ASYMPTOTICS FOR SOME COMBINATORIAL CHARACTERISTICS OF THE CONVEX HULL OF POISSON POINT PROCESS ON THE CLIFFORD TORUS WITH LARGE VALUE OF RATE

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Abstract
Let $P_\lambda$ be a Poisson point process of rate $\lambda$ within the Clifford torus $T^2 \subset \mathbb{E}^d$. Let $(f_0, f_1, f_2, f_3)$ be the $f$-vector for the convex hull of $P_\lambda$ and let $\bar{v}$ be the mean valence of a vertex of the convex hull. Asymptotic expressions for $Ef_1$, $Ef_2$, $Ef_3$ and $E\bar{v}$ as $\lambda \to \infty$ obtained in this paper.

Introduction
Recently Poisson-Voronoi tessellations became an object for extensive investigations. The first non-trivial result concerning Poisson-Voronoi tessellations belongs to J. L. Meijering (see [1]). He proved that the mean value for the number of facets of a Voronoi cell tends to a constant that equals approximately $15.53 \ldots$ as the rate of Poisson process $\mathcal{P}$ within $\mathbb{E}^3$ tends to infinity. Later R. E. Miles, J. Møller, L. Muche and others (see, for example, the survey in [2]) obtained further results concerning Poisson-Voronoi tessellations.

The other important notion concerning discrete point sets is the notion of Delaunay triangulation. Let $A$ be a finite set of points in general position. It is known that the following statements are equivalent.

1. A subset $B \subset A$ spans a face of Delaunay triangulation.
2. A set of points equidistant to all points of $B$ contains a face of Voronoi tessellation.

Therefore the notions of Voronoi tessellation and Delaunay triangulation are dual to each other.

Consider a Delaunay triangulation of the sphere

$$S^d = \{ (\xi_1, \xi_2, \ldots, \xi_{d+1}) \subset \mathbb{E}^{d+1} : \xi_1^2 + \xi_2^2 + \ldots + \xi_{d+1}^2 = r^2 \}$$

for a finite set $A \subset S^d$ of points in general position in $S^d$. A subset $B \subset A$ determines a face of the triangulation if and only if $\text{conv } B$ is a face of $\text{conv } A$.

N. Dolbilin and M. Tanemura ([3], [4]) studied the convex hulls of finite subsets of the Clifford torus $T^2$ in $\mathbb{E}^4$. They have completely described the combinatorial structure of the convex hulls for periodic point sets defined in [4]. Moreover, there was performed a numeric simulation of the convex hull for the Poisson point process within $T^2$. It has shown that the mean value for the mean valence of a vertex of the convex hull has asymptotics $O^*(\ln \lambda)$, where $\lambda$ is the rate of the process. Here and further we write $F_1 = O^*(F_2)$ if

$$\limsup_{\lambda \to \infty} \max_{\lambda \to \infty} \left( \left| \frac{F_1}{F_2} \right|, \left| \frac{F_2}{F_1} \right| \right) < \infty.$$

N. Dolbilin suggested the author to prove the conjecture on the logarithmic growth of the mean valence of a vertex.
In this paper we prove this conjecture and some related theorems.

**Notation and main results**

Consider the four-dimensional Euclidean space $\mathbb{E}^4$. Let

$$T^2 = \{(\cos \phi, \sin \phi, \cos \psi, \sin \psi) : -\pi < \phi, \psi \leq \pi\}$$

be the two-dimensional Clifford torus embedded into the three-dimensional sphere

$$S^3 = \{ (\xi_1, \xi_2, \xi_3, \xi_4) : \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 2 \}$$

of radius $\sqrt{2}$.

$T^2$ has a locally Euclidean planar metric and, consequently, the Lebesgue measure $\text{mes}_2$, where $\text{mes}_2(T^2) = 4\pi^2$.

We will deal with the Poisson point processes $\mathcal{P}$ within $T^2$. For every spatial region $A \subset T^2$ denote by $n(A)$ a stochastic variable equal to the number of points occurred in $A$.

Denote by $\text{Pois}(\nu)$ the Poisson distribution with rate parameter $\nu$, i.e. the distribution of a stochastic variable $\zeta_\nu$ such that

$$P(\zeta_\nu = j) = e^{-\nu} \frac{\nu^j}{j!} \quad \text{for } j = 0, 1, 2, \ldots$$

Say that $\mathcal{P}_\lambda$ is the Poisson point process of rate $\lambda > 0$ if the stochastic variable $n(A)$ is distributed as $\text{Pois}(\lambda \text{mes}_2(A))$ for every spatial region $A \subset T^2$.

The convex hull of $\mathcal{P}_\lambda$ is almost surely a simplicial polytope.

The random event $n(T^2) > 4$ is called non-degenerate case. In this case the convex hull is almost surely a four-dimensional polytope. Denote by $f_i$ the number of $i$-faces of the convex hull of $\mathcal{P}_\lambda$ for $i = 0, 1, 2, 3$.

If $n(T^2) \leq 4$ occurred, the case is called degenerate. In these cases the $f$-vector $(f_0, f_1, f_2, f_3)$ will be also defined. More precisely, for the 3-dimensional simplex (4 points) the $f$-vector will be $(4, 6, 4, 2)$, for the 2-dimensional simplex $- (3, 3, 1, 0)$, for the segment $- (2, 1, 0, 0)$, for one point $- (1, 0, 0, 0)$, and for empty set of points $- (0, 0, 0, 0)$.

Now express the main results of this paper and describe the instruments for proofs.

The main results of this paper are as follows.

**Theorem 1.** The mean number of hyperfaces of convex hull of $\mathcal{P}_\lambda$ is $O^*(\lambda \ln \lambda)$ as $\lambda$ tends to infinity.

**Theorem 2.** The mean number of 1-faces and 2-faces of convex hull of $\mathcal{P}_\lambda$ are both $O^*(\lambda \ln \lambda)$ as $\lambda$ tends to infinity.

**Remark.** We have completely described up to the magnitude the asymptotics of the mean value for $f$-vector of the convex hull of $\mathcal{P}_\lambda$.

The other combinatorial characterization of a polytope is main valence of its vertex. More precisely, consider the stochastic variable

$$\bar{v} = \begin{cases} \frac{2f_3}{f_0}, & \text{if } f_0 \neq 0, \\ 0, & \text{if } f_0 = 0. \end{cases}$$
Then the \( \bar{v} \) is called the **mean valence of vertex** of the convex hull of the Poisson point process.

**Theorem 3.** The mean value of the mean valence of a vertex of convex hull for \( \mathcal{P}_\lambda \) has asymptotics \( E\bar{v} = O^*(\ln \lambda) \) as \( \lambda \) tends to infinity.

**Remark.** Theorem 3 provides an answer to the problem proposed by Dolbilin and Tanemura.

**Construction of a covering of \( X \)**

Let \((T^2)^4\) be the fourth Cartesian power of \( T^2 \) with natural measure \( \text{mes}_8 \). Let \( X \subset (T^2)^4 \) be the set of all points \( x = (x_1, x_2, x_3, x_4) \), where \( x_i \in T^2 \) such that points \( x_1, x_2, x_3, x_4 \) are affinely independent in \( \mathbb{E}^4 \).

For every \( x \in X \) denote by \( p(x) \) a hyperplane spanned by points \( x_1, x_2, x_3, x_4 \). It is obvious that \( X \) is open in \((T^2)^4\). Moreover, it is easily seen that \((T^2)^4 \setminus X\) has a zero measure.

Denote by \( \Pi^+(x) \) and \( \Pi^-(x) \) the two half-spaces determined by \( p(x) \) for every \( x \in X \).

The sets \( C^+(x) = T^2 \cap \Pi^+(x) \) and \( C^-(x) = T^2 \cap \Pi^-(x) \) are called **caps**.

Without loss of generality, assume that for every \( x \in X \)

\[
\text{mes}_2(C^+(x)) \leq \text{mes}_2(C^-(x)).
\]

Let \( G : X \to \mathbb{R} \) be a function determined by

\[
G(x) = \text{mes}_2(C^+(x)).
\]

Clearly, \( G(x) \) is continuous on \( X \).

We will obtain the integral expression for \( Ef_3 \) in the following way. First of all, we construct partitions of \((T^2)^4\). Then compare integral sums for these partitions with estimates for \( Ef_3 \).

Let \( x = (x_1, x_2, x_3, x_4) \in (T^2)^4 \) and \( x_i = (\cos \phi_i, \sin \phi_i, \cos \psi_i, \sin \psi_i) \) for \( i = 1, 2, 3, 4 \).

A set \( Y = Y(n, l_i, m_i) \subset (T^2)^4 \) is called **binary** if there exist non-negative integer \( n \) and \( l_i, m_i \in \mathbb{Z} \cap [0, 2^n - 1] \) such that \( Y \) is a set of all points \( x \in (T^2)^4 \) satisfying the inequalities

\[
\frac{2\pi l_i}{2^n} \leq \phi_i \leq \frac{2\pi (l_i + 1)}{2^n},
\]

\[
\frac{2\pi m_i}{2^n} \leq \psi_i \leq \frac{2\pi (m_i + 1)}{2^n},
\]

where \( i = 1, 2, 3, 4 \).

Let small enough \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) be fixed.

For every \( x \in X \) take a binary set \( Y(x) = Y(n(x), l_i(x), m_i(x)) \subset X \) satisfying the following conditions:
1. \( x \in Y(x) \).

2. \( l_i - l_j \neq 0, \pm 1 \) (mod \( 2^n \)) or \( m_i - m_j \neq 0, \pm 1 \) (mod \( 2^n \)) for each pair of \( i, j \in \{1, 2, 3, 4\}, i \neq j \).

3. \( \frac{2\pi}{2^n} < \varepsilon_1 \).

4. \( \left(\frac{2\pi}{2^n}\right)^2 < \varepsilon_2 G(x) \).

For every \( x' \in Y(x) \) consider the ordered pair of caps determined by \( p(x') \).
Define the order of caps in each pair such that the ordered pair depends continuously on \( x' \). Suppose the first cap determined by \( p(x) \) is \( C^+(x) \).

The first term of the ordered pair will be called smaller cap and denoted by \( C^i_Y(x') \). The second will be called larger cap and denoted by \( C^l_Y(x') \).

Define
\[
U^*_Y = \bigcup_{x' \in Y(x)} C^*_Y(x'), \quad U^l_Y = \bigcup_{x' \in Y(x)} C^l_Y(x'),
\]
\[
I^*_Y = \bigcap_{x' \in Y(x)} C^*_Y(x'), \quad I^l_Y = \bigcap_{x' \in Y(x)} C^l_Y(x').
\]

5. 
\[
1 - \varepsilon_3 \leq \frac{\text{mes}_2(U^*_Y)}{\text{mes}_2(I^*_Y)} \leq 1 + \varepsilon_3 \quad \text{and} \quad 1 - \varepsilon_3 \leq \frac{\text{mes}_2(U^l_Y)}{\text{mes}_2(I^l_Y)} \leq 1 + \varepsilon_3.
\]

Conditions 1 – 5 are called cover conditions.

**Lemma 1.1** A binary set \( Y(x) \) satisfying the cover conditions exists for every \( x \in X \). Since \( x = (x_1, x_2, x_3, x_4) \) and \( x_1, x_2, x_3, x_4 \) are distinct points of \( T^2 \), then there exist squares \( Y_{0,i}(x) \subset T^2 \) such that
\[
Y_0(x) = Y_{0,1}(x) \times Y_{0,2}(x) \times Y_{0,3}(x) \times Y_{0,4}(x)
\]
is a binary set,
\[
Y_0(x) \subset X, \quad x_i \in Y_{0,i} \quad \text{and} \quad Y_i \cap Y_j = \emptyset
\]
for \( i, j = 1, 2, 3, 4, i \neq j \).

Every binary subset of \( Y_0(x) \) satisfies cover condition 2.

\( C^o_Y(x') \) and \( C^l_Y(x') \) can be defined as above for all \( x' \in Y(x) \).

Since \( C^o_Y(x') \) and \( C^l_Y(x') \) depend continuously on \( x' \), then there exists a binary set \( Y_1(x) \subset Y_0(x) \) satisfying cover conditions 1 and 5. Every binary subset of \( Y_1(x) \) satisfies cover condition 5. Moreover, \( \text{mes}_2(I^l_{Y_1}) > 0 \).

Now it is not hard to choose a binary set \( Y(x) \subset Y_1(x) \) satisfying cover conditions 1, 3 and condition
\[
\left(\frac{2\pi}{2^n}\right)^2 < \varepsilon_2 \text{mes}_2(I^l_{Y_1}),
\]
which is stronger than cover condition 4. Since
\[ Y(x) \subset Y_1(x) \subset Y_0(x), \]
cover conditions 2 and 5 are also satisfied.
Consider a family of binary sets
\[ \{Y(x) : x \in X\}. \]
This family is a covering of \( X \) by binary sets.
Every two distinct binary sets \( Y(x), Y(y) \) of this covering sets holds one of the following conditions:
1. \( Y(x) \subset Y(y) \).
2. \( Y(y) \subset Y(x) \).
3. \( \text{int} Y(x) \cap \text{int} Y(y) = \emptyset \).

If \( Y(x) \subset Y(y) \) then exclude \( Y(x) \) from the covering. After all possible exclusions a subfamily \( S = S(\epsilon_1, \epsilon_2, \epsilon_3) \) was obtained. \( S \) is a subcovering because for every \( x \in X \) the biggest binary set including \( x \) from the covering was not excluded.
Obviously, \( \text{int} Y(x) \cap \text{int} Y(y) = \emptyset \) for every two sets \( Y(x), Y(y) \in S \). Consequently, \( S \) is not more than countable.

**Integral expression for** \( E f_3 \)

Take an arbitrary set \( Y(x) \) from \( S \). Denote by \( A_Y \) a stochastic variable equal to the number of \( y = (y_1, y_2, y_3, y_4) \) such that \( y \in Y(x) \) and \( y_1, y_2, y_3, y_4 \) are the vertices of a facet of the convex hull for \( P_\lambda \). Since the vertices of a facet can be ordered in 24 different ways we have
\[
24E f_3 = \sum_{Y(x) \in S} E A_{Y(x)}. \tag{1}
\]

Further, if we deal with one set \( Y(x) \) of \( S \), it will denoted by simply \( Y \).
By the definition of a binary set \( Y \) is a direct product of four non-intersecting closed squares \( Y_1, Y_2, Y_3, Y_4 \subset T^2 \).
Let \( y = (y_1, y_2, y_3, y_4) \) such that \( y_1, y_2, y_3, y_4 \) determine a facet of the convex hull of \( P_\lambda \). Suppose \( C(y) \) is the cap corresponding to \( y \) such that \( C(y) \) is separated from the convex hull of \( P_\lambda \) by \( p(y) \).
The cap \( C(y) \) is called the cap **corresponding to a facet**.

**Proposition 1.** Suppose that for \( i = 1, 2, 3, 4 \) there is exactly one point \( y_i \) of \( P_\lambda \) such that \( y_i \in Y_i \). Assume there are no points of \( P_\lambda \) in \( U^i_\lambda \setminus (\bigcup_{i=1}^4 Y_i) \) (respectively, \( U^i_\lambda \setminus (\bigcup_{i=1}^4 Y_i) \)). If \( y = (y_1, y_2, y_3, y_4) \) then the cap \( C^i_Y(y) \) (respectively, \( C^i_Y(y) \)) has no interior points. Therefore points \( y_1, y_2, y_3, y_4 \) determine a facet for the convex hull of \( P_\lambda \) **corresponding to** \( C^i_Y(y) \) (respectively, \( C^i_Y(y) \)).
Proposition 2. Suppose $I_Y \setminus (\bigcup_{i=1}^{4} Y_i)$ (respectively, $I_Y \setminus (\bigcup_{i=1}^{4} Y_i)$) is non-empty. Then there is no $y \in Y$ such that the cap $C_Y^* (y)$ (respectively, $C_Y^i (y)$) corresponds to a facet of the convex hull of $P_{\lambda}$.

Proposition 3. Suppose $I_Y \setminus (\bigcup_{i=1}^{4} Y_i)$ (respectively, $I_Y \setminus (\bigcup_{i=1}^{4} Y_i)$) is empty and $n(Y_i) = k_i$. Then there exist at most $k_1 k_2 k_3 k_4$ points $y \in Y$ such that caps $C_Y^* (y)$ (respectively, $C_Y^i (y)$) correspond to facets of the convex hull of $P_{\lambda}$.

The proofs of these propositions are trivial and therefore omitted.

Denote by $A_Y^*$ and, respectively, $A_Y^i$ the number of caps $C_Y^* (y)$ (respectively, $C_Y^i (y)$) with $y \in Y$ corresponding to a facet of the convex hull of $P_{\lambda}$. Obviously, $A_Y = A_Y^* + A_Y^i$.

Lemma 2.1.

$$
E A_Y^* \geq mes_{S}(Y) \cdot \lambda^4 \cdot e^{-4\epsilon_2^2} \cdot e^{-\lambda(1+\epsilon_2)G(x)},
$$

$$
E A_Y^i \geq mes_{S}(Y) \cdot \lambda^4 \cdot e^{-4\epsilon_2^2} \cdot e^{-\lambda(1+\epsilon_2)(4\pi^2-G(x))}.
$$

Proof. Prove, for example, the first estimate, as the proof of the second is similar. Since $P_{\lambda}$ is a Poisson point process and $Y$ and $U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i)$ are pairwise non-intersecting sets, the five random events

$n(Y_1) = 1$, $n(Y_2) = 1$, $n(Y_3) = 1$, $n(Y_4) = 1$,

$n((U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i)) = 0$

are independent.

Therefore

$$
P\left( n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n((U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right) =
$$

$$
= \prod_{i=1}^{4} P(n(Y_i) = 1) \cdot P\left( n((U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right).
$$

Since $n(Y_i)$ is distributed as $\text{Pois}(\lambda \cdot \text{mes}_{2}(Y_i))$,

$$
P(n(Y_i) = 1) = \lambda \cdot \text{mes}_{2}(Y_i) \cdot e^{-\lambda \text{mes}_{2}(Y_i)}.
$$

By cover condition 3, $\text{mes}_{2}(Y_i) \leq \epsilon_i^2$. Hence

$$
P(n(Y_i) = 1) \geq \lambda \cdot \text{mes}_{2}(Y_i) \cdot e^{-\lambda \epsilon_i^2}.
$$

Since stochastic variable $n((U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i))$ is distributed as

$$
\text{Pois} \left( \lambda \cdot \text{mes}_{2}(U_Y^* \setminus (\bigcup_{i=1}^{4} Y_i)) \right),
$$

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then
\[ P \left( n(U_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0 \right) = e^{-\lambda \cdot \text{mes}_2 \left( U_Y^s \setminus \bigcup_{i=1}^{4} Y_i \right)} . \]

Obviously,
\[ \text{mes}_2 \left( U_Y^s \setminus \bigcup_{i=1}^{4} Y_i \right) \leq \text{mes}_2 (U_Y) . \]

By cover condition 5,
\[ \text{mes}_2 (U_Y) \leq (1 + \varepsilon_3) \text{mes}_2 (I_Y) \leq (1 + \varepsilon_3) G(x) . \]

Therefore
\[ P \left( n(U_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0 \right) \geq e^{-\lambda(1+\varepsilon_3) G(x)} . \]

Finally, since
\[ \text{mes}_8 (Y) = \prod_{i=1}^{4} \text{mes}_2 (Y_i) , \]
then
\[
P \left( n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n(U_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0 \right) \geq \text{mes}_8 (Y) \cdot \lambda^4 \cdot e^{-4\lambda\varepsilon_3^2} \cdot e^{-\lambda(1+\varepsilon_3) G(x)} .
\]

If random events
\[ n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n(U_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0 \]
occur simultaneously, then, by Proposition 1, there exists exactly one \( y \in Y \) corresponding to a hyperface. Hence
\[ A_Y^s \geq P \left( n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n(U_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0 \right) , \]
and the first estimate is proved.

Lemma 2.2.
\[
E A_Y^s \leq \lambda^4 \cdot \text{mes}_8 (Y) \cdot e^{-\lambda(1-\varepsilon_3-4\varepsilon_2) G(x)} , \\
E A_Y^l \leq \lambda^4 \cdot \text{mes}_8 (Y) \cdot e^{-\lambda(1-\varepsilon_3-4\varepsilon_2)(4\pi^2- G(x))} .
\] (3)

Proof. As above, prove the first estimate. The proof of the second estimate is similar.
By Proposition 2, the conditional expectation

\[ E\left( A_s \mid n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) \neq 0 \right) = 0. \]

Proposition 3 implies the following estimates for conditional expectations:

\[ E\left( A_s \mid n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0, n(Y_i) = k_i \text{ for } i = 1, 2, 3, 4 \right) \leq k_1k_2k_3k_4. \]

By the law of total probability,

\[ EA_s \leq \sum_{k_i=1}^{\infty} k_1k_2k_3k_4 \cdot P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0, n(Y_i) = k_i \text{ for } i = 1, 2, 3, 4 \right). \]

Since the stochastic variables \( n(Y_1), n(Y_2), n(Y_3), n(Y_4), n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) \) are independent,

\[ P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0, n(Y_i) = k_i \text{ for } i = 1, 2, 3, 4 \right) = \prod_{i=1}^{4} P(n(Y_i) = k_i) \cdot P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right). \]

Consequently,

\[ \sum_{k_i=1}^{\infty} k_1k_2k_3k_4 \cdot P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0, n(Y_i) = k_i \text{ for } i = 1, 2, 3, 4 \right) = P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right) \cdot \prod_{i=1}^{4} \sum_{k_i=1}^{\infty} k_i P(n(Y_i) = k_i) = P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right) \cdot \prod_{i=1}^{4} E(n(Y_i)) = P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right) \cdot \prod_{i=1}^{4} (\lambda \cdot mes_2(Y_i)) = P\left( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) = 0 \right) \cdot \lambda^4 \cdot mes_8(Y). \]

Since \( n(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) \) is distributed as

\[ Pois\left( \lambda \cdot mes_2(I_Y^s \setminus (\bigcup_{i=1}^{4} Y_i)) \right), \]
then
\[ P\left(n\left(I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right)\right) = 0\right) = e^{-\lambda \cdot \text{mes}_2\left(I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right)\right)} . \]

Clearly, by covering conditions 4 and 5,
\[ \text{mes}_2\left(I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right)\right) \geq \text{mes}_2(Y_i) - \sum_{i=1}^{4} \text{mes}_2(Y_i) \geq (1 - \varepsilon_3 - 4\varepsilon_2)G(x) , \]
and, finally,
\[ EA \leq \lambda \cdot \text{mes}_8(Y) \cdot e^{-\lambda(1 - \varepsilon_3 - 4\varepsilon_2)G(x)} , \]
which is the first estimate of Lemma 2.2.

**Lemma 2.3.**
\[ Ef_3 = \frac{1}{24} \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) \, dx . \]  

**Proof.** Using \( A_Y = A_1^* + A_1^* \), substitute estimates (2) and (3) into (1). The substitution gives
\[ \sum_{Y(x) \in S} \text{mes}_8(Y(x)) \cdot \lambda^4 \cdot e^{-4\lambda\varepsilon_1^2} \cdot \left( e^{-\lambda(1 + \varepsilon_3)G(x)} + e^{-\lambda(1 + \varepsilon_3)(4\pi^2 - G(x))} \right) \leq \]
\[ \leq 24Ef_3 \]  
(5)

\[ \sum_{Y(x) \in S} \text{mes}_8(Y(x)) \cdot \lambda^4 \cdot \left( e^{-\lambda(1 - \varepsilon_3 - 4\varepsilon_2)G(x)} + e^{-\lambda(1 - \varepsilon_3 - 4\varepsilon_2)(4\pi^2 - G(x))} \right) \geq \]
\[ \geq 24Ef_3 \]  
(6)

Let an arbitrary \( 0 < \delta < 1 \) be fixed. Choose small enough \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) to satisfy the following conditions:

1. 
\[ (1 - \delta) \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) \, dx \leq \]
\[ \leq \sum_{Y(x) \in S(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \text{mes}_8(Y(x)) \cdot \lambda^4 \cdot \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) \leq \]
\[ \leq (1 + \delta) \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) \, dx . \]
2. For every \( x \in X \)
\[
e^{-4\lambda\varepsilon_1^2} \cdot \left( e^{-\lambda(1+\varepsilon_3)G(x)} + e^{-\lambda(1+\varepsilon_3)(4\pi^2-G(x))} \right) \geq (1 - \delta) \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))} \right),
\]

3. For every \( x \in X \)
\[
\left( e^{-\lambda(1-\varepsilon_3-4\varepsilon_2)G(x)} + e^{-\lambda(1-\varepsilon_3-4\varepsilon_2)(4\pi^2-G(x))} \right) \leq (1 + \delta) \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))} \right).
\]

Prove that such \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) exist.
Let \( d(Y) \) be a diameter of a binary set \( Y \) in the natural metric of \( (T^2)^4 \). Let
\[
d(S) = \sup_{Y(x) \in S} d(Y(x))
\]
be a diameter of subcovering \( S \). The function
\[
\lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))} \right)
\]
is bounded and continuous on \( X \) and \( \text{mes}_S((T^2)^4 \setminus X) = 0 \). Therefore
\[
\lim_{d(S) \to 0} \sum_{Y(x) \in S} \text{mes}_S(Y(x)) \cdot \lambda^4 \cdot \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))} \right) =
\]
\[
= \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2-\lambda G(x))} \right) dx.
\]
By covering condition 3, \( d(S) \leq \sqrt{8}\varepsilon_1 \). Consequently, there exists \( \varepsilon_{1,1} > 0 \) such that for every choice of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) with \( \varepsilon_1 < \varepsilon_{1,1} \) condition 1 holds.
Choose some \( \varepsilon_{1,2} > 0 \) and \( \varepsilon_{3,2} > 0 \) such that
\[
e^{-4\lambda\varepsilon_{1,2}^2} \cdot e^{-4\lambda\varepsilon_{3,2}^2} > 1 - \delta.
\]
Since \( 0 < G(x) < 4\pi^2 \), then every choice of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) with \( \varepsilon_1 < \varepsilon_{1,2} \) and \( \varepsilon_3 < \varepsilon_{3,2} \) satisfies condition 2.
Finally, choose some \( \varepsilon_{2,3} > 0 \) and \( \varepsilon_{3,3} > 0 \) such that
\[
e^{4\lambda\varepsilon_{2,3}^2(4\varepsilon_{2,3}+\varepsilon_{3,3})} < 1 + \delta.
\]
Similarly, every choice of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) with \( \varepsilon_2 < \varepsilon_{2,3} \) and \( \varepsilon_3 < \varepsilon_{3,3} \) satisfies condition 3.
Hence every choice of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) satisfying

\[
0 < \varepsilon_1 < \min(\varepsilon_{1,1}, \varepsilon_{1,2}), \\
0 < \varepsilon_2 < \varepsilon_{2,3}, \\
0 < \varepsilon_3 < \min(\varepsilon_{3,2}, \varepsilon_{3,3})
\]
satisfies conditions 1 – 3.

From the condition 2 and (5) follows

\[
24E_f^3 \geq (1 - \delta) \sum_{Y(x) \in S} \text{mes}_S(Y(x)) \cdot \lambda^4 \cdot \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right).
\]

Applying condition 1 we obtain

\[
24E_f^3 \geq (1 - \delta)^2 \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) dx.
\]

Similarly, from (6) and conditions 1 and 3 we obtain

\[
24E_f^3 \leq (1 + \delta)^2 \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) dx.
\]

Since \( \delta \) is an arbitrary positive number, we have

\[
24E_f^3 = \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda(4\pi^2 - \lambda G(x))} \right) dx,
\]

and Lemma 2.3 is proved.

**Structure of caps**

**Lemma 3.1.** For every cap \( C^+(x) \) (respectively, \( C^-(x) \)) there exist \( a, b \geq 0, \phi_0, \psi_0 \) satisfying \( a^2 + b^2 \geq 2 \) and \( -\pi < \phi_0, \psi_0 \leq \pi \) such that

\[
C^+(x) = \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \leq 1 \right\},
\]

and, respectively,

\[
C^-(x) = \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \geq 1 \right\}.
\]

**Proof.** Suppose

\[
p(x) = \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : a_1\xi_1 + a_2\xi_2 + b_1\xi_3 + b_2\xi_4 = c \}
\]

where \( c \geq 0 \).
Then
\[ \partial C^+(x) = \partial C^-(x) = \{ (\phi, \psi) \in T^2 : a_1 \cos \phi + a_2 \sin \phi + b_1 \cos \psi + b_2 \sin \psi = c \}. \]

The equation for \( \partial C^+(x) \) could be rewritten as
\[ a' \cos(\phi - \phi_0) + b' \cos(\psi - \psi_0) = c, \]
where \( a' = \sqrt{a_1^2 + a_2^2} \) and \( b' = \sqrt{b_1^2 + b_2^2} \).

Since
\[ \cos(\phi - \phi_0) = 1 - 2 \sin^2 \frac{\phi - \phi_0}{2} \quad \text{and} \quad \cos(\psi - \psi_0) = 1 - 2 \sin^2 \frac{\psi - \psi_0}{2}, \]
the previous equation is equivalent to
\[ a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} = a' + b' - c. \]

The set \( \partial C^+(x) \) contains infinitely many points, therefore
\[ 0 \leq c < a' + b'. \]

If \( c = 0 \) then \( p(x) \) passes through the origin and therefore divides \( T^2 \) into equal parts. Consequently,
\[ \text{mes}_2 \left( \left\{ (\phi, \psi) \in T^2 : a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} \leq \frac{a' + b'}{2} \right\} \right) = \text{mes}_2 \left( \left\{ (\phi, \psi) \in T^2 : a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} \geq \frac{a' + b'}{2} \right\} \right). \]

Therefore for \( c > 0 \)
\[ \text{mes}_2 \left( \left\{ (\phi, \psi) \in T^2 : a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} \leq \frac{a' + b' - c}{2} \right\} \right) < \text{mes}_2 \left( \left\{ (\phi, \psi) \in T^2 : a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} \geq \frac{a' + b' - c}{2} \right\} \right). \]

Since \( \text{mes}_2(C^+(x)) \leq \text{mes}_2(C^-(x)) \),
\[ C^+(x) = \left\{ (\phi, \psi) \in T^2 : a' \sin^2 \frac{\phi - \phi_0}{2} + b' \sin^2 \frac{\psi - \psi_0}{2} \leq \frac{a' + b' - c}{2} \right\}. \]

Let
\[ a = \sqrt{\frac{2a'}{a' + b' - c}} \quad \text{and} \quad b = \sqrt{\frac{2b'}{a' + b' - c}}. \]

Then
\[ C^+(x) = \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \leq 1 \right\}. \]
Hence,

\[ C^-(x) = \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \geq 1 \right\}, \]

and Lemma 3.1 is proved.

**Remark 1.** \( a, b, \phi_0, \psi_0 \) can be now considered as functions \( a(x), b(x), \phi_0(x), \psi_0(x) \) of argument \( x \in X \).

**Remark 2.** It is not hard to see that for every \( a, b \geq 0 \), \( \phi_0, \psi_0 \) satisfying \( a^2 + b^2 \geq 2 \) and \( -\pi < \phi_0, \psi_0 \leq \pi \) the sets

\[ \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \leq 1 \right\} \]

and

\[ \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \geq 1 \right\} \]

are caps.

**Lemma 3.2.** Let

\[ C^+(x) = \left\{ (\phi, \psi) \in T^2 : a^2 \sin^2 \frac{\phi - \phi_0}{2} + b^2 \sin^2 \frac{\psi - \psi_0}{2} \leq 1 \right\}, \]

where \( a, b \geq 0 \), \( a^2 + b^2 \geq 2 \) and \( -\pi < \phi_0, \psi_0 \leq \pi \). Then

\[ \gamma_1 < (a + 1)(b + 1)G(x) < \gamma_2, \]

where \( \gamma_1, \gamma_2 > 0 \) and do not depend on \( x \).

**Proof.** Without loss of generality assume \( \phi_0 = \psi_0 = 0 \).

Consider the case \( a = 0 \). Then

\[ C^+(x) = \left\{ (\phi, \psi) \in T^2 : b^2 \sin^2 \frac{\phi}{2} \leq 1 \right\}, \]

or, equivalently,

\[ C^+(x) = \left\{ (\phi, \psi) \in T^2 : |\psi| \leq 2 \arcsin \frac{1}{b} \right\}. \]

Consequently,

\[ (a + 1)(b + 1)G(x) = 8\pi(b + 1) \arcsin \frac{1}{b}. \]

Since \( \frac{1}{b} < \arcsin \frac{1}{b} < \frac{\pi}{2} \), \( \frac{1}{b} \) and \( b \geq \sqrt{2} \),

\[ 8\pi < (a + 1)(b + 1)G(x) < 4\pi^2 \left( 1 + \frac{\sqrt{2}}{2} \right), \]

and the case \( a = 0 \) is completely proved. The case \( b = 0 \) is similar.
Now suppose \( a > 0 \) and \( b > 0 \). Since
\[
\frac{|\phi|}{\pi} \leq \left| \frac{\sin \frac{\phi}{2}}{\frac{\phi}{2}} \right| \leq \frac{|\phi|}{2} \quad \text{and} \quad \frac{|\psi|}{\pi} \leq \left| \frac{\sin \frac{\psi}{2}}{\frac{\psi}{2}} \right| \leq \frac{|\psi|}{2},
\]
then the following inclusions hold:
\[
C^+(x) \subset \left\{ (\phi, \psi) \in T^2 : a^2 \left( \frac{\phi}{2} \right)^2 \leq \frac{1}{2} \text{ and } b^2 \left( \frac{\psi}{2} \right)^2 \leq \frac{1}{2} \right\},
\]
\[
C^+(x) \supset \left\{ (\phi, \psi) \in T^2 : a^2 \left( \frac{\phi}{\pi} \right)^2 \leq 1 \text{ and } b^2 \left( \frac{\psi}{\pi} \right)^2 \leq 1 \right\}.
\]
Therefore
\[
\min \left( 2\pi, \frac{2\sqrt{2}}{a} \right) \cdot \min \left( 2\pi, \frac{2\sqrt{2}}{b} \right) \leq G(x) \leq \min \left( 2\pi, \frac{2\pi}{a} \right) \cdot \min \left( 2\pi, \frac{2\pi}{b} \right).
\]
We have
\[
\min \left( 2\pi, \frac{2\sqrt{2}}{a} \right) = \frac{1}{\max \left( \frac{1}{2\pi}, \frac{a}{2\sqrt{2}} \right)} \geq \frac{1}{\frac{1}{2\pi} + \frac{a}{2\sqrt{2}}} = \frac{2\pi\sqrt{2}}{1 + \frac{2\sqrt{2}}{2\pi}},
\]
\[
\min \left( 2\pi, \frac{2\pi}{a} \right) \leq \frac{2}{\frac{1}{2\pi} + \frac{a}{2\pi}} = \frac{4\pi}{a + 1}.
\]
and, similarly,
\[
\min \left( 2\pi, \frac{2\sqrt{2}}{b} \right) \geq \frac{2\sqrt{2}}{\pi a + \sqrt{2}},
\]
\[
\min \left( 2\pi, \frac{\pi\sqrt{2}}{b} \right) \leq \frac{4\pi}{b + 1}.
\]
Finally,
\[
8 \leq \frac{2\pi\sqrt{2}(a + 1)}{\pi a + \sqrt{2}} \cdot \frac{2\pi\sqrt{2}(b + 1)}{\pi b + \sqrt{2}} \leq (a + 1)(b + 1)G(x) \leq 16\pi^2,
\]
and Lemma 3.2 is proved completely.

**Estimates for the measure function**

For every \( t \in \mathbb{R} \) define
\[
M(t) = \text{mes}_8 \{ x \in X : G(x) < t \},
\]
\[
N(t) = \text{mes}_8 \{ x \in X : G(x) < t \text{ and } \min(a(x), b(x)) < 100 \},
\]
\[
L(t) = \text{mes}_8 \{ x \in X : G(x) < t \text{ and } \min(a(x), b(x)) \geq 100 \}.
\]
It is easily seen that \( M(t) = N(t) = L(t) = 0 \) for \( t < 0 \) and \( M(t) = N(t) + L(t) \) for every \( t \in \mathbb{R} \).

**Lemma 4.1.** There exists \( \gamma_3 > 0 \) such that

\[
N(t) < \gamma_3 t^3
\]

for every \( 0 < t < \frac{1}{2} \).

**Proof.** Introduce the functions

\[
N_1(t) = \text{mes}_8 \{ x \in X : G(x) < t \text{ and } a(x) < 100 \},
\]

\[
N_2(t) = \text{mes}_8 \{ x \in X : G(x) < t \text{ and } b(x) < 100 \}.
\]

Obviously, \( N_1(t) = N_2(t) \) and \( N(t) \leq N_1(t) + N_2(t) \).

Suppose \( 0 < t \leq \frac{\gamma_1}{1000\pi} \).

Let \( a(x) < 100 \) and \( G(x) < t \). From Lemma 3.2 follows

\[
b(x) \geq \frac{\gamma_1}{(a + 1)G(x)} - 1 > \frac{\gamma_1}{101t}.
\]

By Lemma 3.1 cap \( C^+ (x) \) is described by the inequality

\[
a(x)^2 \sin^2 \frac{\phi - \phi_0}{2} + b(x)^2 \sin^2 \frac{\psi - \psi_0}{2} \leq 1.
\]

From the last inequality follows

\[
\left| \sin \frac{\psi - \psi_0}{2} \right| \leq \frac{1}{b} < \frac{101t}{\gamma_1}.
\]

Hence

\[
C^+ (x) \subseteq \left\{ (\phi, \psi) \in T^2 : \left| \sin \frac{\psi - \psi_0}{2} \right| < \frac{101t}{\gamma_1} \right\} = S(t, x).
\]

A set \( S \subset T^2 \) is called a stripe if there exist \( \psi_1 \in (-\pi, \pi] \) and \( d \in (-1, 1) \) such that

\[
S = \{(\phi, \psi) \in T^2 : \cos(\psi - \psi_1) \leq d \}.
\]

The centerline of \( S \) is the line \( \psi = \psi_1 \), and 2 arccos \( d \) is the width of \( S \).

\( S(t, x) \) is obviously a stripe of width

\[
w(t) = 4 \arcsin \frac{101t}{\gamma_1} < \frac{1000\pi t}{\gamma_1}
\]

and the centerline of \( S(t, x) \) is described by the equation \( \psi = \psi_0 \).

Let

\[
k = k(t) = \left\lfloor \frac{2\pi}{w(t)} \right\rfloor.
\]
Consider \( k \) stripes \( S_1, S_2, \ldots, S_k \subset T^2 \) of width \( 2w(t) \) each such that \( S_j \) has centerline \( \psi = -\pi + \frac{2\pi}{k}j \).

It is obvious that \( S(t, x) \subset S_j \), where \( j \) is the nearest integer to \( \frac{k(\psi_0 + \pi)}{2\pi} \) and \( S_0 = S_k \).

Let \( x = (x_1, x_2, x_3, x_4) \), where \( x_i \in T^2 \). Obviously, every \( x_i \in \partial C^+(x) \), therefore \( x \in S_j^4 \).

Finally,

\[ N_1(t) = mes_8 \{ x \in X : G(x) < t \text{ and } a(x) < 100 \} \leq \]

\[ \leq mes_8 \left( \bigcup_{j=1}^{k} (t)S_j^4 \right) \leq k(t)(4\pi w(t))^4 \leq \]

\[ \leq (4\pi)^5 w(t)^3 \leq \gamma'_3 t^3. \]

Then

\[ N(t) \leq 2N_1(t) \leq 2\gamma'_3 t^3. \]

The case

\[ 0 < t \leq \frac{\gamma_1}{1000\pi} \]

is proved completely.

Suppose

\[ \frac{\gamma_1}{1000\pi} < t < \frac{1}{2}. \]

Obviously,

\[ N(t) \leq mes_8 ((T^2)^4) = 256\pi^8. \]

Then

\[ N(t) < 256\pi^8 \left( \frac{1000\pi}{\gamma_1} \right)^3 t^3, \]

and Lemma 4.1 is now proved completely.

**Lemma 4.2.** There exist \( \gamma_4, \gamma_5 > 0 \) such that

\[ \gamma_4 t^3 |\ln t| < L(t) < \gamma_5 t^3 |\ln t| \]

for every \( 0 < t < \frac{1}{2} \).

**Proof.** Suppose \( \min(a(x), b(x)) \geq 100 \). Assume

\[ x = (x_1, x_2, x_3, x_4), \text{ where } x_i = (\phi_i, \psi_i) \in T^2 \text{ for } i = 1, 2, 3, 4. \]

Let

\[ \alpha(x) = \frac{1}{a(x)}, \quad \beta(x) = \frac{1}{b(x)}. \]

By assumptions, \( 0 < \alpha(x), \beta(x) < \frac{1}{100} \).

Lemma 3.2 easily implies that there exist \( \gamma'_1, \gamma'_2 > 0 \) such that

\[ \gamma'_1 \alpha(x) \beta(x) < G(x) < \gamma'_2 \alpha(x) \beta(x). \]  

(7)
Since $x_i \in \partial C^+(x)$ for $i = 1, 2, 3, 4$, then
\[
\frac{\sin^2 \frac{\phi_i - \phi_0}{2}}{\alpha^2} + \frac{\sin^2 \frac{\psi_i - \psi_0}{2}}{\beta^2} = 1.
\]

Therefore there exist define parameters $-\pi < \theta_i \leq \pi$ for $i = 1, 2, 3, 4$ such that
\[
\sin \frac{\phi_i - \phi_0}{2} = \alpha \cos \theta_i \quad \text{and} \quad \sin \frac{\psi_i - \psi_0}{2} = \beta \sin \theta_i.
\]

It is easily seen that there is a monomorphism from the set of parameters $(\alpha, \beta, \phi_0, \psi_0, \theta_1, \theta_2, \theta_3, \theta_4)$ into the set $(\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \phi_4, \psi_4)$.

Since there are two parametrizations of a point $x \in (T^2)^4$, consider a Jacobi matrix between these parametrizations. The elements are computed as follows:

\[
\begin{align*}
\frac{\partial \phi_i}{\partial \theta_0} &= 1, \quad \frac{\partial \psi_i}{\partial \theta_0} = 0; \\
\frac{\partial \phi_i}{\partial \psi_0} &= 0, \quad \frac{\partial \psi_i}{\partial \psi_0} = 1; \\
\frac{\partial \phi_i}{\partial \alpha} &= \frac{2 \cos \theta_i}{\cos \frac{\phi_i - \phi_0}{2}}, \quad \frac{\partial \psi_i}{\partial \alpha} = 0; \\
\frac{\partial \phi_i}{\partial \beta} &= 0, \quad \frac{\partial \psi_i}{\partial \beta} = \frac{2 \sin \theta_i}{\cos \frac{\psi_i - \psi_0}{2}}; \\
\frac{\partial \phi_i}{\partial \theta_j} &= -2 \alpha \sin \theta_i, \quad \frac{\partial \psi_i}{\partial \theta_j} = \frac{2 \beta \cos \theta_i}{\cos \frac{\psi_i - \psi_0}{2}}, \quad \frac{\partial \phi_i}{\partial \theta_j} = \frac{\partial \psi_i}{\partial \theta_j} = 0.
\end{align*}
\]

Therefore
\[
J = \begin{vmatrix} D(\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \phi_4, \psi_4) \\ D(\phi_0, \psi_0, \alpha, \beta, \theta_1, \theta_2, \theta_3, \theta_4) \end{vmatrix} =
\begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 \cos \theta_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\cos \frac{2 \theta_1 - \phi_0}{2} & 2 \cos \theta_2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 \cos \theta_3 & 0 & 1 & 0 & 0 & 0 & 0 \\
\cos \frac{2 \theta_3}{2} & 2 \cos \theta_4 & 0 & 1 & 0 & 0 & 0 & 0 \\
-2 \alpha \sin \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cos \frac{2 \alpha \sin \theta_1}{2} & 2 \beta \cos \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 \alpha \sin \theta_3 & 0 & 0 & 0 & 0 & 0 \\
\cos \frac{2 \alpha \sin \theta_3}{2} & 2 \beta \cos \theta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 \alpha \sin \theta_4 & 0 & 0 & 0 \\
\cos \frac{2 \alpha \sin \theta_4}{2} & 2 \beta \cos \theta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 \alpha \sin \theta_4 & 0 \\
\cos \frac{2 \alpha \sin \theta_4}{2} & 2 \beta \cos \theta_4 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}.
\]

Direct computation shows that
\[
J = \sum_{(i j k l)} 64 \text{sign}(i j k l) \alpha^2 \beta^2 \cdot \frac{1}{\prod_{m=1}^{4} \cos \frac{\phi_m - \phi_0}{2} \cos \frac{\psi_m - \psi_0}{2}} \times \\
\times \cos^2 \theta_i \cos \theta_j \sin^2 \theta_k \sin \theta_l \cos \frac{\phi_l - \phi_0}{2} \cos \frac{\psi_l - \psi_0}{2},
\]

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where \((ijkl)\) adopts all permutations of \((1234)\).

From (7) easily follows that

\[
\text{mes}_8 \left\{ x \in (T^2)^4 : \alpha(x)\beta(x) < \frac{t}{\gamma_2} \text{ and } \max(\alpha(x), \beta(x)) < \frac{1}{100} \right\} \leq L(t) \leq \text{mes}_8 \left\{ x \in (T^2)^4 : \alpha(x)\beta(x) < \frac{t}{\gamma_1} \text{ and } \max(\alpha(x), \beta(x)) < \frac{1}{100} \right\}.
\]

Therefore

\[
\int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_2}} d\phi_1 d\psi_1 d\phi_2 d\psi_2 d\phi_3 d\psi_3 d\phi_4 d\psi_4 \leq L(t) \leq \int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_1}} d\phi_1 d\psi_1 d\phi_2 d\psi_2 d\phi_3 d\psi_3 d\phi_4 d\psi_4.
\]

In variables \((\alpha, \beta, \phi_0, \psi_0, \theta_1, \theta_2, \theta_3, \theta_4)\) the last inequality can be written as follows

\[
\int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_2}} |J| d\alpha d\beta d\phi_0 d\psi_0 d\theta_1 d\theta_2 d\theta_3 d\theta_4 \leq L(t) \leq \int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_1}} |J| d\alpha d\beta d\phi_0 d\psi_0 d\theta_1 d\theta_2 d\theta_3 d\theta_4.
\]

Let

\[
J_1 = \frac{J}{\alpha^2 \beta^2} = \sum_{(i,j,k,l)} 64 \text{ sign } (ijkl) \cdot \frac{1}{\prod_{m=1}^{4} \cos^2 \phi_m - \phi_0^2} \cdot \cos \phi_m - \phi_0^2 \cos \psi_m - \psi_0^2 \times \cos^2 \theta_i \cos \theta_j \sin^2 \theta_k \sin \theta_i \cos \frac{\phi_j - \phi_0}{2} \cos \frac{\psi_j - \psi_0}{2}.
\]

Then \(J_1\) can be considered as a function \(J_1(\alpha, \beta, \phi_0, \psi_0, \theta_1, \theta_2, \theta_3, \theta_4)\).

Obviously,

\[
\int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_2}} |J| d\alpha d\beta d\phi_0 d\psi_0 d\theta_1 d\theta_2 d\theta_3 d\theta_4 = \int_{\max(\alpha,\beta) < \frac{1}{8\alpha\beta \gamma_1}} |J_1| d\phi_0 d\psi_0 d\theta_1 d\theta_2 d\theta_3 d\theta_4.
\]
Similarly,

\[
\int_{\max(\alpha, \beta) < \frac{1}{100}} \int_{\alpha, \beta \in (-\pi, \pi]} \int_{\theta, \phi \in (-\pi, \pi]} |J| \, d\theta_1 d\theta_2 d\theta_3 d\theta_4 \, d\phi_0 d\psi_0 d\phi_1 d\psi_1 d\phi_2 d\psi_2 d\phi_3 d\psi_3 d\phi_4 d\psi_4
\]

\[
= \int_{\max(\alpha, \beta) < \frac{1}{100}} \alpha^2 \beta^2 \, d\alpha d\beta \int_{\phi, \psi \in (-\pi, \pi]} |J_1| \, d\phi_0 d\psi_0 d\phi_1 d\psi_1 d\phi_2 d\psi_2 d\phi_3 d\psi_3 d\phi_4.
\]

Since \(\max(\alpha, \beta) < \frac{1}{100}\), then

\[
|\sin \frac{\phi_m - \phi_0}{2}| < \frac{1}{100} \text{ and } |\sin \frac{\psi_m - \psi_0}{2}| < \frac{1}{100}
\]

for \(m = 1, 2, 3, 4\).

Hence

\[
\cos \frac{\phi_m - \phi_0}{2} > \frac{4999}{5000} \text{ and } \cos \frac{\psi_m - \psi_0}{2} > \frac{4999}{5000}.
\]

The last estimates and (8) easily imply the following inequality independent from \(\alpha, \beta\)

\[
J_1 \geq 64 \left| \sum_{(i,j,k)} \text{sign } (i,j,k) \cdot \cos^2 \theta_i \cos \theta_j \sin^2 \theta_k \sin \theta_l \right| - \frac{64 \cdot 2 \cdot 24}{5000}.
\]

Let \(\theta_m = \frac{(m-2)\pi}{2}\). Then

\[
\left| \sum_{(i,j,k)} \text{sign } (i,j,k) \cdot \cos^2 \theta_i \cos \theta_j \sin^2 \theta_k \sin \theta_l \right| = 4.
\]

Consequently, there exists some neighbourhood of point \((-\frac{\pi}{4}, 0, \frac{\pi}{4}, \pi)\) in co-
ordinates \((\theta_1, \theta_2, \theta_3, \theta_4)\) such that \(|J_1| > 64\) in this neighbourhood, and the
neighbourhood is independent from \(\alpha, \beta, \phi_0, \psi_0\).

Therefore there exist positive constants \(\gamma'_4, \gamma'_5\) independent of \(\alpha, \beta\) and sat-
sifying

\[
\gamma'_4 < \int_{\phi, \psi \in (-\pi, \pi]} \int_{\theta, \phi \in (-\pi, \pi]} |J| \, d\phi_0 d\psi_0 d\theta_1 d\theta_2 d\theta_3 d\theta_4 < \gamma'_5
\]

for every \(0 < \alpha, \beta < \frac{1}{100}\).

Therefore

\[
\gamma'_4 \cdot \int_{\max(\alpha, \beta) < \frac{1}{100}} \alpha^2 \beta^2 \, d\alpha d\beta < L(t) < \gamma'_5 \cdot \int_{\max(\alpha, \beta) < \frac{1}{100}} \alpha^2 \beta^2 \, d\alpha d\beta. \quad (9)
\]
If $\tau < \frac{1}{10000}$ then

$$\int_{\max(\alpha, \beta) < \frac{1}{10000}} \alpha^2 \beta^2 \, d\alpha \, d\beta = \frac{\tau^3}{9} - \frac{2 \ln 100}{9} \tau^3 + \frac{1}{9} \tau^3 |\ln \tau|.$$

Consequently, as $t \to 0$, the main terms in left and right parts of (9) have order $t^3 |\ln t|$. Therefore $L(t)$ has order $t^3 |\ln t|$ as $t \to 0$ and the estimates of Lemma 4.2 are proved in some neighbourhood of 0. Then Lemma 4.2 can be proved for every $0 < t < \frac{1}{2}$ by standard arguments.

Lemma 4.1 and Lemma 4.2 together easily imply that there exist positive constants $\gamma_6, \gamma_7$ such that

$$\gamma_6 t^3 |\ln t| < M(t) < \gamma_7 t^3 |\ln t|$$

for every $t > 0$.

**Proof of Theorem 1.**

It is obvious that

$$\int_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} \, dx < \int_{(T^2)^4} \lambda^4 \left( e^{-\lambda G(x)} + e^{-\lambda (4\pi^2 - \lambda G(x))} \right) \, dx < 2 \int_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} \, dx.$$

Then, by Lemma 2.3,

$$E f_3 = O^* \left( \int_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} \, dx \right).$$

The integral

$$\int_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} \, dx = \int_{\mathbb{R}} \lambda^4 e^{-\lambda t} \, dM(t),$$

where the right side is a Stieltjes integral.

Since $G(x)$ is the measure of $C^+(x)$, then for every $x \in X$ holds $0 < G(x) \leq 2\pi^2$. Therefore $M(t)$ is a constant for $t \leq 0$ and for $t \geq 2\pi^2$. Hence

$$\int_{(T^2)^4} \lambda^4 e^{-\lambda G(x)} \, dx = \int_{0}^{2\pi^2} \lambda^4 e^{-\lambda t} \, dM(t).$$
Since $M(t)$ is monotonous, $e^{-M}$ is monotonous and continuous, then

\[
\int_0^{2\pi^2} \lambda^4 e^{-M} M(t) = \lambda^4 M(2\pi^2) e^{-2\lambda^2} + \lambda^5 \int_0^{2\pi^2} e^{-M} M(t) \, dt + \lambda^5 \int_0^{\frac{\pi}{2}} e^{-M} M(t) \, dt.
\]

Obviously, as $\lambda \to \infty$,

\[
\lambda^4 M(2\pi^2) e^{-2\lambda^2} = o(1),
\]

\[
\lambda^5 \int_0^{2\pi^2} e^{-M} M(t) \, dt = o(1),
\]

\[
\lambda^5 \int_0^{\frac{\pi}{2}} e^{-M} M(t) \, dt = O^*(\lambda^5 \int_0^{\frac{\pi}{2}} e^{-M} t^3 |\ln t| \, dt).
\]

Let $u = e^{-\lambda t}$, then

\[
t = -\frac{\ln u}{\lambda} \quad \text{and} \quad dt = -\frac{du}{\lambda u}.
\]

Therefore

\[
\int \frac{e^{-\lambda t^3 |\ln t|}}{\lambda^4} \, dt = \int e^{-\frac{\ln^3 u}{\lambda^4}} \cdot (\ln \lambda - \ln(-\ln u)) \, du = O^*(\lambda^{-4} \ln \lambda).
\]

Hence

\[
\lambda^5 \int_0^{\frac{\pi}{2}} e^{-M} M(t) \, dt = O^*(\lambda \ln \lambda) \quad \text{and} \quad E f_3 = O^*(\lambda \ln \lambda),
\]

which is the statement of Theorem 1.

**Proof of Theorem 2.**

From Dehn-Sommerville equations for a simplicial 4-polytope follows that

\[f_2 = 2f_3 + r_2 \quad \text{and} \quad f_1 = f_3 - f_0 + r_1,
\]

where stochastic variables $r_1$ and $r_2$ are errors of degenerate cases, i.e. $r_1 = r_2 = 0$ almost surely if $n(T^2) > 4$ and $r_1, r_2 < 10$ almost surely. Since

\[
\lim_{\lambda \to \infty} P(n(T^2) \leq 4) = 0,
\]

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then
\[ E r_1 = o(1), \quad E r_2 = o(1), \quad E f_3 = O^*(\lambda \ln \lambda) \]
as \( \lambda \to \infty \) by Theorem 1. Also,
\[ E f_0 = E n(T^2) = 4\lambda \pi^2 \]
as \( \lambda \to \infty \) since \( n(T^2) \) is distributed as \( Pois(4\lambda \pi^2) \).

Finally,
\[ E f_2 = 2E f_3 + E r_2 = O^*(\lambda \ln \lambda), \quad E f_1 = E f_3 - E f_0 + E r_1 = O^*(\lambda \ln \lambda), \]
and Theorem 2 is proved.

Let \( \lambda > 0 \) be a fixed constant. Again consider the Poisson point process \( \mathcal{P}_\lambda \)
within \( T^2 \). Let
\[ \eta = \begin{cases} f_3, & \text{if } f_0 \neq 0, \\ f_0, & \text{if } f_0 = 0. \end{cases} \]

**Integral expression for \( E \eta \)**

Choose arbitrary positive \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \). Consider a covering of \( X \) by binary sets
satisfying covering conditions 1 – 5. Choose a subcovering \( S(\varepsilon_1, \varepsilon_2, \varepsilon_3) \) such
that if \( Y(x), Y(y) \in S \), then either
\[ \text{int} Y(x) \cap \text{int} Y(y) = \emptyset \quad \text{or} \quad Y(x) = Y(y). \]

By Lemma 1.1, such subcovering \( S \) exists.

Introduce stochastic variables
\[ B^s_Y = \begin{cases} \frac{A^s_Y}{f_0}, & \text{if } f_0 \neq 0, \\ 0, & \text{if } f_0 = 0. \end{cases} \]
\[ B^l_Y = \begin{cases} \frac{A^l_Y}{f_0}, & \text{if } f_0 \neq 0, \\ 0, & \text{if } f_0 = 0. \end{cases} \]

Similarly to (1),
\[ 24E \eta = \sum_{Y(x) \in S} E (B^s_Y + B^l_Y). \]

For \( \nu > 0 \) let \( \zeta_\nu \) be distributed as \( Pois(\nu) \). Denote
\[ h(\nu) = E \frac{1}{\zeta_\nu + 4} = \sum_{j=0}^{\infty} \frac{\nu^j}{j!(j+4)} e^{-\nu}. \quad (10) \]

Obviously, \( h(\nu) \) is continuous for \( \nu > 0 \).

Applying the Abel identity to (10), obtain
\[ h(\nu) = \frac{1}{4} - \sum_{j=1}^{\infty} \left( \frac{1}{5-j} - \frac{1}{6-j} \right) P(\zeta_\nu \geq j). \]
From this expression immediately follows $h(\nu) < \frac{1}{4}$ for $\nu > 0$.

Moreover, [5, Problem 3.70] states that $P(\zeta_\nu \geq j)$ is an increasing function of $\nu$ for every $j > 0$. Therefore $h(\nu)$ is decreasing for $\nu > 0$.

Direct computation of the sum in (10) gives

$$h(\nu) = \frac{1}{\nu} - \frac{3}{\nu^2} + \frac{6}{\nu^3} - \frac{6 - 6e^{-\nu}}{\nu^4}. \quad (11)$$

Consider a binary set $Y = Y(x) \in \mathcal{S}$. Let

$$Y = Y_1 \times Y_2 \times Y_3 \times Y_4,$$

where $Y_i \subset T^2$ for $i = 1, 2, 3, 4$.

Proofs of the next two lemmas use Propositions 1 - 3.

Lemma 5.1.

$$EB^\nu \geq mes_8(Y) \cdot \lambda^4 \cdot e^{-4\lambda \varepsilon_1^2} \cdot e^{-\lambda(1+\varepsilon_3)G(x)} \cdot h\left(\lambda(4\pi^2 - G(x))\right),$$

$$EB^\nu \geq mes_8(Y) \cdot \lambda^4 \cdot e^{-4\lambda \varepsilon_1^2} \cdot e^{-\lambda(1+\varepsilon_3)(4\pi^2-G(x))} \cdot h\left(\lambda G(x)\right). \quad (12)$$

Proof. Prove, for example, the first estimate, as the proof of the second is similar. The six random events

$$n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1,$$

$$n\left(\bigcup_{i=1}^{4} Y_i\right) = 0, \quad n\left(\bigcup_{i=1}^{4} I_Y \right) = k$$

are independent since $\mathcal{P}_\lambda$ is a Poisson point process and

$$Y_1, Y_2, Y_3, Y_4, \bigcup_{i=1}^{4} Y_i, \bigcup_{i=1}^{4} I_Y$$

are pairwise non-intersecting sets.

Therefore

$$P\left(n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n\left(\bigcup_{i=1}^{4} Y_i\right) = 0, n\left(\bigcup_{i=1}^{4} I_Y \right) = k\right) =$$

$$= \prod_{i=1}^{4} P(n(Y_i) = 1) \cdot P\left(n\left(\bigcup_{i=1}^{4} Y_i\right) = 0\right) \cdot P\left(n\left(\bigcup_{i=1}^{4} I_Y \right) = k\right).$$

Similarly to the proof of Lemma 2.1 we obtain

$$P(n(Y_i) = 1) \geq \lambda \cdot mes_2(Y_i) \cdot e^{-\lambda \varepsilon_1^2}$$

and

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If random events
\[ n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, n \left( U_Y^4 \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) = 0 \]
occur simultaneously, then, by Proposition 1, there exists exactly one \( y \in Y \) corresponding to a hyperface. Consequently,

\[ n(Y_1) = 1, n(Y_2) = 1, n(Y_3) = 1, n(Y_4) = 1, \]

\[ n \left( \left( U_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) \right) = 0, \quad n \left( \left( U_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) \right) = k \]
together imply

\[ B_Y^* = \frac{1}{k+4}. \]

Then

\[ E B_Y^* \geq \sum_{k=0}^{\infty} \frac{1}{k+4} \cdot \prod_{i=1}^{4} P(n(Y_i) = 1) \times \]

\[ \geq \text{mess}(Y) \cdot \lambda^4 \cdot e^{-4\lambda \varepsilon^2} \cdot e^{-\lambda (1+\varepsilon_3)G(x)} \cdot h \left( \lambda \text{mes} \left( I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) \right) \geq \]

and the first statement of Lemma 5.1 is proved.

**Lemma 5.2.**

\[ E B_Y^* \leq \lambda^4 \cdot \text{mess}(Y) \cdot e^{-\lambda (1-\varepsilon_3-4\varepsilon_2)G(x)} \cdot h \left( \lambda (1-4\varepsilon_2)(4\pi^2 - G(x)) \right), \]

\[ E B_Y^* \leq \lambda^4 \cdot \text{mess}(Y) \cdot e^{-\lambda (1-\varepsilon_3-4\varepsilon_2)(4\pi^2 - G(x))} \cdot h \left( \lambda (1-4\varepsilon_2)G(x) \right). \]

**Proof.** As above, prove the first estimate. The proof of the second estimate is similar.

From Propositions 2 and 3 and the law of total probability follows

\[ E B_Y^* \leq \sum_{k=0}^{\infty} \sum_{k_i=1}^{\infty} \frac{k_1 k_2 k_3 k_4}{k + k_1 + k_2 + k_3 + k_4} \times \]

\[ \times P \left( n(Y_i) = k_i, n \left( I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) = 0, n \left( U_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) = k \right) \leq \]

\[ \leq \sum_{k=0}^{\infty} \sum_{k_i=1}^{\infty} \frac{k_1 k_2 k_3 k_4}{k + 4} \cdot P \left( n(Y_i) = k_i, n \left( I_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) = 0, n \left( U_Y^* \setminus \left( \bigcup_{i=1}^{4} Y_i \right) \right) = k \right). \]
As in the proof of Lemma 2.2,
\[
\sum_{k=0}^{\infty} \sum_{k_i=1}^{\infty} \frac{k_1 k_2 k_3 k_4}{k + 4} \cdot P\left(n(Y_i) = k_i, n(I_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0, n(U_Y^i \setminus \bigcup_{i=1}^{4} Y_i) = k\right) =
\]
\[
= P\left(n(I_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0\right) \cdot \prod_{i=1}^{4} \sum_{k_i=1}^{\infty} k_i P(n(Y_i) = k_i) \times \]
\[
\times \sum_{k=0}^{\infty} \frac{1}{k + 4} P\left(n(U_Y^i \setminus \bigcup_{i=1}^{4} Y_i) = k\right) =
\]
\[
= P\left(n(I_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0\right) \cdot \lambda^4 \cdot mes_8(Y) \cdot h\left(\lambda mes_2(U_Y^i \setminus \bigcup_{i=1}^{4} Y_i)\right).
\]

As it was proved earlier,
\[
P\left(n(I_Y^s \setminus \bigcup_{i=1}^{4} Y_i) = 0\right) \leq e^{-\lambda(1-\varepsilon_3-4\varepsilon_2)G(x)}.
\]

Obviously, by covering condition 4,
\[
mes_2(U_Y^i \setminus \bigcup_{i=1}^{4} Y_i) \geq (1-4\varepsilon_2)(4\pi^2 - G(x)).
\]

Finally,
\[
E B_Y^s \leq \lambda^4 \cdot mes_8(Y) \cdot e^{-\lambda(1-\varepsilon_3-4\varepsilon_2)G(x)} \cdot h(\lambda(1-4\varepsilon_2)(4\pi^2 - G(x))),
\]

which is the first estimate of Lemma 5.2.

**Lemma 5.3.**

\[
E \eta = \frac{1}{24} \int_{(T^2)^s} \lambda^4 \left(e^{-\lambda G(x)} h(4\lambda \pi^2 - \lambda G(x)) + e^{-4\lambda \pi^2 + \lambda G(x)} h(\lambda G(x))\right) dx. \quad (14)
\]

The proof of Lemma 5.3 is easily obtained by applying arguments of Lemma 2.3 to inequalities (12) and (13).

**Asymptotics of** \(E \eta\)

**Lemma 6.1.**

\[E \eta = O^*(\ln \lambda).\]

**Proof.** Since
\[
h(\lambda G(x)) < \frac{1}{4} \quad \text{and} \quad e^{-4\lambda \pi^2 + \lambda G(x)} \leq e^{-2\lambda \pi^2},
\]

\[
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\]
then
\[ \int e^{-4\lambda \pi^2 + \lambda G(x)} h(\lambda G(x)) \, dx = o(1) \]
as \( \lambda \to \infty \).

Further,
\[ 2\lambda \pi^2 \leq 4\lambda \pi^2 - \lambda G(x) < 4\lambda \pi^2. \]

Therefore from (11) follows
\[ h(4\lambda \pi^2 - \lambda G(x)) = O^* (\lambda^{-1}). \]

Consequently,
\[ \int \lambda^4 e^{-\lambda G(x)} h(4\lambda \pi^2 - \lambda G(x)) \, dx = O^* \left( \lambda^3 \int e^{-\lambda G(x)} \, dx \right) = O^* (\ln \lambda). \]

Then Lemma 6.1 easily follows from (14).

**Proof of Theorem 3**
Since in non-degenerate cases \( f_1 = f_3 + f_0 \),
\[ \bar{v} = \eta + 2 + r \quad \text{and} \quad E \bar{v} = 2E\eta + 2 + Er, \]
where \( r < 10 \) almost surely and \( r = 0 \) in non-degenerate cases. Therefore
\[ Er = o(1) \]
as \( \lambda \to \infty \).

By Lemma 6.1,
\[ E\eta = O^*(\ln \lambda), \]
consequently
\[ E \bar{v} = O^*(\ln \lambda), \]
and Theorem 3 is proved.
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