The Standard Conjectures for the Variety of Lines on a Cubic Hypersurface

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Abstract

The purpose of this note is to prove Grothendieck’s standard conjectures for the Fano variety of lines on a smooth cubic hypersurface in projective space.

Introduction

Let \( X \subset \mathbb{P}^{n+1} \) be a smooth cubic hypersurface over a field \( k \) (with \( n \geq 3 \)). The Fano variety of lines on \( X \), \( F := F(X) \), has proved to be useful in understanding the geometry of the cubic. A thorough study of the Fano variety was completed by Altman and Kleiman in \([1]\), in which they show, among other things, that the Fano variety of \( X \) is smooth and has dimension \( 2(n-2) \). In the case of \( n = 3 \), the Fano variety is a surface of general type which possesses a great many remarkable properties, which were used by Clemens and Griffiths (in their well-known paper \([5]\)) to establish that the smooth cubic threefold is not rational (when \( k = \mathbb{C} \)). For \( n = 4 \), Beauville and Donagi showed in \([3]\) that \( F \) has the structure of a hyper-Kähler variety by establishing that it is deformation equivalent to \( S^{[2]} \), the second punctual Hilbert scheme of a \( K3 \) surface. Quite recently, Galkin and Schinder in \([7]\) were able to prove the following relation in the Grothendieck ring of varieties:

\[
[X^{[2]}] = [X][\mathbb{P}^{n+1}] + L^2[F] \in K_0(Var)
\]

where \( X^{[2]} \) denotes the second punctual Hilbert scheme of \( X \). By applying the Hodge realization map \( K_0(Var) \to K_0(\mathbb{Q}\text{-HS}) \) to \([1]\), they also obtain an (abstract) isomorphism of \( \mathbb{Q}\)-Hodge structures:

\[
H^\ast(F, \mathbb{Q}) \cong \text{Sym}^2 H^n_{prim}(X, \mathbb{Q})(2) \oplus \bigoplus_{k=0}^{n-2} H^n_{prim}(X, \mathbb{Q})(-k+1) \oplus \bigoplus_{k=0}^{2(n-2)} \mathbb{Q}(-k) \oplus a_\oplus
\]

where \( H^n_{prim}(X, \mathbb{Q}) \) is the primitive cohomology and \( a_\oplus \) are some positive integers that will be given in the sequel.

Using decomposition \([2]\) as a guide will allow us to prove the standard conjectures for \( F \). These conjectures were first stated by Grothendieck in his paper \([8]\) and concern the existence of certain fundamental algebraic cycles.
Conjecture 0.1. (Grothendieck) Let $Y$ be a smooth projective variety of dimension $d_Y$ over $k$, $H^*$ denote a Weil cohomology and $H^2_{alg}(Y) \subset H^2(Y)$ denote the image of the cycle class map, $CH^j(Y) \to H^2(Y)$.

1. (Lefschetz standard) For an ample divisor $h \in \text{Pic}(Y)$, there exist correspondences

$$\Lambda^j \in CH^j(Y \times Y)$$

such that $\Lambda^j_\ast : H^{2d_Y-j}(Y) \to H^j(Y)$ is the inverse of the Lefschetz isomorphism:

$$L^j_h := h^j : H^j(Y) \to H^{2d_Y-j}(Y)$$

is induced by an algebraic cycle.

2. (K"unneth) The K"unneth components $\delta_j \in H^j(Y) \otimes H^{2d_Y-j}(Y)$ in the decomposition of the diagonal:

$$\Delta_Y \in H^{2d_Y}(Y \times Y) \cong \bigoplus_{j=0}^{2d_Y} H^j(Y) \otimes H^{2d_Y-j}(Y)$$

are induced by algebraic cycles; i.e., $\delta_j \in H^{2d_Y}_{alg}(Y \times Y)$.

3. ($\sim_{num} \sim_{hom}$) For any $\gamma \in H^{2d_Y}_{alg}(Y)$, if $\gamma \cdot \gamma' = 0$ for all $\gamma' \in H^{2d_Y-j}_{alg}(Y)$, then $\gamma = 0$.

The conjectures are known in a few important cases, but in general remain wide open. They are known to be true, for instance, when $X$ is an Abelian variety ([9]), as well as when the cycle class map is an isomorphism ([10]). (This condition holds for varieties admitting a cellular decomposition such as Grassmannians, toric varieties, etc.) It is also known that the conjectures remain true for hyperplane sections, products, and projective bundles. It is noteworthy that when $k$ is of characteristic 0, Conjecture 0.1 implies the other 2 conjectures (see [9] for details). Also, Arapura proves that the standard conjectures for uniruled threefolds, unirational fourfolds, the moduli space of stable vector bundles over a smooth projective curve, and for the Hilbert scheme $S[n]$ of every smooth projective surface ([2], Corollaries 4.3, 7.2 and 7.5). Finally, Charles and Markman have shown in [4] that the conjectures are true for all smooth projective varieties which are deformation equivalent to a Hilbert scheme of $K3$ surfaces.

Theorem 0.1. The standard conjectures hold for the Fano variety of lines of a smooth cubic hypersurface in $\mathbb{P}^{n+1}_C$ when $H^*$ is singular cohomology with $\mathbb{Q}$ coefficients.

The idea will be to show that the direct sum of Hodge classes $\mathbb{Q}(-k)^{a_k}_2$ in ([2]) is algebraic and that the orthogonal complement of this sum is an appropriate Tate twist of $H^a_{prim}(X)$. This will allow us to construct correspondences $\Gamma_k \in CH^k(F \times F)$ which induce isomorphisms:

$$\Gamma_k : H^{4(n-2)-k}(F) \to H^k(F)$$
for $k \leq 2(n - 2)$. Then, arguments of Kleiman from [9] can be used to deduce the standard conjectures.

**Conventions**

Throughout this note, $X$ will denote a smooth cubic hypersurface over $\mathbb{C}$ and $F$ its Fano variety of lines, $G_{1,n+1}$ the Grassmannian of lines in $\mathbb{P}^{n+1}$ and $H = c_1(\mathcal{O}_X(1)) \in \text{Pic}(X)$. We will let $H^*$ denote singular cohomology with coefficients $\mathbb{Q}$ coefficients. (However, most of the results hold over any field with any choice of Weil cohomology.) We will also let $CH^*$ denote the Chow group with $\mathbb{Q}$ coefficients and for a correspondence $\phi \in CH^*(Z \times Y)$, we will let $\phi^*: H^*(Z) \to H^*(Y)$ denote the usual action on cohomology. For convenience, we will abuse notation by omitting Tate twists. We will also let $\lfloor x \rfloor$ denote the greatest integer $\leq x$.

1 **Lemmas**

In this section, we will prove a few geometric facts about the cubic hypersurface and its Fano variety of lines, which will be necessary for the proof of Theorem 0.1.

The universal line over the Fano variety $F$ gives a projective bundle:

$$p : \mathbb{P}(E) \to F$$

and, hence, a natural imbedding $\iota : F \hookrightarrow G_{1,n+1}$ into the Grassmannian of lines on $\mathbb{P}^{n+1}$. We can use the Grassmannian to account for the Hodge classes which appear in (2); more precisely, we have the following:

**Lemma 1.1.** Let $i : Y \hookrightarrow G_{1,n+1}$ be smooth closed subvariety of the Grassmannian of codimension $r$. Then, for $k \leq 2(n + 2) - 2r$, the pull-back

$$\iota^*: H^k(G_{1,n+1}) \to H^k(Y)$$

is injective.

**Proof.** Note that $H^*(G_{1,n+1})$ is (as a graded algebra) generated by $\{c_1(E), c_2(E)\}$. Since the class of $Y$ is effective in $G_{1,n+1}$, it represents a (non-zero) homogeneous degree $2r$ polynomial in the Chern classes. So, suppose that $\alpha \in H^k(G_{1,n+1})$ is a (non-zero) polynomial in the Chern classes such that $\iota^*\alpha = 0 \in H^k(Y)$. Then, from the projection formula, we deduce that

$$0 = \iota_*\iota^*\alpha = \alpha \cdot Y \in H^{k+2r}(Gr(1, n+1))$$

This means that $\alpha \cdot Y$ is a non-zero polynomial in the Chern classes of degree $= k + 2r \leq 2(n + 2)$. But this is already a contradiction thanks to the following claim:
Claim 1.1. There are no relations among the Chern classes in degrees $\leq 2(n + 2)$.

Proof of Claim. For $2d \leq 2(n + 2)$, the polynomials in the Chern classes are generated by the set

$$\{c_1(\mathcal{E})^d, c_2(\mathcal{E}) \cdot c_1(\mathcal{E})^{d-2}, \ldots, c_2(\mathcal{E})^\lfloor \frac{d}{2} \rfloor \cdot c_1(\mathcal{E})^{d-2\lfloor \frac{d}{2} \rfloor}\}$$

We see then that the rank of $H^{2d}(G_{1,n+1})$ is $\leq \lfloor \frac{d+2}{2} \rfloor$. However, by counting Schubert classes, one deduces that the rank is given by the number of solutions to:

$$n_1 + 2n_2 = d$$

where $n_1, n_2$ are non-negative integers satisfying $n_1 + n_2 \leq n + 2$. This latter is no restriction so long as $d \leq n + 2$. It follows that the rank is exactly $\lfloor \frac{d+2}{2} \rfloor$. Thus, the claim.

Corollary 1.1. For $k \leq 2(n - 2)$, the pull-back

$$\iota^* : H^k(G_{1,n+1}) \rightarrow H^k(F)$$

is injective. Moreover, for $0 \leq k \leq 4(n - 2)$ with $k \neq 2(n - 2)$,

$$\text{rank}(\iota^* H^k(G_{1,n+1})) = \begin{cases} \geq a_k^\frac{4}{2} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

Proof. The first statement follows from Lemma 1.1 and the fact that $F$ has codimension 4 in $G_{1,n+1}$. The second statement is trivially true when $k$ is odd. On the other hand, when $k$ is even, we note that by Corollary 5.7 of [7], we have

$$a_k^\frac{4}{2} = \begin{cases} \lfloor \frac{k+4}{4(n-2)-k+4} \rfloor & \text{for } k < 2(n - 2) \\ \lfloor \frac{k+4}{4(n-2)-k+4} \rfloor & \text{for } k > 2(n - 2) \end{cases}$$

and we observe that $a_k^\frac{4}{2} = a_{\frac{4(n-2)-k}{2}}$. Now, the proof of the claim above shows that for $k < 2(n - 2)$ we have

$$\text{rank}(\iota^* H^k(G_{1,n+1})) = \frac{k+4}{4} = a_k^\frac{4}{2}$$

On the other hand, for $k > 2(n - 2)$ then by the Hard Lefschetz theorem, we have

$$\text{rank}(\iota^* H^k(G_{1,n+1})) \geq \text{rank}(\iota^* H^{4(n-2)-k}(G_{1,n+1}) \cdot c_1(\mathcal{E})^{k-2(n-2)}) = \text{rank}(\iota^* H^{4(n-2)-k}(G_{1,n+1})) = a_{\frac{4(n-2)-k}{2}} = a_k^\frac{4}{2}$$
Corollary 1.2. Whenever $k < n - 2$ or $k > 3(n - 2)$ or $n - 2 \leq k \leq 3(n - 2)$, $k \neq 2(n - 2)$ and $k \equiv n \mod 2$,

$$H^k(F) = \iota^* H^k(G_{1,n+1}) \tag{3}$$

Proof. In all these cases, an inspection of (2) shows that

$$\text{rank}(H^k(F)) = \begin{cases} a \frac{k}{2} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

which gives $\text{rank}(H^k(F)) \leq \text{rank}(\iota^* H^k(G_{1,n+1}))$, according to Corollary 1.1.

So, this must in fact be an equality and this gives (3).

There are also natural projection maps:

$$
\begin{array}{c}
\mathbb{P}(\mathcal{E}) & \xrightarrow{p} & F \\
\downarrow q & & \downarrow \\
X & & 
\end{array} \tag{4}
$$

This gives the well-known cylinder correspondence $\Gamma \in CH^{n-1}(X \times F)$ whose action on cohomology is given by $\Gamma_* = p_* q^*$, as well as its transpose $\Gamma^* = q_* p^*$. Moreover, we have the following result:

Lemma 1.2. Let $H^n_{\text{prim}}(X)$ denote the primitive cohomology of $X$. Then,

$$\Gamma_* = p_* q^* : H^n_{\text{prim}}(X) \to H^{n-2}(F)$$

is injective. When $n$ is odd, this is an isomorphism. When $n$ is even, $\Gamma_* H^n_{\text{prim}}(X)$ and $\iota^* H^{3(n-2)}(G_{1,n+1})$ are orthogonal with respect to the cup product.

Proof. The first statement is perhaps well-known. However, since the author could not find a reference, we proceed as in [3]. Indeed, since $p : \mathbb{P}(\mathcal{E}) \to F$ is a $\mathbb{P}^1$-bundle, there is a decomposition:

$$H^n(\mathbb{P}(\mathcal{E})) = p^* H^n(F) \oplus p^* H^{n-2}(F) \cdot h$$

where $h = c_1(\mathcal{O}(1))$ is the class of the (anti-)tautological bundle. From [6] Chapter 3 (or otherwise), we have

$$p_*(p^* \gamma) = 0, \ p_*(p^* \gamma_{n-2} \cdot h) = \gamma_{n-2}$$

for $\gamma_j \in H^j(F)$. Since $q^*$ is injective, it suffices to show that

$$p^* H^n(F) \cap q^* H^n_{\text{prim}}(X) = 0$$

So, suppose there is some $\gamma \in H^n_{\text{prim}}(X)$ and some $\gamma_n \in H^n(F)$ such that

$$p^* \gamma_n = q^* \gamma \in H^n(\mathbb{P}(\mathcal{E}))$$
We then observe that \( h = q^* H \), from which it follows that
\[
p^* \gamma_{n-2} \cdot h = q^* \gamma \cdot q^* H = q^* (\gamma \cdot H)
\] (5)

Since \( \gamma \in H^n_{prim}(X) \), \( \gamma \cdot H = 0 \in H^{n+2}(X) \) and then (5) gives
\[
\gamma_{n-2} = p_*(p^* \gamma_{n-2} \cdot h) = p_*(q^* (\gamma \cdot H)) = 0
\]
This gives the first statement. When \( n \) is odd, an inspection of the decomposition in (2) reveals that \( H^{n-2}(F) \) has the same dimension as \( H^n(X) \), which implies that \( \Gamma_* \) is an isomorphism in this case. When \( n \) is even, we would like to prove:
\[
\iota^* H^{3(n-2)}(G_{1,n+1}) \cdot \Gamma_* H^n_{prim}(X) = 0
\] (6)

To this end, we have the following commutative diagrams:
\[
\begin{array}{cccc}
F & \xrightarrow{p} & \mathbb{P}(\mathcal{E}) & \xrightarrow{q} & X \\
\downarrow \iota & & \downarrow \tau & & \downarrow j \\
G_{1,n+1} & \xrightarrow{\mathbb{P}} & \mathbb{P}(G_{1,n+1}(\mathcal{E})) & \xrightarrow{\pi} & \mathbb{P}^{n+1}
\end{array}
\] (7)

where \( \mathbb{P}(G_{1,n+1}(\mathcal{E})) \to G_{1,n+1} \) denotes the tautological \( \mathbb{P}^1 \)-bundle and \( j : X \hookrightarrow \mathbb{P}^{n+1} \) is the inclusion. Then, (6) becomes
\[
\iota^* H^{3(n-2)}(G_{1,n+1}) \cdot \Gamma_* H^n_{prim}(X) = \iota^* H^{3(n-2)}(G_{1,n+1}) \cdot p_* q^* (H^n_{prim}(X))
\]
\[
= p_*(p^* \iota^* H^{3(n-2)}(G_{1,n+1}) \cdot q^* H^n_{prim}(X))
\]
\[
= p_*(\mathbb{P} \mathbb{P}^* H^{3(n-2)}(G_{1,n+1}) \cdot q^* H^n_{prim}(X))
\]

where the second equality uses the projection formula and where the third equality uses the commutativity of the left square in (7). Thus, (6) reduces to showing that
\[
p_*(\mathbb{P} \mathbb{P}^* H^{3(n-2)}(\mathbb{P}(G_{1,n+1}(\mathcal{E}))) \cdot q^* H^n_{prim}(X)) = 0
\] (8)

For this, we observe that \( \mathbb{P}(G_{1,n+1}(\mathcal{E})) \to \mathbb{P}^{n+1} \) is the projective bundle \( \mathbb{P}(T \mathbb{P}^{n+1}) \to \mathbb{P}^{n+1} \), where \( T \mathbb{P}^{n+1} \) is the tangent bundle of \( \mathbb{P}^{n+1} \). To see this, note that for \( x \in X(\mathbb{C}) \)
\[
qu^{-1}(x) = \{(y, \ell) \in \mathbb{P}^{n+1} \times G_{1,n+1} \mid x, y \in \ell\}
\]

which one realizes as the fiber over \( x \) of the projective bundle \( \mathbb{P}(T \mathbb{P}^{n+1}) \to \mathbb{P}^{n+1} \) using the exact sequence:
\[
0 \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_{\mathbb{P}^{n+1}}(1) \oplus \mathbb{P}^{n+2} \to T \mathbb{P}^{n+1} \to 0
\]

Then, we have the diagram:
\[
\begin{array}{cccc}
\mathbb{P}(\mathcal{E}) & \xrightarrow{q} & X \\
\downarrow \iota & & \downarrow \tau \\
\mathbb{P}(G_{1,n+1}(\mathcal{E})) & \cong \mathbb{P}(T \mathbb{P}^{n+1}) & \xrightarrow{\pi} & \mathbb{P}^{n+1}
\end{array}
\]
To show that
\[ \tau^* H^{3(n-2)}(\mathbb{P} G_{1,n+1}(E)) \cdot q^* H^n_{\text{prim}}(X) = 0 \] (9)
we observe that
\[ H^{3(n-2)}(\mathbb{P} G_{1,n+1}(E)) = H^{3(n-2)}(\mathbb{P}(\mathbb{P}^{n+1})) = \bigoplus_{k=0}^{n+1} \tau^* H^{3(n-2)-2k}(\mathbb{P}^{n+1}) \cdot c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^k \]

Then, for each \( k \), we have
\[
\tau^* (\tau^* H^{3(n-2)-2j}(\mathbb{P}^{n+1}) \cdot c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^k) \cdot q^* (H^n_{\text{prim}}(X)) = q^* (j^* H^{3(n-2)-2j}(\mathbb{P}^{n+1}) \cdot H^n_{\text{prim}}(X)) \cdot \tau^* c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^k = 0
\]

since \( H^n_{\text{prim}}(X) \cdot H = 0 \). This gives (9) and, hence, (8).

The next lemma gives a characterization of the cohomology of \( F \) except in the middle degree. We first introduce the following notation.

**Notation 1.1.** Denote by \((-,-)_k : H^k(F) \otimes H^k(F) \to \mathbb{Q}\) the pairing:
\[ (\alpha, \alpha')_k := \alpha \cdot \alpha' \cdot c_1(E)^{2(n-2)-k} \]

for \( \alpha, \alpha' \in H^k(F) \). By Poincaré duality and the Hard Lefschetz decomposition, \((-,-)_k\) is non-degenerate.

**Lemma 1.3.** For \( 0 \leq k \neq 4(n-2) \) with \( k \neq 2(n-2) \),
\[ H^k(F) := \begin{cases} 
L^* \Gamma_* H^n_{\text{prim}}(X) & \text{for } n \text{ odd and } k = n - 2 + 2s, 0 \leq s \leq n - 2 \\
L^* \Gamma_* H^n_{\text{prim}}(X) \oplus \epsilon^* H^k(G_{1,n+1}) & \text{for } n \text{ even and } k = n - 2 + 2s, 0 \leq s \leq n - 2 \\
\tau^* H^k(G_{1,n+1}) & \text{otherwise}
\end{cases} \]

where \( L \) denotes the Lefschetz operator for \( c_1(E) \) and where \( L^* \Gamma_* H^n_{\text{prim}}(X) \oplus \epsilon^* H^k(G_{1,n+1}) \) is an orthogonal decomposition with respect to \((-,-)_k\).

**Proof.** For the first, note that, by the Hard Lefschetz theorem,
\[ L^* : H^n-2(F) \to H^k(F) \] (10)
is injective. Then, using (2) it follows that \( H^n-2(F) \) and \( H^k(F) \) have the same rank. The third follows from Corollary (5). For the second, first note that \( \tau^* H^k(G_{1,n+1}) \) and \( L^* \Gamma_* H^n_{\text{prim}}(X) \) are orthogonal for \((-,-)_k\). Indeed, let \( \alpha \in L^* \Gamma_* H^n_{\text{prim}}(X) \) and write \( \alpha = \alpha' \cdot c_1(E)^s \) for \( \alpha' \in \Gamma_* H^n_{\text{prim}}(X) \) and let \( \beta \in \tau^* H^k(G_{1,n+1}) \). Then, \( \beta \cdot c_1(E)^s \in \tau^* H^{3(n-2)}(G_{1,n+1}) \) and by (13) the subspaces \( \tau^* H^{3(n-2)}(G_{1,n+1}) \) and \( \Gamma_* H^n_{\text{prim}}(X) \) are orthogonal. It follows that
\[ (\alpha, \beta)_k = \alpha' \cdot (\beta \cdot c_1(E)^s) = 0 \]

Now, from the injectivity of (10), it follows that
\[ \text{rank}(L^* \Gamma_* H^n_{\text{prim}}(X)) = \text{rank}(\Gamma_* H^n_{\text{prim}}(X)) = \text{rank}(H^n_{\text{prim}}(X)) \]
By Corollary 1.1 we have $\text{rank}(\iota^* H^k(G_{1,n+1})) \geq a_{\frac{k}{2}}$. Moreover, using (2) we obtain
\[
\text{rank}(H^k(F)) = \text{rank}(H^n_{\text{prim}}(X)) + a_{\frac{k}{2}}
\]
Thus, it follows that
\[
\text{rank}(\iota^* H^k(G_{1,n+1})) + \text{rank}(L^s \Gamma_* H^n_{\text{prim}}(X)) \geq \text{rank}(H^k(F)) \quad (11)
\]
Since $\iota^* H^k(G_{1,n+1})$ and $L^s \Gamma_* H^n_{\text{prim}}(X)$ are orthogonal with respect to $(-,-)_k$, we deduce that (11) is an equality and, hence, that
\[
H^k(F) = L^s \Gamma_* H^n_{\text{prim}}(X) \oplus \iota^* H^k(G_{1,n+1})
\]
\[
\square
\]

**Corollary 1.3.** For all $0 \leq k \leq 4(n-2)$ with $k \neq 2(n-2)$,
\[
\text{rank}(\iota^* H^k(G_{1,n+1})) = a_{\frac{k}{2}} = \text{rank}(\iota^* H^{4(n-2)-k}(G_{1,n+1}))
\]
Moreover, the pairing:
\[
\iota^* H^k(G_{1,n+1}) \otimes \iota^* H^{4(n-2)-k}(G_{1,n+1}) \rightarrow \mathbb{Q}
\]
induced by the cup product is non-degenerate.

**Proof.** For the first statement, note by Corollary 1.1 we have
\[
\text{rank}(\iota^* H^k(G_{1,n+1})) \geq a_{\frac{k}{2}}
\]
However, the last few sentences of the proof of Lemma 1.3 show that this must be an equality. It suffices to prove the second statement in the case that $k \leq 2(n-2)$. To this end, suppose that $\alpha \neq 0 \in H^k(G_{1,n+1})$. The proof of Corollary 1.3 then shows that
\[
\alpha \cdot F \neq 0 \in H^{k+8}(Gr(1,n+1))
\]
So, there exists $\beta \neq 0 \in H^{4n-(k+8)}(Gr(1,n+1))$ such that $\alpha \cdot \beta \cdot F \neq 0$. Thus, by the projection formula,
\[
\iota_*(\iota^* \alpha \cdot \iota^* \beta) = \alpha \cdot \beta \cdot F \neq 0 \in H^{4n}(Gr(1,n+1))
\]
from which it follows that $\iota^* \alpha \cdot \iota^* \beta \neq 0$. \qed

\section{Proof of Theorem 0.1}
Since we are working over $\mathbb{C}$, it will suffice to prove that for $0 \leq k \leq 2(n-2)$ there exist correspondences $\Gamma_k \in CH^k(F \times F) = Cor^{k-2(n-2)}(F,F)$ for which
\[
\Gamma_k : H^{4(n-2)-k}(F) \rightarrow H^k(F)
\]
is an isomorphism. Indeed, according to Theorem 2.9 of [9], this implies both the Lefschetz standard and the Künneth conjectures. Then, [8] shows that the Lefschetz standard conjecture implies that $\sim_{\text{num}}=\sim_{\text{hom}}$. What remains then is to construct the required correspondences $\Gamma_k$. To this end, let $L \in CH^{2(n-2)+1}(F \times F) = Cor^1(F,F)$ denote the Lefschetz correspondence for $c_1(E)$. Also, let

$$\delta_{\text{prim}} := \Delta_X - \sum_{0 \leq r \leq n} \frac{1}{3} H^r \times H^{n-r} \in CH^n(X \times X) = Cor^0(X,X)$$

be the correspondence for which $\delta_{\text{prim}} \ast H^*(X) = H^n_{\text{prim}}(X)$. Then, we can define the following correspondence in $CH^k(F \times F)$:

$$\Gamma'_k := \begin{cases} L^s \circ \Gamma \circ \delta_{\text{prim}} \circ t \Gamma \circ L^s & \text{for } k = n - 2 + 2s, 0 \leq s \leq \lfloor \frac{n-2}{2} \rfloor \\ 0 & \text{for all other } k \end{cases} \quad (12)$$

**Lemma 2.1.** When $\Gamma'_k \neq 0$, we have

1. $\Gamma'_{k*}(H^k(F)) = L^t \Gamma^n_{\text{prim}}(X)$.
2. $\Gamma'_{k*}(t^* H^{4(n-2)-k}(G_{1,n+1})) = 0$

**Proof.** The statement of (1) will follow from the fact that

$$\delta_{\text{prim}} \circ t \Gamma \circ L^s : H^{4(n-2)-k}(F) \to H^n_{\text{prim}}(X) \quad (13)$$

is surjective for $k = n - 2 + 2s$ (with $0 \leq s \leq \lfloor \frac{n-2}{2} \rfloor$). To this last end, note that the hard Lefschetz theorem implies that $L^s H^{4(n-2)-k}(F) = H^{3(n-2)}(F)$; i.e.,

$$L^s : H^{4(n-2)-k}(F) \to H^{3(n-2)}(F)$$

is surjective. Moreover, $\delta_{\text{prim}} \circ t \Gamma : H^{3(n-2)}(F) \to H^n_{\text{prim}}(X)$ is surjective since its Poincaré dual

$$t(\delta_{\text{prim}} \circ t \Gamma) = \Gamma \circ \delta_{\text{prim}} : H^n_{\text{prim}}(X) \to H^{n-2}(F)$$

is injective by Lemma 1.2. For the statement of (2), it will suffice to show that

$$t^* H^{4(n-2)-k}(G_{1,n+1}) \subset \ker \{ \delta_{\text{prim}} \circ t \Gamma \circ L^s : H^{4(n-2)-k}(F) \to H^n_{\text{prim}}(X) \} \quad (14)$$

or equivalently that

$$t^* H^{3(n-2)}(G_{1,n+1}) \subset \ker \{ \delta_{\text{prim}} \circ t \Gamma : H^{3(n-2)}(F) \to H^n_{\text{prim}}(X) \}$$

For this, note that

$$\ker \{ \delta_{\text{prim}} \circ t \Gamma : H^{3(n-2)}(F) \to H^n_{\text{prim}}(X) \} = (\Gamma H^n_{\text{prim}}(X))^\perp$$

where $(\ )^\perp$ is with respect to the cup product. The statement of (14) then follows from the last statement of Lemma 1.2. \qed
Lemma 2.2. For $0 \leq k \leq 2(n-2)$, there exists a correspondence $\Gamma''_k \in CH^k(F \times F)$ such that:

1. $\Gamma''_{k*}(\iota^*H^{4(n-2)-k}(G_{1,n+1})) = \iota^*H^k(G_{1,n+1})$

2. If $n$ is even and $k = n - 2 + 2s$ for some $s$, $\Gamma''_{k*}(L^{n-2-s}\Gamma_*H^{n}_{prim}(X)) = 0$

Proof. For $k$ odd, set $\Gamma''_k = 0$. For $k$ even, we let

$$\{\alpha_{k,m}\} \subset \iota^*H^k(G_{1,n+1})$$

be a basis, for which there are lifts $\tilde{\alpha}_{k,m} \in \iota^*CH^{2}\hat{H}(G_{1,n+1})$. Then, we can set

$$\Gamma''_k = \sum_{m=1}^{\lfloor k/2 \rfloor + 1} \tilde{\alpha}_{k,m} \times \tilde{\alpha}_{k,m} \in CH^k(F \times F) = Cor^{k-2(n-2)}(F,F)$$

By Corollary 1.3 there exists a dual basis

$$\{\beta_{k,m}\} \subset \iota^*H^{4(n-2)-k}(G_{1,n+1})$$

with respect to the cup product. A standard computation then shows that

$$\Gamma''_{k*}(\beta_{k,n}) = \delta_{m,n} \cdot \alpha_{k,m} \in \iota^*H^k(G_{1,n+1})$$

where $\delta_{m,n} = 1$ if $m = n$ and 0 otherwise. This proves (1). For (2), we note by Lemma 1.3 that $\iota^*H^k(G_{1,n+1})$ and $L^{n-2-s}\Gamma_*H^n_{prim}(X)$ are orthogonal with respect to the cup product (since $\iota^*H^k(G_{1,n+1})$ and $L^{n-2-s}\Gamma_*H^n_{prim}(X)$ are orthogonal with respect to $(\gamma,-\gamma))$. Thus, for all $m$ and all $\gamma \in L^s\Gamma_*H^n_{prim}(X)$, we have $\alpha_{k,m} \cdot \gamma = 0$, from which it follows that $\Gamma''_{k*}(\gamma) = 0$.

To complete the proof, we set

$$\Gamma_k = \Gamma'_k + \Gamma''_k \in CH^k(F \times F) = Cor^{k-2(n-2)}(F,F) \quad (15)$$

Then, we have the following lemma:

Lemma 2.3. $\Gamma_k : H^{4(n-2)-k}(F) \to H^k(F)$ is an isomorphism for $0 \leq k \leq 2(n-2)$.

Proof. It suffices to show that $\Gamma_k$ is surjective (since the ranks of $H^{4(n-2)-k}(F)$ and $H^k(F)$ are the same). To this end, we observe from Lemma 1.3

$$H^k(F) := \begin{cases} 
L^s\Gamma_*H^n_{prim}(X) \oplus \iota^*H^k(G_{1,n+1}) & \text{for } n \text{ even and } k = n - 2 + 2s \\
L^s\Gamma_*H^n_{prim}(X) & \text{for } n \text{ odd and } k = n - 2 + 2s \\
\iota^*H^k(G_{1,n+1}) & \text{otherwise}
\end{cases} \quad (16)$$

When $H^{4(n-2)-k}(F) = \iota^*H^{4(n-2)-k}(G_{1,n+1})$, surjectivity follows from the fact that $\Gamma'_k = 0$ in this case and Lemma 2.2. When $n$ is odd and $k = n - 2 + 2s$, surjectivity follows from the fact that $\Gamma''_k = 0$ and Lemma 2.1. What remains
then is to prove surjectivity in the case that $n$ is even and $k = n - 2 + 2s$. For this, we define
\[ V' = L^{n-2} \Gamma_* H^n_{\text{prim}}(X), \quad V'' = \iota^* H^{4(n-2)-k}(G_{1,n+1}), \]
\[ W' = L^2 \Gamma_* H^n_{\text{prim}}(X), \quad W'' = \iota^* H^k(G_{1,n+1}) \]
Then, we observe that
\[ \Gamma'_k(V') = W', \quad \Gamma'_k(V'') = 0, \quad \Gamma''_k(V') = 0, \quad \Gamma''_k(V'') = W'' \]
Since $H^{4(n-2)-k}(F) = V' \oplus V''$ and $H^k(F) = W' \oplus W''$, the lemma then follows from the following (essentially trivial) fact from linear algebra:

**Claim 2.1.** Let $V = V' \oplus V''$ and $W = W' \oplus W''$ are vector spaces. Suppose that $T', T'' : V \to W$ are linear maps for which $T'(V') = W'$, $T'(V'') = 0$, $T''(V') = 0$ and $T''(V'') = W''$. Then, $T = T' + T'' : V \to W$ is surjective.

\[ \square \]

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