ANTICANONICAL TRANSFORMATIONS AND GRAND JACOBIAN

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An independent (algebraic) proof is given of the factorization property of the Grand Jacobian corresponding to the anticanonical transformations in the BV formalism.

Keywords: BV formalism, anticanonical transformations, Grand Jacobian.

INTRODUCTION

The BV formalism (or the field-antifield formalism) [1, 2] is a powerful method of covariant quantization that can be applied to arbitrary gauge-invariant systems. This method is based on the fundamental concept of global supersymmetry known as the BRST symmetry [3, 4]. One of the most important objects of the field-antifield formalism is its odd symplectic structure, known to mathematicians as the Buttin bracket [5]. The classic master equation and the Ward identities for the generating functional of the vertex functions (effective action) are formulated in terms of the antibracket. One important property of this method is that the antibracket is conserved relative to the anticanonical transformations, which are dual with respect to the canonical transformations for the Poisson bracket. An important role and rich geometric possibilities of general anticanonical transformations were realized in the field-antifield formalism in the procedure of gauge fixing [6]. The original procedure of gauge fixing [1, 2] in fact corresponds to a special type of anticanonical transformation in the action, this transformation being a proper solution of the quantum master equation.

Anticanonical transformations play a key role in the description of the structure of renormalizations and the gauge dependence of the effective action in gauge theories of general type [6]. Another important application is to examine the arbitrariness in the solutions of both the classical master equation [7] and the quantum master equation [8, 9], when the Grand Jacobian of the anticanonical transformations plays a substantial role in the total quantum action. The Grand Jacobian possesses an interesting property, known as the factorization property, enabling it to be expressed in the form of a superdeterminant of a supermatrix in the sector of only anticanonically transformed fields. Another possibility is associated with the use in this representation of the superdeterminant of a supermatrix in the sector of only anticanonically transformed antifields. These properties of the Grand Jacobian were stated, at least in a minimal way, in [10], although a proof was not provided in it (see also [11]). Later we filled this gap by proving the property of factorization of the Grand Jacobian by solving the Lie equation for a one-parameter family of antisymplectic variables subjected to anticanonical transformations [9].

In this paper, we present a simple proof of the property of factorization of the Grand Jacobian corresponding to anticanonical transformation in the field-antifield formalism [1, 2], based only on the use of algebraic properties of the anticanonical transformations. We employ De Witt’s condensed notation [12]. We apply the notation $\epsilon(A)$ for the Grassmann parity of an arbitrary quantity $A$. Functional derivatives with respect to fields and antifields are understood as left. Right functional derivatives are marked by the special symbol “$\leftarrow$”.

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ANTICANONICAL TRANSFORMATIONS

We continue our exposition using antisymmetric Darboux coordinates $z^A$ in the form of an explicit splitting into fields $\phi^i$ and antifields $\phi^*_i$,

$$z^A = \{\phi^i, \phi^*_i\}, \quad \varepsilon(z^A) = \varepsilon_A, \quad \varepsilon(\phi^*_i) = \varepsilon(\phi^i) + 1. \quad (1)$$

For arbitrary functionals $G = G(\phi, \phi^*)$ and $H = H(\phi, \phi^*)$ the antibracket is defined by the rule

$$(G, H) = G \left( \frac{\partial}{\partial \phi^i} - \frac{\partial}{\partial \phi_i^*} \right) H, \quad \varepsilon((G, H)) = \varepsilon(G) + \varepsilon(H) + 1, \quad (2)$$

so that

$$(z^A, z^B) = E^{AB}, \quad \varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad (3)$$

where $E^{AB}$ are elements of the constant invertible antisymmetric metric $E$ with the following block structure:

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad E^{AB} = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix} \quad (4)$$

and antisymmetry property

$$E^{AB} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{BA}. \quad (5)$$

In terms of $z^A$ the antibracket is rewritten in the form

$$(G, H) = G \left( \frac{\partial}{\partial z^A} E^{AB} \frac{\partial}{\partial z^B} \right) H. \quad (6)$$

Let $F = F(\phi, \Phi^*)$ and $\varepsilon(F) = 1$, and let there be a generator of the anticanonical transformation,

$$\Phi^i = \frac{\partial}{\partial \Phi^i} F(\phi, \Phi^*), \quad \phi^*_i = \frac{\partial}{\partial \phi^i} F(\phi, \Phi^*). \quad (7)$$

Any anticanonical transformation preserves the antibracket,

$$(Z^A, Z^B) = E^{AB}, \quad (8)$$

where $Z^A = (\Phi^i, \phi^*_i)$ and $\varepsilon(Z^A) = \varepsilon(z^A) = \varepsilon_A$ are considered as functions of $z^A$, $Z^A = Z^A(z)$, found from Eqs. (2). The condition of solvability for the anticanonical transformation leads to the following relations:
\[
\left( Z^A(z) \frac{\partial}{\partial z^C} \right) \left( z^C(Z) \frac{\partial}{\partial Z^B} \right) = \delta^A_B, \quad \left( z^A(Z) \frac{\partial}{\partial Z^C} \right) \left( Z^C(z) \frac{\partial}{\partial z^B} \right) = \delta^A_B. \quad (9)
\]

Let \( H^A_B \) be elements of the supermatrix \( H \) of anticanonical transformation (7):

\[
H^A_B = \begin{pmatrix} A^i & B^i \\ C^j & D^j \end{pmatrix}, \quad H^A_B = \begin{pmatrix} A^i_{j} & B^j_{i} \\ C^j_{i} & D^j_{i} \end{pmatrix}. \quad (11)
\]

Here we employ the notation

\[
A^i_j = \Phi^i_j \Phi^*_{ij}, \quad B^i_j = \Phi^i_j \Phi^*_{ij}, \quad C^j_i = \Phi^*_i \Phi^*_{ij}, \quad D^j_i = \Phi^*_i \Phi^*_{ij}. \quad (12)
\]

The quantities in expressions (12) and (13) have the following assignment of Grassmann parities:

\[
\varepsilon(A^i_j) = \varepsilon_i + \varepsilon_j, \quad \varepsilon(B^i_j) = \varepsilon_i + \varepsilon_j + 1, \quad (14)
\]

\[
\varepsilon(C^j_i) = \varepsilon_j + \varepsilon_i + 1, \quad \varepsilon(D^j_i) = \varepsilon_i + \varepsilon_j. \quad (15)
\]

In terms of the anticanonical transformation, we have

\[
A^i_j = \frac{\partial}{\partial \Phi^i_j} \frac{\partial}{\partial \Phi^*_{kj}} F + \left( \frac{\partial}{\partial \Phi^i_j} \frac{\partial}{\partial \Phi^*_{kj}} F \right) C^j_i
\]

\[
= \frac{\partial}{\partial \Phi^i_j} \frac{\partial}{\partial \Phi^*_{kj}} F - \left( \frac{\partial}{\partial \Phi^i_j} \frac{\partial}{\partial \Phi^*_{kj}} F \right) D^j_k \left( \frac{\partial}{\partial \Phi^*_{kj}} \frac{\partial}{\partial \Phi^*_{kj}} F \right), \quad (16)
\]

\[
B^j_i = \frac{\partial}{\partial \Phi^*_{kj}} F \left( \Phi^*_k \Phi^*_{ij} \frac{\partial}{\partial \Phi^*_{ij}} F \right) = \left( \frac{\partial}{\partial \Phi^*_{kj}} \frac{\partial}{\partial \Phi^*_{kj}} F \right) D^j_k, \quad (17)
\]

or, employing the notation

\[
\mathcal{A}^j_i = \frac{\partial}{\partial \Phi^i_j} \frac{\partial}{\partial \Phi^*_{kj}} F, \quad \mathcal{K}^j_i = \frac{\partial}{\partial \Phi^*_{kj}} \frac{\partial}{\partial \Phi^*_{kj}} F, \quad (18)
\]
written more compactly as

\[ A = M + K C, \quad B = K D. \]  \hspace{2cm} (19)

We introduce the supermatrix \( R \) with elements \( R^A_B \) as

\[
R^A_B = z(Z)^A \frac{\partial}{\partial Z^B}, \quad \varepsilon(R^A_B) = \varepsilon_A + \varepsilon_B,  \hspace{2cm} (20)
\]

\[
R^A_B = \begin{pmatrix} \mathcal{V}^j_i & \mathcal{W}^j_i \\ \mathcal{Z}^j_i & \mathcal{U}^j_i \end{pmatrix}, \quad R = \begin{pmatrix} \mathcal{V} & \mathcal{W} \\ \mathcal{Z} & \mathcal{U} \end{pmatrix},  \hspace{2cm} (21)
\]

where the quantities

\[
\mathcal{V}^j_i = \phi^i_j(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j}, \quad \mathcal{W}^j_i = \phi^i_j(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j},  \hspace{2cm} (22)
\]

\[
\mathcal{Z}^j_i = \phi^*_i(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j}, \quad \mathcal{U}^j_i = \phi^*_i(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j},  \hspace{2cm} (23)
\]

obey the following assignment of Grassmann parities:

\[
\varepsilon(\mathcal{V}^j_i) = \varepsilon_i + \varepsilon_j, \quad \varepsilon(\mathcal{W}^j_i) = \varepsilon_i + \varepsilon_j + 1,  \hspace{2cm} (24)
\]

\[
\varepsilon(\mathcal{Z}^j_i) = \varepsilon_i + \varepsilon_j + 1, \quad \varepsilon(\mathcal{U}^j_i) = \varepsilon_i + \varepsilon_j.  \hspace{2cm} (25)
\]

In terms of the anticanonical transformation, we have

\[
\mathcal{U}^j_i = (-1)^{e_i(e_j+1)} e_i \left( \frac{\partial}{\partial \Phi^j} \frac{\partial}{\partial \Phi^k} F \right) + \left( \frac{\partial}{\partial \Phi^j} \frac{\partial}{\partial \Phi^k} F \right) \phi^k(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j}  
\]

\[
= \widetilde{M}^j_i + N_{ik} \mathcal{W}^k_i,  \hspace{2cm} (26)
\]

\[
\mathcal{Z}^j_i = \left( \frac{\partial}{\partial \Phi^j} \frac{\partial}{\partial \Phi^k} F \right) \phi^k(\Phi, \Phi^*) \frac{\partial}{\partial \Phi^j} = N_{ik} \mathcal{V}^k_i,  \hspace{2cm} (27)
\]

where \( \widetilde{M} \) is the supermatrix transposed with \( M \) (Eq. (18)),

\[
\widetilde{M}^j_i = (-1)^{e_i(e_j+1)} M^j_i = \left( \frac{\partial}{\partial \Phi^j} \frac{\partial}{\partial \Phi^k} F \right), \quad (\widetilde{M}^{-1})^j_i = (-1)^{e_i(e_j+1)} (M^{-1})^j_i.  \hspace{2cm} (28)
\]
\[ \mathcal{N}_{ij} = \left( \frac{\partial}{\partial \Phi^i} \frac{\partial}{\partial \Phi^j} F \right). \]  

(29)

From the first relations in (9), (11), and (21), we have

\[ \mathcal{A}V + BZ = I, \quad \mathcal{A}W + BU = 0, \]  

(30)

\[ CV + DZ = 0, \quad CW + DU = I. \]  

(31)

From the first relations in (31) and (27), we derive the equation

\[ C = -DN, \]  

(32)

and consequently, \( A \) can be represented in the form

\[ A = M - \mathcal{K}DN. \]  

(33)

From the second relations in (30) and (31) it follows that

\[ \mathcal{U}^{-1} = D - CA^{-1}B. \]  

(34)

In turn, from the second relations in (9), (11), and (21), we have

\[ \mathcal{V}A + WC = I, \quad \mathcal{V}B + WD = 0, \]  

(35)

\[ \mathcal{Z}A + UC = 0, \quad \mathcal{Z}B + UD = I. \]  

(36)

From the second relations in (19) and (35) it follows that

\[ \mathcal{W} = -\mathcal{V}C. \]  

(37)

Employing this relation, the first relation in (35) and the second relation in (36) yield

\[ \mathcal{V} = M^{-1}, \quad \mathcal{D} = \mathcal{M}^{-1} \]  

(38)

along with the following representations for \( A \) and \( U \):

\[ A = M - \mathcal{K}M^{-1} \mathcal{N}, \quad U = \mathcal{M} - MN^{-1} \mathcal{K}. \]  

(39)

Let us now consider

\[ (\mathcal{K}\mathcal{M}^{-1} \mathcal{N})'' = (-1)^{(e_j + 1)e_j} (\mathcal{K}\mathcal{M}^{-1} \mathcal{N})_{ji} = (-1)^{(e_j + 1)e_j} \frac{\partial^2 F}{\partial \Phi^i \partial \Phi^j_k} (\mathcal{M}^{-1})''_{ki} \frac{\partial^2 F}{\partial \Phi^j \partial \Phi^i}. \]  

(40)
We finally obtain the relations

$$A = \tilde{U}, \quad U = \tilde{A},$$

which play a key role in the proof of the property of factorization of the Grand Jacobian.

**GRAND JACOBIAN**

Let $J$ be the Grand Jacobian of anticanonical transformation (3), which is expressed in terms of the superdeterminant of supermatrix $H$ (Eq. (10)),

$$J = s\text{Det} \, H.$$

It is well known [12] that the superdeterminant $H$ can be written in terms of the superdeterminants of its superblocks $A, B, C,$ and $D$ in Eq. (11) as

$$s\text{Det} \, H = (s\text{Det} \, A)s\text{Det} \, X^{-1}, \quad X = D - CA^{-1}B.$$

By virtue of relation (34), $X^{-1} = U$, so that, taking relations (41) into account, we derive the equality

$$J = (s\text{Det} \, A)s\text{Det} \, U = (s\text{Det} \, A)s\text{Det} \, \tilde{A}.$$

Note that the equality

$$s\text{Det} \, Q = s\text{Det} \, \tilde{Q}$$

is valid for arbitrary supermatrix $Q$:

$$\tilde{Q}^{\dagger} = (-1)^{e_{j+1}} Q^{\dagger}.$$

Consequently, we establish the equality

$$J = (s\text{Det} \, A)^2,$$

known in the field-antifield formalism as the factorization of the Grand Jacobian corresponding to the anticanonical transformations. Note that there also exists the possibility of expressing the property of factorization of the Grand Jacobian in terms of the supermatrix $\mathcal{M}$. Toward this end, we use the representation for $J$ in the form [13]

$$J = s\text{Det} \, H = s\text{Det} \, (A - BD^{-1}C)s\text{Det} \, D^{-1}.$$

Taking into account that

$$A = \mathcal{M} - K\tilde{\mathcal{M}}^{-1} \, \mathcal{N}, \quad D^{-1} = \tilde{\mathcal{M}}, \quad B = K\tilde{\mathcal{M}}^{-1}, \quad C = \tilde{\mathcal{M}}^{-1} \, \mathcal{N},$$

we obtain
\[ A - BD^{-1}C = \mathcal{M}, \]  
\[ J = \text{sDet } \mathcal{M} \text{sDet } \tilde{\mathcal{M}} = (\text{sDet } \mathcal{M})^2 \]

– the second representation of the factorization property of the Grand Jacobian.

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