Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux–Crum transformations

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Abstract
A simple derivation is presented of the four families of infinitely many shape-invariant Hamiltonians corresponding to the exceptional Laguerre and Jacobi polynomials. The Darboux–Crum transformations are applied to connect the well-known shape-invariant Hamiltonians of the radial oscillator and the Darboux–Pöschl–Teller potential to the shape-invariant potentials of Odake–Sasaki. Dutta and Roy derived the two lowest members of the exceptional Laguerre polynomials by this method. The method is expanded to its full generality and many other ramifications, including the aspects of the generalized Bochner problem and the bispectral property of the exceptional orthogonal polynomials, are discussed.

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1. Introduction

Here we construct in an elementary way the four sets of infinitely many shape-invariant Hamiltonians and the corresponding exceptional \((X_\ell)\) polynomials [1, 2]. The idea is quite simple. We start with a prepotential \(W_\ell(x; \lambda)\) and define a pair of factorized Hamiltonians \(H_\ell^\pm(\lambda)\) which are intertwined by the Darboux–Crum transformations [3, 4] in terms of \(A_\ell(\lambda)\) and \(A_\ell^\dagger(\lambda)\):
Here, $\lambda$ stands for the set of parameters of the theory. The prepotential $W_{\ell}(x; \lambda)$ is so chosen that $\mathcal{H}^{(+)}_{\ell}(\lambda)$ is the well-known shape-invariant [5] Hamiltonian of the radial oscillator [6, see, for example, a review 7] potential or the Darboux–Poschl–Teller (DPT) potential [8] and $\mathcal{H}^{(-)}_{\ell}(\lambda)$ is the Hamiltonian of the recently derived shape-invariant potentials of Odake–Sasaki [1, 2, 9, 10]. The essential property of the prepotential to achieve the above goal is that $e^{\pm W_{\ell}(x; \lambda)}$ is not square integrable. We know that $\mathcal{H}^{(+)}_{\ell}(\lambda)$ has a well-defined ground state. Thus, these two functions $e^{\pm W_{\ell}(x; \lambda)}$ cannot correspond to the ground states of $\mathcal{H}^{(+)}_{\ell}(\lambda)$ or $\mathcal{H}^{(-)}_{\ell}(\lambda)$. That is, the ground states of $\mathcal{H}^{(\pm)}_{\ell}(\lambda)$ are not annihilated by $A_{\ell}(\lambda)$ or $A_{\ell}^{\dagger}(\lambda)$. This means that $\mathcal{H}^{(+)}_{\ell}(\lambda)$ and $\mathcal{H}^{(-)}_{\ell}(\lambda)$ are exactly iso-spectral including the ground states:

$$\mathcal{H}^{(+)}_{\ell}(\lambda)\phi^{(+)}_{\ell,n}(x; \lambda) = \mathcal{E}^{(+)}_{\ell,n}(\lambda)\phi^{(+)}_{\ell,n}(x; \lambda), \quad \mathcal{H}^{(-)}_{\ell}(\lambda)\phi^{(-)}_{\ell,n}(x; \lambda) = \mathcal{E}^{(-)}_{\ell,n}(\lambda)\phi^{(-)}_{\ell,n}(x; \lambda)$$

$$\mathcal{E}^{(+)}_{\ell,n}(\lambda) = \mathcal{E}^{(-)}_{\ell,n}(\lambda) > 0, \quad \ell = 1, 2, \ldots$$

By construction $\mathcal{H}^{(+)}_{\ell}(\lambda)$ is shape invariant and exactly solvable. That is, the set of eigenvalues $\{\mathcal{E}^{(+)}_{\ell,n}(\lambda)\}$ and the corresponding eigenfunctions $\{\phi^{(+)}_{\ell,n}(x; \lambda)\}$ are exactly known. Throughout this paper we choose all the eigenfunctions to be real. They form a complete set of orthogonal functions:

$$\int \phi^{(+)}_{\ell,m}(x; \lambda)\phi^{(+)}_{\ell,n}(x; \lambda) \, dx = \delta_{m,n}, \quad \mathcal{H}_{\ell,n}(\lambda) > 0. \quad (1.6)$$

Thanks to the intertwining relations (1.2) the eigenfunctions $\{\phi^{(-)}_{\ell,n}(x; \lambda)\}$ of the partner Hamiltonian $\mathcal{H}^{(-)}_{\ell}(\lambda)$ are obtained from the well-known eigenfunctions $\{\phi^{(+)}_{\ell,n}(x; \lambda)\}$ of the radial oscillator or the DPT Hamiltonian $\mathcal{H}^{(\pm)}_{\ell}(\lambda)$ by the Darboux–Crum transformation in terms of $A_{\ell}(\lambda)$:

$$\phi^{(-)}_{\ell,n}(x; \lambda) = A_{\ell}(\lambda)\phi^{(+)}_{\ell,n}(x; \lambda), \quad \ell = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots, \quad (1.7)$$

and vice versa:

$$\phi^{(+)}_{\ell,n}(x; \lambda) = \frac{A_{\ell}(\lambda)}{\mathcal{E}^{(+)}_{\ell,n}(\lambda)}\phi^{(-)}_{\ell,n}(x; \lambda), \quad \ell = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots, \quad (1.8)$$

Of course $\{\phi^{(-)}_{\ell,n}(x; \lambda)\}$ is another complete set of orthogonal functions:

$$\int \phi^{(-)}_{\ell,m}(x; \lambda)\phi^{(-)}_{\ell,n}(x; \lambda) \, dx = \int A_{\ell}(\lambda)\phi^{(+)}_{\ell,m}(x; \lambda) \cdot A_{\ell}(\lambda)\phi^{(+)}_{\ell,n}(x; \lambda) \, dx$$

$$= \int \phi^{(+)}_{\ell,m}(x; \lambda) \cdot A_{\ell}(\lambda)^{\dagger}A_{\ell}(\lambda)\phi^{(+)}_{\ell,n}(x; \lambda)$$

$$= \mathcal{E}^{(+)}_{\ell,n}(\lambda)\int \phi^{(+)}_{\ell,m}(x; \lambda)\phi^{(+)}_{\ell,n}(x; \lambda) \, dx = \mathcal{E}^{(+)}_{\ell,n}(\lambda)\delta_{mn}. \quad (1.9)$$

These imply the completeness of the exceptional orthogonal polynomials and the above relationship (1.7) provides the formula relating the exceptional orthogonal polynomials to the classical orthogonal polynomials (i.e. the Laguerre or Jacobi polynomials) as shown in (2.1) and (2.3) of [10]. Orthogonality (1.9) corresponds to the integration formulas derived in section 7 of [10]. These will be demonstrated in detail in subsequent sections.

The above requirements lead to the following general form of the prepotential $W_{\ell}(x; \lambda)$:

$$W_{\ell}(x; \lambda) = \bar{w}_{0}(x; \lambda + \ell \delta) + \log \xi_{0}(\eta(x); \lambda), \quad (1.10)$$
where $\tilde{w}_0(x; \lambda)$ is obtained by changing the sign of one term of the prepotential $w_0(x; \lambda)$ corresponding to the radial oscillator or the DPT potential. Here $\delta$ is the shift of the parameters. The change of the sign ensures the non-square integrability of $e^{\pm W_\ell(x; \lambda)}$. The additional term is the logarithm of the degree $\ell$ eigenpolynomial, the Laguerre or Jacobi polynomial with twisted parameters or arguments, introduced by Odake–Sasaki [1, 2, 10].

Before going to the details in the subsequent sections, let us make a few remarks about the background. This type of approach of deriving a new exactly solvable Hamiltonian from a known one in terms of the Darboux–Crum [3, 4] transformations has a long history and various aspects [11–13]. The method we are concerned in this paper is, in its essence, based on an alternative factorization of an exactly solvable Hamiltonian (plus a constant), e.g. the radial oscillator and the DPT potential. Some refer to those newly found Hamiltonians as 'conditionally exactly solvable'. Junker and Roy [13] discussed an example of an alternative factorization of the radial oscillator Hamiltonian by using the confluent hypergeometric function $F_1$, which could encompass the results of the L1 exceptional orthogonal polynomials [1] if the parameters and settings are properly chosen. After the introduction of the $X_1$ Laguerre polynomials by Gomez-Ullate et al [14] and Quesne [15, 16] and the $X_\ell$ polynomials by Odake–Sasaki [1, 2], Roy and his collaborator [17] derived the $X_1$ and $X_2$ Laguerre polynomials of the L1 type in this way. This will be mentioned in a later section. A recent report by Gomez-Ullate et al [18] has small overlap with the present work. As far as we know, a 'conditionally exactly solvable' treatment of the fully general (non-symmetric) DPT potential does not exist. Therefore, the present derivation of the $J_1$ and $J_2$ exceptional Jacobi polynomials from the classical Jacobi polynomials based on the Darboux–Crum transformation is new.

The exceptional orthogonal polynomials were originally introduced in [14] by extending Bochner’s theorem [19] for Sturm–Liouville problems. The characterization of the exceptional orthogonal polynomials as polynomial solutions of Sturm–Liouville-type equations under generalized Bochner problems will be discussed in section 4. We will show that the exceptional Laguerre and Jacobi polynomials have the bispectral property [20]. On top of the Sturm–Liouville-type equation, the $X_\ell$ polynomials satisfy $(4\ell + 1)$ recursion relations (4.15) and (4.33), which could be rewritten as an eigenvalue equation (4.16) of a semi-infinite matrix $K$ which acts on the label $n$ of the exceptional polynomials $P_n(x)$ and $L_n(x)$. Simple interpretation of the bispectral property is provided as the characteristic feature of the invariant polynomial subspaces of the Sturm–Liouville-type operator.

This paper is organized as follows. In section 2, the two types (L1 and L2) of infinitely many exceptional Laguerre polynomials are derived by simple Darboux–Crum transformations connecting them with the Laguerre polynomials. The two types (J1 and J2) of infinitely many exceptional Jacobi polynomials are derived in a similar way by using the Darboux–Crum transformations connecting them with the Jacobi polynomials. The aspects of the generalized Bochner problems, in particular, the bispectral property, will be discussed in section 4. Based on the so-called Darboux transformations of the orthogonal polynomials, that is the Christoffel and Geronimus transformations of the Jacobi and Laguerre polynomials [21, 22], the bispectral property of the exceptional polynomials is derived elementarily. In other words, the $X_\ell$ polynomials are shown to satisfy $(4\ell + 1)$-term recurrence relations, which are the generalization of the well-known three-term recurrence relations. The final section is for comments and discussions. It provides a simple proof of shape invariance of the Hamiltonians of the exceptional orthogonal polynomials. It is shown that they also possess the creation/annihilation operators inherited from the radial oscillator/DPT Hamiltonian systems. It is stressed that the Hamiltonians of the radial oscillator/DPT potentials admit infinitely many non-singular factorizations, which induce the Darboux–Crum transformations. The extended
‘three-term recurrence’ relations for the exceptional orthogonal polynomials are also discussed from the Darboux–Crum transformations point of view.

2. Exceptional Laguerre polynomials

Here we will derive the $L_1$ and $L_2$ exceptional Laguerre polynomials as well as the corresponding Hamiltonians, that is, the potentials.

2.1. Radial oscillator

Let us start with the radial oscillator with $\lambda = g > 0$ and $\delta = 1$:

\[ w_0(x; g) \overset{\text{def}}{=} -\frac{1}{2} x^2 + g \log x, \quad 0 < x < \infty, \quad (2.1) \]

\[ R^{(\ell)}_R(g) = p^2 + x^2 + \frac{g(g - 1)}{x^2} - 1 - 2g. \quad (2.2) \]

It is trivial to verify the shape invariance [5]:

\[ H^{(-\ell)}_R(g) = H^{(\ell)}_R(g + 1) + 4, \quad E_1(g) = 4. \quad (2.3) \]

Its eigenvalues and eigenfunctions are

\[ E_n(g) = 4n, \quad (2.4) \]

\[ \phi_n(x; g) = P_n(\eta; g) e^{w_0(x; g)}, \quad \eta \equiv \eta(x) \overset{\text{def}}{=} x^2, \quad P_n(x; g) = L_n^{(g - \frac{1}{2})}(x), \quad (2.5) \]

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial satisfying the differential equation

\[ x \frac{\partial^2}{\partial x^2} L_n^{(\alpha)}(x) + (\alpha + 1 - x) \frac{\partial}{\partial x} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0. \quad (2.6) \]

It should be stressed that the ground-state wavefunction $\phi_0(x; g) = e^{w_0(x; g)} = e^{-x^2/2} x^\ell$ is square integrable and it provides the orthogonality measure of the Laguerre polynomials:

\[ \int_0^\infty e^{2w_0(x; g)} P_n(x^2; g) P_m(x^2; g) \, dx = h_n(g) \delta_{mn}, \quad h_n(g) \overset{\text{def}}{=} \frac{1}{2n!} \Gamma \left( n + g + \frac{1}{2} \right). \quad (2.7) \]

The radial oscillator Hamiltonian system is exactly solvable in the Heisenberg picture, too [23]. The exact annihilation/creation operators are obtained as the positive/negative energy parts of the Heisenberg operator solution (see, for example, (3.8) of [23]):

\[ a^{(\pm)} = \left( \left( \frac{d}{dx} + x \right)^2 - \frac{g(g - 1)}{x^2} \right)^{\frac{1}{4}}. \quad (2.8) \]

The action of these operators are

\[ a^{(-)} \phi_n(x; g) = -\left( n + g - \frac{1}{2} \right) \phi_{n-1}(x; g), \quad a^{(+)\,} \phi_n(x; g) = -(n + 1) \phi_{n+1}(x; g). \quad (2.9) \]

2.2. $L_1$ and $L_2$ exceptional Laguerre polynomials

Here we derive the $L_1$ and $L_2$ exceptional Laguerre polynomials. For each positive integer $\ell = 1, 2, \ldots$, let us consider the pair of Hamiltonians $H^{(\ell)}_R(g)$ and $H^{(\ell - 1)}_R(g)$ corresponding to the following prepotentials ($\eta \equiv \eta(x) \overset{\text{def}}{=} x^2$):

\[ a^{(-)} \phi_n(x; g) = -\left( n + g - \frac{1}{2} \right) \phi_{n-1}(x; g), \quad a^{(+)\,} \phi_n(x; g) = -(n + 1) \phi_{n+1}(x; g). \quad (2.9) \]
The partner Hamiltonians are
\[ H_{\ell}(g) \equiv L_{\ell}^{(g+\ell-\frac{1}{2})}(\eta), \]  
\[ \xi_{\ell}(\eta; g) \equiv L_{\ell}^{(g+\ell-\frac{1}{2})}(\eta), \]  
\[ W_{\ell}(x; g) \equiv -\frac{x^2}{2} - (g + \ell) \log x + \log \xi_{\ell}(\eta; g), \quad g > -1/2, \]  
\[ \xi_{\ell}(\eta; g) \equiv L_{\ell}^{(g-\ell-\frac{1}{2})}(\eta), \]  
\[ \mathcal{H}_{\ell}^{(s)}(g) \equiv A_{\ell}^{s}(g) A_{\ell}(g), \quad \mathcal{H}_{\ell}^{(-)}(g) \equiv A_{\ell}(g) A_{\ell}^{s}(g), \]  
\[ A_{\ell}(g) \equiv \frac{d}{dx} - \frac{dW_{\ell}(x; g)}{dx}, \quad A_{\ell}^{s}(g) = -\frac{d}{dx} - \frac{dW_{\ell}(x; g)}{dx}. \]

By simple calculation using the differential equation for the Laguerre polynomial \( (2.6) \), the Hamiltonian \( \mathcal{H}_{\ell}^{(s)}(g) \) is shown to be equal to the radial oscillator with \( g \to g + \ell + 1 \) for \( L1 \) and with \( g \to g + \ell + 1 + 1 \) for \( L2 \) up to an additive constant:

\[ \mathcal{H}_{\ell}^{(s)}(g) = p^2 + x^2 + \frac{(g + \ell)(g + \ell - 1)}{x^2} + 2g + 6\ell - 1, \]
\[ = \mathcal{H}_{R}(g + \ell - 1) + 2(2g + 4\ell - 1), \]
\[ \mathcal{H}_{\ell}^{(-)}(g) = p^2 + x^2 + \frac{(g + \ell)(g + \ell - 1)}{x^2} + 2(g - \ell) - 1, \]
\[ = \mathcal{H}_{R}^{(-)}(g + \ell + 1) + 2(2g + 1). \]

The partner Hamiltonians are

\[ \mathcal{H}_{\ell}^{(-)}(g) = p^2 + x^2 + \frac{(g + \ell)(g + \ell - 1)}{x^2} + 2(g - \ell) - 3 \]
\[ + 2 \left( \frac{\partial_x \xi_{\ell}(\eta; g)}{\xi_{\ell}(\eta; g)} \right) \left( 2 \left( x + \frac{(g + \ell - 1)}{x} \right) + \frac{\partial_x \xi_{\ell}(\eta; g)}{\xi_{\ell}(\eta; g)} \right), \]
\[ \mathcal{H}_{\ell}^{(-)}(g) = p^2 + x^2 + \frac{(g + \ell)(g + \ell - 1)}{x^2} + 2(g + 3\ell) + 1 \]
\[ + 2 \left( \frac{\partial_x \xi_{\ell}(\eta; g)}{\xi_{\ell}(\eta; g)} \right) \left( -2 \left( x + \frac{(g + \ell)}{x} \right) + \frac{\partial_x \xi_{\ell}(\eta; g)}{\xi_{\ell}(\eta; g)} \right). \]

Up to additive constants, the above Hamiltonians \( (2.20) \) and \( (2.21) \) are equal to the Hamiltonians of the \( L1 \) and \( L2 \) exceptional orthogonal polynomials derived by Odake–Sasaki \( [1, 10] \):

\[ \mathcal{H}_{\ell}^{O-S}(g) \equiv p^2 + \left( \frac{dw_{\ell}(x; g)}{dx} \right)^2 + \frac{d^2 w_{\ell}(x; g)}{dx^2}, \]
\[ w_{\ell}(x; g) \equiv -\frac{x^2}{2} + (g + \ell) \log x + \log \frac{\xi_{\ell}(\eta; g + 1)}{\xi_{\ell}(\eta; g)}, \]
\[ \mathcal{H}_{\ell}^{O-S}(g) = \mathcal{H}_{\ell}^{O-S}(g) + 2(2g + 4\ell - 1), \]
\[ \mathcal{H}_{\ell}^{O-S}(g) = \mathcal{H}_{\ell}^{O-S}(g) + 2(2g + 1). \]
The definition of $\xi_\ell(\eta; g)$ for the L1 and L2 Odake–Sasaki cases is the same as those given in (2.11) and (2.13).

Let us note that $e^{W_\ell(x; \eta; g)}$ is not square integrable at $x = \infty$ and $e^{-W_\ell(x; \eta; g)}$ is not square integrable at $x = 0$ for the L1 case, whereas for the L2 case, $e^{W_\ell(x; \eta; g)}$ is not square integrable at $x = \infty$. In both cases the prepotentials $W_\ell(x; \eta; g)$ (2.10) and (2.12) are regular in the interval $0 < x < \infty$. For this, it is enough to show that $\eta_\ell(\eta(x); g)$ does not have a zero in $0 < x < \infty$. In fact we have

$$L_1 : \quad \xi_\ell(\eta(x); g) = \sum_{k=0}^{\ell} \frac{(g + \ell + k - \frac{1}{2})_{\ell-k}}{k!(\ell - k)!} x^{2k} > 0, \quad (2.26)$$

$$L_2 : \quad (-1)^{\ell} \xi_\ell(\eta(x); g) = \sum_{k=0}^{\ell} \frac{(g + \frac{1}{2})_{\ell-k}}{k!(\ell - k)!} x^{2k} > 0, \quad (2.27)$$

as shown in (2.39) of [9]. Thus, we find that $e^{\pm W_\ell(x; \eta; g)}$ cannot be the ground states of the Hamiltonians $\mathcal{H}_\ell^{(\pm)}(g)$. However, we know quite well that $\mathcal{H}_\ell^{(+)}(g)$, being the radial oscillator Hamiltonian, has a well-defined ground state. This means that the partner Hamiltonians $\mathcal{H}_\ell^{(-)}(g)$, thus the Hamiltonians of the L1 and L2 exceptional Laguerre polynomials, are exactly iso-spectral to the radial oscillator Hamiltonians $\mathcal{H}_\ell^{(+)}(g)$, which have the following eigenvalues and the corresponding eigenfunctions:

$$L_1 : \quad \xi_{\ell, n}^{(+)}(g) = 4n + 2(2g + 4\ell - 1), \quad \phi_{\ell, n}^{(+)}(x; g) = e^{-\frac{\ell}{2} x^{g+\ell+1}} L_n^{(g+\ell+\frac{1}{2})}(\eta), \quad (2.28)$$

$$L_2 : \quad \xi_{\ell, n}^{(+)}(g) = 4n + 2(2g + 1), \quad \phi_{\ell, n}^{(+)}(x; g) = e^{-\frac{\ell}{2} x^{g+\ell+1}} L_n^{(g+\ell+\frac{1}{2})}(\eta). \quad (2.29)$$

The intertwining relation (1.2) implies the simple expressions of the eigenfunctions of the partner Hamiltonians $\mathcal{H}_\ell^{(-)}$ in terms of $A_\ell(g)$:

$$L_1 : \quad \xi_{\ell, n}^{(-)}(g) = 4n + 2(2g + 4\ell - 1), \quad \phi_{\ell, n}^{(-)}(x; g) = e^{O_{-S}}(g) = 4n, \quad (2.30)$$

$$\phi_{\ell, n}^{(-)}(x; g) = \phi_{\ell, n}^{(-)}(x; g) = \left( - \frac{d}{dx} \frac{dW_\ell(x; g)}{dx} \right) e^{-\frac{\ell}{2} x^{g+\ell+1}} L_n^{(g+\ell+\frac{1}{2})}(\eta)$$

$$= 2 e^{-\frac{\ell}{2} x^{g+\ell}} \xi_\ell(\eta; g) \left( -\xi_\ell(\eta; g + 1) L_n^{(g+\ell+\frac{1}{2})}(\eta) + \xi_\ell(\eta; g) \partial_\eta L_n^{(g+\ell+\frac{1}{2})}(\eta) \right), \quad (2.31)$$

$$L_2 : \quad \xi_{\ell, n}^{(-)}(g) = 4n + 2(2g + 1), \quad \phi_{\ell, n}^{(-)}(x; g) = e^{O_{-S}}(g) = 4n, \quad (2.32)$$

$$\phi_{\ell, n}^{(-)}(x; g) = \phi_{\ell, n}^{(-)}(x; g) = \left( - \frac{d}{dx} \frac{dW_\ell(x; g)}{dx} \right) e^{-\frac{\ell}{2} x^{g+\ell+1}} L_n^{(g+\ell+\frac{1}{2})}(\eta)$$

$$= 2 e^{-\frac{\ell}{2} x^{g+\ell}} \xi_\ell(\eta; g) \left( \left( g + 1 + \frac{1}{2} \right) \xi_\ell(\eta; g + 1) L_n^{(g+\ell+\frac{1}{2})}(\eta) + \eta \xi_\ell(\eta; g) \partial_\eta L_n^{(g+\ell+\frac{1}{2})}(\eta) \right). \quad (2.33)$$

These are to be compared with the explicit expressions of the exceptional Laguerre polynomials $P_{\nu, \kappa}(\eta; g)$ derived in [10]:

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\[ \phi_{\ell,n}(x; g) = \psi_{\ell}(x; g) P_{\ell,n}(\eta; g), \quad \psi_{\ell}(x; g) \overset{\text{def}}{=} e^{-x^2/2g} \xi_{\ell}(\eta; g), \]

\[ \xi_{\ell}(\eta; g) \overset{\text{def}}{=} \begin{cases} L_{\ell}^{(g+\ell-1/2)}(-\eta) & : \text{L1} \\ L_{\ell-1}^{(g+\ell-1/2)}(\eta) & : \text{L2}, \end{cases} \]

\[ P_{\ell,n}(\eta; g) \overset{\text{def}}{=} \begin{cases} \xi_{\ell}(\eta; g+1) L_{\ell-n}^{(g+\ell+1/2)}(\eta) - \xi_{\ell}(\eta; g) \partial_\eta L_{\ell-n}^{(g+\ell+1/2)}(\eta) & : \text{L1} \\ (n+g+\frac{1}{2})^{-1} \left( (g+\frac{1}{2}) \xi_{\ell}(\eta; g+1) L_{\ell-n}^{(g+\ell+1/2)}(\eta) + \eta \xi_{\ell}(\eta; g) \partial_\eta L_{\ell-n}^{(g+\ell+1/2)}(\eta) \right) & : \text{L2}. \end{cases} \]

The final expressions of eigenfunctions (2.31) and (2.33) are the same as (2.1) of [10] up to a multiplicative constant. The identities of the Laguerre polynomials (E.11) and (E.12) of [10] are used. Thus, we have derived the Hamiltonians as well as the eigenfunctions, that is, the L1 and L2 exceptional Laguerre polynomials and the weight functions, from those of the radial oscillator by the Darboux–Crum transformations.

3. Exceptional Jacobi polynomials

Here we will derive the J1 and J2 exceptional Jacobi polynomials as well as the corresponding Hamiltonians, that is, the potentials.

3.1. Trigonometric DPT potential

The trigonometric DPT [8] potential has the two parameters \( \lambda = (g, h) \), \( g > 0, h > 0 \), and \( \delta \equiv (1, 1) \):

\[ u_0(x; g, h) \overset{\text{def}}{=} g \log \sin x + h \log \cos x, \quad 0 < x < \frac{\pi}{2}, \]

\[ \mathcal{H}_{\text{DPT}}^{(+)}(g, h) = p^2 + \frac{g(g - 1)}{\sin^2 x} + \frac{h(h - 1)}{\cos^2 x} - (g + h)^2. \]

It is trivial to verify the shape invariance [5]:

\[ \mathcal{H}_{\text{DPT}}^{(-)}(g, h) = \mathcal{H}_{\text{DPT}}^{(+)}(g + 1, h + 1) + 4(g + h + 1), \quad E_1(g, h) = 4(g + h + 1). \]

Its eigenvalues and eigenfunctions are

\[ E_n(g, h) = 4n(n + g + h), \]

\[ \phi_n(x; g, h) = P_n(\eta(x); g, h) e^{u_0(x; g, h)}, \quad \eta \equiv \eta(x) \overset{\text{def}}{=} \cos 2x, \]

\[ P_n(x; g, h) = P_n^{(\frac{1}{2}, h-\frac{1}{2})}(x), \]

where \( P_n^{(\alpha, \beta)}(x) \) is the Jacobi polynomial satisfying the second-order differential equation

\[ (1 - x^2) \frac{d^2}{dx^2} P_n^{(\alpha, \beta)}(x) + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) + n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x) = 0. \]

The ground-state wavefunction \( \phi_0(x; g, h) = e^{u_0(x; g, h)} = (\sin x)^g (\cos x)^h \) is square integrable and it provides the orthogonality measure of the Jacobi polynomials:

\[ \int_0^{\pi/2} \sin^{2n}(\eta(x); g, h) P_n(\eta(x); g, h) dx = h_n(g, h) \delta_{nn}. \]
The trigonometric DPT potential is also exactly solvable in the Heisenberg picture [23]. The annihilation and creation operators are (see, for example, (3.28) of [23])

\[ a^{(\pm)}/2 = a^{(\pm)} 2\sqrt{\mathcal{H}} = \pm \sin 2x \frac{d}{dx} \cos 2x \sqrt{\mathcal{H}} + \frac{\alpha^2 - \beta^2}{\sqrt{\mathcal{H}}} \pm 1, \]

(3.9)

where

\[ \mathcal{H} \equiv \mathcal{H}_{\text{DPT}} + (g + h)^2, \quad \alpha \equiv g - \frac{1}{2}, \quad \beta \equiv h - \frac{1}{2}. \]

(3.10)

When applied to the eigenvector \( \phi_n \) (3.5) as \( E_n(g, h) + (g + h)^2 = (2n + g + h)^2 \), we obtain (see, for example, (3.29) and (3.30) of [23])

\[ a^{(-)} \phi_n(x; g, h)/2 = -\sin 2x \frac{d\phi_n(x; g, h)}{dx} + (2n + g + h) \cos 2x \phi_n(x; g, h) \]

\[ + \frac{\alpha^2 - \beta^2}{2n + \alpha + \beta} \phi_n(x; g, h) \]

\[ = \frac{4(n + \alpha)(n + \beta)}{2n + \alpha + \beta} \phi_{n-1}(x; g, h), \]

(3.11)

\[ a^{(+)} \phi_n(x; g, h)/2 = \sin 2x \frac{d\phi_n(x; g, h)}{dx} + (2n + g + h) \cos 2x \phi_n(x; g, h) \]

\[ + \frac{\alpha^2 - \beta^2}{2n + \alpha + \beta + 2} \phi_n(x; g, h) \]

\[ = \frac{4(n + 1)(n + \alpha + \beta + 1)}{2n + \alpha + \beta + 2} \phi_{n+1}(x; g, h). \]

(3.12)

### 3.2. J1 and J2 exceptional Jacobi polynomials

Here we derive the J1 and J2 exceptional Jacobi polynomials. As explained in [9, 10], the exceptional J1 and J2 orthogonal polynomials are ‘mirror images’ of each other, reflecting the parity property \( P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x) \) of the Jacobi polynomial. Here we present both cases in parallel so that the structure of these polynomials can be better understood by comparison.

For each positive integer \( \ell = 1, 2, \ldots \), let us consider the pair of Hamiltonians \( \mathcal{H}_\ell^{(1)}(g, h) \) and \( \mathcal{H}_\ell^{(-1)}(g, h) \) corresponding to the following prepotential (\( \eta \equiv \eta(x) \equiv \cos 2x \)):

**J1:** \( W_\ell(x; g, h) \equiv (g + \ell - 1) \log s \sin x - (h + \ell) \log \cos x + \log \xi_\ell(\eta; g, h), \)

(3.13)

\[ \xi_\ell(\eta; g, h) \equiv P_\ell^{(g + \ell - 1, -h - \ell - 1)}(\eta), \quad g > h > 0, \]

(3.14)

**J2:** \( W_\ell(x; g, h) \equiv -(g + \ell) \log s \sin x + (h + \ell - 1) \log \cos x + \log \xi_\ell(\eta; g, h), \)

(3.15)

\[ \xi_\ell(\eta; g, h) \equiv P_\ell^{(-g - \ell - 1, h + \ell - 1)}(\eta), \quad h > g > 0, \]

(3.16)

\[ \mathcal{H}_\ell^{(1)}(g, h) \equiv A_\ell(g, h) A_\ell(g, h), \quad \mathcal{H}_\ell^{(-1)}(g, h) \equiv A_\ell(g, h) A_\ell^*(g, h), \]

(3.17)

\[ A_\ell(g, h) \equiv \frac{d}{dx} - dW_\ell(x; g, h), \quad A_\ell^*(g, h) = -\frac{d}{dx} - dW_\ell(x; g, h). \]

(3.18)

By simple calculation using the differential equation for the Jacobi polynomial (3.6), we obtain the trigonometric DPT potential for \( \mathcal{H}_\ell^{(1)}(g, h) \) up to an additive constant:
of the J1 and J2 exceptional Jacobi polynomials derived by Odake–Sasaki [1, 10]: 

\[
\mathcal{H}_J^{(+)}(g, h) = p^2 + \frac{(g + \ell - 1)(g + \ell - 2)}{\sin^2 x} + \frac{(h + \ell)(h + \ell + 1)}{\cos^2 x} - (2\ell + g - h - 1)^2,
\]

\[= \mathcal{H}_D^{(+)}(g + \ell - 1, h + \ell + 1) + (2g + 4\ell - 1)(2h + 1),
\]

\[\mathcal{H}_J^{(+)}(g, h) = p^2 + \frac{(g + \ell)(g + \ell + 1)}{\sin^2 x} + \frac{(h + \ell - 1)(h + \ell - 2)}{\cos^2 x} - (2\ell + h - g - 1)^2,
\]

\[= \mathcal{H}_D^{(+)}(g + \ell + 1, h + \ell - 1) + (2h + 4\ell - 1)(2g + 1).
\]

The partner Hamiltonians are

\[\mathcal{H}_J^{(-)}(g, h) = p^2 + \frac{(g + \ell)(g + \ell - 1)}{\sin^2 x} + \frac{(h + \ell)(h + \ell - 1)}{\cos^2 x}
\]

\[+ 2 \left( \frac{\partial \xi_i(x; g, h)}{\xi_i(x; g, h)} \right) \left( 2 ((g + \ell - 1) \cot x + (h + \ell) \tan x) + \frac{\partial \xi_i(x; g, h)}{\xi_i(x; g, h)} \right)
\]

\[= -(2\ell + g - h - 1)^2 + 8(\ell + g - h - 1),
\]

\[\mathcal{H}_J^{(-)}(g, h) = p^2 + \frac{(g + \ell)(g + \ell - 1)}{\sin^2 x} + \frac{(h + \ell)(h + \ell - 1)}{\cos^2 x}
\]

\[+ 2 \left( \frac{\partial \xi_i(x; g, h)}{\xi_i(x; g, h)} \right) \left( -2 ((g + \ell) \cot x + (h + \ell - 1) \tan x) + \frac{\partial \xi_i(x; g, h)}{\xi_i(x; g, h)} \right)
\]

\[= -(2\ell + h - g - 1)^2 + 8(\ell + h - g - 1).
\]

Up to additive constants, the above Hamiltonian (3.23) and (3.24) are equal to the Hamiltonian of the J1 and J2 exceptional Jacobi polynomials derived by Odake–Sasaki [1, 10]:

\[\mathcal{H}_{0-S}^{(g, h)} \overset{\text{def}}{=} p^2 + \left( \frac{dw_i(x; g, h)}{dx} \right)^2 + \frac{d^2w_i(x; g, h)}{dx^2}.
\]

\[w_i(x; g, h) \overset{\text{def}}{=} (g + \ell) \log \sin x + (h + \ell) \log \cos x + \log \frac{\xi_i(x; g + 1, h + 1)}{\xi_i(x; g, h)},
\]

\[\mathcal{H}_J^{(-)}(g, h) = \mathcal{H}_{0-S}^{(g, h)} + (2g + 4\ell - 1)(2h + 1),
\]

\[\mathcal{H}_J^{(-)}(g, h) = \mathcal{H}_{0-S}^{(g, h)} + (2h + 4\ell - 1)(2g + 1).
\]

The definition of \(\xi_i(x; g)\) for the J1 and J2 Odake–Sasaki cases is the same as those given in (3.14) and (3.16). Again let us note that \(e^{W_i(x; g; h)}\) is not square integrable at \(x = \pi/2\) and \(e^{-W_i(x; g; h)}\) is not square integrable at \(x = 0\) for the J1 case, whereas for the J2 case \(e^{W_i(x; g; h)}\) is not square integrable at \(x = 0\) and \(e^{-W_i(x; g; h)}\) is not square integrable at \(x = \pi/2\). But the prepotentials \(W_i(x; g, h)\) (3.13) and (3.15) are regular in the interval \(0 < x < \pi/2\). For this, it is enough to show that \(\xi_i(x; g, h)\) does not have a zero in \(0 < x < \pi/2\). In fact we have

\[\mathcal{H}_J^{(-)}(x; g, h) = \left( \frac{h + \ell}{\ell} \right) \sum_{k=0}^{\ell} \frac{(-1)^k}{\ell!} (\ell - k + 1)_{\ell} (g - h + \ell - 1)_{\ell} (\cos x)^{2k} > 0,
\]
as shown in (2.40) of [9]. Thus, we find that $e^{\lambda W(x; g, h)}$ cannot be the ground states of the Hamiltonians $\mathcal{H}^{(b)}(g, h)$. However, we know well that $\mathcal{H}^{(e)}(g, h)$, being the trigonometric DPT Hamiltonian, has a well-defined ground state. This means that the partner Hamiltonians $\mathcal{H}^{(c)}(g, h)$, thus the Hamiltonians of the J1 and J2 exceptional Jacobi polynomials, are exactly iso-spectral to the trigonometric DPT Hamiltonians $\mathcal{H}^{(e)}(g, h)$, which have the following eigenvalues and the corresponding eigenfunctions:

\[
\begin{align*}
J_1 : & \quad \mathcal{E}^{(e)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell) + (2g + 4\ell - 1)(2h + 1), \\
& \quad \phi^{(e)}_{\ell,n}(x; g, h) = (\sin x)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta), \\
J_2 : & \quad \mathcal{E}^{(e)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell) + (2h + 4\ell - 1)(2g + 1), \\
& \quad \phi^{(e)}_{\ell,n}(x; g, h) = (\sin x)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta). 
\end{align*}
\] (3.30)

The intertwining relation (1.2) implies the simple expressions of the eigenfunctions of the partner Hamiltonians $\mathcal{H}^{(c)}$ in terms of $A_{\ell}(g)$:

\[
\begin{align*}
J_1 : & \quad \mathcal{E}^{(c)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell) + (2g + 4\ell - 1)(2h + 1), \\
& \quad \phi^{(c)}_{\ell,n}(x; g, h) = (\sin x)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta), \\
J_2 : & \quad \mathcal{E}^{(c)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell) + (2h + 4\ell - 1)(1 + 2g), \\
& \quad \phi^{(c)}_{\ell,n}(x; g, h) = (\sin x)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta).
\end{align*}
\] (3.31-3.35)

\[
\begin{align*}
J_1 : & \quad \mathcal{E}^{(s)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell), \\
& \quad \phi^{(s)}_{\ell,n}(x; g, h) = \phi^{(c)}_{\ell,n}(x; g, h) \\
= & \left( \frac{d}{dx} - \frac{dW(x; g, h)}{dx} \right) \left( \sin x \right)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta) \\
= & -2 \left( \sin x \right)^{\ell+1} \left( \cos x \right)^{h+1} \left( \frac{\xi_{\ell}(\eta)}{\xi_{\ell}(\eta)} \right) \left( \frac{1}{2} \right) P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta) \\
& + (1 + \eta) \xi_{\ell}(\eta; g, h) \partial_{\eta} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta), \\
J_2 : & \quad \mathcal{E}^{(s)}_{\ell,n}(g, h) = 4n(n + g + h + 2\ell) + (2h + 4\ell - 1)(1 + 2g), \\
& \quad \phi^{(s)}_{\ell,n}(x; g, h) = \phi^{(c)}_{\ell,n}(x; g, h) \\
= & \left( \frac{d}{dx} - \frac{dW(x; g, h)}{dx} \right) \left( \sin x \right)^{\ell+1} \left( \cos x \right)^{h+1} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta) \\
= & -2 \left( \sin x \right)^{\ell+1} \left( \cos x \right)^{h+1} \left( \frac{\xi_{\ell}(\eta)}{\xi_{\ell}(\eta)} \right) \left( \frac{1}{2} \right) P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta) \\
& - (1 - \eta) \xi_{\ell}(\eta; g, h) \partial_{\eta} P_{n}^{(g+\frac{1}{2},h+\frac{1}{2})}(\eta).
\end{align*}
\] (3.36-3.40)
These are to be compared with the explicit expressions of the exceptional Jacobi polynomials $P_{c,n}(\eta; g, h)$ derived in [10]:

$$
\phi_{c,n}(x; g, h) = \psi_{c}(x; g, h)P_{c,n}(\eta; g, h), \quad \psi_{c}(x; g, h) \overset{\text{def}}{=} \frac{(\sin x)^{\ell}(\cos x)^{h+\ell}}{\xi_{c}(\eta; g, h)},
$$

$$
\xi_{c}(\eta; g, h) \overset{\text{def}}{=} \begin{cases} P_{c}^{(g+\frac{1}{2}+h, h^{-\ell})}(\eta), & g > h > 0: \ J_{1} \\ P_{c}^{(g-\ell-\frac{1}{2}, h-h^{+\ell})}(\eta), & h > g > 0: \ J_{2}, \end{cases}
$$

$$
P_{c,n}(\eta; g, h) \overset{\text{def}}{=} \begin{cases} (n+h+\frac{1}{2})^{-1}((h+\frac{1}{2})\xi_{c}(\eta; g+1, h+1)P_{n}^{(g+\frac{1}{2}, h-h^{+\ell})}(\eta) \\
\quad + (1+\eta)\xi_{c}(\eta; g, h)\partial_{\eta}P_{n}^{(g+\frac{1}{2}, h-h^{+\ell})}(\eta)) \quad : \ J_{1} \\
(n+g+h\frac{1}{2})^{-1}((g+\frac{1}{2})\xi_{c}(\eta; g+1, h+1)P_{n}^{(g+\frac{1}{2}, h-h^{+\ell})}(\eta) \\
\quad - (1-\eta)\xi_{c}(\eta; g, h)\partial_{\eta}P_{n}^{(g-h^{+\ell}, h^{+\ell})}(\eta)) \quad : \ J_{2}. \end{cases}
$$

The final expressions (3.37) and (3.40) are the same as (2.3) of [10] up to a multiplicative constant. The identity of the Jacobi polynomials (E.22) of [10] is used. Thus, we have derived the Hamiltonians as well as the eigenfunctions of the $J_{1}$ and $J_{2}$ exceptional Jacobi polynomials from those of the trigonometric DPT by the Darboux–Crum transformations.

The exceptional Laguerre and Jacobi polynomials satisfy a second-order linear differential equation in the entire complex $\eta$ plane:

$$
\tilde{\cal{H}}_{c}^{O-S}(\lambda)P_{c,n}(\eta; \lambda) = \tilde{E}_{c,n}(\lambda)P_{c,n}(\eta; \lambda), \quad \tilde{E}_{c,n}(\lambda) = \tilde{E}_{n}(\lambda+\ell\delta). \tag{3.41}
$$

For later use we give the explicit form of the second-order Fuchsian differential operator $\tilde{\cal{H}}_{c}^{O-S}$ which was given in (3.5) of [10]:

$$
\tilde{\cal{H}}_{c}^{O-S}(\lambda) = -4\left(c_{2}(\eta)\frac{d^{2}}{d\eta^{2}} + (c_{1}(\eta, \lambda + \ell\delta) - 2c_{2}(\eta)\partial_{\eta}\log \xi_{c}(\eta; \lambda))\frac{d}{d\eta} + 2d_{1}(\lambda)\frac{c_{2}(\eta)}{d_{2}(\eta)}\frac{\partial_{\eta}\xi_{c}(\eta; \lambda + \delta)}{\xi_{c}(\eta; \lambda)} + \frac{1}{4} \tilde{E}_{c}(\lambda + \delta) \right). \tag{3.42}
$$

where $c_{1}, c_{2}, d_{1}$ and $\tilde{E}_{c}$ are given by

$$
c_{1}(\eta, \lambda) \overset{\text{def}}{=} \begin{cases} g + \frac{1}{2} - \eta : \ L \\ h - g - (g + h + 1)\eta : \ J, \end{cases}, \quad c_{2}(\eta) \overset{\text{def}}{=} \begin{cases} \eta : \ L \\ 1 - \eta^{2} : \ J, \end{cases} \tag{3.43}
$$

$$
d_{1}(\lambda) \overset{\text{def}}{=} \begin{cases} 1 : \ L_{1} \\ g + \frac{1}{2} : \ L_{2}, \ J_{2} \\ h + \frac{1}{2} : \ J_{1}, \end{cases}, \quad d_{2}(\eta) \overset{\text{def}}{=} \begin{cases} 1 : \ L_{1} \\ -\eta : \ L_{2} \\ \mp(1 \pm \eta) : \ J_{1}/J_{2}, \end{cases} \tag{3.44}
$$

$$
\tilde{E}_{c}(\lambda) \overset{\text{def}}{=} \begin{cases} \mp 4\ell : \ L_{1}/L_{2} \\ 4\ell(\ell \pm g \mp h - 1) : \ J_{1}/J_{2}. \end{cases} \tag{3.45}
$$

### 4. Generalized Bochner problem: bispectral property

The exceptional Laguerre polynomials $L_{1}$ and $L_{2}$ as well as the exceptional Jacobi polynomials $J_{1}$ and $J_{2}$ belong to complete orthogonal families of functions (the completeness follows from the well-known properties of the Darboux process; indeed, the Darboux transformation which does not generate new eigenstates preserves the completeness of the transformed system of eigenfunctions as the solutions of the self-adjoint Schrödinger equation). These
polynomials, however, do not belong to the ordinary families of orthogonal polynomials because polynomials of the first \( \ell - 1 \) degrees are absent in these systems \([14]\). Hence these polynomials do not satisfy the three-term recurrence relation which is a characteristic property of nondegenerate orthogonal polynomials (see, e.g., \([24]\)). Nevertheless, as we will show, the exceptional polynomials \( J_1, J_2, L_1 \) and \( L_2 \) do satisfy \((4\ell + 1)\)-term recurrence relations, i.e. they can be considered as eigenvectors of a semi-infinite matrix \( K \) having \( 4\ell + 1 \) diagonals. In this sense the considered exceptional polynomials possess a very important bispectral property \([20]\): they are simultaneously the eigenfunctions of a Sturm–Liouville operator and a matrix \( K \).

4.1. Bispectrality of the exceptional Jacobi polynomials

We consider exceptional orthogonal polynomials of the \( J_1 \) type (the type \( J_2 \) can be considered in the same manner). For simplicity of presentation we slightly change the previous notation, denoting the exceptional \( J_1 \) polynomials as \( \hat{P}_n(x) \) and introducing the standard parameters

\[
a \equiv \frac{g + \ell - \frac{1}{2}}{2}, \quad b \equiv \frac{h + \ell + \frac{1}{2}}{2}.
\]

We also use \( x \) for the sinusoidal coordinate \( \eta \) and denote \( \pi(x) = \xi_\ell(x; g, h) = P^{(a-b)}_\ell(x) \) which is a polynomial of degree \( \ell \). This polynomial will play a crucial role in the following. Then formula (O-S2.3) for the \( J_1 \) polynomials can be presented in the form

\[
\hat{P}_n(x) = \pi(x)\left((1 + x)P^{(a,b)}_n(x) + bP^{(a,b)}_n(x)\right) - \pi'(x)(1 + x)P^{(a,b)}_n(x), \tag{4.1}
\]

which is equal to \( P_{\ell,n}(x; g, h) \) in (O-S2.3) up to a multiplicative constant. Recall that the Jacobi polynomials \( P^{(a,b)}_n(x) \) satisfy the three-term recurrence relation

\[
A_n P^{(a,b)}_{n+1}(x) + B_n P^{(a,b)}_n(x) + C_n P^{(a,b)}_{n-1}(x) = x P^{(a,b)}_n(x), \tag{4.2}
\]

with

\[
A_n = \frac{2(n + 1)(n + a + b + 1)}{(2n + a + b + 1)(2n + a + b + 2)}, \quad B_n = \frac{b^2 - a^2}{(2n + a + b)(2n + a + b + 2)}, \quad C_n = \frac{2(n + a)(n + b)}{(2n + a + b)(2n + a + b + 1)}.
\]

They are orthogonal in the interval \([-1, 1]\)

\[
\int_{-1}^{1} P^{(a,b)}_n(x) P^{(a,b)}_m(x)(1 - x)^a(1 + x)^b dx = h_n \delta_{nm}, \tag{4.3}
\]

where

\[
h_n \equiv \frac{2^{a+b+1}\Gamma(n + a + 1)\Gamma(n + b + 1)}{n!(2n + a + b + 1)\Gamma(n + a + b + 1)}
\]

is the normalization constant. The exceptional \( J_1 \) polynomials are orthogonal on the same interval

\[
(\hat{P}_n, \hat{P}_m) \equiv \int_{-1}^{1} \hat{P}_n(x) \hat{P}_m(x) \hat{w}(x) dx = \hat{h}_n \delta_{nm}, \tag{4.4}
\]

with the weight function

\[
\hat{w}(x) = \frac{(1 - x)^{a+1}(1 + x)^{b-1}}{\pi^a(x)} \tag{4.5}
\]

and some nonzero normalization coefficients \( \hat{h}_n \) (in fact, these coefficients can easily be connected with the coefficients \( h_n \); however, we do not need their explicit expressions here).
Using the elementary properties of the Jacobi polynomials [21], or (2.24) of [10], we also have

\[
\hat{P}_n(x) = (b + n) \pi (x) P_n^{(a+1,b-1)}(x) - \pi'(x) (1 + x) P_n^{(a,b)}(x).
\]  

(4.6)

Moreover, there are classical formulas [21]

\[
(2n + a + b + 1)(1 + x) P_n^{(a,b)}(x) = 2(n + 1) P_{n+1}^{(a,b-1)}(x) + 2(n + b) P_n^{(a,b-1)}(x),
\]

(4.7)

and

\[
(2n + a + b + 1) P_n^{(a,b)}(x) = (n + a + b + 1) P_n^{(a+1,b)}(x) - (n + b) P_{n-1}^{(a+1,b)}(x).
\]

(4.8)

These formulas have a simple interpretation in terms of so-called Darboux transformations of the orthogonal polynomials [22]. Namely, formula (4.7) describes the Christoffel transformation of the Jacobi polynomials \( P_n^{(a,b-1)}(x) \rightarrow P_n^{(a+1,b)}(x) \) while formula (4.8) describes the Geronimus transformation \( P_n^{(a+1,b)}(x) \rightarrow P_n^{(a,b)}(x) \).

Combining formulas (4.7) and (4.8), we obtain

\[
(1 + x) P_n^{(a,b)}(x) = \alpha_n P_{n+1}^{(a+1,b-1)}(x) + \beta_n P_n^{(a+1,b-1)}(x) + \gamma_n P_{n-1}^{(a+1,b-1)}(x),
\]

\[
\alpha_n \overset{\text{def}}{=} \frac{2(n + 1)(n + a + b + 1)}{(2n + a + b + 1)(2n + a + b + 2)}, \quad \beta_n \overset{\text{def}}{=} \frac{2(a + b)(n + b)}{(2n + a + b)(2n + a + b + 2)}, \quad \gamma_n \overset{\text{def}}{=} \frac{2(n + b)(n + b - 1)}{(2n + a + b)(2n + a + b + 1)}.
\]

(4.9)

Using formula (4.9) with the repeated use of the three-term recurrence relation (4.2) for the factors \( \pi(x) \) and \( \pi'(x) \) in (4.6), we conclude that

\[
\hat{P}_n(x) = \sum_{s=-d}^{n+t} \xi_{ns} P_s^{(a+1,b-1)}(x),
\]

(4.10)

with some real coefficients \( \xi_{ns} \).

Formula (4.10) establishes a ‘local’ property of the exceptional polynomials \( \hat{P}_n(x) \) with respect to the basis \( P_n^{(a+1,b-1)}(x), n = 0, 1, 2, \ldots \). Recall that \( \deg(\hat{P}_n(x)) = n + \ell \) and hence degrees of the polynomials on the lhs and rhs of (4.10) coincide. There is a reciprocal ‘local’ relation which expresses the polynomials \( \pi^2(x) P_n^{(a+1,b-1)}(x) \) in terms of a finite linear combination of the exceptional polynomials \( \hat{P}_n(x) \). In order to get this relation, we can expand the product of the polynomials \( \pi^2(x) P_n^{(a+1,b-1)}(x) \) in terms of the basis \( \hat{P}_n(x) \):

\[
\pi^2(x) P_n^{(a+1,b-1)}(x) = \sum_{s=0}^{\infty} \eta_{ns} \hat{P}_s(x).
\]

(4.11)

This sum in general can be infinite, because the polynomials \( \hat{P}_n(x) \) form a basis in a Hilbert space with the scalar product given by formula (4.4) but due to existence of a ‘gap’ in degrees of polynomials \( \hat{P}_n(x) \) for a generic polynomial \( Q(x) \) we should have an expansion

\[
Q(x) = \sum_{s=0}^{\infty} \zeta_{ns} \hat{P}_s(x)
\]

with infinitely many coefficients \( \zeta_{ns} \). However, for some special choices of the polynomial \( Q(x) \) this expansion can contain only a finite number of terms.

In order to find the coefficients \( \eta_{ns} \), let us multiply both sides of (4.11) by \( \hat{w}(x) \hat{P}_s(x) \) (\( \hat{w}(x) \) is given by (4.5)) and integrate over the interval \([-1, 1]\). Due to the orthogonality relation for the polynomials \( \hat{P}_n(x) \), on the rhs after integration we obtain the term \( \hat{h}_n \eta_{ns} \) with the nonzero coefficient \( \hat{h}_n \) in (4.4). On the other hand, on the lhs we have the integral

\[
\int_{-1}^{1} (1 - x)^{a+1} (1 + x)^{b-1} P_n^{(a+1,b-1)}(x) \hat{P}_s(x) \, dx.
\]

(4.12)
Substituting expression (4.10) into (4.12) and using the orthogonality property of the Jacobi polynomials \( P_{n}^{(a+1,b-1)}(x) \), we have the relation
\[
\hat{h}_{n} \eta_{ns} = h_{n} \xi_{ns}.
\]  
(4.13)

Hence only \( 2\ell + 1 \) coefficients \( \eta_{ns} \) are nonzero. We thus have the expansion
\[
\pi^{2}(x) P_{n}^{(a+1,b-1)}(x) = \sum_{s=n-\ell}^{n+\ell} \eta_{ns} \hat{P}_{s}(x),
\]  
(4.14)

where the coefficients \( \eta_{ns} \) are connected with \( \xi_{ns} \) by the ‘mirror’ relation (4.13). Formulas (4.10) and (4.14) can be considered as a generalization for generic \( \ell \) of corresponding formulas obtained for \( \ell = 1 \) in [14].

Eliminating the Jacobi polynomials \( P_{n}^{(a+1,b-1)}(x) \) from these formulas, we arrive at the recurrence relation
\[
\pi^{2}(x) \hat{P}(x) = \sum_{s=n-2\ell}^{n+2\ell} K_{ns} \hat{P}_{s}(x), \quad \hat{P}_{s}(x) = 0, \quad \text{if} \quad s < 0,
\]  
(4.15)

with some real coefficients \( K_{ns} \). This recurrence relation belongs to the class of \( (4\ell + 1) \)-diagonal relations. This means that in the operator form relation (4.15) can be presented as
\[
K \hat{P} = \pi^{2}(x) \hat{P},
\]  
(4.16)

where \( \hat{P} \) is an infinite dimensional vector with the components \( \hat{P}_{s}(x) \), \( n = 0, 1, 2, \ldots \),
\[
\hat{P} = \{ \hat{P}_{0}(x), \hat{P}_{1}(x), \hat{P}_{2}(x), \ldots \},
\]

and \( K \) is a matrix with the entries \( K_{ns} \). This matrix has no more than \( 4\ell + 1 \) nonzero diagonals. The corresponding polynomials \( \hat{P}_{s}(x) \) satisfy the \( (4\ell + 1) \)-term recurrence relation (4.15). The ordinary orthogonal polynomials satisfy the three-term recurrence relation. Hence, we have polynomials \( \hat{P}_{s}(x) \) satisfying more general recurrence relation. Recurrence relations of such a type were studied e.g. by Durán and Van Assche [25] who showed that such polynomials should satisfy a matrix orthogonality relation. Thus, the exceptional Jacobi polynomials belong to the class of polynomials satisfying higher order recurrence relations. Note, however, that in the approach described in [25] it is required that polynomials \( \hat{P}_{s}(x) \) have exactly the degree \( n = 0, 1, 2, \ldots \). In our case the polynomial \( \hat{P}_{s}(x) \) has the degree \( n + \ell, \quad n = 0, 1, 2, \ldots \). This means that the methods described in [25] should be modified if applied to the case of the exceptional polynomials.

Formulas (4.10) and (4.14) admit an interesting algebraic interpretation. Introduce semi-infinite matrices \( \Xi \) and \( H \) by their entries \( \xi_{ns} \) and \( \eta_{ns} \). Then we have
\[
\pi^{2}(x) \hat{P}(x) = H \Xi \hat{P}(x),
\]  
(4.17)

where
\[
\hat{P}(x) = \{ P_{0}^{(a+1,b-1)}(x), P_{1}^{(a+1,b-1)}(x), P_{2}^{(a+1,b-1)}(x), \ldots \},
\]

and
\[
\pi^{2}(x) \hat{P} = \Xi H \hat{P}, \quad K = \Xi H.
\]  
(4.18)

From relation (4.17) it follows that
\[
\pi^{2}(J) = H \Xi,
\]  
(4.19)
where $J$ is a Jacobi (three-diagonal) matrix corresponding to the Jacobi polynomials $P_{n^{(a+1,b-1)}}(x)$, i.e.

$$x \hat{P}(x) = J \hat{P}(x).$$

Thus, the matrices $H$ and $\Xi$ appear under factorization of the $(4\ell + 1)$-diagonal matrix $\pi^2(J)$. The exceptional polynomials $\hat{P}_n(x)$ satisfy the recurrence relation (4.18) which appear after refactorization (permutation) of the factors $H$ and $\Xi$. Such permutation of the matrix factors is known as the Darboux transformation of the matrix $\pi^2(J)$. Similar Darboux transformations were already studied in [20] where refactorization of the quadratic polynomials in $J$ corresponding to the Jacobi and Laguerre polynomials was considered. Clearly, under such refactorization one obtains new polynomials satisfying a five-term recurrence relation. These polynomials are not classical orthogonal polynomials. Nevertheless, the authors of [20] showed that these new polynomials are the eigenfunctions of a linear fourth-order differential operator. In our case we have almost the same construction as in [20] but the resulting polynomials are exceptional (i.e. some degrees are absent) and satisfy a second-order differential equation.

It is also interesting to note that the weight function $\hat{w}(x)$, given by (4.5), differs from the ‘classical’ weight function $(1 - x)^{a+1}(1 + x)^{b-1}$ for the Jacobi polynomials by the factor $\pi^{-2}(x)$. For the ordinary orthogonal polynomials it is known that a rational modification of the weight function $w(x) \to w(x)U(x)^{-1}$ with some polynomial $U(x)$ of degree $M$ is equivalent to the application of $M$ Geronimus transforms [22]. If $P_n(x)$ are the orthogonal polynomials corresponding to the weight function $w(x)$, then the orthogonal polynomials $\tilde{P}_n(x)$ corresponding to the weight function $w(x)U(x)^{-1}$ are given by the linear combination [22]

$$\tilde{P}_n(x) = A^{(0)}_n P_n(x) + A^{(1)}_n P_{n-1}(x) + \ldots + A^{(M)}_n P_{n-M}(x), \quad (4.20)$$

with some coefficients $A^{(i)}_n$, $i = 0, 1, 2, \ldots M$. Formula (4.20) describes a special case of the $M$-time Geronimus transform (a general case of the Geronimus transform allows an addition of $M$ concentrated masses located on the roots of the polynomial $U(x)$ [22]).

If we now compare formulas (4.10) and (4.20) we see that the exceptional J1 Jacobi polynomials $\tilde{P}_n(x)$ are connected with the ordinary Jacobi polynomials $P_n^{(a+1,b-1)}(x)$ by a transformation which resembles the ordinary $M$-time Geronimus transform, where $M = 2\ell$. However, this transformation can be considered as a degeneration of the ordinary Geronimus transform in a sense that the first $\ell$ polynomials $\tilde{P}_i(x), \ i = 0, 1, \ldots, \ell - 1$, become zero.

The case of the J2 exceptional Jacobi polynomials can be considered in the same manner leading to a recurrence relation of the same type (4.15).

### 4.2. Bispectrality of the exceptional Laguerre polynomials

Consider the case of the exceptional L1 Laguerre polynomials. Here we list some well-known formulas for the ordinary Laguerre polynomials (see, e.g., [21]):

recurrence relation: $-(n + 1)L_{n+1}^{(a)}(x) + (2n + a + 1)L_n^{(a)}(x) - (n + a)L_{n-1}^{(a)}(x) = xL_n^{(a)}(x), \quad (4.21)$

differentiation formula: $L_n^{(a)}(x) = -L_{n-1}^{(a+1)}(x), \quad (4.22)$

geronomius transformation: $L_n^{(a)}(x) = L_n^{(a+1)}(x) - L_{n-1}^{(a+1)}(x). \quad (4.23)$
Following (2.11), we have \( \pi(x) = \xi_\ell(x; g) = L_\ell^{(a)}(-x) \), where \( a \stackrel{\text{def}}{=} g + \ell - 3/2 \). We denote the \( L_1 \) exceptional Laguerre polynomials as \( \hat{L}_n(x) \), which are equal to \( P_{\ell,n}(x; g) \) given in (O-S2.1) up to an overall sign

\[
\hat{L}_n(x) = \pi(x) L(a)_{n} \quad (\ell + 1 - x),
\]

where \( a \) is defined as \( g + \ell - 3/2 \). We denote the \( L_1 \) exceptional Laguerre polynomials as \( \hat{L}_n(x) \), which are equal to \( P_{\ell,n}(x; g) \) given in (O-S2.1) up to an overall sign

\[
\hat{L}_n(x) = \pi(x) L(a)_{n} \quad (\ell + 1 - x),
\]

\( \ell \) is the degree of \( \hat{L}_n(x) \). Hence, by the recurrence relation (4.21) we have

\[
\hat{L}_n(x) = \sum_{s=n-\ell}^{n+\ell} \xi_{ns} L_{s}^{(a+1)}(x),
\]

with some real coefficients \( \xi_{ns} \). This is a formula expressing the polynomials \( \hat{L}_n(x) \) as a linear combination of the Laguerre polynomials \( L_{s}^{(a+1)}(x) \). In order to obtain the reciprocal formula let us consider the expression

\[
\pi^2(x) L_{n}^{(a+1)}(x) = \sum_{k=0}^{\infty} \eta_{nk} \hat{L}_k(x),
\]

with some real coefficients \( \eta_{nk} \). Multiply both sides of (4.29) by \( \hat{L}_s(x) \) and integrate with the weight function (4.26) over the interval \((0, \infty)\). We then get

\[
\hat{h}_s \eta_{ns} = \int_0^{\infty} x^{a+1} e^{-x} L_{n}^{(a+1)}(x) \hat{L}_s(x) \frac{d x}{\pi^2(x)}.
\]

Substitute expression (4.28) for \( \hat{L}_s(x) \) in terms of \( L_{k}^{(a+1)}(x) \) into the rhs of (4.30). Using the orthogonality relation for the Laguerre polynomials, we have

\[
\hat{h}_s \eta_{ns} = \hat{h}_n \xi_{ns}.
\]

We already have that \( \xi_{ns} = 0 \) if \( |n - s| > \ell \). Hence \( \eta_{ns} = 0 \) if \( |n - s| > \ell \) and we have the expansion with \( (2\ell + 1) \) terms:

\[
\pi^2(x) L_{n}^{(a+1)}(x) = \sum_{k=n-\ell}^{n+\ell} \eta_{nk} \hat{L}_k(x),
\]

where the coefficients \( \eta_{nk} \) are connected with \( \xi_{ns} \) by (4.31).

Similar to the case of the \( J_1 \) exceptional Jacobi polynomials we obtain a \( (4\ell + 1) \)-term recurrence relation

\[
\pi^2(x) \hat{L}_n(x) = \sum_{k=n-2\ell}^{n+2\ell} K_{nk} \hat{L}_k(x), \quad \hat{L}_s(x) = 0, \quad \text{if } s < 0.
\]
The case of the L2 exceptional Laguerre polynomials can be considered in a similar manner leading to a recurrence relation of the same type (4.33).

One can formulate a natural conjecture that all the exceptional orthogonal polynomials (i.e. polynomials satisfying a linear second-order Sturm–Liouville equation and orthogonal to one another, see the detailed description of this problem in [14]) satisfy a recurrence relation of type (4.15) or (4.33) with an appropriately defined polynomial π(x).

It is worthwhile to stress that the function π(x) plays the important role of the eigenvalue of the recurrence relation (4.15) or (4.33). In the ordinary three-term recurrence relation (4.2) or (4.21), the same role is played by the (sinusoidal) coordinate x itself.

4.3. Invariant polynomial subspaces

Here we will show that the bispectral property of the exceptional Laguerre and Jacobi polynomials (4.15) and (4.33) can easily be understood as the characteristic feature of the invariant polynomial subspaces of the second-order Fuchsian differential operator \( \tilde{\mathcal{H}}_{\ell}^{0-O} (3.41) \). Because of the denominators containing π(x) ≡ ξ\(\ell\)(x; λ), the operator \( \tilde{\mathcal{H}}_{\ell}^{0-O} \) does not map a generic polynomial in x into a polynomial. In other words

\[
\tilde{\mathcal{H}}_{\ell}^{0-O}(\lambda) V_n \not\subseteq V_n, \quad V_n \overset{\text{def}}{=} \text{Span}[1, x, \ldots, x^n], \quad n = 0, 1, 2, \ldots \tag{4.34}
\]

However, it is easy to see that

\[
\tilde{\mathcal{H}}_{\ell}^{0-O}(\lambda) \pi(x)^2 V_n \subseteq V_{n+2\ell}, \quad n = 0, 1, 2, \ldots \tag{4.35}
\]

In other words, \( \pi(x)^2 V_n \) belongs to the degree \( n + 2\ell \) invariant polynomial subspace of the operator \( \tilde{\mathcal{H}}_{\ell}^{0-O} \), see section 6 of [10]. Therefore, \( \pi(x)^2 \tilde{P}_n(x) \) and \( \pi(x)^2 \tilde{L}_n(x) \) also belong to the degree \( n + 3\ell \) invariant polynomial subspace and they can be expressed as a linear combination of \( \{\tilde{P}_n(x)\} \) (\( \{\tilde{L}_n(x)\} \)):

\[
\pi(x)^2 \tilde{P}_n(x) = \sum_{s=0}^{n+2\ell} K_{ns} \tilde{P}_s(x), \quad \pi(x)^2 \tilde{L}_n(x) = \sum_{s=0}^{n+2\ell} K_{ns} \tilde{L}_s(x), \quad n = 0, 1, \ldots \tag{4.36}
\]

Next let us consider the inner products (4.4) and (4.25) for \( s < n - 2\ell \):

\[
(\pi(x)^2 \tilde{P}_n, \tilde{P}_s) = (\tilde{P}_n, \pi(x)^2 \tilde{P}_s), \quad (\pi(x)^2 \tilde{L}_n, \tilde{L}_s) = (\tilde{L}_n, \pi(x)^2 \tilde{L}_s).
\]

Since

\[
\pi(x)^2 \tilde{P}_s(x) = \sum_{j=0}^{s+2\ell} K_{sj} \tilde{P}_j, \quad \pi(x)^2 \tilde{L}_s(x) = \sum_{j=0}^{s+2\ell} K_{sj} \tilde{L}_j,
\]

we find

\[
(\pi(x)^2 \tilde{P}_n, \tilde{P}_s) = 0, \quad (\pi(x)^2 \tilde{L}_n, \tilde{L}_s) = 0, \quad \implies K_{ns} = 0, \quad s < n - 2\ell. \tag{4.37}
\]

This simply means the desired results

\[
\pi(x)^2 \tilde{P}_n(x) = \sum_{s=n-2\ell}^{n+2\ell} K_{ns} \tilde{P}_s(x), \quad \pi(x)^2 \tilde{L}_n(x) = \sum_{s=n-2\ell}^{n+2\ell} K_{ns} \tilde{L}_s(x), \quad n = 0, 1, \ldots \tag{4.38}
\]

This is essentially the same argument for demonstrating the three-term recurrence relation.
5. Comments and discussion

A few remarks are in order. Dutta and Roy [17] derived the first two members of the L1 exceptional Laguerre polynomials in a way shown in section 2. In our language, they used a prepotential:

\[ W_\ell(x; g) \equiv \frac{x^2}{2} + (g + 1) \log x + \log u_\ell(x; g), \quad (5.1) \]
\[ u_\ell(x; g) \equiv \ell^{\scriptstyle(x+\frac{1}{2})}(-x^2), \quad \ell = 1, 2. \quad (5.2) \]

But erroneously they insist that the partner Hamiltonian \( \mathcal{H}_\ell^{(-)}(g) \) is not shape invariant on account of the fact that its partner is the radial oscillator Hamiltonian.

Let us emphasize that the shape invariance is the intrinsic property of the Hamiltonian, or the potential, determined by the unique factorization in terms of the ground-state wavefunction. Allowing for singularities, a generic quantum mechanical Hamiltonian \( \mathcal{H} \) has infinitely many different factorizations. For example, if

\[ \mathcal{H} = p^2 + V(x), \quad \mathcal{H}\phi_n(x) = E_n\phi_n(x), \quad (5.3) \]

then it is trivial to show the following factorization:

\[ \mathcal{H} = \left( -\frac{d^2}{dx^2} - \frac{\partial_x\phi_n(x)}{\phi_n(x)} \right) \left( \frac{d}{dx} - \frac{\partial_x\phi_n(x)}{\phi_n(x)} \right) + E_n, \quad n \geq 1. \quad (5.4) \]

The right-hand side as a whole is non-singular, but each factor is singular, since \( \phi_n(x) \) has \( n \) zeros. Usually the ground state \( n = 0 \) provides the unique non-singular factorization.

The results of the present paper assert that the radial oscillator (DPT) potential admits two families, L1 and L2 (J1 and J2), of infinitely many non-singular factorizations. Perhaps it would be worthwhile to reflect upon the significance of the infinitely many non-singular factorizations in historical perspective. More than three decades ago, Miller [26] embarked on the program to classify and exhaust factorizations of shape-invariant Hamiltonians (\( \mathcal{H} = -\frac{d^2}{dx^2} + V(x) \)). Although he did not employ the word ‘shape invariance’, the essential ingredients were there. He considers the cases in which the Hamiltonian depends on one parameter only, say, \( g \). Based on an assumption, rephrased in our notation, that the derivative of the prepotential has a finite power dependence on \( g \)

\[ \frac{d^j\omega(x; g)}{dx} = \sum_{j=N}^{M} \alpha_j(x) g^j, \quad N, M \in \mathbb{Z}_+, \quad (5.5) \]

he came to the conclusion that the allowed factorization types were the same as those listed by Infeld–Hull [6]. The present cases of the radial oscillator, for example the L2 (2.12),

\[ L_2 : \quad \frac{dW_\ell(x; g)}{dx} = -x - g \frac{\partial_x\xi_\ell(n; g)}{\xi_\ell(n; g)} , \quad (5.6) \]

are not covered by his assumption.

It should be remarked that the present method provides an alternative proof of shape invariance of the Odake–Sasaki Hamiltonians [1, 2, 9, 10]. Here we will show the shape invariance of the Hamiltonians of the exceptional Laguerre polynomials. Starting from the prepotential \( W_\ell(x; g) \), (2.10) for L1 and (2.12) for L2, we obtain the Hamiltonian \( \mathcal{H}_\ell^{(-)}(g) \), (2.16) for L1 and (2.18) for L2 and the partner Hamiltonian \( \mathcal{H}_\ell^{(+)}(g) \), (2.20) for L1 and (2.21) for L2. Then, we can construct another partner Hamiltonian of \( \mathcal{H}_\ell^{(+)}(g) \) by using the factorization in terms of the ground-state wavefunction \( e^{-x^2/2} x^{g+\ell - 1} \) for L1 and \( e^{-x^2/2} x^{g+\ell+1} \) for L2.
for \( L_2 \). By the shape invariance of the radial oscillator Hamiltonian, the result is simply \( \mathcal{H}^{(±)}(g + 1) + 4 \). Thus, it is also obtained by the shifted prepotential \( W_ℓ(x; g + 1) \), which produces the partner Hamiltonian \( \mathcal{H}^{(−)}_ℓ(g + 1) \):

\[
\begin{align*}
\mathcal{H}^{(±)}_ℓ(g + 1) & \quad W_ℓ(x; g + 1) \quad \mathcal{H}^{(−)}_ℓ(g + 1) \\
\text{groundstate wavefunction} & \quad \text{groundstate wavefunction} \\
\mathcal{H}^{(±)}_ℓ(g) & \quad W_ℓ(x; g) \quad \mathcal{H}^{(−)}_ℓ(g)
\end{align*}
\]

Laguerre polynomial \quad L_1, L_2 exceptional Laguerre polynomial

Two ways of proving shape invariance.

The direct proof of shape invariance of \( \mathcal{H}^{(−)}_ℓ(g) \) by using the ground-state wavefunction was given in [9]. Obviously, the above proof of shape invariance is also valid for the Hamiltonians of the \( J_1, J_2 \) exceptional Jacobi polynomials, discussed in section 3.

The annihilation/creation operators of the Hamiltonians of the \( L_1, L_2 \) exceptional Laguerre polynomials are obtained from those of the corresponding radial oscillator Hamiltonians (2.16), (2.18),

\[
A_ℓ(g) a_ℓ^{(±)} A_ℓ^†(g), \quad \text{(5.7)}
\]

where \( a_ℓ^{(±)} \) are defined in (2.8) with the replacement \( g \to g + ℓ - 1 \) for \( L_1 \) and \( g \to g + ℓ + 1 \) for \( L_2 \). Similarly the annihilation/creation operators of the Hamiltonians of the \( J_1, J_2 \) exceptional Jacobi polynomials are obtained from those of the corresponding DPT Hamiltonians (3.11) and (3.12) by proper replacements of the parameters. The creation/annihilation operators for the \( L_1 \) exceptional Laguerre polynomials are mentioned in Dutta–Roy paper [17] for the \( ℓ = 1 \) and 2 cases.

The three-term recurrence relations for the Laguerre and Jacobi polynomials are mapped to those of the exceptional orthogonal polynomials simply by \( A_ℓ \). The explicit forms are given in [10]. This is definitely different from the bispectral property discussed in section 4.

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