Spectral splits of neutrinos as a BCS-BEC crossover type phenomenon

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We show that the spectral split of a neutrino ensemble which initially consists of electron type neutrinos, is analogous to the BCS-BEC crossover already observed in ultra cold atomic gas experiments. Such a neutrino ensemble mimics the deleptonization burst of a core collapse supernova. Although these two phenomena belong to very different domains of physics, the propagation of neutrinos from highly interacting inner regions of the supernova to the vacuum is reminiscent of the evolution of Cooper pairs between weak and strong interaction regimes during the crossover. The Hamiltonians and the corresponding many-body states undergo very similar transformations if one replaces the pair quasispin of the latter with the neutrino isospin of the former.

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I. INTRODUCTION

A core-collapse supernova releases 99% of its energy in the form of neutrinos in the MeV energy scale [1, 2]. Our basic understanding about these neutrinos was confirmed [3, 4] when supernova 1987A exploded in our neighbor galaxy, the Large Magellanic Cloud, and generated 19 neutrino events in Kamiokande [5] and IBM [6] detectors. The next important breakthrough in this field will be the observation of neutrinos from a supernova explosion in our own galaxy which is estimated to generate thousands of neutrino events in current neutrino detectors [7]. Therefore a future galactic supernova presents a unique opportunity to test our understanding of neutrinos. This includes the many-body aspects of their flavor transformations [8, 9] which develop via the neutrino-neutrino (\(\nu\nu\)) interactions in the supernova [10, 11].

Although neutrino cross sections are extremely small, their tiny scattering amplitudes can add up coherently to give rise to a finite effect when neutrinos propagate in the presence of a matter background [12]. This is similar to the refraction of light in matter except that, since neutrinos can interact with each other via neutral current, they can also create a self refraction effect on themselves [11]. Two kinds of diagrams, shown in Fig. 1, add up coherently in self refraction: (a) the forward scattering diagram in which there is no momentum transfer between particles and (b) the exchange diagram in which particles completely swap their momenta [11]. The former gives rise to an ordinary refraction index through the optical theorem [12]. The latter can be viewed as a flavor-exchange diagram between neutrinos and, as such, it couples the flavor transformation of each neutrino to the flavor content of the entire neutrino ensemble. This turns the flavor evolution of neutrinos near the core of a supernova into a many-body problem [13–18].

The correlations between flavor histories of neutrinos with different energies, which are referred to as collective neutrino oscillations, have been extensively studied [19–26]. The large array of resulting nonlinear and emergent behavior displayed by self interacting neutrinos are reminiscent of condensed matter systems. A formal analogy between collective neutrino oscillations and BCS pairing model of superconductivity [27] has recently been pointed out by Pehlivan et al. [15, 16] and further elaborated in [28]. Besides the Cooper pairs of electrons in superconductors, BCS pairing is observed in a broad range of many body systems, including neutron stars and atomic nuclei [29], ultra cold atomic gases [30, 31] and excitonic condensates in semiconductor structures [32–34].

One collective behavior observed in some numerical simulations of neutrinos emerging from supernova is the spectral split or spectral swap phenomenon in which neutrinos in different flavor (or mass) eigenstates completely exchange their spectra around a certain critical energy [22, 24]. In this paper we show that, for a neutrino ensemble which initially consists of only electron type neutrinos, the formation of the spectral split also corresponds to the well known BCS-BEC crossover [35, 36] phenomenon. We describe the neutrinos using the effective two flavor mixing scenario and the neutrino bulb

\[ \nu_\alpha \leftrightarrow \nu_\beta \]

FIG. 1. (Color online) Forward (a) and exchange (b) diagrams which add up coherently in \(\nu\nu\) scattering.
The pairing interaction between fermionic atoms can be transition of Cooper pairs. In ultra cold dilute gas systems, weakly interacting fermions through a coherent superposition of zero center-of-mass momentum pairs and $\Delta$ the distance from the supernova center. Combined with decreasing neutrino density which increases the normal-superfluid transition of Cooper pairs. In ultra cold dilute gas systems, weakly interacting fermions through a coherent superposition of zero center-of-mass momentum pairs and $\Delta$.

We assume that the atoms can occupy a discrete set of energy levels $\varepsilon_k$ and the operators $c^\dagger_{k\uparrow}$ and $c_{k\downarrow}$ annihilate spin-up and spin-down fermions, respectively, in the $k$th time-reversed energy levels. The chemical potential $\mu$ is introduced as a Lagrange multiplier to fix the average particle number. The interactions can be tuned via Feshbach resonances \[35\] which function as a control knob for the coupling constant $g$. The physics is captured by the mean field

$$\tilde{\Delta} = \left(\frac{\Delta^+ + \Delta^-}{2}, \frac{\Delta^+ - \Delta^-}{2}, \Delta^0\right)$$

where

$$\Delta^- = \sum_k \langle c^\dagger_{k\uparrow} c_{k\downarrow}\rangle, \quad \Delta^0 = \sum_k \langle \frac{c^\dagger_{k\uparrow} c_{k\downarrow}^\dagger - c^\dagger_{k\downarrow} c_{k\uparrow}^\dagger}{2} \rangle.$$  \[3\]

The pairing potential $\Delta^- = (\Delta^+)^*$ describes the scattering of zero center-of-mass momentum pairs and $\Delta^0$ corresponds to the Hartree potential which can be included in the definition of $\mu$, i.e., $\mu \rightarrow \mu + g\Delta^0$. In Eq. \[3\], the expectation values are calculated with respect to a state which satisfies the usual mean field self-consistency requirements.

The flavor evolution of self interacting neutrinos near the core of a supernova is described by a mathematically similar Hamiltonian

$$H_{\nu\nu} = \frac{1}{2} \sum_k \left( a^\dagger_{k\uparrow} a^\dagger_{k\downarrow} \right) \left( -\omega_k + \lambda \right) \left( G\nu^+ \omega_k - \lambda \right) \left( a_{k\downarrow} a_{k\uparrow}^\dagger \right).$$  \[4\]

We assume that neutrinos are box quantized in volume $V$ and therefore occupy a discrete set of energy levels $\varepsilon_k$. The operator $a_{k\downarrow}$ annihilates a neutrino in the $k$th energy level, in the mass eigenstate with mass $m_i$ ($i = 1, 2$). To be specific, we assume that $m_1 > m_2$ which corresponds to inverted mass hierarchy. The oscillation frequency for a neutrino with energy $\varepsilon_k$ associated with this mass difference is given by

$$\omega_k = (m_1^2 - m_2^2)/2\varepsilon_k.$$  \[5\]

The Lagrange multiplier $\lambda$ plays an analogous role to the chemical potential $\mu$ in Eq. \[1\] as discussed in Ref. \[15\].

\[\nu\nu\] interaction strength is given by

$$G = \frac{2\sqrt{2} G_F}{V} D(r)$$  \[6\]

where the Fermi constant $G_F$ appears due to our use of the Fermi 4-point interaction as shown in Fig. \[1\]. We also use the neutrino bulb model which approximates the angular dependence of the $\nu\nu$ scattering amplitude \[14\] with an effective geometrical factor $D(r) \propto 1/r^2$ \[22\] where $r$ is the distance from the supernova center. Combined with decreasing neutrino density which increases the normal-
ization volume, G drops as 1/r^4.

Here, we adopt an effective two flavor mixing scenario
\[ a_{\kappa e} = \cos \vartheta a_{k1} + \sin \vartheta a_{k2}, \quad a_{\kappa \mu} = \sin \vartheta a_{k1} - \cos \vartheta a_{k2}, \] (7)
where the effects of the third flavor and the other background particles are absorbed in a single mixing angle \( \vartheta \) and a single mass squared difference (see e.g. [14, 15]). In Eq. (4), \( \nu \nu \) interactions are described by the mean field \( F \) defined by
\[ P^{-} = \sum_{\ell} \langle a_{\ell 1} a_{\ell 2} \rangle \quad P^{0} = \sum_{\ell} \left\langle \frac{a_{\ell 1}^{\dagger} a_{\ell 2} - a_{\ell 1} a_{\ell 2}}{2} \right\rangle. \] (8)
The components \( P^{\pm} \) create the exchange diagrams shown in Fig. 1(a) while \( P^{0} \) creates the forward scattering diagram in Fig. 1(b). \( P^{0} \) contributes to the diagonal of Eq. (4) and plays a similar role to that of an Hartree potential for fermionic pairs. In the case of neutrinos, \( P^{0} \) is always non-zero, and we include it in the definition of \( \lambda \) in order to highlight the resemblance with the BCS model.

The similarity between Eqs. (1) and (4) suggests the mapping
\[ a_{\ell 2} \leftrightarrow c_{k \uparrow} \quad \text{and} \quad a_{\ell 1} \leftrightarrow c_{k \downarrow}^{\dagger}, \] (9)
which reveals the common \( SU(2) \) group structure of these problems. This group is generated by the \textit{quasispin operators} for BCS pairs [14] given by
\[ J_{k}^{+} = c_{k \downarrow}^{\dagger} c_{k \uparrow}, \quad J_{k}^{0} = \frac{1}{2} \left( c_{k \uparrow}^{\dagger} c_{k \downarrow}^{\dagger} - c_{k \downarrow} c_{k \uparrow} \right), \] (10)
and the \textit{mass isospin operators} for neutrinos given by
\[ J_{k}^{-} = a_{\ell 1}^{\dagger} a_{\ell 2}, \quad J_{k}^{0} = \frac{1}{2} \left( a_{k 2}^{\dagger} a_{k 1} - a_{k 1}^{\dagger} a_{k 2} \right). \] (11)
For convenience, we denote both the pair quasispin and the neutrino isospin with the same symbol but it is always clear which one is being referred to from the context. In both cases, components of \( J_{k} \) obey the \( SU(2) \) algebra, i.e.,
\[ [J_{k}^{+}, J_{k}^{-}] = 2\delta_{kk'} J_{k}^{0}, \quad \text{and} \quad [J_{k}^{0}, J_{k'}^{0}] = \pm \delta_{kk'} J_{k}^{\pm}. \] (12)

In terms of these operators, the Hamiltonians describing the BCS pairs and self interacting neutrinos can be written respectively as
\[ H_{\text{BCS}} = \sum_{k} 2(\epsilon_{k} - \mu) J_{k}^{0} - G (\langle J^{-}\rangle J^{+} + \langle J^{+}\rangle J^{-}) \] (13)
\[ H_{\nu \nu} = - \sum_{k} (\omega_{k} - \lambda) J_{k}^{0} + G \left( \langle J^{-}\rangle J^{+} + \langle J^{+}\rangle J^{-} \right) \]
where \( \bar{J} \) is the total quasi-/iso-spin operator for all energy levels, i.e., \( \bar{J} = \sum_{k} J_{k} \). Note that the two Hamiltonians differ by an overall minus sign.

Eq. (10) tells us that for a single energy level \( k \), quasispin up and down states correspond to that level being occupied (\(|\uparrow\rangle\)) or unoccupied (\(|\downarrow\rangle\)) by a pair, respectively. (Levels occupied by unpaired atoms decouple from the pairing dynamics and are ignored here.) In the case of neutrinos, Eq. (11) tells us that the isospin up and down states for energy level \( k \) correspond to the neutrino occupying that energy level being in \( \nu_{2} \) and \( \nu_{1} \) mass eigenstate, respectively. Therefore, the analogous states are
\[ |\uparrow\rangle \leftrightarrow |\nu_{2}\rangle \quad \text{and} \quad |\downarrow\rangle \leftrightarrow |\nu_{1}\rangle. \] (14)

TABLE I. The analogous states and operators in the correspondence between the self interacting neutrinos and BCS pairing.

| Neutrino operators | BCS operators |
|--------------------|---------------|
| \( a_{\ell 1}^{\dagger} \) in flavor basis | Quasi-particle operators |
| \( \lim_{y \to \infty} \tilde{c}_{k \uparrow} \) | in flavor basis |
| \( \lim_{y \to \infty} \tilde{c}_{k \downarrow} \) | (10) |

III. BCS-BEC CROSSOVER AND SPECTRAL SPLITS

The ground state of the pairing Hamiltonian evolves from weakly bound Cooper pairs in the BCS limit of vanishing interactions to the Bose-Einstein condensation of tightly bound diatomic molecules in the limit of strong interactions. This evolution takes place without a phase...
transition as the interaction strength is varied and hence the ground state can be described by the same variational BCS wave function throughout the crossover between BCS and BEC limits [37, 38]. The theoretical prediction has been experimentally observed in ultra cold atomic systems [39–43].

The variational BCS ground state of the Hamiltonian in Eq. (1) can be written as

$$|\text{BCS}\rangle = \prod_k \left( \cos \theta_k + \sin \theta_k c_k \right) |\emptyset\rangle$$

$$= \prod_k \left( \cos \theta_k |\downarrow\rangle_k + \sin \theta_k |\uparrow\rangle_k \right)$$

where the angle \(\theta_k\) is to be found from

$$\cos 2\theta_k = \frac{(\epsilon_k - \mu)}{E_k}$$

with

$$E_k = \sqrt{(|\epsilon_k - \mu|^2 + g^2 \Delta^2 + \Delta^2)}.$$ (19)

The requirement that the state in Eq. (17) should satisfy Eq. (5) gives rise to the self-consistency equations

$$\frac{1}{g} = \sum_k \frac{f_k - \frac{1}{2}}{E_k},$$

$$\sum_k \left( n_p(k) - \frac{1}{2} \right) = \sum_k \frac{\epsilon_k - \mu}{E_k} \left( f_k - \frac{1}{2} \right)$$

which determine the mean field \(\Delta^\pm\) and the chemical potential \(\mu\). The distribution of the quasiparticles is given by the Fermi function

$$f_k = \frac{1}{\exp(E_k/k_BT) + 1}$$ (21)

where \(T\) is the temperature and \(k_B\) is the Boltzmann constant. It is also useful to define the quasiparticle operators

$$\tilde{c}_{k\uparrow} = \cos \theta_k c_{k\uparrow} - \sin \theta_k c_{k\downarrow}^\dagger, \quad \tilde{c}_{k\downarrow} = \sin \theta_k c_{k\uparrow}^\dagger + \cos \theta_k c_{k\downarrow}$$ (22)

which annihilate the BCS ground state so that \(|\text{BCS}\rangle\) can be viewed as a quasiparticle vacuum. Accordingly, pair occupation numbers in the \(|\text{BCS}\rangle\) state are given by

$$n_p(k) = \langle \tilde{c}_{k\uparrow}^\dagger \tilde{c}_{k\downarrow} \rangle = \langle \tilde{c}_{k\downarrow} \tilde{c}_{k\uparrow}^\dagger \rangle = \sin^2 \theta_k.$$ (23)

These occupation numbers can be calculated for any value of the interaction constant \(g\) by first solving the self-consistency equations given in Eq. (20) for the mean field \(\Delta^\pm\) and chemical potential \(\mu\), and then calculating \(\theta_k\) from Eqs. (18) and (19).

In the weak interaction limit, the solution describes the non-interacting Fermi sea with

$$\lim_{\theta_k \to 0} \mu = \epsilon_F$$

$$\lim_{\theta_k \to \pi} \mu = \epsilon_F$$ (25)

is the Fermi energy. Fig. 3 shows the corresponding particle (blue dashed line) and hole (red dotted line) occupation numbers in this limit which follow from substituting Eq. (24) in Eq. (23). The particle pairs fill the levels up to the Fermi energy and the system displays the characteristic distribution of a degenerate ideal Fermi gas at zero temperature.

The interaction strength increases from left to right in the upper panel of Fig. 3 and the distributions are gradually smoothed out as more and more levels start to take part in pairing. In the limit of strong interactions the angle \(\theta_k\) tends to the same value for all pairs, i.e.,

$$\lim_{g \to \infty} \theta_k = \theta.$$ (26)

In this limit, \(|\text{BCS}\rangle\) represents a BEC in the form of a coherent state of atomic pairs occupying the same single-pair quantum state. Fig. 3 displays the occupation numbers of particle and hole pairs which are almost uniform in this limit indicating that all levels are taking part in pairing. Here, quasihole distribution is also shown with a solid black line. Note that the quasihole distribution is equal to unity and remains the same throughout the crossover for any value of \(g\) because \(|\text{BCS}\rangle\) is the quasiparticle vacuum, but it is indicated only in this plot to emphasize its resemblance to the \(\nu_e\) distribution in the strong neutrino self interaction regime (see below).

The typical evolution of the chemical potential in the BCS-BEC crossover from positive values to negative values is shown in the left panel of Fig. 4 as a function of

| Self interacting neutrinos | Fermions with pairing |
|---------------------------|----------------------|
| Oscillation frequency \(\omega_k\) | 2\(\epsilon_k\) | Pair energy |
| Split frequency \(\omega_c\) | 2\(\epsilon_F\) | Fermi energy |
| Lagrange multiplier \(\lambda\) | 2\(\mu\) | Chem. potential |
| Interaction strength \(G\) | 2\(g\) | Pairing strength |
| Neutrino mean field \(P^\pm\) | \(\Delta^\pm\) | Pair mean field |
| Neutrino numbers \(n_l(k) - n_r(k)\) | \(n_p(k) - \frac{1}{2}\) | Pair occupation in mass eigenstates |

| Table II. The list of analogous scalar quantities in the correspondence between the self interacting neutrinos and BCS pairing. The last two lines follow from the analogy between BCS and BEC limits [37, 38]. The theoretical prediction has been experimentally observed in ultra cold atomic systems [39–43]. The variational BCS ground state of the Hamiltonian in Eq. (1) can be written as |
|--------------------------|------------------|
| Oscillation frequency \(\omega_k\) | 2\(\epsilon_k\) | Pair energy |
| Split frequency \(\omega_c\) | 2\(\epsilon_F\) | Fermi energy |
| Lagrange multiplier \(\lambda\) | 2\(\mu\) | Chem. potential |
| Interaction strength \(G\) | 2\(g\) | Pairing strength |
| Neutrino mean field \(P^\pm\) | \(\Delta^\pm\) | Pair mean field |
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The typical evolution of the chemical potential in the BCS-BEC crossover from positive values to negative values is shown in the left panel of Fig. 4 as a function of
the inverse scattering length, the parameter characterizing the strength of the pairing interaction. The vanishing of the chemical potential is accompanied with the shift of excitation energy minimum to zero momentum and is identified as the separation point between the BCS and BEC sides of the crossover.

For self interacting neutrinos, the state which is analogous to [BCS] can be written down using Table [I]

\[ |^{\text{"BCS"}}\rangle = \prod_k \left( \cos \theta_k + \sin \theta_k a_{k2}^\dagger a_{k1} \right) |\nu_1 \nu_1 \nu_1 \ldots \rangle \]

\[ = \prod_k \left( \cos \theta_k |\nu_1 \rangle_k + \sin \theta_k |\nu_2 \rangle_k \right) \]

Here, the angle \( \theta_k \) and the associated self consistency equations are the same as those given in Eqs. (18-20) with the replacements shown in Table [I] and \( \int k \rightarrow \frac{1}{2} \rightarrow -\phi_c / 2 \) where \( \phi_c \) is the Fermi function describing electron neutrino energy distribution. The last replacement follows from the analogy between pair quasispin and neutrino isospin (see Eqs. (10) and (11)). Due to the overall sign difference between the two Hamiltonians given in Eq. (13), \( |^{\text{"BCS"}}\rangle \) is not the ground state of the neutrino Hamiltonian, but its highest energy eigenstate. However, since the energy spectra of the two Hamiltonians are the same apart from an overall sign, \( |^{\text{"BCS"}}\rangle \) should also evolve smoothly between strong and weak interaction regimes without a phase transition (i.e., with no level crossings).

Using Eq. (19), one can also define the analogs of the quasiparticle operators introduced in Eq. (22)

\[ \tilde{a}_{k2} = \cos \theta_k a_{k2} - \sin \theta_k a_{k1} \]

\[ \tilde{a}_{k1} = -\sin \theta_k a_{k2}^\dagger + \cos \theta_k a_{k1}^\dagger \] (28)

which similarly annihilate the \( |^{\text{"BCS"}}\rangle \) state. Note that, unlike the operators in Eq. (22), these operators do not mix particle and hole states which is consistent with the number conserving nature of the neutrino self interactions. Denoting the states associated with the operators \( \tilde{a}_{k1}^\dagger \) and \( \tilde{a}_{k2}^\dagger \) by \( |\tilde{\nu}_1 \rangle \) and \( |\tilde{\nu}_2 \rangle \), respectively, this tells us that the \( |^{\text{"BCS"}}\rangle \) state is a \( |\tilde{\nu}_1 \rangle \) condensate.

For neutrinos, the limit of strong self interactions is
realized near the core of the supernova. In general, the flavor composition of neutrinos released from the core depends on the explosion phase. Here we consider a special case in which all neutrinos are released as νe, which is more relevant for the initial stages of the explosion. For such a configuration, the solution of the consistency equations given in Eq. (6) yield

$$\lim_{G \to \infty} \theta_k = \tilde{\theta}$$  \hspace{1cm} (29)$$
where \(\tilde{\theta}\) is the neutrino mixing angle introduced in Eq. (7). This is similar to the BCS regime of the fermion pairs. Substituting this angle in Eq. (27) and using Eq. (7) gives

$$\langle \nu^e_k \nu^e_k \nu^e_k \cdots \rangle$$  \hspace{1cm} (30)$$
which confirms the self-consistency of the state. In this limit, the quasiparticle operators which annihilate this state become the particle operators in flavor basis, i.e.,

$$\tilde{a}_{k_1}^\dagger = a_{k \mu}$$ and $$\tilde{a}_{k_1} = a_{k \mu}^\dagger.$$

In Fig. 3, we plot neutrino occupation numbers associated with the \(|\nu^e \nu^e \nu^e \cdots \rangle\) state in this limit. Note that, although the pairing Hamiltonian in Eq. (1) describes the atomic pairs at ultra low temperatures, the Hamiltonian in Eq. (4) represents self interactions of neutrinos for any (thermal or non-thermal) energy distribution. The main features of the analogy is independent of the neutrino energy distribution. For illustration, we use a thermal νe distribution with a temperature of 5 MeV which is shown with the solid-black line. The corresponding νe and ν2 occupation numbers which follow from Eq. (7) are shown with red-dotted and blue-dashed lines, respectively.

As the neutrinos move away from the core of the supernova, νν interactions are gradually turned off as described by Eq. (6). Therefore, one expects the \(|\nu^e \nu^e \nu^e \cdots \rangle\) state to evolve in a way which is similar to the BCS-BEC crossover of \(|\nu^e \nu^e \nu^e \cdots \rangle\) state, but reversed in the direction of decreasing interaction strength. The middle panel of Fig. 4 shows the evolution of the neutrino distributions as the neutrino self interaction constant G decreases from right to left. In the dilute, weakly interacting regime where

$$\lim_{G \to 0} \lambda = \omega_e,$$
the state \(|\nu^e \nu^e \nu^e \cdots \rangle\) in Eq. (27) evolves into

$$\langle \nu^e_k \nu^e_k \nu^e_k \cdots \rangle = \prod_{\omega_k < \omega_e} |\nu^e_k\rangle_k \prod_{\omega_k > \omega_e} |\nu^e_k\rangle_k$$  \hspace{1cm} (33)$$
under the adiabatic evolution conditions. This distribution is plotted in Fig. 3f and corresponds to the BCS limit of the fermion pairing. This is a particular example of a spectral split phenomenon, so called because the original νe energy distribution is eventually split between the two mass eigenstates. This phenomenon was observed in numerical simulations of supernova neutrinos by various groups (see Refs. [24, 25] for review).

The similarity between pair distribution of cold atoms which we treat at zero temperature, and the neutrino distribution which we treat at finite temperature becomes pronounced if we focus on the Bogoliubov coefficients

$$z_{k1} = \left| \frac{\langle a_{k_1}^\dagger \tilde{a}_{k_1} \rangle}{\langle \tilde{a}_{k_1} \tilde{a}_{k_1} \rangle} \right| = \cos \theta_k, \quad z_{k2} = \left| \frac{\langle a_{k_2}^\dagger \tilde{a}_{k_1} \rangle}{\langle \tilde{a}_{k_1} \tilde{a}_{k_1} \rangle} \right| = \sin \theta_k.$$  \hspace{1cm} (34)$$
These are the probability amplitudes for the neutrino born in the state \(|\nu^e_k\rangle\) near the core (where it almost overlaps with \(|\nu^e_k\rangle\) to be found in \(|\nu^e_k\rangle\) or \(|\nu^e_k\rangle\) mass eigenstate, respectively. The evolutions of these coefficients are shown in the lower panel of Fig. 3 as a function of the interaction constant. In strongly interacting limit near the center of the supernova Bogoliubov coefficients are uniform with \(z_{k1} \rightarrow \cos \theta\) and \(z_{k2} \rightarrow \sin \theta\) but as the neutrinos move away from the center, \(|\nu^e_k\rangle\) becomes more and more like \(|\nu^e_k\rangle\) (or \(|\nu^e_k\rangle\) for low (high) \(\omega\) values. This is reminiscent of the fact that, during the BCS-BEC crossover, quasihole degrees of freedom at the BCS limit coincide with real particles (holes) for low (high) energies at the BCS limit.

In Fig. 4, we plot the eigenvalues of the fermion pairing (left panel) and self interacting neutrino (right panel) Hamiltonians for three representative values of the respective interaction strengths. For the fermion pairs, these eigenvalues are ±\(E_k\) with \(E_k\) given by Eq. (19). For neutrinos, they are given by the same formula with the replacements from Table [1]. The solid black lines represent the weakly interacting limits where the chemical potential \(\mu\) and the Lagrange multiplier \(\lambda\) is almost coincident with their limiting values which are the Fermi energy \(\epsilon_F\), and the split frequency \(\omega_e\) respectively. The dashed blue lines correspond to the point at which \(\mu\) and \(\lambda\) become zero, and the dotted red lines represent the regime in which they are negative. In the case of fermion pairs, the difference between these eigenvalues (2\(E_k\)) is the energy gap for the creation of a quasiparticle pair excitation in the system. This energy gap is minimized at \(\epsilon_k = \mu\) on the BCS side, i.e. while \(\mu > 0\), and at \(\epsilon_k = 0\) on the BEC.
cold atomic gases. Due to their mathematically equivalent Hamiltonians, these systems undergo identical transformations while they evolve from strong to weak interaction regimes. In particular, the Fermi energy of the BCS model and the critical split frequency of neutrinos play analogous roles. Note that although the pair quasipin and the neutrino isospin obey the same algebra, the latter is number conserving whereas the former is not. As a result, in Eqs. (15-16), BCS pair states live in a Fock space whereas the neutrino states live in a regular Hilbert space. The energy gap plotted in Fig. 5 is a quasipair excitation gap in the case of cold atoms from the ground state, whereas for neutrinos, it is the energy difference between the two mixing eigenstates.

In the BEC limit, all atomic pairs occupy the same state so that the configuration is maximally symmetric. The analog neutrino state should also be maximally symmetric, i.e., all neutrinos should initially have the same flavor. We choose that to be $\nu_e$ to associate the initial state with the neutralization burst of supernova, but any other flavor could have been chosen. Antineutrinos of the opposite flavor ($\bar{\nu}_e$ in our case) could also be added to the picture without breaking the analogy because neutrinos and antineutrinos transform under the conjugate representations, e.g., $\bar{\nu}_e$ transforms in the same way as $\nu_e$. In other words, the BCS-BEC crossover analogy considered here would hold for any maximally symmetric initial state which consists of neutrinos of one type, and antineutrinos of the opposite type. For such a state, the split occurs in the neutrino sector and the behavior of antineutrinos is dictated by the lepton number conservation, i.e., the conservation of $P^0$ defined in Eq. (8) (22, 54). Other neutrino spectral split scenarios involving less symmetric initial configurations, do not correspond to a simple BCS-BEC crossover. Such initial neutrino states may display more complicated behavior including multiple spectral splits [55]. It is an open question whether or not these splits correspond to some other phenomena in the fermion pairing scheme.

In a realistic supernova setting small quantities of non-electron flavor neutrinos would be present in the initial deleptonization phase. Moreover, recent simulations suggest that the neutrino spectral splits are unstable against the inclusion of the multi angle and three flavor effects [23, 56], both of which are important in a real supernova but are omitted in this study. Still, the present analogy can be helpful in understanding some aspects of collective flavor oscillations in supernova. For example, an experimental cold atom system can be used to simulate the possible contribution of entangled many-body states to the collective behaviour of neutrinos [15–17, 57, 58]. Such a contribution will present itself as a deviation of the experimental cold atom system from the results obtained by the mean field approximation which is currently employed by most numerical studies of supernova neutrinos, including the one presented here. Possible departures from adiabaticity [59], the factors affecting the split frequency [60, 62], and even the multi angle

![FIG. 5. Left panel: Eigenvalues of the Hamiltonian matrix for fermion pairs given in Eq. (1) at three different values of the interaction constant $g$ corresponding to three different values of the chemical potential. $\mu/\epsilon_F = 1$ (solid black line) is on the BCS limit, $\mu/\epsilon_F < 0$ is on BEC side (red dotted line), and $\mu = 0$ (blue dashed line) is the boundary between BCS and BEC regimes. Right panel: Eigenvalues of the Hamiltonian matrix for self interacting neutrinos given in Eq. (4) for three different values of the interaction constant $G$ corresponding to positive, zero, and negative values of the Lagrange multiplier $\lambda$. $\lambda = 0$ is the boundary between BCS and BEC regimes.](image)
instability of split behavior mentioned above can be subjects of such an experimental study. Moreover, in the full three flavor mixing case neutrino isospin generalizes to an \(SU(3)\) operator \[16\]. There are pairing scenarios for atomic systems \[63, 64\] and in QCD \[65\] suggesting similar analogies in this case. A possible extension of our analogy in this direction may help us to gain insight about the three flavor instability.

Finally, other quantum many-body systems, in which the relative strength of kinetic and interaction energies is density dependent, might also have been considered in lieu of ultra cold atomic Fermi gases in our discussion.

For instance, in the context of excitonic condensates, the BCS-BEC crossover is driven by density \[66\]. In such electronic systems with long range Coulomb interactions, somewhat counter-intuitively, the low density limit results in a strongly interacting system. For neutrinos, the self interaction term is density dependent because many scattering amplitudes must coherently superpose to generate the effect. Thus, unlike the case of excitonic condensates, the relative strength of neutrino self interactions decreases with density.

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