The mean velocity profile of near-wall turbulent flow: is there anything in between the logarithmic and power laws?

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ABSTRACT
The question of possible analytical forms for the mean velocity profile in a near-wall turbulent flow is addressed. An approach based on the use of dispersion relations for the flow velocity is developed in the context of a two-dimensional channel flow. It is shown that for an incompressible flow conserving vorticity, there exists a decomposition of the velocity field into rotational and potential components, such that the restriction of the former to an arbitrary cross section of the channel is a functional of the vorticity and velocity distributions over that cross section, while the latter is divergence-free and bounded downstream thereof. By eliminating the unknown potential component with the help of a dispersion relation, a nonlinear integro-differential equation for the flow velocity is obtained. It is then analysed within an asymptotic expansion in the small ratio $v^*/U$ of the friction velocity to the mean flow velocity. Upon statistical averaging in the lowest nontrivial order, this equation relates the mean velocity to the cross-correlation function of the velocity fluctuations. Analysis of the equation reveals existence of two continuous families of solutions, one having the power-law near-wall asymptotic $U \sim y^n$, where $y$ is the distance to the wall, $n > 0$, and the other, $U \sim \ln^p(y/y_0)$, with $y_0 = \text{const}$ and $p \geq 1$. In the limit of infinite channel height, the exponent $n$ turns out to be asymptotically a universal function of the Reynolds number, $n \sim 1/\ln \text{Re}$, whereas $p \to 1$. Thus, the logarithmic profile ($p = 1$) is found to be a member of the power-log family whose members with $p > 1$ are intermediate between the power- and logarithmic-law profiles with respect to their slopes at large $y$. These results are discussed in the light of the existing controversy regarding experimental verification of the law of the wall.

1. Introduction

Investigation of the near-wall flows constitutes a major part of the turbulence studies. Its results help clarifying basic features of turbulence on the one hand, and find numerous applications in science and industry, on the other hand. In view of simplicity of their practical realisation, the near-wall flows have been extensively studied experimentally, whereas the relative simplicity of the boundary conditions in channels or pipes and evergrowing computational capabilities allow direct numerical simulation of these flows at still higher
Reynolds numbers. Wall turbulence is also very attractive from a purely theoretical standpoint, as a system characterised by a rather simple mean flow and well-defined direction of the momentum flow. Moreover, spatial separation of the sources and sinks of momentum, and the well-organised hierarchy of turbulent structures ranged according to their distance to the wall make the study of certain aspects of turbulence even simpler than in the isotropic case.

Despite these simplifications, however, analytical description of the wall turbulence remains nearly as scarce as of other, apparently more complicated flows. The reason for this, shared by all turbulent motions, is the necessity to deal with the nonlinearity of an irregular flow. While construction of an exact general solution of the hydrodynamic equations is now as illusory as it was a hundred years ago, perturbative treatment of the flow nonlinearity has not been successful either. In fact, all attempts to assign ‘smallness’ in one way or another to the inertia effects run into fundamental difficulties such as violation of the Galilean invariance. On the other hand, the same nonlinearity is also the reason for incompleteness of the statistical approach, as it requires introduction of a more or less arbitrary model assumption regarding the structure of the correlation functions of the fluctuating flow velocity (the closure problem). In particular, no closed relation following from the basic equations is known that would allow determination of the mean flow velocity profile.

Yet, the above-mentioned specifics of the near-wall turbulent flow has always been a source for the strong belief that its mean velocity profile can be determined indirectly using dimensional analysis and similarity arguments. As is well known, however, the results of these analyses are not unique: there are two main competing variants of the law of the wall – the power law, originally due to Prandtl, and von Kármán’s universal logarithmic velocity profile.[1] Although there are indications that the former is observed in flows characterised by moderately large Reynolds numbers, it is the logarithmic profile that finds overwhelming experimental support as the asymptotic law of the near-wall velocity distribution in the large Reynolds number limit. Specifically, the recent data [2–4] obtained by direct numerical simulations and highly accurate laboratory measurements in channel flows show that while for Re (based on the channel height and bulk mean velocity) up to approximately 60,000–80,000 dependence of the skin-friction coefficient on the Reynolds number is well approximated by a power law, at higher Re, it is best described by a log-law (power-law behaviour of the skin friction implies a power law in the mean velocity profile). Things are similar in the inner layers of pipe flows [5] and free boundary layers, [6] and a pronounced similarity exists between the three types of turbulent flows at very high Reynolds numbers, which goes beyond mere conformity of the mean velocity profiles.[7–9]

It is to be noted, however, that because of the limited measurement accuracy and some vagueness in the spatial extent of the inner layer, the two velocity profiles are not always easy to discern experimentally, so that one and the same data are sometimes interpreted differently. This circumstance, on the one hand, and the formal consistency of the incomplete similarity hypothesis underlying the power law, on the other hand, are the reasons for periodic revivals of theoretical interest to this law, and notable recent ones by Barenblatt [10] and Oberlack [11]. While the former author merely explored consequences of the postulated scaling law for the mean shear, the latter made an attempt to derive the scaling from the Reynolds-averaged Navier–Stokes equation by exploiting its symmetries within the Lie group approach. The conclusion was that both the logarithmic and power law are solutions to this equation, in the sense that they satisfy the Lie group relations. Though this result
does not prove actual realisability of either solution, it confirms their consistency from the standpoint of symmetries of the basic equations.

Thus, in spite of the general belief in the logarithmic law, there are empirical indications on the relevance of the power law at Reynolds numbers so large that the turbulence is fully developed, while no theoretical grounds to abandon it in favour of the log-law. This state of affairs is controversial, because it appears to be commonly accepted that there is place for only one law in a fully developed flow. On this account, the fact that the power law ultimately gives way to the log-law as Re grows is usually interpreted as a good reason to reject it completely. One of the purposes of the present paper is to make a step towards clarifying this issue.

At this point, it is worthwhile to return to the logarithmic law to discuss the underlying hypothesis of complete similarity in some detail. This hypothesis was initially expressed [1] as a rather strong supposition that the structure of turbulent disturbances in the wall-normal direction can be described infinitesimally in terms of the mean shear and its first derivative in that direction. The subsequent development greatly relaxed the basic assumptions, but one of them has been remaining invariant, namely that the analysis, be it a matching procedure in the overlap region [12,13] or dimensional analysis, is to be carried in terms of the mean shear. The dimensional argument, for instance, goes as follows [12,14]: outside the viscous layer, the friction velocity $v_*$ and the distance to the wall $y$ are the only dimensional quantities which the mean shear $dU(y)/dy$ can be built of, so that one can write

$$\frac{dU}{dy} = \frac{v_*}{\chi y},$$

$\chi$ being a numerical coefficient; integration then yields the logarithmic law for the mean flow velocity $U(y)$

$$U = \frac{v_*}{\chi} \ln \frac{y}{y_0},$$

where constant $y_0$ may depend on the fluid kinematic viscosity $\nu$. It is to be noted that the power law can be derived in exactly the same way: the $y$-derivative of the dimensionless combination $\ln \left( \frac{U}{v_*} \right)$ must be of the form $n/y$, $n = \text{const}$, therefore, $U \sim v_*(y/y_0)^n$. This result is usually formulated as the hypothesis of incomplete similarity [10,15]: the dimensionless function $F(yv_*/\nu, \text{Re})$ defined by the relation

$$\frac{dU}{dy} = \frac{v_*}{y} F(yv_*/\nu, \text{Re})$$

behaves at large Reynolds numbers as $F \sim (yv_*/\nu)^n$, where $n = n(\text{Re})$ is small but nonzero. The smallness is required by the experiment, as the measured mean flow velocity grows rather slowly away from the wall outside the viscous layer, and fitting to the experimental results yields [16,17] $n \sim 1/\ln \text{Re}$. But sufficient smallness of the velocity gradient can also be achieved by reducing the asymptotic of the function $F$ to a logarithmic, $F \sim \ln^{p-1}(yv_*/\nu)$, with any real $p > 1$, in which case $U(y) \sim \ln^p(y/y_0)$. This profile can also be obtained directly
by applying the dimensional argument that led to Equation (2), to the function $U^{1/p}$: Equation (1) is then replaced by

$$\frac{dU^{1/p}}{dy} = \frac{v_s^{1/p}}{\varepsilon y},$$

whence

$$U = \frac{u_s}{\varepsilon p} \ln^p \frac{y}{y_0}.$$

Clearly, the dimensional argument gives no reason to prefer the value $p = 1$ to any other, unless one postulates that it is the mean shear that must be independent of viscosity outside the viscous layer. Historically, it is the logarithmic and power laws that have been used to approximate the mean velocity profile, but the above discussion shows that the power-log profiles (4) are as natural as the other two. Moreover, the fact that there is a range of Reynolds numbers where transition from the power to the logarithmic law occurs suggests that the power-log laws can be of direct experimental relevance. Indeed, the power-log distributions with $p > 1$ are intermediate between the log-law and power-law distributions, in the sense that for sufficiently large $y$’s, they grow faster than the former, but slower than the latter. This family, therefore, might be interpolating between the two classic profiles. The main purpose of this paper is to give a more solid ground to this idea.

The analysis will be carried out for a two-dimensional channel flow. Admittedly, restriction to two dimensions suppresses important features of the turbulent flow, most notably the vortex stretching that drives the energy cascade. However, the simple logic underlying this consideration is that once existence of a given velocity profile is established in two dimensions, it is thereby a possible profile in the general case where velocity fluctuations are three-dimensional. Of course, this argument does not prove actual realizability of the profile, but one should notice that it goes beyond the formal statement that the two-dimensional solutions automatically satisfy three-dimensional equations. As will be shown below, the three velocity profiles – the log, power, and power-log – are solutions of the same integro-differential equation, so that practical relevance of the two particular cases suggests the same for the intermediate. It is, perhaps, worth recalling in this connection that the classical derivation [1] of the logarithmic law is also based on an essentially two-dimensional treatment of velocity fluctuations.

As was already mentioned, the stumbling block to the development of analytical description of turbulence has been the virtual impossibility to construct, in any form, explicit general solution of the basic hydrodynamic equations. Of course, this difficulty does not pertain to turbulence only, but plagues many other hydrodynamic problems, particularly those involving free boundaries, e.g., ablation fronts, condensation discontinuities, etc. A general method to cope with this difficulty was found in the context of premixed flame propagation.[18,19] In brief, the method applies to incompressible flows conserving vorticity, and consists in decomposing the flow velocity field into rotational and potential parts in such a way that it becomes possible to explicitly compute the boundary value of the former on some control surface (the flame front, in the case of premixed combustion). The unknown potential component can then be eliminated using a dispersion relation – a
singular-kernel integral equation for its boundary values. The result is a nonlinear integro-differential equation for the flow velocity distribution on the surface. This program will be realised below for the two-dimensional turbulent channel flow.

The paper is organised as follows. Section 2 specifies the channel flow to be studied, displays the governing equations and formulates the boundary conditions in a way which, though somewhat unconventional in the turbulence studies, is most adequate to the integral treatment to be given. The main integro-differential equation for the flow fields is derived in Section 3. The derivation begins in Section 3.1 with a standard representation of the velocity field via the area integral of the vorticity distribution in the channel, which is then gradually simplified with the help of a certain equivalence relation to obtain the rotational component of the flow velocity in the form of an integral of the vorticity distribution over the channel cross section. The unknown potential component is finally projected out in Section 3.2 using the dispersion relation which expresses analyticity and boundedness of the corresponding complex velocity downstream of that cross section. Relations of this sort are widely used in various areas of physics, e.g., Kramers–Kronig dispersion relations in optics, Källén–Lehmann spectral formulas in quantum field theory, etc. In spite of the absence of real dispersion in the present context, for want of a better name, the same term ‘dispersion relation’ will be kept for the main integro-differential equation thus obtained, as well as for its descendant derived in Section 3.3 by expanding in powers of $v_s/U$ and statistical averaging, and further simplified in Section 3.4 in high channels. A detailed analysis of the dispersion relation is carried out in Section 4 where the velocity profiles discussed above are derived as possible near-wall asymptotics of solutions to the dispersion relation, and the form of the respective cross-correlation functions is established. Throughout the paper, the channel height is assumed large enough to justify the approximations made, and Section 4.5 examines the possibility to go over to the limit of infinite height. The paper has an Appendix that contains derivation of an identity involving the Hilbert operator, used in Section 4.3 to determine the near-wall mean velocity asymptotic.

2. Flow conditions

Consider developed turbulent channel flow between two parallel plates at a distance $h$. It will be assumed throughout that $h$ is much larger than the viscous length $\nu/v_*$ (\(\nu\) is the kinematic fluid viscosity), and that the flow is considered far enough downstream of the channel inlet, so that, on average, it is steady and homogeneous in the streamwise direction. In regions sufficiently close to either wall (but outside of the viscous layer), the Reynolds shear stress is approximately constant, but if both walls are at rest and the fluid motion is sustained by a pressure gradient, the shear stress is an odd function of the wall-normal coordinate with respect to the channel midpoint. As constancy of the shear stress is the defining property of the near-wall turbulence, its consideration is technically more convenient in the setting where this property holds throughout the channel cross section. We, therefore, switch to the picture where one of the channel walls moves in its plane with a constant speed $V$, with the mean pressure gradient in the streamwise direction vanishing (Figure 1). The Reynolds number will always be assumed large enough for the turbulence to be fully developed, $Re = Vh/\nu \gg 1$. Then, the shear stress is constant everywhere in the channel except the regions of width $\sim \nu/v_*$ adjacent to the walls. Denoting $x, y$ the streamwise and wall-normal coordinates, $u, v$ the corresponding flow velocity components, and
taking also the fluid of unit density, we thus have

\[ \langle \nu \nu \rangle = -v^2, \]

where the angular brackets denote time averaging. Despite formal equality of the walls, when speaking about the near-wall flow, we will always refer to the wall at rest, taken to be at \( y = 0 \), the other wall playing an auxiliary role which is to provide a source of momentum, and to ensure the flow homogeneity. Of course, the above change in the global flow configuration relies on the assumption that the specifics of the outer (core) flow have negligible influence on the structure of the inner layer, but as the subsequent consideration shows, this change is inconsequential in that the achieved simplification of the shear-stress distribution does not preclude existence of any of the velocity profiles of interest – the log, power, or power-log law.

The velocity field satisfies

\[ \frac{\partial \omega}{\partial t} + v_i \frac{\partial \omega}{\partial x_i} = \nu \Delta \omega, \]

(5)

\[ \frac{\partial v_i}{\partial x_i} = 0, \]

(6)

where \( \omega \) is the vorticity,

\[ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \]

(7)

\((x_1, x_2) \equiv (x, y), (v_1, v_2) \equiv (u, v), \) the subscripts run over 1, 2, and as usual, summation over repeated indices is understood.

An important premise underlying the present approach is that for a given shear stress, the mean velocity distribution near the wall can be calculated neglecting the viscous term in Equation (5) outside the viscous layer. As it is part of the somewhat more general standard assumption that this is allowed in the dynamical equations themselves, whose justification can be found in textbooks,[12–14] it will not be discussed further. We only mention that it
in no way extends over the fields of Reynolds stresses themselves, nor it precludes the possibility of incomplete similarity. Dependence on viscosity resides in the correlation functions of the velocity fluctuations, and the omission made means merely that their structure cannot be inferred from the inviscid equation. We also note for the future that this implicit presence of viscosity guarantees regularity of the flow velocity, considered as a function of complex $x$, in a vicinity of the real axis. Indeed, acting as a short-wavelength cut-off for the flow perturbations, viscosity prohibits any real-$x$ singularity.

With the right-hand side omitted, Equation (5) becomes the statement that $\omega$ is constant along the fluid particle trajectories. Put simply, the aim of the subsequent analysis is to infer the consequences of this conservation law as to the structure of the turbulent flow. It is motivated by the fact that vorticity distribution defines the flow velocity up to a potential, and so the statement of its conservation represents a strong constraint on the flow dynamics. Because of the unknown potential velocity component, this constraint does not fix uniquely the fluid evolution, but the hope is that it will be sufficient to relate the flow characteristics on average once the potential component is properly isolated. Thus, based on the above-mentioned premise, our analysis will lead to a nonlocal relation between the mean velocity distribution in the channel and the cross-correlation function of the velocity fluctuations.

Next, to ensure fulfilment of the boundary condition $v = 0$ at the wall $y = 0$, it is convenient to introduce an auxiliary channel which is a reflection of the initial one (to be referred to as physical in what follows) with respect to the wall $y = 0$. Specifically, the new channel has the same height $h$, occupying the region $y \in (-h, 0)$. It is filled with the same fluid as the physical channel, and its wall $y = -h$ moves with the same speed and in the same direction as the wall $y = h$ (Figure 1). Unlike the usual formulation of the reflection procedure in the case of laminar flows, however, the instantaneous velocity fields do not satisfy the relations

$$u(x, -y) = u(x, y), \quad v(x, -y) = -v(x, -y),$$

which hold only on average. Instead, to formulate the condition of impermeability of the wall $y = 0$, we consider the velocity fields in the regions $y \in (0, h)$ and $y \in (-h, 0)$ as restrictions of a unique field describing the flow of the same fluid in the channel $y \in (-h, h)$ without the wall $y = 0$, and require these velocity fields be statistically independent of each other. This formulation of the boundary condition is sufficient for our purposes, and is admissible indeed, for any nonzero flux through the surface $y = 0$ (that is, $v(x, 0) \neq 0$ for some $x$) would necessarily result in a correlation between the flow velocities to the left and to the right of the surface. As to the other boundary condition at $y = 0$, namely, $u(x, 0) = 0$, it does not apply since our consideration has been restricted to the outside of the viscous layer, but the presence of the wall implies that the function $\langle u(x, y) \rangle \equiv U(y)$, obtained as a solution of the inviscid equations, necessarily has a singularity at $y = 0$ when formally continued over the viscous layer. Here, ‘singularity’ means that the function $U(y)$ and/or its derivatives become unbounded in a vicinity of $y = 0$. The existence of such singularity can be taken as a necessary condition on the inviscid solutions in the extended channel $y \in (-h, h)$, to rule out laminar solutions proper to this channel, e.g., the trivial solution $v = 0, u = V$. Besides $y = 0$, there can be other singular points of $U(y)$, like $y = y_0$ in function (4). All such points must belong to the viscous layer for $U(y)$ to represent a physical velocity distribution. Still, the presence of regions where the function $U(y)$ is formally negative or even complex-valued may be troublesome for the integral relations involving $U(y)$ to be derived below.
By this reason, it will be assumed in what follows that whenever \( U(y) \) is treated integrally, it is continued over the viscous layer in such a way that it becomes constant within the laminar sublayer, say, \( U(y) = v_s \), the true inviscid solution being smoothly matched to this constant within the buffer layer. Though largely arbitrary, this prescription is justified by the narrowness of the viscous layer, and ensuing relative smallness of its contribution to the integral quantities.

By the above construction, all statistical properties of the flows in the regions \( y \in (0, h) \) and \( y \in (-h, 0) \) are identical. More precisely, correlation functions \( \langle \tilde{u}(x_1, y_1) \cdots \tilde{v}(x_n, y_n) \cdots \rangle \) of the velocity fluctuations are invariant up to a sign under the reflection \( (x, y) \to (x, -y) \), the sign being plus or minus depending on whether the number of \( v \)-components in the product is even or odd. In the case when field derivatives are involved in the product, parity of the correlation function is determined by the total number of \( v \)-components and \( y \)-differentiations. In addition to that, statistical properties of the flow in the physical channel are also invariant under the inversion with respect to any point \((x, h/2)\) (or equivalently, under rotation by \( \pi \) around this point). In fact, that the flow statistics inherits the symmetries of the boundary conditions is part of the supposition that the turbulent flow is fully developed. This property implies a condition on the mean flow velocity

\[
U(h/2) = \frac{V}{2},
\]

which replaces the boundary condition \( U(h) = V \) not applicable to inviscid solutions.

Of special importance for the following is the cross-correlation function

\[
R(x_1, y_1; x_2, y_2) = \langle \tilde{u}(x_1, y_1)\tilde{v}(x_2, y_2) \rangle.
\]

Since \( \langle \nu \rangle = 0 \), replacing \( u \) with the fluctuation \( \tilde{u} = u - U \) in this definition makes no difference. Assuming that \( R \) is initially defined for positive \( y_1, y_2 \), its continuation over the whole channel \( y \in (-h, h) \) is, according to the above conditions of statistical independence and parity,

\[
R(x_1, y_1; x_2, y_2) = R(x_1, |y_1|; x_2, |y_2|) \left\{ \theta(y_1)\theta(y_2) - \theta(-y_1)\theta(-y_2) \right\},
\]

where \( \theta(x) \) is the step function: \( \theta(x) = 0 \) for \( x < 0 \), and \( \theta(x) = 1 \) for \( x \geq 0 \).

It remains to guarantee impermeability of the walls at \( y = -h \) and \( y = h \). A general way to achieve this is to use an appropriate Green function to represent the velocity field, but since we deal with a straight channel flow in the absence of external fields, it is sufficient to treat it as part of a \( 2h \)-periodic flow obtained by periodic continuation of the flow in the domain \( y \in (-h, h) \) along the \( y \)-axis.

### 3. Derivation of the dispersion relation

#### 3.1. Flow decomposition

To extract essential information about the vortical structure of the flow, its velocity field will be gradually simplified by stripping it of potential contributions to be collectively denoted
\( v_i^p \), with the aim to express its rotational part \( v_i = v_i - v_i^p \) in the region downstream of some control surface as a functional of the vorticity distribution over that surface. The simplest choice of the control surface will be made, namely, a vertical cross section of the flow by the plane \( x = 0 \). As this procedure has been carried out in detail in more general situations of curved control surfaces and channels of varying height,[18–22] it will be reproduced here more compactly using simplifications admitted by the present case.

One starts with the following Biot–Savart type formula which is a consequence of Equation (6)

\[
v_i = \varepsilon_{ik} \partial_k \int_{\Lambda} d\ell_i \varepsilon_{lm} \frac{\ln r}{2\pi} - \partial_i \int_{\Lambda} d\ell_k \varepsilon_{mk} \frac{\ln r}{2\pi} - \varepsilon_{ik} \partial_k \int_{\Sigma} ds \frac{\ln r}{2\pi} \omega .
\]  

(12)

Here, \( \varepsilon_{ik} = -\varepsilon_{ki}, \varepsilon_{12} = +1, \partial_i = \partial / \partial x_i ; \Sigma \) and \( \Lambda \) denote any part of the flow and its boundary, respectively; \( r \) is the distance between an infinitesimal fluid element \( ds \) at the point \((\tilde{x}, \tilde{y})\) and the point of observation \((x, y) \in \Sigma, r^2 = (x_i - \tilde{x}_i)^2\), and \( d\ell_i \) is the line element normal to \( \Lambda \) and directed outward from \( \Sigma \). We choose \( \Sigma \) be a rectangular set of points with \( y \in [-R, R] \) between the lines \( x = 0 \) and \( x = R \), where \( R \gg h \) is meant to go ultimately to infinity, so that \( \Sigma \) will occupy the whole region downstream of \( x = 0 \).

Two fields \( f_i(x, y) \), \( \tilde{f}_i(x, y) \) are said equivalent, \( f_i(x, y) = \tilde{f}_i(x, y) \), if \( \phi_i(x, y) = f_i(x, y) - \tilde{f}_i(x, y) \) satisfies \( \partial_i D \phi_i = 0, \varepsilon_{ik} \partial_i D \phi_k = 0 \), and \( D \phi_i \) is bounded in \( \Sigma \), where \( D \phi_i \) denotes any first-order spatial derivative of \( \phi_i \). This extra differentiation \( D \) and the condition of boundedness will be found important when taking the limit \( R \to \infty \) and applying a dispersion relation to the potential part of the velocity field. In particular, \( D \)-differentiation makes the line integrals on the right-hand side of Equation (12) convergent in this limit, and hence the corresponding contributions bounded. Since these terms are also irrotational and divergence-free, one has

\[
v_i = -\varepsilon_{ik} \partial_k \int_{\Sigma} ds \frac{\ln r}{2\pi} \omega ,
\]

or with all the field arguments displayed,

\[
v_i(x, y, t) = -\varepsilon_{ik} \partial_k \int_{\Sigma} d\tilde{x}d\tilde{y} \frac{\ln r}{2\pi} \omega(\tilde{x}, \tilde{y}, t).
\]  

(13)

The purpose of the subsequent equivalence transformation is to reduce the right-hand side of Equation (13) to a one-dimensional integral over \( \tilde{y} \). To this end, we will deform the integration contour to extract the singularity of the integral kernel at \( r = 0 \). It is this singularity that determines vortical structure of the flow in a vicinity of the given observation point, because contributions of the fluid elements lying outside of the vicinity are irrotational inside. Since the control surface \((x = 0)\) is planar, there is no need to switch to the Lagrangian time of the fluid elements as was done in [21,22], and the contour deformation can be performed directly in terms of the \( \tilde{x} \)-integration variable. Let \( X(\eta, \tau, t), Y(\eta, \tau, t) \) denote the \( x- \) and \( y- \) coordinates at the current time \( t \) of the fluid element that crossed the point \((0, \eta)\) at instant \( \tau \). Then, the integral appearing in Equation (13) can be written, on
account of the vorticity conservation, as

$$\int_\Sigma ds \omega \ln r = \int_{-R}^R d\tilde{y} \int_0^R d\tilde{x} \omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)] \ln \{ (x - \tilde{x})^2 + (y - \tilde{y})^2 \}^{1/2},$$  \hspace{1cm} (14)

where \( \{ \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t) \} \) is the solution of the equations

$$X(\eta, \tau, t) = \tilde{x}, \quad Y(\eta, \tau, t) = \tilde{y}. \hspace{1cm} (15)$$

This solution exists for any point \((\tilde{x}, \tilde{y})\), except the zero-measure set \(\tilde{y} = 2hn, \ n \in \mathbb{Z}\), because the mean fluid velocity is positive outside of this set. It is also unique, because different trajectories do not intersect. Singularities of the integrand, considered as a function of the complex variable \(\tilde{x}\), are the logarithm branch points

$$x_+ = x + iy, \quad x_- = x - iy,$$  \hspace{1cm} (16)

and those corresponding to the function \(\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)]\). The latter singularities are necessarily present as long as the function \(\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)]\) is not constant, and are expected to be located at distances \(\leq h\) from the real axis, because the largest eddy size is \(h\). It is not difficult to see that the contributions of such singularities are potential. Consider the field

$$\phi_i = -\frac{1}{2\pi} \omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)] \varepsilon_{ijk} \partial_k \ln \{ (x - \tilde{x})^2 + (y - \tilde{y})^2 \}^{1/2},$$

which on account of Equation (14) represents the integrand in Equation (13). It satisfies \(\partial_i \phi_i = 0\), and \(\varepsilon_{ijk} \partial_j \phi_k = 0\) for \((x, y) \neq (\tilde{x}, \tilde{y})\), implying that it is potential for any \(\tilde{x}\) with a non-zero imaginary part. At the same time, the function \(\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)]\) cannot be singular at real \(\tilde{x}\), because \(\eta(\tilde{x}, \tilde{y}, t)\) and \(\tau(\tilde{x}, \tilde{y}, t)\) are real when \(\tilde{x}\) is real, whereas all the flow functions involved are regular for real values of their arguments. Therefore, contribution of any singularity of \(\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)]\) encountered when deforming the contour of \(\tilde{x}\)-integration, such as a pole \(x_0\) of arbitrary order or a cut \(C_0\) connecting branch points, sketched in Figure 2, is potential indeed. Also, since each such singularity can be surrounded by a contour of finite length, the corresponding terms in \(D\nu_i\) are bounded, and therefore, equivalent to zero. In other words, all such contributions can be omitted altogether. On the other hand, the points \(x_\pm\) tend to the real axis as \(\tilde{y} \rightarrow y\), clipping the integration contour (the segment \(\tilde{x} \in [0, R]\)). It is this singularity that determines the vertical structure of the flow. To extract its contribution, we replace the \(\tilde{x}\)-integral over \([0, R]\) by one half of the integral over the contour \(C = C_+ \cup C_-\) shown in Figure 2, compensating the phase change of \(r = \sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2}\) by adding an integral of the jump of the logarithm at the cut connecting the points \(x_-\), \(x_+\). The contour \(C\) can then be moved away from \(x_\pm\) by deforming it into the contour \(\tilde{C} = \tilde{C}_+ \cup \tilde{C}_-\). To embrace \(x_\pm\) for all \(y, \tilde{y} \in \Sigma\), the outer (horizontal) segments of \(\tilde{C}\) should be at a distance \(\tilde{R} > 2R\) from the real axis.

We thus obtain

$$v_i = \frac{i}{4} \varepsilon_{ijk} \partial_k \int_{-R}^R d\tilde{y} \int_{x_-}^{x_+} d\tilde{x} \omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)] + \int_{-R}^R d\tilde{y} \int_{\tilde{C}} d\tilde{x} \phi_i. \hspace{1cm} (17)$$
Figure 2. Singularities of the integrand in Equation (14), and contours of integration over $\tilde{x}$ used to extract the singular contribution to the rotational velocity component. $x_\pm$ are the branch points of the logarithm, $x_0$, $C_0$ are a pole and a cut connecting branch points of the function $\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau (\tilde{x}, \tilde{y}, t)]$.

The last term here is potential in $\Sigma$, but it is difficult to give a general proof that it remains bounded in the limit $R \to \infty$. If it does, then so does the first term, and vice versa, because the left-hand side is bounded as a physical field (more precisely, this statement holds true for the field derivatives, according to the definition of $\dot{\omega}$). In this case, the last term $\dot{\omega} = 0$, and therefore it can be omitted. The problem is that despite seemingly different structure of the two integrals, a possibility cannot be excluded that both are divergent in the limit $R \to \infty$, the divergences vanishing only in their sum. A general way to overcome this difficulty is to ‘reshuffle’ possible divergences by introducing an intermediate regularisation of the integrals that would ensure separate existence of their limits as $R \to \infty$, and then remove the regularisation. The most convenient practical recipe is the following. We observe that since the integrand in the first term is independent of $x$, $y$, by the virtue of Newton–Leibniz theorem, it is a one-dimensional integral over $\tilde{y}$ of a function defined on the control surface. By the construction, the functions $\eta(\tilde{x}, \tilde{y}, t)$, $\tau (\tilde{x}, \tilde{y}, t)$ are $2\pi$-periodic with respect to $\tilde{y}$, hence, so is the function $\omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau (\tilde{x}, \tilde{y}, t)]$. Therefore, the first integral in Equation (17) is periodic for $R = \infty$, if finite. An appropriate regularisation of such integrals is the introduction of an exponential damping $e^{-\mu |y-\tilde{y}|}$ into integrands, where the parameter $\mu$ is sufficiently large. One can then go over to the limit $R \to \infty$, so that Equation (17) is replaced by

$$
\nu_i = \frac{i}{4} \epsilon_{ik} \partial_k \int_{-\infty}^{\infty} d\tilde{y} e^{-\mu |y-\tilde{y}|} \int_{x_-}^{x_+} d\tilde{x} \omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau (\tilde{x}, \tilde{y}, t)] + \int_{-\infty}^{\infty} d\tilde{y} e^{-\mu |y-\tilde{y}|} \int_{C} d\tilde{x} \phi_i.
$$

(18)

Now, the first integral can be analytically continued in $\mu$ to $\mu = 0$, as this continuation exists in view of the above-mentioned $2\pi$-periodicity of the integral. On the other hand, for $\mu = 0$, the right-hand side of Equation (18) coincides with that of Equation (17), so that
up to a constant, it defines the same velocity field. Moreover, it can be proved \[21\] that the procedure just described preserves potentiality of the second term in Equation (18). This equation, thus, reduces to
\[
v_i = \frac{i}{4} \epsilon_{ik} \partial_k \int_{-\infty}^{\infty} d\tilde{y} e^{-\mu |y-\tilde{y}|} \int_{x_n}^{x_p} d\tilde{x} \omega[0, \eta(\tilde{x}, \tilde{y}, t), \tau(\tilde{x}, \tilde{y}, t)] \bigg|_{\mu=0}.
\]
(19)

Concrete realisations of this procedure in application to various problems of premixed combustion can be found in \[21–23\].

Fortunately, the case under consideration admits an important simplification that makes the use of analytic continuation superfluous. Namely, it will be shown in Section 3.3 that the obtained expression for the rotational velocity component turns out to be localisable, that is, at each order of an asymptotic expansion in \(v_p/U\), a finite-order derivative of the integrand in Equation (19) with respect to \(y\) vanishes for \(\mu = 0\) and \(\tilde{y} \neq y\). A rigorous consideration \[22\] shows that the result of formal differentiation of a localisable expression can differ from that obtained using the analytic continuation by a term which is position-dependent for general (curved) control surfaces, but in the case of a plane, this term is constant, equal to the mean of the expression over that plane. Therefore, by switching to a \(y\)-derivative of sufficiently high order, such that the obtained expression is explicitly finite and has zero mean, one can ensure correctness of the result. As this trick greatly simplifies calculations, we use it below omitting the factor \(e^{-\mu |y-\tilde{y}|}\) and the accompanying symbol of analytic continuation to \(\mu = 0\) in Equation (19).

### 3.2. The dispersion relation

Given a potential velocity field \(v_p\), the complex combination \(v_i - iv_p = u_p - iv_p \equiv g^p\) is an analytical function of the complex variable \(x + iy \equiv z\). If, in addition to that, it is periodic in \(y\) and bounded in the domain \(x \geq 0\), its restriction to the control surface \((x = 0)\) satisfies the relation
\[
(1 - i\hat{H}) (g^p)' = 0,
\]
(20)
where prime denotes the differentiation with respect to \(y\), and \(\hat{H}\) is the Hilbert operator defined on functions \(a(y)\) with zero mean over \(y \in (-\infty, \infty)\) by
\[
(\hat{H}a)(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tilde{y} \frac{a(\tilde{y})}{\tilde{y} - y}.
\]
(21)

Equation (20) is a formula of the Sokhotski–Plemelj type which is readily obtained by applying the Cauchy theorem to \(g^p\) in the domain \(x \geq 0\). For a \(2h\)-periodic function \(a(y)\), the definition of \(\hat{H}\) can be rewritten, in view of the well-known expansion of the cotangent, as
\[
(\hat{H}a)(y) = \frac{1}{2h} \int_{-h}^{h} d\tilde{y} a(\tilde{y}) \cot \left\{ \frac{\pi}{2h} (\tilde{y} - y) \right\}.
\]
(22)
Relation (20) holds for higher derivatives $d^n g^\phi/dy^n$ as well, because $d^n g^\phi/dz^n$ satisfies the requirements imposed on $g^\phi$.

These requirements are also met by the field

$$G \equiv g(x, y) - i \frac{1}{4} \int_{-\infty}^{\infty} d\tilde{y} \left\{ \omega[0, \eta(x_+, \tilde{y}, t), \tau(x_+, \tilde{y}, t)] \delta x_+ - \omega[0, \eta(x_-, \tilde{y}, t), \tau(x_-, \tilde{y}, t)] \delta x_- \right\},$$

where $g = u - iv, \mathcal{J} = \partial/\partial \tilde{y} + i \partial/\partial x$. Indeed, according to Equation (19), this field satisfies $G \equiv 0$, hence, it is potential and bounded in the domain $x \geq 0$. Therefore, substituting $\delta x_\pm = i \pm i \chi(y - \tilde{y})$, where $\chi(y)$ is the sign function,

$$\chi(y) = \begin{cases} +1, y > 0, \\ 0, y = 0, \\ -1, y < 0, \end{cases}$$

we find

$$(1 - i \hat{H}) \left[ u - iv + \frac{1}{4} \int_{-\infty}^{\infty} d\tilde{y} \left\{ \omega[\eta(i|y - \tilde{y}|, \tilde{y}, t), \tau(i|y - \tilde{y}|, \tilde{y}, t)][1 + \chi(y - \tilde{y})] \\ -\omega[\eta(-i|y - \tilde{y}|, \tilde{y}, t), \tau(-i|y - \tilde{y}|, \tilde{y}, t)][1 - \chi(y - \tilde{y})] \right\} \right]' = 0. \quad (23)$$

Hereon, $u, v, \omega$ denote the restrictions of the fields $u(x, y, t), v(x, y, t), \omega(x, y, t)$ to the control surface, $u(y, t) \equiv u(0, y, t), \omega(y, t) \equiv \omega(0, y, t), \omega(y, t)$, etc.

Next, we recall that the function $\eta(x, \tilde{y}, t)$ is defined by Equation (15) wherein trajectories $(X(\eta, \tau, t), Y(\eta, \tau, t))$ cross the control surface for $t = \tau$, whence, $X(\eta, \tau, \tau) = 0$, $Y(\eta, \tau, \tau) = \eta$. On the other hand, we know that the vortical structure near given observation point $(x, y)$ is determined by an infinitesimal vicinity of the point $(0, \eta(x, y, t))$ on the control surface. Therefore, the integrand in Equation (23) can be calculated using the following approximation of trajectories, valid for infinitesimal $(t - \tau)$:

$$X(\eta, \tau, t) = (t - \tau)u(\eta, t), \quad Y(\eta, \tau, t) = \eta + (t - \tau)v(\eta, t). \quad (24)$$

Substituting this into Equation (15) yields an equation for the function $\eta(x_\pm, \tilde{y}, t)$

$$\eta + x_\pm \frac{v(\eta, t)}{u(\eta, t)} = \tilde{y}. \quad (25)$$

Once $\eta(x_\pm, \tilde{y}, t)$ is found, the function $\tau(x_\pm, \tilde{y}, t)$ is given by

$$\tau(x_\pm, \tilde{y}, t) = t - \frac{x_\pm}{u[\eta(x_\pm, \tilde{y}, t), t]} \quad (26)$$

Despite Equation (23) is complex, only its real part represents an independent relation. This is because its left-hand side vanishes identically under the action of $(1 + i \hat{H})$ by virtue of the well-known property of the Hilbert operator, $\hat{H}^2 = -1$. Together with Equation (25), it thus yields an integro-differential relation for the functions $u(\eta, t), v(\eta, t)$. 
3.3. Asymptotic expansion of the dispersion relation

Equation (23) is generally rather complicated, but it can be greatly simplified in two respects. One is that the condition \( v_*/V \ll 1 \) allows asymptotic expansion of Equations (23) and (25) with respect to \( v/U, \tilde{u}/U \), because \( v, \tilde{u} = O(v_*) \), and \( v_*/U \ll 1 \) holds outside the viscous layer \( (\tilde{u} \) is the fluctuation of the streamwise velocity component, \( \tilde{u} = u - U \)). This expansion will be carried out below to the second order which is the leading non-trivial order, that is, the one where Equation (23) first becomes a nontrivial relation upon averaging over time. The other simplification is the disregard of the memory effects associated with the non-locality in time of the integrand in Equation (23). It is not difficult to show that the presence of the coordinate dependence in the argument \( \tau \) of the function \( \omega(\eta(\chi_+, \tilde{y}, t), \tau(\chi_+, \tilde{y}, t)) \) does not affect the vorticity distribution. Moreover, the divergenceless of the vorticity component would neither be violated if \( \tau(\chi_+, \tilde{y}, t) \rightarrow t \). However, the boundedness property is not generally preserved, so that the replacement \( \tau(\chi_+, \tilde{y}, t) \rightarrow t \) in \( \omega \) is not an equivalence transformation in the sense of \( \circ \). In this connection, it is perhaps worth to invoke the experience gained from the theory of premixed flame propagation. Discarding the memory effects does not change the general structure of the Darrieus–Landau dispersion relation which describes the unstable behaviour of nearly planar flame fronts, though it does change numerical coefficients therein, and this change turns out to be important in the limit of small gas expansion where the front dynamics becomes slow [21]. In general, the relative role of the memory effects decreases together with the characteristic time of the process, and so neglecting them makes sense in a sufficiently fast flow. Since \( [\tau(\chi_+, \tilde{y}, t) - t] \sim 1/V \) according to Equation (26), this formally means that the memory effects are treated in the zeroth-order approximation with respect to \( v_*/V \). Having in mind to use Equation (23) averaged over time, it is also possible that validity of the replacement \( \tau(\chi_+, \tilde{y}, t) \rightarrow t \) can be justified beyond this approximation, but for now, it will be adopted as a plausible assumption.

We begin with solving Equation (25) approximately. Omitting the time argument for brevity, the first-order solution \( \eta(\chi_+, \tilde{y}, t) \equiv \eta_\pm \) of this equation is

\[
\eta_\pm = \tilde{y} - x_\pm \frac{v(\tilde{y})}{U(\tilde{y})}.
\]

Substituting this in the small second term on the left of Equation (25), and expanding again gives

\[
\eta_\pm = \tilde{y} - x_\pm \left[ \frac{v(\tilde{y})}{U(\tilde{y})} - \frac{v(\tilde{y})\tilde{u}(\tilde{y}) + x_\pm v(\tilde{y})v'(\tilde{y})}{U^2(\tilde{y})} \right].
\] (28)

In all these formulas, \( x_\pm = \pm i|y - \tilde{y}| \). This expression is then used to expand the functions \( \omega(\eta_\pm) \) in the integrand of Equation (23)

\[
\omega(\eta_\pm) = \omega(\tilde{y}) - \omega'(\tilde{y})x_\pm \left[ \frac{v(\tilde{y})}{U(\tilde{y})} - \frac{v(\tilde{y})\tilde{u}(\tilde{y}) + x_\pm v(\tilde{y})v'(\tilde{y})}{U^2(\tilde{y})} \right] + \frac{x_\pm^2}{2} \omega''(\tilde{y}) \frac{v^2(\tilde{y})}{U^2(\tilde{y})}.
\] (29)
The real part of Equation (23) reads
\[
\frac{\partial u}{\partial y} - \frac{H}{2} \frac{\partial v}{\partial y} + \frac{1}{4} \int_{-\infty}^{\infty} dy \left[ \text{Re} \omega(\eta_+)[1 + \chi(y - \tilde{y})] - \text{Re} \omega(\eta_-)[1 - \chi(y - \tilde{y})] \right]' \\
+ \frac{\hat{H}}{4} \int_{-\infty}^{\infty} dy \left[ \text{Im} \omega(\eta_+)[1 + \chi(y - \tilde{y})] - \text{Im} \omega(\eta_-)[1 - \chi(y - \tilde{y})] \right]' = 0.
\]

The real and imaginary parts of \(\omega(\eta_{\pm})\) appearing here can be written, according to Equation (29) and the definition (7) of \(\omega\), as
\[
\text{Re} \omega(\eta_{\pm}) = \omega(\tilde{y}) - (y - \tilde{y})^2 \left[ \omega'(\tilde{y}) \frac{v(\tilde{y})v'(\tilde{y})}{U^2(\tilde{y})} + \frac{\omega''(\tilde{y})}{2} \frac{v^2(\tilde{y})}{U^2(\tilde{y})} \right], \\
\text{Im} \omega(\eta_{\pm}) = \mp |y - \tilde{y}| \omega'(\tilde{y}) \left[ \frac{v(\tilde{y})}{U(\tilde{y})} - \frac{v(\tilde{y})u(\tilde{y})}{U^2(\tilde{y})} \right],
\]
where \(\partial v/\partial x\) stands for \(\partial v(x, \tilde{y})/\partial x\) at \(x = 0\). Using the relation \(\chi'(y - \tilde{y}) = 2\delta(y - \tilde{y})\) \(\delta(y)\) is the Dirac function], \(\tilde{y}\)-integral of the leading-order contribution is readily calculated. The result is that the term \(\partial u/\partial y\) falls off from the equation, whereas the remaining terms take the form
\[
\hat{H} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} + \frac{\hat{H}}{2} \int_{-\infty}^{\infty} dy \chi(y - \tilde{y}) \omega'(\tilde{y}) \left[ \frac{v(\tilde{y})}{U(\tilde{y})} - \frac{v(\tilde{y})u(\tilde{y})}{U^2(\tilde{y})} \right] + \int_{-\infty}^{\infty} dy |y - \tilde{y}| \frac{[\omega'(\tilde{y})v^2(\tilde{y})]}{2U^2(\tilde{y})}' = 0.
\]

As was explained at the end of Section 3.1, the unregularised \(\tilde{y}\)-integrals here have only formal meaning. But the second \(y\)-derivative of this equation is already of strict meaning,
\[
\frac{\partial}{\partial y} \left[ \hat{H} \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} \right] + \hat{H} \frac{\partial}{\partial y} \left( \frac{\omega' v}{U} \right) - \hat{H} \frac{\partial}{\partial y} \left( \frac{\omega' v u}{U^2} \right) + \left[ \frac{(\omega' v)^2}{U^2} - C_0 \right] = 0, \quad (30)
\]
where
\[
C_0 = \frac{1}{2h} \int_{-h}^{h} dy \frac{[\omega' v]^2}{U^2}.
\]

Indeed, all terms in this equation are explicitly finite and have zero mean over the control surface [or, which is the same, over \(y \in (-h, h)\)]. We note for the future that \(\langle C_0 \rangle = 0\), because \(\langle (\omega' v)^2 \rangle\) is an odd function of \(y\) (cf. account of the parity properties at the end of Section 2).

It is important that within its accuracy, Equation (30) can be put in a form that will be shown in Section 3.4 to be free of the closure problem. To this end, we first rewrite Equation (30) as
\[
\frac{\partial}{\partial y} \left[ \hat{H} \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} \right] + \hat{H} \frac{\partial}{\partial y} \left( \frac{\omega' v}{U} \right) = O \left( \omega' v^2 \right),
\]
which integrates to

$$\hat{H} \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} + \hat{H} \left( \frac{\omega^' v}{U} - C \right) = O \left( \frac{\omega^' v^3}{V^2} \right),$$

(31)

the integration constant $C = C(t)$ being the spatial mean of $\omega^' v/U$ over $y \in (-h, h).$ Acting on this equation by $\hat{H},$ using the identity $\hat{H}^2 = -1,$ and multiplying the result by $Uv$ or $U\tilde{u}$ gives, respectively,

$$\omega^' v^2 = CUv - Uv \hat{H} \frac{\partial^2 v}{\partial y \partial x} - Uv \frac{\partial^2 v}{\partial y^2} + O \left( \frac{\omega^' v^3}{V} \right),$$

(32)

$$\omega^' v\tilde{u} = CU\tilde{u} - U\tilde{u} \hat{H} \frac{\partial^2 v}{\partial y \partial x} - U\tilde{u} \frac{\partial^2 v}{\partial y^2} + O \left( \frac{\omega^' v^3}{V} \right).$$

(33)

We then put this into Equation (30) omitting terms of the third order in $v_*/V,$ and average over time

$$\hat{H} \frac{d}{dy} \frac{1}{U} \left( \tilde{u} \hat{H} \frac{\partial^2 v}{\partial y \partial x} + v \frac{\partial^2 v}{\partial y \partial x} + \tilde{u} \frac{\partial^2 v}{\partial y^2} - v \frac{\partial^2 \tilde{u}}{\partial y^2} - C \tilde{u} \right) = \frac{1}{U^2} \frac{d}{dy} \frac{1}{U} \left( \tilde{v} \hat{H} \frac{\partial^2 v}{\partial y \partial x} + v \frac{\partial^2 v}{\partial y^2} - C \tilde{v} \right).$$

(35)

As the time-averaged quantities defined on the control surface depend only on $y,$ the partial $y$-derivatives acting thereupon have been replaced by $d/dy.$ Finally, we note that

$$\left\langle \tilde{u} \frac{\partial^2 v}{\partial y^2} \right\rangle = \left\langle v \frac{\partial^2 \tilde{u}}{\partial y^2} \right\rangle.$$

(34)

Indeed, taking the time-average of Equation (5),

$$\left\langle v_i \frac{\partial \omega}{\partial x_i} \right\rangle = 0,$$

inserting the definition of $\omega,$ and using Equation (6) gives

$$\left\langle \tilde{u} \frac{\partial^2 v}{\partial x^2} - v \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u} \frac{\partial^2 v}{\partial y^2} - v \frac{\partial^2 \tilde{u}}{\partial y^2} \right\rangle = 0.$$

But in view of the mean flow homogeneity in the streamwise direction, the cross-correlation function (10) depends on $x_1, x_2$ only through the difference $(x_1 - x_2),$ therefore,

$$\left\langle \tilde{u} \frac{\partial^2 v}{\partial x^2} \right\rangle = \left\langle v \frac{\partial^2 \tilde{u}}{\partial x^2} \right\rangle.$$

which proves the identity (34). We, thus, arrive at the following equation:

$$\hat{H} \frac{d}{dy} \frac{1}{U} \left( \tilde{u} \hat{H} \frac{\partial^2 v}{\partial y \partial x} + v \frac{\partial^2 v}{\partial y \partial x} - C \tilde{u} \right) = \frac{1}{U^2} \frac{d}{dy} \frac{1}{U} \left( \tilde{v} \hat{H} \frac{\partial^2 v}{\partial y \partial x} + v \frac{\partial^2 v}{\partial y^2} - C \tilde{v} \right).$$

(35)
A detailed analysis of this equation will be carried out in Section 4, but several important points of its derivation are to be discussed beforehand. First, the transformation just performed turned out to be possible because of the special feature of Equation (30) that the terms proportional to \( \omega^r v^2 \) and \( \omega^i (v^2)^i \) had combined into the last term in Equation (30), which involves only the product \( \omega^i v^2 \). As a result, triple correlations of fluctuating fields now enter the equation only spatially averaged, through the terms \( \langle C \tilde{u} \rangle \), \( \langle Cv \rangle \), where \( C = (1/2h) \int_{-h}^{h} dy \omega v / U \). Second, it is to be noted that in the course of asymptotic expansion of Equation (23), the flow velocity itself has not been expanded (neither \( U \), nor \( \tilde{u}, v \)). In other words, solutions to this equations are not treated as asymptotic series. In fact, doing so would be inconsistent, as can be seen from Equation (31): an attempt to write \( v_i = v_i^{(0)} + v_i^{(1)} + \cdots \), where \( v^{(1)}/v^{(0)} = O(v_x/U) \), would give

\[
\begin{align*}
\hat{H} \frac{\partial^2 v^{(0)}}{\partial y^2} - \frac{\partial^2 v^{(0)}}{\partial y \partial x} = 0,
\hat{H} \frac{\partial^2 v^{(1)}}{\partial y^2} - \frac{\partial^2 v^{(1)}}{\partial y \partial x} + \hat{H} \left( \frac{\omega^{(0)}}{U} v^{(0)} - C \right) = 0.
\end{align*}
\]

But the first of these equations means that the field \( v_i^{(0)} \) is potential, whence \( \omega^{(0)} = 0 \), and so the second equation fails to couple \( v_i^{(1)} \) to \( v^{(0)} \) or \( U \), thus merely duplicating the first. It is also worth noting that even if the expansion of \( v_i \) was consistent, it would actually be inadequate to the main purpose of Equation (35), which is to provide a relation between measurable quantities (mean velocity and velocity correlations). Therefore, using the expansion \( v_i = v_i^{(0)} + v_i^{(1)} + \cdots \) would result in a sequence of relations involving quantities such as \( \langle v^{(0)} \rangle \tilde{u}^{(1)} \), which could be determined only by solving the complete system of hydrodynamic equations, a task the present approach is designed to overcome.

Incidentally, it is not difficult to see that in the first-order approximation, there is no turbulence driven by shear. In fact, averaging Equation (31) over time and using \( \langle v \rangle \equiv 0 \), \( \langle C \rangle \equiv 0 \) (the latter follows from the oddness of the combination \( \langle \omega^i v \rangle / U = -\langle v \triangle \tilde{u} \rangle / U \)), one finds \( \hat{H} (\langle v \triangle \tilde{u} \rangle / U) = 0 \), hence, \( \langle v \tilde{u} \rangle = 0 \), which for a realistic cross-correlation function means \( \langle v \tilde{u} \rangle = 0 \), that is \( v_x = 0 \). A nontrivial shear turbulence appears in the next second-order approximation, wherein the cross-correlation function is not zero, but is related to \( U \) by Equation (35).

Finally, it is quite clear that the above procedure of asymptotic expansion of Equation (23) can be carried out to any order desired, because the integrals \( \int_{-\infty}^{\infty} dy \omega [\hat{\eta}(x_\pm, \hat{y}, t)][1 \pm \chi(y - \hat{y})] \) are localisable at any finite order.

### 3.4. **Equation (35) in high channels**

Equation (35) relates the mean flow velocity \( U(y) \), the cross-correlation function (10), and triple correlations of velocity fluctuations. The presence of the latter constitutes a closure problem: in general, additional information regarding the structure of triple correlations is needed to make of Equation (35) a useful relation. It is important, however, that the triple products of field fluctuations enter this equation only in the combinations \( \langle C \tilde{u} \rangle \), \( \langle Cv \rangle \),
where

\[ C = \frac{1}{2h} \int_{-h}^{h} dy \frac{\omega' u}{U}. \]

For fixed \( v_*, v \), this mean rapidly tends to zero as \( h \) increases, because \( U \) increases with \( |y| \), \( v \) remains of order \( v_* \), while \( \omega' \) decreases, for the Taylor length \( \lambda \) appearing in the vorticity estimate \( \omega \sim v_*/\lambda \), and the length scale of the vorticity field (the Kolmogorov length \( \lambda_0 \)) grows with \( h \) (\( \lambda \sim h^{1/2}, \lambda_0 \sim h^{1/4} \)), so that \( \omega' \sim h^{-3/4} \) in the bulk. Therefore, for sufficiently large \( h \), the terms involving \( C \) in Equation (35) can be omitted. This equation, thus, takes the form

\[ \hat{H} \frac{d}{dy} \frac{1}{U} \left( \hat{u} \hat{H} \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 \hat{u}}{\partial x^2} \right) = -\frac{1}{U^2} \frac{d}{dy} U \left( \hat{v} \hat{H} \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y \partial x} \right), \quad (36) \]

where the continuity equation was also used to express all averages via the cross-correlation function (10).

4. Analysis of the dispersion relation

4.1. Cross-correlation spectral density

As was already mentioned, the cross-correlation function depends on the streamwise coordinates only via their difference, and satisfies

\[ R(x, y; x, y) = -v_*^2. \quad (37) \]

In addition to that, the flow continuity requires \( \partial^2 R(x_1, y_1; x_2, y_2)/\partial x_1 \partial y_2 \) be symmetric under the interchange \((x_1, y_1) \leftrightarrow (x_2, y_2)\),

\[ \frac{\partial^2 R(x_1, y_1; x_2, y_2)}{\partial x_1 \partial y_2} = - \left\{ \frac{\partial \hat{u}(x_1, y_1)}{\partial x_1} \frac{\partial \hat{u}(x_2, y_2)}{\partial x_2} \right\} = \frac{\partial^2 R(x_2, y_2; x_1, y_1)}{\partial x_2 \partial y_1}. \quad (38) \]

Otherwise, there are no general restrictions on \( R(x_1, y_1; x_2, y_2) \); in particular, it may depend on \((y_1 + y_2)\) as well as on \((y_1 - y_2)\). The aim of the subsequent consideration is not to identify all possible functions \( R(x_1, y_1; x_2, y_2) \) and \( U(y) \) that satisfy Equation (36), but only a subclass thereof, which is general enough to include all three basic velocity profiles discussed in Section 1, on the one hand, and simple enough to admit complete analysis, on the other hand. We, therefore, specialise to the simplest realisation of condition (37) where \( R(x_1, y_1; x_2, y_2) \) depends on \( y_1, y_2 \) in the physical channel also via their difference only:

\[ R(x_1, y_1; x_2, y_2) = R(x_1 - x_2, y_1 - y_2; 0, 0) \equiv R(x_1 - x_2, y_1 - y_2). \]

Then, \( R(0, 0) = -v_*^2 \), and the spectral density \( \rho \) can be conveniently introduced as

\[ R(x_1 - x_2, y_1 - y_2) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dkdq}{(2\pi)^2} \rho(k, q)e^{ik(x_1 - x_2) + iq(y_1 - y_2)}, \quad y_1, y_2 \in (0, h). \quad (39) \]
As we deal with \( h \gg v/v_* \), decomposition into Fourier integral with respect to the coordinate \( y \) is used here instead of a Fourier series. According to the rule (11), extension of \( R \) over \( y_1, y_2 \in (-h, h) \) reads

\[
R(x_1 - x_2, y_1 - y_2) = -\int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} q(k, q) e^{i(k(x_1 - x_2)+q(y_1 - y_2))} \{ \theta(y_1)\theta(y_2) - \theta(-y_1)\theta(-y_2) \}.
\]

Writing the density as

\[
\rho(k, q) = \rho_+(k, q) + \rho_-(k, q),
\]

where \( \rho_+(k, q) \) is even with respect to \( k \), and \( \rho_-(k, q) \) is odd, the symmetry property (38) requires \( \rho_+(k, q) \) be even also with respect to \( q \), and \( \rho_-(k, q) \) be odd:

\[
\begin{align*}
\rho_+(k, q) &= \rho_+(\bar{k}, q) = \rho_+(k, -q), \\
\rho_-(k, q) &= -\rho_-(\bar{k}, q) = -\rho_-(k, -q).
\end{align*}
\]

Equation \( R(0, 0) = -v_*^2 \) plays the role of normalisation condition for \( \rho_+ \),

\[
\int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} \rho_+(k, q) = v_*^2,
\]

whereas Equation (38) implies a positivity property for \( \rho_- \):

\[
\int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} kq \rho_-(k, q) = -\frac{\partial^2 R(0, 0)}{\partial x_1 \partial y_2} = \left\langle \left( \frac{\partial \tilde{u}}{\partial x} \right)^2 \right\rangle > 0.
\]

In terms of the decomposition (40), the averages appearing in Equation (36) can be written, by virtue of linearity of the Hilbert operator, as

\[
\begin{align*}
\left\langle \hat{u} \hat{H} \frac{\partial^2 \tilde{u}}{\partial y \partial x} \right\rangle &= \int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} kq \rho_-(k, q) \int_{0}^{h} dy e^{iq(|\tilde{y}|-\tilde{y})} \cot \left\{ \frac{\pi}{2h} (\tilde{y} - |\tilde{y}|) \right\} \chi(\tilde{y}) \equiv u H u_x y, \\
\left\langle \frac{\partial^2 \tilde{u}}{\partial x^2} \right\rangle &= \int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} k^2 \rho_+(k, q) \chi(y) \equiv u_x x v, \\
\left\langle v \hat{H} \frac{\partial^2 \tilde{u}}{\partial x^2} \right\rangle &= \int_{-\infty}^{\infty} dk dq \frac{1}{(2\pi)^2} k^2 \rho_+(k, q) \int_{0}^{h} dy e^{iq(|\tilde{y}|-\tilde{y})} \cot \left\{ \frac{\pi}{2h} (\tilde{y} - |\tilde{y}|) \right\} \equiv v H u_x x.
\end{align*}
\]
\[
\left\langle \frac{\partial^2 \tilde{u}}{\partial y \partial x} v \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dkdq}{(2\pi)^2} kqQ_-(k, q) \equiv u_{xy}v.
\] (47)

The symbols on the right-hand side of these equations are merely shorthand for the integrals representing their left-hand sides. We observe that each of these functions depends on either \(Q_+\) or \(Q_-\), that \(u_{xy}v\) is a positive constant (Cf. (43)), whereas \(u_{xx}v\) is constant in the physical and auxiliary channels separately. Rewritten in the abridged notation, Equation (36) reads

\[
\hat{H} \frac{d}{dy} \left[ \frac{uHv_{xy} - u_{xx}v}{U} \right] = -\frac{1}{U^2} \frac{d}{dy} U [vHu_{xx} + u_{xy}v].
\] (48)

### 4.2. Analytical structure of Equation (48)

Equation (48) shows that the mean velocity profile \(U(y)\) depends on the structure of correlation between \(\tilde{u}\) and \(v\) throughout the channel. Its near-wall asymptotic, however, exhibits a great deal of universality: it will be shown to depend only on a few integral characteristics of the cross-correlation spectrum. The subsequent preparations serve this purpose.

#### 4.2.1. An equivalent system for Equation (48)

Given the functions defined by Equations (44)–(47), Equation (48) determines the mean velocity distribution \(U(y)\) up to a multiplicative constant. It is also of interest to reverse the problem and ask to what extent the mean velocity distribution in the channel affects velocity correlations. To address these questions, it is useful to first recast Equation (48) as a system of two coupled equations. The possibility to perform this transformation stems from the fact that the two functions \(Q_+(k, q), Q_-(k, q)\) are independent of each other. This observation suggests to write

\[
uHv_{xy} - u_{xx}v \equiv [f_1(y) + \alpha] \chi(y),
\] (49)

\[
vHu_{xx} + u_{xy}v \equiv [f_2(y) + \beta],
\] (50)

where \(f_{1,2}(y)\) are new independent functions, \(\alpha, \beta\) are as yet arbitrary constants, and to consider the following system:

\[
\hat{H} \left\{ \frac{d}{dy} \frac{f_1 \chi}{U} \right\} = -\frac{1}{U^2} \frac{d}{dy} U \frac{\beta U}{dy},
\] (51)

\[
\hat{H} \left\{ \frac{1}{U^2} \frac{d}{dy} \frac{f_2 \chi}{U} \right\} = \frac{d}{dy} \frac{\alpha \chi}{U}, \quad y \neq 0.
\] (52)

If \(U(y)\) is a solution of this system with \(f_{1,2}(y), \alpha, \beta\) satisfying Equations (49) and (50), then, taking the Hilbert transform of Equation (52) with the help of \(\hat{H}^2 = -1\), and subtracting the result from Equation (51) shows that it is also solution of Equation (48). Thus, solutions of the above system span a subset of solutions of Equation (48). On the other hand,
$U(y)$ always exists such that it satisfies Equations (51) and (52) for some $f_1, f_2$. In fact, Equations (51) and (52) can be viewed as defining $f_1, f_2$ for a given $U(y)$ (this issue will be dealt with in detail in Section 4.4). Therefore, as long as the assumption that the core flow has negligible influence on the structure of the inner layer is an acceptable approximation, the actual structure of the function $R(x, y)$ for $y \sim h$ is immaterial regarding the form of $U(y)$ in the region $|y| \ll h$, provided that the functions $f_{1, 2}(y)$ have the same asymptotics for $y \ll h$ as the functions $uHv_{xy}, vHu_{xx}$, respectively. In this case, therefore, the system (Equations (51) and (52)) is equivalent to Equation (48) (we will have more to say on the role of the velocity correlations at distances $y \sim h$ in Section 4.5.2).

It will now be proved that for $|y| \ll h$, this system of integral relations can be reduced to an ordinary differential equation. To this end, we note that Equations (51) and (52) allow $dU/d|y|$ to be expressed in two different ways

$$\beta \frac{d}{d|y|} \frac{1}{U} = \hat{H} \chi \left\{ \frac{d}{dy} \frac{f_1 \chi}{U} \right\},$$

$$\frac{d}{d|y|} \frac{1}{U} = \frac{1}{\alpha} \hat{H} \left\{ \frac{1}{U^2} \frac{dU f_2}{dy} \right\}, \quad \alpha \neq 0.$$

These expressions can be combined to give

$$\hat{H} \hat{X} \hat{H} \frac{d}{dy} \frac{f_1 \chi}{U} = -\frac{\beta}{\alpha U^2} \frac{dU f_2}{dy}. \quad (53)$$

According to the Appendix, the action of $\hat{H} \hat{X} \hat{H}$ on a $2h$-periodic function $a(y)$ with zero spatial mean reads

$$\left( \hat{H} \hat{X} \hat{H} a \right)(y) = -\chi(y)a(y) + \frac{1}{\pi h} \int_{-h}^{h} d\tilde{y} a(\tilde{y}) \cot \left\{ \frac{\pi}{2h} (y - \tilde{y}) \right\} \ln \left| \frac{\tan(\pi \tilde{y}/2h)}{\tan(\pi y/2h)} \right|. \quad (54)$$

This complicated expression greatly simplifies for $|y| \ll h$ and even $a(y)$. We first observe that the integrand is regular for $\tilde{y} = y$. Therefore, integration over $\tilde{y} \sim y$ gives rise to a contribution which is $O(y/h)$ relative to the first term on the right-hand side of Equation (54). On the other hand, integration over $|\tilde{y}| \sim h$ is also $O(y/h)$ because the integrand becomes an odd function of $\tilde{y}$ on neglecting $y$ in the argument of the cotangent. As the function $(f_1 \chi/U)'$ is even in $y$, Equation (53) gives for $|y| \ll h$

$$\frac{d}{dy} \frac{f_1}{U} = \frac{\beta}{\alpha U^2} \frac{dU f_2}{dy},$$

or

$$(\ln U)' = \frac{\alpha f_1' - \beta f_2'}{\alpha f_1 + \beta f_2}. \quad (55)$$

It remains to determine the near-wall behaviour of the functions $f_{1, 2}$, or equivalently, of $uHv_{xy}, vHu_{xx}$. 
4.2.2. Near-wall asymptotics of $uH_{v_{xy}}$, $vH_{u_{xx}}$

The small-$y$ asymptotics of functions (44) and (46) are determined by that of the integral

$$I \equiv \frac{1}{2h} \int_{0}^{h} \tilde{y} e^{iq|y|} \cot \left\{ \frac{\pi}{2h} (\tilde{y} - |y|) \right\},$$

which is

$$I = \frac{1}{\pi} \ln \frac{2}{|qy|}.$$

It is seen that the leading term of $I$ is singular at $y = 0$. This singularity is due to the pole of the cotangent at $\tilde{y} = |y|$ in the integrand of Equation (56). It is to be noted that the asymptotic (57) is nonuniform with respect to $q$: for a given $y$, it holds for $q$ such that $q|y| \ll 1$. Since the shear stress is directly determined by the cross-correlation function, the primary characteristic length of $R$ is the channel height $h$, because eddies of this size are most efficient in transporting the flow momentum. If this length is unique, then the $q$-integration in functions (44) and (46) is essentially over $q = O(1/h)$, and therefore their leading terms for $y \ll h$ read simply

$$uH_{v_{xy}}|_{\text{lead}} = \frac{1}{\pi} \ln \frac{h}{|y|} \int_{-\infty}^{\infty} \frac{dk dq}{(2\pi)^2} k q Q_{-}(k, q),$$

$$vH_{u_{xx}}|_{\text{lead}} = \frac{1}{\pi} \ln \frac{h}{|y|} \int_{-\infty}^{\infty} \frac{dk dq}{(2\pi)^2} k^2 Q_{+}(k, q).$$

Integration over $q$ in these formulas has been extended over all $-\infty < q < +\infty$, because $q \gg 1/|y|$ do not contribute to $q_{\pm}(k, q)$ anyway. Things change, however, when in addition to $h$ the cross-correlation function is characterised by an ‘inner’ length $\Lambda \ll h$, such as $\lambda$ or $\lambda_0$, as is presumably the case with the function $Q_{-}(k, q)$ which is related to the fluctuating streamwise velocity gradient via Equation (43). In this case, integration over $q = O(1/\Lambda)$ gives rise to a term $\sim \ln |y|$ only for $|y| \ll \Lambda$, whereas at larger $y$s, it is $q = O(1/h)$ which only contribute to the prefactor of the logarithm. Under condition $\Lambda \ll h$, this fact can be taken into account by writing an intermediate ($\Lambda \lesssim |y| \ll h$) asymptotic of, e.g., the function $uH_{v_{xy}}$ as

$$uH_{v_{xy}}|_{\text{lead}} = \frac{1}{\pi} \ln \frac{h}{|y|} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\sqrt{h\Lambda}}^{\sqrt{h\Lambda}} \frac{dq}{2\pi} k q Q_{-}(k, q).$$

Thus, in the presence of the scale $\Lambda$, the logarithmic near-wall asymptotic of this function is compound, switching from (58) to (60) as $y$ increases from $|y| \ll \Lambda$ to $|y| \gtrsim \Lambda$.

4.3. The near-wall asymptotic of mean flow velocity

We are now in a position to show that the three velocity profiles – the power, log, and power-log laws – are possible near-wall asymptotics of the mean flow velocity. To avoid cluttering
of the exposition by similarly looking expressions, the formulas will be written down as if
$h$ was the only length scale of the problem. In view of locality of Equation (55), the same
results will apply also in the presence of inner scale $\Lambda$ separately to $|y| \ll \Lambda$ and to the
intermediate asymptotic region $\Lambda \lesssim |y| \ll h$, provided that the $q$-integrations are extended
over appropriate domains (cf. Equations (58) and (60)).

Depending on the value of the ratio $\beta/\alpha$ in Equation (55), we obtain the following clas-
sification of the near-wall velocity profiles:

**The logarithmic law.** According to Equation (58) or (60), the proportionality coefficient
of the logarithm in the function $f_1$ is nonzero in view of the positivity property (43) of $Q_-$. Equation (55) then shows that the case $\beta = 0, \alpha \neq 0$ corresponds to solutions having
logarithmic profile near the wall,

$$U = \frac{v_s}{\kappa} \ln \frac{|y|}{y_0}. \tag{61}$$

Here, the integration constant is written in the conventional form $v_s/\kappa$, where $\kappa$ is the
Kármán constant. This constant is naturally left undetermined as we deal here only with
the near-wall asymptotic of $U(y)$. The constant $y_0$ is related to $\alpha$ and the characteristics of
the cross-correlation spectrum by Equation (49):

$$\alpha = - \iint \frac{dkdq}{(2\pi)^2} \left\{ \frac{1}{\pi} kqQ_- \ln(|q|y_0/2) + k^2Q_+ \right\}. \tag{62}$$

As $U$ is positive, $\alpha$ must be chosen so as to position the point with the ordinate $y_0$ within
the viscous layer.

**Power laws.** When the parameters $\alpha, \beta$ satisfy

$$\alpha \iint dq dq \frac{kqQ_-}{kqQ_+} = - \beta \iint dq dq \frac{kqQ_+}{kqQ_-}, \tag{63}$$

the proportionality coefficient of the logarithm in the denominator of Equation (55) v a n-
ishes, and the law of the wall becomes a power law

$$U = A|y|^n, \tag{64}$$

where the constant of integration $A$ is again arbitrary, while the exponent

$$n = \frac{2\alpha \iint dq dq \frac{kqQ_-}{kqQ_+}}{\iint dq dq \left\{ kqQ_- \left[ \alpha \ln(|q|y_0) - \beta \pi \right] + k^2Q_+ \left[ \beta \ln(|q|y_0) + \alpha \pi \right] \right\} + 4\pi^3(\alpha^2 + \beta^2)} \tag{65}$$

$y_0$ being a constant of which $n$ is independent by virtue of Equation (63). $n$ must be positive
for $U$ to grow away from the wall. For practical applications, it will suffice to consider the
case $n < 1$.

**Power-log laws.** All other cases fall into the class of power-log laws:

$$U = \frac{v_s}{\kappa} \ln^p \frac{|y|}{y_0}. \tag{66}$$
As before, the integration constant \( \kappa \) is arbitrary, whereas the constants \( p \) and \( y_0 \) are related to the parameters \( \alpha, \beta \) by Equations (49) and (50)

\[
P = \frac{\int \int dk dq (\alpha k q \rho - \beta k^2 \rho)}{\int \int dk dq (\alpha k q \rho + \beta k^2 \rho)}.
\]

\[
\alpha^2 + \beta^2 + \int \int \frac{dk dq}{(2\pi)^2} \left\{ \alpha k^2 \rho - \beta k q \rho + [\alpha k q \rho + \beta k^2 \rho] \frac{1}{\pi} \ln \frac{y_0 |q|}{2} \right\} = 0.
\] (68)

As in the case of the log-law, parameters \( \alpha, \beta \) must be such that \( y_0 \) is less than the viscous layer thickness. Also, \( U \) grows away from the wall only for \( p > 0 \). Less trivial restrictions on the parameters will be established in Section 4.5.2.

Finally, a note on the structure of the obtained solutions is in order. As we have seen, all three basic velocity distributions involve an arbitrary multiplicative constant – the constant of integration of Equation (55). This is, in fact, a general property of solutions to the dispersion relation (48) which is homogeneous in \( U \). In principle, this arbitrary factor can be fixed by the normalisation condition (9) once \( U(y) \) satisfying Equation (48) is found for all \( y \in (0, h) \). In this sense, Equations (9) and (48) allow a complete solution to the problem of finding the mean velocity distribution for a given cross correlation. However, they provide no relation between \( V \) and the friction velocity \( v^* \) that would allow one to fix the value of the Kármán constant \( \kappa \) (or \( \kappa_p \) in the power-log law (66)).

### 4.4. Reconstruction of the cross-correlation spectral density

Equation (48) was used in Section 4.3 to determine the possible profiles of the mean flow velocity near the wall. It is important, however, that the problem can be reversed to address the question as to what extent the mean velocity distribution in the channel affects velocity correlations. Viewing the problem from this standpoint reveals certain duality between these two objects, indicating that the question ‘which velocity profile is the right one’ is not probably the right question to ask. As will now be proved, given a mean velocity distribution \( U \), one can always find a cross-correlation function \( R \) such that \( U, R \) satisfy Equation (48). In fact, there are infinitely many such \( R \)s. To this end, we rewrite the system (51) and (52) as

\[
f_1 = -\beta \chi U \hat{H} \left( \frac{1}{U} - c \right),
\]

\[
f_2 = -\frac{\alpha}{U} \int_0^y d\tilde{y} U^2(\tilde{y}) \frac{d}{d\tilde{y}} \left( \hat{H} \chi \right) (\tilde{y}),
\]

where the constant \( c \) is the mean value of \( 1/U \) across the channel. Once \( U(y) \) is given, these equations define the functions \( f_{1,2}(y) \) for arbitrary \( \alpha, \beta \). Moreover, one can arrange things so as to fix the small-\( y \) asymptotics of \( f_{1,2} \) as desired. It is not difficult to see that in all three cases considered in Section 4.3, the small-\( y \) asymptotics of \( f_{1,2} \) given by Equations (69) and (70) are logarithmic, \( f_{1,2}(y) \sim \ln (|y|/h) \). The proportionality coefficients in these asymptotics can always be assigned any desired value by adjusting the parameters \( \alpha, \beta \), because
according to Equations (69) and (70), \(f_1 \sim \beta, f_2 \sim \alpha\). This proves the result referred to in the beginning of Section 4.3.

Whether or not the asymptotics of \(f_{1,2}\) are fixed, the cross-correlation spectral density can now be reconstructed as follows. One first differentiates and then Fourier-transforms Equations (44)–(47), (49) and (50) with respect to \(y\), which yields two integral relations for the functions \(q_{\pm}(k, q)\)

\[
\int \frac{dk}{2\pi} k^2 q_{+}(k, q) = -\frac{h}{q} \int_{-h}^{h} d\tilde{y} f_{2}'(\tilde{y}) \sin \left(\frac{\pi \tilde{y}}{h}\right) \cos (q\tilde{y}), \tag{71}
\]

\[
\int \frac{dk}{2\pi} kq_{-}(k, q) = -\frac{h}{q} \int_{-h}^{h} d\tilde{y} f_{1}'(\tilde{y}) \sin \left(\frac{\pi \tilde{y}}{h}\right) \cos (q\tilde{y}). \tag{72}
\]

Since the dispersion relation holds only outside the viscous layer, \(|y| \gg v/\nu_\ast\), Equations (71) and (72) are valid only for \(|q| \ll v_\ast/\nu\). Thus, the form of \(q_{\pm}(k, q)\) at \(q \approx v_\ast/\nu\) remains arbitrary. On the other hand, the left-hand side of Equations (49) and (50), and the values of the constants \(u_{xx}, u_{xy}\) in particular, depend essentially on this form. This functional freedom in \(q_{\pm}\) can be used to fix, in infinitely many different ways, the constants that fell off from Equations (49) and (50) as a result of the \(y\)-differentiation. This proves the existence of \(R\) satisfying the dispersion relation together with the given \(U\). It is clear that there is a continuum of such \(Rs\), as the dependence on the argument \(k\) of the spectral density also remains essentially arbitrary.

4.5. ‘Infinitely’ high channels

The near-wall properties of the mean flow velocity are naturally expected to simplify as the channel height grows. A key issue is then existence of their limit as \(h \to \infty\). As it follows from the preceding discussion, this issue cannot be settled without additional assumptions regarding velocity cross correlations. Yet, even the general case admits an interesting simplification leading to a kind of universality in the power law. This will be discussed first, and then the limit of infinite channel height will be addressed.

4.5.1. Large-\(h\) asymptotic of the power law

Consider the case when the odd part of the cross-correlation function is characterised by an inner length \(\Lambda\) that scales with the Reynolds number as \(\Lambda \sim h/Re^a, 0 < a < 1\). Then, its ratio to the viscous layer thickness, \(\Lambda v_\ast/\nu \sim Re^{1-a}\), grows with the channel height, and any fixed point \(y\) will be found obeying \(y \ll \Lambda\) for sufficiently large \(h\). Integrals appearing in expression (65) for the power-law exponent can be evaluated in this case as

\[
\iint_{-\infty}^{\infty} dk dq k^2 q_{+} \ln(|q|\gamma_0) \approx \ln \left(\frac{\gamma_0}{\Lambda}\right) \iint_{-\infty}^{\infty} dk dq k^2 q_{+},
\]

\[
\iint_{-\infty}^{\infty} dk dq kq_{-} \ln(|q|\gamma_0) \approx \ln \left(\frac{\gamma_0}{\Lambda}\right) \iint_{-\infty}^{\infty} dk dq kq_{-}. \tag{73}
\]
Taking into account condition (63), one thus finds, with a logarithmic accuracy,

\[ n \approx \frac{2}{a \ln \text{Re}}. \] (74)

We conclude that the large-\( h \) asymptotic of the power law turns out to be to a large degree universal, depending only on the scaling of the cross-correlation function with the Reynolds number (one should remember that this conclusion holds true only on a subclass of cross-correlation functions, namely that corresponding to a power-law asymptotic of the mean velocity). That the exponent \( n \) slowly decreases with the increase of the Reynolds number was known since the pioneering works of Prandtl and von Kármán, whereas the scaling \( n \sim 1/\ln \text{Re} \) was suggested much later in [10] based on the analysis [16] of the same experimental data [24] that was used earlier to verify the logarithmic law.[1]

**4.5.2. Law of the wall in the limit \( h \to \infty \)**

Let us now explore the possibility to go over to the limit of infinite channel height in the relations obtained in Section 4.3. To this end, we introduce the following natural assumption:

**Assumption (A):** For a given value of the shear stress, the leading term of the near-wall mean velocity asymptotic is independent of the cross-correlation between velocity fluctuations near the wall and in the bulk.

This condition is critical in that it turns out to be a dividing line between the logarithmic and power laws. In application to the system (51) and (52), it means that the function \( R(x, y) \) is required to be such that the functions \( f_1, 2(y) \) decrease away from the wall so that the bulk contribution to the leading term of \( U(y) \) can be neglected. This requirement on \( R \) will be discussed in more detail later on; meanwhile, the main consequence of Assumption (A) will be inferred, namely a reduction in the classification of the near-wall velocity profiles.

First of all, we note that this assumption allows taking the limit \( h \to \infty \) in Equations (51) and (52), with all the functions involved replaced by the leading terms of their small-\( y \) asymptotics, and the Hilbert operator written accordingly in the non-periodic form (21). Then, it is readily seen that the power laws (64) are inconsistent with Assumption (A). Consider, for instance, Equation (52). Substituting Equation (64), the right-hand side is seen to be proportional to \(|y|^{-n - 1}\). However, setting \( f_2(y) \sim \ln |y| \) in the left-hand side, and extracting the leading term yields an expression proportional to

\[ \int_{-\infty}^{\infty} \frac{dy \ln |\tilde{y}|}{|\tilde{y}|^{n+1} (\tilde{y} - y)} = 2 \int_{0}^{\infty} \frac{dy \ln \tilde{y}}{\tilde{y}^{n}(\tilde{y}^2 - y^2)}. \]

The formula [25]

\[ \int_{0}^{\infty} \frac{dy \ln \tilde{y}}{\tilde{y}^{n}(\tilde{y}^2 - y^2)} = \frac{\pi}{4} |y|^{-n-1} \left\{ \pi \cot \frac{(1-n)\pi}{2} - 2 \ln |y| \cot \frac{(1-n)\pi}{2} \right\}, \quad n < 1, \tag{75} \]

then shows that the leading term on the left is \( \sim |y|^{-n-1} \ln |y| \). We conclude that the mean velocity profiles characterised by the power-type near-wall asymptotics can be realised only
in flows with sufficiently strong cross correlation between velocity fluctuations near the wall and in the bulk. Evidently, parameters of such laws must be essentially dependent on the Reynolds number, because dependence on the flow properties at distances $|y| \sim h$, hence on the value of $h$ itself is necessarily present. In particular, it is meaningless to speak about the limit $h \to \infty$ (that is, $\text{Re} \to \infty$) in this situation. On the contrary, a similar calculation shows that Assumption (A) can be accommodated by the log- and the power-log laws. The leading terms of the integrals appearing in Equations (51) and (52) can be extracted by writing

$$
\int_{-\infty}^{\infty} \frac{d\tilde{y}}{\ln^p(\tilde{y}/y_0)(\tilde{y} - y)} = \frac{2}{\gamma} \int_{0}^{\infty} \frac{d\eta}{[\ln \eta + \ln(|y|/y_0)]^p} \approx \frac{-2p}{y \ln^{p+1}(|y|/y_0)} \int_{0}^{\infty} d\eta \ln \eta ,
$$

and applying formula (75) with $n = 0$. The result is that the parameters $\alpha$, $\beta$ are no longer arbitrary, but are related to the log exponent $p$ and the cross-correlation spectral density

$$
\alpha = -\frac{p + 1}{2} \int \int \frac{dk dq}{(2\pi)^2} k^2 q_+, \quad \beta = \frac{p - 1}{2} \int \int \frac{dk dq}{(2\pi)^2} k q_-. \quad (76)
$$

It remains to specify conditions on the cross-correlation function under which Assumption (A) holds. As we know, the leading term of $U(y)$ can be identified as the limit of $U$ as $h \to \infty$. It follows from Equations (51) and (52) that this term is independent of the bulk behaviour of $f_{1,2}(y)$, that is Assumption (A) is met, provided that these functions decrease away from the wall. In view of relations (44), (49) and (46), (50), this implies also certain restrictions on the $y$-dependence of the function $R(x, y)$, but these are strongly interrelated with the properties of the cross correlations in the streamwise direction. This is clearly seen from Equations (71) and (72) showing that the functions $f_{1,2}(y)$ determine $q$-dependence of the integrals $\int dk k q_-(k, q)$ and $\int dk k^2 q_+(k, q)$, rather than of the spectral density itself. Since no uniform conclusion regarding the $y$-dependence of $R(x, y)$ can be drawn under such circumstances, it is instructive to look at two opposite extreme cases, which are though both unrealistic, yet indicative as to the variety of possible cross-correlation functions consistent with Assumption (A). Of primary importance is the function $q_+(k, q)$ as it determines cross correlations in the wall-normal and streamwise directions, $R(0, y)$ and $R(x, 0)$. In the case when its dependence on $k$ and $q$ factorises, $q_+(k, q) = q_+^{(1)}(k) q_+^{(2)}(q)$, it follows easily from Equation (71) that $R(x, y) \sim y^3 f_2^3(y)$, so that the cross correlations decrease away from the wall roughly at the same rate as $f_2$. However, in the opposite case of isotropic cross-correlation spectrum, $q_+(k, q) = q_+(k^2 + q^2)$, the factor $k^2$ in the integrand on the left-hand side of Equation (71) comes into play, bringing in an extra factor $y^{-2}$ when $|y| \gg |x|$, so that $R$ scales with the distance to the wall as $R(x, y) \sim y^3 f_2^3(y)$. In principle, therefore, Assumption (A) can be met even in flows characterised by $R(x, y)$ growing with the distance as fast as $y^b$, $b < 2$.

### 4.5.3. Large-$h$ asymptotic of the power-log exponent

Having identified solutions of the power-log family for which the limit $h \to \infty$ exists, we finally turn to the issue that was addressed in Section 4.5.1 in the context of power laws, namely, the large-$h$ asymptotic of the law of the wall in the case when the cross-correlation function is characterised by an inner length $\Lambda \sim h/\text{Re}^a$. This turns out to be quite revealing as to the role of the purely logarithmic law ($p = 1$). The point is that the logarithmic case is only a member of the continuous family of power-log solutions, and there is no reason a
priori for \( p \) to be equal unity exactly. On the contrary, its value is to be inferred from Equations (76), and (67), (68).

Equation (67) becomes an identity on account of Equation (76), while Equation (68) relates \( p \) to \( y_0 \). Evaluating the \( q \)-integrals as in Section 4.5.1 yields the relation

\[
2 \ln \frac{h}{y_0} - (p + 1) \ln \text{Re}^a + \frac{\pi}{2} \left( p^2 - 1 \right) \left[ \frac{\int \int dk dq \, k^2 \varrho_+}{\int \int dk dq \, k q_+} + \frac{\pi}{2} \left( p - 1 \right) \left( p - 3 \right) \frac{\int \int dk dq \, k q_-}{\int \int dk dq \, k^2 \varrho_+} \right] = 0.
\]

(77)

It is understood that the integrations in this formula are over all \( q \), in accordance with the fact that any given \( y \) becomes \( \ll \Lambda \) in the limit \( h \to \infty \). Using the estimates \( \int \int dk dq \, k q_- = \left( \langle \partial \tilde{u} / \partial x \rangle \right)^2 \sim v_*^2 / \Lambda^2 \), \( \int \int dk dq \, k q_+ \sim v_*^2 / h^2 \) shows that the last term in the above equation is proportional to \( \text{Re}^{2a} \), and so it cannot be compensated by the rest of the equation involving only logarithms of \( \text{Re} \), unless

\[
p = 1 + f(\ln \text{Re}) / \text{Re}^a.
\]

(78)

where the function \( f(\ln \text{Re}) \) is approximately linear for large values of its argument. Neglecting the terms proportional to negative powers of \( \text{Re} \), Equation (77) thus reduces to

\[
f = \frac{2}{\pi} \ln \frac{h}{y_0 \text{Re}^a}.
\]

Finally, noting that \( y_0 \sim v / v_* \), and that the friction Reynolds number \( \text{Re}_s = v_* h / v \) is related to \( \text{Re} \) roughly as \( \text{Re}_s = \pi \text{Re} / \ln \text{Re} \), one finds with logarithmic accuracy

\[
f = \frac{2}{\pi} \ln \left( \frac{\text{Re}^{1-a}}{\ln \text{Re}} \right).
\]

(79)

Since \( a < 1 \), \( f \) is positive for large \( \text{Re} \), and we conclude that \( p \) tends to unity from above.

It is, perhaps, worth reemphasising that the result (78) is more of formal significance as it pertains to the region \( |y| \ll \Lambda \), while it is \( |y| \gtrsim \Lambda \) which are actually important in the analysis of the available experimental data. In this intermediate region, \( p \) is not bound to tend to unity, at least as rapidly as required by Equation (78). In fact, comparison with the measured velocity profiles given below yields \( p \) ranging from 1 to about 2. One should also remember that this asymptotic expression for \( p \) is valid only under conditions specified at the end of Section 4.5.2.

5. Discussion and conclusions

The power-log profiles (66) naturally emerged along with the logarithmic and power-law profiles as possible asymptotics of the near-wall mean velocity distribution. Found as solutions of the dispersion relation considered as an equation for the mean flow velocity, the power-log laws span a continuous family of which the logarithmic profile is just a member. Evidently, complexity of real flows can only enlarge the parameter space of this family, so that this state of things is to be expected to take place in general. At present, there are no theoretical grounds to prefer one law to the others, so that one can only rely on the
experiment to identify the law relevant under the given flow conditions. It is plain that the power-log family provides a more accurate approximation than the logarithmic law, merely because it brings in a new continuous parameter – the exponent $p$; the question is whether the improvement is significant. Comparison with the experimental data shows that it really is; as was alluded in Section 1, the power-log law turns out to be of particular advantage over the pure log at moderately large Reynolds numbers. Namely, solutions with $p > 1$ compete well with the power law in approximating the well-known tendency of the mean velocity semi-log graphs to curve upwards for large $y$, providing at the same time an adequate fit over the conventional ‘logarithmic’ region, $30 \lesssim 60 \nu/v_* \lesssim y \lesssim 0.15 \pm 0.3 h$. Figures 3–5 illustrate this for $Re_*$ ranging from 1000 to 11, 600. Figure 3 compares power- and power-log approximations of the mean flow velocity distribution obtained by DNS of an incompressible channel flow with a comparatively low $Re_* \approx 1000$ [26]. The approximations are best fits as given by the standard Maple Fit procedure based on the least square method [27]. It is seen that the power law gives, in this case, an almost perfect fit to the experimental results. This can be quantified by the mean squared error which is 0.0004, while for the power-log fit it is 0.01. A similar comparison of the power-log law with the other two profiles is made in Figure 4 for $Re_* \approx 2000$ and 6000 using the experimental data obtained at the High Reynolds Number Turbulent Channel Flow facility [2] (the power-log fit is shown throughout by a solid line). Here, the power-log approximation yields an order-of-magnitude smaller mean squared error than the pure log or power fits. Finally, recent PIV data [28] for a zero-pressure gradient flow with $Re_* \approx 11, 600$, obtained in the High Reynolds Number Boundary Layer Wind Tunnel, is fitted in Figure 5 with the help of the power and power-log profiles. It shows that in flows of this type, the power law can be

![Figure 3](image-url)

**Figure 3.** The mean velocity distribution in a channel flow with $Re_* \approx 1000$. Shown are the results of DNS [26] (marks), a power-log best fit (solid line), and a power-law best fit (dashed line) over the region $y \in (60, 800)$. $U$ is measured in units of $v_*$, and $y$ in units of $\nu/v_*$. 

$$U = 2.03(\ln y + 1.95)^{1.11}$$

$$U = 8.3 y^{0.15}$$
superior to the log- and power-log laws even at relatively high Reynolds numbers (cf. [17]).

The results of comparison in the four instances are summarised in Table 1.

Next, comments on the role of Assumption (A) introduced in Section 4.5.2 are in order. This assumption turned out to be critical in the comparison of the power- and power-log families regarding the influence of the bulk flow on the near-wall velocity distribution. As was demonstrated in Section 4.5.2, only the power-log profiles (including the case $p = 1$) are consistent with the assumption that this influence is negligible, in the sense that the bulk
Table 1. Best fits of the experimental data for channel (Ch) and boundary layer (BL) flows shown in Figures 3–5. Number in square brackets below each fit is the respective mean squared error. The experimental uncertainty in U is given explicitly in the cited works only for \( \text{Re}^* = 2000 \) and 6000, in which cases it is 1.5%.

| Flow type | \( \text{Re}^* \) | \( U_{\log} \) | \( U_{\text{power-log}} \) | \( U_{\text{power}} \) |
|-----------|------------------|-----------------|------------------|------------------|
| Ch (DNS)  | 1000             | 2.8ln \( y + 3.42 \) [0.011] | 2.03 (ln \( y + 1.95 \))^1.11 [0.001] | 8.29 \( y^{0.15} \) [0.0004] |
| Ch        | 2000             | 2.6ln \( y + 4.53 \) [0.011] | 0.08 (ln \( y + 8.9 \))^2.03 [0.001] | 8.56 \( y^{0.14} \) [0.010] |
| Ch        | 6000             | 2.6ln \( y + 4.67 \) [0.011] | 0.44 (ln \( y + 6.04 \))^1.54 [0.002] | 9.47 \( y^{0.12} \) [0.032] |
| BL        | 11,600           | 2.3ln \( y + 0.46 \) [0.069] | 0.27 (ln \( y + 4.46 \))^1.69 [0.023] | 5.48 \( y^{0.16} \) [0.008] |

velocity fluctuations do not affect the leading term of the near-wall mean velocity asymptotic. Conversely, the bulk effect on the flow near the wall must be sufficiently strong to sustain the mean velocity distribution obeying a power-law. The required strength of correlations is quantified in Section 4.4 in the form of Equations (71) and (72), which represent the integral equations for reconstructing the cross-correlation spectrum that gives rise to the desired velocity profile. This result shows that from the purely theoretical standpoint, the question ‘which velocity profile is the right one’ is a bit of red herring, because given a mean velocity distribution, one can always find a cross-correlation function so as to satisfy the dispersion relation. There is little doubt that this conclusion also extends to the general three-dimensional case, but the question of whether the flow obeying given law is practically realisable can presently be answered only by the experiment.

The found form of the power-law velocity profile deserves further discussion. The large-\( h \) asymptotic of the exponent \( n \) in Equation (64) turned out to be a universal function of the Reynolds number. The inverse proportionality to ln Re was inferred in [17] as a consequence of the so-called principle of vanishing viscosity, stating that the mean shear must remain finite as \( \nu \to 0. \) This principle was suggested to replace the much stronger requirement that the mean shear be independent of \( \nu, \) which leaves the logarithmic law as the only possibility for near-wall velocity asymptotic. We thus see that the results of Section 4.5.1 lend purely theoretical support to the principle of vanishing viscosity.

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No potential conflict of interest was reported by the author.

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Appendix

To derive formula (54), we first write $\hat{H}\hat{X}\hat{H}$ longhand using definition (22)

$$
(\hat{H}\hat{X}\hat{H}a)(y) = \frac{1}{(2h)^2}\int_{-h}^{h} dy_1 \chi(y_1) \cot\left(\frac{\pi}{2h}(y_1 - y)\right) \int_{-h}^{h} dy_2 a(y_2) \cot\left(\frac{\pi}{2h}(y_2 - y_1)\right).
$$

The integral over $y_1$ cannot be taken directly by changing the order of integration, because of the point $y_1 = y_2 = y$ where the poles of the cotangents merge. To avoid this singularity, it is sufficient to rewrite the right-hand side, for $y > 0$, as

$$
\frac{1}{(2h)^2}\int_{-h}^{h} dy_1 \cot\left(\frac{\pi}{2h}(y_1 - y)\right) \int_{-h}^{h} dy_2 a(y_2) \cot\left(\frac{\pi}{2h}(y_2 - y_1)\right)
$$

$$
- \frac{2}{(2h)^2}\int_{-h}^{0} dy_1 \cot\left(\frac{\pi}{2h}(y_1 - y)\right) \int_{-h}^{h} dy_2 a(y_2) \cot\left(\frac{\pi}{2h}(y_2 - y_1)\right).
$$

The first term here is nothing but $(\hat{H}\hat{H}a)(y) = -a(y)$. Changing now the order of integration in the second, integrating over $y_1$ with the help of the identity

$$
\cot x \cot y = -1 + \cot(x - y)(\cot y - \cot x),
$$

and recalling the condition $\int_{-h}^{h} dy a(y) = 0$, one finds

$$
(\hat{H}\hat{X}\hat{H}a)(y) = -a(y) + \frac{1}{\pi h} \int_{-h}^{h} d\tilde{y} a(\tilde{y}) \cot\left(\frac{\pi}{2h}(y - \tilde{y})\right) \ln\left|\tan(\pi\tilde{y}/2h)\right|/\tan(\pi y/2h).
$$

Combined with the result of a similar calculation for $y < 0$, this is Equation (54).