MACROSCOPIC DIMENSION AND DUALITY GROUPS

ALEXANDER DRANISHNIKOV

Abstract. We show that for a rationally inessential orientable closed $n$-manifold $M$ whose fundamental group $\pi$ is a duality group the macroscopic dimension of its universal cover is strictly less than $n$:

$$\dim_{MC} \tilde{M} < n.$$ 

As a corollary we obtain the following

0.1. Theorem. The inequality $\dim_{MC} \tilde{M} < n$ holds for the universal cover of a closed spin $n$-manifold $M$ with a positive scalar curvature metric if the fundamental group $\pi_1(M)$ is a virtual duality group virtually satisfying the Analytic Novikov Conjecture.

1. Introduction

In his book dedicated to Gelfand’s 80th anniversary [Gr] Gromov introduced the notion of macroscopic dimension and proposed a conjecture that the macroscopic dimension of the universal cover $\tilde{M}$ of a closed $n$-manifold with a positive scalar curvature metric is at most $n-2$. This conjecture was proven by D. Bolotov for 3-manifolds [B]. It was proved recently by Bolotov and myself [BD] for spin $n$-manifolds, $n > 3$ whose fundamental group satisfies the Analytic Novikov Conjecture and the Rosenberg-Stoltz K-theoretic condition [RS]: $ko_\ast(\pi) \to KO_\ast(\pi)$ is a monomorphism. In this paper we show that in the case when $\pi$ is a duality group the inequality $\dim_{MC} \tilde{M} < n$ can be proven without that $K$-theoretic condition.

We prove the inequality $\dim_{MC} \tilde{M} < n$ for all rationally inessential orientable closed $n$-manifold $M$ whose fundamental group is a duality group. The rational inessentiality means that $f_\ast([M]) = 0$ in $H_{\ast}(\pi; \mathbb{Q})$.

Date: May 2, 2014.

2000 Mathematics Subject Classification. Primary 53C23; Secondary 55M10, 55M30.

Supported by NSF grant DMS-0904278.
$H_n(B\pi; \mathbb{Q})$. In the case of a spin manifold with a positive scalar curvature metric the rational inessentiality follows from Rosenberg’s theorem and the KO-homology Chern-Dold character.

2. **Homological obstruction to the inequality**

$$\dim_{MC} \tilde{M} < n$$

A map $f : X \to Y$ of a metric space is *uniformly cobounded* if there is a constant $C > 0$ such that $\text{diam}(f^{-1}(y)) < C$ for all $y \in Y$.

We modified in [Dr] Gromov’s definition of macroscopic dimension as follows.

2.1. **Definition.** A metric space $X$ has the *macroscopic dimension* less than or equal to $k$, $\dim_{MC} X \leq k$, if there is a Lipschitz uniformly cobounded map $f : X \to N^k$ to a $k$-dimensional simplicial complex.

Here we assume that a simplicial complex has a uniform metric, i.e. metric induced from the Hilbert space for the natural imbedding $K \subset \Delta \subset \ell^2$ into the standard simplex in $\ell^2$.

We recall that locally finite homology groups $H_*^{lf}(\tilde{X}; G)$ of a complex $\tilde{X}$ with coefficients in an abelian group $G$ are defined by means of the chain complex $C_*^{lf}(\tilde{X}; G)$ of infinite locally finite chains with coefficients in $G$. Thus,

$$C_*^{lf}(\tilde{X}; G) = \{ \sum_{e \in E_n(\tilde{X})} \lambda_e e \mid \lambda_e \in G \}.$$

Here $E_n(\tilde{X})$ denotes the set of $n$-cells of a CW complex $\tilde{X}$. The local finiteness condition on a chain requires that for every $x \in \tilde{X}$ there is a neighborhood such that the number of $n$-cells $e$ intersecting $U$ for which $\lambda_e \neq 0$ is finite. This condition is satisfied automatically when $\tilde{X}$ is a locally finite complex.

Now let $\tilde{X}$ be the universal cover of a complex $X$ with fundamental group $\pi$. The equivariant locally finite homology groups $H_*^{lf,\pi}(\tilde{X}; G)$ with coefficients in an abelian group $G$ are defined by the subcomplex of infinite *locally finite invariant chains*

$$C_n^{lf,\pi}(\tilde{X}; G) = \{ \sum_{e \in E_n(\tilde{X})} \lambda_e e \mid \lambda_{ge} = \lambda_e, \lambda_e \in G \} \subset C_n^{lf}(\tilde{X}; G).$$

We note that there is an obvious equality $H_*(X; G) = H_*^{lf,\pi}(\tilde{X}; G)$.

A normed abelian group $(G, \| \|)$ consists of an abelian group $G$ and a nonnegative function $\| \| : G \to \mathbb{R}$ that satisfies the axioms (1) $|a| = |-a|$; (2) $|a + b| \leq |a| + |b|$; (3) $|a| = 0$ iff $a = 0$. We consider the natural norm on $\mathbb{Z}$ and the norm with value 1 on $\mathbb{Z}_p \setminus \{0\}$. 
Let $G$ be a normed abelian group. A chain
\[ \sum_{e \in E_n(\tilde{X})} \lambda_e e \in C^d_n(\tilde{X}; G) \]
is called \textit{bounded} if there is a uniform bound on $\lambda_e$. Note that bounded infinite chains form a chain subcomplex $C_n^{(\infty)}(\tilde{X}; G) \subset C^d_n(\tilde{X}; G)$. The homology groups $H_n^{(\infty)}(\tilde{X}; G)$ of that complex are called $\ell^\infty$-homology in view of the analogy with the $\ell^{(\infty)}$-cohomology defined in [Ge]. Note that these homology groups first appeared in [BW] under the name \textit{uniformly finite homology}.

The inclusion of chain groups $C_n^{lf,\pi}(\tilde{X}; G) \to C_n^{(\infty)}(\tilde{X}; G)$ defines a homomorphism
\[ \text{pert}^X_\pi : H_n(X; G) \to H_n^{(\infty)}(\tilde{X}; G). \]

In the case of a discrete group $\pi$ we will use notations
\[ H_n(\pi; G) = H_n(B\pi; G), \quad H^{(\infty)}(\pi; G) = H^{(\infty)}_n(E\pi; G), \]
\[ H^{lf}_n(\pi; G) = H^{lf}_n(E\pi; G), \quad \text{and} \quad \text{pert}^\pi_\pi = \text{pert}^{B\pi}_\pi. \]

Note that $\ell^\infty$-homology is a special case (when coefficient group $G$ is a trivial $\pi$-module) of the almost equivariant homology defined in [Dr]. Thus, Theorem 4.2 of [Dr] in the language of $\ell^\infty$-homology can be rephrased as follows:

\begin{itemize}
  \item 1. \textbf{dim}_{MC} \tilde{M} < n;
  \item 2. $f_*([M]) \in \ker\{\text{pert}^\pi_\pi : H_n(\pi; \mathbb{Z}) \to H_n^{(\infty)}(\pi; \mathbb{Z})\}$ where $f : M \to B\pi$ is a map classifying the universal cover of $M$.
\end{itemize}

\section{Main Result}

We recall [Br] that a group $\pi$ of type FP is called a \textit{duality group} if for all $i$ there are natural isomorphisms
\[ H^i(\pi, -) \to H_{n-i}(\pi, D \otimes -) \]
where $n = \text{cd}(\pi)$, the cohomological dimension of $\pi$, and $D = H^n(\pi, \mathbb{Z}\pi)$. If $D = \mathbb{Z}$ then $\pi$ is called a Poincare duality group. Examples of Poincare duality groups are the fundamental groups of closed aspherical manifolds. Examples of duality groups different from Poincare duality groups include free groups, all knot groups, torsion free arithmetic groups and the products of all the above.
3.1. Proposition. For any duality group \( \pi \) with \( \text{cd}(\pi) = N \), and \( p \in \mathbb{N} \),
\[
H_n^{(\infty)}(\pi; \mathbb{Z}_p) = 0 \quad \text{for} \quad n \neq N
\]
and
\[
H_N^{(\infty)}(\pi; \mathbb{Z}_p) = \oplus \mathbb{Z}_p \neq 0.
\]

Proof. It suffices to consider the case when \( p \) is prime. Since the coefficient group is finite, \( H_n^{(\infty)}(\pi; \mathbb{Z}_p) = H_n^{\text{lf}}(\pi; \mathbb{Z}_p) \). Note that
\[
H_n^{\text{lf}}(\pi; \mathbb{Z}_p) = H_n^{\text{lf}}(E\pi; \mathbb{Z}_p) = H_n^c(E\pi; \mathbb{Z}_p) = H_n^c(\pi; \mathbb{Z}_p \otimes \mathbb{Z}_p) = 0
\]
for all \( n \neq N \). Here the second equality is the duality isomorphism between homology and cohomology with coefficient in a field applied to the one-point compactification of \( E\pi \). For the last two equalities see [Br], Chapter VIII, Lemma 7.4 and Theorem 10.1 respectively.

By the cited Theorem 10.1, \( H_N^c(\pi; \mathbb{Z}_p) \) is torsion free as an abelian group. Therefore, \( H_N^{\text{lf}}(\pi; \mathbb{Z}_p) \otimes \mathbb{Z}_p = \oplus \mathbb{Z}_p \). The Universal Coefficient Formula for the cohomology with compact supports and the equality \( H_c^i(E\pi; \mathbb{Z}) = H^i(\pi; \mathbb{Z}_p) \) imply that \( H_N^c(E\pi; \mathbb{Z}_p) = \oplus \mathbb{Z}_p \). Hence, \( H_N^{(\infty)}(\pi; \mathbb{Z}_p) = \oplus \mathbb{Z}_p \). \( \square \)

3.2. Corollary. For every duality group \( \pi \) with \( \text{cd}(\pi) = N \) and any \( n \neq N - 1 \), the homology group \( H_n^{(\infty)}(\pi) \) does not contain elements of finite order.

Proof. In view of the Universal Coefficient Formula
\[
0 \rightarrow H_n^{(\infty)}(\pi) \otimes \mathbb{Z}_p \rightarrow H_n^{(\infty)}(\pi; \mathbb{Z}_p) \rightarrow \text{Tor}(H_{n-1}^{(\infty)}(\pi), \mathbb{Z}_p) \rightarrow 0
\]
Proposition 3.1 implies that \( H_n^{(\infty)}(\pi) \) does not have \( p \)-torsions for \( n \neq N - 1 \). \( \square \)

REMARK. One can derive that the \( p \)-torsion subgroup of \( H_n^{(\infty)}(\pi) \) is \( \oplus \mathbb{Z}_p^{\infty} \) where \( \mathbb{Z}_p^{\infty} = \lim_{\rightarrow} \mathbb{Z}_{p^k} \). Most likely for duality group this sum is empty. Corollary 3.2 and Theorem 2.2 imply the following.

3.3. Corollary. Suppose that an orientable closed \( n \)-manifold \( M \) with the duality fundamental group \( \pi \) is rationally inessential and \( \text{cd}(\pi) \neq n + 1 \). Then \( \dim_{\mathcal{MC}} \tilde{M} < n \).

We recall that a group \( \pi' \) virtually satisfies some property \( \mathcal{P} \) if it contains a finite index torsion free subgroup \( \pi \) with that property. In particular, a group \( \pi' \) is said to be a virtually duality group if some subgroup \( \pi \subset \pi' \) of finite index is a duality group [Br] and a group \( \pi' \) virtually satisfies the Analytic Novikov Conjecture if it contains a finite index subgroup \( \pi \) that satisfies the Analytic Novikov Conjecture.
3.4. Theorem. Suppose that the fundamental group $\pi'$ of a spin manifold $M'$ with the positive scalar curvature is a virtual duality group which virtually satisfies the Analytic Novikov Conjecture. Then

$$\dim_{MC} \widetilde{M}' < n.$$ 

Proof. The conditions imply that there is a subgroup $\pi \subset \pi'$ of finite index which is a duality group and satisfies the Analytic Novikov Conjecture. Let $p : M \to M'$ be a finite covering that corresponds to the subgroup $\pi$. Clearly, $M$ is spin and a metric of positive scalar curvature on $M'$ lifts to a metric with positive scalar curvature on $M$. We show that $\dim_{MC} \widetilde{M} < n$. The required result would follow from the fact that the manifolds $M$ and $M'$ have the same universal cover $\widetilde{M} \equiv \widetilde{M}'$.

Let $f : M \to B\pi$ be a classifying map of the universal cover $\widetilde{M}$. By Rosenberg’s theorem [R] we have $\alpha \circ f^*([M]_{KO}) = 0$ where

$$\alpha : KO_*(B\pi) \to KO_*(C_r(\pi))$$

is the analytic assembly map and $[M]_{KO}$ is the $KO$-fundamental class defined by the spin structure. The Analytic Novikov Conjecture for $\pi$ says that $\alpha$ is a monomorphism. Therefore, $f^*([M]_{KO}) = 0$. There is a natural isomorphism called the Chern-Dold character

$$KO_n(X) \otimes \mathbb{Q} \to \bigoplus_{i \in \mathbb{Z}} H_{n+4i}(X; \mathbb{Q})$$

(see for example [R], Theorem-Definition 7.13). We note that under the Chern-Dold character isomorphism the rationalized $KO$-fundamental class $[M]_{KO}$ of $M$ is taken to an element $a = (a_i)_{i \in \mathbb{Z}} \in \bigoplus_{i \in \mathbb{Z}} H_{n+4i}(M; \mathbb{Q})$ with $a_0 \neq 0$. This is an obvious fact for $M = S^n$. For general $M$ it can be shown by taking a degree one map of $M$ onto the $n$-sphere $S^n$. Therefore, $f^*([M]_{KO}) = 0$ in $H_n(B\pi; \mathbb{Q})$. Then Corollary 3.3 implies the inequality $\dim_{MC} \widetilde{M} < n$. This completes the proof if $cd(\pi) \neq n + 1$.

The case $cd(\pi) = n + 1$ is treated separately. We claim that in this case $M$ is integrally inessential. Since the complex $B\pi/B\pi^{(n-1)}$ is $(n+1)$-dimensional and $(n-1)$-connected, it is homotopy equivalent to the wedge of spheres and Moore spaces $\vee_i S^n \vee \vee_j M(\mathbb{Z}_m, n) \vee \vee_k S^{n+1}$. We note that any essential map of $S^n$ to $S^n$ or to the Moore space $M(\mathbb{Z}_m, n)$ takes the $KO$-fundamental class $[S^n]_{KO}$ to nonzero. We may assume that $M$ has a CW structure with one top dimensional cell and
the map $f$ is cellular. The commutativity of the diagram

$$
\begin{array}{ccc}
M_0 & \xrightarrow{f} & B\pi \\
\downarrow p & & \downarrow q \\
S^n = M_0/M_0^{(n-1)} & \xrightarrow{\bar{f}} & B\pi/B\pi^{(n-1)}
\end{array}
$$

and the fact that $f_*([M]_{KO}) = 0$ imply that $\bar{f}_*([S^n]_{KO}) = 0$. This implies that $\bar{f}$ is null-homotopic and hence it induces a zero homomorphism of the integral homology groups. Since the quotient map $q : B\pi \to B\pi/B\pi^{(n-1)}$ induces a monomorphism of $n$-dimensional homology groups, $f_*([M]) = 0$ in $H_n(B\pi;\mathbb{Z})$.

□

REFERENCES

[BW] J. Block, S. Weinberger, Aperiodic tilings, positive scalar curvature and amenability of spaces, J. Amer. Math. Soc. 5 no. 4 (1992), 907-921.

[B] D. Bolotov, Macroscopic dimension of 3-Manifolds, Math. Physics, Analysis and Geometry 6 (2003), 291 - 299.

[BD] D. Bolotov, A. Dranishnikov, On Gromov’s scalar curvature conjecture, Proc. of AMS, 138 no. 4 (2010), 1517-1524.

[Br] K. Brown Cohomology of groups, Graduate Texts in Mathematics, 87 Springer, New York Heidelberg Berlin, 1994.

[Dr] A. Dranishnikov, On macroscopic dimension of rationally essential manifolds, Geometry and Topology, 15 (2011), no. 2, 1107-1124.

[Ge] S.M. Gersten, Cohomological lower bounds for isoperimetric functions on groups, Topology, 37 No 5 (1998) 1031-1072.

[Gr] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, Functional analysis on the eve of the 21st century. Vol II, Birkhauser, Boston, MA, 1996.

[R] J. Rosenberg $C^*$-algebras, positive scalar curvature, and the Novikov conjecture, III, Topology 25 (1986), 319 - 336.

[RS] J. Rosenberg, S. Stolz, Metrics of positive scalar curvature and connections with surgery, in: Surveys on Surgery Theory (vol. 2), S. Cappell, A. Ranicki, J. Rosenberg (eds.), Annals of Mathematical Studies 149 (2001), Princeton University Press.

[Ru] Yu. Rudyak, On Thom spectra, orientability, and cobordism, Springer, 1998.

ALEXANDER N. DRANISHNIKOV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, GAINESVILLE, FL 32611-8105, USA

E-mail address: dranish@math.ufl.edu