A structure of the solenoidal 2D vector and 2-tensor fields given in a domain with the conformal Riemannian metric

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Abstract. The Helmholtz decomposition of a vector field on potential and solenoidal parts is much more natural from physical and geometric points of view then representations through the components of the vector in the Cartesian coordinate system of Euclidean space. The structure, representation through potentials and detailed decomposition for 2D symmetric m-tensor fields in a case of the Euclidean metric is known. For the Riemannian metrics similar results are known for vector fields. We investigate the properties of the solenoidal vector and 2-tensor two-dimensional fields given in the Riemannian domain with the conformal metric and establish the connections between the fields and metrics.

1. Preliminary definitions and constructions

The Helmholtz decomposition of a vector field on potential and solenoidal parts is well known [1], [2]. This decomposition is much more natural from physical and geometric points of view then representation through the components over coordinate systems in Euclidean space. A profound generalization of Helmholtz decomposition to a case of the symmetric tensor fields, given in a compact Riemannian manifold, was suggested in [3]. The structure, representation through potentials and detailed decomposition for 2D symmetric m-tensor fields in the case of the Euclidean metric was established in [4], [5]. In a case of the Riemannian metric similar results are known partially only for vector fields. We investigate here the properties of the solenoidal vector and symmetric 2-tensor fields given in the Riemannian domain with the conformal metric and establish the connections between the fields and metrics. The solenoidal fields of general type and special type depending on the scalar potentials are considered.

Let in Euclidean space $\mathbb{R}^n$ the Cartesian rectangular coordinate system be given. Points of $\mathbb{R}^n$ have the coordinates $(x^1, \ldots, x^n)$. We use the symbol $B = \{ x \in \mathbb{R}^n \mid (x^1)^2 + \ldots + (x^n)^2 < 1 \}$ for the unit ball, and $\partial B$ for the unit sphere.

We suppose the ball $B$ to be the Riemannian domain with the metric
\[
ds^2 = g_{ij}(x)dx^idx^j, \quad i, j = 1, \ldots, n. \tag{1}\]

The designation $T^m(B)$ ($S^m(B)$) is used for a set of covariant (symmetric) tensor fields of rank $m$ with components $u_{i_1 \ldots i_m}(x)$, $v_{i_1 \ldots i_m}(x)$, \ldots. It is easy to change the covariant components by contravariant (and back) with usage of the operators of raising and lowering indices,
$$
u^{i_1 \ldots i_m} = g^{imj_1} \ldots g^{imj_m} u_{j_1 \ldots j_m}, \quad u_{i_1 \ldots i_m} = g_{i_1 j_1} \ldots g_{i_m j_m} v^{j_1 \ldots j_m}.$$
We remind the definitions of the tangent bundle, \(TB = \{(x, \xi) \mid x \in B, \xi \in T_xB\}\), the tangent space
\[ T_xB = \{\xi \in \mathbb{R}^n \mid x \in B \text{ is fixed; the origin of a vector } \xi \text{ is at the point } x\}, \]
and the manifold \(\Omega B = \{(x, \xi) \in TB \mid |\xi|^2 \equiv g_{ij}\xi^i\xi^j = 1\}\) of tangent unit vectors (in metric \(g\)). The boundary \(\partial(\Omega B)\) consists from two parts,
\[ \partial_x\Omega B = \{(x, \xi) \in \Omega B \mid \pm\langle \xi, \nu(x) \rangle \geq 0\}, \tag{2} \]
where \(\nu(x)\) is unit vector of outer normal to \(\partial B\) at a point \(x\). Remind also that the components of a tensor field depend on a point \(x \in B\), which is connected with the tangent space \(T_xB\).

For \(u \in T^m\) the operator of symmetrization \(\sigma : T^m \rightarrow S^m\) is defined by an equality
\[ \sigma = \frac{1}{m!} \sum_{\pi \in \Pi_m} \rho_\pi, \tag{3} \]
where \(\Pi_m\) is the group of all permutations of the set \(\{1, \ldots, m\}\).

At the same time with \(L^2(B)\)-spaces we use the spaces \(C^\infty(S_m(B)), C^\infty_c(S_m(B))\), and Sobolev spaces \(H^k(S_m(B)), H^k_0(S_m(B))\) and \(H^k(\partial_x\Omega B), k\) is integer, \(k \geq 0\). The index 0 from below specifies the corresponding subspaces of functions or tensor fields vanishing together with their derivatives on the boundary \(\partial B\). Below we omit the sign \(B\) of unit ball in the designations of sets and spaces of functions and tensor fields.

The operator of inner differentiation \(d : C^\infty(S^m) \rightarrow C^\infty(S^{m+1})\) is defined by a rule \(d = \sigma \nabla\), where \(\nabla\) is the Riemannian connection, and the components of the tensor field \((\nabla u)\) expressed through the components of \(u\) as follows,
\[ (\nabla u)_{j_1 \ldots j_m k} = u_{j_1 \ldots j_m; k} = \frac{\partial}{\partial x^k}u_{j_1 \ldots j_m} - \sum_{l=1}^m \Gamma^p_{kj} u_{j_1 \ldots j_{l-1} p j_{l+1} \ldots j_m}. \tag{4} \]

The divergence \(\delta : C^\infty(S^{m+1}) \rightarrow C^\infty(S^m)\) is defined by the formula
\[ (\delta u)_{i_1 \ldots i_m} = u_{i_1 \ldots i_m; j} g^{jk}. \]

The formula (4) contains Christoffel symbols,
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kp} \left( \frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p} \right), \tag{5} \]
where \(g^{ij}\) are the contravariant components of fundamental tensor. The inner product is defined as follows,
\[ \langle u, v \rangle = u^{i_1 \ldots i_m} v_{i_1 \ldots i_m} = g^{i_1 j_1} \ldots g^{i_m j_m} u_{j_1 \ldots j_m} v_{i_1 \ldots i_m}. \]

We remind that a symmetric \(m\)-tensor field \(u \in C^\infty(S^m)\) is potential if there exists a symmetric \((m - 1)\)-tensor field \(p \in C^\infty(S^{m-1})\) such that \(u = dp\), where the operator \(d = \sigma \nabla\) is defined by (3), (4). A symmetric \(m\)-tensor field \(v \in C^\infty(S^m)\) is solenoidal, if its divergence is equal to 0 in \(B\), \(\delta v = 0\), \((\delta v) \in C^\infty(S^{m-1})\).

An essential generalization of Helmholtz decomposition for a vector field into a case of the symmetric tensor fields, given in a compact Riemannian manifold, was suggested in [3]. The symmetric tensor field \(f \in H^1(S^m)\) can be represented as a sum of the potential \(dv\) and the solenoidal \(f^s\) fields,
\[ f = f^s + dv, \quad \delta f^s = 0, \quad v \mid_{\partial B} = 0, \quad f^s \in H^1(S^m), \quad v \in H^2(S^{m-1}), \]
and the decomposition is unique.

For the vector fields it is valid more detailed decomposition [1], [2]. Namely, every vector field \( w \in H^1(S^1) \) is decomposed into the sum of the potential \( d\varphi \), solenoidal \( v \) and harmonic \( dh \) vector fields,

\[
w = v + dh + d\varphi, \quad \delta v = 0, \quad \varphi |_{\partial B} = 0, \quad \langle v, \nu \rangle |_{\partial B} = 0,
\]

where \( h \) is harmonic in \( B \) function, \( \nu \) is the vector of outer to the boundary \( \partial B \) normal, \( v \in H^1(S^1), \varphi \in H^2 \). The decomposition (6) is unique.

Later in [4] it was established that in 2D case the solenoidal field \( v \) can be represented through potential \( \psi \) in the form similar to the representation of a potential field, \( v = d^\perp \psi \) (an exact definition of \( d^\perp \) will be given below). The condition \( \langle v, \nu \rangle |_{\partial B} = 0 \) can be change then to the condition \( \psi |_{\partial B} = 0 \).

We formulate the definitions of ray transforms here for 2D case only. The longitudinal ray transform (LRT) [3] over a symmetric \( m \)-tensor field \( f \in H^1(S_m) \) is linear operator \( \mathcal{P} : H^1(S_m) \to H^1(\partial_+ \Omega B) \), defined by the equality

\[
\mathcal{P} f(x, \xi) = \int_{\tau_-(x, \xi)}^0 f_{i_1...i_m}(\gamma_{x,\xi}(t)) \gamma^{i_1}_{x,\xi}(t) \ldots \gamma^{i_m}_{x,\xi}(t) dt,
\]

where \( \gamma_{x,\xi}(t) : [\tau_-(x, \xi), 0] \to B \) is the geodesic, satisfying the initial conditions \( \gamma_{x,\xi}(0) = x, \gamma_{x,\xi}'(0) = \xi, \tau_-(x, \xi) \) is the value of the parameter (arc length) at which the geodesic intersects the circle \( \partial_+ \Omega B \) for the second time. The boundary \( \partial_+ \Omega B \) is defined by (2).

Alongside with LRT it is possible to define the transverse ray transform (TRT) over a symmetric \( m \)-tensor field \( f \in H^1(S_m) \). Namely, TRT is the linear operator \( \mathcal{P}^\perp : H^1(S_m) \to H^1(\partial_+ \Omega B) \),

\[
\mathcal{P}^\perp f(x, \xi) = \int_{\tau_-(x, \xi)}^0 f_{i_1...i_m}(\gamma_{x,\xi}(t)) \eta^{i_1}_{x,\xi}(t) \ldots \eta^{i_m}_{x,\xi}(t) dt.
\]

The vectors \( \eta_{x,\xi}(t) \) and \( \gamma_{x,\xi}(t) \) satisfy to the conditions

\[
g_{ij}(x) \gamma^i_{x,\xi}(t) \gamma^j_{x,\xi}(t) = 1, \quad g_{ij}(x) \eta^i_{x,\xi}(t) \eta^j_{x,\xi}(t) = 1, \quad g_{ij}(x) \gamma^i_{x,\xi}(t) \eta^j_{x,\xi}(t) = 0
\]

for one and the same value of the parameter \( t \).

The operator \( \mathcal{P} \) has nonzero kernel consisting of potential fields \( d\varphi \in C^1_0(S^1) \). The kernel of the operator \( \mathcal{P}^\perp \) contains the solenoidal fields \( d^\perp \psi \in C^1_0(S^1) \). The solenoidal \( d\varphi \in C^1_0(S^1) \) (potential \( d^\perp \psi \in C^1_0(S^1) \)) part is uniquely reconstructed by \( \mathcal{P} (\mathcal{P}^\perp) [5] \).

2. The orthogonal vector fields in Riemannian domain

Let in the disk \( B \subset \mathbb{R}^2 \) the Riemannian metric (1) be given. The covariant components of the metric tensor \( g_{ij}(x) \) form the symmetric matrix \( (g_{ij}) \). The contravariant components \( g^{ij} \) form the inverse matrix \( (g^{ij}) \). The symbol \( g \) designates the determinant of the matrix \( (g_{ij}) \), \( g = g_{11}g_{22} - (g_{12})^2 \), and \( \sqrt{g} = \sqrt{g_{11}g_{22} - (g_{12})^2} \).

Let us consider the following problem. By a given vector \( \xi, |\xi|^2 = g_{11}(\xi_1)^2 + 2g_{12}\xi_1\xi_2 + g_{22}(\xi_2)^2 \) it is required to find the vector \( \eta \) such that \( \langle \xi, \eta \rangle = 0, |\eta|^2 = |\xi|^2 \). Considering the covariant components \( \xi_i = g_{11}\xi_1 + g_{22}\xi_2, \eta_j = g_{11}\eta_1 + g_{22}\eta_2, i, j = 1, 2, \) of the vectors \( \xi, \eta \), obtain the system

\[
\eta^1 (g_{11}\xi_1 + g_{12}\xi_2) + \eta^2 (g_{21}\xi_1 + g_{22}\xi_2) = \eta^i \xi_i = 0,
\]

\[
\eta^1 (g_{11}\eta_1 + g_{12}\eta_2) + \eta^2 (g_{21}\eta_1 + g_{22}\eta_2) = \eta^i \eta_i = |\xi|^2.
\]
As ξ^kξ_k = g_{ij}ξ^iξ^j > 0, then, for example, for ξ_2 ≠ 0, η^2 = −ξ_1 η^1, and

\[ |ξ|^2 = \frac{(ξ_1)^2}{(ξ_2)^2} g((g^{22}(ξ_2)^2) + 2g^{12}ξ_1ξ_2 + g^{11}(ξ_1)^2) = g \frac{(ξ_1)^2}{(ξ_2)^2} |ξ|^2. \]

Hence η^1 = ± ξ_2 \frac{g_{21}ξ^1 + g_{22}ξ^2}{\sqrt{g}}. Here and below the solution with a sign ”minus” is chosen.

Finding the second component η^2 of η, we obtain

\[ η^1 = -\frac{ξ_2}{\sqrt{g}} = -\frac{g_{21}ξ^1 + g_{22}ξ^2}{\sqrt{g}}, \quad η^2 = \frac{ξ_1}{\sqrt{g}} = \frac{g_{11}ξ^1 + g_{12}ξ^2}{\sqrt{g}} \] (8)

for the contravariant components of the vector η, and

\[ η_1 = g_{11}η^1 + g_{12}η^2 = -\sqrt{g}ξ^2, \quad η_2 = g_{21}η^1 + g_{22}η^2 = \sqrt{g}ξ^1 \] (9)

for covariant. It is easy to see that |η|^2 = η^i η_i = −ξ_2 (−\sqrt{g}ξ^2) + ξ_1 (\sqrt{g}ξ^1) = ξ_i ξ^i = |ξ|^2.

Besides, we have connections

\[ ξ^1 = \frac{η_2}{\sqrt{g}} = \frac{g_{21}η^1 + g_{22}η^2}{\sqrt{g}}, \quad ξ^2 = -\frac{η_1}{\sqrt{g}} = -\frac{g_{11}η^1 + g_{12}η^2}{\sqrt{g}} \]

for contravariant components of ξ, and

\[ ξ_1 = \sqrt{g}η^2 = \sqrt{g}(g^{21}η_1 + g^{22}η_2), \quad ξ_2 = -\sqrt{g}η^1 = -\sqrt{g}(g^{11}η_1 + g^{12}η_2) \]

for covariant components.

The posed task can be solved also and with a help of discriminant tensor e with contravariant components e^{12} = −e^{21} = 1/\sqrt{g}, e^{11} = e^{22} = 0, and covariant e_{12} = −e_{21} = −\sqrt{g}, e_{11} = e_{22} = 0. In other words,

\[ (e_{ij}) = \sqrt{g} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (e^{ij}) = \frac{1}{\sqrt{g}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (10)

Obviously the components of sought-for vector η, orthogonal to the given ξ, can be found as η_i = e_{ij}ξ^j, or η_1 = −\sqrt{g}ξ^2; η_2 = \sqrt{g}ξ^1. This expressions coincide with (9) obtained above.

Accept the designation ξ^1 = η for the vector, orthogonal (in a sense of Riemannian metric) to the given ξ, |η| = |ξ|. Note two special cases of Riemannian metrics.

— The Euclidean metric has the components g_{11} = g^{11} = g_{22} = g^{22} = 1, g_{12} = g^{12} = g_{21} = g^{21} = 0. Thus η_1 = η^1 = −ξ_2 = −ξ^2; η_2 = η^2 = ξ_1 = ξ^1; g = 1/\sqrt{g} = 1. This ratios are valid for the rectangular Cartesian coordinate systems.

— The conformal metric with components g_{11} = g_{22} = e^{2μ(x,y)}; g^{11} = g^{22} = e^{−2μ(x,y)}; g_{12} = g_{21} = 0; g = e^{2μ(x,y)}; \sqrt{g} = e^{2μ(x,y)};

\[ ds^2 = e^{2μ(x,y)}(dx^2 + dy^2) \] (11)

In this case η^1 = −ξ^2, η^2 = ξ^1, η_1 = −ξ_2, η_2 = ξ_1, but η_1 = e^{2μ}η^1 = −e^{2μ}ξ^2, η_2 = e^{2μ}η^2 = e^{2μ}ξ^1.

The length of ξ is |ξ| = e^{μ((ξ^1)^2 + (ξ^2)^2)^{1/2}}. Below we often omit the signs of variables x, y, on which the function μ depends.
3. The solenoidal 2D vector and 2-tensor fields

We remind the definitions and properties of certain differential operators. Let \( \varphi, \psi \) be the functions of \( C^k \) smoothness, \( k \geq 0 \). An action of the operator of inner differentiation \( d : C^k \to C^{k-1}(S_1) \) on the function \( \varphi \) leads to (covariant) potential vector field \( u \in C^{k-1}(S_1) \),

\[
 u_k = (d \varphi)_k = \varphi, \quad u = (d \varphi) = \left( \frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^2} \right) \quad (12)
\]

The operator \( d \) coincides in this case with the operator of gradient. Let us define the operator \( d^\perp : C^k \to C^{k-1}(S^1) \), \( d^\perp = -ed \), where \( e \) is discriminant tensor (10). The action of \( d^\perp \) on \( \psi \) leads to (contravariant) vector field \( v \in C^{k-1}(S^1) \),

\[
 v^i = (d^\perp \psi)^i = e^i_j (d \psi)_j = \frac{(-1)^i}{\sqrt{g}} \psi_{(3-i)} = \frac{(-1)^i}{\sqrt{g}} (d \psi)_{3-i}
\]

\[
 (v^1, v^2) = \left( (d^\perp \psi)^1, (d^\perp \psi)^2 \right) = \frac{1}{\sqrt{g}} \left( - \frac{\partial \psi}{\partial x^2}, \frac{\partial \psi}{\partial x^1} \right) = \frac{1}{\sqrt{g}} \left( - (d \psi)_2, (d \psi)_1 \right). \quad (13)
\]

For the covariant components of the field \( v = (d^\perp \psi) \) we obtain

\[
 v_1 = (d^\perp \psi)_1 = g_{11} (d^\perp \psi)^1 + g_{12} (d^\perp \psi)^2 = \sqrt{g} (g^{22} (d \psi)_2 + g^{12} (d \psi)_1) = - \sqrt{g} (d \psi)^2,
\]

\[
 v_2 = (d^\perp \psi)_2 = g_{21} (d^\perp \psi)^1 + g_{22} (d^\perp \psi)^2 = - \sqrt{g} (g^{12} (d \psi)_2 + g^{11} (d \psi)_1) = \sqrt{g} (d \psi)^1.
\]

The divergence \( \delta : C^k(S_1) \to C^{k-1} \), acting on the vector field \( v \), yields a scalar,

\[
 \delta v = v_{ij} g^{ij} = g_{ik} v^k_j g^{ij} = g_{ik} g^{ij} v^k_j = u^i_j = \frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{kj} = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} v^i)}{\partial x^j}. \quad (14)
\]

Here we use the properties \( g^{ij}_{\perp} = 0 \) of the metric tensor, and \( \Gamma^i_{ji} = \frac{\partial (\ln \sqrt{g})}{\partial x^j} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} \) of the Christoffel symbols. Direct calculations show that the field \( v \) is solenoidal, \( \delta v = 0 \). The operator \( \delta^\perp : C^k(S^1) \to C^{k-1} \) is defined as follows, \( \delta^\perp u = -\delta u^\perp \), where \( u^\perp \) is the vector field, orthogonal to the field \( u \) (see (8), (9)). With usage of (14), (8) we obtain

\[
 \delta^\perp u = -\delta u^\perp = -u^i_{\perp} = - \left( (u^i)^{\perp}_i \right)_i = - \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} (u^i)^{\perp})}{\partial x^i} = \frac{1}{\sqrt{g}} \left( \frac{\partial u_2}{\partial x^2} - \frac{\partial u_1}{\partial x^1} \right). \quad (15)
\]

Certain properties and connections of vector, 2-tensor fields and the defined above differential operators are formulated below.

**Proposition 1.** Let \( v \in C^k(S^1) \) be solenoidal, \( u \in C^k(S^1(B)) \) be potential vector fields, \( k \) integer, \( k \geq 1 \). Then:

1. there exist \( \varphi, \psi \in C^{k+1} \) such that \( u = d \varphi, v = d^\perp \psi \), where the operators \( d, d^\perp \) are defined by (12), (13);
2. the potentials \( \varphi, \psi \in C^k, k \geq 2 \) satisfy to the relations

\[
 \delta (d^\perp \psi) = 0, \quad \delta^\perp (d \varphi) = 0,
\]

\[
 \delta (d \varphi) = \Delta \varphi, \quad \delta^\perp (d^\perp \psi) = \Delta \psi,
\]

where \( \Delta \) is the Laplace-Beltrami operator, \( \Delta f = g^{ik} f_{;i;k} \) for any \( f \in C^k, k \geq 2 \), and the operators \( \delta, \delta^\perp \) are defined by (14), (15);
3. for a solenoidal \( v \in C^k(S^2) \) and potential \( u \in C^k(S^2(B)) \), \( k \geq 1 \), symmetric 2-tensor fields there exist potentials \( \varphi, \psi, \chi \in C^k_0(B) \), such that \( u = d^2 \varphi + (dd^\perp) \chi \), \( v = (d^2 - \nabla^2)^2 \psi \).

**Proof.** Two of the listed in the proposition properties are the definitions, the other are known ([4], [5]), or can be verified directly.

Let in \( B \) the conformal Riemannian metric (11) be given. We remind that for this metric \( g_{11} = g_{22} = e^{2\mu(x,y)} \), \( g_{12} = g_{21} = 0 \), \( g_{11} = g^{22} = e^{-2\mu(x,y)} \), \( g_{12} = g^{21} = 0 \). The Christoffel symbols of this metric are

\[
\begin{align*}
\Gamma^1_{11} &= \mu_x, \quad \Gamma^1_{12} = \mu_y, \quad \Gamma^1_{22} = -\mu_x, \\
\Gamma^2_{11} &= -\mu_y, \quad \Gamma^2_{12} = \mu_x, \quad \Gamma^2_{22} = \mu_y.
\end{align*}
(16)
\]

It was established (see (14)) that vector field \( v \), defined by (13), is solenoidal. For Euclidean metric the solenoidal field has a form \( v = (\psi - \frac{\partial \psi}{\partial x^2}, \psi_1). \) It turns out, such a field is solenoidal and for the metric (11). Indeed (here and below we often suppose \( x^1 = x, x^2 = y \)), it is enough to find \( v_{j;j}, j = 1, 2, \)

\[
v_{1;1} = \frac{\partial v_1}{\partial x} - v_1 \mu_x + v_2 \mu_y, \quad
v_{2;2} = \frac{\partial v_2}{\partial y} + v_1 \mu_x - v_2 \mu_y,
\]

and make sure that

\[
\delta v = v_{i;j} g^{ij} v = v_{j;i} g^{ij} = e^{-2\mu}(v_{1;1} + v_{2;2}) = 0.
\]

Consider symmetric tensor fields of the second rank. It is known (proposition 1, [4]), that in a case of the Euclidean metric the solenoidal symmetric 2-tensor field depends on one potential, and its components are

\[
\begin{align*}
v_{11} &= \frac{\partial^2 \psi}{\partial y^2}, \quad v_{12} = v_{21} = -\frac{\partial^2 \psi}{\partial x \partial y}, \quad v_{22} &= \frac{\partial^2 \psi}{\partial x^2}.
\end{align*}
(17)
\]

We are interesting in a question when the tensor field (17), given in \( B \) with conformal metric, is solenoidal. We mean of course that instead of partial derivatives we use covariant derivatives.

Let us consider the potential 2-tensor field \( u = (u_{jk}), u_{jk} = \varphi_{;jk} \). As

\[
\varphi_{;jk} = \frac{\partial^2 \varphi}{\partial x^j \partial x^k} - \frac{\partial \varphi}{\partial x^j} \Gamma^i_{jk},
(18)
\]

then \( u \) is symmetric, and its components are

\[
\begin{align*}
u_{11} &= \varphi_{;1;1} = \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \varphi}{\partial x} \Gamma^1_{11} - \frac{\partial \varphi}{\partial y} \Gamma^2_{11} \\
u_{12} &= \varphi_{;1;2} = \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial x} \Gamma^1_{12} - \frac{\partial \varphi}{\partial y} \Gamma^2_{12} \\
u_{22} &= \varphi_{;2;2} = \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial \varphi}{\partial x} \Gamma^1_{22} - \frac{\partial \varphi}{\partial y} \Gamma^2_{22}.
\end{align*}
\]

for arbitrary Riemannian metric. For the metric (11), taking in mind (16), we obtain

\[
\begin{align*}
u_{11} &= \varphi_{xx} - \varphi_x \mu_x + \varphi_y \mu_y, \quad
u_{12} = \varphi_{xy} - \varphi_x \mu_y - \varphi_y \mu_x, \quad
u_{22} = \varphi_{yy} + \varphi_x \mu_x - \varphi_y \mu_y.
\end{align*}
(19)
\]

We define the symmetric 2-tensor field \( v = (v_{jk}) \) as follows

\[
\begin{align*}
v_{11} &= u_{22}, \quad v_{12} = v_{21} = -u_{12} = -u_{21}, \quad v_{22} &= u_{11},
\end{align*}
(20)
\]
and finding out when the field \( v \) is solenoidal. Application of the operator \( \delta \) leads to the vector field \( w = (w_1, w_2) = ((\delta v)_1, (\delta v)_2) \) with the components \( w_j = v_{jk;g^{kl}} \), where

\[
v_{jk;g^{kl}} = \frac{\partial v_{jk}}{\partial x^l} - v_{sk} \Gamma^s_{jl} - v_{js} \Gamma^s_{kl}.
\]

Let us calculate the components \( w_1, w_2 \), using the properties of the metric (11) and relations (20), (16),

\[
w_1 = v_{1k;g^{kl}} = v_{11;1}g^{kl} + v_{12;1}g^{k1} + v_{12;2}g^{12} = e^{2\mu} \left( \frac{\partial u_{22}}{\partial x} - \frac{\partial u_{12}}{\partial y} - 2u_{22}\mu_x - 2u_{12}\mu_y - u_{11}\mu_x + u_{22}\mu_y \right),
\]

\[
w_2 = v_{2k;g^{kl}} = v_{21;1}g^{kl} + v_{21;2}g^{12} + v_{22;2}g^{22} = e^{2\mu} \left( - \frac{\partial u_{22}}{\partial x} + \frac{\partial u_{11}}{\partial y} - 2u_{22}\mu_y + 2u_{21}\mu_x + u_{21}\mu_y - 2u_{12}\mu_x - 2u_{11}\mu_y \right).
\]

Substituting the terms (19) of the components \( u_{ij} \) into the formulas for \( w_k \) and taking in mind the relations

\[
\frac{\partial u_{22}}{\partial x} - \frac{\partial u_{12}}{\partial y} = \varphi_x (\mu_{xx} + \mu_{yy}) + \mu_x (\varphi_{xx} + \varphi_{yy}),
\]

\[
- \frac{\partial u_{22}}{\partial x} + \frac{\partial u_{11}}{\partial y} = \varphi_y (\mu_{xx} + \mu_{yy}) + \mu_y (\varphi_{xx} + \varphi_{yy}),
\]

we obtain

\[
w_1 = e^{-2\mu(x,y)} \varphi_x \Delta \mu,
\]

\[
w_2 = e^{-2\mu(x,y)} \varphi_y \Delta \mu.
\]

Thus it is shown the rightness of the following statement,

**Theorem 1.** Let in the unit disk \( B \) the Riemannian metric \( ds^2 = e^{2\mu(x,y)}(dx^2 + dy^2) \) be given. The symmetric 2-tensor field \( u_{ij} \) with components \( u_{11} = \varphi;2,2, \ v_{12} = v_{21} = -\varphi;1,2 = -\varphi;2,1, \ v_{22} = \varphi;1,1 \) is defined by (18) for any twice continuously differentiable in \( B \) function \( \varphi(x,y) \). Then the field \( v \) is solenoidal if and only if \( \mu(x,y) \) is the harmonic in \( B \) function.

**Acknowledgements**

The work was partially supported by the Program for Fundamental Researches of SB RAS (project No. 0314-2016-0011) and by RFBR according to the RFBR-DFG project No. 19-51-12008.

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