A New Constitutive Model for Alumina Powder Compaction

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Abstract

Based on a systematic experimental investigation of the mechanical properties of alumina powder a new general 3-D constitutive model for compaction of powders has been developed. A significant feature of the model is that it describes the time effects on compaction. These effects have been revealed by creep and relaxation phenomena and by the influence of the strain rate. Although the theory is applicable to general 3-D conditions all the parameters involved in the model can be derived from the results of a few number of conventional triaxial drained compression tests. The procedure to identify the model parameters from data is general. Unlike the models available in the literature, the model developed in the present paper accounts for both compressibility and dilatant behavior of powders. Thus it provides a realistic description and a better understanding of the mechanisms of powder pressing. The theory is able to model important aspects of powder behavior with a degree of accuracy which appears to make it useful in FEM analyses of stress and density distribution in a pressed compact.

1. Introduction

Almost every industry handles powders/bulk solids, either as raw materials, intermediates in the production process, or as final products. Examples include: chemicals and chemical processing, foods, detergents, ceramics, minerals, pharmaceuticals, etc. However, the understanding of powder behavior, despite its economic importance, is yet limited compared to other classes of materials or disperse systems. Particularly, the mechanical behavior of cohesive bulk solids under external loads can be critical to the final performance and quality of the product. To mould an advanced ceramic shape, for example, the principal method is still pressing. High production rates and close tolerances can be achieved with dry pressing. However, important problems such as cracking and pore-formation, non-uniform density have yet to be overcome. Furthermore, flaws may not be visible by routine inspection and thus, the defects are carried out through sintering to the final product. Also the density variation in the green compact causes distortion or cracking during fire. In order to reduce density variation in the compact, moulding is restricted to simple geometrical shapes of relatively small sizes.

Modelling of the mechanics of compaction has thus developed through the necessity of controlling densification kinetics and shape change in order to optimize the design start up and processing of powders. A promising approach for practical applications appears to be to treat the powder as a homogeneous continuum. During the past decades considerable efforts have been undertaken to describe the behavior of granular materials within the framework of classical plasticity theory. Two main methods have been followed in order to formulate constitutive models. The first one, that can be referred to as the empirical procedure, is based on the best fitting of the data obtained from a series of tests by means of simple functions (polynomial, exponential, etc.). The characteristics of these functions are chosen just to meet the shape of the experimentally obtained pressure versus density diagrams. A list of the most used empirical laws can be found in the review paper by Shinohara [14]. The empirical laws lead to accurate results when applied to situations having boundary conditions, and loading paths similar to those used in the tests from which the laws have been formulated. However, the use of these laws for different conditions would in general lead to erroneous results.

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A second procedure consists of developing a constitutive model, represented by a set of stress-strain (time) relationships having a sound mechanical basis. Afterwards, the values of the material parameters are calibrated from data. An adequate constitutive model generally lead to results reasonably close to those experimentally observed, even when applied to situations different from those considered for the determination of its parameters.

During the past decades considerable efforts have been undertaken to adapt existing models for granular materials (mainly soils) to powder-like materials. Generally, the theory used for establishing the yield locus is the Mohr-Coulomb theory:

\[ \sigma = C + \sigma_n \tan \Phi \]  

(1)

where \( \sigma \) is the magnitude of the shear stress on the failure plane, \( \sigma_n \) is the normal stress on that plane (compression stresses are assumed to be positive), \( C \) is the cohesion, and \( \Phi \) is the angle of internal friction. The cohesion and the slope of the yield locus (\( \tan \Phi \)) as well as the wall friction are measured using the Jenike cell. The Jenike Cell is still the most widely used flow-properties measuring device in industry and the yield locus [Eq. (1)] is generally considered for bin design.

The use of the Mohr-Coulomb criterion [Eq. (1)] is based on the assumption that the material is perfectly plastic, i.e., the yield locus is fixed in the stress space. Thus the yield locus coincides with the failure surface (Kamath et al. [9]). However, the experimental evidence suggests that for powders a clear distinction exists between the stress conditions at which the irreversible behavior initiates and those corresponding to failure. Recently, critical-state models have been applied to describe the yield locus (\( \tan \Phi \)) as well as the wall friction are measured using the Jenike cell. The Jenike Cell is still the most widely used flow-properties measuring device in industry and the yield locus [Eq. (1)] is generally considered for bin design.

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The critical-state concept is generally the only one used for the description of steady-state flow and the concept of critical state line. The critical-state concept states that if a sample of material undergoes an increasing deviatoric strain, it tends towards an ultimate condition defined by a zero volume change and a constant ratio of the deviatoric stress \( q \) over mean stress \( p \):

\[ d \varepsilon_v = 0, \quad d \varepsilon_d = 0, \quad q = Np, \quad \frac{\varepsilon_d}{\varepsilon_v} = \lambda \ln \left( \frac{p}{p_o} \right) \]  

(2)

\( N, \lambda \) and \( \lambda \) are material constants which do not depend on the initial volume \( v_o, \nu \) is the current volume of the sample and \( p_o \) is a reference pressure; \( p \) and \( q \) denote the stress invariants, \( \varepsilon_v \) and \( \varepsilon_d \) the strain invariants (see Nomenclature):

\[ p = \frac{1}{3} \text{tr} \sigma, \quad q = \left[ \frac{3}{2} \text{tr} \sigma^2 \right]^{\frac{1}{2}}, \]  

\[ \varepsilon_v = \text{tr} \varepsilon, \quad \varepsilon_d = \frac{2}{3} \varepsilon' : \varepsilon' \]  

(3)

The volume of the material is therefore defined by the set \( (p, q, \nu) \). This concept seemed to be verified at least qualitatively in clays.

However, experimental data on powders reported by various authors \([8, 14]\) suggest that there exists, for each confining pressure, a stress value for which the slope of the octahedric shear stress versus volumetric strain curve changes sign, i.e., above which the volume expands. Thus, the behavior of powders is not described correctly for undrained conditions by the critical state class of models. It appears then necessary to develop a constitutive model of enough generality to properly account for the complex volumetric behavior of powders.

All the models discussed so far neglect the time effect on compaction. However, it is well known that time has an important influence on the mechanical behavior of powders. This effect is revealed by creep and relaxation phenomena, and by the influence of rate of strain. Several points arise: (i) What is the importance of the viscous part of the behavior in comparison with the elasto-plastic behavior, which is generally the only one considered? (ii) How does one measure this effect without performing very long experiments which are inevitably expensive?

In the present paper an attempt is made to address these topics. First, we present the experimental results concerning the influence of time on compaction of alumina powder. Then, based on the data, we formulate a 3-D elastic-viscoplastic model which describes the behavior of the material under general loading conditions. Finally, data from conventional triaxial compression tests are used to demonstrate the capability of the model to reproduce the commonly observed features of powder's response.

2. Experimental studies

The bulk properties of alumina powder were investigated by means of hydrostatic and triaxial compression tests on cylindrical specimens 7 cm in diameter and 15 cm in length. To assure a similar
specimen for each test and in order to minimize the variability in axial deformation and specimen volumetric change the two-layers filling method has been followed. This method consists of filling half of the triaxial mould with material, compressing it with uniform tamping and repeating this operation until the mould is completely filled. Thus for all tests the initial bulk density varied from 1.42 g/cm$^3$ to 1.427 g/cm$^3$. The arithmetic mean particle size varied from 40 $\mu$m to 200 $\mu$m. The specimen was saturated with water so that before testing the water content was 50% per volume.

2.1 Hydrostatic tests.

Figure 1 shows the results of a hydrostatic test with 4 unloading-reloading cycles. Here, and throughout this paper, compressive stresses and strains are assumed to be positive. $\varepsilon_v$ denotes the volumetric strain, while $\sigma$ is the mean stress which coincides with the applied hydrostatic pressure. Before passing from loading to unloading the pressure was held constant for a short period of time (10 minutes). A volume decrease by creep had been observed. After the material had reached a quasi-stable state (nearly zero decrease of volumetric strain) the unloading was performed. It is worthwhile to point out that in this way the viscous effects have been separated from unloading and therefore the obtained unloading curve is elastic. From its slope one can evaluate accurately the bulk modulus $K$. Figure 1 shows that bulk modulus increases as pressure is increased. This nonlinear behavior can be attributed to the presence of pores.

Since the pores close as pressure is raised, at sufficiently high pressure, when most of the pores are closed, one can expect that further increase in pressure does not cause additional change in slope. An exponential type of law of the form

$$K(\sigma) = K^\infty - P_a \exp \left( -b \frac{\sigma}{P_a} \right)$$

(4)

seems to fit well the data. In Eq (4) $K^\infty$ is the constant value towards which the bulk modulus tends for high pressures; $P_a$ is a reference pressure expressed in the same units as $K^\infty$ and the mean stress $\sigma$; $a$ and $b$ are dimensionless parameters. For alumina the numerical values of these parameters are: $a=10^5$, $b=-1.219 \times 10^{-4}$, $K^\infty=1 \times 10^7$ kPa, $P_a=1$ kPa.

The test shows that alumina powder exhibits time-dependent properties (see Figure 1). The question which arises is whether the dominant properties of alumina are visco-elastic or viscoplastic. In other words, are the time-dependent properties influenced by the loading history? In order to answer to this question, three additional hydrostatic tests with step-wise increase of pressure have been performed (Figure 2). The results obtained are revealing. For a stress path consisting of successive stress increments of equal magnitude, the smaller those increments are the larger the deformation is. It is clear that only an elastic/viscoplastic model can describe correctly the material behavior.

![Figure 1](image1.png)

**Fig. 1** Volumetric creep and compressibility of alumina powder. Partial unloadings are used to determine the bulk modulus.

![Figure 2](image2.png)

**Fig. 2** Influence of the loading history on the hydrostatic response.
2.2 Drained triaxial compression tests

The realization of drained triaxial compression tests is presently well mastered in the laboratory. The cylindrical specimen was contained in a latex membrane to isolate it from direct contact with the surrounding water with which the testing cell is filled. The sample sat in the cell between a rigid base and a rigid top cap which is loaded by means of a ram passing out of the cell. The tests were run at a constant displacement rate of 0.1 mm/min. The rigid base is porous so that the pore fluid can drain from the sample. The quantities measured during any test were: the pressure in the cell fluid, the cell pressure \( \sigma_0 \) (which provides an all-around pressure on the lateral surface of the sample), the axial force applied to the loading ram, the change in length of the sample and the change in the volume of the sample. First the hydrostatic pressure was raised to a predetermined level, and afterwards \( \sigma_0 \) was kept constant while the axial stress \( \sigma \) was further increased. Drained compression tests were carried out for 6 different pressures in the range 98-490 kPa. Three experiments were done for each confining pressure. From these experiments one can determine the elastic properties and draw conclusions about the deformation mechanism and the failure strength of the material.

The practical measurement of the Young modulus is not as easy as it seems because the test results can be strongly influenced by the rheological behavior of the material. This problem can be overcome by performing fast experiments where viscous effects have no chance to interfere. Since dynamic tests are difficult to perform, in most cases the elastic properties are measured from experiments with unloading-reloading cycles. However, during unloading-reloading cycles significant hysteresis loops are usually observed. A rather good estimation of the real value of \( E \) can be obtained from the slope of the middle part of the unloading curve which is generally nearly linear. The determination of the \( E \) modulus from such tests yields much smaller values than in dynamic tests. Nevertheless, these reduced values are often used in model computations and in practical applications and thus misleading results may be obtained.

An experimental procedure which allows the experimentator to determine more accurately the elastic properties by static methods has been proposed by one of the authors (see [4]). Thus, before passing from loading to unloading one has to keep the deformation constant for a certain time interval in order to permit the material to reach by relaxation a quasi-stable state. In this way the hysteresis loop can be practically eliminated. Figure 3 shows the axial and radial strains as functions of the octahedral shear stress \( \tau \):

\[
\tau = \frac{\sqrt{3}}{3} (\sigma_1 - \sigma_0)
\]

obtained in such test (\( \sigma_0 = 392 \) kPa). Note that the stress-strain curves are strongly non-linear. Five unloading-reloading cycles were performed. Before passing to unloading, the axial deformation was kept constant and the time dependent evolution of the axial stress \( \sigma \) was measured. A significant stress relaxation was recorded within minutes. At each stress level about half of the stress reduction shown in Figure 3 is due to relaxation. From the unloading slopes we were able to evaluate the Young modulus \( E \). A dependence of this modulus on the mean stress is observed. To account for this variation we propose the following expression

\[
E(\sigma) = E^0 - \rho \beta \exp(-d\sigma)
\]

where \( E^0 \) is a constant limiting value that \( E \) approaches as pressure increases while \( \beta \), and \( d \) are dimensionless constants. For alumina \( E^0 = 7 \times 10^6 \) kPa, \( \beta = 6.95 \times 10^6 \) and \( d = 0.002 \). Figure 3 also shows the developing of irreversible deformation and the steadily increasing elastic limit for alumina. Note also that first the volume decreases as \( \tau \) increases. Thus, additional compaction is obtained in the deviatoric part of the test. However, as the applied load is further increased, the volume starts to expand (dilatancy). This phenomenon was observed for all the tests on alumina. A comparison of all volumetric profiles is shown in Figure 4. For each confining pressure, the intersection of the \( \tau - \varepsilon_r \)
Influence of confining pressure on the volumetric behavior. Arrows show the passage from compressibility to dilatancy.

curve with the horizontal axis represents the volumetric strain reached at the end of the hydrostatic part of the test. It is clearly seen that the stress value for which the slope of the \( r - \varepsilon_v \) curve changes sign depends on the confining pressure. These critical points can be used to define in the \( \sigma-r \) plane the boundary between the compressible and dilatant domains (Cristescu [4,5]). For alumina this boundary can be expressed as

\[
X(\sigma, r) = 0.555\sigma - r
\]

(7)

Stress-states for which \( X > 0 \) will produce compressibility, while stress-states for which \( X < 0 \) produce dilatancy. Eq. (7) approximates this boundary for the range of pressures considered in our tests. For higher pressures this boundary may be nonlinear. The exact determination of the compressibility/dilatancy boundary is of great importance for industrial applications. For a given confining pressure, it provides the exact amount of axial stress to be applied in order to get maximal compaction.

2.3 Relaxation and rate-effect experiments

For each confining pressure relaxation tests have also been performed. The results obtained in such a test are shown in Figure 5. After a preloading up to \( r = 187.6 \) kPa, the relative movement of the endfaces of the specimens is suddenly stopped (\( \varepsilon_1 = 0 \)). Then, the variation of the axial stress under constant deformation and constant lateral pressure is recorded. The axial stress decreases quite fast. After 15 minutes the relaxation practically ceases. The stress reduction is about 18% of the stress before relaxation. Afterwards loading is further performed. This cycle (loading-relaxation-reloading) is repeated eight times.

The fast relaxation observed for alumina is of great significance for industrial problems since it may influence both the compressibility and dilatancy characteristics of the powder. When pressing a powder in a die, for instance, one has to consider that the stresses may relax thus producing a change in the density distribution in the compact.

An experimental study of the influence of the strain rate on the mechanical properties of alumina has also been done. Figure 6 shows the stress-strain curves obtained for two different axial strain rates (of \( 3.3 \times 10^{-4} \) s\(^{-1} \) and, \( 1.1 \times 10^{-5} \) s\(^{-1} \)). Though the two strain rates are not significantly different, the influence of the strain rate on the mechanical response of the alumina powder is obvious. The \( r - \varepsilon_1 \) and \( r - \varepsilon_2 \) curves for the higher rate are above the ones obtained for the slower rate. Also, the curves \( r - \varepsilon_v \).
show that slower rates produce more compressibility and more dilatancy. This aspect is characteristic for powders and may also be important for industrial applications.

From the experimental results presented, we can conclude that the material exhibits irreversible time-dependent properties. Thus, in modelling the material behavior an elastic/viscoplastic approach is adopted. The elastic/viscoplastic constitutive equation which will be presented in the next section, is applicable to general 3-D stress conditions and can describe the loading-history effects. The parameters involved can be obtained easily from the results of a series of conventional drained triaxial compression tests. Compared to other available theories, this model has an improved capability to describe several major aspects of the mechanical behavior of powders including the shear-dilatancy effects, stress-path effects, time-dependent effects.

3. Development of elastic/viscoplastic model for alumina powder

The development of the elastic/viscoplastic constitutive equation, applicable to powders under general 3-D stress conditions is based on the concepts of the viscoplasticity theory. In order to formulate an elastic/viscoplastic model three basic ingredients are required:

1) Yield function;
2) Flow rule;
3) Hardening rule.

The yield surface does not constitute a limit on possible stress states. The viscous effects are related to the stress exceeding the current yield limit. The distance between this stress-state and the current yield surface is called overstress. The concept of overstress is due to Ludwik [10].

The flow rule relates the viscoplastic strain rates to the stresses. The current yield surface is not fixed but its position depends on the amount of hardening experienced by the material. Also, a short-term failure surface defining the boundary of the constitutive domain, is to be defined.

In the model the irreversible stress work per unit of volume $W$ is used as hardening parameter:

$$ W(T) = \int_0^T \sigma(t) : \dot{\varepsilon}^i(t) \, dt $$

where $\dot{\varepsilon}^i$ represents the irreversible strain rate and $\sigma$ the stress tensor. $W$ captures yielding in terms of shear strains as well as volumetric strains. It can be decomposed in two terms:

$$ W(T) = \int_0^T \sigma(t) : \dot{\varepsilon}^i(t) \, dt + \int_0^T \sigma'(t) : \varepsilon'(t) \, dt \quad (9) $$

In Eq. (9) "prime" means deviator and the "·" denotes the tensorial product. The second right-hand side term is always positive and represents the energy related to the change in shape. The first right-hand side term is the energy related to volume change. It increases in the compressibility domain and decreases in the dilatancy one. Thus the irreversible volumetric work can be thought as a damage indicator.

The general form of the elastic/viscoplastic constitutive equation proposed is

$$ \varepsilon = \frac{\sigma}{2G} + \left[ \frac{1}{3K} - \frac{1}{2G} \right] \sigma I + k \langle \frac{1}{H(\sigma)} \rangle N(\sigma) \quad (10) $$

(see Cristescu [4, 5, 6] and Cazacu [2]). In Equation (10) $\varepsilon$ is the rate of total strain, $G$ and $K$ are elastic moduli, $k$ is a viscosity parameter while the symbol $< >$, known as Macaulay bracket, is used to denote the positive part of a function (i.e. $< A > := (1/2)(A + |A|)$). $H(\sigma)$ is the yield function, whereas $N(\sigma)$ is a tensor valued function which defines the orientation of the viscoplastic strain rates.

Let us observe that in contrast to the classical viscoplasticity models no assumption concerning the existence of a viscoplastic potential is done.

There are two main topics to be addressed: the determination of the yield function $H(\sigma)$ and that of the viscoplastic strain rate orientation tensor $N(\sigma)$.

3.1 Determination of the yield function $H(\sigma)$

As the material is isotropic the yield function $H(\sigma)$ depends on stress invariants only. For sake of simplicity, we assume that $H(\sigma)$ depends on the mean stress

$$ \sigma = \frac{1}{3} \langle tr \sigma \rangle \quad (11) $$

and on the octahedral shear stress $r$, only

$$ r = \sqrt{\frac{1}{3} \langle \sigma^2 \rangle : \sigma \cdot \sigma \rangle \quad (12) $$

The yield function $H(\sigma)$ can be determined from data obtained in drained triaxial compression tests. Since in such tests, the first stage is hydrostatic and the second one is deviatoric, we assume that $H(\sigma)$ is the sum of two terms:

$$ H(\sigma) = H_s(\sigma) + H_d(\sigma, r) \quad (13) $$

such that $H_s(\sigma, 0) = 0$. Thus, for hydrostatic conditions the yield function reduces to $H_s$.

The procedure to determine $H_s$ is the following: first compute the irreversible stress work in a hydrostatic creep test (as the one described in the...
Since the stresses are equal in all directions the irreversible stress work reduces to:

\[ W(T) = \int_0^T \sigma(t) \dot{\epsilon}_i'(t) \, dt \]

where \( \dot{\epsilon}_i' \) is the irreversible volumetric rate of deformation, and \( \sigma \) — the mean stress — is equal to the applied pressure. Next, plot the obtained values of \( W(T) \) at creep-stabilization as a function of \( \sigma \).

The irreversible stress work increases monotonically as the pressure increases. A second order polynomial matches well the data for alumina. Thus:

\[ H_h(\sigma) = a_h \left( \frac{\sigma}{p_a} \right)^2 + b_h \left( \frac{\sigma}{p_a} \right) \]

where \( a_h = 5.833 \times 10^{-6} \) and \( b_h = 1 \times 10^{-6} \).

Further, \( H_h(\sigma, \tau) \) can be determined from the deviatoric part of the compression tests. The irreversible stress work is computed using the formula:

\[ W_d(T) = \int_0^T \left( \frac{3}{2} \tau + \sigma \dot{\epsilon}_i' \right) \, dt + \int_\tau \dot{\sigma} \dot{\epsilon}_i' \, dt \]

where \( T_\tau \) represents the beginning of the deviatoric part of the test, \( \dot{\epsilon}_i' \) the axial irreversible rate of deformation.

Figure 7 shows the data for irreversible stress work obtained from 3 triaxial compression tests on alumina with confining pressures of 294, 392 and 490 kPa, respectively. A possible function which approximates accurately the irreversible work contours is:

\[ H_d(\sigma, \tau) = A \left( \frac{\tau}{p_a} \right)^9 + B \left( \frac{\tau}{p_a} \right)^2 \]

It was found that \( A \) depends on the confining pressure and can be expressed as:

\[ A(\sigma_0) = a_1 \left( \frac{\sigma_0}{p_a} + a_2 \right)^{-m} \]

where \( a_1 = 0.27, a_2 = 37 \) and \( m = 8.5 \). \( B \) can be considered to be constant, i.e., \( B = 4.6 \times 10^{-5} \). Since \( \sigma_0 = \sigma - \tau/\sqrt{3} \) it follows that the expression of the yield function in terms of invariants is:

\[ H(\sigma, \tau) = a_h \left( \frac{\sigma}{p_a} \right)^2 + b_h \left( \frac{\sigma}{p_a} \right) + a_1 \left( \frac{\tau}{p_a} \right)^9 + B \left( \frac{\tau}{p_a} \right)^2 \]

Figure 8 shows in a \( \sigma-\tau \)-plane the shape of several surfaces \( H = \text{constant} \). The short-term failure surface is represented as full line while the compressibility/dilatancy boundary is dotted line. The line \( \partial H/\partial \sigma = 0 \) is shown as dotted line. In an associated model this last line would coincide with the compressibility/dilatancy boundary. For a saturated powder the domain between the compressibility/dilatancy boundary and the \( \partial H/\partial \sigma = 0 \) line is a domain of possible loose of stability (Cristescu [3]). The domain between the compressibility/dilatancy boundary and the short-term failure surface is a dilatancy domain. All stress-states under the compressibility/dilatancy boundary produce compaction. The model reproduces accurately the shear-dilatancy effect. From this figure one can conclude that shear (i.e., \( \tau \)) superposed on a hydrostatic pressure (i.e., \( \sigma \)) produces additional compaction. In fact for each value of \( \sigma \) the maximal compaction is obtained for the corresponding value of \( \tau \) located on the compressibility/dilatancy boundary.

### 3.2 Determination of the viscoplastic strain rate orientation tensor \( N(\sigma) \)

In order to describe the orientation of the irreversible strain increments one has to determine
the tensorial function \( N(\sigma) \). As the material is isotropic \( N \) must satisfy the invariance requirement

\[
N(Q^T \sigma Q) = Q N(\sigma) Q^T
\]

for any orthogonal transformation \( Q \). From classical theorems of representation of isotropic tensor functions follows that \( N \) can be represented as

\[
N(\sigma) = N_I I + N_2 \sigma + N_3 \sigma^2
\]

where \( N_i \) are scalar valued functions of the stress invariants. In order to reduce the complexity of the problem we assume that \( N_3 = 0 \) and also we disregard the influence of the third stress invariant. Thus, we consider that \( N(\sigma) \) is of the form

\[
N(\sigma) = N_I I + N_2 \sigma
\]

where \( \sigma' \) is the deviatoric stress and \( N_I = N_I(\sigma, \tau) \). Since

\[
\varepsilon' = k \left( 1 - \frac{W}{H(\sigma)} \right) (N_I I + N_2 \sigma') \tag{23}
\]

it follows that

\[
\varepsilon' = 3k \left( 1 - \frac{W}{H(\sigma)} \right) N_I \tag{24}
\]

Afterwards, using Eq. (23), from the data obtained during the hydrostatic part of the test we determine \( k N_I |_{\sigma=0} \). We use the notation \( k N_I |_{\sigma=0} = \varphi(\sigma) \). A second order polynomial in \( \sigma \) seems to fit well the data. Thus

\[
k N_I |_{\sigma=0} = \alpha \sigma^2 + \beta \sigma \tag{25}
\]

where \( \alpha = -8 \times 10^{-11} \) and \( \beta = 1.641 \times 10^{-4} \). Next, we must determine \( N_I(\sigma, \tau) \) for \( \tau \neq 0 \). From Eq. (23) follows that this function must be positive in the compressibility domain and negative in the dilatancy one.

The simplest possible function satisfying these properties is

\[
k N_I = \varphi(\sigma) + \tau \frac{\sigma}{\rho_a} \text{sgn} X(\sigma, \tau) \Psi(\sigma, \tau) \tag{26}
\]

where \( X(\sigma, \tau) \) is defined by Eq. (7). In Eq. (26) all the functions, except \( \Psi(\sigma, \tau) \), have been already determined. By making use of Eq. (23), from Eq. (26) results

\[
\Psi(\sigma, \tau) = \zeta_c \left[ \frac{\tau}{\rho_a} - \gamma_a \left( \frac{\sigma - \tau}{\rho_a} \right)^3 \right] - \gamma_c \tag{27}
\]

\[
\cdot \exp \left( \frac{z_0 - \sqrt{\tau}}{\rho_a} \right)
\]

where \( \gamma_a = 0.842, \gamma_c = -56.06, \zeta_0 = -0.01 \) and \( z_0 = 1.127 \times 10^{-11} \).

From the flow rule Eq. (22) follows that

\[
\dot{\varepsilon}' = k \left( 1 - \frac{W}{H(\sigma)} \right) \left( N_I(\sigma, \tau) + \frac{N_2(\sigma, \tau)}{\rho_a} \right) \tag{28}
\]

\[
\dot{\varepsilon}' = k \left( 1 - \frac{W}{H(\sigma)} \right) \left( N_I(\sigma, \tau) + \frac{N_2(\sigma, \tau)}{\rho_a} \right)
\]

By subtracting Eq. (28) from Eq. (28), we obtain

\[
k N_2(\sigma, \tau) = \frac{3}{\sqrt{2}} \frac{\varepsilon_2 - \dot{\varepsilon}_2}{W} \tag{29}
\]

Where \( \varepsilon_2 = \frac{3}{\sqrt{2}} \dot{\varepsilon}_2 \). From the deviatoric part of the same triaxial tests performed at \( \sigma_3 = 294 \), 392 and 490 kPa, in conjunction with Eq. (29) we found that \( k N_2(\sigma, \tau) \) can be approximated by

\[
k N_2(\sigma, \tau) = m_1 \left( \frac{\tau}{\rho_a} \right)^4 \exp \left[ m_2 \left( \frac{\sigma - \tau}{\rho_a} \right) \right] + m_3 \left( \frac{\tau}{\rho_a} \right)^2 \exp \left[ m_4 \left( \frac{\sigma - \tau}{\rho_a} \right)^2 \right] + m_5 \tag{30}
\]

where \( m_1 = 2.38 \times 10^{-33}, m_2 = -0.038, m_3 = 4 \times 10^{-7}, m_4 = -0.005 \) and \( \tau = 8 \times 10^{-4} \).

4. Comparison with the experimental results

The accuracy of the elastic/viscoplastic model described previously was evaluated by comparing predicted and measured strains in drained compression tests at different confining pressures. Figure 9 shows a comparison between model prediction and experimental results for \( \sigma_3 = 294 \) kPa. Let us note that this test has not been used for the identification of the constitutive parameters. The axial and radial strains are predicted fairly accurately, both in magnitude and variation. The calcu-
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Nomenclature

\begin{align*}
C & : \text{Cohesion} \quad [\text{Pa}] \\
E & : \text{Young's Modulus} \quad [\text{kPa}] \\
E^\infty & : \text{limit value of } E \quad [\text{kPa}] \\
G & : \text{shear modulus} \quad [\text{kPa}] \\
H(\sigma) & : \text{yield function} \quad [\text{kPa}] \\
K & : \text{bulk modulus} \quad [\text{kPa}] \\
K^\infty & : \text{limit value of } K \quad [\text{kPa}] \\
N(\sigma) & : \text{viscoplastic strain rate orientation} \quad [-] \\
p_\sigma & : \text{mean stress} \quad [\text{kPa}] \\
p_a & : \text{reference pressure} \quad [1\text{kPa}] \\
q & : \text{equivalent stress} \quad [\text{kPa}] \\
\nu & : \text{current volume} \quad [\text{m}^3] \\
\nu_0 & : \text{initial volume} \quad [\text{m}^3] \\
W(t) & : \text{irreversible stress work per unit volume} \quad [\text{Pa}] \\
X(\sigma,\epsilon) & : \text{compressibility/dilatancy boundary function} \quad [\text{Pa}] \\
\epsilon' & : \text{strain deviator tensor} \quad [-] \\
\epsilon_q & : \text{equivalent strain} \quad [-] \\
\epsilon_v & : \text{volumetric strain} \quad [-] \\
\sigma' & : \text{stress deviator tensor} \quad [\text{Pa}] \\
\sigma_n & : \text{normal stress on the failure plane} \quad [\text{Pa}] \\
\sigma_1 & : \text{axial stress} \quad [\text{Pa}] \\
\sigma_3 & : \text{confining pressure} \quad [\text{Pa}] \\
\tau & : \text{octahedral shear stress} \quad [\text{Pa}] \\
\tau_f & : \text{shear stress on the failure plane} \quad [\text{Pa}] \\
\phi & : \text{angle of internal friction} \quad [-]
\end{align*}

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