CLASSIFYING SPACES FOR PROPER
ACTIONS OF LOCALLY-FINITE GROUPS

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For each finite ordinal \( n \), and each locally-finite group \( G \) of cardinality \( \aleph_n \), we construct an \((n+1)\)-dimensional, contractible CW-complex on which \( G \) acts with finite stabilizers. We use the complex to obtain information about cohomology with induced coefficients. Our techniques also give information about the location of some large free abelian groups in the hierarchy \( H \).

Throughout, let \( G \) be a group, and let \( A \) be a \( \mathbb{Z} \)-module with trivial \( G \)-action. We let \( AG \) denote the induced \( \mathbb{Z}G \)-module \( \mathbb{Z} \otimes_\mathbb{Z} A \). Ring-actions on modules and group-actions on sets are tacitly understood to be on the left, if not specified otherwise.

1. Holt’s Conjectures

The purpose of this section is to describe our main algebraic results, Theorems 3.10 and 5.4, and place them in the context of prior work.

1.1 Notation. Let \( \text{rank}(G) \) denote the smallest cardinal \( \kappa \) such that there exists some set of generators of \( G \) of cardinality \( \kappa \).

If \( G \) is not finitely generated, then \( \text{rank}(G) = |G| \), and we define \( \aleph \text{-rank}(G) \) to be the ordinal \( \alpha \) such that \( \text{rank}(G) = \aleph_\alpha \); if \( G \) is finitely generated, then \( \text{rank}(G) < |G| \), and we set \( \aleph \text{-rank}(G) = -1 \).

Recall that, for each ordinal \( \alpha \), \( \omega_\alpha \) denotes the least ordinal of cardinality \( \aleph_\alpha \).

We find it convenient to set \( \aleph_{-1} = 1 \).

Throughout this section, let \( n \in \mathbb{N} (= \omega_0) \). □

D. F. Holt proposed the following description of the cohomology with induced coefficients, for locally-finite groups.

1.2 Conjecture (Holt [6]). If \( G \) is locally finite, then

\[
|H^n(G, AG)| = |A|^{\aleph_{n-1}} \quad \text{if } n = \aleph \text{-rank}(G) + 1,
\]

\[
H^n(G, AG) = 0 \quad \text{if } n \neq \aleph \text{-rank}(G) + 1.
\]
Commentary. In Examples 3.3 we recall that, for any group $G$,

\begin{equation}
H^n(G, AG) = \begin{cases} 
A & \text{if } G \text{ is finite and } n = 0, \\
0 & \text{if } G \text{ is finite or } n = 0, \text{ but not both.}
\end{cases}
\end{equation}

Thus the conjecture really concerns the cases where $n \geq 1$ and $G$ is infinite, and the notation has been artificially contrived to embrace the trivial marginal cases.

For any infinite group $G$, the set of cocycles for $G$ with coefficients in $AG$ is of cardinality $|A|^{|G|}$, so $1 \leq |H^n(G, AG)| \leq |A|^{|G|}$. If $G$ is infinite and locally finite, then the conjecture implies that only the extreme values can be achieved. □

We now briefly state the cases which are known, including those obtained in this paper.

1.4 Notation. We say that $G$ has the finite extension property for proper subgroups if each proper subgroup of $G$ is a proper subgroup of finite index in some subgroup of $G$. For example, abelian torsion groups have this property.

Let us say that $A$ is $o(G)$-inverting if, for every finite subgroup $H$ of $G$, multiplication by $|H|$ gives an automorphism of $A$; equivalently, for each $g \in G$ whose order $o(g)$ is finite, multiplication by $o(g)$ gives an automorphism of $A$.

If $R$ is a ring (associative, with 1), then $R$ is $o(G)$-inverting, as $\mathbb{Z}$-module, if and only if the order of each finite subgroup of $G$ is a unit in $R$. If $R$ is not $o(G)$-inverting, then it is easy to show that $\text{cd}_R G$, the cohomological dimension of $G$ with respect to $R$, is $\infty$, a value which we shall think of as $\omega_0 = \aleph_0$. □

1.5 Known cases of Conjecture 1.2. Let $G$ be a locally-finite group.

1. $H^n(G, AG) = 0$ if $n > \aleph\text{-rank}(G) + 1$.

2. If $G$ has the finite extension property for proper subgroups, then $H^n(G, AG) = 0$ if $n \neq \aleph\text{-rank}(G) + 1$.

3. For $n \in \{0, 1\}$, $H^n(G, AG) = 0$ if $n \neq \aleph\text{-rank}(G) + 1$.

4. For $n \in \{0, 1, 2\}$, $|H^n(G, AG)| = |A|^{|\aleph|}$ if $n = \aleph\text{-rank}(G) + 1$.

5. It is consistent with ZFC that $|H^n(G, AG)| \geq 2^{|\aleph|}$ if $n = \aleph\text{-rank}(G) + 1$ and $A$ is nonzero.

Hence, it is consistent with ZFC that $|H^n(G, AG)| = |A|^{|\aleph|}$ if $n = \aleph\text{-rank}(G) + 1$ and $|A| \leq \aleph_{n-1}$.

Commentary. (1), the “easy” part of Conjecture 1.2, is proved in Theorem 3.10. It was proved in [11] for the case where $A$ is $o(G)$-inverting, and, before that, in [4], [5] for the case where $A$ is $o(G)$-inverting and torsion.

(2) was proved by Holt [5]. We give another proof of the abelian case in Corollary 6.10.

(3). By (1.3), this holds for $n = 0$. It was proved by Holt [6] for $n = 1$; see Theorem 6.4.

(4). By (1.3), this holds for $n = 0$. It is well known for $n = 1$; see Theorem 4.5. In Theorem 5.4, we prove it for $n = 2$; Holt [6] had previously shown this was consistent with ZFC, see [14, Section 1].

(5). Suppose that $n = \aleph\text{-rank}(G) + 1$ and that $A$ is nonzero.

We shall now see that it is consistent with ZFC that $|H^n(G, AG)| \geq 2^{|\aleph|}$. 


For each prime $p$, we write $\mathbb{Z}(p^\infty) := \lim_{m \to \infty} \mathbb{Z}/p^m\mathbb{Z}$, where, for $m \in \mathbb{N}$, the map $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$ is given by multiplication by $p$.

We claim that there exists a $\mathbb{Z}$-module $k$, and a $\mathbb{Z}$-submodule $A'$ of $A$, such that the quotient $A/A'$ is isomorphic to $k$, and either $k = \mathbb{Q}$, or there exists a prime $p$ such that $k = \mathbb{Z}/p\mathbb{Z}$ or $k = \mathbb{Z}(p^\infty)$.

Consider first the case where $A$ is not divisible, so there exists a prime $p$ such that $A/pA$ is nonzero. But $A/pA$ is a direct sum of $\mathbb{Z}$-submodules each of which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Hence $A/pA$ projects onto any such summand.

If $A$ is divisible, then $A$ is a direct sum of $\mathbb{Z}$-submodules each of which is isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}(p^\infty)$ for some prime $p$; see, for example, [3, Theorem IV.23.1]. Hence $A$ projects onto any such summand.

In all cases, we can find $A'$, $k$ as claimed.

Now there is a long exact sequence in cohomology which contains the subsequence

$$H^n(G, AG) \to H^n(G, kG) \to H^{n+1}(G, A'G).$$

By (1), $H^{n+1}(G, A'G) = 0$, so $|H^n(G, AG)| \geq |H^n(G, kG)|$.

Thus, for the first part, it remains to show that it is consistent with ZFC that $|H^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

It is proved in [11] that, if $k$ is an $o(G)$-inverting prime field, then it is consistent with ZFC that $|H^n(G, kG)| \geq 2^{\aleph_{n-1}}$. However, on carefully reading that proof, one sees that all applications of the $o(G)$-inverting hypothesis can be replaced with applications of (1), so in fact, it is proved that it is consistent with ZFC that if $k$ is a prime field then $|H^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

It remains to consider the case where $k = \mathbb{Z}(p^\infty)$ for some prime $p$. Again, it is not difficult to show that the argument in [11] can be further modified to cover this case by interpreting $\dim k^m := m$ for all $m \in \mathbb{N}$. If $M$ is a finitely generated (free) $\mathbb{Z}$-submodule of $\mathbb{Z}G$, and $kM$ denotes the image of the natural map $k \otimes_{\mathbb{Z}} M \to k \otimes_{\mathbb{Z}} \mathbb{Z}G = kG$, then one can show $\dim kM = \text{rank } M$, since $k$ is divisible. Moreover, if $M'$ is a $\mathbb{Z}$-submodule of $M$, then $\dim kM' \leq \dim kM$, and, if equality holds, then $kM' = kM$, since $k$ is divisible. Using these observations, one can verify that the argument in [11] applies with $k = \mathbb{Z}(p^\infty)$.

It follows that, in all cases, it is consistent with ZFC that $|H^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

Now suppose that $|A| \leq \aleph_{n-1}$. Hence $|A|^{\aleph_{n-1}} = 2^{\aleph_{n-1}}$; see [7, p.49] for the case where $n \geq 1$. Thus, it is consistent with ZFC that $|H^n(G, AG)| \geq |A|^{\aleph_{n-1}}$.

We previously observed that $|H^n(G, AG)| \leq |A|^{\aleph_{n-1}}$, so it is consistent with ZFC that $|H^n(G, AG)| = |A|^{\aleph_{n-1}}$.

This proves (5). It had previously been proved by Holt [5] in the case where $G$ has the finite extension property for proper subgroups; see [14, Section 1].

We wish to refine part of Conjecture 1.2.

1.6 Conjecture. If $G$ is locally finite, and $n = \aleph\text{-rank}(G) + 1$, then $H^n(G, AG)$ contains a $\mathbb{Z}$-submodule isomorphic to $A^{\aleph_{n-1}}$, and hence $|H^n(G, AG)| = |A|^{\aleph_{n-1}}$.

Commentary. By (1.3), this holds for $n = 0$. It is probably well known for $n = 1$; see Theorem 4.5. In Theorem 5.4, we prove it for $n = 2$. 

Conjecture 1.2 was preceded by, and motivated by, an earlier proposal, concerning the cohomological dimension of locally-finite groups.
1.7 Conjecture. If $G$ is a locally-finite group, and $R$ is a nonzero, $o(G)$-inverting ring, then $\text{cd}_R G = \min\{\aleph\text{-rank}(G) + 1, \infty\}$.

Commentary. Holt [4] proposed this conjecture with the additional hypothesis that $R$ is a field of prime order, and, in [11], the additional hypothesis was weakened to $R$ being commutative.

Notice that $\min\{\aleph\text{-rank}(G) + 1, \infty\}$ can be expressed as $\inf\{n \in \mathbb{N} \mid \aleph_n > |G|\}$, where the infimum of the empty set is taken to be $\infty$.

The inequality $\text{cd}_R G \leq \aleph\text{-rank}(G) + 1$ follows from a classic result of Auslander [1, Proposition 3]; see [12, Lemma 3.7] or Theorem 3.10 below.

Cohomological dimension cannot increase on passing to a subgroup, so we may assume that $\aleph\text{-rank}(G) < \omega_0$, and let $n = \aleph\text{-rank}(G) + 1$. The conjecture now amounts to the claim that $H^n(G, M) \neq 0$ for some $RG$-module $M$. Notice that, on $RG$-modules, $H^{n+1}(G, -)$ vanishes and (hence) $H^n(G, -)$ is right exact; also, $M$ is a quotient of some free $RG$-module. The conjecture is therefore equivalent to the claim that $H^n(G, AG) \neq 0$ for some free $R$-module $A$. This claim is implied by the claim that $H^n(G, AG) \neq 0$ for some vector space $A$ over the prime subfield of some simple quotient ring of $R$. This means that we may assume that $R$ is a prime field.

The foregoing claims are known to be consistent with ZFC [11], and have been proved in various cases. The case where $G$ is abelian was proved by N. Chen; see [12, Corollary 7.6] or Corollary 6.10 below. Chen’s result has been extended in two directions. Osofsky [12, Corollary 7.5] settled the case where $G$ is generated by finite groups whose pairwise products are subgroups; in particular, $G$ has the finite extension property for proper subgroups. Holt [5] settled the case where $G$ has the finite extension property for proper subgroups and $R$ is a finite prime field. □

2. $G$-complexes

In this section we construct finite-dimensional contractible spaces with locally-finite groups acting on them.

2.1 Definitions. A map $f \colon X_1 \to X_2$ between CW-complexes is cellular if it carries the $d$-skeleton of $X_1$ to the $d$-skeleton of $X_2$ for all $d \in \mathbb{N}$.

A $G$-CW-complex, or $G$-complex for short, is a CW-complex $X$ with a $G$-action such that each element of $G$ acts continuously on $X$, permuting the open cells, and fixing only those cells which it fixes pointwise. It follows that $G$ acts cellularly.

If $X$ is a $G$-complex then, for each $H \leq G$, the set $X^H$, consisting of points fixed by all of $H$, is a CW-subcomplex of $X$, and the set $X/H$ consisting of the $H$-orbits is a quotient CW-complex.

Let $\text{sub}(G)$ denote the set of all subgroups of $G$. A subset $\mathcal{X}$ of $\text{sub}(G)$ is a subgroup-closed $G$-family if each subgroup of each element of $\mathcal{X}$ belongs to $\mathcal{X}$ and, moreover, $\mathcal{X}$ is closed under taking conjugates by elements of $G$.

If $\mathcal{X}$ is a subgroup-closed $G$-family, then by a space of type $E(G, \mathcal{X})$ we mean a $G$-complex $X$ with the properties that, for each $H \in \mathcal{X}$, $X^H$ is contractible, and for each $H \in \text{sub}(G) - \mathcal{X}$, $X^H$ is empty. In this event, $X$ is also said to be a classifying space for $G$-actions with stabilizers in $\mathcal{X}$.

If $\mathcal{X}$ is a class of groups, and $\mathcal{X} \cap \text{sub}(G)$ is a subgroup-closed $G$-family, then by a space of type $E(G, \mathcal{X})$ we mean a space of type $E(G, \mathcal{X} \cap \text{sub}(G))$. 
We let $\mathcal{F}$ denote the class of finite groups. Notice that $\mathcal{F} \cap \text{sub}(G)$ is a subgroup-closed $G$-family. A space of type $E(G, \mathcal{F})$ is called an $E\mathcal{G}$. (It is also called a classifying space for proper $G$-actions, that is, $G$-actions with finite stabilizers.) □

The following is well known.

2.2 Proposition. If $\mathcal{X}$ is a subgroup-closed $G$-family, then there exists a space of type $E(G, \mathcal{X})$, and any $G$-map between two spaces of type $E(G, \mathcal{X})$ is a $G$-homotopy equivalence.

Proof. The first part can be seen by Milnor’s construction. Thus, let $\Delta$ be any $G$-set such that $\mathcal{X}$ is precisely the set of subgroups of $G$ which fix at least one point of $\Delta$. Let $X = \Delta \ast \Delta \ast \Delta \ast \cdots$, the union of iterated joins of $\Delta$. Then $X$ is a space of type $E(G, \mathcal{X})$.

For the second part, see, for example, [15, Proposition II.2.7]. □

2.3 Corollary. If $\mathcal{X}_1 \subseteq \mathcal{X}_2$ are subgroup-closed $G$-families, and $X_1$ (resp. $X_2$) is a space of type $E(G, \mathcal{X}_1)$ (resp. $E(G, \mathcal{X}_2)$), then there exists a cellular $G$-map $X_1 \rightarrow X_2$.

Proof. The join $X_1 \ast X_2$ is a space of type $E(G, \mathcal{X}_2)$, and the inclusions

$$\iota_1: X_1 \rightarrow X_1 \ast X_2, \quad \iota_2: X_2 \rightarrow X_1 \ast X_2$$

are $G$-maps. By Proposition 2.2, $\iota_2$ is a $G$-homotopy equivalence, and the homotopy inverse $X_1 \ast X_2 \rightarrow X_2$ composed with $\iota_1$ gives a $G$-map $X_1 \rightarrow X_2$. This is then $G$-homotopic to a cellular $G$-map $X_1 \rightarrow X_2$; see, for example, [15, Theorem II.2.1]. □

One could give a dual proof, using the projection maps from the Cartesian product $X_1 \times X_2$, which is a space of type $E(G, \mathcal{X}_1)$.

The following is a topological analogue of a classic result of Auslander [1, Proposition 3].

2.4 Theorem. Let $\beta$ be a limit ordinal, let $(G_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subgroups of $G$, and let $(\mathcal{X}_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subsets of $\text{sub}(G)$ such that, for each $\alpha \leq \beta$, $\mathcal{X}_\alpha$ is a subgroup-closed $G_\alpha$-family.

Let $n \in \mathbb{N}$, and suppose that, for each $\alpha < \beta$, there exists an $n$-dimensional space $Y_\alpha$ of type $E(G_\alpha, \mathcal{X}_\alpha)$. Then there exists an $(n+1)$-dimensional space of type $E(G_\beta, \mathcal{X}_\beta)$.

Proof. For each $\alpha < \beta$, $\mathcal{X}_\alpha \subseteq \mathcal{X}_{\alpha+1} \cap \text{sub}(G_\alpha)$ are subgroup-closed $G_\alpha$-families. Also, $Y_\alpha$ is a space of type $E(G_\alpha, \mathcal{X}_\alpha)$, and $Y_{\alpha+1}$ can be viewed as a space of type $E(G_\alpha, \mathcal{X}_{\alpha+1} \cap \text{sub}(G_\alpha))$. By Corollary 2.3, there exists a cellular $G_\alpha$-map $Y_\alpha \rightarrow Y_{\alpha+1}$, and hence a cellular $G_{\alpha+1}$-map $f_\alpha: G_{\alpha+1} \times G_\alpha Y_\alpha \rightarrow Y_{\alpha+1}$. Let $M_\alpha$ denote the mapping cylinder of $f_\alpha$. Since $f_\alpha$ is cellular, $M_\alpha$ has the structure of a CW-complex, and is a space of type $E(G_{\alpha+1}, \mathcal{X}_{\alpha+1})$. Notice that $\dim M_\alpha = n + 1$, since $\dim Y_\alpha = \dim Y_{\alpha+1} = n$.

We recursively construct a continuous chain $(X_\alpha \mid \alpha < \beta)$ where $X_\alpha$ is a space of type $E(G_\alpha, \mathcal{X}_\alpha)$, and, for $\alpha \geq 1$, $\dim X_\alpha = n + 1$.

We take $X_0 = Y_0$, and at limit ordinals we take directed unions.
Suppose $\alpha < \beta$ and that $X_\alpha$ has been constructed.

By Proposition 2.2, since $X_\alpha$ and $Y_\alpha$ are of type $E(G_\alpha, x_\alpha)$ there exists a cellular $G_\alpha$-map $Y_\alpha \to X_\alpha$, and hence a cellular $G_{\alpha+1}$-map $G_{\alpha+1} \times G_\alpha Y_\alpha \to G_{\alpha+1} \times G_\alpha X_\alpha$. Take $X_{\alpha+1}$ to be the identification space, or pushout,

$$G_{\alpha+1} \times G_\alpha Y_\alpha \longrightarrow M_\alpha$$

$$\downarrow$$

$$G_{\alpha+1} \times G_\alpha X_\alpha \longrightarrow X_{\alpha+1}.$$ 

Notice that $\dim X_{\alpha+1} = n + 1$, since $\dim Y_\alpha = n$, $\dim M_\alpha = n + 1$, and $\dim X_\alpha \leq n + 1$. It is not difficult to check that $X_{\alpha+1}$ is of type $E(G_{\alpha+1}, x_{\alpha+1})$.

This completes the proof. □

2.5 Remark. For $n = 0$ and $\beta = \omega_0$, the construction in the above proof gives the Bass-Serre tree of the graph of groups corresponding to the countable ascending chain $(G_\alpha \mid \alpha < \omega_0)$. Here, $x_{\omega_0} = \bigcup_{\alpha < \omega_0} \text{sub}(G_\alpha)$. □

2.6 Theorem. If $n \in \mathbb{N}$, and $G$ is a locally-finite group with $\aleph_0$-rank($G$) < $n$, then there exists an $n$-dimensional $E(G)$.

Proof. We argue by induction on $n$.

If $n = 0$, then $G$ is finite. Here a single point with trivial $G$-action is a $0$-dimensional $E(G)$.

Thus we may assume that $n \geq 1$, and that the result holds for smaller $n$.

We can choose a continuous chain $(G_\alpha \mid \alpha \leq \omega_n)$ of subgroups of $G$ such that $G_{\omega_n} = G$, and, for each $\alpha < \omega_n$, $\aleph_0$-rank($G_\alpha$) < $n - 1$, so, by the induction hypothesis, there exists an $(n - 1)$-dimensional $E(G_\alpha)$. Thus, by Theorem 2.4, there exists an $n$-dimensional $E(G)$.

This completes the proof. □

2.7 Remarks. The foregoing construction applies in greater generality.

Suppose that, for every group $H$, there is specified a subgroup-closed $H$-family $\mathcal{Y}(H)$ satisfying the following three conditions:

- Any group isomorphism $H_1 \to H_2$ induces a bijection $\mathcal{Y}(H_1) \to \mathcal{Y}(H_2)$.
- If $H_1 \leq H_2$, then $\mathcal{Y}(H_1) \subseteq \mathcal{Y}(H_2)$.
- If $H$ is the union of a well-ordered chain of subgroups $H_\alpha$, then $\mathcal{Y}(H)$ is the union of the $\mathcal{Y}(H_\alpha)$.

Let $\mathfrak{S}_0$ denote the class consisting of those groups $H$ such that $\mathcal{Y}(H) = \text{sub}(H)$. For $n < \omega_0$, recursively define $\mathfrak{S}_{n+1}$ to be the class consisting of those groups which can be expressed as the union of a well-ordered chain of subgroups which lie in $\mathfrak{S}_n$.

The above argument then shows that if $G \in \mathfrak{S}_n$, then there exists an $n$-dimensional $E(G, \mathcal{Y}(G))$.

For any $n > 0$ and any $G \in \mathfrak{S}_n$, it can be arranged that all of the spaces involved in the construction of $E(G, \mathcal{Y}(G))$ have distinguished contractible subcomplexes which are transversals for the group actions. In particular, the quotient complex $E(G, \mathcal{Y}(G))/G$ is contractible, although we will not use this information. □

We record one example.
2.8 Theorem. If $n \in \mathbb{N}$, and $\aleph$-rank$(G) < n$, and $\mathfrak{X}$ is the set of all subgroups of all finitely generated subgroups of $G$, then there exists an $n$-dimensional $E(G, \mathfrak{X})$.

Proof. For each group $H$, let $\mathcal{Y}(H)$ be the set consisting of the subgroups of the finitely generated subgroups of $H$. It is easy to see that $\mathcal{Y}(-)$ respects isomorphisms, inclusions and well-ordered unions.

It can be shown, by induction on $n$, that $G \in \mathfrak{G}_n$, in the notation of the previous remark, so, by that remark, there exists an $n$-dimensional $E(G, \mathfrak{Y}(G))$. □

2.9 Example. In the foregoing theorem, if $G$ is abelian (or locally finite), then $\mathfrak{X}$ is the set of finitely generated subgroups of $G$. □

3. Eventual vanishing of cohomology with induced coefficients

In this section, we recall how $H^*(G, -)$ can be computed using an $E(G)$, and apply the method in the case where $G$ is locally finite.

3.1 Definitions. Let $M$ be a $\mathbb{Z}G$-module.

We say that $M$ is $G$-acyclic if $H^n(G, M) = 0$ for all $n \geq 1$.

Any $\mathbb{Z}G$-summand of a $G$-acyclic $\mathbb{Z}G$-module is again $G$-acyclic.

If $\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$, then it is easy to see that $M$ is $G$-acyclic if and only if the sequence

$$0 \to \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \to \text{Hom}_{\mathbb{Z}G}(P_0, M) \to \text{Hom}_{\mathbb{Z}G}(P_1, M) \to \cdots$$

is exact. When this holds, we say that $\text{Hom}_{\mathbb{Z}G}(-, M)$ carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.

For any set $\Delta$, we let $A[[\Delta]]$ denote the set of all functions from $\Delta$ to $A$, and such a function $x \mapsto a_x$ will be written as a formal sum $\sum x \in \Delta a_x.x$, and its support is

$$\text{supp}(\sum x \in \Delta a_x.x) := \{x \in \Delta | a_x \neq 0\}.$$
consistent with the notation for the induced $\mathbb{Z}G$-module $AG$. Here $A[[G]]$ is called a coinduced $\mathbb{Z}G$-module.

There is a natural bijection

$$\text{Hom}_\mathbb{Z}(M, A) \to \text{Hom}_{\mathbb{Z}G}(M, A[[G]]), \quad \psi \mapsto (m \mapsto \sum_{x \in G} \psi(x^{-1}m).x).$$

Since $\text{Hom}_\mathbb{Z}(\_, A)$ carries $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules to $\mathbb{Z}G$-split exact sequences of $\mathbb{Z}G$-modules, we see that co-induced $\mathbb{Z}G$-modules are $G$-acyclic. $\square$

3.3 Examples. Let $G$ be a finite group.

Here, $AG = A[[G]]$, so induced $\mathbb{Z}G$-modules are co-induced, and hence $G$-acyclic.

Suppose that $M$ is an $o(G)$-inverting $\mathbb{Z}G$-module. Then the multiplication map

$$M[[G]] \to M, \quad \sum_{g \in G} m_g g \mapsto \sum_{g \in G} gm_g,$

is $\mathbb{Z}G$-split with right inverse $m \mapsto \frac{1}{|G|} \sum_{g \in G} g^{-1}m.g$. Here, $M$ is a $\mathbb{Z}G$-summand of an induced $\mathbb{Z}G$-module, so $M$ is $G$-acyclic. $\square$

In the following, $G$ acts on tensor products over $\mathbb{Z}$ via the diagonal action.

3.4 Lemma. Let $M$ be a $\mathbb{Z}G$-module.

1. The functor $\text{Hom}_{\mathbb{Z}G}(\_, \mathbb{Z}G, M)$ carries $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules to exact sequences.

2. Let $H$ be a subgroup of $G$. If $M$ is $H$-acyclic, then the functor

$$\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/H] \otimes_\mathbb{Z} \_, M)$$

carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.

Proof. (2). Let $L$ be a $\mathbb{Z}G$-module. There is a natural identification of $\mathbb{Z}G$-modules,

$$\mathbb{Z}[G/H] \otimes_\mathbb{Z} L = \mathbb{Z}G \otimes_{\mathbb{Z}H} L,$$

with $gH \otimes \ell$ corresponding to $g \otimes g^{-1}\ell$.

It follows that we can identify

$$\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/H] \otimes_\mathbb{Z} \_, M) = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G \otimes_{\mathbb{Z}H} \_, M) = \text{Hom}_{\mathbb{Z}H}(\_, M)$$

as functors on $\mathbb{Z}G$-modules. Since $M$ is $H$-acyclic, this functor carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.

(1) is proved similarly. $\square$
3.5 Notation. Let $X$ be a $G$-complex.

We shall treat $X$ as the $G$-set whose elements are the open cells of $X$. The cellular chain complex of $X$ is then the permutation module $\mathbb{Z}X$, with the structure of a differential graded $\mathbb{Z}G$-module, with differential $\partial$ of degree $-1$. Here the grading is that determined by the dimensions of the cells, so the $n$th component $C_n(\mathbb{Z}X)$ has as $\mathbb{Z}$-basis the cells of dimension $n$.

We let $\eta: X \times X \to \mathbb{Z}$ denote the function such that $\partial x = \sum_{y \in X} \eta(x, y) \cdot y$ for each $x \in X$. Thus, if $x$ is an $n$-cell, then $\eta(x, y) = 0$ unless $y$ is one of the finitely many $(n-1)$-cells incident to $x$. \hfill $\Box$

The following is a degenerate case of the equivariant cohomology spectral sequence; see, for example, [2, VII.7.10(7.10)].

3.6 Theorem. Let $X$ be an acyclic $G$-complex. If $M$ is a $\mathbb{Z}G$-module which is $G_x$-acyclic for each $x \in X$, then $H^*(\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, M)) \simeq H^*(G, M)$, as graded abelian groups.

Recall that $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, M)$ denotes the differential graded abelian group with $n$th component $C^n(\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, M)) = \text{Hom}_{\mathbb{Z}G}(C_n(\mathbb{Z}X), M)$.

Proof. The homology of $(\mathbb{Z}X, \partial)$ is $\mathbb{Z}$, concentrated in degree zero.

We choose a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and write it as $(\mathbb{Z}Y, \partial)$ for some $G$-free $G$-set $Y$; for example, we could take $Y$ to be an $EG$, and $\mathbb{Z}Y$ its cellular chain complex. Then $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}Y, M)$ is an additive abelian differential graded group, and its cohomology is $H^*(G, M)$.

We consider the double complex $\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y$ with diagonal $G$-action, and the double complex $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y, M)$. We get a fourth-quadrant commuting diagram which can be schematically represented as

\[
\begin{array}{c}
0 \\
\downarrow \\
\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, M) \\
\downarrow \\
0 \longrightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}Y, M) \longrightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y, M).
\end{array}
\]

To show that the cohomology group of the outer row, $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, M)$, is isomorphic to the cohomology group of the outer column, $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}Y, M)$, it suffices to show that the remaining, or inner, rows and columns of (3.7) are exact. Each inner column is exact because $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} -, M)$ is exact on augmented projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$, by Lemma 3.4(2). Similarly, each inner row is exact because $\text{Hom}_{\mathbb{Z}G}(-, \otimes_{\mathbb{Z}} \mathbb{Z}Y, M)$ is exact on $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules, by Lemma 3.4(1). \hfill $\Box$

3.8 Corollary. Let $M$ be a $\mathbb{Z}G$-module, and let $X$ be a finite-dimensional acyclic $G$-complex. If $M$ is $G_x$-acyclic for each $x \in X$, then $H^n(G, M) = 0$ for all $n > \dim X$. \hfill $\Box$

We record the case of finite stabilizers.
3.9 Corollary. Let $M$ be a $\mathbb{Z}G$-module, and suppose that $M$ is $H$-acyclic for each finite subgroup $H$ of $G$; for example, this holds if $M = AG$, or if $M$ is o($G$)-inverting. Let $X$ be an acyclic $G$-complex with finite stabilizers; for example, this holds if $X$ is an $EG$. Then $H^n(G, M) = 0$ for all $n > \dim X$. □

Here we can apply Theorem 2.6.

3.10 Theorem. Let $G$ be a locally-finite group, and $M$ a $\mathbb{Z}G$-module which is $H$-acyclic for each finite subgroup $H$ of $G$; for example, this holds if $M = AG$, or if $M$ is o($G$)-inverting. Then $H^n(G, M) = 0$ for all $n > \aleph \text{-rank}(G) + 1$. □

3.11 Remark. Theorem 3.10 can also be proved using the argument of the first paragraph of [11, Section 1]. □

4. Locally-finite groups of cardinality $\aleph_0$

In this section, we recall how $H^*(G, AG)$ can be computed using an $EG$, and apply the method in the one-dimensional case.

4.1 Definitions. Let $X$ be an $EG$, or, more generally, any acyclic $G$-complex in which all cell stabilizers are finite, and let Notation 3.5 apply.

We have natural identifications

\[(4.2)\quad A[[X]] = A^X = \text{Hom}_G(\mathbb{Z}X, A).\]

For simplicity, let us suppose that $X$ is finite dimensional.

Then $A[[X]]$ has the structure of a differential graded $\mathbb{Z}G$-module, in which the differential $\partial^*$ has degree +1, and is given by

\[
\partial^* \left( \sum_{x \in X} a_x.x \right) = \sum_{x \in X} \left( \sum_{y \in X} \eta(x, y)a_y \right).x.
\]

The cohomology of $(A[[X]], \partial^*)$ is $A$ concentrated in degree zero.

Let

\[A_G[[X]] := \left\{ \sum_{x \in X} a_x.x \in A[[X]] \mid \{ g \in G \mid a_{gx} \neq 0 \} \text{ is finite, for all } x \in X \right\}.\]

Since $G$-stabilizers are finite, we see that $A_G[[X]]$ consists of all functions from $X$ to $A$ with finite support in each $G$-orbit.

It is straightforward to check that $A_G[[X]]$ is a differential graded $\mathbb{Z}G$-submodule of $A[[X]]$.

We write $C^n(A_G[[X]])$, $B^n(A_G[[X]])$, and $Z^n(A_G[[X]])$ for the $n$-cochains, $n$-coboundaries, and $n$-cocycles, respectively. □

Sometimes the notation $\text{Hom}_c(\mathbb{Z}[X], A)$ is used to denote $A_G[[X]]$; see, for example, [2, Lemma VIII.7.4].

The following is a variation on the usual “compact supports” cohomology; see, for example, [2, Proposition VIII.7.5]. It is particularly useful in the study of ends of groups.
4.3 Theorem. If $X$ is a finite-dimensional acyclic $G$-complex with finite stabilizers, then there is a natural isomorphism $H^*(A_G[[X]]) \simeq H^*(G, AG)$ of graded abelian groups.

Proof. There is a natural identification of $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, A[[G]])$ with $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}X, A)$; see (3.2). There is also a natural identification of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}X, A)$ with $A[[X]]$; see (4.2). It is easy to show that under these identifications, $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, AG)$ corresponds to $A_G[[X]]$. Hence $H^*(A_G[[X]]) \simeq H^*(\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, AG))$. Finally, $H^*(\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, AG)) \simeq H^*(G, AG)$, by Theorem 3.6. \hfill $\square$

There is a natural right $G$-action on $H^*(G, AG)$, arising from the $\mathbb{Z}G$-bimodule structure on $AG$. This agrees with the natural right $G$-action on $A_G[[X]]$ which we have transformed into a left $G$-action.

Let us illustrate how Theorem 4.3 can be used to study $H^*(G, AG)$ when $G$ is locally finite of cardinality $\aleph_0$. To do this we now construct a standardized $\mathbb{P}G$, as in Remark 2.5.

4.4 Definition. Let $G$ be a locally-finite group of cardinality $\aleph_0$.

Index the elements of $G$ with $\omega_0$, so $G = \{h_\alpha \mid \alpha < \omega_0\}$. For each $\beta \leq \omega_0$, let $H_\beta := \langle h_\alpha \mid \alpha < \beta \rangle$. Let $(G_\alpha \mid \alpha < \omega_0)$ be the subsequence of $(H_\alpha \mid \alpha \leq \omega_0)$ obtained by omitting each term which is equal to an earlier term.

Notice that $G_{\omega_0} = G$ and, for each $\alpha < \omega_0$, $|G_\alpha| < \aleph_0$. Moreover, $G_0 = 1$, and, for each $\alpha < \omega_0$, $G_{\alpha+1} = (G_\alpha, g_\alpha)$, where $g_\alpha = h_{\alpha'}$ and $\alpha'$ is the least ordinal such that $h_{\alpha'} \notin G_\alpha$.

We define the line, denoted $\mathbb{R}$, to be the tree with vertices $(v_n \mid n \in \mathbb{Z})$, (oriented) edges $(e_n \mid n \in \mathbb{Z})$, and incidence relations $\iota e_n = v_n$, $\tau e_n = v_{n+1}$, for all $n \in \mathbb{Z}$.

We define the half-line, denoted $\mathbb{R}^+$, to be the subtree of $\mathbb{R}$ with vertices $(v_\alpha \mid \alpha < \omega_0)$, and edges $(e_\alpha \mid \alpha < \omega_0)$.

There is a $G$-tree $X$ with $G$-transversal $\mathbb{R}^+$ such that the $G$-stabilizer of $v_\alpha$, and of $e_\alpha$, is $G_\alpha$, for all $\alpha < \omega_0$. This completely specifies $X$. It is not difficult to see that $X$ is a locally-finite tree with one end.

We denote the set of vertices (resp. edges) of $X$ by $V X$ (resp. $EX$). \hfill $\square$

The following is fairly standard, but we do not know an explicit reference.

4.5 Theorem. If $G$ is a locally-finite group of cardinality $\aleph_0$, then $H^*(G, AG)$ is concentrated in degree 1, $H^1(G, AG)$ contains a $\mathbb{Z}$-submodule isomorphic to $A^{\aleph_0}$, and $|H^1(G, AG)| = |A|^{\aleph_0} = |A|^{|G|}$.

Proof. By (1.3) and Theorem 3.10, $H^*(G, AG)$ is concentrated in degree 1, so it remains to study $H^1(G, AG)$.

Let $X$ be as in Definition 4.4.

By Theorem 4.3, $H^1(G, AG) = H^1(A_G[[X]])$.

Let $Q$ denote $\{e_\alpha \mid \alpha < \omega_0\}$, so we can view

$$A[[Q]] \subseteq A_G[[EX]] = C^1(A_G[[X]]) = Z^1(A_G[[X]])$$.

Let $\mathcal{P}$ be a partition of $\omega_0$ into $\aleph_0$ subsets, each of cardinality $\aleph_0$. We denote the map $\omega_0 \to \mathcal{P}$ by $\alpha \mapsto [\alpha]$.

Consider any $\phi \in A^\mathcal{P}$, $[\alpha] \mapsto \phi[\alpha]$.
Define
\[ \phi^\dagger := \sum_{\alpha < \omega_1} \phi[\alpha].e_\alpha \in A[[Q]] \subseteq Z^1(A_G[[X]]). \]

We claim that if \( \phi^\dagger \in B^1(A_G[[X]]) \) then \( \phi = 0 \).

Suppose not, so there exists \( \psi \in C^0(A_G[[X]]) = A_G[[VX]] \) such that \( \partial^*_\psi = \phi^\dagger \), and there exists \( p \in P \) such that \( \phi(p) \neq 0 \).

There exists \( \mu < \omega_0 \) such that \( \text{supp}(\psi) \cap G\nu_0 \subseteq G_\mu \nu_0 \).

There exists \( \alpha \in p \) such that \( \alpha > \mu \). Notice \( \phi[\alpha] = \phi(p) \neq 0 \).

In \( \mathbb{Z}X \), \( \partial(\sum_{i=0}^{\alpha} e_i) = v_{\alpha+1} - v_0 \). Hence \( \partial((g_\alpha - g_{\alpha-1}) \sum_{i=0}^{\alpha} e_i) = -(g_\alpha - g_{\alpha-1})v_0 \), since \( g_\alpha, g_{\alpha-1} \in G_{\alpha+1} = G_{\nu_0+1} \).

We can view \( \psi \) as an additive map \( \mathbb{Z}X \to A \), and apply it to the foregoing to get \( \phi^\dagger((g_\alpha - g_{\alpha-1}) \sum_{i=0}^{\alpha} e_i) = \psi(-(g_\alpha - g_{\alpha-1})v_0) = 0 \), since \( g_\alpha, g_{\alpha-1} \notin G_{\alpha-1} \supseteq G_\mu \).

Thus
\[ 0 = \phi^\dagger((g_\alpha - g_{\alpha-1}) \sum_{i=0}^{\alpha} e_i) = \phi^\dagger(\sum_{i=0}^{\alpha} g_\alpha e_i) - \phi^\dagger(\sum_{i=0}^{\alpha-1} g_{\alpha-1} e_i) - \phi^\dagger(e_\alpha) = 0 - 0 - \phi(p). \]

Hence \( \phi(p) = 0 \), which is a contradiction.

This proves that the composition
\[ A^P \to A[[Q]] \to H^1(A_G[[X]]) = H^1(G, AG) \]

is injective. Since \( |P| = \aleph_0 \), \( A^{\aleph_0} \) embeds in \( H^1(G, AG) \). □

5. Locally-finite groups of cardinality \( \aleph_1 \)

In this section we study \( H^*(G, AG) \) when \( G \) is locally finite with \( \aleph \text{-rank}(G) = 1 \), topologizing and refining results of D. J. Holt.

We begin by constructing a standardized \( G \).

5.1 Definitions. Let \( G \) be a locally-finite group of cardinality \( \aleph_1 \).

Let \( \omega_1' \) denote the set of limit ordinals less than \( \omega_1 \).

As in Definition 4.4, we start by indexing the elements of \( G \), \( G = \{ h_\alpha \mid \alpha < \omega_1 \} \), set \( H_\beta := \langle h_\alpha \mid \alpha < \beta \rangle \) for each \( \beta \leq \omega_1 \), and let \( (G_\alpha \mid \alpha \leq \omega_1) \) be the subsequence of \( (H_\alpha \mid \alpha \leq \omega_1) \) obtained by omitting each term which is either finite, or equal to an earlier term.

Notice \( (G_\alpha \mid \alpha \leq \omega_1) \) is a continuous chain of subgroups of \( G \), \( G_{\omega_1} = G \), and, for each \( \alpha < \omega_1 \), \( |G_\alpha| = \aleph_0 \) and \( G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle \), where \( g_\alpha = h_\alpha' \), and \( \alpha' \) is the least ordinal such that \( h_\alpha' \notin G_\alpha \).

For each subgroup \( H \) of \( G \), we set \( H^* := G_0 \cap H \).

We shall now construct a family \( (G_{\alpha,n} \mid \alpha < \omega_1, n < \omega_0) \) of finite subgroups of \( G \) such that the following hold:

1. for each \( \alpha < \omega_1 \), \( G_{\alpha,n} \mid n < \omega_0 \) is an increasing chain with union \( G_\alpha \);
2. for each \( \alpha < \omega_1 \), \( n < \omega_0 \), \( G_{\alpha+1,n} = \langle G_{\alpha,n}, g_\alpha \rangle \);
3. for each \( \alpha \in \omega_1', n < \omega_0 \), there is a distinguished element
   \[ g_{\alpha,n} \in G_{\alpha,n+1}^* - G_{\alpha,n+1}^* (= G_0 \cap (G_{\alpha,n+1} - G_{\alpha,n})); \]
4. for each \( \alpha \in \omega_1', n < \omega_0 \), \( G_{\alpha+1,n} \cap G_\alpha = G_{\alpha,n} \).
We proceed as follows.

First, choose an arbitrary chain \((G_{0,n} \mid n < \omega_0)\) of finite subgroups with union \(G_0\).

For \(\alpha \in \omega\), choose an arbitrary chain \((\tilde{G}_{\alpha,m} \mid m < \omega_0)\) of finite subgroups with union \(G_\alpha\). For each \(m < \omega_0\), define \(\tilde{G}_{\alpha+1,m} := (\tilde{G}_{\alpha,m},g_\alpha)\). Choose any increasing function \(\omega_0 \to \omega_0, n \mapsto m_n\), such that the chain \((\tilde{G}_{\alpha+1,m_n} \mid n < \omega_0)\) of finite subgroups of \(G_0\) with union \(G_0\) is strictly increasing. For \(n < \omega_0\), define

\[
G_{\alpha+1,n} := \tilde{G}_{\alpha+1,m_n} \quad \text{and} \quad G_{\alpha,n} := G_\alpha \cap G_{\alpha+1,n}.
\]

Notice that (4) holds, and that we may assume that (3) holds. Also \(G_{\alpha,n} \supseteq \tilde{G}_{\alpha,m_n}\), so

\[
G_{\alpha+1,n} = \tilde{G}_{\alpha+1,m_n} = (\tilde{G}_{\alpha,m_n},g_\alpha) \subseteq (G_{\alpha,n},g_\alpha) \subseteq G_{\alpha+1,n}.
\]

For \(\alpha < \omega\), and \(n < \omega_0\), if \(G_{\alpha,n}\) is defined, set \(G_{\alpha+1,n} := (G_{\alpha,n},g_\alpha)\); this completes the recursive definition of the family \((G_{\alpha,n} \mid \alpha < \omega, n < \omega_0)\), and we see that (1) and (2) also hold.

For \(\alpha < \omega\), let \(Y_\alpha\) denote the \(G_\alpha\)-tree of Definition 4.4 corresponding to the chain \((G_{\alpha,n} \mid n < \omega_0)\).

We define the plane, denoted \(\mathbb{R}^2\), to be the two-dimensional CW-complex with vertices \((v_m,n \mid (m,n) \in \mathbb{Z}^2)\), and edges \((x_{m,n},y_{m,n} \mid (m,n) \in \mathbb{Z}^2)\), and faces \((f_{m,n} \mid (m,n) \in \mathbb{Z}^2)\), and incidence relations given by, for \((m,n) \in \mathbb{Z}^2\),

\[
\ell x_{m,n} = \tau x_{m-1,n} = \ell y_{m,n} = \tau y_{m,n-1} = v_{m,n}
\]

and \(f_{m,n}\) is attached along the path \(x_{m,n},y_{m+1,n},x_{m,n+1},y_{m,n}\).

We define the semi-infinite strip, denoted \([0,1] \times \mathbb{R}^+\), to be the subcomplex of \(\mathbb{R}^2\) with vertices \((v_{0,n},v_{1,n} \mid n < \omega_0)\), edges \((x_{0,n},y_{0,n},y_{1,n} \mid n < \omega_0)\), and faces \((f_{0,n} \mid n < \omega_0)\).

Notice there are two distinguished subcomplexes of \([0,1] \times \mathbb{R}^+\) which are isomorphic to \(\mathbb{R}^+\), and will be denoted \(\{0\} \times \mathbb{R}^+\) and \(\{1\} \times \mathbb{R}^+\).

Let \(\alpha < \omega\). We construct a two-dimensional \(G_{\alpha+1}\)-space \(M_\alpha\), for which \([0,1] \times \mathbb{R}^+\) is a \(G_{\alpha+1}\)-transversal, and the \(G_{\alpha+1}\)-stabilizer of \(v_{0,n},x_{0,n},y_{0,n}\), and \(f_{0,n}\) is \(G_{\alpha,n}\), while the \(G_{\alpha+1}\)-stabilizer of \(v_{1,n}\) and \(y_{1,n}\) is \(G_{\alpha+1,n}\). This completely specifies \(M_\alpha\). Notice that \(M_\alpha\) is the mapping cylinder of a \(G_{\alpha+1}\)-map

\[
G_{\alpha+1} \times G_\alpha \ Y_\alpha \to Y_{\alpha+1}.
\]

If \(\alpha \in \omega\), then \(Y_\alpha\) is a \(G_\alpha\)-subtree of \(Y_{\alpha+1}\).

To construct \(X\) we glue together

\[
(G \times G_\alpha \ M_\alpha \mid \alpha < \omega_1)
\]

amalgamating

\[
(G \times G_\alpha \ Y_\alpha \mid \alpha < \omega_1),
\]

as in the proof of Theorem 2.4. The image in \(X_{\alpha+1} \subseteq X\) of \(f_{0,n} \in M_\alpha\) will be denoted \(f_{\alpha,n}\).
We denote the set of vertices (resp. edges, resp. faces) of \( X \) by \( VX \) (resp. \( EX \), resp. \( FX \)).

Let us recall the notion of a club (= closed unbounded subset). Thus, an \( \omega_1 \)-club is any subset \( S \) of \( \omega_1 \) such that the set of the least upper bounds of the nonempty subsets of \( S \) is precisely \( S \cup \{\omega_1\} \). If \( S \) is an \( \omega_1 \)-club, then so is \( S \cap \omega'_1 \); recall that \( \omega'_1 \) denotes the set of limit ordinals in \( \omega_1 \).

For \( \phi \in A_G[[X]] \) and \( \alpha \leq \omega_1 \), if \( \text{supp}(\phi) \cap GX_\alpha \subseteq X_\alpha \), we say that \( \phi \) respects \( \alpha \). It is straightforward to show that the set of ordinals in \( \omega_1 \) respected by \( \phi \) is an \( \omega_1 \)-club. \( \square \)

**5.2 Lemma** (Holt [6]). Let \( G \) be a periodic group, and let \( H, K \) be proper subgroups of \( G \) which generate \( G \). Let \( X \) be the \( G \)-graph with vertex set the disjoint union of \( G/H \) and \( G/K \), and edge set \( G/(H \cap K) \), with \( g(H \cap K) \) joining \( gH \) to \( gK \), for each \( g \in G \). Then \( X \) is connected, and deleting the two vertices \( H \) and \( K \), and the one edge \( H \cap K \), leaves a connected space.

**Proof.** Collapsing all the edges of \( X \) leaves a transitive \( G \)-set, and one of the points is fixed by \( H \) and \( K \). Since \( H \) and \( K \) generate \( G \), this point is fixed by \( G \), so forms a \( G \)-orbit. Thus we have only one point, so \( X \) is connected.

Let \( Y \) be the subgraph of \( X \) obtained by deleting the vertices \( H \) and \( K \) and all their incident edges. It suffices to show that \( Y \) is a connected graph. Since \( X \) is connected, it suffices to show that each \( X \)-neighbour of \( H \) is \( Y \)-connected to each \( X \)-neighbour of \( K \). Thus, let \( h \in H - K \), and \( k \in K - H \); it suffices to show that the vertices \( hK \) and \( kH \) are \( Y \)-connected.

Let \( L = \langle hk^{-1} \rangle = \langle kh^{-1} \rangle \). Since \( G \) is periodic, \( L \) is finite.

We consider the action of \( L \) on \( X \). Let \( m \) and \( n \) denote the orders of the \( L \)-orbits of the vertices \( H \) and \( K \), respectively. By symmetry, we may assume that \( m \geq n \).

Notice that \( L \) is not contained in \( K \), so \( n \geq 2 \).

Let \( g = kh^{-1} \). In \( X \), there is an edge \( H \cap K \) joining \( H \) to \( K \), and an edge \( k(H \cap K) \) joining \( kh = gH \) to \( kK = K \). Applying powers of \( g \) to these, we get a path in \( X \) with vertices

\[ H, K, gH, gK, g^2H, g^2K, \ldots, g^{n-1}H, g^{n-1}K. \]

By the definition of \( m \) and \( n \), these \( 2n \) vertices are all distinct, so, on deleting the first two, we get a path in \( Y \) connecting \( gH = kH \) to \( g^{n-1}K = g^{-1}K = hK \). \( \square \)

**5.3 Theorem** (Holt [6]). If \( G \) is locally finite, and \( |G| = \aleph_1 \), then \( H^1(G, AG) = 0 \).

**Proof.** Let \( X \) be as in Definitions 5.1.

Consider any \( \phi \in Z^1(A_G[[X]]) \). Thus \( \text{supp}(\phi) \) is a collection of edges of \( X \), with only finitely many in each \( G \)-orbit. A subset of \( X \) which meets (that is, has nonempty intersection with) \( \text{supp}(\phi) \) is said to be broken by \( \phi \). Since \( \phi \) is a 1-cocycle, we get 0 if we sum, in \( A \), the \( \phi \)-labels, with the appropriate signs, around any face, or along any closed path in the 1-skeleton, since \( X \) is simply-connected. Thus there is a well-defined \( \phi \)-sum from any vertex to any other vertex.

Consider any \( \alpha \in \omega'_1 \) such that \( \phi \) respects \( \alpha \) as in the last paragraph of Definitions 5.1.
From Definitions 5.1, there is a cellular \( G_{\alpha+1} \)-map \( M_{\alpha+1} \to X \), so \( \phi \) induces an element \( \phi_{\alpha+1} \in A[[M_{\alpha+1}]] \). Since the \( G \)-stabilizers for \( X \) are finite, \( \phi_{\alpha+1} \) lies in \( A_{G_{\alpha+1}}[[M_{\alpha+1}]] \). Moreover, \( \phi_{\alpha+1} \) respects \( \alpha \) in the obvious sense, since \( Y_\alpha \) is mapped to \( X_\alpha \) by construction.

There exists \( n_0 < \omega_0 \) such that \( \text{supp}(\phi_{\alpha+1}) \cap G_{\alpha+1}x_{0,0} \subseteq G_{\alpha+1,n_0}x_{0,0} \). This means that, for \( g \in G_{\alpha+1} \), if \( \phi_{\alpha+1} \) breaks \( gx_{0,0} \), then the terminal vertex of \( gx_{0,0} \) lies in \( G_{\alpha+1,n_0}v_{1,0} \).

Now consider any \( n \) such that \( n_0 < n < \omega_0 \).

Let \( p_n \) denote the reduced open path in the tree \( \{0\} \times \mathbb{R}^+ \) from \( v_{0,0} \) to \( v_{0,n} \), and \( e \cdot p_n \) the open path obtained by concatenating \( e : = x_{0,0}^{-1} \) and \( p_n \). We are interested in the \( G_{\alpha+1,n} \)-graph \( Z \) generated by the closure \( \overline{e \cdot p_n} \).

Consider any \( g \in G_{\alpha+1,n} \).

If \( \phi_{\alpha+1} \) breaks \( gp_n \) then \( g \in G_\alpha \), since \( \phi_{\alpha+1} \) respects \( \alpha \), and hence \( g \in G_{\alpha,n} \), so \( g v_{0,n} = v_{0,n} \). That is, if \( \phi \) breaks \( gp_n \), then the terminal vertex of \( gp_n \) is \( v_{0,n} \).

We apply Lemma 5.2 to the graph \( Y' \) obtained by taking \( H^- = G_{\alpha+1,0} \) and \( K^- = G_{\alpha,n_0} \), so \( \langle H^-, K^- \rangle = \langle G_{\alpha,n_0} \rangle = G_{\alpha+1,n_0} \). By Lemma 5.2, deleting \( g^{-1}K^- \) from \( Y' \) leaves a connected space containing \( G_{\alpha+1,n_0} / H^- \), where we include the trivial case where \( g^{-1}K^- \) does not belong to \( Y' \).

Let \( Y \) be the \( G_{\alpha+1,n_0} \)-subspace of \( M_{\alpha+1} \) generated by \( \overline{e \cdot p_n} \), and consider the \( G_{\alpha+1,n_0} \)-map from \( Y \) to \( Y' \) which assigns \( v_{1,0} \) to \( H^- \), \( v_{0,n} \) to \( K^- \), and \( e \cdot p_n \) to \( H^- \cap K^- \). It follows that deleting \( g^{-1}K^- \overline{p_n} \) from \( Y \) leaves a connected space containing \( G_{\alpha+1,n_0}v_{1,0} \).

Suppose \( g \notin G_{\alpha+1,n_0} \), so \( g \in G_{\alpha+1,n} - G_{\alpha+1,n_0} \). Then \( \text{supp}(\phi_{\alpha+1}) \cap gY \subseteq K^- p_n \). Hence, on deleting \( \text{supp}(\phi_{\alpha+1}) \) from the 1-skeleton of \( M_{\alpha+1} \), one of the resulting components contains \( g G_{\alpha+1,n_0}v_{1,0} \).

Next, we apply Lemma 5.2 to the graph \( Z' \) obtained by taking \( H = G_{\alpha+1,n_0} \) and \( K = G_{\alpha,n} \), so \( \langle H, K \rangle = G_{\alpha+1,n} \). We conclude that

\[ Z' = (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K)) \]

is a connected graph.

Let \( Z \) be the \( G_{\alpha+1,n} \)-subspace of \( M_{\alpha+1} \) generated by \( \overline{e \cdot p_n} \), and consider the map of \( G_{\alpha+1,n} \)-spaces from \( Z \) to \( Z' \) which assigns \( v_{1,0} \) to \( H \), \( v_{0,n} \) to \( K \), and \( e \cdot p_n \) to \( H \cap K \). There is induced a surjective map

\[ Z - (\{v_{1,0}\} \cup \{v_{0,n}\} \cup (H \cup K)(e \cdot p_n)) \to Z' - (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K)). \]

Notice that \( \phi_{\alpha+1} \) breaks only edges of \( Z \) which lie in \( He \cup Kp_n \), so there is a map from \( Z - (\{v_{1,0}\} \cup \{v_{0,n}\} \cup (H \cup K)(e \cdot p_n)) \) to the set of components of the 1-skeleton of \( M_{\alpha+1} \) - \( \text{supp}(\phi_{\alpha+1}) \). Moreover, we have seen that each subset \( gHv_{1,0} \) maps to a component of the 1-skeleton of \( M_{\alpha+1} \) - \( \text{supp}(\phi_{\alpha+1}) \). Thus the map factors through \( Z' - (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K)) \), which is connected, so maps to a single component. Hence some component \( X' \) of the 1-skeleton of \( M_{\alpha+1} \) - \( \text{supp}(\phi_{\alpha+1}) \) contains \( \langle H, K \rangle \cap Hv_{1,0} \), that is, \( (G_{\alpha+1,n} - G_{\alpha+1,n_0})v_{1,0} \).

Since \( n > n_0 \) was arbitrary, all of \( (G_{\alpha+1} - G_{\alpha+1,n_0})v_{1,0} \) is contained in \( X' \). Thus, for any path between any two elements of \( (G_{\alpha+1} - G_{\alpha+1,n_0})v_{1,0} \), the \( \phi_{\alpha+1} \)-sum, and hence the \( \phi \)-sum, is 0.
Let \( \psi_{\alpha+1} \in C^0(A[[X_{\alpha+1}]]) \) be defined on each vertex \( v \) as the \( \phi \)-sum along any path from any vertex of \( (G_{\alpha+1} - G_{\alpha+1,n}) v_{1,0} \) to \( v \). Then \( \psi_{\alpha+1} \in C^0(A_{G_{\alpha+1}}[[X_{\alpha+1}]]) \) and \( \phi|_{X_{\alpha+1}} = \partial^*(\psi_{\alpha+1}) \).

The \( \alpha < \omega_1 \) which are respected by \( \phi \) converge to \( \omega_1 \), and it follows that the corresponding (unique) \( \psi_{\alpha+1} \) converges to an element \( \psi \in C^0(A_G[[X]]) \) such that \( \phi = \partial^*(\psi) \), so \( \phi \in B^1(A_G[[X]]) \).

Hence \( H^1(A_G[[X]]) = 0 \), so \( H^1(G, AG) = 0 \). \( \square \)

Up until now, in this section, we have given a straightforward topological translation of [6], which we felt illuminated the arguments. We now come to a new result.

**5.4 Theorem.** If \( G \) is locally finite, and \( |G| = \aleph_1 \), then \( H^2(G, AG) \) has a subgroup isomorphic to \( A^{\aleph_1} \), and \( |H^2(G, AG)| = |A^G| = |A|^{\aleph_1} \).

**Proof.** Let \( X \) be as in Definitions 5.1.

Let \( \alpha \in \omega_1^\prime \).

Let \( M'_\alpha \) denote the \( G_\alpha \)-space with \( G_\alpha \)-transversal \([-1, 0] \times \mathbb{R}^+\), where the \( G_\alpha \)-stabilizers of \( v_{-1, n}, y_{-1, n}, x_{-1, n} \) and \( f_{-1, n} \) are \( G_\alpha^\ast = G_0 \cap G_\alpha \), while the \( G_\alpha \)-stabilizers of \( v_{0, n} \) and \( y_{0, n} \) are \( G_\alpha \). This completely specifies \( M'_\alpha \).

Notice that \( M'_\alpha \) is the mapping cylinder of an injective \( G_\alpha \)-map \( G_\alpha \times G_\alpha Y_\alpha^\ast \to Y_\alpha \), where \( Y_\alpha^\ast \) is the \( G_\alpha \)-tree of Definition 4.4 corresponding to the chain \( (G_\alpha^\ast, n \mid n < \omega_0) \), a \( G_0 \)-subtree of \( Y_\alpha \).

We have a cellular \( G_\alpha \)-map \( Y_\alpha \to X_\alpha \), and, similarly, by Corollary 2.3, we can construct a cellular \( G_0 \)-map \( Y_\alpha^\ast \to Y_\alpha \) between spaces of type \( \mathbb{E}G_0 \). These two maps can be extended to a \( G_\alpha \)-map \( M'_\alpha \to X_\alpha \).

Notice that \( Y_\alpha \) is contained in both \( M_\alpha \) and \( M'_\alpha \), and the map \( Y_\alpha \to X_\alpha \) has been extended to \( M_\alpha \to X_{\alpha+1} \) and to \( M'_\alpha \to X_\alpha \).

For each \( n < \omega_0 \), let \( \lambda'_{\alpha,n} \) denote the least ordinal such that the \( X_{\lambda'_{\alpha,n}+1} \) contains the image of the face \( f_{-1, n} \), under the map \( M'_\alpha \to X_\alpha \). Notice that \( \lambda'_{\alpha,n} + 1 < \alpha \), since \( \alpha \) is a limit ordinal. Choose a strictly ascending sequence \( (\lambda_{\alpha,n} \mid n < \omega_0) \) with limit \( \alpha \), such that \( \lambda_{\alpha,n} > \max\{\lambda'_{\alpha,i} \mid 0 \leq i \leq n\} \). Let \( h_{\alpha,n} := g_{\lambda_{\alpha,n}} \in G_{\lambda_{\alpha,n}+1} - G_{\lambda_{\alpha,n}} \).

Let \( Q = \{h_{\alpha,n} f_{\alpha,n} \mid (\alpha, n) \in \omega_1 \times \mathbb{N}\} \), so

\[
A[[Q]] \subseteq A_G[[FX]] = C^2(A_G[[X]]) = Z^2(A_G[[X]]).
\]

Recall that a subset \( S \) of \( \omega_1 \) is said to be \( \omega_1 \)-stationary if \( S \) meets each \( \omega_1 \)-club. Let \( P \) be a partition of \( \omega_1^\prime \) into \( \aleph_1 \) subsets, each being \( \omega_1 \)-stationary; see, for example, [7, Lemma 7.6, p.59].

Consider any \( \phi \in Ap, \{\alpha\} \mapsto \phi(\alpha) \).

Define

\[
\phi^\dagger := \sum_{(\alpha, n) \in \omega_1^\prime \times \mathbb{N}} \phi(\alpha) h_{\alpha,n} f_{\alpha,n} \in A[[Q]] \subseteq Z^2(A_G[[X]]).
\]

It is easy to see that \( \phi^\dagger \) respects all the ordinals in \( \omega_1 \).

We claim that if \( \phi^\dagger \in B^2(A_G[[X]]) \), then \( \phi = 0 \).
Suppose not, so there exists $\psi \in C^1(A_G[[X]]) = A_G[[EX]]$, such that $\partial^* \psi = \phi^+$, and there exists $p \in \mathcal{P}$ such that $\phi(p) \neq 0$. We shall obtain a contradiction.

Since $Y_0$ is countable, there exists $\mu < \omega_1$ such that $\text{supp}(\psi) \cap GY_0 \subseteq G\mu Y_0$.

Since $p$ is $\omega_1$-stationary, and the set of ordinals respected by $\psi$ is an $\omega_1$-club, there exists $\alpha \in p$ such that $\alpha > \mu$ and $\psi$ respects $\alpha$. Notice $\phi[\alpha] = \phi(p) \neq 0$.

Now $\phi^+$ and $\psi$ induce elements $\phi_{\alpha+1}$ and $\psi_{\alpha+1}$, respectively, in $A_{G_{\alpha+1}}[[M_\alpha]]$, and, moreover, $\partial^* \psi_{\alpha+1} = \phi_{\alpha+1}$ in $A_{G_{\alpha+1}}[[M_\alpha]]$. Notice $\phi_{\alpha+1} = \sum_{n<\omega_0} \phi(p)_n f_{\alpha,n}$.

Let $\xi := - \sum_{n<\omega_0} \phi(p)_n h_{\alpha,n} y_{0,n} \in A_G[[EY_\alpha]] \subseteq A_{G_{\alpha+1}}[[M_\alpha]]$. Then $\partial^*(\psi_{\alpha+1} - \xi) = 0$. Moreover, $\psi_{\alpha+1} - \xi$ respects $\alpha$, since both $\xi$ and $\psi_{\alpha+1}$ respect $\alpha$. By the proof of Theorem 5.3, there exists $\nu < \alpha$ such that the $(\psi_{\alpha+1} - \xi)$-sum along any path in $Y_\alpha$ between any two vertices in $(G_\alpha - G_\nu) v_{0,0}$ is zero.

There exists $\kappa < \alpha$ such that $\psi_\alpha$ vanishes on $(G_\alpha - G_\kappa) x_{-1,0}$.

Choose $n < \omega_0$ such that $\lambda_{\alpha,n}$ is greater than $\mu$, $\nu$ and $\kappa$. Choose $g \in G_{\alpha,n+1} - G_{\alpha,n}$. Thus $g \in G_0$, $g$ fixes $x_{-1,n+1}$, and $g$ moves $y_{0,n}$.

In $\mathbb{Z}[M_\alpha]$,

$$\partial(\sum_{i=0}^n f_{-1,i}) = x_{-1,0} - x_{-1,n+1} + \sum_{i=0}^n (y_{0,i} - y_{-1,i}).$$

Hence

$$\partial(h_{\alpha,n}(1-g) \sum_{i=0}^n f_{-1,i}) = h_{\alpha,n}(1-g)(x_{-1,0} + \sum_{i=0}^n (y_{0,i} - y_{-1,i})).$$

We can view $\psi_\alpha$ as an additive map $\mathbb{Z}[EM_\alpha] \to A$, and apply it to the foregoing equation, to get

$$\phi_\alpha(h_{\alpha,n}(1-g) \sum_{i=0}^n f_{-1,i})$$

$$= \psi_\alpha(h_{\alpha,n}(1-g)(x_{-1,0} + \sum_{i=0}^n (y_{0,i} - y_{-1,i}))).$$

Notice that $h_{\alpha,n}G_{\lambda_{\alpha,n}} \cap G_{\lambda_{\alpha,n}} = \emptyset$.

Since $h_{\alpha,n}h_{\alpha,n}g \notin G_{\lambda_{\alpha,i}}$, for $0 \leq i \leq n$, we see $\phi_\alpha(h_{\alpha,n}(1-g) \sum_{i=0}^n f_{-1,i}) = 0$.

Also, $h_{\alpha,n}h_{\alpha,n}g \notin G_\mu$, so $\psi_\alpha(h_{\alpha,n}(1-g)y_{-1,i}) = 0$.

Also, $h_{\alpha,n}h_{\alpha,n}g \notin G_\kappa$, so $\psi_\alpha(h_{\alpha,n}(1-g)x_{-1,0}) = 0$.

It follows that $\psi_\alpha(h_{\alpha,n}(1-g) \sum_{i=0}^n y_{0,i}) = 0$. 


But $h_{α,n}h_{α,n}g \notin G_ν$, so $\left(\psi_α - ξ(h_{α,n}(1 - g)\sum_{i=0}^{n} y_{0,i})\right) = 0$. Hence

$$ξ\left(\sum_{i=0}^{n} h_{α,n}(1 - g)y_{0,i}\right) = 0.$$ 

Now $ξ(h_{α,n}y_{0,n}) = -φ(p)$, and $ξ$ vanishes on all other summands because $h_{α,n}G_0 \cap G_{λ_α,i+1} = ∅$ for $0 \leq i \leq n - 1$, and $gy_{0,n} \neq y_{0,n}$. Hence $φ(p) = 0$, which is a contradiction.

Since $|P| = ℵ_1$, we have an embedding of $A^{ℵ_1}$ in $H^2(G, AG)$. □

6. Cohomology of directed unions

In this section, we recall some known results about cohomology for well-ordered directed unions, with special emphasis on abelian groups.

6.1 Notation. We let $(P(G, \partial))$ denote the bar resolution for $G$, and let $P_n(G)$ denote its $n$th component, for each $n \in \mathbb{Z}$. Thus $(P(G, \partial))$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and, for $n ≥ 0$, $P_n(G)$ has as $\mathbb{Z}$-basis the Cartesian power $G^{n+1}$, with $G$ acting by left multiplication on the first coordinate, and, for $n ≥ 1$,$$
∂_n(g_0, \ldots, g_n) := \sum_{i=0}^{n-1} (-1)^i (g_0, \ldots, g_i-1, g_i, g_{i+1}, g_{i+2}, \ldots, g_n) + (-1)^n (g_0, \ldots, g_{n-1}).$$

As usual, if $n ≤ -1$, $P_n(G) = 0$, and, if $n ≤ 0$, $∂_n = 0$. □

The following is a degenerate case of the cohomology spectral sequence for well-ordered directed unions; see, for example [13, Section 3].

6.2 Lemma (Robinson [13, Proposition 1]). Let $n \in \mathbb{N}$, let $M$ be a $\mathbb{Z}G$-module, let $β$ be a limit ordinal, and let $(G_α | α ≤ β)$ be a continuous chain of subgroups of $G$. If $H^{n-1}(G_α, M) = 0$ for all $α < β$, then $H^n(G_β, M) = \lim_{α < β} H^n(G_α, M)$.

Proof. For each $α ≤ β$, view the bar resolution $P(G_α)$ as a $\mathbb{Z}$-subcomplex of $P(G_β)$, so $(P_n(G_α) | α ≤ β)$ is a continuous chain.

We want to show that the natural map

$$(6.3) \quad H^n(G_β, M) \rightarrow \lim_{α < β} H^n(G_α, M)$$

is bijective.

We begin by showing it is injective.

Consider any element $ξ$ of the kernel of (6.3). Then $ξ$ is represented by an $n$-cocycle $φ_β: P_n(G_β) → M$, so $φ_β$ is $\mathbb{Z}G_β$-linear, and $φ_β \circ ∂_{n+1} = 0$.

Consider any $α ≤ β$. Let $φ_{β, α}$ denote the restriction of $φ_β$ to $P_n(G_α)$. We shall construct, transfinitely, a continuous directed system of maps

$$(ψ_{β, α}: P_{n-1}(G_α) → M | α ≤ β)$$
such that $\psi_{\beta, \alpha}$ is $\mathbb{Z}G_\alpha$-linear, and $\psi_{\beta, \alpha} \circ \partial_n = \phi_{\beta, \alpha}$. It will then follow that $\phi_{\beta} = \phi_{\beta, \beta} = \psi_{\beta, \beta} \circ \partial_n$ is a coboundary, that $\xi = 0$, and that (6.3) is injective.

If $\alpha < \beta$, then $\phi_{\beta, \alpha}$ represents an element $\xi_\alpha$ of $H^n(G_\alpha, M)$, and, since $\xi$ lies in the kernel of (6.3), $\xi_\alpha = 0$, so there exists a $\mathbb{Z}G\alpha$-linear map 

$$\psi_\alpha : P_{n-1}(G_\alpha) \to M$$

such that $\psi_\alpha \circ \partial_n = \phi_{\beta, \alpha}$.

Let $\psi_{\beta, 0} = \psi_0$.

If $\alpha < \beta$ and we have constructed $\psi_{\beta, \alpha}$, then we construct $\psi_{\beta, \alpha+1}$ as follows. Let $\psi_{\alpha+1, \alpha}$ denote the restriction of $\psi_{\alpha+1}$ to $P_{n-1}(G_\alpha)$. Since $\psi_{\alpha+1} \circ \partial_n = \phi_{\beta, \alpha+1}$, we see that $\psi_{\alpha+1, \alpha} \circ \partial_n = \phi_{\beta, \alpha} = \psi_{\beta, \alpha} \circ \partial_n$.

Thus $\psi_{\beta, \alpha} - \psi_{\alpha+1, \alpha}$ is an $(n-1)$-cocycle, so represents an element of $H^{n-1}(G_\alpha, M)$. By hypothesis, this element is zero, so there exists a $\mathbb{Z}G_\alpha$-linear map $\mu_\alpha : P_{n-2}(G_\alpha) \to M$ such that $\psi_{\beta, \alpha} - \psi_{\alpha+1, \alpha} = \mu_\alpha \circ \partial_n$. Let $\mu_{\alpha, \alpha+1} : P_{n-2}(G_{\alpha+1}) \to M$ denote the unique $\mathbb{Z}G_{\alpha+1}$-linear map which is $\mu_\alpha$ on $P_{n-2}(G_\alpha)$, and is zero on $G_{\alpha+1}^{-1} - (G_{\alpha+1} \times G_{\alpha+2}^{-2})$. Now define

$$\psi_{\beta, \alpha+1} : = \psi_{\alpha+1} + \mu_{\alpha, \alpha+1} \circ \partial_n.$$ 

By construction, $\psi_{\beta, \alpha+1}$ acts on $P_{n-1}(G_\alpha)$ as $\psi_{\alpha+1, \alpha} + \mu_\alpha \circ \partial_n = \psi_{\beta, \alpha}$. Also $\psi_{\beta, \alpha+1} \circ \partial_n = \psi_{\alpha+1} \partial_n + 0 = \phi_{\beta, \alpha+1}$.

If $\alpha$ is a limit ordinal in $\beta + 1$ and we have constructed $(\psi_{\beta, \alpha}) | \alpha' < \alpha$, then the latter has a direct limit which we take to be $\psi_{\beta, \alpha}$.

This completes the construction, so (6.3) is injective.

A similar, but easier, argument shows that (6.3) is surjective, and here the hypothesis on $H^{n-1}$ is not needed. □

A special case of this result appeared in the penultimate paragraph of the proof of Theorem 5.3. We can now say even more.

6.4 Theorem (Holt [6]). If $G$ is locally finite, and $\aleph\text{-rank}(G) \neq 0$, then $H^1(G, AG) = 0$.

Proof. By (1.3), we may assume that $\aleph\text{-rank}(G) \geq 1$, and we proceed by induction on $\aleph\text{-rank}(G)$. If $\aleph\text{-rank}(G) = 1$, the assertion holds by Theorem 5.3. Thus we may assume that $\aleph\text{-rank}(G) \geq 2$ and the result holds for smaller groups. There exists an ordinal $\beta$ and a continuous chain $(G_\alpha | \alpha \leq \beta)$ of subgroups of $G$ with $G_\beta = G$, and $1 \leq \aleph\text{-rank}(G_\alpha) < \aleph\text{-rank}(G)$ for all $\alpha < \beta$.

By (1.3) and the induction hypothesis, $H^0(G_\alpha, AG) = H^1(G_\alpha, AG) = 0$ for all $\alpha < \beta$, since $AG$ is an induced $\mathbb{Z}G_\alpha$-module. By Lemma 6.2, $H^1(G, AG) = 0$. □

6.5 Remark. In light of Theorem 3.10, the method of proof of Theorem 6.4 shows that

“If $n \in \mathbb{N}$, $G$ is locally finite, and $n \neq \aleph\text{-rank}(G) + 1$, then $H^n(G, \mathbb{Z}G \otimes \mathbb{Z} -) = 0$,”

is equivalent to

“If $n \in \mathbb{N}$, $G$ is locally finite, and $n = \aleph\text{-rank}(G)$, then $H^n(G, \mathbb{Z}G \otimes \mathbb{Z} -) = 0$.” □
6.6 Definitions. Let $H$ be a subgroup of $G$, and let $L$, $M$ be $ZH$-modules.

Let $N_G(H)$ denote the normalizer of $H$ in $G$, and let $C_G(H)$ denote the centralizer of $H$ in $G$.

There is a natural action of $N_G(H)$ on $\text{Ext}^*_ZH(L, M)$, where $L$, $M$ are viewed as $ZH$-modules by restriction. Perhaps the simplest way to define this action is as follows. Let $(P, \partial)$ be a projective $ZH$-resolution of $L$, and view $(P, \partial)$ as a projective $ZH$-resolution of $L$. Then any $g \in N_G(H)$ gives rise to an action on $\text{Hom}_{ZH}(P, M)$, $\phi \mapsto \phi^g$, where $\phi^g(p) = g^{-1} \phi(gp))$. Since $g$ normalizes $H$, we see that $\phi^g$ is $ZH$-linear. The $N_G(H)$-action respects cocycles and coboundaries, so induces an action on the cohomology, $\text{Ext}^*_ZH(L, M)$.

In the foregoing, if $\phi$ is $ZH$-linear, then $\phi^g = \phi$. It follows that the image of the restriction map

$$\text{Ext}^*_ZH(L, M) \to \text{Ext}^*_ZH(L, M)$$

is contained in the set $\text{Ext}^*_ZH(L, M)^{N_G(H)}$ of points fixed by $N_G(H)$.

Let $g$ be an element of $C_G(H)$. Left multiplication by $g$ determines a $ZH$-linear endomorphism of $M$ and we denote it by $g \cdot |_M$. Similarly, $g \cdot |_P \in \text{End}_{ZH}(P)$ is a lift of $g \cdot |_L \in \text{End}_{ZH}(L)$. Now $\text{Ext}^*_ZH(L, M)$ is a bimodule over $\text{End}_{ZH}(M)$ and $\text{End}_{ZH}(L)$, and we see that the action of $g$ on $\text{Ext}^*_ZH(L, M)$ is given by $\eta \mapsto (g \cdot |_M)^{-1} \circ \eta \circ (g \cdot |_L)$.

If we restrict to the case where $L = Z$ with trivial $G$-action, we see that the image of the restriction map $H^*(G, M) \to H^*(H, M)$ is contained in $H^*(H, M)^{N_G(H)}$, and hence in $H^*(H, M)^{C_G(H)}$, and here $C_G(H)$ acts by left multiplication on $M$. □

The following is now an immediate consequence of Lemma 6.2

6.7 Corollary. Let $n \in \mathbb{N}$, let $M$ be a $ZH$-module, let $\beta$ be a limit ordinal, and let $(G_{\alpha} \mid \alpha \leq \beta)$ be a continuous chain of subgroups of $G$ with $G_{\beta} = G$. If, for each $\alpha < \beta$, $H^n(G_{\alpha}, M)^{N_G(G_{\beta})} = 0$ and $H^{n-1}(G_{\alpha}, M) = 0$, then $H^n(G, M) = 0$. □

We record consequences for abelian groups which seem to be new.

6.8 Theorem. Let $G$ be an abelian group, $\lambda$ an ordinal, and $\Delta$ a $G$-set with stabilizers of $\aleph$-rank strictly less than $\lambda$.

(1) For each $n \in \mathbb{N}$, if $\aleph$-rank($G$) $\geq \lambda + n$ then $H^n(G, A\Delta) = 0$.

(2) If $\aleph$-rank($G$) $\geq \lambda + \omega_0$ then $H^*(G, A\Delta) = 0$.

Proof. (1). We argue by induction on $n$.

If $n = 0$, then all $G$-stabilizers of elements of $\Delta$ have infinite index in $G$, so $\Delta$ has no finite $G$-orbits. Thus $(A\Delta)^G = 0$, that is, $H^0(G, A\Delta) = 0$.

Now suppose that $n \geq 1$, and that the result holds for smaller $n$. Let $\beta$ denote the least ordinal of cardinality rank($G$), so $\beta$ is a limit ordinal. Moreover, there exists a continuous chain of subgroups $(G_{\alpha} \mid \alpha \leq \beta)$ such that $G_{\beta} = G$, and, for each $\alpha < \beta$,

$$\lambda + n - 1 \leq \aleph\text{-rank}(G_{\alpha}) < \aleph\text{-rank}(G).$$

Consider $\alpha < \beta$.

By the induction hypothesis, $H^{n-1}(G_{\alpha}, A\Delta) = 0$, so, by Corollary 6.7, it remains to show that $H^n(G_{\alpha}, A\Delta)^{N_G(G_{\beta})} = 0$. Here $N_G(G_{\alpha}) = C_G(G_{\alpha}) = G$, since $G$ is abelian. Thus, we want to show that $H^n(G_{\alpha}, A\Delta)^G = 0$ where $G$ is acting via multiplication on $A\Delta$. 
Consider any element \( \zeta \in H^n(G_\alpha, A\Delta) \). We can use the bar resolution \( P(G_\alpha) \), and represent \( \zeta \) by a \( \mathbb{Z}G_\alpha \)-linear map \( \phi: P_n(G_\alpha) \to A\Delta \). Since \( |G_\alpha| < |G| \), we see there is a \( G_\alpha \)-subset \( \Delta' \) of \( \Delta \) such that \( A\Delta' \) contains the image of \( \phi \), and \( |\Delta'| < |G| \).

Let \( \Delta'' \) be the complement of \( \Delta' \) in \( \Delta \). Then \( A\Delta = A\Delta' \oplus A\Delta'' \), so

\[
H^n(G_\alpha, A\Delta) = H^n(G_\alpha, A\Delta') \oplus H^n(G_\alpha, A\Delta''),
\]

and, in the corresponding expression \( \zeta = (\zeta', \zeta'') \), we have \( \zeta'' = 0 \).

Now

\[
|\{g \in G \mid g\Delta' \cap \Delta' \neq \emptyset\}| < |G|
\]

because the elements of \( \Delta \) have \( G \)-stabilizers of cardinality strictly less than \( |G| \).

Hence there exists \( g \in G \) such that \( g\Delta' \cap \Delta' = \emptyset \), that is, \( g\Delta' \subseteq \Delta'' \), so

\[
H^n(G_\alpha, A\Delta')^g \subseteq H^n(G_\alpha, A\Delta'').
\]

It follows that if \( \zeta^g = \zeta \) then \( \zeta = 0 \). This proves that \( H^n(G_\alpha, A\Delta)^G = 0 \), and (1) is proved.

(2) follows from (1). \( \square \)

The case where \( \Delta = G \) is of interest; here we can take \( \lambda = 0 \).

6.9 Corollary. Let \( n \in \mathbb{N} \), and let \( G \) be an abelian group. Then, \( H^n(G, AG) = 0 \) if \( n < \aleph \)-rank(\( G \)) + 1. \( \square \)

6.10 Corollary. Let \( n \in \mathbb{N} \), and let \( G \) be an abelian, torsion group.

(1) (Holt [5]). \( H^n(G, AG) = 0 \) if \( n \neq \aleph \)-rank(\( G \)) + 1.

(2) (Chen [12, Corollary 7.6]). If \( R \) is a nonzero, \( o(G) \)-inverting ring, then \( cd_R G = n \) if and only if \( \aleph \)-rank(\( G \)) = \( n - 1 \), that is,

\[
cd_R G = \min\{\aleph \text{-rank}(G) + 1, \infty\}.
\]

(2) follows from the fact that, if \( cd_R(G) < \infty \), then \( H^m(G, \mathbb{Z}G \otimes_{\mathbb{Z}} -) \neq 0 \) for \( m = cd_R(G) \); see the Commentary on Conjecture 1.7. \( \square \)

7. Cardinals, free abelian groups, and \( H_\mathfrak{F} \)

We now recall the hierarchies introduced in [9]; see [10] for more details.

7.1 Notation. Let \( \mathfrak{X} \) denote a class of groups.

All the classes of groups that we consider are closed under isomorphism, for example, the class \( \mathfrak{F} \) of all finite groups.

We let \( L \mathfrak{X} \) denote the class of groups whose finitely generated subgroups all lie in \( \mathfrak{X} \). For example, if \( \mathfrak{X} \) contains all finitely generated abelian groups, then \( L \mathfrak{X} \) contains all abelian groups.

We let \( H_1 \mathfrak{X} \) denote the class of all groups \( G \) for which there exists a finite-dimensional contractible \( G \)-complex with all stabilizers lying in \( \mathfrak{X} \). For example, \( H_1 \mathfrak{F} \) contains all finitely generated abelian groups, since, if \( G \) is finitely-generated and
abelian, then $G$ has a finite subgroup $N$ such that $G/N$ is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, and thus $G/N$ acts freely on $\mathbb{R}^n$ preserving a CW-structure.

If $\mathbf{H}_1 \mathcal{X} = \mathcal{X}$, then $\mathcal{X}$ is said to be $\mathbf{H}_1$-closed.

We let $\mathbf{H} \mathcal{X}$ denote the smallest $\mathbf{H}_1$-closed class of groups which contains $\mathcal{X}$. This class has a hierarchy indexed by the ordinals, where for each ordinal $\beta$, we define $\mathbf{H}_\beta \mathcal{X}$ recursively, by setting

$$
\mathbf{H}_0 \mathcal{X} := \mathcal{X},
\mathbf{H}_\beta \mathcal{X} := \mathbf{H}_1 \mathbf{H}_{\beta-1} \mathcal{X} \quad \text{if } \beta \text{ is a successor ordinal},
\mathbf{H}_\beta \mathcal{X} := \bigcup_{\alpha < \beta} \mathbf{H}_\alpha \mathcal{X} \quad \text{if } \beta \text{ is a limit ordinal}.
$$

□

We can use Theorem 2.8 to get new sufficient conditions for membership in $\mathbf{H} \mathcal{X}$.

7.2 Theorem. Let $\mathcal{X}$ be a subgroup-closed class of groups, and let $G \in \mathbf{L} \mathcal{X}$.

(1) If $\aleph$-rank$(G) = -1$ then $G \in \mathbf{H}_0 \mathcal{X}$.

(2) If $\aleph$-rank$(G) < \omega_0$ then $G \in \mathbf{H}_1 \mathcal{X}$.

(3) If $\aleph$-rank$(G) = \omega_0$ then $G \in \mathbf{H}_2 \mathcal{X}$.

Proof. (1). If $\aleph$-rank$(G) = -1$ then $G$ is a finitely generated element of $\mathbf{L} \mathcal{X}$, so lies in $\mathcal{X} = \mathbf{H}_0 \mathcal{X}$.

(2). If $\aleph$-rank$(G) = n$ for some $n \in \mathbb{N}$ then, by Theorem 2.8, $G$ acts on a contractible $(n + 1)$-dimensional CW-complex with stabilizers contained in finitely generated subgroups of $G$. Since $G$ lies in $\mathbf{L} \mathcal{X}$, and $\mathcal{X}$ is subgroup closed, we see that these stabilizers lie in $\mathcal{X}$. Hence $G \in \mathbf{H}_1 \mathcal{X}$.

(3). If $\aleph$-rank$(G) = \omega_0$, then we can write $G$ as the union of an ascending chain $(G_n \mid n \in \mathbb{N})$ of subgroups, such that, for each $n \in \mathbb{N}$, $\aleph$-rank$(G_n) = n$. As in Remark 2.5, there exists a $G$-tree with each cell stabilizer contained in $G_n$, for some $n \in \mathbb{N}$, so lying in $\mathbf{H}_1 \mathcal{X}$ by (2). Hence, $G \in \mathbf{H}_2 \mathcal{X}$. □

We next consider some necessary conditions for membership in $\mathbf{H} \mathcal{X}$, which are natural generalizations of [8, Lemma 1].

7.3 Lemma. Let $n \in \mathbb{N}$, let $R$ be a ring, let

$$
0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M_{-1} \to 0
$$

be an exact sequence of $R$-modules, and let $L$ be an $R$-module.

Suppose that $\text{Ext}^i_R(L, M_i) = 0$ for $i = 0, \ldots, n$. Then $\text{Ext}^0_R(L, M_{-1}) = 0$.

Proof. Clearly the result holds for $n = 0$. Thus we may assume that $n \geq 1$, and that the result holds with $n - 1$ in place of $n$.

Let $M'_{n-1}$ denote the cokernel of the map $M_n \to M_{n-1}$, so we have exact sequences

$$
0 \to M_{n} \to M_{n-1} \to M'_{n-1} \to 0 \tag{7.4}
$$

$$
0 \to M'_{n-1} \to M_{n-2} \to \cdots \to M_{0} \to M_{-1} \to 0 \tag{7.5}
$$
Now (7.4) gives rise to a long exact sequence which contains the segment
\[ \text{Ext}^{n-1}_R(L, M_{n-1}) \to \text{Ext}^{n-1}_R(L, M'_{n-1}) \to \text{Ext}^n_R(L, M_n). \]
Here the outer terms are zero, by hypothesis, so the inner term is zero. The induction hypothesis can now be applied to (7.5), and we see that $\text{Ext}^0_R(L, M_1) = 0$.

The result follows. □

We record the contrapositive of the case where $R = \mathbb{Z}G$, and $M_{n-1} = L = \mathbb{Z}$ with trivial $G$-action.

7.6 Corollary. If $n \in \mathbb{N}$, and
\[ 0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to \mathbb{Z} \to 0 \]
is an exact sequence of $\mathbb{Z}G$-modules, then there exists $i$ such that $0 \leq i \leq n$ and $H^i(G, M_i) \neq 0$. □

7.7 Proposition. If $\mathcal{X}$ is a class of groups, and $H^*(G, \mathbb{Z} \Delta) = 0$ for every $G$-set $\Delta$ for which all stabilizers lie in $\mathcal{X}$, then $G \notin H_1 \mathcal{X}$.

Proof. Suppose that $G \in H_1 \mathcal{X}$, so there exists a finite-dimensional, contractible CW-complex $X$ on which $G$ acts with all stabilizers lying in $\mathcal{X}$. Let $n$ denote the dimension of $X$. The augmented cellular chain complex of $X$,
\[ 0 \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0, \]
is an exact sequence of $\mathbb{Z}G$-modules, so, by Corollary 7.6, there exists $i$ such that $0 \leq i \leq n$ and $H^i(G, C_i(X)) \neq 0$. Thus $H^i(G, C_i(X)) \neq 0$. But we can write $C_i(X) = \mathbb{Z} \Delta$, where $\Delta$ is the set of $i$-dimensional open cells of $X$, so is a $G$-set with all stabilizers in $\mathcal{X}$. This contradicts the hypothesis. □

Combining Theorem 6.8(2) and Proposition 7.7, we get the following.

7.8 Corollary. Let $\mathcal{X}$ be a class of groups and $\lambda$ be an ordinal. Suppose that $G$ is an abelian group in $H_1 \mathcal{X}$ such that every subgroup of $G$ which lies in $\mathcal{X}$ has $\aleph$-rank strictly less than $\lambda$. Then $\aleph$-rank($G$) $< \lambda + \omega_0$. □

7.9 Theorem. If $\beta$ is any ordinal, then every abelian group in $H_\beta \mathcal{Y}$ has $\aleph$-rank strictly less than $\omega_0^\beta$.

Proof. We argue by induction on $\beta$.

The result holds for $\beta = 0$ by definition of $\mathcal{Y}$.

Thus we may assume that $\beta > 0$, and that the result holds for smaller ordinals.

Consider the case where $\beta$ is a limit ordinal. Here, each abelian group in $H_\beta \mathcal{Y}$ lies in $H_\alpha \mathcal{Y}$ for some $\alpha < \beta$, so, by the induction hypothesis, is of cardinality strictly less than $\aleph_{\omega_0 \alpha} \leq \aleph_{\omega_0^\beta}$.

Now consider the case where $\beta$ is a successor ordinal, and write $\beta = \alpha + 1$. By the induction hypothesis, every abelian group in $H_\alpha \mathcal{Y}$ is of cardinality strictly less than $\aleph_{\omega_0 \alpha}$. If we apply Corollary 7.8, with $\mathcal{X} = H_\alpha \mathcal{Y}$, we see that every abelian group in $H_1 \mathcal{X} = H_\beta \mathcal{Y}$ is of cardinality strictly less than $\aleph_{\omega_0 \alpha + \omega_0} = \aleph_{\omega_0^\beta}$. □

We can now make six statements, of varying profundity and novelty, about how free abelian groups fit into the hierarchy $H \mathcal{Y}$.
7.10 Theorem. For each cardinal $\kappa$, let $A_\kappa$ denote the free abelian group of rank $\kappa$.

1. $A_0 \in H_0 F$.
2. $(A_\kappa \mid 1 \leq \kappa < \aleph_0) \subseteq H_1 F - H_0 F$.
3. $(A_\kappa \mid \aleph_0 \leq \kappa < \aleph_0) \subseteq H_2 F - H_1 F$.
4. $A_{\aleph_0} \subseteq H_3 F - H_2 F$.
5. For each finite ordinal $n$, $A_{\aleph_0 n} \notin H_{n+1} F$; equivalently, every free abelian group in $H_{n+1} F$ has $\aleph$-rank strictly less than $\omega_0 n$.
6. For each infinite ordinal $\beta$, $A_{\aleph_0 \beta} \notin H_\beta F$.

Proof. (6) is a special case of Theorem 7.9.

1. is clear.

2. Consider $n \in \mathbb{N}$. Then $A_n$ acts freely on an $n$-dimensional CW-complex with underlying space $\mathbb{R}^n$, so $A_n \in H_1 F$.

It follows that $A_n \in H_1 F$, and it is clear that, if $n \geq 1$, then $A_n \notin H_0 F$.

5. It is well known that, for each $n \in \mathbb{N}$, $H^n(A_n, \mathbb{Z}) = \mathbb{Z}$. This can be seen by induction, using a long exact sequence in cohomology which gives a short exact sequence of graded groups

$$0 \to H^*(A_{n-1}, \mathbb{Z}) \to H^{*+1}(A_n, \mathbb{Z}) \to H^{*+1}(A_{n-1}, \mathbb{Z}) \to 0;$$

it can also be seen from the fact that the $n$-torus is a $K(A_n, 1)$.

Suppose that $A_{\aleph_0} \in H_1 F$, so, for some $n \in \mathbb{N}$, $A_{\aleph_0} \subseteq H_1 F$ acts freely on a contractible $n$-dimensional CW-complex $X$. Hence $A_{n+1}$ acts freely on $X$, so $H^{n+1}(A_{n+1}, -) = 0$, which contradicts the fact that $H^{n+1}(A_{n+1}, \mathbb{Z}) = \mathbb{Z}$. Hence $A_{\aleph_0} \notin H_1 F$, so every free abelian group in $H_1 F$ has finite rank, that is, has $\aleph$-rank equal to $-1$.

Now suppose that $n \geq 1$, and that every free abelian group in $H_n F$ has $\aleph$-rank strictly less than $\omega_0(n - 1)$. Corollary 7.8, with $X = H_n F$, and $\lambda = \omega_0(n - 1)$, implies that every free abelian group in $H_{n+1} F$ has $\aleph$-rank strictly less than $\omega_0 n$.

Now (5) has been proved by induction.

3. and (4). By (2), $H_1 F$ includes all finitely generated free abelian groups. Hence every free abelian group lies in $L H_1 F$. If we apply Theorem 7.2 and (3), with $X = H_1 F$, we find that the free abelian groups of $\aleph$-rank strictly less than $\omega_0$ lie in $H_2 F$, and that the free abelian group of $\aleph$-rank $\omega_0$ lies in $H_3 F$. Combined with (5), these imply (3) and (4). □

7.11 Remark. Theorem 7.10 gives an interesting new proof that

$$H_0 F \neq H_1 F \neq H_2 F \neq H_3 F.$$  □

7.12 Conjecture. $A_{\aleph_0 n+1} \notin H_3 F$; equivalently, $A_{\aleph_0 n+1} \notin H_3 F$. □

7.13 Conjecture. $H_3 F \neq H F$. □

7.14 Conjecture. There exists an ordinal $\alpha$ such that $H_\alpha F = H F$. □

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