The effect of nonlocal term on the superlinear Kirchhoff type equations in $\mathbb{R}^N$

Juntao Sun$^{a,b}$† Tsung-fang Wu$^c$‡

$^a$School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, PR China

$^b$School of Mathematical Sciences, Qufu Normal University, Shandong 273165, PR China

$^c$Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan

Abstract

We are concerned with a class of Kirchhoff type equations in $\mathbb{R}^N$ as follows:

$$\left\{ \begin{array}{l}
-M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + \lambda V(x) u = f(x,u) \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right.$$  

where $N \geq 1$, $\lambda > 0$ is a parameter, $M(t) = am(t) + b$ with $a, b > 0$ and $m \in C(\mathbb{R}^+, \mathbb{R}^+)$, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying $\lim_{|u| \to \infty} f(x,u)/|u|^{k-1} = q(x)$ uniformly in $x \in \mathbb{R}^N$ for any $2 < k < 2^*(2^* = \infty$ for $N = 1, 2$ and $2^* = 2N/(N-2)$ for $N \geq 3$). Unlike most other papers on this problem, we are more interested in the effects of the functions $m$ and $q$ on the number and behavior of solutions. By using minimax method as well as Caffarelli-Kohn-Nirenberg inequality, we obtain the existence and multiplicity of positive solutions for the above problem.

Key words: Nontrivial solutions, Kirchhoff type equations, Caffarelli-Kohn-Nirenberg inequality, Steep potential well.

1 Introduction

Consider the following nonlinear Kirchhoff type equations:

$$\left\{ \begin{array}{l}
-M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + \lambda V(x) u = f(x,u) \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right. \quad (K_{\lambda,a})$$

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†E-mail address: jtsun@sdut.edu.cn (J. Sun)

‡E-mail address: tfwu@nuk.edu.tw (T.-F. Wu)
where $N \geq 1$, $\lambda > 0$ is a parameter, $M(t) = am(t) + b$ with $a, b > 0$ and $m$ being positive continuous function on $\mathbb{R}^+$, and $f$ is a continuous function on $\mathbb{R}^N \times \mathbb{R}$ such that $f(x, s) \equiv 0$ for all $x \in \mathbb{R}^N$ and $s < 0$. We assume that the potential $V(x)$ satisfies the following hypotheses:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ and there exists $c_0 > 0$ such that the set \( \{ V < c_0 \} := \{ x \in \mathbb{R}^N \mid V(x) < c_0 \} \) has finite positive Lebesgue measure, where $|\cdot|$ is the Lebesgue measure;

(V2) $\Omega = intV^{-1}(0)$ is nonempty and has smooth boundary with $\overline{\Omega} = V^{-1}(0)$.

Kirchhoff type equations, of the form similar to Eq. $(K_{a,\lambda})$, are often referred to as being nonlocal because of the presence of the integral. When $m(t) = t$, Eq. $(K_{a,\lambda})$ is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - \left( a \int_{\Omega} |\nabla u|^2 \, dx + b \right) \Delta u = f(x, u), \quad (1)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$. As an extension of the classical D’Alembert’s wave equation, Eq. (1) was first presented by Kirchhoff [12] in 1883 to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations, where $u$ denotes the displacement, $f$ is the external force and $b$ is the initial tension while $a$ is related to the intrinsic properties of the string, such as Young’s modulus.

Since Lions [11] introduced an abstract framework to the Kirchhoff type equations, the qualitative analysis of nontrivial solutions for such equations with various nonlinear terms, including the existence and multiplicity of positive solutions and of sign-changing solution, has begun to receive much attention; see, for example, [1, 2, 4, 13, 14, 15, 16, 17, 20, 21] for the bounded domain case and [5, 7, 8, 9, 10, 18, 19] for the unbounded domain case.

Let us briefly comment some well-known results for the nonlinear Kirchhoff type equations. Alves-Corrêa-Ma [1] studied the following Kirchhoff type equations in the bounded domain $\Omega$:

$$\begin{cases} -M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u = f(x, u) \quad \text{in } \Omega, \\ u \in H^1_0(\Omega). \end{cases} \quad (2)$$

They concluded the existence of positive solutions of Eq. (2) when $M$ does not grow too fast in a suitable interval near zero and $f$ is locally Lipschitz subject to some prescribed criteria. Bensedik-Bouchekif [2] studied the asymptotically linear case and obtained the existence of positive solutions of Eq. (2) when the function $M$ is a non-decreasing function and $M(t) \geq m_0$ for some $m_0 > 0$, and $f$ is the asymptotically linear satisfying some assumptions about its asymptotic behaviors near zero and infinite. Later, Chen-Kuo-Wu [4] illustrated the difference in the solution behavior which arises from the consideration of the nonlocal effect for Eq. (2) with $M(t) = at + b$ and $f$ being concave-convex nonlinearity.
Compared with the bounded domain case, the unbounded domain case seems to be more delicate. The primary difficulty lies in the lack of the embedding of compactness. Figueiredo-Ikoma-Júnior [7] studied the existence and concentration behaviors of positive solutions to the following Kirchhoff type equations:

\[
\begin{cases}
-\varepsilon^2 M (\varepsilon^2 - N \int_{\mathbb{R}^N} |\nabla u|^2 \, dx) \Delta u + V(x) u = f(u) \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\] (3)

Under suitable conditions on \(M\) and \(f\), a family of positive solutions for Eq. (3) concentrating at a local minimum of \(V\) are constructed. Recently, we [18] introduced the hypotheses \((V_1) - (V_2)\) to Kirchhoff type equations in \(\mathbb{R}^N (N \geq 3)\) and studied the existence of nontrivial solution for Eq. \((K_{\lambda,a})\) with \(m(t) = t\) and \(f\) being asymptotically linear or superlinear at infinity on \(u\).

Inspired by the above facts, in this paper we are likewise interested in looking for nontrivial solutions for Eq. \((K_{a,\lambda})\) in \(\mathbb{R}^N\) with \(N \geq 1\). However, distinguishing from the existing literatures, we are more focus on the interaction between the functions \(m\) and \(f\), leading to the difference in the number of solutions. Specifically, we find that the powers of \(m\) and \(f\) will dominate the number of solutions for Eq. \((K_{a,\lambda})\). We require that the function \(m\) satisfies some asymptotic behaviors near infinite and that \(f\) is \(k\)-asymptotically linear at infinity on \(u\) for any real number \(2 < k < 2^*\), i.e., \(\lim_{|u| \to \infty} f(x, u)/|u|^{k-1} = q(x)\) uniformly in \(x \in \mathbb{R}^N\), while not requiring any assumption about the asymptotic behavior near zero of \(m\) and \(f\).

We wish to point out that in the study of one positive solution, the range of the parameter \(a > 0\) in Eq. \((K_{a,\lambda})\) is dependent on the limiting function \(q\) of \(f\). In other words, the different types of \(q\) will bring about the different ranges of \(a\). Moreover, in the study of two positive solutions, the geometry of the variational structure of Eq. \((K_{a,\lambda})\) is known to have a local minimum and a mountain pass, since the power of \(m\) is greater than the one of \(f\). In view of this, it is clear to use the minimax method to seek two solutions of Eq. \((K_{a,\lambda})\) as critical points of the associated energy functional \(J_{a,\lambda}\). However, since the norms \(\|u\|_{D^{1,2}} = (\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2}\) and \(\|u\|_{H^1} = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx)^{1/2}\) are not equivalent in \(H^1(\mathbb{R}^N)\), we can not apply the standard techniques to verify the boundedness below of \(J_{a,\lambda}\) and the boundedness of the \((PS)\)-sequence.

Based on the analysis above, we suggest some new techniques and introduce new hypotheses on \(m\) and \(q\) in the present paper. By using the minimax method and Caffarelli-Kohn-Nirenberg inequality, we obtain the existence and multiplicity of positive solutions for Eq. \((K_{a,\lambda})\) under the different assumptions on \(m\) and \(f\), respectively.

We now summarize our main results as follows.

**Theorem 1.1** Suppose that \(N \geq 1\) and conditions \((V1) - (V2)\) hold. In addition, for any real number \(2 < k < 2^*\), we assume that the functions \(m\) and \(f\) satisfy the following conditions:
(L1) there exists \( m_\infty > 0 \) such that \( \lim_{t \to \infty} t^{-(k-2)/2} m(t) = m_\infty \) and
\[
\int_\sigma^\eta m(t) dt \geq \frac{2(\eta - \sigma)}{k} m(\eta)
\]
for all \( 0 \leq \sigma < \eta \);

(D1) there exist \( q \in L^\infty(\mathbb{R}^N) \) and \( 0 \leq \mu < N - \frac{k(N-2)}{2} \) satisfying \( q \not\equiv 0 \) on \( \Omega \) and
\[
\liminf_{|x| \to \infty} |x|^\mu q(x) > 0
\]
such that
\[
\int_\sigma^\eta m(t) dt \geq 2(m_\infty) \frac{k}{(\eta - \sigma)} m(\eta)
\]
for all \( 0 \leq \sigma < \eta \);

(D2) \( s \mapsto \frac{f(x,s)}{s^{k-1}} \) is nondecreasing function on \((0, \infty)\) for any fixed \( x \in \mathbb{R} \).

Then there exists \( \bar{\Lambda} > 0 \) such that Eq. \((K_{\lambda,a})\) admits at least one positive solution for all \( \lambda > \bar{\Lambda} \) and \( a > 0 \).

Remark 1.1  
(i) It is not difficult to find such functions \( m \) satisfying condition \((L1)\). For example, let \( m(t) = t^{(k-2)/2} + \theta t^{(k-2)/4} \) for \( t > 0 \), where \( 2 < k < 2^* \) and \( \theta \geq 0 \). Clearly, \( \lim_{t \to \infty} t^{-(k-2)/2} m(t) = 1 \). Moreover, a direct calculation shows that
\[
\int_\sigma^\eta m(t) dt = \frac{2}{k} (\eta^{k/2} - \sigma^{k/2}) + \frac{4\theta}{k+2} (\eta^{(k+2)/4} - \sigma^{(k+2)/4})
\]
\[
\geq \frac{2}{k} (\eta^{k/2} - \eta^{(k-2)/2} \sigma) + \frac{4\theta}{k+2} (\eta^{(k+2)/4} - \eta^{(k-2)/4} \sigma)
\]
\[
> \frac{2(\eta - \sigma)}{k} m(\eta) \text{ for all } 0 \leq \sigma < \eta.
\]

(ii) Under condition \((D1)\), it is not difficult to verify that for any real number \( 2 < k < 2^* \),
\[
\inf_{u \in H^1(\mathbb{R}^N)} \left( \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\int_{\mathbb{R}^N} q(x) |u|^k dx} \right)^{k/2} = 0.
\]

For more details, we refer to the proof of Lemma 3.5 below.

We now assume that the function \( m \) satisfies the following assumptions instead of condition \((L1)\):

(L2) \( m(t) \) is nondecreasing on \( t \geq 0 \);

(L3) there exist three positive numbers \( m_0, \delta \) and \( T_0 \) such that \( m(t) \geq m_0 t^\delta \) for all \( t \geq T_0 \).

Then we have the following result.
Theorem 1.2 Suppose that $N \geq 3$, conditions $(V1)-(V2), (L2)$ and $(L3)$ with $\delta \geq \frac{2}{N-2}$ hold. In addition, for any real number $2 < k < 2^*$, we assume that the function $f$ satisfies conditions $(D1)-(D2)$. Then there exist constants $\tilde{\alpha}_*, \Lambda_* > 0$ such that for every $0 < a < \tilde{\alpha}_*$ and $\lambda > \Lambda_*$, Eq. $(K_{a, \lambda})$ admits at least two positive solutions $u_{a, \lambda}^-$ and $u_{a, \lambda}^+$ satisfying $J_{a, \lambda}(u_{a, \lambda}^-) < 0 < J_{a, \lambda}(u_{a, \lambda}^+)$ In particular, $u_{a, \lambda}^-$ is a ground state solution of Eq. $(K_{a, \lambda})$.

Remark 1.2 Compared with Theorem 1.1, if we raise the power $\delta$ of the function $m$ such that $\delta \geq \frac{2}{N-2} > \frac{k-2}{2}$ for $N \geq 3$ in Theorem 1.2, then two positive solutions of Eq. $(K_{a, \lambda})$ can be obtained.

It is well known that for $N \geq 3$ and $2 < k < 2^*$, the following minimum problem

\[ \overline{\mathcal{D}}_1^{(k)} := \inf_{u \in D^{1,2}({\mathbb{R}}^N)} \left( \frac{\int_{{\mathbb{R}}^N} |\nabla u|^2 dx}{\int_{{\mathbb{R}}^N} |x|^{\frac{k(N-2)}{2}-N} |u|^k dx} \right) > 0 \]  

(4)

is achieved by some $\overline{\varphi}_k \in D^{1,2}({\mathbb{R}}^N)$ by Caffarelli-Kohn-Nirenberg inequality.

We now assume that the following assumption holds:

$(D1)'$ there exist the function $q(x)$ satisfying $q \not\equiv 0$ on $\tilde{\Gamma}$ and $q(x) \leq c_* |x|^{\frac{k(N-2)}{2}-N}$ for some $c_* > 0$ and for all $x \in {\mathbb{R}}^N$ such that

\[ \lim_{s \to \infty} \frac{f(x, s)}{s^{k-1}} = q(x) \] uniformly in $x \in {\mathbb{R}}^N$.

Under condition $(D1)'$ and $(4)$, it is easily seen that for any $2 < k < 2^*$, the minimum problem

\[ \overline{\mathcal{D}}_1^{(k)} := \inf_{u \in H^1({\mathbb{R}}^N)} \left( \frac{\int_{{\mathbb{R}}^N} |\nabla u|^2 dx}{\int_{{\mathbb{R}}^N} q(x)|u|^k dx} \right) \geq \overline{\mathcal{D}}_1^{(k)} > 0. \]

(5)

Then we have the following results.

Theorem 1.3 Suppose that $N \geq 3$ and conditions $(V1)-(V2)$ hold. In addition, for any real number $2 < k < 2^*$, we assume that conditions $(L1), (D1)'$ and $(D2)$ hold. Then for each $0 < a < \frac{1}{m_{\infty} \tilde{\alpha}_*}$, there exists $\overline{\lambda} > 0$ such that Eq. $(K_{\lambda, a})$ admits at least one positive solution for all $\lambda > \overline{\lambda}$.

Theorem 1.4 Suppose that $N \geq 3$ and conditions $(V1)-(V2), (L2)$ hold. In addition, for any real number $2 < k < 2^*$, we assume that conditions $(L3)$ with $\delta > \frac{k-2}{2}$, $(D1)'$ and $(D2)$ hold. Then there exist constants $\overline{\alpha}_*, \overline{\Lambda}_* > 0$ such that for every $0 < a < \overline{\alpha}_*$ and $\lambda > \overline{\Lambda}_*$, Eq. $(K_{a, \lambda})$ admits at least two positive solutions $u_{a, \lambda}^-$ and $u_{a, \lambda}^+$ satisfying $J_{a, \lambda}(u_{a, \lambda}^-) < 0 < J_{a, \lambda}(u_{a, \lambda}^+)$ . In particular, $u_{a, \lambda}^+$ is a ground state solution of Eq. $(K_{a, \lambda})$.

Remark 1.3 Unlike Theorem 1.2 in Theorem 1.4 we only require the power $\delta$ of $m(t)$ is greater than $\frac{k-2}{2}$, also leading to two positive solutions of Eq. $(K_{a, \lambda})$ under condition $(D1)'$.

The remainder of this paper is organized as follows. After giving some preliminaries in Section 2, we prove that $J_{\lambda, a}$ satisfies the mountain pass geometry in Section 3. In Sections 4–6, we give the proofs of Theorems 1.1–1.4.
2 Preliminaries

Let

\[ X = \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2 \, dx < \infty \right\} \]

be equipped with the inner product and norm

\[ \langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) \, dx, \quad \|u\| = \langle u, u \rangle^{1/2}. \]

For \( \lambda > 0 \), we also need the following inner product and norm

\[ \langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv) \, dx, \quad \|u\|_\lambda = \langle u, u \rangle_\lambda^{1/2}. \]

Clearly, \( \|u\| \leq \|u\|_\lambda \) for \( \lambda \geq 1 \). Now we set \( X_\lambda = (X, \|u\|_\lambda) \).

For \( N = 1, 2 \), applying conditions \((V1) - (V2)\), the Hölder, Young and Gagliardo-Nirenberg inequalities, there exists a sharp constant \( A_N > 0 \) such that

\[ \int_{\mathbb{R}^N} u^2 \, dx \leq \frac{1}{c_0} \int_{\{V \geq c_0\}} V(x) u^2 \, dx + \left( \|\{V < c_0\}\| \int_{\mathbb{R}^N} u^4 \, dx \right)^{1/2} \]

\[ \leq \frac{1}{c_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + A_N^2 \|\{V < c_0\}\|^{1/2} \|u\|^{N/2}_{D^{1,2}} \|u\|^{(4-N)/2}_{L^2} \]

\[ \leq \frac{1}{c_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + \frac{NA_N^{8/N}}{4} \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \|u\|^{2}_{D^{1,2}} + \frac{4 - N}{4} \|u\|^2_{L^2}, \]

which shows that

\[ \int_{\mathbb{R}^N} u^2 \, dx \leq \frac{4}{Nc_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx + A_N^{8/N} \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \|u\|^{2}_{D^{1,2}}. \]

This implies that

\[ \|u\|^2_{H^1} \leq \max \left\{ 1 + A_N^{8/N} \|\{V < c_0\}\|^{2/N}_{D^{1,2}}, \frac{4}{Nc_0} \right\} \|u\|^2. \] (6)

Similarly, we also have

\[ \|u\|^2_{H^1} \leq \left( 1 + A_N^{8/N} \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \right) \|u\|^2_\lambda \text{ for } \lambda \geq \frac{4}{Nc_0} \left( 1 + A_N^{8/N} \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \right)^{-1}. \] (7)

For \( N \geq 3 \), following [18], we have

\[ \|u\|^2_{H^1} \leq \max \left\{ 1 + S^2 \|\{V < c_0\}\|^{2/N}_{D^{1,2}}, c_0^{-1} \right\} \|u\|^2 \] (8)

and

\[ \|u\|^2_{H^1} \leq \left( 1 + S^2 \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \right) \|u\|^2_\lambda \text{ for } \lambda \geq c_0^{-1} \left( 1 + S^2 \|\{V < c_0\}\|^{2/N}_{D^{1,2}} \right), \]
where $S$ is the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^2 r(\mathbb{R}^N)$. Set
\[
\alpha_N := \begin{cases} 
\max \left\{ 1 + A_N^{8/N} \left| \{ V < c_0 \} \right|^{2/N}, \frac{4}{N c_0} \right\} & \text{for } N = 1, 2; \\
\max \left\{ 1 + S^r \left| \{ V < c_0 \} \right|^{2/N}, c_0^{-1} \right\} & \text{for } N \geq 3.
\end{cases}
\]

Thus, it follows from (6) and (8) that
\[
\| u \|_{H^1}^2 \leq \alpha_N \| u \|^2,
\]
which implies that the imbedding $X \hookrightarrow H^1(\mathbb{R}^N)$ is continuous.

Since the imbedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad (2 \leq r < +\infty)$ is continuous for $N = 1, 2$, from (7) it follows that for any $r \in [2, +\infty)$,
\[
\int_{\mathbb{R}^N} |u|^r \, dx \leq S_r^{-r} \| u \|_{H^1}^r \leq S_r^{-r} \left( 1 + A_N^{8/N} \left| \{ V < c_0 \} \right|^{2/N} \right)^{r/2} \| u \|_{\lambda}^r
\]
for $\lambda \geq \frac{4}{N c_0} \left( 1 + A_N^{8/N} \left| \{ V < c_0 \} \right|^{2/N} \right)^{-1}$, where $S_r$ is the best Sobolev constant for the imbedding of $H^1(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N) \quad (2 \leq r < +\infty)$. For $N \geq 3$, following the argument in (18) (see pp 1776-1777), for any $r \in [2, 2^*)$ one has
\[
\int_{\mathbb{R}^N} |u|^r \, dx \leq \left( \int_{\{ V \geq c_0 \}} u^2 \, dx + \int_{\{ V < c_0 \}} u^2 \, dx \right)^{2^* - \frac{r}{2}} \left( S^{-1} \| u \|_{D^{1,2}} \right)^{\frac{2^* (2 - r)}{2^* - 2}}
\]
\[
\leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx + S^{-2} \left| \{ V < c_0 \} \right|^{2/N} \| u \|^2_{D^{1,2}} \right)^{\frac{2 - r}{2}} \left( S^{-1} \| u \|_{D^{1,2}} \right)^{\frac{2^* (r - 2)}{2^* - 2}}
\]
\[
\leq \left\{ \left| \{ V < c_0 \} \right|^{(2^* - r)/2^*} S^{-r} \| u \|_{\lambda}^r \right. \text{ for } \lambda \geq c_0^{-1} S^{2^*} \left| \{ V < c_0 \} \right|^{-2/N}.
\]

Now we set
\[
\Theta_{r,N} := \begin{cases} 
S_r^{-r} \left( 1 + A_N^{8/N} \left| \{ V < c_0 \} \right|^{2/N} \right)^{r/2} & \text{if } N = 1, 2; \\
S^{-r} \left| \{ V < c_0 \} \right|^{\frac{2^* - r}{2^* - 2}} & \text{if } N \geq 3,
\end{cases}
\]
and
\[
\Lambda_N := \begin{cases} 
\frac{4}{N c_0} \left( 1 + A_N^{8/N} \left| \{ V < c_0 \} \right|^{2/N} \right)^{-1} & \text{if } N = 1, 2; \\
c_0^{-1} S^2 \left| \{ V < c_0 \} \right|^{-2/N} & \text{if } N \geq 3.
\end{cases}
\]

Thus, by (10)–(11) we have for any $r \in [2, 2^*)$ and $\lambda \geq \Lambda_N$,
\[
\int_{\mathbb{R}^N} |u|^r \, dx \leq \Theta_{r,N} \| u \|_{\lambda}^r.
\]
Eq. \((K_{\lambda,a})\) is variational and its solutions are the critical points of the functional defined in \(X\) by

\[
J_{\lambda,a}(u) = \frac{a}{2} \hat{m}(\|u\|_{D^{1,2}}^2) + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right) - \int_{\mathbb{R}^N} F(x,u) dx,
\]

where \(\hat{m}(t) = \int_0^t m(s) ds\) and \(F(x,u) = \int_0^u f(x,s) ds\). Furthermore, it is not difficult to prove that the functional \(J_{\lambda,a}\) is of class \(C^1\) in \(X\), and that

\[
\langle J'_{\lambda,a}(u), v \rangle = (am(\|u\|_{D^{1,2}}^2) + b) \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} \lambda V(x) uv dx - \int_{\mathbb{R}^N} f(x,u) v dx.
\]

The following theorem is a variant version of the mountain pass theorem, which helps us find a so-called Cerami type \((PS)\)-sequence.

**Theorem 2.1** ([6], Mountain Pass Theorem) Let \(E\) be a real Banach space with its dual space \(E^*\), and suppose that \(I \in C^1(E, \mathbb{R})\) satisfies

\[
\max \{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\| = \rho} I(u),
\]

for some \(\mu < \eta, \rho > 0\) and \(e \in E\) with \(\|e\| > \rho\). Let \(\alpha \geq \eta\) be characterized by

\[
\alpha = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),
\]

where \(\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}\) is the set of continuous paths joining \(0\) and \(e\). Then there exists a sequence \(\{u_n\} \subset E\) such that

\[
I(u_n) \to \alpha \geq \eta \quad \text{and} \quad (1 + \|u_n\|) \|I'(u_n)\|_{E^*} \to 0, \quad \text{as} \; n \to \infty.
\]

### 3 Mountain pass geometry

In this section we prove that the energy functional \(J_{\lambda,a}\) satisfies the mountain pass geometry under the different assumptions on \(m\) and \(f\), respectively.

**Lemma 3.1** Suppose that conditions \((V1) - (V2)\) and \((D1) - (D2)\) hold. Then there exist \(\rho > 0\) and \(\eta > 0\) such that \(\inf \{J_{\lambda,a}(u) : u \in X_\lambda \text{ with } \|u\|_\lambda = \rho\} > \eta\) for all \(\lambda \geq \Lambda_N\).

**Proof.** By conditions \((D1) - (D2)\), we obtain that

\[
f(x,s) \leq q(x)s^{k-1} \quad \text{for all} \; s \geq 0\]

(15)
and
\[ F(x, s) \leq \frac{1}{k} q(x) s^k \quad \text{for all } s \geq 0. \]  

(16)

Then, by (12) and (16), for every \( u \in X \) and \( \lambda \geq \Lambda_N \) one has
\[ \int_{\mathbb{R}^N} F(x,u) \, dx \leq \frac{|q|_\infty}{k} \int_{\mathbb{R}^N} |u|^k \, dx \leq \frac{|q|_\infty \Theta_{k,N}}{k} \| u \|^k_\lambda. \]

This implies that
\[
J_{\lambda,a}(u) = \frac{a}{2} \tilde{m} \left( \| u \|^2_{D^{1,2}} \right) + \frac{1}{2} \left( b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx \right) - \int_{\mathbb{R}^N} F(x,u) \, dx
\geq \frac{1}{2} \min \{ b,1 \} \| u \|^2_\lambda - \frac{|q|_\infty \Theta_{k,N}}{k} \| u \|^k_\lambda.
\]

Thus, letting \( \| u \|_\lambda = \rho > 0 \) small enough, it is easy to obtain that there exists \( \eta > 0 \) such that \( \inf \{ J_{\lambda,a}(u) : u \in X_\lambda \text{ with } \| u \|_\lambda = \rho \} > \eta \) for all \( \lambda \geq \Lambda_N \), since \( 2 < k < 2^* \). The lemma is proved.

Lemma 3.2  Suppose that conditions (V1) – (V2), (D1)' and (D2) hold. Then there exist \( \rho > 0 \) and \( \eta > 0 \) such that \( \inf \{ J_{\lambda,a}(u) : u \in X_\lambda \text{ with } \| u \|_\lambda = \rho \} > \eta \) for all \( a > 0 \) and \( \lambda \geq \Lambda_N \).

Proof. It follows from conditions (D1)' and (D2) that
\[
f(x,s) \leq c_* \left| x \right|^{k(N-2)-N} s^{k-1} \quad \text{for all } s \geq 0
\]
and
\[
F(x,s) \leq \frac{c_*}{k} \left| x \right|^{k(N-2)-N} s^k \quad \text{for all } s \geq 0.
\]

(18)

Then, by (14) and (18), for every \( u \in X \) and \( \lambda \geq \Lambda_N \) one has
\[
\int_{\mathbb{R}^N} F(x,u) \, dx \leq \frac{c_*}{k} \int_{\mathbb{R}^N} \left| x \right|^{k(N-2)-N} |u|^k \, dx \leq \frac{c_*}{k \nu^{(k)}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{k/2}
\leq \frac{c_*}{k \nu^{(k)}} \| u \|^k_\lambda,
\]
which implies that
\[
J_{\lambda,a}(u) \geq \frac{1}{2} \left( b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx \right) - \frac{c_*}{k \nu^{(k)}} \| u \|^k_\lambda
\geq \frac{1}{2} \min \{ b,1 \} \| u \|^2_\lambda - \frac{c_*}{k \nu^{(k)}} \| u \|^k_\lambda.
\]

Thus, letting \( \| u \|_\lambda = \rho > 0 \) small enough, it is easy to obtain that there exists \( \eta > 0 \) such that \( \inf \{ J_{\lambda,a}(u) : u \in X_\lambda \text{ with } \| u \|_\lambda = \rho \} > \eta \) for all \( \lambda \geq \Lambda_N \), since \( 2 < k < 2^* \). The lemma is proved.
Lemma 3.3 Suppose that conditions (V1) – (V2), (L1) and (D1) – (D2) hold. Let \( \rho > 0 \) be as Lemma 3.7. Then there exists \( e \in X \) with \( \|e\|_\lambda > \rho \) such that \( J_{\lambda,a}(e) < 0 \) for all \( a > 0 \) and \( \lambda > 0 \).

Proof. Let \( u \in X \setminus \{0\} \) with \( u > 0 \) and define \( u_n(x) = n^{-N/k}u\left(\frac{x}{n}\right) \). A direct calculation shows that
\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = n^{N-2-\frac{N}{k}} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]
and
\[
\int_{\mathbb{R}^N} q(x)u_n^k \, dx = n^{-N} \int_{\mathbb{R}^N} q(nx)u^k\left(\frac{x}{n}\right) \, dx = \int_{\mathbb{R}^N} q(nx)u^k(x) \, dx.
\]
Thus, it follows from condition (D1) and Fatou’s lemma that
\[
\frac{\|u_n\|_{D,2}^k}{\int_{\mathbb{R}^N} q(x)u_n^k \, dx} = \frac{n^{-\left(N-k\left(N-\frac{N}{2}\right)\right)} \|u\|_{D,2}^k}{\int_{\mathbb{R}^N} q(nx)u^k(x) \, dx} \leq \frac{R_0^k \|u\|_{D,2}^k}{n^{N-k\left(N-\frac{N}{2}\right)} \mu \int_{|x| \leq R_0} |nx|^\mu q(nx)u^k(x) \, dx} \leq \frac{R_0^k \|u\|_{D,2}^k}{Cn^{N-k\left(N-\frac{N}{2}\right)} \mu \int_{|x| \leq R_0} u^k \, dx} \to 0 \text{ as } n \to \infty,
\]
which indicates that
\[
\inf_{u \in X} \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{k/2}}{\int_{\mathbb{R}^N} q(x)|u|^k \, dx} = 0.
\]
Thus, for each \( a > 0 \), there exists \( \phi_k \in X \setminus \{0\} \) with \( \phi_k > 0 \) such that
\[
am_\infty \|\phi_k\|_{D,2}^k - \int_{\mathbb{R}^N} q(x)\phi_k^k \, dx < 0.
\]
Using the above inequality, together with conditions (L1), (D1) – (D2) and Lebesgue’s dominated convergence theorem, leads to
\[
\lim_{t \to +\infty} \frac{J_{\lambda,a}(t\phi_k)}{tk} = \lim_{t \to +\infty} \frac{1}{2tk} \left( b \|\phi_k\|_{D,2}^2 + \int_{\mathbb{R}^N} \lambda V(x)\phi_k^2 \, dx \right) + \lim_{t \to +\infty} \left[ \frac{am}{2tk} \|\phi_k\|_{D,2}^2 - \int_{\mathbb{R}^N} \frac{F(x,t\phi_k)}{t^k\phi_k^k} \phi_k^k \, dx \right] \leq \frac{am_\infty k}{k} \|\phi_k\|_{D,2}^k - \frac{1}{k} \int_{\mathbb{R}^N} q(x)\phi_k^k \, dx
\]
\[
= \frac{1}{k} \left( am_\infty \|\phi_k\|_{D,2}^k - \int_{\mathbb{R}^N} q(x)\phi_k^k \, dx \right) < 0.
\]
This implies that \( J_{\lambda,a}(t\phi_k) \to -\infty \) as \( t \to +\infty \). Therefore, there exists \( e \in X \) with \( \|e\|_\lambda > \rho \) such that \( J_{\lambda,a}(e) < 0 \) and the lemma is proved. □
Lemma 3.4 Suppose that conditions (V1) – (V2) and (D1) – (D2) hold. Let $\rho > 0$ be as Lemma 3.1. Then there exist $\tilde{a}_* > 0$ and $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $0 < a < \tilde{a}_*$ and $\lambda > 0$.

**Proof.** According to the argument of Lemma 3.3, there exists $\phi_k \in X \setminus \{0\}$ with $\phi_k > 0$ such that $\int_{\mathbb{R}^N} q(x)\phi_k^2dx > 0$ by condition (D1). Then using conditions (D1) – (D2), together with Lebesgue’s dominated convergence theorem one has

$$
\lim_{t \to +\infty} \frac{J_{\lambda,0}(t\phi_k)}{t^k} = \lim_{t \to +\infty} \frac{1}{2t^{k-2}} \left( b\|\phi_k\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)\phi_k^2dx \right) - \lim_{t \to +\infty} \int_{\mathbb{R}^N} \frac{F(x,t \phi_k)}{t^k\phi_k} \phi_k^kdx
\leq - \frac{1}{k} \int_{\mathbb{R}^N} q(x)\phi_k^kdx < 0,
$$

where $J_{\lambda,0}(u) = J_{\lambda,a}(u)$ with $a = 0$. Thus, if $J_{\lambda,0}(t\phi_k) \to -\infty$ as $t \to +\infty$, then there exists $e \in X$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,0}(e) < 0$. Since $J_{\lambda,a}(e) \to J_{\lambda,0}(e)$ as $a \to 0^+$, we obtain that there exists $\tilde{a}_* > 0$ such that $J_{\lambda,a}(e) < 0$ for all $0 < a < \tilde{a}_*$ and $\lambda > 0$. ■

**Remark 3.1** We point out that the value of $\tilde{a}_*$ can not be determined in general, but only in some special cases. For example, let us assume that $N = 4$, $m(t) = t$ and

$$
f(x,s) = \begin{cases} q(x)s^2 & \text{if } s \geq 0, \\
0 & \text{if } s < 0,
\end{cases}
$$

where the function $q$ is as in condition (D1). Clearly, the function $m(t)$ does not satisfy condition (L1). Define the minimum problem

$$
\tilde{\mu}_1(\lambda) = \inf_{u \in H^1(\mathbb{R}^N)} \frac{\left( \int_{\mathbb{R}^N} |\nabla u|^2dx \right)^2 \left( b\int_{\mathbb{R}^N} |\nabla u|^2dx + \int_{\mathbb{R}^N} \lambda V(x)u^2dx \right)}{\left( \int_{\mathbb{R}^N} q(x)|u|^3dx \right)^2}.
$$

Then $\tilde{\mu}_1(\lambda) \geq \frac{\min\{1,b\}S^6}{|q|_\infty^2|\{V < c_0\}|^{1/2}} > 0$ for all $\lambda > \frac{S^2}{|c_0|\{V < c_0\}|^{1/2}}$. Indeed, by (9), for every $\lambda > \frac{S^2}{|c_0|\{V < c_0\}|^{1/2}}$ there holds

$$
\geq \frac{\left( \int_{\mathbb{R}^N} |\nabla u|^2dx \right)^2 \left( b\int_{\mathbb{R}^N} |\nabla u|^2dx + \int_{\mathbb{R}^N} \lambda V(x)u^2dx \right)}{\left( \int_{\mathbb{R}^N} q(x)|u|^3dx \right)^2}
\geq \frac{\min\{1,b\}S^6\|u\|_{D^{1,2}}^4\|u\|_\lambda^4}{|q|_\infty^2|\{V < c_0\}|^{1/2}\|u\|_\lambda^4\|u\|_{D^{1,2}}^2}
\geq \frac{\min\{1,b\}S^6}{|q|_\infty^2|\{V < c_0\}|^{1/2}} > 0,
$$

11
which implies that \( \hat{\mu}_1 \geq \frac{\min\{1,b\}s^2}{|q(x_0)|^{2/3}} > 0 \). Thus, for every \( 0 < a < \frac{2}{\gamma\hat{\mu}_1(\lambda)} \) there exists \( u_0 \in X_\lambda \) such that

\[
\frac{1}{9} \left( \int_{\mathbb{R}^N} q(x) |u_0|^3 \, dx \right)^2 > \frac{a}{2} \|u_0\|_{D^{1,2}}^2 \left( b \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_0^2 \, dx \right).
\]

Note that

\[
J_{\lambda,a}(t_0u_0) = \frac{at^4}{4} \|u_0\|_{D^{1,2}}^4 + \frac{t^2}{2} \left( b \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_0^2 \, dx \right) - \int_{\mathbb{R}^N} F(x,t_0u_0) \, dx
\]

\[
= t^2 \left[ \frac{at^2}{4} \|u_0\|_{D^{1,2}}^4 + \frac{1}{2} \left( b \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_0^2 \, dx \right) - \frac{t}{3} \int_{\mathbb{R}^N} q(x) |u_0|^3 \, dx \right]
\]

where

\[
g(t_0) := \frac{at^2}{4} \|u_0\|_{D^{1,2}}^4 + \frac{1}{2} \left( b \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_0^2 \, dx \right) - \frac{t}{3} \int_{\mathbb{R}^N} q(x) |u_0|^3 \, dx.
\]

A direct calculation shows that \( \min_{t>0} g(t_0u_0) < 0 \) by (20). This indicates that there exists \( t_0 > 0 \) such that \( J_{\lambda,a}(t_0u_0) < 0 \). Letting \( e = t_0u_0 \in X_\lambda \). Then \( \|e\|_\lambda > \rho \), since \( \rho > 0 \) small enough. Hence, there exists \( e \in X_\lambda \) with \( \|e\|_\lambda > \rho \) such that \( J_{\lambda,a}(e) < 0 \) for all \( 0 < a < \frac{2}{\gamma\hat{\mu}_1(\lambda)} \) and \( \lambda > \frac{s^2}{c_0|V(x_0)|^{1/2}} \).

**Lemma 3.5** Suppose that conditions (V1) – (V2), (L1), (D1)', and (D2) hold. Let \( \rho > 0 \) be as Lemma 3.2. Then for each \( 0 < a < \frac{1}{\min\{1,b\}(\lambda)} \), there exists \( e \in X \) with \( \|e\|_\lambda > \rho \) such that \( J_{\lambda,a}(e) < 0 \) for all \( \lambda > 0 \).

**Proof.** It follows from (3) that for each \( 0 < a < \frac{1}{\min\{1,b\}(\lambda)} \), there exists \( \psi_k \in H^1(\mathbb{R}^N) \) with \( \psi_k > 0 \) such that

\[
\frac{1}{m^{(k)}} \leq \int_{\mathbb{R}^N} q(x)\psi_k \, dx < \frac{1}{am_\infty},
\]

which implies that

\[
am_\infty \|\psi_k\|_{D^{1,2}}^k - \int_{\mathbb{R}^N} q(x)\psi_k \, dx < \frac{1}{m^{(k)}} \|\psi_k\|_{D^{1,2}}^k - \int_{\mathbb{R}^N} q(x)\psi_k \, dx \leq 0.
\]

Using this, together with conditions (L1), (D1)', (D2) and Lebesgue’s dominated convergence theorem, yields

\[
\lim_{t \to +\infty} \frac{J_{\lambda,a}(t\psi_k)}{t^k} = \lim_{t \to +\infty} \frac{1}{2t^{k-2}} \left( b \|\psi_k\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)\psi_k^2 \, dx \right)
\]

\[
+ \lim_{t \to +\infty} \frac{a m(t^2 \|\psi_k\|_{D^{1,2}}^2)}{2tk} \|\psi_k\|_{D^{1,2}}^k \|\psi_k\|_{D^{1,2}}^k - \int_{\mathbb{R}^N} F(x,t\psi_k) \psi_k \, dx
\]

\[
\leq \frac{am_\infty}{k} \|\psi_k\|_{D^{1,2}}^k - \frac{1}{k} \int_{\mathbb{R}^N} q(x)\psi_k \, dx
\]

\[
= \frac{1}{k} \left( am_\infty \|\psi_k\|_{D^{1,2}}^k - \int_{\mathbb{R}^N} q(x)\psi_k \, dx \right) < 0.
\]
This implies that $J_{\lambda,a}(t\psi_k) \to -\infty$ as $t \to +\infty$. Hence, for each $0 < a < \frac{1}{m_{\infty}^{\lambda,a}}$, there exists $c \in X$ with $\|e\| > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $\lambda > 0$ and the lemma is proved. ■

If condition (L1) is not required, then we also have a conclusion similar to Lemma 3.5 but $a > 0$ must be small.

**Lemma 3.6** Suppose that conditions (V1) – (V2), (D1)$'$ and (D2) hold. Let $\rho > 0$ be as Lemma 3.4. Then there exists $\bar{\alpha}_* > 0$ and $e \in X$ with $\|e\| > \rho$ such that $J_{\lambda,a}(e) < 0$ for all $0 < a < \bar{\alpha}_*$ and $\lambda > 0$.

**Proof.** The proof is similar to that of Lemma 3.4, and we omit it here. ■

**Remark 3.2** Similar to Remark 3.1, the value of $\bar{\alpha}_*$ can also not be determined in general, but only in some special cases. Next, we give an example. For any real number $2 < k < 2^*$, we assume that $m(t) = mt^\delta$ with $\delta > \frac{k-2}{2}$ and

$$f(x,s) = \begin{cases} q(x) s^{k-1} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

where the function $q$ satisfies condition (D1)$'$. It is easily seen that $m(t)$ does not satisfy condition (L1). Let us consider the minimum problem:

$$\bar{\mu}_1^{(k)}(\lambda) := \inf_{u \in H^1(\mathbb{R}^N)} \left( \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} q(x) |u|^k \, dx} \right)^{\frac{2}{25-k+2}} \frac{b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx}{\int_{\mathbb{R}^N} q(x) |u|^k \, dx}.$$  

Then $\bar{\mu}_1^{(k)}(\lambda) \geq b \left( \frac{\bar{\alpha}_*}{\bar{\alpha}_1^{(k)}} \right)^{\frac{25}{25-k+2}} > 0$ for all $\lambda > 0$, where $\bar{\alpha}_1^{(k)}$ is as [4]. Indeed, by condition (D1)$'$ and the Caffarelli-Kohn-Nirenberg inequality one has

$$\int_{\mathbb{R}^N} q(x) |u|^k \, dx \leq \frac{c_*}{\bar{\alpha}_1^{(k)}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{k/2}.$$  

Using the above inequality leads to for all $\lambda > 0$,

$$\frac{\left( \frac{b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} q(x) |u|^k \, dx} \right)^{\frac{25}{25-k+2}} \frac{b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \lambda V(x) u^2 \, dx}{\int_{\mathbb{R}^N} q(x) |u|^k \, dx}}{\left( \frac{c_*}{\bar{\alpha}_1^{(k)}} \right)^{\frac{25}{25-k+2}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{k}{25-k+2}}} \geq b \left( \frac{\bar{\alpha}_1^{(k)}}{c_*} \right)^{\frac{25}{25-k+2}} > 0,$$  

13
which implies that \( \tilde{\mu}_1^{(k)}(\lambda) \geq b \left( \frac{\mu(\lambda)}{c_n} \right)^{\frac{2^*}{2^* - k + 2}} > 0 \). Thus, for every \( 0 < a < \left( \frac{(\delta + 1)(k-2)}{\delta m_0k} \right) \left( \frac{2\delta - k + 2}{k\delta \tilde{\mu}_1^{(k)}(\lambda)} \right)^{\frac{2^* - k + 2}{2^*}} \), there exists \( u_0 \in X_\lambda \) such that

\[
\left( \frac{2\delta - k + 2}{k\delta} \right) \left[ \frac{(\delta + 1)(k-2)}{\delta m_0k} \right] \left( \int_{\mathbb{R}^N} q(x) |u_0|^k \, dx \right)^{\frac{2^* - k + 2}{2^*}} > \left( \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx \right)^{\frac{2^* - k + 2}{2^*}} - \left( b \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx + \int_{\mathbb{R}^N} \lambda V(x) u_0^2 \, dx \right)^{\frac{2^* - k + 2}{2^*}}.
\]

Next, following the argument in Remark 3.1, we obtain that there exists \( e \in X_\lambda \) with \( \|e\|_\lambda > \rho \) such that \( J_{\lambda,a}(e) < 0 \) for all \( 0 < a < \left( \frac{(\delta + 1)(k-2)}{\delta m_0k} \right) \left( \frac{2\delta - k + 2}{k\delta \tilde{\mu}_1^{(k)}(\lambda)} \right)^{\frac{2^* - k + 2}{2^*}} \) and \( \lambda > 0 \).

### 4 Proofs of Theorems 1.1 and 1.2

Define

\[
\alpha_{\lambda,a} = \inf_{\gamma \in \Gamma_\lambda} \max_{0 \leq t \leq 1} J_{\lambda,a}(\gamma(t))
\]

and

\[
\alpha_{0,a}(\Omega) = \inf_{\gamma \in \Gamma_{\lambda}(\Omega)} \max_{0 \leq t \leq 1} J_{\lambda,a}|_{H_0^1(\Omega)}(\gamma(t)),
\]

where

\[
\Gamma_\lambda = \{ \gamma \in C([0,1], X_\lambda) : \gamma(0) = 0, \gamma(1) = e \}
\]

and

\[
\Gamma_{\lambda}(\Omega) = \{ \gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e \}.
\]

Note that \( \alpha_{0,a}(\Omega) \) independent of \( \lambda \). Following the argument in [18], we can take a number \( D_a > 0 \) such that \( 0 < \eta \leq \alpha_{\lambda,a} < \alpha_{0,a}(\Omega) < D_a \) for all \( \lambda \geq \Lambda_N \). Thus, by Lemmas 3.1 and 3.3 (or Lemma 3.4) and the mountain pass theorem [6], we obtain that for each \( \lambda \geq \Lambda_N \) and \( a > 0 \) (or \( 0 < a < \bar{a}_a \), there exists a sequence \( \{ u_n \} \subset X_\lambda \) such that

\[
J_{\lambda,a}(u_n) \to \alpha_{\lambda,a} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda) \|J_{\lambda,a}'(u_n)\|_{X_\lambda^{-1}} \to 0, \quad \text{as} \ n \to \infty, \tag{21}
\]

where \( 0 < \eta \leq \alpha_{\lambda,a} \leq \alpha_{0,a}(\Omega) < D_a \). Furthermore, we have the following results.

**Lemma 4.1** Suppose that conditions (V1) – (V2), (L1) and (D2) hold. Then for each \( a > 0 \) and \( \lambda \geq \Lambda_N \), the sequence \( \{ u_n \} \) defined in (21) is bounded in \( X_\lambda \).

**Proof.** By condition (D2), for \( s > 0 \) one has

\[
F(x,s) - \frac{1}{k} f(x,s) s = \int_0^s \frac{f(x,t)}{t^{k-1}} \, dt - \int_0^s \frac{f(x,s)}{s^{k-1}} \, dt \leq 0.
\]

\[
\int_0^s \left( \frac{f(x,t)}{t^{k-1}} - \frac{f(x,s)}{s^{k-1}} \right) t^{k-1} \, dt \leq 0. \tag{22}
\]
For $n$ large enough, it follows from condition (L1) and (21) – (22) that

$$\alpha_{\lambda,a} + 1 \geq J_{\lambda,a}(u_n) - \frac{1}{k} \langle J'_{\lambda,a}(u_n), u_n \rangle$$

$$= \frac{k - 2}{2k} \left( b \int_{\mathbb{R}^N} \|\nabla u_n\|^2 + \lambda V(x)u_n^2 dx \right)$$

$$+ \frac{a}{2} \left[ \hat{m} \left( \|u_n\|_{D^{1,2}}^2 \right) - \frac{2}{k} \left[ \lambda \|u_n\|_{D^{1,2}}^2 \right] \right]$$

$$- \int_{\mathbb{R}^N} \left[ F'(x,u_n) - \frac{1}{k} f(x,u_n) \right] dx$$

$$\geq \frac{(k - 2) \min \{ b, 1 \}}{2k} \|u_n\|_{\lambda}^2,$$

which implies that $\{u_n\}$ is bounded in $X_{\lambda}$ for each $a > 0$ and $\lambda \geq \Lambda_N$. \[ \blacksquare \]

**Lemma 4.2** Suppose that $N \geq 3$, conditions (V1) – (V2), (L3) with $\delta \geq \frac{2}{N - 2}$ and (D1) – (D2) hold. Then for all $0 < a < \bar{a}$ and

$$\lambda > \bar{\lambda}_0 := \begin{cases} \frac{|q|_{\infty}}{c_0} \max \left\{ \left( \frac{|q|_{\infty}}{am_0 S^2} \right)^{\frac{k - 2}{2 - k}}, \frac{2 - k}{2 - 2} \right\} & \text{if } \delta = \frac{2}{N - 2}, \\
\frac{|q|_{\infty}(2 - k)}{c_0(2 - 2)} & \text{if } \delta > \frac{2}{N - 2}, \end{cases}$$

the sequence $\{u_n\}$ defined in (21) is bounded in $X_{\lambda}$.

**Proof.** (i) $\delta = \frac{2}{N - 2}$ : Note that $2(\delta + 1) = 2^*$. Suppose on the contrary. Then $\|u_n\|_{\lambda} \to +\infty$ as $n \to \infty$. The proof is divided into three separate cases:

Case A: $\|u_n\|_{D^{1,2}} \to \infty$ and

$$\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx \geq \lambda c_0 S^{-2^*} \left( \frac{|q|_{\infty}}{\lambda c_0} \right)^{\frac{2^* - 2}{4 - 2}}. \tag{23}$$

By (21), we have

$$\frac{\langle J'_{\lambda,a}(u_n), u_n \rangle}{\|u_n\|_{D^{1,2}}^{2(\delta + 1)}} = o(1),$$

where $o(1)$ denotes a quantity which goes to zero as $n \to \infty$. Using this, together with condition (L3) and (15), gives

$$o(1) = \frac{am \left( \|u_n\|_{D^{1,2}}^2 \right) \|u_n\|_{D^{1,2}}^2 + b \|u_n\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx - \int_{\mathbb{R}^N} f(x,u_n)u_n dx}{\|u_n\|_{D^{1,2}}^{2(\delta + 1)}}$$

$$\geq am_0 + \frac{b}{\|u_n\|_{D^{1,2}}^{2\delta}} + \int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx - |q|_{\infty} \int_{\mathbb{R}^N} |u_n|^k dx. \tag{24}$$
Note that
\[
\int_{\mathbb{R}^N} |u_n|^k \, dx \leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx + |\{V < c_0\}|^{2/N} S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{\frac{2-\frac{k}{2}}{2-\frac{k}{2}}} \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{-\frac{k-2}{2-\frac{k}{2}}}
\]
\[
\leq S^{-\frac{2(k-2)}{2-\frac{k}{2}}} (\lambda c_0) \frac{2-k}{2-\frac{k}{2}} \left( \int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx \right)^{\frac{2-k}{2}} \|u_n\|_{D^{1,2}}^{\frac{2(k-2)}{2-\frac{k}{2}}}
+ |\{V < c_0\}|^{(2-k)/2} S^{-k} \|u_n\|_{D^{1,2}}^k.
\]
Then there holds
\[
\frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k \, dx}{\|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \geq \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx}{\|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \cdot \left[ 1 - |q|_\infty (\lambda c_0)^{-\frac{2-k}{2-\frac{k}{2}}} \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{-\frac{k-2}{2-\frac{k}{2}}} \right] - \frac{\|q\|_\infty |\{V < c_0\}|^{(2-k)/2}}{S^k \|u_n\|_{D^{1,2}}^{2(\delta+1) - k}}.
\]
(25)

It follows from (23)-(25) that
\[
o(1) \geq a_m + \frac{b}{\|u_n\|_{D^{1,2}}^{2k}} - \frac{|q|_\infty |\{V < c_0\}|^{(2-k)/2}}{S^k \|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \cdot \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx}{\|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \cdot \left[ 1 - |q|_\infty (\lambda c_0)^{-\frac{2-k}{2-\frac{k}{2}}} \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{-\frac{k-2}{2-\frac{k}{2}}} \right]
+ \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx}{\|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \cdot \left[ 1 - |q|_\infty (\lambda c_0)^{-\frac{2-k}{2-\frac{k}{2}}} \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{-\frac{k-2}{2-\frac{k}{2}}} \right] + o(1)
= a_m + \frac{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx}{\|u_n\|_{D^{1,2}}^{2(\delta+1) - k}} \cdot \left[ 1 - |q|_\infty (\lambda c_0)^{-\frac{2-k}{2-\frac{k}{2}}} \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{-\frac{k-2}{2-\frac{k}{2}}} \right] + o(1)
\geq a_m + o(1).
\]

This is a contradiction.

Case $B$: $\|u_n\|_{D^{1,2}} \to \infty$ and
\[
\int_{\mathbb{R}^N} \lambda V(x) u_n^2 \, dx - |q|_\infty (\lambda c_0)^{-\frac{2-k}{2-\frac{k}{2}}} \left( S^{-2} \lambda c_0 \right)^{-\frac{k-2}{2-\frac{k}{2}}} \|u_n\|_{D^{1,2}}^2 \]
(26)
Applying (9) and the Young inequality gives
\[
\int_{\mathbb{R}^N} |u_n|^k dx
\]
\[
\frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx + |\{V < c_0\}|^{2/N} S^{-2} \|u_n\|_{D^{1,2}}^2
\]
\[
\left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{\frac{k}{k-2}}
\]
\[
\leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx + \|\{V < c_0\}\|^{2/N} S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{\frac{k}{k-2}}
\]
\[
\leq \left( S^{-2} \|u_n\|_{D^{1,2}}^2 \right)^{\frac{k}{k-2}}
\]
\[
\leq \frac{|q|_\infty}{\lambda c_0} \left( \frac{q}{\lambda c_0} \right)^{\frac{2^*}{k-2}} + o(1).
\]

By (24) and (27) one has
\[
o (1) = \frac{\langle J_{\lambda,a}^u(u_n), u_n \rangle}{\|u_n\|_{D^{1,2}}^{2(\delta+1)}} \geq am_0 + \frac{b}{\|u_n\|_{D^{1,2}}^{2\delta}} + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx
\]
\[
\|u_n\|_{D^{1,2}}^{2(\delta+1)} - |q|_\infty S^{-2} \left( \frac{|q|_\infty}{\lambda c_0} \right)^{\frac{2^*}{k-2}} + o(1)
\]
\[
\geq am_0 - |q|_\infty S^{-2} \left( \frac{|q|_\infty}{am_0 S^{2^*}} \right)^{\frac{k}{k-2}} + o(1)
\]
which contradicts with
\[
\lambda > |q|_\infty \left( \frac{|q|_\infty}{am_0 S^{2^*}} \right)^{\frac{k}{2^*-k}}.
\]

Case C: \( \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx \to \infty \) and \( \|u_n\|_{D^{1,2}} \leq C_* \) for some \( C_* > 0 \) and for all \( n \). From (9), (13) and condition (L3) it follows that
\[
o (1) = \frac{am_0 \|u\|_{D^{1,2}}^2 \|u_n\|_{D^{1,2}}^2}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} + \frac{b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} - \int_{\mathbb{R}^N} f(x, u_n) u_n dx
\]
\[
\geq \frac{b \|u_n\|_{D^{1,2}}^2}{\int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} + 1 - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx.
\]

Applying (9) and the Young inequality gives
\[
\int_{\mathbb{R}^N} |u_n|^k dx \leq \frac{2^*-k}{2^*-2} \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx + |\{V < c_0\}|^{2/N} S^{-2} \|u_n\|_{D^{1,2}}^2 \right)
\]
\[
+ \frac{k-2}{2^*-2} S^{-2^*} \|u_n\|_{D^{1,2}}^{2^*}
\]
By (29) and the fact of \(\|u_n\|_{D^{1.2}} \leq C_n\) for all \(n\), we obtain that
\[
\frac{\int_{\mathbb{R}^N} |u_n|^k dx}{\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx} \leq \frac{(2^* - k)}{(2^* - 2)\lambda c_0} + \frac{(2^* - k)\|V < c_0\|^{2/2}}{(2^* - 2)\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx}.
\]
Using (28) and (30) yields
\[
\frac{2^* - k}{(2^* - 2)\lambda c_0} + o(1) = \frac{(2^* - k)}{(2^* - 2)\lambda c_0}.
\]
which contradicts with
\[
\lambda > \frac{(2^* - k)|q|_\infty}{(2^* - 2)c_0}.
\]

(ii) \(\delta > \frac{2}{N-2}\) : Clearly, \(2(\delta + 1) > 2^*\). Suppose on the contrary. Then \(\|u_n\|_\lambda \to +\infty\) as \(n \to \infty\). We consider the proof in two separate cases:

Case D : \(\|u_n\|_{D^{1.2}} \to \infty\). It follows from (14)–(15) and condition (L3) that
\[
o(1) = \frac{am}{\|u_n\|^{2^{(\delta+1)}-2^*}} + \frac{b}{\|u_n\|^{2^{*}-2}} + \frac{\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{\|u_n\|^{2^{*}-2}}.
\]
By (29), we deduce that
\[
\frac{\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{\|u_n\|^{2^{*}-2}} \geq \left(1 - \frac{|q|_\infty (2^* - k)}{(2^* - 2)\|u_n\|^{2^{*}-2}}\right) \frac{\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx}{\|u_n\|^{2^{*}-2}} - |q|_\infty S^{-2^*} \left(\frac{k - 2}{2^* - 2}\right).
\]
Using this, together with (31), leads to
\[
o(1) \geq am_0\|u_n\|^{2^{(\delta+1)}-2^*} + \frac{b}{\|u_n\|^{2^{*}-2}} + \frac{\int_{\mathbb{R}^N} \nabla V(x) u_n^2 dx - |q|_\infty \int_{\mathbb{R}^N} |u_n|^k dx}{\|u_n\|^{2^{*}-2}} \geq am_0\|u_n\|^{2^{(\delta+1)}-2^*} - |q|_\infty S^{-2^*} \left(\frac{k - 2}{2^* - 2}\right)
\]
\[
\to \infty \text{ as } n \to \infty,
\]

since $2(\delta + 1) > 2^*$. This is a contradiction.

Case $E: \int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx \to \infty$ and $\|u_n\|_{D^{1,2}} \leq C_*$ for some $C_* > 0$ and for all $n$. It follows from (14) – (15) and condition (L3) that

$$o(1) = \frac{am \left( \|u_n\|_{D^{1,2}} \right) \|u_n\|_{D^{1,2}}^2}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx} + \frac{b \|u_n\|_{D^{1,2}} + \int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx} - \frac{\int_{\mathbb{R}^N} f(x, u_n)u_n dx}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx} \geq \frac{b \|u_n\|_{D^{1,2}}^2}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx} + 1 - \frac{|q|_{\infty}}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx},$$

(32)

By (30) and (32) one has

$$o(1) \geq \frac{b \|u_n\|_{D^{1,2}}}{\int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx} + 1 - \frac{|q|_{\infty}}{\lambda c_0 \lambda \|u_n\|_{D^{1,2}}^2} \geq 1 - \frac{|q|_{\infty} \lambda (2^* - k)}{\lambda c_0 (2^* - 2)} + o(1),$$

which contradicts with

$$\lambda > \frac{|q|_{\infty} \lambda (2^* - k)}{\lambda c_0 (2^* - 2)}.$$

In conclusion, the sequence $\{u_n\}$ is bounded in $X_\lambda$ for all $0 < a < \bar{a}_*$ and $\lambda > \bar{\lambda}_0$. This completes the proof. \[\blacksquare\]

We now investigate the following two compactness results for the functional $J_{\lambda,a}$ under conditions $(D1) - (D2)$.

**Proposition 4.3** Suppose that $N \geq 1$, conditions $(V1) - (V2), (L1)$ and $(D1) - (D2)$ hold. Then for each $D > 0$ there exists $\Lambda = \Lambda(a, D) \geq \Lambda_N > 0$ such that $J_{\lambda,a}$ satisfies the $(C)_\alpha$-condition in $X_\lambda$ for all $\alpha < D$ and $\lambda > \bar{\lambda}$.

**Proof.** Let $\{u_n\}$ be a $(C)_\alpha$-sequence with $\alpha < D$. By Lemma 4.1, we have $\{u_n\}$ is bounded in $X_\lambda$. Then there exists a subsequence $\{u_n\}$ and $u_0 \in X_\lambda$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } X_\lambda \quad \text{and} \quad u_n \to u_0 \quad \text{strongly in } L^r_{loc}(\mathbb{R}^N) \text{ for } 2 \leq r < 2^*.$$

Next, we prove that $u_n \to u_0$ strongly in $X_\lambda$. Let $v_n = u_n - u_0$. Using condition $(V1)$ gives

$$\int_{\mathbb{R}^N} v_n^2 dx \leq \frac{1}{\lambda c_0} \|v_n\|_{\lambda}^2 + o(1), \quad (33)$$

Using this, together with the Hölder and Sobolev inequalities, for any $\lambda > \Lambda_N$, we check the following estimation:

Case (i) $N = 1, 2$:

$$\int_{\mathbb{R}^N} |v_n|^r dx \leq \left( \int_{\mathbb{R}^N} v_n^2 dx \right)^{1/2} \cdot \left( \int_{\mathbb{R}^N} v_n^{2(r-1)} dx \right)^{1/2} \leq \left[ \frac{1}{c_0} \left( 1 + \Lambda_N^{8/\lambda} |\{V < c_0\}|^{2/\lambda} \right)^{-1/2} \right]^{1/2} S_{2(r-1)}^{|v_n|_{\lambda}^2 + o(1)}.$$
Moreover, it follows from the boundedness of the sequence \( \{v_n\} \) that there exists a constant \( A > 0 \) such that
\[
\sup_{n} \left\{ \int_{\mathbb{R}^N} |v_n|^r \, dx \right\} \leq \frac{1}{\lambda c_0} \left[ \left( \int_{\mathbb{R}^N} |v_n|^2 \, dx \right)^{\frac{r}{2}} \right]^{\frac{2}{r}} S^{-\frac{2(r-2)}{2}} \|v_n\|_\lambda^r + o(1).
\]

Set
\[
\Pi_{\lambda,r} = \left\{ \left[ \frac{1}{\lambda c_0} \left( 1 + A^{8/N} \|\{V < c_0\}\|^2 \right)^{\frac{r-1}{2}} \right]^{1/2} S^{1-r} \right\} \quad \text{if } N = 1, 2,
\]
\[
\left( \frac{1}{\lambda c_0} \right)^{\frac{2}{r}} S^{-\frac{2(r-2)}{2}} \quad \text{if } N \geq 3.
\]

Case (ii) \( N \geq 3 \):
\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \left( \int_{\mathbb{R}^N} |v_n|^2 \, dx \right)^{\frac{r}{2}} \left( \int_{\mathbb{R}^N} |v_n|^2 \, dx \right)^{\frac{r-2}{2}} \leq \left( \frac{1}{\lambda c_0} \right)^{\frac{r}{2}} S^{-\frac{2(r-2)}{2}} \|v_n\|_\lambda^r + o(1).
\]

Clearly, \( \Pi_{\lambda,r} \to 0 \) as \( \lambda \to \infty \). Then we have
\[
\int_{\mathbb{R}^N} |v_n|^r \, dx \leq \Pi_{\lambda,r} \|v_n\|_\lambda^r + o(1). \tag{34}
\]

Following the argument of [13], it is easy to verify that
\[
\int_{\mathbb{R}^N} F(x, v_n) \, dx = \int_{\mathbb{R}^N} F(x, u_n) \, dx - \int_{\mathbb{R}^N} F(x, u_0) \, dx + o(1) \tag{35}
\]
and
\[
\sup_{\|u\| = 1} \int_{\mathbb{R}^N} [f(x, v_n) - f(x, u_n) + f(x, u_0)] h(x) \, dx = o(1).
\]

Thus, using (33), (35) and Brezis-Lieb Lemma [3], we deduce that
\[
J_{a,\lambda} (u_n) - J_{a,\lambda} (u_0) = \frac{a}{2} [\tilde{m} (\|u_n\|^2_{D^{1,2}}) - \tilde{m} (\|u_0\|^2_{D^{1,2}})] + \frac{b}{2} \|v_n\|^2_{D^{1,2}}
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x) v_n^2 \, dx - \int_{\mathbb{R}^N} F(x, v_n) \, dx + o(1).
\]

Moreover, it follows from the boundedness of the sequence \( \{u_n\} \) in \( X_\lambda \) that there exists a constant \( A > 0 \) such that \( \|u_n\|^2_{D^{1,2}} \to A \) as \( n \to \infty \), which indicates that for any \( \varphi \in C_0^\infty (\mathbb{R}^N) \), there holds
\[
o(1) = \langle J'_a, \varphi \rangle
\]
\[
= \int_{\mathbb{R}^N} \lambda V(x) u_n \varphi \, dx + (am (\|u_n\|^2_{D^{1,2}}) + b) \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi \, dx - \int_{\mathbb{R}^N} f(x, u_n) \varphi \, dx
\]
\[
\to \int_{\mathbb{R}^N} \lambda V(x) u_0 \varphi \, dx + (am (A) + b) \int_{\mathbb{R}^N} \nabla u_0 \nabla \varphi \, dx - \int_{\mathbb{R}^N} f(x, u_0) \varphi \, dx
\]
as \( n \to \infty \). This implies that
\[
\int_{\mathbb{R}^N} \lambda V(x) u_0^2 \, dx + (am (A) + b) \|u_0\|^2_{D^{1,2}} - \int_{\mathbb{R}^N} f(x, u_0) u_0 \, dx = 0. \tag{36}
\]
Note that
\[ o(1) = \int_{\mathbb{R}^N} \lambda V(x)u_n^2 dx + (am (\|u_n\|_{D^{1,2}}^2) + b) \|u_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, u_n) u_n dx. \]

Combining the above two equalities gives
\[
o(1) &= \int_{\mathbb{R}^N} \lambda V(x)v_n^2 dx + am (\|u_n\|_{D^{1,2}}^2) \|u_n\|_{D^{1,2}}^2 - am (A) \|u_0\|_{D^{1,2}}^2 \\
&\quad + b \|v_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \\
&= \int_{\mathbb{R}^N} \lambda V(x)v_n^2 dx + am (\|u_n\|_{D^{1,2}}^2) \|u_n\|_{D^{1,2}}^2 \\
&\quad - am (\|u_n\|_{D^{1,2}}^2) \|u_0\|_{D^{1,2}}^2 + b \|v_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \\
&= \int_{\mathbb{R}^N} \lambda V(x)v_n^2 dx + (am (\|u_n\|_{D^{1,2}}^2) + b) \|v_n\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx. \tag{37}
\]

In addition, it follows from (36), conditions (L1) and (D3) that
\[
J_{\lambda,a}(u_0) = \frac{a}{2} \tilde{m} (\|u_0\|_{D^{1,2}}^2) + \frac{1}{2} \left( b \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u_0^2 dx \right) - \int_{\mathbb{R}^N} F(x, u_0) dx \\
- \frac{1}{k} \left( \int_{\mathbb{R}^N} \lambda V(x)u_0^2 dx + (am (A) + b) \|u_0\|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, u_0) u_0 dx \right) \\
= \frac{a}{2} \left[ \tilde{m} (\|u_0\|_{D^{1,2}}^2) - \frac{2}{k} m (A) \|u_0\|_{D^{1,2}}^2 \right] \\
+ \frac{(k-2) \min\{b, 1\}}{2k} \|u_0\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \left[ \frac{1}{k} f(x, u_0) u_0 - F(x, u_0) \right] dx \\
\geq \frac{a}{2} \left[ \tilde{m} (\|u_0\|_{D^{1,2}}^2) - \frac{2}{k} m (A) \|u_0\|_{D^{1,2}}^2 \right] \\
\geq \frac{a}{2} \left[ \tilde{m} (\|u_0\|_{D^{1,2}}^2) - \frac{2}{k} m (\|u_0\|_{D^{1,2}}^2) \|u_0\|_{D^{1,2}}^2 \right] \\
+ \frac{a}{k} \|u_0\|_{D^{1,2}}^2 \left[ m (\|u_0\|_{D^{1,2}}^2) - m (A) \right] \\
\geq \frac{a}{k} \|u_0\|_{D^{1,2}}^2 \left[ m (\|u_0\|_{D^{1,2}}^2) - m (A) \right].
\]

Then there exists a constant $K$ satisfying $K = 0$ if $m (\|u_0\|_{D^{1,2}}^2) \geq m (A)$ or $K < 0$ if $m (\|u_0\|_{D^{1,2}}^2) < m (A)$ such that
\[ J_{\lambda,a}(u_0) \geq K. \]
Using this, together with (33), (37), conditions (L1) and (D3) leads to
\[
D - K \geq \alpha - J_{a,\lambda}(u_0) = J_{a,\lambda}(u_n) - J_{a,\lambda}(u_0) + o(1)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx + \frac{a}{2} \left[ \hat{m} \left( \| u_n \|_{D^{1,2}}^2 \right) - \hat{m} \left( \| u_0 \|_{D^{1,2}}^2 \right) \right] + \frac{b}{2} \| v_n \|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} F(x, v_n) dx + o(1)
\]
\[
= \frac{k - 2}{2k} \left( b \| v_n \|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx \right) - \int_{\mathbb{R}^N} \left( F(x, v_n) - \frac{1}{k} f(x, v_n) v_n \right) dx
\]
\[
= \frac{(k - 2) \min \{1, b\}}{2k} \| v_n \|_{\Lambda}^2 + o(1)
\]
which implies that there exists a constant \( \tilde{D} = \tilde{D}(D) > 0 \) such that
\[
\| v_n \|_{\Lambda}^2 \leq \frac{2k \tilde{D}}{(k - 2) \min \{1, b\}} + o(1).
\]  (38)

It follows from the (15), (34), (37) and (38) that
\[
o(1) = \int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx + a m \left( \| u_n \|_{D^{1,2}}^2 \right) \| v_n \|_{D^{1,2}}^2 + b \| v_n \|_{D^{1,2}}^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx
\]
\[
\geq \min \{1, b\} \| v_n \|_{\Lambda}^2 - |q|_{\infty} \int_{\mathbb{R}^N} |v_n|^k dx
\]
\[
\geq \min \{1, b\} \| v_n \|_{\Lambda}^2 - |q|_{\infty} \Pi_{\lambda, k} \| v_n \|_{\Lambda}^k
\]
\[
\geq \min \{1, b\} \| v_n \|_{\Lambda}^2 - |q|_{\infty} \Pi_{\lambda, k} \left[ \frac{2k \tilde{D}}{(k - 2) \min \{1, b\}} \right]^{k/2} + o(1),
\]
which implies that there exists \( \tilde{\Lambda} := \tilde{\Lambda}(a, D) \geq \Lambda_N \) such that for \( \lambda > \tilde{\Lambda} \),
\[
v_n \to 0 \text{ strongly in } X_\lambda.
\]

This completes the proof. \( \blacksquare \)

**Proposition 4.4** Suppose that \( N \geq 3 \), conditions (V1) – (V2), (L2), (L3) with \( \delta \geq \frac{2}{N - 2} \) and (D1) – (D2) hold. Then for each \( D > 0 \) there exists \( \tilde{\Lambda}_1 = \tilde{\Lambda}_1(a, D) \geq \Lambda_N > 0 \) such that \( J_{\lambda, a} \) satisfies the \( (C)_{\alpha} \)-condition in \( X_\lambda \) for all \( \alpha < D \) and \( \lambda > \tilde{\Lambda}_1 \).

**Proof.** The proof is similar to that of Proposition 4.3 in which only some places are adjusted. Now we briefly verify it. By Lemma 4.7 there exists a constant \( K < 0 \) such that
\[
J_{\lambda, a}(u_0) \geq K.
\]  (39)
It follows from (33), (37), (39), conditions (L2) and (D2) that

\[
D - K \geq \alpha - J_{a,j} (u_0) = J_{a,j} (u_n) - J_{a,j} (u_0) + O(1)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx + \frac{a}{2} \left[ \tilde{m} \left( \|u_n\|_{D_1}^2 \right) - \tilde{m} \left( \|u_0\|_{D_1}^2 \right) \right] + \frac{b}{2} \|v_n\|_{D_1}^2 - \int_{\mathbb{R}^N} F(x, v_n) dx + O(1)
\]

\[
= \frac{k - 2}{2k} \left( b \|v_n\|_{D_1}^2 + \int_{\mathbb{R}^N} \lambda V(x) v_n^2 dx \right) - \int_{\mathbb{R}^N} \left( F(x, v_n) - \frac{1}{k} f(x, v_n) v_n \right) dx
\]

\[
+ \frac{a}{2} \left[ \tilde{m} \left( \|u_n\|_{D_1}^2 \right) - \tilde{m} \left( \|u_0\|_{D_1}^2 \right) - \frac{2}{k} m \left( \|u_n\|_{D_1}^2 \right) \|v_n\|_{D_1}^2 \right] + O(1)
\]

\[
\geq \frac{(k - 2) \min \{1, b\}}{2k} \|v_n\|_{\Lambda}^2
\]

\[
- \frac{a}{2} \tilde{m} \left( \|u_0\|_{D_1}^2 \right) - \frac{a}{k} m \left( \|u_0\|_{D_1}^2 \right) \|u_0\|_{D_1}^2 + O(1)
\]

\[
\geq \frac{(k - 2) \min \{1, b\}}{2k} \|v_n\|_{\Lambda}^2
\]

\[
- \frac{a}{2} m \left( \|u_0\|_{D_1}^2 \right) \|u_0\|_{D_1}^2 - \frac{a}{k} m (A^2) A^2 + O(1),
\]

which implies that for each \( \lambda > \Lambda_N \) there exists a constant \( \overline{D} = \overline{D}(D) > 0 \) such that

\[
\|v_n\|_{\Lambda}^2 \leq \frac{2k \overline{D}}{(k - 2) \min \{1, b\}} + O(1).
\]

Next, following the argument of Proposition 4.3 we easily arrive at the conclusion. This completes the proof. ■

**Theorem 4.5** Suppose that \( N \geq 1 \), conditions (V1) – (V2), (L1) and (D1) – (D2) hold. Then for every \( a > 0 \) and \( \lambda > \Lambda \), the energy functional \( J_{a,j} \) has a nontrivial critical point \( u_\lambda \in X_\lambda \) such that \( J_{a,j}(u_\lambda) > 0 \).

**Proof.** By Proposition 4.3 and \( 0 < \eta \leq \alpha_{\lambda,a} \leq \alpha_{0,a}(\Omega) \) for all \( \lambda \geq \Lambda_N \), \( J_{a,j} \) satisfies the \((C)_{\alpha_{\lambda,a}}\)-condition in \( X_\lambda \) for each \( a > 0 \) and \( \lambda > \Lambda \). That is, there exist a subsequence \( \{u_n\} \) and \( u_\lambda \in X_\lambda \) such that \( u_n \rightharpoonup u_\lambda \) strongly in \( X_\lambda \). This implies that \( u_\lambda \) is a nontrivial critical point of \( J_{\lambda,a} \) satisfying \( J_{\lambda,a}(u_\lambda) = \alpha_{\lambda,a} > 0 \). ■

**Theorem 4.6** Suppose that \( N \geq 3 \), conditions (V1) – (V2), (L2), (L3) with \( \delta \geq \frac{2}{N-2} \) and (D1) – (D2) hold. Then there exists a constant \( \overline{\Lambda}_2 \geq \max \{\overline{\Lambda}_0, \overline{\Lambda}_1\} \) such that for every \( 0 < a < \overline{a} \) and \( \lambda > \overline{\Lambda}_2 \), the energy functional \( J_{a,j} \) has a nontrivial critical point \( u^+_{a,j} \in X_\lambda \) satisfying \( J_{a,j}(u^+_{a,j}) > 0 \).

**Proof.** Similar to the argument of Theorem 4.5 by Proposition 4.4 we easily arrive at the conclusion. ■
Lemma 4.7 Suppose that $N \geq 3$, conditions (V1) – (V2), (L3) with $\delta \geq \frac{2}{N-2}$ and (D1) – (D2) hold. Then the energy functional $J_{a,\lambda}$ is bounded below on $X_\lambda$ for all $a > 0$ and $
abla$

$$\lambda > \bar{\Lambda}_3 := \begin{cases} \max \left\{ \Lambda_N, \frac{2|q|_{\infty}}{c_0 k} \left( \frac{2(\delta+1)|q|_{\infty}}{m_0 k S^2} \right)^{\frac{N-2}{2}} \right\} & \text{if } \delta = \frac{2}{N-2}, \\ \Lambda_N & \text{if } \delta > \frac{2}{N-2}. \end{cases}$$

Furthermore, if

$$\lambda > \bar{\Lambda}_4 := \max \left\{ \bar{\Lambda}_3, \frac{2|q|_{\infty} (2^* - k)}{c_0 k(2^* - 2)} \right\},$$

then there exists $\tilde{R}_a > T_0^{1/2}$ such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{\lambda} \geq \tilde{R}_a.$$ 

Proof. If $\|u\|_{D^{1,2}} < T_0^{1/2}$, then by (9), (13) and the Young inequality one has

$$J_{a,\lambda}(u) \geq \frac{\min\{b, 1\}}{2} \|u\|_\lambda^2 - \frac{|q|_\infty}{k} \int_{\mathbb{R}^N} |u|^k dx$$

$$\geq \frac{\min\{b, 1\}}{2} \|u\|_\lambda^2 - \frac{|q|_\infty}{k S^k} \|\{V < c_0\}\|_\lambda^{2(2^*-k)/2} \|u\|_\lambda^{2(2^*-k)} \|u\|_{D^{1,2}}^{2(2^*-k)}$$

$$\geq \frac{2^*(k-2) \min\{b, 1\}}{2p(2^*-2)} \|u\|_\lambda^2$$

$$- \frac{k-2}{k(2^*-2) (\min\{b, 1\})^{2^*-2}} \left( |q|_\infty S^{-k} \|\{V < c_0\}\|_\lambda^{(2^*-k)/2} \right) \|u\|_{D^{1,2}}^{2^*-2}$$

$$\geq \frac{2^*(k-2) \min\{b, 1\}}{2p(2^*-2)} \|u\|_\lambda^2$$

$$- \frac{k-2}{k(2^*-2) (\min\{b, 1\})^{2^*-2}} \left( |q|_\infty S^{-k} \|\{V < c\}\|_\lambda^{(2^*-k)/2} \right) \frac{2^*-2}{2^*-4} T_0^{2^*-2},$$

which shows that $J_{a,\lambda}$ is bounded below on $X_\lambda$ for all $a > 0$ and $\lambda > \Lambda_N$. If $\|u\|_{D^{1,2}} \geq T_0^{1/2}$, then we consider two cases as follows:

(i) $\delta = \frac{2}{N-2}$ and $\|u\|_{D^{1,2}} \geq T_0^{1/2}$: The argument is also divided into two separate cases:

Case A: $\int_{\mathbb{R}^N} \lambda V(x) u^2 dx \geq \lambda c_0 S^{-2^*} \left( \frac{2|q|_{\infty}}{k S c_0} \right)^{\frac{2^*-2}{2^*-4}} \|u\|_{D^{1,2}}^{2^*-2}$. By condition (L3), (9), (13) and
for each $\lambda > 0$ one has

$$J_{a, \lambda}(u) \geq \frac{a}{2} \left( \left\| u \right\|_{D^{1,2}}^2 + \frac{1}{2} \left( b \left\| u \right\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right) \right) - \frac{|q|}{k} \int_{\mathbb{R}^N} |u|^k dx$$

$$\geq \frac{m_0 a}{2(\delta + 1)} \left\| u \right\|_{D^{1,2}}^{2(\delta + 1)} + \frac{1}{2} \left( b \left\| u \right\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right)$$

$$- \frac{|q|}{k} \lambda c_0 \int_{\mathbb{R}^N} \lambda V(x) u^2 dx + S^{-2} \left\{ \left\{ V < c_0 \right\}^{2/N} \left\| u \right\|_{D^{1,2}}^2 \right\}^{2s-k} \left( S^{-1} \left\| u \right\|_{D^{1,2}} \right)^{2s(k-2)2^s-2}$$

where we have used the Young and Sobolev inequalities. This implies that $J_{a, \lambda}(u)$ is bounded below on $X$ for all $a > 0$ and $\lambda > \Lambda_N$.

Case $B : \int_{\mathbb{R}^N} \lambda V(x) u^2 dx < \lambda c_0 S^{-2} \left( \frac{2|q|}{k \lambda c_0} \right)^{\frac{2s-k}{2^s-2}} \left\| u \right\|_{D^{1,2}}^{2s}$. It follows from (9) that

$$\int_{\mathbb{R}^N} |u|^k dx$$

$$\leq \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u^2 dx + S^{-2} \left\{ \left\{ V < c_0 \right\}^{2/N} \left\| u \right\|_{D^{1,2}}^2 \right\}^{2s-k} \left( S^{-1} \left\| u \right\|_{D^{1,2}} \right)^{2s(k-2)2^s-2} \right)^{\frac{2s-k}{2^s-2}}$$

$$\leq S^{-2s} \left( \frac{2|q|}{k \lambda c_0} \right)^{\frac{2s-k}{2^s-2}} \left\| u \right\|_{D^{1,2}}^{2s} + S^{-2} \left\{ \left\{ V < c_0 \right\}^{2/N} \left\| u \right\|_{D^{1,2}}^2 \right\}^{2s-k} \left( S^{-1} \left\| u \right\|_{D^{1,2}} \right)^{2s(k-2)2^s-2} \left\| u \right\|_{D^{1,2}}^{k}$$

$$= S^{-2s} \left( \frac{2|q|}{k \lambda c_0} \right)^{\frac{2s-k}{2^s-2}} \left\| u \right\|_{D^{1,2}}^{2s} + S^{-k} \left\{ \left\{ V < c \right\}^{2/N} \right\}^{\frac{2s-k}{2^s-2}} \left\| u \right\|_{D^{1,2}}^{k}$$
Using this, together with condition (L3) once again, yields

\[
J_{a,\lambda}(u) \geq \frac{a}{2} \hat{m} \left( \|u\|_{D^{1,2}}^2 \right) + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right) - \frac{|q|_\infty}{k} \int_{\mathbb{R}^N} |u|^k dx
\]

\[
\geq \frac{m_0 a}{2(\delta + 1)} \|u\|_{D^{1,2}}^{2(\delta + 1)} + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right)
\]

\[
- \frac{|q|_\infty}{k} \left[ S^{-2} \left( \frac{2|q|_\infty}{k \lambda c_0} \right)^{\frac{k}{k-2}} \|u\|_{D^{1,2}}^2 + S^{-k} |\{ V < c_0 \}| \|u\|_{D^{1,2}}^k \right]
\]

\[
\geq \left[ \frac{m_0 a}{2(\delta + 1)} - \frac{|q|_\infty}{k S^{2*}} \left( \frac{2|q|_\infty}{k \lambda c_0} \right)^{\frac{k}{k-2}} \right] \|u\|_{D^{1,2}}^2
\]

\[
- \frac{|q|_\infty}{k S^{2*}} |\{ V < c_0 \}| \|u\|_{D^{1,2}}^k .
\]

This shows that if

\[
\lambda > \max \left\{ \Lambda_N, \frac{2|q|_\infty}{k c_0} \left( \frac{2(\delta + 1)|q|_\infty}{m_0 a k S^{2*}} \right)^{\frac{k}{k-2}} \right\},
\]

then \( J_{a,\lambda} \) is bounded below on \( X_\lambda \) for all \( a > 0 \) and there exists \( R_a > 0 \) such that

\[
J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{D^{1,2}} \geq R_a.
\]

(ii) \( \delta > \frac{2}{N-2} \) and \( \|u\|_{D^{1,2}} \geq T_0^{1/2} \): It follows from condition (L3), (9), (15) and the Young inequality that

\[
J_{a,\lambda}(u) = \frac{a}{2} \hat{m} \left( \|u\|_{D^{1,2}}^2 \right) + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right) - \int_{\mathbb{R}^N} F(x, u) dx
\]

\[
\geq \frac{m_0 a}{2(\delta + 1)} \|u\|_{D^{1,2}}^{2(\delta + 1)} + \min \{b, 1\} \|u\|_{D^{1,2}}^2
\]

\[
- \frac{|q|_\infty}{k} S^{-k} |\{ V < c_0 \}| \|u\|_{D^{1,2}}^k \|
\]

\[
\geq \frac{m_0 a}{2(\delta + 1)} \|u\|_{D^{1,2}}^{2(\delta + 1)} + \min \{b, 1\} \frac{2^*(k-2)}{2p(2^*-2)} \|u\|_{D^{1,2}}^2
\]

\[
- \frac{k-2}{k(2^*-2) \min \{b, 1\}} \left( |q|_\infty S^{-k} |\{ V < c_0 \}| \|u\|_{D^{1,2}}^{2^*(k-2)} \right) \|u\|_{D^{1,2}}^{2^*}
\]

\[
\geq \frac{m_0 a}{2(\delta + 1)} \|u\|_{D^{1,2}}^{2(\delta + 1)}
\]

\[
- \frac{k-2}{k(2^*-2) \min \{b, 1\}} \left( |q|_\infty S^{-k} |\{ V < c_0 \}| \|u\|_{D^{1,2}}^{2^*(k-2)} \right) \|u\|_{D^{1,2}}^{2^*}.
\]
which implies that $J_{a,\lambda}(u)$ is bounded below on $X$ for all $a > 0$ and $\lambda > \Lambda_N$, since $\delta > \frac{2}{N-2}$. Moreover, for every $a > 0$, there exists

$$R_a > t := \left[ \frac{2(\delta + 1)(k - 2) \left( |q|_\infty S^{-k} \| \{V < c_0\} \|_{(2^* - k)/2^*} \right)^{\frac{2^* - 2}{k(2^* - 2)}}}{km_0a(2^* - 2) (\min\{b, 1\})^{\frac{2}{k - 2}}} \right]^{\frac{1}{2 - 2}}$$

such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{D^{1,2}} \geq R_a = \max\{T_0^{1/2}, R_a\}.$$

Next, we show that there exists a constant $\tilde{R}_a > R_a$ such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{\Lambda} \geq \tilde{R}_a.$$

Let

$$\tilde{R}_a = \left[ \frac{R_a^2 + 2A_0 (R_a)}{1 - \frac{1}{\lambda c_0} \left( \frac{2(2^* - k) |q|_\infty}{k(2^* - 2)} \right)} \right]^{1/2},$$

where

$$A_0 (R_a) = \frac{(2^* - k) |q|_\infty}{k(2^* - 2)} \| \{V < c_0\} \|_{2^*/N} S^{-2} \tilde{R}_a^2 + \frac{(k - 2) |q|_\infty}{k(2^* - 2)} S^{-2} \tilde{R}_a^{2^*}.$$

For $u \in X_\lambda$ with $\|u\|_{\Lambda} \geq \tilde{R}_a$, if $\|u\|_{D^{1,2}} \geq R_a$, then the result holds clearly. If $T_0^{1/2} \leq \|u\|_{D^{1,2}} < \tilde{R}_a$, then it is enough to indicate that $J_{a,\lambda}(u) \geq 0$ when

$$\int_{\mathbb{R}^N} \lambda V(x) u^2 dx \geq 2A_0 (R_a) \left( 1 - \frac{2(2^* - k) |q|_\infty}{\lambda c_0 k(2^* - 2)} \right)^{-1}. $$

Indeed, by (29) we deduce that

$$J_{a,\lambda}(u) \geq \frac{a}{2} \bar{m} \left( \|u\|_{D^{1,2}}^2 \right) + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u^2 dx \right) - \frac{|q|_\infty}{k} \int_{\mathbb{R}^N} |u|^k dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x) u^2 dx - \frac{(k - 2) |q|_\infty}{k(2^* - 2)} S^{-2} \tilde{R}_a^{2^*}$$

$$- \frac{(2^* - p) |q|_\infty}{k(2^* - 2)} \left( \frac{1}{\lambda c_0} \int_{\mathbb{R}^N} \lambda V(x) u^2 dx + |\{V < c_0\}|^{2/N} S^{-2} \tilde{R}_a^{2^*} \right)$$

$$\geq \frac{1}{2} \left[ 1 - \frac{2(2^* - k) |q|_\infty}{\lambda c_0 k(2^* - 2)} \right] \int_{\mathbb{R}^N} \lambda V(x) u^2 dx - A_0 (\tilde{R}_a)$$

$$\geq 0.$$

Hence, we obtain that there exists a constant $\tilde{R}_a > 0$ defined as (40) such that

$$J_{a,\lambda}(u) > 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{\Lambda} \geq \tilde{R}_a.$$

This completes the proof. ■
Lemma 4.8 Suppose that $N \geq 3$, conditions (V1) – (V2), (L3) with $\delta \geq \frac{2}{N-2}$ and (D1) – (D2) hold. Then for every $a > 0$ and $\lambda > \tilde{\Lambda}_4$ one has

$$\tilde{\theta}_a := \inf \left\{ J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \|u\|_\lambda < \tilde{R}_a \right\} < 0.$$ 

Proof. The proof directly follows from Lemmas 3.4 and 4.7. $\blacksquare$

Theorem 4.9 Suppose that $N \geq 3$, conditions (V1) – (V2), (L2), (L3) with $\delta \geq \frac{2}{N-2}$ and (D1) – (D2) hold. Then there exists a constant $\tilde{\Lambda}_5 \geq \max \left\{ \tilde{\Lambda}_1, \tilde{\Lambda}_4 \right\}$ such that for every $a > 0$ and $\lambda \geq \tilde{\Lambda}_5$, $J_{a,\lambda}$ has a nontrivial critical point $u_{a,\lambda}^- \in X_\lambda$ such that $J_{a,\lambda}(u_{a,\lambda}^-) = \tilde{\theta}_a < 0$.

Proof. By Lemma 4.8 and the Ekeland variational principle, there exists a bounded minimizing sequence $\{u_n\} \subset X_\lambda$ with $\|u_n\|_\lambda < \tilde{R}_a$ such that

$$J_{a,\lambda}(u_n) \to \tilde{\theta}_a \quad \text{and} \quad (1 + \|u_n\|_{X_\lambda}) \|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0 \quad \text{as} \ n \to \infty.$$ 

According to Proposition 4.4 there exist a subsequence $\{u_n\}$ and $u_{a,\lambda}^- \in X_\lambda$ such that $u_n \to u_{a,\lambda}^-$ strongly in $X_\lambda$. This indicates that $u_{a,\lambda}$ is a nontrivial critical point of $J_{a,\lambda}$ satisfying $J_{a,\lambda}(u_{a,\lambda}^-) = \tilde{\theta}_a < 0$. $\blacksquare$

We are now ready to prove Theorems 1.1 and 1.2. Theorems 1.1 directly follows from Theorem 4.5. By using Theorems 4.6 and 4.9 there exists a positive constant $\tilde{\Lambda}_* \geq \max \left\{ \tilde{\Lambda}_2, \tilde{\Lambda}_5 \right\}$ such that for every $0 < a < \tilde{a}_*$ and $\lambda \geq \tilde{\Lambda}_*$, Eq. (K$_{a,\lambda}$) admits two positive solutions $u_{a,\lambda}^-$ and $u_{a,\lambda}^+$ satisfying $J_{a,\lambda}(u_{a,\lambda}^-) < 0 < J_{a,\lambda}(u_{a,\lambda}^+)$. In particular, $u_{a,\lambda}^-$ is a ground state solution of Eq. (K$_{a,\lambda}$). Hence, we arrive at Theorem 1.2.

5 Proofs of Theorems 1.3 and 1.4

Following the argument at the begining of Section 4, by virtue of Lemmas 3.2, 3.3 (or Lemma 3.6) and the mountain pass theorem [6], we obtain that for each $\lambda \geq \Lambda_N$ and $0 < a < \frac{1}{m_{\infty}^2}$ (or $0 < a < \bar{a}_*$), there exists a sequence $\{u_n\} \subset X_\lambda$ such that

$$J_{\lambda,a}(u_n) \to \alpha_{\lambda,a} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda) \|J'_{\lambda,a}(u_n)\|_{X_\lambda^{-1}} \to 0, \quad \text{as} \ n \to \infty, \quad (41)$$ 

where $0 < \eta \leq \alpha_{\lambda,a} \leq \alpha_{0,a}(\Omega) < D_a$. Furthermore, we have the following result.

Lemma 5.1 Suppose that $N \geq 3$, conditions (V1) – (V2), (L3) with $\delta \geq \frac{2}{N-2}$, (D1) and (D2) hold. Then for each $0 < a < \frac{1}{m_{\infty}^2}$, the sequence $\{u_n\}$ defined in (41) is bounded in $X_\lambda$ for all $\lambda \geq \Lambda_N$. 

28
Proof. Suppose on the contrary. Then \( \|u_n\|_\lambda \to +\infty \) as \( n \to \infty \). We consider the proof in two separate cases:

Case \( A : \|u_n\|_{D_{1,2}} \to \infty \). It follows from (14), (17), condition (L3) and the Caffarelli-Kohn-Nirenberg inequality that

\[
o(1) = \frac{am (\|u_n\|_{D_{1,2}}^2 \|u_n\|_{D_{1,2}}^2 + b \|u_n\|_{D_{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx)}{\|u_n\|_{D_{1,2}}^k} - \frac{\int_{\mathbb{R}^N} f(x, u_n) u_n dx}{\|u_n\|_{D_{1,2}}^k} \geq am_0 \frac{\|u_n\|_{D_{1,2}}^{2(\delta + 1) - k}}{\|u_n\|_{D_{1,2}}^k} - \frac{c_* \|u_n\|_{D_{1,2}}^k}{\nu^{(k)}_1} \to \infty \quad \text{as} \quad n \to \infty,
\]

since \( 2(\delta + 1) > k \). This is a contradiction.

Case \( B : \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx \to \infty \) and \( \|u_n\|_{D_{1,2}} \leq C_* \) for some \( C_* > 0 \) and for all \( n \). It follows from (14), (17) and condition (L3) that

\[
o(1) = \frac{am (\|u_n\|_{D_{1,2}}^2 \|u_n\|_{D_{1,2}}^2 + b \|u_n\|_{D_{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx)}{\|u_n\|_{D_{1,2}}^k} - \frac{\int_{\mathbb{R}^N} f(x, u_n) u_n dx}{\|u_n\|_{D_{1,2}}^k} \geq 1 - \frac{c_* \|u_n\|_{D_{1,2}}^k}{\nu^{(k)}_1 \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} \geq 1 - \frac{c_* C_*^k}{\nu^{(k)}_1 \int_{\mathbb{R}^N} \lambda V(x) u_n^2 dx} = 1 + o(1).
\]

This is a contradiction. In conclusion, the sequence \( \{u_n\} \) is bounded in \( X_\lambda \) for all \( 0 < a < \frac{1}{m_{\infty} R_1^{(k)}} \) and \( \lambda \geq \Lambda_N \). This completes the proof. \( \blacksquare \)

Similar to Propositions 4.3 and 4.4, we likewise obtain the following two compactness lemmas for the functional \( J_{\lambda,a} \) under conditions (D1)' and (D2).

**Proposition 5.2** Suppose that \( N \geq 3 \), conditions (V1) – (V3), (L1), (D1)' and (D2) hold. Then for each \( D > 0 \) there exists \( \overline{\lambda} = \overline{\lambda}(a, D) \geq \Lambda_N > 0 \) such that \( J_{\lambda,a} \) satisfies the \((C)_{\alpha}\)-condition in \( X_\lambda \) for all \( \alpha < D \) and \( \lambda > \overline{\lambda} \).

**Proposition 5.3** Suppose that \( N \geq 3 \), conditions (V1) – (V3), (L2), (L3) with \( \delta > \frac{k-2}{2} \), (D1)' and (D2) hold. Then for each \( D > 0 \) there exists \( \overline{\lambda}_* = \overline{\lambda}_*(a, D) \geq \Lambda_N \) such that \( J_{\lambda,a} \) satisfies the \((C)_{\alpha}\)-condition in \( X_\lambda \) for all \( \alpha < D \) and \( \lambda > \overline{\lambda}_* \).

By Proposition 5.2 and 5.3 we now give the following two existence results.

**Theorem 5.4** Suppose that \( N \geq 3 \), conditions (V1) – (V3), (L1), (D1)' and (D2) hold. Then for each \( 0 < a < \frac{1}{m_{\infty} R_1^{(k)}} \) the energy functional \( J_{a,\lambda} \) admits a nontrivial critical point \( u_\lambda \in X_\lambda \) such that \( J_{a,\lambda}(u_\lambda) > 0 \) for all \( \lambda > \overline{\lambda} \).
Proof. Similar to the proof of Theorem 4.5, it is easily proved by using (11), Lemma 4.1 and Proposition 5.2.

Theorem 5.5 Suppose that $N \geq 3$, conditions (V1) – (V3), (L2), (L3) with $\delta > \frac{k-2}{2}$, (D1)' and (D2) hold. Then for every $0 < a < a^*$ and $\lambda > a^*$, the energy functional $J_{a,\lambda}$ has a nontrivial critical point $u^+_{a,\lambda} \in X_\lambda$ satisfying $J_{a,\lambda}(u^+_{a,\lambda}) > 0$.

Proof. Similar to the argument of Theorem 4.6 we easily arrive at the conclusion by Lemma 5.1 and Proposition 5.3.

Lemma 5.6 Suppose that $N \geq 3$, conditions (V1) – (V3), (L3) with $\delta > \frac{k-2}{2}$, (D1)' and (D2) hold. Then the energy functional $J_{a,\lambda}$ is bounded below on $X_\lambda$ for all $a > 0$ and $\lambda > 0$. Furthermore, there exists $\overline{R}_a > 0$ such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_\lambda \geq \overline{R}_a.$$ 

Proof. If $\|u\|_{D^{1,2}} < T_0^{1/2}$, then by (13) and (19) one has

$$J_{a,\lambda}(u) \geq \frac{1}{2} \left( b \|u\|^2_{D^{1,2}} + \int_{\mathbb{R}^N} \lambda V(x)u^2\,dx \right) - \frac{c_*}{k_{D^{1,2}}} \|u\|^k_{D^{1,2}}$$

which implies that $J_{a,\lambda}$ is bounded below on $X_\lambda$ for all $a > 0$ and $\lambda > 0$.

If $\|u\|_{D^{1,2}} \geq T_0^{1/2}$, then it follows from condition (L3) with $\delta > \frac{k-2}{2}$, (13) and (19) that

$$J_{a,\lambda}(u) \geq \frac{a}{2} \hat{m} \left( \|u\|^2_{D^{1,2}} \right) + \frac{1}{2} \left( b \|u\|^2_{D^{1,2}} + \int_{\mathbb{R}^N} \lambda V(x)u^2\,dx \right) - \frac{c_*}{k_{D^{1,2}}} \|u\|^k_{D^{1,2}}$$

which implies that $J_{a,\lambda}(u)$ is bounded below on $X$ for all $a > 0$ and $\lambda > 0$, since $\delta > \frac{k}{2} - 1$.

Moreover, for every $a > 0$, there exists $R_a > t_B := \left( \frac{2(\delta+1)c_*}{k_{D^{1,2}} m_0a} \right)^{1/(2\delta+2-k)}$ such that

$$J_{a,\lambda}(u) \geq 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_{D^{1,2}} \geq R_a.$$ 

Next, we show that there exists a constant $\overline{R}_a > 0$ such that $J_{a,\lambda}(u) \geq 0$ for all $u \in X_\lambda$ with $\|u\|_\lambda \geq \overline{R}_a$. Let

$$\overline{R}_a = \left( R_a^2 + \frac{2c_*}{k_{D^{1,2}}} R_a^k \right)^{1/2} > 0. \tag{42}$$
For $u \in X_\lambda$ with $\|u\|_\lambda \geq \overline{R}_a$, If $\|u\|_{D^{1,2}} \geq R_a$, then the result holds clearly. If $\|u\|_{D^{1,2}} < R_a$, then it is enough to indicate that $J_{a,\lambda}(u) \geq 0$ when $\int_{\mathbb{R}^N} \lambda V(x)u^2 \, dx \geq \frac{2c_*}{k^{(k)}_1} R_a^k$. Indeed, we have

$$J_{a,\lambda}(u) = \frac{a}{2} \hat{m}(\|u\|_{D^{1,2}}^2) + \frac{1}{2} \left( b \|u\|_{D^{1,2}}^2 + \int_{\mathbb{R}^N} \lambda V(x)u^2 \, dx \right) - \frac{c_*}{k^{(k)}_1} \|u\|_{D^{1,2}}^k \geq 0.$$ 

Hence, we obtain that there exists a constant $\overline{R}_a > 0$ defined as (42) such that

$$J_{a,\lambda}(u) > 0 \text{ for all } u \in X_\lambda \text{ with } \|u\|_\lambda \geq \overline{R}_a.$$ 

This completes the proof. ■

**Lemma 5.7** Suppose that $N \geq 3$, conditions (V1) – (V3), (L3) with $\delta > \frac{k-2}{2}, (D1)'$ and (D2) hold. Then for every $a > 0$ and $\lambda > 0$ one has

$$\overline{\theta}_a =: \inf \{ J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \|u\|_\lambda < \overline{R}_a \} < 0.$$  

(43)

**Proof.** The proof directly follows from Lemmas 3.6 and 5.6. ■

**Theorem 5.8** Suppose that $N \geq 3$, conditions (V1) – (V3), (L2), (L3) with $\delta > \frac{k-2}{2}, (D1)'$ and (D2) hold. Then for every $a > 0$ and $\lambda > \overline{R}_s$, $J_{a,\lambda}$ has a nonzero critical point $u^-_{a,\lambda} \in X_\lambda$ such that

$$J_{a,\lambda}(u^-_{a,\lambda}) = \overline{\theta}_a < 0,$$

where $\overline{\theta}_a$ is as in (43).

**Proof.** By Lemma 5.7 and the Ekeland variational principle, we obtain that there exists a minimizing bounded sequence $\{u_n\} \subset X_\lambda$ with $\|u_n\|_\lambda < \overline{R}_a$ such that

$$J_{a,\lambda}(u_n) \to \overline{\theta}_a \text{ and } J'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty.$$ 

Then from Proposition 5.3 it follows that there exist a subsequence $\{u_n\}$ and $u^-_{a,\lambda} \in X_\lambda$ with $\|u^-_{a,\lambda}\|_\lambda < \overline{R}_a$ such that $u_n \to u^-_{a,\lambda}$ strongly in $X_\lambda$. This indicates that $J'_{a,\lambda}(u^-_{a,\lambda}) = 0$ and $J_{a,\lambda}(u^-_{a,\lambda}) = \overline{\theta}_a < 0$. The proof is complete. ■

We are now ready to prove Theorems 1.3 and 1.4: Theorems 1.3 directly follows from Theorem 5.4. By virtue of Theorems 5.5 and 5.8 for every $0 < a < \overline{R}_a$ and $\lambda > \overline{R}_a$, Eq. $(K_{a,\lambda})$ admits two positive solutions $u^-_{a,\lambda}$ and $u^+_{a,\lambda}$ satisfying $J_{a,\lambda}(u^-_{a,\lambda}) < 0 < J_{a,\lambda}(u^+_{a,\lambda})$. In particular, $u^-_{a,\lambda}$ is a ground state solution of Eq. $(K_{a,\lambda})$. Hence, Theorem 1.4 is proved.
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