**SOME CHARACTERIZATIONS OF GRADIENT YAMABE SOLITONS**

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**Abstract.** In this article we have proved that a gradient Yamabe soliton satisfying some additional conditions must be of constant scalar curvature. Later, we have showed that in a gradient expanding or steady Yamabe soliton with non-negative Ricci curvature if the potential function satisfies some integral condition then it is subharmonic, in particular, for steady case the potential function becomes harmonic. Also we have proved that, in a compact gradient Yamabe soliton, the potential function agrees with the Hodge-de Rham potential upto a constant.

**Introduction**

Yamabe solitons are special solutions to the Yamabe flow which is introduced by R. S. Hamilton [6] in the late 20th century. Let \((M, g)\) be a Riemannian manifold of dimension \(n\), \((n \geq 2)\), such that \(\{g(t)\}\) is the 1-parameter family of metrics and \(R(t)\) is its scalar curvature. Then the equation of Yamabe flow is given by

\[
\frac{\partial g(t)}{\partial t} = -R(t)g(t).
\]

A Riemannian manifold \((M, g, \lambda)\) of dimension \(n\), \((n \geq 2)\), which is connected, is called a Yamabe soliton if it satisfies

\[
\frac{1}{2} \mathcal{L}_X g = (R - \lambda)g,
\]

where \(\mathcal{L}_X g\) is the Lie derivative of the metric \(g\) with respect to the smooth vector field \(X\), \(\lambda\) is a constant and \(R\) denotes the scalar curvature of \(M\). A Yamabe soliton \((M, g, \lambda)\) is called shrinking, steady, and expanding Yamabe soliton if \(\lambda > 0\), \(\lambda = 0\), and \(\lambda < 0\), respectively. If there exists a smooth function \(f : M \rightarrow \mathbb{R}\) such that \(X = \nabla f\), then the Yamabe soliton is called a gradient Yamabe soliton. We write a gradient Yamabe soliton by \((M, g, f, \lambda)\), and in

\(2020\) Mathematics Subject Classification: 53C20; 53C21.

Key words and phrases: gradient Yamabe soliton; scalar curvature; harmonic function; Hodge-de Rham; Riemannian manifold.
In this case, (1) takes the form

\[ \nabla^2 f = (R - \lambda)g, \]

where \( \nabla^2 f \) denotes the Hessian of the potential function \( f \). Moreover, when the vector field \( X \) is trivial or the potential function \( f \) is constant, then Yamabe soliton will be called trivial, otherwise it will be a non-trivial Yamabe soliton. Taking the trace of equation (2) we get

\[ \Delta f = (R - \lambda)n. \]

In tensors of local coordinate system, the equation (2) can be written as

\[ \nabla_i \nabla_j f = (R - \lambda)g_{ij}. \]

For a given vector field \( X \) on a compact oriented Riemannian manifold \( M \), the Hodge-de Rham decomposition theorem, (see e.g. [10]) states that we may decompose \( X \) as the sum of a divergence free vector field and gradient of a function \( h \). Hence we set

\[ X = Y + \Delta h, \]

where \( \text{div}Y = 0 \). Here the function \( h \) is known as Hodge-de Rham potential. Recently many authors have been studied Yamabe soliton, (see e.g. [2, 3, 5, 7, 8, 12] and also the references therein). It is proved, in [5], that a compact gradient Yamabe soliton \((M, g, f, \lambda)\) is of constant scalar curvature. Also in [7], Shu-Yu Hsu provides an alternative proof of this result. In the present paper we omit the compactness condition and with the help of some other conditions, we have proved that the scalar curvature is constant. Zhu-Hong Zhang [11] proved a result depending on scalar curvature and potential function for gradient Ricci solitons. Following the way of Zhu-Hong Zhang [11], in this paper we have proved a result depending on scalar curvature and potential function for gradient Yamabe soliton.

At first we have proved that in a gradient Yamabe soliton with potential function bounded below and the manifold satisfying linear volume growth has constant scalar curvature. In the last section we have showed that the potential function in a gradient expanding or steady Yamabe soliton satisfying some integral condition is subharmonic.
1. Gradient Yamabe soliton with constant scalar curvature

Lemma 1.1. Let $f$ be a non-negative subharmonic function in $B(q, 2r)$ then the following inequality holds

$$
\int_{B(q, r)} |\nabla f|^2 \leq \frac{C}{r^2} \int_{B(q, 2r)} f^2,
$$

where $B(q, r)$ is a ball with radius $r > 0$ and center at $q$ and $C$ is a real constant.

Theorem 1.2. Let $(M, g, f, \lambda)$ be a gradient Yamabe soliton of dimension $n$, with the potential function $f \geq K$ for some constant $K > 0$, $R \leq \lambda$ and $M$ be of linear volume growth. Then $M$ must be of constant scalar curvature.

Proof. Since $R \leq \lambda$, it follows from equation (3) that $\Delta f \leq 0$. Now

$$
\left(\frac{1}{f}\right)_j = -\left(\frac{1}{f^2}\right)f_j,
$$

also

$$
\left(\frac{1}{f}\right)_{jj} = \left(\frac{2}{f^3}\right)f_j^2 - \left(\frac{1}{f^2}\right)f_{jj}.
$$

Hence

$$
\Delta \left(\frac{1}{f}\right) = \left(\frac{2}{f^3}\right)|\nabla f|^2 - \frac{\Delta f}{f^2}.
$$

As $\Delta f \leq 0$, it follows that $\Delta \left(\frac{1}{f}\right) \geq 0$. Since the manifold is of linear volume growth, so the volume of $B(q, r)$, i.e., $V(B(q, r)) \leq C_1 r$, for some constant $C_1 > 0$. Therefore from the equation (6), we obtain

$$
\int_{B(q, r)} |\nabla \frac{1}{f}|^2 \leq \left(\frac{C}{r^2}\right) \int_{B(q, 2r)} \left(\frac{1}{f^2}\right) \leq \left(\frac{C}{r^2 K^2}\right)V(B(q, 2r)) \leq \left(\frac{C}{r^2 K^2}\right)C_1 2r \leq \frac{2CC_1}{rK^2} \rightarrow 0
$$

as $r \rightarrow \infty$. Therefore

$$
\int_M |\nabla \frac{1}{f}|^2 = 0,
$$

(8)
which follows that the function \( \frac{1}{f} \) is constant. Consequently the potential function \( f \) is constant. Then it follows from (3) that \( R = \lambda \), which proves the result. \( \square \)

**Proposition 1.3.** Let \((M, g, f, \lambda)\) be a gradient Yamabe soliton of dimension \( n \). Then the following relation holds;

\[
\Delta R = -\frac{1}{n-1}\left\{\frac{1}{2}\nabla_i R \nabla_l f + R(R - \lambda)\right\},
\]

where \( R \) denotes the scalar curvature of \( M \).

**Proof.** Taking the covariant derivative in (4) and using the commutating formula for covariant derivative, we have

\[
g_{jk} \nabla_i R - g_{ik} \nabla_j R = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = -R_{ijkl} \nabla_l f.
\]

Taking the trace on \( j \) and \( k \), we get

\[
g_{ij} \nabla_i R - g_{ij} \nabla_j R = -R_{ijjl} \nabla_l f.
\]

This implies

\[
n \nabla_i R - \nabla_i R = -R_{il} \nabla_l f,
\]

which yields

\[
\nabla_i R = -\frac{1}{n-1} \nabla_i \nabla_l f.
\]

Now with the help of equation (10) and contracted second Bianchi identity \( \nabla_i R = 2\nabla_j R_{ij} \), we get

\[
\Delta R = g^{ij} \nabla_i \nabla_j R
\]

\[
= -g^{ij} \nabla_i \left(\frac{1}{n-1} R_{jl} \nabla_l f\right)
\]

\[
= -\frac{1}{n-1} g^{ij} \nabla_i (R_{jl} \nabla_l f)
\]

\[
= -\frac{1}{n-1} g^{ij} \left\{\nabla_i (R_{jl}) \nabla_l f + R_{jl} \nabla_i \nabla_l f\right\}
\]

\[
= -\frac{1}{n-1} \left\{\frac{1}{2} \nabla_l R \nabla_l f + g^{ij} R_{jl} (Rg_{il} - \lambda g_{il})\right\}
\]

\[
= -\frac{1}{n-1} \left\{\frac{1}{2} \nabla_l R \nabla_l f + R(R - \lambda)\right\}.
\]
Hence, we get our result. □

2. Yamabe soliton and potential functions

**Theorem 2.1.** Let $(M, g, f, \lambda)$ be a gradient expanding or steady Yamabe soliton of dimension $n$ with non-negative scalar curvature. If the potential function $f$ satisfies the condition

\[(11) \quad \int_{M-B(q,r)} \frac{f}{d(x,q)^2} < \infty,\]

where $B(q,r)$ is a ball with radius $r > 0$ and center at $q$, then $f$ is subharmonic.

**Proof.** Taking trace in (2), we get

\[(12) \quad (R - \lambda)n = \Delta f.\]

Let us consider the cut-off function, introduced in [4], $\psi_r \in C^\infty_0(B(q,2r))$ for $r > 0$ such that

\[
\begin{align*}
0 &\leq \psi_r \leq 1 & \text{in } B(q,2r) \\
\psi_r & = 1 & \text{in } B(q,r) \\
|\nabla \psi_r|^2 &\leq \frac{C}{r^2} & \text{in } B(q,2r) \\
\Delta \psi_r &\leq \frac{C}{r^2} & \text{in } B(q,2r),
\end{align*}
\]

where $C > 0$ is a constant. Then as $r \to \infty$, we get $\Delta \psi_r \to 0$ as $\Delta \psi_r \leq \frac{C}{r^2}$. Now using (12) and integration by parts, we obtain

\[
0 \leq \int_{B(q,2r)} \psi_r R = \int_{B(q,2r)} \psi_r (\lambda + \frac{1}{n} \Delta f) = \int_{B(q,2r)} \psi_r \lambda + \frac{1}{n} \int_{B(q,2r)} \psi_r \Delta f
\]

\[
\leq \frac{1}{n} \int_{B(q,2r)-B(q,r)} f \Delta \psi_r
\]

\[
\leq \frac{1}{n} \int_{B(q,2r)-B(q,r)} C \frac{f}{r^2} \to 0,
\]

as $r \to \infty$. Since $\psi_r = 1$ in $B(q,r)$, so from above inequality we have $R = 0$. Then from equation (3), we get

\[\Delta f \geq 0,\]

This completes the proof. □

By the Lemma 1.1 and Theorem 2.1 we get the following corollary:
Corollary 2.1.1. Let $(M, g, f, \lambda)$ be a gradient expanding or steady Yamabe soliton of dimension $n$ with non-negative scalar curvature. If a non-negative potential function $f$ satisfies the condition

\begin{align*}
\int_{M-B(q,r)} \frac{f}{d(x,q)^2} < \infty,
\end{align*}

where $B(q,r)$ is a ball with radius $r > 0$ and center at $q$, then $f$ satisfies the inequality

\begin{align*}
\int_{B(q,r)} |\nabla f|^2 \leq \frac{C}{r^2} \int_{B(q,2r)} f^2.
\end{align*}

Corollary 2.1.2. Let $(M, g, f, \lambda)$ be a gradient steady Yamabe soliton of dimension $n$ with non-negative scalar curvature. If the potential function $f$ satisfies

\begin{align*}
\int_{M-B(q,r)} \frac{f}{d(x,q)^2} < \infty,
\end{align*}

where $B(q,r)$ is a ball with radius $r > 0$ and center at $q$, then $f$ is harmonic.

Proof. In the above theorem we get the scalar curvature $R = 0$, hence for steady Yamabe soliton we have $\Delta f = 0$, i.e., $f$ is harmonic. \qed

Corollary 2.1.3. Let $(M, g, f, \lambda)$ be a gradient shrinking Yamabe soliton of dimension $n$ with non-negative scalar curvature and $\lambda \leq \Delta \psi$, for some smooth function $\psi$. If the potential function $f$ satisfies

\begin{align*}
\int_{M-B(q,r)} \frac{f}{d(x,q)^2} < \infty,
\end{align*}

where $B(q,r)$ is a ball with radius $r > 0$ and center at $q$, then $f$ is superharmonic.

Proof. Consider the cut-off function as in Theorem 2.1. Now using (12) and integration by parts, we get
\[ 0 \leq \int_{B(q,2r)} \psi_r R = \int_{B(q,2r)} \psi_r (\lambda + \frac{1}{n} \Delta f) = \int_{B(q,2r)} \psi_r \lambda + \frac{1}{n} \int_{B(q,2r)} \psi_r \Delta f \]
\[ \leq \int_{B(q,2r)} \psi_r \Delta \psi + \frac{1}{n} \int_{B(q,2r)} \psi_r \Delta f \]
\[ \leq \int_{B(q,2r) - B(q,r)} \psi \Delta \psi + \frac{1}{n} \int_{B(q,2r) - B(q,r)} f \Delta \psi \]
\[ \leq \int_{B(q,2r) - B(q,r)} \frac{C}{r^2} \left( \frac{f}{n} + \psi \right) \to 0, \]
as \( r \to \infty \). Since \( \psi_r = 1 \) in \( B(q,r) \), so from above inequality we have \( R = 0 \). Then from equation \( (3) \), we get
\[ \Delta f \leq 0, \]
which proves the result.

In [1] Aquino et. al proved a result for gradient Ricci soliton. We have proved the same result for gradient Yamabe soliton as follows:

**Theorem 2.2.** Let \((M, g, X, \lambda)\) be a gradient Yamabe soliton of dimension \( n \) which is compact, with potential function \( f \). Then up to a constant, \( f \) agrees with the Hodge-de Rham potential \( h \).

**Proof.** For Yamabe soliton \((M, g, X, \lambda)\), it follows that
\[ (R - \lambda)n = \text{div} X. \]
Now with the help of Hodge-de Rham decomposition we get \( \text{div} X = \Delta h \), and hence
\[ (R - \lambda)n = \Delta h. \]
Again from \( (2) \) we have
\[ \Delta f = (R - \lambda)n. \]
Now, from equations \( (17) \) and \( (18) \), we get
\[ \Delta (f - h) = 0. \]
Therefore, we have \( f = h + C \), for some constant \( C \), which completes the proof of the theorem.

\[ \Box \]
3. ACKNOWLEDGMENT

The second author gratefully acknowledges to the CSIR (File No.: 09/025(0282)/2019-EMR-I), Govt. of India for financial assistance.

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