Interesting features of a general class of higher derivative theories of quantum gravity

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ABSTRACT: We investigate some classical and quantum aspects of a general class of higher derivative theories of gravity. We propose a generalized version of the so-called Teyssandier gauge condition and we investigate its implications on the linearized field equations. An exhaustive investigation on the particle spectra is done, including a discussion on the appearance of ghost-like particles. We investigate the UV properties and we determine the power-counting renormalizability of the theory. Finally we probe a conjecture which relates renormalizability with the cancellation of Newtonian singularities.

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1 Introduction

A complete theory of quantum gravity remains one of the most important problems of theoretical physics. In fact, one of the biggest challenges in the construction of a quantum theory for the gravitational interaction is the lack of experimental hints on what we should expect from gravity in the microscopic level. However, even without experimental evidences, several approaches of quantum gravity were proposed in the last few decades, for instance: string theories, loop quantum gravity, causal dynamical triangulations, causal sets and induced quantum gravity [1, 2]. Nevertheless, at the time of this writing, none of the aforementioned theories give the final word in quantum gravity.

At the classical level, however, the gravitational interaction is very well described in terms of Einstein’s general relativity (GR), which produces excellent results in comparison with experimental tests (e.g. solar system tests, cosmological observations and the recent confirmation of gravitational waves). Therefore, a natural path for the construction of a quantum theory of gravity involves, somehow, performing the quantization of GR.

In this vein, since the fundamental field in GR is the spacetime metric, a possible way to perform its quantization is by means of the functional integration over all metric fluctuations around some vacuum configuration [1, 3]. For instance, we may consider the metric splitting
$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and, then, perform the path integral quantization of the fluctuation field $h_{\mu\nu}$. In this approach we can derive Feynman rules of the theory and apply the standard techniques of perturbative QFT, however, we have to face some theoretical problems. At tree-level approximation we may compute scattering process and even reproduce important results from classical GR, such as the gravitational light bending \[4\]. The real problem arises when we reach loop diagrams. In this case, the UV divergences are not treatable by means of perturbative renormalization and, as a consequence, the theory is UV incomplete. In fact, by means of simple power counting arguments it can be verified that the superficial degree of divergence of an arbitrary Feynman diagram increases with the number of vertices and, therefore, the theory is non-renormalizable.

A possible way to deal with the problem of UV divergences is the introduction of higher derivative terms\(^1\). In fact, the existence of higher derivative terms improves the behavior of the tree-level propagator in such a way that it compensates the “nasty” UV behavior of the vertices containing derivative couplings. In such a case, the power counting argument indicates that the superficial degree of divergence is independent, or even decreasing, with the number of vertices. This heuristic argument indicates that theories containing higher derivatives are power counting (super-)renormalizable.

In a seminal paper \[6\], Stelle investigated deeply the renormalizability of higher derivative theories of quantum gravity. In fact, Stelle proved that a fourth derivative theory described by an action containing curvature squared terms is renormalizable (in 4-dimensions) up to all orders in perturbation theory. However, there is a price to be paid for renormalizability: the spectra of the theory exhibits a massive ghost-like particle which can cause unitarity violation. In addition, from the classical point of view, higher derivative may lead to Ostrogradsky’s instabilities. Unfortunately, non-unitarity is a problem so undesirable as non-renormalizability and, therefore, we have to found some way to work around it if we want to follow the route of higher derivative theories of quantum gravity\(^2\).

In the last few decades a lot of effort has been made in attempt to conciliate unitarity and renormalizability of higher derivative theories. For instance, it was verified that if we allow terms with six or higher derivatives in the action, then the theory becomes super-renormalizable \[7–9\]. In addition, when we deal with super-renormalizable theories it is possible to find some region in the parameter space where all the non-trivial poles of the tree-level propagator are complex and, in this case, the theory may be formulated as unitary in the Lee-Wick sense \[10, 11\]. On the other hand, the presence of complex poles may lead to problems with causality \[12\] and a careful analysis of this question is still missing in the literature of higher derivative quantum gravities.

Another possibility arises if we allow non-local terms in the action \[13–15\]. In such a case it is possible to choose a form factor in such a way that the only pole of the propagator correspond to the usual physical graviton and, as a consequence, we may escape from the

\(^1\)An alternative route to deal with this problem is the so-called Asymptotic Safety program\[5\]. In the context of Asymptotically safe quantum gravity the problem of UV divergences may treated with the concept of non-perturbative renormalization. In this paper, we shall restrict ourselves to pertubative techniques.

\(^2\)Remarkably, an Euclidean lattice formulation of the fourth derivative quantum gravity points out that unitarity may be restored at non-perturbative level.
unitarity violation. Also, a typical non-local term improves the tree-level propagator in such a way that the theory becomes super-renormalizable or even UV finite\textsuperscript{3}.

In this paper we shall investigate some classical and quantum aspects of a general class of $D$-dimensional higher derivative gravities. On the classical side, we present a generalization for the so-called Teyssandier gauge condition. In fact, the Teyssandier gauge condition is very useful in the context of fourth-derivative gravities [17] and it was also extended for the case of sixth derivative theories [18]. Here, we shall extend this gauge condition for a general class of higher derivative theories. As an application of the generalized Teyssandier gauge we work on the linearized field solution for a point-like source at rest.

On the quantum side of this general class of $D$-dimensional higher derivative theories we have performed a detailed study on the tree-level particle spectra and, as we shall see, the presence of massive ghosts is unavoidable, except for two very particular cases to be discussed. In addition, we investigate some aspects of the UV behavior of this general class of theories and we discuss a necessary condition for perturbative renormalizability. As we shall see, those higher derivative theories which are ghost-free turns out to be power counting non-renormalizable.

Finally, we explore a very interesting connection between classical and quantum features of higher derivative gravities. In fact, it was pointed by Stelle that a renormalizable theory of quantum gravity should have a regular classical interparticle potential for small distances [6], i.e. the potential energy does not present the so-called Newtonian singularity. This connection was first established in the context of fourth derivative theories in 4-dimensional spacetime [6] and it as later conjectured that this is a general property of $D$-dimensional higher derivative gravities [19]. Recently this conjecture was probed in the context of fourth and sixth derivative gravities in $D$-dimensions [20, 21]. In this paper we intend to probe this conjecture for a general class of $D$-dimensional higher derivative theory. In this vein, we compute a general expression for the interparticle potential energy and we analyze its behavior for small distance. We found a sufficient condition for a regular potential at small distances and, as we shall see, this condition turns out to be automatically satisfied for power-counting renormalizable theories. In addition, we discuss the role of ghost-like particles on the cancellation of Newtonian singularity [22, 23].

This paper is organized as follows: In section 2, we present a general class of higher derivative gravities and investigate some of its classical properties; in section 3, we investigate the particle spectra of the theory and its implications for tree-level unitarity; in section 4, we investigate the UV behavior of the theory and its renormalizability properties; in section 5, we compute a general expression for the interparticle potential and we investigate its behavior for small distances; in section 6, we probe the conjecture that renormalizable theories do not present Newtonian singularities. Finally, in section 7 we present our conclusions.

Throughout this paper we use the conventions $c = \hbar = 1$, $\eta_{\mu\nu} = \text{diag}(+,-,\cdots,-)$, $R^{\mu}_{\nu\alpha\beta} = \partial_{\alpha} \Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\nu\beta} - (\alpha \leftrightarrow \beta)$, $R_{\mu\nu} = R^\rho_{\mu\nu\rho}$ and $R = g^{\mu\nu} R_{\mu\nu}$.

\textsuperscript{3}It is important to emphasise that there are some classes of non-local theories where both renormalizability and tree-level unitarity are respected, however, after quantum corrections it is possible that they become non-unitary [16].
2 General class of higher derivative gravities

Let us start by establishing a general class of higher derivative gravities that we are going to work with. Although the gravitational action has an infinite amount of compatible terms with the symmetries under general coordinates, throughout this paper we shall consider only the quadratic sector of the action\(^4\). In such cases the most general \(D\)-dimensional \((D \geq 3)\) action is given by

\[
S[g_{\mu\nu}] = \int d^Dx \sqrt{|g|} \left( \frac{2\sigma}{\kappa^2} R + \frac{1}{2\kappa^2} R F_1(\Box) R + \frac{1}{2\kappa^2} R_{\mu\nu} F_2(\Box) R^{\mu\nu} \right),
\]

(2.1)

where \(\kappa^2 = 32\pi G\) is the gravitational coupling constant and \(F_1(\Box)\) and \(F_2(\Box)\) are functions of the covariant d’Alembertian operator \((\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu)\). Moreover, we shall assume that the functions \(F_1(\Box)\) and \(F_2(\Box)\), the so-called form factors, have a finite polynomial representation.

At this point some comments regarding the action above are in order:

- At first glance we could ask about a possible contribution coming from the invariant term

\[
R_{\mu\nu\alpha\beta} F_3(\Box) R^{\mu\nu\alpha\beta}. \quad (2.2)
\]

Indeed, this is a legitimate term from the point of view of spacetime symmetries and it apparently contributes to the quadratic sector. However, looking closer this is not completely true. In fact, taking into account small fluctuations around the minkowskian background, \(g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\), we arrive at the following result

\[
R_{\mu\nu\alpha\beta} F_3(\Box) R^{\mu\nu\alpha\beta} = 4R_{\mu\nu} F_3(\Box) R^{\mu\nu} - RF_3(\Box) R + \partial \Omega + \mathcal{O}(h^3). \quad (2.3)
\]

Since we are mainly interested in the quadratic part of the action we can discard the contribution of \(R_{\mu\nu\alpha\beta} F_3(\Box) R^{\mu\nu\alpha\beta}\) by a simple redefinition of the functions \(F_1(\Box)\) and \(F_2(\Box)\). To be precise, if we start with

\[
S[g_{\mu\nu}] = \int d^Dx \sqrt{|g|} \left( \frac{2\sigma}{\kappa^2} R + \frac{1}{2\kappa^2} R F_1(\Box) R + \frac{1}{2\kappa^2} R_{\mu\nu} F_2(\Box) R^{\mu\nu} \right) + \frac{1}{2\kappa^2} R_{\mu\nu\alpha\beta} F_3(\Box) R^{\mu\nu\alpha\beta}
\]

(2.4)

and then consider equation (2.3), we may rewrite the last expression as follows

\[
S[g_{\mu\nu}] = \int d^Dx \sqrt{|g|} \left[ \frac{2\sigma}{\kappa^2} R + \frac{1}{2\kappa^2} R \left( F_1(\Box) - F_3(\Box) \right) R + \frac{1}{2\kappa^2} R_{\mu\nu} \left( F_2(\Box) + 4F_3(\Box) \right) R^{\mu\nu} \right] + \int d^Dx \partial \Omega, \quad (2.5)
\]

\(^4\)Naturally the discussion of renormalizability will take into account non-quadratic terms. However the specific form of these contributions will not be relevant for our purposes.

\(^5\)It is interesting to mention that \(\kappa^2\) is related with the \(D\)-dimensional Einstein constant, \(k_D\), by means of \(\kappa^2 = 4k_D\). In addition, for \(D \geq 4\) one can express \(k_D\) in term of the \(D\)-dimensional Newton constant, namely \(k_D = \left( \frac{D-2}{2(D-4)} \right)^{\frac{1}{2}} \kappa G_D\). For more details, see Appendix A of ref. [24].
where we have discarded out the contribution $\mathcal{O}(h^3)$. Taking into account that
\[ \int d^Dx \partial \Omega = 0 \]
with the proper boundary condition and using the following redefinitions
\[ F_1(\Box) - F_3(\Box) \mapsto F_1(\Box) \quad \text{and} \quad F_2(\Box) + 4F_3(\Box) \mapsto F_2(\Box) \] (2.6)
we recover our original action (2.1).

- As was mentioned before, there are an infinity number of compatible terms with the symmetries of the gravitational interaction. These terms may be obtained through all possible invariant combinations of the Riemann tensor, the Ricci tensor and the Ricci scalar, e.g. $R^3$, $R_{\mu\nu}R^{\mu\nu}R$, $R_{\mu\nu\alpha\beta}R^{\mu\nu}R^{\alpha\beta}$, $R^4$, $(R_{\mu\nu}R^{\mu\nu})^2$ and so on. However, it is not difficult to conclude that the aforementioned contributions brings only terms $\mathcal{O}(h^3)$, which are not relevant for the quadratic part of the action.

- Finally, we have introduced the constant parameter $\sigma$ in order to explore some interesting features related with the unitarity of 3-dimensional higher derivative models. In addition, without loss of generality we will take this parameter to be either $\sigma = +1$ or $\sigma = -1$.

According with equation (2.1) the classical space-time dynamics is governed by a set of fully non-linear differential equations, which has a very complicate nature. Instead of dealing with the exact equations of motion, we will restrict ourselves to linear perturbations around the minkowskian background. In this linearized regime\(^6\) we may derive the following equation of motion resulting from action (2.1)
\[ \left( 2\sigma - \frac{1}{2}F_2(\Box)\Box \right) \left( R_{\mu\nu}^{(\text{lin})} - \frac{1}{2}\eta_{\mu\nu}R^{(\text{lin})} \right) + \left( F_1(\Box) + \frac{1}{2}F_2(\Box) \right) \left( \eta_{\mu\nu}\Box - \partial_\mu \partial_\nu \right) R^{(\text{lin})} = -\frac{\kappa^2}{2} T_{\mu\nu}, \] (2.7)
where $R_{\mu\nu}^{(\text{lin})}$ and $R^{(\text{lin})}$ represent, respectively, the linearized version of the Ricci tensor and the curvature scalar, namely
\[ R_{\mu\nu}^{(\text{lin})} = \kappa \left( \Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial_\alpha h_{\nu\alpha} - \partial_\nu \partial_\alpha h_{\mu\alpha} \right), \] (2.8a)
and
\[ R^{(\text{lin})} = \kappa \left( \Box h - \partial_\alpha \partial_\beta h^{\alpha\beta} \right). \] (2.8b)

By taking the trace of (2.7) we arrive at the following result
\[ \left( F_1(\Box) + \frac{1}{2}F_2(\Box) \right) \Box R^{(\text{lin})} = \frac{\kappa^2}{2(D-1)} T - \frac{D - 2}{2D - 1} \left( 2\sigma - \frac{1}{2}F_2(\Box)\Box \right) R^{(\text{lin})}, \] (2.9)

\(^6\)In this regime the covariant d’Alembertian reduces to the flat operator $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$ and we use the Minkowski metric to lower and raise spacetime indexes.
where \( T = \eta^{\mu\nu}T_{\mu\nu} \). Using the last equation back into (2.7), we obtain
\[
\left( 2\sigma - \frac{1}{2} F_2(\square) \right) \left( F_{\mu\nu}^{(\text{lin})} - \frac{1}{2(D-1)} \eta_{\mu\nu} R^{(\text{lin})} \right) + \\
+ \left( F_1(\square) + \frac{1}{2} F_2(\square) \right) \partial_\mu \partial_\nu R^{(\text{lin})} = \frac{k^2}{2} \left( \frac{1}{D-1} \eta_{\mu\nu} T - T_{\mu\nu} \right).
\]

Now, bearing in mind equation (2.8a) and defining
\[
\Gamma_\mu = \left( \sigma - \frac{1}{4} F_2(\square) \right) \partial^\alpha \gamma_{\alpha\mu} - \frac{1}{2k} \left( F_1(\square) + \frac{1}{2} F_2(\square) \right) \partial_\mu R^{(\text{lin})},
\]
where \( \gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \), we may recast equation (2.10) in the following way
\[
\left( \sigma - \frac{1}{4} F_2(\square) \right) \left( - \frac{1}{2} \Box h_{\mu\nu} + \frac{1}{2k(D-1)} \eta_{\mu\nu} R^{(\text{lin})} \right) + \\
+ \frac{1}{2} \left( \partial_\nu \Gamma_\mu + \partial_\mu \Gamma_\nu \right) = \frac{k}{4} \left( T_{\mu\nu} - \frac{1}{D-1} \eta_{\mu\nu} T \right).
\]

Although the last result is just another way to write the original equation of motion (2.7) it will be very useful in what follows.

### 2.1 Generalized Teyssandier gauge condition

Since we are considering a theory constructed upon the assumption of invariance under general coordinate transformation, there is a gauge freedom that must be fixed. In the linearized regime this gauge freedom manifest itself in terms of the diffeomorphism transformation \( h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \) which makes the linearized action invariant. In the context of linearized General Relativity the usual gauge fixing choice is the de-Donder one, i.e. \( \partial^\mu \gamma_{\mu\nu} = 0 \). However, when we deal with higher derivative gravities there are extensions of the de-Donder gauge which are more convenient. For instance, in the context of fourth-derivative gravity (Stelle’s theory) the so-called Teyssandier gauge condition plays an interesting role. In what follows we shall see that there is a generalized Teyssandier gauge condition which is very useful in the context of general higher derivative gravities. Indeed, we may express a generalized version of the Teyssandier gauge condition as follows
\[
\Gamma_\mu = 0.
\]

The first point that should verify is that the condition above is achievable. In fact, letting \( \Gamma_\mu \neq 0 \), it is not difficult to see that a gauge transformation \( h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \) implies
\[
\Gamma'_\mu = \Gamma_\mu - \left( \sigma - \frac{1}{4} F_2(\square) \right) \Box \xi_\mu.
\]

Thus, we may achieve \( \Gamma'_\mu = 0 \) by choosing \( \xi_\mu \) such that
\[
\left( \sigma - \frac{1}{4} F_2(\square) \right) \Box \xi_\mu = \Gamma_\mu.
\]
Taking into consideration that the differential operator \( \sigma - \frac{1}{4} F_2(\Box) \Box \) is non-singular, we may conclude that the gauge condition (2.13) is achievable. Hence the equation of motion which governs the spacetime dynamics in the linearized regime may be recast as follows

\[
\left( \sigma - \frac{1}{4} F_2(\Box) \Box \right) \left( - \frac{1}{2} \Box h_{\mu\nu} + \frac{1}{2\kappa(D-1)} \eta_{\mu\nu} R^{(\text{lin})} \right) = \frac{\kappa}{4} \left( T_{\mu\nu} - \frac{1}{D-1} \eta_{\mu
u} T \right),
\]

(2.16)
along with the gauge condition

\[
\Gamma_{\mu} = \left( \sigma - \frac{1}{4} F_2(\Box) \Box \right) \partial^\alpha \gamma_{\alpha\mu} - \frac{1}{2\kappa} \left( F_1(\Box) + \frac{1}{2} F_2(\Box) \right) \partial_\mu R^{(\text{lin})} = 0.
\]

(2.17)

Now, let us explore some remarkable features arising from this gauge choice. First of all, let us define the following function of the d’Alembertian operator

\[
\mu_2^2(\Box) = - \frac{4\sigma}{F_2(\Box)}.
\]

(2.18)

By using the last equation in (2.16), we arrive at the following result

\[
\left( \Box + \mu_2^2(\Box) \right) \left( - \frac{1}{\mu_2^2(\Box)} \Box h_{\mu\nu} + \frac{1}{\kappa(D-1)} \eta_{\mu\nu} R^{(\text{lin})} \right) = \frac{\kappa}{2\sigma} \left( T_{\mu\nu} - \frac{1}{D-1} \eta_{\mu\nu} T \right).
\]

(2.19)

Observe that we may recast the last equation as follows

\[
\left( \Box + \mu_2^2(\Box) \right) \psi_{\mu\nu} = \frac{\kappa}{2\sigma} \left( T_{\mu\nu} - \frac{1}{D-1} \eta_{\mu\nu} T \right),
\]

(2.20)

where we have defined

\[
\psi_{\mu\nu} = \frac{1}{\mu_2^2(\Box)} \left( - \Box h_{\mu\nu} + \frac{1}{\kappa(D-1)} \eta_{\mu\nu} R^{(\text{lin})} \right).
\]

(2.21)

On the other hand, taking into account the trace of equation (2.16), we obtain

\[
\left( \sigma - \frac{1}{4} F_2(\Box) \Box \right) \left[ - \frac{1}{2} \Box h + \frac{D}{2\kappa(D-1)} R^{(\text{lin})} \right] = - \frac{\kappa}{4(D-1)} T,
\]

(2.22)

and taking the divergence of (2.17) we are lead to

\[
\partial^\mu \Gamma_{\mu} = \left( \sigma - \frac{1}{4} F_2(\Box) \Box \right) \partial^\mu \partial^\nu \gamma_{\mu\nu} - \frac{1}{2\kappa} \left( F_1(\Box) + \frac{1}{2} F_2(\Box) \right) \Box R^{(\text{lin})} = 0.
\]

(2.23)

Combining the last two equations, we get the following result

\[
R^{(\text{lin})} = \frac{\kappa^2}{2\sigma(D-2)} T - \frac{1}{\mu_0^2(\Box)} \Box R^{(\text{lin})},
\]

(2.24)

where we have defined

\[
\mu_0^2(\Box) = \frac{4\sigma(D-2)}{4(D-1) F_1(\Box) + DF_2(\Box)}.
\]

(2.25)
Using equation (2.24), along with (2.20), we arrive at
\[
\Box \left( h_{\mu \nu} - \psi_{\mu \nu} + \frac{\eta_{\mu \nu}}{\kappa (D - 1) \mu_0^2(\Box)} R^{(\text{lin})} \right) = -\frac{\kappa}{2\sigma} \left( T_{\mu \nu} - \frac{D - 1}{(D - 1)(D - 2)} \eta_{\mu \nu} T \right). 
\] (2.26)

Defining
\[
H_{\mu \nu} = h_{\mu \nu} - \psi_{\mu \nu} + \frac{1}{\kappa (D - 1) \mu_0^2(\Box)} \eta_{\mu \nu} R^{(\text{lin})},
\] (2.27)
we may recast equation (2.26) in the convenient form
\[
\Box H_{\mu \nu} = \frac{\kappa}{2\sigma} \left( 1 - \frac{2}{D - 2} \eta_{\mu \nu} T - T_{\mu \nu} \right). 
\] (2.28)

Finally, let us define
\[
\phi = \frac{1}{\kappa (D - 1) \mu_0^2(\Box)} R^{(\text{lin})}.
\] (2.29)

Using the last equation along with (2.24), we obtain the following result
\[
\Box + \mu_0^2(\Box) \frac{\kappa}{2\sigma} \left( T_{\mu \nu} - \frac{1}{D - 1} \eta_{\mu \nu} T \right).
\] (2.30)

Using the fact that
\[
\phi = \frac{1}{\kappa (D - 1) \mu_0^2(\Box)} R^{(\text{lin})},
\] (2.29)

we may express the metric fluctuation \( h_{\mu \nu} \) in the following way
\[
h_{\mu \nu} = H_{\mu \nu} + \psi_{\mu \nu} - \eta_{\mu \nu} \phi.
\] (2.31)

Hence the general solution of the linearized field equation (2.16) with the generalized Teyssandier gauge condition decouples as a sum of the respective solutions of (2.20), (2.28) and (2.30). In addition, the gauge condition (2.17) can also be separated in terms of independent gauge conditions on the fields \( H_{\mu \nu} \) and \( \psi_{\mu \nu} \). Indeed, after some algebraic manipulation one can derive the following gauge fixing condition
\[
\partial_{\mu} \gamma^{(H)}_{\mu \nu} = 0 \quad \text{and} \quad \partial_{\mu} \partial_{\nu} \psi_{\mu \nu} = \Box \psi,
\] (2.32)
where we have defined \( \gamma^{(H)}_{\mu \nu} = H_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} H \) and \( \psi = \eta^{\mu \nu} \psi_{\mu \nu} \).

Summing up, the general solution for the field equation (2.16) satisfying the gauge condition (2.17) can be cast as follows
\[
h_{\mu \nu} = H_{\mu \nu} + \psi_{\mu \nu} - \eta_{\mu \nu} \phi,
\] (2.33)
where \( H_{\mu \nu} \), \( \psi_{\mu \nu} \) and \( \phi \) satisfies
\[
\Box H_{\mu \nu} = \frac{\kappa}{2\sigma} \left( T_{\mu \nu} - \frac{1}{D - 2} \eta_{\mu \nu} T \right), \quad \text{with} \quad \partial_{\mu} \gamma^{(H)}_{\mu \nu} = 0,
\] (2.34a)
\[
\left( \Box + \mu_0^2(\Box) \right) \psi_{\mu \nu} = \frac{\kappa}{2\sigma} \left( T_{\mu \nu} - \frac{1}{D - 1} \eta_{\mu \nu} T \right), \quad \text{with} \quad \partial_{\mu} \partial_{\nu} \psi_{\mu \nu} = \Box \psi.
\] (2.34b)
and

\[
\left( \Box + \mu_0^2(\Box) \right) \phi = \frac{\kappa}{2\sigma(D-1)(D-2)} T. \tag{2.34c}
\]

The above set of equations is completely analogous to the result of the usual Teyssandier gauge condition in the context of fourth-derivative gravity. As one can see, the general solution decouples in terms of tensorial and scalar fields satisfying completely independent differential equations with independent gauge condition. In particular, the field \( H_{\mu\nu} \) satisfies exactly the same differential equation of the linear metric perturbations in the context of General Relativity with the de-Donder gauge condition.

### 2.2 Linearized field solution for a point-like source at rest

As direct application of the content developed in the previous section we shall compute the linearized field solution associated with a point-like source at rest. In such a case the energy-momentum tensor associated with a point-like body with mass \( M \) located at \( r = 0 \) is given by

\[
T_{\mu\nu} = M \eta_{\mu0} \eta_{\nu0} \delta^{(D-1)}(r). \tag{2.35}
\]

Taking into account the above expression for the energy-momentum tensor and considering only static solution, we are led to the following set of equations

\[
- \nabla^2 H_{\mu\nu} = \frac{\kappa M}{2\sigma} \left( \frac{1}{D-2} \eta_{\mu0} - \eta_{\mu0} \right) \delta^{(D-1)}(r), \tag{2.36a}
\]

\[
\left( - \nabla^2 + \mu_2^2(-\nabla^2) \right) \psi_{\mu\nu} = \frac{\kappa M}{2\sigma} \left( \eta_{\mu0} \eta_{\nu0} - \frac{1}{D-1} \eta_{\mu\nu} \right) \delta^{(D-1)}(r), \tag{2.36b}
\]

and

\[
\left( - \nabla^2 + \mu_0^2(-\nabla^2) \right) \phi = \frac{\kappa M}{2\sigma(D-1)(D-2)} \delta^{(D-1)}(r). \tag{2.36c}
\]

Considering the Fourier representations given by

\[
H_{\mu\nu}(r) = \frac{1}{(2\pi)^{D-1}} \int d^{D-1} k \bar{H}_{\mu\nu}(k) e^{i k \cdot r}, \tag{2.37a}
\]

\[
\psi_{\mu\nu}(r) = \frac{1}{(2\pi)^{D-1}} \int d^{D-1} k \bar{\psi}_{\mu\nu}(k) e^{i k \cdot r}, \tag{2.37b}
\]

and

\[
\phi(r) = \frac{1}{(2\pi)^{D-1}} \int d^{D-1} k \bar{\phi}(k) e^{i k \cdot r}, \tag{2.37c}
\]

and than using the above set of equations in (2.34a), (2.34b) and (2.34c), we get the following results

\[
\bar{H}_{\mu\nu}(k) = \frac{\kappa M}{2\sigma k^2} \left( \frac{1}{D-2} \eta_{\mu\nu} - \eta_{\mu0} \eta_{\nu0} \right), \tag{2.38a}
\]

\[
\bar{\psi}_{\mu\nu}(k) = \frac{\kappa M}{2\sigma} \left( \eta_{\mu0} \eta_{\nu0} - \frac{1}{D-1} \eta_{\mu\nu} \right), \tag{2.38b}
\]

\[
\bar{\phi}(k) = \frac{\kappa M}{2\sigma(D-1)(D-2)} \delta(k^2), \tag{2.38c}
\]
\[ \hat{\psi}_{\mu\nu}(k) = \frac{\kappa M}{2\sigma(k^2 + \mu^2_2(k^2))} \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right). \] (2.38b)

and

\[ \hat{\phi}(k) = \frac{\kappa M}{2\sigma(D - 1)(D - 2)} \frac{1}{k^2 + \mu^2_0(k^2)}. \] (2.38c)

Therefore the general solution for a point-like mass at rest is given by (in the Fourier space)

\[ \hat{h}_{\mu\nu}(k) = \frac{\kappa M}{2\sigma k^2} \left( \frac{1}{D - 2}\eta_{\mu\nu} - \eta_{\mu0}\eta_{\nu0} \right) + \frac{\kappa M}{2\sigma(k^2 + \mu^2_2(k^2))} \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right) + \frac{\kappa M}{2\sigma(D - 1)(D - 2)} \frac{1}{k^2 + \mu^2_0(k^2)}\eta_{\mu\nu}. \] (2.39)

As a consequence, in the usual configuration space, we get the following solutions

\[ h_{\mu\nu}(r) = \frac{\kappa M}{2\sigma(2\pi)^{D-3}} \int_0^{\Gamma(2) \left( \frac{D - 3}{2} \right) \left( \frac{1}{D - 2}\eta_{\mu\nu} - \eta_{\mu0}\eta_{\nu0} \right) + \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right) \int_0^\infty dx \frac{x^{D-1}}{x^2 + r^2\mu^2_2(x^2/r^2)} J_{D-3}(x) + \int_0^\infty dx \frac{x^{D-1}}{x^2 + r^2\mu^2_2(x^2/r^2)} J_{D-3}(x) \left\{ \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right) \int_0^\infty dx \frac{x}{x^2 + r^2\mu^2_2(x^2/r^2)} J_0(x) \right\}, \] (2.40a)

and

\[ h_{\mu\nu}(r) = \frac{\kappa M}{4\pi\sigma} \left\{ \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right) \ln \left( \frac{r}{r_0} \right) + \left( \eta_{\mu0}\eta_{\nu0} - \frac{1}{D - 1}\eta_{\mu\nu} \right) \int_0^\infty dx \frac{x}{x^2 + r^2\mu^2_2(x^2/r^2)} J_0(x) + \frac{1}{2}\eta_{\mu\nu} \int_0^\infty dx \frac{x}{x^2 + r^2\mu^2_2(x^2/r^2)} J_0(x) \right\}, \] (2.40b)

where we have defined \( x \equiv |k|, \) \( r_0 \) denotes an infrared regulator. We also have used the following results

\[ \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{ikr}}{k^2} = \frac{1}{(2\pi)^{D/2}} \frac{2^{D-5}}{r^{D-3}} \Gamma \left( \frac{D - 3}{2} \right), \quad D \geq 4, \] (2.41a)

\[ \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\cdot x}}{k^2} = -\frac{1}{2\pi} \ln \left( \frac{r}{r_0} \right), \] (2.41b)

and

\[ \int \frac{d^{D-1}k}{(2\pi)^{D-1}} f(|k|) e^{ik\cdot r} = \frac{1}{(2\pi)^{D/2}} \frac{1}{r^{D/2}} \int_0^\infty x^{D-1} f(x) J_{D-3}(xr) dx, \quad D \geq 3, \] (2.42)

for an arbitrary function \( f(|k|) \). Evidently we cannot go perform the remaining integrations in (2.40a) without an specific choice of \( \mu^2_2(k^2) \) and \( \mu^2_0(k^2) \).
3 Free propagator, particle spectra and tree level unitarity

One of the most intriguing problems with the formulation of quantum gravity as being the theory of fluctuations around the Minkowski spacetime is the incompatibility between unitarity and renormalizability. In what follows we shall investigate both unitarity and renormalizability for the general higher derivative theory under consideration. Let us start with the tree-level unitarity. As usual, the study of the tree-level unitarity can be done in terms of the pole structure of the saturated propagator. After some standard calculations the free propagator (in momentum space) associated with the general higher derivative gravity (2.1) may be cast as follows

\[ D_{\mu\nu,\alpha\beta}(k) = \frac{1}{\sigma k^2 Q_2(k^2)} P_{\mu\nu,\alpha\beta}^{(2)} \frac{1}{(D-2)} \frac{1}{\sigma k^2 Q_0(k^2)} P_{\mu\nu,\alpha\beta}^{(0-s)} + \frac{2\lambda}{k^2} P_{\mu\nu,\alpha\beta}^{(1)} + \left( \frac{4\lambda}{k^2} - \frac{(D-1)}{\sigma(D-2)k^2 Q_0(k^2)} \right) P_{\mu\nu,\alpha\beta}^{(0-w)} - \frac{\sqrt{D-1}}{\sigma(D-2)k^2 Q_0(k^2)} \left( P_{\mu\nu,\alpha\beta}^{(0-sw)} + P_{\mu\nu,\alpha\beta}^{(0-ws)} \right), \] (3.1)

where \( \{ P^{(2)}, \ldots, P^{(0-ws)} \} \) denotes the set of Barnes-Rivers operators, \( \lambda \) is the gauge fixing parameter\(^7\) and we have defined

\[ Q_2(k^2) = 1 + \frac{1}{4\sigma} k^2 F_2(-k^2), \] (3.2)

and

\[ Q_0(k^2) = 1 - \frac{k^2}{\sigma(D-2)} \left( (D-1) F_1(-k^2) + \frac{D}{4} F_2(-k^2) \right). \] (3.3)

In order to investigate the tree-level unitarity we compute the so-called saturated propagator, namely

\[ SP(k) = i T^\ast_{\mu\nu}(k) D_{\mu\nu,\alpha\beta}(k) T_{\alpha\beta}(k), \] (3.4)

where \( T^\mu_{\nu} \) stands for an external conserved current. Using equation (3.1), we arrive at the following result

\[ SP(k) = \frac{i}{\sigma k^2 Q_2(k^2)} \left( T^\ast_{\mu\nu} T^{\mu\nu} - \frac{1}{D-1} |T|^2 \right) - \frac{i}{\sigma k^2 Q_0(k^2)} \left( (D-1) F_1(-k^2) + \frac{D}{4} F_2(-k^2) \right). \] (3.5)

At this point, in order to go further, we need to specify the form factors \( F_1(\Box) \) and \( F_2(\Box) \). As previously stated, we shall consider form factors given by polynomial functions of the d’Alembertian operator, namely

\[ F_1(\Box) = \sum_{n=0}^{p} \alpha_n (-\Box)^n \quad \text{and} \quad F_2(\Box) = \sum_{n=0}^{q} \beta_n (-\Box)^n, \] (3.6)

where \( \alpha_n \) and \( \beta_n \) are real coefficients with canonical mass dimension \( M^{-2(n+1)} \). Recasting the above functions in the momentum space and then plugging the result in (3.2) and (3.3), we arrive at the following expressions

\[ Q_2(k^2) = 1 + \frac{1}{4\sigma} \sum_{n=0}^{q} \beta_n k^{2n+2}, \] (3.7a)

\(^7\)We have considered the de-Donder gauge condition when we compute the free propagator.
and

\[ Q_0(k^2) = 1 - \frac{1}{\sigma(D - 2)} \left( (D - 1) \sum_{n=0}^{p} \alpha_n k^{2n+2} - \frac{D}{4} \sum_{n=0}^{q} \beta_n k^{2n+2} \right). \]  (3.7b)

Let

\[ \{m_{(2),1}^2, m_{(2),2}^2, \ldots, m_{(2),\tilde{N}+1}^2\} \quad \text{and} \quad \{m_{(0),1}^2, m_{(0),2}^2, \ldots, m_{(0),\tilde{N}+1}^2\}, \]  (3.8)

be, respectively, the set of real roots of the polynomial functions \( Q_2(k^2) \) and \( Q_0(k^2) \), while

\[ \{\eta_{(2),1}^2, \eta_{(2),1}^*, \ldots, \eta_{(2),r}^2, \eta_{(2),r}^*\} \quad \text{and} \quad \{\eta_{(0),1}^2, \eta_{(0),1}^*, \ldots, \eta_{(0),s}^2, \eta_{(0),s}^*\}, \]  (3.9)

denotes, respectively, the sets of complex roots of the polynomials \( Q_2(k^2) \) and \( Q_0(k^2) \). Given how the total number of real and complex roots correspond to the polynomial’s degree, we arrive at the following constraints

\[ q = \tilde{q} + 2r \quad \text{and} \quad \max\{p,q\} = \tilde{N} + 2s \equiv N. \]  (3.10)

By means of the fundamental theorem of algebra we may recast \( Q_2(k^2) \) and \( Q_0(k^2) \) as follows

\[ Q_2(k^2) = \prod_{i=1}^{\tilde{q}+1} \left( \frac{m_{(2),i}^2 - k^2}{m_{(2),i}^2} \right) \prod_{i=1}^{r} \left( \frac{(\eta_{(2),i}^2 - k^2)(\eta_{(2),i}^* - k^2)}{\eta_{(2),i}^* \eta_{(2),i}^2} \right), \]  (3.11a)

and

\[ Q_0(k^2) = \prod_{i=1}^{\tilde{N}+1} \left( \frac{m_{(0),i}^2 - k^2}{m_{(0),i}^2} \right) \prod_{i=1}^{s} \left( \frac{(\eta_{(0),i}^2 - k^2)(\eta_{(0),i}^* - k^2)}{\eta_{(0),i}^* \eta_{(0),i}^2} \right). \]  (3.11b)

As a consequence, using equation (3.1), along with (3.11b) and (3.11a), we arrive at the following result\(^8\)

\[ SP(k) = \frac{i}{\sigma} k^2 \left( T_{\mu\nu}^{*} T_{\mu\nu} - \frac{1}{D - 2} |T|^2 \right) + \]

\[ + \frac{|T|^2}{\sigma(D - 1)(D - 2)} \left\{ \sum_{i=1}^{\tilde{q}+1} \frac{i \xi_{(0),i}}{k^2 - m_{(0),i}^2} + \sum_{i=1}^{s} \frac{i \xi_{(2),i}}{k^2 - \eta_{(2),i}^2} + \sum_{i=1}^{s} \frac{i \xi_{(0),i}^*}{k^2 - \eta_{(0),i}^2} \right\} + \]

\[ - \frac{1}{\sigma} \left( T_{\mu\nu}^{*} T_{\mu\nu} - \frac{1}{D - 1} |T|^2 \right) \left\{ \sum_{i=1}^{\tilde{q}+1} \frac{i \zeta_{(0),i}}{k^2 - m_{(2),i}^2} + \sum_{i=1}^{r} \frac{i \zeta_{(2),i}}{k^2 - \eta_{(2),i}^2} + \sum_{i=1}^{r} \frac{i \zeta_{(0),i}^*}{k^2 - \eta_{(0),i}^2} \right\}. \]  (3.12)

where we have defined

\[ \zeta_{(0),i} = \prod_{j=1}^{s} \frac{\eta_{(0),j}^2}{\eta_{(0),i}^* - \eta_{(0),i}^2} \prod_{j=1}^{s} \frac{\eta_{(0),j}^*}{\eta_{(0),j}^2 - \eta_{(0),i}^2} \prod_{j=1}^{\tilde{N}+1} \frac{m_{(0),j}^2}{m_{(2),j}^2 - \eta_{(0),i}^2}. \]  (3.13a)

\(^8\)For the sake of simplicity we restrict ourselves to the case where all roots of \( Q_2(k^2) \) and \( Q_0(k^2) \) are simple.
\[ \xi_{(0),i} = \prod_{j=1}^{s} \frac{\eta_{(0),j}^2 - m_{(0),j}^2}{\eta_{(0),j}^2 - m_{(0),j}^2,} \prod_{j=1}^{s} \frac{\eta_{(0),j}^2 - m_{(0),j}^2}{\eta_{(0),j}^2 - m_{(0),j}^2,} \prod_{j=1}^{s} \frac{m_{(0),j}^2}{m_{(0),j}^2 - m_{(0),j}^2}, \] (3.13b)

\[ \zeta_{(2),i} = \prod_{j=1}^{r} \frac{\eta_{(2),j}^2 - \eta_{(2),i}^2}{\eta_{(2),j}^2 - \eta_{(2),i}^2,} \prod_{j=1}^{r} \frac{\eta_{(2),j}^2 - \eta_{(2),i}^2}{\eta_{(2),j}^2 - \eta_{(2),i}^2,} \prod_{j=1}^{r} \frac{m_{(2),j}^2}{m_{(2),j}^2 - m_{(2),i}^2}, \] (3.13c)

\[ \tilde{\zeta}_{(2),i} = \prod_{j=1}^{r} \frac{\eta_{(2),j}^2 - \eta_{(2),i}^2}{\eta_{(2),j}^2 - \eta_{(2),i}^2,} \prod_{j=1}^{r} \frac{\eta_{(2),j}^2 - \eta_{(2),i}^2}{\eta_{(2),j}^2 - \eta_{(2),i}^2,} \prod_{j=1}^{r} \frac{m_{(2),j}^2}{m_{(2),j}^2 - m_{(2),i}^2}. \] (3.13d)

As usual, in order to investigate whether the theory exhibits ghosts on its spectrum, we compute the imaginary part of residues of the saturated propagator. In fact, using equation (3.12) we arrive at the following results

\[ \text{Im}\left[ \text{Res} SP(k^2 = 0) \right] = \sigma^{-1} \left( T_{\mu\nu} T^{\mu\nu} - \frac{1}{D-2} |T|^2 \right) \bigg|_{k^2 = 0}, \] (3.14a)

\[ \text{Im}\left[ \text{Res} SP(k^2 = m_{(2),i}^2) \right] = -\sigma^{-1} \xi_{(2),i} \left( T_{\mu\nu} T^{\mu\nu} - \frac{1}{D-1} |T|^2 \right) \bigg|_{k^2 = m_{(2),i}^2}, \] (3.14b)

\[ \text{Im}\left[ \text{Res} SP(k^2 = \eta_{(2),i}^2) \right] = -\sigma^{-1} \text{Im}(\zeta_{(2),i}) \left( T_{\mu\nu} T^{\mu\nu} - \frac{1}{D-1} |T|^2 \right) \bigg|_{k^2 = \eta_{(2),i}^2}, \] (3.14c)

\[ \text{Im}\left[ \text{Res} SP(k^2 = \eta_{(2),i}^2) \right] = -\sigma^{-1} \text{Im}(\zeta_{(2),i}) \left( T_{\mu\nu} T^{\mu\nu} - \frac{1}{D-1} |T|^2 \right) \bigg|_{k^2 = \eta_{(2),i}^2}, \] (3.14d)

\[ \text{Im}\left[ \text{Res} SP(k^2 = m_{(0),i}^2) \right] = \frac{\sigma^{-1} \xi_{(0),i}}{(D-1)(D-2)} |T|^2 \bigg|_{k^2 = m_{(0),i}^2}, \] (3.14e)

\[ \text{Im}\left[ \text{Res} SP(k^2 = \eta_{(0),i}^2) \right] = \frac{\sigma^{-1} \text{Im}(\zeta_{(0),i})}{(D-1)(D-2)} |T|^2 \bigg|_{k^2 = \eta_{(0),i}^2}, \] (3.14f)

\[ \text{Im}\left[ \text{Res} SP(k^2 = \eta_{(0),i}^2) \right] = \frac{\sigma^{-1} \text{Im}(\zeta_{(0),i})}{(D-1)(D-2)} |T|^2 \bigg|_{k^2 = \eta_{(0),i}^2}. \] (3.14g)

Assuming that the real masses obey the following hierarchy\(^9\)

\[ m_{(2),1}^2 < m_{(2),2}^2 < \cdots < m_{(2),\tilde{q}+1}^2 \quad \text{and} \quad m_{(0),1}^2 < m_{(0),2}^2 < \cdots < m_{(0),\tilde{q}+1}^2, \] (3.15)

we arrive at the following conclusion

\[
\begin{cases}
\xi_{(2),i} > 0, & \text{if } i \text{ is odd}, \\
\xi_{(2),i} < 0, & \text{if } i \text{ is even},
\end{cases}
\quad \text{and} \quad
\begin{cases}
\xi_{(0),i} > 0, & \text{if } i \text{ is odd}, \\
\xi_{(0),i} < 0, & \text{if } i \text{ is even}.
\end{cases}
\]

(3.16)

From now one let us separate our investigation in two cases of interest: \( D \geq 4 \) and \( D = 3 \).

\(^9\)It is important to emphasize that this ordering can always be achieved by relabeling the masses.
3.1 Case I - $D \geq 4$:

We first consider the case where space-time dimensionality obeys $D \geq 4$. Taking into account the following set of inequalities (valid for $D \geq 4$)

$$
\left( T_{\mu\nu}^* T^{\mu\nu} - \frac{1}{D-2} |T|^2 \right)_{k^2=0} > 0,
$$

and

$$
\left( T_{\mu\nu}^* T^{\mu\nu} - \frac{1}{D-1} |T|^2 \right)_{k^2=\mu^2} > 0, \quad \mu^2 = m^2_{(2),i}, \eta^2_{(2),i}, \eta^*_{(2),i} (3.17a)
$$

we obtain the following results

$$
\text{Im} \left[ \text{Res } SP(k^2 = 0) \right] \begin{cases} 
> 0, & \text{for } \sigma = +1 \\
< 0, & \text{for } \sigma = -1
\end{cases},
$$

$$
\text{Im} \left[ \text{Res } SP(k^2 = m^2_{(2),i}) \right] \begin{cases} 
< 0, & \text{for } \sigma = +1 \\
> 0, & \text{for } \sigma = -1
\end{cases}, \quad \text{if } i = \text{odd}, (3.18b)
$$

$$
\text{Im} \left[ \text{Res } SP(k^2 = m^2_{(2),i}) \right] \begin{cases} 
> 0, & \text{for } \sigma = +1 \\
< 0, & \text{for } \sigma = -1
\end{cases}, \quad \text{if } i = \text{even}, (3.18c)
$$

$$
\text{Im} \left[ \text{Res } SP(k^2 = m^2_{(0),i}) \right] \begin{cases} 
> 0, & \text{for } \sigma = +1 \\
< 0, & \text{for } \sigma = -1
\end{cases}, \quad \text{if } i = \text{odd}, (3.18d)
$$

$$
\text{Im} \left[ \text{Res } SP(k^2 = m^2_{(0),i}) \right] \begin{cases} 
< 0, & \text{for } \sigma = +1 \\
> 0, & \text{for } \sigma = -1
\end{cases}, \quad \text{if } i = \text{even}. (3.18e)
$$

In addition, considering $\text{Im}(\zeta_{(I),i}) = -\text{Im}(\zeta^*_{(I),i})$ (where $I = 0, 2$), we may conclude the following:

$$
\text{if } \text{Im} \left[ \text{Res } SP(k^2 = \eta^2_{(I),i}) \right] > 0 \implies \text{Im} \left[ \text{Res } SP(k^2 = \eta^*_{(I),i}) \right] < 0, (3.19a)
$$

and

$$
\text{if } \text{Im} \left[ \text{Res } SP(k^2 = \eta^2_{(I),i}) \right] < 0 \implies \text{Im} \left[ \text{Res } SP(k^2 = \eta^*_{(I),i}) \right] > 0. (3.19b)
$$

Taking into account the set of inequalities above we may extract some conclusions:

- Although our former discussion apply both for $\sigma = +1$ and $\sigma = -1$, relation (3.18a) imply $\sigma = +1$ since we expect a physical massless spin-2 particle in the spectrum of the theory corresponding to the usual graviton.
The usual drama of higher derivative theories persists as long as real poles are present. In this case, the alternating signs of the parameters $\xi^{(2),i}$ and $\xi^{(0),i}$ ensure the existence of at least one ghost-like particle. As usual the existence of such particles in the spectrum may lead to a non-unitary $S$-matrix in the context of perturbation theory.

Since the complex poles always appear in pairs, one of them being the complex conjugate of the other, we may conclude that for each complex “physical” particle (not a ghost-like state) there will be a complex ghost corresponding to the complex conjugated of the former. Although complex ghosts may appear in the particle spectrum of the theory they may not cause problems with the unitarity of the $S$-matrix. In fact, higher derivative gravities with complex ghosts have been recently studied by Modesto and Shapiro [10, 11] and there is hope that this kind of theories may be formulated as unitary in the Lee-Wick sense.

The situation now is clear: as long as higher derivatives are implemented by means of polynomial functions of the d’Alembertian operator the presence of at least one massive ghost-like particle appears to be unavoidable. In fact, there are only three cases of higher derivative gravities, constructed with polynomial functions like (3.6), where the particle spectrum do not exhibits a massive ghost like state. The first one occurs with the choice $F_1(\Box) = \alpha_0$ and $F_2(\Box) = 0$ - in this case the propagator has only two poles, $k^2 = 0$ and $k^2 = m^2_{(0),1}$ and we conclude that both poles corresponds to physical particles, i.e. the theory is ghost-free (at least in tree-level analysis). The other two cases of ghost-free higher derivative theories occur in $D = 3$ and will be considered in the next section.

3.2 Case II - $D = 3$:

Now we consider the 3-dimensional case. In fact, when $D = 3$ the energy-momentum tensor satisfies the following relations

\[ \left( T^\mu_\nu T^{\mu\nu} - |T|^2 \right) \bigg|_{k^2 = 0} = 0, \]  

and

\[ \left( T^\mu_\nu T^{\mu\nu} - \frac{1}{2} |T|^2 \right) \bigg|_{k^2 = \mu^2} > 0, \quad \mu^2 = m^2_{(2),i}, \eta^2_{(2),i}, \eta^*_{(2),i}. \]  

First of all, equation (3.20a) implies the following result

\[ \text{Im} \left[ \text{Res } SP(k^2 = 0) \right] = 0. \]  

The last equation tell us that there is no massless propagating mode in the 3-dimensional theory. Indeed, this result was already expected since it is well known that 3-dimensional pure Einstein-Hilbert do not propagate in the vacuum. As a consequence of the last equation we cannot fix the parameter $\sigma$ as we have done in the $D \geq 4$ case. Regarding the massive poles (real and complex) the inequality (3.20b) implies that those relations of the previous case, namely (3.18b)-(3.19b), remains valid and thus those comments about massive ghosts are still valid in the 3-dimensional case.
As was mentioned previously in 3-dimensional space-time there are two ghost-free higher derivative theories. The first one consists in the choice of coefficients which makes $F_1(\Box) = \alpha_0$ and $F_2(\Box) = 0$. In this case the only particle in the spectrum corresponds to the pole $k^2 = m_{(0),1}^2$ and the theory is ghost-free, since it can be verified that the residue at this pole is positive.

The other one is the so-called New Massive Gravity (NMG) [25]. This theory is characterized by the choice $F_1(\Box) = \alpha_0$ and $F_2(\Box) = \beta_0$ with the additional constraint

$$8\alpha_0 + 3\beta_0 = 0.$$ \hspace{1cm} (3.22)

In order to made the theory ghost-free we have to consider the “wrong” sign in the Einstein-Hilbert sector, i.e. $\sigma = -1$, so that the NMG action becomes

$$S_{NMG} = \int d^3 x \sqrt{|g|} \left( -\frac{2}{\kappa^2} R + \frac{\beta_0}{2\kappa^2} \left( R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right).$$ \hspace{1cm} (3.23)

In the present case the free propagator exhibits two simple poles at $k^2 = 0$ and $k^2 = m_{(2),1}^2 \equiv 4/\beta_0$. Differently from what happens in dimensions other than 3, the massless pole do not propagate as a physical particle, since $\text{Res} \, SP(\kappa^2 = 0)|_{D=3} = 0$. Usually the second pole $k^2 = m_{(2),1}^2$ would propagate as a massive ghost, but here the “wrong” sign of the Einstein-Hilbert terms leads to a positive valued residue:

$$\text{Im} \left[ \text{Res} \, SP(k^2 = m_{(2),1}^2) \right] \bigg|_{NMG} = \left. \left( T^{\ast}_{\mu\nu} T^{\mu\nu} - \frac{1}{2} |T|^2 \right) \right|_{k^2 = m_{(2),1}^2} > 0.$$ \hspace{1cm} (3.24)

Consequently the spectrum of the NMG theory has only a single massive physical particle with spin-2 and there is no ghost-like state in the tree level propagator.

### 4 UV Properties and Renormalizability

Undoubtedly, we may attribute most of the interest in dealing with higher derivative theories of quantum gravity to their good UV properties. In fact, as it was mentioned in the introduction, Stelle has proved that fourth derivative theory of quantum gravity is renormalizable (in 4-dimension) to all orders in perturbation theory [6]. Furthermore, theories containing sixth or higher derivatives may be formulated as super-renormalizable [7–9].

In this section we shall investigate the UV properties of the general class of $D$-dimensional higher derivative theories described by the action (2.1). First of all we notice that the propagators and vertices associated with this general class of higher derivative theories have the following UV behavior\(^{10}\)

$$\text{Propagators} \sim \frac{1}{k^{2q+4}} \quad \text{and} \quad \text{Vertices} \sim k^{2N+4}. \hspace{1cm} (4.1)$$

\(^{10}\)Recall that the parameters $q$ and $N$ were introduced in section (3) and they are related with the number of derivatives contained in the action.
As a consequence, for an arbitrary Feynman diagram we found the following UV behavior for the loop integrations

\[ I^{\text{Loops}}_{\text{UV}} \sim \int (d^D k)^L \frac{(k^{2N+4})^V}{(k^{2q+4})^I}, \]  

(4.2)

where \( I \) is the number of internal lines, \( L \) stands for the number of loops and \( V \) denotes the total number of vertices. The superficial degree of divergence associated with the above integral may be cast as follows

\[ \delta = DL + (2N + 4)V - (2q + 4)I. \]  

(4.3)

Now, bearing in mind the following topological relations

\[ L - 1 = I - V, \]  

(4.4a)

and

\[ 2I + E = \sum_{n=3}^{\infty} nV_n, \]  

(4.4b)

where \( E \) is the number of external lines and \( V_n \) denotes the number of vertices connecting \( n \)-lines, we may recast the superficial degree of divergence in the following way

\[ \delta = D - \sum_{n=3}^{\infty} \left[ \frac{n-2}{2}(2q + 4 - D) - 2\lambda \right] V_n + \left( \frac{2q + 4 - D}{2} \right) E, \]  

(4.5)

where the parameter \( \lambda \) is defined to be

\[ \lambda = \begin{cases} p - q, & \text{if } q < p, \\ 0, & \text{if } q \geq p. \end{cases} \]  

(4.6)

As is well known, the power-counting criteria for renormalizability is that the superficial degree of divergence should not depend on the number of vertices, therefore, we arrive at the following conditions for power-counting renormalizability

\[ 2q + 4 - D = 0 \quad \text{and} \quad \lambda = 0. \]  

(4.7)

The first condition relates the number of derivatives in the Ricci squared sector with the dimension of spacetime. As one see, in odd-dimensions we cannot have a power counting renormalizable theory, otherwise it should be necessary to have fractional powers of the d’Alembertian operator. The second condition implies \( q \geq p \), essentially, this inequality tell us that the number of derivatives in the scalar curvature squared sector should not be greater than the number of derivatives in the Ricci squared sector.

Furthermore, the theory can also be formulated as power counting super-renormalizable. In fact, the condition for super-renormalizability is that the superficial degree of divergence decreases with the number of vertices. Taking into account this condition along with (4.5), then, the strongest inequality that we get from this is the following

\[ 2q + 4 - D \geq 4\lambda. \]  

(4.8)
As in the previous case, this above inequality relates the number of derivatives in the Ricci squared sector with the dimensionality of spacetime. In addition, if \( p > q \) the above inequality determines a lower bound on the parameter \( q \) in terms of the spacetime dimension and the number of derivatives in the scalar curvature squared sector.

It is important to stress out that the above discussion relies in the assumption that the Lagrangian parameters, \( \alpha \)'s and \( \beta \)'s, are unrelated. In fact, if there is some relation between the Lagrangian parameters, the power counting could be slightly changed. To illustrate this, let us take as an example the New Massive Gravity case. At a first look, applying the power counting condition above developed, the NMG appears to be classified as super-renormalizable. If this was true, NMG would be an example of ghost-free and super-renormalizable, however, this is not the case. Indeed, in this case the constraint \( 3\alpha_0 + 8\beta_0 \) should be considered in such a way that the UV behavior of the tree-level propagator is given by \( \sim 1/k^2 \). Taking this into account, the correct power counting for the NMG is given by

\[
\delta_{\text{NMG}} = 3 - \frac{1}{2} E + \frac{1}{2} \sum_{n=3}^{\infty} (n + 2)V_n. \tag{4.9}
\]

As one can see, the superficial degree of divergence increases with the number of vertices and, as a consequence, the theory is power counting non-renormalizable. Furthermore, it is important to emphasize that a complete proof for the non-renormalizability of the NMG was performed in reference \[26\].

Since we have discussed the renormalizability of the NMG case, which is a tree-level ghost free model, it is worthwhile to mention the other example of ghost-free higher derivative theory discussed in the previous section. Let us recall that it is defined by the choice \( F_1(\Box) = \alpha_0 \) and \( F_2(\Box) = 0 \). In fact, this choice corresponds to a particular case of the so-called \( f(R) \)-theories and it can be readily verified from our power counting that this theory is classified as non-renormalizable.

5 Non-relativistic potential energy

Since we are motivated by the conjecture that relates renormalizability with the cancellation of Newtonian singularities, let us compute the interparticle potential energy for the general higher derivative gravity theory given by (2.1) and investigate its behavior for small distances. For this purpose we shall employ the prescription presented by Accioly et. al., which is based on the path integral formulation of quantum field theory. This prescription states that, in order to determine the potential energy of gravitational models, we only need to compute

\[
E_D(r) = \kappa D \frac{M_1 M_2}{(2\pi)^{D-1}} \int d^{D-1}k e^{ikr} P_{00,00}(k)|_{k_0=0}, \tag{5.1}
\]

where \( P_{00,00} \) is the \( \mu = \nu = \alpha = \beta = 0 \) component of the modified propagator \( P_{\mu\nu,\alpha\beta} = D_{\mu\nu,\alpha\beta} - D_{\mu\nu,\alpha\beta}^\perp \), with \( D_{\mu\nu,\alpha\beta}^\perp \) being the contribution to the propagator that is orthogonal.
to the energy-momentum tensor, while

$$\kappa_D = \left( \frac{D - 2}{D - 3} \right) G_D \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$$  \hspace{1cm} (5.2)$$

is the $D$-dimensional Einstein constant for $D > 3$, where $G_D$ is the Newton constant in $D$-dimensions. For $D = 3$ general relativity does not have a newtonian limit, so we cannot relate $\kappa_3$ to $G_3$.

Now, since we have already determined the propagator in (3.1), it is straightforward to see that

$$P_{00,00}(k) = \frac{D - 2}{D - 1} \frac{1}{\kappa^2 Q_2(k^2)^{\frac{D}{2}}} - \frac{1}{(D - 1)(D - 2)} \frac{1}{\sigma k^2 Q_0(k^2)^{\frac{D}{2}}}.$$  \hspace{1cm} (5.3)$$

In what follows we shall consider again that $F_1(\Box)$ and $F_2(\Box)$ are polynomial functions of the d’alembertian. Similarly to the previous section, by simple algebraic manipulations we obtain

$$P_{00,00}(k)_{k_0=0} = -\frac{1}{\sigma} \left( \frac{D - 3}{D - 2} \right) \frac{1}{k^2} - \frac{1}{\sigma} \left( \frac{D - 2}{D - 1} \right) \sum_{i=1}^{q+1} q+1 \prod_{j \neq i}^{\mu_2(2),j} - \frac{1}{\mu_2(2),i} k^2 + \frac{1}{\mu_2(2),i} +$$

$$+ \frac{1}{\sigma} \left( \frac{D - 1}{D - 2} \right) \sum_{i=1}^{N+1} N+1 \prod_{j \neq i}^{\mu_2(0),j} - \frac{1}{\mu_2(0),i} k^2 + \frac{1}{\mu_2(0),i},$$  \hspace{1cm} (5.4)$$

where we have defined

$$\mu(2),i = \begin{cases} m(2),i , i = 1, \cdots , \hat{q} + 1, \\ \eta(2),i , i = \hat{q} + 2, \cdots , \hat{q} + r + 1, \\ \eta^*(2),i , i = \hat{q} + r + 2, \cdots , \hat{q} + 2r + 1, \end{cases}$$  \hspace{1cm} (5.5)$$

and

$$\mu(0),i = \begin{cases} m(0),i , i = 1, \cdots , \hat{N} + 1, \\ \eta(0),i , i = \hat{N} + 2, \cdots , \hat{N} + s + 1, \\ \eta^*(0),i , i = \hat{N} + s + 2, \cdots , \hat{N} + 2s + 1. \end{cases}$$  \hspace{1cm} (5.6)$$

Substituting (5.4) into (5.1) and taking into account the following integrals:

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1} k^2 + \mu^2} = \frac{1}{(2\pi)^{\frac{D-1}{2}}} \Gamma(\frac{\mu}{r}) \frac{2^{\frac{D-3}{2}}}{(D - 3)(D - 3)} K_{\frac{D-3}{2}}(\mu r),$$  \hspace{1cm} for $D \geq 3$  \hspace{1cm} (5.7a)$$

and

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1} k^2} = \frac{2^{\frac{D-3}{2}}}{(2\pi)^{\frac{D-1}{2}}} \Gamma\left(\frac{D - 3}{2}\right),$$  \hspace{1cm} for $D \geq 4$,  \hspace{1cm} (5.7b)$$
we find that the $D$-dimensional gravitational potential is given by (for $D \geq 4$)

$$E_D(r) = -\frac{\kappa_D M_1 M_2}{\sigma(2\pi)^{\frac{D-1}{2}}} \left\{ \left( \frac{D-3}{D-2} \right)^{\frac{D-3}{2}} \frac{1}{r^{D-3}} + \left( \frac{D-2}{D-1} \right) \sum_{i=1}^{q+1} \frac{\mu_i^2}{\mu_i(2)_{,j}^2} \left( \mu_i(2)_{,i} - \frac{\mu_i(2)_{,i}}{r} \right) \frac{D-3}{2} K_{D-3}^\nu(\mu_i(2)_i r) + \frac{1}{(D-1)(D-2)} \sum_{i=1}^{N+1} \prod_{j=1}^{N+1} (\mu_j(2)_i - \frac{\mu_j(2)_i}{r}) \frac{D-3}{2} K_{D-3}^\nu(\mu_j(0)_i r) \right\}. \tag{5.8}$$

Similarly, for $D = 3$ the interparticle gravitational potential is determined to be

$$E_3(r) = \frac{\kappa_3 M_1 M_2}{2\sigma(2\pi)} \left\{ \sum_{i=1}^{q+1} \frac{\mu_i^2}{\mu_i(2)_{,j}^2} \left( \mu_i(2)_{,i} - \frac{\mu_i(2)_{,i}}{r} \right) K_0(\mu(2)_j r) + \sum_{i=1}^{N+1} \prod_{j=1}^{N+1} (\mu_j(2)_i - \frac{\mu_j(2)_i}{r}) K_0(\mu(0)_i r) \right\}. \tag{5.9}$$

Our next step will be to analyse the behavior of this gravitational potential for small distances. Defining $\nu = \frac{D-3}{2}$, we shall make a distinction between $D$ odd and even, since a modified Bessel function of the second kind $K_\nu(x)$ has a Taylor’s series expansion which depends if $\nu$ is integer or half-integer.

### 5.1 Regularity of the potential at the origin for $D > 3$ - $D$ even

If the potential is defined in a spacetime with even dimensions, we can expand the modified Bessel function of the second kind according to

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\nu-\frac{1}{2}} \frac{1}{k!} \frac{1}{2^k} \frac{1}{\Gamma(\nu - k + 1/2)} \Gamma(\nu + k + 1/2). \tag{5.10}$$

and, substituting (5.10) in (5.8), we can determine the gravitational potential for small distances to be given by

$$E_D(r) = -\frac{\kappa_D M_1 M_2}{\sigma(2\pi)^{\frac{D-1}{2}}} \left\{ \frac{1}{r^{2\nu}} \Delta^\text{even}_q(r, q, N) - \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\nu-\frac{1}{2}} \frac{(-1)^{\nu+k+1/2}}{2^{k+1} (\nu + k + 1/2) \Gamma(\nu + k + 1/2)} \times \right. \left[ 2\nu + 1 + \sum_{i=1}^{q+1} \frac{\mu_i^2}{\mu_i(2)_{,j}^2} - \mu_i(2)_{,i} \right] + O(r) \right\}. \tag{5.11}$$
noting that we could obtain finite results at sector. We shall discuss in the next section the reasons for this being so. It is also worthwhile
the finiteness of the potential near the origin is independent of the scalar curvature squared
However, we would need to adjust the parameters of the Lagrangian.

\[
\Delta^\text{even}_\nu (r; q, N) = \frac{2^\nu \Gamma(\nu + 1)}{2\nu + 1} - \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\nu-\frac{1}{2}} \sum_{n=0}^{\nu+k-1/2} \frac{(-1)^n \nu^{\nu+n-k-1/2} \Gamma(\nu + k + 1/2)}{2^k n! k!} \Gamma(\nu - k + 1/2) \times \\
\left[ 2\nu + 2 \sum_{i=1}^{q+1} \prod_{j=1 \atop j \neq i}^{q+1} \mu^2_{(2),j} - \mu^2_{(2),i} \right] \left[ \prod_{j=1}^{N+1} \mu^2_{(0),j} - \mu^2_{(0),i} \right] \nu^\nu-n-k-1/2 \\
\prod_{i=1}^{N+1} \prod_{j=1 \atop j \neq i}^{N+1} \mu^2_{(0),j} - \mu^2_{(0),i} \nu^\nu-n-k-1/2 \right].
\]

(5.12)

Therefore, the cancellation of the Newtonian singularity is dependent on the behavior of the function (5.12). With the help of a computer algebra system we may verify that this function will be null for all values of \( r \) if the following condition is satisfied

\[
2q + 4 - D \geq 0.
\]

(5.13)

At first glance this result might seem surprising, after all, we are concluding that the finiteness of the potential near the origin is independent of the scalar curvature squared sector. We shall discuss in the next section the reasons for this being so. It is also worthwhile noting that we could obtain finite results at \( r = 0 \) even if the above condition is not met. However, we would need to adjust the parameters of the Lagrangian.

5.2 Regularity of the potential at the origin for \( D > 3 - D \) odd

For a spacetime with odd dimensions and \( D > 4 \), we can expand the modified Bessel function of the second kind according to

\[
K_\nu(z) = (-1)^{\nu-1} \ln \left( \frac{z}{2} \right) \sum_{k=0}^{\infty} \frac{1}{k!(\nu+k)!} \left( \frac{z}{2} \right)^{\nu+2k} + \frac{1}{2} \left( \frac{2}{z} \right)^\nu \sum_{k=0}^{\nu-1} \frac{(-1)^k (\nu-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k} + \\
\frac{(-1)^\nu}{2} \sum_{k=0}^{\infty} \psi(k+1) + \psi(k+\nu+1) \left( \frac{z}{2} \right)^{\nu+2k},
\]

(5.14)

and, substituting the above expression in (5.8), we found that the gravitational potential for small distances is

\[
E_D(r) = -\frac{\kappa_D M_1 M_2}{\sigma(2\pi)^{D-3}} \left\{ \frac{1}{2^{2\nu}} \Delta^\text{odd}_\nu (r; q, N) + \frac{(-1)^{\nu+1}}{2^{\nu+1} \nu!} \left( \psi(1) + \psi(\nu+1) \right) \times \\
\left( \frac{2\nu}{2\nu + 2} \sum_{i=1}^{q+1} \prod_{j=1 \atop j \neq i}^{q+1} \mu^2_{(2),j} - \mu^2_{(2),i} \right) \left( \prod_{j=1}^{N+1} \mu^2_{(0),j} - \mu^2_{(0),i} \right) + \\
\left( \frac{2\nu}{2\nu + 2} \sum_{i=1}^{q+1} \prod_{j=1 \atop j \neq i}^{q+1} \mu^2_{(2),j} - \mu^2_{(2),i} \right) \ln(\mu^2_{(2),i} \mu^2_{(0),i}) + \\
\left( \frac{1}{2\nu + 2} \sum_{i=1}^{q+1} \prod_{j=1 \atop j \neq i}^{q+1} \mu^2_{(2),j} - \mu^2_{(2),i} \right) \ln(\mu^2_{(0),i} \mu^2_{(0),i}) + \mathcal{O}(r) \right\},
\]

(5.15)
where we have defined

\[
\Delta^{\text{odd}}_\nu(r; q, N) = \frac{2^\nu \Gamma(\nu + 1)}{2\nu + 1} - \sum_{k=0}^{\nu-1} \frac{(-1)^k 2^{\nu-1}(\nu - k - 1)!}{k!} \left( \frac{r}{2} \right)^{2k} \times \\
\times \left( \frac{2\nu + 1}{2\nu + 2} \sum_{i=1}^{q+1} \prod_{j \neq i} \frac{\mu^2_{(2),j}}{\mu^2_{(2),j} - \mu^2_{(2),i}} - \frac{1}{(2\nu + 2)(2\nu + 1)} \sum_{i=1}^{N+1} \prod_{j \neq i} \frac{\mu^2_{(0),j}}{\mu^2_{(0),j} - \mu^2_{(0),i}} \right) \\
+ \frac{(-1)^\nu}{2^\nu \nu!} r^{2\nu} \ln \left( \frac{r}{2} \right) \left( \frac{2\nu + 1}{2\nu + 2} \sum_{i=1}^{q+1} \prod_{j \neq i} \frac{\mu^2_{(2),j}}{\mu^2_{(2),j} - \mu^2_{(2),i}} - \frac{2\nu}{\nu} \right) + \\
- \frac{1}{(2\nu + 2)(2\nu + 1)} \sum_{i=1}^{N+1} \prod_{j \neq i} \frac{\mu^2_{(0),j}}{\mu^2_{(0),j} - \mu^2_{(0),i}} \right). 
\] (5.16)

Analogously to the above section, the finiteness of the potential energy at \( r = 0 \) will depend on the condition \( \Delta^{\text{odd}}_\nu(r; q, N) = 0 \) for every value of the coordinate \( r \). With the help of a computer algebra system we can verify that a sufficient condition to get \( \Delta^{\text{odd}}_\nu(r; q, N) = 0 \) is given by

\[
2q + 3 - D \geq 0, \tag{5.17}
\]

and the same conclusion made in the above section is valid.

### 5.3 Regularity of the potential at the origin for \( D = 3 \)

Now let us investigate the 3-dimensional case. Considering the expansion

\[
K_0(z) = -\ln \left( \frac{z}{2} \right) - \gamma + \mathcal{O}(z^2), \tag{5.18}
\]

where \( \gamma \) is the Euler-Machteroni constant, it is straightforward to see that, for \( D = 3 \), the gravitational potential for small distances is given by

\[
E_3(r) = -\frac{\kappa_3 M_1 M_2}{4\pi \sigma} \left\{ \left( \sum_{i=1}^{q+1} \prod_{j \neq i} \frac{\mu^2_{(2),j}}{\mu^2_{(2),j} - \mu^2_{(2),i}} - \sum_{i=1}^{N+1} \prod_{j \neq i} \frac{\mu^2_{(0),j}}{\mu^2_{(0),j} - \mu^2_{(0),i}} \right) \left( \ln \left( \frac{r}{2} \right) + \gamma \right) + \\
+ \sum_{i=1}^{q+1} \prod_{j \neq i} \frac{\mu^2_{(2),j}}{\mu^2_{(2),j} - \mu^2_{(2),i}} \ln \mu_{(2),i} - \sum_{i=1}^{N+1} \prod_{j \neq i} \frac{\mu^2_{(0),j}}{\mu^2_{(0),j} - \mu^2_{(0),i}} \ln \mu_{(0),i} + \mathcal{O}(r) \right\}. \tag{5.19}
\]

In such a case we can use the identity\(^{11}\)

\[
\sum_{i=1}^{n+1} \prod_{j=1}^{n+1} \frac{a_j}{a_j - a_i} = 1, \tag{5.20}
\]

\(^{11}\)For a rigorous proof of this identity we refer to [23].
valid for any set of complex numbers \( \{a_1, a_2, \ldots, a_{n+1}\} \) with \( n \geq 0 \), in order to verify the following equations

\[
\sum_{i=1}^{q+1} \prod_{j=1, j \neq i}^{N+1} \frac{\mu_i^2}{\mu_j^2 - \mu_i^2} = 1 \quad \text{and} \quad \sum_{i=1}^{q+1} \prod_{j=1, j \neq i}^{N+1} \frac{\mu_i^2}{\mu_j^2 - \mu_i^2} = 1, \tag{5.21}
\]

for \( q \geq 0 \) (note that \( q \geq 0 \) automatically implies \( N \geq 0 \)). Using the result above we can recast the interparticle potential energy for small distances as follows

\[
E_3(r) = -\frac{\kappa_3 M_1 M_2}{4\pi \sigma} \left\{ \sum_{i=1}^{q+1} \prod_{j=1, j \neq i}^{N+1} \frac{\mu_i^2 \ln \mu_i^2}{\mu_j^2 - \mu_i^2} - \sum_{i=1}^{q+1} \prod_{j=1, j \neq i}^{N+1} \frac{\mu_i^2 \ln \mu_i^2}{\mu_j^2 - \mu_i^2} + O(r) \right\}, \tag{5.22}
\]

Therefore, a sufficient condition for cancellation of the Newtonian singularity in 3-dimensional higher derivative gravities is given by \( q \geq 0 \). In fact this condition tells us that the existence of the Ricci squared sector is sufficient for the cancellation of Newtonian singularities.

Finally, it is important to emphasize that above discussion is valid for higher derivative theories with unrelated parameters, otherwise the conditions for the cancellation of Newtonian singularity may be insufficient. Once again the New Massive Gravity is a good example of this statement. In such a case, taking into account the constraint \( 3\alpha_0 + 8\beta_0 = 0 \), the interparticle potential energy for the NMG may be written as follows

\[
E_{\text{NMG}}(r) = -\frac{\kappa_3 M_1 M_2}{4\pi} K_0(m_{(2),1} r), \tag{5.23}
\]

where \( m_{(2),1}^2 = 4/\beta_0 \). Taking into account the expansion of the Bessel function for small arguments we arrive at the following result

\[
E_{\text{NMG}}(r) = \frac{\kappa_3 M_1 M_2}{4\pi} \left[ \gamma + \ln \left( \frac{m_{(2),1} r}{2} \right) \right] + O(r). \tag{5.24}
\]

Therefore, we found a divergent potential energy at \( r = 0 \). In the next section we will discuss the mechanism for the cancellation of Newtonian singularities and it will be clear why there is no cancellation of Newtonian singularity in the case of the NMG theory.

### 6 Relating renormalizability, unitarity and potential energy

In his seminal paper about renormalizability of higher derivative quantum gravity [6], Stelle pointed out a very interesting connection between renormalizability of higher derivatives theories of quantum gravity and the cancellation of Newtonian singularities of the interparticle potential energy. It was later proposed that this connection is a general property of \( D \)-dimensional higher derivative theories of quantum gravity [19]. Essentially this conjecture states that a renormalizable theory of quantum gravity should not present the so-called Newtonian singularity. Recently this conjecture was probed in the case of fourth and sixth derivative theories of gravity [20, 21]. Now we are able to verify this conjecture for a general class of \( D \)-dimensional higher derivative theories of quantum gravity.
Let us recall two important results from our last two sections. First of all, investigating the UV properties of the general class of higher derivative theories under consideration, we found the following necessary conditions for (super-)renormalizability:

- \( 2q + 4 - D = 0 \) \( \sim \) power counting renormalizability;
- \( 2q + 4 - D \geq 4\lambda \) \( \sim \) power counting super-renormalizability.

Furthermore, after an exhaustive investigation on the behavior of the interparticle potential energy for small distances, we found the following sufficient conditions for the cancellation of Newtonian singularities:

- \( 2q + 4 - D \geq 0 \), for even dimensions;
- \( 2q + 3 - D \geq 0 \), for odd dimensions.

Putting all this information together, it is not difficult to conclude that the necessary conditions for (super-)renormalizability automatically implies the sufficient condition for the cancellation of Newtonian singularities. Summing up:

Power counting (super-)renormalizability \( \Rightarrow \) Finite potential energy at \( r = 0 \).

This completes our examination of the aforementioned conjecture for the general class of theories under consideration. It should be emphasized that the inverse order is not necessarily true, \textit{i.e.} the cancellation of Newtonian singularities does not implies renormalizability. In fact, as it was pointed out by Giacchini, it is not difficult to construct an example of higher derivative model with finite potential energy at \( r = 0 \) and power counting non-renormalizable [23].

Finally, let us talk about the role of (non-)unitarity on this connection between renormalizability and Newtonian singularities. As we can see from the previous section, ghost-free higher derivative (local) theories are not compatible with renormalizability. The reason for this point relies in the crucial role played by the ghost-like particles in the improvement of the tree-level propagator. Furthermore, ghost-like particles are also necessary in the cancellation of Newtonian singularities [22, 23]. For instance, is not difficult to verify that those higher derivative theories which are ghost-free, have a divergent potential energy at \( r = 0 \).

Indeed, the fact that ghost-like particles are necessary for the cancellation Newtonian singularities has an interesting explanation. From the classical point of view, ghost-like particles correspond to negative energy propagation modes (which give rises to Ostrogradsky instabilities, for instance). Taking into account, from an heuristic point of view, that the interparticle potential energy is given by the sum of the individual potential energy associated with each propagation mode, it is necessary to have parts with opposite signs in order to have some kind of cancellation. Therefore, negative energy propagation modes are necessary for the cancellation of Newtonian singularities.
The role of ghost-free particles in the mechanism for the cancellation of Newtonian singularities was recently explored in the literature. In fact, Modesto and collaborators demonstrated that 4-dimensional theories described by (2.1), with \( F_1(\Box) \) and \( F_2\Box \) being polynomial functions with the same degree, \textit{i.e.} with the same number of ghosts and physical particles, the cancellation of Newtonian singularities occurs [22]. Later, Giacchini demonstrated the equal number of ghosts and physical particle is not a necessary condition for the cancellation of Newtonian singularities in 4-dimensional higher derivative gravities [23]. The only necessary condition is that the particle spectra of the theory should contain \textit{at least} a massive ghost and a massive physical particle (besides the usual massless graviton). In the case of 3-dimensional theories, it is not difficult to adapt Giacchini’s demonstration in order to get the same conclusion. However, in the case of spacetime with dimension higher than 4, the situation is more subtle. Although we have no demonstration, the above argumentation appears to be valid as well, nevertheless, the minimal number of both massive ghosts and physical particles increases with the dimension of the spacetime.

7 Final remarks

In this paper we investigate some interesting classical and quantum aspects of a general class of local higher derivative theories of gravity. From the classical point of view we proposed a new gauge condition which generalizes the so-called Teyssandier gauge. This generalized gauge choice has an interesting property when applied to the linearized equations of motion of a general class of higher derivative gravities. Essentially it decouples the linearized equation of motion in terms of three different sectors with different gauge conditions. One of these sectors reproduces the dynamics of the dynamics of General Relativity. Although for the most part of this paper we dealt with polynomial form factors, all the discussion regarding the generalized condition can be perform in the context of non-polynomial (non-local) form factors also.

During the rest of this paper we investigated some aspects related with tree level unitarity, renormalizability properties and their relation with the classical connection. First of all we performed an exhaustive investigation of the particle spectra and the presence of ghost-like particles. As it was verified, the appearance of ghost-like particles in higher derivative theories appears to be unavoidable. However, there are two exception for this statement - one of them is a special case of \( f(R) \)-theories and the other one is the so-called New Massive Gravity.

Following our investigation we studied the UV properties of the general class of higher derivative theories under consideration. We found a couple of necessary conditions for power-counting (super-)renormalizability, relating the number of derivatives of the model with the spacetime dimension. Unfortunately, the models which are tree-level ghost-free turns out to be non-renormalizable. This fact points out to the impossibility of conciliating renormalizability with unitarity in quantum gravity. A possible way to evade this problem takes the route of non-local theories of quantum gravity [14, 15] or Lee-Wick formulation of unitarity [10, 11]. In two recent papers, Anselmi and Piva developed a new formulation
of Lee-Wick theories [27, 28] and it certainly deserves some investigation in the context of higher derivative quantum gravities.

We also verified a conjecture which states that (super-)renormalizable theories of quantum gravity do not present the so-called Newtonian singularity in the potential energy. As we have seen, a necessary condition for power counting (super-)renormalizability automatically implies a sufficient condition for the cancellation of Newtonian singularities. We also discussed the role of ghost-like particle in the mechanism of cancellation of Newtonian singularities.

It is worth mentioning that so far we have not found a complete proof of the sufficient condition for the cancellation of Newtonian singularities in arbitrary dimensions. This condition was obtained by inspection with the help of an algebraic manipulation system and its analytical derivation can be pointed out as an future perspective.

Finally, our discussion regarding the connection between (super-)renormalizability and the cancellation of Newtonian singularities are based in the assumption of polynomial (local) form factors. We hope that our conclusions may be generalized for non-local theories, however, the mechanism on the cancellation of Newtonian singularities depends on the specific form of the functions $F_1(\Box)$ and $F_2(\Box)$.

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