Convergence of quantum electrodynamics on the Poincaré group

V. V. Varlamov*

Abstract

Extended particles are considered in terms of the fields on the Poincaré group. Dirac-like wave equations for extended particles of any spin are defined on the various homogeneous spaces of the Poincaré group. Free fields of the spin 1/2 and 1 (Dirac and Maxwell fields) are considered in detail on the eight-dimensional homogeneous space, which is equivalent to a direct product of Minkowski spacetime and two-dimensional complex sphere. It is shown that a massless spin-1 field, corresponding to a photon field, should be defined within principal series representations of the Lorentz group. Interaction between spin-1/2 and spin-1 fields is studied in terms of a trilinear form. An analogue of the Dyson formula for $S$-matrix is introduced on the eight-dimensional homogeneous space. It is shown that in this case elements of the $S$-matrix are defined by convergent integrals.

Keywords: extended particles, fields on the Poincaré group, homogeneous spaces, wave equations, quantum electrodynamics

PACS numbers: 02.30.Gp, 02.60.Lj, 03.65.Pm, 12.20.-m

1 Introduction

It is well-known that the representation of a particle as an idealized point has been used in physics for a long time. Historically, such a description follows from classical (celestial) mechanics, where the distances between explored objects are much greater than their sizes. However, in quantum field theory a point-like particle description meets with some conceptual difficulties. For example, ultra-violet divergences follow directly from the point-like description. Refusing from the point-like idealization, we come to extended objects.

Our consideration based on the concept of generalized wave functions introduced by Ginzburg and Tamm in 1947 [24], where the wave function depends both coordinates $x_\mu$ and additional internal variables $u_\mu$ which describe spin of the particle, $\mu = 0, 1, 2, 3$. In 1955, Finkelstein showed [18] that elementary particles models with internal degrees of freedom can be described on manifolds larger than Minkowski spacetime (homogeneous spaces of the Poincaré group). The quantum field theories on the Poincaré group were discussed in the papers [38, 35, 12, 5, 36, 61, 39, 16, 25, 29]. A consideration of the field models on the homogeneous spaces leads naturally to a generalization of the concept of wave function (fields on the Poincaré group). The general form of these fields relates closely with the

---

*Siberian State Industrial University, Kirova 42, Novokuznetsk 654007, Russia
structure of the Lorentz and Poincaré group representations \cite{20, 43, 11, 25} and admits the following factorization $f(x, z) = \phi^n(z)\psi_n(x)$, where $x \in T_4$ and $\phi^n(z)$ form a basis in the representation space of the Lorentz group. At this point, four parameters $x^\mu$ correspond to position of the point-like object, whereas remaining six parameters $z \in \text{Spin}_+(1, 3)$ define orientation in quantum description of orientable (extended) object \cite{26, 27} (see also \cite{33}). It is obvious that the point-like object has no orientation, therefore, orientation is an intrinsic property of the extended object. Taking it into account, we come to consideration of physical quantity as an extended object, the generalized wave function of which is described by the field
\[ \psi(\alpha) \equiv \langle x, g | \psi \rangle \]
on the homogeneous space of some orthogonal group $\text{SO}(p, q)$, where $x \in T_n$ (position) and $g \in \text{Spin}_+(p, q)$ (orientation), $n = p + q$. So, in \cite{54, 55} Segal and Zhou proved convergence of quantum field theory, in particular, quantum electrodynamics, on the homogeneous space $R^1 \times S^3$ of the conformal group $\text{SO}(2, 4)$, where $S^3$ is the three-dimensional real sphere.

On the other hand, measurements in quantum field theory lead to extended objects. As is known, loop divergences emerging in the Green functions in quantum field theory originate from correspondence of the Green functions to unmeasurable (and hence unphysical) point-like quantities. This is because no physical quantity can be measured in a point, but in a region, the size of which (or ‘diameter’ of the extended object) is constrained by the resolution of measuring equipment \cite{2}. Ordinary quantum field theory defines the field function $\phi(x)$ as a scalar product of the state vector of the system and the state vector corresponding to the localization at the point:
\[ \phi(x) \equiv \langle x | \phi(x) \rangle. \]
This field describes the point-like object. On the other hand, resolution-dependent fields
\[ \phi_a(x) \equiv \langle x, a; g | \phi(x) \rangle, \]
introduced in the work \cite{2}, describe in essence extended objects, here $\langle x, a; g |$ is the bra-vector corresponding to localization of the measuring device around the point with the spatial resolution $a$, $g$ labels the apparatus function of the equipment (an aperture). It is easy to see that the resolution-dependent fields have the same mathematical structure as the fields on the Poincaré group (more generally, fields on the groups).

The present paper is organized as follows. Basic facts concerning fields on the Poincaré group are considered in the section 2. Dirac-like wave equations for extended objects are presented in the section 3. Solutions for the fields $\psi(\alpha) = \langle x, g | \psi \rangle$ of spin 1/2 and 1 (Dirac and Maxwell fields) are given in terms of associated hyperspherical functions defined on the two-dimensional complex sphere, where $g \in \text{Spin}_+(1, 3)$. Interaction between these fields is introduced in the section 4 in terms of a trilinear form.

2 Fields on the Poincaré group

As it mentioned above, Ginzburg and Tamm proposed to consider new internal continuous variables $u_\mu$. It allows one to generalize the Klein-Gordon wave equation. The Ginzburg-
The Tamm equation has the form

\[
\left[ \Box - m^2 + \frac{\beta}{2} M_{\mu\nu} M_{\mu\nu} \right] \Psi(x_\mu, u_\gamma) = 0,
\]

where

\[
M_{\mu\nu} = -i \left( u_\mu \frac{\partial}{\partial u_\nu} - u_\nu \frac{\partial}{\partial u_\mu} \right) \quad (\mu, \nu = 0, 1, 2, 3)
\]

are rotation generators of the four-dimensional space, the constant \( \beta \) is analogous to the momentum \( J \) in the rotator Hamiltonian \( H = (2J)^{-1} L_{ik} L_{ik} \), where \( L_{ik} \) are generators of the Lorentz group, \( m^2 \) is a constant. The wave function \( \Psi \) depends on coordinates \( x_\mu \) of mass centre of the particle and internal variables \( u_\mu \). The 4-vector \( u_\mu \) is a 4-vector of relative motion of the structural component of the particle around its mass centre. The wave function \( \Psi \) can be factored:

\[
\Psi(x_\mu, u_\mu) = \Psi(x_\mu) \Phi(u_\mu).
\]

Then the Ginzburg-Tamm equation is decomposed in the following two equations:

\[
(\Box - m^2 + \lambda_1 \beta) \Psi(x_\mu) = 0, \quad L_1 \Phi(u_\mu) = \lambda_1 \Phi(u_\mu).
\]

At this point, the function \( \Phi(u_\mu) \) is an eigenfunction of the operator \( L_1 \),

\[
L_1 = M_{\mu\nu} M_{\mu\nu}/2, \quad L_1 \Phi = \lambda_1 \Phi.
\]

The first equation from (2) has the following solution:

\[
\Psi(x_\mu) = C \exp(-im_0 t),
\]

where

\[
m_0^2 = m^2 - \lambda_1 \beta.
\]

Thus, a mass spectrum, described by the equation (1), is defined by an eigenvalue of the operator \( L_1 \). The mass operator \( L_1 \) in spherical coordinates on the one-sheeted hyperboloid \( H^3 \),

\[
u_0 = r \sinh \chi; \quad u_2 = r \cosh \chi \sin \theta \sin \phi; \\
u_1 = r \cosh \chi \sin \theta \cos \phi; \quad u_3 = r \cosh \chi \cos \theta,
\]

is defined by an expression

\[
L_1 = \frac{1}{r} \frac{\partial}{\partial r} \left( r^3 \frac{\partial}{\partial r} \right) - \frac{1}{\cosh^2 \chi} \frac{\partial}{\partial \chi} \left( \cosh^2 \chi \frac{\partial}{\partial \chi} \right) - \\
\quad \quad - \frac{1}{\cosh^2 \chi} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] - \left( r \frac{\partial}{\partial r} \right)^2 - 2r \frac{\partial}{\partial r}.
\]

The hyperboloid \( H^3 \) is a homogeneous space of the Lorentz group. Solutions of the Ginzburg-Tamm equation have the form of the fields on the Poincaré group:

\[
\Psi(x_\mu, u_\mu) = C \exp(-im_0 t) Y_{l,m}(\theta, \phi) P_i^j(\tanh \chi),
\]

where \( Y_{l,m}(\theta, \phi) \) is a spherical function, and \( P_i^j(\tanh \chi) \) is a Legendre polynomial.
Further, fields on the Poincaré group present itself a natural generalization of the concept of wave function. These fields (generalized wave functions) were introduced independently by several authors [24, 8, 70, 56] mainly in connection with constructing relativistic wave equations. In essence, this generalization consists in replacing the Minkowski space by a larger space on which the Poincaré group acts. If this action is to be transitive, one is lead to consider the homogeneous spaces of the Poincaré group. All the homogeneous spaces of this type were listed by Finkelstein [18] and by Bacry and Kihlberg [6] and the fields on these spaces were considered in the works [38, 7, 45, 34, 35, 61, 25].

A homogeneous space $\mathcal{M}$ of a group $G$ has the following properties:
a) It is a topological space on which the group $G$ acts continuously, that is, let $y$ be a point in $\mathcal{M}$, then $gy$ is defined and is again a point in $\mathcal{M}$ ($g \in G$).
b) This action is transitive, that is, for any two points $y_1$ and $y_2$ in $\mathcal{M}$ it is always possible to find a group element $g \in G$ such that $y_2 = gy_1$.

There is a one-to-one correspondence between the homogeneous spaces of $G$ and the coset spaces of $G$. Let $H_0$ be a maximal subgroup of $G$ which leaves the point $y_0$ invariant, $gy_0 = y_0$, $g \in H_0$, then $H_0$ is called the stabilizer of $y_0$. Representing now any group element of $G$ in the form $g = g_c y_0$, where $g_0 \in H_0$ and $g_c \in G/H_0$, we see that, by virtue of the transitivity property, any point $y \in \mathcal{M}$ can be given by $y = g_c y_0 y_0 = g_c y$. Hence it follows that the elements $g_c$ of the coset space give a parametrization of $\mathcal{M}$. The mapping $\mathcal{M} \leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on $\mathcal{M}$ is continuous by definition. The stabilizers $H$ and $H_0$ of two different points $y$ and $y_0$ are conjugate, since from $H_0 y_0 = g_0 y_0 = g^{-1} y$, it follows that $gH_0 g^{-1} y = y$, that is, $H = gH_0 g^{-1}$.

Returning to the Poincaré group $\mathcal{P}$, we see that the enumeration of the different homogeneous spaces $\mathcal{M}$ of $\mathcal{P}$ amounts to an enumeration of the subgroups of $\mathcal{P}$ up to a conjugation. Following to Finkelstein, we require that $\mathcal{M}$ always contains the Minkowski space $\mathbb{R}^{1,3}$ which means that four parameters of $\mathcal{M}$ can be denoted by $x (x^\mu)$. This means that the stabilizer $H$ of a given point in $\mathcal{M}$ can never contain an element of the translation subgroup of $\mathcal{P}$. Thus, the stabilizer must be a subgroup of the homogeneous Lorentz group $\mathfrak{G}_+$. In such a way, studying different subgroups of $\mathfrak{G}_+$, we obtain a full list of homogeneous spaces $\mathcal{M} = \mathcal{P}/H$ of the Poincaré group. In the present paper we restrict ourselves by a consideration of the following four homogeneous spaces:

\[
\begin{align*}
\mathcal{M}_{10} &= \mathbb{R}^{1,3} \times \mathfrak{L}_6, \quad H = 0; \\
\mathcal{M}_8 &= \mathbb{R}^{1,3} \times S^2_c, \quad H = \Omega^c_c; \\
\mathcal{M}_7 &= \mathbb{R}^{1,3} \times H^3, \quad H = \text{SU}(2); \\
\mathcal{M}_6 &= \mathbb{R}^{1,3} \times S^2, \quad H = \{\Omega^c_\psi, \Omega^r, \Omega^c_c\}.
\end{align*}
\]

Hence it follows that a group manifold of the Poincaré group, $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathfrak{L}_6$, is a maximal homogeneous space of $\mathcal{P}$, $\mathfrak{L}_6$ is a group manifold of the Lorentz group. The fields on the manifold $\mathcal{M}_{10}$ were considered by Lurçat [33]. These fields depend on all the ten parameters of $\mathcal{P}$:

\[
\psi(\alpha) = \langle x, g | \psi \rangle = \psi(x_0, x_1, x_2, x_3)\psi(g_1, g_2, g_3, g_4, g_5, g_6),
\]

where an explicit form of $\psi(x)$ is given by the exponentials, and the functions $\psi(g)$ are expressed via the generalized hyperspherical functions $\mathfrak{M}^m_{\nu m}(g)$. As is known, the universal covering of the proper Poincaré group is isomorphic to a semidirect product $\text{SL}(2; \mathbb{C}) \circ T_4$ or $\text{Spin}_+(1,3) \circ T_4$. Since the group $T_4$ is Abelian, then all its representations are one-dimensional. Thus, all the finite-dimensional representations of the proper Poincaré group
in essence are equivalent to the representations \( \mathfrak{c} \) of the group \( \text{Spin}_+(1, 3) \). In the case of finite-dimensional representations of \( \text{Spin}_+(1, 3) \),

\[
T_\theta q(\xi, \bar{\xi}) = (\gamma \xi + \delta)^{l_{0}+l_{1}-(\gamma \xi + \delta)^{l_{0}-l_{1}+1}} q \left( \frac{\alpha \xi + \beta}{\gamma \xi + \delta} \right), \tag{3}
\]

basic functions (matrix elements) of the finite-dimensional representation of \( \mathcal{P} \) have the form

\[
t^{l}_{mn}(\alpha) = e^{-im\pi} \mathfrak{m}^{l}_{mn}(\mathfrak{g}),
\]

where

\[
\mathfrak{m}^{l}_{mn}(\mathfrak{g}) = e^{-im\pi} \mathfrak{z}^{l}_{mn}(\cos \theta^{c}) e^{-im\pi}. \tag{65}
\]

Hyperspherical functions \( \mathfrak{z}^{l}_{mn}(\cos \theta^{c}) \) can be expressed via the hypergeometric functions \( \mathfrak{65} \). So, at \( m \geq n \) and \( m \geq k, k \geq n \) we have

\[
\mathfrak{z}^{l}_{mn}(\cos \theta^{c}) = i^{m-n} \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}} \cos^{2l} \theta \cosh^{2l} \frac{\tau}{2} \times
\]

\[
\times \sum_{k=-l}^{l} \tan^{m-k} \frac{\theta}{2} \tanh^{k-n} \frac{\tau}{2} \times
\]

\[
\times 2F_1 \left( \begin{array}{c} m-l, -k-l \\ m-k+1 \end{array} \right) \frac{-\tan^{2} \frac{\theta}{2}}{2F_1 \left( \begin{array}{c} k-l, -n-l \\ k-n+1 \end{array} \right) \frac{\tan^{2} \frac{\tau}{2}}{2} } \right). \tag{4}
\]

There also exist three expressions of the hypergeometric type for the functions \( \mathfrak{z}^{l}_{mn}(\cos \theta^{c}) \) with the index values \( m \geq n, m \geq k, n \geq k, n \geq m, k \geq m, n \geq k \) and \( n \geq m, k \geq m, k \geq n \). For the unitary representations, that is, in the case of principal series representations of the group \( \text{SO}_0(1, 3) \), there exists an analogue of the formula \( \mathfrak{63} \),

\[
V_{a} f(z) = (a_{12}z + a_{22})^{\lambda+\frac{i\pi}{2}+1} (a_{12}z + a_{22})^{-\frac{i\pi}{2}} f \left( \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}} \right), \tag{5}
\]

where \( f(z) \) is a measurable function of the Hilbert space \( L_{2}(Z) \), satisfying the condition \( \int |f(z)|^{2} dz < \infty, z = x + iy \). A totality of all representations \( a \rightarrow \mathcal{T}^{a} \), corresponding to all possible pairs \( \lambda, \rho \), is called a principal series of representations of the group \( \text{SO}_0(1, 3) \) and denoted as \( \mathfrak{S}_{\lambda, \rho} \). At this point, a comparison of \( \mathfrak{5} \) with the formula \( \mathfrak{63} \) for the spinor representation \( \mathfrak{S}_{1} \) shows that the both formulas have the same structure; only the exponents at the factors \( (a_{12}z + a_{22}) \) and the functions \( f(z) \) are different. In the case of spinor representations the functions \( f(z) \) are polynomials \( p(z, \bar{z}) \) in the spaces \( \text{Sym}_{(k,r)} \), and in the case of a representation \( \mathfrak{S}_{\lambda, \rho} \) of the principal series \( f(z) \) are functions from the Hilbert space \( L_{2}(Z) \). Further, we know that a representation \( S_{k} \) of the group \( \text{SU}(2) \) is realized in terms of the functions \( t^{l}_{mn}(u) = e^{-im\pi} \mathcal{P}^{l}_{mn}(\cos \theta)e^{-im\pi} \mathfrak{68} \). It is well known that the representation \( S_{k} \) of \( \text{SU}(2) \) is contained in \( \mathfrak{S}_{\lambda, \rho} \) not more then one time \( \mathfrak{43} \). At this point, \( S_{k} \) is contained in \( \mathfrak{S}_{\lambda, \rho} \), when \( \frac{1}{2} \) is one from the numbers \( -k, -k+1, \ldots, k \). Matrix elements
of the principal series representations of the group \(SO_0(1, 3)\) have the form \[65\]

\[
\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g) = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)}3_{mn}^{-\frac{1}{2}+i\rho,l_0} = e^{-m(\epsilon+i\varphi)-n(\epsilon+i\psi)} \times
\]

\[
\sum_{t=-l_0}^{l_0} i^{m-t} \sqrt{\Gamma(l_0 - m + 1)\Gamma(l_0 + m + 1)\Gamma(l_0 - t + 1)\Gamma(l_0 + t + 1)} \times
\]

\[
\cos^{2l_0} \theta \tan^{m-t} \theta \times
\]

\[
\min(l_0-m,l_0+t) \sum_{j=\max(0,t-m)}^{l_0} i^{2j} \tan^{2j} \frac{\theta}{2} \Gamma(j+1)\Gamma(l_0 - m - j + 1)\Gamma(l_0 + t - j + 1)\Gamma(m - t + j + 1) \times
\]

\[
\sqrt{\Gamma(\frac{1}{2} + i\rho - n)\Gamma(\frac{1}{2} + i\rho + n)\Gamma(\frac{1}{2} + i\rho - t)\Gamma(\frac{1}{2} + i\rho + t) \cosh^{-1+2i\rho} \frac{\tau}{2} \tanh^{n-t} \frac{\tau}{2} \times
\]

\[
\sum_{s=\max(0,t-n)}^{\infty} \tan^{2s} \frac{\tau}{2} \Gamma(s+1)\Gamma(\frac{1}{2} + i\rho - n - s)\Gamma(\frac{1}{2} + i\rho + t - s)\Gamma(n - t + s + 1), \quad (6)
\]

where \(l_0 = \left| \frac{1}{2} \right|\) and \(\frac{1}{2}\) is one from the numbers \(-k, -k + 1, \ldots, k\). It is obvious that \(\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g)\) cannot be attributed as matrix elements to single irreducible representation. From the latter expression it follows that relativistic spherical functions \(f(g)\) of the principal series can be defined by means of the function

\[
\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g) = e^{-m(\epsilon+i\varphi)}3_{mn}^{-\frac{1}{2}+i\rho,l_0}(\cos \theta^c) e^{-n(\epsilon+i\psi)}, \quad (7)
\]

where

\[
3_{mn}^{-\frac{1}{2}+i\rho,l_0}(\cos \theta^c) = \sum_{t=-l_0}^{l_0} P^{l_0}_{m}\cos \theta^c \mathcal{M}_{tn}^{-\frac{1}{2}+i\rho}(\cosh \tau).
\]

In turn, the functions \(\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g)\) can be expressed via the hypergeometric functions. So, at \(m \geq t, t \geq n\) we have \[65\]

\[
\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g) = \sum_{t=-l_0}^{l_0} \frac{P^{l_0}_{m}\cos \theta^c \mathcal{M}_{tn}^{-\frac{1}{2}+i\rho}(\cosh \tau)}{\Gamma(l_0 + m + 1)\Gamma(l_0 - m + 1)\Gamma(i\rho - n + \frac{1}{2})}\times
\]

\[
\cos^{2l_0} \theta \cosh^{-1+2i\rho} \frac{\tau}{2} \sum_{t=-l_0}^{l_0} \tan^{m-t} \theta \tan^{t-n} \frac{\tau}{2} \times
\]

\[
2F_1 \left( \begin{array}{c} m - l_0, -t - l_0 \cr -\tau \end{array} \right) - \tan^{2} \frac{\theta}{2} 2F_1 \left( \begin{array}{c} t - i\rho + \frac{1}{2}, -n - i\rho + \frac{1}{2} \cr t - n + 1 \end{array} \right) \tan^{2} \frac{\tau}{2} \right). \quad (8)
\]

There also exist three expressions of the hypergeometric type for the functions \(\mathcal{M}_{mn}^{-\frac{1}{2}+i\rho,l_0}(g)\) with the index values \(m \geq t, n \geq t, t \geq m, n \geq t\) and \(t \geq m, t \geq n\).

The following eight-dimensional homogeneous space \(\mathcal{M}_8 = \mathbb{R}^{1,3} \times S_c^2\) is a direct product of the Minkowski space \(\mathbb{R}^{1,3}\) and the complex two-sphere \(S_c^2\). In this case the stabilizer \(H\) consists of the subgroup \(\Omega_{\psi}^c\) of the diagonal matrices \(\begin{pmatrix} e^{\frac{i\omega e}{2}} & 0 \\ 0 & e^{-\frac{i\omega e}{2}} \end{pmatrix}\). Bacry and Kihlberg
claimed that the space $\mathcal{M}_8$ is the most suitable for a description of both half-integer and integer spins. The fields, defined in $\mathcal{M}_8$, depend on the eight parameters of $\mathcal{P}$:

$$\langle x, \varphi^c, \theta^c | \psi \rangle = \psi(x) \psi(\varphi^c, \theta^c) = \psi(x_0, x_1, x_2, x_3) \psi(\varphi, c, \theta, \tau),$$

where the functions $\psi(\varphi^c, \theta^c)$ are expressed via the associated hyperspherical functions defined on the surface of the complex two-sphere $S^2_2$. The sphere $S^2_2$ can be constructed from the quantities $z_k = x_k + iy_k, \bar{z}_k = x_k - iy_k (k = 1, 2, 3)$ as follows:

$$S^2_2: z_1^2 + z_2^2 + z_3^2 = x^2 - y^2 + 2ixy = r^2.$$  

(10)

The complex conjugate (dual) sphere $\hat{S}^2_2$ is

$$\hat{S}^2_2: \hat{z}_1^2 + \hat{z}_2^2 + \hat{z}_3^2 = x^2 - y^2 - 2ixy = \hat{r}^2.$$  

(11)

For more details about the two-dimensional complex sphere see [30, 31, 59]. The surface of $S^2_2$ is homeomorphic to the space of the pairs $(z_1, z_2)$, where $z_1$ and $z_2$ are the points of a complex projective line, $z_1 \neq z_2$. This space is a homogeneous space of the Lorentz group [21]. In Euler parametrization we have $(z_1, z_2) \sim (\varphi^c, \theta^c) = (\varphi - ic, \theta - it)$. It is well-known that both quantities $x^2 - y^2, xy$ are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant (Casimir operators of the Lorentz group are constructed from such quantities). Moreover, since the real and imaginary parts of the complex two-sphere transform like the electric and magnetic fields, respectively, the invariance of $z^2 \sim (E + iB)\hat{z}$ under proper Lorentz transformations is evident. At this point, the quantities $x^2 - y^2, xy$ are similar to the well known electromagnetic invariants $E^2 - B^2, EB$. This intriguing relationship between the Laplace-Beltrami operators, Casimir operators of the Lorentz group and electromagnetic invariants $E^2 - B^2 \sim x^2 - y^2, EB \sim xy$ leads naturally to a Riemann-Silberstein representation of the electromagnetic field (see, for example, [69, 57, 10]). Further, associated hyperspherical functions

$$\mathcal{M}_l^m(\varphi^c, \theta^c, 0) = e^{-im\varphi^c} \mathfrak{Z}_l^m(\cos \theta^c),$$

(12)

where

$$\mathfrak{Z}_l^m(\cos \theta^c) = \sum_{k=-l}^l P_{mk}^l(\cos \theta) \mathfrak{P}_l^k(\cosh \tau),$$

are defined on the surface of the two-dimensional complex sphere (10). In turn, the functions $\mathcal{M}_l^m(\varphi^c, \theta^c, 0) = e^{im\varphi^c} \mathfrak{Z}_l^m(\cos \theta^c)$ are defined on the surface of the dual sphere (11). Explicit expressions of the hypergeometric type for the functions $\mathfrak{Z}_l^m(\cos \theta^c)$ and $\mathfrak{Z}_l^m(\cos \theta^c)$ follow directly from the previous expressions (4) and (8) at $n = 0$. In the case of the principal series representations of the group $SO_0(1, 3)$ we have

$$\mathfrak{Z}_{-\frac{l}{2} + i\rho, l_0}^m(\cos \theta^c) = \sum_{t=-l_0}^{l_0} P_{mt}^{l_0}(\cos \theta) \mathfrak{P}_{-\frac{l}{2} + i\rho}^t(\cosh \tau),$$

(13)

where $\mathfrak{P}_{-\frac{l}{2} + i\rho}(\cosh \tau)$ are conical functions (see [9]).

In turn, a seven-dimensional homogeneous space $\mathcal{M}_7 = \mathbb{R}^{1,3} \times H^3$ is a direct product of $\mathbb{R}^{1,3}$ and a three-dimensional timelike (two-sheeted) hyperboloid $H^3$. The stabilizer $H$
consists of the subgroup of three-dimensional rotations, SU(2). Quantum field theory on the space \( \mathcal{M}_7 \) was studied by Boyer and Fleming \[12\]. They showed that the fields built over \( \mathcal{M}_7 \) are in general nonlocal and become local only when the finite dimensional representations of the Lorentz group are used. It is easy to see that the fields \( \psi \in \mathcal{M}_7 \) depend on the seven parameters of the Poincaré group:

\[
\langle x, \tau, \epsilon, \varepsilon | \psi \rangle = \psi(x_0, x_1, x_2, x_3)\psi(\tau, \epsilon, \varepsilon),
\]

where the functions \( \psi(\tau, \epsilon, \varepsilon) \) are expressed via \( e^{-i\eta} \mathcal{P}_m(\cosh \tau) \) in the case of finite dimensional representations, and also via \( e^{-i\eta} \mathcal{P}_m^{\pm}(\cosh \tau) \) in the case of principal series of unitary representations, and via \( e^{-i\eta} \mathcal{P}_{mn}(\cosh \tau) \) in the case of the discrete representation series of the subgroup SU(1, 1). The spherical function \( \mathcal{P}_{mn}(\cosh \tau) \) on the group SU(1, 1) has the form

\[
\mathcal{P}_{mn}(\cosh \tau) = \sqrt{\Gamma(l + m + 1)\Gamma(l - m + 1)\Gamma(l - n + 1)\Gamma(l + n + 1)} \times \\
\cos^{2i} \frac{\tau}{2} \tan^{n-m} \frac{\tau}{2} \times \\
\sum_{s=\max(0, m-n)}^{\min(l-n, l+m)} \frac{\tan^{2s} \frac{\tau}{2}}{\Gamma(s + 1)\Gamma(l - n - s + 1)\Gamma(n - m + s + 1)\Gamma(l - m + s + 1)}. \quad (14)
\]

Further, a six-dimensional space \( \mathcal{M}_6 = \mathbb{R}^{1,3} \times S^2 \) is a minimal homogeneous space of the Poincaré group, since the real two-sphere \( S^2 \) has a minimal possible dimension among the homogeneous spaces of the Lorentz group. In this case, the stabilizer \( H \) consists of the subgroup \( \Omega_\psi \) and the subgroups \( \Omega_\tau \) and \( \Omega_\epsilon \) formed by the matrices \((\cosh \frac{\tau}{2}, \sinh \frac{\tau}{2}; \sinh \frac{\tau}{2}, \cosh \frac{\tau}{2})\) and \((e^{\frac{\tau}{2}} 0; 0 e^{-\frac{\tau}{2}})\), respectively. Field models on the homogeneous space \( \mathcal{M}_6 \) have been considered in the works \[36, 39, 16\]. In the paper \[16\] Drechsler considered the real two-sphere as a ‘spin shell’ \( S^2_{r=2s} \) of radius \( r = 2s \), where \( s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \). The fields, defined on \( \mathcal{M}_6 \), depend on the six parameters of \( \mathcal{P} \):

\[
\langle x, \varphi, \theta | \psi \rangle = \psi(x_0, x_1, x_2, x_3)\psi(\varphi, \theta),
\]

where the functions \( \psi(\varphi, \theta) \) are expressed via the generalized spherical functions of the type \( e^{-im\varphi}P_m^{l}(\cos \theta) \) or via the Wigner D-functions, here

\[
P_m^l(\cos \theta) = e^{-i(m\varphi + n\psi)}\sqrt{\Gamma(l - m + 1)\Gamma(l + m + 1)\Gamma(l - n + 1)\Gamma(l + n + 1)} \times \\
\cos^{2i} \frac{\theta}{2} \tan^{m-n} \frac{\theta}{2} \times \\
\sum_{j=\max(0, n-m)}^{\min(l-n, l+m)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j + 1)\Gamma(l - m - j + 1)\Gamma(l + n - j + 1)\Gamma(m - n + j + 1)}. \quad (15)
\]

is a spherical function on the group SU(2).

Further, let \( \mathcal{H} \) be a Hilbert space and let \( U(g) \) be a square integrable representation of a locally compact Lie group \( G \) acting transitively on \( \mathcal{H} \), \( \forall \phi \in \mathcal{H}, g \in G : U(g)\phi \in \mathcal{H} \). Let there exist such a vector \( | \psi \rangle \in \mathcal{H} \) that satisfies the admissibility condition:

\[
C_\psi = ||\psi||^{-2} \int_{g \in G} ||U(g)\psi||^2 d\mu(g) < \infty,
\]

8
where \( d\mu(g) \) is the left-invariant Haar measure on \( G \). Then, any vector \( |\phi\rangle \in \mathcal{H} \) can be represented in the form

\[
|\phi\rangle = C_\psi^{-1} \int_G |U(g)\psi\rangle d\mu(g)\langle \psi^* | \phi \rangle.
\]

At the restriction of \( G \) on the affine subgroup \( x' = ax + b \) (the group of translations and dilations of the real axis) a harmonic analysis on the group \( G \) is reduced to the wavelet-transformation:

\[
\phi(x) = \frac{1}{C_\psi} \int \frac{1}{a^d} \psi\left(\frac{x-b}{a}\right) \phi_a(b) \frac{dadb}{a},
\]

\[
\phi_a(b) = \int \frac{1}{a^d} \psi\left(\frac{x-b}{a}\right) \phi(x) d^d x,
\]

\[
C_\psi = \int_0^\infty |\tilde{\psi}(ak)|^2 \frac{da}{a} < \infty.
\]

At this restriction the field \( \psi(\alpha) = \langle x, g | \psi \rangle \) on the group \( G \) is reduced to the field \( \psi_a(\alpha) = \langle x, a | \psi \rangle \sim \langle x, a; g | \phi(x) \rangle \) on the affine group, where \( a \) is a scale factor which can be associated with the resolution of measuring equipment [2]. It should be noted that further restriction of \( G \) on the translation subgroup leads to the restriction of \( \psi(\alpha) = \langle x, g | \psi \rangle \) to the field \( \psi(x) = \langle x | \psi \rangle \) describing the point-like object, and the wavelet-transformation in this case is reduced to the Fourier-transformation. Thus, the affine group is a minimal group which can be used for description of the extended object. The restriction of \( G \) to the affine subgroup is equivalent formally to the Faddeev-Popov method considering translation and dilatation invariance in quantum field calculations related with the renormalization group, at this point, \( a = 1/\Lambda \), where \( \Lambda \) is a cut-off parameter. Moreover, in the framework of wavelet-analysis, loop Green functions emerging in \( \phi^4 \)-model, are free from ultra-violet divergences [1, 2].

3 Free fields on the two-dimensional complex sphere

We will start with a more general homogeneous space of the group \( \mathcal{P} \), \( \mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathbb{C}_6 \) (group manifold of the Poincaré group). Let \( \mathcal{L}(\alpha) \) be a Lagrangian on the group manifold \( \mathcal{M}_{10} \) (in other words, \( \mathcal{L}(\alpha) \) is a 10-dimensional point function), where \( \alpha \) is the parameter set of this group. Then an integral for \( \mathcal{L}(\alpha) \) on some 10-dimensional volume \( \Omega \) of the group manifold we will call an action on the Poincaré group:

\[
A = \int_\Omega d\alpha \mathcal{L}(\alpha),
\]

where \( d\alpha \) is a Haar measure on the group \( \mathcal{P} \).

Let \( \psi(\alpha) = \langle x, g | \psi \rangle \) be a function on the group manifold \( \mathcal{M}_{10} \) (now it is sufficient to assume that \( \psi(\alpha) \) is a square integrable function on the Poincaré group) and let

\[
\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial \alpha} \frac{\partial \mathcal{L}}{\partial \psi_{\alpha}} = 0
\]

(17)
be Euler-Lagrange equations on $\mathcal{M}_{10}$ (more precisely speaking, the equations (17) act on the tangent bundle $T \mathcal{M}_{10} = \bigcup_{\alpha \in \mathcal{M}_{10}} T_{\alpha} \mathcal{M}_{10}$ of the manifold $\mathcal{M}_{10}$, see [4]). Let us introduce a Lagrangian $\mathcal{L}(\alpha)$ depending on the field function $\psi(\alpha)$ as follows

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \psi^*(\alpha) B_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \psi^*(\alpha)}{\partial \alpha_\mu} B_\mu \psi(\alpha) \right) - \kappa \psi^*(\alpha) B_{11} \psi(\alpha),$$

where $B_\nu (\nu = 1, 2, \ldots, 10)$ are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\psi(\alpha)$, $\kappa$ is a non-null real constant.

Further, if $B_{11}$ is non-singular, then we can introduce the matrices

$$\Gamma_\mu = B_{11}^{-1} B_\mu, \quad \mu = 1, 2, \ldots, 10,$$

and represent the Lagrangian $\mathcal{L}(\alpha)$ in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Gamma_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \overline{\psi}(\alpha) \Gamma_\mu \psi(\alpha) \right) - \kappa \overline{\psi}(\alpha) \psi(\alpha),$$

where

$$\overline{\psi}(\alpha) = \psi^*(\alpha) B_{11}.$$

Varying independently $\psi(x)$ and $\overline{\psi}(x)$, we obtain from (18) in accordance with (17) the following equations:

$$\Gamma_i \frac{\partial \psi(x)}{\partial x_i} + \kappa \psi(x) = 0, \quad (i = 0, \ldots, 3) \quad (19)$$

$$\Gamma_i^T \frac{\partial \overline{\psi}(x)}{\partial x_i} - \kappa \overline{\psi}(x) = 0.$$

The matrix $\Gamma_0$ in equations (19) can be written in the form (see [20, 3, 48])

$$\Gamma_0 = \text{diag} \left( C^0 \otimes I_1, C^1 \otimes I_3, \ldots, C^s \otimes I_{2s+1}, \ldots \right)$$

(20)

for integer spin and

$$\Gamma_0 = \text{diag} \left( C^\frac{1}{2} \otimes I_2, C^\frac{3}{2} \otimes I_4, \ldots, C^s \otimes I_{2s+1}, \ldots \right)$$

(21)

for half-integer spin, where $C^s$ is a spin block. If the spin block $C^s$ has non-null roots, then the particle possesses the spin $s$. The spin block $C^s$ in (20)–(21) consists of the elements $c^s_{\tau \tau'}$, where $\tau_1, \tau_2$ and $\tau'_1, \tau'_2$ are interlocking irreducible representations of the Lorentz group, that is, such representations, for which $l'_1 = l_1 \pm \frac{1}{2}$, $l'_2 = l_2 \pm \frac{1}{2}$. At this point, the block $C^s$ contains only the elements $c^s_{\tau \tau'}$ corresponding to such interlocking representations $\tau_1, \tau_2$, $\tau'_1, \tau'_2$ which satisfy the conditions

$$|l_1 - l_2| \leq s \leq l_1 + l_2, \quad |l'_1 - l'_2| \leq s \leq l'_1 + l'_2.$$

The two most full schemes of the interlocking irreducible representations of the Lorentz group for integer and half-integer spins are shown on the Fig. 1 and Fig. 2. As follows from Fig. 1 the simplest field is the scalar field

$$(0, 0).$$
This field is described by the Fock-Klein-Gordon equation. In its turn, the simplest field from the Fermi-scheme (Fig. 2) is the electron-positron (spinor) field corresponding to the following interlocking scheme:

\[
\begin{pmatrix}
\frac{1}{2}, 0 \\
0, \frac{1}{2}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0, \frac{1}{2}
\end{pmatrix}.
\]

This field is described by the Dirac equation. Further, the next field from the Bose-scheme (Fig. 1) is a photon field (Maxwell field) defined within the interlocking scheme

\[
(1, 0) \rightarrow \begin{pmatrix}
\frac{1}{2}, \frac{1}{2}
\end{pmatrix} \rightarrow (0, 1).
\]

This interlocking scheme leads to the Maxwell equations. The fields \((1/2, 0) \oplus (0, 1/2)\) and \((1, 0) \oplus (0, 1)\) (Dirac and Maxwell fields) are particular cases of fields \(\psi(\alpha) = \langle x, g | \psi \rangle\) of the
Fig. 2: Interlocking representation scheme for the fields of half-integer spin (Fermi-scheme).

type \((l, 0) \oplus (0, l)\), where \(g \in \text{Spin}^+_1(1, 3)\). Wave equations for such fields and their general solutions were found in the works \([62, 63, 64]\).

It is easy to see that the interlocking scheme, corresponding to the Maxwell field, contains the field of tensor type:

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

Further, the next interlocking scheme (see Fig. 2)

\[
\begin{pmatrix}
\frac{3}{2} & 0 \\
\frac{1}{2} & 1
\end{pmatrix} \longrightarrow \begin{pmatrix}1 & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix}\frac{1}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix}0 & \frac{3}{2} \end{pmatrix},
\]

corresponding to the Pauli-Fierz equations \([19]\), contains a chain of the type

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1
\end{pmatrix} \longrightarrow \begin{pmatrix}\frac{1}{2} & 1 \end{pmatrix}.
\]
In such a way we come to wave equations for the fields \( \psi(\alpha) = \langle x, g | \psi \rangle \) of tensor type \((l_1, l_2) \oplus (l_2, l_1)\). Wave equations for such fields and their general solutions were found in the work \cite{67}.

Further, varying independently \( \psi(g) \) and \( \bar{\psi}(g) \) one gets from (18) the following equations:

\[
\begin{align*}
\Gamma_k \frac{\partial \psi(g)}{\partial g_k} + \kappa \psi(g) &= 0, \\
\Gamma_k^* \frac{\partial \psi(g)}{\partial g_k} - \kappa \bar{\psi}(g) &= 0, \quad (k = 1, \ldots, 6)
\end{align*}
\]

where

\[
\psi(q) = \begin{pmatrix}
\psi(q) \\
\bar{\psi}(q)
\end{pmatrix}, \quad \Gamma_k = \begin{pmatrix}
0 & \Lambda_k^* \\
\Lambda_k & 0
\end{pmatrix}.
\]

Non-zero elements of the matrices \( \Lambda_k \) and \( \Lambda_k^* \) have the form (for more details see \cite{62})

\[
\Lambda_1: \quad \begin{cases}
d_{l-1,m-1,m}^{l+k} = -\frac{c_l-1}{2} \sqrt{(l+m)(l+m-1)}, \\
d_{l,l,m-1,m}^{l+k} = \frac{c_l}{2} \sqrt{(l+m)(l+m+1)}, \\
d_{l+1,m-1,m}^{l+k} = \frac{c_l}{2} \sqrt{(l-m+1)(l-m+2)}, \\
d_{l-1,m+1,m}^{l+k} = \frac{c_l}{2} \sqrt{(l-m)(l-m-1)}, \\
d_{l,l,m+1,m}^{l+k} = \frac{c_l}{2} \sqrt{(l+m+1)(l-m)}, \\
d_{l+1,m+1,m}^{l+k} = -\frac{c_l}{2} \sqrt{(l+m+1)(l+m+2)}.
\end{cases}
\]

\[
\Lambda_2: \quad \begin{cases}
b_{l-1,m-1,m}^{l+k} = -\frac{i c_l-1}{2} \sqrt{(l+m)(l+m-1)}, \\
b_{l,l,m-1,m}^{l+k} = \frac{i c_l}{2} \sqrt{(l+m)(l+m+1)}, \\
b_{l+1,m-1,m}^{l+k} = \frac{i c_l}{2} \sqrt{(l-m+1)(l-m+2)}, \\
b_{l-1,m+1,m}^{l+k} = -\frac{i c_l}{2} \sqrt{(l-m)(l-m-1)}, \\
b_{l,l,m+1,m}^{l+k} = -\frac{i c_l}{2} \sqrt{(l+m+1)(l-m)}, \\
b_{l+1,m+1,m}^{l+k} = \frac{i c_l}{2} \sqrt{(l+m+1)(l+m+2)}.
\end{cases}
\]

\[
\Lambda_3: \quad \begin{cases}
c_l^{l,k}_{l-1,l,m} = c_{l-1,l}^{l,k} \sqrt{t^2 - m^2}, \\
c_l^{l,k}_{l,l,m} = c_{l,l}^{l,k} m, \\
c_l^{l,k}_{l+1,l,m} = c_{l+1,l}^{l,k} \sqrt{(l+1)^2 - m^2}.
\end{cases}
\]

\[
\Lambda_1^*: \quad \begin{cases}
d_{l-1,m-1,m}^{l+k} = -\frac{c_l-1}{2} \sqrt{\hat{l} \hat{m} (\hat{l} - \hat{m} - 1)}, \\
d_{l,l,m-1,m}^{l+k} = \frac{c_l}{2} \sqrt{\hat{l} \hat{m} (\hat{l} - \hat{m} + 1)}, \\
d_{l+1,m-1,m}^{l+k} = \frac{c_l}{2} \sqrt{\hat{l} \hat{m} (\hat{l} - \hat{m} + 2)}, \\
d_{l-1,m+1,m}^{l+k} = \frac{c_l}{2} \sqrt{\hat{l} \hat{m} (\hat{l} - \hat{m} - 1)}, \\
d_{l,l,m+1,m}^{l+k} = \frac{c_l}{2} \sqrt{\hat{l} \hat{m} (\hat{l} - \hat{m})}, \\
d_{l+1,m+1,m}^{l+k} = -\frac{c_l}{2} \sqrt{\hat{l} \hat{m} (\hat{l} + \hat{m} + 1) (\hat{l} + \hat{m} + 2)}.
\end{cases}
\]
matrices \( \Lambda_1 \) where \( \Lambda_1 \) wave functions which we write the decomposition (29), the wave function also decomposes into a direct sum of component relations between indecomposable wave equations and composite particles will be studied in As is known, the indecomposable wave equations correspond to composite particles. A indecomposable relativistic wave equations. Otherwise, we have decomposable equations. \( \star \)

\[
\begin{align*}
\Lambda_2^* : & \left\{ 
\begin{array}{lll}
\lambda_{i, j, m-1, \bar{m}} &=& -\frac{\kappa_i - 1}{2} \sqrt{(i + \bar{m})(i - m - 1)}, \\
\lambda_{i, j, m-1, \bar{m}} &=& \frac{\kappa_i}{2} \sqrt{(i + \bar{m})(i - m + 1)}, \\
\lambda_{i, j, m+1, \bar{m}} &=& \frac{\kappa_i}{2} \sqrt{(i - m + 1)(i - m + 2)}, \\
\lambda_{i, j, m+1, \bar{m}} &=& -\frac{\kappa_i}{2} \sqrt{(i - m)(i - m - 1)}, \\
\lambda_{i, j, m+1, \bar{m}} &=& -\frac{\kappa_i}{2} \sqrt{(i - m)(i - m)}, \\
\lambda_{i, j, m+1, \bar{m}} &=& -\frac{\kappa_i}{2} \sqrt{(i - m)(i - m + 1)}.
\end{array}
\right.
\end{align*}
\]

(27)

\[
\Lambda_3^* : \left\{ \begin{array}{ll}
f_{l-1, l, \bar{m}} &=& c_{l-1, l, \bar{m}} \sqrt{l^2 - \bar{m}^2}, \\
f_{l, l, \bar{m}} &=& c_{l, l, \bar{m}}, \\
f_{l+1, l, \bar{m}} &=& c_{l+1, l, \bar{m}} \sqrt{(l + 1)^2 - \bar{m}^2}.
\end{array} \right.
\]

(28)

In general, the matrix \( \Lambda_3 \) must be a reducible representation of the proper Lorentz group \( \mathfrak{g}_+ \), and can always be written in the form

\[
\Lambda_3 = \begin{bmatrix} \Lambda_3^{11} & 0 & & \\ \Lambda_3^{12} & \Lambda_3^{22} & 0 & \\ & & \ddots & \ddots \\ 0 & & & \Lambda_3^{nn} \end{bmatrix},
\]

(29)

where \( \Lambda_3^{ij} \) is a spin block (the matrix \( \Lambda_3^* \) has the same decompositions). It is obvious that the matrices \( \Lambda_1, \Lambda_2 \) and \( \Lambda_1^*, \Lambda_2^* \) admit also the decompositions of the type \( (29) \) by definition. If the spin block \( \Lambda_3^{ij} \) has non-null roots, then the particle possesses the spin \( i \). Corresponding to the decomposition \( (28) \), the wave function also decomposes into a direct sum of component wave functions which we write

\[
\psi = \psi_{l_1 m_1} + \psi_{l_2 m_2} + \psi_{l_3 m_3} + \ldots.
\]

According to a de Broglie theory of fusion \([14]\), interlocking representations give rise to indecomposable relativistic wave equations. Otherwise, we have decomposable equations. As is known, the indecomposable wave equations correspond to composite particles. A relation between indecomposable wave equations and composite particles will be studied in a separate work.

Returning to the equations \( (22) \), we see that the first equation can be written in the form

\[
\sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} + i \sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} + k \psi = 0,
\]

\[
\sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} - i \sum_{j=1}^{3} \Lambda_j \frac{\partial \psi}{\partial a_j} + k \psi = 0,
\]

(30)

where \( g_1 = a_1, g_2 = a_2, g_3 = a_3, g_4 = ia_1, g_5 = ia_2, g_6 = ia_3, a_1^* = -ig_4, a_2^* = -ig_5, a_3^* = -ig_6, \) and \( \tilde{a}_j, \tilde{a}_j^* \) are the parameters corresponding to the dual basis. In essence, the
equations (30) are defined in a three-dimensional complex space $\mathbb{C}^3$. In turn, the space $\mathbb{C}^3$ is isometric to a 6-dimensional bivector space $\mathbb{R}^6$ (a parameter space of the Lorentz group [17]). The bivector space $\mathbb{R}^6$ is a tangent space of the group manifold $\mathcal{L}_6$ of the Lorentz group, that is, the manifold $\mathcal{L}_6$ in each its point is equivalent locally to the space $\mathbb{R}^6$. Thus, for all $g \in \mathcal{L}_6$ we have $T_g \mathcal{L}_6 \simeq \mathbb{R}^6$. There exists a close relationship between the metrics of the Minkowski spacetime $\mathbb{R}^{1,3}$ and the metrics of $\mathbb{R}^6$ (see Appendix). In the case of $\mathbb{R}^{1,3}$ with the metric tensor

$$
 g_{\alpha \beta} = \begin{pmatrix}
 -1 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 \\
 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 1 \\
 \end{pmatrix}
$$

in virtue of (33) for the metric tensor of $\mathbb{R}^6$ we obtain

$$
 g_{ab} = \begin{pmatrix}
 -1 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 \end{pmatrix},
$$

(31)

where the order of collective indices in $\mathbb{R}^6$ is $23 \to 0, 10 \to 1, 20 \to 2, 30 \to 3, 31 \to 4, 12 \to 5$. The system (30) in the bivector space is written as follows:

$$
 \sum_i \left[ g_{i1} A_1 T_g^{-1} \frac{\partial \psi'}{\partial a_i} + g_{i2} A_2 T_g^{-1} \frac{\partial \psi'}{\partial a_i} + g_{i3} A_3 T_g^{-1} \frac{\partial \psi'}{\partial a_i} - i g_{i1} A_1 T_g^{-1} \frac{\partial \psi'}{\partial a_i'} - i g_{i2} A_2 T_g^{-1} \frac{\partial \psi'}{\partial a_i'} - i g_{i3} A_3 T_g^{-1} \frac{\partial \psi'}{\partial a_i'} \right] + \kappa c T_g^{-1} \psi' = 0,
$$

$$
 \sum_i \left[ g_{i1} A_1^* T_g^{-1} \frac{\partial \psi'}{\partial a_i} + g_{i2} A_2^* T_g^{-1} \frac{\partial \psi'}{\partial a_i} + g_{i3} A_3^* T_g^{-1} \frac{\partial \psi'}{\partial a_i} + i g_{i1} A_1^* T_g^{-1} \frac{\partial \psi'}{\partial a_i'} + i g_{i2} A_2^* T_g^{-1} \frac{\partial \psi'}{\partial a_i'} + i g_{i3} A_3^* T_g^{-1} \frac{\partial \psi'}{\partial a_i'} \right] + \kappa c T_g^{-1} \psi' = 0,
$$

where

$$
 \frac{\partial}{\partial a_1} = \frac{\sin \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\cos \varphi^c \cos \theta^c}{r} \frac{\partial}{\partial \theta} + \frac{\cos \varphi^c \sin \theta^c}{r} \frac{\partial}{\partial r},
$$

(32)

$$
 \frac{\partial}{\partial a_2} = \frac{\cos \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\sin \varphi^c \cos \theta^c}{r} \frac{\partial}{\partial \theta} + \frac{\sin \varphi^c \sin \theta^c}{r} \frac{\partial}{\partial r},
$$

(33)

$$
 \frac{\partial}{\partial a_3} = -\frac{\sin \theta^c}{r} \frac{\partial}{\partial \theta} + \frac{\cos \theta^c}{r} \frac{\partial}{\partial r}.
$$

(34)

$$
 \frac{\partial}{\partial a_1^*} = i \frac{\partial}{\partial a_1} = -\frac{\sin \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\cos \varphi^c \sin \theta^c}{r} \frac{\partial}{\partial \theta} + \frac{i \cos \varphi^c \sin \theta^c}{r} \frac{\partial}{\partial r},
$$

(35)

$$
 \frac{\partial}{\partial a_2^*} = i \frac{\partial}{\partial a_2} = \frac{\cos \varphi^c}{r \sin \theta^c} \frac{\partial}{\partial \varphi} + \frac{\sin \varphi^c \cos \theta^c}{r} \frac{\partial}{\partial \theta} + i \sin \varphi^c \sin \theta^c \frac{\partial}{\partial r},
$$

(36)

$$
 \frac{\partial}{\partial a_3^*} = i \frac{\partial}{\partial a_3} = -\frac{\sin \theta^c}{r} \frac{\partial}{\partial \theta} + i \cos \theta^c \frac{\partial}{\partial r}.
$$

(37)
\[
\begin{align*}
\frac{\partial}{\partial a_1} &= -\frac{\sin \theta}{r^* \sin \theta \partial \varphi} + \frac{\cos \theta \cos \varphi}{r^*} \frac{\partial}{\partial \theta} + \frac{\cos \theta \sin \varphi}{r^*} \frac{\partial}{\partial r^*}, \\
\frac{\partial}{\partial a_2} &= \frac{\cos \theta}{r^* \sin \theta \partial \varphi} + \frac{\sin \theta \cos \varphi}{r^*} \frac{\partial}{\partial \theta} + \frac{\sin \theta \sin \varphi}{r^*} \frac{\partial}{\partial r^*}, \\
\frac{\partial}{\partial a_3} &= -\frac{\sin \theta}{r^*} \frac{\partial}{\partial \theta} + \cos \theta \frac{\partial}{\partial r^*}.
\end{align*}
\]

\[
\frac{\partial}{\partial a_1}^* = -i \frac{\partial}{\partial a_1} - \frac{\sin \theta}{r^* \sin \theta \partial \varphi} - \frac{\cos \theta \cos \varphi}{r^*} \frac{\partial}{\partial \theta} - i \cos \theta \sin \varphi \frac{\partial}{\partial r^*}, \\
\frac{\partial}{\partial a_2}^* = -i \frac{\partial}{\partial a_2} - \frac{\cos \theta}{r^* \sin \theta \partial \varphi} - \frac{\sin \theta \cos \varphi}{r^*} \frac{\partial}{\partial \theta} - i \sin \theta \sin \varphi \frac{\partial}{\partial r^*}, \\
\frac{\partial}{\partial a_3}^* = -i \frac{\partial}{\partial a_3} - \frac{\sin \theta}{r^*} \frac{\partial}{\partial \theta} - i \cos \theta \frac{\partial}{\partial r^*}
\]

are derivatives defined on the spheres (10) and (11). Solutions of wave equations for extended object are found in the form of series in associated hyperspherical functions which defined on the spheres $S_\varphi^2$ and $S_\epsilon^2$:

\[
\begin{align*}
\psi_{lm;n}^k &= f_{lnk}(r)m_{ln}(\varphi, \epsilon, \theta, \tau, 0, 0), \\
\psi_{lm;n}^k &= f_{lnk}(r^*)m_{ln}(\varphi, \epsilon, \theta, \tau, 0, 0),
\end{align*}
\]

where $l_0 \geq l$, $-l_0 \leq m$, $n \leq l_0$ and $l_0 \geq \hat{l}$, $-l_0 \leq \hat{m}$, $\hat{n} \leq \hat{l}_0$.

We claim that a system of Dirac like wave equations

\[
\begin{cases}
\Gamma_i \frac{\partial \psi(x)}{\partial x_i} + \kappa \psi(x) = 0, \\
\Gamma_i^* \frac{\partial \psi(x^*)}{\partial x_i} - \kappa \psi(x^*) = 0, \ (i = 0, \ldots, 3); \\
\Gamma_k \frac{\partial \psi(g)}{\partial g_k} + \kappa \psi(g) = 0, \\
\Gamma_k^* \frac{\partial \psi(g^*)}{\partial g_k} - \kappa \psi(g^*) = 0 \ (k = 1, \ldots, 6)
\end{cases}
\]

describes extended particles of any spin on the group manifold $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times S_6$ of the Poincaré group $\mathcal{P}$ (in particular, on the homogeneous space $\mathcal{M}_8 = \mathbb{R}^{1,3} \times S_6^2$). In its turn, the Ginzburg-Tamm system of Klein-Gordon like wave equations (2) describes extended particles on the homogeneous space $\mathcal{M} = \mathbb{R}^{1,3} \times H^3$ of $\mathcal{P}$, where $H^3$ is a 3-dimensional one-sheeted hyperboloid. A relation between the systems (44) and (2) is similar to a relation between usual Dirac and Klein-Gordon wave equations for the point-like particles. Namely, equations (44) (differential equations of the first order) can be understood as ‘square roots’ of the equations (2) (differential equations of the second order).
3.1 The Dirac field

In accordance with the general Fermi-scheme (Fig. 1) of the interlocking representations of $\mathfrak{g}_+$ the field $(1/2, 0) \oplus (0, 1/2)$ is defined within the following chain:

$$
\begin{pmatrix}
\frac{1}{2} \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
\frac{1}{2}
\end{pmatrix}.
$$

We start with the Lagrangian (18) on the group manifold $\mathcal{M}_{10}$:

$$
\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Gamma_{\mu} \frac{\partial \psi(\alpha)}{\partial x_{\mu}} - \frac{\partial \overline{\psi}(\alpha)}{\partial x_{\mu}} \Gamma_{\mu} \psi(\alpha) \right) - \frac{1}{2} \left( \overline{\psi}(\alpha) \gamma_{\nu} \frac{\partial \psi(\alpha)}{\partial g_{\nu}} - \frac{\partial \overline{\psi}(\alpha)}{\partial g_{\nu}} \gamma_{\nu} \psi(\alpha) \right) - \kappa \overline{\psi}(\alpha) \psi(\alpha),
$$

where $\psi(\alpha) = \psi(x) \psi(g) (\mu = 0, 1, 2, 3, \nu = 1, \ldots, 6)$, and

$$
\gamma_0 = \begin{pmatrix}
\sigma_0 & 0 \\
0 & -\sigma_0
\end{pmatrix}, \quad \gamma_1 = \begin{pmatrix}
0 & \sigma_1 \\
-\sigma_1 & 0
\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}
0 & \sigma_2 \\
-\sigma_2 & 0
\end{pmatrix}, \quad \gamma_3 = \begin{pmatrix}
0 & \sigma_3 \\
-\sigma_3 & 0
\end{pmatrix},
$$

$$
\gamma_4 = \begin{pmatrix}
0 & \Lambda_1 \iota \\
\iota \Lambda_1 & 0
\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}
0 & \Lambda_2 \iota \\
\iota \Lambda_2 & 0
\end{pmatrix}, \quad \gamma_6 = \begin{pmatrix}
0 & \Lambda_3 \iota \\
\iota \Lambda_3 & 0
\end{pmatrix},
$$

where $\sigma_i$ are the Pauli matrices, and the matrices $\Lambda_j$ and $\Lambda_j^*$ are derived from (23)–(25) and (26)–(28) at $l = 1/2$:

$$
\Lambda_1 = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \Lambda_2 = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \Lambda_3 = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
$$

$$
\Lambda_1^* = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \Lambda_2^* = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}, \quad \Lambda_3^* = \frac{1}{2} \hat{c}_{1/2} \begin{pmatrix}
1 & 1 \\
0 & -1
\end{pmatrix}.
$$

It is easy to see that these matrices coincide with the Pauli matrices $\sigma_i$ when $c_{1/2} = 2$. The Dirac bispinor $\psi = (\psi_1, \psi_2, \psi_1^\dagger, \psi_2^\dagger)^T$ is defined on $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times S_c^2$ by the following components [63]:

$$
\psi_{1n}^l(\alpha) = \psi_1^l(x) \psi_{1n}^l(g) = u_1(p) e^{-ipx} f_{1/2}^l (\text{Re} r) \mathcal{M}_{1/2,n}^l (\varphi, \epsilon, \theta, \tau, 0, 0),
$$
$$
\psi_{2n}^l(\alpha) = \psi_2^l(x) \psi_{2n}^l(g) = \pm u_2(p) e^{-ipx} f_{1/2}^l (\text{Re} r) \mathcal{M}_{-1/2,n}^l (\varphi, \epsilon, \theta, \tau, 0, 0),
$$
$$
\psi_{1n}^l(\alpha) = \psi_1^l(x) \psi_{1n}^l(g) = \mp v_1(p) e^{ipx} f_{-1/2}^l (\text{Re} r^*) \mathcal{M}_{1/2,n}^l (\varphi, \epsilon, \theta, \tau, 0, 0),
$$
$$
\psi_{2n}^l(\alpha) = \psi_2^l(x) \psi_{2n}^l(g) = v_2(p) e^{ipx} f_{-1/2}^l (\text{Re} r^*) \mathcal{M}_{-1/2,n}^l (\varphi, \epsilon, \theta, \tau, 0, 0),
$$
where
\[ u_1(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_z + E+m}{p_z - E+m} \end{pmatrix}, \quad u_2(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_z - E+m}{p_z + E+m} \end{pmatrix}, \]
\[ v_1(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} \frac{p_z + E+m}{1} \\ \frac{p_z - E+m}{1} \end{pmatrix}, \quad v_2(p) = \left( \frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & \frac{p_z + E+m}{p_z - E+m} \end{pmatrix}, \]

here \( p_{\pm} = p_x \pm ip_y \). Radial functions have the form
\[ f_{l,\pm}^i(Re \, r) = C_1 \sqrt{\kappa c} Re \, J_l \left( \sqrt{\kappa c} Re \, r \right) + C_2 \sqrt{\kappa c} Re \, J_{l+1} \left( \sqrt{\kappa c} Re \, r \right), \]
\[ f_{l,\mp}^i(Re \, r^*) = C_1 \frac{1}{2} \sqrt{\kappa c} Re \, r^* J_{l+1} \left( \sqrt{\kappa c} Re \, r^* \right) - C_2 \frac{1}{2} \sqrt{\kappa c} Re \, r^* J_{l-1} \left( \sqrt{\kappa c} Re \, r^* \right), \]

where \( J_l \left( \sqrt{\kappa c} Re \, r \right) \) are the Bessel functions of half-integer order, and
\[ l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots; \]
\[ i = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots; \]
\[ \mathfrak{M}_i^\frac{1}{2}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{i \frac{\tau}{2}(\epsilon + i \varphi)} \mathfrak{Y}_i^{\frac{1}{2}}(\theta, \tau), \]
\[ \mathfrak{Y}_i^{\frac{1}{2}}(\theta, \tau) = \cos^2 \frac{\theta}{2} \cosh^2 \frac{\tau}{2} \sum_{k=-l}^{l} i^{\frac{1}{2} - k} \tan^{\frac{1}{2} - k} \theta \tan^{-k} \frac{\tau}{2} \times \]
\[ _2F_1 \left( \begin{array}{c} \pm \frac{1}{2} - l + 1, 1 - l - k \\ \pm \frac{1}{2} - k + 1 \end{array} \middle| i^2 \tan^2 \frac{\theta}{2} \right) _2F_1 \left( \begin{array}{c} -l + 1, 1 - l - k \\ -k + 1 \end{array} \middle| \tanh^2 \frac{\tau}{2} \right), \]
\[ \mathfrak{M}_i^\frac{1}{2}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{-i \frac{\tau}{2}(\epsilon + i \varphi)} \mathfrak{Y}_i^{\frac{1}{2}}(\theta, \tau), \]
\[ \mathfrak{Y}_i^{\frac{1}{2}}(\theta, \tau) = \cos^2 \frac{\theta}{2} \cosh^2 \frac{\tau}{2} \sum_{k=-i}^{i} i^{\frac{1}{2} - k} \tan^{\frac{1}{2} - k} \theta \tan^{-k} \frac{\tau}{2} \times \]
\[ _2F_1 \left( \begin{array}{c} \pm \frac{1}{2} - \hat{i} + 1, 1 - \hat{i} - \hat{k} \\ \pm \frac{1}{2} - \hat{k} + 1 \end{array} \middle| i^2 \tan^2 \frac{\theta}{2} \right) _2F_1 \left( \begin{array}{c} -\hat{i} + 1, 1 - \hat{i} - \hat{k} \\ -\hat{k} + 1 \end{array} \middle| \tanh^2 \frac{\tau}{2} \right). \]
are the associated hyperspherical functions defined on the spheres $S_2^c$ and $\tilde{S}_2^c$. General solutions are found in the form of generalized Fourier integrals

$$
\psi_1(\alpha) = \sum_{l=\frac{1}{2}}^{\infty} \sum_{n=-l}^{l}(\text{Re} a) \int_{-\frac{1}{2}T^4} F_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x,
$$
$$
\psi_2(\alpha) = \sum_{l=\frac{1}{2}}^{\infty} \sum_{n=-l}^{l}(\text{Re} a) \int_{-\frac{1}{2}T^4} F_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x,
$$
$$
\dot{\psi}_1(\alpha) = \sum_{l=\frac{1}{2}}^{\infty} \sum_{n=-l}^{l}(\text{Re} a^*) \int_{-\frac{1}{2}T^4} F_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x,
$$
$$
\dot{\psi}_2(\alpha) = \sum_{l=\frac{1}{2}}^{\infty} \sum_{n=-l}^{l}(\text{Re} a^*) \int_{-\frac{1}{2}T^4} \bar{F}_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x,
$$

where

$$
\alpha_{l,n}^\pm = \frac{(-1)^n(2l+1)(2\dot{l}+1)}{32\pi^4 \int_{-\frac{1}{2}T^4} (\text{Re} a) d^4x} \int_{-\frac{1}{2}T^4} F_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x d^4y,
$$

$$
\alpha_{l,n}^\pm = \frac{(-1)^n(2l+1)(2\dot{l}+1)}{32\pi^4 \int_{-\frac{1}{2}T^4} (\text{Re} a^*) d^4x} \int_{-\frac{1}{2}T^4} \bar{F}_{\frac{1}{2}l-n}(\varphi, \psi, \tau, 0, 0) d^4x d^4y.
$$

### 3.2 The Maxwell field

In accordance with the general Bose-scheme of the interlocking representations of $\mathfrak{G}_+$ (see Fig.1), the field $(1, 0) \oplus (0, 1)$ is defined within the following interlocking scheme:

$$
(1, 0) \longleftrightarrow \left( \frac{1}{2}, \frac{1}{2} \right) \longleftrightarrow (0, 1).
$$

We start with the Lagrangian (18) on the group manifold $\mathcal{M}_{10}$. Let us rewrite (18) in the form

$$
\mathcal{L}(\alpha) = -\frac{1}{2} \left( \frac{\partial \phi(\alpha)}{\partial x^\mu} \Gamma_{\mu} - \frac{\partial \phi(\alpha)}{\partial x^\mu} \Gamma_{\mu} \phi(\alpha) \right) - \frac{1}{2} \left( \frac{\partial \phi(\alpha)}{\partial q^\nu} \Gamma_{\nu} - \frac{\partial \phi(\alpha)}{\partial q^\nu} \Gamma_{\nu} \phi(\alpha) \right),
$$

where $\phi(\alpha) = \phi(x) \phi(g)$ ($\mu = 0, 1, 2, 3, \nu = 1, \ldots, 6$), and

$$
\Gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & -\alpha_1 \\ \alpha_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & -\alpha_3 \\ \alpha_3 & 0 \end{pmatrix},
$$

$$
\gamma_1 = \begin{pmatrix} 0 & \Lambda_1^* \\ \Lambda_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \Lambda_2^* \\ \Lambda_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \Lambda_3^* \\ \Lambda_3 & 0 \end{pmatrix},
$$

19
\[ \gamma_4 = \begin{pmatrix} 0 & i\Lambda_1^* \\ i\Lambda_1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & i\Lambda_2^* \\ i\Lambda_2 & 0 \end{pmatrix}, \quad \gamma_6 = \begin{pmatrix} 0 & i\Lambda_3^* \\ i\Lambda_3 & 0 \end{pmatrix}, \] (53)

where
\[ \alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (54)

and the matrices \( \Lambda_j \) and \( \Lambda_j^* \) are derived from (23)–(25) and (26)–(28) at \( l = 1 \):
\[ \Lambda_1 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_3 = c_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \] (55)
\[ \Lambda_1^* = \frac{\sqrt{2}}{2} \epsilon_{11} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2^* = \frac{\sqrt{2}}{2} \epsilon_{11} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_3^* = \epsilon_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \] (56)

At this point, electromagnetic field should be defined in the Riemann-Silberstein representation \[69, 57, 10]. The Riemann-Silberstein (Majorana-Oppenheimer) representation considered during long time by many authors \[40, 46, 28, 42, 51, 41, 13, 23\]. The interest to this formulation of electrodynamics has grown in recent years \[32, 58, 22, 17\]. One of the main advantages of this approach lies in the fact that Dirac and Maxwell fields are derived similarly from the Dirac-like Lagrangians. These fields have the analogous mathematical structure, namely, they are the functions on the Poincaré group. This circumstance allows us to consider the fields \((1/2,0) \oplus (0,1/2)\) and \((1,0) \oplus (0,1)\) on an equal footing, from the one group theoretical viewpoint.

In 1949, Newton and Wigner \[44\] showed that for the photon there exist no localized states. Therefore, a massless field \((1,0) \oplus (0,1)\) should be considered within the unitary infinite-dimensional representation of the Lorentz group. In accordance with the Naimark theorem \[13\], the representation \( S_k \) of the subgroup \( SU(2) \) is contained in \( \mathfrak{G}_{\lambda,\rho} \), not more than one time. Matrix elements of the representation \( \mathfrak{G}_{\lambda,\rho} \) are defined by the functions \( \Phi \).

We suppose that the photon field is described within the infinite-dimensional representation \( \mathfrak{G}_{\lambda,\rho} \). This representation contains a finite-dimensional representation associated with the field \((1,0) \oplus (0,1)\).

Since longitudinal solutions \( \phi_{0,n}^{-\frac{1}{2}+i\rho,\lambda} (\alpha) \) and \( \phi_{0,\bar{n}}^{-\frac{1}{2}-i\rho,\lambda} (\alpha) \) do not contribute to a real photon due to their transversality conditions, then particular solutions for the Majorana-Oppenheimer bispinor \( \Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T \) on \( \mathcal{M}_8 \) are defined by the following expressions \[64\]:
\[ \phi_{\pm,1,n}^{-\frac{1}{2}+i\rho,\lambda} (\alpha) = \phi_\pm (k; x, t) \phi_{\pm,1,n}^{-\frac{1}{2}+i\rho,\lambda} (g) = \{2(2\pi)^3\}^{-\frac{1}{2}} \frac{1}{\epsilon_\pm (k)} \exp[i(k \cdot x - \omega t)] f_{1,\pm,1}^{-\frac{1}{2}+i\rho,\lambda} (r) \mathfrak{M}_{\pm,1,n}^{-\frac{1}{2}+i\rho,\lambda} (\varphi, \epsilon, \theta, \tau, 0, 0), \]
\[ \phi_{\pm,1,n}^{-\frac{1}{2}-i\rho,\lambda} (\alpha) = \phi_\pm^* (k; x, t) \phi_{\pm,1,n}^{-\frac{1}{2}-i\rho,\lambda} (g) = \{2(2\pi)^3\}^{-\frac{1}{2}} \frac{1}{\epsilon_\pm^* (k)} \exp[-i(k \cdot x - \omega t)] f_{1,\pm,1}^{-\frac{1}{2}-i\rho,\lambda} (r^*) \mathfrak{M}_{\pm,1,n}^{-\frac{1}{2}-i\rho,\lambda} (\varphi, \epsilon, \theta, \tau, 0, 0), \]
where

$$\varepsilon_{\pm}(\mathbf{k}) = \left\{ \begin{array}{ll} 2|\mathbf{k}|^2(k_1^2 + k_2^2) \end{array} \right\}^{-\frac{1}{2}} \left[ \begin{array}{c} -k_1k_3 \pm ik_2|\mathbf{k}| \\ -k_2k_3 \mp ik_1|\mathbf{k}| \\ k_1^2 + k_2^2 \end{array} \right]$$

are the polarization vectors of a photon. Radial functions have the form

$$f_{1,1}^{\frac{1}{2}+i\rho, l_0}(r) = C\sqrt{r} + \sqrt{\left( l_0 + i\rho + \frac{1}{2} \right) \left( \frac{l_0}{2} + \frac{i\rho}{2} + \frac{5}{4} \right) r},$$

$$f_{1,1}^{\frac{1}{2}-i\rho, l_0}(r^*) = C'\sqrt{r^*} + \sqrt{\left( l_0 - i\rho + \frac{3}{2} \right) \left( \frac{l_0}{2} - \frac{i\rho}{2} + \frac{7}{4} \right) r^*},$$

and

$$M_{\pm 1,n}^{\frac{1}{2}+i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp(i+\varphi)\frac{1}{2}+i\rho, l_0}(\varphi, \epsilon, \theta, \tau),$$

$$M_{\pm 1,n}^{\frac{1}{2}-i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp(i-\varphi)\frac{1}{2}-i\rho, l_0}(\varphi, \epsilon, \theta, \tau),$$

are associated hyperspherical functions defined on the spheres $S^2_\epsilon$ and $S^2_\epsilon$. General solutions are found via the following generalized Fourier integrals:

$$\Phi_{\pm 1}(\alpha) = \left\{ 2(2\pi)^3 \right\}^{\frac{1}{2}} \sum_{l_0=1}^{\infty} f_{1,\pm 1}^{\frac{1}{2}+i\rho, l_0}(r) \sum_{n=-l_0}^{l_0} \int \left( \frac{\varepsilon_{\pm}(\mathbf{k})}{\varepsilon_{\pm}(\mathbf{k})} \right) e^{i\kappa x} \alpha_{l_0,n}^{\frac{1}{2}+i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) d^4x,$$

$$\dot{\Phi}_{\pm 1}(\alpha) = \left\{ 2(2\pi)^3 \right\}^{\frac{1}{2}} \sum_{l_0=1}^{\infty} f_{1,\pm 1}^{\frac{1}{2}-i\rho, l_0}(r^*) \sum_{\hat{n}=-l_0}^{l_0} \int \left( \frac{\varepsilon_{\pm}^*(\mathbf{k})}{\varepsilon_{\pm}^*(\mathbf{k})} \right) e^{-i\kappa x} \alpha_{l_0,n}^{\frac{1}{2}-i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) d^4x,$$

where

$$\alpha_{l_0,n}^{\frac{1}{2}+i\rho} = \frac{(-1)^n(2l_0 + 1)^2}{32\pi^4 f_{1,\pm 1}^{\frac{1}{2}+i\rho, l_0}(a)} \int_{S^2_\epsilon} \int F_{1,1}(\alpha) e^{-i\kappa x} M_{\pm 1,n}^{\frac{1}{2}+i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) d^4x d^4g,$$

$$\alpha_{l_0,n}^{\frac{1}{2}-i\rho} = \frac{(-1)^n(2l_0 + 1)^2}{32\pi^4 f_{1,\pm 1}^{\frac{1}{2}-i\rho, l_0}(a^*)} \int_{S^2_\epsilon} \int \dot{F}_{1,1}(\alpha) e^{i\kappa x} M_{\pm 1,n}^{\frac{1}{2}-i\rho, l_0}(\varphi, \epsilon, \theta, \tau, 0, 0) d^4x d^4g.$$
4 Interaction

Up to now we analyze free Dirac and Maxwell fields. Let us consider an interaction between these fields. As usual, interactions between the fields are described by an interaction Lagrangian \( \mathcal{L}_I \). In our case we take the following Lagrangian \([60]\):

\[
\mathcal{L}_I (\alpha) = \mu (\overline{\psi}(\alpha) \sigma^{D}_{\mu \nu} \psi(\alpha) ) (\Xi^M_{\mu \nu \kappa} \phi(\alpha)),
\]

where \( \sigma^{D}_{\mu \nu} = \frac{1}{2} (\Xi^D_{\mu \nu} - \Xi^D_{\nu \mu}) \) and \( \Xi^D = (\Gamma^D_0, \Gamma^D_1, \Gamma^D_2, \Gamma^D_3, \gamma^D_1, \gamma^D_2, \gamma^D_3, \gamma^D_4, \gamma^D_5, \gamma^D_6) \), \( \Xi^M = (\Gamma^M_0, \Gamma^M_1, \Gamma^M_2, \Gamma^M_3, \gamma^M_1, \gamma^M_2, \gamma^M_3, \gamma^M_4, \gamma^M_5, \gamma^M_6) \), here \( \Gamma^D \) and \( \gamma^D \) are the matrices \([46]\) and \([47]\)–\([48]\), and \( \Gamma^M \) and \( \gamma^M \) are the matrices \([51]\) and \([52]\)–\([53]\).

The full Lagrangian of interacting Dirac and Maxwell fields equals to a sum of the free field Lagrangians and the interaction Lagrangian:

\[
\mathcal{L}(\alpha) = \mathcal{L}_D(\alpha) + \mathcal{L}_M(\alpha) + \mathcal{L}_I(\alpha),
\]

where \( \mathcal{L}_D(\alpha) \) and \( \mathcal{L}_M(\alpha) \) are of the type \([45]\) and \([50]\), respectively. Or,

\[
\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Xi^D_{\mu \nu} \frac{\partial \psi(\alpha)}{\partial \alpha^\mu} - \frac{\partial \overline{\psi}(\alpha)}{\partial \alpha^\mu} \Xi^D_{\mu \nu} \psi(\alpha) \right) - \frac{1}{2} \left( \overline{\phi}(\alpha) \Xi^M_{\mu \nu} \frac{\partial \phi(\alpha)}{\partial \alpha^\mu} - \frac{\partial \overline{\phi}(\alpha)}{\partial \alpha^\mu} \Xi^M_{\mu \nu} \phi(\alpha) \right) - \kappa \overline{\psi}(\alpha) \psi(\alpha) + \mu (\overline{\psi}(\alpha) \sigma^{D}_{\mu \nu} \psi(\alpha) ) (\Xi^M_{\mu \nu \kappa} \phi(\alpha)).
\]

Since the Lagrangian \([57]\) does not contain derivatives on the field functions, then for a Hamiltonian density we have \( \mathcal{H}_I(\alpha) = -\mathcal{L}_I(\alpha) \).

As is known, in the standard quantum field theory the S-matrix is expressed via the Dyson formula \([53]\)

\[
S = T \left[ \exp \left( -\frac{i}{\hbar c} \int_{-\infty}^{+\infty} \mathcal{H}_I(x) d^4 x \right) \right],
\]

where \( T \) is the time ordering operator.

In our case, the electron-positron and photon fields are defined on the space \( \mathcal{M}_8 = \mathbb{R}^{1,3} \times S^2_\epsilon \) which larger then the Minkowski space \( \mathbb{R}^{1,3} \). With a view to define a formula similar to the equation \([58]\) it is necessarily to replace \( d^4 x \) by the following invariant measure on \( \mathcal{M}_8 \):

\[
d^8 \mu = d^4 x d^4 \mathbf{g},
\]

where

\[
d^4 \mathbf{g} = \sin \theta^c d\theta d\tau d\varphi d\epsilon.
\]

Therefore, an analogue of the Dyson formula \([58]\) on the manifold \( \mathcal{M}_8 \) can be written as follows

\[
S = T \left[ \exp \left( -\frac{i}{\hbar c} \int_{T_\epsilon} \int_{S^2_\epsilon} \mathcal{H}_I(\alpha) d^4 x d^4 \mathbf{g} \right) \right].
\]

22
Further, using an explicit expression for the associated hyperspherical function

\[ 3^n_l(\cos \theta^c) = i^n \sqrt{\frac{\Gamma(l - m + 1)}{\Gamma(l + m + 1)}} \cos^m \frac{\theta^c}{2} \sin^m \frac{\theta^c}{2} \times \sum_{j=0}^{l-m} \frac{(-1)^j \Gamma(l + m + j + 1)}{\Gamma(j + 1) \Gamma(m + j + 1) \Gamma(l - m - j + 1)} \sin^{2j} \frac{\theta^c}{2}, \]

that is, to investigate convergence of the integral

\[ I_1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(m_e \varphi^e - \hat{m}_e \varphi^e - m_f \varphi^f)} 3_{l_e}^{\hat{m}_e}(\cos \hat{\theta}^e) 3_{l_e}^{m_e}(\cos \theta^e) 3_{l_f}^{m_f}(\cos \theta^c) \sin \theta^c d\theta^c d\varphi^c. \]

Here the symbols \( l_e, m_e \) correspond to the electron field \( \psi(g) \), \( \hat{l}_e, \hat{m}_e \) correspond to the positron field \( \overline{\psi}(g) \), and \( m_f \) corresponds to the photon field \( \phi(g) \). It is obvious that convergence of the integral

\[ I_2 = \int_{-\infty}^{+\infty} e^{-i(m_e \varphi^e - \hat{m}_e \varphi^e - m_f \varphi^f)} d\varphi^c \]

is not difficult to investigate. Let us consider the following integral:

\[ I_3 = \int_{-\infty}^{+\infty} 3_{l_e}^{\hat{m}_e}(\cos \hat{\theta}^e) 3_{l_e}^{m_e}(\cos \theta^c) 3_{l_f}^{m_f}(\cos \theta^c) \sin \theta^c d\theta^c. \]

Rewriting hyperspherical functions via the hypergeometric functions, we obtain

\[ I_3 = \frac{\hat{l}_e^{m_e + \hat{m}_e + m_f}}{\Gamma(m_e + 1) \Gamma(m_e + 1) \Gamma(m_f + 1)} \left[ \frac{\Gamma(l_e + m_e + 1) \Gamma(l_e + \hat{m}_e + 1) \Gamma(l_e - m_f + 1)}{\Gamma(l_e - m_e + 1) \Gamma(l_e - \hat{m}_e + 1) \Gamma(l_e - m_f + 1)} \right] \times \left( \frac{\hat{m}_e + 1 - \hat{l}_e}{\hat{m}_e + 1} \right) \sin \frac{\theta^c}{2} \times \]

\[ _2F_1 \left( \begin{array}{c} l_e + m_e + 1, m_e - l_e \\ m_e + 1 \end{array} \right) \sin^2 \frac{\theta^c}{2} \times _2F_1 \left( \begin{array}{c} i\rho + m_f + 1, m_f - i\rho + 1 \\ m_f + 1 \end{array} \right) \sin^2 \frac{\theta^c}{2} \sin \theta^c d\theta^c. \]
we rewrite the first two hypergeometric functions in the integral (59). Then

\[ I_3 = \frac{i^{m_e + \tilde{m}_e + m_f}}{\Gamma(m_f + 1)} \sqrt{\frac{\Gamma(l_e - m_e + 1)\Gamma(\tilde{l}_e - \tilde{m}_e + 1)\Gamma(m_f + i\rho + \frac{1}{2})}{\Gamma(l_e + m_e + 1)\Gamma(\tilde{l}_e + \tilde{m}_e + 1)\Gamma(i\rho - m_f + \frac{1}{2})}} \]

\[
\sum_{j_1=0}^{l_e - m_e} \sum_{j_2=0}^{\tilde{l}_e - \tilde{m}_e} \frac{(-1)^{j_1 + j_2}\Gamma(l_e + m_e + j_1 + 1)}{\Gamma(j_1 + 1)\Gamma(j_2 + 1)\Gamma(m_e + j_1 + 1)\Gamma(m_e + j_2 + 1)} \times \\
\int_{-\infty}^{+\infty} \cos^{m_e + \tilde{m}_e + m_f} \frac{\theta^c}{2} \sin^{m_e + \tilde{m}_e + m_f + 2j_1 + 2j_2} \frac{\theta^c}{2} \times \\
_2F_1\left(\frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1}, \sin^2 \frac{\theta^c}{2}\right) \sin \theta^c d\theta^c.
\]

Making the substitution \( z = \cos \theta^c \) in the integral

\[ I_4 = \frac{1}{2^{m_e + \tilde{m}_e + m_f}} \int_{-\infty}^{+\infty} \sin^{m_e + \tilde{m}_e + m_f} \frac{\theta^c}{2} \sin^{2j_1 + 2j_2} \frac{\theta^c}{2} \times \\
_2F_1\left(\frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1}, \sin^2 \frac{\theta^c}{2}\right) \sin \theta^c d\theta^c,
\]

we obtain

\[ I_4 = \frac{1}{2^{m_e + \tilde{m}_e + m_f} + j_1 + j_2} \int_{-\infty}^{+\infty} \left(1 - z^2\right)^{m_e + \tilde{m}_e + m_f} \frac{\theta^c}{2} \times \\
_2F_1\left(\frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1}, \frac{1 - z}{2}\right) dz.
\]

Or,

\[ I_4 = \frac{1}{2^{m_e + \tilde{m}_e + m_f} + j_1 + j_2} \sum_{q=0}^{m_e + \tilde{m}_e + m_f} (-1)^q \frac{(m_e + \tilde{m}_e + m_f)!}{q!(m_e + \tilde{m}_e + m_f - q)!} \times \\
\int_{-\infty}^{+\infty} z^k \left(1 - z\right)^{j_1 + j_2} \frac{\theta^c}{2} \times \\
_2F_1\left(\frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1}, \frac{1 - z}{2}\right) dz.
\]

Introducing a new variable \( t = (1 - z)/2 \), we find

\[ I_4 = \frac{1}{2^{m_e + \tilde{m}_e + m_f} + j_1 + j_2} \sum_{q=0}^{m_e + \tilde{m}_e + m_f} (-1)^{q+1} \frac{(m_e + \tilde{m}_e + m_f)!}{q!(m_e + \tilde{m}_e + m_f - q)!} \times \\
\int_{-\infty}^{+\infty} (1 - 2t)^q t^{j_1 + j_2} \frac{\theta^c}{2} \times \\
_2F_1\left(\frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1}, \frac{1 - z}{2}\right) dt.
\]

24
Decomposing \((1 - 2t)^q\) via the Newton binomial, we obtain

\[
I_4 = \frac{1}{2^{m_e + \hat{m}_e + m_f + j_1 + j_2}} \sum_{q=0}^{m_e + \hat{m}_e + m_f} \sum_{p=0}^{q} (-1)^{q+p+1} \frac{(m_e + \hat{m}_e + m_f)!}{(m_e + \hat{m}_e + m_f - q)!} \times \]
\[
\frac{2^p}{p!(k - p)!} \int_{-\infty}^{+\infty} t^{j_1 + j_2 + p} _2F_1 \left( \frac{i\rho + m_f + \frac{1}{2}, m_f - i\rho + \frac{1}{2}}{m_f + 1} \right) dt.
\]

With the aim to calculate the latter integral we use the following formula \[^{49} \]:

\[
I_5 = \int t^n _2F_1 \left( \frac{a, b}{c} \left| t \right. \right) dt =
\]
\[
= n! \sum_{k=1}^{n+1} (-1)^{k+1} \frac{(c - k)_k t^{n-k+1}}{(a - k + 1)! (a - k_k) (b - k)_k} _2F_1 \left( \frac{a - k, b - k}{c - k} \left| t \right. \right).
\]

Then

\[
I_4 = \frac{1}{2^{m_e + \hat{m}_e + m_f + j_1 + j_2}} \sum_{q=0}^{m_e + \hat{m}_e + m_f} \sum_{p=0}^{q} (j_1 + j_2 + p)! \times
\]
\[
\sum_{k=1}^{j_1 + j_2 + p} \frac{(-1)^{q+p+k} 2^p (m_e + \hat{m}_e + m_f)!}{p!(k - p)! (m_e + \hat{m}_e + m_f - q)! (m_f + i\rho + \frac{3}{2} - k)!} \times
\]
\[
\frac{(m_f - k + 1)_k t^{j_1 + j_2 + p - k + 1}}{(m_f + i\rho - k + \frac{3}{2})_k (m_f - k + 1)_k} \times
\]
\[
_2F_1 \left( \frac{i\rho + m_f - k + \frac{3}{2}, m_f - i\rho - k + \frac{3}{2}}{m_f - k + 1} \right) dt.
\]

With the aim to investigate the convergence of \((59)\) let us apply the following asymptotic expansion for the hypergeometric function \[^{9} \]:

\[
_2F_1 \left( \frac{a, b}{c} \left| t \right. \right) = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} (-t)^{-a} _2F_1 \left( \frac{a, 1 - c + a}{1 - b + a} \left| \frac{1}{t} \right. \right) +
\]
\[
+ \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} (-t)^{-b} _2F_1 \left( \frac{b, 1 - c + b}{1 - a + b} \left| \frac{1}{t} \right. \right).
\]

Thus,

\[
I_3 = \frac{\Gamma(m_e + \hat{m}_e + m_f)}{\Gamma(m_f + 1)} \sqrt{\frac{\Gamma(l_e - m_e + 1) \Gamma(l_e - \hat{m}_e + 1) \Gamma(m_f + i\rho + \frac{3}{2})}{\Gamma(l_e + m_e + 1) \Gamma(l_e + \hat{m}_e + 1) \Gamma(i\rho - m_f + \frac{3}{2})}} \times
\]
\[
\sum_{j_1=0}^{l_e - m_e} \sum_{j_2=0}^{l_e - \hat{m}_e} \frac{(-1)^{j_1 + j_2} \Gamma(l_e + m_e + j_1 + 1) \Gamma(l_e + \hat{m}_e + j_2 + 1)}{j_1 + j_2 \Gamma(j_1 + 1) \Gamma(j_2 + 1) \Gamma(m_e + j_1 + 1) \Gamma(\hat{m}_e + j_2 + 1)} \times
\]
Appendix: Spinor groups and bivector spaces

As is known, rotations of pseudo-Euclidean spaces \( \mathbb{R}^{p,q} \) are defined by spinor groups [37]

\[
\text{Spin}(p,q) = \{ s \in \Gamma^+_{p,q} \mid N(s) = \pm 1 \},
\]

Our next paper will be devoted to these questions.

1. At this point, the spin-1 field (Maxwell field) is defined within an infinite-dimensional space of homogeneous space \( M \), and, therefore, in accordance with (58) the elements of \( S \)-matrix are defined by convergent expressions.

5 Summary

We have proved convergence of quantum electrodynamics on the Poincaré group in the case of homogeneous space \( \mathcal{M}_8 = \mathbb{R}^{1,3} \times S^2_c \). We considered Dirac like equations for extended objects on \( \mathcal{M}_8 \) and their particular solutions corresponding to the fields of spin 1/2 and 1. At this point, the spin-1 field (Maxwell field) is defined within an infinite-dimensional representation of the Lorentz group. We showed that an analogue of the Dyson formula for \( S \)-matrix in the case of \( \mathcal{M}_8 \) is defined by convergent integrals. It would be interesting to consider quantum field models and their convergence on other homogeneous spaces of the Poincaré group, such as \( \mathcal{M}_6 = \mathbb{R}^{1,3} \times S^2 \), \( \mathcal{M}_7 = \mathbb{R}^{1,3} \times H^3 \) and \( \mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{L}_6 \). It would be interesting also to consider wave equations and field models for the fields \( \psi(\alpha) = \langle x, q \mid \psi \rangle \) on the de Sitter group, where \( x \in \mathcal{T}_5 \) and \( q \in \text{Spin}_+(1,4) \simeq \text{Sp}(1,1) \), and for the fields \( \psi(\alpha) = \langle x, c \mid \psi \rangle \) on the conformal group, where \( x \in \mathcal{T}_6 \) and \( c \in \text{Spin}_+(2,6) \simeq \text{SU}(2,2) \).

Our next paper will be devoted to these questions.
where $\Gamma_{p,q}^+ = \Gamma_{p,q} \cap C_{p,q}^+$ is a special Lipschitz group, $s \in C_{p,q}^+$ is an even invertible element of the real Clifford algebra $C_{p,q}^+$. In more detail, the element $s$ is a linear combination of even basic elements, that is,

$$s = \sum_k a^{i_1 \ldots i_{2k}} e_{i_1 \ldots i_{2k}}.$$  

The condition $N(s) = \pm 1$ means

$$\sum_k \sigma(i_1) \ldots \sigma(i_{2k}) (a^{i_1 \ldots i_{2k}})^2 = \pm 1. \quad (60)$$

On the other hand, from the fact that the elements $s \in \Gamma_{p,q}^+$ are even products of $\nu = \sum \nu^i e_i$, it is easy to derive that coordinates of $s \in \text{Spin}(p, q)$ are related by the following conditions [50]:

$$a^{i_1 i_2 \ldots i_{2k}} (a)^{k-1} = (2k - 1)!! a^{[i_1 i_2 i_3 i_4] \ldots a^{i_{2k-1} i_{2k}}}. \quad (61)$$

Conditions (61) can be rewritten in the form

$$\begin{aligned}
a a^{i_1 i_2 i_3 i_4} &= 3!! a^{[i_1 i_2 i_3 i_4]}, \\
a a^{i_1 i_2 i_3 i_4 i_5 i_6} &= 5!! a^{[i_1 i_2 i_3 i_4 i_5 i_6]}, \\
&\vdots \\
a a^{i_1 i_2 \ldots i_{2k}} &= (2k - 1)!! a^{[i_1 i_2 \ldots i_{2k-1} i_{2k}]}.
\end{aligned} \quad (62)$$

Conditions (60) and (61) express all coordinates of the elements $s$, belonging to the spinor group $\text{Spin}(p, q)$, via $n(n-1)/2$ coordinates $a^{ij}$, where $n = p + q$. The number of the coordinates $a^{ij}$ coincides with the number of parameters of the rotation group of the space $\mathbb{R}^{p,q}$. This fact shows that expressions (60) and (61) form a full system of the conditions separating the spinor group $\text{Spin}(p, q)$ from the algebra $C_{p,q}^+$.

Further, parameters of the rotation group of the space $\mathbb{R}^{p,q}$ form a bivector space $\mathbb{R}^N$, where $N = \frac{n(n-1)}{2}$. Indeed, let $\mathbb{R}^{p,q}$ be the $n$-dimensional pseudo-Euclidean space, $p + q = n$. Let us evolve in $\mathbb{R}^{p,q}$ all the tensors satisfying the following two conditions: 1) a rank of the tensors is even; 2) covariant and contravariant indexes are divided into separate skewsymmetric pairs. Such tensors can be exemplified by bivectors (skewsymmetric tensors of the second rank). The set of all bivector tensor fields in $\mathbb{R}^{p,q}$ is called a bivector set, and its representation in a given point of $\mathbb{R}^{p,q}$ is called a local bivector set. In any tensor from the bivector set we take the each skewsymmetric pair $\alpha \beta$ as one collective index. At this point, from the two possible pairs $\alpha \beta$ and $\beta \alpha$ we fix only one, for example, $\alpha \beta$. The number of all collective indexes is equal to $N = \frac{n(n-1)}{2}$. In a given point, the bivector set of the space $\mathbb{R}^{p,q}$ with the contravariant components defines in the collective indexes a vector set, and the each vector of this set has $N$ components. Identifying these vectors with the points of the $N$-dimensional manifold, we come to an affine manifold $E^N$ if and only if this manifold admits a Klein geometry with the group

$$\eta^{a'} = A_a^{a'} \eta^a, \quad \eta^a = A_a^a \eta^{a'},$$

$$\det A_a^{a'} \neq 0, \quad A_a^a A_b^b = \delta_c^c,$$

where

$$A_a^{a'} \rightarrow A_{[\alpha}^{a'} A_{\beta]}^a.$$  

Thus, any local bivector set of the space $\mathbb{R}^{p,q}$ ($p + q = n$) can be mapped onto the affine space $E^N$. Therefore, $E^N$ is related with the each point of the space $\mathbb{R}^{p,q}$. The space $E^N$
is called a bivector space. It should be noted that the bivector space is a particular case of the most general mathematical construction called a Grassmannian manifold (a manifold of $m$-dimensional planes of the affine space). In the case $m = 2$ the manifold of two-dimensional planes is isometric to the bivector space, and the Grassmann coordinates in this case are called Pluecker coordinates.

The metrization of the bivector space $E^N$ is given by the formula (see [17])

$$g_{ab} ightarrow g_{a_{\beta}\gamma_{\delta}} \equiv g_{\alpha_{\gamma}} g_{\beta_{\delta}} - g_{\alpha_{\delta}} g_{\beta_{\gamma}},$$

where $g_{\alpha\beta}$ is a metric tensor of the space $\mathbb{R}^{p,q}$, and the collective indexes are skewsymmetric pairs $\alpha\beta \rightarrow a$, $\gamma\delta \rightarrow b$. After introduction of $g_{ab}$, the bivector affine space $E^N$ is transformed to a metric space $\mathbb{R}^N$.

As is known, any transformation from the rotation group of the space $\mathbb{R}^{p,q}$ can be represented via $\frac{n(n-1)}{2}$ transformations in the planes $(x_1, x_2)$, $(x_1, x_3)$, ..., $(x_{p+q}, x_1)$. The full number of the planes $(x_i, x_j)$ is equal to $N = \frac{n(n-1)}{2}$, and the each plane $(x_i, x_j)$ corresponds to the coordinate $a_{ij}$ of the spinor group $\text{Spin}(p,q)$.

References

[1] Altaisky, M. V.: Wavelet-Based Quantum Field Theory. SIGMA 3, 105–118 (2007); [arXiv:0711.1671] [hep-th] (2007).

[2] Altaisky, M. V.: Quantum field theory without divergences. Phys. Rev. D. 81, 125003 (2010); [arXiv:1002.2566] [hep-th] (2010).

[3] Amar, V., Dozzio, U.: Gel’fand-Yaglom Equations with Charge or Energy Density of Definite Sign. Nuovo Cimento A11, 87–99 (1972).

[4] Arnold, V. I.: Mathematical Methods of Classical Mechanics. Nauka, Moscow (1989) [in Russian].

[5] Arodz, H.: Metric tensors, Lagrangian formalism and Abelian gauge field on the Poincaré group. Acta Phys. Pol. B. 7, 177–190 (1976).

[6] Bacry, H., Kihlberg, A.: Wavefunctions on homogeneous spaces. J. Math. Phys. 10, 2132–2141 (1969).

[7] Bacry, H., Nuyts, J.: Mass-Spin Relation in a Lagrangian Model. Phys. Rev. 157, 1471–1472 (1967).

[8] Bargmann, V., Wigner, E. P.: Group theoretical discussion of relativistic wave equations. Proc. Nat. Acad. USA 34, 211–223 (1948).

[9] Bateman, H., Erdélyi, A.: Higher Transcendental Functions. vol. I. Mc Grow-Hill Book Company, New York (1953).

[10] Bialynicki-Birula, I.: Photon wave function. Progress in Optics, Vol. XXXVI, Ed. E. Wolf, Elsevier, Amsterdam (1996); [arXiv:quant-th/0508202] (2005).
[11] Biedenharn, L. C., Braden, H. W, Truini, P., van Dam, H.: *Relativistic wavefunctions on spinor spaces*. J. Phys. A: Math. Gen. **21**, 3593–3610 (1988).

[12] Boyer, C. P., Fleming, G. N.: *Quantum field theory on a seven-dimensional homogeneous space of the Poincaré group*. J. Math. Phys. **15**, 1007–1024 (1974).

[13] Da Silveira.: *Dirac-like equations for the photon*. Z. Naturforsh **A34**, 646–647 (1979).

[14] de Broglie L. Théorie Generale des Particules a Spin (Méthode de Fusion). Gauthier-Villars, Paris (1943).

[15] de Broglie, L.: *La théorie de la mesure en mécanique ondulatorie*. Gauthier-Villars, Paris (1957).

[16] Drechsler, W.: *Geometro-stohastically quantized fields with internal spin variables*. J. Math. Phys. **38**, 5531–5558 (1997); arXiv:gr-qc/9610046 (1996).

[17] Esposito, S.: *Covariant Majorana Formulation of Electrodynamics*. Found. Phys. **28**, 231–244 (1998); arXiv:hep-th/9704144 (1997).

[18] Finkelstein, D.: *Internal Structure of Spinning Particles*. Phys. Rev. **100**, 924–931 (1955).

[19] Fierz, M., Pauli, W.: *On Relativistic Wave Equations of Particles of Arbitrary Spin in an Electromagnetic Field*. Proc. Roy. Soc. (London) A. **173**, 211–232 (1939).

[20] Gel’fand, I. M., Minlos, R. A., Shapiro, Z. Ya.: Representations of the Rotation and Lorentz Groups and their Applications. Pergamon Press, Oxford (1963).

[21] Gel’fand, I. M., Graev, M. I., Vilenkin, N. Ya.: *Generalized Functions Vol. 5. Integral Geometry and Representation Theory*. Academic Press, New York, London (1986).

[22] Gersten, A.: *Maxwell equations as one-photon quantum equation*. Found. Phys. Lett. **12**, 291–298 (1998); arXiv:quant-ph/9911049 (1999).

[23] Giannetto, E.: *A Majorana-Oppenheimer Formulation of Quantum Electrodynamics*. Lettere al Nuovo Cimento **44**, 140–144 (1985).

[24] Ginzburg, V. L., Tamm, I. E.: *On the theory of spin*. Zh. Ehksp. Teor. Fiz. **17**, 227–237 (1947).

[25] Gitman, D. M., Shelepin, A. L.: *Fields on the Poincaré Group: Arbitrary Spin Description and Relativistic Wave Equations*. Int. J. Theor. Phys. **40**, 3, 603–684 (2001); arXiv:hep-th/0003146 (2000).

[26] Gitman, D. M., Shelepin, A. L.: *Field on the Poincaré group and quantum description of orientable objects*. Eur. Phys. J. C. **61**, 111–139 (2009); arXiv:0901.2537 [hep-th] (2009).

[27] Gitman, D. M., Shelepin, A. L.: *Classification of quantum relativistic orientable objects*. Phys. Scr. **83**, 015103 (2011); arXiv: 1001.5290 [hep-th] (2010).

[28] Good, R. H.: *Particle aspect of the electromagnetic field equations*. Phys. Rev. **105**, 1914 (1957).
[29] Grandpeix, J.-Y., Lurçat, F.: *Particle description of zero energy vacuum*. Found. Phys. **32**, 109–158 (2002); [arXiv:hep-th/0106229](https://arxiv.org/abs/hep-th/0106229) (2001).

[30] Huszar, M.: *Angular Momentum and Unitary Spinor Bases of the Lorentz Group*. Preprint JINR No. E2-5429, Dubna (1970).

[31] Huszar, M., Smorodinsky, J.: *Representations of the Lorentz Group on the Two-Dimensional Complex Sphere and Two-Particle States*. Preprint JINR No. E2-5020, Dubna (1970).

[32] Inagaki, T.: *Quantum-mechanical approach to a free photon*. Phys. Rev. **A49**, 2839–2843 (1994).

[33] Kaiser, G.: *Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis*. arXiv: 0910.0352 [math-ph] (2009).

[34] Kihlberg, A.: *Internal Co-ordinates and Explicit Representations of the Poincaré Group*. Nuovo Cimento **A53**, 592–609 (1968).

[35] Kihlberg, A.: *Fields on a homogeneous space of the Poincaré group*. Ann. Inst. Henri Poincaré. **13**, 57–76 (1970).

[36] Kuzenko, S. M., Lyakhovich, S. L., Segal, A. Yu.: *A geometric model of the arbitrary spin massive particle*. Int. J. Mod. Phys. A. **10**, 1529–1552 (1995); [arXiv:hep-th/9403196](https://arxiv.org/abs/hep-th/9403196) (1994).

[37] Lipschitz R.: Untersuchungen über die Summen von Quadraten. Max Cohen und Sohn, Bonn (1886).

[38] Lurçat, F.: *Quantum field theory and the dynamical role of spin*. Physics. **1**, 95 (1964).

[39] Lyakhovich, S. L., Segal, A. Yu., Sharapov, A. A.: *Universal model of a D = 4 spinning particles*. Phys. Rev. D. **54**, 5223–5238 (1996); [arXiv:hep-th/9603174](https://arxiv.org/abs/hep-th/9603174) (1996).

[40] Majorana, E.: *Scientific Papers*, unpublished, deposited at the “Domus Galileana”, Pisa, quaderno **2**, p.101/1; **3**, p.11, 160; **15**, p.16; **17**, p.83, 159.

[41] Mignani, R., Recami, E., Baldo, M.: *About a Dirac-Like Equation for the Photon according to Ettore Majorana*. Lettere al Nuovo Cimento **11**, 568–572 (1974).

[42] Moses, H. E.: *Solution of Maxwell’s Equations in Terms of a Spinor Notation: the Direct and Inverse Problem*. Phys. Rev. **113**, 1670–1679 (1959).

[43] Naimark, M. A.: *Linear Representations of the Lorentz Group*. Pergamon, London (1964).

[44] Newton, T. D., Wigner, E. P.: *Localized states for elementary systems*. Rev. Mod. Phys. **21**, 400 (1949).

[45] Nilsson, J., Beskow, A.: *The concept of wave function and irreducible representations of the Poincaré group*. Arkiv för Fysik **34**, 307–324 (1967).
[46] Oppenheimer, J. R.: *Note on light quanta and the electromagnetic field*. Phys. Rev. **38**, 725 (1931).

[47] Petrov, A. Z.: Einstein Spaces. Pergamon Press, Oxford (1969).

[48] Pletyukhov, V. A., Strazhev, V. I.: *On Dirac-like relativistic wave equations*. Russian J. Phys. n.12, 38-41 (1983).

[49] Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I.: Integrals and Series: Supplementary Chapters [in Russian]. Nauka, Moscow (1981).

[50] Rozenfeld, B. A.: Geometry of Lie groups. Dordrecht-Boston-London (1997).

[51] Sachs, M., Schwebel, S. L.: *On covariant formulation of the Maxwell-Lorentz theory of electromagnetism*. J. Math. Phys. **3**, 843–848 (1962).

[52] Sastry, R. R.: *Quantum Mechanics of Extended Objects*. arXiv:quant-ph/9903025 (1999).

[53] Schweber, S. S.: An Introduction to Relativistic Quantum Field Theory. Harper & Row, New York (1961).

[54] Segal, I. E., Zhou, Z.: *Convergence of nonlinear massive quantum field theory in the Einstein universe*. Ann. Phys. **218**, 2, 279–292 (1992).

[55] Segal, I. E., Zhou, Z.: *Convergence of Quantum Electrodynamics in a Curved Deformation of Minkowski Space*. Ann. Phys. **232**, 1, 61–87 (1994).

[56] Shirokov, Yu. M.: *Relativistskaia teoria spina*. Zh. Eksp. Teor. Fiz. **21**, 748–760 (1951).

[57] Silberstein, L.: *Elektromagnetische Grundgleichungen in bivectorieller Behandlung*. Ann. d. Phys. **22**, 579 (1907).

[58] Sipe, J. E.: *Photon wave functions*. Phys. Rev. **A52**, 1875–1883 (1995).

[59] Smorodinsky, Ya. A., Huszar, M.: *Representations of the Lorentz group and the generalization of helicity states*. Teor. Mat. Fiz. **4**, 3, 328–340 (1970).

[60] Todorov, N. S.: *Extended Particles Part I: Reformulation and Reinterpretation of the Dirac and Klein-Gordon Theories*. Annales de la Fondation Louis de Broglie **25**, 41–66 (2000).

[61] Toller, M.: *Free quantum fields on the Poincaré group*. J. Math. Phys. **37**, 2694–2730 (1996); arXiv:gr-qc/9602031 (1996).

[62] Varlamov, V. V.: *General Solutions of Relativistic Wave Equations*. Int. J. Theor. Phys. **42**, 3, 583–633 (2003); arXiv:math-ph/0209030 (2002).

[63] Varlamov, V. V.: *Relativistic wavefunctions on the Poincaré group*. J. Phys. A: Math. Gen. **37**, 5467–5476 (2004); arXiv:math-ph/0308038 (2003).

[64] Varlamov, V. V.: *Maxwell field on the Poincaré group*. Int. J. Mod. Phys. A. **20**, 17, 4095–4112 (2005); arXiv:math-ph/0310051 (2003).
[65] Varlamov V. V.: *Relativistic spherical functions on the Lorentz group*. J. Phys. A: Math. Gen. **39**, 805–822 (2006); [arXiv:math-ph/0507056](http://arxiv.org/abs/math-ph/0507056) (2005).

[66] Varlamov V. V.: *Towards the Quantum Electrodynamics on the Poincaré Group*. New Topics in Mathematical Physics Research (Ed. C. V. Benton) New York. – Nova Science Publishers, 2006. – P. 109–179; [arXiv:hep-th/0403070](http://arxiv.org/abs/hep-th/0403070) (2004).

[67] Varlamov, V. V.: *General Solutions of Relativistic Wave Equations II: Arbitrary Spin Chains*. Int. J. Theor. Phys. **46**, 4, 741–805 (2007); [arXiv:math-ph/0503058](http://arxiv.org/abs/math-ph/0503058) (2005).

[68] Vilenkin, N. Ya.: Special Functions and the Theory of Group Representations. AMS, Providence (1968).

[69] Weber, H.: Die partiellen Differential-Gleichungen der mathematischen Physik nach Riemann’s Vorlesungen. Friedrich Vieweg und Sohn, Braunschweig (1901).

[70] Yukawa, H.: *Quantum theory of non-local fields. I. Free fields*. Phys. Rev. **77**, 219–226 (1950).