolds at Lagrangian submanifolds; for them, however, action on functions on Lagrangian submanifolds is not considered. See a discussion in [4].)

We shall show that thick morphisms of (super)manifolds are the classical limits of certain ‘quantum thick morphisms’ which are given by oscillatory integral operators of a special kind. The connection between oscillatory integral operators or Fourier integral operators and canonical relations between cotangent bundles is well known. Each such relation determines a class of Fourier integral operators [2]. In this class we consider a distinguished integral operator which generalizes the pullback operator with respect to a smooth map. It is defined by a ‘quantum’ (depending on $\hbar$) generating function. The action of such operators on oscillatory wave functions in the classical limit reproduces the non-linear pullback introduced in [3], [4]. Similarly, the composition of operators reproduces in this limit the composition of thick morphisms.

Our point of departure is the following observation. Consider invertible thick morphisms of a (super)manifold $M$ to itself. We arrive at a formal (super)group of ‘thick diffeomorphisms’ of $M$. What is its Lie (super)algebra?

**Theorem 1.** The Lie superalgebra of the formal supergroup of thick diffeomorphisms of a supermanifold $M$ can be identified with the Lie superalgebra $C^\infty(T^*M)$ with respect to the canonical Poisson bracket. The infinitesimal action of the Lie superalgebra $C^\infty(T^*M)$ on even functions on $M$ which corresponds to non-linear pullbacks is the ‘Hamilton–Jacobi action’ $f(x) \mapsto f(x) + \varepsilon H(x, \partial f/\partial x)$.

**Proof.** Let us check the second statement. The generating function of a thick morphism $\Phi: M \to M$ close to the identity is $S(x, p) = x^a p_a + \varepsilon H(x, p)$, where $\varepsilon^2 = 0$. The pullback of $\Phi$ maps $f(x)$ to $f(y) + S(x, p) - y^a p_a$, where $y$ and $p$ are found from the equations $y^a = \frac{\partial S}{\partial p_a}(x, p) = x^a + \varepsilon \frac{\partial H}{\partial p_a}(x, p)$ and $p_a = \frac{\partial f}{\partial x^a}(y)$. The substitution into $f(y) + S(x, p) - y^a p_a$, after some simplification, gives $f(x) + \varepsilon H\left(x, \frac{\partial f}{\partial x}(x)\right)$, as claimed.

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We now make two remarks. 1) Since $C^\infty(T^*M)$ is the Lie algebra of the group of canonical transformations of $T^*M$, there is an isomorphism of the corresponding formal supergroups (of thick diffeomorphisms of $M$ and of formal canonical transformations of $T^*M$), and it can also be seen directly in the language of generating functions; canonical transformations of $T^*M$ act, at least infinitesimally, on even functions on $M$, which is reminiscent of spinor representations, and we expect that there is a connection here. 2) Since the Hamilton–Jacobi equation is the classical analogue of the Schrödinger equation, there should be a ‘Schrödinger’ or ‘quantum’ version of thick morphisms and their action on functions.

**Definition.** A quantum thick morphism $\tilde{\Phi} : M_1 \to M_2$ is defined by an integral operator (quantum pullback) $\tilde{\Phi}^*$ that maps functions on $M_2$ to functions on $M_1$ by the following rule (to simplify the notation, the formula is given for ordinary manifolds, but the modification for the super case is obvious):

$$\tilde{\Phi}^*(w)(x) = \frac{1}{(2\pi)^{n_2}} \int_{T^*M_2} Dy Dq \exp \left\{ \frac{i}{\hbar} (S_h(x, q) - y^i q_i) \right\} w(y), \quad n_2 = \dim M_2. \quad (1)$$

The function $S_h(x, q)$ in (1) is an analogue of a generating function $S(x, q)$ specifying a thick morphism [3], [4]. It is a power series in the $q_i$ and $\hbar$.

**Theorem 2.** On the phases of oscillatory wave functions, a quantum pullback $\tilde{\Phi}^*$ in the classical limit $\hbar \to 0$ induces the non-linear pullback $\tilde{\Phi}^*$ with respect to the thick morphism with the generating function $S(x, q) = S_0(x, q)$.

A similar statement holds for the composition of quantum thick morphisms. The resulting integral formula gives in the zeroth order the composition law for generating functions [4], and then the higher corrections in $\hbar$. This explains why one should assume from the start that the functions $S_h(x, q)$ depend on $\hbar$: even if there is no such dependence initially, it arises in the higher-order terms of the ‘quantum composition law’ (cf. the star-product in deformation quantization).

**Proof of Theorem 2.** For a function of the form $w(y) = e^{i\varphi(y)/\hbar}$ on $M_2$, we have $\tilde{\Phi}^*(w)(x) = \frac{1}{(2\pi)^{n_2}} \int_{T^*M_2} Dy Dq \exp \left\{ \frac{i}{\hbar} (g(y) + S_h(x, q) - y^i q_i) \right\}$. According to the stationary phase method, for $\hbar \to 0$ the integral is asymptotically equal to $e^{i f(x)/\hbar}$, where $f(x)$ is the value of the phase function at the stationary point (as a function of $y$, $q$). By differentiating and letting $\hbar$ go to zero, we get that $\frac{\partial}{\partial y^j}(g(y) + S_0(x, q) - y^i q_i) \equiv \frac{\partial g}{\partial y^j}(y) - q_j$ and $\frac{\partial}{\partial q_j}(g(y) + S_0(x, q) - y^i q_i) \equiv \frac{\partial S_0}{\partial q_j}(x, q) - y^i$, which gives exactly the system of equations $q_j = \frac{\partial g}{\partial y^j}(y)$ and $y^i = \frac{\partial S}{\partial q_j}(x, q)$ in the definition of $\tilde{\Phi}^*$. Therefore, $f = \tilde{\Phi}^*[g]$. The theorem is proved.

Bearing in mind applications of thick morphisms of supermanifolds to $L_\infty$-morphisms of algebras of functions, we expect that quantum thick morphisms may provide a certain ‘quantum version’. We hope to elaborate on this question elsewhere.

**Bibliography**

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**Theodor Th. Voronov**  
University of Manchester, Manchester, UK;  
Tomsk State University  
*E-mail:* theodore.voronov@manchester.ac.uk  
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