THE EXISTENCE PROBLEM FOR STEINER NETWORKS
IN STRICTLY CONVEX DOMAINS
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ABSTRACT

We consider the existence problem for ‘Steiner networks’ (trivalent graphs with 2\(\pi/3\) angles at each junction) in strictly convex domains, with ‘Neumann’ boundary conditions. For each of the three possible combinatorial possibilities, sufficient conditions on the domain are derived for existence; in addition, in each case explicit examples of nonexistence are given.

0. Introduction.

About 20 years ago, motivated by dynamical models in materials science describing phase separation and the motion of interfaces separating phases, [Bronsard-Reitich] introduced the problem of motion of networks of curves in a planar domain with normal velocity proportional to the curvature (at regular points), and fixed angle conditions at the junctions. They derived (formally) from the underlying model (an Allen-Cahn parabolic system with two-dimensional order parameter and a three-well potential) the geometric evolution, as well as the boundary conditions: the angles formed by the curves at a ‘triple junction’ (where three arcs meet) is constant throughout the evolution, and the arcs meeting the boundary of the domain do so orthogonally at all times. In the same paper, short-time existence was proved for the simplest network, three arcs meeting at an interior (moving) point, with all angles equal to 2\(\pi/3\) radians. More recently, [Mantegazza et al.] undertook a thorough study of this system of geometric evolution equations, which is distinguished by non-standard boundary conditions at the junctions. In particular (again for this simplest network, but also for fixed-endpoint boundary conditions) they made explicit the natural continuation criterion for the short-time solution, and developed the rescaling analysis of singularity formation. This enabled them to rule out finite-time curvature blowup at certain rates, which goes a long way towards the natural global existence theorem in this setting.

With the goal of understanding the global non-linear evolution of this system, it is natural to consider the existence, classification and linearized
stability properties of steady-state solutions. (The flow is the gradient of total length of the network, at least as long as the solution is classical, that is, does not go through a topological change.) These have a very simple variational description: consider embedded networks of curves in a bounded domain \( D \subset \mathbb{R}^2 \), \( C^1 \) up to each node and to \( \partial D \), and with all nodes trivalent. If we consider variations which preserve the combinatorics of the network, but allow the boundary vertices to move freely on \( \partial D \), then the critical points of total length are exactly those networks where (i) all edges are straight line segments; (ii) the angles between edges at each node are \( 2\pi/3 \); (iii) the edges that meet \( \partial D \) do so orthogonally. For brevity (and by analogy with the fixed-boundary node case) we call these ‘Steiner networks’. In a recent paper, Ikota-Yanagida settle the linearized stability question for Steiner networks that are trees, without the assumption of convexity for the domain.

It might seem at first that the existence of such networks of arbitrary complexity, at least in the case of convex domains, presents no greater subtlety. But already the first attempt at a simple geometric construction to produce hexagonal cells in a given domain led the author (a little over a year ago) to discover a phenomenon akin to a simple kind of ‘holonomy’ (section 5). Once this case appeared settled, it was natural to try to identify all the possible networks, and consider their existence in a given convex domain. This is in principle a simple geometric problem, but it leads to results that may seem surprising (at least, they were surprising to the author.)

For example, consider complexity. One sometime sees optimistic drawings of ‘honeycombs’ of adjacent hexagonal cells tiling planar domains. But, in fact, it is not too hard to show (Section 2) that in a strictly convex domain, there are only three possible Steiner networks, excluding critical chords (Fig. 8): a single triple junction, or ‘triode’ (in the terminology of Mantegazza et al.); a single ‘double triode’; or a single hexagonal cell, ‘anchored’ to the boundary by six edges. In addition, in such domains a Steiner network is necessarily connected.

For the simplest case, the triode, one might hope (on general ‘variational’ grounds) that they always exist in smooth convex domains. Indeed there have been such claims in the literature (e.g. Tabachnikov) based on the fact that the vertices of a triode (when it exists) are contact points of a circumscribed equilateral triangle which has critical perimeter among such circumscribed triangles. But this ignores the fact that, for such a ‘critical’ configuration of boundary points to define a triode, the inner normals at
these points must meet inside $D$. And it is easy to construct examples of strictly convex domains in which some critical configurations violate this.

Finding convex domains for which all critical configurations have the property that the normals meet outside $D$ proved harder; after spending some time trying to prove this cannot happen, the author became aware of the class of convex curves of ‘constant height’, beautifully described in [Yaglom-Boltianskii]. For these curves the perimeter of a circumscribed equilateral triangle is constant along the curve, so the three inner normals at the contact points always meet at a single point; and in one case, the constant-height biangle (which has two corners), the locus of these normal intersections is entirely outside the domain. This example can be smoothed, and although the constant-height smoothings do support triodes, the notion of ‘Minkowski sum’ of convex domains suggests a way to perturb them so as to destroy all triodes. Eventually it turned out to be more efficient to implement all the constructions completely analytically, using the support function in the tangent-angle parametrization; this setup is described in Section 1. For the case of triodes, perhaps the most striking non-existence result is Corollary 3.5: let $C$ be a smooth, strictly convex curve of constant height, with the property that the three normals at contact points meet on the outside, for some circumscribed equilateral triangle. Then arbitrarily close to $C$, one finds (i) strictly convex curves which stably support triodes; (ii) strictly convex curves which support no triodes, also stably. (Here ‘arbitrarily close’ and ‘stably’ refer to the strong $C^r$ topology in the tangent-angle parametrization, for some $r \geq 2$.) On the positive side, as expected, under a ‘pinching condition’ for the curvature, the normals always intersect internally, so at least two triodes exist (Proposition 3.2).

The situation for double triodes (Section 4) is somewhat similar; in addition to a ‘criticality’ condition and the requirement that normals meet internally, a third condition plays a role (note that the circle does not support double triodes.) For non-existence results, curves of ‘constant width’ exhibit a similar ‘instability property’ as constant-height curves for triodes (Proposition 4.4). If $C_0$ is a strictly convex curve of constant width, arbitrarily close to $C_0$ one finds: (i) strictly convex curves $C$ for which all sufficiently far outer parallel curve support double triodes; (ii) strictly convex curves $C$ for which neither $C$ itself nor any outer parallel curve supports double triodes. It is harder to state a satisfactory sufficient criterion for existence (the case of the circle rules out ‘curvature pinching’). Section 4 contains three existence results. Proposition 4.3 states that if $C$ is a strictly convex curve with only two critical chords, making an angle greater than $\pi/3$, some

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outer parallel curve stably supports a double triode. In fact, if \(C\) satisfies, in addition, the ‘curvature pinching’ condition of Prop. 3.2, \(C\) itself already supports double triodes.

Section 5 deals with existence for hexagonal cells. Here the criticality condition is vanishing of the ‘holonomy’, and one has to consider a configuration of 24 points, at least some of which must be inside the domain to guarantee existence. It turns out that existence in a sufficiently far outer parallel curve of a given strictly convex curve \(C\) always holds, and existence of hexagonal cells for \(C\) itself is guaranteed under a ‘curvature pinching’ condition (Proposition 5.8). One gets examples exhibiting neither hexagonal cells nor triodes by a slight modification of the construction in section 3 (Corollary 5.10).

A natural question we are unable to resolve at this point is whether there are strictly convex domains supporting no Steiner networks at all (except for critical chords.) The problem is that our construction of domains without triodes relies on the existence of convex curves of constant height with exterior intersections, and such examples are rare- we have been unable to further ‘engineer’ their width function so as to rule out double triodes.

The results in the paper suggest that, also for the dynamical problem, results obtained for evolution in the standard disk may not carry over unchanged to more general convex domains; in addition, we expect some of the techniques introduced here may be useful to obtain estimates or constraints for the evolution problem, at least in case the arcs remain convex. It is easy to think of slightly more general settings in which some of the phenomena should persist: for example, what if one removes the requirement of strict convexity? What about existence of doubly-periodic Steiner networks (i.e., networks on a flat torus)? Finally, one could look at the problem of geodesic Steiner networks on surfaces of positive curvature. It is well-known that those on the standard sphere have been classified, and the result plays a role in the study of minimal surface sheets meeting at a 4-junction in the unit ball. If one looks at smooth ovaloids in \(\mathbb{R}^3\) (boundaries of smooth, strictly convex sets), it is natural to wonder whether ovaloids exist which admit no geodesic Steiner networks at all (closed geodesics don’t count.)

The study of Steiner networks with Dirichlet boundary conditions (that is, critical-length graphs spanning a given set of points) has a long history, both in the euclidean plane and on Riemannian surfaces (see e.g. [Ivanov-Tuzhilin] for a survey.) For Steiner networks of surfaces, a parametric version of the singular Plateau problem was addressed recently via
energy functionals in [Mese-Yamada]. Existence for the free-boundary case addressed in the present paper (networks with endpoints moving freely on a given boundary) does not seem to have been considered previously, with the exceptions noted earlier.

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1. Strictly convex curves.

1.1 Generalities. We consider strictly convex oriented Jordan curves \( \mathcal{C} \), boundary of convex domains \( D \) in \( \mathbb{R}^2 \). ‘Strictly convex’ means that, for each \( \theta \) in the unit circle \( S = \mathbb{R} \mod 2\pi \mathbb{Z} \), the oriented support line of \( D \) with unit normal \( N(\theta) = (-\sin \theta, \cos \theta) \) meets \( \mathcal{C} \) at exactly one point. This defines a continuous surjective map \( B : S \to \mathcal{C} \), as well as a continuous function \( p(\theta) = -\langle B(\theta), N(\theta) \rangle \). We are primarily interested in the case where \( B \) is injective and smooth (or at least piecewise \( C^2 \)), and thus defines a diffeomorphism \( S \to \mathcal{C} \), the ‘tangent angle parametrization’; for each \( \theta \in S \), the unit tangent vector to \( \mathcal{C} \) at \( B(\theta) \) is \( T(\theta) = (\cos \theta, \sin \theta) \), while \( N(\theta) \) is the inner unit normal. When \( B \) is \( C^1 \), so is \( p \) (the ‘support function’), and \( B \) can be recovered from \( p \) via:

\[
B(\theta) = -p(\theta)N(\theta) + p'(\theta)T(\theta).
\]  

(1.1)

If the origin \( 0 \in \mathbb{R}^2 \) is strictly inside \( D \) (as will be assumed throughout the paper), from \( \langle B(\theta), N(\theta) \rangle < 0 \) for all \( \theta \) follows \( p(\theta) > 0 \). Since we only use the tangent angle parametrization, relation (1.1) justifies describing classes of curves, and indeed carrying out all of the analysis, in terms of properties of \( p \). If \( p \in C^2(S) \), differentiating (1.1) we obtain:

\[
B'(\theta) = r(\theta)T(\theta), \quad \text{where } r(\theta) = p''(\theta) + p(\theta).
\]

Thus, where \( r(\theta) > 0 \), \( B \) is an immersion with unit tangent vector \( T(\theta) \); geometrically, \( r(\theta) \) is the radius of curvature at \( B(\theta) \). Note that the tangent-angle parametrization is less regular than arc length: if the arc length parametrization \( \Gamma(s) \) is \( C^2 \) with positive curvature \( k = \frac{d\theta}{ds} = \langle \Gamma_{ss}, \Gamma_s^\perp \rangle \), we have \( r = 1/k \) positive and continuous, so \( B \) is only \( C^1 \). The support function \( p \) can be recovered from \( r \) by integration (assuming \( r \), say, piecewise \( C^0 \)):

\[
p(\theta) = p(0)\cos(\theta) + p'(0)\sin(\theta) + \int_0^\theta \sin(\theta - \tau)r(\tau)d\tau. \quad (1.2)
\]

With \( r > 0 \) of class \( C^k \) and \( 2\pi \)-periodic, (1.1) and (1.2) will always define a parametrization of class \( C^{k+2} \) of a strictly convex Jordan curve, provided only \( p \) given by (1.2) is \( 2\pi \)-periodic, for which it suffices to impose:

\[
\int_0^{2\pi} r(\tau)\cos(\tau)d\tau = \int_0^{2\pi} r(\tau)\sin(\tau)d\tau = 0.
\]

(This is automatic if \( r = p'' + p \) with \( p \) \( 2\pi \)-periodic and piecewise \( C^2 \).) Our main interest is in smooth (or \( C^r \) for some \( r \geq 2 \)) strictly convex curves,
identified with the set:

\[ \mathcal{P}_{\text{smooth}} = \{ p \in C^2(\mathbb{R}), 2\pi\text{-periodic}; p > 0, r = p'' + p > 0 \} \]

We will also need to consider piecewise \( C^2 \) strictly convex curves without corners:

\[ \mathcal{P}_{\text{pw}} = \{ p \in C^2_{\text{pw}}(\mathbb{R}), 2\pi\text{-periodic}; p > 0, r = p'' + p > 0 \} \]

as well as piecewise \( C^2 \) strictly convex curves with corners:

\[ \mathcal{P}_{\text{corner}} = \{ p \in C^2_{\text{pw}}(\mathbb{R}), 2\pi\text{-periodic}; p > 0, r = p'' + p \geq 0, Z(r) = \bigcup_{i=1}^{N} [a_i, b_i] \} \]

where the last condition means the zero set \( Z(r) \) of \( r \) in \([0, 2\pi]\) is assumed to be a finite union of closed non-degenerate intervals.

If \( p \in \mathcal{P}_{\text{corner}} \), let \( I = [a, b] \) be a \( \theta \) interval on which \( r = 0 \). Then \( B'\theta(\theta) = 0 \) for \( \theta \in I \), and \( B \equiv B(a) \) in \( I \). Thus \( B \) is not, strictly speaking, a parametrization on the interval \( I \), and the unit normals \( \{ N(\theta); \theta \in I \} \) correspond to the set of support lines at \( B(a) \), the ‘corner’ corresponding to \( I \). Adding a positive constant \( c \) to an element \( p \) of \( \mathcal{P}_{\text{corner}} \) yields an element of \( \mathcal{P}_{\text{pw}} \), that is, erases the corners. Geometrically, this corresponds to considering the ‘exterior parallel curves’ \( B_c(\theta) \) to the convex curve defined by \( p \): from (1.1), we see that:

\[ B_c(\theta) = -(p + c)N(\theta) + p'(\theta) = B(\theta) - cN(\theta) \]

More generally, it is clear that each of the three classes \( \mathcal{P}_{\text{smooth}} \subset \mathcal{P}_{\text{pw}} \subset \mathcal{P}_{\text{corner}} \) is closed under linear combinations with positive coefficients. This corresponds to the well-known fact that convexity is preserved by homothety (with a fixed center 0) and Minkowski sum; the latter operation is conveniently described in terms of the tangent-angle parametrization (or ‘pseudo-parametrization’, in the case of \( \mathcal{P}_{\text{corner}} \): if \( B_1(\theta), B_2(\theta) \) parametrize \( C_1 \) (resp. \( C_2 \), the Minkowski sum \( D_1 + D_2 \) of the domains they bound has support function \( p_1(\theta) + p_2(\theta) \), and its boundary is (pseudo)parametrized by \( B_1(\theta) + B_2(\theta) \). Adding \( c > 0 \) to \( p \) corresponds to taking Minkowski sum with a disk of radius \( c \).

We can use the support function in the tangent-angle parametrization to introduce a strong topology in the space of strictly convex curves: the uniform \( C^2 \) topology induced in the convex cone \( \mathcal{P}_{\text{smooth}} \) from \( C^2(S) \). Accordingly, we say something happens ‘stably’ for a given \( p_0 \in \mathcal{P}_{\text{smooth}} \) (or even
\( p_0 \in \mathcal{P}_{\text{corner}} \) if it also happens for all \( p \in \mathcal{P}_{\text{smooth}} \) in some \( C^2 \)-neighborhood of \( p_0 \).

### 1.2 Special classes of convex curves.

#### 1.2.1 Width and constant width.

Let \( \mathcal{C} \) be a strictly convex curve in \( \mathbb{R}^2 \). The width \( b(\theta) \) in direction \( \theta \in S \) is the unoriented distance between the supporting lines with unit normals \( N(\theta), N(\theta + \pi) \). Clearly, since \( 0 \in D \):

\[
b(\theta) = p(\theta) + p(\theta + \pi) = \langle B(\theta + \pi) - B(\theta), N(\theta) \rangle > 0.
\]

Width behaves linearly under homothety/Minkowski sum (at corresponding directions). Denote by \( w \in C^1 \) the ‘derived width’:

\[
w(\theta) = -b'(\theta) = -p'(\theta) - p'(\theta + \pi) = \langle B(\theta + \pi) - B(\theta), T(\theta) \rangle.
\]

We say \( \mathcal{C} \) is symmetric if \( B(\theta + \pi) = -B(\theta) \) for all \( \theta \); this is equivalent to \( \pi \)-periodicity for the support function \( p(\theta) \), so in this case \( b(\theta) = 2p(\theta) \). It is easy to see that one can always construct symmetric convex curves with prescribed ‘derived width’:

**Proposition 1.1.** Let \( w \) be \( C^1 \), \( \pi \)-periodic and satisfy \( \int_0^\pi w(\tau)d\tau = 0 \). There exists a one-parameter family of ‘parallel’ strictly convex symmetric curves with ‘derived width function’ \( w \).

**Proof.** Define \( b(\theta) = C - \int_0^\theta w; b \) is \( C^2 \), \( \pi \)-periodic, positive if \( \max_{\theta} \int_0^\theta w < C \) and satisfies \( b'' + b > 0 \) if \( \max_{\theta} \{ \int_0^\theta w(\tau)d\tau + w'(\theta) \} < C \). Both conditions will hold for \( C \) in some interval \( (C_0, \infty) \), and then \( p(\theta) = (1/2)b(\theta) \in \mathcal{P}_{\text{smooth}} \) and defines a strictly convex symmetric curve with ‘derived width’ \( w \).

**Example 1.1.** The Reuleaux triangle of constant width \( b \) is obtained from an equilateral triangle \( B_1B_2B_3 \) of side length \( b \) by drawing three circular arcs with center \( B_i \), radius \( b \) and aperture \( \pi/3 \) radians (Fig. 1). Placing the origin of \( \mathbb{R}^2 \) at the barycenter of the triangle, and fixing \( B(0) = (p'(0), -p(0)) = (0, -b(\theta)) \) gives the support function \( p \in \mathcal{P}_{\text{corner}} \):

\[
p(\theta) = \begin{cases} 
    b - b\frac{\sqrt{3}}{3} \cos \theta, & \theta \in [-\pi/6, \pi/6] \\
    b \frac{\sin \theta + b\frac{\sqrt{3}}{2} \cos \theta}{3}, & \theta \in [\pi/6, \pi/2],
\end{cases}
\]  

extended to \( \mathbb{R} \) with period \( 2\pi/3 \).

The arc of the curve corresponding to \( \theta \in [-\pi/6, \pi/6] \) is regular, while \( B(\theta) \equiv B_1 = b(1/2, -\sqrt{3}/6) \) (a vertex) for \( \theta \in [\pi/6, \pi/2] \). It is geometrically clear that \( p \) is \( 2\pi/3 \)-periodic, and therefore \( B(\theta + 2\pi/3) = R_{2\pi/3}B(\theta) \) for all \( \theta \) (\( \mathcal{C} \) is ‘3-symmetric’, but not symmetric; the only symmetric curve of
constant width is the circle.) One checks easily that \( p \in C^1 \), and \( B(\theta) \) maps \([\pi/2, 5\pi/6]\) and \([\pi/6, 3\pi/2]\) to regular arcs and \([5\pi/6, 7\pi/6], [3\pi/2, 11\pi/6]\) to the vertices \( B_2 = (0, b \sqrt{3}/3) \) and \( B_3 = b(-1/2, -\sqrt{3}/6) \).

The radius of curvature \( r(\theta) \) is constant (equal to \( b \)) on the regular intervals, and vanishes on the singular intervals. Adding an arbitrary positive constant \( c \) to \( p \) yields a support function in \( \mathcal{P}_{pw} \); the corresponding \( B_c(\theta) \) parametrizes the \( c \)-parallel curve, which is of constant width \( b+c \). Although this curve has no corners, the parametrization \( B_c(\theta) \) is only piecewise \( C^1 \).

It will be of interest to consider explicit examples of curves of constant width with smooth support function, and it is natural to use truncated Fourier series. In general, write:

\[
p(\theta) = C + \sum_{n \geq 1} a_n \cos n \theta + b_n \sin n \theta,
\]

where \( C, a_n, b_n \) are Fourier coefficients. We have the following simple observations:

(i) \( \mathcal{C} \) is centrally symmetric \((p(\theta) = p(\theta + \pi), B(\theta + \pi) = -B(\theta)) \) iff \( a_n \) and \( b_n \) vanish for \( n \) odd;

(ii) \( \mathcal{C} \) is 3-symmetric \((p \text{ is } 2\pi/3\text{-periodic}) \) iff \( a_n \) and \( b_n \) vanish unless \( n \) is a multiple of 3;
(iii) \( C \) has constant width iff \( a_n \) and \( b_n \) vanish for \( n \) even.

(iv) Suppose the \( x \)-axis is an axis of symmetry of \( C \). With \( B(\theta) = (b_1(\theta), b_2(\theta)) \), this means \( b_2(\theta) + b_2(\pi - \theta) \equiv 0 \), or equivalently \( p(\pi - \theta) = p(\theta) \) for all \( \theta \). In terms of Fourier coefficients, this means: \( a_n = 0, n \) odd; \( b_n = 0, n \) even. (In this case the \( x \)-axis is a critical chord, and there is a second critical chord perpendicular to it.)

(v) Now suppose the \( y \)-axis is an axis of symmetry of \( C \). Then \( b_1(\theta) + b_1(2\pi - \theta) \equiv 0 \). Equivalently, \( p \) is even in \( \theta \), so \( b_n = 0 \) for all \( n \).

Example 1.1, cont. Thus the Fourier coefficients for the Reuleaux triangle vanish unless \( n \) is an odd multiple of 3 (\( n = 3, 9, 15, \ldots \)), and all sine coefficients \( b_n \) vanish. For the first few nonzero terms we find:

\[
p(\theta) = \frac{1}{2} - \frac{1}{4\pi} \cos 3\theta + \frac{1}{120\pi} \cos 9\theta - \frac{1}{560\pi} \cos 15\theta + \ldots
\]

We may obtain a smooth curve of constant width by taking a finite number of terms; but note that, since \( r(\theta) \) is only piecewise continuous, the radius of curvature for the approximation may fail to be positive, no matter how many terms we take (Gibbs phenomenon); so we have to add a constant to \( p \). For example, taking the first three terms given above we find that the minimum curvature (attained at \( \theta = \pi/4 \)) is

\[
\frac{1}{2} - \frac{4\sqrt{2}}{3\pi} = -0.1002\ldots
\]

and subtracting this number from \( p \) we obtain:

\[
p_\epsilon(\theta) = \frac{4\sqrt{2}}{3\pi} - \frac{1}{4\pi} \cos 3\theta + \frac{1}{120\pi} \cos 9\theta + \epsilon. \quad (1.4)
\]

For any \( \epsilon > 0 \), \( p_\epsilon \in \mathcal{P}_{smooth} \) has constant width, and in addition is 3-symmetric and symmetric with respect to the \( y \)-axis; the corresponding \( B_\epsilon \) parametrizes a ‘smoothed Reuleaux triangle’ (shown in Fig.1).

1.2.2. Height and constant height.

Given an oriented convex curve \( C \) and a direction \( \theta \), there is a unique smallest oriented equilateral triangle \( T(\theta) \) enclosing \( C \) with sides parallel to \( T(\theta), T(\theta + 2\pi/3), T(\theta + 4\pi/3) \). If \( C \) is strictly convex, possibly with corners, \( T(\theta) \) has exactly three points in common with \( C \): \( B_1 = B(\theta), B_2 = B(\theta + 2\pi/3), B_3 = B(\theta + 4\pi/3) \); each side of the triangle is contained in a supporting line.

Referring to Fig.9 (section 3), we have for the vertices of \( T(\theta) \):

\[
Q_1 = B_1 + \frac{2}{\sqrt{3}} s_1 T_1 = B_2 + \frac{2}{\sqrt{3}} b_2 T_2.
\]
Taking inner products with $T_1$ and with $T_2$, we find:

\[ s_1 = -\langle B_2 - B_1, N_2 \rangle, \quad t_2 = \langle B_1 - B_2, N_1 \rangle, \]

and cyclically mod 3:

\[ Q_3 = B_1 + \frac{2}{\sqrt{3}}t_1T_1, \text{ with } t_1 = \langle B_1 - B_2, N_1 \rangle. \]

Thus the side length of $T(\theta)$ is given by:

\[ l_{\text{tri}}(\theta) = \langle Q_1 - Q_3, T_1 \rangle = \frac{2}{\sqrt{3}}(s_1 - t_1) = -\frac{2}{\sqrt{3}}(\langle B_2 - B_1, N_2 \rangle + \langle B_3 - B_1, N_3 \rangle) \]
\[ = -\frac{2}{\sqrt{3}}(\langle B_2, N_2 \rangle + \langle B_3, N_3 \rangle + \langle B_1, N_1 \rangle), \]

using $N_1 + N_2 + N_3 = 0$. We use the notation $N_i = N(\theta + 2(i - 1)\pi/3)$ throughout the paper, with $i = 1, 2, 3$.

Thus we see that the height of the triangle $T(\theta)$ is:

\[ h(\theta) = p(\theta) + p(\theta + 2\pi/3) + p(\theta + 4\pi/3). \]

We call $h(\theta)$ the height function of $C$ (or of $p$) and define the ‘derived height’ by:

\[ \omega(\theta) = h'(\theta) = \langle B_1, T_1 \rangle + \langle B_2, T_2 \rangle + \langle B_3, T_3 \rangle \]
\[ = \langle B_2 - B_1, T_2 \rangle - \langle B_1 - B_3, T_3 \rangle, \quad (1.5) \]

since $T_1 + T_2 + T_3 = 0$.

In complete analogy with Proposition 1.1, one can always find strictly convex curves with given ‘derived height’. The proof is completely analogous.

**Proposition 1.2.** Let $\omega$ be $C^1$ and $2\pi/3$-periodic, and satisfy $\int_0^{2\pi/3} \omega(\tau)d\tau = 0$. There exists a one-parameter family of parallel, 3-symmetric strictly convex curves with derived height function $\omega$.

One checks easily that to the list of characterizations of properties of $C$ by the Fourier coefficients of $p$ one may add:

(vi) $C$ is a curve of constant height ($\omega \equiv 0$) iff $a_n = b_n = 0$ whenever $n$ is a multiple of 3.

The only 3-symmetric curve of constant height is therefore the circle.

**Example 1.2.** There are also strictly convex curves of constant height with corners, consisting of any number of circular arcs (except for multiples of 3) with the same radius (see Yaglom-Boltianskiii). We describe in detail the simplest of them, the constant height biangle of height $h$. 

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Figure 2: The constant-height biangle and two smoothed parallel curves; also shown is the locus $OMPN$ of normal intersections for all three curves.

Starting from the equilateral triangle $B_1B_2B_3$ of height $h$ (with $B_1 - B_3 = hT(0)$, see Fig. 2), we draw a $\pi/3$ arc with center $B_2$ and radius $h$ (which intersects the triangle at $M, N$), then reflect this arc on the segment $MN$. Taking as the origin of $\mathbb{R}^2$ the midpoint of $MN$, we compute the support function $p$:

$$p(\theta) = \begin{cases} 
  h(1 - \frac{\sqrt{3}}{2}, \cos \theta), & \theta \in [-\pi/6, \pi/6], \\
  \frac{h}{2} \sin \theta, & \theta \in [\pi/6, 5\pi/6],
\end{cases} \quad (1.6)$$

extended to the real line as a $\pi$-periodic function. One checks easily that $p \in P_{\text{corner}}$ (in particular, $p \in C^1$), and the corresponding $B(\theta)$ maps $[-\pi/6, \pi/6]$ and $[5\pi/6, 7\pi/6]$ to the regular arcs $MB(0)N$, $NB(\pi)M$ (resp.) and $[\pi/6, 5\pi/6], [7\pi/6, 11\pi/6]$ to the vertices $N = (\frac{h\sqrt{3}}{2}, 0), M = (-\frac{h\sqrt{3}}{2}, 0)$ (resp.).

Adding positive constants to $p$ we obtain the support functions of the outer parallel curves of the biangle, which are constant-height curves without corners (but only piecewise $C^2$).

As before, to obtain smooth examples we consider Fourier series. Since the biangle is symmetric with respect to the $x$ and $y$ axes, and of constant
height, from (iv), (v) and (vi) above we see that its support function has a Fourier cosine series, with \( a_n \) non-vanishing only for \( n \) even, and not a multiple of 3. The first few terms are (we set \( h = 1 \)):

\[
p(\theta) = \frac{1}{3} - \frac{\sqrt{3}}{3\pi} \cos(2\theta) - \frac{\sqrt{3}}{30\pi} \cos(4\theta) + \frac{\sqrt{3}}{252\pi} \cos(8\theta) + \ldots
\]

The radius of curvature corresponding to the first three terms is:

\[
r(\theta, 4) = \frac{1}{3} + \frac{\sqrt{3}}{\pi} \left( \cos(2\theta) + \frac{1}{2} \cos(4\theta) \right),
\]

attaining the minimum value \( \frac{1}{3} - \frac{3\sqrt{3}}{4\pi} = -0.0802 \ldots \) at \( \theta = \pi/3 \). We subtract this value from \( p \) to obtain the support function of a ‘smoothed biangle’:

\[
p_\epsilon(\theta) = \frac{\sqrt{3}}{\pi} \left( \frac{3}{4} - \frac{1}{3} \cos(2\theta) - \frac{1}{30} \cos(4\theta) \right) + \epsilon. \quad (1.7)
\]

For any \( \epsilon > 0 \), this defines a support function in \( P_{\text{smooth}} \), and the corresponding curve is strictly convex, of constant height, and has the symmetries of the biangle. (See Fig.2).

The constant-height biangle and its smoothed versions will play an important role in the construction of examples.
2. Steiner networks in strictly convex domains.

A network in a bounded convex domain $D \subset \mathbb{R}^2$ is an embedded (undirected) graph $\mathcal{N}$ intersecting $\mathcal{C} = \partial D$ at finitely many points (all univalent vertices of $\mathcal{N}$), and with all interior vertices trivalent. We assume each edge admits a regular $C^1$ parametrization up to its end-vertices and/or the boundary; in particular, the total length $L$ of $\mathcal{N}$ is well-defined. The critical points of $L$ (with respect to variations that don’t change $\mathcal{N}$ combinatorially, but allow the boundary vertices to move on the boundary) are ‘Steiner networks’ with free (Neumann) boundary conditions. By definition, this means the edges are line segments, meeting at each interior vertex with three angles equal to $2\pi/3$ radians, and meeting $\partial D = \mathcal{C}$ orthogonally at the boundary vertices. The main problem addressed in this paper is the existence and classification of such networks in strictly convex planar domains. (A special case is a ‘critical chord’ in $D$, a one-edge network with no interior vertices.)

In this section we show that the combinatorial possibilities for such networks in a strictly convex domain are in fact rather limited—other than critical chords, only three different types may occur (Fig.8): the triple junction or ‘triode’ (in the terminology of [Mantegazza et al.]); the ‘double triode’; and a single ‘hexagonal cell’, anchored to the boundary by six edges. Furthermore, a Steiner network is necessarily connected, although this is not assumed a priori.

In the following, $\mathcal{N}$ denotes a Steiner network in a strictly convex bounded domain $D \subset \mathbb{R}^2$.

2.1 Chains. A chain $C$ is a connected subgraph of $\mathcal{N}$, including at least two edges, without branching (each internal vertex of $C$ is adjacent to two edges of $C$) and such that all exterior angles between two consecutive edges (in principle either $\pi/3$ or $-\pi/3$) have the same sign. We may choose a consistent orientation of the edges of $C$ so that all exterior angles are equal to $\pi/3$; the chain then has an initial vertex $v_0$ and a final vertex $v_N$. A chain is ‘inextendible’ if it is not a subset of another chain. Any chain can be continued to an inextendible one, in a unique way.

(1) Claim: An inextendible chain either starts and ends at two different boundary vertices of $\mathcal{N}$ or is closed ($v_N = v_0$), and then consists of six edges and six interior vertices, making up a convex equiangular hexagon.

Proof. Assume, by contradiction, $v_N$ is an interior vertex and $C$ is not closed. Since the chain is not forward-extendible past $v_N$, $v_N$ must be adjacent (in $\mathcal{N}$) to another vertex $v$ of $C$. $v_N$, $v$, the edge $f$ connecting them
and the edge $e$ of $C$ arriving at $v$ together define a configuration as shown in Fig. 3.

From the definition of ‘chain’, there is an arc of $C$ (i.e., a subchain $C_1$) which (together with the edge $f$ of $N$ connecting $v_N$ and $v$) bounds an open region $U$ containing an edge $e$ of $C$ arriving at $v$ (Fig. 3).

The backward continuation of the chain from $e$ is contained in $U$, and does not meet $C_1$; in particular, it never meets the boundary of $D$, so the initial vertex $v_0$ is also interior. Since $C$ cannot be backward-continued further past $v_0$, $v_0$ is adjacent in $N$ to a vertex $\hat{v}$ of $C$ preceding $v$ in $C$, and we must have the following configuration including $v_0$, $\hat{v}$ and two edges $g, \tilde{e}$ as shown in Fig.4.

But then there must be a second sub-arc $C_2$ of $C$ (contained in $U$) from $v_0$ to $\hat{v}$, which (together with the edge $g$ of $N$ connecting $v_0$ and $\hat{v}$) bounds an open set $V \subset U$ containing the edge $\tilde{e}$ of $C$ leaving $\hat{v}$ (Fig. 4). Forward continuation of $C_2$ past $\tilde{e}$ can never meet a vertex of $C_2$, hence is entirely contained in the open set $V$, and will not meet the arc $C_1$ of the chain- a contradiction.

The case when the contradiction hypothesis is that $v_0$ is interior is completely analogous (or just reverse the orientations.)

If the chain is closed, since each interior angle is $2\pi/3$, it must be an equiangular hexagon.

(2) If $D$ is strictly convex, a maximal chain connecting two boundary vertices cannot contain more than three edges.
A fourth edge would give a total ‘turning angle’ of at least $\pi$ for the chain, which is impossible for a chain connecting two boundary points (by strict convexity). Note that (1) and (2) imply that a chain including a boundary vertex has at most two interior vertices (since forward continuation from a third interior vertex would yield an inextendible chain with at least four edges, beginning and ending at boundary vertices.)

2.2 Classification of connected components of $\mathcal{N}$. If we exclude the case of critical chords, each boundary vertex of $\mathcal{N}$ is adjacent to a unique vertex, necessarily an interior one. Let $b_0$ be a boundary vertex of a connected component $\hat{\mathcal{N}}$, adjacent to the interior vertex $v_1$. There are only three (unoriented) directions for edges in the network, so by rotating $D$ we may assume $b_0$ is ‘vertically above’ $v_1$, and then phrases such as ‘upper left’, ‘lower right’, ‘vertically below’ have a well-defined meaning. Three cases are possible.

(i) If $v_1$ connects to two other boundary vertices $b_1, b_2$, the connected component $\hat{\mathcal{N}}$ is a triode (Fig.5).

(ii) (Fig.6) If exactly one of the vertices (other than $b_0$) adjacent to $v_1$ in $\mathcal{N}$ is interior (say, the lower-right vertex $v_2$, while the lower-left vertex $b_1$ is a boundary vertex), both the remaining vertices adjacent to $v_2$ must be boundary vertices; otherwise, there would be a chain beginning at a boundary vertex ($b_0$ or $b_1$) and containing at least three interior vertices.
Thus, in this case the connected component $\hat{N}$ is a ‘double triode’.

(iii) Assume now both vertices (other than $b_0$) adjacent to $v_1$ in $N$ are interior ($v_L$ to the left, $v_R$ to the right; Fig.7). The remaining ‘upper’ adjacent vertices of $v_L$ and $v_R$ are both boundary vertices ($b_L$ and $b_R$, respectively), to avoid ‘long’ boundary chains starting at $b_0$. The ‘lower’ adjacent vertices $v'_L$, $v'_R$ must be interior vertices; otherwise one would have ‘long’ chains connecting a boundary vertex ($v'_L$ or $v'_R$) to an interior vertex ($v_R$ or $v_L$, resp.) Of the two lower adjacent vertices to $v'_L$ (resp. $v'_R$) the one on the left (resp. right) must be a boundary vertex ($b'_L$, resp. $b'_R$), to avoid ‘long’ chains connecting the boundary vertex $b_L$ (resp. $b_R$) to an interior vertex.

The remaining adjacent vertices of $v'_L$, $v'_R$ must be interior (were either of them a boundary vertex, it would be part of a ‘long’ chain including $v_1$); call them $v'_L$, $v'_R$. We claim that, in fact, we must have $v'_L = v'_R := v_2$; equivalently, the chain $C = v_R v'_R v_L v'_L v_L v'_L v_2$ is closed.

Consider the inextendible chain $\hat{C}$ containing $C$. If $C$ is not closed, $\hat{C}$ must begin and end on the boundary of $D$, and, as seen in (2) above, in this case $\hat{C}$ has at most three edges. Since the chain $C$ has 6 edges, this cannot happen; thus $C$ is closed, and therefore an equiangular hexagon.

Finally, the lower adjacent vertex to $v_2$ must be a boundary vertex $b_1$, to avoid long chains starting at $b'_L$ or $b'_R$. Thus this connected component of $N$ is a hexagonal cell ‘anchored’ to the boundary.

2.3 Connectedness of the network.
Suppose $\mathcal{N}$ has two non-intersecting connected components $\hat{\mathcal{N}}_1$ and $\hat{\mathcal{N}}_2$. In particular, the boundary vertices of $\hat{\mathcal{N}}_1$ and $\hat{\mathcal{N}}_2$ are ‘non-interlacing’, that is, $\partial D$ is partitioned into two arcs (disjoint except for their endpoints), each containing all the boundary vertices of $\hat{\mathcal{N}}_1$ or $\hat{\mathcal{N}}_2$. Each of $\hat{\mathcal{N}}_1$ and $\hat{\mathcal{N}}_2$ is of one of the four types listed above (including critical chords). For any of the four types, one sees directly that there is a connected arc of $C$ starting and ending at boundary vertices $B, \bar{B}$ of that connected sub-graph, with a total turning angle from $B$ to $\bar{B}$ (for the unit tangent to $C$) of at least $\pi$ ($\pi$ for a critical chord, $4\pi/3$ for a triode, $5\pi/3$ for a double triode or a hexagonal cell.) For a strictly convex curve $C$, it is not possible for two such arcs to be disjoint. Hence there is only one connected component.

We summarize the conclusion in the following proposition (Fig. 8):

**Proposition 2.1.** Let $\mathcal{N}$ be a Steiner network in a strictly convex domain in $\mathbb{R}^2$, with ‘free’ (Neumann) boundary conditions. Then $\mathcal{N}$ is connected, and is one of: (i) a critical chord; (ii) a triode; (iii) a double triode; (iv) a closed equiangular hexagon, anchored to the boundary.
3. Existence of triodes.

3.1 Preliminary remarks. Given a strictly convex curve $C$ and a direction $\theta \in S$, let $T(\theta)$ be the circumscribed equilateral triangle with one side parallel to $T(\theta)$, touching $C$ at the points $B_1 = B(\theta)$, $B_2 = B(\theta + 2\pi/3)$, $B_3 = B(\theta + 4\pi/3)$; for definiteness we take $\theta$ in $[0, 2\pi/3)$. The inner normals $n_1, n_2, n_3$ at the points of contact intersect pairwise:

$$n_1 \cap n_2 = \{P_1\}, \quad n_2 \cap n_3 = \{P_2\}, \quad n_3 \cap n_1 = \{P_3\}$$

(Fig.9). $\theta$ defines a triode configuration exactly when $P_1 = P_2 = P_3 = P$ and $P$ is in the interior of $D$. To express this analytically, define functions $u(\theta) = u_1, v(\theta) = v_1$ via:

$$P_1 = B_1 + \frac{2}{\sqrt{3}}u_1N_1 = B_2 + \frac{2}{\sqrt{3}}v_2N_2.$$

(Here $v_2 = v(\theta + 2\pi/3)$.) Taking inner products with $T_1$ and $T_2$ one finds:

$$u_1 = \langle B_2 - B_1, T_1 \rangle, \quad v_2 = \langle B_2 - B_1, T_1 \rangle.$$

This gives the explicit definitions of the ‘forward and backward triangle functions’ $u(\theta), v(\theta)$:

$$u(\theta) = \langle B(\theta + 2\pi/3) - B(\theta), T(\theta + 2\pi/3) \rangle,$$

$$v(\theta) = \langle B(\theta) - B(\theta + 4\pi/3), T(\theta + 4\pi/3) \rangle.$$

Using $T_1 + T_2 + T_3 = 0$, we see that $u - v$ is $2\pi/3$-periodic:

$$u(\theta) - v(\theta) = \langle B(\theta), T(\theta) \rangle + \langle B(\theta + 2\pi/3), T(\theta + 2\pi/3) \rangle + \langle B(\theta + 4\pi/3), T(\theta + 4\pi/3) \rangle.$$

Recalling (1.5) in section 1, we see that $u - v = \omega$, the ‘derived height’:

$$u(\theta) - v(\theta) = \omega(\theta) = h'(\theta).$$

This immediately implies the following.

**Proposition 3.1.** The three inner normals at the points of contact of a circumscribed equilateral triangle $T(\theta)$ intersect at a single point exactly when $\theta \in S$ is a critical point of the height function; this happens for at least two geometrically distinct configurations. A ‘critical configuration’ ($\omega(\theta) = 0$) defines a triode iff $u(\theta), u(\theta + 2\pi/3), u(\theta + 4\pi/3)$ are all positive.

It is useful to express $u(\theta)$ in terms of the support function $p$. From $\langle B_2, T_2 \rangle = p'(\theta + 2\pi/3)$ and:

$$\langle B_1, T_2 \rangle = -p(\theta)\langle N_1, T_2 \rangle + p'(\theta)\langle T_1, T_2 \rangle = -\frac{\sqrt{3}}{2}p(\theta) - \frac{1}{2}p'(\theta),$$

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Figure 9: The forward and backward triangle functions

we have:
\[ u(\theta) = p'(\theta + 2\pi/3) + \frac{1}{2} p'(\theta) + \frac{\sqrt{3}}{2} p(\theta). \]  
(3.1)

A similar calculation yields:
\[ v(\theta) = -p'(\theta + 4\pi/3) - \frac{1}{2} p'(\theta) + \frac{\sqrt{3}}{2} p(\theta). \]

**Definition 3.1.** It is useful to observe that given a (sufficiently differentiable) $2\pi$-periodic function $p$ one can always define the ‘radius of curvature’
\[ r = p'' + p, \]
the ‘derived height’:
\[ \omega(\theta) = p'(\theta) + p'(\theta + 2\pi/3) + p'(\theta + 4\pi/3) \]
and the ‘triangle function’ $u$ (by (3.1)), even when $p$ is not the support function of a strictly convex curve.

**Remark 3.1.** Let $L(\theta) = B(\theta + 2\pi/3) - B(\theta)$. Then $\langle L(\theta), N(\theta) \rangle > 0$ for all $\theta$ (by strict convexity). Observing that $\sqrt{3}N_1 = T_2 - T_3$, and using $u(\theta) = \langle L(\theta), T_1 \rangle$, $v(\theta + 2\pi/3) = \langle L(\theta), T_1 \rangle$, we have:
\[ u(\theta) > \langle L(\theta), T_3 \rangle = -\langle L(\theta), T_1 \rangle - \langle L(\theta), T_2 \rangle = -v(\theta + 2\pi/3) - u(\theta), \]

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or \(2u(\theta) + v(\theta + 2\pi/3) > 0\) for all \(\theta\). By \(2\pi/3\) periodicity of \(u - v\), this is equivalent to:

\[u(\theta) + v(\theta) + u(\theta + 2\pi/3) > 0 \quad \forall \theta.\]

A similar argument yields the equivalent inequalities:

\[2v(\theta) + u(\theta + 4\pi/3) > 0, \quad u(\theta) + v(\theta) + v(\theta + 2\pi/3) > 0 \quad \forall \theta.\]

This shows that, for any \(\theta\), at most two of \(u_1, u_2, u_3\) can be negative; and at a zero of \(\omega\), at most one of \(u_1, u_2, u_3\) may be negative (for a triode, all must be positive.)

### 3.1 Existence of triodes

It is natural to look for conditions of ‘curvature pinching’ type that guarantee \(u > 0\) everywhere.

For each \(\theta\), define \(\theta_* \in (\theta, \theta + 2\pi)\) by requiring \(B(\theta_*)\) to be the (other) intersection of the normal through \(B(\theta)\) with \(C\) (Fig. 9). Set \(d(\theta) = (B(\theta_*) - B(\theta), N(\theta))\). All intersections of inner normals (at contact points of a circumscribed triangle \(T(\theta)\)) will be internal if we require, for all \(\theta\):

\[0 < u_1 < \frac{\sqrt{3}}{2}d_1, 0 < u_2 < \frac{\sqrt{3}}{2}d_2, 0 < u_3 < \frac{\sqrt{3}}{2}d_3. \quad (3.2)\]

For \(C^2\) curves, we have:

\[d(\theta) = \int_0^{\theta_* - \theta} r(\tau + \theta) \sin \tau d\tau\]

and:

\[u(\theta) = \int_0^{2\pi/3} r(\tau + \theta) \cos(2\pi/3 - \tau) d\tau,\]

so we want:

\[0 < -\frac{1}{2} \int_0^{2\pi/3} r(\tau + \theta) \cos \tau d\tau + \frac{\sqrt{3}}{2} \int_0^{2\pi/3} r(\tau + \theta) \sin \tau d\tau < \frac{\sqrt{3}}{2} \int_0^{\theta_* - \theta} r(\tau + \theta) \sin \tau d\tau.\]

This suggests the condition:

\[\theta + 2\pi/3 < \theta_* < \theta + 4\pi/3 \quad \forall \theta. \quad (3.3)\]

(Note that, for the circle, \(\theta_* = \theta + \pi \forall \theta\).) This condition implies, in particular:

\[\theta + 4\pi/3 < (\theta + 2\pi/3)_* < \theta + 2\pi, \quad \theta + 2\pi < (\theta + 4\pi/3)_* < \theta + 8\pi/3 \quad \forall \theta.\]
It is clear geometrically that:

\[ u_1 < 0 \Rightarrow (\theta + 2\pi/3)_* > \theta + 2\pi \]

and

\[ u_1 > (\sqrt{3}/2)d(\theta) \Rightarrow (\theta + 2\pi/3)_* < \theta_* < \theta + 4\pi/3; \]

thus, (3.3) indeed implies (3.2).

It is easy to express (3.3) as a condition on the radius of curvature. Since \( \theta_* \) is characterized by:

\[
\int_{\theta}^{\theta_*} r(\tau)(T(\tau), T(\theta))d\tau = \int_{\theta}^{\theta_*} \cos(\tau - \theta)r(\tau)d\tau = 0,
\]

(2.3) is equivalent to the two conditions:

\[
\int_0^{2\pi/3} r(\tau + \theta)\cos\tau d\tau > 0, \quad \int_{\pi/2}^{4\pi/3} r(\tau + \theta)\cos\tau d\tau < 0. \tag{3.4}
\]

Assume \( 0 < r_{\min} \leq r(\theta) \leq r_{\max} \) in \([0, 2\pi]\). The first inequality in (3.4) is equivalent to:

\[
\int_0^{\pi/2} r(\tau + \theta)\cos\tau d\tau > \int_{\pi/2}^{2\pi/3} r(\tau + \theta)\cos\tau d\tau,
\]

which would follow from \( r_{\min} > (1 - \sqrt{3}/2)r_{\max} \). Similarly, the second inequality in (3.4) is equivalent to:

\[
\int_0^{\pi/2} r(\tau + \theta)d\tau < \int_{\pi/2}^{4\pi/3} r(\tau + \theta)\cos\tau d\tau,
\]

which would follow from: \( r_{\max} < (1+\sqrt{3}/2)r_{\min} \). This is the more restrictive inequality, and gives the sufficient condition for existence of triodes recorded in the following proposition.

**Proposition 3.2.** Let \( p \in \mathcal{P}_{\text{smooth}} \) satisfy \( r_{\max} < (1 + \sqrt{3}/2)r_{\min} \). Then the three inner normals at the contact points of a circumscribed equilateral triangle always intersect inside \( \mathcal{C} \). Thus \( \mathcal{C} \) supports at least two geometrically distinct triodes (stably).

**Example 2.1.** The constant-height biangle has \( u < 0 \) on the ‘regular intervals’ \((-\pi/6, \pi/6)\) and \((5\pi/6, 7\pi/6)\) (Fig.2). Indeed its ‘forward triangle function’ \( u_{bi}(\theta) \) is even and \( \pi \)-periodic, given in \([-\pi/6, 5\pi/6]\) by:

\[
u_{bi}(\theta) = \begin{cases} \frac{\sqrt{3}}{2} - \cos \theta, & -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}, \\ \frac{1}{2}(\sqrt{3}\sin \theta - \cos \theta), & \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}. \end{cases}\]

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Figure 10: Proof of proposition 3.3

$u$ increases from the minimum $\frac{\sqrt{3}}{2} - 1$ at $\theta = 0$ to the maximum $\sqrt{3}/2$ at $\theta = \pi/2$, and is positive exactly on the ‘singular intervals’ $(\frac{\pi}{6}, \frac{5\pi}{6})$ and $(\frac{7\pi}{6}, \frac{11\pi}{6})$, where $B(\theta)$ maps to one of the endpoints of the maximal diameter. Thus the biangle does not support triodes. We can remedy this by adding $c > 0$ to the support function, producing $p_c \in \mathcal{P}_{pw}$ (without corners) with ‘triangle function’ $u_c(\theta) = u_{bi}(\theta) + c$. For $0 < c < 1 - \frac{\sqrt{3}}{2}$, we have $u_c < 0$ on the interval $(-\delta, \delta)$, for some $0 < \delta < \frac{\pi}{6}$. ($u_c(\delta) = 0$). (Geometrically, the biangle and each of its outer parallel curves have the same locus of normal intersections, shown in Fig.2) But if $\theta \in (\delta, \frac{\pi}{6})$, we have $\theta + \frac{2\pi}{3} \in (\frac{2\pi}{3} + \delta, \frac{7\pi}{6})$ and $\theta + \frac{4\pi}{3} \in (\frac{4\pi}{3} + \delta, \frac{2\pi}{3})$, and $u_c > u_{bi} > 0$ on both intervals; so $u_1, u_2, u_3$ are all positive for these values of $\theta$, and since $\omega \equiv 0$ each value of $\theta$ in $(\delta, \frac{\pi}{6})$ corresponds to a triode. More generally, we have the following proposition.

**Proposition 3.3.** Any $p \in \mathcal{P}_{pw}$ of constant height supports triodes.

**Proof.** (Fig.10.) Since $C$ is a $C^1$ strictly convex curve, we may use a topological argument. For a given $\theta_0$, consider the triangle of tangency points $\Delta(\theta_0) = \{B_1(\theta_0), B_2(\theta_0), B_3(\theta_0)\} \subset \bar{D}$; if all interior angles of $\Delta(\theta_0)$ are less than $2\pi/3$, the normals to $C$ at these points intersect at a single point in $D$, and we are done. Assume, instead, that the interior angle of $\Delta(\theta_0)$ at $B_1(\theta_0)$ is greater than $2\pi/3$, while the angle at $B_2(\theta_0)$ is smaller than $\pi/2$. By continuity, moving $B_1$ towards $B_2$ we find $\theta$ so that the angle of $\Delta(\theta)$ at $B_1(\theta)$ is exactly $\pi/2$, so that all interior angles of $\Delta(\theta)$ are smaller than $2\pi/3$. Hence the normals to $C$ at the $B_i(\theta)$ intersect internally, and since $\omega \equiv 0$ we see $\theta$ defines a triode.

**Example 3.2.** For the Reuleaux triangle, with support function $p \in \mathcal{P}_{corner}$ given by (1.3), one checks easily that $u \geq 0$ and vanishes only at $\theta = \pi/2, 3\pi/2$. On the other hand, $\omega$ vanishes exactly at $0$ and $\pi/3$ (in $[0, 2\pi/3]$), and both are transversal zeros of $\omega$. $\theta = 0, 2\pi/3, 4\pi/3$ all map to regular points of $C$, and hence define a triode, while $\theta = \pi/3, \pi, 5\pi/3$ map to the three vertices of the Reuleaux triangle, and again define a triode (if we
relax the definition slightly). For the ‘smoothed Reuleaux triangles’ defined by \( p_\varepsilon \) in (1.4), the corresponding \( \omega_\varepsilon \) vanishes at the same points, and \( u_\varepsilon \) is everywhere positive; so these curves also support two triodes. Note that the ratio \( r_{\max}/r_{\min} \) can be made arbitrarily large, by taking \( \varepsilon > 0 \) small.

We end this subsection by recording four simple observations regarding existence for triodes: (i) any \( C \) corresponding to \( p \in P_{puw} \) with an axis of reflection symmetry supports triodes. (ii) If \( C \) supports a triode, it extends to a triode in any exterior parallel curve; (iii) For any strictly convex \( C \), all exterior parallel curves \( C_c \) for \( c \) sufficiently large will support triodes (since adding a sufficiently large positive constant to \( p \) makes \( u \) everywhere positive.) (iv) Curves of constant width \((w \equiv 0)\) always support triodes (see Remark 5.5 in section 5.)

3.3 Strictly convex curves without triodes.

We use the nonexistence criterion: if \( u < 0 \) at every zero of \( \omega \) in a ‘fundamental domain’ (i.e., an interval of length \( 2\pi/3 \)), then \( C \) supports no triodes. We can achieve this by starting with a support function for a strictly convex domain of ‘constant height’ satisfying \( u < 0 \) in some interval, and taking convex combinations with another ‘support function’ for which the zeros of \( \omega \) have the desired property. For the second one, we don’t even need \( r > 0 \).

**Proposition 3.4.** Let \( p_0 \in P_{\text{smooth}} \) satisfy \( \omega_0 \equiv 0 \) (‘constant height’) and \( u_0 < 0 \) in some interval \( I \subset (-\pi, \pi) \). Let \( \tilde{p} \) be \( 2\pi \)-periodic and \( C^2 \), with the property that all zeros of the corresponding ‘derived height’ \( \tilde{\omega} \) in a fundamental domain are contained in an interval \( J \subset I \). Then for all \( \lambda \in (0, 1) \) sufficiently close to 1, the function:

\[
p_\lambda(\theta) = (1 - \lambda)\tilde{p}(\theta) + \lambda p_0(\theta)
\]

is in \( P_{\text{smooth}} \), and is the support function of a \( C^2 \) strictly convex curve which does not support triodes. If the zeros of \( \tilde{\omega} \) are transversal, this is true stably.

**Proof.** Since \( r_\lambda = (1 - \lambda)r + \lambda r_0 \), we clearly have \( p_\lambda \in P_{\text{smooth}} \) (i.e., \( p > 0 \) and \( r > 0 \)) for \( \lambda \) sufficiently close to 1. Also, \( u = (1 - \lambda)\tilde{u} + \lambda u_0 \), so for \( \lambda \) close to 1 we have \( u_\lambda < 0 \) in \( I \); while \( \omega_\lambda = (1 - \lambda)\tilde{\omega} \), so for all \( \lambda \in (0, 1) \) we have \( \omega_\lambda < 0 \) in \( J \subset I \). So for \( \lambda \) close to 1, all zeros of \( \omega_\lambda \) in a fundamental domain are contained in an interval where \( u_\lambda < 0 \). Hence \( p_\lambda \) cannot support a triode.

It is not hard to construct functions \( \tilde{p} \) satisfying the conditions of the
proposition. For example, for any $0 < m < 1$, the function:

$$q(\theta) = \frac{\cos(3\theta)}{1 - m \sin(3\theta)}$$

is $2\pi/3$-periodic, with critical points exactly where $\sin(3\theta) = m$. Thus the zeros of $q'$ in $[0, 2\pi/3)$ are exactly $\frac{1}{3}\arcsin(m)$ and $\frac{2}{3} - \frac{1}{3}\arcsin(m)$ (where $\arcsin(m) \in (0, \pi/2)$). The distance between these zeros is $\frac{2}{3}(\frac{2}{3} - \arcsin(m))$, which can be made arbitrarily small by taking $m$ close enough to 1. Thus, given $I \subset (-\pi, \pi)$ (defined by $p_0$ as in the proposition), a suitable translate of $q$ can be used as $\tilde{p}$. (Note that, since $q$ is $2\pi/3$-periodic, the corresponding derived height function is simply $3q'$.)

**Corollary 3.5.** Let $p_0 \in \mathcal{P}_{\text{smooth}}$ be any curve of constant height, such that $u_0 < 0$ somewhere. Then arbitrarily $C^r$ close to $p_0$ (if $p_0 \in C^r$, where $r \geq 2$), one finds: (i) $C^r$ strictly convex curves supporting triodes (stably); (ii) $C^r$ strictly convex curves which do not support any triodes (also stably).

**Proof.** Part (ii) follows directly from the proposition; for stability of the no-triode property, we just need to remark that the zeros of $q$ given above are transversal. Clearly, by taking $\lambda$ sufficiently close to 1, $p_\lambda$ can be made arbitrarily $C^r$-close to $p_0$. For (i), recall that by Prop. (3.3), one may find $\bar{\theta}$ so that $u_{10}, u_{20}, u_{30}$ (computed for $p_0$) are all positive at $\bar{\theta}$. As explained above, we find $\tilde{p}$ smooth, $2\pi/3$-periodic, so that all the zeros of the corresponding $\tilde{\omega}$ in a fundamental domain are found in a small neighborhood of $\tilde{\theta}$, where $u_{10}, u_{20}, u_{30}$ are still positive. Taking now:

$$p_\lambda = (1 - \lambda)\tilde{p} + \lambda p_0,$$

we have, for $\lambda$ sufficiently close to 1: (i) $p_\lambda \in \mathcal{P}_{\text{smooth}}$ and is as close to $p_0$ as desired; (ii) the zeros of $\omega_\lambda = (1 - \lambda)\tilde{\omega}$ in a fundamental domain are all found in a neighborhood of $\bar{\theta}$ where $u_{1\lambda}, u_{2\lambda}, u_{3\lambda}$ are all positive. Thus $p_\lambda$ supports triodes.

**Example 3.3.** An explicit example of $p_0 \in \mathcal{P}_{\text{smooth}}$ of constant height with $u < 0$ somewhere is the ‘smoothed biangle’ of (1.7):

$$p_\epsilon(\theta) = \frac{\sqrt{3}}{\pi} \left( \frac{3}{4} - \frac{1}{3} \cos(2\theta) - \frac{1}{30} \cos(4\theta) \right) + \epsilon$$

($\epsilon > 0$ arbitrary). The ‘triangle function’ $u_\epsilon$ computed from $p_\epsilon$ is:

$$u_\epsilon(\theta) = \frac{3}{2\pi}(-\cos(2\theta) + \frac{1}{10} \cos(4\theta) + \frac{3}{4} + \frac{\sqrt{3}}{2} \epsilon.$$
For $\epsilon = 0$, this is negative in $(-x_0, x_0)$ and vanishes at

$$x_0 = (1/2) \arccos(5/2 - \sqrt{3}) = 0.3476\ldots$$

Since $u_\epsilon = u_0 + \frac{\sqrt{3}}{2} \epsilon$ and $u_0(0.34) \sim -\frac{0.01}{\pi}$, we find that for $\epsilon = 0.02/(\pi \sqrt{3})$, we have $u_\epsilon < 0$ in $I = (-0.34, 0.34)$.

Turning to parameters in $\tilde{p}$, we let $m = \sin(3\pi/4) = \sqrt{2}/2$; then the critical points of $q(x)$ in $[0, 2\pi/3]$ are $\pi/4 = 0.785\ldots$ and $\pi/12 = 0.262\ldots$; letting $\tilde{p}(\theta) = q(\theta + \pi/6)$, $\tilde{\omega}$ has in the fundamental domain $[-\pi/6, \pi/2]$ only the zeros $-\pi/12 = -0.262\ldots$ and $\pi/12 = 0.262\ldots$, both in the interval $I$.

**Remark 3.2.** One checks numerically that the Fourier series truncated at $n = 9$ of $\tilde{p}(\theta) = q(\theta + \pi/6) = -\sin(3\theta)/(1 - (\sqrt{2}/2) \cos(3\theta))$ already has the desired property. Writing $p_\lambda$ in the form (for $p_\epsilon$ and $\epsilon$ given above:)

$$p_\lambda(\theta) = \frac{\lambda}{1 + \lambda} p_\epsilon(\theta) + \frac{1}{(1 + \lambda)} \tilde{p}(\theta),$$

one finds that for $\lambda = 5,000$ we already have $p_\lambda > 0$ and $r_\lambda > 0$ for all $\theta$. The resulting convex curve is visually indistinguishable from the biangle.

**Remark 3.3.** The construction depends essentially on the existence of convex curves of constant height for which the ‘triangle function’ $u$ is negative somewhere. While this happens for the constant height biangle, the other ‘curved regular polygons’ of constant height have $u > 0$ everywhere, and any convex curve of constant height may be uniformly approximated by such polygons. So our construction of examples of convex curves without triodes ultimately depends on the existence of this atypical example of a convex constant-height curve.

### 3.4. Convergence of triode configurations.

**Definition 3.2.** $\theta \in S$ defines a **boundary triode** if $\omega(\theta) = 0$ and $u(\theta) = 0$; geometrically, one of the edges of the triode collapsed to a boundary point of $D$.

As long as the limit convex curve is without corners, this is the only kind of degeneration allowed under convergence in $\mathcal{P}_{smooth}$ with the $C^2$ topology.

**Lemma 3.6.** If $p \in \mathcal{P}_{pw}$ with (piecewise continuous) radius of curvature function $r(\theta) \geq r_0 > 0$, then for each $\theta \in \mathbb{R}$:

$$||B(\theta + 2\pi/3) - B(\theta)|| \geq (3/2) \min\{r(\tau); \tau \in [\theta, \theta + 2\pi/3]\} \geq (3/2)r_0.$$
Proof. Since:

\[ B(\theta+2\pi/3) - B(\theta) = \left( \int_0^{2\pi/3} r(\tau+\theta) \cos \tau \, d\tau \right) T(\theta) + \left( \int_0^{2\pi/3} r(\tau+\theta) \sin \tau \, d\tau \right) N(\theta), \]

we have:

\[ ||B(\theta+2\pi/3) - B(\theta)||^2 \geq \left( \int_0^{2\pi/3} r(\tau+\theta) \sin \tau \, d\tau \right)^2 \geq \left( \frac{3}{2} \right)^2 \left( \min_{[\theta,\theta+2\pi/3]} r \right)^2. \]

Proposition 3.7. Let \( p_i, p \in P_{\text{smooth}}, p_i \to p \) uniformly in \( C^2(S) \). Let \( T_i = (B_{i1}, B_{i2}, B_{i3}, P_i) \) be a triode or boundary triode in \( C_i \), where \( P_i \in \bar{D}_i \). Then (up to passing to subsequences) \( T_i \to T = (B_1, B_2, B_3, P) \) (meaning \( B_{i1} \to B_1, P_i \to P \), etc.)

Proof. This is practically self-evident, since \( \omega_i(\theta) = 0 \) implies \( \omega(\theta) = 0 \). From the lemma, \( ||B_a^i - B_b^i|| \) is uniformly bounded below for \( a \neq b \) in \( \{1, 2, 3\} \), so the only ‘collapse’ allowed is \( ||B_a^i - P^i|| \to 0 \) for some \( a \) (and then \( P^i \to P \in C \)), which gives a boundary triode in the limit.

Remark 3.4. If \( p \in P_{\text{corner}} \) and \( C \) has corners, it is conceivable that \( p_i \to p \) (say, uniformly in \( C^1 \)), but the \( T_i \) collapse to a segment. This happens, for example, when triodes in the outer parallel curves of a biangle converge to the diameter of the biangle. They cannot collapse to a point, however, since the total length \( L_i \) is uniformly bounded below, as long as \( p_i(\theta) \geq p_0 > 0 \).
4. Double triode configurations.

4.1 Basic construction. We adopt for the remainder of the paper the notation $\theta = \theta + \pi$. Let $C$ be strictly convex. Given any direction $\theta \in S$, the circumscribed triangles (given by their vertices) $T(\theta) = \{Q_1, Q_2, Q_3\}$ and $T(\bar{\theta}) = \{\bar{Q}_1, \bar{Q}_2, \bar{Q}_3\}$ define three parallelograms:

\[ P(\theta) = \{Q_1, S_2, \bar{Q}_1, \bar{S}_2\} = P(\bar{\theta}) \] (Fig. 11) and also (see Fig. 15):

\[ P(\theta + 2\pi/3) = \{Q_2, S_3, \bar{Q}_2, \bar{S}_3\}, \quad P(\theta + 4\pi/3) = \{Q_3, S_1, \bar{Q}_3, \bar{S}_1\}. \]

For each $\theta \in S$, intersections of normals define a four-point configuration $(P(\theta), U(\theta), P(\bar{\theta}), U(\bar{\theta}))$, collapsing to two points when $\theta$ defines a double triode. Here $P(\theta) = n(\theta) \cap n(\theta + 2\pi/3)$, as in section 3. To define $U(\theta)$, we consider ‘projections’. Since there are only three unoriented directions involved in any potential network configuration (once $\theta$ is fixed), given any two non-parallel normal lines $n, \bar{n}$ along two of the directions, there is a well-defined projection from $n$ to $\bar{n}$ along the third normal direction, denoted $pr_{n \to \bar{n}}$. For a double triode configuration, we consider the points:

\[ U(\theta) = pr_{\bar{n}_2 \to n_1}(P(\bar{\theta})), \quad \bar{U}(\theta) = U(\bar{\theta}) = pr_{n_2 \to \bar{n}_1}(P(\theta)) \]
(where \( n_1 = n(\theta), \bar{n}_2 = n(\theta + 2\pi/3 + \pi) \), etc.) We write this in simplified notation as follows:

\[
U_1 = pr_{21}(\bar{P}_1), \quad \bar{U}_1 = pr_{21}(P_1).
\]

We have a double triode when (i) \( P_1 = U_1 \) (and then \( \bar{P}_1 = \bar{U}_1 \)); and (ii) \( P_1 \) and \( \bar{P}_1 \) are inside \( C \). (A third necessary condition will be described soon).

Then the parallelogram \( P_1U_1 \bar{P}_1 \bar{U}_1 \) collapses to the ‘bridge’ of the double triode \( X = \{ B(\theta), B(\theta + 2\pi/3), B(\theta), B(\theta + 2\pi/3) \} \).

We let \( u(\theta) = u_1 \) have the same meaning as in section 3, and define \( \sigma(\theta) = \sigma_1, \sigma(\bar{\theta}) = \bar{\sigma}_1 \):

\[
P_1 = B_1 + \frac{2}{\sqrt{3}} u_1 N_1, \quad U_1 = B_1 + \frac{2}{\sqrt{3}} \sigma_1 N_1, \quad \bar{P}_1 = \bar{B}_1 + \frac{2}{\sqrt{3}} \bar{u}_1 \bar{N}_1, \quad \bar{U}_1 = \bar{B}_1 + \frac{2}{\sqrt{3}} \bar{\sigma}_1 \bar{N}_1.
\]

Since \( U_1 \) is the projection of \( \bar{P}_1 \) along the direction \( N_3 \), we also have:

\[
U_1 = \bar{P}_1 + \alpha N_3 \text{ for some } \alpha \in \mathbb{R} \quad \text{taking inner products with } T_3,
\]

we find:

\[
\sigma_1 = \langle B_1 - \bar{B}_1, T_3 \rangle - \bar{u}_1.
\]

We are interested in the difference \( \sigma - u \):

\[
\sigma - u = \langle B_1 - \bar{B}_1, T_3 \rangle - \langle B_2 - \bar{B}_1, T_2 \rangle - \langle B_2 - B_1, T_2 \rangle = \langle B_1 - B_1, T_1 \rangle + \langle B_2 - B_2, T_2 \rangle = w(\theta) + w(\theta + 2\pi/3),
\]

where the derived width function \( w \) was defined in section 1. We record the expression for \( \sigma \):

\[
\sigma(\theta) = u(\theta) + w(\theta) + w(\theta + 2\pi/3) \quad (4.1)
\]

Condition (i) above for existence of a double triode is \( u = \sigma \), or:

\[
y(\theta) := w(\theta) + w(\theta + 2\pi/3) = 0.
\]

We can relate this to the perimeter \( L_{par} \) of the circumscribed parallelogram \( P(\theta) \):

\[
L_{par}(\theta) = 2(\langle S_2 - Q_1, T_2 \rangle + \langle \bar{Q}_1 - S_2, \bar{T}_1 \rangle) = \frac{4}{\sqrt{3}}(b(\theta) + b(\theta + 2\pi/3)).
\]

Since \( b'(\theta) = -w(\theta) \), we conclude:

\[
y(\theta) = 0 \Leftrightarrow \theta \text{ is a critical point of the perimeter of } P(\theta).
\]
It is useful to observe that, since \( w \) is \( \pi \)-periodic, one can recover \( w \) from \( y \) via:
\[
y(\theta) + y(\theta + \pi/3) - y(\theta + 2\pi/3) = 2w(\theta), \quad (4.2)
\]
with a similar relation between \( L_{par} \) and \( b \). Thus the class ‘curves of constant width’ is the same as ‘curves of constant parallelogram perimeter’.

The third condition for a double triode arises from the fact that two types of configurations are possible if \( y(\theta) = 0 \). (Fig.12).

Case (I) (good): \( w(\theta) < 0 \) and \( w(\theta + 2\pi/3) > 0 \);
Case (II) (bad): \( w(\theta) > 0 \) and \( w(\theta + 2\pi/3) < 0 \).

In fact, one easily computes that the (oriented) length of the ‘bridge’ of a critical configuration (which is positive for a double triode) is: \( \langle P_1 - \bar{P}_1, N_3 \rangle = -(2/\sqrt{3})w(\theta) \), and the total length of the network is:
\[
L = \frac{2}{\sqrt{3}}(u_1 + \bar{u}_1 + v_2 + \bar{v}_2 - w).
\]

Using the already computed expressions for the various terms, we find:
\[
L = b(\theta) + b(\theta + 2\pi/3),
\]
in words: the length of the double triode $X(\theta)$ is $\sqrt{3}/4$ times the perimeter of the parallelogram $P(\theta)$.

From (4.2) we see that a double triode corresponds to $y(\theta) = 0$ with $y(\theta + \pi/3) < y(\theta + 2\pi/3)$. It follows that 3-symmetric curves (for which $w$ and $y$ are both $\pi$- and $2\pi/3$-periodic, hence $\pi/3$-periodic) cannot support non-degenerate double triodes. The same holds for curves of constant width ($w \equiv 0$).

To summarize: $\theta$ corresponds to a double triode configuration if (i) $y(\theta) := w(\theta) + w(\theta + 2\pi/3) = 0$; (ii) $u(\theta) > 0$ and $u(\theta) > 0$; (iii) $w(\theta) < 0$. Note that if $\theta$ is a transversal zero of $y$, any sufficiently $C^2$ close curve will also support a double triode.

4.2 Sufficient conditions for existence. It is natural to consider convex curves admitting an axis of reflection symmetry, and then look for symmetric double triodes. Such an axis is always a critical chord of $C$ - its endpoints correspond to critical points of the width function $b$, which will be assumed to be non-degenerate, hence a local max (‘maximal chord’) or local min (‘minimal chord’) of $b$. Symmetry implies the existence of a second critical chord orthogonal to the axis of symmetry. For our first existence result we assume these are the only critical chords of $C$.

Proposition 4.1. Let $C$ be a $C^1$ strictly convex curve ($p \in P_{pw}$) Assume $C$ has a non-degenerate maximal chord which is a line of reflection symmetry and, except for the orthogonal minimal chord (also assumed non-degenerate), no other critical chords. Then $C$ stably supports a ‘double triode’.

Examples are given by ellipses and outer parallel curves of the constant-height biangle.

Proof. (Fig. 13) Position $C$ so that the line of symmetry has direction $T(0)$. Then the maximal chord is $B(\frac{3\pi}{2})B(\frac{\pi}{2})$, the minimal chord $B(0)B(\pi)$. 

Figure 13: Existence of a double triode with symmetry (Prop.4.1)
w = −b′ is negative on \((0, \pi/2) \cup (\pi, 3\pi/2)\), in particular at \(\theta_0 = \pi/6\). It is easy to check that \(\theta_0 = \pi/6\) satisfies the two other conditions for a double triode. First, \(p'(\pi - \theta) = -p'(\theta)\) (symmetry) implies \(w(\pi - \theta) = -w(\theta)\), and hence \(y(\pi/2 - \theta) = -y(\theta)\), so \(y(\pi/6) = 0\). Second, symmetry implies \(B(5\pi/6) - B(\pi/6) = cN(0)\), for some \(c > 0\). Hence \(u(\pi/6) = c\cos(5\pi/6 - \pi/2) > 0\), and similarly for \(u(7\pi/6)\). Stability follows from the fact that \(\pi/6\) is a transversal zero of \(y\).

The next result assumes reflection symmetry only near the endpoints of a maximal chord, and makes quantitative the intuition that double triodes are easier to find near ‘sharp tips’. A motivating example is a strictly convex domain with two corners, so that at each corner no wedge with aperture greater than or equal to \(\pi/6\) about the chord joining the corners fits inside \(D\); it is easy to see that any outer parallel curve \(C_\epsilon\) supports a double triode.

To set up the notation, let \(KK\) be a non-degenerate maximal chord with direction \((1, 0)\), so \(K = B(\pi/2), \bar{K} = B(3\pi/2)\). We seek a double triode with ‘bridge’ along \(KK\) (that is, defined by \(\theta = \pi/6\)), and assume \(y(\pi/6) = 0\) (this would follow from global reflection symmetry, as seen above.) Set:

\[
\begin{align*}
    r_K &= \sup\{r(\theta); \theta \in [\pi/6, 5\pi/6]\}; \\
    r_{\bar{K}} &= \sup\{r(\theta); \theta \in [7\pi/6, 11\pi/6]\}.
\end{align*}
\]

**Proposition 4.2.** Let \(C\) be strictly convex (say, defined by \(p \in P_{pw}\)), with a non-degenerate maximal chord \(KK\) of length \(L_{chord}\). Assume (i) \(p(\pi - \theta) = p(\theta)\) for \(\theta \in [\pi/6, 5\pi/6] \cup [7\pi/6, 11\pi/6]\) (reflection symmetry near \(K, \bar{K}\)); (ii) \(\theta_0 = \pi/6\) is a transversal zero of \(y\); (iii) with \(c_0 = (4/3)(1 + 4\pi^2/3)^{1/2}\), assume: \(r_K + r_{\bar{K}} < \frac{1}{c_0}L_{chord} (c_0^{-1} \sim 0.2)\). Then \(C\) stably supports a double triode.

**Proof.** (Fig. 14.) \(u(\pi/6) > 0\) and \(u(7\pi/6) > 0\) follow from symmetry exactly as before, so we only need to check \(w(\pi/6) < 0\). Letting \(P, \bar{P}\) be the points on the chord \(KK\) where the normals through \(B(\pi/6), B(5\pi/6)\) (resp.
through \(B(7\pi/6), B(11\pi/6)\) meet, this is equivalent to \(|KP| + |\bar{K}P| < L_{\text{chord}}\); so let \(x = x_K = |KP|\); we estimate \(x\) in terms of \(r_K\). First note (Fig.14):
\[
\frac{\pi}{3} = \int_{K}^{B} \frac{1}{r} ds \geq \frac{1}{r_K} \text{arclength}_C(KB) \geq \frac{1}{r_K} |KB|.
\]
Then with \(r_0 := |BP| = -\langle B-P, N(5\pi/6) \rangle\) and \(B = P + (x, 0) + \int_{\pi/2}^{5\pi/6} r(\tau) T(\tau) d\tau\), we have:
\[
r_0 = \frac{x}{2} + \int_{\pi/2}^{5\pi/6} r(\tau) \sin(5\pi/6 - \tau) d\tau,
\]
so \(\frac{x}{2} < r_0 < \frac{1}{2}(x + r_K)\) and (using the triangle \(BPK\), where the interior angle at \(P\) is \(\pi/3\)):
\[
r_K^2 \frac{\pi^2}{9} \geq |KB|^2 = r_0^2 + x^2 - xr_0 \geq \frac{3x^2}{4} - \frac{r_K^2}{2},
\]
which gives the upper bound: \(x \leq c_0 r_K\). Repeating the argument at the other end for \(x_{\bar{K}} = |\bar{K}P|\), we get \(x_{\bar{K}} \leq c_0 r_{\bar{K}}\), so:
\[
x_K + x_{\bar{K}} \leq c_0 (r_K + r_{\bar{K}}) < L_{\text{chord}}.
\]

Our last existence result in this section makes no symmetry assumptions, but yields a weaker conclusion.

**Proposition 4.3.** Let \(C\) be a strictly convex curve (defined by \(p \in \mathcal{P}_{pw}\)). Assume \(C\) has only two critical chords (both non-degenerate), making an angle greater than \(\pi/3\) (unoriented angles, taking values in \((0, \pi/2]\)). Then the parallel curves \(C_\epsilon\) support a double triode stably, for all \(\epsilon > 0\) sufficiently large (and possibly for \(\epsilon = 0\) already).

**Proof.** We only need to find \(\theta \in [0, \pi]\) so that \(y(\theta) = 0\) and \(w(\theta) < 0\); then add a sufficiently large constant to \(p\) to guarantee \(u(\theta), u(\hat{\theta})\) are positive (adding constants to \(p\) leaves \(w\) and \(y\) unchanged.) Let the minimal chord be \(B(0)B(\pi)\) and the maximal \(B(\alpha)B(\alpha + \pi)\), with \(\alpha \in (\pi/3, \pi/2]\). Then \(w < 0\) in \((0, \alpha) \cup (\pi, \alpha + \pi)\) and \(w > 0\) in \((\alpha, \pi) \cup (\alpha + \pi, 2\pi)\). Letting \(\tilde{w}(\theta) := w(\theta + 2\pi/3)\), this means: \(\tilde{w} < 0\) in \((\alpha + \pi, \alpha + 4\pi/3) \cup (\pi, \alpha + 4\pi/3)\), \(\tilde{w} > 0\) in \((\alpha + \pi, 4\pi/3) \cup (\alpha + 4\pi/3, 7\pi/3)\). Since \(\alpha \in (\pi/3, \pi/2]\), there are arcs in \(S\) where both \(w\) and \(\tilde{w}\) are positive, as well as arcs where both are negative, determining the sign of \(y\):
\[
y = w + \tilde{w} < 0\text{ in } (\pi/3, \alpha) \cup (4\pi/3, \alpha + \pi);\]
Thus \( y \) must have a zero \( \theta \in (0, \frac{\pi}{3}) \), where \( w < 0 \). (There is also a zero in the interval \( (\alpha, \alpha + \frac{\pi}{3}) \), where \( w > 0 \), but we’ll ignore it.)

**Remark 4.1.** We can guarantee \( u > 0 \) already for \( C \) by imposing a ‘curvature pinching condition’, as in Proposition 3.2 (e.g. \((1 + \sqrt{3}/2) \min \theta r > \max \theta r\).

**Remark 4.2.** The restriction to two critical chords is made just to simplify the statement; a more general result follows from the same argument (assuming all critical chords are non-degenerate), but is cumbersome to state. Consider the oriented angular distance from a minimal chord to the next maximal chord (moving counterclockwise on \( S \)), with values in \((0, \pi)\). If this distance is always greater than \( 2\pi/3 \), the same conclusion follows. Another sufficient condition is: the smallest oriented distance from a maximal chord to the next minimal chord is less than \( 2\pi/3 \), and if chords achieving this least distance define the arc \((\theta_{\max}, \theta_{\min})\) in \( S \) (where \( w > 0 \)), the union of this arc with the two adjacent arcs where \( w < 0 \) has angular measure at least \( 2\pi/3 \). We omit the proof.

### 4.3 Examples without double triodes.

The critical points of parallelogram length are solutions of \( y(\theta) := w(\theta) + w(\theta + 2\pi/3) = 0 \). To define a double triode, we must in addition have \( w(\theta) < 0 \). Thus if at every zero of \( y \) (in a fundamental domain, such as \([0, \pi]\)) we have \( w > 0 \) (or, equivalently, \( y(\theta + \pi/3) > y(\theta + 2\pi/3) \)), then \( C \) does not support a double triode (nor does any of its outer parallel curves.)

The idea to construct examples is to start from \( p_0 \) of constant width \((w_0 \equiv 0; \text{e.g., the circle})\), then perturb it by adding \( \tilde{p} \) (not necessarily convex!), constructed so that \( \tilde{y} \) has the desired property. This leads to the following ‘instability property’ for curves of constant width.

**Proposition 4.4.** Let \( C_0 \) (with support function \( p_0 \in \mathcal{P}_{\text{smooth}} \)) be a strictly convex curve of constant width. Then arbitrarily \( C^2 \) close to \( p_0 \) one finds (i) support functions \( p \) so that (stably) neither the convex curve \( C \) nor its outer parallel curves support double triodes; (ii) support functions \( p \) of curves for which (stably) some outer parallel curve carries double triodes.

**Proof.** Let \( \tilde{w} \) (\( \pi \)-periodic, with \( \int_0^\pi \tilde{w}(\tau)d\tau = 0 \)) have the property: \( \tilde{w} > 0 \) at each zero of \( \tilde{w}(. + \pi/3) \), and these zeros are all transversal. Find \( \tilde{p} \) \( 2\pi \)-periodic so that \( \tilde{w}(\theta) = -(\tilde{p}(\theta) + \tilde{p}(\theta + \pi)) \) (for example, \( \tilde{p} = (-1/2) \int_\theta^\theta \tilde{w} \)).
Then the support function:

\[ p(\theta) = \frac{1}{\lambda + 1} \tilde{p}(\theta) + \frac{\lambda}{\lambda + 1} p_0(\theta) \]

has, for all \( \lambda > 0 \) large enough, the properties: (a) \( p \) is positive and strictly convex (meaning \( \rho > 0 \)); (b) neither \( p \) nor \( p + c \) (for any \( c > 0 \)) supports double triodes, since \( w > 0 \) at each zero of \( w(\cdot) + w(\cdot + 2\pi/3) \) (given that this is true for \( \tilde{w} \) and \( w = \frac{1}{\lambda + 1} \tilde{w} \)). Since \( p \) can be made arbitrarily close to \( p_0 \) in \( C^2 \) norm, this proves (i). To show (ii), consider the support function:

\[ \bar{p}(\theta) = -\frac{1}{\lambda + 1} \tilde{p}(\theta) + \frac{\lambda}{\lambda + 1} p_0(\theta), \]

for which \( \bar{w} = -(\lambda + 1)^{-1} \tilde{w} \), so \( \bar{w} < 0 \) at each zero of \( \bar{w}(\cdot) + \bar{w}(\cdot + 2\pi/3) \). Again, for large enough \( \lambda \), \( \bar{p} \) is positive and \( \bar{r} > 0 \), so \( \bar{p} \) defines a strictly convex curves which supports double triodes, possibly after adding a sufficiently large constant (this is not needed if \( u_0 > 0 \) everywhere.)

It remains to exhibit a \( \pi \)-periodic function \( \tilde{w} \) with the desired properties. The \( 2\pi \)-periodic function:

\[ z(\theta) = \frac{M - M \cos(\theta - \bar{\theta})}{M - M \cos(\theta - \bar{\theta}) + (1 - \cos \theta)(1 - \cos \bar{\theta})} - (1 - \cos \theta)(1 - \cos \bar{\theta}) \]

(where \( M \) and \( \bar{\theta} \) are parameters) has 0 and \( \bar{\theta}/2 = \pi/12 \) as its only critical points \( (z(0) = 1, z(\bar{\theta}) = -1 \) and \( -1 < z(\theta) < 1 \) otherwise). Choosing \( M = 1, \theta = \pi/6 \), one finds that both at \( \theta = 0 \) and \( \theta = \bar{\theta} \):

\[ z'(\theta + \frac{2\pi}{3}) > z'(\theta + \frac{4\pi}{3}). \]

Let \( Y(\theta) = z(2\theta) \). Then 0 and \( \bar{\theta}/2 = \pi/12 \) are the only critical points of \( Y \) in \([0, \pi]\), and at each of them: \( Y'(\theta + \frac{\pi}{3}) > Y'(\theta + \frac{2\pi}{3}) \); thus setting \( \bar{y} = Y' \) and \( \bar{w}(\theta) = (1/2)(\bar{y}(\theta) + \bar{y}(\theta + \pi/3) - \bar{y}(\theta + 2\pi/3)) \), we have \( \bar{w} > 0 \) when \( \bar{y} = 0 \), as desired.

**Example 4.1.** The construction can be made completely explicit. For the given values \( M = 1, \theta = \pi/6 \), one finds \( \bar{w}(0) \sim 0.0635 \) and \( \bar{w}(\pi/12) \sim 0.0125 \) (both positive). We may take \( \bar{p}(\theta) = (-1/2)(Y(\theta) + Y(\theta + \pi/3) - Y(\theta + 2\pi/3)) \) (then \( \bar{w} = -\bar{p}' \)).

Adding a large constant to this \( \bar{p} \) (which corresponds to perturbing a circle by \( \bar{p} \)), we get an explicit example of a strictly convex curve supporting no double triodes (one finds numerically that \( \bar{p} + 300 \) already corresponds
to a convex curve). Or one can start from the smoothed Reuleaux triangle \((p_\epsilon \text{ in (1.4))})\) and let:

\[
p = \frac{1}{\lambda + 1} \tilde{p} + \frac{\lambda}{\lambda + 1} p_\epsilon, \quad \bar{p} = -\frac{1}{\lambda + 1} \tilde{p} + \frac{\lambda}{\lambda + 1} p_\epsilon.
\]

For \(\epsilon = 1\) and \(\lambda = 1,000\), both represent convex curves; \(\tilde{p}\) stably supports double triodes, while \(p\) stably doesn’t- and both can be taken arbitrarily close to \(p_\epsilon\).

Remark 4.3. For future reference, we record here the following non-existence criterion: if \(p \in \mathcal{P}_{\text{pw}}\) has the property that, for all \(\theta\) so that \(y(\theta) = 0\), either \(w(\theta) > 0\) or \(u(\theta) < 0\), then \(\mathcal{C}\) supports no double triodes (stably.)

Remark 4.4. In example 4.1, we could take \(p_\epsilon\) to have constant width and height (say, a finite linear combination of Fourier components with frequency an odd integer not divisible by 3.) With a little more care, the perturbation \(\tilde{p}\) can also be chosen to have vanishing height function; this gives a construction of examples of convex constant-height curves which do not support double triodes (however, all known examples do support triodes.)
5. Hexagonal cells.

5.1 The hexagonal cell configuration; holonomy. An hexagonal cell in $D$ (when it exists) is defined by a choice of six points on $C$ (with angular separation $\pi/3$), and by an additional parameter taking values in an open interval, measuring how far the cell is from the boundary. The existence of a cell for a given $\theta \in S$ is determined by: (i) a criticality condition, depending only on the configuration of normal lines defined by $\theta$ and its $\pi/3$ translates (i.e., unchanged for parallel curves); (ii) inequalities ruling out certain critical configurations of normal lines (analogous to $w < 0$ for double triodes); (iii) requiring points of the configuration to be inside $C$, which can always be achieved by taking outer parallel curves.

The configuration of tangent polygons and normal lines to be considered for hexagons includes those considered for triodes and double triodes, and it is useful to preserve the notation used in those two cases. We describe the notational conventions used in this section (see Fig.15.)

(1) Indices 1,2,3 denote positive $2\pi/3$ shifts from $\theta$, and a bar is used for $\pi$ shifts. Thus the boundary points are: $B_1, B_3, B_2, \bar{B}_1, B_3, \bar{B}_2$ (in cyclic order, with $B_1 = B(\theta)$), and the unit tangent and normal vector at these points are denoted accordingly.

(2) In general, the cyclic order of a generic ‘index’ $I$ taking 6 values will be: $I = (1, 3, 2, 1, 3, 2)$. Advancing one step in this cyclic order corresponds to shifting $\theta$ by $\pi/3$.

(3) From the circumscribed equilateral triangles $T(\theta), T(\bar{\theta})$ we have the 6 ‘triangular normal intersections’ $P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3$. (These are indicated by dots in Fig. 15; to avoid encumbering the figure, only $P_1$ and $\bar{P}_2$ are labelled.) The derived height $\omega$ and ‘triangle functions’ $u, v$ are defined exactly as before (and e.g. $\bar{u}_2 = u(\theta + \frac{2\pi}{3} + \pi) = u(\theta + \frac{5\pi}{3})$).

(4) In addition to the ‘projections’ of the $P_i$ considered in section 4:

$$U_1 = pr_{21}(\bar{P}_1), \quad \bar{U}_1 = pr_{21}(P_1)$$

(and cyclic, e.g. $\bar{U}_3 = pr_{13}(P_3)$), we also need ‘backward’ projections such as:

$$V_1 = pr_{\bar{n}_3 \rightarrow n_1}(\bar{P}_3) = pr_{31}(\bar{P}_3)$$

(and cyclic: $\bar{V}_3 = pr_{23}(P_3), V_2 = pr_{12}(\bar{P}_1)$, etc. ).

Recall $U_1 = B_1 + (2/\sqrt{3})\sigma_1 N_1$, where in (4.1) we found an expression for $\sigma_1$. In the same way we let:

$$V_1 = B_1 + \frac{2}{\sqrt{3}} \tau_1 N_1,$$
Figure 15: The configuration for an hexagonal cell
with $\tau_1 = \tau(\theta)$, and use the fact that also $V_1 = \bar{P}_3 + \beta N_2$ (for some $\beta \in \mathbb{R}$) to compute $\tau_1$, by taking inner products with $T_2$. We find:

$$\tau_1 = \langle \bar{B}_3 - B_1, T_2 \rangle + \bar{u}_3.$$ 

Using $\bar{u}_3 = \langle \bar{B}_1 - \bar{B}_3, \bar{T}_1 \rangle$ and rearranging:

$$\tau_1 = \langle B_1 - B_3, T_3 \rangle - \langle \bar{B}_3 - B_3, T_3 \rangle - \langle \bar{B}_1 - B_1, T_1 \rangle,$$

which can also be written in the form:

$$\tau(\theta) = v(\theta) - w(\theta) - w(\theta + \frac{4\pi}{3}) \quad (5.1)$$

Note that there is a total of 12 points defined by both types of projection (6 of type $U_I$, 6 of type $V_I$). (Only $\bar{U}_1$ and $V_3$ are shown in Fig.15.)

(5) The configuration includes also 6 new points, ‘hexagonal normal intersections’: $R_1, \bar{R}_3, R_2, \bar{R}_1, R_3, \bar{R}_2$ (in cyclic order), where: $\{R_1\} = n_1 \cap \bar{n}_3$ and cyclic (e.g. $\{\bar{R}_3\} = \bar{n}_3 \cap n_2$, $\{\bar{R}_2\} = \bar{n}_2 \cap n_1$, etc.) These points are indicated by white squares in Fig.15, where only $R_1$ is labelled. Define functions $s(\theta) = s_1, t(\theta) = t_1$ by:

$$R_1 = B_1 + \frac{2}{\sqrt{3}}s_1N_1, \quad \bar{R}_2 = B_1 + \frac{2}{\sqrt{3}}t_1N_1.$$

We compute $s_1$ and $t_1$ in the usual way and find:

$$s_1 = \langle \bar{B}_3 - B_1, \bar{T}_3 \rangle, \quad s(\theta) = \langle B(\theta + \frac{\pi}{3}) - B(\theta), T(\theta + \frac{\pi}{3}) \rangle,$$

$$t_1 = \langle B_1 - \bar{B}_2, T_2 \rangle, \quad t(\theta) = \langle B(\theta) - B(\theta - \frac{\pi}{3}), T(\theta - \frac{\pi}{3}) \rangle.$$

It turns out $s$ and $t$ can be expressed in terms of previously defined functions. We have:

$$s_1 = \langle B_3 - B_3, T_3 \rangle + \langle B_3 - B_1, T_3 \rangle = -w_3 + \langle B_1 - B_3, T_3 \rangle = -w_3 + v_1,$$

or:

$$s(\theta) = v(\theta) - w(\theta + \frac{4\pi}{3}) \quad (5.2)$$

A similar computation yields:

$$t(\theta) = u(\theta) + w(\theta + \frac{2\pi}{3}) \quad (5.3)$$
In total, the configuration for hexagonal cells includes 24 intersection points, defined by the configuration of normals (at \( \theta \) and its \( \pi/3 \) translates.)

Remark 5.1. From (5.3) and expression (4.1) for \( \sigma \), we find:

\[
\sigma(\theta) = t(\theta) + w(\theta) \quad (5.4),
\]

and combining (5.2) and (5.1), we similarly find:

\[
\tau(\theta) = s(\theta) - w(\theta) \quad (5.5).
\]

In particular, the functions \( s, t, \sigma \) and \( \tau \) can be written as simple combinations of \( u, \omega \) and their translates:

\[
\begin{align*}
    s &= -\omega(\theta) - w(\theta + \frac{2\pi}{3}) + u(\theta); \\
    t &= w(\theta + \frac{2\pi}{3}) + u(\theta); \\
    \sigma &= w(\theta) + w(\theta + \frac{2\pi}{3}) + u(\theta); \\
    \tau &= -\omega(\theta) - w(\theta) - w(\theta + \frac{2\pi}{3}) + u(\theta).
\end{align*} \quad (5.6)
\]

We see immediately that by adding a positive constant to the support function \( p(\theta) \) (which adds a constant to \( u \) without changing the other functions)- i.e., by considering outer parallel curves- we can make all these functions positive.

The intersections of the equilateral triangles \( T(\theta) = (Q_1, Q_2, Q_3), T(\bar{\theta}) = (\bar{Q}_1, \bar{Q}_2, \bar{Q}_3) \) (for a given \( \theta \)) define six points \( (S_1, \bar{S}_3, S_2, S_1, S_3, \bar{S}_2) \), vertices of a circumscribed equiangular hexagon \( H(\theta) \). To construct a hexagonal cell \( H(\theta) = (H_1 \bar{H}_3 \bar{H}_1 \bar{H}_3 H_2) \), we pick a small positive \( x = x_1 \) and define:

\[
H_1 = B_1 + \frac{2}{\sqrt{3}} x_1 N_1,
\]

and successively:

\[
\bar{H}_3 = pr_{13}(H_1) = \bar{B}_3 + \frac{2}{\sqrt{3}} \bar{N}_3, \ldots, H_{I+1} = pr_{n_I \rightarrow n_{I+1}}(H_I), \ldots
\]

and finally:

\[
H_1^* = pr_{n_2 \rightarrow n_1}(H_2) = B_1 + \frac{2}{\sqrt{3}} x_1^* N_1.
\]

In general \( H_1^* \neq H_1 \), and instead of a ‘cell’ we have only a hexagonal ‘chain’ with seven vertices, beginning with \( H_1 \) and ending with \( H_1^* \), both on \( n_1 \).
(Fig. 16; points $H_3$, $H_3$ and $H_1^*$ are not labelled, to avoid overloading the figure.) We call this ‘defect’ the holonomy of $\theta$:

$$\text{hol}(\theta) = x_1^* - x_1;$$

we will soon see that it depends only on the configuration of normals. Its vanishing is the ‘criticality condition’ mentioned above. To compute the holonomy, beginning with:

$$\langle R_1 - H_1, N_1 \rangle = \langle R_1 - \bar{H}_3, \bar{N}_3 \rangle$$

we obtain:

$$\bar{x}_3 = x_1 + \bar{t}_3 - s_1,$$

and successively in cyclic order:

$$x_2 = \bar{x}_3 + t_2 - s_3, \ldots$$

$$x_1^* = \bar{x}_2 + t_1 - s_2.$$

Adding the results, we have:

$$\text{hol}(\theta) = x_1^* - x_1 = \sum_{i=1}^{3} (t_i - s_i) + \sum_{i=1}^{3} (\bar{t}_i - \bar{s}_i).$$

Using the expressions given above for $s_I$ and $t_I$, one easily computes that:

$$\text{hol}(\theta) = \langle \bar{B}_1 - B_1, T_1 \rangle + \langle \bar{B}_2 - B_2, T_2 \rangle + \langle \bar{B}_3 - B_3, T_3 \rangle$$

$$= w(\theta) + w(\theta + 2\pi/3) + w(\theta + 4\pi/3).$$

**Remark 5.2.** Recalling that $w(\theta) = -(p'(\theta) + p'(\bar{\theta}))$, we have the equivalent expression:

$$\text{hol}(\theta) = -\sum_{j=0}^{5} p'(\theta + j\pi/3) = -\left(\omega(\theta) + \omega(\bar{\theta})\right).$$

We now relate the holonomy to the perimeter $L_{\text{hex}}(\theta)$ of the circumscribed hexagon $H(\theta)$. Proceeding as usual, we find:

$$S_1 = B_1 + \frac{2}{\sqrt{3}}\langle B_1 - \bar{B}_3, \bar{N}_3 \rangle T_1 = \bar{B}_3 + \frac{2}{\sqrt{3}}\langle B_1 - \bar{B}_3, N_1 \rangle \bar{T}_3,$$

$$\bar{S}_3 = \bar{B}_3 + \frac{2}{\sqrt{3}}\langle \bar{B}_3 - B_2, N_2 \rangle \bar{T}_3,$$

which gives for the length of the side $S_1 \bar{S}_3$:

$$\langle \bar{S}_3 - S_1, \bar{T}_3 \rangle = -\frac{2}{\sqrt{3}}(\langle B_1, N_1 \rangle + \langle B_2, N_2 \rangle + \langle B_3, N_3 \rangle).$$
Repeating this for the other sides and adding the results (using $\bar{N}_i = -N_i$), we obtain:

$$L_{\text{hex}}(\theta) = \frac{2}{\sqrt{3}}(\langle B_1 - B_1, N_1 \rangle + \langle B_2 - B_2, N_2 \rangle + \langle B_3 - B_3, N_3 \rangle)
= \frac{2}{\sqrt{3}}(b(\theta) + b(\theta + \frac{2\pi}{3}) + b(\theta + \frac{4\pi}{3})).$$

This immediately implies the following (since $w = -b'$):

**Proposition 5.1** $\text{hol}(\theta) = -(\sqrt{3}/2)L'_{\text{hex}}(\theta)$. Hence there are always at least two directions $\theta \in S$ along which the ‘holonomy’ vanishes.

**Remark 5.3.** We can also relate the hexagonal perimeter and the height function: from $h(\theta) = p(\theta) + p(\theta + 2\pi/3) + p(\theta + 4\pi/3)$ and $b(\theta) = p(\theta) + p(\bar{\theta})$ follows:

$$L_{\text{hex}}(\theta) = \frac{2}{\sqrt{3}}(h(\theta) + h(\bar{\theta})).$$

**Remark 5.4 (Total length of the network.)** The total length of a closed cell $\mathbb{H} = (H_1\bar{H}_2H_2\bar{H}_1H_3\bar{H}_2)$ is:

$$L(\theta) = \langle \bar{H}_3 - H_1, \bar{N}_2 \rangle + \langle H_2 - \bar{H}_3, N_1 \rangle + \langle \bar{H}_1 - H_2, \bar{N}_3 \rangle + \langle H_3 - \bar{H}_1, N_2 \rangle + \langle \bar{H}_2 - H_3, N_1 \rangle + \langle H_1 - \bar{H}_2, N_2 \rangle
+ \frac{2}{\sqrt{3}}(x_1 + \bar{x}_3 + \ldots + \bar{x}_2).$$
We rewrite the terms on the right-hand side in the form:

\[
\langle \bar{H}_3 - H_1, \bar{N}_2 \rangle = \langle \bar{B}_3 - B_1, \bar{N}_2 \rangle - \frac{1}{\sqrt{3}}(x_1 + \bar{x}_3)
\]

(and cyclic), and then combine them in pairs and rearrange:

\[
\langle \bar{B}_3 - B_1, \bar{N}_2 \rangle + \langle B_3 - \bar{B}_1, N_2 \rangle = \langle B_1 - \bar{B}_1, N_2 \rangle + \langle B_3 - B_3, N_2 \rangle
\]

(and cyclic). Adding up the results, we find:

\[
L(\theta) = \langle \bar{B}_1 - B_1, N_1 \rangle + \langle \bar{B}_2 - B_2, N_2 \rangle + \langle \bar{B}_3 - B_3, N_3 \rangle,
\]

so we see that the total length of the network is independent of the \(x_i\), and in fact:

\[
L(\theta) = \frac{\sqrt{3}}{2} L_{hex}(\theta).
\]

5.2. Existence in a parallel curve.

Vanishing of the holonomy is not the whole story. First, some vertices of the cell could end up being outside the domain; second, if we are not
careful the ‘successive projection’ construction could lead to a ‘twisted cell’. (Fig.17; this example has non-zero holonomy, but one sees easily that the problem may also occur when the holonomy vanishes.)

The parameter \( x = x_1 \) of a hexagonal cell plays a secondary role in existence considerations. Since consecutive interior vertices of a hexagonal cell lie on consecutive normals, it is always possible to ‘slide’ the vertices along their normals until one obtains a degenerate 5-vertex cell with one vertex at a ‘hexagonal intersection’ \( R_1 \) (where the interior angle is \( \pi/3 \)) and four vertices with interior angles \( 2\pi/3 \); two ‘triangular intersections’ \( P_I, P_J \) and two ‘parallelogram points’ \( U_I, V_J \). After relabelling, we may assume the 5-cell is the chain (shown in Fig. 15 in the case of nonzero holonomy):

\[
V_3 \xrightarrow{pr_{32}} \overline{P}_2 \xrightarrow{pr_{21}} R_1 \xrightarrow{pr_{32}} P_1 \xrightarrow{pr_{23}} \overline{U}_1 \xrightarrow{pr_{13}} V_3,
\]

obtained by successive projection along consecutive normals (taking \( R_1 \) to map to itself under \( pr_{n_1 \rightarrow n_3} \)). We also used the fact that \( V_3 := pr_{23}(\overline{P}_2) = pr_{13}(\overline{U}_1) \), which follows from \( hol(\theta) = 0 \). Existence of a 5-cell of this form (entirely contained in \( D \)) is both necessary and sufficient for the existence of a hexagonal cell in \( D \).

Now, examination of Fig.17 shows the reason we get a ‘twisted’ cell is that \( \overline{H}_3 \) and \( H_2 \) are ‘ahead’ of \( \overline{R}_3 \) (on \( n_3 \), resp. \( n_2 \)); unlike, say, \( H_3 \) and \( \overline{H}_2 \) which are ‘behind’ \( R_3 \) (on \( n_3 \), resp. \( \overline{n}_2 \)). The normal lines are oriented (by the inner unit normal vectors to \( C \)), so ‘ahead’ and ‘behind’ have meaning. Thus we need to make sure the successive projections are always ‘behind’ the \( R_I \) along the corresponding normals.

More precisely, in a ‘chain’ obtained by projection on consecutive normals, the angles between consecutive edges of the chain (\( \pi/3 \) or \( 2\pi/3 \)) depend on the relative orientations of the corresponding consecutive projections. Define the mapping \( pr_{n_I \rightarrow n_{I+1}} : X_I \mapsto X_{I+1} \) to be positive if \( X_{I+1} - X_I \) has the direction of \( N_{I-1} \), negative if it has the direction of \( -N_{I-1} \).

This depends on considering three consecutive normals, which in Fig.18 we denote by \( n_1, \overline{n}_2, n_2 \). Three examples of 3-vertex chains are shown (from a point on \( n_1 \) to a point on \( n_2 \)). For \( X_1 \rightarrow \overline{X}_3 \rightarrow X_2 \), a (+) projection is followed by a (-) projection, resulting in an angle \( \pi/3 \) at \( \overline{X}_3 \); the reason \( \overline{X}_3 \rightarrow X_2 \) is (-) is that \( \overline{X}_3 \) and \( X_2 \) are both ‘ahead’ of \( R_3 \) on their respective normals. For \( X'_1 \rightarrow \overline{X}'_3 \rightarrow X'_2 \), both \( X'_1 \) and \( X'_3 \) are ‘ahead’ of \( R_3 \); hence both projections are (-), and the angle at \( X'_2 \) is \( 2\pi/3 \). Finally, with \( X''_1, \overline{X}''_3, X''_2 \) both ‘behind’ \( R_1 \) and \( \overline{X}''_3 \) ‘behind’ \( \overline{R}_3 \), the chain \( X''_1 \rightarrow \overline{X}''_3 \rightarrow X''_2 \) results from two (+) projections, and the angle at \( \overline{X}''_3 \) is \( 2\pi/3 \).
A sufficient condition for a 5-cell obtained by consecutive projection from $R_1$ (backwards to $V_3$ and forward to $\bar{U}_1$) to have the correct angles is that its vertices always lie ‘behind’ the intersections of normals in question; then all projections will be (+), and all angles $2\pi/3$ (except at $R_1$), as desired. (The condition is not necessary; having all projections be (-) would achieve the same result.) This requirement can be expressed in terms of the functions describing the positions of points on the chain along their normals, as follows.

\[
\begin{align*}
R_1 &\leq \bar{R}_3(\bar{n}_3), \quad P_1 \leq \bar{R}_3(n_2) \iff \bar{t}_3 \leq \bar{s}_3 \text{ and } v_2 \leq t_2 \\
P_1 &\leq R_2(n_2), \quad \bar{U}_1 \leq R_2(\bar{n}_1) \iff v_2 \leq s_2 \text{ and } \sigma_1 \leq t_1 \\
P_2 &\leq \bar{R}_2(\bar{n}_2), \quad R_1 \leq \bar{R}_2(n_1) \iff \bar{u}_2 \leq \bar{s}_2 \text{ and } s_1 \leq t_1 \\
V_3 &\leq R_3(n_3), \quad \bar{P}_2 \leq R_3(\bar{n}_2) \iff \tau_3 \leq s_3 \text{ and } \bar{u}_2 \leq \bar{t}_2 \\
\bar{U}_1 &\leq \bar{R}_1(\bar{n}_1), \quad V_3 \leq \bar{R}_1(n_3) \iff \bar{\sigma}_1 \leq \bar{s}_1 \text{ and } \tau_3 \leq t_3
\end{align*}
\]

(equality is always allowed, since it just means the minimal 5-cell has ‘collapsed’ to fewer than five vertices- the circle is an extreme example). The two inequalities on the right in each of the first four lines are equivalent to each other, even if $\text{hol}(\theta) \neq 0$ (then we have an open 5-chain, from $V_3$ to $\bar{U}_1$); if $\text{hol}(\theta) = 0$, the same is true for the last line.

We now use the previously computed expressions (5.6) for the functions $u_I, v_I$, etc. appearing on the right-hand side to express the inequalities in
terms only of $\omega$ and $w$; this yields six inequalities (two for the last line, since we don’t assume $\text{hol}(\theta) = 0$ at this point.) To simplify them, we use the easily verified identity:

$$w(\theta) + w(\theta + \frac{\pi}{3}) + w(\theta + \frac{2\pi}{3}) = -\omega(\theta) - \omega(\theta + \frac{\pi}{3}).$$

The result is (in the same order as above);

- \((A)\) \(w(\theta + \frac{\pi}{3}) + \omega(\theta) \geq 0\)
- \((B)\) \(w(\theta) \leq 0\)
- \((C)\) \(w(\theta) + \omega(\theta + \frac{\pi}{3}) \leq 0\)
- \((D)\) \(w(\theta + \frac{2\pi}{3}) \geq 0\)
- \((E)\) \(\omega(\theta) \geq 0\) and \((F)\) \(w(\theta + \frac{2\pi}{3}) \leq 0\).

Note that \((B),(D),(E),(F)\) is a ‘minimal set’ (i.e., imply \((A)\) and \((C)\)).

If we now use the condition

$$\text{hol}(\theta) = w(\theta) + w(\theta + \frac{\pi}{3}) + w(\theta + \frac{2\pi}{3}) = -\omega(\theta) - \omega(\theta + \frac{\pi}{3}) = 0,$$

\((E)\) and \((F)\) are equivalent; so in this case \((B),(D)\) and \((E)\) are sufficient to guarantee the projection construction based at \(R_1 = R(\theta)\) yields a convex 5-cell with the correct angles.

In addition, rather than starting at \(R_1\) (corresponding to \(\theta\)), we could have started at any other \(R_I\), corresponding to the \(\pi/3\)-translation orbit of \(\theta\). That is, we only need \((B),(D)\) and \((E)\) to hold for some \(\pi/3\) translate of \(\theta\). And it turns out (somewhat surprisingly, given the experience with double triodes) that this is always true (assuming \(\text{hol}(\theta) = 0\)), as verified in the combinatorial proposition 5.4 below. We summarize the conclusion, bearing in mind that only the configuration of normals plays a role.

**Proposition 5.2.** Let \(n_1, \bar{n}_3, n_2, \bar{n}_1, n_3, \bar{n}_2\) be six oriented lines in \(\mathbb{R}^2\), with unit direction vectors between consecutive lines differing by \(\pi/3\) rotations; assume the configuration has zero holonomy. Then it supports a one-parameter family of convex equiangular hexagonal cells, with one vertex on each line.

**Corollary 5.3.** For any strictly convex curve \(C\) in \(\mathbb{R}^2\) (possibly with corners), all sufficiently far outer parallel curves \((C_d\) for \(d \geq d_0 \geq 0\)) support hexagonal cells.

**Proposition 5.4.** Let \(w\) and \(\omega\) be \(\pi\)-periodic and \(2\pi/3\)-periodic functions on \(S\) (resp.) Suppose \(\theta_0 \in S\) is a solution of the equations:

$$\omega(\theta_0) + \omega(\theta_0 + \frac{\pi}{3}) = 0, \quad w(\theta_0) + w(\theta_0 + \frac{\pi}{3}) + w(\theta_0 + \frac{2\pi}{3}) = 0.$$
Then some $\pi/3$ translate $\hat{\theta}$ of $\theta_0$ satisfies, in addition to these two equations, also the inequalities:

$$w(\hat{\theta}) \leq 0, \quad w(\hat{\theta} + \frac{\pi}{3}) \geq 0, \quad \omega(\hat{\theta}) \geq 0. \quad (5.7)$$

**Proof.** (By exhaustive listing of cases.) The sign of $\omega(\theta_0)$ determines that of $\omega(\theta)$ for all $\pi/3$-translates $\theta$ of $\theta_0$, given $2\pi/3$ periodicity and the first equation. For $w$, given the sign of $w(\theta_0)$ the second equation implies 3 possible sign combinations for $w(\theta_0 + \pi/3)$ and $w(\theta_0 + 2\pi/3)$, which then determine the signs at the other translates. This gives 12 cases, and in each case we can find translates satisfying all three inequalities desired.

We proceed to list the 12 cases. On each half-line, the four signs are those of $\omega(\theta_0)$, $w(\theta_0)$, $w(\theta_0 + \pi/3)$, $w(\theta_0 + 2\pi/3)$ (in this order); the last entry is the translate $\hat{\theta}$ of $\theta_0$ satisfying the sign conditions in the proposition.

$$
\begin{array}{cccc}
+ & - & + & + \\
- & + & - & + \\
+ & - & - & + \\
- & + & + & - \\
- & - & + & + \\
- & - & - & - \\
+ & + & + & - \\
- & - & + & - \\
+ & + & - & + \\
- & + & - & - \\
- & - & + & + \\
- & - & - & + \\
\end{array}
\begin{array}{c}
\theta_0 \\
\theta_0 + 4\pi/3 \\
\theta_0 + 2\pi/3 \\
\theta_0 + \pi \\
\theta_0 + \pi/3 \\
\theta_0 + 5\pi/3 \\
\theta_0 + 2\pi/3 \\
\theta_0 + 5\pi/3 \\
\theta_0 + \pi/3 \\
\theta_0 + 2\pi/3 \\
\theta_0 + 3\pi/2 \\
\theta_0 + 5\pi/2 \\
\end{array}
$$

5.3 Existence under curvature conditions. In this subsection we show that sufficiently strong ‘pinching conditions’ on the radius of curvature imply all 24 points of the configuration are inside the domain; combined with the conclusion of the previous sections, this shows hexagonal cells exist in this case.

The points of the configuration on the normal $n_1 = n(\theta)$ are:

$$P_1, P_2, R_1, \bar{R}_2, U_1, V_1,$$

and all can be written in the form $B_1 + (2/\sqrt{3}) f_I N_1$, where $f_I$ is, respectively:

$$u(\theta), v(\theta), s(\theta), t(\theta), \sigma(\theta), \tau(\theta).$$

Recall from section 3 the chord $n_1 \cap D$ has endpoints $B(\theta), B(\theta_*)$ (with $\theta_* \in (\theta, \theta + 2\pi)$) and length:

$$d(\theta) = \langle B(\theta_*) - B(\theta), N(\theta) \rangle.$$

Thus, we seek conditions that imply $0 < f_I(\theta) < (\sqrt{3}/2)d(\theta)$, for each $f_I$ given above and all $\theta$. 

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For $u$ and $v$, this was done in Proposition 3.2: we showed that $r_{\text{max}} < (1 + \sqrt{3}/2)r_{\text{min}}$ implies $\theta_* \in [\theta + 2\pi/3, \theta + 4\pi/3]$ for all $\theta$, and that this condition implies both $u(\theta)$ and $v(\theta)$ are in $(0, (\sqrt{3}/2)d(\theta))$.

Our first observation is that $s(\theta)$ and $t(\theta)$ are always positive:

$$
\begin{align*}
    s(\theta) &= \langle B(\theta + \frac{\pi}{3}) - B(\theta), T(\theta + \frac{\pi}{3}) \rangle = \int_0^{\pi/3} r(\tau + \theta) \cos(\tau - \pi/3)d\tau > 0, \\
    t(\theta) &= \langle B(\theta) - B(\theta - \frac{\pi}{3}), T(\theta - \frac{\pi}{3}) \rangle = \int_{-\pi/3}^{\pi/3} r(\tau + \theta) \cos(\tau + \pi/3)d\tau > 0.
\end{align*}
$$

(5.8)

Remark 5.5. From relations (5.2),(5.3), we see that this implies $v(\theta) > w(\theta + \pi/3)$ and $u(\theta) > -w(\theta + 2\pi/3)$ for each $\theta$, and we conclude: (i) curves of constant width ($w \equiv 0$) always support triodes, and (ii) at a zero of hol satisfying inequalities (5.7), we automatically have $v(\theta) > 0$ (but not necessarily $u(\theta) > 0$).

The following two lemmas deal with the pairs $(s, t)$ and $(\sigma, \tau)$.

**Lemma 5.6** Assume $\theta^* \in [\theta + 2\pi/3, \theta + 4\pi/3]$, for all $\theta$. Then if $r_{\text{max}} < (4/3)r_{\text{min}}$, we have $s(\theta), t(\theta)$ both (positive and) less than $(\sqrt{3}/2)d(\theta)$, for all $\theta$.

**Proof.** (i) From (5.8) we see that, to show $s < (\sqrt{3}/2)d$, we need:

$$
\begin{align*}
    \frac{1}{2} \int_0^{\pi/3} r(\tau + \theta) \cos \tau d\tau + \sqrt{3} \frac{\pi}{2} \int_0^{\pi/3} r(\tau + \theta) \sin \tau d\tau < \frac{\sqrt{3}}{2} \int_0^{\theta_* - \theta} r(\tau + \theta) \sin \tau d\tau,
\end{align*}
$$

or:

$$
\begin{align*}
    \int_0^{\pi/3} r(\tau + \theta) \cos \tau d\tau < \sqrt{3} \int_{\pi/3}^{2\pi/3} r(\tau + \theta) \sin \tau d\tau + \int_{2\pi/3}^{\theta_* - \theta} r(\tau + \theta) \sin \tau d\tau.
\end{align*}
$$

We estimate the left-hand side from above by $\frac{\sqrt{3}}{2}r_{\text{max}}$. If $\theta_* - \theta \leq \pi$, estimate the right-hand side from below by $\sqrt{3}r_{\text{min}}$ (using only the first integral on the right). If $\theta_* - \theta \geq \pi$, the right-hand side is bounded below by:

$$
\sqrt{3}[r_{\text{min}} + r_{\text{min}} \int_{2\pi/3}^{\pi} \sin \tau d\tau - r_{\text{max}} \int_0^{4\pi/3} \sin \tau d\tau] = \sqrt{3}[\frac{3}{2}r_{\text{min}} - \frac{1}{2}r_{\text{max}}].
$$

This gives the condition $r_{\text{max}} < (3/2)r_{\text{min}}$.

(ii) To show $t < (\sqrt{3}/2)d$, from (5.8) we need the condition:

$$
\begin{align*}
    \int_{-\pi/3}^{\pi/3} r(\tau + \theta) \cos \tau d\tau + \sqrt{3} \int_{-\pi/3}^{\pi/3} r(\tau + \theta) \sin \tau d\tau < \sqrt{3} \int_{2\pi/3}^{\pi} r(\tau + \theta) \sin \tau d\tau + \int_{2\pi/3}^{\theta_* - \theta} r(\tau + \theta) \sin \tau d\tau.
\end{align*}
$$

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One verifies easily that the left-hand side is bounded above by $\sqrt{3}r_{\text{max}}$. For the right-hand side, there are two cases: if $\theta_* \in \left(\frac{2\pi}{3}, \pi\right]$, the right-hand side is greater than $\sqrt{3}r_{\text{min}}[1 - \cos(\theta_* - \theta)] > (3\sqrt{3}/2)r_{\text{min}}$. If $\theta_* - \theta \in [\pi, 4\pi/3)$, the right-hand side is bounded below by:

$$\sqrt{3}[r_{\text{min}} \int_0^\pi \sin \tau d\tau - r_{\text{max}} \int_\pi^{4\pi/3} |\sin \tau| d\tau] = \sqrt{3}[2r_{\text{min}} - \frac{1}{2}r_{\text{max}}].$$

This gives the condition $(3\sqrt{3}/2)r_{\text{max}} < 2\sqrt{3}r_{\text{min}}$, or $r_{\text{max}} < (4/3)r_{\text{min}}$, which is more stringent than that in part (i) above.

**Lemma 5.7.** Assume $\theta_* \in [\theta + 2\pi/3, \theta + \pi]$, for all $\theta$. Then if $(4 + 3\sqrt{3})r_{\text{max}} < (4 + 4\sqrt{3})r_{\text{min}}$, we have: $0 < \tau < (\sqrt{3}/2)d$ and $0 < \sigma < (\sqrt{3}/2)d$.

**Proof.** Using $\tau(\theta) = s(\theta) - w(\theta)$, to prove the statement for $\tau$ we need to verify:

$$0 < \frac{1}{2} \int_0^{\pi/3} r(\tau + \theta) \cos \tau d\tau - \frac{\sqrt{3}}{2} \int_0^{\pi/3} r(\tau + \theta) \sin \tau d\tau - \int_0^{\pi/2} r(\tau + \theta) \cos \tau d\tau + \int_{\pi/2}^{\pi/3} r(\tau + \theta) |\cos \tau| d\tau,$$

For $\tau > 0$, we see it is enough to show that:

$$0 < \frac{\sqrt{3}}{4}r_{\text{min}} - \frac{\sqrt{3}}{4}r_{\text{min}} - r_{\text{max}} + r_{\text{min}},$$

so the condition is $r_{\text{max}} < (1 + \sqrt{3}/2)r_{\text{min}}$. For the upper bound on $\tau$, we again consider two cases. If $\theta_* \in [\theta + 2\pi/3, \theta + \pi]$, we need:

$$\frac{\sqrt{3}}{4}r_{\text{max}} + \frac{\sqrt{3}}{4}r_{\text{max}} - r_{\text{min}} + r_{\text{max}} < \frac{\sqrt{3}}{2}[\frac{3}{2}r_{\text{min}} + r_{\text{min}}]\cos(\theta_* - \theta) + \frac{1}{2}],$$

or $(4 + 2\sqrt{3})r_{\text{max}} < (4 + 4\sqrt{3})r_{\text{min}}$.

If $\theta_* \in [\theta + \pi, \theta + 4\pi/3]$, we need:

$$\frac{\sqrt{3}}{2}r_{\text{max}} - r_{\text{min}} + r_{\text{max}} < \frac{\sqrt{3}}{2}[\frac{3}{2}r_{\text{min}} + r_{\text{min}}] \int_0^{4\pi/3} |\sin \tau|;$$

this gives the condition $(4 + 3\sqrt{3})r_{\text{max}} < (4 + 4\sqrt{3})r_{\text{min}}$, the more stringent of the three.

The proof for $\sigma$ is almost identical (and gives the same constant), so we omit it.
We summarize the discussion in the following proposition.

**Proposition 5.8.** Let $C$ be a strictly convex curve ($p \in P_{pw}$) satisfying the radius of curvature bounds $\frac{r_{\max}}{r_{\min}} < \frac{4+4\sqrt{3}}{4+3\sqrt{3}} \sim 1.188\ldots$ Then for each $\theta \in S$, all 24 points of the ‘hexagonal configuration’ are inside $C$. Thus $C$ (stably) supports at least two geometrically distinct ‘bands’ of hexagonal cells (corresponding to the global max and min of the ‘hexagonal perimeter’ $L_{\text{hex}}$).

5.4. Examples of nonexistence.

If $C$ supports a hexagonal cell, in particular it must support a minimal 5-cell as described in section 5.2: for some ‘hexagonal intersection’ $R_1 = R(\theta)$, the 5-cell has the form $V_3\overline{P_2}R_1P_1\overline{U_1}$, where the chain is obtained by backwards and forward projection on consecutive normals, beginning at $R_1$; in particular, $P_1$ and $\overline{P_2}$ are ‘triangle intersections’. All the points in the 5-chain must be inside $C$. Thus we have the necessary condition: if $\theta$ with $\text{hol}(\theta) = 0$ corresponds to a hexagonal cell, then for some $\hat{\theta}$ among the $\pi/3$-translates of $\theta$ we must have $u(\hat{\theta}) > 0$ (so $P_1(\hat{\theta}) \in D$) and $u(\hat{\theta} - \pi/3) > 0$ (so $\overline{P_2}(\hat{\theta}) \in D$).

Suppose $\omega = u - v$ is $\pi$-periodic, and therefore in fact $\pi/3$-periodic. Then $\text{hol} \equiv 2\omega$, so at a zero of $\text{hol}$ we have $P_1 = P_2 = P_3 := P$ and $\overline{P_1} = \overline{P_2} = \overline{P_3} := \overline{P}$. If, in addition, we find that $u(\theta) < 0$ and $u(\theta + \pi/3) < 0$ for every zero of $\omega$ (or $\text{hol}$) in some $\pi/3$ fundamental domain, then at every such zero both $P$ and $\overline{P}$ are outside the domain; thus $C$ does not support a hexagonal cell (or a triode). This leads to the following construction.

**Proposition 5.9.** Assume $p_0$ has the properties: (i) strict convexity ($p_0 \in P_{\text{smooth}}$); (ii) constant height ($\omega_0 \equiv 0$); (iii) $u_0 < 0$ in some interval $I = (-\delta, \delta)$, and also in $I + \pi = (-\delta + \pi, \delta + \pi)$, for some $0 < \delta < \pi/6$.

Let $\tilde{p}$ have the properties: (iv) $\tilde{p}$ is $\pi/3$-periodic; (v) all the zeros of $\tilde{\omega}$ in a $\pi/3$-fundamental domain (say, $[-\pi/6, \pi/6]$) are contained in $I$. Then:

$$p := \tilde{p} + \lambda p_0$$

has the properties, for $\lambda$ sufficiently large: (a) strict convexity; (b) at each zero of $\text{hol}$ (equivalently, of $\omega$, since $\omega = \tilde{\omega}$ is $\pi/3$-periodic), $P$ and $\overline{P}$ are outside $D$. In particular, $C$ supports no hexagonal cells, and no triodes. (c) Any support function sufficiently $C^2$-close to $p$ also satisfies (a) and (b).

**Proof.** If $\lambda$ is large enough, the zeros of $\omega = \tilde{\omega}$ in a fundamental domain containing $\theta = 0$ are all in $I$ (where $u < 0$ for $\lambda$ large), so for any $\theta$ with zero holonomy in this fund. domain we have $P \notin D$; while the zeros of $\omega$
(or hold in a fund. domain containing π are all in I + π, where again u < 0, so P ∗ D. From the discussion above, (b) follows. (a) and (c) are clear.

Example 5.1 To get an explicit example, we modify Example 3.3; take for p0 the support function of a constant height ε-biangle, for ε > 0 sufficiently small, or a smoothed version, such as  with in example 3.3; then I = (−0.34, 0.34). With set:

\[ q(\theta) = \frac{\cos(6\theta)}{(1 - m \sin(6\theta))}. \]

Then q is π/3-periodic, with critical points π/24 and π/8 in [0, π/3). Defining \( \tilde{p} = q(\theta + \pi/12) \), we find \( \tilde{\omega} = 3\tilde{p}' \) has zeros at ±π/24 ∼ ±0.1309 in the fundamental domain [−π/12, π/4), both in I so \( \tilde{p} \) satisfies (iv) and (v) of the proposition.

Corollary 5.10. If \( p_0 \in \mathcal{P}_{\text{smooth}} \) has constant height and satisfies \( u_0 < 0 \) on some interval I and on its translate \( I + \pi \), then arbitrarily \( C^2 \)-close to \( p_0 \) one finds strictly convex curves \( p \in \mathcal{P}_{\text{smooth}} \) with the property that curves in an open \( C^2 \)-neighborhood of \( p \) support no hexagonal cells or triodes.

Remark 5.6- multiple non-existence. It is natural to try to refine this construction, so as to obtain domains supporting no triodes, double triodes, or hexagonal cells. Unfortunately our examples of non-existence for double triodes (section 4) rely on control of the derived width function \( w(\theta) \), while the class of convex curves of constant height with \( u < 0 \) somewhere (on which the examples without triodes are based) appears to be too small to allow fine control of \( w \). While the construction of section 4 may be used to find smooth convex curves of constant height without double triodes, all known examples have \( u > 0 \) everywhere (Remark 4.4). On the other hand, there seems to be no fundamental reason why a convex domain without any Steiner networks would be an impossibility.

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