Quantum Landau-Lifshitz model at four loops: $1/J$ and $1/J^2$ corrections to BMN energies

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Abstract

In a previous paper [hep-th/0510080] the effective Landau-Lifshitz (LL) Lagrangians in the $SU(2)$ sector coming from string theory and gauge theory have been found to three loops in the effective expansion parameter $\tilde{\lambda} = \lambda/J^2$. In this paper we continue this study and find the effective Landau-Lifshitz Lagrangians to four loops. We extend to four-loops $\tilde{\lambda}^4$ the computations of $1/J$ and $1/J^2$ corrections to BMN energies done in [hep-th/0510080] to three-loops. We compare these corrections obtained from quantum “gauge-theory” LL action with the corrections obtained from the conjectured Bethe ansatz for the long range spin chain representing perturbative large $N \mathcal{N} = 4$ Super Yang-Mills in the $SU(2)$ sector and find perfect matching to four loops $\tilde{\lambda}^4$. We compare also the $1/J$ and $1/J^2$ corrections obtained from quantum “string-theory” LL action with those obtained from the “quantum string” Bethe ansatz and again find perfect matching to four-loops.
1 Introduction

AdS/CFT correspondence implies that the quantum energy of a string state matches the quantum dimension of a corresponding operator in the dual $\mathcal{N} = 4$ Super Yang-Mills theory. Under certain conditions a class of string quantum states can be treated as semiclassical solutions and their classical energy can be compared with the dimension of corresponding operators on the gauge theory side. Going beyond semiclassical approximation it turns out that quantum corrections to semiclassical solutions of strings propagating in $AdS_5 \times S^5$ play an important role in the AdS/CFT duality [1, 2, 3].

The “three-loop discrepancy” between gauge and string theory predictions is by now a well known fact: it is present [4] for semiclassical spinning string solutions [5], and also in the computation of the $1/J$ correction to the two-impurity BMN state [6]. The computations of $1/J$ corrections to BMN in [6] and to circular string solution in [5] were done using full superstring theory, i.e. using the full set of world sheet fields, both bosons and fermions. Attempts to compute the next correction, i.e. $1/J^2$, to the BMN energies proved to be difficult [7], and have not yet been obtained from a full superstring computation. An alternative and much easier approach to compute $1/J^n$ corrections to BMN energies have been developed in [8] for the $SU(2)$ sector. It was shown in [9, 8] that by using the effective Landau-Lifshitz (LL) action one can ignore all fermions and bosonic modes outside the $SU(2)$ sector. The only effect of these modes in the effective LL theory in the $SU(2)$ sector is the proper regularization, which one needs to consider. As shown in [8] this can be accomplished with a combination of normal ordering and $\zeta$-function regularization. Let us point out that this discussion is valid only for analytic terms in $\tilde{\lambda}$. Non-analytic terms, i.e non-integer powers in $\tilde{\lambda}$ terms were found in [10, 11] (see also [12] for an analysis of the validity of zeta-function regularization), and it was shown in [13] that they cannot be obtained correctly in the LL approach regardless of the regularization used. In this paper we will consider only analytic terms.

On string side an LL action in the $SU(2)$ sector can be obtained by considering fast moving strings propagating in the $S^3$ subspace of $S^5$ [14, 15, 16]. The resulting string LL action have been used in [13] to compute the $1/J^2$ corrections to BMN energies to three-loops in $\tilde{\lambda}^3$. Results from the string LL action have been compared with results from the string “quantum” Bethe ansatz of Arutyunov, Frolov and Staudacher (AFS) [17], and for the $1/J^2$ corrections for $M$-impurity BMN states perfect agreement has been found. In this paper we continue this comparison to four loops $\tilde{\lambda}^4$. However, in order to do this we need to find first the string LL at four-loops. We find it indirectly: first at 4-loops we construct the LL action by including all possible eight derivative terms; secondly we fix the coefficients so that the energy as computed from LL matches the string energy for circular solution, and also the $1/J$ and $1/J^2$ corrections to BMN energies as obtained from quantum string LL match those obtained by string Bethe ansatz in [17, 13]. All these conditions are consistent and determine uniquely the string LL Lagrangian.
On gauge theory side the gauge LL action at one loop in the $SU(2)$ sector is the effective action for the ferromagnetic Heisenberg spin chain in the continuum limit, with higher derivative counterterms to account for lattice effects. Going to higher orders in $\tilde{\lambda}$ corresponds in terms of spin chain to going beyond nearest neighbor interactions. An all-loop Bethe ansatz that preserves BMN scaling to all loops in the thermodynamic limit was proposed by Beisert, Dippel and Staudacher (BDS) [18]. In order to compare results between this BDS Bethe ansatz and an effective action calculation one needs to find extensions of gauge LL action to higher loops. The LL action from gauge theory was derived to two-loops in [15] where it was shown to be the same as the string LL action derived from string theory. In [13] the gauge LL action to three-loops was found by indirect methods. In this paper we extend that method to four loops and proceed as for the string LL: we compare results from gauge LL action for circular string, $1/J$ and $1/J^2$ corrections to BMN energies obtained from quantum gauge LL to results obtained from BDS Bethe ansatz for operators that are duals to circular strings. This allows us to determine uniquely the gauge LL Lagrangian to four-loops. Also we find perfect matching between the $1/J$ and $1/J^2$ corrections to BMN energies as obtained from quantum gauge LL and BDS Bethe ansatz.

The manifestation of the “three-loop discrepancy” at the classical level of LL actions was obtained in [13] where gauge and string LL actions were found to differ at three-loops $\tilde{\lambda}^3$. In this paper we find that the “three-loop discrepancy” continues at four-loops, as expected. The $1/J$ and $1/J^2$ corrections to BMN energies, which can be found from quantum LL, also start differing at three-loops and continue to differ at four-loops.

This paper is organized as follows: In Section 2 we describe the gauge and string LL actions and find some of the unknown coefficients by using the circular string solution. In Section 3 after reviewing the quantization of LL action developed in [8], we compute the $1/J$ and $1/J^2$ corrections in Section 3.1 and 3.2, respectively. In section 4 we collect the results and find completely both the gauge and string LL actions. In Appendix A we describe details of the computation of energy of circular string by using LL action, string theory and BDS Bethe ansatz. In Appendix B we present some details of the computation of $1/J^2$ corrections to BMN energies from quantum LL.

## 2 Classical LL action to $\tilde{\lambda}^4$ order

The main steps as well as notations in this section and throughout the paper are as in [13]. We start with the string LL action in the $SU(2)$ sector [13] (for a review see also [19, 8])

$$S = J \int dt \int_0^{2\pi} d\sigma \frac{d\sigma}{2\pi} L ,$$  \hspace{1cm} (2.1)
In this paper we want to fix the coefficients by integrating out all superstring world-sheet fields except $\vec{n}$. The first term in this expansion can be also obtained from the classical string action becomes clearer when we introduce fluctuations in the next section.

The coefficients $a$, $b$, $c$ were fixed in [13] for both string and gauge LL. Let us recall the values of the “3-loop” coefficients in the string and gauge theory LL Lagrangians

$$a_s = -\frac{7}{4}, \quad b_s = -\frac{25}{2}, \quad c_s = \frac{13}{16},$$

$$a_g = -\frac{7}{4}, \quad b_g = -\frac{23}{2}, \quad c_g = \frac{3}{4}.$$  

In this paper we want to fix the coefficients $a_i$, $i = 1...7$. To do this we consider again, as in [13], the circular solution with two unequal spins ($J_1 \neq J_2$). For this solution
we compare the energy at four loops ($\tilde{\lambda}^4$) obtained from the string LL with the energy obtained from string theory at this order. We present the details in Appendix A. We obtain the following string LL coefficients

$$a_s^{(s)} = \frac{119}{4096}, \quad a_7^{(s)} = \frac{-323}{32768}, \quad a_1^{(s)} - a_2^{(s)} + a_3^{(s)} + a_4^{(s)} = -\frac{59}{2048} \quad (2.6)$$

On the other hand we can compare the energy for the circular solution obtained from LL to the corresponding solution on gauge theory side by using gauge BDS Bethe ansatz. We obtain then the gauge LL coefficients

$$a_s^{(g)} = \frac{111}{4096}, \quad a_7^{(g)} = \frac{-267}{32768}, \quad a_1^{(g)} - a_2^{(g)} + a_3^{(g)} + a_4^{(g)} = -\frac{59}{2048} \quad (2.7)$$

We see that in both gauge and string LL we need more equations among the coefficients to find them. Following the method in [13] we compare the $\tilde{\lambda}^4/J$ and $\tilde{\lambda}^4/J^2$ corrections to BMN vacuum obtained from quantum LL to the corresponding results obtained from gauge (BDS) and string (AFS) Bethe ansatze.

As in [13, 8], let us now rewrite the LL Lagrangian (2.2) in terms of two independent fields. Solving the constraint $|\vec{n}|^2 = 1$ as $n_3 = \sqrt{1 - n_1^2 - n_2^2}$ we get the following $SO(2)$ invariant expression for the Lagrangian in terms of $n_1$ and $n_2$ ($a, b = 1, 2; \ n^2 = n_a n_a$)

$$L = h^2(n) \epsilon_{ab} \dot{n}_a n_b - H(n_1, n_2), \quad h^2(n) = \frac{1 - \sqrt{1 - n^2}}{2n^2} = \frac{1}{4} + \frac{1}{16} n^2 + ... \quad (2.8)$$

We omit the complicated form of $H(n_1, n_2)$ for simplicity. To simplify the quantization of the LL Lagrangian near a particular solution it is useful to put it into the standard canonical form [8] by doing the field redefinition $n_a \rightarrow z_a$

$$z_a = h(n) n_a, \quad n_a = 2\sqrt{1 - z^2} \ z_a \quad (2.9)$$

to obtain

$$L = \epsilon_{ab} \dot{z}_a z_b - H(z_1, z_2). \quad (2.10)$$

Having the Lagrangian in the standard form $L = p\dot{q} - H(p, q)$, the quantization is straightforward: we promote $z_a$ to operators, impose the canonical commutation relation (cf. (2.1))

$$[z_1(t, \sigma), z_2(t, \sigma')] = iJ^{-1} \pi \delta(\sigma - \sigma') \quad (2.11)$$

and then decide how to order the “coordinate” and “momentum” operators in $H(z_1, z_2)$.

### 3 Quantization near BPS vacuum: corrections to BMN spectrum from LL Hamiltonian

Following [8, 13] we use the LL action to compute quantum $1/J$ and $1/J^2$ corrections to the BMN spectrum of fluctuations near the BPS vacuum solution

$$\vec{n} = (0, 0, 1) \quad (3.1)$$
representing the massless geodesic in $R_t \times S^3$. The $1/J$ corrections can be found from the Bethe ansatz on the spin chain \[20, 21\] or from a direct superstring quantization \[22, 6\]. As explained in \[8\], the derivation from the LL action turns out to be much simpler than the string-theory derivation. Here we will extend the method of \[8, 13\] to $\tilde{\lambda}^4/J$ and $\tilde{\lambda}^4/J^2$ orders.

Expanding near this vacuum corresponds to expansion near $n_a = 0$ in (2.8) or $z_a = 0$ in (2.10). Observing that the factor $J$ in front of the LL action (2.1) plays the role of the inverse Planck constant, it is natural to rescale $z_a$ as

$$z_1 = \frac{1}{\sqrt{J}} f, \quad z_2 = \frac{1}{\sqrt{J}} g,$$

so that powers of $1/J$ will play the role of coupling constants for the fluctuations in the non-linear LL Hamiltonian. Expanding the Hamiltonian in (2.8), (2.10) to sixth order in the fluctuation fields $f, g$ we get

$$S = \int dt \int_0^{2\pi} d\sigma \frac{2g f - H}{2\pi}, \quad H = H_2 + H_4 + H_6 + \ldots,$$

$$H_2 = f \left( \sqrt{1 - \lambda \partial_t^2} - 1 \right) f + g \left( \sqrt{1 - \lambda \partial_t^2} - 1 \right) g,$$

$$H_4 = \frac{1}{J} \left\{ (f^2 + g^2)(\sqrt{1 - \lambda \partial_t^2} - 1)(f^2 + g^2) - f(f^2 + g^2)(\sqrt{1 - \lambda \partial_t^2} - 1)f ight. \
- g(f^2 + g^2)(\sqrt{1 - \lambda \partial_t^2} - 1)g + \frac{3\lambda^2}{8} (f^2 + g^2)^2 \
+ \frac{\lambda^3}{4} \left[ b(f'f'' + g'g'')^2 + a(f^2 + g^2)(f'' + g'') + 16\lambda^4 \left( a_1(f'' + g'')^2 + a_2(f'' + g'')(f'' + g'') + a_3(f^2 + g^2)(f'f'')^2 + a_4(f^2 + g^2)(f'' + g'') \right) \right] \right\} + O \left( \frac{\lambda^5}{J} \right),$$

$$H_6 = \frac{1}{J^2} \left\{ \frac{1}{4} f(f^2 + g^2)(\sqrt{1 - \lambda \partial_t^2} - 1)[f(f^2 + g^2)] \
+ \frac{1}{4} g(f^2 + g^2)(\sqrt{1 - \lambda \partial_t^2} - 1)[g(f^2 + g^2)] \
- \frac{1}{4} f(f^2 + g^2)^2(\sqrt{1 - \lambda \partial_t^2} - 1)f - \frac{1}{4} g(f^2 + g^2)^2(\sqrt{1 - \lambda \partial_t^2} - 1)g \
+ \frac{3\lambda^2}{4} \left[ 2(f^2 + g^2)(f f' + g g') - (f^2 + g^2)(f'^2 + g'^2) \right] \right\}$$

\[2\]Unfortunately, the exact (all order in $\tilde{\lambda}$) form of the $\tilde{n}^4$, $\tilde{n}^6$ and $\tilde{n}^8$ terms in the LL action is not known, preventing us from computing the $1/J$ and $1/J^2$ corrections to all orders in $\tilde{\lambda}$. 

6
Let us first consider the quadratic approximation. The linearized equations of motion for the fluctuations are

\[ \dot{f} = -(1 - \sqrt{1 - \tilde{\lambda}\partial_\tau^2}) g, \quad \dot{g} = (1 - \sqrt{1 - \tilde{\lambda}\partial_\tau^2}) f, \] (3.7)
and their solution may be written as
\[
\begin{align*}
  f(t, \sigma) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( a_n e^{-i\omega_n t + in\sigma} + a_n^\dagger e^{i\omega_n t - in\sigma} \right), \quad \omega_n = \sqrt{1 + \tilde{\lambda} n^2} - 1, \quad (3.8) \\
  g(t, \sigma) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( -ia_n e^{-i\omega_n t + in\sigma} + ia_n^\dagger e^{i\omega_n t - in\sigma} \right), \quad (3.9)
\end{align*}
\]
for real \( f \) and \( g \). Upon quantization (3.7) becomes the equations of motion for the operators \( f, g \)
\[
\dot{f} = i[\bar{H}_2, f], \quad \dot{g} = i[\bar{H}_2, g], \quad \bar{H}_2 \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} H_2,
\]
provided we use the canonical commutation relations in (2.11)
\[
[f(t, \sigma), f(t, \sigma')] = 0, \quad [g(t, \sigma), g(t, \sigma')] = 0, \quad [f(t, \sigma), g(t, \sigma')] = i\pi\delta(\sigma - \sigma'). \quad (3.11)
\]
Then the coefficients in (3.8),(3.9) satisfy
\[
[a_n, a_m^\dagger] = \delta_{n-m}, \quad (3.12)
\]
so that \( a_n \) and \( a_n^\dagger \) can be interpreted as annihilation and creation operators, with the vacuum state \( |0\rangle \) defined by \( a_n|0\rangle = 0 \), for all integer \( n \). A general oscillator state is
\[
|\Psi\rangle = \prod_{n=-\infty}^{\infty} \frac{(a_n^\dagger)^{k_n}}{\sqrt{k_n!}} |0\rangle. \quad (3.13)
\]
The integrated Hamiltonian \( \bar{H}_2 \) then becomes
\[
\bar{H}_2 = \sum_{n=-\infty}^{\infty} \omega_n a_n^\dagger a_n, \quad (3.14)
\]
where we have used the normal ordering to ensure that the vacuum energy is zero, since the BMN vacuum is a BPS state in both gauge theory and string theory.

One also needs to impose the extra constraint that the total \( \sigma \)-momentum is zero \([8]\). For physical oscillator states we get
\[
\sum_{n=-\infty}^{\infty} na_n^\dagger a_n|\Psi\rangle = 0, \quad \sum_{n=-\infty}^{\infty} nk_n = 0. \quad (3.15)
\]
Below we shall consider the “\( M \)-impurity” states as oscillator states with \( k_n = 1 \):
\[
|M\rangle = a_{n_1}^\dagger \ldots a_{n_M}^\dagger |0\rangle, \quad (3.16)
\]
where for simplicity we shall assume that all \( n_j \) are different (generalization to states with several equal \( n_j \) is straightforward, at least for \( 1/J \) corrections). Then the zero-momentum condition (3.15) gives

\[
\sum_{j=1}^{M} n_j = 0 , \tag{3.17}
\]

and the leading term in the energy of an \( M \)-impurity state takes the familiar form

\[
E^{(0)} = \langle M| \hat{H}_2 |M \rangle = \sum_{j=1}^{M} (\sqrt{1 + \bar{\lambda} n_j^2} - 1) . \tag{3.18}
\]

Let us also compute the difference of spins and obtain

\[
J_1 - J_2 = J - 2 \sum_{n=-\infty}^{\infty} a_n^\dagger a_n . \tag{3.19}
\]

Applied to \( M \)-impurity state the above relation gives \( J_1 - J_2 = J - 2M \). Since \( J_1 + J_2 = J \) we have

\[
J_1 = J - M , \quad J_2 = M . \tag{3.20}
\]

In the LL approach we use \( J = J_1 + J_2 \) as a natural total angular momentum, corresponding to a “fast” collective coordinate. \( M \) is a characteristic of a particular state, while \( J \) enters into the background-independent form of the LL action (2.1). This is in line with gauge/spin chain intuition, where the use of total \( J \) or spin chain length as the state-independent parameter is natural. The corresponding gauge-theory states are \( \text{Tr}(\Phi_1^{J_1} \Phi_2^{J_2}) + ... \), and \( J \) plays the role of the length of the spin chain and \( M \) is the number of magnons.

### 3.1 \( 1/J \) corrections to the BMN spectrum

In this section we fix more unknown coefficients by comparing \( 1/J \) correction to BMN vacuum obtained from the quantum string LL, to the corresponding correction at \( \bar{\lambda}^4 \) obtained in [17] from string Bethe ansatz. For the gauge LL coefficients we compare with \( 1/J \) corrections obtained from gauge Bethe ansatz. The \( \bar{\lambda}^4/J \) correction obtained in [17] is

\[
\frac{\bar{\lambda}^4}{16J} \left[ 3 \sum_{k=1}^{M} n_k^3 \sum_{j=1}^{M} n_j^5 + \left( \sum_{j=1}^{M} n_j^4 \right)^2 + 2 \sum_{k=1}^{M} n_k^6 \sum_{j=1}^{M} n_j^2 - 16 \sum_{k=1}^{M} n_k^8 \right] . \tag{3.21}
\]

Note that this matches the result obtained in [24] from string theory.

On the other hand the \( 1/J \) correction to BMN on the gauge theory side at four loops (\( \bar{\lambda}^4 \)) can be obtained from the BDS Bethe ansatz

\[
\frac{\bar{\lambda}^4}{16J} \left[ 6 \sum_{k=1}^{M} n_k^3 \sum_{j=1}^{M} n_j^5 - 16 \sum_{k=1}^{M} n_k^8 \right] . \tag{3.22}
\]
We compute now the $1/J$ correction to BMN vacuum by using the quantum LL. Following the method developed in [8] we add the four loop contribution to the three loop one already computed in [13]. We recall that at quartic order in fluctuations one is to use normal ordering. We obtain then the following normal ordered Hamiltonian

$$
\bar{H}_4 = \frac{1}{J} \sum_{n,m} h_{nm} a_n^+ a_m^+ a_n a_m ,
$$

(3.23)

$$
h_{nm} = 1 + \sqrt{1 + \tilde{\lambda}(n - m)^2} - 2\sqrt{1 + \tilde{\lambda}n^2}
+ \frac{3\tilde{\lambda}^2}{4} n^2 m^2 + \frac{\tilde{\lambda}^3}{16} n^2 m^2 [2a(n + m)^2 + b(n - m)^2]
- 16\tilde{\lambda}^4 \left[ n^4 m^4 \left( 2a_1 + a_4 \right) + n^6 m^2 \left( -a_2 + 2a_3 + a_4 \right) \right]
- n^5 m^3 a_2] + O(\tilde{\lambda}^5) .
$$

(3.24)

Computing the expectation value of $\bar{H}_4$ and expanding to order $\tilde{\lambda}^4$ we obtain

$$
E^{(1)} = \langle M | \bar{H}_4 | M \rangle = \frac{1}{J} \left\{ \tilde{\lambda} \sum_{i=1}^{M} n_i^2 - \tilde{\lambda}^2 \sum_{i=1}^{M} n_i^4 + \frac{1}{8} (1 - 4a) \tilde{\lambda}^3 \sum_{i=1}^{M} n_i^6 
+ \frac{3\tilde{\lambda}^2}{4} \sum_{i,j=1}^{M} n_i^2 n_j^2 \left[ (2a + b + 15)(n_i^2 + n_j^2) + 2(2a - b - 10)n_i n_j \right] 
+ 16\tilde{\lambda}^4 \left[ - (2a_1 + a_4 + \frac{175}{1024}) \left( \sum_{i=1}^{M} n_i^4 \right)^2 
+ (a_2 - 2a_3 - a_4 - \frac{35}{256}) \sum_{i=1}^{M} n_i^6 \sum_{k=1}^{M} n_k^2 + (a_2 + \frac{35}{128}) \sum_{i=1}^{M} n_i^6 \sum_{k=1}^{M} n_k^3 
+ (2a_1 - 2a_2 + 2a_3 + 2a_4 - \frac{5}{1024}) \sum_{j=1}^{M} n_j^8 \right] + O(\tilde{\lambda}^5) \right\} .
$$

(3.25)

Comparing this expression at four loops with the result in (3.21) we obtain $a_2^{(s)}$ and two independent equations

$$
a_2^{(s)} = -\frac{67}{256}
$$

(3.26)

$$
2a_1^{(s)} + a_4^{(s)} = -\frac{179}{1024} , \quad 2a_3^{(s)} + a_4^{(s)} = -\frac{104}{256} .
$$

(3.27)

We see that combining the above equations we obtain the last equation in (2.6), so that in fact we have only 2 independent equations relating $a_1^{(s)}$, $a_3^{(s)}$ and $a_4^{(s)}$.

Comparing (3.25) at four loops with the result in (3.22) we again obtain $a_2^{(g)}$ and two independent equations

$$
a_2^{(g)} = -\frac{64}{256}
$$

(3.28)
\( 2a_1^{(g)} + a_4^{(g)} = -\frac{175}{1024}, \quad 2a_3^{(g)} + a_4^{(g)} = -\frac{99}{256}. \)  
(3.29)

We see that combining the above equations we obtain the last equation in (2.7), so that, as in the string case, we have only 2 independent equations relating \( a_1^{(g)}, a_3^{(g)} \) and \( a_4^{(g)} \).

In both string and gauge cases we need 2 more equations to determine the still unknown coefficients \( a_1, a_3, a_4 \) and \( a_6 \).

### 3.2 \( 1/J^2 \) corrections to the BMN spectrum

To find \( 1/J^2 \) corrections we follow the method in [8, 13] where order \( \tilde{\lambda}/J^2 \) terms were computed to three loops (\( \tilde{\lambda}^3 \)). We need to combine the second order perturbation theory correction for the quartic Hamiltonian (3.5) with the first order perturbation theory correction for the sixth order Hamiltonian in (3.6). The regularization issues were discussed in detail in [8]: to match string/gauge results we should use the normal-ordered form of the Hamiltonians and apply \( \zeta \)-function regularization for intermediate-state sums. We shall also need to add a local higher-derivative “counterterm” which (on gauge side) is a lattice correction to the continuum limit of the LL action (see [8] and below). We present details of this computation in Appendix B.

We find the following \( \tilde{\lambda}^4/J^2 \) corrections to BMN energies

\[
E^{(2)} = \frac{\tilde{\lambda}^4}{J^2} \left[ \left( \frac{429}{16} + \frac{3a_1}{2} - 512 a_1 - 864 a_2 + 1664 a_3 + 576 a_4 - 256 a_5 \right) + 64 a_6 - \frac{3b}{4} \right] \sum_{i=1}^{M} n_i^2 \left( \sum_{j=1}^{M} n_j^2 \right)^2
+ \left( -\frac{1087}{32} + \frac{5a_1}{4} + 128 a_1 + 1632 a_2 - 2496 a_3 - 1312 a_4 - 128 a_5 \right) - 64 a_6 + \frac{7b}{8} \sum_{i=1}^{M} n_i^4 \left( \sum_{j=1}^{M} n_j^2 \right)^2
+ \left( \frac{1}{16} - 704 a_1 - 1472 a_2 + 2496 a_3 + 896 a_4 + 128 a_5 \right) + 64 a_6 - \frac{3b}{2} \left( \sum_{i=1}^{M} n_i^4 \right)^2
+ \left( -\frac{477}{32} + \frac{13a_1}{4} + 64 a_1 + 416 a_2 - 704 a_3 - 320 a_4 - 768 a_5 + \frac{3b}{8} \right) \sum_{i=1}^{M} n_i^8
+ \left( \frac{189}{8} - 4a + 256 a_1 - 1984 a_2 + 2688 a_3 + 1728 a_4 + 512 a_5 \right) + 64 a_6 - b \right] \sum_{i=1}^{M} n_i^6 \sum_{j=1}^{M} n_j^2
\]
\[
+ \left( -\frac{23}{8} - 2a + 768a_1 + 2240a_2 - 3584a_3 - 1536a_4 + 512a_5 \\
- 128a_6 + 2b \right) \sum_{i=1}^{M} n_i^5 \sum_{j=1}^{M} n_j^3 \\
+ \frac{1}{4} \left( -\frac{21}{4} + 4a - 512(a_1 - a_2 + a_3 + a_4) \right) \sum_{i \neq j} \frac{n_i^5 n_j^5}{(n_i - n_j)^2} \\
+ \frac{5\pi^2}{96} \sum_{i=1}^{M} n_i^{10} \right],
\]

(3.30)

where, as we mentioned in section 2, we added as in [8, 13] the expansion to four loops of the second term in kinetic term (2.3).

4 Comparison with Bethe ansatz computation and fixing the LL Lagrangians

In this section we are matching the $\tilde{\lambda}^4 / J^2$ corrections obtained above from the quantum LL and the corresponding corrections obtained from Bethe ansatz. We will thus fix the coefficients at four loops for both string and gauge LL Lagrangians.

In [13] the gauge Bethe ansatz (BDS) and string Bethe ansatz (AFS) were used to find the $1/J^2$ corrections to all orders in $\tilde{\lambda}$. We present here the results to four loops. In the case of the gauge theory we find

\[
E_g^{(2)} = \frac{\tilde{\lambda}}{J^2} \left[ -\frac{\pi^2}{6} \sum_{i=1}^{M} n_i^4 + 2 \sum_{i=1}^{M} n_i^2 - \sum_{i \neq j}^{M} \frac{2n_i^2 n_j^2}{(n_i - n_j)^2} \right] \\
+ \frac{\tilde{\lambda}^2}{J^2} \left[ \frac{\pi^2}{12} \sum_{i=1}^{M} n_i^6 - \frac{13}{2} \sum_{i=1}^{M} n_i^4 + \frac{3}{2} \left( \sum_{i=1}^{M} n_i^2 \right)^2 + \sum_{i \neq j}^{M} \frac{n_i^3 n_j^3}{(n_i - n_j)^2} \right] \\
+ \frac{\tilde{\lambda}^3}{J^2} \left[ -\frac{\pi^2}{16} \sum_{i=1}^{M} n_i^8 + \frac{49}{4} \sum_{i=1}^{M} n_i^6 - \frac{9}{4} \sum_{i=1}^{M} n_i^4 \sum_{j=1}^{M} n_j^2 - \frac{9}{2} \left( \sum_{i=1}^{M} n_i^3 \right)^2 \\
- \frac{1}{4} \left( \sum_{i=1}^{M} n_i^3 \right)^3 - \frac{3}{4} \sum_{i \neq j}^{M} \frac{n_i^4 n_j^4}{(n_i - n_j)^2} \right] \\
+ \frac{\tilde{\lambda}^4}{J^2} \left[ \frac{5\pi^2}{96} \sum_{i=1}^{M} n_i^{10} - \frac{305}{16} \sum_{i=1}^{M} n_i^8 + \frac{15}{8} \sum_{i=1}^{M} n_i^6 \sum_{j=1}^{M} n_j^2 + \frac{41}{4} \sum_{i=1}^{M} n_i^5 \sum_{j=1}^{M} n_j^3 \\
+ \frac{25}{16} \left( \sum_{i=1}^{M} n_i^4 \right)^2 - \frac{7}{8} \left( \sum_{i=1}^{M} n_i^3 \right)^2 \sum_{j=1}^{M} n_j^2 + \frac{5}{8} \sum_{i=1}^{M} n_i^4 \left( \sum_{j=1}^{M} n_j^2 \right)^2 \\
+ \frac{5}{8} \sum_{i \neq j}^{M} \frac{n_i^5 n_j^5}{(n_i - n_j)^2} \right] + O\left(\frac{\tilde{\lambda}^5}{J^2}\right).
\]

(3.31)
For the string case, the result is

\[ E^{(2)}_s = \frac{\tilde{\lambda}}{J^2} \left[ -\frac{\pi^2}{6} \sum_{i=1}^{M} n_i^4 + 2 \sum_{i=1}^{M} n_i^2 - \frac{\sum_{j \neq i} (2n_i^2 n_j^2)}{(n_i - n_j)^2} \right] + \frac{\tilde{\lambda}^2}{J^2} \left[ -\frac{\pi^2}{12} \sum_{i=1}^{M} n_i^6 - \frac{13}{2} \sum_{i=1}^{M} n_i^4 + \frac{3}{2} \left( \sum_{i=1}^{M} n_i^2 \right)^2 + \sum_{i \neq j}^M \frac{n_i^2 n_j^2}{(n_i - n_j)^2} \right] + \frac{\tilde{\lambda}^3}{J^2} \left[ \frac{\pi^2}{16} \sum_{i=1}^{M} n_i^8 + \frac{49}{4} \sum_{i=1}^{M} n_i^6 - \frac{31}{8} \sum_{i=1}^{M} n_i^4 \sum_{j=1}^{M} n_j^2 - 3 \left( \sum_{i=1}^{M} n_i^3 \right)^2 \right. \\
+ \frac{1}{8} \left( \sum_{i=1}^{M} n_i^3 \right)^3 - 3 \sum_{i \neq j} \frac{n_i^4 n_j^4}{(n_i - n_j)^2} \right] + \frac{\tilde{\lambda}^4}{J^2} \left[ \frac{5\pi^2}{96} \sum_{i=1}^{M} n_i^{10} - \frac{305}{16} \sum_{i=1}^{M} n_i^8 + \frac{75}{16} \sum_{i=1}^{M} n_i^6 \sum_{j=1}^{M} n_j^2 + \frac{51}{8} \sum_{i=1}^{M} n_i^5 \sum_{j=1}^{M} n_j^3 \right. \\
+ 23 \sum_{i=1}^{M} n_i^4)^2 - \frac{7}{16} \left( \sum_{i=1}^{M} n_i^3 \right)^2 \sum_{j=1}^{M} n_j^2 - 5 \sum_{i=1}^{M} n_i^4 \left( \sum_{i=1}^{M} n_i^2 \right)^2 \right. \\
+ 5 \sum_{i \neq j} \frac{n_i^5 n_j^5}{(n_i - n_j)^2} \left.+ O\left( \frac{\tilde{\lambda}^5}{J^2} \right) \right]. \quad (3.32)

Comparing (3.30) with the gauge part (3.31) except for the pole type term we obtain six equations with six unknowns, which solved give the gauge LL coefficients

\[ a_1^{(g)} = 0, \quad a_2^{(g)} = -\frac{64}{256} = -\frac{1}{4}, \quad a_3^{(g)} = -\frac{221}{2048}, \quad a_4^{(g)} = -\frac{175}{1024} \quad (3.33) \]

\[ a_5^{(g)} = \frac{111}{4096}, \quad a_6^{(g)} = \frac{141}{256} \quad (3.34) \]

We see that the \( \tilde{\lambda}^4/J^2 \) corrections contain enough details to fix almost completely the gauge LL Lagrangian except for the coefficient \( a_7^{(g)} \). As a consistency check we see that these match the coefficients and equations between coefficients obtained from comparing \( \tilde{\lambda}^4/J \), and the energy of circular string (2.7, 3.23, 3.29). The conclusion is that besides finding the full gauge LL Lagrangian at four loops, we also find perfect agreement for the \( \tilde{\lambda}^4/J \) and \( \tilde{\lambda}^4/J^2 \) corrections as obtained from quantum gauge LL and gauge Bethe ansatz.

Similarly, for the string part comparing (3.30) with (3.32) we find the coefficients

\[ a_1^{(s)} = 0, \quad a_2^{(s)} = -\frac{67}{256}, \quad a_3^{(s)} = -\frac{237}{2048}, \quad a_4^{(s)} = -\frac{179}{1024}, \quad (3.35) \]

\[ a_5^{(s)} = \frac{119}{4096}, \quad a_6^{(s)} = \frac{649}{1024}. \quad (3.36) \]
These are again consistent with our findings in (2.6), (3.26), (3.27). We find again the string LL Lagrangian at four loops and also perfect agreement for the $\lambda^4/J$ and $\tilde{\lambda}^4/J^2$ corrections as obtained from quantum string LL and string Bethe ansatz.

Let us now look at the pole type term. This is the same as obtained from both string and gauge Bethe ansatze. Matching the pole term we obtain

$$\frac{1}{4} \left( -\frac{21}{4} + 4a - 512(a_1 - a_2 + a_3 + a_4) \right) = \frac{5}{8},$$

which gives

$$a_1 - a_2 + a_3 + a_4 = -\frac{59}{2048},$$

which is consistent with the equations (2.6), (2.7). Note that the pole type term depends only on the coefficient $a$ and the combination $a_1 - a_2 + a_3 + a_4$, which are the same for both gauge and string LL.

As expected we found that the disagreement between string LL and gauge LL actions continues also at four loops. The difference between the two LL Lagrangians or Hamiltonians is

$$L_s - L_g = -(H_s - H_g) = \frac{\tilde{\lambda}^3}{64} \left[ (\partial_1 \tilde{n} \cdot \partial_1^2 \tilde{n})^2 - \frac{1}{16} (\partial_1 \tilde{n} \cdot \partial_1 \tilde{n})^3 \right]$$

$$- \frac{\lambda^4}{256} \left[ \frac{3}{2} (\partial_1 \tilde{n})^2 (\partial_1 \tilde{n} \partial_1^2 \tilde{n}) + 2 (\partial_1 \tilde{n} \partial_1^3 \tilde{n})(\partial_1 \tilde{n})^2 + (\partial_1^3 \tilde{n})^2 (\partial_1 \tilde{n})^2 \right]$$

$$- \frac{1}{2} (\partial_1 \tilde{n})^4 (\partial_1^2 \tilde{n})^2 - \frac{85}{4} (\partial_1 \tilde{n} \partial_1^2 \tilde{n})^2 (\partial_1 \tilde{n})^2 + \frac{7}{16} (\partial_1 \tilde{n})^8 \right] + O(\tilde{\lambda}^5)$$

We observe that both at three-loops and four-loops only one of the terms has the same coefficients in both gauge and string LL Lagrangians, i.e. $a_s = a_g$ at three-loops and $a_1^{(s)} = a_1^{(g)}$ at four-loops. The disagreement between the two LL Lagrangians is not unexpected\(^3\) and can be explained by the order of limits taken on the two sides\([\text{[4, 18]}]\). On string theory side one first takes $J$ large with $\tilde{\lambda}=$fixed to suppress quantum corrections and then expands in small $\tilde{\lambda}$, while on gauge theory side $\lambda$ small is taken first, and then large $J$ expansion to isolate contributions depending on $\tilde{\lambda}$.

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\(^3\)It is rather remarkable that the gauge and string LL actions are the same at one and two-loops.
Appendix A: Fixing coefficients in 4-loop gauge and string LL actions: circular string example

In this section we recall the details of the rational circular string solution in the \( SU(2) \) sector \cite{25} and compute its classical energy to order \( \tilde{\lambda}^4 \). The solution is given by

\[
X_r = b_r e^{i(w_r \tau + m_r \sigma)}, \quad r = 1, 2
\]  

(A.1)

where \( X_2^r = 1 \) are \( S^3 \) coordinates and

\[
b_1^2 + b_2^2 = 1, \quad w_r = \sqrt{m_r^2 + \nu^2}, \quad J_r = \sqrt{\lambda} J_r = \sqrt{\lambda} b_r^2 w_r.
\]  

(A.2)

\( \nu \) is a parameter to be determined from the conformal gauge constraints

\[
E^2 = 2(w_1 J_1 + w_2 J_2) - \nu^2, \quad m_1 J_1 + m_2 J_2 = 0,
\]  

(A.3)

where the energy is

\[
E = \sqrt{\lambda} \mathcal{E}.
\]

Introducing the notation

\[
m \equiv m_1, \quad n \equiv m_1 - m_2, \quad \mathcal{J} = J_1 + J_2,
\]

one can solve one of the constraints for \( \nu \) at large \( \mathcal{J} \) or small \( \tilde{\lambda} = \frac{1}{\mathcal{J}} \) to obtain

\[
\nu^2 = \mathcal{J}^2 + m(m - n) - \frac{3m(m - n)(2m - n)^2}{4 \mathcal{J}^2} + \frac{5m(m - n)(2m - n)^4}{8 \mathcal{J}^4}
- \frac{7}{64 \mathcal{J}^6} m(n - 2m)^4 (24m^3 - 48mn^2 + 29n^2m - 5n^3)
+ \frac{9}{128 \mathcal{J}^8} m(n - 2m)^6 (44m^3 - 88mn^2 + 51n^2m - 7n^3) + O(\frac{1}{\mathcal{J}^{10}}).
\]

(A.4)

Then the string energy to \( \tilde{\lambda}^4 \) order is found to be

\[
E_s = J \left[ 1 + \frac{1}{2} \tilde{\lambda} m(n - m) - \frac{1}{8} \tilde{\lambda}^2 m(n - m)(n^2 - 3mn + 3m^2)
+ \frac{1}{16} \tilde{\lambda}^3 m(n - m)(n^4 - 7mn^3 + 20m^2n^2 - 26m^3n + 13m^4)
+ \frac{1}{128} \tilde{\lambda}^4 m(n - m)(323m^6 - 969mn^5 + 1207n^2m^4 - 799n^3m^3 + 297n^4m^2
- 59n^5m + 5n^6) + O(\tilde{\lambda}^5) \right].
\]

(A.5)

Starting now with the LL Lagrangian (2.2) let us find the energy for the corresponding solution with \( J_1 \neq J_2 \) which is given to leading order by \( \tilde{n} = (n_1, n_2, n_3) \) where

\[
n_1 = 2 \sqrt{\frac{m}{n}} \left( 1 - \frac{m}{n} \right) \cos n\sigma + O(\tilde{\lambda}), \quad n_2 = 2 \sqrt{\frac{m}{n}} \left( 1 - \frac{m}{n} \right) \sin n\sigma + O(\tilde{\lambda}),
\]

(A.6)
Comparing (A.5) with (A.8) we obtain the coefficients

\[ n_3 = 1 - \frac{2m}{n} + O(\lambda) \quad \text{(A.7)} \]

This solution can also be found by expanding the full string solution at large $J$.\(^4\)

Plugging this solution into the Hamiltonian in (2.2) we find for its LL energy

\[
E_{LL} = J \left[ 1 + \frac{\lambda}{2} m(n-m) - \frac{\lambda^2}{8} m(n-m)(3m^2 + n^2 - 3mn) \right. \\
+ \frac{\lambda^3}{16} m(n-m) \left[ n^4 - 7n^2m(n-m) + 13m^2(n-m)^2 \right] \\
+ \frac{\lambda^4}{128} m(n-m) \left[ -32768 \ a_7 \ m^6 + 98304 \ a_7 \ m^5n - 98304 \ a_7 \ m^4n^2 + 8192 \ a_5 \ m^4n^2 \\
+ 32768 \ a_7 \ m^3n^3 - 16384 \ a_5 \ m^3n^3 - 2048 \ a_1 \ m^2n^4 - 2048 \ a_3 \ m^2n^4 \\
+ 8192 \ a_5 \ m^2n^4 - 2048 \ a_4 \ m^2n^4 + 2048 \ a_2 \ m^2n^4 + 2048 \ a_1 \ mn^5 \\
+ 2048 \ a_3 \ mn^5 + 2048 \ a_4 \ mn^5 - 2048 \ a_2 \ mn^5 + 5 \ n^6 \right] + O(\lambda^5) \quad \text{(A.8)}
\]

Comparing (A.3) with (A.8) we obtain the coefficients

\[ a_5^{(s)} = \frac{119}{4096}, \quad a_7^{(s)} = -\frac{323}{32768}, \quad a_1^{(s)} - a_2^{(s)} + a_3^{(s)} + a_4^{(s)} = -\frac{59}{2048}. \quad \text{(A.9)} \]

We want now to compare the four-loop LL energy for the circular solution with the energy of the corresponding state on the gauge theory side obtained from the Bethe ansatz. One can compute the latter as in [26], but now using BDS Bethe ansatz, and the result is\(^5\)

\[
E_\gamma = J \left[ 1 + \frac{\lambda}{2} m(n-m) - \frac{\lambda^2}{8} m(n-m)(n^2 - 3mn + 3m^2) \right. \\
+ \frac{1}{16} \lambda^3 m(n-m)(n^2 - 3mn + 3m^2)(n-2m)^2 \\
- \frac{1}{128} \lambda^4 m(n-m)(267 m^6 - 801 mn^5 + 1023 n^2m^4 - 711 n^3m^3 \\
+ 281 n^4m^2 - 59 n^5m + 5 n^6) + O(\lambda^5) \right]. \quad \text{(A.10)}
\]

Comparing (A.8) with (A.10) we obtain the following coefficients and equations between them

\[ a_5^{(g)} = \frac{111}{4096}, \quad a_7^{(g)} = -\frac{267}{32768}, \quad a_1^{(g)} - a_2^{(g)} + a_3^{(g)} + a_4^{(g)} = -\frac{59}{2048}. \quad \text{(A.11)} \]

\(^4\)The unit vector $\vec{n}$ can be written as $\vec{n} = (\sin 2\psi \cos 2\varphi, \sin 2\psi \sin 2\varphi, \cos 2\psi)$. In terms of global angular coordinates of $S^5$ with the metric $ds^2 = dt^2 + d\gamma^2 + cos^2 \gamma \ d\varphi_1^2 + \sin^2 \gamma \ (dv^2 + cos^2 \psi \ d\varphi_2^2 + sin^2 \psi \ d\varphi_2^2)$ we have $\varphi = \frac{\varphi_1 - \varphi_2}{2}$. Note also that the cartesian coordinates are $X_1 = \cos \psi \ e^{i\varphi_1}$, $X_2 = \sin \psi \ e^{i\varphi_2}$.

\(^5\)We are grateful to J. Minahan for sharing these unpublished results with us. Let us also mention that the computation of energy for this solution has also been computed using the string Bethe ansatz and the results match (A.5), as expected.
Appendix B: More on the $1/J^2$ corrections to BMN energy from quantum LL

In this appendix we present some details of the computation of $\tilde{\lambda}^4/J^2$ corrections to the BMN energies. We start with the second-order perturbation ("exchange") contribution. Starting with the quartic Hamiltonian \(^{(3.23)}\) we need to compute

$$
\langle M | (\tilde{H}_4)^{(2)} | M \rangle = \sum_{M \neq M'} \frac{\langle M | \tilde{H}_4 | M' \rangle \langle M' | \tilde{H}_4 | M \rangle}{E_M - E_{M'}} ,
$$

where $| M' \rangle$ is any possible intermediate state, and $| M \rangle = a_{n_1}^\dagger ... a_{n_M}^\dagger | 0 \rangle$. Since $\tilde{H}_4$ in \(^{(3.23)}\) contains only terms of the form $(a^\dagger)^2 a^2$, the only non-trivial intermediate states can be the $M' = M$ -particle states of the form $a_{n_1}^\dagger ... a_{n_M}^\dagger | 0 \rangle$. Then in order for the matrix element $\langle 0 | a_{n_1}...a_{n_M} | \tilde{H}_4 a_{n_1}^\dagger ... a_{n_M}^\dagger | 0 \rangle$ to be non-zero, there should be a $j$ and $k$ such that $n'_j = n_j + q$ and $n'_k = n_k - q$, with all other $n'_i = n_i$, $i \neq j, k$. In order for $| M \rangle$ to be distinct from $| M' \rangle$, we require that $0 \neq q \neq n_k - n_j$. With these conditions, we then find that if $n_k \neq n_j$

$$
E^{(1)}_4 \equiv \langle M | \tilde{H}_4 | M' \rangle = \frac{1}{2} \left\{ 2\sqrt{1 + \tilde{\lambda} q^2} + 2\sqrt{1 + \tilde{\lambda}(n_k - n_j - q)^2} - \sqrt{1 + \tilde{\lambda}n_k^2} - \sqrt{1 + \tilde{\lambda}n_j^2} - \sqrt{1 + \tilde{\lambda}(q + n_j)^2} - \sqrt{1 + \tilde{\lambda}(n_k - q)^2} + \frac{3\tilde{\lambda}^2}{2} n_k n_j (q + n_j) (n_k - q) \left[ 1 + \frac{\tilde{\lambda}}{12} \left( 2a(n_j + n_k)^2 + b[(n_k - n_j)^2 - 2q(n_k - n_j - q)] \right) \right] - \tilde{\lambda}^4 \left[ 8a_1 n_j^2 n_k^2 (n_j + q)^2 (n_k - q)^2 + a_2 n_j n_k (n_j + q)(n_k - q)[2(q + n_j)^2 (n_k - q)^2 + 2n_j^2 n_k^2 + (n_j^2 + n_k^2)((n_j + q)^2 + (n_k - q)^2)] - 2a_3 n_j^2 n_k^2 (n_j + q)(n_k - q)[n_k^2 (n_j + n_k) + (q + n_j)^3 + (n_k - q)^3] + 2a_4 n_j n_k (q + n_j)(n_k - q)[n_k^2 + (q + n_j)^4 + (n_k - q)^4] + 4a_5 n_j n_k^2 (n_j + q)(n_k - q)[(n_j + q)^2 + (n_k - q)^2] \right] \right\} ,
$$

(B.2)

where $n_j + q$ and $n_k - q$ are not equal to one of the other $n'_i$'s. The energy difference in (B.1) is

$$
W_1 \equiv E_M - E_{M'} = \sqrt{1 + \tilde{\lambda}n_j^2} + \sqrt{1 + \tilde{\lambda}n_k^2} - \sqrt{1 + \tilde{\lambda}(n_j + q)^2} - \sqrt{1 + \tilde{\lambda}(n_k - q)^2} .
$$

(B.3)

If $n_j + q = n_l$, and so $| M' \rangle$ has two impurities with the same momenta, then the matrix element is

$$
E^{(2)}_4 \equiv \langle M | \tilde{H}_4 | M' \rangle = \sqrt{2} \frac{E^{(1)}_4}{J} |_{q = n_l - n_j}
$$

and the energy difference is

$$
W_2 \equiv E_M - E_{M'} = \sqrt{1 + \tilde{\lambda}n_j^2} + \sqrt{1 + \tilde{\lambda}n_k^2} - \sqrt{1 + \tilde{\lambda}(n_j + n_k - n_l)^2} .
$$

(B.5)
Then the “exchange” contribution is given by

\[
\langle M | (\tilde{H}_4)^{(2)} | M \rangle = \frac{1}{4} \sum_{j \neq k}^M \left[ \sum_{q=-\infty}^{\infty} \frac{(E_1^{(1)})^2}{W_1} + \sum_{i \neq j}^M \frac{(E_4^{(2)})^2}{W_2} \right].
\]  

(B.6)

It was seen in [13] that if one looks at this expression as a string-theory expression (i.e. non-perturbative in \( \tilde{\lambda} \), with sums done before the expansion in \( \tilde{\lambda} \)), the sum over the “virtual” momentum \( q \) produces a contribution which is non-analytic in \( \tilde{\lambda} \). This phenomenon, which was absent at order \( 1/J \), was first observed [10]. As we already pointed out, it was shown in [13] that the correct coefficients of the non-analytic terms cannot be determined correctly in the Landau-Lifshitz approach as other modes outside the \( SU(2) \) sector contribute to non-analytic terms. In this paper we focus only on the analytic terms obtained by expanding in \( \tilde{\lambda} \) before doing the sum over \( q \). Our aim here is to extend the computation of [8,13] to the order \( \tilde{\lambda}^4/J^2 \) and to show that the results match the gauge and string Bethe ansatz results. Computing the sums in (B.6) we obtain the “exchange” contribution to the \( \tilde{\lambda}^4/J^2 \) correction to the BMN energies

\[
\tilde{\lambda}^4/J^2 \left[ \left( \frac{127}{8} + \frac{3a}{2} - 384 a_1 - 96 a_2 - 128 a_3 + 544 a_4 - 32 a_5 \right) \right.
\]
\[
- \frac{3b}{4} \left( \sum_{i=1}^M n_i^2 \left( \sum_{j=1}^M n_j^3 \right)^2 \right)
\]
\[
+ \left( - \frac{599}{64} + \frac{5a}{4} + 64 a_1 - 144 a_2 + 288 a_3 - 576 a_4 - 160 a_5 \right)
\]
\[
+ \frac{7b}{8} \left( \sum_{i=1}^M n_i^4 \right)^2
\]
\[
+ \left( \frac{525}{128} - \frac{349}{16} - 384 a_1 + 32 a_2 - 416 a_3 + 896 a_4 + 64 a_5 \right)
\]
\[
- \frac{3b}{2} \left( \sum_{i=1}^M n_i^4 \right)^2
\]
\[
+ \left( \frac{5M^2}{128} - \frac{477}{32} + \frac{13a}{4} - 320 a_1 - 512 a_2 + 800 a_3 - 1088 a_4 - 704 a_5 \right)
\]
\[
+ \frac{3b}{8} - 5M \left( \sum_{i=1}^M n_i^8 \right)
\]
\[
+ \left( \frac{105M}{32} - \frac{217}{16} - 4a + 256 a_1 + 352 a_2 - 544 a_3 + 640 a_4 + 448 a_5 \right)
\]
\[
- b \left( \sum_{i=1}^M n_i^8 \sum_{j=1}^M n_j^2 \right)
\]
\[
+ \left( \frac{327}{8} - 2a + 768 a_1 + 352 a_2 - 448 a_4 + 384 a_5 + 2b \right)
\]
\[-\frac{35M}{16} \sum_{i=1}^{M} n_i^5 \sum_{j=1}^{M} n_j^3 \\
+ \frac{1}{4} \left( -\frac{21}{4} + 4a - 512(a_1 - a_2 + a_3 + a_4) \right) \sum_{i \neq j} n_i^5 n_j^5 \\
+ \frac{5\pi^2}{96} \sum_{i=1}^{M} n_i^{10} \right], \quad (B.7)

Let us now compute the sixth-order “contact” contribution coming from the expectation value of the sixth order term in the LL Hamiltonian (3.6). The normal ordered form for this term can be written as

\[ \tilde{H}_6 = \frac{1}{{\cal J}^2} \sum_{n,m,k} h_{nmk} a_n^\dagger a_m^\dagger a_k^\dagger a_n a_m a_k , \quad (B.8) \]

where for simplicity we do not write down the complicated form of \( h_{nmk} \). After a lengthy but straightforward computation we obtain the expectation value of \( \tilde{H}_6 \)

\[ \tilde{\lambda}^4 \left[ \left( \frac{175}{16} - 128 \ a_1 + 112 \ a_2 - 736 \ a_3 + 1120 \ a_4 + 608 \ a_5 \\
- 256 \ a_6 + 64 \ a_7 \right) \sum_{i=1}^{M} n_i^2 \left( \sum_{j=1}^{M} n_j^3 \right)^2 \\
+ \left( \frac{1575}{64} + 64 \ a_1 - 576 \ a_2 + 1344 \ a_3 - 1920 \ a_4 - 1152 \ a_5 \\
- 128 \ a_6 - 64 \ a_7 \right) \sum_{i=1}^{M} n_i^4 \left( \sum_{j=1}^{M} n_j^2 \right)^2 \\
+ \left( \frac{175}{8} - 320 \ a_1 - 1056 \ a_3 + 1600 \ a_4 + 832 \ a_5 + 128 \ a_6 \\
+ 64 \ a_7 - \frac{525M}{128} + 48M a_2 \right) \left( \sum_{i=1}^{M} n_i^4 \right)^2 \\
+ \left( -\frac{5M^2}{128} + 384 \ a_1 + 384 \ a_2 - 384 \ a_3 + 384 \ a_4 + 384 \ a_5 - 768 \ a_6 \\
+ 5M - 64M a_2 \right) \sum_{i=1}^{M} n_i^8 \\
+ \frac{595}{16} + 896 \ a_2 - 1440 \ a_3 + 2048 \ a_4 + 1280 \ a_5 + 512 \ a_6 \right] \]

As in [13] the only regularization we use here is the assumption that the sixth order term in the Hamiltonian is normal ordered.
\[
+ 64 \mathbf{a}_7 - \frac{105M}{32} + 16Ma_2 \sum_{i=1}^{M} n_i^6 \sum_{j=1}^{M} n_j^2 \\
+ \left( -\frac{175}{4} - 800 \mathbf{a}_2 + 2240 \mathbf{a}_3 - 3136 \mathbf{a}_4 - 1920 \mathbf{a}_5 \\
+ 512 \mathbf{a}_6 - 128 \mathbf{a}_7 + \frac{35M}{16} \right) \sum_{i=1}^{M} n_i^5 \sum_{j=1}^{M} n_j^3, 
\]

Putting together (B.7) and (B.9) we see that the explicit dependence on the number of impurities \(M\) cancels, and we obtain the result (3.30).

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