VEECH SURFACES ASSOCIATED WITH RATIONAL BILLIARDS

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Abstract. A nice trick for studying the billiard flow in a rational polygon is to unfold the polygon along the trajectories. This gives rise to a translation or half-translation surface tiled by the original polygon, or equivalently an Abelian or quadratic differential.

Veech surfaces are a special class of translation surfaces with a large group of affine automorphisms, and interesting dynamical properties. The first examples of Veech surfaces came from rational billiards.

We first present the mathematical objects and fix some vocabulary and notation. Then we review known results about Veech surfaces arising from rational billiards. The interested reader will find annex tables on the author’s web page.

1. The objects

1.1. Polygons. We consider billiards in Euclidean (plane) polygons. A polygon is called rational if the angles made by its sides are all rational in unit $\pi$. (Note concerning non simply connected polygons: this condition is not only for adjacent sides.) A rational billiard is a billiard in a rational polygon. All billiards in this paper are rational.

A triangle is determined by its angles up to a scaling factor. Thus the parameter space for rational triangles is discrete. This is not the case for other polygons.

A triangle is acute if all its angles are acute, right if it has a right angle, obtuse if it has an obtuse angle.

A triangle is equilateral if its three angles are equal, isosceles if two of its angles are equal, scalene otherwise.

1.2. Translation and half-translation surfaces. A translation structure on a surface is an atlas on this surface with finitely many punctures, whose transition functions are translations. A surface which admits a translation structure is called a translation surface.

A half-translation structure on a surface is an atlas on this surface with finitely many punctures, whose transition functions are translations possibly composed with a central symmetry. A surface which admits a half-translation structure is called a half-translation surface.

A translation surface is a particular case of a half-translation surface. A half-translation surface is not a translation surface in general. But from a half-translation surface one can always obtain a translation surface by a branched double cover.
There is a correspondence between translation surfaces and Abelian differentials, and between half-translation surfaces and quadratic differentials.

1.3. Surfaces associated with a rational billiard.

Translation surface. There is a classical construction of a translation surface from a rational billiard (see [FoxKsh36, KtkZml75]). Starting with a polygon with angles $(m_i/n_i)\pi$, and letting $N = \text{lcm}(n_i)$, to each vertex of the polygon correspond $N/n_i$ singular points on the surface with multiplicity $m_i$ (cone points with angle $m_i2\pi$ or, in other words, zeros of order $m_i - 1$ of the Abelian differential).

Half-translation surface. One can consider a reduced construction in order to obtain a half-translation surface. Starting with a polygon with angles $(m_i/n_i)(\pi/2)$, and letting $N = \text{lcm}(n_i)$, to each vertex of the polygon correspond $N/n_i$ singular points on the surface with multiplicity $m_i/2$ (cone points with angle $m_i\pi$ or, in other words, zeros of order $m_i - 2$ of the quadratic differential).

Comparison. Both constructions are the same if the classical $N$ is odd. Otherwise the translation surface is a branched double cover of the half-translation surface.

Remark. For each $k$ we can construct a “$1/k$-translation” surface, endowed with a differential of order $k$. In particular, it is a flat surface with cone type singularities, whose angles are multiples of $2\pi/k$. The information for cone points can be obtained from angles of the polygon as above, by expressing them in unit $\pi/k$.

In particular, the minimal construction consists in gluing just two copies of the initial polygon.

However, translation and half-translation surfaces (or Abelian and quadratic differentials) play a distinguished role because there is an action of $\text{SL}(2, \mathbb{R})$ on the space of these surfaces (or of these differentials).

1.4. Veech group. On a translation surface we can consider affine diffeomorphisms (diffeomorphisms on the surface punctured at its singularities, that are affine in the charts of the translation structure, and extend to a homeomorphism of the whole surface). The derivatives (or linear parts) of these diffeomorphisms form a subgroup of $\text{SL}^\pm(2, \mathbb{R})$. If we consider only those diffeomorphisms which preserve orientation, we get a subgroup of $\text{SL}(2, \mathbb{R})$.

For half-translation surfaces we work modulo $-\text{Id}$ so that we obtain subgroups of $\text{PSL}^\pm(2, \mathbb{R})$ or $\text{PSL}(2, \mathbb{R})$.

We call Veech group the subgroup of $\text{PSL}(2, \mathbb{R})$ obtained for the translation surface with only non-removable singularities (those that correspond to zeros of the Abelian differential).

Triangle groups. We denote by $\triangle(p, q, r)$ the $(p, q, r)$-triangle group. See [BrdS8] for more information about these groups. This notation really refers to a conjugation class of subgroups of $\text{PSL}(2, \mathbb{R})$. Note that the class of $\text{PSL}(2, \mathbb{Z})$ is $\triangle(2, 3, \infty)$.

Lattice property. A surface has the lattice property (or Veech property) if its Veech group has finite covolume in $\text{PSL}(2, \mathbb{R})$. A polygon is said to have the lattice property if its associated translation surface does.

The lattice property is interesting in that it implies good ergodic properties, namely the Veech dichotomy: in each direction the flow is either periodic or uniquely ergodic.
2. The results

In this section we review polygons (mainly triangles) known to have or not to have the lattice property. We begin with the arithmetic examples, those that are the closest to the basic example of the square. We then move to regular polygons, and then to right and acute triangles for which the Veech property is characterized. For scalene triangles, only partial answers are known.

2.1. Arithmetic case.

The translation surface associated with the billiard in a rectangle, in a right isosceles triangle, in an equilateral triangle or in the \((\pi/6, \pi/3, \pi/2)\) triangle is a flat torus. Its Veech group is \(\triangle(2,3,\infty)\).

In the following subsections we always skip the torus case.

Gutkin and Judge proved the following in [GutJdg96, GutJdg00].

A translation surface has an arithmetic (conjugate to a subgroup of \(\text{PSL}(2,\mathbb{Z})\)) Veech group if and only if it is (translation) tiled by a Euclidean parallelogram, or in other words if it is a (translation) cover of a one-punctured flat torus.

2.2. Regular polygons. All regular polygons have the lattice property.

The Veech group of the translation surface associated with the regular \(n\)-gon for \(n \geq 5\) is a subgroup of \(\triangle(2,n,\infty)\). Its index can be given as follows [Vch92]. Let \(\varepsilon(n) = \gcd(2,n)\), let \(N = n/\varepsilon(n)\), let \(\sigma(n) = \gcd(4,n)\).

Let \(\omega(n) = n \prod_{p \mid n} (1 + 1/p)\). Then the index is \(\frac{\omega(N)}{\omega(\sigma(N))} \varepsilon(n)\).

2.3. Right triangles.

A right triangle has the lattice property if and only if its smallest angle is \(\pi/n\) for some \(n \geq 4\).

For \(n \geq 5\), the Veech group of the corresponding translation surface is \(\triangle(2,n,\infty)\) if \(n\) is odd, and \(\triangle(m,\infty,\infty)\) if \(n = 2m\).

Vorobets showed that the condition is sufficient and gave the corresponding Veech group [Vrb96, section 4]. Kenyon and Smillie showed that the condition is necessary [KenSmi00, section 6].

2.4. Acute triangles.

2.4.1. Isosceles.

An acute isosceles triangle has the lattice property if and only if its apex angle is \(\pi/n\) for some \(n \geq 3\).

For \(n \geq 4\), the Veech group of the associated translation surface is \(\triangle(n,\infty,\infty)\).

Showing that the condition is sufficient reduces to the right triangle case by an unfolding construction, see [Vrb94, section 5] or [GutJdg00]. Kenyon and Smillie showed that the condition is necessary [KenSmi00, section 6]. The Veech group was given for an example in [EarGrd97], and for the general case in [HbtSch00].
2.4.2. Scalene.

An acute scalene triangle has the lattice property if and only if it is one of the exceptional triangles

\[
\left(\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}\right), \quad \left(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}\right),
\]

The Veech groups of the associated translation surfaces are

\[\triangle(9, \infty, \infty), \quad \triangle(6, \infty, \infty), \quad \triangle(15, \infty, \infty).\]

T(3, 4, 5) appeared in [Vch89], T(3, 4, 5) and T(3, 5, 7) in [Vrb96]: T(2, 3, 4) first appeared in [KenSmi00] where Kenyon and Smillie conjectured that the three mentioned triangles were the only lattice examples amongst all rational acute triangles, and gave a proof with a bound on the common denominator of the angles (10000). Puchta made this bound useless [Pch01]. Hubert and Schmidt gave the precise Veech groups [HbtSch01].

2.5. Obtuse triangles. Ward [Wrd98] defines sharp triangles of type \((\frac{\pi}{m}, \frac{p\pi}{m}, \frac{q\pi}{m})\) with \(p < q\) and \(4p \leq m\), and proves the following.

A sharp triangle with \(p\) or \(m\) odd has the lattice property if and only if \(p = 1\) (in which case it is isosceles).

2.5.1. Isosceles. The following result is due to Veech.

The obtuse isosceles triangle with two angles \(\frac{\pi}{n}\) for \(n \geq 5\) has the lattice property. The Veech group of the associated translation surface is \(\triangle(2, n, \infty)\) if \(n\) is odd, and \(\triangle(m, \infty, \infty)\) if \(n = 2m\).

Hubert and Schmidt [HbtSch00] show that the lattice property is lost in this example if we mark the points that come from the vertices of angle \(\frac{\pi}{n}\) of the triangle.

2.5.2. Scalene. Examples by Vorobets [Vrb96] and Ward [Wrd98] show that among obtuse triangles with two angles of type \(\frac{\pi}{n}\) and \(\frac{\pi}{m}\), some have the lattice property and some do not.

In particular they studied those triangles with angles \(\frac{\pi}{2n}\) and \(\frac{\pi}{n}\).

For \(n \geq 4\), the triangle \(\left(\frac{\pi}{2n}, \frac{\pi}{n}, \frac{(2n - 3)\pi}{2n}\right)\) has the lattice property. The Veech group of the associated surface is \(\triangle(3, n, \infty)\).

Vorobets proved the lattice property, Ward [Wrd98 Theorem A] identified the precise Veech group.

Vorobets shows that the triangles T(1, 3, 8) and T(2, 3, 7) do not have the lattice property. For T(1, 3, 8), it also follows from Ward’s sharp triangles criterion.

2.6. Final comments. Other polygons that have been studied include rectangles, squares with a wall, rhombi, L-shaped polygons.

Rational billiards were the primary source of examples for Veech surfaces. They furnished discrete series and some isolated examples, but in all this only gave a finite number of examples in each genus. Other techniques recently provided infinitely many examples in genus two [Ch, Men].
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