In the course of studying group theory, it soon becomes apparent that abelian groups are far nicer than nonabelian groups. As such, here we will be studying a particular class of abelian groups, and so the term group will be used exclusively to mean an abelian group. As is customary when dealing with abelian groups, the group operation will be denoted by + and the identity by 0. When multiplicative notation is used, it will mean repeated addition, \( nx = x + x + \cdots + x \). Thus, for \( x \in G \), we say that \( n \mid x \) if some \( y \in G \) satisfies \( ny = x \). In \( \mathbb{Q} \) and \( \mathbb{R} \), \( n \mid x \) for all \( n \in \mathbb{N} \) and \( x \) in the group; if this condition holds for an arbitrary group \( D \), we say that \( D \) is divisible. It is readily seen that divisibility is equivalent to the condition \( nD = \{ nd | d \in D \} = D \) for all \( n \).

1 Some Definitions

We have already noted that \( \mathbb{Q} \) and \( \mathbb{R} \) are divisible, other examples include \( \mathbb{Q}/\mathbb{Z} \), and a class of groups called the quasicyclic or Prüfer groups, and denoted \( \mathbb{Z}(p^\infty) \). Essentially, \( \mathbb{Z}(p^\infty) \) is the subgroup of \( \mathbb{Q}/\mathbb{Z} \) consisting of all elements whose order is a power of \( p \), which are precisely the rationals with denominator a power of \( p \). Another way of viewing \( \mathbb{Z}(p^\infty) \) is as the subset of \( S^1 = \{ z \in \mathbb{C}, |z| = 1 \} \) consisting of all numbers of the form \( e^{2\pi i/p^n} \). It is not immediately obvious from the definition that \( \mathbb{Z}(p^\infty) \) is divisible, but it is easy to prove, and left as an exercise to the reader. Later, it will be useful to have at our disposal the following formal definition of \( \mathbb{Z}(p^\infty) \). We say that \( \mathbb{Z}(p^\infty) = \langle c_n \rangle_{n \in \mathbb{N}} \), where \( c_n \) satisfy the property \( pc_{n+1} = c_n, pc_1 = 0 \). Thus, \( \mathbb{Z}(p^\infty) \) can be thought of as the limit of an ascending chain of subgroups \( 0 = \langle c_1 \rangle \leq \langle c_2 \rangle \leq \cdots \leq \langle c_n \rangle \cdots \).

In the study of finite abelian groups, the central theorem is the fundamental theorem of finite abelian groups, which states that any finite abelian group can be written uniquely (up to rearrangement) as a direct sum of cyclic groups of prime power order. Here, our main result is similar; any divisible group can be written as a direct sum of copies of \( \mathbb{Z}(p^\infty) \) and \( \mathbb{Q} \), with this representation being essentially unique. Before this can be stated formally, the notion of direct sum must be clarified. Unlike the text \[3\], which defines \( B \oplus C \) as the cartesian product of \( B \) and \( C \), here we say that for subgroups \( B \) and \( C \) of \( A \), \( A = B \oplus C \) if \( A = B + C = \{ b + c | b \in B, c \in C \} \), and if moreover \( B \cap C = 0 \). The subgroups \( B \) and \( C \) are then called direct summands of \( A \), and \( B \) is called the complement
of $C$. This essentially says that any element of $A$ can be written uniquely as the sum of an element of $B$ and an element of $C$. Given an arbitrary collection of subgroups of $A$ indexed by some set $I$, we say that $A = \bigoplus_{i \in I} A_i$ if any element of $A$ can be written as a sum of finitely many elements of various $A_i$, and if $A_i \cap \sum_{j \neq i} A_j = 0$ for all $i \in I$, where $\sum_{j \neq i} A_j$ is the set of all sums of finitely many elements from the $A_j$. Essentially, this means that any element of $A$ can be written uniquely as a sum of finitely many elements from the set of $A_i$. For example, the group of all polynomials of $x$ with coefficients in $\mathbb{Z}$ is isomorphic to $\bigoplus_{n} \mathbb{Z}$.

Given a group $G$, the set of all elements of $G$ with finite order is denoted $T(G)$, and called the torsion part of $G$. In the text [3], it is left as an exercise to show that $T(G)$ is in fact a subgroup of $G$, and that $G/T(G)$ is torsion-free. If for all $g \in G$, $|g| = p^k$ for some $k$, then $G$ is called a $p$-group. Another important, but less obvious subgroup is $S(G)$, the socle of $G$. The subgroup $S(G)$ is composed of all elements of $G$ with the order a square-free integer. Since $|a - b|$ divides lcm$(|a|, |b|)$, and any divisor of a square-free integer is square-free, $S(G)$ is indeed a subgroup. What makes $S(G)$ important is that it is an essential subgroup of $T(G)$, meaning that any non-trivial subgroup of $T(G)$ has a non-trivial intersection with $S(G)$. To see why this is the case, simply note that $S(G)$ contains all elements of prime order, and any subgroup of $T(G)$ must contain an element of prime order.

There is a final pair of concepts that are crucial to what will take place. Given a group $G$, we say that a system $\{a_i\}_{i \in I}$ of elements of $G$ is linearly independant if $\sum_{k=1}^{n} m_k a_{i_k} = 0$ implies $m_k a_{i_k} = 0$ for all $k$, i.e., no linear combination of elements from $\{a_i\}_{i \in I}$ is 0. If $\{a_i\}_{i \in I}$ is a linearly independant system in $G$, and no other linearly independant system $\{b_j\}_{j \in J}$ satisfies $\{a_i\}_{i \in I} \subseteq \{b_j\}_{j \in J}$, then $\{a_i\}_{i \in I}$ is called maximal. The existence of maximal linearly independant systems is guaranteed by the famous Zorn’s Lemma, which can be explained as follows. Given an arbitrary set $S$, a partial order on $S$ is a binary relation, denoted $\leq$, that is reflexive and transitive, and such that $a \leq b$ and $b \leq a$ imply $a = b$. In our case, the relation $\leq$ is usually set-theoretic inclusion. A subset $T$ of $S$ is called totally ordered if for all $a, b \in T$, either $a \leq b$ or $b \leq a$. If every totally ordered subset of $S$ has an upper bound in $S$, then Zorn’s Lemma says that $S$ itself has an upper bound, an element $s \in S$ such that $t \leq s$ for all $t \in S$. Thus, in the case of linearly independant systems, given any ascending chain $\{a_{1,i}\} \subseteq \{a_{2,i}\} \subseteq \cdots$, it is evident that the union $\bigcup_{n \in \mathbb{N}} \{a_{n,i}\}$ is also linearly independant and an upper bound, and so by Zorn’s Lemma, any group $G$ has a maximal linearly independant system.

## 2 Proof of the Main Result

To prove our main result, we will first show that any divisible subgroup is a direct summand. This implies that the torsion subgroup is a direct summand, with its complement being torsion-free. We will then show that the torsion part can be written as a direct sum of $p$-components, and each of those $p$-components
is a direct sum of copies of $\mathbb{Z}(p^{\infty})$. Finally, we will show that the complement of the torsion subgroup can be written as a direct sum of copies of $\mathbb{Q}$. This gives the main result: If $D$ is a divisible group, then

$$D \cong \bigoplus_{m_p} \mathbb{Z}(p^{\infty}) \bigoplus \mathbb{Q}$$

with the cardinal numbers $m_p$ and $n$ being a complete set of invariants for $D$.

**Lemma 1.** If $D$ is a divisible subgroup of $G$, then $G = D \oplus E$ for some $E \leq G$.

**Proof.** We let $\mathcal{E} = \{E_i\}_{i \in I}$ be the set of all subgroups of $G$ such that $D \cap E_i = 0$. The trivial subgroup is in $\mathcal{E}$, so it is non-empty, and any ascending chain of subgroups $\{E_i\}_{i \in \mathbb{N}}$ with $E_i \subseteq E_{i+1}$ satisfies $E_i \subseteq \bigcup_{i \in \mathbb{N}} E_i$ and $(\bigcup_{i \in \mathbb{N}} E_i) \cap D = 0$. By Zorn’s Lemma, there exists some $E \in \mathcal{E}$ such that $D \cap E = 0$, and that no other subgroup of $G$ disjoint to $D$ properly contains $E$. We wish to show that $G = D \oplus E$. If there is some $x_0 \in G$ such that $x_0 \notin D \oplus E$, then we create a sequence $\{x_i\}$ such that if $x_i \in D \oplus E$, then $x_j \in D \oplus E$ for all $j \leq i$. Since there will eventually be an $x_i$ in $D \oplus E$, we will derive a contradiction to $x_0 \notin D \oplus E$.

This sequence is created as follows: Since $E$ is maximal, $(x_0) \oplus E$ and $D$ cannot be disjoint, which means that $n_0 x_0 + e_0 = d_0 \in D$ for some $n_0 \in \mathbb{N}$, $e_0 \in E$. Obviously, $n_0$ cannot be either 0 or 1, for if $n_0 = 0$, then $e_0 = d_0 \in D$, which is a contradiction, and if $n_0 = 1$, then $x_0 = d_0 - e_0 \in D \oplus E$. Since $D$ is divisible, $d_0 = n_0 d_1$ for some $d_1 \in D$, which means that $n_0 x_0 + e_0 = n_0 d_1$, or $n_0(x_0 - d_1) = -e_0$. We let $x_1 = x_0 - d_1$. If $x_1 \in D \oplus E$, then $x_0 = x_1 + d_1 \in D \oplus E$, so we may assume that $x_1 \notin D \oplus E$. This means we can form $\langle x_1 \rangle \oplus E$, the elements of which are of the form $mx_1 + e$ with $0 \leq m < n_0$, because $n_0 x_1 = -e_0$.

Given arbitrary $i \in \mathbb{N}$, since $E$ is maximal, there must be a non-zero element in common with $\langle x_i \rangle \oplus E$ and $D$, i.e. $n_i x_i + e_i = d_i \in D$. Since $D$ is divisible, $d_i = n_i d_{i+1}$, and $n_i(x_i - d_{i+1}) = -e_i$. We define $x_{i+1} = x_i - d_{i+1}$. If $n_i = 1$, then $x_i = d_{i+1} - e_i \in D \oplus E$. Also, if $x_{i+1} \in D \oplus E$, then $x_i = x_{i+1} - d_{i+1} \in D \oplus E$, so we can assume that $x_{i+1} \notin D \oplus E$. But then we can form $\langle x_{i+1} \rangle \oplus E$, the elements of which are of the form $mx_{i+1} + e$, where $0 \leq m < n_i$. This means that $n_{i+1}$ will be strictly less than $n_i$, and so there must eventually be some $n_i = 1$, which forces all the $x_i \in D \oplus E$, and so $G = D \oplus E$. \hfill \Box

**Lemma 2.** If $G = T(G)$, then $G = \bigoplus_p T_p$, where $T_p$ are $p$-groups.

**Proof.** We define $T_p$ to be the set of all elements of $G$ whose order is a power of $p$. To see that $T_p$ is a subgroup, note that $p^{\lcm(a_i, b)}(a - b) = 0$, so $a - b \in T_p$ whenever $a, b \in T_p$. Since $|a_1 + \cdots + a_n|$ divides lcm($a_1, \ldots, a_n$), it must be that $T_{p_i} \cap \bigoplus_{j \neq i} T_{p_j} = 0$, so the $T_{p_i}$ generate a direct sum $\bigoplus_p T_p \leq G$. If $g \in G$ and $|g| = m = \prod_p p_i^{r_i}$, then let $m_i = mp_i^{-r_i}$. Obviously, $\gcd(m_1, m_2, \cdots) = 1$, so there exist $s_i$ such that $\sum_i s_i m_i = 1$, which means that $g = \sum_i s_i m_i g$. Since $p_i^{r_i} m_i g = mg = 0$, we have that $m_i g \in T_{p_i}$ for all $p_i$, which means that $g \in \bigoplus_p T_p$, and so $G = \bigoplus_p T_p$. \hfill \Box

**Lemma 3.** If $D$ is a divisible $p$-group, then $D = \bigoplus_m \mathbb{Z}(p^{\infty})$. 

3
Proof. We begin by selecting a maximal linearly independent system \( \{a_i\}_{i \in I} \) in \( S(D) \), the socle of \( D \). Since \( D \) is divisible, for every \( a_i \), there is a sequence \( \{a_{i,j}\}_{j \in \mathbb{N}} \) such that \( a_{i,1} = a_i \), \( pa_{i,n+1} = a_{i,n} \). Thus, each \( a_i \) can be embedded in a subgroup isomorphic to \( \mathbb{Z}(p^\infty) \). If any of those subgroups intersect, \( ma_{i,k} = na_{j,l} \), where \( \gcd(m,p) = \gcd(n,p) = 1 \), then if \( k = l \), \( p^{k-1}(ma_{i,k} - na_{j,l}) = ma_{i,1} - na_{j,1} = 0 \), which contradicts the linear independence of \( a_{i,1} \) and \( a_{j,1} \). If \( k \neq l \), then without loss of generality we may assume that \( l > k \). This means that \( p^k(ma_{i,k} - na_{j,l}) = m0 - na_{j,l-k} = -na_{j,l-k} = 0 \). But \( na_{j,l-k} = 0 \) only when \( n \) is a multiple of a power of \( p \), which contradicts \( \gcd(n,p) = 1 \). Thus, \( \{a_i\}_{i \in I} \) can be embedded in a direct sum \( A = \bigoplus_m \mathbb{Z}(p^\infty) \leq D \). Since \( A \) is divisible, \( D = A \oplus B \), and if \( B \neq 0 \), then the system \( \{a_i\}_{i \in I} \) would not be maximal. Thus, \( D \cong \bigoplus_n \mathbb{Q} \). \( \square \)

Lemma 4. If \( D \) is torsion-free and divisible, then \( D \cong \bigoplus_n \mathbb{Q} \).

Proof. As before, select a maximal linearly independent system \( \{b_i\}_{i \in I} \) in \( D \). Since there is exactly one solution to the equation \( ny = x \) because \( D \) is torsion-free, the expression \( y = \frac{1}{n} x \) is well-defined, and thus \( \frac{1}{n} x = m \left( \frac{1}{n} x \right) \) is well-defined. We can then form the set \( \text{Q}b_i = \{rb_i|r \in \mathbb{Q}\} \) for each \( b_i \). Since \( rb_i - qb_i = (r-q)b_i \), each of the sets \( \text{Q}b_i \) is a subgroup of \( D \). If \( \text{Q}b_i \cap \text{Q}b_j \neq 0 \) for some \( i \neq j \), then \( \frac{a_i}{n}b_i - \frac{a_j}{n}b_j = 0 \), which implies \( mtb_i + snb_j = 0 \), which contradicts the linear independance of \( b_i \) and \( b_j \). Thus, we can form the direct sum \( B = \bigoplus_{i \in I} \text{Q}b_i \leq D \). The map \( \frac{1}{n}b_i \to \frac{1}{n} \) is readily seen to be an isomorphism, so we write \( B = \bigoplus_{n=1}^\infty \mathbb{Q} \leq D \). The subgroup \( B \) is divisible, so \( D = B \oplus C \) for some \( C \), and if \( C \neq 0 \), then \( \{b_i\}_{i \in I} \) is not maximal, so \( D \cong \bigoplus_n \mathbb{Q} \). \( \square \)

Theorem 1. If \( G \) is divisible, then \( G \cong \bigoplus_{m} \mathbb{Z}(p^\infty) \bigoplus_n \mathbb{Q} \).

Proof. If \( x \in T(G) \), then for any \( n \in \mathbb{N} \), since \( G \) is divisible, there exists \( y \) such that \( ny = x \). But then \( |x|ny = |x|n = 0 \), so \( y \in T(G) \). This means that \( T(G) \) is divisible, and so by lemma 1, \( G = T(G) \oplus E \) for some \( E \). It is left as an exercise to show that \( E \cong G/T(G) \), and thus \( E \) is torsion-free. The subgroup \( T(G) \) is a torsion group, so by lemma 2, \( T(G) \cong \bigoplus_p T_p \). Another fact, left as an exercise, is that a direct sum is divisible if and only if the direct summands are divisible, so \( T_p \) are divisible. By lemma 3, \( T_p \cong \bigoplus_{n=1}^\infty \mathbb{Z}(p^n) \). This means that \( T(G) \cong \bigoplus_{m=1}^\infty \mathbb{Z}(p^{\infty}) \). By lemma 4, \( E \cong \bigoplus \mathbb{Q} \). Putting together the expressions for \( T(G) \) and \( E \) gives the main result. \( \square \)

3 Conclusion

Theorem 1 shows us that there is much less variety in the class of divisible groups than one might initially assume. For instance, the general euclidean space \( \mathbb{R}^n \) has cardinality \( 2^{2^n} \) for all \( n \), and is torsion-free, so \( \mathbb{R}^n \cong \bigoplus_{2^{2^n}} \mathbb{Q} \). This means that \( \mathbb{R}, \mathbb{R}^2, \text{ etc. are all isomorphic as groups. Furthermore, there is the surprising result that } C^t = C \setminus \{0\} \text{ and } S^1 = \{z \in \mathbb{C}, |z| = 1\} \text{ are isomorphic as groups. To see why, note that in both } C^t \text{ and } S^1, \text{ there are exactly } p^k \text{
solutions to the equation \( x^{pk} = z \), which means that \( m_p = 1 \) for all \( p \) in the decompositions of both groups. Both \( \mathbb{C}^* \) and \( S^1 \) have cardinality \( 2^{\aleph_0} \), so \( n = 2^{\aleph_0} \) in both decompositions. Thus, we have the isomorphism

\[
\mathbb{C}^* \cong S^1 \cong \bigoplus_{p \in P} \mathbb{Z}(p^\infty) \bigoplus \mathbb{Q}.
\]

It is instructive to try and understand what the isomorphism \( \phi : \mathbb{C}^* \to S^1 \) actually does. The elements of finite order in \( \mathbb{C}^* \) coincide with the elements of finite order in \( S^1 \), so \( \phi \) sends any \( z \in T(\mathbb{C}^*) \) to itself. We can then select \( 2^{\aleph_0} \) elements of \( \mathbb{C}^* \) which have infinite order, and \( \prod_i z_i^{n_i} = 1 \) implies \( z_i^{n_i} = 1 \) for all \( i \). An example of such a collection that is of cardinality \( \aleph_0 \) is the set \( e^{i\sqrt{2}}, e^{i\sqrt{3}}, e^{i\sqrt{5}}, \ldots \). If \( e^{mi\sqrt{p}} = e^{ni\sqrt{q}} \) and \( p \neq q \), then \( e^{m(n\sqrt{p} - m\sqrt{q})} = 1 \), which only occurs when \( m\sqrt{p} - n\sqrt{q} = 2\pi k, k \in \mathbb{Z} \), which is impossible, as \( \pi \) and its nonzero multiples are transcendental, while \( m\sqrt{p} - n\sqrt{q} = 0 \) implies \( m/n = \sqrt{q/p} \), which is impossible since \( \sqrt{q/p} \) is irrational. Since both \( \mathbb{C}^* \) and \( S^1 \) contain collections of such elements of size \( 2^{\aleph_0} \), \( \phi \) simply need be any bijection between the respective maximal linearly independent systems of \( \mathbb{C}^* \) and \( S^1 \). The isomorphism \( \xi : \mathbb{R}^n \to \mathbb{R} \) is very similar, but simpler. Given a point \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), form the subgroup \( \mathbb{Q}x = \{ rx : r \in \mathbb{Q} \} \). For any \( y \notin \mathbb{Q}x \), it is easy to see that \( \mathbb{Q}y \cap \mathbb{Q}x = \{0 \} \). The same holds for any \( z \notin \mathbb{Q}x \oplus \mathbb{Q}y \), and so on. Eventually, one can select \( 2^{\aleph_0} \) elements of \( \mathbb{R}^n \) that are all linearly independent. But the same can be done for \( \mathbb{R} \), and so \( \xi \) is just any bijection between the respective maximal linearly independent systems.

In general, given two divisible groups \( A \) and \( B \), if both \( A \) and \( B \) have the same number of solutions to the equation \( p^kx = y \) for all \( p \) prime, \( k \in \mathbb{N} \), and if \( A \) and \( B \) have the same cardinalities, it must be that \( A \cong B \). Thus, the divisible groups are completely classified, which means that most current research on abelian groups focuses on reduced groups, which are groups with no divisible subgroups.

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