DYNAMICS OF A SPHERICAL ACCRETION SHOCK WITH NEUTRINO HEATING AND ALPHA-PARTICLE RECOMBINATION

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ABSTRACT

We investigate the effects of neutrino heating and \( \alpha \)-particle recombination on the hydrodynamics of core-collapse supernovae. Our focus is on the nonlinear dynamics of the shock wave that forms in the collapse and the assembly of positive energy material below it. To this end, we perform time-dependent hydrodynamic simulations with FLASH2.5 in spherical and axial symmetry. These generalize our previous calculations by allowing for bulk neutrino heating and for nuclear statistical equilibrium between \( n, p, \) and \( \alpha \). The heating rate is freely tunable, as is the starting radius of the shock relative to the recombination radius of \( \alpha \)-particles. An explosion in spherical symmetry involves the excitation of an overstable mode, which may be viewed as the \( \ell = 0 \) version of the ”Standing Accretion Shock Instability.” In two-dimensional simulations, nonspherical deformations of the shock are driven by plumes of material with positive Bernoulli parameter, which are concentrated well outside the zone of strong neutrino heating. The nonspherical modes of the shock reach a large amplitude only when the heating rate is also high enough to excite convection below the shock. The critical heating rate that causes an explosion depends sensitively on the initial position of the shock relative to the recombination radius. Weaker heating is required to drive an explosion in two dimensions than in one, but the difference also depends on the size of the shock. Forcing the infalling heavy nuclei to break up into \( n \) and \( p \) below the shock only causes a slight increase in the critical heating rate, except when the shock starts out at a large radius. This shows that heating by neutrinos (or some other mechanism) must play a significant role in pushing the shock far enough out that recombination heating takes over.

Key words: hydrodynamics – instabilities – nuclear reactions, nucleosynthesis, abundances – shock waves – supernovae: general

Online-only material: color figures, animations

1. INTRODUCTION

Although tremendous progress has been made on the mechanism of core-collapse supernovae in recent years, we still do not have a clear picture of a robust path to an explosion in stars that form iron cores. Heating by the absorption of electron-type neutrinos significantly modifies the settling flow below the bounce shock but—in spite of early positive results (Bethe & Wilson 1985)—explosions are obtained in spherical collapse calculations only if the progenitor star is lighter than about \( 10–12 \, M_\odot \) (Kitaura et al. 2006). More massive stars fail to explode in spherical symmetry (Liebendörfer et al. 2001).

Two-dimensional collapse calculations show strong deformations of the shock and convective motions below it (Burrows et al. 1995, 2006, 2007; Janka & Mueller 1996; Buras et al. 2006a, 2006b; Marek & Janka 2009). It has long been recognized that convection increases the residency time of settling material in the zone of strong neutrino heating (Herant et al. 1992). It is also becoming clear that multidimensional explosions require the assembly of a smaller amount of material with positive energy, but the details of how this happens remain murky.

An early treatment of shock breakout by Bethe (1997) focused on the strong gradient in the ram pressure of the infalling material, but implicitly assumed that the shocked material had already gained positive energy. If large-scale density inhomogeneities are present below the shock, they will trigger a finite-amplitude, dipolar instability, thereby allowing accretion to continue simultaneously with the expansion of positive-energy fluid (Thompson 2000). The accretion shock is also capable of a dipolar oscillation which leads, above a critical amplitude, to a bifurcation between freshly infalling material and material shocked at earlier times (Blondin et al. 2003). Such an oscillation is easily excited in a spherical flow composed of a zero-energy, polytropic fluid (Blondin & Mezzacappa 2006) via a linear feedback between ingoing vortex and entropy waves and outgoing sound waves (Foglizzo et al. 2007). It can also be excited indirectly by neutrino heating, which if sufficiently strong will drive large-scale buoyant motions below the shock (Herant et al. 1994; Foglizzo et al. 2006). For relatively weak heating, the dipolar oscillation can decrease the damping effect of advection and trigger convective motions that would otherwise be suppressed (Foglizzo et al. 2006; Scheck et al. 2008).

A significant sink of thermal energy in the accretion flow arises from the dissociation of heavy nuclei. The heavy elements that flow through the shock are broken up into \( \alpha \)-particles and nucleons when exposed to the high temperature (\( >1 \, \text{MeV} \)) of the post-shock region. The Bernoulli parameter \( b \) of the shocked fluid then becomes substantially negative. A significant fraction of this dissociation energy can be recovered if nucleons recombine into \( \alpha \)-particles (Bethe 1996). But for this to occur, a decrease in the temperature is required and thus the shock must expand significantly beyond the radius at which it typically stalls (\( \sim 100–150 \, \text{km} \)).

One of the primary goals here, and in a previous paper (Fernández & Thompson 2009; hereafter Paper I), is to gauge the relative importance of these effects in setting the stage for a successful explosion. The persistence and amplitude of a dipolar oscillation fully can only be reliably measured in fully
three-dimensional simulations (there are preliminary indications that it is less prominent in three dimensions than in axial symmetry; Iwakami et al. 2008). On the other hand, the interplay between $\alpha$-particle recombination and hydrodynamical instabilities has received little attention. Although recombination is certainly present in previous numerical studies which employ finite-temperature equations of state (EOSs), it should be kept in mind that considerable uncertainties in the EOS remain at finite-temperature equations of state (EOSs), it should be kept in mind that considerable uncertainties in the EOS remain at

Supranuclear densities. A softening or hardening of the EOS

In this paper, we study the interplay between nonspherical shock oscillations, neutrino heating, and $\alpha$-particle recombination, when the heating rate is pushed high enough for an explosion to occur. Our focus is on the stalled shock phase, between $\sim 100$ ms and 1 s after bounce. In a one-dimensional calculation, the accretion flow reaches a quasi-steady state during this interval, and the shock gradually recedes (e.g., Liebendörfer et al. 2001; Buras et al. 2006a).

Our approach is to introduce EOSs of increasing complexity into one- and two-dimensional, time-dependent hydrodynamic simulations. To this end, we use the code FLASH2.5 (Fryxell et al. 2000), which is well tested in problems involving nuclear energy release in compressible fluids (Calder et al. 2002). We adopt a steady-state model as our initial condition, and a constant mass accretion rate, neutrino luminosity, and fixed inner boundary. The steady-state approximation to the stalled shock phase was first introduced by Burrows & Goshy (1993), and has recently been used by Ohnishi et al. (2006) to study the nonlinear development of the shocked flow with a semi-realistic equation of state and neutrino heating. In Paper I, we modeled the accretion flow as a polytropic fluid, from which a fixed dissociation energy is removed immediately below the shock, and allowed for neutrino cooling but not heating.

Here we generalize this model to allow both for heating, and for nuclear statistical equilibrium (NSE) between neutrons, protons, and $\alpha$-particles. Heating is introduced in a simple, parameterized way, without any attempt at simulating neutrino transport. The nuclear abundances are calculated as a function of pressure $p$ and density $\rho$, using a complete finite-temperature, partially degenerate EOS. Our model for the shocked material retains one significant simplification from Paper I: we do not allow the electron fraction $Y_e$ to vary with position below the accretion shock. Here there are two competing effects: electron captures tend to reduce $Y_e$, whereas absorption of $\nu_e$ and $\bar{\nu}_e$ tends to drive high-entropy material below the shock toward $Y_e \simeq 0.5$. Since we are interested especially in the dynamics of this high-entropy material, we set $Y_e = 0.5$ when evaluating the $\alpha$-particle abundance. To obtain a realistic density profile, we continue to approximate the internal energy of the fluid as that of a polytropic fluid with a fixed index $\gamma = \frac{4}{3}$. In reality, the equation of state between the neutrinosphere and the shock depends in a complicated way on the degeneracy of the electrons and the effects of electron captures. The consequences of introducing these additional degrees of freedom will be examined in future work.

In spite of these simplifications, our results already show many similarities with more elaborate collapse calculations. Spherical explosions are due to a global instability resembling the one-dimensional standing accretion shock instability (SASI), but modified by heating. As in Paper I, we find that the period of the $\ell = 0$ mode remains close to twice the post-shock advection time. Strong deformations of the shock in two-dimensional simulations are driven by material with positive Bernoulli parameter, which generally resides outside the radius $r_w$ where the gravitational binding energy of an $\alpha$-particle is equal to its nuclear binding energy. The recombination of $\alpha$-particles can play a major role in creating this positive-energy material, but for this to happen the shock must be pushed beyond $\sim 200$ km from the neutrinosphere.

In this paper, we consider only neutrino heating as the impetus for the initial expansion of the shock, rather than more exotic effects such as rotation or magnetic fields. We find that the critical heating rate is a strong function of the initial position of the shock with respect to $r_w$, which implies that a much higher neutrino luminosity is needed to revive a shock that stalls well inside $r_w$. The difference in the critical heating rate between one-dimensional and two-dimensional simulations also depends on the size of the shock, and hence on the structure of the forming neutron star.

We find that the shock develops a dipolar oscillation with a large amplitude only when the heating rate is also high enough to trigger a strong convective instability. We therefore surmise that buoyant motions driven by neutrino heating play a major role in driving the dipolar oscillations that are seen in more complete simulations of core collapse. Some evidence is found that acoustic wave emission by the convective motions also plays a role. We investigate the possibility of a heat engine within the gain layer (the region with a net excess of neutrino heating over cooling). We find that most of the heat deposition by neutrinos is concentrated in lateral flows at the base of the prominent convective cells. At the threshold for an explosion, neutrino heating plays a key role in pushing material to positive Bernoulli parameter, but only if the shock starts well inside $r_w$.

The plan of the paper is as follows. Section 2 presents our numerical setup, treatment of heating, cooling, and nuclear dissociation, and outlines the sequences of models. Sections 3 and 4 show results from one-dimensional and two-dimensional simulations, respectively. We focus on the relative effectiveness of neutrino heating and $\alpha$-particle recombination in driving an explosion, and the relation between Bernoulli parameter and large-scale deformations of the shock. The critical heating rate for an explosion is analyzed in Section 5, and the competition between advective-acoustic feedback and convective instability is discussed in Section 6. We summarize our findings in Section 7. The appendices contain details about our EOS and the numerical setup.

2. NUMERICAL MODEL

As in Paper I, the initial configuration is a steady, spherically symmetric flow onto a gravitating point mass $M$. The flow contains a standing shock wave, and the settling flow below the shock cools radiatively in a narrow layer outside the inner boundary of the simulation volume. The space of such models is labeled basically by three parameters: accretion rate $\dot{M}$, luminosity $L$, in electron neutrinos and anti-neutrinos, and the radius $r_w$ of the base of the settling flow, which corresponds roughly to the neutrinosphere radius. The mass $M$ of the
collapsed material represents a fourth parameter, but it covers a narrower range than the other three. The infalling material is significantly de-leptonized before hitting the shock only in the first 50 ms or so of the collapse (e.g., Liebendorfer et al. 2001).

We explore a two-dimensional surface through this three-dimensional parameter space by (1) fixing the ratio of $r_s$ to the initial shock radius $r_{s0}$ in the absence of heating ($r_s/r_{s0} = 0.4$); (2) allowing $r_{s0}$ to vary with respect to an appropriately chosen physical radius; and then (3) increasing the level of heating until an explosion is uncovered. In the full problem, the shock radius at zero heating is a unique function of $M$ and $r_{s0}$ with a small additional dependence on $M$ and the composition of the flow outside the shock (Houch & Chevalier 1992). The secular cooling of the collapsed core forces a gradual decrease in $r_s$, and $M$ also varies with time and with the progenitor model.

Given the important role that $\alpha$-particle recombination plays in the final stages of an explosion, we implement (2) by referencing $r_{s0}$ to the radius where the gravitational binding energy of an $\alpha$-particle equals its nuclear binding energy,

$$r_a = \frac{G M m_a}{Q_\alpha} \approx 254 M_{1.3} \text{ km.} \quad (1)$$

Here $Q_\alpha \approx 28.30$ MeV is the energy needed to break up an $\alpha$-particle into $2n$ and $2p$, $m_a$ the mass of an $\alpha$-particle, and $M_{1.3} = M/(1.3 M_\odot)$. Choice (1) allows us to consider models that have, implicitly, both a range of physical values of $r_s$ and a range of $M$. It is, of course, made partly for computational simplicity (the limited size of the computational domain) and also to facilitate a comparison between models that have different values of $r_{s0}/r_a$. Nuclear dissociation is taken into account either by removing a fixed specific energy $\varepsilon$ right below the shock, or by enforcingNSE between $n$, $p$, and $\alpha$ throughout the settling flow. Once this choice is made, the normalization of the cooling function is adjusted to give $r_s/r_{s0}$ = 0.4. The heating rate remains freely adjustable thereafter.

We adopt this simplification because we do not intend to find the precise value of the critical neutrino luminosity, but instead to probe the behavior of the system around this critical point, whatever its absolute value.

We now describe the key components of this model in more detail and explain the setup of the hydrodynamic calculations. As in Paper I, the time evolution is carried out using the second-order, Godunov-type, adaptive-mesh-refinement code FLASH2.5 (Fryxell et al. 2000).

2.1. Initial Conditions

The introduction of heating causes a change in the structure of the initial flow configuration. The radius $r_s$ of the shock in the time-independent solution to the flow equations increases with heating rate; that is, $r_s \geq r_{s0}$. The material above the shock is weakly bound to the protoneutron star, and in practice can be taken to have a zero Bernoulli parameter

$$b = \frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{G M}{r}. \quad (2)$$

Here $v$ is the total fluid velocity, which is radial in the initial condition, $p$ is the pressure, $\rho$ is the mass density, and $G$ is Newton’s constant. The flow upstream of the shock is adiabatic and has a Mach number $M_{1} = 5$ at a radius $r = r_{s0}$.

The composition of the fluid is very different upstream and downstream of the shock. Changes in internal energy due to nuclear dissociation and recombination are taken into account using the two models described in Section 2.1.1. For the internal energy density of the fluid $e$, we continue to use the polytropic relation $e = p/(\gamma - 1)$; hence, the second term on the right-hand side of Equation (2). Because we are not explicitly including changes in electron fraction due to weak processes, we keep $\gamma = 4/3$ for both models of nuclear dissociation. This largely determines the density profile inside a radius $\sim 1/2 r_a$, where the $\alpha$-particle abundance is very low.

The upstream and downstream flow profiles are connected through the Rankine–Hugoniot jump conditions, which are modified so as to allow a decrement $\varepsilon$ in $b$ across the shock. The resulting compression factor is (Paper I)

$$\kappa \equiv \frac{\rho_2}{\rho_1} = (\gamma + 1) \left[ \left( 1 + M_{1}^{-2} \right)^{\gamma - 1} - \frac{\gamma}{2} \frac{2 e}{v_{x}^2} \right]^{-1}, \quad (3)$$

which reduces to $\kappa \rightarrow (\gamma + 1)/(\gamma - 1)$ for $M_{1} \rightarrow \infty$ and $\varepsilon = 0$. Throughout this paper, the specific nuclear dissociation energy $\varepsilon$ is defined to be positive. The subscripts 1 and 2 denote upstream and downstream variables, respectively.

All flow variables are made dimensionless by scaling radii to $r_{s0}$, velocities to $v_{x0} = (2GM/(r_{s0})^{1/2}$, timescales to $t_{x0} = r_{s0}/v_{x0}$, and densities to the upstream density at $r = r_{s0}, \rho_1(r_{s0})$ (Equation (B13)). See Paper I for further details. Throughout the paper, we denote the average of a function $F(X, \ldots)$ over some variable $X$ by $\langle F \rangle_X$.

2.1.1. Nuclear Dissociation

We model nuclear dissociation in two ways. First, we remove a fixed specific energy $\varepsilon$ right below the shock, as done in Paper I. This represents the prompt and complete breakup of whatever heavy nuclei are present in the upstream flow. The main limitation of this approximation is that the dissociation energy does not change with the radius (or inclination) of the shock. The main advantage is simplicity: $\varepsilon$ is independent of any dimensional parameters and can be expressed as a fraction of $v_{x0}^2$.

We also use a more accurate dissociation model which allows forNSE between $\alpha$-particles and nucleons.3 During the stalled shock phase of core-collapse supernovae, the shock sits at $r \sim 100–200$ km, with a post-shock temperature $T > 1$ MeV and density $\rho > 10^9$ g cm$^{-3}$. In these conditions, the heavy nuclei flowing through the shock are broken up into $\alpha$, $p$, and $n$.

A range of isotopes are present in the iron core of a massive star as well as in nuclear burning shells (Woosley et al. 2002), but since the binding energy per nucleon varies only by $\sim 10\%$ we simply assume a single type of nucleus in the upstream flow. We focus here on the later stages of the stalled shock phase, during which the oxygen shell is accreted. An energy $Q_0 \approx 14.44$ MeV must be injected to dissociate an $^{16}$O nucleus into four $\alpha$-particles (Audi et al. 2003), which corresponds to the specific dissociation energy

$$\varepsilon_0 = \frac{Q_0}{m_0} \approx 0.038 M_{1.3} \left( \frac{r}{150 \text{ km}} \right) v_{x0}^2. \quad (4)$$

3 Although heavier nuclei can begin to recombine once the shock moves significantly beyond $r_{s0}$, this generally occurs only after the threshold for an explosion has been reached, and makes a modest additional contribution to the recombination energy.
Here $m_O \simeq 16m_n$ is the mass of an oxygen nucleus, with $m_n$ being the atomic mass unit. The smallness of this number indicates that light oxygen survives in the post-shock flow, and so we set the equilibrium mass fraction of oxygen to zero below the shock, $X^\text{eq}_O = 0$. The binding energy of an $\alpha$-particle is of course much larger, giving
\[
e_\alpha = \frac{Q_\alpha}{m_\alpha} \simeq 0.297M_{1.5}\left(\frac{r}{150\text{ km}}\right) v_{\text{f1}}^2(r). \tag{5}
\]

We find that $\alpha$-particles appear in significant numbers only at relatively large radii ($\geq 0.5r_\alpha$) and in material that has either (1) been significantly heated by electron neutrinos closer to the neutrinosphere or (2) been freshly shocked outside $r_\alpha$. The electrons are only mildly degenerate in material that has a high entropy and $\alpha$-particle content, so that neutrino heating drives $Y_e$ close to $0.5$ (or even slightly above; see; e.g., Buras et al. 2006b). We therefore set $Y_e = 0.5$ in the Saha equation that determines the equilibrium mass fractions $X_n^\text{eq}$, $X_p^\text{eq}$, and $X^\text{eq}_\alpha = 1 - X_n^\text{eq} - X_p^\text{eq}$. These quantities are tabulated as functions of $p$ and $\rho$ using an ideal, finite-temperature and partially degenerate equation of state for electrons and nucleons; see Appendix A for details. Specific choices must then be made for the parameters $r_{\alpha0}$, $M$, and $\dot{M}$; we generally take $M = 1.3M_\odot$ and $\dot{M} = 0.3M_\odot\text{s}^{-1}$, but allow $r_{\alpha0}$ to vary. An investigation of how changes in $Y_e$ feed back onto the formation of $\alpha$-particles is left for future work.

A specific energy
\[
e_{\text{nuc}} = -X_O(e_O + e_\alpha) - \left(X_n - X_n^\text{eq}[\rho, p]\right)e_\alpha \tag{6}
\]
is either released or absorbed within a single time step (it can be of either sign). Here $X_O$ is nonvanishing only for fluid elements that have just passed across the shock, and we have set $X^\text{eq}_O = 0$. The quantity (6) is introduced as an energy source term in FLASH, and from it one readily obtains a rate of release of nuclear binding energy per unit mass,
\[
de_{\text{nuc}} \equiv \frac{de_{\text{nuc}}}{dt} = \frac{e_{\text{nuc}}}{\Delta t} \tag{7}
\]
where $\Delta t$ is the simulation time step.

In the initial condition, the dissociation energy at the shock is obtained from Equation (6) using $X_O = 1$ and $X_\alpha = 0$ upstream of the shock:
\[
e(t = 0) = e_O + \left(1 - X_n^\text{eq}[p_2, p_2]\right)e_\alpha. \tag{8}
\]

Figure 1 shows how $e(t = 0)$ and $X_\alpha^\text{eq}$ depend on the shock radius $r_\alpha$, for upstream flows composed$^5$ of pure $^{16}\text{O}$ and $^{56}\text{Fe}$, and for different values of $M$. The dissociation energy is approximately constant inside $\sim 75\text{ km}$, where the downstream flow is composed of free nucleons, but decreases at greater distances, remaining $\sim 40\%$ of the gravitational binding energy at the shock. The mass fraction of $\alpha$-particles reaches $50\%$ at $r \approx 150-175\text{ km}$, with a weak dependence on $M$.

2.1.2. Heating and Cooling in the Post-shock Flow

To allow direct comparison with our previous results, we employ a cooling rate per unit volume of the form
\[
\mathcal{L}_C = C p^a \rho^{-b-a}, \tag{9}
\]
with $a = 1.5$, $b = 2.5$, and $C$ a normalization constant. As in Paper I, we include a Gaussian entropy cutoff to prevent runaway cooling. The exponents in Equation (9) represent cooling dominated by the capture of relativistic, nondegenerate electrons and positrons on free nucleons (e.g., Bethe 1990). Inside the radius where the electrons become strongly degenerate, and $\alpha$-particles are largely absent, one has $\mathcal{L}_C \propto p^3 \rho \propto (Y_e\rho)^3$. This gives essentially the same dependence of $\mathcal{L}_C$ on $r$ as Equation (9) when $Y_e = \text{constant}$ and $Y = \frac{1}{3}$ (corresponding to $\rho \propto r^{-3}$ in a nearly adiabatic settling flow). In more realistic collapse calculations, $Y_e$ grows with radius between the neutrinosphere and the shock, but $\rho$ tends to decrease more rapidly than $\sim r^{-3}$ (e.g., Buras et al. 2006a). Our chosen form for the cooling function results in a slightly wider gain region and, therefore, a slightly lower critical heating rate for an explosion. The bulk of the cooling occurs in a narrow layer close to the accretor at $r = r_\alpha$, and the accreted material accumulates in the first few computational cells adjacent to the inner boundary without a major effect on the rest of the flow.

We model neutrino heating as a local energy generation rate per unit volume of the form
\[
\mathcal{L}_H = H(1 - X_\alpha) p/\rho^2. \tag{10}
\]

The normalization constant $H$ measures the strength of the heating. The factor $(1 - X_\alpha)$ accounts for the fact that the cross section for neutrino absorption by $\alpha$-particles is much smaller than that for free nucleons (Bethe 1990). For simplicity, we do not include the flux factor due to the transition between diffusion and free-streaming. Our focus here is on the nature of the instabilities occurring in the flow near the threshold for an explosion, and we do not attempt a numerical evaluation of the critical heating rate.

An additional energy source term arises from the change in the equilibrium fraction of $\alpha$-particles as they are advected in the steady-state initial solution. The instantaneous adjustment of $X_\alpha$ to its equilibrium value, combined with Equation (6), yields

\[
\frac{\partial X_\alpha}{\partial t} = \frac{\partial X_{\alpha,0}}{\partial t} + \frac{1}{\Delta t} \left( X_{\alpha,0}^\text{eq} - X_\alpha \right). \tag{11}
\]

The change in $X_\alpha$ measured at the instant of release follows the condition on $\alpha$-particle production at the neutrinosphere or the simple approximations made in the description of the cooling process.
an energy generation rate per unit volume

$$\mathcal{L}_\alpha = \rho v e_\alpha \frac{dX_{eq}^\alpha}{dr} = \rho v e_\alpha \left[ \frac{\partial X_{eq}^\alpha}{\partial \rho} \frac{d \rho}{dr} + \frac{\partial X_{eq}^\alpha}{\partial \rho^2} \frac{d \rho^2}{dr} \right].$$  (11)

This energy generation rate is negative, as the temperature increases inward and thus the $\alpha$-particle fraction decreases with decreasing radius ($\rho$ is negative).

### 2.1.3. Numerical Setup

In our time-dependent calculations, we use one-dimensional and two-dimensional axisymmetric spherical coordinates with baseline resolution $\Delta r_{\text{base}} = r_0/320$ and $\Delta \theta_{\text{base}} = \pi/192$, with one extra level of mesh refinement inside $r = r_s + 0.1(r_0 - r_s)$ to better resolve the steep density gradient that arises in the cooling layer. We do not employ a hybrid Riemann solver because we do not see the appearance of the odd–even decoupling instability (Quirk 1994).

We employ a reflecting inner boundary condition at $r = r_s$ for the sake of simplicity; we do not attempt to model the protoneutron star (as done by Murphy & Burrows 2008) or for the sake of simplicity; we do not attempt to model the outer boundary condition is set fixed at $r = 7r_0$, and is set by the upstream flow at that position.

To trigger convection below the shock, we introduce random cell-to-cell velocity perturbations in $v_r$ and $v_\theta$ at $t = 0$, with an amplitude $1\%$ of the steady-state radial velocity. To study the interplay between shock oscillations and convection, we also drop overdense shells with a given Legendre index $\ell$, as done in Paper I, without random velocity perturbations.

In order to track the residency time of the fluid in the gain region, we assign a scalar to each spherically symmetric mass shell in the upstream flow. This scalar is passively advected by FLASH2.5. Through this technique, we are able to assign a “fluid” time to each element in the domain, corresponding to the time at which the mass shell would cross the instantaneous angle-averaged shock position if advected from the outer boundary at the upstream velocity:

$$t_F = t_{OB} + \int_{r_\theta(t)_{OB}}^{r_\theta(t)_{OB}} \frac{dr}{|v_r|}. \quad (12)$$

Here $t_{OB}$ is the time at which the fluid enters through the outer radial boundary at $r = r_{OB}$, and $\langle r_\theta(t)_{OB} \rangle$ is the angle-averaged shock position. Initially, $t_{OB} = 0$ and all the fluid below the shock is set to $t_F = 0$. This prescription works well for statistical studies (Section 4.2), tracing large-scale fluid patches, despite some inevitable turbulent mixing on small scales.

An explosion is defined as either (1) a collision between the shock and the outer boundary of the simulation volume ($r = 7r_0$) within $1000t_{OB}$ of the start of the simulation; or (2) in the special case of the one-dimensional constant-\$e$ models, a transient expansion that breaks a quasi-steady pattern within the same time frame. Even in the one-dimensional simulations, very small changes in heating rate can lead to dramatic changes in shock behavior, and so this definition of explosion is good enough for our purposes.

### 2.2. Model Sequences

We choose six sequences of models, each with a range of heating parameters $H \geq 0$, and each evolved both in spherical (one-dimensional) and axial (two-dimensional) symmetry. Their parameters are summarized in Table 1. In each sequence, the normalization of the cooling function is chosen so that $r_s/r_0 = 0.4$ at zero heating. Three sequences have a constant dissociation energy, which take the values $\varepsilon/v_{r0}^2 = \{0.1, 0.15, 0.2\}$. The other three sequences assume NSE below the shock, and have shock radii $r_0 = \{50, 75, 125\}$ km at zero heating. This means that the physical value of the cooling radius also takes on different values, namely $\{20, 30, 50\}$ km. In effect, our models are probing different sizes for the neutrinosphere, and different times following the collapse. The other parameters in the NSE models are $M = 1.3 M_\odot$ and $M = 0.3 M_\odot$ s$^{-1}$.

Table 1 samples some properties of a few models from each sequence: one with zero heating, another with $H$ close to the critical value for an explosion, and a third with the largest heating parameter that will allow a steady solution. Note that the shock starts out at $\sim 1.3r_0$ in the time-independent, spherical flow solution, and quickly saturates at $\sim (1.8-2)r_0$ in the two-dimensional models with heating just below threshold for an explosion. The quantity $\varepsilon/v_r^2$ references the dissociation energy to (twice) the kinetic energy of the upstream flow, and is the key free parameter determining the compression rate $\kappa$ across the shock (Equation (3)).

When examining how the prescription for nuclear dissociation influences the results, we will focus on the $\varepsilon = 0.15v_{r0}^2$ sequence and the NSE sequence with $r_0 = 75$ km, which have similar initial density profiles (due to the low initial $\alpha$-particle abundance in the NSE model).

The six initial models at zero heating are shown in Figure 2(a). Panel (b) shows the sequence of initial models with $r_0 = 75$ km and a range of heating parameters. The model with $H = 0.007v_{r0}^3$ is close to the threshold for an explosion, while the one with $H = 0.009v_{r0}^3$ is well above threshold. At higher values of $H$, cooling by $\alpha$-particle dissociation (Equation (11)) can be significant in a layer below the shock, causing the density profile to steepen slightly.
Figure 2. Sample initial density profiles, which are solutions to the spherically symmetric and time-independent flow equations. Panel (a) shows the zero-heating configurations for all the sequences listed in Table 1. Other parameters are \( \gamma = 4/3, M = 1.3 M_{\odot}, \rho = 3 \rho_{\odot} \) for all configurations, and \( M = 0.3 M_{\odot}, M_{\odot} \) for the NSE models. Panel (b) shows a sequence with a fixed cooling function and range of heating rates (\( H \) is given in units of \( r_{\odot}v_{\odot}C_{\odot} \)). The dashed line shows the upstream flow. Panel (c) shows a sequence with different EOS, \( r_{\odot} = 125 \text{ km}, H = 0 \). The labels “\( +\alpha \)” and “\( -\alpha \)” mean with and without \( \alpha \)-particles included in the EOS, while “full” means that the EOS explicitly includes finite temperature and partially degenerate electrons, black body photons, and ideal-gas ions. All other parameters are the same as in (a). Only the \( \gamma = 4/3 \) upstream flow is shown. See Appendix B for further details.

(A color version of this figure is available in the online journal.)

3. ONE-DIMENSIONAL SIMULATIONS

3.1. Shock Oscillations and Transition to Explosion

An explosion in spherical symmetry involves the development of an unstable \( \ell = 0 \) SASI mode. We showed in Paper I that, in the absence of neutrino heating, the period of this mode is essentially twice the post-shock advection time. As heating is introduced into the flow, we find that this relation is maintained. The \( \ell = 0 \) mode is damped until the heating rate is pushed above a critical value, which we now discuss.

It should be emphasized that this critical heating rate is generally lower than that defined by Burrows & Goshy (1993), which marked the disappearance of a steady, spherically symmetric solution to the flow equations. Large amplitude one-dimensional shock oscillations have been witnessed near the threshold for explosion in calculations by Ohnishi et al. (2006) and Murphy \& Burrows (2008). Both calculations employed a realistic EOS, but like us included neutrino heating as a local source term in the energy equation. Oscillations have also been seen by Buras et al. (2006b) in more elaborate calculations with Boltzmann neutrino transport.

The origin of the one-dimensional SASI oscillation can be briefly summarized as follows. An initial outward shock displacement generates an entropy perturbation, which is negative for \( \gamma \lesssim 5/3 \). This entropy perturbation is advected down to the cooling layer, where it causes, at constant ambient pressure, an increase in the cooling rate, \( \delta S / C = (\nu - 1) / \gamma (b - a) \delta > 0 \). The resulting negative pressure perturbation is rapidly communicated to the shock, which recedes and generates an entropy perturbation of the opposing sign. One more iteration results in a shock displacement of the same sign as the initial displacement, and allows the cycle to close. The duration of the \( \ell = 0 \) mode is therefore nearly twice the advection time from the shock to the cooling layer,

\[
\frac{2\pi}{\omega_{osc}} \simeq 2 \int_{r_s}^{r_{\odot}} \frac{dr}{|v_r|} \tag{13}
\]

The cycle is stable for \( \gamma = 4/3 \) and \( r_s/r_{\odot} = 0.4 \).

When heating is added, the density profile flattens. Increasing \( \gamma \) has the same effect, and has been found to push up the growth rate of linear SASI modes (e.g., Paper I). There is a critical heating rate for which the damping effect of the one-dimensional SASI is neutralized and there is no net growth. We find that, once the heating rate exceeds this critical value, the system always explodes.

We therefore define the critical heating rate in our one-dimensional simulations to be the minimum heating rate for growing shock oscillations. \( ^5 \) Figure 3 shows real and imaginary eigenfrequencies as a function of heating rate for our one-dimensional initial configurations with constant \( \epsilon \). The curves were obtained by solving the differential system of Foglizzo et al. (2007), modified to account for a constant rate of nuclear dissociation (Paper I) as well as incorporating the heating function in Equation (10). The runs marked by stars explode within a time \( 1000 t_{\text{rfm}} \), and so require a small, but finite, positive growth rate.

In an exploding run, the expansions become longer and contractions shorter as the shock oscillation develops a large amplitude. Eventually the accretion flow is halted during a

\( ^5 \) We define our critical heating parameter \( H_c \) to be the average of the values in the exploding and nonexploding runs that are closest to the threshold for explosion, within our fiducial \( 1000 t_{\text{rfm}} \) cutoff.
contraction. This marks the point of explosion, beyond which the feedback between the shock and the cooling layer is broken. Material then tends to pile up in the gain region, is further heated, and more material reverses direction. The net effect is to push the shock outward. Movie 1 in the online material illustrates this chain of events.

3.2. Effects of Alpha-particle Recombination

Shock breakout is controlled by the build-up of positive energy fluid downstream of the shock, and therefore is sensitive to the density profile below the shock. Heating by neutrinos is concentrated fairly close to the protoneutron star, inside a distance $\sim (2-3)r_\star$. Heating by $\alpha$-particle recombination is concentrated at a greater distance $\sim r_\alpha$ (Equation (1)), but still can reach a comparable amplitude.

The dependence of shock breakout on heating rate is displayed in Figure 4 for two accretion models and several values of $H$ close to $H_{cr}$ (see Table 1). The initial expansion of the shock during the explosion phase is very similar for models with constant dissociation energy $\varepsilon$ and with NSE in the shocked fluid. However, the time evolution bifurcates near the radius $r_\alpha$.

Figure 5 shows successive profiles of the shocked flow in the exploding run with $H = 1.09 H_\star$ and $r_{\alpha 0} = 75$ km. The $\alpha$-particle fraction approaches unity as the shock reaches the radius $r_\alpha$. The second panel shows the specific nuclear energy generation rate (Equation (7)) normalized to the adiabatic rate of change of the enthalpy,

$$w_{\text{ad}} = \frac{1}{\rho} \frac{dp}{dt} = -c_s^2 \nabla \cdot \mathbf{v}. \quad (14)$$

Here $c_s = (\gamma p/\rho)^{1/2}$ is the sound speed. The third panel compares the amplitude and distribution of neutrino and recombination heating, and the bottom panel plots the radial velocity in the post-shock region.

We can summarize this behavior as follows: during the initial expansion phase, fluid below the shock continues to move inward, and the dissociation of $\alpha$-particles removes energy from the flow (as expected from Equation (11)). Some fluid behind the shock begins to move outward around $300 H_\star$, but nuclear dissociation still causes a net loss of internal energy. However, the recombination of $\alpha$-particles sets in above $r_\alpha$, especially in regions where $X_{\alpha} \lessapprox 0.5$. By the time the shock hits the outer boundary, $d(e_{\text{nuc}})/dt$ exceeds one-half of $|w_{\text{ad}}|$. The dependence of the density contrast $\kappa$ (Equation (3)) on radius also has an influence on the details of breakout. When the dissociation energy $\varepsilon$ is held fixed, $\kappa$ increases toward larger radius. This has the effect of creating a dense layer of fluid below the shock when the shock has reached a radius where $\varepsilon \sim \nu_\alpha^2/2$. In spherical symmetry, the breakout of the shock is then impeded by this dense layer, which cannot exchange position with the lighter material below it. It can happen that...
the energy in the expanding region is no longer able to sustain the heavier material above, and the shock collapses, as shown in Figure 4 for the constant-\(\varepsilon\) run with \(H = 1.08H_c\). This obstruction is avoided when statistical equilibrium between \(n\), \(p\), and \(\alpha\) is maintained below the shock, because \(\varepsilon / v_\alpha^2\) and \(\kappa\) both decrease gradually as the shock expands to distances much larger than \(r_s\) (Figure 1). This limit to the shock expansion does not occur in two dimensions, as the superposition of dense fluid over lighter fluid is Rayleigh–Taylor unstable on the dynamical time \(t_{\text{m}}\).

Figure 5. Radial profiles of various quantities during shock breakout in the NSE model with \(H = 1.09H_c\) and \(r_{s0} = 75\ \text{km}\) (Figure 4). Top panel: mass fraction of \(\alpha\)-particles. Second panel: rate of release of specific nuclear binding energy \(d\nu_{\text{nu}} / dt\) compared with the (adiabatic) rate of change of enthalpy \(w_{\text{ad}}\) (Equation (14)). Third panel: net neutrino heating rate per unit volume \(L_{\nu} - L_C\) (thin solid curves) and \(d\nu_{\text{nu}} / dt\) (thick dashed curves), both normalized to the local value of \(v_\text{\alpha}\) at radius \(r_{s0}\). Bottom panel: radial velocity normalized to \(v_\text{\alpha}\) at radius \(r_{s0}\). Both \(d\nu_{\text{nu}} / dt\) and \(w_{\text{ad}}\) are smoothed in radius for clarity.

(A color version of this figure is available in the online journal.)

4. TWO-DIMENSIONAL SIMULATIONS

Extending the flow calculation to two dimensions reveals some subtle patterns of behavior. The time evolution of the shock is shown in the left panel of Figure 6 for a range of heating rates near the threshold for explosion. In contrast with the one-dimensional runs, the breakout of the shock looks similar in models with constant dissociation energy and with NSE between \(n\), \(p\), and \(\alpha\) below the shock. Both types of models are subject to buoyancy-driven instabilities, which allow cold material below the shock to interchange position with hotter material within the gain region. As a result, the shock is highly asymmetric at breakout in both cases. In Section 5, we compare the critical heating rate for explosion in one- and two-dimensional simulations, and examine how it is influenced by \(\alpha\)-particle recombination.

Around the threshold for explosion, all of our runs develop vigorous convective motions before the SASI has a chance to undergo even a few oscillations. At high heating rates, we find that the convective instability is driven by the negative entropy gradient within the layer of maximal neutrino heating. In nonexploding runs, the shock settles to a quasi-equilibrium state with oscillations taking place over a range of angular (Legendre) index \(\ell\), as previously seen by Ohnishi et al. (2006), Scheck et al. (2008), and Murphy & Burrows (2008). The amplitude of the \(\ell = 1\) and 2 modes remains small until the heating parameter \(H\) has begun to exceed about one-half the critical value for an explosion. The competition between SASI growth and convective instability is examined in detail in Section 6.

One gains considerable insight into the mechanism driving shock breakout by examining the distribution of Bernoulli parameter (Equation (2)) in the shocked fluid. We first consider the NSE runs with \(r_{s0} = 50\ \text{km}\) and 125 km, with the heating parameter \(H\) just above the threshold for an explosion. Two snapshots from each of these runs are shown in Figure 7. In the first case, the initial equilibrium shock radius is only \(~r_{s0}/4\) km, and \(\alpha\)-particles are essentially absent below the shock. In the second, the shock starts at \(~2r_{s0}/3\) and \(X_{\alpha} \sim 0.5\) initially in the post-shock flow.

Large deformations of the outer shock are caused by convective plumes that carry positive energy. Strong neutrino heating is generally concentrated within an inner zone where the material is gravitationally bound \((b < 0)\). The degree of symmetry of this bound material depends on the \(\alpha\)-particle abundance. In the \(r_{s0} = 125\ \text{km}\) run, it is spherically stratified and the material with \(b > 0\) is generally excluded from it. Strong recombination heating is present both below and above the surface where \(b \approx 0\), indicating that it is mainly responsible for imparting positive energy to the shocked material. The mean shock radius expands by a factor \(~2.5\) between the upper two frames in Figure 7, but the growth in the volume of positive-energy material is not accompanied by a significant expansion of the inner bound region, whose outer radius remains fixed at \(r \approx r_{s0}\).

This segregation of bound from unbound material is broken when the shock is more compact initially, as is seen in the lower two panels of Figure 7. A single dominant accretion plume is continuously present, which funnels cold and dense material into the zone of strong neutrino heating. Alpha-particles are present only well outside the boundary between \(b < 0\) and \(b > 0\). The competition between \(\alpha\)-particle and neutrino heating is discussed in more detail in Section 4.1, and the influence of \(\alpha\)-particles on the threshold heating rate for an explosion is examined in Section 5.
The accumulation of a bubble of hot material right behind the shock is a consequence of the balance of the buoyancy force acting within the bubble, and the ram pressure of the pre-shock material. The ratio of force densities is (Thompson 2000)

$$\frac{F_{\text{buoy}}}{F_{\text{ram}}} \simeq \left( \frac{\rho - \rho_{\text{bubble}}}{\rho} \right) \left( \frac{2GM/r_s}{v_r^2} \right) \Delta \Omega_{\text{bubble}},$$ (15)

where $r_s$ is the shock radius, $v_r$ is the ambient radial flow speed, $\rho$ is the ambient density, $\rho_{\text{bubble}}$ the density of the bubble, and $\Delta \Omega_{\text{bubble}}$ is its angular size. A low-density bubble ($\rho - \rho_{\text{bubble}} \sim \rho$) can resist being entrained by the convective flow once it grows to a size $\Delta \Omega_{\text{bubble}} \sim M_{\text{conv}}^2$ Sr, where $M_{\text{conv}}$ is the convective Mach number. On the other hand, the bubble must attain a much larger angular size $\Delta \Omega_{\text{bubble}} \sim 1$ Sr if the buoyancy force is to overcome the upstream ram ($v_r \sim v_{ff}$) and force a significant expansion of the shock surface. Figure 7 shows that the extent of the shock expansion is indeed correlated with the angular width of the region where hot material accumulates.

Another interesting feature of Figure 7 is the presence of secondary shocks, which are triggered once the outer shock becomes significantly nonspherical. Their locations are marked by discrete jumps in the rate of recombination heating. Secondary shocks are also prevalent throughout the nonlinear phase in the constant-$\epsilon$ models. Figure 8 shows the normalized pressure gradient $(r/p)\nabla p$ for collapse models of both types, when $H$ is just above threshold for an explosion (right before the shock hits the outer boundary of the simulation volume). The online version.

**Figure 6.** Left panel: angle-averaged shock radius (solid lines) and maximum shock radius (dotted lines) for various two-dimensional models around the threshold for explosion. Upper panel shows runs with constant dissociation energy $\epsilon = 0.15v_{ff}^2$, while lower panels displays NSE runs with $r_{s0}$ as labeled. Critical heating rates $H_{\text{cr}}$ are different for each configuration, and can be found in Figure 15. Right panel: black lines show expansion timescale of maximum shock radius $t_{\text{exp}} \sim r_{\text{max}} / |dr_{\text{max}}/dt|$, computed using a polynomial fit for the runs just above the threshold for explosion (corresponding to the black dashed lines on the left panels). Green and red lines show the average residency time over the 50% and 10% of the gain region volume with highest $t_{\text{res}}$, respectively (see Section 4.2 for the definition of this timescale). Shock breakout occurs whenever $t_{\text{exp}} \sim \langle t_{\text{res}} \rangle_{\text{vol}}$, except in the model where recombination heating is dominant ($r_{s0} = 125$ km).

(A color version of this figure is available in the online journal.)
Figure 7. Snapshots of two separate models, with heating parameter $H$ just above the threshold for an explosion, and initial shock radius either well inside $r_0$ ($r_0 = 50$ km, without heating) or close to $r_0$ ($r_0 = 125$ km, without heating). Within each panel, the top figure displays Bernoulli parameter $b$; the middle figure the rate of nuclear energy generation; and the bottom figure the net rate of neutrino heating. Top left: the early development of an asymmetric plume with positive $b$; top right: the same run just before the shock hits the outer boundary. In this $r_0 = 125$ km run, the heating by $\alpha$-particle recombination is enhanced with respect to neutrino heating due to the large $X_\alpha$ in the initial stationary model. The central zone with $b < 0$ maintains a nearly spherical boundary near the radius $r_\alpha$ $(\approx 2.0 r_0)$, and recombination heating straddles the boundary. Bottom left: $\alpha$-particles begin to form as the shock approaches $r_\alpha$ in the $r_0 = 50$ km run, but neutrino heating remains much stronger than recombination heating. Bottom right: the same run just before the shock hits the outer boundary. When the shock starts off well inside $r_\alpha$, neutrino heating dominates the initial expansion, and material with $b > 0$ forms well inside $r_\alpha$ (see Section 4.3). (A color version and animations of this figure are available in the online journal.)

of this paper contains an animated version of Figure 8 showing the complete evolution. In both cases, secondary shocks extend over the whole post-shock domain, signaling the dissipation of supersonic turbulence which is stirred by accretion plumes that penetrate into the gain region.

4.1. Distribution of Alpha-particle Recombination Heating

Heat input by neutrino absorption and by $\alpha$-particle recombination have very different distributions within the shocked fluid:

- strong neutrino heating is concentrated inside $r_\alpha$, whereas recombination heating of a comparable amplitude is distributed throughout the settling flow.
- Strong recombination heating quite naturally extends below the zone where $\alpha$-particles are present in significant numbers, as is seen in Figure 9. The first and third panels of this figure depict the pre-explosion steady state of the $r_0 = 75$ km model with $H = 1.02H_c$, while the second and fourth panels show the last time before the shock hits the outer boundary. At the latter time, one sees that the strongest recombination heating is concentrated in a layer where $X_\alpha \lesssim 0.5$,
at the base of the extended $\alpha$-rich plumes. Just as in the one-dimensional simulations (e.g., Figure 5), $X_\alpha$ approaches unity during shock breakout.

The relative strength of neutrino heating and recombination heating depends on the initial radius of the shock, and on the Bernoulli parameter of the post-shock material. Figure 10 separates out cooling by $\alpha$-particle dissociation from heating by recombination and neutrino radiation during the pre-explosion quasi-steady state (leftmost panels), at the onset of explosion (second panel left to right), and during breakout (two rightmost panels). See Figure 6 for comparison. The colored curves show the positive, negative, and net contributions from nuclear energy generation. The sharp negative spike near $b = 0$ represents $\alpha$-particle dissociation in fresh, cold downflows. The formation of material with $b > 0$ is primarily due to $\alpha$-particle recombination in the $r_{\text{fo}} = 125$ km run. As the initial radius of the shock is reduced with respect to $r_{\text{fo}}$, neutrino heating makes a proportionately larger contribution near breakout.

The strength of the boost given to the shock by recombination heating can be gauged by comparing $\frac{dE_{\text{nuc}}}{dt}$ to the adiabatic rate of change $\frac{dE_{\text{ad}}}{dt}$ of the enthalpy of the flow (Equation (14)). Figure 11 shows the result for all three NSE sequences with $H$ just above $H_\alpha$. In all cases, $\frac{dE_{\text{nuc}}}{dt} \approx \frac{dE_{\text{ad}}}{dt}$ in various parts of the shocked fluid once the shock extends beyond a radius $\sim r_{\text{fo}}$. Most of the heat input by recombination is concentrated where $X_\alpha \sim 0$-0.5, just as in spherical symmetry.

Histograms of $\frac{dE_{\text{nuc}}}{dt}$ versus $b$ and $X_\alpha$ are shown in the right panel of Figure 12. The rapid dissociation of $\alpha$-particles in fresh downflows is represented by the long tail toward large negative values of $\frac{dE_{\text{nuc}}}{dt}$, showing that the overall contribution of nuclear energy generation is negative. The $\alpha$-particle concentration is very stratified, with higher $X_\alpha$ occurring at larger radius. Most of the mass with positive Bernoulli parameter is located at large radii.

### 4.2. Residency Time

A long residency time of material in the gain region is commonly viewed as a key ingredient in a successful neutrino-driven explosion. To calculate $t_{\text{res}}$, we use the method described in Section 2.1.3: we first assign a unique “fluid time” $t_F$ (Equation (12)) to each infalling radial mass shell in the simulation, which is effectively the time at which it passes the shock. We then define $t_{\text{res}}$ the residency time of the fluid as

$$t_{\text{res}} = t - t_F,$$

where $t$ is the present time. A related method (tracer particles) is used by Murphy & Burrows (2008) to calculate the residency time in collapse simulations with a more realistic EOS.

As material with positive Bernoulli parameter accumulates below the shock, we indeed find that its $t_{\text{res}}$ grows larger. The shock starts running outward if the energy of this unbound material grows on a timescale shorter than the convective time. The right panel of Figure 6 shows the characteristic expansion time of the shock $t_{\text{exp}} \sim \frac{r_{\text{exp}}}{d^2 r_{\text{exp}}/dt^2}$ (as measured at its outermost radius), alongside $t_{\text{res}}$ (as measured within the material comprising the upper part of the residency time distribution). The final breakout of the shock seen in the left panel of Figure 6 corresponds to the time when $t_{\text{exp}} \sim t_{\text{res}}$. This lengthening of the mean residency time can largely be

---

6 Since $t_F$ is defined in terms of the angle-averaged radius of the shock, there is a modest error in $t_F$ due to nonradial deformations of the shock. Given the lack of substantial large-scale mixing between the single accretion funnel and convective cells, this prescription serves well as a tracer of different fluid populations.
The expansion time to the dynamical time. What changes most dramatically during breakout is the ascribed to the increased dynamical time of the expanding shock.

The breakout is a bit more gradual in the $r_{\text{io}} = 125$ km model with heating just above threshold for an explosion ($H = 1.04H_{0.5}$; see animated version of Figures 7(a) and 7(b) in the online material). In this case, the expansion time of the shock remains somewhat longer than the residency time of material below the shock, which implies that the breakout depends on the continuing release of nuclear binding energy.

Figure 10. Heating rate of material, as distributed with respect to Bernoulli parameter $b$. This illustrates the relative importance of neutrino heating and nuclear dissociation/recombination in hot and cold parts of the flow. We restrict attention to material in the gain region (defined by $\rho r > \rho r_{\text{c}}$) in the three two-dimensional NSE runs just above the threshold for explosion. Four snapshots are shown: the pre-explosion quasi-steady state (leftmost), onset of explosion (second from left to right), and breakout (third and fourth). 

Figure 10. Heating rate of material, as distributed with respect to Bernoulli parameter $b$. This illustrates the relative importance of neutrino heating and nuclear dissociation/recombination in hot and cold parts of the flow. We restrict attention to material in the gain region (defined by $\rho r > \rho r_{\text{c}}$) in the three two-dimensional NSE runs just above the threshold for explosion. Four snapshots are shown: the pre-explosion quasi-steady state (leftmost), onset of explosion (second from left to right), and breakout (third and fourth). Black curves: net heating rate resulting from neutrino absorption and emission. Red/green curves: heating/cooling rate by $\alpha$-particle recombination and dissociation in material with $\frac{\text{d}E_{\text{nucl}}}{\text{d}t} > 0$ and $\frac{\text{d}E_{\text{nucl}}}{\text{d}t} < 0$, respectively. Blue curves: net heating/cooling rate due to changing $\alpha$-particle abundance. The sharp negative spike near $b = 0$ represents $\alpha$-particle dissociation in fresh, cold downflows. The formation of material with $b > 0$ is primarily due to $\alpha$-particle recombination in the $r_{\text{io}} = 125$ km run. As the initial radius of the shock is reduced with respect to $r_{\text{io}}$, neutrino heating makes a proportionately larger contribution near breakout.

(A color version of this figure is available in the online journal.)

There are large differences in the distribution of $t_{\text{res}}$ are concentrated in regions of positive $b$. In the left panel of Figure 12, we plot the distribution of $b$ and $t_{\text{res}}$ in the $\varepsilon = 0.15\nu_{\text{ff0}}$ run that is just above the threshold for an explosion. Regions with small or negative residency time represent freshly injected fluid. The distribution is stratified in $b$ around $t_{\text{res}} \sim 40t_{\text{ff0}}$ (which is approximately an overturn period of a convective cell, see Section 4.3 and Figure 13). Material with more negative $b$ resides on average at a smaller radius. Fluid with a longer residency time has mostly positive $b$, corresponding to material transported upwards by convective cells.

It is also apparent from the lower panel of Figure 12 that material with a longer residency time tends to have lower $X_{\alpha}$, as is expected because it also tends to have a higher temperature.

4.3. Heat Engine in a Two-dimensional Explosion

Fresh material that is accreted through an oblique shock has a relatively low entropy, but once it reaches the base of the gain region it is exposed to an intense flux of electron-type neutrinos. Some of this heated material rises buoyantly, and forces an overturn of the fluid below the shock. Material with a longer residency time may therefore undergo multiple episodes of heating. On this basis, Herant et al. (1992, 1994) suggested that a convective flow would mediate a heat engine below the shock that would drive a secular increase in the energy of the shocked fluid.

We now investigate whether a heat engine operates in our simulations, and how it depends on the heating parameter $H$. We focus on a model with a constant nuclear dissociation energy, $\varepsilon = 0.15\nu_{\text{ff0}}$. In this class of models, the infalling heavy nuclei are completely broken up below the shock, and no heating by the reassembly of $\alpha$-particles is allowed. As a first step, we average the convective flow over windows of width $50\ t_{\text{ff0}}$, which de-emphasizes short-term fluctuations in the averaged velocity field $\langle \nu \rangle$, Figure 13 shows $\langle \nu \rangle$, and $b_i$ (Equation (2)) at four different times in the run with $H$ just above the threshold for explosion ($H = 1.02H_{0.5}$). The radius of maximum heating ($r \approx 0.66r_{\text{io}}$) coincides with the lower boundary of the convective cells, across which material flows horizontally. The overturn period in these large-scale cells is $\sim 40-50t_{\text{ff0}}$, as found by integrating streamlines of the mean flow (an example is shown on the lower left panel of Figure 13). Heated fluid accumulates in the region in between the top of convective cells and the shock. A strong deformation of the shock allows a plume of fresh material to descend diagonally between the convective cells. The tilt of this cold downflow intermittently flips in sign, and the averaged circulation pattern typically has an “∞” shape. The heating of fluid parcels in the two hemispheres is also intermittent, and sometimes two circulation flows are established simultaneously, thereby causing a bipolar expansion of the shock.

We have identified a useful figure of merit which connects a secular increase in the shock radius to the strength of neutrino heating at the base of the gain region. Figure 14 shows the absolute value of the Bernoulli parameter $b$ at the radius of maximum heating ($r_{H,max} \approx 0.66r_{\text{io}}$) as a function of polar angle.
To identify the range of angles comprising the lower boundary of a solid line is the quantity analog to the nonexploding run. Overplotted as thick lines in the four snapshots of Figure 13, and the bottom panel shows the \( \theta \).

(A color version of this figure is available in the online journal.)

\[ \Theta = \frac{\mathcal{L}_H - \mathcal{L}_C}{\rho} \bigg|_{r,\theta=\epsilon \rightarrow r,\theta} \frac{\rho_{H_{\max}}}{|v_{01}|} \]  

This measures the specific energy that is absorbed from neutrinos by the material that flows laterally along the lower boundary of the convective cells. In Equation (17), the angular average of the heating rate and meridional velocity is restricted to a single convective cell.

In nonexploding models, the circulation in the gain region settles to a quasi-steady state, with no net amplification of the mass in material with positive \( b \). The heat absorbed at the base of the convective cells is of the same order of the Bernoulli parameter of the fluid, that is, \( \Theta \lesssim |b| \). In an exploding model, \( \Theta \) will often exceed \(|b|\) by a factor 2–3. As is shown in the upper panel of Figure 14, \( \Theta \) grows with time as the system approaches the explosion.

The stability of the averaged flow pattern appears to be, in part, an artifact of the axisymmetry of the flow. This imposes strong restrictions on the motion of convective cells, causing vorticity to accumulate on the largest spatial scales.

Our observation that the bulk of the neutrino heating takes place within horizontal flows suggests that the ratio of heating timescale to radial advection time in the gain layer may be a less precise diagnostic of the conditions for explosion in axisymmetry: the horizontal convective velocity is typically low compared with the downward velocity of the main accretion plume. We do observe that the main accretion plume becomes strongly distorted near the threshold for an explosion, so that a significant fraction of the plume material enters one of the convection cells. This effectively decreases the amount of material that accretes to the protoneutron star and thus increases the overall advection timescale across the gain region.

5. CRITICAL HEATING RATE FOR EXPLOSION

An explosion occurs when the heating parameter \( H \) is raised above a critical value \( H_{cr} \). We now explore how \( H_{cr} \) depends on the details of the EOS and the initial radius of the shock.

One can express \( H \) simply in terms of the ratio of the heating rate \( (4\pi r^3)\mathcal{L}_H \) to the accretion luminosity \( (GM\dot{M}/r) \), in the idealized (but unrealistic) case where the flow is composed only of free nucleons and moves hypersonically. Then this ratio depends on \( H \) but not on the accretion rate \( \dot{M} = (4\pi r^2 \rho(r)|v_{|\theta|}(r)|) \). The precise value of the reference radius is unimportant; we choose \( r_0 \) as the shock radius in the time-independent, spherical flow solution at \( H = 0 \). Then

\[ \frac{4\pi r_0^3 \mathcal{L}_H|v_{\rho}(r_0)|}{GM\dot{M}/r_0} \bigg|_{X_\rho=0; \dot{M}=\infty} = \frac{2H}{r_0^3\epsilon v_1^2}. \]

This quantity is \( \sim 10^{-3} \) to \( 10^{-2} \) in the models we examine, which are below or near the threshold for explosion.

Note that the cooling in our model is concentrated at the base of the settling flow. As a result, the width of the gain region (relative to the shock radius) does not change significantly between different models. The critical heating parameter is therefore only indirectly related to the amplitude of the cooling function through the structure of the settling flow below the shock. Our purpose here is to explore how the critical heating rate depends on the strength of the gravitational binding of the shocked fluid to the collapsed core, and on the abundance of alpha particles.

Figure 15 displays \( H_{cr} \) for all of our model sequences. The abscissa is \( \epsilon/v_1^2 \), where \( \epsilon \) is the nuclear dissociation energy and \( v_1 \) is the flow speed upstream of the shock in the initial configuration (that is, in the time-independent, spherical flow solution). In the case of the NSE equation of state, this quantity can be translated into an initial value of the shock radius using Figure 1. (Note that \( \epsilon/v_1^2 \) has a weak dependence on \( r_r \) in the NSE sequence.)

A few interesting features of Figure 15 deserve comment. First, a comparison with Table 1 shows that the critical heating rate for explosion is \( \sim 50\% \)–\( 70\% \) of the maximum heating rate for which a steady-state flow solution can be found. The maximal heating parameter \( H_{\text{steady}} \) for a steady flow corresponds directly to the one first determined by Burrows & Goshy (1993) using a more realistic EOS. Note also that the values of \( H_{cr} \) in the one-dimensional and two-dimensional models are much closer to each other than they are to \( H_{\text{steady}} \). This result is perhaps not surprising, given that the explosion is not immediate, but is approached through a series of transient fluid motions.

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7 To identify the range of angles comprising the lower boundary of a convective cell \( (r = H_{\max}) \), we first average \( (v_\rho) \) over all polar angles, and then define a single cell as a zone where \( |(v_\rho)_{\theta}| < |(v_\rho)_{\theta_\max}| \) and \( v_\theta \) maintains a constant sign. Once the convective cells have been so identified, the angular average is repeated within each cell. The quantity \( |(v_\rho)_{\theta_\max}| \) appearing in Equation (17) represents this more restricted average, which typically covers \( \sim 1 \) rad in the polar direction (e.g., Figure 13).

8 Our method for determining \( H_{cr} \) is discussed in Section 3.1.
Second, $H_{cr}$ is lower when NSE between $n$, $p$, and $\alpha$ is maintained below the shock. In this case, the dissociation energy at the shock is not fixed, but is (roughly) inversely proportional to radius. However, the difference between the NSE models and the constant-$\epsilon$ models is only $\sim$10% in $H_{cr}$ when the shock starts out well below $r_\alpha$ (Equation (1)). An explosion is significantly easier when the fluid below the shock starts out with a significant population of $\alpha$ particles, as in the models with $r_\alpha = 125$ km.

Third, $H_{cr}$ tends to decrease with increasing $\epsilon/v_{2,0}$: a slightly lower heating rate per unit mass is required to explode a flow with a larger density contrast $\kappa$ across the shock. Because almost all the gravitating mass is in the collapsed core, the gravitational binding energy of the gain region is approximately proportional to $\kappa$, whereas the net heat absorbed over the advection time is a stronger function of density, $t_{adv} \int (\mathcal{L}_{H} - \mathcal{L}_{C}) d^3r \propto \kappa^2$. (One factor of $\kappa$ comes from the advection time $t_{adv}$ as given by Equation (13), and the other from the density dependence of $\mathcal{L}_{H}$.) For example, Table 1 shows that $\kappa$ is $\sim$1.6 times larger for $\epsilon/v_{2,0}^2 = 0.2$ than for $\epsilon/v_{2,0}^2 = 0.1$, and that $H_{cr}$ is smaller by the inverse of the same factor.
Critical heating parameter \( H_\alpha \) that yields an explosion, for all the model sequences explored in this paper (Table 1). The abscissa is the ratio of \( \epsilon \) to \( v_1^2 \) in the initial flow configuration (\( v_1 \) being the radial flow velocity just upstream of the shock). Error bars show the separation between exploding and nonexploding models, with the points marking the average.

(A color version of this figure is available in the online journal.)

Fourth, the two-dimensional runs all require less heating than their one-dimensional counterparts to explode. A major reason for this is that all two-dimensional configurations explode along one or both poles (see Figures 11 and 13), so that less material must be lifted through the gravitational field than in a fully spherical explosion. We have found that the precise value of the difference between the critical heating rate in the one-dimensional and two-dimensional models also decreases. (A color version of this figure is available in the online journal.)
Figure 17. Development of a convective instability is strongly limited when the parameter $\chi \lesssim 3$. These panels show snapshots of entropy (normalized to initial post-shock value) in a NSE run ($r_{so} = 75$ km) with two different heating rates. When $\chi = 2.1$, convective cells of a limited extent are triggered in the layer where the net heating rate is strongest, but they do not propagate into the upper parts of the gain region. Convection becomes much more vigorous and widespread when $\chi = 5.5$. Note that both of these models are nonexploding. (An animation of this figure is available in the online journal.)

6. CONVECTION AND THE SASI

Overturns of the fluid below the shock can be triggered in two distinct ways: through the development of Ledoux convection in the presence of a strong negative entropy gradient, or via the nonlinear development of the SASI (a linear feedback between ingoing entropy and vortex waves, and an outgoing sound wave). We now show that the amplitude of the dipolar mode that is excited in the shock is strongly tied to the level of neutrino heating, and so thermal forcing plays a crucial role in maintaining the oscillation. To a certain extent, this distinction is of secondary importance, in the sense that memory of the linear phase of the instability is lost once the inflow of fresh material below the shock bifurcates from older shocked fluid. Nonetheless, the origin of the convective motions does have implications for the stability of $\ell = 1$ and 2 modes in three-dimensional simulations: one expects that large-scale oscillations will change shape and direction more rapidly if they are triggered primarily by neutrino heating.

We can ask whether a heating parameter $H$ that yields an explosion will also form an unstable entropy gradient below the shock. Convection develops through a competition between inward advection and neutrino heating. A detailed analysis by must exceed a critical value $\sim 3$ for unstable convective plumes to grow before being advected downward through the gain region below the shock. Here $\omega_{BV}$ is the Brunt–Väisälä frequency.

Using our initial flow models, we can translate $H$ into a value for $\chi$, and find (Table 1) that typically $\chi \sim 5–20$ at the threshold for a neutrino-driven explosion. The implication for convection below the shock is illustrated in Figure 17, which shows two snapshots for models with $\chi = 2.1$ and 5.5, neither of which explodes. At the lower heating rate, the time required for convection to develop depends on the strength of the seed perturbation, whereas at the higher heating rate convective overturns develop rapidly within the layer of strong neutrino heating and spread throughout the post-shock region over a few dozen dynamical times. (The figure shows the result in the case where the seed perturbation is dominated by a small spherical startup error in the initial model.)

We conclude that, near the threshold for a neutrino-driven explosion and for our given set of physical assumptions, convection is driven primarily through the development of a strong, negative entropy gradient within the gain region, rather than through the nonlinear development of SASI modes. The growth of the SASI requires at least a few oscillations, each
with a period comparable to the advection time. The SASI is therefore subdominant when $\chi \gtrapprox 3$. It is worth comparing this with the results of Scheck et al. (2008), who forced the inner boundary of the simulation volume to move inward to model core contraction, thereby generating large advection velocities. The net result was that the flow barely reached $\chi \approx 3$ in exploding configurations. While this effect may be important for relatively prompt explosions, it should be kept in mind that the exploding configurations. While this effect may be important for relatively prompt explosions, it should be kept in mind that the exploding configurations.

In Paper I we considered the nonlinear, saturated state of the SASI in the absence of neutrino heating, and showed that the amplitude of the shock oscillations drops significantly as the dissociation energy $\varepsilon$ is increased. We now explore how the rms amplitude of the shock oscillations correlates with the strength of heating. To eliminate the effect of secular shock motions around or above the threshold for explosion, we first calculate the running average $\langle a_{\ell} \rangle_{50t}$ of the shock Legendre coefficients $a_{\ell}$ over an interval $50t_{\text{ff}0}$, and then calculate the rms of $\hat{a}_{\ell} \equiv a_{\ell} - \langle a_{\ell} \rangle_{50t}$ over the duration of each simulation. The result is plotted in Figure 18 as a function of $H$ for two model sequences ($\varepsilon = 0.15v_{\text{ff}0}^2$ and NSE with $r_{s0} = 75$ km).

There is a clear trend of increasing $\ell = 1, 2$ mode amplitude with increasing heating rate. This confirms our previous suggestion that large-amplitude shock oscillations require strong heating when the dissociation energy behind the shock exceeds $\sim 0.15v_{\text{ff}0}^2$. The models with $H = 0$ reveal a slight exception to the overall trend: the rms amplitude of the shock oscillations appears larger than it does in models with small but finite heating rate, because the oscillations are strongly intermittent at $H = 0$ (see Paper I). The shock oscillations grow much stronger just below the threshold for explosion (exploding runs are marked by stars), above which they seem to saturate. Note also that their amplitudes do not vary much with the choice of the dissociation model. The rms amplitudes relative to the running average of $a_{\ell}$ at the threshold for explosion are $\{5\%, 12\%, 8\%\}$ for the $\ell = 1, 2, 3$ modes in the $\varepsilon/v_{\text{ff}0}^2 = 0.15$ sequence, and $\{6\%, 12\%, 7\%\}$ in the NSE $r_{s0} = 75$ km sequence.

We have performed an additional sequence of runs in which we drop an overdense shell with a given Legendre index $\ell$ through the shock (see also Paper I). This has the effect of selectively triggering individual SASI modes. Figures 19 and 20 display the Legendre coefficients of the shock alongside the angle-averaged entropy gradient. We find that the amplitude of the $\ell = 1$ and 2 modes is strongly tied to the strength of convection. For both types of dissociation models, convection is quenched by the accretion flow when $\chi < 3$: it grows intermittently in strength, but never reaches large enough
amplitudes to significantly distort the shock surface. As a consequence, the entropy gradient remains shallow and negative most of the time. Coherent shock oscillations are also seen, but they have a low amplitude due to the large dissociation energy.

Convection grows much more rapidly when $\chi > 3$, distorting the shock surface before the SASI has the chance to execute a lateral sound wave oscillation. Indeed, the $\ell = 1$ mode is marked by the vertical solid lines in Figure 21, and indeed encompasses most of the mass.

7. SUMMARY

We have investigated the effects of $\alpha$-particle recombination and neutrino heating on the hydrodynamics of core-collapse supernovae, when the heating rate is pushed high enough to reach the threshold for an explosion. The effect of dimensionality has been probed by comparing one-dimensional (spherical) and two-dimensional (axisymmetric) time-dependent hydrodynamic calculations. Our main results can be summarized as follows:

1. The critical heating parameter that yields an explosion depends sensitively on the starting position of the shock relative to $r_\alpha$. This means that the critical neutrino luminosity depends sensitively on the stall radius of the shock and, in turn, on the core structure of the progenitor star and the density profile in the forming neutron star. Within the framework explored in this paper, we find two extreme types of explosion. In the first, neutrino heating does most of the work, with a significant final boost from $\alpha$-particle recombination. In the second, neutrino heating is generally less important at promoting material below the shock to positive energies.

2. During the final stages of an explosion, the heat released by $\alpha$-particle recombination is comparable to the work done by adiabatic expansion. This heat is concentrated in material that has previously been heated by neutrinos. Significantly more energy is lost through $\alpha$-particle dissociation in fresh downflows, so that nuclear dissociation remains on balance an energy sink within the accretion flow.

3. The large-amplitude oscillations that are seen in one-dimensional runs near an explosion are the consequence of the $\ell = 0$ SASI as modified by heating. In contrast with the $\ell = 1, 2$ modes of a laminar accretion flow, the period of these oscillations is close to twice the post-shock advection time. The critical heating rate for an explosion (assuming constant mass accretion rate, neutrino luminosity, and inner boundary) corresponds to neutral stability for the $\ell = 0$ mode.

4. The critical heating parameter $H_\alpha$ for an explosion is generally lower in axial than in spherical symmetry, but the difference becomes smaller as the starting radius of the shock approaches $r_\alpha$. The difference depends somewhat on the ratio $r_\alpha/r_\Sigma$ and thus on the cooling efficiency and equation of state.

5. Nonspherical deformations of the shock are tied to the formation of large-scale plumes of material with positive energy. Our two-dimensional explosions with a super-critical heating rate involve a large-scale convective instability that relies on the accumulation of vorticity on the largest spatial scales. Volume-filling convective cells are apparent in a time-averaged sense. Transient heating events create positive-energy material that accumulates in between the convective cells and the shock. A significant fraction of the heating occurs in horizontal flows at the base of the convective cells, which are fed by a dominant equatorial accretion.
plume. If the heating parameter is large enough, this results in an amplifying cycle and explosion.

6. The amplitude of the \( \ell = 1 \) and \( \ell = 2 \) modes correlates strongly with the value of the heating parameter, and is coupled to the appearance of vigorous neutrino-driven convection below the shock. In agreement with the work of Foglizzo et al. (2006) and Scheck et al. (2008), we find that \( \chi \approx 3 \) marks the transition from a strong linear instability in a nearly laminar flow below the shock, to a volume-filling convective instability. In all of our simulations, the threshold for explosion lies well within the latter regime. This highlights a basic difference between one-dimensional and two-dimensional explosions: the mechanism is fundamentally nonlinear in two dimensions.

7. We have explored essentially one ratio of cooling radius to shock radius, namely \( r_\alpha / r_\odot = 0.4 \) at zero heating (corresponding to \( r_\alpha / r_\odot \approx 0.2 \) near the threshold for an explosion). The growth of the \( \ell = 1 \) SASI mode is strongest for this particular aspect ratio when \( \varepsilon = 0 \) (see Figure 12 of Paper I). As dissociation is introduced into the flow, we found that the peak growth rate moves to larger values of \( r_\alpha / r_\odot \). On the other hand, detailed collapse calculations indicate ratios of neutrinosphere radius to shock radius that are even smaller than \( \sim 0.2 \) following \( \sim 100 \) ms after collapse (e.g., Marek & Janka 2009). We conclude that the \( \ell = 1 \) SASI mode is not being artificially suppressed by our choice of initial shock size.

8. Vortical motions with a Mach number \( \sim 0.3\text{--}0.5 \) first appear at the onset of convective instability around the radius of maximal neutrino heating, but before the dipolar mode of the shock reaches its limiting amplitude. These vortices are a source of acoustic waves, which have a similar period to the large-scale oscillation of the shock. Near the threshold for an explosion, the turbulence in the gain region becomes supersonic, as the existence of widespread secondary shocks attests. These shocks convert turbulent kinetic energy to internal energy, increasing the effective heating rate.

There are at least two reasons why explosions by the mechanism investigated here may be more difficult in fully three-dimensional simulations. First, the existence of more degrees of freedom for the low-order modes of the shock in three dimensions implies that the amplitude of individual shock oscillations is lower. As a result, it is more difficult for the shock to extend out to the radius where \( \alpha \)-particle recombination gives it the final push. Second, an axisymmetric explosion that is driven by neutrino heating involves the accumulation of vorticity on the largest spatial scales, an effect that is special to two dimensions. A full resolution of these issues is possible only with high-resolution three-dimensional simulations.

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APPENDIX A

ALPHA-PARTICLE ABUNDANCE IN NUCLEAR STATISTICAL EQUILIBRIUM

We calculate the \( \alpha \)-particle mass fraction \( X_\alpha \) in nuclear statistical equilibrium by limiting the nuclear species to \( \alpha \)-particles and free nucleons, and fixing the electron fraction \( Y_e = 0.5 \). We tabulate \( X_\alpha \) and temperature \( T \) as a function of pressure \( p \) and density \( \rho \), and then use these tables to calculate the rate of release of nuclear binding energy by the method described in Section 2.1.1. The temperature does not appear explicitly in the FLASH hydrodynamic solver, and only enters the flow equations indirectly through \( X_\alpha \).

We include the contributions to \( p \) from radiation, relativistic and partially degenerate electron–positron pairs, and nonrelativistic \( \alpha \)-particles and nucleons. When \( k_B T > m_e c^2 / 2 \), it can be written as (Bethe et al. 1980)

\[
p = \frac{1}{12} \left( \frac{k_B T}{hc} \right)^4 \left[ \frac{11 \pi^2}{15} + 2 \eta^2 + \frac{1}{\pi^2} \eta^4 \right] + \left[ 1 - \frac{3}{4} \frac{X_\alpha}{m_\alpha} \right] \rho \frac{k_B T}{m_\alpha},
\]

where \( \eta = \mu_e / (k_B T) \) the normalized electron chemical potential, also known as degeneracy parameter, and \( h, c, \) and \( m_\alpha \) are Planck’s constant, the speed of light, and the atomic mass unit, respectively. The density and degeneracy parameter are further related by

\[
\rho = \frac{m_\alpha}{3 \pi^2 Y_e} \left( \frac{k_B T}{hc} \right)^3 \eta (\pi^2 + \eta^2),
\]

as supplemented by the conditions of mass and charge conservation,

\[
X_\alpha + X_\beta + X_e = 1 \quad (A4)
\]

\[
X_\beta + \frac{1}{2} X_e = Y_e. \quad (A5)
\]

In Equations (A3)–(A5), \( X_\alpha \) and \( X_\beta \) are the mass fractions of free neutrons and protons, respectively, and \( Q_{\alpha} = 28.3 \text{ MeV} \) is the binding energy of an \( \alpha \)-particle. Combining Equations (A1) and (A2) gives \( \eta \) and \( T \) in terms of \( p \) and \( \rho \). The equilibrium mass fraction \( X_\alpha^{eq} \) is calculated from \( \rho \) and \( T \). For numerical calculations, we tabulate \( X_\alpha^{eq} \), \( \partial X_\alpha^{eq} / \partial \ln \rho \), and \( \partial X_\alpha^{eq} / \partial \ln p \) for a grid of density and pressure. In addition, we tabulate partial derivatives of \( T \) to substitute into Equations (B10) and (B11).

Figure 22 shows contours of constant \( X_\alpha^{eq} \) and constant entropy for different variables as a function of density. The entropy per nucleon is obtained by adding the contributions from the different components (e.g., Bethe et al. 1980).

\[
S = \pi^2 Y_e \left( \frac{11 \pi^2}{15} + \frac{1}{\pi^2} \eta^4 \right) + \left[ 1 - \frac{3}{4} X_\alpha \right] \times \left[ \frac{5}{2} + \ln \left( \frac{m_\alpha n_Q(T)}{H} \right) \right] - X_\beta \ln X_\beta - X_e \ln X_e
\]

\[
- \frac{1}{4} X_\alpha \ln (X_\alpha / 32).
\]

(A6)
The post-shock density in the initial configuration is typically \( n < 10^9 \, \text{g cm}^{-3} \) with an entropy \( \sim 10^{11} \, \text{g cm}^{-3} \) MeV. In this regime, the electrons are nondegenerate and the pressure in photons and pairs begins to exceed the pressure (dashed lines, negative). Panel (e): ratio of relativistic pressure (photons and pairs) to material pressure (\( \alpha \)-particles and nucleons). Panel (f): adiabatic index \( \gamma \) for different adiabats (dotted line: \( \gamma = 4/3 \)).

(Approximation version of this figure is available in the online journal.)

The post-shock density in the initial configuration is typically \( n \approx 10^9 \, \text{g cm}^{-3} \) with an entropy \( \sim 10^{11} \, \text{g cm}^{-3} \) MeV. The formation of \( \alpha \)-particles that is seen in Figure 1 results from an expansion of the shock into the part of the thermodynamic plane in Figures 22(a) and 22(b) where \( n < 10^9 \, \text{g cm}^{-3} \) and \( T < 1 \, \text{MeV} \). In this regime, the electrons are nondegenerate and the pressure in photons and pairs begins to exceed the nucleon pressure. The dip in the adiabatic index seen in Figure 22(f) results from \( \alpha \)-particle dissociation/recombination, which partially compensates the change in internal energy due to compression/expansion.

APPENDIX B

TIME-INDEPENDENT FLOW EQUATIONS DESCRIBING INITIAL MODELS

We write down the ordinary differential equations that are used to compute the initial flow, and the density profiles in Figures 2(a)–2(c). The steady-state Euler equations in spherical symmetry are

\[
\frac{1}{r} \frac{dv_r}{dr} + \frac{1}{\rho} \frac{dp}{dr} + \frac{2}{r} = 0, \quad (B1)
\]

\[
v_r \frac{dv_r}{dr} + \frac{1}{\rho} \frac{dp}{dr} + g = 0, \quad (B2)
\]

\[
\rho v_r \frac{d\epsilon_{\text{int}}}{dr} - \frac{p v_r}{\rho} \frac{dp}{dr} = \mathcal{L}_H - \mathcal{L}_C + \mathcal{L}_a, \quad (B3)
\]

where \( \epsilon_{\text{int}} \) is the internal energy per unit mass, \( \mathcal{L}_H, \mathcal{L}_C, \) and \( \mathcal{L}_a \) the source terms described in Equations (9)–(11), and \( g = GM/r^2 \). Since two variables suffice to describe the thermodynamic state of a system, we write

\[
\frac{d\epsilon_{\text{int}}}{dr} = E_\rho \frac{dp}{dr} + E_p \frac{d\rho}{dr}, \quad (B4)
\]

and

\[
E_\rho = \frac{1}{(\gamma - 1) \rho} \quad \text{(constant \( \gamma \))}, \quad (B7)
\]

\[
E_p = -\frac{1}{(\gamma - 1) \rho^2} \quad \text{(constant \( \gamma \))}. \quad (B8)
\]

The coefficients in Equations (B4) and (B5) work out to

\[
\frac{dp}{dr} = \frac{(\rho v_r E_\rho - A_\rho) \left( \rho g - 2\rho v_r^2/r \right) + (\mathcal{L}_H - \mathcal{L}_C)}{(\rho v_r E_\rho - \rho v_r/\rho - A_\rho) + v_r^2 (\rho v_r E_\rho - A_\rho)}. \quad (B6)
\]

For a constant-\( \gamma \) equation of state, \( \epsilon_{\text{int}} = p / [(\gamma - 1) \rho] \). The pressure (Equation (A1)) in the NSE model described in Appendix A can be decomposed into contributions from relativistic particles and from nucleons, \( p = p_{\text{rel}} + p_{\text{mat}} \), and the specific internal energy is

\[
\epsilon_{\text{int}} = \frac{1}{\rho} \left( 3p_{\text{rel}} + \frac{3}{2} p_{\text{mat}} \right) = \frac{3}{2} \frac{p}{\rho} - \frac{3}{4} X_\alpha \frac{k_B T}{m_\mu}, \quad (B9)
\]
One therefore finds

\[ E_\rho = \frac{3}{\rho} + \frac{9}{8} \frac{k_B T}{m_u} \frac{\partial X_a}{\partial \rho} - \frac{3}{2} \left( 1 - \frac{3}{4} X_a \right) \frac{1}{m_u} \frac{\partial (k_B T)}{\partial \rho}, \quad \text{(NSE)} \]

\[ E_\rho = - \frac{3 \rho}{\rho^2} + \frac{9}{8} \frac{k_B T}{m_u} \frac{\partial X_a}{\partial \rho} - \frac{3}{2} \left( 1 - \frac{3}{4} X_a \right) \frac{1}{m_u} \frac{\partial (k_B T)}{\partial \rho}. \quad \text{(NSE)} \]

The initial post-shock solution is obtained by integrating the above equations from \( r_s \) to an inner radius \( r_a \) at which the flow stagnates. We iterate the normalization of the cooling function in Equation (9) so that \( r_s = 0.4 r_\odot \) in the absence of heating. When adding heating, the cooling normalization and \( r_s \) are kept fixed, which results in an expansion of the shock from its initial position to \( r_s > r_\odot \) (Figures 2(a) and 2(b)). The initial Mach number at the inner boundary is chosen so as to satisfy

\[ \sum_{i} \left( \mathcal{P}_{H,i} - \mathcal{P}_{C,i} + \mathcal{P}_{nuc,i} \right) V_i \simeq 0.995 \left[ \frac{GM}{r_s} - \epsilon(t = 0) \right] |\dot{M}|, \quad \text{(B12)} \]

where the sum is taken over the computational cells below the shock at our fixed resolution (see Section 2.1.3). \( V_i \) is the volume of each computational cell, and the source terms are evaluated at the inner radial cell face. The numerical coefficient on the right-hand side depends on the radial resolution, and is chosen empirically to prevent runaway cooling due to discreteness effects in time-dependent calculations. The resulting inner Mach number is typically \( 10^{-3} \) to \( 10^{-2} \).

When including \( \alpha \)-particles in the EOS, one needs to calculate self-consistently the value of \( X_a^{\text{eq}} \) below the shock, the corresponding dissociation energy \( \epsilon(t = 0) \) (Equation (8)), and compression factor \( \kappa \) (Equation (3)). The density upstream of the shock is obtained from

\[ \rho_1(r_s) = \frac{M}{4\pi r_s^2 |v_1(r_s)|}, \quad \text{(B13)} \]

\[ v_1(r_s) = - \frac{v_{\text{th}}(r_s)}{\sqrt{1 + 2M_{\odot}^2(r_s)/(\gamma - 1)}} \quad \text{(B14)} \]

is the upstream velocity at \( r = r_s \), while the upstream pressure satisfies

\[ \rho_1(r_s) = \frac{\rho_1(r_s) |v_1(r_s)|^2}{\gamma M_{\odot}^2(r_s)}. \quad \text{(B15)} \]

Equations (B13) and (B15) are transformed to physical units for input to the NSE model by adopting \( M_1(r_\odot) = 5, M = 0.3 M_\odot \) \( \text{s}^{-1} \), \( M = 1.3 M_\odot \), and a particular value for the shock radius \( r_\odot \) in the absence of heating.

**APPENDIX C**

**TIME EVOLUTION**

Here we give some further details of the time evolution of our initial models using FLASH2.5. Heating and cooling are applied in an operator split way in between hydrodynamic sweeps. Nuclear dissociation in the constant-\( \epsilon \) model is implemented through the fuel+ash module in FLASH, with the modifications described in Paper I. The rate of change of specific internal energy is computed using the current hydrodynamic variables and timestep, after which the EOS subroutine is called to ensure that the variables are thermodynamically consistent.

In the NSE nuclear dissociation module, numerical stability is maintained using an implicit update of the pressure in between hydro sweeps,

\[ p_{\text{new}} = p_{\text{cur}} + (\gamma - 1) \rho \epsilon_{\text{nuc}}(\rho, p_{\text{new}}), \quad \text{(C1)} \]

where \( \epsilon_{\text{nuc}} \) is the energy generation per timestep in Equation (6), and the subscripts \( \text{cur} \) and \( \text{new} \) refer to the current and new value of the pressure, respectively. The density is kept constant across this step, so as to be consistent with the other source terms. Equation (C1) usually converges in 3–4 Newton iterations, adding a negligible overhead to our execution time. We restrict the timestep of the simulation so that, in addition to the standard Courant–Friedrichs–Levy condition, it enforces \(|\epsilon_{\text{nuc}}| < 0.8(\rho/\rho)/(\gamma - 1)\). To prevent \( \alpha \)-particle recombination in the cooling layer (due to the decrease of internal energy), we adopt a cutoff in density, so that \( X_a = 0 \) if \( \rho > 3 \times 10^{10} \text{ g cm}^{-3} \).

Both the constant-\( \epsilon \) and the NSE dissociation modules require that the Mach number remain below a fixed value \( M_{\text{burn}} \) for burning, which has the effect of preventing dissociation or recombination upstream of the shock (Paper I). This results in a small amount of incomplete burning in the presence of strong shock deformations, a phenomenon which is also encountered in the full collapse problem. We set the threshold to \( M_{\text{burn}} = 2 \) in most of our simulations. The critical heating parameter \( H_4 \) depends weakly on \( M_{\text{burn}} \); changes in \( M_{\text{burn}} \) cause small changes in the amount of unburnt material with zero Bernoulli parameter, and only slightly alters the net energy of the gain region. At the outer boundary of the simulation volume, \( M_{\text{burn}} \) is just below the Mach number of the upstream flow. We have tried to expand the outer boundary to \( r = 9r_\odot \) (with a somewhat smaller \( M_{\text{burn}} \)) and found that runs that did hit \( r = 7r_\odot \) still hit the new outer boundary.

Aside from the inclusion of heating and NSE dissociation, the numerical setup for our runs is identical to that of Paper I. In that paper, the numerical output was verified by comparing the measured growth rates of linearly unstable modes of the shock with the solution to the eigenvalue problem. We have tested the implementation of our NSE dissociation model by verifying that, in the absence of initial perturbations, our steady-state initial conditions remain steady. Spherical transients present in the initial data die out in a few \( \ell \) oscillation cycles, and are present even when nuclear burning is omitted. (See Paper I for a more extended description.)

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