THE NON-PARABOLICITY OF INFINITE VOLUME ENDS

M. P. CAVALCANTE, H. MIRANDOLA, AND F. VITÓRIO

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Abstract. Let $M^m$, with $m \geq 3$, be an $m$-dimensional complete non-compact manifold isometrically immersed in a Hadamard manifold $\bar{M}$. Assume that the mean curvature vector has finite $L^p$-norm, for some $2 \leq p \leq m$. We prove that each end of $M$ must either have finite volume or be non-parabolic.

1. Introduction

Let $(M^m, \langle , \rangle)$ be a complete non-compact Riemannian manifold without boundary. We recall that $M$ is parabolic if it does not admit a non-constant positive superharmonic function. Otherwise, it is said to be non-parabolic. There exist equivalent definitions for parabolic manifolds (see for instance Theorem 5.1 of [8]). Let $E \subset M$ be an end of $M$, that is, an unbounded connected component of $M - \overline{\Omega}$, for some compact subset $\Omega \subset M$. The property of parabolicity can be localized on each end of $M$. Namely, we say that an end $E$ is parabolic (see Definition 2.4 of [10]) if it does not admit a harmonic function $f : E \to \mathbb{R}$ satisfying:

1. $f|_{\partial E} = 1$;
2. $\liminf_{y \to \infty} f(y) < 1$.

Otherwise, we say that $E$ is a non-parabolic end of $M$. It is well known that $M$ is non-parabolic if and only if it admits a non-parabolic end. Furthermore, ends with finite volume are parabolic (see for instance Section 14.4 of [8]). In this direction we recall the following result due to Li and Wang:

**Theorem A** (Corollary 4 of [12] and Corollary 2.9 of [10]). Let $E$ be an end of a complete manifold. Suppose that, for some constants $\nu \geq 1$ and $C > 0$, $E$ satisfies a Sobolev-type inequality of the form

$$\left( \int_E |u|^{2\nu} \right)^{\frac{1}{\nu}} \leq C \int_E |\nabla u|^2,$$

for all compactly supported Sobolev function $u \in W^{1,2}_c(E)$. Then $E$ must either have finite volume or be non-parabolic. Moreover, in the case $\nu > 1$, $E$ must be non-parabolic.

Note that if a complete manifold $M$ that satisfies a Sobolev inequality as in Theorem A with $\nu = 1$ (that is just the Dirichlet Poincaré inequality), then the first eigenvalue $\lambda_1(M)$ of the Laplace-Beltrami operator is positive; hence $M$ must
be non-parabolic (see Proposition 10.1 of [8]). Example 1.1 below exhibits a complete manifold that contains a finite volume end and that also satisfies a Sobolev inequality as in Theorem A with $\nu = 1$.

Cao, Shen and Zhu [2] showed that if $M^m$, with $m \geq 3$, is a complete manifold, then each end of $M$ is non-parabolic provided that $M$ can be realized as a minimal submanifold in a Euclidean space $\mathbb{R}^n$. The same conclusion also was obtained by Fu and Xu [7] provided that there exists an isometric immersion of $M$ in a Hadamard manifold $\bar{M}$ with finite total mean curvature, that is, the mean curvature vector field $H$ of the immersion satisfies $\|H\|_{L^\infty(M)} < \infty$. In both cases, they observed that $M$ admits a Sobolev-type inequality as in Theorem A with $\nu > 1$.

Our main result states the following:

**Theorem 1.1.** Let $x : M^m \rightarrow \bar{M}$, with $m \geq 3$, be an isometric immersion of a complete non-compact manifold $M$ in a Hadamard manifold $\bar{M}$. Let $E$ be an end of $M$ such that the mean curvature vector satisfies $\|H\|_{L^p(E)} < \infty$, for some $2 \leq p \leq m$. Then $E$ must either have finite volume or be non-parabolic.

Example 1.3 below exhibits an example of a complete non-compact hypersurface $M^m$ in $\mathbb{R}^{m+1}$, with $m \geq 3$, of finite volume and mean curvature vector with finite $L^p$-norm, for all $2 \leq p < m - 1$. This example shows that Theorem 1.1 is not a consequence of Theorem A (except when $p = m$). Note also that the catenoids in $\mathbb{R}^3$ are parabolic minimal surfaces whose ends have infinite area, which shows that the hypothesis $m \geq 3$ is essential.

In the present paper we also give a unified proof of the following fact:

**Theorem B.** Let $x : M \rightarrow \bar{M}$ be an isometric immersion of a complete non-compact manifold $M$ in a manifold $\bar{M}$ with bounded geometry (i.e., $M$ has sectional curvature bounded from above and injectivity radius bounded from below by a positive constant). Let $E$ be an end of $M$ and assume that the mean curvature vector of $x$ satisfies $\|H\|_{L^p(E)} < \infty$, for some $m \leq p \leq \infty$. Then $E$ must have infinite volume.

The fact above was proved by Frensel [4] and by do Carmo, Wang and Xia [3] for the case where the mean curvature vector field is bounded in norm (the case $p = \infty$), by Fu and Xu [7] for the case where the total mean curvature is finite (the case $p = m$) and by Cheung and Leung [1] for the case where the mean curvature vector has finite $L^p$-norm for some $p > m$. Since the cylinders of the form $M^m = S^{m-1} \times \mathbb{R}$, where $S^{m-1}$ is the unit Euclidean $(m - 1)$-dimensional sphere, are examples of complete parabolic hypersurfaces in $\mathbb{R}^{m+1}$ we conclude that boundedness of the mean curvature vector does not imply that $M$ admits a Sobolev-type inequality. Furthermore, for all $m \geq 3$, we exhibit an example of a parabolic complete non-compact hypersurface $M^m$ in $\mathbb{R}^{m+1}$ such that the mean curvature vector has finite $L^p$-norm, for all $p > 2(m - 1)$. These examples show that Theorem B is not a consequence of Theorem A.

Two questions arise in this paper: is there an example of a complete non-compact submanifold $M^m$, with $m \geq 3$, in a Euclidean space satisfying one of the conditions below?

1. $M$ has finite volume and $\|H\|_{L^p(M)} < \infty$, for some $m - 1 \leq p < m$;
2. $M$ is parabolic and $\|H\|_{L^p(M)} < \infty$, for some $m < p \leq 2(m - 1)$.
2. Proof of Theorem 1.1

Choose \( r_0 > 0 \) so that the geodesic ball \( B_{r_0} \subset M \) of radius \( r_0 \) and center at some point \( \xi_0 \in M \) satisfies \( \partial E \subset B_{r_0} \). For each \( r > r_0 \), consider \( E_r = E \cap B_r \) and let \( f_r : \overline{E}_r \rightarrow \mathbb{R} \) be a solution of the Dirichlet Problem:

\[
\begin{cases}
    \Delta_M f_r = 0 & \text{in } E_r, \\
    f_r = 1 & \text{in } \partial E, \\
    f_r = 0 & \text{on } E \cap \partial B_r.
\end{cases}
\]

It follows from the maximum principle that \( 0 < f_r \leq f_s < 1 \) in \( E_r \), for all \( s \geq r \). Hence, by standard gradient estimates it follows that \( \{f_r\} \) is an equicontinuous family which converges uniformly on compact subsets, when \( r \) goes to infinity, to a function \( f : E \rightarrow \mathbb{R} \) satisfying

\[
\begin{cases}
    \Delta_M f = 0 & \text{in } E, \\
    0 \leq f \leq 1 & \text{in } E, \\
    f = 1 & \text{on } \partial E.
\end{cases}
\]

If \( f \not\equiv 1 \), then it follows from the maximum principle that \( \liminf_{x \to E(\infty)} f(x) < 1 \), which shows that \( E \) is non-parabolic. Furthermore, it is well known that an end of finite volume is parabolic (see section 14.4 of [8]). Hence, to prove Theorem 1.1 it is sufficient to show the following:

Claim 2.1. Either \( f \not\equiv 1 \) or \( \text{vol}(E) \) is finite.

Suppose, by contradiction, that \( f \equiv 1 \) and \( \text{vol}(E) \) is infinite. This implies that, given any \( L > 1 \), there exists \( r_1 > r_0 \) such that \( \text{vol}(E_{r_1} - E_{r_0}) > 2L \). Since \( f_r \rightarrow 1 \) uniformly on compact subsets, there exists \( r_2 > r_1 \) such that \( f_r^{2m-2} > \frac{1}{2} \) everywhere in \( E_{r_1} \), for all \( r > r_2 \). Thus, defining \( h(r) = \int_{E_r - E_{r_0}} f_r^{2m-2} \), with \( r > r_0 \), we obtain

\[
(2.1) \quad h(r) \geq \int_{E_r - E_{r_0}} f_r^{2m-2} > L,
\]

for all \( r > r_2 \). In particular, we have that \( \lim_{r \to \infty} h(r) = \infty \).

Now, for each \( r > r_0 \), let \( \varphi = \varphi_r \in C_0^\infty(E) \) be a cut-off function satisfying:

1. \( 0 \leq \varphi \leq 1 \) everywhere in \( E \);
2. \( \varphi \equiv 1 \) in \( E_r - E_{r_0} \).

By the Hoffmann-Spruck Inequality [9] we have

\[
S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_r} |\nabla (\varphi f_r)|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2,
\]

where \( S \) is a positive constant.

Note that

\[
|\nabla (\varphi f_r)|^2 = f_r^2 |\nabla \varphi|^2 + \varphi^2 |\nabla f_r|^2 + \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle
\]

and

\[
\varphi^2 |\nabla f_r|^2 = \text{div}_M ((f_r \varphi^2) \nabla f_r) - \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle,
\]

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since \( f_r \) is harmonic. Using that \( f_r \varphi \) vanishes on \( \partial E_r \) we obtain

\[
S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} \text{div}_M((f_r \varphi^2) \nabla f_r) + \int_{E_r} (\varphi f_r)^2 |H|^2
\]

and

\[
= \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2.
\]

Thus, since \( 0 \leq \varphi \leq 1 \) in \( E \) and \( \varphi \equiv 1 \) in \( E_r - E_{r_0} \), we obtain

\[
S^{-1} h(r)^{\frac{m-2}{m}} \leq S^{-1} \left( \int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} f_r^2 |\nabla \varphi|^2 + \int_M |H|^2.
\]

First, assume that \( \|H\|_{L^p(E)} \) is finite. Then, since \( 0 \leq f_r \leq 1 \), we have

\[
S^{-1} h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_M |H|^2.
\]

Thus, \( \lim_{r \to \infty} h(r) < \infty \), which is a contradiction. Now, assume that \( \|H\|_{L^p(E)} \) is finite, for some \( 2 < p \leq m \). Note that \( \frac{m}{m-2} \leq \frac{p}{p-2} \). Since \( 0 \leq f_r \leq 1 \) and \( h(r) > 1 \), for all \( r > r_2 \), we have:

1. \( f_r^{\frac{2p}{p-2}} \leq f_r^{\frac{2m}{m-2}} \);
2. \( h(r)^{\frac{m-2}{m}} \leq h(r)^{\frac{m-2}{m}} \), for all \( r > r_2 \).

Thus, using the Hölder Inequality, we have

\[
\int_{E_r - E_{r_0}} f_r^2 |H|^2 \leq \|H\|_{L^p(E_r - E_{r_0})}^2 \left( \int_{E_r - E_{r_0}} f_r^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}} \leq \|H\|_{L^p(E - E_{r_0})}^2 h(r)^{\frac{m-2}{m}};
\]

for all \( r > r_2 \).

Choose \( r_0 > 0 \) large enough so that \( \|H\|^2_{L^p(E - E_{r_0})} < \frac{1}{2S} \). Using (2.2) and (2.3) we obtain the following:

\[
S^{-1} h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_{E_{r_0}} |H|^2 + \frac{S^{-1}}{2} h(r)^{\frac{m-2}{m}}.
\]

This shows that \( \lim_{r \to \infty} h(r) < \infty \), which is a contradiction. Therefore, Claim 2.1 and Theorem 1.1 are proved.

### 3. Proof of Theorem B

Since \( \bar{M} \) has bounded geometry, the sectional curvature \( \bar{K} \) and the injectivity radius \( \bar{i}(M) \) of \( M \) satisfy:

\[
\bar{K} < b^2 \text{ and } \bar{i}(\bar{M}) > r_0,
\]

for some positive constants \( b \) and \( r_0 \). Let \( E \) be an end of \( M \) and assume that \( \|H\|_{L^p(E)} \) is finite, for some \( m \leq p \leq \infty \). Fix \( 0 < R_0 < \min\{r_0, \frac{\pi}{2b} \} \), take \( \xi_0 \in M \) and consider \( B_R = B_R(\xi_0) \), for all \( R > 0 \). Choose \( R_1 > R_0 \), sufficiently large, so that \( \partial E \subset B_{R_1} \) and the distance \( d_M(\partial E, x) > R_0 \), for all \( x \) in \( E - B_{R_1} \). Let \( q \in \bar{E} = E - B_{2R_1} \) and \( 0 < R < R_0 \). Since \( B_R(q) \subset E - B_{R_1}(\xi_0) \) we obtain, by
the isoperimetric inequality for submanifolds (Theorem 2.2 of [9]) and the Hölder Inequality, the following:

\[ S \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)) + \|H\|_{L^p(E - B_{R_1}(\xi_0))} \operatorname{vol}(B_R(q))^{\frac{p-1}{p}} \]

where \( S > 0 \) is a constant that depends only on \( m \).

Assume, by contradiction, that \( \operatorname{vol}(E) \) is finite. Take \( R_1 > 0 \) sufficiently large so that

\[ \operatorname{vol}(E - B_{R_1}) < 1 \quad \text{and} \quad \|H\|_{L^p(E - B_{R_1})} < \frac{S}{2} \]

Since \( p \geq m \) we have that \( \frac{p-1}{p} \geq \frac{m-1}{m} \). By \([9,3]\) and using that \( B_R(q) \subset E - B_{R_1} \), we have that \( \operatorname{vol}(B_R(q))^{\frac{p-1}{p}} \leq \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \). Thus, using \((3.2)\) and \((3.3)\), we obtain

\[ \frac{S}{2} \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)) \]

By the coarea formula, we have that \( \operatorname{vol}(\partial B_R(q)) = \frac{d}{dR} \operatorname{vol}(B_R(q)) \). Using \((3.4)\), we obtain \( \frac{d}{dR} \operatorname{vol}(B_R(q)) \frac{m}{p} \geq \frac{S}{2m} \). This implies that

\[ \operatorname{vol}(B_R(q)) \geq \frac{S}{2m} R^m, \]

for all \( q \in E_1 \) and \( 0 < R < R_0 \).

Since \( M \) is complete and \( E \subset M \) is connected and non-compact, there exists a sequence \( p_2, p_3, \ldots \) in \( E \) such that

\[ p_k \in E \cap (B_{2kR_1} - B_{(2k-1)R_1}) \]

Note that \( B_{R_0/2}(p_k) \subset E - B_{R_1} \) and \( B_{R_0/2}(p_k) \cap B_{R_0/2}(p_{k'}) = \emptyset \), for all \( k \neq k' \). Since

\[ \operatorname{vol}(E) \geq \operatorname{vol}(E - B_{R_1}) \geq \sum_{k=2}^{\infty} \operatorname{vol}(B_{R_0/2}(p_k)), \]

it follows from \((3.3)\) that \( \operatorname{vol}(E) \) is infinite. Theorem B is proved.

4. Examples

**Example 4.1.** Consider the warped product manifold \( M^m = \mathbb{R} \times_{\kappa^t} P \), where \( P \) is any complete \((m - 1)\)-dimensional manifold with finite volume. The metric of \( M \) is complete and the end \( E = (-\infty, 0) \times P \subset M \) has finite volume given by

\[ \operatorname{vol}(E) = \int_{-\infty}^{0} \int_P e^{m-1} dt dP = \frac{\operatorname{vol}(P)}{m - 1}. \]

Fix \( k \in \mathbb{R} \) and let \( h_\kappa : M \to \mathbb{R} \) be the function defined by \( h_\kappa(t, x) = \kappa t \). The gradient vector field of \( h_\kappa \) satisfies

\[ \nabla h_\kappa = \kappa \frac{\partial}{\partial t}, \]

where \( \frac{\partial}{\partial t}(t, x) = \frac{d}{ds} \bigg|_{s=t} (s, x) \in T_{(t, x)} M \). It is simple to show that \( \nabla_Z \frac{\partial}{\partial t} = Z - \langle Z, \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t} \). This implies that the Laplacian of \( h_\kappa \) satisfies

\[ \Delta h_\kappa = \kappa \operatorname{div}(\frac{\partial}{\partial t}) = \kappa (m - 1). \]
Fix $\eta \in C^\infty_0(M)$. Using (4.1) and (4.2) we obtain
\[
\kappa(m - 1)\int_M \eta^2 = \int_M \eta^2 \Delta h_\kappa = \int_M (\text{div}(\eta^2 \nabla h_\kappa) - 2\eta \langle \nabla \eta, \nabla h_\kappa \rangle)
\]
\[
= -2\int_M \langle \nabla \eta, \eta \nabla h_\kappa \rangle \geq -\int_M \left| \nabla \eta \right|^2 - \eta^2 |\nabla h_\kappa|
\]
\[
= -\int_M \left| \nabla \eta \right|^2 - \kappa^2 |\eta|^2.
\]
Thus, it holds that $\int_M |\nabla \eta|^2 + \kappa(\kappa + (m - 1))\eta^2 \geq 0$, for all $k \in \mathbb{R}$. In particular, if we take $\kappa = -\frac{m-1}{2}$ we obtain
\[
\int_M |\nabla \eta|^2 - \frac{(m - 1)^2}{4} \eta^2 \geq 0.
\]
Hence $M$ satisfies a Sobolev inequality as in Theorem A with $\nu = 1$.

**Example 4.2.** Let $f : (-\infty, \infty) \to (0, \infty)$ be a positive smooth function satisfying that $f(t) = f(-t)$ and $f(t) = t^{\frac{1}{m-1}}$, for all $t \geq 1$. Consider the immersion $x : S^{m-1} \times \mathbb{R} \to \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ given by $x(v, t) = (f(t)v, t)$. Consider $M$ the product $S^{m-1} \times \mathbb{R}$ endowed with the metric induced by $x$. The metric of $M$ is given by
\[
(4.3) \quad \langle \cdot, \cdot \rangle_{(v, t)} = (1 + f'(t)^2)dt^2 + f(t)^2 \langle \cdot, \cdot \rangle_v,
\]
where $\langle \cdot, \cdot \rangle_v$ denotes the metric of $S^{m-1}$. Note that $M$ is a complete manifold with two ends.

We claim that $M$ is parabolic. To do this, it is sufficient to prove that the following ends of $M$:
\[
E_+ = (1, \infty) \times S^{m-1} \quad \text{and} \quad E_- = (-\infty, -1) \times S^{m-1}
\]
are parabolic (see Proposition 14.1 of [8]). In fact, we define:
\[
V_+(s) = \text{vol}_M \left( \{ q \in E_+ \mid d(q, \partial E_+) \leq s \} \right)
\]
and
\[
V_-(s) = \text{vol}_M \left( \{ q \in E_- \mid d(q, \partial E_-) \leq s \} \right).
\]
Using (4.3) and that $f(t) = t^{\frac{1}{m-1}}$, for all $|t| \geq 1$, we obtain that
\[V_+(s) = V_-(s) \leq Ds^2,
\]
for some constant $D > 0$ and for all $s \geq 1$. In particular,
\[
\int_{-\infty}^{\infty} \frac{s}{V_+(s)} ds = \int_{-\infty}^{\infty} \frac{s}{V_-(s)} ds = \infty.
\]
This implies that $M$ is parabolic (see section 14.4 of [8]).

We claim that the mean curvature vector $H$ of the isometric immersion $x$ has finite $L^p$-norm, for all $p > m$. In fact, a simple computation shows that
\[
(4.4) \quad mH(x(v, t)) = \frac{(m - 1)}{f(t)^{1 + f'(t)^2}} - \frac{f''(t)}{(1 + f'(t)^2)^{\frac{3}{2}}}.
\]
Using that $f(t) = t^{\frac{1}{m-1}}$, for all $|t| \geq 1$, we obtain that $|H(x(v, t))| \leq Ct^{-\frac{1}{m-1}}$, for some $C > 0$ and for all $x(v, t) \in E_+ \cup E_-$. Thus, we obtain
\[
\int_M |H|^p dM \leq D \int_{-\infty}^{\infty} t^{1 - \frac{p}{m-1}} dt.
\]
for some $D > 0$. This implies that $\|H\|_{L^p(M)}$ is finite when $p > 2(m-1)$.

**Example 4.3.** Let $x : S^{m-1} \times \mathbb{R} \to \mathbb{R}^{m+1}$ be the immersion given by $x(v, t) = (e^{-t^2}v, t)$ and consider $M = S^{m-1} \times \mathbb{R}$ endowed with the metric induced by $x$. The metric of $M$ is complete and the volume of $M$ is given by

$$\text{vol}(M) = \omega_{m-1} \int_{-\infty}^{\infty} (1 + 4t^2 e^{-2t^2})^{\frac{1}{2}} e^{-(m-1)t^2} dt,$$

where $\omega_{m-1}$ is the volume of $S^{m-1}$. This implies that $\text{vol}(M)$ is finite, since the integral $\int_{-\infty}^{\infty} e^{-(m-1)t^2} dt$ is finite and the function $t \in \mathbb{R} \mapsto 1 + 4t^2 e^{-2t^2}$ is bounded. In particular, $M$ is parabolic since it has finite volume (see Theorem 7.3 of [8]).

The mean curvature vector $H$ of the isometric immersion $x$ is given by

$$H(x(v, t)) = h(t) = \frac{2e^{-t^2}(1 - 2t^2)}{m(4t^2 e^{-2t^2} + 1)^{\frac{3}{2}}} + \frac{(m - 1)e^2}{m(4t^2 e^{-2t^2} + 1)^{\frac{1}{2}}}.$$

Using that $\lim_{t \to \infty} e^{-t^2}(1 - 2t^2) = \lim_{t \to \infty} 4t^2 e^{-2t^2} = 0$ we obtain that

$$\lim_{t \to \pm \infty} h(t) e^{-t^2} = \frac{m - 1}{m}.$$

Thus the integral

$$\int_M |H|^p = \omega_{n-1} \int_{-\infty}^{\infty} \left( |h(t)|^p(1 + 4t^2 e^{-2t^2})^{\frac{p}{2}} e^{-(m-1)t^2} \right) dt$$

converges if $0 \leq p < m - 1$ and diverges if $p \geq m - 1$.

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Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, CEP 57072-970, Brazil.

E-mail address: marcos.petrucio@pq.cnpq.br

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, CEP 21945-970, Brasil.

E-mail address: mirandola@im.ufrj.br

Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, CEP 57072-970, Brazil.

E-mail address: feliciano.vitorio@pq.cnpq.br