\( \mathcal{N} = 2 \) SUSY symmetries for a moving charged particle under influence of a magnetic field: Supervariable approach

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Abstract: We exploit the supersymmetric invariant restrictions (SUSYIRs) on the supervariables to derive the nilpotent \( \mathcal{N} = 2 \) SUSY transformations for the supersymmetric quantum mechanical model of the motion of a charged particle in the X-Y plane (where the magnetic field \( B_z \) is applied along the Z-direction). The supervariables are defined on a \((1, 2)\)-dimensional supermanifold parametrized by a bosonic "time" variable \( t \) and a pair of Grassmannian variables \( \theta \) and \( \bar{\theta} \) (with \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)). We take the (anti-)chiral supervariables for our purpose so that the nilpotency property of the \( \mathcal{N} = 2 \) SUSY symmetry transformations could be captured within the framework of supervariable approach. We express the Lagrangian as well as supercharges in terms of the supervariables (that are obtained after the application of the appropriate SUSYIRs) and provide geometrical basis, within the framework of our supervariable approach, for \((i)\) the nilpotency property of the above SUSY transformations, and \((ii)\) the SUSY invariance of the Lagrangian.

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1 Introduction

The principle of local gauge invariance has played a pivotal role in providing theoretical basis for three out of four fundamental interactions of nature. There is a stunning degree of agreement between theory and experiment as far as the standard model of particle physics (based on the (non-)Abelian 1-form gauge theories) is concerned. The quantization of such theories is a bit involved in the sense that these theories are described by the singular Lagrangian densities (which are always endowed with constraints). The well-known Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the mathematically rich and theoretically useful approaches to covariantly quantize such kind of gauge theories where the local gauge symmetries of the original theory are traded with the (anti-)BRST symmetries.

Two of the abstract mathematical properties associated with the (anti-)BRST symmetries are the nilpotency and absolute anticommutativity. In the language of theoretical physics, the former property implies the fermionic nature of (anti-)BRST symmetries (and their corresponding charges) and the latter property establishes their linear independence. In other words, given a local classical gauge symmetry of a gauge theory, there exist two fermionic quantum symmetries that are independent of each-other. These quantum gauge symmetries have been christened as the BRST and anti-BRST symmetries whose existence itself enables the covariant canonical quantization of a given gauge theory.

The geometrical origin and interpretations for the above mentioned nilpotency and anticommutativity properties are provided by the superfield formalism [1-8]. In particular, the Bonora-Tonin (BT) superfield approach [4,5] has been very successful in the context of gauge theories where the horizontality (HC) condition plays a very crucial role. The latter condition leads to the derivation of (anti-)BRST symmetries for the gauge and corresponding (anti-)ghost fields which turn out to be nilpotent of order two and absolutely anticommuting. These properties owe their origin to the translational generators along the Grassmannian directions of the supermanifold on which the ordinary gauge theory is generalized. The HC alone, however, does not say anything about the (anti-)BRST symmetries of the matter fields that are also present in an interacting gauge theory.

In a set of papers [9-13], we have extended the BT-superfield formalism where, in addition to the HC, we have exploited the gauge invariant restrictions (GIRs) to derive the (anti-)BRST symmetry transformations for the matter fields, too, in an interacting gauge theory. The symmetries (and their geometrical interpretations) turn out to be consistent with one-another when the HC and GIRs are tapped together within the framework of the augmented version of BT superfield formalism [9-13]. It has been a long-standing problem to apply the above superfield formalism [1-13] to derive the SUSY transformations for the SUSY systems where the nilpotency property exists but the anticommuting property does not. In a very recent set of papers [14,15], however, we have applied the augmented version of superfield formalism [9-13] to the $\mathcal{N} = 2$ SUSY quantum mechanical models to derive the $\mathcal{N} = 2$ SUSY symmetry transformations in a consistent and cogent manner.

The purpose of our present investigation is to exploit the theoretical tools and techniques of our earlier works on supervariable approach [14,15] to derive the $\mathcal{N} = 2$ SUSY transformations for the SUSY system of a moving charged particle in the X-Y plane under influence of a magnetic field that is applied in the Z-direction (i.e. perpendicular to the X-Y plane). We express the conserved charges and the Lagrangian in the language of
supervariables and provide the geometrical interpretations for the SUSY invariance of the Lagrangian as well as the nilpotency of the conserved charges in terms of the translational generators along the Grassmanian directions \((\bar{\theta})\theta\) of the (anti-)chiral super-submanifold, respectively. These generators are defined on the \((1, 1)\)-dimensional super-submanifolds of the general \((1, 2)\)-dimensional supermanifold on which our starting theory of (a moving charged particle under influence of a magnetic field) is generalized.

Our present investigation has been motivated by the following key factors. First and foremost, to put our central ideas \([14,15]\) on a firmer-footing, it is essential that we have to apply the supervariable approach to models with superpotentials that are completely different from the superpotentials of the \(\mathcal{N} = 2\) SUSY free particle, harmonic oscillator (HO) and the generalized version of the HO \([16,17]\). This is the reason that, in our present investigation, we have taken the SUSY example of the motion of a charged particle under influence of a magnetic field\(^2\) and have demonstrated the utility of our supervariable approach. Second, it has been a long-standing problem to apply some form of the superfield approach \([1-13]\) to capture the nilpotency of the SUSY symmetries and provide a geometrical meaning to it. We have accomplished this goal in our present investigation and in our earlier works \([14,15]\). Finally, our method of application of superfield formalism might turn out to be useful in the context of SUSY gauge theories, too. Thus, our present investigation is our modest step towards our main goal of discussion about the 4D SUSY gauge theories.

The contents of our present investigation are organized as follows. First of all, we discuss the bare essentials of \(\mathcal{N} = 2\) SUSY transformations for the motion of a charged particle under influence of a magnetic field in Sec. 2. We exploit the virtues of anti-chiral supervariables to derive one of the two nilpotent \(\mathcal{N} = 2\) SUSY transformations in Sec. 3. Our Sec. 4 is devoted to the derivation of other \(\mathcal{N} = 2\) SUSY transformations by invoking the chiral supervariables. In our Sec. 5, we discuss about the SUSY invariance of the Lagrangian of the theory and nilpotency of the \(\mathcal{N} = 2\) SUSY charges within the framework of supervariable formalism. Our Sec. 6 deals with the cohomological aspects of the \(\mathcal{N} = 2\) SUSY transformations and corresponding symmetry generators. Finally, in Sec. 7, we make some concluding remarks and point out a few future directions.

We provide the logical reasons behind our choice of the (anti-)chiral supervariables in Appendix A and show that this choice avoids the property of absolute anticommutativity.

## 2 Preliminaries: \(\mathcal{N} = 2\) SUSY symmetries

We begin with the following Lagrangian for the motion of a charged particle (of mass \(m = 1\) and charge \(e = 1\)) in the X-Y plane (see, e.g. \([16,17]\) for details):

\[
L_0 = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (\dot{x} A_x + \dot{y} A_y) + i \bar{\psi} \dot{\psi} + B_z \bar{\psi} \psi,
\]

where the magnetic field \(B_z = \partial_x A_y (x, y) - \partial_y A_x (x, y)\) is in the Z-direction and the whole trajectory of the particle is parametrized by the evolution “time” parameter \(t\). As a

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\(^2\)The motion of a charged particle under influence of an electromagnetic field has been a physically very important problem because this is the starting point for the idea of an interacting gauge theory. Thus, its study, in the context of the \(\mathcal{N} = 2\) SUSY quantum mechanical model, is interesting in its own right.
consequence, we have the “generalized” instantaneous velocities of the SUSY particle as: 
\( \dot{x} = dx/dt, \dot{y} = dy/dt \) and \( \psi = d\psi/dt \). The instantaneous position variables \( x(t) \) and \( y(t) \) are bosonic in nature and variables \( \psi(t) \) and \( \bar{\psi}(t) \) are fermionic \((\dot{\psi}^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0)\) at the classical level. The X-Y components of the vector potentials \((A_x, A_y)\) have no explicit “time” dependence and they are only function of the instantaneous position of the particle \((i.e. A_x(x, y), A_y(x, y))\). In the above, we have four bosonic variables \( x(t), y(t), A_x(x, y) \) and \( A_y(x, y) \) and two fermionic variables. These are superpartners of one-another as is clear from a close look at their supersymmetric transformations [cf. (2) below].

It can be readily checked that the starting Lagrangian \((1)\) respects the following \( \mathcal{N} = 2 \) SUSY transformations \((i.e. s_1 \) and \( s_2)\) for the relevant variables, namely:

\[
\begin{align*}
& s_1 x = \psi, \quad s_1 y = -i \psi, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = i (\dot{x} - i \dot{y}), \\
& s_1 A_x = (\partial_x A_x - i \partial_y A_x) \psi, \quad s_1 A_y = (\partial_x A_y - i \partial_y A_y) \psi, \\
& s_2 x = \bar{\psi}, \quad s_2 y = i \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = i (\dot{x} + i \dot{y}), \\
& s_2 A_x = \bar{\psi} (\partial_x A_x + i \partial_y A_x), \quad s_2 A_y = \bar{\psi} (\partial_x A_y + i \partial_y A_y),
\end{align*}
\]

(2)

because the Lagrangian transforms as:

\[
\begin{align*}
& s_1 L_0 = -\frac{d}{dt} \left[ (A_x - i A_y) \psi \right], \quad s_2 L_0 = +\frac{d}{dt} \left[ \bar{\psi} \left\{ \dot{x} + i \dot{y} - (A_x + i A_y) \right\} \right].
\end{align*}
\]

(3)

This shows that the action integral \( S = \int dt \ L_0 \) remains invariant under the transformations \( s_1 \) and \( s_2 \) for the physically well-defined variables which vanish off at infinity.

We point out that the above infinitesimal transformations are off-shell nilpotent of order two \((i.e. s_1^2 = s_2^2 = 0)\) which establishes the fermionic nature of these transformations. This is the reason that the above transformations change bosonic variables into fermionic variables and vice-versa. Furthermore, we note that the anticommutator of the fermionic transformations \( s_1 \) and \( s_2 \) leads to a bosonic transformation \((i.e. s_\omega = \{s_1, s_2\})\)

\[
\begin{align*}
& s_\omega \Phi = \bar{\Phi}, \quad \Phi = x(t), \quad y(t), \quad \psi(t), \quad \bar{\psi}(t), \quad A_x(x, y), \quad A_y(x, y),
\end{align*}
\]

(4)

modulo a factor of \((2i)\). In the derivation of the above bosonic symmetry transformations, we have used \(d\) for obvious reasons the following inputs:

\[
\begin{align*}
& \partial_x \psi(t) = 0, \quad \partial_y \psi(t) = 0, \quad \partial_x \bar{\psi}(t) = 0, \quad \partial_y \bar{\psi}(t) = 0, \\
& \frac{d}{dt} A_x(x, y) = \dot{x} \partial_x A_x + \dot{y} \partial_y A_x, \quad \frac{d}{dt} A_y(x, y) = \dot{x} \partial_x A_y + \dot{y} \partial_y A_y.
\end{align*}
\]

(5)

Under the above transformations \((4)\), the starting Lagrangian \(L_0\) transforms to a total “time” derivative (of itself) as follows:

\[
\begin{align*}
& s_\omega L_0 = \frac{d}{dt} [L_0],
\end{align*}
\]

(6)

which demonstrates the invariance of the action integral \( S = \int dt \ L_0 \). Thus, we note that there are three continuous symmetries in our present theory where two of them are fermionic and one of them is bosonic in nature. It is straightforward to check that \( s_\omega \) commutes with both the fermionic transformations \( s_{(1)2} \) \((i.e. [s_\omega, s_1] = 0, [s_\omega, s_2] = 0)\) in the operator form.
According to Noether’s theorem, the above continuous symmetry transformations lead to the derivation of conserved charges $Q_i$ (with $i = 1, 2, 3$) as

\[
Q_1 \equiv Q = \left( p_x + A_x, A_x \right) \psi,
\]
\[
Q_2 \equiv \tilde{Q} = \tilde{\psi} \left( p_x + A_x + i (p_y + A_y) \right),
\]
\[
Q_3 \equiv Q_\omega = \left( \frac{(p_x + A_x)^2}{2} + \frac{(p_y + A_y)^2}{2} - B_z \tilde{\psi} \psi \right) \equiv H.
\] (7)

The conservation ($\dot{Q}_i = 0$) of the above charges $Q_i$ can be proven directly by using the following Euler-Lagrange equations of motion:

\[
\dot{\psi} - i B_z \psi = 0, \quad \ddot{\psi} + i B_z \psi = 0,
\]
\[
\dot{\psi} + i B_z \psi = 0, \quad \ddot{\psi} - i B_z \psi = 0,
\] (8)

which are derived from the Lagrangian $L_0$. The above conserved charges are the generators of the infinitesimal symmetry transformations listed in (2) and (4). This can be explicitly checked by the following general relationship between the infinitesimal continuous symmetry transformations and their generators (for the generic variable $\Phi$), namely;

\[
s_r \Phi = \pm i [\Phi, Q_r], \quad r = 1, 2, \omega,
\] (9)

where the ($\pm$) signs, expressed as the subscripts, on the square bracket correspond to the (anti)commutator for the generic variable $\Phi = x, y, A_x, A_y, \psi, \tilde{\psi}$ being (fermionic) bosonic in nature. It is straightforward to note that $Q^2 = \tilde{Q}^2 = 0$ because using equations (9), (7) and (2), it is clear that $s_1 Q = i \{Q, Q\} = 0$, $s_2 \tilde{Q} = i \{\tilde{Q}, \tilde{Q}\} = 0$. Furthermore, we also note that $s_1 Q = i \{Q, Q\} = 2iH$ and $s_2 \tilde{Q} = i \{\tilde{Q}, \tilde{Q}\} = 2iH$ which imply $\{Q, \tilde{Q}\} = 2H$. We shall comment more on this algebraic structure in our Sec. 6. In the forthcoming sections, we wish to capture most of the material, discussed in this section, within the framework of supervariable approach and provide geometrical basis for them.

3 Anti-chiral supervariables: Derivation of one of the two SUSY transformations

We have observed, in the previous section, that the nilpotent ($s_1^2 = s_2^2 = 0$) $\mathcal{N} = 2$ SUSY transformations ($s_1, s_2$) do not anticommute (i.e. $\{s_1, s_2\} \neq 0$). In this section, we wish to capture the derivation and nilpotency of the first of the two SUSY transformations (i.e. $s_1$) within the framework of supervariable approach. First of all, we generalize the evolution parameter $t$ to its superspace counterpart $Z^M = (t, \theta, \tilde{\theta})$ which parametrizes a (1, 2)-dimensional supermanifold. Here a pair of Grassmannian variables $\theta$ and $\tilde{\theta}$ (with $\theta^2 = \tilde{\theta}^2 = 0$, $\theta \tilde{\theta} + \tilde{\theta} \theta = 0$) are required, in addition to the bosonic evolution parameter $t$, for the complete characterization of the (1, 2)-dimensional supermanifold. To derive the transformations $s_1$ and its nilpotency, we choose the anti-chiral supervariables (corresponding to all the ordinary dynamical variables of the starting Lagrangian $L_0$) on the
(1, 1)-dimensional super-submanifold (of the general (1, 2)-dimensional supermanifold on which our present SUSY theory is generalized). In other words, first of all, we generalize the simple variables \((x(t), y(t), \psi(t), \bar{\psi}(t))\) onto the (1, 1)-dimensional anti-chiral super-submanifold as (see, e.g. [14,15]):

\[
\begin{align*}
    x(t) & \longrightarrow X(t, \theta, \bar{\theta}) \mid_{\theta=0} \equiv X(t, \bar{\theta}) = x(t) + \bar{\theta} f_1(t), \\
y(t) & \longrightarrow Y(t, \theta, \bar{\theta}) \mid_{\theta=0} \equiv Y(t, \bar{\theta}) = y(t) + \bar{\theta} f_2(t), \\
\psi(t) & \longrightarrow \Psi(t, \theta, \bar{\theta}) \mid_{\theta=0} \equiv \Psi(t, \bar{\theta}) = \psi(t) + i \bar{\theta} b_1(t), \\
\bar{\psi}(t) & \longrightarrow \bar{\Psi}(t, \theta, \bar{\theta}) \mid_{\theta=0} \equiv \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + i \bar{\theta} b_2(t),
\end{align*}
\]

where the secondary variables \((b_1, b_2)\) and \((f_1, f_2)\) are bosonic and fermionic in nature, respectively. It is elementary to note that the bosonic (i.e. \(x, y, b_1, b_2\)) and fermionic \((\psi, \bar{\psi}, f_1, f_2)\) d.o.f. do match on the r.h.s. of the above anti-chiral expansions (cf. (10)) which is one of the key requirements of a SUSY theory. We shall determine the exact form of the secondary variables by invoking the appropriate set of SUSY IRs.

A decisive feature of the augmented version of BT-superfield formalism [9-13] and our earlier works [14,15] is the requirement that all the gauge/SUSY invariant quantities must remain independent of the Grassmannian variables \(\theta\) and \(\bar{\theta}\) when they are generalized onto a specific supermanifold. We observe that such invariant quantities, w.r.t. \(s_1\), are:

\[
\begin{align*}
s_1[\psi(t)] = 0, \quad s_1[x(t) \psi(t)] = 0, \quad s_1[y(t) \psi(t)] = 0, \quad s_1[\dot{x}(t) \dot{\psi}(t)] = 0, \\
s_1[\dot{y}(t) \dot{\psi}(t)] = 0, \quad s_1\left[\frac{1}{2} \left(\dot{x}^2(t) + \dot{y}^2(t)\right) + i \bar{\psi}(t) \dot{\psi}(t)\right] = 0.
\end{align*}
\]

As per prescription laid down in [14,15], we have the following equalities:

\[
\begin{align*}
\Psi(t, \bar{\theta}) = \psi(t), \quad X(t, \bar{\theta}) \Psi(t, \bar{\theta}) = x(t) \psi(t), \quad Y(t, \bar{\theta}) \Psi(t, \bar{\theta}) = y(t) \psi(t), \\
\dot{X}(t, \bar{\theta}) \Psi(t, \bar{\theta}) = \dot{x}(t) \dot{\psi}(t), \quad \dot{Y}(t, \bar{\theta}) \Psi(t, \bar{\theta}) = \dot{y}(t) \dot{\psi}(t), \\
\frac{1}{2} \left[\dot{X}^2(t, \bar{\theta}) + \dot{Y}^2(t, \bar{\theta})\right] + i \bar{\Psi}(t, \bar{\theta}) \dot{\Psi}(t, \bar{\theta}) = \frac{1}{2} \left[\dot{x}^2(t) + \dot{y}^2(t)\right] + i \bar{\psi}(t) \dot{\psi}(t),
\end{align*}
\]

which lead to the following relationships amongst the secondary variables \((b_1, b_2, f_1, f_2)\) of the expansions (10) and the basic variables \((x, y, \psi, \bar{\psi})\), namely:

\[
\begin{align*}
b_1(t) = 0, \quad f_1(t) \psi(t) = 0, \quad \dot{f}_1(t) \dot{\psi}(t) = 0, \quad f_2(t) \psi(t) = 0, \\
\dot{f}_2(t) \dot{\psi}(t) = 0, \quad \dot{x}(t) \dot{f}_1(t) + \dot{y}(t) \dot{f}_2(t) - b_2(t) \dot{\psi}(t) = 0.
\end{align*}
\]

The non-trivial solution of the above restrictions are \(f_1(t) \propto \psi(t)\) and \(f_2(t) \propto \psi(t)\). For the algebraic convenience, however, we choose \(f_1(t) = \psi(t)\) and \(f_2(t) = -i \psi(t)\). It is evident that if we take the help of these relationships, we obtain \(b_2 = \bar{x} - i\bar{y}\).

\[1\text{We are theoretically compelled to choose the (anti-)chiral supervariables because the nilpotent } \mathcal{N} = 2 \text{ SUSY transformations do not anticommute (i.e. } \{s_1, s_2\} \neq 0). \text{ This should be contrasted with the (anti-) BRST symmetry transformations which absolutely anticommute (see, e.g. [9-13] for details). Within the framework of superfield approach to the (anti-)BRST symmetries, the superfields are expanded along the Grassmannian directions } (\theta, \bar{\theta}) \text{ in their full blaze of glory (see, e.g. Appendix for more discussion).}\]
The explicit substitution of \((f_1, f_2, b_1, b_2)\) into the original expansions (10) leads to the following final expansions of the (anti-)chiral supervariables of the theory, namely:

\[
\begin{align*}
X^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} &= X^{(1)}(t, \bar{\theta}), \\
X^{(1)}(t, \bar{\theta}) &= x(t) + \bar{\theta} \left(\psi \psi\right) \equiv x(t) + \bar{\theta} (s_1 x), \\
Y^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} &= Y^{(1)}(t, \bar{\theta}), \\
Y^{(1)}(t, \bar{\theta}) &= y(t) + \bar{\theta} \left(-i \psi \psi\right) \equiv y(t) + \bar{\theta} (s_1 y), \\
\Psi^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} &= \Psi^{(1)}(t, \bar{\theta}), \\
\Psi^{(1)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} (0) \equiv \psi(t) + \bar{\theta} (s_1 \psi), \\
\bar{\Psi}^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} &= \bar{\Psi}^{(1)}(t, \bar{\theta}), \\
\bar{\Psi}^{(1)}(t, \bar{\theta}) &= \bar{\psi}(t) + \bar{\theta} \left[i (\bar{x} - i \bar{y})\right] \equiv \bar{\psi}(t) + \bar{\theta} (s_1 \bar{\psi}),
\end{align*}
\]

where the superscript \((1)\) on the supervariables denotes the expansions of the supervariables after the application of the SUSYIRs (12). It is evident now that the following geometrical relationship between the SUSY transformations \(s_1\) and the translational generators \(\partial_{\theta}\) (on the \((1, 1)\)-dimensional anti-chiral supervariables) exist in an explicit fashion:

\[
\frac{\partial}{\partial \theta} \left[ \Omega^{(1)}(t, \theta, \bar{\theta}) \right] |_{\theta=0} = s_1 \Omega(t) \equiv \pm i [\Omega(t), Q]_{\pm},
\]

where \(\Omega^{(1)}(t) = x(t), y(t), \psi(t), \bar{\psi}(t)\) is the generic variable of the starting Lagrangian \(L_0\) and \(\Omega^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0}\) stands for the generic supervariables (14) that have been obtained after application of the SUSYIRs (12). A close look at (15) and (14) explains clearly that we have already obtained the SUSY transformations \(s_1 \psi, y, \bar{s}_1 \bar{\psi}, y\) which are present in (2). Their nilpotency is also clear because of the relationship in (15) which states that \(s_1^2 = 0\) and \((\partial_{\theta})^2 = 0\) are inter-related in a meaningful manner. In other words, the nilpotency of \(s_1\) (i.e. \(s_1^2 = 0\)) is intimately connected with the nilpotency \((\partial_{\theta})^2 = 0\) of the translational generator \(\partial_{\theta}\) on the anti-chiral super-submanifold.

Now we focus on the SUSY transformations for \(A_x\) and \(A_y\) and point out the derivation of \(s_1 A_x\) and \(s_1 A_y\) within the framework of our supervariable approach. Towards this goal in mind, first of all, we generalize the ordinary potentials \(A_x(x, y)\) and \(A_y(x, y)\) onto their counterpart anti-chiral supervariables on the anti-chiral super-submanifold as

\[
\begin{align*}
A_x(x, y) &\rightarrow \bar{A}_x X^{(1)}, Y^{(1)} \equiv \bar{A}_x (x + \bar{\theta} \psi, y - i \bar{\theta} \psi) \\
&= A_x(x, y) + \bar{\theta} \left[\left(\partial_x A_x(x, y) - i \partial_y A_x(x, y)\right) \psi\right] \\
&\equiv A_x(x, y) + \bar{\theta} \left(s_1 A_x(x, y)\right), \\
A_y(x, y) &\rightarrow \bar{A}_y X^{(1)}, Y^{(1)} \equiv \bar{A}_y (x + \bar{\theta} \psi, y - i \bar{\theta} \psi) \\
&= A_y(x, y) + \bar{\theta} \left[\left(\partial_x A_y(x, y) - i \partial_y A_y(x, y)\right) \psi\right] \\
&\equiv A_y(x, y) + \bar{\theta} \left(s_1 A_y(x, y)\right).
\end{align*}
\]

\(^{\text{It will be noted that our supervariable approach allows us to choose the secondary variables as has been done in (14) \textit{modulo a constant factor}. This freedom would be exploited in our Sec. 6.}\)
We note that we have to use the expansions, obtained in (14), for the derivation of SUSY transformations $s_1A_x(x,y)$ and $s_1A_y(x,y)$ which are

\[ s_1A_x = (\partial_xA_x - i \partial_yA_x) \psi, \quad s_1A_y = (\partial_xA_y - i \partial_yA_y) \psi. \] (17)

It is clear that our above results match with the ones listed in (2). Finally, we mention that we have derived all the SUSY transformations corresponding to the dynamical variables of $L_0$ within the framework of our supervariable approach. We have also provided the geometrical interpretation for the nilpotency ($s_1^2 = 0$) of the SUSY transformations ($s_1$) and shown that it is intimately connected with the nilpotency ($\partial_\theta^2 = 0$) of the translational generator ($\partial_\theta$) on the anti-chiral super-submanifold. In other words, we find that the relationship (15) establishes a very intimate relationship between the transformations $s_1$, their generators $Q$ and the translational generator ($\partial_\theta$) on the anti-chiral super-submanifold.

4 Chiral supervariables: Derivation of the nilpotent second SUSY transformations

To derive the other SUSY transformations $s_2$, beside the first one (i.e. $s_1$), we take recourse to the chiral supervariables that are generalization of, first of all, the simple dynamical variables $(x(t), y(t), \psi(t), \bar{\psi}(t))$ of the starting Lagrangian $L_0$. In other words, we generalize the ordinary SUSY theory onto a $(1, 1)$-dimensional chiral super-submanifold (of the general $(1, 2)$-dimensional supermanifold) as [14,15]

\[
\begin{align*}
    x(t) &\longrightarrow X(t, \theta, \bar{\theta}) \mid_{\bar{\theta}=0} \equiv X(t, \theta) = x(t) + \theta \bar{f}_1(t), \\
y(t) &\longrightarrow Y(t, \theta, \bar{\theta}) \mid_{\bar{\theta}=0} \equiv Y(t, \theta) = y(t) + \theta \bar{f}_2(t), \\
\psi(t) &\longrightarrow \Psi(t, \theta, \bar{\theta}) \mid_{\bar{\theta}=0} \equiv \Psi(t, \theta) = \psi(t) + i \theta \bar{b}_1(t), \\
\bar{\psi}(t) &\longrightarrow \bar{\Psi}(t, \theta, \bar{\theta}) \mid_{\bar{\theta}=0} \equiv \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i \theta \bar{b}_2(t),
\end{align*}
\] (18)

where $(\bar{b}_1, \bar{b}_2)$ and $(\bar{f}_1, \bar{f}_2)$ are the bosonic and fermionic secondary variables, respectively. It is crystal clear that the bosonic $(x, y, \bar{b}_1, \bar{b}_2)$ and fermionic $(\psi, \bar{\psi}, \bar{f}_1, \bar{f}_2)$ d.o.f. do match on the r.h.s. of the expansions (18) which is a key requirement of a SUSY theory.

As proposed in the augmented version of superfield formalism [9-13] and in our earlier works [14,15], we have to find out the SUSY invariant quantities under $s_2$ and demand that they should be independent of the Grassmannian variables $\theta$ and $\bar{\theta}$ when they are generalized onto the appropriate supermanifold. In this regards, we note that the following quantities are invariant under $s_2$, namely:

\[
\begin{align*}
s_2[\bar{\psi}(t)] &= 0, \quad s_2[x(t) \bar{\psi}(t)] = 0, \quad s_2[y(t) \bar{\psi}(t)] = 0, \quad s_2[\dot{x}(t) \bar{\psi}(t)] = 0, \\
s_2[\bar{\psi}(t)] &= 0, \quad s_2\left[\frac{1}{2} \left(\dot{x}^2(t) + \dot{y}^2(t)\right) - i \dot{\psi}(t) \psi(t)\right] = 0.
\end{align*}
\] (19)

As a consequence, we have the following SUSYIRs on the (super)variables, namely:

\[
\begin{align*}
\dot{\Psi}(t, \theta) &= \bar{\psi}(t), \quad \dot{X}(t, \theta) \bar{\Psi}(t, \theta) = x(t) \bar{\psi}(t), \quad \dot{Y}(t, \theta) \bar{\Psi}(t, \theta) = y(t) \bar{\psi}(t), \\
\dot{X}(t, \theta) \bar{\Psi}(t, \theta) &= \dot{x}(t) \bar{\psi}(t), \quad \dot{Y}(t, \theta) \bar{\Psi}(t, \theta) = \dot{y}(t) \bar{\psi}(t), \\
\frac{1}{2} \left[\dot{X}^2(t, \theta) + \dot{Y}^2(t, \theta)\right] - i \dot{\Psi}(t, \theta) \bar{\Psi}(t, \theta) &= \frac{1}{2} \left[\dot{x}^2(t) + \dot{y}^2(t)\right] - i \dot{\psi}(t) \bar{\psi}(t).
\end{align*}
\] (20)
Using the expansions from (18), we obtain (from the above SUSYIRs) the following:
\[
\begin{align*}
\bar{b}_1(t) &= 0, \quad \bar{f}_1(t) \psi(t) = 0, \quad \bar{f}_1(t) \dot{\psi}(t) = 0, \quad \bar{f}_2(t) \dot{\psi}(t) = 0, \\
\bar{f}_2(t) \dot{\psi}(t) &= 0, \quad \dot{x}(t) \bar{f}_1(t) + \dot{y}(t) \bar{f}_2(t) - \bar{b}_1(t) \dot{\psi}(t) = 0. \\
\end{align*}
\]
(21)

The non-trivial solution of the above restrictions are $\bar{f}_1(t) \propto \dot{\psi}(t)$ and $\bar{f}_2(t) \propto \ddot{\psi}(t)$. For the algebraic convenience, however, we choose $\bar{f}_1(t) = \dot{\psi}(t)$ and $\bar{f}_2(t) = i \ddot{\psi}(t)$. Using these values (i.e. $\bar{f}_1 = \dot{\psi}$, $\bar{f}_2 = i \ddot{\psi}$), we obtain $\bar{b}_1 = \dot{x} + i \dot{y}$. The substitution of the above secondary variables in the equation (18) of the supervariable expansions leads to
\[
\begin{align*}
X^{(2)}(t, \theta, \bar{\theta}) |_{\bar{\theta} = 0} &= X^{(2)}(t, \theta), \\
X^{(2)}(t, \theta) &= x(t) + \theta (\dot{\psi}) \equiv x(t) + \theta (s_2 x), \\
Y^{(2)}(t, \theta, \bar{\theta}) |_{\bar{\theta} = 0} &= Y^{(2)}(t, \theta), \\
Y^{(2)}(t, \theta) &= y(t) + \theta (i \ddot{\psi}) \equiv y(t) + \theta (s_2 y), \\
\bar{\Psi}^{(2)}(t, \theta, \bar{\theta}) |_{\bar{\theta} = 0} &= \bar{\Psi}^{(2)}(t, \theta), \\
\bar{\Psi}^{(2)}(t, \theta) &= \dot{\psi}(t) + \theta (0) \equiv \dot{\psi}(t) + \theta (s_2 \ddot{\psi}), \\
\Psi^{(2)}(t, \theta, \bar{\theta}) |_{\bar{\theta} = 0} &= \Psi^{(2)}(t, \theta), \\
\Psi^{(2)}(t, \theta) &= \psi(t) + \theta \left[ i (\dot{x} + i \dot{y}) \right] \equiv \psi(t) + \theta (s_2 \psi),
\end{align*}
\]
(22)

where the superscript (2) stands for the expansions obtained after the application of SUSYIRs (20). It is clear from the above expansions that we have already derived the transformations $s_2$ for the variables $(x(t), y(t), \psi(t), \dot{\psi}(t))$ as given in our equation (2).

We now dwell a bit on the derivation of the SUSY transformations $s_2$ for the potential functions $A_x(x, y)$ and $A_y(x, y)$. First of all, we generalize these ordinary variables onto the $(1, 1)$-dimensional chiral super-submanifold as:
\[
\begin{align*}
A_x(x, y) &\rightarrow \bar{A}_x(X^{(2)}, Y^{(2)}) \equiv \bar{A}_x(x + \theta \ddot{\psi}, y + i \theta \ddot{\psi}) \\
&= A_x(x, y) + \theta \left[ \ddot{\psi} (\partial_x A_x(x, y) + i \partial_y A_x(x, y)) \right] \\
&\equiv A_x(x, y) + \theta (s_2 A_x(x, y)), \\
A_y(x, y) &\rightarrow \bar{A}_y(X^{(2)}, Y^{(2)}) \equiv \bar{A}_y(x + \theta \ddot{\psi}, y + i \theta \ddot{\psi}) \\
&= A_y(x, y) + \theta \left[ \ddot{\psi} (\partial_x A_y(x, y) + i \partial_y A_y(x, y)) \right] \\
&\equiv A_y(x, y) + \theta (s_2 A_y(x, y)).
\end{align*}
\]
(23)

A close look at the expansions (22) and (23) demonstrates that we have already obtained the SUSY transformations $s_2$ [cf. (2)] for all the relevant variables of the theory.$^5$ We further note that the following mappings do exist, namely:
\[
\frac{\partial}{\partial \theta} [\Omega^{(2)}(t, \theta, \bar{\theta})] |_{\bar{\theta} = 0} = s_2 \Omega(t) \equiv \pm i [\Omega(t), \bar{Q}]_{\pm},
\]
(24)

$^5$We would like to emphasize that all our transformations can be modified by a constant factor without violating the sanctity of our method. We have used such kind of modifications in Sec. 6.
where $\Omega^{(2)}(t)$ is the generic variable of the Lagrangian (1) [i.e. $\Omega^{(2)}(t) = x(t), y(t), \psi(t), \bar{\psi}(t), A_x(x, y), A_y(x, y)$] and $\Omega^{(2)}(t, \theta, \bar{\theta})|_{\theta = 0}$ denotes the supervariables [cf. (22), (23)] that have been obtained after the application of the SUSYIRs (20). We note (from (22)) that the nilpotent symmetry transformations $s_2$ and corresponding charge $Q$ are intimately related to the translational generator $\partial_\theta$ along the Grassmannian direction of the chiral super-submanifold. We further lay emphasis on the fact that the mapping in (24) provides the geometrical meaning of SUSY transformations $s_2$ as well as its nilpotency.

### 5 Invariance of Lagrangian and nilpotency of supercharges: Supervariable approach

In this section, first of all, we capture the SUSY invariance of starting Lagrangian (1) in the terminology of the supervariable approach. After this, we shall discuss about the nilpotency of the supercharges within the framework of the same formalism.

As far as the invariance of the Lagrangian $L_0$ of (1), under the SUSY symmetry transformations $s_1$ is concerned, we observe that the starting Lagrangian $L_0$ can be generalized onto a (1, 1)-dimensional anti-chiral supermanifold in the following manner:

$$L_0 \implies \tilde{L}_0^{(ac)} = \frac{1}{2} \left[ \dot{X}^{(1)}(t, \bar{\theta}) \dot{X}^{(1)}(t, \bar{\theta}) + \dot{Y}^{(1)}(t, \bar{\theta}) \dot{Y}^{(1)}(t, \bar{\theta}) \right] + i \bar{\Psi}^{(1)}(t, \bar{\theta}) \dot{\Psi}^{(1)}(t, \bar{\theta})$$

$$- \left[ \dot{X}^{(1)}(t, \bar{\theta}) \dot{A}_x(X^{(1)}, Y^{(1)}) + \dot{Y}^{(1)}(t, \bar{\theta}) \dot{A}_y(X^{(1)}, Y^{(1)}) \right]$$

$$+ \left[ \partial_x \left( \dot{A}_y(X^{(1)}, Y^{(1)}) \right) - \partial_y \left( \dot{A}_x(X^{(1)}, Y^{(1)}) \right) \right] \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}), \quad (25)$$

where all the supervariables, present in the Lagrangian $\tilde{L}_0^{(ac)}$, are the ones that have been derived in (14) as well as (16) and the superscript $(ac)$ stands for the anti-chiral behavior of the Lagrangian $\tilde{L}_0^{(ac)}$. In view of the mapping (15), the invariance of the starting Lagrangian $L_0$ under $s_1$ can be captured within the framework of the supervariable approach as:

$$\frac{\partial}{\partial \bar{\theta}} \tilde{L}_0^{(ac)} = - \frac{d}{dt} \left[ (A_x - i A_y) \psi \right] \iff s_1 L_0 = - \frac{d}{dt} \left[ (A_x - i A_y) \psi \right]. \quad (26)$$

The above equation encapsulate the geometrical meaning of the invariance of the starting Lagrangian $L_0$. This can be stated in the language of the translation along the Grassmannian direction $\bar{\theta}$. In fact, the above equation (26) demonstrates that the Lagrangian $\tilde{L}_0^{(ac)}$ of the theory is a sum of composite supervariables such that its translation along the Grassmannian $\bar{\theta}$-direction produces a total time derivative in the ordinary spacetime.

Exactly the above kind of analysis can be performed for the invariance of the starting Lagrangian $L_0$ under the SUSY transformations $s_2$. For instance, it can be checked that the $L_0$ can be generalized onto the (1, 1)-dimensional chiral super-submanifold as

$$L_0 \implies \tilde{L}_0^{(c)} = \frac{1}{2} \left[ \dot{X}^{(2)}(t, \theta) \dot{X}^{(2)}(t, \theta) + \dot{Y}^{(2)}(t, \theta) \dot{Y}^{(2)}(t, \theta) \right] + i \bar{\Psi}^{(2)}(t, \theta) \dot{\Psi}^{(2)}(t, \theta)$$

$$- \left[ \dot{X}^{(2)}(t, \theta) \dot{A}_x(X^{(2)}, Y^{(2)}) + \dot{Y}^{(2)}(t, \theta) \dot{A}_y(X^{(2)}, Y^{(2)}) \right]$$

$$+ \left[ \partial_x \left( \dot{A}_y(X^{(2)}, Y^{(2)}) \right) - \partial_y \left( \dot{A}_x(X^{(2)}, Y^{(2)}) \right) \right] \bar{\Psi}^{(2)}(t, \theta) \Psi^{(2)}(t, \theta), \quad (27)$$
where all the supervariables of \( \tilde{L}_0^{(c)} \) owe their origin to the superexpansions (22) and (23) and the superscript \((c)\) on the Lagrangian shows its chiral behavior. In view of the relationship in (24), it is obvious that

\[
\frac{\partial}{\partial \theta} \tilde{L}_0^{(c)} = \frac{d}{dt} [\bar{\psi} \left\{ \dot{x} + i \dot{y} - (A_x + i A_y) \right\}] \iff s_1 L_0 = \frac{d}{dt} [\bar{\psi} \left\{ \dot{x} + i \dot{y} - (A_x + i A_y) \right\}].
\] (28)

The above relationship provides the geometrical meaning for the quasi-invariance of the starting Lagrangian \( L_0 \) in the ordinary space under \( s_2 \). Geometrically, the invariance of the action \( S = \int dt L_0 \) can be captured within the framework of the supervariable approach.

The translation of the chiral Lagrangian \( \tilde{L}_0^{(c)} \) (which is a sum of the composite supervariables), along the \( \theta \)-direction of the (1, 1)-dimensional chiral super-submanifold, produces a total “time” derivative in the ordinary space. This is what has been done in (28).

Now we concentrate on the geometrical interpretation of the nilpotency of the supercharges \( Q(\bar{Q}) \) in the language of the translational generators \( (\partial_\theta, \partial_\bar{\theta}) \) along the \( (\theta, \bar{\theta}) \) directions of the (1, 1)-dimensional super-submanifolds of the general (1, 2)-dimensional supermanifold. Towards this goal in mind, we note that we can express the supercharge \( Q \) in terms of the anti-chiral supervariables, obtained after the application of SUSY restrictions (12), in three different ways as:

\[
Q = \frac{\partial}{\partial \bar{\theta}} \left[ -i \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} \left[ -i \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}) \right],
\]

\[
Q = \frac{\partial}{\partial \bar{\theta}} \left[ \left( \dot{x}(t) - i \dot{y}(t) \right) X^{(1)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} \left[ \left( \dot{x}(t) - i \dot{y}(t) \right) X^{(1)}(t, \bar{\theta}) \right],
\]

\[
Q = \frac{\partial}{\partial \bar{\theta}} \left[ i \left( \dot{x}(t) - i \dot{y}(t) \right) Y^{(1)}(t, \bar{\theta}) \right] \equiv \int d\bar{\theta} \left[ i \left( \dot{x}(t) - i \dot{y}(t) \right) Y^{(1)}(t, \bar{\theta}) \right].
\] (29)

where the ordinary variables are from (1) and the supervariables are from (14) and (16).

In view of the mapping (15), the above charge \( Q \) can be also expressed as follows:

\[
Q = s_1 \left[ -i \bar{\psi} \psi \right], \quad Q = s_1 \left[ (\dot{x} - i \dot{y}) x \right], \quad Q = s_1 \left[ i (\dot{x} - i \dot{y}) y \right].
\] (30)

In the ordinary space. Now the nilpotency of the charge \( Q \) becomes pretty trivial in the sense that it is connected with the nilpotency of the transformations \( s_1 \) through the relationship: \( s_1 Q = +i \left\{ Q, Q \right\} = 0 \) due to \( s_1^2 = 0 \). This observation could be also captured in the language of the translational generator \( \partial_\bar{\theta} \) because we observe that \( \partial_\bar{\theta} Q = 0 \) due to expressions of \( Q \) listed in (29) where we note that it is the nilpotency of the translational generator \( \partial_\bar{\theta} \) (i.e. \( \partial_\bar{\theta}^2 = 0 \)) which is responsible for the proof of the nilpotency of \( Q \). Thus, we emphasize that the nilpotency properties of \( Q, s_1 \) and \( \partial_\bar{\theta} \) are intertwined together in a geometrical meaningful manner (because we find that the quantities of ordinary space turn out to be connected with the translational generators along the Grassmannian directions of the supermanifold when a specific combination of the supervariables is translated).

We focus on the nilpotency of \( Q \) in the language of geometry on the chiral super-submanifold. Towards this goal in mind, we can also express the supercharge \( \bar{Q} \) in terms of the chiral supervariables, obtained after the application of SUSY restrictions (20), in three
different ways as:

\[ \bar{Q} = \frac{\partial}{\partial \theta} \left[ i \bar{\Psi}^{(2)}(t, \theta) \Psi^{(2)}(t, \theta) \right] \equiv \int d\theta \left[ i \bar{\Psi}^{(2)}(t, \theta) \Psi^{(2)}(t, \theta) \right], \]

\[ \bar{Q} = \frac{\partial}{\partial \theta} \left[ X^{(2)}(t, \theta) \left( \dot{x}(t) + i \dot{y}(t) \right) \right] \equiv \int d\theta \left[ X^{(2)}(t, \theta) \left( \dot{x}(t) + i \dot{y}(t) \right) \right], \]

\[ \bar{Q} = \frac{\partial}{\partial \theta} \left[ -i Y^{(2)}(t, \theta) \left( \dot{x}(t) + i \dot{y}(t) \right) \right] \equiv \int d\theta \left[ -i Y^{(2)}(t, \theta) \left( \dot{x}(t) + i \dot{y}(t) \right) \right], \]

where the ordinary variables are from (1) and the supervariables are from (22) and (23). In view of the mapping (24), we can express (31) in the ordinary space because the above charge (\( \bar{Q} \)) can be also expressed in terms of \( s_2 \) as follows:

\[ \bar{Q} = s_2 \left[ i \tilde{\psi} \psi \right], \quad \bar{Q} = s_2 \left[ x (\dot{x} + i \dot{y}) \right], \quad \bar{Q} = s_2 \left[ -i y (\dot{x} + i \dot{y}) \right]. \]

The above two equations (31) and (32) show that the charge \( \bar{Q} \) can be expressed in terms of nilpotent (\( s_2^2 = 0 \)) transformations \( s_2 \) and nilpotent (\( \bar{\partial}_\theta = 0 \)) translational generator (\( \partial_\theta \)).

A close look at (31) and (32) clarify the nilpotency of the charge \( \bar{Q} \) which is beautifully intertwined with the nilpotency of \( s_2 \) (i.e. \( s_2^2 = 0 \)) and/or nilpotency (\( \bar{\partial}_\theta = 0 \)) of the translational generator \( \partial_\theta \) on the chiral super-submanifold. This can be verified by the observation that \( s_2 \bar{Q} = +i \{ \bar{Q}, \bar{Q} \} = 0 \) due to nilpotency of \( s_2 \). Similarly, we note that \( \partial_\theta \bar{Q} = 0 \) because of the nilpotency (\( \bar{\partial}_\theta = 0 \)) of the generator \( \partial_\theta \). In view of the mapping in (24), we draw the conclusion that \( s_2 \bar{Q} = 0 \) which clearly proves that \( \{ \bar{Q}, \bar{Q} \} = 0 \). The latter relation emerges from \( s_2 \bar{Q} = +i \{ \bar{Q}, \bar{Q} \} = 0 \). Ultimately, we draw the important conclusion that the nilpotency of \( s_2 \), \( \bar{Q} \) and \( \partial_\theta \) blend together in a beautiful fashion within the framework of supervariable approach to SUSY system of the \( \mathcal{N} = 2 \) quantum mechanical model under consideration in our present endeavor.

### 6 Cohomological aspects: Continuous symmetries

For the sake of completeness of our paper, we concisely point out the mathematical meaning of the symmetry transformation operators (\( s_1, s_2, s_2 \)) that have been mentioned in equations (2) and (4). Towards this goal in mind, we modify these transformations by a constant factor\(^5\) as follows (see, e.g. [17] for details)

\[
\begin{align*}
    s_1 x &= \frac{\psi}{\sqrt{2}}, \\
    s_1 y &= \frac{-i \psi}{\sqrt{2}}, \\
    s_1 \psi &= 0, \\
    s_1 \bar{\psi} &= \frac{i}{\sqrt{2}} [\dot{x} - i \dot{y}], \\
    s_1 A_x &= \frac{1}{\sqrt{2}} (\partial_x A_x - i \partial_y A_x) \psi, \\
    s_1 A_y &= \frac{1}{\sqrt{2}} (\partial_x A_y - i \partial_y A_y) \psi, \\
    s_2 x &= \frac{\tilde{\psi}}{\sqrt{2}}, \\
    s_2 y &= \frac{i \tilde{\psi}}{\sqrt{2}}, \\
    s_2 \tilde{\psi} &= 0, \\
    s_2 \psi &= \frac{i}{\sqrt{2}} [\dot{x} + i \dot{y}], \\
    s_2 A_x &= \frac{\tilde{\psi}}{\sqrt{2}} (\partial_x A_x + i \partial_y A_x), \\
    s_2 A_y &= \frac{\tilde{\psi}}{\sqrt{2}} (\partial_x A_y + i \partial_y A_y).
\end{align*}
\]

\(^5\)We have taken a factor of \((1/\sqrt{2})\) in the overall transformations so that the corresponding charges would be able to satisfy one of simplest form of the \( sl(1/1) \) algebra of \( \mathcal{N} = 2 \) SUSY (cf. (40) below).
The above transformations are nilpotent \((s^2_1 = s^2_2 = 0)\) of order two and they are symmetry of the Lagrangian \(L_0\). It is straightforward to check that the algebra obeyed by the transformation operators \((s_1, s_2, s_ω)\) is

\[
s^2_1 = 0, \quad s^2_2 = 0, \quad \{s_1, s_2\} = s_ω = (s_1 + s_2)^2, \\
[s_ω, s_1] = 0, \quad [s_ω, s_2] = 0, \quad \{s_1, s_2\} \neq 0. \tag{34}
\]

At the algebraic level the above algebra is exactly like the algebra obeyed by the de Rham cohomological operators of differential geometry (see, e.g. [19-22] for details) where we have a set of three operators \((d, δ, ∆)\) which satisfy

\[
d^2 = 0, \quad δ^2 = 0, \quad \{d, δ\} = ∆ = (d + δ)^2, \\
[∆, d] = 0, \quad [∆, δ] = 0, \quad \{d, δ\} \neq 0. \tag{35}
\]

In the above, the operators \((δ)d\) are the (co-)exterior derivatives (with \(d^2 = δ^2 = 0\)) and \(∆ = (d + δ)^2\) is the Laplacian operator.

In the realm of differential geometry, one knows that the (co-)exterior derivatives are connected by the relation \(δ = ± * d * \) where \((*)\) is the Hodge duality operation on a given compact manifold on which the set \((d, δ, ∆)\) is defined. In our theory, the \((*)\) operation is replaced by a discrete set of symmetry transformations, namely;

\[
x \rightarrow \mp x, \quad ψ \rightarrow \mp ψ, \quad A_x \rightarrow \pm A_x, \quad t \rightarrow -t, \\
y \rightarrow \pm y, \quad \bar{ψ} \rightarrow \mp ψ, \quad A_y \rightarrow \mp A_y, \quad B_z \rightarrow B_z. \tag{36}
\]

under which the Lagrangian (1) remains invariant and we observe that the nilpotent transformations \(s_2\) and \(s_1\) are connected by the following relationship (see, e.g. [17] for details)

\[
s_2Φ_1 = + * s_1 * Φ_1 \Rightarrow s_2 = + * s_1 * , \quad Φ_1 = x, y, A_x, A_y, \\
s_2Φ_2 = - * s_1 * Φ_2 \Rightarrow s_2 = - * s_1 * , \quad Φ_2 = ψ, \bar{ψ}. \tag{37}
\]

where \((±)\) signs for the above relationship are governed by the application of two consecutive discrete symmetry transformations on a given dynamical variable as

\[
* [ * ] Φ_1 = + Φ_1, \quad Φ_1 = x, y, A_x, A_y, \\
* [ * ] Φ_2 = - Φ_2, \quad Φ_2 = ψ, \bar{ψ}. \tag{38}
\]

The above is the rule laid down by the requirements of a perfect duality invariant theory [23]. We note, from equation (38), that it is the interplay of continuous and discrete symmetry transformations of the theory which provide the physical realization of the relationship between the (co-)exterior derivatives: \(δ = ± * d * \) of differential geometry.

According to Noether’s theorem, the existence of the continuous symmetries lead to the derivation of the conserved charges. For our model under consideration, we note that

\[\text{It is elementary to note that the } s_ω \text{ transformations (8) would be now expressed modulo an } i \text{ factor.}\]
$(s_1, s_2, s_\omega)$ lead to the following expressions for the conserved charges $Q_i$ (with $i = 1, 2, 3$)

\begin{align*}
Q_1 &\equiv Q = \frac{1}{2} \left[ (p_x + A_x) - i (p_y + A_y) \right] \psi, \\
Q_2 &\equiv \bar{Q} = \frac{\bar{\psi}}{2} \left[ (p_x + A_x) + i (p_y + A_y) \right], \\
Q_3 &\equiv Q_\omega = \left[ \frac{(p_x + A_x)^2}{2} + \frac{(p_y + A_y)^2}{2} - B_z \bar{\psi} \psi \right] \equiv H. \tag{39}
\end{align*}

It can be checked readily that the above charges obey one of the simplest $\mathcal{N} = 2$ SUSY quantum mechanical algebra (without any central extension), namely;

\begin{align*}
Q^2 &= 0 \quad \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = H, \quad \dot{Q} = -i [Q, H] = 0, \quad \dot{\bar{Q}} = -i [\bar{Q}, H] = 0. \tag{40}
\end{align*}

which provides the physical realization of the Hodge algebra. The comparison between (35) and (40) shows that $\Delta$ and $H$ behave like the Casimir operators for the whole algebra, respectively, because they commute with the rest of relevant operators.

To establish the perfect analogy with the de Rham cohomological operators $(d, \delta, \Delta)$ of differential geometry, one has to prove other properties that should be obeyed by the charges $Q$, $\bar{Q}$ and $H$. For instance, when the (co-)exterior derivatives operate on a differential form, they (lower)raise the degree of the form by one whereas the degree of a form remains intact when it is acted upon by the Laplacian operator $\Delta$. This analogy has also been established in our earlier work where the algebra (40) for our present model plays a key role in the Fock space of the total quantum Hilbert states [17]. Finally, we conclude that our present SUSY model is a perfect model for the Hodge theory where the symmetry properties (36), (33) and (8) of the Lagrangian (1) provide the physical realization of the de Rham cohomological operators (i.e. $d, \delta, \Delta$) of the differential geometry.

### 7 Conclusions

In our present endeavor, we have taken an example of the $\mathcal{N} = 2$ SUSY quantum mechanical model whose superpotential is totally different from the cases of $\mathcal{N} = 2$ SUSY free particle and HO (and the generalization of HO) [14,15,17]. This has been done purposely so that our idea of supervariable approach [14,15] could be put on a firmer-footing. We have derived the proper $\mathcal{N} = 2$ transformations for the SUSY system under consideration by exploiting the idea of SUSYIRs and provided the geometrical basis for the nilpotency of SUSY transformations and SUSY invariance of the Lagrangian in the language of translational generators $(\partial_\theta, \partial_{\bar{\theta}})$ on the chiral and anti-chiral super-submanifolds of the general (1, 2)-dimensional supermanifold (on which our SUSY system is generalized).

We have demonstrated, in our present investigation, that the nilpotency of a SUSY transformation of an ordinary dynamical variable of the starting Lagrangian (1) is intimately connected with a set of two successive translations of the corresponding supervariable along $(\bar{\theta})\theta$ directions of the (1, 1)-dimensional (anti-)chiral super-submanifolds of the general (1, 2)-dimensional supermanifold on which our starting theory is generalized (cf. Sec. 5). Similarly, we have established that the SUSY invariance of the Lagrangian (1) is
equivalent to the translation of a sum of composite supervariables (that are present in the (anti-)chiral Lagrangians) along the $\bar{\theta}\theta$ directions of the (anti-)chiral super-submanifolds such that it produces a total time derivative in the ordinary space thereby rendering the action integral invariant under the $\mathcal{N} = 2$ SUSY transformations (cf. Sec. 5).

Our present work and earlier works [14,15] are our modest first few steps towards our main goal of deriving the SUSY transformations with the minimal knowledge about the classical Lagrangian and its symmetries. Such expectations and intuitions have emerged due to our experiences in the application of superfield formalism [4,5,9-13] to the gauge systems (described within the framework of BRST approach). In fact, in the realm of BRST formalism, if one knows the (anti-)BRST symmetries, there is absolutely no problem in obtaining the gauge-fixing and Faddeev-Popov ghost terms (see, e.g. [9-13]). Our central ideology is to develop theoretical tools and techniques so that we could derive the whole structure of the SUSY Lagrangian from the knowledge of SUSY symmetry transformations. Our plan is to devote more time on the supervariable approach to delve deep into the basic structure of SUSY theories where the symmetries would play an important role.

So far, we have applied our supervariable approach to the derivation of $\mathcal{N} = 2$ SUSY symmetries for some explicit examples, viz., $\mathcal{N} = 2$ SUSY free particle and HO. Our main goal is to apply the augmented version of superfield approach to SUSY gauge theories that have become important because of their relevance to the modern developments in (super)string theories. In fact, our aim is to study the $\mathcal{N} = 2, 4$ and 8 SUSY gauge theories within the framework of BRST formalism where, we are sure, our augmented version of BT-superfield formalism [9-13] is going to play very important role. As far as SUSY gauge gauge theories are concerned, we have already taken the first step and supersymmetrized the HC in the context of free spinning relativistic particle and obtained the proper (i.e. nilpotent and absolutely anticommuting) (anti-)BRST transformations [18]. Thus, we firmly believe that the superfield formalism [4-13] is going to play a decisive role in the BRST description of SUSY gauge theories which we envisage to work on in the future.

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Appendix A: Choice of the (anti-)chiral supervariables

In this Appendix, we explain the reasons behind our choice of (anti-)chiral supervariables for the derivation of the SUSY transformations within the framework of supervariable approach. We compare and contrast our supervariable approach with the BT-superfield formalism [4,5] applied for the derivation of (anti-)BRST symmetries ($s_{(a)b}$) for a given $p$-form ($p = 1, 2, 3...$) gauge theory which are nilpotent of order two (i.e. $s^2_{(a)b} = 0$) and absolutely anticommuting (i.e. $s_bs_{ab} + s_{ab}s_b = 0$) in nature.

As we have mentioned in the main body of our text, one of the key differences between the $\mathcal{N} = 2$ SUSY transformations and (anti-)BRST symmetry transformations is the fact that whereas the latter are absolutely anticommuting, the former are not. Thus, within
the framework of BT-superfield approach to (anti-)BRST symmetry transformations, a
generic superfield (defined on a (D, 2)-dimensional supermanifold) is expanded along both
the Grassmannian directions (θ and ¯θ) of the supermanifold, namely;

\[ \Sigma(x, \theta, \bar{\theta}) = \sigma(x) + \theta \bar{R}(x) + \bar{\theta} R(x) + i \theta \bar{\theta} S(x), \]  

(A1)

where σ(x) is an ordinary D-dimensional field and Σ(x, θ, ¯θ) is the corresponding superfield
on the (D, 2)-dimensional supermanifold that is characterized by the superspace coordinates
\( Z^M = (x^\mu, \theta, \bar{\theta}) \) where \( x^\mu \) is the bosonic coordinate (with \( \mu = 0, 1, 2, \ldots D - 1 \)).

It is evident from (A1) that if \( \sigma(x) \) is a bosonic ordinary field, then, \( \Sigma(x, \theta, \bar{\theta}) \) would be
also bosonic (i.e. secondary fields \( (\bar{R}, R) \) would be fermionic and \( S(x) \) bosonic). On the
contrary, if \( \sigma(x) \) is fermionic, then, \( \Sigma(x, \theta, \bar{\theta}) \) and \( S(x) \) would be fermionic, too. The pair
\((R, \bar{R})\) would become bosonic in the case of \( \sigma(x) \) being fermionic. A natural consequence
of the expansion in (A1) is the observation that

\[ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left( \Sigma(x, \theta, \bar{\theta}) \right) = i S(x) \quad \iff \quad s_b s_{ab} \sigma(x), \]

\[ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left( \Sigma(x, \theta, \bar{\theta}) \right) = -i S(x) \quad \iff \quad s_{ab} s_b \sigma(x), \]  

(A2)

where \( s_{(a)b} \) are the (anti-)BRST symmetries and they are identified with the translational
generators \( (\partial_\theta)\partial_{\bar{\theta}} \) along the Grassmannian direction \( (\theta)\bar{\theta} \) [4-13]. It is clear from (A2) that

\[ (\partial_\theta \partial_\theta + \partial_\theta \partial_{\bar{\theta}}) \Sigma(x, \theta, \bar{\theta}) = 0 \quad \iff \quad (s_b s_{ab} + s_{ab} s_b) \sigma(x) = 0, \]  

(A3)

which establishes the absolute anticommutativity of the (anti-)BRST symmetry transfor-
mations. Furthermore, it is also obvious that the nilpotency (i.e. \( \partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0 \)) of above
translational generators \( (\partial_\theta)\partial_{\bar{\theta}} \) implies the nilpotency of the (anti-)BRST transformations
(i.e. \( s_{(a)b}^2 = 0 \)). Thus, whenever we consider the full expansions (like (A1)) of the superfield,
the nilpotency as well as absolute anticommutativity properties are automatically captured
within the framework of superfield formalism (see, e.g. [4-13] for details). A key signature
of a gauge theory is the existence of the (anti-)BRST invariant Curci-Ferrari (CF) type
restrictions when this theory is discussed within the framework of BRST formalism [4-13].
If the general expansion of the superfield is taken like (A1), this condition also emerges
very naturally within the framework of superfield formalism (see, e.g. [4-13]). We have
been able to establish the connection of CF-type restrictions with the geometrical objects
called gerbes in the context of free Abelian 2-form and 3-form gauge theories [24,25].

The application of our supervariable approach to a SUSY system theoretically compels
us to choose the (anti-)chiral supervariables so that we could capture only the nilpotency
property but avoid the absolute anticommutativity of the \( N = 2 \) SUSY transformations.
This is what precisely we have done in our present endeavor.

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