Design of microelectromechanical systems for variability via chance-constrained optimization

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Abstract. In this paper we describe a robust design method for microelectromechanical systems (MEMS) subject to inherent geometric and material uncertainties. We propose to formulate the robust MEMS design with variability problem as a specific optimization problem, namely chance-constrained posynomial programming, in which the statistical variations of both the process parameters and design variables can be easily incorporated. Using such constrained optimization approach, automated robust design can be obtained with very suboptimal design cost, and parametric yield of each specification can be guaranteed. Finally, the method is illustrated by considering the example of a simple crab-leg resonator.

1. Introduction
MEMS is a fast growing research area enabled by integrating a number of functions, including optics, mechanics, fluidics and electronics, on a single silicon chip using traditional integrated circuit process technology. Manufacturing yield, which is concerned with deviations in device performance due to parameter variations induced by fluctuations in the manufacturing process, should be taken into consideration in the early MEMS design phase. Although several CAD tools for MEMS are appearing to help the design both at the device-level and at the system-level, most of them, however, cannot take into account possible variation of design after fabrication process, which may lead to non-negligible effect on the overall structure behavior.

Uncertainty is particularly important in MEMS design for several reasons. First, with MEMS feature sizes approaching micron and submicron levels, the MEMS performance metric is rendered more sensitive to process variations. Second, the materials used in MEMS devices might be poorly characterized. This is mainly due to the number of new materials and processes being used. We are well aware that metals, plastics, and other material properties are largely dependent on how they are manufactured, similarly the properties of MEMS materials are dependent on how they are fabricated. Further complicating the characterization process is the difficulty of making measurements at the micrometer scale. For these reasons, most microfabrication materials have large uncertainties associated with their properties. Therefore, a robust design method is needed to ensure the highest product performance, which is greatly influenced by manufacturing process variations due to the small dimensions and high feature complexity.

In this paper we describe a robust design method for MEMS devices subject to inherent geometric and material uncertainties. We formulate the MEMS “design under uncertainty” [1] (or “design for variability”) problem as a special type of constrained optimization problem,
called *chance-constrained posynomial programming*. In this method, the variations in both the process parameters and design variables can be readily considered in the early design phase. Keeping with the primary object of introducing a framework of MEMS design for variability, the example of a simple crab-leg is adopted to illustrate the proposed method. This numerical example reveals that automated robust design can be obtained with very suboptimal design cost and that parametric yield of each specification can be guaranteed.

2. **Posynomial programming**

Let $\mathbb{R}^n_+ \equiv \{\mathbb{x} \in \mathbb{R}^n | x_i > 0, i = 1, \ldots, n\}$ denote the set of real $n$-vectors whose components are positive. A function $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, defined as

$$f(x) = \sum_{k=1}^{K} c_k \prod_{j=1}^{n} x^{a_{jk}},$$

where $c_k \geq 0$ and $a_{jk} \in \mathbb{R}$, is called a *posynomial*. An optimization problem of the form

$$\text{minimize } f_0(x) \quad \text{subject to } f_s(x) \leq 1, \quad s = 1, \ldots, m,$$

where $f_0, \ldots, f_m$ are posynomials, is called a *geometric program in posynomial form*, or a *posynomial program*. Here, the constraints $x_i > 0, i = 1, \ldots, n$ are implicit.

The most important feature of posynomial programs is that they can be reformulated as convex optimization problems and, therefore, globally optimal solutions can be computed with great efficiency. (See [2] for more details.)

3. **Chance-constrained posynomial programming**

3.1. **Motivations & assumptions**

In many posynomial programs (especially those resulting from integrated circuit designs), the coefficients $c_k$ in (1) are often posynomials in other process/physical/system parameters, say, $p_i > 0, i = 1, \ldots, n_p$. Therefore, here we assume

$$c_k = d_k \prod_{i=1}^{n_p} p_i^{b_{ik}}, \quad k = 1, \ldots, K,$$

where $d_k \geq 0$ and $b_{ik} \in \mathbb{R}$, and rewrite (1) as

$$f(x;p) = \sum_{k=1}^{K} d_k \prod_{i=1}^{n_p} p_i^{b_{ik}} \prod_{j=1}^{n_x} x_j^{a_{jk}},$$

which is a posynomial in both $x \in \mathbb{R}^{n_x}$ and $p \in \mathbb{R}^{n_p}$.

These parameters $p_i, i = 1, \ldots, n_p$ must be assigned values before optimization techniques can be applied to obtain numerical solutions. Precise values of these involved parameters are often impossible to determine due to either random noises, measurement errors or other technical difficulties. In addition, after applying optimization techniques, the optimal solutions of $x_j, j = 1, \ldots, n_x$, even if computed very accurately, may be difficult to implement accurately. To account for these uncertainties, throughout this paper we assume $x \in \mathbb{R}^{n_x}$ and $p \in \mathbb{R}^{n_p}$ have “additive” normal variations (with zero means and given standard deviations):

$$\delta p_i \sim \mathcal{N}(0, \sigma_{p_i}^2), \quad i = 1, \ldots, n_p$$

$$\delta x_j \sim \mathcal{N}(0, \sigma_{x_j}^2), \quad j = 1, \ldots, n_x,$$
where $\delta p_i, \ i = 1, \ldots, n_p$ and $\delta x_j, \ j = 1, \ldots, n_x$ are mutually independent. In addition, we assume

$$p_i - 3\sigma_{p_i} > 0, \ i = 1, \ldots, n_p$$

$$x_j - 3\sigma_{x_j} > 0, \ j = 1, \ldots, n_x.$$  \hfill (6) \hfill (7)

(Note that in general we can verify if the assumption (7) holds since in many cases, e.g., in integrated circuit designs, it is easy to determine reasonable range of values for each design variable $x_j$ before applying optimization techniques.)

Directly incorporating $\delta p_i$ and $\delta x_j$ into (3) leads to

$$\hat{f}(x; p) = \sum_{k=1}^{K} d_k \prod_{i=1}^{n_p} (p_i + \delta p_i)^{b_{ik}} \prod_{j=1}^{n_x} (x_j + \delta x_j)^{a_{jk}},$$

i.e., a posynomial with (additive) normal variations (in $p_i$ and $x_j$). We will call (3) a nominal posynomial to distinguish it from its perturbed version (8).

In this paper, we also assume in (8) that the normal variations (4)--(5) are narrow, i.e.,

$$\sigma_{p_i} \ll p_i, \ i = 1, \ldots, n_p$$

$$\sigma_{x_j} \ll x_j, \ j = 1, \ldots, n_x.$$ \hfill (9) \hfill (10)

(Similar to (7), in general we can easily verify if the assumption (10) holds.)

3.2. Formulation

Now we introduce the formulation of the chance-constrained programming for robust MEMS design. First we suppose that the underlying nominal design optimization problem (i.e., the case with $\delta p_i = 0$ and $\delta x_j = 0$, for all $i, j$) can be cast as a (nominal) posynomial program:

$$\min f_0(x; p)$$

subject to

$$f_s(x; p) \leq 1, \ s = 1, \ldots, m,$$

where $x \in \mathbb{R}^{n_x}$ are the optimization variables. When the uncertainties are taken into account, we consider the following chance-constrained posynomial program instead:

$$\min \mathbb{E} \hat{f}_0(x; p)$$

subject to

$$\text{Prob} \left( \hat{f}_s(x; p) > 1 \right) \leq q_s, \ s = 1, \ldots, m,$$

where $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$ have the form of (8) and satisfy the narrow variation assumptions (9)--(10), and $1 > q_s > 0$ is given to bound the probability that the $s$th constrain is violated (due to the uncertainties $\delta p_i$ and $\delta x_j$). (Typically $q_s$ are chosen to be less than 0.1, e.g., $q_s = 0.05$, for high yields.) In addition, given a feasible solution of (12), say $\hat{x}$, we call

$$1 - \text{Prob} \left( \hat{f}_s(\hat{x}; p) > 1 \right) = \text{Prob} \left( \hat{f}_s(\hat{x}; p) \leq 1 \right)$$

the parametric yield (of the solution $\hat{x}$) for the $s$th constraint.
3.3. A relaxation

Consider the following (stochastic) optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \mathbb{E}\hat{f}_0(x;p) \\
\text{subject to} & \quad \mathbb{E}\hat{f}_s(x;p) \leq 1 - t_s\sqrt{\beta_s}, \quad s = 1, \ldots, m
\end{align*}
\]

(13)

where \(x \in \mathbb{R}^{n_x}\) are the optimization variables, \(\beta_s > 0\) is a given upper bound for the variance of \(\hat{f}_s(x;p)\), and \(t_s > 0\) satisfies \(1 - t_s\sqrt{\beta_s} > 0, \quad s = 1, \ldots, m\).

We observe that the statistics of (8) tends to be distributed unimodally. Therefore, by the Vysochanskiï-Petunin Inequality [3, 4], the above stochastic optimization problem with values of \(t_s\) given by

\[
t_s = \frac{2}{3\sqrt{q_s}}, \quad s = 1, \ldots, m
\]

(14)

is a relaxation of the chance-constrained posynomial program (12), provided that the specifications \(q_s\) given in (12) satisfy

\[
q_s \leq \frac{1}{6} \approx 0.167, \quad s = 1, \ldots, m.
\]

(15)

To be more specific, if specifications \(q_s\) in (12) satisfy (15) and \(\hat{x}^\star\) is an optimal solution of (13) with values of \(t_s\) assigned as in (14), then \(\hat{x}^\star\) is feasible to (12).

3.4. Approximation as nominal posynomial programs

Under the assumptions of narrow normal variations (9) and (10), the first moment of the posynomial (8) with normal variations (4)–(5) can be approximated as a nominal posynomial in optimization variables \(x \in \mathbb{R}^{n_x}\):

\[
\mathbb{E}\hat{f}(x;p) \simeq \sum_{k=1}^{K} \left( d_k \left( \prod_{i=1}^{n_p} p_i^{b_{ik}} \prod_{j=1}^{n_x} x_j^{a_{jk}} \right) \left( 1 + \sum_{i=1}^{n_p} \frac{b_{ik}^2}{2} \left( \frac{\sigma_{p_i}}{p_i} \right)^2 + \sum_{j=1}^{n_x} \frac{a_{jk}^2}{2} \left( \frac{\sigma_{x_j}}{x_j} \right)^2 \right) \right).
\]

(16)

We would like to emphasize that, although (16) looks complicated, it is a nominal posynomial in \(x \in \mathbb{R}^{n_x}\). Similarly, it can be shown that

\[
\text{Var}\hat{f}(x;p) \lesssim \sum_{k=1}^{K} d_k^2 \left( \prod_{i=1}^{n_p} p_i^{2b_{ik}} \prod_{j=1}^{n_x} x_j^{2a_{jk}} \right) \left( \sum_{i=1}^{n_p} \frac{b_{ik}^2}{2} \left( \frac{\sigma_{p_i}}{p_i} \right)^2 + \sum_{j=1}^{n_x} \frac{a_{jk}^2}{2} \left( \frac{\sigma_{x_j}}{x_j} \right)^2 \right) + \sum_{1 \leq k \leq l \leq K} \left( d_k d_l \prod_{i=1}^{n_p} p_i^{b_{ik}+b_{il}} \prod_{j=1}^{n_x} x_j^{a_{jk}+a_{jl}} \right) \left( \sum_{b_{ik}b_{il} > 0} \frac{b_{ik}b_{il}}{2} \left( \frac{\sigma_{p_i}}{p_i} \right)^2 + \sum_{a_{jk}a_{jl} > 0} \frac{a_{jk}a_{jl}}{2} \left( \frac{\sigma_{x_j}}{x_j} \right)^2 \right),
\]

(17)

i.e., the variance of \(\hat{f}(x;p)\) can be upper bounded (approximately) by a (complex) nominal posynomial in \(x \in \mathbb{R}^n\).

We have shown that, with narrow normal variations (9) and (10), the expectation of the posynomial (8) can be approximated as a nominal posynomial (16). Similarly, the variance of (8) can be upper bounded (approximately) by a nominal posynomial (17). Therefore, the stochastic optimization problem (13) can be readily approximated as a nominal posynomial program (in the form of (11)) simply by replacing the objective function and the constraint functions with their corresponding posynomial approximations (in the form of (16) and (17) respectively). Then the resulting nominal posynomial program, if feasible, can be solved to obtain an approximation solution for (13).
4. **Robust design example: a crab-leg resonator**

In this section we discuss a six variable crab-leg resonator. This example, taken from [5, 6], is sufficiently complex that numerical optimization is necessary to find a robust design.

The crab-leg resonator is shown in Figure 1. The structure exhibits four-fold symmetry and thus there are six design variables of interest: \( h_1, h_2, L_1 \) and \( L_2 \) for the four legs, and \( h_m \) and \( b_m \) for the height and width of the proof mass, respectively. The legs that support the proof mass are anchored to the substrate. We assume that this resonator will be fabricated using a surface micro-machining technology and therefore the entire structure will have a fixed thickness \( d \). We also neglect the mass of the four legs and therefore the mass of the proof mass is expressed as \( M = \rho h_m b_m d \), where \( \rho \) is the density.

\[
\omega_n = \sqrt{k_x/M} = \sqrt{\frac{16Eh_1^3 L_1^3}{\rho b_m L_3^1 (4h_1^2 L_2 + h_2^2 L_1)}}.
\]

(Our goal is to design the resonant frequency in the \( x \)-direction, and thus \( k_x \) is the stiffness of interest.)

To satisfy design rules for MEMS processes such as MUMPs, we first impose constraints that specify minimum line and space requirements:

\[
\begin{align*}
    h_1 & \geq h_{\text{min}}, & h_2 & \geq h_{\text{min}}, & b_m & \geq h_{\text{min}}, & h_m & \geq 2h_2 + h_{\text{min}}. \\

\end{align*}
\]

(18)

To ensure that the legs can be modelled as thin-beams, we also impose

\[
\begin{align*}
    L_1 & \geq 10h_1, & L_2 & \geq 10h_2.

\end{align*}
\]

(19)
The next two are requirements imposed by the designer on the overall size of the structure:

\[ X_{\text{max}} \geq 2L_2 + 2h_1 + b_m, \quad Y_{\text{max}} \geq 2L_1 + h_m. \]  

(20)

The next constraint is in place to separate the two resonant frequencies:

\[ k_y \geq 16k_x. \]  

(21)

(We would like the parasitical \( y \)-direction resonant frequency to be much greater than that in the \( x \)-direction to reduce motion in the \( y \)-direction.) The last constraint is to enforce that the stress, \( \beta \), at the joint (where the two beams join to form a leg) does not exceed a maximum, \( \beta \):

\[ \beta_{\text{max}} \geq \beta. \]  

(22)

Here the nonlinear expression for \( \beta \) is given by [6]:

\[
\beta = \frac{12Eh_1^3h_2^3L_1D}{h_2^2L_1^2(4h_1^3L_2 + h_2^3L_1)},
\]

where \( D \) is the maximum allowable deflection of the structure in the \( x \)-direction.

In the robust design, we consider sixth variance-not-linked-to-mean independent narrow normal variations in design variables: \( \delta L_1, \delta L_2, \delta h_1, \delta h_2, \delta h_m, \) and \( \delta b_m \). (To simplify the formulation, we do not consider the variations in process parameters.) Since we are considering geometric process variations, we will assume that the standard deviation of each variation is equal. We let \( \sigma_{L_1} = \sigma_{L_2} = \sigma_{h_1} = \sigma_{h_2} = \sigma_{h_m} = \sigma_{b_m} = 0.1 \mu m \).

Let \( \omega_{\text{target}} \) represent the target resonant frequency. The design cost to be minimized is the expectation of the squared error between \( \omega^2_n \) and \( \omega^2_{\text{target}} \), i.e., \( E(\omega^2_n - \omega^2_{\text{target}})^2 \). Then by introducing suitable auxiliary variables, the proposed robust design can be relaxed as a chance-constrained posynomial program in the form of (12), in which we let \( q_s = 0.05 \) for each constraint.

As discussed in §3.3, the resulting chance-constrained posynomial program can be further relaxed as a stochastic optimization problem in the form of (13), in which, by (14), \( t_s = 3 \) for all \( s \). We then solve this stochastic optimization problem by solving its (nominal) posynomial programming approximation, as discussed in §3.4, to obtain a robust design for this crab-leg resonator. A Monte Carlo analysis with 10,000 sample points is used to evaluate the performance variability of the robust design by computing the parametric yield of each constraint. We found that, in accordance with \( q_s = 0.05 \) for all \( s \), in our robust design each constraint has parametric yield no less than 95%. Moreover, most of the constraints are essentially active. This would imply that this robust design is very suboptimal (in the sense of minimizing the design cost).

5. Conclusions

In this paper we formulate the robust MEMS design under uncertainty problem as a chance-constrained posynomial program, which can be relaxed as a stochastic optimization that can be further approximated as a nominal posynomial program and hence solved efficiently. The method is illustrated by considering the numerical example of a simple crab-leg resonator, which shows that automated robust design can be obtained, with very suboptimal design cost as well as guaranteed parametric yield for each specification.

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