Exact Solution for a Rectangle with Rigidly Clamped Horizontal Sides

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Abstract. In this paper, in the form of expansions in Papkovich–Fadle eigenfunctions, an exact solution to the boundary value problem of the theory of elasticity is constructed for a rectangle whose horizontal sides are rigidly clamped. The expansion coefficients are determined with the help of biorthogonal functions by simple formulas.

1. Introduction
An example of an exact solution to the biharmonic problem of the theory of elasticity is considered for a thin elastic plate of rectangular shape (the plane stress state) in which the horizontal sides are rigidly clamped and normal stresses act at the ends (for simplicity, we assume that the tangential stresses are equal to zero). We will seek the solution to the problem in the form of expansions in Papkovich–Fadle eigenfunctions:

\[ U(x, y) = \sum_{k=1}^{\infty} a_k \xi(\lambda_k, y) \sinh \lambda_k x + \bar{a}_k \xi(\bar{\lambda}_k, y) \sinh \bar{\lambda}_k x, \]
\[ V(x, y) = \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y) \cosh \lambda_k x + \bar{a}_k \chi(\bar{\lambda}_k, y) \cosh \bar{\lambda}_k x. \]
\[ \sigma_x(x, y) = \sum_{k=1}^{\infty} \alpha_k s_k(\lambda_k, y) \cosh \lambda_k x + \beta_k s_k(\lambda_k, y) \cosh \lambda_k x, \]

\[ \sigma_y(x, y) = \sum_{k=1}^{\infty} \alpha_k s_k(\lambda_k, y) \cosh \lambda_k x + \beta_k s_k(\lambda_k, y) \cosh \lambda_k x, \]

\[ \tau_{xy}(x, y) = \sum_{k=1}^{\infty} \alpha_k t_{xy}(\lambda_k, y) \sinh \lambda_k x + \beta_k t_{xy}(\lambda_k, y) \sinh \lambda_k x, \]

where \( U(x, y) = Gu(x, y) \), \( V(x, y) = Gv(x, y) \), \( G \) is the shear modulus, \( u(x, y) \) and \( v(x, y) \) are the longitudinal and transverse displacements, respectively. The numbers \( a_k, \alpha_k \) \((k = 1, 2, \ldots)\) are the unknown expansion coefficients. The numbers \( \pm\lambda_k, \pm\tilde{\lambda}_k \) \((k = 1, 2, \ldots)\) are the set of all the roots of the characteristic equation, and the first root \( \lambda_1 \) is real \((\nu = \frac{1}{3} \text{ is Poisson's ratio})\):

\[ L(\lambda) = \frac{(3-\nu)\sin 2\lambda}{8\lambda} - \frac{1+\nu}{4} = 0. \]

Since \( \xi(\lambda_k, \pm1) = \chi(\lambda_k, \pm1) = 0 \), the boundary conditions on the horizontal sides of the rectangle are satisfied automatically. The unknown expansion coefficients are determined from the boundary conditions at \( x = \pm d \) by using the functions

\[ x_k(y) = -\frac{\cos x_k y}{4x_k \sin x_k} (|y| \leq 1) \]

that are the finite parts of the functions \( X_k(y) \) biorthogonal to the functions \( s_k(\lambda_k, y) \). The functions \( X_k(y) \) are found by solving the equations \((\lambda \text{ is a real parameter})\)

\[ \int_{-\infty}^{\infty} s_k(\lambda, y) X_k(y) dy = \frac{\lambda L(\lambda)}{\lambda^2 - \lambda_k^2}. \]

The functions \( X_k(y) \) satisfy the following biorthogonality relations:

\[ \int_{-\infty}^{\infty} s_k(\lambda_k, y) X_m(y) dy = \begin{cases} M_k = \frac{L(\lambda_k)}{2} (\lambda_k = \lambda_m); \\ 0 (\lambda_k \neq \lambda_m), \end{cases} \]

\[ \int_{-\infty}^{\infty} s_k(\lambda_k, y) \overline{X}_m(y) dy = \begin{cases} \overline{M}_k = \frac{L(\lambda_k)}{2} (\lambda_k = \lambda_m); \\ 0 (\lambda_k \neq \lambda_m). \end{cases} \]

and \((k \text{ and } m \text{ are any natural numbers})\)

\[ \int_{-\infty}^{\infty} s_k(\lambda_k, y) X_m(y) dy = \int_{-\infty}^{\infty} s_k(\lambda_k, y) \overline{X}_m(y) dy = 0. \]

They are obtained directly from formula (4).

In the case when tangential stresses are given at the ends of the rectangle, the following functions are used to determine the coefficients \( a_k \):

\[ t_k(y) = \frac{\sin x_k y}{4\sin x_k} (|y| \leq 1) \]

that are the finite parts of the functions \( T_k(y) \) biorthogonal to the Papkovich–Fadle eigenfunctions \( t_{xy}(\lambda_k, y) \). They are determined by solving the equations

\[ \int_{-\infty}^{\infty} t_k(\lambda, y) T_k(\lambda) dy = \frac{\lambda^2 L(\lambda)}{\lambda^2 - \lambda_k^2}. \]
For them, the biorthogonality relations will be as follows:

\[
\int_{-\infty}^{\infty} t_{\nu}(\lambda_k, y) T_m(y) \, dy = \begin{cases} \lambda_k M_k (\lambda_k = \lambda_m); \\ 0 (\lambda_k \neq \lambda_m). \end{cases}
\]

\[
\int_{-\infty}^{\infty} t_{\nu}(\lambda_k, y) \overline{T}_m(y) \, dy = \begin{cases} \overline{\lambda}_k \overline{M}_k (\overline{\lambda}_k = \overline{\lambda}_m); \\ 0 (\overline{\lambda}_k \neq \overline{\lambda}_m). \end{cases}
\]

and \((k \text{ and } m \text{ are any natural numbers})\)

\[
\int_{-\infty}^{\infty} t_{\nu}(\lambda_k, y) T_m(y) \, dy = \int_{-\infty}^{\infty} t_{\nu}(\lambda_k, y) \overline{T}_m(y) \, dy = 0.
\]

The integrals \((4) – (12)\) do not generally exist for complex values of the parameter \(\lambda\), but they can be made existing due to the corresponding contour deformation (more on this in [2, 5]).

We give the expressions for the functions generating the Papkovich–Fadle eigenfunctions when \(\lambda = \lambda_k\):

\[
\xi(\lambda, y) = \left(\frac{3 - \nu}{4\alpha}\sin \lambda - \frac{1 + \nu}{4}\cos \lambda\right) \cos \lambda y - \frac{1 + \nu}{4} \sin \lambda \sin \lambda y,
\]

\[
\chi(\lambda, y) = \frac{1 + \nu}{4}(\cos \lambda \sin \lambda y - \sin \lambda \cos \lambda y),
\]

\[
s_x(\lambda, y) = \left(\frac{3 + \nu}{2}\sin \lambda - \frac{1 + \nu}{2}\lambda \cos \lambda\right) \cos \lambda y - \frac{1 + \nu}{2} \lambda \sin \lambda \sin \lambda y,
\]

\[
s_y(\lambda, y) = \left(-\sin \lambda + \frac{1 + \nu}{2}\lambda \cos \lambda\right) \sin \lambda y - \frac{1 + \nu}{2} \lambda \sin \lambda \cos \lambda y.
\]

### 3. Solution of the boundary value problem

Satisfying the boundary conditions at the ends of the rectangle, we obtain a system of two equations for determining the unknown expansion coefficients:

\[
\sigma(y) = \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) \cosh \lambda_k d + \overline{a}_k s_x(\overline{\lambda}_k, y) \cosh \overline{\lambda}_k d;
\]

\[
0 = \sum_{k=1}^{\infty} a_k t_{\nu}(\lambda_k, y) \sinh \lambda_k d + \overline{a}_k t_{\nu}(\overline{\lambda}_k, y) \sinh \overline{\lambda}_k d.
\]

Following, for example, [1–4] and projecting \((14)\) onto biorthogonal directions, based on the biorthogonality relations \((5)-(7)\) and \((10)-(12)\), for each number \(k = 1, 2, \ldots\) we obtain the system of two algebraic equations

\[
\begin{align*}
\sigma_k + \overline{\sigma}_k &= a_k M_k \cosh \lambda_k d + \overline{a}_k \overline{M}_k \cosh \overline{\lambda}_k d; \\
0 &= a_k \lambda_k M_k \sinh \lambda_k d + \overline{a}_k \lambda_k \overline{M}_k \sinh \overline{\lambda}_k d,
\end{align*}
\]

where the numbers \(\sigma_k\) (the Lagrange coefficients) are as follows:

\[
\sigma_k = \int_{-1}^{1} \sigma(y) x_k(y) \, dy.
\]

Solving \((15)\) and substituting \(a_k\) into formulas \((1)\), after separating the zero-series (for example, [2]), we obtain the final formulas for the displacements and stresses:

\[
U(x, y) = U'(x, y) + \sum_{k=2}^{\infty} 2 \text{Re} \left( \sigma_k \frac{\xi(\lambda_k, y)}{\lambda_k M_k} \right) \frac{\text{Im}(\overline{\lambda}_k d \sinh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k d)}.
\]
\[
V(x, y) = -V^1(x, y) - \sum_{k=1}^{\infty} 2 \text{Re} \left( \sigma_k \frac{\chi(\lambda_k, y)}{M_k} \right) \frac{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k d)}
\]

\[
\sigma_k(x, y) = \sigma_k^1(x, y) + \sum_{k=1}^{\infty} 2 \text{Re} \left( \sigma_k \frac{s_k(\lambda_k, y)}{M_k} \right) \frac{\text{Im}(\lambda_k^2 \lambda_k \sinh \lambda_k d \cosh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k d)}
\]

\[
\tau_k(x, y) = \tau_k^1(x, y) + \sum_{k=1}^{\infty} 2 \text{Re} \left( \sigma_k \frac{t_k y(\lambda_k, y)}{\lambda_k^2 M_k} \right) \frac{\text{Im}(\lambda_k \lambda_k \sinh \lambda_k d \sinh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k d)}
\]

where

\[
U^1(x, y) = \left[ (\lambda d \cosh \lambda d + \sinh \lambda d) \sinh \lambda x - \lambda x \cosh \lambda x \sinh \lambda d \right] \frac{\xi(\lambda, y)}{\lambda d M_\Delta}
\]

\[
V^1(x, y) = \left[ (\lambda d \cosh \lambda d + \sinh \lambda d) \cosh \lambda x - \lambda x \sinh \lambda x \cosh \lambda d \right] \frac{\sigma(\lambda, y)}{\lambda d M_\Delta}
\]

\[
\sigma^1_k(x, y) = \left[ (\lambda d \cosh \lambda d + \sinh \lambda d) \cosh \lambda x - \lambda x \sinh \lambda x \cosh \lambda d \right] \frac{s_k(\lambda_k, y)}{\lambda_k M_\Delta}
\]

\[
\sigma^1_k(x, y) = \left[ (\lambda d \cosh \lambda d + \sinh \lambda d) \cosh \lambda x - \lambda x \sinh \lambda x \cosh \lambda d \right] \frac{s_k(\lambda_k, y)}{\lambda_k M_\Delta}
\]

\[
\tau^1_k(x, y) = [d \cosh \lambda d \sinh \lambda x - x \cosh \lambda x \sinh \lambda d] \frac{\sigma^1_k(\lambda_k, y)}{\lambda_k^2 M_1}
\]

\[
\Delta = \lambda d + \cosh \lambda d \sinh \lambda d.
\]

The summands (18) standing in front of infinite sums correspond to the real root \( \lambda_k \). They are obtained from the general representations for the complex roots \( \lambda_k \) as a result of passing to the limit when the imaginary part of the root tends to zero.

### 4. Example

Let

\[
\sigma(y) = \begin{cases} 
0.5, & |y| \leq \alpha (0 < \alpha < 1) \; ; \\
0, & \alpha < |y| < 1).
\end{cases}
\]

We assume that \( \alpha = 0.5, \; d = 2 \) and first consider the solution for the self-balanced function

\[
\sigma(y) = \begin{cases} 
0.25, & |y| \leq 0.5; \\
-0.25, & 0.5 < |y| < 1.
\end{cases}
\]

Using formula (16), we find the Lagrange coefficients

\[
\sigma_k = \frac{1}{8 \lambda_k^3 \sin \lambda_k} \left( \sin \lambda_k - 2 \sin \frac{\lambda_k}{2} \right)
\]

for it.

In equation (4), we pass to the limit as \( \lambda \to 0 \). Then we obtain

\[
\int_{-\infty}^{\infty} 1X_k(y) dy = \frac{\nu - 1}{2 \lambda_k^2}.
\]

In this expression, on the right is the Lagrange coefficient of expansion of a unit into a series in the system of functions \( s_k(\lambda_k, y) \). Then the sum
\[
\sigma_k = \frac{1}{8 \lambda_k^3} \sin \lambda_k \left( \sin \lambda_k - 2 \sin \frac{\lambda_k}{2} \right) + \frac{1}{4} \nu - \frac{1}{2} \lambda_k^3 \quad (23)
\]

will be the Lagrange coefficient for the function (19). Substituting (23) into formulas (17), we obtain the solution to the problem.

Figure 1 shows the change in the tangential stresses at \( y = 1 \). Figure 2 shows the change in the normal stresses at the ends \( x = \pm d \) of the rectangle.

Figures 3 and 4 show changes in the normal stresses in the clamp and displacements at the end \( x = -d \) of the rectangle, respectively.

5. Conclusion
1. According to figure 1, the tangential stresses suffer a discontinuity at the angular points, varying from zero to a finite value. This is due to violating the law of parity. For this reason, the stresses \( \sigma_\tau(x+\kappa, y) \) also have finite discontinuities (figure 2) at the ends of the rectangle.

2. The law of parity for the tangential stresses at the angular points of the rectangle is not violated if the normal stresses \( \sigma(x) \) at the ends of the rectangle are self-balanced and vanish in the vicinity of the angular points. In this case, the tangential stresses in the clamp change smoothly from zero at the angular points.

3. In the obtained exact solution, there is no singularity for stresses at the corner points of a rectangle, in contrast to the corresponding solutions for an infinite wedge. The reason for this is that, in the solution for a wedge, the type of boundary conditions changes at a point (apex of the wedge) lying on one coordinate line, namely along the wedge boundary. In the article [6] it was shown that in problems for a half-strip or a rectangle, in this case, functions biorthogonal to Papkovich–Fadle functions should have the corresponding singularity. Therefore, the solution for a wedge cannot be considered as asymptotic in the corresponding problems for a half-strip or rectangle, since these are both physically and mathematically different problems.

6. References

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