In this brief review, I summarize the current theoretical knowledge of heavy quarkonium inclusive decays, with emphasis on recent progress made in the framework of QCD effective field theories. In appendix, I list the imaginary parts of the matching coefficients of the dimension 6 and dimension 8 NRQCD four-fermion operators as presently known.

**Keywords:** QCD; Effective Field Theories; Heavy Quarkonium; Decays.

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1. Introduction

Heavy quarkonia (charmonium, bottomonium, ...) provide an ideal set of observables to probe properties of low-energy QCD in a controlled way. The reason is the following. Heavy quarkonia are non-relativistic bound states and, therefore, characterized by a set of energy scales hierarchically ordered: \( m, mv, mv^2 \), ... where \( m \) is the heavy-quark mass and \( v \ll 1 \) the relative heavy-quark velocity. For heavy quarkonia, \( m \) is much larger than the scale of non-perturbative physics, \( \Lambda_{\text{QCD}} \), and, therefore, degrees of freedom associated with that scale can be treated perturbatively and calculations done order by order in \( \alpha_s \). The non-relativistic hierarchy of scales also survives below \( \Lambda_{\text{QCD}} \). Therefore, for any heavy quarkonium state the low-energy dynamics is organized in matrix elements ordered in powers of \( v \) (and, in general, \( \Lambda_{\text{QCD}}/m \)). To any given order in \( \alpha_s \) and \( v \), only a finite number of Feynman diagrams and matrix elements respectively have to be calculated.

The way to implement rigorously these expansions in QCD is provided by the non-relativistic effective field theories (EFTs) of QCD. The first has been Non-Relativistic QCD, NRQCD\(^1\). It is obtained from QCD by integrating out degrees of freedom of energy \( m \). NRQCD still contains the lower energy scales as dynamical degrees of freedom. In the last few years, the problem of integrating out the remaining dynamical scales has been addressed by several groups and has now reached a solid level of understanding (lists of references may be found in\(^2\)). The ultimate EFT obtained by subsequent matchings from QCD, where only the lightest degrees of freedom of energy \( mv^2 \) are left dynamical, is potential NRQCD, pNRQCD\(^3\). This EFT is close to a quantum-mechanical description of the bound system and,
therefore, as simple. It has been systematically explored in the dynamical regime \( \mu v^2 \gtrsim \Lambda_{\text{QCD}} \) in \cite{5,6} and in the regime \( \mu v^2 \ll \Lambda_{\text{QCD}} \) in \cite{5,6,7,8,9,10}. An alternative approach to pNRQCD has been suggested in \cite{11}. This approach has been developed in the dynamical situation \( \mu v^2 \gg \Lambda_{\text{QCD}} \), but not been extended, so far, to \( \mu v^2 \gg /\Lambda_{\text{QCD}} \) where most of the heavy quarkonium states are believed to lie.

In this letter, I will review the theory status of heavy quarkonium inclusive and electromagnetic decays into light particles in the framework of QCD non-relativistic EFTs. The main mechanism of heavy quarkonium decay into light particles is quark–anti-quark annihilation. Since this happens at a scale \( 2m \), which is perturbative, the heavy quarks annihilate into the minimal number of gluons allowed by symmetry. Experimentally this fact is reflected by the narrow width of the heavy quarkonia below the open flavour threshold. In an EFT language, once the scale \( m \) has been integrated out, the information on decays is carried by contact terms (four-fermion operators) whose matching coefficients develop an imaginary part. The low-energy dynamics is in the matrix elements of the four-fermion operators evaluated on the heavy quarkonium states. If one assumes that only heavy quarkonium states with quark–anti-quark in a singlet configuration can exist, then only singlet four-fermion operators contribute and the matrix elements reduce to heavy quarkonium wave functions (or derivatives of them) calculated in the origin. This assumption is known as the “singlet model”. Explicit calculations show that at higher order the singlet matching coefficients develop infrared divergences (for \( P \) waves this happens at next-to-leading order \cite{12} compare with the expressions in Appendix A). In the singlet model, these do not cancel in the expression of the decay widths. It has been the first success of NRQCD to show that, due to the non-Abelian nature of QCD, the Fock space of a heavy quarkonium state may contain a small component of quark–anti-quark in an octet configuration bound with some gluonic degrees of freedom (the component is small because operators coupling transverse gluons with quarks are suppressed by powers of \( v \)), due to this component, matrix elements of octet four-fermion operators contribute and, finally, exactly these contributions absorb the infrared divergences of the singlet matching coefficients in the decay widths, giving rise to finite results \cite{13,12}. NRQCD is now the standard framework to study heavy quarkonium decays. From the theoretical side in recent years the main effort has gone into two obvious directions: (1) improving the knowledge of the perturbative series of the matching coefficients either by fixed order calculations or by resumming large contributions (renormalons or large logs); (2) improving the knowledge of the NRQCD matrix elements either by direct evaluation, which may be obtained by fitting the experimental data, by lattice calculations, and by models, or by exploiting the hierarchy of scales still entangled in NRQCD and constructing EFTs of lower energy. I already mentioned that pNRQCD is the ultimate of these EFTs. In such context a new factorization can be achieved that allows to reduce, under some dynamical circumstances, the number of non-perturbative parameters in the expression of the decay widths.
Experimentally several facilities have operated and are operating in an energy range relevant for heavy quarkonium. I refer to the pages of the Quarkonium Working Group for a broad overview. Here, I would like to mention only some of the data produced in the last few years relevant for heavy quarkonium inclusive and electromagnetic decays. They come from the E835 experiment at FNAL, where heavy quarkonium is produced from $p\bar{p}$ annihilation (E835 operated in the charmonium energy region) and from the B-factories (BABAR at SLAC and BELLE at KEKB, which operate at the $\Upsilon(4S)$ resonance), BES at BEPC (operating in the charmonium energy region) and CLEO at CESR (CLEOIII took data in the bottomonium energy region, CLEO-c is taking data in the charmonium one) where heavy quarkonium is produced from $e^+e^-$ collisions. In the bottomonium system, CLEO has provided the first experimental values (still affected by large uncertainties) for the ratios of the inclusive decay widths of $2P_J$ bottomonium states, extracted from the data on two-photons transitions from $\Upsilon(3S)$ decays. In the charmonium system, new determinations of the $\eta_c$ resonance parameters came from the experiments E835, BABAR (where the $\eta_c$ is produced from two-photon interactions), BELLE, $(B \to \eta_c K)$ and BES $(J/\psi \to \eta_c \gamma)$. I refer to 21 for comparison and discussion of the data. E835 also provides $\Gamma(\eta_c \to \gamma \gamma)$. The branching ratios for $\psi(2S) \to e^+e^-$ and $\psi(2S) \to \mu^+\mu^-$ have been measured by BABAR and CLEO in both cases from two-photon interactions. For what concerns the $L = 1$ charmonium states, the $\chi_{c0}$ resonance parameters have been newly measured at E835. The same collaboration provides a determination of $\Gamma(\chi_{c0} \to \gamma \gamma)$ and $\Gamma(\chi_{c2} \to \gamma \gamma)$. New values for $\Gamma(\chi_{c0} \to \gamma \gamma)$ and $\Gamma(\chi_{c2} \to \gamma \gamma)$ have been provided by CLEO and for $\Gamma(\chi_{c2} \to \gamma \gamma)$ by BELLE, 32 in both cases from two-photon production processes.

The letter is organized as follows. In Sec. 2 I review the NRQCD factorization formulas for heavy quarkonium decay widths. In Sec. 3 I briefly discuss the perturbative series of the matching coefficients. Appendix A contains a complete list of all the imaginary parts of the matching coefficients of the dimension 6 and 8 operators (hadronic and electromagnetic) at their present accuracy. In Sec. 4 I discuss the NRQCD matrix elements and in Sec. 5 the pNRQCD factorization. Some final remarks are given in Sec. 6.

2. NRQCD

The NRQCD factorization formulas are obtained by separating contributions coming from degrees of freedom of energy $m$ from those coming from degrees of freedom of lower energy. In the case of heavy quarkonium decay widths, they have been rig-
orously prove\textsuperscript{2} High-energy contributions are encoded into the imaginary parts of the four-fermion matching coefficients, $f, g_{1,8,ee,\gamma \gamma} (2S+1L_J)$ and are ordered in powers of $\alpha_s$. Low-energy contributions are encoded into the matrix elements of the four-fermion operators on the heavy quarkonium states $|H\rangle$ ($\langle \ldots | H \equiv \langle H | \ldots | H\rangle$). These are, in general, non-perturbative objects, which can scale as powers of $\Lambda_{\text{QCD}}, m_v, m_v^2, \ldots$ (i.e. of the low-energy dynamical scales of NRQCD). Therefore, matrix elements of higher dimensionality are suppressed by powers of $v$ or $\Lambda_{\text{QCD}}/m_v$.

Including up to the NRQCD four-fermion operators of dimension 8, the NRQCD factorization formulas for inclusive decay widths of heavy quarkonia into light hadrons ($LH$) read\textsuperscript{2,13}

\[
\Gamma(V_Q(nS) \to LH) = \frac{2}{m^2} \left( \text{Im} \ f_1 (3S_1) \langle O_1 (3S_1) \rangle_{VQ(nS)} 
+ \text{Im} f_8 (3S_1) \langle O_8 (3S_1) \rangle_{VQ(nS)} + \text{Im} f_{8} (1S_0) \langle O_8 (1S_0) \rangle_{VQ(nS)} 
+ \text{Im} g_1 (3S_1) \langle P_1 (3S_1) \rangle_{VQ(nS)} + \text{Im} f_8 (3P_0) \langle O_8 (3P_0) \rangle_{VQ(nS)} 
+ \text{Im} f_8 (3P_1) \langle O_8 (3P_1) \rangle_{VQ(nS)} \right),
\]

\[
\Gamma(P_Q(nS) \to LH) = \frac{2}{m^2} \left( \text{Im} f_1 (1S_0) \langle O_1 (1S_0) \rangle_{PQ(nS)} 
+ \text{Im} f_8 (1S_0) \langle O_8 (1S_0) \rangle_{PQ(nS)} + \text{Im} f_8 (3S_1) \langle O_8 (3S_1) \rangle_{PQ(nS)} 
+ \text{Im} g_1 (1S_0) \langle P_1 (1S_0) \rangle_{PQ(nS)} + \text{Im} f_8 (1P_1) \langle O_8 (1P_1) \rangle_{PQ(nS)} \right),
\]

\[
\Gamma(\chi_Q(nJS) \to LH) = \frac{2}{m^2} \left( \text{Im} f_1 (2S+1P_J) \langle O_1 (2S+1P_J) \rangle_{\chi_Q(nJS)} 
+ \text{Im} f_8 (2S+1S_0) \langle O_8 (1S_0) \rangle_{\chi_Q(nJS)} \right).
\]

At the same order the electromagnetic decay widths are given by:

\[
\Gamma(V_Q(nS) \to e^+ e^-) = \frac{2}{m^2} \left( \text{Im} f_{ee} (3S_1) \langle O_{\text{EM}} (3S_1) \rangle_{VQ(nS)} 
+ \text{Im} g_{ee} (3S_1) \langle P_{\text{EM}} (3S_1) \rangle_{VQ(nS)} \right),
\]

\[
\Gamma(P_Q(nS) \to \gamma \gamma) = \frac{2}{m^2} \left( \text{Im} f_{\gamma \gamma} (1S_0) \langle O_{\text{EM}} (1S_0) \rangle_{PQ(nS)} \right).
\]

\textsuperscript{a}Such a proof is still lacking for the NRQCD factorization of heavy quarkonium production cross sections.
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\[ + \text{Im} g_{\gamma\gamma}(^1S_0) \frac{\langle \mathcal{O}_{\text{EM}}(^1S_0) \rangle_{\chi_Q(nS)}}{m^2}, \quad (5) \]

\[ \Gamma(\chi_Q(nJ_1) \rightarrow \gamma\gamma) = 2 \text{Im} f_{\gamma\gamma}(^3P_J) \frac{\langle \mathcal{O}_{\text{EM}}(^3P_J) \rangle_{\chi_Q(nJ_1)}}{m^4}, \quad J = 0, 2. \quad (6) \]

The symbols \( V_Q \) and \( P_Q \) indicate respectively the vector and pseudoscalar S-wave heavy quarkonium and the symbol \( \chi_Q \) the generic P-wave quarkonium (the states \( \chi_Q(n10) \) and \( \chi_Q(nJ_1) \) are usually called \( h_Q((n-1)P) \) and \( \chi_QJ((n-1)P) \), respectively).

The operators \( O, P_{1,8,EM}^{(2S+1)L_J} \) are the dimension 6 and 8 four-fermion operators of the NRQCD Lagrangian. They are classified in dependence of their transformation properties under colour as singlets (1) and octets (8) and under spin (S), orbital (L) and total angular momentum (J). The operators with the subscript EM are the singlet operators projected on the QCD vacuum. The explicit expressions of the operators can be found in [2] (or listed in Appendix A of [9]).

3. The perturbative expansion

The imaginary parts of the four-fermion matching coefficients have been calculated over the past twenty years by different authors and to different levels of precision. Since the results are scattered over a large number of papers, some of them being difficult to collect, some having been corrected in subsequent publications and some still being in disagreement with each other, I have listed all the imaginary parts of the matching coefficients of the dimension 6 and 8 operators (hadronic and electromagnetic) at the present accuracy in Appendix A. The tree-level matching of the dimension 10 S-wave operators can be found in [33]. The tree-level matching of the dimension 9 electromagnetic P-wave operators can be found in [34].

The convergence of the perturbative series of the four-fermion matching coefficients is often bad. Let us consider, for instance, the following matching coefficients \((n_f = 3, \mu_R = 2m)\) [35]:

\[ \text{Im} f_1(^1S_0) = (\ldots) \times \left( 1 + 11.1 \frac{\alpha_s}{\pi} \right), \]
\[ \text{Im} f_8(^1S_0) = (\ldots) \times \left( 1 + 13.7 \frac{\alpha_s}{\pi} \right), \]
\[ \text{Im} f_8(^3S_1) = (\ldots) \times \left( 1 + 10.3 \frac{\alpha_s}{\pi} \right), \]
\[ \text{Im} f_1(^3P_0) = (\ldots) \times \left( 1 + \left( 13.6 - 0.44 \log \frac{\mu}{2m} \right) \frac{\alpha_s}{\pi} \right), \]
\[ \text{Im} f_1(^3P_2) = (\ldots) \times \left( 1 - \left( 0.73 + 1.67 \log \frac{\mu}{2m} \right) \frac{\alpha_s}{\pi} \right). \]

Apart from the case of \( \text{Im} f_1(^3P_2) \), the series in \( \alpha_s \) of the other coefficients does not show convergence. This behaviour cannot be adjusted by a suitable choice of the factorization scale \( \mu \), which enters only in \( \text{Im} f_1(^3P_{0,2}) \). A solution may be provided by the resummation of the large contributions in the perturbative series coming from
bubble-chain diagrams. This analysis has been successfully carried out for $S$-wave annihilation decays.\cite{36} A treatment that includes $P$-wave decays is still missing.

4. The relativistic expansion

The NRQCD matrix elements may be fitted on the experimental decay data\cite{37,29} or calculated on the lattice\cite{38}. The matrix elements of singlet operators can be linked at leading order to the Schrödinger wave functions in the origin\cite{2} and therefore may be evaluated by means of potential models.\cite{39} In general, however, NRQCD matrix elements, in particular of higher dimensionality, are poorly known or completely unknown.

It has been discussed in\cite{34} and\cite{33}, that higher-order operators, not included in the formulas of Sec. 2, even if parametrically suppressed, may turn out to give sizeable contributions to the decay widths. This may be the case, in particular, for charmonium, where $v^2 \sim 0.3$, so that relativistic corrections are large, and for $P$-wave decays where the above formulas provide, indeed, only the leading-order contribution in the velocity expansion. In fact it was pointed out in\cite{34,35} that if no special cancellations among the matrix elements occur, then the order $v^2$ relativistic corrections to the electromagnetic decays $\chi_c^0 \rightarrow \gamma \gamma$ and $\chi_c^2 \rightarrow \gamma \gamma$ may be as large as the leading terms.

In\cite{37} it was also noted that the numerical relevance of higher-order matrix elements may be enhanced by the multiplying matching coefficients. This is, indeed, the case for the decay width of $S$-wave vector states, where the matching coefficients multiplying the octet matrix elements (with the only exception of $\text{Im} f_8(3P_1)$) are enhanced by $\alpha_s$ with respect to the coefficient $\text{Im} f_1(3S_1)$ of the leading singlet matrix element (see Eq. (1) and Appendix A).

In the bottomonium system, 14 $S$- and $P$-wave states lie below the open flavour threshold ($\Upsilon(nS)$ and $\eta_b(nS)$ with $n = 1, 2, 3$; $h_b(nP)$ and $\chi_{bJ}(nP)$ with $n = 1, 2$ and $J = 0, 1, 2$) and in the charmonium system 8 ($\psi(nS)$ and $\eta_c(nS)$ with $n = 1, 2$; $h_c(1P)$ and $\chi_{cJ}(1P)$ with $J = 0, 1, 2$). For these states Eqs. 1-6 describe the decay widths into light hadrons and into photons or $e^+e^-$ in terms of 46 NRQCD matrix elements (40 for the $S$-wave decays and 6 for the $P$-wave decays). More matrix elements are needed if, as discussed above, higher-order operators have to be included.

4.1. pNRQCD

The number of non-perturbative parameters may be reduced by integrating out from NRQCD degrees of freedom with energy lower than $m$, since each degree of freedom that is integrated out leads to a new factorization. Eventually, one ends up with pNRQCD, where only degrees of freedom of energy $mv^2$ are left dynamical. In the context of pNRQCD, the NRQCD four-fermion matrix elements can be written either as convolutions of Coulomb amplitudes with non-local correlators (in the dynamical situation $mv^2 \sim \Lambda_{QCD}$) or products of wave-functions in the origin by
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non-local correlators (in the dynamical situation \( m v^2 \ll \Lambda_{\text{QCD}} \)). The first situation is believed to be the relevant one at least for the bottomonium ground state\(^5\)\(^6\)\(^7\)\(^8\). In the limiting case \( m v^2 \gg \Lambda_{\text{QCD}} \), the correlators reduce to local condensates and explicit formulas have been worked out for the electromagnetic decay of the \( \Upsilon(1S) \) in\(^9\)\(^10\). The last situation is expected to be the relevant one for most of the existing heavy quarkonia (with the possible exception of the bottomonium ground state) and has been studied in\(^8\)\(^9\)\(^10\). However, a general consensus on the above assignations of heavy quarkonium states to dynamical regions has not been reached yet. As an example, I mention that in\(^44\) it is suggested that also some of the higher bottomonium states may be Coulombic bound states while in practically all potential models\(^45\)\(^46\) the bottomonium ground state is described by means of confining potentials.

In the situation \( m v^2 \ll \Lambda_{\text{QCD}} \) and under the condition that: (a) all higher gluonic excitations between the two heavy quarks develop a mass gap of order \( \Lambda_{\text{QCD}} \), (b) threshold effects are small, and (c) contributions coming from virtual pairs of quark-antiquark with three momentum of order \( \sqrt{m \Lambda_{\text{QCD}}} \) are subleading, \(^c\) the NRQCD octet matrix elements relevant for Eqs. \(^1\)\(^1\)\(^1\) can be written at leading order in the \( v \) and \( \Lambda_{\text{QCD}}/m \) expansion as \(8\)\(^1\)\(^0\):

\[
\langle O_8^8(S_1) \rangle_{VQ(nS)} = \langle O_8^1(S_0) \rangle_{PQ(nS)} = C_A \frac{|R_{n0}^{(0)}(0)|^2}{2\pi} \left( -\frac{2(C_A/2-C_F)\mathcal{E}_3^{(2)}}{3m^2} \right), \quad (7)
\]
\[
\langle O_8^1(S_0) \rangle_{VQ(nS)} = \langle O_8^1(S_1) \rangle_{PQ(nS)} = \frac{3}{C_A} \frac{|R_{n0}^{(0)}(0)|^2}{2\pi} \left( -\frac{(C_A/2-C_F)c_F^2B_1}{3m^2} \right), \quad (8)
\]
\[
\langle O_8^3(P_J) \rangle_{VQ(nS)} = \langle O_8^1(P_J) \rangle_{PQ(nS)} = \frac{3}{2\pi} \left( 2J + 1 \right) \frac{|R_{n0}^{(0)}(0)|^2}{2\pi} \left( -\frac{(C_A/2-C_F)\mathcal{E}_1}{9} \right), \quad (9)
\]
\[
\langle O_8^1(S_0) \rangle_{\chi Q(nJS)} = \frac{T_F}{3} \frac{|R_{n0}^{(1)}(0)|^2}{\pi m^2} \mathcal{E}_3, \quad (10)
\]

where \( c_F \) stands for the chromomagnetic matching coefficient, which is known at next-to-leading order\(^13\). Therefore, at the considered order, the octet matrix ele-

\(^c\)Concerning the perturbative calculation of the electromagnetic decay width of the \( \Upsilon(1S) \) a renormalization group improved expression can be found in\(^12\) and the wave function in the origin at next-to-next-to-leading order in\(^13\).

\(^b\)Conditions (a) and (b) select the simplest version of pNRQCD with only one degree of freedom: the heavy quarkonium singlet field. Condition (a) is supported by lattice data on the excitation spectrum of the gluon field around a static quark-antiquark pair\(^47\). Condition (b) may be problematic for the \( \psi(2S) \), whose mass is very close to the \( DD \bar{D} \) production threshold. Condition (c) is more technical and affects the matching to NRQCD. I refer to\(^10\) for a discussion of its validity.
ments factorize into the product of the zeroth-order radial part of the heavy quarkonium wave function, $R_{n\ell}^{(0)}$, which may be calculated from the real part of the pNRQCD Hamiltonian, and some chromoelectric and chromomagnetic correlators:

$$
\mathcal{E}_n = \frac{1}{N_c} \int_0^\infty dt \, t^n \langle gE(t) \cdot gE(0) \rangle,
$$

$$
\mathcal{B}_n = \frac{1}{N_c} \int_0^\infty dt \, t^n \langle gB(t) \cdot gB(0) \rangle,
$$

$$
\mathcal{E}_3^{(2)} = \frac{1}{4N_c} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \, (t_2 - t_3)^3 \left\{ \langle gE(t_1) \cdot gE(t_2) \rangle \{ gE(t_3) \cdot gE(0) \} \right\}_c
$$

$$
- \frac{4}{N_c} \langle \text{Tr}(gE(t_1) \cdot gE(t_2)) \text{Tr}(gE(t_3) \cdot gE(0)) \rangle_c,
$$

where

$$
\langle gE(t_1) \cdot gE(t_2) \rangle \{ gE(t_3) \cdot gE(0) \} \rangle_c = \langle gE(t_1) \cdot gE(t_2) \rangle \langle gE(t_3) \cdot gE(0) \rangle
$$

$$
- \frac{1}{N_c} \langle gE(t_1) \cdot gE(t_2) \rangle \{ gE(t_3) \cdot gE(0) \} \rangle_c.
$$

These correlators are universal in the sense that they do not depend on the heavy quarkonium state and, hence, may be calculated once for ever, either by means of lattice simulations or specific models of the QCD vacuum or extracted from some set of experimental data.

At leading order in the $v$ and $\Lambda_{\text{QCD}}/m$ expansion the singlet matrix elements can be expressed in terms of the wave functions in the origin only:

$$
\langle O_1(3S_1) \rangle_{\mathcal{V}_Q(nS)} = \langle O_1(1S_0) \rangle_{\mathcal{P}_Q(nS)} = \langle O_{\text{EM}}(3S_1) \rangle_{\mathcal{V}_Q(nS)}
$$

$$
= \langle O_{\text{EM}}(1S_0) \rangle_{\mathcal{P}_Q(nS)} = C_A \frac{|R_{n0}^{(0)}(0)|^2}{2\pi},
$$

$$
\langle O_1(2S+1P_J) \rangle_{\chi_Q(nJS)} = \langle O_{\text{EM}}(2S+1P_J) \rangle_{\chi_Q(nJS)} = \frac{3}{2} \frac{C_A}{\pi} |R_{n1}^{(0)}(0)|^2.
$$

At leading order the matrix elements of the $\mathcal{P}_1$ operators involve also the correlator $\mathcal{E}_1$:

$$
\langle \mathcal{P}_1(3S_1) \rangle_{\mathcal{V}_Q(nS)} = \langle \mathcal{P}_1(1S_0) \rangle_{\mathcal{P}_Q(nS)} = \langle \mathcal{P}_{\text{EM}}(3S_1) \rangle_{\mathcal{V}_Q(nS)}
$$

$$
= \langle \mathcal{P}_{\text{EM}}(1S_0) \rangle_{\mathcal{P}_Q(nS)} = C_A \frac{|R_{n0}^{(0)}(0)|^2}{2\pi} \left( mE_{n0}^{(0)} - \mathcal{E}_1 \right),
$$

where $E_{n0}^{(0)} \simeq M - 2m$ is the leading-order binding energy. Equation (17) reduces to the formula obtained in [10] if the heavy quarkonium state satisfies also the condition $mv \gg \Lambda_{\text{QCD}}$.

The leading corrections to the above formulas come from quark-antiquark pairs of three momentum of order $\sqrt{m\Lambda_{\text{QCD}}}$. The existence of this degree of freedom in the heavy quarkonium system has been pointed out in [10] where the leading correction to Eq. (15) has been calculated.
The pNRQCD factorization formulas reduce the number of non-perturbative parameters needed in order to describe heavy quarkonium decay widths. In particular, they have been used to calculate bottomonium matrix elements from charmonium data. This is useful since, at the moment, bottomonium data are less abundant than charmonium ones. In this way $P$-wave bottomonium inclusive decay widths have been calculated before the first data by CLEO-II were made available. One has to stress, however, that the theoretical uncertainties associated to $P$-wave heavy quarkonium decays are rather large, due to the large corrections either in the perturbative series (as discussed in Sec. 3) or in the relativistic expansion (as discussed in Sec. 4). For the inclusive decay width of $P$-wave heavy quarkonium neither the resummation of large perturbative corrections, nor the computation of operators appearing at next-to-leading order in the $v$ and $\Lambda_{QCD}/m$ expansion has been done yet.

5. Conclusions

In this letter, I have reviewed some general aspects of the theory of inclusive heavy quarkonium decays. The standard framework is provided by NRQCD and more generally by non-relativistic EFTs of QCD. These have put the study of heavy quarkonium observables on the solid ground of QCD. Models and phenomenological approaches have not necessarily become obsolete: they may provide estimates of the non-perturbative parameters that appear in the EFTs. In particular, potential models may still be useful to estimate the heavy quarkonium wave functions. However, also potentials are parameters of the EFT and have a precise expression in terms of the original degrees of freedom (gluons and quarks) of QCD. Lattice gauge theories provide the most natural and well founded tool to calculate non-perturbative quantities. In fact several lattice determinations of matrix elements entering in the heavy quarkonium decay width expression at the level of NRQCD, as well as of correlators and Wilson loops entering at the level of pNRQCD already exist.

Experimentally, heavy quarkonium decay data have been produced in large amount in the last years and have improved the accuracy of several of the measured widths and branching ratios. They call for comparable precise theoretical determinations. The relevance is twofold. On one hand we may extract from heavy quarkonium data several of the non-perturbative parameters that characterize the low-energy dynamics of QCD. This is possible, because we have simple and exact expressions that factorize the non-perturbative physics. As an example, I mention that the correlators entering in the expression of the decay widths in pNRQCD give information on the masses of the heavy quarkonium exotic hybrid states, describe the behaviour of the QCD static potential at intermediate distances and contribute to the heavy quarkonium levels. On the other hand we may use heavy quarkonium data to extract some of the fundamental parameters of the Standard Model (e.g. the heavy-quark masses and $\alpha_s$). In the case of $\alpha_s$, this is not yet possible from heavy
quarkonium decay data with an accuracy comparable with other determinations due to the difficulties discussed in this review\textsuperscript{d}. This is one of the many challenges in the present and future of heavy quarkonium physics.

**Appendix A. Imaginary parts of the dimension 6 and dimension 8 four-fermion matching coefficients**

The imaginary parts of the four-fermion matching coefficients are known to different levels of precision. In the following, I will indicate their most updated values. The symbols stand for: $C_A = N_c = 3$, $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, $B_F = (C_A/2 - C_F)(C_A^2/2 - 2) = 5/12$, $T_F = 1/2$, $\beta_0 = 11 N_c/3 - 4 T_F n_f/3$, $e_Q$ the electrical charge of the quark $Q$ ($e_b = -1/3$, $e_c = 2/3$, ...), $\alpha$ the electromagnetic coupling constant, $\alpha_s$ the strong coupling constant in the $\overline{\text{MS}}$ scheme and $n_f$ the number of active light flavours (typically 3 and 4 for charmonium and bottomonium respectively).

The scale $\mu$ is the factorization scale and the scale $\mu_R$ the renormalization scale. In a physical quantity, like the decay width, the $\mu$ dependence will be canceled by low-energy matrix elements and the $\mu_R$ dependence by higher-order terms in the perturbative expansion. The strong coupling constant $\alpha_s(\mu_R)$ has to be understood as running with $n_f$ flavours:\textsuperscript{e}

\begin{equation}
\text{Im } f_1(S_0) \equiv C_F \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s(\mu_R)^2 \\
\times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -5 + \frac{\pi^2}{4} C_F + \left( \frac{199}{18} + \frac{13}{24}\pi^2 \right) C_A \\
- \frac{16}{9} n_f T_F + \beta_0 \log \frac{\mu_R}{2m} \right] \right\},
\end{equation}

\begin{equation}
\text{Im } f_1(S_1) \equiv 2 \frac{\pi^2}{9} (\pi^2 - 9) C_F (C_A^2 - 4) \left( \frac{C_A}{2} - C_F \right)^2 \alpha_s(\mu_R)^3 \\
\times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -9.46(2) C_F + 4.13(17) C_A - 1.161(2) n_f + \frac{3}{2} \beta_0 \log \frac{\mu_R}{m} \right] \right\} \\
+ \pi e_Q \left( \sum_{i=1}^{n_f} e_{Q_i}^2 \right) \alpha^2 \left\{ 1 - \frac{13}{4} C_F - \pi \alpha_s \right\}.
\end{equation}

\textsuperscript{d}The value quoted by the PDG\textsuperscript{33} seems to underestimate some of the uncertainty\textsuperscript{34}.

\textsuperscript{e}Note that from

$$\alpha_s(\mu_R) = \alpha_s(\mu'_R) \left( 1 + \frac{\alpha_s}{\pi} \beta_0 \log \frac{\mu'_R}{\mu_R} \right),$$

and $\alpha_s^{(n_f)}(m) = \alpha_s^{(n_f+1)}(m)$ it follows that

$$\alpha_s^{(n_f)}(\mu_R) = \alpha_s^{(n_f+1)}(\mu_R) \left( 1 - \frac{2}{3} \frac{\alpha_s}{\pi} T_F \log \frac{\mu_R}{m} \right).$$
\[ \text{Im } f_1(^1P_1) \overset{\text{LSL}}{=} \frac{8B_F C_F}{3} \left( \frac{C_A}{2} - C_F \right) \alpha_s^3 \left( -\frac{7}{3} + \frac{7}{48}\pi^2 - \log \frac{\mu}{2m} \right), \quad (A.3) \]

\[ \text{Im } f_1(^3P_0) \overset{\text{LSL}}{=} 3C_F \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s(\mu_R)^2 \]
\[ \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( -\frac{7}{3} + \frac{\pi^2}{4} \right) C_F + \left( \frac{427}{81} - \frac{\pi^2}{144} \right) C_A \right. \\
\left. + \frac{4}{27} n_f \left( -\frac{29}{6} - \log \frac{\mu}{2m} + \beta_0 \log \frac{\mu_R}{2m} \right) \right] \right\}, \quad (A.4) \]

\[ \text{Im } f_1(^3P_0) \overset{\text{LSL}}{=} 3C_F \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s(\mu_R)^2 \]
\[ \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( -\frac{7}{3} + \frac{\pi^2}{4} \right) C_F + \left( \frac{454}{81} - \frac{\pi^2}{144} \right) C_A \right. \\
\left. + \frac{4}{27} n_f \left( -\frac{29}{6} - \log \frac{\mu}{2m} + \beta_0 \log \frac{\mu_R}{2m} \right) \right] \right\}, \quad (A.5) \]

\[ \text{Im } f_1(^3P_1) \overset{\text{LSL}}{=} \frac{C_F}{2} \left( \frac{C_A}{2} - C_F \right) \alpha_s^3 \left[ \left( \frac{587}{27} - \frac{317}{144}\pi^2 \right) C_A \right. \\
\left. + \frac{8}{9} n_f \left( -\frac{4}{3} - \log \frac{\mu}{2m} \right) \right], \quad (A.6) \]

\[ \text{Im } f_1(^3P_2) \overset{\text{LSL}}{=} \frac{4}{5} C_F \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s(\mu_R)^2 \]
\[ \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -4C_F + \left( \frac{2185}{216} - \frac{337}{384}\pi^2 + \frac{5}{3} \log 2 \right) C_A \right. \\
\left. + \frac{5}{9} n_f \left( -\frac{29}{15} - \log \frac{\mu}{2m} + \beta_0 \log \frac{\mu_R}{2m} \right) \right] \right\}, \quad (A.7) \]

\[ \text{Im } f_1(^3P_2) \overset{\text{LSL}}{=} \frac{4}{5} C_F \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s(\mu_R)^2 \]
\[ \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -4C_F + \left( \frac{2239}{216} - \frac{337}{384}\pi^2 + \frac{5}{3} \log 2 \right) C_A \right. \\
\left. + \frac{5}{9} n_f \left( -\frac{29}{15} - \log \frac{\mu}{2m} + \beta_0 \log \frac{\mu_R}{2m} \right) \right] \right\}, \quad (A.8) \]

\[ \text{Im } g_1(^1S_0) \overset{\text{LSL}}{=} -\frac{4C_F}{3} \left( \frac{C_A}{2} - C_F \right) \pi \alpha_s^2, \quad (A.9) \]
\[
\text{Im} g_1(^3S_1) \equiv -\frac{19\pi^2 - 132C_F(C_A^2 - 4)}{54} \left(\frac{C_A}{2} - C_F\right)^2 \alpha_s^3, \quad (A.10)
\]

\[
\text{Im} f_8(^1S_0) \equiv B_F\pi\alpha_s(\mu_R)^2 \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( -5 + \frac{\pi^2}{4}\right) C_F + \left( \frac{122}{9} - \frac{17}{24}\pi^2\right) C_A - \frac{16}{9}n_f T_F + \beta_0 \log \frac{\mu_R}{2m} \right] \right\}, \quad (A.11)
\]

\[
\text{Im} f_8(^3S_1) \equiv n_f \frac{\pi\alpha_s(\mu_R)^2}{6} \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( -\frac{13}{4}\right) C_F + \left( \frac{133}{18} + \frac{2}{3} \log 2 - \frac{\pi^2}{4}\right) C_A - \frac{10}{9}n_f T_F + \left( -\frac{73}{4} + \frac{67}{36}\pi^2\right) \frac{5}{n_f} + \beta_0 \log \frac{\mu_R}{2m} \right] \right\}, \quad (A.12)
\]

\[
\text{Im} f_8(^1P_1) \equiv \frac{C_A}{12} \pi\alpha_s^2, \quad (A.13)
\]

\[
\text{Im} f_8(^3P_0) \equiv 3B_F\pi\alpha_s(\mu_R)^2 \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ \left( -\frac{7}{3} + \frac{\pi^2}{4}\right) C_F + \left( \frac{463}{81} + \frac{35}{27} \log 2 - \frac{17}{216}\pi^2\right) C_A + \frac{4}{27}n_f \left( -\frac{29}{6} - \log \frac{\mu}{2m} \right) + \beta_0 \log \frac{\mu_R}{2m} \right] \right\}, \quad (A.14)
\]

\[
\text{Im} f_8(^3P_1) \equiv B_F\alpha_s \left[ \left( \frac{1369}{108} - \frac{23}{18}\pi^2\right) C_A + \frac{4}{9}n_f \left( -\frac{4}{3} - \log \frac{\mu}{2m} \right) \right], \quad (A.15)
\]

\[
\text{Im} f_8(^3P_2) \equiv \frac{4}{5} B_F\pi\alpha_s(\mu_R)^2 \times \left\{ 1 + \frac{\alpha_s}{\pi} \left[ -4C_F + \left( \frac{4955}{431} - \frac{43}{72}\pi^2 + \frac{7}{9} \log 2\right) C_A + \frac{5}{9}n_f \left( -\frac{29}{15} - \log \frac{\mu}{2m} \right) + \beta_0 \log \frac{\mu_R}{2m} \right] \right\}, \quad (A.16)
\]
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Im $f_{\gamma\gamma}(1S\,0)$ \cite{5556} \quad $\pi e_Q^4 \alpha^2 \left\{ 1 + \frac{\alpha_s}{\pi} \left( -5 + \frac{\pi^2}{4} \right) C_F \right\}$, \quad (A.17)

Im $f_{ee}(3S\,1)$ \cite{61} \quad $\frac{\pi e_Q^2 \alpha^2}{3} \left\{ 1 - 4 C_F \frac{\alpha_s(\mu_R)}{\pi} \right\}$

$\left[ \left( \frac{79}{18} \pi^2 - 2 \pi^2 \log \frac{\mu}{m} + 2 \pi^2 \log 2 + \frac{39}{4} - \zeta_3 \right) C_F 
+ \left( \frac{89}{18} \pi^2 - \pi^2 \log \frac{\mu}{m} - \frac{5}{3} \pi^2 \log 2 - \frac{151}{36} \frac{13}{2} \zeta_3 - \frac{22}{3} \log \frac{\mu}{m} \right) C_A
+ \left( \frac{11}{9} + \frac{8}{3} \log \frac{\mu R}{m} \right) T_{F,n_f} + \left( \frac{4}{9} \pi^2 + \frac{44}{9} \right) T_F \right\}$, \quad (A.18)

Im $f_{\gamma\gamma\gamma}(3S\,1)$ \cite{5557} \quad $4 \frac{\pi^2}{9} - \frac{9}{9} e^6 \alpha^3 \left\{ 1 - 9.46(2) C_F \frac{\alpha_s}{\pi} \right\}$, \quad (A.19)

Im $f_{\gamma\gamma}(3P\,0)$ \cite{5556} \quad $3 \pi e_Q^4 \alpha^2 \left\{ 1 + \frac{\alpha_s}{\pi} \left( -\frac{7}{3} + \frac{\pi^2}{4} \right) C_F \right\}$, \quad (A.20)

Im $f_{\gamma\gamma}(3P\,2)$ \cite{5556} \quad $4 \frac{\pi e_Q^4 \alpha^2}{5} \left\{ 1 - 4 C_F \frac{\alpha_s}{\pi} \right\}$, \quad (A.21)

Im $g_{\gamma\gamma}(1S\,0)$ \cite{2} \quad $- \frac{4}{3} \pi e_Q^4 \alpha^2$, \quad (A.22)

Im $g_{ee}(3S\,1)$ \cite{2} \quad $- \frac{4}{5} \pi e_Q^2 \alpha^2$, \quad (A.23)

Im $g_{ee}(3S\,1,3\,D\,1)$ \cite{2} \quad $- \frac{\pi^2}{3} e_Q^2 \alpha^2$. \quad (A.24)

The number over the equal sign indicates the reference/references where the most updated value of the matching coefficient can be found. Some comments are in order. The order $\alpha_s$ corrections to Im $f_1(3S\,1)$ and Im $f_{\gamma\gamma}(3S\,1)$, given in Eqs. (A.19) and (A.22) respectively, are known only numerically and, therefore, affected by a numerical error. The last line of Eq. (A.22), proportional to $\alpha^2$, comes from the annihilation of the quark-antiquark pair into a virtual photon, which then decays...
into light hadrons. For the order $\alpha_s$ corrections to $\text{Im} f_1(^3P_0)$ and $\text{Im} f_1(^3P_2)$ there are at the moment two (numerically slightly) different determinations in the literature. Since the question of which one, if any, is correct has not been settled yet I have reported both determinations (Eqs. (A.4)-(A.5) and Eqs. (A.7)-(A.8)). The expression of $\text{Im} f_8(^3S_1)$ given in Eq. (A.12) and taken from 58 is different from that one reported in 56 (there $n_f \rightarrow n_f/3$). According to one of the authors this is the correct one 62. The electromagnetic matching coefficients (A.17)-(A.24) refer to annihilation processes where the final states consist of two photonics (Eqs. (A.17), (A.20), (A.21) and (A.22)), three photons (Eq. (A.19)) and 2 massless fermions (Eqs. (A.18), (A.23) and (A.24)). Note that the matching coefficient $\text{Im} f_{ee}(^3S_1)$, having been calculated at order $\alpha_s^2$, is the most accurately known.

The running equations for the imaginary parts of the matching coefficients of the four-fermion NRQCD operators of dimension 6 and 8 have been obtained in 9 and can be read there in Appendix C.

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