AN EXTENSION OF THE PAIRING THEORY
BETWEEN DIVERGENCE–MEASURE FIELDS AND BV FUNCTIONS

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ABSTRACT. In this paper we introduce a nonlinear version of the notion of Anzellotti’s pairing between divergence–measure vector fields and functions of bounded variation, motivated by possible applications to evolutionary quasilinear problems. As a consequence of our analysis, we prove a generalized Gauss–Green formula.

1. Introduction

In recent years the pairing theory between divergence–measure vector fields and BV functions, initially developed by Anzellotti [6, 7], has been extended to more general situations (see e.g. [10–15, 27, 29, 30] and the references therein). These extensions are motivated, among others, by applications to hyperbolic conservation laws and transport equations [2, 3, 10–14, 17, 19–21], problems involving the 1-Laplace operator [5, 26], the prescribed mean curvature problem [27, 28] and to lower semicontinuity problems in BV [8, 22, 23].

Another major related result concerns the Gauss–Green formula and its applications (see e.g. [6, 9, 15, 16, 18, 27]).

Let us describe the problem in more details. Let $\mathcal{DM}^\infty$ denote the class of bounded divergence–measure vector fields $A: \mathbb{R}^N \to \mathbb{R}^N$, i.e. the vector fields with the properties that $A$ is bounded and $\text{div } A$ is a finite Radon measure. If $A \in \mathcal{DM}^\infty$ and $u$ is a function of bounded variation with precise representative $u^*$, then Chen and Frid [10] proved that

$$\text{div}(uA) = u^* \text{div } A + \mu,$$

where $\mu$ is a Radon measure, absolutely continuous with respect to $|Du|$. This measure $\mu$ has been denoted by Anzellotti [6] with the symbol $(A, Du)$ and by Chen and Frid with $A \cdot Du$, and it is called the pairing between the divergence–measure field $A$ and the gradient of the BV function $u$. The characterization of the decomposition of this measure into absolutely continuous, Cantor and jump parts has been studied in [10, 18]. In particular, the analysis of the jump part has been considered in [18] using the notion of weak normal traces of a divergence–measure vector field on oriented countably $H^{N-1}$-rectifiable sets given in [2].

In view to applications to evolutionary quasilinear problems, our aim is to extend the pairing theory from the product $uA$ to the mixed case $B(x, u)$. Our main assumptions on $B$ are that $B(\cdot, w) \in \mathcal{DM}^\infty$ for every $w \in \mathbb{R}$, $B(x, \cdot)$ is of class $C^1$ and the least upper bound

$$\sigma := \bigvee_{w \in \mathbb{R}} |\text{div}_x \partial_w B(\cdot, w)|$$

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is a Radon measure. (See Section 3.1 for the complete list of assumptions.) We remark that, in the case $B(x, w) = w A(x)$, with $A \in \mathcal{D}\mathcal{M}^\infty$, these assumptions are automatically satisfied with $\sigma = |\text{div} A|$. More precisely, we will prove that, if $u \in BV \cap L^\infty$, then the composite function $v(x) := B(x, u(x))$ belongs to $\mathcal{D}\mathcal{M}^\infty$ and

$$\text{div}[B(x, u(x))] = \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma + \mu,$$

in the sense of measures. Here $F(\cdot, w)$ denotes the Radon–Nikodym derivative of the measure $\text{div}_x B(\cdot, w)$ with respect to $\sigma$, $u^\pm$ are the approximate limits of $u$ and $\mu \equiv (\partial_w B(\cdot, u), Du)$ is again a Radon measure, absolutely continuous with respect to $|Du|$ (see Theorem 3.3). We recall that the analogous chain rule for BV vector fields has been proved in [1]. Notice that, when $B(x, w) = w A(x)$, then $\mu = (A, Du)$ is exactly the Anzellotti’s pairing between $A$ and $Du$. Even for general vector fields $B(x, w)$ we can prove the following characterization of the decomposition $\mu = \mu^{ac} + \mu^c + \mu^j$ of this measure into absolutely continuous, Cantor and jump parts (see Theorem 4.6):

$$\mu^{ac} = \langle \partial_w B(x, u(x)), \nabla u(x) \rangle \mathcal{L}^N,$$

$$\mu^c = \left\langle \partial_w B(x, \tilde{u}(x)), Du \right\rangle,$$

$$\mu^j = [\beta^+(x, u^+(x)) - \beta^-(x, u^-(x))] \mathcal{H}^{N-1} J_u,$$

where, for every $w \in \mathbb{R}$, $\beta^\pm(\cdot, w)$ are the normal traces of $B(\cdot, w)$ on $J_u$ and $\beta^\pm(\cdot, w) := [\beta^+(\cdot, w) + \beta^-(\cdot, w)]/2$, and $\mu^c$ is the Cantor part of the measure. We remark that, to prove the representation formula for $\mu^c$, we need an additional technical assumption (see (3.1) in Theorem 4.6).

We recall that a similar characterization of $\text{div}[B(x, u(x))]$ has been proved in [19] under stronger assumptions, including the existence of the strong traces of $B(\cdot, w)$.

As a consequence of our analysis we prove that, if $E \subset \mathbb{R}^N$ is a bounded set with finite perimeter, then the following Gauss–Green formula holds:

$$\int_{E^1} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2} \; d\sigma(x) + \mu(E^1) = - \int_{\partial^* E} \beta^+(x, u^+(x)) \; d\mathcal{H}^{N-1},$$

where $E^1$ is the measure theoretic interior of $E$, and $\partial^* E$ is the reduced boundary of $E$ (see Theorem 4.1). In the particular case $B(x, w) = w A(x)$, the analogous formula has been proved in [18 Theorem 5.1]. We recall that the map $x \mapsto \beta^+(x, u^+(x))$ in the last integral coincides with the interior weak normal trace of the vector field $x \mapsto B(x, u(x))$ on $\partial^* E$ (see Proposition 4.3).

The plan of the paper is the following. In Section 2 we recall some known results on functions of bounded variation, divergence–measure vector fields and their normal traces, and the Anzellotti’s pairing.

In Section 3 we list the assumptions on $B$ and we prove a number of its basic properties, including some regularity result of the weak normal traces. Finally, we prove that for every $u \in BV \cap L^\infty$, the composite function $x \mapsto B(x, u(x))$ belongs to $\mathcal{D}\mathcal{M}^\infty$.

In Section 4 we prove our main results on the pairing measure $\mu$ and its properties.

Finally, in Section 5 we describe some gluing construction and we prove an extension theorem that will be used in Section 6 to prove a Gauss–Green formula for weakly regular domains.
2. Preliminaries

In the following, $\Omega$ will always denote a nonempty open subset of $\mathbb{R}^N$.

Let $u \in L^1_{\text{loc}}(\Omega)$. We say that $u$ has an approximate limit at $x_0 \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N(B_r(x_0))} \int_{B_r(x_0)} |u(x) - z| \, dx = 0. \tag{1}$$

The set $S_u \subset \Omega$ of points where this property does not hold is called the approximate discontinuity set of $u$. For every $x_0 \in \Omega \setminus S_u$, the number $z$, uniquely determined by (1), is called the approximate limit of $u$ at $x_0$ and denoted by $\tilde{u}(x_0)$.

We say that $x_0 \in \Omega$ is an approximate jump point of $u$ if there exist $a, b \in \mathbb{R}$ and a unit vector $\nu \in \mathbb{R}^N$ such that $a \neq b$ and

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N(B_r^+(x_0))} \int_{B_r^+(x_0)} |u(y) - a| \, dy = 0,$$

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N(B_r^-(x_0))} \int_{B_r^-(x_0)} |u(y) - b| \, dy = 0,$$

where $B_r^\pm(x_0) := \{ y \in B_r(x_0) : \pm(y - x_0) \cdot \nu > 0 \}$. The triplet $(a, b, \nu)$, uniquely determined by (2) up to a permutation of $(a, b)$ and a change of sign of $\nu$, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. The set of approximate jump points of $u$ will be denoted by $J_u$.

The notions of approximate discontinuity set, approximate limit and approximate jump point can be obviously extended to the vectorial case (see [4, §3.6]).

In the following we shall always extend the functions $u^\pm$ to $\Omega \setminus (S_u \setminus J_u)$ by setting

$$u^\pm \equiv \tilde{u} \text{ in } \Omega \setminus S_u.$$

**Definition 2.1** (Strong traces). Let $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and let $\mathcal{J} \subset \mathbb{R}^N$ be a countably $\mathcal{H}^{N-1}$-rectifiable set oriented by a normal vector field $\nu$. We say that two Borel functions $u^\pm: \mathcal{J} \to \mathbb{R}$ are the strong traces of $u$ on $\mathcal{J}$ if for $\mathcal{H}^{N-1}$-almost every $x \in \mathcal{J}$ it holds

$$\lim_{r \to 0^+} \int_{B_r^\pm(x)} |u(y) - u^\pm(x)| \, dy = 0,$$

where $B_r^\pm(x) := B_r(x) \cap \{ y \in \mathbb{R}^N : \pm(y - x, \nu(x)) \geq 0 \}$.

Here and in the following we will denote by $\rho \in C^\infty(\mathbb{R}^N)$ a symmetric convolution kernel with support in the unit ball, and by $\rho_\varepsilon(x) := \varepsilon^{-N} \rho(x/\varepsilon)$.

In the sequel we will use often the following result.

**Proposition 2.2.** Let $u \in L^1_{\text{loc}}(\Omega)$ and define

$$u_\varepsilon(x) = \rho_\varepsilon \ast u(x) := \int_\Omega \rho_\varepsilon(x-y) u(y) \, dy.$$  

If $x_0 \in \Omega \setminus S_u$, then $u_\varepsilon(x_0) \to \tilde{u}(x_0)$ as $\varepsilon \to 0^+$.

**Proposition 2.3.** Let $E$ be a Lebesgue measurable subset of $\mathbb{R}^N$ and $G$ a Borel subset of $\mathbb{R}^M$. Let $g: E \times G \to \mathbb{R}$ be a Borel function such that for $L^N$–a.e. $x \in E$ the function $g(x, \cdot)$ is continuous on $G$. Then there exists an $L^N$–null set $\mathcal{M} \subset \mathbb{R}^N$ such that for every $t \in G$ the function $g(\cdot, t)$ is approximately continuous in $E \setminus \mathcal{M}$.
Proof. (See the proof of Theorem 1.3, p. 539 in [23].) By the Scorza–Dragoni theorem (see [25] Theorem 6.35), for every \( i \in \mathbb{Z}^+ \) there exists a closed set \( K_i \subset E \), with \( \mathcal{L}^N(E \setminus K_i) < 1/i \), such that the restriction of \( g \) to \( K_i \times G \) is continuous. Let \( K_i^* \) be the set of points with density \( 1 \) of \( K_i \), and define

\[
\mathcal{E} := \bigcup_{i=1}^\infty (K_i \cap K_i^*).
\]

Since \( \mathcal{L}^N(K_i) = \mathcal{L}^N(K_i \cap K_i^*) \), it holds

\[
\mathcal{L}^N(E \setminus \mathcal{E}) \leq \mathcal{L}^N(E \setminus K_i) \leq \frac{1}{i} \quad \forall i \in \mathbb{Z}^+,
\]

and so \( \mathcal{L}^N(E \setminus \mathcal{E}) = 0 \).

Let us fix \( t \in G \) and let us prove that the function \( u(x) := g(x,t) \) is approximately continuous on \( \mathcal{E} \). Let \( \varepsilon > 0 \) and \( x_0 \in \mathcal{E} \). By definition of \( \mathcal{E} \), there exists an index \( i \in \mathbb{Z}^+ \) such that \( x_0 \in K_i \cap K_i^* \). Since the restriction of \( u \) to \( K_i \) is continuous, there exists \( \delta > 0 \) such that

\[
|u(x) - u(x_0)| < \varepsilon \quad \forall x \in B_\delta(x_0) \cap K_i.
\]

As a consequence, for every \( r \in (0, \delta) \), it holds

\[
\mathcal{L}^N(B_r(x_0) \cap \{ x \in E : |u(x) - u(x_0)| \geq \varepsilon \}) \leq \mathcal{L}^N(B_r(x_0) \setminus K_i),
\]

hence the conclusion follows since \( x_0 \) is a Lebesgue point of \( K_i \). \qed

Corollary 2.4. Let \( E \) be a Lebesgue measurable subset of \( \mathbb{R}^N \) and \( G \) a Borel subset of \( \mathbb{R}^M \). Let \( g : E \times G \to \mathbb{R} \) be a Borel function such that for \( \mathcal{L}^N \)-a.e. \( x \in E \) the function \( g(x, \cdot) \) is continuous on \( G \). Let us define

\[
g_\varepsilon (x, t) := \int_{\mathbb{R}^N} \rho_\varepsilon (x - y) g(y, t) \, dy.
\]

Then there exists an \( \mathcal{L}^N \)-null set \( Z \subset \mathbb{R}^N \) such that for every \( t \in G \) and for every \( x \in E \setminus Z \) we have \( g_\varepsilon(x_0, t) \to g(x_0, t) \), as \( \varepsilon \to 0^+ \).

2.1. Functions of bounded variation and sets of finite perimeter. We say that \( u \in L^1(\Omega) \) is a function of bounded variation in \( \Omega \) if the distributional derivative \( Du \) of \( u \) is a finite Radon measure in \( \Omega \). The vector space of all functions of bounded variation in \( \Omega \) with respect to \( Du \) is a finite Radon measure in \( \Omega \). The vector space of all functions of bounded variation in \( \Omega \) with respect to \( Du \) is a finite Radon measure in \( \Omega \). Moreover, we will denote by \( BV_{loc}(\Omega) \) the set of functions \( u \in L^1_{loc}(\Omega) \) that belong to \( BV(A) \) for every open set \( A \subset \Omega \) (i.e., the closure \( \overline{A} \) of \( A \) is a compact subset of \( \Omega \)).

If \( u \in BV(\Omega) \), then \( Du \) can be decomposed as the sum of the absolutely continuous and the singular part with respect to the Lebesgue measure, i.e.,

\[
Du = D^a u + D^s u, \quad D^a u = \nabla u \mathcal{L}^N,
\]

where \( \nabla u \) is the approximate gradient of \( u \), defined \( \mathcal{L}^N \)-a.e. in \( \Omega \). On the other hand, the singular part \( D^s u \) can be further decomposed as the sum of its Cantor and jump part, i.e.,

\[
D^s u = D^c u + D^j u, \quad D^c u := D^s u \mathcal{L}(\Omega \setminus S_u), \quad D^j u := D^s u \mathcal{L} J_u,
\]

where the symbol \( \mu \mathcal{L} B \) denotes the restriction of the measure \( \mu \) to the set \( B \). We will denote by \( D^2 u := D^a u + D^s u \) the diffuse part of the measure \( Du \).

In the following, we will denote by \( \theta_u : \Omega \to S^{N-1} \) the Radon–Nikodým derivative of \( Du \) with respect to \( |Du| \), i.e. the unique function \( \theta_u \in L^1(\Omega, |Du|)^N \) such that the polar
decomposition \( Du = \theta_u |Du| \) holds. Since all parts of the derivative of \( u \) are mutually singular, we have
\[
\begin{align*}
D^a u = \theta_u |D^a u|, \quad D^j u = \theta_u |D^j u|, \quad D^e u = \theta_u |D^e u|
\end{align*}
\]

as well. In particular \( \theta_u(x) = \nabla u(x)/|\nabla u(x)| \) for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \) such that \( \nabla u(x) \neq 0 \) and \( \theta_u(x) = \text{sign}(u^+(x) - u^-(x)) \nu_u(x) \) for \( \mathcal{H}^{N-1} \)-a.e. \( x \in J_u \).

Let \( E \) be an \( \mathcal{L}^N \)-measurable subset of \( \mathbb{R}^N \). For every open set \( \Omega \subset \mathbb{R}^N \) the perimeter \( P(E, \Omega) \) is defined by
\[
P(E, \Omega) := \sup \left\{ \int_E \text{div} \varphi \, dx : \varphi \in C^1_c(\Omega, \mathbb{R}^N), \|\varphi\|_{\infty} \leq 1 \right\}.
\]

We say that \( E \) is of finite perimeter in \( \Omega \) if \( P(E, \Omega) < +\infty \).

Denoting by \( \chi_E \) the characteristic function of \( E \), if \( E \) is a set of finite perimeter in \( \Omega \), then \( D\chi_E \) is a finite Radon measure in \( \Omega \) and \( P(\Omega, \Omega) = |D\chi_E| (\Omega) \).

If \( \Omega \subset \mathbb{R}^N \) is the largest open set such that \( E \) is locally of finite perimeter in \( \Omega \), we call reduced boundary \( \partial^r E \) of \( E \) the set of all points \( x \in \Omega \) in the support of \( |D\chi_E| \) such that the limit
\[
\tilde{\nu}_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E| (B_\rho(x))}
\]
events in \( \mathbb{R}^N \) and satisfies \( |\tilde{\nu}_E(x)| = 1 \). The function \( \tilde{\nu}_E : \partial^r E \to S^{N-1} \) is called the measure theoretic unit normal to \( E \).

A fundamental result of De Giorgi (see [4, Theorem 3.59]) states that \( \partial^r E \) is countably \((N - 1)\)-rectifiable and \( |D\chi_E| = \mathcal{H}^{N-1} \cap \partial^r E \).

Let \( E \) be an \( \mathcal{L}^N \)-measurable subset of \( \mathbb{R}^N \). For every \( t \in [0, 1] \) we denote by \( E^t \) the set
\[
E^t := \left\{ x \in \mathbb{R}^N : \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} = t \right\}
\]
of all points where \( E \) has density \( t \). The sets \( E^0, E^1, \partial^r E := \mathbb{R}^N \setminus (E^0 \cup E^1) \) are called respectively the measure theoretic exterior, the measure theoretic interior and the essential boundary of \( E \). If \( E \) has finite perimeter in \( \Omega \), Federer’s structure theorem states that \( \partial^r E \cap \Omega \subset E^{1/2} \subset \partial^r E \) and \( \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^r E \cup E^1)) = 0 \) (see [4, Theorem 3.61]).

2.2. **Divergence–measure fields.** We will denote by \( \mathcal{DM}^\infty(\Omega) \) the space of all vector fields \( A \in L^\infty(\Omega, \mathbb{R}^N) \) whose divergence in the sense of distributions is a bounded Radon measure in \( \Omega \). Similarly, \( \mathcal{DM}^\infty_{\text{loc}}(\Omega) \) will denote the space of all vector fields \( A \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N) \) whose divergence in the sense of distributions is a Radon measure in \( \Omega \). We set \( \mathcal{DM}^\infty = \mathcal{DM}^\infty(\mathbb{R}^N) \).

We recall that, if \( A \in \mathcal{DM}^\infty_{\text{loc}}(\Omega) \), then \( |\text{div} A| \ll \mathcal{H}^{N-1} \) (see [10, Proposition 3.1]). As a consequence, the set
\[
\Theta_A := \left\{ x \in \Omega : \limsup_{r \to 0^+} \frac{\text{div} A(B_r(x))}{r^{N-1}} > 0 \right\},
\]
is a Borel set, \( \sigma \)-finite with respect to \( \mathcal{H}^{N-1} \), and the measure \( \text{div} A \) can be decomposed as
\[
\text{div} A = \text{div}^a A + \text{div}^e A + \text{div}^j A,
\]
where \( \text{div}^a A \) is absolutely continuous with respect to \( \mathcal{L}^N \), \( \text{div}^e A(B) = 0 \) for every set \( B \) with \( \mathcal{H}^{N-1}(B) < +\infty \), and
\[
\text{div}^j A = h \mathcal{H}^{N-1} \cup \Theta_A
\]
for some Borel function \( h \) (see [3, Proposition 2.5]).
2.3. Anzellotti’s pairing. As in Anzellotti [6] (see also [10]), for every $A \in \mathcal{DM}_\infty^0(\Omega)$ and $u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ we define the linear functional $(A, Du)$: $C_\infty^0(\Omega) \rightarrow \mathbb{R}$ by

\begin{equation}
\langle (A, Du), \varphi \rangle := - \int_\Omega u^* \varphi \, d\text{div} A - \int_\Omega u \cdot \nabla \varphi \, dx.
\end{equation}

The distribution $(A, Du)$ is a Radon measure in $\Omega$, absolutely continuous with respect to $|Du|$ (see [6, Theorem 1.5] and [10, Theorem 3.2]), hence the equation

\begin{equation}
\text{div}(uA) = u^* \text{div} A + (A, Du)
\end{equation}

holds in the sense of measures in $\Omega$. (We remark that, in [10], the measure $(A, Du)$ is denoted by $A \cdot Du$.) Furthermore, Chen and Frid in [10] proved that the absolutely continuous part of this measure with respect to the Lebesgue measure is given by $(A, Du)^a = A \cdot \nabla u L^N$.

3. Assumptions on the vector field and preliminary results

As we have explained in the Introduction, we are willing to compute the divergence of the composite function $v(x) := B(x, u(x))$ with $u \in BV$, where $B(\cdot, t) \in \mathcal{DM}_\infty^0$ and $B(x, \cdot) \in C^1$. Nevertheless, it will be convenient to state our assumptions on the vector field $b(x, t) := \partial_t B(x, t)$.

In Section 3.1 we list the assumptions on $b$ and we prove a number of basic properties of $b$ and $B$.

Then, in Section 3.2 we prove some regularity result of the weak normal traces of $B$.

Finally, in Section 3.3 we prove that $v \in \mathcal{DM}_\infty^0$.

3.1. Assumptions on the vector field $B$. In this section we list and comment all the assumptions on the vector field $B(x, t)$.

Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set. Let $b: \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N$ be a function satisfying the following assumptions:

(i) $b$ is a locally bounded Borel function;
(ii) for $\mathcal{L}^N$-a.e. $x \in \Omega$, the function $b(x, \cdot)$ is continuous in $\mathbb{R}$;
(iii) for every $t \in \mathbb{R}$, $b(\cdot, t) \in \mathcal{DM}^\infty_{\text{loc}}(\Omega)$;
(iv) the least upper bound

\[ \sigma := \bigvee_{t \in \mathbb{R}} |\text{div}_x b(\cdot, t)| \]

is a Radon measure. (See [4, Def. 1.68] for the definition of least upper bound of measures.)

We remark that, since $\text{div}_x b(\cdot, t) \ll \mathcal{H}^{N-1}$ for every $t \in \mathbb{R}$, then also $\sigma \ll \mathcal{H}^{N-1}$.

From (i) and Proposition 2.3 it follows that there exists a set $Z_1 \subset \mathbb{R}^N$ such that $\mathcal{L}^N(Z_1) = 0$ and, for every $t \in \mathbb{R}$, the function $x \mapsto b(x, t)$ is approximately continuous on $\mathbb{R}^N \setminus Z_1$.

By definition of least upper bound of measures, we have that $\text{div}_x b(\cdot, t) \ll \sigma$ for every $t \in \mathbb{R}$. If we denote by $f(\cdot, t)$ the Radon–Nikodym derivative of $\text{div}_x b(\cdot, t)$ with respect to $\sigma$, we have

\[ \text{div}_x b(\cdot, t) = f(\cdot, t) \sigma, \quad f(\cdot, t) \in L^1(\sigma), \quad \forall t \in \mathbb{R}. \]

Moreover, since $|\text{div}_x b(\cdot, t)| \leq \sigma$, we have

\begin{equation}
\forall t \in \mathbb{R} : \quad |f(x, t)| \leq 1 \quad \text{for } \sigma\text{-a.e. } x \in \Omega.
\end{equation}
Let us extend \( b \) to \( 0 \) in \( (\mathbb{R}^N \setminus \Omega) \times \mathbb{R} \), so that the vector field

\[
B(x, t) := \int_0^t b(x, s) \, ds, \quad x \in \mathbb{R}^N, \ t \in \mathbb{R},
\]

is defined for all \( (x, t) \in \mathbb{R}^N \times \mathbb{R} \). Moreover \( B(x, 0) = 0 \) for every \( x \in \mathbb{R}^N \) and, from (ii), for every \( x \in \mathbb{R}^n \) one has \( b(x, t) = \partial_t B(x, t) \) for every \( t \in \mathbb{R} \).

**Lemma 3.1.** For every \( t \in \mathbb{R} \) it holds \( B(\cdot, t) \in \mathcal{DM}_\text{loc}^\infty (\Omega) \) and \( \text{div}_x B(\cdot, t) \ll \sigma \). If we denote by \( F(\cdot, t) \) the Radon–Nikodým derivative of \( \text{div}_x B(\cdot, t) \) with respect to \( \sigma \), we have that

\[
F(x, t) = \int_0^t f(x, s) \, ds, \quad \text{for } \sigma-\text{a.e. } x \in \Omega,
\]

and, for every \( t, s \in \mathbb{R} \),

\[
|F(x, t) - F(x, s)| \leq |t - s| \quad \text{for } \sigma-\text{a.e. } x \in \Omega.
\]

**Proof.** Let \( \varphi \in C^1_c (\Omega) \). We have that

\[
\int_\Omega \nabla \varphi(x) \cdot B(x, t) \, dx = \int_\Omega \nabla \varphi(x) \cdot \int_0^t b(x, s) \, ds \, dx
\]

\[
= - \int_0^t \left( \int_\Omega \varphi(x) f(x, s) \, d\sigma(x) \right) \, ds
\]

\[
= - \int_0^t \varphi(x) \left( \int_0^t f(x, s) \, ds \right) \, d\sigma(x).
\]

Hence \( \text{div}_x B(\cdot, t) \) is a Radon measure, it is absolutely continuous with respect to \( \sigma \) and its Radon–Nikodým derivative with respect to \( \sigma \) is \( F(\cdot, t) \).

For every non-negative \( \varphi \in C^1_c (\Omega) \) and for every \( t, s \in \mathbb{R} \), taking into account that for every \( s \in \mathbb{R} \), \( |f(x, s)| \leq 1 \) for \( \sigma-\text{a.e. } x \in \Omega \), an analogous integration gives

\[
\left| \int_\Omega \nabla \varphi(x) \cdot [B(x, t) - B(x, s)] \, dx \right| = \left| \int_s^t \left( \int_\Omega \varphi(x) f(x, w) \, d\sigma(x) \right) \, dw \right|
\]

\[
\leq |t - s| \int_\Omega \varphi(x) \, d\sigma(x),
\]

so that

\[
|\text{div}_x B(\cdot, t)(A) - \text{div}_x B(\cdot, s)(A)| \leq |t - s| \sigma(A)
\]

for every Borel set \( A \subseteq \Omega \), hence [9] follows.

**Lemma 3.2.** There exists a set \( Z_2 \subset \Omega \), with \( \sigma(Z_2) = 0 \) and \( \mathcal{H}^{N-1}(Z_2) = 0 \), such that every \( x_0 \in \Omega \setminus Z_2 \) is a Lebesgue point of \( F(\cdot, t) \) with respect to the measure \( \sigma \) for every \( t \in \mathbb{R} \).

**Proof.** Let \( S \subset \mathbb{R} \) be a countable dense subset of \( \mathbb{R} \) and, for every \( s \in S \), let \( \Omega_s \) be the set of the points \( x_0 \in \Omega \) such that the Radon-Nikodým derivative \( F(\cdot, s) \) of \( \text{div}_x B(\cdot, s) \) exists in \( x_0 \) and \( x_0 \) is a Lebesgue point of the function \( F(\cdot, s) \) with respect to the measure \( \sigma \).

Define

\[
Z_2 := \mathbb{R}^N \setminus \bigcap_{s \in S} \Omega_s.
\]

Clearly \( \sigma(Z_2) = 0 \) and \( \mathcal{H}^{N-1}(Z_2) = 0 \). We claim that every \( x_0 \in \Omega \setminus Z_2 \) is a Lebesgue point of the function \( F(\cdot, w) \) for all \( w \in \mathbb{R} \).
Fix $x_0 \in \Omega \setminus Z_2$ and $w \in \mathbb{R}$. Since the set $S$ is dense in $\mathbb{R}$ we may find a sequence $(w_n)$ of points in $S$ such that $w_n \to w$. Then, by (3),

$$|F(x_0, w) - F(x, w)| \leq |F(x_0, w) - F(x_0, w_n)| + |F(x_0, w_n) - F(x, w_n)| + |F(x, w_n) - F(x, w)| \leq |F(x_0, w_n) - F(x, w_n)| + 2|w - w_n|.$$ 

By averaging over $B_r(x_0)$ and letting $r \to 0^+$ in the previous inequality we get

$$\limsup_{r \to 0^+} \frac{1}{\sigma(B_r(x_0))} \int_{B_r(x_0)} |F(x_0, w) - F(x, w)| \, d\sigma \leq 2|w - w_n|,$$

where we have used the fact that $x_0$ is a Lebesgue point of $F(\cdot, w_n)$ w.r.t. the measure $\sigma$. Since $w_n \to w$ letting $n \to \infty$, the previous inequality proves that every $x_0 \in \Omega \setminus Z_2$ is a Lebesgue point of the function $F(\cdot, w)$ for all $w \in \mathbb{R}$. \hfill $\Box$

### 3.2. Weak normal traces of $B$.

Let $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$-rectifiable set. By Lemma 3.1 it follows that, for almost every $t \in \mathbb{R}$, the traces of the normal component of the vector field $B(\cdot, t)$ can be defined as distributions $\text{Tr}^\pm(B(\cdot, t), \Sigma)$ in the sense of Anzellotti (see [2],[6],[10]). It turns out that these distributions are induced by $L^\infty$ functions on $\Sigma$, still denoted by $\text{Tr}^\pm(B(\cdot, t), \Sigma)$, and

$$\|\text{Tr}^\pm(B(\cdot, t), \Sigma)\|_{L^\infty(\Sigma, \mathcal{H}^{N-1})} \leq \|B(\cdot, t)\|_{L^\infty(\Omega)}.$$

More precisely, let us briefly recall the construction given in [2] (see Proposition 3.4 and Remark 3.3 therein). Since $\Sigma$ is countably $\mathcal{H}^{N-1}$-rectifiable, we can find countably many oriented $C^1$ hypersurfaces $\Sigma_i$, with classical normal $\nu_{\Sigma_i}$, and pairwise disjoint Borel sets $N_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$.

Moreover, it is not restrictive to assume that, for every $i$, there exist two open bounded sets $\Omega_i, \Omega_i'$ with $C^1$ boundary and outer normal vectors $\nu_{\Omega_i}$ and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial \Omega_i \cap \partial \Omega_i'$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \quad \forall x \in N_i.$$

At this point we choose, on $\Sigma$, the orientation given by $\nu_{\Sigma}(x) := \nu_{\Sigma_i}(x)$ $\mathcal{H}^{N-1}$-a.e. on $N_i$.

Using the localization property proved in [2, Proposition 3.2], for every $t \in \mathbb{R}$ we can define the normal traces of $B(\cdot, t)$ on $\Sigma$ by

$$\beta^- (\cdot, t) := \text{Tr}(B(\cdot, t), \partial \Omega_i), \quad \beta^+ (\cdot, t) := -\text{Tr}(B(\cdot, t), \partial \Omega_i'), \quad H^{N-1} \text{- a.e. on } N_i.$$

These two normal traces belong to $L^\infty(\Sigma)$ (see [2, Proposition 3.2]) and

$$\text{div}_x B(\cdot, t) |\Sigma = [\beta^+(\cdot, t) - \beta^-(\cdot, t)] |\Sigma.$$

**Lemma 3.3.** The maps $t \mapsto \beta^\pm(\cdot, t)$ are Lipschitz continuous from $\mathbb{R}$ to $L^\infty(\Sigma)$. More precisely

$$\|\beta^\pm (\cdot, t) - \beta^\pm (\cdot, s)\|_{L^\infty(\Sigma)} \leq \|b\|_{\infty} |t - s| \quad \forall t, s \in \mathbb{R}.$$

**Proof.** It is enough to observe that, for every $i$, the map $X \mapsto \text{Tr}(X, \partial \Omega_i)$ is linear in $\mathcal{D}M^\infty$ and, by Proposition 3.2 in [2], it holds

$$\|\text{Tr}(X, \partial \Omega_i)\|_{L^\infty(\partial \Omega_i)} \leq \|X\|_{L^\infty(\Omega_i)} \quad \forall X \in \mathcal{D}M^\infty,$$

and the same inequality holds when $\Omega_i$ is replaced by $\Omega_i'$. Hence (12) follows from Assumption (i). \hfill $\Box$
In Step 6 of the proof of Theorem 4.3 we will use these normal traces on the set $\Sigma := J_u$, where $J_u$ is the jump set of a $BV$ function $u$. More precisely, we will use a slightly stronger property, stated in the following proposition.

**Proposition 3.4.** There exist a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, and representatives $\overline{\beta}^\pm$ of $\beta^\pm$ (in the sense that, for every $t \in \mathbb{R}$, $\overline{\beta}^\pm(\cdot, t) = \beta^\pm(\cdot, t)$ $\mathcal{H}^{N-1}$-a.e. in $\Sigma$) such that

\[
|\beta^\pm(x, t) - \overline{\beta}^\pm(x, s)| \leq \|b\|_\infty |t - s| \quad \forall x \in \Sigma', \ t, s \in \mathbb{R}.
\]

**Proof.** Let $Q \subset \mathbb{R}$ be a countable dense subset of $\mathbb{R}$. There exists a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, such that

\[
|\beta^\pm(x, p) - \beta^\pm(x, q)| \leq M|p - q| \quad \forall x \in \Sigma', \ p, q \in Q,
\]

where $M := \|b\|_\infty$. Let us define the functions $\overline{\beta}^\pm$ in the following way. If $q \in Q$, we define $\overline{\beta}^\pm(x, q) = \beta^\pm(x, q)$ for every $x \in \Sigma$. If $t \in \mathbb{R} \setminus Q$, let $(q_j) \subset Q$ be a sequence converging to $t$, and define

\[
\overline{\beta}^\pm(x, t) := \begin{cases} 
\lim_{j \to +\infty} \beta^\pm(x, q_j), & \text{if } x \in \Sigma', \\
\beta^\pm(x, t), & \text{if } x \in \Sigma \setminus \Sigma'.
\end{cases}
\]

(We remark that, for every $x \in \Sigma'$, by (13) $(\beta^\pm(x, q_j))_j$ is a Cauchy sequence, hence it is convergent, in $\mathbb{R}$. Moreover, its limit is independent of the choice of the sequence $(q_j) \subset Q$ converging to $t$.) From Lemma 3.3 we have that

\[
|\beta^\pm(x, q_j) - \beta^\pm(x, t)| \leq M|q_j - t| \quad \text{for } \mathcal{H}^{N-1}$-a.e. $x \in \Sigma.
\]

Passing to the limit as $j \to +\infty$, it follows that $\overline{\beta}^\pm(\cdot, t) = \beta^\pm(\cdot, t)$ $\mathcal{H}^{N-1}$-a.e. on $\Sigma$.

Let $t, s \in \mathbb{R}$, and let $(t_j), (s_j) \subset Q$ be two sequences in $Q$ converging respectively to $t$ and $s$. From (13) we have that

\[
|\beta^\pm(x, t_j) - \beta^\pm(x, t_s)| \leq M|t_j - t_s| \quad \forall x \in \Sigma', \ j, s \in \mathbb{N},
\]

hence (13) follows passing to the limit as $j \to +\infty$. \hfill \Box

**Remark 3.5.** In what follows we will always denote by $\beta^\pm$ the representatives $\overline{\beta}^\pm$ of Proposition 3.4.

### 3.3. Basic estimates on the composite function

In this section we shall use a regularization argument to prove that the composite function $v(x) := B(x, u(x))$ belongs to $\mathcal{D}^1_0(\Omega)$.

**Lemma 3.6.** Let $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ satisfy assumptions (i), (ii), extended to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$, and let $B : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ be defined by (14). For every $\varepsilon > 0$ and every $t \in \mathbb{R}$ let $b_\varepsilon(\cdot, t) := \rho_\varepsilon \ast b(\cdot, t)$ and $B_\varepsilon(\cdot, t) := \rho_\varepsilon \ast B(\cdot, t)$. Then it holds:

(a) there exists an $\mathcal{L}^N$-null set $Z \subset \Omega$ such that

\[
\lim_{\varepsilon \to 0^+} b_\varepsilon(x, t) = b(x, t), \quad \lim_{\varepsilon \to 0^+} B_\varepsilon(x, t) = B(x, t), \quad \forall (x, t) \in (\Omega \setminus Z) \times \mathbb{R};
\]

(b) $\partial_1 B_\varepsilon(x, t) = b_\varepsilon(x, t)$ for every $(x, t) \in \Omega_\varepsilon \times \mathbb{R}$, where $\Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \}$.

**Proof.** The conclusion (a) for $b_\varepsilon$ follows directly from Corollary 2.4.

By the Fubini–Tonelli theorem we have that

\[
B_\varepsilon(x, t) = \int_0^t b_\varepsilon(x, s) \, ds, \quad \forall (x, t) \in \Omega_\varepsilon \times \mathbb{R},
\]
hence (b) follows. Moreover, for every \( x \in \mathbb{R}^N \setminus Z \) and every \( t \in \mathbb{R} \), the conclusion (a) for \( B_\varepsilon \) follows from the Dominated Convergence Theorem. \( \square \)

**Lemma 3.7.** Let \( b \) satisfy assumptions (i)--(iv) and let \( B \) be defined by (17). Then, for every \( u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \), the function \( v : \Omega \to \mathbb{R}^N \), defined by

\[
v(x) := B(x, u(x)), \quad x \in \Omega,
\]

belongs to \( DM^\infty_{loc}(\Omega) \) and

\[
|\nabla v|(K) \leq \|u\|_{L^\infty(K)}|\sigma(K) + \|b\|_{\infty}|Du|(K) \quad \forall K \subset \Omega, \ K \text{ compact.}
\]

Moreover, the functions \( v_\varepsilon(x) := B_\varepsilon(x, u(x)) \) converge to \( v \) a.e. in \( \Omega \) and in \( L^1_{loc}(\Omega) \).

**Remark 3.8.** In the following we shall use the notation

\[
\nabla_x B(x, u(x)) := (\nabla_x B)(x, u(x)) = \nabla_x B(x, t)|_{t=u(x)}.
\]

The divergence of the composite function \( x \mapsto B(x, u(x)) \) will be denoted by \( \nabla [B(x, u(x))] \).

**Proof.** Using the same notation of Lemma 3.6 one has

\[
\nabla_x B_\varepsilon(\cdot, t) = F(\cdot, t) \rho_\varepsilon * \sigma.
\]

Since \( F(x, t) = \int_0^t f(x, s) \, ds \) and \( |f(x, s)| \leq 1 \), we have that, for every \( t \in \mathbb{R} \),

\[
|\nabla_x B_\varepsilon(x, t)| \leq |t| \rho_\varepsilon * \sigma(x), \quad \text{for } \sigma\text{-a.e. } x \in \Omega.
\]

Let us define \( v_\varepsilon(x) := B_\varepsilon(x, u(x)) \). Consider first the case \( u \in C^1(\Omega) \cap L^\infty(\Omega) \). We have that

\[
\nabla v_\varepsilon(x) = \nabla_x B_\varepsilon(x, u(x)) + \langle b_\varepsilon(x, u(x)), \nabla u(x) \rangle,
\]

hence, from (18), for every compact subset \( K \) of \( \Omega \) and for every \( \varepsilon > 0 \) small enough it holds

\[
\int_K |\nabla v_\varepsilon| \, dx \leq \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x) \, dx + \|b\|_{\infty} \int_K |\nabla u(x)| \, dx.
\]

By an approximation argument (see [31] Theorem 5.3.3), when \( u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \) we get

\[
\int_K |\nabla v_\varepsilon| \, dx \leq \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x) \, dx + \|b\|_{\infty} |Du|(K).
\]

Finally, (17) follows observing that, by Lemma 3.6(a), \( B_\varepsilon(x, u(x)) \rightarrow B(x, u(x)) \) for \( L^N \)-a.e. \( x \in \Omega \), hence \( v_\varepsilon \rightarrow v \) in \( L^1_{loc}(\Omega) \).

\( \square \)

4. Main results

The main results of the paper are stated in Theorems 4.3 and 4.6. As a preliminary step, we will prove Theorems 4.3 under the additional regularity assumption \( u \in W^{1,1} \).

**Theorem 4.1** (\( DM^\infty \)-dependence and \( u \in W^{1,1} \)). Let \( b \) satisfy assumptions (i)--(iv) and let \( B \) be defined by (17). Then, for every \( u \in W^{1,1}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \), the function \( v : \Omega \to \mathbb{R}^N \), defined by

\[
v(x) := B(x, u(x)), \quad x \in \Omega,
\]

belongs to \( DM^\infty_{loc}(\Omega) \) and the following equality holds in the sense of measures:

\[
\nabla v = F(\cdot, t) \sigma + \langle \partial_t B(x, u(x)), \nabla u(x) \rangle \, d^N,
\]

where \( F(\cdot, t) \) is the Radon–Nikodým derivative of \( \nabla_x B(\cdot, t) \) with respect to \( \sigma \) (see Section 3.7).
Then by (ii) and the Dominated Convergence Theorem we have
\[
\lim_{\varepsilon \to 0} \langle \nabla B(x, u(x)), \nabla u(x) \rangle = I^N.
\]

By Corollary 3.80 in [4] it holds
\[
I^N = \int_{\Omega} \langle \nabla \phi(x), \nabla u(x) \rangle \, dx.
\]

**Proof.** By Lemma 3.7 the function \( v \) belongs to \( \mathcal{DM}_\infty^N(\Omega) \) and satisfies (17).

Since all results are of local nature in the space variables, it is not restrictive to assume that \( \Omega = \mathbb{R}^N \), \( b \) is a bounded Borel function, and \( b(\cdot, t) \in \mathcal{DM}_\infty^N \) for every \( t \in \mathbb{R} \).

We will use a regularization argument as in [22, Theorem 3.4]. More precisely, as in Lemma 3.7, let \( B_\varepsilon(\cdot, t) := \rho_\varepsilon \ast B(\cdot, t) \) and \( v_\varepsilon(x) := B_\varepsilon(x, u(x)) \). Since \( B_\varepsilon \) is a Lipschitz function in \((x, t)\), by using the chain rule formula of Ambrosio and Dal Maso (see [4, Theorem 3.101]) one has
\[
\int_{\mathbb{R}^N} \langle \nabla \phi(x), v_\varepsilon(x) \rangle \, dx = - \int_{\mathbb{R}^N} \phi(x) \partial_t B_\varepsilon(x, u(x)) \, dx
\]
\begin{equation}
- \int_{\mathbb{R}^N} \phi(x) \langle b_\varepsilon(x, u(x)), \nabla u(x) \rangle \, dx,
\end{equation}
and the claim will follow by passing to the limit as \( \varepsilon \to 0^+ \).

Namely, by Lemma 3.7, \( v_\varepsilon(x) \to v(x) \) for \( L^N \)-a.e. \( x \in \mathbb{R}^N \). Then by the Dominated Convergence Theorem we have
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \langle \nabla \phi(x), v_\varepsilon(x) \rangle \, dx = \int_{\mathbb{R}^N} \langle \nabla \phi(x), v(x) \rangle \, dx.
\]
This is equivalent to say
\[
\text{div}[B_\varepsilon(x, u(x))] \mathcal{L}^N \rightharpoonup \text{div} v(x), \quad \text{as } \varepsilon \to 0^+
\]
in the weak* sense of measures.

Similarly, from Lemma 3.6(a) we have that, for \( L^N \)-a.e. \( x \in \mathbb{R}^N \),
\[
\lim_{\varepsilon \to 0^+} b_\varepsilon(x, u(x)) = b(x, u(x)).
\]

Then by (ii) and the Dominated Convergence Theorem we have
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) \langle b_\varepsilon(x, u(x)), \nabla u(x) \rangle \, dx = \int_{\mathbb{R}^N} \phi(x) \langle b(x, u(x)), \nabla u(x) \rangle \, dx.
\]

It remains to prove that
\[
I_\varepsilon := \int_{\mathbb{R}^N} \phi(x) \partial_x B_\varepsilon(x, u(x)) \, dx \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^N} \phi(x) F(x, \tilde{u}(x)) \, d\sigma(x).
\]

Assume first that \( u \geq 0 \) and let \( C > \|u\|_\infty \). Let us rewrite \( I_\varepsilon \) in the following way:
\[
I_\varepsilon = \int_{\mathbb{R}^N} \phi(x) \int_0^{\tilde{u}(x)} \partial_x b_\varepsilon(x, t) \, dt \, dx
\]
\[
= \int_0^C dt \int_{\mathbb{R}^N} \phi(x) \chi_{\{u > t\}}(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x - y) \, d\text{div}_y b(y, t)
\]
\[
= \int_0^C dt \int_{\mathbb{R}^N} \rho_\varepsilon \ast (\phi \chi_{\{u > t\}})(y) \, d\text{div}_y b(y, t).
\]

By Corollary 3.80 in [4] it holds
\[
\rho_\varepsilon \ast (\phi \chi_{\{u > t\}})(x) \to \phi(x) \chi_{\{u > t\}}(x) = \phi(x) \chi_{\{\tilde{u} > t\}}(x) \quad \text{for } \mathcal{H}^{N-1} \text{-a.e. } x
\]
(hence for \( \text{div}_x b(\cdot, t) \)-a.e. \( x \)). Passing to the limit as \( \varepsilon \to 0 \) we get

\[
\lim_{\varepsilon \to 0} I_\varepsilon = \int_0^C dt \int_{\mathbb{R}^N} \phi(x) \chi_{\{t \geq t\}}(x) d \text{div}_x b(x, t) \\
= \int_{\mathbb{R}^N} \phi(x) \int_0^C f(x, t) dt d\sigma(x) \\
= \int_{\mathbb{R}^N} \phi(x) F(x, \bar{u}(x)) d\sigma(x),
\]

hence (25) is proved in the case \( u \geq 0 \).

The general case can be handled similarly. Namely, the integral \( I_\varepsilon \) can be written as

\[
I_\varepsilon = \int_{-C}^C dt \int_{\mathbb{R}^N} \phi(x) \chi_{u, t}(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x - y) d \text{div}_y b(y, t) \\
= \int_{-C}^C dt \int_{\mathbb{R}^N} \rho_\varepsilon * (\phi \chi_{u, t})(y) d \text{div}_y b(y, t),
\]

where \( \chi_{u, t} \) is the characteristic function of the set

\[
\{ x \in \mathbb{R}^N : t \text{ belongs to the segment of endpoints } 0 \text{ and } u(x) \},
\]

and the limit as \( \varepsilon \to 0 \) can be computed exactly as in the previous case. \( \Box \)

We now state the main results of the paper; the proofs are collected at the end of the section.

**Theorem 4.3** (\( \mathcal{DM}^\infty \)-dependence and \( u \in BV \)). Let \( b \) satisfy assumptions (i)–(iv), let \( B \) be defined by (17), and let \( u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \). Then the distribution \( (b(\cdot, u), Du) \), defined by

\[
\langle (b(\cdot, u), Du), \varphi \rangle := -\frac{1}{2} \int_\Omega \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \varphi(x) d\sigma(x) \\
- \int_\Omega B(x, u(x)) \cdot \nabla \varphi(x) dx, \quad \forall \varphi \in C^\infty_c(\Omega),
\]

is a Radon measure in \( \Omega \), and satisfies

\[
|\langle (b(\cdot, u), Du) \rangle(E)| \leq \|b\|_\infty |Du|(E), \quad \text{for every Borel set } E \subset \Omega.
\]

In other words, the composite function \( v : \Omega \to \mathbb{R}^N \), defined by \( v(x) := B(x, u(x)) \), belongs to \( \mathcal{DM}^\infty_{\text{loc}}(\Omega) \), and the following equality holds in the sense of measures:

\[
\text{div } v = \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma + (b(\cdot, u), Du).
\]

**Remark 4.4.** The measure \( (b(\cdot, u), Du) \) extends the notion of pairing defined by Anzellotti \([6]\), in the case \( b(x, u) = A(x) \), with \( A \in \mathcal{DM}^\infty \).

**Proposition 4.5** (Traces of the composite function). Let the assumptions of Theorem 4.3 hold, and let \( \Sigma \subset \Omega \) be a countably \( \mathcal{H}^{N-1} \)-rectifiable set, oriented as in Section 3.2. Then the normal traces on \( \Sigma \) of the composite function \( v \in \mathcal{DM}^\infty_{\text{loc}}(\Omega) \), defined at (19), are given by

\[
\text{Tr}^+(v, \Sigma) = \begin{cases} 
\beta^+(x, u^+(x)), & \text{for } \mathcal{H}^{N-1} \text{-a.e. } x \in J_u, \\
\beta^+(x, \bar{u}(x)), & \text{for } \mathcal{H}^{N-1} \text{-a.e. } x \in \Sigma \setminus J_u,
\end{cases}
\]
where, for every $t \in \mathbb{R}$, $\beta^\pm(\cdot, t)$ are the normal traces of $B(\cdot, t)$ on $\Sigma$ (see (11)). With our convention $u^\pm(x) = \tilde{u}(x)$ if $x \in \Omega \setminus S_u$, (30) can be written as $\text{Tr}^+(\nu, \Sigma) = \beta^+(x, u^+(x))$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma$.

**Theorem 4.6** (Representation of the pairing measure). Let the assumptions of Theorem 4.3 hold, and consider the standard decomposition of the measure $\mu := (b(\cdot, u), Du)$ as

$$\mu = \mu^{ac} + \mu^c + \mu^j, \quad \mu^d := \mu^{ac} + \mu^c.$$

Then

$$\mu^{ac} = \langle b(x, \tilde{u}(x)), \nabla u(x) \rangle \mathcal{L}^N,$$

$$\mu^j = [\beta^+(x, u^+(x)) - \beta^*(x, u^-(x))] \mathcal{H}^{N-1} \mathbf{1}_{J_u},$$

where, for every $t \in \mathbb{R}$, $\beta^\pm(\cdot, t)$ are the normal traces of $B(\cdot, t)$ on $J_u$ and $\beta^*(\cdot, t) := \lfloor \beta^+(\cdot, t) + \beta^-(\cdot, t) \rfloor / 2$.

Moreover, if there exists a countable dense set $Q \subset \mathbb{R}$ such that

$$|D^c u|(S_{b(t)}, t) = 0 \quad \forall t \in Q,$$

then

$$\mu^d = \left\langle \tilde{b}(x, \tilde{u}(x)), D^d u \right\rangle.$$

Therefore, under this additional assumption the following equality holds in the sense of measures:

$$\text{div} \nu = \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma$$

$$+ \left\langle \tilde{b}(x, \tilde{u}(x)), D^d u \right\rangle + [\beta^+(x, u^+(x)) - \beta^*(x, u^-)] \mathcal{H}^{N-1} \mathbf{1}_{J_u}.$$

**Remark 4.7.** Since $\mathcal{L}^N(S_{b(t)}, t) = 0$ for every $t \in \mathbb{R}$, assumption (31) is equivalent to $|D^d u|(S_{b(t)}, t) = 0$ for every $t \in Q$. In particular, it is satisfied, for example, if $S_{b(t)}$ is $\sigma$–finite with respect to $\mathcal{H}^{N-1}$, for every $t \in Q$ (see [4, Proposition 3.92(c)])]. This is always the case if $b(\cdot, t) \in BV_{loc}(\Omega, \mathbb{R}^N) \cap L^\infty_{loc}(\Omega, \mathbb{R}^N)$. Another relevant situation for which (31) holds happens when $D^c u = 0$, i.e. if $u$ is a special function of bounded variation, e.g. if $u$ is the characteristic function of a set of finite perimeter.

**Remark 4.8.** For $u \in BV_{loc}(\Omega)$ we introduce the following notation:

$$\text{div}_x B(\cdot, t)_{|t=u(x)} := \frac{1}{2} \left[ \text{div}_x B(\cdot, u^+(x)) + \text{div}_x B(\cdot, u^-(x)) \right]$$

$$:= \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma$$

(see also Remark 4.2). Then, with some abuse of notation, equation (32) can be written as

$$\text{div} \nu = \text{div}_x B(x, t)_{|t=u(x)}$$

$$+ \left\langle \tilde{b}(x, \tilde{u}(x)), D^d u \right\rangle + [\beta^+(x, u^+(x)) - \beta^*(x, u^-)] \mathcal{H}^{N-1} \mathbf{1}_{J_u}.$$

**Remark 4.9** (Anzellotti’s pairing). In the special case $B(x, t) = tA(x)$, with $A \in \mathcal{D}M_{loc}^\infty(\Omega)$, we have that

$$b^1(x, t) = A(x), \quad \sigma = |\text{div} A|, \quad f(x, t) = \frac{d}{dt} \text{div} A, \quad F(x, t) = t \frac{d}{dt} \text{div} A,$$
Theorem 4.1.

that the general case can be handled as it has been illustrated at the end of the proof of [28].

In the remaining part of the proof, for the sake of simplicity we assume $x$ constant independent of $u$.

Indeed, to the limit in (37) we obtain

$$
\lim_{\varepsilon \to 0^+} \left( \int_{\mathbb{R}^N} (\nabla \phi(x), \varphi(x)) \, dx \right) = - \int_{\mathbb{R}^N} \phi(x) F(x, u_\varepsilon(x)) \, d\sigma(x)
$$

(36)

Convergence Theorem.

Since $\phi \in C^1_0(\mathbb{R}^N)$ we get

$$
\int_{\mathbb{R}^N} (\nabla \phi(x), \varphi(x)) \, dx = - \int_{\mathbb{R}^N} \phi(x) F(x, u_\varepsilon(x)) \, d\sigma(x)
$$

(34)

Now we will pass to the limit as $\varepsilon \to 0^+$ in each term.

STEP 1. Firstly, we note that

$$
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} (\nabla \phi(x), \varphi(x)) \, dx = \int_{\mathbb{R}^N} (\nabla \phi(x), \varphi(x)) \, dx.
$$

(35)

Indeed, $u_\varepsilon(x) \to u(x)$, as $\varepsilon \to 0^+$, for a.e. $x$, $B(x, \cdot)$ is Lipschitz continuous with Lipschitz constant independent of $x$ and $B$ is locally bounded. Thus (35) holds by the Dominated Convergence Theorem.

STEP 2. We will prove that

$$
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) F(x, u_\varepsilon(x)) \, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[ \int_0^{u_\varepsilon(x)} f(x, w) \, dw \right] \, d\sigma(x).
$$

(36)

From (35) it holds

$$
\int_{\mathbb{R}^N} \phi(x) F(x, u_\varepsilon(x)) \, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[ \int_0^{u_\varepsilon(x)} f(x, w) \, dw \right] \, d\sigma(x).
$$

(37)

Since $u_\varepsilon(x) \to u^*(x)$ for $\mathcal{H}^{N-1}$-a.e. $x$, and so also for $\sigma$-a.e. $x$ (since $\sigma \ll \mathcal{H}^{N-1}$), passing to the limit in (37) we obtain

$$
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) F(x, u_\varepsilon(x)) \, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[ \int_0^{u^*(x)} f(x, w) \, dw \right] \, d\sigma(x) =: I.
$$

(38)

In the remaining part of the proof, for the sake of simplicity we assume $u \geq 0$. We remark that the general case can be handled as it has been illustrated at the end of the proof of Theorem 4.1.

Let $C > \|u\|_{L^\infty(K)}$, where $K$ is the support of $\phi$. The integral $I$ can be rewritten as

$$
I = \int_0^C \left[ \int_{\mathbb{R}^N} \phi(x) \chi_{\{u^* > w\}}(x) f(x, w) \, d\sigma(x) \right] \, dw.
$$

(39)
On the other hand, for $\mathcal{L}^1$-a.e. $w \in \mathbb{R}$ we have that
\[ \chi_{\{u^* > w\}} = \frac{1}{2} \left[ \chi_{\{u^* > w\}} + \chi_{\{u^- > w\}} \right], \quad \mathcal{H}^{N-1}$-a.e. (hence $\sigma$-a.e.) in $\mathbb{R}^N$ (see \cite{22} Lemma 2.2). Hence we get
\[
I = \int_0^C \left[ \int_{\mathbb{R}^N} \phi(x) \frac{1}{2} \left[ \chi_{\{u^* > w\}}(x) + \chi_{\{u^- > w\}}(x) \right] f(x, w) d\sigma(x) \right] dw
\]
\[
= \int_{\mathbb{R}^N} \phi(x) \left[ \int_0^C \frac{1}{2} \left[ \chi_{\{u^* > w\}}(x) + \chi_{\{u^- > w\}}(x) \right] f(x, w) dw \right] d\sigma(x)
\]
\[
= \int_{\mathbb{R}^N} \phi(x) \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] d\sigma(x),
\]
so that (30) is proved.

**STEP 3.** We claim that the distribution $(b(\cdot, u), Du)$ defined at (27) is a Radon measure, satisfying (28) (and hence absolutely continuous with respect to $|Du|$).

For simplicity, let us denote by $\mu$ the distribution $(b(\cdot, u), Du)$ defined at (27). Since
\[
\mu = \text{div} \, v - \frac{1}{2} \left[ F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma,
\]
by Lemma 3.4 it is clear that $\mu$ is a Radon measure and (29) holds. Moreover, by (34), (35) and (36) we have that, for every $\phi \in C_c(\mathbb{R}^N),\n\]
\[
\langle \mu, \phi \rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) \left( b(x, u_\epsilon(x)), \nabla u_\epsilon(x) \right) dx.
\]
Let us prove that (28) holds. Namely, let $U \subset \mathbb{R}^N$ be an open set, let $K \subset U$ be a compact set, and let $\phi \in C_c(\mathbb{R}^N)$ be a function with support contained in $K$. There exists $r_0 > 0$ such that $K_r := K + B_r(0) \subset U$ for every $r \in (0, r_0)$. Let $r \in (0, r_0)$ be such that $|Du|(\partial K_r) = 0$ (this property holds for almost every $r$). Then
\[
| \langle \mu, \phi \rangle | \leq \| \phi \|_\infty \| b \|_\infty \liminf_{\epsilon \to 0} \int_{K_r} |\nabla u_\epsilon| dx = \| \phi \|_\infty \| b \|_\infty |Du|(K_r) \leq \| \phi \|_\infty \| b \|_\infty |Du|(U),
\]
hence
\[
|\mu|(K) \leq \| b \|_\infty |Du|(U),
\]
so that (28) follows by the regularity of the Radon measures $|\mu|$ and $|Du|$. \hfill \Box

**Proof of Proposition 4.5.** We will use the same notations of Section 3.2. It is not restrictive to assume that $J_u$ is oriented with $\nu_{\Sigma}$ on $J_u \cap \Sigma$.

Since, by Theorem 4.3, $v \in \mathcal{D}\mathcal{M}^\infty$, there exist the weak normal traces of $v$ on $\Sigma$. Let us prove (30) for $\text{Tr}^\Sigma$.

Let $x \in \Sigma$ satisfy:

(a) $x \in (\mathbb{R}^N \setminus S_u) \cup J_u$, $x \in N_i$ for some $i$, the set $N_i$ has density 1 at $x$ and $x$ is a Lebesgue point of $\beta^{-}(\cdot, t)$, with respect to $\mathcal{H}^{N-1} |_{\partial \Omega_t}$, for every $t \in \mathbb{R}$;
(b) $\sigma \mathcal{L} \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ as $\varepsilon \to 0$;
(c) $|\text{div} \, v| \mathcal{L} \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$. 


We remark that $H^{N-1}$-a.e. $x \in \Sigma$ satisfies these conditions. In particular, (a) is satisfied thanks to Proposition 3.4 whereas (b) and (c) follow from [11, Theorem 2.56 and (2.41)].

In order to simplify the notation, in the following we set $u^-(x) := \tilde{u}(x)$ if $x \in \Omega \setminus S_u$.

Let us choose a function $\varphi \in C_c^\infty(\mathbb{R}^N)$, with support contained in $B_1(0)$, such that $0 \leq \varphi \leq 1$. For every $\varepsilon > 0$ let $\varphi_\varepsilon(y) := \varphi\left(\frac{y-x}{\varepsilon}\right)$.

By the very definition of normal trace, the following equality holds for every $\varepsilon > 0$ small enough:

$$
\frac{1}{\varepsilon^{N-1}} \int_{\partial \Omega_i} \left[ \text{Tr}(v, \partial \Omega_i) - \text{Tr}(B(\cdot, u^-(x)), \partial \Omega_i) \right] \varphi_\varepsilon(y) \, dH^{N-1}(y)
= \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla \varphi_\varepsilon(y) : [v(y) - B(y, u^-(x))] \, dy
+ \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon(y) \, d[\text{div} v - \text{div}_x B(\cdot, u^-(x))](y).
$$

(39)

Using the change of variable $z = (y - x)/\varepsilon$, as $\varepsilon \to 0$ the left hand side of this equality converges to

$$
\int_{\Omega_i} \nabla \varphi(z) : \left[ B(x + \varepsilon z, u(x + \varepsilon z)) - B(x + \varepsilon z, u^-(x)) \right] \, dz,
$$

where $\Pi_x$ is the tangent plane to $\Sigma_i$ at $x$. Clearly $\varphi$ can be chosen in such a way that $\int_{\Pi_x} \varphi \, dH^{N-1} > 0$.

In order to prove (39) for $\text{Tr}^-$ it is then enough to show that the two integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ at the right hand side of (39) converge to $0$ as $\varepsilon \to 0$.

With the change of variables $z = (y - x)/\varepsilon$ and by the very definition of $v$ we have that

$$
I_1(\varepsilon) = \int_{\Omega_i^\varepsilon} \nabla \varphi(z) : \left[ B(x + \varepsilon z, u(x + \varepsilon z)) - B(x + \varepsilon z, u^-(x)) \right] \, dz,
$$

where

$$
\Omega_i^\varepsilon := \frac{\Omega_i - x}{\varepsilon}.
$$

As $\varepsilon \to 0$, these sets locally converge to the half space $P_x := \{z \in \mathbb{R}^N : \langle z, \nu(x) \rangle < 0\}$, hence

$$
\lim_{\varepsilon \to 0} \int_{\Omega_i^\varepsilon \cap B_1} |u(x + \varepsilon z) - u^-(x)| \, dz = \lim_{\varepsilon \to 0} \int_{P_x \cap B_1} |u(x + \varepsilon z) - u^-(x)| \, dz = 0
$$

(see [11, Remark 3.85]) and, by (ii),

$$
|I_1(\varepsilon)| \leq \|b\|_\infty \|\nabla \varphi\|_\infty \int_{\Omega_i^\varepsilon \cap B_1} |u(x + \varepsilon z) - u^-(x)| \, dz \to 0.
$$

From (b) we have that

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon(y) \, d\text{div}_x B(\cdot, u^-(x))(y) \leq \lim_{\varepsilon \to 0} \frac{\sigma(B_\varepsilon(x))}{\varepsilon^{N-1}} = 0.
$$

In a similar way, using (c), we get

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon \, d\text{div} v = 0,
$$

so that $I_2(\varepsilon)$ vanishes as $\varepsilon \to 0$. 

The proof of (30) for $\text{Tr}^+$ is entirely similar. □

**Proof of Theorem 4.6.** We shall divide the proof into several steps.

**STEP 1.** We are going to prove that

$$\mu^{ac} = \langle b(x, u(x)), \nabla u(x) \rangle L^N.$$ 

Let us choose $x \in \mathbb{R}^N$ such that

(a) there exists the limit $\lim_{r \to 0} \frac{\mu(B_r(x))}{r^N};$

(b) $\lim_{r \to 0} \frac{|D^s u|(B_r(x))}{r^N} = 0;$

(c) $\lim_{r \to 0} \frac{1}{r^N} \int_{B_r(x)} |\langle b(y, u(y)), \nabla u(y) \rangle - \langle b(x, u(x)), \nabla u(x) \rangle| \, dy = 0.$

We remark that these conditions are satisfied for $L^N$-a.e. $x \in \mathbb{R}^N$.

Let $r > 0$ be such that

$$|D^s u|(\partial B_r(x)) = 0.$$ 

Observe that

$$\nabla u_\varepsilon = \rho_\varepsilon \ast Du + \rho_\varepsilon \ast \nabla u + \rho_\varepsilon \ast D^s u.$$ 

Hence for every $\phi \in C_0(\mathbb{R}^N)$ with support in $B_r(x)$ it holds

$$\left| \frac{1}{r^N} \int_{B_r(x)} \phi(y) |\langle b(y, u_\varepsilon(y)), (\rho_\varepsilon \ast Du)(y) - \langle b(x, u(x)), \nabla u(x) \rangle| \, dy \right|$$

$$\leq \frac{1}{r^N} \int_{B_r(x)} \phi(y) |\langle b(y, u_\varepsilon(y)), (\rho_\varepsilon \ast \nabla u)(y) - \langle b(x, u(x)), \nabla u(x) \rangle| \, dy$$

$$+ \frac{1}{r^N} \|\phi\|_\infty \|b\|_\infty \int_{B_r(x)} \rho_\varepsilon \ast |D^s u| \, dy,$$

where in the last inequality we use that $|\rho_\varepsilon \ast D^s u| \leq \rho_\varepsilon \ast |D^s u|$. We note that by (40)

$$\lim_{\varepsilon \to 0} \int_{B_r(x)} \rho_\varepsilon \ast |D^s u| \, dy = |D^s u|(B_r(x)).$$

Hence taking the limit as $\varepsilon \to 0$ we obtain

$$\left| \frac{1}{r^N} \int_{B_r(x)} \phi(y) \, d\mu(y) - \frac{1}{r^N} \int_{B_r(x)} \phi(y) \langle b(x, u(x)), \nabla u(x) \rangle \, dy \right|$$

$$\leq \frac{1}{r^N} \int_{B_r(x)} \phi(y) |\langle b(y, u(y)), \nabla u(y) \rangle - \langle b(x, u(x)), \nabla u(x) \rangle| \, dy$$

$$+ \frac{1}{r^N} \|\phi\|_\infty \|b\|_\infty |D^s u|(B_r(x)).$$
When \( \phi(y) \to 1 \) in \( B_r(x) \), with \( 0 \leq \phi \leq 1 \), we get
\[
\frac{1}{\omega_{NT}^N} \mu(B_r(x)) - \langle b(x, u(x)), \nabla u(x) \rangle \\
\leq \frac{1}{\omega_{NT}^N} \int_{B_r(x)} |\langle b(y, u(y)), \nabla u(y) \rangle - \langle b(x, u(x)), \nabla u(x) \rangle| \, dy
\\
+ \frac{1}{\omega_{NT}^N} \|b\|_{\infty} |D^s u|(B_r(x)).
\]

Now the conclusion is achieved by taking the limit for \( r \to 0 \) and using (b) and (c) above.

**STEP 2.** For the jump part of the measure \( \mu \) it holds:
\[
\mu^J = [\beta^+(x, u^+) - \beta^-(x, u^-)] |H^{N-1}_u| \mathbb{L} J_u
\]
Namely, this is a direct consequence of Proposition 4.5 in the particular case \( \Sigma = J_u \).

**STEP 3.** From now to the end of the proof, we shall assume that the additional assumption \((31)\) holds.

Let \( S := \bigcup_{q \in Q} S_{b, q} \). By assumption \((31)\) we have that \( |D^c u|(S) = 0 \).

We claim that, for every \( x \in \mathbb{R}^N \setminus S \) and every \( t \in \mathbb{R} \), there exists the approximate limit of \( b \) at \( (x, t) \) and
\[
\tilde{b}(x, t) = \lim_{j} \tilde{b}(x, q_j), \quad \forall (q_j) \subset Q, \ q_j \to t.
\]
Namely, let us fix a point \( x \in \mathbb{R}^N \setminus S \). By assumptions (i), (ii) and the Dominated Convergence Theorem, the map
\[
\psi(t) := \lim_{r \to 0} \int_{B_r(x)} b(y, t) \, dy, \quad t \in \mathbb{R},
\]
is continuous, and \( \psi(q) = \tilde{b}(x, q) \) for every \( q \in Q \). Hence the limit in \((43)\) exists for every \( t \) and it is independent of the choice of the sequence \((q_j) \subset Q\) converging to \( t \).

Let \( t \in \mathbb{R} \) be fixed, let us denote by \( c \in \mathbb{R}^N \) the value of the limit in \((43)\) and let us prove that \( c = \tilde{b}(x, t) \). We have that
\[
\int_{B_r(x)} |b(y, t) - c| \, dy \leq \int_{B_r(x)} |b(y, t) - b(y, q_j)| \, dy + \int_{B_r(x)} |b(y, q_j) - \tilde{b}(x, q_j)| \, dy + |\tilde{b}(x, q_j) - c|.
\]
As \( j \to +\infty \), the first integral at the r.h.s. converges to 0 by (i), (ii) and the Dominated Convergence Theorem. The second integral converges to 0 since \( x \in \mathbb{R}^N \setminus S \) and \((q_j) \subset Q\). Finally, \( \lim_j |\tilde{b}(x, q_j) - c| = 0 \) by the very definition of \( c \), so that the claim is proved.

**STEP 4.** We are going to prove that
\[
\mu^d = \left\langle \tilde{b}(x, \tilde{u}(x)), D^d u \right\rangle
\]
in the sense of measures. We remark that, by Step 3, the approximate limit \( \tilde{b}(x, t) \) exists for every \( (x, t) \in (\mathbb{R}^N \setminus S) \times \mathbb{R} \), with \( |D^c u|(S) = 0 \). As a consequence, the function \( x \mapsto \tilde{b}(x, \tilde{u}(x)) \) is well-defined for \( |D^d u|\)-a.e. \( x \in \mathbb{R}^N \), and it belongs to \( L^\infty(\mathbb{R}^N, |D^d u|) \).

If we consider the polar decomposition \( D^d u = \theta |D^d u| \), this equality is equivalent to
\[
\frac{d\mu}{d|D^d u|}(x) = \frac{d\mu^d}{d|D^d u|}(x) = \left\langle \tilde{b}(x, \tilde{u}(x)), \theta(x) \right\rangle
\]
for \( |D^d u|\)-a.e. \( x \in \mathbb{R}^N \). Let us choose \( x \in \mathbb{R}^N \) such that
(a) $x$ belongs to the support of $D^d u$, that is, $|D^d u|(B_r(x)) > 0$ for every $r > 0$;
(b) there exists the limit $\lim_{r \to 0} \frac{\mu^d(B_r(x))}{|D^d u|(B_r(x))}$;
(c) $\lim_{r \to 0} \frac{|D^d u|(B_r(x))}{|D u|(B_r(x))} = 0$;
(d) $\lim_{r \to 0} \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \left| \langle \tilde{b}(y, \tilde{u}(y)) , \theta(y) \rangle - \langle \tilde{b}(x, \tilde{u}(x)) , \theta(x) \rangle \right| d|D^d u|(y) = 0$.

We remark that these conditions are satisfied for $|D^d u|$-a.e. $x \in \mathbb{R}^N$. In particular, (d) follows from the fact that the map $y \mapsto \tilde{b}(y, \tilde{u}(y))$ belongs to $L^\infty(\mathbb{R}^N, |D^d u|)$.

Let $r > 0$ be such that

$\lim_{r \to 0} \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \langle b(y, u_\varepsilon(y)), (\rho_\varepsilon * D u)(y) \rangle dy \leq \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \langle \tilde{b}(y, \tilde{u}(y)) , \theta(y) \rangle d|D^d u|(y)$

$\leq \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \langle \tilde{b}(x, \tilde{u}(x)) , \theta(x) \rangle d|D^d u|(y)$

$+ \frac{1}{|D^d u|(B_r(x))} \|\phi\|_\infty \|\tilde{b}\|_\infty \int_{B_r(x)} \rho_\varepsilon * |D^j u| dy,$

where in the last inequality we use that $|\rho_\varepsilon * D^j u| \leq \rho_\varepsilon * |D^j u|$. We note that by (44)

$\lim_{\varepsilon \to 0} \int_{B_r(x)} \rho_\varepsilon * |D^j u| dy = |D^j u|(B_r(x)).$

Hence by taking the limit as $\varepsilon \to 0$ in (45) we obtain

$\lim_{\varepsilon \to 0} \int_{B_r(x)} \phi(y) d\mu(y)$

$\leq \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \langle \tilde{b}(x, \tilde{u}(x)) , \theta(x) \rangle d|D^d u|(y)$

$\leq \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left| \langle \tilde{b}(y, \tilde{u}(y)) , \theta(y) \rangle - \langle \tilde{b}(x, \tilde{u}(x)) , \theta(x) \rangle \right| d|D^d u|(y)$

$+ \frac{1}{|D^d u|(B_r(x))} \|\phi\|_\infty \|\tilde{b}\|_\infty |D^j u|(B_r(x)).$
When $\phi(y) \to 1$ in $B_r(x)$, with $0 \leq \phi \leq 1$, we get
\[
\left| \frac{1}{|D^d u|(B_r(x))} \mu(B_r(x)) - \left\langle \hat{b}(x, \tilde{u}(x)) , \theta(x) \right\rangle \right| \leq \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \left| \left\langle \hat{b}(y, \tilde{u}(y)) , \theta(y) \right\rangle - \left\langle \hat{b}(x, \tilde{u}(x)) , \theta(x) \right\rangle \right| d|D^d u|(y)
\]
\[+ \frac{1}{|D^d u|(B_r(x))} \|b\|_\infty |D^j u|(B_r(x)).\]

The conclusion is achieved now by taking $r \to 0$ and by using (c) and (d). \qed

5. Gluing constructions and extension theorems

A direct consequence of Theorems 4.3, 4.6 and [15] Theorems 5.1 and 5.3 are the following gluing constructions and extension theorems (the proofs are entirely similar to that of [14] Theorems 8.5 and 8.6 and [15] Theorems 5.1 and 5.3).

**Theorem 5.1** (Extension). Let $W \subseteq \text{int}(E) \subset E \subset U \subset \Omega$, where $\Omega, U, W \subset \mathbb{R}^N$ are open sets and $E$ is a set of finite perimeter in $\Omega$. Let
\[ b_1 : U \times \mathbb{R} \to \mathbb{R}^N, \quad b_2 : (\Omega \setminus \overline{W}) \times \mathbb{R} \to \mathbb{R}^N \]
satisfy assumptions (i)–(iv) in Theorem 5.1 (Extension) $\text{div} v = \chi_{E^1} \text{div} v_1 + \chi_{E^2} \text{div} v_2 + [\text{Tr}^+(v_1, \partial^* E) - \text{Tr}^-(v_2, \partial^* E)] \mathcal{H}^{N-1} \mathbb{1} \partial^* E.$

**Theorem 5.2** (Gluing). Let $U \subseteq \Omega \subset \mathbb{R}^N$ be open sets with $\mathcal{H}^{N-1}(\partial U) < \infty$, and let
\[ b_1 : U \times \mathbb{R} \to \mathbb{R}^N, \quad b_2 : (\Omega \setminus \overline{U}) \times \mathbb{R} \to \mathbb{R}^N \]
satisfy assumptions (i)–(iv) in $U \times \mathbb{R}$ and $\Omega \setminus \overline{U}$ respectively. Let $B_1, B_2$ be the corresponding integral functions with respect to the second variable. Given $u_1 \in BV_{loc}(U) \cap L^\infty(U)$ and $u_2 \in BV_{loc}(\Omega \setminus \overline{W}) \cap L^\infty(\Omega \setminus \overline{W})$, let $v_i(x) := B_i(x, u_i(x))$, $i = 1, 2$. Then the function

\[ v(x) := \begin{cases} v_1(x), & \text{if } x \in E, \\ v_2(x), & \text{if } x \in \Omega \setminus E, \end{cases} \]

belongs to $DM_{loc}^\infty(\Omega)$ and

\[ \text{div} v = \chi_{E^1} \text{div} v_1 + \chi_{E^2} \text{div} v_2 + [\text{Tr}^+(v_1, \partial^* E) - \text{Tr}^-(v_2, \partial^* E)] \mathcal{H}^{N-1} \mathbb{1} \partial^* E. \]

6. The Gauss–Green formula

Let $E \subseteq \Omega$ be a set of finite perimeter. Using the conventions of Section 4.2 we will assume that the generalized normal vector on $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$ with the measure–theoretic interior unit normal vector $\nu_E$ to $E$.

We recall that, if $u \in BV_{loc}(\Omega)$, then we will understand $u^\pm(x) = \tilde{u}(x)$ for every $x \in \Omega \setminus S_u$. 

\[ \]
The following result has been proved in [18] in the case $B(x, w) = w A(x)$ (see also [15] for related results). To simplify the notation, we will denote by $\mu := (b(\cdot, u), Du)$ the Radon measure introduced in [27].

**Theorem 6.1** (Gauss–Green formula). Let $b$ satisfy assumptions (i)–(iv) and let $B$ be defined by (7). Let $E \Subset \Omega$ be a bounded set with finite perimeter and let $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$. Then the following Gauss–Green formulas hold:

\[
(47) \quad \int_{E^1} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2} \, d\sigma(x) + \mu(E^1) = - \int_{\partial^* E} \beta^+(x, u^+(x)) \, d\mathcal{H}^{N-1},
\]
\[
(48) \quad \int_{E \cup \partial E} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2} \, d\sigma(x) + \mu(E^1 \cup \partial^* E)
\]
\[
\quad = - \int_{\partial^* E} \beta^-(x, u^-(x)) \, d\mathcal{H}^{N-1},
\]

where $E^1$ is the measure theoretic interior of $E$, and $\beta^+(\cdot, t) := \text{Tr}^+(B(\cdot, t), \partial^* E)$ are the normal traces of $B(\cdot, t)$ when $\partial^* E$ is oriented with respect to the interior unit normal vector.

**Proof.** Since $E$ is bounded we can assume, without loss of generality, that $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By Theorem 4.3, the composite function $v(x) := B(x, u(x))$ belongs to $\mathcal{D}M^\infty$. Since $E$ is a bounded set of finite perimeter, the characteristic function $\chi_E$ is a compactly supported $BV$ function, so that $\text{div}(\chi_E v)(\mathbb{R}^N) = 0$ (see [15, Lemma 3.1]).

We recall that, for every $w \in BV \cap L^\infty$ and every $A \in \mathcal{D}M^\infty$, it holds

\[
\text{div}(w A) = w^+ \text{div} A + (A, Dw),
\]

where $(A, Dw)$ is the Anzellotti pairing between the function $w$ and the vector field $A$ (see [6]). Hence, using the above formula with $w = \chi_E$ and $A = v$, it follows that

\[
(49) \quad 0 = \text{div}(\chi_E v)(\mathbb{R}^N) = \int_{\mathbb{R}^N} \chi^*_E \, d\text{div} v + (v, D\chi_E)(\mathbb{R}^N).
\]

Since

\[
(\nu, D\chi_E) = [\text{Tr}^+(\nu, \partial^* E) - \text{Tr}^-(\nu, \partial^* E)] \mathcal{H}^{N-1} \subseteq \partial^* E,
\]

from Proposition 4.5 we get

\[
(50) \quad (\nu, D\chi_E)(\mathbb{R}^N) = \int_{\partial^* E} [\beta^+(x, u^+(x)) - \beta^-(x, u^-(x))] \, d\mathcal{H}^{N-1}(x).
\]

Since $\chi_E^* = \chi_{E^1} + \frac{1}{2} \chi_{\partial^* E}$, using again Proposition 4.5 and (29) it holds

\[
\int_{\mathbb{R}^N} \chi^*_E \, d\text{div} v = \text{div}(v(E^1)) + \frac{1}{2} \int_{\partial^* E} \frac{[\beta^+(x, u^+(x)) + \beta^-(x, u^-(x)]}{2} \, d\mathcal{H}^{N-1}(x)
\]
\[
= \int_{E^1} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2} \, d\sigma(x) + \mu(E^1)
\]
\[
+ \frac{1}{2} \int_{\partial^* E} \frac{[\beta^+(x, u^+(x)) + \beta^-(x, u^-(x)]}{2} \, d\mathcal{H}^{N-1}(x).
\]

Formula (47) now follows from (49), (50) and (51).

The proof of (18) is entirely similar. \qed
It is worth to mention a consequence of the gluing construction given in Theorem 5.2 and the Gauss–Green formula (17). To this end, following [27], any bounded open set $\Omega \subset \mathbb{R}^N$ with finite perimeter, such that $\mathcal{H}^{N-1}(\partial \Omega) = \mathcal{H}^{N-1}(\partial^* \Omega)$, will be called weakly regular. For weakly regular sets we have the following version of the Gauss–Green formula (see [15] Corollary 5.5) for a similar statement for autonomous vector fields.

**Theorem 6.2** (Gauss–Green formula for weakly regular sets). Let $\Omega \subset \mathbb{R}^N$ be a weakly regular set. Let $b$ satisfy assumptions (i)--(iv), let $B$ be defined by (17) and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the following Gauss–Green formula holds:

$$\int_{\partial \Omega} F(x, u^+(x)) + F(x, u^-(x)) \, d\sigma(x) + \mu(\Omega) = - \int_{\partial \Omega} \beta^+(x, u^+(x)) \, d\mathcal{H}^{N-1}. \tag{52}$$

**Proof.** Since $\Omega$ is a set of finite perimeter, it holds $\partial^* \Omega \subseteq \partial \Omega$, hence the assumption $\mathcal{H}^{N-1}(\partial \Omega) = \mathcal{H}^{N-1}(\partial^* \Omega)$ of weak regularity implies that $\mathcal{H}^{N-1}(\partial \Omega \setminus \partial^* \Omega) = 0$. Consequently,

$$\mathcal{H}^{N-1}(\partial \Omega) = \mathcal{H}^{N-1}(\partial^* \Omega), \quad \mathcal{H}^{N-1}(\Omega \setminus \partial^* \Omega) = 0, \quad \mathcal{H}^{N-1}(\Omega \setminus (\mathbb{R}^N \setminus \Omega)) = 0. \tag{53}$$

Let us consider the vector field

$$\mathbf{v}(x) := \begin{cases} \mathbf{v}_1(x) := B(x, u(x)), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

By Theorem 5.2 and (53) we have that $\mathbf{v} \in \mathcal{D}M^\infty$ and

$$\text{div } \mathbf{v} = \chi_\Omega \text{div } \mathbf{v}_1 + \text{Tr } \mathbf{v}_1, (\mathbf{v}, \partial \Omega) \mathcal{H}^{N-1}(\partial \Omega),$$

hence (52) follows reasoning as in the proof of (17). \qed

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