Supersymmetric Inflation of Dynamical Origin

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Abstract

Dynamical models of inflation are given with composite inflatons by means of massive supersymmetric gauge theory. Nearly flat directions and stable massive ones in the potential are identified and slow-roll during inflation is examined. This kind of dynamical inflations may be ubiquitous in fundamental unified theory with supersymmetry, which should contain gauge theories for interactions of elementary particles.
1 Introduction

Cosmological inflation [1] is expected to be a key ingredient to realize our present universe. Various possible candidates have been considered as origins of inflaton fields in particle-physics models. Weakly coupled fields such as moduli fields in supersymmetric models may be plausible candidates to yield nearly flat potential for slow-roll inflation.

In contrast, strongly coupled fields might naively seem inappropriate to achieve flatness of the potential due to large dynamical corrections to the potential. However, if dynamical scale as large as fundamental cutoff scale (or genuinely strong coupling) is to be considered, low-energy degrees of freedom may just behave as weakly coupled fields in effective theory description below the cutoff scale, which we assume to be the reduced Planck scale, for simplicity.

In this paper, we pursue dynamical origin of inflaton fields by means of composite inflaton models of massive supersymmetric gauge theory in four spacetime dimensions. Although the supergravity corrections are important along with those originated from the strong dynamics, we ignore them in the present analysis as the first step, anticipating cancellation among them. We consider a possibility that such an inflaton is identified with the primordial one, though primary inflations [1, 4] proceeding the primordial one might be the appropriate stage for dynamical inflation. Anyway, in both cases, the dynamical inflation may be relevant for vacuum selection processes in the very early universe.

2 Preliminary notes

Let us recapitulate an effective theory description of supersymmetric $Sp(N)$ gauge theory [5] with $2N + 4$ chiral superfields $Q_i$ of the fundamental representation, where we adopt the notation $Sp(1) = SU(2)$ and $i = 1, \cdots , 2N + 4$ denotes a flavor index with the gauge index omitted.

The effective superpotential which may describe the dynamics of the gauge interaction
is given by

\[ W_{\text{eff}} = \frac{1}{\Lambda^{N-1}} \text{Pf} \, M \quad (1) \]

in terms of gauge-invariant low-energy degrees of freedom

\[ M_{ij} = -M_{ji} \sim \frac{1}{\Lambda} Q_i Q_j, \quad (2) \]

where \( \Lambda \) denotes a dynamical scale of the gauge interaction and \((\text{Pf} \, M)^2 = \det M\).

As for the effective Kähler potential, we adopt the minimal one with the higher-order corrections comparable to the supergravity ones ignored in this paper.

3 Dynamic inflation

For concreteness, we first present an inflationary model based on the effective theory description of supersymmetric \( Sp(2) \) gauge theory with 8 chiral superfields \( Q_i \) of the fundamental representation, where \( i = 1, \cdots, 8 \).

3.1 The model

Introduction of mass terms for the superfields \( Q_i \) yields the effective superpotential for \( M_{ij} \) as

\[ W_{\text{eff}} = \frac{1}{\Lambda} \text{Pf} \, M - m_{ij} \Lambda M_{ij}, \quad (3) \]

where \( m_{ij} \) denote mass parameters. Let us adopt the Planck unit where the reduced Planck scale is unity and set \( \Lambda \sim 1 \) as explained in the Introduction. Without loss of generality, by field redefinitions of the \( M_{ij} \), the mass matrix can be block-diagonalized as follows\(^3\)

\[
\begin{bmatrix}
0 & m_{12} \\
-m_{12} & 0 \\
0 & m_{34} \\
-m_{34} & 0 \\
0 & m_{56} \\
-m_{56} & 0 \\
0 & m_{78} \\
-m_{78} & 0
\end{bmatrix},
\]

\(^3\)Any antisymmetric matrix \( A \) can be block-diagonalized as \( A = U \hat{A} U^T \) where \( U \) is a unitary matrix and \( \hat{A} \) is block-diagonal \([6]\).
where \( m_{12}, m_{34}, m_{56}, \) and \( m_{78} \) are real and non-negative.

In the following, we consider the case with the common positive mass, \( m_{12} = m_{34} = m_{56} = m_{78} \equiv m^2/2\Lambda \). The superpotential is then given by

\[
W_{\text{eff}} = \lambda \text{Pf} M - m^2(M_{12} + M_{34} + M_{56} + M_{78}),
\]

where \( \lambda \) is a positive coupling of order one.

In order to examine the form of the potential from the viewpoint of slow-roll inflation, we concentrate on the block-diagonal direction in \( M_{ij} \), namely, \( M_{12}, M_{34}, M_{56}, \) and \( M_{78} \), since there are no unstable directions\(^4\) in all the other components of \( M_{ij} \) around this block-diagonal direction (see the Appendix).

In terms of new variables \( \phi_i \equiv M_{2i-1,2i} \) with \( i = 1, \cdots, 4 \) and the other components vanishing, the scalar potential is given by

\[
V = \sum_i \left| \lambda \frac{\phi_1 \phi_2 \phi_3 \phi_4}{\phi_i} - m^2 \right|^2,
\]

under the minimal Kähler potential.

Along the phases of the fields \( \phi_i \), the potential has a minimum at \( \arg(\phi_i) = 0 \), and hence hereafter we set \( \phi_i = \sqrt{2} \phi_i \) to be positive so that

\[
V = \sum_i \left( \frac{\lambda}{\sqrt{8}} \frac{\varphi_1 \varphi_2 \varphi_3 \varphi_4}{\varphi_i} - m^2 \right)^2.
\]

The supersymmetric vacuum in the chosen direction is given by

\[
\varphi_i = \varphi_0 \equiv \sqrt{2} \left( \frac{m^2}{\lambda} \right)^{1/3}.
\]

### 3.2 Inflationary dynamics

Let us regard \( \chi = \varphi_4 \) as an inflaton and adopt slow-roll approximation to investigate the inflationary dynamics.

For \( \chi \gg \varphi_0 \), the other fields are stabilized at the following values:

\[
\varphi_i = \varphi_0 f(\chi/\varphi_0); \quad f(x) = \frac{1}{\sqrt{x}} + \frac{1}{4x^2} + \mathcal{O}\left(\frac{1}{x^{3.5}}\right),
\]

\(^4\)Although the masses of such stable directions are not necessarily larger than the Hubble mass during inflation, we neglect their effects in the following analyses.

\(^5\)We simply do not consider the regime \( \chi < \varphi_0 \) in this paper.
where \(i = 1, 2, 3\). Substituting these values back into the potential, the scalar potential for \(\chi \gg \varphi_0\) is given by

\[
V(\chi) \simeq m^4 \left[ 1 - 2 \left( \frac{\varphi_0}{\chi} \right)^{\frac{4}{3}} + O \left( \left( \frac{\varphi_0}{\chi} \right)^{\frac{2}{3}} \right) \right].
\]

(10)

The inflationary regime is determined by the slow-roll conditions \(\epsilon(\chi) \leq 1, \ |\eta(\chi)| \leq 1\), where

\[
\epsilon(\chi) = \frac{1}{2} \left( \frac{V'(\chi)}{V(\chi)} \right)^2 \simeq \frac{9}{2} \frac{\varphi_0^3}{\chi^5},
\]

(11)

\[
\eta(\chi) = \frac{V''(\chi)}{V(\chi)} \simeq -\frac{15}{2} \frac{\varphi_0^4}{\chi^7},
\]

(12)

which are satisfied for \(\chi \geq \chi_f \simeq 1.8 \varphi_0^{3/7}\).

The field value \(\chi_{N_e}\) corresponding to the \(e\)-fold number \(N_e\) is given by

\[
N_e \simeq \int_{\chi_f}^{\chi_{N_e}} d\chi \frac{V(\chi)}{V'(\chi)} \simeq \frac{2}{21} \frac{\varphi_0^{-\frac{4}{7}}}{\chi_{N_e}^{\frac{5}{7}}} - \frac{5}{7},
\]

(13)

which leads to

\[
\chi_{N_e} \simeq \left( \frac{21}{2} N_e \right)^{\frac{7}{5}} \varphi_0^{\frac{4}{7}}.
\]

(14)

The spectral index of the density fluctuations is thus given by

\[
n_s \simeq 1 - 6\epsilon(\chi_{N_0}) + 2\eta(\chi_{N_0})
\]

\[
\simeq 1 - \frac{10}{7N_0},
\]

(15)

where \(N_0\) is the \(e\)-fold number corresponding to the present horizon.

If this is the primordial inflation, its scale is determined by the COBE normalization

\[
\left| \frac{V(\chi_{N_0})^{\frac{2}{7}}}{V'(\chi_{N_0})} \right| = \frac{1}{3} m^2 \varphi_0 \left( \frac{\chi_{N_0}}{\varphi_0} \right)^{\frac{2}{7}} \simeq 5.3 \times 10^{-4},
\]

(16)

which implies

\[
m \simeq 1.9 \times 10^{-3} \left( \frac{50}{N_0} \right)^{\frac{2}{7}} \left( \frac{1}{\lambda} \right)^{\frac{1}{7}}.
\]

(17)
For $\lambda \simeq 1$ and $N_0 \simeq 50$, we obtain $m \simeq 1.9 \times 10^{-3}$, $\phi_0 \simeq 0.021$, $\chi_f \simeq 0.35$, and $\chi_{N_0} \simeq 1.2$. Although the initial value of the inflaton exceeds the dynamical scale $\Lambda$, or the Planck scale, this could be reduced by means of supergravity corrections, which we not discuss in the present analysis. The reheating via the inflaton decay may also be caused by higher dimensional operators.

4 Higher-rank models

For the general case of massive supersymmetric $Sp(N)$ gauge theory with $2N + 4$ chiral superfields $Q_i$ of the fundamental representation ($i = 1, \cdots, 2N + 4$ with $N$ of order one), we start from

$$W_{\text{eff}} = \frac{1}{\Lambda^{N-1}} \text{Pf} M - m_{ij} \Lambda M_{ij}. \quad (18)$$

Then, as is the case in the previous section corresponding to $N = 2$, we are led to consider

$$W_{\text{eff}} = \lambda \prod_i \phi_i - m^2 \sum_i \phi_i, \quad (19)$$

as the effective superpotential in terms of block-diagonal variables $\phi_i \equiv M_{2i-1,2i}$ for $i = 1, \cdots, N + 2$. The scalar potential is given by

$$V = \sum_i \left| \lambda \frac{\prod_j \phi_j}{\phi_i} - m^2 \right|^2 \quad (20)$$

under the minimal Kähler potential.

We again set $\varphi_i = \sqrt{2} \phi_i$ to be positive so that

$$V = \sum_i \left( \frac{\lambda}{\sqrt{2}^{N+1}} \frac{\prod_j \varphi_j}{\varphi_i} - m^2 \right)^2. \quad (21)$$

The supersymmetric vacuum in the chosen direction is given by

$$\varphi_i = \varphi_0 \equiv \sqrt{2} \left( \frac{m^2}{\lambda} \right)^{\frac{1}{N+1}}. \quad (22)$$

Let us regard $\chi = \varphi_{N+2}$ as an inflaton. For $\chi \gg \varphi_0$, the other fields are stabilized at the following values:

$$\varphi_i = \varphi_0 f (\chi/\varphi_0); \quad f(x) = x^{-\frac{1}{N}} + \frac{1}{N^2} x^{-\frac{N+2}{N}} + \mathcal{O} \left( x^{-\frac{2N+3}{N}} \right), \quad (23)$$
where \( i = 1, \ldots, N + 1 \). Substituting these values back into the potential, the scalar potential for \( \chi \) is given by

\[
V(\chi) \simeq m^4 \left[ 1 - 2 \left( \frac{\varphi_0}{\chi} \right)^\frac{N+1}{N} + \mathcal{O} \left( \left( \frac{\varphi_0}{\chi} \right)^\frac{2(N+1)}{N} \right) \right].
\] (24)

The inflationary regime is determined by the slow-roll conditions \( \epsilon(\chi) \leq 1, \ |\eta(\chi)| \leq 1 \), where

\[
\epsilon(\chi) = \frac{1}{2} \left( \frac{V'(\chi)}{V(\chi)} \right)^2 \simeq 2 \left( \frac{N+1}{N} \right)^2 \frac{1}{\varphi_0^2} \left( \frac{\varphi_0}{\chi} \right)^\frac{4N+2}{N},
\] (25)

\[
\eta(\chi) = \frac{V''(\chi)}{V(\chi)} \simeq -2 \frac{(N+1)(2N+1)}{N^2} \frac{1}{\varphi_0^2} \left( \frac{\varphi_0}{\chi} \right)^\frac{3N+1}{N},
\] (26)

which are satisfied for \( \chi \geq \chi_f \simeq \varphi_0^{N+1/3N+1} \).

The field value \( \chi_{N_e} \) corresponding to the \( e \)-fold number \( N_e \) is given by

\[
N_e \simeq \int_{\chi_f}^{\chi_{N_e}} d\chi \frac{V(\chi)}{V'(\chi)} \simeq \frac{N^2}{2(N+1)(3N+1)} \frac{\varphi_0^2}{\chi_{N_e}^{3/2}} \left( \frac{\varphi_0}{\chi_{N_e}} \right)^\frac{N+1}{N},
\] (27)

which leads to

\[
\chi_{N_e} \simeq \left( \frac{2(N+1)(3N+1)}{N^2} N_e \right)^\frac{N}{3N+1} \varphi_0^{\frac{N+1}{N+1}}.
\] (28)

The spectral index of the density fluctuations is given by

\[
n_s \simeq 1 - 6\epsilon(\chi_{N_0}) + 2\eta(\chi_{N_0}) \simeq 1 - \frac{4N+2}{(3N+1)N_0},
\] (29)

where \( N_0 \) is again the \( e \)-fold number corresponding to the present horizon, while \( N \) is the rank of the gauge group \( \text{Sp}(N) \).

The COBE normalization requires, for the primordial case,

\[
\left| \frac{V(\chi_{N_0})^{3/2}}{V'(\chi_{N_0})} \right| = \frac{1}{2} \frac{N}{N+1} m^2 \varphi_0 \left( \frac{\chi_{N_0}}{\varphi_0} \right)^\frac{2N+1}{N} \simeq 5.3 \times 10^{-4},
\] (30)

which implies

\[
m \simeq 2^{\frac{N+1}{2N}} \left( \frac{2(N+1)}{N} \times 5.3 \times 10^{-4} \right)^\frac{3N+1}{6N} \left( \frac{1}{\chi} \right)^\frac{1}{6N} \left( \frac{(2N+1)(3N+1)}{N^2} N_0 \right)^\frac{2N-1}{6N}.
\] (31)
For $\lambda \simeq 1$ and $N_0 \simeq 50$, we obtain, for $N = 3, 4, \text{and } 5$,

\begin{align}
  m &\simeq 2.6 \times 10^{-3}, \quad \phi_0 \simeq 0.073, \quad \chi_f \simeq 0.4, \quad \chi_{N_0} \simeq 2.2 \quad (N = 3), \quad (32) \\
  m &\simeq 3.1 \times 10^{-3}, \quad \phi_0 \simeq 0.14, \quad \chi_f \simeq 0.5, \quad \chi_{N_0} \simeq 3.0 \quad (N = 4), \quad (33) \\
  m &\simeq 3.5 \times 10^{-3}, \quad \phi_0 \simeq 0.21, \quad \chi_f \simeq 0.6, \quad \chi_{N_0} \simeq 3.6 \quad (N = 5). \quad (34)
\end{align}

5 Conclusion

We have investigated simple\footnote{That is, without recourse to certain tunings such as those in Ref.\cite{7}.} massive supersymmetric gauge theories which dynamically provide flat potentials for cosmological inflation.

The $Sp(N)$ gauge group is utilized with $2N + 4$ chiral superfields of the fundamental representation to result in inflation of the hybrid type with composite inflatons. More general gauge theories might be of interest along similar lines, and also inflation of the new type might be realized around the origin of the field space in certain gauge theories.

The gauge theory is assumed to have genuinely strong coupling with its dynamical scale around the fundamental cutoff scale in the supergravity theory where it is immersed.\footnote{Such strong coupling may come into play in circumstances implied, for instance, by heterotic M-theory with a sizable bulk.} The expected ubiquity of (massive) gauge theories in fundamental unified theory with gauge interactions of elementary particles may indicate reality of the dynamical inflation, at least, as primary inflations.

In this paper, we have not analyzed supergravity corrections as well as dynamical ones in the Kähler potential, which should be examined in a future work.

Acknowledgements

This work is supported by the Grant-in-Aid for Yukawa International Program for Quark-Hadron Sciences and the Grant-in-Aid for the 21st Century COE “Center for Diversity and Universality in Physics” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. The work by KH was supported by JSPS (18840012).
Appendix

In this Appendix, we show that at the points along the block-diagonal direction $M_{12}$, $M_{34}$, \ldots, $M_{2N+3,2N+4}$, there are no unstable directions in all the other components of $M_{ij}$ for the massive $Sp(N)$ gauge theory with the common mass.

For a superpotential

$$W = \lambda \text{Pf} M - m_{ij} M_{ij}$$

(35)

with the minimal Kähler potential, the scalar potential of the fields $M_{ij}$ is given by

$$V = \sum_{i<j} \left| \frac{\partial W}{\partial M_{ij}} \right|^2 = \frac{1}{2} \sum_{i,j} \left| \frac{1}{2} \lambda \text{Pf} (M^{-1})_{ji} - m_{ij} \right|^2$$

$$= \frac{1}{8} \lambda^2 | \text{det} M | \text{Tr} (M^{-1} M^{-1\dagger}) + \frac{1}{2} \text{Tr} (mm^\dagger) - \frac{1}{4} \lambda \left[ \text{Pf} M \text{Tr} (M^{-1} m^* ) + \text{h.c.} \right],$$

(36)

where we have used $2 \text{Pf} M \delta(\text{Pf} M) = \delta(\text{det} M) = \text{det} M \text{Tr} (M^{-1} \delta M)$. Note that the parameters $m_{ij}$ here stand for $m_{ij} \Lambda$ in the main text and $m$ denotes the corresponding matrix.

Now let us decompose the fields $M_{ij}$ into the block-diagonal direction and the other directions:

$$M = U \hat{M} U^T,$$

(37)

where

$$\hat{M} = \begin{pmatrix}
\phi_1 \\
-\phi_1 \\
\phi_2 \\
-\phi_2 \\
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\cdot & \<sup>8</sup>N is the rank of the gauge group with $N = 2$ in section 2.
for the present purposes. Here $\phi_i$ are positive variables and $X_a$ are hermitian matrices with real variables $\theta_a$.

We would like to show that the potential has no unstable direction at $\theta = 0$, or the point with all the $\theta_a$ vanishing, when the block-diagonal direction $\phi_i$ develops nonvanishing values. In the scalar potential Eq.(36), only the last term depends on the $\theta_a$:

$$V(\theta) = -\frac{1}{4} \lambda \text{Pf} \left( U \hat{M} U^T \right) \text{Tr} \left( U^* \hat{M}^{-1} U^\dagger m^* \right) + \text{h.c.}$$

Hereafter, we utilize the form of the common mass as assumed in the main text, that is,

$$m_{ij} = m_0 \Omega_{ij}; \quad \Omega = \begin{pmatrix}
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
\end{pmatrix}, \quad m_0 > 0. \quad (41)$$

Firstly, the dependence on the overall phase direction $U = e^{i\theta_0}$ is given by

$$V(\theta_0) = -\frac{1}{4} \lambda \text{Pf} \hat{M} e^{2(N+2)i\theta_0} m_0 \text{Tr} \left( \hat{M}^{-1} \Omega \right) e^{-2i\theta_0} + \text{h.c.} \quad (42)$$

$$= -\frac{1}{2} \lambda \text{Pf} \hat{M} m_0 \text{Tr} \left( \hat{M}^{-1} \Omega \right) \cos(2(N+1)\theta_0). \quad (43)$$

Note that $\text{Pf} \hat{M} > 0$ and $\text{Tr} \left( \hat{M}^{-1} \Omega \right) > 0$, and hence $\theta_0 = 0$ is a minimum. Thus we take $\det U = 1$ henceforth, which results in

$$V(\theta) = -\frac{1}{4} \lambda \text{Pf} \hat{M} m_0 \text{Tr} \left( U^* \hat{M}^{-1} U^\dagger \Omega \right) + \text{h.c.}, \quad (44)$$

since $\text{Pf} (U \hat{M} U^T) = \text{Pf} \hat{M} \det U$. It is easy to see that $\theta = 0$ is a stationary point:

$$\left. \frac{\partial V(\theta)}{\partial \theta_a} \right|_{\theta=0} = 0. \quad (45)$$

In order to examine its stability, let us decompose the off-diagonal directions into the following two sets of directions: $X_a = X_a^S + X_a^A$, where

$$X_a^S = \Omega X_a^{ST} \Omega^{-1}, \quad X_a^A = -\Omega X_a^{AT} \Omega^{-1}. \quad (46)$$
It is straightforward to show that the second derivative with respect to the direction $X_a$ is given by

$$\left. \frac{\partial^2 V(\theta)}{\partial \theta_a^2} \right|_{\theta=0} = m_0 \lambda \text{Tr} \left[ \hat{M}^{-1} \Omega \left( X_a^S X_a^{S^2} + X_a^{S^2} X_a^S \right) \right] \geq 0,$$

(47)

where the equality holds if and only if $X_a^S = 0$. Note that the directions with $X_a^S = 0$ constitute symmetry algebra of the massive gauge theory with the common mass so that they are completely flat. Therefore the scalar potential along the block-diagonal direction $\phi_i$ has no instability toward the other directions $\theta_a$.

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