ULTRALOCALLY CLOSED CLONES

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Abstract. Given a clone $C$ on a set $A$, we characterize the clone of operations on $A$ which are local term operations of every ultrapower of the algebra $\langle A; C \rangle$.

1. Introduction

The Baker–Pixley Theorem asserts that if a clone $C$ on a finite set $A$ contains a $d$-ary near unanimity operation ($d \geq 3$), then every operation that preserves all compatible relations of arity $< d$ of the algebra $\langle A; C \rangle$ belongs to $C$. This theorem does not extend in unmodified form to clones on infinite sets. Rather, the result is that if a clone $C$ on an infinite set $A$ contains a $d$-ary near unanimity operation, then every operation that preserves all compatible relations of arity $< d$ of the algebra $\langle A; C \rangle$ belongs to the local closure of $C$.

“Local closure” is a closure operator on the lattice of clones on $A$. We denote the local closure of a clone $C$ by $\Lambda_\omega(C)$, where we use capital Lambda to stand for “local”. This closure operator is useful for translating results about clones on finite sets to locally closed clones on arbitrary sets.

The drawbacks of passing from a clone to its local closure are that (i) there are relatively few locally closed clones on any infinite set, and (ii) the local closure of a clone is a coarse approximation to the clone. Regarding (i), every clone on a finite set is locally closed, but on an infinite set of cardinality $\nu$ there are $2^{2^\omega}$-many clones, and only $2^{\omega}$-many are locally closed (see, e.g., [2, p. 396]). Regarding (ii), the local closure of a simple $R$-module always agrees with the End($V$)-module structure on a vector space $V$. This may be regarded as a ‘coarse’ approximation to the $R$-module structure since, for example, the ring End($V$) typically has many nontrivial idempotents while $R$ need not have any.

In this paper, we introduce a collection of finer closure operators on clone lattices, the most interesting of which is called “ultralocal closure”. We denote the ultralocal closure of a clone $C$ by $\Upsilon_\omega(C)$, with capital Upsilon to stand for “ultralocal”. The

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concept of ultralocal closure is inspired by the work of Vaggione in [10]. We shall find that

- $C \subseteq \Upsilon_\omega(C) \subseteq \Lambda_\omega(C)$ (the ultralocal closure of $C$ is contained in the local closure of $C$),
- the number of ultralocally closed clones on an infinite set of cardinality $\nu$ is large ($= 2^{2^\nu}$), and
- $\Upsilon_\omega(C)$ can replace the use of $\Lambda_\omega(C)$ in some arguments that extend results about clones on finite sets to clones on infinite sets (e.g., the Baker–Pixley Theorem).

In fact, our work here covers a little more than we have described so far. Namely, for every set $A$ and every cardinal $\kappa$ we shall define the $\kappa$-ultraclosure of a clone $C$ on $A$, written $\Upsilon_\kappa(C)$. We say a clone is $\kappa$-ultraclosed if $\Upsilon_\kappa(C) = C$. It will follow from the definitions that $\Upsilon_1(C)$ is the clone of all operations on $A$ and

$$\Upsilon_1(C) \supseteq \Upsilon_2(C) \supseteq \cdots \supseteq \Upsilon_\omega(C) \supseteq \Upsilon_{\omega_1}(C) \supseteq \cdots \supseteq C.$$ 

Then, our main results are:

1. A characterization of the $\omega$-ultraclosure (i.e., the ultralocal closure) of a clone, $\Upsilon_\omega(C)$ (Theorem 3.1 and Corollary 3.2).
2. A proof, using the characterization theorem, of a version of the Baker–Pixley Theorem: every clone containing a $d$-ary near unanimity operation ($d \geq 3$) satisfies $\Upsilon_d(C) = C$ (Theorem 4.1). (The original proof of this statement, using different arguments and terminology, is due to Vaggione in [10].)
3. A proof, using the characterization theorem, that the clone of any simple module is ultralocally closed (Theorem 5.1).
4. We exhibit examples of clones that are, or are not, ultralocally closed (Section 6).

2. Preliminaries

Throughout this paper, $A$ and $I$ will denote nonempty sets. By a clone we will mean a clone of operations on some set $A$, that is, a set of finitary operations on $A$ that contains the projection operations and is closed under superposition. The largest clone on $A$ is the clone $O_A$ of all operations on $A$.

Fix $A$ and $I$. For any ultrafilter $U$ on $I$, the ultrapower $A^I/U$ of $A$ consists of the equivalence classes $a/U$ ($a = (a_i)_{i \in I} \in A^I$) of the equivalence relation $\equiv_U$ on $A^I$ defined by

$$(a_i)_{i \in I} \equiv_U (b_i)_{i \in I} \text{ if and only if } \{i \in I : a_i = b_i\} \in U.$$ 

The diagonal map $\delta : A \to A^I/U$, $a \mapsto (a)_{i \in I}/U$ is injective, therefore $A^I/U$ may be viewed as an extension of $A$, via $\delta$. 
For every $n$-ary operation $f : A^n \to A$ on $A$, and for every ultrafilter $\mathcal{U}$ on some set $I$, $f$ has an extension $f_\mathcal{U}$ to the ultrapower $A^I/\mathcal{U}$ of $A$, defined as follows:

$$f_\mathcal{U}(a_1/\mathcal{U}, \ldots, a_n/\mathcal{U}) = f(a_1, \ldots, a_n)/\mathcal{U} \quad \text{for all } a_1, \ldots, a_n \in A^I,$$

where $f$ on the right hand side acts coordinatewise on elements of $A^I$. For any clone $\mathcal{C}$ on $A$ and ultrafilter $\mathcal{U}$ in $I$, we get a clone $\mathcal{C}_\mathcal{U}$ on $A^I/\mathcal{U}$ by defining

$$\mathcal{C}_\mathcal{U} := \{ t_\mathcal{U} : t \in \mathcal{C} \}.$$

This is the clone of the ultrapower $\langle A; \mathcal{C} \rangle^I/\mathcal{U}$ of the algebra $\langle A; \mathcal{C} \rangle$. The diagonal map $\delta : A \to A^I/\mathcal{U}$ is an elementary embedding $\langle A; \mathcal{C} \rangle \to \langle A; \mathcal{C} \rangle^I/\mathcal{U} = \langle A^I/\mathcal{U}; \mathcal{C}_\mathcal{U} \rangle$, therefore the algebra $\langle A; \mathcal{C} \rangle^I/\mathcal{U} = \langle A^I/\mathcal{U}; \mathcal{C}_\mathcal{U} \rangle$ may be viewed as an elementary extension of $\langle A; \mathcal{C} \rangle$.

Let $f$ be an $n$-ary operation on $A$ and let $\mathcal{C}$ be an arbitrary clone on $A$. Furthermore, let $\kappa > 0$ and $\lambda$ be cardinals. We say that $f$ is $\lambda$-interpolable by $\mathcal{C}$, if whenever $S \subseteq A^n (= \text{dom}(f))$ satisfies $|S| \leq \lambda$, there is some $n$-ary $t \in \mathcal{C}$ such that $f|_S = t|_S$. (See Figure 1 for the case when $\lambda$ is finite.) We define the $\kappa$-closure, $\Lambda_\kappa(\mathcal{C})$, of $\mathcal{C}$ to consist of all operations on $A$ that are $\lambda$-interpolable by $\mathcal{C}$ for every $\lambda < \kappa$. (Notice the strict $<$ here!) The clone $\mathcal{C}$ is called $\kappa$-closed if $\mathcal{C} = \Lambda_\kappa(\mathcal{C})$. In the special case $\kappa = \omega$, the $\omega$-closure $\Lambda_\omega(\mathcal{C})$ of $\mathcal{C}$ is called the local closure of $\mathcal{C}$, and $\mathcal{C}$ is called locally closed if $\mathcal{C} = \Lambda_\omega(\mathcal{C})$.

![Figure 1](image-url)  

**Figure 1.** $f$ is $\lambda$-interpolable ($\lambda < \omega$)

For $f$, $\mathcal{C}$, and $\kappa$, $\lambda$ as before, we will say that $f$ is $\lambda$-ultrainterpolable by $\mathcal{C}$, if $f_\mathcal{U}$ is $\lambda$-interpolable by $\mathcal{C}_\mathcal{U}$ for every ultrafilter $\mathcal{U}$ on any set $I$. That is, $f$ is $\lambda$-ultrainterpolable by $\mathcal{C}$, if for any ultrafilter $\mathcal{U}$ on any set $I$, we have that whenever $S \subseteq (A^I/\mathcal{U})^n$ satisfies $|S| \leq \lambda$, there is some $n$-ary $t \in \mathcal{C}$ such that $(f_\mathcal{U})|_S = (t_\mathcal{U})|_S$. We define the $\kappa$-ultraclosure, $\Upsilon_\kappa(\mathcal{C})$, of $\mathcal{C}$ to consist of all operations on $A$ that are $\lambda$-ultrainterpolable by $\mathcal{C}$ for every $\lambda < \kappa$. (Strict $<$ here, too!) The clone $\mathcal{C}$ is called $\kappa$-ultraclosed if $\mathcal{C} = \Upsilon_\kappa(\mathcal{C})$. In the special case $\kappa = \omega$, the $\omega$-ultraclosure $\Upsilon_\omega(\mathcal{C})$ of $\mathcal{C}$ is called the ultralocal closure of $\mathcal{C}$, and $\mathcal{C}$ is called ultrlocally closed if $\mathcal{C} = \Upsilon_\omega(\mathcal{C})$.

If $f$ is $\lambda$-ultrainterpolable by $\mathcal{C}$, then $f$ is $\lambda$-interpolable by $\mathcal{C}$, for the following reason. Assume that $f$ is $\lambda$-ultrainterpolable by $\mathcal{C}$, and that $\mathcal{U}$ is a principal ultrafilter
on some set $I$ with $\{u\} \in U$ ($u \in I$). Since $f$ is $\lambda$-ultrainterpolable by $C$, $f_{U}$ is $\lambda$-interpolable by $C_{U}$. Since $U$ is generated by $\{u\}$, the equivalence relation $\equiv_{U}$ is the kernel of the projection $A^{I} \to A$ onto the $u$-th coordinate, so $\delta: A \to A^{I}/U$ is a bijection. Therefore, up to renaming elements of the base sets via $\delta$, $C_{U}$ and $C$ are the same clone, and $f_{U}$ and $f$ are the same operation. Hence, $f$ is $\lambda$-interpolable by $C$.

The argument just given proves statement (1) of the lemma below. Statement (2) is an immediate consequence of the definitions. Statement (3) follows from the fact that for a finite set $A$, the elementary embedding $\delta: A \to A^{I}/U$ is an isomorphism for any ultrafilter $U$ on any set $I$.

**Lemma 2.1.** For arbitrary clone $C$ on a set $A$, and for any cardinals $\mu, \nu (>0)$,

1. $C \subseteq \Upsilon_{\mu}(C) \subseteq \Lambda_{\mu}(C)$, and
2. $C \subseteq \Upsilon_{\nu}(C) \subseteq \Upsilon_{\mu}(C)$ if $\mu \leq \nu$.
3. For finite $A$,
   - $\Upsilon_{\mu}(C) = \Lambda_{\mu}(C)$, moreover,
   - $C = \Upsilon_{\mu}(C) = \Lambda_{\mu}(C)$ if $\mu$ is infinite.

Statement (3) of the lemma shows that for clones on finite sets the closure operators $\Upsilon_{\mu}$ ($\mu > 0$) are not new. Therefore our results in the forthcoming sections are interesting only for clones on infinite sets.

Since every operation $f$ on a set $A$ is 0-interpolable by any clone $C$ on $A$, we have that $\Upsilon_{1}(C) = \Lambda_{1}(C) = O_{A}$. Hence, statements (1)–(2) of Lemma 2.1 can be summarized as follows:

$$
O_{A} = \Lambda_{1}(C) \supseteq \Lambda_{2}(C) \supseteq \Lambda_{3}(C) \supseteq \cdots \supseteq \Lambda_{\omega}(C) \supseteq \Lambda_{\omega_{1}}(C) \supseteq \cdots \supseteq C \supseteq \Upsilon_{1}(C) \supseteq \Upsilon_{2}(C) \supseteq \Upsilon_{3}(C) \supseteq \cdots \supseteq \Upsilon_{\omega}(C) \supseteq \Upsilon_{\omega_{1}}(C) \supseteq \cdots \supseteq C.
$$

For any cardinal $\kappa > 0$, the property that a clone $C$ is $\kappa$-closed can be rephrased in terms of invariant relations, as stated in Lemma 2.2 below. For $\kappa = \omega$ the results of this lemma are due to Romov, [7]. The statements carry over from $\kappa = \omega$ to arbitrary cardinals $\kappa > 0$ without any essential changes.

For any set $R$ of (finitary or infinitary) relations on a set $A$, we will use the notation $\text{Pol}(R)$ for the clone consisting of all (finitary) operations on $A$ that preserve every relation in $R$.

**Lemma 2.2.** (cf. [7]) Let $\kappa$ be a nonzero cardinal, $C$ a clone on a set $A$, and let $R$ be a set of relations of arity $< \kappa$ on $A$.

1. $\text{Pol}(R)$ is a $\kappa$-closed clone on $A$.
2. If $C \subseteq \text{Pol}(R)$ (that is, if $R$ consists of invariant relations of $C$), then
   $$
   C \subseteq \Lambda_{\kappa}(C) \subseteq \text{Pol}(R).
   $$
(3) $\Lambda_\kappa(C) = \text{Pol}(R_C)$ for the set $R_C$ of all invariant relations of arity $< \kappa$ of $C$.

Using Lemma 2.1(1) one can expand the sequence of inclusions in (2) to

$$C \subseteq \Upsilon_\kappa(C) \subseteq \Lambda_\kappa(C) \subseteq \text{Pol}(R).$$

This will be useful for us, because it shows that if a property of clones is expressible by the preservation of some invariant relation, then this property is passed on from $C$ to $\Lambda_\kappa(C)$, and hence to $\Upsilon_\kappa(C)$, for large enough $\kappa$.

Next we discuss special cases of Lemma 2.2 when $C$ is an essentially unary clone, a module clone, or a product clone. The effect of $\Upsilon_\kappa$ on unary clones and product clones will be employed in Section 6 to construct large families of clones on infinite sets that are not ultralocally closed, while the effect of $\Upsilon_\kappa$ on module clones will be used in Section 5 where we show that the clone of every simple module is ultralocally closed.

In our first corollary to Lemma 2.2 a clone $C$ is called essentially unary if every operation in $C$ depends on at most one of its variables.

**Corollary 2.3.** Let $C$ be a clone and $\kappa$ a nonzero cardinal.

1. If $C$ is essentially unary, then so are $\Lambda_\kappa(C)$ and $\Upsilon_\kappa(C)$ for every $\kappa \geq 4$.

2. If all unary operations in $C$ are injective, then $\Lambda_\kappa(C)$ and $\Upsilon_\kappa(C)$ have the same property for every $\kappa \geq 3$.

**Proof.** For (1), we use the following easily proved fact.

**Claim 2.4.** An operation $f$ on a set $A$ is essentially unary if and only if $f$ preserves the ternary relation $\rho_3 := \{(a, b, c) \in A^3 : a = b \text{ or } b = c\}$.

**Proof of Claim 2.4.** Let $A$ be an arbitrary set. It is proved in [6, Lemma 1.3.1] that an operation $f$ on $A$ is essentially unary if and only if $f$ preserves the 4-ary relation $\pi_4 := \{(a, b, c, d) \in A^4 : a = b \text{ or } c = d\}$. In other words, $\text{Pol}(\pi_4)$ is the clone of all essentially unary operations.

To prove that the relation $\pi_4$ here can be replaced by $\rho_3$, notice that $\text{Pol}(\rho_3)$ contains all essentially unary operations; therefore it suffices to show that $\text{Pol}(\rho_3) \subseteq \text{Pol}(\pi_4)$. This can be done by exhibiting a primitive positive definition for $\pi_4$ in terms of $\rho_3$. (See, e.g., [6, Chapter 2] for why this is sufficient.)

We claim that the primitive positive formula

$$\Phi(x_0, x_1, x_2, x_3) \equiv \Psi(x_0, x_1, x_2, x_3) \land \Psi(x_1, x_0, x_2, x_3) \quad \text{with}$$

$$\Psi(x_0, x_1, x_2, x_3) \equiv \exists y \left( \rho_3(x_0, x_1, y) \land \rho_3(y, x_2, x_3) \right)$$

defines $\pi_4$. Indeed, it is easy to verify that the relation defined by $\Psi(x_0, x_1, x_2, x_3)$ is $\{(a, b, c, d) \in A^4 : a = b \text{ or } c = d \text{ or } b = c\}$. Hence the relation defined by $\Phi(x_0, x_1, x_2, x_3)$ is $\{(a, b, c, d) \in A^4 : a = b \text{ or } c = d \text{ or } a = b = c\} = \pi_4$.

This claim is also proved in [1, Lemma 5.3.2].
Corollary 2.5. Let $C$ be a clone and $\kappa$ a nonzero cardinal. If $C$ is the clone of an $R$-module, for some ring $R$, with underlying abelian group $\hat{A} = \langle A; +, -, 0 \rangle$, then so are $\Lambda_\kappa(C)$ and $\Upsilon_\kappa(C)$ for every $\kappa \geq 4$.

Proof. Let $R\hat{A}$ be an $R$-module with underlying abelian group $\hat{A}$, and let $C$ be the clone of term operations of $R\hat{A}$. It is known (for example, it follows from [3 Proposition 2.1]) that

- the graph of $+$, that is, the ternary relation $\gamma(+) := \{(a, b, a + b) : a, b \in A\}$ is preserved by every operation in $C$; moreover,

- the clone $\text{Pol}(\gamma(+))$ of all operations that preserve $\gamma(+)$. coincides with the clone of the module $\text{End}(\hat{A})\hat{A}$, which is $\hat{A}$ as a module over its endomorphism ring $\text{End}(\hat{A})$.

Consequently, every subclone $S$ of the clone of $\text{End}(\hat{A})\hat{A}$ such that $S$ contains the clone of $\hat{A}$, is the clone of a module $S\hat{A}$ with underlying abelian group $\hat{A}$ for some subring $S$ of $\text{End}(\hat{A})$; namely, $S$ is the ring of all unary operations in $S$. By Lemma 2.2(2), each $\Upsilon_\kappa(C)$ ($\kappa \geq 4$) is one of these clones, therefore each $\Upsilon_\kappa(C)$ ($\kappa \geq 4$) is the clone of a module with underlying abelian group $\hat{A}$, as claimed.

Corollary 2.6. Let $C$ be a clone on a set $A \times B$, and let $\kappa$ be a nonzero cardinal. If $C$ is a product clone on $A \times B$, then so are $\Lambda_\kappa(C)$ and $\Upsilon_\kappa(C)$ for every $\kappa \geq 4$.

Proof. Let $*$ denote the binary operation on $A \times B$ defined as follows:

$$(a_1, b_1) * (a_2, b_2) = (a_1, b_2)$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. It follows from Claim 2.4 that if $C$ is an essentially unary clone, then $C \subseteq \text{Pol}(\rho_3)$. Hence, by applying Lemma 2.2(2) with $R = \{\rho_3\}$, we get that $C \subseteq \Upsilon_\kappa(C) \subseteq \Lambda_\kappa(C) \subseteq \text{Pol}(\rho_3)$ for $\kappa \geq 4$. This shows that the clone $\Lambda_\kappa(C)$ and its subclone, $\Upsilon_\kappa(C)$, are also essentially unary if $\kappa \geq 4$. The proof of (1) is complete.

A unary operation $f : A \to A$ is injective exactly when it preserves the binary “not equal” relation $\{(a, b) \in A^2 : a \neq b\}$. Now, statement (2) follows in the same way as statement (1). $\square$
This operation is known as the binary diagonal operation or the rectangular band operation of the product $A \times B$. Notice that $\ast$ is the product operation $p_1^A \times p_2^B \in \mathcal{O}_A \times \mathcal{O}_B$ where $p_1^A$ is binary projection to the first variable on $A$, and $p_2^B$ is binary projection to the second variable on $B$. We will also use the graph of the operation $\ast$, which is the following ternary relation:

$$\gamma(\ast) := \{(u, v, u \ast v) \in (A \times B)^3 : u, v \in A \times B\}.$$  

We will need the following facts.

**Claim 2.7.** Let $A, B$ be arbitrary sets.

1. The following conditions on an $n$-ary operation $f$ on $A \times B$ are equivalent:
   - $f = f_A \times f_B$ for some $n$-ary operations $f_A$ on $A$ and $f_B$ on $B$;
   - $f$ commutes with $\ast$;
   - $f$ preserves the graph $\gamma(\ast)$ of the operation $\ast$.

2. A clone $C$ on $A \times B$ is a product clone if and only if
   - (i) $C \subseteq \text{Pol}(\gamma(\ast))$, i.e., every operation in $C$ commutes with $\ast$, and
   - (ii) $\ast$ is a member of $C$.

**Proof of Claim 2.7.** For (1), let $f$ be an $n$-ary operation on $A \times B$, i.e., $f : (A \times B)^n \to A \times B$. We will write an $n$-tuple of pairs from $A \times B$ as an $n \times 2$ matrix $[\overline{a} \; \overline{b}]$ with columns $\overline{a} \in A^n$ and $\overline{b} \in B^n$. The rows are the pairs $(a_i, b_i)$ $(i < n)$ where $\overline{a} = [a_0 \ldots a_{n-1}]^T$ and $\overline{b} = [b_0 \ldots b_{n-1}]^T$. Thus, when $\ast$ is applied coordinatewise (down columns) to two $n$-tuples, $[\overline{a} \; \overline{b}]$ and $[\overline{a'} \; \overline{b'}]$ in $(A \times B)^n$, we get

$$[\overline{a} \; \overline{b}] \ast [\overline{a'} \; \overline{b'}] = [\overline{a} \; \overline{b'}].$$  

Let $\widetilde{f}_A$ denote the function $\widetilde{f}_A : (A \times B)^n \to A$ obtained from $f$ by composing it with the function $A \times B \to A$, $(a, b) \mapsto a$, and similarly, let $\widetilde{f}_B : (A \times B)^n \to B$ be obtained from $f$ by composing it with the function $A \times B \to B$, $(a, b) \mapsto b$. We have

$$f([\overline{a} \; \overline{b}]) = (\widetilde{f}_A([\overline{a} \; \overline{b}]), \widetilde{f}_B([\overline{a} \; \overline{b}])) \quad \text{for all } [\overline{a} \; \overline{b}] \in (A \times B)^n.$$  

Now we are ready to prove the equivalence of the three conditions in (1). The last two of these conditions are different ways of stating the same relationship between $f$ and $\ast$, and they are easily seen to be implied by the first condition. Therefore it remains to prove that the first condition follows from the second. The second condition is the statement that

$$f([\overline{a} \; \overline{b}] \ast [\overline{a'} \; \overline{b'}]) = f([\overline{a} \; \overline{b}]) \ast f([\overline{a'} \; \overline{b'}]) \quad \text{for all } [\overline{a} \; \overline{b}], [\overline{a'} \; \overline{b'}] \in (A \times B)^n.$$  

By applying (2.1) and (2.2) we see that the left hand side of the equality in (2.3) is

$$f([\overline{a} \; \overline{b}] \ast [\overline{a'} \; \overline{b'}]) = f([\overline{a} \; \overline{b}]) = (\widetilde{f}_A([\overline{a} \; \overline{b}]), \widetilde{f}_B([\overline{a} \; \overline{b}])),$$

while the right hand side is

$$f([\overline{a} \; \overline{b}]) \ast f([\overline{a'} \; \overline{b'}]) = (\widetilde{f}_A([\overline{a} \; \overline{b}]), \widetilde{f}_B([\overline{a'} \; \overline{b'}])).$$
Thus, (2.3) is equivalent to the condition that \( \bar{f}_A \) does not depend on the second column of the input matrix \([\bar{a} \; \bar{b}]\), and \( \bar{f}_B \) does not depend on the first column of the input matrix \([\bar{a}′ \; \bar{b}]\). That is, there exist \( f_A : A^n \to A \) and \( f_B : B^n \to B \) such that

\[
f(\bar{a}, \bar{b}) = (f_A(\bar{a}), f_B(\bar{b})) \quad \text{for all } \bar{a} \in A^n \text{ and } \bar{b} \in B^n,
\]

or equivalently, there exist \( n \)-ary operations \( f_A \) on \( A \) and \( f_B \) on \( B \) such that \( f = f_A \times f_B \). This finishes the proof of (1).

For the forward implication of statement (2), if \( C \) is a product clone on \( A \times B \), then (i) holds by part (1) of this claim and (ii) holds by the observation made in the paragraph preceding Claim 2.7 that \( * \) is a product operation where each factor is a projection.

For the backward implication of statement (2), assume that \( C \) is a clone on \( A \times B \) such that conditions (i)–(ii) are met. By statement (1) above, (i) implies that every operation \( f \in C \) is a product operation: \( f = f_A \times f_B \) for some operations \( f_A \) on \( A \) and \( f_B \) on \( B \), of the same arity as \( f \). Let \( P := \{ f_A : f \in C \} \) and \( Q := \{ f_B : f \in C \} \). It follows that \( P \) is a clone on \( A \), \( Q \) is a clone on \( B \), and \( C \) is a subclone of \( P \times Q \). We claim that \( C = P \times Q \). Let \( n \geq 1 \), and consider arbitrary \( n \)-ary operations \( g \in P \) and \( h \in Q \). By the definitions of \( P \) and \( Q \), there exist \( n \)-ary operations \( g′, h′ \in C \) such that \( g = g_A′ \) and \( h = h_B′ \); that is, \( g′ = g \times g_B′ \) and \( h′ = h_A′ \times h \). By condition (ii) we have \( * \in C \), therefore \( g \times h = (g \times g_B′) \times (h_A′ \times h) = g′ \times h′ \in C \). This shows that \( C \supseteq P \times Q \), which completes the proof of (2).

It follows from Claim 2.7 that if \( C \) is a product clone on \( A \times B \), then \( * \in C \subseteq \text{Pol}(\gamma(*)) \). Therefore, by applying Lemma 2.2(2) with \( R = \{ \gamma(*) \} \), we obtain that \( * \in C \subseteq \Upsilon_\kappa(C) \subseteq \Lambda_\kappa(C) \subseteq \text{Pol}(\gamma(*)) \) for \( \kappa \geq 4 \). Hence, Claim 2.7(2) yields that \( \Lambda_\kappa(C) \) and \( \Upsilon_\kappa(C) \) are both product clones for \( \kappa \geq 4 \).

\[\Box\]

### 3. Characterizing ultralocal closure

Our main goal in this section is to characterize the \( \kappa \)-ultraclosure of a clone \( C \) for each nonzero cardinal \( \kappa \leq \omega \). The main ingredient is the following characterization of the operations that are \( \lambda \)-ultrainterpolable by \( C \) for some \( \lambda < \omega \).

In what follows, a **cover** of a set \( X \) is a set \( C \subseteq \mathcal{P}(X) \) of subsets of \( X \) such that \( \bigcup C = X \). So, \( C \) is a **finite cover** of \( X \) if \( C \) is a finite set and a cover of \( X \).

**Theorem 3.1.** Let \( C \) be a clone on a set \( A \), and let \( f : A^n \to A \) be an \( n \)-ary operation on \( A \) \((0 < n < \omega)\). The following conditions are equivalent for any \( \lambda < \omega \).

\[
(\dagger)_\lambda^n f \text{ is } \lambda\text{-ultrainterpolable by } C.
\]

\[
(\ddagger)_\lambda^n A^n (= \text{ dom}(f)) \text{ has a finite cover } C_\lambda \subseteq \mathcal{P}(A^n) \text{ such that whenever } B \subseteq C_\lambda \text{ satisfies } |B| \leq \lambda, \text{ there exists an } n\text{-ary } t[B] \in C \text{ such that } f|\bigcup B = t[B]|\bigcup B.
\]
Note that condition \((\ddagger)_\lambda\) and the finite cover \(C_\lambda\) involved both depend on the choice of \(f\), \(C\), and \(\lambda\). Dependence on \(f\) and \(C\) is suppressed in the notation, but when we apply condition \((\ddagger)_\lambda\), the choice of \(f\) and \(C\) will be clear from the context.

Condition \((\ddagger)_\lambda\) is illustrated by Figure 2. The figure indicates that \(A^n\) has a finite cover \(C_\lambda\) such that for any set \(B = \{B_0, B_1, \ldots, B_{\lambda-1}\} \subseteq C_\lambda\), there is a \(t[B] \in C\) such that for all \(B_i \in B\), \(f|_{B_i} = t[B]|_{B_i}\).

**Figure 2.** \(f\) is \(\lambda\)-ultrainterpolable \((\lambda < \omega)\)

**Corollary 3.2.** Let \(C\) be a clone on a set \(A\), and let \(\kappa \leq \omega\) be a nonzero cardinal. The \(\kappa\)-ultraclosure, \(\Upsilon_\kappa(C)\), of \(C\) consists of all operations \(f : A^n \rightarrow A\) \((0 < n < \omega)\) which satisfy condition \((\ddagger)_\lambda\) from Theorem 3.1 for every \(\lambda < \kappa\).

We will focus primarily on the case \(\kappa = \omega\), that is, on the \(\omega\)-ultraclosure of clones \(C\) on infinite sets, which we also call the ultralocal closure of \(C\). In Section 6 we will give examples to show that there exist clones on infinite sets that are not ultralocally closed (see Theorems 6.1 and 6.5).

The rest of this section is devoted to the proof of Theorem 3.1. We start by introducing some terminology and notation, that will allow us to restate condition \((\ddagger)_\lambda\) of Theorem 3.1 in a form that is more convenient for our proof.

Let \(C\), \(f\) with \(\text{dom}(f) = A^n\), and \(\lambda\) be as in Theorem 3.1. It will be convenient to think of the elements of \(A^n\) as columns of length \(n\), and the elements of the set \((A^n)_\lambda\) as \(\lambda\)-sequences of column vectors in \(A^n\), or equivalently, as \(n \times \lambda\) matrices where each one of the \(\lambda\) columns is an element of \(A^n\). Now, for each \(n\)-ary operation \(t \in C\) define

\[
E_t := \{[a_i]_{i < \lambda} \in (A^n)_\lambda : f(a_i) = t(a_i) \text{ for all } i < \lambda\}
\]

to be the set of all \(n \times \lambda\) matrices where \(f\) is equal to \(t\). Hence,

\[
N_t := (A^n)_\lambda \setminus E_t
\]
is the set of all $n \times \lambda$ matrices where $f$ is not equal to $t$.

Let $\mathcal{F}_\lambda$ denote the collection of all subsets of $(A^n)^\lambda$ of the form $N_t$ ($t \in \mathcal{C}$) defined above. We will say that $\mathcal{F}_\lambda$ has the finite intersection property if the intersection of any finite subfamily of $\mathcal{F}_\lambda$ is nonempty.

**Lemma 3.3.** Let $\mathcal{C}$ be a clone on a set $A$, and let $f: A^n \to A$ be an $n$-ary operation on $A$ ($0 < n < \omega$). The following conditions are equivalent for every nonzero $\lambda < \omega$.

(i) The condition below from Theorem 3.1:

\[
(\exists)_\lambda A^n (= \text{dom}(f)) \text{ has a finite cover } \mathcal{C}_\lambda \subseteq \mathcal{P}(A^n) \text{ such that whenever } \mathcal{B} \subseteq \mathcal{C}_\lambda \text{ satisfies } |\mathcal{B}| \leq \lambda, \text{ there exists an } n\text{-ary } t^{[\mathcal{B}]} \in \mathcal{C} \text{ such that } f|_{\bigcup \mathcal{B}} = t^{[\mathcal{B}]}|_{\bigcup \mathcal{B}}.
\]

(ii) $(A^n)^\lambda$ has a finite cover $\mathcal{D}_\lambda \subseteq \mathcal{P}((A^n)^\lambda)$ such that for every $D \in \mathcal{D}_\lambda$ there exists $s[D] \in \mathcal{C}$ such that we have

\[
(3.1) \quad f(a_i) = s[D](a_i) \text{ for all } i < \lambda \text{ whenever } [a_i]_{i<\lambda} \in D.
\]

(iii) $\mathcal{F}_\lambda$ fails to have the finite intersection property.

**Proof.** Suppose (i) holds. Since $\mathcal{C}_\lambda$ is finite, so is

\[
\mathcal{D}_\lambda := \left\{ \left( \bigcup \mathcal{B} \right)^\lambda : \mathcal{B} \subseteq \mathcal{C}_\lambda, |\mathcal{B}| \leq \lambda \right\}.
\]

Since $\mathcal{C}_\lambda$ covers $A^n$, it follows that $\mathcal{D}_\lambda$ covers $(A^n)^\lambda$. Moreover, our assumption $(\exists)_\lambda$ yields that for every member $D = (\bigcup \mathcal{B})^\lambda$ of $\mathcal{D}_\lambda$ the operation $s[D] := t^{[\mathcal{B}]} \in \mathcal{C}$ satisfies the requirement in (ii). This finishes the proof of $(i) \Rightarrow (ii)$.

Conversely, assume (ii), and for each $D \in \mathcal{D}_\lambda$ and each $j < \lambda$ define $D^{(j)}$ to be the projection of $D$ onto its $j$-th coordinate; that is, $D^{(j)} := \{a_j : [a_i]_{i<\lambda} \in D\}$. Furthermore, let $\tilde{D} := \bigcup\{D^{(j)} : j < \lambda\}$. Condition (3.1) from assumption (ii) implies that for each $\tilde{D}$ with $D \in \mathcal{D}_\lambda$,

\[
(3.2) \quad s[D] \in \mathcal{C} \text{ satisfies } f(a) = s[D](a) \text{ for all } a \in \tilde{D}.
\]

Since $D \subseteq \tilde{D}^\lambda$ for every $D \in \mathcal{D}_\lambda$ and $\mathcal{D}_\lambda$ is a finite cover of $(A^n)^\lambda$, it follows that the set $\mathcal{E} := \{D^{(j)} : D \in \mathcal{D}_\lambda, j < \lambda\}$ is a finite cover of $A^n$. Let $\mathcal{A}$ denote the Boolean algebra of sets generated by $\mathcal{E}$. Clearly, $\mathcal{A}$ is finite, and the set $\mathcal{C}_\lambda$ of all atoms of $\mathcal{A}$ is a finite cover of $A^n$ which partitions $A^n$ into nonempty subsets. Our goal is to show that $\mathcal{C}_\lambda$ satisfies the requirements in condition $(\exists)_\lambda$.

Let $\mathcal{B} = \{C_0, \ldots, C_{\lambda-1}\}$ be any subset of $\mathcal{C}_\lambda$ of size \leq \lambda. Choose $a_i \in C_i$ for each $i < \lambda$. Since $[a_i]_{i<\lambda} \in (A^n)^\lambda$, there must exist $D \in \mathcal{D}_\lambda$ with $[a_i]_{i<\lambda} \in D$. Hence, $[a_i]_{i<\lambda} \in \prod_{i<\lambda}(C_i \cap D^{(i)})$, showing that each $C_i \cap D^{(i)}$ is a nonempty member of $\mathcal{A}$ contained in an atom $C_i$. This forces $C_i \subseteq D^{(i)}$ for all $i < \lambda$. Hence,

\[
C_0 \times \cdots \times C_{\lambda-1} \subseteq D^{(0)} \times \cdots \times D^{(\lambda-1)} \subseteq \tilde{D}^\lambda,
\]
which implies that $\bigcup \mathcal{B} \subseteq \bigcup \{D^{(i)} : i < \lambda\} = \tilde{D}$. Thus, (3.2) implies that $f(a) = s^{(D)}(a)$ holds for all $a \in \bigcup \mathcal{B}$. This completes the proof of (ii) $\Rightarrow$ (i).

It remains to prove that (ii) $\iff$ (iii). Condition (iii) holds, i.e., $\mathcal{F}_\lambda$ fails to have the finite intersection property, if and only if $\mathcal{C}$ contains finitely many $n$-ary operations $t_1, \ldots, t_r$ such that $N_{t_1} \cap \cdots \cap N_{t_r} = \emptyset$, or equivalently, $E_{t_1} \cup \cdots \cup E_{t_r} = (A^n)^\lambda$. Thus, if (iii) holds, then (ii) also holds with the choice $\mathcal{D}_\lambda = \{E_{t_j} : j = 1, \ldots, r\}$. Conversely, if (ii) holds, then we have $D \subseteq E_{s[D]}$ for every $D \in \mathcal{D}_\lambda$. Hence we have finitely many operations $s^{(D)} (D \in \mathcal{D}_\lambda)$ in $\mathcal{C}$ such that $\bigcup \{E_{s[D]} : D \in \mathcal{D}_\lambda\} = (A^n)^\lambda$. As we explained at the beginning of this paragraph, this proves (iii).

\textit{Proof of Theorem 3.1.} Let $\mathcal{C}$ be a clone on a set $A$, and let $f : A^n \to A$ be an $n$-ary operation on $A$ ($0 < n < \omega$). Theorem 3.1 states for every $\lambda < \omega$, the property that

$$(\dagger)_\lambda \text{ $f$ is $\lambda$-ultra-interpolable by $\mathcal{C}$}$$

is characterized by the condition

$$(\ddagger)_\lambda \text{ $A^n (= \text{dom}(f))$ has a finite cover $\mathcal{C}_\lambda (\subseteq \mathcal{P}(A^n))$ such that whenever $\mathcal{B} \subseteq \mathcal{C}_\lambda$ satisfies $|\mathcal{B}| \leq \lambda$, there exists an $n$-ary $t^{[\mathcal{B}]} \in \mathcal{C}$ such that $f|_{\bigcup \mathcal{B}} = t^{[\mathcal{B}]}|_{\bigcup \mathcal{B}}$.}$$

The statement of the theorem is vacuously true for $\lambda = 0$, because both conditions $(\ddagger)_0$ and $(\ddagger)_0$ hold for $f$ for the following reason. For $(\ddagger)_0$, notice that $f$ is 0-ultra-interpolable by any $n$-ary projection in $\mathcal{C}$, since any two $n$-ary operations (on any set) agree on $\emptyset$. For $(\ddagger)_\lambda$, if we choose $\mathcal{C}_0 := \{A^n\}$, the same reasoning yields the required equality for $\mathcal{B} = \emptyset$ and any $n$-ary projection $t^{[\emptyset]}$.

Therefore, we will assume from now on that $\lambda > 0$. First, we will prove $(\ddagger)_\lambda \Rightarrow (\ddagger)_\lambda$. To obtain a contradiction, assume that $f$ is $\lambda$-ultra-interpolable by $\mathcal{C}$, but $(\ddagger)_\lambda$ fails. By Lemma 3.3, the latter assumption means that the family $\mathcal{F}_\lambda$ of subsets of $I := (A^n)^\lambda$ has the finite intersection property. It follows that there exists an ultrafilter $\mathcal{U}$ on $I$ such that $\mathcal{F}_\lambda \subseteq \mathcal{U}$. Each member $\alpha \in I = (A^n)^\lambda$ is an $n \times \lambda$ matrix $\alpha = [\alpha^{(\ell)}]_{j<\lambda}^{\ell=n}$. For each $j < n$ and $\ell < \lambda$ define an element $\overline{\alpha}_{j}^{(\ell)}$ of $A^I$ as follows: $\overline{\alpha}_{j}^{(\ell)} := (\alpha^{(\ell)}_{j})_{\alpha \in I}$. This yields a subset

$$(3.3) \quad S := \{(\overline{\alpha}_{0}^{(\ell)}/\mathcal{U}, \ldots, \overline{\alpha}_{n-1}^{(\ell)}/\mathcal{U}) : \ell < \lambda\}$$

of $(A^I/\mathcal{U})^n$ with $|S| \leq \lambda$.

Our assumption is that $f$ is $\lambda$-ultra-interpolable by $\mathcal{C}$. Hence, for the ultrafilter $\mathcal{U}$ and set $S \subseteq (A^I/\mathcal{U})^n$ of size $\leq \lambda$ just constructed, $f_{\mathcal{U}}$ is interpolated on $S$ by $t_{\mathcal{U}}$ for some $n$-ary operation $t \in \mathcal{C}$; that is, $f_{\mathcal{U}}$ and $t_{\mathcal{U}}$ satisfy

$$(3.4) \quad f_{\mathcal{U}}(\overline{\alpha}_{0}^{(\ell)}/\mathcal{U}, \ldots, \overline{\alpha}_{n-1}^{(\ell)}/\mathcal{U}) = t_{\mathcal{U}}(\overline{\alpha}_{0}^{(\ell)}/\mathcal{U}, \ldots, \overline{\alpha}_{n-1}^{(\ell)}/\mathcal{U}) \quad \text{for all } \ell < \lambda.$$}

Thus, the set

$$E := \{\alpha \in I = (A^n)^\lambda : f(\alpha^{(\ell)}_0, \ldots, \alpha^{(\ell)}_{n-1}) = t(\alpha^{(\ell)}_0, \ldots, \alpha^{(\ell)}_{n-1}) \text{ for all } \ell < \lambda\}$$

is a subset of $S$ and $f_{\mathcal{U}}$ is interpolated by $t_{\mathcal{U}}$ on $E$. Therefore, $f_{\mathcal{U}}$ is $\lambda$-ultra-interpolable by $\mathcal{C}$, which contradicts our assumption. This completes the proof of $(\ddagger)_\lambda \Rightarrow (\ddagger)_\lambda$. Therefore, $(\ddagger)_\lambda$ holds for every $\lambda < \omega$.
is a member of \( \mathcal{U} \). Clearly, \( E \subseteq E_t \), so \( E_t \in \mathcal{U} \). However, by the construction of \( \mathcal{U} \) we have that \( N_t = I \setminus E_t \in \mathcal{F}_\lambda \subseteq \mathcal{U} \), so \( E_t \notin \mathcal{U} \). This contradiction finishes the proof of \((\ddagger)_\lambda \Rightarrow (\ddagger)_\lambda \).

To prove the implication \((\ddagger)_\lambda \Rightarrow (\ddagger)_\lambda \), assume that \((\ddagger)_\lambda \) holds, let \( A^f/\mathcal{U} \) be an arbitrary ultrapower of \( A \), and consider a subset \( S \) of \( (A^f/\mathcal{U})^n \) of size \( \leq \lambda \). Although the set \( I \) is now different from the set \( I \) in the preceding paragraphs, we may write \( S \) in the form \((\ref{3.3}) \) where \( \mathbf{a}^{(j)}_i = (a^{(j)}_{j,i})_{i \in I} \in A^f \) for all \( j < n \) and \( \ell < \lambda \). We have to show that there exists an \( n \)-ary operation \( t \in \mathcal{C} \) such that \( t_\mathcal{U} \) interpolates \( f_\mathcal{U} \) on \( S \), i.e., such that \((\ref{3.4}) \) holds.

Let \( \mathcal{C}_\lambda = \{C_0, \ldots, C_{r-1}\} \) be a finite cover of \( A^r \) (of size \( r \)) provided by our assumption \((\ddagger)_\lambda \); i.e., \( \mathcal{C}_\lambda \) has the property that whenever \( \mathcal{B} \subseteq \mathcal{C}_\lambda \) satisfies \( |\mathcal{B}| \leq \lambda \), there exists an \( n \)-ary \( t[\mathcal{B}] \in \mathcal{C} \) such that \( f|_{\bigcup \mathcal{B}} = t[\mathcal{B}]|_{\bigcup \mathcal{B}} \). For each \( \lambda \)-tuple \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_{\lambda-1}) \in r^\lambda \) of subscripts of members of \( \mathcal{C}_\lambda \) define

\[
I_\varepsilon := \{i \in I : (a^{(j)}_{0i}, \ldots, a^{(j)}_{n-1,i}) \in C_{\varepsilon_i} \text{ for all } j \leq \lambda \}.
\]

These sets form a finite cover \( \mathcal{I} := \{I_\varepsilon : \varepsilon \in r^\lambda \} \) of \( I \) (with possibly some of the sets \( I_\varepsilon \) empty). Since \( \mathcal{U} \) is an ultrafilter on \( I \), there exists \( \varepsilon \in r^\lambda \) such that \( I_\varepsilon \in \mathcal{U} \). Now let \( \mathcal{B} := \{C_{\varepsilon_0}, \ldots, C_{\varepsilon_{\lambda-1}}\} \). We have \( \mathcal{B} \subseteq \mathcal{C}_\lambda \) and \( |\mathcal{B}| \leq \lambda \), therefore there is a corresponding \( n \)-ary operation \( t[\mathcal{B}] \in \mathcal{C} \) satisfying \( f|_{\bigcup \mathcal{B}} = t[\mathcal{B}]|_{\bigcup \mathcal{B}} \). Since \( C_{\varepsilon_0} \times \cdots \times C_{\varepsilon_{\lambda-1}} \subseteq (\bigcup \mathcal{B})^\lambda \), it follows that the set

\[
\{i \in I : (a^{(j)}_{0i}, \ldots, a^{(j)}_{n-1,i}) = t[\mathcal{B}](a^{(j)}_{0i}, \ldots, a^{(j)}_{n-1,i}) \text{ for all } j \leq \lambda \}
\]

contains \( I_\varepsilon \), and hence belongs to \( \mathcal{U} \). This establishes \((\ref{3.4}) \) for \( t := t[\mathcal{B}] \), and hence completes the proof of \((\ddagger)_\lambda \Rightarrow (\ddagger)_\lambda \). \( \square \)

4. Clones containing near unanimity operations

Recall that for any integer \( d \geq 3 \), a \( d \)-ary operation \( h \) on a set \( A \) is called a \( d \)-ary near unanimity operation if it satisfies

\[
(4.1) \quad h(a, \ldots, a, b, a, \ldots, a) = a \quad \text{for all } a, b \in A \text{ and } 1 \leq i \leq d,
\]

where the sole occurrence of the letter \( b \) is in the \( i \)-th position.

In \( [10] \), Vaggione proved the following infinitary version of the Baker–Pixley Theorem: Let \( \mathcal{C} \) be the clone of term operations of an algebra \( A \) with universe \( A \), and assume that \( \mathcal{C} \) contains a \( d \)-ary near unanimity operation. If \( f \) is an operation on \( A \) such that for every ultrafilter \( \mathcal{U} \) on any set \( I \),

\((\diamond)\) the extension \( f_\mathcal{U} \) of \( f \) to \( A^f/\mathcal{U} \) preserves all subalgebras of \( (A^f/\mathcal{U})^{d-1} \),

then \( f \in \mathcal{C} \).

Since the clone of term operations of \( A \) is \( \mathcal{C} \), the clone of term operations of the ultrapower \( A^f/\mathcal{U} \) is \( \mathcal{C}_\mathcal{U} \). Therefore, by Lemma \( 2.2 \)\((3) \), \((\diamond) \) is equivalent to the
condition that \( f_U \) is \((d - 1)\)-interpolable by operations in \( \mathcal{C}_U \). Since \((\diamond)\) is required to hold for every ultrafilter \( U \), the assumption on \( f \) in Vaggione’s result is equivalent to saying that \( f \) is \((d - 1)\)-ultrainterpolable by \( C \). Hence, Vaggione’s main result in [10] states, in our terminology, that every clone that contains a \( d \)-ary near unanimity operation is \( d \)-ultraclosed. We now derive this result from Corollary 3.2.

**Theorem 4.1 ([10])**. Every clone that contains a \( d \)-ary near unanimity operation \((d \geq 3)\) is \( d \)-ultraclosed, and hence is also ultracomponent closed.

**Proof.** Let \( C \) be a clone on a set \( A \) such that \( C \) contains a \( d \)-ary near unanimity operation \( h \) \((d \geq 3)\). Our goal is to show that \( C = \Upsilon_d(C) \). By Lemma 2.1(2), this will also imply that \( C = \Upsilon_\omega(C) \). By Corollary 3.2 to establish \( C = \Upsilon_d(C) \), it suffices to prove that every operation \( f : A^n \to A \) \((0 < n < \omega)\) which satisfies condition \((\frac{1}{d})_{d-1}\) from Theorem 3.1 is a member of \( C \). So, assume that condition \((\frac{1}{d})_{d-1}\) holds for \( f \).

Thus, there is a finite cover \( \mathcal{C}_{d-1} \) of \( A^n \) and there exist \( n \)-ary operations \( t^{[B]} \in C \) for every set \( B \subseteq \mathcal{C}_{d-1} \) with \(|B| \leq d - 1\) such that \( f|_B = t^{[B]}|_B \). If \(|\mathcal{C}_{d-1}| \leq d - 1\), the last equality holds for \( \bigcup \mathcal{C}_{d-1} = A^n \), so \( f = t^{[C_{d-1}]} \in C \).

Assume from now on that \(|\mathcal{C}_{d-1}| \geq d \). We will apply the usual Baker–Pixley argument to the regions in \( \mathcal{C}_{d-1} \) to show, by induction on \( m \), that for every \( m \geq d - 1 \),

\[(*)_m \quad f|_{\bigcup B_i} = t^{[B_i]}|_{\bigcup B_i} \quad \text{for every} \quad i < m.\]

This will complete the proof, because then by choosing \( m = |\mathcal{C}_{d-1}| \) and \( B = \mathcal{C}_{d-1} \), we will have \( \bigcup \mathcal{C}_{d-1} = A^n \) and hence \( f = t^{[C_{d-1}]} \in C \).

To prove \((*)_m\) for \( m \geq d - 1 \), notice first that \((*)_{d-1}\) is exactly the condition that is forced by \((\frac{1}{d})_{d-1}\). Assume therefore that \( m \geq d \) and \((*)_{m-1}\) holds. Let \( B = \{C_0, \ldots, C_{m-1}\} \) be a subset of \( \mathcal{C}_{d-1} \) of cardinality \( \leq m \). For each \( i < m \), let \( B_i := B \setminus \{C_i\} \). By the induction hypothesis \((*)_{m-1}\), there exist \( n \)-ary operations \( t^{[B_i]} \in C \) such that

\[(4.2) \quad f|_{\bigcup B_i} = t^{[B_i]}|_{\bigcup B_i} \quad \text{for every} \quad i < m.\]

We claim that the operation

\[(4.3) \quad t^{[B]} := h(t^{[B_1]}, \ldots, t^{[B_d]}) \in C\]

satisfies the equality

\[(4.4) \quad f|_B = t^{[B]}|_B\]

required by \((*)_m\). Indeed, if \( a \in \bigcup B \), then \( a \in C_j \) for some \( j < m \), so \( a \in \bigcup B_i \) for all \( i < m \) with \( i \neq j \). Thus, by \((4.2)\), \( t^{[B_i]}(a) = f(a) \) for all \( i < m \), \( i \neq j \). Hence, when evaluating the operation on the right hand side of \((4.3)\) at \( a \), all but possibly one of the arguments of \( h \) are equal to \( f(a) \), therefore the near unanimity identities in \((4.1)\) force \( t^{[B]}(a) = f(a) \). This proves \((4.4)\), and finishes the proof of the theorem. \(\square\)
5. Simple Modules

Our goal in this section is to prove that the clone of any simple module is ultralo-cally closed. We do not know whether simplicity is a necessary hypothesis for this result.

Theorem 5.1. The clone of any simple module is 4-ultraclosed, and hence is also ultralocally closed.

Proof. Let \( R^A \) be a simple \( R \)-module, and let \( C \) denote its clone. It follows from Corollary 2.5 that for all \( \kappa \geq 4 \), the \( \kappa \)-closure \( \Lambda_\kappa(C) \) as well as the \( \kappa \)-ultraclosure \( \Upsilon_\kappa(C) \) of \( C \) are clones of modules on the set \( A \) which share the underlying abelian group \( \hat{A} \) of \( R^A \). Therefore, to determine these clones it suffices to determine the rings of scalars of the corresponding modules. Let \( R \) and \( S \) denote the scalar rings of the modules with clones \( \Upsilon_4(C) \) and \( \Lambda_\omega(C) \), respectively. We may assume without loss of generality that the actions of the rings \( R, \overline{R} \), and \( S \) are faithful, and identify each scalar in \( R, \overline{R}, \) or \( S \) with its action as an endomorphism of the underlying abelian group \( \hat{A} \). Upon this identification \( R, \overline{R}, \) and \( S \) become the set of all unary operations in \( C, \Upsilon_4(C), \) and \( \Lambda_\omega(C) \), respectively. Hence \( R \subseteq \overline{R}, R \subseteq S, \) and showing that \( C \) is 4-ultraclosed amounts to showing that \( R = \overline{R} \).

It follows from Jacobson’s Density Theorem that the scalar ring \( S \) of the local closure \( \Lambda_\omega(C) \) of \( C \) is the double centralizer ring of \( R \). As a reminder, if \( R^A \) is a simple left \( R \)-module and \( D = \End(R^A) \) is the (single) centralizer ring, then by Schur’s Lemma, \( D \) is a division ring. We let \( D \) act on \( A \) on the right, making \( A_D \) a right \( D \)-vector space. The double centralizer ring is the ring \( \End(A_D) \) of \( D \)-linear maps, which will act on the left. It is clear that \( R \subseteq \End(A_D) \). The Density Theorem asserts that the ring \( R \) of \( D \)-linear maps is dense in the ring \( \End(A_D) \) of all \( D \)-linear maps in the sense that every map \( f \in \End(A_D) \) can be interpolated on each finite subset of \( A_D \) by a map in \( R \). In our language this asserts that the local closure \( \Lambda_\omega(C) \) of the clone \( C \) of \( R^A \) is the clone of the module \( \End(A_D)A \). Thus, \( S = \End(A_D) \).

Next we want to show that \( \Lambda_\omega(C) = \Lambda_4(C) \). Let \( \mathcal{R} \) be the set consisting of the following relations on \( A \): the graph \( \gamma(+) \) of the binary operation \(+\) (addition of the module \( R^A \), and the graphs \( \gamma(d) \) of all unary operations \( d \in D \) (endomorphisms of the module \( S^A \)). All relations in \( \mathcal{R} \) have arity \( \leq 3 \), therefore the clone \( \Pol(\mathcal{R}) \) is 4-closed by Lemma 2.2(1). Using the fact (see the proof of Corollary 2.5) that \( \Pol(\gamma(+)) \) is the clone of the module \( \End(A_D^A)A \) one can easily check that \( \Pol(\mathcal{R}) \) coincides with the clone of the module \( \End(A_D^A)A \). Since the clone of \( \End(A_D^A)A \) is \( \Lambda_\omega(C) \), we get that \( \Lambda_\omega(C) \) is 4-closed. This implies that \( \Lambda_\omega(C) = \Lambda_4(C) \), as claimed.

It follows now from Lemma 2.2(2) that

\[
\Clone(R^A) = C \subseteq \Upsilon_4(C) \subseteq \Lambda_4(C) = \Lambda_\omega(C) = \Clone(S^A)
\]
where the leftmost term is the clone of $\mathcal{R}A$ and the rightmost term is the clone of $\mathcal{S}A$, $\mathcal{S} = \text{End}(\mathcal{A}_D)$. Hence, for the unary components of these clones we have that $\mathcal{R} \subseteq \mathcal{R} \subseteq \mathcal{S}$. Consequently, to establish that $\mathcal{C}$ is 4-ultraclosed, i.e., $\mathcal{R} = \mathcal{R}$, it remains to show for every $D$-linear map $f \in \mathcal{S} = \text{End}(\mathcal{A}_D)$ that if $f$ is in the 4-ultraclosure of $\mathcal{C}$, then $f \in \mathcal{R}$. There is nothing to prove if the set $\mathcal{A}$ is finite, because then $\mathcal{C} = \Lambda_n(\mathcal{C})$ (see Lemma 2.11(3)), and hence by the last displayed line $\mathcal{C} = \Upsilon_4(\mathcal{C})$.

Assume from now on that $\mathcal{A}$ is infinite, let $f \in \mathcal{S} = \text{End}(\mathcal{A}_D)$, and suppose $f$ is in the 4-ultraclosure of $\mathcal{C}$. Our goal is to prove that $f \in \mathcal{R}$. We will apply to $f$ the criterion of Corollary 3.2 for $\kappa = 4$ in the case $n = \lambda = 1$. By condition $(\dagger)_1$, for $n = 1$, the set $\mathcal{A}$ has a finite cover $\mathcal{C}_1 = \{B_0, \ldots, B_{m-1}\}$ such that whenever $B_i \in \mathcal{C}_1$ ($i < m$), there is an element $r_i \in \mathcal{R}$ that interpolates $f$ on $B_i$ (that is, $f|_{B_i} = r_i|_{B_i}$). Since $f$ and $r_i$ are both $D$-linear mappings, the kernel of $f - r_i$ is a $D$-subspace of $\mathcal{A}_D$ containing $B_i$. Hence, we may enlarge each set $B_i$ to $B'_i = \ker(f - r_i)$ and still have a finite cover $\{B'_0, \ldots, B'_{m-1}\}$ of $\mathcal{A}$ such that $f|_{B'_i} = r_i|_{B'_i}$ for each $i < m$, but now we have that the sets $B'_i$ ($i < m$) are $D$-subspaces of $\mathcal{A}_D$. Replacing each $B_i$ with $B'_i$ and dropping the primes, we now assume that our original set $\mathcal{C}_1$ consisted of $D$-subspaces of $\mathcal{A}_D$.

We may, in fact, assume more. Recall that our goal is to prove that the $D$-linear map $f$ is in $\mathcal{R}$. But the $D$-linear map $f$ is in $\mathcal{R}$ iff the $D$-linear map $f - r_0$ is in $\mathcal{R}$. Therefore, we may replace each of $f, r_0, r_1, \ldots, r_{m-1}$ with $f' := f - r_0, r'_0 := r_0 - r_0, r'_1 := r_1 - r_0, \ldots, r'_{m-1} := r_{m-1} - r_0$ and prove the desired statement in the setting where the first scalar $r'_0 = r_0 - r_0$ is zero. Dropping the primes we henceforth assume that $f|_{B_i} = r_i|_{B_i}$ for all $i < m$, and that the first ring element on the list, $r_0$, equals 0.

If $D$ is infinite, then there is nothing more to do. It is known that a vector space $\mathcal{A}_D$ over an infinite division ring $D$ cannot be expressed as a finite union of proper subspaces, so $\mathcal{A} = B_j$ must hold for some $j < m$. In this case, $f = f|_{A} = r_j|_{A} = r_j$, so $f \in \mathcal{R}$, as desired.

Henceforth we assume that $D$ is a finite field. Since the vector space $\mathcal{A}_D$ is infinite, $\mathcal{A}_D$ must be infinite dimensional. In this situation we use Neumann’s Lemma [4, 5], which asserts that if a group $G$ is expressible as a finite, irredundant union of cosets of subgroups, $G = \bigcup_{i < n} g_i H_i$, then the index $[G : \bigcap_{i < n} H_i]$ is finite. Here we take $G = \mathcal{A}$ and $g_i H_i = B_i$ to obtain (after discarding some of the $B_i$’s, if the cover $\mathcal{C}_i$ is redundant) that the intersection $I := \bigcap B_i$ is a $D$-subspace of $\mathcal{A}$ that has finite group-theoretic index in $A$. Since $f|_{I} = r_0|_{I} = \cdots = r_{m-1}|_{I}$ and $r_0 = 0$, we derive that each of the $D$-linear maps $f, r_0, \ldots, r_{m-1}$ contains $I$ in its kernel. Since $I$ has finite group-theoretic index in $\mathcal{A}$, the images of the maps $f, r_0, \ldots, r_{m-1}$ are all finite. In particular, the $D$-subspaces $r_0 A, \ldots, r_{m-1} A$ are finite subspaces of the infinite dimensional $D$-space $\mathcal{A}_D$.

Choose $m$ independent subspaces of $\mathcal{A}, V_0, \ldots, V_{m-1}$, for which there exist $D$-linear isomorphisms $\sigma_i: r_i A \to V_i$ ($i < m$). This is possible since each $r_i A$ is a
finite dimensional subspace of the infinite dimensional space $A_D$. By the facts that $R$ is dense in $S = \text{End}(A_D)$ and that each $r_iA$ ($i < m$) is finite dimensional, there exist $s_i \in R$ such that $s_i|_{r_iA} = \sigma_i$ for all $i < m$. Consider the ring element $t = s_0r_0 + \cdots + s_{m-1}r_{m-1}$ in $R$.

**Claim 5.2.** The $D$-linear map $t$ has kernel contained in $\ker(f)$.

**Proof of Claim 5.2.** Choose a vector $v \in A$ and assume that $0 = tv = \sum_{i<m} s_ir_iv$. Since the $s_i$’s have independent ranges, it follows that $s_ir_iv = 0$ for all $i < m$. But since $s_i$ is an isomorphism defined on the range of $r_i$, we even get that $r_iv = 0$ for all $i < m$. This implies that $v \in \bigcap_{i<m} \ker(r_i)$. Now, since $v \in A = \bigcup_{i<m} B_i$, there is some $i < m$ such that $v \in B_i$, and for this $i$ we have $f(v) = r_iv = 0$. Hence $v \in \ker(f)$.

At this point we know that $t$ and $f$ are $D$-linear endomorphisms of the space $A_D$, and that $\ker(t) \subseteq \ker(f)$. It follows from the First Isomorphism Theorem of linear algebra that there is a $D$-linear map $u$ such that $ut = f$. Since the image of $t$, $tA \subseteq \sum_{i<m} V_i$, is finite dimensional, the Density Theorem allows us to interpolate $u$ on $tA$ by an element $u' \in R$. In fact, since $u'$ is itself $D$-linear, there is no harm in assuming that $u = u'$, so that $u \in R$. With this choice $f = ut \in R$.

To summarize, we argued that if an operation $f: A \to A$ belongs to the unary component of the 4-ultraclosure of the clone of $R\Lambda$, then in fact $f$ equals an operation in the unary component of the clone of $R\Lambda$. This establishes that the clone of $R\Lambda$ is 4-ultraclosed. By Lemma 2.1(2) it follows also that the clone of $R\Lambda$ is ultralocally closed.

\[ \square \]

6. $\Lambda_\omega$ versus $\Upsilon_\omega$

In this final section we discuss some similarities and dissimilarities between local closure and ultralocal closure. Since both $\Lambda_\omega$ and $\Upsilon_\omega$ equal the identity operator on the lattice of clones on a finite set, we will assume throughout that the base set $A$ is infinite.

It is known (see, e.g., [3], [2, p. 367], or Subsection 6.C below) that there are $2^{2^\nu}$ clones on an infinite set $A$ of cardinality $\nu$. Among these, only $2^\nu$ are locally closed (see, e.g. [2, p. 396]), which shows that the range of the closure operator $\Lambda_\omega$ on the lattice of clones on $A$ is small. One of our goals in this section is to prove the theorem below, which shows that, in contrast to $\Lambda_\omega$, the range of the closure operator $\Upsilon_\omega$ on the lattice of clones on $A$ is large.

**Theorem 6.1.** If $A$ is an infinite set of cardinality $\nu$, then the lattice of clones on $A$ contains

1. an interval $[C_1, D_1]$ of size $2^{2^\nu}$ such that every clone in the interval is ultralocally closed, and
(2) an interval $[C_2, D_2]$ of size $2^\nu$ such that no clone in the interval is ultralocally closed.

In fact, the interval $[C_1, D_1]$ can be chosen so that $D_1 = O_A$, $C_1$ is generated by a single operation, and the interval contains $2^{2\nu}$ clones that are maximal in $D_1 = O_A$. The interval $[C_2, D_2]$ can be chosen so that it is isomorphic to the lattice of all clones on $A$, hence it also contains $2^{2\nu}$ clones that are maximal in $D_2$.

Another well-known fact (noted, e.g., in [2, p. 395]) is that if $A$ is an infinite set, then the lattice of all locally closed clones on $A$ is not algebraic. Equivalently, the closure operator

$$\Lambda_\omega(-) : \mathcal{P}(O_A) \to \mathcal{P}(O_A), \quad F \mapsto \Lambda_\omega(\langle F \rangle)$$

on $O_A$, which assigns to each set of operations the least locally closed clone containing it, is not an algebraic closure operator. Here we say that a closure operator on a set $S$ is algebraic if for any set $X \subseteq S$, $X$ is closed if and only if $X$ is the set-theoretic union of the closures of its finite subsets.

Analogously, given an infinite cardinal $\kappa$, we will say that a closure operator $\neg: \mathcal{P}(S) \to \mathcal{P}(S), X \mapsto \overline{X}$ on $S$ is $\kappa$-algebraic if for any set $X \subseteq S$,

$$(6.1) \quad X = \overline{X} \iff X = \bigcup \{Y : Y \subseteq X, |Y| < \kappa\}.$$ 

So, a closure operator $\neg$ on $S$ is $\kappa$-algebraic if for any set $X \subseteq S$, $X$ is closed if and only if $X$ is the union of the closures of its subsets of size less than $\kappa$. In this terminology ‘algebraic’ is the same as ‘$\omega$-algebraic’.

**Theorem 6.2.** For arbitrary infinite set $A$, the closure operator

$$\Upsilon_\omega(-) : \mathcal{P}(O_A) \to \mathcal{P}(O_A), \quad F \mapsto \Upsilon_\omega(\langle F \rangle),$$

which assigns to each set of operations on $A$ the least ultralocally closed clone containing it,

(1) is not algebraic, but

(2) it is $\omega_1$-algebraic.

Thus, a clone $C$ on $A$ is ultralocally closed if and only if $C$ contains the ultralocal closure of every countably generated subclone of $C$.

Of course, for each set $A$, the local closure operator $\Lambda_\omega(-)$ on $O_A$ is $\kappa$-algebraic for large enough $\kappa$, say for $\kappa > 2^{|A|}$, because every clone on $A$ has size $\leq 2^{|A|}$. But there is no fixed $\kappa$ for which the local closure operator $\Lambda_\omega(-)$ is $\kappa$-algebraic for all infinite $A$, as the next theorem asserts.

**Theorem 6.3.** If $A$ is an infinite set of cardinality $\nu$, then the closure operator $\Lambda_\omega(\langle - \rangle)$ is not $\kappa$-algebraic for any infinite regular cardinal $\kappa \leq \nu$. 
Our last theorem answers a question posed by the referee: "Is it obvious that the full clone on a countable set is (or is not) the ultralocal closure of a finite (countable) clone?" The theorem implies that the full clone on a countable set is not the ultralocal closure of any set of functions of cardinality less than $2^\omega$.

**Theorem 6.4.** Let $A$ be an infinite set of cardinality $\omega$, and let $F \subseteq O_A$. If $\Upsilon_\omega(\langle F \rangle)$ is an uncountable clone that contains a near unanimity operation, then $|F| = |\Upsilon_\omega(\langle F \rangle)|$. In particular, if $\Upsilon_\omega(\langle F \rangle) = O_A$, then $|F| = 2^\omega$.

Before proving these results in Subsection 6.D, we discuss some examples.

### 6.A. Alternating groups and their clones.

For an arbitrary set $A$ and for any permutation $\pi$ of $A$ the support of $\pi$ is defined to be the set $\text{supp}(\pi) := \{ a \in A : \pi(a) \neq a \}$. We will denote the group of all permutations of $A$ of finite support by $\text{Sym}_\omega(A)$. The alternating group on $A$ is the subgroup $\text{Alt}(A)$ of $\text{Sym}_\omega(A)$ consisting of all even permutations. The essentially unary clones generated by the groups $\text{Alt}(A)$ and $\text{Sym}_\omega(A)$ will be denoted by $\text{Alt}(A)$ and $\text{Sym}_\omega(A)$, respectively. They are different when $|A| > 1$, since $\text{Sym}_\omega(A)$ will contain odd permutations of $A$ and $\text{Alt}(A)$ will not.

The next theorem describes an example of a clone that is not ultralocally closed. This example will also play a role in the proof of Theorem 6.1(2).

**Theorem 6.5.** If $A$ is an infinite set, then the clone $\text{Alt}(A)$ is not ultralocally closed. Its ultralocal closure is the clone $\text{Sym}_\omega(A)$. In fact,

1. $\Upsilon_d(\text{Alt}(A)) = \text{Sym}_\omega(A)$ for all $4 \leq d \leq \omega$; while
2. $\Lambda_d(\text{Alt}(A))$ is the essentially unary clone generated by the monoid of all injective unary operations $A \to A$, for all $4 \leq d \leq \omega$.

**Proof.** The first two sentences of the claim, which assert that $\text{Sym}_\omega(A)$ is the ultralocal closure of $\text{Alt}(A)$, follow from (i) when $d = \omega$. To prove (i)–(ii), fix $d$ such that $4 \leq d \leq \omega$. It follows from Corollary 2.3(1)–(2) that both clones $\Upsilon_d(\text{Alt}(A))$ and $\Lambda_d(\text{Alt}(A))$ are essentially unary, and every unary operation $f : A \to A$ in them is injective. Thus, in both statements (i)–(ii), the clone equalities follow if we establish that the clones involved contain the same injective unary operations $A \to A$.

Now, to finish the proof of (ii), it is enough to observe that every injective unary operation $A \to A$ is $k$-interpolable by permutations in $\text{Alt}(A)$ for every $k < d$.

For the proof of (i) recall that our assumption $d \leq \omega$ implies, by Lemma 2.1(2), that $\Upsilon_d(\text{Alt}(A)) \supseteq \Upsilon_\omega(\text{Alt}(A))$. Hence, the equality in (i) will follow if we prove that for all injective unary operations $f : A \to A$,

\begin{equation}
 f \in \Upsilon_d(\text{Alt}(A)) \Rightarrow f \in \text{Sym}_\omega(A) \Rightarrow f \in \Upsilon_\omega(\text{Alt}(A)).
\end{equation}

To prove the first implication in (6.2) assume that $f \in \Upsilon_d(\text{Alt}(A))$ is injective. Applying Corollary 3.2 with $\kappa = d$ and $\lambda = 1$ we see that $A$ has a finite cover $\mathcal{C}_1$ with the property that for each $C \in \mathcal{C}_1$ there exists $t^{[C]} \in \text{Alt}(A)$ such that
that moves at most finitely many elements of $A$ in the set of unary members of $\Upsilon$. Since $\Upsilon$ is closed under composition, and since $\text{Sym}_{\omega}(A)$ is generated under composition by all transpositions, it suffices to verify that $\Upsilon(\text{Alt}(A))$ contains every transposition. To conclude that the transposition $f = (a \ b)$ ($a, b \in A$, $a \neq b$) belongs to $\Upsilon(\text{Alt}(A))$ we need to show, by Corollary 3.2, that condition (‡) holds for all $k < \omega$. There is nothing to prove for $k = 0$, so assume that $k$ is a positive integer. Choose $C_k$ to be any partition of $A$ into $k + 1$ blocks $C_0, C_1, \ldots, C_k$ such that $a, b \in C_0$ and every block $C_i$ ($i \leq k$) has size $\geq 2$. Clearly, such a partition exists, since $A$ is infinite. For every block $C$ with $|C| \leq k$ we have $(a b)|_C = \text{id}$ if $C_0 \notin C$, and $(a b)|_C = (a b)(c d)|_C$ if $C_0 \in C$ and $c, d$ are distinct elements of some $C_i \notin C$. This proves that $(a b) \in \Upsilon(\text{Alt}(A))$, as claimed. \hfill $\blacksquare$

For every finite subset $B$ of $A$ let $\text{Alt}_B(A)$ denote the subgroup of $\text{Alt}(A)$ consisting of all permutations $\pi \in \text{Alt}(A)$ with $\text{supp}(\pi) \subseteq B$. Let $\text{Alt}_B(A)$ denote the essentially unary clone generated by the group $\text{Alt}_B(A)$.

We will now show that these clones $\text{Alt}_B(A)$, unlike $\text{Alt}(A)$, are ultralocally closed. We will use these clones in the proof of Theorem 6.2(1).

**Lemma 6.6.** If $B$ is a finite subset of an infinite set $A$, then the clone $\text{Alt}_B(A)$ is locally closed, and hence is ultralocally closed; that is,

$$\text{Alt}_B(A) = \Upsilon(\text{Alt}_B(A)) = \Lambda(\text{Alt}_B(A)).$$

**Proof.** By Corollary 2.3(1), all three clones here are essentially unary. Hence, by Lemma 2.1(1), it suffices to show for every unary operation $f \in \Lambda(\text{Alt}_B(A))$ that $f \in \text{Alt}_B(A)$. So, let $f : A \to A$ be a unary operation in $\Lambda(\text{Alt}_B(A))$. Then $f$ is interpolated by a permutation $\pi_C \in \text{Alt}(A)$ for any finite set $C = B \cup \{a\}$ where $a \in A \setminus B$. Since $a \notin B$, we have $a \notin \text{supp}(\pi_C)$, so $f(a) = \pi_C(a) = a$. Letting $a \in A$ vary, we conclude that $f$ is the identity function off of $B$, while $f$ agrees with $\pi_C \in \text{Alt}_B(A)$ on $B$. Hence, $f \in \text{Alt}_B(A)$.

\hfill $\blacksquare$

**6.B. Product Clones.** Product clones were defined in Section 2 in the paragraph preceding Corollary 2.6. Here we want to show that for large enough $\kappa$, both closure operators $\Upsilon_\kappa$ and $\Lambda_\kappa$ commute with the formation of product clones. The special case $\kappa = \omega$ will be applied in the proof of Theorem 6.1(2).

**Lemma 6.7.** Let $P$ be a clone on $A$ and $Q$ a clone on $B$.

(i) $\Upsilon_\kappa(P \times Q) = \Upsilon_\kappa(P) \times \Upsilon_\kappa(Q)$ for all $\kappa \geq 4$, and

(ii) $\Lambda_\kappa(P \times Q) = \Lambda_\kappa(P) \times \Lambda_\kappa(Q)$ for all $\kappa \geq 4$. 


Proof. Let $\kappa \geq 4$. We know from Corollary 2.6 that both clones $\Upsilon_{\kappa}(P \times Q)$ and $\Lambda_{\kappa}(P \times Q)$ are product clones on $A \times B$. Hence the equalities in statements (i)–(ii) will follow if we prove the following fact for all $0 < n < \omega$ and all cardinals $\lambda < \kappa$:

$(\diamond_{n,\lambda})$ a product operation $f \times g$, where $f$ is an $n$-ary operation on $A$ and $g$ is an $n$-ary operation on $B$, is $\lambda$-ultrainterpolable [\(\lambda\)-interpolable] by $P \times Q$ if and only if $f$ is $\lambda$-ultrainterpolable [\(\lambda\)-interpolable] by $P$ and $g$ is $\lambda$-ultrainterpolable [\(\lambda\)-interpolable] by $Q$.

The part of $(\diamond_{n,\lambda})$ that refers to $\lambda$-interpolability is an immediate consequence of the definitions. Alternatively, one can use Lemma 2.2(3) and the extension of [6, Satz 2.3.7(vi)] to relations of arbitrary (possibly infinite) arity. This proves the equality in Lemma 6.7(ii).

Now we prove the part of $(\diamond_{n,\lambda})$ that refers to $\lambda$-ultrainterpolability. This will prove the equality in Lemma 6.7(i). Let $f \times g$ be a product operation as in $(\diamond_{n,\lambda})$. Recall from the definition that $f \times g$ is $\lambda$-ultrainterpolable by $P \times Q$ if and only if

(1) the operation $(f \times g)_U$ is $\lambda$-interpolable by the clone $(P \times Q)_U$ for every ultrafilter $U$ on any nonempty set $I$.

Similarly, $f$ is $\lambda$-ultrainterpolable by $P$ and $g$ is $\lambda$-ultrainterpolable by $Q$ if and only if

(2) $f_U$ is $\lambda$-interpolable by $P_U$ and $g_U$ is $\lambda$-interpolable by $Q_U$ for every ultrafilter $U$ on any nonempty set $I$.

The proof of $(\diamond_{n,\lambda})$ for $\lambda$-ultrainterpolability will be complete if we show that conditions (1) and (2) are equivalent.

Fix $I$ and $U$. The following map is a bijection between the sets $(A \times B)^I/U$ and $(A^I/U) \times (B^I/U):

$$
\varepsilon: (A \times B)^I/U \to (A^I/U) \times (B^I/U), \quad ((a_i, b_i))_{i \in I}/U \mapsto ((a_i)_{i \in I}/U, (b_i)_{i \in I}/U).
$$

By identifying $(A \times B)^I/U$ and $(A^I/U) \times (B^I/U)$ via $\varepsilon$ we see that for any two $n$-ary operations $h_A$ on $A$ and $h_B$ on $B$ we have that $(h_A \times h_B)_U$ and $(h_A)_U \times (h_B)_U$ are the same operation. In particular, this implies that $(f \times g)_U$ and $f_U \times g_U$ are the same operation, and $(P \times Q)_U$ and $P_U \times Q_U$ are the same clone.

Thus, (1) is equivalent to the condition that $f_U \times g_U$ is $\lambda$-interpolable by $P_U \times Q_U$ for every ultrafilter $U$ on any nonempty set $I$. As we observed in the proof (ii), this condition is equivalent to condition (2), which completes the proof.

Corollary 6.8. Let $P$ be a clone on $A$, $Q$ a clone on $B$, and let $\kappa \geq 4$ be a cardinal. The product clone $P \times Q$ is $\kappa$-ultraclosed if and only if both $P$ and $Q$ are $\kappa$-ultraclosed.

6.C. Goldstern–Shelah clones. Given an infinite set $A$ and a maximal ideal $I$ of the Boolean algebra $P(A)$, Goldstern and Shelah define in [3, Definition 2.1] a clone $C(I)$ by specifying that $f \in C(I)$ iff for each $S \in I$ we have $f(S, S, \ldots, S) \in I$. 
They prove that \( C(I) \) is a maximal clone on \( A \), and that if \( I \) and \( J \) are distinct maximal ideals of \( P(A) \), then \( C(I) \) and \( C(J) \) are distinct maximal clones on \( A \). It is known that there exist \( 2^{2^{|A|}} \)-many maximal ideals in \( P(A) \), so this construction produces \( 2^{2^{|A|}} \)-many maximal clones on \( A \). This number is the same as the number of all clones on \( A \).

We now derive from Theorem 4.1 that all the Goldstern–Shelah clones are 3-ultraclosed, and hence are ultralocally closed.

**Corollary 6.9.** Every Goldstern–Shelah clone \( C(I) \) contains a ternary near unanimity operation. Consequently, every such clone is 3-ultraclosed; that is, it satisfies \( C(I) = \Upsilon_3(C(I)) \).

**Proof.** An operation \( f : A^n \to A \) is called conservative if \( f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\} \) for every tuple \( (a_1, \ldots, a_n) \in A^n \). If \( f \) is a conservative operation on \( A \), \( I \) is a maximal ideal of \( P(A) \), and \( S \in I \), then \( f(S, \ldots, S) \subseteq S \in I \), so \( f \in C(I) \). Since any set supports a conservative ternary near unanimity operation, any Goldstern–Shelah clone \( C(I) \) contains a ternary near unanimity operation. By Theorem 4.1 we have \( C(I) = \Upsilon_3(C(I)) \). \( \square \)

**6.D. Proofs of Theorems 6.1–6.4.** We start with Theorem 6.1, which is about the existence of large intervals in the clone lattice such that either all clones in the interval are ultralocally closed, or none of them are ultralocally closed.

**Proof of Theorem 6.1.** Let \( A \) be an infinite set of cardinality \( \nu \).

To construct a large interval \([C_1, D_1]\) of ultralocally closed clones on \( A \) let \( D_1 := O_A \). By the discussion at the beginning of Subsection 6.C there are \( 2^{2^\nu} \) Goldstern–Shelah clones \( C(I) \) on \( A \), each one a maximal subclone of \( O_A \), where \( I \) runs over all maximal ideals of the Boolean algebra \( P(A) \). We also saw in the proof of Corollary 6.9 that if \( f \) is a conservative ternary near unanimity operation on \( A \) then \( f \in C(I) \) for all \( I \). Therefore, if we let \( C_1 := \langle f \rangle \) for a fixed such \( f \), then we get that all Goldstern–Shelah clones \( C(I) \) are in the interval \([C_1, D_1]\). Every clone in this interval contains \( f \), and hence is ultralocally closed by Theorem 4.1. The Goldstern–Shelah clones in \([C_1, D_1]\) witness that the interval \([C_1, D_1]\) contains \( 2^{2^\nu} \) clones that are maximal in \( D_1 \), and hence \([C_1, D_1]\) has size \( 2^{2^\nu} \). This proves statement (1) of Theorem 4.1 and all properties of the interval \([C_1, D_1]\) claimed in the last paragraph of the theorem.

To show that there also exists a large interval of clones on \( A \) such that none of the clones in the interval are ultralocally closed, we will first work in the lattice of clones on the set \( A \times A \). Consider the product clones \( \text{Alt}(A) \times C \) where \( \text{Alt}(A) \) is the essentially unary clone generated by the alternating group on \( A \) (see Subsection 6.A), and \( C \) is an arbitrary clone on \( A \). It follows from Claim 2.7(2) that the product clones on \( A \times A \) form an interval in the lattice of clones on \( A \times A \). So, the product clones
$\text{Alt}(A) \times C$ with fixed first coordinate $\text{Alt}(A)$ form a subinterval, namely

\[(6.3) \quad [\text{Alt}(A) \times \langle \rangle, \text{Alt}(A) \times O_A],\]

where $\langle \rangle$ is the clone of projections on $A$ (i.e., the subclone of $O_A$ generated by the empty set of operations). Clearly, the interval $(6.3)$ is isomorphic to the lattice of all clones on $A$; in particular, there are $2^{2^\nu}$ clones in the interval that are maximal in the clone $\text{Alt}(A) \times O_A$ at the top. Moreover, Corollary \ref{cor:6.8} implies that none of the clones in the interval $(6.3)$ are ultralocally closed, because by Theorem \ref{thm:6.5}, $\text{Alt}(A)$ is not ultralocally closed.

Since $|A \times A| = |A| = \nu$, the clone of all operations on $A \times A$ is isomorphic to the clone of all operations on $A$. Hence the result proved in the preceding paragraph completes the proof of the existence of an interval $[C_2, D_2]$ with the properties stated in part (2) and the last paragraph of Theorem \ref{thm:6.1}. \hfill \Box

**Remark 6.10.** The proof of Theorem \ref{thm:6.1}(1) shows how to find $2^{2^\nu}$ ultralocally closed, near unanimity clones on an infinite set $A$ of cardinality $\nu$. The referee of this paper suggested the following idea for constructing $2^{2^\nu}$ ultralocally closed, essentially unary clones on $A$.

Assume $0, 1$ are distinct elements of $A$, and let $\mathcal{A}$ be an *independent family* of subsets of $A \setminus \{0, 1\}$ (i.e., $\mathcal{A}$ is a free generating set of the Boolean subalgebra of $\mathcal{P}(A \setminus \{0, 1\})$ generated by $\mathcal{A}$). For each set $X \in \mathcal{A}$ let $f_X : A \to A$ be the characteristic function of $X$ (i.e., $f_X(a) = 1$ if $a \in X$ and $f_X(a) = 0$ if $a \in A \setminus X$). The claim is that for every subset $S$ of $\mathcal{A}$,

$$f_X \in \Upsilon_\omega((f_S : S \in S)) \quad \text{if and only if} \quad X \in S,$$

hence the ultralocally closed clones $\Upsilon((f_S : S \in S))$ ($S \subseteq \mathcal{A}$), which are essentially unary by Corollary \ref{cor:2.3}, form an ordered subset in the clone lattice on $A$ that is order isomorphic to the power set of $\mathcal{A}$. Since for every infinite set of size $\nu$ there exists an independent family $\mathcal{A}$ of subsets such that $|\mathcal{A}| = 2^\nu$, this construction yields an ordered set of essentially unary, ultralocally closed clones on $A$ that is order isomorphic to the power set of a $2^\nu$-element set.

Our second result to be proved here is Theorem \ref{thm:6.2} which is about the algebraicity degree of $\Upsilon_\omega(\langle \rangle)$.

**Proof of Theorem \ref{thm:6.2}.** Let $A$ be any infinite set. For the proof of statement (1), which asserts that the closure operator $\Upsilon_\omega(\langle \rangle)$ is not algebraic, we will use the clones $\text{Alt}(A)$ and $\text{Alt}_B(A)$ discussed in Subsection 6.A. It is clear from the definition of $\text{Alt}(A)$ that every finite subset of $\text{Alt}(A)$ is contained in $\text{Alt}_B(A)$ for some finite $B \subseteq A$. We also
know from Lemma 6.6 that each such clone $\text{Alt}_B(A)$ is ultralocally closed. Therefore

$$
\bigcup \{ \Upsilon_\omega(\langle F \rangle) : F \subseteq \text{Alt}(A), |F| < \omega \} \subseteq \bigcup \{ \Upsilon_\omega(\text{Alt}_B(A)) : B \subseteq A, |B| < \omega \}
$$

$$
= \bigcup \{ \text{Alt}_B(A) : B \subseteq A, |B| < \omega \}
$$

$$
= \text{Alt}(A).
$$

Actually, $=$ holds in place of $\subseteq$ above, because every term $\Upsilon_\omega(\text{Alt}_B(A)) (|B| < \omega)$ in the union on the right hand side can be rewritten as $\Upsilon_\omega(\langle \text{Alt}_B(A) \rangle)$, where $\text{Alt}_B(A)$ is a finite set of permutations of $A$. Hence, every term in the union on the right hand side of $\subseteq$ appears as a term in the union on the left hand side as well, proving that $\supseteq$ also holds. This implies that

$$
\text{Alt}(A) = \bigcup \{ \Upsilon_\omega(\langle F \rangle) : F \subseteq \text{Alt}(A), |F| < \omega \}.
$$

On the other hand, we have by Theorem 6.5 that

$$
\text{Alt}(A) \subsetneq \text{Sym}_\omega(A) = \Upsilon_\omega(\text{Alt}(A)).
$$

This proves that the closure operator $\Upsilon_\omega(-)$ is not algebraic.

For claim (2), which states that the closure operator $\Upsilon_\omega(-)$ is $\omega_1$-algebraic, it suffices to show that the following equality holds for any set $G$ of operations on $A$:

$$(6.4) \quad \Upsilon_\omega(\langle G \rangle) = \bigcup \{ \Upsilon_\omega(\langle F \rangle) : F \subseteq G, |F| \leq \omega \}.$$  

Indeed, (6.4) immediately implies that for any set $G$ of operations on $A$,

$$
G = \Upsilon_\omega(\langle G \rangle) \iff G = \bigcup \{ \Upsilon_\omega(\langle F \rangle) : F \subseteq G, |F| \leq \omega \},
$$

which is the defining property for $\Upsilon_\omega(-)$ to be $\omega_1$-algebraic. (See (6.1).)

Now we prove (6.4). The inclusion $\supseteq$ holds because $\Upsilon_\omega(-)$ is a closure operator. For the reverse inclusion, let $f$ be an operation in $\Upsilon_\omega(\langle G \rangle)$, say $f$ is $n$-ary. By Corollary 3.2, this means that

$$
(\dagger) \quad \text{for every } k < \omega, A^n (= \text{dom}(f)) \text{ has a finite cover } C_k (\subseteq \mathcal{P}(A^n)) \text{ such that whenever } B \subseteq C_k \text{ satisfies } |B| \leq k, \text{ there exists an } n\text{-ary } t^{[B]} \in \langle G \rangle \text{ such that } f|_{\bigcup B} = t^{[B]}|_{\bigcup B}.
$$

For each fixed $k < \omega$, there are finitely many choices for $B$, and for each choice of $B$, the operation $t^{[B]} \in \langle G \rangle$ is generated by a finite subset of $G$. Therefore there exists a finite subset $F_k$ of $G$ such that condition $(\dagger)$ holds for that $k$ with $F_k$ in place of $G$. Hence, by letting $F := \bigcup \{ F_k : k < \omega \}$, we see that $|F| \leq \omega$, and $(\dagger)$ holds for $F$ in place of $G$. This shows that $f \in \Upsilon_\omega(\langle F \rangle)$, and completes the proof of (6.4) and statement (2).

The last statement of Theorem 6.2 is a reformulation of the statement that the closure operator $\Upsilon_\omega(-)$ is $\omega_1$-algebraic. □

Next we prove Theorem 6.3 about the algebraicity degree of $\Lambda_\omega(-)$. 
Proof of Theorem 6.3. Let $A$ be an infinite set of cardinality $\nu$. Our task is to show that the closure operator $\Lambda_\omega(\langle \cdot \rangle)$ on $O_A$ is not $\kappa$-algebraic for any infinite regular cardinal $\kappa \leq \nu$. For each subset $B$ of $A$ let $\text{Inj}_B(A)$ denote the set of all injective functions $f : A \to A$ with ‘support’ in $B$, by which we mean all injective functions $f : A \to A$ satisfying $f(a) = a$ for all $a \in A \setminus B$. It is clear that, for each subset $B \subseteq A$, $\text{Inj}_B(A)$ is closed under composition, and hence it generates an essentially unary clone $\text{Inj}_B(A)$ with unary part $\text{Inj}_B(A)$. We claim that the clone $\text{Inj}_B(A)$ is locally closed. Indeed, by Corollary 2.3, $\Lambda(\text{Inj}_B(A))$ is an essentially unary clone, and every unary operation in it is injective. Furthermore, since every injective function $g : A \to A$ in $\Lambda_\omega(\text{Inj}_B(A))$ agrees, on each singleton set $\{a\} \subseteq A \setminus B$, with some function in $\text{Inj}_B(A)$, we get that $g \in \text{Inj}_B(A)$. This implies that $\text{Inj}_B(A)$ is a locally closed clone for every set $B \subseteq A$.

We will use these clones to show that $\Lambda_\omega(\langle \cdot \rangle)$ is not $\kappa$-algebraic for any infinite regular $\kappa \leq \nu$. Fix such a $\kappa \leq \nu = |A|$. In what follows, a set $X$ is called $\kappa$-small if $|X| < \kappa$. Let

$$G := \bigcup \{\text{Inj}_B(A) : B \subseteq A, |B| < \kappa\}$$

(which is a union of clones), and let $G := \bigcup \{\text{Inj}_B(A) : B \subseteq A, |B| < \kappa\}$ (which is a union of sets of unary functions). $G$ is closed under composition, for if $f_i \in \text{Inj}_B(A)$ with $|B_i| < \kappa$ ($i < 2$), then $f_1 \circ f_0 \in \text{Inj}_{B_0 \cup B_1}(A)$, and $|B_0 \cup B_1| < \kappa$. It follows that $G$ is an essentially unary clone with unary part $G$. Our goal is to show that

$$G = \bigcup \{\Lambda_\omega(\langle F \rangle) : F \subseteq G, |F| < \kappa\}$$

and

$$G \subseteq \Lambda_\omega(G) = \Lambda_\omega(\langle G \rangle).$$

This will prove that the closure operator $\Lambda_\omega(\langle \cdot \rangle)$ on $O_A$ is not $\kappa$-algebraic. (See (6.1).)

In (6.5) the inclusion $\subseteq$ clearly holds, because $f \in G$ implies that $f \in \Lambda_\omega(\langle F \rangle)$ for $F = \{f\} \subseteq G$ with $|F| = 1 < \kappa$. To prove the reverse inclusion, recall that each operation $f \in G$ is a member of $\text{Inj}_B(A)$ for some $\kappa$-small subset $B \subseteq A$. Therefore for every $\kappa$-small set $F \subseteq G$ which appears on the right hand side of (6.5) there exists a ‘support selecting function’ $f \mapsto B_f$ such that $f \in \text{Inj}_{B_f}(A)$ and $|B_f| < \kappa$ for all $f \in F$. Since $\text{Inj}_B(A) \subseteq \text{Inj}_{B'}(A)$ whenever $B \subseteq B'$ ($\subseteq A$), we see that $F \subseteq \text{Inj}_{B_F}(A)$ holds for the set $B_F := \bigcup\{B_f : f \in F\}$. Since $F$ is $\kappa$-small, each $B_f$ is $\kappa$-small, and $\kappa$ is regular, $B_F$ is also $\kappa$-small. Thus, $\langle F \rangle \subseteq \text{Inj}_{B_F}(A)$. We proved earlier that the clone $\text{Inj}_{B_F}(A)$ is locally closed, therefore we obtain that $\Lambda_\omega(\langle F \rangle) \subseteq \Lambda_\omega(\text{Inj}_{B_F}(A)) = \text{Inj}_{B_F}(A)$. This inclusion holds for every $\kappa$-small set $F \subseteq G$ on the right hand side of (6.5), and so does the inequality $|B_F| < \kappa$. Hence, the right hand side of (6.5) is contained in $G$.

In (6.6) the equality $=$ holds, because $G$ is a clone, and hence $G = \langle G \rangle$. For the inclusion $\subseteq$ recall that $\kappa$ is an infinite cardinal such that $\kappa \leq \nu = |A|$. Furthermore, by its definition, $G$ is an essentially unary clone whose unary part $G$ consists of all
injective functions $A \to A$ of $\kappa$-small support. Therefore, $G$ does not contain all injections $A \to A$. By Corollary 2.3, the clone $\Lambda_\omega(G)$ is also essentially unary, and its unary part consists of injections $A \to A$. However, the unary part of $\Lambda_\omega(G)$ does contain all injections $A \to A$, because every injective function $A \to A$ is interpolable, on each finite set $S \subseteq A$, by injections of $\kappa$-small support. □

Finally, we prove Theorem 6.4.

**Proof of Theorem 6.4.** Let $A$ be an infinite set of cardinality $\nu$, and let $F \subseteq O_A$. The main statement of the theorem is that if the clone $\Upsilon_\omega(\langle F \rangle)$ is uncountable and contains a near unanimity operation, then $|F| = |\Upsilon_\omega(\langle F \rangle)|$. The second statement concerns the special case when $\Upsilon_\omega(\langle F \rangle) = O_A$. In this special case $\Upsilon_\omega(\langle F \rangle)$ clearly contains a near unanimity operation and $|\Upsilon_\omega(\langle F \rangle)| = |O_A| = 2^\nu$, so $\Upsilon_\omega(\langle F \rangle)$ is uncountable. Therefore the main statement yields the desired conclusion that $|F| = |\Upsilon_\omega(\langle F \rangle)| = 2^\nu$.

To prove the main statement, assume $C := \Upsilon_\omega(\langle F \rangle)$ is uncountable and $h$ is a near unanimity operation in $C$. In particular, we have that $C = \Upsilon_\omega(C)$ and $\langle F \cup \{h\} \rangle \subseteq C$. Thus,

$$C = \Upsilon_\omega(\langle F \rangle) \subseteq \Upsilon_\omega(\langle F \cup \{h\} \rangle) \subseteq \Upsilon_\omega(C) = C,$$

which implies that $C = \Upsilon_\omega(\langle F \cup \{h\} \rangle)$. However, since the clone $\langle F \cup \{h\} \rangle$ contains a near unanimity operation, we know from Theorem 4.1 that it is ultralocally closed, that is $\Upsilon_\omega(\langle F \cup \{h\} \rangle) = \langle F \cup \{h\} \rangle$. Hence, $C = \langle F \cup \{h\} \rangle$, and therefore $|C| = |F| + |h|$. Now the assumption that $C = \Upsilon_\omega(\langle F \rangle)$ is uncountable implies that $|F| = |C| = |\Upsilon_\omega(\langle F \rangle)|$, as claimed. □

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