EXTREMAL FUNCTIONS FOR THE SECOND-ORDER SOBOLEV INEQUALITY ON GROUPS OF POLYNOMIAL GROWTH

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Abstract. In this paper, we prove the second-order Sobolev inequalities on Cayley graphs of groups of polynomial growth. We use the discrete Concentration-Compactness principle to prove the existence of extremal functions for best constants in supercritical cases. As applications, we get the existence of positive ground state solutions to the $p$-biharmonic equations and the Lane-Emden systems.

1. Introduction

Sobolev inequalities are important in the studies of partial differential equations and Riemannian geometry etc. For $N, k, p \geq 1, kp < N, \frac{1}{p'} = \frac{1}{p} - \frac{k}{N}$, we have the classical Sobolev inequality,

$$\|u\|_{p'} \leq C_p \|u\|_{D^{k,p}} \quad \forall u \in D^{k,p}_0(\mathbb{R}^N),$$

where $C_p$ is a constant depending on $N$ and $p$, we omit the dependence of constants $N$ for convenience, and $D^{k,p}_0(\mathbb{R}^N)$ denotes the completion of $C^\infty_0(\mathbb{R}^N)$ in the norm $\|u\|_{D^{k,p}} := \sum_{|\alpha|=k} \int |D^\alpha u|^p dx$. Define

$$D^{k,p}(\mathbb{R}^N) := \{ u \in L^{\frac{Nkp}{N-kp}}(\mathbb{R}^N) : \|u\|_{D^{k,p}} < \infty \},$$

and it is well-known that $D^{k,p}_0(\mathbb{R}^N) = D^{k,p}(\mathbb{R}^N)$. It is sufficient to establish the inequality (1.1) for $k = 1$ as its validity for higher $k$ can be obtained by induction on $k$ [2].

Whether the best constant is attained by some $u \in D^{k,p}_0(\mathbb{R}^N)$, which is called the extremal function, has been intensively investigated in the literature. When $k = 1, p = 1$ the best constant is the isoperimetric constant and the extremal function is the characteristic function of a ball [23, 24]. For $k = 1, p > 1$, the best constants and extremal functions were obtained in [4, 70, 71, 61, 15]. For $k > 1$, P. L. Lions [44, 45, 46, 47] established the Concentration-Compactness method, which provided a new idea for proving the existence of extremal functions. The general idea is as follows. The best constant in the Sobolev inequality (1.1) is given by

$$K := \inf_{u \in D^{k,p}(\mathbb{R}^N)} \|u\|_{D^{k,p}}^p > 0.$$

Take a minimizing sequence $\{u_n\}$ and regard $\{|u_n|^p dx\}$ as a sequence of probability measures. He proved in [44, 45] that there are three cases of the limit of the sequence: compactness, vanishing and dichotomy. Vanishing and dichotomy
are ruled out by the rescaling trick and subadditivity inequality. Therefore, the extremal function exists by the compactness. Since the Concentration-Compactness principle requires weak convergence \( u_n \rightharpoonup u \) in \( D^{k,p}(\mathbb{R}^N) \), this method does not apply the case of \( p = 1 \). And Lieb proved the existence of extremal functions by a compactness technique [22, Lemma 2.7] which is induced by the Brézis-Lieb lemma [10].

In recent years, people paid attention to the analysis on discrete groups. Since the Sobolev inequalities are useful analytical tools, they have been extended to the discrete setting [14, 58]. For finite graphs, Sobolev inequalities and sharp constants have been obtained by [55, 56, 67, 69, 80, 81]. In this article, we generalize our previous results on the existence of extremal functions for the first-order Sobolev inequality to higher order inequalities on Cayley graphs of groups of polynomial growth.

Let \((G, S)\) be a Cayley graph of a group \( G \) with a finite symmetric generating set \( S \), i.e. \( S = S^{-1} \). There is a natural metric on \((G, S)\) called the word metric, denoted by \( d^G \). Let \( B^S_p(n) := \{ x \in G \mid d^G(p, x) \leq n \} \) denote the closed ball of radius \( n \) centered at \( p \in G \) and denote \( | B^S_p(n) | := 2B^S_p(n) \) as the volume (i.e. cardinality) of the set \( B^S_p(n) \). When \( e \) is the unit element of \( G \), the volume \( \beta_S(n) := | B^S_e(n) | \) of \( B^S_e(n) \) is called the growth function of the group, see [49, 50, 78, 28, 26, 27, 72]. A group \( G \) is called of polynomial growth, or of polynomial volume growth, if \( \beta_S(n) \leq C n^A \), for any \( n \geq 1 \) and some \( A > 0 \), which is independent of the choice of the generating set \( S \) since the metrics \( d^G \) and \( d^{S_1} \) are bi-Lipschitz equivalent for different finite generating sets \( S \) and \( S_1 \). By Gromov’s theorem and Bass’ volume growth estimate of nilpotent groups [6], for any group \( G \) of polynomial growth there are constants \( C_1(S), C_2(S) \) depending on \( S \) and \( N \in \mathbb{N} \) such that for any \( n \geq 1 \),

\[
C_1(S)n^N \leq \beta_S(n) \leq C_2(S)n^N,
\]

where the integer \( N \) is called the homogeneous dimension or the growth degree of \( G \). Since \( N \) is sort of dimensional constant of \( G \), we always omit the dependence of \( N \) in various constants.

In this paper, we consider the Cayley graph \((G, S)\) of a group of polynomial growth with the homogeneous dimension \( N \geq 3 \). We denote by \( \ell^p(G) \) the \( \ell^p \)-summable functions on \( G \) and by \( D_0^{k,p}(G) \) (\( k = 1, 2 \)) \( (\text{resp. } \tilde{D}_0^{2,p}(G)) \) the completion of finitely supported functions in the \( D^{k,p} \) \( (\text{resp. } \tilde{D}^{2,p}) \) norm, where \( \|u\|_{D^{k,p}(G)} := \|\nabla^k u\|_{\ell^p(G)} \), \( \|u\|_{\tilde{D}^{2,p}(G)} := \|\Delta u\|_{\ell^p(G)} \) and \( \|u\|_{\tilde{D}^{2,p}(G)} := \|\nabla^2 u\|_{\ell^p(G)} \), see Section 2 for details. Using Dungey’s result [22, Theorem 1], we prove the second-order norm defined by the Hessian and Laplace operators are equivalent on a nilpotent group \( G \), hence \( D_0^{2,p}(G) = \tilde{D}_0^{2,p}(G) \), see Lemma [6]. Analogous to the continuous setting, set

\[
D^{k,p}(G) := \left\{ u \in \ell^{N-k,p}(G) : \|u\|_{D^{k,p}(G)} < \infty \right\},
\]

\[
\tilde{D}^{2,p}(G) := \left\{ u \in \ell^{N-2,p}(G) : \|u\|_{\tilde{D}^{2,p}(G)} < \infty \right\}.
\]

For \( \mathbb{R}^N \), the equivalence \( D_0^{k,p}(\mathbb{R}^N) = D^{k,p}(\mathbb{R}^N) \) follows from the approximation of \( f \in D^{k,p}(\mathbb{R}^N) \) by compact supported functions using nice cutoff functions. Inspired by the continuous setting, for a concrete example, integer lattice graph \( \mathbb{Z}^N \), the equivalence \( D_0^{2,p}(\mathbb{Z}^N) = D^{2,p}(\mathbb{Z}^N) \) can be proved by constructing cutoff functions. We prove that any \( D^{1,p}(\mathbb{Z}^N) \) function can be approximated by a "linear"
cutoff function of the logarithmic function (Theorem 7) and Cosco, Nakajima and Schweiger give a more explicit estimate in [17]. However, the "linear" cutoff function does not work for $D^{2,p}(\mathbb{Z}^N)$ since the Laplacian is "bad" when it comes to the sharp corner. Different from the continuous setting which can be mollified locally, we find a way to fix this by taking a "close to linear" third-order polynomial function being smooth at the sharp corner. Moreover, the Euclidean distance is necessary for cutoff functions since its Laplacian is decreasing while the combinatorial distance is not. Then using the discrete chain rule (Lemma 9) we can get a good estimate of the second-order norm, and finally construct desired cut-off functions, see Theorem 11. Then we prove the equivalence $D^{k,p}_0(G) = D^{k,p}(G)$ on general Cayley graphs $(G,S)$, see Theorem 8 and Theorem 12. First $D^{k,p}_0(G) \subseteq D^{k,p}(G)$ follows from Sobolev inequalities, then we prove the converse direction via the parabolicity theory on graphs [33, Corollary 2.6] for $k = 1$ and the functional analysis method by checking the Laplace operator is an isometry from $D^{0,p}_0(G)$ (also $D^{2,p}(G)$) to $\ell^p(G)$.

For special case $p = 2$, we extend the results to the Cayley graph $(G,S)$ satisfying $\beta_S(n) \geq C(S)n^N, \forall n \geq 1$. We prove the Hodge decomposition theorem on 1-forms on edges (Theorem 18) and use it to prove $D^{1,2}_0 = D^{1,2}$ on $G$ (Corollary 19). Moreover, we prove $D^{2,2} = D^{2,2}$ on the groups satisfying the second-order Sobolev inequality including polynomial growth groups and non-amenable groups.

Using the boundedness of Riesz transforms [22] and the functional calculus in Banach spaces [30] we get the following discrete second-order Sobolev inequality by induction on the first-order Sobolev inequality:

\begin{equation}
\|u\|_{\ell^q} \leq C_{p,q}\|u\|_{D^{2,p}}, \forall u \in D^{2,p}(G),
\end{equation}

where $N \geq 3, 1 < p < \frac{N}{2}$, $q = p^{**} := \frac{Np}{N - 2p}$, see Theorem 10. Since $\ell^p(G)$ embeds into $\ell^q(G)$ for any $q > p$, one verifies that the inequality (1.2) holds for $q \geq p^{**}$. Recalling the continuous setting, it is called subcritical for $q < p^{**}$, critical for $q = p^{**}$ and supercritical for $q > p^{**}$ for Sobolev inequalities. Therefore, (1.2) holds in both critical and supercritical cases.

The optimal constant in the Sobolev inequality (1.2) is given by

\begin{equation}
K := \inf_{u \in D^{2,p}(G)} \frac{\|u\|_{D^{2,p}}}{\|u\|_q}.
\end{equation}

In order to prove that the infimum is achieved, we consider a minimizing sequence $\{u_n\} \subset D^{2,p}(G)$ satisfying

\begin{equation}
\|u_n\|_q = 1, \|u_n\|_{D^{2,p}} \to K, n \to \infty.
\end{equation}

We want to prove $u_n \to u$ strongly in $D^{2,p}(G)$, which yields that $u$ is a minimizer.

We prove the following main results.

**Theorem 1.** For $N \geq 3, 1 < p < \frac{N}{2}, q > p^{**} = \frac{Np}{N - 2p}$, let $\{u_n\} \subset D^{2,p}(G)$ be a minimizing sequence satisfying (1.4). Then there exists a sequence $\{x_n\} \subset G$ and $v \in D^{2,p}(G)$ such that the sequence after translation $\{u_n(x_n) := u_n(x_nx)\}$ contains a convergent subsequence that converges to $v$ in $D^{2,p}(G)$. And $v$ is a minimizer for $K$.

**Remark.** (1) This result implies that best constant can be obtained in the supercritical case.
(2) If we define the second-order norm by the Hessian operator, i.e., $\tilde{D}^2_p$ (see Section 2), then the second-order Sobolev inequalities also hold, and the strong convergence and existence of minimizer in $\tilde{D}^2_p(G)$ follows from a similar argument.

(3) In particular, for $p = 1$, we can get the same results in $\tilde{D}^2_1(\mathbb{Z}^N)$, see Section 7.

We will provide two proofs for Theorem 1. In the continuous setting, Lions proved the existence of extremal functions by Concentration-Compactness principle [46, Lemma I.1.] and a rescaling trick [46, Theorem I.1, (17)]. And Lieb in [42] used a compactness technique and the rearrangement inequalities. Following Lions, the main idea of proof I is to prove a discrete analog of Concentration-Compactness principle, see Lemma 15. However, we don’t know proper notion of rescaling and rearrangement tricks on graphs to exclude the vanishing case of the limit function. Inspired by [35], for the supercritical case, we prove that the translation sequence has a uniform positive lower bound at the unit element, see Lemma 16, which excludes the vanishing case. The idea of proof II is based on a compactness technique by Lieb [42, Lemma 2.7] and the nonvanishing of the limit of translation sequence.

As applications, we get the existence of positive ground state solutions for the discrete nonlinear $p$-biharmonic equations and the Lane–Emden systems in supercritical cases. They have certain physical backgrounds, such as traveling waves in a suspension bridge [21] and the static deflection of an elastic plate [1], and have been well studied in continuous setting, see for example [43, 43, 74, 66, 65, 12, 59, 52, 68, 33, 48, 27, 19, 73, 75, 31, 41, 54, 53, 57, 29, 36, 37, 38] and references therein. For discrete setting, there are few results of the fourth order nonlinear equations [32, 79].

First by Theorem 1, we can get the existence of positive solutions to Euler-Lagrange equation of (1.2) as follows.

**Corollary 2.** For $N \geq 3, 1 < p < \frac{N}{2}, q > p^*$, there is a positive ground state solution of the nonlinear $p$-biharmonic equation

$$\Delta (|\Delta u|^{p-2} \Delta u) - |u|^{q-2} u = 0, \forall u \in D^2_p(G),$$

where $\Delta$ is the Laplace on graphs defined as $\Delta u(x) := \sum_{y \sim x} (u(y) - u(x))$.

Consider the Lane-Emden system

$$\begin{cases} 
-\Delta u = |v|^{p'-2} v \\
-\Delta v = |u|^{q-2} u \\
u \in D^2_p(G), v \in D^{2, q'}(G),
\end{cases}$$

where $N \geq 3, p, q > 1, p' = \frac{p}{p-1}, q' = \frac{q}{q-1}$, $(p', q)$ lies on the critical hyperbola, that is,

$$\frac{1}{p'} + \frac{1}{q} = \frac{N - 2}{N}.$$ 

Indeed, $(u, v)$ is a solution of (1.6) if and only if $u$ is a solution of (1.5) with $v := -|\Delta u|^{p-2} \Delta u$ in $\mathbb{R}$ critical cases. This relation is sometimes called reduction-by-inversion, see [16] Lemma 2.1 [9, 20, 62] for continuous setting. We can also prove the equivalence in supercritical cases for discrete setting, see Lemma 17.

Hence, by Corollary 2 we have a positive solution of the Lane-Emden system.
Corollary 3. For \( N \geq 3, \ p, q > 1, \ \frac{1}{p} + \frac{1}{q} < \frac{N-2}{N} \), there is a pair of positive solution \((u, v)\) for the Lane-Emden system \((E)\).

Conjecture 4. According to the results in continuous cases \[46\] Corollary 1.2 
\[43, 13, 74, 64, 12, 59, 52, 68, 19, 54, 75, 73\], we conjecture that \((E')\) and \((E)\) have positive solutions in critical cases and the non-negative solutions in subcritical cases are trivial.

The paper is organized as follows. In Section 2, we recall some basic facts and prove some useful lemmas. In Section 3, we study the important equivalence of Sobolev spaces and prove the discrete second-order Sobolev inequality using the boundedness of Riesz transforms and the functional calculus. In Section 4, we introduce the Brézis-Lieb lemma and prove the Concentration-Compactness principle on \(G\). In Section 5, we prove a key lemma to exclude the vanishing case and give two proofs for Theorem \(1\). As applications, we get the existence results for \(p\)-biharmonic equations and Lane-Emden systems in Section 6. In Section 7, we prove the Hodge decomposition theorem on 1-forms on edges for \(p = 2\) and study the existence of extremal functions for \(p = 1\).

2. Preliminaries

Let \(G\) be a countable group. It is called finitely generated if it has a finite generating set \(S\). We always assume that the generating set \(S\) is symmetric, i.e. \(S = S^{-1}\). The Cayley graph of \((G, S)\) is a graph structure \((V, E)\) with the set of vertices \(V = G\) and the set of edges \(E\) where for any \(x, y \in G, xy \in E\) (also denoted by \(x \sim y\)) if \(x = ys\) for some \(s \in S\). The Cayley graph of \((G, S)\) is endowed with a natural metric, called the word metric \[11\]: For any \(x, y \in G\), the distance between them is defined as the length of the shortest path connecting \(x\) and \(y\) by assigning each edge of length one,

\[
d^S(x, y) = \inf\{k : x = x_0 \sim \cdots \sim x_k = y\}.
\]

One easily sees that for two generating sets \(S \) and \(S_1\) the metrics \(d^S\) and \(d^{S_1}\) are bi-Lipschitz equivalent, i.e. there exist two constants \(C_1(S, S_1), C_2(S, S_1)\) such that for any \(x, y \in G\)

\[
C_1(S, S_1)d^{S_1}(x, y) \leq d^S(x, y) \leq C_2(S, S_1)d^{S_1}(x, y).
\]

Let \(B^S_p(n) := \{x \in G \mid d^S(p, x) \leq n\}\) denote the closed ball of radius \(n \) centered at \(p \in G\). By the group structure, it is obvious that \(|B^S_p(n)| = |B^S_q(n)|\), for any \(p, q \in G\). The growth function of \((G, S)\) is defined as \(\beta_S(n) := |B^S_e(n)|\) where \(e\) is the unit element of \(G\). A group \(G\) is called of polynomial growth if there exists a finite generating set \(S\) such that \(\beta_S(n) \leq Cn^A\) for some \(C, A > 0\) and any \(n \geq 1\). One checks that this definition is independent of the choice of the generating set \(S\). Thus, the polynomial growth is indeed a property of the group \(G\). In this paper, we consider the Cayley graph \((G, S)\) of a group of polynomial growth

\[
C_1(S)n^N \leq \beta_S(n) \leq C_2(S)n^N,
\]

for some \(N \in \mathbb{N}\) and any \(n \geq 1\), where \(N\) is called the homogenous dimension of \(G\).

We denote by \(C(G)\) the space of functions on \(G\). The support of \(u \in C(G)\) is defined as \(\text{supp}(u) := \{x \in G : u(x) \neq 0\}\). Let \(C_0(G)\) be the set of all functions
with finite support. For any \( u \in C(G) \), the \( \ell^p \) norm of \( u \) is defined as

\[
\|u\|_{\ell^p(G)} := \begin{cases} 
\left( \sum_{x \in G} |u(x)|^p \right)^{1/p} & 0 < p < \infty, \\
\sup_{x \in G} |u(x)| & p = \infty,
\end{cases}
\]

and we shall write \( \|u\|_{\ell^p(G)} \) as \( \|u\|_p \) for convenience. The \( \ell^p(G) \) space is defined as

\[
\ell^p(G) := \{ u \in C(G) : \|u\|_{\ell^p(G)} < \infty \}.
\]

For any \( u \in C(G) \), the difference operator is defined as for any \( x \sim y \)

\[
\nabla_{xy} u = u(y) - u(x).
\]

In particular, for the Cayley graph \((G, S)\) with symmetric generating set \( S := \{s_1, s_2, \cdots, s_m\} \), we can define the difference in \( i \)-th direction as

\[
\partial_i u(x) := u(xs_i) - u(x).
\]

Let

\[
|\nabla u(x)|_p := \left( \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p}
\]

be the \( p \)-norm of the gradient of \( u \) at \( x \).

We define the Laplace operator as

\[
\Delta u(x) := \sum_{y \sim x} (u(y) - u(x)).
\]

We write \( p \)-norm of the Hessian of \( u \) as

\[
\|\nabla^2 u\|_p := \left( \sum_{i,j=1}^m \sum_{x \in G} |\partial_j \partial_i u(x)|^p \right)^{1/p} = \left( \sum_{i,j=1}^m \|\partial_j \partial_i u(x)\|_p^p \right)^{1/p}.
\]

The \( D^{k,p} \) \((k = 1, 2)\) norms of \( u \) are given by

\[
\|u\|_{D^{1,p}(G)} := \left( \sum_{x \in G} \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p},
\]

\[
\|u\|_{D^{2,p}(G)} := \left( \sum_{x \in G} \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p} = \left( \sum_{x \in G} \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p}.
\]

We use the Hessian operator to define \( D^{2,p} \) norm as

\[
\|u\|_{{\tilde D}^{2,p}(G)} := \|\nabla^2 u\|_p.
\]

Since \( G \) is a regular graph, that is, the degree \( d_x := \# \{y : y \sim x\} \) is constant for \( x \in G \), we have

\[
\|\nabla u\|_p := \|\nabla u(x)\|_p \sim \|\nabla u(x)\|_1 \sim \|\nabla u(x)\|_1^{1/p} = \|u\|_{D^{1,p}}.
\]

In this paper, we use \( a \lesssim b \) to denote \( a \leq Cb \) for some \( C > 0 \) and \( a \sim b \) to denote \( a \lesssim b \lesssim a \).
We define $D^{k,p}_0(G)$ ($k = 1, 2$) and $\tilde{D}^{2,p}_0(G)$ as the completion of $C_0(G)$ in $D^{k,p}$ norm and $\tilde{D}^{2,p}$ norm respectively. And define

$$D^{k,p}(G) := \left\{ u \in \ell^{N_k,p}(G) : \|u\|_{D^{k,p}(G)} < \infty \right\},$$

$$\tilde{D}^{2,p}(G) := \left\{ u \in \ell^{N_2,p}(G) : \|u\|_{\tilde{D}^{2,p}(G)} < \infty \right\}.$$

Let $c_0(G)$ be the completion of $C_0(G)$ in $\ell^\infty$ norm. It is well-known that $\ell^1(G) = (c_0(G))^\ast$. We set

$$\|\mu\| := \sup_{u \in c_0(G), \|u\| = 1} \langle \mu, u \rangle, \quad \forall \mu \in \ell^1(G).$$

By definition,

$$\mu_n \overset{w^*}{\longrightarrow} \mu \text{ in } \ell^1(G) \text{ if and only if } \langle \mu_n, u \rangle \longrightarrow \langle \mu, u \rangle, \forall u \in c_0(G).$$

In the proof, we will use the following results, see [18].

**Fact.** (a) Every bounded sequence of $\ell^1(G)$ contains a $w^*$-convergent subsequence.

(b) If $\mu_n \overset{w^*}{\longrightarrow} \mu$ in $\ell^1(G)$, then $\mu_n$ is bounded and

$$\|\mu\| \leq \lim_{n \to \infty} \|\mu_n\|.$$

(c) If $\mu \in \ell^1(G) := \{ \mu \in \ell^1(G) : \mu \geq 0 \}$, then

$$\|\mu\| = \langle \mu, 1 \rangle.$$

Next we give a discrete chain rule.

**Lemma 5.** For a weighted graph $G = (V, w, m)$, where $w_{xy}, m(x)$ denote the weights assigned to the edge $xy \in E$ and to the vertex $v$, respectively. Let $\phi \in C^2(\mathbb{R})$, $f \in C(V)$, then for some $\xi \in [m, M]$,

$$\Delta \phi(f)(x) = \phi'(f(x))\Delta f(x) + \phi''(\xi)\Gamma f(x),$$

where $\Delta f(x) := \sum_{y \sim x} \frac{w_{xy}}{m(x)} (f(y) - f(x))$, $\Gamma f(x) := \frac{1}{2} \sum_{y \sim x} \frac{w_{xy}}{m(x)} | f(y) - f(x) |^2$ and

$$m := \min_{y \in E} f(y), \quad M := \max_{y \in E} f(y).$$

**Proof.** With $s = f(x)$ and $t_y = f(y) - f(x)$, Taylor’s theorem gives

$$\phi(s + t_y) = \phi(s) + t_y \phi'(s) + \frac{1}{2} t_y^2 \phi''(\xi_y)$$

for some $\xi_y \in [s \wedge (s + t_y), s \vee (s + t_y)]$. Summing up gives

$$\Delta \phi(f)(x) = \sum_{y \sim x} \frac{w_{xy}}{m(x)} (\phi(s + t_y) - \phi(s))$$

$$= \sum_{y \sim x} \frac{w_{xy}}{m(x)} \left( t_y \phi'(s) + \frac{1}{2} t_y^2 \phi''(\xi_y) \right)$$

$$= \sum_{y \sim x} \frac{w_{xy}}{m(x)} \left( t_y \phi'(s) + \frac{1}{2} t_y^2 \phi''(\xi) \right)$$

$$= \phi'(f(x))\Delta f(x) + \phi''(\xi)\Gamma f(x).$$
where in the third identity, we can choose some \( \xi \) between minimum and maximum of the \( \xi_y \) by continuity of \( \phi'' \), that is,

\[
m \leq \xi \leq M.
\]

This finishes the proof. \( \square \)

3. Sobolev spaces and second-order Sobolev inequalities

In this section, inspired by the results in the continuous setting, we study some important properties of Sobolev spaces defined in Section 2. By induction on the discrete first-order Sobolev inequalities, we prove the second-order Sobolev inequalities using the boundedness of Riesz transforms and the functional calculus.

First, we introduce a key lemma of norm equivalence using Dungey’s result [22, Theorem 1].

**Lemma 6.** Let Cayley graph \((G, S)\) is a nilpotent group of polynomial growth with symmetric generating set \( S := \{s_1, s_2, \ldots, s_m\} \), for \( p > 1 \), if \( u \in C_0(G) \), then

\[
\|u\|_{\widetilde{D}^2,p(G)} \sim \|u\|_{D^2,p(G)}.
\]

Moreover, \( D^2_p(G) = \widetilde{D}^2_p(G) \).

**Proof.** Since \(-\Delta u(x) = \frac{1}{2} \sum_{i=1}^{m} \partial_{s_i} u(x)\), then

\[
\|\Delta u\|_p = \left( \sum_x \left( \frac{1}{2} \sum_{i=1}^{m} \partial_{s_i} u(x) \right)^p \right)^{1/p} \\
\leq \left( \sum_x \sum_{i=1}^{m} \left| \partial_{s_i} u(x) \right|^p \right)^{1/p} \\
\leq \left( \sum_x \sum_{i,j=1}^{m} \left| \partial_{s_i} \partial_{s_j} u(x) \right|^p \right)^{1/p} = \|\nabla^2 u\|_p.
\]

Using Dungey’s result in [22, Theorem 1], we know for any \( i, j \),

\[
\|\partial_{s_i} \partial_{s_j} u(x)\|_p \lesssim \|\Delta u\|_p,
\]

which implies

\[
\|\nabla^2 u\|_p \lesssim \|\Delta u\|_p.
\]

Hence the norms defined by the Laplace and Hessian operator are equivalent. So that the spaces defined by the completion of finitely supported functions in the equivalent norms coincide. \( \square \)

**Remark.** The assumption of nilpotent group is necessary, if \( G \) is not nilpotent then, (3.1) may fail, see [3, Section 1].

For a Cayley graph it is well known that if \( \beta_S(n) \geq C(S)n^N, \forall n \geq 1 \) for \( N \geq 3 \), then the first-order Sobolev inequality holds [34, Theorem 3.6],

\[
\|u\|_{\ell^p} \leq C_{p,q} \|u\|_{D^{1,p}}, \forall u \in D^1_p(G)
\]

where \( 1 \leq p < N, q \geq p^* := \frac{Np}{N-p} \). In fact, this follows from a standard trick and the isoperimetric estimate [77, Theorem 4.18].
We consider a concrete example of Cayley graphs of polynomial growth with the homogeneous dimension $N$, the integer lattice graph $\mathbb{Z}^N$, which serve as the discrete counterpart of $\mathbb{R}^N$. It consists of the set of vertices $V = \mathbb{Z}^N$ and the set of edges

$$E = \left\{ \{x, y\} : x, y \in \mathbb{Z}^N, \sum_{i=1}^{N} |x_i - y_i| = 1 \right\}.$$  

Analogous the continuous setting, we construct cutoff functions to prove the results on $\mathbb{Z}^N$.

**Theorem 7.** If $N \geq 3$, $1 < p < N$, then $D_{0,1}^{1,p}(\mathbb{Z}^N) = D_{0}^{1,p}(\mathbb{Z}^N)$.

**Proof.** For any $u \in D_{0,1}^{1,p}(\mathbb{Z}^N)$, there exists a sequence $\{u_n\} \subset C_0(\mathbb{Z}^N)$ such that $u_n \rightarrow u$ in $D_{0}^{1,p}$.

And $u \in \ell^{\frac{N}{p}}(\mathbb{Z}^N)$ by the Sobolev inequality (3.2). Hence, $D_{0,1}^{1,p}(\mathbb{Z}^N) \subseteq D_{0}^{1,p}(\mathbb{Z}^N)$.

In the other direction, the key is to find suitable cutoff functions $\eta_n(x) \in C_0(\mathbb{Z}^N)$. For any $u \in D_{0}^{1,p}(\mathbb{Z}^N)$, set $u_n := u\eta_n \in C_0(\mathbb{Z}^N)$, then by Hölder inequalities

$$\|u_n - u\|^p_{D_{0,1}^{1,p}(\mathbb{Z}^N)} = \sum_{x \in \mathbb{Z}^N} \sum_{y \sim x} |\nabla_{xy}(u\eta_n) - \nabla_{xy}u|^p$$

$$= \sum_{x \in \mathbb{Z}^N} \sum_{y \sim x} |\nabla_{xy}u\eta_n + \nabla_{xy}\eta_n u(x) - \nabla_{xy}u|^p$$

$$\lesssim \sum_{x \in \mathbb{Z}^N} |\nabla u(x)|^p \max_{y \sim x} |\eta_n(y) - 1|^p + \|\nabla \eta_n\|^p_{\ell^N} \|u\|^p_{\ell^p}.$$  

Hence, if the cutoff functions satisfy

$$\eta_n$$ is uniformly bounded,

$$\eta_n \rightarrow 1$$ pointwise on $\mathbb{Z}^N$

and

$$\|\nabla \eta_n\|^p_{\ell^N(\mathbb{Z}^N)} \rightarrow 0,$$

then we can prove the other direction by the dominated convergence theorem.

Let $r > 1$, and $R \gg r$ be large enough. Define

$$\eta(x) := 1 \wedge \frac{\log R - \log |x|}{\log R - \log r} \vee 0,$$

where $|x|$ stands for the Euclidean distance. Then

$$\|\nabla \eta\|^N_{\ell^N(\mathbb{Z}^N)} = \sum_{r \leq |x| \leq R} \sum_{y \sim x} |\nabla_{xy}\eta|^N$$

$$\lesssim \left( \log \frac{R}{r} \right)^{-N} \sum_{r - 1 \leq |x| \leq R + 1} \sum_{y \sim x} |\log |x| - \log |y||^N$$

$$\lesssim \left( \log \frac{R}{r} \right)^{-N} \sum_{r - 1 \leq |x| \leq R + 1} |x|^{-N},$$
where the second inequality follows from the mean value theorem. In the following, we can estimate the summation on $\mathbb{Z}^N$ by the integral on $\mathbb{R}^N$,

$$\sum_{r-1 \leq |x| \leq R+1} |x|^{-N} \lesssim \sum_{r-1 \leq |x| \leq R+1} \int |x|^{-N} dt$$

$$\lesssim \sum_{r-1 \leq |t| \leq R+1} \int |t|^{-N} dt$$

$$\lesssim \int_{\tilde{B}(r+2) \setminus \tilde{B}(r-2)} |t|^{-N} dt,$$

where $S_x(\frac{1}{2}) := \{ t \in \mathbb{R}^N : |t_i - x_i| < \frac{1}{2}, 1 \leq i \leq N \}$ is the Euclidean cube, and $\tilde{B}(r)$ is the Euclidean ball in $\mathbb{R}^N$ with radius of $r$ and centered at the origin. Hence,

$$\|\nabla \eta\|_{l^N(\mathbb{Z}^N)} \lesssim \left( \frac{\log R}{r} \right)^{-N} \log \frac{R}{r} = O \left( \left( \frac{\log R}{r} \right)^{1-N} \right).$$

For fixed $r$, letting $R \to \infty$ when $N \geq 3$, we have that $\|\nabla \eta\|_N \to 0$. Since $\eta(x)$ is uniformly bounded and tends to 1 pointwise, we prove $u \in D_0^{1,p}(\mathbb{Z}^N)$ by (3.3). $\square$

**Remark.** For $D_1^{1,p}(\mathbb{Z}^N)$, one can replace the Euclidean distance $| \cdot |$ in the definition of cutoff function $\eta$ by any norm $\| \cdot \|_r$ with $r > 0$, and Cosco, Nakajima and Schweiger give a more explicit estimate of the $p$-capacity [17].

For a Cayley graph $(G, S)$ of polynomial growth with the homogeneous dimension $N$, it is difficult to construct desired cutoff functions. We find an alternative method to prove the result by the parabolicity theory [64].

**Theorem 8.** If $N \geq 3$, $1 < p < N$, then $D_0^{1,p}(G) = D^{1,p}(G)$.

**Proof.** By the same argument as Theorem 7, $D_0^{1,p}(G) \subseteq D^{1,p}(G)$ follows from the Sobolev inequalities (3.2).

In the other direction, by the parabolicity theory on graphs [63, Corollary 2.6] we know that $G$ is $N$-parabolic which implies the $N$-capacity is zero. Hence, we can get a uniformly bounded sequence $\eta_n(x)$ with finite support satisfying $\eta_n(x) \to 1$ pointwise and $\|\nabla \eta_n\|_{l^N(G)} \to 0$. Set $u_n := u \eta_n \in C_0(G)$, then by Hölder inequalities

$$\|u_n - u\|_{D^{1,p}(G)}^p = \sum_{x \in G} \sum_{y \sim x} |\nabla u_n(x) + \nabla \eta_n u(x) - \nabla \eta_n u(x)|^p$$

$$\lesssim \sum_{x \in G} |\nabla u(x)|^p \max_{y \sim x} |\eta_n(y)| - 1 |^p + \|\nabla \eta_n\|_{l^N}^p \|u\|_{l^p}^p \to 0.$$

That is, $D^{1,p}(G) \subseteq D_0^{1,p}(G)$. $\square$

**Remark.** For $N \geq 3$, if the Cayley graph $(G, S)$ satisfies $\beta_S(n) \geq C(S)n^N$, $\forall n \geq 1$, then we can prove $D_0^{1,2}(G) = D^{1,2}(G)$ using the Hodge decomposition theorem on 1-forms on edges, see Theorem 18 and Corollary 19.

By the properties of semigroup [3, Chapter 5], we have the following lemma.
Lemma 9. For any $u \in \ell^p(G)$, $p \in (1, \infty)$,

$$Mu := \frac{1}{|F(-\frac{1}{2})|} \int_0^\infty (e^{t\Delta} u - u) t^{-\frac{3}{2}} dt,$$

where $e^{t\Delta}$ is the semigroup of $\Delta$. Then $M$ is a bounded linear operator in $\ell^p(G)$, and

$$L^\frac{1}{2} u := (-\Delta)^{\frac{1}{2}} u = Mu, \forall u \in \ell^p(G).$$

Proof. Obviously, $M$ is well-defined and linear. And for any $u \in C_0(G)$,

$$Mu \sim \int_0^1 \left( \int_0^t e^{s\Delta} \Delta u ds \right) t^{-\frac{3}{2}} dt + \int_1^\infty (e^{t\Delta} u - u) t^{-\frac{3}{2}} dt.$$

Then,

$$\|Mu\|_p \lesssim \int_0^1 \left( \int_0^t \|e^{s\Delta}\|_p \|\Delta u\|_p ds \right) t^{-\frac{3}{2}} dt + \int_1^\infty \left( \|e^{t\Delta}\|_p \|u\|_p + \|u\|_p \right) t^{-\frac{3}{2}} dt \lesssim \|u\|_p \int_0^1 t^{-\frac{3}{2}} dt + \|u\|_p \int_1^\infty t^{-\frac{3}{2}} dt \lesssim \|u\|_p.$$

Hence, $M$ is a bounded operator in $\ell^p(G)$. And these definitions are consistent for different $p$, i.e. two of them agree on their common domain since the extensions of semigroup $e^{t\Delta}$ are consistent in different $\ell^p(G)$, see [40].

By the spectral mapping theorem in Banach spaces [30, Proposition 3.1.1], we know that $L^\frac{1}{2}$ is a bounded operator in $\ell^p(G)$. For $p = 2$, using the functional calculus in Hilbert spaces, for any $u \in \ell^2(G)$, we have

$$L^\frac{1}{2} u = Mu,$$

which is also true for any $u \in C_0(G)$. Hence by the Bounded Linear Transformation theorem [60, Theorem I.7], we get

$$L^\frac{1}{2} u = Mu, \forall u \in \ell^p(G).$$

□

Now we are ready to prove the discrete second-order Sobolev inequality.

Theorem 10. For $N \geq 3, 1 < p < \frac{N}{2}, p^* := \frac{Np}{N-2p}$, we have the second-order Sobolev inequalities

(3.5) $\|u\|_{\ell^{p^{**}}} \leq C_p \|\nabla u\|_{D^2,p}, \forall u \in D^2_{0,p}(G)$,

and for $1 \leq p < \frac{N}{2},$

(3.6) $\|u\|_{\ell^{p^{**}}} \leq C_p \|u\|_{\tilde{D}^2_{0,p}}, \forall u \in \tilde{D}^2_{0,p}(G)$.

Proof. Using the completion trick, it suffices to prove that the second-order Sobolev inequalities (3.5) and (3.6) hold for any $u \in C_0(G)$. By the first-order Sobolev inequality (3.2) we have

(3.7) $\|u\|_{\ell^{p^{**}}} \leq C_p \|\nabla u\|_{p^{*}}$.

By the boundedness of Riesz transforms [22], we know that

(3.8) $\|\nabla u\|_{p^{*}} \lesssim \|L^\frac{1}{2} u\|_{p^{*}}$. 

And $L^2 u \in \ell_p(G)$ since $L^2$ is a bounded linear operator in $\ell_p(G)$ by Lemma 9. Then $L^2 u \in D_0^{1,p}(G)$ follows from $\ell_p(G)$ embeds into $D_1^{1,p}(G)$ and $D_0^{1,p}(G) = D^{1,p}(G)$. Hence by the first-order Sobolev inequality again we know

$$(3.9) \quad \|L^2 u\|_{p^*} \leq C_p \|\nabla L^2 u\|_p.$$  

By the boundedness of Riesz transforms, we get

$$(3.10) \quad \|\nabla L^2 u\|_p \leq \|L^2 L^2 u\|_{\ell_2} = \|u\|_{D^2,2}.$$  

The inequality (3.9) is proved by (3.7)-(3.10). Next by the first-order Sobolev inequality (3.2),

$$\|u\|_{p^*} \leq \|\nabla u\|_{p^*} = \left( \sum_{i=1}^m \|\partial_i u\|_{\ell_p}^p \right)^{1/p^*} \leq \left( \sum_{i=1}^m \|\nabla \partial_i u\|_{\ell_p}^p \right)^{1/p^*} \approx \|\nabla^2 u\|_{\ell_p}.$$  

Hence the Sobolev inequality (3.6) holds.  

\[ \square \]

Remark. (1) Since $\ell^q(G)$ embeds into $\ell^q(G)$ for any $q > p$, see [35, Lemma 2.1], we get the second-order Sobolev inequalities (3.2) and (3.6) in supercritical cases $q > p^*$.  

(2) The second-order Sobolev inequality for $u \in D_0^{2,1}(G)$ can not be proved directly since the Riesz transforms is weak type $(1,1)$ for the nilpotent group $G$ [22, Theorem 1], and we don’t know whether the inequality

$$\|u\|_{p^*} \approx \|\nabla u\|_{p^*} := \sup_{t > 0} t \{ \{x \in G : |\nabla u(x) | > t\} \}$$  

holds or not.

Then we prove the equivalence of higher-order Sobolev spaces.

**Theorem 11.** If $N \geq 3$, $1 < p < \frac{N}{2}$, then

$$(3.11) \quad D_0^{2,p}(\mathbb{Z}^N) = D^{2,p}(\mathbb{Z}^N),$$  

and for $1 \leq p < \frac{N}{2}$,

$$\widetilde{D}_0^{2,p}(\mathbb{Z}^N) = \widetilde{D}^{2,p}(\mathbb{Z}^N).$$  

**Proof:** By the same argument as Theorem [7], $D_0^{2,p}(G) \subseteq D^{2,p}(G)$ and $\widetilde{D}_0^{2,p}(\mathbb{Z}^N) \subseteq \widetilde{D}^{2,p}(\mathbb{Z}^N)$ follow from the Sobolev inequalities (3.5) and (3.6).

In the other direction, the key is to find suitable cutoff functions $\eta_n(x) \in C_0(\mathbb{Z}^N)$. For any $u \in D^{2,p}(\mathbb{Z}^N)$ (resp. $\widetilde{D}^{2,p}(\mathbb{Z}^N)$), set $u_n := u \eta_n \in C_0(\mathbb{Z}^N)$, then by Hölder inequalities

$$(3.12) \quad \|u_n - u\|_{D^{2,p}(\mathbb{Z}^N)} \lesssim \sum_{x \in \mathbb{Z}^N} |\Delta u(x) |^p \max_{y \sim x} \eta_n(y) - 1 |^p + \|\Delta \eta_n\|_{\ell_2^2} \|u\|_{p^*}.$$
and
\[
\|u_n - u\|_{D^{2,p}(\mathbb{Z}^N)}^p \lesssim \sum_{x \in \mathbb{Z}^N} \sum_{i,j=1}^m \{|\partial_j \partial_i u(x)|^p \eta_n(x_i s_j) - 1|^p + |\partial_j u(x)|^p |\partial_i \eta_n(x_i s_j)|^p + |\partial_i u(x)|^p |\partial_j \eta_n(x_i s_j)|^p \}
\]
(3.13)
\[
\lesssim \sum_{x \in \mathbb{Z}^N} \sum_{i,j=1}^m |\partial_j \partial_i u(x)|^p \max_{i,j} |\eta_n(x_i s_j) - 1|^p + \|\nabla \eta_n\|_{\mathbb{L}^p(\mathbb{Z}^N)} \|\nabla u\|_{\mathbb{L}^p(\mathbb{Z}^N)}^p
\]
\[
+ \|\nabla^2 \eta_n\|_{\mathbb{L}^p(\mathbb{Z}^N)}^p \|u\|_{\mathbb{L}^p(\mathbb{Z}^N)}^p .
\]
Hence, if the cutoff functions satisfy
\[
(3.14) \quad \eta_n \to 1 \text{ pointwise on } \mathbb{Z}^N
\]
and
\[
\left\{ \begin{array}{l}
\|\Delta \eta_n\|_{\mathbb{L}^p(\mathbb{Z}^N)} \to 0, \text{ for } D^{2,p}(\mathbb{Z}^N), \\
\|\nabla \eta_n\|_{\mathbb{L}^p(\mathbb{Z}^N)}, \|\nabla^2 \eta_n\|_{\mathbb{L}^p(\mathbb{Z}^N)} \to 0, \text{ for } D^{2,p}(\mathbb{Z}^N),
\end{array} \right.
\]
then we can prove the other direction by the dominated convergence theorem. We distinguish two cases and construct cutoff functions respectively:

**Figure 3.1.** The figure shows the cutoff functions constructed for $D^{1,p}(\mathbb{Z}^N)$ (solid line) and $D^{2,p}(\mathbb{Z}^N)$ (dotted line), respectively.

**Case 1.** For $D^{2,p}(\mathbb{Z}^N)$, define $f(x) := \sum_i x_i^2$ and
\[
\phi(s) := 1 \wedge \frac{\log(1 - (1 - \frac{s}{R})^3)}{\log(1 - (1 - \frac{s}{R})^3)} \wedge 0.
\]
Let $\eta(x) := \phi(f(x)), X := \{x \in \mathbb{Z}^N : f(y) > r \text{ for all } y \sim x\}$. Then $\Delta f = 2N$, $\Gamma f = 4f + N$. The numbers $r$ and $R$ rather stand for the square of the distance as $f$ is the square of the Euclidean distance. By definition $\phi \in C^2_r(\mathbb{R}, \infty)$ and
\[
-(\log(\frac{R}{c_1 r}))-1 \frac{c_2}{s} \leq \phi'(s) \leq 0 \leq \phi''(s) \leq (\log(\frac{R}{c_1 r}))-1 \frac{c_3}{s^2}, \forall s \in (r, R].
\]
By the discrete chain rule (Lemma 3) on $X$, we have
\[
\Delta \eta(x) = \Delta \phi(f)(x) = 2N \phi'(f(x)) + (4f(x) + N)\phi''(\xi^2)
\]
with \( f(x) - 2\sqrt{f(x)} + 1 \leq \xi \leq f(x) + 2\sqrt{f(x)} + 1 \). And thus if \( 100 < r < f(y) \) for all \( y \sim x \), we have \( 2\xi^2 \geq f^2(x) \) giving

\[-(\log(\frac{R}{c_1r}))^{-1}2N \frac{c_2}{f(x)} \leq \Delta \eta(x) \leq (\log(\frac{R}{c_1r}))^{-1}(4f(x) + N) \frac{2c_3}{f^2(x)}.
\]

Then taking \( c := \max \{2c_2, 9c_3\} \), we get

\[|\Delta \eta(x)| \leq \frac{cN}{f(x)\log(\frac{R}{c_1r})} \sim \frac{1}{f(x)\log \frac{R}{r}}.\]

Thus,

\[(3.15) \quad \sum_{x \in \mathcal{X}} |\Delta \eta(x)| \frac{r}{x} \leq \left( \frac{1}{\log \frac{R}{r}} \right)^{\frac{r}{x}} \sum_{r < f(x) \leq R} f(x) \frac{r}{x} \leq \left( \frac{1}{\log \frac{R}{r}} \right)^{\frac{r}{x}} \int_{\tilde{B}_{r+1} \setminus \tilde{B}_{r-1}} |t|^{-N} \, dt \leq O \left( \left( \log \frac{R}{r} \right)^{-\frac{r}{x}} \right).\]

This goes to 0 as \( R \to \infty \) for \( N \geq 3 \). The boundary at \( \sqrt{r} \) does not matter as we can keep \( r \) constant and let \( R \to \infty \). At the neighbourhood of boundary \( \partial \sqrt{r} \), i.e. \( Y := \{x \in \mathbb{Z}^N : \sqrt{r} - 2 \leq |x| \leq \sqrt{r} + 2\} \), by the mean value theorem we have

\[|\Delta \eta(x)| = |\sum_{y \sim x} (\phi(f(y)) - \phi(f(x)))| \leq \frac{r}{x} |\phi'(r)| |f(y) - f(x)| \lesssim \frac{1}{\sqrt{\log \frac{R}{r}}}.
\]

Since the cardinality of \( Y \) is finite, then

\[(3.16) \quad \sum_{x \in Y} |\Delta \eta(x)| \frac{r}{x} \lesssim \sum_{x \in Y} \left( \frac{1}{\sqrt{\log \frac{R}{r}}} \right) \to 0 \text{ as } R \to \infty.
\]

Hence, by (3.15) and (3.16) we know

\[\|\Delta \eta(x)\|_{L^\infty(\mathbb{Z}^N)} \to 0 \text{ as } R \to \infty,
\]

and we prove \( u \in D^{2,p}_0(\mathbb{Z}^N) \) by (3.12).

Case 2. For \( \tilde{D}^{2,p}(\mathbb{Z}^N) \), set \( \eta(x) := \phi(f(x)) \) as in Case 1. Then we can check that

\[\|\nabla \eta\|^2_{L^p(\mathbb{Z}^N)} \lesssim O \left( \left( \log \frac{R}{r} \right)^{1-N} \right).\]

By the norm equivalence (Lemma 3) and the result in Case 1, we know

\[\|\nabla^2 \eta\|_{L^p(\mathbb{Z}^N)} \sim \|\Delta \eta\|_{L^p(\mathbb{Z}^N)} \lesssim O \left( \left( \log \frac{R}{r} \right)^{1-N} \right).
\]

Hence, \( \tilde{D}^{2,p}(\mathbb{Z}^N) \subseteq \tilde{D}^{2,p}_0(\mathbb{Z}^N) \) by (3.13). This finishes the proof. 

\[\square\]

**Remark.** (1) The cutoff function \( \eta \) defined for \( D^{1,p}(\mathbb{Z}^N) \) can not work for \( D^{2,p}(\mathbb{Z}^N) \) directly since the Laplacian is "bad" when it comes to the sharp corner \( R \). We find a way to fix this by taking a "close to linear" function \( \phi \) being smooth at the boundary \( R \). Now the Euclidean distance is necessary for constructing the cutoff.
function since its Laplacian is decreasing while the combinatorial distance is not. Then using the chain rule we can get a good estimate.

(2) For \(1 < p < \frac{N}{2}\), by Lemma 9 and Theorem 11 we know \(D^{2,p}(\mathbb{Z}^N) = D^{2,p}_0(\mathbb{Z}^N) = \tilde{D}^{2,p}_0(\mathbb{Z}^N) = \tilde{D}^{2,p}(\mathbb{Z}^N)\).

Then we prove the higher-order equivalence on a Cayley graph \(G\) using the functional analysis.

**Theorem 12.** If \(N \geq 3\), \(1 < p < \frac{N}{2}\), then \(D^{2,p}_0(G) = D^{2,p}(G)\).

**Proof.** First \(D^{2,p}_0(G) \subseteq D^{2,p}(G)\) by the Sobolev inequalities (3.5). The other direction can be proved by checking the Laplace operator \(\Delta\) is an isometry from \(D^{2,p}_0(G)\) to \(\ell^p(G)\). The injection follows from the \(\ell^p\)–Liouville property of graphs [5, Section 1.7]. The range of \(\Delta\), denoted by \(\text{Ran}(\Delta)\), is closed by the completeness of \(D^{2,p}_0(G)\).

For any \(h \in \ell^q(G)\), the dual space of \(\ell^p(G)\),

\[
0 = \langle h, \Delta u \rangle, \quad \forall f \in \text{Ran}(\Delta),
\]

which implies

\[
0 = h(f), \quad \forall f \in \text{Ran}(\Delta),
\]

Hence \(\Delta h = 0\) and \(h = 0\) by the \(\ell^p\)–Liouville property, then the surjection follows from the closed range theorem. Similarly, \(\Delta\) is also an isometry from \(D^{2,p}_0(G)\) to \(\ell^p(G)\). Then by the fact \(D^{2,p}_0(G) \subseteq D^{2,p}(G)\) we know \(D^{2,p}_0(G) = D^{2,p}(G)\). \(\square\)

**Remark.** For the special case \(p = 2\), \(D^{2,2}_0 = D^{2,2}\) on general groups satisfying the second-order Sobolev inequality, for example, polynomial growth groups and non-amenable groups.

### 4. The Concentration-Compactness Principle

In this section, we prove the discrete Concentration-Compactness principle. We first introduce a key lemma as in [10, Theorem 1].

Consider a measure space \((\Omega, \Sigma, \mu)\), which consists of a set \(\Omega\) equipped with a \(\sigma\)-algebra \(\Sigma\) and a Borel measure \(\mu: \Sigma \to [0, \infty]\).

**Lemma 13.** (Brézis-Lieb lemma) Let \((\Omega, \Sigma, \mu)\) be a measure space, \(\{u_n\} \subset L^p(\Omega, \Sigma, \mu)\), and \(0 < p < \infty\). If

(a) \(\{u_n\}\) is uniformly bounded in \(L^p\), and

(b) \(u_n \to u, n \to \infty\) \(\mu\)-almost everywhere in \(\Omega\), then

\[
\lim_{n \to \infty} (\|u_n\|_{L^p}^p - \|u_n - u\|_{L^p}^p) = \|u\|_{L^p}^p.
\]

**Remark.** (1) The preceding lemma is a refinement of Fatou’s Lemma.

(2) Since \(\{u_n\}\) is uniformly bounded in \(L^p\), passing to a subsequence if necessary, we have

\[
\lim_{n \to \infty} \|u_n\|_p^p = \lim_{n \to \infty} \|u_n - u\|_p^p + \|u\|_p^p.
\]

(3) If \(\Omega\) is countable and \(\mu\) is a positive measure defined on \(\Omega\), then we get a discrete version of Lemma 13.
Corollary 14. Let \( \Omega \subset G, \{u_n\} \subset D^{2,p}(G), \) and \( 1 < p < \infty. \) If
\( (a)' \) \( \{u_n\} \) is uniformly bounded in \( D^{2,p}(G), \) and
\( (b)' \) \( u_n \to u, n \to \infty \) pointwise on \( G, \) then
\[
\lim_{n \to \infty} \left( \sum_{x \in \Omega} |\Delta u_n(x)|^p - \sum_{x \in \Omega} |\Delta u_n(x) - u(x)|^p \right) = \sum_{x \in \Omega} |\Delta u(x)|^p.
\]

Proof. Since \( \{|\Delta u_n|\} \) is uniformly bounded in \( \ell^p(G) \) and \( \Delta u_n \to \Delta u, n \to \infty \) pointwise on \( G, \) by Lemma 13 we get
\[
\lim_{n \to \infty} \left( |\Delta u_n|_{\ell^p(\Omega)}^p - |\Delta u_n - u|_{\ell^p(\Omega)}^p \right) = |\Delta u|_{\ell^p(\Omega)}^p,
\]
which is equivalent to the equation (4.2).

Next, we establish the Concentration-Compactness principle on \( G. \)

Lemma 15. (Discrete Concentration-Compactness lemma) For \( N \geq 3, 1 < p < \frac{N}{q}, q \geq p^*, \) if \( \{u_n\} \) is uniformly bounded in \( D^{2,p}(G) \). Then passing to a subsequence, still denoted by \( \{u_n\}, \) we have
\[
\lim_{n \to \infty} \lim_{R \to \infty} \sum_{d^2(x, e) > R} |\Delta u_n(x)|^p := \mu_\infty, \quad \lim_{n \to \infty} \lim_{R \to \infty} \sum_{d^2(x, e) > R} |u_n(x)|^q := \nu_\infty,
\]
exist. For the above \( \{u_n\}, \) we have
\[
|\Delta (u_n - u)|^p \xrightarrow{w^*} 0 \quad \text{in } \ell^1(G),
\]
\[
|u_n - u|^q \xrightarrow{w^*} 0 \quad \text{in } \ell^1(G),
\]
\[
\nu_\infty^{p/q} \leq K^{-1} \mu_\infty,
\]
\[
\lim_{n \to \infty} \|u_n\|_{D^2,p} = \|u\|_{D^2,p} + \mu_\infty,
\]
\[
\lim_{n \to \infty} \|u_n\|_q = \|u\|_q + \nu_\infty.
\]

Proof. Since \( \{u_n\} \) is uniformly bounded in \( \ell^q(G) \), they are bounded in \( \ell^\infty(G). \) By diagonal principle, passing to a subsequence we get (4.3). Since \( \{|\Delta u_n|^p\} \) is uniformly bounded in \( \ell^1(G) \), we get (4.4) by the Banach-Alaoglu theorem and (4.3). For every \( R \geq 1, \) passing to a subsequence if necessary,
\[
\lim_{n \to \infty} \sum_{d^2(x, e) > R} |\Delta u_n(x)|^p, \quad \lim_{n \to \infty} \sum_{d^2(x, e) > R} |u_n(x)|^q.
\]
exist. Then we can define \( \mu_\infty, \nu_\infty \) by the monotonicity in \( R. \)

Let \( v_n := u_n - u, \) then \( v_n \to 0 \) pointwise on \( G \) and \( \{|\Delta v_n|^p\} \) is uniformly bounded in \( \ell^1(G). \) Then any subsequence of \( \{|\Delta v_n|^p\} \) contains a subsequence, still denoted by \( \{|\Delta v_n|^p\}, \) that \( w^*-\)converges to 0 in \( \ell^1(G), \) which follows from
\[
\sum h|\Delta v_n|^p \to 0, \quad \forall h \in C_0(G).
\]
Hence we obtain (4.5). Similarly, we get (4.6).
For $R \geq 1$, let $\Psi_R \in C(G)$ such that $\Psi_R(x) = 1$ for $d^S(x, e) \geq R + 1$, $\Psi_R(x) = 0$ for $d^S(x, e) \leq R$. By the discrete Sobolev inequality (1.2), we have
\[
(\sum |\Psi_R v_n|^q)^{p/q} \leq K^{-1} \sum |\Delta (\Psi_R v_n)|^p = K^{-1} \sum (\nabla_{xy} \Psi_R v_n(y) + \Psi_R(x) \nabla_{xy} v_n)^2.
\]

For any $\epsilon > 0$, there exists $C_{\epsilon} > 0$ such that
\[
\left| \sum_{y \sim x} \nabla_{xy} \Psi_R v_n(y) + \sum_{y \sim x} \Psi_R(x) \nabla_{xy} v_n \right|^p \leq C_{\epsilon} \sum_{y \sim x} \nabla_{xy} \Psi_R v_n(y) + (1 + \epsilon) |\Delta v_n|^p |\Psi_R(x)|^p.
\]

Since $v_n \to 0$ pointwise on $G$, by $\epsilon \to 0^+$ we obtain
\[
\lim_{n \to \infty} \left( \sum_{d^S(x, e) > R} |v_n|^q \right)^{p/q} \leq K^{-1} \lim_{n \to \infty} \sum |\Delta v_n|^p |\Psi_R|^p.
\]
From the definition of $\Psi_R$,
\[
\lim_{n \to \infty} \left( \sum_{d^S(x, e) > R} |v_n|^q \right)^{p/q} \leq K^{-1} \lim_{n \to \infty} \sum |\Delta v_n|^p.
\]
By Lemma 13 and Corollary 13, we have
\[
\lim_{n \to \infty} \left( \sum_{d^S(x, e) > R} |u_n(x)|^q - \sum_{d^S(x, e) > R} |\Delta v_n|^q \right) = \sum_{d^S(x, e) > R} |\Delta u(x)|^p,
\]
\[
\lim_{n \to \infty} \left( \sum_{d^S(x, e) > R} |u_n(x)|^q - \sum_{d^S(x, e) > R} |v_n(x)|^q \right) = \sum_{d^S(x, e) > R} |u(x)|^q.
\]
Hence by letting $R \to \infty$, we get
\[
\lim_{R \to \infty} \lim_{n \to \infty} \sum_{d^S(x, e) > R} |\Delta v_n(x)|^p = \mu_{\infty},
\]
\[
\lim_{R \to \infty} \lim_{n \to \infty} \sum_{d^S(x, e) > R} |v_n(x)|^q = \nu_{\infty}.
\]
Combining the equations (4.11)–(4.13), we get
\[
\nu_{\infty}^{p/q} \leq K^{-1} \mu_{\infty}.
\]
Since $u_n \to u$ pointwise on $G$, then for every $R \geq 1$,
\[
\lim_{n \to \infty} \sum |\Delta u_n|^p = \lim_{n \to \infty} \left( \sum |\Psi_R| |\Delta u_n|^p + \sum (1 - \Psi_R) |\Delta u_n|^p \right) = \lim_{n \to \infty} \sum |\Psi_R| |\Delta u|^p + \sum (1 - \Psi_R) |\Delta u|^p,
\]
and
\[
\lim_{n \to \infty} \sum |u_n|^q = \lim_{n \to \infty} \left( \sum |\Psi_R| |u_n|^q + \sum (1 - \Psi_R) |u_n|^q \right) = \lim_{n \to \infty} \sum |\Psi_R| |u|^q + \sum (1 - \Psi_R) |u|^q.
\]
Letting $R \to \infty$, we obtain
\[
\lim_{n \to \infty} \sum |\Delta u_n|^p = \mu_{\infty} + \sum |\Delta u|^p = \mu_{\infty} + \|u\|_{L^2, p}^p,
\]
\[
\lim_{n \to \infty} \sum |u_n|^q = \nu_{\infty} + \sum |u|^q = \nu_{\infty} + \|u\|_q^q.
\]
Remark. (1) In the continuous setting, P. L. Lions [10], Bianchi et al. [8] and Ben-Naoum et al. [7] proved that the limit of the minimizing sequence norm can be divided into three parts, i.e. the norm of the limit function, the norm of the limit of the difference between the sequence and the limit function, and the norm of the sequence at infinity. The corresponding parts still satisfy the Sobolev inequality, see [76, Lemma 1.40].

(2) The difference between the sequence and the limit function \( w^* \)-converges to 0 in \( l^1(G) \), i.e. (1.5) and (4.6), which is not true in continuous setting. For example, consider the sequence of probability measures \( \{ \delta_n \} \) in \([0,1] \), where \( \delta_n(x) := n\chi_{[0,\frac{1}{n}]}(dx) \), then \( \delta_n \to 0 \) almost everywhere in \([0,1] \). However, \( \delta_n \xrightarrow{w^*} \delta_0 \) in \( (C[0,1])^* \) and the Dirac measure \( \delta_0 \) is non-zero. This is the advantage of the discrete setting.

5. Proof of Theorem 1

In this section, we will prove the existence of the extremal function for the discrete second-order Sobolev inequality (1.2). Firstly, we prove that the minimizing sequence after translation has a uniform positive lower bound at the unit element \( e \in G \). This is crucial to rule out the vanishing case of the limit function.

**Lemma 16.** For \( N \geq 3, 1 < p < \frac{N}{2}, q > p^* \), let \( \{ u_n \} \subset D^{2,p}(G) \) be a minimizing sequence satisfying (1.2). Then \( \lim_{n \to \infty} \| u_n \|_{\ell^\infty} > 0 \).

**Proof.** Choosing \( q' \) such that \( p^* < q' < q < \infty \), by interpolation inequality we have

\[
1 = \| u_n \|_2^2 \leq \| u_n \|_{q'}^q \| u_n \|_{q'}^{-q'} \leq C_{q',p} \| u_n \|_{D^{2,p}} \| u_n \|_{q'}^{-q'},
\]

where \( C_{q',p} \) is the constant in the Sobolev inequality (1.2).

By taking the limit, we obtain

\[
1 \leq C_{q',p} K \sup_{n \to \infty} \| u_n \|_{q'}^{-q'}.
\]

This proves the lemma. \( \square \)

**Remark.** The maximum of \( |u_n| \) is attainable since \( \| u_n \|_q = 1 \). Define \( v_n(x) := u_n(x_n, x) \), where \( |u_n(x_n)| = \max_x |u_n(x)| \). Then the translation sequence \( \{ v_n \} \) is uniformly bounded in \( D^{2,p}(G) \), \( \| v_n \|_{q} = 1 \) and \( |v_n(e)| = \| u_n \|_{\ell^\infty} \), where \( e \) is the unit element of \( G \). By Lemma 16 passing to a subsequence if necessary, we have

\[
v_n \to v \quad \text{pointwise on } G,
\]

\[
|v(e)| = \lim_{n \to \infty} \| u_n \|_{\ell^\infty} > 0.
\]

Next, we give the proof I of Theorem 1

**Proof I of Theorem 1** Let \( \{ u_n \} \subset D^{2,p}(G) \) be a minimizing sequence satisfying (1.4). And the translation sequence \( \{ v_n \} \) is defined in the Remark after Lemma 16.

By equalities (1.9) and (1.8) in Lemma 16 passing to a subsequence, we get

\[
K = \lim_{n \to \infty} \| v_n \|_{D^{2,p}}^p = \| v \|_{D^{2,p}}^p + \mu_{\ell^\infty},
\]

\[
1 = \lim_{n \to \infty} \| v_n \|_{q}^q = \| v \|_{q}^q + \nu_{\ell^\infty}.
\]

From the Sobolev inequality (1.2), (4.7) and the inequality

\[
(a^q + b^q)^{p/q} \leq a^p + b^p, \forall a, b \geq 0,
\]

we have

\[
(a^q + b^q)^{p/q} \leq a^p + b^p, \forall a, b \geq 0.
\]
we get

\[ K = \|v\|^p_{D^{2,p}} + \mu_\infty \]
\[ \geq K((\|v\|^q_q)^{p/q} + \nu_\infty^{p/q}) \]
\[ \geq K((\|v\|^q_q + \nu_\infty)^{p/q}) = K. \]

Since \((a^q + b^q)^{p/q} < a^p + b^p\) unless \(a = 0\) or \(b = 0\), we deduce from (5.1) that \(\|v\|^q_q = 1\). By

\[ \|v\|^p_{D^{2,p}} \geq K\|v\|^p_q, \]
we get

\[ \|v\|^p_{D^{2,p}} = K = \lim_{n \to \infty} \|v_n\|^p_{D^{2,p}}. \]

That is, \(v\) is a minimizer. \(\square\)

Then we give another proof for Theorem 1 using the discrete Brézis-Lieb lemma.

**Proof II of Theorem 1.** Using Lemma 16, by the translation and taking a subsequence if necessary as before, we can get a minimizing sequence \(\{u_n\}\) satisfying (1.4), \(u_n \to u\) pointwise on \(G\), and \(|u(e)| > 0\).

By Lemma 13, the inequality (5.2) and the Sobolev inequality, passing to a subsequence, we have

\[ K = \lim_{n \to \infty} \|u_n\|^p_{D^{2,p}} = \lim_{n \to \infty} \|u_n\|^p_{D^{2,p}} = \lim_{n \to \infty} \|u_n - u\|^p_{D^{2,p}} + \|u\|^p_{D^{2,p}} \]
\[ \geq \lim_{n \to \infty} \|u_n - u\|^q_q + \|u\|^q_q \]
\[ \geq \lim_{n \to \infty} K\|u_n - u\|^q_q + \|u\|^q_q. \]

Since \(u \neq 0\), by the Sobolev inequality we have that

\[ \|u\|^p_{D^{2,p}} \leq K\|u\|^p_q, \]

which implies

\[ \|u\|^p_{D^{2,p}} = K\|u\|^p_q. \]

By (5.3), passing to a subsequence, we get

\[ \lim_{n \to \infty} \|u_n - u\|^p_{D^{2,p}} = \lim_{n \to \infty} \|u_n - u\|^p_q. \]

Since \(0 < \|u\|_q \leq \lim_{n \to \infty} \|u_n\|_q = 1\), it suffices to show that \(\|u\|_q = 1\). Suppose that it is not true, i.e. \(0 < \|u\|_q = D < 1\), then by Lemma 13

\[ \lim_{n \to \infty} \|u_n - u\|_q^q = \lim_{n \to \infty} \|u_n\|_q^q - \|u\|_q^q = 1 - D^q > 0. \]

However, \((a^q + b^q)^{p/q} < a^p + b^p\) if \(a, b > 0\). This yields a contradiction by (5.3).

Thus, \(\|u\|_q = 1\) and \(u\) is a minimizer. \(\square\)
6. Proofs for corollaries

As applications, we prove corollaries in the introduction. The following lemma establishes the equivalence between solutions to the higher-order quasilinear problem (1.5) and to the system (1.6).

**Lemma 17.** For \( N \geq 3 \), \( p, q > 1 \), \( \frac{1}{p'} + \frac{1}{q} < \frac{N-2}{N} \), \( p' = \frac{p}{p-1} \), \( q' = \frac{q}{q-1} \), \((u, v)\) is a solution of (1.6) if and only if \( u \) is a solution of (1.5) with \( v := - |\Delta u|^{p-2} \Delta u \).

**Proof.** Since \( \frac{1}{p'} + \frac{1}{q} = \frac{N-2}{N} \), \( \frac{1}{p'} + \frac{1}{q} < \frac{N-2}{N} \) if and only if \( q > p^{**} \). Suppose that \( u \in D^{2,p}(G) \) is a solution of (1.5). Set \( v := - |\Delta u|^{p-2} \Delta u \). Then \( v \in \ell^{q'}(G) \) and \( u \in \ell^{p^{**}}(G) \) by the Sobolev inequality. And \( u \in \ell^{q}(G) \), since \( \ell^{p^{**}}(G) \) embeds into \( \ell^{q}(G) \), see Theorem 12. By the uniqueness theorem for harmonic functions we know that \( w \) is a solution of

\[-\Delta w = |u|^{q-2} u.\]

Hence, \( \Delta w \in \ell^{q'} \) and \( w \in D^{2,q'}(G) \) by the fact that Laplace operator \( \Delta \) is an isometry from \( D^{2,q'}(G) \) to \( \ell^{q'}(G) \), see Theorem 12. By the uniqueness theorem for harmonic functions we know that

\[v = w \in D^{2,q'}(G).\]

Note that \( |v|^{p'-2}v = -\Delta u. \) That is, \((u, v)\) is a solution of (1.6). On the other hand, if \((u, v)\) is a pair of solution for (1.6), we have \( - |\Delta u|^{p-2} \Delta u = v \). This proves the lemma. \( \square \)

By Theorem 1, we can prove Corollary 2.

**Proof of Corollary 2.** By Theorem 1 there exists a minimizer \( u \) for the problem (1.3). Let \( v \) be a solution of

\[-\Delta v = |u|^{q-2} u.\]

Then \( \|v\|_q \leq \|\Delta u\|_p \) since Laplace operator \( \Delta \) is an isometry from \( D^{2,p}(G) \) to \( \ell^{p}(G) \), and \( D^{2,p}(G) \) embeds into \( \ell^{q}(G) \) for \( q \geq p^{**} \). Note that \( u \leq v \) by the maximum principle. Replacing \( u \) by \(-u\), we get \( -u \leq v \) similarly. Hence,

\[0 \leq |u| \leq v.\]

In particular we have \( 1 = \|u\|_q \leq \|v\|_q \) and \( \|\Delta u\|_p = \|\Delta v\|_p \). Therefore, we know that \( \frac{v}{\|v\|_q} \) is a non-negative minimizer. It follows from the Lagrange multiplier that \( \frac{v}{\|v\|_q} \) is a non-negative solution of (1.5). The maximum principle yields that it is positive. \( \square \)

Finally, we can prove Corollary 3 by Corollary 2 and Lemma 17.

**Proof of Corollary 3.** By Corollary 2 there exists a positive solution \( u \) of equation (1.5). Set \( v := - |\Delta u|^{p-2} \Delta u \) as Lemma 17 and we know

\[-\Delta v = |u|^{q-2} u > 0.\]

Hence \( v \) is positive by the maximum principle. \( \square \)
7. The cases of $p = 2$ and $p = 1$

In this section, we study the special cases of $p = 2$ and $p = 1$. First, for $p = 2$, consider a Cayley graph $(G, S)$ satisfying $\beta_S(n) \geq C(S)n^{1/n}, \forall n \geq 1$. By the first-order Sobolev inequality \[\text{(3.2)}\] we know $D^{1,2}_0(G) \subseteq D^{1,2}(G)$. We fix an orientation for edge set $E$. Then $D^{1,2}_0(G)$ and $D^{1,2}(G)$ are Hilbert spaces equipped with the inner product

$$\langle \nabla_e u, \nabla_e v \rangle_E := \sum_{e \in E} (u(e_+) - u(e_-)) (v(e_+) - v(e_-)),$$

where $e_-$ and $e_+$ are the initial and terminal endpoints of $e$ respectively. For any $\alpha \in C(E)$,

$$\text{div} \alpha(x) := \sum_{e=(y,x) \in E} \alpha(e) - \sum_{e=(x,y) \in E} \alpha(\bar{e}), \ x \in V.$$

Then we can prove the Hodge decomposition theorem on 1-forms on edges on the Cayley graph.

**Theorem 18.** If the Cayley graph $(G, S)$ satisfies $\beta_S(n) \geq C(S)n^{1/n}, \forall n \geq 1$, then we have decompositions

$$\ell^2(E) = \nabla D^{1,2}_0(G) \oplus H$$

and

$$\ell^2(E) = \nabla D^{1,2}(G) \oplus H,$$

where $H := \{ u \in \ell^2(E) : \text{div}u = 0 \}$.

**Proof.** For any $u \in \ell^2(E)$, we define a bounded linear operator on $D^{1,2}_0(G)$ as

$$L(v) := -\langle \text{div}u, v \rangle.$$

A nondegenerate bilinear functional $B : D^{1,2}_0(G) \times D^{1,2}_0(G) \to \mathbb{R}$ is defined as

$$B(v, w) := \langle \nabla_e v, \nabla_e w \rangle_E.$$

By the Lax–Milgram theorem, there exists $f \in D^{1,2}_0(G)$ such that

$$B(f, v) = L(v), \ \forall v \in D^{1,2}_0(G).$$

Hence, for any $v \in C_0(G)$,

$$\langle \Delta f, v \rangle = -\langle \nabla_e f, \nabla_e v \rangle_E = \langle \text{div}u, v \rangle,$$

which implies that $\Delta f = \text{div}u$.

Since $\|\nabla f\|_{\ell^2(E)} = \frac{1}{2} \|f\|_{D^{1,2}(G)}$, $\nabla f \in \ell^2(E)$. Let $h = u - \nabla f \in \ell^2(E)$. Then

$$\text{div} h = \text{div}u - \text{div}\nabla f = \text{div}u - \Delta f = 0.$$

That is $h \in H$. The decomposition is proved. If $u \in \nabla D^{1,2}_0(G) \cap H$, then there exists $v \in D^{1,2}_0(G)$ such that $u = \nabla v$ and $\text{div}u = \Delta v = 0$. Hence $v = 0$ and $u = 0$. The property of the direct sum is proved. And we can get the decomposition $\ell^2(E) = \nabla D^{1,2}(G) \oplus H$ by the same argument. \qed

**Corollary 19.** If the Cayley graph $(G, S)$ satisfies $\beta_S(n) \geq C(S)n^{1/n}, \forall n \geq 1$, then $D^{1,2}_0(G) = D^{1,2}(G)$. 
Proof. By the Sobolev inequality we know $D^{1,2}_0(G) \subseteq D^{1,2}(G)$. Then for any $u \in D^{1,2}(G)$, by the Hodge decomposition theorem we get
\[ \nabla u = \nabla \tilde{u} + h, \quad \tilde{u} \in D^{1,2}_0(G), \quad h \in H. \]
And $h = \nabla u - \nabla \tilde{u} \in \nabla D^{1,2}(G)$, hence $h = 0$, that is $\nabla (u - \tilde{u}) = 0$, which implies that $u = \tilde{u} \in D^{1,2}_0(G)$. Hence $D^{1,2}(G) \subseteq D^{1,2}_0(G)$. \hfill \Box

Remark. For the groups satisfying the second-order Sobolev inequality, for example, polynomial growth groups and non-amenable groups, we can prove $D^{2,2}_0 = D^{2,2}$ by checking the Laplace operator is an isometry from $D^{2,2}_0(G)$ (also $D^{2,2}(G)$) to $\ell^2(G)$ as Theorem 12.

For the special case $p = 1$, since $\tilde{D}^{2,1}_0(\mathbb{Z}^N) = \tilde{D}^{2,1}(\mathbb{Z}^N)$ by Lemma 11, we can consider the optimal constant $\tilde{K}$ of Sobolev inequality \[ (3.6) \] in the supercritical case $q > 1^{**} = \frac{N}{N-2}$ on $\mathbb{Z}^N$:
\[ \tilde{K} := \inf_{u \in \tilde{D}^{2,1}(\mathbb{Z}^N)} \frac{\|u\|_{\tilde{D}^{2,1}}}{\|u\|_q = 1}. \]
In order to prove that the infimum is achieved, consider a minimizing sequence $\{u_n\} \subset \tilde{D}^{2,1}(\mathbb{Z}^N)$ satisfying
\[ \|u_n\|_q = 1, \quad \|u_n\|_{\tilde{D}^{2,1}(\mathbb{Z}^N)} \to \tilde{K}, \quad n \to \infty. \]
Then by the same argument as the proof of Theorem 11, we can get the following result for $\tilde{D}^{2,1}(\mathbb{Z}^N)$.

Theorem 20. For $N \geq 3$, $q > 1^{**} = \frac{N}{N-2}$, let $\{u_n\} \subset \tilde{D}^{2,1}(\mathbb{Z}^N)$ be a minimizing sequence satisfying (7.2). Then there exists a sequence $\{x_n\} \subset \mathbb{Z}^N$ and $v \in \tilde{D}^{2,1}$ such that the sequence after translation $\{v_n(x) := u_n(x_n + x)\}$ contains a convergent subsequence that converges to $v$ in $\tilde{D}^{2,1}$. And $v$ is a minimizer for $\tilde{K}$.

Remark. This result implies that the best constant can be obtained in the supercritical case.

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