AN ALGEBRAIC PERSPECTIVE OF GROUP RELAXATIONS

REKHA R. THOMAS

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1. Introduction

Group relaxations of integer programs were introduced by Ralph Gomory in the 1960s [13], [15]. Given a general integer program of the form

$$\text{minimize } \{ c \cdot x : Ax = b, \ x \geq 0, \ \text{integer} \},$$

its group relaxation is obtained by dropping non-negativity restrictions on all the basic variables in the optimal solution of its linear relaxation. In this paper, we survey recent results on group relaxations obtained from the algebraic study of integer programming using Gröbner bases of toric ideals [36]. No knowledge of these methods is assumed, and the exposition is self-contained and hopefully accessible to a person familiar with the traditional methods of integer programming. Periodic comments on the algebraic origins, motivations and counterparts of many of the described results — which the reader may pursue if desired — offer a more complete picture of the theory.

For the sake of brevity, we will bypass a detailed account of the classical theory of group relaxations. A short expository account can be found in [33, §24.2], and a detailed set of lecture notes on this topic in [25]. We give a brief synopsis of the essentials based on the recent survey article by Aardal et. al [3] and refer the reader to any of the above sources for further details and references on the classical theory of group relaxations.

Assuming that all data in (1) are integral and that $A_B$ is the optimal basis of the linear relaxation of (1), Gomory’s group relaxation of (1) is the problem

$$\text{minimize } \{ \tilde{c} \cdot x_N : A_B^{-1}A_Nx_N \equiv A_B^{-1}b \ (\text{mod } 1), \ x_N \geq 0, \ \text{integer} \}.$$
Here $B$ and $N$ are the index sets for the basic and non-basic columns of $A$ corresponding to the optimal solution of the linear relaxation of (1). The vector $x_N$ denotes the non-basic variables and the cost vector $\tilde{c} = c_N - c_B A_B^{-1} A_N$ where $c = (c_B, c_N)$ is partitioned according to $B$ and $N$. The notation $A_B^{-1} A_N x_N \equiv A_B^{-1} b \,(mod\,1)$ indicates that $A_B^{-1} A_N x_N - A_B^{-1} b$ is a vector of integers. Problem (2) is called a “group relaxation” of (1) since it can be written in the canonical form

\[
\text{minimize } \{ \tilde{c} \cdot x_N : \sum_{j \in N} g_j x_j \equiv g_0 \,(mod\,G), \, x_N \geq 0, \, \text{integer}\}
\]  

(3)

where $G$ is a finite abelian group and $g_j \in G$. Problem (3) can be viewed as a shortest path problem in a graph on $|G|$ nodes which immediately furnishes algorithms for solving it. Once the optimal solution $x_N^*$ of (3) is found, it can be uniquely lifted to a vector $x^* = (x_B^*, x_N^*) \in \mathbb{Z}^n$ such that $Ax^* = b$. If $x_B^* \geq 0$ then $x^*$ is the optimal solution of (1). Otherwise, $c \cdot x^*$ is a lower bound for the optimal value of (1). Several strategies are possible when the group relaxation fails to solve the integer program. See [4], [17], [31] and [43] for work in this direction. A particular idea due to Wolsey [42] that is very relevant for this paper is to consider the extended group relaxations of (1). These are all the possible group relaxations of (1) obtained by dropping non-negativity restrictions on all possible subsets of the basic variables $x_B$ in the optimum of the linear relaxation of (1). Gomory’s group relaxation (2) of (1) and (1) itself are therefore among these extended group relaxations. If (2) does not solve (1), then one could resort to other extended relaxations to solve the problem. At least one of these extended group relaxations (in the worst case (1) itself) is guaranteed to solve the integer program (1).

The convex hull of the feasible solutions to (2) is called the corner polyhedron [14]. A major focus of Gomory and others who worked on group relaxations was to understand the polyhedral structure of the corner polyhedron. This was achieved via the master polyhedron of the group $G$ [15] which is the convex hull of the set of points

\[
\{z : \sum_{g \in G} g z_g \equiv g_0 \,(mod\,G), \, z \geq 0, \, \text{integer}\}.
\]

Facet-defining inequalities for the master polyhedron provide facet inequalities of the corner polyhedron [15]. As remarked in [3], this landmark paper [15] introduced several of the now standard ideas in polyhedral combinatorics like projection onto faces, subadditivity, master polytopes, using automorphisms to generate one facet from another, lifting techniques and so on. See [16] for further results on generating facet inequalities.

In the algebraic approach to integer programming, one considers the entire family of integer programs of the form (1) as the right hand side vector $b$ varies. Definition 2.6 defines a set of group relaxations for each program in this family. Each relaxation is indexed by a face of a simplicial complex called a regular triangulation (Definition 2.1). This complex encodes all the optimal bases of the linear programs arising from the coefficient matrix $A$ and cost vector $c$ (Lemma 2.3). The main result of Section 2 is Theorem 2.8 which states that the group relaxations in Definition 2.6 are precisely all the bounded group relaxations of all programs in the family. In particular, they include all the extended group relaxations of all programs in the family and typically contain more relaxations for each program. This
theorem is proved via a particular reformulation of group relaxations which is crucial for the rest of the paper. This and other reformulations are described in Section 2.

The most useful group relaxations of an integer program are the “least strict” ones among all those that solve the program. By this we mean that any further relaxation of non-negativity restrictions will result in group relaxations that do not solve the problem. The faces of the regular triangulation indexing all these special relaxations for all programs in the family are called the \textit{associated sets} of the family (Definition 3.1). In Section 3 we develop tools to study associated sets. This leads to Theorem 3.10 which characterizes associated sets in terms of \textit{standard pairs} and \textit{standard polytopes}. Theorem 3.11 shows that one can “read off” the “least strict” group relaxations that solve a given integer program in the family from these standard pairs.

The results in Section 3 lead to an important invariant of the family of integer programs being studied called its \textit{arithmetic degree}. In Section 4 we discuss the relevance of this invariant and give a bound for it based on a result of Ravi Kannan (Theorem 4.8). His result builds a bridge between our methods and those of Kannan, Lenstra, Lovasz, Scarf and others that use geometry of numbers in integer programming.

Section 5 examines the structure of the poset of associated sets. The main result in this section is the \textit{chain theorem} (Theorem 5.2) which shows that associated sets occur in saturated chains. Theorem 5.4 bounds the length of a maximal chain.

In Section 6 we define a particular family of integer programs called a \textit{Gomory family}, for which all associated sets are maximal faces of the regular triangulation. Theorem 6.2 gives several characterizations of Gomory families. We show that this notion generalizes the classical notion of \textit{total dual integrality} in integer programming [33, §22]. We conclude in Section 7 with constructions of Gomory families from matrices whose columns form a Hilbert basis. In particular, we recast the existence of a Gomory family as a \textit{Hilbert cover} problem. This builds a connection to the work of Sebő [31], Bruns & Gubeladze [7] and Firla & Ziegler [11] on \textit{Hilbert partitions} and \textit{covers} of polyhedral cones. We describe the notions of super and $\Delta$-\textit{normality} both of which give rise to Gomory families (Theorems 7.7 and 7.14).

The majority of the material in this paper is a translation of algebraic results from [21], [22], [29], [38] and §12.D, [38] and [39]. The translation has sometimes required new definitions and proofs. Kannan’s theorem in Section 4 has not appeared elsewhere.

We will use the letter $\mathbb{N}$ to denote the set of non-negative integers, $\mathbb{R}$ to denote the real numbers and $\mathbb{Z}$ for the integers. The symbol $P \subseteq Q$ denotes that $P$ is a subset of $Q$, possibly equal to $Q$, while $P \subset Q$ denotes that $P$ is a proper subset of $Q$.

\section{Group Relaxations}

Throughout this paper, we fix a matrix $A \in \mathbb{Z}^{d \times n}$ of rank $d$, a cost vector $c \in \mathbb{Z}^n$ and consider the family $IP_{A,c}$ of all integer programs

\[ IP_{A,c}(b) := \text{minimize} \ \{ c \cdot x : Ax = b, \ x \in \mathbb{N}^n \} \]

as $b$ varies in the semigroup $\mathbb{N}A := \{ Au : u \in \mathbb{N}^n \} \subseteq \mathbb{Z}^d$. This family is precisely the set of all feasible integer programs with coefficient matrix $A$ and cost vector $c$. The semigroup $\mathbb{N}A$ lies in the intersection of the $d$-dimensional polyhedral cone $\text{cone}(A) := \{ Au : u \geq 0 \} \subseteq \mathbb{R}^d$ and the $d$-dimensional lattice $\mathbb{Z}A := \{ Au : u \in \mathbb{Z}^n \} \subseteq \mathbb{Z}^d$. For simplicity, we will assume
that \(\text{cone}(A)\) is pointed and that \(\{u \in \mathbb{R}^n : Au = 0\}\), the kernel of \(A\), intersects the non-negative orthant of \(\mathbb{R}^n\) only at the origin. This guarantees that all programs in \(\text{IP}_{A,c}\) are bounded. In addition, the cost vector \(c\) will be assumed to be generic in the sense that each program in \(\text{IP}_{A,c}\) has a unique optimal solution.

The linear relaxation of \(\text{IP}_{A,c}(b)\) is the linear program

\[
\text{LP}_{A,c}(b) := \text{minimize} \{ \ c \cdot x : Ax = b, \ x \geq 0 \}.
\]

We denote by \(\text{LP}_{A,c}\) the family of all linear programs of the form \(\text{LP}_{A,c}(b)\) as \(b\) varies in \(\text{cone}(A)\). These are all the feasible linear programs with coefficient matrix \(A\) and cost vector \(c\). Since all data are integral and all programs in \(\text{IP}_{A,c}\) are bounded, all programs in \(\text{LP}_{A,c}\) are bounded as well.

In the classical definitions of group relaxations of \(\text{IP}_{A,c}(b)\), one assumes knowledge of the optimal basis of the linear relaxation \(\text{LP}_{A,c}(b)\). In the algebraic set up, we define group relaxations for all members of \(\text{IP}_{A,c}\) at one shot and, analogously to the classical setting, assume that the optimal bases of all programs in \(\text{LP}_{A,c}\) are known. This information is carried by a polyhedral complex called the regular triangulation of \(\text{cone}(A)\) with respect to \(c\).

A polyhedral complex \(\Delta\) is a collection of polyhedra called cells (or faces) of \(\Delta\) such that:

(i) every face of a cell of \(\Delta\) is again a cell of \(\Delta\) and,

(ii) the intersection of any two cells of \(\Delta\) is a common face of both.

The set-theoretic union of the cells of \(\Delta\) is called the support of \(\Delta\). If \(\Delta\) is not empty, then the empty set is a cell of \(\Delta\) since it is a face of every polyhedron. If all the faces of \(\Delta\) are cones, we call \(\Delta\) a cone complex.

For \(\sigma \subseteq \{1, \ldots, n\}\), let \(A_{\sigma}\) be the submatrix of \(A\) whose columns are indexed by \(\sigma\), and let \(\text{cone}(A_{\sigma})\) denote the cone generated by the columns of \(A_{\sigma}\). The regular subdivision \(\Delta_c\) of \(\text{cone}(A)\) is a cone complex with support \(\text{cone}(A)\) defined as follows.

**Definition 2.1.** For \(\sigma \subseteq \{1, \ldots, n\}\), \(\text{cone}(A_{\sigma})\) is a face of the regular subdivision \(\Delta_c\) of \(\text{cone}(A)\) if and only if there exists a vector \(y \in \mathbb{R}^d\) such that \(y \cdot a_j = c_j\) for all \(j \in \sigma\) and \(y \cdot a_j < c_j\) for all \(j \notin \sigma\).

The regular subdivision \(\Delta_c\) can be constructed geometrically as follows. Consider the cone in \(\mathbb{R}^{d+1}\) generated by the lifted vectors \((a_i^t, c_i) \in \mathbb{R}^{d+1}\) where \(a_i\) is the \(i\)th column of \(A\) and \(c_i\) is the \(i\)th component of \(c\). The lower facets of this lifted cone are all those facets whose normal vectors have a negative \((d + 1)\)th component. Projecting these lower facets back onto \(\text{cone}(A)\) induces the regular subdivision \(\Delta_c\) of \(\text{cone}(A)\) (see [37]). Note that if the columns of \(A\) span an affine hyperplane in \(\mathbb{R}^d\), then \(\Delta_c\) can also be seen as a subdivision of \(\text{conv}(A)\), the \((d - 1)\)-dimensional convex hull of the columns of \(A\).

The genericity assumption on \(c\) implies that \(\Delta_c\) is in fact a triangulation of \(\text{cone}(A)\) (see [37]). We call \(\Delta_c\) the regular triangulation of \(\text{cone}(A)\) with respect to \(c\). For brevity, we may also refer to \(\Delta_c\) as the regular triangulation of \(A\) with respect to \(c\). Using \(\sigma\) to label \(\text{cone}(A_{\sigma})\), \(\Delta_c\) is usually denoted as a set of subsets of \(\{1, \ldots, n\}\). Since \(\Delta_c\) is a complex of simplicial cones, it suffices to list just the maximal elements (with respect to inclusion) in this set of sets. By definition, every one dimensional face of \(\Delta_c\) is of the form \(\text{cone}(a_i)\) for some column \(a_i\) of \(A\). However, not all cones of the form \(\text{cone}(a_i)\), \(a_i\) a column of \(A\), need appear as a one dimensional cell of \(\Delta_c\).
Example 2.2. (i) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $c = (1, 0, 0, 1)$. The four columns of $A$ are the four dark points in Figure 1 labeled by their column indices $1, \ldots, 4$. Figure 1 (a) shows the cone generated by the lifted vectors $(a_i^t, c_i) \in \mathbb{R}^3$. The rays generated by the lifted vectors have the same labels as the points that were lifted. Projecting the lower facets of this lifted cone back onto $cone(A)$, we get the regular triangulation $\Delta_c$ of $cone(A)$ shown in Figure 1 (b). The same triangulation is shown as a triangulation of $conv(A)$ in Figure 1 (c). The faces of the triangulation $\Delta_c$ are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\emptyset$. Using only the maximal faces, we may write $\Delta_c = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$.

(ii) For the $A$ in (i), $cone(A)$ has four distinct regular triangulations as $c$ varies. For instance, the cost vector $c' = (0, 1, 0, 1)$ induces the regular triangulation $\Delta_{c'} = \{\{1, 3\}, \{3, 4\}\}$ shown in Figure 2 (b) and (c). Notice that $\{2\}$ is not a face of $\Delta_{c'}$.

(iii) If $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $c = (1, 0, 0, 1)$, then $\Delta_c = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. However, in this case, $\Delta_c$ can only be seen as a triangulation of $cone(A)$ and not of $conv(A)$. \qed
Proposition 2.3. \cite[Lemma 1.4]{37} An optimal solution of $LP_{A,c}(b)$ is any feasible solution $x^*$ such that $\text{supp}(x^*) = \tau$ where $\tau$ is the smallest face of the regular triangulation $\Delta_c$ such that $b \in \text{cone}(A_\tau)$.

Proposition 2.3 implies that $\sigma \subseteq \{1, \ldots, n\}$ is a maximal face of $\Delta_c$ if and only if $A_\sigma$ is an optimal basis for all $LP_{A,c}(b)$ with $b$ in $\text{cone}(A_\sigma)$. For instance, in Example 2.2 (i), if $b = (4, 1)^t$ then the optimal basis of $LP_{A,c}(b)$ is $[a_1, a_2]$ where as if $b = (2, 2)^t$, then the optimal solution of $LP_{A,c}(b)$ is degenerate and either $[a_1, a_2]$ or $[a_2, a_3]$ could be the optimal basis of the linear program. (Recall that $a_i$ is the $i$th column of $A$.) All programs in $LP_{A,c}$ have one of $[a_1, a_2], [a_2, a_3]$ or $[a_3, a_4]$ as its optimal basis.

Given a polyhedron $P \subseteq \mathbb{R}^n$ and a face $F$ of $P$, the normal cone of $F$ at $P$ is the cone $N_P(F) := \{ \omega \in \mathbb{R}^n : \omega \cdot x' \geq \omega \cdot x, \text{ for all } x' \in F \text{ and } x \in P \}$. The normal cones of all faces of $P$ form a cone complex in $\mathbb{R}^n$ called the normal fan of $P$.

Proposition 2.4. The regular triangulation $\Delta_c$ of $\text{cone}(A)$ is the normal fan of the polyhedron $P_c := \{ y \in \mathbb{R}^d : yA \leq c \}$.

Proof. The polyhedron $P_c$ is the feasible region of $\text{maximize } \{ y \cdot b : yA \leq c, y \in \mathbb{R}^d \}$, the dual program to $LP_{A,c}(b)$. The support of the normal fan of $P_c$ is $\text{cone}(A)$, since this is the polar cone of the recession cone $\{ y \in \mathbb{R}^d : yA \leq 0 \}$ of $P_c$. Suppose $b$ is any vector in the interior of a maximal face $\text{cone}(A_\sigma)$ of $\Delta_c$. Then by Proposition 2.3 $LP_{A,c}(b)$ has an optimal solution $x^*$ with support $\sigma$. By complementary slackness, the optimal solution $y$ to the dual of $LP_{A,c}(b)$ satisfies $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j \leq c_j$ otherwise. Since $\sigma$ is a maximal face of $\Delta_c$, $y \cdot a_j < c_j$ for all $j \notin \sigma$. Thus $y$ is unique, and $\text{cone}(A_\sigma)$ is contained in the normal cone of $P_c$ at the vertex $y$. If $b$ lies in the interior of another maximal face $\text{cone}(A_\tau)$ then $y'$, (the dual optimal solution to $LP_{A,c}(b)$) satisfies $y' \cdot A_\tau = c_\tau$ and $y' \cdot A_\tau < c_\tau$ where $\tau \neq \sigma$. As a result, $y'$ is distinct from $y$, and each maximal cone in $\Delta_c$ lies in a distinct maximal cone in the normal fan of $P_c$. Since $\Delta_c$ and the normal fan of $P_c$ are both cone complexes with the same support, they must therefore coincide.

Example 2.2 continued.

Figure 3 (a) shows the polyhedron $P_c$ for Example 2.2 (i) with all its normal cones. The normal fan of $P_c$ is drawn in Figure 3 (b). Compare this fan with that in Figure 1 (b).

Corollary 2.5. The polyhedron $P_c$ is simple if and only if the regular subdivision $\Delta_c$ is a triangulation of $\text{cone}(A)$.

Regular triangulations were introduced by Gel’fand, Kapranov and Zelevinsky \cite{12} and have various applications. They have played a central role in the algebraic study of integer programming \cite{37, 38}, and we use them now to define group relaxations of $IP_{A,c}(b)$.

A subset $\tau$ of $\{1, \ldots, n\}$ partitions $x = (x_1, \ldots, x_n)$ as $x_\tau$ and $x_{\bar{\tau}}$ where $x_\tau$ consists of the variables indexed by $\tau$ and $x_{\bar{\tau}}$ the variables indexed by the complementary set $\bar{\tau}$. Similarly, the matrix $A$ is partitioned as $A = [A_\tau, A_{\bar{\tau}}]$ and the cost vector as $c = (c_\tau, c_{\bar{\tau}})$. If $\sigma$ is a maximal face of $\Delta_c$, then $A_\sigma$ is nonsingular and $Ax = b$ can be written as $x_\sigma = A_\sigma^{-1}(b-A_\sigma x_\sigma)$. 
Then $c \cdot x = c_{\sigma}(A^{-1}_{\sigma}(b - A_{\sigma}x_{\sigma}))+c_{\sigma}x_{\sigma} = c_{\sigma}A_{\sigma}^{-1}b + (c_{\sigma} - c_{\sigma}A_{\sigma}^{-1}A_{\sigma})x_{\sigma}$. Let $\tilde{c}_{\sigma} := c_{\sigma} - c_{\sigma}A_{\sigma}^{-1}A_{\sigma}$ and, for any face $\tau$ of $\sigma$, let $\tilde{c}_{\tau}$ be the extension of $\tilde{c}_{\sigma}$ to a vector in $\mathbb{R}^{|\pi|}$ by adding zeros.

We now define a group relaxation of $IP_{A,c}(b)$ with respect to each face $\tau$ of $\Delta_c$.

**Definition 2.6.** The group relaxation of the integer program $IP_{A,c}(b)$ with respect to the face $\tau$ of $\Delta_c$ is the program:

$$G^{\tau}(b) = \text{minimize } \{ \tilde{c}_{\tau} \cdot x_{\tau} : A_{\tau}x_{\tau} + A_{\tau}x_{\tau} = b, \ x_{\tau} \geq 0, (x_{\tau}, x_{\tau}) \in \mathbb{Z}^n \}.$$ 

Equivalently, $G^{\tau}(b) = \text{minimize } \{ \tilde{c}_{\tau} \cdot x_{\tau} : A_{\tau}x_{\tau} \equiv b \ (\text{mod } ZA_{\tau}), \ x_{\tau} \geq 0, \text{ integer} \}$ where $ZA_{\tau}$ is the lattice generated by the columns of $A_{\tau}$. Suppose $x_{\tau}^*$ is an optimal solution to the latter formulation. Since $\tau$ is a face of $\Delta_c$, the columns of $A_{\tau}$ are linearly independent, and therefore the linear system $A_{\tau}x_{\tau} + A_{\tau}x_{\tau}^* = b$ has a unique solution. Solving this system for $x_{\tau}$, the optimal solution $x_{\tau}^*$ of $G^{\tau}(b)$ can be uniquely lifted to the solution $(x_{\tau}^*, x_{\tau}^*)$ of $Ax = b$. The formulation of $G^{\tau}(b)$ in Definition 2.6 shows that $x_{\tau}^*$ is an integer vector. The group relaxation $G^{\tau}(b)$ solves $IP_{A,c}(b)$ if and only if $x_{\tau}^*$ is also non-negative.

The group relaxations of $IP_{A,c}(b)$ from Definition 2.6 contain among them the classical group relaxations of $IP_{A,c}(b)$ found in the literature. The program $G^{\sigma}(b)$, where $A_{\sigma}$ is the optimal basis of the linear relaxation $LP_{A,c}(b)$, is precisely Gomory’s group relaxation of $IP_{A,c}(b)$ [13]. The set of relaxations $G^{\tau}(b)$ as $\tau$ varies among the subsets of $\sigma$ are the extended group relaxations of $IP_{A,c}(b)$ defined by Wolsey [12]. Since $\emptyset \in \Delta_c$, $G^{\emptyset}(b) = IP_{A,c}(b)$ is a group relaxation of $IP_{A,c}(b)$, and hence $IP_{A,c}(b)$ will certainly be solved by one of its
extended group relaxations. However, it is possible to construct examples where a group relaxation $G^r(b)$ solves $IP_{A,c}(b)$, but $G^r(b)$ is neither Gomory’s group relaxation of $IP_{A,c}(b)$ nor one of its nontrivial extended Wolsey relaxations (see Example 4.2). Thus, Definition 2.6 typically creates more group relaxations for each program than in the classical situation. This has the obvious advantage that it increases the chance that $IP_{A,c}(b)$ will be solved by some non-trivial relaxation, although one may have to keep track of many more relaxations for each program. In Theorem 2.8, we will prove that Definition 2.6 is the best possible in the sense that the relaxations of $IP_{A,c}(b)$ defined there are precisely all the bounded group relaxations of the program.

The goal in the rest of this section is to describe a useful reformulation of the group problem $G^r(b)$ which is needed in the rest of the paper and in the proof of Theorem 2.8. Given a sublattice $\Lambda$ of $\mathbb{Z}^n$, a cost vector $w \in \mathbb{R}^n$ and a vector $v \in \mathbb{N}^n$, the lattice program defined by this data is

$$\text{minimize } \{ w \cdot x : x \equiv v \text{ (mod } \Lambda), \ x \in \mathbb{N}^n \}.$$  

Let $\mathcal{L}$ denote the $(n-d)$-dimensional saturated lattice $\{ x \in \mathbb{Z}^n : Ax = 0 \} \subseteq \mathbb{Z}^n$ and $u$ be a feasible solution of the integer program $IP_{A,c}(b)$. Since $IP_{A,c}(b) = \text{minimize } \{ c \cdot x : Ax = b (= Au), \ x \in \mathbb{N}^n \}$ can be rewritten as $\text{minimize } \{ c \cdot x : x - u \in \mathcal{L}, \ x \in \mathbb{N}^n \}$, $IP_{A,c}(b)$ is equivalent to the lattice program

$$\text{minimize } \{ c \cdot x : x \equiv u \text{ (mod } \mathcal{L}), \ x \in \mathbb{N}^n \}.$$  

For $\tau \subseteq \{1, \ldots, n\}$, let $\pi_\tau$ be the projection map from $\mathbb{R}^n \rightarrow \mathbb{R}^{|\tau|}$ that kills all coordinates indexed by $\tau$. Then $\mathcal{L}_\tau := \pi_\tau(\mathcal{L})$ is a sublattice of $\mathbb{Z}^{|\tau|}$ that is isomorphic to $\mathcal{L}$: Clearly, $\pi_\tau : \mathcal{L} \rightarrow \mathcal{L}_\tau$ is a surjection. If $\pi_\tau(v) = \pi_\tau(v')$ for $v, v' \in \mathcal{L}$, then $A_\tau v_\tau + A_\tau v_\tau = 0 = A_\tau v'_\tau + A_\tau v'_\tau$, implies that $A_\tau (v_\tau - v'_\tau) = 0$. Then $v_\tau = v'_\tau$ since the columns of $A_\tau$ are linearly independent. Using this fact, $G^r(b)$ can also be reformulated as a lattice program:

$$G^r(b) = \text{minimize } \{ \bar{c}_\tau \cdot x_\tau : A_\tau x_\tau + A_\tau x_\tau = b, \ x_\tau \geq 0, (x_\tau, x_\tau) \in \mathbb{Z}^n \}$$

$$= \text{minimize } \{ \bar{c}_\tau \cdot x_\tau : (x_\tau, x_\tau) - (u_\tau, u_\tau) \in \mathcal{L}, \ x_\tau \in \mathbb{N}^{|\tau|} \}$$

$$= \text{minimize } \{ \bar{c}_\tau \cdot x_\tau : x_\tau - u_\tau \in \mathcal{L}_\tau, \ x_\tau \in \mathbb{N}^{|\tau|} \}$$

$$= \text{minimize } \{ \bar{c}_\tau \cdot x_\tau : x_\tau \equiv \pi_\tau(u) \text{ (mod } \mathcal{L}_\tau), \ x_\tau \in \mathbb{N}^{|\tau|} \}.$$  

Lattice programs were shown to be solved by Gröbner bases in [39]. Theorem 5.3 in [39] gives a geometric interpretation of these Gröbner bases in terms of corner polyhedra. This paper was the first to make a connection between the theory of group relaxations and commutative algebra (see [39, §6]). Special results are possible when the sublattice $\Lambda$ is of finite index. In particular, the associated Gröbner bases are easier to compute.

Since the $(n-d)$-dimensional lattice $\mathcal{L} \subset \mathbb{Z}^n$ is isomorphic to $\mathcal{L}_\sigma \subset \mathbb{Z}^{|\sigma|}$ for $\sigma \in \Delta_c$, $\mathcal{L}_\sigma$ is of finite index if and only if $\sigma$ is a maximal face of $\Delta_c$. Hence the group relaxations $G^r(b)$ as $\sigma$ varies over the maximal faces of $\Delta_c$ are the easiest to solve among all group relaxations of $IP_{A,c}(b)$. They contain among them Gomory’s group relaxation of $IP_{A,c}(b)$. We give them a collective name in the following definition.

**Definition 2.7.** The group relaxations $G^r(b)$ of $IP_{A,c}(b)$, as $\sigma$ varies among the maximal faces of $\Delta_c$, are called the *Gomory relaxations of $IP_{A,c}(b)$.*
It is useful to reformulate $G^\tau(b)$ once again as follows. Let $B \in \mathbb{Z}^{n \times (n-d)}$ be any matrix such that the columns of $B$ generate the lattice $\mathcal{L}$, and let $u$ be a feasible solution of $IP_{A,c}(b)$ as before. Then

$$IP_{A,c}(b) = \text{minimize } \{c \cdot x : x - u \in \mathcal{L}, \ x \in \mathbb{N}^n\}$$

$$= \text{minimize } \{c \cdot x : x = u - Bz, \ x \geq 0, \ z \in \mathbb{Z}^{n-d}\}.$$  

The last problem is equivalent to $\text{minimize } \{c \cdot (u - Bz) : Bz \leq u, \ z \in \mathbb{Z}^{n-d}\}$ and, therefore $IP_{A,c}(b)$ is equivalent to the problem

$$(4) \quad \text{minimize } \{(-cB) \cdot z : Bz \leq u, \ z \in \mathbb{Z}^{n-d}\}.$$  

There is a bijection between the set of feasible solutions of (4) and the set of feasible solutions of $IP_{A,c}(b)$ via the isomorphism $z \mapsto u - Bz$. In particular, $0 \in \mathbb{R}^{n-d}$ is feasible for (4) and it is the pre-image of $u$ under this map.

If $B^\tau$ denotes the $|\tau| \times (n-d)$ submatrix of $B$ obtained by deleting the rows indexed by $\tau$, then $\mathcal{L}_\tau = \pi_\tau(\mathcal{L}) = \{B^\tau z : z \in \mathbb{Z}^{n-d}\}$. Using the same techniques as above, $G^\tau(b)$ can be reformulated as

$$\text{minimize } \{(-\tilde{c}_\tau B^\tau) \cdot z : B^\tau z \leq \pi_\tau(u), \ z \in \mathbb{Z}^{n-d}\}.$$  

Since $\tilde{c}_\tau = \pi_\tau(c - c_\sigma A_\sigma^{-1}A)$ for any maximal face $\sigma$ of $\Delta_c$ containing $\tau$ and the support of $c - c_\sigma A_\sigma^{-1}A$ is contained in $\tau$, $\tilde{c}_\tau B^\tau = (c - c_\sigma A_\sigma^{-1}A)B = cB$ since $AB = 0$. Hence $G^\tau(b)$ is equivalent to

$$(5) \quad \text{minimize } \{(-cB) \cdot z : B^\tau z \leq \pi_\tau(u), \ z \in \mathbb{Z}^{n-d}\}.$$  

The feasible solutions to (4) are the lattice points in the rational polyhedron $P_u := \{z \in \mathbb{R}^{n-d} : Bz \leq u\}$, and the feasible solutions to (5) are the lattice points in the relaxation $P_u^\tau := \{z \in \mathbb{R}^{n-d} : B^\tau z \leq \pi_\tau(u)\}$ of $P_u$ obtained by deleting the inequalities indexed by $\tau$.

In theory, one could define group relaxations of $IP_{A,c}(b)$ with respect to any $\tau \subseteq \{1, \ldots, n\}$. The following theorem illustrates the completeness of Definition 2.6.

**Theorem 2.8.** The group relaxation $G^\tau(b)$ of $IP_{A,c}(b)$ has a finite optimal solution if and only if $\tau \subseteq \{1, \ldots, n\}$ is a face of $\Delta_c$.

**Proof.** Since all data are integral it suffices to prove that the linear relaxation

$$\text{minimize } \{(-cB) \cdot z : z \in P_u^\tau\}$$

is bounded if and only if $\tau \in \Delta_c$.

If $\tau$ is a face of $\Delta_c$ then there exists $y \in \mathbb{R}^d$ such that $yA_\tau = c_\tau$ and $yA_\tau < c_\tau$. Using the fact that $A_\tau B^\tau + A_\tau B^\tau = 0$ we see that $cB = c_\tau B^\tau + c_\tau B^\tau = yA_\tau B^\tau + c_\tau B^\tau = y(-A_\tau B^\tau) + c_\tau B^\tau = (c_\tau - yA_\tau)B^\tau$. This implies that $cB$ is a positive linear combination of the rows of $B^\tau$ since $c_\tau - yA_\tau > 0$. Hence $cB$ lies in the polar of $\{z \in \mathbb{R}^{n-d} : B^\tau z \leq 0\}$ which is the recession cone of $P_u^\tau$ proving that the linear program $\text{minimize } \{(-cB) \cdot z : z \in P_u^\tau\}$ is bounded.

The linear program $\text{minimize } \{(-cB) \cdot z : z \in P_u^\tau\}$ is feasible since 0 is a feasible solution. If it is bounded as well then $\text{minimize } \{c_\tau x_\tau + c_\tau x_\tau : A_\tau x_\tau + A_\tau x_\tau = b, \ x_\tau \geq 0\}$ is feasible and bounded. As a result, the dual of the latter program $\text{maximize } \{y \cdot b : yA_\tau = c_\tau, \ yA_\tau \leq c_\tau\}$ is feasible. This shows that a superset of $\tau$ is a face of $\Delta_c$ which implies that $\tau \in \Delta_c$ since $\Delta_c$ is a triangulation. \qed
3. Associated Sets

The group relaxation \(G^\tau(b)\) (seen as \([3]\)) solves the integer program \(IP_{A,c}(b)\) (seen as \([4]\)) if and only if both programs have the same optimal solution \(z^* \in \mathbb{Z}^{n-d}\). If \(G^\tau(b)\) solves \(IP_{A,c}(b)\) then \(G^\tau(b)\) also solves \(IP_{A,c}(b)\) for every \(\tau' \subset \tau\) since \(G^\tau(b)\) is a stricter relaxation of \(IP_{A,c}(b)\) than \(G^\tau(b)\). For the same reason, one would expect that \(G^\tau(b)\) is easier to solve than \(G^\tau(b)\). Therefore, the most useful group relaxations of \(IP_{A,c}(b)\) are those indexed by the maximal elements in the subcomplex of \(\Delta_c\) consisting of all faces \(\tau\) such that \(G^\tau(b)\) solves \(IP_{A,c}(b)\). The following definition isolates such relaxations.

**Definition 3.1.** A face \(\tau\) of the regular triangulation \(\Delta_c\) is an associated set of \(IP_{A,c}\) (or is associated to \(IP_{A,c}\)) if for some \(b \in \mathbb{N}^A\), \(G^\tau(b)\) solves \(IP_{A,c}(b)\) but \(G^\tau(b)\) does not for all faces \(\tau' \subset \tau\).

The associated sets of \(IP_{A,c}\) carry all the information about all the group relaxations needed to solve the programs in \(IP_{A,c}\). In this section we will develop tools to understand these sets. We start by considering the set \(\mathcal{O}_c \subset \mathbb{N}^n\) of all the optimal solutions of all programs in \(IP_{A,c}\). A basic result in the algebraic study of integer programming is that \(\mathcal{O}_c\) is an order ideal or down set in \(\mathbb{N}^n\), i.e., if \(u \in \mathcal{O}_c\) and \(v \leq u\), then \(v \in \mathcal{O}_c\). One way to prove this is to show that the complement \(\mathcal{N}_c := \mathbb{N}^n \setminus \mathcal{O}_c\) has the property that if \(v \in \mathcal{N}_c\) then \(v + \mathbb{N}^n \subseteq \mathcal{N}_c\). Every lattice point in \(\mathbb{N}^n\) is a feasible solution to a unique program in \(IP_{A,c}\) \((u \in \mathbb{N}^n\) is feasible for \(IP_{A,c}(Au)\)). Hence, \(\mathcal{N}_c\) is the set of all non-optimal solutions of all programs in \(IP_{A,c}\). A set \(P \subseteq \mathbb{N}^n\) with the property that \(P + \mathbb{N}^n \subseteq P\) whenever \(P \in \mathbb{N}^n\) has a finite set of minimal elements. Hence there exists \(\alpha_1, \ldots, \alpha_t \in \mathcal{N}_c\) such that

\[
\mathcal{N}_c = \bigcup_{i=1}^t (\alpha_i + \mathbb{N}^n).
\]

As a result, \(\mathcal{O}_c\) is completely specified by the finitely many “generators” \(\alpha_1, \ldots, \alpha_t\) of its complement \(\mathcal{N}_c\). See [10] for proofs of these assertions.

**Example 3.2.** Consider the family of knapsack problems:

\[
\text{minimize} \quad \{10000x_1 + 100x_2 + x_3 : 2x_1 + 5x_2 + 8x_3 = b, \quad (x_1, x_2, x_3) \in \mathbb{N}^3\}
\]

as \(b\) varies in the semigroup \(\mathbb{N}[2 5 8]\). The set \(\mathcal{N}_c\) is generated by the vectors

\[
(0, 8, 0), (1, 0, 1), (1, 6, 0), (2, 4, 0), (3, 2, 0), \text{ and } (4, 0, 0)
\]

which means that \(\mathcal{N}_c = ((0, 8, 0) + \mathbb{N}^3) \cup \cdots \cup ((4, 0, 0) + \mathbb{N}^3)\). Figure [3] is a picture of \(\mathcal{N}_c\) (created by Ezra Miller). The white points are its generators. One can see that \(\mathcal{O}_c\) consists of finitely many points of the form \((p, q, 0)\) where \(p \geq 1\) and the eight “lattice lines” of points \((0, i, *)\), \(i = 0, \ldots, 7\).

For the purpose of computations, it is most effective to think of \(\mathcal{N}_c\) and \(\mathcal{O}_c\) algebraically. A monomial \(x^u\) in the polynomial ring \(S := k[x_1, \ldots, x_n]\) is a product \(x^u = x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}\) where \(u = (u_1, \ldots, u_n) \in \mathbb{N}^n\). We assume that \(k\) is a field, say the set of rational numbers. For a scalar \(k_u \in k\) and a monomial \(x^u\) in \(S\), we call \(k_u x^u\) a term of \(S\). A polynomial \(f = \sum k_u x^u\) in \(S\) is a combination of finitely many terms in \(S\). A subset \(I\) of \(S\) is an ideal of \(S\) if (1) \(I\) is closed under addition, i.e., \(f, g \in I \Rightarrow f + g \in I\) and (2) if \(f \in I\) and
g ∈ S then fg ∈ I. We say that I is generated by the polynomials f₁, . . . , fₜ, denoted as 
I = ⟨f₁, . . . , fₜ⟩, if I = {∑ᵢ₌₁^ₜ fᵢgᵢ : gᵢ ∈ S}. By Hilbert’s basis theorem, every ideal in S 
has a finite generating set. An ideal M in S is called a monomial ideal if it is generated by 
monomials, i.e., M = ⟨xᵥ¹, . . . , xᵥᵢ⟩ for monomials xᵥⁱ, . . . , xᵥᵢ in S. The monomials that do 
not lie in M are called the standard monomials of M. The cost of a term kᵥxᵥ with respect 
to a vector c ∈ ℜⁿ is the dot product c · u. The initial term of a polynomial f = ∑ kᵥxᵥ ∈ S 
with respect to c, denoted as inᶜ(f), is the sum of all terms in f of maximal cost. For any 
ideal I ⊂ S, the initial ideal of I with respect to c, denoted as inᶜ(I), is the ideal generated 
by all the initial terms inᶜ(f) of all polynomials f in I. These concepts come from the theory 
of Gröbner bases for polynomial ideals. See [9] for an introduction.

The toric ideal of the matrix A, denoted as IA, is the binomial ideal in S defined as:

\[ I_A := ⟨x^u - x^v : u, v ∈ ℤ^n \text{ and } Au = Av⟩. \]

Toric ideals provide the link between integer programming and Gröbner basis theory. See

[36] and [41] for an introduction to this area of research. This connection yields the following 
basic facts that we state without proofs. (Recall that the cost vector c of IPₐ,c was assumed 
to be generic in the sense that each program in IPₐ,c has a unique optimal solution.)

Lemma 3.3. [36] (i) If c is generic, then the initial ideal inᶜ(Iₐ) is a monomial ideal.
(ii) A lattice point u is non-optimal for the integer program IPₐ,c(Au), or equivalently, 
u ∈ N_c, if and only if xᵘ lies in the initial ideal inᶜ(Iₐ). In other words, a lattice point u 
lies in O_c if and only if xᵘ is a standard monomial of inᶜ(Iₐ).
(iii) The reduced Gröbner basis G_c of Iₐ with respect to c is the unique minimal test set 
for the family of integer programs IPₐ,c.

Figure 4. The set of non-optimal solutions N_c for Example 3.2.
(iv) If \( u \) is a feasible solution of \( IP_{A,c}(b) \), and \( x^u \) is the unique normal form of \( x^u \) with respect to \( G_c \), then \( u^* \) is the optimal solution of \( IP_{A,c}(b) \).

We do not elaborate on parts (iii) and (iv) of Lemma 3.3. They are not needed for what follows and are included for completeness. Since \( c \) is generic, Lemma 3.3 (ii) implies that there is a bijection between the lattice points of \( O_c \) and the semigroup \( \mathbb{N}A \) via the map \( \phi_A : O_c \to \mathbb{N}A \) such that \( u \mapsto Au \). The inverse of \( \phi_A \) sends a vector \( b \in \mathbb{N}A \) to the optimal solution of \( IP_{A,c}(b) \).

**Example 3.2 continued.** In this example, the toric ideal \( I_A = \langle x_1^4 - x_3, x_2^2 - x_1x_3 \rangle \) and its initial ideal with respect to the cost vector \( c = (10000, 100, 1) \) is

\[
in_c(I_A) = \langle x_8^2, x_1x_3, x_1x_2^6, x_2^3x_2, x_1^4 \rangle.
\]

Note that the exponent vectors of the generators of \( in_c(I_A) \) are the generators of \( N_c \). \( \square \)

We will now describe a certain decomposition of the set \( O_c \) which in turn will shed light on the associated sets of \( IP_{A,c} \). For \( u \in \mathbb{N}^n \), consider \( Q_u := \{ z \in \mathbb{R}^n : Bz \leq u, \ (-cB) \cdot z \leq 0 \} \) and its relaxation \( Q_u^r := \{ z \in \mathbb{R}^n : B^r z \leq \pi_r(u), \ (-cB) \cdot z \leq 0 \} \) where \( B, B^r \) are as in (i) and (ii) and \( \tau \in \Delta_c \). By Theorem 2.3, both \( Q_u \) and \( Q_u^r \) are polytopes. Notice that if \( \pi_r(u) = \pi_r(u') \) for two distinct vectors \( u, u' \in \mathbb{N}^n \), then \( Q_u^r = Q_u^r \).

**Lemma 3.4.** (i) A lattice point \( u \) is in \( O_c \) if and only if \( Q_u \cap \mathbb{Z}^{n-d} = \{0\} \).
(ii) If \( u \in O_c \), then the group relaxation \( G^r(Au) \) solves the integer program \( IP_{A,c}(Au) \) if and only if \( Q_u^r \cap \mathbb{Z}^{n-d} = \{0\} \).

**Proof.** (i) The lattice point \( u \) belongs to \( O_c \) if and only if \( u \) is the optimal solution to \( IP_{A,c}(Au) \) which is equivalent to \( 0 \in \mathbb{Z}^{n-d} \) being the optimal solution to the reformulation (i) of \( IP_{A,c}(Au) \). Since \( c \) is generic, the last statement is equivalent to \( Q_u \cap \mathbb{Z}^{n-d} = \{0\} \). The second statement follows from (i) and the fact that (iii) solves (i) if and only if they have the same optimal solution. \( \square \)

In order to state the coming results, it is convenient to assume that the vector \( u \) in (i) and (ii) is the optimal solution to \( IP_{A,c}(b) \). For an element \( u \in O_c \) and a face \( \tau \) of \( \Delta_c \) let \( S(u, \tau) \) be the affine semigroup \( u + \mathbb{N}(e_i : i \in \tau) \subseteq \mathbb{N}^n \) where \( e_i \) denotes the \( i \)th unit vector of \( \mathbb{R}^n \). Note that \( S(u, \tau) \) is not a semigroup if \( u \neq 0 \) (since \( 0 \notin S(u, \tau) \)), but is a translation of the semigroup \( \mathbb{N}(e_i : i \in \tau) \). We use the adjective affine here as an affine subspace which is not a subspace but the translation of one. Note that if \( v \in S(u, \tau) \), then \( \pi_\tau(v) = \pi_\tau(u) \).

**Lemma 3.5.** For \( u \in O_c \) and a face \( \tau \) of \( \Delta_c \), the affine semigroup \( S(u, \tau) \) is contained in \( O_c \) if and only if \( G^r(Au) \) solves \( IP_{A,c}(Au) \).

**Proof.** Suppose \( S(u, \tau) \subseteq O_c \). Then by Lemma 3.3 (i), for all \( v \in S(u, \tau) \),

\[
Q_v = \{ z \in \mathbb{R}^{n-d} : B^r z \leq \pi_\tau(v), B^r z \leq \pi_\tau(u), (-cB) \cdot z \leq 0 \} \cap \mathbb{Z}^{n-d} = \{0\}.
\]

Since \( \pi_\tau(v) \) can be any vector in \( \mathbb{N}^{|\tau|} \), \( Q_u^r \cap \mathbb{Z}^{n-d} = \{0\} \). Hence, by Lemma 3.3 (ii), \( G^r(Au) \) solves \( IP_{A,c}(Au) \).
If \( v \in S(u, \tau) \), then \( \pi_\tau(u) = \pi_\tau(v) \), and hence \( Q^T_u = Q^T_v \). Therefore, if \( G^T(Au) \) solves \( IP_{A,c}(Au) \), then \( \{0\} = Q^T_u \cap \mathbb{Z}^{n-d} = Q^T_v \cap \mathbb{Z}^{n-d} \) for all \( v \in S(u, \tau) \). Since \( Q^T_v \) is a relaxation of \( Q_v \), \( Q_v \cap \mathbb{Z}^{n-d} = \{0\} \) for all \( v \in S(u, \tau) \) and hence by Lemma 3.4 (i), \( S(u, \tau) \subseteq \mathcal{O}_c \).

**Lemma 3.6.** For \( u \in \mathcal{O}_c \) and a face \( \tau \) of \( \Delta_c \), \( G^T(Au) \) solves \( IP_{A,c}(Au) \) if and only if \( G^T(Av) \) solves \( IP_{A,c}(Av) \) for all \( v \in S(u, \tau) \).

**Proof.** If \( v \in S(u, \tau) \) and \( G^T(Au) \) solves \( IP_{A,c}(Au) \), then as seen before, \( \{0\} = Q^T_u \cap \mathbb{Z}^{n-d} = Q^T_v \cap \mathbb{Z}^{n-d} \) for all \( v \in S(u, \tau) \). By Lemma 3.4 (ii), \( G^T(Av) \) solves \( IP_{A,c}(Av) \) for all \( v \in S(u, \tau) \). The converse holds for the trivial reason that \( u \in S(u, \tau) \).

**Corollary 3.7.** For \( u \in \mathcal{O}_c \) and a face \( \tau \) of \( \Delta_c \), the affine semigroup \( S(u, \tau) \) is contained in \( \mathcal{O}_c \) if and only if \( G^T(Av) \) solves \( IP_{A,c}(Av) \) for all \( v \in S(u, \tau) \).

Since \( \pi_\tau(u) \) determines the polytope \( Q^T_u = Q^T_v \) for all \( v \in S(u, \tau) \), we could have assumed that \( supp(u) \subseteq \bar{\tau} \) in Lemmas 3.3 and 3.6.

**Definition 3.8.** For \( \tau \in \Delta_c \) and \( u \in \mathcal{O}_c \), \( (u, \tau) \) is called an admissible pair of \( \mathcal{O}_c \) if

(i) the support of \( u \) is contained in \( \bar{\tau} \), and

(ii) \( S(u, \tau) \subseteq \mathcal{O}_c \) or equivalently, \( G^T(Av) \) solves \( IP_{A,c}(Av) \) for all \( v \in S(u, \tau) \).

An admissible pair \( (u, \tau) \) is a standard pair of \( \mathcal{O}_c \) if the affine semigroup \( S(u, \tau) \) is not properly contained in \( S(v, \tau') \) where \( (v, \tau') \) is another admissible pair of \( \mathcal{O}_c \).

**Example 3.2 continued.** From Figure 4, one can see that the standard pairs of \( \mathcal{O}_c \) are

\[
\begin{align*}
(1,0,0) & \quad (1,3,0) & \quad (0,0,0) & \quad (3) \\
(2,0,0) & \quad (2,3,0) & \quad (1,0) & \quad (3) \\
(3,0,0) & \quad (1,4,0) & \quad (0,2) & \quad (3) \\
(1,1,0) & \quad (1,5,0) & \quad (3) \\
(2,1,0) & \quad (4,0) & \quad (3) \\
(3,1,0) & \quad (5,0) & \quad (3) \\
(1,2,0) & \quad (6,0) & \quad (3) \\
(2,2,0) & \quad (7,0) & \quad (3) \\
\end{align*}
\]

and

\[
\begin{align*}
(0,3) & \quad (3) \\
(0,4) & \quad (3) \\
(0,5) & \quad (3) \\
(0,6) & \quad (3) \\
(0,7) & \quad (3) \\
\end{align*}
\]

**Definition 3.9.** For a face \( \tau \) of \( \Delta_c \) and a lattice point \( u \in \mathbb{N}^n \), we say that the polytope \( Q^T_u \) is a standard polytope of \( IP_{A,c} \) if \( Q^T_u \cap \mathbb{Z}^{n-d} = \{0\} \) and every relaxation of \( Q^T_u \) obtained by removing an inequality in \( B^\tau z \leq \pi_\tau(u) \) contains a non-zero lattice point.

Figure 5 is a diagram of a standard polytope \( Q^T_u \). The dashed line is the boundary of the half space \( (-cB) \cdot z \leq 0 \) while the other lines are the boundaries of the halfspaces given by the inequalities in \( B^\tau z \leq \pi_\tau(u) \). The origin is the only lattice point in the polytope and if any inequality in \( B^\tau z \leq \pi_\tau(u) \) is removed, a lattice point will enter the relaxation.

We re-emphasize that if \( Q^T_u \) is a standard polytope, then \( Q^T_{u'} \) is the same standard polytope if \( \pi_\tau(u) = \pi_\tau(u') \). Hence the same standard polytope can be indexed by infinitely many \( u \in \mathbb{N}^n \). We now state the main result of this section which characterizes associated sets in terms of standard pairs and standard polytopes.
Figure 5. A standard polytope.

Theorem 3.10. The following statements are equivalent:

(i) The admissible pair \((u, \tau)\) is a standard pair of \(O_c\).

(ii) The polytope \(Q^r_u\) is a standard polytope of \(IP_{A,c}\).

(iii) The face \(\tau\) of \(\Delta_c\) is associated to \(IP_{A,c}\).

Proof. (i) \(\Leftrightarrow\) (ii): The admissible pair \((u, \tau)\) is standard if and only if for every \(i \in \tau\), there exists some positive integer \(m_i\) and a vector \(v \in S(u, \tau)\) such that \(v + m_i e_i \in N_c\). (If this condition did not hold for some \(i \in \tau\), then \((u', \tau \cup \{i\})\) would be an admissible pair of \(O_c\) such that \(S(u', \tau \cup \{i\})\) contains \(S(u, \tau)\) where \(u'\) is obtained from \(u\) by setting the \(i\)th component of \(u\) to zero. Conversely, if the condition holds for an admissible pair then the pair is standard.) Equivalently, for each \(i \in \tau\), there exists a positive integer \(m_i\) and a \(v \in S(u, \tau)\) such that \(Q^r_{v+m_ie_i} = Q^r_{u+m_ie_i}\) contains at least two lattice points. In other words, the removal of the inequality indexed by \(i\) from the inequalities in \(B^rz \leq \pi(u)\) will bring an extra lattice point into the corresponding relaxation of \(Q^r_u\). This is equivalent to saying that \(Q^r_u\) is a standard polytope of \(IP_{A,c}\).

(i) \(\Leftrightarrow\) (iii): Suppose \((u, \tau)\) is a standard pair of \(O_c\). Then \(S(u, \tau) \subseteq O_c\) and \(G^r(Au)\) solves \(IP_{A,c}(Au)\) by Lemma 3.3. Suppose \(G^r(Au)\) solves \(IP_{A,c}(Au)\) for some face \(\tau' \subseteq \Delta_c\) such that \(\tau \subseteq \tau'\). Lemma 3.3 then implies that \(S(u, \tau')\) lies in \(O_c\). This contradicts the fact that \((u, \tau)\) was a standard pair of \(O_c\) since \(S(u, \tau)\) is properly contained in \(S(\hat{u}, \tau')\) corresponding to the admissible pair \((\hat{u}, \tau')\) where \(\hat{u}\) is obtained from \(u\) by setting \(u_i = 0\) for all \(i \in \tau' \setminus \tau\).

To prove the converse, suppose \(\tau\) is associated to \(IP_{A,c}\). Then there exists some \(b \in N.A\) such that \(G^r(b)\) solves \(IP_{A,c}(b)\) but \(G^r(b)\) does not for all faces \(\tau' \subseteq \Delta_c\) containing \(\tau\). Let \(u\) be the unique optimal solution of \(IP_{A,c}(b)\). By Lemma 3.3, \(S(u, \tau) \subseteq O_c\). Let \(\hat{u} \in N^n\) be obtained from \(u\) by setting \(u_i = 0\) for all \(i \in \tau\). Then \(G^r(A\hat{u})\) solves \(IP_{A,c}(A\hat{u})\) since \(Q^r_u = Q^r_{\hat{u}}\). Hence \(S(\hat{u}, \tau) \subseteq O_c\) and \((\hat{u}, \tau)\) is an admissible pair of \(O_c\). Suppose there exists another admissible pair \((w, \sigma)\) such that \(S(\hat{u}, \tau) \subseteq S(w, \sigma)\). Then \(\tau \subseteq \sigma\). If \(\tau = \sigma\) then \(S(\hat{u}, \tau)\) and \(S(w, \sigma)\) are both orthogonal translates of \(N(e_i : i \in \tau)\) and hence \(S(\hat{u}, \tau)\) cannot be properly contained in \(S(w, \sigma)\). Therefore, \(\tau\) is a proper subset of \(\sigma\) which implies...
that $S(\hat{u}, \sigma) \subseteq O_c$. Then, by Lemma 3.3, $G^\tau(A\hat{u})$ solves $IP_{A,c}(A\hat{u})$ which contradicts that $\tau$ was an associated set of $IP_{A,c}$.

\[\square\]

**Example 3.2 continued.** In Example 3.2 we can choose $B$ to be the $3 \times 2$ matrix

$$B = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ -1 & -1 \end{bmatrix}.$$ 

The standard polytope defined by the standard pair $((1, 0, 0), \emptyset)$ is hence

$$\{(z_1, z_2) \in \mathbb{R}^2 : -z_1 + 4z_2 \leq 1, 2z_1 \leq 0, -z_1 - z_2 \leq 0, 9801z_1 - 40001z_2 \leq 0\}$$

while the standard polytope defined by the standard pair $((0, 2, 0), \{3\})$ is

$$\{(z_1, z_2) \in \mathbb{R}^2 : -z_1 + 4z_2 \leq 0, 2z_1 \leq 2, 9801z_1 - 40001z_2 \leq 0\}.$$ 

The associated sets of $IP_{A,c}$ in this example are $\emptyset$ and $\{3\}$. There are twelve quadrangular and eight triangular standard polytopes for this family of knapsack problems. 

\[\square\]

Standard polytopes were introduced in [22], and the equivalence of parts (i) and (ii) of Theorem 3.10 was proved in [22, Theorem 2.5]. Under the linear map $\phi_A : \mathbb{N}^n \to \mathbb{N}A$ where $u \mapsto Au$, the affine semigroup $S(u, \tau)$ where $(u, \tau)$ is a standard pair of $O_c$ maps to the affine semigroup $Au + NA_r$ in $\mathbb{N}A$. Since every integer program in $IP_{A,c}$ is solved by one of its group relaxations, $O_c$ is covered by the affine semigroups corresponding to its standard pairs. We call this cover and its image in $\mathbb{N}A$ under $\phi_A$ the **standard pair decompositions** of $O_c$ and $\mathbb{N}A$, respectively. Since standard pairs of $O_c$ are determined by the standard polytopes of $IP_{A,c}$, the standard pair decomposition of $O_c$ is unique. The terminology used above has its origins in [38] which introduced the **standard pair decomposition of a monomial ideal**. The specialization to integer programming appear in [22, [23] and [36 §12.D]. The following theorem shows how the standard pair decomposition of $O_c$ dictates which group relaxations solve which programs in $IP_{A,c}$.

**Theorem 3.11.** Let $v$ be the optimal solution of the integer program $IP_{A,c}(b)$. Then the group relaxation $G^\tau(Av)$ solves $IP_{A,c}(Av)$ if and only if there is some standard pair $(u, \tau')$ of $O_c$ with $\tau \subseteq \tau'$ such that $v$ belongs to the affine semigroup $S(u, \tau')$.

**Proof.** Suppose $v$ lies in $S(u, \tau')$ corresponding to the standard pair $(u, \tau')$ of $O_c$. Then $S(v, \tau') \subseteq O_c$ which implies that $G^{\tau'}(Av)$ solves $IP_{A,c}(Av)$ by Lemma 3.3. Hence $G^{\tau'}(Av)$ also solves $IP_{A,c}(Av)$ for all $\tau \subseteq \tau'$.

To prove the converse, suppose $\tau'$ is a maximal element in the subcomplex of all faces $\tau$ of $\Delta_c$ such that $G^{\tau}(Av)$ solves $IP_{A,c}(Av)$. Then $\tau'$ is an associated set of $IP_{A,c}$. In the proof of (iii) $\Rightarrow$ (i) in Theorem 3.10, we showed that $(\hat{v}, \tau')$ is a standard pair of $O_c$ where $\hat{v}$ is obtained from $v$ by setting $v_i = 0$ for all $i \in \tau'$. Then $v \in S(\hat{v}, \tau')$.

\[\square\]

**Example 3.2 continued.** The eight standard pairs of $O_c$ of the form (*, \{3\}), map to the eight affine semigroups:

$$(N[8], (5 + N[8]), (10 + N[8]), (15 + N[8]), (20 + N[8]), (25 + N[8]), (30 + N[8]) \text{ and } (35 + N[8]))$$
For all right hand side vectors $b$ in the union of these sets, the integer program $IPA_c(b)$ can be solved by the group relaxation $G^{(3)}(b)$. The twelve standard pairs of the form $(*, \emptyset)$ map to the remaining finitely many points

\[ 2, 4, 6, 7, 9, 11, 12, 14, 17, 19, 22 \text{ and } 27 \]

of $\mathbb{N}[2, 5, 8]$. If $b$ is one of these points, then $IPA_c(b)$ can only be solved as the full integer program. In this example, the regular triangulation $\Delta_c = \{3\}$. Hence $G^{(3)}(b)$ is a Gomory relaxation of $IPA_c(b)$.

For most $b \in NA$, the program $IPA_c(b)$ is solved by one of its Gomory relaxations, or equivalently, by Theorem 3.11, the optimal solution $v$ of $IPA_c(b)$ lies in $S(*, \sigma)$ for some standard pair $(*, \sigma)$ where $\sigma$ is a maximal face of $\Delta_c$. For mathematical versions of this informal statement, see [36, Proposition 12.16] and [13, Theorems 1 and 2]. Roughly speaking, these right hand sides are away from the boundary of $\text{cone}(A)$. (This was seen in Example 3.2 above, where for all but twelve right hand sides, $IPA_c(b)$ was solvable by the Gomory relaxation $G^{(3)}(b)$. Further, these right hand sides were toward the boundary of $\text{cone}(A)$, the origin in this one-dimensional case.) For the remaining right hand sides, $IPA_c(b)$ can only be solved by $G^\tau(b)$ where $\tau$ is a lower dimensional face of $\Delta_c$ - possibly even the empty face. An important contribution of the algebraic approach here is the identification of the minimal set of group relaxations needed to solve all programs in the family $IPA_c$ and of the particular relaxations necessary to solve any given program in the family.

4. Arithmetic Degree

For an associated set $\tau$ of $IPA_c$ there are only finitely many standard pairs of $O_c$ indexed by $\tau$ since there are only finitely many standard polytopes of the form $Q^\tau_c$. Borrowing terminology from [38], we call the number of standard pairs of the form $(*, \tau)$ the multiplicity of $\tau$ in $O_c$ (abbreviated as $\text{mult}(\tau)$). The total number of standard pairs of $O_c$ is called the arithmetic degree of $O_c$. Our main goal in this section is to provide bounds for these invariants of the the family $IPA_c$ and discuss their relevance. We will need the following interpretation from Section 3.

**Corollary 4.1.** The multiplicity of the face $\tau$ of $\Delta_c$ in $O_c$ is the number of distinct standard polytopes of $IPA_c$ indexed by $\tau$, and the arithmetic degree of $O_c$ is the total number of standard polytopes of $IPA_c$.

**Proof.** This result follows from Theorem 3.10. \qed

**Example 3.2 continued.** The multiplicity of the associated set $\{3\}$ is eight while the empty set has multiplicity twelve. The arithmetic degree of $O_c$ is hence twenty. \qed

If the standard pair decomposition of $O_c$ is known, then we can solve all programs in $IPA_c$ by solving (arithmetic degree)-many linear systems as follows. For a given $b \in NA$ and a standard pair $(u, \tau)$, consider the linear system

\[ A_\tau \pi(\tau) + A_\tau x = b, \text{ or equivalently, } A_\tau x = b - A_\tau \pi(\tau). \]
As $\tau$ is a face of $\Delta_c$, this linear system can be solved uniquely for $x$. Since the optimal solution of $IP_{A,c}(b)$ lies in $S(w, \sigma)$ for some standard pair $(w, \sigma)$ of $O_c$, at least one non-negative and integral solution for $x$ will be found as we solve the linear systems (3) obtained by varying $(u, \tau)$ over all the standard pairs of $O_c$. If the standard pair $(u, \tau)$ yields such a solution $v$, then $(\pi_\tau(u), v)$ is the optimal solution of $IP_{A,c}(b)$. This pre-processing of $IP_{A,c}$ has the same flavor as [27]. The main result in [27] is that given a coefficient matrix $A \in \mathbb{R}^{m \times n}$ and cost vector $c$, there exists floor functions $f_1, \ldots, f_k : \mathbb{R}^m \rightarrow \mathbb{Z}^n$ such that for a right hand side vector $b$, the optimal solution of the corresponding integer program is the one among $f_1(b), \ldots, f_k(b)$ that is feasible and attains the best objective function value. The crucial point is that this algorithm runs in time bounded above by a polynomial in the length of the data for fixed $n$ and $j$, where $j$ is the affine dimension of the space of right hand sides. Given this result, it is interesting to bound arithmetic degree.

The second equation in (6) suggests that one could think of the first arguments $u$ in the standard pairs $(u, \tau)$ of $O_c$ as “correction vectors” that need to be applied to find the optimal solutions of programs in $IP_{A,c}$. Thus the arithmetic degree of $O_c$ is the total number of correction vectors that are needed to solve all programs in $IP_{A,c}$. The multiplicities of associated sets give a finer count of these correction vectors, organized by faces of $\Delta_c$. If the optimal solution of $IP_{A,c}(b)$ lies in the affine semigroup $S(w, \sigma)$ given by the standard pair $(w, \sigma)$ of $O_c$, then $w$ is a correction vector for this $b$ as well as all other $b$’s in $(Au + NA_c)$. One obtains all correction vectors for $IP_{A,c}$ by solving the (arithmetic degree)-many integer programs with right hand sides $Au$ for all standard pairs $(u, \tau)$ of $O_c$. See [44] for a similar result from the classical theory of group relaxations.

In Example 3.2, $\Delta_c = \{\{3\}\}$ and both its faces $\{3\}$ and $\emptyset$ are associated to $IP_{A,c}$. In general, not all faces of $\Delta_c$ need be associated sets of $IP_{A,c}$ and the poset of associated sets can be quite complicated. (We will study this poset in Section 5.) Hence, for $\tau \in \Delta_c$, $\text{mult}(\tau) = 0$ unless $\tau$ is an associated set of $IP_{A,c}$. We will now prove that all maximal faces of $\Delta_c$ are associated sets of $IP_{A,c}$. Further, if $\sigma$ is a maximal face of $\Delta_c$ then $\text{mult}(\sigma)$ is the absolute value of $\text{det}(A)$ divided by the g.c.d. of the maximal minors of $A$. This g.c.d. is non-zero since $A$ has full row rank. If the columns of $A$ span an affine hyperplane, then the absolute value of $\text{det}(A)$ divided by the g.c.d. of the maximal minors of $A$ is called the normalized volume of the face $\sigma$ in $\Delta_c$. We first give a non-trivial example.

**Example 4.2.** Consider the rank three matrix

$$A = \begin{bmatrix}
5 & 0 & 0 & 2 & 1 & 0 \\
0 & 5 & 0 & 1 & 4 & 2 \\
0 & 0 & 5 & 2 & 0 & 3
\end{bmatrix}$$

and the generic cost vector $c = (21, 6, 1, 0, 0, 0)$. The first three columns of $A$ generate $\text{cone}(A)$ which is simplicial. The regular triangulation

$$\Delta_c = \{\{1, 3, 4\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}, \{4, 5, 6\}\}$$

is shown in Figure 3 as a triangulation of $\text{conv}(A)$. The six columns of $A$ have been labeled by their column indices. The arithmetic degree of $O_c$ in this example is 70. The following table shows all the standard pairs organized by associated sets and the multiplicity of each associated set. Note that all maximal faces of $\Delta_c$ are associated to $IP_{A,c}$. The g.c.d. of the
maximal minors of $A$ is five. Check that $\text{mult}(\sigma)$ is the normalized volume of $\sigma$ whenever $\sigma$ is a maximal face of $\Delta_c$.

| $\tau$  | standard pairs ($\cdot, \tau$)                                      | $\text{mult}(\tau)$ |
|---------|--------------------------------------------------------------------|----------------------|
| $\{1, 3, 4\}$ | $(0, \cdot), (e_5, \cdot), (e_6, \cdot), (e_5 + e_6, \cdot), (2e_6, \cdot)$ | 5                    |
| $\{1, 4, 5\}$ | $(0, \cdot), (e_2, \cdot), (e_3, \cdot), (e_6, \cdot), (e_2 + e_3, \cdot), (2e_2, \cdot)$ | 8                    |
| $\{2, 5, 6\}$ | $(0, \cdot), (e_3, \cdot), (2e_3, \cdot)$ | 3                    |
| $\{3, 4, 6\}$ | $(0, \cdot), (e_5, \cdot), (2e_5, \cdot), (3e_5, \cdot)$ | 4                    |
| $\{4, 5, 6\}$ | $(0, \cdot), (e_3, \cdot), (2e_3, \cdot), (3e_3, \cdot), (4e_3, \cdot)$ | 5                    |
| $\{1, 4\}$ | $(e_3 + 2e_5 + e_6, \cdot), (2e_3 + 2e_5 + e_6, \cdot), (2e_3 + 2e_5, \cdot), (2e_3 + 3e_5, \cdot), (2e_3 + 4e_5, \cdot)$ | 5                    |
| $\{1, 5\}$ | $(e_2 + e_6, \cdot), (2e_2 + e_6, \cdot), (3e_2 + e_6, \cdot)$ | 3                    |
| $\{2, 5\}$ | $(e_3 + e_4, \cdot), (e_4, \cdot), (2e_4, \cdot)$ | 3                    |
| $\{3, 4\}$ | $(e_2, \cdot), (e_1 + e_2, \cdot), (e_1 + 2e_5, \cdot), (e_1 + 2e_5 + e_6, \cdot), (e_2 + e_5, \cdot)$ | 5                    |
| $\{3, 6\}$ | $(e_2, \cdot), (e_2 + e_5, \cdot)$ | 2                    |
| $\{4, 5\}$ | $(e_2 + 2e_3, \cdot), (e_2 + 3e_3, \cdot), (2e_2 + 2e_3, \cdot), (3e_2 + e_3, \cdot), (4e_2, \cdot)$ | 5                    |
| $\{5, 6\}$ | $(e_2 + 3e_3, \cdot)$ | 1                    |
| $\{1\}$ | $(e_2 + e_3 + e_6, \cdot), (e_2 + e_3 + e_5 + e_6, \cdot), (e_2 + 2e_6, \cdot), (e_2 + e_3 + 2e_6, \cdot), (2e_2 + 2e_6, \cdot), (2e_2 + e_6, \cdot), (e_2 + e_3 + 2e_6 + e_6, \cdot)$ | 6                    |
| $\{3\}$ | $(e_1 + e_2 + e_6, \cdot), (e_1 + e_2 + 2e_6, \cdot)$ | 2                    |
| $\{4\}$ | $(e_1 + e_2 + 2e_3 + e_5, \cdot), (e_1 + e_2 + 2e_3 + 2e_5, \cdot), (e_1 + e_2 + 2e_3 + 3e_5, \cdot), (e_1 + e_2 + 2e_3 + 4e_5, \cdot), (e_1 + 3e_3 + 3e_5, \cdot), (e_1 + 3e_3 + 4e_5, \cdot)$ | 6                    |
| $\{\emptyset\}$ | $(e_1 + e_2 + 2e_3 + e_5 + e_6, \cdot), (e_1 + e_2 + 2e_3 + 2e_5 + e_6, \cdot), (e_1 + e_2 + e_3 + e_6, \cdot), (e_1 + 2e_2 + e_3 + e_5 + e_6, \cdot), (e_1 + 2e_2 + e_3 + 2e_5 + e_6, \cdot), (e_1 + 2e_2 + e_3 + 2e_6, \cdot), (e_1 + 3e_2 + 2e_6, \cdot)$ | 7                    |

Arithmetic Degree 70
Observe that the integer program $IP_{A,c}(b)$ where $b = A(e_1 + e_2 + e_3)$ is solved by $G^\tau(b)$ with $\tau = \{1, 4, 5\}$. By Proposition 2.3, Gomory’s relaxation of $IP_{A,c}(b)$ is indexed by $\sigma = \{4, 5, 6\}$ since $b$ lies in the interior of the face $cone(A_\sigma)$ of $\Delta_c$. However, neither this relaxation nor any nontrivial extended relaxation solves $IP_{A,c}(b)$ since the optimal solution $e_1 + e_2 + e_3$ is not covered by any standard pair $(\cdot, \tau)$ where $\tau$ is a non-empty subset of $\{4, 5, 6\}$. 

**Theorem 4.3.** For a set $\sigma \subseteq \{1, \ldots, n\}$, $(0, \sigma)$ is a standard pair of $O_c$ if and only if $\sigma$ is a maximal face of $\Delta_c$.

**Proof.** If $\sigma$ is a maximal face of $\Delta_c$, then by Definition 2.1, there exists $y \in \mathbb{R}^d$ such that $yA_\sigma = c_\sigma$ and $yA_\bar{\sigma} < c_\bar{\sigma}$. Then $p = c_\bar{\sigma} - yA_\bar{\sigma} > 0$ and $pB^\sigma = (c_\bar{\sigma} - yA_\bar{\sigma})B^\sigma = c_\bar{\sigma}B^\sigma + yA_\bar{\sigma}B^\sigma = c_\bar{\sigma}B^\sigma + c_\bar{\sigma}B^\sigma = cB$. Hence there is a positive dependence relation among $(-cB)$ and the rows of $B^\sigma$. Since $\sigma$ is a maximal face of $\Delta_c$, $|det(A_\sigma)| \neq 0$. However, $|det(B^\sigma)| = |det(A_\sigma)|$ which implies that $|det(B^\sigma)| \neq 0$. Therefore, $(-cB)$ and the rows of $B^\sigma$ span $\mathbb{R}^{n-d}$ positively. This implies that $Q_0^\sigma = \{z \in \mathbb{R}^{n-d} : B^\sigma z \leq 0, (-cB) \cdot z \leq 0\}$ is a polytope consisting of just the origin. If any inequality defining this simplex is dropped, the resulting relaxation is unbounded as only $n - d$ inequalities would remain. Hence $Q_0^\sigma$ is a standard polytope of $IP_{A,c}$ and by Theorem 3.10, $(0, \sigma)$ is a standard pair of $O_c$.

Conversely, if $(0, \sigma)$ is a standard pair of $O_c$, then $Q_0^\sigma$ is a standard polytope of $IP_{A,c}$. Since every inequality in the definition of $Q_0^\sigma$ gives a halfspace containing the origin and $Q_0^\sigma$ is a polytope, $Q_0^\sigma = \{0\}$. Hence there is a positive linear dependence relation among $(-cB)$ and the rows of $B^\sigma$. If $|\bar{\sigma}| > n - d$, then $Q_0^\sigma$ would coincide with the relaxation obtained by dropping some inequality from those in $B^\sigma z \leq 0$. This would contradict that $Q_0^\sigma$ was a standard polytope and hence $|\sigma| = d$ and $\sigma$ is a maximal face of $\Delta_c$. 

**Corollary 4.4.** Every maximal face of $\Delta_c$ is an associated set of $IP_{A,c}$.

For Theorem 4.3 and Corollary 4.6 below we assume that the g.c.d. of the maximal minors of $A$ is one which implies that $ZA = \mathbb{Z}^d$.

**Theorem 4.5.** If $\sigma$ is a maximal face of $\Delta_c$ then the multiplicity of $\sigma$ in $O_c$ is $|det(A_\sigma)|$.

**Proof.** Consider the full dimensional lattice $L_\sigma = \pi_\sigma(L) = \{B^\sigma z : z \in \mathbb{Z}^{n-d}\}$ in $\mathbb{Z}^{n-d}$. Since the g.c.d. of the maximal minors of $A$ is assumed to be one, the lattice $L_\sigma$ has index $|det(B^\sigma)| = |det(A_\sigma)|$ in $\mathbb{Z}^{n-d}$. Since $L_\sigma$ is full dimensional, it has a strictly positive element which guarantees that each equivalence class of $\mathbb{Z}^{n-d}$ modulo $L_\sigma$ has a non-negative member. This implies that there are $|det(A_\sigma)|$ distinct equivalence classes of $\mathbb{N}^{n-d}$ modulo $L_\sigma$. Recall that if $u$ is a feasible solution to $IP_{A,c}(b)$ then 

$$G^\sigma(b) = \text{minimize } \{\bar{c}_\sigma \cdot x_\sigma : x_\sigma \equiv u_\sigma \pmod{L_\sigma}, x_\sigma \in \mathbb{N}^{n-d}\}.$$ 

Since there are $|det(A_\sigma)|$ equivalence classes of $\mathbb{N}^{n-d}$ modulo $L_\sigma$, there are $|det(A_\sigma)|$ distinct group relaxations indexed by $\sigma$. The optimal solution of each program becomes the right hand side vector of a standard polytope (simplex) of $IP_{A,c}$ indexed by $\sigma$. Since no two optimal solutions are the same (as they come from different equivalence classes of $\mathbb{N}^{n-d}$ modulo $L_\sigma$), there are precisely $|det(A_\sigma)|$ standard polytopes of $IP_{A,c}$ indexed by $\sigma$. 

Corollary 4.6. The arithmetic degree of $O_c$ is bounded below by the sum of the absolute values of $\det(A_\sigma)$ as $\sigma$ varies among the maximal faces of $\Delta_c$.

A primary ideal $J$ in $k[x_1, \ldots, x_n]$ is a proper ideal such that $fg \in J$ implies either $f \in J$ or $g^t \in J$ for some positive integer $t$. A prime ideal $J$ of $k[x_1, \ldots, x_n]$ is a proper ideal such that $fg \in J$ implies that either $f \in J$ or $g \in J$. A primary decomposition of an ideal $I$ in $k[x_1, \ldots, x_n]$ is an expression of $I$ as a finite intersection of primary ideals in $k[x_1, \ldots, x_n]$. Lemma 3.3 in [38] shows that every monomial ideal $M$ in $k[x_1, \ldots, x_n]$ admits a primary decomposition into irreducible primary ideals that are indexed by the standard pairs of $M$. The radical of an ideal $I \subset k[x_1, \ldots, x_n]$ is the ideal $\sqrt{I} := \{ f \in S : f^t \in I, \text{ for some positive integer } t \}$. Radicals of primary ideals are prime. The radicals of the primary ideals in a minimal primary decomposition of an ideal $I$ are called the associated primes of $I$. This list of prime ideals is independent of the primary decomposition of the ideal. The minimal elements among the associated primes of $I$ are called the minimal primes of $I$ while the others are called the embedded primes of $I$. The minimal primes of $I$ are precisely the defining ideals of the isolated components of the zero-set or variety of $I$ while the embedded primes cut out embedded subvarieties in the isolated components. See a textbook in commutative algebra like [10] for more details.

A face $\tau$ of $\Delta_c$ is an associated set of $IP_{A,c}$ if and only if the monomial prime ideal $p_\tau := \langle x_j : j \notin \tau \rangle$ is an associated prime of the ideal $in_c(I_A)$. Further, $p_\sigma$ is a minimal prime of $in_c(I_A)$ if and only if $\sigma$ is a maximal face of $\Delta_c$. Hence the lower dimensional associated sets of $IP_{A,c}$ index the embedded primes of $in_c(I_A)$. The standard pair decomposition of a monomial ideal was introduced in [38] to study its associated primes. The multiplicity of an associated prime $p_\sigma$ of $in_c(I_A)$ is an algebraic invariant of $in_c(I_A)$, and [38] shows that this is exactly the number of standard pairs indexed by $\tau$. Similarly, the arithmetic degree of $in_c(I_A)$ is a refinement of the geometric notion of degree and [38] shows that this number is the total number of standard pairs of $in_c(I_A)$. These connections explain our choice of terminology. Theorem 4.3 is a translation of the specialization of Lemma 3.5 in [38] to toric initial ideals. We refer the interested reader to [30], §8 and §12.D and [38], §3 for the algebraic connections. Theorem 4.5 is a staple result of toric geometry and also follows from [13, Theorem 1]. It is proved via the algebraic technique of localization in [20, Theorem 8.8].

Theorem 4.5 gives a precise bound on the multiplicity of a maximal associated set of $IP_{A,c}$, which in turn provides a lower bound for the arithmetic degree of $O_c$ in Corollary 4.6. No exact result like Theorem 4.5 is known when $\tau$ is a lower dimensional associated set of $IP_{A,c}$. Such bounds would provide a bound for the arithmetic degree of $O_c$. Bounds on the arithmetic degree of a general monomial ideal in terms of its dimension and minimal generators can be found in [38, Theorem 3.1]. One hopes that stronger bounds are possible for toric initial ideals. We close with a first attempt at bounding the arithmetic degree of $O_c$ (under certain non-degeneracy assumptions). This result is due to Ravi Kannan, and its simple arguments are along the lines of proofs in [20] and [28].

Suppose $S \in \mathbb{Z}^{m \times n}$ and $u \in \mathbb{N}^n$ are fixed and $K_u := \{ x \in \mathbb{R}^n : Sx \leq u \}$ is such that $K_u \cap \mathbb{Z}^n = \{0\}$ and the removal of any inequality defining $K_u$ will bring in a non-zero lattice point into the relaxation. Let $s^{(i)}$ denote the $i$th row of $S$, $M := \max \|s^{(i)}\|_1$ and $\Delta_k(S)$ and $\delta_k(S)$ be the maximum and minimum absolute values of the $k \times k$ subdeterminants of $S$. We
By LP duality, there are \( n \) body K translations leave the quantities in the lemma invariant, we may prove the lemma for the

Without loss of generality we may assume that

Proof. Clearly, \( K_u \) is bounded since otherwise there would be a non-zero lattice point on an unbounded edge of \( K_u \) due to the integrality of all data. Suppose \( \text{width}_{s(i)}(K_u) > M(n+2) \) for all rows \( s(i) \) of \( S \). Let \( p \) be the center of gravity of \( K_u \). Then by a property of the center of gravity, for any \( x \in K_u \), \((1/(n+1))th \) of the vector from \( p \) to the reflection of \( x \) about \( p \) is also in \( K_u \), i.e., \((1 + \frac{1}{n+1})p - \frac{1}{n+1}x \in K_u \). Fix \( i \), \( 1 \leq i \leq m \) and let \( x_0 \) minimize \( s(i) \cdot x \) over \( K_u \). By the definition of width, we then have \( u_i - s(i) \cdot x_0 > M(n+2) \) which implies that

\[
\tag{7}
s(i) \cdot x_0 < u_i - M(n+2).
\]

Now \( s(i) \cdot ((1 + \frac{1}{n+1})p - \frac{1}{n+1}x_0) \leq u_i \) implies that

\[
\tag{8}
s(i) \cdot p \leq u_i(\frac{n + 1}{n + 2}) + \frac{s(i) \cdot x_0}{n + 2}.
\]

Combining (7) and (8) we get

\[
\tag{9}
s(i) \cdot p < u_i - M.
\]

Let \( q = \lfloor p \rfloor \) be the vector obtained by rounding down all components of \( p \). Then \( p = q + r \) where \( 0 \leq r_j < 1 \) for all \( j = 1, \ldots, n \), and by (9), \( s(i) \cdot (q + r) < u_i - M \) which leads to \( s(i) \cdot q + (s(i) \cdot r + M) < u_i \). Since \( M = \max ||s(i)||_1 \),

\[
\tag{10}
-M \leq s(i) \cdot r \leq M.
\]

and hence, \( s(i) \cdot q < u_i \). Repeating this argument for all rows of \( S \), we get that \( q \in K_u \). Similarly, if \( q' = \lceil p \rceil \) is the vector obtained by rounding up all components of \( p \), then \( p = q' - r \) where \( 0 \leq r_j < 1 \) for all \( j = 1, \ldots, n \). Then (9) implies that \( s(i) \cdot (q' - r) < u_i - M \) which leads to \( s(i) \cdot q' + (M - s(i) \cdot r) < u_i \). Again by (9), \( s(i) \cdot q' < u_i \) and hence \( q' \in K_u \). Since \( q \neq q' \), at least one of them is non-zero which contradicts that \( K_u \cap \mathbb{Z}^n = \{0\} \). \( \square \)

Lemma 4.10. For any two rows \( s(i), s(j) \) of \( S \), \( \text{width}_{s(i)}(K_u) \leq 2\frac{\Delta_n(S)}{\delta_n(S)} \text{width}_{s(j)}(K_u) \).

Proof. Without loss of generality we may assume that \( j = n + 1 \). Since \( K_u \) is bounded, \( \text{width}_{s(j)}(K_u) \) is finite. Suppose the minimum of \( s(j) \cdot x \) over \( K_u \) is attained at \( v \). Since translations leave the quantities in the lemma invariant, we may prove the lemma for the body \( K_{u'} \) obtained by translating \( K_u \) by \(-v\). Now \( s(j) \cdot x \) is minimized over \( K_{u'} \) at the origin. By LP duality, there are \( n \) linearly independent constraints among the \( m \) defining \( K_{u'} \) such
that the minimum of \( s^{(n+1)} \cdot x \) subject to just these \( n \) constraints is attained at 0. After renumbering the inequalities if necessary, assume these \( n \) constraints are the first \( n \). Let
\[
D = \{ x : s^{(l)} \cdot x \leq u'_l, \ l = 1, 2, \ldots, n + 1 \}
\]
where of course \( u'_1 = u'_2 = \cdots = u'_n = 0 \). Then by the above, \( D \) is a bounded simplex.

Since \( D \) contains \( K_{u'} \), it suffices to show that for each \( i \),
\[
\text{width}_{s^{(i)}}(D) \leq 2\left( \frac{\Delta_n(S)}{\delta_n(S)} \right) \text{width}_{s^{(n+1)}}(K_{u'}) = 2\left( \frac{\Delta_n(S)}{\delta_n(S)} \right) u'_{n+1}.
\]
We show that for each vertex \( q \) of \( D \), \(| s^{(i)} \cdot q | \leq \left( \frac{\Delta_n(S)}{\delta_n(S)} \right) u'_{n+1} \) which will prove (11). This is clearly true for \( q = 0 \). Without loss of generality assume that vertex \( q \) satisfies \( s^{(i)} \cdot q = u'_l \) for \( l = 2, 3, \ldots, n + 1 \). Since the determinant of the submatrix of \( S \) consisting of the rows \( s^{(2)}, \ldots, s^{(n+1)} \) is not zero, for any \( i \) there exists rationals \( \lambda_l \) such that \( s^{(i)} = \sum_{l=2}^{n+1} \lambda_l s^{(l)} \). By Cramer’s rule, \(| \lambda_l | \leq \left( \frac{\Delta_n(S)}{\delta_n(S)} \right) \). Therefore, \( s^{(i)} \cdot q = \sum_{l=2}^{n+1} \lambda_l s^{(l)} \cdot q = \sum_{l=2}^{n+1} \lambda_l u'_l = \lambda_{n+1} u'_{n+1} \)

\[
| s^{(i)} \cdot q | = | \lambda_{n+1} u'_{n+1} | = | \lambda_{n+1} | u'_{n+1} \leq \left( \frac{\Delta_n(S)}{\delta_n(S)} \right) u'_{n+1}.
\]

\[\square\]

**Proof of Theorem 4.8.** From Lemmas 4.9 and 4.10 it follows that for any \( i, 1 \leq i \leq m \),
\[
\text{width}_{s^{(i)}}(K_u) \leq 2\left( \frac{\Delta_n(S)}{\delta_n(S)} \right) M(n + 2) = 2M(n + 2)\left( \frac{\Delta_n(S)}{\delta_n(S)} \right).
\]
Since \( 0 \in K_u \), \( \min \{ s^{(i)} \cdot x : x \in K_u \} \leq 0 \) while \( \max \{ s^{(i)} \cdot x : x \in K_u \} = u_i \). Therefore, \( u_i = u_i - 0 \leq \text{width}_{s^{(i)}}(K_u) \) and hence, \( 0 \leq u_i \leq 2M(n + 2)\left( \frac{\Delta_n(S)}{\delta_n(S)} \right) \) for all \( 1 \leq i \leq m \).

Reverting back to our set up, let \( B = \left[ \begin{array}{c} B \\ -cB \end{array} \right] \). Suppose \( K_u \) is the standard polytope \( Q_u^n \). By Theorem 4.3, \( 0 \leq u_i \leq 2M(n - d + 2)\left( \frac{\Delta_n(B)}{\delta_n(B)} \right) \).

**Corollary 4.11.** If no maximal minor of \( B \) is zero, then the arithmetic degree of \( \mathcal{O}_c \) is at most \( \left( 2M(n - d + 2)\left( \frac{\Delta_n(B)}{\delta_n(B)} \right) \right)^n \).

The above arguments do not use the condition that the removal of an inequality from \( K_u \) will bring in a lattice point into the relaxation. Further, the bound is independent of the number of facets of \( K_u \), and Corollary 4.11 is straightforward. Thus, further improvements may be possible with more effort. However, apart from providing a bound for arithmetic degree, these proofs have the nice feature that they build a bridge to techniques from the geometry of numbers that have played a central role in theoretical integer programming as seen in the work of Kannan, Lenstra, Lovász, Scarf and others. See [23] for a survey.

5. **The Chain Theorem**

We now examine the structure of the poset of associated sets of \( IP_{A,c} \) which we denote as \( Assets(IP_{A,c}) \). All elements of \( Assets(IP_{A,c}) \) are faces of the regular triangulation \( \Delta_c \) and the partial order is set inclusion. Theorem 4.3 provides a first result.
Corollary 5.1. The maximal elements of \( \text{Assets}(IP_{A,c}) \) are the maximal faces of \( \Delta_c \).

Example 4.2 continued. The lower dimensional associated sets of this example (except the empty set) are the thick faces of \( \Delta_c \) shown in Figure 7.

Despite the seemingly chaotic structure of \( \text{Assets}(IP_{A,c}) \) beyond its maximal elements, it has an important structural property that we now explain.

Theorem 5.2. [The Chain Theorem] If \( \tau \in \Delta_c \) is an associated set of \( IP_{A,c} \) and \( |\tau| < d \) then there exists a face \( \tau' \in \Delta_c \) that is also an associated set of \( IP_{A,c} \) with the property that \( \tau \subset \tau' \) and \( |\tau' \setminus \tau| = 1 \).

Proof. Since \( \tau \) is an associated set of \( IP_{A,c} \), by Theorem 3.10, \( O_c \) has a standard pair of the form \( (v, \tau) \) and \( Q^\tau_v = \{ z \in \mathbb{R}^{n-d} : B^\tau z \leq \pi_\tau(v), (-cB) \cdot z \leq 0 \} \) is a standard polytope of \( IP_{A,c} \). Since \( |\tau| < d \), \( \tau \) is not a maximal face of \( \Delta_c \) and hence by Theorem 4.3, \( v \neq 0 \). For each \( i \in \bar{\tau} \), let \( R^i \) be the relaxation of \( Q^\tau_v \) obtained by removing the \( i \)th inequality \( b_i \cdot z \leq v_i \) from \( B^\tau z \leq \pi_\tau(v) \), i.e.,

\[
R^i := \{ z \in \mathbb{R}^{n-d} : B^\tau \{i\} z \leq \pi_{\tau \cup \{i\}}(v), (-cB) \cdot z \leq 0 \}.
\]

Let \( E^i := R^i \setminus Q^\tau_v \). Clearly, \( E^i \cap Q^\tau_v = \emptyset \), and, since the removal of \( b_i \cdot z \leq v_i \) introduces at least one lattice point into \( R^i \), \( E^i \cap \mathbb{Z}^{n-d} \neq \emptyset \). Let \( z^*_i \) be the optimal solution to minimize \( \{ (-cB) \cdot z : z \in E^i \cap \mathbb{Z}^{n-d} \} \) if the program is bounded. This integer program is always feasible since \( E^i \cap \mathbb{Z}^{n-d} \neq \emptyset \), but it may not have a finite optimal value. However, there exists at least one \( i \in \bar{\tau} \) for which the above integer program is bounded. To see this, pick a maximal simplex \( \sigma \in \Delta_c \) such that \( \tau \subset \sigma \). The polytope \( \{ z \in \mathbb{R}^{n-d} : B^\sigma z \leq \pi_\sigma(v), (-cB) \cdot z \leq 0 \} \) is a simplex and hence bounded. This polytope contains all \( E^i \) for \( i \in \sigma \setminus \tau \), and hence all these \( E^i \) are bounded and have finite optima with respect to \( (-cB) \cdot z \). We may assume that the inequalities in \( B^\tau z \leq \pi_\tau(v) \) are labeled so that the finite optimal values are ordered as \( (-cB) \cdot z_1^* \geq (-cB) \cdot z_2^* \geq \cdots \geq (-cB) \cdot z_p^* \) where \( \{1, 2, \ldots, p\} \subseteq \bar{\tau} \).
Claim: Let \( N^1 := \{ z \in \mathbb{R}^{n-d} : B^x(1)z \leq \pi_{\tau \cup \{1\}}(v), (-cB) \cdot z \leq (-cB) \cdot z^*_1 \} \). Then \( z^*_1 \) is the unique lattice point in \( N^1 \) and the removal of any inequality from \( B^x(1)z \leq \pi_{\tau \cup \{1\}}(v) \) will bring in a new lattice point into the relaxation.

Proof. Since \( z^*_1 \) lies in \( R^1 \), \( 0 = (-cB) \cdot 0 \geq (-cB) \cdot z^*_1 \). However, \( 0 > (-cB) \cdot z^*_1 \) since otherwise, both \( z^*_1 \) and \( 0 \) would be optimal solutions to \( \min\{( -cB ) \cdot z : z \in R^1 \} \) contradicting that \( c \) is generic. Therefore,

\[
N^1 = R^1 \cap \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z^*_1 \} \\
= (E^1 \cup Q^*_v) \cap \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z^*_1 \} \\
= (E^1 \cap \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z^*_1 \}) \\
\quad \bigcup (Q^*_v \cap \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z^*_1 \}).
\]

Since \( c \) is generic, \( z^*_1 \) is the unique lattice point in the first polytope and the second polytope is free of lattice points. Hence \( z^*_1 \) is the unique lattice point in \( N^1 \). The relaxation of \( N^1 \) got by removing \( b_j \cdot z \leq v_j \) is the polyhedron \( N^1 \cup (E^j \cap \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z^*_1 \}) \) for \( j \in \varpi \) and \( j \neq 1 \). Either this is unbounded, in which case there is a lattice point \( z \) in this relaxation such that \( (-cB) \cdot z^*_1 \geq (-cB) \cdot z \), or (if \( j \leq p \)) we have \( (-cB) \cdot z^*_1 \geq (-cB) \cdot z^*_j \) and \( z^*_j \) lies in this relaxation.

Translating \( N^1 \) by \(-z^*_1\) we get \( Q^x_v(1) := \{ z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq 0, B^x(1)z \leq v' \} \) where \( v' = \pi_{\tau \cup \{1\}}(v) - B^x(1)z^*_1 \geq 0 \) since \( z^*_1 \) is feasible for all inequalities except the first one. Now \( Q^x_v(1) \cap \mathbb{Z}^{n-d} = \{0\} \), and hence \((v', \tau \cup \{1\})\) is a standard pair of \( O_c \).

Example 4.2 continued. The empty set is associated to \( IP_{A,c} \) and \( \emptyset \subset \{1\} \subset \{1, 4\} \subset \{1, 4, 5\} \) is a saturated chain in \( Assets(IP_{A,c}) \) that starts at the empty set.

In algebraic language, the chain theorem says that the associated primes of \( in_c(I_A) \) occur in saturated chains. This was proved in [22, Theorem 3.1]. When the cost vector \( c \) is not generic, \( in_c(I_A) \) is no longer a monomial ideal, and its associated primes need not come in saturated chains. See [22, Remark 3.3] for such an example. An important open question in the algebraic study of integer programming is to characterize all monomial ideals that can appear as the initial ideal (with respect to some generic cost vector) of a toric ideal.

In our set up this amounts to characterizing all down sets in \( \mathbb{N}^n \) that can appear as the set of optimal solutions to a family \( IP_{A,c} \), where \( A \) and \( c \) satisfy the assumptions from Section 2. Theorem 5.2 imposes the necessary condition that the poset of sets indexing the standard pairs of the down set have the chain property. Unfortunately, this is not sufficient to characterize down sets of the form \( O_c \). See [30] for another class of monomial ideals that also have the chain property.

Since the elements of \( Assets(IP_{A,c}) \) are faces of \( \Delta_c \), a maximal face of which is a \( d \)-element set, the length of a maximal chain in \( Assets(IP_{A,c}) \) is at most \( d \). We denote the maximal length of a chain in \( Assets(IP_{A,c}) \) by \( length(Assets(IP_{A,c})) \). When \( n - d \) (the corank of \( A \)) is small compared to \( d \), \( length(Assets(IP_{A,c})) \) has a stronger upper bound than \( d \). We use the following result of Bell and Scarf to prove the bound.

**Theorem 5.3.** [33, Corollary 16.5 a] Let \( Ax \leq b \) be a system of linear inequalities in \( n \) variables, and let \( c \in \mathbb{R}^n \). If \( \max \{ c \cdot x : Ax \leq b, x \in \mathbb{Z}^n \} \) is a finite number, then max
\( \{ c \cdot x : Ax \leq b, x \in \mathbb{Z}^n \} = \max \{ c \cdot x : A'x \leq b', x \in \mathbb{Z}^n \} \) for some subsystem \( A'x \leq b' \) of \( Ax \leq b \) with at most \( 2^n - 1 \) inequalities.

**Theorem 5.4.** The length of a maximal chain in the poset of associated sets of \( IP_{A,c} \) is at most \( \min(d, 2^{n-d} - (n - d + 1)) \).

**Proof.** As seen earlier, \( \text{length}(\text{Assets}(IP_{A,c})) \leq d \). If \( v \) lies in \( O_c \), then the origin is the optimal solution to the integer program minimize \( \{ (-cB) \cdot z : Bz \leq v, z \in \mathbb{Z}^{n-d} \} \). By Theorem 5.3, we need at most \( 2^{n-d} - 1 \) inequalities to describe the same integer program which means that we can remove at least \( n - (2^{n-d} - 1) \) inequalities from \( Bz \leq v \) without changing the optimum. Assuming that the inequalities removed are indexed by \( \tau \), \( Q^\tau \) will be a standard polytope of \( IP_{A,c} \). Therefore, \( |\tau| \geq n - (2^{n-d} - 1) \). This implies that the maximal length of a chain in \( \text{Assets}(IP_{A,c}) \) is at most \( d - (n - (2^{n-d} - 1)) = 2^{n-d} - (n - d + 1) \). \( \square \)

**Corollary 5.5.** The cardinality of an associated set of \( IP_{A,c} \) is at least \( \max(0, n - (2^{n-d} - 1)) \).

**Corollary 5.6.** If \( n - d = 2 \), then \( \text{length}(\text{Assets}(IP_{A,c})) \leq 1 \).

**Proof.** In this situation, \( 2^{n-d} - (n - d + 1) = 4 - (4 - 2 + 1) = 4 - 3 = 1 \). \( \hfill \square \)

We conclude this section with a family of examples for which \( \text{length}(\text{Assets}(IP_{A,c})) = 2^{n-d} - (n - d + 1) \). This is adapted from [22, Proposition 3.9] which was modeled on a family of examples from [32].

**Proposition 5.7.** For each \( m > 1 \), there is an integer matrix \( A \) of corank \( m \) and a cost vector \( c \in \mathbb{Z}^n \) where \( n = 2^m - 1 \) such that \( \text{length}(\text{Assets}(IP_{A,c})) = 2^m - (m + 1) \).

**Proof.** Given \( m > 1 \), let \( B' = (b_{ij}) \in \mathbb{Z}^{(2^m-1) \times m} \) be the matrix whose rows are all the \( \{1, -1\} \)-vectors in \( \mathbb{R}^m \) except \( v = (-1, -1, \ldots, -1) \). Let \( B \in \mathbb{Z}^{(2^m+m-1) \times m} \) be obtained by stacking \( B' \) on top of \(-I_m \) where \( I_m \) is the \( m \times m \) identity matrix. Set \( n = 2^m + m - 1 \), \( d = 2^m - 1 \) and \( A' = [I_d | B'] \in \mathbb{Z}^{d \times n} \). By construction, the columns of \( B \) span the lattice \( \{ u \in \mathbb{Z}^m : A'u = 0 \} \). We may assume that the first row of \( B' \) is \((1, 1, \ldots, 1) \in \mathbb{R}^m \). Adding this row to all other rows of \( A' \) we get \( A \in \mathbb{N}^{d \times n} \) with the same row space as \( A' \). Hence the columns of \( B \) are also a basis for the lattice \( \{ u \in \mathbb{Z}^n : Au = 0 \} \). Since the rows of \( B \) span \( \mathbb{Z}^m \) as a lattice, we can find a cost vector \( c \in \mathbb{Z}^n \) such that \( -(cB) = v \).

For each row \( b_i \) of \( B' \) set \( r_i := |\{ b_{ij} : b_{ij} = 1 \}| \), and let \( r \) be the vector of all \( r_i \)'s. By construction, the polytope \( Q := \{ z \in \mathbb{R}^m : B'z \leq r, -(cB) \cdot z \leq 0 \} \) has no lattice points in its interior, and each of its \( 2^m \) facets has exactly one vertex of the unit cube in \( \mathbb{R}^m \) in its relative interior. If we let \( w_i = r_i - 1 \), then the polytope \( \{ z \in \mathbb{R}^m : B'z \leq w, -(cB) \cdot z \leq 0 \} \) is a standard polytope \( Q^r \) of \( IP_{A,c} \) where \( \tau = \{ d + 1, d + 2, \ldots, d + m = n \} \) and \( w = \pi_\tau(u) \). Since a maximal face of \( \Delta_c \) is a \( d = (2^m - 1) \)-element set and \( |\tau| = m \), Theorem 5.2 implies that \( \text{length}(\text{Assets}(IP_{A,c})) \geq 2^m - 1 - m = 2^m - (m + 1) \). However, by Theorem 5.4, \( \text{length}(\text{Assets}(IP_{A,c})) = \min(2^m - 1, 2^m - (m + 1)) = 2^m - (m + 1) \) since \( m > 1 \) by assumption. \( \square \)
Example 5.8. If we choose \( m = 3 \) then \( n = 2^m + m - 1 = 10 \) and \( d = 2^m - 1 = 7 \). Constructing \( B' \) and \( A \) as in Proposition 5.7, we get

\[
B' = \begin{bmatrix}
1 & 1 & 1 \\
-1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1 \\
-1 & 1 & -1 \\
1 & -1 & -1
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\
1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 \\
1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0
\end{bmatrix}
\]

The vector \( c = (11, 0, 0, 0, 0, 0, 10, 10, 10) \) satisfies \((-cB) = (-1, -1, -1)\). The associated sets of \( IP_{A,c} \) along with their multiplicities are given below.

| \( \tau \) | Multiplicity | \( \tau \) | Multiplicity |
|----------|-------------|----------|-------------|
| \{4, 5, 6, 7, 8, 9, 10\} | 4 | \{2, 3, 7, 8, 9, 10\} | 2 |
| \{1, 5, 6, 7, 8, 9, 10\} | 4 | \{5, 6, 7, 8, 9, 10\} | 1 |
| \{3, 4, 5, 6, 7, 8, 9, 10\} | 4 | \{4, 5, 6, 7, 8, 9\} | 1 |
| \{3, 4, 5, 6, 7, 9, 10\} | 4 | \{2, 3, 4, 8, 9, 10\} | 1 |
| \{3, 4, 5, 6, 7, 10\} | 2 | \{2, 4, 7, 8, 9, 10\} | 2 |
| \{2, 3, 4, 7, 8, 9, 10\} | 4 | \{1, 5, 7, 8, 9, 10\} | 1 |
| \{3, 4, 5, 6, 7, 8, 10\} | 2 | \{2, 3, 4, 8, 9, 10\} | 1 |
| \{3, 4, 5, 6, 7, 9, 10\} | 1 | \{4, 5, 6, 8, 9, 10\} | 2 |
| \{3, 4, 5, 6, 8, 10\} | 1 | \{1, 5, 6, 8, 9, 10\} | 1 |
| \{2, 4, 5, 7, 8, 9, 10\} | 2 | \{3, 4, 5, 6, 8, 9, 10\} | 1 |
| \{1, 6, 7, 8, 9, 10\} | 1 | \{6, 7, 8, 9, 10\} | 1 |
| \{3, 5, 6, 7, 8, 10\} | 1 | \{7, 8, 9, 10\} | 1 |
| \{3, 6, 7, 8, 9, 10\} | 2 | \{8, 9, 10\} | 1 |

The elements in the unique maximal chain in \( Assets(IP_{A,c}) \) are marked with a * and \( \text{length}(Assets(IP_{A,c})) = 2^3 - (3 + 1) = 4 \) as predicted by Proposition 5.7.

6. Gomory Integer Programs

Recall from Definition 2.7 that a group relaxation \( G^\sigma(b) \) of \( IP_{A,c}(b) \) is called a Gomory relaxation if \( \sigma \) is a maximal face of \( \Delta_c \). As discussed in Section 2, these relaxations are the easiest to solve among all relaxations of \( IP_{A,c}(b) \). Hence it is natural to ask under what conditions on \( A \) and \( c \) would all programs in \( IP_{A,c} \) be solvable by Gomory relaxations. We study this question in this section. The majority of the results here are taken from [21].

Definition 6.1. The family of integer programs \( IP_{A,c} \) is a Gomory family if, for every \( b \in NA \), \( IP_{A,c}(b) \) is solved by a group relaxation \( G^\sigma(b) \) where \( \sigma \) is a maximal face of the regular triangulation \( \Delta_c \).

Theorem 6.2. The following conditions are equivalent:
(i) \( IP_{A,c} \) is a Gomory family.
(ii) The associated sets of \( IP_{A,c} \) are precisely the maximal faces of \( \Delta_c \).
(iii) \((*, \tau)\) is a standard pair of \(O_c\) if and only if \(\tau\) is a maximal face of \(\Delta_c\).

(iv) All standard polytopes of \(IP_{A,c}\) are simplices.

Proof. By Definition 6.1, \(IP_{A,c}\) is a Gomory family if and only if for all \(b \in NA\), \(IP_{A,c}(b)\) can be solved by one of its Gomory relaxations. By Theorem 3.11, this is equivalent to saying that every \(u \in O_c\) lies in some \(S(*, \sigma)\) where \(\sigma\) is a maximal face of \(\Delta_c\) and \((*, \sigma)\) a standard pair of \(O_c\). Definition 3.1 then implies that all associated sets of \(IP_{A,c}\) are maximal faces of \(\Delta_c\). By Theorem 4.3, every maximal face of \(\Delta_c\) is an associated set of \(IP_{A,c}\) and hence (i) \(\Leftrightarrow\) (ii). The equivalence of statements (ii), (iii) and (iv) follow from Theorem 3.10. \(\square\)

If \(c\) is a generic cost vector such that for a triangulation \(\Delta\) of \(cone(A)\), \(\Delta = \Delta_c\), then we say that \(\Delta\) supports the order ideal \(O_c\) and the family of integer programs \(IP_{A,c}\). No regular triangulation of the matrix \(A\) in Example 4.2 supports a Gomory family. Here is a matrix with a Gomory family.

Example 6.3. Consider the \(3 \times 6\) matrix

\[
A = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 2 & 3 & 4 
\end{bmatrix}.
\]

In this case, \(cone(A)\) has 14 distinct regular triangulations and 48 distinct sets \(O_c\) as \(c\) varies among all generic cost vectors. Ten of these triangulations support Gomory families; one for each triangulation. For instance, if \(c = (0, 0, 1, 1, 0, 3)\), then

\[
\Delta_c = \{\sigma_1 = \{1, 2, 5\}, \sigma_2 = \{1, 4, 5\}, \sigma_3 = \{2, 5, 6\}, \sigma_4 = \{4, 5, 6\}\}
\]

and \(IP_{A,c}\) is a Gomory family since the standard pairs of \(O_c\) are:

\((0, \sigma_1), (e_3, \sigma_1), (e_4, \sigma_1), (0, \sigma_2), (0, \sigma_3), \text{ and } (0, \sigma_4)\). \(\square\)

Algebraically, \(IP_{A,c}\) is a Gomory family if and only if the initial ideal \(in_c(I_A)\) has no embedded primes and hence Theorem 6.2 is a characterization of toric initial ideals without embedded primes. A sufficient condition for an ideal in \(k[x_1, \ldots, x_n]\) to not have embedded primes is that it is Cohen-Macaulay \([10]\). In general, Cohen-Macaulayness is not necessary for an ideal to be free of embedded primes. However, empirical evidence seemed to suggest for a while that for toric initial ideals, Cohen-Macaulayness might be equivalent to being free of embedded primes. A counterexample to this was found recently by Laura Matusevich. The algebraic approach to integer programming allows one to compute all down sets \(O_c\) of a fixed matrix \(A\) as \(c\) varies among the set of generic cost vectors. See \([24]\), \([36]\) and \([37]\) for details. The software package TiGERS \([2]\) is custom-tailored for this purpose.

We now compare the notion of a Gomory family to the classical notion of total dual integrality \([33, \S 22]\). It will be convenient to assume that \(ZA = \mathbb{Z}^d\) for these results.

Definition 6.4. The system \(yA \leq c\) is totally dual integral (TDI) if \(LP_{A,c}(b)\) has an integral optimal solution for each \(b \in cone(A) \cap \mathbb{Z}^d\).

Definition 6.5. The regular triangulation \(\Delta_c\) is unimodular if \(ZA_\sigma = \mathbb{Z}^d\) for every maximal face \(\sigma \in \Delta_c\).
Example 6.6. The regular triangulation in Example 2.2 (i) is unimodular while those in Example 2.2 (ii) and (iii) are not.

Lemma 6.7. The system $yA \leq c$ is TDI if and only if the regular triangulation $\Delta_c$ is unimodular.

Proof. The regular triangulation $\Delta_c$ is the normal fan of $P_c$ by Proposition 2.2, and it is unimodular if and only if $ZA_\sigma = Z^d$ for every maximal face $\sigma \in \Delta_c$. This is equivalent to every $b \in cone(A_\sigma) \cap Z^d$ lying in $NA_\sigma$ for every maximal face $\sigma$ of $\Delta_c$. By Lemma 2.3, this happens if and only if $LP_{A,c}(b)$ has an integral optimum for all $b \in cone(A) \cap Z^d$.

Corollary 8.4 in [36] shows that $\Delta_c$ is unimodular if and only if the monomial ideal in $c(I_{A_\sigma})$ is generated by square-free monomials. Hence, by computing in $c(I_{A_\sigma})$, one can determine whether $yA \leq c$ is TDI. Such computations can be carried out on computer algebra systems like CoCoA [1] or MACAULAY 2 [18] for moderately sized examples. See [36] for algorithms.

Theorem 6.8. If $yA \leq c$ is TDI then $IP_{A,c}$ is a Gomory family.

Proof. By Theorem 4.3, $(0, \sigma)$ is a standard pair of $\mathcal{O}_c$ for every maximal face $\sigma$ of $\Delta_c$. Lemma 6.7 implies that $cone(A_\sigma)$ is unimodular (i.e., $ZA_\sigma = Z^d$), and therefore $NA_\sigma = cone(A_\sigma) \cap Z^d$ for every maximal face $\sigma$ of $\Delta_c$. Hence the semigroups $NA_\sigma$ arising from the standard pairs $(0, \sigma)$ as $\sigma$ varies over the maximal faces of $\Delta_c$ cover $NA$. Therefore the only standard pairs of $\mathcal{O}_c$ are $(0, \sigma)$ as $\sigma$ varies over the maximal faces of $\Delta_c$. The result then follows from Theorem 6.2.

When $yA \leq c$ is TDI, the multiplicity of a maximal face of $\Delta_c$ in $\mathcal{O}_c$ is one (from Theorem 4.3). By Theorem 6.8, no lower dimensional face of $\Delta_c$ is associated to $IP_{A,c}$. While this is sufficient for $IP_{A,c}(b)$ to be a Gomory family, it is far from necessary. TDI-ness guarantees local integrality in the sense that $LP_{A,c}(b)$ has an integral optimum for every integral $b$ in $cone(A)$. In contrast, if $IP_{A,c}$ is a Gomory family, the linear optima of the programs in $LP_{A,c}$ may not be integral.

If $A$ is unimodular (i.e., $ZA_\sigma = Z^d$ for every nonsingular maximal submatrix $A_\sigma$ of $A$), then the feasible regions of the linear programs in $LP_{A,c}$ have integral vertices for each $b \in cone(A) \cap Z^d$, and $yA \leq c$ is TDI for all $c$. Hence if $A$ is unimodular, then $IP_{A,c}$ is a Gomory family for all generic cost vectors $c$. However, just as integrality of the optimal solutions of programs in $LP_{A,c}$ is not necessary for $IP_{A,c}$ to be a Gomory family, unimodularity of $A$ is not necessary for $IP_{A,c}$ to be a Gomory family for all $c$.

Example 6.9. Consider the seven by twelve integer matrix

$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}$
of rank seven. The maximal minors of $A$ have absolute values zero, one and two and hence $A$ is not unimodular. This matrix has 376 distinct regular triangulations supporting 418 distinct order ideals $O_c$ (computed using TiGERS). In each case, the standard pairs of $O_c$ are indexed by just the maximal simplices of the regular triangulation $\Delta_c$ that supports it. Hence $IP_{A,c}$ is a Gomory family for all generic $c$.

The above discussion shows that $IP_{A,c}$ being a Gomory family is more general than $yA \leq c$ being TDI. Similarly, $IP_{A,c}$ being a Gomory family for all generic $c$ is more general than $A$ being a unimodular matrix.

7. Gomory Families and Hilbert Bases

As we just saw, unimodular matrices or more generally, unimodular regular triangulations lead to Gomory families. A common property of unimodular matrices and matrices $A$ such that $cone(A)$ has a unimodular triangulation is that the columns of $A$ form a Hilbert basis for $cone(A)$, i.e., $NA = cone(A) \cap \mathbb{Z}^d$ (assuming $ZA = \mathbb{Z}^d$).

**Definition 7.1.** A $d \times n$ integer matrix $A$ is normal if the semigroup $NA$ equals $cone(A) \cap \mathbb{Z}^d$.

The reason for this (highly over used) terminology here is that if the columns of $A$ form a Hilbert basis, then the zero set of the toric ideal $I_A$ (called a toric variety) is a normal variety. See [36, Chapter 14] for more details. We first note that if $A$ is not normal, then $IP_{A,c}$ need not be a Gomory family for any cost vector $c$.

**Example 7.2.** The matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$ is not normal since $(1, 2)^t$ which lies in $cone(A) \cap \mathbb{Z}^2$ cannot be written as a non-negative integer combination of the columns of $A$. This matrix gives rise to 10 distinct order ideals $O_c$ supported on its four regular triangulations $\{\{1, 4\}\}, \{\{1, 2\}, \{2, 4\}\}, \{\{1, 3\}, \{3, 4\}\}$ and $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Each $O_c$ has at least one standard pair that is indexed by a lower dimensional face of $\Delta_c$.

The matrix in Example 4.2 is also not normal and has no Gomory families. While we do not know whether normality of $A$ is sufficient for the existence of a generic cost vector $c$ such that $IP_{A,c}$ is a Gomory family, we will now show that under certain additional conditions, normal matrices do give rise to Gomory families.

**Definition 7.3.** A $d \times n$ integer matrix $A$ is $\Delta$-normal if $cone(A)$ has a triangulation $\Delta$ such that for every maximal face $\sigma \in \Delta$, the columns of $A$ in $cone(A_\sigma)$ form a Hilbert basis.

**Remark 7.4.** If $A$ is $\Delta$-normal for some triangulation $\Delta$, then it is normal. To see this note that every lattice point in $cone(A)$ lies in $cone(A_\sigma)$ for some maximal face $\sigma \in \Delta$. Since $A$ is $\Delta$-normal, this lattice point also lies in the semigroup generated by the columns of $A$ in $cone(A_\sigma)$ and hence in $NA$.

Observe that $A$ is $\Delta$-normal with respect to all the unimodular triangulations of $cone(A)$. Hence triangulations $\Delta$ with respect to which $A$ is $\Delta$-normal generalize unimodular triangulations of $cone(A)$.
Examples 7.5 and 7.6 show that the set of matrices where \( \text{cone}(A) \) has a unimodular triangulation is a proper subset of the set of \( \Delta \)-normal matrices which in turn is a proper subset of the set of normal matrices.

**Example 7.5.** Examples of normal matrices with no unimodular triangulations can be found in \([3]\) and \([11]\). If \( \text{cone}(A) \) is simplicial for such a matrix, \( A \) will be \( \Delta \)-normal with respect to its coarsest (regular) triangulation \( \Delta \) consisting of the single maximal face with support \( \text{cone}(A) \). For instance, consider the following example taken from \([36, \text{Example 13.17}]\):

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\
0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{bmatrix}.
\]

Here \( \text{cone}(A) \) has 77 regular triangulations and no unimodular triangulations. Since \( \text{cone}(A) \) is simplicial, \( A \) is \( \Delta \)-normal with respect to its coarsest regular triangulation \( \{\{1, 2, 3, 8\}\} \).

**Example 7.6.** There are normal matrices \( A \) that are not \( \Delta \)-normal with respect to any triangulation of \( \text{cone}(A) \). To see such an example, consider the following modification of the matrix in Example 7.5 that appears in \([36, \text{Example 13.17}]\):

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]

This matrix is again normal and each of its nine columns generate an extreme ray of \( \text{cone}(A) \). Hence the only way for this matrix to be \( \Delta \)-normal for some \( \Delta \) would be if \( \Delta \) is a unimodular triangulation of \( \text{cone}(A) \). However, there are no unimodular triangulations in this example.

**Theorem 7.7.** If \( A \) is \( \Delta \)-normal for some regular triangulation \( \Delta \) then there exists a generic cost vector \( c \in \mathbb{Z}^n \) such that \( \Delta = \Delta_c \) and \( IP_{A,c} \) is a Gomory family.

**Proof.** Without loss of generality we can assume that the columns of \( A \) in \( \text{cone}(A_\sigma) \) form a minimal Hilbert basis for every maximal face \( \sigma \) of \( \Delta \). If there were a redundant element, the smaller matrix obtained by removing this column from \( A \) would still be \( \Delta \)-normal.

For a maximal face \( \sigma \in \Delta \), let \( \sigma_{in} \subset \{1, \ldots, n\} \) be the set of indices of all columns of \( A \) lying in \( \text{cone}(A_\sigma) \) that are different from the columns of \( A_\sigma \). Suppose \( a_{i_1}, \ldots, a_{i_k} \) are the columns of \( A \) that generate the one dimensional faces of \( \Delta \), and \( c' \in \mathbb{R}^n \) a cost vector such that \( \Delta = \Delta_{c'} \). We modify \( c' \) to obtain a new cost vector \( c \in \mathbb{R}^n \) such that \( \Delta = \Delta_c \) as follows. For \( j = 1, \ldots, k \), let \( c_j := c'_j \). If \( j \in \sigma_{in} \) for some maximal face \( \sigma \in \Delta \), then \( a_j = \sum_{i \in \sigma} \lambda_i a_i \), \( 0 \leq \lambda_i < 1 \) and we define \( c_j := \sum_{i \in \sigma} \lambda_i c_i \). Hence, for all \( j \in \sigma_{in} \), \((a_j, c_j) \in \mathbb{R}^{d+1} \) lies in \( C_\sigma := \text{cone}((a_i, c_i) : i \in \sigma) = \text{cone}((a'_i, c'_i) : i \in \sigma) \) which was a facet of \( C = \text{cone}((a_i, c_i) : i = 1, \ldots, n) \). If \( y \in \mathbb{R}^d \) is a vector as in Definition 2.1 showing that \( \sigma \) is a maximal face of \( \Delta_{c'} \) then \( y \cdot a_i = c_i \) for all \( i \in \sigma \cup \sigma_{in} \) and \( y \cdot a_j < c_j \) otherwise. Since \( \text{cone}(A_\sigma) = \text{cone}(A_{\sigma \cup \sigma_{in}}) \), we conclude that \( \text{cone}(A_\sigma) \) is a maximal face of \( \Delta_c \).
If \( b \in \text{NA} \) lies in \( \text{cone}(A_\sigma) \) for a maximal face \( \sigma \in \Delta_c \), then \( IP_{A,c}(b) \) has at least one feasible solution \( u \) with support in \( \sigma \cup \sigma_{in} \) since \( A \) is \( \Delta \)-normal. Further, \( (b',c \cdot u) = ((Au)^t,c \cdot u) \) lies in \( C_\sigma \) and all feasible solutions of \( IP_{A,c}(b) \) with support in \( \sigma \cup \sigma_{in} \) have the same cost value by construction. Suppose \( u \in \mathbb{N}^n \) is any feasible solution of \( IP_{A,c}(b) \) with support not in \( \sigma \cup \sigma_{in} \). Then \( c \cdot u < c \cdot \omega \) since \((a_i',c_i) \in C_\sigma \) if and only if \( i \in \sigma \cup \sigma_{in} \) and \( C_\sigma \) is a lower facet of \( C \). Hence the optimal solutions of \( IP_{A,c}(b) \) are precisely those feasible solutions with support in \( \sigma \cup \sigma_{in} \). The vector \( b \) can be expressed as \( b = b' + \sum_{i \in \sigma} z_i a_i \) where \( z_i \in \mathbb{N} \) are unique and \( b' \in \{ \sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1 \} \cap \mathbb{Z}^d \) is also unique. The vector \( b' = \sum_{j \in \sigma_{in}} r_j a_j \) where \( r_j \in \mathbb{N} \). Setting \( u_i = z_i \) for all \( i \in \sigma \), \( u_j = r_j \) for all \( j \in \sigma_{in} \) and \( u_k = 0 \) otherwise, we obtain all feasible solutions \( u \) of \( IP_{A,c}(b) \) with support in \( \sigma \cup \sigma_{in} \).

If there is more than one such feasible solution, then \( c \) is not generic. In this case, we can perturb \( c \) to a generic cost vector \( c'' = c + \epsilon \omega \) by choosing \( 1 \gg \epsilon > 0 \), \( \omega_j \ll 0 \) whenever \( j = i_1, \ldots, i_k \) and \( \omega_j = 0 \) otherwise. Suppose \( u_1, \ldots, u_t \) are the optimal solutions of the integer programs \( IP_{A,c''}(b') \) where \( b' \in \{ \sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1 \} \cap \mathbb{Z}^d \). (Note that \( t = |\{ \sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1 \} \cap \mathbb{Z}^d| \) is the index of \( \mathbb{Z}A_\sigma \in \mathbb{Z}A \).) The support of each such \( u_i \) is contained in \( \sigma_{in} \). For any \( b \in \text{cone}(A_\sigma) \cap \mathbb{Z}^d \), the optimal solution of \( IP_{A,c''}(b) \) is hence \( u = u_i + z \) for some \( i \in \{1, \ldots, t\} \) and \( z \in \mathbb{N}^n \) with support in \( \sigma \). This shows that \( \text{NA} \) is covered by the affine semigroups \( \phi_A(S(u_i, \sigma)) \) where \( \sigma \) is a maximal face of \( \Delta \) and \( u_i \) as above for each \( \sigma \). By construction, the corresponding admissible pairs \((u_i, \sigma)\) are all standard for \( \mathcal{O}_{c''} \). Since all data is integral, \( c'' \in \mathbb{Q}^n \) and hence can be scaled to lie in \( \mathbb{Z}^n \). Renaming \( c'' \) as \( c \), we conclude that \( IP_{A,c} \) is a Gomory family.

**Corollary 7.8.** Let \( A \) be a normal matrix such that \( \text{cone}(A) \) is simplicial, and let \( \Delta \) be the coarsest triangulation whose single maximal face has support \( \text{cone}(A) \). Then there exists a cost vector \( c \in \mathbb{Z}^n \) such that \( \Delta = \Delta_c \) and \( IP_{A,c} \) is a Gomory family.

**Example 7.9.** Consider the normal matrix in Example 6.3. Here \( \text{cone}(A) \) is generated by the first, second and sixth columns of \( A \) and hence \( A \) is \( \Delta \)-normal with respect to the regular triangulation \( \{ \{1, 2, 6\} \} \). There are 13 distinct sets \( \mathcal{O}_c \) supported on \( \Delta \). Among the 13 corresponding families of integer programs, only one is a Gomory family. A representative cost vector for this \( IP_{A,c} \) is \( c = (0, 0, 4, 4, 1, 0) \). The standard pair decomposition of \( \mathcal{O}_c \) is the one constructed in Theorem 7.4. The affine semigroups \( S(\cdot, \sigma) \) from this decomposition are:

\[
S(0, \sigma), S(e_3, \sigma), S(e_4, \sigma), \text{ and } S(e_5, \sigma).
\]

Note that \( A \) is not \( \Delta \)-normal with respect to the regular triangulation supporting the Gomory family \( IP_{A,c} \) in Example 6.3. The columns of \( A \) in \( \text{cone}(A_{\sigma_1}) \) are the columns of \( A_{\sigma_1} \) and \( A_3 \). The vector \( (1, 2, 2) \) is in the minimal Hilbert basis of \( \text{cone}(A_{\sigma_1}) \) but is not a column of \( A \). This example shows that a regular triangulation \( \Delta \) of \( \text{cone}(A) \) can support a Gomory family even if \( A \) is not \( \Delta \)-normal. The Gomory families in Theorem 7.4 have a very special standard pair decomposition.

**Problem 7.10.** If \( A \in \mathbb{Z}^{d \times n} \) is a normal matrix, does there exist a generic cost vector \( c \in \mathbb{Z}^n \) such that \( IP_{A,c} \) is a Gomory family?
While we do not know the answer to this question, we will now show that stronger results are possible for small values of $d$.

**Theorem 7.11.** If $A \in \mathbb{Z}^{d \times n}$ is a normal matrix and $d \leq 3$, then there exists a generic cost vector $c \in \mathbb{Z}^n$ such that $IP_{A,c}$ is a Gomory family.

**Proof.** It is known that if $d \leq 3$ then $\text{cone}(A)$ has a regular unimodular triangulation $\Delta_c$ [34]. The result then follows from Corollary 6.8.

Before we proceed, we rephrase Problem 7.10 in terms of covering properties of $\text{cone}(A)$ and $NA$ along the lines of [6], [7], [8], [11] and [34]. To obtain the same set up as in these papers we assume in this section that $A$ is normal and the columns of $A$ form the unique minimal Hilbert basis of $\text{cone}(A)$. Using the terminology in [7], the free Hilbert cover problem asks whether there exists a covering of $NA$ by semigroups $NA_\tau$ where the columns of $A_\tau$ are linearly independent. The unimodular Hilbert cover problem asks whether $\text{cone}(A)$ can be covered by full dimensional unimodular subcones $\text{cone}(A_\tau)$ (i.e., $\mathbb{Z}A_\tau = \mathbb{Z}^d$), while the stronger unimodular Hilbert partition problem asks whether $\text{cone}(A)$ has a unimodular triangulation. (Note that if $\text{cone}(A)$ has a unimodular Hilbert cover or partition using subcones $\text{cone}(A_\tau)$, then $NA$ is covered by the semigroups $NA_\tau$.) All these problems have positive answers if $d \leq 3$ since $\text{cone}(A)$ admits a unimodular Hilbert partition in this case [6], [34]. Normal matrices (with $d = 4$) such that $\text{cone}(A)$ has no unimodular Hilbert partition can be found in [6] and [11]. Examples (with $d = 6$) that admit no free Hilbert cover and hence no unimodular Hilbert cover can be found in [7] and [8].

When $yA \leq c$ is TDI, the standard pair decomposition of $NA$ induced by $c$ gives a unimodular Hilbert partition of $\text{cone}(A)$ by Theorem 6.7. An important difference between Problem 7.10 and the Hilbert cover problems is that affine semigroups cannot be used in Hilbert covers. Moreover, affine semigroups that are allowed in standard pair decompositions come from integer programming. If there are no restrictions on the affine semigroups that can be used in a cover, $NA$ can always be covered by full dimensional affine semigroups: for any triangulation $\Delta$ of $\text{cone}(A)$ with maximal subcones $\text{cone}(A_\tau)$, the affine semigroups $b + NA_\sigma$ cover $NA$ as $b$ varies in $\{\sum_{i \in \sigma} \lambda_i a_i : 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ and $\sigma$ varies among the maximal faces of the triangulation. A partition of $NA$ derived from this idea can be found in [19, Theorem 5.2]. We recall the notion of supernormality introduced in [19].

**Definition 7.12.** A matrix $A \in \mathbb{Z}^{d \times n}$ is supernormal if for every submatrix $A'$ of $A$, the columns of $A$ that lie in $\text{cone}(A')$ form a Hilbert basis for $\text{cone}(A')$.

**Proposition 7.13.** For $A \in \mathbb{Z}^{d \times n}$, the following are equivalent:

(i) $A$ is supernormal,
(ii) $A$ is $\Delta$-normal for every regular triangulation $\Delta$ of $\text{cone}(A)$,
(iii) Every triangulation of $\text{cone}(A)$ in which all columns of $A$ generate one dimensional faces is unimodular.

**Proof.** The equivalence of (i) and (iii) was established in [19, Proposition 3.1]. Definition 7.12 shows that (i) $\Rightarrow$ (ii). Hence we just need to show that (ii) $\Rightarrow$ (i). Suppose that $A$ is $\Delta$-normal for every regular triangulation of $\text{cone}(A)$. In order to show that $A$ is supernormal
we only need to check submatrices $A'$ where the dimension of $cone(A')$ is $d$. Choose a cost
vector $c$ with $c_i > 0$ if the $i$th column of $A$ does not generate an extreme ray of $cone(A')$, and $c_i = 0$ otherwise. This gives a polyhedral subdivision of $cone(A)$ in which $cone(A')$ is a
maximal face. There are standard procedures that will refine this subdivision to a regular triangulation $\Delta$ of $cone(A)$. Let $T$ be the set of maximal faces $\sigma$ of $\Delta$ such that $cone(A_\sigma)$
lies in $cone(A')$. Since $A$ is $\Delta$-normal, the columns of $A$ that lie in $cone(A_\sigma)$ form a Hilbert basis for $cone(A_\sigma)$ for each $\sigma \in T$. However, since their union is the set of columns of $A$ that lie in $cone(A')$, this union forms a Hilbert basis for $cone(A')$.

It is easy to catalog all $\Delta$-normal and supernormal matrices, of the type considered in this paper, for small values of $d$. We say that the matrix $A$ is graded if its columns span an affine hyperplane in $\mathbb{R}^d$. If $d = 1$, $cone(A)$ has $n$ triangulations $\{\{i\}\}$ each of which has the unique maximal subcone $cone(a_i)$ whose support is $cone(A)$. If we assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then $A$ is normal if and only if either $a_1 = 1$, or $a_n = -1$. Also, $A$ is normal if and only if it is supernormal. If $d = 2$ and the columns of $A$ are ordered counterclockwise around the origin, then $A$ is normal if and only if $det(a_i, a_{i+1}) = 1$ for all $i = 1, \ldots, n - 1$. Such an $A$ is supernormal since it is $\Delta$-normal for every triangulation $\Delta$ — the Hilbert basis of a maximal subcone of $\Delta$ is precisely the set of columns of $A$ in that subcone. If $d = 3$ then as mentioned before, $cone(A)$ has a unimodular triangulation with respect to which $A$ is $\Delta$-normal. However, not every such $A$ needs to be supernormal: the matrix in Example 6.3 is not $\Delta$-normal for the $\Delta$ supporting the Gomory family in that example. If $d = 3$ and $A$ is graded, then without loss of generality we can assume that the columns of $A$ span the hyperplane $x_1 = 1$. If $A$ is normal as well, then its columns are precisely all the lattice points in the convex hull of $A$. Conversely, every graded normal $A$ with $d = 3$ arises this way — its columns are all the lattice points in a polygon in $\mathbb{R}^2$ with integer vertices. In particular, every triangulation of $cone(A)$ that uses all the columns of $A$ is unimodular. Hence, by Proposition 7.13, $A$ is supernormal, and therefore $\Delta$-normal for any triangulation of $A$.

**Theorem 7.14.** Let $A \in \mathbb{Z}^{d \times n}$ be a normal matrix of rank $d$.

(i) If $d = 1, 2$ or $A$ is graded and $d = 3$, every regular triangulation of $cone(A)$ supports at least one Gomory family.

(ii) If $d = 2$ and $A$ is graded, every regular triangulation of $cone(A)$ supports exactly one Gomory family.

(iii) If $d = 3$ and $A$ is not graded, or if $d = 4$ and $A$ is graded, then not all regular triangulations of $cone(A)$ may support a Gomory family. In particular, $A$ may not be $\Delta$-normal with respect to every regular triangulation.

**Proof.** (i) If $d = 1, 2$ or $A$ is graded and $d = 3$, $A$ is supernormal and hence by Proposition 7.13 and Theorem 7.7, every regular triangulation of $cone(A)$ supports at least one Gomory family.

(ii) If $d = 2$ and $A$ is graded, then we may assume that

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n-1 \end{bmatrix}.$$ 

In this case, $A$ is supernormal and hence every regular triangulation $\Delta$ of $cone(A)$ supports a Gomory family by Theorem 7.7. Suppose the maximal cones of $\Delta$, in counter-clockwise
order, are \( C_1, \ldots, C_r \). Assume the columns of \( A \) are labeled such that \( C_i = cone(a_{i-1}, a_i) \) for \( i = 1, \ldots, r \), and the columns of \( A \) in the interior of \( C_i \) are labeled in counter-clockwise order as \( b_{i1}, \ldots, b_{ik_i} \). Hence the \( n \) columns of \( A \) from left to right are:

\[
\begin{align*}
& a_0, b_{11}, \ldots, b_{1k_1}, a_1, b_{21}, \ldots, a_{r-1}, b_{r1}, \ldots, b_{rk_r}, a_r.
\end{align*}
\]

Indexing the columns of \( A \) by their labels, the maximal faces of \( \triangle \) are \( \sigma_i = \{ i-1, i \} \) for \( i = 1, \ldots, r \). Let \( e_i \) be the unit vector of \( \mathbb{R}^n \) indexed by the true column index of \( a_i \) in \( A \) and \( e_{ij} \) be the unit vector of \( \mathbb{R}^n \) indexed by the true column index of \( b_{ij} \) in \( A \). Since the columns of \( A \) form a minimal Hilbert basis of \( cone(A) \), \( e_i \) is the unique solution to \( IP_{A,c}(a_i) \) for all \( c \) and \( e_{ij} \) is the unique solution to \( IP_{A,c}(b_{ij}) \) for all \( c \). Hence the standard pairs of Theorem 7.7 are \((0, \sigma_i)\) and \((e_{ij}, \sigma_i)\) for \( i = 1, \ldots, r \) and \( j = 1, \ldots, k_i \).

Suppose \( \triangle \) supports a second Gomory family \( IP_{A,w} \). Then every standard pair of \( \mathcal{O}_w \) is also of the form \((*, \sigma_i)\) for \( \sigma_i \in \Delta \), and \( r \) of them are \((0, \sigma_i)\) for \( i = 1, \ldots, r \). The remaining standard pairs are of the form \((e_{ij}, \sigma_i)\). To see this, consider the semigroups in \( NA \) arising from the standard pairs of \( \mathcal{O}_w \). The total number of standard pairs of \( \mathcal{O}_c \) and \( \mathcal{O}_w \) are the same. Since the columns of \( A \) all lie on \( x_1 = 1 \), no two \( b_{ij} \)'s can be covered by a semigroup coming from the same standard pair and none of them are covered by a semigroup \((0, \sigma_i)\). We show that if \((e_{ij}, \sigma_k)\) is a standard pair of \( \mathcal{O}_w \) then \( k = i \) and thus \( \mathcal{O}_w = \mathcal{O}_c \).

If \( r = 1 \), the standard pairs of \( \mathcal{O}_w \) are \((0, \sigma_1), (e_{11}, \sigma_1), \ldots, (e_{1k_1}, \sigma_1)\) as in Theorem 7.7. If \( r > 1 \), consider the last cone \( C_r = cone(a_{r-1}, a_r) \). If \( a_{r-1} \) is the second to last column of \( A \), then \( C_r \) is unimodular and the semigroup from \((0, \sigma_r)\) covers \( C_r \cap \mathbb{Z}^2 \). The subcomplex comprised of \( C_1, \ldots, C_{r-1} \) is a regular triangulation \( \Delta' \) of \( cone(A') \) where \( A' \) is obtained by dropping the last column of \( A \). Since \( A' \) is a normal graded matrix with \( d = 2 \) and \( \Delta' \) has less than \( r \) maximal cones, the standard pairs supported on \( \Delta' \) are as in Theorem 7.7 by induction. If \( a_{r-1} \) is not the second to last column of \( A \) then \( b_{rk_r} \), the second to last column of \( A \) is in the Hilbert basis of \( C_r \) but is not a generator of \( C_r \). So \( \mathcal{O}_w \) has a standard pair of the form \((e_{rk_r}, \sigma_i)\). If \( \sigma_i \neq \sigma_r \), then the lattice point \( b_{rk_r} + a_r \) cannot be covered by the semigroup from this or any other standard pair of \( \mathcal{O}_w \). Hence \( \sigma_i = \sigma_r \). By a similar argument, the remaining standard pairs indexed by \( \sigma_r \) are \((e_{r(k_r-1)}, \sigma_r), \ldots, (e_{r1}, \sigma_r)\) along with \((0, \sigma_r)\). These are precisely the standard pairs of \( \mathcal{O}_c \) indexed by \( \sigma_r \). Again we are reduced to considering the subcomplex comprised of \( C_1, \ldots, C_{r-1} \) and by induction, the remaining standard pairs of \( \mathcal{O}_w \) are as in Theorem 7.7.

(iii) The \( 3 \times 6 \) normal matrix \( A \) of Example 6.3 has 10 distinct Gomory families supported on 10 out of the 14 regular triangulations of \( cone(A) \). Furthermore, the normal matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 2 & 1 & 1 \\
0 & 0 & 4 & 3 & 2 & 1 \\
0 & 0 & 4 & 3 & 2 & 1
\end{bmatrix}
\]

has 11 distinct Gomory families supported on 11 out of its 19 regular triangulations. \( \square \)

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