ABELIAN POWERS IN PAPER-FOLDING WORDS

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ABSTRACT. We show that paper-folding words contain arbitrarily large abelian powers.

1. INTRODUCTION

Study of abelian powers in infinite words dates back to Erdős’s question whether there is an infinite word avoiding abelian squares ([6]). Abelian patterns, their presence or avoidability in infinite words, are a natural generalization of analogous questions for ordinary patterns. Both variants are amply studied. Thue’s famous square-free word over three letters (see [4]) has its counterpart in Keränen’s construction of an abelian square-free word ([9]). In [5], it is shown that all long abelian patterns are avoidable over two letters.

Paper-folding words are (infinite) words that can be represented by repeated folding of a paper strip. Some interesting properties of these words and further references can be found in [1, 2], see also [8].

During the workshop Outstanding Challenges in Combinatorics on Words at BIRS, James Currie raised the question (asked already in 2007 by Manuel Silva, [7]) whether paper-folding words contain arbitrarily large abelian powers. In this paper, we answer the question positively for all paper-folding words.

2. PRELIMINARIES

Paper-folding words can be defined in several equivalent ways. They are binary words that can be represented as limits of an infinite process of folding a strip of paper. The process is governed by another binary word \( b = b_0 b_1 b_2 \cdots \) called the sequence of instructions that tells whether the corresponding fold should be a hill or a valley. We shall follow the notation from [3] and use the binary alphabet \( \{1, -1\} \) for both the sequence of instructions and the resulting paper-folding word \( f = f_1 f_2 f_3 \cdots \). Note that the sequence of instructions is indexed starting from 0, while the paper-folding word starting from 1, which is justified by the following defining formula:

\[
f_i = (-1)^i b_k, \quad \text{where } i = 2^k(2j + 1).
\]

An equivalent formulation is that

\[
f_i = \begin{cases} -1 & \text{iff } i \equiv (2 + b_k) \cdot 2^k \mod 2^{k+2}, \\ 1 & \text{otherwise}. \end{cases}
\]

If \( i \equiv (2 + b_k) \cdot 2^k \mod 2^{k+2} \), then we say that \( f_i \) is \(-1\) of order \( k \). The regular paper-folding word is defined by \( b_i = 1 \) for all \( i \).

A word \( w \) is said to be an abelian \( m \)-power if \( w = w_1 w_2 \cdots w_m \), and the word \( w_j \) can be obtained from \( w_i \) by permutation of its letters for each \( i, j \in \{1, 2, \ldots, m\} \). In other words, \( |w_i|_a = |w_j|_a \) for each letter \( a \), where \( |u|_a \) denotes the number of
occurrences of the letter $a$ in $u$. We say that $f$ contains an abelian $m$-power if $f_if_{i+1} \cdots f_j$ is an abelian $m$-power for some $0 < i \leq j$.

An important ingredient of our considerations will be inequalities modulo a given number (in fact, always a power of two). This requires some clarification which should prevent possible confusion. By $(a \mod n)$ we denote the unique integer in $\{0, 1, \ldots, n-1\} \cap (a + n \cdot \mathbb{Z})$. We write

$$a \succ b \mod n$$

if and only if

$$(a \mod n) > (b \mod n)$$

in the standard integer order. Similarly we define $a \succeq b \mod n$.

Note the following fact.

**Fact 1.** Let $a, b, c, d \in \mathbb{Z}$. If

$$(a \mod n) + c, (b \mod n) + d \in \{0, 1, \ldots, n-1\},$$

then

$$a + c \succ b + d \mod n$$

if and only if

$$(a \mod n) + c > (b \mod n) + d,$$

$$a + c \succeq b + d \mod n$$

if and only if

$$(a \mod n) + c \geq (b \mod n) + d.$$

Note also that while $a \succ b \mod n$ always implies $a \succeq b + 1 \mod n$, the inverse implication never holds if $b + 1 \equiv 0 \mod n$.

### 3. The Problem

We are given a paper folding word $f$ defined by a sequence of instructions $b$. Our task is to find, for each $m$, numbers $s \geq 0$ and $d \geq 1$ such that the word

$$w_0w_2 \ldots w_{m-1},$$

where $w_j, j = 0, 1, \ldots, m-1$, is defined by

$$w_j = f_{s+jd+1}f_{s+jd+2} \cdots f_{s+(j+1)d},$$

is a abelian $m$-power. That is,

$$(|w_0|_{-1}, |w_2|_{-1}, \ldots, |w_{m-1}|_{-1})$$

is a constant vector.

For $b \in \{-1, 1\}$, $k \geq 0$ and $\ell, n \geq 1$, denote

$$D_{k,b}(\ell, n) := \# \{i \mid \ell < i \leq n, i \equiv (2 + b) \cdot 2^k \mod 2^{k+2}\}.$$ 

Then we have

$$|f_{\ell+1}f_{\ell+2} \cdots f_n|_{-1} = \sum_{k=0}^{\infty} D_{k,b_k}(\ell, n).$$

Note that $D_{k,b}(\ell, n)$ counts the number of $-1$s of order $k$ present in the word $f_{\ell+1}f_{\ell+2} \cdots f_n$. This “stratification” is very useful, since $D_{k,b}(\ell, n)$ can be well estimated from the length of the interval.

**Lemma 2.**

$$D_{k,b}(\ell, n) = \left\lceil \frac{n - \ell}{2^{k+2}} \right\rceil + \varepsilon_{k,b}(\ell, n),$$
where

\[
\varepsilon_{k,b}(\ell, n) = \begin{cases} 
1 & \text{if } n - \ell > (2 + b) \cdot 2^{k} - (\ell + 1) \mod 2^{k+2}, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Let

\[ q = \left\lfloor \frac{n - \ell}{2^{k+2}} \right\rfloor \quad \text{and} \quad r \equiv n - \ell \mod 2^{k+2}. \]

Divide the interval \((\ell, n]\) of integers into \(q + 1\) subintervals

\[
[\ell + j \cdot 2^{k+2} + 1, \ell + (j + 1) \cdot 2^{k+2}], \quad j = 0, \ldots, q - 1,
\]

\[
[\ell + q \cdot 2^{k+2} + 1, \ell + q \cdot 2^{k+2} + q].
\]

First \(q\) intervals have length \(2^{k+2}\) whence each of them contains exactly one number equal to \((2 + b) \cdot 2^{k} \mod 2^{k+2}\). The last interval, possibly empty, may or may not contain such a number. One readily verifies that the value of \(\varepsilon_{k,b}(\ell, n)\) defined by (2) indicates the presence of that additional \((2 + b) \cdot 2^{k} \mod 2^{k+2}\).

\[ \square \]

**Example.** An interval \((\ell, \ell + 6]\) of length six is supposed to contain one number \(3 \mod 4\) and no number \(6 \mod 8\). However, if \(\ell \mod 4\) is 1 or 2, then the interval contains two numbers \(3 \mod 4\) and \(\varepsilon_{0,1}(\ell, \ell + 6) = 1\). Moreover, if \(\ell \mod 8\) is not 6 or 7 then the interval contains a number \(6 \mod 8\) and \(\varepsilon_{1,1}(\ell, \ell + 6) = 1\) (see Figure 1).

It is easy to see that for each \(\ell, u \geq 0\) and \(b \in \{1, -1\}\)

\[
\varepsilon_{k,b}(\ell, \ell + 2^{u}) = \begin{cases} 
0 & \text{if } k \leq u - 2, \\
D_{k,b}(\ell, \ell + 2^{u}) & \text{otherwise}.
\end{cases}
\]

The whole problem concentrates in the distribution of those “additional” minus ones given by the mapping \(\varepsilon\). Let’s therefore define

\[ \mathcal{E}_{k,b}(s, d, m) := (\varepsilon_{k,b}(s, s + d), \varepsilon_{k,b}(s + d, s + 2d), \ldots, \varepsilon_{k,b}(s + (m - 1)d, s + md)). \]
and

$$\Delta(s, d, m) := \sum_{k=0}^{\infty} \varepsilon_{k, b_{k}}(s, d, m).$$

Note that $\Delta(s, d, m)$ is just the vector $\mathbf{1}$ scaled down by a constant vector corresponding to the number of $-1$s expected given the length $d$. Our aim is to find (for a given $m$) numbers $s$ and $d$ such that $\Delta(s, d, m)$ is a constant vector.

4. Additivity

The key tool in the solution of the problem is the possibility to add two $\Delta$-vectors, formulated in the following lemma.

**Lemma 3** (Additivity of $\Delta$-vectors). Let $s$, $s' \geq 0$, and $d$, $d' \geq 1$ be positive integers such that $s'$ and $d'$ are even. Let $r$ be such that

$$2^r > s + md,$$

and for each $i \geq 0$ the following implication holds:

$$\text{if } \varepsilon_{i, 1}(s', d', m) \neq \varepsilon_{i-1}(s', d', m) \text{ then } b_i = b_{i+r}.$$

Then

$$\Delta(s, d, m) + \Delta(s', d', m) = \Delta(s + 2^r s', d + 2^r d', m).$$

**Proof.** Denote $s'' = s + 2^r s'$ and $d'' = d + 2^r d'$.

Let firstly $k \leq r - 1$. Since

$$s'' + jd'' \equiv s + jd \mod 2^{k+2}$$

holds for each $j$ (recall that $s'$ and $d'$ are even), we have

$$\varepsilon_{k, b_{k}}(s, d, m) = \varepsilon_{k, b_{k}}(s'', d'', m).$$

Let now $k = r + i$ with $i \geq 0$. Let $j \in \{0, 1, \ldots, m - 1\}$ and $b \in \{-1, 1\}$. If

$$d' \geq (2 + b) \cdot 2^i - (s' + jd') - 1 \mod 2^{i+2},$$

then

$$d' \geq (2 + b) \cdot 2^i - (s' + jd') \geq 1 \mod 2^{i+2},$$

which implies

$$2^r d' \geq (2 + b) \cdot 2^k - 2^r (s' + jd') \geq 2^r \mod 2^{k+2}.$$

From (1) we deduce (see Fact 1)

$$2^r d' + d \geq (2 + b) \cdot 2^k - 2^r (s' + jd') - s - jd - 1 \mod 2^{k+2}.$$

Indeed, both sides of the inequality (7) are divisible by $2^r$ whence adding $d$ to the left side and subtracting $s + jd + 1$ from the right side keeps both sides in the interval $\{0, 1, \ldots, 2^{k+2} - 1\}$. We have shown

$$\varepsilon_{i, b}(s' + jd', s' + (j + 1)d') = \varepsilon_{k, b}(s'' + jd'', s'' + (j + 1)d'') = 1.$$

On the other hand, if

$$(2 + b) \cdot 2^i - (s' + jd') - 1 \geq d' \mod 2^{i+2},$$

then

$$2^r d' \geq (2 + b) \cdot 2^k - 2^r (s' + jd') \geq 2^r \mod 2^{k+2}.$$
The inequality (4) implies $2^r - s - jd - 1 \geq d$, and, as above, we deduce
\[(2 + b) \cdot 2^k - 2^r(s' + jd') - s - jd - 1 \geq 2^r d' + d \mod 2^{k+2}.
\]
In this case, we have
\[
(2 + b) \cdot 2^k - 2^r(s' + jd') - s - jd - 1 \geq 2^r d' + d.
\]
Using the assumption (5), we deduce from (8) and (10)
\[
E_{k,b}(s'', d'', m) = E_{r+1,b,r+1}(s'', d'', m).
\]
Since the inequality (4) implies $E_{k,b}(s, d, m) = 0$ for $k \geq r$, we have
\[
\Delta(s'', d'', m) = \sum_{k=0}^{r-1} E_{k,b}(s'', d'', m) + \sum_{i=0}^{\infty} E_{r+i,b,r+i}(s'', d'', m) = \Delta(s, d, m) + \Delta(s', d', m),
\]
and the proof is complete. \( \square \)

5. PROOF OF THE MAIN CLAIM

Additivity of $\Delta$-vectors implies that in order to solve the problem it is enough to find $\Delta$-vectors that sum to a constant vector and the corresponding parts of the sequence of instructions are synchronized.

Next lemma indicates an interval which is free of $-1$s of high orders.

**Lemma 4.** For each $t \geq 0$ there is a number $\ell_t$ such that
\[
D_{k,b}(\ell_t, \ell_t + 2t - 1) = 0
\]
for all $k \geq t$.

**Proof.** The number $\ell_t$ depends on values of $b_t, b_{t+1}, b_{t+2}$ and $b_{t+3}$ and we will give it explicitly. Let $\ell = \ell(x_0, x_1, x_2, x_3)$, with $x_i \in \{1, -1\}$, be given by Figure 2. It can be directly checked that
\[
D_{0,x_0}(\ell, \ell + 3) = D_{1,x_1}(\ell, \ell + 3) = D_{2,x_2}(\ell, \ell + 3) = D_{3,x_3}(\ell, \ell + 3) = 0
\]
and also
\[
D_{k,b}(\ell, \ell + 3) = 0
\]
for both $b \in \{1, -1\}$ and each $k \geq 4$. Consequently, $\ell_0 = \ell(b_0, b_1, b_2, b_3)$ has desired properties. Note that this is equivalent to $f_{\ell_0+1} = f_{\ell_0+2} = f_{\ell_0+3} = 1$.

In general, we define
\[
\ell_t = 2^t \ell(b_t, b_{t+1}, b_{t+2}, b_{t+3}).
\]
Suppose that there is a number $n \equiv (2 + b_k) \cdot 2^k \mod 2^{k+2}$ with $k \geq t$ and $\ell_t < n < \ell_t + 2^{t+2}$. Then $n' = 2^{-t} n$ satisfies
\[
n' \equiv (2 + b_k) \cdot 2^{k-t} \mod 2^{k-t+2}
\]
and
\[
\ell(b_t, b_{t+1}, b_{t+2}, b_{t+3}) < n' < \ell(b_t, b_{t+1}, b_{t+2}, b_{t+3}) + 4,
\]
a contradiction with the first part of the proof. \( \square \)
Remark 5. The interval \((\ell_t, \ell_t + 2^t - 1]\) from the previous lemma has length \(2^t - 1\) and contains no number equivalent to \((2 + b) \cdot 2^t \mod 2^{t+2}\). Therefore, it spreads between two consecutive occurrences of such numbers. Moreover, the word \(f_{\ell_t+1} f_{\ell_t+2} \cdots f_{\ell_t+2^{t+2}-1}\) has a period \(2^{t+1}\), and it is of the form \(w1w\). Lemma 4 is therefore a slightly stronger form of \([8, \text{Proposition 5}]\).

Lemma 4 has the following important consequence.

Lemma 6. Let \(t \geq 1\), \(0 \leq u \leq t\) and \(0 \leq p \leq 2^t - 1\). Then

1. for each \(k \geq 0\) such that \(k \leq u - 2\) or \(k \geq t - 1\), we have
   \[ \Delta_{k,b}(\ell_{t-1} + 2^u p, 2^u, 2^{t-u}) = 0; \]

2. moreover,
   \[ \Delta_{k,b}(\ell_{t-1} + 2^u p, 2^u, 2^{t-u}) = 0 \]
   for both \(b \in \{-1,1\}\) and each \(k \geq 0\) such that \(k \leq u - 2\) or \(k \geq t + 3\).

Proof. All intervals covered by the vector \(\Delta_{k,b}(\ell_{t-1} + 2^u p, 2^u, 2^{t-u})\) lie in \((\ell_{t-1}, \ell_{t-1} + 2^{t+1})\). The claim now follows from \([3]\), Lemma 4 and from the fact that \(\ell_{t-1}\) depends on values \(b_{t-1}, b_t, b_{t+1}\) and \(b_{t+2}\) only. \(\square\)

We now indicate vectors with a constant sum.

Lemma 7. For each \(t \geq 2\) and \(1 \leq u < t\)

\[ D := \sum_{p=0}^{2^t - u - 1} \Delta(\ell_{t-1} + 2^u p, 2^u, 2^{t-u}), \]

is a constant vector.
Figure 3. Illustration of Lemma 7. Green intervals sum to $D_2$.

Proof. From the definition of $\Delta$ and Lemma 6(1), we deduce

$D = (D_0, D_1, \ldots, D_{2^t-u-1}) = \sum_{p=0}^{2^t-u-1} \sum_{k=u}^{t-2} \varepsilon_{k,b_k}(\ell_{t-1} + 2^u p, 2^u, 2^t-u),$

and

$D_i = \sum_{p=0}^{i-2} \sum_{k=0}^{t-1} \varepsilon_{k,b_k}(\ell_{t-1} + 2^u(p+i), \ell_{t-1} + 2^u(p+i+1)).$

By (3), we have (see Figure 3)

$D_i = \sum_{k=u}^{t-1} D_{k,b_k}(\ell_{t-1} + 2^u i, \ell_{t-1} + 2^t + 2^u i).$

Since the length of the interval $(\ell_{t-1} + 2^u i, \ell_{t-1} + 2^t + 2^u i)$ is $2^t$, which is divisible by $2^k+2$ for all $k = \{0, 1, \ldots, t-2\}$, we finally obtain

$D_i = \sum_{k=0}^{t-2} \frac{2^t}{2^k+2} = 2^t-u - 1,$

which is independent of $i$ as claimed. □

NB: the previous lemma holds also for $u = 0$. We omit this case for sake of simplicity.

We are ready to prove the main claim of the paper.

Theorem 8. All paper-folding words contain arbitrarily large abelian powers.

Proof. Let $f$ be a paper-folding word defined by a sequence of instructions $b$. Let $m = 2^q$ for some $q \in \mathbb{N}$. We shall find an abelian $m$-power in $f$. Choose $u \geq 1$ and $t \geq 2$ such that $t-u = q$ and the factor

$b_{u-1}b_u b_{u+1} \cdots b_{t+1}b_{t+2}$

occurs infinitely many times in $b$. Using Lemma 3 inductively, we construct numbers $s_j$ and $d_j$, with $j \in \{1, 2, \ldots, m\}$, such that

$\Delta(s_j, d_j) = \sum_{p=0}^{j-1} \Delta(\ell_{t-1} + 2^u p, 2^u, m).$
Clearly, $s_1 = \ell_{t-1}$ and $d_1 = 2^u$. Numbers $s_{j+1}, d_{j+1}$ are defined by

\[ s_{j+1} = s_j + 2^{r_j}(\ell_{t-1} + 2^u j), \quad d_{j+1} = d_j + 2^{r_j}2^u, \]

where $r_j$ is chosen to satisfy $2^{r_j} > s_j + md_j$ and

\[ b_i = b_{i+1} \text{ for each } i = u-1, u, \ldots, t+1, t+2. \]

Since $u \geq 1$ and $t \geq 2$, all $d_j$ and $s_j$ are even. Lemma 6(2) implies that the assumption (5) of Lemma 3 is fulfilled, whence

\[ \Delta(s_{j+1}, d_{j+1}, m) = \Delta(s_j, d_j, m) + \Delta(\ell_{t-1} + 2^u j, 2^u, m) = \sum_{p=0}^{j} \Delta(\ell_{t-1} + 2^u p, 2^u, m) \]

as required. By Lemma 7, the vector $\Delta(s_m, d_m, m)$ is a constant vector, which means that

\[ f_{s_m+1}f_{s_m+2}\cdots f_{s_m+md_m}, \]

is an abelian $m$-power. \qed

6. Examples

Let us use Theorem 8 to find an abelian fourth power in the regular paper-folding sequence. Regularity allows to ignore the condition (5) of Lemma 3.

Choose $u = 1$ and $t = 3$. We have $\ell_2 = 4 \cdot \ell_0(1, 1, 1, 1) = 4 \cdot 7 = 28$. Lemma 4 claims that the word $f_{29}f_{30}\cdots f_{43}$ does not contain $-1$s of order higher than one. This, in particular, means that the word has period 8.

| $i$ | 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 |
|-----|------------------|
| $f_i$ | 1 -1 -1 1 1 -1 1 1 -1 1 1 1 -1 |

We want to sum vectors

\[
\Delta(28, 2, 4) = (1, 1, 0, 1), \\
\Delta(30, 2, 4) = (1, 0, 1, 1), \\
\Delta(32, 2, 4) = (0, 1, 1, 1), \\
\Delta(34, 2, 4) = (1, 1, 1, 0).
\]

In Lemma 3 we choose $r = r_2 = 6$ since $2^6 > 28 + 4 \cdot 2 = 36$, and obtain

\[
\Delta(28, 2, 4) + \Delta(30, 2, 4) = \Delta(28 + 2^6 \cdot 30, 2 + 2^6 \cdot 2, 4) = \Delta(1948, 130, 4) = (2, 1, 1, 2).
\]

Since $2^{12} > 1948 + 4 \cdot 130 > 2^{11}$, we put $r_2 = 12$, and get

\[
\Delta(1948, 130, 4) + \Delta(32, 2, 4) = \Delta(1948 + 2^{12} \cdot 32, 130 + 2^{12} \cdot 2, 4) = \\
= \Delta(133020, 8322, 4) = (2, 2, 2, 3).
\]

Finally, with $r_3 = 18$, we have

\[
\Delta(133020, 8322, 4) + \Delta(34, 2, 4) = \Delta(9045916, 532610, 4) = (3, 3, 3, 3).
\]
The following tables illustrate how the addition works.

| $k$ | $\mathcal{E}_{k,1}(28, 2, 4)$ | $k$ | $\mathcal{E}_{k,1}(30, 2, 4)$ |
|-----|-------------------------------|-----|-------------------------------|
| 0   | (0, 1, 0, 1)                  | 0   | (1, 0, 1, 0)                  |
| 1   | (1, 0, 0, 0)                  | 1   | (0, 0, 0, 1)                  |

| $k$ | $\mathcal{E}_{k,1}(32, 2, 4)$ | $k$ | $\mathcal{E}_{k,1}(34, 2, 4)$ |
|-----|-------------------------------|-----|-------------------------------|
| 0   | (0, 1, 0, 1)                  | 0   | (1, 0, 1, 0)                  |
| 1   | (0, 0, 1, 0)                  | 1   | (0, 1, 0, 0)                  |

| $k$ | $\mathcal{E}_{k,1}(9,045,916, 532,610, 4)$ |
|-----|-------------------------------------------|
| 0   | (0, 1, 0, 1)                              |
| 1   | (1, 0, 0, 0)                              |
| 2   | (0, 0, 0, 0)                              |
| 3   | (0, 0, 0, 0)                              |
| 4   | (0, 0, 0, 0)                              |
| 5   | (0, 0, 0, 0)                              |
| 6   | (1, 0, 1, 0)                              |
| 7   | (0, 0, 0, 1)                              |
| 8   | (0, 0, 0, 0)                              |
| 9   | (0, 0, 0, 0)                              |
| 10  | (0, 0, 0, 0)                              |
| 11  | (0, 0, 0, 0)                              |
| 12  | (0, 1, 0, 1)                              |
| 13  | (0, 0, 1, 0)                              |
| 14  | (0, 0, 0, 0)                              |
| 15  | (0, 0, 0, 0)                              |
| 16  | (0, 0, 0, 0)                              |
| 17  | (0, 0, 0, 0)                              |
| 18  | (1, 0, 1, 0)                              |
| 19  | (0, 1, 0, 0)                              |

Lemma 3, however, can be used to find more reasonable fourth power. One easily verifies that $\Delta(6, 1, 4) = (1, 0, 0, 0)$ and $\Delta(0, 2, 4) = (0, 1, 1, 1)$. Since $2^4 > 6 + 4 \cdot 1$, we have

$$\Delta(6, 1, 4) + \Delta(0, 2, 4) = \Delta(6 + 2^4 \cdot 0, 1 + 2^4 \cdot 2, 4) = \Delta(6, 33, 4) = (1, 1, 1, 1).$$

Corresponding tables are as follows.

| $k$ | $\mathcal{E}_{k,1}(6, 1, 4)$ | $k$ | $\mathcal{E}_{k,1}(0, 2, 4)$ |
|-----|-------------------------------|-----|-------------------------------|
| 0   | (1, 0, 0, 0)                  | 0   | (0, 1, 0, 1)                  |
| 1   | (0, 0, 0, 0)                  | 1   | (0, 0, 1, 0)                  |

| $k$ | $\mathcal{E}_{k,1}(6, 33, 4)$ |
|-----|-------------------------------|
| 0   | (1, 0, 0, 0)                  |
| 1   | (0, 0, 0, 0)                  |
| 2   | (0, 0, 0, 0)                  |
| 3   | (0, 0, 0, 0)                  |
| 4   | (0, 1, 0, 1)                  |
| 5   | (0, 0, 1, 0)                  |
Looking at the zero vectors for $k = 1, 2, 3$ in the resulting table, one may be tempted to think that $2^4$ is unnecessarily large scaling ratio. However, this is not true, since we have
\[
\Delta(6, 1, 4) + \Delta(0, 2, 4) \neq \Delta(6 + 2^3 \cdot 0, 1 + 2^3 \cdot 2, 4) = \Delta(6, 17, 4) = (1, 1, 2, 0).
\]

| $k$ | $\mathcal{E}_{k,1}(6,17,4)$ |
|-----|-----------------------------|
| 0   | (1, 0, 0, 0)                |
| 1   | (0, 0, 0, 0)                |
| 2   | (0, 0, 0, 0)                |
| 3   | (0, 1, 1, 0)                |
| 4   | (0, 0, 1, 0)                |

Acknowledgments

I am grateful to James Currie, Narad Rampersad and, in particular, to Thomas Stoll for inspiring discussion about the problem and for useful comments. I also thank Jeffrey Shallit for his remarks.

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