ARITHMETIC APPLICATIONS OF ARAKELOV SELF-INTERSECTION NUMBERS FOR MODULAR CURVES $X_0(p^2)$

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Abstract. In this paper, we compute the semi-stable models of modular curves of the forms $X_0(p^2)$ for an odd prime $p$ and compute the Arakelov self-intersection numbers for these models. We give two applications of our computations. In particular, we give an effective version of the Bogomolov’s conjecture following the strategy outlined by Zhang and find the stable Faltings heights of the arithmetic surfaces corresponding to these modular curves.

Contents

1. Introduction 1
2. Semistable models 3
3. Arakelov self-intersection of the canonical sheaf 15
4. Dual graphs of special fibers 21
5. Effective Bogomolov’s conjecture 23
6. Stable Faltings Heights 25
Appendix A. Small primes 26
References 26

1. Introduction

Let $K$ be a number field and $X$ a smooth projective curve over $K$ with genus $g_X > 0$. Let $\mathcal{O}_K$ be the ring of integers of $K$ and $\mathcal{X}/\mathcal{O}_K$ the minimal regular model for $X/K$. The model $\mathcal{X}/\mathcal{O}_K$ is a regular scheme of dimension 2 with a map $\mathcal{X} \to \text{Spec} \mathcal{O}_K$ hence an arithmetic surface in the sense of Liu [17]. There is an intersection theory for metrized invertible sheaves on arithmetic surfaces due to Arakelov (see Faltings [10], or [4] Section 2 for a brief summary). We denote this intersection pairing by $\langle \cdot, \cdot \rangle$. Let $\mathcal{O}_{\mathcal{X}/\mathcal{O}_K}$ be the relative dualizing sheaf on $\mathcal{X}/\mathcal{O}_K$ equipped with the Arakelov metric. The quantity

$$\mathcal{O}_{\mathcal{X}} = \frac{\langle \mathcal{O}_{\mathcal{X}/\mathcal{O}_K}, \mathcal{O}_{\mathcal{X}/\mathcal{O}_K} \rangle}{[K : \mathbb{Q}]}$$

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is an invariant for \( X \) independent of the field \( K \) if \( X \) is semi-stable. This means for any field extension \( K'/K \) if we take \( X' = X \times_{\text{Spec} K} \text{Spec} K' \) and \( X'/\mathcal{O}_{K'} \) is the minimal regular model of \( X'/K' \) then \( \omega_{X'}^2 = \omega_X^2 \). We call this quantity the stable arithmetic self-intersection number for \( X \).

For the rest of the paper, we fix \( K \) to be the number field
\begin{equation}
K = \frac{\mathbb{Q}[x]}{(x^{(p^2-1)/2} - p)},
\end{equation}
and \( \mathcal{O}_K \) will denote its ring of integers. This is a totally ramified finite extension of \( \mathbb{Q} \) of degree \( \frac{p^2-1}{2} \).

The main objective of this paper is to calculate the stable arithmetic self-intersection \( \omega_{p^2}^2 \) for the modular curves \( X_0(p^2)/K \). The modular curve \( X_0(p^2) \) is the compactified moduli space of elliptic curves along with some extra structure (see [8]).

In Section 2 following Edixhoven [9], we show that the minimal regular model \( X_0(p^2)/\mathcal{O}_K \) is semi-stable. Stable models for these curves over \( \mathbb{Q}_p \) were already computed by Edixhoven. However, the stable models are not regular. We resolve singularities and blow down all possible rational components in the special fiber, without introducing singularities, and thus obtain the minimal regular models. The cases of \( p \in \{5, 7, 13\} \) are dealt with separately in the appendix. We expect that the semi-stable models will be useful to compute the index of the Eisenstein ideals, and prove the Ogg’s conjecture for the modular curves of the form \( X_0(p^2) \) following the strategy of Mazur [19].

From the geometry of the semi-stable models and the results of [1], we prove the following asymptotic formula [Theorem 1, Section 3]:
\[
\omega_{p^2}^2 = 2g_{p^2} \log p^2 + o(p^2 \log(p^2)).
\]

We remark that the basic strategy to prove the above mentioned theorem is same as that of similar theorems proved for modular curves of the form \( X_0(N), X_1(N) \) or \( X(N) \) with \( N \) square-free [1, 21, 18, 11]. For Fermat’s curves, similar theorems were proved in [6]. Note that effective bounds on self-intersection numbers are given for general arithmetic surfaces in [15, 16].

We give two arithmetic applications of the above theorem similar to Abbes-Ullmo [1] and Mayer [18]. First we prove effective Bogomolov conjecture for the particular case of modular curves of the form \( X_0(p^2) \) with \( p \) — an odd prime — by using Zhang’s proof of the general effective Bogomolov conjecture [27, Theorem 5.6]. In the above mentioned fundamental paper, the bounds are achieved in terms of the admissible self-intersection number. This number, however, depends on the semi-stable model of the particular modular curve \( X_0(p^2) \). The computation of this number involves the geometry of the special fibers of \( X_0(p^2)/\mathcal{O}_K \) which is quite complicated. We wish to draw the attention of the reader to the seminal work of L. Szpiro [25, Theorem 3, page 241], [24] for the introduction to this subject.

Let \( K_{X_0(p^2)} \) be the canonical divisor for the modular curve \( X_0(p^2) \) and let \( D = \infty \) be the divisor of degree 1 corresponding to the cusp \( \infty \in X_0(p^2)(\mathbb{Q}) \). This divisor gives an embedding of the curve into its Jacobian \( \varphi_D : X_0(p^2) \rightarrow J_0(p^2) \). Let \( h_{\text{NT}} \) be the Néron-Tate height on \( J_0(p^2) \).

**Theorem 1.** For a sufficiently large prime \( p \), the set
\[
\left\{ x \in X_0(p^2)(\overline{\mathbb{Q}}) \mid h_{\text{NT}}(\varphi_D(x)) < \left( \frac{1}{2} - \epsilon \right) \log(p^2) \right\}
\]
is finite but the set
\[ \{ x \in X_0(p^2)(\overline{\mathbb{Q}}) \mid h_{\text{NT}}(\varphi_D(x)) \leq (1 + \epsilon) \log(p^2) \} \]
is infinite.

The general Bogomolov’s conjecture has been proved by Ullmo [26] using ergodic theory; however the proof is not effective.

We also find an asymptotic formula for the stable Faltings height \( h_{\text{Fal}} \) of the Jacobian \( J_0(p^2)/\mathbb{Q} \) of \( X_0(p^2)/\mathbb{Q} \). This is also the arithmetic degree of the direct image of \( \mathcal{X}_{X_0(p^2)}/\mathcal{O}_K \) onto Spec \( \mathcal{O}_K \).

**Theorem 2.** The stable Faltings heights of the modular curves \( X_0(p^2) \) satisfy the following asymptotic estimate:
\[
h_{\text{Fal}}(J_0(p^2)) = \frac{2}{3} g_{p^2} \log(p^2) + o(g_{p^2} \log(p^2)).
\]

In [12], various bounds are provided for the Arakelov self-intersection numbers, admissible self-intersection numbers, and stable Faltings heights for any smooth projective curve. In this article, we provide a precise asymptotic expression in the particular case of modular curves of the form \( X_0(p^2) \). In a recent article [23], Pierre Parent gave a bound on the height of points of the \( X_0(p^2)(K) \) (for a quadratic field \( K \)) assuming Brumer’s conjecture. It will be intriguing to relate our results with that of Parent’s.

In a future direction, we hope to generalize our work to \( X_0(p^3) \) using the semi-stable models constructed by McMurdy-Coleman [20]. We hope to use the work of de-Shalit [7] to achieve this.

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## 2. Semistable models

Let \( K \) be the number field as in equation (1.1) and \( p \) be the prime ideal of \( \mathcal{O}_K \) such that \( p = p^{(p^2-1)/2} \). Since the prime ideal is totally ramified, we have the following isomorphism of the residue field: \( \mathcal{O}_K/(p) \cong \mathbb{F}_p \). Although it is not made explicit, the following theorem is a direct consequence of the results of Section 2 of Edixhoven [9].

**Theorem 3 (Edixhoven).** The minimal regular model \( X_0(p^2)/\mathcal{O}_K \) of \( X_0(p^2) \) is semi-stable.

This section is devoted to the study of this minimal regular model \( X_0(p^2)/\mathcal{O}_K \). Let \( g_{p^2} \) be the genus of \( X_0(p^2) \). For any non-zero prime \( q \neq p \) of \( \mathcal{O}_K \) the fiber \( X_0(p^2)(q) \) is a smooth curve of genus \( g_{p^2} \) over the residue field \( \mathcal{O}_K/(q) \). Hence we just need to determine the special fiber.

We start with the regular model \( \tilde{X}_0(p^2)/\mathbb{Z} \) of the modular curve, constructed by Edixhoven [9]. To this we apply the procedure of section 2 in Edixhoven to obtain a stable model, and finally we resolve the singularities.

Let us spell out the procedure explicitly. There are points of triple intersection in the special fiber
\[ \tilde{X}_0(p^2) \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{F}_p. \]
We blow up at these points to get an arithmetic surface $\mathcal{Y}/\mathcal{Z}$ with normal crossings. Each of the $k$ triple intersection points gives us a projective line of multiplicity $p + 1$ after blow up; we name these new components $L_1, \ldots, L_k$. Below is a picture of the special fiber of $\mathcal{Y}$ for different primes $p$. Each irreducible component is a projective line.

$$\begin{array}{cccc}
L_1 & L_2 & L_3 & \ldots \ldots \\
E & E & E & \ldots \ldots \\
F & F & F & \ldots \ldots \\
p = 12k + 1 & p = 12k + 5 & p = 12k + 7 & p = 12k + 11
\end{array}$$

Next we base change to $\mathcal{O}_K$, that is we let $\mathcal{Z} = \mathcal{Y} \times_{\text{Spec} \mathcal{Z}} \text{Spec} \mathcal{O}_K$.

Note that $\mathcal{Z}/\mathcal{O}_K$ has only one special fiber, the fiber over $(p)$ and

$$\mathcal{Z}(p) \cong \mathcal{Z} \times_{\text{Spec} \mathcal{O}_K} \text{Spec} \mathcal{F}_p \cong \mathcal{Y} \times_{\text{Spec} \mathcal{Z}} \text{Spec} \mathcal{F}_p \cong \mathcal{Y}(p).$$

Let $\tilde{\mathcal{Z}}/\mathcal{O}_K$ be the normalisation of $\mathcal{Z}/\mathcal{O}_K$. $\tilde{\mathcal{Z}}$ is not regular, so we calculate the desingularization and call it $W/\mathcal{O}_K$. Finally we blow down the rational components with self-intersection -1 in the special fiber of $W/\mathcal{O}_K$ to get the minimal regular model $X_0(p^2)/\mathcal{O}_K$.

We first do the calculation locally, that is we determine the complete local rings of $\tilde{\mathcal{Z}}$. Let $z \in \mathcal{Z}(p)$ be a closed point of the special fiber and $\tilde{\mathcal{O}}_{\mathcal{Z},z}$ the complete local ring of that point. Then we use the following result from EGA

$$\tilde{\mathcal{Z}} \times_{\mathcal{Z}} \text{Spec} \tilde{\mathcal{O}}_{\mathcal{Z},z} = \text{Spec}(\text{normalization of } \tilde{\mathcal{O}}_{\mathcal{Z},z}).$$

Calculations are dependent on the class of $p$ in $\mathbb{Z}/12\mathbb{Z}$.

2.1. **Case 1:** $p = 12k + 1$. Let $\tilde{\mathcal{O}}_K$ be the completion of $\mathcal{O}_K$ at its prime ideal $(p)$. Let $z$ be a closed point of the special fiber of $\mathcal{Z}$ then

$$\tilde{\mathcal{O}}_{\mathcal{Z},z} \cong \frac{\tilde{\mathcal{O}}_K[[x,y]]}{(x^ay^b - p^{(p^2-1)/2})},$$

where at the double points of the special fiber

$$(a,b) \in \{(p + 1, 1) \cup \left\{ p + 1, \frac{p - 1}{2}, \frac{p - 1}{3} \right\} \times \{p - 1\}$$

and at smooth points of the special fiber

$$(a,b) \in \left\{ 1, p + 1, p - 1, \frac{p - 1}{2}, \frac{p - 1}{3} \right\} \times \{0\}.$$
We want to calculate the complete local rings of the pre-image of these points under the normalisation morphism. All the cases have been discussed in [9] section 2.2 except when \((a, b) \in \{ \frac{p-1}{2}, \frac{p-1}{3} \} \times \{0, p-1\} \). We quote the relevant results from Edixhoven.

- If \((a, b) = (1, 0)\) the complete local ring is regular hence normal.

- If \((a, b) = (p-1, 0)\)

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} = \bigsqcup_{\zeta \in \mu_{p-1}(\hat{O}_K)} \text{Spec } \hat{O}_K[[x,y]]/(x - \zeta p^{(p+1)/2}).
\]

- If \((a, b) = (p+1, 0)\)

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} = \bigsqcup_{\zeta \in \mu_{p+1}(\hat{O}_K)} \text{Spec } \hat{O}_K[[x,y]]/(x - \zeta p^{(p-1)/2}).
\]

- If \((a, b) = (p+1, 1)\),

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} = \text{Spec } \hat{O}_K[[x,y]]/(xu - p^{(p-1)/2})
\]

with \(y = u^{p+1}\). The normalisation is not regular.

- If \((a, b) = (p+1, p-1)\)

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} = \text{Spec } \hat{O}_K[[u,v]]/(uv - p) \bigsqcup \text{Spec } \hat{O}_K[[u,v]]/(uv - p).
\]

- When \((a, b) = \left(\frac{p-1}{2}, 0\right)\), after factorising

\[
\left(x^{(p-1)/2} - p^{(p^2-1)/2}\right) = \prod_{\zeta \in \mu_{(p-1)/2}(\hat{O}_K)} \left(x - \zeta p^{p+1}\right)
\]

we see that

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} \cong \bigsqcup_{\zeta \in \mu_{(p-1)/2}(\hat{O}_K)} \text{Spec } \hat{O}_K[[x,y]]/(x - p^{p+1}).
\]

There are \(\frac{p-1}{2}\) pre-images of this point in the normalisation.

- For \((a, b) = \left(\frac{p-1}{3}, 0\right)\) similarly as above

\[
\tilde{Z} \times_Z \text{Spec } \hat{O}_{Z,z} \cong \bigsqcup_{\zeta \in \mu_{(p-1)/3}(\hat{O}_K)} \text{Spec } \hat{O}_K[[x,y]]/(x - p^{3(p+1)/2}).
\]

There are \(\frac{p-1}{3}\) pre-images of this point in the normalisation.

- Next when \((a, b) = \left(\frac{p-1}{2}, p-1\right)\)

\[
\left(x^{(p-1)/2}y^{p-1} - p^{(p^2-1)/2}\right) = \prod_{\zeta \in \mu_{(p-1)/2}(\hat{O}_K)} (xy^2 - \zeta p^{p+1}).
\]
Let \( u = \frac{p^{(p+1)/2}}{y} \) then the normalisation of \( \hat{O}_K[[x, y]]/(xy^2 - p^{p+1}) \) is \( \hat{O}_K[[x, y, u]]/(x - u^2, uy - p^{(p+1)/2}) \simeq \hat{O}_K[[u, y]]/(uy - p^{(p+1)/2}) \). Hence
\[
\tilde{Z} \times \text{Spec } \hat{O}_K \cong \bigsqcup_{\zeta \in \mu_{p-1}/3(\hat{O}_K)} \text{Spec } \hat{O}_K[[u, y]]/(uy - p^{(p+1)/2}).
\]

Here \( x = u^2 \). There are \( \frac{p-1}{2} \) pre-images of this point in the normalisation.

- If \( (a, b) = \left( \frac{p-1}{3}, p - 1 \right) \) by an analogous computation as the previous case we have
\[
\tilde{Z} \times \text{Spec } \hat{O}_K \cong \bigsqcup_{\zeta \in \mu_{p-1}/3(\hat{O}_K)} \text{Spec } \hat{O}_K[[u, y]]/(uy - p^{(p+1)/2}),
\]
where \( x = u^3 \). There are \( \frac{p-1}{3} \) pre-images of this point in the normalisation.

From this and the calculations of Edixhoven [9, section 2] it is clear that the special fiber of \( \tilde{Z} \) is given by the following figure. The number adjacent to each component is its geometric genus.

![Figure 1. \( \tilde{Z} \times \text{Spec } \hat{O}_K \text{ Spec } \mathbb{F}_p \), when \( p = 12k + 1 \).](image)

The map \( \tilde{Z} \times \text{Spec } R \text{ Spec } \mathbb{F}_p \to \tilde{Z} \times \text{Spec } R \text{ Spec } \mathbb{F}_p \) is described as follows.

1. The components \( \tilde{C}_{2,0} \) and \( \tilde{C}_{0,2} \) map to \( C_{2,0} \) and \( C_{0,2} \) isomorphically.
2. The component \( \tilde{L}_i \) maps to \( L_i \) with degree \( p + 1 \). Over the points of intersection of \( L_i \) with \( C_{2,0} \) and \( C_{0,2} \) the map is totally ramified; at the points of intersection of \( \tilde{L}_i \) with \( \tilde{C}_{1,1} \) and \( \tilde{C}_{1,1}' \) the ramification index is \( \frac{p-1}{2} \).
3. The components \( \tilde{E}_i \) and \( \tilde{F}_i \) map isomorphically to \( E \) and \( F \) respectively.
(4) On the other hand, $\tilde{C}_{1,1}^j$ maps to $C_{1,1}$ with degree $(p - 1)/2$. At the points of intersection with $\tilde{L}_i$ the map is totally ramified. At the points of intersection with $\tilde{E}_i^j$, the ramification index is 2 and at the points of intersection with $\tilde{F}_i^j$ the ramification index is 3.

Only possible points where $\tilde{Z}$ can fail to be regular are the singular points of the special fiber. In the following table we list the local equation of all the double points of the special fiber.

|       | $\tilde{C}_{2,0}$ | $\tilde{C}_{1,1}^1$ | $\tilde{C}_{2,1}^2$ | $\tilde{C}_{0,2}$ |
|-------|-------------------|-------------------|-------------------|-------------------|
| $\tilde{L}_i$ | $ux - p^{(p-1)/2}$ | $uv - p$ | $uv - p$ | $ux - p^{(p-1)/2}$ |
| $\tilde{E}_i^j$ | $uy - p^{(p+1)/2}$ | | | |
| $\tilde{F}_i^j$ | | | $uy - p^{(p+1)/2}$ | |
| $\tilde{F}_i^j$ | | | | $uy - p^{(p+1)/2}$ |

Hence the singular points are: the intersections of $\tilde{L}_i$ with $\tilde{C}_{2,0}$ and $\tilde{C}_{0,2}$, which we call $\alpha_i$ and $\beta_i$ respectively; the intersection points of $\tilde{E}_i^j$ and $\tilde{F}_i^j$ with $\tilde{C}_{1,1}^j$ which we call $\sigma_i^j$ and $\tau_i^j$ respectively.

We resolve the singularities by successive blow ups as in Liu [17]. Let $W/\mathcal{O}_K$ be the de-singularisation. The pre-image of $\alpha_i$ in $W$ is a tail consisting of projective lines, $A_{1,i}, \ldots, A_{(p-1)/2-1,i}$, where $A_{1,i}$ meets $\tilde{C}_{2,0}$ and $A_{2,i}$; $A_{j-1,i}$ meets $A_{j-1,i}$ and $A_{j+1,i}$ for $1 < j < (p - 1)/2 - 1$; $A_{(p-1)/2-1,i}$ meets $A_{(p-1)/2-2,i}$ and $\tilde{L}_i$.

Similarly, the pre-image of $\beta_i$ is a tail of projective lines $B_{1,i}, \ldots, B_{(p-1)/2-1,i}$ with analogous intersections.

The pre-image of $\sigma_i^j$ is again a union of $\frac{p+1}{2} - 1 = \frac{p-1}{2}$ projective lines $S_{1,i}^j, \ldots, S_{(p-1)/2,i}^j$ each meeting its successor, $S_{1,i}^j$ intersects $\tilde{E}_i^j$ and $S_{(p-1)/2,i}^j$ intersects $\tilde{C}_{1,1}^j$.

Finally the pre-image of $\tau_i^j$ is again a union of $\frac{p+1}{2} - 1 = \frac{p-1}{2}$ projective lines $T_{1,i}^j, \ldots, T_{(p-1)/2,i}^j$ each meeting its successor, $T_{1,i}^j$ intersects $\tilde{F}_i^j$ and $T_{(p-1)/2,i}^j$ intersects $\tilde{C}_{1,1}^j$.

The model $W/\mathcal{O}_K$ is regular and semi-stable. However $\tilde{E}_i^j$ are projective lines with self-intersection $-1$ and can be blown-down, and similarly we can blow down $\tilde{F}_i^j$. Successively we can blow-down all $S_{1,i}^j$ and $T_{1,i}^j$.

Let $X_0(p^2)/\mathcal{O}_K$ be the resultant arithmetic surface; then it is the minimal regular model of $X_0(p^2)/K$ if $k > 1$ (see figure 2). We deal with the case $p = 13$ separately in the appendix.

2.2. Case 2: $p = 12k + 5$. Here we encounter the complete local rings

$$\hat{\mathcal{O}}_{Z,z} \cong \frac{\hat{\mathcal{O}}_K[[x,y]]}{(x^ay^b - p^{(p^2-1)/2})},$$

where at the double points of the special fiber

$$(a, b) \in \{(p + 1, 1), (p + 1, p - 1), (\frac{p+1}{2}, p - 1), (\frac{p+1}{3}, p - 1), (\frac{p+1}{5}, 1)\}$$
Figure 2. Special fiber of $A_0(p^2)/\mathcal{O}_K$ when $p = 12k + 1$, $k > 1$.

and at smooth points of the special fiber

$$(a, b) \in \{1, p + 1, p - 1, \frac{p+1}{3}, \frac{p+1}{3}, p - 1\} \times \{0\}.$$

We only discuss the cases that have not been dealt already. The new cases that we encounter presently are $(a, b) \in \{(\frac{p+1}{3}, 0), (\frac{p+1}{3}, 1), (\frac{p+1}{3}, p - 1)\}$.

- When $(a, b) = (\frac{p+1}{3}, 0)$ we have

$$\left(x^{(p+1)/3} - p^{(p^2-1)/2}\right) = \prod_{\zeta \in \mu(p+1)/3(\hat{\mathcal{O}}_K)} \left(x - \zeta p^{3(p-1)/2}\right).$$

Hence

$$\tilde{Z} \times_{\tilde{Z}} \text{Spec} \tilde{\mathcal{O}}_{Z,z} \cong \bigcup_{\zeta \in \mu(p+1)/3(\hat{\mathcal{O}}_K)} \text{Spec} \tilde{\mathcal{O}}_K[[x, y]]/(x - p^{3(p-1)/2}).$$

There are $\frac{p+1}{3}$ pre-images of this point in the normalisation.

- When $(a, b) = (\frac{p+1}{3}, 1)$ we have to calculate the normalisation of $\tilde{\mathcal{O}}_K[[x, y]]/(x^{(p+1)/3}y - p^{(p^2-1)/2})$.

We can in this case take $u = p^{3(p-1)/2}/x$, then the normalisation is

$$\tilde{\mathcal{O}}_K[[x, y, u]]/(y - u^{(p+1)/3}, ux - p^{3(p-1)/2}) \cong \tilde{\mathcal{O}}_K[[x, u]]/(ux - p^{3(p-1)/2}).$$

Thus

$$\tilde{Z} \times_{\tilde{Z}} \text{Spec} \tilde{\mathcal{O}}_{Z,z} \cong \text{Spec} \tilde{\mathcal{O}}_K[[x, u]]/(ux - p^{3(p-1)/2})$$

with $y = u^{(p+1)/3}$. There is only one pre-image of this point in the normalisation and clearly the normalisation is not regular at that point.
Finally when \((a, b) = (\frac{p+1}{3}, p-1)\), we can factorise
\[
\left(x^{(p+1)/3}y^{p-1} - p^{(p^2-1)/2}\right) = \left(x^{(p+1)/6}y^{(p-1)/2} - p^{(p^2-1)/4}\right) \left(x^{(p+1)/6}y^{(p-1)/2} + p^{(p^2-1)/4}\right).
\]

Consider
\[\widehat{O}_K[[x, y]]/(x^{(p+1)/6}y^{(p-1)/2} - p^{(p^2-1)/4})\]
taking \(u = p^{(p+1)/2}/y\) we get an integral extension
\[\widehat{O}_K[[x, y, u]]/(x^{(p+1)/6} - u^{(p-1)/2}, uy - p^{(p+1)/2}),\]
again taking \(v = x/u^2\) we get the integral extension
\[\widehat{O}_K[[y, u, v]]/(v^{(p+1)/6} - u^{(p-5)/6}, uy - p^{(p+1)/2}).\]

Taking \(t = u/v\) we get the integral extension
\[\widehat{O}_K[[y, v, t]]/(v - t^{(p-5)/6}, yvt - p^{(p+1)/2}) \cong \widehat{O}_K[[y, t]]/(yt^{(p+1)/6} - p^{(p+1)/2}).\]

Finally we can take \(s = p^3/t\) to get the extension
\[\widehat{O}_K[[y, s, t]]/(st - p^3, y - s^{(p+1)/6}) \cong \widehat{O}_K[[s, t]]/(st - p^3)\]
which is normal but not regular. We have
\[\tilde{Z} \times_Z \text{Spec} \widehat{O}_{Z, z} \cong \text{Spec} \widehat{O}_K[[s, t]]/(st - p^3) \bigcup \text{Spec} \widehat{O}_K[[s, t]]/(st - p^3).\]

There are thus 2 pre-images of this point in the normalization and at both points the surface is not regular.

From these calculations and those of Edixhoven, section 2 it is clear that the special fiber of \(\tilde{Z}\) is the following.

The map of the special fibers \(\tilde{Z} \times_{\text{Spec}O_K} \text{Spec} \mathbb{F}_p \to Z \times_{\text{Spec}O_K} \text{Spec} \mathbb{F}_p\) is described as follows:

1. The components \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) map to \(C_{2,0}\) and \(C_{0,2}\) isomorphically.
2. The component \(\tilde{L}_i\) maps to \(L_i\) with degree \(p+1\). Over the points of intersection of \(L_i\) with \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) the map is totally ramified; at the points of intersection of \(\tilde{L}_i\) with \(\tilde{C}_{1,1}^1\) and \(\tilde{C}_{1,1}^2\) the ramification index is \(\frac{p+1}{2}\).
3. The component \(\tilde{E}_i^1\) maps isomorphically to \(E\).
4. The component \(\tilde{F}\) maps to \(F\) with degree \(\frac{p+1}{3}\), the point on \(F\) at which it intersects \(C_{2,0}\) has only one pre-image where we have total ramification and same for the point where it intersects \(C_{0,2}\). The point where \(F\) intersects \(C_{1,1}\) has 2 pre-images and the ramification index is \(\frac{p+1}{6}\) at both of these points.
5. The component \(\tilde{C}_{1,1}^1\) maps to \(C_{1,1}\) with degree \((p-1)/2\). At the points of intersection with \(\tilde{L}_i\) the map is totally ramified. At the points of intersection with \(\tilde{E}_i^1\), the ramification index is 2 and at the points of intersection with \(\tilde{F}\) the map is again totally ramified.
The local equations at the double points of the special fiber are listed in the following table.

|   | \( \tilde{C}_{2,0} \) | \( \tilde{C}_{1,1}^1 \) | \( \tilde{C}_{1,1}^2 \) | \( \tilde{C}_{0,2} \) |
|---|-----------------|-----------------|-----------------|-----------------|
| \( \tilde{L}_i \) | \( ux - p^{(p-1)/2} \) | \( uv - p \) | \( uv - p \) | \( ux - p^{(p-1)/2} \) |
| \( \tilde{E}_i^1 \) | \( uy - p^{(p+1)/2} \) | \( uy - p^{(p+1)/2} \) | \( uy - p^{(p+1)/2} \) |
| \( \tilde{E}_i^2 \) | \( st - p^3 \) | \( st - p^3 \) | \( st - p^3 \) | \( st - p^3 \) |
| \( \tilde{F} \) | \( ux - p^{3(p-1)/2} \) | \( st - p^3 \) | \( st - p^3 \) | \( st - p^3 \) |

When \( k > 0 \) the special fiber of the minimal regular model \( X_0(p^2)/\mathcal{O}_K \) is shown in Figure 4 and \( X_0(25) \) has genus 0.

2.3. Case 3: \( p = 12k + 7 \). Now we encounter the complete local rings

\[
\hat{\mathcal{O}}_{Z,z} \cong \frac{\hat{\mathcal{O}}_K[[x, y]]}{(x^a y^b - p^{(p-1)/2})},
\]

where at the double points of the special fiber

\[(a, b) \in \{(p + 1, 1), (p + 1, p - 1), (p^{1/2} + 1, p - 1), (p^{1/2} + 1, 1), (p^{-1/3}, p - 1)\}\]

and at smooth points of the special fiber

\[(a, b) \in \{1, p + 1, p - 1, p^{1/2}, p^{-1/3}\} \times \{0\}.\]
The new cases are \((a, b) \in \{(\frac{p+1}{2}, 0), (\frac{p+1}{2}, 1), (\frac{p+1}{2}, p-1)\} \).

- When \((a, b) = (\frac{p+1}{2}, 0)\) we have

\[
(x^{(p+1)/2} - p^{(p^2-1)/2}) = \prod_{\zeta \in \mu(p+1)/2(\hat{O}_K)} (x - \zeta p^{p-1}).
\]

Hence

\[
\tilde{Z} \times Z \text{Spec} \hat{O}_{Z,z} \cong \bigsqcup_{\zeta \in \mu(p+1)/2(\hat{O}_K)} \text{Spec} \hat{O}_K[[x, y]]/(x - p^{p-1}).
\]

- When \((a, b) = (\frac{p+1}{2}, 1)\) we have to calculate the normalisation of \(\hat{O}_K[[x, y]]/(x^{(p+1)/2} y - p^{(p^2-1)/2})\).

We can in this case take \(u = p^{p-1}/x\), then the normalisation is

\[
\hat{O}_K[[x, y, u]]/(y - u^{(p+1)/2}, ux - p^{p-1}) \cong \hat{O}_K[[x, u]]/(ux - p^{p-1}).
\]

Thus

\[
\tilde{Z} \times Z \text{Spec} \hat{O}_{Z,z} \cong \text{Spec} \hat{O}_K[[x, u]]/(ux - p^{p-1})
\]

with \(y = u^{(p+1)/2}\). There is only one pre-image of this point in the normalisation and clearly the normalisation is not regular at that point.
Finally, when \((a, b) = (\frac{p+1}{2}, p - 1)\) we can factorise
\[
\left( x^{(p+1)/2}y^{p-1} - p^{(p^2-1)/2} \right) = \left( x^{(p+1)/4}y^{(p-1)/2} - p^{(p^2-1)/4} \right) \left( x^{(p+1)/4}y^{(p-1)/2} + p^{(p^2-1)/4} \right).
\]
Consider
\[
\hat{O}_K[[x, y]]/(x^{(p+1)/4}y^{(p-1)/2} - p^{(p^2-1)/4})
\]
and taking \(u = p^{(p+1)/2}/y\) we get an integral extension
\[
\hat{O}_K[[x, y, u]]/(x^{(p+1)/4} - u^{(p-1)/2}, u^y - p^{(p+1)/2}),
\]
again taking \(v = x/u\) we get the integral extension
\[
\hat{O}_K[[y, u, v]]/(v^{(p+1)/4} - u^{(p-3)/4}, u^y - p^{(p+1)/2}).
\]
Taking \(t = u/v\) we get the integral extension
\[
\hat{O}_K[[y, v, t]]/(v - t^{(p-3)/4}, yvt - p^{(p+1)/2}) \cong \hat{O}_K[[y, t]]/(yt^{(p+1)/4} - p^{(p+1)/2}).
\]
Finally we can take \(s = p^2/t\) to get the extension
\[
\hat{O}_K[[y, s, t]]/(st - p^2, y^s - s^{(p+1)/4}) \cong \hat{O}_K[[s, t]]/(st - p^2)
\]
which is normal but not regular. We have thus have
\[
\hat{Z} \times \mathbb{Z} \text{Spec } \hat{O}_{\mathbb{Z}, z} \cong \text{Spec } \hat{O}_K[[s, t]]/(st - p^2) \bigsqcup \text{Spec } \hat{O}_K[[s, t]]/(st - p^2).
\]

There are thus 2 pre-images of this point in the normalisation, at both points the surface is not regular.

These together give the special fiber of \(\hat{Z}/\mathcal{O}_K\) which we describe below.

The map of the special fibers \(\hat{Z} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathbb{F}_p \to \hat{Z} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathbb{F}_p\) is described as follows.

1. The components \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) map to \(C_{2,0}\) and \(C_{0,2}\) isomorphically.
2. The components \(\tilde{L}_i\) maps to \(L_i\) with degree \(p + 1\). At the points of intersection with \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) the map is totally ramified; at the points of intersection with \(\tilde{C}^1_{1,1}\) and \(\tilde{C}^2_{1,1}\) the ramification index is \(\frac{p+1}{2}\).
3. The component \(\tilde{F}^1_i\) maps isomorphically to \(F\).
4. The component \(\tilde{E}\) maps to \(E\) with degree \(\frac{p+1}{2}\), the point on \(E\) at which it intersects \(C_{2,0}\) has only one pre-image where we have total ramification and same for the point where it intersects \(C_{0,2}\). The point where \(E\) intersects \(C_{1,1}\) has 2 pre-images and the ramification index is \(\frac{p+1}{4}\) at both of these points.
5. The component \(\tilde{C}^1_{1,1}\) map to \(C_{1,1}\) with degree \((p - 1)/2\). At the points of intersection with \(\tilde{L}_i\) the map is totally ramified. At the points of intersection with \(\tilde{F}^1_i\), the ramification index is 3 and at the points of intersection with \(\tilde{E}\) the map is again totally ramified.
Below we give the local equations at the nodes of the special fiber.

\[
\begin{array}{c|c|c|c|c}
\tilde{C}_{2,0} & \tilde{C}_{1,1} & \tilde{C}_{1,1} & \tilde{C}_{0,2} \\
L_i & u_x - p^{(p-1)/2} & u_v - p & u_v - p & u_x - p^{(p-1)/2} \\
E & u_x - p^{p-1} & s t - p^2 & s t - p^2 & u_x - p^{p-1} \\
F_i^1 & u y - p^{(p+1)/2} & & & \\
F_i^2 & & u y - p^{(p+1)/2} & & \\
\end{array}
\]

When \( k > 0 \) the special fiber of the minimal regular model \( X_0(p^2)/\mathcal{O}_K \) is depicted by Figure 6. We deal with the case \( p = 7 \) separately in the appendix.

2.4. Case 4: \( p = 12k + 11 \). We encounter the complete local rings

\[
\tilde{\mathcal{O}}_{Z,z} \cong \frac{\tilde{\mathcal{O}}_K[[x,y]]}{(x^a y^b - p^{(p^2 - 1)/2})},
\]

where at the double points of the special fiber

\[
(a, b) \in \left\{ p + 1, \frac{p + 1}{2}, \frac{p + 1}{3} \right\} \times \{1, p - 1\}.
\]
and at smooth points of the special fiber

\[(a, b) \in \left\{1, p + 1, p - 1, \frac{p + 1}{2}, \frac{p + 1}{3}\right\} \times \{0\}.

Since all these cases have already been dealt with we just draw the special fiber below.

The map of the special fibers \(\tilde{Z} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathcal{F}_p \to Z \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathcal{F}_p\) is described as follows.

1. The components \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) map to \(C_{2,0}\) and \(C_{0,2}\) isomorphically.

2. The component \(\tilde{L}_i\) maps to \(L_i\) with degree \(p + 1\). At the points of intersection with \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) the map is totally ramified; at the points of intersection with \(\tilde{C}_{1,1}\) and \(\tilde{C}_{1,1}'\) the ramification index is \(\frac{p + 1}{2}\).

3. The component \(\tilde{E}\) maps to \(E\) with degree \(\frac{p + 1}{2}\). The point on \(E\) at which it intersects \(C_{2,0}\) has only one pre-image where we have total ramification and same for the point where it intersects \(C_{0,2}\). The point where \(E\) intersects \(C_{1,1}\) has 2 pre-images and the ramification index is \(\frac{p + 1}{4}\) at both of these points.

4. The component \(\tilde{F}\) maps to \(F\) with degree \(\frac{p + 1}{3}\), the point on \(F\) at which it intersects \(C_{2,0}\) has only one pre-image where we have total ramification and same for the point where it intersects
The point where $F$ intersects $C_{1,1}$ has 2 pre-images and the ramification index is $\frac{p+1}{6}$ at both of these points.

(5) The component $\tilde{C}_{1,1}$ map to $C_{1,1}$ with degree $(p-1)/2$. At the points of intersection with $\tilde{L}_i$, $\tilde{E}$ and $\tilde{F}$ the map is totally ramified.

The local equations at the nodes of the special fiber are the following.

|        | $\tilde{C}_{2,0}$ | $\tilde{C}_{1,1}^{-1}$ | $\tilde{C}_{1,1}^{-2}$ | $\tilde{C}_{0,2}$ |
|--------|-------------------|-------------------------|------------------------|-------------------|
| $\tilde{L}_i$ | $ux - p^{(p-1)/2}$ | $uv - p$ | $uv - p$ | $ux - p^{(p-1)/2}$ |
| $\tilde{E}$   | $ux - p^{p-1}$    | $st - p^2$ | $st - p^2$ | $ux - p^{p-1}$ |
| $\tilde{F}$   | $ux - p^{3(p-1)/2}$ | $st - p^3$ | $st - p^3$ | $ux - p^{3(p-1)/2}$ |

The special fiber of the minimal regular model $X_0(p^2)/\mathcal{O}_K$ is shown in Figure 8.

3. ARAKELOV SELF-INTERSECTION OF THE CANONICAL SHEAF

Recall that the cusps $0, \infty$ of $X_0(p^2)$ are both $\mathbb{Q}$ points of the curve. Let $H_0$ and $H_\infty$ be the corresponding sections (horizontal divisors) in $X_0(p^2)/\mathcal{O}_K$. Here we assume $p \neq 5, 7, 13$.

From [4], section 6, we know that the horizontal divisor $H_0$ intersects exactly one of the curves $\tilde{C}_{2,0}$ or $\tilde{C}_{0,2}$ of the special fiber — at an $\mathbb{F}_p$ rational point — transversally [cf. Liu [17 Chapter 9, Proposition 1.30 and Corollary 1.32, p. 388]]. We call that component $\tilde{C}_0$. It follows from the cusp and component
Figure 8. Special fiber of $X_0(p^2)/\mathcal{O}_K$ when $p = 12k + 11$.

labelling of Katz and Mazur [14] p. 296] that $H_\infty$ meets the other component transversally and we call it $\tilde{C}_\infty$. 
Let us define the vertical divisors $V_{0,p}$ and $V_{\infty,p}$ of $X_0(p^2)/\mathcal{O}_K$, supported on the special fiber. When $p = 12k + 1$ let $x = -(k - 2)/k = -(p - 25)/(p - 1)$, define

$$V_{0,p} = (12 - 12g_{p^2}) \tilde{C}_0 + 7\tilde{C}_{1,1} + 7\tilde{C}_{1,1} + \sum_{i=1}^{k} \frac{p - 1}{2} x L_i$$

$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k - 1} \left[ lx + \frac{p - 1 - 2l}{p - 1} (12 - 12g_{p^2}) \right] A_{l,i} + \sum_{i=1}^{k} \sum_{l=1}^{6k - 1} (lx) B_{l,i},$$

$$V_{\infty,p} = (12 - 12g_{p^2}) \tilde{C}_\infty + 7\tilde{C}_{1,1} + 7\tilde{C}_{1,1} + \sum_{i=1}^{k} \frac{p - 1}{2} x L_i$$

$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k - 1} \left[ lx + \frac{p - 1 - 2l}{p - 1} (12 - 12g_{p^2}) \right] B_{l,i} + \sum_{i=1}^{k} \sum_{l=1}^{6k - 1} (lx) A_{l,i}.$$

If $p = 12k + 5$ let $x = -3(k - 1)/(3k + 1) = -(p - 17)/(p - 1)$, define

$$V_{0,p} = (12 - 12g_{p^2}) \tilde{C}_0 + 3\tilde{C}_{1,1} + 3\tilde{C}_{1,1} + \sum_{i=1}^{k} \frac{p - 1}{2} x L_i + \frac{p - 1}{2} x F$$

$$- (4k - 5)S_1 - (4k - 5)T_1 - (2k - 4)S_2 - (2k - 4)T_2 + \sum_{i=1}^{k} \sum_{l=1}^{6k + 1} (lx) B_{l,i} + \sum_{j=1}^{6k + 1} \frac{j x}{3} H_j$$

$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k + 1} \left[ lx + \frac{p - 1 - 2l}{p - 1} (12 - 12g_{p^2}) \right] A_{l,i} + \sum_{j=1}^{6k + 1} \frac{j x}{3} + \frac{3(p - 1) - 2j}{3(p - 1)} (12 - 12g_{p^2}) G_j$$

$$V_{\infty,p} = (12 - 12g_{p^2}) \tilde{C}_\infty + 3\tilde{C}_{1,1} + 3\tilde{C}_{1,1} + \sum_{i=1}^{k} \frac{p - 1}{2} x L_i + \frac{p - 1}{2} x F$$

$$- (4k - 5)S_1 - (4k - 5)T_1 - (2k - 4)S_2 - (2k - 4)T_2 + \sum_{i=1}^{k} \sum_{l=1}^{6k + 1} (lx) A_{l,i} + \sum_{j=1}^{6k + 1} \frac{j x}{3} G_j$$

$$+ \sum_{i=1}^{k} \sum_{l=1}^{6k + 1} \left[ lx + \frac{p - 1 - 2l}{p - 1} (12 - 12g_{p^2}) \right] B_{l,i} + \sum_{j=1}^{6k + 1} \frac{j x}{3} + \frac{3(p - 1) - 2j}{3(p - 1)} (12 - 12g_{p^2}) H_j.$$
If \( p = 12k + 11 \) let \( x = -6k/(6k + 5) = -(p - 11)/(p - 1) \) then

\[
V_{0,p} = (12 - 12g_\nu) \tilde{C}_0 + \sum_{i=1}^{k} \frac{p-1}{2} x \tilde{L}_i + \frac{p-1}{2} x \tilde{E} + \frac{p-1}{2} x \tilde{F} \\
- 3kU - 3kV - 4kS_1 - 4kT_1 - 2kS_2 - 2kT_2 + \sum_{i=1}^{k} \sum_{l=1}^{6k+4} (lx) B_{i,l} + \sum_{j=1}^{12k+9} \frac{jx}{2} N_j + \sum_{j=1}^{18k+14} \frac{jx}{3} H_j \\
+ \sum_{i=1}^{k} \sum_{l=1}^{6k+4} \left[ lx + \frac{p-1 - 2l}{p-1} (12 - 12g_\nu) \right] A_{i,l} + \sum_{j=1}^{12k+9} \left[ \frac{jx}{2} + \frac{p-1 - j}{p-1} (12 - 12g_\nu) \right] M_j \\
+ \sum_{j=1}^{18k+14} \left[ \frac{jx}{3} + \frac{3(p-1) - 2j}{3(p-1)} (12 - 12g_\nu) \right] G_j.
\]

\[
V_{\infty,p} = (12 - 12g_\nu) \tilde{C}_\infty + \sum_{i=1}^{k} \frac{p-1}{2} x \tilde{L}_i + \frac{p-1}{2} x \tilde{E} + \frac{p-1}{2} x \tilde{F} \\
- 3kU - 3kV - 4kS_1 - 4kT_1 - 2kS_2 - 2kT_2 + \sum_{i=1}^{k} \sum_{l=1}^{6k+4} (lx) A_{i,l} + \sum_{j=1}^{12k+9} \frac{jx}{2} M_j + \sum_{j=1}^{18k+14} \frac{jx}{3} G_j \\
+ \sum_{i=1}^{k} \sum_{l=1}^{6k+4} \left[ lx + \frac{p-1 - 2l}{p-1} (12 - 12g_\nu) \right] B_{i,l} + \sum_{j=1}^{12k+9} \left[ \frac{jx}{2} + \frac{p-1 - j}{p-1} (12 - 12g_\nu) \right] N_j \\
+ \sum_{j=1}^{18k+14} \left[ \frac{jx}{3} + \frac{3(p-1) - 2j}{3(p-1)} (12 - 12g_\nu) \right] H_j.
\]

Let \( K_{X_0(p^2)/\mathcal{O}_K} \) be a canonical divisor on \( X_0(p^2)/\mathcal{O}_K \).

**Proposition 4.** For \( m \in \{0, \infty\} \), the divisors \( D_{m,p} = K_{X_0(p^2)/\mathcal{O}_K} - (2g_\nu - 2)H_m + V_{m,p} \) are orthogonal to all vertical divisors with respect to the Arakelov intersection pairing.

**Proof.** We note that \( H_0 \) intersects \( \tilde{C}_0 \) transversally and does not meet any other component of the special fiber. Similarly \( H_\infty \) intersects \( \tilde{C}_\infty \) transversally and no other component. We shall only discuss the case when \( p = 12k + 1 \) and \( k > 1 \), since all other cases are analogous.

If \( V \) is any fiber other than the special fiber, then \( V \) is a prime vertical divisor, and we have \( \langle K_{X_0(p^2)/\mathcal{O}_K}, V \rangle = (2g_\nu - 2) \log(p) \), by adjunction formula for Arakelov theory. The sections \( H_m \) intersect \( V \) in a \( \mathbb{F}_p \) rational point, hence \( \langle H_m, V \rangle = \log(p) \). Finally as \( \langle V, V_{m,p} \rangle = 0 \), the result follows. It remains to check for prime vertical divisors supported on the special fiber.

If \( S \) is the special fiber then for any prime vertical divisor \( V \), \( \langle V, S \rangle = 0 \). From this we have the following: (see figures 1 and 2), noting that \( \mathcal{O}_K/(p) = \mathbb{F}_p \), so \( \log(\#\mathcal{O}_K/(p)) = \log(p) \)

\[
\tilde{C}_0^2 = \tilde{C}_\infty^2 = (\tilde{C}_{1,1})^2 = (\tilde{C}_{1,1})^2 = -k \log(p), \\
\tilde{L}_i^2 = -4 \log(p), \quad A_{i,i}^2 = B_{i,i}^2 = -2 \log(p).
\]
Using the adjunction formula now

\[
\langle D_{0,p}, \tilde{C}_0 \rangle = [(2p_a(\tilde{C}_0) - 2) \log(p) - \tilde{C}_0^2] - (2g_{p^*} - 2) \log(p) + (12 - 12g_{p^*})\tilde{C}_0^2
\]

\[
+ \sum_{i=1}^{k} \left[ -\frac{k-2}{k} + \frac{p-3}{p-1} (12 - 12g_{p^*}) \right] \langle A_{1,i}, \tilde{C}_0 \rangle
\]

\[
= [(k - 2) - (2g_{p^*} - 2) - (k - 2) + (p - 1)(g_{p^*} - 1) - (p - 3)(g_{p^*} - 1)] \log(p)
\]

\[
= 0.
\]

\[
\langle D_{0,p}, \tilde{C}_\infty \rangle = [(2p_a(\tilde{C}_0) - 2) \log(p) - \tilde{C}_0^2] + \sum_{i=1}^{k} -\frac{k-2}{k} (B_{1,i}, \tilde{C}_0)
\]

\[
= [(k - 2) - (k - 2)] \log(p) = 0.
\]

\[
\langle D_{0,p}, \tilde{C}_{1,1} \rangle = [(2p_a(\tilde{C}_{1,1}) - 2) \log(p) - (\tilde{C}_{1,1}^2)] + 7(\tilde{C}_{1,1}^2) + \sum_{i=1}^{k} -6(k - 2) \langle \tilde{L}_i, \tilde{C}_{1,1} \rangle
\]

\[
= [(6k^2 - 6k + 2 - 2 + k) - 7k - 6k(k - 2)] \log(p) = 0
\]

\[
\langle D_{0,p}, \tilde{L}_i \rangle = [(2p_a(\tilde{L}_i) - 2) \log(p) - \tilde{L}_i^2] - 6(k - 2)\tilde{L}_i^2 + 7(\tilde{C}_{1,1}^i, \tilde{L}_i) + 7(\tilde{C}_{2,1}^i, \tilde{L}_i)
\]

\[
\left[ -\frac{(6k - 1)(k - 2)}{k} + \frac{2}{p-1} (12 - 12g_{p^*}) \right] \langle A_{6k-1,i}, \tilde{L}_i \rangle - \frac{(6k - 1)(k - 2)}{k} \langle B_{6k-1,i}, \tilde{L}_i \rangle
\]

\[
= \left[ (12k - 2 + 4) + 24(k - 2) + 14 - 2 \frac{(6k - 1)(k - 2)}{k} - 2 \frac{12k^2 - 3k - 2}{k} \right] \log(p) = 0.
\]

The rest are just similar calculations observing that \( A_{l,i} \) and \( B_{l,i} \) are projective lines and keeping track of which components they intersect.

Proposition 5. With the notation from the previous proposition, the following equality holds.

\[
\nu_{X_0(p^2)/\mathcal{O}_\mathcal{K}} = -4g_{p^2}(g_{p^2} - 1)\langle H_0, H_\infty \rangle + \frac{1}{g_{p^2} - 1} \left[ g_{p^2} \langle V_{0,p}, V_{\infty,p} \rangle - \frac{V_{0,p}^2 + V_{\infty,p}^2}{2} \right] + e_p.
\]

where

\[
e_p = \begin{cases} 0 & \text{if } p \equiv 11 \pmod{12}, \\ O(\log(p)) & \text{if } p \not\equiv 11 \pmod{12}. \end{cases}
\]
This translates to
(3.2)
\[
\varphi_{X_0(p^2)/\mathcal{O}_K} = -4g_p^2(g_p^2 - 1)\langle H_0, H_\infty \rangle + \frac{3p^3 - 51p^2 + 137p - 89}{24} \log(p) + O(\log(p)),
\]
\[p \equiv 1 \pmod{12},\]
\[
\varphi_{X_0(p^2)/\mathcal{O}_K} = -4g_p^2(g_p^2 - 1)\langle H_0, H_\infty \rangle + \frac{3p^3 - 19p^2 - 23p - 217}{24} \log(p) + O(\log(p)),
\]
\[p \equiv 5 \pmod{12},\]
\[
\varphi_{X_0(p^2)/\mathcal{O}_K} = -4g_p^2(g_p^2 - 1)\langle H_0, H_\infty \rangle + \frac{3p^3 - 27p^2 + 17p - 209}{24} \log(p) + O(\log(p)),
\]
\[p \equiv 7 \pmod{12},\]
\[
\varphi_{X_0(p^2)/\mathcal{O}_K} = -4g_p^2(g_p^2 - 1)\langle H_0, H_\infty \rangle + \frac{3p^3 + 5p^2 - 143p - 145}{24} \log(p),
\]
\[p \equiv 11 \pmod{12}.
\]

Proof. The proof of formula (3.1) is completely analogous to the proof of Lemma 33 of [4]. This proof depends upon the fact that Néron-Tate heights are bounded by the heights of the Heegner points of $X_0(N)$ associated to to the CM fields $\mathbb{Q}(\sqrt{-d})$ with $d \in \{1, 3\}$ (see Proposition 32 of [4]). Here, we use an argument similar to [21, Section 6].

From Proposition 3.1 it follows that $V_{0,p}^2 = \langle (2g_p^2 - 2)H_0 - \omega, V_0 \rangle$ and $\langle V_{0,p}, V_{\infty,p} \rangle = \langle (2g_p^2 - 2)H_0 - \omega, V_{\infty,p} \rangle$. Using this we have the following:

| $p \equiv 1 \pmod{12}$ | $V_{0,p}^2 = V_{\infty,p}^2$ | $\frac{4p^4 - 43p^3 + 39p^2 + 423p + 729}{24}$ | $\frac{3p^3 - 99p^2 + 377p + 871}{24}$ |
|--------------------------|------------------|-----------------|-----------------|
| $\langle V_{0,p}, V_{\infty,p} \rangle$ | | | |

| $p \equiv 5 \pmod{12}$ | $V_{0,p}^2 = V_{\infty,p}^2$ | $\frac{4p^4 - 43p^3 + 71p^2 + 263p + 217}{24}$ | $\frac{3p^3 - 67p^2 + 217p + 359}{24}$ |
|--------------------------|------------------|-----------------|-----------------|
| $\langle V_{0,p}, V_{\infty,p} \rangle$ | | | |

| $p \equiv 7 \pmod{12}$ | $V_{0,p}^2 = V_{\infty,p}^2$ | $\frac{4p^4 - 43p^3 + 63p^2 + 303p + 321}{24}$ | $\frac{3p^3 - 75p^2 + 257p + 463}{24}$ |
|--------------------------|------------------|-----------------|-----------------|
| $\langle V_{0,p}, V_{\infty,p} \rangle$ | | | |

| $p \equiv 11 \pmod{12}$ | $V_{0,p}^2 = V_{\infty,p}^2$ | $\frac{-4p^4 - 43p^3 + 95p^2 + 143p + 1}{24}$ | $\frac{3p^3 - 43p^2 + 97p + 143}{24}$ |
|--------------------------|------------------|-----------------|-----------------|
| $\langle V_{0,p}, V_{\infty,p} \rangle$ | | | |

Now $3.2$ just follows from $3.1$. □

**Theorem 6.** The asymptotic expression of the stable arithmetic self-intersection of $X_0(p^2)/K$ is

\[
\varphi_{p^2} = 2g_p^2 \log p^2 + \frac{p \log(p^2)}{8} + O(p \log(p^2)).
\]

Proof. We have

\[
\varphi_{p^2} = \frac{1}{(p^2 - 1)/2} \varphi_{X_0(p^2)/\mathcal{O}_K}.
\]
Hence from the previous proposition we have
\[
\varpi_{p^2}^2 = -4g_{p^2}(g_{p^2} - 1)g_{\text{can}}(0, \infty) + \left(\frac{p}{4} - \frac{51}{12} + \frac{35}{3(p+1)}\right)\log(p) + O(\log(p)),
\]
\[
p \equiv 1 \pmod{12}
\]
\[
\varpi_{p^2}^2 = -4g_{p^2}(g_{p^2} - 1)g_{\text{can}}(0, \infty) + \left(\frac{p}{4} - \frac{19}{12} - \frac{5}{3(p-1)} - \frac{18}{p^2 - 1}\right)\log(p) + O(\log(p)),
\]
\[
p \equiv 5 \pmod{12}
\]
\[
\varpi_{p^2}^2 = -4g_{p^2}(g_{p^2} - 1)g_{\text{can}}(0, \infty) + \left(\frac{p}{4} - \frac{27}{12} + \frac{5}{3(p+1)} - \frac{18}{p^2 - 1}\right)\log(p) + O(\log(p)),
\]
\[
p \equiv 7 \pmod{12}
\]
\[
\varpi_{p^2}^2 = -4g_{p^2}(g_{p^2} - 1)g_{\text{can}}(0, \infty) + \left(\frac{p}{4} + \frac{5}{12} - \frac{35}{3(p-1)}\right)\log(p),
\]
\[
p \equiv 11 \pmod{12}.
\]

Now from [4, Proposition 23],
\[
\langle H_0, H_\infty \rangle = g_{\text{can}}(0, \infty) = -\frac{\log p}{g_{p^2}} + o\left(\frac{\log p}{g_{p^2}}\right)
\]
which completes the proof. \square

4. Dual graphs of special fibers

Let \(G_p\) be the dual graph of the special fiber \(X_0(p^2)(p)\). This is a metrized graph in the sense of Zhang [28]. Let \(V(G_p)\) be the set of vertices, and \(E(G_p)\) the set of edges of \(G_p\). We denote by \(l(G_p)\) the sum of lengths of all the edges. For any \(x \in V(G_p)\), \(g(x)\) is the geometric genus of the corresponding component of \(X_0(p^2)(p)\).

4.1. Case \(p = 12k + 1\). Recall the diagram for the special fiber is given by Figure [2]. We draw the dual graph of the special fiber here. In the dual graph we denote the vertices corresponding to the components by small letters with the same indices but drop the tildes. We ignore the vertices with genus 0 and valence 2.
Each black edge has length $6k$ and green edge has length 1. None of the edges intersect. As a consequence, we conclude that $l(G_p) = 12k^2 + 2k$. We have $g(\ell_i) = 6k$, $g(c_{2,0}) = g(c_{0,2}) = 0$, $g(c_{1,1}) = 3k^2 - 3k + 1$.

4.2. **Case** $p = 12k + 5$. In this case, diagram for the special fiber is given by Figure 4. Further $g(c_{2,0}) = g(c_{0,2}) = 0$, $g(c_{1,1}) = 3k^2 - k$, $g(\ell_i) = 6k + 2$, $g(f) = 2k$. The dual graph is the following.

![Diagram for the special fiber for $p = 12k + 5$](image)

Black edges have length $6k + 2$, orange edges length $18k + 6$, blue edges length 3 and green are of length 1. As a consequence, we conclude that $l(G_p) = 12k^2 + 42k + 18$.

4.3. **Case** $p = 12k + 7$. In this case, diagram for the special fiber is given by Figure 6. Further $g(c_{2,0}) = g(c_{0,2}) = 0$, $g(c_{1,1}) = 3k^2$, $g(\ell_i) = 6k + 3$, $g(e) = 3k + 1$. The dual graph is the following.

![Diagram for the special fiber for $p = 12k + 7$](image)
Black edges have length $6k + 3$, cyan edges length $12k + 6$, red edges length $2$ and green are of length $1$. As a consequence, we conclude that $l(G_p) = 12k^2 + 32k + 16$.

4.4. **Case** $p = 12k + 11$. In this case, diagram for the special fiber is given by Figure 8. Further $g(c_{2,0}) = g(c_{0,2}) = 0$, $g(c_{1,1}) = 3k^2 + 2k$, $g(t_i) = 6k + 5$, $g(c) = 3k + 2$, $g(f) = 2k + 1$. The dual graph is the following.

![Diagram](image)

Black edges have length $6k+5$, orange edges length $18k+15$, cyan edges length $12k+10$, blue edges length $3$, red edges length $2$ and green are of length $1$. As a consequence, we conclude that $l(G_p) = 12k^2 + 72k + 60$.

5. **Effective Bogomolov’s conjecture**

We will prove Theorem 1 in this section. First we start with two propositions about the asymptotic behaviour of some quantities involved as $p \to \infty$. Let $\tau(\Gamma)$ be the tau constant associated to the metrized graph $\Gamma$ as defined in [5]. In this article, we are interested in the metrized graph $G_p$ — the dual graph of the special fiber — as in Section 4.

**Proposition 7.** We have the following asymptotic estimate for $\tau(G_p)$

$$
\frac{8(g_{p^2} - 1)}{(p^2 - 1)g_{p^2}^2} \tau(G_p) \to 0.
$$

**Proof.** From Section 4, we conclude that $l(G_p) = 12k^2 + ak + b$.

with $a, b \in \mathbb{N}$. Recall by [4], page 6, remark 3, we have

$$
g_{p^2} = 1 + \frac{(p + 1)(p - 6) - 12c}{12}
$$

with $c \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$. From the above expression, it is easy to see that $\frac{8(g_{p^2} - 1)}{(p^2 - 1)g_{p^2}^2} l(G_p) \to 0$ as $p \to \infty$. 

Recall that by [3], Equation 14.3, p. 37 if \( \Gamma \) is a graph with \( n \) edges, then we have a bound on \( \tau(\Gamma) \):

\[
\frac{1}{16n} l(\Gamma) \leq \tau(\Gamma) \leq \frac{1}{4} l(\Gamma).
\]

Hence, we obtain the asymptotic bound as in the proposition.

For any two vertices \( x \) and \( y \) of \( G_p \), let \( r(x, y) \) be the resistance between them. The resistance function is defined in Zhang [28, Section 3]; see Proposition 3.3 of Zhang for properties of \( r \). Let us denote

\[
\theta(x, y) = (v(x) - 2 + 2g(x))(v(y) - 2 + 2g(y))r(x, y).
\]

We also consider the following quantity associated to any metrized graph \( \Gamma \), as defined in [5]

\[
\tilde{\theta}(\Gamma) = \sum_{x, y \in V(\Gamma)} \theta(x, y).
\]

**Proposition 8.** We have the following asymptotic bound

\[
\frac{1}{(p^2 - 1)g_{p^2}} \tilde{\theta}(G_p) = O\left( \frac{1}{p^2} \right).
\]

**Proof.** For any two vertices \( x, y \) of a metrized graph \( \Gamma \), \( r(x, y) \) is bounded above by the length of the shortest path between the vertices [2, p. 15, Exercise 12]. Let \( p = 12k + 1 \), then

\[
\theta(l_i, l_j) \leq 2(12k + 2)^2,
\]

\[
\theta(l_i, c_{2,0}) = \theta(l_i, c_{0,2}) \leq (k - 2)(12k + 2)(6k),
\]

\[
\theta(l_i, c_{1,1}) = \theta(l_i, c_{1,1}^2) \leq (12k + 2)(6k^2 - 5k),
\]

\[
\theta(c_{2,0}, c_{0,2}) \leq (12k)(k - 2)^2,
\]

\[
\theta(c_{1,1}^2, c_{0,2}) = \theta(c_{1,1}^2, c_{2,0}) \leq (6k - 1)(k - 2)(6k^2 - 5k),
\]

\[
\theta(c_{1,1}^2, c_{1,1}) \leq 2(6k^2 - 5k)^2.
\]

The result now follows by summing everything up and dividing by \( g_{p^2}(p^2 - 1) \) and noting that \( g_{p^2} = O(p^2) \).

In all other cases the proof is similar. \( \square \)

Let \( \varpi_{a,p^2} \) be the admissible metrized relative dualising sheaf of the curve \( \mathcal{X}(p^2)/\mathcal{O}_K \); see Zhang [28] for definition.

**Lemma 9.** We have the following asymptotic expression for admissible self-intersection numbers for the modular curve of the form \( X_0(p^2) \):

\[
\varpi_{a,p^2} = 2g_{p^2} \log(p) + o(g_{p^2} \log(p)).
\]

**Proof.** Recall the following Theorem that connects admissible self-intersection number with the arithmetic self-intersection number [5, Theorem 4.45, 28, 27, Theorem 5.5]:

\[
\varpi_{a,p^2} = \varpi_{p^2} - \frac{1}{[K: \mathbb{Q}]} \left( \frac{4(g_{p^2} - 1)}{g_{p^2}} \tau(G_p) + \frac{1}{2g_{p^2}} \tilde{\theta}(G_p) \right) \log p.
\]

Hence the result now follows from Theorem [6] and Propositions [7,8]
Proof of Theorem 1. Since $h_{NT}(x) \geq 0$ for all $x$, we note that by [28, Theorem 5.6]

$$a'(D) \geq \frac{\sigma_{a,p^2}}{4(g_{p^2} - 1)} + \frac{2g_{p^2} - 2}{g_{p^2}} h_{NT}\left(D - \frac{K_{X_0(p^2)}}{2g_{p^2} - 2}\right) \geq \frac{1}{2} a'(D)$$

for the divisor $D = \infty$ as in the introduction. Let $h = h_{NT}\left(D - \frac{K_{X_0(p^2)}}{2g_{p^2} - 2}\right)$, then by an argument similar to section 6 of Michel-Ullmo [21], we can show that

$$(5.2) \quad h = \frac{1}{g_{p^2}^2} O(\log p).$$

By Zhang’s theorem

$$h_{NT}(\varphi_D(x)) \leq a'(D) - \epsilon$$

holds for only finitely many $x$. We deduce that

$$F_2 = \left\{ x \in X_0(p^2)(\mathbb{Q}) \mid h_{NT}(\varphi_D(x)) < \frac{\sigma_{a,p^2}}{4(g_{p^2} - 1)} + h - \epsilon \right\}$$

is finite.

On the other hand, the set

$$F_1 = \left\{ x \in X_0(p^2)(\mathbb{Q}) \mid h_{NT}(\varphi_D(x)) \leq \frac{\sigma_{a,p^2}}{2(g_{p^2} - 1)} + 2h + \epsilon \right\}$$

is infinite. From Lemma 9 we conclude that for large enough $p$:

$$2 - \epsilon < \frac{\sigma_{a,p^2}}{g_{p^2} \log(p^2)} < 2 + \epsilon.$$

The result now follows because of the bound on $h$ given by (5.2).

6. Stable Faltings Heights

Let $J_0(N)/\mathbb{Q}$ be the Jacobian variety of the modular curve $X_0(N)/\mathbb{Q}$. We denote by $h_{Fal}(J_0(N))$ the stable Faltings height of $J_0(N)/\mathbb{Q}$. The arithmetic Noether formula, (see Moret-Bailly [22], Theorem 2.5), implies

$$(6.1) \quad 12 h_{Fal}(J_0(p^2)) = \sigma_{p^2} + \frac{2s_p \log(p)}{p^2 - 1} + \delta_{Fal}(X_0(p^2)) - 4g_{p^2} \log(2\pi).$$

Here $s_p$ denotes the number of singular points in the special fiber of $X_0(p^2)/\mathcal{O}(K)$ and $\delta_{Fal}$ is the Faltings delta invariant, as defined in Faltings [10, Theorem 1].

Proof of Theorem 2. From Section 2 we calculate $s_p$:

$$s_p = \frac{p^2 - 1}{12}, \quad p \equiv 1 \pmod{12},$$

$$s_p = \frac{(p + 1)(p + 31)}{12}, \quad p \equiv 5 \pmod{12},$$

$$s_p = \frac{(p + 1)(p + 17)}{12}, \quad p \equiv 7 \pmod{12},$$

$$s_p = \frac{(p + 1)(p + 49)}{12}, \quad p \equiv 11 \pmod{12}. $$
Thus, we deduce that:

\[
\frac{2s_p \log(p)}{p^2 - 1} = \frac{\log(p)}{6} + o(\log(p)).
\]

By \([13, \text{Theorem 5.6}]\), we have \(\delta_{\text{Fal}}(X_0(p^2)) = O(g_p^2) = O(p^2)\) if \(g_p^2 > 1\) (valid for \(p > 7\)). Hence the result follows from (6.1) and Theorem 6.

\[
\square
\]

Appendix A. Small primes

For the prime 5, \(X_0(25)\) has genus 0. The next prime is 7 in which case \(g(X_0(49)) = 1\). From the calculations of section 2.3 it follows that the special fiber of the minimal regular model of \(X_0(49)\) over \(\mathcal{O}_K\) is a single genus 1 curve hence in fact smooth, this is the component \(\tilde{E}\), all other components can be blown down.

For \(p = 11\) the regular minimal model of \(X_0(121)/\mathcal{O}_K\) is as described in 2.4.

Finally when \(p = 13\) we see that the special fiber of \(X_0(169)/\mathcal{O}_K\) is described by the following diagram.

We have to contract \(\tilde{C}_{2,0}\) and \(\tilde{C}_{0,2}\) since these are genus 0 components with self-intersection \(-1\).

\[
\begin{array}{c}
\tilde{C}_{1,1}^0 \\
\downarrow \\
\tilde{L}_1 \\
\end{array}
\quad
\begin{array}{c}
\tilde{C}_{1,1}^2 \\
\downarrow \\
6 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
1 \\
\end{array}
\]

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